The Spectrum of an Adelic Markov Operator

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Abstract

With the help of the representation of $SL(2, \mathbb{Z})$ on the rank two free module over the integer adeles, we define the transition operator of a Markov chain. The real component of its spectrum exhibits a gap, whereas the non-real component forms a circle of radius $1/\sqrt{2}$.

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1 Introduction and Setup

In this article we consider unitary representations of $SL(2, \mathbb{Z})$. With $L$ and $R$ representing $\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ respectively $\left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$, we analyze the Markov operator

$$T := \frac{1}{2}(L + R).$$ (1.1)

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As we indicate in this introductory section, its spectrum depends on the representation chosen and controls various number theoretical equipartition rates.

In Section 2 we shall then do the spectral analysis for relevant representations.

The Number-Theoretical Spin Chain

The Euler product and Dirichlet series

\[ Z(s) := \frac{\zeta(s-1)}{\zeta(s)} = \prod_{p \in \mathbb{P}} \frac{1 - p^{-s}}{1 - p} = \sum_{n=1}^{\infty} \varphi(n) n^{-s} \quad (1.2) \]

(with Euler’s \( \varphi \)-function) converge in the half-plane \( \Re(s) > 2 \).

On the discrete abelian group

\[ G := \bigoplus_{N} (\mathbb{Z}/2\mathbb{Z}) \]

with \( \mathbb{Z}/2\mathbb{Z} \cong \{0, 1\}, + \) we define \( h : G \to \mathbb{N} \) by \( h(0) := 1 \) and

\[ h(g_1, \ldots, g_{n-1}, 1, 0, \ldots) := h(g_1, \ldots, g_{n-1}, 0, \ldots) + h(1-g_1, \ldots, 1-g_{n-1}, 0, \ldots). \quad (1.3) \]

Denoting by \( h_k \) the restriction of \( h \) to the subgroup \( (\mathbb{Z}/2\mathbb{Z})^k \cong (\bigoplus_{\ell=1}^k \mathbb{Z}/2\mathbb{Z}) \oplus (\bigoplus_{\ell=k+1}^{\infty} \{0\} ) \) of \( G \), e.g. for \( k = 3 \) one obtains

\[
\begin{array}{cccccccccc}
g & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
h_3(g) & 1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 \\
\end{array}
\]

The Dirichlet series equals

\[ Z(s) = \sum_{g \in G} h(g)^{-s} = \sum_{g \in G} \exp(-sH(g)), \]

with \( H := \ln(h) : G \to [0, \infty) \), see [Kn1]. \( Z \) is approximated by

\[ Z_k(s) := \sum_{g \in (\mathbb{Z}/2\mathbb{Z})^k} h_k(g)^{-s} = \sum_{g \in (\mathbb{Z}/2\mathbb{Z})^k} \exp\left(-sH_k(g)\right) \quad (s \in \mathbb{C}; \ k \in \mathbb{N}_0), \]

for \( H_k := \ln(h_k) \). Writing \( Z_k(s) = \sum_{n=1}^{\infty} \varphi_k(n)n^{-s} \) defines the \( \varphi_k : \mathbb{N} \to \mathbb{N}_0 \).

Since \( \varphi_k \leq \varphi \), but \( Z \) has a pole at 2, \( Z_k \) absolutely converges to \( Z \) exactly in the half plane \( \Re(s) > 2 \).

Although the Pontryagin dual \( \hat{G} \) of the direct sum \( G \) is the compact direct product \( \prod_{N} (\mathbb{Z}/2\mathbb{Z}) \), it is impossible to define the Fourier transform of \( H \) in the sense of locally compact abelian groups, since \( H \) is not bounded, and \( \ell^p(G) \subseteq \)
$C^\infty(G)$ for $1 \leq p < \infty$. However, the Fourier transform of $-H$ exists in the sense
\[
j : G \setminus \{0\} \to \mathbb{R}, \quad j(t) := - \lim_{k \to \infty} 2^{-k} \sum_{g \in (\mathbb{Z}/2\mathbb{Z})^k} H_k(g) (-1)^{(g,t)}
\]
(in fact it can be defined in the same way on $\hat{G} \setminus \{0\}$, but then vanishes outside $G \subseteq \hat{G}$).

1.1 Remark (Statistical mechanics interpretation) In the language of statistical mechanics and thermodynamic formalism (see, e.g. Ruelle [Ru]), $Z$ is the partition function of an infinite spin chain with energy function $H$. Similarly, $Z_k$ is the partition function of $k$ spins.

The interaction $j$ has the following properties (see [Kn1, Kn2, GK]):

- it is asymptotically translation invariant,
- of long range (leading to phase transition),
- and is non-negative.

The last property is called ferromagnetism. Ferromagnetic spin systems are known to obey many specific inequalities.

For ferromagnetic Ising systems (that is, spin systems that unlike our $j$ possesses only pair interaction), by the Lee-Young Theorem the partition function zeroes are on the imaginary line of the complex plane of magnetization.

Since $\varphi(n) = |\{k \in \{1, \ldots, n\} \mid \gcd(k, n) = 1\}| \ (n \in \mathbb{N})$, we can consider the Dirichlet series $Z$ as a sum over $\Lambda := \{(a, b) \in \mathbb{Z}^2 \mid \gcd(a, b) = 1\}$:
\[
Z(s) = 1 + \sum_{(a, b) \in \Lambda \cap \mathbb{N}^2} (a + b)^{-s} \quad (\Re(s) > 2).
\]

But with the unitary representation of $\text{SL}(2, \mathbb{Z})$ on the Hilbert space $\ell^2(\Lambda)$ we may also write
\[
Z(s) = 1 + \sum_{k=0}^{\infty} \langle (2T) \cdot (\cdot)^{-s} \rangle_{\ell^2(\Lambda)},
\]
with $T$ generally defined in (1.1). This first representation is analyzed in Subsection 2.3.
Twisting the Dirichlet Series

For the Dirichlet series $Z$ the half-plane of convergence is not larger than the half-plane \( \{ \Re(s) > 2 \} \) of absolute convergence. To have a chance to look into the critical strip, we now twist the Euler product of $Z$ in (1.2) to obtain a Dirichlet series 

$$
\tilde{Z}(s) := \prod_{p \in \mathbb{P}} \frac{1 + p^{-s}}{1 + p^{1-s}} = \sum_{n=1}^{\infty} \lambda(n) \varphi(n) n^{-s} = \frac{\zeta(s) \zeta(2(s - 1))}{\zeta(s - 1) \zeta(2s)} \quad (\Re(s) > 2)
$$

(1.5)

with the Liouville function, given by $\lambda(p_{i_1}^{a_1} \cdot \ldots \cdot p_{i_k}^{a_k}) = (-1)^{a_1+\ldots+a_k}$ for $p_i \in \mathbb{P}$. This has the following properties:

- Of the four zeta functions appearing in (1.5), only $s \mapsto \zeta(s - 1)$ is not absolutely convergent for $\Re(s) > 3/2$.
- The pole of $\zeta$ at $s = 1$, leading to the pole of $Z$ at $s = 2$, gives rise to $\tilde{Z}(2) = 0$.
- The non-trivial zeros of $\zeta$, shifted by 1 for $Z$, now turn into poles of $\tilde{Z}$.
- $\tilde{Z}$ has an additional pole at $3/2$.

So the Dirichlet series $\tilde{Z}$ converges in the half-plane $\{ s \in \mathbb{C} \mid \Re(s) > s_0 + 1 \}$ if and only if there are no zeros of $\zeta$ with real part strictly larger than $s_0 \geq 1/2$.

Instead of considering convergence of the partial sums 

$$
\hat{Z}_N(s) := \sum_{n=1}^{N} \lambda(n) \varphi(n) n^{-s} \quad (s \in \mathbb{C}),
$$

(1.6)

to $\hat{Z}(s) = \sum_{g \in G} \lambda \circ h(g) h(g)^{-s}$, one may also look at the convergence of 

$$
\tilde{Z}_k(s) := \sum_{g \in (\mathbb{Z}/2\mathbb{Z})^k} \lambda \circ h_k(g) h_k(g)^{-s} = \sum_{n=1}^{\infty} \lambda(n) \varphi_k(n) n^{-s}.
$$

(1.7)

If we set $N \equiv N(k) := \lfloor \pi \sqrt{2k/3} \rfloor$ in (1.6), then both sums include asymptotically the same number of terms $n^{-s}$ (since $\sum_{n=1}^{N} \varphi(n) \sim 3/\pi^2 N^2$).

The numerics of their moduli (see Figure 1.1, and [Kn3]) suggests that if there is convergence in the half-plane $\Re(s) > 3/2$ (that being equivalent to the truth of RH), the convergence of $\tilde{Z}_k$ may be even better than for $\tilde{Z}_N(k)$.

$\tilde{Z}_k$ numerically converges better, since it has better self-averaging properties than $\hat{Z}_N$ (which is of course a very non–smooth truncation of $Z$).

A heuristic reason for such a supposed convergence of $\tilde{Z}_N$ is to compare the terms $\lambda(n)$ appearing in $\tilde{Z}_N$ to i.i.d. random variables which take the values $\pm 1$
Figure 1.1: Bottom graph: Modulus of the twisted Dirichlet function $\tilde{Z}$ in its critical (half-) strip $\Re(s) \in [3/2, 2]$. The real (respectively imaginary) part of $s$ is plotted on the $y$-axis ($x$-axis). Poles (shifted Riemann zeroes) of $\tilde{Z}$ are located at the points $3/2 + 14.1 i$ and $3/2 + 21.0 i$ of the abscissa. The approximants $\tilde{Z}_k$ (middle) respectively $\hat{Z}_N(k)$ (up) of $\tilde{Z}$, for $k = 25$.

with equal probability $\frac{1}{2}$. For the case of $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ a well-known similar heuristic goes back to Denjoy (1931), and is described in Section 12.3 of Edwards [Ed].

Although that is obviously absurd in the literal sense, we show in this article that there is some truth to the argument. In the case $\tilde{Z}_k$ we thus consider the ensembles $g \mapsto \lambda \circ h(g)$ for $g \in (\mathbb{Z}/2\mathbb{Z})^k$. Let us begin with a simpler question.

1.2 Example (divisibility) To convey the idea, we ask about the divisibility properties of the ensemble $h_k(g)$ ($g \in (\mathbb{Z}/2\mathbb{Z})^k$) of $2^k$ integers. For division by $n = 3$ a statistic is given in Table 1. The set

$$\Lambda(n) := \{(a, b) \in (\mathbb{Z}/n\mathbb{Z})^2 \mid \gcd(a, b, n) = 1\} \quad (n \in \mathbb{N})$$

has the cardinality of Jordan’s totient function $J_2(n) = n^2 \prod_{p \mid n} (1 - p^{-2})$, whereas its anti-diagonal $\{(a, b) \in \Lambda(n) \mid a + b = 0\}$ is of size $\phi(n)$. So for the uniform distribution on $\Lambda(n)$ the expectation value of divisibility by $n$ equals the quotient $1/\psi(n)$ of these numbers, with Dedekind’s psi function $\psi$. For $n \in \mathbb{P}$
Table 1: Statistic for divisibility by 3 of $h_k$. Upper row: exponent $k$ of the group cardinality $2^k$. Second row: Number of $g \in (\mathbb{Z}/2\mathbb{Z})^k$ with $3|h_k(g)$. Third row: After subtraction of expectation value $\frac{1}{4}2^k$. Last row, for comparison: $2^{k/2}$, the square root of $|(\mathbb{Z}/2\mathbb{Z})^k|$.

this simplifies to $1/(n+1)$. Indeed, line 2 of Table 1 is compatible with the presumption that the quotient $1/4$ is approached as $k \to \infty$.

Moreover, considering lines 3 and 4 of Table 1, the deviation from the expectation $|(\mathbb{Z}/2\mathbb{Z})^k|/\psi(n) = 2^{k-2}$ seems to be of the order $2^{k/2}$, the square root of the number of elements. This is similar to the sum of $2^k$ i.i.d. random variables.

Both facts can be proven easily in the example at hand, using the matrix $t(3) := \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, (1.8)

• This acts on $\ell^2((\mathbb{Z}/3\mathbb{Z})^2)$, is indexed lexicographically and is the Markov matrix for the recursion (1.3), taken (mod 3). So we obtain the second line of Table 1 as matrix entries or, equivalently, $\ell^2$ scalar products:

$$\left\| \left\{ g \in (\mathbb{Z}/2\mathbb{Z})^k \mid h_k(g) = 0 \text{ (mod 3)} \right\} \right\| = \left\langle (2t(3))^k \mathbb{I}_{(1)} + \mathbb{I}_{(2)} + \mathbb{I}_{(3)} \right\rangle.$$  

• The spectrum of $t(3)$ equals

$$\left\{ 1, 1, \frac{1}{4}(-1 + i\sqrt{7}), \frac{1}{4}(-1 - i\sqrt{7}), \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}.$$  

The expectation $\frac{1}{2}2^k$ equals $\left\langle (2t(3))^k P_{PF} \mathbb{I}_{(1)} + \mathbb{I}_{(2)} \right\rangle$, with the orthogonal projection $P_{PF}$ to the Perron-Frobenius eigenspace $\text{span}(\mathbb{I}_{A(3)})$.

• On the other hand, apart from the two Perron-Frobenius eigenvalues 1 in (1.9), whose eigenfunctions are constant on the two parts of the $t(3)$--invariant decomposition $(\mathbb{Z}/3\mathbb{Z})^2 = \{(0)\} \cup A(3)$ of the state space, the spectral radius
Figure 1.2: The digraph with state space $\Lambda(3)$ for the Markov matrix $t(3)$. The chain starts in the state $\left(\frac{1}{0}\right)$. Divisibility by 3 of $h_k$ has a proportion equal to the sum of probabilities for the states $\left(\frac{2}{1}\right)$ and $\left(\frac{3}{2}\right)$ after $k - 1$ iterations of $t(3)$.

of $t(3)$ is $1/\sqrt{2}$. As $t(3)$ is semisimple, this leads to $\|(t(3) - P_{PF})^k\|_{\Lambda(3)} \leq c 2^{-k/2}$, and thus to the exponential estimate

$$\left|\left| \{ g \in \left(\mathbb{Z}/2\mathbb{Z}\right)^k \mid h_k(g) = 0 \pmod{3} \} \right| - \frac{1}{2} 2^k \right| \leq c 2^{k/2} \quad (k \in \mathbb{N})$$

for the deviation from the mean. $\diamond$

Generalizing the example, the representations relevant for divisibility by $n \in \mathbb{N}$ will be dealt with in Subsection 2.4.

But the purpose of this article is to further generalize that kind of Markov estimate, to gain some control on joint divisibility properties of the values of $h_k$ ($k \in \mathbb{N}$). Such a control is clearly necessary for estimating the function $\lambda \circ h_k$ appearing in (1.7). The natural language for this question is the one of adeles, and we consider the corresponding representations in Subsections 2.5 and 2.6.

Methods from the theory of expander graphs are used. Some (like the one of Figure 1.2) but not all graphs arising here relate to Ramanujan graphs, as defined by Lubotzky, Phillips, and Sarnak in [LPS]. There is now a large literature on such expanders, see, e.g. [BHV, DSV, HLW] for surveys.

Notation

- Miscellaneous: The positive integers are $\mathbb{N} = \{1, 2, 3\ldots\}$, whereas $\mathbb{N}_0 := \{0, 1, 2\ldots\}$. With the primes $\mathbb{P} := \{2, 3, 5\ldots\}$ we set $\mathbb{P}_\infty := \{\infty\} \cup \mathbb{P}.$
The imaginary unit is typeset as \( i \). \( \pm 1 \) is abbreviated by \( \pm \). Column vectors are sometimes written as row vectors, when this does no harm.

We often apply a function to subsets \( A \) of its domain: \( f(A) := \{ f(a) \mid a \in A \} \).

The spectrum of an operator \( O \) is denoted by \( \text{spec}(O) \). So its spectral radius \( \sup(|\text{spec}(O)|) \) is bounded by its \( L^2 \) operator norm \( \|O\| \).

We use the notation \( \ell^2(M) \) for the \( C\text{-Hilbert space} \) over an infinite set \( M \), equipped with counting measure, and similarly for the Banach spaces \( \ell^p(M) \).

For finite sets \( M \), in \( \ell^p(M) \) the counting measure is often normalized to be a probability measure.

- **Rings:** We index the absolute values \( |\cdot|_v : \mathbb{Q}_v \to \mathbb{R} \) by \( v \in \mathbb{P}_\infty \), with \( \mathbb{Q}_\infty = \mathbb{R} \). Whereas the ring of integers \( \mathbb{Z}_\infty = \mathbb{Z} \) is metrically discrete, for \( p \in \mathbb{P} \) the ring \( \mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \} \) of \( p \)–adic integers is compact.

  For a subset \( S \subseteq \mathbb{P}_\infty \) the ring \( \mathbb{Z}_S := \prod_{v \in S} \mathbb{Z}_v \) is equipped with the product topology. \( \mathbb{Q} \) embeds diagonally in \( \mathbb{Z}_S \). For \( S \subseteq \mathbb{P} \) this embedding is dense in the compact ring \( \mathbb{Z}_S \). This holds in particular for \( \mathbb{Z}_\mathbb{P} \) (also denoted by \( \widehat{\mathbb{Z}} \)).

- **Haar measures:** The groups \( \mathbb{Z}_S \) are locally compact abelian and thus carry Haar measures: \( m_p \) on \( (\mathbb{Z}_p,+) \), normalized by \( m_p(\mathbb{Z}_p) = 1 \) \( (p \in \mathbb{P}) \), \( m_\infty \) (counting measure) on \( (\mathbb{Z},+) \) and

  \[
  m_S := \bigotimes_{p \in S} m_p \quad \text{on} \quad (\mathbb{Z}_S,+). \tag{1.10}
  \]

  So \( m_S(\mathbb{Z}_S) = 1 \) if \( S \subseteq \mathbb{P} \).

- **Modules and Hilbert Spaces:** Our main objects will be the \( \mathbb{Z}_v \)–modules \( \mathbb{Z}_v^2 \), and \( \mathbb{Z}_S \)–module \( \mathbb{Z}_S^2 \) for \( S \subseteq \mathbb{P}_\infty \), with Haar measure \( m_S^2 := m_S \otimes m_S \).

  Since \( \text{SL}(2,\mathbb{Z}) \subseteq \text{SL}(2,\mathbb{Z}_v) \) is a subgroup, we obtain \((\text{via } O_S f := f \circ O^{-1})\) unitary representations of \( \text{SL}(2,\mathbb{Z}) \) on the \( C\text{-Hilbert spaces} \)

  \[
  \mathcal{H}_S := L^2(\mathbb{Z}_S^2,m_S^2) \quad (S \subseteq \mathbb{P}_\infty).
  \]

**The Spectral Problem**

Given such a unitary representation on a Hilbert space \( \mathcal{H} \), rotation and inversion

\[
J, I := J^2 \in \text{SL}(2,\mathbb{Z}) \quad J \left( \begin{array}{c} \ell \\ r \end{array} \right) = \left( \begin{array}{c} -r \\ \ell \end{array} \right), \quad I \left( \begin{array}{c} \ell \\ r \end{array} \right) = \left( \begin{array}{c} -r \\ -\ell \end{array} \right),
\]

left and right addition

\[
L, R \in \text{SL}(2,\mathbb{Z}) \quad L \left( \begin{array}{c} \ell \\ r \end{array} \right) = \left( \begin{array}{c} \ell + r \\ r \end{array} \right), \quad R \left( \begin{array}{c} \ell \\ r \end{array} \right) = \left( \begin{array}{c} \ell r \\ r \end{array} \right),
\]

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give rise to unitary operators $J, I, L, R$ on $H$.

Only when considering specific properties of a given unitary representation, we will provide the operators and spaces with a subindex, like $J_S$ and $H_S$ for $S \subseteq \mathbb{P}_\infty$. We now temporarily omit that index for ease of notation.

We are to analyze the operators

$$T \in \mathcal{B}(H), \quad T = \frac{1}{2}(L + R).$$

(1.11)

Being convex combinations of unitaries, they are of operator norm $\|T\| \leq 1$.

Since we exclusively consider unitary representations of $O \in \text{SL}(2, \mathbb{Z})$ given by measure preserving actions of the group via $O f = f \circ O^{-1}$, $T$ shares with $L$ and $R$ (and all above $O$) the invariance of $\text{spec}(T)$ under complex conjugation.

### 2 Spectral Analysis of the Operators $T$

We now analyze the operators (1.11) on Hilbert spaces $H$ of unitary representations, more precisely on subspaces $H^\pm$ of $H$, defined by parity under inversion.

In Section 2.1 we find regions of $\mathbb{C}$ which are independent of the representation of $\text{SL}(2, \mathbb{Z})$ and contain the spectrum (Proposition 2.2). On the (more important) subspace $H^+$ we relate the non-normal operator $T$ to a conjugate pair of projections, thus preparing for a more precise spectral analysis (Prop. 2.4).

The rest of the section is about concrete representations:

For the regular representation (Sect. 2.2) the operator has a (minimal possible) spectral radius $1/\sqrt{2}$ (Prop. 2.7). This is shown using the Laplacians on a regular tree, and it leads to a similar statement for the defining representation of $\text{SL}(2, \mathbb{Z})$ (Prop. 2.9 of Sect. 2.3).

To prepare for the finite adeles, Section 2.4 considers representation of the finite groups $\text{SL}(2, \mathbb{Z}/n\mathbb{Z})$, $n \in \mathbb{N}$. Here a result by Bourgain and Varjú $[BV]$ of 2012 is used to show the existence of an $n$-independent spectral gap. Although $T$ is non-normal, the spectrum of the operator on the finite adeles is the closure of the union of these spectra (see Prop. 2.17 of Sect. 2.1). The final Section 2.6 then combines these results and states that the adelic operator $T$ has a gap.

#### 2.1 Algebraic Restrictions

In this subsection we continue to temporarily omit the subindex of the representation. Since $I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ is a nontrivial involution, $\text{spec}(I) \subseteq \{-1, 1\}$ for the spectrum of a unitary representation $I$. We thus obtain an orthogonal decomposition

$$H = H^+ \oplus H^-$$

(2.1)
of the Hilbert space into eigenspaces of $I$. Since $I$ is in the center of $\text{SL}(2, \mathbb{Z})$, the representations $O$ of all $O \in \text{SL}(2, \mathbb{Z})$ restrict to operators $O^\pm$ on $H^\pm$.

In this article we mainly analyze the operators $T^\pm$.

We first derive algebraic identities between operators $O^\pm$ on $H^\pm$, that impose restrictions on $\text{spec}(T^\pm)$ and are valid for all unitary representations of $\text{SL}(2, \mathbb{Z})$.

The restrictions of the representation $J$ of $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ to $H^\pm$ are (anti-)selfadjoint, with $\text{spec}(J^+) \subseteq \{ -1, 1 \}$ (and $\text{spec}(J^-) \subseteq \{ -i, i \}$).

We use the identities

$$J = LR^{-1}L = R^{-1}LR^{-1} \quad (2.2)$$

for the representations of $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, exhibited in Figure 2.1. So

$$J^*LJ = R^{-1}, \quad J^*RJ = L^{-1} \quad \text{and thus} \quad T^* = J^*TJ \quad \text{on} \quad H. \quad (2.3)$$

The equations $(2.2)$ are complemented by

$$J = IL^{-1}L = IRL^{-1} \quad (2.4)$$

In order to proceed in the spectral analysis of $T$, we introduce the operators

$$Y_+ := \frac{1}{3}(\frac{3}{2}I - \frac{1}{2}L + R^{-1}L + L^{-1}R) \quad \text{and} \quad Y_- := J^*Y_+J \quad (2.5)$$

on $H$. Note that the signs of $Y_\pm$ do not refer to the splitting (2.1), but instead

$$Y_\pm = Y_\pm^+ \oplus Y_\pm^- \quad \text{on} \quad H = H^+ \oplus H^-.$$  

2.1 Lemma $Y_\pm$ are orthogonal projections, and $T^*T = \frac{1}{6}(I + I + 6Y_+)$.

Proof. $Y_\pm$ are self-adjoint, since $L$ and $R$ are unitary and $I$ is an involution. To show that $Y_\pm^2 = Y_\pm$, one uses the relation $LR^{-1}L = IRL^{-1}R$, see Fig. 2.1. The second claim follows from

$$8T^*T = 2(L^{-1} + R^{-1})(L + R) = 4I + 2(R^{-1}L + L^{-1}R) = I + I + 6Y_+.$$  

For the self-adjoint operators $A := Y_+ - Y_-$ and $B := I - Y_+ - Y_-$ (see Avron, Seiler and Simon [ASS])

$$A^2 + B^2 = I, \quad AB + BA = 0 \quad \text{and} \quad [A^2, Y_\pm] = [B^2, Y_\pm] = 0. \quad (2.6)$$

Like $Y_\pm$, $A$ and $B$ are complexifications of real self-adjoint operators, so that, e.g. eigenfunctions may be chosen to be real.

Lemma 2.1 implies that the kernels of $T^-$ and of $Y_+^-$ coincide. A more important use of the lemma is to localize the spectrum of $T^\pm$:
2.2 Proposition  For all unitary representations of $SL(2, \mathbb{Z})$

1. $T^+$ is invertible, with $(T^+)^{-1} = 2(T^+)^* - J^+$, and $2T^+ + (T^+)^{-1}$ is self-adjoint:

\[
2T^+ + (T^+)^{-1} = -3J^+B^+ = -3B^+J^+.
\] (2.7)

So with the circle $C := \{c \in \mathbb{C} \mid |c| = 1/\sqrt{2}\}$ and $I := [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$

\[
\text{spec}(T^+) \subseteq C \cup I.
\]

2. $(T^-)^*T^- = \frac{3}{4}Y^+ \text{ so that with the disk } D := \{c \in \mathbb{C} \mid |c| \leq \sqrt{3}/2\}$

\[
\text{spec}(T^-) \subseteq D.
\]

Proof. For the sake of brevity we partly omit the superscript $\pm$.

1. • We show that $T(2T^* - J) = \mathbb{1}$ on $\mathcal{H}^+$, the identity $(2T^* - J)T = \mathbb{1}$ being similar: $2TT^* = \mathbb{1} + \frac{1}{2}(LR^{-1} + RL^{-1})$, and, using (2.2) and (2.4) on $\mathcal{H}^+$

\[
2TJ = LJ + RJ = L(L^{-1}RL^{-1}) + R(R^{-1}LR^{-1}) = RL^{-1} + LR^{-1}.
\]

• This implies (2.7), since its left hand side equals

\[
2T^+ + (T^+)^{-1} = 2(T^+ + (T^+)^*) - J^+ = L^+ + R^+ + (L^+)^{-1} + (R^+)^{-1} - J^+.
\]
whereas, using (2.2) and (2.4), \(-3JB = J(-I + R^{-1}L + L^{-1}R + LR^{-1}) = IR + L + IL^{-1} + R^{-1} - IJ = -3BJ\) on \(\mathcal{H}^+\) since \(I^+ = I_{\mathcal{H}^+}\).

- Since the preimage of \([-3, 3]\) for the map \(t \mapsto 2t + t^{-1}\) equals \(C \cup I\), we thus have to show that the resolvent set of \(2T^+ + (T^+)^{-1}\) contains \(\mathbb{C} \setminus [-3, 3]\). This follows from (2.7) since \(\|BJ\| \leq \|B\|\|J\| \leq 1\), as \(\|B\|^2 = \|B^2\| \leq 1\) by the first identity in (2.6).

2. With Lemma 2.1 we get \((T^-)^*T^- = \frac{3}{4}Y_+^*\). As \(\|T^-\|^2 = \|(T^-)^*T^-\| = \frac{3}{4}\|Y_+^*\| \leq \frac{3}{4}\), the spectral radius of \(T^-\) is smaller than \(\sqrt{3}/2\). \(\square\)

It might be instructive to have a look at the concrete spectra of \(T^\pm\) for a certain representation, depicted in Figure 2.3 on Page 32.

The two projections \(Y_\pm\) are unitarily equivalent and commute on the intersections of their kernels and ranges. We thus obtain (see Halmos in [Ha2]) an orthogonal decomposition of the Hilbert space as a direct sum

\[ \mathcal{H} = C_{0, 0} \oplus C_{0, 1} \oplus C_{1, 0} \oplus C_{1, 1} \oplus \tilde{N}, \]

with the closed commuting subspaces \(C_{j,k} := \ker(Y_+ - jI) \cap \ker(Y_+ - kI)\).

As we are analyzing \(T\) instead of \(Y_\pm\), we modify that decomposition. From the formulae \(T^*T = \frac{1}{2}(I + I + 6Y_+^*)\) (Lemma 2.1) and \(TT^* = \frac{1}{2}(I + I + 6Y_-)\) we see that \(T\) is normal on the domains \(C_{0, 0}\) and \(C_{1, 1}\), but not on \(C_{0, 1}\) and \(C_{1, 0}\). Thus we set

\[ \mathcal{K} := C_{0, 0}, \quad \mathcal{R} := C_{1, 1} \quad \text{and obtain} \quad \mathcal{H} = \mathcal{K} \oplus \mathcal{R} \oplus \mathcal{N} \quad \text{(2.8)} \]

in the sense of an orthogonal direct sum, with \(\mathcal{N} = C_{0,1} \oplus C_{1,0} \oplus \tilde{N}\).

In Example 2.11 we exhibit a representation for which \(T^- |_{\mathcal{N}^+}\) is not semisimple (with \(\mathcal{N}^\pm := \mathcal{N} \cap \mathcal{H}^\pm\)). As noted in Remark 2.5.2 below, \(T^\pm |_{\mathcal{N}^\pm}\) need not be semisimple, either.

Indeed, the spectral analysis of \(T^+\) on the joint kernel respectively range subspaces \(\mathcal{K}^+ = \mathcal{K} \cap \mathcal{H}^+\) and \(\mathcal{R}^+ = \mathcal{R} \cap \mathcal{H}^+\) of \(Y_\pm\) is simple:

2.3 Lemma

- \(T^+ = \frac{1}{2}(3Y_+ - I_{\mathcal{H}^+})J^+ = \frac{1}{2}J^+(3Y_+^* - I_{\mathcal{H}^+})\).
- The operators \(J^+\) and \(T^+\) leave the decomposition \(\mathcal{H}^+ = \mathcal{K}^+ \oplus \mathcal{R}^+ \oplus \mathcal{N}^+\) invariant.
- The operator \(T^+\) restricts to the mutually orthogonal eigenspaces

\[ \mathcal{K}^{+,\pm} := \{ v \in \mathcal{K}^+ | J^+ v = \pm v\} \quad \text{and} \quad \mathcal{R}^{+,\pm} := \{ v \in \mathcal{R}^+ | J^+ v = \pm v\} \]

with \(T^+ |_{\mathcal{K}^{+,\pm}} = \pm \frac{1}{2} I |_{\mathcal{K}^{+,\pm}}\) and \(T^+ |_{\mathcal{R}^{+,\pm}} = \pm I \mathcal{R}^{+,\pm}\).

- Furthermore \(T |_{\mathcal{K}^-} = T^* |_{\mathcal{K}^-} = 0\).
Proof. • On \( \mathcal{H}^+ \) we have \( 3Y_+ - \mathbb{I} = R^{-1}L + L^{-1}R \). Thus by (2.2) and (2.4) \( J^+(3Y_+ - \mathbb{I}) = I^+R + L^+ = 2T^+ \), using \( I^+ = \mathbb{I}_{\mathcal{H}^+} \).

The definition (2.5) of \( Y_+ \) then implies the other identity \( T^+ = \frac{1}{2}(3Y_+ - \mathbb{I})J^+ \).

• We can present \( K \) as \( \ker(Y_+ + Y_-) \) and \( R \) as \( \ker(2\mathbb{I} - Y_+ - Y_-) \). As \( Y_+ + Y_- \) commutes with \( J \), we see that both subspaces are \( J \)-invariant. Since \( J \) is self-adjoint, this also follows for the orthogonal complement \( \mathcal{N} \) of \( K \oplus R \). \( T^+ \) invariance of \( K^+ \) and \( R^+ \) then follows from the formula \( T^+ = \frac{1}{2}(3Y_+ - \mathbb{I})J^+ \).

• With the same formula \( T^+|_{K^+, \pm} = \frac{1}{2}(3Y_+ - \mathbb{I})J^+|_{K^+, \pm} = \pm \frac{1}{2} \mathbb{I}|_{K^+, \pm} \) and \( T^+|_{R^+, \pm} = \frac{1}{2}(3Y_- - \mathbb{I})J^+|_{R^+, \pm} = \pm \mathbb{I}|_{R^+, \pm} \).

• Finally, since \( I^+ = \mathbb{I}_{\mathcal{H}^+} \), the formulae \( T^*T = \frac{1}{8}(I + I + 6Y_+) \) and \( TT^* = \frac{1}{8}(I + I + 6Y_-) \) imply \( T^*T|_{K^-} = TT^*|_{K^-} = 0 \) so that \( T|_{K^-} = T^*|_{K^-} = 0 \). □

To analyze the rest of the spectrum one works on the last direct summand in (2.8), the non-commuting subspace \( \mathcal{N} \). Note again that on \( \mathcal{N}^+ \) the operators \( T^+ \) and \( J^+ \) do not commute, and hence \( T^+|_{\mathcal{N}^+} \) is not normal. However, the deviation from normality can be controlled.

To that aim we note that, as \([B, J] = 0\), the self-adjoint operator \( B^+ \) decomposes as

\[
B^+ = B^{+,+} \oplus B^{+,-}
\]

on \( \mathcal{H}^+ = \mathcal{H}^{+,+} \oplus \mathcal{H}^{+,−} \) for the subspaces

\[
\mathcal{H}^{+,\pm} := \{ v \in \mathcal{H}^+ \mid J^+ v = \pm v \}.
\]

Since \( A^+J^+ + J^+A^+ = 0 \), \( A^+ \) maps \( \mathcal{H}^{+,\pm} \) into \( \mathcal{H}^{+,\mp} \). Actually, the restrictions

\[
A^+|_{\mathcal{N}^{+,\pm}} : \mathcal{N}^{+,\pm} \to \mathcal{N}^{+,\mp}
\]

for the orthogonal direct sum

\[
\mathcal{N}^+ = \mathcal{N}^{+,+} \oplus \mathcal{N}^{+,−}
\]

are injective, since \( A^+v = 0 \) implies \( B^2v = v \), using (2.6).

In the polar decomposition \( O = U |O| \) of a self-adjoint operator \( O \), the partial isometry \( U \) commutes with \(|O|\), and is also called the sign of \( O \) and written as \( \text{sign}(O) \), since \( \text{spec}(U) \subseteq \{-1, 0, 1\} \).

We now use the operator-valued spectral measure

\[
E := E(O) \colon \mathcal{B}(\text{spec}(O)) \to \mathcal{B}(\mathcal{H})
\]

of a normal operator \( O \in \mathcal{B}(\mathcal{H}) \), defined on the Borel sigma algebra of its spectrum, whose unique existence is assured by the spectral theorem, see e.g. Chapter IX, §2 of Conway [Co]. Since \( B^+ \) and \( J^+ \) commute, we can use instead of \( E(B^+) \) its refinement \( E(B^{+,+}) \oplus E(B^{+,-}) \) (see Exercise 17 on p. 273 of [Co]).

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2.4 Proposition • \( A^+ \) transforms the spectral measures \( E = E(B^{+,\pm}) \) of the operators \( B^{+,\pm} \) as \( A^+ E(B) = E(-B) A^+ \) \( (B \in \mathcal{B}(\text{spec}(B^{+,\pm}))). \)

• The operator \( A^+ |_{\mathcal{N}^+} \) splits orthogonally into
\[
A^+ = A^+_+ \oplus A^+_- \quad \text{on} \quad \mathcal{N}^+ = \mathcal{N}^+_+ \oplus \mathcal{N}^-_+
\]
with \( \mathcal{N}^+_\pm := P_\pm(\mathcal{N}^+) \) for the family \( (P_+ + P_- = 1_{\mathcal{N}^+}) \) of orthogonal projectors
\[
P_\pm := \frac{1}{2}(1 \pm \text{sign}(A^+ |_{\mathcal{N}^+})) \quad \text{on} \quad \mathcal{N}^+.
\]

• The spectral measure of \( A^+_\pm \) on a measurable set \( 0 \notin A \in \mathcal{B}(\text{spec}(A^+_\pm)) \) equals
\[
E(A^+_\pm)(A) = P_\pm E(|B^{+,\pm}|)(\sqrt{1 - A^2_\pm}), \quad \text{with} \quad A_\pm := \{ x \in A \mid \pm x \geq 0 \}.
\]

• Denoting by \( b \mapsto dE^{+,\pm}(b) \) integration w.r.t. the spectral measure of \( B^{+,\pm} \),
\[
T^+ |_{\mathcal{N}^+} = \int_{(-1,1)} \frac{1}{4} \left( \frac{1-3b}{-3\sqrt{1-b^2}} \frac{3\sqrt{1-b^2}}{-1-3b} \right) dE^{+,+}(b) \oplus dE^{-,-}(-b).
\]

This relates \( \{(b, \pm 1)\} \subseteq \text{spec}(B^+, J^+) \) to \( \{ \frac{1}{4}(-3b \pm \sqrt{9b^2 - 8}) \} \subseteq \text{spec}(T^+) \).

In particular, possibly excluding the values \( \pm 1 \) and \( \pm \frac{1}{2} \), \( \text{spec}(T^+) \) is symmetric w.r.t. inversion at the circle \( C \) of radius \( 1/\sqrt{2} \), including multiplicities.

Proof. • The formula \( A^+ E(B) = E(-B) A^+ \) follows with (2.6) from
\[
A^+ f(B) = A^+ \left( f_+(B^+) + f_-(B^+) \right) = \left( f_+(B^+) - f_-(B^+) \right) A^+
\]
for \( f : \mathbb{R} \to \mathbb{R} \) Borel measurable and \( f_\pm(x) := \frac{1}{2}(f(x) \pm f(-x)) \), by choosing \( f := 1_B \) and noting that \( (1_B)_+ - (1_B)_- = 1_B \).

• (2.12) is an orthogonal projector family, since \( P_+ + P_- = 1_{\mathcal{N}^+} \) and \( A^+ |_{\mathcal{N}^+} \) is injective, so that
\[
\left( \text{sign}(A^+ |_{\mathcal{N}^+}) \right)^2 = 1_{\mathcal{N}^+} \quad \text{and thus} \quad P^2_\pm = P_\pm \quad \text{and} \quad P_\pm P_\mp = 0.
\]

• Eq. (2.13) can be verified by merely using the relations \( (A^+)^2 + (B^+)^2 = 1 \) and \( A^+ B^+ + B^+ A^+ = 0 \) on \( \mathcal{N}^+ \):
\[
|B^+| = \sqrt{(B^+)^2} \quad \text{implies} \quad |A^+|, |B^+| = 0 = [P_\pm, |B^+|]. \quad \text{So} \quad P_\pm E(|B^+|)(B) \quad \text{is self-adjoint for any Borel set} \ B \ \text{and respects the decomposition given by (2.12).}
\]
Since \( P_\pm \) and \( E(|B^+|)(B) \) are orthogonal projections, the right hand side of (2.13), applied to \( B \), is a projection. By the above, since \( E(|B^{+,\pm}|) \) is a spectral measure, \( P_\pm E(|B^+|) \) are spectral measures, too.

To verify (2.13), we substitute \( |B^+| = \sqrt{1 - (A^+)^2} \). Since \( x \mapsto \sqrt{1 - x^2} \) is strictly monotone on \([0,1]\) and on \([-1,0]\), \( E(|B^+|)(\sqrt{1 - A^2_\pm}) = E(A^+_\pm)(A) \)
on $\mathcal{N}_\pm^+$. 

- The representation (2.14) of $T^+$ based on the spectral measure of $B^\pm$ follows from

\[
T^+ = \frac{1}{4}(1 - 3A^+ - 3B^+)J^+ = \frac{1}{4}J^+ (1 + 3A^+ - 3B^+) \quad (2.15)
\]

(which is a consequence of the formula $T^+ = \frac{1}{2}(3Y^+ + 1)J^+ = \frac{1}{2}J^+ (3Y^+ + 1)$ in Lemma 2.3). The term $\frac{1}{4}(1 - 3B^+)J^+$ of (2.15) is diagonal in the common spectral resolution of $(B^+, J^+)$ and thus gives rise to the diagonal of

\[
\frac{1}{4} \left( \begin{array}{cc} 1 - 3b & 3\sqrt{1 - b^2} \\ -3\sqrt{1 - b^2} & 1 - 3b \end{array} \right).
\]

The operator $A^+$ interchanges the two subspaces, and $(A^+)^2 = 1 - (B^+)^2$ by (2.6), hence the off-diagonal entries.

\[ \square \]

2.5 Remarks 1. The splitting (2.10) is diagonal w.r.t. $\mathcal{N}^+ = \mathcal{N}_+^+ \oplus \mathcal{N}_-^+$, whereas the splitting (2.11) is diagonal w.r.t. $\mathcal{N}^+ = \mathcal{N}_+^+ \oplus \mathcal{N}_-^,$.

2. For the spectral parameter $b \in (-1, 1) \setminus \{ \pm \frac{\sqrt{3}}{3} \}$ the matrix $\frac{1}{4} \left( \begin{array}{cc} 1 - 3b & 3\sqrt{1 - b^2} \\ -3\sqrt{1 - b^2} & 1 - 3b \end{array} \right)$ appearing in (2.14) has eigenvalues $\frac{1}{4} \left( -3b \pm \sqrt{9b^2 - 8} \right)$ with eigenvectors

\[
\left( \frac{1 \pm \sqrt{9b^2 - 8}}{3} \right).
\]

We note that the vector entries $\frac{1 \pm \sqrt{9b^2 - 8}}{3}$ are of modulus one in the interval $|b| < \sqrt{3}/3$. For $b = \pm \frac{\sqrt{3}}{3}$ the degenerate eigenvalues $\pm 1/\sqrt{2}$ are the points of the circle $C$ on the real axis, and Jordan blocks arise. \[ \diamond \]

2.2 The Regular Representation and Laplacians on Trees

Before we come to the analytic estimates of $T_S$ for sets $S \subseteq \mathbb{P}_\infty$ of places, we consider the corresponding problem for the operator $T_{SL}$ on the Hilbert space

\[
\mathcal{H}_{SL} := l^2(SL(2, \mathbb{Z}))
\]

with counting measure. $T_{SL}$ is defined by the unitary left regular representation of $SL(2, \mathbb{Z})$. Following the notation in (2.1) and (2.8) we split into the orthogonal subspaces

\[
\mathcal{H}_{SL} = \mathcal{H}_{SL}^+ \oplus \mathcal{H}_{SL}^- \quad \text{and} \quad \mathcal{H}_{SL}^+ = \mathcal{K}_{SL}^+ \oplus \mathcal{R}_{SL}^+ \oplus \mathcal{N}_{SL}^+.
\]

$T_{SL}^+$ is related to a graph Laplacian. So we first introduce some graph-theoretic notation, already used in [Kn4] for finite graphs. In the present context infinite graphs like regular trees arise. See, e.g. Mohar and Woess [MW] for an overview of Laplacians on infinite graphs, and Figà-Talamanca and Nebbia [FTN] for trees.
For unoriented graphs \((V, E)\) with bounded vertex degree, we consider the Hilbert space \(\mathcal{H} = \mathcal{H}_V \oplus \mathcal{H}_E\) with vertex Hilbert space \(\mathcal{H}_V := \ell^2(V)\), \(\mathcal{H}_E := \{f \in \ell^2(E) \mid f((v, v)) = -f((v, w))\}\), with inner product \(\langle f, g \rangle_E := \frac{1}{2} \sum_{e \in E} f(e)\overline{g(e)}\).

Then the adjoint of \(d : \mathcal{H}_V \to \mathcal{H}_E\), \(df((v, w)) := f(w) - f(v)\) equals \(d^* : \mathcal{H}_E \to \mathcal{H}_V\), \(d^* g(v) = -\sum_{w : (v, w) \in E} g((v, w))\).

One defines the operator \(\Delta : \mathcal{H} \to \mathcal{H}\) by
\[
\Delta := d^* d \oplus d d^* = Q^2 \quad \text{with} \quad Q := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix},
\]
so that \(\Delta = \Delta_V \oplus \Delta_E\) with vertex Laplacian
\[
\Delta_v f(v) = \sum_{w : (v, w) \in E} (f(v) - f(w)) \quad (f \in \mathcal{H}_V)
\]
and edge Laplacian
\[
\Delta_E g((v, w)) = \sum_{x : (v, x) \in E} g((v, x)) + \sum_{x : (x, w) \in E} g((x, w)) \quad (g \in \mathcal{H}_E). \tag{2.17}
\]

Under the assumption of bounded degree \(Q\) and thus \(\Delta\) are bounded, self-adjoint operators, with operator norm \(\|\Delta\| \leq 2 \sup_{v \in V} \deg(v)\).
In fact \((\Delta, \mathbb{1} \oplus (-\mathbb{1}), Q)\) is a supersymmetric triple (i.e. all operators are self-adjoint with \(\Delta = Q^2\) and \((\mathbb{1} \oplus (-\mathbb{1}))^2 = \mathbb{1}\), see, e.g. Sect. 6.3 of Cycon, Froese, Kirsch, and Simon [CFKS]). Apart from zero eigenvalues, the spectra of \(\Delta_E\) and \(\Delta_V\) coincide (including multiplicities, when finite), since there \(Q\) is invertible, with
\[
\Delta_E d = d \Delta_V \quad \text{and} \quad d^* \Delta_E = \Delta_V d^*.
\]

2.6 Remarks 1. For the bipartite graphs \((V, E)\) with vertex set \(V = V_+ \cup V_-\) considered in this article, for \(\ell^2(E)\) with counting measure the map
\[
\mathcal{I} : \ell^2(E) \to \mathcal{H}_E, \quad \mathcal{I}(f)((v^\pm, v^\mp)) := \pm f(\{v^-, v^+\}) \quad (v^\pm \in V^\pm)
\]
is an isomorphism of Hilbert spaces. We will use \(\mathcal{I}\) implicitly.
2. For connected graphs \((V, E)\), shortest paths define metrics
\[
\text{dist} \equiv \text{dist}_V : V \times V \to \mathbb{N}_0 \quad \text{and} \quad \text{dist}_E : E \times E \to \mathbb{N}_0.
\]
We now determine analytically the spectrum \(\text{spec}(T^+_{\text{SL}})\). This is a proper subset of the set in \(\mathbb{C}\) which was determined algebraically in Proposition 2.2.

2.7 Proposition \(\text{spec}(T^+_{\text{SL}}) = \{-\frac{1}{2}, \frac{1}{2}\} \cup C\). The multiplicities of the eigenvalues \(\pm \frac{1}{2}\) are infinite. The spectrum of \(T^+_{\text{SL}}\) on the circle \(C\) is absolutely continuous.

Proof. • We first construct the relevant graph \((V, E)\) for \(T^+_{\text{SL}}\). The orbits generated by \(-R^{-1}L = (\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix})\) are of cardinality three in \(\text{SL}(2, \mathbb{Z})\), and similarly for \(-LR^{-1} = (\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix})\).

So the images of these orbits in \(\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm I\}\), (that is, the orbits \(v_+ \) of \(\pm R^{-1}L\), and the orbits \(v_- \) of \(\pm LR^{-1}\)) are of order three, too.

The vertex set of the graph is \(V := V_+ \cup V_-\), with the disjoint sets \(V_\pm\) of orbits \(v_\pm \subseteq \text{PSL}(2, \mathbb{Z})\).

The edge set \(E\) is defined as the set of \(\{v_+, v_-\}\) with \(v_\pm \in V_\pm\) and \(v_+ \cap v_- \neq \emptyset\). For \(\{v_+, v_-\} \in E\) the intersection \(v_+ \cap v_-\) consists of a unique element of the group \(\text{PSL}(2, \mathbb{Z})\).

Taken together, \(\pm R^{-1}L\) and \(\pm LR^{-1} = \pm J R^{-1}L J\) generate an index two subgroup of \(\text{PSL}(2, \mathbb{Z})\), consisting of the elements which \((\text{mod } 2)\) are in the cyclic subgroup
\[
\{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\} \cong C_3 \quad \text{of} \quad \text{SL}(2, \mathbb{Z}/2\mathbb{Z}). \quad (2.18)
\]

The coset space disjoint from this subgroup arises by multiplication with \(\pm J\) and corresponds \((\text{mod } 2)\) to the set \(\{(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix})\}\) of \(\text{SL}(2, \mathbb{Z}/2\mathbb{Z})\)-matrices.

Since \(\text{PSL}(2, \mathbb{Z}) \cong C_2 \ast C_3\) is a free product, the graph \((V, E)\) is thus a disjoint union of two copies of the three-regular tree.

• We now use the above identification \(E \cong \text{PSL}(2, \mathbb{Z})\). Dropping the index \(\text{SL}\), on \(\mathcal{H}^+\) the operator
\[
\Delta_E := 3(\mathbb{I} - B) = 3(Y_+ + Y_-) = 3\mathbb{I} - I + R^{-1}L + L^{-1}R + LR^{-1} + RL^{-1}\]
\[
(2.19)
\]
is the edge Laplacian for the graph \((V, E)\) since \((R^{-1}L)^3 = (L^{-1}R)^3 = I\) and \(I^+ = I_{\mathcal{H}^+}\) (note that \(g((v, w))\) appears twice on the right hand side of (2.17)).

• The subspace \(\mathcal{R}^+_{\text{SL}}\) in (2.16) is zero-dimensional, since it consists of the constant functions that are in \(l^2(\text{SL}(2, \mathbb{Z}))\). So by Lemma 2.3, \(\pm 1\) are not eigenvalues of \(T^+_{\text{SL}}\). We will see that \(\pm 1 \not\in \text{spec}(T^+_{\text{SL}})\), when analyzing the operator on \(\mathcal{N}^+_{\text{SL}}\).

• Thus, using Lemma 2.3, the multiplicities of the eigenvalues \(\pm \frac{1}{2}\) of \(T^+_{\text{SL}}\) equal the dimensions of the \(Y_\pm\)-kernel subspaces \(\mathcal{K}^+_{\text{SL}, \pm}\).
So by (2.7) we must construct the eigenfunctions for $B_{SL}^{\pm}$ with eigenvalue 1. This is done by considering the eigenspace for the eigenvalue 0 of $\Delta_E$. For the edge metric $\text{dist}_E$, and an arbitrary edge $g \in E \cong \text{PSL}(2, \mathbb{Z})$, one defines the functions $v_g \in \ell^2(E) \cong \mathcal{H}_E$ (Remark 2.6.1),

$$v_g(h) := \begin{cases} \frac{(-2)^{-\text{dist}_E(g,h)}}{\sqrt{3}} & \text{if } h \text{ is in the connected component of } g \\ 0 & \text{otherwise} \end{cases}. \quad (2.20)$$

As there are exactly $2^{D+1}$ edges $g \in E$ of distance $\text{dist}_E(g, h) = D \in \mathbb{N}$, the $v_g$ are indeed square integrable (with $\|v_g\|_E = \sqrt{3}$). Notice that $d^*v_g = 0 \in \mathcal{H}_V$.

So $v_g$ is an eigenfunction with eigenvalue zero of $\Delta_E = dd^*$ (a harmonic function) which does not correspond to an eigenfunction of $\Delta_V$.

By (anti–)symmetrizing $v_g$ w.r.t. the $J$–action (that is, w.r.t. the two connected components of the graph) we show $\pm \frac{1}{2} \in \text{spec}(T_{SL}^+)$. The multiplicity of these eigenvalues is infinite, although the set $\{v_g \mid g \in \text{PSL}(2, \mathbb{Z})\}$ is linear dependent.

- The rest of $\text{spec}(T_{SL}^+)$ relates to the subspace $\mathcal{N}_{SL}^+$. Since $\mathcal{R}_{SL}^+ = \{0\}$, this corresponds to $\text{ran}(\Delta_E)$, or by supersymmetry, to $\text{ran}(\Delta_V)$.

It is known (see, e.g. [FTN], Chapter II.6) that the spectrum of the vertex Laplacian on the $(q + 1)$–regular tree equals

$$\text{spec}(\Delta_V) = [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \quad (2.21)$$

and is absolutely continuous.

However, we repeat the proof for the Laplacian at hand, since we need it in the real and the adelic cases below. Instead of (2.21) we show the estimate

$$\text{spec}(\text{Ad}) = [-\sqrt{8}, \sqrt{8}] \quad (2.22)$$

for the adjacency matrix $\text{Ad} \in \mathbf{B} (\mathcal{H}_V)$ which equals $\text{Ad} f(v) = \sum_{w : \{v, w\} \in E} f(w)$. (2.22) is equivalent to (2.21), since $(V, E)$ is three-regular. We prove that the spectrum is contained in (2.22) by constructing the resolvent $(\text{Ad} - \lambda \mathbb{I})^{-1}$ for all $\lambda \in \mathbb{C} \setminus [-\sqrt{8}, \sqrt{8}]$. We define the family of operators

$$D(k) \equiv D_{SL}(k) \in \mathbf{B} (\mathcal{H}_V) \quad , \quad (D(k)f)(v) := \sum_{w : \text{dist}(v, w) = k} f(w) \quad (k \in \mathbb{N}_0). \quad (2.23)$$

So in particular $D(0) = \mathbb{I}_{\mathcal{H}_V}$ and $D(1) = \text{Ad}$. The $D(k)$ are self-adjoint.

- Their norm is estimated as follows for $k \in \mathbb{N}$. Given an arbitrary tree root $u \in V$, one sets $d(v) := \text{dist}(u, v)$ and orthogonally decomposes the Hilbert space as

$$\mathcal{H}_V = \bigoplus_{\ell \in \mathbb{Z}/k\mathbb{Z}} \mathcal{H}_\ell \quad \text{with} \quad \mathcal{H}_\ell := \{f \in \mathcal{H}_V \mid \text{supp} f \subseteq C_\ell\}$$
for $C_\ell := \{ v \in V \mid [d(v)]_k = \ell \}$, with $[x]_k := x \pmod k$.

Accordingly, $D(k) = (D(k)_{\ell,m})$ with restricted operators $D(k)_{\ell,m} : \mathcal{H}_\ell \to \mathcal{H}_m$.

Their operator norms are bounded by

$$
\|D(k)_{\ell,m}\| \leq 3 \, 2^{k/2} \quad (k \in \mathbb{N}, \ \ell, m \in \mathbb{Z}/k\mathbb{Z}).
$$

To see this, we notice that for vertices $v \in C_\ell$ and $w \in C_m$ with $\text{dist}(v, w) = k$ there is a unique vertex $x \in V$ in the intersection of the connecting paths $[u, v], [u, w]$ and $[v, w] \subseteq V$, and $2d(x) = \ell + m \pmod k$. By the triangle inequality $\text{dist}(x, w) \leq k$, so given $\ell$ and $m$ there occur at most three values of $\text{dist}(x, w) = d(w) - d(x)$. Let $W \equiv W(\ell, m) \subseteq \mathbb{N}_0$ be the set of these values.

The image of $f \in \mathcal{H}_\ell$ has the squared norm $\|D(k)_{\ell,m}f\|^2 =$

$$
\sum_{w \in C_m} \left| \sum_{v \in C_\ell, \text{dist}(v, w) = k} f(v) \right|^2 = \sum_{w \in W} \sum_{x \in C_m, \text{dist}(x, w) = k} \left| \sum_{v \in C_\ell, \text{dist}(x, v) = k - w} f(v) \right|^2 \leq \sum_{w \in W} \frac{3}{2} \, 2^w \sum_{x \in C_m, \text{dist}(x, w) = k} \left| f(v) \right|^2 \leq \frac{27}{4} \, 2^k \|f\|^2,
$$

the second to last inequality being Hölder’s. This proves (2.24). Since the $L^2$–matrix norm is bounded above by the Hilbert-Schmidt norm, we thus get

$$
\|D_{SL}(k)\| \leq \left( \sum_{\ell, m \in \mathbb{Z}/k\mathbb{Z}} \|D(k)_{\ell,m}\|^2 \right)^{1/2} \leq 3 \, k \, 2^{k/2} \quad (k \in \mathbb{N}).
$$

• For parameters $c, d \in \mathbb{C}$ the series

$$
D(c) \equiv D_{SL}(c) := d \mathbb{I} + \sum_{k=1}^{\infty} D(k) \, 2^{-ck}
$$

thus converges to a bounded operator, if $\Re(c) > \frac{1}{2}$.

Conversely, the estimate for the partial sums $\|(d \mathbb{I} + \sum_{k=1}^{K} D(k) \, 2^{-ck})\delta_u\| =$

$$
\left( d^2 + \frac{3}{2} \sum_{k=1}^{K} 2^{(1-2c)k} \right)^{1/2} = \left( d^2 + 3 \frac{2^{(1-2c)K} - 1}{2 - 2c} \right)^{1/2}
$$

for all $K \in \mathbb{N}$ shows divergence for $\Re(c) < \frac{1}{2}$, and similarly for $\Re(c) = \frac{1}{2}$.
For $\Re(c) > \frac{1}{2}$ the operator $D_{SL}(c)$ is a resolvent for $Ad$:

$$\begin{align*}
(Ad - \lambda \mathbb{I}) D(c) &= (3 2^{-c} - \lambda d) \mathbb{I} - \sum_{k=1}^{\infty} \lambda D(k) 2^{-ck} \\
&\quad + \sum_{k=2}^{\infty} 2D(k-1) 2^{-ck} + \sum_{k=0}^{\infty} D(k+1) 2^{-ck} \\
&= (3 2^{-c} - \lambda d) \mathbb{I} + \sum_{k=1}^{\infty} (2^{1-c} - \lambda + 2^c) D(k) 2^{-ck} = \mathbb{I}
\end{align*}$$

for $d := \frac{32^{-c-1}}{\lambda}$ and $2^{1-c} + 2^c = \lambda$ or $c_{\pm}(\lambda) := \log_2 \left( \frac{1}{2} (\lambda \pm \sqrt{\lambda^2 - 8}) \right)$. Since $\max(\Re(c_{\pm}(\lambda))) > \frac{1}{2}$ if and only if $\lambda \in \mathbb{C} \setminus [-\sqrt{8}, \sqrt{8}]$, this proves (2.22), and thus (2.21) for $q = 2$.

- In our case the spectral interval

$$\text{spec}(\Delta_E) \setminus \{0\} = \text{spec}(\Delta_V) = [3 - \sqrt{8}, 3 + \sqrt{8}]$$

is, using (2.7) and $\Delta_E = 3(\mathbb{I} + B^+)$, the image of the circle $C$ under the maps

$$x \mapsto 3 \pm (2x + 1/x).$$

The sign depends on the $J^+$ subspace. Each point $y \in (3 - \sqrt{8}, 3 + \sqrt{8})$ has the two complex conjugate preimages

$$\frac{1}{4} \left( 3 - y \pm i\sqrt{6y - y^2 - 1} \right) \text{ respectively } -\frac{1}{4} \left( 3 - y \pm i\sqrt{6y - y^2 - 1} \right). \tag{2.27}$$

As for all representations considered in this article, the spectrum of $T_{SL}^+$ is invariant under complex conjugation. So $C$ belongs to $\text{spec}(T_{SL}^+)$, but the only real points of the spectrum are $\pm \frac{1}{2}$ and $\pm 1/\sqrt{2}$.

- As the inverse transformations (2.27) are analytic and non–constant on the open interval $(3 - \sqrt{8}, 3 + \sqrt{8})$, absolute continuity of $\text{spec}(\Delta_V)$ leads to absolute continuity of $\text{spec}(T_{SL}^+)$.

We are particularly interested in the contractions $T_S^+$, for the real 'infinite' case $S = \{\infty\}$, the finite adele case $S = \mathbb{P}$ and the adelic case $S = \mathbb{P}_\infty$, which we now analyze in succession.

The first steps will always consist in considering the orbits of $SL(2, \mathbb{Z})$ in $\mathbb{Z}_S^2$, since $SL(2, \mathbb{Z})$ is generated by left and right addition $L = (\frac{1}{0} \frac{1}{1})$, $R = (\frac{1}{1} \frac{1}{0})$, and their inverses.
2.3 The Real Case

We begin with the simplest of our three sets of places, \( S = \{\infty\} \) and thus analyze \( T_{\infty} \) on \( \ell^2(\mathbb{Z}^2) \). The \( \text{SL}(2, \mathbb{Z}) \)-orbits in \( \mathbb{Z}^2 \) are of the form
\[
T_{\infty, n} := T_{\infty} | \ell^2(n \Lambda)
\]
with \( n \Lambda := \{(a, b) \in \mathbb{Z}^2 \mid \gcd(a, b) = 1\} \) and \( n \in \mathbb{N}_0 \).

Except for \( 0 \Lambda = \{(0, 0)\} \), the actions on these orbits are mutually isomorphic. Thus in the orthogonal decomposition
\[
\ell^2(\mathbb{Z}^2) = \bigoplus_{n \in \mathbb{N}_0} \ell^2(n \Lambda), \quad T_{\infty} = \bigoplus_{n \in \mathbb{N}_0} T_{\infty, n}
\]
we need only consider the trivial case \( n = 0 \) and \( n = 1 \), setting
\[
\mathcal{H}_\Lambda := \ell^2(\Lambda) \quad \text{and} \quad T_\Lambda := T_{\infty, 1}.
\]

The action of \( \text{SL}(2, \mathbb{Z}) \) on \( \Lambda \) is isomorphic to its left action on the coset space by the parabolic subgroup
\[
L = \{(1, n, 0, 1) \mid n \in \mathbb{Z}\},
\]
via
\[
\text{SL}(2, \mathbb{Z}) / L \cong \Lambda, \quad \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right],
\]
fitting with the left regular representation.

Thus the action of \( \text{SL}(2, \mathbb{Z}) \) on \( \Lambda \) is not free, unlike the one on itself. However we will see in Lemma 2.8 that in a precise sense it is not far from being free, a fact that is vital for the spectrum of \( T_\Lambda \).

The typical fiber of the bundle
\[
\Pi : \text{SL}(2, \mathbb{Z}) \to \Lambda
\]
defined by (2.30) is isomorphic to \( \mathbb{Z} \). We define a section \( \Lambda \to \text{SL}(2, \mathbb{Z}) \) by choosing for \( \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Lambda \) the element \( \left[\begin{smallmatrix} a & -ka \\ c & -kc \end{smallmatrix}\right] \) of the fiber over \( \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \) with minimal \( k \in \mathbb{Z} \) so that
\[
\langle \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right], \left[\begin{smallmatrix} a & -ka \\ c & -kc \end{smallmatrix}\right] \rangle \leq 0.
\]
Thus we obtain a trivialization \( \Pi \times F : \text{SL}(2, \mathbb{Z}) \to \Lambda \times \mathbb{Z} \) of the bundle with second factor
\[
F : \text{SL}(2, \mathbb{Z}) \to \mathbb{Z}, \quad \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \lceil (ab + cd)/(a^2 + c^2) \rceil,
\]
(the argument of the ceil function being the unipotent parameter of the Iwasawa decomposition of \( \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \text{SL}(2, \mathbb{R}) \)).

The matrices appearing in the following lemma are \( \pm R^{-1}L, \pm L^{-1}R, \pm RL^{-1} \) and \( \pm LR^{-1} \). Apart from the identity, it is their unitary representations that are the terms of the operator \( B^+ \), see (2.19).

2.8 Lemma For \( A \in \text{SL}(2, \mathbb{Z}) \)
\[
|F(MA) - F(A)| \leq 1 \quad \text{if} \quad M \in \pm \left\{ \left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \right\}.
\]
Proof. We set \( A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \tilde{F}(A) := (ab + cd)/(a^2 + c^2) \). Since \( \tilde{F} \) is even, we need only consider the positive sign in the list of \( M \). These \( M \) are elliptic of order three, with \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \) and \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \).

\[ |\tilde{F}(\left( \begin{smallmatrix} -1 & 0 \\ 1 & 0 \end{smallmatrix} \right)A) - \tilde{F}(A)| = |(a^2 - ac - c^2)/(a^2 + c^2)(2a^2 + 2ac + c^2)| \]

is smaller than \( \frac{c}{2}/(a^2 + c^2) \) and thus \( < 1 \) if \( a^2 + c^2 > 1 \). It is \( \leq 1 \) if \( a^2 + c^2 = 1 \). The case \( |\tilde{F}(\left( \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right)A) - \tilde{F}(A)| = |(a^2 + ac - c^2)/(a^2 + c^2)(2a^2 - 2ac + c^2)| \)

is similar. \( \Box \)

According to (2.1) and (2.8) we split the Hilbert space \( \mathcal{H}_A \) into the orthogonal subspaces

\[ \mathcal{H}_A = \mathcal{H}_A^+ \oplus \mathcal{H}_A^- \quad \text{with} \quad \mathcal{H}_A^+ = \mathcal{K}_A^+ \oplus \mathcal{R}_A^+ \oplus \mathcal{N}_A^+. \quad (2.33) \]

2.9 Proposition \( \text{spec}(T_{\infty,0}) = \{1\}, \text{spec}(T_{\infty,n}) = \text{spec}(T_\Lambda) \) \((n \in \mathbb{N})\), \text{and} \( \text{spec}(T_\Lambda^+) = \{-\frac{1}{2}, \frac{1}{2}\} \cup C. \)

Like for the case \( T_{SL}^+ \), the spectrum of \( T_\Lambda^+ \) on \( C \) is absolutely continuous.

Proof. • Since \( \ell^2(0\Lambda) \cong \mathbb{C} \), the first statement is obvious.

• The pull-back \( \ell^2(n\Lambda) \rightarrow \ell^2(\Lambda) \) induced by multiplication \( \Lambda \rightarrow n\Lambda \) by \( n \) is unitary for \( n \in \mathbb{N} \) and then conjugates \( T_{\infty,n} \) with \( T_\Lambda \).

• We first fix the function spaces involved. The operator induced by the map \( \Pi \) from (2.31) is defined in two steps. First we define on the subspace

\[ \tilde{U} := \{ \phi \in C^{\text{SL}(2,\mathbb{Z})} \mid \forall x \in \Lambda : \phi|_{\Pi^{-1}(x)} \in \ell^1(\Pi^{-1}(x)) \} \]

with absolutely integrable fibers the operator

\[ \tilde{\Pi} : \tilde{U} \rightarrow C^\Lambda, \quad (\tilde{\Pi}\phi)(x) = \sum_{y \in \Pi^{-1}(x)} \phi(y). \quad (2.34) \]

Then we set

\[ U := \{ \phi \in \tilde{U} \mid |\tilde{\Pi}|(\ell^2(\Lambda)) \} \quad \text{and} \quad \hat{\Pi} := \tilde{\Pi}|_U. \]

\( U \) is a subspace of \( \ell^2(\text{SL}(2,\mathbb{Z})) \), since \( \sum_{y \in \Pi^{-1}(x)} |\phi(y)|^2 \leq (\sum_{y \in \Pi^{-1}(x)} |\phi(y)|)^2. \)

\( \hat{\Pi} \) defines an unbounded operator between Hilbert spaces, since the fibers of the bundle projection \( \Pi \) are isomorphic to \( \mathbb{Z} \), \text{i.e.} infinite. Nevertheless, on certain subspaces this norm is finite: For all \( N \in \mathbb{N}_0 \), with \( F \) defined in (2.32)

\[ U_N := \{ \phi \in \ell^2(\text{SL}(2,\mathbb{Z})) \mid F(\text{supp} \phi) \subseteq \{ -N, \ldots, N \} \} \subseteq U, \]
and by Hölder’s inequality
\[ \| \hat{\Pi} \phi \| \leq \sqrt{2N + 1} \| \phi \| \quad (\phi \in U_N). \] (2.35)

So \( U \) is a dense subspace of \( \ell^2(\text{SL}(2, \mathbb{Z})) \).

- Conversely, the embedding \( E : \Lambda \cong \Lambda \times \{0\} \to \text{SL}(2, \mathbb{Z}) \) defined by the bijection \( \Pi \times F : \text{SL}(2, \mathbb{Z}) \to \Lambda \times \mathbb{Z} \) induces an isometric embedding

\[ \hat{E} : \ell^2(\Lambda) \to U, \quad (\hat{E} \phi)(y) = \begin{cases} \phi(E^{-1}(y)) & \text{if } F(y) = 0 \\ 0 & \text{else} \end{cases} \]

of Hilbert spaces (and \( \hat{E}(\ell^2(\Lambda)) = U_0 \)).

- We begin with the subspace \( K^+ + \Lambda \) in (2.33). Due to Lemma 2.3 the spectra of \( T_{\Lambda}^+ \) on the \( J \)-subspaces \( K^+ \), \( \pm \Lambda \) equal \( \{-\frac{1}{2}\} \) respectively \( \{\frac{1}{2}\} \). \( K^+ \), \( \pm \Lambda \) are non-trivial, since the eigenfunctions \( v_g \) of \( T_{\Lambda}^+ \), defined in (2.20) are in the domain \( U \), but not in the kernel of the projection operator \( \hat{\Pi} \):

- It is absolutely summable over the fibers of (2.31) at any \( h \in \text{SL}(2, \mathbb{Z}) \), since \( \sum_{m=-\infty}^{\infty} 2^{-|m|} = 3 < \infty \) for \( h = g \), and similar else.

- It is non-vanishing, since the projection at \( g \) is \( (\hat{\Pi} v_g)(y) = \sum_{m \in \mathbb{Z}} (-2)^{-|m|} = \frac{1}{3} \).

- To show that the spectrum of \( T_{\Lambda}^+ \) on \( \mathcal{N}_{\Lambda}^+ \) is contained in the circle \( C \), we use (2.7) and prove that \( \text{spec}(B_{\Lambda}^+ \mathcal{N}_{\Lambda}^+) \subseteq \left[ -\sqrt{8}/3, \sqrt{8}/3 \right] \). For that we compare \( B_{\Lambda}^+ \) with \( B_{\text{SL}}^+ \), whose resolvent on \( \mathcal{N}_{\text{SL}}^+ \) was given in (2.25). \( B_{\Lambda}^+ \) on the Hilbert space \( \mathcal{N}_{\Lambda}^+ \) is analyzed with the help of the commutative diagram (indeed, \( B_{\text{SL}} \) restricts to \( U \))

\[
\begin{array}{c}
U \\ \hat{\Pi} \\
\downarrow \\
\ell^2(\Lambda)
\end{array}
\xrightarrow{B_{\text{SL}}}
\begin{array}{c}
U \\ \hat{\Pi} \\
\downarrow \\
\ell^2(\Lambda)
\end{array}
\] (2.36)

The operators \( D_{\text{SL}}(k) \) from (2.23) map by Lemma 2.8 the subspace \( U_0 \) into \( U_k \).

Therefore, by (2.35) for \( \Re(c) > \frac{1}{2} \) the resolvent \( D_{\text{SL}}(c) = (B_{\text{SL}}^+ - \lambda(c) \mathbb{1})^{-1} \) of \( B_{\text{SL}}^+ \) has the property

\( D_{\text{SL}}(c)(U_0^+) \subseteq U^+ \) and \( (B_{\Lambda}^+ - \lambda(c) \mathbb{1})^{-1} = \hat{\Pi} \circ (B_{\text{SL}}^+ - \lambda(c) \mathbb{1})^{-1} \circ \hat{E} \)

is bounded. This shows that \( \text{spec}(B_{\Lambda}^+ \mathcal{N}_{\Lambda}^+) \subseteq \left[ -\sqrt{8}/3, \sqrt{8}/3 \right] \). Prop. 2.4 then implies that \( \text{spec}(T_{\Lambda}^+ \mathcal{N}_{\Lambda}^+) \subseteq C \).

- The converse inclusion \( \text{spec}(T_{\Lambda}^+ \mathcal{N}_{\Lambda}^+) \supseteq C \) is provided by (2.26), together
with Lemma 2.8. They imply divergence of the partial sums for the resolvent $(B^+_\Lambda - \lambda(c))^{-1}$ in the case $\Re(c) < \frac{1}{2}$.

- The absolute continuity of $\text{spec}(B^+_\text{SL} \upharpoonright \Lambda^+_\text{SL})$ follows from absolute continuity of $\text{spec}(B^+_\Lambda \upharpoonright \Lambda^+_\Lambda)$; the one of $\text{spec}(T^+_\Lambda \upharpoonright \Lambda^+_\Lambda)$ then follows using the general formula (2.14).

2.4 Representations of $\text{SL}(2, \mathbb{Z}/n\mathbb{Z})$

We denote the binary modular congruence groups by

$$G_n := \text{SL}(2, \mathbb{Z}/n\mathbb{Z}) \quad (n \in \mathbb{N}).$$

Via the surjective homomorphism $\text{SL}(2, \mathbb{Z}) \to G_n$ the group $\text{SL}(2, \mathbb{Z})$ acts on the Hilbert space

$$h(n) := \ell^2((\mathbb{Z}/n\mathbb{Z})^2)$$

by the permutation representation. As remarked above, the $G_n$–orbit

$$\Lambda(n) = \{ (a, b) \in (\mathbb{Z}/n\mathbb{Z})^2 \mid \gcd(a, b, n) = 1 \} \quad (n \in \mathbb{N}) \quad (2.37)$$

has cardinality $|\Lambda(n)| = J_2(n) = n^2 \prod_{p \mid n}(1 - p^{-2})$. To it we associate the $|\Lambda(n)|$–dimensional, $\text{SL}(2, \mathbb{Z})$-invariant subspace

$$\tilde{h}(n) := \{ f \in h(n) \mid \text{supp}(f) \subseteq \Lambda(n) \}.$$

We use small bold letters to abbreviate operators on $h(n)$. So with $T$ from (1.11)

$$t(n) := T_{(\mathbb{Z}/n\mathbb{Z})^2} \in B(h(n)) \quad \text{and} \quad \tilde{t}(n) := T_{\Lambda(n)} \in B(\tilde{h}(n)). \quad (2.38)$$

The spectral theory of the operators $\tilde{t}(n)$ on $\tilde{h}(n)$ is related to the representation theory of the group $G_n$, since the action of $\text{SL}(2, \mathbb{Z})$ on $\Lambda(n)$ gives rise to a representation of $G_n$ on $\tilde{h}(n)$.

Unitary Fourier transform is denoted by

$$F_n : h(n) \to h(n) \; , \; (F_n f)(\ell) = n^{-1} \sum_{k \in (\mathbb{Z}/n\mathbb{Z})^2} \exp(-2\pi i (k, \ell)/n) f(k) \quad (n \in \mathbb{N}).$$

2.10 Lemma On the Hilbert spaces $h(n)$, with $\tilde{j}(n) := J_{(\mathbb{Z}/n\mathbb{Z})^2}$

$$[F_n, \tilde{j}(n)] = 0 \; , \; [F_n, b(n)] = 0 \; \text{and} \; [\tilde{j}(n), F_n, \tilde{t}(n)] = 0.$$
Proof. • In general for $O \in \text{SL}(2, \mathbb{Z})$ represented by $o(n)$ we get
\[ F_n o(n) f(k) = n^{-1} \sum_{\ell \in (\mathbb{Z}/n\mathbb{Z})^2} \exp \left( -2\pi i \langle k, \ell \rangle / n \right) f \circ O^{-1}(\ell) \]
\[ = n^{-1} \sum_{m \in (\mathbb{Z}/n\mathbb{Z})^2} \exp \left( -2\pi i \langle k, O(m) \rangle / n \right) f(m) \]
\[ = n^{-1} \sum_{m \in (\mathbb{Z}/n\mathbb{Z})^2} \exp \left( -2\pi i \langle O^\top(k), m \rangle / n \right) f(m) \]
\[ = (F_n f)(O^\top(k)) = ((o(n)^\top)^{-1} F_n f)(k). \]

• As $(J^\top)^{-1} = J$, $[F_n, j(n)] = 0$,
• and as $L^\top = R$, $F_n$ permutes the representations $l(n)$ and $r(n)^*$. So $F_n$ conjugates $t(n)$ and $t(n)^*$. But the same does $j(n)$, see (2.3). So $[j(n) F_n, t(n)] = 0$.
• The case of $b(n) \equiv B = \frac{1}{2}(I - R^{-1}L - L^{-1}R - LR^{-1} - RL^{-1})$ is similar. \( \square \)

$j(n) F_n$ is not a multiple of the identity if $n \in \mathbb{N} \setminus \{1\}$, although both $j(n)$ and $F_n$ have the same square $i(n)$. For an eigenfunction $f \in \hat{h}(n)$ of $t(n)$ by Lemma 2.10 we either obtain a linearly independent eigenfunction $j(n) F_n f$ or a non-trivial symmetry of $f$. The second alternative applies, e.g., to the mean zero eigenfunctions $f \in \hat{h}(n)$ of $i(n)$ if $n \in \mathbb{P}$.

For the prime case $n \in \mathbb{P}$ the spectral theory of the operators $i(n)$ is partly done in [Kn4]. So we review here the theory of $G_n$ representations for arbitrary $n \in \mathbb{N}$ and then apply it to the operators $i(n)$.

The following example shows that the operators $t(n)$ need not be semisimple.

2.11 Example The operator $t(6)$ is defective. The vector $v \in h^{-}(6)$,
\[ v := (0 1 0 0 0 1 -1 0 0 0 0 0 -1 0 1 0 0 1 0 -1 0 1 0 0 1 0 0 0 0 0) \]
(in lexicographical order of $(\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ in the least residue system modulo 6), is a generalized $t(6)$-eigenvector of eigenvalue 0, with $t(6)^2 v = 0$, but $t(6)v \neq 0$. \( \diamond \)

2.4.1 The Graph of the Regular $G_n$-Representation

By general wisdom the regular representation of $G_n$ contains all irreducible representations, with multiplicity given by cardinality of conjugacy classes. The operator $T_{G_n}$ on $\ell^2(G_n)$ thus restricts to these subrepresentations.

For $n > 2$ the operator $T_{G_n}^+$ is related to the Laplacians of a graph $(V_n, E_n)$ in a way that is analogous to the case of $T_{SL}^+$ treated in Section 2.2.

• Again the bipartite vertex set $V_n = V_n^+ \cup V_n^-$ is composed of the set $V_n^+$ of orbits of $-R^{-1}L \in G_n$ and the orbit set $V_n^+$ of $-L^{-1}R \in G_n$. 

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• The edge set $E_n$ equals the group, the edges $v^-, v^+$ connecting orbits $v^±$ with $v^- \cap v^+ = \{g\}$ with $g \in G_n$.

Figure 2.2 shows the graph of the group $G_5$.

2.12 Proposition For $n \in \mathbb{N}$, $n \geq 3$ with prime decomposition $n = \prod_{i=1}^{s} p_i^{k_i}$
• the graph $(V_n, E_n)$ is three-regular and thus has $|V_n| = \frac{2}{3}|E_n|$ vertices, with the number $|E_n| = |G_n| = \prod_{i=1}^{s} p_i^{3k_i-2}(p_i^2 - 1)$ of edges.
• For odd $n$ it is connected. If $n/2$ is odd, it has two components and for $4|n$ it has four components. All components are isomorphic.
• The girth of a graph $(V, E)$ being the length of a shortest cycle,

$$\text{girth}(V_n, E_n) \geq 2 \left[ \cosh^{-1}\left( \frac{\sqrt{5n}}{4} \right) \right] - 2 \sim 2 \log_\Phi(n) \quad \text{with golden ratio } \Phi.$$

Proof. • For $n \geq 3$ the matrices $-R^{-1}L = (-1 -1, -1 1)$, $-LR^{-1} = (1 -1, 0 1)$ and $(-LR^{-1})^2 = (1 1, 0 1)$ are all different in $G_n$. Thus all orbits are of order three, and $(V_n, E_n)$ is three-regular.
• The group $\text{GL}(2, \mathbb{Z}/n\mathbb{Z})$ of invertible matrices over the residue class ring $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\text{GL}(2, \mathbb{Z}/p_1^{k_1}\mathbb{Z}) \times \ldots \times \text{GL}(2, \mathbb{Z}/p_s^{k_s}\mathbb{Z})$. Moreover,

$$|\text{GL}(2, \mathbb{Z}/p^k\mathbb{Z})| = p^{4k-1}|\text{GL}(2, \mathbb{Z}/p\mathbb{Z})|, \quad \text{with } |\text{GL}(2, \mathbb{Z}/p\mathbb{Z})| = (p^2-1)(p^2-p),$$

the formula going back to Camille Jordan. $\text{SL}(2, \mathbb{Z}/n\mathbb{Z})$ is the kernel of the determinant homomorphism of $\text{GL}(2, \mathbb{Z}/n\mathbb{Z})$. So with $\varphi(p^k) = (p-1)p^{k-1}$ the cardinality equals

$$|\text{SL}(2, \mathbb{Z}/p^k\mathbb{Z})| = |\text{GL}(2, \mathbb{Z}/p^k\mathbb{Z})|/\varphi(p^k) = p^{3k-2}(p^2-1). \quad (2.39)$$

We also have the group isomorphism

$$\text{SL}(2, \mathbb{Z}/n\mathbb{Z}) \cong \text{SL}(2, \mathbb{Z}/p_1^{k_1}\mathbb{Z}) \times \ldots \times \text{SL}(2, \mathbb{Z}/p_s^{k_s}\mathbb{Z}), \quad (2.40)$$

since the above homomorphism for $\text{GL}(2, \mathbb{Z}/n\mathbb{Z})$ respects the determinant 1.
• We first show that for $2|n$ there are at least two graph components in $(V_n, E_n)$. Then the homomorphism $G_n \to G_2$ shows, like in (2.18), that the identity $(1 0, 0 1) \in G_n$ lies in a component projecting onto the cyclic subgroup $C_3$ of $G_2$ of index 2, missing the coset space of $J = (0 1, 1 0)$.

For $4|n$ the homomorphism $G_n \to G_4$ shows additionally that the image of the identity component does not contain $I = (-1 0, 0 -1) = J^2 \in G_4$. Unlike in $G_2$, this is not the identity. So the graph has at least four components, corresponding to the powers of $J$. 

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We begin with the case \( \text{SL}(2, \mathbb{Z}) \) and thus is covered by a graph with edge set \( \text{PSL}(2, \mathbb{Z}) \), consisting of four three-regular trees. As the latter graph covers \((V_n, E_n)\), the latter has no more components.

- Multiplication by \( J^k \) provides graph isomorphisms of the components.
- The norms of the generators \(-R^{-1}L, -LR^{-1} \in \text{SL}(2, \mathbb{Z})\) and their inverses equal the golden ratio \( \Phi = \frac{1}{2}(\sqrt{5}+1) \). So all their \( k \)-fold products \( X = (x_{1,1} x_{1,2}) \) have norms \( \|X\| \leq \Phi^k \). By induction \( \max(|x_{i,j}|) \leq f_{k+1}, f_{\ell} \) being the \( \ell \)-th Fibonacci number, and the next to maximal \( |x_{i,j}| \leq f_k \). So for values of \( k \) with \( f_k \leq n/2 \), all \( X \) are different mod \( n \). Solving for \( k \), using Binet’s formula \( f_k = (\Phi^k - (-\Phi)^{-k})/\sqrt{5} \) gives the result. \( \square \)

### 2.4.2 Representations of \( \text{SL}(2, \mathbb{Z}/n\mathbb{Z}) \) for Primes \( n \)

We begin with the case \( n = p \in \mathbb{P} \) of primes, with the field \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \). It is then known (see Naimark and Štern [NS], Section II, §5, and Lafferty and Rockmore [LR], Section 2) that the irreducible representations of \( \text{SL}(2, \mathbb{F}_p) \) are divided into the classes of principal respectively discrete (or cuspidal) series. Denoting by

\[
B_n := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in G_n \right\} \quad \text{and} \quad U_n := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in B_n \right\} \quad (n \in \mathbb{N})
\]

the Borel respectively unipotent subgroups, the discrete series representations of \( G_p \) are characterized by the property that their restriction to \( U_p \) does not contain the trivial representation.

The principal series representations of \( G_p \) are the irreducible subrepresentations of those induced from the Borel subgroup \( B_p \).

#### 2.13 Remark (Principal series reps and the Hilbert space \( \tilde{h}(p) \))

The appearance of the principal series representation for the Hilbert space \( \tilde{h}(p) = \ell^2(\Lambda(p)) \) is explained as follows.

1. First, we consider \( \tilde{h}(n) \) for \( n \in \mathbb{N} \) as the subspace of \( \ell^2(G_n) \), invariant under the right action of the unipotent subgroup \( U_n \). As such, it is invariant under the left action \( \hat{L}_g \) of the \( g \in G_n \) on \( \ell^2(G_n) \): If \( \hat{R} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f = f \), then

\[
\hat{R} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \hat{L}_g f(x) = \hat{L}_g f \left( x \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = f \left( g^{-1} x \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = f(g^{-1} x) = \hat{L}_g f(x).
\]

2. For a generator \( \alpha \) of the cyclic group \( (\mathbb{F}_p^\times, \cdot) \) the characters on \( \mathbb{F}_p^\times \) are

\[
\psi_j : \mathbb{F}_p^\times \to S^1, \quad \psi_j(\alpha^k) = \exp \left( 2\pi i j k/(p-1) \right) \quad (j \in \{0, \ldots, p-2\}).
\]

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Figure 2.2: The graph of $SL(2, \mathbb{F}_5)$. The 120 edges are the group elements, the identity being shown in black. Its four neighbors (grey) are $-R^{-1}L$, $-L^{-1}R$ and their inverses. Green: Borel subgroup. The graph is bipartite, with $V^+$ and $V^-$ colored differently. It has the (large) girth = 10 and is Ramanujan.

Considering $\ell^2(B_n)$ as the subspace of $\ell^2(G_n)$ given by the functions vanishing outside the Borel subgroup, for all $j$ and primes $p$ the character

$$\tilde{\psi}_j : B_p \to S^1 , \quad \tilde{\psi}_j\left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \psi_j(a)$$

is in $\ell^2(B_p) \cap \tilde{h}(p)$.

3. More generally, the subspaces

$$\text{Ind}_{p,j} := \{ f \in \ell^2(G_p) \mid \forall g \in G_p, b \in B_p : f(gb) = \tilde{\psi}_j(b)f(g) \} \quad (2.41)$$

are contained in $\tilde{h}(p)$, and give rise to the induced representations

$$\rho_{p,j} : G_p \to \text{GL(Ind}_{p,j}) \quad , \quad (\rho_{p,j}(g)f)(x) = f(g^{-1}x)$$

of the Borel group $B_p$.

4. As the $(p+1)$–dimensional subspaces $\text{Ind}_{p,j}$, $j \in \{0, \ldots, p-2\}$ of $\tilde{h}(p)$ are mutually orthogonal, and $\dim(\tilde{h}(p)) = p^2 - 1$, we obtain the orthogonal decomposition

$$\tilde{h}(p) = \bigoplus_{j=0}^{p-2} \text{Ind}_{p,j}$$

\footnote{Observe the non-standard use of left and right actions.}
The above family of induced representations contains every irreducible principal series representation derived from the character \( \psi_j \) with multiplicity two if \( \psi_{p-1-j}^2 \neq 1 \) (since \( \psi_{p-1-j} = \psi_j^{-1} \)), and with multiplicity one otherwise.

2.4.3 Representations of \( SL(2, \mathbb{Z}/n\mathbb{Z}) \) for Prime Powers \( n = p^k \)

In Nobs and Wolfart [NWI, NWII] all irreducible representations for \( n = p^k, p \in \mathbb{P} \), are determined. The method (going back to Weil, to Kloosterman and to Tanaka) to find the representations is to consider transformation properties of theta functions associated to binary quadratic forms. The case \( p = 2 \) needs special treatment. This is an issue to be clarified, since [BV], see below, use in their Section 2 Lemma 7.1 of [BG2], which only refers to odd prime powers.

Inspection of the tables in Section 9 of [NWII], however, shows that the dimensions of all such new representations (the old ones being those already arising for \( p^\ell, 0 \leq \ell < k \)) are bounded below by \( 3n/16 \), including the case \( n = 2^k \). Although [BV] use a lower bound \( n/3 \) (valid for the odd prime powers), this does not change the argument, since linear growth in \( n \) implies the existence of an \( n \)-independent spectral gap.

2.4.4 Representations of \( SL(2, \mathbb{Z}/n\mathbb{Z}) \) for Arbitrary Integers \( n \)

Using (2.40), \( G_n \) is isomorphic to a direct product of the \( G_q \), with the prime power \( q \) appearing in the factorization of \( n \). So the irreducible (unitary) representations of \( G_n \) arise as tensor products of irreducible representations.

As a consequence of (2.39) and Sect. 2.4.3, the dimensions of faithful representations are bounded below by \( c_\alpha n^\alpha \), for any \( \alpha \in (0, 1) \). This follows from the estimate \( \omega(n) = o(\log(n)) \) for the number of distinct prime factors of \( n \), and can be used in going from the case of prime powers \( n \) to general \( n \in \mathbb{N} \), see the proof of Theorem 1 in Bourgain and Varjú [BV].

A spectral estimate going back to Sarnak and Xue and used in Bourgain and Gamburd [BG] uses that each irreducible representation appears in the regular representation with multiplicity its dimension. [BV] then show, given a finite symmetric generating set \( S \subseteq SL(d, \mathbb{Z}) \) for its image \( S_n \subseteq SL(d, \mathbb{Z}/n\mathbb{Z}) \) the expansion property, uniform in \( n \in \mathbb{N} \). For our context, this implies that the vertex Laplacians \( \Delta_{V(n)} \) have a spectral gap uniform in \( n \in \mathbb{N} \).

2.14 Proposition There exists an \( \varepsilon > 0 \), such that for \( G_n = SL(2, \mathbb{Z}/n\mathbb{Z}) \)

\[ \text{spec}(T_{G_n}^+) \subseteq C \cup I_\varepsilon \cup \{ \pm \frac{1}{2}, 1 \} \quad \text{and} \quad \text{spec}(\tilde{t}^+(n)) \subseteq C \cup I_\varepsilon \cup \{ \pm \frac{1}{2}, 1 \} \quad (n \in \mathbb{N}), \]

with the circle \( C = S^1/\sqrt{2} \) and \( I_\varepsilon := \{ x \in \mathbb{R} \mid |x| \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon] \setminus \{ 1/\sqrt{2} \} \}. \]
Proof. • The first inclusion follows from the result on $\Delta_{V(n)}$ via the relation $\Delta_{E(n)} = 3(1 - \mathbb{B}_{G_n}^\ell)$ and (2.14).

• By considering the Hilbert space $\tilde{h}^+(n)$ as a subspace of $\ell^2(G_n)$ as in Remark 2.13,

$$\tilde{h}^+(n) = T^+_G|_{\tilde{h}(n)} \quad (n \in \mathbb{N}).$$

So the statement about the spectra of the operators $\tilde{h}^+(n)$ follows from the one for $T^+_G$. \hfill \Box

2.5 Finite Adeles

We now consider the set $S = \mathbb{P}$ of places, i.e. the ring $\hat{\mathbb{Z}} = \mathbb{Z}_\mathbb{P}$ of finite integral adeles. The inverse limit

$$\mathbb{Z}_\mathbb{P} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$$

gives rise to natural homomorphisms

$$\pi_n : \mathbb{Z}_\mathbb{P} \to \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad \pi^2_n : \mathbb{Z}^2_\mathbb{P} \to (\mathbb{Z}/n\mathbb{Z})^2 \quad (n \in \mathbb{N}).$$

Functions $f \in h(n) = \ell^2((\mathbb{Z}/n\mathbb{Z})^2)$ pull back to $f \circ \pi^2_n \in \mathcal{H}_\mathbb{P}$, the Hilbert space for the finite integral adeles. These are locally constant (or Schwartz–Bruhat) functions, and with our normalization convention (Haar measure having total mass one) the embeddings $h(n) \hookrightarrow \mathcal{H}_\mathbb{P}$ are isometric.

If we thus consider $h(n)$ as a subspace of $\mathcal{H}_\mathbb{P}$, the latter Hilbert space is a (non–direct) sum of the $h(n)$ ($n \in \mathbb{N}$), defined in (2.38). This is implied by the facts that

• every Schwartz–Bruhat function on $\hat{\mathbb{Z}}^2$ is a finite linear combination of characteristic functions $\mathbb{1}_{(a+b)^2}$ with $n \in \mathbb{N}$ and $a,b \in \hat{\mathbb{Z}}$ (following e.g. from Lemma 5.4.7 in Deitmar [De]),

• the space of Schwartz–Bruhat functions on $\hat{\mathbb{Z}}^2$ is dense in $\mathcal{H}_\mathbb{P} = L^2(\hat{\mathbb{Z}}^2, m^2_\mathbb{P})$.

To obtain a direct sum decomposition, we first note that the natural homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m|n$ give rise to isometric embeddings $h(m) \hookrightarrow h(n)$.

Then the orthogonal decompositions

$$h(n) = \bigoplus_{m \in \mathbb{N} : m|n} \tilde{h}(m)$$

(Lemma 3 of [Kn4]) lead to the orthogonal direct sum

$$\mathcal{H}_\mathbb{P} = \bigoplus_{m \in \mathbb{N}} \tilde{h}(m). \quad (2.42)$$
Proof. Eigenvectors \( f \in h(n) \) of \( t(n) \) pull back to eigenvectors \( f \circ \pi_n^2 \in \mathcal{H}_n^+ \) of \( T_n^+ \), with the same eigenvalue.

We split the Hilbert spaces and operators orthogonally into

\[
h(n) = h^+(n) \oplus h^-(n) \quad \text{respectively} \quad t(n) = t^+(n) \oplus t^-(n).
\]

\( h^\pm(n) \) can be considered as finite dimensional subspaces of \( \mathcal{H}_n^\pm \), and \( t^\pm(n) \) the restriction of \( T_n^\pm \).

In analogy with the \textit{oldform / newform} calculus of the theory of modular forms
we need only determine the action of \( T_n^\pm \) on the subspaces \( \tilde{h}^\pm(n) := h(n) \cap h^\pm(n) \)
of \( h^\pm(n) \).

Then the restricted operators \( \tilde{t}^\pm(n) := t^\pm(n)|_{\tilde{h}^\pm(n)} \) have the property that

\[
\bigoplus_{k \in \mathbb{N} : k | n} \tilde{h}^\pm(k) \cong h^\pm(n) \quad \text{and} \quad \bigoplus_{k \in \mathbb{N} : k | n} \tilde{t}^\pm(k) \cong t^\pm(n). \tag{2.43}
\]

So the operators \( \tilde{t}^\pm(n) := \bigoplus_{k \in \mathbb{N} : k \leq n} \tilde{t}^\pm(k) \) \((n \in \mathbb{N})\) converge strongly to \( T_n^\pm \), and

\[
\text{spec}(t^\pm(n)) \subseteq \text{spec}(t^\pm(n + 1)) \subseteq \text{spec}(T_n^\pm). \tag{2.44}
\]

Similar to the previous sections, we consider the operators \( B_{n}^{\pm,\pm} \) and \( b^{+,\pm}(n) \)
\((n \in \mathbb{N})\), defined by (2.9) and related to \( T_n^\pm \) respectively \( t^\pm(n) \) via (2.7):

\[
B_{n}^{\pm,\pm} = \mp \frac{1}{2}(2T_{n}^{\pm,\pm} + (T_{n}^{\pm,\pm})^{-1}) \quad \text{and} \quad b^{+,\pm}(n) = \mp \frac{1}{2}(2t_{n}^{\pm,\pm} + (t_{n}^{\pm,\pm})^{-1}). \tag{2.45}
\]

2.15 Remark \textbf{(Spectra for direct sums)} The following lemma is not \textit{a priori} obvious, since the operator \( T_n^\pm \) is not normal. \textit{E.g.} there is the example of the direct sum \( \bigoplus_{k=2}^\infty N_k \) of nilpotent Jordan matrices \( N_k \) of size \( k \) which has the closed unit disk as its spectrum (Problem 98 in Halmos, [Ha1]). \hfill \diamond

2.16 Lemma \( \bigcup_{n \in \mathbb{N}} \text{spec}(t^\pm(n)) = \text{spec}(T_n^\pm). \)

\textbf{Proof.} \bullet Closedness of spectra and (2.44) imply \( \bigcup_{n \in \mathbb{N}} \text{spec}(t^\pm(n)) \subseteq \text{spec}(T_n^\pm). \)

\bullet By (2.7) the operators \( b^{+,\pm}(n) \) and \( B_{n}^{\pm,\pm} \) are self-adjoint and bounded by 1. Holomorphic functional calculus shows that their spectra have inclusion properties analogous to (2.44), and \( s - \lim_{n \to \infty} b^{+,\pm}(n) = B_{n}^{\pm,\pm} \). So by using the resolvent estimate for the normal operators \( b^{+,\pm}(n) \) with \( \lambda \in \mathbb{C} \) in the resolvent set

\[
\| (b^{+,\pm}(n) - \lambda \mathbb{I})^{-1} \| = 1 / \text{dist}(\lambda, \text{spec}(b^{+,\pm}(n))),
\]

we obtain \( \bigcup_{n \in \mathbb{N}} \text{spec}(b^{+,\pm}(n)) = \text{spec}(B_{n}^{\pm,\pm}). \tag{2.14} \) implies the lemma. \hfill \square

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2.17 Proposition

The numbers $\pm \frac{1}{2}$ and 1 are eigenvalues of infinite multiplicity of the operator $T_P^+$ on $H_P^+$, and there is a $\varepsilon > 0$ with

$$\text{spec}(T_P^+) \subseteq \{-\frac{1}{2}, \frac{1}{2}, 1\} \cup C \cup I_\varepsilon \quad \text{and} \quad \text{spec}(T_P^+) \cap I_\varepsilon \neq \emptyset$$

with $C = S^1/\sqrt{2}$ and $I_\varepsilon = \{x \in \mathbb{R} \mid |x| \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon] \setminus \{1/\sqrt{2}\}\}$.

**Proof.**

1. Since for all integers $n \in \mathbb{N}$, $\Lambda(n) \subseteq (\mathbb{Z}/n\mathbb{Z})^2$ is a single $\text{SL}(2, \mathbb{Z})$-orbit, and $\text{SL}(2, \mathbb{Z})$ is generated by $L$ and $R$, 1 is an eigenvalue of multiplicity one for the operator $t^+(n)$ on the subspace $\tilde{h}^+(n) \subseteq \ell^2(\Lambda(n))$.

So 1 is of multiplicity $\sigma_0(n)$ for $t^+(n)$ (with the divisor function $\sigma_0$), of multiplicity $n$ for $t^+(n)$ and of infinite multiplicity for $T_P^+$.

2. That $\pm \frac{1}{2}$ are eigenvalues of $T_P^+$ can already be inferred from the case $n = 2$ with Hilbert space $\tilde{h}^+(2) = h(2)$. The matrix $t(2) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ (w.r.t. the basis given by the lexical listing $(00, 01, 10, 11)$ of $(\mathbb{Z}/2\mathbb{Z})^2$) has the eigenvalues $1, -\frac{1}{2}$ with eigenvector $(0 -1 -1 1)^T$ and $\frac{1}{2}$ with eigenvector $(0 -1 1 0)^T$.

In Proposition 11 of [Kn4] the multiplicities of the eigenvalues $\pm \frac{1}{2}$ were calculated for all $n \in \mathbb{P}$ in ‘projective’ (that is, dilation invariant) subspaces of $\tilde{h}(n)$, using quadratic reciprocity. In particular these multiplicities are positive for both
signs and all \( n \in \mathbb{P} \setminus \{3, 7\} \).

This shows that the eigenvalues \( \pm \frac{1}{2} \) of \( T_0^+ \) are of infinite multiplicity.

- The first nontrivial (that is, on \( I_\varepsilon \)) real eigenvalues of \( h^+(n) \) arise for \( n = 34 \).
- They are the real roots of the polynomial \( 64x^{12} - 64x^{11} + 64x^{10} - 64x^9 + 36x^8 - 26x^7 + 18x^6 - 13x^5 + 9x^4 - 8x^3 + 4x^2 - 2x + 1 \) and equal to \( 0.819427 \ldots \) respectively. 

- \( \frac{0.610182 \ldots}{\frac{1}{2}} = 1/(2 \times 0.819427 \ldots) \), symmetric w.r.t. the circle \( C \).

- That the part \( (\text{spec}(T_0^+)) \setminus \{\pm \frac{1}{2}, 1\} \cap \mathbb{R} \) of the spectrum is included in \( I_\varepsilon \) for some \( \varepsilon > 0 \), is the generalization of Proposition 15 of \cite{Kn4} from the projective subspace to the Hilbert space \( \tilde{h}^+ + (n) \), and from primes to general integers \( n \).

The argument in \cite{Kn4} used expander estimates for the Laplacians \( 3(1 - b^+ \pm (n)) \), \( n \in \mathbb{P} \), based on Selberg's Theorem for the congruence subgroups of \( \text{SL}(2, \mathbb{Z}) \) (see Lubotzky \cite{Lu}, Section 4.3 and 4.4). Here we use for \( n \in \mathbb{N} \) Proposition 2.14, based on the estimates by Bourgain, Gamburd and Varjú, as explained in Section 2.4.4 above.

\[ \square \]

**2.18 Remark (Spectral gap)** Proposition 2.17 says that \( T_0^+ \) has a spectral gap, but the gap is not of maximal possible size, since \( \text{spec}(T_0^+) \cap I_\varepsilon \neq \emptyset \), that is, eigenvalues \( \lambda \neq 1 \) of modulus \( |\lambda| > 1/\sqrt{2} \) occur.

One mechanism to reconcile this with RH could be that eigenvalues of \( t^+(n) \), which are not equal \( \pm \frac{1}{2} \) and do not already occur for \( t(m) \) with \( m|n \), have modulus going to \( 1/\sqrt{2} \) as \( n \to \infty \).

By Proposition 2.17 the operator \( T_0^+ \) has a highly degenerate spectrum. We now partly lift that degeneracy without changing the spectrum, by restricting it to the \( \text{SL}(2, \mathbb{Z}) \)-invariant subspace

\[ \mathcal{H}_{\Lambda_P} := L^2(\Lambda_P, m_P^2|_{\Lambda_P}) \quad \text{of} \quad \mathcal{H}_P = L^2(\mathbb{Z}^2, m_P^2). \]

Here

\[ \Lambda_P := \prod_{p \in \mathbb{P}} \Lambda_p \quad \text{with} \quad \Lambda_p := \mathbb{Z}_p^2 \setminus p \mathbb{Z}_p^2 \quad \text{equals} \quad \lim_{n \to \infty} \Lambda(n), \quad (2.46) \]

the inverse limit being defined w.r.t. the homomorphisms

\[ \pi_{n,m} : (\mathbb{Z}/n\mathbb{Z})^2 \to (\mathbb{Z}/m\mathbb{Z})^2, \quad (\ell + n\mathbb{Z}) \mapsto (\ell + m\mathbb{Z}) \quad (m|n) \in \mathbb{N}, \]

restricted to \( \Lambda(n) \). By the product formulas (2.46) for \( \Lambda_P \) and (1.10) for \( m_P^2 \) the restricted Haar measure \( m_P^2|_{\Lambda_P} \) has total mass \( 1/\zeta(2) \). By \( \text{SL}(2, \mathbb{Z}) \)-invariance of \( \mathcal{H}_{\Lambda_P} \) the operator

\[ T_{\Lambda_P} := T_P|_{\mathcal{H}_{\Lambda_P}} \]

acts on the Hilbert space \( \mathcal{H}_{\Lambda_P} \).

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It is related to the (non-discrete) Markov chain with state space \( \Lambda_P \) and stochastic kernel
\[
\kappa : \Lambda_P \times B(\Lambda_P) \to [0, 1], \quad \kappa(x, A) = \frac{1}{2}(\delta_{L(x)} + \delta_{R(x)})(A).
\]
The weak Markov property is standard (see, e.g. Klenke [Kle], Theorem 17.11).

The chain is not irreducible in the sense of Nummelin [Num]. However, the SL(2, \( \mathbb{Z} \))–action on \( (\Lambda_P, m_P^2 |_{\Lambda_P}) \) is ‘irreducible-aperiodic’ in the weak sense that its projections to \( \Lambda(n) (n \in \mathbb{N}) \) are, by Lemma 4 of [Kn4].

Grigorchuk showed in [Gr], Theorem 1 an individual ergodic theorem for the Cesàro means. In our context it says that for \( f \in L^p(\Lambda_P) \) with \( p \in [1, \infty) \)
\[
\bar{f} := \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^{i}_{\Lambda_P} f \in L^p(\Lambda_P),
\]
with the same \( L^1(\Lambda_P) \) expectation as \( f \), and \( \bar{f} \) is invariant under the action of the semigroup generated by \( L \) and \( R \) on \( L^p(\Lambda_P) \). Thus \( \bar{f} \) is constant \( m_P^2 |_{\Lambda_P} \)-a.e.

2.19 Lemma \( \text{spec}(T^+_{\Lambda_P}) = \text{spec}(T^+_{P}) \).

The multiplicity of the Perron-Frobenius eigenvalue 1 of \( T^+_{\Lambda_P} \) is one.

Proof. • We only need to show that \( \text{spec}(T^+_{\Lambda_P}) \supseteq \text{spec}(T^+_{P}) \). For that we lift the eigenfunctions of the \( \hat{\ell}^\pm(m) \) in the decomposition (2.43) to \( \hat{\ell}^\pm(n) \), using \( \pi_{n,m} \).

• Clearly the function \( \Lambda_P \to \{1\} \) is eigenfunction of \( T^+_{\Lambda_P} \) with eigenvalue 1. If, on the other hand, \( f : \Lambda_P \to \mathbb{C} \) is an eigenfunction of \( T^+_{\Lambda_P} \) with eigenvalue 1, then it is left-and right-invariant: \( L^+_{\Lambda_P} f = f \) and \( R^+_{\Lambda_P} f = f \). So it is SL(2, \( \mathbb{Z} \))–invariant, and by the above constant a.e. \( \Box \)

This restricted operator is important, since \( \Lambda_P \) is the closure of the SL(2, \( \mathbb{Z} \))–orbit of our initial point \( (1) \in \hat{\mathbb{Z}}^2 \).

2.6 The Adelic Case

We now consider for \( P_\infty = \{\infty\} \cup P \) the adelic Markov operator \( T_{P_\infty} \) on the Hilbert space
\[
\mathcal{H}_{P_\infty} = L^2(\mathbb{Z}^2 \times \hat{\mathbb{Z}}^2, m^2_{P_\infty}).
\]

The following statement is the main result of this article. It characterizes the spectrum of \( T_{P_\infty} \) as the union of spectra of the operator \( T_P \) analyzed in Proposition 2.17 and of an operator \( T_{\Lambda \times \mathbb{Z}^2} \), which in turn has the same spectrum as \( T_{\Lambda} \) from Proposition 2.9.
2.20 Theorem \( \text{spec}(T^+_{\mathcal{P}_\infty}) = \text{spec}(T^+_{\mathcal{P}}) \cup \text{spec}(T^+_{\Lambda \times \hat{Z}^2}) \) with

\[
\text{spec}(T^+_{\Lambda \times \hat{Z}^2}) = \bigcup_{n \in \mathbb{N}} \text{spec}(T^+_{\Lambda \times \Lambda(n)}) = \{-\frac{1}{2}, \frac{1}{2}\} \cup C.
\]

In particular the adelic Markov operator \( T^+_{\mathcal{P}_\infty} \) has a spectral gap.

Proof. \( \ast \) Because of (2.29) and (2.42) the Hilbert space

\[ \mathcal{H}_{\mathcal{P}_\infty} = L^2(\mathbb{Z}^2 \times \hat{\mathbb{Z}}^2, m_\mathcal{P}^2) \cong \ell^2(\mathbb{Z}^2) \otimes L^2(\hat{\mathbb{Z}}^2, m_\mathcal{P}^2) \]

splits into a non-orthogonal direct sum

\[ \mathcal{H}_{\mathcal{P}_\infty} \cong \bigoplus_{m \in \mathbb{N}_0, n \in \mathbb{N}} \ell^2(m \Lambda) \otimes \ell^2(\Lambda(n)) \cong \bigoplus_{m \in \mathbb{N}_0, n \in \mathbb{N}} \ell^2(m \Lambda \times \Lambda(n)). \]

The operator \( T^+_{\mathcal{P}_\infty} \) splits accordingly into a direct sum of \( T^+_{\Lambda \times \Lambda(n)} \).

Since \( \ell^2(0 \Lambda) \cong \mathbb{C} \) and \( \ell^2(m \Lambda) \cong \ell^2(\Lambda) \) (see the proof of Proposition 2.9)

\[ T_{0\Lambda \times \Lambda(n)} \cong t(n) \quad \text{and} \quad T_{m\Lambda \times \Lambda(n)} \cong T^+_{\Lambda \times \Lambda(n)} \quad (m, n \in \mathbb{N}). \quad (2.47) \]

\( \ast \) The first identity follows from the direct sum decomposition of \( T^+_{\mathcal{P}_\infty} \) and (2.47).

\( \ast \) \( T^+_{\Lambda \times \Lambda(n)} = \frac{1}{2}(L^+_{\Lambda \times \Lambda(n)} + R^+_{\Lambda \times \Lambda(n)}) \cong \frac{1}{2}(L^+_{\Lambda} \otimes L^+_{\Lambda(n)} + R^+_{\Lambda} \otimes R^+_{\Lambda(n)}) \) has for all \( n \in \mathbb{N} \) the eigenvalues \( \pm \frac{1}{2} \), since according to Prop. 2.9 \( T^+_{\Lambda} = \frac{1}{2}(L^+_{\Lambda} + R^+_{\Lambda}) \) has such eigenfunctions \( \phi_{\pm} \), and for the constant eigenfunction \( I_{\Lambda(n)} \in h(n) \) of \( L^+_{\Lambda(n)} \) and \( R^+_{\Lambda(n)} \) with eigenvalue one \( \phi_{\pm} \otimes I_{\Lambda(n)} \) are eigenfunctions of \( T^+_{\Lambda \times \Lambda(n)} \) with eigenvalues \( \pm \frac{1}{2} \).

\( \ast \) By a converse argument \( T^+_{\Lambda \times \Lambda(n)} \) does not have the eigenvalues \( \pm 1 \), since these would imply that \( T^+_{\Lambda} \) had these eigenvalues, contradicting Proposition 2.9.

\( \ast \) To show that \( \text{spec}(T^+_{\Lambda \times \Lambda(n)}) \subseteq \{-\frac{1}{2}, \frac{1}{2}\} \cup C \), we first consider the resolvent of \( B^+_{\Lambda \times \Lambda(n)} \). Arguing in a way analogous to the one in the proof of Prop. 2.7, we obtain the estimate

\[ \| D_{\Lambda \times \Lambda(n)}(k) \| \leq 3n k^{2k/2} \quad (k \in \mathbb{N}) \quad (2.48) \]

for the operators (indexed by \( k \in \mathbb{N}_0 \))

\[ D(k) \equiv D_{\Lambda \times \Lambda(n)}(k) \in \mathcal{B}(\mathcal{H}_{V,\Lambda \times \Lambda(n)}) \quad , \quad (D(k)f)(v) := \sum_{w: \text{dist}(v,w)=k} f(w) \]

on the vertex Hilbert space \( \mathcal{H}_{V,\Lambda \times \Lambda(n)} \) of the \( Y_\pm \) action on \( \text{SL}(2, \mathbb{Z}) \times \Lambda(n) \). The constant factor \( n \) in the estimate (2.48) of the \( D_{\Lambda \times \Lambda(n)}(k) \) does not change the convergence properties of the resolvent of \( B^+_{\Lambda \times \Lambda(n)} \) on \( \mathcal{N}^+ \), compared to the one
for $B_{SL}^+$. So $\text{spec}(B_{SL \times \Lambda(n)}^+) \subseteq [-\sqrt{8}/3, \sqrt{8}/3]$. We employ the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{B_{SL \times \Lambda(n)}} & U \\
\uparrow E & & \uparrow \hat{E} \\
\ell^2(\Lambda \times \Lambda(n)) & \xrightarrow{B_{\Lambda \times \Lambda(n)}} & \ell^2(\Lambda \times \Lambda(n))
\end{array}
$$

defined like (2.36) to conclude that $\text{spec}(B_{\Lambda \times \Lambda(n)}^+) \subseteq [-\sqrt{8}/3, \sqrt{8}/3]$, too.  

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