SMALL BOUND FOR BIRATIONAL AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES
(WITH AN APPENDIX BY YUJIRO KAWAMATA)

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ABSTRACT. We give an effective upper bound of $|\text{Bir}(X)|$ for the birational automorphism group of an irregular $n$-fold (with $n = 3$) of general type in terms of the volume $V = V(X)$ under an "albanese smoothness and simplicity" condition. To be precise, $|\text{Bir}(X)| \leq d_3 V^{10}$. An optimum linear bound $|\text{Bir}(X)| \leq \frac{1}{3} \times 42^3 V$ is obtained for those 3-folds with non-maximal albanese dimension. For all $n \geq 3$, a bound $|\text{Bir}(X)| \leq d_n V^{10}$ is obtained when alb$_X$ is generically finite, alb$(X)$ is smooth and Alb$(X)$ is simple.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. Let $X$ be a normal projective $n$-fold of general type. $X$ is minimal if the canonical divisor $K_X$ is nef and $X$ has at worst terminal singularities (see Kawamata-Matsuda-Matsuki [31], Kollár-Mori [34]). It is known that $|\text{Aut}(X)| \leq 42 \deg(K_X)$ when $\dim X = 1$ (Hurwitz), and $|\text{Aut}(X)| \leq (42K_X)^2$ when $X$ is a minimal surface of general type (Xiao [49], [50]). See also works of Andreotti, Howard-Sommese [27], Huckleberry - Sauer [28], Corti [13] and probably many others.

For Gorenstein minimal $n$-folds $X$ of general type, Szabo [42] gave a polynomial upper bound of $|\text{Bir}(X)|$, though its degree is quite huge. See also works of Catanese - Schneider [9], Xiao [51], Cai [8] and Kovacs [35].

In this paper, we try to get a more realistic upper bound (of order) for the whole birational automorphism group Bir$(X)$. If fact, we consider irregular varieties $X$ and, without assuming the minimality of $X$, obtain a linear (resp. degree 10) bound in the volume $V(X)$ when $n = 3$ (resp. for all $n \geq 3$) if the albanese map is not generically finite onto a variety of general type (resp. is generically finite onto a smooth variety contained in the simple Albanese variety); for details, see below.

Our approach here of considering the albanese morphism alb$_X : X \to$ Alb$(X)$ is sort of the generalization of the idea as in Xiao [48] and Catanese-Schneider [9] to find a $G$-equivariant pencil. The universal property of Alb$(X)$ guarantees that any action of a group $G$ on $X$
induces a canonical action on $\text{Alb}(X)$ such that $\text{alb}_X$ is $G$-equivariant though $G$ might not act faithfully on the latter. When $\text{alb}_X$ is not generically finite onto a variety of general type, Ueno’s result \cite{ueno} says that $Y := \text{alb}_X(X)/B_0$ is of general type, where $B_0$ is the identity connected component of the subtorus of $\text{Alb}(X)$ stabilizing $\text{alb}(X)$ with respect to its translation action. So one can apply the weak positivity of the relative dualizing sheaf due to Fujita \cite{fujita}, Kawamata \cite{kawamata}, and Viehweg \cite{viehweg}, then use the volume $V(X)$ to give an optimum upper bound of $V(Y)V(F)$ and finally reduce to the cases of the fibre and base ($F$ being the general fibre of $X \to Y$).

The ”best” case where $\text{alb}_X$ is generically finite onto $W = \text{alb}_X(X)$ of general type, turns out to be the hardest one. It is supposed to be the ”best” situation because then $|6K_X|$ is birational according to Chen - Hacon \cite{chen-hacon} Corollary 5.3. It may then improve the coefficient in Szabo’s bound, but not the power of $V = V(X)$.

Therefore, we have to explore closely good properties of the abelian variety $\text{Alb}(X)$ as a torus. Certainly, after quotient away the translations, the image $\overline{G}$ of $G = \text{Aut}(X) \to \text{Aut}_{\text{variety}}(\text{Alb}(X))$ can be thought of as a subgroup of $\text{GL}_{2q}(\mathbb{Z})$ with $q = q(X) = \dim \text{Alb}(X)$, via the rational representation on the first integral homology of $\text{Alb}(X)$.

But Feit’s upper bound for the order of a finite subgroup in $\text{GL}_{2q}(\mathbb{Z})$ with $2q > 10$, is a huge number $(2q)!2^{2q}$ (see Friedland \cite{friedland}) and it is even attainable by the orthogonal group $O_{2q}(\mathbb{Z})$. Or one can apply Jordan’s Lemma \cite{jordan} below, together with Xiao’s linear bound for abelian subgroups, but then the constant $J_{2q}$ depends on $q$ and hence on the volume of $X$, rather than on the dimension of $X$.

So we have to catch the missing information of $X$ when passing to $\text{Alb}(X)$. To do so, we first give a linear bound of Betti numbers of $X$ in terms of the volume of $X$ in Theorem \ref{thm:betti} below, then bound the exponent $\exp(G)$ using the classical Lefschetz fixed point formula in Lemma \ref{lem:lfpf} below, and finally bound $|G|$ itself. In the process, one needs a technical finiteness condition on the fixed locus $\text{Alb}(X)^G$ as in Theorem \ref{thm:finite}. We remark that this condition is automatically satisfied when $\text{Alb}(X)$ is simple. Hopefully, one will be able to remove this restriction and eventually have the desired small bound for all irregular varieties.

One reason of our considering irregular varieties is that these ones are expected to have a bigger $\text{Aut}(X)$ compared with the general ones.

The advantage of using the albanese map for irregular varieties is that one could bound $\text{Aut}(X)$ inductively by reducing either to the cases of the fibre and base of a fibration, or to the case where $\text{alb}_X$ is generically finite onto a variety of general type (which is covered by Theorem \ref{thm:finite} to some extent), since $\text{alb}_X$ always carries an action of $G$ on $X$ over to actions on the base and fibre of $\text{alb}_X$. This method is applicable in all dimensions. See Theorem \ref{thm:inductive} below for details.
We now state the main results of the paper. When \( \text{alb}_X \) is not generically finite onto a variety of general type, one obtains the optimum linear bound; see the Remark in the Appendix; see \ref{1.7}.

**Theorem 1.1.** Let \( X \) be a smooth projective 3-fold of general type with irregularity \( q(X) \geq 4 \). Let \( V = V(X) \) be the volume of \( X \) and \( G = \text{Bir}(X) (\geq \text{Aut}(X)) \) the birational automorphism group of \( X \).

1. Suppose that \( \text{alb}_X \) is not of maximal albanese dimension, i.e., \( \dim \text{alb}_X(X) < \dim X \). Then \( |G| \leq \frac{1}{3} \times 42^3 V \).
2. More generally, suppose only that \( \text{alb}_X : X \to \text{Alb}(X) \) is not generically finite onto a 3-fold of general type in \( \text{Alb}(X) \), i.e., the Kodaira dimension \( \kappa(\text{alb}_X(X)) < \dim X \). Then \( |G| \leq \frac{1}{3} \times 42^3 V \).

When \( \text{alb}_X \) is generically finite onto a variety of general type, i.e., \( \kappa(\text{alb}_X(X)) = \dim X \), one has the following slightly bigger bound, but the degree of the bound is still a constant independent of \( \dim X \).

**Theorem 1.2.** Let \( X \) be a smooth projective \( n \)-fold (\( n \geq 3 \)) of general type. Let \( V = V(X) \) be the volume of \( X \) and \( G = \text{Bir}(X) \). Assume the following conditions:

(i) \( \text{alb}_X \) is generically finite onto \( W (= \text{alb}(X)) \) of general type,
(ii) \( W \) is smooth, and
(iii) either \( A := \text{Alb}(X) \) is a simple abelian variety; or \( G \) induces an action on \( A \) such that the fixed locus \( A^g \) for every \( \text{id} \neq g \in G \) is a non-empty finite set unless \( g \) is a translation of \( A \).

Then there is a constant \( d_n \) (indepedent of \( X \)) such that \( |G| \leq d_n V^{10} \).

Combining Theorems \ref{1.1} and \ref{1.2}, one obtains a reasonably small upper bound of \( |\text{Bir}(X)| \) for some irregular 3-folds of general type.

One ingredient for the proofs of results above is the following linear bound of Betti numbers in terms of the volume.

**Theorem 1.3.** Let \( X \) be a smooth projective \( n \)-fold with ample \( K_X \). Then there is a constant \( a_n \), independent of \( X \), such that the Betti numbers \( B_i(X) \leq a_n K_X^n \) for all \( i \).

**Remark 1.4.** (1) The constants \( d_n \) and \( a_n \) in the results above are computable from the proof. The main contributors towards them are some of the existing constants: \( x_n \) in Xiao’s Theorem \ref{2.11}, \( J_n \) the Jordan constant in Lemma \ref{2.7}, \( r_n \) of Angehrn - Siu in Lemma \ref{2.1} and \( h_n \) of Heier in Lemma \ref{4.2}.
(2) Note that the degree (= 10) of the polynomal in \( V(X) \) in Theorem \ref{1.2} is independent of the dimension \( n \) of \( X \). We propose the following, which is somewhat more general than the one in Xiao \ref{49} for smooth and minimal \( X \) and for \( \text{Aut}(X) \):
Conjecture 1.5. There is a constant $\delta_n$ such that the following holds for every smooth projective $n$-fold of general type, where $V = V(X)$ is the volume:

$$|\text{Bir}(X)| \leq \delta_n V.$$ 

When $n \leq 2$, Conjecture 1.5 is confirmed by Hurwitz, Xiao [49] and Xiao [50], and one can take $\delta_n = (42)^n$. Theorem 1.1 confirms the conjecture for $n = 3$ with some additional assumption. The result below is a further evidence for arbitrary dimension.

Theorem 1.6. Let $X$ and $Y$ be smooth projective varieties of general type and let $f : X \to Y$ be a surjective morphism with connected general fibre $F$. Suppose that $G = \text{Bir}(X)$ acts regularly and faithfully on $X$ and $G$ acts on $Y$ so that $f$ is $G$-equivariant.

If Conjecture 1.5 holds for both $Y$ and $F$, then it also holds for $X$.

1.7. Terminology and Notation.

For a normal projective variety $X$ of dimension $n$, we say that $X$ is minimal if the canonical divisor $K_X$ is nef and if $X$ has at worst terminal singularities; see [31], [34].

For a divisor $D$ on $X$, we define the volume of $D$ as $V(D) = \limsup_{s \to \infty} h^0(X, sD)/(s^n/n!)$; see [36]. $V(X) := V(K_X)$ is called the volume of $X$, a birational invariant. Note that $V(X) = K_X^n$ when $X$ is minimal.

Let $X$ be a variety and $G \leq \text{Aut}(X)$. Let $G_x = \{g \in G \mid g(x) = x\}$ be the stabilizer subgroup. For $H \leq G$, define the fixed locus $X^H := \{x \in X \mid h(x) = x \text{ for some } id \neq h \in H\}$. For $g \in G$, define the fixed locus $X^g := \{x \in X \mid g(x) = x\}$. Thus normally, $X^g \supseteq X^H$.

For an abelian variety $A$, denote by $\text{Aut}_{\text{variety}}(A)$ the set of automorphisms of $A$ as a variety, and $\text{Aut}_{\text{group}}(A)$ the subgroup of bijective homomorphisms.

When $H \leq G$ and $g \in G$, denote by $C_H(g) = \{h \in H \mid gh = hg\}$ the centralizer. Define the exponent $\exp(G) = \text{lcm}\{\text{ord}(g) \mid g \in G\}$.

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2. Preliminary results

We first show how $K_X^n$ controls (or is controlled by) other invariants linearly. For its generalization to arbitrary $n$, see Proposition 4.1. We
remark that in the assertion (4) below, one can take \( r = 2 + n(n+1)/2 \)
or bigger according to Angehrn-Siu [2], or Kollar [33] Theorem 5.8.

**Lemma 2.1.** Let \( X \) be a Gorenstein minimal projective \( n \)-fold \((n \geq 2)\)
of general type and \( f : X' \to X \) a resolution.

1. Suppose \( n = 2 \). Then \( q(X) \leq p_g(X) \leq \frac{1}{2} K_X^2 + 2 \), and \( K_X^2 \leq 9(1 + p_g(X)) \).

2. Suppose \( n = 3 \). Then \( p_g(X) \leq K_X^3 + 2 \) and \( q(X) \leq 2 + 50K_X^3 \). Also \( \chi(\mathcal{O}(K_X)) = f^*K_X.c_2(X')/24 \geq K_X^3/72 > 0 \) and

\[
P_m(X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 + (2m-1)\chi(\mathcal{O}(K_X)) \quad (m \geq 2).
\]

3. Suppose \( n = 3 \) and \( p_g(X) > 0 \). Then \( K_X^3 \leq 144p_g(X) \).

4. Suppose \( n \geq 2 \). Then for every \( r \geq 4 \) such that \(|rK_X|\) is base point free and \(|(r+1)K_X|\) is non-empty we have \( P_r(X) \leq n + r^nK^*_X/2 \).

**Proof.** (1) Now \( X \) is smooth for “terminal” means smooth in dimension 2. Note that \( 0 < \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) \) for surfaces of general type. Then the first two inequalities follow from the Noether inequality. The last one follows from the Miyaoka-Yau inequality in [33] Theorem 1.1 and [52]: \( K_X^2 \leq 3c_2(X) \) and the calculation: \( 1 + p_g(X) \geq \chi(\mathcal{O}_X) = (K_X^2 + c_2(X))/12 \geq K_X^3/9 \).

(2) By Chen [11] Theorem 3, we have \( p_g(X) \leq 2 + K_X^3 \). By Lee [17], \(|4K_X|\) is base point free and we take its general member \( S \) which is smooth (noting that \( \text{Sing}(X) \) is finite because \( X \) is terminal). Consider the exact sequence

\[
0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0.
\]

Note that \( H^i(X, \mathcal{O}_X(-S)) = 0 \) for \( i = 1, 2 \) by Kawamata-Viehweg vanishing. Taking cohomology of the exact sequence, we get \( q(X) = q(S) \). Now by (1), \( q(S) \leq p_g(S) \leq \frac{1}{2} K_S^2 + 2 \). Substituting in \( K_S = (K_X + S)|S = 5K_X|S \) and \( K_S^2 = 100K_X^3 \), we get the first part of (2).

The second part of (2) follows from the Riemann-Roch formula in Reid [11] and Miyaoka-Yau inequality.

(3) By the proof of Hacon [22] page 6, under the assumption that \( p_g(X) > 0 \), one has \( \chi(\omega_X) \leq 2p_g(X) \). Now (3) follows from (2).

(4) Consider first the case \( \dim X = 2 \). The system \(|rK_X|\) is base point free for all \( r \geq 4 \), by Bombieri [7]. By (1), \( \chi(\mathcal{O}_X) \leq 1 + p_g(X) \leq 3 + K_X^2/2 \). The pluri-genus formula says that \( P_r(X) = (r-1)K_X^2/2 + \chi(\mathcal{O}_X) \leq 3 + \frac{1}{2}(1 + r(r-1))K_X^2 \). One can verify that (4) is true.

Now assume that \( n \geq 3 \). Let \( X_{n-1} \) be a general member of \(|rK_X|\). By Kollar-Mori [23] Lemma 5.17, \( X_{n-1} \) is again terminal with \( K_{X_{n-1}} = (1+r)K_X|X_{n-1} \). Let \( X_i \) be the the intersection of \( n-i \) general members of \(|rK_X|\). Inductively, we see that each \( X_i \) is terminal with nef and big \( K_{X_i} = (1 + (n-i)r)K_X|X_i \). Also \( C = X_1 \) is a connected smooth curve; note that \( C \) is linearly equivalent to the nef and big divisor \( rK_X|X_2 \) on the terminal, which is smooth in dimension 2, surface \( X_2 \);
one may also refer to Hartshorne [25] Ch III, Exercise 11.3, for the connectedness of each \( X_i \). Inductively, we obtain \( h^0(X, rK_X) \leq 1 + h^0(X_{n-1}, rK_X|X_{n-1}) \leq \cdots \leq (n-1) + h^0(C, rK_X|C) \), by considering the exact sequence of cohomologies coming from

\[ 0 \to \mathcal{O}_{X_{i+1}} \to \mathcal{O}_{X_{i+1}}(rK_X|X_{i+1}) \to \mathcal{O}_{X_i}(rK_X|X_i) \to 0. \]

The divisor \( D = rK_X|C \) on \( C \) is special in the sense of Griffiths-Harris [12] page 251, because \( K_C - D = (1 + (n-2)r)K_X|C \geq (r + 1)K_X|C \geq 0 \). Applying Clifford’s Theorem [ibid], one has \( h^0(C, D) \leq 1 + \deg(D)/2 \). Substituting this and \( \deg(D) = rK_X\cdot(rK_X)^{n-1} = r^nK^n_X \) into the above, we obtain (4).

In the case of generically finite surjective map between varieties of general type, the volume upstairs can control the volume downstairs and sometimes even the degree of the map. See also Lemma 6.1.

**Lemma 2.2.** Let \( X \) and \( Y \) be smooth projective \( n \)-folds of general type and \( f : X \to Y \) a generically finite surjective morphism.

1. We have \( V(X) \geq \deg(f) V(Y) \).
2. Suppose further that \(|K_Y|\) defines a generically finite rational map \( \Phi \). Then \( V(Y) \geq \deg \Phi \deg \Phi(Y) \geq 1 \) and \( \deg(f) \leq V(X) \).

**Proof.** Let \( \varepsilon > 0 \). By Fujita’s approximation as in Lazarsfeld [36] Theorem 11.4.4, after smooth modifications of \( X \) and \( Y \), we may assume that \( K_Y \sim_{\mathbb{Q}} H + E \) with \( H \) an ample \( \mathbb{Q} \)-divisor and \( E \) an effective \( \mathbb{Q} \)-divisor, such that \( H^n = V(H) > V(Y) - \varepsilon \). By the ramification divisor formula, we have \( V(X) \geq V(f^*H) = (f^*H)^n = \deg(f)H^n > \deg(f)(V(Y) - \varepsilon) \).

Now (1) follows by letting \( \varepsilon \) tend to zero.

The first inequality in (2) is by the proof of Hacon-McKernan [22] Lemma 2.2; the rest now follows.

The assertion (1) below means that the volume controls the order of a free-acting group, while (2) says that sometimes \( |G| \) can be bounded by its exponent \( \exp(G) \). We denote by cartier\((Y)\) the Cartier index, i.e., the smallest positive integer such that \( cK_Y \) is a Cartier divisor.

**Lemma 2.3.** Let \( X \) be a smooth minimal projective \( n \)-fold of general type and \( f : X \to Y \) a finite surjective morphism.

1. Suppose that \( f \) is unramified. Then \( \deg(f) = K^n_X/\text{cartier}(Y) \leq K^n_X \). In particular, if a subgroup \( H \leq \text{Aut}(X) \) acts freely on \( X \), then \( |H| = K^n_X/\text{cartier}(Y) \leq K^n_X \), where \( Y = X/H \).
2. Suppose that a subgroup \( G \leq \text{Aut}(X) \) has isolated fixed locus \( X^G \).

Let \( f : X \to Y := X/G \) be the quotient map and \( c = \text{cartier}(Y) \) the Cartier index. Then \( c|\exp(G) \) and \( |G| \leq cK^n_X \leq \exp(G) K^n_X \).

**Proof.** We note that \( K_X \) is nef and big and that \( K^n_X \) is a positive integer. (1) follows from the fact that \( K_X = f^*K_Y \).
(2) Denote by \( G_x = \{ g \in G \mid g(x) = x \} \) the stabilizer subgroup at the point \( x \in X \). We may regard \( G_x \) as a subgroup of \( \text{GL}_n(T_{X,x}) \cong \text{GL}_n(C) \). For the image \( \overline{\tau} \in Y \) of \( x \), the local Cartier index \( c(\overline{\tau}) \) equals the index of the quotient group \( G_x/G_x \cap \text{SL}_n(C) \subset \text{GL}_n(C)/\text{SL}_n(C) \cong C^* \), which is cyclic and generated by the image \( \overline{g} \) of some \( g \in G_x \). Thus \( c(\overline{\tau}) \) divides \( \exp(G_x) \) and also \( \exp(G) \). Hence \( c = \text{cartier}(Y) \), which is the lcm of \( c(\overline{\tau}) \), also divides \( \exp(G) \). Since \( cK_Y \) is Cartier, it can be written as \( \text{H}_1 - \text{H}_2 \) with very ample divisors \( H_j \) not passing through the isolated set \( \text{Sing}(Y) \) (the image of \( X^{G} \)). Then \( cK_Y^n = (\text{H}_1 - \text{H}_2).K_Y^{n-1} = (K_Y|H_1)^{n-1} - (K_Y|H_2)^{n-1} \) is an integer. Since \( f : X \to Y \) is etale outside the finite set \( \text{Sing}(Y) \), one has \( K_X = f^*K_Y \). Hence \( |G| = \deg(f) = K_X^n/K_Y^n = cK_X^n/cK_Y^n \leq cK_X^n \). □

An application of the result below is given in Remark 1.4. We remark that the finiteness assumption below on \( X^g \) can be removed for surfaces as shown in Ueno [44].

**Lemma 2.4.** Suppose that \( X \) is a smooth projective \( n \)-fold and \( g \) is an automorphism of finite order. Assume that \( X^g \) is finite.

1. The Euler number of the fixed locus \( X^g \) is given by \( e(X^g) = \sum_{i=0}^{2n} (-1)^i \text{Tr} g^*H^i(X,C)/(\text{torsion}) \).
2. One has \( |e(X^g)| \leq \sum_{i=0}^{2n} B_i(X) \), where \( B_i(X) \) is the Betti number.

**Proof.** (1) is the classical Lefschetz fixed point formula. For (2), let \( \text{ord}(g) = m \) and diagonalize \( g^*H^i(X,C) = \text{diag}[\zeta_{m_1}^{b_1}, \ldots, \zeta_{m_k}^{b_k}] \) where \( b_i = B_i(X) \) and \( \zeta_{m_i} \) is an \( m \)-th root of 1. Thus \( |\text{Tr} g^*H^i(X,C)/(\text{torsion})| = |\sum_{j=1}^{b_i} \zeta_{m_i}^{b_j}| \leq b_i \). This proves the lemma. □

The following is the relation between two stabilizers \( G_x \) and \( (G/H)_{\overline{\tau}} \).

**Lemma 2.5.** Suppose that a normal subgroup \( H \leq G \) acts freely on a variety \( X \). Set \( \overline{G} = G/H \) and \( \overline{X} = X/H \).

1. For every \( x \in X \) and its image \( \overline{\tau} \in \overline{X} \), the homomorphism between stabilizers \( \varphi : G_x \to G_{\overline{\tau}} \) which is given by \( g \mapsto \overline{g} = gH \), is an isomorphism.
2. In particular, the fixed locus \( X^G \) is empty (resp. finite) if and only if \( \overline{X}^{\overline{G}} \) is empty (resp. finite).

**Proof.** To show that \( \varphi \) is surjective, suppose that \( \overline{g}(\overline{\tau}) = \overline{\tau} \). Then \( g(x) = h(x) \) for some \( h \in H \). Thus \( h^{-1}g \in G_x \) and \( \overline{g} = \varphi(h^{-1}g) \). To show that \( \text{Ker}(\varphi) \) is trivial, suppose that \( \varphi(g) = \text{id} \in \overline{G}_{\overline{\tau}} \leq \overline{G} \). So \( g \in H \). Hence \( g \in H \cap G_x = \{ \text{id} \} \) by the freeness of the \( H \) action on \( X \). So \( \text{Ker}(\varphi) \) is trivial and the lemma is proved. □

Next are about nilpotent groups and relation between centralizers.

**Lemma 2.6.** Let \( G \) be a finite group.

1. If \( G \) is nilpotent, then \( \exp(G) \) equals \( \mu(G) := \max \{ \text{ord}(g) \mid g \in G \} \).
(2) Consider quotient groups inclusion \( K/H \trianglelefteq G/H \). Let \( \tau \in K \) and \( \overline{\tau} = \tau H \in G/H \). Then \(|C_{K/H}(\overline{\tau})| \leq |C_K(\tau)| \leq |C_G(\tau)|\).

Proof. (1) is true for \( p \)-groups while \( G \), being nilpotent, is a direct product of its Sylow subgroups (see Gorenstein [18] Theorem 3.5). So (1) follows.

(2) Write \( C_{K/H}(\tau) = L/H \). Consider the map: \( \phi : L \to \tau H \) \( (g \mapsto g^{-1}\tau g) \). Then \(|L/C_L(\tau)| = |\text{Im}(\phi)| \leq |H| \), so \(|L/H| \leq |C_L(\tau)|\). □

When \( \text{Aut}(X) \) fixes a point \( x \), Jordan’s lemma and its consequences below, together with Xiao’s Theorem 2.11, will produce a linear bound \(|\text{Aut}(X)| \leq J_n x_n K^n_X\).

Lemma 2.7. The following are true.

(1) (Jordan’s lemma) Let \( J_n = (n+2)! \) (resp. \( J_n = n^4(n+2)! \)) for \( n > 63 \) (resp. for \( n \leq 63 \)). Then every finite group in \( \text{GL}_n(\mathbb{C}) \) contains an abelian normal subgroup of index \( \leq J_n \).

(2) Suppose that \( X \) is an \( n \)-fold and \( G \leq \text{Aut}(X) \) is a finite subgroup fixing a smooth point \( x \in X \). Then \( G \) contains an abelian normal subgroup of index \( \leq J_n \).

(3) Suppose that \( X \) is a projective \( n \)-fold of general type and \( x \in X \) a smooth point. If \( G \leq \text{Aut}(X) \) fixes \( x \), then \( G \) contains an abelian normal subgroup of index \( \leq J_n \).

Proof. For (1), see Weisfeiler [47] page 5279 (for better bound) and Aljadeff- Sonn [3] page 353. The original number produced by Jordan was \( J(n) = (49n)^{n^2} \). For (2), \( G \) can be regarded as a subgroup of \( \text{GL}_n(T_{X,x}) \cong \text{GL}_n(\mathbb{C}) \), so apply (1). For (3), \( \text{Aut}(X) \) and hence \( G \) are finite since \( X \) is of general type. Then apply (2). □

The result below has been proved by many authors. We only mention that the nefness of \( \Omega^1_X \) comes from the exact sequence below and the fact that quotient bundles of a nef bundle are again nef (see Hartshorne [23] Ch II, Theorem 8.17; Lazarsfeld [36] Proposition 6.1.2):

\[
0 \to N^\vee_{X/A} \to \Omega^1_A|X \to \Omega^1_X \to 0.
\]

Theorem 2.8. Let \( A \) be an abelian variety and \( X \subset A \) a subvariety of general type.

(1) \( |K_X^\natural| \) defines a generically finite map, where \( \hat{X} \to X \) is any desingularization.

(2) Suppose that \( X \) is smooth. Then \( \Omega^1_X \) is nef and \( K_X \) is ample.

Moreover, \( |K_X^\natural| \) is base point free and \( \Phi|_{|K_X^\natural|} \) is a finite morphism.

Proof. For the proof, see Ran [40] Corollary 2, or Abramovich [1]. See also Hartshorne [23], Ueno [43] and Griffiths-Harris [20] for earlier development. □

Here are some properties of simple abelian varieties to be used in §6.
Lemma 2.9. Let \( A \) be a simple abelian variety.

1. Every proper subvariety of \( A \) is of general type.
2. If \( \text{id} \neq g \in \text{Aut}_{\text{variety}}(A) \) is not a translation, then \( A^g \) is a non-empty finite set.
3. If \( X \to A \) is a non-constant morphism from a projective variety \( X \), then \( q(\tilde{X}) \geq \dim(A) \), where \( \tilde{X} \to X \) is any desingularization.

Proof. (1) is just Ueno \[43\] Corollary 10.10.

(2) Suppose that \( g \) is not a translation. By Birkenhake-Lange \[6\] Lemma 13.1.1, we may assume that \( \text{id} \neq g \in \text{Aut}_{\text{group}}(A) \). Now \( \text{Ker}(g-\text{id}) \) is a proper subgroup of \( A \), whence it is non-empty and 0-dimensional by the simplicity of \( A \). So (2) follows (see \[6\] Formula 13.1.2).

(3) By the universal property of \( \text{Alb}(\tilde{X}) \), the non-constant morphism \( \tilde{X} \to A \) factors as \( \tilde{X} \to \text{Alb}(\tilde{X}) \to A \), where the latter map is a homomorphism (modulo a translation) and has image the translation of a non-trivial subtorus of \( A \). This image is \( A \) by the simplicity of \( A \). This implies (3) : \( \dim(A) \leq \dim \text{Alb}(\tilde{X}) = q(\tilde{X}) \). \(\square\)

We end §2 with Xiao’s linear bound for abelian subgroups:

Theorem 2.11. (Xiao \[51\] Theorem 1) Let \( X \) be a smooth projective \( n \)-fold with nef and big \( K_X \). Let \( G \leq \text{Aut}(X) \) be an abelian subgroup. Then there exists a constant \( x_n \), independent of \( X \), such that \( |G| \leq x_n K^n_X \).

3. \( G \)-equivariant fibrations; the proof of Theorem 1.6

In this section we consider \( G \)-equivariant fibration \( f : X \to Y \). We will prove simultaneously Theorem 3.1 below and Theorem 1.6 in the Introduction.

Theorem 3.1. Let \( X \) and \( Y \) be smooth projective varieties of general type and dimensions \( n \) and \( k \), respectively. Suppose that a group \( G \) acts faithfully on \( X \) and acts anyhow on \( Y \). Let \( f : X \to Y \) be a \( G \)-equivariant surjective morphism with connected general fibre \( F \). Suppose further that \( |\text{Bir}(Y)| \leq a_1 V(Y)^d \) and \( |\text{Bir}(F)| \leq a_2 V(F)^d \) for some positive integers \( a_i \) and \( d \).

Then we have:

\[ |G| \leq a V(X)^d, \quad \text{with} \quad a = \frac{a_1 a_2}{\left(\binom{n}{k}ight)^d}. \]
We now prove first Theorem 3.1 and late Theorem 1.6 as an application. Note that the general fibre $F$ is also of general type since the Kodaira dimension $\kappa(X) = \dim X$ and by Iitaka's easy addition: $\kappa(X) \leq \kappa(F) + \dim Y$. Since $f$ is $G$-equivariant, we have the natural exact sequence:

$$1 \to K \to G \to \text{Aut}(Y).$$

The group $K$ acts trivially on $Y$ and hence faithfully on the fibre $F$, so $K$ can be regarded as a subgroup of $\text{Aut}(F)$, where $F$ is smooth and of general type. Now Theorem 3.1 follows from the Theorem in the Appendix:

$$|G| \leq |K| |\text{Aut}(Y)| \leq |\text{Aut}(Y)| |\text{Aut}(F)| \leq$$

$$|\text{Bir}(Y)| |\text{Bir}(F)| \leq a_1a_2 V(Y)^d V(F)^d \leq a V(X)^d.$$

Finally, Theorem 1.6 is a consequence of Theorem 3.1 and the inductive assumption, where we set $a_1 = \delta_{\dim Y}$, $a_2 = \delta_{\dim F}$ and $d = 1$.

**Remark 3.2.** Weaker versions of Theorems 1.6 and 3.1 can also be proved using Lemma 2.1, Kollár [32] Theorem 3.5 (ii) and Catanese-Schneider [9] pages 10-11, which are strengthened to the current form, thanks to the Appendix - an application of the weak positivity due to Fujita [10], Kawamata [30] and Viehweg [40].

### 4. Bound of Betti numbers; the proof of Theorem 1.3

In this section, for a smooth projective variety $X$, we shall bound the invariants like Betti numbers $B_i(X)$ and the Euler number $e(X)$, in terms of the volume of $X$. We first bound the irregularity and geometric genus.

**Proposition 4.1.** Let $X$ be a smooth projective $n$-fold ($n \geq 2$) with ample $K_X$. Then $p_g(X) \leq e_n K_X^n$ and $q(X) \leq e_n K_X^n$, where $e_n = \max(n + r^n, 2 + \frac{1}{2}r^{n-2}(1 + (n-2)r)^2)$ with $r = 2 + n(n+1)/2$.

**Proof.** For $r = 2 + n(n+1)/2$, as in Lemma 2.1, $|rK_X|$ is base point free. Since $K_X$ is ample, $\Phi = \Phi_{rK_X} : X \to \mathbb{P}^N$ with $N = P_r(X) - 1$, is a finite morphism. Hence $(rK_X)^n = \deg(\Phi) \deg \Phi(X) \geq \deg \Phi(X) \geq N - n + 1$ (see Griffiths-Harris [19] page 173). Thus $p_g(X) \leq P_r(X) = N + 1 \leq n + r^n K_X^n \leq (n + r^n) K_X^n$.

As in Lemma 2.1, take the ladder $X = X_n \supset X_{n-1} \supset \cdots \supset X_2$, where $X_i$ is the intersection of $n-i$ general members of $|rK_X|$, and a smooth $i$-fold with ample $K_{X_i} = (1 + (n-i)r)K_X|X_i$. Each $X_{i-1}$ is a smooth ample divisor on $X_i$. By Lefschetz hyperplane section theorem, we have $q(X) = q(X_{n-1}) = \cdots = q(X_2)$. By Lemma 2.1, we conclude the proposition: $q(X_2) \leq \frac{1}{2}K_{X_2}^2 + 2 = 2 + \frac{1}{2}r^{n-2}(1 + (n-2)r)^2 K_X^n$. \(\square\)

To bound the Betti numbers, we need the bound of Euler number $e(X)$ below.
Lemma 4.2. Let $X$ be a smooth projective $n$-fold.

(1) If the cotangent bundle $\Omega^1_X$ is nef, then $0 \leq (-1)^ne(X) = c_n(\Omega^1_X) \leq K^n_X$.

(2) If $K_X$ is ample then $|e(X)| \leq h_nK^n_X$, where $h_n$ is the constant, independent of $X$, in Heier \cite{Heier} Proposition 1.7.

Proof. (2) is proved in Heier \cite{Heier} Proposition 1.7. For (1), by the nefness of $\Omega^1_X$ and Demailly-Peternell-Schneider \cite{DPS} page 31, we have $0 \leq c_n(\Omega^1_X) \leq c_1(\Omega^1_X)^n = K^n_X$. Note also that $c_n(\Omega^1_X) = (-1)^nc_n(T_X) = (-1)^nc_n(X) = (-1)^ne(X)$, where $T_X$ is the tangent bundle; see for instance, Fulton \cite{Fulton} Example 3.2.13. This proves the lemma. \hfill \Box

4.3. Proof of Theorem 1.3

We will prove by induction on $n$. The case $n = 1$ is clear. When $n = 2$, it follows from Lemma 2.1 or 4.1. Indeed, note that $B_1(X) = 2q(X)$, $\chi(O_X) \leq 1 + p_g(X)$, $c_2(X) \leq K^2_X + c_2(X) = 12\chi(O_X)$ and $B_2(X) = c_2(X) - 2 + 4q(X)$.

Suppose $n \geq 3$ and the theorem is true for all such $k$-folds with $2 \leq k < n$. Let $X_{n-1}$ be a general member of $|rK_X|$ with $r = 2 + n(n+1)/2$ (see Lemma 2.1). Note that $K_{X_{n-1}} = (1+r)K_X|X_{n-1}$ is ample and $K^{n-1}_{X_{n-1}} = r(1+r)^{n-1}K^n_X$. By induction, $B_i(X_{n-1}) \leq a_{n-1}K^{n-1}_{X_{n-1}} = a_nK^n_X$ where $a_n = a_{n-1}r(1+r)^{n-1}$. By Lefschetz hyperplane section theorem, $B_i(X) = B_i(X_{n-1})$ for all $i \leq n - 2$ and $B_{n-1}(X) \leq B_{n-1}(X_{n-1})$ (see Lazarsfeld \cite{Lazarsfeld} Theorem 3.1.17). Note also that $B_j(X) = B_{2n-j}(X)$ by Lefschetz duality. Thus $B_j(X) \leq B_j(X_{n-1}) \leq a_nK^n_X$ for all $j \neq n$.

On the other hand, by Lemma 4.2, $h_nK^n_X \geq |\sum_{i=0}^{2n}(-1)^iB_i(X)|$. So $|B_n(X)| \leq h_nK^n_X + |\sum_{i \neq n}(-1)^iB_i(X)| \leq a_nK^n_X$ where $a_n = h_n + 2na_n$. We are done by the induction. This proves Theorem 1.3.

Remark 4.4. For the $X$ in Theorem 1.3 it might be possible to give a second proof of Xiao’s linear bound (in terms of $K^n_X$) of $\text{ord}(g)$ for every $g \in \text{Aut}(X)$ with $\text{ord}(g)$ prime and $X^g$ finite. Indeed, note that the quotient map $X' = X - X^g \rightarrow X'/\langle g \rangle$ is unramified, and hence $\text{ord}(g)$ divides the Euler number $e = e(X) - e(X^g)$, so $e$ is bounded by $|e(X)| + |e(X^g)|$ (provided that $e \neq 0$) while the latter is bounded linearly by $K^n_X$ (see Lemmas 2.1 and 4.2 and Theorem 1.3).

5. Automorphism groups of subvarieties of abelian varieties

In this section, we shall prove the following two results:

Theorem 5.1. Let $X$ be a smooth $n$-fold ($n \geq 3$) of general type contained in an abelian variety $A$ of dimension $q$. Let $G$ be a subgroup of $\{g \in \text{Aut}_\text{variety}(A) \mid g(X) = X\}$ such that the fixed locus $A^g$ for every $g \neq 0$ is a non-empty finite set unless $g$ is a translation of $A$. Set $V = K^n_X$ (see Theorem 2.3).
Then there is a constant $d_n$, independent of $X$ and $A$, such that $|G| \leq d_n q^{10-b} V^b \leq d_n S^{10}$, where $S = \max\{V, q\}$ and $5 \leq b \leq 10$.

**Corollary 5.2.** Let $A$ be a simple abelian variety and $X \subset A$ a smooth $n$-dimensional ($n \geq 3$) proper subvariety. Let $G = \{g \in \text{Aut}_\text{variety}(A) \mid g(X) = X\}$ be the stabilizer. Set $V = K^n_X$ (see Theorem 2.8). Then there is a constant $d''_n$ (independent of $X$ and $A$) such that $|G| \leq d''_n V^{10}$.

We fix an abelian variety $A$ of dimension $q$ and an $n$-dimensional smooth subvariety $X \subset A$ of general type. Let $G$ be a subgroup of the group $\{g \in \text{Aut}_\text{variety}(A) \mid g(X) = X\}$. Note that $\text{Aut}_\text{variety}(A) = T \rtimes A_0$ (split extension) where $T = \{T_t \mid t \in A\}$ is the normal subgroup of translations and $A_0 = \text{Aut}_\text{group}(A)$ is the subgroup of bijective group-homomorphisms.

5.3. **Assumption.**

(i) $X \subset A$ is a smooth projective $n$-fold of general type in the abelian variety $A$ of dimension $q$,

(ii) the subgroup $G \subseteq \{g \in \text{Aut}_\text{variety}(A) \mid g(X) = X\}$ contains no translations of $A$ (we note that $G \leq \text{Aut}(X)$ is finite because $X$ is of general type), and

(iii) the fixed locus $A^g$ for every id $\neq g \in G$ is a non-empty finite set.

We collect some information on the structure of $G$.

**Lemma 5.4.** Suppose that $G \subseteq \text{Aut}(X)$ and $X \subset A$ satisfy conditions in 5.3. Let $id \neq g \in G$. Then we have:

1. Let $G_0$ be the image of the composition $G \subseteq \text{Aut}_\text{variety}(A) \to A_0$. Then $G \to G_0$ is an isomorphism.
2. For $g \in G$ and $g_0$ its image in $G_0$, we have $|A^g| = |A^{g_0}| \geq |\{0\}| = 1$.
3. If $H \leq G$ is abelian, then $H$ is cyclic.
4. For $H \leq G$, if $|H| = p^2$ for some prime $p$, then $H$ is cyclic.
5. If $|X^g| \geq 2$ for some $g \in G$, then $\text{ord}(g) = p^s$ for some prime $p$.
6. If $g$ has order $m$ in $G$ and if $\zeta_m$ denotes a primitive $m$-th root of 1, then there is a diagonalization $g^*|H^0(A, \Omega^1_A) = \text{diag}[\zeta_m^{\alpha_1}, \ldots, \zeta_m^{\alpha_n}]$ where each $\zeta_m^{\alpha_i}$ is a primitive $m$-th root of 1.
7. If $g \in G$, then the Euler function $\varphi(\text{ord}(g))$ divides $2q$.
8. If $p^2$ divides $|G|$ for some prime $p$, then there is an element $g \in G$ such that $\text{ord}(g) = p^2$.
9. If $g \in G$, then $\text{ord}(g)$ divides $4q^2 \alpha(g)$, where
$$\alpha(g) = \prod_{\text{prime } p \mid \text{ord}(g), p^2 \mid |G|} p.$$  

10. If $H \leq G$, then the exponent $\exp(H)$ divides $4q^2 \alpha(H)$, where
$$\alpha(H) = \prod_{\text{prime } p \mid |H|, p^2 \mid |G|} p.$$
Every prime factor \(\exp(C_{13.2.2}, 13.2.1\) and \(13.2.5\). Our (4) is because a group of order \(p^2\) is abelian and by (3). Now (5) follows from (1) and Birkenhake - Lange Corollary 13.2.4. Also (8) is a consequence of (4) together with Sylow's theorem and a basic result on \(p\)-groups. For (9), write \(\text{ord}(g) = p_1^{t_1}p_2^{t_2}\ldots p_u^{t_u}\) with \(t_i \geq 1\). Then \(\varphi(\text{ord}(g)) = p_1^{t_1-1}(p_1 - 1)\ldots p_u^{t_u-1}(p_u - 1)\). If \(t_1 \geq 2\), then \(2(t_1 - 1) \geq t_1\), and \(p_1^{t_1}\) divides \((\varphi(\text{ord}(g)))^2\) as well as \((2q)^3\) by (7). If \(p^2 \mid |G|\), then \(p \mid 2q\) by (8) and (7). Thus \(\text{ord}(g) \mid 4q^2\alpha(g)\) and (9) is proved. Our (9) implies (10). This proves the lemma.

The centralizer lemma below is a key to the proof of Theorem 5.1.

Lemma 5.5. (centralizer lemma) Suppose that \(G \subseteq \text{Aut}(X)\) and \(X \subset A\) satisfy conditions in 2.5. Let \(\tau \in G\) be of prime order \(p\) such that the fixed locus \(X^\tau\) is non-empty. Let \(V = K^n_X\) (see Theorem 2.8).

(1) Every prime factor \(p_1 \neq p\) of the order \(|C_G(\tau)|\) of the centralizer, divides \(|X^\tau|(|X^\tau| - 1)\).

(2) \(\exp(C_G(\tau))\) divides \(4p^\varepsilon q^2|X^\tau|(|X^\tau| - 1)\), where \(\varepsilon = 0\) (resp. 1) when \(p^2\) divides \(|G|\) (resp. otherwise).

(3) One has \(|C_G(\tau)| \leq k_nq^2V^{3+\varepsilon}\), where \(k_n = \max\{J_n(x_n, 4a_n^2\varepsilon)\}\); see Lemma 2.7 and Theorems 1.6 and 2.11.

Proof. (1) Suppose \(g \in C_G(\tau)\) has order equal to a prime \(p_1 \neq p\). Write \(|X^\tau| \equiv r\) with \(0 \leq r < p_1\). If \(r = 0, 1\), then we are done. Suppose that \(r \geq 2\). Then \(|X^\tau g| \geq r \geq 2\). By Lemma 5.4 \(pp_1 = \text{ord}(\tau g)\) equals a prime power, a contradiction.

(2) follows from (1) and Lemma 5.4 (10).

(3) If \(|X^\tau| = 1\), then \(C_G(\tau)\) fixes the unique point \(x\) in \(X^\tau\). Hence \(C_G(\tau) \leq \text{GL}_n(T_{X,x})\) and we apply Lemma 2.7 and Theorem 2.11 and obtain \(|C_G(\tau)| \leq J_n(x_nK^n_X)\). Suppose now that \(|X^\tau| \geq 2\). By (2), Lemmas 2.3 and 2.3 and Theorem 1.3, we have \(|C_G(\tau)| \leq \exp(C_G(\tau)V \leq 4p^\varepsilon q^2|X^\tau|(|X^\tau| - 1)V \leq 4p^\varepsilon q^2(\sum_{i=0}^{2n}B_i(X))^2V \leq 4a_n^2p^\varepsilon q^2V^3\). By Theorem 2.11 \(p \leq x_nV\) and hence (3) follows. This proves the lemma.

Theorem 5.1 should follow from the crucial proposition below.

Proposition 5.6. Suppose that \(X \subset A\) and \(G \subseteq \text{Aut}(X)\) satisfy the conditions in 2.5. Let \(q = \dim(A)\) and \(V = K^n_X\) (see Theorem 2.8). Then the there is a constant \(d_n\), independent of \(X\) and \(A\), such that \(|G| \leq d_nq^{10-b}V^b \leq d_nS^{10}\), where \(S = \max\{V, q\}\) and \(5 \leq b \leq 10\).

For the proof, we argue closely along the lines of Huckleberry-Sauer and Szabo instead. Let \(H \leq G\) be a maximum among normal subgroups of \(G\) which acts freely on \(X\). Take a normal subgroup \(K/H \leq G/H\). Fix some \(\tau \in K\) such that the fixed locus \(X^\tau \neq \emptyset\). We may assume that \(\text{ord}(\tau) = \text{ord}(\tau) = p\) is a prime (see 5.3 (iii) and Lemma 2.5).
Claim 5.7. Let $G_1 = C_G(\tau)$. Then $|G| \leq |G_1|||K||$.

Proof. Note that $\varphi : G \to K (g \mapsto g^{-1}\tau g)$ is a well-defined map. Then the claim follows from that $|G/G_1| = |\varphi(G)| \leq |K|$. □

Suppose that we can choose $K/H$ to be abelian. Note that $K/H$ acts on the smooth minimal $n$-fold $X/H$ of general type. By Theorem 2.11 we have $|K/H| \leq x_nK^n_{X/H} = x_nV/|H|$. Thus by Claim 5.7 and Lemmas 5.4 and 2.3 we have $|G| \leq |G_1||K/H||H| \leq k_nq^{2V^3+\varepsilon}x_nV \leq d_nq^2V^5$ with $d_n = k_nx_n$.

We may assume that there is no such abelian $K/H$. Then $G/H$ is semi-simple, i.e., it has no non-trivial abelian normal subgroup. Let $M/H$ be a non-trivial minimal normal subgroup. Then $M/H = \prod E/H$, a $k$-fold product of the same non-abelian simple group $E/H$. Let $M_i/H$ be all distinct non-trivial minimal normal subgroups of $G/H$ and let $S/H = \prod M_i/H$ be the socle of $G/H$. We write $M_i/H = \prod_{i=1}^k E_i/H$ with non-abelian simple group $E_i/H$.

Suppose that $M/H = \prod_{i=1}^k E_i/H$ with $k \geq 2$. Then $\alpha(M)||H|$ is of Lie type. By the proof of Huckleberry-Sauer Proposition 7, there exist a universal constant $d$ and a Sylow $p_1$-subgroup $U/H$ of $M/H$ such that $|M/H| \leq d|U/H|^{5/2}$. Note that $U/H$ acts on the smooth minimal $n$-fold $X/H$ of general type and the fixed locus $(X/H)^{U/H}$ is finite by 5.3 (iii) and Lemma 2.3. By Lemmas 2.3 and 2.6 and Theorem 2.11 we have $|U/H| \leq \exp(U/H)k^\alpha_{X/H} = \mu(U/H)V/|H| \leq x_nK^n_{X/H}V/|H| = x_n(V/|H|)^2$. Thus $|M/H| \leq dx_n^{5/2}(V/|H|)^5$. By Lemma 5.5 we have $|G| \leq |G_1||M| \leq k_nq^{2V^3+\varepsilon}dx_n^{5/2}V^5 = d_nq^{2V^8+\varepsilon}$ with $d_n = k_ndx_n^{5/2}$. If $p^2||G|$ then $\varepsilon = 0$ and we are done.

If the fixed locus $X^U = \emptyset$, then $|U| \leq V$ by Lemma 2.3 and we will even have a better bound. We may assume that the fixed locus $X^U \neq \emptyset$ (equivalently $(X/H)^{U/H} \neq \emptyset$ by Lemma 2.3). Then our initial $\tau$ (of order $p$) can be taken from $U/H$ so that $p_1 = p$ and $U/H$ is a $p$-Sylow subgroup of $M/H$. If $p^2$ divides $|U/H|$ then it also divides $|G|$, whence $\varepsilon = 0$ and we are done. Otherwise, $|U/H| = p \leq x_nK^n_{X/H} = x_nV/|H|$ by Theorem 2.11 and we will be done again.
We are left with the case where each $M_i/H$ is an alternating group $A_{k_i}$. Suppose that $M/H = A_k$ is the smallest among them. If there are two $M_i/H$ then we will be done as in the case $M/H = \prod_k E_i/H$ with $k \geq 2$ because the fact that $|M/H|$ divides $|M_i/H|$ for two $i$ implies that $\alpha(M) \mid |H|$ as well.

Therefore, we assume that the sockel $S/H = M/H = A_k$. Then as noticed by Huckleberry and Sauer, the conjugation map induces an injection $1 \to G/H \to \text{Aut}(S/H)$. Hence we have the following, where the first factor 2 is needed only when $k = 6$ (so that $\text{Aut}(A_6) / A_6 = (\mathbb{Z}/2)^{\oplus 2}$; see Atlas [12]):

$$|G/H| \leq |\text{Aut}(S/H)| \leq 2 \times 2 |A_k| = 4|M/H|.$$  

First, treat the case $k \leq 58$. Then $|G| \leq 4|A_k||H| \leq 4|A_k|V \leq 4(58!)V$ by Lemma 2.3 and we are done. Or as noticed by Szabo [12], $M/H$ contains a Sylow $p_1$-subgroup $U/H$ such that (as above) $|M/H| \leq |U/H|\leq (x_n(V/H))^{\beta} \leq (x_7(V/H)^2)^5$, whence $|G| \leq d_nV^{10}$ with $d_n = 4x_n^5$.

We next deal with the case $k > 59$. We use the approach of Szabo [12], but we apply the centralizer Lemma 2.10 instead. As in Lemma 2.10, set $\ell = [k/4]$ and we have $A_\ell \times A_{3\ell} \leq A_k \leq A_{4\ell+3}$.

Suppose that the subgroup $A_\ell < A_k = M/H$ acts freely on $X/H$. Then $A_\ell < G$ (with $A_\ell / H = A_\ell$) also acts freely on $X$ by Lemma 2.3, whence $|A_\ell| \leq K^n_X = V$ by Lemma 2.3. By Lemma 2.10 $|A_k| \leq |A_\ell|^8 \leq (V/|H|)^8$, so $|G| = |G/H||H| \leq 4|A_k||H| \leq 4V^8$.

Suppose that $A_\ell$ does not act freely on $X/H$. We may take $\overline{\tau} \in G/H$ to be from $A_\ell$ and $\tau \in M$ with $\text{ord}(\tau) = p = \text{ord}(\overline{\tau})$. Now $A_{3\ell} \leq C_{M/H}(\overline{\tau}) \leq C_{G/H}(\overline{\tau})$. This and Lemma 2.3 imply $|A_{3\ell}| \leq |C_G(\tau)| = |G_1|$. By Lemmas 2.10, 2.3 and 5.3 $|G| \leq 4|A_k||H| \leq 4|A_{3\ell}| |H| \leq 4|G_1| \leq 4(k_n^2V^{3+\epsilon})^{1.7} \leq 4(k_n^{3.4}V^{6.1+1.7\epsilon})^{1.7} = d_n^{4k_n^{1.7}}$. Since $p$ divides $|A_\ell|$, our $p^2$ divides $A_k = M/H$ and also $|G|$. So $\epsilon = 0$ and we are done. This completes the proof of Proposition 5.6.

### 5.8. Proof of Theorem 5.1

Since $X$ is of general type, $G \leq \text{Aut}(X)$ is finite. Write $\text{Aut}_{\text{variety}}(A) = T \times A_0$ as at the beginning of the section. Set $T_G = T \cap G$ which acts freely on $A$ and $X$. If $G = T_G$, then $|G| \leq K^n_X$ by Lemma 2.3 and we are done. So assume that $T_G < G$. Note that $A \to A/T_G$ is an isogeny of abelian varieties, $X \to X/T_G$ is etale and $G/T_G \leq \{g \in \text{Aut}_{\text{variety}}(A/T_G) \mid g(X/T_G) = X/T_G\}$. We shall check the conditions in 5.3 for $G/T_G \leq \text{Aut}(X/T_G)$ and $X/T_G \subset A/T_G$. For every $g \in G \setminus T_G$, we have $A^g \neq 0$ by the assumption on $G$, so $(A/T_G)^g \neq 0$ by Lemma 2.5 and the assumption on $G$. By Proposition 5.6 we have $|G/T_G| \leq d_n(q(A/T_G)^{10-b}(K^n_X/T_G))^b = d_nq^{10-b}(V/|T_G|)^b$. Now the theorem follows because $b \geq 5$. This proves Theorem 5.1.
5.9. Proof of Corollary 5.2

By Lemma 2.9, \( X \) is of general type and the fixed locus \( A^g \) for every \( \text{id} \neq g \in G \) is a non-empty finite set unless \( g \) is a translation of \( A \). Thus we can apply Theorem 5.1. Now Corollary 5.2 with \( d'_n = d_ne^{5n} \), follows from Theorem 5.1, Lemma 2.9, Proposition 4.1 and Theorem 2.8.

6. Proofs of Theorems 1.1 and 1.2

In this section we shall prove Theorems 1.1 and 1.2. We start with an observation which bounds \( \deg(\text{alb}_X) \) in terms of the volume of \( X \). Note that \( \text{Alb}(X) = \text{Alb}(X') \) if \( X \) and \( X' \) are smooth and birational. The lemma below follows from Lemma 2.2 and Theorem 2.8 (1) after resolving singularities of \( W \), since the volumes are birational invariants.

Lemma 6.1. Let \( X \) be a smooth projective \( n \)-fold. Suppose that \( \text{alb}_X : X \to W := \text{alb}(X) \subset \text{Alb}(X) \) is generically finite onto \( W \), and \( W \) is of general type. Then \( V(X) \geq \deg(\text{alb}_X) V(W) \geq \deg(\text{alb}_X) \).

6.2. Proof of Theorem 1.2

Let \( X \) and \( G = \text{Bir}(X) \) be as in Theorem 1.2. By Hanamura [23], Lemma 2.4, after a smooth modification of \( X \), we may assume that \( G \) acts regularly on \( X \). The universal property of \( \text{Alb}(X) \) implies that there is an induced action (not necessarily faithful) of \( G \) on \( \text{Alb}(X) \) such that \( \text{alb}_X : X \to \text{Alb}(X) \) is \( G \)-equivariant. Let \( K \) (resp. \( \overline{G} \)) be the kernel (resp. image) of the homomorphism \( G = \text{Aut}(X) \to \text{Aut}_{\text{variety}}(\text{Alb}(X)) \) so that we have the exact sequence

\[
1 \to K \to G \to \overline{G} \to 1.
\]

Since \( K \) acts trivially on \( W = \text{alb}_X(X) \), we can factor \( \text{alb}_X \) as \( X \to X/K \to W \). In particular, \( |K| \leq \deg(\text{alb}_X) \). By Lemma 2.9 and the assumption in Theorem 1.2, the fixed locus \( \text{Alb}(X)^\overline{G} \) for every \( \text{id} \neq \overline{g} \in \overline{G} \) is a non-empty finite set unless \( \overline{g} \) is a translation of \( \text{Alb}(X) \). Thus, by Theorem 2.8 (2), we can apply Proposition 4.1 and Theorem 5.1 to \( W \). Setting \( V = V(X) \) and \( V(W) = K^n_W \) and noting that \( q := q(W) = q(X) = \dim \text{Alb}(X) \) by the universal property and definition of \( \text{Alb}(X) \), one obtains the inequalities below with \( d''_n = d_ne^{10-b} \):

\[
|\overline{G}| \leq d_nq^{10-b}V(W)^b \leq d''_nV(W)^{10}.
\]

Now by Lemma 6.1 or 2.2 we conclude Theorem 1.2

\[
|G| = |K||\overline{G}| \leq \deg(\text{alb}_X)|\overline{G}| \leq d''_nV \cdot V(W)^9 \leq d''_nV^{10}.
\]

6.3. Proof of Theorem 1.1

Theorem 1.1 (1) is a special case of Theorem 1.2 (2). Therefore, we have only to show Theorem 1.1 (2).
So suppose that \( \text{alb}_X \) is not generically finite onto a 3-fold of general type. Set \( W = \text{alb}_X(X) \subset \text{Alb}(X) \). Then either \( \dim W < 3 \), or \( \kappa(W) < \dim W \leq 3 \). As above, after a smooth modification of \( X \), we may assume that \( G = \text{Bir}(X) \) acts regularly on \( X \) and of course on \( \text{Alb}(X) \) (not necessarily faithful on the latter) so that \( \text{alb}_X : X \to \text{Alb}(X) \) is \( G \)-equivariant. Since \( \dim \text{Alb}(X) = q(X) \geq 4 > \dim X \geq \dim W \) by the assumption, our \( W \) is a proper subvariety of \( \text{Alb}(X) \). Hence by Ueno [43] Lemma 9.14 and Corollary 10.4, the Kodaira dimension \( \kappa(W) \geq 1 \). As in Ueno [43] Theorem 10.9 or Mori [39] Theorem 3.7, let \( B \) be the identity connected component of \( \{ a \in \text{Alb}(X) \mid a + W \subseteq W \} \). Then \( W/B \) is of general type, \( W \to W/B \) is an etale fibre bundle with fibre \( B \) and is birational to the Iitaka fibring of \( W \). Since the pluri-canonical systems of \( W \) are \( G \)-stable, we may take \( G \)-equivariant desingularizations \( X' \to X \) and \( Y' \to W/B \) such that \( X' \to Y' \) is a well-defined \( G \)-equivariant morphism, though \( G \) might not act faithfully on \( Y' \). Rewrite \( X' \) as \( X \). Take a Stein factorization \( X \to Y \to Y' \) so that \( f : X \to Y \) has connected general fibre \( F \) and is necessarily \( G \)-equivariant. We may also assume that \( Y \) is already smooth (or do equivariant modifications again). Note that \( G = \text{Bir}(X) = \text{Aut}(X) \) now. We have \( \dim Y = \dim Y' = \kappa(W) = 1, 2 \). Since the subvariety \( W/B < \text{Alb}(X)/B \) is of general type, both \( Y \) and \( Y' \) are of general type. Apply Theorem 3.1 to the \( G \)-equivariant map \( f : X \to Y \) with connected general fibre \( F \) say. Note that \( 3 = \dim X = \dim Y + \dim F \) and \( |\text{Bir}(Z)| = |\text{Aut}(Z_{\text{min}})| \leq (42)^{\dim Z}V(Z) \) for both \( Z = Y \) and \( Z = F \), thanks to Hurwitz and Xiao. Here \( Z_{\text{min}} \) is the unique smooth minimal model of \( Z \), since \( \dim Z \leq 2 \). By Theorem 3.1 we have \( |G| \leq a V(X) \) with \( a = (42)^3/3 \). This proves Theorem 1.1 (2).

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A PRODUCT FORMULA FOR VOLUMES OF VARIETIES

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The volume $v(X)$ of a smooth projective variety $X$ is defined by

$$v(X) = \lim \sup \frac{\dim H^0(X, mK_X)}{m^d/d!}$$

where $d = \dim X$. This is a birational invariant.

**Theorem 7.1.** Let $f : X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Assume that both $Y$ and the general fiber $F$ of $f$ are varieties of general type. Then

$$\frac{v(X)}{d_X!} \geq \frac{v(Y) v(F)}{d_Y! d_F!}$$

where $d_X = \dim X$, $d_Y = \dim Y$ and $d_F = \dim F$.

**Proof.** Let $H$ be an ample divisor on $Y$. There exists a positive integer $m_0$ such that $m_0 K_Y - H$ is effective.

Let $\epsilon$ be a positive integer. By Fujita’s approximation theorem ([1]), after replacing a birational model of $X$, there exists a positive integer $m_1$ and ample divisors $L$ on $F$ such that $m_1 K_F - L$ is effective and $v(\frac{1}{m_1} L) > v(F) - \epsilon$.

By Viehweg’s weak positivity theorem ([2]), there exists a positive integer $k$ such that $S^k(f_* \mathcal{O}_X(m_1 K_{X/Y}) \otimes \mathcal{O}_Y(H))$ is generically generated by global sections for a positive integer $k$. $k$ is a function on $H$ and $m_1$.

We have

$$\text{rank } \text{Im}(S^m S^k(f_* \mathcal{O}_X(m_1 K_{X/Y}) \to f_* \mathcal{O}_X(km_1 m K_{X/Y})))$$

$$\geq \dim H^0(F, kmL)$$

$$\geq (v(F) - 2\epsilon) \frac{(km_1 m)^{d_F}}{d_F!}$$

for sufficiently large $m$. 


Then
\[
\dim H^0(X, km_1mK_X) \\
\geq \dim H^0(Y, k(m_1 - m_0)mK_Y) \times (v(F) - 2\epsilon) \frac{(km_1m)^{d_F}}{d_F!}
\]
\[
\geq (v(Y) - \epsilon) \frac{(k(m_1 - m_0)m)^{d_Y}}{d_Y!} (v(F) - 2\epsilon) \frac{(km_1m)^{d_F}}{d_F!}
\]
\[
\geq (v(Y) - 2\epsilon)(v(F) - 2\epsilon) \frac{(km_1m)^{d_Y}}{d_Y!d_F!}
\]
if we take \(m_1\) large compared with \(m_0\) such that
\[
\frac{(v(Y) - \epsilon)}{(v(Y) - 2\epsilon)} \geq \left( \frac{m_1}{m_1 - m_0} \right)^{d_Y}.
\]

\[\square\]

Remark 7.2. If \(X = Y \times F\), then we have an equality in the formula. We expect that the equality implies the isotriviality of the family.

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