ESCAPE PROBABILITY FOR STOCHASTIC DYNAMICAL SYSTEMS WITH JUMPS*

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Abstract. The escape probability is a deterministic concept that quantifies some aspects of stochastic dynamics. This issue has been investigated previously for dynamical systems driven by Gaussian Brownian motions. The present work considers escape probabilities for dynamical systems driven by non-Gaussian Lévy motions, especially symmetric α-stable Lévy motions. The escape probabilities are characterized as solutions of the Balayage-Dirichlet problems of certain partial differential-integral equations. Differences between escape probabilities for dynamical systems driven by Gaussian and non-Gaussian noises are highlighted. In certain special cases, analytic results for escape probabilities are given.

Dedicated to Professor David Nualart on the occasion of his 60th birthday

1. Introduction

Stochastic dynamical systems arise as mathematical models for complex phenomena in biological, geophysical, physical and chemical sciences, under random fluctuations. A specific orbit (or trajectory) for such a system could vary wildly from one realization to another, unlike the situation for deterministic dynamical systems. It is desirable to have different concepts for quantifying stochastic dynamical behaviors. The escape probability is such a concept.

Brownian motions are Gaussian stochastic processes and thus are appropriate for modeling Gaussian random fluctuations. Almost all sample paths of Brownian motions are

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continuous in time. For a dynamical system driven by Brownian motions, almost all orbits (or paths or trajectories) are thus continuous in time. The escape probability is the likelihood that an orbit, starting inside an open domain $D$, exits this domain first through a specific part $\Gamma$ of the boundary $\partial D$. This concept helps understand various phenomena in sciences. One example is in molecular genetics [26]. The frequency of collisions of two single strands of long helical DNA molecules that leads to a double-stranded molecule is of interest and can be computed by virtue of solving an escape probability problem. It turns out that the escape probability satisfies an elliptic partial differential equation with properly chosen boundary conditions [26, 18, 25, 5].

Non-Gaussian random fluctuations are widely observed in various areas such as physics, biology, seismology, electrical engineering and finance [29, 16, 20]. Lévy motions are a large class of non-Gaussian stochastic processes whose sample paths are discontinuous in time. For a dynamical system driven by Lévy motions, almost all the orbits $X_t$ are discontinuous in time. In fact, these orbits are càdlàg (right-continuous with left limit at each time instant), i.e., each of these orbits has countable jumps in time. Due to these jumps, an orbit could escape an open domain without passing through its boundary. In this case, the escape probability is the likelihood that an orbit, starting inside an open domain $D$, exits this domain first by landing in a target domain $U$ in $D^c$ (the complement of domain $D$).

As we see, the escape probability is defined slightly differently for dynamical systems driven by Gaussian or non-Gaussian processes. Although the escape probability for the former has been investigated extensively, the characterization for the escape probability for the latter has not been well documented as a dynamical systems analysis tool for applied mathematics and science communities. See our recent works [6, 11] for numerical analysis of escape probability and mean exit time for dynamical systems driven by symmetric $\alpha$-stable Lévy motions.

In this paper, we carefully derive a partial differential-integral equation to be satisfied by the escape probability for a class of dynamical systems driven by Lévy motions, especially symmetric $\alpha$-stable Lévy motions. Namely the escape probability is a solution of a nonlocal differential equation. We highlight the differences between escape probabilities for dynamical systems driven by Gaussian and non-Gaussian processes. These are illustrated in a few examples.

More precisely, let $\{X_t, t \geq 0\}$ be a $\mathbb{R}^d$-valued Markov process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let $D$ be an open domain in $\mathbb{R}^d$. Define the exit time

$$\tau_{D^c} := \inf\{t > 0 : X_t \in D^c\},$$

where $D^c$ is the complement of $D$ in $\mathbb{R}^d$. Namely, $\tau_{D^c}$ is the first time when $X_t$ hits $D^c$.

When $X_t$ has almost surely continuous paths, i.e., $X_t$ is either a Brownian motion or a solution process for a dynamical system driven by Brownian motions, a path starting at $x \in D$ will hit $D^c$ by hitting $\partial D$ first (assume for the moment that $\partial D$ is smooth). Thus $\tau_{D^c} = \tau_{\partial D}$. Let $\Gamma$ be a subset of the boundary $\partial D$. The likelihood that $X_t$, starting at
$x$, exits from $D$ first through $\Gamma$ is called the escape probability from $D$ to $\Gamma$, denoted as $p(x)$. That is,

$$p(x) = \mathbb{P}\{X_{t_{\partial D}} \in \Gamma\}.$$ 

We will verify that (Section 3.2) the escape probability $p(x)$ solves the following Dirichlet boundary value problem:

$$
\begin{aligned}
\mathcal{L}p &= 0, \quad x \in D, \\
p|_{\partial D} &= \psi,
\end{aligned}
$$

(1)

where $\mathcal{L}$ is the infinitesimal generator of the process $X_t$ and the boundary data $\psi$ is defined as follows

$$\psi(x) = \begin{cases} 
1, & x \in \Gamma, \\
0, & x \in \partial D \setminus \Gamma.
\end{cases}$$

When $X_t$ has càdlàg paths which have countable jumps in time, i.e., $X_t$ could be either a Lévy motion or a solution process of a dynamical system driven by Lévy motions, the first hitting of $D^c$ may occur somewhere in $D^c$. For this reason, we take a subset $U$ of $D^c$, and define the likelihood that $X_t$ exits firstly from $D$ by landing in the target set $U$ as the escape probability from $D$ to $U$, also denoted by $p(x)$. That is,

$$p(x) = \mathbb{P}\{X_{t_{D^c}} \in U\}.$$ 

We will demonstrate that (Section 3.4) the escape probability $p(x)$ solves the following Balayage-Dirichlet boundary value problem:

$$
\begin{aligned}
Ap &= 0, \quad x \in D, \\
p|_{D^c} &= \varphi,
\end{aligned}
$$

(2)

where $A$ is the characteristic operator of $X_t$ and $\varphi$ is defined as follows

$$\varphi(x) = \begin{cases} 
1, & x \in U, \\
0, & x \in D^c \setminus U.
\end{cases}$$

Therefore by solving a deterministic boundary value problem (1) or (2), we obtain the escape probability $p(x)$.

This paper is arranged as follows. In Section 2, we introduce Balayage-Dirichlet problem for discontinuous Markov processes, and also define Lévy motions. The main result is stated and proved in Section 3. In Section 4, we present analytic solutions for escape probabilities in a few special cases.

2. Preliminaries

In this section, we recall basic concepts and results that will be needed throughout the paper.
2.1. Balayage-Dirichlet problem for discontinuous Markov processes. The following materials are from [3, 17, 27, 7, 19, 12]. Let \( G \) be a locally compact space with a countable base and \( \mathcal{G} \) be the Borel \( \sigma \)-field of \( G \). Also, \( \varsigma \) is adjoined to \( G \) as the point at infinity if \( G \) is noncompact, and as an isolated point if \( G \) is compact. Furthermore, let \( G_{\varsigma} = G \cup \{ \varsigma \} \).

**Definition 2.1.** A Markov process \( Y \) with state space \((G, \mathcal{G})\) is called a Hunt process provided:

(i) The paths functions \( t \to Y_t \) are right continuous on \([0, \infty)\) and have left-hand limits on \([0, \zeta)\) almost surely, where \( \zeta := \inf\{ t : Y_t = \varsigma \} \).

(ii) \( Y \) is strong Markov.

(iii) \( Y \) is quasi-left-continuous: whenever \( \{ \tau_n \} \) is an increasing sequence of \( \mathcal{F}_t \)-stopping times with limit \( \tau \), then almost surely \( Y_{\tau_n} \to Y_{\tau} \) on \( \{ \tau < \infty \} \).

**Definition 2.2.** Let \( G \) be an open subset of \( G \) and \( Y_t(x) \) be a Hunt process starting at \( x \in G \). A nonnegative function \( h \) defined on \( G \) is said to be harmonic with respect to \( Y_t \) in \( G \) if for every compact set \( K \subset G \),

\[
\mathbb{E}[h(Y_{\tau_{K^c}}(x))] = h(x), \quad x \in G.
\]

**Definition 2.3.** Let \( f \) be nonnegative on \( G^c \). We say \( h \) defined on \( G \) solves the Balayage-Dirichlet problem for \( G \) with “boundary value” \( f \), denoted by \((G, f)\), if \( h = f \) on \( G \) and further satisfies the following boundary condition:

\[
\forall z \in \partial G, \quad h(y) \to f(z), \quad \text{as} \ y \to z \text{ from inside} \ G.
\]

A point \( z \in \partial G \) is called regular for \( G \) with respect to \( Y_t(z) \) if

\[
\mathbb{P}\{\tau_{G^c} = 0\} = 1.
\]

Here \( G \) is said to be regular if any \( z \in \partial G \) is regular for \( G^c \).

Let \( \rho \) be a metric on \( G \) compatible with the given topology. Let \( \mathcal{I}_G \) be the family of functions \( g \geq 0 \) bounded on \( G \) and lower semicontinuous in \( G \) such that \( \forall x \in G \), there is a number \( A g(x) \) satisfying

\[
\frac{\mathbb{E}[g(Y_{\tau_{\varepsilon}}(x))] - g(x)}{\mathbb{E}[\tau_{\varepsilon}]} \to A g(x), \quad \text{as} \ \varepsilon \downarrow 0,
\]

where \( \tau_{\varepsilon} := \inf\{ t > 0 : \rho(Y_{t}(x), x) > \varepsilon \} \). We call \( A \) with domain \( \mathcal{I}_G \) the characteristic operator of \( Y_t \) relative to \( G \). If \( L \) with domain \( \mathcal{D}_G \) is the infinitesimal generator of \( Y_t \) relative to \( G \), \( \mathcal{D}_G \subseteq \mathcal{I}_G \) and

\[
A f = L f, \quad f \in \mathcal{D}_G.
\]

(c.f. [10])

We quote the following result about the existence and regularity of the solution for the Balayage-Dirichlet problem.

**Theorem 2.4.** ([17])

Suppose that \( G \) is relatively compact and regular, and \( f \) is nonnegative and bounded on
If $f$ is continuous at any $z \in \partial G$, then $h(x) = \mathbb{E}[f(Y_{\tau_G}(x))]$ is the unique solution to the Balayage-Dirichlet problem $(G, f)$, and $Ah(x) = 0$ for $h \in I_G$.

2.2. Lévy motions.

**Definition 2.5.** A process $L_t$, with $L_0 = 0$ a.s. is a $d$-dimensional Lévy process or Lévy motion if

(i) $L_t$ has independent increments; that is, $L_t - L_s$ is independent of $L_v - L_u$ if $(u, v) \cap (s, t) = \emptyset$;

(ii) $L_t$ has stationary increments; that is, $L_t - L_s$ has the same distribution as $L_v - L_u$ if $t - s = v - u > 0$;

(iii) $L_t$ is stochastically continuous;

(iv) $L_t$ is right continuous with left limit.

The characteristic function for $L_t$ is given by

$$
\mathbb{E}(\exp\{i\langle z, L_t \rangle\}) = \exp\{t\Psi(z)\}, \quad z \in \mathbb{R}^d,
$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. The function $\Psi : \mathbb{R}^d \to \mathbb{C}$ is called the characteristic exponent of the Lévy process $L_t$. By the Lévy-Khintchine formula, there exist a nonnegative-definite $d \times d$ matrix $Q$, a measure $\nu$ on $\mathbb{R}^d$ satisfying

$$
\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d \setminus \{0\}} ((|u|^2 \wedge 1)\nu(du) < \infty,
$$

and $\gamma \in \mathbb{R}^d$ such that

$$
\Psi(z) = i\langle z, \gamma \rangle - \frac{1}{2}\langle z, Qz \rangle
+ \int_{\mathbb{R}^d \setminus \{0\}} (e^{i\langle z, u \rangle} - 1 - i\langle z, u \rangle 1_{|u| \leq 1})\nu(du). \tag{3}
$$

The measure $\nu$ is called the Lévy measure of $L_t$, $Q$ is the diffusion matrix, and $\gamma$ is the drift vector.

We now introduce a special class of Lévy motions, i.e., the symmetric $\alpha$-stable Lévy motions $L_t^\alpha$.

**Definition 2.6.** For $\alpha \in (0, 2)$. A $d$-dimensional symmetric $\alpha$-stable Lévy motion $L_t^\alpha$ is a Lévy process with characteristic exponent

$$
\Psi(z) = -C|z|^{\alpha}, \quad z \in \mathbb{R}^d, \tag{4}
$$

where

$$
C = \pi^{-1/2}\Gamma((1 + \alpha)/2)\Gamma(d/2) \frac{\Gamma((d + \alpha)/2)}{\Gamma((d + \alpha)/2)}.
$$

(c.f. [24, Page 115] for the above formula of $C$.)
Thus, for a $d$-dimensional symmetric $\alpha$-stable Lévy motion $L_t^\alpha$, the diffusion matrix $Q = 0$, the drift vector $\gamma = 0$, and the Lévy measure $\nu$ is given by

$$\nu(du) = \frac{C_{d,\alpha}}{|u|^{d+\alpha}} du,$$

where

$$C_{d,\alpha} = \frac{\alpha \Gamma((d + \alpha)/2)}{2^{1-\alpha} \pi^{d/2} \Gamma(1 - \alpha/2)}.$$

(c.f. [8, Page 1312] for the above formula of $C_{d,\alpha}$) Moreover, comparing [4] with (3), we obtain

$$-C|z|^\alpha = \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{i\langle z, u \rangle} - 1 - i \langle z, u \rangle 1_{|u| \leq 1} \right) \frac{C_{d,\alpha}}{|u|^{d+\alpha}} du.$$

Let $C_0(\mathbb{R}^d)$ be the space of continuous functions $f$ on $\mathbb{R}^d$ satisfying $\lim_{|x| \to \infty} f(x) = 0$ with norm $\| f \|_{C_0(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |f(x)|$. Let $C^2_0(\mathbb{R}^d)$ be the set of $f \in C_0(\mathbb{R}^d)$ such that $f$ is two times differentiable and the first and second order partial derivatives of $f$ belong to $C_0(\mathbb{R}^d)$. Let $C_c^\infty(\mathbb{R}^d)$ stand for the space of all infinitely differentiable functions on $\mathbb{R}^d$ with compact supports. Define

$$(\mathcal{L}_\alpha f)(x) := \int_{\mathbb{R}^d \setminus \{0\}} (f(x + u) - f(x) - \langle \partial_x f(x), u \rangle 1_{|u| \leq 1}) \frac{C_{d,\alpha}}{|u|^{d+\alpha}} du$$

on $C^2_0(\mathbb{R}^d)$. And then for $\xi \in \mathbb{R}^d$

$$(\mathcal{L}_\alpha e^{i\langle \cdot, \xi \rangle})(x) = e^{i\langle x, \xi \rangle} \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{i\langle u, \xi \rangle} - 1 - i \langle \xi, u \rangle 1_{|u| \leq 1} \right) \frac{C_{d,\alpha}}{|u|^{d+\alpha}} du.$$

By Courrègge’s second theorem ([11, Theorem 3.5.5, p.183]), for every $f \in C_c^\infty(\mathbb{R}^d)$

$$(\mathcal{L}_\alpha f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle z, x \rangle} \left[ e^{-i\langle x, z \rangle} (\mathcal{L}_\alpha e^{i\langle \cdot, z \rangle})(x) \right] \hat{f}(z) dz$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle z, x \rangle} \left[ \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{i\langle u, z \rangle} - 1 - i \langle z, u \rangle 1_{|u| \leq 1} \right) \frac{C_{d,\alpha}}{|u|^{d+\alpha}} du \right] \hat{f}(z) dz$$

$$= -\frac{C}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle z, x \rangle} |z|^\alpha \hat{f}(z) dz$$

$$= C \cdot [-(\Delta)^{\alpha/2} f](x).$$

Set $p_t := L_t - L_{t-}$. Then $p_t$ defines a stationary $(\mathcal{F}_t)$-adapted Poisson point process with values in $\mathbb{R}^d \setminus \{0\}$ ([13]). And the characteristic measure of $p$ is the Lévy measure $\nu$. Let $N_p((0, t], du)$ be the counting measure of $p_t$, i.e., for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$

$$N_p((0, t], B) := \# \{ 0 < s \leq t : p_s \in B \},$$

where $\#$ denotes the cardinality of a set. The compensator measure of $N_p$ is given by

$$\tilde{N}_p((0, t], du) := N_p((0, t], du) - t\nu(du).$$
The Lévy-Itô theorem states that for a symmetric $\alpha$-stable process $L_t$,

(i) for $1 \leq \alpha < 2$,

\[ L_t = \int_0^t \int_{|u| \leq 1} u\tilde{N}_p(ds, du) + \int_0^t \int_{|u| > 1} uN_p(ds, du), \]

(ii) for $0 < \alpha < 1$,

\[ L_t = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} uN_p(ds, du). \]

3. Boundary value problems for escape probability

In this section, we formulate boundary value problems for the escape probability associated with Brownian motions, SDEs driven by Brownian motions, Lévy motions and SDEs driven by Lévy motions. For Lévy motions, in particular, we consider symmetric $\alpha$-stable Lévy motions. We will see that the escape probability can be found by solving deterministic partial differential equations or partial differential-integral equations, with properly chosen boundary conditions.

Figure 1. A particle executing unbiased random walk in a bounded interval

3.1. Boundary value problem for escape probability of Brownian motions. Suppose that a particle executes an unbiased random walk on a straight line. Let $D = (a, b)$. Figure 1 shows the random walk scenario. That is, a particle moves according to the following rules (LS):

(i) During the passage of a certain fixed time interval, a particle takes 1 step of a certain fixed length $\delta$ along the $x$ axis.

(ii) It is equally probable that the step is to the right or to the left.

If the particle starting from $x \in D$ eventually escapes $D$ by crossing the boundary $b$, then it must have moved to one of the two points adjacent to $x$ first and then crossed the boundary. Thus

\[ p(x) = \frac{1}{2}[p(x - \delta) + p(x + \delta)], \]

for $x \in D$. By Taylor expansion on the right hand side to the second order, we have

\[ \frac{1}{2}p''(x) = 0. \]

The boundary conditions are

\[ \lim_{x \to b} p(x) = 1, \quad \lim_{x \to a} p(x) = 0, \]
since the nearer the particle starts to b, the more likely it will first cross the boundary through b. 

Note that the limit of the random walk is a standard Brownian motion \( W_t \), that is, 

(i) \( W \) has independent increments; 
(ii) for \( 0 < s < t \), \( W_t - W_s \) is a Gaussian random variable with mean zero and variance \( (t-s) \).

Thus, the escape probability \( p(x) \) of a standard Brownian motion from \( D \) through the boundary \( b \) satisfies 

\[
\begin{aligned}
\frac{1}{2} \Delta p(x) &= 0, \\
p(b) &= 1, \\
p(a) &= 0,
\end{aligned}
\]

where \( \frac{1}{2} \Delta = \frac{1}{2} \partial_{xx} \) is the infinitesimal generator for a scalar standard Brownian motion \( W_t \).

3.2. Boundary value problem for escape probability of SDEs driven by Brownian motions. Some results in this subsection can be found in [21, Chapter 9].

Let \( \{W(t)\}_{t \geq 0} \) be an \( m \)-dimensional standard \( \mathcal{F}_t \)-adapted Brownian motion. Consider the following stochastic differential equation (SDE) in \( \mathbb{R}^d \):

\[
X_t(x) = x + \int_0^t b(X_s(x)) \, ds + \int_0^t \sigma(X_s(x)) \, dW_s. 
\]

(5)

We make the following assumptions about the drift \( b : \mathbb{R}^d \to \mathbb{R}^d \) and the diffusion coefficient \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m \).

\( (H_{b, \sigma}^1) \)

\[
|b(x) - b(y)| \leq \lambda(|x - y|), \\
|\sigma(x) - \sigma(y)| \leq \gamma(|x - y|).
\]

Here \( \lambda \) and \( \gamma \) are increasing concave functions with the properties \( \lambda(0) = \gamma(0) = 0, \int_0^\infty \frac{1}{\lambda(u)} \, du = \int_0^\infty \frac{1}{\gamma^2(u)} \, du = \infty \).

Under \( (H_{b, \sigma}^1) \), it is well known that there exists a unique strong solution to Eq.(5) [30].

This solution is denoted by \( X_t(x) \).

We also make the following assumption.

\( (H_{\sigma}^2) \) There exists a \( \xi > 0 \) such that for any \( x, y \in D \)

\[
\langle y, \sigma \sigma^*(x) y \rangle \geq \xi |y|^2.
\]

This condition guarantees that the infinitesimal generator

\[
\mathcal{L} := \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}
\]

for Eq.(5) is uniformly elliptic in \( D \), since then the eigenvalues of \( \sigma \sigma^* \) are away from 0 in \( D \). Here the matrix \( [a_{ij}] := \frac{1}{2} \sigma (x) \sigma^*(x) \).
Figure 2. Escape probability for SDEs driven by Brownian motions: An annular open domain $D$ with a subset $\Gamma$ of its boundary $\partial D$

Let $D$ be an open annular domain as in Figure 2. In one dimensional case, it is just an open interval. Let $\Gamma$ be its inner (or outer) boundary. Taking

$$
\psi(x) = \begin{cases} 
1, & x \in \Gamma, \\
0, & x \in \partial D \setminus \Gamma,
\end{cases}
$$

we have

$$
\mathbb{E}[\psi(X_{\tau_D}(x))] = \int_{\{\omega : X_{\tau_D}(x) \in \Gamma\}} \psi(X_{\tau_D}(x))d\mathbb{P}(\omega)
$$

$$
+ \int_{\{\omega : X_{\tau_D}(x) \in \partial D \setminus \Gamma\}} \psi(X_{\tau_D}(x))d\mathbb{P}(\omega)
$$

$$
= \mathbb{P}\{\omega : X_{\tau_D}(x) \in \Gamma\}
$$

$$
= p(x).
$$

This means that, for this specific $\psi$, $\mathbb{E}[\psi(X_{\tau_D}(x))]$ is the escape probability $p(x)$, which we are looking for.

We need to use [21, Theorem 9.2.14] or [9] in order to see that the escape probability $p(x)$ is closely related to a harmonic function with respect to $X_t$. This requires that the boundary data $\psi$ to be bounded and continuous on $\partial D$. For the domain $D$ taken as in Figure 2, with $\Gamma$ the inner boundary (or outer) boundary, the above chosen $\psi$ in (6) is indeed bounded and continuous on $\partial D$. Thus, we have the following result by [21, Theorem 9.2.14].

**Theorem 3.1.** The escape probability $p(x)$ from an open annular domain $D$ to its inner (or outer) boundary $\Gamma$, for the dynamical system driven by Brownian motions (5), is the
solution to the following Dirichlet boundary value problem

\[
\begin{cases}
Lp = 0, \\
p|\Gamma = 1, \\
p|_{\partial D \setminus \Gamma} = 0.
\end{cases}
\]

Figure 3. A particle executing Lévy motion in a bounded interval

3.3. Boundary value problem for escape probability of symmetric \( \alpha \)-stable Lévy motions. Assume a particle is taking a one-dimensional Lévy flight, where the distribution of step sizes is a symmetric \( \alpha \)-stable distribution (Figure 3). Let \( p(x) \) denote the escape probability of the particle starting at \( x \) in \( D = (a, b) \) and then first escapes \( D \) over the right boundary \( b \). It could first move to somewhere inside \( D \), say \( x + y \in D \), and then achieve its goal by jumping over the right boundary \( b \) from the new starting point \( x + y \). More precisely,

\[
p(x) = \int_{\mathbb{R}\setminus\{0\}} P\{\text{the first step length is } y\} p(x + y) dy.
\]

According to [2], the symmetric \( \alpha \)-stable probability density function is the following

\[
f_{\alpha,0}(y) = \begin{cases} -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(\alpha k+1)}{k! |y|^\alpha} \sin[k\left(\frac{\alpha \pi}{2} - \alpha \arg y\right)], & 0 < \alpha < 1, \\
\frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha k)}{\Gamma(\alpha)} y^k \cos[k\left(\frac{\alpha \pi}{2}\right)], & 1 < \alpha < 2,
\end{cases}
\]

where \( \arg y = \pi \) when \( y < 0 \).

For \( 0 < \alpha < 2 \), the asymptotic expansion has also been given by [2] as follows

\[
f_{\alpha,0}(y) = \begin{cases} -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(\alpha k+1)}{k! |y|^\alpha} \sin[k\left(\frac{\alpha \pi}{2} - \alpha \arg y\right)] + o(|y|^{-\alpha(n+1)}) - 1, & |y| \to \infty, \\
\frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha k)}{\Gamma(\alpha)} y^k \cos[k\left(\frac{\alpha \pi}{2}\right)] + o(|y|^{n+1}), & |y| \to 0,
\end{cases}
\]

where \( C_1(\alpha) = \frac{\Gamma(1+\alpha)}{\pi} \) and \( C_2(\alpha) = \frac{\Gamma(\frac{1}{\alpha})}{\pi^\alpha} \). Take \( N > 0 \) large enough and fix it. Thus,

\[
0 = \int_{\mathbb{R}\setminus\{0\}} f_{\alpha,0}(y)[p(x + y) - p(x)] dy \\
= \int_{(-N,N)\setminus\{0\}} f_{\alpha,0}(y)[p(x + y) - p(x)] dy
\]
Thus, putting (8), (9) and (10) together, we have for $0 < \alpha < 1$

$$I_1 = \int_{(-N,N) \setminus \{0\}} f_{\alpha,0}(N \cdot \frac{y}{N}) [p(x + y) - p(x)]dy$$

$$= \int_{(-N,N) \setminus \{0\}} f_{\alpha,0}(N) (\frac{y}{N})^\alpha [p(x + y) - p(x)]dy$$

$$= \int_{(-N,N) \setminus \{0\}} C_1(\alpha) N^{\alpha} (\frac{y}{N})^\alpha [p(x + y) - p(x)]dy$$

$$= \int_{(-N,N) \setminus \{0\}} C_1(\alpha) y^\alpha [p(x + y) - p(x)]dy. \quad (8)$$

For $I_2$, we calculate

$$I_2 = \int_{N} f_{\alpha,0}(y)[1 - p(x)]dy + \int_{-N} f_{\alpha,0}(y)[0 - p(x)]dy$$

$$= \int_{N} \left[ \frac{C_1(\alpha)}{y^{1+\alpha}} + o \left( \frac{1}{y^{1+2\alpha}} \right) \right] [1 - p(x)]dy$$

$$- \int_{-N} \left[ \frac{C_1(\alpha)}{(-y)^{1+\alpha}} + o \left( \frac{1}{(-y)^{1+2\alpha}} \right) \right] p(x)dy$$

$$= \int_{N} \frac{C_1(\alpha)}{y^{1+\alpha}} dy - \int_{\mathbb{R} \setminus [-N,N]} \frac{C_1(\alpha)}{|y|^{1+\alpha}} p(x)dy$$

$$= \int_{\mathbb{R} \setminus [-N,N]} \frac{C_1(\alpha) p(x + y)}{|y|^{1+\alpha}} dy - \int_{\mathbb{R} \setminus [-N,N]} \frac{C_1(\alpha)}{|y|^{1+\alpha}} p(x)dy \quad (9)$$

Note that for $0 < \alpha < 1$, by the fact that the integral of an odd function on a symmetric interval is zero, it holds that

$$\int_{\mathbb{R} \setminus \{0\}} p'(x)yI_{|y|\leq 1} \frac{C_1(\alpha)}{|y|^{1+\alpha}} dy = p'(x)C_1(\alpha) \int_{|y|\leq 1 \setminus \{0\}} \frac{y}{|y|^{1+\alpha}} dy = 0. \quad (10)$$

Thus, putting (8), (9) and (10) together, we have for $0 < \alpha < 1$

$$\int_{\mathbb{R} \setminus \{0\}} [p(x + y) - p(x) - p'(x)yI_{|y|\leq 1}] \frac{C_1(\alpha)}{|y|^{1+\alpha}} dy = 0.$$

Moreover, $C_1(\alpha) = C_{1,\alpha}$.

For $\alpha \in [1, 2)$, we only divide $I_1$ into two parts $I_{11}$ and $I_{12}$, where

$$I_{11} := \int_{|y|\leq \epsilon \setminus \{0\}} f_{\alpha,0}(y)[p(x + y) - p(x)]dy,$$
\[ I_{12} := \int_{(-N,N) \setminus (-\rho,\rho)} f_{\alpha,0}(y)[p(x + y) - p(x)] dy, \]

and \( \rho > 0 \) is a small enough constant.

For \( I_{11} \), by Taylor expansion and self-affine property in [26], we get

\[
\begin{align*}
\int_{\{|y| \leq \rho\} \setminus \{0\}} f_{\alpha,0}(y)[p(x + y) - p(x)] dy & = \int_{\{|y| \leq \rho\} \setminus \{0\}} f_{\alpha,0}\left(\frac{1}{\rho} \cdot \rho y\right) p'(x) dy \\
& = \int_{\{|y| \leq \rho\} \setminus \{0\}} f_{\alpha,0}\left(\frac{1}{\rho} y\right) \frac{1}{(\rho |y|)^{1+\alpha}} p'(x) dy \\
& = \int_{\{|y| \leq \rho\} \setminus \{0\}} \left(\frac{1}{\rho}\right)^{1+\alpha} \cdot \frac{1}{(\rho |y|)^{1+\alpha}} p'(x) dy \\
& = \int_{\mathbb{R} \setminus \{0\}} p'(x) y I_{\{|y| \leq \rho\}} \frac{C_1(\alpha)}{|y|^{1+\alpha}} dy.
\end{align*}
\]

For \( I_{12} \), we apply the same technique as that in dealing with \( I_1 \) for \( \alpha \in (0, 1) \).

Next, by the similar calculation to that for \( \alpha \in (0, 1) \), we obtain for \( \alpha \in [1, 2) \)

\[
\int_{\mathbb{R} \setminus \{0\}} \left[p(x + y) - p(x) - p'(x) y I_{\{|y| \leq 1\}}\right] \frac{C_1(\alpha) dy}{|y|^{1+\alpha}} = 0.
\]

Since the limit of the Lévy flight is a symmetric \( \alpha \)-stable Lévy motion \( L^\alpha_t \), the escape probability \( p(x) \) of a symmetric \( \alpha \)-stable Lévy motion, from \( D \) to \( [b, \infty) \) satisfies

\[
\begin{cases}
-(\Delta)_{\mathbb{R}^d}^{\frac{\alpha}{2}} p(x) = 0, \\
p(x)|_{[b, \infty)} = 1, \\
p(x)|_{(-\infty, a]} = 0.
\end{cases}
\]

Note that \( -(\Delta)_{\mathbb{R}^d}^{\frac{\alpha}{2}} \) is the infinitesimal generator for a scalar symmetric \( \alpha \)-stable Lévy motion \( L^\alpha_t \).

### 3.4. Boundary value problem for escape probability of SDEs driven by general Lévy motions

Let \( L_t \) be a Lévy process independent of \( W_t \). Consider the following SDE in \( \mathbb{R}^d \)

\[
X_t(x) = x + \int_0^t b(X_s(x)) \, ds + \int_0^t \sigma(X_s(x)) \, dW_s + L_t.
\]

Assume that the drift \( b \) and the diffusion \( \sigma \) satisfy the following conditions:

\textbf{(H}_b\textbf{)} there exists a constant \( C_b > 0 \) such that for \( x, y \in \mathbb{R}^d \)

\[
|b(x) - b(y)| \leq C_b |x - y| \cdot \log(|x - y|^{-1} + e);
\]

\textbf{(H}_\sigma\textbf{)} there exists a constant \( C_\sigma > 0 \) such that for \( x, y \in \mathbb{R}^d \)

\[
|\sigma(x) - \sigma(y)|^2 \leq C_\sigma |x - y|^2 \cdot \log(|x - y|^{-1} + e).
\]
Under \((H_b)\) and \((H_\sigma)\), it is well known that there exists a unique strong solution to Eq. (11) (see [23]). This solution will be denoted by \(X_t(x)\). Moreover, \(X_t(x)\) is continuous in \(x\).

**Lemma 3.2.** The solution process \(X_t(x)\) of the SDE (11) is a strong Markov process.

**Proof.** Let \(\eta\) be a \((\mathcal{F}_t)_{t\geq 0}\)-stopping time. Set
\[
\mathcal{G}_t := \sigma\{W_{\eta+t} - W_\eta, L_{\eta+t} - L_\eta\} \cup \mathcal{N}, \quad t \geq 0,
\]
where \(\mathcal{N}\) is of all \(P\)-zero sets. That is, \(\mathcal{G}_t\) is a completed \(\sigma\)-algebra generated by \(W_{\eta+t} - W_\eta\) and \(L_{\eta+t} - L_\eta\). Besides, \(\mathcal{G}_t\) is independent of \(\mathcal{F}_t\). Let \(X(x, \eta, \eta + t)\) denote the unique solution of the following SDE
\[
X(x, \eta, \eta + t) = x + \int_\eta^{\eta+t} b(X(x, \eta, s)) \, ds + \int_\eta^{\eta+t} \sigma(X(x, \eta, s)) \, dW_s + L_{\eta+t} - L_\eta. \tag{12}
\]
Moreover, \(X(x, \eta, \eta + t)\) is \(\mathcal{G}_t\)-measurable and \(X(x, 0, t) = X_t(x)\). By the uniqueness of the solution to (12), we have
\[
X(x, 0, \eta + t) = X(x, 0, \eta, \eta + t), \quad \text{a.s.}
\]

For any bounded measurable function \(g\),
\[
\mathbb{E}[g(X_{\eta+t}(x)) | \mathcal{F}_\eta] = \mathbb{E}[g(X(x, 0, \eta + t)) | \mathcal{F}_\eta] = \mathbb{E}[g(X(x, 0, \eta), \eta, \eta + t)) | \mathcal{F}_\eta] = \mathbb{E}[g(X(y, \eta, \eta + t)) | y = X(x, 0, \eta)] = \mathbb{E}[g(X(y, 0, t)) | y = X(x, 0, \eta)]. \tag{13}
\]

Here the last equality holds because the distribution of \(X(y, \eta, \eta + t)\) is the same to that of \(X(y, 0, t)\). The proof is completed since (13) implies that
\[
\mathbb{E}[g(X_{\eta+t}(x)) | \mathcal{F}_\eta] = \mathbb{E}[g(X_{\eta+t}(x)) | X_\eta(x)].
\]

Because \(L_t\) has càdlàg and quasi-left-continuous paths ([24]), \(X_t(x)\) also has càdlàg and quasi-left-continuous paths. Thus by Lemma 3.2 and Definition 2.1 above, we see that \(X_t(x)\) is a Hunt process. Let \(D\) be a relatively compact and regular open domain (Figure 4 or Figure 5). Theorem 2.4 implies that \(\mathbb{E}[\varphi(X_{\tau_D^c}(x))]\) is the unique solution to the Balayage-Dirichlet problem \((D, \varphi)\), under the condition that \(\varphi\) is nonnegative and bounded on \(D^c\). Set
\[
\varphi(x) = \begin{cases} 
1, & x \in U, \\
0, & x \in D^c \setminus U.
\end{cases}
\]
Then \(\varphi\) is nonnegative and bounded on \(D^c\). We observe that
\[
\mathbb{E}[\varphi(X_{\tau_{D^c}}(x))] = \int_{\{\omega : X_{\tau_{D^c}}(x) \in U\}} \varphi(X_{\tau_{D^c}}(x)) \, d\mathbb{P}(\omega)
\]
Figure 4. Escape probability for SDEs driven by Lévy motions: an open annular domain $D$, with its inner part $U$ (which is in $D^c$) as a target domain.

Figure 5. Escape probability for SDEs driven by Lévy motions: a general open domain $D$, with a target domain $U$ in $D^c$

$$\int \{ \omega : X_{\tau_{D^c}}(x) \in D^c \setminus U \} \varphi(X_{\tau_{D^c}}(x))dP(\omega)$$

$$= P\{ \omega : X_{\tau_{D^c}}(x) \in U \}$$
This means that, for this specific $\varphi$, $\mathbb{E}[\varphi(X_{\tau_D}(x))]$ is the escape probability $p(x)$ that we are looking for. By the definition of the characteristic operator, $p \in I_D$ and by Theorem 2.4 $Ap(x) = 0$. Thus we obtain the following theorem.

**Theorem 3.3.** Let $D$ be a relatively compact and regular open domain, and let $U$ be a set in $D^c$. Then the escape probability $p(x)$, for the dynamical system driven by Lévy motions (11), from $D$ to $U$, is the solution of the following Balayage-Dirichlet problem

\[
\begin{cases}
Ap = 0, \\
p|_U = 1, \\
p|_{D^c \setminus U} = 0,
\end{cases}
\]

where $A$ is the characteristic operator for this system.

**Remark 3.4.** Unlike the SDEs driven by Brownian motions, a typical open domain $D$ here could be a quite general open domain (Figure 4), as well as an annular domain (Figure 5). This is due to the jumping properties of the solution paths. It is also due to the fact that, in Theorem 2.4, the function $f$ is only required to be continuous on the boundary $\partial D$ (not on the domain $D^c$).

Finally we consider the representation of the characteristic operator $A$, for a SDE driven by a symmetric $\alpha$-stable Lévy process $L^\alpha_t$, with $\alpha \in (0, 2)$:

\[
X_t(x) = x + \int_0^t b(X_s(x)) \, ds + \int_0^t \sigma(X_s(x)) \, dW_s + L^\alpha_t. \tag{14}
\]

Let us first consider the case of $1 \leq \alpha < 2$. For $f \in C^2_0(\mathbb{R}^d)$, applying the Itô formula to $f(X_{\tau_e}(x))$, we obtain

\[
f(X_{\tau_e}(x)) - f(x) = \int_0^{\tau_e} \langle \partial_y f(X_s), b(X_s) \rangle \, ds + \int_0^{\tau_e} \langle \partial_y f(X_s), \sigma(X_s) \rangle \, dW_s
\]

\[+ \int_0^{\tau_e} \int_{|u| \leq 1} (f(X_s + u) - f(X_s)) \, N_p(ds, du)\]

\[+ \int_0^{\tau_e} \int_{|u| > 1} (f(X_s + u) - f(X_s)) \, \tilde{N}_p(ds, du)\]

\[+ \frac{1}{2} \int_0^{\tau_e} \left( \frac{\partial^2}{\partial y_i \partial y_j} f(X_s) \right) \sigma_{ik}(X_s) \sigma_{kj}(X_s) \, ds\]

\[+ \int_0^{\tau_e} \int_{|u| \leq 1} (f(X_s + u) - f(X_s)) \langle \partial_y f(X_s), u \rangle^\alpha \frac{C_{d, \alpha}}{|u|^{d+\alpha}} \, du \, ds.
\]
Here and hereafter, we use the convention that repeated indices imply summation from 1 to \(d\). Taking expectation on both sides, we get

\[
\begin{align*}
\mathbb{E}[f(X_{\tau_\varepsilon}(x))] & - f(x) \\
= & \mathbb{E} \int_0^{\tau_\varepsilon} \langle \partial_y f(X_s), b(X_s) \rangle ds + \frac{1}{2} \mathbb{E} \int_0^{\tau_\varepsilon} \left( \frac{\partial^2}{\partial y_i \partial y_j} f(X_s) \right) \sigma_{ik}(X_s) \sigma_{kj}(X_s) ds \\
& + \mathbb{E} \int_0^{\tau_\varepsilon} \int_{\mathbb{R}^d \setminus \{0\}} (f(X_s + u) - f(X_s) - \langle \partial_y f(X_s), u \rangle) \frac{C_{\alpha d}}{|u|^{d+\alpha}} duds.
\end{align*}
\]

The infinitesimal generator \(\mathcal{L}\) of Eq.(11) is as follows [1]:

\[
(\mathcal{L} f)(x) = \langle \partial_x f(x), b(x) \rangle + \frac{1}{2} \left( \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right) \sigma_{ik}(x) \sigma_{kj}(x) \\
+ \int_{\mathbb{R}^d \setminus \{0\}} (f(x + u) - f(x) - \langle \partial_x f(x), u \rangle) \frac{C_{\alpha d}}{|u|^{d+\alpha}} du.
\]

So,

\[
\left| \frac{\mathbb{E}[f(X_{\tau_\varepsilon}(x))] - f(x)}{\mathbb{E}[\tau_\varepsilon]} - (\mathcal{L} f)(x) \right| = \left| \frac{\mathbb{E} \int_0^{\tau_\varepsilon} (\mathcal{L} f)(X_s) ds}{\mathbb{E}[\tau_\varepsilon]} - \frac{\mathbb{E} \int_0^{\tau_\varepsilon} (\mathcal{L} f)(x) ds}{\mathbb{E}[\tau_\varepsilon]} \right| \\
\leq \frac{\mathbb{E} \int_0^{\tau_\varepsilon} |(\mathcal{L} f)(X_s) - (\mathcal{L} f)(x)| ds}{\mathbb{E}[\tau_\varepsilon]} \\
\leq \sup_{|y-x|<\varepsilon} |(\mathcal{L} f)(y) - (\mathcal{L} f)(x)|.
\]

Because \((\mathcal{L} f)(x)\) is continuous in \(x\),

\[
Af(x) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}[f(X_{\tau_\varepsilon}(x))] - f(x)}{\mathbb{E}[\tau_\varepsilon]} = (\mathcal{L} f)(x).
\]

Similarly, we also have \(A = \mathcal{L}\) for \(0 < \alpha < 1\).

**Remark 3.5.** The above deduction tells us \(Af = \mathcal{L} f\) for \(f \in C^2_0(\mathbb{R}^d)\). If the considered driving process is not a symmetric \(\alpha\)-stable Lévy motion, the domain of \(\mathcal{L}\) is unclear and thus \(A = \mathcal{L}\) may not be true. The corresponding escape probability \(p(x)\) is the solution of the following Balayage-Dirichlet problem (in terms of operator \(\mathcal{L}\), instead of \(A\)):

\[
\begin{align*}
\mathcal{L} p &= 0, \\
p|_{U} &= 1, \\
p|_{D^c \setminus U} &= 0.
\end{align*}
\]

4. **Examples**

In this section we consider a few examples.
Example 4.1. In 1-dimensional case, take $D = (-r, r)$ and $\Gamma = \{r\}$. For each $x \in D$, the escape probability $p(x)$ of $X_t = x + W_t$ from $D$ to $\Gamma$ satisfies the following differential equation

$$
\begin{cases}
\frac{1}{2}p''(x) = 0, & x \in (-r, r), \\
p(r) = 1, & \\
p(-r) = 0.
\end{cases}
$$

We obtain that $p(x) = \frac{x + r}{2r}$ for $x \in [-r, r]$. It is a straight line (See Figure 6).

In 2-dimensional case, take $D = \{x \in \mathbb{R}^2; r < |x| < R\}$ and $\Gamma = \{x \in \mathbb{R}^2; |x| = r\}$. For every $x \in D$, the escape probability $p(x)$ of $X_t = x + W_t$ from $D$ to $\Gamma$ satisfies the following elliptic partial differential equation

$$
\begin{cases}
\frac{1}{2}\Delta p(x) = 0, & x \in D, \\
p(x)|_{|x|=r} = 1, & \\
p(x)|_{|x|=R} = 0.
\end{cases}
$$

By solving this equation, we obtain that $p(x) = \frac{\log R - \log |x|}{\log R - \log r}$. It is plotted in Figure 7.

Example 4.2. Consider the following SDE driven by Brownian motions:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where $b$ and (non-zero) $\sigma$ are real functions. When $b$ and $\sigma$ satisfy $H_{b, \sigma}^1$, the equation has a unique solution which is denoted as $X_t$. We take $D = (-r, r)$ and $\Gamma = \{r\}$. For
each \( x \in D \), under the condition \((H_\sigma^2)\), the escape probability \( p(x) \) satisfies

\[
\begin{align*}
\frac{1}{2} \sigma^2(x) p''(x) + b(x) p'(x) &= 0, \quad x \in (-r, r), \\
p(r) &= 1, \\
p(-r) &= 0.
\end{align*}
\]

The solution is

\[
p(x) = \frac{\int_{-r}^{x} e^{-2 \int_{-r}^{y} \frac{b(z)}{\sigma^2(z)} dz} dy}{\int_{-r}^{r} e^{-2 \int_{-r}^{y} \frac{b(z)}{\sigma^2(z)} dz} dy}
\]

for \( x \in [-r, r] \). See Figure 8.

**Example 4.3.** In 1-dimensional case, take \( D = (-r, r) \) and \( U = [r, \infty) \). For each \( x \in D \) and a symmetric \( \alpha \)-stable Lévy process \( L^\alpha_t \), the escape probability \( p(x) \) of \( X_t = x + L^\alpha_t \) from \( D \) to \( U \) satisfies the following differential-integral equation

\[
\begin{align*}
-(\Delta)^\frac{\alpha}{2} p(x) &= 0, \quad x \in (-r, r), \\
p(x)|_{[r, \infty)} &= 1, \\
p(x)|_{(-\infty, -r]} &= 0.
\end{align*}
\]

It is difficult to deal with this equation because of the fractional Laplacian operator. But we can solve it via Poisson kernel. From [14], for \( x \in (-r, r) \),

\[
p(x) = \frac{\sin \frac{\pi \alpha}{2}}{\pi} \int_{r}^{\infty} \frac{(r^2 - x^2)^{\alpha/2}}{(y^2 - r^2)^{\alpha/2}} \frac{1}{(y - x)} dy.
\]
Figure 8. Escape probability in Example 4.2: $b(x) = -x, \sigma(x) = 1, r = 2$

Obviously, $p(-r) = 0$. To justify $p(r) = 1$, we apply the substitution $y = (r^2 - xv)(x - v)^{-1}$ to obtain

\[
p(r) = \sin \frac{\pi \alpha}{2} \int_{-r}^{r} (r - v)^{\alpha-1} (r^2 - v^2)^{-\frac{\alpha}{2}} dv
\]
\[
= \sin \frac{\pi \alpha}{2} \int_{0}^{1} (1 - v)^{\frac{\alpha}{2} - 1} v^{1 - \frac{\alpha}{2} - 1} dv
\]
\[
= \frac{\sin \frac{\pi \alpha}{2}}{\pi} B\left(\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right)
\]
\[
= \frac{\sin \frac{\pi \alpha}{2}}{\pi} \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1 - \frac{\alpha}{2}\right)
\]
\[
= 1,
\]

where the Beta and Gamma functions and their properties are used in the last two steps. The escape probability $p(x)$ is plotted in Figure 9 for various $\alpha$ values.

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Figure 9. Escape probability in Example 4.3: $r = 2$

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