ON THE MOMENT-ANGLE MANIFOLDS
OF POSITIVE RICCI CURVATURE
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Abstract: We construct Riemannian metrics of positive Ricci curvature on some moment-angle manifolds. In particular, we construct a nonformal moment-angle Riemannian manifold of positive Ricci curvature.

Keywords: positive Ricci curvature, moment-angle manifold, quasitoric manifold

1. Introduction

An interesting question in Riemannian geometry is the topological complexity of the manifolds admitting a Riemannian metric of positive Ricci curvature. For instance, Sha and Yang constructed in [1] a metric of positive Ricci curvature on the connected sum of however many copies of $S^n \times S^m$ for fixed $n$ and $m$, which implies the absence of an a priori estimate on the Betti number. Wraith generalized in [2] the examples of Sha and Yang, showing the existence of a Riemannian metric of positive Ricci curvature on the connected sums of $\#_{i=1}^{N} S^{n_i} \times S^{m_i}$.

In this article we construct Riemannian metrics of positive Ricci curvature on some moment-angle manifolds. A moment-angle manifold $Z_P$ is constructed from a polyhedron $P$, and a torus $T$ acts freely on $Z_P$ in a canonical fashion so that $Z_P/T = P$. We describe the construction of moment-angle manifolds in detail in the next section. The main result of this article is the following

Theorem. Take an octahedron $P$ obtained from a 3-dimensional cube by cutting off small neighborhoods of two edges lying on skew lines. The 11-dimensional moment-angle manifold $Z_P$ admits a Riemannian metric of positive Ricci curvature.

In addition to this example we consider other moment-angle manifolds with metrics of positive Ricci curvature, but the space in the theorem is of a particular interest since it has rather complicated topology. For instance, as shown in [3, 4], its cohomology contains nontrivial Massey products, and so $Z_P$ with a metric of positive Ricci curvature is a nonformal moment-angle manifold.

In closing the introduction, we state two natural questions.

Question 1. Does there exist a metric of positive Ricci curvature on all moment-angle manifolds?

We think that the answer to this question is positive. To give a reason for that (other than the results of this article) we can point out the following: If $P$ is either a two-dimensional polygon or a polyhedron obtained from a higher dimensional tetrahedron by iterating the operation of cutting off a neighborhood of some vertex then $Z_P$ is diffeomorphic [5] to a certain connected sum of products of spheres of various dimensions, and Wraith’s article [2] provides a positive answer to Question 1 in this case.

Closely related to moment-angle manifolds is the concept of a quasitoric manifold [4]. The second question seems much more complicated and uncertain.

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Question 2. Does there exist a metric of positive Ricci curvature on all quasitoric manifolds?

To suggest a likely positive answer to this question, we mention only that some Riemannian metrics of positive Ricci curvature are constructed in [6] on 4-dimensional simply-connected quasitoric manifolds, while it is proved in [7] that these metrics can be chosen invariant under arbitrarily specified action of $T^2$.

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2. Moment-Angle Manifolds

The concept of a moment-angle manifold is introduced in [8]. The article [4] gives a detailed presentation of moment-angle manifolds and closely related quasitoric manifolds appears. We recall here only the needed definitions and properties.

In the Euclidean space $\mathbb{R}^n$, take the $n$-dimensional polyhedron $P$ determined by the system of inequalities

$$\sum_{i=1}^{n} a_{ij} x^j + b_i \geq 0, \ i = 1, \ldots, m,$$

in general position. Denote the faces of $P$ by $P_1, \ldots, P_m$. Given $p \in P$ denote by $G(p)$ the inclusion-least face containing $p$. Consider $X = P \times T^m$, where $T^m = \{(z_1, \ldots, z_m) \mid z_i \in \mathbb{C}, |z_i| = 1\}$ is the standard $m$-dimensional torus, in which the enumeration of coordinates corresponds to the enumeration of the faces of $P$. Denote by $T^F$ the circle $S^1 \subseteq T^m$ corresponding to the $i$th face. Now, given a face $G$ of $P$ put

$$T^G = \prod_{G \subseteq F_i} T^F_i \subset T^m.$$

Identify the points of $X$ by the rule

$$(p, z_i) \sim (p', z'_i), \text{ if } p = p' \text{ and } z_i, z'_i \in T^G(p).$$

It can be shown [4, 8] that the quotient space $Z_P = (P \times T^m)/\sim$ carries the canonical structure of a topological manifold, while the natural translation action of $T^m$ on itself induces a continuous action on $Z_P$. The stabilizer of a point $(p, z) \in Z_P$ is the subgroup $T^G(p)$, and $Z_P/T^m = P$. Moreover, $Z_P$ can be endowed with the structure of a smooth manifold such that the natural action of $T^m$ is smooth [4]. The manifold $Z_P$ (with some $T^m$-invariant smoothness) is called a moment-angle manifold.

Consider now some torus $T^{m-n} \subset T^m$ acting freely on $Z_P$. Then $M = Z_P/T^{m-n}$ is a quasitoric manifold, and, conversely, every quasitoric manifold can be constructed from some moment-angle manifold [4, 8]. Meanwhile, a principal bundle $\pi : Z_P \to M$ arises with the structure group $T^{m-n}$. Similarly, if the quotient space $Z_P/T^{m-n}$ is an orbifold then we call it a quasitoric orbifold. In this case $\pi$ is a principal toric bundle in the orbifold sense.

Example 0. Suppose that $P = I = [0, 1]$ is a segment of the real line. In this case it is easy to see that $Z_P = S^3 = \{(u, v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}$. The torus $T^2 = \{(z_1, z_2) \mid |z_1| = |z_2| = 1\}$ acts on $S^3$ in the standard fashion,

$$(z_1, z_2) \in T^2 : (u, v) \mapsto (uz_1, vz_2).$$

Example 1. Suppose that $P = I^3$ is a 3-dimensional cube. Then $Z_P = S^3 \times S^3 \times S^3$. The torus $T^6 = T^2 \times T^2 \times T^2$ acts on $Z_P$. Consider the diagonally embedded circle $S^1$ in $T^2$. Then $S^1 \times S^1 \times S^1$ acts freely on $Z_P$, and we obtain the quasitoric manifold $M_1 = S^2 \times S^2 \times S^2$.

We can modify $M_1$ as follows: The torus $T^3 = T^6/(S^1 \times S^1 \times S^1)$ acts on $M_1$ (each circle of this torus acts by rotations about the poles of its sphere $S^2$). Naturally, $M_1/T^3 = P$. Consider the subgroup $\Gamma = \mathbb{Z}_3$ in $T^3$ generated by $(\omega, \omega, \omega) \in T^3$, where $\omega = e^{2\pi i/3}$. Then $M_1/\Gamma$ is a quasitoric orbifold with eight singular points, whose neighborhoods look like $\mathbb{C}^3/\mathbb{Z}_3$. Resolving each singularity using a blowup (cutting out some neighborhood of a singular point and gluing in its place the cube of the canonical complex line bundle over $CP^2$; the details of this operation appear in the next subsection), we obtain
a quasitoric manifold $N_1$. The polyhedron $Q_1 = N_1/T^3$ corresponding to $N_1$ results from the cube $P$ by chopping off all vertices. We arrive at a principal $T^{11}$-bundle $\pi_1 : Z_{Q_1} \to N_1$.

**Example 2.** Consider the polyhedron $Q_2$ resulting from the cube $P$ by chopping off a neighborhood of one vertex. In order to describe $Z_{Q_2}$, consider the following construction. Suppose that the torus $T^3$ acts on $Z_P = S^3 \times S^3 \times S^3$ as

$$(z_1, z_2, z_3) \in T^3 : \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1 z_2^2 u_1 & z_1 v_1 \\ z_2 z_3^2 u_2 & z_2 v_2 \\ z_3 z_1^2 u_3 & z_3 v_3 \end{pmatrix},$$

where the rows of the matrices contain the coordinates of a point on the corresponding sphere $S^3$.

**Lemma 1.** The action described above is free everywhere on $Z_P$ except for the submanifolds

$$F = \left\{ \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \mid |u_1| = |u_2| = |u_3| = 1 \right\} = T^3.$$ 

The points of $F$ have the stabilizer $\text{Fix}(F) = \Gamma = Z_3$ generated by $(\omega, \omega, \omega)$, where $\omega = e^{2\pi i/3}$.

Therefore, the quotient space $M_2' = Z_P/T^3$ is a quasitoric orbifold with one singular point $p = F/T^3$, whose neighborhood is diffeomorphic to $\mathbb{C}^3/Z_3$. Resolving the singular point $p$, we obtain the quasitoric manifold $N_2$ corresponding to the polyhedron $N_2/T^3 = Q_2$, and the principal $T^4$-bundle $\pi_2 : Z_{Q_2} \to N_2$. The topology of $Z_{Q_2}$ is studied in [5] (using the techniques of [4]). In particular, $Z_{Q_2}$ cannot be expressed as the connected sum of products of spheres. Some further properties of $Z_{Q_2}$ are studied in [9].

**Example 3.** Consider the polyhedron $Q_3$ resulting from the cube $P$ by cutting off neighborhoods of two nonskew edges. In order to describe $Z_{Q_3}$, consider the following construction. Suppose that the torus $T^3$ acts on $Z_P = S^3 \times S^3 \times S^3$ as

$$(z_1, z_2, z_3) \in T^3 : \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1 u_1 & z_1 z_2^2 z_3 v_1 \\ z_2 z_3^2 u_2 & z_2 z_3^2 z_2 v_2 \\ z_3 z_1^2 u_3 & z_3 v_3 \end{pmatrix},$$

where the rows of the matrices contain the coordinates of a point on the corresponding sphere $S^3$.

**Lemma 2.** The action described above is free everywhere on $Z_P$ except for the two submanifolds

$$F_1 = \left\{ \begin{pmatrix} u_1 & v_1 \\ u_2 & 0 \\ u_3 & 0 \end{pmatrix} : |u_1|^2 + |v_1|^2 = |u_2|^2 = |u_3|^2 = 1 \right\},$$

$$F_2 = \left\{ \begin{pmatrix} 0 & v_1 \\ 0 & v_2 \\ u_3 & v_3 \end{pmatrix} : |v_1|^2 = |v_2|^2 = |u_3|^2 + |v_3|^2 = 1 \right\}.$$ 

The points of $F_1$ and $F_2$ have the stabilizers

$$\text{Fix}(F_1) = \Gamma_1 = \{(1, \pm(1, 1))\} = Z_2, \quad \text{Fix}(F_2) = \Gamma_2 = \{\pm(1, 1, 1)\} = Z_2.$$

Therefore, the quotient space $M_3' = Z_P/T^3$ is a quasitoric orbifold with two singular submanifolds $F_1/T^3 = S^2$ and $F_2/T^3 = S^2$. Tubular neighborhoods of $F_1/T^3$ and $F_2/T^3$ are fibrations with the fibers diffeomorphic to $\mathbb{C}^2/Z_2$. Resolving $M_3'$ along the singular submanifolds (blowing up along the fibers of tubular neighborhoods) we obtain the quasitoric manifold $N_3$ corresponding to the polyhedron $N_3/T^3 = Q_3$, and the principal $T^3$-bundle $\pi_3 : Z_{Q_3} \to N_3$. As we mention in the introduction, the manifold $Z_{Q_3}$ in this example is not formal [4].

**Theorem 1.** The moment-angle manifolds $Z_{Q_1}$, $Z_{Q_2}$, and $Z_{Q_3}$ of Examples 1–3 admit $T^k$-invariant Riemannian metrics of positive Ricci curvature (with $k = 11, 4, 5$ respectively) such that the principal bundles $\pi_1$, $\pi_2$, and $\pi_3$ are Riemannian submersions.

The rest of the article is devoted to a proof of this theorem.
3. Blowups of Manifolds of Positive Ricci Curvature at Singular Points

In the complex space \( \mathbb{C}^n \) equipped with the standard Hermitian product consider the “round” \((2n+1)\)-sphere \( S^{2n+1}_R \) of radius \( R \) with the induced Riemannian metric \( R^2 ds^2_{2n+1} \). The circle \( S^1 \) acts freely on \( S^{2n+1}_R \) by isometries: \( u \in S^1 : (z_0, \ldots, z_n) \mapsto (uz_0, \ldots, uz_n) \). The quotient space of this action is the complex projective space \( \mathbb{C}P^n_R \) with the Fubini–Study metric (dilated by a factor of \( R \)). As usual, we write \( \mathbb{C}P^n_1 = \mathbb{C}P^n \).

Recall the construction of a (topological) blowup of the complex projective space \( \mathbb{C}P^n \). Consider the canonical complex bundle over \( \mathbb{C}P^{n-1} \) with fiber \( \mathbb{C} \). Its spherical subbundle is known to be isomorphic to the Hopf bundle \( S^{2n-1} \rightarrow \mathbb{C}P^{n-1} \). Take the total space \( E \) of the ball subbundle in the canonical bundle; thus, \( \partial E = S^{2n-1} \). Cut some geodesic ball \( B \) of radius \( \varepsilon \) out of \( \mathbb{C}P^n \) and identify \( \mathbb{C}P^n \setminus B \) with \( E \) along the common boundary. The resulting manifold is endowed with the canonical smoothness and is diffeomorphic to \( \mathbb{C}P^n \# \mathbb{C}P^m \). Consider now the orbifold \( \mathbb{C}P^n / \mathbb{Z}_p \), where the group \( \mathbb{Z}_p \) acts with an isolated fixed point \( p \) and repeat the previous construction taking the geodesic ball centered at \( p \). In this case \( \partial((\mathbb{C}P^n / \mathbb{Z}_p) \setminus B) = S^{2n-1}/\mathbb{Z}_p \). Thus, we have to consider the \( p \)th tensor power of the canonical \( \mathbb{C} \)-bundle \( \mathcal{E}^p \), with \( \partial(\mathcal{E}^p) = S^{2n-1}/\mathbb{Z}_p \), and identify \((\mathbb{C}P^n / \mathbb{Z}_p) \setminus B \) with \( \mathcal{E}^p \) along the common boundary.

The resulting manifold \( M \) models the blowup at a singular point. Observe that \( \mathbb{C}P^n / \mathbb{Z}_p \) and \( \mathcal{E}^p \) carry actions of \( T^n \) which agree on the boundaries. Therefore, \( M \) carries a canonical action of \( T^n \).

We are interested in the following concrete action of \( \mathbb{Z}_n \) on \( \mathbb{C}P^n \): take \( \omega = e^{2\pi i/n} \) and let \( \mathbb{Z}_n \) act on \( \mathbb{C}^{n+1} \) as

\[ \omega : (z_0, \ldots, z_n) \mapsto (z_0, \omega z_1, \ldots, \omega z_n). \]

This action clearly induces some isometric action of \( \mathbb{Z}_n \) on \( \mathbb{C}P^n \) that is free outside two subsets \( \bar{p} = \{1:0: \ldots : 0\} \) and \( \mathbb{C}P^{n-1} = \{z_0 = 0\} \). Therefore, \( \bar{p} \) is an isolated fixed point, and we can make a blowup of its neighborhood. The goal of this section is to prove

**Theorem 2.** There exists \( \sigma > 0 \) such that for every \( \varepsilon > 0 \) the manifold \( M \) obtained from \( \mathbb{C}P^n / \mathbb{Z}_n \) by blowing up the singular point \( \bar{p} \) as described above admits a Riemannian metric \( g \) with Ricci curvature at least \( \sigma \) coinciding with the Fubini–Study metric outside the geodesic ball of radius \( \varepsilon \) centered at \( \bar{p} \). Moreover, \( T^n \) acts on \( M \) by isometries.

We begin our proof of Theorem 2 with

**Lemma 3.** The Fubini–Study metric on \( \mathbb{C}P^n_R \) can be expressed as

\[ g_R = dt^2 + R^2 \sin^2 \frac{t}{R} \cos^2 \frac{t}{R} ds_v^2 + R^2 \sin^2 \frac{t}{R} ds_h^2, \]

where \( 0 \leq t \leq R\pi/2 \) is the distance to \( \bar{p} \), while \( ds_v^2 = ds_{2n-1}^2|_V \) and \( ds_h^2 = ds_{2n-1}^2|_H \) are restrictions of the standard spherical metric on the distribution of vertical and horizontal subspaces under the diagonal action of \( S^1 \) on \( S^{2n-1}_R \).

**Proof.** The tangent space to the sphere at \( p \in S^{2n+1}_R \) decomposes into the horizontal and vertical subspaces, \( T_pS^{2n+1}_R = V_p \oplus H_p \), with respect to the Riemannian submersion \( \pi_n : S^{2n+1}_R \rightarrow \mathbb{C}P^n_R \):

\[ V_p = \{ t \cdot ip \mid t \in \mathbb{R} \}, \quad H_p = \{ u \in \mathbb{C}^{n+1} \mid \langle u, p \rangle_{\mathbb{C}} = 0 \}. \]

Fix \( p_0 = (R, 0, \ldots, 0) \in S^{2n+1}_R \) and \( \bar{p}_0 = \pi_n(p_0) \). Given \( 0 \leq t \leq R\pi/2 \), put

\[ S_t = \{ (z_0, z_1, \ldots, z_n) \mid |z_0| = R \cos(t/R) \} \subset S^{2n+1}_R. \]

It is obvious that \( S_0 \) is the \( S^1 \)-orbit of \( p_0 \), which is a closed geodesic on the sphere; \( S_{R\pi/2} \) is the equatorial sphere in \( S^{2n-1}_R \) corresponding to the embedding \( \mathbb{C}^n \subset \mathbb{C}^{n+1} \) as the complex hyperplane \( \{ z_0 = 0 \} \). For \( 0 < t < R\pi/2 \) the submanifold \( S_t \) amounts to the tubular hypersurface of radius \( t \) around \( S_0 \).
(or, equivalently, the tubular hypersurface of radius $R \pi/2 - t$ around $S_{R \pi/2}$) and is isometric to the product $S^1 \times S^{2n-1}$ with the metric

$$g = R^2 \left( \cos^2 \frac{t}{R} \, ds_1^2 + \sin^2 \frac{t}{R} \, ds_{2n-1}^2 \right).$$

Every shortest normal geodesic going from $S_0$ to $S_t$ projects to a geodesic in $\mathbb{C}P^n_R$ of the same length; hence, $S_t$ projects to the geodesic sphere $\mathbb{S}_t$ in $\mathbb{C}P^n_R$ of radius $t$ centered at $\bar{p}_0$, which is the quotient space of $S_t$ by the action of $S^1$. It is easy to see that $\mathbb{S}_t$ (for $0 < t < R \pi/2$) is diffeomorphic to $S^2_{2n-1}$ with a “distorted” metric. Denoting by $ds_2^2 = ds_{2n-1}^2 |_V$ and $ds_2^2 = ds_{2n-1}^2 |_H$ the restrictions of the standard spherical metric on the distribution of vertical and horizontal subspaces in $\mathbb{S}_t$, we can verify directly that the metric on $\mathbb{S}_t$ is

$$\tilde{g} = R^2 \left( \sin^2 \frac{t}{R} \, \cos^2 \frac{t}{R} \, ds_v^2 + \sin^2 \frac{t}{R} \, ds_h^2 \right).$$

Thus, we express the Fubini–Study metric on $\mathbb{C}P^n_R$ as

$$g_R = dt^2 + R^2 \sin^2 \frac{t}{R} \, \cos^2 \frac{t}{R} \, ds_v^2 + R^2 \sin^2 \frac{t}{R} \, ds_h^2,$$

where $0 \leq t \leq R \pi/2$. The proof of the lemma is complete.

Henceforth we consider metrics on $\mathbb{R} \times S^{2n-1}$ of the general form

$$g = dt^2 + h(t)^2 \, ds_v^2 + f(t)^2 \, ds_h^2. \quad (2)$$

The formulas of the next lemma are well-known and easy to verify.

**Lemma 4.** Take a unit vector $X_0 = \frac{\partial}{\partial t}$ in the radial direction, a unit vector $X_1$ in the vertical direction (i.e., $X_1 \in V$), and a unit vector $X_2$ in the horizontal direction ($X_2 \in H$). The Ricci curvature can be computed by the formulas:

$$\text{Ric}(X_i, X_j) = 0 \text{ for } i \neq j, \quad \text{Ric}(X_0, X_0) = -\frac{h''}{h} - (2n - 2) \frac{f''}{f},$$

$$\text{Ric}(X_1, X_1) = -\frac{h''}{h} - (2n - 2) \frac{f'h'}{fh} + (2n - 2) \frac{h^2}{f^4}, \quad (3)$$

$$\text{Ric}(X_2, X_2) = -\frac{f''}{f} - \frac{f'h'}{fh} - (2n - 3) \frac{(f')^2}{f^2} + 2n \frac{h^2}{f^4} - \frac{h''}{f^4}.$$

**Proof of Theorem 2.** Start with the metric $g$ of the particular form

$$g = \frac{dr^2}{1 - \phi(r)} + r^2 (1 - \phi(r)) \, ds_v^2 + r^2 \, ds_h^2 \quad (4)$$

with some smooth function $\phi(r)$. Then (3) becomes

$$\text{Ric}(X_0, X_0) = \text{Ric}(X_1, X_1) = \frac{1}{2} \left( \phi'' + (2n + 1) \frac{\phi'^2}{r} \right), \quad \text{Ric}(X_2, X_2) = \frac{\phi'}{r} + 2n \frac{\phi}{r^2}.$$

For instance, the Euclidean metric on $\mathbb{C}^n$ corresponds to $\phi(r) = 0$, while the metric $g_R$ on $\mathbb{C}P^n_R$ considered above results when

$$\phi(r) = \phi_R(r) = \frac{r^2}{R^2}, \quad 0 \leq r \leq R.$$

Observe that $g_R$ is an Einstein metric with the cosmological constant $\frac{2(n+1)}{R^2}$. Put $\psi(r) = r \phi' + 2n \phi$. The Ricci tensor is then exceptionally simple:

$$\text{Ric}(X_0, X_0) = \text{Ric}(X_1, X_1) = \frac{\psi'}{2r}, \quad \text{Ric}(X_2, X_2) = \frac{\psi}{r^2}.$$

It is obvious that the metric on $\mathbb{C}P^n_R$ corresponds to $\psi_R(r) = 2(n+1) \frac{r^2}{R^2}$.
Remark 1. Putting \( \psi(r) \equiv 0 \), we obtain a Ricci-flat Riemannian metric

\[
g_n = \frac{dr^2}{1 - \frac{\kappa}{r^{2n}}} + r^2 \left( 1 - \frac{1}{r^{2n}} \right) ds_0^2 + r^2 ds_h^2,
\]

which amounts to the metric with holonomy group \( SU(n) \) which was found by Calabi in [10]. Precisely the Calabi metric led us to the ansatz (4).

Put \( r_1 = \sqrt{R} \). Define a function \( \psi_n(r) \) for \( 1 \leq r \leq R \). Suppose that

\[
\psi_n(r) = \psi_R(r) \quad \text{for} \quad r_1 \leq r.
\]

Furthermore, \( \psi'_n(r_1) = 4(n+1)r_1/R^2 > 0 \). Thus, we can extend \( \psi_n \) smoothly inside the interval \([1, r_1]\) so that it satisfies on this interval the conditions

\[
\psi'_n(r) > \frac{\kappa}{R^2} \quad \text{for} \quad r \geq 1, \quad \psi_n(r) \geq \frac{\kappa r^2 - 1}{R^2} \quad \text{for} \quad 1 \leq r \leq r_1,
\]

\[
\psi_n(1) = 0, \quad \int_1^{r_1} \psi_n(s) s^{2n-1} ds = \eta,
\]

where \( \kappa > 0 \) is independent of \( R \), while \( \eta \) is an arbitrarily specified number satisfying

\[
\frac{\kappa r_1^{2n}}{R^2} \left( \frac{r_1^2}{2n+2} - \frac{1}{2n} \right) < \eta < \frac{\psi_n(r_1)}{2n} \left( r_1^{2n} - 1 \right).
\]

Choose

\[
\eta = \frac{r_1^{2(n+1)}}{R^2} - 1.
\]

Indeed, the inequalities for \( \eta \) in this case become

\[
\kappa R^{n-1} \left( \frac{1}{2n+2} - \frac{1}{2nR} \right) < R^{n-1} - 1, \quad \left( R^{n-1} - 1 \right) < \frac{n+1}{n} \left( R^{n-1} - \frac{1}{R} \right).
\]

Thus, for \( R > 2 \) both inequalities are fulfilled, and we obtain the required function \( \psi_n \). Now we can put

\[
\phi_n(r) = \frac{1}{r^{2n}} \int_1^r \psi_n(s) s^{2n-1} ds + \frac{1}{r^{2n}}.
\]

It is obvious that the metric \( g_n \), constructed from this function, has the nonnegative Ricci curvature which is strictly positive for \( r > 1 \). Moreover, the Ricci curvature at \( r = 1 \) vanishes only in the direction of \( X_2 \). Verify that \( \phi_n \) coincides with \( \phi_R \) on the interval \( r_1 \leq r \leq R \). Indeed, if \( r \geq r_1 \) then

\[
\phi_n(r) = \frac{1}{r^{2n}} + \frac{1}{r^{2n}} \int_{r_0}^r \psi_n(s) s^{2n-1} ds + \frac{1}{r^{2n}} \int_{r_1}^r 2(n+1) s^{2n+1} ds
\]

\[
= \frac{1}{r^{2n}} \left( 1 + \eta - \frac{r_1^{2(n+1)}}{R^2} \right) + \frac{r_1^{2n+1}}{R^2} = \frac{r_1^{2n+1}}{R^2}.
\]

Thus, the metric \( g_n \) determined by \( \phi_n \) coincides with \( g_R \) on \( \mathbb{C} P^n_R \) for \( r \geq r_1 \). Observe that

\[
\phi_n(1) = 1, \quad \phi'_n(1) = -2n.
\]

In order to avoid the vanishing of the Ricci curvature for \( r = 1 \), consider the modified metric

\[
g'_n = \frac{dr^2}{1 - \phi_n(r)} + r^2 (1 - \phi_n(r)) ds_0^2 + (1 - \delta(r)) r^2 ds_h^2
\]
with some smooth function \( \delta(r) \geq 0 \). Take a sufficiently small \( \nu > 0 \). Choose a nonincreasing function \( \delta_\nu(r), 1 \leq r \leq R \), so that

\[
\delta_\nu(r) = 0 \quad \text{for} \quad r \geq r_1, \quad \delta_\nu(r) > 0 \quad \text{for} \quad 1 \leq r < r_1,
\]

\( \delta_\nu(r) \) is constant for \( 1 \leq r \leq 2 \), \( \| \delta_\nu(r) \|_{C^\infty} \leq \nu \).

It is clear that if we take \( R > 4 \) then for every \( \nu > 0 \) the function \( \delta_\nu(r) \) with these properties exists. Then (3) shows that for \( 1 \leq r \leq 2 \) we have

\[
\text{Ric}'(X_2, X_2) = \frac{\psi_n}{r^2} + \frac{2\delta_\nu(1)}{(1 - \delta_\nu(1)) r^2} \left( n - 2 - \frac{2 - \delta_\nu(1)}{1 - \delta_\nu(1)} (1 - \phi_n) \right).
\]

Since \( \phi_n \) is strictly positive everywhere on \( [1, R] \), we may assume that \( \phi_n \geq c(R) > 0 \). Consequently, taking \( \nu > 0 \) with

\[
\delta_\nu(r) \leq \delta_\nu(1) \leq \frac{2c(R)}{1 + c(R)} \quad \text{for} \quad 1 \leq r \leq 2
\]

we verify immediately that

\[
\text{Ric}'(X_2, X_2) \geq \frac{\psi_n}{r^2} + \frac{2\delta_\nu(1)}{(1 - \delta_\nu(1)) r^2} (n - 2) \geq \frac{2\delta_\nu(1)}{(1 - \delta_\nu(1))} \frac{n - 2}{4} > 0. \tag{5}
\]

Furthermore, for \( 2 \leq r \leq R \) the curvature of \( g_n \) satisfies

\[
\text{Ric}(X_2, X_2) = \frac{\psi_n}{r^2} \geq \frac{\kappa}{r^2} \geq \frac{3\kappa}{R^2}.
\]

By continuity, the lower bound

\[
\text{Ric}'(X_2, X_2) \geq \frac{2\kappa}{R^2} \tag{6}
\]

on the Ricci curvature of \( g'_n \) for \( 2 \leq r \leq R \) holds for all sufficiently small \( \nu > 0 \) (recall that \( \nu \) is chosen independently of \( R \)). On the other hand, since the Ricci curvature \( \text{Ric}(X_0, X_0) = \text{Ric}(X_1, X_1) \) in the metric \( g_n \) is bounded below by the constant \( \frac{\kappa}{R^2} \), the similar estimate

\[
\text{Ric}'(X_0, X_0) = \text{Ric}'(X_1, X_1) \geq \frac{\kappa}{2R^2} \tag{7}
\]

for \( g'_n \) also holds for all sufficiently small \( \nu \). Thus, fixing a sufficiently small number \( \nu \), we deduce from (5)–(7) the general lower bound

\[
\text{Ric}'(X, X) \geq \frac{\sigma}{R^2} \tag{8}
\]

on the Ricci curvature, where \( |X| = 1 \), while the constant \( \sigma > 0 \) is chosen independently of \( R > 4 \). Observe that, in particular,

\[
\frac{2\delta_\nu(1)}{(1 - \delta_\nu(1))} \frac{n - 2}{4} \geq \sigma. \tag{9}
\]

Let us discuss the smoothness of \( g'_n \) in a neighborhood of \( r = 1 \). Firstly, since \( h \) vanishes at the initial point \( r = 1 \), while \( f \) is strictly positive, the integral circles of the vertical distribution \( V \) (the vertical fibers of the submersion) collapse to a point as \( r \to 1 \), and the metric \( g'_n \) is defined on the blowup of \( \mathbb{C}P^r_R \) at \( \bar{p} \). The smoothness of metrics of this type is studied, for instance, in [11]. A smoothness criterion is that

\[
\begin{align*}
\frac{d}{dt} \bigg|_{r=1} \ f(t) = 0, & \quad \frac{d}{dt} \bigg|_{r=1} \ h(t) = \pm 1.
\end{align*}
\]
Strictly speaking, these relations imply $C^1$-smoothness, but the quasilinearity of the Ricci curvature operator enables us to make this metric infinitely smooth while preserving the qualitative estimate (8).

Direct calculations show that

\[ \frac{d}{dt} \bigg|_{t=1} \frac{\partial}{\partial r} f(t) = 0, \quad \frac{d}{dt} \bigg|_{t=1} \frac{\partial}{\partial r} h(t) = -\frac{\phi'(1)}{2} = n. \]

We see that the smoothness requires us to shrink the vertical circles of the submersion “by a factor of $n$”; thus, $g_n$ is a smooth metric of positive Ricci curvature on the space that results from resolving the singularities of $\mathbb{C}P^n/\mathbb{Z}_n$.

Now multiply $g_n$ by $1/R^2$. This yields a Riemannian metric $g$ on $\mathbb{C}P^n/\mathbb{Z}_n$ blown up at the singular point, while the blowup occurs inside the neighborhood of the radius

\[ \frac{r_1}{R} = \frac{1}{\sqrt{R}} = \varepsilon \to 0 \]

as $R \to \infty$. The estimate (8) becomes $\text{Ric}(X,X) \geq \sigma$, where $\sigma$ is independent of $\varepsilon$. It remains to observe that all operations with the metric preserve its $T^n$-invariance. The proof of the theorem is complete.

**Remark 2.** Using (9), we can estimate

\[ f(1) = \sqrt{1 - \delta(1)} \leq \sqrt{\frac{n - 2}{n - 2 + 2\sigma}}. \]

Shrinking the metric by a factor of $R^2$, we estimate the “size” $D$ of the space $\mathbb{C}P^{n-1}$ glued instead of the singular point as

\[ D \leq \frac{1}{R} \sqrt{\frac{n - 2}{n - 2 + 2\sigma}} = \varepsilon^2 \sqrt{\frac{n - 2}{n - 2 + 2\sigma}}. \] (10)

This estimate is useful below.

### 4. Construction of Riemannian Metrics of Positive Ricci Curvature

We need several available results. Firstly, we have to lift a Riemannian metric of positive Ricci curvature from the base to the total space of a Riemannian submersion while preserving the positivity of the Ricci curvature. Constructions of this kind are considered in [12–16]. The possibility of this lifting is proved in [14] for principal toric bundles (which is all we need here), but without stating the invariance of the final metric. Thus, in order to obtain invariant metrics on moment-angle manifolds, we use the following stronger result.

**Theorem 3** [16]. Take a compact connected Riemannian manifold $(Y, g_Y)$ of positive Ricci curvature. Take a principal bundle $P$ over $Y$ with compact connected structure group $G$ such that $\pi_1(P)$ is finite. Then there exists a $G$-invariant metric $g_P$ on $P$ of positive Ricci curvature such that $\pi: (P, g_P) \to (Y, g_Y)$ is a Riemannian submersion.

The following theorem enables us to replace a Riemannian metric in a small neighborhood of a manifold with some model metric.

**Theorem 4** [16]. On the ball $U(0, \rho_0) = \{ x \in \mathbb{R}^n \mid |x| \leq \rho_0 \}$ consider two Riemannian metrics $g_0$ and $g_1$ of positive Ricci curvature with the same 1-jets $J^1(g_0)$ and $J^1(g_1)$ at 0. Then there exist a Riemannian metric $\bar{g}$ on $U(0, \rho_0)$ of positive Ricci curvature, and $0 < \rho_2 < \rho_1 < \rho_0$ such that $\bar{g} = g_1$ for $|x| < \rho_2$ and $\bar{g} = g_0$ for $|x| > \rho_1$.

**Remark 3.** In [17] Theorem 4 is proved for negative Ricci curvature, but all arguments carry over without effort to the positive case. In addition, for our goals it is important to know a bit more about the construction of the proof of this theorem. Firstly, we seek the metric $\bar{g}$ as $\bar{g} = (1-s)g_0 + sg_1$, where $s = \psi(|x|)$ for some smooth function $\psi: \mathbb{R} \to [0,1]$. This immediately leads us to conclude that if $g_0$
and $g_1$ are invariant under the action of a group $G$ preserving the function $|x|$ then the final metric $\tilde{g}$ is invariant under $G$ as well. Secondly, the coefficients of the Levi-Civita connection and the Ricci tensor of $g_1, g_2$, and $\tilde{g}$ can be chosen arbitrarily close to each other independently of the constant $\rho_0$.

Now we deal in succession with the examples of Section 2.

EXAMPLES 1 and 2. As we describe in Section 2, the orbifolds $M'_1$ and $M'_2$ have isolated singular points, in whose neighborhoods they look like $\mathbb{C}^3/\mathbb{Z}_3$. Applying Theorem 4 (to a $\mathbb{Z}_3$-cover of a small neighborhood of each singular point), we may assume that a neighborhood of each singular point is isometric to a geodesic ball in $\mathbb{C}P^3/\mathbb{Z}_3$. Using Theorem 2, we arrive at invariant metrics of positive Ricci curvature on $N_1$ and $N_2$ obtained by blowing up the singular points. Finally, Theorem 3 yields the required metrics on $Z_{Q_1}$ and $Z_{Q_2}$.

EXAMPLE 3. Suppose that $\varepsilon > 0$. Construct a smooth function $f(t)$ on $[0, \pi/2]$ satisfying

\[
 f(t) = \begin{cases} 
 1 & \text{for } 0 \leq t \leq \varepsilon, \\
 \cos t & \text{for } \pi/2 - \varepsilon \leq t \leq \pi/2, \\
 f'(t) < 0, & f''(t) < 0 \text{ for } \varepsilon < t < \pi/2 - \varepsilon.
\end{cases}
\]

It is obvious that this function exists for all sufficiently small $\varepsilon$. On the space $[0, \pi/2]^3 \times (S^1)^6$ with coordinates $(t_1, t_2, t_3, \phi_1, \phi_2, \psi_2, \phi_3, \psi_3)$ consider the Riemannian metric

\[
 g = dt_1^2 + \cos(t_1)^2 d\phi_1^2 + f(\pi/2 - t_1)^2 d\psi_1^2 + dt_2^2 + f(t_2)^2 d\phi_2^2 + d\hat{\theta}_2^2 + f(t_2)^2 d\phi_3^2 + \sin(t_3)^2 d\psi_3^2.
\]

It is not difficult to see that for each $i = 1, 2, 3$ the coordinates $(t_i, \phi_i, \psi_i)$ are defined on the corresponding sphere $S^3$, and $g$ is a smooth metric on $Z_P = S^3 \times S^3 \times S^3$ of nonnegative sectional curvature in view of the restrictions on $f(t)$.

In the chosen coordinates the subsets $F_1$ and $F_2$ are determined by $t_2 = t_3 = 0$ and $t_1 = t_2 = \pi/2$ respectively.

REMARK 4. It is clear that the transformation

\[
 (t_1, t_2, t_3, \phi_1, \phi_2, \psi_2, \phi_3, \psi_3) \mapsto (\pi/2 - t_3, \pi/2 - t_2, \pi/2 - t_1, \phi_3, \phi_2, \psi_2, \phi_1, \psi_1)
\]

is an isometry of $Z_P$ switching $F_1$ and $F_2$. Thus, it suffices to make the subsequent arguments (which are local) for $F_1$.

Take the neighborhood $U \subset Z_P$ of $F_1$ defined by $t_2 < \varepsilon$ and $t_3 < \varepsilon$. It is obvious that $U$ is isometric to the direct product $S^3 \times S^1 \times S^1 \times D^2 \times D^2$ equipped with the metric

\[
 g|_U = (dt_1^2 + \cos(t_1)^2 d\phi_1^2 + f(\pi/2 - t_1)^2 d\psi_1^2) + (d\phi_2^2 + d\phi_3^2) + (dt_2^2 + \sin(t_2)^2 d\psi_2^2) + (dt_3^2 + \sin(t_3)^2 d\psi_3^2).
\]

(Therefore, each disk $D^2$ is isometric to a geodesic circle on the unit two-dimensional sphere.) Put $S = D^2 \times D^2$ with the induced metric. Below we also consider in $S$ the coordinates $(t, \theta, \psi, \phi)$ satisfying $t_2 = t \cos(\theta/2), t_3 = \sin(\theta/2), \psi_2 = \psi + \phi$, and $\psi_3 = \psi - \phi$.

It is clear that $S$ is of nonnegative sectional curvature and strictly positive Ricci curvature (to be precise, the Ricci curvature of $S$ is equal to 1). By Theorem 4 we can deform the metric on $S$ so that it remains the same outside of the $\varepsilon/2$-neighborhood of $p = \{t = 0\} \in S$, but becomes isometric inside the $\varepsilon/4$-neighborhood to the geodesic ball of radius $\varepsilon/4$ in $\mathbb{C}P^2$, while the positivity of the Ricci curvature is preserved. By Theorem 2 we can blow up $S/\mathbb{Z}_2$ inside the neighborhood of radius $\varepsilon/4$ without losing the positivity of the Ricci curvature, and obtain a manifold $S'$. Replacing each factor $S$ in $U$ by a two-sheeted cover over $S'$ (ramified along the vertical circles of the Hopf bundle), we obtain $U' = S^3 \times S^1 \times S^1 \times S'$. Since outside the neighborhood of radius $\varepsilon/2$ the Riemannian manifold $S$ and the two-sheeted cover over $S'$ are isometric, glue $U'$ in place of $U$ into $Z_P$, obtaining $Z'$. Strictly speaking, $U'$ is doubly ramified over $F_1$, but the quotient by the action (1) of $T^3$ is a smooth manifold. Observe that the whole construction works for arbitrarily small $\varepsilon.$
Lemma 4. For sufficiently small \( \varepsilon \) the quotient space \( \tilde{U} = U' / T^3 \) with the induced Riemannian metric is of strictly positive Ricci curvature.

Proof. The induced metric on \( \tilde{U} \) is characterized by the property that the quotient mapping \( U' \to \tilde{U} \) is a Riemannian submersion. Use the analog of O’Neil’s formula for the Ricci curvature [18]:

\[
\tilde{R}(\tilde{X}, \tilde{X}) = R(X, X) + 2(A X, A X) + (T X, T X) - (D_X N, X).
\]

Here \( \tilde{R} \) and \( R \) are the Ricci tensors of \( \tilde{U} \) and \( U' \); \( A \) and \( T \) are the fundamental tensors of the Riemannian submersion; \( \tilde{X} \) is a tangent vector field in \( \tilde{U} \) with the horizontal lifting \( X \) in \( U' \); \( N \) is the mean curvature vector of the fibers of the submersion: \( N = \sum_i T_{U_i} U_i \), where \( U_i \) is an orthonormal basis in the vertical space of the submersion. As a basis (not orthonormal) for vertical vector fields we can choose

\[
V_1 = \partial_{\psi_1} + 2\partial_{\psi_2} + \partial_{\psi_3}, \quad V_2 = \partial_{\psi_1} + \partial_{\psi_2} + \partial_{\psi_3}, \quad V_3 = -\partial_{\psi_1} - \partial_{\psi_2} + \partial_{\psi_3},
\]

Therefore,

\[
\tilde{R}(\tilde{X}, \tilde{X}) \geq R(X, X) - (D_X N, X). \tag{11}
\]

Observe to begin with that \( Z_P \) is of nonnegative Ricci curvature. Meanwhile, \( R(X, X) = 0 \) in exactly in two cases: (1) \( t_2 \leq \varepsilon, \ t_3 \leq \varepsilon \), and \( X = \alpha \partial_{\phi_2} + \beta \partial_{\phi_2} \); (2) \( t_1 \geq \pi/2 - \varepsilon, \ t_2 \geq \pi/2 - \varepsilon \), and \( X = \alpha \partial_{\psi_1} + \beta \partial_{\psi_2} \), where \( \alpha \) and \( \beta \) are some coefficients. Verify directly that these nonzero vectors \( X \) cannot be horizontal. Consequently, there exists a constant \( \sigma > 0 \) such that the Ricci curvature of \( Z_P \) in the horizontal direction is at least \( \sigma \). By Remark 3, applying Theorem 4 to deform the metric, we can make the Ricci curvature of the resulting metric in the horizontal direction to be at least \( \sigma / 2 \) independently of a small \( \varepsilon \). Finally, by Theorem 2, once we apply the construction of a blowup of the Ricci curvature, the resulting metric in the horizontal direction is still bounded below by some constant \( \kappa > 0 \) independent of \( \varepsilon \). Thus, we can bound the first term in (11) from below:

\[
R(X, X) \geq \kappa. \tag{12}
\]

Suppose now that in the space of vertical vector fields of the submersion we choose an orthonormal basis \( V_i' = Y_i + Z_i \), where \( Y_i \) and \( Z_i \) are fields tangent to \( S^3 \times T^2 \) and \( S' \) respectively, for \( i = 1, 2, 3 \). Then

\[
N = \sum_{i=1}^3 (T_{Y_i} Y_i + T_{Z_i} Z_i) = N_1 + N_2
\]

is the corresponding decomposition of the mean curvature vector. We see that the quadratic form \( R(X, X) - (D_X N, X) \) decomposes into three blocks corresponding to the subspaces tangent to \( S^3 \), \( T^2 \) (the flat torus), and \( S \).

When we apply the construction of Theorem 4, by Remark 3 the resulting metric \( \tilde{g} \) on \( S \) linearly interpolates between the metrics \( g_0 \) and \( g_1 \). Thus, in our case it is of the form

\[
\tilde{g} = (dt^2 + f(t)^2 d\theta^2) + \sum_{i,j=1}^2 g_{ij}(t, \theta) \, d\xi^i d\xi^j,
\]

where \( \psi = \xi^1 \) and \( \phi = \xi^2 \). Simple calculation shows that

\[
D_{\xi^i} \xi^j = -\frac{1}{2} \text{grad}(g_{ij})
\]

(the gradient is taken with respect to the metric \( dt^2 + h^2 d\theta^2 \)). Put

\[
Z_i = \sum_{j=1}^2 \alpha_i^j \partial_{\xi^j}.
\]
Since the torus $T^3$ fixes the pole $p \in S$, the fields $Z_i$ vanish at $t = 0$. Consequently, $\alpha_i^j(p) = 0$ for all $i$ and $j$. Furthermore,

$$
N_2 = -\frac{1}{2} \sum_{i=1}^{3} \sum_{j,k=1}^{2} \alpha_i^j \alpha_i^k \text{grad}(g_{jk}).
$$

Hence,

$$
-(D_X N_2, X)|_{t=0} = (-X(N_2, X) + (N_2, D_X X))|_{t=0}
$$

$$
= \frac{1}{2} \sum_{i=1}^{3} \sum_{j,k=1}^{2} (2\alpha_i^j g_{jk}(\alpha_i^k) + \alpha_i^j \alpha_i^k X(g_{jk}))|_{t=0} = 0.
$$

Therefore, taking a sufficiently small $\varepsilon > 0$, we can make $\bar{g}$ satisfy

$$
|(D_X N_2, X)| \leq \kappa/2 \tag{13}
$$

in $U$. Inspect what happens to $(D_X N_2, X)$ under the blowup, as we describe in Theorem 2. The metric is deformed in the class of metrics of the more concrete form

$$
\bar{g} = dt^2 + \frac{1}{4} h(t)^2 (d\psi + \cos(\theta) d\phi)^2 + \frac{1}{4} f(t)^2 (d\theta^2 + \sin(\theta)^2 d\phi^2)
$$

(this metric amounts to (2) for $n = 2$ in the coordinates $(t, \theta, \psi, \phi)$). Recall that here $h(0) = 0$, $f'(0) = 0$, and $f(0) = f_0 > 0$ can be chosen arbitrarily small as $\varepsilon$ decreases (see (10) and Remark 2). In addition, the coefficient of $\partial_\psi$ in $V'_i$ vanishes at $t = 0$ since the sphere $\{t = 0\}$ is fixed under the action of $\partial_\psi$. Therefore, $\alpha_i^1(p) = 0$. Furthermore, since the radius of the sphere $t = 0$ is equal to $f_0$, we can also make the sphere arbitrarily small; hence, $\alpha_i^2$ at $t = 0$ can be made arbitrarily small as $\varepsilon$ decreases. Thus, in this case

$$
-(D_X N_2, X)|_{t=0} = (-X(N_2, X) + (N_2, D_X X))|_{t=0}
$$

$$
= \frac{1}{2} \sum_{i=1}^{3} \sum_{j,k=1}^{2} (2\alpha_i^j g_{jk}(\alpha_i^k) + \alpha_i^j \alpha_i^k X(g_{ij}) + \alpha_i^j \alpha_i^k (\text{grad}(g_{jk}), D_X X))|_{t=0}.
$$

Since all derivatives of $f(t)$ and $h(t)$ are certainly bounded independently of $\varepsilon$, all derivatives of $g_{ij}$ along the unit fields are bounded. Hence, decreasing $\varepsilon$, we can achieve (13) in this case as well.

Consider now the component $N_1$ of the mean curvature vector. In the space of vertical fields choose the basis

$$
V_1 = 2\partial_{\phi_3} + \partial_{\psi_3} + \partial_{\psi_3}, \quad V_2 = \partial_{\psi_1} + \partial_{\phi_2} + \partial_{\phi_3} + \partial_{\psi_2}, \quad V_3 = \partial_{\phi_1} - \partial_{\phi_2} - 2\partial_{\phi_1} - 2\partial_{\psi_2}.
$$

Using a standard orthogonalization method, pass at each point to an orthonormal basis consisting of vertical vector fields $V'_1, V'_2$, and $V'_3$, with

$$
V'_1 = Z'_1, \quad V'_2 = y_1 \partial_{\psi_1} + Z'_2, \quad V'_3 = z_1 \partial_{\phi_1} + z_2 \partial_{\psi_1} + Z'_3,
$$

where $Z'_i$ are tangent to $T^2 \times S'$, while $y_1, z_1, and$ $z_2$ are functions of $(t_1, t_2, t_3)$. Consequently,

$$
N_1 = (z_1^2 \sin(t_1) \cos(t_1) + w^2 f(\pi/2 - t_1) f'(\pi/2 - t_1)) \partial_{t_1}
$$

for some $w$ expressible explicitly in terms of $y_1, z_1, and$ $z_2$. As above, we see that $(D_X N_1, X) \neq 0$ only if $X = \partial_{t_1}$. Therefore, it remains to verify the positivity of the Ricci curvature only for $X = \partial_{t_1}$. But for all horizontal vectors $Y$ orthogonal to $\partial_{t_1}$ the sectional curvature $K(\partial_{t_1}, Y)$ is nonnegative, while $K(\partial_{t_1}, \partial_{\phi_1}) = 1$. Hence, O’Neil’s formula yields

$$
\bar{R}(\partial_{t_1}, \partial_{t_1}) = \sum_{\tilde{Y} \perp \partial_{t_1}} K(\partial_{t_1}, \tilde{Y}) \geq \sum_{\tilde{Y} \perp \partial_{t_1}, V_1, V_2, V_3} K(\partial_{t_1}, Y) > 0 \tag{14}
$$

since there exist horizontal vectors $Y$ with everywhere nonvanishing coefficient of $\partial_{\phi_1}$.

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In order to complete the proof of the lemma, consider an arbitrary horizontal vector

\[ X = \alpha_1 \partial_{t_1} + \alpha_2 \partial_{\phi_1} + \alpha_3 \partial_{\psi_1} + \alpha_4 \partial_{\phi_2} + \alpha_5 \partial_{\phi_3} + X_2, \]

where \( X_2 \) is a tangent vector to \( S' \). By (11)–(14),

\[ \tilde{R}(\tilde{X}, \tilde{X}) \geq \alpha_1^2 \tilde{R}(\tilde{\partial}_{t_1}, \tilde{\partial}_{t_1}) + R(\alpha_2 \partial_{\phi_1} + \alpha_3 \partial_{\psi_1} + \alpha_4 \partial_{\phi_2}, \alpha_2 \partial_{\phi_1} + \alpha_3 \partial_{\psi_1} + \alpha_4 \partial_{\phi_2}) + |X_2|^2 \frac{K}{2}. \]

It is clear that if \( \alpha_1 = 0 \) and \( X_2 = 0 \) then \( \alpha_2 \partial_{\phi_1} + \alpha_3 \partial_{\psi_1} + \alpha_4 \partial_{\phi_2} \) is a horizontal vector, and the middle term in the last inequality is bounded below by \( \sigma \). But if \( \alpha_1 \) (respectively \( |X_2| \)) is nonzero then the middle term is at least nonnegative, while the first (respectively second) term in the last inequality is strictly positive. The proof of the lemma is complete.

Making this construction also for a neighborhood of \( P_2 \), we obtain a quasitoric manifold \( N_3 \) of positive Ricci curvature on the corresponding moment-angle manifolds.

**Remark 5.** The authors managed to construct other actions of \( T^3 \) on \( Z_P \) corresponding to cutting off various combinations of vertices. It is clear that in all cases we can construct by similar arguments Riemannian metrics of positive Ricci curvature on the corresponding moment-angle manifolds.

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