FGM Copula's Extension Via Polynomial Function

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Abstract. This study is concerned with generalizing the well-known family of FGM copula in terms of a polynomial function. A general form of the FGM copula via a copula function is proposed in order to obtain a new FGM copula family. Various particular cases have been presented according to the degree of the polynomial function. Proofs of necessary and sufficient conditions of the created copulas have been explained so that we guarantee that the desire function is copula. We firstly propose and define a general form that associate the FGM copula with a polynomial function. Graphs and some extra properties are presented as explanatory tool of some degrees of the obtain copula. Finally, we present some important calculations related to the most popular dependences by means of the FGM copula extension.

Keywords: Copula Concepts, FGM Copula, Joint Distribution Function, Nonparametric Measures.

1. Introduction
In this paper, we concerned with a type of generalization to the FGM copula family via a polynomial function of degree n. The polynomial function follows from the modification of FGM copula that has been presented by Sriboonchitta-Kreinovich, see [6]. Consequently, we have obtained a function that depend on the general form of the FGM copula. This function satisfies the boundary conditions of copula \( C(t, 0) = 0, C(t, 1) = t \). The only remaining condition that we need to figure out is the 2-increasing condition. This can be performed by providing all the properties which are related to the joint density of copula in terms of the proposed structure. Indeed, we have to provide properties related to the joint density of copula with a convex set of all the coefficients of the polynomial function so that we ensure the region of that coefficients. First of all, we have to test the convex set of all the coefficients of the polynomial function and find its conditions. Indeed, this is not an easy issue and need several necessary and sufficient conditions so that we obtain the 2-increasing condition that verify the proposed FGM function to be a copula. Furthermore, there are some special cases of the convex set have been explained by using some graphs. Afterwards, we connect the obtained copula to the measures of association that their general forms will be presented in the next section. We have used Mathematica software to perform almost required graphs, and calculations. Honestly, we have to
mention that there are several studies that had equivalent extensions to the FGM copula, see [2, 3, 4, 5]. Finally, we refer to the contents of each section of this study: section two deals with some basic notions. Section three is devoted to present the FGM copula family with its extension, show some necessary, and sufficient conditions. While section four will be dedicated to demonstrate the calculations of measures of association with proving of the intervals of each dependence. Eventually, we show some conclusions and future works in section five.

2. Basic notions

This section is dedicated to review the basic notions of copulas and their properties. Basically, it is important to recall the basic definition of the bivariate copula.

**Definition 2.1** [4] A bivariate copula is function \( C : [0, 1]^2 \rightarrow [0, 1] \) that holds the following conditions

(I) For all \( t \in [0, 1] \), \( C(t, 0) = C(0, t) = 0 \);

(II) For all \( t \in [0, 1] \), \( C(t, 1) = t = C(1, t) \);

(III) \( C \) is 2-increasing for all \( t_1, t_2, h_1, h_2 \in [0, 1] \) such that \( t_1 \leq t_2 \), and \( h_1 \leq h_2 \). Then \( C(t_1, h_1) + C(t_2, h_2) \geq C(t_1, h_1) + C(t_2, h_2) \).

In association with a definition of bivariate copula we can refer to Sklar's theorem, which represents the core of copula's notion.

\[
F(x, y) = C(F_1(x), F_2(y))
\]  

where \( F \) is the joint distribution function, \( F_1 \), and \( F_2 \) are the marginal distribution functions of \( F \), respectively, whose connected to their joint via copula function. We note that when the random variables \( X, Y \) are continuous, then the bivariate copula is unique, see [2, 4].

Moreover, we recall a notion of FGM copula that has been founded by Farlie (1960), Gumbel (1958), and Morgenstern (1956). When the FGM copula family is symmetric function, then it has the following form. For all \( t, h \in [0, 1] \).

\[
C(t, h) = th + \rho_1(t)h(1 - h)
\]  

where \( \rho_1 \in [-1, 1] \).

It is clear that \( C \) is copula only within the interval \([-1, 1]\). There are other various concepts related to this copula family can be found in [5]. Associating with a notion of FGM copula, we also need to recall a notion of Sriboonchita-Kreinovich of FGM copula that has the formula

\[
C(t, h) = th + \theta(t, h)t(1 - t)h(1 - h)
\]  

They have explained that \( C(t, h) \) in (2.3) is the widest copula between others within three different methods, see [7].

Since this study consists of some calculations and properties of some dependences, so we also have to recall their essential forms that are

\[
\rho_C(t, h) = 12 \int_0^1 \frac{\partial^2}{\partial t \partial h} [C(t, h) - th] dt dh
\]  

\[
\gamma_C(t, h) = 4 \int_0^1 \frac{\partial^2}{\partial (t, h)} C(t, h) dt dh - 1
\]  

\[
\beta_C(t, h) = 4C \left( \frac{1}{2}, \frac{1}{2} \right) - 1
\]

Indeed, these measures are also known by measures of concordance and discordance see [1, 2, 3, 4, 5].

3. FGM Copula's Extension Via Polynomial Function

This part is devoted to present the main proposal that involving the combination of the standard FGM copula to the polynomial function of Sriboonchita-Kreinovich. Suppose that \( \theta(t, h) \) is a polynomial of degree \( n \) with horizontal section. This means \( \theta(t, h) \) can be written as \( \theta(h) \). Then

\[
\theta(h) = a_n h^n + \cdots + a_1 h + a_0
\]  

(3.1)
Now, substitute (3.1) in (2.3) so, we obtain the following function. That is
\[ C(a_0, a_1, ..., a_n)(t, h) = th + (a_n h^n + \cdots + a_1 h + a_0)(t - t^2)(h - h^2) \] (3.2)
Rearranging the formula in (3.2) with respect to the polynomial function leads us to the following general form
\[ C(a_0, a_1, ..., a_n)(t, h) = th + (t - t^2)f(h) \] (3.3)
where \( f(h) = (a_0 h + (a_1 - a_0) h^2 + \cdots + (a_n - a_{n-1}) h^{n+1} - a_n h^{n+2}) \).
Conversely, when we suppose that \( \theta(t, h) \) is a polynomial function with respect to the vertical section, so one can write the function in (2.3) as the following
\[ C(a_0, a_1, ..., a_n)(t, h) = th + (a_0 t + (a_1 - a_0) t^2 + \cdots + (a_n - a_{n-1}) t^{n+1} - a_n t^{n+2})(h - h^2) \]
We only focus on the situation of \( \theta(t, h) \) with horizontal section because the second situation (\( \theta(t, h) \) with vertical section) is equivalence and has the same routine.
Note that, if \( \theta(h) \) is a polynomial of degree zero, then we simply have the standard FGM copula.
Also, we have to mention that the function in (3.3) is a polynomial of degree \( (n + 4) \). In other words, the function in (3.3) can be rewritten as a polynomial function form of degree \( (n + 4) \).
\[ C(a_0, a_1, ..., a_n)(t, h) = a(h)t^2 + b(h)t + c(h) \] (3.4)
where \( a(h) = -(a_0 h + (a_1 - a_0) h^2 + \cdots + (a_n - a_{n-1}) h^{n+1} - a_n h^{n+2}), b(h) = (h + a_0 h + (a_1 - a_0) h^2 + \cdots + (a_n - a_{n-1}) h^{n+1} a_n h^{n+2}), c(h) = 0 \).
1. \( a(0) = a(1) = 0 \), \( c(h) = 0 \)
2. \( a(h) = h - a(h) \)
3. Let \( h_1, h_2 \in [0,1], \exists h_1 \leq h_2 \Rightarrow |a(h_2) - a(h_1)| \leq |h_2 - h_1| \) (Liptiz condition).
Now, we are in position to test whether the function in (3.3) is copula or not. Indeed, there are more transparent notions that we have to discuss so that we can ensure that this function fulfills the conditions of copula. One can easily see that \( C(a_0, a_1, ..., a_n)(t, h) \) holds the boundary conditions of copula.
Mathematically speaking, \( C(a_0, a_1, ..., a_n)(t, 0) = C(a_0, a_1, ..., a_n)(0, h) = 0 \), and \( C(a_0, a_1, ..., a_n)(t, 1) = C(a_0, a_1, ..., a_n)(1, t) = t \). The only remaining condition that we need to verify is the 2-increasing condition. In fact, this condition need more properties to be handled so that we can decide whether \( C(a_0, a_1, ..., a_n)(t, h) \) has the 2-increasing property or not. In advance, the following necessary conditions of \( C(a_0, a_1, ..., a_n)(t, h) \) need to be examined to have a 2-increasing property. Obviously, those conditions are associated with a density of the function \( C(a_0, a_1, ..., a_n)(t, h) \).
This can be presented by the two conditions below
\[ C(a_0, a_1, ..., a_n)(t, h) \geq 0 \] for all \( t, h \in [0; 1] \);
\[ C(a_0, a_1, ..., a_n)(t, h) = \int_0^h \int_0^t c(a_0, a_1, ..., a_n)(s, w)dsdw \]
where \( c(a_0, a_1, ..., a_n)(t, h) = \frac{\partial^2 C(a_0, a_1, ..., a_n)(t, h)}{\partial t \partial h} \), and it is known as the joint density of copula. A simple differentiation of the function in (3.3) yields the joint density. That is
\[ c(a_0, a_1, ..., a_n)(t, h) = 1 + (1 - 2t)f'(h) \] (3.5)
Where, \( f'(h) = a_0 + 2(a_1 - a_0) h + 3(a_1 - a_0) h^2 + \cdots + (n + 1)(a_n - a_{n-1}) h^n - (n + 2)a_n h^{n+1} \).
To prove condition (C1), we need to show that for all \( t, h \in [0,1] \), the joint density in (4) with respect to the constants \( a_0, a_1, ..., a_n \) is non-negative. To perform that, it is necessary to find all the solutions of a convex set for which \( a_0, a_1, ..., a_n \) belong to that set, let say \( T \in R^n \).
Graphically, we can see the convex set, for example, for the constants \( (a_0, a_1) \) polynomial function of degree 1 that has the following shape
This notion can be illustrated and shown by means of the following lemma

**Lemma 3.1** A function in (3.3) be a copula, if its joint density \( c(a_0, a_1, \ldots, a_n)(t, h) \) is absolutely continuous, differentiable, and non-negative, if and only if, for all the constants \(-1 \leq a_0 \leq 1\), and \(-1 \leq a_0 + a_1 + \cdots + a_n \leq 1\)

**Proof**

i. According to Lemma 2.1 in [5], the joint density in (3.4) satisfies the following conditions
   - Lipschitz condition
   - Absolutely continuous and differentiable everywhere over \([0,1]\)

ii. If \( a_1 = a_2 = \cdots = a_n = 0 \), and \( a_0 \in [-1,1] \), then this is nothing more than the standard FGM copula, and the prove over that is verified and trivial.

iii. A solution that guarantees \((a_0, a_1, \ldots, a_n) \in T \subseteq \mathbb{R}^n\) (convex set) is only the interval \(|a_0 + a_1 + \cdots + a_n| \leq 1\) where \( a_0 \in [-1,1] \).

v. For \(-1 \leq a_0 + a_1 + \cdots + a_n \leq 1\), such that \( a_0 \in [-1,1] \), we have
   \[
   \text{Ran}(1 + 2t(a_0 + 2(a_1 - a_0)h + 3(a_2 - a_1)h^2 + \cdots + (n + 1)(a_n - a_{n-1})h^n - (n + 2(a_n h^{n+1}))
   \]
   \[
   = |a_0 + a_1 + \cdots + a_n|, |a_0 + a_1 + \cdots + a_n| \] .

Thus, the range of the joint density of \( C(a_0, a_1, \ldots, a_n)(t, h) \) in (3.3) can be performed by the following way
\[
\text{Ran}(1 + |a_0 + a_1 + \cdots + a_n|, |a_0 + a_1 + \cdots + a_n|) = [1 - |a_0 + a_1 + \cdots + a_n|, 1 + |a_0 + a_1 + \cdots + a_n|].
\]

Therefore, for all \( t, h \in [0,1] \), \( c(a_0, a_1, \ldots, a_n)(t, h) \geq 0 \) if and only if \( a_0 \in [-1,1] \), and if and only if \(|a_0 + a_1 + \cdots + a_n| \leq 1\).

This complete the proof.

The proof of **Lemma 3.1** is only shown necessary conditions, but it is not sufficient to decide that our function \( C(a_0, a_1, \ldots, a_n)(t, h) \) has the 2-increasing property. Indeed, there are more conditions need to be verified in order to ensure that property.

**Theorem 3.1** Let \( C(a_0, a_1, \ldots, a_n)(t, h) \) be a function that satisfies the boundary conditions of copula \( C(a_0, a_1, \ldots, a_n)(t, 0) = C(a_0, a_1, \ldots, a_n)(0, h) = 0, C(a_0, a_1, \ldots, a_n)(t, 1) = t = C(a_0, a_1, \ldots, a_n)(1, t) \), and it has the properties of **Lemma 3.1**. The function \( C(a_0, a_1, \ldots, a_n)(t, h) \) be a copula, if and only if, it holds the 2-increasing property, if and only if, \( V_C([t_1, t_2] \times [y_1, y_2]) \geq 0, t_1, t_2, y_1, y_2 \in [0,1] \).

**Proof**

It is sufficient to show that \( C(a_0, a_1, \ldots, a_n)(t, h) \) in (3.3) holds the 2-increasing property. For simplicity, we can write the function (3.3) by the following form

\[
C(a_0, a_1, \ldots, a_n)(t, h) = th + (a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n)t(1 - t)h(1 - h)
\]

Thus
\[ V_C([t_1, t_2] \times [h_1, h_2]) = (t_2 - t_1)(h_2 - h_1) + [a_0 + a_1(h_2 - h_1) + a_2(h_2 - h_1)^2 + \cdots + a_n(h_2 - h_1)^n] \]

\[ [(t_2 - t_1)(1 - h_2 - h_1)[1 + [a_0 + a_1(h_2 - h_1) + a_2(h_2 - h_1)^2 + \cdots + a_n(h_2 - h_1)^n] (1 - t_2 - t_1)(1 - h_2 - h_1)] \]

Hence, \((t_2 - t_1)(h_2 - h_1) \geq 0\).

Let \(g((t_2 - t_1), (h_2 - h_1)) = [a_0 + a_1(h_2 - h_1) + a_2(h_2 - h_1)^2 + \cdots + a_n(h_2 - h_1)^n](1 - t_2 - t_1)(1 - h_2 - h_1) \)

\[ \text{Ran}(g((t_2 - t_1), (h_2 - h_1))) = [-1, 1], \text{For all } t_1, t_2, h_1, h_2 \in [0, 1] \]

Thus, \(\text{Ran}(1 + [-1, 1]) = [0, 2]\), implies that \(V_C([t_1, t_2] \times [h_1, h_2]) \geq 0\).

Therefore, \(C_{(a_0, a_1, \ldots, a_n)}(t, h)\) has 2-increasing property, and it is copula, if and only if \(|a_0 + a_1 + \cdots + a_n| \leq 1\).

This complete the proof.

4. Determination of Particular Cases for The Coefficients of Polynomial Function

The number of constants of \(\theta(h)\) in the FGM copula has several general properties that we can determine in this part.

**Corollary 4.1** If \(\theta(h)\) is a polynomial of degree \(n\) such that \(|a_0| \leq 1, \) and \(|a_0 + a_1 + \cdots + a_n| \leq 1\), then

\[-2^0 \leq a_0 \leq 2^0, -2^1 \leq a_1 \leq 2^1, \ldots, -2^{n-1} \leq a_{n-1} \leq 2^{n-1}, -2^n \leq a_n \leq 2^n\]

**Proof**

It is clear that for \(n = 0\) (polynomial of degree zero), we have the standard FGM copula, and its proof is trivial, see [4]. Then

For \(n = 1\), we have 

\(-1 \leq a_0 + a_1 \leq 1 \Rightarrow -2 \leq a_1 \leq 2, \) when \(-1 \leq a_0 \leq 1\).

For \(n = 1\), we have 

\(-1 \leq a_0 + a_1 + a_2 \leq 1 \Rightarrow -2^2 \leq a_2 \leq 2^2, \) when \(-1 \leq a_0 \leq 1, \) and \(-2 \leq a_1 \leq 2\).

Repeating this process to the degree \(n\), will lead us to the main claim of this corollary.

Therefore, 

\[-1 \leq a_0 \leq 1, -2 \leq a_1 \leq 2, \ldots, -2^{n-1} \leq a_{n-1} \leq 2^{n-1} \Rightarrow -2^n \leq a_n \leq 2^n.\]

The proof is complete.

For instance, there are several FGM copulas with different polynomial degree can be shown.

Let \(n = 1\), such that \(-1 \leq a_0 + a_1 \leq 1, \) and \(-1 \leq a_0 \leq 1\), implies that the vertices with the extreme values \((a_0, a_1) = (1, -2), (-1, 2),\) and all other pairs between them yield copulas.

Some of the FGM copula family of this degree can be shown in Table 1

**Table 1:** FGM copula with degree \(n=1\)

| \((a_0, a_1)\) | \(\sum_{i=0}^{1} a_i\) | \(C_{(a_0, a_1)}(t, h)\) |
|--|--|--|
| (1, -2) | -1 | \(th + (t - t^2)(h - 3h^2 + 2h^3)\) |
| (1, -1) | 0 | \(th + (t - t^2)(h - 2h^2 + h^3)\) |
| (-1,1) | 0 | \(th - (t - t^2)(h - 2h^2 + h^3)\) |
| (-1,2) | 1 | \(th - (t - t^2)(h - 3h^2 + 2h^3)\) |

In fact, there are other copulas like \(C_{(1,0,0.5)}(t, h), C_{(-0.5,1)}(t, h),\) and so on.

In the tables below \((Table 2, Table 3)\), we only show FGM copula of degree 2, and 3. Indeed, there are many other copulas with higher degrees can be illustrated too.
Table 2: FGM copula with degree n = 2

| \((a_0, a_1, a_2)\) | \(\sum_{i=0}^{2} a_i\) | \(C_{(a_0,a_1,a_2)}(t,h)\) |
|-------------------|------------------|------------------|
| \((1,2,-2)\)      | 1                | \(th + (t - t^2)(h + h^2 - 4h^3 + 2h^4)\) |
| \((-1,-2,2)\)     | -1               | \(th - (t - t^2)(h + h^2 - 4h^3 + 2h^4)\) |
| \((1,2,-3)\)      | 0                | \(th + (t - t^2)(h + h^2 - 5h^3 + 3h^4)\) |
| \((-1,-2,3)\)     | 0                | \(th - (t - t^2)(h + h^2 - 5h^3 + 3h^4)\) |
| \((1,2,-4)\)      | -1               | \(th + (t - t^2)(h + h^2 - 6h^3 + 4h^4)\) |
| \((-1,-2,4)\)     | 1                | \(th - (t - t^2)(h + h^2 - 6h^3 + 4h^4)\) |

Table 3: FGM copula with degree n = 3

| \((a_0, a_1, a_2, a_3)\) | \(\sum_{i=0}^{3} a_i\) | \(C_{(a_0,a_1,a_2,a_3)}(t,h)\) |
|--------------------------|------------------|------------------|
| \((1,2,4,-6)\)          | 1                | \(th + (t - t^2)(h + h^2 + 2h^3 - 10h^4 + 6h^5)\) |
| \((-1,-2,-4,6)\)        | -1               | \(th - (t - t^2)(h + h^2 + 2h^3 - 10h^4 + 6h^5)\) |
| \((1,2,-7)\)            | 0                | \(th + (t - t^2)(h + h^2 + 2h^3 - 11h^4 + 7h^5)\) |
| \((-1,-2,-4,7)\)        | 0                | \(th - (t - t^2)(h + h^2 + 2h^3 - 11h^4 + 7h^5)\) |
| \((1,2,-8)\)            | -1               | \(th - (t - t^2)(h + h^2 + 2h^3 - 12h^4 + 8h^5)\) |
| \((-1,-2,-4,8)\)        | 1                | \(th - (t - t^2)(h + h^2 + 2h^3 - 12h^4 + 8h^5)\) |

On the other hand, we can present several examples of the proposed extension via the FGM copula family when \(|a_0 + a_1 + \cdots + a_n| > 1\), for which \(C_{(a_0,a_1,\ldots,a_n)}(t,h)\) does not hold the condition of copula. This situation can be demonstrated in the following table.

Table 4: Functions withe \(n = 1, \ldots, 6\) that are not copulas

| \(n\) | \((a_0, a_1, a_2, a_3)\) | \(\sum_{i=0}^{3} a_i\) | \(C_{(a_0,a_1,a_2,a_3)}(t,h)\) |
|-------|--------------------------|------------------|------------------|
| 1     | \((1,2)\)                | 3                | \(th + (t - t^2)(h + h^2 - 2h^3)\) |
| 2     | \((-1,1,9,4)\)           | 4.9              | \(th + (t - t^2)(-h + 2.9h^2 + 2.1h^3 - 4h^4)\) |
| 3     | \((1,2,4,8)\)            | 15               | \(th + (t - t^2)(h + h^2 + 2h^3 + 4h^4 - 8h^5)\) |
| 4     | \((-1,-2,-4,-8,-16)\)    | -31              | \(th + (t - t^2)(-h - h^2 - 2h^3 - 4h^4 - 8h^5 + 16h^6)\) |
| 5     | \((-1,2,-3.5,8,12,-32)\) | -12.5            | \(th + (t - t^2)(-h + 3h^2 - 5.5h^3 + 11.5h^4 + 4h^5 - 44h^6 + 32h^7)\) |
| 6     | \((-1,1.5,-4,2.5,-11,-31,40)\) | -3               | \(th + (t - t^2)(-h + 2.5h^2 - 5.5 h^3 + 6.5h^4 - 13.5h^5 - 20h^6 + 71h^7 - 40h^8)\) |

It is clear that each function \(C_{(a_0,a_1,\ldots,a_n)}(t,h)\) in Table 4 is not copula. For instance, for \(n = 2\), the vector of constants \((-1,1.9,4)\) has the sum \(-1+1.9+4=4.9\), which is greater than 1. The joint density \(c_{(-1,1.9,4)}(t,h)\) over the pair \((t,h) = (0,1)\) yields...
\[ c_{(-1,1,9,0)}(0,1) = 1 + (1)(-1 + 5.8 + 6.3 - 16) = -3.9 < 0 \]

This gives a negative joint density, which means that the 2-increasing condition does not fulfil.

In general, for any degree \( n \), if the absolute sum of the assigned constants is greater than 1, then this will yield a function which is not copula.

Moreover, we can graphically show some particular cases of each illustrated FGM copula. Then, for \( n = 1, 2, 3 \), we obtain the following graphs of each degree.

**Figure 1: \( n = 1 \)**

\[ C_{(-1,2)}(t,h) \quad C_{(1,-2)}(t,h) \]

**Figure 2: \( n = 2 \)**

\[ C_{(-1,-2,4)}(t,h) \quad C_{(1,2,-4)}(t,h) \]

**Figure 3: \( n = 3 \)**

\[ C_{(-1,-2,-4,8)}(t,h) \quad C_{(1,2,4,-8)}(t,h) \]

5. **Computations of the Dependences Via the FGM Copula's Extension**

In this part, we present several calculations of the dependences through the extended FGM copula with different degree of the polynomial function. Therefore, the implementing of the FGM copula in (3.3) with respect to the formulas of the measures of association in (2.4), (2.5), (2.6), and (2.7) will yield the following results that can be shown in the following theorem.
**Theorem 5.1** For all $t, h \in [0,1]$, and $a_n \in [-2^n, 2^n], n = 0, 1, \ldots$, for some integer $n$ Where $|a_0 + a_1 + \cdots + a_n| \geq 1$. Then the yielding dependences $\rho_{C(a_0, a_1, \ldots, a_n)}$, $\tau_{C(a_0, a_1, \ldots, a_n)}$, $\gamma_{C(a_0, a_1, \ldots, a_n)}$, and $\beta_{C(a_0, a_1, \ldots, a_n)}$, respectively, in terms of the general copula in (3.3) are the following

$$
\rho_{C(a_0, a_1, \ldots, a_n)} = \frac{2}{n+3} \left( \frac{a_n}{2} + \frac{a_1-a_0}{3} + \cdots + \frac{a_n-a_{n-1}}{n+2} - \frac{a_n}{n+3} \right) \quad (5.1)
$$

$$
\tau_{C(a_0, a_1, \ldots, a_n)} = \frac{2}{n+3} \rho_{C(a_0, a_1, \ldots, a_n)} \quad (5.2)
$$

$$
\gamma_{C(a_0, a_1, \ldots, a_n)} = 4 \int_0^t \left( C(a_0, a_1, \ldots, a_n)(t, 1-t) + C(a_0, a_1, \ldots, a_n)(t, t) - t \right) dt \quad (5.3)
$$

$$
\beta_{C(a_0, a_1, \ldots, a_n)} = 2^{n-1} a_n + 2^{n-1} a_{n-1} + \cdots + 2 a_1 + a_0 \quad (5.4)
$$

The proof of **Theorem 5.1** directly follows from the calculations within each general formula of each dependence (Spearman, Kendall's tau, Gini's gamma, Blomqvist beta).

In **Table 5**, we show specific values of each correlation with degree $1 \leq n \leq 4$ for the extremes values of the coefficients $(a_0, a_1, \ldots, a_n)$. Indeed, there are infinite number of the values of these measures of association for the coefficients values that lies in between the extremes of each interval at each degree $n$.

**Table 5: Some values of dependences with degree $n = 1, 2, 3, 4$**

| $n$ | $(a_0, a_1, \ldots, a_n)$ | $\rho_{C(a_0, a_1, \ldots, a_n)}$ | $\tau_{C(a_0, a_1, \ldots, a_n)}$ | $\gamma_{C(a_0, a_1, \ldots, a_n)}$ | $\beta_{C(a_0, a_1, \ldots, a_n)}$ |
|-----|--------------------------|-------------------------------|------------------------------|--------------------------------|--------------------------|
| 1   | (1, -2)                  | 0.333                         | 0.222                        | 0.266                         | 0.25                     |
| 1   | (1, 0)                   | 0.333                         | 0.222                        | 0.266                         | 0.25                     |
| 2   | (1, 2, -2)               | 0.466                         | 0.311                        | 0.380                         | 0.375                    |
| 2   | (-1, -2, 2)              | 0.466                         | 0.311                        | 0.380                         | 0.375                    |
| 3   | (1, 2, -6)               | 0.666                         | 0.444                        | 0.552                         | 0.562                    |
| 3   | (-1, -2, 4, 6)           | 0.666                         | 0.444                        | 0.552                         | 0.562                    |
| 4   | (1, 2, 4, -14)           | 0.933                         | 0.622                        | 0.774                         | 0.781                    |

**Table 6** shows the general maximum values of the intervals that each dependence is belong to over the FGM copula's extension.

**Table 6: Extremes values of each dependence**

| $n$ | $\rho_{C(a_0, a_1, \ldots, a_n)}$ | $\tau_{C(a_0, a_1, \ldots, a_n)}$ | $\gamma_{C(a_0, a_1, \ldots, a_n)}$ | $\beta_{C(a_0, a_1, \ldots, a_n)}$ |
|-----|---------------------------------|-------------------------------|--------------------------------|--------------------------|
| 1   | $[0.333]$                       | $[0.222]$                     | $[0.266]$                       | $[0.25]$                |
| 2   | $[0.466]$                       | $[0.311]$                     | $[0.380]$                       | $[0.375]$               |
| 3   | $[0.666]$                       | $[0.444]$                     | $[0.552]$                       | $[0.562]$               |
| 4   | $[0.933]$                       | $[0.622]$                     | $[0.774]$                       | $[0.781]$               |

**Remark 5.1** The obtained values of each dependences in **Table 6** is greater than the values of the dependences that have been shown in [4, 5].

6. Conclusion

This work can be summarized by the following statements:

A necessary condition $(|a_0 + a_1 + \cdots + a_n| \geq 1)$ is crucial in the constructing of the extension of the FGM copula. Several intervals of the coefficients of the polynomial function that belong to the convex set have been founded. At each degree of the polynomial $\theta(h)$, there are infinite number of copulas.
that has various properties, conditions, and shapes were presented. For $n = 1,2,3,4$, we have extended the intervals of each dependence that reach the maximum value of the correlation coefficient whether $-1$ or $1$. There are many open problems related to this extension need to be investigated like studying the case of the extension of the FGM copula with degree $n \geq 5$.

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