A smooth summation of Ramanujan expansions

Giovanni Coppola

Abstract. We studied Ramanujan series \[ \sum_{q=1}^{\infty} G(q)c_q(a) \], where \( c_q(a) \) is the well-known Ramanujan sum and the complex numbers \( G(q) \), as \( q \in \mathbb{N} \), are the Ramanujan coefficients; of course, we mean, implicitly, that the series converges pointwise, in all natural \( a \), as its partial sums \( \sum_{q \leq Q} G(q)c_q(a) \) converge in \( \mathbb{C} \), when \( Q \to \infty \). Motivated by our recent study of infinite and finite Euler products for the Ramanujan series, in which we assumed \( G \) multiplicative, we look at a kind of (partial) smooth summations. These are \( \sum_{q \in (P)} G(q)c_q(a) \), where the indices \( q \) in \( (P) \) means that all prime factors \( p \) of \( q \) are up to \( P \) (fixed); then, we pass to the limit over \( P \to \infty \). Notice that this kind of partial sums over \( P \)-smooth numbers (i.e., in \( (P) \), see the above) make up an infinite sum, themselves, \( \forall P \in \mathbb{P} \) fixed, in general; however, our summands contain \( c_q(a) \), that has a vertical limit, i.e. it’s supported over indices \( q \in \mathbb{N} \) for which the \( p \)-adic valuations of, resp., \( q \) and \( a \), namely \( v_p(q) \), resp., \( v_p(a) \) satisfy \( v_p(q) \leq v_p(a) + 1 \) and this is true \( \forall p \leq P \) (\( P \)'s fixed).

In other words, \( \forall G: \mathbb{N} \to \mathbb{C} \), here, \( \sum_{q \in (P)} G(q)c_q(a) \) is a finite sum, \( \forall a \in \mathbb{N} \), \( \forall P \in \mathbb{P} \) fixed: we will call \( \sum_{q=1}^{\infty} G(q)c_q(a) \) a Ramanujan smooth series if and only if \( \exists \lim_{P} \sum_{q \in (P)} G(q)c_q(a) \in \mathbb{C} \), \( \forall a \in \mathbb{N} \).

Notice a very important property: Ramanujan smooth series and Ramanujan series need not to be the same.

We prove: Ramanujan smooth series converge under Wintner Assumption. (This is not necessarily true for Ramanujan series.) We apply this to correlations and to the Hardy–Littlewood “2k-Twin Primes” Conjecture.

1. Introduction. Main results for: arithmetic functions, correlations and 2k−twin primes

We pursue our study of Ramanujan expansions with smooth moduli, started in [C1]. There, we obtained pointwise converging Ramanujan expansions, for some arithmetic functions having Eratosthenes transform supported over smooth numbers: say, the \( F: \mathbb{N} \to \mathbb{C} \) with “smooth divisors”; we then applied this general result (see [C1], Theorem 1), to the correlations satisfying a reasonable hypothesis (see [C1], Corollary 1).

Here, a new kind of summation Ramanujan expansions will give us a “new world”, of elementary results about convergence and, notably, for more general arithmetic functions (no restriction on their divisors, here).

Among these, following Theorem 1, a completely unexpected, new version, say, of Delange Theorem [De] about the convergence of Ramanujan expansions: if we confine to the summation of partial sums on smooth numbers, we can get their convergence, but with a weaker hypothesis with respect to Delange’s (i.e., (DH)), following) and this, actually, is the assumption in the Wintner’s Criterion (i.e., (2.1) in (ii), see Theorem 2.1 in Chapter VIII of [ScSp]): that we’ll call the Wintner Assumption, abbreviated (WA), for \( F: \mathbb{N} \to \mathbb{C} \); having [W] Eratosthenes transform \( F' \equiv F \ast \mu \), where \( \mu \) is Möbius function and \( \ast \) is Dirichlet product [T]:

\[
(WA) \quad \sum_{d=1}^{\infty} \frac{|F'(d)|}{d} < \infty.
\]

However, (WA) is not sufficient for the convergence of classical partial sums (see §5.1). In fact, for this we need Delange Hypothesis (compare (6) in [De]), next (DH): in which \( \omega(d) \equiv |\{p \in \mathbb{P}, p | d\}| \) is the number of prime factors of \( d \in \mathbb{N} \) (whence, see [T], \( 2^{\omega(d)} = \sum_{t \in \mathbb{N}} a^2(t) \) is the number of square-free divisors of \( d \)):

\[
(DH) \quad \sum_{d=1}^{\infty} \frac{2^{\omega(d)}|F'(d)|}{d} < \infty.
\]

Like we did in [C1], we write \( \{n \in \mathbb{N} : (n, p) = 1, \forall p > V\} \) for the set of \( V \)-smooth numbers, while \( \{n \in \mathbb{N} : (n, p) = 1, \forall p \leq V\} \) is the set of \( V \)-sifted numbers. Notice that: \( \{V\} \cap \{V\} = \{1\}, \forall V \in \mathbb{N} \).

We write \( V = P \in \mathbb{P} \) hereafter, so that \( (P) \) and \( \{P\} \) avoid the trivial case \( 1 \equiv 1 \equiv \{1\} \).

MSC 2010: 11N05, 11P32, 11N37 - Keywords: Ramanujan expansion, correlation, 2k−twin primes
In the following, we use the classical notation \( \ll \) of Vinogradov (\( A \ll B \) means \( |A| \leq C \cdot B \), for some constant \( C > 0 \)), with \( \ll \) indicating a dependence on \( \varepsilon > 0 \), arbitrarily small usually, in the \( \ll \) constant.

As usual, we say that \( F : \mathbb{N} \to \mathbb{C} \) satisfies the Ramanujan Conjecture, by definition, when: \( \forall \varepsilon > 0, \exists C = C(\varepsilon) : |F(n)| \leq C \cdot n^\varepsilon, \forall n \in \mathbb{N} \) (large enough), i.e., in Vinogradov notation,

\[
(\text{Ramanujan Conjecture}) \quad \forall \varepsilon > 0, \quad F(n) \ll n^\varepsilon, \quad \text{as } n \to \infty.
\]

(\text{In other papers}, we write \( F \ll 1 \) for that, also calling \( F \) “essentially bounded”: compare [C3] and [CM].)

We rely, here and in [C1], on the fact that all \( F : \mathbb{N} \to \mathbb{C} \) satisfying Ramanujan Conjecture and having \( F' \) supported on smooth numbers, say \( (P) \), have a nice behavior for the convergence issues related to Ramanujan expansions and their coefficients. This is based, at last, on the following bound (compare [C1], Lemma 3, for all the details), in which \( \varepsilon > 0 \) is arbitrarily small:

\[
(1) \quad \sum_{m \in (P)} m^{\varepsilon-1} = \prod_{p \leq P} \sum_{K=0}^{\infty} (p^{\varepsilon-1})^K = \prod_{p \leq P} \frac{1}{1 - p^{-\varepsilon}} < \infty,
\]

and notice that the same series, but without the condition “\( m \in (P) \)”, of course, is a diverging one.

This elementary estimate (coming from multiplicativity of \( m^{\varepsilon-1} \), w.r.t. \( m \in \mathbb{N} \)) seems to be not so powerful; however, it implies that \( F \) satisfying Ramanujan Conjecture, with \( P \)-smooth divisors, satisfy Delange Hypothesis (see (DH) above), that (thanks to [De] main result) implies: Carmichael coefficients \( \text{Car} \ F \), see the following, equal Wintner coefficients \( \text{Win} \ F \), see the following (compare Theorem 1 in [C1]).

The (\text{WA}) is called Wintner Assumption, because Wintner [W] was the first to work with it for the Ramanujan expansions; first of all, by positivity it implies the existence of all the “Wintner coefficients”, say, of our \( F \), namely:

\[
\text{Win}_q F \overset{\text{def}}{=} \sum_{d \equiv 0 \mod q} F'(d) \frac{d}{d}, \quad \forall q \in \mathbb{N},
\]

converging (even absolutely) from (WA); this also implies the existence of all the following limits in all the, say, “Carmichael coefficients”, of our \( F \), where \( c_q(n) \) is the Ramanujan sum [R] of modulus \( q \) & argument \( n \) (we recall soon after), namely:

\[
\text{Car}_q F \overset{\text{def}}{=} \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} F(n)c_q(n), \quad \forall q \in \mathbb{N},
\]

with \( \varphi(q) \overset{\text{def}}{=} \{|n \leq q : (n, q) = 1\} \) the Euler totient function. Wintner [W] proved that (WA) \( \Rightarrow \) \( \text{Car} F = \text{Win}_q F \), namely \( \text{Car}_q F = \text{Win}_q F \), \( \forall q \in \mathbb{N} \) here. If all these \( q \)-coefficients exist, these two, \( \text{Car} F \), resp., \( \text{Win} F \), may be called, resp., CARMICHAEL TRANSFORM, resp., WINTNER TRANSFORM, of our \( F \). (Of course, existence implies uniqueness, for both these transforms; that are arithmetic functions, themselves.)

The Ramanujan smooth expansion of our \( F \), where \( c_q(a) \overset{\text{def}}{=} \sum_{j \leq \phi(j, q) = 1} \cos \frac{2\pi ja}{q} \) is the well-known Ramanujan sum [R], [M], holds with these coefficients, under (WA) (see next Theorem 1): \( \forall a \in \mathbb{N} \), fixed,

\[
F(a) = \lim_{P} \sum_{q \in (P)} (\text{Car}_q F)c_q(a) = \lim_{P} \sum (\text{Win}_q F)c_q(a).
\]

We will call hereafter \( \sum_{q=1}^{\infty} G(q)c_q(a) \) a Ramanujan smooth series, say, of coefficient \( G : \mathbb{N} \to \mathbb{C} \), by definition, when the limit \( \lim_{P} \sum_{q \in (P)} G(q)c_q(a) \) exists in \( \mathbb{C} \), for all natural \( a \).

A big warning is that the classical Ramanujan series, defined if \( \exists \lim_{Q} \sum_{q \leq Q} G(q)c_q(a) \in \mathbb{C} \), is A PRIORI different from this. (Compare \$5.1 \text{\ for the example of } G \text{\ constant}.)

Thus, all the results we consider (Lemmas, Theorems & Corollaries) are about this “smooth summation.” We write: \( \text{supp}(F) \overset{\text{def}}{=} \{n \in \mathbb{N} : F(n) \neq 0\} \) the support of any \( F : \mathbb{N} \to \mathbb{C} \). We start with our main results.
1.1. General Theorems for arithmetic functions

We give a kind of improvement, of Delange main result [De], inasmuch our partial sums are smooth: instead of (DH), we need (WA). In the following, QED is the end of a part of a Proof, ending with a □

Theorem 1. (Wintner’s “Dream Theorem”) Let $F : \mathbb{N} \rightarrow \mathbb{C}$ satisfy Wintner Assumption (WA). Then

$$
\forall a \in \mathbb{N}, \quad F(a) = \lim_{P \rightarrow \infty} \sum_{q \in (P)} (\text{Win}_q F) c_q(a) = \lim_{P \rightarrow \infty} \sum_{q \in (P)} (\text{Car}_q F) c_q(a).
$$

The additional hypothesis that $\text{Win} F$ is, say, smooth supported: $\text{supp}(\text{Win} F) \subseteq (Q)$ for some prime $Q$, gives

$$
\forall a \in \mathbb{N}, \quad F(a) = \sum_{q \leq Q} (\text{Win}_q F) c_q(a) = \sum_{q \leq Q} (\text{Car}_q F) c_q(a).
$$

In particular, in case $\text{supp}(\text{Win} F)$ is finite, say

$$
\exists Q \in \mathbb{N} : \text{Win}_q F = 0, \quad \forall q > Q,
$$

we have

$$
\forall a \in \mathbb{N}, \quad F(a) = \sum_{q \leq Q} (\text{Win}_q F) c_q(a) = \sum_{q \leq Q} (\text{Car}_q F) c_q(a).
$$

Proof. Fix $a \in \mathbb{N}$, take $P \geq a$, $P \in \mathbb{P}$, getting from Lemma 1, (3),

$$
F(a) = \sum_{d \in (P)} \frac{F'(d)}{d} \sum_{q \mid d} c_q(a);
$$

then Lemma 1, (4), together with Wintner assumption gives the following double series absolute convergence:

$$
\sum_{d \in (P)} \frac{|F'(d)|}{d} \sum_{q \mid d} |c_q(a)| \leq \sum_{d=1}^{\infty} \frac{|F'(d)|}{d} \sum_{q \in (P)} |c_q(a)| < \infty,
$$

allowing the exchange of these $d, q$ sums:

$$
F(a) = \sum_{q \in (P)} \left( \sum_{d \in (P)} \frac{F'(d)}{d} \right) c_q(a) = \sum_{q \in (P)} (\text{Win}_q F) c_q(a) - \sum_{q \in (P)} \left( \sum_{d \in (P)} \frac{F'(d)}{d} \right) c_q(a);
$$

another exchange, for these other sums, is possible for the same reason:

$$
\sum_{q \in (P)} \left( \sum_{d \in (P)} \frac{F'(d)}{d} \right) c_q(a) = \sum_{d \in (P)} \frac{F'(d)}{d} \sum_{q \in (P)} c_q(a),
$$

implying, from Lemma 1, (5):

$$
\left| \sum_{q \in (P)} \left( \sum_{d \in (P)} \frac{F'(d)}{d} \right) c_q(a) \right| \leq \sum_{d \in (P)} \frac{|F'(d)|}{d} \sum_{q \in (P)} c_q(a) \leq a \sum_{d \in (P)} \frac{|F'(d)|}{d} \rightarrow 0,
$$

completing the first part. QED The case of $\text{supp}(\text{Win} F) \subseteq (Q)$ follows from : $(Q) \subseteq (P), \forall P > Q$. QED

In particular, when $\text{supp}(\text{Win} F) \subseteq [1, Q]$, use : $q \leq Q \Rightarrow q \in (Q)$ and previous case. □

Remark 1. A shorter alternative proof (see §3) follows from Lemma 2.

\diamond
In what follows, we use the expression fixed length Ramanujan expansion to indicate a finite Ramanujan expansion \( \sum_{q \leq Q} G(q)c_q(a) \) where \( Q \in \mathbb{N} \) is an absolute constant (not \( a \)-dependent, in particular). In this paper, we will not use the expression finite Ramanujan expansion, since its length may depend on \( a \in \mathbb{N} \).

**Remark 2.** We proved in [C3] : \( F \) has a fixed length Ramanujan expansion \( \iff \supp(F') \) is finite. \( \diamond \)

In [CM], we gave many characterizations, for correlations \( F(a) = C_{f,g}(N,a) \overset{def}{=} \sum_{n \leq N} f(n)g(n+a) \), of the condition: \( \supp(F') \) is finite.

Even if we are considering smooth partial sums, in case they have fixed length, of course, they are the same of classical partial sums. In other words, fixed length partial sums, of course, do converge in any of the summation methods we choose! This trivial remark is applied, in next result : it characterizes the finiteness of Ramanujan series partial sums, say, whenever the Wintner coefficients are, in turn, finitely supported.

We recall the notation \( 0(n) \overset{def}{=} 0, \forall n \in \mathbb{N} \), for the null-function.

Two new characterizations arise, for FIXED LENGTH Ramanujan expansions; first one is a little bit technical, in next result, where second equivalence implies: \( F' \) finitely supported \( \iff F' \) smooth-supported.

**Theorem 2.** Let \( F : \mathbb{N} \to \mathbb{C} \) have finite \( \supp(\text{Win } F) \). Then

\[
\supp(F') \text{ is finite } \iff \lim_{P} \sum_{r \in P \atop r > 1} \frac{F'(dr)}{r} = 0(d) \iff \exists Q \in \mathbb{N} : \supp(F') \subseteq (Q).
\]

**Remark 3.** The series here is defined as

\[
\sum_{r \in P \atop r > 1} \frac{F'(dr)}{r} = \lim_x \sum_{r \in P \atop 1 \leq r \leq x} \frac{F'(dr)}{r},
\]

that exists in \( \mathbb{C} \) when \( \exists \text{Win } F \), as proved in Lemma 2, §2 (there, compare Remark 5).

**Proof.** We prove the first and the second equivalence in both directions, considering a large prime \( P \).

Since \( r \in P \) and \( r > 1 \) implies \( r > P \), whence \( dr > P \), \( \forall d \in \mathbb{N} \), first “\( \Rightarrow \)” follows.

From (7) of Lemma 2, also first “\( \Leftarrow \)” follows.

Second “\( \Leftarrow \)” follows from: \( r \in P \) and \( r > 1 \Rightarrow \exists p > P, p|r \Rightarrow F'(dr) = 0, \forall d \in \mathbb{N} \).

Finally, second “\( \Rightarrow \)” follows from first “\( \Leftarrow \)” and the triviality: \( \supp(F') \subseteq [1,Q] \Rightarrow \supp(F') \subseteq (Q) \). \( \Box \)

We give an important “summary”, for sufficient conditions to get Ramanujan smooth expansions, with Wintner coefficients.

**Remark 4.** Let \( F : \mathbb{N} \to \mathbb{C} \) have Win \( F \). If at least one of the following three hypotheses holds:

\[
F' \text{ has finite support } \quad \text{OR} \quad F' \text{ has smooth support } \quad \text{OR} \quad F \text{ satisfies Wintner Assumption},
\]

then \( F \) has a Ramanujan smooth expansion, with Wintner coefficients. We give an immediate justification, for this. From Theorem 1, (WA) \( \Rightarrow \) the thesis, while finite support implies, trivially, smooth support, too; then we restrict to smoothness of \( F' \) support: use Lemma 3, §3 and Remark 6.

Actually, Lemma 3 in §3 gives an equivalent condition for the Ramanujan smooth expansion, with Wintner coefficients.

In the forthcoming subsections we present:

\( \diamond \) in next subsection, an application to “correlations”, that satisfy a “reasonable hypothesis”;

\( \diamond \) then, in subsection 1.3, a particular, but noteworthy case of “reasonable correlation”: the 2k–twin primes correlation, in Hardy-Littlewood Conjecture; this is proved under Wintner Assumption (giving a new Conditional Proof stronger than the one we gave in [C0], under Delange Hypothesis).
A short glance to the following sections:

◇ Section 2, “Lemmata for the Theorems”, supplies the Lemmas for Theorems 1 & 2 Proofs: Lemma 1 gives elementary calculations; while, Lemma 2 is the core of present paper: it presents a kind of “arithmetic orthogonality”, realizing WINTNER’S P—ORTHOGONALITY DECOMPOSITION, after a decomposition into two orthogonal sets of indices, namely, the P—smooth and the P—sifted (here P is any fixed prime).

◇ Section 3, “A deeper look into Ramanujan smooth expansions: Ramanujan-Wintner smooth expansions”, gives a characterization of arithmetic functions having the Ramanujan smooth expansion, with Wintner coefficients, in Lemma 3. Also, it provides a shorter Proof for Theorem 1.

◇ Section 4, starting from an idea in [C1], gives “LOCAL EXPANSIONS” which have P—smooth coefficients (both Wintner’s & Carmichael’s) that converge to the coefficients in RAMANUJAN smooth expansions, compare “Theorem 1(Smooth Version)”. (A kind of stronger Theorem 1, under (WSA), Wintner’s Smooth Assumption, weaker than Wintner Assumption.) The properties of these P—smooth coefficients are then studied in three sets of Arithmetic Functions. From 8th-version onwards, we add Properties 1 and 2.

◇ Section 5, continuing to expose & generalize our elementary methods. Speaking about: “RAMANUJAN CLOUDS”: generalizations of Wintner Assumption (like the (WSA), in §4, quoted above, and beyond) & of the (R.E.E.F.), that we introduce for correlations in next §1.2; and further generalizations: of the RREF for arithmetic functions F with finite support for Win F, a kind of decomposition for F in two parts that are ANALYTIC (an entire function!) and IRREGULAR (from “Irregular Series”), a brief study of irregular series of multiplicative functions.

Then, we deepen two important issues, expanding previous version 5.
First, §5.6, we study the Counterexample 1 in third version of [C1], that proves: (BH) for correlations doesn’t imply the (R.E.E.F.), providing some interesting details for this very simple correlation. We add, from version 8 onwards, Curiosity 1.

Second, §5.7, we explicitly calculate P—smooth Carmichael-Wintner coefficients for the imaginary exponentials, whence for (BH)—correlations, proving that they all converge to classical Carmichael-Wintner coefficients, as P → ∞ in primes. A very important difference, from version 6 to 7, is a correction, i.e. q” definition.

◇ Last but not least: a glance at Euler products, links between Eratosthenes & Wintner Transforms “in Wintner’s style”, [W], with further Remarks, and a brief coming soon for future work, are in Section 6. Version 9 adds new results, “Crossing HORIZONTAL AND VERTICAL LIMITS”, in 6.3, to get the RREF.

1.2. Applications for the correlations satisfying Basic Hypothesis

Given two arithmetic functions f, g : N → C, for their CORRELATION $C_{f,g}(N,a) \overset{\text{def}}{=} \sum_{n \leq N} f(n)g(n + a)$, that has Eratosthenes Transform $C'_{f,g}(N,t) \overset{\text{def}}{=} \sum_{a | t} C_{f,g}(N,a)\mu(t/a)$, we assume [C1] the BASIC HYPOTHESIS:

\[(BH) \quad g(m) \overset{\text{def}}{=} \sum_{q \mid m, q \leq Q} g'(q), \forall m \in \mathbb{N}, \text{ with } Q \leq N, \text{ and } C_{f,g}(N,a) \text{ is fair,}\]

where the condition to be fair for $C_{f,g}(N,a)$ means that the dependence on a is only in the argument of $g(n+a)$ (not inside f, nor in g). The main consequences are given in Proposition 1 of [C1]; in particular, (BH) for $C_{f,g}(N,a)$ implies that $C'_{f,g}(N,d)$ satisfies Ramanujan Conjecture (from the boundedness of $C_{f,g}(N,a)$) and Carmichael-Wintner coefficients (i.e., Carmichael & Wintner coefficients are the same) of $C_{f,g}(N,a)$ are

\[\hat{g}(q) = \frac{\phi(q)}{\varphi(q)} \sum_{n \leq N} f(n)c_q(n), \text{ with } \hat{g}(q) \overset{\text{def}}{=} \sum_{d \leq Q; d \equiv 0 \mod q} \frac{g'(d)}{d}, \forall q \in \mathbb{N}.\]
(Since \(\text{supp}(\hat{g}) \subseteq [1, Q]\), also for these coefficients \([1, Q]\) contains their support: outside \([1, Q]\) they vanish !) The Ramanujan expansion with these coefficients (given in \((iii)\) of Theorem 1 [CM]) is called (see [C3], §4) the Ramanujan exact explicit formula:

\[
(\text{R.E.E.F.}) \quad C_{f,g}(N, a) = \sum_{q \leq Q} \left( \frac{\hat{g}(q)}{\varphi(q)} \sum_{n \leq N} f(n)c_q(n) \right)c_q(a), \quad \forall a \in \mathbb{N}.
\]

We start with our first application, the strongest, for the Correlations.

**Corollary 1.** (the (R.E.E.F.) follows from Basic Hypothesis and Wintner Assumption)

Let the correlation \(C_{f,g}(N, a)\) satisfy \((\text{BH})\) and \((\text{WA})\). Then, the (R.E.E.F.) holds.

**Proof.** From \((iii)\) of Proposition 1 in [C1], \((\text{BH})\) gives the finitely-supported Carmichael-Wintner coefficients above. Apply Theorem 1 to \(F(a) = C_{f,g}(N, a)\).

New characterizations follow, for the correlations with \((\text{BH})\) having the R.e.e.f., from Theorem 2.

**Corollary 2.** Let the correlation \(C_{f,g}(N, a)\) satisfy \((\text{BH})\). Then

\[
\text{supp}(C_{f,g}'(N, \cdot)) \subseteq (Q), \quad \text{for some} \quad Q \in \mathbb{N}
\]

and

\[
\lim_{r \to 0} \sum_{r \leq r \leq 1} C_{f,g}'(N, dr) = 0(d)
\]

are properties both equivalent to the R.E.E.F., of \(C_{f,g}(N, a)\).

**Proof.** Straightforward, from Theorem 2 for \(F(a) = C_{f,g}(N, a)\).

1.3. Another conditional Proof of Hardy-Littlewood Conjecture, under Wintner Assumption

The classical “Hardy-Littlewood Conjecture”, for \(2k\)-twin primes, is the asymptotic given, once fixed an even number \(2k\) \((k \geq 1)\), for the autocorrelation of von Mangoldt function \(\Lambda\) (see [T]) of shift \(2k\), namely \(C_{\Lambda,\Lambda}(N, 2k)\) (compare Conjecture B, page 42, in [HL]):

\[
(\text{H-L}) \quad C_{\Lambda,\Lambda}(N, 2k) \sim \mathcal{G}(2k)N, \quad \text{as} \quad N \to \infty,
\]

where the classical Singular Series is defined as:

\[
\mathcal{G}(2k) \overset{\text{def}}{=} \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)}c_q(2k) = 2 \prod_{p|2k, p > 2} \left( 1 + \frac{1}{p-1} \right) \prod_{p|2k} \left( 1 - \frac{1}{(p-1)^2} \right), \quad \forall k \in \mathbb{N}.
\]

As a very simple consequence of our Corollary 1, we get the following result, whose Proof we sketch here, closely following the Proof of Corollary 2 in [C0].

However, as we did in [C0], we first have to, say, “truncate”, the function \(g(m) = \sum_{d|m} g'(d)\) (by \(g'\) definition), with the \(N\)-truncated divisor sum called \(g_N(m) \overset{\text{def}}{=} \sum_{d|m, d \leq N} g'(d)\) because, then, for the correlation \(C_{f,g_N}(N, a)\), \((\text{BH})\) holds (but not for \(C_{f,g}(N, a)\), in general); we apply this to \(f = g = \Lambda\), getting \(g_N(m) = \Lambda_N(m) \overset{\text{def}}{=} - \sum_{d|m, d \leq N} \mu(d) \log d\), but for general \(f, g : N \to \mathbb{C}\) the equation \((1)\) in [C0] entails

\[
C_{f,g}(N, a) = C_{f,g_N}(N, a) + O\left( a \cdot \max_{n \leq N} |f(n)| \cdot \max_{N \leq q \leq N+a} |g'(q)| \right), \quad \forall a \in \mathbb{N},
\]

whence in particular

\[
(\text{T}) \quad C_{\Lambda,\Lambda}(N, a) = C_{\Lambda,\Lambda_N}(N, a) + O\left( a (\log N) (\log(N + a)) \right), \quad \forall a \in \mathbb{N}.
\]
We’ll use hereafter the \( O \)-notation of Landau \([D]\), equivalent to Vinogradov’s (in fact, \( A = O(B) \) amounts to \( A \ll B \), same for \( A = O_\epsilon(B) \) and \( A \ll_\epsilon B \)).

In fact, actually, Wintner Assumption (WA), instead of (DH), suffices to prove even more than (H-L).

**Corollary 3.** Assuming (WA) for \( C_{\Lambda,N} (N,a) \), i.e.,

\[
\sum_{d=1}^{\infty} \frac{1}{d} |C'_{\Lambda,N} (N,d)| < \infty,
\]

we get a kind of Hardy-Littlewood asymptotic formula, with an absolute constant \( c > 0 \), once \( k \in \mathbb{N} \) is fixed

\[
C_{\Lambda,N}(N,2k) = \mathcal{O}(2kN) + O \left( N e^{-c \sqrt{\log N}} \right).
\]

**Proof. (Sketch)** We first get the (R.E.E.F.) for \( C_{\Lambda,N}(N,a) \), from Corollary 1 above. Then, (T) above, say “Truncation Formula”, reduces a known calculation, performed in \([C0]\) Corollary 2 Proof, for the RHS (Right Hand Side) of the (R.E.E.F.) to the RHS above.

2. Lemmata for the Theorems

We recall hereafter \( 1_\wp \overset{\text{def}}{=} 1 \) if and only if the property \( \wp \) is true and otherwise \( \overset{\text{def}}{=} 0 \), in the formula (compare \([D]\) and \([M]\)):\( \forall n \in \mathbb{N}, \forall a \in \mathbb{Z} \)

\[
\sum_{q|n} c_q(a) = \sum_{q|n} \sum_{d|q} \mu \left( \frac{q}{d} \right) = \sum_{d|n} \mu \left( \frac{n}{d} \right) = \sum_{d|n} \mu(K) = 1_{n|a \cdot n},
\]

from Kluyver’s formula: \( \sum_{d|a,d|q} d \left( \frac{q}{d} \right) = c_q(a) [K] \) and Möbius inversion: \( \sum_{K|m} \mu(K) = 1_{m=1} \) (see \([T]\)).

Next elementary Lemma is most of our first Proof of Theorem 1, see §1.1 above. Recall \( p \)-adic valuation: as usual, \( v_p(a) \overset{\text{def}}{=} \max \{ K \in \mathbb{N}_0 : p^K|a \} \), where \( \mathbb{N}_0 \overset{\text{def}}{=} \mathbb{N} \cup \{0\} \). Recall also: \( \pi(x) \overset{\text{def}}{=} |\{ p \in \mathbb{P} : p \leq x \}| \).

**Lemma 1.** (Elementary Properties of Smooth Divisors)

Let \( P \) be a prime number and \( F : \mathbb{N} \rightarrow \mathbb{C} \) be any arithmetic function, with Eratosthenes transform \( F' \) (recall, \( F' \overset{\text{def}}{=} F \ast \mu \)). Then, \( \forall a \in \mathbb{N} \) FIXED,

\[
P \geq a \quad \Rightarrow \quad F(a) = \sum_{d \in (P)} \frac{F'(d)}{d} \sum_{q \in (P) \atop q|d} c_q(a);
\]

\[
\forall d \in \mathbb{N}, \quad \sum_{q \in (P) \atop q|d} |c_q(a)| \leq \sum_{q \in (P)} |c_q(a)| \leq 2^{(P)} a < \infty;
\]

\[
\forall d \in \mathbb{N}, \quad \sum_{q \in (P) \atop q|d} c_q(a) = \sum_{q \in (P) \atop q|d} c_d(a) = 1_{d|a} \cdot c_d(a), \text{ with } d(a) \overset{\text{def}}{=} \prod_{p \leq P} p^{v_p(a)} \text{, whence } 0 \leq \sum_{q \in (P) \atop q|d} c_q(a) \leq a.
\]

**Proof.** Fix \( a \in \mathbb{N} \), take \( P \geq a, \ P \in \mathbb{P} \), write \( 1_{d|a} \) from (2), getting

\[
F(a) = \sum_{d \in (P) \atop d|a} F'(d) = \sum_{d \in (P)} \frac{F'(d)}{d} \sum_{q \in (P) \atop q|d} c_d(a) = \sum_{d \in (P)} \frac{F'(d)}{d} \sum_{q \in (P) \atop q|d} c_q(a),
\]
Lemma 2

Both vital, for our arguments.

Let

\[ \sum_{q \in (P)} |c_q(a)| = \prod_{p \leq P} \left( \sum_{K \geq 0} \varphi(p^K) + p^{\nu_p(a)} \right) = \prod_{p \leq P} \left( 2p^{\nu_p(a)} \right) \leq 2^{\pi(P)} a, \]

providing (4).

QED

The condition “\( q \in (P) \) and \( q \mid d' \)”, by definition of \( d(P) \), is equivalent to the single condition \( q \) divides \( d(P) \), so (2) with \( n = d(P) \) entails (5).

The Lemma is completely settled.

QED

See that the main reason why our Theorem 1 works for Ramanujan smooth expansions but not for Ramanujan expansions is inside property (5) above; in fact, if we wish, say, to get the same \( d \)–independent bound for usual partial sums, we should consider (as Wintner does explicitly, see [W] page 31)

\[ \sum_{q \leq Q \atop q \mid d} c_q(a), \]

which has not a closed expression similar to the one in (5): this time, the multiplicative structure is, say, broken by the interval constraint.

Note the, say, very simple structure of Lemma 1: once added (WA), the proof of Theorem 1 is immediate.

We wish to prove a kind of equivalence condition, for the convergence for Ramanujan smooth series with Wintner coefficients (see Lemma 3, next section). So, in next Lemma we, say, decompose in a regular part (over \( P \)–smooth numbers), containing Wintner coefficients, and an irregular part (over \( P \)–sifted numbers), containing Eratosthenes transform. We do apply this decomposition in §3: an alternative (much) shorter Proof of Theorem 1, then, is immediate. We might say that Wintner’s Dream Theorem is a straightforward application of Wintner’s (\( P \)–)Orthogonal Decomposition, i.e., next Lemma 2.

From Möbius inversion \([T]\) quoted above, abbreviating \( \mu \) def \( \prod_{p \leq P} p \), we get the useful formulae:

\[ 1_{(a, b) = 1} = \sum_{K \mid a \atop K \mid b} \mu(K), \quad \forall a, b \in \mathbb{N} \Rightarrow 1_{r \in P} = \sum_{K \mid r \atop K \in (P)} \mu(K) = \sum_{K \mid r \atop K \in (P)} \mu(K). \]

A kind of “arithmetic orthogonality among indices”, say, allows to decompose \( F' \) in (7), then \( F \) in (8): both vital, for our arguments.

Lemma 2. (WINTNER ORTHOGONAL DECOMPOSITION)

Let \( F : \mathbb{N} \to \mathbb{C} \) have all Wintner orthogonal coefficients, say \( \exists \text{Win } F : \mathbb{N} \to \mathbb{C} \). Then

\[ \forall d \in \mathbb{N}, \forall P \in \mathbb{P}, \quad F'(d) = d \sum_{K \in (P)} \mu(K) (\text{Win}_{dK} F) - \sum_{r \in P \atop r > 1} \frac{F'(dr)}{r}, \]

whence

\[ \forall d \in \mathbb{N}, \quad F'(d) = \lim_{P} \left( d \sum_{K \in (P)} \mu(K) (\text{Win}_{dK} F) - \sum_{r \in P \atop r > 1} \frac{F'(dr)}{r} \right). \]

If we join the hypothesis: \( \text{Win } F \) smooth-supported, say \( \text{supp}(\text{Win } F) \subseteq (Q) \), we get

\[ \forall d \in \mathbb{N}, \quad F'(d) = \mathbf{1}_{d \in (Q)} \cdot d \cdot \sum_{K \in (Q)} \mu(K) (\text{Win}_{dK} F) - \lim_{P} \sum_{r \in P \atop r > 1} \frac{F'(dr)}{r}, \]
whence in particular for finite support, say, $\text{supp}(\text{Win} F) \subseteq [1,Q]$, this entails

$$
\forall d \in \mathbb{N}, \quad F'(d) = d \sum_{K \leq \frac{Q}{d}} \mu(K) \left( \text{Win}_{dK} F \right) - \lim_{P \to \infty} \sum_{r \in \mathbb{P} \cap \left( \mathbb{N} \cup \left\{ 0 \right\} \right)} \frac{F'(dr)}{r}.
$$

Summing (7) over the divisors $d$ of $a$, we obtain (however $P \in \mathbb{P}$, here)

$$
(8) \quad \forall a \in \mathbb{N}, \forall P \geq a, \quad F(a) = \sum_{q \in (P)} (\text{Win}_q F) c_q(a) - \sum_{d|a} \sum_{r \in \mathbb{P} \cap \left( \mathbb{N} \cup \left\{ 0 \right\} \right)} \frac{F'(dr)}{r},
$$

whence

$$
\forall a \in \mathbb{N}, \quad F(a) = \lim_{P} \left( \sum_{q \in (P)} (\text{Win}_q F) c_q(a) - \sum_{d|a} \sum_{r \in \mathbb{P} \cap \left( \mathbb{N} \cup \left\{ 0 \right\} \right)} \frac{F'(dr)}{r} \right).
$$

This time, $\text{supp}(\text{Win} F) \subseteq (Q)$ gives

$$
\forall a \in \mathbb{N}, \quad F(a) = \sum_{q \in (Q)} (\text{Win}_q F) c_q(a) - \lim_{P} \sum_{d|a} \sum_{r \in \mathbb{P} \cap \left( \mathbb{N} \cup \left\{ 0 \right\} \right)} \frac{F'(dr)}{r},
$$

in particular $\text{supp}(\text{Win} F) \subseteq [1,Q]$ entails

$$
\forall a \in \mathbb{N}, \quad F(a) = \sum_{q \leq Q} (\text{Win}_q F) c_q(a) - \lim_{P} \sum_{d|a} \sum_{r \leq \frac{Q}{d}} \frac{F'(dr)}{r}.
$$

**Remark 5.** The $r$–series above (defined in Remark 3, §1) is called the **Irregular series**, $\text{Irr}_d^{(P)} F$, of argument $d \in \mathbb{N}$, over the prime $P \in \mathbb{P}$, relative to $F : \mathbb{N} \to \mathbb{C}$, and the following Proof implies it converges in $\mathbb{C}$, when $\text{Win} F$ exists. \[ \Diamond \]

**Proof.** In order to prove (7), we fix $d \in \mathbb{N}$ and $P \in \mathbb{P}$, considering

$$
\sum_{r \in \mathbb{P}(x)} \frac{F'(dr)}{r} = \lim_{x} \sum_{r \leq x} \frac{F'(dr)}{r} = \lim_{x} \sum_{r \leq x} \frac{F'(dr)}{r} - \sum_{K \not\in (P)} \mu(K) = d \lim_{x} \sum_{K \in (P)} \mu(K) \sum_{r \leq \frac{Q}{d} \mod K} \frac{F'(dr)}{dr},
$$

thanks to (6); the $K$–sum, thanks to $\mu(K)$, is over the square-free $K$ and, furthermore, the condition that $K$ divides the $P$–primorial (abbreviated $P^*$) amounts to $K \in (P)$, from: $K$ square-free; in all, thanks to the fact: $\mu$ is supported in square-free numbers, say *Möbius vertical limit*, this $K$–sum is **finite and clearly NOT DEPENDING ON** $x$, giving:

$$
\sum_{r \in \mathbb{P}(x)} \frac{F'(dr)}{r} = d \sum_{K \in (P)} \mu(K) \lim_{x} \sum_{r \leq \frac{Q}{d} \mod K} \frac{F'(dr)}{dr} = d \sum_{K \in (P)} \mu(K) (\text{Win}_{dK} F),
$$

thanks to the definition of Wintner coefficients (all series converging for them, since $\exists \text{Win} F$).

Separating the contribute of $r = 1$ in the $r$–series settles (7) proof. \[ \text{QED} \]

Joining $\text{supp}(\text{Win} F) \subseteq (Q)$, whenever $P \geq Q$, then (7), in particular for the case $\text{supp}(\text{Win} F) \subseteq [1,Q]$, entails both the two particular formulae, after (7). \[ \text{QED} \]

Then, (8) comes from (7) summing over $d|a$, with Kluhver formula and: $d|a, P \geq a \Rightarrow d \in (P)$. \[ \text{QED} \]

The two particular formulae after (8) follow from (8), as we saw for (7), above. \[ \Box \]

**Remark 6.** As it’s clear from the Proof, in case $\text{supp}(\text{Win} F) \subseteq (Q)$ (in particular, whenever we have $\text{supp}(\text{Win} F) \subseteq [1,Q]$, too) we get that the **Irregular series** (defined in Remark 3), $\text{Irr}_d^{(P)} F$, is constant $\forall P \geq Q$, *W.R.T. the prime $P$, uniformly in the argument $d \in \mathbb{N}$*

$$
(9) \quad \sum_{r \in \mathbb{P}(Q)} \frac{F'(dr)}{r} = \sum_{r \leq \frac{Q}{d}} \frac{F'(dr)}{r}, \quad \forall P > Q \Rightarrow \lim_{P} \sum_{r \in \mathbb{P}(Q)} \frac{F'(dr)}{r} = \sum_{r \leq \frac{Q}{d}} \frac{F'(dr)}{r},
$$

i.e., the LHS (Left Hand Side) of (9), as a function of $P \in \mathbb{P}$, is constant $\forall P \geq Q$, uniformly $\forall d \in \mathbb{N}$. Then, notice that (assuming, as we can, that our $Q$ is prime) when a fortiori $\text{supp}(F') \subseteq (Q)$, we have from (9): $\lim_{P} \text{Irr}_d^{(P)} F = \text{Irr}_d^{(Q)} F = 0(d)$, because $F'(dr) = 0(d), \forall r \in (Q \setminus \left\{ 1 \right\})$.

\[ \Diamond \]
3. A deeper look into Ramanujan smooth expansions: Ramanujan-Wintner smooth expansions

A more careful analysis yields in fact the more general result, for THE Ramanujan smooth expansion, with Wintner coefficients. We write THE to highlight its uniqueness, clear from the choice of coefficients $G(q) := \text{Win}_q F$. As we’ll see in section 5.1, once fixed $F$ (esp., $F = 0$), we may have many $G$, in a Ramanujan smooth expansion.

**Lemma 3. (characterizing $F$ having Ramanujan-Wintner smooth expansion)**

Let $F : \mathbb{N} \to \mathbb{C}$ have all the Wintner coefficients. Then,

$$F'(d) = \lim_{P} \sum_{r \in \mathbb{N}} \frac{F'(dr)}{r} = 0,$$

whence, $\forall a \in \mathbb{N}$ fixed,

$$F(a) = \lim_{P} \sum_{q \in \mathbb{N}} (\text{Win}_q F) c_q(a) \iff \lim_{P} \sum_{d|a} \sum_{r \in \mathbb{N}} \frac{F'(dr)}{r} = 0.$$

**Proof.** From Lemma 2, passing to the limit over $P \in \mathbb{P}$, we get first equivalence from (7) and second one from (8).

We give an easy property (next Proposition), connecting $|F'| \ast 1$ to $F$.

From above Lemma 3 and the trivial implication, $\forall d \in \mathbb{N}$,

$$\lim_{P} \sum_{r \in \mathbb{N}} \frac{|F'(dr)|}{r} = 0 \Rightarrow \lim_{P} \sum_{r \in \mathbb{N}} \frac{F'(dr)}{r} = 0$$

we easily prove the following. We abbreviate “RWE”, for “Ramanujan-Wintner expansion”: Ramanujan expansion with Wintner coefficients. Joining “smooth”, hereafter, amounts, as above, to requiring smooth partial sums.

**Proposition 1.** Given any $F : \mathbb{N} \to \mathbb{C}$, we have

$$|F'| \ast 1 \text{ has smooth RWE} \Rightarrow F \text{ has smooth RWE}.$$

Notice that, actually, this can also be proved following Theorem 1 proof in §1.

By the way, we give now a shorter proof of this Theorem.

**Alternative proof of Wintner’s Dream Theorem**

**Proof.** Using (8) of Lemma 2, i.e., applying Wintner Orthogonal Decomposition to $F$, it suffices to prove:

$$(WA) \Rightarrow \lim_{P} \sum_{r \in \mathbb{N}} \frac{F'(dr)}{r} = 0(d),$$

so, fix $d \in \mathbb{N}$ and consider:

$$\sum_{r \in \mathbb{N}} \frac{|F'(dr)|}{dr} \leq \sum_{m \geq dP} \frac{|F'(m)|}{m} \leq \sum_{n \geq P} \frac{|F'(n)|}{n}$$

is infinitesimal, as $P \to \infty$. □
4. Smooth coefficients in Ramanujan expansions

In our “A smooth shift approach for a Ramanujan expansion”, [C1], we introduced the smooth restriction, to $P$—smooth numbers (here $P \in \mathbb{P}$ is fixed), of any given arithmetic function $F : \mathbb{N} \rightarrow \mathbb{C}$,

$$F_{(P)}(a) \overset{\text{def}}{=} \sum_{d|a} F'(d), \quad \forall a \in \mathbb{N},$$

which is, so to speak, the origin of Carmichael’s & Wintner’s “$P$—smooth coefficients”:

$$\text{Car}_q^{(P)} F \overset{\text{def}}{=} \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{a \leq x} \left( \sum_{d|a} F'(d) \right) c_q(a) = \text{Car}_q F_{(P)}, \quad \forall q \in \mathbb{N}$$

and

$$\text{Win}_q^{(P)} F \overset{\text{def}}{=} \sum_{d \in (P)_q} \frac{F'(d)}{d} = \text{Win}_q F_{(P)}, \quad \forall q \in \mathbb{N}.$$

We say in the following that they exist, whenever the relative limits exist in $\mathbb{C}$ (for Wintner’s, the limit of partial sums, i.e., the series converges in $\mathbb{C}$). The existence of all coefficients (for $P$ fixed), $\forall q \in \mathbb{N}$, is expressed saying : $\exists \text{Car}_q^{(P)} F$ or, resp., $\exists \text{Win}_q^{(P)} F$. They are, resp., the Carmichael $P$—smooth transform and Wintner $P$—smooth transform; when they both exist and are the same, we indicate them as $F^{(P)}$, say the Carmichael-Wintner $P$—smooth transform. In [C1] we gave Theorem 1 & Corollary 1, which we will generalize here to the following Theorem 1’[C1] & Corollary 1’[C1].

The interest in these $P$—smooth transforms comes from the following Theorem 1’[C1], giving a kind of “limit expansion”, for any “reasonable”, say, arithmetic function $F : \mathbb{N} \rightarrow \mathbb{C}$ we mean that $\exists \text{Win}_q^{(P)} F$, $\forall P \in \mathbb{P}$.

It’s based on next elementary Lemma, an immediate application of “RAMANUJAN VERTICAL LIMIT”:

(RVl)\[c_q(a) \neq 0 \Rightarrow v_p(q) \leq v_p(a) + 1, \quad \forall p|q.\]

**Lemma 4.** Let $F : \mathbb{N} \rightarrow \mathbb{C}$ have all the $P$—smooth $q$—th Wintner coefficients: $\exists \text{Win}_q^{(P)} F$, $\forall P \in \mathbb{P}, \forall q \in \mathbb{N}$. Then

$$\forall a \in \mathbb{N}, \forall P \in \mathbb{P}, P \geq a, \quad F(a) = \sum_{q \in (P)} \left( \text{Win}_q^{(P)} F \right) c_q(a),$$

whence

$$\forall a \in \mathbb{N}, \quad F(a) = \lim_{P \to \infty} \sum_{q \in (P)} \left( \text{Win}_q^{(P)} F \right) c_q(a).$$

**Proof.** Fix $a \in \mathbb{N}$ and choose a prime $P \geq a$ so that $d|a \Rightarrow d \in (P)$ and from (2)

$$F(a) = \lim_{x \to \infty} \sum_{d|a, d \leq x} F'(d) = \lim_{x \to \infty} \sum_{d \in (P), d \leq x} \frac{F'(d)}{d} \sum_{q|d} c_q(a),$$

whence $d \in (P), q|d \Rightarrow q \in (P)$ proves, from (RVl), that the sum over $q$ is both finite (in terms of $a \in \mathbb{N}$ and $P \geq a, P \in \mathbb{P}$) and is not dependent on $x$ (going to infinity); giving

$$F(a) = \sum_{q \in (P)} c_q(a) \lim_{x \to \infty} \sum_{d \in (P), d \leq x \text{ d\div q}} \frac{F'(d)}{d} = \sum_{q \in (P)} \left( \text{Win}_q^{(P)} F \right) c_q(a),$$

from the definition of $P$—smooth $q$—th Wintner coefficient. □

**Remark 7.** See that, usually, the exchange of two summations, typically over $d$ and $q$ like in the above, needs a double series (over $d, q$) absolute convergence, while here the (RVl) property allows weaker hypotheses. Also, notice that (whatever $P \in \mathbb{P}$ is fixed) the condition: $\exists \text{Win}_q^{(P)} F$, $\forall q \notin (P)$, **is not strictly required.**
This Lemma is very powerful: each time we have hypotheses ensuring the existence of all the $P$-smooth $q$-th Wintner coefficients, we get a kind of “Ramanujan-Wintner local expansion” (with a smooth summation of partial sums & $P$-smooth Wintner coefficients). The only problem is the, say, “local nature of coefficients”, that usually are unknown; while, of course, Wintner coefficients have better chances to be easily calculated: for example, under suitable hypotheses, they are exactly the Carmichael coefficients. This happens under (WA) above, as proved by Wintner (see [C3]).

The same (WA) ensures that

$$\lim_{P} \text{Win}_q^{(P)} F = \text{Win}_q F, \quad \text{uniformly } \forall q \in \mathbb{N},$$

thanks to the absolute convergence for the series inside (WA).

Wintner Assumption, actually, suffices (see Theorem 1) for $F : \mathbb{N} \to \mathbb{C}$ to get THE RAMANUJAN-WINTNER SMOOTH EXPANSION for $F$.

Can we get the same expansion under a weaker hypothesis? Well, our Theorem 1 proof reveals this “at once”. From the point of view of Wintner coefficients, next result is, in fact, a generalization of our Theorem 1 above. We give it here, as its hypotheses are a bit more technical than Theorem 1 ones.

WINTNER’S SMOOTH ASSUMPTION (WSA), following, is a less general constraint than (WA) above:

$$(\text{WSA}) \quad \lim_{P} \sum_{d \in \mathcal{P}(P)} \frac{|F'(d)|}{d} = 0$$

where we implicitly agree that: given our $F : \mathbb{N} \to \mathbb{C}$, there exists a prime $P_0$ (depending ONLY on $F$), such that each series above over $d \notin (P)$ converges $\forall P > P_0$ and, then, above limit over $P$ exists and vanishes. This (WSA) alone proves that $F$ converges, with smooth partial summations, to its Ramanujan-Wintner Smooth expansion and this is already proved, in Theorem 1 proof (see its end)!

Joining two technical hypotheses about (classic & smooth) Wintner coefficients we also get, say for free, other two informations: see next result.

**Theorem 1 (Smooth version).** Let $F : \mathbb{N} \to \mathbb{C}$ have all the $q$-th Wintner coefficients (i.e., $\exists \text{Win} F$) and all the $P$-smooth $q$-th Wintner coefficients (i.e., $\exists \text{Win}^{(P)} F, \forall P$), $\forall P \in \mathbb{P}, \forall q \in \mathbb{N}$. Assume (WSA). Then

$$(*) \quad \lim_{P} \text{Win}_q^{(P)} F = \text{Win}_q F, \quad \text{uniformly } \forall q \in \mathbb{N}$$

and

$$(**) \quad \lim_{P} \text{Irr}_d^{(P)} F = 0(d), \quad \text{pointwisely } \forall d \in \mathbb{N},$$

whence

$$(***) \quad F(a) = \lim_{P} \sum_{q \in \mathcal{P}(P)} (\text{Win}_q F) c_q(a), \quad \text{pointwisely } \forall a \in \mathbb{N}.$$

**Proof.** Above (*) follows immediately from

$$\left| \text{Win}_q F - \text{Win}_q^{(P)} F \right| \leq \sum_{d \in \mathcal{P}(P)} \frac{|F'(d)|}{d} \leq \sum_{d \in \mathcal{P}(P)} \frac{|F'(d)|}{d};$$

for (**) use Lemma 2 to prove the convergence, whence existence of $\text{Irr}_d^{(P)} F$ for all $P > P_0$ and $\forall d \in \mathbb{N}$, then $\forall d \in \mathbb{N}$ FIXED

$$\left| \text{Irr}_d^{(P)} F \right| \leq \sum_{r \in \mathcal{P}(P)} \frac{|F'(dr)|}{r} \leq d \sum_{m \in \mathcal{P}(P)} \frac{|F'(m)|}{m},$$

as $m = dr$ with $r \in P(\text{ and } r > 1)$ imply $\exists p > P : p|m \Rightarrow m \notin (P)$; for (***) use (**) just proved and the characterization of Lemma 3.

\[ \Box \]
Remark 8. The main “defect”, so to speak, is the fact that the coefficients may change, as P changes. In particular, all \( F \) supported over \( P \) are “Carmichael-Wintner” \( P \)-smooth coefficients (i.e., \( \hat{F}(P)(q) \) above).

**Proof.** First of all, the explicit formula above for \( \hat{F}(P)(q) \) was proved in Th.m 1, (i) [C1] (in which these coefficients were born). The formula for \( F \), then, was proved in Th.m 1 of Theorem 1 proof, in case \( F' \) is supported over \( P \)-smooth numbers: this is implicit here, assuming \( P \geq a \). So, present second part is more general than Theorem 1 in [C1].

Its immediate application follows, to the Correlations. Again, we join [C1] to distinguish from present Corollary 1.

**Corollary 1’[C1].** Let \( C_{f,g}(N,a) \), the correlation of any couple \( f,g : \mathbb{N} \to \mathbb{C} \), satisfy Ramanujan Conjecture. Then

\[
\forall a \in \mathbb{N}, \forall P \in \mathbb{P}, P \geq a, F(a) = \sum_{q \in (P)} \hat{F}(P)(q)c_q(a),
\]

whence

\[
\forall a \in \mathbb{N}, F(a) = \lim_{P \to \infty} \sum_{q \in (P)} \hat{F}(P)(q)c_q(a).
\]

In particular, all \( F : \mathbb{N} \to \mathbb{C} \) satisfying Ramanujan Conjecture are pointwise limits, over primes \( P \to \infty \), of “finite Ramanujan expansions”, with “Carmichael-Wintner” \( P \)-smooth coefficients (i.e., \( \hat{F}(P)(q) \) above).

**Remark 9.** The main “defect”, so to speak, is the fact that the coefficients may change, as \( P \) changes. ☣

**Proof.** Apply Theorem 1'[C1] to \( F(a) = C_{f,g}(N,a) \).

More in general, Ramanujan Conjecture for \( F \) is not required, if we wish to get the existence of all \( \hat{F}(P)(q) \) above. However, the existence of all \( \hat{F}(P)(q) \) also follows from another hypothesis for \( F \):

\[
\exists \delta < 1 : F(a) \ll \delta a^d, \quad a \to \infty.
\]

This is, by Möbius inversion [T], equivalent to the same for \( F' \), the *Eratosthenes Transform* of our \( F \).

This property of Neat Sub-Linearity, actually, implies even more than the existence of all \( \hat{F}(P)(q) \) above.

\[
\sum_{d \mid q \text{ odd}} \frac{|F'(d)|}{d} \ll \delta \sum_{d \in (P)} d^{\delta - 1} \ll \delta, P 1, \quad \text{UNIFORMLY } \forall q \in \mathbb{N}.
\]

This is another application of (1) above.
i.e. $F$ is square-free supported, say, $F$ “ignores Prime-Powers”. Equivalently, $F(a)$ depends ONLY on $a = \prod_{p | a} p$ (with $\kappa(1) \equiv 1$ for the void product), the square-free kernel of $a \in \mathbb{N}$: we express this as $F = F \circ \kappa$ (with “o”, here, the usual composition of functions), namely $F(a) = F(\kappa(a)), \forall a \in \mathbb{N}$.

In fact, (IPP) implies that

$$\forall q \in \mathbb{N}, \sum_{d \leq x \atop d \equiv 0 \mod q} \frac{F'(d)}{d} = \sum_{d \leq x \atop d \equiv 0 \mod q} \frac{\mu^2(d)F'(d)}{d}$$

has a finite limit in complex numbers, as $x \to \infty$, since previous summation’s support is bounded, having cardinality bounded uniformly $\forall q \in \mathbb{N}$ as

$$\left| \left\{ d \in (P) : \mu^2(d) = 1 \right\} \right| = 2^{\pi(P)}.$$

(Recall: $\pi(P) =$ number of primes $\leq P$ and all square-free $d$ with prime factors $\leq P$ are $2^{\pi(P)}$, of course.)

For classic Carmichael & Wintner coefficients, Wintner discovered their coincidence, whenever His (WA) holds.

Actually, a little bit more generally, under the following hypothesis:

(EQT) $$\lim_{x \to \infty} \frac{1}{x} \sum_{d \leq x} |F'(d)| = 0,$$

say, “ERATOSTHENES TRANSFORM DECAY”, equivalent to the vanishing of $|F'(d)|$ mean-value (esp., see [C3], Remark 7), we get Car $F = $ Win $F$ again, from following Lemma (compare the proof of (5) in [C3]).

See that (WA) $\Rightarrow$ (ETD) (from quoted proof), but the converse implication doesn’t hold (esp., we may take $F'(d) = 1/\log d, \forall d > 1$).

However, just like (ETD) implies coincidence of Carmichael & Wintner coefficients, say a classic consequence, it also implies, for all fixed primes $P$, the coincidence, say, of Carmichael & Wintner $P$–smooth transforms: Win$_{(P)}^F F = $ Car$_{(P)}^F F$. These two consequences for $F$, under (ETD), hold thanks to the next Lemma. (In which we express the proximity of partial sums up to $x \in \mathbb{N}$, say; in fact, the coefficients exist, by definition, if and only if the $x$–limit exists in complex numbers.)

**Lemma 5.** (Links between classic & smooth Carmichael/Wintner coefficients). Given any $F : \mathbb{N} \to \mathbb{C}, \forall P \in \mathbb{P}, \forall q \in \mathbb{N}, \exists C(q) > 0$ such that, $\forall x \in \mathbb{N},$

$$\left| \frac{1}{x} \sum_{a \leq x} F(a) \frac{\zeta(a)}{\varphi(q)} - \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d} \right| \leq \frac{C(q)}{x} \sum_{d \leq x} |F'(d)|$$

and

$$\left| \frac{1}{x} \sum_{a \leq x} \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d} \frac{\zeta(a)}{\varphi(q)} - \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d} \right| \leq \frac{C(q)}{x} \sum_{d \leq x} |F'(d)|.$$

**Remark 10.** Notice that the positive constant $C(q)$ depends only on $q \in \mathbb{N}$. \n
We briefly prove this Lemma from the following elementary “fact” (a kind of short Lemma).

**Fact 1.** Once fixed $d, q \in \mathbb{N}$, we get $\sum_{m \leq x} c_q(dm) = \mathbf{1}_{q | d} \cdot \varphi(q) \cdot \frac{x}{d} + O_q(1), \forall x \in \mathbb{N}$.

In fact, use $c_q(dm) = \sum_{j \in \mathbb{Q}^*} c_q(jdm)$, whence $q \nmid d \Rightarrow \sum_{m \leq x} c_q(dm) = O \left( \sum_{j \in \mathbb{Q}^*} \sum_{d \leq x} \frac{1}{(jdm)^2} \right) = O_q(1).$
**Proof (Lemma 5).** We prove second inequality (first is similar), exchanging sums & applying Fact 1:

\[
\frac{1}{x} \sum_{a \leq x} \sum_{d|a} F'(d) \varphi(a) = \frac{1}{x} \sum_{d \leq x} F'(d) \cdot \frac{1}{\varphi(q)} \sum_{m \leq \frac{x}{d}} c_q(dm) = \sum_{d \leq x} \frac{F'(d)}{d} + O_q \left( \frac{1}{x} \sum_{d \leq x} |F'(d)| \right).
\]

From previous Lemma, we get, for \(P\)-smooth coefficients, using \(\exists \text{Win}(P)\), \(\forall P \in \mathbb{P}\), as we saw, for the \(F : \mathbb{N} \to \mathbb{C}\) satisfying (NSL) or (IPP), the equation \(\text{Car}(P) F = \text{Win}(P) F\), \(\forall P \in \mathbb{P}\), too. In fact, the remainder in previous lemma goes to 0 as \(x \to \infty\): under (NSL) by (1), while under (IPP) because the \(d - \text{sum}\) is bounded (w.r.t. \(x\)). Thus, previous Lemma 5 implies next Lemma 6.

**Lemma 6.** (Two conditions for coincidence of \(P\)-smooth Carmichael/Wintner transforms). Let \(F : \mathbb{N} \to \mathbb{C}\) satisfy (NSL) or (IPP). Then, \(\forall P \in \mathbb{P}\), \(\text{Car}(P) F = \text{Win}(P) F\).

The hypothesis (ETD) is only able to prove \(\text{Car}(P) F = \text{Win}(P) F\) (from Lemma 5 above), when we already know that \(\exists \text{Car}(P) F \or \exists \text{Win}(P) F\). Notwithstanding its greater generality w.r.t. (WA), our (ETD) can only prove \(\text{Car} F = \text{Win} F\) (from quoted Lemma) if we know, again, that at least one of these transforms exists. (Of course, instead, (WA) \(\Rightarrow \exists \text{Win} F\), immediately.)

Applying Lemma 6, we easily prove (leaving as exercises) the following two properties of uniqueness, for the \(P\)-smooth Carmichael-Wintner coefficients, relative to the two classes of functions: (IPP) & (NSL).

We start with the more “analytic”, so to speak, class, namely the (IPP) functions.

**Property 1.** Let \(F\) be (IPP). Then, fix a prime \(P\), getting:

1. \(\exists \text{Win}(P) F, \exists \text{Car}(P) F\) and \(\text{Car}(P) F = \text{Win}(P) F\) is SQUARE-FREE SUPPORTED
2. COEFFICIENTS IN \((*)_{(P)}\) ARE UNIQUE:

\[
G_P : \mathbb{N} \to \mathbb{C} \text{ with } \text{supp}(G_P) \subseteq (P), G_P = \mu^2 \cdot G_P \text{ and}
\]

\[
(*) : \forall a \in \mathbb{N}, \quad F_{(P)}(a) = \sum_{q \leq \prod_{r \leq P} q_r} G_P(q) c_q(a)
\]

ENTAIL \(G_P = \text{Win}(P) F\).

We come to the more “arithmetical”, so to speak, class, namely the (NSL) functions.

**Property 2.** Let \(F\) be (NSL). Then, fix a prime \(P\), getting:

1. \(\exists \text{Win}(P) F, \exists \text{Car}(P) F, \text{Car}(P) F = \text{Win}(P) F\) and for \(q \in (P), \text{Car}_q(P) F = \text{Win}_q(P) F = O_{\varepsilon}(q^{-\varepsilon})\)
2. COEFFICIENTS IN \((*)_{(P)}\) ARE UNIQUE:

\[
G_P : \mathbb{N} \to \mathbb{C} \text{ with } \text{supp}(G_P) \subseteq (P), G_P(q) = O_{\varepsilon}(q^{-\varepsilon}), \forall q \in (P) \text{ and}
\]

\[
(*) : \forall a \in \mathbb{N}, \quad F_{(P)}(a) = \sum_{q \in (P)} G_P(q) c_q(a)
\]

ENTAIL \(G_P = \text{Win}(P) F\).

(Hint: both Proofs use that in an absolutely converging double-series we may exchange summations.)

We wish, here, to introduce next section, with new elementary methods.
5. General elementary methods introducing new ideas

We gather some complementary results, having elementary proofs, which supply standard new methods for the study of Ramanujan expansions: especially the ones with smooth summation and/or Wintner coefficients.

5.1. Ramanujan Clouds

We start with a very easy result that connects absolutely convergent and smooth summation convergent Ramanujan expansions. We recall, for this reason, the notation (compare [C3]) for Ramanujan Clouds:

\[< F > \overset{df}{=} \left\{ G : \mathbb{N} \to \mathbb{C} \mid \forall a \in \mathbb{N}, F(a) = \sum_{q=1}^{\infty} G(q)c_q(a) \right\} \]

is the Ramanujan cloud of our \( F \), namely the set of “classic”, say, Ramanujan coefficients, for a fixed \( F : \mathbb{N} \to \mathbb{C} \); then, we have another set of Ramanujan coefficients for \( F \), constituting the Ramanujan smooth cloud of our \( F \):

\[\subset F \supset \overset{df}{=} \left\{ G : \mathbb{N} \to \mathbb{C} \mid \forall a \in \mathbb{N}, F(a) = \lim_{P} \sum_{q \in (P)} G(q)c_q(a) \right\} \]

where, in fact, we take (for \( P \in \mathbb{P} \)) the \( P \)-smooth partial sums’ limit over \( P \in \mathbb{P} \). We complete the notation with the Ramanujan absolute cloud of our \( F \):

\[< F >_{\text{ABS}} \overset{df}{=} \left\{ G \in< F > \mid \forall a \in \mathbb{N}, \sum_{q \in \mathbb{N}} |G(q)c_q(a)| < \infty \right\} , \]

the set of classic Ramanujan coefficients of our \( F \), in ABSOLUTELY converging \( F \) Ramanujan expansions.

We start noticing that, for the null-function \( 0 \) we have \( < 0 > \neq \subset 0 \supset \):

\[G = C \text{ is constant } \Rightarrow \sum_{q \in (P)} G(q)c_q(a) = C \prod_{p \leq P} \sum_{K=0}^{v_p(a)+1} c_pK(p^{v_p(a)}) = 0(a) \Rightarrow G \in \subset 0 \supset , \]

compare: Main Lemma in [C2], for the calculation of present \( p \)-Euler factors (the \( K \)-sum here). However, a constant function \( G \neq 0 \) can’t be a Ramanujan coefficient of ANY \( F : \mathbb{N} \to \mathbb{C} \), as, for example at \( a = 1 \), we don’t have convergence for the “classic”, say, series:

\[C \neq 0, a = 1 \Rightarrow \sum_{q=1}^{\infty} G(q)c_q(a) = C \sum_{q=1}^{\infty} \mu(q) \text{ doesn’t converge in } \mathbb{C}. \]

(The same coefficients, with summation over \( P \)-smooth partial sums, give convergence, to 0 here, see above.)

In particular, it doesn’t converge absolutely, too. We now know that: Ramanujan smooth clouds are NOT contained in Ramanujan absolute clouds (of course for the same \( F \)). The converse is true since:

\[\sum_{q \in (P)} |G(q)c_q(a)| \leq \sum_{q > P} |G(q)c_q(a)|, \]

whatever is \( G : \mathbb{N} \to \mathbb{C} \) and \( \forall P \in \mathbb{P} \). Actually, we have proved that \( < 0 >_{\text{ABS}} \) is STRICTLY CONTAINED in \( \subset 0 \supset \); for a general \( F : \mathbb{N} \to \mathbb{C} \) it is also true : it follows from the fact that \( 1 \in \subset 0 \supset \) and \( G \in< F >_{\text{ABS}} \) imply \( G + 1 \in \subset F \supset \), but \( G + 1 \notin< F >_{\text{ABS}} \). By the way, given any \( F \), \( < F >_{\text{ABS}} \neq \emptyset \) because it contains Hil \( F \), the Hildebrand coefficient ([ScSp], page 166) of our \( F \). In all, we have proved the following.

**Proposition 2.** Given any \( F : \mathbb{N} \to \mathbb{C} \), we have Hil \( F \in< F >_{\text{ABS}} \) (i.e., \( F \) Ramanujan Expansion with Hildebrand Coefficient converges absolutely) and \( < F >_{\text{ABS}} \) is strictly contained in \( \subset F \supset \).

In particular, we know that all Ramanujan smooth clouds are non-empty.
5.2. Wintner Assumption, Wintner Smooth Assumption and beyond

An even more general hypothesis, starting from (WA), than Wintner Smooth Assumption (WSA) above, is of course (compare the caveat soon after (WSA) above) the following “WINTNER WEAK ASSUMPTION”:

\[(\text{WWA})\quad \exists P_F \in \mathbb{P} : \sum_{d \in (P)} \frac{|F'(d)|}{d} < \infty, \quad \forall P > P_F.\]

Trivially (WA) \(\Rightarrow\) (WSA) \(\Rightarrow\) (WWA). Unexpectedly, for “softly decaying”, say, Wintner coefficients (compare next (DD) in next result with general definition [C3]), we have (WWA) \(\Rightarrow\) (WA).

**Proposition 3.** Let \(F : \mathbb{N} \to \mathbb{C}\) have \(\text{Win } F\), with the following, say, “DELANGE DUAL HYPOTHESIS”:

\[(\text{DD})\quad \sum_{q=1}^{\infty} 2^{\omega(q)} |\text{Win}_q F| < \infty.\]

Then, (WWA) \(\Rightarrow\) (WA).

**Proof.** Use, \(\forall P \in \mathbb{P}\) fixed, the P-ORTHOGONAL WINTNER DECOMPOSITION for \(F'\), i.e. (7) above:

\[
\frac{F'(d)}{d} = \sum_{K \in (P)} \mu(K) \text{Win}_{dK} F - \sum_{r \geq 1} \frac{F'(dr)}{dr} + \sum_{d \in (P)} \frac{|F'(d)|}{d} \leq \sum_{d \in (P)} \mu^2(K) |\text{Win}_{dK} F| + \sum_{m \in (P)} \frac{|F'(m)|}{m}
\]

and, passing to \(\lim_{P}\) and applying (DD) above with \(dK = q\), we get (WWA) \(\Rightarrow\) (WA).

On the same lines of Corollary 1, it follows next stronger Corollary: simply from (BH) implying finiteness of \(\text{Win } F\) support (giving (DD) trivially).

(\text{WWA}) – Corollary 1. Correlations \(F(a) := C_{f,a}(N, a)\) with (BH) and (WWA) have the (R.E.E.F.).

Of course, for all fixed \(P \in \mathbb{P}\), any function \(F : \mathbb{N} \to \mathbb{C}\) satisfying (IPP) has

\[
\sum_{d \in (P)} \frac{|F'(d)|}{d} < \infty, \text{ being a finite sum,}
\]

whence it has (WA) IFF (if & only if) it has (WWA). The same property, for all fixed \(P \in \mathbb{P}\), is shared by any \(F\) with (NSL), from (1).

In view of this last property, since (BH)—correlations satisfy Ramanujan Conjecture (see [C1] Proposition 1 for this, quoted in §1.2), whence (NSL), previous Corollary is actually prefectly equivalent to above Corollary 1. In other words, the difference in between (WA) and (WWA) may be appreciated only in very general so-to-speak environments for \(F\).

5.3. The REEF in general

We saw the applications to (BH)—CORRELATIONS (esp., Corollary 1) in §1.2, of our results for general \(F\), in order to get the (R.E.E.F.). We warn the reader that, in this subsection, \(F \neq 0\). Also, see the following, the case \(\text{Win } F = 0\) is, say, a “singular one”.

We wish to generalize the CONCEPT of (R.E.E.F.), that REGARDS CORRELATIONS; for any general \(F : \mathbb{N} \to \mathbb{C}\) we say (notice the notational difference : no dots)

\[
F \text{ has the REEF } \iff \forall a \in \mathbb{N}, \ F(a) = \sum_{q \leq Q} (\text{Win}_q F) c_q(a),
\]

for some FIXED CONSTANT \(Q \in \mathbb{N}\). From this property, we get that

\[
F' \text{ has the REEF } \iff \forall d \in \mathbb{N}, \ F'(d) = d \sum_{K \leq \frac{d}{Q}} \mu(K) \text{Win}_{dK} F,
\]

17
for the SAME $Q$ AS ABOVE. In fact, applying Eratosthenes Transform to the, say, $F -$ REEF, we get the $F' -$ REEF, simply by KLÜYVER’S FORMULA (after (2) above):

$$F(a) = \sum_{d | a} F'(d) = \sum_{q \leq Q} (\text{Win}_{q} F) \sum_{d|a \atop d|q} d\mu(q/d) = \sum_{d} \sum_{K \leq Q} \mu(K) \text{Win}_{dK} F,$$

after Möbius Inversion [T].

On the other hand, summing over the divisors $d \in \mathbb{N}$ of $a \in \mathbb{N}$, we get the $F -$ REEF, from the $F' -$ REEF. This idea, of connecting the Ramanujan expansion of a fixed $F$ to an expansion for its Eratosthenes Transform $F'$, goes back to Lucht (see [C3], Proposition 2).

In particular, the $F -$ REEF implies that Win $F$ has support $\text{supp}(\text{Win } ) \subseteq [1, Q]$, apart from the trivial, implicit property: $\exists \text{ Win } F$.

On the converse, we ask: ONCE WE KNOW THAT $\text{supp}(\text{Win } ) \subseteq [1, Q]$, for some $Q \in \mathbb{N}$, UNDER WHICH CONDITIONS WE GET THE $F -$ REEF ABOVE?

For example, Theorem 1 ensures that (WA) gives the RAMANUJAN-WINTNER SMOOTH EXPANSION; once we join to this: $\text{supp}(\text{Win } )$ is finite, we get the REEF. We similarly prove the following.

**Theorem 3.** ($P$-INFINITESIMAL IRREGULAR SERIES & DEFINITIVELY VANISHING WIN IMPLY THE REEF)

Let $F : \mathbb{N} \to \mathbb{C}$ have Win $F$ and assume $\text{Ir}^{(P)} F \to 0$, as $P \to \infty$ in primes. Then,

$$|\text{supp}(\text{Win } F)| < \infty \implies F \text{ has the REEF.}$$

We supply a complete and explicit Proof, gathering above properties. (Alternatively use Th.2 & Lemma 3.)

**Proof.** From Lemma 2, the existence of Win $F$ implies the existence of our $F$ irregular series, compare Remark 3 & Remark 5. From Lemma 3, the vanishing hypothesis for the irregular series entails (being equivalent to) the Ramanujan-Wintner smooth expansion, for $F$; which, under the finiteness for Win $F$ support, implies the $F -$ REEF.

Of course, the main hypothesis in this Theorem, like also in applications to correlations, is the one for the vanishing of our $F$ irregular series over $P$, as $P \to \infty$ (in the primes). See that, while in previous approaches we rely on less general hypotheses, here a kind of top-generality-hypothesis, so to speak, like this irregular-series-vanishing stops any quest for (WA) generalizations, that we briefly described in previous subsection. In fact, $\text{Ir}^{(P)} F \to 0$ as $P \to \infty$ is EQUIVALENT TO THE RVS expansion; this last ingredient only needs finiteness of non-zero Wintner coefficients to produce the $F -$ REEF as Theorem 3 illustrates.

Recall that our previous study, regarding finite Ramanujan expansions [CM], proves that in case of FIXED LENGTH Ramanujan Expansions (like the (R.E.E.F.) & the REEF for general $F$) our arithmetic function is a TRUNCATED DIVISOR SUM (with divisors $d \leq Q$, for Reefs over $q \leq Q$ : see [C3] Theorem 3).

For the fixed length Ramanujan expansion $F(a) = \sum_{q} G_{F}(q)c_{q}(a), \forall a \in \mathbb{N}$, with coefficients $G_{F}$, we set

$$\ell_{F} \overset{\text{def}}{=} \sup\{q \in \mathbb{N} : G_{F}(q) \neq 0\}, \text{ hereafter assuming } G_{F} \neq 0,$$

which, of course, is finite IFF the Ramanujan expansion of our $F$ with coefficients $G_{F}$ has a fixed length; however, it’s $+\infty$ IFF such Ramanujan expansion has NOT fixed length. See that, for example, we might have a length depending on $a \in \mathbb{N}$, say $\ell_{F}(a)$, getting $\ell_{F} \overset{\text{def}}{=} \sup_{a \in \mathbb{N}} \ell_{F}(a)$. Compare Theorem 1'[C1] in §4.

See that, of course, $\ell_{F} \in \mathbb{N}$ IFF our $F$ has the REEF, from: $F'$ has the REEF $\Rightarrow G_{F} = \text{Win } F$.

Analogously, for a general $F \neq 0$, we may define $d_{F} \overset{\text{def}}{=} \sup\{d \in \mathbb{N} : F'(d) \neq 0\}$ and this, say, “top divisor”, in finite case (otherwise it’s $+\infty$, “for almost all arithmetic functions”), is linked to $\ell_{F}$ as $\ell_{F} = d_{F}$ (true even in not finite case, as $\ell_{F} = +\infty = d_{F}$, then); this follows from quoted Theorem 3 [C3]. See that, of course, $d_{F}$ is finite IFF our $F$ is a TRUNCATED DIVISOR SUM, with top divisor $d_{F} \in \mathbb{N}$. Compare next subsection’s definition of $Q_{F}$ in case $F = 0$ : accordingly, we may define $d_{0} \overset{\text{def}}{=} 0$ (but NOT $d_{0} \overset{\text{def}}{=} 0$ !).

We conclude this brief ride on the $F$-REEFs highlighting the ABSOLUTE CONVERGENCE of fixed length Ramanujan expansions, whence of THE $F$-REEF.

However, we saw above, there are constant functions $G$ in the Ramanujan smooth cloud of 0, while (apart from $G = 0$ itself) there are none in the Ramanujan clouds!

Needless to say, the Panorama of Ramanujan Clouds is very different from Smooth Ramanujan Clouds Landscape...!
5.4. Analytic part and irregular part of arithmetic functions

In this subsection, we further generalize previous approach and we study the set

$$\mathbb{C}^{\mathbb{N}}_{\text{FIN-WIN}} := \{ F : \mathbb{N} \to \mathbb{C}, \exists \text{ Win } F \& \exists Q \in \mathbb{N} : \text{supp}(\text{Win } F) \subseteq [1, Q] \}$$

of arithmetic functions with finitely-supported Wintner Transform (i.e., only a finite number of Wintner coefficients doesn’t vanish). For all such $F$ with $\text{Win } F \neq 0$, we define WINTNER’S RANGE $Q_F := \text{sup} \text{supp}(\text{Win } F)$, but this definition also says $Q_F = +\infty$ IFF our $F \notin \mathbb{C}^{\mathbb{N}}_{\text{FIN-WIN}}$. While, in case $F$ has $\text{Win } F = 0$ we set $Q_F = 0$. In other words, $\forall Q \in \mathbb{C}^{\mathbb{N}}_{\text{FIN-WIN}}$, $Q_F$ is the maximum $q$ with $\text{Win}_q F \neq 0$.

Then for these functions $F$, from $3\text{Win } F$ and $\text{supp}(\text{Win } F) \subseteq [1, Q_F]$, Lemma 2 equation (8) entails

\[
\text{(FAI)} \quad F(a) = \sum_{q \leq Q_F} (\text{Win}_q F)(a) - \sum_{d|a} \text{Irr}_d^{(Q_F)} F, \quad \forall a \in \mathbb{N},
\]

where now $Q_F \in \mathbb{N}$ might be non-prime; in this case, we may substitute $Q_F$ with biggest prime $P \leq Q_F$, say $P_F$, using the property of the irregular series, compare Remark 6, of being constant w.r.t. $P \in \mathbb{P}$ as long as $P \geq P_F$. Notice : if $\text{Win } F = 0$, then $Q_F = 0$ gives the expected empty sum over $q$ inside (FAI).

We call this equation (FAI) from the $F = A_F - I_F$ ANALYTIC-IRREGULAR DECOMPOSITION of our fixed $F \in \mathbb{C}^{\mathbb{N}}_{\text{FIN-WIN}},$

\[
A_F(a) := \sum_{q \leq Q_F} (\text{Win}_q F)(a) = \sum_{q \leq Q_F} (\text{Win}_q F) \sum_{j \in \mathbb{Z}_q} e^{2\pi i ja/q}, \quad \forall a \in \mathbb{C}
\]

is the, say, $F-$ANALYTIC PART, that’s in fact a Holomorphic function of $a \in \mathbb{C} : A_F \in \mathbb{H}(\mathbb{C})$; while, $I_F$ is the, say, $F-$IRREGULAR PART, defined $\forall a \in \mathbb{N}$ in terms of irregular series over the prime $P_F \in \mathbb{P}$, using the property of (CM) a kind of “rarity”, say, is the $F - \text{REEF} !!!$

Thanks to (FAI) we might think about the Ramanujan-Wintner Smooth expansion, say RWSE, for a fixed ARBITRARY $F \in \mathbb{C}^{\mathbb{N}}$, as a process of asymptotic approximations, as $Q \to \infty$, by functions $F \in \mathbb{C}^{\mathbb{N}}_{\text{FIN-WIN}},$ each with Wintner coefficients vanishing after $Q_F$ ! From this point of view, (FAI), itself, is a kind of “APPROXIMATE REEF”. (Compare page 8 in [C1, version 3].)

See that having $Q_F$ doesn’t suffice to get the REEF for $F$ (compare Counterexample 1 [C1] in §5.6).

Inside our FIN-WIN set of arithmetic functions $F$, previous Theorem 3 is now very clear: the $F - \text{REEF}$ is equivalent to having $I_F = 0$ in (FAI)! This also reveals that the functions $F$ in our set, having the REEF, are entire functions and, by Liouville Theorem, $F(a)$ is bounded $\forall a \in \mathbb{C}$ IFF our $F$ is a constant !! So, once again (compare quoted property from [CM]) a kind of “rarity”, say, is the $F - \text{REEF} !!!$

Theorem 4. Let $F : \mathbb{N} \to \mathbb{C}$ have finite $Q_F$. Then

$F$ has the RWSE $\iff F$ has the REEF $\iff F$ satisfies the (WA).

We leave the Proof to the interested reader.

Remark 11. In case $Q_F = +\infty$, we may even have $F$ with the RWSE, but without the (WA). (Hint: esp., $F_\alpha$ having a completely multiplicative $F'_{\alpha}$, with $F'_{\alpha}(p) := e(\alpha p)$, $\forall p \in \mathbb{P}$, with a fixed irrational $0 < \alpha < 1$).

Last, but not least, notice that, when coming back to correlations, the Basic Hypothesis makes our $F(a) := C_{f_\alpha}(N, a)$ (again, from Proposition 1 [C1, version 3]) have a finitely-supported Wintner Transform, i.e. : $F \in \mathbb{C}^{\mathbb{N}}_{\text{FIN-WIN}}$. Thus (FAI) can turn into a practical & effective formula for estimating the remainder for $-I_F = F - A_F$, where $A_F$ now’s nothing but the fixed-length Ramanujan expansion in the (R.E.E.F.) ! In other words, even if we might, as it’s “too rare”, not have the (R.E.E.F.), we might, from (FAI), try to estimate the remainder, in terms of our $F-$IRREGULAR SERIES (in $I_F$, here), for the “HARDY-LITTLEWOOD ASYMPTOTICS”, for general (BH)–correlations $F$. (Compare [C0] formulae.)
5.5. Irregular series of multiplicative arithmetic functions

The Irregular Series, Irr\textsuperscript{(P)}\textsubscript{d} F, for general F having Win F, even assuming all the hypotheses above, remains a kind of mystery.

However, for F : N \to C a multiplicative arithmetic function, it simplifies a lot, as we see now:

\[ F \text{ multiplicative } \Rightarrow \text{ Irr}\textsuperscript{(P)}\textsubscript{d} F = F'(d) \text{ Irr}\textsuperscript{(P)}\textsubscript{1} F, \quad \forall P \in \mathbb{P}, \forall d \in (P). \]

5.6. Correlations with Basic Hypothesis, but without Reef: studying Counterexample 1

The Counterexample 1 [C1] shows that the Basic Hypothesis, implying that the Wintner transform is finitely-supported, is NOT sufficient to get the Reef.

We recall briefly that Counterexample 1, see [C1], is the correlation of two arithmetic functions, say \( f, g : N \to \mathbb{C} \), chosen this way:

\[ \text{FIX } N, Q \in \mathbb{N} \text{ with } Q \leq N \text{ and two integers } 1 \leq n_0 \leq N \text{ and } 2 < q_0 \leq Q. \]

Choose

\[ f_0(n) \overset{\text{def}}{=} 1_{\{n_0\}}(n), \quad \forall n \in \mathbb{N} \quad \text{and} \quad g_0(m) \overset{\text{def}}{=} c_{g_0}(m), \quad \forall m \in \mathbb{N} \]

whence:

\[ n_0 \equiv -1(\text{mod } q_0) \]

implies: we can’t have the Reef for \( C_{f_0,g_0}(N,a), \forall a \in \mathbb{N} \), since in particular for \( a = 1 \) Reef’s LHS and RHS are DIFFERENT. (See page 8,[C1], for details).

We profit, here, to gather some properties of our Counterexample 1, we’ll call \( F_0(a) \), in the more manageable case that modulus \( q_0 \) is prime \( q_0 = p_0 \in \mathbb{P} \) and \( n_0 = q_0 - 1 = p_0 - 1 \): hereafter, with \( p_0 > 2 \),

\[ F_0(a) \overset{\text{def}}{=} c_{p_0}(a - 1), \quad \forall a \in \mathbb{N}. \]

(It might seem that this is not a correlation, but please gather above definitions!)

This correlation satisfies, as usual, our Basic Hypothesis and, by the way, has Wintner Transform Win \( F_0 \) simply given by \( q\text{-th coefficient } \frac{1}{\varphi(p_0)}c_{p_0}(p_0 - 1) = \frac{1}{\varphi(p_0)}\mu(p_0) = -1/\varphi(p_0) \), if and only if \( q = p_0 \), vanishing otherwise. (In particular, this Transform is finitely-supported, of course.) We start calculating Eratosthenes Transform:

\[ F'_0(1) = F_0(1) = \varphi(p_0) = p_0 - 1, \quad \text{while} \quad d > 1 \Rightarrow F'_0(d) = p_0 S_{p_0}(d), \]

where we set

\[ S_{p_0}(d) \overset{\text{def}}{=} \sum_{a \equiv 1 \text{ mod } p_0} \mu\left(\frac{d}{a}\right), \quad \forall d \in \mathbb{N}, \]

because: \( \forall d \in \mathbb{N} \) we have

\[ F'_0(d) = \sum_{a \equiv 1 \text{ mod } p_0} \varphi(p_0)\mu\left(\frac{d}{a}\right) + \sum_{a \equiv 1 \text{ mod } p_0} \mu(p_0)\mu\left(\frac{d}{a}\right) = (p_0 - 1) \sum_{a \equiv 1 \text{ mod } p_0} \mu\left(\frac{d}{a}\right) - \sum_{a \equiv 1 \text{ mod } p_0} \mu\left(\frac{d}{a}\right), \]

which is \( p_0 S_{p_0}(d), \forall d > 1 \), from Möbius inversion:

\[ \sum_{a \equiv 1 \text{ mod } p_0} \mu\left(\frac{d}{a}\right) = \sum_{a \equiv d \text{ mod } p_0} \mu\left(\frac{d}{a}\right) - \sum_{a \equiv 1 \text{ mod } p_0} \mu\left(\frac{d}{a}\right), \quad \forall d > 1. \]

Dirichlet characters modulo \( p_0 \) allow to write

\[ (\ast)_{p_0} \quad S_{p_0}(d) = \frac{1}{\varphi(p_0)} \sum_{\chi(\text{mod } p_0)} \sum_{a \equiv d \text{ mod } p_0} \chi(a)\mu\left(\frac{d}{a}\right), \quad \forall d \in \mathbb{N}\setminus\{1\}. \]
We may distinguish three cases, for the integers $d > 1$:

(0) $v_{p_0}(d) = 0$ and $d > 1$

(1) $v_{p_0}(d) = 1$

(2) $v_{p_0}(d) \geq 2$

Last case (2) is the simplest, since, setting $K := d/p^{v_{p_0}(d)} \in \mathbb{Z}_{p_0}^*$,

$$S_{p_0}(d) = S_{p_0}(p_0^{v_{p_0}(d)} \cdot K) = \sum_{a \equiv K \mod p_0} \mu \left( p_0^{v_{p_0}(d)} \frac{K}{a} \right) = \mu \left( p_0^{v_{p_0}(d)} \right) S_{p_0}(K) = 0,$$

in case (2).

Similarly, setting in case (1) $K := d/p_0 \in \mathbb{Z}_{p_0}^*$,

$$S_{p_0}(d) = S_{p_0}(p_0 \cdot K) = \sum_{a \equiv K \mod p_0} \mu \left( p_0 \cdot \frac{K}{a} \right) = \mu(p_0)S_{p_0}(K) = -S_{p_0}(d/p_0),$$

in case (1). In particular, we may omit the single $d = p_0$, as $S_{p_0}(p_0) = -S_{p_0}(1) = -1$.

Everything boils down to case (0), in which formula $(*)_{p_0}$ at previous page, with Dirichlet characters, becomes:

$$S_{p_0}(d) = \frac{1}{\varphi(p_0)} \sum_{\chi(p_0) \equiv \chi(K) \equiv 0 \pmod{p_0}} \mu(K)\chi(d)\chi(K) = \frac{1}{\varphi(p_0)} \sum_{\chi(p_0) \equiv 0 \pmod{p_0}} \chi(d) \prod_{p \mid d} \left( 1 - \chi(p) \right),$$

in case (0), because the flipping $K := \frac{d}{a}$ of divisors $a \mid d$ has

$$\chi \left( \frac{d}{K} \right) = \chi(d) \frac{\chi(K)}{\chi(K)} = \chi(d)\chi(K), \quad \forall K \mid d \ (\text{recall } d \in \mathbb{Z}_{p_0}^*).$$

In this formula, the finite product over primes $p$ dividing $d$ (from $p \equiv 1 \mod p_0 \Rightarrow \chi(p) = 1, \forall \chi \mod p_0$) immediately entails the property

$$(*)_0 \quad \exists p \mid d : p \equiv 1 \mod p_0 \quad \Rightarrow \quad S_{p_0}(d) = 0.$$

We may so to speak summarize these properties of $F'_0$, giving a glance to (without calculating it) the mean value of $|F'_0|$, i.e.:

$$\lim_{x} \frac{1}{x} \sum_{d \leq x} |F'_0(d)| = p_0 \lim_{x} \frac{1}{x} \left( \sum_{1 < d \leq x} |S_{p_0}(d)| + \sum_{1 < d \leq x} |S_{p_0}(d)| \right) = p_0 \lim_{x} \frac{1}{x} \left( \sum_{1 < d \leq x} |S_{p_0}(d)| + \sum_{1 < d \leq x} |S_{p_0}(d)| \right) = (p_0 + 1) \lim_{x} \frac{1}{x} \sum_{1 < d \leq x} |S_{p_0}(d)|,$$

where the first equation comes from distinguishing cases (0) and (1), while second one introduces the $\flat$ notation, from $(*)_0$ property, that means: any prime $p \mid d$ is NEITHER 0 NOR 1 modulo $p_0$; finally, last equation, so to speak, comes from the change of variable in second limit passing from $x$ to $p_0 x$.  

21
We still have two properties of our $F_0$ that are noteworthy to see: namely, we give a brief look at the behavior of $S_{p_0} (d)$, respectively on square-free $d > 1$ and on the powers of primes $p$ different from $p_0$.

First of all, see that on square-free $d > 1$ we have

$$S_{p_0} (d) = \mu(d) \sum_{a \equiv d \mod p_0} \mu(a),$$

from the trivial remark that these $d$ have $\mu(d/a) = \mu(d)\mu(a)$, because $a$ is square-free, too, and $1/\mu(a) = \mu(a)$ in this case. Hence,

$$|S_{p_0} (d)| = \left| \sum_{a \equiv d \mod p_0} \mu(a) \right|, \quad \forall d > 1, \mu^2(d) = 1.$$

This may be of some help in above calculations for $|S_{p_0} (d)|$ averages; also, Dirichlet characters modulo $p_0$ simplify above $(\ast)_{p_0}$ as:

$$(\bar{s})_{p_0} \quad \tilde{S}_{p_0} (d) \overset{df}{=} \sum_{a \equiv d \mod p_0} \mu(a) = \frac{1}{\varphi(p_0)} \sum_{\chi(K \equiv 0 \mod p_0)} \prod_{p|d} (1 - \chi(p)) , \quad \forall d \in \mathbb{N}, \mu^2(d) = 1.$$

Above cases (0), (1) and (2) for $S_{p_0}$ become, for $\tilde{S}_{p_0} (d)$ on square-free $d > 1$, only the two possibilities

(0) \quad $v_{p_0} (d) = 0$ and $d > 1, \mu^2(d) = 1$

(1) \quad $v_{p_0} (d) = 1$ and $\mu^2(d) = 1$

becoming, for $\tilde{S}_{p_0}$, on the same lines as above, in only one occurrence:

i.e. in case (1), setting $K := d/p_0 \in \mathbb{Z}^+_p,$ to get

$$\tilde{S}_{p_0} (d) = \tilde{S}_{p_0} (p_0 \cdot K) = \sum_{a \equiv K \mod p_0} \mu(p_0 \cdot K) = \mu(p_0) \tilde{S}_{p_0} (K) = -\tilde{S}_{p_0} (d/p_0).$$

Turning back to our $F_0$, we prove now that its Eratosthenes Transform $F'_0 (d)$ is NOT infinitesimal as $d \to \infty$; simply, calculating $S_{p_0} (d)$ on $d = p^K$, powers, with infinitely many $K \in \mathbb{N}$, of primes $p \neq p_0$ with $p \equiv 1(\mod p_0)$, it follows, from next formula, that $F'_0 (p^K) = \pm p_0$, for infinitely many $K \in \mathbb{N}$, because:

$$S_{p_0} (p^K) = \sum_{p_0 \equiv K \mod p_0} \mu(p^{K-j}) = 1_{p^K \equiv 1(\mod p_0)} - 1_{p^K - 1 \equiv 1(\mod p_0)} \neq 0, \quad \text{as} \quad K \to \infty,$$

from the definition of $S_{p_0}$ above (recalling $p_0 > 2$ here), since Fermat’s little Theorem implies that it’s 1 on the $K \equiv 0(\mod p_0 - 1)$ and -1 on the $K \equiv 1(\mod p_0 - 1)$. Of course, these give two subsequences for $F'_0 (d)$ not infinitesimal on $d = p^K$, as $d \to \infty$.

In particular, saying that $F'_0 (d)$ doesn’t go to 0 as $d \to \infty$ proves once again that the Reef doesn’t hold: in fact, the Reef holds if and only if our Eratosthenes Transform has finite support!

We will study in deeper details: in order to prove whether (ETD) holds or not for our $F_0$ above (recall, an instance of [C1] Counterexample 1) we found some technical difficulties we hope to overcome in the future.

Last but not least, we propose an exercise to interested readers. Recall $p_0 > 2$ in the above $F_0$ definition.

**Curiosity 1. Our $F_0$ is not (IPP): taking $p \equiv -1(\mod p_0)$, with $p > p_0$, we have, when $a = p^2$,

$$F_0 (\kappa(a)) = F_0 (p) = -1 \neq p_0 - 1 = F_0 (p^2) = F_0 (a).$$

22
5.7. Smooth/classic Carmichael-Wintner coefficients for imaginary exponentials & applications

We make, say, a kind of exercises in computing resp., the Classic Carmichael Car \( F_{j,q} \) and all the Smooth Carmichael \( \text{Car}_{(P)}^{(P)} \) \( F_{j,q} \) coefficients, \( \forall P \in \mathbb{P} \), for the remarkable \( F_{j,q}(a) := e_q(ja) \) : the imaginary exponential function, where the two parameters \( q \in \mathbb{N} \) and \( j \in \mathbb{Z}_q^* \) are FIXED. We'll use this notation, recalling: from Lemma 6, since our \( F_{j,q} \) is bounded (whence, (NSL), too), we have \( \text{Win}^{(P)}_{j,q} = \text{Car}^{(P)}_{j,q} \) \( \forall P \in \mathbb{P} \).

We also know, after finding Car \( F_{j,q} \), that it equals Win \( F_{j,q} \), from Proposition 3 in [C3], that is a kind of reformulation of a 1987 result of Delange (see [C3] for the bibliography). We'll indicate \( F_{j,q} = e_q(j \bullet) \in \mathbb{C}^N \).

Thus

**Lemma 7.** *(CARMICHAEL COEFFICIENTS OF IMAGINARY EXPONENTIAL FUNCTION)*

Fix \( q \in \mathbb{N} \) and \( j \in \mathbb{Z}_q^* \). Then, \( \forall \ell \in \mathbb{N} \),

\[
\text{Car}_\ell \ e_q(j \bullet) = 1, \quad \frac{1}{\varphi(q)}.
\]

**Proof.** Carmichael coefficient definition and Kluvyer formula (see soon after (2) above)

\[
\text{Car}_\ell \ e_q(j \bullet) = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \sum_{a \leq x} e_q(ja) c_\ell(a) = \frac{1}{\varphi(\ell)} \sum_{d|\ell} d\mu\left(\frac{\ell}{d}\right) \lim_{x \to \infty} \sum_{m \leq x/d} e_q(jdm),
\]

with the cancellation in exponential sums, i.e., as \( x \to \infty \),

\[
\sum_{m \leq x/d} e_q(jdm) = 1_{d \equiv 0(\text{mod } q)} \left(\frac{x}{d}\right) + 1_{d \equiv 0(\text{mod } q)} O\left(\frac{1}{\sqrt{|d|}}\right) = 1_{d \equiv 0(\text{mod } q)} \cdot \frac{x}{d} + O_q(1),
\]

give soon the thesis

\[
\text{Car}_\ell \ e_q(j \bullet) = \frac{1}{\varphi(\ell)} \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) = \frac{1}{\varphi(\ell)} \cdot 1_{q|\ell} \cdot \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) = 1_{\ell = q} \cdot \frac{1}{\varphi(\ell)},
\]

by Möbius inversion (quoted after (2) above).

**Remark 12.** The main idea is the **resonance of moduli** \( q \) and \( \ell \). Writing \( c_\ell(a) \) with the exponentials and applying soon exponential sums cancellation, as an alternative proof, renders this more transparent.

This result is so easy that we may have called it a “Fact”. In case of our \( F_{j,q} \) transform \( \text{Car}^{(P)}_{j,q} \), we need more small ideas combined together: the main anthem is a kind of writing averages over \( P \)--smooth numbers involving imaginary exponentials in term of same averages over Dirichlet characters, that have a multiplicative structure, instead.

In fact, we start calculating \( P \)--smooth Carmichael coefficients of a general class of arithmetic functions \( F \), the (NSL) ones, in terms of \( P \)--smooth numbers averages, with Ramanujan sums; this will be applied to our imaginary exponential function \( F = F_{j,q} \), but the following result is quite general. Proof follows [C1].

**Lemma 8.** *(CARMICHAEL P--SMOOTH COEFFICIENTS OF (NSL) FUNCTIONS)*

Let \( F : \mathbb{N} \to \mathbb{C} \) be (NSL). Then, \( \forall P \in \mathbb{P} \),

\[
\text{Car}^{(P)}_{\ell} F = \frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \cdot \sum_{t \in (P)} F(t) c_\ell(t), \quad \forall \ell \in (P).
\]

**Proof.** Carmichael \( P \)--smooth \( \ell \)--th coefficient definition and Lemma 1 of [C1] (“Möbius Switch”) give

\[
\text{Car}^{(P)}_{\ell} F = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{a \leq x} c_\ell(a) \sum_{d \in (P)} F'(d) = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{a \leq x} c_\ell(a) \sum_{t \in (P) \atop d|\ell} F(t),
\]

23
where the sums exchange, the property \( \ell \in (P), m \in P( \Rightarrow c_\ell(tm) = c_\ell(t) \) and Lemma 2 [C1], a kind of Eratosthenes-Legendre sieve, give

\[
\sum_{a \leq x} c_\ell(a) \sum_{t \in (P) \atop \ell | a} F(t) = \sum_{t \in (P)} F(t) \sum_{m \leq x/t \atop m \in P} c_\ell(tm) = \sum_{t \in (P)} F(t)c_\ell(t) \sum_{m \leq x/t \atop m \in P} 1
\]

\[
= \sum_{t \in (P)} F(t)c_\ell(t) \left( \prod_{p \leq P} \left( 1 - \frac{1}{p} \right) \frac{x}{t} + O_P(1) \right) = \prod_{p \leq P} \left( 1 - \frac{1}{p} \right) \frac{x}{t} \sum_{t \in (P)} F(t)c_\ell(t) + O_{P,\ell,F}(1)
\]

and recalling (for details, see [C1]: Proposition 2 Proof start)

\[
\text{Next Lemma is a Corollary of previous one, plus a switch of harmonics: from imaginary exponentials to Dirichlet characters. Gauss sums } \tau(\chi) \text{ definition } |D| \text{ is recalled in the Proof.}
\]

**Lemma 9. (Imaginary exponentials’ Carmichael } P{-smooth coeff.s: Switch to characters)**

**Fix } q \in \mathbb{N} \text{ and } j \in \mathbb{Z}^*_q. \text{ Then, } \forall P \in \mathbb{P}, \text{ with } P \geq q,

\[
\text{Car}_\ell^{(P)} e_q(j \bullet) = \frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \sum_{b/q' \equiv b \pmod{q'}} \frac{1}{b \varphi(q')} \sum_{\ell \pmod{q'}} \tau(\ell)\chi(j) \sum_{t \in (P)} \chi(t) c_\ell(bt), \quad \forall \ell \in (P).
\]

**Proof.** Straight from previous Lemma for \( F = F_{j,q} = e_q(j \bullet), \)

\[
\text{Car}_\ell^{(P)} e_q(j \bullet) = \frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \sum_{t \in (P)} \frac{e_q(jt)}{t} c_\ell(t), \quad \forall \ell \in (P).
\]

We switch from imaginary exponentials to Dirichlet characters of modulus \( q' := q/b \) by the inversion formula (see [D]) with the **Gauss sum**

\[
\tau(\chi) \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}^*_q} \chi(m)e_{q'}(m) \Rightarrow e_q'(k) = \frac{1}{\varphi(q')} \sum_{\chi(\ell \pmod{q'})} \tau(\ell)\chi(k), \quad \forall k \in \mathbb{Z}^*_q
\]

giving at once, from hypothesis \( P \geq q \) entailing \( b \in (P) \) \( \forall b|q, \) the following:

\[
\sum_{t \in (P)} \frac{e_q(jt)}{t} c_\ell(t) = \sum_{b/q \in (P) \atop (t,q) = b} \frac{e_{q/b}(j(t/b))}{t} c_\ell(t) = \sum_{b/q \in (P) \atop (t,q) = b} \frac{1}{b} \sum_{t' \in (P) \atop (t',q/b) = 1} \frac{e_{q/b}(jt')}{t'} c_\ell(bt') =
\]

\[
= \sum_{b/q \in (P) \atop (t,q) = b} \frac{1}{b} \sum_{t' \in (P) \atop (t',q/b) = 1} \frac{e_{q/b}(jt')}{t'} c_\ell(bt) = \sum_{b/q \in (P) \atop (t,q) = b} \frac{1}{b \varphi(q')} \sum_{\chi(\ell \pmod{q'})} \tau(\ell)\chi(j) \sum_{t \in (P)} \chi(t) c_\ell(bt) \frac{t}{t},
\]

from \( j \in \mathbb{Z}^*_q \) and the property: \((t,q') = 1\) is implicit in presence of \( \chi(t), \) whence the formula.  

\[
\square
\]
We have a kind of two small problems to face, for an explicit formula in terms of characters and partial Euler products. First, we have to get rid of the “extra factor”, so to speak, in the Ramanujan sum of modulus \(\ell\) in the above formula: we solve this in next Lemma, with a small idea (we will “kill \(b\), say).

**Lemma 10.** (Absorbing Extra Factors in Ramanujan Sums) Choose any \(\ell, b, t \in \mathbb{N}\). Then

\[
c_{\ell}(bt) = \frac{\varphi(\ell)}{\varphi(\ell/(\ell, b))} c_{\ell/(\ell, b)}(t).
\]

**Proof.** Ramanujan sums Explicit Formula [M, page 22 : Hölder’s 1936 formula], applied twice:

\[
c_{\ell}(bt) = \varphi(\ell) \cdot \frac{\mu(\ell/(\ell, bt))}{\varphi(\ell/(\ell, bt))} = \varphi(\ell) \cdot \frac{\mu(\ell/(\ell', b't))}{\varphi(\ell/(\ell', b't))} = \frac{\varphi(\ell)}{\varphi(\ell')} c_{\ell'}(b') = \frac{\varphi(\ell)}{\varphi(\ell')} c_{\ell'}(t),
\]

using now \(\ell/(\ell, bt) = \ell/(\ell, b't)\), where

\[
\ell' := \ell/(\ell, b), \ b' := b/(\ell, b),
\]

together with \(b' \in \mathbb{Z}_{\ell'}\). \(\Box\)

Just like we have, say, separated \(b\) from other variables, we need now to separate the prime factors of a fixed modulus \(q'\) from other variables, in next Lemma with Dirichlet characters modulo \(q'\). In fact, when we want to “flip”, say, a Dirichlet character \(\chi(d)\), over divisors \(d|n\), into \(\chi(n/K)\), with complementary divisor \(K := n/d\), we may then write \(\chi(n/K) = \chi(n)/\chi(K)\) only if we know that \(K\) is coprime to \(q'\) (our \(\chi\) modulus); in other words, we have to separate the prime-factors of \(n\) dividing modulus \(q'\). As we see soon.

**Lemma 11.** (Separating Modulus Prime-Factors Before Flipping Dirichlet Characters) Choose any \(\ell', q' \in \mathbb{N}\). Then, setting \(q'' := \prod_{p | \ell', p | q} p^{v_p(\ell')}\), \(\ell'' := \ell'/q''\), we have \(\forall \chi \pmod{q'}\)

\[
\chi'(\ell') = \sum_{d|\ell'} \chi(d) \mu \left( \frac{\ell'}{d} \right) = \mu(q'') \chi(\ell'') \prod_{p | \ell''} (1 - \overline{\chi}(p)).
\]

**Proof.** In the sum over \(d\), in LHS, the factor \(\chi(d)\) implies \((d, q') = 1\) and \(\ell'' \in \mathbb{Z}_{q'}^*\) by construction:

\[
\sum_{d|\ell'} \chi(d) \mu \left( \frac{\ell'}{d} \right) = \sum_{d|\ell''} \chi(d) \mu \left( \frac{\ell''}{d} \right) = \sum_{d|\ell''} \chi(d) \mu(q'') \mu(K) \chi \left( \frac{\ell''}{K} \right) =
\]

\[
= \mu(q'') \chi(\ell'') \sum_{K|\ell''} \mu(K) \overline{\chi}(K) = \mu(q'') \chi(\ell'') \prod_{p | \ell''} (1 - \overline{\chi}(p)),
\]

flipping, say, the divisors \(d\) as: \(K := \ell''/d\), having used \(K|\ell'' \Rightarrow \chi(K) \neq 0 \Rightarrow \chi(\ell''/K) = \chi(\ell'') \overline{\chi(K)}\) and the general formula [T]

\[
\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p)),
\]

for all multiplicative functions \(f\). \(\Box\)
Before gathering all these Lemmas together to compute Carmichael $P$-smooth coefficients of our imaginary exponential function, in next Theorem, we need to look at the corresponding Carmichael coefficients: we express them as the $\chi = \chi_0$ part of Lemma 9 formula, for ALL the principal characters modulo $q'$, $\forall q' \in \mathbb{N}$ (they’re the only $\chi$ modulo $q'$, of course, in cases $q' = 1, 2$).

**Lemma 12. (Imaginary exponentials’ Carmichael coefficients: principal characters)**

*Fix $q \in \mathbb{N}$ and $j \in \mathbb{Z}_q$. Then, $\forall P \in \mathbb{P}$, with $P \geq q$,*

$$\text{Car}_\ell e_q(j \bullet) = \frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \sum_{\delta \in \mathbb{N}} \frac{1}{b_\varphi(q')} \tau(\bar{\chi}_0) \chi_0(j) \sum_{t \in (P)} \frac{\chi_0(t)}{t} c_t(bt), \quad \forall \ell \in (P).$$

**Proof.** Straight from: $\tau(\bar{\chi}_0) = \tau(\chi_0) = c_{q'}(1) = \mu(q')$ [D], $\mu(q')/\varphi(q') = c_q(bt)/\varphi(q)$ (from quoted Hölder 1936 formula, in [M]) and $\chi_0(j) = 1$ (recall $(j, q) = 1 = (j, q')$ here), rendering RHS

$$\frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \sum_{\delta \in \mathbb{N}} \frac{c_q(bt)}{b_\varphi(q')} \frac{\chi_0(t)}{t} c_t(bt) = \frac{1}{\varphi(q)} \sum_{m \in (P)} \frac{1}{m} \sum_{\delta \in \mathbb{N}} \frac{c_q(bt)}{bt},$$

because $\chi_0(t) \neq 0 \Leftrightarrow (t, q/b) = 1 \Leftrightarrow (bt, q) = b$, while our RHS is, from $P \geq q \Rightarrow q \in (P) \Rightarrow b \in (P), \forall b | q$,

$$\frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \sum_{\delta \in \mathbb{N}} \frac{c_t(bt)}{bt} c_q(bt) = \frac{1}{\varphi(\ell)} \frac{1}{\varphi(q)} \sum_{m \in (P)} \frac{1}{m} \sum_{u \in (P)} \frac{c_t(u) c_q(u)}{u} = \frac{1}{\varphi(q)} \cdot 1_{\ell = q},$$

following from the “Smooth Twisted Orthogonality”, see Proposition 2 in 3rd version of [C1]:

$$\frac{1}{\varphi(\ell)} \sum_{m \in (P)} \frac{1}{m} \sum_{u \in (P)} \frac{c_t(u) c_q(u)}{u} = 1_{\ell = q},$$

whence by Lemma 7 the thesis.\[\Box\]

**Remark 13.** Once fixed $q' \in \mathbb{N}$, in case $\exists \chi \neq \chi_0(\mod q')$, writing $\asymp$ for both $\ll$ and $\gg$, as $P \in \mathbb{P}$, $P \rightarrow \infty$,

$$\sum_{m \in (P)} \frac{1}{m} = \prod_{p \leq P} \left(1 - \frac{1}{p}\right)^{-1} \asymp \log P, \quad \text{while} \quad \sum_{m \in (P)} \frac{\chi(m)}{m} = \prod_{p \leq P} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \asymp 1, \forall \chi \neq \chi_0(\mod q'),$$

as these last products converge, for $P \in \mathbb{P}$, $P \rightarrow \infty$, to $\prod_p (1 - \chi(p)/p)^{-1} = L(1, \chi) \neq 0$. These two partial Euler products, from Lemma 9, will appear in next Theorem, in the way its sketchy Proof suggests.\[\Diamond\]

**Remark 14.** By the way, more precisely, next Theorem’s bound comes from (now, $\forall q' \in \mathbb{N}$ fixed):

$$d | n \Rightarrow \prod_{p | d} (1 - \chi(p)) = \prod_{p | d} \left|1 - \chi(p)\right| \leq 2^{\omega(d)} \leq 2^{\omega(n)} \leq 2^{\varphi(n)}, \quad \forall \chi(\mod q'),$$

because $\forall n \in \mathbb{N}$,

$$\frac{2^{\omega(n)}}{\varphi(n)} = \prod_{p | n} \frac{2}{\varphi(p^\nu(p(n)))} = \prod_{p | n} \frac{2}{2^\nu_p(n) - 1} \cdot \prod_{p > 2} \frac{2}{(p - 1)p^\nu_p(n) - 1} \leq 2,$$

an absolute constant.\[\Diamond\]
We are ready to state and prove our most interesting result about Carmichael coefficients, both smooth and classic, for the imaginary exponential function. We may abbreviate $P$–Carmichael Transform to mean: $P$–smooth Carmichael Transform. Also, “to”, hereafter, may shorten “converges to”.

**Theorem 5.** (Imaginary exponentials’ $P$–Carmichael Transform to Carmichael Transform)
Fix $q \in \mathbb{N}$ and $j \in \mathbb{Z}_q^*$, choose $P \in \mathbb{P}$ with $P \geq q$ and take $\ell \in (P)$. Then, the explicit formula holds:

$$
\text{Car}_\ell^{(P)} e_q(j\bullet) = \text{Car}_\ell e_q(j\bullet) + \frac{1}{b_q p(q')} \sum_{x \equiv q (\text{mod } q')} \tau \chi(j) \frac{\mu(q')}{\varphi(p')} \prod_{p \leq P} \frac{1 - \chi(p)}{p} ^{-1} \prod_{p \leq P} \frac{1 - \frac{\chi(p)}{p}}{p} ^{-1},
$$

Abbreviating $q' := q/b$, $\ell' := \frac{\ell}{(q,\ell)}$, $q'' := \prod_{p|\ell',p|q'} p^{\nu(p')}$ and $\ell'' := \ell'/q''$. As a consequence, the bound:

$$
\text{Car}_\ell^{(P)} e_q(j\bullet) = \text{Car}_\ell e_q(j\bullet) + O_q \left( \frac{1}{\log P} \right) = \left( 1 - \frac{1}{q} \right) + O_q \left( \frac{1}{\log P} \right),
$$

uniformly $\forall \ell \in \mathbb{N}$ (see Remark 14), with the constant depending at most on the fixed $q \in \mathbb{N}$.

**Proof (Sketch).** Gather: Lemmas 9,10, Kluvyer Formula for $e_q(t)$, Lemma 11,12 and Remarks 13,14.

Since any Correlation, say $C_{f,g_q}(N,a)$, satisfying Basic Hypothesis is a linear combination of imaginary exponentials $e_q(ja)$ as follows:

$$
C_{f,g_q}(N,a) = \sum_{q \leq Q} \hat{g}_Q(q) \sum_{j \in \mathbb{Z}_q^*} S_f \left( \frac{j}{q} \right) e_q(ja),
$$

where we’ll abbreviate henceforth

$$
S_f(\alpha) \overset{\text{def}}{=} \sum_{n \leq N} f(n)e(n\alpha), \quad \forall \alpha \in [0,1],
$$

previous Theorem for imaginary exponentials has the following Corollary for (BH)–correlations, of two fixed $f,g_Q : \mathbb{N} \to \mathbb{C}$. (For the details about truncated $g_Q$ and its Ramanujan coefficients $\hat{g}_Q$, see the above §1.2.)

**Corollary 4.** (All (BH)–correlations’ $P$–Carmichael Transform to Carmichael Transform)
Fix $Q, N \in \mathbb{N}$, with $Q \leq N$, and abbreviate $F(a) := C_{f,g_q}(N,a)$, $\forall a \in \mathbb{N}$, for the $f$ and $g_Q$ (BH)–correlation. Choose $P \in \mathbb{P}$ and take $\ell \in (P)$. Then, the explicit formula holds:

$$
\text{Car}_\ell^{(P)} F = \text{Car}_\ell F + \sum_{q \leq Q} \hat{g}_Q(q) \sum_{b_q} \frac{1}{b_q p(q')} \sum_{x \equiv q (\text{mod } q')} \tau \chi(j) S_f \left( \frac{j}{q} \right) \times
$$

$$
\frac{\mu(q')}{\varphi(p')} \prod_{p \leq P} \frac{1}{1 - \chi(p)} ^{-1} \prod_{p \leq P} \frac{1 - \frac{\chi(p)}{p}}{p} ^{-1},
$$

abbreviating $q' := q/b$, $\ell' := \frac{\ell}{(q,\ell)}$, $q'' := \prod_{p|\ell',p|q'} p^{\nu(p')}$ and $\ell'' := \ell'/q''$. As a consequence, the bound:

$$
\text{Car}_\ell^{(P)} F = \text{Car}_\ell F + O_{Q,N,f,g} \left( \frac{1}{\log P} \right) = \hat{g}_Q(\ell) \sum_{j \in \mathbb{Z}_q^*} S_f \left( \frac{j}{\ell} \right) \frac{1}{\varphi(\ell)} + O_{Q,N,f,g} \left( \frac{1}{\log P} \right),
$$

uniformly $\forall \ell \in \mathbb{N}$, with an absolute constant depending at most on the fixed $Q, N \in \mathbb{N}$, $f,g \in \mathbb{C}^\mathbb{N}$.

**Remark 15.** Since $\sum_{j \in \mathbb{Z}_q^*} S_f \left( \frac{j}{\ell} \right) = \sum_{n \leq N} f(n)e(\ell n)$, the RHS explicit part here is nothing but the $\ell$–th coefficient in REEF’s RHS. We explicitly highlight: CONVERGENCE OF COEFFICIENTS DOESN’T IMPLY CONVERGENCE OF EXPANSIONS! (Compare Counterexample 1 in §5.6, for this.)
Of course we might consider a kind of \textit{Approximate Reef} for (BH)−correlations, once defined the Error Term:

$$E_{f,g,q}(N,a) \overset{\text{def}}{=} C_{f,g,q}(N,a) - \sum_{q \leq Q} \tilde{g}_Q(q) \sum_{n \leq N} f(n)c_q(n) \frac{1}{\varphi(q)} c_q(a)$$

arising (as we saw in previous Corollary and Remark) as our (BH)−correlation minus its REEF RHS. With this definition, previous Corollary may be written more explicitly, for the Correlation, as

\textbf{Corollary 5. (Explicit Formula For Error Term of (BH)−correlations)}

Let $F(a) := C_{f,g,q}(N,a)$, $\forall a \in \mathbb{N}$, represent a (BH)−correlation and define its Error Term as above. Then $\forall a \in \mathbb{N}$ fixed, choosing $P \in \mathbb{P}$ with $P \geq \max(Q, a)$, we get:

$$E_{f,g,q}(N,a) = \sum_{\ell \in \mathbb{P}} c_{\ell}(a) \left( \sum_{q \leq Q} \tilde{g}_Q(q) \sum_{b|q} \frac{1}{b\varphi(q')} \sum_{x \neq 0 \pmod{q'}} \tau(\chi) \sum_{j \in \mathbb{Z}_q^*} \chi(j) S_f \left( \frac{j}{q} \right) \times \right.$$

$$\times \frac{\mu(q'') \chi(\ell'')}{\varphi(\ell'')} \prod_{p|\ell''} \left( 1 - \frac{\chi(p)}{p} \right)^{-1} \left( \frac{1 - \frac{\chi(p)}{p}}{1 - 1/p} \right)^{-1},$$

where this quantity in brackets is $\text{Car}_{\ell}(P) F - \text{Car}_{\ell} F = \text{Win}_{\ell}^{(P)} F - \text{Win}_{\ell} F$ and we abbreviate as above

$q' := q/b$, $\ell' := \ell/\ell''$, $q'' := \prod_{p|\ell''} p^{\nu(p)}$ and $\ell'' := \ell'/q''$.

We leave the Proof as an exercise, for the interested reader.

As we also leave the other following “Exercise”, arising from the question: what if we introduce Dirichlet characters AT ONCE from the imaginary exponential $e_{q}(ja)$ in the (BH)−correlation?

\textbf{Theorem 6. (Dirichlet Characters Explicit Formula For (BH)−correlations Error Term)}

Let $F(a) := C_{f,g,q}(N,a)$, $\forall a \in \mathbb{N}$, represent a (BH)−correlation and define its Error Term as above. Then $\forall a \in \mathbb{N}$ fixed, abbreviate now $q' := q/(q,a)$ and $a' := a/(q,a)$, to get:

$$E_{f,g,q}(N,a) = \sum_{q \leq Q} \tilde{g}_Q(q) \cdot \frac{1}{\varphi(q')} \sum_{x \neq 0 \pmod{q'}} \tau(\chi) \chi(a') \sum_{j \in \mathbb{Z}_q^*} \chi(j) S_f \left( \frac{j}{q} \right).$$
6. Odds & ends. Recent work. Further remarks & future work

We start with some complementary results, about Euler products, in next subsection §6.1.

Then, subsection §6.2 of remarks on the connections between $F'$ and Win $F$, for general $F : \mathbb{N} \to \mathbb{C}$. In present Version 9 it is expanded, from new results described in next subsection.

Recent work, leading to a few considerable new results, is in fact given in §6.3.

Last but not least, we give further remarks and a kind of “short coming soon” on future work: last subsection §6.4.

6.1. Euler products

We give a very short proof of a property coming from the equivalence in Theorem 2.

Proposition 4. If $F : \mathbb{N} \to \mathbb{C}$ has Win $F$, is multiplicative, satisfying

$$F' = \mu^2 \cdot F', \quad \lim_x \sum_{r \in \mathbb{P}(1 < r \leq x)} \frac{F'(r)}{r} = \lim_x \sum_{r \in \mathbb{P}(r \geq 1)} \frac{F'(r)}{r}, \quad \forall \mathbb{P} \in \mathbb{P}, \quad \text{and} \quad \sum_p \log \left(1 + \frac{F'(p)}{p}\right) \text{ converges,}$$

then $F$ has an Eratosthenes transform with finite support: $|\text{supp}(F')| < \infty$.

Proof(Sketch). From the equivalence of Theorem 2, using the hypothesis over the limits for $x \to \infty$, we can express the $r$–series as an infinite Euler product: (the case $F = 0$ has a trivial proof, so we know $F \neq 0$, that implies $F'(1) = F(1) = 1$ here)

$$\sum_{r \in \mathbb{P}(1 < r \leq z)} \frac{F'(r)}{r} = \prod_{p > P} \left(1 + \frac{F'(p)}{p}\right) = \exp \left( \sum_{p > P} \log \left(1 + \frac{F'(p)}{p}\right) \right) \to 1,$$

from the hypothesis of convergence for the log–series over primes.

An immediate application to $F(a) := C_{f,g}(N, a)$, thanks to (BH) consequences (see above), gives our

Corollary 6. If $C_{f,g}(N, a)$ satisfies (BH) and the following hypotheses: $C_{f,g}(N, \cdot)$ is multiplicative, with $C'_{f,g}(N, \cdot)$ square-free supported,

$$\lim_x \sum_{r \in \mathbb{P}(1 < r \leq z)} \frac{C'_{f,g}(N, r)}{r} = \lim_x \sum_{r \in \mathbb{P}(r \geq 1)} \frac{C'_{f,g}(N, r)}{r}, \quad \forall \mathbb{P} \in \mathbb{P}, \quad \text{and} \quad \sum_p \log \left(1 + \frac{C'_{f,g}(N, p)}{p}\right) \text{ converges,}$$

then $C_{f,g}(N, a)$ has the (R.E.F.).

Correlations like in (H-L), of course, are not multiplicative. However, future work can be devoted to this specific hypothesis, for general $F$ (namely, applying Proposition 4).

6.2. Eratosthenes Transforms and their averages (following Wintner)

We give here some remarks, about $F'$ and Win $F$ links.

By definition, from Eratosthenes transform (always existing), when $\exists$ Win $F$, we know Wintner transform. Even in case $\exists$ Win $F$, on the other hand, we can’t identify $F'$ from the knowledge of Win $F$.

Actually, this is not completely true: with delicate assumptions, our formula (7) can help, say, “to rebuild $F'$ from Win $F'$”. Namely, the Wintner Orthogonal Decomposition for $F'$ helps knowing $F'$ from Win $F$. (7)
Wintner Orthogonal Decomposition comes from a kind of “arithmetic orthogonality”:

$$\forall P \in \mathbb{P}, \forall d \in \mathbb{N}, \quad d = d(P) \cdot d_j p_i, \quad \text{where} \quad d(P) = \prod_{p \leq P} p^{v_p(d)} \quad \text{and} \quad d_j p_i \overset{def}{=} \prod_{p > P} p^{v_p(d)}$$

are, say, the $P$—smooth, resp., the $P$—sifted part of $d$ (and $d(P)$ defined in (5) above, with the usual $p$—adic valuation recalled soon before Lemma 1).

From this, in fact, once we consider (compare Theorem 1 proof in §1)

$$\sum_{d \in (P)} F'(d) \sum_{d \in (P)} c_q(d) = \sum_{d \in (P)} F'(d) \cdot d(P) = \sum_{d \in (P)^+} F'(d) \cdot d_j p_i = \sum_{d \in (P)^+} \sum_{r > 1} F'(d r)$$

it is clear that we are separating $P$—smooth indices, involving Wintner transform, from $P$—sifted indices, involving Eratosthenes transform.

Once we know Wintner coefficients, philosophically speaking (say, without assumptions), in order to know $F'$ at a fixed $d \in \mathbb{N}$, we only require knowledge of our $F'$ at natural numbers with “arbitrarily large” prime factors.

In fact, compare §5.4, the Wintner Assumption for $F$, thanks to Theorem 1, allows to calculate not only $F$, but also $F'$. In some sense, (WA) constraint on Win $F$ allows to “rebuild”, say, $F'$ from Win $F$. However we can not do this, in general, since two functions with the same Wintner Transform may differ a lot: for example, both 0 and the error term for $F'$, have Wintner Transform 0, but these error terms are not always 0 (the null-function), as testified above in §5.6.

In this present version 9, we add new, recent work.

We have found another requirement on $F$ that allows to recover $F'$ from Win $F$.

It is not as easy as (WA) above, since it involves TWO HYPOTHESES, on our $F$ : the FIRST regards, so to speak, the SMOOTHNESS OF Win $F$, while the SECOND is, little by little, more and more technical (from next Theorem 7 to Theorem 8 and Theorem 9) and may be called a kind of VERTICAL CONSTRAINT so to speak. In fact, see the following, we start asking (for 2nd hypothesis) an easy condition: $F$ (IPP), see Th.m 7; then, after an easy definition before Th.m 8, we ask more generally that $F'$ is supported over numbers $d$ with prime-power factors $p^j$ having $j \leq K$ for a fixed $K$ (generalizing previous condition $K = 1$), $K$ in natural numbers, and we express this saying that $F'$ has VERTICAL LIMIT $K \in \mathbb{N}$, see Th.m 8; then, we ask an even MORE GENERAL condition on $F$, while keeping Win $F$ smooth-supported, in Theorem 9: a kind of VERTICAL CONSTRAINT that involves the Irregular Series of our $F$.

We explicitly warn the reader that we give Theorems 7,8,9 in order of increasing generality, to keep a kind of “historic discovery order”, so to speak. Also, our exposition starts from easier second hypothesis, keeping first hypothesis constant, for a kind of clarity unfolding, as concepts become more and more general. The final Theorem 9 being most general, it has Theorem 8 as a Corollary, whereas Theorem 7 is a kind of particular case ($K = 1$) of Theorem 8 (general $K \in \mathbb{N}$), then.

Finally, see that, actually the Vertical Constraint, Theorem 9 second hypothesis, is rather cumbersome and I guess not so easy to check, see the comments soon after Theorem 9 Proof.

We give a short coming soon of next subsection results, because we wish to underline that, as above, they are in the spirit of, so to speak, rebuilding $F'$ from Win $F$. In fact, Theorems 7 to 9 are, actually, able to imply the $P_0$—smoothness of $F'$ support from that of Win $F$ support.

They do it, somehow, “CROSSING”, so to speak, the HORIZONTAL LIMIT on Wintner Transform, since prime-factors of moduli $q$ with Win$_q F \neq 0$ are $p \leq P_0$, and the VERTICAL LIMIT (not on Win $F$ support, but) on $F'$ support, since all prime-powers factors $p^j$ of divisors $d$ with $F'(d) \neq 0$ have $j \leq K \in \mathbb{N}$, fixed. Informally speaking, next Theorems realize, say, a kind of “Wintner’s Crossing Property”!
6.3. Arithmetic functions’ vertical limits and smooth-supported Win F entail the REEF for F

We start with the easiest vertical limit for $F'$ (it’s square-free supported) of our $F : \mathbb{N} \to \mathbb{C}$, when $F$ (IPP); this, together with the hypothesis that Wintner coefficients for $F$ vanish outside the $P_0$—smooth numbers, for a certain fixed $P_0 \in \mathbb{P}$, gives the REEF: see next Theorem 7.

Then, we keep this hypothesis on Win $F$, while generalizing the vertical limit, from $K = 1$ corresponding to $F$ (IPP), to general $K \in \mathbb{N}$: see the definitions, soon after next Theorem 7 & we apply them in its generalization, Theorem 8. Even this Theorem is actually, technically speaking, a Corollary of subsequent Theorem 9; that generalizes the concept of vertical limit, through a kind of vertical constraint, not expressed in terms of prime-powers limits, but assuming a technical convergence condition, on $F$ Irregular Series.

Since we are going to assume the same “HORIZONTAL LIMIT”, say, on the Wintner coefficients in all of our subsequent results, we profit to give, in next Proposition, the resulting properties of our irregular series for $F$, that we’ll use in all of next results’ Proofs.

We recall that the following “$P$—stability” property has already been exposed in previous sections (esp., compare Remark 6) and follows immediately from (7) in Lemma 2, like “$P$—switching” too.

**Proposition 5.** ($P$—stability & $P$—switching for Irr$(P)$, from Win F horizontal limit)

Let $F : \mathbb{N} \to \mathbb{C}$ have Win $F$ smooth-supported, namely

$$(\text{WIN})_{P_0} \quad \exists P_0 \in \mathbb{P} : \text{supp(Win } F) \subseteq (P_0).$$

Then

$$\forall P \in \mathbb{P}, P \geq P_0, \quad \text{ Irr}_d^{P_0} F = \text{ Irr}_d^{P_0} F, \quad \text{UNIFORMLY } \forall d \in \mathbb{N}.$$  

Furthermore, this $P$—stability can be combined with the other property, say, $P$—switching, next:

$$\text{ Irr}_d^{P_0} F = -F'(d), \quad \text{UNIFORMLY } \forall d \notin (P_0).$$

Our first result follows, to get the REEF. We avoid trivial case: $F$ constant.

**Theorem 7.** Let non-constant $F : \mathbb{N} \to \mathbb{C}$ have supp(Win $F) \subseteq (P_0)$, for some $P_0 \in \mathbb{P}$, assuming $F$ (IPP). Then, $\forall a \in \mathbb{N}, F(a) = \sum_{q \in (P_0)} (\text{Win}_q F) c_q(a)$, whence the $F$—REEF.

**Proof.** We start quoting Wintner Orthogonality Decomposition for $F'$, namely (7) in Lemma 2: $\forall P \in \mathbb{P},$

$$F'(d) = d \sum_{s \in (P)} \mu(s) \text{ Win}_d s F - \text{ Irr}_d^{(P)} F, \quad \forall d \in \mathbb{N},$$

whence $(\text{WIN})_{P_0}$ in Proposition 5 gives the “$P$—stability” of Irr$(P)$ $F$ from $P = P_0$ onwards:

$$(*) \quad \text{Irr}^{(P)} F = \text{ Irr}^{(P_0)} F, \quad \forall P \geq P_0 (P \in \mathbb{P}).$$

Then, numbering consecutive primes from $P_0$ onwards as : $P_0 < P_1 < P_2 < \cdots < P_m < \cdots$,

$$\forall d \in \mathbb{N}, \quad \text{ Irr}_d^{(P_k)} F = \text{ Irr}_d^{(P_0)} F = \sum_{r \in (P_k)} F'(dr) + \sum_{r \in (P_{k+1})} F'(dP_k r) P_k r = \text{ Irr}_d^{(P_k)} F + \frac{1}{P_k} \text{ Irr}_d^{(P_k)} F,$$

(we used here $F$ (IPP), ignoring $P_1$ prime powers), whence, since our hypothesis $(\text{WIN})_{P_0}$ gives, from both properties in Proposition 5, $\text{ Irr}_d^{(P_k)} F = -F'(P_k d)$, we get: $F'(P_k) = 0, \forall d \in \mathbb{N}$; iterating on $m \in \mathbb{N}$, in the same way

$$\forall m \in \mathbb{N}, \forall d \in \mathbb{N}, \quad \text{ Irr}_d^{(P_m)} F = \text{ Irr}_d^{(P_{m-1})} F = \text{ Irr}_d^{(P_m)} F + \frac{1}{P_m} \text{ Irr}_d^{(P_m)} F,$$

31
the REEF following from: \( F(\text{IPP}) \Rightarrow F' = \mu^2 F' \Rightarrow \text{Win} F = \mu^2 \cdot \text{Win} F \Rightarrow |\text{supp} (\text{Win} F)| \leq 2^{\pi(P_0)}. \)

See that, apart from the properties that come only from the smooth support of our Win \( F \), the other property of our irregular series we are applying, here, is a kind of recursion which simplifies a lot, from the other hypothesis, namely no prime-power-factors in \( F' \) support!

In fact, without a specific hypothesis on our \( F \), this recursion is not so simple. Before we generalize (IPP) arithmetic functions, we give a Lemma to show how this general recursion goes, for the irregular series.

By the way, we need a hypothesis for this series to converge, namely: existence of Wintner Transform.

This is, so to speak, contained in next Lemma. The Proof comes from \( \text{Irr}^{(P)} F \) definition.

**Lemma 13. (Recursion for the irregular series)**

Let \( F : \mathbb{N} \to \mathbb{C} \) have Win \( F \). Then, given a sequence of consecutive primes \( P_0 < P_1 < \cdots < P_m < \cdots \), once fixed any \( m \in \mathbb{N} \),

\[
\forall d \in \mathbb{N}, \quad \text{Irr}_d^{(P_m-1)} F = \text{Irr}_d^{(P_m)} F + \sum_{r \in (P_m)_{r \geq 2}} \sum_{j=1}^{\infty} \frac{F'(dP_0^jr)}{P_m^jr}.
\]

Notice: the \( r \)–series and the \( j \)–series may not be exchanged, in general. Furthermore, any bound on \( F' \) modulus in this double series ruins the convergence of present \( r \)–series! However, if the \( j \)–summation is finite, we can exchange summations very easily: for this reason, we introduce a generalization of (IPP) functions, that have \( j \leq 1 \), to \( j \leq K \), with fixed \( K \in \mathbb{N} \), here.

We write, \( \forall n \in \mathbb{N} \),

\[
V(n) \overset{\text{def}}{=} \max \{ v_p(n) : p \in \mathbb{P} \}
\]

for the, say, (global) Valuation of a natural \( n \in \mathbb{N} \). Then, by abuse of notation, we use the same symbol for the (global) Valuation of any non-zero arithmetic function \( G : \mathbb{N} \to \mathbb{C}, G \neq 0 \), which might be infinite this time:

\[
V(G) \overset{\text{def}}{=} \sup \{ V(n) : n \in \text{supp}(G) \}
\]

and we call \( G : \mathbb{N} \to \mathbb{C} \) a (KVL) arithmetic function, when this sup is finite: \( V(G) \in \mathbb{N}_0 \) (the case \( V(G) = 0 \) holding IFF the only non-zero value of \( G(n) \) is at \( n = 1 \)),

\[
G \text{ (KVL)} \overset{\text{def}}{\iff} V(G) \in \mathbb{N}_0.
\]

Here, (KVL) abbreviates “\( K \)–Vertically Limited”, as we may write for these functions: \( V(G) = K \). For example, \( F \) (IPP) if and only if: \( V(F') \leq 1 \) (i.e., \( F' \) is square-free supported): recall, \( V(F') = 0 \) exactly for constant \( F = F(1) \neq 0 \). However, this \( F' \) is (KVL) but how do we call the corresponding \( F \)? Well, it Ignores Prime Powers, being “\( K \)–th powers independent”, recalling \( K = 1 \) for \( F \) (IPP), and we introduce, say, the “\( K \)–Vertically Independent” arithmetic functions

\[
F \text{ (KVI)} \overset{\text{def}}{\iff} F' \text{ (KVL)}.
\]
From previous, next Lemma. Again, for the Proof recall $\text{Irr}^{(P)} F$ definition. We avoid $F$ constant, now.

**Lemma 14.** (Recursion for the Irregular series of $K$—Vertically independent Arith. Funcs)

Let $F : \mathbb{N} \to \mathbb{C}$ have $\text{Win} F$ and let $F$ be (KVI), with $V(F') = K \in \mathbb{N}$. Then, given a sequence of consecutive primes $P_0 < P_1 < \cdots < P_m < \cdots$, once fixed any $m \in \mathbb{N}$,

$$\forall d \in \mathbb{N}, \quad \text{Irr}_d^{(P_m)} F = \text{Irr}_d^{(P_{m-1})} F + \sum_{j \leq K} \frac{\text{Irr}_d^{(P_m)} F}{P_m^j}.$$

Now, we generalize hypothesis $F$ (IPP) to hypothesis $F$ (KVI), here, with $F$ non-constant.

**Theorem 8.** Let $F : \mathbb{N} \to \mathbb{C}$ have $\text{supp} (\text{Win} F) \subseteq (P_0)$, for a certain $P_0 \in \mathbb{P}$, and assume $F$ (KVI), with $V(F') = K \in \mathbb{N}$. Then, $\forall a \in \mathbb{N}$, $F(a) = \sum_{q \in (P_0)} (\text{Win}_q F) c_q(a)$, whence the $F$ — REEF.

**Proof.** The case of second hypothesis : non-constant $F$ (IPP) is equivalent to $V(F') = 1 = K$. We start for next cases $K \geq 2$, getting the same consecutive primes $P_0 < P_1 < \cdots < P_m$, together with the $P$—Stability of $\text{Irr}^{(P)} F$, from $P = P_0$ on; but now, for general $K \in \mathbb{N}$ we need Lemma 14, to get :

$$\forall m \in \mathbb{N}, \forall d \in \mathbb{N}, \quad \text{Irr}_d^{(P_m)} F = \text{Irr}_d^{(P_0)} F + \sum_{j \leq K} \frac{\text{Irr}_d^{(P_m)} F}{P_m^j} \Rightarrow \sum_{j \leq K} \frac{F'(dP_m^j)}{P_m^j} = 0 \Rightarrow \sum_{j \leq K} \frac{F'(dP_m^j)}{P_m^j} = 0,$$

after using $P$—Stability and $P$—Switching of Proposition 5, from the hypothesis $(\text{WIN})_{P_0}$.

Thus

$$(**): \quad \forall m \in \mathbb{N}, \forall d \in \mathbb{N}, \quad F'(dP_m) = -\sum_{j \leq K-1} \frac{F'(dP_m^{j+1})}{P_m^j},$$

after renaming the $j$—variable, here. This $(**)$ is recursion on $P_m$—Powers. In fact, fix $m \in \mathbb{N}$ and this recursion, together with $V(F') = K$, say, “kills powers” from the highest:

$$F'(dP_m) = -\sum_{j \leq K-1} \frac{F'(dP_m^{j+1})}{P_m^j}, \forall d \in \mathbb{N} \text{ (set } d := P_m^{K-1} t) \Rightarrow F'(tP_m^K) = -\sum_{j \leq K-1} \frac{F'(dP_m^{j+K})}{P_m^j} = 0, \forall t \in \mathbb{N},$$

whence

$$F'(dP_m) = -\sum_{j \leq K-2} \frac{F'(dP_m^{j+1})}{P_m^j}, \forall d \in \mathbb{N} \text{ (set } d := P_m^{K-2} t) \Rightarrow F'(tP_m^{K-1}) = -\sum_{j \leq K-2} \frac{F'(dP_m^{j+K-1})}{P_m^j} = 0, \forall t \in \mathbb{N},$$

where this time we combine $V(F') = K$ with previous vanishing above. Iterating, we get

$$F'(tP_m^2) = -\frac{F'(dP_m^3)}{P_m^2} = 0, \forall t \in \mathbb{N}$$

from vertical limit and previous vanishing values, whence

$$F'(dP_m) = -\frac{F'(dP_m^2)}{P_m} = 0, \forall d \in \mathbb{N}.$$

In all, $P_m | t \Rightarrow F'(t) = 0$ and this holds $\forall m \in \mathbb{N}$.

In other words, we get back to previous Proof last part, as $\text{supp}(F') \subseteq (P_0)$ & so on: the REEF’s from $|\text{supp}(\text{Win} F)| \leq (K+1)^{\pi(P_0)}$.

**Remark 16.** We see the irony of fate at work on $(**)$, as a *posteriori* it becomes completely trivial. \(\diamond\)
Next, we give present, most general hypothesis on $F'$, here.

**Theorem 9.** Let $F : \mathbb{N} \to \mathbb{C}$ have $\text{supp}(\text{Win} F) \subseteq (P_0)$, for a certain $P_0 \in \mathbb{P}$, and

$$\forall P \geq P_0, \forall d \in \mathbb{N}, \quad \sum_{r \geq 1} \sum_{j=1}^{\infty} \frac{F'(dP^j r)}{P^j r} = \sum_{j=1}^{\infty} P^{-j} \text{Irr}_{dP^j} (P) F.$$

Then, $\text{supp}(F') \subseteq (P_0)$, whence the Ramanujan-Wintner Smooth Expansion.

**Proof.** The Lemma 13 above gives, together with Proposition 5 like in previous Proofs, with the same consecutive primes $P_0 < P_1 < \cdots < P_m < \cdots$, fixing $m \in \mathbb{N}$, after changing $j$ variable,

$$\forall d \in \mathbb{N}, \quad \text{Irr}_d^{(P_0)} F = \text{Irr}_d^{(P_0)} F + \sum_{j=1}^{\infty} \frac{\text{Irr}_{dP^j}^{(P_0)} F}{P_m^j} \Rightarrow F'(dP_m) = -\sum_{j=1}^{\infty} P_m^{-j} F'(dP_m^{j+1}).$$

Thus

$$(***) \quad F'(dP_m) = -P_m^{-1} F'(dP_m^2) - \sum_{j=1}^{\infty} P_m^{-j-1} F'(dP_m^{j+2}), \ \forall d \in \mathbb{N},$$

whence, setting $d = tP_m$ and back with $d$ instead of $t$,

$$F'(dP_m^2) = -P_m^{-1} F'(dP_m^3) - \sum_{j=1}^{\infty} P_m^{-j-1} F'(dP_m^{j+3}), \ \forall d \in \mathbb{N},$$

which we plug into $(***)$ to get

$$F'(dP_m) = -P_m^{-1} \left( -P_m^{-1} F'(dP_m^3) - \sum_{j=1}^{\infty} P_m^{-j-1} F'(dP_m^{j+3}) \right) - \sum_{j=1}^{\infty} P_m^{-j-1} F'(dP_m^{j+2}) = 0, \ \forall d \in \mathbb{N},$$

true $\forall m \in \mathbb{N}$, whence $\text{supp}(F') \subseteq (P_0)$.

Notice: the condition on exchanging double summation in the double series, say, in Theorem 9 “Vertical Constraint”, is very technical and doesn’t allow easy shortcuts, as the double series doesn’t converge absolutely due to the lack of absolute convergence for the Irregular series!

Also, this most general result doesn’t supply the REEF because it has no explicit request on $F'$ vertical LIMIT: this, in Theorems 8,7 allows to estimate explicitly the cardinality of non-vanishing Wintner coefficients, whence the REEF. (Compare Theorems 8,7 Proofs final parts.)

Going back to applications, for (BH)—correlations $F(a) := C_{f,g_0} (N, a), \forall a \in \mathbb{N}$, see that the condition (WIN)$_{P_0}$ follows from (BH) (compare §1.2 above), with $P_0 \overset{\text{def}}{=} \max \{ p \in \mathbb{P} : p \leq Q \}$ and this, thanks to Counterexample 1 studied in §5.6 above, renders crystal clear that we need a kind of vertical constraint. In fact, since no (R.E.E.F.) holds for it (see quoted §5.6) we see that, not only it is not (IPP) (compare §5.6, Curiosity 1), but it has neither the much lighter vertical constraint, in Theorem 9 above.
6.4. Further remarks and future work

We were looking, in previous versions, for a kind of “supplementary hypothesis”, which, added to (BH), gives the (R.E.E.F.) : we thought (ETD) could be the right one. Actually, (BH) alone doesn’t give the (R.E.E.F.), as we proved in third version of [C1] with Counterexample 1 there, compare §5.6. As our Theorem 1 shows, (WA) is a good hypothesis of this kind : it “gives the (R.E.E.F.)”, to (BH)—correlations (in Corollary 1). Finally, the “missing hypothesis”, say, is given by the vertical constraints, more and more general, of above Theorems 7,8,9 : in fact, from (BH) we know that supp(Win F) ⊆ [1,Q] ⊆ (P₀), with Q ≤ P₀ ∈ ℙ, abbreviating with F(a) our correlation of shift a ∈ ℕ. Actually, Theorem 9 generality doesn’t supply the (R.E.E.F.), but only the Ramanujan-Wintner Smooth Expansion. Our Theorem 8 and its particular case Theorem 7, here, give the (R.E.E.F.) to (BH)—correlations, but at a high price so to speak: a vertical limit on the divisors d|a of correlation’s shift a ∈ ℕ. This is not so natural, for a correlation; however, it points in the “heuristically right direction”, say, i.e.: (BH) correlations with shift-factor g which is (IPP) have square-free supported Wintner Transforms, entailing that (R.E.E.F.)’s main term (that’s A_F, see §5.4) is (IPP) itself, with smooth-supported Wintner Transform, in full concordance with Theorem 7, say!

There are two main directions where to look at in future work: the (BH)—correlations world, both for its own sake & for the inspiration for finding (as we did in present work!) new general results; and the theoretical fascination coming from the “new Ramanujan clouds”: mainly the Ramanujan smooth clouds, as for Ramanujan clouds we already started, with Luca Ghidelli, a kind of structural description (beginning with multiplicative Ramanujan coefficients, compare [CG1] and [CG2]).

Last but not least we will, in future papers, give other explicit formulæ, for the correlations satisfying (BH), coming from the elementary approach (compare Theorem 6 above) with the so-called Dirichlet characters explicit formulæ.
Bibliography

[C0] G. Coppola, *An elementary property of correlations*, Hardy-Ramanujan J. 41 (2018), 65–76.

[C1] G. Coppola, *A smooth shift approach for a Ramanujan expansion*, ArXiv:1901.01584v3 (Third Version)

[C2] G. Coppola, *Finite and infinite Euler products of Ramanujan expansions*, ArXiv:1910.14640v2 (Second Version)

[C3] G. Coppola, *Recent results on Ramanujan expansions with applications to correlations*, Rend. Sem. Mat. Univ. Pol. Torino 78.1 (2020), 57–82.

[CG1] G. Coppola and L. Ghidelli, *Multiplicative Ramanujan coefficients of null-function*, ArXiV:2005.14666v2 (Second Version)

[CG2] G. Coppola and L. Ghidelli, *Convergence of Ramanujan expansions, I [Multiplicativity on Ramanujan clouds]*, ArXiV:1910.14640v1

[CM] G. Coppola and M. Ram Murty, *Finite Ramanujan expansions and shifted convolution sums of arithmetical functions, II*, J. Number Theory 185 (2018), 16–47.

[D] H. Davenport, *Multiplicative Number Theory*, 3rd ed., GTM 74, Springer, New York, 2000.

[De] H. Delange, *On Ramanujan expansions of certain arithmetical functions*, Acta Arith., 31 (1976), 259–270.

[HL] G.H. Hardy and J.E. Littlewood, *SOME PROBLEMS OF ‘PARTITIO NUMERORUM’: III: ON THE EXPRESSION OF A NUMBER AS A SUM OF PRIMES*. Acta Mathematica 44 (1923), 1–70.

[K] J.C. Kluyver, *Some formulae concerning the integers less than n and prime to n*, Proceedings of the Royal Netherlands Academy of Arts and Sciences (KNAW), 9(1):408–414, 1906.

[M] M. Ram Murty, *Ramanujan series for arithmetical functions*, Hardy-Ramanujan J. 36 (2013), 21–33. Available online

[R] S. Ramanujan, *On certain trigonometrical sums and their application to the theory of numbers*, Transactions Camb. Phil. Soc. 22 (1918), 259–276.

[ScSp] W. Schwarz and J. Spilker, *Arithmetical Functions*, Cambridge University Press, 1994.

[T] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Studies in Advanced Mathematics, 46, Cambridge University Press, 1995.

[W] A. Wintner, *Eratosthenian averages*, Waverly Press, Baltimore, MD, 1943.

Giovanni Coppola - Università degli Studi di Salerno (affiliation)
Home address : Via Partenio 12 - 83100, Avellino (AV) - ITALY
e-mail : giocop70@gmail.com
e-page : www.giovannicoppola.name
e-site : www.researchgate.net