Towards exact solutions of electron-phonon interaction in metals

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Electron-phonon interaction (EPI) plays a fundamental role in metals. It affects various physical quantities and can induce superconductivity and other instabilities. Previous theoretical studies on EPI are largely based on Migdal-Eliashberg formalism, which neglects all the vertex corrections and is valid only for weak EPI. Here, we go beyond Migdal-Eliashberg formalism and develop an efficient Dyson-Schwinger equation approach to investigate EPI. Remarkably, we prove that the fully renormalized electron and phonon propagators can be entirely determined by skilfully using the symmetry-imposed constraints on correlation functions. In particular, the full vertex corrections are incorporated self-consistently via the longitudinal and transverse Ward-Takahashi identities derived from a global U(1) symmetry without discarding any Feynman diagram. Our approach is non-perturbative and valid for both weak and strong EPIs. As an application, we study high-$T_c$ superconductivity mediated by interfacial optical phonon in one-unit-cell FeSe/SrTiO$_3$ and compute $T_c$ with unprecedented precision.

I. INTRODUCTION

In crystalline solids, atoms are arranged in a highly ordered pattern, forming periodic lattices. Due to thermal and quantum fluctuations, atoms vibrate around equilibrium positions. Phonons are the quanta of such vibration modes. For metals, electrons get scattered by phonons as they travel inside metals. The electron-phonon interaction (EPI) plays a major role in metals [1–3], and affect all the thermodynamical, spectral, and transport properties. EPI is also the key ingredient of Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [4–6], and triggers Cooper pairing in a large number of superconductors. Metals cannot be thoroughly understood without a detailed knowledge about EPI.

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From theoretical perspective, EPI is not easy to handle. In the absence of mutual interaction, electrons and phonons propagate freely, described by free propagators $G_0(p)$ and $F_0(q)$, respectively. As interaction is tuned on, electrons and phonons influence each other dramatically, and their behaviors are described by fully dressed propagators $G(p)$ and $F(q)$. The standard approach to EPI is weak-coupling perturbation \cite{1, 3}. At the leading order, $G(p)$ and $F(q)$ are computed by using free propagators $G_0(p)$ and $F_0(q)$. Apparently, this approximation is oversimplified since electrons and phonons are tightly coupled and never free. Such a crude approximation can be more or less improved by employing the Migdal-Eliashberg (ME) theory \cite{1–8}, which computes $G(p)$ in a self-consistent way by using $F_0(q)$. The validity of ME formalism is crucially based on Migdal theorem, which states that all the quantum corrections to EPI vertex function $\Gamma_{\nu}(q, p)$ are suppressed by small factor $\omega_D/E_F$, where $\omega_D$ is phonon frequency and $E_F$ is Fermi energy, and thus can be entirely ignored.

In the past sixty years, ME formalism has been widely adopted to study EPI in various metals and superconductors \cite{1–3}. However, it has long been known that Migdal theorem is not always valid \cite{9–11}. There exist many classes of realistic systems in which the ratio $\omega_D/E_F$ is not small and traditional ME theory breaks down. Notable examples include low carrier-density superconductors such as SrTiO$_3$ \cite{12, 13} and Moiré superconductor \cite{14, 15}, cuprate superconductors \cite{7, 16}, and one-unit-cell (1UC) FeSe/SrTiO$_3$ system \cite{17–20}. The ME results are especially unreliable for strong EPIs. This fact has been discussed \cite{9–11} for decades, and was recently re-confirmed by a determinant quantum Monte Carlo (DQMC) study \cite{11}. In order to acquire a quantitatively more reliable knowledge of EPIs, it is necessary to go beyond ME theory and develop a more generic approach that can incorporate all the omitted contributions and is applicable to strong EPIs.

Dyson-Schwinger (DS) equations treat interacting electrons and phonons on an equal footing in the outset, and thus might provide a more powerful tool than traditional weak-coupling perturbation. Within this framework, all the $n$-point correlation functions ($n=2$ for propagators) are self-consistently connected \cite{21, 22}, constituting an infinite number of equations. Unfortunately, these equations are usually not closed. This seriously hinders their applicability to real systems. To make DS equations closed, one might invoke a hard truncation (e.g., choosing special Feynman diagrams), or introduce an Ansatz for the vertex. But such treatments are based on unjustified assumptions and cannot be really trusted.

Here, we go beyond ME theory and develop an efficient DS equation approach to accurately treat EPI. Remarkably, we will prove that the self-consistent DS equations are indeed closed and fully solvable. The exact DS equations for renormalized electron and phonon propagators, repre-
sented by $G(p)$ and $F(q)$ respectively, can be written down without making any approximation or ignoring any Feynman diagram. This is achieved by properly utilizing the symmetry-imposed constraints on various correlation functions. Using our approach, one can start from free electrons and free phonons and entirely determine $G(p)$ by solving its self-consistent integral equation. Then the obtained $G(p)$ can be inserted into an integral to compute $F(q)$. Once $G(p)$ and $F(p)$ are known, they can be used to calculate various observable quantities. Our approach is universal and applicable to metals defined in any spatial dimension. Moreover, our approach does not involve any small expansion parameter and hence works equally well for weak and strong EPIs.

The most important advance achieved in our approach is that the full vertex function $\Gamma_v(q,p)$ for EPI is completely determined by the Ward-Takahashi identities (WTIs) derived from the U(1) symmetry of the system. Notice that the ordinary longitudinal WTI by itself is not sufficient to uniquely specify $\Gamma_v(q,p)$. We derive a new transverse WTI, which has never been revealed previously, within the functional integral framework, and then combine both the longitudinal and transverse WTIs to express $\Gamma_v(q,p)$ purely in terms of electron propagators. After doing so, we find that the DS equation of electron propagator $G(p)$ is made self-closed and decoupled from the DS equations of all the other correlation functions.

As a direct application of our approach, we investigate the high-$T_c$ superconductivity induced by interfacial optical phonons (IOPs) in FeSe/SrTiO$_3$ system, and compute $T_c$ with an unprecedented precision. It is shown that neglecting vertex corrections significantly underestimates the value of $T_c$. This result would help ascertain whether the coupling of electrons in FeSe film to IOPs by itself is able to produce the observed high $T_c$.

II. MODEL AND DYSON-SCHWINGER EQUATION ANALYSIS

To illustrate our approach, let us first define several quantities. In quantum many-particle theory, one studies various $n$-point correlation functions $\langle O_1 O_2...O_n \rangle$, where $O$’s are Heisenberg operators and $\langle ... \rangle$ stands for taking mean value on the ground state of total Hamiltonian. To handle Cooper pairing, we define Nambu spinor $\Psi^\dagger(p) = (\psi^\dagger_\uparrow(p), \psi^\dagger_\downarrow(-p))$ and then write down the Lagrangian density [9]

$$\mathcal{L} = \Psi^\dagger(p)(\epsilon\sigma_0 - \xi_p\sigma_3)\Psi(p) + \frac{1}{2}\phi^\dagger(q)F_{0}^{-1}(q)\phi(q) - g\phi(q)\Psi^\dagger(p+q)\sigma_3\Psi(p), \quad (1)$$

where $\xi_p = \frac{p^2}{2m}$ is kinetic energy of non-relativistic electrons, $g$ is coupling constant for EPI. $\sigma_{1,2,3}$ are standard Pauli matrices, and $\sigma_0$ denotes unit $2 \times 2$ matrix. Shorthand notations are $p \equiv (\epsilon, \mathbf{p})$,
\[ q \equiv (\omega, \mathbf{q}), \text{ and } z \equiv (t, \mathbf{z}). \] Theoretical description of EPI involves three quantities: full electron propagator \( G(p) = -i\langle \Psi \Psi^\dagger \rangle \), full phonon propagator \( F(q) = -i\langle \phi \phi^\dagger \rangle \), and full electron-phonon vertex function \( \Gamma_\nu(q,p) \) that is defined by the relation \( F(q)G(p+q)\Gamma_\nu(q,p)G(p) = \langle \phi \Psi \Psi^\dagger \rangle \). In the non-interacting limit, \( G(p) \rightarrow G_0(p) = \frac{1}{\omega_\sigma - \xi_\sigma \mathbf{q}} \), and \( F(q) \rightarrow F_0(q) = \frac{2\eta_\sigma}{\omega_\sigma - \Omega_\mathbf{q}} \). Here, \( F_0(q) \) can be identified as the Fourier transformed expression of \( \mathbb{D}^{-1} \), where \( \mathbb{D} = -(\partial_t^2 + \Omega_\mathbf{q}) \) is the dynamical operator for phonon field \( \phi \) satisfying \( \mathbb{D}\phi(t,\mathbf{q}) = 0 \) for free phonons. The phonon dispersion \( \Omega_\mathbf{q} \) is strongly material dependent and can be computed numerically. Our basic results are independent of the concrete expression of \( \Omega_\mathbf{q} \), and even independent of whether \( \phi \) is phonon or other types of bosonic mode.

The framework of functional integral will be adopted throughout the paper. To generate various \( n \)-point correlation functions, we introduce three external sources \( \eta^\dagger, \eta, \text{ and } J \) for the field operators \( \Psi, \Psi^\dagger, \text{ and } \phi \), respectively, and then write the partition function in the form \( Z(\eta^\dagger, \eta, J) = \int D\Psi D\Psi^\dagger D\phi \exp \left[ i \int (L + \eta^\dagger \Psi + \Psi^\dagger \eta + J \phi) \right] \). From \( Z \), one can derive the following DS equations \( [21, 22] \)

\[
G^{-1}(p) = G_0^{-1}(p) - ig \int_q \sigma_3 G(p + q)F(q)\Gamma_\nu(q,p), \tag{2}
\]

\[
F^{-1}(q) = F_0^{-1}(q) + ig \int_p \text{Tr} [\sigma_3 G(p + q)\Gamma_\nu(q,p)G(p)], \tag{3}
\]

\[
\Gamma_\nu(q,p) = g\sigma_3 - \int_{p'} G(p' + q)\Gamma_\nu(q,p')G(p')K_4(p,p',q). \tag{4}
\]

Here, integration over \( d \)-dimensional energy-momenta is abbreviated as \( \int_q \equiv \frac{d^d q}{(2\pi)^d} \). \( K_4(p,p',q) \) denotes the kernel function defined via 4-point correlation function \( \langle \phi \phi^\dagger \Psi \Psi^\dagger \rangle \). These equations are formally exact and contain all the effects caused by EPI. But they appear to be too complicated to handle. Furthermore, they are not closed, because \( K_4(p,p',q) \) satisfies an equation that is coupled to 5-, 6-, and higher-points correlation functions. Indeed, there are an infinite number of coupled equations, making it hard to extract reliable solutions.

ME theory assumes that \( F(q) \rightarrow F_0(q) \) and \( \Gamma_\nu(q,p) \rightarrow g\sigma_3 \). Then one is left with only the DS equation of \( G(p) \). Taking \( \Gamma_\nu(q,p) = g\sigma_3 \) is reliable when \( \omega_D/E_F \) is small enough. However, as aforementioned, this condition is not satisfied in many important cases. A frequently used method to include vertex corrections is to perturbatively compute a special class of diagrams. This method has several drawbacks. Computing such diagrams is normally difficult and becomes out of control in strong-coupling regime. The feedback effects of \( G(p) \) and \( F(q) \) on \( \Gamma_\nu(q,p) \) are difficult to examine. What is worse, this method is ad hoc if there is no guarantee that the chosen class of diagrams dominate over the omitted ones.
Here, we are not intended to compute specific Feynman diagrams. Motivated by the studies of quantum gauge field theories \cite{26,28}, we will resort to generic field-theoretic analysis and manage to unveil the intrinsic relations between different correlation functions based on symmetry considerations. The model has a global U(1) symmetry: $\mathcal{L}$ is invariant under the gauge transformation $\Psi \to e^{i\alpha \sigma_3} \Psi$ with $\alpha$ being an arbitrary constant. This symmetry implies the conservation of electric charge, and, according to Noether’s theorem, induces a conserved current $j_\mu \equiv (j_t, j)$, where

$$j_t = \Psi^\dagger \sigma_3 \Psi, \quad j = \frac{1}{2m}[(\nabla \Psi^\dagger)\sigma_0 \Psi - \Psi^\dagger \sigma_0 (\nabla \Psi)].$$

(5)

This current $j_\mu$ can be used to define a vector function $\Gamma_\mu \equiv (\Gamma_t, \Gamma)$ via

$$\langle j_\mu(z)\Psi(z_1)\Psi^\dagger(z_2) \rangle = -\int dz_3 dz_4 G(z_1, z_3) \Gamma_\mu(z, z_3, z_4) G(z_4, z_2).$$

(6)

where $z = (t, \mathbf{z})$ is $d$-dimensional coordinate and $dz_3 = dt d^{d-1}\mathbf{z}_3$.

$Z$ should be invariant under infinitesimal variation of phonon field $\delta \phi$, i.e.,

$$\int \mathcal{D} \Psi^\dagger \mathcal{D} \phi \frac{\delta}{\delta \phi} \exp[i \int \mathcal{L}] = 0.$$  

(7)

This then leads to $g(\Psi^\dagger \sigma_3 \Psi) = \langle \mathcal{D} \phi \rangle$. Performing functional derivative of both sides with respect to $\eta$ and $\eta^\dagger$ in order, one can verify that

$$g(\langle j_t \Psi \Psi^\dagger \rangle) = \langle \mathcal{D} \phi \Psi \Psi^\dagger \rangle.$$  

(8)

In momentum space, the l.h.s. of this relation is $gG(p + q)\Gamma_t(q, p)G(p)$ and the r.h.s. is $F_0^{-1}(q)F(q)G(p + q)\Gamma_v(q, p)G(p)$, which gives rise to an identity \cite{26}

$$gF_0(q)\Gamma_t(q, p) = F(q)\Gamma_v(q, p).$$

(9)

This identity is of ultimate importance in that it can be used to greatly simplify DS equations. Indeed, inserting this identity into Eq. (2), we get

$$G^{-1}(p) = G_0^{-1}(p) + ig^2 \int_q \sigma_3 G(p + q)F_0(q)\Gamma_t(q, p).$$

(10)

Now, it is the free phonon propagator $F_0(q)$, rather than the full one $F(q)$, that enters into the DS equation for $G(p)$. The next step would be to determine $\Gamma_t(q, p)$.

In the absence of external sources, Noether’s theorem indicates that $\partial_\mu j_\mu = 0$. In the presence of external sources, $\partial_\mu j_\mu$ does not vanish. Instead, current $j_\mu$ satisfies a more generic Slavnov-Taylor (ST) identity

$$\text{Tr} \left[ \sigma_m (i\sigma_0 \partial_t - \sigma_3 \xi_{\partial x}) + (i\sigma_0 \partial_t + \sigma_3 \xi_{\partial x}) \sigma_m \right] \langle \Psi(z)\Psi^\dagger(z_3) \rangle = -\int g(\phi(z_1)\Psi(z_3)\langle \sigma_m \sigma_3 - \sigma_3 \sigma_m \rangle \Psi(z) + \langle \Psi^\dagger(z_3)\rangle \sigma_m \eta(z) - \eta^\dagger(z_3)\sigma_m \langle \Psi(z) \rangle. \right]$$
The derivation of this identity is presented in Appendix A. Here, we have introduced an arbitrary 2 × 2 matrix \( \sigma_m \), which might be \( \sigma_0 \) or any Pauli matrix \( \sigma_{1,2,3} \).

We first assume \( \sigma_m = \sigma_3 \). Then it is easy to see that the first term of the r.h.s. of ST identity vanishes. After performing functional derivative of both sides of the ST identity with respect to \( \eta \) and \( \eta^\dagger \) in order, followed by Fourier transformation, we convert the ST identity into

\[
\omega \Gamma_t(q,p) - (\xi_{p+q} - \xi_p) \Gamma_s(q,p) = G^{-1}(p + q)\sigma_3 - \sigma_3 G^{-1}(p),
\]

where the function \( \Gamma_s(q,p) \) is connected to \( \Gamma(q,p) \) via the relation \( \Gamma(q,p) = \frac{2p + q}{2m} \Gamma_s(q,p) \). This is the well-known ordinary WTI [6, 9], which is said to be longitudinal because it is derived by taking the divergence of the d-dimensional current, namely \( \partial \cdot j \equiv \partial \mu j_\mu = \partial_t j_t + \nabla \cdot j \), and then replacing \( j_\mu \) with \( \partial \mu j_\mu \) in Eq. (1). While this WTI is absolutely exact and should be satisfied by any physically reliable result, it alone cannot uniquely determine \( \Gamma_t(q,p) \). This is because there are two unknown independent functions, i.e., \( \Gamma_t(q,p) \) and \( \Gamma_s(q,p) \). To determine two independent functions, we need one more constraint. Fortunately, the U(1) symmetry and the rich field-theoretic structure do give us one more intrinsic constraint on \( \Gamma_t(q,p) \) and \( \Gamma_s(q,p) \).

Notice that the conserved current \( j_\mu \) is defined via two matrices \( \sigma_3 \) and \( \sigma_0 \), corresponding to \( \Gamma_t \) and \( \Gamma_s \) respectively. If we assume \( \sigma_m = \sigma_0 \) and then substitute it into the ST identity, it would be easy to derive the following new identity

\[
\omega \Gamma_s(q,p) - (\xi_{p+q} - \xi_p) \Gamma_t(q,p) = G^{-1}(p + q)\sigma_0 - \sigma_0 G^{-1}(p).
\]

We call this identity the transverse WTI. This can be understood as follows. The term \( \omega \Gamma_s(q,p) \) is derived by taking the time derivative of the spatial part of current \( j_\mu \), whereas the term \( (\xi_{p+q} - \xi_p) \Gamma_t(q,p) \) comes from applying the Laplace operator to the time-component of \( j_\mu \). To the best of our knowledge, this transverse WTI has not been reported previously.

The longitudinal WTI (12) is coupled to the transverse WTI (13) in a self-consistent way, allowing one to unambiguously determine both \( \Gamma_t(q,p) \) and \( \Gamma_s(q,p) \). This seems to be analogous to the determination of electric and magnetic fields by solving the coupled Maxwell equations. \( \Gamma_s(q,p) \) seems to be useless (since EPI is described by \( \phi \Psi^\dagger \sigma_3 \Psi \)). A remarkable achievement is that \( \Gamma_t(q,p) \) is now expressed purely in terms of full electron propagators \( G(p + q) \) and \( G(p) \):

\[
\Gamma_t(q,p) = \frac{\omega [G^{-1}(p + q)\sigma_3 - \sigma_3 G^{-1}(p)] + (\xi_{p+q} - \xi_p)[G^{-1}(p + q)\sigma_0 - \sigma_0 G^{-1}(p)]}{\omega^2 - (\xi_{p+q} - \xi_p)^2}.
\]

Inserting this \( \Gamma_t(q,p) \) into Eq. (10), one can see that the DS equation of \( G(p) \) no longer couples to the DS equations of other correlation functions and becomes entirely self-closed, which is the main new result of the present work.
Generically, the full electron propagator $G(p)$ can be formally written as

$$G^{-1}(\epsilon, p) = A_1(\epsilon, p)\epsilon\sigma_0 - A_2(\epsilon, p)\xi_p\sigma_3 + \Delta_1(\epsilon, p)\sigma_1 + \Delta_2(\epsilon, p)\sigma_2,$$

(15)

where $A_1(\epsilon, p)$ is mass renormalization function, $A_2(\epsilon, p)$ is chemical potential renormalization, and $\Delta_{1,2}(\epsilon, p)$ are two pairing functions. Upon inserting Eq. (15) into Eq. (10), the DS equation for $G(p)$ can be readily decomposed into four self-consistent equations for $A_{1,2}(\epsilon, p)$ and $\Delta_{1,2}(\epsilon, p)$, which are amenable to numerical studies. The function $\Delta_{1,2}(\epsilon, p)$ vanishes in the normal state, and acquires nonzero values in the superconducting state. Thus, the DS equation of $G(p)$ can be applied to investigate metallic and superconducting states as well as the transition between them.

After solving the equations of $A_{1,2}(\epsilon, p)$ and $\Delta_{1,2}(\epsilon, p)$, the solutions can be inserted to the DS equation for phonon propagator $F(q)$. Using the previously derived identities, we obtain

$$F(q) = F_0(q) + g^2F_0^2(q)\int_p \text{Tr}[\sigma_3G(p+q)\Gamma_t(q,p)G(p)].$$

(16)

Since $\Gamma_t$ is expressed in terms of $G(p)$ and $G(p+q)$, one can extract the full information about phonons by directly integrate over $p \equiv (\epsilon, p)$, which is quite easy comparing to solving self-consistent integral equations.

The above WTI identities and DS equations are derived by carrying out generic field-theoretical calculations, and thus are essentially non-perturbative. No small expansion parameter is employed in the derivation. Comparing to traditional perturbative expansion method, the biggest advantage of our approach is that we do not discard any physical process. This guarantees that all the results are valid for any value of $\lambda_m$, the dimensionless coupling constant for EPI, and can be used to access the strong EPI regime. In contrast, the ME equations are derived based on perturbative expansion in powers of small $\lambda_m\omega_D/E_F$, and thus are invalid for large $\lambda_m$, especially when $\omega_D/E_F \approx 1$.

Different from quantum Monte Carlo simulation, our approach is not plagued with fermion sign problem and also free of finite-size effects. To what extend the results about $G(p)$ and $F(q)$ are exact is solely determined by the errors generated in numerical integration, which can be gradually reduced by costing reasonably more computer resources.

### III. HIGH-$T_c$ SUPERCONDUCTIVITY DUE TO OPTICAL PHONONS

Our DS equation approach is applicable to any metal hosting EPI. To examine its efficiency, we now apply it to a concrete example. Here we choose to study 1UC FeSe/SrTiO$_3$.

Bulk FeSe has $T_c \approx 8$K [20]. When 1UC FeSe is placed on SrTiO$_3$ substrate [17], its $T_c$ is dramatically promoted. This discovery has opened a new route to engineering interfacial high-$T_c$
superconductors. An important issue is to determine what mechanism causes such a high $T_c$. It is revealed that, although charge carrier doping and K-intercalation also enhance $T_c$, $T_c$ could be higher than 70K only when 1UC FeSe is at interface to SrTiO$_3$ or other similar substrate. Thus, interfacial coupling must play a unique role. Angle-resolved photoemission spectroscopy (ARPES) experiments have provided strong evidence that the coupling of electrons of FeSe-layer to IOPs generated by oxygen ions of SrTiO$_3$ may account for both replica bands and high-$T_c$. Motivated by these experiments, the IOP-induced superconductivity has been theoretically investigated by solving ME equations of $A_1(\epsilon, p)$ and $\Delta(\epsilon, p)$. However, to date there is still no consensus on the exact value of $T_c$ caused by IOPs. Some theorists propose that IOPs cannot induce such a high $T_c$ by themselves and that an additional pairing mechanism, be it magnetic or nematic fluctuation, needs to be considered. To be honest, such a proposal is hard to verify or deny if there is not a well-controlled tool to precisely compute $T_c$ induced by neither IOPs nor order-parameter fluctuations. Magnetic and nematic fluctuations usually lead to strong non-Fermi liquid (NFL) behaviors, and thus are much more difficult to handle than EPI. Without an accurate result of $T_c$ produced purely by IOPs, one can never judge whether it is necessary to consider an additional pairing mechanism. In this regard, going beyond ME theory is of paramount importance.

Experiments and first principles calculations revealed that IOPs are nearly dispersionless, and the frequency $\Omega(q) \equiv \Omega \approx 81$meV. The coupling of electrons to IOPs is dominated by small-$q$ forward scattering, described by 

$$g(q) = g_0 e^{-|q|/a_0}. \quad (17)$$

Migdal theorem breaks down in this system since $\omega_D/E_F \approx 1$. One might argue that vertex corrections are suppressed by certain account if the dimensionless coupling constant $\lambda_m = g_0^2 \Omega^{-2}$ is small enough. However, without exact solutions in hands, it is not possible to examine the importance of omitted contributions. Our approach turns out to be currently the only available tool to investigate how $T_c$ is influenced by full vertex corrections.

The full set of self-consistent integral equations of $A_{1,2}(\epsilon, p)$ and $\Delta(\epsilon, p)$ are complicated. To get a rapid glimpse of the influence of vertex corrections on $T_c$, here we assume that $A_2(\epsilon, p) = 1$. Moreover, since the EPI is dominated by small-$q$ scattering, we only retain the leading contribution of small-$q$ processes. Please see Appendix D for more details.

The numerical results of $T_c$ are shown in Fig. 1. The upper, middle, lower curves correspond to results obtained by using BCS mean-field approximation, our DS equation approach, and ME
FIG. 1: Superconducting $T_c$ induced by IOPs. Figure (a) shows the result obtained by using a delta potential (corresponding to $a_0 \to 0$ limit), and figure (b) shows that by using more realistic exponential coupling function $g(q)$. If we choose $\Omega = 81\text{meV}$, then $0.01\Omega \approx 9.5\text{K}$.

theory, respectively. It is clear that mean-field calculations always overestimate $T_c$, whereas ME theory always underestimates $T_c$. For small values of $\lambda_m$, mean-field and ME results are pretty good. However, as $\lambda_m$ grows, the deviation from exact results (middle curve) becomes progressively more significant. For $\lambda_m = 0.5$, mean-field result of $T_c$ is about 32K higher, and ME result of $T_c$ is about 20K lower than our DS result. When $\lambda_m$ further grows, mean-field and ME treatments cannot be trusted. In contrast, our DS equation results are still reliable even if $\lambda_m \gg 1$.

IV. DISCUSSION

Our DS equation approach provides a reliable framework to study EPI in any metal or superconductor. However, EPI is not the only interaction in realistic systems. Coulomb interaction and the static and dynamical screening effects may modify the results obtained by considering only EPI, and hence need to be properly taken into account. It is also interesting to examine ordering instabilities other than superconductivity, such as charge density wave \cite{34, 35}.

In this paper, we focus on metals with a finite Fermi surface. For band-touching Dirac and Weyl semimetals, the fermionic excitations have more degrees of freedom (valley, sublattice, etc.), and the conserved current would lead to much more complicated WTI's. The basic idea and the essential calculational steps of our approach could be generalized to investigate the interaction between phonons and Dirac/Weyl fermions \cite{36}.

The applicability of our approach is certainly not restricted to EPI systems. The DS equations
can be similarly constructed and solved if the phonon is replaced by another sort of bosonic mode, such as magnon or gauge boson. For instance, we expect that our approach can be applied to study the NFL behaviors in models that exhibit strong interaction between U(1) gauge boson and fermions excited on the Fermi surface [37–41]. Such models may be applied [37–41] to describe the anomalous properties of cuprate superconductors and quantum spin liquids. In these models, the vertex corrections would have more significant effects on physical quantities than EPI systems, because the importance of vertex function relies on the energy-momenta dependence of $A_1(\epsilon, \mathbf{p})$ and $A_2(\epsilon, \mathbf{p})$. In the case of EPI, both $A_1(\epsilon, \mathbf{p})$ and $A_2(\epsilon, \mathbf{p})$ are smooth functions. However, $A_{1,2}(\epsilon, \mathbf{p})$ could display singular behaviors at small $\epsilon$ and $\mathbf{p}$ due to strong gauge interaction. Therefore, the quantum corrections to fermion-gauge boson coupling would play a more important role than EPI systems.

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Appendix A: Derivation of two Ward-Takahashi identities

The total Lagrangian density for electron-phonon interaction (EPI) is given by

$$\mathcal{L} = \mathcal{L}_e + \mathcal{L}_p + \mathcal{L}_{ep}$$

$$= \Psi^\dagger(\epsilon, \mathbf{p})(\epsilon\sigma_0 - \xi_\mathbf{p}\sigma_3)\Psi(\epsilon, \mathbf{p}) + \frac{1}{2}\phi^\dagger(\omega, \mathbf{q})F_0^{-1}(\omega, \mathbf{q})\phi(\omega, \mathbf{q})$$

$$-g\phi(\omega, \mathbf{q})\Psi^\dagger(\epsilon + \omega, \mathbf{p} + \mathbf{q})\sigma_3\Psi(\epsilon, \mathbf{p}) + J\phi + \Psi^\dagger\eta + \eta^\dagger\Psi,$$

where the electron energy $\xi_\mathbf{p} = \frac{\mathbf{p}^2}{2m} - \mu$, and $\Psi(\epsilon, \mathbf{p})$ is the Nambu spinor

$$\Psi = \begin{pmatrix} \psi^\dagger(p) \\ \psi^\dagger(-p) \end{pmatrix}.$$  

The free boson propagator $F_0(\omega, \mathbf{q})$ is

$$F_0(\omega, \mathbf{q}) = \frac{2\Omega_\mathbf{q}}{\omega^2 - \Omega_\mathbf{q}^2}.$$  

In the coordinate space, the Lagrangian density has the following form

$$\mathcal{L} = \Psi^\dagger(t, \mathbf{z})(i\partial_t\sigma_0 - \xi_\mathbf{p}\sigma_3)\Psi(t, \mathbf{z}) + \frac{1}{2}\phi^\dagger(t, \mathbf{z})\mathcal{D}_\mathbf{z}\phi(t, \mathbf{z})$$

$$-g\phi(t, \mathbf{z})\Psi^\dagger(t, \mathbf{z})\sigma_3\Psi(t, \mathbf{z}) + J\phi + \Psi^\dagger\eta + \eta^\dagger\Psi,$$
where \( \mathbb{D}_z = -\frac{\partial^2 + \Omega^2_{\partial z}}{2m_{\partial z}} \), where \( \Omega_{\partial z} \) stands for the real-space correspondence of phonon dispersion \( \Omega_q \), and \( \xi_{\phi} = -\frac{\nabla^2}{2m} \). Here we have introduced three sources \( J, \eta, \) and \( \eta^\dagger \) and couple them to field operators \( \phi, \Psi^\dagger, \) and \( \Psi \), respectively.

We will work in the framework of function integral \([21]\). Adopting this framework, the Dyson-Schwinger (DS) equations and the Ward-Takahashi identities (WTIs) can be derived in a compact and elegant manner. There are three essential quantities: partition function \( Z(\eta^\dagger, \eta, J) \), generating functional \( W(\eta^\dagger, \eta, J) \), and generating functional \( \Xi(\Psi^\dagger, \Psi, \phi) \). They are defined as follows:

\[
Z(\eta^\dagger, \eta, J) = \int \mathcal{D}\phi\mathcal{D}\Psi^\dagger\mathcal{D}\Psi \exp \left(i \int \mathcal{L}\right),
\]

\[
W(\eta^\dagger, \eta, J) = -i \ln Z(\eta^\dagger, \eta, J),
\]

\[
\Xi(\Psi^\dagger, \Psi, \phi) = W(\eta^\dagger, \eta, J) - \left[ \int J(z)\phi(z) + \eta^\dagger(z)\Psi(z) + \Psi^\dagger(z)\eta(z) \right].
\]

The following identities will be frequently used:

\[
\frac{\delta W}{\delta J} = \langle \phi \rangle, \quad \frac{\delta W}{\delta \eta} = -\langle \Psi^\dagger \rangle, \quad \frac{\delta W}{\delta \eta^\dagger} = \langle \Psi \rangle, \quad (A5)
\]

\[
\frac{\delta \Xi}{\delta \phi} = -J, \quad \frac{\delta \Xi}{\delta \Psi^\dagger} = \eta^\dagger, \quad \frac{\delta \Xi}{\delta \Psi} = -\eta. \quad (A9)
\]

It is known that \( W(\eta^\dagger, \eta, J) \) generates all the connected Green’s functions and \( \Xi(\eta^\dagger, \eta, J) \) generates all the irreducible proper vertices of electron-phonon coupling. For instance, the full electron propagator \( G(z - z') \) and full phonon propagator \( F(z - z') \) are given by

\[
G(z - z') \equiv -i\langle \Psi(z)\Psi^\dagger(z') \rangle = -\left( \frac{\delta^2 \Xi}{\delta \Psi^\dagger(z')\delta \Psi(z)} \right)^{-1}, \quad (A10)
\]

\[
F(z - z') \equiv -i\langle \phi(z)\phi^\dagger(z') \rangle = -\left( \frac{\delta^2 \Xi}{\delta \phi^\dagger(z')\delta \phi(z)} \right)^{-1}. \quad (A11)
\]

With the help of Eqs. \((A5, A9)\), the above two expressions are calculated by the following steps:

\[
\frac{\delta^2 W}{\delta \eta^\dagger(z)\delta \eta(z')} = -\delta^2 \Psi^\dagger(z') = -\left( \frac{\delta^2 \Xi}{\delta \Psi^\dagger(z')\delta \Psi(z)} \right)^{-1}, \quad \frac{\delta^2 W}{\delta \eta(z')\delta \eta(z)} = -\left( \frac{\delta^2 \Xi}{\delta \Psi^\dagger(z')\delta \Psi(z)} \right)^{-1}, \quad (A12)
\]

\[
\frac{\delta^2 W}{\delta \phi(z')\delta \phi^\dagger(z')} = -\left( \frac{\delta^2 \Xi}{\delta \phi^\dagger(z')\delta \phi(z)} \right)^{-1}, \quad (A13)
\]

Here it is important to emphasize that \( \frac{\delta^2 W}{\delta \eta^\dagger\partial \eta} \) and \( \frac{\delta^2 W}{\delta \phi \partial \phi} \) only involve connected Feynman diagrams for the electron and phonon propagators. To understand this, we take phonon field \( \phi \) as an example,
and perform functional derivatives:

\[
\frac{\delta^2 W}{\delta J \delta J} = -i \frac{\delta}{\delta J} \left( \frac{1}{Z} \int D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\} \right)
= -i \int \frac{D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\}}{Z} + i \int \frac{D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\}}{Z} \int D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\}
= i \int \frac{D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\}}{Z} - i \int \frac{D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\}}{Z} \int D\phi D\Psi D\Psi^\dagger i\phi \exp\{i\mathcal{L}\}. \tag{A14}
\]

The first term of the r.h.s. of this equation contains all the connected and disconnected diagrams, whereas the second term contains only disconnected diagrams. Hence, the phonon propagator \( F = -\frac{\delta^2 W}{\delta J \delta J} \) contains only connected diagrams. The same is true for electron propagator, namely \( G = \frac{\delta^2 W}{\delta \eta \delta \eta^\dagger} \) contains only connected diagrams.

\[
\begin{align*}
\frac{-\delta^3 W}{\delta J \delta \eta \delta \eta^\dagger} &= \Gamma_v \\
\frac{\delta^3 \Xi}{\delta \phi \delta \Psi^\dagger \delta \Psi} &\equiv -FG \frac{\delta^3 \Xi}{\delta \phi \delta \Psi^\dagger \delta \Psi} G.
\end{align*}
\tag{A15}
\]

The 3-point correlation function \( \langle \phi \Psi \Psi^\dagger \rangle \) is computed as follows

\[
\langle \phi \Psi \Psi^\dagger \rangle \equiv \frac{\delta^3 W}{\delta J \delta \eta \delta \eta^\dagger} = -FG \frac{\delta^3 \Xi}{\delta \phi \delta \Psi^\dagger \delta \Psi} G,
\tag{A15}
\]

which we have defined a truncated (external legs dropped) vertex function \( \Gamma_v \)

\[
\Gamma_v(z_1; z_2; z_3) \equiv \frac{\delta^3 \Xi}{\delta \phi(z_1) \delta \Psi^\dagger(z_2) \delta \Psi(z_3)}.
\tag{A16}
\]

These two formulae are easy to understand with the help of Fig. 2. Analogous to electron and phonon propagators, here the vertex function \( \Gamma_v \) receives contributions solely from connected diagrams. It is straightforward to derive 4-point and higher-point correlation functions by means of similar operations. But for our purposes it is only necessary to consider 2-point and 3-point correlation functions, including \( G, F, \) and \( \Gamma_v \). For a more comprehensive illustration of functional integral techniques, please read the standard textbooks of quantum field theory, such as the one of Itzykson and Zuber [21].
1. Derivation of DS equations

An apparent fact is that the partition function should be invariant under an infinitesimal variation of $\Psi^\dagger$, that is

$$\int \frac{\delta}{\delta \Psi^\dagger} \exp\{iS\} = 0.$$ (A17)

It is easy to get

$$\langle (i\sigma_0 \partial_t - \sigma_3 \xi \partial_z) \Psi(z) - g\phi(z)\sigma_3 \Psi(z) + \eta(z) \rangle = 0,$$ (A18)

which then leads to

$$-\eta(z) = (i\sigma_0 \partial_t - \sigma_3 \xi \partial_z) \frac{\delta W}{\delta \eta(z)} + ig\sigma_3 \frac{\delta^2 W}{\delta J(z) \delta \eta(z)} - g \frac{\delta W}{\delta J(z)} \sigma_3 \frac{\delta W}{\delta \eta(z)}.$$ (A19)

The last term of the r.h.s. vanishes upon removing the external sources, and, for simplicity, can be directly omitted. Operating functional derivative of both sides with respect to $\eta(z)$ yields

$$\delta(z - z_3) \sigma_0 = (i\sigma_0 \partial_t - \sigma_3 \xi \partial_z) \frac{\delta^2 W}{\delta \eta(z) \delta \eta(z)} + ig\sigma_3 \frac{\delta^3 W}{\delta J(z) \delta \eta(z)} \frac{\delta W}{\delta J(z)} \sigma_3 \frac{\delta W}{\delta \eta(z)}.$$ (A20)

This expression can be re-written as

$$G^{-1}(z - z_3) = (i\sigma_0 \partial_t - \sigma_3 \xi \partial_z) \frac{\delta W}{\delta \eta(z)} - ig \int dz_1' dz_2' \sigma_3 \times F(z - z_1') G(z - z_2') \frac{\delta^3 \Xi}{\delta \phi(z_1') \delta \Psi^\dagger(z_1') \delta \Psi(z_3)}.$$ (A21)

Making use of the definition

$$\Gamma_v(z_1'; z'; z_3) \equiv \frac{\delta^3 \Xi}{\delta \phi(z_1') \delta \Psi^\dagger(z_1') \delta \Psi(z_3)},$$ (A23)

we derive the Fourier-transformed DS equation

$$G^{-1}(p) = G_0^{-1}(p) - ig \int dq \sigma_3 G(p + q) F(q) \Gamma_v(q, p),$$ (A24)

where

$$\Gamma_v(q, p) = \int dz_1' dz_2' e^{ip(z_1' - z_3)} e^{-i(p + q)(z_1' - z')} \Gamma_v(z_1', z', z_3).$$ (A25)

The DS equation of full phonon propagator $F(q)$ and that of vertex function $\Gamma_v(q, p)$ can be similarly derived [21, 22], which will not be explicitly presented here. For a diagrammatic illustration of the coupled DS equations, please refer to Fig. 3.
2. Relation between $\Gamma_v(q,p)$ and $\Gamma_t(q,p)$

There are two sorts of vertex functions. One is the EPI vertex function defined through the mean value $\langle \phi \Psi \Psi^\dagger \rangle$, as shown in Eq. (A15), and the other is defined in terms of conserved current $j_\mu \equiv (j_t, j)$ in the following way

$$
\langle j_\mu(z) \Psi(z_1) \Psi^\dagger(z_2) \rangle = - \int dz_3 dz_4 G(z_1, z_3) \Gamma_\mu(z, z_3, z_4) G(z_4, z_2),
$$

(A26)

where the minus sign appearing in the r.h.s. comes from $i^2$ since $\langle \Psi \Psi^\dagger \rangle = iG$. The former is a scalar function $\Gamma_v$, whereas the latter is a $d$-dimensional vector $\Gamma_\mu$. These two sorts of vertex functions are closely related, but are apparently not the same thing. We now would like to derive the relation between them.

Under an infinitesimal variation of phonon field $\phi$, the partition function should be invariant. Thus, $0 = \int \frac{\delta}{\delta \phi_0} \exp\{iS\}$, which gives rise to

$$
0 = \langle D_\phi \phi(z) - g \Psi^\dagger(z) \sigma_3 \Psi(z) + J(z) \rangle,
$$

(A27)

The above formula is re-written in the form

$$
g \langle \Psi^\dagger(z) \sigma_3 \Psi(z) \rangle = D_z \frac{\delta W}{\delta J(z)} + J(z),
$$

(A28)

which, after making functional derivative with respect to $\eta^\dagger$ and $\eta$ in order, leads to

$$
\frac{\delta^2}{\delta \eta^\dagger(z_1) \delta \eta(z_2)} \langle \Psi^\dagger(z) \sigma_3 \Psi(z) \rangle = \langle j_t(z) \Psi(z_1) \Psi^\dagger(z_2) \rangle = g^{-1} D_z \frac{\delta^3 W}{\delta J(z) \delta \eta^\dagger(z_1) \delta \eta(z_2)}.
$$

(A29)
Notice that the time-component of conserved current \( j_{\mu} \), i.e., \( j_t = \Psi^* \sigma_3 \Psi \) appears in the l.h.s. of this equation.

To proceed, we define two vertex functions \( \Gamma_t \) and \( \Gamma_s \):

\[
- \int dz_3 dz_4 G(z_1 - z_3) \Gamma_t(z, z_3, z_4) G(z_4 - z_2) \equiv \frac{\delta^2}{\delta \eta^\dagger(z_1) \delta \eta(z_2)} \langle \Psi^\dagger(z) \sigma_3 \Psi(z) \rangle,
\]

\[
- \int dz_3 dz_4 G(z_1 - z_3) \Gamma_s(z, z_3, z_4) G(z_4 - z_2) \equiv \frac{\delta^2}{\delta \eta^\dagger(z_1) \delta \eta(z_2)} \langle \Psi^\dagger(z) \sigma_0 \Psi(z) \rangle.
\]

Using the relation

\[
\frac{\delta^3 W}{\delta J \delta \eta^\dagger \delta \eta} = -FG \frac{\delta^3 \Xi}{\delta \phi \delta \Psi^\dagger \delta \Psi} G,
\]

we obtain

\[
\int dz_3 dz_4 G(z_1 - z_3) \Gamma_t(z, z_3, z_4) G(z_4 - z_2) = \int dz_5 dz_3 dz_4 g^{-1} D F(z; z_5) G(z_1 - z_2) \Gamma_v(z_5; z_3; z_4) G(z_4 - z_2)
\]

The Fourier transformation of \( \Gamma_t(z, z_3, z_4) \) is defined as

\[
\Gamma_t(q, p) \equiv \int dz_3 dz_4 e^{-i(p+q)(z-z_3)} e^{ip(z-z_4)} \Gamma_t(z, z_3, z_4),
\]

which leads us to an identity

\[
\Gamma_t(q, p) = \frac{1}{g} F_0^{-1}(q) F(q) \Gamma_v(q, p).
\]

Taking advantage of this identity, the product \( F(q) \Gamma_v(q, p) \) appearing in the DS equation of fermion propagator \( G(p) \) can be replaced with the product \( g F_0(q) \Gamma_t(q, p) \). Finally, we find that \( G(p) \) satisfies the following integral equation

\[
G^{-1}(p) = G_0^{-1}(p) - ig^2 \int dq \sigma_3 G(p + q) F_0(q) \Gamma_t(q, p).
\]

The DS equation for \( G(p) \) would be entirely self-closed if the function \( \Gamma_t \) can be expressed only in terms of electron propagators. Below we will show that this is indeed achievable.

### 3. Determining \( \Gamma_t(q, p) \) based on Ward-Takahashi identities

As defined previously, the vector function \( \Gamma_{\mu} \equiv (\Gamma_t, \Gamma) \). The \( d - 1 \) spatial components are equal, so \( \Gamma \) can be decomposed as \( \Gamma = \frac{2p + q}{2m} \Gamma_s \), where \( \Gamma_s \) is given by Eq. \( \text{(A31)} \). Clearly, \( \Gamma_{\mu} \) could be uniquely determined once \( \Gamma_t \) and \( \Gamma_s \) are specified. For this purpose, we now would derive their relations with the help of the global U(1) symmetry.
Once again, we employ the identity \(0 = \int \frac{\delta}{\delta \Psi^\dagger} \exp\{iS\}\) and write it in the form
\[
(A37)
\]
\[
\langle (i\sigma_0 \partial_t - \sigma_3 \xi \partial_k) \Psi(z) - g\phi(z)\sigma_3 \Psi(z) + \eta(z) \rangle = 0.
\]
Multiplying a generic \(2 \times 2\) matrix \(\sigma_m\) to both sides of this equation and then performing the variation \((-i)\frac{\delta}{\delta \eta}\), we find
\[
-(-i) \text{Tr}[\sigma_m] \delta(z - z_3) = -\langle \Psi^\dagger(z_3) \sigma_m (i\sigma_0 \partial_t - \sigma_3 \xi \partial_k) \Psi(z) \rangle + g\phi(z) \langle \Psi^\dagger(z_3) \sigma_m \sigma_3 \Psi(z) \rangle - \eta^\dagger(z_3) \sigma_m \sigma_3 \eta(z).
\]
This equation is re-written as
\[
\text{Tr} \left[ \sigma_m (i\sigma_0 \partial_t - \sigma_3 \xi \partial_k) \langle \Psi(z) \Psi^\dagger(z_3) \rangle \right] + (-i) \text{Tr}[\sigma_m] \delta(z - z_3) = -g\phi(z) \langle \Psi^\dagger(z_3) \sigma_m \sigma_3 \Psi(z) \rangle + \langle \Psi^\dagger(z_3) \rangle \sigma_m \sigma_3 \eta(z).
\]
(A38)

Similar computational steps can be done as follows. We start the identity \(0 = \int \frac{\delta}{\delta \Psi^\dagger} \exp\{iS\}\), multiply both sides by the generic matrix \(\sigma_m\), and perform functional derivative \((-i)\frac{\delta}{\delta \eta}\) on both sides. After straightforward analytical calculations, we obtain
\[
\text{Tr} \left[ (i\sigma_0 \partial_t + \sigma_3 \xi \partial_k) \sigma_m \langle \Psi(z) \Psi^\dagger(z_3) \rangle \right] - (-i) \text{Tr}[\sigma_m] \delta(z - z_3) = g\phi(z) \langle \Psi(z) \sigma_3 \sigma_3 \Psi(z_3) \rangle - \eta^\dagger(z_3) \sigma_m \langle \Psi(z) \rangle.
\]
(A40)

Adding equation (A39) to equation (A40) gives rise to a generalized Slavnov-Taylor identity:
\[
\text{Tr} \left[ \sigma_m (i\sigma_0 \partial_t - \sigma_3 \xi \partial_k) + (i\sigma_0 \partial_{t_3} + \sigma_3 \xi \partial_{k_3}) \sigma_m \rangle \langle \Psi(z) \Psi^\dagger(z_3) \rangle \right] = -g\phi(z) \langle \Psi(z) \rangle \sigma_3 \sigma_3 \sigma_3 \eta(z) - \eta^\dagger(z_3) \sigma_m \langle \Psi(z) \rangle.
\]
(A41)

The matrix \(\sigma_m\) could be unity matrix \(\sigma_0\) or any Pauli matrix \(\sigma_{1,2,3}\). Since the conserved current \(j_\mu\) is defined via \(\sigma_0\) and \(\sigma_3\), here we choose \(\sigma_m = \sigma_0\) and \(\sigma_m = \sigma_3\) respectively. It is easy to see that the first term of the r.h.s. vanishes in both cases. Then the above equation is simplified to
\[
\text{Tr} \left[ (i\sigma_0 \partial_t - \sigma_0 \xi \partial_k) + (i\sigma_0 \partial_{t_3} + \sigma_0 \xi \partial_{k_3}) \rangle \langle \Psi(z) \Psi^\dagger(z_3) \rangle \right] = \langle \Psi^\dagger(z_3) \rangle \sigma_3 \eta(z) - \eta^\dagger(z_3) \sigma_3 \langle \Psi(z) \rangle
\]
(A42)

and
\[
\text{Tr} \left[ [(i\sigma_0 \partial_t - \sigma_3 \xi \partial_k) + (i\sigma_0 \partial_{t_3} + \sigma_3 \xi \partial_{k_3}) \rangle \langle \Psi(z) \Psi^\dagger(z_3) \rangle \right] = \langle \Psi^\dagger(z_3) \rangle \sigma_0 \eta(z) - \eta^\dagger(z_3) \sigma_0 \langle \Psi(z) \rangle.
\]
(A43)
Then take the limit \( \lim_{z_3 \to z} \), and get

\[
\begin{align*}
i \partial_t & \text{Tr}[\sigma_3(\Psi(z)\Psi^\dagger(z))] - \lim_{z_3 \to z} (\xi_{\delta_k} - \xi_{\delta_{k_3}}) \text{Tr}[\sigma_0(\Psi(z)\Psi^\dagger(z))] = -\frac{\delta W}{\delta \eta(z)} \sigma_3 \eta(z) - \eta^\dagger(z) \sigma_3 \frac{\delta W}{\delta \eta^\dagger(z)}, \\
i \partial_t & \text{Tr}[\sigma_0(\Psi(z)\Psi^\dagger(z))] - \lim_{z_3 \to z} (\xi_{\delta_k} - \xi_{\delta_{k_3}}) \text{Tr}[\sigma_3(\Psi(z)\Psi^\dagger(z))] = -\frac{\delta W}{\delta \eta(z)} \sigma_0 \eta(z) - \eta^\dagger(z) \sigma_0 \frac{\delta W}{\delta \eta^\dagger(z)}.
\end{align*}
\]

(A44)

To derive WTIs, we now carry out functional derivative with respect to \( \eta \) and then to \( \eta^\dagger \) on both sides. The calculations go as follows:

\[
\begin{align*}
i \partial_t & \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}[-\langle \Psi^\dagger(z) \sigma_3 \Psi(z) \rangle] - \lim_{z_3 \to z} (\xi_{\delta_k} - \xi_{\delta_{k_3}}) \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}[-\langle \Psi^\dagger(z_3) \sigma_0 \Psi(z) \rangle] = \frac{\delta^2 W}{\delta \eta^\dagger(z_1) \delta \eta(z)} \sigma_3 \delta(z - z_2) - \delta(z - z_1) \sigma_3 \frac{\delta^2 W}{\delta \eta^\dagger(z) \delta \eta(z)} \\
&= G(z_1 - z) \sigma_3 \delta(z - z_2) - \delta(z - z_1) \sigma_3 G(z - z_2), \tag{A45}
\end{align*}
\]

\[
\begin{align*}
i \partial_t & \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}[-\langle \Psi^\dagger(z) \sigma_0 \Psi(z) \rangle] - \lim_{z_3 \to z} (\xi_{\delta_k} - \xi_{\delta_{k_3}}) \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}[-\langle \Psi^\dagger(z_3) \sigma_3 \Psi(z) \rangle] = \frac{\delta^2 W}{\delta \eta^\dagger(z_1) \delta \eta(z)} \sigma_0 \delta(z - z_2) - \delta(z - z_1) \sigma_0 \frac{\delta^2 W}{\delta \eta^\dagger(z) \delta \eta(z)} \\
&= G(z_1 - z) \sigma_0 \delta(z - z_2) - \delta(z - z_1) \sigma_0 G(z - z_2). \tag{A46}
\end{align*}
\]

After performing Fourier transformations, we find that

\[
\begin{align*}
i \partial_t & \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}[-\langle \Psi^\dagger(z) \sigma_3 \Psi(z) \rangle] \to (-\omega) G(p + q) \Gamma_\eta(q, p) G(p), \tag{A47}
\end{align*}
\]

\[
\begin{align*}
i \partial_t & \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}[-\langle \Psi^\dagger(z) \sigma_0 \Psi(z) \rangle] \to (-\omega) G(p + q) \Gamma_\sigma(q, p) G(p), \tag{A48}
\end{align*}
\]

and that

\[
\begin{align*}
\lim_{z_3 \to z} (\xi_{\delta_k} - \xi_{\delta_{k_3}}) \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}(\Psi^\dagger(z_3) \sigma_0 \Psi(z)) & \to (\xi_{p+q} - \xi_p) G(p + q) \Gamma_\sigma(q, p) G(p), \tag{A49}
\end{align*}
\]

\[
\begin{align*}
\lim_{z_3 \to z} (\xi_{\delta_k} - \xi_{\delta_{k_3}}) \frac{\delta^2}{\delta \eta^\dagger(z_1) \eta(z_2)}(\Psi^\dagger(z_3) \sigma_3 \Psi(z)) & \to (\xi_{p+q} - \xi_p) G(p + q) \Gamma_\eta(q, p) G(p). \tag{A50}
\end{align*}
\]

From Eqs. (A47 A50), we eventually get two WTIs:

\[
\begin{align*}
\omega \Gamma_\eta(q, p) - (\xi_{p+q} - \xi_p) \Gamma_\sigma(q, p) & = G^{-1}(p + q) \sigma_3 - \sigma_3 G^{-1}(p), \tag{A51}
\end{align*}
\]

\[
\begin{align*}
\omega \Gamma_\sigma(q, p) - (\xi_{p+q} - \xi_p) \Gamma_\eta(q, p) & = G^{-1}(p + q) \sigma_0 - \sigma_0 G^{-1}(p). \tag{A52}
\end{align*}
\]

Eq. (A51) is the well-known ordinary WTI that was first derived by Engelsberg and Schrieffer in coupled electron-phonon system. Eq. (A52) is a new WTI that, to the best of our knowledge,
has never been obtained previously. The ordinary and new WTI s are coupled to each other, and
the function $\Gamma_t(q,p)$ can be determined by solving these two WTI s. Indeed, we find that

$$\Gamma_t(q,p) = \frac{\omega[G^{-1}(p + q)\sigma_3 - \sigma_3G^{-1}(p)] + (\xi_{p+q} - \xi_p)[G^{-1}(p + q)\sigma_0 - \sigma_0G^{-1}(p)]}{\omega^2 - (\xi_{p+q} - \xi_p)^2}. \quad (A53)$$

We end this appendix by briefly remarking on a subtle issue. The WTI s are derived from U(1)
symmetry. In the superconducting phase, Cooper pairing leads to spontaneous breaking of U(1)
symmetry. One natural question is: are WTI s changed in the superconducting state? This issue
has been addressed in the context of high-energy physics. For gauge field theories, the (ordinary)
WTI was proved to be the same in symmetric and symmetry-broken phases [42]. We suppose the
same conclusion holds in the case of EPI.

**Appendix B: Self-consistent equations of $A_{1,2}$ and $\Delta_{1,2}$**

After obtaining the exact expression of the vertex function $\Gamma_t$, we are now ready to study
superconducting transition. Since it is often necessary to study finite-temperature properties,
here we adopt Matsubara formalism and define imaginary frequency ($i\epsilon = i2n\pi/\beta$ for boson and
$i\omega = i(2n + 1)\pi/\beta$ for fermion).

At finite $T$, the free electron propagator is

$$G_0(p) = \frac{1}{i\epsilon\sigma_0 - \xi_p\sigma_3}, \quad (B1)$$

where $p \equiv (\epsilon, p)$ and $\xi_p = \frac{p^2}{2m}$. Free phonon propagator is

$$F_0(q) = -\frac{2\Omega_q}{\omega^2 + \Omega_q^2}, \quad (B2)$$

where $q \equiv (\omega, q)$ and $\Omega_q$ can be obtained from first-principle calculations.

The exact (fully renormalized) electron propagator $G(p)$ satisfies the following DS equation

$$G^{-1}(p) = G_0^{-1}(p) + \int \frac{d^dq}{(2\pi)^d} g^2(q)\sigma_3G(p + q)F_0(q)\Gamma_t(q,p), \quad (B3)$$

where the momentum dependence of coupling constant is given by $g(q)$. The full electron propa-
gator is formally written as

$$G(\epsilon, p) = \frac{1}{iA_1(\epsilon, p)\epsilon\sigma_0 - A_2(\epsilon, p)\xi_p\sigma_3 + \Delta_1(\epsilon, p)\sigma_1 + \Delta_2(\epsilon, p)\sigma_2}. \quad (B4)$$
The function $\Gamma_t(q,p)$ is expressed in terms of $A_{1,2}(\epsilon,p)$ and $\Delta_{1,2}(\epsilon,p)$ as follows

$$
\Gamma_t(q,p) = \frac{i\omega [iA_1(p + q)(\epsilon + \omega) - iA_1(p)e] - (\xi_{p+q} - \xi_p)[A_2(p + q)\xi_{p+q} - A_2(p)\xi_p]}{(i\omega)^2 - (\xi_{p+q} - \xi_p)^2}.
$$

Inserting Eq. (B4) and Eq. (B5) into Eq. (B3), one will be able to obtain four self-consistent integral equations for $A_1$, $A_2$, $\Delta_1$ and $\Delta_2$. These four equations are exact and contain all the EPI-induced effects.

In practice, we need to employ some further approximations to simplify numerical computation. Our main interest here is in the impact of vertex corrections on the value of $T_c$. Since the pairing function $\Delta_{1,2}$ vanishes continuously as $T \to T_c$, here we drop the dependence of $\Gamma_t(q,p)$ on $\Delta_{1,2}(p)$ and solve the DS equations of $A_{1,2}(\epsilon,p)$ and $\Delta_1(\epsilon,p)$, which is taken to be real as usual, by utilizing the expression of $\Gamma_t(q,p)$ obtained in the limit of $T \to T_c$, namely

$$
\Gamma_t(q,p) = \frac{i\omega [iA_1(p + q)(\epsilon + \omega) - iA_1(p)e] - (\xi_{p+q} - \xi_p)[A_2(p + q)\xi_{p+q} - A_2(p)\xi_p]}{(i\omega)^2 - (\xi_{p+q} - \xi_p)^2}.
$$

Around $T_c$, there are already numerous Cooper pairs, but the long-range phase coherence is absent. Therefore, the two WTIs are reliable and one does not need to worry about whether the WTIs are modified by U(1) gauge symmetry breaking. Nevertheless, if one is willing to compute pairing functions deep inside the superconducting state, such an approximation is not reliable.

The fully renormalized phonon propagator can be computed by integrating over energy and momenta, without the need to solve any self-consistent equation. In particular, we find that

$$
F(q) = F_0(q) + F_0^2(q)g^2(q) \int \frac{d^d p}{(2\pi)^d} \text{Tr}[\sigma_3 G(p + q)\Gamma_t(q,p)G(p)]
$$

and

$$
F(q) = F_0(q) + F_0^2(q)g^2(q) \int \frac{d^d p}{(2\pi)^d} \text{Tr} \left[ i\omega G(p) - \sigma_3 G(p + q)\sigma_3 + (\xi_{p+q} - \xi_p)(\xi_{p+q} - \xi_p) \right].
$$
Appendix C: Detailed derivation of Eqs. (A47-A50)

The Fourier transformations employed in Eqs. (A47-A50) are a little tricky \[6, 9, 28\]. Below we present the calculational details.

The first one is:

\[
\begin{align*}
&i\partial_t \frac{\delta^2}{\delta \eta(\eta_1) \eta(\eta_2)} \left[ -\langle \psi^\dagger(z) \sigma_3 \psi(z) \rangle \right] \\
&= i\partial_t \int dz_4 dz_5 G(z_1 - z_4) \Gamma_t(z_4 - z, z - z_5) G(z_5 - z_2) \\
&= i\partial_t \int dz_4 dz_5 \int dp_1 G(p_1) e^{-ip_1(z_1 - z_4)} \int dpdq \Gamma_t(q, p) e^{-i(p + q)(z_4 - z) - ip(z - z_5)} \\
&\quad \times \int dp_2 G(p_2) e^{-ip_2(z_5 - z_2)} \\
&= \int dpdq(i\partial_t) \int dp_1 p_2 \int dz_4 dz_5 G(p_1) e^{-ip_1(z_1 - z_4)} \Gamma_t(q, p) e^{-i(p + q)(z_4 - z) - ip(z - z_5)} \\
&\quad \times G(p_2) e^{-ip_2(z_5 - z_2)} \\
&= \int dpdq(i\partial_t) \int dp_1 p_2 G(p_1) e^{-i(p + q)z_1} \Gamma_t(q, p) e^{i(p + q)z_2} \Gamma_t(q, p) e^{ipz_2} \delta(p_1 - p - q) \delta(p - p_2) \\
&= \int dpdq(i\partial_t) e^{-i(p + q)z_1} G(p_1) \Gamma_t(q, p) e^{iqz} G(p_2) e^{ipz_2} \\
&= \int dpdq(-\omega) e^{-i(p + q)z_1} G(p_1) \Gamma_t(q, p) e^{iqz} G(p_2) e^{ipz_2} \\
&= \int dpdq(-\omega) e^{-i(p + q)(z_1 - z)} G(p_1) \Gamma_t(q, p) G(p_2) e^{-ip(z - z_2)}.
\end{align*}
\]

(C1)

Now one can readily identify that \(-\omega G(p_1) \Gamma_t(q, p) G(p_2)\) is the Fourier transformed expression of

\[i\partial_t \frac{\delta^2}{\delta \eta(\eta_1) \eta(\eta_2)} \left[ -\langle \psi^\dagger(z) \sigma_3 \psi(z) \rangle \right].\]

Another one is:

\[
\begin{align*}
&\lim_{\eta_3 \to z} \frac{(\xi_{\eta_k} - \xi_{\eta_k})}{\delta \eta(\eta_1) \eta(\eta_2)} \langle \psi^\dagger(\eta_3) \sigma_3 \psi(\eta_3) \rangle \\
&= \lim_{\eta_3 \to z} (\xi_{\eta_k} - \xi_{\eta_k})(-1) \int dz_4 dz_5 G(z_1 - z_4) \Gamma_t(z_4 - z_3, z - z_5) G(z_5 - z_2) \\
&= \lim_{\eta_3 \to z} (\xi_{\eta_k} - \xi_{\eta_k})(-1) \int dz_4 dz_5 \int dp_1 G(p_1) e^{-ip_1(z_1 - z_4)} \\
&\quad \times \int dpdq \Gamma_t(q, p) e^{-i(p + q)(z_4 - z_3) - ip(z - z_5)} \int dp_2 G(p_2) e^{-ip_2(z_5 - z_2)} \\
&= (-1) \int dpdp \lim_{\eta_3 \to z} \int dp_1 dp_2 \int dz_4 dz_5 (\xi_{\eta_k} - \xi_{\eta_k}) G(p_1) e^{-ip_1(z_1 - z_4)} \\
&\quad \times \Gamma_t(q, p) e^{-i(p + q)(z_4 - z_3) - ip(z - z_5)} G(p_2) e^{-ip_2(z_5 - z_2)} \\
&= (-1) \int dpdp \lim_{\eta_3 \to z} \int dp_1 dp_2 (\xi_{\eta_k} - \xi_{\eta_k}) G(p_1) e^{-ip_1(z_1 - z_4)} \\
&\quad \times \Gamma_t(q, p) e^{i(p + q)z_3 - ipz_2} G(p_2) e^{ipz_2} \delta(p_1 - p - q) \delta(p - p_2) \\
&= (-1) \int dpdp \lim_{\eta_3 \to z} (\xi_{\eta_k} - \xi_{\eta_k}) G(p_1) e^{-i(p + q)z_1} \Gamma_t(q, p) e^{i(p + q)z_3 - ipz_2} G(p_2) e^{ipz_2}
\end{align*}
\]
\[
\begin{align*}
&= (-1) \int dq dp \lim_{z_3 \to z} (\xi_p - \xi_{p+q}) G(p_1) e^{-i(p+q)z_1} \Gamma_t(q,p) e^{i(p+q)z_3-ipz} G(p_2) e^{ipz} \\
&= \int dq dp (\xi_{p+q} - \xi_p) G(p_1) e^{-i(p+q)(z_1-z_3)} \Gamma_t(q,p) G(p_2) e^{i(p-z_2)} \\
&= \int dq dp (\xi_{p+q} - \xi_p) e^{-i(p+q)(z_1-z_3)} G(p_1) \Gamma_t(q,p) G(p_2) e^{i(p-z_2)}. \quad (C2)
\end{align*}
\]

After Fourier transformation, the expression \( \lim_{z_3 \to z} (\xi_{\partial_k} - \xi_{\partial_p}) \delta^2(\varphi_{z_1}) \delta^2(\varphi_{z_2}) \langle \Psi^{\dagger}(z_3) \sigma_3 \Psi(z) \rangle \) is converted into \( (\xi_{p+q} - \xi_p) G(p_1) \Gamma_t(q,p) G(p_2) \).

Appendix D: Application to 1UC FeSe/SrTiO\(_3\)

In the case of 1UC FeSe/SrTiO\(_3\), the electrons of FeSe film couple to the interfacial optical phonons (IOPs). Different from ordinary (acoustic) phonons, the IOPs are nearly dispersionless, so \( \Omega_q \) can be taken as a constant. This types of EPI is sharply peaked at \( q = 0 \). Taking advantage of this unique feature, we assume \( q = 0 \) and then use it to simplify the function \( \Gamma_t(q,p) \) given by Eq. (B5) so as to reduce the time of numerical work. Moreover, if one is mainly interested in the computation of superconducting \( T_c \), it is reasonable to linearize the coupled DS equations, i.e., \( \Delta \to 0 \), near \( T_c \).

Similar to Ref. \[20\], here we consider two different forms of \( g_q \): \( \delta \) function and exponential function. Because the EPI is extreme forward scattering, the initial and final states of electrons are all located near the Fermi surface. This allows us to further set \( \xi_{p_F} = 0 \). As a result, the coupled equations are independent of momenta. There are three approximations: BCS mean-field theory, ME theory, and our DS approach. The corresponding gap equations are presented below.

The numerical results of \( T_c \) are shown and discussed in the main context of the paper.

1. \( \delta \) function

At mean-field level, the BCS gap equation is

\[
\Delta(\epsilon_n) = \frac{\lambda_m \Omega^2}{\beta} \sum_m \frac{2 \Omega}{\Omega^2 + \omega_m^2} \frac{\Delta(\epsilon_n + \omega_m)}{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)}. \quad (D1)
\]

The ME equations are

\[
\begin{align*}
\Delta(\epsilon_n) &= \frac{\lambda_m \Omega^2}{\beta} \sum_m \frac{2 \Omega}{\Omega^2 + \omega_m^2} A_1^2(\epsilon_n + \omega_m)(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m), \quad (D2) \\
A_1(\epsilon_n) &= 1 + \frac{\lambda_m \Omega^2}{\beta} \sum_m \frac{2 \Omega}{\epsilon_n + \omega_m^2} \frac{\epsilon_n + \omega_m}{A_1^2(\epsilon_n + \omega_m)(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)}. \quad (D3)
\end{align*}
\]
Including both $A_1$ and the vertex function $\Gamma_t$ leads to the following two equations

$$\Delta(\epsilon_n) = \frac{\lambda_m \Omega^2}{\beta} \sum_m \frac{2\Omega}{\Omega^2 + \omega_n^2} A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m),$$

(D4)

$$\times A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m) - A_1(\epsilon_n)\epsilon_n,$$

(D5)

$$A_1(\epsilon_n) = 1 + \frac{1}{\epsilon_n} \frac{\lambda_m \Omega^2}{\beta} \sum_m \frac{2\Omega}{\Omega^2 + \omega_n^2} A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)$$

$$\times A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m) - A_1(\epsilon_n)\epsilon_n.$$  

(D6)

2. Exponential function

BCS gap equation is given by

$$\Delta(\epsilon_n) = \left(\frac{2}{r}\right)^2 \lambda_m \Omega^2 T \sum_m \int_0^1 dx \exp \left(-\frac{2x}{r}\right) \frac{2\Omega}{\Omega^2 + \omega_n^2} \frac{\Delta(\epsilon_n + \omega_m)}{\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m) + \zeta^2}}}.$$  

(D7)

The ME equations are

$$\Delta(\epsilon_n) = \left(\frac{2}{r}\right)^2 \lambda_m \Omega^2 T \sum_m \int_0^1 dx \exp \left(-\frac{2x}{r}\right) \frac{2\Omega}{\Omega^2 + \omega_n^2} \frac{\Delta(\epsilon_n + \omega_m)}{\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m) + \zeta^2}}}.$$  

(D8)

$$A_1(\epsilon_n) = 1 + \frac{1}{\epsilon_n} \left(\frac{2}{r}\right)^2 \lambda_m \Omega^2 T \sum_m \int_0^1 dx \exp \left(-\frac{2x}{r}\right) \frac{2\Omega}{\Omega^2 + \omega_n^2} \frac{A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m)}{\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m) + \zeta^2}}}.$$  

(D9)

After including both $A_1$ and vertex corrections, the coupled DS equations are

$$\Delta(\epsilon_n) = \left(\frac{2}{r}\right)^2 \lambda_m \Omega^2 T \sum_m \int_0^1 dx \exp \left(-\frac{2x}{r}\right) \frac{2\Omega}{\Omega^2 + \omega_n^2} \frac{\Delta(\epsilon_n + \omega_m)}{\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m) + \zeta^2}}} \times A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m) - A_1(\epsilon_n)\epsilon_n.$$  

(D10)

$$A_1(\epsilon_n) = 1 + \frac{1}{\epsilon_n} \left(\frac{2}{r}\right)^2 \lambda_m \Omega^2 T \sum_m \int_0^1 dx \exp \left(-\frac{2x}{r}\right) \frac{2\Omega}{\Omega^2 + \omega_n^2} \frac{A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m)}{\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m)\sqrt{(\epsilon_n + \omega_m)^2 + \Delta^2(\epsilon_n + \omega_m) + \zeta^2}}} \times A_1(\epsilon_n + \omega_m)(\epsilon_n + \omega_m) - A_1(\epsilon_n)\epsilon_n.$$  

(D11)
The screening of EPI due to the Coulomb interaction also needs to be incorporated. However, the latter influence may be approximately taken into account by employing a pseudopotential $\mu^*$, which is relatively easy. After including pseudopotential [20], the value of $T_c$ would be reduced by roughly 20%.

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