Global solutions to the tangential Peskin problem in 2-D

Jiajun Tong

Beijing International Center for Mathematical Research, Peking University, Beijing 100871, People’s Republic of China
E-mail: tongj@bicmr.pku.edu.cn

Received 31 March 2023; revised 31 October 2023
Accepted for publication 13 November 2023
Published 5 December 2023

Recommended by Dr Yao Yao

Abstract
We introduce and study the tangential Peskin problem in 2D, which is a scalar drift-diffusion equation with a nonlocal drift. It is derived with a new Eulerian perspective from a special setting of the 2D Peskin problem where an infinitely long and straight 1D elastic string deforms tangentially in the Stokes flow induced by itself in the plane. For initial datum in the energy class satisfying natural weak assumptions, we prove existence of its global solutions. This is considered as a super-critical problem in the existing analysis of the Peskin problem based on Lagrangian formulations. Regularity and long-time behaviour of the constructed solution is established. Uniqueness of the solution is proved under additional assumptions.

Keywords: Peskin problem, fluid–structure interaction, global solution, drift-diffusion equation, nonlocal drift
Mathematics Subject Classification numbers: 35A01, 35A02, 35D30, 35K55, 35Q74, 35R11

1. Introduction
1.1. Problem formulation and main results

Consider the following scalar drift-diffusion equation on \( \mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z}) = [-\pi, \pi) \),

\[
\partial_t f - \mathcal{H} f \cdot \partial_x f - f (-\Delta)^{\frac{1}{2}} f, \quad f(x, 0) = f_0(x).
\]

(1.1)

Here \( \mathcal{H} \) denotes the Hilbert transform on \( \mathbb{T} \),

\[
\mathcal{H}f(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{T}} \frac{f(y)}{2 \tan \left( \frac{y-x}{2} \right)} \, dy,
\]

(1.2)
and
\[
(-\Delta)^2 f(x) := \frac{1}{\pi} \text{p.v.} \int_\mathbb{T} \frac{f(x) - f(y)}{4\sin^2 \left(\frac{x-y}{2}\right)} \, dy = \mathcal{H}f'(x). \tag{1.3}
\]

In the paper, we want to study positive global solutions \(f = f(x,t)\) to (1.1) under rather weak assumptions on the initial data.

In section 2 of the paper, we will show that the drift-diffusion equation (1.1) arises as a scalar model of the 2D Peskin problem. In general, the 2D Peskin problem, also known as the 2D Stokes immersed boundary problem in the literature, describes coupled motion of a 2D Stokes flow and 1D closed elastic string immersed in it [46, 48, 54, 62]. Its mathematical formulation and related analytical works will be reviewed in section 2.1. The equation (1.1) stems from a special setting of it, where an infinitely long and straight elastic string deforms and moves only \textit{tangentially} in a Stokes flow in \(\mathbb{R}^2\) induced by itself. If we assume the string lies along the \(x\)-axis, then \(f = f(x,t)\) in (1.1) represents how much the string segment at the spatial point \((x,0)\) gets stretched in the horizontal direction (see (2.9)). Thus, we shall call (1.1) the \textit{tangential Peskin problem} in 2D. Analysis of such simplified models may improve our understanding of well-posedness and possible blow-up mechanism of the original 2D Peskin problem. See section 2.2 for more detailed discussions.

Suppose \(f\) is a strictly positive smooth solution to (1.1). It is straightforward to find that \(F := \frac{1}{f}\) verifies a conservation law
\[
\partial_t F = \partial_x (\mathcal{H}f \cdot F), \quad F(x,0) = F_0(x) := (f_0(x))^{-1}. \tag{1.4}
\]

This leads to the following definition of (positive) weak solutions to (1.1).

\textbf{Definition 1.1 (weak solution).} Let \(f_0 \in L^1(\mathbb{T})\) such that \(f_0 > 0\) almost everywhere, and \(F_0 := \frac{1}{f_0} \in L^1(\mathbb{T})\). For \(T \in [0, +\infty]\), we say that \(f = f(x,t)\) is a \((\text{positive})\) weak solution to (1.1) on \(\mathbb{T} \times [0,T]\), if the followings hold.

\begin{enumerate}[(a)]
  \item \(f > 0\) a.e. in \(\mathbb{T} \times [0,T]\), and \(F \in L^\infty([0,T];L^1(\mathbb{T}))\), where \(F \equiv \frac{1}{f}\) a.e. in \(\mathbb{T} \times [0,T]\);
  \item \(\mathcal{H}f \cdot F \in L^\infty_{\text{loc}}(\mathbb{T} \times [0,T])\);
  \item For any \(\varphi \in C^\infty_0(\mathbb{T} \times [0,T])\),
\end{enumerate}

\[
\int_\mathbb{T} \varphi(x,0)F_0(x) \, dx + \int_{\mathbb{T} \times [0,T]} \partial_t \varphi \cdot F \, dx \, dt = \int_{\mathbb{T} \times [0,T]} \partial_x \varphi \cdot \mathcal{H}f \cdot F \, dx \, dt. \tag{1.5}
\]

In this case, we also say \(F\) is a \((\text{positive})\) weak solution to (1.4).

If \(T = +\infty\), the weak solution is said to be global.

\textbf{Remark 1.1.} Given (1.1), it is tempting to define the weak solutions by requiring

\[
\int_\mathbb{T} \varphi(x,0)f_0(x) \, dx + \int_{\mathbb{T} \times [0,T]} \partial_t \varphi \cdot f \, dx \, dt
= \int_{\mathbb{T} \times [0,T]} \partial_x \varphi \cdot \mathcal{H}f \cdot f \, dx \, dt + 2 \int_{\mathbb{T} \times [0,T]} (-\Delta)^{\frac{1}{2}}(\varphi f)(-\Delta)^{\frac{1}{2}}f \, dx \, dt
\]

to hold for any \(\varphi \in C^\infty_0(\mathbb{T} \times [0,T])\). Although this formulation imposes no regularity assumptions on \(1/f\), it does not effectively relocate the derivative to the test function, so we choose not to work with it.
We introduce the $H^p$-semi-norms for $s > 0$,
$$\|f\|^2_{H^p(\mathbb{T})} := \sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{f}_k|^2,$$
where the Fourier transform of a function $f$ in $L^1(\mathbb{T})$ is defined by
$$\mathcal{F}(f)_k = \hat{f}_k := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} \, dx. \quad (1.6)$$

Our first result is on the existence of a global weak solution with nice properties.

**Theorem 1.1.** Suppose $f_0 \in L^1(\mathbb{T})$ satisfies $f_0 > 0$ almost everywhere, with $F_0 := \frac{1}{\alpha} f_0 \in L^1(\mathbb{T})$. Then (1.1) admits a global weak solution $f = f(x, t)$ in the sense of definition 1.1. It has the following properties.

1. **(Energy relation)** $f \in L^\infty([0, +\infty); L^1(\mathbb{T})) \cap L^2([0, +\infty); H^{1/2}(\mathbb{T}))$. For any $t \geq 0$,
   $$\|f(\cdot, t)\|_{L^1} + 2 \int_0^t \|f(\cdot, \tau)\|_{H^{1/2}}^2 \, d\tau \leq \|f_0\|_{L^1},$$
   while for any $0 < t_1 \leq t_2$,
   $$\|f(\cdot, t_2)\|_{L^1} + 2 \int_{t_1}^{t_2} \|f(\cdot, \tau)\|_{H^{1/2}}^2 \, d\tau = \|f(\cdot, t_1)\|_{L^1}. \quad (1.7)$$

2. **(Instant positivity and smoothness)** For any $t_0 > 0$, the solution $f$ is $C^\infty$ and strictly positive in $\mathbb{T} \times [t_0, +\infty)$, so $f$ is a positive strong solution to (1.1) for all positive times (see definition 3.1). To be more precise, define $F = \frac{1}{\alpha}$ pointwise on $\mathbb{T} \times (0, +\infty)$. Then $\|f(\cdot, t)\|_{L^\infty}$ and $\|F(\cdot, t)\|_{L^\infty}$ are non-increasing in $t \in (0, +\infty)$. They satisfy
   $$\|f(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq C t^{-\frac{1}{2}} \|f_0\|_{L^1}^{\frac{1}{2}} \quad \forall t \in (0, \|f_0\|_{L^1}^{-1}], \quad (1.8)$$
   where $C$ is a universal constant, and that
   $$\|F(\cdot, t)\|_{L^\infty} \leq \frac{1}{8} \|F_0\|_{L^1} \exp \left[ \coth \left( \frac{4}{\alpha} \|F_0\|_{L^1}^{-1} t \right) \right] \quad \forall t > 0.$$  
   Moreover, for any $\alpha \in (0, \frac{1}{2})$, $\|f(\cdot, t)\|_{C^\alpha} \in L^1([0, +\infty))$, satisfying that, for $t \in (0, \|f_0\|_{L^1}^{-1}]$,
   $$\int_0^t \|f(\cdot, \tau)\|_{C^\alpha} \, d\tau \leq C_\alpha t^{\frac{1-\alpha}{2}} \left( \|f_0\|_{L^1} - 2\pi f_\infty \right)^{\frac{1-\alpha}{2}} \|f_0\|_{L^1} \|F_0\|_{L^1}^{\frac{\alpha}{2}},$$
   where $C_\alpha > 0$ is a universal constant depending on $\alpha$, and where $f_\infty$ is defined in (1.9) below.

3. **(Conservation law)** $\|F(\cdot, t)\|_{L^1} = \|F_0\|_{L^1}$ for all $t \geq 0$, and the map $t \mapsto F(\cdot, t)/\|F_0\|_{L^1}$ is continuous from $[0, +\infty)$ to $\mathcal{P}_1(\mathbb{T})$. Here $\mathcal{P}_1(\mathbb{T})$ denotes the space of probability densities on $\mathbb{T}$ equipped with $1$-Wasserstein distance $W_1$ (see e.g. [65, chapter 6]).
(4) (Long-time exponential convergence) Define

\[ f_\infty := \left( \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{f_0(x)} \, dx \right)^{-1}. \] (1.9)

Then for arbitrary \( j, k \in \mathbb{N}, \| \partial_x^j \partial_t^k (f(\cdot, t) - f_\infty) \|_{L^2(\mathbb{T})} \) is finite for any \( t > 0 \), and it decays to 0 exponentially as \( t \to +\infty \). It admits an upper bound that only depends on \( j, k, \| f_0 \|_{L^1}, \) and \( \| F_0 \|_{L^1}. \) See proposition 5.2.

(5) (Large-time analyticity and sharp decay) There exists \( T_* > 0 \) depending only on \( \| f_0 \|_{L^1} \) and \( \| F_0 \|_{L^1}. \), such that \( f(x, t) \) is analytic in the space-time domain \( \mathbb{T} \times (T_*, +\infty) \). In addition, for any \( t > 0 \),

\[ \| f(\cdot, t + T_*) \|_{\mathcal{F}_{0,1}} \leq C f_\infty, \]

with

\[ \nu(t) \geq \frac{1}{2} \ln \left[ 1 + \exp (2f_\infty t) \right] - \frac{1}{2} \ln 2, \]

and \( C > 0 \) being a universal constant. Here the \( \mathcal{F}_{\nu(t)} \) semi-norms for Wiener-algebra-type spaces will be defined in section 7.

**Remark 1.2.** We remark that \( L^1(\mathbb{T}) \) is considered to be the energy class for the initial data \( f_0 \) (see remark 3.1), while the condition \( F_0 \in L^1(\mathbb{T}) \) is natural given (1.4) (also see remark 2.2). Such initial data \( f_0 \) may be unbounded from above. It can also touch zero, but not smoothly due to the assumption \( F_0 = 1/f_0 \in L^1(\mathbb{T}) \); yet, it can touch zero with a cusp. If in addition \( F_0 \in L^p(\mathbb{T}) \) for some \( p > 1 \), the energy equality holds up to \( t = 0 \) for the constructed solution, i.e. for any \( t \geq 0 \),

\[ \| f(\cdot, t) \|_{L^1} + 2 \int_0^t \| f(\cdot, \tau) \|_{H^1}^2 \, d\tau = \| f_0 \|_{L^1}. \]

Note that we do not claim the above properties for all weak solutions, but only for the constructed one. Regularity of general weak solutions to (1.1) is wildly open.

It will be clear in section 2 that the equation (1.1) and theorem 1.1 are formulated with a Eulerian perspective. In corollary 2.1 in section 2.2, we will recast these results in the corresponding Lagrangian coordinate, which is more commonly used in the analysis of the Peskin problem. Roughly speaking, in the Lagrangian framework, for arbitrary initial data in the energy class satisfying some weak assumptions but no restriction on its size, we are able to well define a global solution to the 2D tangential Peskin problem with nice properties. Let us note that, in the existing studies on the Peskin problem with the Lagrangian perspectives, this is considered as a super-critical problem. See more discussions in section 2.

We also would like to highlight the following very interesting property of (1.1), which is helpful when proving theorem 1.1. It states that band-limited initial datum give rise to band-limited solutions, with the band width being uniformly bounded in time. This allows us to easily construct global solutions from such initial datum. This property is found by taking the Fourier transform of (1.1). We will justify it in section 6.1.

**Proposition 1.1.** Suppose \( f_0 > 0 \) is band-limited, namely, there exists some \( K \in \mathbb{N} \), such that \( \mathcal{F}(f_0)_k = 0 \) for all \( |k| > K \). Then (1.1) has a global strong solution \( f = f(x, t) \) that is also strictly positive and band-limited, such that \( f(t) = 0 \) for all \( |k| > K \) and \( t \geq 0 \). Such a solution is
For any convex function $f$, we let $f_k(t) \equiv 0$ for all $|k| > K$. Denote $\hat{f}(\cdot, t)_0 = \frac{1}{\pi} \int_{\mathbb{T}} f(x, t) \mathrm{d}x$. Then for $k \in [0, K]$, $\hat{f}_k(t) = \mathcal{F}(f(\cdot, t))_k$ solves

$$
\frac{d}{dt} \hat{f}_k(t) = -k\hat{f}(t)\hat{f}_k(t) - \sum_{j>0} 2(k + 2j)\hat{f}_{k+j}(t)\hat{f}_j(t), \quad \hat{f}_k(0) = \mathcal{F}(f_0)_k,
$$

where $\hat{f}_0(t)$ should be understood as $\hat{f}(t)$. Finally, let $\hat{f}_k(t) = \hat{f}_{-k}(t)$ for $k \in [-K, 0)$.

Uniqueness of the weak solution to (1.1) is also an open question given the very weak assumptions on the initial data and the solution. To achieve positive results on the uniqueness, we introduce a new notion of weak solutions. We inherit the notations from definition 1.1.

**Definition 1.2 (dissipative weak solution).** Under the additional assumption $\ln f_0 \in H^{1/2}(\mathbb{T})$, a weak solution $f = f(x, t)$ to (1.1) on $\mathbb{T} \times [0, T)$ (in the sense of definition 1.1) is said to be a dissipative weak solution, if it additionally verifies

(a) $\ln f \in L^\infty([0, T]; H^{1/2}(\mathbb{T}))$ and $\partial_x \sqrt{f} \in L^2(\mathbb{T} \times [0, T])$, satisfying that (see lemma 5.2)

$$
\|\ln f(\cdot, t)\|^2_{H^{1/2}} + 4 \int_0^t \int_\mathbb{T} \left(\partial_x \sqrt{f}\right)^2 \mathrm{d}x \mathrm{d}t \leq \|\ln f_0\|^2_{H^{1/2}}
$$

for any $t \in [0, T]$.

(b) For any convex function $\Phi \in C^1_{\text{loc}}((0, +\infty))$ such that $\Phi(F_0(x)) \in L^1(\mathbb{T})$, it holds

$$
\int_\mathbb{T} \Phi(F(x, t)) \mathrm{d}x + \int_0^t \int_\mathbb{T} (\Phi(F) - F\Phi'(F)) (-\Delta)^{\frac{1}{2}} f \mathrm{d}x \mathrm{d}t \leq \int_\mathbb{T} \Phi(F_0(x)) \mathrm{d}x \quad (1.10)
$$

for all $t \in [0, T)$ (see remark 6.1).

Then we state uniqueness of the dissipative weak solutions under additional assumptions.

**Theorem 1.2.** Suppose $\ln f_0 \in H^{1/2}(\mathbb{T})$. Further assume $f_0 \in L^\infty(\mathbb{T})$ and $f_0 \geq \lambda$ almost everywhere with some $\lambda > 0$. Then (1.1) admits a unique dissipative weak solution.

In particular, the solution coincides with the one constructed in theorem 1.1, and thus it satisfies all the properties there.

### 1.2. Related studies

The equation (1.1) is derived from a special setting of the 2D Peskin problem, but we choose to review related works in that direction in section 2.1. Beyond that, (1.1) is reminiscent of many classic partial differential equations (PDEs) arising in fluid dynamics and other subjects. We list a few, but we note that the list of equations and literature below is by no means exhaustive.

(I) In [16], Córdoba, Córdoba, and Fontelos (CCF) studied

$$
\partial_t \theta - \mathcal{H}\theta \cdot \partial_x \theta = -\nu (-\Delta)^{\frac{1}{2}} \theta
$$

on $\mathbb{R}$. They showed finite-time blow-up for the inviscid case $\nu = 0$, and yet well-posedness when $\nu > 0$ and $\alpha \geq 1$, with smallness of initial data assumed for the critical case $\alpha = 1$. See [23, 42, 44, 58] for related results.
When $\nu > 0$, the CCF model is known as a 1D analogue of the dissipative quasigeostrophic equation in $\mathbb{R}^2$

$$
\partial_t \theta + u \cdot \nabla \theta = -(\Delta)^{\frac{\alpha}{2}} \theta, \quad u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta).
$$

Here $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the Riesz transforms on $\mathbb{R}^2$. The case $\alpha = 1$ is again critical, for which global well-posedness results are established in e.g. \cite{8, 15, 38}; also see \cite{24}.

Compared with the CCF model with $\nu > 0$ and $\alpha = 1$, our equation \eqref{1.1} features a fractional diffusion with variable coefficient. The diffusion is enhanced when $f$ is large, while it becomes degenerate when $f$ is almost zero. The latter turns out to be one of the main difficulties in the analysis of \eqref{1.1}. See lemma 4.1 and section 5.1 for how we handle such possible degeneracy.

(II) The equation

$$
\partial_t \rho + \partial_s (\rho \mathcal{H} \rho) = 0, \quad (1.11)
$$

or equivalently (at least for nice solutions $\rho = \rho(x,t)$),

$$
\partial_t \rho = -\partial_s \rho \cdot \mathcal{H} \rho - \rho (\Delta)^{\frac{s}{2}} \rho, \quad (1.12)
$$

has been studied extensively in several different contexts, such as fluid dynamics \cite{2, 10, 12}, and dislocation theory \cite{3, 21, 32–34}. Interestingly, \eqref{1.11} has similarities with both \eqref{1.1} and \eqref{1.4}. Indeed, \eqref{1.12} differs from \eqref{1.1} only in the sign of the drift term. On the other hand, both \eqref{1.4} and \eqref{1.11} are conservation laws; the transporting velocity in \eqref{1.11} is $\mathcal{H} \rho$, while it is $-\mathcal{H}(\frac{1}{2})$ in \eqref{1.4}. It is not immediately clear how such differences lead to distinct features of these equations as well as different difficulties in their analysis, but apparently, it is less straightforward to propose and study the weak formulation of \eqref{1.1} than that of \eqref{1.11}.

A generalisation of \eqref{1.11} reads

$$
\partial_t \rho + \partial_s (\rho \mathcal{H} \rho) + (1 - \theta) \mathcal{H} \rho \cdot \partial_s \rho = 0
$$

where $\theta \in [0, 1]$ is a fixed parameter \cite{12, 20, 49}. Formally, \eqref{1.1} corresponds to the case $\theta = -1$, up to a change of variable $\rho = -f$, which is not covered by existing works to the best of our knowledge.

Another generalisation of \eqref{1.11} is

$$
\partial_t \rho - \text{div} \left( \rho \nabla (\Delta)^{-s} \rho \right) = 0.
$$

The case $s = 1$ arises in the hydrodynamics of vortices in the Ginzburg–Landau theory \cite{26}; its well-posedness was proved in \cite{47}. When $s \in (0,1)$, it was introduced as a nonlinear porous medium equation with fractional potential pressure \cite{4–7}. Its well-posedness, long-time asymptotics, regularity of the solutions, as well as other properties has been investigated. If we set the equation in 1D and denote $\alpha = 2(1-s)$, then it can be written as

$$
\partial_t \rho + \left( \partial_x^{-1} (\Delta)^{\frac{s}{2}} \rho \right) \partial_s \rho = -\rho (\Delta)^{\frac{s}{2}} \rho.
$$

This turns out to be a special case of the 1D Euler alignment system for studying flocking dynamics \cite{22, 56, 57, 61}. Clearly, \eqref{1.11} corresponds to the case $s = \frac{1}{2}$, or equivalently, $\alpha = 1$ in 1D.
(III) It was derived in [59] that the dynamics of real roots of a high-degree polynomial under continuous differentiation can be described by

$$\partial_t u + \partial_x \left( \arctan \left( \frac{H u}{u} \right) \right) = 0,$$

or equivalently (at least for nice solutions $u = u(x,t)$),

$$\partial_t u = \frac{H u \cdot \partial_x u - u (-\Delta)^{1/2} u}{u^2 + (H u)^2}. \tag{1.13}$$

Also see [55, 60]. Recently, its well-posedness was studied in [1, 31, 39, 40]. Although (1.13) looks similar to (1.1) if one ignores the denominator of the right-hand side, its solutions can behave differently from those to (1.1) because of very different form of nonlinearity and degeneracy. Even when $u$ is sufficiently close to a constant $\bar{u}$, [31] formally derives that $u - \bar{u}$ should solve

$$\partial_t v = -(-\Delta)^{1/2} v + \partial_x (v H v)$$

up to suitable scaling factors and higher-order errors. This equation becomes (1.11) formally after a change of variable $\rho = 1 - v$.

(IV) Letting $\omega := -f$, we may rewrite (1.1) as

$$\partial_t \omega + u \partial_x \omega = \omega \partial_x u, \quad u = H \omega. \tag{1.14}$$

Then it takes a similar form as the De Gregorio model [18, 19],

$$\partial_t \omega + u \partial_x \omega = \omega \partial_x u, \quad u = H \omega. \tag{1.15}$$

The latter is known as a 1D model for the vorticity equation of the 3D Euler equation, playing an important role in understanding possible blow-ups in the 3D Euler equation as well as many other PDEs in fluid dynamics. Its analytic properties are quite different from (1.1); see e.g. [11, 13, 27, 35, 36, 43, 50] and the references therein. Nevertheless, we refer the readers to lemma 5.3 below, which exhibits an interesting connection of these two models due to their similar algebraic forms.

1.3. Organisation of the paper

In section 2, we review the Peskin problem in 2D and derive (1.1) formally in a special setting of it with a new Eulerian perspective. We highlight corollary 2.1 in section 2.2, which recasts the results in theorem 1.1 in the Lagrangian formulation. Sections 3–5 establish a series of $a$ priori estimates for nice solutions. We start with some basic estimates for (1.1) in section 3, and then prove in section 4 that smooth solutions to (1.1) admit finite and positive $a$ priori upper and lower bounds at positive times. In order to make sense of the weak formulation, we prove in section 5.1 that certain Hölder norms of $f$ are time-integrable near $t = 0$. Section 5.2 further shows that the solutions enjoy smoothing estimates at all positive times, and they converge to a constant equilibrium exponentially as $t \to +\infty$. Section 5.3 addresses the issue of the time continuity of the solutions at $t = 0$. In section 6, we prove existence of the desired global weak solutions, first for band-limited and positive initial data in section 6.1, and then for general initial data in section 6.2. Corollary 2.1 will be justified there as well. Section 6.3 proves
that the solution becomes analytic when $t$ is sufficiently large, and we also establishes sharp rate of exponential decay of the solution. Finally, in appendix A, we discuss an $H^1$-estimate for $F$ that is of independent interest.

2. The tangential Peskin problem in 2D

2.1. The 2D Peskin problem

The 2D Peskin problem, also known as the Stokes immersed boundary problem in 2D, describes a 1D closed elastic string immersed and moving in a 2D Stokes flow induced by itself. Let the string be parameterised by $X = X(s, t)$, where $s \in \mathbb{T}$ is the Lagrangian coordinate (as opposed to the arclength parameter). Then the 2D Peskin problem is given by

\begin{equation}
- \Delta u + \nabla p = \int_\mathbb{T} F_X(s, t) \delta(x - X(s, t)) \, ds, \tag{2.1}
\end{equation}

\begin{equation}
\text{div} u = 0, \quad |u|, |p| \to 0 \quad \text{as} \quad |x| \to \infty, \tag{2.2}
\end{equation}

\begin{equation}
\frac{\partial X}{\partial t}(s, t) = u(X(s, t), t), \quad X(s, 0) = X_0(s). \tag{2.3}
\end{equation}

Here $u = u(x, t)$ and $p = p(x, t)$ denote the flow field and pressure in $\mathbb{R}^2$, respectively. The right-hand side of (2.1) is the singular elastic force applied to the fluid that is only supported along the string. $F_X$ is the elastic force density in the Lagrangian coordinate, determined by the string configuration $X$. In general, it is given by [52]

\[ F_X(s, t) = \partial_s \left( T(|X'(s, t)|, s, t) \frac{X'(s, t)}{|X'(s, t)|} \right). \]

Here and in what follows, we write $\partial_s X(s, t)$ as $X'(s, t)$ for brevity. $T$ denotes the tension along the string. In the simple case of Hookean elasticity, $T(|X'(s, t)|, s, t) = k_0|X'(s, t)|$, where $k_0 > 0$ is the Hooke’s constant, and thus $F_X(s, t) = k_0X''(s, t)$.

The Peskin problem is closely related to the well-known numerical immersed boundary method [51, 52]. As a model problem of fluid–structure interaction, it has been extensively studied and applied in numerical analysis for decades. Nonetheless, analytical studies of it started only recently. In the independent works [46] and [48], the authors first studied well-posedness of the problem with $F_X = X''(s, t)$. Using the 2D Stokeslet

\[ G_0(x) = \frac{1}{4\pi} \left( -\ln|x| + \frac{x_k y_j}{|x|^2} \right), \quad i, j = 1, 2, \tag{2.4} \]

the problem was reformulated into a contour dynamic equation in the Lagrangian coordinate

\[ \partial_s X(s, t) = \int_{\mathbb{T}} G(X(s, t) - X(s', t)) X''(s', t) \, ds'. \tag{2.5} \]

Based on this, well-posedness results of the string evolution were proved in [46] and [48] in subcritical spaces $H^2(\mathbb{T})$ and $C^{1,\alpha}(\mathbb{T})$, respectively, under the so called well-stretched condition on the initial configuration—a string configuration $Y(s)$ is said to satisfy the well-stretched condition with constant $\lambda > 0$, if for any $s_1, s_2 \in \mathbb{T}$,

\[ |Y(s_1) - Y(s_2)| \geq \lambda|s_1 - s_2|. \tag{2.6} \]
It is worthwhile to mention that [48] also established a blow-up criterion, stating that if a solution fails at a finite time, then either $\|X(\cdot, t)\|_{C^{1,\alpha}}$ goes to infinity, or the best well-stretched constant shrinks to zero.

Since then, many efforts have been made to establish well-posedness under the scaling-critical regularity—in this problem, it refers to $W^{4,\infty}(T)$ and other regularity classes with the same scaling. García-Juárez, Mori, and Strain considered the case where the fluids in the interior and the exterior of the string can have different viscosities, and they showed global well-posedness for initial data of medium size in the critical Wiener space $W^{1,1}(T)$ [30]. In [29], Gancedo, Granero-Belinchón, and Scrobogna introduced a toy scalar model that captures the string motion in the normal direction only, for which they proved global well-posedness for small initial data in the critical Lipschitz class $W^{1,\infty}(T)$. Recently, Chen and Nguyen established well-posedness of the original 2D Peskin problem in the critical space $\dot{B}^{4}_{1,\infty}(T)$ [14]. However, in the Hookean case, the total energy of the system is $\frac{1}{4}\|X(\cdot, t)\|_{H^1}^2$. Hence, the Cauchy problem with the initial data belonging to the energy class is considered to be super-critical.

In addition to these studies, the author introduced a regularised Peskin problem in [63], and proved its well-posedness and its convergence to the original singular problem as the regularisation diminishes. Aiming at handling general elasticity laws beyond the Hookean elasticity, [54] and the very recent work [9] extended the analysis to a fully nonlinear Peskin problem. Readers are also referred to [45] for a study on the case where the elastic string has both stretching and bending energy.

2.2. Derivation of the tangential Peskin problem in 2D

In this subsection, we will present a formal derivation of the tangential Peskin problem in 2D, without paying too much attention to regularity and integrability issues. It is not the main point of this paper to rigorously justify the derivation, although it clearly holds for nice solutions.

Consider an infinitely long and straight Hookean elastic string lying along the x-axis, which admits tangential deformation in the horizontal direction only. By (2.5), this feature gets preserved for all time in the flow induced by itself. With abuse of notations, we still use $X(s,t)$ to denote the actual physical position (i.e. the x-coordinate) of the string point with the Lagrangian label $s \in \mathbb{R}$; note that $X(\cdot, t)$ is now a scalar function. Assume $X : s \mapsto X(s, t)$ to be suitably smooth and strictly increasing, so that it is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Under the assumption of no transversal deformation, (2.5) reduces to (thanks to (2.4))

$$\partial_t X(s,t) = \frac{1}{4\pi} \int_{\mathbb{R}} \left(-\ln |X(s,t) - X(s',t)| + 1 \right) X''(s',t) \mathrm{d}s'. \quad (2.7)$$

By integration by parts,

$$\partial_t X(s) = -\frac{1}{4\pi} \mathrm{p.v.} \int_{\mathbb{R}} \frac{X'(s')^2}{X(s) - X(s')} \mathrm{d}s' = -\frac{1}{4\pi} \mathrm{p.v.} \int_{\mathbb{R}} \frac{X'(s')}{X(s) - X(s')} \mathrm{d}X(s'). \quad (2.8)$$

Here we omitted the $t$-dependence for conciseness.

We define $f = f(x,t)$ such that

$$f(X(s,t), t) = X'(s, t). \quad (2.9)$$
Physically, \( f(x,t) \) represents the extent of local stretching of the elastic string at the spatial position \( x \). Then (2.8) becomes

\[
\partial_t X(s,t) = \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(X(s',t),t)}{X(s,t) - X(s',t)} \, dx(s',t)
\]

\[
= \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y,t)}{X(s,t) - y} \, dy - \frac{1}{4} \mathcal{H} f(s,t)|_{s=X(s,t)}.
\]

(2.10)

Here \( \mathcal{H} \) denotes the Hilbert transform on \( \mathbb{R} \). Differentiating (2.10) with respect to \( s \) yields

\[
\partial_s (X'(s,t)) = \frac{1}{4} (-\Delta)^{\frac{1}{4}} f(s,t)|_{s=X(s,t)} \cdot X'(s,t).
\]

Using (2.9), this can be further recast as

\[
\partial f(X(s,t),t) + \partial_t X(s,t) \cdot \partial f(X(s,t),t) = -f(X(s,t),t) \cdot \frac{1}{4} (-\Delta)^{\frac{1}{4}} f(s,t)|_{s=X(s,t)}.
\]

Using (2.10) again on the left-hand side and letting \( x = X(s,t) \), we obtain a scalar equation on \( \mathbb{R} \) for \( f = f(x,t) \)

\[
\partial f = \frac{1}{4} \left( \mathcal{H} f \cdot \partial f - (-\Delta)^{\frac{1}{4}} f \right).
\]

(2.11)

Now assuming \( f(x,t) \) to be spatially \( 2\pi \)-periodic for all time, and discarding the coefficient \( \frac{1}{4} \) by a change of the time variable, we obtain (1.1) on \( \mathbb{T} \). Note that, in this periodic setting, the Hilbert transform and the fractional Laplacian in (1.1) should be re-interpreted as those on \( \mathbb{T} \) (see (1.2) and (1.3) for their definitions).

We introduce this special setting of the 2D Peskin problem, mainly motivated by the interest of understanding global behaviour of the 2D Peskin problem with general initial data, which is a challenging problem. Given the blow-up criterion in [48] mentioned above, it is natural to ask whether finite-time singularity would occur in some typical scenarios. For example, if a local string segment is smooth at the initial time, and if it is known to ‘never gets close to other parts of the string’, one may ask whether or not it would spontaneously develop finite-time singularity, such as loss of regularity and aggregation of Lagrangian points, etc. The tangential Peskin problem proposed here can be the first step in this direction.

We state a few remarks on (2.11) and the above derivation.

**Remark 2.1.** Equation (2.11) provides the exact string dynamics in the special setting considered here, as no approximation is made in the derivation. This is in contrast to the scalar model in [29], where the tangential deformation of the string should have made a difference to the evolution but gets ignored purposefully.

Although the tangential problem (2.11) (or equivalently, (2.8)) is derived in a simplified geometric setting, different from that of the original 2D Peskin problem (2.5) where the string is a closed curve, we revealed a surprising connection between the two problems in a recent paper [64]. We proved that, in the original 2D Peskin problem, if the closed elastic string initially exhibits a perfectly circular shape, then it should remain to be a circle of the same radius for all time, while the centre of the circle can move. After normalising the position of the circle centre, we showed that the tangential deformation along the circle is described *exactly* by the tangential model. This allows us to fully characterise the global dynamics of such circular configurations under rather weak assumptions (also see corollary 2.1 below).
Beyond that, analysis in this paper also provides useful guidance for studying the original 2D Peskin problem there with more general initial data.

**Remark 2.2.** Given the physical interpretation of \( f, F(x,t) := 1/f(x,t) \) represents the spatial density of Lagrangian material points along the elastic string. Thus, there should be no surprise that \( F \) satisfies a conservation law (1.4), and it is natural to assume \( F \) to have \( L^1 \)-regularity at least.

**Remark 2.3.** If \( f = f(x,t) \) is a sufficiently smooth solution to (2.11) on \( \mathbb{R} \) with initial data \( f(x,0) = f_0(x) \), then for any \( \lambda > 0 \),

\[
\lambda f(x, \lambda t), \ f(\lambda x, \lambda t), \ \lambda f(\frac{x}{\lambda}, t)
\]

are all solutions, corresponding to initial datum \( \lambda f_0(x), f_0(\lambda x) \), and \( \lambda f_0(x/\lambda) \), respectively.

**Remark 2.4.** We imposed periodicity in (2.11) mainly for convenience of analysis. Many results in this paper should still hold in the case of \( \mathbb{R} \), given suitable decay conditions at the spatial infinity. In that case, one has to pay extra attention to integrability issues from time to time. We thus choose to work on \( \mathbb{T} \) to avoid such technicality.

If the spatial periodicity is to be imposed, it is enough to assume the \( 2\pi \)-periodicity, as one can always rescale in space and time. Also note that the \( 2\pi \)-periodicity is imposed in the Eulerian coordinate. In the Lagrangian coordinate, \( X'(\cdot, t) \) is periodic as well given the monotonicity and bijectivity assumptions of \( X(\cdot, t) \), but the period is \( \frac{2\pi}{\hat{T}} \), with \( \hat{T}_\infty \) defined in (1.9). Indeed,

\[
X^{-1}(x + 2\pi, t) - X^{-1}(x, t) = \int_{X^{-1}(x,t)}^{X^{-1}(x+2\pi,t)} \frac{1}{f(X(s,t),t)} \, dX(s,t) = \int_{x}^{x+2\pi} \frac{1}{f(y,t)} \, dy.
\]

Observe that the last integral is a constant independent of \( x \) and \( t \) thanks to the periodicity of \( f \) and the conservation law (1.4) of \( \frac{1}{\hat{T}} \).

**Remark 2.5.** One may consider general elasticity laws. Assume the tension takes the simple form \( T = T([X'(s,t)]) \), with \( T(\cdot) \) satisfying suitable assumptions. Then we can follow the above argument to obtain (again with the coefficient \( \frac{1}{4} \) discarded)

\[
\partial_t f = H(T(f)) \cdot \partial_x f - f(\Delta)^{\frac{1}{2}} T(f).
\] (2.12)

Here \( f \) is still defined by (2.9). In this case, \( F := \frac{1}{f} \) solves

\[
\partial_t F = \partial_x (H[T(f)] \cdot F).
\] (2.13)

Suppose \( T : [0, +\infty) \to [0, +\infty] \) is sufficiently smooth and strictly increasing. Define \( g(x,t) := T(f(x,t)) \). Then (2.12) can be written as

\[
\partial_t g = Hg \cdot \partial_x g - \mathcal{N}(g)(\Delta)^{\frac{1}{2}} g.
\] (2.14)

Here \( \mathcal{N} \) is a function that sends \( T(f) \) to \( T'(f)f \), i.e.

\[
\mathcal{N}(g) := T'(T^{-1}(g)) \cdot T^{-1}(g).
\]
If \( T(f) = f^\gamma \) with \( \gamma > 0 \), then \( N(g) = \gamma g \). Denoting \( \omega := -\gamma g \), we may write (2.14) as
\[
\partial_t \omega + \gamma^{-1} u \partial_x \omega = \omega \partial_x u, \quad u = H \omega.
\]
This provides a formal analogue of Okamoto–Sakajo–Wunsch generalisation [50] of the De Gregorio model (also see section 1.2).

Most of, if not all, the previous analytical studies of the 2D Peskin problem are based on the Lagrangian formulations, such as (2.5), (2.7), and (2.8), whereas (2.11) (or equivalently (1.1)) is formulated in the Eulerian coordinate. We will thus recast our main result theorem 1.1 in the Lagrangian coordinate. Even though it is concerned with a special setting of the 2D Peskin problem, compared with the existing results, it significantly weakens the assumption on the initial data for which a global solution can be well-defined.

Recall that \( X = X(s,t) \) represents the evolving configuration of the infinitely long elastic string lying along the \( x \)-axis. It is assumed to be periodic in space. It is supposed to solve (2.8) in the Lagrangian coordinate with the initial condition \( X(s,0) = X_0(s) \). Regarding the existence of such an \( X(s,t) \), we have the following result, which will be proved in section 6.2.

**Corollary 2.1.** Suppose \( X_0 : s \mapsto X_0(s) \in \mathbb{R} \) is a function defined on \( \mathbb{R} \), satisfying that

(i) \( X_0(s) \) is strictly increasing in \( s \), and \( X_0(s + 2 \pi) = X_0(s) + 2 \pi \) for any \( s \in \mathbb{R} \);

(ii) When restricted on \( [-\pi, \pi] \), \( X_0 \in H^1([-\pi, \pi]) \);

(iii) The inverse function of \( X_0 \) on any compact interval of \( \mathbb{R} \) is absolutely continuous.

Then there exists a function \( X = X(s,t) \) defined on \( \mathbb{R} \times [0, +\infty) \), satisfying that

1. \( X(s,t) \) is a smooth strong solution of (2.8) in \( \mathbb{R} \times (0, +\infty) \), i.e.
\[
\partial_t X(s,t) = -\frac{1}{4\pi} \text{p.v.} \int_\mathbb{R} \frac{X'(s',t)^2}{X(s,t) - X(s',t)} \, ds'
\]

holds pointwise in \( \mathbb{R} \times (0, +\infty) \).

2. For any \( \alpha \in (0, \frac{1}{2}) \), \( X(s,t) - s \in C^\alpha(\mathbb{R} \times [0, +\infty)) \). As a consequence, \( X(\cdot,t) \) converges uniformly to \( X_0(\cdot) \) as \( t \to 0^+ \).

3. For any \( t \geq 0 \), \( X(\cdot,t) \) verifies the above three assumptions on \( X_0(\cdot) \), with \( ||X(\cdot,t)||_{H^1([-\pi,\pi])} \) being non-increasing in \( t \in [0, +\infty) \). It additionally satisfies the well-stretched condition when \( t > 0 \), i.e. there exists \( \lambda = \lambda(t) > 0 \), such that
\[
|X(s_1,t) - X(s_2,t)| \geq \lambda(t) |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}.
\]

In fact, \( \frac{1}{\lambda(t)} \) satisfies an estimate similar to (1.7).

4. There exists a constant \( c_\infty \), such that as \( t \to +\infty \), \( X(s,t) \) converges uniformly to \( X_\infty(s) := s + c_\infty \) with an exponential rate. The exponential convergence also holds for higher-order norms.

For simplicity, we assumed in (i) that the periodicity of \( X_0 \) is \( 2\pi \) in both the Lagrangian and the Eulerian coordinate (see remark 2.4). The assumption (ii) only requires \( X_0 \) to belong to the energy class (see the discussion in section 2.1 and also remark 3.1 below), without imposing any condition on its size. The assumption (iii) is a relaxation of the well-stretched assumption that is commonly used in the literature. The function \( X(s,t) \) found in corollary 2.1 can be naturally defined as a global solution to (2.8) with the initial data \( X_0 \). For brevity, we choose not to elaborate the notion of the solution.
3. Preliminaries

In sections 3–5, we will establish a series of \textit{a priori} estimates for nice solutions to (1.1) (and also (1.4)), which prepares us for proving existence of weak solutions later. Throughout these sections, we will always assume \( f = f(x,t) \) is a strictly positive strong solution to (1.1). The definition of the strong solution is as follows.

**Definition 3.1.** Given \( T \in [0, +\infty) \), \( f(x,t) \) is said to be a strong solution to (1.1) on \( \mathbb{T} \times [0, T) \), if

(a) \( f \in C^1_{\text{loc}}(\mathbb{T} \times [0, T)) \);
(b) \((-\Delta)^2 f\) can be defined pointwise as a function in \( C_{\text{loc}}(\mathbb{T} \times [0, T)) \);
(c) Equation (1.1) holds pointwise in \( \mathbb{T} \times [0, T) \).

Here the time derivative of \( f \) at \( t = 0 \) is understood as the right derivative.

When discussing strong solutions, we always implicitly assume that the initial data \( f_0 \) is correspondingly smooth.

We start with a few basic properties of positive strong solutions.

**Lemma 3.1.** Suppose \( f \) is a strictly positive strong solution to (1.1) on \( \mathbb{T} \times [0, T) \). Then

1. (Energy estimate) For any \( t \in [0, T) \),
   \[
   \frac{1}{2} \|f(\cdot,t)\|_{L^2} + \int_0^t \|f(\cdot,\tau)\|_{L^2}^2 \, d\tau = \frac{1}{2} \|f_0\|_{L^2}. \tag{3.1}
   \]
2. (Decay of \( L^p \)-norms of \( f \) and \( F \)) For any \( p \in [1, +\infty) \), \( \|f(\cdot,t)\|_{L^p} \) and \( \|F(\cdot,t)\|_{L^p} \) are non-increasing in \( t \). In particular, \( \|F(\cdot,t)\|_{L^1} \) is time-invariant, and \( \|f(\cdot,t)\|_{L^p} \geq 2\pi f_\infty \), where \( f_\infty \) is defined in (1.9).

**Proof.** Equation (3.1) can be readily proved by integrating (1.1) and integration by parts.

For arbitrary \( \alpha \in \mathbb{R} \),
\[
\partial_t f^\alpha = \partial_x (\mathcal{H} f \cdot f^\alpha) - (1 + \alpha) f^\alpha (-\Delta)^2 f. \tag{3.2}
\]

Integrating (3.2) yields
\[
\frac{d}{dt} \int_\mathbb{T} f^\alpha(x,t) \, dx = -\frac{1 + \alpha}{\pi} \int_\mathbb{T} f^\alpha(x,t) \cdot \text{p.v.} \int_\mathbb{T} \frac{f(x,t) - f(y,t)}{4 \sin^2 \left( \frac{x-y}{2\pi} \right)} \, dy \, dx.
\]

Given the regularity assumption on \( f(\cdot,t) \), we may exchange the \( x \)- and \( y \)-variables to obtain
\[
\frac{d}{dt} \int_\mathbb{T} f^\alpha(x,t) \, dx = -\frac{1 + \alpha}{2\pi} \lim_{\delta \to 0^+} \int_{|x-y| \geq \delta} (f^\alpha(x,t) - f^\alpha(y,t)) \cdot \frac{f(x,t) - f(y,t)}{4 \sin^2 \left( \frac{x-y}{2\pi} \right)} \, dx \, dy.
\]

It is clear that, for any \( a, b \geq 0 \in \mathbb{R} \),
\[
(a^\alpha - b^\alpha)(a - b) \begin{cases} 
\geq 0, & \text{if } \alpha > 0, \\
= 0, & \text{if } \alpha = 0, \\
\leq 0, & \text{if } \alpha < 0.
\end{cases}
\]
Hence, for all \( \alpha \in (-\infty, -1] \cup [0, +\infty) \),
\[
\frac{d}{dt} \int_{\mathbb{T}} f^\alpha(x, t) \, dx \leq 0.
\]

This complete the proof of the second claim. The time-invariance of \( \|F(\cdot, t)\|_{L^2} \) follows directly from (1.4). We can further prove the lower bound for \( \|f(\cdot, t)\|_{L^2} \) by the Cauchy–Schwarz inequality.

**Remark 3.1.** Recall that (1.1) is derived from the Hookean elasticity case with \( k_0 = 1 \), where the elastic energy density is \( E(p) = \frac{1}{2} p^2 \) as \( E'(p) = T(p) \). We also find
\[
\frac{1}{2}\|f(\cdot, t)\|_{L^2(\mathbb{T})}^2 = \frac{1}{2} \int_{X^{-1}(2\pi, 0)} f(X(s, t), t) \, dX(s, t) = \int_{X^{-1}(0, 0)} \frac{1}{2} |X'(s, t)|^2 \, ds,
\]
which is exactly the total elastic energy of the string in one period in the Lagrangian coordinate. This is why we called (3.1) the energy estimate. Hence, we shall also call \( L^2(\mathbb{T}) \) the energy class for (1.1). In general, for (2.12), the energy estimate writes
\[
\frac{d}{dt} \int_{\mathbb{T}} \frac{E(f)}{f} \, dx = -\|T(f)\|_{q^2}^2, \quad \text{where} \quad E(p) := \int_0^p T(q) \, dq.
\]

**Lemma 3.2.** Assume \( f \) to be a strictly positive strong solution to (1.1) on \( \mathbb{T} \times [0,T) \). Then \( f \) satisfies maximum and minimum principle, i.e. for all \( 0 \leq t_1 \leq t_2 < T \),
\[
\inf_{x} f(x, t_1) \leq \inf_{x} f(x, t_2) \leq \sup_{x} f(x, t_2) \leq \sup_{x} f(x, t_1).
\]

If we additionally assume \( \partial_t f \in C^1_{\text{loc}}(\mathbb{T} \times [0,T)) \) and that \( (-\Delta)^{\frac{1}{2}}(\partial_t f) \) can be defined pointwise in \( \mathbb{T} \times [0,T) \) and is locally continuous, then \( \partial_t f \) also satisfies the maximum and minimum principle, i.e.
\[
\inf_{x} \partial_t f(x, t_1) \leq \inf_{x} \partial_t f(x, t_2) \leq \sup_{x} \partial_t f(x, t_2) \leq \sup_{x} \partial_t f(x, t_1).
\]

**Proof.** It follows immediate from (1.1) that \( f \) satisfies the maximum/minimum principle, since \( f \geq 0 \). Differentiating (1.1) yields
\[
\partial_t (\partial_t f) = Hf \cdot \partial_t (\partial_t f) - f (-\Delta)^{\frac{1}{2}} (\partial_t f).
\]
Hence, \( \partial_t f \) also satisfies the maximum/minimum principle.

**Remark 3.2.** In the context of the Peskin problem, the well-stretched assumption (2.6) (with constant \( \lambda > 0 \)) corresponds to \( f(\cdot, t) \geq \lambda \) in (1.1) and (2.11). As a result, strong solutions to (2.7) satisfy well-stretched condition for all time with the constant \( \lambda \) as long as it is initially this case. Besides, since
\[
\partial_t f(X(s, t), t) = \frac{X''(s, t)}{X'(s, t)} = \frac{1}{2} \partial_t \ln \left( \left| X'(s, t) \right|^2 \right),
\]
if \( f(x, 0) \geq \lambda > 0 \) and \( \|X''\|_{L^\infty} < +\infty \), we have \( X'' \) to be uniformly bounded in time.
4. A priori lower and upper bounds at positive times

In this section, we want to prove that, if \( f = f(x,t) \) is a strictly positive strong solution on \( T \times [0, +\infty) \), then for any given \( t > 0 \), \( f(\cdot, t) \) admits a priori lower and upper bounds, which are positive and finite, and which only depend on \( \|f_0\|_{L^1} \) and \( \|F_0\|_{L^1} \).

**Lemma 4.1.** Suppose \( f = f(x,t) \) is a strictly positive strong solution on \( T \times [0, +\infty) \). Denote \( f_*(t) := \min_{x \in \mathbb{T}} f(x,t) \). Then for all \( t > 0 \),

\[
f_*(t) \geq 8 \|F_0\|_{L^1}^{-1} \exp \left[ - \coth \left( \frac{4}{\pi} \|F_0\|_{L^1}^{-1} t \right) \right].
\]

(4.1)

or equivalently,

\[
\|F(\cdot, t)\|_{L^\infty} \leq \frac{1}{8} \|F_0\|_{L^1} \exp \left[ \coth \left( \frac{4}{\pi} \|F_0\|_{L^1}^{-1} t \right) \right].
\]

In particular, when \( t \leq \|F_0\|_{L^1} \),

\[
\|F(\cdot, t)\|_{L^\infty} \leq \frac{1}{8} \|F_0\|_{L^1} \exp \left( C \|F_0\|_{L^1} t^{-1} \right),
\]

(4.2)

where \( C \) is a universal constant.

**Proof.** Denote \( A := \sqrt{8\|F_0\|_{L^1(\mathbb{T})}^{-1/2}} > 0 \). We first prove that

\[
f_*(t) \geq \exp \left[ - \frac{1}{A} \coth \left( \frac{A}{2\pi t} \right) \right].
\]

(4.3)

Suppose the minimum \( f_*(t) \) is attained at \( x_*(t) \). Since \( \coth \left( \frac{A}{2\pi t} \right) > 1 \), we may assume \( f_*(t) < \exp \left( \frac{A}{2\pi t} \right) \) for all time of interest, as otherwise (4.3) holds automatically. We first derive a lower bound for \( -(\Delta)^{1/2} f(x_*(t), t) \). With \( \delta \in (0, \pi) \) to be determined,

\[
-(\Delta)^{1/2} f(x_*(t)) \geq \frac{1}{\pi} \int_{T \setminus [-\delta, \delta]} \frac{f(x_*(t) + y) - f(x_*)}{4 \sin^2 \left( \frac{y}{2} \right)} \, dy
\]

\[
= \frac{1}{\pi} \left( \int_{T \setminus [-\delta, \delta]} \frac{f(x_*(t) + y)}{4 \sin^2 \left( \frac{y}{2} \right)} \, dy \right) \left( \int_{T \setminus [-\delta, \delta]} \frac{1}{f(x_*(t) + y)} \, dy \right) \left( \int_{T \setminus [-\delta, \delta]} F(x_*(t) + y) \, dy \right)^{-1}
\]

\[
- \frac{f(x_*)}{\pi \tan \left( \frac{\delta}{2} \right)}
\]

\[
\geq \frac{1}{\pi} \left( \int_{s} \frac{1}{\sin \left( \frac{2y}{\delta} \right)} \, dy \right)^2 \|F\|_{L^1(\mathbb{T})}^{-1} - \frac{f(x_*)}{\pi \tan \left( \frac{\delta}{2} \right)}
\]

\[
\geq \frac{4}{\pi} \left| \frac{\ln \tan \delta}{4} \right|^2 \|F_0\|_{L^1(\mathbb{T})}^{-1} - \frac{f(x_*)}{2\pi \tan \left( \frac{\delta}{4} \right)}.
\]

Here we used the Cauchy–Schwarz inequality and the fact \( \|F(\cdot, t)\|_{L^1(\mathbb{T})} \equiv \|F_0\|_{L^1(\mathbb{T})} \). Since \( f_*(t) < \exp \left( -\frac{1}{A} \right) < 1 \), we may take \( \delta \in (0, \pi) \) such that \( \tan \left( \frac{\delta}{4} \right) = f(x_*) \). This gives

\[
-(\Delta)^{1/2} f(x_*(t)) \geq \frac{4}{\pi} \|F_0\|_{L^1(\mathbb{T})}^{-1} \ln f(x_*(t))^2 - \frac{1}{2\pi},
\]
Therefore, in (4.3) we may argue as in e.g. [17, theorem 3.1] to find that
\[
\frac{d}{dt} f_{\tau}(t) = -f(x_{\tau}, t) (-\Delta)^{\frac{1}{2}} f(x_{\tau}, t) \geq f_{\tau}(t) \left( \frac{4}{\pi} \|F_0\|_{L^1(T)}^{-1} \right) \ln f_{\tau}(t) \left( \frac{4}{\pi} \|F_0\|_{L^1(T)}^{-1} \right) \left( \frac{4}{\pi} \|F_0\|_{L^1(T)}^{-1} \right) - \frac{1}{2\pi} \right].
\]
Hence, with \( A = \sqrt{8} \|F_0\|_{L^1(T)}^{-1/2} > 0, \)
\[
\frac{d}{dt} \left( \frac{1}{\ln f_{\tau}(t)} \right) \leq -\frac{1}{2\pi} \left[ A^2 - (\ln f_{\tau}(t))^{-2} \right].
\]
Denote \( g_{\tau}(t) = (\ln f_{\tau}(t))^{-1}. \) Since \( f_{\tau}(t) < \exp(-\frac{1}{2}), \) we have \( g_{\tau}(t) \in (-A, 0) \) for all time of interest (including \( t = 0). \) Then we write the above inequality as
\[
\frac{d}{dt} \left( \frac{1}{A + g_{\tau}(t)} \right) \leq -\frac{A}{\pi}.
\]
Since \( g_{\tau}(0) \in (-A, 0), \)
\[
\ln \left( \frac{A + g_{\tau}(t)}{A - g_{\tau}(t)} \right) \leq -\frac{A}{\pi} t + \ln \left( \frac{A + g_{\tau}(0)}{A - g_{\tau}(0)} \right) \leq -\frac{A}{\pi} t,
\]
Therefore,
\[
\frac{1}{\ln f_{\tau}(t)} \leq A \cdot \frac{\exp \left( -\frac{A}{\pi} t \right) - 1}{\exp \left( -\frac{A}{\pi} t \right) + 1}.
\]
Then (4.3) follows.
Recall that if \( f = f(x, t) \) is a strong solution to (1.1) on \( T, \) then for any \( \lambda > 0, f_{\lambda}(x, t) := \lambda f(x, \lambda t) \) is a solution corresponding to initial condition \( \lambda f_0(x). \) Now applying (4.3) to \( f_{\lambda}, \) we deduce that,
\[
f_{\lambda, \tau}(t) := \min_{x \in T} f_{\lambda}(x, t) \geq \exp \left( -\frac{A}{\lambda^{1/2} \cdot \coth \left( \frac{\lambda^{1/2} A}{2\pi} t \right)} \right).
\]
Here we still defined \( A = \sqrt{8} \|F_0\|_{L^1(T)}^{-1/2}. \) Since \( \min_{x \in T} f_{\lambda}(x, t) = \lambda \min_{x \in T} f(x, \lambda t), \) for all \( \lambda, \tau > 0, \)
\[
\min_{x \in T^*} f(x, \tau) \geq \lambda^{-1} \exp \left( -\frac{1}{\lambda^{1/2} A} \cdot \coth \left( \frac{\lambda^{-1/2} A}{2\pi} \right) \right).
\]
Taking \( \lambda = A^{-2} \) yields (4.1).

**Remark 4.1.** We chose (4.1) instead of (4.3) as the main estimate, because (4.1) is scaling-invariant while (4.3) is not. Note that \( \|F_0\|_{L^1}^{-1} t \) is a dimensionless quantity under the scaling \( f_{\lambda}(x, \lambda t). \)

Next we turn to study the upper bound for \( f(\cdot, t). \) In the following lemma, we will prove a ‘from \( L^1 \) to \( L^\infty \) type estimate, which states that the \( \|f(\cdot, t)\|_{L^\infty} \) should enjoy a decay like \( t^{-1/2} \) when \( t \ll 1. \) This may be reminiscent of similar boundedness results in, e.g. [7, 8], which are proved by considering cut-offs of the solution and applying a De Giorgi–Nash–Moser-type iteration (without the part of proving Hölder regularity). Here we provide a different argument.
Lemma 4.2. Suppose \( f = f(x, t) \) is a strong solution to (1.1) on \( \mathbb{T} \times [0, +\infty) \). Then for \( 0 < t \leq \|f_0\|_{L^1}^{-1} \),

\[
\|f(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq C t^{-\frac{1}{2}} \|f_0\|_{L^1}^{\frac{3}{2}},
\]

where \( C > 0 \) is a universal constant. Consequently (see lemma 3.2), for \( t > \|f_0\|_{L^1}^{-1} \),

\[
\|f(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq C \|f_0\|_{L^1}.
\]

Remark 4.2. The exponents of \( t \) and \( \|f_0\|_{L^1} \) in the above estimate are sharp in the view of dimension analysis (see the energy estimate in lemma 3.1).

Proof. Without loss of generality, let us assume \( \|f_0\|_{L^1} \geq 1 \) up to suitable scaling. We proceed in several steps.

Step 1 (basic setup). Let \( W : [0, +\infty) \rightarrow [0, +\infty) \) be a convex increasing function, which is to be specified later. For the moment, we assume \( W \) is finite and locally \( C^{1,1} \) on \( [0, +\infty) \), \( W(0) = 0 \), and \( W'(0) = 1 \). Define

\[
I(t) = \begin{cases} 
\|f_0\|_{L^1}, & \text{if } t = 0, \\
\frac{1}{\pi^{\frac{1}{2}}} \int_\mathbb{T} W(\hat{t}^{1/2} f(x, t)) \, dx, & \text{if } t > 0.
\end{cases}
\]

Thanks to the assumption on \( W \) and that \( f \) is a strong solution, \( I(t) \) is locally finite near \( t = 0 \), continuous at \( 0 \), and continuously differentiable on \( [0, +\infty) \) whenever it is finite.

Let us explain the idea in the rest of the proof. We introduce the above functional mainly to study the size of the set

\[
S_\Lambda(t) := \left\{ x \in \mathbb{T} : f(x, t) > \Lambda \hat{t}^{1/2} \right\}
\]

for large \( \Lambda > 0 \). We will show that, one can suitably choose \( W \), which will depend on the Lebesgue measure of \( S_\Lambda(t) \) (denoted by \( |S_\Lambda(t)| \)), such that \( I(t) \) stays bounded for all time. This in turn provides an improved bound for \( |S_\Lambda(t)| \) for sufficiently large \( \Lambda \), which gives rise to an improved choice of \( W \). Then we bootstrap to find \( |S_\Lambda(t)| = 0 \) for all \( \Lambda \)'s that are sufficiently large.

Step 2 (the growth of \( I(t) \)). Fix \( t \in [0, T] \). By direct calculation and integration by parts,

\[
\frac{d}{dt} I(t) = \frac{1}{2 t^{3/2}} \int_\mathbb{T} W' \left( \hat{t}^{1/2} f \right) \cdot \hat{t}^{1/2} f - W \left( \hat{t}^{1/2} f \right) \, dx
\]

\[
- \frac{1}{t} \int_\mathbb{T} \left[ W' \left( \hat{t}^{1/2} f \right) \cdot \hat{t}^{1/2} f - W \left( \hat{t}^{1/2} f \right) \right] \cdot \left( -\Delta \right)^{\frac{3}{2}} \left( \hat{t}^{1/2} f \right) \, dx
\]

\[
- \frac{2}{t} \int_\mathbb{T} W \left( \hat{t}^{1/2} f \right) \cdot \left( -\Delta \right)^{\frac{3}{2}} \hat{t}^{1/2} f \, dx.
\]

Denote \( g(x) := \hat{t}^{1/2} f(x, t) \) and \( V^2(y) := W'(y)y - W(y) \). Then this becomes

\[
\frac{d}{dt} I(t) = \frac{1}{2 t^{3/2}} \int_\mathbb{T} V^2(g(x)) \, dx - \frac{1}{t} \int_\mathbb{T} V^2(g(x)) \cdot \left( -\Delta \right)^{\frac{3}{2}} g(x) \, dx
\]

\[
- \frac{2}{t} \int_\mathbb{T} W(g(x)) \cdot \left( -\Delta \right)^{\frac{3}{2}} g(x) \, dx.
\]
With \( \Lambda > 0 \) to be determined, we define
\[
g_\Lambda(x) := \min\{g(x), \Lambda\}, \quad \tilde{g}_\Lambda(x) := \max\{g(x), \Lambda\}.
\]

In what follows, we shall first take \( V \) properly and then determine \( W \).

Assume \( V \) to be increasing. Then for the first term on the right-hand side of (4.5), we have
\[
\int_T V^2(g(x)) \, dx = \int_T [V(g_\Lambda(x)) + (V(\tilde{g}_\Lambda(x)) - V(\Lambda))]^2 \, dx
\leq 2 \int_T V^2(g_\Lambda(x)) \, dx + 2 \int_T (V(\tilde{g}_\Lambda(x)) - V(\Lambda))^2 \, dx.
\]

Assume \( V^2(y) \leq y^3 \) for \( y \in [0, \Lambda] \). Then
\[
\int_T V^2(g_\Lambda(x)) \, dx \leq \int_T \min\{t^{1/2}f(x, t), \Lambda\}^3 \, dx \leq t^2 \int_T f(x, t)^3 \, dx.
\]

For the second term in (4.6), we first assume
\[
\Lambda \geq 2\|f_0\|_{L^1}^{1/2}, \tag{4.8}
\]
and then apply Hölder’s inequality and interpolation (see e.g. [41]) to derive that
\[
\int_T (V(\tilde{g}_\Lambda(x)) - V(\Lambda))^2 \, dx \leq \|V(\tilde{g}_\Lambda(x)) - V(\Lambda)\|_{L^1(T)}^2 \|\{x \in T : g(x, t) > \Lambda\}\|_1^{1/2}
\leq C\|V(\tilde{g}_\Lambda(x)) - V(\Lambda)\|_{L^1(T)} \|\tilde{g}_\Lambda(x) - V(\Lambda)\|_{H^{1/2}(T)} |S_\Lambda(t)|.
\]

Here we can use \( H^{1/2}\)-semi-norm because, for \( \Lambda \geq 2\|f_0\|_{L^1}^{1/2}, V(\tilde{g}_\Lambda(x)) - V(\Lambda) \) is zero on a set of positive measure. Indeed, this is because, when \( t \leq \|f_0\|_{L^1}^{-1} \),
\[
\int_T g(x, t) \, dx = t^2\|f(\cdot, t)\|_{L^1} \leq \|f_0\|_{L^1} \|f_0\|_{L^1} \leq \frac{1}{2}\Lambda.
\]

Hence, (4.9) gives
\[
\int_T (V(\tilde{g}_\Lambda(x)) - V(\Lambda))^2 \, dx \leq C_1 \|V(\tilde{g}_\Lambda(x))\|_{H^{1/2}(T)}^2 |S_\Lambda(t)|, \tag{4.10}
\]
where \( C_1 > 0 \) is a universal constant. Summarising (4.6), (4.7), and (4.10), we obtain that
\[
\frac{1}{2t^{1/2}} \int_T V^2(g(x)) \, dx \leq \|f(x, t)\|_{L^1}^3 + \frac{C_1}{t^{1/2}} \|V(\tilde{g}_\Lambda(x))\|_{H^{1/2}(T)}^2 |S_\Lambda(t)|. \tag{4.11}
\]
Now we handle the second term on the right-hand side of (4.5). Since \( V^2 \) is increasing,

\[
\int_T V^2(\varphi(x)) \cdot (-\Delta)^{1/2} \varphi(x) \, dx \\
\geq \frac{1}{2\pi} \int_T \int_T \frac{(V(\tilde{g}_\Lambda(x)) - V(\tilde{g}_\Lambda(y))) (\tilde{g}_\Lambda(x) - \tilde{g}_\Lambda(y))}{4 \sin^2 \left( \frac{\pi}{4T} \right)} \, dx \, dy \\
= \frac{1}{2\pi} \int_T \int_{\{\tilde{g}_\Lambda(x) \neq \tilde{g}_\Lambda(y)\}} \frac{(V(\tilde{g}_\Lambda(x)) + V(\tilde{g}_\Lambda(y))) (\tilde{g}_\Lambda(x) - \tilde{g}_\Lambda(y))}{V(\tilde{g}_\Lambda(x)) - V(\tilde{g}_\Lambda(y))} \cdot \frac{(V(\tilde{g}_\Lambda(x)) - V(\tilde{g}_\Lambda(y)))^2}{4 \sin^2 \left( \frac{\pi}{4T} \right)} \, dx \, dy.
\]

Suppose \( V \) is increasing and convex on \([0, +\infty)\), such that for all \( z > \Lambda \) and all \( t \in [0, T] \),

\[
\frac{V''(z)}{V(z)} \leq t^4 \left(C_1 |S_\Lambda(t)| \right)^{-1}.
\]

Then by the mean value theorem,

\[
\frac{(V(\tilde{g}_\Lambda(x)) + V(\tilde{g}_\Lambda(y))) (\tilde{g}_\Lambda(x) - \tilde{g}_\Lambda(y))}{V(\tilde{g}_\Lambda(x)) - V(\tilde{g}_\Lambda(y))} \geq t^{-\frac{1}{2}} C_1 |S_\Lambda(t)|.
\]

Therefore,

\[
\frac{1}{T} \int_T V^2(\varphi(x)) \cdot (-\Delta)^{1/2} \varphi(x) \, dx \geq \frac{C_1}{T^{3/2}} \|V(\tilde{g}_\Lambda(x))\|_{H^{1/2}(\Omega)}^2 |S_\Lambda(t)|. \tag{4.12}
\]

The last term on the right-hand side of (4.5) is clearly non-positive, since \( W \) is non-decreasing. Hence, combining (4.5), (4.11), and (4.12) yields

\[
\frac{d}{dt} I(t) \leq \|f(x, t)\|_{L^2}^3.
\]

Thanks to the energy estimate (3.1) and interpolation, for all \( t > 0 \),

\[
I(t) \leq I(0) + \int_0^t \|f(\cdot, \tau)\|_{L^2}^3 \, d\tau \leq \|f_0\|_{W^1} + C\|f_0\|_{L^2}^2 =: C_* \tag{4.13}
\]

Note that \( C_* \) is a constant only depending on \( \|f_0\|_{W^1} \). Actually, when deriving (4.13), we first obtained

\[
I(t) \leq I(\delta) + \int_\delta^t \|f(\cdot, \tau)\|_{L^2}^3 \, d\tau,
\]

and then sent \( \delta \to 0^+ \).

**Step 3 (the choice of \( W \)).** Summarising our assumptions on \( V \), we need

- \( V \) is increasing and convex on \([0, +\infty)\).
- For \( y \in [0, \Lambda] \), \( V^2(y) \leq y^3 \).
For $y \in (\Lambda, +\infty)$, 

$$\frac{V'(y)}{V(y)} \leq \inf_{t \in [0, T]} t^{\frac{1}{2}} \left( C_{1} |S_{\Lambda}(t)| \right)^{-1} =: \beta.$$  

(4.14)

We remark that $\beta$ admits an \textit{a priori} lower bound that only depends on $\Lambda$ and $\|f_{0}\|_{L^{2}}$. In fact, by the energy estimate (3.1) and interpolation, 

$$\|f\|_{L^{2}(\mathbb{T})} \leq C \|f_{0}\|_{L^{2}}^{\frac{1}{2}}.$$  

Since $\|f(\cdot, t)\|_{L^{2}}$ is non-increasing in time (see lemma 3.1), this implies 

$$\|f(\cdot, t)\|_{L^{2}} \leq C \|f_{0}\|_{L^{2}}^{\frac{1}{2}} t^{-\frac{1}{4}},$$  

which gives 

$$|S_{\Lambda}(t)| = \left| \left\{ x \in \mathbb{T} : f(x, t) > \Lambda / t^{1/2} \right\} \right| \leq C \|f_{0}\|_{L^{2}}^{\frac{1}{2}} t^{-\frac{1}{4}} \Lambda^{2}.$$

Therefore, 

$$\beta = \inf_{t \in [0, T]} t^{\frac{1}{2}} \left( C_{1} |S_{\Lambda}(t)| \right)^{-1} \geq C_{2} \|f_{0}\|_{L^{2}}^{\frac{1}{2}} \Lambda^{2},$$  

(4.15)

where $C_{2}$ is a universal constant.

We define $V$ to be the unique continuous function on $[0, +\infty)$ that satisfies 

$$V(y) = y^{\frac{3}{2}} \quad \text{on} \quad [0, \Lambda], \quad \text{and} \quad V'(y) = \beta V(y) \quad \text{on} \quad y \in (\Lambda, +\infty),$$

i.e. 

$$V(y) = \begin{cases} y^{3/2}, & \text{if } y \in [0, \Lambda], \\ \Lambda^{3/2} e^{2y(y-\Lambda)}, & \text{if } y \in [\Lambda, +\infty). \end{cases}$$  

(4.16)

We shall suitably choose $\Lambda$ in order to guarantee that $V$ is increasing and convex. In fact, we only need $V'(\Lambda^{-}) \leq V'(\Lambda^{+})$, i.e. $\frac{1}{2} \Lambda^{1/2} \leq \beta \Lambda^{1/2}$. By virtue of (4.15), it is safe to take 

$$\Lambda \geq C_{3} \|f_{0}\|_{L^{2}}^{\frac{1}{2}},$$  

(4.17)

where $C_{3}$ is a universal constant. Without loss of generality, we may assume $C_{3} \geq 2$ so that (4.17) is no weaker than (4.8).

With this choice of $V$, we thus define $W$ by 

$$\frac{d}{dy} \left( \frac{W(y)}{y} \right) = \frac{V^{2}(y)}{y}, \quad \lim_{y \to 0^{+}} \frac{W(y)}{y} = 1.$$ 

For future use, we calculate it explicitly 

$$W(y) = \begin{cases} y + \frac{1}{2} y^{3}, & \text{if } y \in [0, \Lambda], \\ y \left( 1 + \frac{1}{2} \Lambda^{2} + \Lambda^{2} \int_{\Lambda}^{y} e^{-2\beta(z-\Lambda)} dz \right), & \text{if } y \in [\Lambda, +\infty). \end{cases}$$  

(4.18)
This resulting \( W \) is convex on \([0, +\infty)\). Indeed, on \((0, \Lambda) \cup (\Lambda, +\infty)\),
\[
W''(y) = \frac{1}{y} (W'(y) y - W(y))' = \frac{2}{y} V(y) V'(y) \geq 0,
\]
and \(W'(\Lambda^-) = W'(\Lambda^+)\) because \(V^2(y)\) and \(W(y)\) are continuous at \(y = \Lambda\).

By virtue of (4.13), for any \(W\) constructed in the above way, we have
\[
\int_T W\left(\tau^{1/2} f(x, t)\right) \, dx \leq C_* t^{1/2}.
\]

This implies, for any \(\lambda > 0\),
\[
|S_\lambda(t)| = \left| \left\{ x \in T : f(x, t) \geq \lambda/\tau^{1/2} \right\} \right| \leq C_* t^{1/2} W(\lambda)^{-1},
\]
and thus,
\[
\inf_{r \in [0, T]} t^{1/2} \left( C_1 |S_\lambda(t)| \right)^{-1} \geq (C_1 C_*)^{-1} W(\lambda). \tag{4.19}
\]

Here \(C_1\) and \(C_*\) are defined in (4.10) and (4.13), respectively. We will treat (4.19) as an improvement of (4.15).

**Step 4 (bootstrap).** Next we introduce a bootstrap argument. Let \(C_1, C_2,\) and \(C_3\) be the universal constants defined in (4.10), (4.15), and (4.17), respectively. With \(C_0 \geq C_3 \geq 2\) to be chosen, we let
\[
\Lambda_0 = C_0 \|f_0\|_{L_1}^{1/2}, \quad \beta_0 = C_2 \|f_0\|_{L_1}^{-1/2} \Lambda_0^2 = C_2 C_0^2 \|f_0\|_{L_1}^{-1/2}.
\]

Let \(W_0\) be defined by (4.18), with \(\Lambda\) and \(\beta\) there replaced by \(\Lambda_0\) and \(\beta_0\). For \(j \in \mathbb{Z}_+ \cup \{\infty\}\), let
\[
\Lambda_j = \left( C_0 + \sum_{k=1}^{j-1} k^{-2} \right) \|f_0\|_{L_1}^{1/2}.
\]

In the view of (4.19) and also (4.14), iteratively define
\[
\beta_j = (C_1 C_*)^{-1} W_{j-1}(\Lambda_j), \tag{4.20}
\]
while \(W_j\) is then constructed by (4.18), with \(\Lambda\) and \(\beta\) replaced by \(\Lambda_j\) and \(\beta_j\).

Observe that \(\{\Lambda_j\}_j\) is increasing, and
\[
C_0 \|f_0\|_{L_1}^{1/2} \leq \Lambda_j \leq C_0' \|f_0\|_{L_1}^{1/2}, \tag{4.21}
\]
where \(C_0'/C_0\) is bounded by some universal constant. With \(C_0\) being suitably large, we claim that, for all \(j \in \mathbb{Z}_+\),
\[
\beta_{j-1} \geq j \|f_0\|_{L_1}^{-1/2}. \tag{4.22}
\]
We prove this by induction. First, choose \( C_0 \geq \max \{ C_2^{-1/2}, C_3 \} \), so that (4.22) holds for \( j = 1 \). For \( j \geq 2 \), by the definitions of \( \Lambda_j \) and \( W_j \), we derive by (4.18) that for \( j \in \mathbb{Z}_+ \),

\[
W_{j-1}(\Lambda_j) \geq \Lambda_j \left( 1 + \frac{1}{2} \Lambda_j^2 + \frac{\Lambda_{j-1}^3}{\Lambda_j} \int_{0}^{\Lambda_j - \Lambda_{j-1}} e^{2\beta_j - 1} \, dz \right).
\]

By (4.20) and (4.21), and using the definition of \( C_* \) in (4.13), we obtain that

\[
\beta_j \geq (C_1 C_*)^{-1} \frac{\Lambda_{j-1}^3}{\Lambda_j} \frac{e^{2\beta_j - 1}\|f\|_{L^1}^{1/2} - 1}{2\beta_j - 1} 
\geq C \left( \frac{C_0 \|f\|_{L^1}^{1/2}}{\|f\|_{L^1} + C \|f\|_{L^1}^{1/2}} \right)^2 j^{-2} \|f\|_{L^1}^{1/2} \frac{e^{2\beta_j - 1}\|f\|_{L^1}^{1/2} - 1}{2\beta_j - 1},
\]

where \( C \) is a universal constant. Using the induction hypothesis (4.22) and the assumption \( \|f\|_{L^1} \geq 1 \),

\[
\beta_j \geq C C_0 \cdot j^{-2} \cdot \frac{e^{2j} - 1}{2j} \|f\|_{L^1}^{-1/2} \geq C C_0^3 (j + 1)^3 \|f\|_{L^1}^{-1/2},
\]

where \( C \) is a universal constant. In the second inequality, we used the trivial fact that for all \( j \in \mathbb{Z}_+ \), \( \frac{e^{2j} - 1}{2j} \geq C^2 (j + 1)^3 \) for some universal \( C > 0 \). Now we choose \( C_0 \) to be suitably large but universal, so that \( C C_0^3 \geq 1 \) in the above estimate. This yields (4.22) with \( j \) replaced by \( (j + 1) \). Therefore, by induction, (4.22) holds for all \( j \in \mathbb{Z}_+ \).

Using (4.19) and (4.20), we find for all \( j \in \mathbb{Z}_+ \),

\[
\sup_{t \in [0,T]} t^{-\frac{1}{2}} \cdot C_1 \left\{ x \in T : f(x,t) \geq \Lambda_\infty / t^{1/2} \right\} \leq \beta_j^{-1}.
\]

By (4.22) and the arbitrariness of \( j \), we conclude that for all \( t \in [0,T] \),

\[
f(\cdot,t) \leq \Lambda_\infty t^{-\frac{1}{2}} \quad \text{a.e. on } T.
\]

Since

\[
\Lambda_\infty = \left( C_0 + \sum_{k=1}^{\infty} k^{-2} \right) \|f\|_{L^1}^{1/2},
\]

where \( C_0 > 0 \) is universal, the desired estimate follows.

\[ \square \]

5. Smoothing and decay estimates

5.1. Higher regularity with time integrability at \( t = 0 \)

With the estimates established so far, we are still not able to make sense of the integral

\[
\int_{T \times [0,T]} \partial_x \varphi \cdot \mathcal{H} f \cdot F \, dx \, dt
\]

\[ \int \]
in the weak formulation (1.5). Indeed, with initial data \( f_0, F_0 \in L^1(\mathbb{T}) \), we only have

\[
f \in L^2_T L^1(\mathbb{T}) \cap L^2_T H^1(\mathbb{T}) \cap L^\infty_T L^2(\mathbb{T}), \quad \text{and} \quad F \in L^2_T L^1(\mathbb{T}),
\]

whereas \( H^{1/2}(\mathbb{T}) \not\subset L^\infty(\mathbb{T}) \), and the Hilbert transform is not bounded in \( L^1(\mathbb{T}) \) either. A natural idea to resolve this integrability issue is to prove that \( f \) has some spatial regularity higher than \( H^{1/2} \), with the corresponding norm being time-integrable. This seems not trivial because of the nonlinearity and degeneracy of the equation (1.1). In particular, the very bad lower bound for \( f \) in lemma 4.1 indicates extremely weak smoothing effect in some part of \( \mathbb{T} \), so the higher-order norms of \( f \) could be terribly singular near \( t = 0 \). Fortunately, by studying the special properties of (1.1), we can prove that certain Hölder norms of \( f \) are time-integrable near \( t = 0 \) (see corollary 5.1). This is the theme of this subsection.

For convenience, we assume \( f \) to be a strictly positive smooth solution to (1.1) on \( \mathbb{T} \times [0, +\infty) \). Define \( h := \ln f \). Clearly, \( h \) is a smooth solution to

\[
\partial_t h = \mathcal{H} f \cdot \partial_x h - (-\Delta)^{1/2} f. \tag{5.1}
\]

We start from the entropy estimate for \( F \).

**Lemma 5.1.**

\[
\frac{d}{dt} \int_{\mathbb{T}} F \ln F \, dx + \|h\|_{H^{1/2}}^2 \leq 0.
\]

**Proof.** We derive with (1.4) that

\[
\frac{d}{dt} \int_{\mathbb{T}} F \ln F \, dx = \int_{\mathbb{T}} (1 + \ln F) \partial_t F \, dx = -\int_{\mathbb{T}} \partial_x (1 + \ln F) \cdot \mathcal{H} f \cdot F \, dx
\]

\[
= -\int_{\mathbb{T}} \mathcal{H} f \cdot \partial_x F \, dx = \int_{\mathbb{T}} F (-\Delta)^{1/2} f \, dx. \tag{5.2}
\]

The right-hand side can be written as

\[
\int_{\mathbb{T}} F (-\Delta)^{1/2} f \, dx = \int_{\mathbb{T}} F(x) \cdot \frac{1}{\pi} \text{p.v.} \int_{\mathbb{T}} \frac{f(x) - f(y)}{4 \sin^2 \left( \frac{y^2}{4} \right)} \, dy \, dx
\]

\[
= -\frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{T}} \frac{(F(x) - F(y))(f(y) - f(x))}{4 \sin^2 \left( \frac{y^2}{4} \right)} \, dy \, dx.
\]

Since

\[
(F(x) - F(y))(f(y) - f(x)) = \frac{(F(x) - F(y))^2}{F(x)F(y)} = \left( \sqrt{\frac{F(x)}{F(y)}} - \sqrt{\frac{F(y)}{F(x)}} \right)^2
\]

\[
\geq \left( 2 \ln \sqrt{\frac{F(x)}{F(y)}} \right)^2 = (\ln F(x) - \ln F(y))^2,
\]

we obtain the desired estimate. \( \square \)

Then we state an \( H^{1/2} \)-estimate for \( h \).
Lemma 5.2.

\[ \frac{d}{dt} \| h \|_{H^1}^2 + 4 \| \sqrt{f} \|_{L^2}^2 + \int_T \left( -\Delta \frac{1}{2} h \right)^2 \, dx = 0. \]

**Proof.** Recall that for a mean-zero smooth function \( u \) defined on \( T \), we have the Cotlar’s identity

\[(\mathcal{H} u)^2 - u^2 = 2 \mathcal{H} (u \mathcal{H} u). \quad (5.3)\]

We calculate with (5.1) and (5.3) that

\[ \frac{d}{dt} \| h \|_{H^1}^2 = 2 \int_T \partial_t h \cdot (-\Delta)^{\frac{1}{2}} h \, dx \]

\[ = -2 \int_T f \cdot \mathcal{H} (\partial_t h (-\Delta)^{\frac{1}{2}} h) \, dx - 2 \int_T \left( -\Delta \right)^{\frac{1}{2}} f \cdot (-\Delta)^{\frac{1}{2}} h \, dx \]

\[ = \int_T \left( (\partial_t h)^2 - \left( -\Delta \right)^{\frac{1}{2}} h \right) \, dx - 2 \int_T \partial_t (e^h) \cdot \partial_t h \, dx \]

\[ = -\int_T \left( (\partial_t h)^2 + \left( -\Delta \right)^{\frac{1}{2}} h \right) \, dx. \quad (5.4)\]

Observing that \( f(\partial_t h)^2 = 4 (\partial_t \sqrt{f})^2 \), we obtain the desired result. \( \square \)

We further show \( \| \sqrt{f} \|_{H^1} \) satisfies another simple estimate, which implies it is non-increasing as well.

Lemma 5.3.

\[ \frac{d}{dt} \| \sqrt{f} \|_{H^1}^2 + \frac{1}{2} \| f \|_{H^2}^2 = 0. \]

**Proof.** Denoting \( \omega = f \) and \( u = -\mathcal{H} f \), we recast (1.1) as

\[ \partial_t \omega + u \partial_x \omega = \omega \partial_x u. \]

Note that in order to keep \( \omega \) positive, here we take a different change of variable from that in (1.14). Hence,

\[ \partial_t \partial_x \sqrt{\omega} + u \partial_{xx} \sqrt{\omega} + \frac{1}{2} \partial_x u \partial_x \sqrt{\omega} = \frac{1}{2} \sqrt{\omega} \partial_x u. \]

Then we derive that

\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{w} \|_{H^1}^2 = \int_T -u \partial_{xx} \sqrt{\omega} \partial_x \sqrt{\omega} - \frac{1}{2} \partial_x u \partial_x \sqrt{\omega} \partial_x \sqrt{\omega} + \frac{1}{2} \sqrt{\omega} \partial_x \sqrt{\omega} \partial_x u \, dx \]

\[ = \frac{1}{2} \int_T -u \partial_x (\partial_x \sqrt{\omega})^2 - \partial_x u (\partial_x \sqrt{\omega})^2 + \frac{1}{2} \partial_x \omega \partial_x u \, dx \]

\[ = \frac{1}{4} \int_T \partial_x \omega \partial_x u \, dx. \]

Plugging in \( u = -\mathcal{H} \omega \) yields the desired result. \( \square \)
Remark 5.1. For the De Gregorio model (1.15), non-negative solutions $\omega = \omega(x,t)$ that are sufficiently smooth are known to enjoy conservation of $\|\sqrt{\omega}\|_{H^p}$ [43]. Interestingly, we can recover that result by following the derivation above and plugging in $ic = H\omega$ (see (1.15)) in the last line.

By lemmas 4.1 and 4.2, $f(\cdot,t)$ is positive and bounded for all positive $t$. Hence, whenever $t > 0$, that $f(\cdot,t) \in H^{1/2}(T)$ implies $h(\cdot,t) \in H^{1/2}(T)$ and vice versa. This is because

$$\|f\|^2_{H^{1/2}(T)} = \frac{1}{2\pi} \int_{T \times T} \frac{(f(x) - f(y))^2}{4\sin^2\left(\frac{x-y}{2}\right)} \, dx \, dy = \frac{1}{2\pi} \int_{T \times T} \frac{(e^{h(x)} - e^{h(y)})^2}{4\sin^2\left(\frac{x-y}{2}\right)} \, dx \, dy,$$

and

$$\min f^2 \cdot (h(x) - h(y))^2 \leq \left(\frac{e^{h(x)} - e^{h(y)}}{x-y}\right)^2 \leq \max f^2 \cdot (h(x) - h(y))^2. \quad (5.6)$$

Hence,

$$\|h(\cdot,t)\|_{H^{1/2}(T)} \leq \|F(\cdot,t)\|_{L^\infty} \|f(\cdot,t)\|_{H^{1/2}(T)}. \quad (5.7)$$

On the other hand, by lemma 3.1, for any $t > 0$, there exists $t_\ast \in \left[\frac{t}{2}, t\right]$, such that $\|f(\cdot,t_\ast)\|_{H^{1/2}(T)} \leq Ct^{-1/2} \|f_0\|_{L^2}^{1/2}$. Combining this with (5.7), lemmas 4.1, and 5.2, we find

$$\|h(\cdot,t)\|_{H^{1/2}(T)} \leq \|h(\cdot,t_\ast)\|_{H^{1/2}} \leq Ct^{-\frac{1}{2}} \|f_0\|_{L^2}^{1/2} \|F_0\|_{L^\infty}\exp\left[\coth\left(C \|F_0\|_{L^1}^{1/4}\right)\right].$$

This bound is too singular near $t = 0$ for future analysis. In what follows, we establish an improved estimate by applying lemmas 5.1–5.3.

Proposition 5.1. For any $t > 0$,

$$\|h(\cdot,t)\|_{H^{1/2}} \leq Ct^{-\frac{1}{2}} \|F_0\|_{L^2}^{1/2} \left[\coth\left(\frac{2\pi}{\pi} \|F_0\|_{L^1}^{-1/2} t\right)\right]^{1/2}, \quad (5.8)$$

and for any $0 < t \leq \|F_0\|_{L^2}^{-1}$,

$$\|f(\cdot,t)\|_{H^{1/2}} \leq Ct^{-\frac{1}{2}} \|F_0\|_{L^2}^{1/2} \|F_0\|_{L^\infty}, \quad (5.9)$$

where $C$’s are universal constants.

Remark 5.2. These estimates are good enough for future analysis, though we do not know whether they are sharp in terms of the singularity at $t = 0$. However, the large-time decay of (5.8) is not optimal. We will prove exponential decay of $h$ and $f$ in sections 5.2 and 7.

Proof. By lemmas 3.1 and 4.1, for all $t > 0$,\n
$$\int_T F \ln F \, dx \leq \|F_0\|_{L^1} \ln \left[\frac{1}{8 \|F_0\|_{L^1}} + \coth\left(\frac{4}{\pi} \|F_0\|_{L^1}^{-1} t\right)\right].$$

On the other hand, since $x \mapsto x \ln x$ is convex on $(0, +\infty)$, by Jensen’s inequality,

$$\int_T F \ln F \, dx \geq \|F_0\|_{L^1} \ln \left(\frac{1}{2\pi} \|F_0\|_{L^1}\right).$$
Lemma 5.2 shows that \( \|h\|_{H^1} \) is non-increasing. Hence, by lemma 5.1, for any \( t > 0 \),
\[
\|h(\cdot, t)\|_{H^2}^2 \leq Ct^{-1} \left[ \int_T F \ln F \left( x, \frac{t}{2} \right) \, dx - \|F_0\|_{L^1} \ln \left( \frac{1}{2\pi} \|F_0\|_{L^1} \right) \right] 
\leq Ct^{-1} \|F_0\|_{L^1} \coth \left( \frac{2}{\pi} \|F_0\|_{L^1}^{-1} t \right).
\]
This gives (5.8). Similarly, since lemma 5.3 shows \( \|\sqrt{f}\|_{H^1} \) is non-increasing, by lemma 5.2, we find for any \( t > 0 \),
\[
\|\sqrt{f(\cdot, t)}\|_{H^1}^2 \leq Ct^{-1} \|h(\cdot, t/2)\|_{H^2}^2 \leq Ct^{-\frac{3}{2}} \|F_0\|_{L^1} \coth \left( \frac{1}{\pi} \|F_0\|_{L^1}^{-1} t \right). \tag{5.10}
\]
Since
\[
\|f\|_{H^1} \leq C \|\sqrt{f}\|_{L^\infty} \|\sqrt{f}\|_{H^1} = C \|f\|_{L^\infty} \|\sqrt{f}\|_{H^1}, \tag{5.11}
\]
we apply lemma 4.2 and (5.10) to derive that, for \( 0 < t \leq \|F_0\|_{L^1}^{-1} \),
\[
\|f(\cdot, t)\|_{H^1} \leq Ct^{-\frac{1}{2}} \|F_0\|_{L^1} \|F_0\|_{L^1} \left[ \coth \left( \frac{1}{\pi} \|F_0\|_{L^1}^{-1} t \right) \right]^\frac{1}{2}.
\]
In this case, thanks to the Cauchy–Schwarz inequality,
\[
t^{-\frac{1}{2}} \|F_0\|_{L^1} \geq \|F_0\|_{L^1} \|F_0\|_{L^1} \geq 4\pi^2,
\]
so \( t \leq \frac{1}{4\pi^2} \|F_0\|_{L^1} \). Then we can further simplify the above estimate to obtain (5.9). 

\[\square\]

**Corollary 5.1.** For \( 0 < t \leq \|F_0\|_{L^1}^{-1} \) and \( \alpha \in (0, \frac{1}{2}) \), with \( f_\infty \) defined in (1.9),
\[
\int_0^t \|f(\cdot, \tau)\|_{C^\alpha} \, d\tau \leq C_\alpha \alpha^{-\frac{1-2\alpha}{2}} \left( \|F_0\|_{L^1} - 2\pi f_\infty \right) \alpha^{-\frac{1-2\alpha}{2}} \|F_0\|_{L^1}^{2\alpha}, \tag{5.12}
\]
where \( C_\alpha > 0 \) is a universal constant depending on \( \alpha \). To be simpler, we have
\[
\int_0^t \|f(\cdot, \tau)\|_{C^\alpha} \, d\tau \leq C_\alpha t^{-\frac{1-2\alpha}{2}} \|F_0\|_{L^1}^{1-2\alpha} \|F_0\|_{L^1}^{2\alpha}.
\]

**Proof.** With \( \alpha \in (0, \frac{1}{2}) \) and \( \delta > 0 \) to be determined, by Sobolev embedding, interpolation, and the Young’s inequality,
\[
\|f(\cdot, t)\|_{C^\alpha} \leq C_\alpha \|f(\cdot, t)\|_{H^{\frac{1}{2}+\alpha}} \leq C_\alpha \|f(\cdot, t)\|_{H^{\frac{1}{2}}} \|f(\cdot, t)\|_{L^1}^{2\alpha} \leq 2\delta \|f(\cdot, t)\|_{H^{\frac{1}{2}}}^2 + C_\alpha \delta^{-\frac{1-2\alpha}{2}} \|f(\cdot, t)\|_{H^{\frac{1}{2}}}^{\frac{2\alpha}{1-2\alpha}}.
\]
Here \( C_\alpha > 0 \) is a universal constant depending only on \( \alpha \). Hence, by lemma 3.1 and (5.9), for \( 0 < t \leq \|F_0\|_{L^1}^{-1} \),
\[
\int_0^t \|f(\cdot, \tau)\|_{C^\alpha} \, d\tau \leq 2\delta \int_0^t \|f(\cdot, \tau)\|_{H^{\frac{1}{2}}}^2 \, d\tau + C_\alpha \delta^{-\frac{1-2\alpha}{2}} \|F_0\|_{L^1}^{\frac{2\alpha}{1-2\alpha}} \|F_0\|_{L^1}^{-\alpha} \int_0^t \|f(\cdot, \tau)\|_{H^{\frac{1}{2}}} \, d\tau 
\leq \delta \left( \|F_0\|_{L^1} - 2\pi f_\infty \right) + C_\alpha \delta^{-\frac{1-2\alpha}{2}} \|F_0\|_{L^1}^{\frac{2\alpha}{1-2\alpha}} \|F_0\|_{L^1}^{-\alpha} \int_0^t \tau^{-\frac{1-2\alpha}{2}} \, d\tau.
\]
The last integral is finite as long as $\alpha \in (0, \frac{1}{2})$. Calculating the integral and optimising over $\delta > 0$ yields (5.12).

**Remark 5.3.** By better incorporating the estimate for $\|f\|_{l^p}$ from lemma 5.3, one may slightly improve the power of $t$ in (5.12) to become $t^{\frac{k-\alpha}{2}}$, up to corresponding changes in the other factors. However, we omit the details as that is not essential for the future analysis.

### 5.2. Instant smoothing and global bounds

Now we derive *a priori* estimates that show the solution $f$ would instantly become smooth, and that it converges exponentially to the constant $f_\infty$ (see (1.9)) as $t \to +\infty$. We start with a higher-order generalisation of lemmas 5.2 and 5.3 for positive smooth solutions.

**Lemma 5.4.** Suppose $f = f(x, t)$ is a strictly positive smooth solution on $\mathbb{T} \times [0, +\infty)$. Let $f_\star (t)$ be defined in lemma 4.1. For any $k \in \mathbb{Z}_+$ and any $t > 0$,

$$\frac{d}{dt} \|h\|^2_{l^p} + f_\star (t) \|h\|^2_{l^p} \leq \|h\|^2_{l^p} \cdot C_k \|\|f\|_{L^\infty}\| F\|_{L^\infty} \| \sqrt{f}\|^2_{l^p} \sum_{m=0}^{k-1} \|h\|^{2m}_{l^p},$$

(5.13)

and

$$\frac{d}{dt} \|F^2 \partial^k h\|_{L^2}^2 + \|f\|^2_{l^p} \leq \|F^2 \partial^k h\|_{L^2}^2 \cdot C_k 1_{\{t > 1\}} \|f\|_{L^\infty} \|F\|_{L^\infty} \| \sqrt{f}\|^2_{l^p},$$

(5.14)

where $C_k$’s are universal constants only depending on $k$.

**Proof.** We first prove (5.13). With $k \in \mathbb{Z}_+$, we differentiate (5.1) in $x$ for $k$-times, integrate it against $(-\Delta)^{1/2} \partial^k h$, and derive as in lemma 5.2 that

$$\frac{1}{2} \frac{d}{dt} \|h\|^2_{l^p} = \int_T \partial^k (\mathcal{H}f \partial \partial h) \cdot (-\Delta)^{1/2} \partial^k h \ dx - \int_T (-\Delta)^{1/2} \partial^k f \cdot (-\Delta)^{1/2} \partial^k h \ dx$$

$$= -\int_T f \cdot \mathcal{H} \left[ \partial^{k+1} h \cdot (-\Delta)^{1/2} \partial^k h \right] \ dx - \int_T \partial^{k+1} (\mathcal{H}) \cdot \partial^k h \ dx$$

$$+ \int_T \left[ \partial^k (\mathcal{H}f \partial \partial h) - \mathcal{H}f \partial^k h \right] (-\Delta)^{1/2} \partial^k h \ dx
$$

$$= \frac{1}{2} \int_T \left[ \partial^{k+1} h \right]^2 - \left[ (-\Delta)^{1/2} \partial^k h \right]^2 \ dx - \int_T \partial^{k+1} h \cdot \partial^k h \ dx$$

$$+ \int_T \left[ \partial^k (\mathcal{H}f \partial h) - \mathcal{H}f \partial^k h \right] (-\Delta)^{1/2} \partial^k h \ dx
$$

$$- \int_T \left[ \partial^k (\mathcal{H}f \partial h) - \mathcal{H}f \partial^k h \right] \partial^k \partial^k h \ dx$$

$$= -\frac{1}{2} \int_T \left[ \partial^{k+1} h \right]^2 + \left[ (-\Delta)^{1/2} \partial^k h \right]^2 \ dx + R_k(t),$$

where

$$R_k(t) := \int_T \left[ \partial^k (\mathcal{H}f \partial h) - \mathcal{H}f \partial^k h \right] (-\Delta)^{1/2} \partial^k h - \left[ \partial^k (\mathcal{H}f \partial h) - \mathcal{H}f \partial^k h \right] \partial^k \partial^k h \ dx.$$
We bound $R_k$ as follows.

$$|R_k| \leq C_k \sum_{n=1}^{k} \int_{\mathbb{T}} \left| \partial_x^n Hf \right| \left| \partial_t^{k+1-n} h \right| \left| -\Delta \right|^{\frac{1}{2}} \partial_t^k h + \left| \partial_x^n f \right| \left| \partial_t^{k+1-n} h \right| \left| \partial_t^{k+1} h \right| \, dx$$

$$\leq C_k \left\| \partial_t^{k+1} h \right\|_{L^2} \sum_{n=1}^{k} \left\| \partial_t^{k+1-n} h \right\|_{L^{2k/(2k+1)}} \left\| \partial_x^n f \right\|_{L^{2k/(2k+1)}},$$

where $C_k > 0$ is a universal constant depending on $k$. Since

$$\partial_t^n f = \partial_t^n (e^t) = e^t \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha_1, \ldots, \alpha_m} \prod_{j=1}^{m} \partial_{\alpha_j} h,$$

with $c_{\alpha_1, \ldots, \alpha_m}$ depending on $(a_1, \ldots, a_m)$, we have that

$$|R_k| \leq C_k \left\| \partial_t^{k+1} h \right\|_{L^2} \sum_{n=1}^{k} \left\| \partial_t^{k+1-n} h \right\|_{L^{2k/(2k+1)}} \left\| \partial_x^n f \right\|_{L^{2k/(2k+1)}},$$

By Sobolev embedding and interpolation, for $l \in \{1, \ldots, k\}$,

$$\left\| \partial_t^l h \right\|_{L^{2l/(2l+1)}} \leq C_k \left\| h \right\|_{H^{l+\frac{1}{2}}} \leq C_k \left( \left\| h \right\|_{H^{l+\frac{1}{2}}} \left\| h \right\|_{H^m} \right)^{\frac{1}{m+1}} \left\| h \right\|_{H^m}^{\frac{m}{m+1}}.$$ 

Hence, (5.17) becomes

$$|R_k| \leq C_k \left\| f \right\|_{L^\infty} \left\| \partial_t^{k+1} h \right\|_{L^2} \sum_{m=1}^{k} \left\| h \right\|_{H^{l+\frac{1}{2}}} \left\| h \right\|_{H^m} \left\| h \right\|_{H^m}^{m-1}$$

$$\leq \frac{1}{2} f_s(t) \left\| \partial_t^{k+1} h \right\|_{L^2} + \left\| h \right\|_{H^{l+\frac{1}{2}}}^2 \cdot C_k f_s(t)^{-1} \left\| f \right\|_{L^\infty} \left\| h \right\|_{H^m} \sum_{m=1}^{k} \left\| h \right\|_{H^m}^{2(m-1)}.$$

Combining this with (5.15) and using the fact $\left\| h \right\|_{H^m} \leq C_f(t)^{-1/2} \left\| f \right\|_{L^m}$, we obtain (5.13). We also remark that, if $k = 0$ in (5.15), we obtain $R_0(t) = 0$ and thus the estimate in lemma 5.2.

The proof of (5.14) is similar. With $k \in \mathbb{Z}_+$,

$$\frac{d}{dt} \int_{\mathbb{T}} F (\partial_t^k f)^2 \, dx = \int_{\mathbb{T}} \partial_t (Hf \cdot F) (\partial_t^k f)^2 \, dx + 2 \int_{\mathbb{T}} F \partial_t^k f \cdot \partial_t (Hf \cdot \partial_t^k f - f(-\Delta)^{\frac{1}{2}} f) \, dx$$

$$= \int_{\mathbb{T}} \partial_t (Hf \cdot F) (\partial_t^k f)^2 \, dx + 2 \int_{\mathbb{T}} F \partial_t^k f \cdot \partial_t (Hf \cdot \partial_t^k f - f(-\Delta)^{\frac{1}{2}} f) \, dx$$

$$+ 2 \int_{\mathbb{T}} F \partial_t^k f \left( \partial_t^k \left( Hf \cdot \partial_t^k f - f(-\Delta)^{\frac{1}{2}} f \right) - \left( Hf \cdot \partial_t^k f - f(-\Delta)^{\frac{1}{2}} f \right) \right) \, dx$$

$$= \int_{\mathbb{T}} \partial_t (Hf \cdot F) (\partial_t^k f)^2 + F \cdot Hf \cdot \partial_t (\partial_t^k f)^2 - \partial_t^k f \cdot (-\Delta)^{\frac{1}{2}} \partial_t^k f \, dx + \tilde{R}_k$$

$$= -2 \left\| h \right\|_{H^{l+\frac{1}{2}}}^2 + \tilde{R}_k,$$  \hspace{1cm} (5.18)
where
\[ \tilde{\mathcal{R}}_k(t) := 2 \int_T F \partial^k_x f \left( \partial^{k-1}_x (\mathcal{H} f \cdot \partial_x \partial_t f - f(-\Delta)^{\frac{1}{2}} \partial_t f) - \left( \mathcal{H} f \cdot \partial_x \partial^k_t f - f(-\Delta)^{\frac{1}{2}} \partial^k_t f \right) \right) dx. \]

Deriving analogously as before, we can show that
\[
|\tilde{\mathcal{R}}_k| \leq C_k |F|_{L^\infty} \int_T \left| \partial^k_x f \right| \sum_{n=1}^{k-1} \left( \left| \partial^k_x \mathcal{H} f \cdot \partial^{k-1-n}_x \partial_x \partial_t f \right| + \left| \partial^k_x f \cdot (-\Delta)^{\frac{1}{2}} \partial^{k-1-n}_x \partial_x \partial_t f \right| \right) dx
\leq C_k |F|_{L^\infty} \left\| \partial^k_x f \right\|_{L^2} \sum_{n=1}^{k-1} \left\| \partial^{k-1-n}_x \partial_x \partial_t f \right\|_{L^2} \left\| \partial^{k-1-n}_x \partial_x \partial_t f \right\|_{L^2}.
\]

By Sobolev embedding and interpolation, for all \( n \in \{1, \ldots, k-1\} \),
\[
\left\| \partial^k_x f \right\|_{L^{\frac{2(k+1)}{k+n}}} \left\| \partial^{k-1-n}_x \partial_x \partial_t f \right\|_{L^{\frac{2(k+1)}{k+n}}} \leq C_k \|f\|_{H^{k+\frac{1}{2}} \to \frac{k+n}{2(k+1)-k}} \|f\|_{H^{(k+1)-k} \to \frac{k+n}{2(k+1)-k}} \leq C_k \|f\|_{H^{k+\frac{1}{2}}} \|f\|_{H^\infty}.
\]

Combining these estimates with (5.18) yields
\[
\frac{d}{dt} \left\| F^2 \partial^k_x f \right\|_{L^2}^2 + 2 \|f\|_{H^{k+\frac{1}{2}}}^2 \leq C_k \|f\|_{L^\infty} \left\| \partial^k_x f \right\|_{L^2} \|f\|_{H^{k+\frac{1}{2}}} \|f\|_{H^\infty}.
\]
and thus, by Young’s inequality,
\[
\frac{d}{dt} \left\| F^2 \partial^k_x f \right\|_{L^2}^2 + \|f\|_{H^{k+\frac{1}{2}}}^2 \leq \left\| \partial^k_x f \right\|_{L^2}^2 + C_k \|f\|_{L^\infty} \left\| F^2 \right\|_{L^2} \|f\|_{H^\infty}.
\]
Then (5.14) follows. Note that (5.19) with \( k = 1 \) exactly corresponds to the estimate in lemma 5.3.

The quantities involved in the estimates in lemma 5.4 can be linked as follows.

**Lemma 5.5.** For any \( k \in \mathbb{Z}_+ \) and any \( t > 0 \),
\[
\|h\|_{H^{k+\frac{1}{2}}} \leq \|f\|_{H^{k+\frac{1}{2}}} \cdot C_k \left[ \|f\|_{L^\infty} \|F\|_{L^2}^{\frac{1}{2}} \sum_{n=1}^{k-1} \|h\|_{H^\infty}^{2n} \|\mathcal{F}\|_{L^2}^{2n} \right] f^*_\star (t)^{-\frac{1}{2}} \sqrt{\|f\|_{H^\infty}}
\]
and
\[
\|F^2 \partial^k_x f\|_{L^2} \leq \|h\|_{H^\infty} \cdot C_k \|f\|_{L^\infty} \sum_{m=0}^{k-1} \|h\|_{H^{k+\frac{1}{2}}}^{2m},
\]
where \( C_k \)’s are universal constants only depending on \( k \).
Proof. For $k \in \mathbb{Z}_+$,

$$
\|h\|_{H^{k+\frac{1}{2}}} = \left\| \partial_{\xi}^{-1} (F \partial_{\xi} f) \right\|_{L^2}
\leq C_k \|F \partial_{\xi} f\|_{H^{k+\frac{1}{2}}} + C_k \sum_{n=1}^{k+1} \sum_{j=1}^{n} \left( \prod_{j=1}^{m} \left\| \partial_{\xi}^{m-j} f \right\|_{H^{k+\frac{1}{2}}} \right)
\leq C_k \|F\|_{H^k} \|f\|_{H^{k+\frac{1}{2}}} + C_k \sum_{n=1}^{k+1} \sum_{j=1}^{n} \left( \prod_{j=1}^{m} \left\| \partial_{\xi}^{m-j} f \right\|_{H^{k+\frac{1}{2}}} \right)
\|\partial_{\xi}^{m-k} f\|_{H^{k+\frac{1}{2}}}.
$$

Here we used the inequality that, for any $s_1 > \frac{1}{2}$ and $s_2 > 0$ with $s_1 \geq s_2$,

$$
\|fg\|_{H^{s_2}} \leq C_{s_1, s_2} \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}.
$$

Since $a_j \in [1, k - 1]$, we derive with interpolation and the condition $\sum_j a_j = n$ that

$$
\|F\|_{H^{k+\frac{1}{2}}} \leq \left( \sum_{j=1}^{m} \left\| \partial_{\xi}^{m-j} f \right\|_{H^{k+\frac{1}{2}}} \right)^{\frac{1}{m}} \left( \prod_{j=1}^{m} \left\| \partial_{\xi}^{m-j} f \right\|_{H^{k+\frac{1}{2}}} \right)^{\frac{1}{m}}
\leq \left( f_* (t)^{-1} \|h\|_{H^{k+\frac{1}{2}}} \right)^{\frac{1}{m}} \left( f_* (t)^{-2} \|f\|_{H^k} \right)^{\frac{1}{m}} \left( \prod_{j=1}^{m} \left\| \partial_{\xi}^{m-j} f \right\|_{H^{k+\frac{1}{2}}} \right)^{\frac{1}{m}}
\leq f_* (t)^{-\left(\frac{m+1}{2}\right)} \left\| h \right\|_{L^{\infty}} \left\| f \right\|_{L^{\infty}} \left\| f \right\|_{H^{k+\frac{1}{2}}}.
$$

In the second inequality, we used the definition of $H^{1/2}$-semi-norm and the fact $F = e^{-h}$ (see (5.5) and (5.6)) to obtain $\|F\|_{H^{1/2}} \leq f_* (t)^{-1} \|h\|_{H^{1/2}}$. In the last inequality, we used $\|f\|_{H^{1/2}} \leq \|f\|_{L^\infty} \|h\|_{H^{1/2}}$ that is also derived from (5.6). Hence,

$$
\|h\|_{H^{k+\frac{1}{2}}} \leq C_k \left[ f_* (t)^{-2} \sum_{n=1}^{k+1} \sum_{j=1}^{n} f_* (t)^{-\left(\frac{m+1}{2}\right)} \left\| h \right\|_{L^{\infty}} \left\| f \right\|_{L^{\infty}} \left\| f \right\|_{H^{k+\frac{1}{2}}} \right].
$$

This implies the desired inequality.

On the other hand, thanks to (5.16), we derive using Sobolev embedding and interpolation that
It decays exponentially to 0 as $t \to \infty$. We do not pursue precise form of estimates for the above claims, as they are too complicated to handle. Suppose that

\[ (5.21) \]

\[ \text{It is non-increasing in } t, \text{ and non-decreasing in } k, \text{ and stays bounded as } t \to \infty. \]

Throughout this proof, we will use the notation $\mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1})$ to denote a generic function of $k, t, ||f_0||_{L^1},$ and $||F_0||_{L^1}$, which satisfies the properties (i) and (ii) above, and yet stays bounded as $t \to \infty$.

Now we may bootstrap with lemmas 5.2, 5.4, and 5.5 to prove finiteness of $||f||_{H^s}$ for all $k \in \mathbb{Z}_+$ at all positive times, as well as their exponential decay as $t \to \infty$.

**Proposition 5.2.** Suppose $f = f(x, t)$ is a strictly positive smooth solution on $\mathbb{T} \times [0, +\infty)$. For all $k \in \mathbb{Z}_+$ and $t > 0$,

\[ ||h(\cdot, t)||_{H^{s+\frac{1}{2}}} \cdot ||F^k \partial_t^s f(\cdot, t)||_{L^2} \leq \mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1}). \quad (5.20) \]

Here $\mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1})$ denotes a generic function of $k, t, ||f_0||_{L^1},$ and $||F_0||_{L^1}$, which satisfies:

(i) It is finite whenever $t > 0$;
(ii) It is non-increasing in $t$, and non-decreasing in $||f_0||_{L^1}$ and $||F_0||_{L^1}$;
(iii) It decays exponentially to 0 as $t \to +\infty$ with an explicit rate that only depends on $||f_0||_{L^1}$ and $||F_0||_{L^1}$, i.e. there exists some $\alpha(||f_0||_{L^1}, ||F_0||_{L^1}) > 0$ and constant $\mathcal{C}_k(||f_0||_{L^1}, ||F_0||_{L^1}) > 0$, such that for all $t \geq t_0$,

\[ \mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1}) \leq \mathcal{C}_k(||f_0||_{L^1}, ||F_0||_{L^1}) \exp[-\alpha(||f_0||_{L^1}, ||F_0||_{L^1}) t]. \]

As a result, for any $t_0 > 0$, $f(x, t)$ is a smooth solution to (1.1) on the space-time domain $\mathbb{T} \times [0, +\infty)$. As $t \to +\infty$, $f(x, t)$ converges exponentially to $f_\infty$ defined in (1.9). To be more precise, for arbitrary $j, k \in \mathbb{N}$, and $t > 0$,

\[ ||\partial_t^j \partial_x^k (f(\cdot, t) - f_\infty)||_{L^2(\mathbb{T})} \leq \mathcal{C}_{j,k}(t, ||f_0||_{L^1}, ||F_0||_{L^1}). \quad (5.21) \]

Here $\mathcal{C}_{j,k}(t, ||f_0||_{L^1}, ||F_0||_{L^1})$ is a function having similar properties as $\mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1})$ above.

**Remark 5.4.** We do not pursue precise form of estimates for the above claims, as they are too complicated but may not be optimal. Nevertheless, in section 7 below, we will prove sharp decay estimates for the solution when $t \gg 1$.

**Proof.** Throughout this proof, we will use the notation $\mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1})$ to denote a generic function of $k, t, ||f_0||_{L^1},$ and $||F_0||_{L^1}$, which satisfies the properties (i) and (ii) above, and yet stays bounded as $t \to +\infty$. 

(1.) $\mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1})$ stays bounded as $t \to +\infty$. 

\[ \mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1}) \leq \mathcal{C}_k(||f_0||_{L^1}, ||F_0||_{L^1}), \]

\[ \mathcal{C}_k(t, ||f_0||_{L^1}, ||F_0||_{L^1}) \leq \mathcal{C}_k(||f_0||_{L^1}, ||F_0||_{L^1}) \exp[-\alpha(||f_0||_{L^1}, ||F_0||_{L^1}) t]. \]
For brevity, we will write it as $C_k(t)$ in what follows. Like the commonly-used notation $C$ for universal constants, its precise definition may vary from line to line. If it turns out to not depend on $k$, we simply write it as $C(t)$.

By lemmas 4.1, 4.2, and proposition 5.1, we find

$$\|f(\cdot, t)\|_{L^\infty}, \|F(\cdot, t)\|_{L^\infty}, \|h(\cdot, t)\|_{H^\frac{1}{4}} \leq C(t).$$

As a result, lemmas 5.4 and 5.5 imply that, for any $k \in \mathbb{Z}_+$ and any $t > 0$,

$$\frac{d}{dt} \|h\|^2_{H^\frac{1}{4}} + f_*(t) \|h\|^2_{H^{\frac{1}{4}+\frac{1}{4}}} \leq \|h\|^2_{H^{\frac{1}{4}+\frac{1}{4}}} \cdot C_k(t) \|\sqrt{F}\|^2_{H^{\frac{1}{8}}}, \quad (5.22)$$

$$\frac{d}{dt} \|F^\frac{1}{2} \partial_t f\|^2_{L^2} + \|f\|^2_{H^{\frac{1}{4}+\frac{1}{4}}} \leq \|F^\frac{1}{2} \partial_t f\|^2_{L^2} \cdot C_k(t) \|F\|^2_{H^{\frac{1}{8}}}, \quad (5.23)$$

and

$$\|h\|_{H^{\frac{1}{4}+\frac{1}{4}}} \leq \|f\|_{H^{\frac{1}{4}+\frac{1}{4}}} \cdot C_k(t) \|\sqrt{F}\|_{H^{\frac{1}{8}}}, \quad (5.24)$$

$$\|F^\frac{1}{2} \partial_t f\|_{L^2} \leq \|h\|_{H^{\frac{1}{4}}} \cdot C_k(t). \quad (5.25)$$

**Step 1 (preliminary).** We start from bounding $\|h(\cdot, t)\|_{H^\frac{1}{4}}$ and $\|\sqrt{F}\|_{H^{\frac{1}{8}}}$.

By (5.4) in lemma 5.2,

$$\frac{d}{dt} \|h\|^2_{H^\frac{1}{4}} \leq -f_*(t) \int_T (\partial_t h)^2 + \left( (-\Delta)^\frac{1}{2} h \right)^2 \, dx = -2f_*(t) \|h\|^2_{H^\frac{1}{4}}.$$

Hence, for arbitrary $t > t_* > 0$, we have

$$\|h(\cdot, t)\|_{H^\frac{1}{4}}^2 \leq \|h(\cdot, t_*)\|_{H^\frac{1}{4}}^2 \exp \left( -2 \int_{t_*}^t f_*(\tau) \, d\tau \right),$$

This together with lemma 4.1 and proposition 5.1 implies $\|h(\cdot, t)\|_{H^\frac{1}{4}}$ verifies the properties in (5.20), although it is not included there.

Thanks to (5.10), for any $t > 0$,

$$\|F^\frac{1}{2} \partial_t f(\cdot, t)\|_{L^2}^2 = \|\sqrt{f(\cdot, t)}\|_{H^{\frac{1}{8}}}^2 \leq Cr^{-1} \|h(\cdot, t/2)\|_{H^\frac{1}{4}}^2.$$

Hence, $\|\sqrt{f(\cdot, t)}\|_{H^{\frac{1}{4}}}$ satisfies (5.20) as well. Consequently, given a $C_k(t)$ satisfying the assumptions at the beginning of the proof, we have

$$C_k(t) \|\sqrt{f(\cdot, t)}\|_{H^{\frac{1}{4}}}^2 \leq \frac{1}{2} f_*(t) \quad (5.26)$$

for all $t > 1$, where the required largeness depends on the form of $C_k(\cdot)$, and thus on $k$, $\|f_0\|_{L^1}$, and $\|F_0\|_{L^1}$ essentially.

In the next two steps, we prove the claim (5.20) by induction.
Step 2 (base step). Let us verify (5.20) with \( k = 1 \). We have just proved \( \| F^1 \partial_t f(t) \|_{L^2} \) satisfies (5.20). On the other hand, by lemma 5.3, for arbitrary \( t > 0 \), there exists \( t_\ast \in [\frac{1}{2} t, \frac{3}{2} t] \), such that

\[
\| f(\cdot, t_\ast) \|_{H^\frac{1}{2}}^2 \leq C t^{-1} \| F^1 \partial_t f(\cdot, t/2) \|_{L^2}^2.
\]

Then (5.24) implies

\[
\| h(\cdot, t_\ast) \|_{H^\frac{1}{2}}^2 \leq C t^{-1} \| F^1 \partial_t f(\cdot, t/2) \|_{L^2} \cdot C(t_\ast) \| \sqrt{f(\cdot, t_\ast)} \|_{H^\frac{1}{2}} \leq C(t).
\]

Now, applying (5.22) with \( k = 1 \), we find

\[
\begin{align*}
\| h(\cdot, t) \|_{H^\frac{1}{2}}^2 + \int_{t_\ast}^t \exp \left( \int_{t_\ast}^\tau C(\tau) \| \sqrt{f(\cdot, \tau)} \|_{H^\frac{1}{2}}^2 \, d\tau \right) f_\ast(\cdot, \tau') \| h(\cdot, \tau') \|_{H^\frac{1}{2}}^2 \, d\tau' \\
\leq \exp \left( \int_{t_\ast}^t C(\tau) \| \sqrt{f(\cdot, \tau)} \|_{H^\frac{1}{2}}^2 \, d\tau \right) \| h(\cdot, t_\ast) \|_{H^\frac{1}{2}}^2.
\end{align*}
\] (5.27)

Notice that

\[
\int_{t_\ast}^\infty C(\tau) \| \sqrt{f(\cdot, \tau)} \|_{H^\frac{1}{2}}^2 d\tau \leq C(t_\ast) \leq C(t),
\] (5.28)

so \( \| h(\cdot, t) \|_{H^\frac{1}{2}} \leq C(t) \). Also, (5.22) and (5.26) imply that, for sufficiently large \( t \),

\[
\frac{d}{dt} \| h(\cdot, t) \|_{H^\frac{1}{2}}^2 + \frac{1}{2} \| f(\cdot, t) \|_{H^\frac{1}{2}}^2 \leq 0.
\]

Hence, \( \| h(\cdot, t) \|_{H^\frac{1}{2}} \) decays exponentially as \( t \to +\infty \) with an explicit rate (see (4.1)). Therefore, \( \| h(\cdot, t) \|_{H^\frac{1}{2}} \) satisfies (5.20).

We also derive from (5.27) that there exists \( t'_\ast \in [\frac{1}{2} t, \frac{3}{2} t] \), such that

\[
\| h(\cdot, t'_\ast) \|_{H^\frac{1}{2}}^2 \leq C t^{-1} f_\ast(t'_\ast)^{-1} \exp \left( \int_{t_\ast}^{t'_\ast} C(\tau) \| \sqrt{f(\cdot, \tau)} \|_{H^\frac{1}{2}}^2 \, d\tau \right) \| h(\cdot, t_\ast) \|_{H^\frac{1}{2}}^2 \leq C(t).
\] (5.29)

Step 3 (induction step). Suppose that (5.20) holds for the case \( k \), and that for any \( t > 0 \), there exists \( t_\ast \in [t - 2^{-k} t, t - 2^{-k+1} t] \), such that \( \| h(\cdot, t_\ast) \|_{H^{k+1}} \leq C_k(t) \) (see (5.29)). In this step, we verify (5.20) for \( \| h(\cdot, t) \|_{H^{k+1}} \) and \( \| F^2 \partial_t^{k+1} f(\cdot, t) \|_{L^2} \) by arguing as above.

Thanks to (5.25),

\[
\| F^2 \partial_t^{k+1} f(\cdot, t_\ast) \|_{L^2} \leq \| h(\cdot, t_\ast) \|_{H^{k+1}} \cdot C_{k+1}(t_\ast) \leq C_{k+1}(t).
\]

Then (5.23) yields

\[
\begin{align*}
\| F^2 \partial_t^{k+1} f(\cdot, t) \|_{L^2}^2 + \int_{t_\ast}^t \exp \left( \int_{t_\ast}^\tau C_{k+1}(\tau) \| \sqrt{f(\cdot, \tau)} \|_{H^\frac{1}{2}}^2 \, d\tau \right) \| f(\tau, t') \|_{H^\frac{1}{2}}^2 \, d\tau' \\
\leq \exp \left( \int_{t_\ast}^t C_{k+1}(\tau) \| \sqrt{f(\cdot, \tau)} \|_{H^\frac{1}{2}}^2 \, d\tau \right) \| F^2 \partial_t^{k+1} f(\cdot, t_\ast) \|_{L^2}^2.
\end{align*}
\] (5.30)
which implies \( \| F^\frac{1}{2} \partial_t^{k+1} f(\cdot, t) \|_{L^2} \leq C_{k+1}(t) \) (see (5.28)). In addition, since
\[
\| f \|_{H^{k+\frac{1}{2}}}^2 \geq \| F^\frac{1}{2} F^k \partial_t^{k+1} f \|_{L^2}^2 \geq f_c(t) \| F^\frac{1}{2} \partial_t^{k+1} f \|_{L^2}^2,
\]
(5.23) also implies that, for \( t \gg 1 \) (see (5.26)),
\[
\frac{d}{dt} \| F^\frac{1}{2} \partial_t^{k+1} f \|_{L^2}^2 + \frac{1}{2} f_c(t) \| F^\frac{1}{2} \partial_t^{k+1} f \|_{L^2}^2 \leq 0.
\]
This gives the exponential decay of \( \| F^\frac{1}{2} \partial_t^{k+1} f(\cdot, t) \|_{L^2} \) as \( t \to +\infty \). Therefore, \( \| F^\frac{1}{2} \partial_t^{k+1} f(\cdot, t) \|_{L^2} \) satisfies (5.20).

Besides, (5.30) also implies that there exists \( t'_* \in [t - 2^{-2k-1} t, t - 2^{-2k-2} t] \), such that
\[
\| f(\cdot, t'_*) \|_{H^{k+\frac{1}{2}}} \leq C 2^k r^{-1} \exp \left( \int_{t_*}^{t'_*} C_{k+1}(t) \| \sqrt{f} \|_{L^\infty} \frac{d\tau}{\sqrt{\tau}} \right) \| F^\frac{1}{2} \partial_t^{k+1} f(\cdot, t) \|_{L^2} \leq C_{k+1}(t).
\]
Then we can use (5.22) and (5.24) and argue as in the previous step to show that \( \| \delta(\cdot, t) \|_{H^{k+\frac{1}{2}}} \) satisfies (5.20). Moreover, there exists \( t''_* \in [t - 2^{-2k-1} t, t - 2^{-2k-3} t] \), such that \( \| \delta(\cdot, t''_*) \|_{H^{k+\frac{1}{2}}} \leq C_{k+1}(t) \). We omit the details.

By induction, we thus prove (5.20) for all \( k \in \mathbb{Z}_+ \).

**Step 4 (convergence to \( f_\infty \)).** For all \( k \in \mathbb{Z}_+ \),
\[
\| f \|_{H^k}^2 = \| F^\frac{1}{2} \partial_t^k f(\cdot, t) \|_{L^2}^2 \leq \| f \|_{L^\infty} \| F^\frac{1}{2} \partial_t^k f(\cdot, t) \|_{L^2}^2.
\]
By lemma 4.2, \( \| f(\cdot, t) \|_{H^k} \) is bounded on \([t_0, +\infty)\) for arbitrary \( t_0 > 0 \), and it decays exponentially as \( t \to +\infty \). Hence, \( \| f(\cdot, t) \|_{H^k} \) satisfies the same properties. Combining this with lemma 3.1, we find that \( \| f(\cdot, t) \|_{L^2} \) is non-increasing in time and its convergence at \( t \to +\infty \) is exponential. Hence, \( f(\cdot, t) \) is smooth in space-time in \( \mathbb{T} \times [t_0, +\infty) \) for any \( t_0 > 0 \), and it converges exponentially to some constant as \( t \to +\infty \) in \( H^k(\mathbb{T}) \)-norms for all \( k \in \mathbb{N} \). Recall that \( \| F \|_{L^2} \) is conserved (see lemma 3.1), so the constant must be \( f_\infty \) defined in (1.9). This proves (5.21) with \( j = 0 \).

The claim (5.21) with \( j \in \mathbb{Z}_+ \) immediately follows from the case \( j = 0 \) and equation (1.1).

\[ \square \]

### 5.3. Time-continuity at \( t = 0 \)

We end this section by showing the time-continuity of the solution at \( t = 0 \) in a suitable sense. Note that the time-continuity at all positive times is already covered in proposition 5.2.

**Lemma 5.6.** Suppose \( f \) is a strictly positive strong solution of (1.1). Then for any \( 0 < t \leq \| f_0 \|_{L^2}^{-1} \) and \( \alpha \in (0, \frac{1}{3}) \),
\[
W_1(F_0, f(\cdot, t)) \leq C_{\alpha} \| F_0 \|_{L^2} \left( t \| F_0 \|_{L^2}^{-1} \right)^{1-\frac{5\alpha}{6}} \left( \| f_0 \|_{L^2} \| F_0 \|_{L^2} \right)^{1-\frac{\alpha}{3}}.
\]
Here \( W_1(\cdot, \cdot) \) denotes the 1-Wasserstein distance [65, chapter 6], and \( C_{\alpha} > 0 \) is a universal constant depending on \( \alpha \).
Proof. Given $t_* \in (0, \|f_0\|_{L^1}^{-1})$ and $\delta < t_*$, let $\eta_\delta(t)$ be a smooth non-increasing cutoff function on $[0, +\infty)$, such that $\eta_\delta \equiv 1$ on $[0, t_* - \delta]$ and $\eta_\delta \equiv 0$ on $[t_*, +\infty)$. Let $f \in C^\infty(\mathbb{T})$. We take $\phi(x, t) = \phi(x) \eta_\delta(t)$ in (1.5) and derive that

$$
\int_\mathbb{T} \phi(x) F_0(x) \, dx + \int_0^{t_*} \eta_\delta'(t) \int_\mathbb{T} \phi(x) F(x, t) \, dx \, dt = \int_0^{t_*} \eta_\delta(t) \int_\mathbb{T} \phi'(x) \cdot F \, dx \, dt.
$$

Sending $\delta \to 0^+$ and using the smoothness of $f$ and $F$ at positive times, we obtain

$$
\int_\mathbb{T} \phi(x) (F_0(x) - F(x, t_*)) \, dx = \int_0^{t_*} \int_\mathbb{T} \phi'(x) \cdot F \, dx \, dt.
$$

By corollary 5.1 and the fact that $\|F(\cdot, t_*)\|_{L^1}$ is conserved,

$$
\left| \int_\mathbb{T} \phi(x) (F_0(x) - F(x, t_*)) \, dx \right| \leq C_\alpha \|\phi'\|_{L^\infty} \|F_0\|_{L^1} \int_0^{t_*} \|f(\cdot, t)\|_{C^\alpha} \, dt \leq C_\alpha \|\phi'\|_{L^\infty} \|F_0\|_{L^1} \left( t_* \|F_0\|_{L^1}^{-1} \right)^{\frac{\alpha}{1-\alpha}} \left( \|f_0\|_{L^1} \|F_0\|_{L^1} \right)^{\frac{\alpha}{1-\alpha}}.
$$

By the equivalent definition of the 1-Wasserstein distance (see [65, chapter 6])

$$
W_1(F_0, F(\cdot, t_*)) = \sup_{\|\phi\|_{L^1} \leq 1} \int_\mathbb{T} \phi(x) (F_0(x) - F(x, t_*)) \, dx,
$$

we conclude. \hfill \Box

6. Global well-posedness

6.1. Well-posedness for band-limited positive initial data

First we consider strictly positive and band-limited initial data $f_0$; the latter assumption means that $f_0$ is a finite linear combination of Fourier modes. To prove the well-posedness in this case, it might be feasible to turn the equation (1.1) back to its Lagrangian formulation (see (2.7) and (2.8)), and argue as in [46, 48]. Indeed, we can recover the string configuration $X_0(x)$ from $f_0(x)$ through (2.9) (also see the proof of corollary 2.1 in section 6.2), and $X_0$ should be smooth and satisfy the well-stretched assumption (see remark 3.2). Nonetheless, here we present a more straightforward approach by taking advantage of the special nonlinearity of (1.1).

Define the Fourier transform of $f \in L^1(\mathbb{T})$ as in (1.6). Denote $\hat{f} := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \, dx = \mathcal{F}(f)_0$. We write (1.1) as

$$
\partial_t f + \hat{f}(t) (-\Delta)^{\frac{1}{4}} f = \mathcal{H} f \cdot \partial_t \hat{f} - (f - \hat{f}) \cdot (-\Delta)^{\frac{1}{4}} f.
$$

Taking the Fourier transform on both sides yields, for any $m \in \mathbb{Z},$

$$
\frac{d}{dt} \hat{f}_m + m\hat{f}(t) \hat{f}_m = \sum_{n \neq m} -\text{sgn} (m-n) \hat{f}_{m-n} \cdot \text{inf}_n - \hat{f}_{m-n} \cdot |n| \hat{f}_n
$$

$$
= - \sum_{n \neq 0, m} (1 - \text{sgn} (m-n) \text{sgn} (n)) \hat{f}_{m-n} \cdot |n| \hat{f}_n.
$$

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Given \( n \neq 0, m \), we observe that \( \text{sgn}(m - n)\text{sgn}(n) \neq 1 \) only when \((m - n)\) and \(n\) have opposite signs, which implies \( \max\{|m - n|, |n|\} > |m| \). Hence, for \( m \geq 0 \),

\[
\frac{d}{dt}f_m = -mf(t)\hat{f}_m - \sum_{\max\{|m-n|,|n|\} > m} 2\hat{f}_{m-n} \cdot |n|\hat{f}_n \\
= -mf(t)\hat{f}_m - \sum_{n < 0} 2(|n| + |m - n|)\hat{f}_{m-n}\hat{f}_n,
\]

i.e.

\[
\frac{d}{dt}f_m = -mf(t)\hat{f}_m - \sum_{n > 0} 2(m + 2n)\hat{f}_{m+n}\hat{f}_n. \tag{6.1}
\]

The equation for \( \hat{f}_m \) with \( m < 0 \) can be obtained by taking the complex conjugate. Formally, this immediately implies that, if there exists some \( K \in \mathbb{N} \) such that \( \mathcal{F}(f_0)_{|k| > K} = 0 \) for all \(|k| > K\), then \( \hat{f}_k(t) \equiv 0 \) for all \( t \geq 0 \) and \(|k| > K\), and moreover, \( \hat{f}_{k,K}(t) = -\hat{K}f(t)\hat{f}_{K,K}(t) \). Hence, we have the following well-posedness result.

**Proposition 6.1 (proposition 1.1).** Suppose \( f_0 > 0 \) is band-limited, i.e. there exists some \( K \in \mathbb{N} \), such that \( \mathcal{F}(f_0)_{|k| > K} = 0 \) for all \(|k| > K\). Then \((1.1)\) has a global strong solution \( f = f(x,t) \) that is also strictly positive and band-limited, such that \( f_k(t) = 0 \) for all \(|k| > K\) and \( t \geq 0 \). Such a solution is unique, and it can be determined by a finite family of ODEs among \((6.1)\) with the initial condition \( f_k(0) = \mathcal{F}(f_0)_k \).

**Proof.** As we pointed out, it is enough to solve the ODEs for \( \hat{f}_k(t) \) with \( k \in [-K,K] \) (see \((6.1)\)) and simply put \( f_k(t) \equiv 0 \) if \(|k| > K\). The local well-posedness of the ODEs is trivial. The local ODE solution can be extended to the time interval \([0, +\infty)\) thanks to the boundedness of \( ||f(t)\|_{L^2} \) (see lemma \(3.1\)) as well as the Paserval’s identity. Positivity of \( f \) follows from lemma \(3.2\). \( \square \)

### 6.2. Weak solutions for initial data in the energy class

Now we turn to prove theorem \(1.1\). Recall that we call \( f_0 \in L^1(\mathbb{T}) \) an initial data in the energy class in the view of remark \(3.1\).

**Proof of theorem 1.1 (part).** In this part of the proof, we will establish existence of a global weak solution to \((1.1)\), and then show it enjoys the properties \((1)-(4)\) in theorem \(1.1\). The remaining property \((5)\) will be studied in section \(7\) (see corollary \(7.1)\).

To obtain weak solutions corresponding to general initial data, we shall first approximate them by smooth solutions, and then pass to the limit in the weak formulation \((1.5)\). In this process, we will need the Fejér kernel on \(\mathbb{T}\) \([25, \text{chapter 1}]\)

\[
\mathcal{F}_N(x) := \frac{1}{2\pi N} \frac{\sin^2 \left( \frac{N}{2} x \right)}{\sin^2 \frac{x}{2}} = \frac{1}{2\pi} \sum_{|k| < N} \left( 1 - \frac{|k|}{N} \right) e^{ikx}. \tag{6.2}
\]

For any given \( N \in \mathbb{Z}_+ \), \( \mathcal{F}_N \) is non-negative and band-limited, having integral \(1\) on \(\mathbb{T}\). \( \{\mathcal{F}_N\}_{N=1}^{\infty} \) is known to be an approximation of identity on \(\mathbb{T}\).

We proceed in several steps.
Step 1 (approximate solutions). Given $f_0 \in L^1(\mathbb{T})$ with $F_0 = \frac{1}{f_0} \in L^1(\mathbb{T})$, define their cut-offs for $M \in \mathbb{Z}_+$ that
\[
f_{0,M} := \max \left\{ \min \{f_0, M\}, M^{-1} \right\}, \quad F_{0,M} = (f_{0,M})^{-1}.
\]
Then $f_{0,M}, F_{0,M} \in [M^{-1}, M]$ on $\mathbb{T}$. Clearly, as $M \to +\infty$, $f_{0,M} \to f_0$ and $F_{0,M} \to F_0$ in $L^1(\mathbb{T})$.

Next, for $N \in \mathbb{Z}_+$, we let
\[
f_{0,M,N} := \mathcal{F}_N * f_{0,M}, \quad F_{0,M,N} := (f_{0,M,N})^{-1}.
\]
Then $f_{0,M,N}$ is positive and band-limited; $f_{0,M,N} \in [M^{-1}, M]$ on $\mathbb{T}$. As $N \to +\infty$, $f_{0,M,N} \to f_0$ in $L^1(\mathbb{T})$. In particular, we have $\|f_{0,M,N}\|_{L^1} = \|f_{0,M}\|_{L^1}$ for all $N \in \mathbb{Z}_+$. Moreover, $F_{0,M,N} \to F_{0,M}$ in $L^1(\mathbb{T})$ as $N \to +\infty$, because $|F_{0,M,N} - F_{0,M}| \leq M^2 |f_{0,M,N} - f_0|$. We also have $\|F_{0,M,N}\|_{L^1} \leq \|F_{0,M}\|_{L^1}$ for all $N \in \mathbb{Z}_+$. This is because, by Jensen’s inequality,
\[
F_{0,M,N}(x) = \frac{1}{\mathcal{F}_N * f_{0,M}}(x) \leq \left[ \mathcal{F}_N * \left( \frac{1}{f_{0,M}} \right) \right](x) = \mathcal{F}_N * F_{0,M}(x).
\]

By proposition 6.1, there exists a positive and band-limited $f_{M,N}$ that is a strong solution to (1.1) with initial data $f_{0,M,N}$. Define $F_{M,N} = (f_{M,N})^{-1}$. Hence, for any $\varphi \in C_0^\infty(\mathbb{T} \times [0, +\infty))$ and arbitrary $0 < \delta < 1$,
\[
\int_{\mathbb{T}} \varphi(x, 0) F_{0,M,N}(x) \, dx = \int_{\mathbb{T} \times [0, \delta]} + \int_{\mathbb{T} \times [\delta, +\infty]} (\partial_t \varphi \cdot \mathcal{H} f_{M,N} - \partial_x \varphi) F_{M,N} \, dx \, dr. \quad (6.3)
\]

Step 2 (taking the limit in $N$). Since $\|f_{0,M,N}\|_{L^1}$ and $\|F_{0,M,N}\|_{L^1}$ are uniformly bounded in $N$, by lemma 3.1 and proposition 5.2, there exist smooth functions $f_M$ and $F_M$ defined on $\mathbb{T} \times (0, +\infty)$, and subsequences $\{f_{M,j}\}_{j=1}^\infty$ and $\{F_{M,j}\}_{j=1}^\infty$, such that, as $j \to +\infty$, $f_{M,j} \to f_M$ and $F_{M,j} \to F_M$ in $C^4_{\text{loc}}(\mathbb{T} \times (0, +\infty))$ (all $k \in \mathbb{N}$), and moreover, $f_{M,N} \to f_M$ in $L^2([0, +\infty); H^2(\mathbb{T}))$.

It is then not difficult to verify the following facts.

(a) $F_M = \frac{1}{f_M}$ on $\mathbb{T} \times (0, +\infty)$; by lemma 3.2, $f_M, F_M \in [M^{-1}, M]$ on $\mathbb{T} \times (0, +\infty)$.
(b) For all $t > 0$,
\[
\|F_M(\cdot, t)\|_{L^1} = \lim_{j \to +\infty} \|F_{M,j}(\cdot, t)\|_{L^1} = \lim_{j \to +\infty} \|F_{0,M,j}\|_{L^1} = \|F_{0,M}\|_{L^1}.
\]
(c) For any $0 < t_1 \leq t_2$,
\[
\frac{1}{2} \|f_M(\cdot, t_2)\|_{L^1} + \int_{t_1}^{t_2} \|f_M(\cdot, \tau)\|_{H^1}^2 \, d\tau = \frac{1}{2} \|f_M(\cdot, t_1)\|_{L^1} \leq \frac{1}{2} \|f_{0,M}\|_{L^1}. \quad (6.4)
\]

Indeed, one first proves an identical energy relation for $f_{M,j}$ (see (3.1)), and then send $j \to +\infty$. This in particular implies the energy inequality, i.e. for all $t > 0$,
\[
\frac{1}{2} \|f_M(\cdot, t)\|_{L^1} + \int_t^T \|f_M(\cdot, \tau)\|_{H^1}^2 \, d\tau \leq \frac{1}{2} \|f_{0,M}\|_{L^1}.
\]

We will show later that the energy equality is actually achieved (see (6.7) below).
(d) With $M \in \mathbb{Z}_+$ fixed, the estimates in corollary 5.1 hold uniformly for all $f_{M,N}$ and $f_M$, where $f_0$ and $F_0$ there should be replaced by $f_{0,M}$ and $F_{0,M}$ respectively. Let us explain the estimates for $f_M$. Indeed, (5.9) follows from the convergence $f_{M,N} \to f_M$ at all positive times, and then the proof of (5.12) for $f_M$ uses the property (c) above and a similar argument as in corollary 5.1. As a consequence, $\mathcal{H}_M \cdot F_M \in L^1_{\text{loc}}(\mathbb{T}^1 \times [0,\infty))$.

(e) With $M \in \mathbb{Z}_+$ fixed, the estimate (5.21) in proposition 5.2 holds uniformly for all $f_{M,N}$ and $f_M$, again with $f_0$ and $F_0$ there replaced by $f_{0,M}$ and $F_{0,M}$. Observe that $F_{0,M,N} \to F_{0,M}$ in $L^1(\mathbb{T}^1)$ implies $f_{\infty,M,N} \to f_{\infty,M}$ (they are defined by (1.9) in terms of $f_{0,M,N}$ and $f_{0,M}$ respectively). Hence, $f_M$ converges to $f_{\infty,M}$ exponentially as $t \to +\infty$.

Now in (6.3), we take $N$ to be $N_j$ and send $j \to +\infty$. Note that $\delta$ is arbitrary in (6.3). By virtue of (d) above, as $\delta \to 0^+$,

$$
\sup_{j \in \mathbb{Z}_+} \int_{T \times [0,\delta]} \left( \partial_t \varphi \cdot \mathcal{H}_M \varphi - \partial_t \varphi \right) F_{M,N} \, dx \, dt \quad \text{and} \quad \int_{T} \left( \partial_t \varphi \cdot \mathcal{H}_M - \partial_t \varphi \right) F_M \, dx \, dt
$$

both converge to 0. Hence,

$$
\int_{T} \varphi(x,0) F_{0,M}(x) \, dx = \int_{T \times [0,\infty)} \partial_t \varphi \cdot \mathcal{H}_M \cdot F_M - \partial_t \varphi \cdot F_M \, dx \, dt, \quad (6.5)
$$

Therefore, $f_M$ is a global weak solution to (1.1) with initial data $f_{0,M}$.

We further confirm that $F_M$ takes the ‘initial data’ $F_{0,M}$ in a more quantitative sense. For $t > 0$,

$$
W_1 \left( \frac{F_{0,M}}{\|F_{0,M}\|_{L^1}}, \frac{F_M(\cdot,t)}{\|F_{0,M}\|_{L^1}} \right) \leq W_1 \left( \frac{F_{0,M}}{\|F_{0,M}\|_{L^1}}, \frac{F_{0,M,N_j}}{\|F_{0,M,N_j}\|_{L^1}} \right) + W_1 \left( \frac{F_{M,N_j}(\cdot,t)}{\|F_{0,M,N_j}\|_{L^1}}, \frac{F_{M,N_j}(\cdot,t)}{\|F_{0,M,N_j}\|_{L^1}} \right) + W_1 \left( \frac{F_{0,M,N_j}}{\|F_{0,M,N_j}\|_{L^1}}, \frac{F_{0,M,N_j}(\cdot,t)}{\|F_{0,M,N_j}\|_{L^1}} \right).
$$

The first two terms on the right-hand side converge to zero as $j \to +\infty$ because of strong $L^1$-convergence. Thanks to lemma 5.6 with e.g. $\alpha = \frac{1}{10}$, the last term above satisfies that, for arbitrary $j \in \mathbb{Z}_+$ and $t \in (0,\|f_{0,M}\|_{L^1}^{-1})$ (recall that $\|f_{0,M,N_j}\|_{L^1} = \|f_{0,M}\|_{L^1}$ for all $N \in \mathbb{Z}_+$),

$$
W_1 \left( \frac{F_{0,M,N_j}}{\|F_{0,M,N_j}\|_{L^1}}, \frac{F_{M,N_j}(\cdot,t)}{\|F_{0,M,N_j}\|_{L^1}} \right) \leq C (t\|f_{0,M,N_j}\|_{L^1}^{-1})^{-\frac{\alpha}{2}} (\|f_{0,M}\|_{L^1} \|f_{0,M,N_j}\|_{L^1})^{\frac{1}{2}}.
$$

Therefore,

$$
W_1 \left( \frac{F_{0,M}}{\|F_{0,M}\|_{L^1}}, \frac{F_M(\cdot,t)}{\|F_{0,M}\|_{L^1}} \right) \leq C (t\|F_{0,M}\|_{L^1}^{-1})^{\frac{\alpha}{2}} (\|f_{0,M}\|_{L^1} \|F_{0,M}\|_{L^1})^{\frac{1}{2}} \quad (6.6)
$$

for all $t \in (0,\|f_{0,M}\|_{L^1}^{-1})$. This further implies [65], as $t \to 0^+$,

$$
\frac{F_M(\cdot,t)}{\|F_{0,M}\|_{L^1}} \quad \text{converges weakly in} \quad \mathcal{P}(\mathbb{T}) \quad \text{to} \quad \frac{F_{0,M}}{\|F_{0,M}\|_{L^1}},
$$

where $\mathcal{P}(\mathbb{T})$ denotes the space of probability measure on $\mathbb{T}$. Since $\|F_M\| \leq M$, we further have that, for any $p \in [1,\infty)$, $F_M(\cdot,t)$ converges weakly in $L^p(\mathbb{T})$ to $F_{0,M}$ as $t \to 0^+$. Since $x \to \frac{1}{x}$
is convex on \((0, +\infty), \|f_{0,M}\|_{L^1} \leq \liminf_{t \to 0^+} \|f_M(\cdot, t)\|_{L^1}\) (see [28, chapter 2]). This together with (6.4) implies that \(\lim_{t \to 0^+} \|f_M(\cdot, t)\|_{L^1} = \|f_{0,M}\|_{L^1}\) and therefore, for any \(t > 0\),
\[
\frac{1}{2} \|f_M(\cdot, t)\|_{L^1} + \int_0^t \|f_M(\cdot, \tau)\|_{L^2}^2 \, d\tau = \frac{1}{2} \|f_{0,M}\|_{L^1}.
\]

(6.7)

**Step 3 (taking the limit in \(M\)).** By virtue of the properties of \(f_{0,M}\) and \(F_{0,M}\), as well as the uniform bounds for \(f_M\) and \(F_M\) proved above, there exist smooth functions \(f\) and \(F\) defined on \(T \times (0, +\infty)\), and subsequences \(\{f_{M_i}\}_{i=1}^{\infty}\) and \(\{F_{M_i}\}_{i=1}^{\infty}\), such that, as \(i \to +\infty\), \(f_{M_i} \to f\) and \(F_{M_i} \to F\) in \(C^k_{\text{loc}}(T \times (0, +\infty))\) \((\forall k \in \mathbb{N})\), and moreover, \(f_{M_i} \to f\) in \(L^2([0, +\infty); H^1(T))\). One can similarly verify:

(a') \(F = \frac{1}{2}\) on \(T \times (0, +\infty)\).

(b') For all \(t > 0\), \(\|F(\cdot, t)\|_{L^1} = \|F_0\|_{L^1}\).

(c') For any \(t > 0\),
\[
\frac{1}{2} \|f(\cdot, t)\|_{L^1} + \int_0^t \|f(\cdot, \tau)\|_{L^2}^2 \, d\tau \leq \frac{1}{2} \|f_0\|_{L^1}.
\]

(6.8)

(d') The estimates in corollary 5.1 hold for \(f\). Hence, \(\mathcal{H}f \cdot F \in L^1_{\text{loc}}(T \times [0, +\infty))\).

(e') The estimate (5.21) in proposition 5.2 holds for \(f\), i.e. \(f\) converges to \(f_\infty\) exponentially as \(t \to +\infty\), where \(f_\infty\) is defined in terms of \(f_0\).

Now taking \(M\) in (6.5) to be \(M_i\) and sending \(i \to +\infty\), we may similarly show that
\[
\int_T \varphi(x, 0) F_0(x) \, dx = \int_{\mathbb{T} \times [0, +\infty)} \partial_\nu \varphi \cdot \mathcal{H}f \cdot F - \partial \varphi \cdot F \, dx \, dr,
\]
i.e. \(f\) is a global weak solution to (1.1) with initial data \(f_0\). Finally, with the help of (6.6), we can argue analogously as before to show
\[
W_1 \left( \frac{F_0}{\|F_0\|_{L^1}}, \frac{F(\cdot, t)}{\|F_0\|_{L^1}} \right) \leq C \left( t \|f_0\|_{L^1}^{-1} \right)^{\frac{1}{2}} \left( \|f_0\|_{L^1} \|F_0\|_{L^1} \right)^{\frac{n}{2}}
\]
for all \(t \in (0, \frac{1}{2} \|f_0\|_{L^1}^{-1}]\). Hence, as \(t \to 0^+\),
\[
\frac{F(\cdot, t)}{\|F_0\|_{L^1}} \text{ converges weakly in } \mathcal{P}(\mathbb{T}) \text{ to } \frac{F_0}{\|F_0\|_{L^1}}.
\]
This time we do not necessarily have \(F(\cdot, t) \to F_0\) in \(L^1(\mathbb{T})\) as \(t \to 0^+\). It is then not clear whether one can refine (6.8) to become the energy equality. However, if \(F_0 \in L^p\) for some \(p > 1\), one can still prove the weak convergence as \(t \to 0^+\) (see part (2) of lemma 3.1) and then the energy equality as before.

Other properties of \(f\) and \(F\), such as their upper and lower bounds at positive times, can be proved similarly by first applying the *a priori* estimates (see section 4) to the approximate solutions and then taking the limits. We skip the details.

This completes this part of the proof. □
Now let us prove corollary 2.1. The main idea is as follows. In the view of (2.10), formally, for each \( s \in \mathbb{R} \), \( X(s,t) \) should satisfy the ODE
\[
\frac{d}{dt} X(s, t) = -\frac{1}{4} \mathcal{H}(\tilde{f}(X(s, t), t)),
\]
where \( \tilde{f} \) solves (2.11) on \( \mathbb{R} \) with the periodic initial condition \( f_0 \) being suitably defined by \( X_0 \). Here we used a different notation \( f \) so that it is distinguished from \( f \) that solves (1.1) on \( \mathbb{T} \).

Hence, our task in the following proof is to define \( f_0 \) from \( X_0 \), find \( \tilde{f} \) with the help of theorem 1.1, and then determine \( X(s,t) \) by solving the ODE above.

**Proof of corollary 2.1.** Denote the inverse function of \( s \mapsto X_0(s) \) to be \( G_0(x) \). Both \( X_0 \) and \( G_0 \) are absolutely continuous on any compact interval of \( \mathbb{R} \) and strictly increasing on \( \mathbb{R} \). Define \( F_0(x) = G_0(x) \) a.e. on \( \mathbb{R} \). Then \( F_0 \) is non-negative and 2\( \pi \)-periodic, and \( \int_{-\pi}^{\pi} F_0(x) \, dx = 2\pi \).

Moreover, since \( G_0 \circ X_0(s) = s \) and \( X_0 \circ G_0(x) = x \) with \( G_0 \) and \( X_0 \) being absolutely continuous on any compact interval of \( \mathbb{R} \), we have \( F_0(X_0(s))X_0'(s) = 1 \) a.e. and equivalently, \( X_0'(G_0(x))G_0'(x) = 1 \) a.e. Hence,
\[
\int_{-\pi}^{\pi} |X_0'(s)|^2 \, ds = \int_{G_0(\pi)}^{G_0(-\pi)} \frac{1}{F_0(X_0(s))} \, ds = \int_{X_0(-\pi)}^{X_0(\pi)} \frac{1}{F_0(x)} \, dx.
\]

Define \( f_0 = \frac{1}{\pi} \). Hence, \( f_0 \) is 2\( \pi \)-periodic, and it is integrable on any interval of length 2\( \pi \), with the integral being equal to \( \int_{-\pi}^{\pi} |X_0'(s)|^2 \, ds \).

Now we take \( f_0 \) and \( F_0 \) as the initial data and solve (2.11) on \( \mathbb{T} \). Since (2.11) contains the extra coefficient \( \frac{1}{4} \) compared with (1.1) (see section 2.2), we first take \( f \) and \( F \) to be the global weak solution to (1.1) that was constructed above with the initial data \( f_0 \) and \( F_0 \), and then let
\[
\tilde{f}(x,t) = f \left( x, \frac{1}{4} t \right), \quad \tilde{F}(x,t) = F \left( x, \frac{1}{4} t \right).
\]

Clearly, they are global weak solutions to (2.11) on \( \mathbb{T} \times [0, +\infty) \). We extend them in space 2\( \pi \)-periodically to the entire real line, still denoting them as \( \tilde{f} \) and \( \tilde{F} \) respectively.

In the view of (2.10), we shall use the transporting velocity field \( -\frac{1}{4} \mathcal{H}(\tilde{f}) \) to define a flow map on \( \mathbb{R} \). Here \( \mathcal{H}(\tilde{f}) \) denotes the 2\( \pi \)-periodic extension on \( \mathbb{R} \) of the Hilbert transform (on \( \mathbb{T} \)) of the original \( \tilde{f} \) defined on \( \mathbb{T} \); or equivalently, one may treat it as the Hilbert transform (on \( \mathbb{R} \)) of the extended \( \tilde{f} \). To avoid the low regularity issue of \( \tilde{f} \) at \( t = 0 \), we specify the ‘initial data’ of the flow map at \( t = 1 \). Given \( x \in \mathbb{R} \), let \( \Psi_t(x) \) solve
\[
\frac{d}{dt} \Psi_t(x) = -\frac{1}{4} \mathcal{H}(\tilde{f}(\Psi_t(x), t)), \quad \Psi_1(x) = x.
\]

Thanks to the properties of \( \tilde{f} \) (see theorem 1.1 and the proof in section 6.2), it is not difficult to show that

- \( \Psi_t(x) \) can be uniquely defined for all \( (x, t) \in \mathbb{R} \times (0, +\infty) \). In \( \mathbb{R} \times (0, +\infty) \), the map \( (x, t) \mapsto \Psi_t(x) \) is smooth.
- For any \( t > 0 \), the map \( x \mapsto \Psi_t(x) \) is bijective, strictly increasing, and is a smooth diffeomorphism from \( \mathbb{R} \) to itself. \( x \mapsto \Psi_t(x) - x \) is 2\( \pi \)-periodic.


• By (5.12) in corollary 5.1, for any $x \in \mathbb{R}$, $t \mapsto \Psi_t(x)$ is Hölder continuous for sufficiently small $t$. The Hölder exponent can be arbitrarily taken in $(0, \frac{1}{2})$, and the corresponding Hölder semi-norm is uniformly bounded in $x$. Hence, $\lim_{t \to 0^+} \Psi_t(x) = \Psi_0(x)$ exists, denoted by $\Psi_0(x)$. The convergence $\lim_{t \to 0^+} \Psi_t(x) = \Psi_0(x)$ is uniform in $x \in \mathbb{R}$. This further implies $x \mapsto \Psi_0(x)$ is continuous, non-decreasing, and $2\pi$-periodic in $\mathbb{R}$. Hence, $x \mapsto \Psi_0(x)$ is surjective from $\mathbb{R}$ to itself.

• By the estimates in proposition 5.2, for each $x \in \mathbb{R}$, as $t \to +\infty$, $\Psi_t(x)$ converges to some $\Phi_{\infty}(x)$ exponentially. Hence, the function $t \mapsto \Psi_t(x)$ is Hölder continuous on $[0, +\infty)$, with the Hölder norm being uniformly bounded in $x$. In addition, $\text{osc}_{t \geq 0} \Psi_t(x) := \max_{t \geq 0} \Psi_t(x) - \min_{t \geq 0} \Psi_t(x)$ is uniformly bounded for all $x \in \mathbb{R}$.

• Thanks to the $F$-equation, for any $t > 0$, $F(\cdot, t)$ is the push forward of $\tilde{F}(\cdot, 1)$ under the map $x \mapsto \Psi_t(x)$, i.e. for any continuous function $\varphi(x)$ on $\mathbb{R}$ with compact support,

$$\int_{\mathbb{R}} \varphi(x) \tilde{F}(x, t) \, dx = \int_{\mathbb{R}} \varphi(\Psi_t(x)) \tilde{F}(x, 1) \, dx.$$  

Send $t \to 0^+$. Because of (6.9) and the uniform convergence from $\Psi_t$ to $\Psi_0$, we find

$$\int_{\mathbb{R}} \varphi(x) F_0(x) \, dx = \int_{\mathbb{R}} \varphi(\Psi_0(x)) \tilde{F}(x, 1) \, dx,$$

i.e., $F_0$ is the pushforward of $\tilde{F}(x, 1)$ under the map $x \mapsto \Psi_0(x)$.

• The previous fact further implies $x \mapsto \Psi_0(x)$ is injective. This is because otherwise, the facts that $x \mapsto \Psi_0(x)$ is continuous and non-decreasing, and that $\tilde{F}(x, 1)$ has a positive lower bound (see (1.7)) imply that $F_0$ must contain Dirac $\delta$-masses, which is a contradiction. We can further show that $\Psi_0$ is a homeomorphism from $\mathbb{R}$ to itself.

With the above $\{\Psi_t(x)\}_{t \geq 0}$, we define $X(s, t) := \Psi_t \circ (\Psi_0)^{-1} \circ X_0(s)$. We claim that this gives the desired function $X(s, t)$ in corollary 2.1.

We first verify that, for any $s \in \mathbb{R}$,

$$\int_{X(0, 1)}^{X(s, 1)} \tilde{F}(y, 1) \, dy = s. \tag{6.10}$$

With $0 < \epsilon \ll 1$, let $\chi_\epsilon(y)$ be a continuous function supported on $[X(0, 1) + \epsilon, X(s, 1) - \epsilon]$, and linear on $[X(0, 1), X(0, 1) + \epsilon]$ and $[X(s, 1) - \epsilon, X(s, 1)]$. By the monotone convergence theorem,

$$\int_{X(0, 1)}^{X(s, 1)} \tilde{F}(y, 1) \, dy = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \chi_\epsilon(y) \tilde{F}(y, 1) \, dy = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \chi_\epsilon \left( \Psi_0^{-1}(y) \right) F_0(y) \, dy \quad \text{dy}$$

$$= \int_{\mathbb{R}} \chi_{X(0, 1), X(s, 1)}(y) (\Psi_0^{-1}(y)) F_0(y) \, dy$$

By definition, $X(s, 1) = (\Psi_0)^{-1} \circ X_0(s)$, and $F_0 = G_0'$ a.e. where $G_0$ is absolutely continuous on any compact interval of $\mathbb{R}$ and is the inverse function of $X_0$. So

$$\int_{X(0, 1)}^{X(s, 1)} \tilde{F}(y, 1) \, dy = \int_{\mathbb{R}} \chi_{X(0, 1), X(s, 1)}(y) F_0(y) \, dy = G_0'(X_0(s)) - G_0'(X_0(0)) = s.$$
In general, for any \(s_1, s_2 \in \mathbb{R}\),
\[
\int_{X(s_1)}^{X(s_2)} \hat{F}(y, 1) \, dy = s_2 - s_1.
\]  \hfill (6.11)

The equation (6.10) implies that
\[
s \mapsto X(s, 1) \quad \text{is the inverse function of the map} \quad x \mapsto \int_{X(0, 1)}^{x} \hat{F}(y, 1) \, dy.
\]

By theorem 1.1, \(\hat{F}(\cdot, 1)\) is smooth, positive, 2\(\pi\)-periodic, and satisfies \(\int_{-\pi}^{\pi} \hat{F}(x, 1) \, dx = 2\pi\). Hence, \(X(\cdot, 1)\) is strictly increasing and smooth on \(\mathbb{R}\), with \(s \mapsto X(s, 1) - s\) being 2\(\pi\)-periodic. Then using the fact \(X(s, t) = \Psi_i(X(s, 1))\), (6.11), and the above-mentioned properties of \(\{\Psi_i(x)\}_{i \geq 0}\), it is not difficult to show that \(X(s, t)\) satisfies all the claims in corollary 2.1; also see the derivation in section 2.2 and remark 3.1. In particular, we may take \(c_\infty = \Psi_\infty \circ (\Psi_0)^{-1} \circ X_0(0)\), and \(\frac{1}{X(0, t)} = \|\hat{F}(\cdot, t)\|_{L^\infty}\). We omit the details. \(\square\)

### 6.3. Uniqueness of the dissipative weak solutions

As is mentioned before, to prove uniqueness, we need to focus on the dissipative weak solutions that satisfy more assumptions.

**Remark 6.1.** The inequality (1.10) in the definition of the dissipative weak solutions is motivated by the following calculation. Suppose \(f\) is a positive strong solution to (1.1). Then \(F := \frac{1}{f}\) verifies
\[
\frac{d}{dt} \int_\mathbb{T} \Phi(F(x, t)) \, dx = \int_\mathbb{T} \Phi'(F) \partial_t F \, dx = - \int_\mathbb{T} \partial_t (\Phi'(F)) \cdot \mathcal{H}F \cdot F \, dx
\]
\[
= - \int_\mathbb{T} F \Phi''(F) \partial_t F \cdot \mathcal{H}f \, dx
= - \int_\mathbb{T} \partial_t (F \Phi'(F) - \Phi(F)) \cdot \mathcal{H}f \, dx
\]
\[
= - \int_\mathbb{T} (\Phi(F) - F \Phi'(F)) (-\Delta)^{\frac{3}{2}} f \, dx.
\]

For the smooth positive solution \(F\), taking \(\Phi(y) = \frac{1}{y}\) yields the energy estimate (3.1). Taking \(\Phi(y) = y \ln y\) yields the entropy estimate (5.2). In general, when \(\Phi \in C^1_{\text{loc}}((0, +\infty))\) is convex, \(\Phi(y) - y \Phi'(y)\) is non-increasing in \(y\). Hence, using \(F = \frac{1}{f}\) and a derivation similar to the previous equation, we find
\[
\int_\mathbb{T} (\Phi(F) - F \Phi'(F)) (-\Delta)^{\frac{3}{2}} f \, dx \geq 0.
\]

We first show that the dissipative weak solutions are bounded from above and from below if the initial data has such properties.

**Lemma 6.1.** Under the assumption of theorem 1.2, let \(f = f(x, t)\) be a dissipative weak solution to (1.1) on \(\mathbb{T} \times [0, T]\). Then for all \(t \in (0, T)\),
\[
\|f(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq \|f_0\|_{L^\infty(\mathbb{T})}, \quad \|F(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq \|F_0\|_{L^\infty(\mathbb{T})}.
\]

As a result, \(\partial_t f, \partial_t F \in L^2(\mathbb{T} \times [0, T])\) with a priori estimates depending on \(\|f_0\|_{L^\infty}, \|F_0\|_{L^\infty}\), and \(\|\ln f_0\|_{H^1}\).
\textbf{Proof.} We choose in (1.10)

\[ \Phi(y) = \begin{cases} (y - \|f_0\|_{L^\infty})^2, & \text{if } y < \|f_0\|_{L^\infty}, \\ 0, & \text{if } y \in [\|f_0\|_{L^\infty}, \|f_0\|_{L^\infty}], \\ (y - \|f_0\|_{L^\infty})^2, & \text{if } y > \|f_0\|_{L^\infty}. \end{cases} \]

Apparently, \( \Phi \in C^1_{\text{loc}}((0, +\infty)) \) is convex, so (1.10) gives that, for any \( t \in (0, T) \),

\[ \int_T^t \Phi(F(x,t)) \, dx + \int_0^t \int_T^t (\Phi(F) - F\Phi'(F)) (-\Delta)^{\frac{1}{2}} f \, dx \, dt \leq 0. \]

The second term above is non-negative due to the convexity of \( \Phi \) (see remark 6.1). Hence,

\[ \int_T^t \Phi(F(x,t)) \, dx \leq 0, \]

which further implies \( F(\cdot, t) \in \|[f_0]\|_{L^\infty}^{-1}, \|f_0\|_{L^\infty} \] a.e. on \( T \).

The last claim follows from the first claim and the definition of the dissipative weak solution (also see (5.11)).

Now we are ready to prove theorem 1.2.

\textbf{Proof of theorem 1.2.} Under the assumptions on \( f_0 \) in theorem 1.2, we can construct a global dissipative weak solution by following the argument in section 6.2. Indeed, denote \( h_0 = \ln f_0 \) and we first consider the solution \( f_i \) to (1.1) with the approximate initial data \( f_{0,i} := \exp(\mathcal{F}_i \ast h_0) \) that is positive and smooth. Such an \( f_i \) can be obtained by following the second step in the proof in section 6.2. \( f_i \) is smooth for all positive times; one can verify that \( f_i \) is a dissipative weak solution. In particular, (1.10) follows from the convexity of \( \Phi \) and the weak \( L^1 \)-convergence \( (p \in [1, +\infty]) \) of \( F_i(\cdot, t) \) to \( F_0 \) as \( t \to 0^+ \). Then we take the limit \( j \to +\infty \) to obtain the global dissipative weak solution \( f \) corresponding to the given initial data \( f_0 \).

Next, we focus on the uniqueness. With abuse of notations, suppose \( f_1 \) and \( f_2 \) are two dissipative weak solutions to (1.1) on \( T \times [0, T) \) for some \( T > 0 \), both starting from the initial data \( f_0 \). We want to show \( f_1 = f_2 \) in \( T \times [0, T) \). Without loss of generality, we may assume \( f_2 \) to be the global dissipative weak solution constructed above. Recall that \( f_2 \) is a strong solution to (1.1) at all positive times. Let \( F_i := \frac{1}{t} (i = 1, 2) \). In what follows, the proof of \( f_1 = f_2 \) relies on a relative entropy estimate between \( F_1 \) and \( F_2 \).

Given \( \tau_2 \in (0, T) \) and \( 0 < \delta \ll 1 \), let \( \rho_3(t) \) to be a smooth cutoff function on \( [\delta, +\infty) \), such that \( \rho_3(t) \equiv 0 \) on \([0, \frac{\delta}{2}] \cup [\tau_2 - \frac{\delta}{2}, +\infty) \), \( \rho_3(t) \equiv 1 \) on \([\delta, \tau_2 - \delta] \), and in addition, \( \rho_3 \) is increasing on \([\frac{\delta}{2}, \delta] \) but decreasing on \([\tau_2 - \delta, \tau_2 - \frac{\delta}{2}] \). Since \( F_1 \) is a weak solution to (1.4), we may take \( \varphi = \rho_3(t) \ln F_2 \) as the test function in (1.5) and derive that

\[ \int_0^{\tau_2} \rho_3'(t) \int_{\mathbb{T}} F_i \ln F_2 \, dx \, dt = \int_0^{\tau_2} \rho_3(t) \int_{\mathbb{T}} \partial_t \ln F_2 \cdot \mathcal{H} f_1 \cdot F_1 - F_1 \partial_t \ln F_2 \, dx \, dt. \]

The right-hand side can be further simplified thanks to the smoothness of \( F_2 \)

\[ \int_0^{\tau_2} \rho_3'(t) \int_{\mathbb{T}} F_i \ln F_2 \, dx \, dt = \int_0^{\tau_2} \rho_3(t) \int_{\mathbb{T}} \frac{F_1}{F_2} \partial_t \ln F_2 \cdot \mathcal{H} f_1 - \partial_t (\mathcal{H} f_2) \, dx \, dt \]

\[ = \int_0^{\tau_2} \rho_3(t) \int_{\mathbb{T}} \partial_t \mathcal{H} (f_1 - f_2) - F_1 (-\Delta)^{\frac{1}{2}} f_2 \, dx \, dt. \]
Next we will send $\delta \to 0^+$. Recall that lemma 6.1 gives $f_i, F_i \in L^\infty([0, T) \times [0, T])$ and $\partial_t f_i, \partial_t F_i \in L^2(\mathbb{T} \times [0, T])$. This will be enough to pass to the limit on the right-hand side. To handle the left-hand side, we want to show $\int_T F_1 \ln F_2\, dx$ is continuous on $[0, t_s]$. In fact, by the definition of dissipative weak solution, $\ln F_i \in L^\infty([0, T]; H^2(\mathbb{T}))$. The upper and lower bounds for $F_i$ implies $F_i \in L^\infty([0, T); L^\infty(\mathbb{T}))$ (see (5.6)). This together with (1.4) and $f_i \in L^2_{(0, T); H^1(\mathbb{T})}$ implies $\partial_t F_i \in L^2([0, T); H^{-1}(\mathbb{T}))$, which in turn gives e.g. $F_i \in C([0, T]; L^2(\mathbb{T}))$. Then we obtain the time-continuity of $\int_T F_1 \ln F_2 \, dx$. Therefore, taking $\delta \to 0^+$, we obtain

$$
\int_T F_0 \ln F_0 \, dx - \int_T F_1(x, t_s) \ln F_2(x, t_s) \, dx
= \int_0^{t_s} \int_T \frac{\partial F_2}{F_2} : F_1 \cdot H(f_1 - f_2) - F_1 (-\Delta)\frac{1}{2} f_2 \, dx \, dt.
$$

(6.12)

On the other hand, by the definition of the dissipative weak solution, with $\Phi(y) = y \ln y$ in (1.10),

$$
\int_T F_1(x, t_s) \ln F_1(x, t_s) \, dx - \int_T F_0 \ln F_0 \, dx \leq \int_T \int_0^{t_s} \int_T F_1(-\Delta)\frac{1}{2} f_1 \, dx \, dt.
$$

Adding this and (6.12) yields an estimate for the relative entropy between $F_1$ and $F_2$

$$
\int_T F_1(x, t_s) \ln \left( \frac{F_1(x, t_s)}{F_2(x, t_s)} \right) \, dx \leq \int_0^{t_s} \int_T (f_1 - f_2)(-\Delta)\frac{1}{2} (F_1 - F_2) \, dx \, dt
+ \int_0^{t_s} \int_T H(f_1 - f_2) \cdot (F_1 - F_2) \frac{\partial F_2}{F_2} \, dx \, dt.
$$

(6.13)

We then derive that

$$
\int_T (f_1 - f_2)(-\Delta)\frac{1}{2} (F_1 - F_2) \, dx
= \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{T}} [(f_1 - f_2)(x) - (f_1 - f_2)(y)] \cdot \frac{(F_1 - F_2)(x) - (F_1 - F_2)(y)}{4 \sin^2 \left( \frac{x - y}{2} \right)} \, dy \, dx
$$

$$
= -\frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{T}} f_1 f_2(x) \cdot \frac{[(F_1 - F_2)(x) - (F_1 - F_2)(y)]^2}{4 \sin^2 \left( \frac{x - y}{2} \right)} \, dy \, dx
$$

$$
- \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{T}} (F_1 - F_2)(y) [f_1 f_2(x) - f_1 f_2(y)] \cdot \frac{(F_1 - F_2)(x) - (F_1 - F_2)(y)}{4 \sin^2 \left( \frac{x - y}{2} \right)} \, dy \, dx
$$

$$
\leq -\frac{1}{2\pi} \| F_0 \|_{L^2(\mathbb{T})}^2 \int_{\mathbb{T} \times \mathbb{T}} \frac{[(F_1 - F_2)(x) - (F_1 - F_2)(y)]^2}{4 \sin^2 \left( \frac{x - y}{2} \right)} \, dy \, dx
$$

$$
- \frac{1}{4\pi} \int_{\mathbb{T} \times \mathbb{T}} [f_1 f_2(x) - f_1 f_2(y)] \cdot \frac{(F_1 - F_2)(x)^2 - (F_1 - F_2)(y)^2}{4 \sin^2 \left( \frac{x - y}{2} \right)} \, dy \, dx.
$$
To obtain the last term above, we exchanged the \( x \)- and \( y \)-variables in the integral. Then (6.13) becomes
\[
\int_T F_1(x,t) \ln \left( \frac{F_1(x,t)}{F_2(x,t)} \right) \, dx + \| F_0 \|_{L^\infty} \| (F_1 - F_2) \|_{L^1_t[U^4]}^2 \\
\leq -\frac{1}{2} \int_0^T \int_T (F_1 - F_2)^2 (-\Delta)^{3/2} f f dt - 2 \int_0^T \int_T \mathcal{H} (F_1 - F_2) \cdot (F_1 - F_2) \partial_x \sqrt{F_2} \, dx \, dt \\
\leq \frac{1}{2} \| F_1 - F_2 \|_{L^1_t[U^1]}^2 \| f f \|_{L^4_t[U^4]}^* + C \| f f \|_{L^2_t[U^2]} \| F_1 - F_2 \|_{L^2_t[U^2]} \| F_2 \|_{L^2_t[U^2]} \| \partial_x \sqrt{F_2} \|_{L^2_t[U^2]} \\
\leq C \| f f \|_{L^\infty[U^2]}^2 \left( \| \partial_x \sqrt{F_1} \|_{L^2[U^2]} + \| \partial_x \sqrt{F_2} \|_{L^2[U^2]} \right) \| F_1 - F_2 \|_{L^2_t[U^2]} \| F_1 - F_2 \|_{L^1_t[U^4]}^2.
\]

Here \( L^p_t \) denotes the \( L^p \)-norm on the time interval \([0,t_*]\). Applying Young’s inequality, we find that
\[
\left\| \int_T F_1 \ln \left( \frac{F_1}{F_2} \right) \, dx \right\|_{L^\infty_t[U^4]} + \| F_0 \|_{L^\infty[U^4]} \| F_1 - F_2 \|_{L^1_t[U^4]}^2 \\
\leq C \| f f \|_{L^\infty[U^2]} \| F_0 \|_{L^\infty[U^2]} \| F_1 - F_2 \|_{L^2_t[U^2]} \int_0^T \| \partial_x \sqrt{F_1} \|_{L^2[U^2]}^2 + \| \partial_x \sqrt{F_2} \|_{L^2[U^2]}^2 \, dt. \tag{6.14}
\]

Now we need a variation of Pinsker’s inequality for the relative entropy, under the condition that \( F_1 \) and \( F_2 \) with identical \( L^1 \)-norms are both positive, bounded, and bounded away from zero. The following argument is adapted from that in [37] (also see [53]). Let \( R := \frac{F_2}{F_1} - 1 \). Observe that \( (1 + x) \ln (1 + x) - x \geq C x^2 \) on any bounded interval in \((-1, +\infty)\), where \( C \) depends on the upper bound of the interval. Since \( \| f f \|_{L^\infty[U^2]} \leq F_i \leq \| F_0 \|_{L^\infty[U^2]} \) \((i = 1, 2)\), we derive that
\[
\int_T F_1 \ln \left( \frac{F_1}{F_2} \right) \, dx = \int_T F_2 (1 + R) \ln (1 + R) - F_2 R \, dx \geq C \int_T F_2 R^2 \, dx \\
= C \int_T F_2^{-1} (F_1 - F_2)^2 \, dx \geq C \| F_1 - F_2 \|_{L^2[U^2]}^2,
\]
where \( C \) essentially depends on \( \| f f \|_{L^\infty[U^2]} \) and \( \| F_0 \|_{L^\infty[U^2]} \).

Applying this to (6.14) yields
\[
\left\| \int_T F_1 \ln \left( \frac{F_1}{F_2} \right) \, dx \right\|_{L^\infty_t[U^4]} + \| F_0 \|_{L^\infty[U^4]} \| F_1 - F_2 \|_{L^1_t[U^4]}^2 \\
\leq C \| f f \|_{L^\infty[U^2]} \| F_0 \|_{L^\infty[U^2]} \left\| \int_T F_1 \ln \left( \frac{F_1}{F_2} \right) \, dx \right\|_{L^\infty_t[U^4]} \int_0^T \| \partial_x \sqrt{F_1} \|_{L^2[U^2]}^2 + \| \partial_x \sqrt{F_2} \|_{L^2[U^2]}^2 \, dt.
\]

Since \( \| \partial_x \sqrt{F} \|_{L^2((0,T);L^2(T))} < +\infty \), we take \( t_* \) to be sufficiently small to conclude
\[
\int_T F_1 \ln \left( \frac{F_1}{F_2} \right) \, dx = 0 \quad \text{on} \quad [0, t_*],
\]
and therefore, \( F_1 = F_2 \) on \([0, t_*]\). Repeating this argument for finitely many times yields the uniqueness on the entire time interval \([0, T]\).

Therefore, we conclude the uniqueness of the dissipative weak solutions on \( \mathbb{T} \times [0, +\infty) \).
7. Analyticity and sharp decay estimates for large times

Let us introduce some notations. Define the Fourier transform of \( f \in L^1(T) \) as in (1.6). For \( m \in \mathbb{N} \), let

\[
\|f\|_{\mathcal{F}^m} := \sum_{k \neq 0} |k|^m \hat{f}_k.
\]

With \( \nu \geq 0 \), we additionally define

\[
\|f\|_{\mathcal{F}^\nu} := \sum_{k \neq 0} e^{\nu |k|} |k|^m \hat{f}_k.
\]

Clearly, \( \|f\|_{\mathcal{F}^\nu} = \|f\|_{\mathcal{F}^0} \).

If \( \|f_0\|_{\mathcal{F}^\nu} \) is suitably small compared to \( f_\infty \), we can prove that the solution will become analytic in space-time domain \( \mathbb{T} \times (t_0, +\infty) \).

**Proposition 7.1.** There exists a universal number \( \varepsilon > 0 \) such that the following holds. Suppose \( f_0 \) in (1.1) is positive and smooth on \( T \), such that \( \|f_0\|_{\mathcal{F}^\nu} \leq \varepsilon f_\infty \). Let \( f \) be the unique strong solution to (1.1). Then for all \( t_0 > 0 \), \( f \) is analytic in the space-time domain \( \mathbb{T} \times (t_0, +\infty) \). In particular, for any \( t \geq 0 \),

\[
\|f(\cdot, t)\|_{\mathcal{F}^\nu} \leq 2 \|f_0\|_{\mathcal{F}^\nu}.
\]

**Remark 7.1.** Note that \( \nu(t) \sim f_\infty t \) for large \( t \) as long as \( \varepsilon < C^{-1}_\ast \). In the view of the linearisation of (1.1) around the equilibrium \( f_\infty \), i.e. \( \partial f = -f_\infty (\Delta)^{1/2} f \), the estimate (7.1) provides the sharp rate of analyticity and decay.

**Proof.** First suppose \( f_0 \) is band-limited (i.e. only finitely many Fourier coefficients of \( f_0 \) are non-zero), and \( f_0(x) \not= f_\infty \). It has been shown in proposition 6.1 that there is a unique band-limited global solution \( f(\cdot, t) \), which is automatically analytic. In the view of theorem 1.2, it is also the unique strong solution starting from \( f_0 \).

To be more quantitative, we recall (6.1): for all \( k \geq 0 \),

\[
\frac{d}{dt} \hat{f}_k = -k \hat{f}(t) \hat{f}_k - \sum_{j \geq 1} (2k + 2j) \hat{f}_{k+j} \hat{f}_j =: -k \hat{f}(t) \hat{f}_k - \hat{N}_k.
\]

The case \( k = 0 \) corresponds to the energy estimate (3.1) in lemma 3.1, while for all \( k > 0 \),

\[
\frac{d}{dt} |\hat{f}_k| = \frac{1}{2|\hat{f}_k|} \left( \hat{f}_k \frac{d}{dt} \hat{f}_k + \hat{f}_k \frac{d}{dt} \hat{f}_k \right) \leq -k |\hat{f}(t)| |\hat{f}_k| + |\hat{N}_k|.
\]
This derivation is valid for $|\hat{f}_k| > 0$, but it is not difficult to see the resulting inequality also holds when $|\hat{f}_k| = 0$. A similar inequality holds for negative $k$’s. Summing over $k \neq 0$, with $\nu(t) \geq 0$ to be determined, we derive that

$$
\frac{d}{dt} \|f(\cdot, t)\|_{\mathcal{F}_{v(t)}} + (\hat{f}(t) - \nu(\cdot) t) \|f(\cdot, t)\|_{\mathcal{F}_{v(t)}} \\
\leq C \sum_{k \geq 1} e^{\nu(t)} \sum_{j \geq 1} (k + 2j) \hat{f}_{k+j}(t) \|\hat{f}_j(t)\|_{\mathcal{F}_{v(t)}} \\
\leq C \sum_{j \geq 1} e^{-\nu(t)} \|\hat{f}_j(t)\|_{\mathcal{F}_{v(t)}} \sum_{k \geq 1} (k + j) e^{(k+j)\nu(t)} |\hat{f}_{k+j}(t)| \\
\leq \frac{1}{2} C e^{-2\nu(t)} \|f(\cdot, t)\|_{\mathcal{F}_{v(t)}} \|f(\cdot, t)\|_{\mathcal{F}_{v(t)}},
$$

(7.2)

where $C_0 > 0$ is a universal constant. Note that, because $f$ is a band-limited strong solution, $\|f(\cdot, t)\|_{\mathcal{F}_{v(t)}}$ is finite at all time and it is smooth in $t$ as long as $\nu(t)$ is finite and smooth.

Now let $\nu(t)$ solve

$$
f_\infty - \nu'(t) = C e^{-2\nu(t)} ||f_0||_{\mathcal{F}_{\nu,1}}, \quad \nu(0) = 0.
$$

(7.3)

Denote $\theta := C e^{-\nu(t)} ||f_0||_{\mathcal{F}_{\nu,1}}$. Then $\mu(t) := e^{2\nu(t)}$ solves

$$
\mu'(t) = 2 f_\infty (\mu(t) - \theta), \quad \mu(0) = 1.
$$

As long as $\theta < 1$, or equivalently $||f_0||_{\mathcal{F}_{\nu,1}} < C e^{-1} f_\infty$, this equation has a unique positive solution $\mu(t) = \theta + (1 - \theta) \exp(2 f_\infty t)$ on $[0, +\infty)$. Then $\nu(t) := \frac{1}{2} \ln \mu(t)$ satisfies (7.3).

With this $\nu(t)$, we claim that

$$
\|f(\cdot, t)\|_{\mathcal{F}_{v(t)}} \leq 2 ||f_0||_{\mathcal{F}_{\nu,1}} \quad \forall t \geq 0,
$$

(7.4)

which further implies spatial analyticity of $f(\cdot, t)$ for all $t > 0$. Indeed, if not, we let $t_*$ denote the infimum of all times where (7.4) does not hold. By the time-continuity of $\|f(\cdot, t)\|_{\mathcal{F}_{v(t)}}$, $t_*$ is positive, and the equality in (7.4) is achieved at time $t_*$. This together with (7.3) and the fact $\hat{f}(t) \geq f_\infty$ implies

$$
\hat{f}(t) - \nu'(t) \geq \frac{1}{2} C e^{-2\nu(t)} \|f(\cdot, t)\|_{\mathcal{F}_{v(t)}} \quad \forall t \in [0, t_*].
$$

Combining this with (7.2), we find

$$
\|f(\cdot, t_*)\|_{\mathcal{F}_{v(t_*)}} \leq \|f_0\|_{\mathcal{F}_{\nu,1}} < 2 ||f_0||_{\mathcal{F}_{\nu,1}},
$$

which contradicts with the definition of $t_*$. For smooth $f_0$ that is not band-limited, we let $f$ be the unique global solution corresponding to the initial data $f_0$, which is constructed in section 6.2. Recall the Fejér kernel $\mathcal{F}_n$ was defined in (6.2). Let $f_n$ be the unique band-limited global solution corresponding to the band-limited initial data $\mathcal{F}_n * f_0$. Then with $\nu(t)$ defined above, for arbitrary $m \in \mathbb{Z}$ and all $t \geq 0$,

$$
\sum_{j \neq 0} e^{\nu(t) \min(|j|, m)} |\mathcal{F}(f_n)_j(t)| \leq ||f_n(\cdot, t)\|_{\mathcal{F}_{\nu(t)}} < 2 ||\mathcal{F}_n * f_0\|_{\mathcal{F}_{\nu,1}}.
$$
Similar to section 6.2, we can prove that, as $n \to +\infty$, $f_n \to f$ in $C^k_{\text{loc}}(T \times (0, +\infty))$ for any $k \in \mathbb{N}$. In fact, we first obtain convergence along a subsequence, and then show the convergence should hold for the whole sequence due to the uniqueness of the solution $f$. Hence, sending $n \to +\infty$ and then letting $m \to +\infty$ yields (7.4) again in this general case.

The space-time analyticity follows from the spatial analyticity of $f$ and the equation (1.1).

**Corollary 7.1 (property (5) of the weak solution constructed in theorem 1.1).** Under the assumptions of theorem 1.1, there exists $T_\ast > 0$ depending only on $\|f_0\|_{L^1}$ and $\|F_0\|_{L^1}$, such that the constructed solution $f$ of (1.1) is analytic in the space-time domain $T \times (T_\ast, +\infty)$. It satisfies that, for any $t > 0$,

$$
\|f(\cdot, t + T_\ast)\|_{\mathcal{F}^{0,1}(T)} \leq C f_\infty,
$$

with

$$
\nu(t) \geq \frac{1}{2} \ln \left[ 1 + \exp \left( 2f_\infty t \right) \right] - \frac{1}{2} \ln 2,
$$

and $C > 0$ being a universal constant.

**Proof.** Using the notations in proposition 7.1, we take $\varepsilon$ there to satisfy $C_\ast \varepsilon \leq \frac{1}{2}$. By the property (4) of the solution (also see proposition 5.2), there is $T_\ast > 0$ depending only on $\|f_0\|_{L^1}$ and $\|F_0\|_{L^1}$, such that

$$
\|f(\cdot, T_\ast)\|_{\mathcal{F}^{0,1}(T)} \leq C \|f(\cdot, T_\ast)\|_{\mathcal{H}^2} \leq C f_\infty.
$$

Here we used the embedding $H^2(T) \hookrightarrow \mathcal{F}^{0,1}(T)$. Then we apply proposition 7.1 to conclude. In particular, since $\theta \leq \frac{1}{2}$ in this case, the desired estimate for $\nu(t)$ follows.

**Data availability statement**

No new data were created or analysed in this study.

**Acknowledgment**

The author is supported by the National Key R&D Program of China under the Grant 2021YFA1001500. The author would like to thank Dongyi Wei, Zhenfu Wang, De Huang, and Fanghua Lin for helpful discussions.

**Appendix. An $H^{-\frac{1}{2}}$-estimate for $F$**

It is worth mentioning another interesting estimate for $F$, although it is not used in the other parts of the paper. For convenience, we assume $f$ to be a strictly positive strong solution to (1.1) on $\mathbb{T} \times [0, T)$. Define

$$
F := \frac{1}{2\pi} \int_{\mathbb{T}} F(x, t) \, dx.
$$
By (1.9) and lemma 3.1, \( F = \frac{1}{f_0} \), which is a time-invariant constant. Then we have the following estimate.

**Lemma A.1.**

\[
\frac{d}{dt} \| \left( \Delta \right)^{-\frac{1}{2}} (F - \bar{F}) \|_{L^2}^2 + \int_T f \cdot (\mathcal{H}F)^2 \, dx = 2\pi F - \bar{F}^2 \int_T f \, dx \leq 0.
\]

**Proof.** By the Cotlar’s identity (5.3) for mean-zero functions

\[
(\mathcal{H}F)^2 - (F - \bar{F})^2 = 2\mathcal{H}(\langle F \rangle \mathcal{H}F).
\]

Hence,

\[
(\mathcal{H}F)^2 - F^2 - \bar{F}^2 + 2FF = 2\mathcal{H}(F \mathcal{H}F) - 2\mathcal{H}H \mathcal{H}F = 2\mathcal{H}(F \mathcal{H}F) + 2\bar{F}(F - \bar{F}),
\]

which simplifies to

\[
(\mathcal{H}F)^2 - F^2 + \bar{F}^2 = 2\mathcal{H}(F \mathcal{H}F).
\]

Using this identity and the \( F \)-equation, we find that

\[
\frac{d}{dt} \| \left( \Delta \right)^{-\frac{1}{2}} (F - \bar{F}) \|_{L^2}^2 = 2 \int_T \partial_t F \cdot \left( \Delta \right)^{-\frac{1}{2}} (F - \bar{F}) \, dx
\]

\[
= -2 \int_T \mathcal{H}f \cdot F \cdot \partial_t \left( \Delta \right)^{-\frac{1}{2}} (F - \bar{F}) \, dx = -2 \int_T f \cdot \mathcal{H}(F \mathcal{H}F) \, dx
\]

\[
= -\int_T f \cdot (\mathcal{H}F)^2 \, dx + \int_T F \, dx - F^2 \int_T f \, dx.
\]

Then the desired estimate follows by rearranging the equation and applying the Cauchy–Schwarz inequality.

Under the additional assumption \( F_0 - \bar{F} \in H^{-\frac{1}{2}}(\mathbb{T}) \), lemma A.1 implies \( F - \bar{F} \in L^2 L^2 \) and \( f^2 \mathcal{H}F \in L^2 L^2 \). Interestingly, this provides another way of making sense of the nonlinear term

\[
\int_{T \times [0,T]} \partial_t \varphi \cdot \mathcal{H}f : F \, dx \, dt
\]

in the weak formulation (1.5) (see section 5.1). Note that the additional assumption \( F_0 - \bar{F} \in H^{-\frac{1}{2}} \) has the same scaling as the assumption \( F_0 \in L^1 \) that we have been using.

Indeed, by Cotlar’s identity (A.1) and the Cauchy–Schwarz inequality,

\[
\int_T f \cdot (\mathcal{H}f)^2 \, dx = \int_T F \cdot \left( (\mathcal{H}f)^2 - f^2 + \bar{F}^2 \right) \, dx + \int_T f \, dx - \bar{F}^2 \int_T F \, dx
\]

\[
\leq \int_T F \cdot 2\mathcal{H}(f \mathcal{H}f) \, dx = -2 \int_T \mathcal{H}F \cdot f \mathcal{H}f \, dx
\]

\[
\leq 2 \| f^2 \mathcal{H}F \|_{L^2} \| f \|^2 \| f \| \| \mathcal{H}f \|_{L^1}
\]

\[
\leq C \| f^2 \mathcal{H}F \|_{L^2} \| f \|^2 \| f \|_{L^1}.
\]
By the interpolation inequality \[ \|f\|_{L^2_T L^2}^{3/2} \leq C \|f\|_{L^6_T L^6} \|\tilde{f}\|_{H^2_T}^{1/2}, \]
Hence, by lemmas 3.1, A.1, and the assumption \( F_0 - \tilde{F} \in H^{-1/2} \), we obtain that \( F^2 \tilde{H} f \in L^2_T L^2 \).

Then the facts \( F^2 \in L^\infty_T L^2 \) and \( F^2 \tilde{H} f \in L^2_T L^2 \) allow us to well define the above nonlinear term in the weak formulation. Alternatively, the facts \( F - \tilde{F} \in L^\infty_T H^{-1/2} \) and \( \tilde{H} f \in L^2_T H^1 \) (see lemma 3.1) also suffice. In either case, one can show that
\[
\int_0^t \int_T \partial_x \varphi \cdot \tilde{H} f \cdot F \, dx \, dt = O \left( 1^{1/2} \right),
\]
which may lead to a simpler proof of existence of the weak solutions.

References

[1] Alazard T, Lazar O and Nguyen Q H 2022 On the dynamics of the roots of polynomials under differentiation J. Math. Pures Appl. 162 1–22
[2] Baker G R, Li X and Morlet A C 1996 Analytic structure of two 1D-transport equations with nonlocal fluxes Physica D 91 349–75
[3] Biler P, Karch G and Monneau R 2010 Nonlinear diffusion of dislocation density and self-similar solutions Commun. Math. Phys. 294 145–68
[4] Caffarelli L and Vázquez J 2011 Nonlinear porous medium flow with fractional potential pressure Arch. Ration. Mech. Anal. 202 537–65
[5] Caffarelli L and Vázquez J 2016 Regularity of solutions of the fractional porous medium flow with exponent 1/2 St. Petersburg Math. J. 27 437–60
[6] Caffarelli L and Vázquez J-L 2011 Asymptotic behaviour of a porous medium equation with fractional diffusion Discrete Contin. Dyn. Syst. 29 1393–404
[7] Caffarelli L A, Soria F and Vázquez J L 2013 Regularity of solutions of the fractional porous medium flow J. Eur. Math. Soc. 15 1701–46
[8] Caffarelli L A and Vasseur A 2010 Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation Ann. Math. 171 1903–30
[9] Cameron S and Strain R M 2023 Communications on Pure and Applied Mathematics Critical local well-posedness for the fully nonlinear Peskin problem accepted (https://doi.org/10.1002/cpa.22139)
[10] Castro A and Córdoba D 2008 Global existence, singularities and ill-posedness for a nonlocal flux Adv. Math. 219 1916–36
[11] Castro A and Córdoba D 2010 Infinite energy solutions of the surface quasi-geostrophic equation Adv. Math. 225 1820–9
[12] Chae D, Córdoba A, Córdoba D and Fontelos M A 2005 Finite time singularities in a 1D model of the quasi-geostrophic equation Adv. Math. 194 203–23
[13] Chen J, Hou T Y and Huang D 2021 On the finite time blowup of the De Gregorio model for the 3D Euler equations Commun. Pure Appl. Math. 74 1282–350
[14] Chen K and Nguyen Q-H 2023 The Peskin problem with \( H^{1,\infty}_{0,0} \) initial data SIAM J. Math. Anal. 55 6262–304
[15] Constantin P, Córdoba D and Wu J 2001 On the critical dissipative quasi-geostrophic equation Indiana Univ. Math. J. 50 97–107
[16] Córdoba A, Córdoba D and Fontelos M A 2005 Formation of singularities for a transport equation with nonlocal velocity Ann. Math. 162 1377–89
[17] Córdoba D and Gancedo F 2009 A maximum principle for the Muskat problem for fluids with different densities Commun. Math. Phys. 286 681–96
[18] De Gregorio S 1990 On a one-dimensional model for the three-dimensional vorticity equation J. Stat. Phys. 59 1251–63
[19] De Gregorio S 1996 A partial differential equation arising in a 1D model for the 3D vorticity equation Math. Methods Appl. Sci. 19 1233–55
[20] de la Hoz F and Fontelos M A 2008 The structure of singularities in nonlocal transport equations J. Phys. A: Math. Theor. 41 185204
[21] Deslauriers J, Tedstrom R, Daw M S, Chrzan D, Neeraj T and Mills M 2004 Dynamic scaling in a simple one-dimensional model of dislocation activity Phil. Mag. 84 2445–54
[22] Do T, Kiselev A, Ryzhik L and Tan C 2018 Global regularity for the fractional Euler alignment system Arch. Ration. Mech. Anal. 228 1–37
[23] Dong H 2008 Well-posedness for a transport equation with nonlocal velocity J. Funct. Anal. 255 3070–97
[24] Dong H 2010 Dissipative quasi-geostrophic equations in critical Sobolev spaces: smoothing effect and global well-posedness Discrete Contin. Dyn. Syst. 26 1197–211
[25] Duoandikoetxea J 2001 Fourier Analysis vol 29 (American Mathematical Society)
[26] Evans L C 1990 Weak Convergence Methods for Nonlinear Partial Differential Equations vol 74 (American Mathematical Society)
[27] Gancedo F, Granero-Belinchón R and Scrobogna S 2023 Global existence in the Lipschitz class for the N-Peskin problem Indiana Univ. Math. J. 72 553–602
[28] Garcia-Juárez E, Mori Y and Strain R M 2023 The Peskin problem with viscosity contrast Anal. PDE 16 785–838
[29] Granero-Belinchón R 2020 On a nonlocal differential equation describing roots of polynomials under differentiation Commun. Math. Sci. 18 1643–60
[30] Head A K 1972 Dislocation group dynamics I. Similarity solutions of the n-body problem Phil. Mag. 26 43–53
[31] Head A K 1972 Dislocation group dynamics II. General solutions of the n-body problem Phil. Mag. 26 55–63
[32] Head A K 1972 Dislocation group dynamics III. Similarity solutions of the continuum approximation Phil. Mag. 26 65–72
[33] Huang D, Tong J and Wei D 2023 On self-similar finite-time blowups of the De Gregorio model on the real line Commun. Math. Phys. 402 2791–829
[34] Kizomono H and Wadade H 2008 Remarks on Gagliardo–Nirenberg type inequality with critical Sobolev space and BMO Math. Z. 259 935–50
[35] Li D and Rodrigo J 2008 Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation Adv. Math. 217 2563–8
[36] Li H 2021 Stability of the Stokes immersed boundary problem with bending and stretching energy J. Funct. Anal. 281 109204
[37] Lin F and Zhang P 2000 On the hydrodynamic limit of Ginzburg-Landau vortices Discrete Contin. Dyn. Syst. 6 121–42
[38] Morlet A C 1998 Further properties of a continuum of model equations with globally defined flux J. Math. Anal. Appl. 221 132–60
[39] Okamoto H, Sakajo T and Wunsch M 2008 On a generalization of the Constantin–Lax–Majda equation Nonlinearity 21 2447
[40] Peskin C S 1972 Flow patterns around heart valves: a numerical method J. Comput. Phys. 10 252–71
[41] Peskin C S 2002 The immersed boundary method Acta Numer. 11 479–517
[53] Pollard D 2022 It’s just calculus and convexity (available at: http://www.stat.yale.edu/~pollard/Books/Mini/Calculus.pdf)
[54] Rodenberg A 2018 2D Peskin problems of an immersed elastic filament in Stokes flow PhD Thesis University of Minnesota
[55] Shlyakhtenko D and Tao T 2022 With an appendix by David Jekel. Fractional free convolution powers Indiana Univ. Math. J. 71 2551–94
[56] Shvydkoy R and Tadmor E 2017 Eulerian dynamics with a commutator forcing Trans. Math. Appl. 1 tnx001
[57] Shvydkoy R and Tadmor E 2018 Eulerian dynamics with a commutator forcing III. Fractional diffusion of order $0 < \alpha < 1$ Physica D 376 131–7
[58] Silvestre L and Vicol V 2016 On a transport equation with nonlocal drift Trans. Am. Math. Soc. 368 6159–88
[59] Steinerberger S 2019 A nonlocal transport equation describing roots of polynomials under differentiation Proc. Am. Math. Soc. 147 4733–44
[60] Steinerberger S 2021 Free Convolution Powers Via Roots of Polynomials Exp. Math. (https://doi.org/10.1080/10586458.2021.1980751)
[61] Tan C 2019 Singularity formation for a fluid mechanics model with nonlocal velocity Commun. Math. Sci. 17 1779–94
[62] Tong J 2018 On the Stokes immersed boundary problem in two dimensions PhD Thesis New York University
[63] Tong J 2021 Regularized Stokes immersed boundary problems in two dimensions: well-posedness, singular limit and error estimates Commun. Pure Appl. Math. 74 366–449
[64] Tong J and Wei D 2023 Geometric properties of the 2-D Peskin problem (arXiv:2304.09556)
[65] Villani C 2009 Optimal Transport: Old and New vol 338 (Springer)