Tri-bimaximal neutrino mixing from discrete subgroups of $SU(3)$ and $SO(3)$ family symmetry

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Abstract

It has recently been shown how tri-bimaximal neutrino mixing can be achieved, using the see-saw mechanism with constrained sequential dominance, through the vacuum alignment of a broken non-Abelian gauged family symmetry such as $SO(3)$ or $SU(3)$. Generalising the approach of Altarelli and Feruglio developed for an $A_4$ model we show how the reduction of the underlying symmetry to a discrete subgroup of $SO(3)$ or $SU(3)$ renders this alignment a generic property of such models. This means near tri-bimaximal mixing can be quite naturally accommodated in a complete unified theory of quark and lepton masses.
1 Introduction

Current neutrino oscillation results [1] are consistent with so-called tri-bimaximal lepton mixing in which the lepton mixing matrix takes the approximate Harrison, Perkins, Scott [2] form:

\[
U_{HPS} \approx \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\] (1)

Given the uncertainties in the current measured values of the neutrino mixing angles, and the theoretical corrections inherent in any model of lepton mixing, it is likely that tri-bimaximal mixing, if at all relevant, is realised only approximately. Nevertheless, given the symmetrical nature of tri-bimaximal mixing, it is of interest to see if it can be reproduced, at least approximately, in models of quark and lepton masses and mixings, in particular those based on the see-saw mechanism where the smallness of neutrino masses emerges most elegantly.

The fact that the MNS mixing matrix in Eq.(1) involves square roots of simple ratios motivates models in which the mixing angles are independent of the mass eigenvalues. One such class of models are see-saw models with sequential dominance (SD) of right-handed neutrinos [3]. In SD, the atmospheric and solar neutrino mixing angles are determined in terms of ratios of Yukawa couplings involving those right-handed neutrinos which give the dominant and subdominant contributions, respectively, to the see-saw mechanism. If the Yukawa couplings involving different families are related in some way, then it is possible for neutrino mixing angle relations to emerge in a simple way, independently of the neutrino mass eigenvalues. For example, maximal atmospheric neutrino mixing results from the Yukawa couplings involving second and third families having equal Yukawa couplings (up to a phase) to the dominant right-handed neutrino. Tribimaximal neutrino mixing then follows if, in addition, the Yukawa couplings involving all three families couple democratically to the leading subdominant right-handed neutrino, providing the couplings are relatively real and the second or third coupling is in anti-phase relative to those of the dominant couplings [4, 5, 6]. If the dominant and subdominant right-handed neutrinos dominate the see-saw mechanism by virtue of their lightness, then they may have the smallest Yukawa couplings, and such democratic relations between different families would not be readily apparent in the charged fermion Yukawa matrices.

The above picture in which Yukawa couplings of different families are equal (up to phases) strongly suggests a non-Abelian family symmetry which is acting behind the scenes to relate all three families together, as emphasised in [4, 7]. In the charged fermion sector, the presence of such a non-Abelian family symmetry is well hidden from view since the masses of the three families of charged fermions are strongly hierarchical,
and thus any non-Abelian family symmetry must be strongly and hierarchically broken. Even though the family symmetry is strongly broken, it is possible for the required equalities of Yukawa couplings to emerge if the several scalar fields which break the family symmetry (called flavons) have their vacuum expectation values (VEVs) carefully (mis)aligned along special directions in family space. Then, if these flavons appear in the effective operators responsible for the Yukawa couplings, the equality of the Yukawa couplings in the SD picture may be due to the particular vacuum alignment of the flavons responsible for that particular operator. These ingredients have been recently used as the basis for models of quark and lepton masses and mixings, incorporating tri-bimaximal neutrino mixing, based on $SO(3)$ \cite{5} and $SU(3)$ \cite{6} family symmetry. However it must be admitted that in these models the vacuum alignment is not realised in the most elegant or efficient manner, and it is one of the purposes of this note to show that the physics of vacuum alignment simplifies if the continuous family symmetry is replaced by a discrete family symmetry subgroup.

In this letter, then, we discuss how the vacuum (mis)alignment needed for tri-bimaximal mixing proceeds quite readily in the case that the theory is invariant under a discrete subgroup of either $SO(3)$ or $SU(3)$ family symmetry. Our vacuum alignment mechanism is a related to that of Altarelli and Feruglio who analysed the spontaneous breaking of $A_4$ \cite{8}, and indeed we show that it immediately allows for a 4-dimensional version of the $A_4$ model\footnote{This has been noted by Altarelli and Feruglio\cite{9} in a recent paper that was issued during the completion of this paper.} without supernatural fine tuning. However our main focus is concerned with simplifying the $SO(3)$ and $SU(3)$ models of refs \cite{5,6}. An important distinction between these models is whether they allow the quadratic invariant $\Sigma_i \phi_i \phi_i$, as is the case for $SO(3)$ \cite{5} or $A_4$ \cite{8}, or forbid it as is the case for $SU(3)$ \cite{6}. The reason that this is important is that, in the former case, viable models of fermion mass require that the left-handed $SU(2)_L$ doublet fermions, $\psi_i$, transform differently from the left-handed charge conjugate $SU(2)_L$ singlet fermions, $\psi_i^c$. As a result it is not straightforward to implement an underlying $SO(10) \otimes G_{\text{family}}$ symmetry. If, as is the case for $SU(3)$, the bilinear invariant is absent then it is possible to achieve this unification \cite{6}. We present three examples, two which apply to the “$SO(3)$—like” case (including $A_4$) and one which applies to the “$SU(3)$—like” case.

2 Constrained sequential dominance

To see how tri-bimaximal neutrino mixing could emerge from SD, we begin by writing the right-handed neutrino Majorana mass matrix $M_{RR}$ in a diagonal basis as

$$M_{RR} = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix},$$
where we shall assume
\[ X \lesssim Y \ll Z. \] (2)

In this basis we write the neutrino (Dirac) Yukawa matrix \( Y_{LR}^{\nu} \) in terms of \((1, 3)\) column vectors \( A_i, B_i, C_i \) as
\[ Y_{LR}^{\nu} = (A \ B \ C) \] (3)
in the convention where the Yukawa matrix corresponds to the Lagrangian coupling \( \bar{L} H_u Y_{LR}^{\nu} \nu_R \), where \( L \) are the left-handed lepton doublets, \( H_u \) is the Higgs doublet coupling to up-type quarks and neutrinos, and \( \nu_R \) are the right-handed neutrinos. The Dirac neutrino mass matrix is then given by \( m_{LR}^{\nu} = Y_{LR}^{\nu} v_u \), where \( v_u \) is the vacuum expectation value (VEV) of \( H_u \). The effective Lagrangian resulting from integrating out the massive right handed neutrinos is
\[ L_{\text{eff}} = \left( \nu_i^T A_i \right) \left( A_j^T \nu_j \right) X + \left( \nu_i^T B_i \right) \left( B_j^T \nu_j \right) Y + \left( \nu_i^T C_i \right) \left( C_j^T \nu_j \right) Z \] (4)
where \( \nu_i, i = 1, 2, 3 \) are the left handed neutrino fields.

The case of interest here is the one in which these terms are ordered, due to the ordering in Eq.(2), with the third term negligible, the second term subdominant and the first term dominant - “light sequential dominance” (LSD)\(^3\), “light” because the lightest right-handed neutrino makes the dominant contribution to the see-saw mechanism. LSD is motivated by unified models in which only small mixing angles are present in the Yukawa sector, and implies that the heaviest right-handed neutrino of mass \( Z \) is irrelevant for both leptogenesis and neutrino oscillations (for a discussion of all these points see\(^10\)).

In \([4, 5]\) we proposed the following set of conditions which are sufficient to achieve tri-bimaximal mixing within the framework of sequential dominance “constrained sequential dominance (CSD)”:

\[ |A_1| = 0, \]
\[ |A_2| = |A_3|, \] (5a)
\[ |B_1| = |B_2| = |B_3|, \] (5b)
\[ A^\dagger B = 0. \] (5c)
\[ A^\dagger B = 0. \] (5d)

The condition in Eqs.(5a,5b) gives rise to bi-maximal mixing in the atmospheric neutrino sector, \( \tan \theta_{23}^{\nu} = 1 \). The remaining conditions in Eq.5 give tri-maximal mixing in the solar neutrino sector, \( \tan \theta_{12}^{\nu} = 1/\sqrt{3} \) and to \( \theta_{13}^{\nu} = 0 \).

With this it is straightforward to build theories which generate tri-bimaximal mixing. A very simple example example is provided by a supersymmetric theory in which the lepton doublets \( L \) are triplets of an \( SO(3) \) family symmetry, but the CP conjugates of

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the right-handed neutrinos, $\nu^c_i$, and Higgs doublets, $H_{u,d}$, are singlets under the family symmetry\(^2\). To generate hierarchical charged lepton masses we need spontaneous breaking of the family symmetry

$$SO(3) \longrightarrow SO(2) \longrightarrow \text{Nothing.}$$

To achieve this symmetry breaking we introduce the additional $SO(3)$ triplet “flavon” fields $\phi_3, \phi_{23}, \phi_{123}$ whose VEVs, $<\phi>$ break the $SO(3)$ family symmetry. The vacuum alignment of the flavon VEVs plays a crucial role in this model, as follows. Suppose that symmetries of the model allow only the Yukawa couplings associated with the superpotential terms of the form:

$$y'' LH_u \nu_1 \frac{\phi_{23}}{M} + y LH_u \nu_2 \frac{\phi_{123}}{M} + y' LH_u \nu_3 \frac{\phi_3}{M}$$

where $y, y', y''$ are complex Yukawa couplings, $M$ is a mass scale. These generate Dirac neutrino mass terms of the form given in Eq.(3) with

$$A_i = <\phi_{23}>_i, \quad B_i = <\phi_{123}>_i, \quad C_i = <\phi_3>_i.$$ 

Provided the vacuum alignment of the VEVs of $\phi_3, \phi_{23}, \phi_{123}$ satisfy Eqs.(5a, 5b, 5c, 5d) one achieves tri-bimaximal mixing with sequential dominance.

This example clearly illustrates the importance of this pattern of vacuum (mis)alignment of the flavon VEVs in achieving tri-bimaximal mixing and the remainder of this paper is concerned with achieving such a vacuum (mis)alignment using discrete family symmetries.

3 \quad A_4

We start with a discussion of the vacuum structure for the potential of a model of fermion masses based on the discrete symmetry $A_4$. The group $A_4$ (or $\Delta(12)$)\(^3\) is a discrete subgroup of $SO(3)$ and $SU(3)$ and so it is relevant to the generalisation of the $SO(3)$ and $SU(3)$ family symmetry models. In this model the symmetry breaking is generated by two $A_4$ triplet fields $\varphi$ and $\varphi'$ with VEVs $\varphi = (v, v, v)$ and $\varphi' = (0, 0, v')$. Although the notation is different, these correspond to the flavons $\phi_{123}$ and $\phi_3$ discussed earlier, and this alignment leads to a model of tri-bimaximal mixing. The alignment is naturally generated along the $F$–flat direction in a specific model with the superpotential constrained by an additional $Z_3 \otimes U(1)_R$ symmetry under which the fields transform as in Table 1. In addition the model uses the triplet “driving” fields $\varphi_0, \varphi'_0$, as well as two\(^4\)

\(^2\)SO(3) has been previously used as a family symmetry in e.g. [12].

\(^3\)A$_4$ $\equiv \Delta(12)$ is one of the family of dihedral like $\Delta(3n^2)$ finite subgroups of $SU(3)$, whose irreducible representations are either 1 or 3 dimensional [13].

\(^4\)The original model uses just one, c.f. [8].
$A_4$ singlets $\xi_1$, $\xi_2$ that acquire VEVs and $\xi_0$ to drive these. Their charge assignments under $Z_3 \otimes U(1)_R$ are listed in Table 1, where $\omega$ is the cube root of unity.

The most general renormalisable superpotential allowed by these symmetries is given by

$$w_d = M(\varphi_0 \varphi) + g(\varphi_0 \varphi \varphi) + g_1(\varphi_0 \varphi' \varphi') + (f_1 \xi_1 + f_2 \xi_2)\varphi_0 \varphi' + f_3 \xi_0 \xi_i \xi_j \varphi' \varphi' \varphi'$$

where the 3-triplet invariant $\varphi_0 \varphi \varphi_0$ stands for $\varphi_1 \varphi_2 \varphi_3$ and cyclic permutations.

The vacuum minimisation conditions correspond to the vanishing of the $F-$terms. For the $\varphi$ field this corresponds to

$$\frac{\partial w}{\partial \varphi_0} = M \varphi_1 + g \varphi_2 \varphi_3 = 0$$
$$\frac{\partial w}{\partial \varphi_2} = M \varphi_2 + g \varphi_3 \varphi_1 = 0$$
$$\frac{\partial w}{\partial \varphi_3} = M \varphi_3 + g \varphi_1 \varphi_2 = 0$$

These are solved by

$$\varphi = (v, v, v), \quad v = \frac{M}{g}. \quad (10)$$

For the $\varphi'$ field the minimisation conditions are given by

$$\frac{\partial w}{\partial \varphi'_0} = g_1 \varphi'_2 \varphi'_3 + (f_1 \xi_1 + f_2 \xi_2)\varphi'_1 = 0$$
$$\frac{\partial w}{\partial \varphi'_2} = g_1 \varphi'_3 \varphi'_1 + (f_1 \xi_1 + f_2 \xi_2)\varphi'_2 = 0$$
$$\frac{\partial w}{\partial \varphi'_3} = g_1 \varphi'_1 \varphi'_2 + (f_1 \xi_1 + f_2 \xi_2)\varphi'_3 = 0$$

And also

$$\frac{\partial w}{\partial \xi_0} = f_3 (\varphi' \varphi') + f_{ij} \xi_i \xi_j \varphi' \varphi' \varphi' = 0 \quad (11)$$

which sets the magnitude of $\varphi' \varphi'$. 

| Field | $\varphi$ | $\varphi'$ | $\xi_1$ | $\xi_2$ | $\varphi_0$ | $\varphi'_0$ | $\xi_0$ |
|-------|----------|-----------|--------|--------|-------------|------------|--------|
| $Z_3$ | 1        | $\omega$  | $\omega$ | $\omega$ | 1           | $\omega$ | $\omega$ |
| $U(1)_R$ | 0 | 0 | 0 | 0 | 2 | 2 | 2 |

Table 1: Transformation property of the fields in the $A_4$ model.
To be able to satisfy eqs. (11) while having the magnitude of $\varphi'$ fixed by eq(11), the VEVs of the singlets must be such to make $f_1 \xi_1 + f_2 \xi_2$ vanish. That leaves us with the solution

$$\varphi' = (0, 0, v'),$$

(12)

where at tree level $v'$ is undetermined but will be induced through dimensional transmutation at radiative order if radiative corrections drive the $\varphi'$ mass squared negative. Such radiative correction are generic and occur if the field $\varphi'$ has significant Yukawa couplings such as the $g_1$ term in Eqs.(8, 10,12) generates the required vacuum alignment.

Note that the potential presented here has an important advantage over the potential considered in [8] in that the associated $A_4$ model does not require the vanishing of any coupling allowed by the symmetry - at the cost of including one extra singlet field. Such “supernatural” vanishing was necessary in the supersymmetric model constructed by Altarelli and Feruglio and led them to construct a five dimensional model in order to obtain a fully natural theory. Our example here shows that this version of the four dimensional model is also fully natural - the remainder of the model is identical to that presented in [8] - and leads to tri-bimaximal mixing. This model has recently been constructed by Altarelli and Feruglio in a paper we received while completing this work [9].

3.1 $Z_3' \ltimes Z_2$

The group $A_4$ has the structure of the semi-direct product group $Z_3' \ltimes Z_2$ and its structure can help to understand the properties of the group and the nature of the group invariants. Under it a generic $A_4$ “triplet” field $\phi_i$ transforms in the manner given in Table 2. From this it is clear that the only low order invariants are $\phi^2 = \phi_i \phi_i$ and $\phi^3 = \phi_1 \phi_2 \phi_3$ as used above. The $Z_3'$ and $Z_2$ factors are clearly discrete subgroups of $SO(3)$ and thus one sees that the $Z_3' \ltimes Z_2$ non-Abelian group is also a subgroup of $SO(3)$.

Given this it is easy to generalise the $SO(3)$ model discussed above by reducing the symmetry group to $Z_3' \ltimes Z_2$ and identifying the fields $\phi_{123} = \varphi$ and $\phi_3 = \varphi'$. To get the full model it is also necessary to generate $\phi_{23} = (0, v'', -v'')$ for the remaining flavon field. Its alignment is readily obtained. Introduce the singlet driving fields $\chi_0$ and $\chi_1$ which transform as in Table 3. The allowed superpotential terms are

$$w' = h_1 \chi_0 \phi_{23} \phi_{123} \phi_{123} + h_2 \chi_1 \phi_3 \phi_{23} \phi_{23}.$$

(13)

If radiative corrections drive the mass squared of the field $\phi_{23}$ negative at the scale $\Lambda$, it will acquire a VEV of $O(\Lambda)$. The condition $F_{\chi_0} = 0$ forces this vev to be orthogonal.

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5This example demonstrates that the vacuum structure is natural but if one wants to embed the symmetry breaking sector into the model of [8] it will be necessary to extend the additional symmetry slightly.
Table 2: Transformation properties of a generic triplet field $\phi$ under the semi direct product group $Z'_3 \ltimes Z_2$.

| Field | $\phi_{123}$ | $\phi_3$ | $\phi_{23}$ | $\varphi_0$ | $\varphi'_{0}$ | $\chi_0$ | $\chi_1$ |
|-------|-------------|--------|-------------|-----------|-------------|--------|--------|
| $Z_3$ | 1           | $\omega$ | 1           | $\omega$  | 1           | $\omega^2$ |
| $U(1)_R$ | 0         | 0     | 1           | 2         | 2           | 1      | 0      |

Table 3: Transformation property of the fields in the $Z'_3 \ltimes Z_2$ model.

Now $\phi_3$, $\phi_{123}$ and $\phi_{23}$ generate tri-bimaximal mixing using the strategy illustrated in section 2 and developed in [5].

\[ \phi_{23} = (0, -v'', v''), \quad v'' \simeq \Lambda \quad (14) \]

4 \quad $Z'_3 \ltimes Z''_3 \ltimes Z_2$ ($\Delta(108)$)

In the model based on $Z'_3 \ltimes Z_2$ the left-handed $SU(2)_L$ doublet fermions, $\psi_i$, are triplets under the $Z'_3$ while the left-handed charge conjugate $SU(2)_L$ singlet fermions, $\psi_i^c$, are singlets. As a result it is not straightforward to embed the model in an underlying $SO(10)$ theory. In this Section we show how vacuum alignment through a non-Abelian discrete symmetry can readily be consistent with an underlying $SO(10)$ structure.

As a simple example consider the discrete group $Z'_3 \ltimes Z''_3 \ltimes Z_2$ \footnote{This group is $\Delta(108)$, i.e. the dihedral like discrete subgroup of $SU(3)$ with $n = 6$ [13]} in which triplet fields $\phi_i$ transform as shown in Table 4 where $\omega$ is the cube root of unity. In this case the only low order invariant allowed by this symmetry is $\phi^3 = \phi_1 \phi_2 \phi_3$. The reason this is important is because an underlying $SO(10)$ gauge group requires that $\psi_i$ and $\psi_i^c$ should be assigned to the same triplet representation. In order to build a viable model of masses it is necessary to forbid the invariant $\psi_i \psi_i^c$. This is possible with discrete subgroups of $SU(3)$ family symmetry as this example shows (but is not possible for discrete subgroups of $SO(3)$ family symmetry as the previous example demonstrated).
Apart from this difference, the model is quite similar to the previous example with fields \( \phi_3, \phi_{23}, \phi_{123} \) as in the previous example which transform under the same symmetry with the same charges as in Table 3. In this case the superpotential takes the form

\[
w = g(\varphi_0\phi_{123}\phi_{123}) + g_1(\varphi'_0\phi_3\phi_3) + \frac{h_1}{M^3}(\varphi_0\phi_{123}\phi_{123})(\phi_{123}\phi_{123}\phi_{123}) + \frac{h_2}{M^3}(\varphi_{0,1}\phi_{123,1}^5 + \varphi_{0,2}\phi_{123,2}^5 + \varphi_{0,3}\phi_{123,3}^5) \tag{15}\]

Here we have allowed for the two possible higher dimension terms of the form \((\varphi\phi\phi)(\phi\phi\phi)\) and \((\varphi_1\phi_1^5 + \varphi_2\phi_2^5 + \varphi_3\phi_3^5)\) because, unlike the first example, the vacuum structure is sensitive to such higher order terms in leading order. The scale \( M \) is the messenger mass scale, possibly the Planck scale \( M_{\text{Planck}} \), responsible for generating these operators.

Clearly the vacuum structure of \( \phi_3 \) is still determined by the Eq.(11) so, allowing for radiative breaking we have

\[
\phi_3 = (0, 0, v') \tag{16}
\]

However the minimisation conditions for \( \phi_{123} \) change and are now given by

\[
\begin{align*}
\frac{\partial w}{\partial \varphi_{0,1}} &= \Phi_2\Phi_3(g + h_1(\Phi\Phi\Phi)) + h_2\Phi_1^5 = 0 \\
\frac{\partial w}{\partial \varphi_{0,2}} &= g\Phi_3\Phi_1(g + h_1(\Phi\Phi\Phi)) + h_2\Phi_2^5 = 0 \\
\frac{\partial w}{\partial \varphi_{0,3}} &= g\Phi_1\Phi_2(g + h_1(\Phi\Phi\Phi)) + h_2\Phi_3^5 = 0
\end{align*}
\tag{17}
\]

where on the right-hand side we have written \( \Phi = \phi_{123} \). This is solved by

\[
\phi_{123} = (v, v, v), \quad v^3 = -\frac{gM^3}{(h_1 + h_2)} \tag{18}
\]

Once again we see, by suitable choice of parameters, that it is easy to obtain the vacuum alignment needed for tri-bimaximal mixing. The vev of the field \( \phi_{23} \) in the direction given by Eq.(11) may be aligned in the same way as the \( Z_3 \times Z_2 \) model thorough the introduction of the singlet driving fields \( \chi_0 \) and \( \chi_1 \) which transform as in Table 3, giving the the allowed superpotential terms, of Eq.(13). The full model based on this discrete symmetry subgroup of \( SU(3) \) is a simplification of the model given in [6], and will be discussed in a future publication [14].

In summary, tri-bimaximal mixing in the neutrino sector occurs quite naturally in CSD models in which vacuum alignment follows from a discrete non-Abelian subgroup of the \( SU(3) \) maximal family group commuting with an underlying GUT. In our examples the tri-maximal mixing is directly related to the existence of the underlying \( Z_3 \) factor.
\[ \begin{array}{c|c|c|c} \phi_i & Z'_3 \phi_i |_{i} & Z''_3 \phi_i |_{i} & Z_2 \phi_i |_{i} \\ \hline \phi_1 & \rightarrow \phi_2 & \rightarrow \phi_1 & \rightarrow \phi_1 \\ \phi_2 & \rightarrow \phi_3 & \rightarrow \omega \phi_2 & \rightarrow -\phi_2 \\ \phi_3 & \rightarrow \phi_1 & \rightarrow \omega^2 \phi_3 & \rightarrow -\phi_3 \\ \end{array} \]

Table 4: Transformation properties of a generic triplet field \( \phi \) under the semi direct product group \( Z'_3 \ltimes Z''_3 \ltimes Z_2 \).

while the bi-maximal mixing is due to the \( Z_2 \) factor, giving a very intuitive origin for the structure. The strategy we have detailed here allows for the extension of a Grand Unified Theory to include a non-Abelian family symmetry of this type. While it seems impossible to incorporate the full \( SU(3) \) family symmetry in heterotic or D-brane string constructions, such discrete non-Abelian groups readily appear as symmetries of the underlying compactification manifold. This is encouraging for the prospect of building a viable superstring theory of fermion masses.

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