Localization in coalgebras. Applications to finiteness conditions.

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Introduction

Let $k$ be a field. Given two finite-dimensional right comodules $N$ and $M$ over a $k$–coalgebra $C$, the $k$–vector spaces $\text{Ext}^n_C(N, M)$ need not to be finite-dimensional. This is due to the fact that the injective right comodules appearing in the minimal injective resolution of $M$ need not to be of finite dimension or even quasi-finite. The obstruction here is that factor comodules of quasi-finite comodules are not in general quasi-finite. This note is mainly devoted to the study of coalgebras for which the class of all quasi-finite right comodules is closed under factor comodules. A major tool here is the local study, in the sense of abstract localization [6], of the comodules which have all their factors quasi-finite.

1 Localization in coalgebras

Let $C$ be a coalgebra over a field $k$, with comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to k$. We will use a variation of Heynemann-Sweedler’s sigma-notation, namely,

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Consider the dual algebra $C^* = \text{Hom}_k(C, k)$ with the convolution product $f \ast g(c) = \sum f(c_1)g(c_2)$. Recall from [12, Lemma 6] that every idempotent $e = e^2 \in C^*$ gives a $k$–coalgebra $eCe$ with comultiplication defined as $\Delta(ece) = \sum ece_1 \otimes ece_2$. Its counit is the restriction of $e$. By [4, Lemma 1.2] we have a $C–eCe$–bicomodule $eCe$ (resp. an $eCe–C$–bicomodule $Ce$), with structure maps defined in a straightforward way. This leads (see [4, Theorem 1.5, Corollary 1.6]) to an exact functor $-\square_{eCe} : \mathcal{M}^C \to \mathcal{M}^{eCe}$ with a right adjoint $-\square_{eCe} : \mathcal{M}^{eCe} \to \mathcal{M}^C$. In fact the functor $-\square_{eCe}$ is naturally isomorphic to the co-hom functor $h_C(Ce, -)$ and also to the functor $e(-)$ that sends $M \in \mathcal{M}^C$ onto $eM$. The fundamental properties of co-hom functors may be found in [13]. The counit of this adjunction is, by [4, Proposition 1.4], an isomorphism, so $\mathcal{M}^{eCe}$ becomes a quotient category of $\mathcal{M}^C$. By [16, Proposition 3.8], every quotient category of $\mathcal{M}^C$ is of this form.

We will give here an alternative approach, more appropriate for our purposes, to this fact.

Let $\{S_i : i \in I\}$ be a complete set of representatives of the isomorphism types of simple right $C$–comodules, which we fix from now on. For each subset $J \subseteq I$ we denote by $\mathcal{A}_J$ the smallest localizing subcategory of $\mathcal{M}^C$ that contains the set $\{S_j : j \in J\}$. Clearly, $M \in \mathcal{A}_J$ if and only if for every submodule $M' \subseteq M$, the quotient $M/M'$ contains a simple submodule $S$ with $S \cong S_j$ for some $j \in J$. Since $\mathcal{M}^C$ is a locally finite category, every localizing subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is of the form $\mathcal{A}_J$, where $J \subseteq I$ consists of those $j \in I$ such that $S_j \in \mathcal{T}$.

Let $\{e_i : i \in I\}$ be a basic set of idempotents for $C$, that is, it is a set of orthogonal idempotents of $C^*$ such that $E(S_i) = Ce_i$ for every $i \in I$ (see [11, Section 3]). The notation $E(M)$ stands for an injective cogenerator (in the category of comodules) of a comodule $M$. Most part of the following proposition was essentially given, with a different proof, in [16, Proposition 3.8].

**Proposition 1.1.** Let $\mathcal{T}$ be a localizing subcategory of $\mathcal{M}^C$ and let $J \subseteq I$ such that $\mathcal{T} = \mathcal{A}_J$. Let $e = \sum_{i \notin J} e_i$ be the idempotent acting as zero on $Ce_j$ for $j \in J$, and as $e_i$ on $Ce_i$ for $i \notin J$. Then

1. $\mathcal{T} = \{M \in \mathcal{M}^C : e_i M = 0 \text{ for all } i \notin J\} = \{M \in \mathcal{M}^C : eM = 0\}$.

2. $\mathcal{M}^C / \mathcal{T} = \mathcal{M}^{eCe}$ and $-\square_{eCe} : \mathcal{M}^C \to \mathcal{M}^{eCe}$ is the localizing functor with right adjoint $-\square_{eCe} : \mathcal{M}^{eCe} \to \mathcal{M}^C$. The localizing functor is naturally isomorphic to the co-hom functor $h_C(Ce, -)$ and also to the functor $e(-)$ that sends $M \in \mathcal{M}^C$ onto $eM$.

**Proof.** (1) The injective right $C$–comodule $\bigoplus_{i \notin J} E(S_i)$ is an injective cogenerator for the class of all $\mathcal{T}$–torsionfree comodules, that is, $M \in \mathcal{T}$ if and only if $\text{Com}_C(M, \bigoplus_{i \notin J} E(S_i)) = \{0\}$.
Proposition 1.3. Let $S$ be a simple right $C$-comodule and assume that $S$ has a projective cover $P$ (e. g. $C$ is right semiperfect). Then

$$\mathcal{T}_{E(S)} = \{ M \in \mathcal{M}^C \mid \text{Com}_C(P, M) = 0 \}$$

and $\mathcal{M}^C/\mathcal{T}_{E(S)}$ is equivalent to $\mathcal{M}_R$, where $R = \text{Com}_C(P, P)$ is the endomorphism ring of $P$ in $\mathcal{M}^C$. Hence, $R$ is a finite-dimensional local ring, and the colocal coalgebra $e_iCe_i$ given in Corollary 1.2 is isomorphic to $R^*$. 

Proof. Let $i \in I$ such that $S \cong S_i$. By Corollary 1.2 $\mathcal{T}_{E(S)} = \mathcal{A}_{I\setminus \{i\}}$, the smallest localizing category containing all $S_j$ with $j \neq i$. Let $j \in I \setminus \{i\}$ and assume $\text{Com}_C(P, S_j) \neq 0$. Then there is an epimorphism $g : P \to S_j$. Since $P$ is local, this would imply that
there exists an epimorphism from \( S_j \) onto \( S \). Therefore, \( S_j \cong S \), a contradiction. Hence, \( \text{Com}_C(P, S_j) = 0 \) for every \( j \neq i \), which implies that \( \mathcal{A}_{\{i\}} \subseteq \{ M \in \mathcal{M}^C \mid \text{Com}_C(P, M) = 0 \} \). Conversely, let \( M \) be a right \( C \)-comodule such that \( \text{Com}_C(P, M) = 0 \), and let \( M' \) be the largest subcomodule of \( M \) such that \( M' \in \mathcal{A}_{\{i\}} \). If \( M/M' \neq 0 \), then it contains a simple subcomodule isomorphic to \( S_i \) and, hence, to \( S \). So, \( \text{Com}_C(P, M/M') \neq 0 \) and, since \( P \) is projective, this implies that \( \text{Com}_C(P, M) \neq 0 \), a contradiction. Therefore, \( M/M' = 0 \) and \( M = M' \in \mathcal{A}_{\{i\}} \). Arguing as in [1] Proposition 8.6], we have that \( T(P) \) is a projective generator of the quotient category \( \mathcal{M}^C/\mathcal{T}_E(S) \), where \( T \) denotes the localization functor. Moreover, \( P \) is finite-dimensional, whence \( T(P) \) is of finite length. Therefore, \( \mathcal{M}^C/\mathcal{T}_E(S) \) is equivalent to the category of left modules over the endomorphism ring \( \text{Hom}_{\mathcal{M}^C/\mathcal{T}_E(S)}(T(P), T(P)) \cong \text{Com}_C(P, P) \). \( \square \)

### 2 Strictly quasi-finite comodules

A right comodule \( M \) over a coalgebra \( C \) is said to be quasi-finite [13] if \( \text{Com}_C(S, M) \) is a finite-dimensional \( k \)-vector space for every simple right \( C \)-comodule \( S \) or, equivalently, \( \text{Com}_C(N, M) \) is finite-dimensional for every comodule \( N \) of finite dimension. Every subcomodule and every essential extension of a quasi-finite comodule is quasi-finite. However, factor comodules of quasi-finite comodules are not in general quasi-finite.

**Example 2.1.** Let \( V \) be a vector space over \( k \), and consider \( C = kg \oplus V \) be the co-commutative \( k \)-coalgebra with coproduct given by \( \Delta(g) = g \otimes g \) and \( \Delta(x) = g \otimes x + x \otimes g \) for \( x \in V \), and counit defined by \( \epsilon(g) = 1 \) and \( \epsilon(x) = 0 \) for \( x \in V \). Its coradical is given by \( C_0 = kg \) and \( C = C_0 \wedge C_0 \). Hence, \( C \) has a unique simple comodule \( kg \) and \( C \) is a colocal coalgebra. Moreover, \( C/C_0 \cong V \) is a semisimple comodule. Therefore, if \( V \) is not finite-dimensional, then \( C/C_0 \) is not a quasi-finite right \( C \)-comodule.

**Definition 2.2.** A right comodule \( M \) is said to be strictly quasi-finite if \( M/M' \) is quasi-finite for every subcomodule \( M' \) of \( M \). The comodule \( M \) is said to be co-noetherian if \( M/M' \) embeds in a finite direct sum of copies of \( C_0 \) for every subcomodule \( M' \).

Co-noetherian comodules were investigated in [15] and [14]. In [14] Definition 3.2], strictly quasi-finite comodules are also called co-notetherian.

The coalgebra itself is a quasi-finite right comodule. As a consequence, every co-noetherian comodule is strictly quasi-finite. The converse is not true in general, as the following example shows.

**Example 2.3.** Let \( (V_n)_{n \geq 1} \) be a sequence of \( k \)-vector spaces such that \( \text{dim}_k(V_n) = d_n \) with \( d_1 < d_2 < \cdots \). Consider the coalgebras \( C_n = kg_n \oplus V_n \) as in Example 2.1. Since \( C_n \) is finite dimensional for every \( n \geq 1 \), we have in particular that \( C = \oplus_{n \geq 1} C_n \) is a semiperfect coalgebra. We will see (Theorem 2.10] that \( C \) is then strictly quasi-finite as a \( C \)-comodule. Let us show that \( C \) is not co-noetherian as a comodule. Indeed, if \( C_C \) were co-noetherian, we had an exact sequence \( 0 \rightarrow C/C_0 \rightarrow C^t \) for some \( t \geq 1 \). From this, we have an exact sequence of \( C_n \)-comodules \( 0 \rightarrow C_n/(C_n)_0 \rightarrow C_n^t \) for every \( n \geq 1 \).
Hence, \( \dim_k(\text{Com}_{C_n}(k g_n, C_n/(C_n)_0)) \leq \dim_k(\text{Com}_{C_n}(k g_n, C'_n)) = t \) for every \( n \geq 1 \). Since \( C_n/(C_n)_0 \cong V_n \) is semi-simple as a \( C_n \)-comodule, we get that \( d_n \leq t \) for every \( n \geq 1 \), which is not possible by the choice of the dimensions \( d_n \). Note that this example also shows that an infinite direct sum of co-noetherian coalgebras need not to be co-noetherian.

We will say that a class of comodules \( \mathcal{C} \) is a Serre class whenever for any short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) of comodules one has that \( M \in \mathcal{C} \) if and only if \( M', M'' \in \mathcal{C} \). The first statement of the following proposition was given without proof in [14, Proposition 3.1]. The second and third statements were observed in [3]. We give proofs for the convenience of the reader. We recall from [8] that a coalgebra \( C \) is said to be almost connected if its coradical \( C_0 \) is finite-dimensional.

**Proposition 2.4.** [14, 3]. Let \( C \) be a coalgebra.

1. The class of all strictly quasi-finite right \( C \)-comodules is a Serre class in \( M^C \).

2. The class of all co-noetherian right \( C \)-comodules is a Serre subclass of the class of all strictly quasi-finite right \( C \)-comodules.

3. If \( C \) is almost connected, then both classes coincide.

**Proof.** (1) Let

\[
\begin{array}{cccccc}
0 & \to & M' & \to & M & \to & M'' & \to & 0
\end{array}
\]

be an exact sequence in \( M^C \). Clearly, if \( M \) is strictly quasi-finite, then \( M' \) and \( M'' \) are strictly quasi-finite. Assume now that \( M' \) and \( M'' \) are strictly quasi-finite, and let \( X \leq M \) be any subcomodule. Consider the diagram with exact rows

\[
\begin{array}{cccccc}
M'' = M/M' & \to & 0
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \to & M'/X \cap M' & \to & M/X & \to & M/(M' + X) & \to & 0
\end{array}
\]

By hypothesis, \( M'/X \cap M' \) and \( M/(M' + X) \) are quasi-finite. Therefore, we can assume in (2) that \( X = 0 \) and we have to prove that \( M \) is quasi-finite in (1). This follows easily from the definition of quasi-finite comodule.

(2) Assume in the sequence (1) that \( M \) is co-noetherian. Clearly, \( M' \) is co-noetherian. Now, if \( X \leq M' \) is any subcomodule, then there exists \( t \geq 0 \) such that \( M/X \) embeds in \( C^t \). Since \( M'/X \) is a subcomodule of \( M/X \) we get that \( M'/X \) embeds in \( C^t \) and so \( M' \) is co-noetherian. Assume now that \( M' \) and \( M'' \) are co-noetherian and let \( X \leq M \) be any subcomodule. From (2) we can assume that \( X = 0 \). We have exact sequences
0 \to M' \to C^r and 0 \to M'' \to C^s for some r, s \geq 0. Using that \( C^r \) is an injective comodule, we can construct an exact diagram

\[
\begin{array}{ccc}
0 & \to & M' \\
\downarrow & & \downarrow \\
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & M'' \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

(3) Over an almost connected coalgebra \( C \), every quasi-finite comodule embeds in a finite direct sum of copies of \( C \).

\[\square\]

**Proposition 2.5.** Let \( C \) be an almost connected coalgebra. The following statements are equivalent for a right \( C \)-comodule \( M \).

(i) \( M \) is strictly quasi-finite,

(ii) \( M \) is co-noetherian,

(iii) \( M^* \) is a noetherian right \( C^r \)-module,

(iv) \( M \) is an artinian object of \( MC \).

**Proof.** (i) \( \Leftrightarrow \) (ii) follows from Proposition 2.4.

(i) \( \Rightarrow \) (iv) Let \( X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots \) be a descending chain of subcomodules of \( M \), and put \( X = \bigcap_{i=1}^{\infty} X_i \). Since \( M/X \) is quasi-finite and \( C \) is almost connected, it follows that the socle of \( M/X \), which is essential, is finite-dimensional. Arguing as in [2, Proposition 10.10], there is a positive integer \( n \) such that \( X = X_1 \). Hence, \( X_1 = X_2 = \cdots \).

(iii) \( \Rightarrow \) (iv) With notations as above, if \( X_i^\perp = \{ f \in M^* \mid f(X_i) = 0 \} \) then we have an ascending chain \( X_1^\perp \subset X_2^\perp \subset \cdots \subset X_n^\perp \subset \cdots \) of \( C^r \)-submodules of \( M^* \). Since \( M^* \) is noetherian, we get \( X_n^\perp = X_{n+1}^\perp = \cdots \) for some positive integer \( n \). But \( (X_n/X_{n+1})^* \cong X_n^\perp/X_{n+1}^\perp = 0 \), whence \( X_n/X_{n+1} = 0 \).

(iv) \( \Leftrightarrow \) (iii) follows from [2, Corollary 4.3].

(iv) \( \Rightarrow \) (i) Since \( M \) is artinian, every factor comodule \( M/X \) of \( M \) has finite-dimensional socle and, in particular, \( M/X \) is quasi-finite.

We will see later (Example 3.8) that a strictly quasi-finite comodule over an arbitrary coalgebra need not to be artinian. It shares, however, some properties with artinian objects. An example is the following.

**Proposition 2.6.** Let \( M \) be a strictly quasi-finite right comodule over a coalgebra \( C \). If \( f : M \to M \) is a monomorphism in \( MC \) then \( f \) is an isomorphism.
Proof. Let $M_0 \subset M_1 \subset \cdots$ be the Loewy series of $M$, that is, $M_{n+1}/M_n$ is the socle of $M/M_n$ for $n > 0$ and $M_0$ is the socle of $M$. It suffices to prove that $f(M_n) = M_n$ for every $n \geq 0$, as $M = \bigcup_{n \geq 0} M_n$. On the other hand, $I = C_0^{+}$ is the Jacobson radical of $C^*$, and $M_n = ann_M(I^{n+1})$ for every $n \geq 0$ (see [5] Lemma 3.1.9). We proceed by induction on $n$. For $n = 0$ we have that $f(M_0) \subseteq M_0$. Since $M$ is quasi-finite, every isotypic component of $M$ is finite-dimensional, which gives $f(M_0) = M_0$. Assume inductively that $f(M_n) = M_n$. Then we have $f(M_{n+1}) \subseteq M_{n+1}$, and we can consider the induced morphism $\overline{f} : M_{n+1}/M_n \to M_{n+1}/M_n$. If $\overline{f}(x + M_n) = 0$, then $f(x) \in M_n$, so $I^{n+1}f(x) = 0$. Hence, $f(I^{n+1}x) = 0$ which implies that $I^{n+1}x = 0$, as $f$ is injective. Therefore, $x \in M_n$ and $\overline{f}$ is injective. This implies that $\overline{f}$ is bijective because $M/M_n$ is quasi-finite. So, $f(M_{n+1}) = M_{n+1}$ which completes the induction. \hfill \qed

Our next aim is to characterize strictly quasi-finite comodules in terms of their localizations at the simple comodules.

Lemma 2.7. Let $T$ be any localizing subcategory of the category $\mathcal{M}^C$ of right comodules over a coalgebra $C$, and let $T : \mathcal{M}^C \to \mathcal{M}^C/T$ be the localizing functor. If $X$ is a simple object in the $\mathcal{M}^C/T$, then there exists a simple right $C$–comodule $Y$ such that $T(Y) \cong X$.

Proof. Let $S : \mathcal{M}^C/T \to \mathcal{M}^C$ be the right adjoint functor to $T$. Then $S(X)$ is a nonzero right $C$–comodule, so there is an exact sequence $0 \to Y \to S(X)$, where $Y$ is a simple comodule. Since $T$ is exact, we get the exact sequence $0 \to S(Y) \to TS(X) \cong X$, which gives an isomorphism $T(Y) \cong X$, as $X$ is a simple object. \hfill \qed

Proposition 2.8. Let $T$ be a localizing subcategory of the category $\mathcal{M}^C$ of right comodules over a coalgebra $C$, and let $T : \mathcal{M}^C \to \mathcal{M}^C/T$ be the localization functor.

(1) If $M$ is a strictly quasi-finite right $C$–comodule, then $T(M)$ is strictly quasi-finite.

(2) If $T = T_{E(S)}$ for some simple right $C$–comodule $S$ and $M$ is a strictly quasi-finite right $C$–comodule, then $T(M)$ is an artinian object of $\mathcal{M}^C/T_{E(S)}$.

Proof. (1) Since $T$ is an exact functor, it suffices to prove that $T(M)$ is quasi-finite. Let $X$ be a simple object of $\mathcal{M}^C/T$. By Lemma 2.7 there is a simple right $C$–comodule $Y$ such that $T(Y) \cong X$. Thus,

$$\text{Hom}_{\mathcal{M}^C/T}(X, T(M)) \cong \text{Hom}_{\mathcal{M}^C/T}(T(Y), T(M)) \cong \text{Com}_C(Y, ST(M)),$$

where $S : \mathcal{M}^C/T \to \mathcal{M}^C$ denotes the right adjoint to $T$. The kernel of the canonical map $\alpha : M \to ST(M)$ is an object of $T$, which gives an exact sequence of vector spaces $0 \to \text{Com}_C(Y, M) \to \text{Com}_C(Y, Im(\alpha))$. On the other hand, $Im(\alpha)$ is essential in $ST(M)$ and, since $Im(\alpha)$ is quasi-finite, we have that $ST(M)$ is quasi-finite too. Therefore, $\text{Com}_C(Y, ST(M))$ is finite-dimensional and so is $\text{Hom}_{\mathcal{M}^C/T}(X, T(M))$.

(2) This follows from Corollary 1.2 and Proposition 2.8 in conjunction with part (1). \hfill \qed
Let \( \{ S_i : i \in I \} \) be a complete set of representatives of the isomorphism types of simple right \( C \)-comodules. For every \( i \in I \) let \( T_i : \mathcal{M}^C \to \mathcal{M}^C / \mathcal{T}_{E(S_i)} \) denote the localization functor.

**Theorem 2.9.** A right \( C \)-comodule \( M \) is strictly quasi-finite if and only if \( T_i(M) \) is artinian for every \( i \in I \).

**Proof.** If \( M \) is quasi-finite, then \( T_i(M) \) is artinian for every \( i \in I \) by Proposition 2.8. Conversely, assume \( T_i(M) \) artinian for every \( i \in I \). Since the functors \( T_i \) are exact, it is enough to prove that \( M \) is quasi-finite. Let \( S \) be a simple right \( C \)-comodule and consider the unique \( i \in I \) such that \( S \cong S_i \). If \( M_i \) denotes the largest subcomodule of \( M \) such that \( M_i \subset \mathcal{T}_{E(S_i)} \), then we have an exact sequence \( 0 \to M_i \to M \to M/M_i \to 0 \). Then \( \text{Com}_C(S_i, M_i) = 0 \) and \( \text{Com}_C(S_i, M) \) is isomorphic to a vector subspace of \( \text{Com}_C(S_i, M/M_i) \). Thus, we can assume that \( M_i = 0 \), that is, \( M \) is \( \mathcal{T}_{E(S_i)} \)-torsionfree. The socle \( \text{soc}(M) \) is then a direct sum of copies of \( S_i \) and, since \( T_i(M) \) is artinian, we have that \( T_i(\text{soc}(M)) \subset T_i(M) \) is artinian as well. Therefore, \( \text{soc}(M) \) consists of a direct sum of finitely many copies of \( S_i \cong S \) and, hence, \( \text{Com}_C(S, M) \) is finite-dimensional. \( \square \)

Recall from [9] that a coalgebra is said to be **right semiperfect** if every simple right comodule has a projective cover.

**Theorem 2.10.** Let \( C \) be a right semiperfect coalgebra. The following statements are equivalent for a right \( C \)-comodule \( M \).

(i) \( M \) is quasi-finite;

(ii) \( M \) is strictly quasi-finite;

(iii) \( T_i(M) \) is finite-dimensional for every \( i \in I \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( S \) be a simple right \( C \)-comodule. The projective cover \( P \to S \to 0 \) of \( S \) gives, for every subcomodule \( M' \leq M \), two exact sequences of vector spaces \( 0 \to \text{Com}_C(S, M/M') \to \text{Com}_C(P, M/M') \) and \( \text{Com}_C(P, M) \to \text{Com}_C(P, M/M') \to 0 \). Therefore, \( \text{Com}_C(S, M/M') \) is finite-dimensional, as \( P \) has finite dimension and \( M \) is quasi-finite. This proves that \( M/M' \) is quasi-finite and, so, \( M \) is strictly quasi-finite.

(ii) \( \Rightarrow \) (iii) This follows from Proposition 1.3 and Theorem 2.9.

(iii) \( \Rightarrow \) (i) By Theorem 2.9 \( \square \)

We can now give an alternative proof of one of the most useful characterizations of right semiperfect coalgebras from [9].

**Corollary 2.11. (Lin)** A coalgebra \( C \) is right semiperfect if and only if \( T_i(C) (= e_i C) \) is finite-dimensional for every \( i \in I \).
Proof. If $C$ is right semiperfect then, by Theorem 2.10 $T_i(C) = e_iC$ is finite-dimensional for every $i \in I$, as $C$ is quasi-finite. Conversely, given any simple right $C$–comodule $S$, pick $i \in I$ such that $E(S^*) = e_iC$. Then the essential inclusion $S^* \subseteq e_iC$ gives an epimorphism with small kernel $(e_iC)^* \rightarrow S^{**} \cong S$, where $(e_iC)^*$ is a right $C$–comodule because $e_iC$ is a finite-dimensional left $C$–comodule. By [4, Lemma 1.2], $(e_iC)^* \cong C^*e_i$, and therefore it is a projective cover of $S$.

3 Strictly quasi-finite coalgebras

Recall that $\{S_i : i \in I\}$ is a complete set of representatives of the isomorphism types of simple right comodules over a coalgebra $C$. For each $i \in I$, let $T_i : M^C \rightarrow M^C/T_iE(S_i)$ denote the canonical localization functor. The following is the main theorem of this section.

Theorem 3.1. The following statements are equivalent for a coalgebra $C$.

(i) $C$ is strictly quasi-finite as a right $C$–comodule,

(ii) for every $i \in I$, $E(S_i)$ is strictly quasi-finite and $T_i(E(S_j)) \neq 0$ only for finitely many $j \in I$,

(iii) every quasi-finite right $C$–comodule is strictly quasi-finite,

(iv) $T_i(C)$ is an artinian object for every $i \in I$.

Proof. (i) ⇒ (ii) Given $i \in I$, $E(S_i)$ embeds in $C$, so $E(S_i)$ is strictly quasi-finite. Now, decompose the injective right comodule $C$ as $C = \bigoplus_{j \in I} e_iC$, where the $m_j$ are positive integers. By Theorem 2.9 $T_i(C) = \bigoplus_{j \in I} T_i(E(S_j))^{m_j}$ is an artinian object. Therefore, only finitely many nonzero direct summands should appear.

(ii) ⇒ (iii) Let $M$ be any quasi-finite right $C$–comodule and consider $i \in I$. Then $E(M)$ is quasi-finite and, thus, $E(M) \cong \bigoplus_{j \in I} E(S_j)^{n_j}$ for some set of non-negative integers $\{n_j : j \in I\}$. Therefore, $T_i(E(M)) = \bigoplus_{j \in I} T_i(E(S_j))^{m_j}$ has finitely many nonzero artinian summands (Theorem 2.9) and, thus, $T_i(E(M))$ is artinian. Using Theorem 2.9 once more, we obtain that $E(M)$ is strictly quasi-finite and so is $M$.

(iii) ⇒ (i) This is clear, as $C_C$ is quasi-finite.

(i) ⇔ (iv) By Theorem 2.9. □

Definition 3.2. A coalgebra $C$ satisfying the equivalent conditions of Theorem 2.9 will be said to be right strictly quasi-finite. Every right semiperfect coalgebra is right strictly quasi-finite (Theorem 2.10).

Remark 3.3. The implication (i) ⇒ (iii) was left open in [14, p. 460].

Let $\{e_i : i \in I\}$ be a complete set of orthogonal primitive idempotents for $C$ (see Section 1). Then right strictly quasi-finite coalgebras can be characterized as follows.
Corollary 3.4. The coalgebra \( C \) is right strictly quasi-finite if and only if for every fixed \( i \in I \) the right \( e_j C e_j \)–comodule \( e_j C e_i \) is artinian for every \( j \in I \) and \( e_i C e_j \neq 0 \) only for finitely many \( j \in I \).

Of course, not every coalgebra is strictly quasi-finite (see Example 2.1). The following example shows that the class of strictly quasi-finite coalgebras contains strictly the class of semiperfect coalgebras.

Example 3.5. Let \( C = k[X] \) the Hopf algebra of polynomials in one indeterminate \( X \); its structure of coalgebra is given by \( \Delta(X) = X \otimes 1 + 1 \otimes X \) and \( \epsilon(X) = 0 \). Since \( C^* \cong k[[X]] \) is a noetherian algebra, it follows from 2.5 that \( C \) is right semiperfect, as \( C \) is an artinian comodule. Having just one type of simple, this coalgebra is then strictly quasi-finite. However, it is not semiperfect.

The notion of a right strictly quasi-finite coalgebra is not left-right symmetric, as the following example shows.

Example 3.6. [5] Example 3.2.9] Let \( C \) be the \( k \)–coalgebra with basis \( \{ g_n, d_n : n \geq 1 \} \) and structure maps given by \( \Delta(g_n) = g_n \otimes g_n, \delta_{nm} + d_n \otimes g_{n+1}, \epsilon(g_n) = 1 \) and \( \epsilon(d_n) = 0 \). Define \( g^*_n \in C^* \) by \( g^*_n(g_m) = \delta_{nm} \) and \( g^*_n(s_m) = 0 \) for every \( m \geq 1 \). We have that \( C = \oplus_{n \geq 1} g_n C \), and \( g^*_n C \) is the injective envelope of the simple left \( C \)–comodule \( kg_n \) for every \( n \geq 1 \). Thus, \( \{ g^*_n : n \geq 1 \} \) is a complete set of primitive orthogonal idempotents for the coalgebra \( C \). An easy computation gives that \( g^*_n Cg_1 = kd_{n+1} \neq 0 \) for every \( n > 1 \) which implies, by Corollary 3.4 that \( C \) is not left strictly quasi-finite. On the other hand, \( C \) is right semiperfect, as \( g^*_n C = kg_n \oplus kd_{n-1} \) for \( n > 1 \) and \( g^*_1 C = kg_1 \).

The following proposition collects a number of properties of right strictly quasi-finite coalgebras. Recall that a \( k \)–coalgebra \( D \) is said to be Morita-Takeuchi equivalent to a \( k \)–coalgebra \( C \) if there is a \( k \)–linear equivalence of categories \( F : \mathcal{M}^D \to \mathcal{M}^C \).

Proposition 3.7. (1) Every subcoalgebra of a right strictly quasi-finite coalgebra is right strictly quasi-finite.

(2) If \( C \) is a right strictly quasi-finite coalgebra and \( A \subseteq C \) is a finite-dimensional subcoalgebra, then \( A_\infty = \bigcup_{n \geq 1} \wedge^n A \) is right artinian as a right comodule. When \( C \) is right semiperfect, \( A_\infty \) becomes finite-dimensional.

(3) If \( C \) is a right strictly quasi-finite coalgebra and \( D \) is a coalgebra Morita-Takeuchi equivalent to \( C \), then \( D \) is right strictly quasi-finite.

(4) Any direct sum of right strictly quasi-finite coalgebras is right strictly quasi-finite.

Proof. (1) Easy.

(2) Let \( \rho_M : M \to M \otimes C \) denote the structure map of a right \( C \)–comodule \( M \). If \( \mathcal{C}_A = \{ M \in \mathcal{M}^C : \rho_M(M) \subseteq M \otimes A \} \) denotes the closed subcategory associated to \( A \), then \( \mathcal{C}_A_\infty \) is the smallest localizing subcategory of \( \mathcal{M}^C \) containing \( \mathcal{C}_A \) (see 10). Clearly, \( \mathcal{C}_A_\infty \) contains only finitely many isomorphism types of simple comodules. Thus \( A_\infty \) is an
almost connected coalgebra, so, by Proposition 2.5, $A_{\infty}$ is artinian as a right comodule. The statement for the semiperfect case follows from [5, Corollary 3.2.11, Exercise 3.3.13].

(3) Let $C$ and $D$ be Morita-Takeuchi equivalent coalgebras and assume $C$ to be right strictly quasi-finite. Let $F : \mathcal{M}^D \to \mathcal{M}^C$ denote an equivalence of categories. Let $M \leq D$ be any right subcomodule. Now, $F(D)$ is a quasi-finite right $C$–comodule, which implies, by Theorem 2.9, that $F(D/M) = F(D)/F(M)$ is quasi-finite, as $C$ is right strictly quasi-finite. Therefore, $D/M$ is quasi-finite, i.e., $D$ is right strictly quasi-finite.

(4) Assume $C = \bigoplus_{\alpha \in \Lambda} C_{\alpha}$, as a coalgebra. By [5, Exercise 2.2.18], $\mathcal{M}^C = \prod_{\alpha \in \Lambda} \mathcal{M}^{C_{\alpha}}$, which means that every right $C$–comodule $M$ decomposes uniquely as $M = \bigoplus_{\alpha \in \Lambda} M_{\alpha}$, where $M_{\alpha} \in \mathcal{M}^{C_{\alpha}}$ for every $\alpha \in \Lambda$. This implies that every factor comodule of $M$ is of the form $M/N = \bigoplus_{\alpha \in \Lambda} M_{\alpha}/N_{\alpha}$ for appropriate $C_{\alpha}$–subcomodules $N_{\alpha}$ of the $M_{\alpha}$’s. If $X$ is a simple right $C$–comodule, then $X \in \mathcal{M}^{C_{\alpha}}$ for an uniquely determined $\alpha$. It follows that $\text{Com}_C(X, M/N) = \text{Com}_{C_{\alpha}}(X, M_{\alpha}/N_{\alpha})$. The statement follows now taking $M = C$.

Example 3.8. Let $C = \bigoplus_{\alpha \in \Lambda} C_{\alpha}$ any infinite direct sum of right strictly quasi-finite coalgebras. By Proposition 3.7, $C_{C}$ is a strictly quasi-finite right comodule, and it is not artinian.

The following consequence of Corollary 3.4 shows that co-commutative strictly quasi-finite coalgebras are just the direct sums of artinian co-commutative coalgebras.

Corollary 3.9. The following statements are equivalent for a co-commutative coalgebra $C$.

(i) $C$ is strictly quasi-finite;

(ii) $e_i Ce_i$ is artinian for every $i \in I$;

(iii) $C$ is a direct sum of (co-commutative) artinian colocal coalgebras;

(iv) $(e_i Ce_i)^*$ is noetherian for every $i \in I$.

Proof. (i) $\Rightarrow$ (ii) This follows from Corollary 3.4

(ii) $\Rightarrow$ (iii) Since $C$ is co-commutative, we get $C = \bigoplus_{i \in I} e_i Ce_i$.

(iii) $\Rightarrow$ (i) Apply Proposition 3.7.4.

(ii) $\Leftrightarrow$ (iv) By Proposition 2.5.

Remark 3.10. The equivalence between (i) and (iv) was obtained in [3] by different methods.

4 Some homological coalgebra

If $M$ is a quasi-finite right comodule over a right strictly quasi-finite coalgebra $C$, then all terms in the minimal injective resolution

\[ 0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} Q_0 \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_n \longrightarrow \cdots \]  

(3)
are quasi-finite. This is easily deduced from the fact that every factor of a quasi-finite comodule is quasi-finite (Theorem 2.9) and that the injective envelope of a quasi-finite comodule is quasi-finite. Now, every quasi-finite injective right \( C \)-comodule \( Q \) decomposes as \( Q = \bigoplus_{i \in I} E(S_i)^{n_{Q,i}} \), where \( n_{Q,i} \) is a non negative integer for every \( i \in I \). By Azumaya-Krull-Remak-Schmidt’s Theorem, the numbers \( n_{Q,i} \) are uniquely determined by \( Q \), and hence they are invariants of the comodule \( Q \). As a consequence, the numbers \( n_{Q,n,S} \) for \( i \in I \) and \( n \geq 0 \) are invariants of the quasi-finite comodule \( M \). If \( S \sim S_i \), then we use the notation \( n_{Q,n,S} \) to refer to \( n_{Q,n,i} \). They can be computed using \( \text{Ext} \)-functors, as we shall show.

The notation \( \text{Ext}^n_C(N,-) \) stands for the \( n \)-th derived functor of \( \text{Com}_C(N,-) \) : \( M \to \mathcal{M}_k \), where \( N \) is a right \( C \)-comodule and \( n \) is a non negative integer.

**Corollary 4.1.** Let \( M \) be a quasi-finite right comodule over a right strictly quasi-finite coalgebra \( C \) with minimal injective resolution as in (3).

(1) \( \text{Ext}^n_C(N,M) \) is finite-dimensional for every finite-dimensional right \( C \)-comodule \( N \).

(2) If \( S \) is a simple right \( C \)-comodule, then

\[
\dim_k \text{Ext}_C^1(S,M) = \dim_k \text{Com}_C(S,S) \times n_{Q,1,S},
\]

for every \( n \geq 0 \).

**Proof.** (1) It is clear, since all terms \( Q_n \) are quasi-finite.

(2) Consider the short exact sequence

\[
0 \to M \to Q_0 \to K_1 \to 0,
\]

where \( K_1 \) is the cokernel of \( \varepsilon \). We have an exact sequence

\[
0 \to \text{Com}_C(S,M) \to \text{Com}_C(S,Q_0) \to \text{Com}_C(S,K_1) \to \text{Ext}_C^1(S,M) \to 0
\]

Since \( \text{Com}_C(S,M) = \text{Com}_C(S,Q_0) \) we get \( \text{Com}_C(S,K_1) \cong \text{Ext}_C^1(S,M) \). Now, \( \text{Com}_C(S,Q_1) = \text{Com}_C(S,Q_1) \), as \( Q_1 = E(K_1) \), so \( \text{Com}_C(S,Q_1) \cong \text{Ext}_C^1(S,M) \). On the other hand, \( \text{Com}_C(S,Q_1) = \text{Com}_C(S,S)^{n_{Q,1,S}} \), and therefore \( \dim_k \text{Ext}_C^1(S,M) = n_{Q_1,S} \times \dim_k \text{Com}_C(S,S) \).

An easy induction, using the isomorphism \( \text{Ext}_C^{n-1}(S,K_1) \cong \text{Ext}_C^n(S,M) \) for \( n > 1 \), gives the desired equality for every \( n \).

**Remark 4.2.** If \( C \) is right semiperfect, then every projective object of \( \mathcal{M}^C \) is projective as a left \( C^* \)-module [7, Lemma 2.1], [5, Corollary 2.4.22]. We thus have \( \text{Ext}_C^n(N,M) = \text{Ext}_C^n(N,M) \) for all right \( C \)-comodules \( N, M \).

A coalgebra is said to be right hereditary [11] if every factor comodule of an injective right comodule is injective. Our closing result says that a right hereditary right semiperfect coalgebra is “locally” semisimple. An example that shows that the pertinence of the semiperfect hypothesis is included.
Proposition 4.3. Let $C$ be a right semiperfect right hereditary coalgebra. Then $e_i C e_i$ is dual to a division $k$–algebra for every $i \in I$.

Proof. Some standard arguments show that if $A \to A/\mathcal{T}$ is a localization of a Grothendieck category $A$ such that factors of injective objects are injective, then the quotient category $A/\mathcal{T}$ inherits such a property. Therefore, if $C$ is a right hereditary coalgebra, then so is $C_i$ (Proposition 2.11). According to Corollary 2.12, $C_i = e_i C e_i$ is finite-dimensional for every $i \in I$. Being a colocal coalgebra [4, Proposition 1.20], there is an epimorphism of right $C_i$–comodules from $C_i$ onto its coradical $(C_i)_0$, which is the dual of a (finite dimensional) division algebra. This implies that $(C_i)_0$ is an injective right $C_i$–comodule, as $C_i$ is right hereditary. Therefore, $C_i = (C_i)_0$, which finishes the proof.

Example 4.4. The coalgebra $k[X]$ given in Example 3.5 is strictly quasi-finite, colocal and hereditary, by it is obviously not the dual of a division algebra.

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