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A reduced parallel transport equation on Lie Groups with a left-invariant metric

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Abstract. This paper presents a derivation of the parallel transport equation expressed in the Lie algebra of a Lie group endowed with a left-invariant metric. The use of this equation is exemplified on the group of rigid body motions $SE(3)$, using basic numerical integration schemes, and compared to the pole ladder algorithm. This results in a stable and efficient implementation of parallel transport. The implementation leverages the python package geomstats and is available online.

Keywords: Parallel transport · Lie Groups

1 Introduction

Lie groups are ubiquitous in geometry, physics and many application domains such as robotics [3], medical imaging [14] or computer vision [9], giving rise to a prolific research avenue. Structure preserving numerical methods have demonstrated significant qualitative and quantitative improvements over extrinsic methods [10]. Moreover, machine learning [2] and optimisation methods [11] are being developed to deal with Lie group data.

In this context, parallel transport is a natural tool to define statistical models and optimisation procedures, such as the geodesic or spline regression [12,18], or to normalise data represented by tangent vectors [20,4].

Different geometric structures are compatible with the group structure, such as its canonical Cartan connection, whose geodesics are one-parameter subgroups, or left-invariant Riemannian metrics. In this work we focus on the latter case, that is fundamental in geometric mechanics [13] and has been studied in depth since the foundational papers of Arnold [1] and Milnor [16]. The fundamental idea of Euler-Poincaré reduction is that the geodesic equation can be expressed entirely in the Lie algebra thanks to the symmetry of left-invariance [15], alleviating the burden of coordinate charts.

However, to the best of our knowledge, there is no literature on a similar treatment of the parallel transport equation. We present here a derivation of the parallel transport equation expressed in the Lie algebra of a Lie group endowed with a left-invariant metric. We exemplify the use of this equation on the group of rigid body motions $SE(3)$, using common numerical integration schemes, and compare it to the pole ladder approximation algorithm. This results in a stable and efficient implementation of parallel transport. The implementation leverages the python package geomstats and is available online at http://geomstats.ai.
In section 2, we give the general notations and recall some basic facts from Lie group theory. Then we derive algebraic expressions of the Levi-Civita connection associated to the left-invariant metric in section 3. The equation of parallel transport is deduced from this expression and its integration is exemplified in section 4.

2 Notations

Let $G$ be a Lie group of (finite) dimension $n$. Let $e$ be its identity element, $\mathfrak{g} = T_e G$ be its tangent space at $e$, and for any $g \in G$, let $L_g : h \in G \mapsto gh$ denote the left-translation map, and $dL_g$ its differential map. Let $\mathfrak{g}^L$ be the Lie algebra of left-invariant vector fields of $G$: $X \in \mathfrak{g}^L$ if and only if $L_g$ preserves $X|_e$ for all $g \in G$.

$\mathfrak{g}$ and $\mathfrak{g}^L$ are in one-to-one correspondence, and we will write $\tilde{x}$ the left-invariant field generated by $x \in \mathfrak{g}$: $\forall g \in G$, $\tilde{x}|_g = dL_g x$. The bracket defined on $\mathfrak{g}$ by $[x, y] = [\tilde{x}, \tilde{y}]_e$ turns $\mathfrak{g}$ into a Lie algebra that is isomorphic to $\mathfrak{g}^L$. One can also check that this bracket coincides with the adjoint map defined by $\text{ad}_x(y) = d_e(h \mapsto \text{Ad}_g y)$, where $\text{Ad}_g = d_e(h \mapsto ghg^{-1})$. For a matrix group, it is the commutator.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathfrak{g}$, and the associated left-invariant vector fields $X^L_i = \tilde{e}_i = g \mapsto dL_g e_i$. As $dL_g$ is an isomorphism, $(X^L_1|_g, \ldots, X^L_n|_g)$ form a basis of $T_g G$ for any $g \in G$, so one can write $X|_g = f^i(g) X^L_i|_g$ where for $i = 1, \ldots, n$, $g \mapsto f^i(g)$ is a smooth real-valued function on $G$. Any vector field on $G$ can thus be expressed as a linear combination of the $X^L_i$ with function coefficients.

Finally, let $\theta$ be the Maurer-Cartan form defined on $G$ by:

$$\forall g \in G, \forall v \in T_g G, \theta|_g(v) = (dL_g)^{-1} v \in \mathfrak{g} \quad (1)$$

It is a $\mathfrak{g}$-valued 1-form and for a vector field $X$ on $G$ we write $\theta(X)|_g = \theta|_g(X|_g)$ to simplify the notations.

3 Left-invariant metric and connection

A Riemannian metric $\langle \cdot, \cdot \rangle$ on $G$ is called left-invariant if the differential map of the left translation is an isometry between tangent spaces, that is

$$\forall g, h \in G, \forall u, v \in T_g G, \langle u, v \rangle_g = \langle dL_h u, dL_h v \rangle_{hg}.$$  

It is thus uniquely determined by an inner product on the tangent space at the identity $T_e G = \mathfrak{g}$ of $G$. Furthermore, the metric dual to the adjoint map is defined such that

$$\forall a, b, c \in \mathfrak{g}, \langle \text{ad}_a^* (b), c \rangle = \langle b, \text{ad}_a(c) \rangle = \langle [a, c], b \rangle. \quad (2)$$

As the bracket can be computed explicitly in the Lie algebra, so can $\text{ad}^*$ thanks to the orthonormal basis of $\mathfrak{g}$. Now let $\nabla$ be the Levi-Civita connection associated...
to the metric. It is also left-invariant and can be characterised by a bi-linear form on $\mathfrak{g}$ that verifies [19,6]:

$$\forall x, y \in \mathfrak{g}, \quad \alpha(x, y) := \langle \nabla_x y \rangle_e = \frac{1}{2} \left( [x, y] - \text{ad}_y^* (x) - \text{ad}_x^* (y) \right)$$ (3)

Indeed by the left-invariance, for two left-invariant vector fields $X = \tilde{x}, Y = \tilde{y} \in \mathfrak{g}^L$, the map $g \mapsto \langle X, Y \rangle_g$ is constant, so for any vector field $Z = \tilde{z}$ we have $Z(\langle X, Y \rangle)_g = 0$. Koszul formula thus becomes

$$2\langle \nabla_X Y, Z \rangle_g = \langle [X, Y], Z \rangle_g - \langle [Y, Z], X \rangle_g - \langle [X, Z], Y \rangle_g$$ (4)

Note however that this formula is only valid for left-invariant vector fields. We will now generalise to any vector fields defined along a smooth curve on $G$, using the left-invariant basis $(X_1^L, \ldots, X_n^L)$.

Let $\gamma : [0, 1] \to G$ be a smooth curve, and $Y$ a vector field defined along $\gamma$. Write $Y = g^t X_i^L$, $\gamma = f^i X_i^L$. Let's also define the left-angular velocities $\omega(t) = \theta|_\gamma(t) \gamma(t) = (f^i \circ \gamma)(t) e_i \in \mathfrak{g}$ and $\zeta(t) = \theta(Y)|_{\gamma(t)} = (g^i \circ \gamma)(t) e_j \in \mathfrak{g}$. Then the covariant derivative of $Y$ along $\gamma$ is

$$\nabla_{\omega(t)} Y = (f^i \circ \gamma)(t) \nabla_{X_i^L} (g^j X_j^L)$$

$$= (f^i \circ \gamma)(t) X_i^L (g^j) X_j^L + (f^i \circ \gamma)(t) (g^j \circ \gamma)(t)(\nabla_{X_i^L} X_j^L)_{\gamma(t)}$$

$$dL_{\omega(t)}^{-1} \nabla_{\omega(t)} Y = (f^i \circ \gamma)(t) X_i^L (g^j) e_j + (f^i \circ \gamma)(t) (g^j \circ \gamma)(t) dL_{\gamma(t)}^{-1} (\nabla_{X_i^L} X_j^L)_{\gamma(t)}$$

$$= (f^i \circ \gamma)(t) X_i^L (g^j) e_j + (f^i \circ \gamma)(t) (g^j \circ \gamma)(t) \nabla_{e_i} e_j$$

where Leibniz formula and the invariance of the connection is used in $(\nabla_{X_i^L} X_j^L) = dL_{\gamma(t)} \nabla_{e_i} e_j$. Therefore for $k = 1 \ldots n$

$$\langle dL_{\omega(t)}^{-1} \nabla_{\omega(t)} Y, e_k \rangle = (f^i \circ \gamma)(t) X_i^L (g^j) \langle e_j, e_k \rangle$$

$$+ (f^i \circ \gamma)(t) (g^j \circ \gamma)(t) \langle \nabla_{e_i} e_j, e_k \rangle$$ (5)

but on one hand

$$\zeta(t) = \theta(Y)|_{\gamma(t)} = \theta|_{\gamma(t)} \left( ((g^i \circ \gamma)(t) X_j^L)_{\gamma(t)} \right)$$

$$= (g^i \circ \gamma)(t) e_j$$ (6)

$$\dot{\zeta}(t) = (g^i \circ \gamma)'(t) e_j = d_{\gamma(t)} g^i \dot{\gamma}(t) e_j$$

$$= d_{\gamma(t)} g^i \left( (f^i \circ \gamma)(t) X_j^L \right)_{\gamma(t)} e_j$$

$$= (f^i \circ \gamma)(t) d_{\gamma(t)} g^j X_j^L \mid_{\gamma(t)} e_j$$

$$= (f^i \circ \gamma)(t) X_j^L (g^j) e_j$$ (7)
and on the other hand, using (4):

\[
(f^i \circ \gamma)(g^j \circ \gamma)(\nabla_{e_i} e_j, e_k) = \frac{1}{2} (f^i \circ \gamma)(g^j \circ \gamma)(\langle [e_i, e_j], e_k \rangle
\]

\[
- \langle [e_j, e_k], e_i \rangle - \langle [e_i, e_k], e_j \rangle
\]

\[
= \frac{1}{2} \langle (f^i \circ \gamma)e_i, (g^j \circ \gamma)e_j, e_k \rangle
\]

\[
- \langle (g^j \circ \gamma)e_j, (f^i \circ \gamma)e_i \rangle - \langle (f^i \circ \gamma)e_i, (g^j \circ \gamma)e_j \rangle \rangle
\]

\[
= \frac{1}{2} ([\omega, \zeta] - \text{ad}_\omega^* \zeta - \text{ad}_\zeta^* \omega) = \alpha(\omega, \zeta) \quad (8)
\]

Thus, we obtain an algebraic expression for the covariant derivative of any vector field \( Y \) along a smooth curve \( \gamma \). It will be the main ingredient of this paper.

\[
dL_{\gamma(t)}^{-1} \nabla_{\dot{\gamma}(t)} Y(t) = \dot{\zeta}(t) + \alpha(\omega(t), \zeta(t)) \quad (9)
\]

A similar expression can be found in [1,7]. As all the variables of the right-hand side are defined in \( \mathfrak{g} \), they can be computed with matrix operations and an orthonormal basis.

4 Parallel Transport

We now focus on two particular cases of (9) to derive the equations of geodesics and of parallel transport along a curve.

4.1 Geodesic equation

The first particular case is for \( Y(t) = \dot{\gamma}(t) \). It is then straightforward to deduce from (9) the Euler-Poincaré equation for a geodesic curve [13,5]. Indeed in this case, recall that \( \omega = \theta|_{\gamma(t)} \dot{\gamma}(t) \) is the left-angular velocity, \( \zeta = \omega \) and \( \alpha(\omega, \omega) = - \text{ad}_\omega^* \omega \). Hence \( \gamma \) is a geodesic if and only if \( dL_{\gamma(t)}^{-1} \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \) i.e. setting the left-hand side of (9) to 0. We obtain

\[
\begin{aligned}
\dot{\gamma}(t) &= dL_{\gamma(t)} \omega(t) \\
\dot{\omega}(t) &= \text{ad}_{\omega(t)}^* \omega(t).
\end{aligned} \quad (10)
\]

Remark 1. One can show that the metric is bi-invariant if and only if the adjoint map is skew-symmetric (see [19] or [6, Prop. 20.7]). In this case \( \text{ad}_{\omega}^* \omega = 0 \) and (10) coincides with the equation of one-parameter subgroups on \( G \).

4.2 Reduced Parallel Transport Equation

The second case is for a vector \( Y \) that is parallel along the curve \( \gamma \), that is, \( \forall t, \nabla_{\dot{\gamma}(t)} Y(t) = 0 \). Similarly to the geodesic equation, we deduce from (9) the parallel transport equation expressed in the Lie algebra.
Theorem 1. Let $\gamma$ be a smooth curve on $G$. The vector $Y$ is parallel along $\gamma$ if and only if it is solution to

$$
\begin{align*}
\omega(t) &= dL_{\gamma(t)}^{-1}\dot{\gamma}(t) \\
Y(t) &= dL_{\gamma(t)}\dot{\xi}(t) \\
\dot{\zeta}(t) &= -\alpha(\omega(t),\zeta(t))
\end{align*}
$$

(11)

Note that in order to parallel transport along a geodesic curve, (10) and (11) are solved jointly.

4.3 Application

We now exemplify Theorem 1 on the group of isometries of $\mathbb{R}^3$, $SE(3)$, endowed with a left-invariant metric $g$. $SE(3)$ is the semi-direct product of the group of three-dimensional rotations $SO(3)$ with $\mathbb{R}^3$, i.e. the group multiplicative law for $R,R' \in SO(3), t,t' \in \mathbb{R}^3$ is given by

$$(R,t) \cdot (R',t') = (RR', t + Rt').$$

It can be seen as a subgroup of $GL(4)$ and represented by homogeneous coordinates:

$$(R,t) = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix},$$

and all group operations then correspond to the matrix operations. Let the metric matrix at the identity be diagonal: $G = \text{diag}(1,1,1,\beta,1,1)$ for some $\beta > 0$, the anisotropy parameter. An orthonormal basis of the Lie algebra $\mathfrak{se}(3)$ is

$$
\begin{align*}
e_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
e_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
e_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
e_4 &= \frac{1}{\sqrt{\beta}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
e_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
e_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
$$

Define the corresponding structure constants $C_{ij}^k = \langle [e_i,e_j],e_k \rangle$, where the Lie bracket $[\cdot,\cdot]$ is the usual matrix commutator. It is straightforward to compute

$$
C_{ij}^k = \frac{1}{\sqrt{2}} \text{ if } ijk \text{ is a direct cycle of } \{1,2,3\};
$$

$$
C_{15}^6 = -C_{16}^5 = -\sqrt{\beta}C_{24}^6 = \frac{1}{\sqrt{\beta}}C_{26}^4 = \sqrt{\beta}C_{34}^5 = -\frac{1}{\sqrt{\beta}}C_{35}^4 = \frac{1}{\sqrt{2}}.
$$

(13)
and all others that cannot be deduced by skew-symmetry of the bracket are equal to 0. The connection can then easily be computed using

\[ \alpha(e_i, e_j) = \nabla_{e_i}e_j = \frac{1}{2} \sum_k (C^k_{ij} - C^k_{ji} + C^k_{ij})e_k, \]

For \( \beta = 1 \), \((SE(3), G)\) is a symmetric space and the metric corresponds to the direct product metric of \( SO(3) \times \mathbb{R}^3 \). However, for \( \beta \neq 1 \), the geodesics cannot be computed in closed-form and we resort to a numerical scheme to integrate (10). According to [8], the pole ladder can be used with only one step of a fourth-order scheme to compute the exponential and logarithm maps at each rung of the ladder. We use a Runge-Kutta (RK) scheme of order 4. The Riemannian logarithm is computed with a gradient descent on the initial velocity, where the gradient of the exponential is computed by automatic differentiation. All of these are available in the `InvariantMetric` class of the package geomstats [17].

We now compare the integration of (11) to the pole ladder [8] for \( \beta = 1.5, 2 \) to parallel transport a tangent vector along a geodesic. The results are displayed on Figure 1 in a log-log plot. As expected, we reach convergence speeds of order two.

![Fig. 1. Comparison of the integration of the reduced equation with the pole ladder](image-url)
for the pole ladder and the RK2 scheme, while the RK4 schemes is of order four. Both integration methods are very stable, while the pole ladder is less stable for $n \geq 200$.

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