Dynamics of two-qubit entanglement in a self-interacting spin-bath

Francesca Fassioli Olsen, Alexandra Olaya-Castro and Neil F Johnson
Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford, OX1 3PU, UK
E-mail: f.fassioli1@physics.ox.ac.uk

Abstract. It has recently been shown that interactions among the spins of a bath can result in an effective suppression of the decoherence of a single qubit embedded in a multi-spin environment. However, the effects of such intrabath interaction on the dynamics of entanglement are still not well understood. Here we study the dynamics of a system composed of two entangled qubits coupled to (1) independent self-interacting spin baths, and (2) a common self-interacting spin bath. We find that, as in the case of a single qubit, strong interactions among the spins in the bath may induce an effective decoupling between the bath and the system – this in turn suppresses the loss of entanglement that a non-, or weakly, interacting bath would typically cause. However we find that the intrabath interaction strength needed to achieve such effects is much larger than in the single qubit case, especially when the two qubits share the same environment. Our results illustrate the system’s size effects in the entanglement sharing condition, where the entanglement within the bath limits the entanglement between the bath and the system.

1. Introduction

Entanglement occurs as a result of quantum interference of states of composite systems, giving rise to non-classical correlations between spatially separated quantum systems. Understanding and manipulating dynamics of these quantum features are of great importance for both fundamental physics and new emerging quantum technologies. Research in the past decade indicates that it is possible to use entanglement to achieve a new form of computation and communication using quantum information [1]. The versatility of entanglement is however limited by the loss of quantum coherence due to the unavoidable interaction between a quantum system and an environment with many degrees of freedom [2]. Therefore many efforts have been devoted to understand the decoherence process of a basic quantum system represented by a two-level system (qubit) in the presence of a bath of harmonic oscillators [3] or a bath of spins [4, 5]. Recent works have however recognized that the knowledge of individual or local processes of decoherence doesn’t necessarily imply a complete understanding of the dynamics of disentanglement of the state of a composite system [6–9]. In these works the problem of interest has been the entanglement dynamics in the presence of system-environment interaction, providing insights into the understanding of how entanglement decay scales with the system’s size. For example, it has been shown that disentanglement of two qubits can occur in a finite time scale, despite of the fact that the single qubit decoherence decays asymptotically [6]. Here we are interested in the entanglement dynamics of a two-qubit system embedded in a spin-bath with self-interactions [10]. The spin-bath model of decoherence has particularly attracted recent attention because of its relevance to solid state systems [11]. Interestingly, Tessieri and Wilkie [10] showed that decoherence of a single spin coupled to a self-interacting environment is suppressed by strong interactions among the spins of the bath,
contrary to the effects of an otherwise non interacting bath where decoherence is fast and irreversible. Dawson et al [12], have shown that such effects can be explained in terms of entanglement sharing where the intrabath interaction implies an intrabath entanglement that limits the entanglement between the system and the bath, and consequently limits its decohering effects. In this work we extend the Tessieri-Wilkie model to a two-qubit system in the presence of the self-interacting spin environment. We study the dynamics of the two-qubit entanglement in terms of the intrabath coupling, controlled by a single parameter $\lambda$, under different system-bath configurations. We therefore consider the cases where the two qubits are coupled to independent and common baths. We show that, as in the single qubit system, strong self-interactions in the bath can suppress decoherence of the system, and most importantly we qualify the system’s size effects. We find that the strength of the intrabath coupling required to suppress disentanglement is greater than in the case of a single qubit, and such effect is more noticeable in the common bath case. We believe such results can be explained in terms of entanglement sharing inequalities.

This paper is organized as follows. In section 2 we give the microscopic approach to treat an open quantum system. In section 3 we describe the system-bath model and the different cases we are going to study. In section 4 we present our results.

2. Description of an open quantum system

One way to study the dynamics of an open quantum system, i.e a system coupled to an environment with a microscopic theory as follows. The Hamiltonian of the total system, i.e open system plus environment will be of the form of

$$H = H_S \otimes I_B + I_S \otimes H_B + H_{SB}$$  

where the Hamiltonian of the open system and the environment or bath are denoted by $H_S$ and $H_B$ respectively. $H_{SB}$ is the Hamiltonian describing the interaction between the system and the bath, and $I_S$ and $I_B$ are the identities in the Hilbert spaces of the system and the environment respectively. The evolution of the total system will be given by

$$\rho(t) = e^{-iHt} \rho(0) e^{iHt}$$

and the density matrix of the open system is obtained by tracing out the degrees of freedom of the bath

$$\rho_S = tr_B \rho$$

Once we have the density matrix of the system we can measure its entanglement, which in the case of a pair of qubits is known exactly [13]. This can be done by measuring an alternative quantity called the tangle which varies between 0 (no entanglement) and 1 (completely entangled) and increases monotonically with the entanglement. The tangle is given by

$$\tau = (\max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\})^2$$

where the $\lambda_i$ are the square roots in decreasing order of the matrix $\rho \tilde{\rho}$ and $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$, $\sigma_y$ is the Pauli $y$ matrix and $\rho$ is the two-particle density matrix. One question regarding entanglement that naturally arises is whether or not the entanglement between two particles limits the entanglement one of them can have with a third one, or in other words, if we consider qubits A, B and C, does the entanglement between A and B limit the entanglement A and C can have? The answer is yes, and the condition to be satisfied is that the total tangle between qubit A and the rest, $\tau_{A(BC)}$, has to be larger than or equal to the sum of the tangle between qubit A and B, $\tau_{AB}$, plus the tangle between qubit A and C, $\tau_{AC}$, $(\tau_{A(BC)} \geq \tau_{AB} + \tau_{AC})$ [14]. If we consider that the tangle is a positive and equal or less than one quantity this relation immediately yields that if qubit A is completely entangled with qubit B ($\tau_{AB} = 1$), then it cannot be entangled with qubit C at the same time. This has recently been extended to an arbitrary number of qubits [15].
3. Two-qubit system in a self-interacting spin-bath

As mentioned earlier we are going to extend the Tessieri-Wilkie model and consider two central spins coupled to the same bath and to two independent baths, in either cases the Hamiltonian of each bath will be of the form

\[ H_B = \sum_{i=1}^{N} \frac{\omega_i}{2} \sigma_z^i + \beta \sum_{i=1}^{N} \sigma_x^i + \lambda \sum_{i=1}^{N} \sum_{j=i+1}^{N} \sigma_x^i \sigma_x^j \]  

(5)

\( \sigma_x \) and \( \sigma_z \) represent the Pauli \( x \) and \( z \) matrices respectively and \( N \) is the number of spins in each bath.

The interaction between each of the central spins and its bath (common or independent) is described by

\[ H_{SB} = \lambda_0 \sum_{i=1}^{N} \sigma_x^i \sigma_s^x \]  

(6)

and the Hamiltonian of each of the central spins is defined as

\[ H_S = \frac{\omega_0}{2} \sigma_z^s + \beta \sigma_x^s \]  

(7)

The values of the different parameters are \( \omega_i = 1 \) for all \( i \), \( \omega_0 = 0.8288 \), \( \lambda_0 = 1 \) and \( \beta = 0.01 \). The number of spins that will be considered in the common bath case is \( N = 8 \) and for independent baths each one will contain \( N_1 = N_2 = 4 \) spins. The parameter \( \lambda \), see (5), controls the strength of the intrabath coupling and in order to study its effect we will choose it to range from 0 to 3. Units are in terms of \( \omega_i \) and such that \( \hbar = 1 \).

3.1. Initial Conditions

To obtain numerically the time evolution of the total (system-bath) density matrix according to (2), we consider the following initial conditions. The system and environment are separable, which means that the state can be written as

\[ \rho(0) = \rho_S(0) \otimes \rho_B(0) \]  

(8)

The spin bath is assumed to start in thermal equilibrium at low temperature \( T \), \( kT = 0.02 \), so that

\[ \rho_B(0) = \frac{\exp(-H_B/kT)}{Tr_B[\exp(-H_B/kT)]} \]  

(9)

In all cases we are going to consider different initially entangled states for the two qubit system. First we take \( |\psi_s\rangle \) as Bell states of the form \( |\psi_s\rangle = \) 

\[ a) |e_1\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}} \quad b) |e_2\rangle = \frac{|\uparrow\downarrow\rangle + |\down\uparrow\rangle}{\sqrt{2}} \]  

in which case \( \rho_S(0) = |\psi_s\rangle \langle \psi_s| \). Next we choose a mixed state, \( \rho_S(0) = \)  

\[ c) \rho_m = \frac{1}{3} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \]
4. Summary of main results

In all the cases studied, i.e. two-qubit system coupled to two independent baths and to a common one, we find that when there are no interactions present among the spins in the bath ($\lambda = 0$) the environment has a strong disentangling effect over the system (figures 1, 2). On the contrary, when we include an environmental self-interaction, decoherence of the system decreases while increasing $\lambda$ tending to the situation where the system is isolated. We can also see a difference between the two configurations studied: when the system is interacting with a common bath (figure 2) it is not as sensitive to $\lambda$ as when it is coupled to independent baths (figure 1).

The effective decoupling between the system and the environment for a strong self-interacting bath can be understood, as Dawson et al. [12] pointed out for the single-qubit system, as an entanglement sharing condition: as $\lambda$ increases so does the intrabath tangle $\tau_B$ which quantifies the entanglement between any two spins in the bath and consequently the bath is less entangled with the system. In figure 3 we show both the intrabath tangle for the case studied by Dawson and for one of the cases we studied: two-qubit system coupled to independent baths initialized in the pure state $|\psi_s\rangle = |e_1\rangle$, and we observe that the values are very similar.

The main result here is the fact that the strength of the intrabath coupling needed to avoid decoherence for the two-qubit case is greater than for the single-qubit one (see reference [12] for decoherence of a single qubit).

As a greater intrabath interaction is needed to obtain the same effects in a two-qubit system than in a single qubit one, this suggests that for a given $\tau_B$, the environment is less entangled with the system in the latter case than in the former. One way to examine the entanglement between the system and the bath
Figure 2. Time evolution of the system's tangle $\tau_S$ of a system of two qubits sharing a common bath, for different initial states of the system as described in 3.1.

Figure 3. Time evolution of the intrabath tangle $\tau_B$ for the case of a single qubit coupled to a self-interacting bath composed of four spins (left) and for the case of a system of two qubits each one of them coupled to an independent bath (right).
is through the linear entropy $1 - tr(\rho_S^2)$ that measures the mixedness of a state (0 for pure states and 1 for completely mixed ones) because mixedness in the system arises from entanglement with the bath. In figure 4 we show that as anticipated the linear entropy is less for a single-qubit than a two-qubit system. This result shows system’s size (two qubits instead of one) effects in the entanglement sharing condition.

Summarizing, in the cases studied we find that

- Strong interactions among the spins in the bath (controlled by $\lambda$) may translate into an effective decoupling between the bath and the system, thereby suppressing the disentangling effects that a non or weakly interacting bath has over the system.
- Comparing between the cases when the system shares a common bath or each qubit is coupled to two independent baths we find that the system is more sensitive to the interactions within the environment in the case of independent baths.
- We show system’s size effect, finding that a particular intrabath coupling strength limits more the decoherence decay in the case of a one qubit system than in the case of a two qubit system.

**Acknowledgments**

F.F.O. and A.O.-C. thank Conicyt, Government of Chile and Trinity College, Oxford respectively, for financial support.

**References**

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[2] Breuer H -P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)

[3] Leggett A J, Chakravarty S, Dorsey A T, Fisher M P A, Garg A and Zwerger W 1987 Rev. Mod. Phys. 59 1

[4] Cucchietti F M, Paz J P and Zurek W H 2005 Phys. Rev. A 72 052113

[5] Breuer H -P, Burgarth D and Petruccione F 2004 Phys. Rev. B 70 045323
[6] Yu T and Eberly J H 2004 Phys. Rev. Lett. 93 140404
[7] Gedik Z 2006 Solid State Commun. 138 82
[8] Hamdouni Y, Fannes M and Petruccione F 2006 Phys. Rev. B 73 245323
[9] Lucamarini M, Paganelli S and Mancini S 2004 Phys. Rev. A 69 062308
[10] Tessieri L and Wilkie J 2003 J. Phys. A 36 12305
[11] Schliemann J, Khaetskii A and Loss D 2003 J. Phys: Condens. Matter 15 R1809
[12] Dawson C M, Hines A P, McKenzie R H and Milburn G J 2005 Phys. Rev. A 71 052321
[13] Wootters W 1998 Phys. Rev. Lett. 80 2245
[14] Coffman V, Kundu J and Wootters W 2000 Phys. Rev. A 61 052306
[15] Osborne T J and Verstraete F 2006 Phys. Rev. Lett. 96 220503