Neutrino spin oscillations in gravitational fields

Maxim Dvornikov  
Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation (IZMIRAN)  
142190, Troitsk, Moscow region, Russia  
and  
Graduate School of Science, Hiroshima University, Higashi-Hiroshima, Japan  
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We study neutrino spin oscillations in gravitational fields. The quasi-classical approach is used to describe the neutrino spin evolution. First we examine the case of a weak gravitational field. We obtain the effective Hamiltonian for the description of neutrino spin oscillations. We also receive the neutrino transition probability when a particle propagates in the gravitational field of a rotating massive object. Then we apply the general technique to the description of neutrino spin oscillations in the Schwarzschild metric. The neutrino spin evolution equation for the case of the neutrino motion in the vicinity of a black hole is obtained. The effective Hamiltonian and the transition probability are also derived. We examine the neutrino oscillations process on different circular orbits and analyze the frequencies of spin transitions. The validity of the quasi-classical approach is also considered.

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I. INTRODUCTION

Neutrino physics becomes one the most interesting fields of elementary particle physics, especially in the wake of the recent experimental achievements in studying of solar and reactor neutrinos (see, e.g., Refs. [1,2]). One of the most intriguing puzzles in neutrino physics in the neutrino oscillations problem. Neutrinos of one type can be converted into another type if there is a mixing between different neutrino eigenstates. Presently three major opportunities for neutrino oscillations are discussed in literature. The first one is neutrino flavor oscillations. The example of this oscillations type are \( \nu_e \rightarrow \nu_{\mu,\tau} \) or \( \nu_{\mu} \rightarrow \nu_{\tau} \) conversions that are the possible explanations of the solar and atmospheric neutrino problems. The second type of neutrino oscillations is the transitions between neutrino helicity states within one flavor, or neutrino spin oscillations. The last opportunity is the combination of the two first neutrino oscillations types, i.e. the transition when both flavor and helicity states can change. This type of neutrino oscillations is called neutrino spin-flavor oscillations. It can be important in solving of the solar neutrino problem. In this paper we restrict ourselves to studying of neutrino spin oscillations.

It was realized more than twenty years ago that external fields drastically change the process of vacuum neutrino oscillations. In Refs. [3, 4] it was established that neutrino weak interactions with background matter result in the resonant enhancement of neutrino flavor oscillations. The electromagnetic interactions are of great importance in studying of neutrino spin and spin-flavor oscillations. For example, the neutrino interaction with an external electromagnetic field provides one of the mechanisms for the mixing between different helicity eigenstates. It is also interesting to study the influence of gravitational fields on neutrino oscillations (neutrino does not seem to participate in strong interactions). Despite the gravitational interaction is relatively weak compared to electromagnetic and weak interactions there are rather strong gravitational fields in the Universe. Thus neutrino oscillations in gravitational fields are of interest in astroparticle physics and cosmology.

The influence of the gravitational interaction on neutrino oscillations has been studied in many publications (see, e.g., Refs. [5, 6, 7, 8, 9, 10, 11, 12]). First we mention Refs. [7, 8] where the formalism for the description of neutrino oscillations in gravitational fields in presence of matter and external electromagnetic fields was worked out. One of the approaches to the description of the spin dynamics of a neutrino in a gravitational field consists in decomposing of the Dirac Hamiltonian (in presence of an external gravitational field) and establishing terms that mix different chirality components of the neutrino wave function. This method was used in Ref. [5]. To describe the helicity evolution in a gravitational field one can use the evolution equation in Heisenberg representation with the Hamiltonian in Foldy-Wouthuysen representation. This approach was implemented in Ref. [11]. Neutrino spin oscillations and spin light of a neutrino in weak gravitational fields were studied in our work [12]. Gravity induced neutrino flavor oscillations were examined in Refs. [10, 11].

There is, however, another approach for the treatment of the spinning particle dynamics in an external gravitational field. The system of equations for the description of the spinning particle motion in gravitational fields was derived by A. Papapetrou in Ref. [13]. Since then plenty of works where Papapetrou equations were analysed have been published. The major difficulties in solving Papapetrou...
petrosov equations are that these equations are non-linear in the particle spin as well as the motion of a spinning particle deviates from the geodesical. Thus to describe the motion of a finite size spinning particle in a gravitational field one has to take into account the contribution of the spin term into the particle motion law. It was established in Ref. [14] that Papapetrou equations can be significantly simplified when one uses the quasi-classical approach and linear in spin approximation of these equations for a point particle.

In this paper we study neutrino spin oscillations in gravitational fields within the quasi-classical approach. Note that the quasi-classical treatment of neutrino spin oscillations in moving and polarized matter under the influence of electromagnetic fields was considered in Refs. [10,11,12]. In Sec. II we formulate the main equations necessary for the description of the neutrino motion and spin evolution. The limit of a weak gravitational field is examined in Sec. III. On the basis of the general equation presented in Sec. II we obtain the effective Hamiltonian for neutrino spin oscillations. This effective Hamiltonian is valid for arbitrary neutrino velocity. We also compare the neutrino spin evolution equation with the analogous one derived in our previous work [12] and find out that the equation obtained in the present work correctly accounts for the neutrino velocity dependence. Then we receive neutrino transition probability when a particle propagates in the gravitational field of a rotating massive object. In Sec. IV we apply the general technique to the description of neutrino spin oscillations in the Schwarzschild metric. We obtain the neutrino spin evolution equation which is valid for the neutrino motion even in the vicinity of a black hole. The effective Hamiltonian and the transition probability are also derived. We examine neutrino oscillations process on different circular orbits and analyze the frequencies of the spin transitions. In Sec. V we discuss our results. The calculation of covariant derivatives of the vierbein vectors (Appendix A) and the basic elements of $SL(2,C)$ group (Appendix B) are also presented.

II. MOTION OF A SPINNING POINT PARTICLE IN GRAVITATIONAL FIELDS

The motion of a spinning particle in gravitational fields was described in Ref. [15]. The evolution of the spin tensor of a particle $S^\mu{}\nu$ and particle’s momentum $p^\mu$ is described by the following equations,

$$\frac{DS^\mu{}\nu}{D\tau} = p^\rho v^\nu \varepsilon_{\rho\mu},$$

$$\frac{Dp^\mu}{D\tau} = -\frac{1}{2} R^\mu{}_{\rho\nu\sigma}v^\nu S^\rho{}_{\sigma},$$

where $v^\mu$ is the unit tangent vector to the center-of-mass world line, $\tau$ is the parameter (not necessarily the proper time) which changes along the world line, $D/D\tau$ denotes the covariant derivative along the world line and $R^\mu{}_{\rho\nu\sigma}$ is the Riemann tensor. The spin vector is introduced in the following way (see, e.g., Ref. [13]),

$$S_\rho = \frac{1}{2m} \sqrt{-g} \varepsilon_{\mu\nu\lambda\rho} p^\mu S^\nu{}_{\lambda},$$

where $\varepsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor density, $g = \det(g_{\mu\nu})$ and $m^2 = p_\mu p^\mu$ is the constant of the particle’s motion.

When we study the motion of point particles in a gravitational field, Eqs. (2.1) and (2.2) can be significantly simplified. It was shown in Ref. [19] that using only the Principle of General Covariance the equations for the description of particle’s spin vector $S^\mu$ and four-velocity $U^\mu = dx^\mu/d\tau$ evolution take the form,

$$\frac{DS^\mu}{D\tau} = 0,$$

$$\frac{DU^\mu}{D\tau} = 0.$$  

Here the covariant derivatives are taken with respect to the proper time $\tau$. It follows from Eqs. (2.3) and (2.4) that particle’s spin and four-velocity are parallel transported along its world-line. It should be noted that usual properties of the spin vector $S^\mu$ in Minkowski space-time, namely $S^\mu U_\mu = 0$ and $S^\mu S_\mu$ is the constant value, remain unchanged when a particle moves in curved space-time. One can verify these properties using Eqs. (2.3) and (2.4). The basic Eqs. (2.3) and (2.4) can be also rewritten with help of the Christoffel symbols of the second kind $\Gamma^\alpha{}_{\beta\gamma}$,

$$\frac{dS^\mu}{d\tau} = -\Gamma^\mu{}_{\alpha\beta} U^\alpha S^\beta,$$

$$\frac{dU^\mu}{d\tau} = -\Gamma^\mu{}_{\alpha\beta} U^\alpha U^\beta.$$  

Eq. (2.5) describes the spin evolution in a general coordinate system. However the particle’s properties are determined by the spin vector components measured with respect to the particle’s rest frame. The particle’s rest-frame-spin precession in gravitational fields was discussed in Ref. [14]. In order to proceed in our analysis of particle’s spin evolution, we should rewrite Eqs. (2.5) and (2.6) in a locally minkowskian frame. One can implement this coordinates transformation with help of the vierbein vectors $e^a{}^\mu$. We recall the basic vierbein vectors properties,

$$g_{\mu\nu} = e^a{}^\mu e^b{}^\nu \eta_{ab}, \quad \delta^\nu{}_{\nu} = e^a{}^\mu e^a{}_{\nu},$$

$$\eta_{ab} = e^a{}^\mu e^b{}^\mu, \quad \delta^a{}_{b} = e^a{}^\mu e^\mu{}_{\nu}$$

where $e^a{}^\mu = \eta_{ab} g^{\mu\nu} e^b{}^\nu$ is the inverse vierbein and $\eta_{ab} = \text{diag}(+1,-1,-1,-1)$ is the metric tensor in a locally minkowskian frame.

The components of the spin and four-velocity with respect to a locally minkowskian frame are,

$$s^a = e^a{}^\mu S^\mu, \quad u^a = e^a{}^\mu U^\mu.$$
The equations for the description of the $s^a$ and $u^a$ evolution have the form (see also Ref. [14]),

\[
\frac{ds^a}{dt} = \frac{1}{\gamma} G^{ab} s_b, \quad (2.8)
\]

\[
\frac{du^a}{dt} = \frac{1}{\gamma} G^{ab} u_b, \quad (2.9)
\]

where $G^{ab} = \eta^{ac} \eta^{bd} \gamma_{cde} u^e = - G^{ba}$ is the analog of the electromagnetic field tensor and $\gamma_{abc} = \eta_{ad} \eta^{be} \epsilon_{abc}$ are the Ricci rotation coefficients. In Eqs. (2.8) and (2.9) we keep the usual notation where $\gamma \neq 0$. The derivates in the right-handed sides of these equations are taken with respect to the laboratory time.

To make the coordinate transformation into the particle’s rest frame one uses a boost within a locally minkowskian frame. The relation between the spin vector $s^a$ and the three dimensional spin in the particle’s rest frame $\zeta$ is given by following expression,

\[
s^a = \left( \zeta u, \zeta + \frac{u(\zeta u)}{1 + u^0} \right). \quad (2.10)
\]

Here $u^0$ and $u$ are the time and space components of the four-velocity $u^a$. We remind that $u^a$ is the four velocity vector in the vierbein frame.

Using Eqs. (2.8) and (2.10) we can readily derive the equation for the description of the three dimensional spin vector evolution,

\[
\frac{d\zeta}{dt} = \frac{2}{\gamma} [\zeta \times G], \quad (2.11)
\]

where

\[
G = \frac{1}{2} \left( B + \frac{1}{1 + u^0} [E \times u] \right). \quad (2.12)
\]

In deriving of Eqs. (2.11) and (2.12) we use the fact that any antisymmetric tensor in four dimensional (minkowskian) space-time can be expressed in terms of the two three dimensional vectors (analogs of electric and magnetic fields), i.e. $G_{ab} = (E, B)$, where $G_{0i} = E_i$ and $G_{ij} = -\epsilon_{ijk} B_k$. Eqs. (2.11) and (2.12) describe particle’s spin precession in arbitrary gravitational field. These equations are linear in spin vector. One can, however, discuss further (non-linear in spin) terms contributing to the particle’s spin evolution (see Ref. [14]).

At the end of this section we mention that Eqs. (2.8) and (2.9) or alternatively Eqs. (2.11) and (2.12) are similar to the equations for the description of the spin and four-velocity evolution of a charged particle with gyromagnetic ratio $g = 2$ interacting with an external electromagnetic field (see also Ref. [14]). The spin evolution of a charged particle with $g = 2$ was studied in Refs. [21] [22]. It was shown that the spin dynamics was completely determined by the particle’s motion law, i.e. by the solution of the Lorentz equation (2.5).

III. NEUTRONI SPIN EVOLUTION IN WEAK GRAVITATIONAL FIELDS

In this section we study the particle spin precession in a weak gravitational field. We describe the neutrino spin oscillations and discuss the obtained results as well as we consider analogous approach elaborated in our previous work.

Weak gravitational fields can be found, for instance, at great distances from a finite massive object under study. In this case we can always choose the quasi-minkowskian coordinate system,

\[
y_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (3.1)
\]

where $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is the Minkowski metric tensor. The metric perturbation $h_{\mu\nu}$ has to vanish at the infinity. One of the possible examples of the metric given in Eq. (3.1) is the gravitational field created by a massive rotating object at great distances. Using post-Newtonian approximation we obtain for the components of the tensor $h_{\mu\nu}$ (see, e.g., Ref. [22]),

\[
h_{00} = 2 \varphi, \quad h_{ij} = 2 \varphi \delta_{ij}, \quad h_{0i} = - h_i, \quad (3.2)
\]

\[
\varphi = - \frac{M}{r}, \quad h = \frac{2}{r^3} [r \times J], \quad (3.3)
\]

where $M$ is the mass of the object, $J$ is its total angular momentum and $r$ is the distance from the object.

First let us consider the particle’s spin evolution when an observer is in the particle’s rest frame. For this purpose we can use directly Eq. (2.5). In our case the four dimensional spin vector is reduced to the three dimensional one, i.e. $S^\mu \rightarrow (0, \zeta)$. Eq. (2.5), written for the spatial components of the spin vector, takes the following form,

\[
\frac{d\zeta_i}{dt} = - \Gamma^i_{0j} \zeta_j, \quad (3.4)
\]

Here we assume that $U^0 = 1$ and $U = 0$ in particle’s rest frame. Note that we do not distinguish between upper and lower indexes when we use three dimensional vectors. Christoffel symbols $\Gamma^i_{0j}$ can be calculated with help of Eq. (3.3),

\[
\Gamma^i_{0j} = \frac{1}{2} \left( \frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} \right) = - \frac{1}{2} \epsilon_{ijk} [\nabla \times h]_k. \quad (3.5)
\]

Using Eqs. (3.4) and (3.5) we obtain the equation for the particle’s spin evolution in its rest frame,

\[
\frac{d\zeta}{dt} = \frac{1}{2} [\zeta \times (\nabla \times h)]. \quad (3.6)
\]

One can see that Eq. (3.6) coincides (to within the sign) with the analogous equation derived in our previous work (see Ref. [12]) if we set there $\gamma = 1$ and $\beta = 0$.

Now let us discuss the particle’s spin precession in a weak gravitational field, with an observer being placed
in the laboratory frame, i.e. a particle has the non-zero velocity with respect to him. In this case we should use Eqs. (2.11) and (2.12) to describe particle’s spin evolution. The Ricci rotation coefficients were calculated in Ref. [14] in the weak gravitational field approximation. They can be expressed in the following way,

\[ \gamma_{abc} = \frac{1}{2} \left( \frac{\partial h_{bc}}{\partial x^a} - \frac{\partial h_{ac}}{\partial x^b} \right), \]  

(3.7)

where \( h_{ab} \) and \( x^a \) are the components of the tensor \( h_{\mu\nu} \) and vector \( x^\mu \) in the vierbein frame. Note that we do not distinguish the vierbein and world indexes in the weak gravitational field approximation.

Using Eqs. (3.2) and (3.7) as well as the definition of the tensor \( G_{ab} \) we receive the expressions for the “electric” and ”magnetic” fields,

\[ E = -\gamma \nabla \varphi + \frac{\gamma}{2} v_i \nabla h_i, \quad B = \gamma [v \times \nabla] \varphi + \frac{\gamma}{2} [\nabla \times h]. \]

One can readily find the vector \( \mathbf{G} \), which determines the particle’s spin precession, in the explicit form,

\[ \mathbf{G} = \gamma \left( \frac{1}{2} [\nabla \times \mathbf{h}] + \frac{2\gamma + 1}{\gamma + 1} [v \times \nabla] \varphi \right) \]

\[ - \frac{\gamma}{2(\gamma + 1)} v_i [v \times \nabla] h_i, \]

(3.8)

where \( \gamma = (1 - v^2)^{-1/2} \). Note that if we set \( v = 0 \) in Eq. (3.8), we obtain the expression consistent with Eq. (3.6) obtained directly from Eq. (2.8). It proves the validity of the used technique.

Eqs. (2.11) and (2.12) govern particle’s spin evolution in a weak gravitational field. These equations are valid for arbitrary particle velocities. Using Eqs. (2.11) and (3.8) we can describe neutrino spin oscillations in the weak gravitational field. For example, it is possible to study the particle’s spin dynamics of a neutrino emitted in the vicinity of a rotating black hole and then propagating faraway from the massive object (see Ref. [12]). Supposing that a neutrino has the velocity directed along the \( \mathbf{e}_z \) base vector we obtain from Eq. (3.5) the expression for the effective Hamiltonian (see Ref. [15])

\[ H_{\text{eff}} = -\frac{1}{\gamma} (\sigma \mathbf{G}), \]  

(3.9)

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices.

Let us consider neutrino spin oscillations when a neutrino interacts with the gravitational field having the properties given in Eq. (3.5). Using Eqs. (3.8) and (3.9) we receive for the vector \( \mathbf{G} \) the following expression,

\[ \mathbf{G} = \gamma \left( \frac{1}{r^3} \left[ r^2 \mathbf{J} - 3r \mathbf{r} \mathbf{J} \right] \right) - \frac{\gamma}{\gamma + 1} \left\{ \frac{1}{r^3} \left[ v^2 \mathbf{J} - v (v \mathbf{J}) \right] - \frac{3}{r^3} [v \times \mathbf{r}] (v [v \times \mathbf{J}]) \right\} \]

\[ + \frac{2\gamma + 1}{\gamma + 1} \frac{M}{r^3} [v \times \mathbf{r}]. \]  

(3.10)

It can be seen that this expression for the vector \( \mathbf{G} \) agrees with the analogous formula obtained in our previous work [12] only for slow neutrinos having \( v \ll 1 \) (see also above). This discrepancy results from the incorrect account of the spin-orbital interaction in Ref. [12]. However, if we study only the radial neutrino motion, the main contribution to the neutrino spin evolution comes from the interaction with the angular momentum of the massive object and we reach an agreement with our recent work.

We consider therefore a neutrino propagating along the radius, i.e. \( \mathbf{v} = v (\mathbf{r}/r) \). In this case we can rewrite Eq. (3.10) in the form,

\[ \mathbf{G} = \frac{\gamma}{2r^3} \left\{ \frac{1}{\gamma} \mathbf{J} - n (Jn) \right\} \left( 2 + \frac{1}{\gamma} \right), \]  

(3.11)

where \( n \) is the unit vector along the neutrino velocity. One can see that the vector \( \mathbf{G} \) given in Eq. (3.11) coincides with that derived in our previous work (see Ref. [12]). Thus all results concerning the neutrino spin light obtained in Ref. [12] are valid only for the radial neutrino motion.

Now let us write down the effective Hamiltonian for neutrino spin oscillations in our case. We can always choose the coordinate system so that,

\[ \mathbf{n} = (0, 0, 1), \quad \mathbf{J} = (J_1, 0, J_3), \]

(3.12)

\[ J_1 = J \sin \vartheta, \quad J_3 = J \cos \vartheta, \]

where \( \vartheta \) is the angle between vectors \( \mathbf{n} \) and \( \mathbf{J} \). Using Eqs. (3.10), (3.11) and (3.12) we obtain the expression for the neutrino spin oscillations effective Hamiltonian,

\[ H_{\text{eff}} = \frac{J}{r^3} \left( \cos \vartheta \sin \vartheta / (2\gamma) \right). \]  

(3.13)

Here \( \mathbf{r} = \mathbf{r}_0 + vt \), where \( \mathbf{r}_0 \) is the initial neutrino coordinate. It follows from Eq. (3.13) that transitions between different polarization states of a neutrino are suppressed by the factor \( 1/\gamma \) which is small for ultrarelativistic neutrinos. The most intensive neutrino spin oscillations take place when a neutrino is propagating in the direction perpendicular to the vector \( \mathbf{J} \) (\( \vartheta = \pi/2 \)). There are no oscillations at all when a neutrino is propagating along the vector \( \mathbf{J} \) (\( \vartheta = 0 \)).
The transition probability for the $\nu_L \leftrightarrow \nu_R$ oscillations can be obtained with help of the effective Hamiltonian given in Eq. (3.13). Let us suppose that $\nu_R(0) = 0$ and $\nu_L(0) = 1$, then we receive the expression for the probability to find a right-handed neutrino in this neutrino beam,

$$P(r) = \frac{\sin^2 \vartheta}{4 \gamma^2 \cos^2 \vartheta + \sin^2 \vartheta} \sin^2 [\Phi(r)],$$

where

$$\Phi(r) = \frac{J}{4 \sqrt{\gamma^2 - 1}} \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) \times \sqrt{4 \gamma^2 \cos^2 \vartheta + \sin^2 \vartheta}.$$  

Here $r_0$ is the distance between a neutrino and the center of the massive object at $t = 0$. Eqs. (3.14) and (3.15) coincide with the similar expression for the transition probability derived in our previous work [12].

IV. NEUTRINO SPIN OSCILLATIONS IN SCHWARZSCHILD METRIC

General equations derived in Sec. II allow one to describe particle’s spin evolution not only in weak gravitational fields. In this section we discuss the neutrino spin evolution and oscillations when a particle interacts with a non-rotating black hole in the case when a neutrino propagates even in the vicinity of a black hole.

When we study the gravitational field of a non-rotating black hole, the interval is known to be expressed with help of the spherical coordinates,

$$dr^2 = A^2 dt^2 - A^{-2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$A = \sqrt{1 - \frac{r_g}{r}},$$

and $r_g$ is the Schwarzschild radius. In Eq. (4.1) we use the spherical coordinate system for the world (not for the vierbein) coordinates. We can decompose the four-velocity $U^\mu$ with help of the spherical coordinates,

$$U^\mu = (U^0, U_r, U_\theta, U_\phi).$$

In order to construct the ”electric” and ”magnetic” fields we should choose the appropriate vierbein vectors. One can verify that the following vectors, one can verify that the following vectors,

$$e^0_\mu = (A, 0, 0, 0),$$
$$e^1_\mu = (0, A^{-1}, 0, 0),$$
$$e^2_\mu = (0, 0, r, 0),$$
$$e^3_\mu = (0, 0, 0, r \sin \theta),$$
satisfy the general properties of the vierbein vectors given in Eq. (2.7).

Using Eqs. (4.2)- (4.7) we can find vierbein components of the vector $u^a$,

$$u^a = (\gamma A, U_r A^{-1}, U_\theta r, U_\phi r \sin \theta).$$

With help of Eqs. (A3)-(A7) as well as using the definition of the tensor $G_{ab}$ one can write down the components of the ”electric” and ”magnetic” fields,

$$E = \left(-\frac{r_g}{2r^2}, 0, 0 \right),$$
$$B = (U_\phi \cos \theta, -U_r A \sin \theta, U_\theta A).$$

Neutrino spin precession is determined by the vector $\Omega = G/\gamma$. The components of this vector can be found on the basis of Eqs. (2.12) and (4.8),

$$\Omega_1 = \frac{1}{2} \nu_\phi \cos \theta,$$
$$\Omega_2 = \nu_\phi \sin \theta \left( -A + \frac{\gamma}{1 + \gamma A} \frac{r_g}{2r} \right),$$
$$\Omega_3 = \frac{1}{2} \nu_\phi \left( A - \frac{\gamma}{1 + \gamma A} \frac{r_g}{2r} \right),$$

where $\nu = (\nu_r, \nu_\theta, \nu_\phi)$ are the components of the world (not the vierbein) velocity. In deriving of Eqs. (4.9)- (4.11) we use the fact that $U = \gamma v$.

Let us discuss a neutrino orbiting a black hole. For simplicity we consider only circular orbits with the radius $R$. We may restrict ourselves to the studying of the orbits lying only in the equatorial plane ($\theta = \pi/2$ or equivalently $\nu_\theta = 0$) because of the spherical symmetry of the gravitational field. In this case $\Omega_1 = 0$ and $\Omega_3 = 0$ in Eqs. (4.9) and (4.11). The expressions for the neutrino angular velocity and $\gamma$ are presented in Ref. [23].

$$v_\phi = \frac{d\phi}{dt} = \sqrt{\frac{r_g}{2R^3}},$$

and

$$\gamma^{-1} = \frac{d\tau}{dt} = \sqrt{1 - \frac{3r_g}{2R}}.$$  

It should be noted that the vierbein four velocity now takes the form, $u^a = (\gamma A, 0, 0, \gamma v_\phi r)$. One can verify that $u^a u_a = 1$ with help of Eqs. (1.2), (1.12) and (4.13). We also mention that Eq. (2.9) is also identically satisfied. Indeed $G_{ab} u^b = 0$ since $E_1 u^0 = B_2 u^3$ and hence $du^a/d\tau = 0$. Therefore a neutrino has constant four velocity with respect to the vierbein frame. Using Eqs. (1.2), (1.10) and (4.13) we can rewrite $\Omega_2$ in the more simple form,

$$\Omega_2 = -\gamma^{-1} v_\phi \frac{\dot{\nu}_\phi}{2}.$$ 

This equation has very clear physical meaning. It is well known that the spin precession of a charged particle with
g = 2 in the external electromagnetic field is completely determined by the particle’s motion law (see Sec. II or Ref. 20). Therefore Eq. (1.14) is nothing else as the Lorentz transformation of the spin rotation frequency from the rest frame to the laboratory frame.

We suppose that initially a neutrino is left-handed, i.e. its initial spin vector is antiparallel to the particle’s velocity. According to Eqs. (2.11) and (1.9)-(1.11) the neutrino spin rotates around the second axis. Therefore using Eq. (2.11) we can construct the effective Hamiltonian for the neutrino spin oscillations in the gravitational field of a non-rotating black hole,

\[ H_{\text{eff}} = \begin{pmatrix} 0 & -i\Omega_2 \\ i\Omega_2 & 0 \end{pmatrix}. \]  (4.15)

It is interesting to note that in Eq. (4.15) the parameter \(|\Omega_2|\neq0\) if \(\gamma^{-1}\neq0\) (see below). One can see it directly in Eq. (4.14). Using Eq. (4.15) we write the expression for the neutrino transition probability,

\[ P(t) = \sin^2(\Omega_2 t). \]  (4.16)

We can see that there is a full mixing in our case and neutrino transition probability can achieve a unit value.

It is possible to describe the neutrino spin evolution using formalism elaborated in Ref. 21 where the the Michel-Telegdi equations was described. It follows from using formalism elaborated in Ref. [21] where the the neutrino transition probability can achieve a unit value.

It is possible to describe the neutrino spin evolution using formalism elaborated in Ref. [21] where the the Michel-Telegdi equations was described. It follows from the results of that work that the spin dynamics is completely determined by the particle’s motion law when \(g = 2\). Eq. (4.9) can be rewritten in the equivalent form,

\[ \frac{d}{dt} \Lambda = FA, \quad F = \frac{i}{2}\sigma(B-iE), \]

and the operator \(\Lambda\) implements the shift of the particle’s velocity along the trajectory. We present the main properties of the \(SL(2, C)\) group in Appendix B. If one has found the operator \(\Lambda\), the resolvent of the Eq. (2.8) takes the form \((g = 2)\),

\[ R(\tau, \tau_0) = L^{-1}(\tau)\Lambda L(\tau_0), \]  (4.17)

where the matrix \(L\) implements the boost from the rest frame to the laboratory frame.

If we consider the neutrino motion in the equatorial plane from the very beginning, the operator \(\Lambda\) is expressed in the following way,

\[ \Lambda = \exp(E\tau) = \begin{pmatrix} \cos \alpha t & \alpha_+ \sin \alpha t \\ -\alpha_- \sin \alpha t & \cos \alpha t \end{pmatrix}, \]  (4.18)

where \(\alpha = \sqrt{B^2 - E^2}/(2\gamma)\) and

\[ \alpha_\pm = \frac{E \pm B}{\sqrt{B^2 - E^2}}. \]

Note that using Eqs. (4.2), (4.8), (4.12) and (4.13) we can show that \(\alpha = v_\phi/(2\gamma)\). It can be verified that \(\Lambda u\Lambda^\dagger = u\), where

\[ u = \begin{pmatrix} u_0 + u^3 & 0 \\ 0 & u_0 - u^3 \end{pmatrix}, \]

i.e. a neutrino moves along a strait line (in the veirbein frame). This fact also agrees with the result obtained above in present paper. The operator \(L\) does not depend on time since the neutrino’s four velocity is constant. This matrix is expressed in the following way (see Appendix B),

\[ L = \begin{pmatrix} \sqrt{u_0^2 + u^3} & 0 \\ 0 & \sqrt{u_0^2 - u^3} \end{pmatrix}. \]  (4.19)

Using Eqs. (4.17)-(4.19) we receive the resolvent \(R(t)\) in the form (here we assume that \(t_0 = 0\)),

\[ R(t) = \begin{pmatrix} \cos \alpha t & \sin \alpha t \\ -\sin \alpha t & \cos \alpha t \end{pmatrix}. \]  (4.20)

In order to find the spin rotation axis one adopts the unit vector \(k\) along this axis, so that \([R, \sigma_k] = 0\). Here use the fact that \(R \in SO(3)\) and \(\Gamma^l = R^{-1}\). It can be easily derived with help of Eq. (4.20) that \([R, \sigma_\mu] = 0\). Thus the spin precession axis is the second axis in the complete agreement with the above obtained results.

One can also plot the frequency of neutrino spin oscillations versus the radius of the orbit. It is possible to see on Fig. 1 that \(|\Omega_2| = 0\) at \(R = 1.5r_g\) and \(|\Omega_2| \rightarrow 0\) at \(R \rightarrow \infty\). One can conclude that \(|\Omega_2|\) has its maximal value (equal to \(6.25 \times 10^{-2}r_g^{-1}\)) at \(R = 2r_g\) [see also Eq. (4.18)]. Let us evaluate the number of revolutions \(N\), that a neutrino should make, necessary for the total spin flip. Using Eqs. (4.12) and (4.13) we can easily find for \(N\),

\[ N = \frac{v_\phi}{2|\Omega_2|} = \gamma|_{R=2r_g} = 2. \]

It is interesting to evaluate the characteristic period of the neutrino spin oscillations (oscillations length). For \(M = 10M_\odot\) at \(R = 2r_g\) we get for \(T = \pi/|\Omega_2| \approx 4.94 \times 10^{-5}\) s.

![FIG. 1: Neutrino spin oscillations frequency versus the radius of the neutrino orbit.](image)
We can discuss another orbit, namely $R = 1.5r_g$. It follows from Eq. (4.14) (see also Fig. 1) that the frequency of the transitions between two polarization states vanishes. It should be noted that $dt' = 0$ at $R = 1.5r_g$ [see Eq. (4.13)]. Therefore this orbit corresponds to massless particles. We reveal that the property of a massless particle in the Minkowski space-time to keep unchanged its spin direction with respect to the momentum remains unaffected in the Schwarzschild metric. It is interesting to note that the absence of the spin rotation for massless particles is valid only for the Schwarzschild metric. It was shown in Refs. [24, 25] that for a background metric with at least one off-diagonal space component, e.g. the Kerr metric, in the limit $m_e \rightarrow 0$ still there are transitions between neutrino and antineutrino, i.e. neutrino spin flip. This phenomenon is used in Refs. [24, 26] to explain the neutrino-antineutrino asymmetry problem.

It is also worth examining the behaviour of $\Omega_2$ for large orbits. If a neutrino moves on a remote orbit, the gravitational field is weak. Thus we again do not distinguish between world and vierbein indexes. We define the coordinate system so that the angular velocity is expressed in the following way, $\omega_{\text{frame}} = v_\phi e_z$, where $e_z$ is the unit vector. Using Eq. (4.17) in the weak field limit we obtain for the spin rotation frequency,

$$\omega_{\text{abs}} = -2\Omega_2 e_z = v_\phi \left( 1 - \frac{3}{4} \frac{r_g}{R} + \ldots \right) e_z,$$

since the second vierbein coordinate axis is nothing else as the $\theta$-axis and for the orbit in the equatorial plane we have $e_\theta = -e_z$. Note that the effective Hamiltonian in Eq. (4.14) determines the evolution of the neutrino rest-frame-spin vector. However an observer is in the laboratory frame ($t'$ is time in the laboratory frame). Thus $\Omega_2$ is the oscillations frequency measured in the laboratory frame. In order to obtain the spin rotation frequency measured by an observer in the co-moving frame we should subtract $\omega_{\text{frame}}$ from $\omega_{\text{abs}}$ since $\omega_{\text{frame}}$ is the angular velocity of the orbital motion, i.e. the angular velocity of the rest frame rotation with respect to the laboratory frame. Finally we get for $\omega_{\text{rel}} = \omega_{\text{abs}} - \omega_{\text{frame}}$ the following expression,

$$\omega_{\text{rel}} = -2\Omega_2 e_z = v_\phi \left( 1 - \frac{3}{4} \frac{r_g}{R} + \ldots \right) e_z,$$

which agrees with the classical result for the spin rotation frequency in the co-moving frame (see, e.g., Ref. [27] or Eq. (3.10) in the limit $\gamma \rightarrow 1$).

At the end of this section we briefly examine the applicability of the quasi-classical approach to the description of the neutrino spin evolution in the gravitational field of a black hole. Basing on the similarity between particle spin precession in gravitational and electromagnetic fields we can apply the results of Ref. [28] where the quasi-classical approximation for the electron spin precession in an external magnetic field was considered. It was found in that paper that one can use the quasi-classical limit if the corresponding inequality (we rewrite it for our purposes) is satisfied,

$$\frac{\hbar}{2\mathcal{E}} \left| \frac{d\xi}{dt'} \right| \ll 1 \quad (4.21)$$

where $\mathcal{E}$ is the neutrino energy in the vierbein frame, the derivative is also taken with respect to the vierbein time $t'$. We again study the case of circular orbits. For the neutrino energy one finds, $\mathcal{E} = m_\nu u^0 = m_\nu \gamma A$, where $m_\nu$ is the neutrino mass. Using Eq. (4.4) we obtain that $dt' = Adt$. Let us discuss an electron neutrino with mass $m_\nu \approx 2\text{eV}$ rotating around a black hole with $M = 10M_\odot$ on the orbit with radius $R = 2r_g$. Then Eq. (4.21) reads,

$$\frac{\hbar}{2\mathcal{E}} \left| \frac{d\xi}{dt'} \right| = \frac{\hbar}{2m_\nu \gamma^2 A^2} v_\phi \approx 4.19 \times 10^{-13} \ll 1.$$  

We can see that the condition of the validity of the quasi-classical approach is satisfied almost in all reasonable astrophysical objects.

V. CONCLUSION

In conclusion we note that neutrino spin oscillations in gravitational fields within the quasi-classical approach have been studied. We have started from the particle spin evolution equation in the vierbein frame. On the the basis of this approach we have investigated neutrino spin oscillations in a weak gravitational field created by a massive rotating object. The three dimensional spin evolution equation has been obtained. We have improved analogous calculation performed in our previous work [12]. The neutrino spin evolution equation derived in the present work is valid for arbitrary neutrino velocities and correctly takes into account the neutrino velocity dependence. Then we have written down the neutrino oscillations effective Hamiltonian [Eq. (4.11)] and transition probability [Eqs. (4.14) and (4.15)] when a neutrino propagates along the radial direction. The neutrino spin conversion rate has been analyzed for different neutrino velocity directions. Then we have examined neutrino spin oscillations in the gravitational field of a non-rotating black hole. We have studied a neutrino moving on circular orbits, with our calculations being valid for the neutrino motion in the vicinity of a black hole. We have derived the effective Hamiltonian [Eq. (4.16)] and transition probability [Eq. (4.18)] in this case. It has been demonstrated that neutrino spin oscillations occur in the Schwarzschild metric. We have also analyzed the dependence of the neutrino spin oscillations frequency on the radius of the orbit. The validity of the quasi-classical approach to the description of neutrino spin oscillations in a gravitational field has been analyzed in our work.

It should be noted that the method used in the present paper for the studying of neutrino spin oscillations is valid not only for neutrinos but also for any spin-1/2 particles. The basic equations (2.11) and (2.12) were demonstrated in Ref. [29] to be applicable to a massive
Dirac particle in a weak static gravitational field. Therefore we can study, e.g., electron spin evolution with help of this approach. It is known that rather frequently massive gravitational objects have inherent magnetic fields and are surrounded by the moving matter (accretion disks). Thus, if we discuss the particle spin oscillations process comprehensively, i.e. take into account all factors, the cases of neutrinos and electrons are different because these particles have diverse types of the interaction with magnetic field and with the background matter. In principle massless particles can be also included into consideration (see Sec. IV). Despite of the fact that the mass of a particle is absent in Eqs. 2.11 and 2.12, the law of motion (i.e. the function \( u^\alpha(r,t) \), which is involved in the spin evolution equations) of massless particles differs from that of massive particles. However in the case of massless particles the helicity and chirality are the same and one can apply the methods used in the recent work 26.

According to Eqs. 2.21-2.11 there is no neutrino spin flip in case of radial neutrino motion in the Schwarzschild metric. This our result agrees with the deduction of Ref. 6 where the radial propagation of neutrinos from active galactic nuclei was studied. However one can hardly agree with the claims of Ref. 9 that a spherically symmetric, static Schwarzschild space-time cannot cause the particle spin flip. That statement confront both our results and the analysis of Ref. 26 in which it was shown by means of direct calculations that in the Schwarzschild metric the spin of a Dirac fermion can precess and change its direction with respect to the particle momentum. The calculations in that paper were based on Dirac equation in curved space. That result also agrees with the statement of our paper that the gravitational field of a non-rotating black hole can cause neutrino spin oscillations.

Note that the results of our paper can be applied to the description of the neutrino spin evolution in various astrophysical media. It is known that relic massive neutrinos can cluster into a halo around a galaxy contributing to cold dark matter. Despite of the fact that the average size of a halo ranges between \( 10^2 \) and \( 10^3 \) kpc for a typical galaxy like the Milky Way (see, e.g., Ref. 21), there is a possibility for a neutrino to be gravitationally captured on orbits close to the central massive object. In this case strong gravitational field can essentially contribute to the neutrino spin oscillations process.

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**APPENDIX A: COVARIANT DERIVATIVES OF THE VIERBEIN VECTORS IN SCHWARZSCHILD METRIC**

The covariant derivatives of the vierbein vectors enter in the definition of the tensor \( G_{ab} \). In this appendix we present the main formulae necessary for the covariant derivatives calculation. The covariant derivative of \( e_{a\mu} \) is equal to

\[
\nabla_{\nu} e_{a\mu} = \partial e_{a\mu} / \partial x^\nu - \Gamma^\lambda_{\mu\nu} e_{a\lambda},
\]

where \( \Gamma^\lambda_{\mu\nu} \) are the Christoffel symbols. The non-zero Christoffel symbols for the Schwarzschild metric are (see, e.g., Ref. 31),

\[
\Gamma^r_{rr} = \frac{r_g}{2r(r_g - r)}, \quad \Gamma^r_{\theta\theta} = r_g - r, \quad \Gamma^r_{tt} = \frac{r_g(r_g - r)}{2r^3}, \quad \Gamma^\theta_{\theta\theta} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\phi} = \frac{1}{r}, \quad \Gamma^t_{rr} = \frac{r_g}{2r(r_g - r)}.
\]

Using Eqs. (A1)-(A2) and (A3) we obtain the expressions for the covariant derivatives in the following form,

\[
e_{1\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{ (A4)}
\]

\[
e_{2\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -r \sin^2 \theta & 0
\end{pmatrix}, \quad \text{ (A5)}
\]
where the first index ($\mu$) numbers lines and the second one ($\nu$) numbers the rows of the matrix.

It is convenient to rewrite the definition of the tensor $G_{ab}$ in the following way,

$$G_{ab} = e_{a\mu} e_{b\nu} U^\nu. \quad (A7)$$

Here we used the properties of the vierbein vectors [see Eq. (2.7)]. Eq. (A7) is more convenient for the further calculations since it allows one to express the tensor $G_{ab}$ directly in terms of the components of the four vector $U^\nu$ rather than $u^\mu$.

APPENDIX B: UNIVERSAL COVERING OF THE Lorentz GROUP

It is well known that $SL(2, \mathbb{C})$ is the universal covering group for the orthochronous proper Lorentz subgroup $L_+^1$. (det $\Lambda = 1$ and $\Lambda^0_0 > 1$ for $\Lambda \in L_+^1$). In this appendix we briefly discuss the local isomorphism between these groups (see also Ref. [32]).

For any element $x^\mu$ of the minkowskian space we set the matrix $x = x^\mu e_\mu$, where $e_0$ is the $2 \times 2$ unit matrix and $e = \sigma$. The Lorentz transformation acts on $x$ as $A x = x \Lambda \Lambda^\dagger$, where $\Lambda^\dagger$ is the Hermitian conjugate of the matrix $\Lambda$. For instance, the Lorentz transformation of the form,

$$y^0 = x^0 \cosh \chi + (x^1) \sinh \chi,$n
$$y = x - (x^1) t + [(x^1) \cosh \chi + x^0 \sinh \chi],$$

corresponds to the matrix $H(1, \chi) = \exp [\chi (\sigma 1)/2]$. Here $1$ is the three dimensional unit vector and $\chi$ is the real parameter.

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