KLEINIAN GROUPS WHICH ARE ALMOST FUCHSIAN

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ABSTRACT. We consider the space of all quasifuchsian metrics on the product of a surface with the real line. We show that, in a neighborhood of the submanifold consisting of fuchsian metrics, every non-fuchsian metric is completely determined by the bending data of its convex core.

Let $S$ be a surface of finite topological type, obtained by removing finitely many points from a compact surface without boundary, and with negative Euler characteristic. We consider complete hyperbolic metrics on the product $S \times ]-\infty, \infty[$.

The simplest ones are the fuchsian metrics defined as follows. Because of our hypothesis that the Euler characteristic of $S$ is negative, $S$ admits a finite area hyperbolic metric, for which $S$ is isometric to the quotient of the hyperbolic plane $\mathbb{H}^2$ by a discrete group $\Gamma$ of isometries. The group $\Gamma$ uniquely extends to a group of isometries of the hyperbolic 3–space $\mathbb{H}^3$ respecting the transverse orientation of $\mathbb{H}^2 \subset \mathbb{H}^3$, for which the quotient $\mathbb{H}^3/\Gamma$ has a natural identification with $S \times ]-\infty, \infty[$. A fuchsian metric is any metric on $S \times ]-\infty, \infty[$ obtained in this way. Note that the image of $\mathbb{H}^2$ in $\mathbb{H}^3$ provides in this case a totally geodesic surface in $S \times ]-\infty, \infty[$, isometric to the original metric on $S$.

These examples can be perturbed to more complex hyperbolic metrics on $S \times ]-\infty, \infty[$. See for instance [Th1], [Mas]. A quasifuchsian metric on $S \times ]-\infty, \infty[$ is one which is obtained by quasi-conformal deformation of a fuchsian metric. Equivalently, a quasifuchsian metric is a geometrically finite hyperbolic metric on $S \times ]-\infty, \infty[$ whose cusps exactly correspond to the ends of $S$. These also correspond to the interior points in the space of all hyperbolic metrics on $S \times ]-\infty, \infty[$ for which the ends of $S$ are parabolic, [Mas], [Su].

If $m$ is a quasifuchsian metric on $S \times ]-\infty, \infty[$, the totally geodesic copy of $S$ which occurred in the fuchsian case is replaced by the convex core $C(m)$, defined as the smallest non-empty closed $m$–convex subset of $S \times ]-\infty, \infty[$. If $m$ is not fuchsian, $C(m)$ is 3–dimensional and its boundary consists of two copies of $S$, each facing an end of $S \times ]-\infty, \infty[$. The geometry of $\partial C(m)$ was investigated by Thurston [Th]; see also [EpM]. The component of $\partial C(m)$ that faces the end $S \times \{+\infty\}$ is a pleated surface, totally geodesic almost everywhere, but bent along a family of simple geodesics; this bending is described and quantified by a measured geodesic lamination $\beta^+(m)$ on $S$. Similarly, the bending of the negative component of $\partial C(m)$, namely the one facing $S \times \{-\infty\}$, is determined by a measured geodesic lamination $\beta^-(m)$.

If $Q(S)$ denotes the space of isotopy classes of quasifuchsian metrics on $S \times ]-\infty, \infty[$ and if $\mathcal{ML}(S)$ is the space of measured geodesic laminations on $S$, the
rule \( m \mapsto (\beta^+(m), \beta^-(m)) \) defines a map \( \beta : \mathcal{Q}(S) \to \mathcal{ML}(S) \times \mathcal{ML}(S) \). By definition, \( \beta(m) = (0, 0) \) is the metric \( m \) is fuchsian, in which case the convex core \( C(m) \) is just a totally \( m \)-geodesic copy of \( S \). The image \( \beta(m) \in \mathcal{ML}(S)^2 \), interpreted as a measured geodesic lamination on two copies of \( S \), is the bending measured geodesic lamination of the quasifuchsian metric \( m \).

The space \( \mathcal{Q}(S) \) is a manifold of dimension \( 2\theta \) where, if \( \chi(S) \) is the Euler characteristic of \( S \) and \( p \) is its number of ends, \( \theta = -3\chi(S) - p \geq 0 \). It contains the space \( \mathcal{F}(S) \) as a proper submanifold of dimension \( \theta \). To some extend, the bending measured lamination \( \beta(m) \) measures how far the metric \( m \in \mathcal{Q}(S) \) is from being fuchsian. Finally, recall that the space \( \mathcal{ML}(S) \) of measured geodesic laminations is a piecewise linear manifold of dimension \( \theta \).

Thurston conjectured that the restriction of the bending map to \( \beta \) could be used to parametrize \( \mathcal{Q}(S) \to \mathcal{F}(S) \), namely that it induces a homeomorphism between \( \mathcal{Q}(S) \to \mathcal{F}(S) \) and an open subset of \( \mathcal{ML}(S)^2 \). The image of \( \beta \) was determined in [BoO].

The goal of the current paper is to prove Thurston’s conjecture on a neighborhood of the space of Fuchsian metrics.

**Theorem 1.** There exists an open neighborhood \( V \) of the fuchsian submanifold \( \mathcal{F}(S) \) in \( \mathcal{Q}(S) \) such that the bending map \( \beta : \mathcal{Q}(S) \to \mathcal{ML}(S)^2 \) induces a homeomorphism between \( V - \mathcal{F}(S) \) and its image.

There are well-known restrictions for \( (\mu, \nu) \in \mathcal{ML}(S)^2 \) to be in the image of \( \beta \); see for instance [BoO]. In particular, if \( (\mu, \nu) \neq (0, 0) \) is the bending measured lamination of some \( m \in \mathcal{Q}(S) \), then the measured geodesic laminations \( \mu \) and \( \nu \) must fill up the surface \( S \), in the sense that every non-trivial measured geodesic lamination has non-zero geometric intersection number with at least one of \( \mu, \nu \). This is equivalent to the condition that every component of \( S - \mu \cup \nu \) is, either a topological disk bounded by the union of finitely many geodesic arcs, or a topological annulus bounded on one side by the union of finitely many geodesic arcs and going to a cusp on the other side.

Let \( \mathcal{FML}(S) \) denote the open subset of \( \mathcal{ML}(S)^2 \) consisting of those \( (\mu, \nu) \) where \( \mu \) and \( \nu \) fill up \( S \). Note that \( \mathcal{FML}(S) \) is endowed with an action of \( \mathbb{R}^+ \), defined by \( t(\mu, \nu) = (t\mu, t\nu) \), which decomposes \( \mathcal{FML}(S) \) as the union of pairwise disjoint rays \((=\text{orbits}) \] 0, \( \infty \] \( (\mu, \nu) \).

**Theorem 2.** In Theorem 1, the neighborhood \( V \) of \( \mathcal{F}(S) \) and its image \( U = \beta(V) \) can be chosen so that \( U - \{ (0, 0) \} \) is an open subset of \( \mathcal{FML}(S) \) which intersects each ray \( ]0, \infty [ (\mu, \nu) \) in an interval \( ]0, \varepsilon_{\mu\nu} [ (\mu, \nu) \).

Theorems 1 and 2 are proved later as Theorem 3. The main idea of the proof is to construct an inverse \( \beta^{-1} : U - \{ (0, 0) \} \to V - \mathcal{F}(S) \), and splits into two steps: an infinitesimal part, and a transversality argument based on the infinitesimal part. The infinitesimal part is now relatively classical; see for instance [Sel]. There are restrictions on which bending data can be realized by an infinitesimal deformation of \( m_0 \in \mathcal{F}(S) \). Through the complex structure of \( \mathcal{Q}(S) \), where multiplication by \( i = \sqrt{-1} \) converts bending to shearing, these restrictions can be expressed in a purely 2–dimensional context; see Section 3. The main part of the proof is to show by a transversality argument that any infinitesimal bending data can actually be realized by a deformation. The only significant idea of the paper is to apply the transversality argument, not in \( \mathcal{Q}(S) \) where the necessary hypotheses are not
realized, but in the manifold-with-boundary $\mathcal{Q}(S)$ obtained by blowing up $\mathcal{Q}(S)$ along the fuchsian submanifold $\mathcal{F}(S)$.

Switching to the blow-up manifold $\mathcal{Q}(S)$ actually provides a better understanding of the restriction of $\beta$ on a neighborhood of $\mathcal{F}(S)$ and of its inverse. See Theorem 3 for a precise statement.

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1. The earthquake section

We consider the Teichmüller space $T(S)$, namely the space of isotopy classes of finite area complete hyperbolic metrics on the surface $S$. Recall that $T(S)$ is diffeomorphic to $\mathbb{R}^\theta$, where $\theta = -3\chi(S) - p \geq 0$ if $\chi(S)$ is the Euler characteristic of $S$ and $p$ is its number of ends.

A standard deformation of a metric $m \in T(S)$ is the left earthquake $E_m^\mu \in T(S)$ along the measured geodesic lamination $\mu$, as constructed in [Th2, Ke1, EpM]. We consider the infinitesimal left earthquake vector $e_m^\mu = \frac{d}{dt}E_m^\mu |_{t=0} \in T_m T(S)$. This provides a section $e^\mu : T(S) \to T T(S)$ of the tangent bundle of $T(S)$, defined by $m \mapsto e_m^\mu$.

There similarly exists a right earthquake $E_m^{-\mu}$ along $\mu$; the notation is justified by the fact that $m \mapsto E_m^{-\mu}$ is the inverse of $m \mapsto E_m^\mu$. We can then consider the infinitesimal right earthquake vector $e_m^{-\mu} = \frac{d}{dt}E_m^{-\mu} |_{t=0} \in T_m T(S)$ and the corresponding section $e^{-\mu} : T(S) \to T T(S)$ of the tangent bundle of $T(S)$. Note that $e_m^{-\mu} = -e_m^\mu$, but it will be convenient to keep a separate notation.

Recall that the measured geodesic laminations $\mu, \nu \in ML(S)$ fill up the surface $S$ if every non-trivial measured geodesic lamination has non-zero geometric intersection number with at least one of $\mu, \nu$. This is equivalent to the condition that every component of $S - \mu \cup \nu$ is, either a topological disk bounded by the union of finitely many geodesic arcs, or a topological annulus bounded on one side by the union of finitely many geodesic arcs and going to a cusp on the other side.

Proposition 3. Let $\mu, \nu \in ML(S)$ be two non-zero measured geodesic laminations. The intersection of the two sections $e^\mu, e^{-\nu} : T(S) \to T T(S)$ of the tangent bundle of $T(S)$ is transverse. These sections meet in exactly one point if $\mu$ and $\nu$ fill up the surface $S$, and are otherwise disjoint.

Proof. We first translate the problem in terms of the length functions $l_\mu, l_\nu : T(S) \to \mathbb{R}$ which to a metric $m \in T(S)$ associate the $m$–lengths of the measured geodesic lamination $\mu$ and $\nu$. The Weil-Petersson symplectic form on $T(S)$ induces an isomorphism between its tangent bundle $TT(S)$ and its cotangent bundle $T^* T(S)$. A celebrated result of Scott Wolpert [Wo1] asserts that this isomorphism sends the section $e^\mu$ of $TT(S)$ to the section $dl_\mu$ of $T^* T(S)$. Therefore, Proposition 3 is equivalent to showing that the sections $dl_\mu$ and $-dl_\nu$ transversely meet in 1 or 0 point, according to whether $\mu$ and $\nu$ fill up the surface $S$ or not.

First consider the case where $\mu$ and $\nu$ fill up $S$. The intersection of the section $dl_\mu$ and $-dl_\nu$ of $T^* T(S)$ correspond to the points $m \in T(S)$ where $dl_\mu = -dl_\nu$, namely to the critical points of the function $l_\mu + l_\nu : T(S) \to \mathbb{R}$. It is proved in
Lemma 4.

Let \( u = d_{m_0} l_\mu = -d_{m_0} l_\nu \in T'_{m_0} \mathcal{T}(S) \). The intersection of the tangent spaces of the sections \( dl_\mu \) and \( -dl_\nu \) at \( u \) consists of all the vectors of the form \( T_{m_0}(dl_\mu)(v) = T_{m_0}(-dl_\nu)(v) \) for some \( v \in T_{m_0} \mathcal{T}(S) \), where \( T_{m_0}(dl_\mu), T_{m_0}(-dl_\nu) : T_{m_0} \mathcal{T}(S) \to T_u T^* \mathcal{T}(S) \) denote the tangent maps of the sections \( dl_\mu, -dl_\nu : \mathcal{T}(S) \to T^* \mathcal{T}(S) \).

Recall that the fibers \( \mathcal{T}(S) \) are sections to the space \( S \) at the point \( u \).

A consequence of this analysis is that, if \( \mu \) and \( \nu \) fill up the surface \( S \), then there is a unique critical point of the length function \( l_\mu + l_\nu \). By an (easy) infinitesimal version of [Th2] (see also [Ke2]), an infinitesimal left earthquake completely determines the measured geodesic lamination along \( \mu \) and \( \nu \).

We now consider the case where \( \mu \) and \( \nu \) do not fill up the surface. In this case, the function \( l_\mu + l_\nu \) has no critical point. Let \( \kappa(\mu, \nu) \) denote the (unique) critical point of the length function \( l_\mu + l_\nu : \mathcal{T}(S) \to \mathbb{R} \). As above, \( \kappa(\mu, \nu) \) is also the unique \( m \in \mathcal{T}(S) \) such that \( e_m^\mu = e_{-m}^\nu \).

Lemma 4. If \( \kappa(\mu', \nu) = \kappa(\mu, \nu) \), then \( \mu' = \mu \).

Proof. By an (easy) infinitesimal version of [Th2] (see also [Ke2]), an infinitesimal left earthquake completely determines the measured geodesic lamination along which it is performed. If \( e_m^\mu = e_m^\nu = e_n^\mu \), it follows that \( \mu' = \mu \).

We will need a result similar to Proposition 3 in the unit tangent bundle \( T^1 \mathcal{T}(S) \). Recall that the fibers \( T^1_0 \mathcal{T}(S) \) of this bundle are the quotient of \( T_m \mathcal{T}(S) \) by \( \{0\} \) under the equivalence relation which identifies \( v \) to \( tv \) when \( t \in |0, \infty| \). In particular, \( T^1 \mathcal{T}(S) \) is a manifold of dimension \( 2\theta - 1 \). Let \( \overline{\mathcal{T}}^\mu, \overline{\mathcal{T}}^\nu : \mathcal{T}(S) \to T^1 \mathcal{T}(S) \) be the sections induced by \( e^\mu \) and \( e^\nu \).

Proposition 5. Let \( \mu, \nu \in \mathcal{ML}(S) \) be two non-zero measured geodesic laminations. The intersection of the two sections \( \overline{\mathcal{T}}^\mu, \overline{\mathcal{T}}^\nu : \mathcal{T}(S) \to T^1 \mathcal{T}(S) \) of the unit tangent bundle of \( \mathcal{T}(S) \) is transverse. If \( \mu \) and \( \nu \) fill up the surface \( S \), these sections meet along a section above a line \( K(\mu, \nu) \) properly embedded in \( \mathcal{T}(S) \). If \( \mu, \nu \) do not fill up \( S \), the intersection is empty.

Proof. The two sections meet above \( m \in \mathcal{T}(S) \) if \( e_m^\mu = e_m^\nu \), namely if there is a \( t > 0 \) such that \( e_m^\nu = te_m^\mu = e_m^\mu \). By Proposition 3, this can occur only when \( t \mu \)
and \( \nu \) fill the surface, namely only when \( \mu \) and \( \nu \) fill the surface. Consequently, the two sections have empty intersection if \( \mu \) and \( \nu \) do not fill up the surface.

If \( \mu \) and \( \nu \) fill up the surface then, for every \( t > 0 \), Proposition 3 shows that there is a unique \( m = \kappa(t\mu, \nu) \in T(S) \) such that \( e_m^\mu = e_m^-\nu \). As a consequence, the two sections \( \overline{\nu}, \overline{\nu} : T(S) \to T^1T(S) \) meet exactly above the image \( K(\mu, \nu) \) of the map \( [0, \infty[ \to T(S) \) defined by \( t \mapsto \kappa(t\mu, \nu) \).

If \( m = \kappa(t\mu, \nu) \) so that \( e_m^\mu = e_m^-\nu \), the tangent space \( T_{e_m}^\mu TT(S) \) is the sum of the tangent spaces \( T_m e^\mu (T_m T(S)) \) and \( T_m e^-\nu (T_m T(S)) \) of the sections \( e^\mu \) and \( e^-\nu \), by the transversality property of Proposition 3. Therefore, in the unit tangent bundle, the tangent space \( T_m T^1T(S) \) is the sum of \( T_m \overline{\nu} (T_m T(S)) \) and \( T_m \overline{\nu} (T_m T(S)) \). As a consequence, the intersection of the two sections \( \overline{\nu} \) and \( \overline{\nu} \) is transverse above the point \( m = \kappa(t\mu, \nu) \).

By transversality, the intersection of the sections is a submanifold of the image of \( \overline{\nu} \). Its dimension is equal to 1, by consideration of the dimensions of \( T(S) \) and \( T^1T(S) \). Since the projection \( \overline{\nu} (T(S)) \to T(S) \) is a diffeomorphism, it follows that the projection \( K(\mu, \nu) \) of this projection is a 1–dimensional submanifold of \( T(S) \).

By definition, \( \kappa(t\mu, \nu) \) is the unique minimum of the convex function \( t dl_\mu + dl_\nu \), which has positive hessian at this minimum. It follows that \( \kappa(t\mu, \nu) \) is a continuous function of \( t \). Conversely, \( t \) is completely determined by \( m = \kappa(t\mu, \nu) \) by Lemma 4. As a consequence, if \( m = \kappa(t\mu, \nu) \) stays in a bounded subset of \( T(S) \), then \( t \) stays in a compact subset of \( [0, \infty[ \). In other words, the map \( [0, \infty[ \to T(S) \) defined by \( t \mapsto \kappa(t\mu, \nu) \) is injective, continuous and proper. It follows that its image \( K(\mu, \nu) \), which we already know is a 1–dimensional submanifold of \( T(S) \), is a line properly embedded in \( T(S) \).

Following the terminology of [Se1] (motivated by [Ke2]), let the Kerckhoff line be the proper 1–dimensional submanifold \( K(\mu, \nu) \subset T(S) \), consisting of all the \( \kappa(t\mu, \nu) \) with \( t > 0 \).

2. Necessary condition for small bending

Let \( t \mapsto m_t, t \in [0, \varepsilon[ \), be a small differentiable curve in \( Q(S) \). If \( \beta(m_t) = \beta^+(t), \beta^-(t) \in M\mathcal{L}(S)^2 \) is its bending measured geodesic lamination, it is shown in [Bo3] that the right derivative \( \frac{d}{dt} \beta(m_t)|_{t=0} \) exists, as an element of the tangent space of \( M\mathcal{L}(S)^2 \) at \( \beta(m_0) \). In general, because \( M\mathcal{L}(S) \) is not a differentiable manifold, this tangent space consists of geodesic laminations with a transverse structure which is less regular than a transverse measure [Bo2]. However, if we assume in addition that the starting point \( m_0 \) of the curve is fuchsian, the tangent space of \( M\mathcal{L}(S)^2 \) at \( \beta(m_0) = (0,0) \) is just \( M\mathcal{L}(S)^2 \); see [Bo2].

We can therefore consider the converse problem: Given a fuchsian metric \( m_0 \in F(S) \) and a pair \( (\mu, \nu) \in M\mathcal{L}(S)^2 \) of measured geodesic laminations, does there exist a small differentiable curve \( t \mapsto m_t \in Q(S), t \in [0, \varepsilon[ \), originating from \( m_0 \) and such that \( \frac{d}{dt} \beta(m_t)|_{t=0} = (\mu, \nu) \)? The following result shows that \( m_0 \) is completely determined by \( \mu \) and \( \nu \).

Note that, by construction, there is a natural identification between the submanifold \( F(S) \subset Q(S) \) consisting of all fuchsian metrics and the Teichmüller space \( T(S) \).
Proposition 6. Let \( \mu, \nu \in \mathcal{ML}(S) \) be two measured geodesic laminations, and let \( t \mapsto m_t, \ t \in [0, \varepsilon[ \) be a differentiable curve in \( \mathcal{Q}(S) \), originating from a fuchsian metric \( m_0 \) and such that the derivative \( \frac{d}{dt} \beta(m_t) |_{t=0} \) of the bending measured lamination is equal to \( (\mu, \nu) \). Then \( \mu \) and \( \nu \) fill up the surface \( S \), and \( m_0 \in \mathcal{F}(S) = \mathcal{T}(S) \) is equal to the minimum \( \kappa(\mu, \nu) \) of the length function \( l_\mu + l_\nu : \mathcal{T}(S) \to \mathbb{R} \).

Proof. We consider two other curves in \( \mathcal{Q}(S) \).

The first one is the pure bending \( t \mapsto B^\mu_{m_0} \), obtained by bending the surface \( S \) along the measured geodesic lamination \( t\mu \) while keeping the metric induced on this pleated surface equal to \( m_0 \). For \( t \geqslant 0 \) small enough, \( B^\mu_{m_0} \) is a quasifuchsian metric for which the positive side of the boundary \( \partial C(B^\mu_{m_0}) \) is a pleated surface with induced metric \( m_0 \) and with bending measured geodesic lamination \( t\mu \). See [EpM, §3] or [Bo1] for the construction of \( B^\mu_{m_0} \), and [Mar, §9] to guarantee that it is quasifuchsian for \( t \) sufficiently small. In addition, it is proved in [EpM, §3.9] [Bo1] that this curve is differentiable in \( \mathcal{Q}(S) \), and in particular admits a tangent vector \( b^\mu_{m_0} = \frac{d}{dt} B^\mu_{m_0} |_{t=0} \in T_{m_0} \mathcal{Q}(S) \). This tangent vector \( b^\mu_{m_0} \) is the infinitesimal pure bending of \( m_0 \in \mathcal{F}(S) \) along the measured geodesic lamination \( \mu \).

The second curve will use the shear-bend coordinates associated, as in [Bo1], to a maximal geodesic lamination \( \lambda \) containing the support of \( \mu \). These coordinates provide a local parametrization of \( \mathcal{Q}(S) \) in terms of the geometry of a pleated surface with pleating locus \( \lambda \). Let \( m'_t \) correspond to a pleated surface whose induced metric is equal to the metric \( m^+_t \in \mathcal{T}(S) \) induced on the positive component of the boundary \( \partial C(m_t) \) of the convex core, and whose bending data is equal to \( t\mu \). By [Mar, §9], \( m_t \) is a quasifuchsian metric for \( t \) small.

It is proved in [Bo3] that, because the curve \( t \mapsto m_t \) is differentiable and because \( \frac{d}{dt} \beta^+(m_t) |_{t=0} = \mu \), the right derivative \( \dot{m}_0^+ = \frac{d}{dt} m_t^+ |_{t=0} \in T_{m_0} \mathcal{T}(S) \) exists and the two curves \( t \mapsto m_t \) and \( t \mapsto m'_t \) have the same tangent vector at \( t = 0 \). In particular, by differentiability of the shear-bend coordinates, the tangent vector \( \dot{m}_0 = \frac{d}{dt} m_t |_{t=0} = \frac{d}{dt} m'_t |_{t=0} \in T_{m_0} \mathcal{Q}(S) \) is the sum of \( b^\mu_{m_0} \) and of \( \dot{m}_0^+ = \frac{d}{dt} m_t^+ |_{t=0} \in T_{m_0} \mathcal{T}(S) = T_{m_0} \mathcal{F}(S) \).

Similarly, bending \( S \) in the negative direction, we can define the infinitesimal pure bending vector \( b^-_{m_0} = -b^\mu_{m_0} \in T_{m_0} \mathcal{Q}(S) \) of \( m_0 \) along \( -\nu \). We can also consider the metric \( m^-_t \in \mathcal{T}(S) \) induced on the negative side of \( \partial C(m_t) \). Then, as above, the vector \( \dot{m}_0 \) is the sum of \( b^-_{m_0} \) and of \( \dot{m}_0^- = \frac{d}{dt} m^-_t |_{t=0} \in T_{m_0} \mathcal{T}(S) = T_{m_0} \mathcal{F}(S) \).

Finally, we will use the complex structure of \( \mathcal{Q}(S) \) coming from the fact that the isometry group of \( \mathbb{H}^3 \) is \( \text{PSL}_2(\mathbb{C}) \). Indeed, considering the holonomy of hyperbolic metrics embeds \( \mathcal{Q}(S) \) into the space \( \mathcal{R}(S) \) of (conjugacy classes) of representations \( \pi_1(S) \to \text{PSL}_2(\mathbb{C}) \). This representation space \( \mathcal{R}(S) \) is a complex manifold near the image of \( \mathcal{Q}(S) \), and admits this image as a complex submanifold (an open subset if \( S \) is compact). The shear-bend local coordinates are well behaved with respect to this complex structure; see [Bo1]. In particular, multiplication by \( i = \sqrt{-1} \) exchanges shearing and bending. We will use the following two consequences of this. First of all, at a fuchsian metric \( m_0 \), the tangent space \( T_{m_0} \mathcal{Q}(S) \) is the direct sum of \( T_{m_0} \mathcal{F}(S) \) and of \( i T_{m_0} \mathcal{F}(S) \). In addition, the infinitesimal pure bending vector \( b^\mu_{m_0} \) belongs to \( i T_{m_0} \mathcal{F}(S) \) and is equal to \( i e^\mu_{m_0} \), where \( e^\mu_{m_0} \in T_{m_0} \mathcal{T}(S) = T_{m_0} \mathcal{F}(S) \) is the infinitesimal earthquake vector along \( \mu \).

Applying the decomposition \( T_{m_0} \mathcal{Q}(S) = T_{m_0} \mathcal{F}(S) \oplus i T_{m_0} \mathcal{F}(S) \) to the vector \( \dot{m}_0 = \dot{m}_0^+ + b^\mu_{m_0} = \dot{m}_0^- + b^-_{m_0} \), we conclude that \( b^\mu_{m_0} = b^-_{m_0} \). Multiplying by \( -i \), it
follows that $\kappa_{m_0} = \kappa_{-m_0}$, As in the proof of Proposition 3, this is equivalent to the property that $m_0 = \kappa(\mu, \nu)$.

3. REALIZING SMALL BENDING

The goal of this section is to prove a converse to Proposition 3, by constructing in Proposition 4 a small curve of quasifuchsian metrics $t \mapsto m_t \in Q(S)$, $t \in [0, \varepsilon]$, such that $\beta(m_t) = (t\mu, tu\nu)$ for every $t$.

For the measured geodesic lamination $\mu \in M(L)(S)$, let $P^+(\mu)$ (resp. $P^-(\mu)$) be the space of quasifuchsian metrics $m$ such that the positive (resp. negative) component of the convex core boundary $\partial C(m)$ has bending measured geodesic lamination $\mu$, for some $t \in [0, \infty]$.

Recall that $\theta = -3\chi(S) + p$ denotes the dimension of the Teichmüller space $T(S)$.

**Lemma 7.** The space $P^\pm(\mu)$ is a submanifold-with-boundary of $Q(S)$, with dimension $\theta + 1$ and with boundary $F(S)$.

**Proof.** We can use the coordinates developed in [Bo1], and associated to a maximal geodesic lamination $\lambda$ containing the support of $\mu$. These provide an open differentiable embedding $\varphi : Q(S) \to T(S) \times H_0(\lambda; \mathbb{R}/2\pi \mathbb{Z})$. The first component of $\varphi(m)$ is the hyperbolic metric induced on the unique $m$–pleated surface $f_m$ with pleating locus $\lambda$. The second component is the bending transverse cocycle of $f_m$, which belongs to the topological group $H_0(\lambda; \mathbb{R}/2\pi \mathbb{Z}) \cong (\mathbb{R}/2\pi \mathbb{Z})^\theta \oplus \mathbb{Z}/2$ of all $(\mathbb{R}/2\pi \mathbb{Z})$–valued transverse cocycles for $\lambda$ that satisfy a certain cusp condition. In general, the bending of the pleated surface $f_m$ is measured by a transverse cocycle and not by a measured geodesic lamination because $f_m$ is not necessarily locally convex.

For notational convenience, it is useful to lift $\varphi$ to an embedding $\psi : Q(S) \to T(S) \times H_0(\lambda; \mathbb{R})$, such that $\psi$ sends $F(S) \cong T(S) \times \{0\}$ by the identity. Here, $H_0(\lambda; \mathbb{R}) \cong \mathbb{R}^\theta$ denotes the space of $\mathbb{R}$–valued transverse cocycles satisfying the cusp condition. Such a $\psi$ exists and is unique because $Q(S)$ is simply connected.

The vector space $H_0(\lambda; \mathbb{R})$ contains the transverse measure (also denoted by $\mu$) of the measured geodesic lamination $\mu$, and therefore it contains the two rays $[0, \infty] \mu$ and $]-\infty, 0] \mu$, consisting of all positive (resp. negative) real multiples of $\mu$.

We claim that $P^+(\mu)$ locally corresponds, under $\psi$, to the intersection of $\psi(Q(S))$ with $T(S) \times [0, \infty] \mu$. Clearly, $\psi$ sends an element of $P^+(\mu)$ to $T(S) \times [0, \infty] \mu$. Conversely, it is proved in [Ka1] that the map $\eta : Q(S) \to T(S) \times ML(S)$ which, to a quasifuchsian metric $m \in Q(S)$, associates the induced metric $m^+ \in T(S)$ and the bending measured lamination $\beta^+(m)$ of the positive boundary component of the convex core $C(m)$ is a local homeomorphism. In addition, interpreting the ray $[0, \infty] \mu$ as a subset of both $H_0(\lambda; \mathbb{R})$ and $ML(S)$, the local inverse for $\eta$ constructed in [Ka1] coincides with $\psi$ on $T(S) \times [0, \infty] \mu$. It follows that, if $m \in Q(S)$ is sufficiently close to $m_0 \in P^+(\mu)$ and if $\psi(m) \in T(S) \times [0, \infty] \mu$, the bending measured lamination $\beta^+(\mu)$ is equal to the second component $t\mu$ of $\psi(m)$. Therefore, a metric $m$ near $m_0 \in P^+(\mu)$ is in $P^+(\mu)$ if and only if $\psi(m) \in T(S) \times [0, \infty] \mu$.

A similar property holds for $P^-(\mu)$ by symmetry. Therefore, under the diffeomorphism $\psi$, $P^+(\mu)$, $P^-(\mu)$ and $F(S)$ locally correspond to the intersection of $\psi(Q(S))$ with $T(S) \times [0, \infty] \mu$, $T(S) \times [-\infty, 0] \mu$ and $T(S) \times \{0\}$, respectively.
Given two measured geodesic laminations $\mu, \nu \in \mathcal{ML}(S)$, we want to consider the intersection of $\mathcal{P}^+ (\mu)$ and $\mathcal{P}^- (\nu)$. Note that this intersection is far from being transverse, since these two $(\theta + 1)$–dimensional submanifolds both contain the $\theta$–dimensional submanifold $\mathcal{F}(S)$ as their boundary. For this reason, we will consider the manifold-with-boundary $\hat{\mathcal{Q}}(S)$ obtained by blowing-up $\mathcal{Q}(S)$ along the submanifold $\mathcal{F}(S)$. Namely, $\mathcal{Q}(S)$ is the union of $\mathcal{Q}(S) - \mathcal{F}(S)$ and of the unit normal bundle $N^1\mathcal{F}(S)$ with the appropriate topology. Recall that the normal bundle $N\mathcal{F}(S) \to \mathcal{F}(S)$ is intrinsically defined as the bundle whose fiber $N_m\mathcal{F}(S)$ at $m \in \mathcal{F}(S)$ is the quotient $T_m\mathcal{Q}(S)/T_m\mathcal{F}(S)$, and that the fiber $N^1_{m\mathcal{F}}(S)$ of the unit normal bundle $N^1\mathcal{F}(S) \to \mathcal{F}(S)$ is the quotient of $N_m\mathcal{F}(S) - \{0\}$ under the multiplicative action of $\mathbb{R}^+$. Considering a tubular neighborhood of $\mathcal{F}(S)$, $\hat{\mathcal{Q}}(S)$ is easily endowed with a natural structure of differentiable manifold with boundary $N^1\mathcal{F}(S)$.

Exploiting the complex structure of $\mathcal{Q}(S)$, identify the normal bundle $N\mathcal{F}(S)$ to $iT\mathcal{F}(S)$. The inclusion map $\mathcal{P}^+ (\mu) - \mathcal{F}(S) \to \mathcal{Q}(S) - \mathcal{F}(S)$ uniquely extends to an embedding $\mathcal{P}^+ (\mu) \to \mathcal{Q}(S)$, which to $m \in \mathcal{F}(S) = \partial\mathcal{P}^+ (\mu)$ associates the unit normal vector $\overrightarrow{\mathcal{N}}_m \in N^1\mathcal{F}(S)$ which is in the direction of the infinitesimal pure bending vector $b^\mu_m$ of $m$ along $+\mu$. We can similarly define an embedding $\mathcal{P}^- (\nu) \to \hat{\mathcal{Q}}(S)$, by associating to $m \in \mathcal{F}(S)$ the unit normal vector $-\overrightarrow{\mathcal{N}}_m \in N^1\mathcal{F}(S)$ which is in the direction of the infinitesimal pure bending vector $b^-\nu_m = -b^\mu_m$ of $m$ along $-\nu$. Let $\hat{\mathcal{P}}^+ (\mu)$ and $\hat{\mathcal{P}}^- (\nu)$ be the respective images of these embeddings.

**Lemma 8.** The subspaces $\hat{\mathcal{P}}^+ (\mu)$ and $\hat{\mathcal{P}}^- (\nu)$ are submanifolds of $\hat{\mathcal{Q}}(S)$, with boundary contained in $\partial\hat{\mathcal{Q}}(S)$.

*Proof.* The embedding $\psi : \mathcal{Q}(S) \to \mathcal{T}(S) \times \mathcal{H}_0(\lambda; \mathbb{R})$ of the proof of Lemma 3 sends the vector $b^\mu_m$ to the vector $(0, \mu)$ in the tangent space of $\mathcal{T}(S) \times \mathcal{H}_0(\lambda; \mathbb{R})$. The result then immediately follows from the fact that $\psi$ locally identifies $\mathcal{P}^+ (\mu)$, $\mathcal{P}^- (\mu)$, and $\mathcal{T}(S)$ to $\mathcal{T}(S) \times [0, \infty) [\mu, \mathcal{T}(S) \times ] -\infty, 0] \mu$ and $\mathcal{T}(S) \times \{0\}$, respectively. This proves Lemma 3. □

**Proposition 9.** The boundaries $\partial\mathcal{P}^+ (\mu)$ and $\partial\mathcal{P}^- (\nu)$ have a non-empty intersection if and only if $\mu$ and $\nu$ fill up the surface $S$. If $\mu$ and $\nu$ fill up $S$, the intersection of $\partial\mathcal{P}^+ (\mu)$ and $\partial\mathcal{P}^- (\nu)$ in $\partial\mathcal{Q}(S)$ is transverse, and is equal to the image of the Kerckhoff line $K(\mu, \nu)$ under the section $m \to \overrightarrow{\mathcal{N}}_m = -\overrightarrow{\mathcal{N}}_m \in \partial\hat{\mathcal{Q}}(S)$.

*Proof.* The diffeomorphism $T^1\mathcal{T}(S) = T^1\mathcal{F}(S) \to iT^1\mathcal{F}(S) = N^1\mathcal{F}(S) = \partial\hat{\mathcal{Q}}(S)$ defined by multiplication by $i$ sends $\sigma^\mu_m$ to $\overrightarrow{\mathcal{N}}_m$ and $\sigma^-\nu_m$ to $\overrightarrow{\mathcal{N}}_m$. This translates Proposition 3 to a simple rephrasing of Proposition 3. □

An immediate consequence of Proposition 3 is that the intersection of the two $(\theta + 1)$–dimensional submanifolds $\mathcal{P}^+ (\mu)$ and $\mathcal{P}^- (\nu)$ in $\mathcal{Q}(S)$ is transverse near the boundary $\partial\mathcal{Q}(S)$. In particular, the intersection $\mathcal{P}^+ (\mu) \cap \mathcal{P}^- (\nu)$ is a 2–dimensional submanifold of $\mathcal{Q}(S)$ near $\partial\mathcal{Q}(S)$, with boundary $\partial\mathcal{P}^+ (\mu) \cap \partial\mathcal{P}^- (\nu)$ contained in $\partial\mathcal{Q}(S)$.

By definition, a metric $m \in \mathcal{P}^+ (\mu) \cap \mathcal{P}^- (\nu)$ has bending measured lamination $\beta(m) = (t\mu, u\nu)$ for some $t, u \geq 0$. This gives a differentiable map $\pi : \mathcal{P}^+ (\mu) \cap \mathcal{P}^- (\nu) \to \mathbb{R}^2$, defined by $m \to (t, u)$.

Let $\pi$ be obtained by blowing up $\mathbb{R}^2$ along $\{0\}$. Because $\pi^{-1}(0) = \mathcal{F}(S)$, the map $\pi$ lifts to a differentiable map $\tilde{\pi} : \hat{\mathcal{P}}^+ (\mu) \cap \hat{\mathcal{P}}^- (\nu) \to \tilde{\mathbb{R}}^2$. 


Lemma 10. The map $\tilde{\pi} : \partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu) \to \mathbb{R}^2$ is a local diffeomorphism near $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$.

Proof. We will prove that, at any point $p_0 \in \partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$, the (linear) tangent map $T_{p_0}\tilde{\pi} : T_{p_0}\partial\mathcal{P}^+(\mu) \cap T_{p_0}\partial\mathcal{P}^-(\nu) \to T_{\tilde{\pi}(p_0)}\mathbb{R}^2$ is injective.

Let $v \in T_{p_0}\partial\mathcal{P}^+(\mu) \cap T_{p_0}\partial\mathcal{P}^-(\nu)$ be such that $T_{p_0}\tilde{\pi}(v) = 0$. Considering the map $\partial\mathcal{P}^+(\mu) \to [0, \infty[$ which to $m \in \partial\mathcal{P}^+(\mu)$ associates $\beta^+(m)/\mu \in [0, \infty[$ and the induced map $\partial\mathcal{P}^+(\mu) \to [0, \infty[$, we see that $v$ must necessarily be in the tangent space of the boundary $\partial\mathcal{P}^+(\mu)$. Symmetrically, it must be in $T_{p_0}\partial\mathcal{P}^-(\nu)$. Therefore, $v$ is tangent to the intersection $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$.

Let us analyze the restriction of $\tilde{\pi}$ to the boundary $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$. By Proposition 3, $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$ is equal to the image of the Kerckhoff line $K(\mu, \nu)$ under the section $m \mapsto \overline{b}_m = -\overline{b}_m \in \partial\mathcal{Q}(S)$. Recall that an element of the Kerckhoff line $K(\mu, \nu)$ is of the form $m = \kappa(t_\mu, \nu)$ for some $t > 0$, which is equivalent to the property that $b^\mu_m = -b^\nu_m$. We will also need a coordinate chart for $\mathbb{R}^2$ near $\tilde{\pi}(p_0)$. Noting that the image of $\pi$ is contained in the quadrant $[0, \infty[^2$, we can use for this the chart $\varphi : [0, \infty[ \times [0, \infty[ \to \mathbb{R}^2$ defined on the interior by $(x, y) \mapsto (xy, y)$.

We claim that, if $m = \kappa(t_\mu, \nu) \in K(\mu, \nu)$, then $\varphi^{-1} \circ \tilde{\pi}(m) = (0, 0)$ is just equal to $(0, 0)$. To see this, choose, in the 2-dimensional manifold $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$, a small curve $s \mapsto \tilde{m}_s$, $s \in [0, \varepsilon]$, such that $\tilde{m}_0 = \overline{b}_m \in \partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$ and such that $\frac{d}{ds}\tilde{m}_s|_{s=0}$ is not tangent to the boundary. The curve $s \mapsto \tilde{m}_s$ projects to a differentiable curve $t \mapsto m_t \in \mathcal{Q}(S)$ with $m_0 = m$. By definition of $\partial\mathcal{P}^+(\mu)$ and $\partial\mathcal{P}^-(\nu)$, the bending measured lamination $\beta(m_s)$ is of the form $(t(s)\mu, u(s)\nu)$ for two differentiable functions $t(s)$ and $u(s)$ with $t(0) = u(0) = 0$. Since $\frac{d}{ds}\tilde{m}_s|_{s=0}$ points away from the boundary, the curve $s \mapsto \tilde{m}_s$ is not tangent to $\mathcal{F}(S)$ at $s = 0$, and it follows that at least one of the derivatives $t'(0)$, $u'(0)$ is non-trivial. If we apply Proposition 3, we conclude that $m = \kappa(t_\mu, \nu, u'(0)\nu)$. Since $m = \kappa(t_\mu, \nu)$, it follows that $t'(0)/u'(0) = t$ by Lemma 3. In particular, $t(s)/u(s)$ tends to $t$ as $s$ tends to 0. Therefore, $\varphi^{-1} \circ \tilde{\pi}(m) = \varphi^{-1} \circ \tilde{\pi}(m_s) = \varphi^{-1} \circ \beta((t(s), u(s)) = (t(s)/u(s), u(s)))$ as $s > 0$ tends to 0, is equal to $(0, 0)$.

This computation shows that the restriction of $\tilde{\pi}$ to the boundary $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$ is a diffeomorphism onto its image. In particular, if $v \in T_{p_0}\partial\mathcal{P}^+(\mu) \cap T_{p_0}\partial\mathcal{P}^-(\nu)$ is such that $T_{p_0}\tilde{\pi}(v) = 0$, then necessarily $v = 0$.

This concludes the proof that the tangent map $T_{p_0}\tilde{\pi} : T_{p_0}\partial\mathcal{P}^+(\mu) \cap T_{p_0}\partial\mathcal{P}^-(\nu) \to T_{\tilde{\pi}(p_0)}\mathbb{R}^2$ is injective. Since $\tilde{\pi}$ sends the boundary of the 2-dimensional manifold $\partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$ to the boundary of the 2-dimensional manifold $\mathbb{R}^2$, this proves that $\tilde{\pi} : \partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu) \to \mathbb{R}^2$ is a local diffeomorphism near $p_0 \in \partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$. □

This immediately gives the following converse to Proposition 3.

Proposition 11. Let $\mu, \nu \in \mathcal{ML}(S)$ be two measured geodesic laminations which fill up the surface $S$, and let $m_0$ be the minimum $\kappa(\mu, \nu)$ of the length function $l_\mu + l_\nu$. Then there is a small differentiable curve $t \mapsto m_t \in \mathcal{Q}(S)$, $t \in [0, \varepsilon]$, beginning at $m_0$ and such that the bending measured lamination $\beta(m_t)$ is equal to $(t_\mu, t_\nu)$ for every $t$.

Proof. Consider the curve $t \mapsto (t, t)$, $t \in [0, \varepsilon]$, in $\mathbb{R}^2$. By Lemma 3 for $\varepsilon$ small enough, there is a curve $t \mapsto \tilde{m}_t \in \partial\mathcal{P}^+(\mu) \cap \partial\mathcal{P}^-(\nu)$ such that $t \mapsto \tilde{\pi}(\tilde{m}_t)$ coincides
with the lift of \( t \mapsto (t, t) \) to \( \mathbb{R}^2 \). By definition of the map \( \pi \), this just means that the projection \( m_t \in Q(S) \) of \( \tilde{m}_t \in \tilde{Q}(S) \) is such that \( \beta(m_t) = (t\mu, tv) \).

### 4. Parametrizing quasi-fuchsian groups by their small bending

Recall that \( \mathcal{FML}(S) \) denotes the open subset of \( \mathcal{ML}(S)^2 \) consisting of those pairs \((\mu, \nu)\) such that \( \mu \) and \( \nu \) fill up the surface \( S \).

Let \( \mathcal{FML}(S) \) be obtained by blowing up \( \mathcal{FML}(S) \cup \{(0, 0)\} \) along \{(0, 0)\}. Namely, \( \mathcal{FML}(S) \) is formally obtained from \( \mathcal{FML}(S) \) by extending each ray \([0, \infty[ (\mu, \nu)\) to a semi-open ray \([0, \infty[ (\mu, \nu)\), with the obvious topology. Note that the boundary \( \partial \mathcal{FML}(S) \) is just the quotient space of \( \mathcal{FML}(S) \) under the multiplicative action of \( \mathbb{R}^+ \).

For every \((\mu, \nu) \in \mathcal{FML}(S)\), Proposition \[1\] provides a maximal ray \( R_{\mu\nu} = [0, \varepsilon_{\mu\nu}(\mu, \nu) \in \mathcal{FML}(S) \cup \{(0, 0)\}) \) and a differentiable map \( \Phi_{\mu\nu} : R_{\mu\nu} \to Q(S) \) such that \( \Phi_{\mu\nu}(\mu', \nu') \) has bending measured lamination \((\mu', \nu')\) for every \((\mu', \nu') \in R_{\mu\nu}\) and such that \( \mathcal{P}^+(\mu) \) and \( \mathcal{P}^-(\nu) \) meet transversely along \( \beta(R_{\mu\nu}) \). Here, the statement that \( R_{\mu\nu} \) is maximal means that \( \varepsilon_{\mu\nu} \in [0, \infty) \) is maximal for this property.

Note that \( R_{\mu\nu} \) and \( \varphi_{\mu\nu} \) depend only on the orbit of \((\mu, \nu)\) under the action of \( \mathbb{R}^+ \), namely on the corresponding point of \( \partial \mathcal{FML}(S) \). Let \( \tilde{R}_{\mu\nu} \) be the lift of \( R_{\mu\nu} \) in \( \mathcal{FML}(S) \), and lift \( \Phi_{\mu\nu} \) to \( \tilde{\Phi}_{\mu\nu} : \tilde{R}_{\mu\nu} \to \tilde{Q}(S) \). In particular, \( \tilde{\Phi}_{\mu\nu} \) sends the initial point of \( \tilde{R}_{\mu\nu} \) to the bending vector \( \overline{\mathbb{D}}_{\mu\nu} = \overline{\mathbb{D}}_{\mu\nu} \in N^1 \mathcal{F}(S) = \partial \tilde{Q}(S) \) with \( m = \kappa(\mu, \nu) \).

Let \( \tilde{U} \subset \mathcal{FML}(S) \) denote the union of all the \( \tilde{R}_{\mu\nu} \), and let \( \tilde{\Phi} : \tilde{U} \to \mathcal{FML}(S) \) restrict to \( \tilde{\Phi}_{\mu\nu} \) on each \( \tilde{R}_{\mu\nu} \). Note that the property that \( \Phi_{\mu\nu}(\mu', \nu') \) has bending measured lamination \((\mu', \nu')\) implies that the \( \tilde{R}_{\mu\nu} \) are pairwise disjoint, so that \( \tilde{\Phi} \) is well-defined. We want to show that \( \tilde{\Phi} \) is continuous.

**Lemma 12.** As the measured geodesic lamination \( \mu \) tends to \( \mu_0 \) for the topology of \( \mathcal{ML}(S) \), the submanifold \( \mathcal{P}^\pm(\mu) \) tends to \( \mathcal{P}^\pm(\mu_0) \) for the topology of \( C^\infty \) convergence on compact subsets.

**Proof.** We will use the tools developed in [Bo1].

Let \( \mu_n \in \mathcal{ML}(S) \), \( n \in \mathbb{N} \), be a sequence converging to \( \mu_0 \). Let \( \lambda_n \) be a maximal geodesic lamination containing the support of \( \mu_n \). Passing to a subsequence if necessary, we can assume that, for the Hausdorff topology, the geodesic lamination \( \lambda_n \) converges to a geodesic lamination \( \lambda_0 \), which is necessarily maximal and contains the support of \( \mu_0 \).

The shear-bend coordinates associated to \( \lambda_n \) provide an open biholomorphic embedding \( \Phi_n : Q(S) \to \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}). \) Here \( \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}) \) is the topological group of \( \mathbb{C}/2\pi i\mathbb{Z} \)-valued transverse cocycles for the maximal geodesic lamination \( \lambda_n \) which satisfy the cusp condition, and is isomorphic to \( (\mathbb{C}/2\pi i\mathbb{Z})^2 \oplus \mathbb{Z}/2 \). For a metric \( m \in Q(S) \), the real part of \( \Phi_n(m) \in \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}) = \mathcal{H}_0(\lambda_n; \mathbb{R}) \oplus i\mathcal{H}_0(\lambda_n; \mathbb{R}/2\pi i) \) measures the induced metric of the unique \( m \)-pleated surface with bending locus \( \lambda_n \), and the imaginary part measures its bending. In particular, \( \mathcal{P}^\pm(\mu_n) \) locally corresponds to \( \Phi_n^{-1}(\mathcal{H}_0(\lambda_n; \mathbb{R}) \oplus i\mathcal{H}_0(\lambda_n; \mathbb{R}/2\pi i)) \), or more precisely to a branch of the immersion which is the composition of the projection \( \mathcal{H}_0(\lambda_n; \mathbb{R}) \oplus i\mathcal{H}_0(\lambda_n; \mathbb{R}/2\pi i) \to \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}) \) and of \( \Phi_n^{-1} \).

To compare the various \( \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}) \) pick a train track \( \tau \) carrying \( \lambda_0 \). Since \( \lambda_n \) converges to \( \lambda_0 \) for the Hausdorff topology, \( \tau \) also carries the \( \lambda_n \) for \( n \) large enough. Then there is a well-defined isomorphism \( \Psi_n : \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}) \to \mathcal{H}_0(\lambda_n; \mathbb{C}/2\pi i\mathbb{Z}) \)
follows from the definition of \( \tilde{\text{map}} \). (Note that this is false for the projection \( \tilde{\text{map}} \).)

Because \( \lambda_0 \) converges to \( \lambda_0 \) for the Hausdorff topology, if follows from the explicit construction of [Boa] \( \S 5, \S 8 \) that \( \Phi_0^{-1} \circ \Psi_0^{-1} \) converges to \( \Phi_0^{-1} \circ \Psi_0^{-1} \), uniformly on compact subsets of the image of \( \Psi_0 \circ \Phi_0 \). Because these maps are holomorphic, the convergence is actually \( C^\infty \). Since \( \mu_n \) converges to \( \mu_0 \) for the topology of \( \mathcal{ML}(S) \), the edge weight system \( \Psi_n(\mu_n) \) converges to \( \Psi_0(\mu_0) \) in \( W_0(\tau, \mathbb{R}) \). It follows that \( \mathcal{P}^\pm(\mu_n) \), which locally corresponds to \( \Phi_0^{-1} \circ \Psi_0^{-1} (W_0 (\tau; \mathbb{R}) \oplus i\mathbb{R}\Psi_n(\mu_n)) \), converges to \( \mathcal{P}^\pm(\mu_0) \), which locally corresponds to \( \Phi_0^{-1} \circ \Psi_0^{-1} (W_0 (\tau; \mathbb{R}) \oplus i\mathbb{R}\Psi_0(\mu_0)) \), in the topology of \( C^\infty \)-convergence on compact subsets.

Note that we actually proved real analytic convergence in Lemma \( \S 12 \). However, we will only need \( C^2 \) convergence.

**Theorem 13.** The subset \( \check{U} \) is an open neighborhood of \( \partial \check{FM}(S) \) in \( \check{FM}(S) \), and \( \check{\Phi} \) is a homeomorphism from \( \check{U} \) to an open neighborhood \( \check{V} \) of \( \partial \check{Q}(S) \) in \( \check{Q}(S) \).

**Proof.** The restriction \( \check{\Phi}_{\mu_0} \) of \( \check{\Phi} \) to \( \check{R}_{\mu_0} \) was constructed by considering the transverse intersection of the submanifolds \( \mathcal{P}^+(\mu) \) and \( \mathcal{P}^-(\nu) \) near the boundary of \( \check{Q}(S) \). By Lemma \( \S 12 \), \( \mathcal{P}^+(\mu) \) and \( \mathcal{P}^-(\nu) \) depend continuously on \( (\mu, \nu) \) for the topology of \( C^1 \) convergence. (Note that one needs the \( C^2 \) continuity of \( \mathcal{P}^+(\mu) \) and \( \mathcal{P}^-(\nu) \) to guarantee the \( C^1 \) continuity of \( \mathcal{P}^+(\mu) \) and \( \mathcal{P}^-(\nu) \) near the boundary.) It follows that the length \( \varepsilon_{\mu_0} \) of \( R_{\mu_0} = [0, \varepsilon_{\mu_0}](\mu, \nu) \) is a lower semi-continuous function of \( (\mu, \nu) \), and that \( \check{\Phi}_{\mu_0} \) depends continuously on \( (\mu, \nu) \). This proves that the union \( \check{U} \) of the \( \check{R}_{\mu_0} \) is open in \( \check{FM}(S) \), and that \( \check{\Phi} \) is continuous.

Let \( p : \check{Q}(S) \to Q(S) \) be the natural projection and, as usual, let \( \beta : Q(S) \to ML(S)^2 \) be the bending map. By construction, \( \beta \circ p \circ \check{\Phi} \) is the identity map on \( \check{U} - \partial \check{FM}(S) \subseteq ML(S)^2 \). It follows that \( \check{\Phi} \) is injective on \( \check{U} - \partial \check{FM}(S) \), and therefore on all of \( \check{U} \) by Proposition \( \S 3 \).

The two spaces \( \check{U} \subseteq \check{FM}(S) \) and \( \check{Q}(S) \) are topological manifolds-with-boundary of the same dimension \( 29 \). The map \( \check{\Phi} : \check{U} \to \check{FM}(S) \) is continuous and injective, and sends boundary points to boundary points. By the Theorem of Invariance of the Domain, it follows that its image \( \check{V} = \check{\Phi}(\check{U}) \) is open in \( \check{Q}(S) \), and that \( \check{\Phi} \) restricts to a homeomorphism \( \check{U} \to \check{V} \).

**Theorem 14.** There exists an open neighborhood \( V \) of the fuchsian submanifold \( \mathcal{F}(S) \) in \( Q(S) \) such that the bending map \( \beta : \mathcal{Q}(S) \to ML(S)^2 \) induces a homeomorphism between \( V - \mathcal{F}(S) \) and its image.

**Proof.** For the canonical identifications between \( \check{Q}(S) - \partial \check{Q}(S) \) and \( Q(S) - \mathcal{F}(S) \), and between \( \check{FM}(S) - \partial \check{FM}(S) \) and \( \mathcal{FM}(S) \), the inverse of the restriction of \( \check{\Phi} \) to \( \check{U} - \partial \check{Q}(S) \) coincides with \( \beta \). Therefore, the only part which requires some checking is that the image \( V \) of \( \check{\Phi}(\check{U}) \) under the canonical projection \( p : \check{Q}(S) \to \mathcal{Q}(S) \) is open in \( \mathcal{Q}(S) \). However, this immediately follows from the fact that the preimages of points under \( p \) are all compact, which implies that \( p \) is an open map. (Note that this is false for the projection \( \mathcal{FM}(S) \to \mathcal{FM}(S) \cup \{0\} \).

Theorem \( \S 14 \) is just Theorem \( \S 13 \) stated in the introduction. Theorem \( \S 14 \) immediately follows from the definition of \( \check{U} \) and from the fact that \( \check{U} \) is open (Theorem \( \S 13 \)).

We conclude with a few remarks.
A recent result of Caroline Series [Se2] shows that we can restrict the neighborhood $V$ of Theorem [4] so that $\beta^{-1}(\beta(V)) = V$.

When $\mu$ and $\nu$ are multicurves, namely when their supports consist of finitely many closed geodesics, it follows from [HoK] that the submanifolds $\mathcal{P}^+(\mu)$ and $\mathcal{P}^-(\nu)$ are everywhere transverse, by a doubling argument as in [BoO]. Consequently, the open subset $V$ of Theorem [4] can be chosen so that the image $U = \beta(V) \subset \mathcal{ML}(S)^2$ contains all rays of the form $[0, \infty)[(\mu, \nu)$ where $\mu$ and $\nu$ are multicurves.

As indicated in the introduction, it is conjectured that we can take $V$ equal to the whole space $Q(S)$. See [BoO] for a characterization of the image of $Q(S)$ under $\beta$.

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