Formulations of Elastodynamics in Anisotropic Multiphase Porous Media from the Principle of Energy Conservation

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Elastodynamics has been usually formulated with either Newton’s second law of motion, Lagrange’s equation, or Hamilton’s variational principle for over 150 years. In this work, contrary to classical mechanics, a novel strategic methodology is proposed for formulating elastodynamics by using the principle of energy conservation. Firstly, based on Hamilton’s variational principle, Hamilton’s canonical equations, Lagrange’s equation, and the elastodynamic equation of motion are derived in arbitrarily anisotropic and multiphasic porous elastic media, for the first time. Secondly, Lagrange’s equation, Hamilton’s canonical equations, and the elastodynamic equation of motion are all reformulated by using the principle of energy conservation for the related media. Both formulations from Hamilton’s variational principle and the principle of energy conservation are validated by each other. The advantages of our methodology lie in that, elastodynamics is directly formulated using a simple constraint of energy conservation without introducing variational concepts. It is easy to understand and has clear physical meanings. Our methodology unlocks the physics essences of Hamilton’s variational principle in continuum mechanics which is a consequence of the principle of energy conservation. Our methodology also paves an alternative way of treating other complex continuous dynamical systems in a broad sense. In addition, the continuity conditions at various medium interfaces are also revisited and extended using our proposed approach, which explains the law of reflections and refractions as an application.

Keywords: Energy conservation, multiphase porous media, Hamilton’s principle, elastodynamic equations, boundary conditions.

I. INTRODUCTION

The elastodynamic behaviors in inhomogeneous anisotropic and fully elastic equivalent media have been treated thoroughly, and the elastodynamic equations are essentially based on Newton’s second law of motion \cite{1–10} in homogenous equivalent elastic media, or Hamilton’s variational principle \cite{11–21} in various rather complex media. In the latter, all impressed forces of the monogenic type are absorbed in the usual way into the Lagrangian density function, while the polygenic forces are given by their virtual work \cite{11–14}.

Porous media, composed of solid skeleton and pore fluids exist widely in nature, such as underground reservoir media containing oil, gas, and water, permafrost in the Arctic region, quick-frozen foods, etc. It is extremely important to understand wave propagation in such media for oil exploration, earthquake disaster prediction, hydrological environment monitoring, and frozen food monitoring. The commonly used method to construct an elastodynamic equation of motion for such complex media is Hamilton’s variational principle, or least action principle, such as for piezoelectric materials \cite{22}, and porous media \cite{23–33}.

However, the achievement of a unified theory of multiphase continuum poroelasticity, capable of addressing multiphase systems with any range of compressibility of the constituents, still represents a challenge for theoretical and applied continuum mechanics, even for a simpler two-phase problem \cite{33}.

Biot paved the way to tackle poroelasticity problems in two-phase porous media filled with a single fluid \cite{23–26}. He assumed that there is relative motion between pore fluid and solid phases, and this coupling is characterized by an apparent density to generate kinetic energy included in Lagrange’s density function in the case of wavelength is much larger than the pore space or grain sizes such that in any field point there would be fluid and solid components \cite{23}, which seems unrealistic. Biot’s assumptions of the coupling between fluid and solid phases were confirmed by Piana who observed the slow compressional waves that are theoretically predicted by Biot \cite{34}, which means Biot’s theory can be used in describing wave propagation in two-phase porous media for low-frequency range to some extent.

Using Biot’s theory, Pride et al. proposed a dual porous medium model to describe the propagation in a variety of
complex rock structures [35]. Furthermore, Ba et al. derived the Biot-Rayleigh’s equation in a dual-porous medium, which is a combination of one fluid and two types of solid matrixes in dual-porous media. They generalized the theory to the special case of two fluids and one type of solid skeleton [32]. Bedford and Stern proposed the theory of liquid-saturated porous media containing bubbles based on the idea of mixture theory [36], which is consistent with Biot’s theory except that the inertial effect caused by the bubble vibration is included. Tuncay and Corapcioğlu used the volume average theory to investigate the wave propagation in two immiscible Newtonian fluid-saturated poroelastic media [37]. Santos et al. established the wave theory of two viscous immiscible fluid-saturated porous media using the principle of compensated virtual work and Lagrange’s approach [29,30]. Leclaire et al. derived momentum conservation equations for frozen porous media by using the Lagrange equation [38]. Furthermore, they studied the permafrost layer problems with one solid and one fluid in the pores. They predicted three types of longitudinal waves and two types of shear waves. Carcione and Seriani proposed the Carcione–Leclaire three-phase model, in which direct contact between solid particles and ice particles is considered [31]. Based on the uniform bubble distribution model, Carcione et al. studied the propagation of elastic waves in partially saturated porous media by using Biot’s theory [39], it is believed that the attenuation and dispersion of elastic waves are mainly caused by the energy transfer between different mode waves.

From the above reviews, it is seen that Biot’s theory is generally believed to be semi-phenomenological, and more rigorous acoustic wave theories are based on the homogenization method [40] and the volume averaging method [41,42], which link the thermal microscopic and macroscopic situations. However, although the latter two methods are relatively rigorous, a lot of parameters are involved, and difficult to obtain experimentally. Thus, it is popular today that, dynamical equations of wave propagation in a porous media filled with different matrixes and porous fluids are derived from the Biot’s theory framework based on Hamilton’s variational principle or Lagrange’s equation.

On the one hand, Hamilton’s variational principle, or the principle of least action [43], although dominating the heart of physics and used in poroelasticity, was not explained well clearly in Lagrange’s analytical mechanics, and this principle is presented in almost all textbooks which is beyond comprehension [12]. Recently, rather than using the existing principles, such as Hamilton’s variational principle, Lagrange’s equation, or Newton’s second law of motion, a methodology for tackling dynamical problems in many-particle systems, is proposed, i.e., starting from the equation of conservation of energy, Lagrange’s equation, Hamilton’s canonical equations, and Hamilton-Jacobi’s equation are derived [44]. The work is also seen for an equivalent continuous system in identifying wave equation of motion in a piezoelectric medium [45,46].

On the other hand, most of the existing wave theories in porous media are limited to the case of two-phase three-component media [33,38] and the two-fluid and one-solid media [29,30]. In practice, multiphase and multicomponent media, such as the acoustic monitoring for frozen foods, and oil, gas, cement material, sandstone, and sandstone-bearing reservoir characterizations are encountered. Therefore, unified poroelasticity in an arbitrary multiphasic porous medium is still left unresolved [33].

In this work, firstly we will give the formulation of the elastodynamic equations in arbitrarily anisotropic and multiphasic porous media from Hamilton’s variational principle and analyze the treatments of the boundary conditions. Furthermore, contrary to the existing Hamiltonian mechanics, we will formulate the general elastodynamics, including Lagrange’s equation and Hamilton’s canonical equations for an arbitrarily anisotropic and multiphasic porous medium under the framework of energy conservation. Finally, as an application example, we will study the stress and displacement continuity at various interfaces by using energy conservation constraints to explain the existing reflection and refraction laws.

II. PHYSICAL MODEL DESCRIPTIONS

A. Assumptions

The assumptions are stated as in the following [38,48]. The displacement, strain, and particle velocity in an elastic medium are all small, so the Eulerian and Lagrangian formulations are consistent. The strain equation, dissipative force, kinetic energy, and momentum are linear (elastic strain energy, dissipative potential, and kinetic energy are of the quadratic type); the principles of continuum mechanics can be applied to measurable macroscopic values, while the macroscopic quantities are the volume averages of the corresponding microscopic quantities; the wavelength related to the macroscopic fundamental volume is large compared to the size of the base volume, that is, compared to the size of the basic volume, and this volume has well-defined properties such as porosity, permeability, and elastic modulus; the condition is adiabatic, and no heat exchange is considered in the process; the stress distribution in the fluid is hydrostatic; the liquid phase is continuous, while the matrix consists of solid and connected pores with disconnected pores being no contributions to the porosity.

B. Model and variable descriptions
Suppose that there are wave motions in a two-phase (solid and fluid) and three-component porous medium (for example, water and gas hydrate-bearing reservoirs or formations, with water and gas hydrate filled in their pores), we arbitrarily extract a volume $\Omega$ from this porous medium, the closed boundary surface is $\partial \Omega$. For medium composition $\alpha$, $u_i^{(\alpha)} = u_i^{(\alpha)}(x,t)$ is particle displacement components, and
$$u_i^{(\alpha)} = du_i^{(\alpha)} / dt = \partial u_i^{(\alpha)}(x,t) / \partial t,$$
which are the particle velocity components. The above equation is consistent with that in [8] for one component bearing media. The cosine directional of the displacement is defined as $l_i$, $i = 1,2,3$. The elastic strain tensor components are defined as [38,48]
$$e^{\alpha}_{ij} = (u^{\alpha}_{ij} + u^{\alpha}_{ji}) / 2,$$
in Composition $\alpha$, and $\alpha, i, j = 1,2,3$. It should be noted that in our later discussions $\alpha$ could be larger than 3, which means our treatments can be extended to an arbitrarily multiphase porous medium. In the above strain formula [37,47],
$$u_i^{\alpha} = \partial x_i / \partial x_j, i,j = 1,2,3,$$
where $x_i$ is coordinate of $i$ direction for a Cartesian coordinate system. For example, the first medium is a solid skeleton; the second is liquid in the pores, and the third is solid in the pores of the model. We denote $\rho_{\alpha m} = \rho_{\alpha m}, m, n = 1,2,3$, the apparent densities with Medium $m$ and Medium $n$ being coupled with each other; while $\rho_{11}, \rho_{22}, \rho_{33}$ are the densities of Media 1, 2, and 3, respectively. As the number of material compositions or components is larger than 3, we may denote the density as $\rho_{11}, \rho_{22}, \rho_{33}$. In this case, the number of material compositions can be larger than 3, so that the coexisting compositions in the media can be any number as expected, i.e., $\alpha = 1,2,3,\ldots, M \geq 3$.

### III. FORMULATIONS OF GENERAL MECHANICAL EQUATIONS

Firstly, we start from Hamilton’s variational principle and investigate the conventional mechanics to formulate Hamilton’s canonical equations, Lagrange’s equation, and elastodynamic equation of motion in arbitrarily anisotropic and multiphase porous media. Secondly, we will formulate the existing mechanical equations from the principle of energy conservation in arbitrarily anisotropic and multiphase porous media.

The density functions of kinetic energy and elastic potential energy in a continuum are defined respectively as [6–8]:
$$K_\rho = \rho \dot{u}_i \dot{u}_i / 2,$$
(1a)
$$V_\rho = V_\rho(u_i).$$
(1b)
For multiphase or multicomponent porous media, such as permafrost and gas hydrate reservoirs, there are two-solid and one-liquid (three-component) media. According to our previous assumptions, a volume $\Omega$ is arbitrarily chosen in the continuous medium of interest with the boundary $\partial \Omega$. These two energy density functions are related to bulk density, particle vibration displacements, and velocities of the two-solid and one-liquid media and the equivalent deformations of each component.

Different from the treatment of the kinetic energy density function in previous work [48], we assume that there may be relative coupled motions between the three components of the porous medium that generate kinetic energy. Therefore, the kinetic energy density function has a similar form as in each medium component. So that the equivalent kinetic energy density function for the whole porous medium can be written as
$$2T_\rho = \rho_{\alpha m}u^{(\alpha)}_{\dot{i}}u_{\dot{i}}^{(\alpha)} + \rho_{\alpha m}u^{(\alpha)}_{\dot{\nu}}u_{\dot{\nu}}^{(\alpha)} + \rho_{\alpha m}u^{(\alpha)}_{\dot{\alpha}}u_{\dot{\alpha}}^{(\alpha)}$$
$$+ \rho_{\alpha m}^2(u^{(\alpha)}_{\nu\nu} - u^{(\alpha)}_{\nu\nu})(u^{(\alpha)}_{\nu\nu} - u^{(\alpha)}_{\nu\nu}) + \rho_{\alpha m}^2(u^{(\alpha)}_{\nu\nu} - u^{(\alpha)}_{\nu\nu})(u^{(\alpha)}_{\nu\nu} - u^{(\alpha)}_{\nu\nu})$$
$$+ \rho_{\alpha m}^2(u^{(\alpha)}_{\nu\nu} - u^{(\alpha)}_{\nu\nu})(u^{(\alpha)}_{\nu\nu} - u^{(\alpha)}_{\nu\nu})$$
(2a)
where $\rho_{\alpha m}, \alpha = 1,2,3$, is the density of Medium $\alpha$, $\rho_{\alpha m}^{(\alpha)} \alpha, \beta = 1,2,3$, is the apparent density produced when Media $\alpha$ and $\beta$ interact with each other [5,7,28], and $\rho_{\alpha m}^{(\alpha)} = \rho_{\alpha m}^{(\alpha)} [48]$. The superscript $\alpha$ in $\rho_{\alpha m}^{(\alpha)}$ represents the apparent density. Rearranging Eq. (2a), we have
$$2T_\rho = (\rho_1 + \rho_{21}^2 + \rho_{31}^3)u^{(1)}_{\nu\nu} + (\rho_2 + \rho_{22}^2 + \rho_{23}^3)u^{(2)}_{\nu\nu}$$
$$+ (\rho_3 + \rho_{31}^3 + \rho_{33}^3)u^{(3)}_{\nu\nu}$$
(2b)
$$- 2\rho_{12}^2u^{(1)}_{\nu\nu}u^{(2)}_{\nu\nu} - 2\rho_{23}^2u^{(2)}_{\nu\nu}u^{(3)}_{\nu\nu} - 2\rho_{31}^3u^{(3)}_{\nu\nu}u^{(1)}_{\nu\nu}.$$  

Denote
$$\rho_1 = \rho_1 + \rho_{21}^2 + \rho_{31}^3, \rho_2 = \rho_2 + \rho_{22}^2 + \rho_{23}^3, \rho_3 = \rho_3$$
(2c)
$$\rho_{12}^2 = \rho_{12}^2 + \rho_{22}^2, \rho_{23}^3 = \rho_{23}^3 + \rho_{33}^3, \rho_{31}^3 = -\rho_{31}^3,$$
where $\rho_{\alpha m} = \rho_{\alpha m}, m, n = 1,2,3$. Rearranging Eq. (2b), the kinetic energy density function can be simplified as
$$2T_\rho = \rho_1u^{(1)}_{\nu\nu}u^{(1)}_{\nu\nu} + \rho_2u^{(2)}_{\nu\nu}u^{(2)}_{\nu\nu} + \rho_3u^{(3)}_{\nu\nu}u^{(3)}_{\nu\nu}$$
$$+ 2\rho_{12}^2u^{(1)}_{\nu\nu}u^{(2)}_{\nu\nu} + 2\rho_{23}^3u^{(2)}_{\nu\nu}u^{(3)}_{\nu\nu} + 2\rho_{31}^3u^{(3)}_{\nu\nu}u^{(1)}_{\nu\nu}.$$  

By using this treatment, it is easy to obtain Eq. (2d) directly, as was usually done from the previous relations using the average value of macroscopic parameters and microscopic parameters [23–26,29–32,38,48].

According to Einstein’s summation convention, the total kinetic energy density of multiphase porous media can be briefly written as
$$2T_\rho = \rho_{\alpha m}u^{(\alpha)}_{\dot{i}}u^{(\alpha)}_{\dot{i}}.$$  

In the above formula, $\alpha, \beta$ are both the upper index and the lower index. If the index is repeated, it conforms to the Einstein summation convention. Expanding the above formula, we get...
Further expansion of the above equation yields
\[ 2T_p = \rho_{ij}u_i^{(2)}u_j + \rho_{ij}u_i^{(3)}u_j. \]

Considering \( \rho'_{ij} = \rho_{ij} \), Eq. (3b) is consistent with Eq. (2d). Eq. (3b) can be extended to multiphasic porous media with arbitrary material components, which means \( \alpha, \beta \in [2, N] \). with \( N \) could be larger than 3. Later, it is seen that the three-component model can be extended to any more than 3 components together with the other variables.

For a porous model, the equation of motion of porous media can be obtained from Lagrange’s equation once the kinetic and potential energy densities are identified [21–30]. In our case, the potential energy density of a solid-filled porous material can be expressed as [38]
\[ V_{\rho}(\alpha, i, j) = 1, 2, 3, \]
where \( e^{(a)}_{ij} \) is, as defined before, the elastic strain components in Composition \( \alpha \) for a multiphasic porous medium.

The density function of potential energy can be used to identify the stress-strain relationship or constitutive relationship of the medium of interest. Here we only consider the case where the constitutive relationship is linear, then the elastic coefficients are determined by using the strain energy density function \( V_{\rho} \) [6,8].

Firstly, we expand the elastic potential energy into Taylor’s series concerning the strain components and keep only the second-order term for linear problems, that is,
\[ V_{\rho} = 2V_{\rho}(0,0,0) + [\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(1)})]e_{ij}^{(1)}e_{kl}^{(1)} + [\nabla^3 V_{\rho} / (\partial e_{ij}^{(2)} \partial e_{kl}^{(2)})]e_{ij}^{(2)}e_{kl}^{(2)} + [\nabla^3 V_{\rho} / (\partial e_{ij}^{(3)} \partial e_{kl}^{(3)})]e_{ij}^{(3)}e_{kl}^{(3)} + 2[\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(2)})]e_{ij}^{(1)}e_{kl}^{(2)} + 2[\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(3)})]e_{ij}^{(1)}e_{kl}^{(3)} + 2[\nabla^3 V_{\rho} / (\partial e_{ij}^{(2)} \partial e_{kl}^{(3)})]e_{ij}^{(2)}e_{kl}^{(3)} \]
\[ = 2V_{\rho} = 2V_{\rho}(0,0,0) + [\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(1)})]e_{ij}^{(1)}e_{kl}^{(1)} + [\nabla^3 V_{\rho} / (\partial e_{ij}^{(2)} \partial e_{kl}^{(2)})]e_{ij}^{(2)}e_{kl}^{(2)} + [\nabla^3 V_{\rho} / (\partial e_{ij}^{(3)} \partial e_{kl}^{(3)})]e_{ij}^{(3)}e_{kl}^{(3)} + 2[\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(2)})]e_{ij}^{(1)}e_{kl}^{(2)} + 2[\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(3)})]e_{ij}^{(1)}e_{kl}^{(3)} + 2[\nabla^3 V_{\rho} / (\partial e_{ij}^{(2)} \partial e_{kl}^{(3)})]e_{ij}^{(2)}e_{kl}^{(3)} \]

The influence of the elastic potential energy on wave motion is about the variation of the space deformation. Therefore, we might as well take \( V_{\rho}(0,0,0) \) to be zero for a reference value.

If the Einstein summation convention is used for the repeated upper and lower indices, the elastic potential energy can be briefly written as
\[ 2V_{\rho} = [\nabla^3 V_{\rho} / (\partial e_{ij}^{(1)} \partial e_{kl}^{(1)})]e_{ij}^{(1)}e_{kl}^{(1)}. \]
To examine its validity, expanding Eq. (4b), yields
\[ 2V_{\rho} = \sigma_{\alpha i}^{(\rho)} e_{i\alpha}^{(\rho)}, k, l, \beta = 1, 2, 3. \] (8)

A. Hamilton’s canonical equations

It is proclaimed that the variational principle is the general law of nature and perhaps the most fundamental law of nature revealed up to now [12], and it continues to hold its ground in the description of all the phenomena of nature [49,50]. Hamilton’s variational principle and its variational variants become so important that they have constructed a founding stone of present-day theoretical physics [12]. Surely, elastodynamics in porous media should be under the umbrella of Hamilton’s variational principle or its variants [33].

In the followings, starting from Hamilton’s variational principle, we will formulate Hamilton’s canonical equations, Lagrange’s equation, and elastodynamic equation of motion in arbitrarily anisotropic and multiphasic porous media, and these formulated equations will be used for comparisons with those derived by our proposed methodology, which validates the derived equations with each other. In addition, we will clarify some ambiguous usages of boundary conditions during the equation formulations. The roadmap for formulating elastodynamics using Hamilton’s variational principle is shown in Fig. 1.

\[ \int_{0}^{t} \delta L dt = 0 \]

FIG. 1. Roadmap for formulating elastodynamic equations from Hamilton’s variable principle.

1. Canonical equations from Hamilton’s principle

In this section, we extend Hamilton’s canonical equations for arbitrarily anisotropic and multiphasic porous continuous media by using Hamilton’s variational principle. Also, we focus our attention on boundary treatments as there have been some ambiguous treatments on them. For example, the virtual displacements on the boundary are taken to be arbitrary [14,22,51], while they are also taken to be zero [6,13,52,53], which seems confused to some extent.

The Lagrange density function is usually not time-explicit for a conservative system [4] so that Hamilton’s variational principle can be directly used. In this case, the Lagrange and Hamilton density functions are denoted by \( L_{\rho} \) and \( H_{\rho} \) respectively and the Lagrange’s density function can be defined as [12,13]

\[ L_{\rho} = L_{\rho}(u_i, \dot{u}_i, u_{ij}) = T_{\rho} - V_{\rho}. \] (9)

By using the relationship between the generalized momentum density function \( \pi_i \), generalized velocity \( \dot{q}_i \), and the Hamilton density function through Legendre transformation is identified, namely [12,13]

\[ H_{\rho} = H_{\rho}(u_i, \pi_i, u_{ij}) = \pi_i \dot{q}_i - L_{\rho}, \] (10a)

where the generalized momentum density is defined by [12,13]

\[ \pi_i = \partial L_{\rho} / \partial \dot{q}_i. \] (10b)

For a given volume in a conservative system, Hamilton’s variational principle states that [12,51], one may use \( \int_{0}^{t} \delta L dt = 0 \) as a constraint to identify Lagrange’s equation or elastodynamic equation of motion, where the Lagrange’s function or Lagrangian is obtained by \( L = \iiint L dv \). Therefore, for a given continuous medium, from

\[ \int_{0}^{t} d\delta \iiint L dv = 0, \] (11)

the correspondent Lagrange’s equation and Hamilton’s canonical equations are determined with the assumed Lagrange density function.

In continuum mechanics, the particle vibration displacement, velocity, and strain components are usually chosen as the independent generalized variables [6,25,32,38].

For any given anisotropic and multiphasic porous medium, considering Legendre transformation shown in Eq. (10a), the Lagrange density function is written as

\[ L_{\rho} = u^{(\alpha)}_{i} \pi^{(\alpha)}_{i} - H_{\rho}, \alpha, i = 1, 2, 3. \]

The Lagrange density function in Eq. (11) is replaced by Hamilton’s one, namely

\[ \int_{0}^{t} dt \iiint \delta (u^{(\alpha)}_{i} \pi^{(\alpha)}_{i} - H_{\rho}) dv = 0. \] (12a)

According to the variational principle, we have

\[ \delta (u^{(\alpha)}_{i} \pi^{(\alpha)}_{i} - H_{\rho}) = \dot{u}^{(\alpha)}_{i} \delta \pi^{(\alpha)}_{i} + \pi^{(\alpha)}_{i} \delta \dot{u}^{(\alpha)}_{i} - \partial H_{\rho} / \partial \dot{u}^{(\alpha)}_{i} \delta \pi^{(\alpha)}_{i}. \] (12b)

Therefore, Eq. (12a) can be written as

\[ \int_{0}^{t} dt \iiint \left( [\dot{u}^{(\alpha)}_{i} - \partial H_{\rho} / \partial \dot{u}^{(\alpha)}_{i}] \delta \pi^{(\alpha)}_{i} + \pi^{(\alpha)}_{i} \delta \dot{u}^{(\alpha)}_{i} - \partial H_{\rho} / \partial \dot{u}^{(\alpha)}_{i} \delta \pi^{(\alpha)}_{i} \right) dv = 0. \] (13a)

In the above equation, since

\[ \int_{0}^{t} \iiint \pi^{(\alpha)}_{i} \delta \dot{u}^{(\alpha)}_{i} dv = \iiint \left( \pi^{(\alpha)}_{i} \delta \dot{u}^{(\alpha)}_{i} \right) \bigg|_{0}^{t} dv \]

\[- \int_{0}^{t} \iiint \dot{u}^{(\alpha)}_{i} \delta \pi^{(\alpha)}_{i} dv = - \int_{0}^{t} \iiint \delta \dot{u}^{(\alpha)}_{i} \pi^{(\alpha)}_{i} dv, \] (13b)
where the constraints $\delta u^{(a)}(t_0) = \delta u^{(a)}(t_f) = 0$ have been used in Hamilton’s variational principle. Furthermore, by using the following identity formula,

$$
(\partial H / \partial u^{(a)}_i) \delta u^{(a)}_i = [(\partial H / \partial u^{(a)}_i) \delta u^{(a)}_i]_j - [(\partial H / \partial u^{(a)}_j) \delta u^{(a)}_j],
$$

and substituting it into Eq. (13a), and considering Eq. (13b), we have

$$
\int_\Omega \int_0^1 \left( (\bar{u}^{(a)} - \partial H / \partial \pi^{(a)}_i) \delta \pi^{(a)}_i - \pi^{(a)}_i \delta u^{(a)}_i 
- \partial H / \partial u^{(a)}_i \right) \delta u^{(a)}_i \, dV = 0.
$$

By using Gauss’s theorem [14], the volume integral

$$
\int_\Omega \int_0^1 \left( \partial (\bar{u}^{(a)} - \partial H / \partial \pi^{(a)}_i) / \pi^{(a)}_i \delta \pi^{(a)}_i - \pi^{(a)}_i \delta u^{(a)}_i 
- \partial H / \partial u^{(a)}_i \right) \delta u^{(a)}_i \, dV = 0.
$$

Comparing Eqs. (14a) and (14b), yields

$$
\int_\Omega \int_0^1 \left( (\partial H / \partial u^{(a)}_i) + \partial H / \partial \pi^{(a)}_i \delta \pi^{(a)}_i 
- \partial H / \partial u^{(a)}_i \right) \delta u^{(a)}_i \, dV = 0.
$$

Considering the first equation in Equation group (13f), again, it is easy to show that arbitrary choice of integral volume in Eq. (14c) leads to Eq. (14d) leads to their coefficients of the independent variables $\delta u^{(a)}_i$ and $\delta u^{(a)}_j$ being zero, respectively, i.e.,

$$
\begin{align*}
\int_\Omega \int_0^1 \left( \partial \bar{H}_{i} / \partial \bar{u}^{(a)}_i + \partial \bar{H}_{j} / \partial \bar{u}^{(a)}_j \right) \delta \bar{u}^{(a)}_i \, dV &= 0, \\
\int_\Omega \int_0^1 \left( \partial \bar{H}_{i} / \partial \bar{u}^{(a)}_i + \partial \bar{H}_{j} / \partial \bar{u}^{(a)}_j \right) \delta \bar{u}^{(a)}_j \, dV &= 0.
\end{align*}
$$

The above equations must be held during Legendre transformation, which will be used later for deriving Lagrange’s equation from Hamilton’s canonical equations.

Eqs. (13f) are derived with Hamilton’s variational principle in a continuous conservative multiphasic porous medium system. The conservative system is guaranteed from the surface boundary conditions with $\delta u^{(a)}(0) = 0$ on $\partial \Omega$, which means there is no energy exchange through $\partial \Omega$. However, it is noted that, arbitrary choices of $\delta u^{(a)}_i |_{\partial \Omega} = 0$, make it impossible to mount the stress boundary conditions on $\partial \Omega$ in Eq. (13d) because whatever stress boundary conditions are, Eq. (15) makes all the stress boundary effects vanish. Therefore, this treatment should be avoided if there are stress-loaded boundary conditions. These constraints are naturally equivalent to traction-free boundary conditions as will be seen that $\sigma_{ij}^{(a)} = \partial \bar{H}_{i} / \partial \bar{u}^{(a)}_j$, and $\sigma_{ij}^{(a)}$ is the normal stress components on the surface of $\partial \Omega$ later.

2. Lagrange’s equations from canonical equations

Lagrange’s equation can be directly deduced from Hamilton’s canonical equations in Eqs. (13f) and (14e) for the porous medium model of interest.

Substituting the first and second equations in Equation group (14e) into the second equation in Equation group (13f) which are Hamilton’s canonical equations, and considering the definition of generalized momentum $\pi^{(a)}_i = \partial \bar{H}_{i} / \partial \bar{u}^{(a)}_i$, yields

$$
d(\partial \bar{H}_{i} / \partial \bar{u}^{(a)}_i) / dt - \partial \bar{H}_{i} / \partial \bar{u}^{(a)}_i + (\partial \bar{L}_{i} / \partial \bar{u}^{(a)}_i) = 0.
$$

which is Lagrange’s equation and it will degenerate into Lagrange’s equation for three-phase porous media [31,39]. Eq. (16) is a general Lagrange’s equation for arbitrary anisotropic multiphasic porous media.

3. Elastodynamic equations from Lagrange’s equation

According to the Lagrange density function in the arbitrarily anisotropic and multiphasic porous medium, the generalized momentum density is obtained and shown as

\[ \pi^i_{(m)} = \partial L / \partial u^i_{(m)} = \partial T / \partial \dot{u}^i_{(m)}. \] (17a)

Substituting Eq. (3a) into Eq. (17a), yields

\[ \pi^i_{(m)} = \partial \left[ \rho u_{ijm} \dot{u}^i_{(m)} \right] / \partial \dot{u}^i_{(m)} / 2 \]
\[ = \rho u_{ijm} \dot{u}^i_{(m)} / 2 + \rho u_{ijm} \dot{u}^i_{(m)} / 2. \]

By using a \( \rho u_{ijm} = \rho u_{ijm} \), further simplification of the above equation, yields

\[ \pi^i_{(m)} = \rho u_{ijm} \dot{u}^i_{(m)}, i, j, m, \alpha, \beta = 1, 2, 3. \] (17b)

The above equation is expressed as the component of the equivalent generalized momentum density function in the \( j \)-direction for Component \( m \) of the porous medium.

As mentioned above, the upper and lower indices in the above formula are repeated, and the Einstein summation convention for a two-solid and one-fluid porous medium is followed

\[ \pi^1_{(m)} = \rho_1 \dot{u}^1_{(m)} + \rho_2 \dot{u}^2_{(m)} + \rho_3 \dot{u}^3_{(m)}, \]
\[ \pi^2_{(m)} = \rho_2 \dot{u}^1_{(m)} + \rho_3 \dot{u}^2_{(m)} + \rho_3 \dot{u}^3_{(m)}, \]
\[ \pi^3_{(m)} = \rho_3 \dot{u}^1_{(m)} + \rho_2 \dot{u}^2_{(m)} + \rho_3 \dot{u}^3_{(m)}. \]

From Eq. (17b), the derivatives of the generalized momentum density function with time for an arbitrary composition medium can be written as

\[ \dot{\pi}^i_{(m)} = \rho u_{ijm} \dot{u}^i_{(m)}. \] (18)

Because \( \partial L / \partial u^i_{(m)} = 0, \alpha, i = 1, 2, 3 \), and

\[ \left[ \partial L / \partial \dot{u}^i_{(m)} \right] = -\left[ \partial V / \partial \dot{u}^i_{(m)} \right] = -\sigma_{ij}. \] (19)

Substituting Eqs. (18) and (19) into Eq. (16), yields

\[ \rho u_{ijm} \dot{u}^i_{(m)} - \sigma_{ij} = 0. \] (20a)

The above equation is the elastodynamic equation of motion expressed in terms of displacement and stress in arbitrarily anisotropic and multiphasic porous media. As mentioned before, the \( \alpha \) represents the number of components or compositions that make up the multiphasic porous medium, which is not limited to 3 and can be extended to any number of compositions of the media.

For the two-solid one-fluid (three-component in two-phase) porous medium, unfolding Eq. (20a), we have

\[ \sigma_{ij}^{(1)} - \rho_1 \dot{u}^1_{(1)} - \rho_2 \dot{u}^2_{(2)} - \rho_3 \dot{u}^3_{(3)} = 0, \]
\[ \sigma_{ij}^{(2)} - \rho_2 \dot{u}^1_{(1)} - \rho_2 \dot{u}^2_{(2)} - \rho_3 \dot{u}^3_{(3)} = 0, \]
\[ \sigma_{ij}^{(3)} - \rho_3 \dot{u}^1_{(1)} - \rho_2 \dot{u}^2_{(2)} - \rho_3 \dot{u}^3_{(3)} = 0. \] (20b)

If the acceleration of the above equation is rewritten as the derivative of the velocity with time, then the above equation is the first-order velocity-stress equation of motion under the conservative system, which is consistent with Carcione and Liu’ s results [31,58]. Comparing Eq. (20a) with (20b), Eq. (20a) is concise, which provides a concise mathematical expression for numerically solving the equation of wave motion, such as with the finite difference approach.

B. General mechanical equations from the principle of energy conservation

In all published works in continuum mechanics, it is claimed that various energy conservation formulae are ‘derived’ from Newton’s law, Lagrange’s equation, or Hamilton’s variational principle. Even in quantum mechanics, Ehrenfest’s theorem, related to energy conservation is derived by using Schrödinger’s equation [59]. However, it is believed that the principle of conservation of energy is a law governing all the natural phenomena known to date and there is no known exception to this law [60]. Unfortunately, the principle of energy conservation is not like Hamilton’s variational principle and its variants, such that although being lack of concrete physical meanings and having been questioned in serval aspects [61,62], it is the heart of theoretical physics, and is still dominating 21st century mostly in classical mechanics [63,64], fluid mechanics [65,66], elastodynamics [67], electrodynamics [68] and in quantum mechanics [68–74], etc.

In this section, totally different from the conventional treatments shown in Sec. III, we will directly formulate Hamilton’s canonical equations, Lagrange’s equation, and elastodynamic equation of motion by using the principle of energy conservation as an axiom, with the even less conditions than those used in Hamilton’s variational principle, and then we will compare the obtained equations with those given by using Hamilton’s variational principle. Furthermore, we will formulate the elastodynamic equations of motion in arbitrarily anisotropic and multiphasic porous media, i.e., we will establish the unified dynamical equation formulations of poroelasticity for arbitrary anisotropic and multiphasic porous media.

The roadmap for constructing elastodynamic equations based on the principle of energy conservation is shown in Fig. 2.
During our formulations of the elastodynamics under the umbrella of the principle of energy conservation, two continuous systems are considered, i.e., the conservative and nonconservative systems. The formulation for the conservative system is used to compare with the results given by Hamilton’s variational principle in validation. Nevertheless, it does not make a difference whether the system is conservative and nonconservative for the equation formulations from the perspective of energy conservation.

In mechanics, the principle of energy conservation states that if in a system the total mechanical energy is \( E \), and the work done by external forces applied to the system is \( W \), the rate of total mechanical energy with time is equal to the rate of work with time, and can be quantified as

\[
dE/dt = dW/dt \tag{21}
\]

in a multiparticle mechanical system [52,75], or

\[
d \left( \int \Omega E \, dv \right)/dt = \int \int \Omega f^{(i)} u^{(i)} dv + \int \int \Omega (\sigma^{(i)} / l) \dot{u}^{(i)} ds, \tag{22a}
\]

in a continuous system, with material component number of \( N \), where \( E_p \) is the total mechanical energy density function. If \( N = 1 \), then Eq. (22a) degenerates into the case for one material component, and is consistent with, for example, those given by Achenbach and Miklowitz without body forces [6,8]. The term on the left-hand side of Eq. (22a) is the rate of the total mechanical energy with time in the volume \( \Omega \) of interest, and those on the right-hand side are the work rate with time done by external body force in the volume and stress on the surface \( \partial \Omega \) of the volume \( \Omega \). If there are any other energy or work engaged in the system, they should be considered accordingly to satisfy Eq. (21).

Since the Hamilton density function \( H \) in the system is not an explicit function of time in the conservative system, it can be written as the sum of kinetic energy and potential energy densities. Therefore,

\[
dE/dt = dH/dt. \tag{22b}
\]

In this case, Eq. (22b) in the conservative system is written as

\[
dH/dt = d\left( \int \int \Omega E_p \, dv \right)/dt = 0. \tag{22c}
\]

However, it is noted that if \( \partial H_p / \partial t \neq 0 \), the system is nonconservative, and Eq. (22b) will not be held. In this situation, one still can link the total energy density with the Hamilton density function through the Lagrange density function [44]. Also, we will discuss this situation in Sec. III for nonconservative system.

### 1. Hamilton’s canonical equations from energy conservation for a conservative system

We start from the law of conservation of energy and deduce Hamilton’s canonical equations in a continuous conservative system.

Using Eqs. (3a) and (4b), the Hamilton density function can be expressed as

\[
H_p = \rho \ddot{u}^{(a)} + \frac{1}{2!} [\dot{\nabla} V_p / (\dot{\nabla} q^{(a)} / \dot{\nabla} q^{(b)})] q^{(a)} q^{(b)}/2. \tag{23}
\]

According to Reynolds’ transport theorem [31], neglecting higher-order variable contributions, the first term on the left-hand side of Eq. (22a) can be written as

\[
dH/dt = \int \Omega [(\partial H_p / \partial u^{(a)}) \dot{u}^{(a)} + (\partial H_p / \partial \dot{u}^{(a)}) \ddot{u}^{(a)} + (\partial H_p / \partial \dot{u}^{(a)}) \dddot{u}^{(a)} + \partial H_p / \partial t] \, dv. \tag{24a}
\]

On the other hand, using the Legendre transformation shown in Eq. (10a), the Hamilton density function is represented by the Lagrange density function. Therefore,

\[
dH/dt = d \int \Omega \left( \dot{u}^{(a)} \pi^{(a)} - L_p \right) \, dv/dt = \int \Omega [(\partial L_p / \partial u^{(a)}) \dot{u}^{(a)} -(\partial L_p / \partial \dot{u}^{(a)}) \ddot{u}^{(a)} - (\partial L_p / \partial \ddot{u}^{(a)}) \dddot{u}^{(a)} - \partial L_p / \partial t] \, dv. \tag{24b}
\]

According to \( \pi^{(a)} = L_p , \) the definition of generalized momentum density shown in Eq. (10b), Eq. (24b) can be simplified as

\[
dH/dt = \int \Omega [(\partial H_p / \partial \ddot{u}^{(a)}) \dddot{u}^{(a)} - (\partial L_p / \partial u^{(a)}) \dot{u}^{(a)} - (\partial L_p / \partial \dot{u}^{(a)}) \ddot{u}^{(a)} - \partial L_p / \partial \dot{u}^{(a)} \, dv. \tag{24c}
\]

Subtracting Eq. (24a) by Eq. (24c) and rearranging, they yields

\[
\int \Omega [(\partial H_p / \partial \ddot{u}^{(a)}) \dddot{u}^{(a)} + (\partial H_p / \partial \dot{u}^{(a)}) \dot{u}^{(a)} + (\partial H_p / \partial u^{(a)}) \ddot{u}^{(a)} + (\partial L_p / \partial \dot{u}^{(a)}) \dot{u}^{(a)} + (\partial L_p / \partial \ddot{u}^{(a)}) \ddot{u}^{(a)} + (\partial L_p / \partial u^{(a)}) \dddot{u}^{(a)} + \partial L_p / \partial t] \, dv = 0. \tag{24d}
\]

Since \( \Omega \) is arbitrarily selected, the necessary condition for the above formula to be held is [14,54]

\[
(\partial H_p / \partial \ddot{u}^{(a)} + \partial L_p / \partial \ddot{u}^{(a)}) \dddot{u}^{(a)} + (\partial H_p / \partial \dot{u}^{(a)} - \ddot{u}^{(a)}) \dot{u}^{(a)} + (\partial L_p / \partial \dot{u}^{(a)}) \dot{u}^{(a)} + (\partial H_p / \partial \dot{u}^{(a)}) \dddot{u}^{(a)} + (\partial L_p / \partial \ddot{u}^{(a)}) \dddot{u}^{(a)} + (\partial L_p / \partial \dot{u}^{(a)}) \dot{u}^{(a)} + \partial L_p / \partial t] \, dv = 0. \tag{25a}
\]

Multiplying both sides of the above equation by \( dt \), we have
Since we assume that the variables \( u^{(a)}_i, \pi^{(a)}_i, t \) and \( u^{(a)}_{ij}, i, j, \alpha = 1,2,3 \), are generalized coordinate components independent of each other, and also \( du^{(a)}_i, \alpha \), \( d\pi^{(a)}_i, \alpha \), \( dt \) and \( du^{(a)}_{ij} \) are arbitrarily independent variables, followed by Arnold treatments \cite{57}, the necessary condition for Eq. (25b) to be held is

\[
\begin{align*}
&\dot{u}^{(a)}_i = \partial H_p / \partial \pi^{(a)}_i, \\
&\partial L_p / \partial u^{(a)}_i = -\partial H_p / \partial u^{(a)}_i, \\
&\partial L_p / \partial \pi^{(a)}_{ij} = -\partial H_p / \partial \pi^{(a)}_{ij}, \\
&\partial L_p / \partial t = -\partial H_p / \partial t.
\end{align*}
\] (26a)

For a conservative system, the total mechanical energy density is the Hamilton’s density function, and \( \partial H_p / \partial t = -\partial L_p / \partial t = 0 \). (26b)

As is assumed, the principle of energy conservation, like Hamilton’s variational principle, may be seen as variable constraints, so that it could be used to identify independent variable couplings, which is the kernel of our proposed methodology.

Substitute Eq. (24a) into Eq. (22c), or the energy conservation equation, we obtain that

\[
\int_\Omega \left[ (\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i + (\partial H_p / \partial \pi^{(a)}_i) \dot{\pi}^{(a)}_i \right] \, dv = 0.
\] (27a)

By using the first equation in Equation group (26a), we have

\[
\int_\Omega \left[ (\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i + (\partial H_p / \partial \pi^{(a)}_i) \dot{\pi}^{(a)}_i \right] \, dv = 0.
\] (27b)

Using the identity equation

\[
(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i = [(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i],
\] (27c)

and substituting it into Eq. (27b), we have

\[
\int_\Omega [(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i + (\partial H_p / \partial \pi^{(a)}_i) \dot{\pi}^{(a)}_i] \, dv = 0.
\] (27d)

In the above equation, by using Gauss’s theorem to convert volume integral into enclosed surface integral, and rearranging it, we have

\[
\int_\Omega [(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i + (\partial H_p / \partial \pi^{(a)}_i) \dot{\pi}^{(a)}_i] \, dv = -\int_{\partial \Omega} (\partial H_p / \partial \pi^{(a)}_i) \dot{u}^{(a)}_i \, ds.
\] (27e)

Now, we prove that

\[
(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i = 0.
\] (28a)

From Eq. (26a), we know that

\[
\partial H_p / \partial u^{(a)}_i = -\partial L_p / \partial u^{(a)}_i = \partial V_p / \partial u^{(a)}_i.
\]

Furthermore, by using the relationship between stress and strain energy density function, it is easy to show that

\[
(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i = \partial V_p / \partial u^{(a)}_i.
\] (28b)

By using Eq. (28a), Eq. (27e) becomes

\[
\int_{\partial \Omega} (\partial H_p / \partial \pi^{(a)}_i) \dot{u}^{(a)}_i \, ds = -\int_{\partial \Omega} \sigma^{(a)}_i \dot{u}^{(a)}_i \, ds.
\]

As was discussed in Sec. III, the same surface boundary conditions are employed, i.e., the traction-free conditions are considered to guarantee the system to be conservative, the above equation is simplified as

\[
\int_{\partial \Omega} (\partial H_p / \partial \pi^{(a)}_i) \dot{u}^{(a)}_i \, ds = 0.
\]

Arbitrary choice of integral volume \( \Omega \) in the above equation implies that its integrand must vanish \cite{14,54}, namely

\[
(\partial H_p / \partial \pi^{(a)}_i) \dot{u}^{(a)}_i = 0.
\] (29a)

Or

\[
(\partial H_p / \partial \pi^{(a)}_i) \dot{u}^{(a)}_i = 0.
\] (29b)

Following the previous treatments \cite{57}, from Eq. (29a) we know that the coefficients of the independent variable \( du^{(a)}_i \) must vanish, i.e.,

\[
\pi^{(a)}_i = -\partial H_p / \partial u^{(a)}_i + (\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i = 0.
\] (29c)

Writing the first formula in Equation group (26a) and Eq. (29b) together, we have

\[
\begin{align*}
\dot{u}^{(a)}_i &= \partial H_p / \partial \pi^{(a)}_i, \\
\pi^{(a)}_i &= -\partial H_p / \partial u^{(a)}_i + (\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i,
\end{align*}
\] (30)

where \( \alpha, i, j = 1,2,3 \), and \( \alpha \) can be extended to any number of medium compositions as expected.

Equation group (30) is Hamilton’s canonical equations we formulate Lagrange’s equation of anisotropic for an arbitrary number of material compositions in the continuous conservative system, derived directly from the principle of conservation of energy. It is Equation group (13f) derived by using Hamilton’s variational principle, both of which are validated with each other.

2. Lagrange’s equation from energy conservation for a conservative system

Next, we formulate Lagrange’s equation of anisotropic and multiphasic porous media from the perspective of energy conservation in a conservative system.

Starting from the principle of conservation of energy, considering Eq. (26b), and substituting Eq. (24c) into Eq. (22c), we have

\[
\int_{\Omega} \int_{\Omega} [(\partial H_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i - (\partial L_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i] \, dv = 0.
\] (31a)

With the help of the following identity formula

\[
[(-\partial L_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i] - [(-\partial L_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i] = (\partial L_p / \partial u^{(a)}_i) \dot{u}^{(a)}_i, \]

(31b)
Eq. (31a) is rewritten as
\[ \iiint_{V} \left( \left( \frac{\partial g}{\partial \xi} \right) \frac{\partial g}{\partial u} \right) + \left( \frac{\partial g}{\partial \nu} \right) \frac{\partial g}{\partial u} \right) du = 0. \] 

By using Gauss’s theorem, converting the volume integral of the third term on the left-hand side of the above equation into surface integral, yields
\[ \iint_{S} \left[ \left( \frac{\partial g}{\partial \xi} \right) \frac{\partial g}{\partial u} \right] dS = 0. \] 

Considering the third equation in Equation group (26a) and using Eq. (28a), we have
\[ \sigma_{ij}^{(a)} = -\partial \sigma_{ij}^{(a)} / \partial u. \]

Substituting the above equation into Eq. (31c), and using the traction free conditions as is done before for guaranteeing the system to be conservative, we have
\[ \iint_{V} \left[ \left( \frac{\partial g}{\partial \xi} \right) \frac{\partial g}{\partial u} \right] dV = 0. \] 

Following the previous procedure, the integrand in Eq. (31d) must vanish because of the arbitrary choice of integral volume \( V \). Therefore, \( \sigma_{ij}^{(a)} = -\partial \sigma_{ij}^{(a)} / \partial u \) or
\[ \sigma_{ij}^{(a)} = -\partial \sigma_{ij}^{(a)} / \partial u + (\partial g / \partial u_{ij})_{j} du_{i}^{(a)} = 0. \] 

Also, following the previous treatments [57], the fact that the above equation is held with the independent variables \( du_{i}^{(a)} \) implies that their coefficients must vanish, i.e.,
\[ \sigma_{ij}^{(a)} = -\partial \sigma_{ij}^{(a)} / \partial u + (\partial g / \partial u_{ij})_{j} du_{i}^{(a)} = 0. \] 

Finally, substituting the definition of generalized momentum density shown in (17a) into the above equation, yields
\[ d(\partial \sigma_{ij}^{(a)} / \partial u) / dt - \partial \sigma_{ij}^{(a)} / \partial u + (\partial g / \partial u_{ij})_{j} = 0. \] 

Comparing the above equation with Eq. (16), we know that Eq. (32), directly derived from the principle of energy conservation is Lagrange’s equation derived from Hamilton’s variational principle for a multiphase porous continuous conservative medium with an arbitrary number of material compositions.

3. Elastodynamic equation of motion from energy conservation for a conservative system

In this section, we will formulate the dynamical equation of motion in a multiphase porous medium directly from the perspective of energy conservation for a conservative system.

Substituting Eqs. (3a) and (6) for kinetic energy and potential energy density functions into the energy conservation equation in Eq. (22c) by neglecting body forces, yields
\[ d\iint_{V} \left[ \rho_{ij}^{(a)} \bar{u}_{ij}^{(a)} / 2 + C_{ijkl}^{(a)} \bar{e}_{ij}^{(a)} \bar{e}_{kl}^{(a)} / 2 \right] dv / dt = 0. \] 

As is assumed, because the system is conservative there are only kinetic energy and potential energy in the system. To guarantee Eq. (33a), we need to consider the conservative conditions, i.e., there is no energy transport across the surface of \( \partial \Omega \). As is done in Sec. III, we consider the enclosed surface a traction-free surface, i.e., \( T_{i} = 0 \) on \( \partial \Omega \). Otherwise, the energy conservation equation should be written as
\[ d\iint_{V} \left[ \rho_{ij}^{(a)} \bar{u}_{ij}^{(a)} / 2 + C_{ijkl}^{(a)} \bar{e}_{ij}^{(a)} \bar{e}_{kl}^{(a)} / 2 \right] dv / dt = \iint_{\partial V} (\sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)} + \sigma_{ij}^{(a)} \bar{u}_{ij}^{(a)}) dv. \] 

The term on the right-hand side of Eq. (33b) should be considered as the contributions to the potential energy since the stresses are related to strains with Hooke’s law. It would be seen that this term will be cancelled with the one on the left-hand side. The advantage of formulating dynamical equation from energy conservation does not make a big difference for the conservative or nonconservative system. As is done before, one may suppose the term on the right-hand side of Eq. (33b) vanishes so that Eq. (33b) becomes the energy conservation equation in a conservative system. In this case, the corresponted term on the left-hand side is naturally zero because of the traction free surface boundary conditions. In both cases, it turns out that they are equivalent with each other.

Starting from Eq. (33b), following the previous treatment, using Reynolds’s transport theorem [31], Eq. (33b) becomes
\[ \iint_{V} \left( \rho_{ij}^{(a)} \bar{u}_{ij}^{(a)} + \rho_{ij}^{(b)} \bar{u}_{ij}^{(b)} + \rho_{ij}^{(c)} \bar{u}_{ij}^{(c)} \right) dv = 0. \] 

By using generalized Hook’s law, \( \sigma_{ij}^{(b)} = C_{ijkl}^{(b)} \bar{e}_{ij}^{(b)} \) shown in Eq. (7), or \( \sigma_{ij}^{(b)} = C_{ijkl}^{(b)} \bar{u}_{ij}^{(b)} \), the above equation is written as
\[ \iint_{V} \left( \rho_{ij}^{(a)} \bar{u}_{ij}^{(a)} + \sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)} \right) dv = 0. \] 

By using the identity formula
\[ \sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)} = (\sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)})_{j} - \sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)}, \]
Eq. (33c) can be written as
\[ \iint_{V} \left( \rho_{ij}^{(a)} \bar{u}_{ij}^{(a)} + \sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)} \right) dt = \iint_{\partial V} (\sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)})_{j} dv = 0. \] 

According to Gauss’s theorem, the volume integral of the second term on the left-hand side of the above equation is converted into the surface integral. Therefore, Eq. (33c) becomes
\[ \iint_{V} \left( \rho_{ij}^{(a)} \bar{u}_{ij}^{(a)} - \sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)} \right) dv + \iint_{\partial V} (\sigma_{ij}^{(b)} \bar{u}_{ij}^{(b)})_{j} dv = 0. \]
For a conservative system, as is assumed before, the traction-free boundary conditions are considered, i.e.,
\[
\iint_{\Omega}(\rho_{ij} \ddot{u}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \ddot{u}^{(\beta)} \, dv = 0.
\]  
(33g)

Following the previous procedures [14,54], to guarantee Eq. (33f) to be held for arbitrary choice of integral volume \( \Omega \), its integrand must vanish, i.e.,
\[
(\rho_{ij} \ddot{u}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \ddot{u}^{(\beta)} = 0.
\]
(34a)

Also, following the previous treatments [57], from Eq. (34a), we have
\[
(\rho_{ij} \ddot{u}^{(\alpha)} - \sigma_{ij}^{(\beta)}) du^{(\beta)} = 0.
\]
(34b)

Hence,
\[
\rho_{ij} \ddot{u}^{(\alpha)} - \sigma_{ij}^{(\beta)} = 0.
\]
(34c)

Exchanging \( \alpha \) and \( \beta \) in the above formula, and considering the symmetry of \( \rho_{ij} = \rho_{ji} \), we have
\[
\rho_{ij} \ddot{u}^{(\alpha)} - \sigma_{ij}^{(\beta)} = 0, \alpha, \beta = 1, 2, 3, \ldots, N; i, j = 1, 2, 3, \]
(34d)

where \( N \) represents the total number of material compositions in the porous medium of interest.

By examining Eq. (34d), we find that it is exactly the elastodynamic equation of motion in multiphase porous media given by Lagrange’s equation in Eq. (20a).

**C. General equations from the principle of energy conservation for a nonconservative system**

If there are non-conservative physical and dissipative forces in the medium, we can also establish the equation of motion in the same way as before. For non-conservative systems, Hamilton’s variational principle cannot be used directly, while the proposed methodology based on the principle of energy conservation is not subject to this restriction. That is, the principle of energy conservation can also be used for formulating general mechanical equations even if \( \partial H_p / \partial t \neq 0 \) and \( \partial L_p / \partial t \neq 0 \). If there exists this situation, usually \( \partial \lambda_i / \partial t \neq 0 \), generalized coordinate transformation may be introduced to get rid of constraints of displacements as is done in our previous work [44]. As is known [56],
\[
du_i(x, t) / dt = (\partial \lambda_i / \partial t) \partial x_i / \partial t + \partial \lambda_i / \partial t.
\]

In our discussions, as assumed, only the linear case is considered. Therefore, high-order contributions are neglected such that
\[
(\partial \lambda_i / \partial t) \partial x_i / \partial t = 0.
\]

In our discussions, since space coordinate variables and time are independently for solid media, we consider \( \partial \lambda_i / \partial t = 0 \) [8] so that
\[
du_i(x, t) / dt = \partial \lambda_i / \partial t,
\]

which is assumed in Sec. II. Therefore, both \( \partial \lambda_i / \partial t \) and \( \partial L_p / \partial t \) vanish.

In this case, the energy conservation equation in Eq. (21) can be written as
\[
\begin{align*}
 d(\iint_{\Omega} E \, dv) / dt =& \iint_{\Omega} (\sigma_{ij}^{(\alpha)}) \dddot{u}^{(\alpha)} \, ds \\
+ \iint_{\Omega} f_i^{(\alpha)} \dddot{u}^{(\alpha)} \, dv + \iint_{\Omega} F_i^{(\alpha)} \dddot{u}^{(\alpha)} \, dv,
\end{align*}
\]
(35a)

where \( f_i^{(\alpha)} \) and \( F_i^{(\alpha)} \) are the body and dissipation forces in Composition \( \alpha \) in the porous media, respectively. The first, second, and third terms on the right-hand side of the above equation are the work done by the surface forces, body forces, and dissipation forces per unit time, respectively.

On the one hand, for arbitrary anisotropic multiphase porous media, the dissipative energy density function can be written as [48]
\[
\phi = b_{\alpha \beta} \dddot{u}_i^{(\alpha \beta)} \dddot{u}_i^{(\alpha \beta)} / 2, i, j, \alpha, \beta = 1, 2, 3.
\]
(35b)

The friction coefficient represents the friction force generated by the relative motion when the velocity vectors of the two media are inconsistent, and it is assumed that the friction coefficients of the relative velocities in the three directions are consistent. If they are inconsistent, the friction coefficient is anisotropic, so the friction coefficients should be treated differently in different directions. Also, the friction coefficients are related to the direction of the relative velocity, so it should be written as \( b_{\alpha \beta}^{(\alpha \beta)} \). In our work, only the isotropic friction coefficients are concerned and \( b_{\alpha \beta}^{(\alpha \beta)} \) is denoted simply with \( b^{(\alpha \beta)} \). From Eq. (35b), it is seen that the dissipation forces can be written as [48]
\[
F_i^{(\alpha)} = -b^{(\alpha \beta)} \dddot{u}_i^{(\alpha \beta)}, \alpha, \beta, \gamma = 1, 2, 3.
\]
(35c)

From the energy conservation equation shown in Eq. (35a), we have
\[
\begin{align*}
\iint_{\Omega} (dT_p / dt + dV_p / dt) \, dv =& \iint_{\Omega} (\sigma_{ij}^{(\alpha)}) \dddot{u}^{(\alpha)} \, ds \\
+ \iint_{\Omega} f_i^{(\alpha)} \dddot{u}^{(\alpha)} \, dv + \iint_{\Omega} F_i^{(\alpha)} \dddot{u}^{(\alpha)} \, dv.
\end{align*}
\]
(35d)

The above equation can be used to derive Lagrange’s equation and the elastodynamic equation of motion in a continuous nonconservative system.

On the one hand, the rate of kinetic energy density with time is written as
\[
\frac{dT_p}{dt} = (\partial T_p / \partial t) \ddot{u}^{(\alpha)} + (\partial T_p / \partial \ddot{u}^{(\alpha)}) \dddot{u}^{(\alpha)} + (\partial T_p / \partial \dot{u}^{(\alpha)}) \dddot{u}^{(\alpha)}
\]
(35e)

By using the identity formula
\[
\frac{\partial (T_p / \partial u_i^{(a)})u_i^{(a)}}{dt} = d\left(\frac{\partial (T_p / \partial u_i^{(a)})u_i^{(a)}}{dt}\right) + \frac{d}{dt}\left(\frac{\partial (T_p / \partial u_i^{(a)})u_i^{(a)}}{dt}\right)
\]

Eq. (35e) can be written as
\[
dT_p / dt = \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \frac{d}{dt}\left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \frac{d}{dt}\left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

According to the definition of kinetic energy, the above equation can be simplified as
\[
dT_p / dt = \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \frac{d}{dt}\left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

Therefore,
\[
dT_p / dt = \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

Since the elastic potential energy density is not a function of particle velocity and time explicitly, the above equation can be written as
\[
dT_p / dt = \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

On the other hand, the rate of elastic potential energy density with time can be written as
\[
dV_p / dt = \left(\frac{\partial V_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \left(\frac{\partial V_p}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

From Eqs. (35h) and (35i), we have
\[
d(T_p + V_p) / dt = \left(\frac{\partial (T_p + V_p)}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial (T_p + V_p)}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

By using Eq. (10), for the multiphase porous media the Lagrange’s density function can be written as
\[
L_p(u_i^{(a)}, u_j^{(a)}, t) = \pi_i^{(a)} u_i^{(a)} - H(u_i^{(a)}, \pi_i^{(a)}, t), i, j = 1, 2, 3.
\]

Following Arnold’s treatments [57], the time derivative of Hamilton’s density function can be written as
\[
dL_p / dt = -\left(\frac{\partial L_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \left(\frac{\partial L_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \left(\frac{\partial L_p}{\partial \pi_i^{(a)}}\right)\pi_i^{(a)}.
\]

On the other hand,
\[
dL_p / dt = \pi_i^{(a)} \pi_i^{(a)} - \left(\frac{\partial L_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial L_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial L_p}{\partial \pi_i^{(a)}}\right)\pi_i^{(a)} dt.
\]

Or
\[
dL_p / dt = \pi_i^{(a)} \pi_i^{(a)} - \left(\frac{\partial L_p}{\partial u_i^{(a)}}\right)u_i^{(a)} - \left(\frac{\partial L_p}{\partial \pi_i^{(a)}}\right)\pi_i^{(a)} dt.
\]

Comparing Eq. (36c) with Eq. (36d), yields
\[
\dot{u}_i^{(a)} = \frac{\partial L_p}{\partial \pi_i^{(a)}} - \left(\frac{\partial L_p}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

Substituting the second and third equations in Equation group (36e) into Eq. (36a), yields
\[
d(T_p + V_p) / dt = \dot{u}_i^{(a)} + \left(\frac{\partial (T_p + V_p)}{\partial u_i^{(a)}}\right)u_i^{(a)} + \left(\frac{\partial (T_p + V_p)}{\partial u_i^{(a)}}\right)u_i^{(a)}.
\]

Therefore, the energy conservation equation in Eq. (35a) can be rewritten as
\[
\int_{\Omega} \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} + \left(\frac{\partial (T_p + V_p)}{\partial u_i^{(a)}}\right)u_i^{(a)} dv = \int_{\Omega} \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} dv + \int_{\Omega} \left(\frac{\partial (T_p + V_p)}{\partial u_i^{(a)}}\right)u_i^{(a)} dv.
\]

Now, substituting identity formula in Eq. (27c) into the above equation and converting part of the volume integral into surface one by using the Gauss’s theorem, i.e.,
\[
\int_{\Omega} \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} dv = \int_{\partial \Omega} \left(\frac{\partial u_i^{(a)}}{\partial n}\right) dv,
\]

Eq. (37b) can be simplified as
\[
\int_{\Omega} \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} dv = \int_{\Omega} \left(\frac{\partial T_p}{\partial u_i^{(a)}}\right)u_i^{(a)} dv + \int_{\partial \Omega} \left(\frac{\partial u_i^{(a)}}{\partial n}\right) dv,
\]

Using Eq. (28a), the above equation can be further simplified as
\[
\int_{\partial \Omega} \left(\frac{\partial u_i^{(a)}}{\partial n}\right) dv = 0.
\]

Following the previous procedures [14,54], it is easy to show that one Hamilton’s canonical equation in a non-conservative multiphase porous medium system is formulated as
\[
\pi_i = -\frac{\partial H}{\partial u_i^{(a)}} + \left(\frac{\partial H}{\partial \pi_i^{(a)}}\right)\pi_i^{(a)}.
\]

With Equation group (36e) and Eq. (37e), Hamilton’s canonical equations are formulated, and can be rewritten as
\[
\dot{u}_i^{(a)} = \left(\frac{\partial L_p}{\partial \pi_i^{(a)}}\right),
\]

From Eq. (37e), and the definition of generalized momentum density, the Lagrange’s equation is obtained in the form of
\[
d\left(\frac{\partial L_p}{\partial u_i^{(a)}}\right) / dt - \left(\frac{\partial L_p}{\partial \pi_i^{(a)}}\right) + \left(\frac{\partial H}{\partial \pi_i^{(a)}}\right)\pi_i^{(a)} = 0.
\]

Substituting the definition formula of generalized momentum density with Lagrange density function and Eq. (31b) into the above equation, considering \(\partial L_p / \partial u_i^{(a)} = 0\), it is reformulated as
\[
\rho_{dd} \dot{u}_i^{(a)} - \sigma_{ij}^{(a)} f_i^{(a)} + b_{ij}^{(a)} \dot{\pi}_i^{(a)} = 0.
\]

which is the general dynamical equation of motion in continuous nonconservative and multiphase porous media. If the body forces and dissipative forces are neglected, Equation will degenerate into Eq. (34c) for the conservative system.
\[
\begin{align*}
\rho_i \dddot{u}_i^{(1)} + \rho_i \dddot{u}_i^{(2)} + \rho_i \dddot{u}_i^{(3)} - \sigma_{i,j}^{(1)} - f_i^{(1)} = 0, \\
+ b_3(u_i^{(2)} - u_i^{(1)}) + b_3(u_i^{(3)} - u_i^{(2)}) = 0, \\
\rho_i \dddot{u}_i^{(1)} + \rho_i \dddot{u}_i^{(2)} + \rho_i \dddot{u}_i^{(3)} - \sigma_{i,j}^{(2)} - f_i^{(2)} = 0, \\
+ b_3(u_i^{(2)} - u_i^{(1)}) + b_3(u_i^{(3)} - u_i^{(2)}) = 0, \\
\rho_i \dddot{u}_i^{(1)} + \rho_i \dddot{u}_i^{(2)} + \rho_i \dddot{u}_i^{(3)} - \sigma_{i,j}^{(3)} - f_i^{(3)} = 0, \\
+ b_3(u_i^{(2)} - u_i^{(1)}) + b_3(u_i^{(3)} - u_i^{(2)}) = 0.
\end{align*}
\]

(39a) \hspace{1cm} (39b) \hspace{1cm} (39c)

In the above equations, since \( b_{33}^{(a)} = b_{33}^{(a)} \), which is an isotropic case for dissipation in the porous medium, its value is only identified by Medium compositions \( n \) and \( a \). Therefore, its superscript \( a \) can be omitted. If the body forces are not considered, the above equation degenerates to the dynamical equations of motion consistent with those in the work by Leclaire and Liu et al [31, 58].

Eq. (37g) is Lagrange’s equation formulated from the energy conservation equation for the non-conservative multiphase porous medium system; while Eqs. (39a), (39b), and (39c) are the corresponding general elastodynamic equations of motion or Cauchy’s equations of motion in a three-composition porous medium.

If the external body forces and dissipation forces are not considered, Eq. (37g) will degenerate into Lagrange’s equation, seen in Eq. (32), which is the same as the one given by Hamilton’s variational principles in Eq. (16) in a conservative system.

Until now, Hamilton’s canonical equations in a continuous conservative multiphase porous medium system, Lagrange’s equations, and elastodynamic equations of motion in a continuous nonconservative multiphasic porous medium system have been formulated by only using the principle of energy conservation, while the derived Lagrange’s equation from energy conservation equation implies Hamilton’s variational principle. Hence, elastodynamics is formulated directly from the axiom of energy conservation.

Hamilton’s variational principle and the principle of energy conservation in a nonconservative continuous system are compared in the followings:

**No.1.** Hamilton’s variational principle is written as [6]

\[
\int_{t_1}^{t_2} dt \iint_{\Omega} \delta(K_\rho - V_\rho) + \delta W_i dV = 0,
\]

with the time terminal constrains of \( \delta \dot{u}_i = 0 \) at \( t_1 \) and \( t_2 \) respectively, where the work done for the system by external forces is denoted as \( W_i \).

**NO.2.** The principle of energy conservation is written as

\[
dt \iint_{\Omega} (K_\rho - V_\rho - W_e) dV / dt = 0.
\]

It is apparent that the proposed methodology is much simpler with clear physical meanings and without time terminal constraints. All the other assumptions are the same.

**IV. BOUNDARY CONDITIONS AT AN INTERFACE FROM THE PRINCIPLE OF ENERGY CONSERVATION**

The kinematic and dynamical boundary conditions of multiphase porous media are important for correctly describing the reflection and transmission of acoustic waves on such boundaries. Previously, the boundary conditions of elastic media are determined by the physical states of stress and displacement. For a multiphase porous model, it is much more complicated to directly use the concepts of force and displacements to determine the conditions at an interface because the compositions and contents of the two media divided by the interface may be different. The interface conditions of Deresiewicz and Skalak [76] have been now widely used in wave propagation studies. However, some issues related to the interface conditions in porous media are still under discussion [77], and more comprehensive reviews on the boundary conditions for porous media can be seen in Carcione’s work [48].

As is done before for dynamical equation formulations, the principle of energy conservation can also be used to determine the boundary conditions at an interface [76], and it can be extended to an arbitrary multiphase porous medium interface.

For any given volume domain \( \Omega \) in a multiphase porous medium, regardless of the physical force, the rate of the total energy with time in the volume can be written as

\[
I = d \iint_{\Omega} \{ T_\rho + V_\rho + f_\rho \} dV / dt = (1/2) \iint_{\Omega} ( \rho_0 |\dot{u}^{(a)}_{i} - \dot{u}^{(b)}_{j}|^2 + |\dot{e}^{(a)}_{ij} - \dot{e}^{(b)}_{ij}|^2 ) dV,
\]

where the dissipation energy is concerned. According to the principle of energy conservation, the rate of the energy with time in the volume domain is equal to the rate of work done by the surface force on the area of the volume enclosed. Its differential form in the two- and three-component porous medium can be obtained with Eq. (36b) without body forces, i.e.,

\[
\rho_0 |\dot{u}^{(a)}_{i} - \dot{u}^{(b)}_{j}|^2 + |\dot{e}^{(a)}_{ij} - \dot{e}^{(b)}_{ij}|^2 ) dV = 0,
\]

(40a)

Now, we consider two volumes \( \Omega_1 \) and \( \Omega_2 \) as shown in Fig. 3. \( \Omega_1 \) and \( \Omega_2 \) share the same surface \( S_2 \), while \( \Omega_1 \) is enclosed by \( S_1 \), like a solid ball with its surface; while \( \Omega_2 \) is the volume between the outer closed surface \( S_1 \) and the inner closed surface \( S_2 \).
Consider the situation on the fluid surface $S_2$, since we consider the boundary problem on the surface $S_1$, it can be obtained from the above equation that
\[
\sigma^{(1)}_{ij} u^{(1)}_{ij} - \sigma^{(2)}_{ij} u^{(2)}_{ij} = 0.
\]
Since $S_2$ is arbitrarily selected in $\Omega_2$, it can be obtained from the above equation that $\sigma^{(1)}_{ij} u^{(1)}_{ij} - \sigma^{(2)}_{ij} u^{(2)}_{ij} = 0$. (41b)

Therefore, the general stress and velocity continuity boundary conditions for two different multiphase porous media are determined by the energy conservation equation. The boundary conditions for special cases are discussed in detail in the following sections.

A. Boundary conditions for solid media

Consider that the two solid media have no relative motion on the boundary, so the displacement is continuous. In this case, Eq. (39b) can be written as
\[
\sigma^{(1)}_{ij} u^{(1)}_{ij} - \sigma^{(2)}_{ij} u^{(2)}_{ij} = 0,
\]
and by using the condition $u^{(1,1)}_i = u^{(2,2)}_i = 0$ on the boundary, noting that $F^{(0)}_i = l_i$, we have
\[
(\sigma^{(0)}_{ij} - \sigma^{(2)}_{ij}) l_i = 0 \quad (42a)
\]
Following the previous procedures, the above equation is written as
\[
(\sigma^{(1)}_{ij} - \sigma^{(2)}_{ij}) l_i = 0 \quad (42b)
\]
Hence, $\sigma^{(1,1)}_{ij} = \sigma^{(2,2)}_{ij}$. Let $T^{(1)}_i = \sigma^{(1)}_{ij} l_j$, $T^{(2)}_i = \sigma^{(2)}_{ij} l_j$.

Eventually, the boundary conditions for stress and displacement at the interface are identified as
\[
\begin{align*}
T^{(1)}_i &= T^{(2)}_i, \\
u^{(1)}_i &= u^{(2)}_i, 
\end{align*}
\]
which are the typical stress and displacement boundary conditions [8].

B. Boundary conditions for fluid and solid media

Now, we consider the situation on the fluid-solid boundary. In this case, Eq. (42b) will be satisfied if the following equations are held.
\[
\begin{align*}
T^{(1)}_i &= T^{(2)}_i, \\
u^{(1)}_i &= u^{(2)}_i, 
\end{align*}
\]
where $T^{(\gamma)}_i = \sigma^{(\gamma)}_{ij} l_j$, and $u^{(\gamma)}_i = u^{(\gamma)}_{ij} l_j, \gamma = 1, 2$, are the stress and displacement components in the outer normal direction $l_i$ on $S_1$, and $\beta$ is omitted because there is only one medium composition on each side of $S_1$.

Now let's discuss the conditions for satisfying Eq. (42d), which are the same as those in Eq. (42c). Since it is a fluid-solid interface, the displacement of the fluid and the solid is not necessarily continuous as it can no longer be assumed that the fluid-solid moves together in all directions on the interface. Therefore, the solid-solid boundary condition determined according to Eq. (42c) or (42d) is no longer applicable.

As we know, on the solid-solid boundary since the stress and displacement components are respectively continuous, the mass of the medium is naturally conserved, while on the fluid-solid interface, the displacement components are not necessarily equal one-to-one, otherwise, the mass is not conserved. Therefore, mass conservation can be used to give the constraint relations between displacements on the boundary [76].
When it comes to fluids (especially gases), the density of which is time dependent. However, mass conservation requires that the density within the volume element changes over time and that any unbalanced mass flows into and out of the balance of the accumulation of mass inside the volume element. The mass conservation can be expressed as [55,56]

\[
\int_{\Omega} (\partial \rho / \partial t) \, dv = -\int_{\partial \Omega} \rho \mathbf{u} \cdot d\mathbf{s}. 
\]

Using Gauss’s theorem, the above equation can be simplified to

\[
\int_{\Omega} (\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u})) \, dv = 0.
\]

We may obtain the differential equation of mass conservation in a continuum from the arbitrary choice of the integral volume domain in the above formula [32]

\[
\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u}) = 0.
\]

From the above equations, we conclude that the density of both fluid and solid is independent of time, so \(\partial \rho / \partial t = 0\). Therefore, the mass continuity boundary conditions can be written as

\[
\nabla \cdot (\rho \mathbf{u}) = 0.
\]

Since the particle vibration of the medium has the same velocity on both sides of the interface in the tangential direction, it does not need to be considered. On the other hand, since the densities of the two media are constant, it can be seen from the above formula that in the direction normal to the interface, the particle velocities of the two media have the difference between the projections of the vibration velocities on the interface and normal to it, and the difference should be zero, i.e.,

\[
(\mathbf{u}_1^{(1)} - \mathbf{u}_2^{(1)}),
\]

which is expressed as

\[
\mathbf{u}_1^{(1)} - \mathbf{u}_2^{(1)} = \mathbf{u}_1^{(2)} - \mathbf{u}_2^{(2)} = 0,
\]

where \(\mathbf{u}_1^{(1)}\) and \(\mathbf{u}_1^{(2)}\) are respectively the components of the particle vibration velocity of the two media normal to the interface. We may change them into displacements, so the particle displacements should have a constraint

\[
\mathbf{u}_1^{(1)} = \mathbf{u}_2^{(1)}, \quad \mathbf{u}_1^{(2)} = \mathbf{u}_2^{(2)}.
\]

Substitute Eq. (40h) into Eq. (40d), and we have

\[
[\sigma_y^{(1)} - \delta_y \sigma^{(2)}],
\]

Therefore, the stress at the fluid-solid interface is

\[
T_i^{(1)} = \delta_y \sigma^{(2)},
\]

where \(i = 1\) is set to the normal direction of the interface, and \(\sigma^{(2)}\) is the stress in the fluid, or

\[
T_1^{(1)} = 0, \quad T_2^{(1)} = 0, \quad T_3^{(1)} = 0,
\]

where \(T_1^{(1)}\) is assumed to be the normal component of stress at the interface, \(P\) is the fluid pressure, \(T_2^{(1)}\) and \(T_3^{(1)}\) are the two tangential stress components on the interface. Eqs. (42i) and (42k) are our commonly used solid-fluid boundary conditions for stress and displacement at the interface[10]. These boundary conditions satisfy both energy and mass conservations.

### C. Boundary conditions for porous media

If there are two different porous media saturated with a single fluid, it is known from Eq. (41b) that

\[
T_i^{(1)} = T_i^{(2)},
\]

In the above formula \(T_i^{(1)} = T_i^{(2)}\) are the stress components in the outward normal direction \(T_i\) on \(S_2\).

Now let’s discuss the condition for satisfying Eq. (41a). The first condition is that Eq. (43a) will be satisfied if \(T_i^{(1)} = T_i^{(2)}\), and \(\mathbf{u}_i^{(1)} = \mathbf{u}_i^{(2)}\). These conditions imply that there is no relative motion between the pore fluid and solid, which is not true. Therefore, the constraint of the fluid velocity is incorrect [33]. Due to the relative motion of the velocity between the fluid-solid medium, the motion of the constraint condition on the boundary should satisfy the conservation of mass as is just discussed for the case of fluid-solid boundary conditions. In this case, Eq. (43a) can be written as

\[
\sigma_y^{(1)} + \delta_y \sigma^{(2)} = -\mathbf{\sigma}_y^{(1)},
\]

In the above equation, the superscript \(\beta\) is omitted, and \(\gamma = 1, 2, \sigma_y^{(1)}\) and \(\sigma_y^{(2)}\) represent the stress components in the skeleton and the fluid of the porous media, respectively, and \(\mathbf{\sigma}_y^{(1)}\) and \(\mathbf{\sigma}_y^{(2)}\) represent the stress on the skeleton velocity and the velocity of the solid relative to the pore fluid, respectively. Following the analyses of the case for a fluid-solid interface, under the condition of open-pore condition, that is, when the porous fluid of Composition 1 and Composition 2 in the medium are connected, it is easy to obtain the following boundary conditions [48,76]

\[
\begin{cases}
T_1^{(1)} = T_1^{(2)}, \\
\sigma_1^{(1)} = \sigma_2^{(2)},
\end{cases}
\]

and

\[
\begin{cases}
\mathbf{u}_1^{(1)} = \mathbf{u}_1^{(2)}, \\
\mathbf{w}_n^{(1)} = \mathbf{w}_n^{(2)}.
\end{cases}
\]

In Eq. (43c), \(n\) stands for the normal direction of the interface. Eqs. (43b) and (43c) can also be extended to multiphase porous media. If there is no relative motion, then the boundary conditions are consistent with the case of solid-solid media by considering all the other displacements to be continuous.

So far, based on energy conservation and mass conservation, the boundary conditions for the stress and displacement on an interface are formulated in porous media under the open pore conditions. Similarly, the
treatment for boundary condition determinations can be easily extended to any combination of multiphase porous media. It allows us to use energy and mass conservations study the wave propagation in arbitrary multiphase porous media and the reflection and refraction at various interfaces. The deduced continuation conditions are consistent with the existing ones.

V. CONCLUSIONS AND DISCUSSIONS

In continuum mechanics, energy conservation equation, i.e., the rate of total mechanical energy with time equals to the rate of the work done by external forces as an axiom, is spelt out. From this axiom, Lagrange’s equation, Hamilton’s canonical equations, and the elastodynamic equation of motion for arbitrarily anisotropic and multiphasic media for the first time, which are consistent with the equation formulations using Hamilton’s variational principle, also for the first time. The boundary conditions for various elastic media, including porous ones are directly derived and extended from the principle of energy conservation as an application example of our methodology.

The advantages of the formulations by our methodology are that it does not have to introduce the concepts of variational principles, neither is there a need for time terminal constraints. It is simple and easy to understand, and its physical meaning is clear, especially suitable for the formulations of weak form dynamic equations in multiphysics coupling systems.

It is pointed out that, contrary to the existing Hamiltonian mechanics, Hamilton’s variational principle, Lagrange’s equation, and Newton’s second law of motion, are all the consequences of energy conservation in continuum mechanics. The energy conservation not only can be used to explain the behaviors of the time-space evolution of wave motion in a broad sense, but also it can be used to elaborate the wave motion mechanisms on complex boundaries, such as wave reflections and refractions at an interface, i.e., Snell’s law of acoustic wave reflection and refraction is also the consequence of energy conservation.

Our proposed methodology unlocks the essences of Hamilton’s variational principle in classical mechanics, and endows Hamilton’s variational principle a new physical explanation, i.e., the reason why the true motion trajectories of the particles predicted by Hamilton’s variational principle, is energy conservation. Our methodology may be extended to the other areas for equation formulations, such as electrodynamics, fluid mechanics, etc.

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