ROTATIONALLY SYMMETRIC SOLUTIONS TO THE
CAHN-HILLIARD EQUATION

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Abstract. This paper is devoted to construction of new solutions to the Cahn-
Hilliard equation in \( \mathbb{R}^d \). Starting from the Delaunay unduloid \( D_\tau \) with parame-
ter \( \tau \in (0, \tau^*) \) we find for each sufficiently small \( \varepsilon \) a solution \( u \) of this equation
which is periodic in the direction of the \( x_d \) axis and rotationally symmetric with
respect to rotations about this axis. The zero level set of \( u \) approaches as \( \varepsilon \to 0 \)
the surface \( D_\tau \). We use a refined version of the Lyapunov-Schmidt reduction
method which simplifies very technical aspects of previous constructions for
similar problems.

1. Introduction. The Cahn-Hilliard equation

\[
\begin{align*}
  u_t &= -\Delta (\varepsilon^2 \Delta u - F'(u)) \quad \text{in } \Omega, \\
  \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega, \\
  \frac{\partial}{\partial \nu} (\varepsilon^2 \Delta u - F'(u)) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( F \) is a double-well potential, is a model that describes the process of phase
separation of two components of a binary alloy. Here \( \Omega \subset \mathbb{R}^d \), \( d \geq 1 \), is a bounded
domain represents the place where the isolation of the components takes place, and
\( \nu \), as usual, denotes the outer normal on \( \partial \Omega \). The function \( u \) represents the concen-
tration of one of the components and \( \varepsilon \) is the range of intermolecular forces. The

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The Cahn-Hilliard equation and related to it the Allen-Cahn equation and the phase field model
have been a subject of extensive research of many mathematicians for more than 30 years. We
have been a part of this group and we owe it to Paul Fife whose papers in the early 90ties were for
us an introduction to the area and an inspiration for the present work. For this reason we think
it is appropriate to dedicate it to his memory.
double-well potential $F(u)$ corresponds to the free energy density at low temperatures, and in this paper we will take

$$F(u) = \frac{1}{4} (1 - u^2)^2, \quad F'(u) = u^3 - u.$$  

From now on we will denote $F'(u) = -f(u)$.

Equation (1) can be derived from the gradient flow of the Helmholtz-free energy functional

$$E_\varepsilon(u) = \int_\Omega \left( F(u(x)) + \frac{1}{2} \varepsilon^2 |\nabla u(x)|^2 \right) dx$$

in $H^{-1}(\Omega)$ subject to the average concentration to be constant, i.e.

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = m,$$

where $m \in [-1, 1]$ (see [18], [19] for details). Note that constant functions $u \equiv \pm 1$ are minimizers of this functional subject to $m = \pm 1$.

Stationary solutions of (1) satisfy the Euler-Lagrange equation (with $f(u) = -F'(u)$)

$$\varepsilon^2 \Delta u + f(u) = \delta_\varepsilon \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = m$$

where $\delta_\varepsilon$ is a Lagrange multiplier.

Using $\Gamma$-convergence approach Modica [27] showed that minimizers $u_\varepsilon$ of (2) subject to constraint (3) $\Gamma$-converge, as $\varepsilon \to 0$, to the function $1 - 2\chi_{A_0}$, where $\chi_{A_0}$ is the characteristic function of an open set $A_0 \subset \Omega$. Moreover $\partial A_0 \cap \Omega$ is locally a surface of constant mean curvature (CMC surface for short). Geometrically the set $A_0$ minimizes the perimeter functional $\text{Per}_\Omega(A)$ among the sets $A \subset \Omega$ whose volume is fixed. A generalisation of these results was given by Sternberg [31]. Furthermore Hutchinson and Tonegawa [22] studied limits of general critical points (2) and showed that their limits are locally minimal or CMC surfaces. On the other hand it is known [25] that if a set $A \subset \Omega$ is an isolated minimizer of the perimeter functional subject to the constant volume constraint then there exists a sequence of minimizers $u_\varepsilon$ of (2) which $\Gamma$ converges to $A$. This result can be used to construct solutions to (4) at least in dimension 2, see [8]. The most complete construction is due to Pacard and Ritoré [28] who proved the following: if $M$ is a compact Riemannian manifold and $N$ is a non degenerate minimal or CMC sub manifold of $M$ which divides $M$ into 2 disjoint components then for all sufficiently small $\varepsilon$ there exist critical points of (2) whose 0 level set converges to $N$. The counterpart of this theory for the time dependent problem (1) was developed among others by Alikakos, Bates and Chen [1] who proved that as $\varepsilon \to 0$ the time evolution of interfaces is governed by the Helle-Shaw problem—of course CMC surfaces are stationary points of the flow. More detailed description of the Cahn-Hilliard flow and key spectral tools can be found for instance in [3], [5], [4], [2], [7] and the references therein. Additional examples of stationary solutions for the singular perturbation problem in a bounded domain have been constructed in [33], [32], [6].
Scaling variables \( x \mapsto x/\varepsilon \) in (4) and letting \( \varepsilon \) tend to 0 leads in a natural way to the following problem:

\[
\Delta u + f(u) = \delta, \quad \text{in } \mathbb{R}^d.
\]  

In dimension \( d = 1 \) there is an obvious solution of this problem when \( \delta = 0 \), namely the unique odd and monotonically increasing heteroclinic solution \( H \) of the ODE, which satisfies

\[
H'' + f(H) = 0, \quad \text{in } \mathbb{R},
\]

\[
H(\pm \infty) = \pm 1.
\]

If \( a \in \mathbb{R}^d \) is a unit vector and \( b \in \mathbb{R} \) then the function

\[
u(x) = H(a \cdot x + b),
\]

is also a solution of (5) with \( \delta = 0 \). When \( \delta \neq 0 \) there exist radially symmetric solutions to (5), see \cite{30}. In both cases the level sets of the solutions are CMC surfaces, in the former case their mean curvature is 0 and in the latter case it is a positive number equal to \((d - 1)/R_0\), where \( R_0 \) is the radius of the level set of the solution. The radially symmetric solutions in \( \mathbb{R}^{d-1} \) can be lifted trivially to \( \mathbb{R}^d \) giving solutions whose nodal sets are cylinders, which again are CMC surfaces.

Dilating of the independent variable by a (large) factor \( \varepsilon^{-1} > 0 \)

\[
\Delta u + \frac{1}{\varepsilon} f(u) = \ell_{\varepsilon}, \quad \text{in } \mathbb{R}^d.
\]

where we have denoted \( \delta/\varepsilon = \ell_{\varepsilon} \). Clearly, if \( u_{\varepsilon} \) is a solution of (7) then \( v(x) = u_{\varepsilon}(\varepsilon x) \) is a solution of (5). On the other hand, if \( v \) is a solution of (5) then \( u_{\varepsilon}(x) = v(\varepsilon x) \) is a solution of (7). In particular this means that while phase transition of the solutions of (5) are of order 1, for the solutions of (7) they are of order \( \varepsilon \). Thus the latter are more “concentrated”. In the sequel we will focus on solving (7). From what we have said above about the singular perturbation problem it is clear that level sets of these solutions should converge, as \( \varepsilon \) tends to 0, to CMC surfaces in \( \mathbb{R}^3 \). In fact we expect (on the basis of formal calculations in Section 2.3) that the Lagrange multiplier

\[
\ell_{\varepsilon} = -\frac{1}{2} H_{\Sigma} \int_{\mathbb{R}} H'(s)^2 ds + \mathcal{O}(\varepsilon),
\]

where \( \Sigma \) is the surface of the phase transition and \( H_{\Sigma} \) is its mean curvature.

We will now introduce a family of embedded CMC surfaces which are good candidates to be the limits of the nodal surfaces. We recall that Delaunay unduloids \cite{16}, \cite{17} are a one parameter family \( D_\tau \), \( \tau \in (0, 1) \) of embedded, periodic CMC surfaces of revolution in \( \mathbb{R}^3 \). When the real parameter \( \tau \) tends to 1 the surfaces \( D_\tau \) approach the straight cylinder while when \( \tau \to 0 \) they become an array of identical spheres arranged along the \( x_3 \) axis. It turns out that Delaunay surfaces can be constructed in any dimension \( d > 3 \) and from now on by \( D_\tau \), \( \tau \in (0, \tau_*) \) we will denote the family of Delaunay surfaces in \( \mathbb{R}^d \). We note that the parameter \( \tau_* \) satisfies:

\[
\tau_* = \frac{(d - 2)^{d/2}}{(d - 1)^{d/2}}.
\]

Again, in the limit \( \tau \to \tau_* \) the surfaces \( D_\tau \) approach the straight cylinder, see \cite{23} or Section 2.1 for details.
It is convenient to “normalize” the Delaunay surface and suppose that the mean curvature of $D_\tau$ is 1 for all $\tau \in (0, \tau_*)$. We will also denote by $N_\tau$ the vector field normal to $D_\tau$. Let us notice that the surface $D_\tau$ divides the space into two disjoint components $\Omega^\pm_\tau$, such that $\mathbb{R}^d \setminus D_\tau = \Omega^+_\tau \cup \Omega^-_\tau$, where $N_\tau$ points towards $\Omega^+_\tau$. By changing the orientation of $D_\tau$ if necessary we can chose $N_\tau$ in such a way that $\Omega^+_\tau$ contains the $x_d$ axis.

Our result is:

**Theorem 1.1.** For all $\tau \in (0, \tau_*)$ when $d = 3$ and with a possible exception of a finite set of $\tau$ when $d > 3$, there exits $\varepsilon_\tau > 0$ such that for all $\varepsilon \in (0, \varepsilon_\tau)$ the problem

$$\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell_\varepsilon \quad \text{in } \mathbb{R}^d \quad (9)$$

has a solution $u_{\tau, \varepsilon}$, which is one-periodic in the direction of the $x_d$-axis and rotationally symmetric with respect to rotations about the same axis. As $\varepsilon \to 0$ we have

$$\ell_\varepsilon = 1 + \mathcal{O}(\varepsilon),$$

and $u_{\tau, \varepsilon}$ satisfies

$$u_{\tau, \varepsilon} \to 1 \quad \text{as } \varepsilon \to 0 \quad \text{in } \Omega^+_\tau,$$

$$u_{\tau, \varepsilon} \to -1 \quad \text{as } \varepsilon \to 0 \quad \text{in } \Omega^-_\tau,$$

uniformly over compacts.

**Remark 1.** In this paper we took $f(u) = u - u^3$, which is the standard nonlinearity for the Cahn-Hilliard equation. Theorem 1.1 holds for more general nonlinearities of bistable, balanced type, namely $f \in C^3$ such that $f(u) = -F'(u)$ where $F$ is a double well, even potential with non degenerate wells at $\pm 1$. Rather straightforward modifications required in the proof of the more general setting are left to the reader.

**Remark 2.** In the statement of the Theorem we assume that $\tau \neq 0, \tau_*$. In fact solutions for these extreme values of the Delaunay parameter are known: when $\tau = 0$ they are simply the radially symmetric solutions in $\mathbb{R}^d$ and when $\tau = \tau_*$ they are radial symmetric solutions in $\mathbb{R}^{d-1}$ lifted to $\mathbb{R}^d$. However our construction does not cover the boundary values of $\tau$. On the one hand it has to do with the difficulty of finding an approximate solution which will give uniformly small error when $\tau \to 0$ and on the other hand with the extra degeneracy of the linearized operator when $\tau \to \tau_*$. The latter case could be possibly dealt with within our construction but we have not pursued this since this would not give any new result.

**Remark 3.** Solutions we construct here are rotationally symmetric, periodic with period $T_\tau$ and also symmetric with respect to the hyperplane $x_d = T_\tau/2$. This could be used to show existence of solutions to the Cahn-Hilliard equation obeying these symmetries using for instance variational methods in the spirit similar to [10] or [15]. However, without further analysis it is not immediately clear how to make sure that the zero level set of such a solution would be a Delaunay surface for $\varepsilon$ sufficiently small. This is important if one wants to use them as a basis of a connected sum construction of solutions of the Cahn-Hilliard equation whose zero level set is a non degenerate, non compact constant mean surface with $k$ Delaunay ends [20].

We will explain now the implementation of the Lyapunov-Schmidt reduction we used in this paper and discuss the differences between our approach and the older implementations which can be found in [28] and [12], [13]. Let us first recall the standard Lyapunov-Schmidt reduction method in its abstract version. Given
Banach spaces $X, Y$ and a linear operator $A: X \to Z$ and a continuous, nonlinear operator $N: X \to Z$, we are to solve the problem:

$$Ax - N(x) = 0. \quad (10)$$

Let

$$\mathcal{N}(A) = Y \subset X, \quad \mathcal{R}(A) = W \subset Z,$$

and let $\pi_Y, \pi_W$ be the projections on the corresponding subspaces. According to Lemma 4.1 in chapter 2 of [9] the following is true: there exists a bounded linear operator $K: W \to \mathcal{R}(I - \pi_Y)$ (the right inverse of $A$) such that $AK = I$ on $W$ and $KA = I - \pi_Y$, and moreover the equation (10) is equivalent to the equation

$$x = y + z, \quad y \in Y, \quad z \in \mathcal{R}(I - \pi_Y)$$

$$z - K\pi_W N(y + z) = 0,$$

$$(I - \pi_W)N(y + z) = 0. \quad (11)$$

In applications the Lyapunov-Schmidt method consists of reducing (10) to (11), solving the first equation for $z$ with $y$ given (which usually can be done by a fixed point argument) and replacing this solution in the second equation to obtain the reduced problem

$$(I - \pi_W)N(y + z(y)) = 0. \quad (12)$$

In practice several complications may arise and we will illustrate this considering a related to our problem which was treated by Pacard and Ritoré [28], and in many aspects it is similar to problem we consider in this paper. Let $M$ be a compact, closed manifold of dimension $n$ and $N \subset M$ a minimal $n-1$ dimensional sub manifold which divides $M$ into two disjoint components. Consider the problem

$$\varepsilon^2 \Delta_M u + u(1-u^2) = 0, \quad \text{on } M. \quad (13)$$

We say that $N$ is non degenerate if the Jacobi operator of $N$

$$J_N = \Delta_N + |A_N|^2 + \text{Ric}_g(\nu_N, \nu_N)$$

has empty kernel ($\Delta_N$ is the Laplace-Beltrami operator on $N$, $|A_N|$ is the norm of the second fundamental form, $\text{Ric}_g$ is the Ricci tensor on $M$ and $\nu_N$ is the normal vector to $N$). The result proven in [28] is: given a non degenerate, minimal sub manifold $N$ of $M$ for each sufficiently small $\varepsilon$ there exists a solution $u_\varepsilon$ of (13) such that the zero level set of $u_\varepsilon$ approaches $N$ as $\varepsilon \to 0$. Moreover, $u_\varepsilon$ converges to $\pm 1$ uniformly over compacts of the two disjoint components of $M \setminus N$.

Let us explain now the implementation of the Lyapunov-Schmidt reduction in [28]. It is expected that for $x \in M$ near $N$ we should have

$$u_\varepsilon(x) = H(\varepsilon^{-1} \text{dist}(x, N)) + \varphi,$$

where $\text{dist}(\cdot, N)$ is the signed distance function on $M$, $H$ is the unique odd, monotonically increasing solution of $-H'' = H(1-H^2)$ in $\mathbb{R}$ and $\varphi$ is a small perturbation. The problem to solve for $\varphi$ amounts to inverting the linearized operator around $H(\varepsilon^{-1} \text{dist}(x, N))$ which has form

$$L = \Delta_M + f'(H(\varepsilon^{-1} \text{dist}(\cdot, N))).$$

It is known that the norm of $L^{-1}$ is large due to local translational invariance of the problem. Thus we need to perturb $N$ as well. To describe this perturbation we consider a manifold $N_h$ to be a normal graph over $N$ described by a smooth and
small function $h: N \to \mathbb{R}$. Furthermore we let $t_h(x) = \text{dist}(x, N_h)$ to be the signed distance from $N_h$. Then we look for a solution of the form

$$u = H\left(\frac{t_h}{\varepsilon}\right) + \varphi.$$  

Now both $h$ and $\varphi$ are unknowns. The problem to solve for $\varphi$ is

$$L_h \varphi = \mathcal{F}(h, \varphi),$$

where $L_h$ is the linearized operator around $H\left(\frac{t_h}{\varepsilon}\right)$. The Lyapunov-Schmidt reduction strategy amounts to projection of the above equation onto the function $H\left(\frac{t_h}{\varepsilon}\right)$ and its complement, denote this last projection by $\pi_h$. This leads to a problem for $\varphi$

$$\pi_h L_h \varphi = \pi_h \mathcal{F}(h, \varphi),$$

which we solve first for a given $h$, and the problem for $h$

$$J_{N_h} h = G(h), \quad (14)$$

which we solve next ($J_{N_h}$ is the Jacobi operator of $N_h$). Let us discuss (14). We notice that the expression of $J_{N_h}$ in local coordinates will depend in general on $h$ and its derivatives up to order 3, while the Jacobi operator is itself only a second order operator. This loss of regularity was dealt with in [28] using a regularisation procedure. In a series of papers [12], [13], [14] del Pino, Kowalczyk and Wei introduced a slightly different approach to circumvent this problem. It amounts to considering perturbation in the normal direction of the fixed manifold $N$ so that $u = H\left(\frac{t + h}{\varepsilon}\right) + \ldots$, where now $t$ is the signed distance from $N$ and $h$ is a smooth, unknown function defined on $N$. Equation (14) takes form

$$J_{N_h} h = G(h), \quad (15)$$

and the problem of the loss of regularity is thus avoided. The problem is now reduced to finding a fixed point of $J_{N_h} \circ G(h)$, using for example Banach fixed point theorem. To do this we need to know that $G$ is at least Lipschitz in $h$. In both implementations of the Lyapunov-Schmidt reduction described above this is rather complicated technical point since $G$ depends in a non explicit, non local and non linear way on $h$. This is mainly due to the fact that the linearized operator $L_h$ still depends on $h$ through the potential $f'(H\left(\frac{t + h}{\varepsilon}\right))$. Thus difficulty is to some extend circumvented in [28] where the presentation of the Lyapunov-Schmidt reduction is state of the art. In this approach modifying the nonlinear problem by composing it (twice) with a carefully chosen diffeomorphism (and its inverse) both the loss of regularity and the nonlocal dependence on the perturbation $h$ are avoided, in fact $h$ appears only algebraically in the problem.

In this paper we propose still another modification to the method. The idea is simple: instead of working with an approximation of the form $u = H\left(\frac{t + h}{\varepsilon}\right) + \ldots$ with $h$ unknown we will improve the initial approximation to $w(t, y) = H\left(\frac{t}{\varepsilon}\right) + \ldots$, $t$ being the signed distance to $N$ and $y \in N$ in such a way that we do not need to “move” $N$ anymore. In other words $h$ will be determined with some sufficient precision before setting up the Lyapunov-Schmidt reduction, which with this modification will look like the abstract setting described at the beginning. This way we avoid both the loss of regularity and technical difficulties due to complicated character of the nonlinear function $G(h)$. This is described in detail in Section 3.1.
2. Preliminaries.

2.1. The surfaces of Delaunay. The Delaunay unduloids $D_\tau$, $\tau \in (0, \tau_*)$ are CMC surfaces of revolution in $\mathbb{R}^d$. Thus for instance in $\mathbb{R}^3$ one can parametrize them in the form

$$x(t, \theta) = (\rho(t) \cos \theta, \rho(t) \sin \theta, t),$$

where $\rho(t)$ solves

$$\rho_{tt} - \frac{1}{\rho} (1 + \rho^2_t) - (1 + \rho^2) \frac{3}{2} = 0.$$

However, in this paper we will use mostly isothermal coordinates of $D_\tau \subset \mathbb{R}^d$: $X_\tau(s, \Theta) = \frac{1}{2} (\tau e^{\sigma_\tau(s)} \Theta, \kappa_\tau(s))$, $(s, \Theta) \in \mathbb{R} \times S^{d-2}$, (16)

where functions $(\sigma_\tau, \kappa_\tau)$ are the unique solutions of the following system of ODEs:

$$(\partial_s \sigma_\tau)^2 + \frac{1}{4} \tau^2 (e^{\sigma_\tau} + e^{(2-d)\sigma_\tau})^2 = 0, \quad \partial_s \sigma_\tau(0) = 0, \quad \sigma_\tau(0) < 0,$$

$$\partial_s \kappa_\tau - \frac{1}{4} \tau^2 (e^{\sigma_\tau} + e^{(2-d)\sigma_\tau}) = 0, \quad \kappa_\tau(0) = 0.$$

(17)

We will now summarize some basic facts about the Delaunay surfaces and their (isothermal when $d = 3$) parametrization. The function $\sigma_\tau$ is periodic, and consequently the surfaces $D_\tau$ are one-periodic along the $x_d$-axis: namely if $2T_\tau$ denotes the minimal period then $D_\tau = D_\tau + 2T_\tau e_d$.

Clearly we have the relation

$$T_\tau = \frac{1}{4} \kappa_\tau(2s_\tau),$$

where $2s_\tau$ is the minimal period of $\sigma_\tau$.

The Jacobi operator $J_\tau$ of $D_\tau$ is defined by:

$$J_\tau := \Delta_{D_\tau} + |A_\tau|^2,$$

(18)

where $\Delta_{D_\tau}$ is the Laplace-Beltrami operator on $D_\tau$ and $|A_\tau|^2$ is the square of the norm of the second fundamental form of $D_\tau$. The Jacobi operator is of fundamental importance in this paper and to understand well its properties we will first consider the special case $d = 3$. In the isothermal coordinates $(s, \theta) \in \mathbb{R} \times S^1$ its expression is given by:

$$L_\tau = \frac{1}{\tau^2 e^{2\sigma_\tau}} \left\{ \partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma_\tau) \right\}.$$  

(19)

The Jacobi fields on $D_\tau$, which are elements of the kernel of $L_\tau$ are of three types:

1. **Jacobi fields arising from infinitesimal translations.** For any $e \in \mathbb{R}^3$, $|e| = 1$ the constant Killing field associated to translations

$$x \mapsto e$$

induces the following Jacobi fields

$$\Phi_\tau^{Te} = e \cdot N_\tau,$$

where $N_\tau$ is the unit normal vector to $D_\tau$. The coordinate vectors $e_j$, $j = 1, 2, 3$ generate three linearly independent Jacobi fields $\Phi_\tau^{Te_j}$ corresponding to translations of $D_\tau$ in the directions of the coordinate axis. We note that in the isothermal coordinates

$$\Phi_\tau^{Te_1} = \Phi_\tau^{Te_2}(s), \quad \Phi_\tau^{Te_j} = \Phi_\tau^{Te_j}(s, \theta), \quad j = 1, 2.$$
It is important to notice that the Jacobi fields $\Phi_{T,e_j}^\tau$ are bounded.

2. *Jacobi fields arising from infinitesimal rotations.* Let $e \in \mathbb{R}^3$, $|e| = 1$ be such that $e \cdot e_3 = 0$. The Killing vector field corresponding to the rotation about the vector $e$ is:

$$x \mapsto (x \wedge e).$$

We define the Jacobi field associated to this vector field by:

$$\Phi^R_e = (x \wedge e) \cdot N_{\tau}.$$ 

There are clearly two linearly independent Jacobi fields associated to the rotations. They are:

$$\Phi^R_{e_1}, \Phi^R_{e_2},$$ 

and they correspond to rotations about the coordinate axis. Note that in isothermal coordinates functions $\Phi^R_{e_j}, j = 1, 2$ grow linearly as functions of $s$.

3. *Jacobi field associated with the variation of the Delaunay parameter.* We define:

$$\Phi^D_{\tau} = -\partial_\tau X_{\tau} \cdot N_{\tau}.$$ 

This Jacobi field is somewhat harder to write explicitly however it can be shown that the function $\Phi^D_{\tau}(s)$ is linearly growing as long as $\partial_\tau T_{\tau} > 0$, which is proven in [24] for all $\tau \in (0, 1)$.

In summary, the Jacobi operator $L_{\tau}$ has at least 6 explicit Jacobi fields which are either linearly growing or bounded. By a result of Mazzeo and Pacard [26] and Jleli and Pacard [24] we know that these are all Jacobi fields with at most linear growth. To explain this let us observe that by separation of variables the equation $J_{\tau}\varphi = 0$ separates into a sequence of problems

$$L_{\tau,j}\varphi = 0, \quad L_{\tau,j} = \frac{1}{\tau^2 e^{2\sigma_\tau}} \left\{ \partial^2_s + \tau^2 \cosh(2\sigma_\tau) - j^2 \right\}, \quad |j| = 0, 1, \ldots.$$

Then we have:

**Proposition 1** ([26]). *The homogeneous problem $L_{\tau,j}\varphi = 0$ has the following solutions:*

1. one periodic and one linearly growing solution when $j = 0$ or $|j| = 1$;
2. two solutions $\varphi^{\pm}_{\tau,j}(s)$ which satisfy:

$$\varphi^{\pm}_{\tau,j}(s + s_{\tau}) = e^{\pm \zeta_{\tau,j}s_{\tau}} \varphi^{\pm}_{\tau,j}(s),$$

with

$$\gamma_{\tau,j} = \text{Re} \zeta_{\tau,j} > 0,$$

when $|j| > 2$. The numbers $\zeta_{\tau,j}$ are called the indicial roots of the operators $L_{\tau,j}$ and correspond to the rate of exponential growth or decay of the solutions of the homogeneous problem at $\pm\infty$.

This basic facts can be generalized for the Jacobi operator of Delaunay surfaces in $\mathbb{R}^d$, $d > 3$ [21]. We will summarize them now and refer the reader to [23] for details. We have the following at most linearly growing Jacobi fields:

1. The are $d$ bounded, periodic Jacobi fields arising from infinitesimal translations. They will be denoted by $\Phi^T_{T,e_j}, j = 1, \ldots, d$. 


(2) There are \( d - 1 \) Jacobi fields arising from infinitesimal rotations of the axis of \( D_r \). We will denote them by \( \Phi_{\tau}^{R,\sigma} \), \( j = 1, \ldots, d - 1 \), and they correspond to rotations about the coordinate axis. Note that in isothermal coordinates functions \( \Phi_{\tau}^{R,\sigma} \), grow linearly as functions of \( \tau \).

(3) There is one Jacobi field associated with the variation of the Delaunay parameter \( \Phi_{\tau}^D = -\partial_\tau X_r \cdot N_\tau \) and it is linearly growing for generic values of \( \tau \) (indeed for all \( \tau \) except possibly a finite set we have \( \partial_\tau T_\tau > 0 \), by analytic dependence of solutions of (17) on the parameters).

### 2.2. Fermi coordinates and shifted Fermi coordinates near a CMC surface.

Let \( \Sigma \) be an embedded CMC surface in \( \mathbb{R}^d \) and let \( H_\Sigma \) denote its mean curvature. By \( N \) we will denote its unit normal. We will assume that there exists a tubular neighborhood \( N_\delta \) of \( \Sigma \) of width \( 2\delta \) in which we can introduce local system of coordinates (Fermi coordinates) \( (y, z) \in \Sigma \times (-\delta, \delta) \) by setting:

\[
x \mapsto (y, z), \quad \text{where } x = y + zN(y).
\]

We suppose that this map, which we denote by \( Y \), is a diffeomorphism from \( N_\delta \) to \( \Sigma \times (-\delta, \delta) \) whenever \( \delta \) is taken sufficiently small. In the sequel we will use the inverse of this map

\[
Y^{-1} : \Sigma \times (-\delta, \delta) \rightarrow N_\delta
\]

\[
(y, z) \mapsto x.
\]

Given a function \( w : N_\delta \rightarrow \mathbb{R}^d \) we define its pullback \( Y^*w \) to \( \Sigma \times (-\delta, \delta) \) by the diffeomorphism \( Y \) as:

\[
Y^*w(y, z) = w \circ Y^{-1}(y, z).
\]

For technical reasons we will chose later the size of the tubular neighbourhood \( \delta \) depending on \( \epsilon \) but for now on we just take \( \delta \) small.

Next we will define **shifted Fermi coordinates**. To do this we let \( h : \Sigma \rightarrow \mathbb{R} \) be a given smooth function such that the map

\[
x \mapsto (y, t), \quad \text{where } x = y + (t + h(y))N(y),
\]

is a diffeomorphism from \( N_\delta \) into \( \Sigma \times (-\delta, \delta) \). We will denote this map by \( Y_h \) and by \( Y_h^{-1} \) we will denote its inverse, finally by \( Y_h^*w \) we will denote the pullback of \( w : N_\delta \rightarrow \mathbb{R}^d \) by \( Y_h \):

\[
Y_h^*w(y, t) = w \circ Y_h^{-1}(y, t).
\]

It will be convenient to have at hand expressions for the Laplacian in Fermi and shifted Fermi coordinates. To derive them by \( \Sigma_z \) we will denote the surface \( \Sigma + zN \) i.e. the original surface \( \Sigma \) shifted in the direction of the normal by \( z \). Locally near \( \Sigma \) have

\[
\Delta = \Delta_{\Sigma_z} + \partial_z^2 - H_{\Sigma_z} \partial_z.
\]

We denote \( zB_{\Sigma_z} = \Delta_{\Sigma_z} - \Delta_{\Sigma} \). The operator \( B_{\Sigma_z} \) is a second order differential operator. To expand the curvature term we use the well known formula:

\[
H_{\Sigma_z} = \sum_{j=1}^{d-1} \frac{k_j}{1 - zk_j} = H_\Sigma + z|A_\Sigma|^2 + z^2 \sum_{j=1}^{d-1} k_j^3 + O(z^3) = H_\Sigma + z|A_\Sigma|^2 + z^2 Q_{\Sigma,z}.
\]

where \( k_j \) are the principal curvatures of \( \Sigma \). In summary we have:

\[
\Delta = \Delta_{\Sigma} + \partial_z^2 - (H_{\Sigma} + z|A_\Sigma|^2)\partial_z + zB_{\Sigma,z} + z^2 Q_{\Sigma,z}.
\]
The reason we expanded the Laplacian in this way will become clear later on. From this it is easy to obtain a formula for the Laplacian in the shifted Fermi coordinates:

\[
\Delta = \Delta_\Sigma + (1 + |\nabla_\Sigma h|^2)\partial_t^2 - (H_\Sigma + \Delta_\Sigma h + (t + h)|A_\Sigma|)^2\partial_t \\
+ (t + h)B_{\Sigma,t+h} + (t + h)^2Q_{\Sigma,t+h}.
\]  

(20)

Anticipating the content of the next section we introduce the stretched shifted Fermi coordinate

\[ t = \frac{t}{\varepsilon}, \quad y = y. \]

Formal consideration will show that an approximate solution \( w_\varepsilon \) of the Cahn-Hilliard can be obtained if we assume that it is a function of the form:

\[ Y_\varepsilon w_\varepsilon(y, t) = H\left(\frac{t}{\varepsilon}\right) + o(1), \]

where \( H \) is the heteroclinic solution of (6).

As before we have a diffeomorphism \( Y_{\varepsilon,h} \) and its inverse \( Y_{\varepsilon,h}^{-1} : \Sigma \times (-\frac{2}{\varepsilon}, \frac{2}{\varepsilon}) \rightarrow \mathcal{N}_\delta \), and for any function \( w : \mathcal{N}_\delta \rightarrow \mathbb{R}^k \) we define its pullback by \( Y_{\varepsilon,h} \) by:

\[ Y_{\varepsilon,h} w(y, t) = w \circ Y_{\varepsilon,h}^{-1}(y, t). \]

Taking onto account formula (20) we get

\[
\Delta = \Delta_\Sigma + \varepsilon^{-2}(1 + |\nabla_\Sigma h|^2)|\partial_t^2 - \varepsilon^{-1}(H_\Sigma + \Delta_\Sigma h + (\varepsilon t + h)|A_\Sigma|)^2\partial_t \\
+ (\varepsilon t + h)B_{\Sigma,\varepsilon t+h} + (\varepsilon t + h)^2Q_{\Sigma,\varepsilon t+h}.
\]

(21)

2.3. Formal expansion of the solution of the Cahn-Hilliard equation concentrating on \( \Sigma \). For the purpose of formal calculations we will assume that the solution of (7) near \( \Sigma \) is a function \( w \), which depends on the stretched and shifted Fermi coordinates \((y, t)\), in the following way

\[ Y_{\varepsilon,h} w(y, t) = U(t) + \varepsilon^2 \psi_0(y, t), \]

(22)

for some functions \( U \) and \( \psi_0 \) which we will determine. Moreover, we will assume that

\[ h = \varepsilon^2 h_0, \]

where \( h_0 \) is a constant to be chosen.

To determine \( U \) and \( \psi_0 \) we write the error \( N_\varepsilon(w) - \ell_\varepsilon := \varepsilon(\Delta w + \frac{1}{\varepsilon}w(1 - w^2)) - \ell_\varepsilon \) in the local coordinates

\[
N_\varepsilon(w) - \ell_\varepsilon = \varepsilon\left\{ \varepsilon^{-2}\partial_t^2 U - \varepsilon^{-1}H_\Sigma\partial_t U + \varepsilon^{-2}f(U) - \varepsilon^{-1}\ell_\varepsilon \\
+ \partial_t^2 \psi_0 - \varepsilon H_\Sigma \partial_t \psi_0 + f'(U)\psi_0 - (\varepsilon t + h_0)|A_\Sigma|^2\partial_t U \\
+ \frac{1}{\varepsilon}f'(U) - \varepsilon f'(U)\psi_0 \\
+ \varepsilon^2 \Delta_\Sigma \psi_0 - \varepsilon^2(\varepsilon t + h_0)|A_\Sigma|^2\partial_t \psi_0 \\
+ \left( (\varepsilon t + h)B_{\Sigma,\varepsilon t+h} + (\varepsilon t + h)^2Q_{\Sigma,\varepsilon t+h} \right)(U + \varepsilon^2 \psi_0) \right\}. 
\]

(23)

In order to get as small as possible this approximation we have to get rid of the first three terms of the right hand side of the expression above, since they show the lower powers in \( \varepsilon \). To write things compactly let:

\[
S_0(w) := \partial_t^2 w - \varepsilon H_\Sigma \partial_t w + f'(U), \\
L_0 w := \partial_t^2 w - \varepsilon H_\Sigma \partial_t w + f'(U)w.
\]
With this notation and the ansatz (22) we can write the problem in the form:
\[ S_0(U) + \varepsilon^2 L_0 \psi_0 + Q_0(U + \varepsilon^2 \psi_0) = \varepsilon \ell_\varepsilon. \]
where \( Q(U + \varepsilon^2 \psi_0) \) represents the rest of the terms in (23), and we also have to determine the Lagrange multiplier \( \ell_\varepsilon \).

Thus we take the Lagrange multiplier \( \ell_\varepsilon \) to be a number such that the following ODE
\[ S_0(U) + U'' - \varepsilon H \Sigma U' + f(U) = \varepsilon \ell_\varepsilon, \quad \text{in } \mathbb{R}, \]
\[ f(U(\pm \infty)) = \varepsilon \ell_\varepsilon. \] (24)
has solution \( U \) such that we have \( U(\pm \infty) = \pm 1 + \sigma_\varepsilon \), where
\[ f(\pm 1 + \sigma_\varepsilon) = \varepsilon \ell_\varepsilon. \]

Also, we have
\[ \ell_\varepsilon = \ell_0 + \mathcal{O}(\varepsilon), \quad \ell_0 = -\frac{1}{2} H \Sigma \int_{\mathbb{R}} H'(s)^2 \, ds, \] (25)
and
\[ U(t) = H(t) + \mathcal{O}(\varepsilon). \]

The function \( U \) is rather easy to find by perturbing the heteroclinic solution \( H \).
In fact, if we consider the ansatz \( U = H + \varepsilon \phi \), where \( H \) is the heteroclinic solution of (6), and \( \phi \) the corresponding perturbation, we find that \( \phi \) must satisfy
\[ H'' + \varepsilon \phi'' - \varepsilon H \Sigma H' - \varepsilon^2 H \Sigma \phi' + f(H + \varepsilon \phi) = \varepsilon \ell_\varepsilon \quad \text{in } \mathbb{R} \] (26)
Taking into account that we can write \( f(H + \varepsilon \phi) = f(H) + \varepsilon f'(H) \phi + [f(H + \varepsilon \phi) - f(H) - \varepsilon f'(H) \phi] \), we have that (26) is equivalent to
\[ L \phi := \phi'' + f'(H) \phi = \ell_\varepsilon + H \Sigma (H' + \varepsilon \phi') + \mathcal{O}(\varepsilon \phi^2) \quad \text{in } \mathbb{R}. \] (27)

Since the linear operator \( L \) has as \( H' \) as the only bounded element of its kernel it follows that the the right hand side of this equation has to be orthogonal to \( H' \) and this condition reads
\[ \int_{\mathbb{R}} \ell_\varepsilon H'(s) \, ds + \int_{\mathbb{R}} H \Sigma (H'(s) + \varepsilon \phi'(s)) H'(s) \, ds + \int_{\mathbb{R}} \mathcal{O}(\varepsilon \phi^2(s)) \, ds = 0 \]
or equivalently
\[ \ell_\varepsilon = -\frac{1}{\int_{\mathbb{R}} H'(s) \, ds} H \Sigma \int_{\mathbb{R}} H'(s)^2 \, ds + \mathcal{O}(\varepsilon) \int_{\mathbb{R}} \phi'(s) H'(s) \, ds \]
\[ + \mathcal{O}(\varepsilon) H \Sigma \int_{\mathbb{R}} \phi'(s) H'(s) \, ds \]
\[ = -\frac{1}{2} H \Sigma \int_{\mathbb{R}} H'(s)^2 \, ds + \mathcal{O}(\varepsilon), \] (28)
as we anticipated. We already know that \( H' \) is an element of the kernel of the second order linear operator \( L \), the other one can be found with the ansatz \( vH' \), plugging this ansatz into the corresponding equation we find that \( v = \int_{\mathbb{R}} \frac{1}{H'} \Sigma \), and it easy to see that \( vH' \) is exponentially increasing in fact \( vH' = \mathcal{O}(\pm e^{-t^2/2}) \) as \( x \to \infty \).
Therefore, in order to find a exponentially decaying particular solution of \( L \phi = g \) we first notice that the Wronskian \( W(H', vH') = 1 \), therefore using variation of parameters, a particular solution has the form
\[ y_p(t) = -H(t) \int_{-\infty}^{t} v(s) H'(s)g(s) \, ds + v(t) H'(t) \int_{-\infty}^{t} H'(s)g(s) \, ds \]
and the exponentially decaying condition is guaranteed accordingly to the following orthogonality conditions.

\[ \int_\mathbb{R} v(t)H'(t)g(t) \, dt = 0 = \int_\mathbb{R} H'(t)g(t) \, dt. \]

Thus (27) can be written as the integral equation

\[ \phi(t) = -H'(t) \int_{-\infty}^t v(s)H'(s)[\ell_\varepsilon + H_\Sigma(H'(s) + \varepsilon \phi'(s)) + O(\varepsilon \phi^2(s))] \, ds \]

\[ + v(t)H'(t) \int_{-\infty}^t H'(s)[\ell_\varepsilon + \varepsilon H_\Sigma(H'(s)\phi'(s)) + O(\varepsilon \phi^2(s))] \, ds \]  

Equations (28)–(29) allow to rephrase problem (27) as a fixed point problem for a nonlinear operator. The presence of the \( \varepsilon \) factor on the non-linearity of the right hand side of (29) allows an implementation of the Banach fixed point theorem which is left to the reader.

Next, we will determine the \( O(\varepsilon^2) \) correction \( \psi_0 \). Ignoring terms of order \( O(\varepsilon^3) \) we get the following equation to solve:

\[ \partial_t^2 \psi_0 - \varepsilon H_\Sigma \partial_t \psi_0 + (1 - 3U^2) \psi_0 = (t + \varepsilon h_0)|A_2|^2 \partial_t U. \]  

(30)

It is convenient to consider, more generally, an ODE (with the right hand side possibly depending on the variable \( y \)) of the form:

\[ \partial_t^2 \varphi - \varepsilon H_\Sigma \partial_t \varphi + (1 - 3U^2) \varphi = g(y, t). \]  

(31)

A solution of this problem can be found by the variation of parameters formula. Indeed, the fundamental set of the ODE is spanned by the functions

\[ \partial_t U = O((\cosh t)^{\eta^{\pm}}), \quad W(t) = O((\cosh t)^{\nu^{\pm}}), \quad t \to \pm \infty. \]  

(32)

with

\[ \eta^{\pm} = \frac{1}{2}(\varepsilon H_\Sigma - \sqrt{-4\iota(\pm \infty) + \varepsilon^2 H_\Sigma^2}), \quad \nu^{\pm} = \frac{1}{2}(\varepsilon H_\Sigma + \sqrt{-4\iota(\pm \infty) + \varepsilon^2 H_\Sigma^2}), \]  

and

\[ \iota(\infty) = 1 - 3(1 + \sigma_2)^2, \quad \iota(-\infty) = 1 - 3(-1 + \sigma_2)^2. \]  

(33)

We can assume that the Wronskian at 0 is 1. If the right hand side of (31) satisfies

\[ \int_\mathbb{R} g(t, y)\partial_t U(t)e^{-\varepsilon H_\Sigma t} \, dt = 0, \quad \forall y \in \Sigma \]  

(34)

we can write \( \varphi = \mathcal{G}(g) \) where

\[ \mathcal{G}(v)(t, y) \]

\[ = -\partial_t U(t) \int_0^t W(s)e^{-\varepsilon H_\Sigma s}v(s, y) \, ds + W(t) \int_{-\infty}^t \partial_s U(s)e^{-\varepsilon H_\Sigma s}v(s, y) \, ds. \]  

(35)

Note that the orthogonality condition (34) guarantees that the function \( \mathcal{G}(g) \) is exponentially decaying whenever \( g \) is exponentially decaying (in \( t \)). To be more precise let us assume for instance that

\[ |g(y, t)|(\cosh t)^\mu \leq C, \]

with \( \mu \in (\eta + \varepsilon H_\Sigma, -\eta] \), where \( \eta = \max\{\eta^{+}, \eta^{-}\} < 0 \). Then we have

\[ |\varphi(y, t)|(\cosh t)^\mu \leq C, \]

as well.
Using this we can determine the function $\psi_0$. To do this we need to choose $h_0$ such that (34) is satisfied, in other words:

$$\int_\mathbb{R} (t + \varepsilon h_0) \partial_t U(t)^2 e^{-\varepsilon H x^t} dt = 0,$$

hence

$$h_0 = -\frac{\int_\mathbb{R} t \partial_t U(t)^2 e^{-\varepsilon H x^t} dt}{\varepsilon \int_\mathbb{R} \partial_t U(t)^2 e^{-\varepsilon H x^t} dt} = O(1),$$

With this choice we define:

$$\psi_0(y, t) = G ((t + \varepsilon h_0) \partial_t U) |A_\Sigma|^2.$$

Note that we have:

$$(|t| + \varepsilon |h_0|) |\partial_t U(t)| (\cosh t)^\mu < C, \quad 0 < \mu < -\eta,$$

and as a consequence

$$|\psi_0(t, y)| (\cosh t)^\mu < C, \quad 0 < \mu < -\eta.$$ Sometimes it is convenient to derive a more refined estimate taking into account the fact that the leading order term on the right hand side is

$$t \partial_t U(t) = O(|t| (\cosh t)^{\eta^+}), \quad |t| \to \pm \infty.$$ Thus, we consider assuming that

$$|g(y, t)|(1 + |t|)^{\beta}(\cosh t)^\mu \leq C,$$

with $\mu \in (\eta + \varepsilon H_{\Sigma}, -\eta]$, and $\beta \in \mathbb{R}$. By a simple argument we have as well:

$$|G(g)(t, y)|(1 + |t|)^{\beta}(\cosh t)^\mu \leq C.$$}

3. Delaunay solutions of the Cahn-Hilliard equation.

3.1. The Lyapunov-Schmidt reduction. While our formal considerations in the above section were valid for any embedded CMC surface $\Sigma$ in $\mathbb{R}^d$ in what follows we will focus on a special example when $\Sigma = D_\tau$, i.e. it is a Delaunay unduloid. Since we are interested in functions which are periodic in the direction of the $x_d$ axis with the minimal period equal to that of $D_\tau$ we will introduce the manifold $\tilde{D}_\tau$ which is obtained by identifying the set $D_\tau \cap \{x_d = 0\}$ with the set $D_\tau \cap \{x_d = 2\}$. The set $\tilde{D}_\tau$ is homeomorphic to the $d-1$ dimensional torus $\mathbb{T}^{d-1}$. Furthermore we will assume that $\tau \in (0, \tau^*)$ is such that the Jacobi field $\Phi^R_{\tau, e_j}$ associated with the change of the Delaunay parameter is linearly growing, or in other words $\partial_\tau T_\tau > 0$.

As we have pointed out this is always true for $d = 3$ and for $d > 3$ it is true except possibly a finite set.

First we note that the approximate solution $w = U + \varepsilon^2 \psi_0$ is so far only defined in $N_\delta$, which is a tubular neighborhood of $\tilde{D}_\tau$. To extend $w$ to the whole space let us define

$$\mathbb{H}(x) = \begin{cases} 1 + \sigma_\varepsilon & \text{if } x \in \tilde{D}^+_\tau, \\ -1 + \sigma_\varepsilon & \text{if } x \in \tilde{D}^-_\tau, \end{cases}$$

where $\tilde{D}^+_\tau, \tilde{D}^-_\tau$ denote, respectively, the interior and the exterior of $\tilde{D}_\tau$.

Let us notice that the function $w$ approaches $\mathbb{H}$ exponentially. Indeed, we have

$$|Y_{\varepsilon, h}^* w(y, t) - Y_{\varepsilon, h}^* \mathbb{H}(y, t)| \leq C e^{-\mu |t|}, \quad y \in \tilde{D}_\tau, \quad t \in \left(\frac{\delta}{\varepsilon}, -\frac{\delta}{\varepsilon}\right).$$
for any $\mu \in (0, |\eta|)$. To make the definition of $w^*$ precise we let $\chi$ to be a cutoff function such that $\chi(s) = 1$, when $|s| \leq \frac{1}{2}$ and $\chi(s) = 0$, when $|s| \geq 1$. Next, we define a cutoff function $\chi^*$ supported in $N_\delta$ by:

$$Y_{\varepsilon,h}^* \chi^*(t) = \chi \left( \frac{\varepsilon t}{\delta} \right).$$

We can define a global approximate solution $w^*$ by

$$w^*(x) = w(x) \chi^*(x) + \mathbb{H}(x) \left[ 1 - \chi^*(x) \right].$$

(38)

Now we look for a solution of the equation (7) in the form

$$u = w^* + \varphi$$

where $\varphi$ is a small (in a way to be specified) function. Thus our problem can be stated: find a function $\varphi: \mathbb{R}^{d-1} \times S^1_{2T} \to \mathbb{R}$, that is one-periodic with period $2T$, in the $x_d$ direction and such that

$$N_\varepsilon (w^* + \varphi) = \ell_\varepsilon, \quad \text{in } \mathbb{R}^{d-1} \times S^1_{2T},$$

(39)

where $S^1_{2T}$ is a circle of radius $2T$.

$$N_\varepsilon (u) = \varepsilon \Delta u + \frac{1}{\varepsilon} u (1 - u^2),$$

and $\ell_\varepsilon$ is the Lagrange multiplier defined in (25). Let us recall that we want our solution to be rotationally symmetric. That is, if by $R_\theta$ we denote the rotation of $\mathbb{R}^d$ about the $x_d$ axis by angle $\theta$ then we should have:

$$(w^* + \varphi)(x) = (w^* + \varphi)(R_\theta x).$$

Since we already have (by definition)

$$w^*(x) = w^*(R_\theta x),$$

then as a result we will have $\varphi(x) = \varphi(R_\theta x)$ as well, as can be seen easily from the proceeding construction.

Since the function $\varphi$ appearing in (39) is expected to be small it is natural to expand the nonlinear operator $N_\varepsilon$ and write:

$$L_\varepsilon \varphi = -N_\varepsilon (w^*) - Q_\varepsilon (\varphi) + \ell_\varepsilon$$

where

$$L_\varepsilon \varphi = DN_\varepsilon (w^*) \varphi, \quad Q_\varepsilon (\varphi) = N (w^* + \varphi) - N_\varepsilon (w^*) - L_\varepsilon \varphi.$$

The strategy based on the Lyapunov-Schmidt reduction is clear. Indeed, we expect that due to the $d$ dimensional bounded (and periodic) kernel of the Jacobi operator $J_\tau$ which is associated to translations of $D_\tau$ in the directions of the coordinate axis $e_j$, $j = 1, \ldots, d$, the linear operator $L_\varepsilon$ should have $d$ dimensional kernel spanned, roughly speaking, by the functions $Z^{\tau, e_j}_\varepsilon$ where

$$Y_{\varepsilon,h}^* Z^{\tau, e_j}_\varepsilon (y, t) = V(y, t) \Phi^{\tau, e_j}_\varepsilon (y), \quad j = 1, \ldots, d,$$

$$V(y, t) = \partial_t w = \partial_t U(t) + \varepsilon^2 \partial_t \psi_0 (y, t).$$

(40)

Notice also that in general any function $Z_\varepsilon$ such that

$$Y_{\varepsilon,h}^* Z_\varepsilon (y, t) = \Phi(y) V(y, t),$$
is also “almost” in the kernel of \( L_\varepsilon \), in the sense that \( Y_{\varepsilon,h}^* (L_\varepsilon Z_\varepsilon) = o(1) \). We introduce a linear subspace of \( L^2(\tilde{D}_\varepsilon \times \mathbb{R}) \) of functions that are orthogonal to \( Z_\varepsilon \) by:

\[
\mathcal{X} = \left\{ \varphi \in L^2(\tilde{D}_\varepsilon \times \mathbb{R}) \mid \int_\mathbb{R} \varphi(y,t)\mathcal{V}(y,t)\,dt = 0 \right\}.
\]

(41)

By \( \Pi \) we denote the orthogonal projection on \( \mathcal{X} \). We set \( \varphi = \varphi^\| + \varphi^\perp \), where

\[
Y_{\varepsilon,h}^* \varphi^\| \in \mathcal{X}, \quad Y_{\varepsilon,h}^* \varphi^\perp = Z \varphi \in \mathcal{X}^\perp.
\]

Finally we split our problem into two equations:

\[
\Pi \circ Y_{\varepsilon,h}^* [N_\varepsilon (w^* + \varphi^\| + \varphi^\perp) - \varepsilon \ell] = 0, \quad (42)
\]

(\( \text{Id} - \Pi \)) \circ Y_{\varepsilon,h}^* [N_\varepsilon (w^* + \varphi^\| + \varphi^\perp) - \varepsilon \ell] = 0. \quad (43)

When solving (42) we use the fact that the associated linear operator is coercive on \( \mathcal{X} \). To solve (43) we will make use of the theory of solvability of the Jacobi operator \( \mathcal{J}_{\tilde{D}_\varepsilon} \). An additional, somewhat technical, step which we have omitted in this informal discussion is to “transfer” the original problem from the space of functions defined on \( \mathbb{R}^{d-1} \times S^1_{2\varepsilon} \) to a space of functions defined on \( \tilde{D}_\varepsilon \times \mathbb{R} \). We will explain these details in Section 5.3 but first we will introduce and study a linear operator which is essentially the expression of \( L_\varepsilon \) in the Fermi coordinates of \( \tilde{D}_\varepsilon \).

3.2. Linear theory for a model problem. In this section we will develop the necessary theory to deal with the operator \( L_\varepsilon \). To this end we will consider the operator

\[
\mathbb{L}_\varepsilon = \varepsilon \Delta_{\tilde{D}_\varepsilon} - (H_{\tilde{D}_\varepsilon} + \varepsilon (t + \varepsilon h_0))\chi(\varepsilon t/\delta)|A_{\tilde{D}_\varepsilon}|^2 \partial_t + \varepsilon^{-1} \partial_t^2 + \varepsilon^{-1} f^\prime(w), \quad (44)
\]

where \( \chi(s) \) is a cutoff function supported in \((-1, 1)\) and equal to 1 in \((-1/2, 1/2)\).

Note that this operator is defined for functions \( \phi: \tilde{D}_\varepsilon \times \mathbb{R} \to \mathbb{R} \) (and not just functions defined on \( \tilde{D}_\varepsilon \times (- \frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}) \)). It is clear that \( Y_{\varepsilon,h}^* L_\varepsilon \approx \mathbb{L}_\varepsilon \). Although the function \( w = U + \varepsilon^2 \psi_0 \) depends on both variables \( y, t \) in some sense the operator \( \mathbb{L}_\varepsilon \) separates variables. To see this, with \( \partial_t w = \partial_t(U + \varepsilon^2 \psi_0) = \mathcal{V} \), we consider functions of the form:

\[
\varphi(y,t) = \mathcal{V}(y,t)Z(y).
\]

Observe that, by construction of \( w = U + \varepsilon^2 \psi_0 \), combining equations (24) and (30) multiplied by \( \varepsilon^2 \) we get

\[
\varepsilon^{-1} \partial_t^2 \psi_0 (H_{\tilde{D}_\varepsilon} + \varepsilon (t + \varepsilon h_0)|A_{\tilde{D}_\varepsilon}|^2 \partial_t \psi_0 + \varepsilon^{-1} f^\prime(w))
\]

\[
= \varepsilon^3 (t + \varepsilon h_0)|A_{\tilde{D}_\varepsilon}|^2 \partial_t \psi_0 + \varepsilon^{-1} Q(\varepsilon^2 \psi_0),
\]

where \( Q(v) = f(U + v) - f(U) - f^\prime(U)v \). Differentiating this equation in \( t \) we get for \( v = \partial_t w : \)

\[
\varepsilon^{-1} \partial_t^2 \mathcal{V} (H_{\tilde{D}_\varepsilon} + \varepsilon (t + \varepsilon h_0)|A_{\tilde{D}_\varepsilon}|^2 \partial_t \mathcal{V} - \varepsilon |A_{\tilde{D}_\varepsilon}|^2 \mathcal{V} + \varepsilon^{-1} f^\prime(w)\mathcal{V})
\]

\[
= -\varepsilon^3 |A_{\tilde{D}_\varepsilon}|^2 \partial_t \psi_0 - \varepsilon^3 (t + \varepsilon h_0)|A_{\tilde{D}_\varepsilon}|^2 \partial_t^2 \psi_0 - \varepsilon^{-1} \partial_t Q(\varepsilon^2 \psi_0).
\]

(45)

From this, using the definition of \( \mathbb{L}_\varepsilon \) in (44) we get:

\[
\mathbb{L}_\varepsilon (\varphi) = \mathbb{L}_\varepsilon (\mathcal{V} Z) = \varepsilon \mathcal{V} \mathcal{J}_{\tilde{D}_\varepsilon} Z + B_\varepsilon (Z)
\]

(46)
where $\mathcal{J}_{\hat{D}_\tau}$ is the Jacobi operator on $\hat{D}_\tau$ and

$$B_\varepsilon(Z) = -\varepsilon(t + \varepsilon h_0)(1 - \chi^*)|A_{\hat{D}_\tau}|^2 Z \partial_t V + 2\varepsilon \nabla_{\hat{D}_\tau} \nabla \cdot \nabla_{\hat{D}_\tau} Z$$

$$+ \varepsilon Z \Delta_{\hat{D}_\tau} V - \varepsilon^3 \left[(t + \varepsilon h_0)|A_{\hat{D}_\tau}|^2 \partial_t^2 \psi_0 + |A_{\hat{D}_\tau}|^2 \partial_t \psi_0 \right] Z$$

$$+ \varepsilon^{-1} \left[f'(w) \partial_t w - f'(U) \partial_t w - \varepsilon^2 f''(U) \partial_t \psi_0 \right] Z.$$

We note that

$$B_\varepsilon(Z) = O(\varepsilon^3)\|Z\|_{C^1(D_\tau)}.$$  \hspace{1cm} (47)

Identity \hspace{1cm} (45) and its consequence \hspace{1cm} (46) is the key calculation which allows to use the usual Lyapunov-Schmidt reduction scheme, as w explained in the introduction. Indeed, if we had taken as the approximate solution only the function $U$ then differentiating the equation (24) for $U' = \partial_t U$ we would have gotten

$$\varepsilon^{-1} \partial_t^2 U' - H_{\hat{D}_\tau} \partial_t U' + \varepsilon^{-1} f'(U) U' = 0.$$  \hspace{1cm} (48)

This equation, unlike (45), does not carry any information about the geometry of $\hat{D}_\tau$ besides its mean curvature which is constant. Following the method of \hspace{1cm} [28] or \hspace{1cm} [12], \hspace{1cm} [13], \hspace{1cm} [14], \hspace{1cm} [29] we would have to perturb the surface $\hat{D}_\tau$ additionally introducing new unknown functions in our problem. With the approach presented here this is no longer necessary and the Lyapunov-Schmidt procedure in this version is in this sense simpler. Recalling that the linearization of the mean curvature operator is the Jacobi operator which depends on the second fundamental form, we see that the operator $L_\varepsilon$ is naturally compatible with the geometric context of our problem. To put it differently: the operator $L_\varepsilon$ is, up to negligible terms, the correct linearization of the Cahn-Hilliard operator near a solution whose zero level set is the constant curvature surface $\hat{D}_\tau$.

To develop invertibility theory for $L_\varepsilon$ we will we employ two basic facts. First, we observe that on the subspace:

$$\mathcal{Y} := \left\{ \varphi(y, t) = V(y, t)Z(y) \mid \int_{\hat{D}_\tau} Z(y) \Phi_{\tau, \varepsilon}^T(y) dy = 0, j = 1, \ldots, d \right\},$$

we have

$$\langle \parallel \varepsilon \varphi, \varphi \rangle \geq C\varepsilon \parallel \varphi \parallel_{L^2(\hat{D}_\tau, \mathbb{R})}^2.$$  \hspace{1cm} (49)

Second, when we consider $\varphi \in \mathcal{X}$ (space $\mathcal{X}$ is defined in \hspace{1cm} (41)) and $g \in L^2(\hat{D}_\tau \times \mathbb{R})$ such that $\varphi$ is a bounded solution of the problem

$$L_\varepsilon \varphi = g, \quad L_\varepsilon = \varepsilon \Delta_{\hat{D}_\tau} + \varepsilon^{-1} \partial_t^2 + \varepsilon^{-1} f'(H),$$

then we have

$$\|\varphi\|_{L^2(\hat{D}_\tau \times \mathbb{R})} \leq C\varepsilon \|g\|_{L^2(\hat{D}_\tau \times \mathbb{R})}.$$  \hspace{1cm} (48)

To prove this estimate we use a contradiction argument which relies on the fact that from

$$\int_\mathbb{R} |v' - f'(H)v|^2 \geq C\|v\|^2_{L^2(\mathbb{R})}, \quad \text{if} \quad \int_\mathbb{R} v(t)H'(t) dt = 0,$$

it follows that the bilinear form

$$B_\varepsilon(\varphi) := \langle L_\varepsilon \varphi, \varphi \rangle,$$

is coercive on $\mathcal{X}$. By a well known argument using \hspace{1cm} (48) is can be shown:

$$\varepsilon^2 \|\nabla_{\hat{D}_\tau} \varphi\|_{L^2(\hat{D}_\tau \times \mathbb{R})} + \|\partial_t \varphi\|_{L^2(\hat{D}_\tau \times \mathbb{R})} + \|\varphi\|_{L^2(\hat{D}_\tau \times \mathbb{R})} \leq C\|g\|_{L^2(\hat{D}_\tau \times \mathbb{R})}.$$  \hspace{1cm} (49)
We refer the reader to [12] or [14] where results similar to estimates (48) and (49) were proven.

At the same time we can use (49) and the coercivity of the bilinear form $B_\varepsilon(\varphi)$ to solve, for $\varphi \in \mathcal{X}$, the equation

$$\Pi_{\mathcal{X}} L_{\varepsilon} \varphi = g,$$

where $\Pi_{\mathcal{X}}$ is the projection on the subspace $\mathcal{X}$ of functions orthogonal to $\mathcal{V}$ defined in (41). To do this we write $L_{\varepsilon} = L_{\varepsilon} + (L_{\varepsilon} - L_{\varepsilon})$ and use a perturbation argument. The solution will still satisfy estimate (49). The perturbation argument is as follows:

for $g \in \mathcal{X}$ we solve

$$L_{\varepsilon} \phi = g + c H,'$$

where $c = \int g H' \, dt$. Then we define a map

$$G_{\mathcal{X}}(g) = \phi - \frac{1}{H'} \int_0^t \phi \, dt.$$

Note that $G_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$. Next we check

$$\| \Pi_{\mathcal{X}} L_{\varepsilon} G_{\mathcal{X}}(g) - g \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})} \leq o(1) \| g \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})}.$$\n
Indeed, since $\mathcal{V} = H' + O(\varepsilon)$ by the construction of $w = H + O(\varepsilon)$ in Section 2.3 we have

$$|c| \leq C \varepsilon \| \phi \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})} \leq C \varepsilon \| g \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})}.$$\n
Moreover, from

$$L_{\varepsilon} - L_{\varepsilon} = -(H_{\mathcal{D}_\varepsilon} + \varepsilon (t + \varepsilon h_0) \chi(\varepsilon t/\delta) \int A_{D_{\varepsilon}}^2 \, dt + \varepsilon^{-1} [f'(w) - f'(H)]$$

we find

$$\| (L_{\varepsilon} - L_{\varepsilon}) \phi \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})} \leq C (\| \partial_t \phi \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})} + \| \phi \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})}) \leq C \varepsilon \| g \|_{L^2(\mathcal{D}_\varepsilon \times \mathbb{R})}$$

by estimate (49). Therefore $\Pi_{\mathcal{X}} L_{\varepsilon} G_{\mathcal{X}}$ is invertible as a map from $\mathcal{X}$ to itself and we can define

$$(\Pi_{\mathcal{X}} L_{\varepsilon})^{-1} = G_{\mathcal{X}}(\Pi_{\mathcal{X}} L_{\varepsilon} G_{\mathcal{X}})^{-1}.$$\n
Moreover it is rather straightforward to show that a solution to (50) will satisfy estimate (49).

We will use these observations to solve the following model equation:

$$L_{\varepsilon} \varphi = g(y, t),$$

where we will assume initially that $g \in L^2(\mathcal{D}_\varepsilon \times \mathbb{R})$. We look for a solution in the form $\varphi = \varphi^\parallel + \varphi^\perp$, where

$$\varphi^\parallel \in \mathcal{X}, \quad \varphi^\perp = Z \psi.$$\n
We write

$$L_{\varepsilon} \varphi = \Pi_{\mathcal{X}} L_{\varepsilon} (\varphi^\parallel + \varphi^\perp) + \Pi_{\mathcal{X}^\perp} L_{\varepsilon} (\varphi^\parallel + \varphi^\perp),$$

and then we need to solve

$$\Pi_{\mathcal{X}} L_{\varepsilon} \varphi^\parallel = \Pi_{\mathcal{X}} g - \Pi_{\mathcal{X}} L_{\varepsilon} \varphi^\perp,$$

$$\Pi_{\mathcal{X}} L_{\varepsilon} \varphi^\perp = \Pi_{\mathcal{X}^\perp} \varphi^\parallel - \Pi_{\mathcal{X}^\perp} L_{\varepsilon} \varphi^\parallel.$$\n
The idea is that terms $\Pi_{\mathcal{X}} L_{\varepsilon} \varphi^\perp$ and $\Pi_{\mathcal{X}^\perp} L_{\varepsilon} \varphi^\parallel$ are of smaller order because $L_{\varepsilon} \mathcal{V} = o(1)$ so that the coupling between the two equations is rather weak. Another important point is that

$$L_{\varepsilon} \varphi^\perp = \mathcal{V} \int_{\mathcal{D}_\varepsilon} Z + B_{\varepsilon}(Z).$$
where $B_{\varepsilon}(Z)$ is small (see (47)). We decompose accordingly $g = g^\parallel + g^\perp$, $g^\perp = \Xi V$ and $B_{\varepsilon}(Z) = B^\parallel_{\varepsilon}(Z) + B^\perp_{\varepsilon}(Z)$, $B^\perp_{\varepsilon}(Z) = T_{\varepsilon}(Z)V$ and look for a solution of the system:

$$
\Pi_X L_{\varepsilon} \phi^\parallel = g^\parallel - B^\parallel_{\varepsilon}(Z),
\varepsilon J_{D_\varepsilon} Z + T_{\varepsilon}(Z) = \Xi - \int_\mathbb{R} (L_{\varepsilon} \phi^\parallel) V dt + \sum_{j=1}^d c_j \Phi^T e_j.
$$

(52)

Note that in the second equation we have introduced Lagrange multipliers $c_j$ to be determined.

Although the two equations in (52) are coupled but this coupling is weak and we can solve the system without any difficulty using invertibility of $\Pi_X$ and $J_{D_\varepsilon}$, a fixed point argument and estimates (47), (48) and (49). An alternative approach is to use a perturbation argument similar in the spirit to the one presented above. Thus we should first solve for given $g^\parallel \in X$ and $g^\perp = \Xi V \in X_\perp$ the system

$$
\Pi_X L_{\varepsilon} \phi^\parallel = g^\parallel,
\varepsilon J_{D_\varepsilon} Z = \Xi + \sum_{j=1}^d c_j \Phi^T e_j.
$$

This would define a map $G(g^\parallel, g^\perp) = (\phi^\parallel, \phi^\perp)$, $\phi^\perp = Z V$. Then we check that

$$
(\Pi_X L_{\varepsilon}, \varepsilon V J_{D_\varepsilon}) \circ G(g^\parallel, g^\perp) = -(\Pi_X L_{\varepsilon} \phi^\parallel, \Pi_X L_{\varepsilon} \phi^\perp),
$$

is a small perturbation of identity. Finally we define the inverse of the operator in (52) by

$$
G \circ \left( (\Pi_X L_{\varepsilon}, \varepsilon V J_{D_\varepsilon}) \circ G \right)^{-1}.
$$

We leave a choice of the method and the details to the reader.

Given that we can solve (52) our purpose is to find suitable estimates for the solution of the problem

$$
L_{\varepsilon} \phi = g(y, t)
$$
on $X$ assuming that

$$
g(y, t) = \mathcal{O}(e^{-\mu |t|}), \quad |t| \to \infty.
$$

In particular we would like to know that $\phi(t, y) = \mathcal{O}(e^{-\mu |t|})$ as well. This is straightforward by comparison principle once we know for example that $\phi$ is bounded. Thus the main issue is to obtain $L_\infty$ control for $\phi$. We will go a little further now and show how to control $a priori$ certain weighted Hölder norms of $\phi^\parallel$ and $\phi^\perp$ (see decomposition in (52)).

In order to simplify the argument and avoid keeping track of negative powers of $\varepsilon$ appearing on the right hand side of various estimates we will rescale the $y$ variable. Thus we will introduce:

$$
\hat{y} = \frac{y}{\varepsilon}, \quad \hat{t} = t.
$$

We will denote $\hat{D}_{\tau, \varepsilon} = \frac{1}{\varepsilon} D_\tau$. We consider the manifold $\hat{D}_{\tau, \varepsilon} \times \mathbb{R}$ equipped with the product metric and the associated Levi-Civita connection. We define the weighted
Hölder norms on $\tilde{D}_{r,\varepsilon} \times \mathbb{R}$:

\[
\|u\|_{C^{0,\alpha}_{\mu}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} = \sup_{\tilde{t} \in \mathbb{T}} \overline{(\cosh \tilde{t})^\mu} \|u\|_{C^{0,\alpha}(\tilde{D}_{r,\varepsilon} \times (\tilde{t} - 1, \tilde{t} + 1))},
\]

\[
\|u\|_{C^{1,\alpha}_{\mu}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} = \|u\|_{C^{0,\alpha}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} + \|\nabla_{\tilde{D}_{r,\varepsilon} \times \mathbb{R}} u\|_{C^{0,\alpha}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})},
\]

\[
\|u\|_{C^{2,\alpha}_{\mu}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} = \|u\|_{C^{0,\alpha}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} + \|\nabla_{\tilde{D}_{r,\varepsilon} \times \mathbb{R}} u\|_{C^{0,\alpha}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} + \|\nabla^2_{\tilde{D}_{r,\varepsilon} \times \mathbb{R}} u\|_{C^{0,\alpha}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})}. 
\]

(53)

Above $\nabla_{\tilde{D}_{r,\varepsilon} \times \mathbb{R}}$ and $\nabla^2_{\tilde{D}_{r,\varepsilon} \times \mathbb{R}}$ respectively denote the gradient and the Hessian on the manifold $\tilde{D}_{r,\varepsilon} \times \mathbb{R}$.

Given functions $u$ and $g$ on $\tilde{D}_{r} \times \mathbb{R}$ and we introduce functions

\[
\tilde{u}(\tilde{y}, \tilde{\varepsilon}) = u(\varepsilon \tilde{y}, \tilde{\varepsilon}), \quad \tilde{w}(\tilde{y}, \tilde{\varepsilon}) = w(\varepsilon \tilde{y}, \tilde{\varepsilon}), \quad \tilde{g}(\tilde{y}, \tilde{\varepsilon}) = \varepsilon g(\varepsilon \tilde{y}, \tilde{\varepsilon}).
\]

and set

\[
\tilde{\mu}_\varepsilon \tilde{u} = \Delta_{\tilde{D}_{r,\varepsilon}} \tilde{u} + \partial^2_\varepsilon \tilde{u} + f'(\tilde{w})\tilde{u} + \varepsilon \tilde{g} \partial_\varepsilon \tilde{u},
\]

where

\[
\tilde{g} := H_{\tilde{D}_{r}} + \varepsilon (\tilde{\varepsilon} + \varepsilon h_0) \chi(\varepsilon \tilde{\varepsilon}/\delta) |A_{\tilde{D}_{r}}(\varepsilon \tilde{y})|^2
\]

is a bounded function. The linear problem we consider is:

\[
\Pi X \tilde{\mu}_\varepsilon \tilde{u} = \tilde{g}, \quad \text{in } \tilde{D}_{r,\varepsilon} \times \mathbb{R},
\]

(54)

where now we assume simply that the right hand side satisfies the orthogonality condition (c.f. (35)):

\[
\int_{\mathbb{R}} \tilde{g}(\tilde{y}, \tilde{\varepsilon}) \Psi(\varepsilon \tilde{y}, \tilde{\varepsilon}) d\tilde{\varepsilon} = 0.
\]

For this problem we can derive a priori estimates using the method of [12, 13, 14].

From this, by a perturbation argument, we will be able get corresponding estimates for $\varphi^0$ satisfying the first equation in (52). Let us explain briefly the main steps.

**Step 1.** We consider a problem of the form:

\[
\Delta g \varphi + \partial^2_\varepsilon \varphi + f'(H(t))\varphi = 0, \quad \text{in } \mathbb{R}^{d-1} \times \mathbb{R}.
\]

The following result is known ([12, 13]):

**Lemma 3.1.** Let $\varphi$ be a bounded solution of (55). Then $\varphi = cH'(t)$, with some constant $c$.

**Step 2.** Consider now equation (54) and assume that $\tilde{g} \in C^{0,\alpha}_{\mu}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})$, with $\mu \in (0, |\eta|)$. The we have:

**Lemma 3.2.** There exists a constant $C > 0$ such that for all sufficiently small $\varepsilon$ any bounded solution of (54) satisfies:

\[
||\tilde{u}||_{C^{2,\alpha}_{\mu}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})} \leq C ||\tilde{g}||_{C^{0,\alpha}_{\mu}(\tilde{D}_{r,\varepsilon} \times \mathbb{R})},
\]

(56)

A proof of this lemma, which relies on Lemma 3.1 and a contradiction argument, follows the same lines as the proof of Lemma 5.2 in [14] (see also similar results in [12, 13]).
Now we should go back to the original variables. We define weighted Hölder norms on $\hat{D}_\tau \times \mathbb{R}$ similarly as in [53]:

\begin{align*}
\|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} &= \sup_{t \in \mathbb{R}} (\cosh t)^\mu \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times (t-1,t+1))}, \\
\|u\|_{\mathcal{C}^{1,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} + \|\nabla \hat{D}_\tau \times u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})}, \\
\|u\|_{\mathcal{C}^{2,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} + \|\nabla \hat{D}_\tau \times u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} + \|\nabla^2 \hat{D}_\tau \times u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})},
\end{align*}

(57)

We note that if for a given function $u : \hat{D}_\tau \times \mathbb{R} \to \mathbb{R}$ we set $\tilde{u}(\tilde{y},\tilde{t}) = u(\tilde{y},\tilde{t})$ then we have

\begin{align*}
\|\tilde{u}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} &= \sum_{0 \leq k + m \leq \ell} \epsilon^n \|\partial^k \hat{D}_\tau^m u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} \\
&\quad + \sum_{0 \leq k + m \leq \ell} \epsilon^{n+\alpha} \|\partial^k \hat{D}_\tau^m u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})},
\end{align*}

(58)

where $[\cdot]_{\alpha,\mu,\hat{D}_\tau \times \mathbb{R}}$ is the weighted Hölder seminorm. Consequently by $\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})$ we denote the space of functions on $\hat{D}_\tau \times \mathbb{R}$ with the norm

$$
\|u\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} := \sum_{0 \leq k + m \leq \ell} \epsilon^n \|\partial^k \hat{D}_\tau^m u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})}.
$$

With this definition we have

$$
\|u\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})},
$$

while with the notation of [53] we get

$$
C^{-1} \|\tilde{u}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} \leq \|u\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} \leq C \epsilon^{-\alpha} \|\tilde{u}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})}.
$$

(59)

From this and Lemma 3.2 it follows for $\ell = 0, 1, 2$:

$$
\|u\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} \leq C \epsilon^{1-\alpha} \|g\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})}.
$$

The procedure described above shows that we can control the size of $\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})$ norm of $\varphi$ in [52] obtaining:

$$
\|\varphi\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} \leq C \epsilon^{1-\alpha} (\|g\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} + \epsilon^2 \|g^\perp\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} + \epsilon \|g^\perp\|_{\mathcal{E}^{\epsilon,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})}).
$$

(60)

Finally, we notice that for the second equation in [52] using elliptic theory we can get Hölder estimates and since $\varphi^\perp = Z\nu$ we find:

$$
\|\varphi^\perp\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})} \leq C \epsilon^{-1} \|g^\perp\|_{\mathcal{C}^{0,\alpha}_{\mu}(\hat{D}_\tau \times \mathbb{R})}.
$$

(61)

### 3.3. The linear problem in the whole space.

Now we will use the theory outlined above to solve the following problem:

$$
\varepsilon \Delta \varphi + \frac{1}{\varepsilon} f'(u^*) \varphi = g(x), \quad \text{in } \mathbb{R}^{d-1} \times S_{T\tau},
$$

(62)

From what we have said above it is in general not possible to find a solution with a reasonably bounded norm unless the right hand side satisfies some extra conditions, or equivalently, we need to introduce Lagrange multipliers that correspond to natural invariances of the problem. Thus, we will solve

$$
\varepsilon \Delta \varphi + \frac{1}{\varepsilon} f'(u^*) \varphi = g(x) + \chi^* \sum_{j=1}^d c_j Z_{\tau,\varepsilon}^T \delta_j, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T\tau},
$$

(63)
where
\[ Y_{c,h}^+(t) = \chi(\varepsilon t/\delta), \quad Y_{c,h}^+Z_{T,c}^j(y, t) = V(y, t)\Phi^j_{T,c}(y), \quad j = 1, \ldots, d \]

The idea is to solve \([63]\) by gluing a solution defined near \(\dot{D}_\tau\) and another one defined away from \(\dot{D}_\tau\). To describe this construction rigorously we need some preparation. We introduce the function \(q(x)\) as follows:
\[
q(x) = \begin{cases} 
    f'(1 + \sigma_\varepsilon), & \text{dist}(x, D_\tau) > \delta/2, \\
    f'(-1 + \sigma_\varepsilon), & \text{dist}(x, D_\tau) < -\delta/2,
\end{cases}
\]
and otherwise \(q(x)\) is a smooth function such that \(\min\{f'(1 + \sigma_\varepsilon), f'(-1 + \sigma_\varepsilon)\} < q(x) \leq \max\{f'(1 + \sigma_\varepsilon), f'(-1 + \sigma_\varepsilon)\}\). Note that \(q(x) = -2 + O(\varepsilon)\). Finally, we need another cutoff function \(\tilde{\chi}\) such that \(\tilde{\chi}\chi^* = \chi^*\) (take for instance \(Y_{c,h}^+\tilde{\chi}(t) = \chi(\varepsilon t/2\delta)\)) and chose \(\delta\) smaller so that the Fermi coordinates are defined in \(N_{2\delta}\). We want to find a solution of \([63]\) in the form \(\varphi = \chi^*\tilde{\varphi} \circ Y_{c,h}^+ + \psi\), where the function \(\tilde{\varphi}\) solves:
\[
L_c\tilde{\varphi} = \left(\tilde{\chi}(g + \chi^* \sum_{j=1}^{d} c_j Z_{T,c}^j) + (L_c - L_c)\tilde{\varphi} - [\chi^*, L_c]\tilde{\varphi} + \varepsilon^{-1}[q - f'(w^*)]\psi\right), \quad \text{in } \dot{D}_\tau \times \mathbb{R},
\]
(64)
and the function \(\psi\) solves
\[
\varepsilon\Delta \psi + \varepsilon^{-1}[(1 - \chi^*)f'(w^*) + \chi^* q]\psi = (1 - \chi^*)\left(g + \chi^* \sum_{j=1}^{d} c_j Z_{T,c}^j\right) - [\chi^*, L_c]\tilde{\varphi}, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T}^c.
\]
(65)
It is clear that multiplying \([64]\) by \(\chi^*\), adding the equations \((64)-(65)\) and using the fact that \(\tilde{\chi}\chi^* = \chi^*\) we get the solution to our problem. In the above and in what follows we abuse slightly notation writing for instance \(\tilde{\varphi}\) as a function defined on \(\dot{D}_\tau \times \mathbb{R}\) and as a function defined on \(\mathbb{R}^{d-1} \times S_{2T^C}\). It is understood that in the latter case we take \(\tilde{\varphi} \circ Y_{c,h}^+\). To avoid complicated notions we will omit the composition with \(Y_{c,h}^+\) or \(Y_{c,h}^{-1}\) whenever it does not cause confusion. Thus the commutator
\[
[\chi^*, L_c]\tilde{\varphi} \in \mathbb{R}^{d-1} \times S_{2T^C},
\]
is
\[
[\chi^*, L_c]\tilde{\varphi} = 2\varepsilon \nabla \tilde{\varphi} \circ Y_{c,h}^+ \nabla \chi^* + \varepsilon \tilde{\varphi} \circ Y_{c,h}^+ \Delta \chi^* ,
\]
in \(\dot{D}_\tau \times \mathbb{R}\). We have to first express \(L_c\) in local coordinate \((y, t)\) (written as \(Y_{c,h}^+ L_c \) and calculate \([\chi^*, Y_{c,h}^+ L_c]\tilde{\varphi}\).

The function \(g\) on the right hand side of this equation satisfies the following general assumptions on its asymptotic behaviour:
\[
\|(g\chi^*)^\|_{W^{0,\infty}(\dot{D}_\tau \times \mathbb{R})} \leq C, \\
\|(g\chi^*)^{-1}\|_{W^{0,\infty}(\dot{D}_\tau \times \mathbb{R})} \leq C, \\
\|(1 - \chi^*)g\|_{W^{0,\infty}(\mathbb{R}^{d-1} \times S_{2T^C})} \leq Ce^{-\alpha/\varepsilon}.
\]
(66)
In addition we assume that \(g\) is rotationally symmetric about the \(x_d\) axis, namely if by \(R_\theta\) we denote the rotation of \(\mathbb{R}^d\) about the \(x_d\) axis by angle \(\theta\) then \(g(R_\theta x) = g(x)\).

In order to solve this coupled system we need to make sure that all terms on the right hand side that involve \(\tilde{\varphi}\) and \(\psi\) are small in suitable weighted Hölder and Hölder norms respectively. It is at this point that we need to choose the parameter \(\delta\) in the definition of the tubular neighbourhood \(N_\delta\) small and dependent on \(\varepsilon\). Thus we take \(\delta(\varepsilon) = \varepsilon^{2/3}\). This means in particular that for \(x \in N_\delta\) we have
\(\varepsilon t(x) = O(\varepsilon^{2/3})\). For reasons that will become clear soon we will also chose the Hölder exponent \(\alpha\) in the definition of \(C^\alpha_{\mu}(D_\tau \times \mathbb{R})\) and \(C^{0,\alpha}(D_\tau \times \mathbb{R})\) to be in the interval \((0, \frac{1}{10})\). Finally, the parameter \(\mu\) will be always taken in the interval \((0, |\eta|)\).

Considering equation (65) we have the following:

**Lemma 3.3.** If \(u\) is a solution of
\[
\varepsilon \Delta u - \frac{1}{\varepsilon} [(1 - \chi^*) f'(w^*) + \chi^* q] u = g, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau},
\]
then we have an a priori estimate:
\[
\|u\|_{C^{\ell,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \leq C \varepsilon^{1-\ell-\alpha} \|g\|_{C^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})},
\]
(67)

The proof of this lemma is straightforward and it is omitted, for similar results see for instance Lemma 4.1 in [11]. From this we get readily an a priori estimate for (65):
\[
\|\psi\|_{C^{\ell,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \leq C \varepsilon^{1-\ell-\alpha} \|1 - \chi^*\|g\|_{C^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} + \varepsilon^{-1/4} \sum_{j=1}^d |c_j| + \|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})}. \tag{68}
\]

From the theory developed in the previous section we can also obtain an a priori estimate for (64). If we write
\[
g = \tilde{\chi}\left\{g + \chi^* \sum_{j=1}^d c_j \mathcal{P}_j + (\mathbb{L}_\tau - L_\tau) \hat{\phi} - [\chi^*, L_\tau] \hat{\phi} + \varepsilon^{-1}[q - f'(w^*)] \psi\right\},
\]
then we have using (60)
\[
\|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C \varepsilon^{-1-\alpha} (\|g\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} + \varepsilon^{2} \|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})}), \tag{69}
\]
and using (61)
\[
\|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C \varepsilon^{-1} \|g\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})}. \tag{70}
\]

We note that the weighted norms we use for \(\hat{\phi}\|\) and \(\hat{\phi}\) are scaled differently with \(\varepsilon\). This slight nuisance is a result of our choice of the original scaling of the Cahn-Hilliard equation. We observe as well that with our definitions \(\|\cdot\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} = \|\cdot\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})}\),

We will now estimate \(\|g\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})}\). To do this we observe that, with \(\hat{\phi} = \hat{\phi} + \hat{\phi}\), we have:
\[
\left\| (\tilde{\chi}(\mathbb{L}_\tau - L_\tau) \hat{\phi}) \right\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C \delta(\varepsilon) (\varepsilon^{-1} \|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} + \varepsilon \|\hat{\phi}\|_{C^{1,\alpha}_{\mu}(D_\tau \times \mathbb{R})})
\]
\[
\left\| (\tilde{\chi}[\chi^*, L_\tau] \hat{\phi}) \right\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C \delta(\varepsilon) (\varepsilon^{-1} \|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})}
\]
\[
+ O(\varepsilon^{-1/3}) \|\hat{\phi}\|_{C^{1,\alpha}_{\mu}(D_\tau \times \mathbb{R})}),
\]
\[
\left\| (\varepsilon^{-1}[q - f'(w^*)] \psi) \right\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C \varepsilon^{-1} \|\psi\|_{C^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})}.
\]

Next we estimate the orthogonal complement of these functions
\[
\left\| (\tilde{\chi}(\mathbb{L}_\tau - L_\tau) \hat{\phi}) \right\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C (\|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} + \varepsilon^{2} \|\hat{\phi}\|_{C^{1,\alpha}_{\mu}(D_\tau \times \mathbb{R})})
\]
\[
\left\| (\tilde{\chi}[\chi^*, L_\tau] \hat{\phi}) \right\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq O(\varepsilon^{-1/3}) (\|\hat{\phi}\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} + \|\hat{\phi}\|_{C^{1,\alpha}_{\mu}(D_\tau \times \mathbb{R})}),
\]
\[
\left\| (\varepsilon^{-1}[q - f'(w^*)] \psi) \right\|_{C^{0,\alpha}_{\mu}(D_\tau \times \mathbb{R})} \leq C \varepsilon^{-1} \|\psi\|_{C^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})}.
\]
We can estimate the parameters $c_j$ by projection of $\gamma$ onto $Z_{r, \epsilon}^{T, s_j}$. Using the above estimate we get:

$$|c_j| \leq C \left\{ \| (\tilde{x} g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) + \left\| \left( \tilde{x} (L_{\epsilon} - L_{\epsilon}) \tilde{\varphi} \right) \right\| + \left\| (\tilde{x} (\varphi^* \tilde{L}_{\epsilon}) \tilde{\varphi} \right\| + \left\| (\tilde{x} (\varphi^* \tilde{L}_{\epsilon}) \tilde{\varphi} \right\| \right\} \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\}$$

Now we use estimates (69)–(70). After rearranging terms suitably and using

$$\epsilon^{1-\alpha} \delta^{-1}(\epsilon) = o(1)$$

to absorb $\tilde{\varphi}$ in the first inequality below we get

$$\| \tilde{\varphi} \| \leq C \epsilon^{1-\alpha} \left\{ \| (\tilde{x} g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) + \epsilon^2 \| (\tilde{x} g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\} + \epsilon^{-1} \| \tilde{\varphi} \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\} \right\}, (71)$$

From (68) we get as well for $\ell = 0, 1, 2$:

$$\| \tilde{\varphi} \| \leq C \epsilon^{1-\ell} \left\{ \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) + \epsilon^{-1} \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\} + \epsilon^{-3-\alpha} \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\}, (72)$$

Using the fact that $\delta(\epsilon) \epsilon^{-\alpha} = o(1)$ to absorb term $\delta(\epsilon) \| \tilde{\varphi} \| \mathcal{C}_0 (\tilde{D}, \epsilon)$ appearing on the right hand side of the first inequality in (71) and combining these estimates we get

$$\| \tilde{\varphi} \| \leq C \epsilon^{1-\alpha} \left\{ \| (\tilde{x} g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) + \epsilon^{-1} \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\} + \epsilon^{-3-\alpha} \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\}, (73)$$

and

$$\| \tilde{\varphi} \| \leq C \epsilon^{1-\alpha} \left\{ \| (\tilde{x} g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) + \epsilon^{-1} \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\} + \epsilon^{-3-\alpha} \| (1 - \chi^* g)^\parallel \| \mathcal{C}_0 (\tilde{D}, \epsilon) \right\}, (74)$$

Using these $a priori$ estimates we can solve the system (64)–(65) by a standard fixed point argument. To do this we replace the functions $\tilde{\varphi}, \tilde{\varphi}^-, \tilde{\varphi}^+$ on the right hand side of the system by known functions $\bar{\varphi}, \bar{\varphi}^-, \bar{\varphi}^+$ which satisfy estimates of the
same type as (73)–(74) but with constants bigger than those appearing in (73)–(74). Then we have a map

$$(\tilde{\phi}^\parallel, \tilde{\phi}^\perp, \psi) \mapsto (\tilde{\phi}^\parallel, \tilde{\phi}^\perp, \psi),$$

from a certain ball in the space $E^\parallel_2(D_\tau \times \mathbb{R}) \times C^{2,0}_\mu(D_\tau \times \mathbb{R}) \times C^2(\mathbb{R}^{d-1} \times S_{2T_r})$ into itself. This and the Lipschitz character of this map being evident from the way we have derived a *priori* estimates allows for an application of the Banach fixed point theorem. We leave the details to the reader and simply state this as a result for (63).

**Lemma 3.4.** For each sufficiently small $\varepsilon$ there exists a solution of (63) in the form $\varphi = \chi^* \varphi \circ Y_{\varepsilon,h} + \psi$ such that estimates (73)–(74) hold.

### 3.4. Proof of Theorem 1.1

Now we can finish solving the nonlinear problem

$$L_\varepsilon \varphi = \ell_\varepsilon - N_\varepsilon (w^*) - Q_\varepsilon(\varphi).$$

(75)

As we saw above we need to modify this equation by introducing Lagrange multipliers. Thus we will consider:

$$L_\varepsilon \varphi = \ell_\varepsilon - N_\varepsilon (w^*) - Q_\varepsilon(\varphi) + \chi^* \sum_{j=1}^{d} c_j Z_{\tau,r}^j.$$  

(76)

To solve this problem we use a fixed point argument and the linear theory in Lemma 3.4 above. The first task is to calculate the size of the error of the approximation $\ell_\varepsilon - N_\varepsilon (w^*)$. This is straightforward using the definition of $w^*$ and formula (21). We recall here that $h = \varepsilon^2 h_0$, where $h_0$ is a constant and consequently this last formula simplifies significantly. We can write:

$$\ell_\varepsilon - N_\varepsilon (w^*) = \chi^*[\ell_\varepsilon - N_\varepsilon (w)] + [N(w^*) - \chi^* N(w) - (1 - \chi^*) N(\varepsilon)] \equiv A_1 + A_2,$$

since $\ell_\varepsilon = N_\varepsilon (\varepsilon)$ in $\text{supp} (1 - \chi^*)$. Using exponential decay of $w = (\pm 1 + \sigma_\varepsilon)$ when $t \to \pm \infty$ we get easily:

$$\|Y_{\varepsilon,h}^* A_2\|_{C^{0,\alpha}_\mu(\partial \tau \times \mathbb{R})} \leq C_0 e^{-c_\mu \varepsilon^{-1/3}},$$

$$\|(1 - \chi^*) A_2\|_{C^{0,\alpha}(\mathbb{R}^{d-1} \times \partial \tau)} \leq e^{-\theta \varepsilon^{-1}}.$$  

(77)

To estimate $A_1$ some standard calculations which we will omit are needed (c.f Section 2.3). As a result we get

$$\|Y_{\varepsilon,h}^* \tilde{\chi} A_1\|_{C^{0,\alpha}_\mu(\partial \tau \times \mathbb{R})} \leq C_0 e^{d},$$

$$\|Y_{\varepsilon,h}^* \tilde{\chi} A_1\|_{C^{0,\alpha}_\mu(\partial \tau \times \mathbb{R})} \leq C_0 e^{d},$$

$$\|(1 - \chi^*) A_1\|_{C^{0,\alpha}(\mathbb{R}^{d-1} \times \partial \tau)} \leq C_0 e^{-\theta \varepsilon^{-1}},$$

(78)

where $C_0$, $c_\mu$ and $\theta$ are positive constants. Now we use the linear theory developed in the previous section to solve the nonlinear problem (76). Thus we write $\varphi = \chi^* \bar{\varphi} \circ Y_{\varepsilon,h} + \psi$, and further decompose $\bar{\varphi} = \bar{\varphi}^\parallel + \bar{\varphi}^\perp$ where $\bar{\varphi}^\parallel \in X \cap C^{2,0}_\mu(\partial \tau \times \mathbb{R})$, $\bar{\varphi}^\perp \in Y \cap C^{2,0}_\mu(\partial \tau \times \mathbb{R})$ and $\psi \in C^{2,0}(\mathbb{R}^{d-1} \times \partial \tau)$. To set up a fixed point scheme we fix functions $\bar{\varphi}^\parallel$, $\bar{\varphi}^\perp$ and $\tilde{\psi}$ in these sets such that

$$\|\bar{\varphi}^\parallel\|_{C^{2,0}_\mu(\partial \tau \times \mathbb{R})} \leq K \varepsilon^{-1/3},$$

$$\|\bar{\varphi}^\perp\|_{C^{2,0}_\mu(\partial \tau \times \mathbb{R})} \leq K \varepsilon^2,$$

$$\|\tilde{\psi}\|_{C^{2,0}(\mathbb{R}^{d-1} \times \partial \tau)} \leq K e^{-\theta \varepsilon^{-1}},$$

(79)
where $K$ is a large constant to be chosen and $\tilde{\theta} \in (\theta/2, \theta)$ is a constant. Let us denote the right hand side of (76) by $g$. It is evident that under the assumptions (79), and with a suitable choice of the constants $\alpha > 0$ and $\mu \in (0, |\eta|)$ we can solve the problem (76) for functions $(\check{\varphi}^\parallel, \check{\varphi}^\perp, \psi)$ which again satisfy (79). Thus we have a non-linear map

$$(\check{\varphi}^\parallel, \check{\varphi}^\perp, \psi) \mapsto (\tilde{\varphi}^\parallel, \tilde{\varphi}^\perp, \psi),$$

of this set into itself. To show that this map is a contraction is straightforward, using the quadratic nature of the nonlinear function $Q(\varphi)$. At the end we have a solution of the problem:

$$\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell \varepsilon + \sum_{j=1}^{d} \chi^* c_j Z_{r,\varepsilon}^T, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T}, \quad (80)$$

where $Z_{r,\varepsilon}^T$ is the (approximate) element of the kernel of the linear operator $L_{\varepsilon}$ associated with translation in the direction of the $x_j$ axis, see (40). To show that in fact

$$c_j = 0, \quad j = 1, \ldots, d,$$

we need:

**Lemma 3.5 (Balancing formula).** Let $X = \sum a_j \partial_{x_j}$ be the infinitesimal generator of translations or rotations in $\mathbb{R}^d$. For any $\mathcal{C}^2(\mathbb{R}^d)$ function it holds:

$$\text{div} \left( \frac{\varepsilon}{2} |\nabla u|^2 - \frac{1}{\varepsilon} F(u) \right) X(u) - \varepsilon X(u) \nabla u = -[\varepsilon \Delta u + \frac{1}{\varepsilon} F'(u)] X(u). \quad (81)$$

We will take $X_j = \partial_{x_j}$ for some $1 \leq j \leq d$ and integrate the balancing formula over the cylinder $C_R = B_R \times S_{2T}$. Using (80) and Green’s theorem we get:

$$\int_{\partial C_R} \left( \frac{\varepsilon}{2} |\nabla u|^2 - \frac{1}{\varepsilon} F(u) + \ell \varepsilon u \right) n_j dS - \int_{\partial C_R} \partial_{x_j} u \partial_n u dS = - \int_{C_R} \left( \sum_{j'=1}^{d} \chi^* c_{j'} Z_{r,\varepsilon}^{T_{j'}} \right) \partial_{x_j} u. \quad (III_R)$$

The first integral $I_R$ is 0 on the top and the bottom of $C_R$ and on the other hand, using the asymptotic behavior of the solution we get finally $\lim_{R \to \infty} I_R = 0$. In the second integral the integrals over the top and the bottom of $C_R$ cancel because $u$ is periodic. Then, from exponential decay of the derivatives of $u$ we get $\lim_{R \to \infty} II_R = 0$. Finally, we note that

$$\partial_{x_j} u \approx Z_{r,\varepsilon}^{T_{j'}},$$

hence

$$III_R = c_j \int_{\tilde{D} \times \mathbb{R}} |Z_{r,\varepsilon}^{T_{j'}}|^2 + o(1) \sum_{j'=1}^{d} c_{j'},$$

from which we get immediately $c_j = 0, j = 1, \ldots, d$.

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REFERENCES

[1] N. D. Alikakos, P. W. Bates and X. Chen, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model, Arch. Rational Mech. Anal., 128 (1994), 165–205.
[2] N. D. Alikakos, X. Chen and G. Fusco, Motion of a droplet by surface tension along the boundary, Calc. Var. Partial Differential Equations, 11 (2000), 233–305.
[3] N. D. Alikakos and G. Fusco, The spectrum of the Cahn-Hilliard operator for generic interface in higher space dimensions, Indiana Univ. Math. J., 42 (1993), 637–674.
[4] N. D. Alikakos and G. Fusco, Slow dynamics for the Cahn-Hilliard equation in higher space dimensions: The motion of bubbles, Arch. Rational Mech. Anal., 141 (1998), 1–61.
[5] N. D. Alikakos, G. Fusco and V. Stefanopoulos, Critical spectrum and stability of interfaces for a class of reaction-diffusion equations, J. Differential Equations, 126 (1996), 106–167.
[6] P. W. Bates and G. Fusco, Equilibria with many nuclei for the Cahn-Hilliard equation, J. Differential Equations, 160 (2000), 283–356.
[7] X. Chen, Spectrum for the Allen-Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces, Comm. Partial Differential Equations, 19 (1994), 1371–1395.
[8] X. Chen and M. Kowalczyk, Existence of equilibria for the Cahn-Hilliard equation via local minimizers of the perimeter, Arch. Rational Mech. Anal., 141 (1998), 1–61.
[9] S. N. Chow and J. K. Hale, Methods of Bifurcation Theory, volume 251 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], Springer-Verlag, New York-Berlin, 1982.
[10] H. Dang, P. C. Fife and L. A. Peletier, Saddle solutions of the bistable diffusion equation, Z. Angew. Math. Phys., 43 (1992), 984–998.
[11] M. del Pino, M. Kowalczyk, F. Pacard and J. Wei, The Toda system and multiple-end solutions of autonomous planar elliptic problems, Adv. Math., 224 (2010), 1462–1516.
[12] M. del Pino, M. Kowalczyk and J. Wei, On De Giorgi’s conjecture in dimension $N \geq 9$, Ann. of Math. (2), 174 (2011), 1485–1569.
[13] M. del Pino, M. Kowalczyk and J. Wei, Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $R^3$, Journ. Diff. Geometry, 93 (2013), 67–131.
[14] M. del Pino, M. Kowalczyk and J. Wei, Traveling waves with multiple and non-convex fronts for a bistable semilinear parabolic equation, Comm. Pure Appl. Math., 66 (2013), 481–547.
[15] M. del Pino, F. Pacard and M. Musso, Solutions of the Allen-Cahn equation which are invariant under screw-motion, Manuscripta Math., 138 (2012), 273–286.
[16] C. Delaunay, Sur la surface de revolution dont la courbure moyenne est constante, J. Math. Pures Appl., 6 (1841), 309–320.
[17] J. Eells, The surfaces of Delaunay, Math. Intelligencer, 9 (1987), 53–57.
[18] P. C. Fife, Models for phase separation and their mathematics, Electron. J. Differential Equations, 2000, pages 48, 26 pp. (electronic).
[19] P. C. Fife, Pattern formation in gradient systems, In Handbook of dynamical systems, North-Holland, Amsterdam, 2 (2002), 677–722.
[20] Á. Hernández and M. Kowalczyk, Delaunay end solutions of the cahn-hilliard equation in, in preparation.
[21] W.-y. Hsiang and W. C. Yu, A generalization of a theorem of Delaunay, J. Differential Geom., 16 (1981), 161–177.
[22] J. E. Hutchinson and Y. Tonegawa, Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory, Calc. Var. Partial Differential Equations, 10 (2000), 49–84.
[23] M. Jleli, End-to-end gluing of constant mean curvature hypersurfaces, Ann. Fac. Sci. Toulouse Math. (6), 18 (2009), 717–737.
[24] M. Jleli and F. Pacard, An end-to-end construction for compact constant mean curvature surfaces, Pacific J. Math., 221 (2005), 81–108.
[25] R. V. Kohn and P. Sternberg, Local minimisers and singular perturbations, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), 69–84.
[26] R. Mazzeo and F. Pacard, Bifurcating nodoids, In Topology and geometry: Commemorating SISTAG, volume 314 of Contemp. Math., pages 169–186. Amer. Math. Soc., Providence, RI, 2002.
[27] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, Arch. Rational Mech. Anal., 98 (1987), 123–142.
[29] F. Pacard and M. Ritoré. From constant mean curvature hypersurfaces to the gradient theory of phase transitions, *J. Differential Geom.*, 64 (2003), 359–423.

[29] F. Pacard and J. Wei. Stable solutions of the Allen-Cahn equation in dimension 8 and minimal cones *J. Funct. Anal.*, 264 (2013), 1131–1167.

[30] L. A. Peletier and J. Serrin. Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$ *Arch. Rational Mech. Anal.*, 81 (1983), 181–197.

[31] P. Sternberg. The effect of a singular perturbation on nonconvex variational problems *Arch. Rational Mech. Anal.*, 101 (1988), 209–260.

[32] J. Wei and M. Winter. On the stationary Cahn-Hilliard equation: Bubble solutions *SIAM J. Math. Anal.*, 29 (1998), 1492–1518 (electronic).

[33] J. Wei and M. Winter. Stationary solutions for the Cahn-Hilliard equation *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15 (1998), 459–492.

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