Closed orbits of Reeb fields on compact Sasakian manifolds

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Abstract
A contact structure on a manifold can be defined as a \( \mathbb{R}^+ \)-automorphic symplectic structure on the cone of that manifold. The associated Reeb field is uniquely defined by the contact structure and transversal to it. A Sasakian manifold is a manifold with an \( \mathbb{R}^+ \)-automorphic Kähler structure on its cone; clearly, this structure is defined on top of a contact structure. Weinstein conjecture predicts that any compact contact manifold admits at least one closed Reeb orbit. We prove this conjecture for Sasakian manifolds, and show that any \( 2n+1 \)-dimensional Sasakian manifold \( M \) admits at least \( n+1 \) closed Reeb orbits. We also show that the number of closed Reeb orbits is either infinite or equal to the sum of all Betti numbers of a Kähler orbifold obtained as an \( S^1 \)-quotient of \( M \). We obtain a similar estimate for the number of elliptic curves on a compact Vaisman manifold.

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1 Introduction

Initially stated for periodic orbits of Hamiltonian flows on hypersurfaces of contact type in symplectic manifolds, Weinstein conjecture ([We]) can be intrinsically formulated in the following form:

**Weinstein Conjecture:** On any closed contact manifold \((N, \eta)\) the Reeb field has at least one closed orbit.

For history and background on Weinstein conjecture, see the survey [P].

The conjecture has been solved in the affirmative for contact hypersurfaces in Euclidean space by C. Viterbo ([Vi]), on \(S^3\), with a different method, by H. Hofer ([Ho]) and, more generally, on closed 3-manifolds, by C.H. Taubes ([Ta]). Several extensions of these results are also available. But in full generality, the conjecture is still open and remains one of the most challenging problems in symplectic and contact topology.

The dynamic of the Reeb flow on a contact manifold became a separate subfield in symplectic geometry, called the Reeb dynamics ([Ge2]). One of the central questions of this subfield is finding an estimate for the number of closed Reeb orbits on contact manifolds under different geometric conditions.

Most research in this direction is based on contact homology, which is a Morse-type cohomology theory expressing the Reeb orbits as fixed point of a certain gradient action on an appropriate loop space. As in Morse (and Floer) theory, the number of closed Reeb orbits bounds the sum of Betti numbers of the relevant cohomology theory.

The most striking achievement in this direction is a result of Hutchins and Cristofaro-Gardiner, who proved that any 3-dimensional compact contact manifolds admits at least 2 distinct closed Reeb orbits ([CH]).

However, for dimension \(>3\) this approach fails, and in general nothing is known.

Adding geometric structures compatible with the given contact structure puts an interesting twist on the Reeb dynamics. In this paper we study the Reeb dynamics on Sasakian manifolds.

These are contact manifolds equipped with a Riemannian metric, in such a way that their symplectic cone is equipped with a Kähler structure which is automorphic with respect to the dilations along the generators of the cone (see Subsection 2.2). In this sense, Sasakian manifolds can be understood as odd-dimensional counterparts of Kähler manifolds. They are ubiquitous and their geometry is a well established subject, see [BG].
The main result of this paper is a confirmation of Weinstein conjecture for compact Sasakian manifolds:

**Theorem 1.1:** Let $M$ be a compact Sasakian manifold of dimension $2n+1$. Then its Reeb field has at least $n+1$ closed orbits.

For the proof, we first observe that the Reeb field can be approximated by a quasi-regular one, i.e. with a Reeb field such that its space of orbits is a projective orbifold. We then relate the closed orbits of the initial Reeb field to the fixed points of the action of the group generated by its flow on this projective orbifold. Finally, we apply the orbifold version, \([\text{Fo}]\), of a celebrated result of A. Białynicki-Birula, \([\text{Bi}]\), counting fixed points of algebraic groups.

Our proof does not explicitly use the contact or Sasakian geometry, but transfers the problem to the framework of complex geometry.

From this argument we could also obtain an explicit expression for the number of the Reeb orbits on a Sasakian manifold associated with a projective orbifold $X$ admitting holomorphic vector fields with isolated zeros. In that case the number of Reeb orbits (for an appropriate Sasakian structure) is equal to the number of fixed points of any of these vector fields. It is also equal to the sum of Betti numbers of $S$. This always gives the lower estimate, and gives an upper estimate when the Sasakian structure is sufficiently general.

The compact Sasakian manifolds are known to be closely related to Vaisman manifolds (see Section 4 for their definition and properties). Indeed, Vaisman manifolds are mapping tori of circles with fibres compact Sasakian manifolds. Diagonal Hopf manifolds and their complex compact submanifolds are typical examples.

Exploiting this relation, we are able to count the elliptic curves on compact Vaisman manifolds:

**Theorem 1.2:** let $V$ be a compact Vaisman manifold of complex dimension $n$. Then $V$ contains at least $n$ elliptic curves.

This theorem is proven in Section 4. Note that this result is elementary for the diagonal Hopf manifolds.
2 CR, contact and Sasakian geometries

We present the necessary notions concerning Sasakian geometry. For details and examples, see [BG]. We start by recalling the definition of CR and contact structures. Both of these geometric structures are subjancent to the Sasakian structures.

2.1 CR manifolds

Definition 2.1: Let $M$ be a smooth manifold, $B \subset TM$ a sub-bundle in the tangent bundle, and $I : B \to B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then $(B, I)$ is called a CR-structure on $M$.

Definition 2.2: Let $(M, B)$ be a CR manifold and $\Pi_{TM/B} : TM \to TM/B$ be the projection to the normal bundle of $B$ in $TM$. The tensor field $B \otimes B \to TM/B$ mapping vector fields $X, Y \in B$ to $\Pi_{TM/B}([X, Y])$ is called the Frobenius form of $B$. It is the obstruction to the integrability of the distribution given by $B$.

Remark 2.3: Let $S$ be a CR manifold, with the bundle $B$ of codimension 1, and almost complex structure $I \in \text{End}(B)$. Since the Frobenius form vanishes when both arguments are from $B^{0,1}$ and $B^{1,0}$, it is a pairing between $B^{0,1}$ and $B^{1,0}$. Indeed, $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$. This proves that the Frobenius form is a Hermitian form taking values in a trivial rank 1 bundle $TM/B$. The Frobenius form on a CR manifold $(M, B, I)$ with codim $B = 1$ is called the Levi form.

Remark 2.4: If, in addition, $B \subset TM$ is a contact bundle, its Levi form is non-degenerate. Therefore, it has constant rank. If it is positive or negative definite, the CR manifold $(M, B, I)$ is called strictly pseudoconvex. In this case we fix the orientation on the trivial bundle $TM/B$ in such a way that the Levi form is positive definite.

Example 2.5:

(i) A complex manifold $(X, I)$ is CR, with $B = TX$. Indeed $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$ is equivalent to the Newlander-Nirenberg theorem.

(ii) Let $(X, I)$ be a complex manifold and $M \subset X$ a real hypersurface. Then $B := TX \cap I(TX)$ is a distribution of dimension $\text{dim}_\mathbb{C} X - 1$.
which gives $M$ the structure of a CR manifold.

(iii) Let $X$ be a complex manifold and $\varphi : X \to \mathbb{R}$ a strictly plurisubharmonic function. Then $\omega := \partial \bar{\partial} \varphi$ is positive definite and all level sets $M_c := \varphi^{-1}(c)$ are strictly pseudoconvex CR manifolds, with Frobenius forms $\omega\big|_{M_c}$.

The following result was proved by D. Burns, but never published by him, see [Le]; another proof, valid also for dim $M = 3$, is given in [Sc].

**Theorem 2.6: (D. Burns)** Suppose $M$ is a compact, connected, strictly pseudoconvex CR manifold of dimension $2n + 1 \geq 5$. The full CR automorphism group $\text{Aut}(M, B, I)$ is compact unless $M$ is globally CR equivalent to $S^{2n+1}$ with its standard CR structure.

**Remark 2.7:** CR automorphisms are also called **CR holomorphic diffeomorphisms**. A vector field whose flow consists in CR automorphisms is called a **CR holomorphic vector field**.

### 2.2 Contact manifolds

**Definition 2.8:** Let $S$ be a manifold. Then $C(S) := S \times \mathbb{R}^{>0}$ is called the **cone over** $S$. The multiplicative group $\mathbb{R}^{>0}$ acts on $C(S)$ by dilations along the generators: $h_\lambda(x, t) \mapsto (x, \lambda t)$.

**Definition 2.9:** A differential $k$-form $\alpha$ on a cone is called **automorphic** if $h_\lambda^* \alpha = q^k \alpha$.

**Definition 2.10:** A **contact manifold** is a manifold $S$ such that its cone $C(S)$ is endowed with an automorphic symplectic form $\omega$, i.e. $h_\lambda^* \omega = \lambda^2 \omega$.

The following characterization explains the geometry of a contact manifold and its relation to CR geometry:

**Theorem 2.11:** Let $S$ be a differentiable manifold. The following are equivalent:

(i) $S$ is contact.

(ii) $S$ is odd-dimensional and there exists an oriented sub-bundle of codimension 1, $B \subset TS$, with non-degenerate Frobenius form $\Lambda^2 B \overset{\Phi}{\to} TS/B$. This $B$ is called the **contact bundle**.
(iii) $S$ is odd-dimensional and there exists an oriented sub-bundle of codimension 1, $B \subset TS$, such that for any nowhere degenerate 1-form $\eta \in \Lambda^1 S$ which annihilates $B$, the form $\eta \wedge (d\eta)^k$ is a non-degenerate volume form (where $\dim S = 2k + 1$). Every such $\eta$ is called a contact form.

**Proof:** Well known (see [MS]). ■

**Definition 2.12:** Let $S$ be a contact manifold with a contact form $\eta$. The vector field $R$ defined by the equations:

$$R \cdot \eta = 1, \quad R \cdot d\eta = 0.$$ 

is called the Reeb or characteristic field.

### 2.3 Sasakian manifolds

**Definition 2.13:** The Riemannian cone of a Riemannian manifold $(M, g_M)$ is $C(M) := M \times \mathbb{R}^{>0}$ endowed with the metric $g = t^2 g_M + dt \otimes dt$, where $t$ is the coordinate on $\mathbb{R}^{>0}$.

**Definition 2.14:** A Sasakian structure on a Riemannian manifold $(M, g_M)$ is a Kähler structure $(g, I, \omega)$ on its Riemannian cone $C(M), g = t^2 g_M + dt \otimes dt$ such that:

- $I$ is $h_\lambda$-invariant, and
- $\omega$ is automorphic: $h_\omega^* \omega = \lambda^2 \omega$.

In this case, $(C(M), g, I, \omega)$ is called the cone of the Sasakian manifold.

**Remark 2.15:**

(i) It can be seen that $\varphi = t^2$ is a Kähler potential for $\omega$, that is, $\omega = dd^c \varphi$, where $d^c = IdI^{-1}$. The manifold $M$ can be identified with a level of $\varphi$. The contact bundle on $M$ can be obtained as $TM \cap I(TM)$, where $TM$ is considered as a sub-bundle of $TC(M)$, and $I$ the complex structure on $TC(M)$. This gives a CR-structure on $M$. An easy calculation implies that the restriction of $dd^c \varphi$ to the contact bundle is equal to the Levi form. Therefore, the CR-structure on the Sasakian manifolds is strictly pseudoconvex.
(ii) Let $\xi := \frac{t}{dt}$ be the Euler field on $C(M)$. The complex structure of the cone is $h_\lambda$ invariant, thus the Euler field acts on the cone by holomorphic homotheties.

(iii) Since $\omega$ is, in particular, a symplectic form, a Sasakian manifold is, in particular, a contact manifold. Its contact form is defined as follows. Let $\eta := \xi \lrcorner \omega$. Then, by Cartan’s formula, $d\eta = \text{Lie}_\xi \omega = \omega$, and hence $(d\eta)^n \wedge \eta = \frac{1}{n+1} \xi \lrcorner \omega^{n+1}$ is non-degenerate, thus each slice $M \times \{t_0\} \subset C(M)$ is contact, with contact form the restriction of $\eta$. We may consider $\eta$ as a 1-form on $M$.

(iv) The above CR structure is given by the distribution $B := \text{ker} \eta$.

(v) Let $R := I\xi$. Clearly, $R$ is tangent to $M$ and transverse to the CR distribution. One then verifies that $\eta(R) = 1$ and $R \lrcorner d\eta = 0$, and hence $R$ is the Reeb field of the contact manifold $(M, \eta)$. Note that on a Sasakian manifold, the Reeb field is the $g_M$-dual of the contact form: $R = \eta^\sharp$. Moreover, its flow consists of contact isometries: $\text{Lie}_R g_M = 0$, $\text{Lie}_R \eta = 0$.

The relation between CR structures and Sasakian structures is clarified in the following result:

**Theorem 2.16:** ([OV2]) Let $(B, I)$ be a CR structure on an odd-dimensional compact manifold $M$. Then there exists a compatible Sasakian metric on $M$ if and only if $M$ admits a CR-holomorphic vector field which is transversal to $B$. Moreover, for every such field $v$, there exists a unique Sasakian metric such that $v$ or $-v$ is its Reeb field.

**Example 2.17:** Odd-dimensional spheres with the round metric are equipped with a natural Sasakian structure, since their cones are $\mathbb{C}^n \setminus 0$ on which the standard flat Kähler form $\sqrt{-1} dz_i \wedge d\bar{z}_i$ is automorphic.

**Definition 2.18:** A Sasakian manifold is called quasi-regular if all orbits of its Reeb field are compact.

If $M$ is a compact quasi-regular Sasakian manifold, with Reeb field $R$, one can consider the space of orbits $X := M/R$ which is the same as $C(M)/\langle \xi, R \rangle$. In this case $C(M)$ is the total space of a principal holomorphic $\mathbb{C}^*$-bundle over $X$ and the corresponding line bundle has positive curvature $dd^c \log \varphi$. Therefore, Kodaira’s theorem implies ([OV1]):

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**Theorem 2.19:** Let $M$ be a compact, quasi-regular Sasakian manifold, with Reeb field $R$. Then the space of orbits $X := M/R$ is a projective orbifold.

**Example 2.20:**

1. Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $C(X) \subset \mathbb{C}^{n+1} \setminus 0$ the corresponding cone. The cone $C(X)$ is obviously Kähler and its Kähler form is automorphic, hence the intersection $C(X) \cap S^{2n-1}$ is Sasakian. This intersection is an $S^1$-bundle over $X$. This construction gives many interesting contact manifolds, including Milnor’s exotic 7-spheres, which happen to be Sasakian.

2. In general, every link of homogeneous singularity is Sasakian. All quasi-regular Sasakian manifolds are obtained this way.

3. All 3-dimensional Sasakian manifolds are quasi-regular, see [Be, Ge1].

The following result shows that on a compact manifold, a Sasakian structure can be approximated with quasi-regular ones.

**Theorem 2.21:** ([OV1]) Let $M$ be a compact Sasakian manifold with Reeb field $R$. Then $R = \lim_{i} R_i$, where $R_i$ are the Reeb fields of quasi-regular Sasakian structures on $M$.

**Proof:** We present a proof slightly different from the original one, without making use of Theorem 2.6. Let $(B, I)$ be the subjacent CR structure, which is strictly pseudoconvex, Remark 2.15, (i). Let $G$ be the closure of the Lie subgroup generated by the flow of $R$ in the group of CR automorphisms $\text{Aut}(M, B, I)$. Since $R$ acts on $M$ by isometries, $G$ is compact. Being also commutative, $G$ is a torus. Now, any vector field $R'$ sitting in the Lie algebra of $G$ sufficiently close to $R$ will still be transversal to $B$, and hence it will be the Reeb field of another Sasakian structure (Theorem 2.16).

But a Reeb field is quasi-regular if and only if it generates a compact subgroup, i.e. it is rational with respect to the rational structure of the Lie algebra of $G$. However, rational points are dense in this Lie algebra. ■

**Remark 2.22:** Note that during the above approximation process, the CR structure remains unchanged. In particular, for each quasi-regular Reeb field $R_i$ approximating $R$, the complex structure on the projective manifold $M/R_i$ is the same, and corresponds to the one on the contact distribution on $M$. 

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3 Closed orbits of Reeb fields

We can prove now our main result, Theorem 1.1.

Let $M$ be a $(2n + 1)$-dimensional Sasakian manifold, with Reeb field $R$ and subjacent CR structure $(B, I)$. As in the proof of Theorem 2.21, denote with $G$ the closure of the group generated by the flow of $R$ in the group of CR automorphisms $\text{Aut}(M, B, I)$. i.e. $G := \langle e^{tR} \rangle$, for $t \in \mathbb{R}$. The following statement is then clear:

Claim 3.1: There exists a one-to-one correspondence between 1-dimensional orbits of $G$ and closed orbits of the Reeb field $R$.

Let $R'$ be a quasi-regular approximation of $R$ (see Theorem 2.21) and $X = M/R'$ the projective quotient orbifold. The key observation is:

Remark 3.2: The group $G$ acts on $X$ by holomorphic isometries. Moreover, there is a one-to-one correspondence between 1-dimensional orbits of the action of $G$ on $M$ and fixed points of the action of $G$ on $X$.

We thus reduced the problem of counting closed orbits of the Reeb field on $M$ to counting fixed points of a group acting by holomorphic isometries on a projective orbifold. This, in turn, is equal to the number of zeros of a generic vector field $r \in \text{Lie}(G)$. For compact projective manifolds, this number is computed by a celebrated theorem of A. Białynicki-Birula, [Bi]. The orbifold version that we need is due to E. Fontanari:

Theorem 3.3: ([Fo]) Let $r$ be a holomorphic vector field on a compact projective orbifold $X$ of complex dimension $k$. Then the number of zeros of $r$ is equal to $\sum_{i=0}^{2k} b_i(X)$, the sum of all Betti numbers of $X$.

Note that by Lefschetz theorem, the sum of all Betti numbers is at least $\dim_{\mathbb{C}} X + 1$, and in our case $\dim_{\mathbb{C}} X = n$, which completes the proof of Theorem 1.1. $

Remark 3.4: Note that in the number of zeros exhibited in Białynicki-Birula’s and Fontanari’s theorems do not count multiplicities, that is, there exist at least $n + 1$ distinct closed orbits.
4 Vaisman manifolds

4.1 Definition and the canonical foliation

Vaisman manifolds are a significant and much studied subclass of locally conformally Kähler manifolds, see [DO]. Here we give a definition which is suitable for our purpose:

Definition 4.1: Let \((V, I, g_V)\) be a Hermitian manifold such that the fundamental form \(\omega_V(\cdot, \cdot) = g_V(\cdot, I\cdot)\) satisfies the equation \(d\omega_V = \theta \wedge \omega_V\) for a \(\nabla^{g_V}\)-parallel 1-form \(\theta\), where \(\nabla^{g_V}\) is the Levi-Civita connection of \(g_V\). Then \((V, I, g_V)\) is a Vaisman manifold and \(\theta\) is its Lee form.

Remark 4.2: By [OV3, Subsection 1.3], all compact Vaisman manifolds, considered as complex manifolds, are obtained in the following way. Let \(C(M)\) be the Kähler cone of a compact Sasakian manifold and \(q\) a non-trivial holomorphic homothety of \(C(M)\). Then the compact complex manifold \(C(M)/(q)\) is Vaisman.

Example 4.3: Diagonal Hopf manifolds \((\mathbb{C}^n \setminus 0)/\langle A \rangle\) where \(A \in \text{GL}(n, \mathbb{C})\) is diagonalizable, with eigenvalues of absolute value strictly greater than 1, are Vaisman. All compact submanifolds of a Vaisman manifold are Vaisman. Non-Kähler elliptic surfaces are Vaisman ([Be]).

Remark 4.4:

(i) The 2-form \(\omega_0 := dd^c \log t\) on \(C(M) = M \times \mathbb{R}^{>0}\) is \(q\)-invariant. Moreover, one can see that \(\omega_0 = \frac{1}{t^2} (\omega_V - dt \wedge I(dt))\), hence \(\omega_0\) is positive-definite in the directions transversal to \(\langle \frac{\partial}{\partial t}, I(\frac{\partial}{\partial t}) \rangle\) and zero on \(\langle \frac{\partial}{\partial t}, I(\frac{\partial}{\partial t}) \rangle\).

(ii) The 1-form \(d\log t\), which lives on the cone, is already \(q\)-invariant. Therefore, the 2-form \(\omega_0\) descends to an exact form on the Vaisman manifold \(V\).

Definition 4.5: The foliation \(\Sigma := \ker \omega_0\) on \(V\) is called the canonical foliation of the Vaisman manifold.

Remark 4.6: The foliation \(\Sigma\) is generated by the \(g_V\)-duals of \(\theta\) and \(I\theta\), which are commuting, Killing and real holomorphic vector fields.

The name of this foliation is motivated by the following:
Proposition 4.7: The foliation $\Sigma$ is independent on the choice of the cone $C(M)$ and of the choice of the holomorphic homothety $q$.

Proof: Suppose we have two different exact and (semi-)positive forms $\omega_0$ and $\omega'_0$. Then the sum $\omega_1 := \omega_0 + \omega'_0$ is still exact and (semi-)positive. If $\ker \omega_0 \neq \ker \omega'_0$, the form $\omega_1$ is strictly positive, which is impossible because $\omega_0$ and $\omega'_0$ are exact and then Stokes theorem implies $\int_V \omega_1^{\dim V} = 0$. $lacksquare$

4.2 Complex curves on Vaisman manifolds

Complex curves are very particular on compact Vaisman manifolds: they have to be elliptic, as shown by the next result.

Theorem 4.8: Let $C$ be a complex curve on a compact Vaisman manifold $V$. Then $C$ is a leaf of the canonical foliation. In particular, $C$ is an elliptic curve.

Proof: As above, $\int_C \omega_0 = 0$, by Stokes, hence $C$ is tangent to $\Sigma = \ker \omega_0$. But all compact leaves of $\Sigma$ are elliptic since the tangent bundle $T\Sigma$ is trivial by construction. $lacksquare$

Remark 4.9: We proved in [OV1] that a compact Vaisman manifold $V$ can be holomorphically embedded in a diagonal Hopf manifold $H$ (see Example 4.3). Intersecting $V$ with two complementary flags of Hopf submanifolds (which exist due to a result of Ma. Kato, [Ka]), we see that $V$ contains at least two elliptic curves. In fact, there are many more elliptic curves, as follows from Corollary 4.10 below.

Let now $V = C(M)/(q)$ where $q = h_\lambda$, the dilation along the generators of the cone, for some fixed $\lambda > 1$. The canonical foliation $\Sigma$ on $V$ will correspond to the vector fields $\xi$ and $R = I\xi$ of the cone. In particular, a leaf of $\Sigma$ will be compact if and only if the corresponding orbit of $R$ is closed. Since $R$ is the Reeb field on $M$, we have proven the following:

Corollary 4.10: Let $M$ be a Sasakian manifold and $V = C(M)/(q)$ the compact Vaisman manifold corresponding to $q = h_\lambda$ with $\lambda > 1$. Then the number of closed Reeb orbits is equal to the number of elliptic curves in $V$.

This proves Theorem 1.2. $lacksquare$
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