I. INTRODUCTION

The Dicke model\textsuperscript{1}, which describes a system of $N$ qubits coupled to a single-mode spatially-uniform field confined in a cavity of volume $V$, plays a central role in quantum optics and cavity quantum electrodynamics (QED)\textsuperscript{2–5}. In 1973 Hepp and Lieb\textsuperscript{6} and subsequently Wang and Hioe\textsuperscript{7} pointed out that for sufficiently strong light-matter coupling the Dicke model in the thermodynamic limit ($N \to \infty$, $V \to \infty$, with $N/V = \text{const}$) has a finite temperature second-order equilibrium phase transition between a normal and “superradiant” state. In the latter, the ground state contains a macroscopically large number of coherent photons, i.e. $(\hat{a} + \hat{a}^\dagger)$ destroys (creates) a cavity photon. Equilibrium superradiance was shown to be robust against the addition of counter-rotating terms\textsuperscript{8,9} neglected in Refs. 6 and 7, but not against restoration of an additional neglected term proportional to $(\hat{a} + \hat{a}^\dagger)^2$ (Ref. 10). This quadratic term is naturally generated by applying minimal coupling $\hat{p} \to \hat{p} + eA/c$ to the electron kinetic energy $\hat{p}^2/(2m)$. Rzażewski et al.\textsuperscript{10} were the first to show that the Thomas-Reiche-Kuhn (TRK) sum rule\textsuperscript{11,12} poses an insurmountable obstacle against equilibrium superradiance in a spatially-uniform quantum cavity field. Physically, this sum rule originates from gauge invariance,\textsuperscript{13,14} and in particular from the property that a system cannot respond to a spatially-uniform and time-independent vector potential. The link between gauge invariance and quadratic terms emerges as following. The quadratic term is responsible for the appearance of a diamagnetic contribution to the current operator\textsuperscript{13,14}. Only when paramagnetic and diamagnetic contributions are considered on equal footing, does one have a precisely gauge-invariant Hamiltonian satisfying the TRK sum rule. Recent advances in technology have reinvigorated interest in equilibrium superradiance\textsuperscript{15,16}, inspiring a literature thread in which the obstacle presented by quadratic terms was periodically resurrected\textsuperscript{17,18}. Complications due to the presence of a superconducting condensate in circuit QED setups were also discussed\textsuperscript{18–22}.

In the Dicke model direct interactions between two-level systems are neglected. Effective long-range interactions between qubits are solely mediated by the common cavity field. Recent experimental progress has created opportunities to study light-matter interactions in an entirely new regime. For example, two-dimensional (2D) electron systems (ESs) can be embedded in cavities or exposed to the radiation field of metamaterials, making it possible to study strong light-matter interactions in the regime where direct electron-electron interactions may play a pivotal role, as in the quantum Hall regime\textsuperscript{23–29}.

Similarly, one can imagine cavity QED in which matter exhibits strongly correlated phenomena\textsuperscript{30–35} such as exciton condensation, superconductivity, magnetism, or Mott insulating states. For all these exciting new possibilities, the paradigmatic Dicke model needs of course to be transcended. The degrees of freedom of microscopic many-body Hamiltonians—such as the one of the jellium model\textsuperscript{14} or the Hubbard model\textsuperscript{36} to name two—need to be coupled to the cavity modes. As the Dicke model story has instructed us, theories of the equilibrium properties of these intriguing new systems must be fully gauge invariant. This has not always been the case in the literature. For example, the case of materials with a low-energy linear energy-momentum dispersion relation, such as graphene and Weyl semimetals, is particularly
tricky. In this case, the low-energy continuum model Hamiltonian needs to be accompanied by an ultraviolet cut-off, which breaks gauge invariance\textsuperscript{37}. Using this model to study superradiant quantum phase transitions, e.g. in graphene\textsuperscript{38}, incorrectly implies photon condensation because a dynamically generated quadratic term is missed\textsuperscript{39,40}. We therefore conclude that low-energy truncations of the Hilbert space must be carried out carefully in order to preserve gauge invariance\textsuperscript{37,41,42}. Another example is that of Ref. 43, where the coupling of the matter degrees of freedom of a two-band Hubbard model to the spatially-uniform vector potential of the cavity was carried out via a paramagnetic current operator not satisfying the continuity equation (see Ref. 44 for further details). A no-go theorem for superradiant quantum phase transitions which is applicable to generic interacting many-body systems in a cavity has been recently demonstrated in Ref. 44, under the strong but almost universally made assumption of a spatially-uniform cavity field.

The term “superradiance” is used to describe a plethora of different collective phenomena, ranging from the amplification of radiation due to coherence in the emitting medium\textsuperscript{4} to the Zel’dovich-Misner-Unruh\textsuperscript{15} amplification of radiation by rotating black holes. To avoid confusion, we will therefore refer to the equilibrium superradiant phase as a photon condensate. Given the impossibility of achieving photon condensation in a spatially-uniform quantum cavity field, in this Article we relax this strong assumption. We lay down a theory of photon condensation in a spatially-varying quantum cavity field that does not rely on the smallness of the electron-electron-interaction coupling constant. As such, our theory is applicable to strongly correlated ESs.

We separately study three cases:

i) We first consider a three-dimensional (3D) ES embedded in a 3D cavity field. In this case, we reach a condition for the occurrence of photon condensation which is universal, in that it does not depend on the cavity material parameters. Indeed, our criterion depends only on a non-local linear response function of the 3DES, namely the static non-local orbital magnetic susceptibility $\chi_{\text{orb}}(q)$. This quantity describes the response of the electron system to a static but spatially-oscillating magnetic field:

$$\chi_{\text{orb}}(q) = -\frac{e^2}{c^2} \frac{\chi_T(q, 0)}{q^2}.$$  \hspace{1cm} (1)

Here, $-e$ is the electron charge, $c$ is the speed of light in vacuum, and $\chi_T(q, 0)$ is the transverse current response function of the interacting ES\textsuperscript{13,14}. We find that photon condensation occurs if and only if $\chi_{\text{orb}}(q) \geq 1/(4 \pi)$.

ii) We then study the role of spin degrees of freedom, by including in the treatment the Zeeman coupling between the electron spin and the spatially-varying cavity field. We also discuss the combined effects of orbital and spin couplings.

iii) Finally, we consider the case of a 2DES embedded in a quasi-2D cavity of extension $L_z$ in the direction perpendicular to the plane hosting the 2DES, i.e. the $\hat{x}$-$\hat{y}$ plane. In this case, the criterion for photon condensation depends on $L_z$, and not only on the intrinsic orbital magnetic properties of the 2DES.

Our Article is organized as follows. Photon condensation in 3D in the presence of purely orbital coupling between the cavity electromagnetic field and matter degrees of freedom is discussed in Sect. II. The role of spin and combined orbital-spin effects (always in 3D) is reported in Sect. III. Finally, the case of 2DESes embedded in quasi-2D cavities is discussed in Sect. IV. A brief summary and our main conclusions are finally presented in Sect. V. A number of cumbersome mathematical proofs and useful technical details are reported in Appendices A-D.

II. 3D PHOTON CONDENSATION

We consider a 3DES interacting with a spatially-varying quantized electromagnetic field. For the sake of concreteness, we assume that the 3DES is described by the jellium model Hamiltonian\textsuperscript{13,14}:

$$\hat{H} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \frac{1}{2} \sum_{ij} v(|\hat{r}_i - \hat{r}_j|).$$  \hspace{1cm} (2)

This model describes $N$ electrons of mass $m$ interacting via an arbitrary\textsuperscript{46} central potential $v(r)$. Charge neutrality (and therefore stability) of the system is guaranteed by a positive background of uniform charge. Electron-background and background-background interactions have not been explicitly written in $\hat{H}$. For future reference, we denote by $|\psi_0\rangle$ and $E_0$ the exact eigenstates and eigenvalues\textsuperscript{13,14} of $\hat{H}$, with $|\psi_0\rangle$ and $E_0$ denoting the ground state and ground-state energy, respectively. We also introduce the 3D Fourier transforms of the density and paramagnetic (number) current operators\textsuperscript{13,14}:

$$\hat{n}(q) = \sum_{i=1}^{N} e^{-i\mathbf{q} \cdot \mathbf{r}_i},$$  \hspace{1cm} (3)

$$\hat{j}_p(q) = \frac{1}{2m} \sum_{i=1}^{N} (\hat{p}_i e^{-i\mathbf{q} \cdot \mathbf{r}_i} + e^{-i\mathbf{q} \cdot \mathbf{r}_i} \hat{p}_i),$$  \hspace{1cm} (4)

with $\hat{n}(-q) = \hat{n}^\dagger(q)$ and $\hat{j}_p(-q) = \hat{j}_p^\dagger(q)$.

We treat the spatially-varying cavity electromagnetic field $\mathbf{A}(\mathbf{r})$ in a quantum fashion\textsuperscript{47,48}. We consider a cavity of volume $V = L_x L_y L_z$, impose periodic boundary conditions on the cavity field, and represent it in terms of plane waves:

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{q}, \sigma} A_{\mathbf{q}} u_{\mathbf{q}, \sigma} (\hat{a}_{\mathbf{q}, \sigma} e^{i\mathbf{q} \cdot \mathbf{r}} + \hat{a}_{\mathbf{q}, \sigma}^\dagger e^{-i\mathbf{q} \cdot \mathbf{r}}).$$  \hspace{1cm} (5)

Here, $\mathbf{q} = (2\pi n_x/L_x, 2\pi n_y/L_y, 2\pi n_z/L_z)$ with $(n_x, n_y, n_z)$ relative integers, $\sigma = 1, 2$ is the polarization index, $u_{\mathbf{q}, \sigma}$ is the linear polarization vector,
\[ A_q = \sqrt{2m\hbar^2/(V\omega_q^2)}, \quad \omega_q = cq/\sqrt{\epsilon}, \quad \text{and} \quad \epsilon_q \text{ is a relative dielectric constant.} \] The following properties hold: \[ \omega_q - \omega_q = \omega_q, \quad u(q) A_q = u(q), \quad A_q - A_q = A_q, \quad \text{and} \quad u(q), u(q') = \delta_{q,q'}. \] In the Coulomb gauge, the transversality condition \[ u_{q,q} = u_{q,q'} = \delta_{q,q'}. \] The photon annihilation and creation operators in Eq. (5) satisfy bosonic commutation relations, \[ [u_{q,q'}, u_{q,q'}^\dagger] = \delta_{q,q'}\delta_{q,q'}. \]

Being a quantum object, the field \( \hat{A}(r) \) has its own dynamics, which is determined by the photon Hamiltonian

\[ H_{ph} = \sum_{q,\sigma} \hbar \omega_q \left( \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} + \frac{1}{2} \right). \] (6)

The full Hamiltonian, including light-matter interactions, is therefore given by

\[ \hat{H}_A = \hat{H} + \hat{H}_{ph} + \sum_{i=1}^N \frac{e}{mc} \hat{A}(r_i) \cdot \hat{p}_i + \sum_{i=1}^N \frac{e^2}{2mc^2} \hat{A}^2(r_i). \] (7)

The third and fourth terms in Eq. (7) are often referred to respectively as the paramagnetic and diamagnetic contributions to the light-matter coupling Hamiltonian.

With the aim of studying the potential existence of a quantum phase transition to a photon condensate and make therefore general statements about the ground state \(|\Psi\rangle\) of \( \hat{H}_A \), the model (7) must be extrapolated to the thermodynamic limit \( N \to \infty, V \to \infty \), with constant \( N/V \). As shown in Appendix A, in this limit, \(|\Psi\rangle\) does not contain light-matter entanglement, i.e. we can take \(|\Psi\rangle = |\psi\rangle |\Phi\rangle\), where \(|\psi\rangle\) and \(|\Phi\rangle\) are matter and light states. We can therefore introduce the effective Hamiltonian for the photonic degrees of freedom, \( \hat{H}_{ph}^{\text{eff}}[\psi] \equiv \langle \psi | \hat{H}_A | \psi \rangle \).

Explicitly,

\[ \hat{H}_{ph}^{\text{eff}}[\psi] = \hat{H}_{ph} + \langle \psi | \hat{H} | \psi \rangle + \sum_{q,\sigma} \frac{e}{c} A_q [\hat{a}_{q,\sigma} \hat{a}_{q,\sigma}^\dagger \cdot u_{q,\sigma} + \text{h.c.}] + \frac{e^2}{2mc^2} \sum_{q,q',\sigma} A_q A_{q'} u_{q,\sigma} u_{q',\sigma} \times \left[ \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} n(q - q') + \hat{a}_{q,\sigma} \hat{a}_{q,\sigma}^\dagger n(q - q') + \hat{a}_{q,\sigma} \hat{a}_{q,\sigma}^\dagger n(q + q') + \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} n(q + q') \right]. \] (8)

where we have used the transversality condition, \( u_{q,\sigma} \cdot q = 0 \), and introduced \( n(q) \equiv \langle \psi | \hat{n}(q) | \psi \rangle \) and \( j(q) \equiv \langle \psi | \hat{j}(q) | \psi \rangle \).

In the Coulomb gauge, 3D photon condensation is manifested by a non-zero zero of the order parameter \( a_{q,\sigma} \equiv \langle \Phi | \hat{a}_{q,\sigma} | \Phi \rangle \) emerging at a critical value of a suitable light-matter coupling constant. At the quantum critical point (QCP), \( a_{q,\sigma} \) is small. Note also that, near the QCP, the matter state can be written as \( |\psi\rangle = |\psi_0\rangle + \sum_{q,\sigma} \hat{a}_{q,\sigma} |\delta\psi_{q,\sigma}\rangle + O(a_{q,\sigma}^2). \) Since the diamagnetic term in Eq. (8) is quadratic in \( a_{q,\sigma} \), we can approximate the quantity \( n(q) \) in the last two lines of this equation with its value in the absence of light-matter interactions, i.e. we can safely take \( n(q) \simeq \langle \psi_0 | \hat{n}(q) | \psi_0 \rangle \). We now assume that the ground state \(|\Psi_0\rangle\) of the 3DES in the absence of light-matter interactions is homogenous and isotropic, i.e. \( \langle \psi_0 | \hat{n}(q) | \psi_0 \rangle = N \delta_{q,0} \). The reason why this assumption was made is obvious from the form of the diamagnetic term in Eq. (8): inhomogeneous ground states with \( \langle \psi_0 | \hat{n}(q) | \psi_0 \rangle \neq N \delta_{q,0} \) would couple modes with \( q \neq q' \), rapidly leading to a problem that is intractable with purely analytical methods. Under this assumption, the effective Hamiltonian reduces to:

\[ \hat{H}_{ph}^{\text{eff}}[\psi] = \langle \psi | \hat{H} | \psi \rangle + \sum_{q,\sigma} \frac{e}{c} A_q [\hat{a}_{q,\sigma} \hat{a}_{q,\sigma}^\dagger \cdot u_{q,\sigma} + \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} \cdot u_{q,\sigma} + \frac{1}{2} \left[ \hbar \omega_q + \hbar \omega_q \left( \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} + \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} \right) + 2 \Delta_q \left( \hat{a}_{q,\sigma} \hat{a}_{q,\sigma}^\dagger + \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} \right) \right]. \] (9)

where \( \Delta_q \equiv N e^2 A_q^2/(2mc^2) \) with \( \Delta_q = \Delta_q - q \), and \( \hbar \omega_q = \omega_q + 2 \Delta_q \). The term \( \sum_{q,\sigma} \hbar \omega_q \) is a vacuum contribution. Eq. (9) is a quadratic function of the photonic operators and can be diagonalized via the following Bogoliubov transformation:

\[ \hat{a}_{q,\sigma}^\dagger = \cosh(x_q) \hat{b}_{q,\sigma}^\dagger - \sinh(x_q) \hat{b}_{-q,\sigma} , \] (10)

where \( \cosh(x_q) = (\lambda_q + 1)/(2\sqrt{\lambda_q}), \sinh(x_q) = (\lambda_q - 1)/(2\sqrt{\lambda_q}) \), and \( \lambda_q = \sqrt{1 + 4 \Delta_q/\hbar \omega_q} \). In terms of the new bosonic operators \( \hat{b}_{q,\sigma}^\dagger, \hat{b}_{q,\sigma} \) the effective Hamiltonian reads as following:

\[ \hat{H}_{ph}^{\text{eff}}[\psi] = \langle \psi | \hat{H} | \psi \rangle + \sum_{q,\sigma} \hbar \Omega_q \left( \hat{b}_{q,\sigma}^\dagger \hat{b}_{q,\sigma}^\dagger + \frac{1}{2} \right) + \sum_{q,\sigma} \frac{e}{c \sqrt{\lambda_q}} [\hat{a}_{q,\sigma} \hat{a}_{q,\sigma} + \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma}^\dagger] + \text{H.c.} \] (11)

where \( \hbar \Omega_q = \hbar \omega_q \lambda_q \).

Being a sum of displaced harmonic oscillators, the ground state \(|\Phi\rangle\) of \( \hat{H}_{ph}[\psi] \), for every matter state \(|\psi\rangle\), is a tensor product \( |\beta\rangle \equiv \otimes_{q,\sigma} |\beta_{q,\sigma}\rangle \) of coherent states of the \( \hat{b}_{q,\sigma} \) operators. In terms of the new bosonic operators \( \hat{b}_{q,\sigma}^\dagger, \hat{b}_{q,\sigma} \) and the order parameter \( \beta_{q,\sigma} \), the order parameter, which can again be considered small at the QCP.

We now introduce the following energy functional, obtained by taking the expectation value of \( \hat{H}_{ph}[\psi] \) over \(|\beta\rangle\): \( E(|\beta\rangle; |\psi\rangle) \equiv \langle \psi | \hat{H}_A | \psi \rangle = \langle \beta | \hat{H}_{ph}^{\text{eff}}[\psi] | \beta \rangle \).
\[
E[\{\beta_{q,\sigma}\}, \psi] = \langle \psi|\hat{H}|\psi \rangle + \sum_{q,\sigma} \hbar \Omega_q \left( |\beta_{q,\sigma}|^2 + \frac{1}{2} \right) \\
+ \sum_{q,\sigma} \frac{e A_q}{c \sqrt{\lambda_q}} \left[ j_p(-q) \cdot u_{q,\sigma}\beta_{q,\sigma} + \text{c.c.} \right].
\]

(12)

This needs to be minimized with respect to \( \{\beta_{q,\sigma}\} \) and \( |\psi\rangle \). The minimization with respect to \( \{\beta_{q,\sigma}\} \) can be done analytically by imposing the condition \( \partial_{\beta_{q,\sigma}} E[\{\beta_{q,\sigma}\}, \psi] = 0 \). We find that the optimal value of \( \{\beta_{q,\sigma}\} \) is given by:

\[
\tilde{\beta}_{q,\sigma} = -\frac{A_q}{\hbar \omega_q \lambda_q^{3/2}} \frac{e}{c} j_p(q) \cdot u_{q,\sigma}.
\]

(13)

Note that this equation can be written in terms of the operator

\[
\hat{B}_{q,\sigma} \equiv -\frac{A_q}{\hbar \omega_q \lambda_q^{3/2}} \frac{e}{c} j_p(q) \cdot u_{q,\sigma},
\]

(14)

i.e. \( \beta_{q,\sigma} = \langle \psi|\hat{B}_{q,\sigma}|\psi \rangle \).

Using Eq. (13) into Eq. (12), we finally find the energy functional that needs to be minimized with respect to \( |\psi\rangle \):

\[
E[\{\tilde{\beta}_{q,\sigma}\}, \psi] = \langle \psi|\hat{H}|\psi \rangle - \sum_{q,\sigma} \hbar \Omega_q \left( |\tilde{\beta}_{q,\sigma}|^2 - \frac{1}{2} \right).
\]

(15)

As in the case of a spatially-uniform cavity field\(^{14}\), we are therefore left with a constrained minimum problem

\[ \langle \psi|\hat{H}|\psi \rangle - \langle \psi_0|\hat{H}|\psi_0 \rangle = -\frac{1}{2} \sum_{q,\sigma} \sum_{q',\sigma'} \chi_{\hat{B}_{q,\sigma},\hat{B}_{q',\sigma'}}^{-1}(0) \tilde{\beta}_{q,\sigma} \tilde{\beta}_{q',\sigma'} \]

(17)

where \( \chi_{\hat{B}_{q,\sigma},\hat{B}_{q',\sigma'}}^{-1}(0) \) is the inverse of the static response function \( \chi_{\hat{B}_{q,\sigma},\hat{B}_{q',\sigma'}}(0) \), the operator \( \hat{B}_{q,\sigma} \) has been introduced in Eq. (14), and we have used the notation of Ref. 14. Since the ground state of the 3DES has been taken to be homogenous and isotropic\(^{14}\),

\[
\chi_{\hat{B}_{q,\sigma},\hat{B}_{-q',\sigma'}}(0) = \chi_{\hat{B}_{q,\sigma},\hat{B}_{-q',\sigma'}}(0) \delta_{q,q'} \delta_{\sigma,\sigma'}.
\]

(18)

As any other response function, \( \chi_{\hat{B}_{q,\sigma},\hat{B}_{-q,\sigma}}(0) \) has a Lehmann representation\(^{13,14}\) in terms of the exact eigenstates of the Hamiltonian (2),

\[
\chi_{\hat{B}_{q,\sigma},\hat{B}_{-q,\sigma}}(0) = \frac{-2 A_q^2}{\hbar^2 \omega_q^2 \lambda_q^2} \frac{e^2}{c^2} \sum_{n \neq 0} \left| \langle \psi_n|j_p(q) \cdot u_{q,\sigma}|\psi_0 \rangle \right|^2 \frac{E_n - E_0}{E_n} < 0.
\]

(19)

We readily recognize \( \chi_{\hat{B}_{q,\sigma},\hat{B}_{-q,\sigma}}(0) \) to be intimately linked to the static, paramagnetic current-current response
that the "diamagnetic term" enters Eq. (24), and we now return to the result of the stiffness theorem. Inserting Eq. (17) inside Eq. (16), we finally find the condition for photon condensation in a 3DES embedded in a spatially-varying electromagnetic field:

\[
- \sum_{q, \sigma} \left[ \frac{1}{2 \chi_{q, \sigma, -q, \sigma}} + \hbar \Omega_q \right] [\hat{\beta}_{q, \sigma}]^2 < 0 .
\]  

(25)

Since we want to minimize the energy difference \( E[\{\hat{\beta}_{q, \sigma}\}, \psi] - E[0, \psi_0] \), the optimal choice of \( \hat{\beta}_{q, \sigma} \) is constructed as following: i) modes with momentum \( q \) and polarization \( \sigma \) such that Eq. (25) is satisfied acquire a finite displacement \( \hat{\beta}_{q, \sigma} \neq 0 \), since this choice lowers the energy difference; ii) on the other hand, modes for which Eq. (25) is not satisfied, are forced to be unpopulated, i.e. to have \( \hat{\beta}_{q, \sigma} = 0 \). A finite occupation of these modes would indeed increase the energy difference. Hence, we can analyze the inequality (25) for a fixed \( q \):

\[
- \chi_{q, \sigma, -q, \sigma}(0) > \frac{1}{2 \hbar \Omega_q} .
\]  

(26)

Using Eq. (24) and the microscopic expression of \( A_q \), we can rewrite Eq. (26) as following:

\[
- 4\pi \left( \frac{e^2}{\omega_q^2 \epsilon_r} \right)^2 \frac{n}{m} + \frac{\Delta_0}{\hbar \omega_q} > 1 .
\]  

(27)

Before further simplifying Eq. (27), we wish to make a few observations on the special case of a single-mode spatially-uniform field:

i) No-go theorem in the presence of the diamagnetic term. Let us consider the standard situation in the literature, in which matter degrees of freedom are minimally coupled to a quantum field, which is assumed to be single mode and spatially uniform, with angular frequency \( \omega_0 \) and amplitude \( A_q = A_0 = \sqrt{2 \pi \hbar c^2 / (V \omega_0 \epsilon_r)} \). Consistently, if the assumption of spatial uniformity is done from the very beginning, by setting \( q = 0 \) in Eq. (5), one has to replace \( \chi^J_q(0,0) \) with \( \lim_{q \to 0} \chi^J_q(0,0) \) inside the square bracket in Eq. (27). In systems with no long-range order (i.e. in systems that do not become superconducting), it is well known that the "diamagnetic sum rule" holds true: \( \lim_{q \to 0} \chi^J_q(0,0) = 0 \). In this case, Eq. (27) reduces to:

\[
\frac{4 \pi \left( \frac{e^2}{\omega_q^2 \epsilon_r} \right)^2 \frac{n}{m} + 1} + 4 \frac{\Delta_0}{\hbar \omega_q} ,
\]  

(28)

with \( \Delta_0 = e^2 N A_0^2 / (2mc^2) \). The left-hand-side of Eq. (28) can be easily seen to be equal to \( 4\Delta_0 / (\hbar \omega_q) \) and this inequality therefore reduces to \( 0 > 1 \), which is clearly absurd. This is the no-go theorem for photon condensation in a single-mode spatially-uniform quantum field.

ii) Spurious "go theorem" in the absence of the diamagnetic term. Neglecting artificially the diamagnetic contribution to Eq. (7) is equivalent to setting \( \Delta_0 = 0 \) in the right-hand-side of Eq. (28). In this case a photon condensate occurs provided that the Drude weight \( D = \pi e^2 n / m \) of the 3DES satisfies the following inequality:

\[
D > \frac{\omega_0^2 \epsilon_r}{4} .
\]  

(29)
Returning to Eq. (27) and using in it the microscopic expressions for $\omega_\varepsilon$ and $\Delta_\mathbf{q}$ given above, we finally conclude that a photon condensate phase occurs if and only if the following inequality is satisfied:

$$-\frac{e^2}{\epsilon_\varepsilon^2} \chi_\varepsilon^J(q,0) > \frac{1}{4\pi} .$$

(30)

The left-hand-side of Eq. (30) has a very clear physical interpretation. It is the non-local orbital magnetic susceptibility

$$\chi_{\text{orb}}(q) = -\frac{e^2}{\epsilon_\varepsilon^2} \chi_\varepsilon^J(q,0) ,$$

(31)

which, in the long-wavelength $q \rightarrow 0$ limit, reduces to the thermodynamic (i.e. macroscopic) orbital magnetic susceptibility (OMS)

$$\chi_{\text{OMS}} \equiv \lim_{q \rightarrow 0} \chi_{\text{orb}}(q) = \frac{\partial M_O}{\partial B} \bigg|_{B=0} .$$

(32)

Here, $M_O$ is the orbital contribution to the magnetization. This limit exists in systems with no long-range order: indeed, $\chi_\varepsilon^J(q,0)$ vanishes like $q^2$ in the long-wavelength $q \rightarrow 0$ limit, in agreement with the diamagnetic sum rule.

In summary, introducing $\chi_{\text{orb}}(q)$, we can write Eq. (30) as

$$\chi_{\text{orb}}(q) > \frac{1}{4\pi} .$$

(33)

Eq. (33) is the most important result of this Section, representing a rigorous criterion for the occurrence of photon condensation in a 3DES.

A. Discussion

A few comments are now in order.

i) In 3D, as clear from Eq. (33), $\chi_{\text{orb}}(q)$ is dimensionless. It therefore naturally plays the role of a coupling constant determining the strength of light-matter interactions. Only when it exceeds the value $1/(4\pi) \sim 0.08$ can photon condensation take place.

ii) The criterion (33) does not depend explicitly on $\epsilon_\varepsilon$ but only implicitly, through the $\epsilon_\varepsilon$-dependence of the e-e interaction potential $v(r)$. The latter, in turn, has an impact on $\chi_{\text{orb}}(q)$.

iii) Note that, while $\chi_{\mathbf{B}_\alpha,\mathbf{B}_\gamma,\mathbf{B}_{-\alpha,\gamma}}(0)$ in Eq. (19) and (24) is negative definite, the transverse contribution $\chi_T(q,0)$ to the current-current response function satisfies the inequality $\chi_T(q,0) < n/m$ and can therefore be both positive or negative. In turn, this implies that, for a given 3DES, $\chi_{\text{OMS}}$ can be positive or negative (and perhaps change sign with microscopic parameters such as the electron density $n$). Broadly speaking, materials can be divided into two groups, from the point of view of their orbital response: a) orbital diamagnets, those which have $\chi_{\text{OMS}} < 0$, are most common. They will not display photon condensation, according to our criterion (33); b) orbital paramagnets, those for which $\chi_{\text{OMS}} > 0$, are much more rare in nature but, as discussed below, do exist. Only orbital paramagnets with $\chi_{\text{OMS}} > 1/(4\pi)$ can display photon condensation.

Just as an example, we remind the reader that for free (i.e. non-interacting) parabolic-band fermions in 3D,

$$\chi_{\text{OMS}}^{(0)} = \frac{\alpha^2}{r_s} \left( \frac{1}{768\pi^5} \right)^{1/3} < 0 ,$$

(34)

where $r_s = \left[ 3/(4\pi n a_B^3) \right]^{1/3}$ is the so-called Wigner-Seitz or gas parameter, $a_B = h^2/(me^2)$ is the Bohr radius, and $\alpha = e^2/(\hbar c)$ is the fine structure constant.

iv) The result in Eq. (33) can be understood as the condition for the occurrence of a static magnetic instability. Indeed, let us consider the energy functional of a material subject to a magnetic field $\mathbf{H}(r)$:

$$E[\mathbf{B}(r)] = \frac{1}{2} \int d^3r \mathbf{H}(r) \cdot \mathbf{B}(r) ,$$

(35)

where $\mathbf{B}(r)$ is the magnetic induction. The latter is related to the magnetic field via the orbital magnetization $M(r)$, i.e. $\mathbf{B}(r) = \mathbf{H}(r) + 4\pi M(r)$. The difference between $\mathbf{H}$ and $\mathbf{B}$ stems from the flow of charges in response to $\mathbf{H}$, which creates an orbital magnetization $M$. In the realm of linear response theory, we can relate the orbital magnetization to the magnetic induction, $M(r) = \int d^3r' \chi_{\text{orb}}(\mathbf{r}-\mathbf{r}') \mathbf{B}(r')$. We can therefore write the energy as a quadratic function of $\mathbf{B}(r)$:

$$E[\mathbf{B}(r)] = \frac{1}{2} \int d^3r \int d^3r' \left[ \delta(\mathbf{r}-\mathbf{r}') - 4\pi \chi_{\text{orb}}(|\mathbf{r}-\mathbf{r}'|) \right] \mathbf{B}(r') \cdot \mathbf{B}(r) .$$

(36)

An instability occurs if $E[\mathbf{B}(r)] < 0$, i.e. if and only if $\mathbf{B}(r) < 4\pi \int d\mathbf{r}' \chi_{\text{orb}}(|\mathbf{r}-\mathbf{r}'|) \mathbf{B}(r')$. Fourier transforming with respect to $\mathbf{r}$ yields Eq. (33).

Magnetostatic instabilities and the criterion (33) have been discussed long ago. In a 3D metal, the de Haas-van Alphen effect (oscillations of the magnetization in response to an applied magnetic field) can lead to a thermodynamic instability of the electron gas. The magnetization is a function of the magnetic induction and when the orbital magnetic susceptibility $\chi_{\text{OMS}}$ obeys the inequality (33), the magnetic induction is a multi-valued function of the field. Condon first pointed out that Maxwell’s construction yields phase coexistence and the formation of (paramagnetic and diamagnetic) domains. These “Condon domains”, although first predicted for Be, were first unambiguously observed in Ag. Since then, Condon domains have been observed also in Be, Sn, and also Al, Pb, and In (for a recent review see, for example, Ref. 54). They have also been observed in Br2-intercalated graphite, which is a layered compound with quasi-2D character.
Our derivation in Sect. II shows that 3D photon condensation and Condon domain formation are the same phenomenon. In essence, the proof reported in Sect. II is a fully quantum mechanical derivation of the condition for the occurrence of Condon domains, which transcends the usual semiclassical approximations used to derive (33).

v) For the remainder of this Article (particularly for Sect. IV), it is useful to derive Eq. (9) in an alternative way.

Instead of determining the exact photonic state, as we did above, we now follow a much more humble approach. We evaluate the expectation value of the Hamiltonian (9) on a trial photonic wavefunction of the form \(|\alpha'\) ≡ \(\otimes q,\sigma |\alpha_q,\sigma\rangle\), namely a tensor product of coherent states of the \(a_q,\sigma\) operators, i.e. \(\hat{a}_{q',\sigma'} |\alpha'\rangle = \alpha_{q',\sigma'} |\alpha'\rangle\).

(We know that the exact eigenstate is not of this form, i.e. it is a tensor product \(|\beta\) ≡ \(\otimes q,\sigma |\beta_q,\sigma\rangle\) of coherent states of the \(b_q,\sigma\) operators. Momentarily, we will understand what error is made in using \(|\alpha'\) rather than \(|\beta\).) Such expectation value is easily obtained by replacing the photonic operators in Eq. (9) with c-numbers, i.e. by replacing \(\hat{a}_{q,\sigma} \rightarrow \alpha_{q,\sigma}\). Up to a constant factor, we find

\[
\hat{E} [|\alpha_{q,\sigma}\rangle, \psi] \equiv \langle \alpha' | \hat{H}_{\text{ph}} | \psi \rangle |\alpha'\rangle = \langle \psi | \hat{H} | \psi \rangle + \sum_{q,\sigma} \frac{e A_q}{c} [\alpha_{q,\sigma} \hat{J}_p (-q) \cdot \hat{u}_{q,\sigma} + \alpha_{q,\sigma}^* \hat{J}_p (q) \cdot \hat{u}_{q,\sigma}] \\
+ \frac{1}{2} \sum_{q,\sigma} [\hbar \omega_q (\alpha_{q,\sigma}^* \alpha_{q,\sigma} + \alpha_{q,\sigma}^* \alpha_{-q,\sigma} + 1) + 2 \Delta_q (\alpha_{-q,\sigma} \alpha_{q,\sigma} + \alpha_{q,\sigma}^* \alpha_{-q,\sigma}^*)].
\]

Performing in Eq. (37) the linear transformation \(\alpha_{q,\sigma}^* = \cosh (x_q) \beta_{q,\sigma}^* - \sinh (x_q) \beta_{-q,\sigma}\), analogous to Eq. (10), we get:

\[
\hat{E} [|\beta_{q,\sigma}\rangle, \psi] = \langle \psi | \hat{H} | \psi \rangle + \sum_{q,\sigma} \frac{\hbar \omega_q}{2} + \frac{\hbar \Omega_q |\beta_{q,\sigma}\rangle^2}{2}
+ \sum_{q,\sigma} \frac{e A_q}{c \sqrt{\hbar \omega_q}} [\hat{J}_p (-q) \cdot \hat{u}_{q,\sigma} \beta_{q,\sigma} + \text{c.c.}].
\]

III. THE ROLE OF ZEEMAN COUPLING AND COMBINED ORBITAL-SPIN EFFECTS

In this Section we investigate the role of the Zeeman coupling. To begin with, we consider (Sect. III A) the case in which the 3DES couples to the radiation field only via the Zeeman term. In the second part of this Section (Sect. III B), we consider the combined role of orbital and Zeeman couplings. The derivation of the corresponding criteria for photon condensation closely follows the case of pure orbital coupling discussed in Sect. II.

A. Light-matter interactions via the Zeeman term

If the 3DES couples to the spatially-varying cavity electromagnetic field only via the Zeeman term, the full Hamiltonian is:

\[
\hat{H}_{\text{B}} = \hat{H} + \hat{H}_{\text{ph}} + \frac{\mu_B g}{2} \sum_{i=1}^N \hat{\sigma}_i \cdot \hat{B}(r_i),
\]

where \(g\) is the Landé g-factor, \(\mu_B\) is the Bohr magneton, \(\hat{\sigma}_i\) is the spin operator of the \(i\)-th electron, and \(\hat{B}(r) = \nabla \times \hat{A}(r)\) is the magnetic component of the cavity electromagnetic field, \(\hat{A}(r)\) being given in Eq. (5). Explicitly, the magnetic field reads as following:

\[
\hat{B}(r) = \sum_{q,\sigma} i q A_q u_{T,q,\sigma} (\hat{a}_q e^{i q \cdot r} - \hat{a}_q^\dagger e^{-i q \cdot r}),
\]

where \(u_{T,q,\sigma} \equiv (q/\sqrt{2}) \otimes q,\sigma\). (Note that \(\{q, u_{q,\sigma}, u_{T,q,\sigma}\}\) is a set of orthogonal vectors.)

As shown in Appendix B, the ground state \(|\Psi\rangle\) of \(\hat{H}_{\text{B}}\) does not contain light-matter entanglement in the thermodynamic limit, i.e. we can take \(|\Psi\rangle = |\psi\rangle |\Phi\rangle\), where \(|\psi\rangle\) and \(|\Phi\rangle\) are matter and light states. As in Sect. II, we are therefore led to introduce an effective Hamiltonian for the photonic degrees of freedom, \(\hat{H}_{\text{ph}}[\psi] \equiv \langle \psi | \hat{H}_{\text{B}} | \psi \rangle:\n
\[
\hat{H}_{\text{ph}}[\psi] = \langle \psi | \hat{H} | \psi \rangle + \hat{H}_{\text{ph}} + \sum_{q,\sigma} \frac{\mu_B g \hbar}{2} [\hat{S} (-q) \hat{a}_{q,\sigma} - \hat{S} (q) \hat{a}_{q,\sigma}^\dagger] \cdot i q u_{T,q,\sigma},
\]

where the 3D Fourier transform of the spin density \(\hat{S}(q) = \sum_{i=1}^N \hat{\sigma}_i \delta (r - r_i)\) and \(\hat{S}(q) = \langle \psi | \hat{S}(q) | \psi \rangle\).

Since Eq. (41) is a sum of displaced harmonic oscillators, we can assume without loss of generality that the ground state \(|\Phi\rangle\) of \(\hat{H}_{\text{ph}}[\psi]\) is a tensor product \(|\alpha'\) ≡ \(\otimes q,\sigma |\alpha_q,\sigma\rangle\).
\( \langle \Omega_{q,\sigma} | \alpha_{q,\sigma} \rangle \) of coherent states of the \( \hat{a}_{q,\sigma} \) operators, i.e., \( \hat{a}_{q',\sigma'} | \alpha_{q',\sigma'} \rangle \).

The total energy, defined as \( E[\{\alpha_{q,\sigma}\}, \psi] \equiv \langle \psi | \hat{H} | \psi \rangle = \sum_\alpha \langle \alpha_{q,\sigma} | \hat{a}_{q,\sigma}^\dagger \hat{a}_{q,\sigma} \rangle | \alpha_{q,\sigma} \rangle \), is given by:

\[
E[\{\alpha_{q,\sigma}\}, \psi] = \langle \psi | \hat{H} | \psi \rangle + \sum_\alpha \left( \alpha_{q,\sigma}^2 + \frac{1}{2} \right) + \sum_\alpha \frac{gJ_{B,\sigma}^2}{2 \hbar} \left[ S(-q) \alpha_{q,\sigma}^2 - S(q) \alpha_{q,\sigma}^2 \right] \cdot i q u_{q,\sigma}.
\]

(43)

Minimization can be performed with respect to \( \{\alpha_{q,\sigma}\} \) analytically by imposing the condition \( \partial_{\alpha_{q,\sigma}} E[\{\alpha_{q,\sigma}\}, \psi] = 0 \). We find that the optimal value of \( \{\alpha_{q,\sigma}\} \) is given by:

\[
\tilde{\alpha}_{q,\sigma} = \frac{gJ_{B,\sigma}^2}{2 \hbar \omega_c} \langle \psi | \hat{S}(q) | \psi \rangle \cdot i q u_{q,\sigma}.
\]

(44)

where \( \chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}^{-1}(0) \) is the inverse of the static response function \( \chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0) \) and the operator \( \hat{C}_{q,\sigma} \) has been introduced in Eq. (45). Inserting Eq. (48) inside Eq. (47) we find:

\[
-\sum_\alpha \left[ \frac{1}{2 \chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0)} + \hbar \omega_q \right] | \tilde{\alpha}_{q,\sigma} |^2 < 0.
\]

(49)

Following the same logical steps discussed in Sect. II, we can consider the previous inequality for a fixed \( q \):

\[
-\chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0) > \frac{1}{2 \hbar \omega_q}.
\]

(50)

We now observe that the homogenous and isotropic nature of the ground state of the 3DES implies that \( \chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0) = \chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0) \delta_{q,q'} \delta_{\sigma,\sigma'} \). We readily recognize \( \chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0) \) to be intimately linked to the static, spin-spin response tensor \( \chi_S^{i,k}(q,0) \). Indeed, it is easy to show that

\[
\chi_{\hat{C}_{q,\sigma},\hat{C}_{q'-\sigma'}}(0) = \frac{q g^2 g_{B,\sigma}^2 A_T^2 V}{4 \hbar^2 \omega_q^2} \sum_{i,k} u_{q,\sigma}^{(i)} v_{q',\sigma'}^{(k)} \chi_S^{i,k}(q,0),
\]

where

\[
\chi_S^{i,k}(q,0) = -\frac{1}{V} \sum_{n \neq 0} \frac{\langle \psi_n | \hat{S}_i(-q) | \psi_n \rangle \langle \psi_n | \hat{S}_i(q) | \psi_n \rangle}{E_n - E_0} - \frac{1}{V} \sum_{n \neq 0} \frac{\langle \psi_n | \hat{S}_i(q) | \psi_n \rangle \langle \psi_n | \hat{S}_i(-q) | \psi_n \rangle}{E_n - E_0},
\]

and \( \hat{S}_i(q) \), with \( i = x, y, z \), denotes the \( i \)-th Cartesian component of \( \hat{S}(q) \).

Isotropy, translational- and spin-rotational invariance imply that the rank-2 tensor \( \chi_S^{i,k}(q,0) \) can be decomposed in terms of the longitudinal, \( \chi_L^{i,k}(q,0) \), and transverse, \( \chi_T^{i,k}(q,0) \), spin-spin response functions:

\[
\chi_S^{i,k}(q,0) = \chi_L^{i,k}(q,0) \frac{q_i q_k}{q^2} + \chi_T^{i,k}(q,0) \left( \delta_{i,k} - \frac{q_i q_k}{q^2} \right).
\]

(53)
Replacing Eq. (53) into Eq. (51), we finally find
\[ \chi_{q,\sigma,C_{-\sigma},0} = \frac{g^2 g^2 \mu_B A_q^2 V}{4 h^2 \omega_q^2} \chi_T^S(q, 0). \] (54)

Using Eqs. (51) and (53) and the microscopic expressions of \( \omega_q = cq/\sqrt{\epsilon_\tau} \) and \( A_q = \sqrt{2\pi \hbar c^2/(V \omega_q \epsilon_\tau)} \) given above, Eq. (50) can be written as following:
\[ -\frac{g^2 \mu_B^2}{4} \chi_T^S(q, 0) > \frac{1}{4\pi}. \] (55)

Again, the left-hand-side of Eq. (55) has a very clear physical interpretation. It is the non-local transverse spin susceptibility\cite{14}:
\[ \chi_{\text{spin}}(q) = -\frac{g^2 \mu_B^2}{4} \chi_T^S(q, 0), \] (56)
which, in the long-wavelength \( q \to 0 \) limit, reduces to the thermodynamic (i.e. macroscopic) spin magnetic susceptibility (SMS)
\[ \chi_{\text{SMS}} \equiv \lim_{q \to 0} \chi_{\text{spin}}(q) = \frac{\partial M_S}{\partial B} \bigg|_{B=0}. \] (57)

Here, \( M_S \) is the spin contribution to the magnetization. For free (i.e. non-interacting) parabolic-band fermions in 3D, \( \chi_{\text{SMS}} \) reduces to the well-known Pauli spin susceptibility\cite{14}, i.e.
\[ \chi_{\text{SMS}}^{(0)} = \frac{g^2}{r_s} \left( \frac{9}{256\pi^3} \right)^{1/3} > 0, \] (58)
where we have used a Landé g-factor \( g = 2 \). Comparing Eq. (58) with Eq. (34), we find the very well-known result,
\[ \chi_{\text{SMS}}^{(0)} = -3 A_{\text{OMS}}. \] (59)

In summary, the condition for the occurrence of photon condensation in a 3DES, when the cavity electromagnetic field couples to matter degrees of freedom via the Zeeman coupling only, is:
\[ \chi_{\text{spin}}(q) > \frac{1}{4\pi}. \] (60)

**B. Combined orbital and Zeeman couplings**

In general, when both orbital and spin light-matter interactions are taken into account the total Hamiltonian is:
\[
\hat{H}_{A+B} = \hat{H} + \hat{H}_{ph} + \frac{\mu_B}{2} \sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{B}(r_i) \\
+ \sum_{i=1}^{N} \frac{e}{mc} \hat{A}(r_i) \cdot \hat{p}_i + \sum_{i=1}^{N} \frac{e^2}{2mc^2} \hat{A}^2(r_i). \tag{61}
\]

Following the same steps discussed in Sects. II and III A, one reaches the following condition for the occurrence of photon condensation in a 3DES:
\[ -\chi_{\hat{B}_{q,\sigma} + \hat{C}_{q,\sigma} + \hat{C}_{-q,\sigma}}(0) > \frac{1}{2 \hbar \Omega_q}. \] (62)
Now, the key point is that, in the absence of spin-orbit coupling, cross response functions vanish:
\[ \chi_{\hat{C}_{q,\sigma} + \hat{B}_{-q,\sigma}}(0) = \chi_{\hat{B}_{q,\sigma} + \hat{C}_{-q,\sigma}}(0) = 0. \] (63)

This is due to the following facts. Consider for example \( \chi_{\hat{C}_{q,\sigma} + \hat{B}_{-q,\sigma}}(0) \). We have\cite{14}
\[
\chi_{\hat{C}_{q,\sigma} + \hat{B}_{-q,\sigma}}(\omega) = -i \hbar \times \\
\times \lim_{\eta \to 0} \int_0^\infty d\tau \{\hat{C}_{q,\sigma}(\tau), \hat{B}_{-q,\sigma}^\dagger(\omega^{\text{in}} + i\eta)\}. \tag{64}
\]
Since the operators \( \hat{C}_{q,\sigma}(t) \) and \( \hat{B}_{-q,\sigma} \) have disjoint supports, the former acting on the spin degrees of freedom while the latter on the charge degrees of freedom, we have \( [\hat{C}_{q,\sigma}(t), \hat{B}_{-q,\sigma}] = 0 \). We therefore conclude that
\[
\chi_{\hat{B}_{q,\sigma} + \hat{C}_{q,\sigma} + \hat{C}_{-q,\sigma}}(0) = \chi_{\hat{B}_{q,\sigma} + \hat{C}_{-q,\sigma}}(0) \\
+ \chi_{\hat{B}_{q,\sigma} + \hat{C}_{q,\sigma}}(0). \tag{65}
\]

Using Eqs. (65), (24), and (54) inside Eq. (62), we find that the condition for occurrence of photon condensation is:
\[
2A_q^2 V \left[ \chi_{\text{orb}}(q) + \chi_{\text{spin}}(q) \right] q^2 > \hbar \omega_q, \tag{66}
\]
which, upon substitution of \( \omega_q = cq/\sqrt{\epsilon_\tau} \) and \( A_q = \sqrt{2\pi \hbar c^2/(V \omega_q \epsilon_\tau)} \), becomes
\[
\chi_{\text{orb}}(q) + \chi_{\text{spin}}(q) > \frac{1}{4\pi}. \tag{67}
\]

This is the most important result for 3DESs: in the absence of spin-orbit coupling in the matter degrees of freedom—or other microscopic mechanisms that are responsible for non-zero cross response function such as \( \chi_{\hat{B}_{q,\sigma} + \hat{C}_{-q,\sigma}}(0) \) and \( \chi_{\hat{B}_{q,\sigma} + \hat{C}_{q,\sigma}}(0) \)—the condition for the occurrence of photon condensation involves the sum of the orbital and spin transverse static response functions.

When electron-electron interactions are negligible (i.e. \( r_s \ll 1 \), the condition (67) for the occurrence of 3D photon condensation (i.e. formation of Condon domains) can be made more explicit. Indeed, consider for example the case of a non-interacting parabolic-band 3D Fermi gas. Using the long-wavelength expression (34) and (58) inside Eq. (67), we immediately see that photon condensation can occur in the absence of electron-electron interactions provided that
\[
r_s < \left( \frac{2}{3\pi} \right)^{1/3} \alpha^2. \tag{68}
\]
or, equivalently, provided that the electron density is sufficiently high,
\[ n > n_c = \frac{9\pi}{8a_0^6} \frac{1}{a_{1s}^3}. \]  
(69)

Unscreened current-current interactions at low temperatures under strong magnetic fields, which may result in non-Fermi-liquid behavior\(^{66}\), lead to the occurrence of long-range magnetic orbital order even at low densities\(^{53}\).

**IV. 2D PHOTON CONDENSATION**

In this Section, we consider the problem of a 2DES embedded in a quasi-2D cavity.

Similarly to the 3D case discussed above in Sect. II, we describe the 2DES with the jellium model Hamiltonian
\[ \hat{H}_2 \text{D} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} e_i (\| \hat{r}_{i,j} - \hat{r}_{i,j} \|), \]  
(70)

where \( \hat{r}_{i,j} \) and \( \hat{p}_{i,j} \) denote respectively the position and momentum operators of the \( i \)-th electron moving in the \( \hat{x} \)-\( \hat{y} \) plane. For future use, we introduce the 2D Fourier transforms of the density and paramagnetic (number) current operators:
\[ \hat{n}(q) = \sum_{i=1}^{N} e^{-i q \cdot \hat{r}_{i,j}}, \]  
(71)
\[ \hat{j}_p(q) = \frac{1}{2m} \sum_{i=1}^{N} (\hat{p}_{i,j} e^{-i q \cdot \hat{r}_{i,j}} + e^{-i q \cdot \hat{r}_{i,j}} \hat{p}_{i,j} \|), \]  
(72)

with the usual properties \( \hat{n}(-q) = \hat{n}^\dagger(q) \) and \( \hat{j}_p(-q) = \hat{j}_p^\dagger(q) \).

We consider a cavity with length \( L_z \) in the \( \hat{z} \) direction, satisfying the quasi-2D condition \( L_z \ll L_x, L_y \). The walls of the cavity in the \( \hat{z} \) direction are assumed to perfectly conducting. Accordingly, the tangential component of the magnetic field and the normal component of the magnetic field must vanish at the cavity boundaries\(^{61}\), \( z = \pm L_z/2 \). In addition, we impose periodic boundary conditions along the \( \hat{x} \) and \( \hat{y} \) directions. The vector potential, fulfilling all boundary conditions, is expressed as
\[ \hat{A}(r) = \sum_{q_{l},\sigma,n_z} A_{q_{l},\sigma,n_z} u_{q_{l},\sigma} \sin \left( \frac{\pi n_z}{L_z} \left( z + \frac{L_z}{2} \right) \right) \]
\[ \times (\hat{a}_{q_{l},\sigma,n_z} e^{i q \cdot \hat{r}_{l}} + \hat{a}_{q_{l},\sigma,n_z}^\dagger e^{-i q \cdot \hat{r}_{l}}). \]  
(73)

Here, \( n_z \) is an integer index, \( q_{l} = (2\pi n_x/L_x, 2\pi n_y/L_y) \) with \((n_x, n_y)\) relative integers, \( \sigma = 1, 2 \) is the polarization index, \( u_{q_{l},\sigma} \) is the linear polarization vector laying in the \( \hat{x} \)-\( \hat{y} \) plane, \( A_{q_{l},\sigma} = \sqrt{4\pi \hbar^2 / (L_x S \omega_{q_{l},\sigma} \epsilon_t)}, \) \( S = L_x L_y, \) \( \epsilon_t \) is the cavity relative dielectric constant, and \( \omega_{q_{l},n_z} = (\epsilon_t / \epsilon_r) \sqrt{q_{l}^2 + (\pi n_z / L_z)^2}. \) Similarly to the 3D case, the following properties hold true: \( \omega_{q_{l},n_z} = \omega_{q_{l},n_z}, \) \( u_{q_{l},\sigma} \) and \( u_{q_{l},\sigma}^\dagger \) are the momentum operators of the \( q_{l} \)-th electron in the \( \hat{x} \)-\( \hat{y} \) plane, fulfilling all boundary conditions along the \( \hat{x} \) and \( \hat{y} \) directions.

This needs to be compared with the 3D one in Eq. (7). Once again, the third and the fourth term in Eq. (74) are the paramagnetic and diamagnetic contributions, respectively. A constant term in Eq. (75) has been dropped, since below we will be only interested in energy differences. From now on, we will follows steps similar to those described in Sect. II. We will therefore mainly highlight differences between the 3D case discussed there and the 2D case discussed in this Section and cut short on the algebraic steps that are identical in the two cases. On purpose, and with notational abuse, we will denote by the same symbols quantities that in both cases have an identical physical meaning.

As in the 3D case, we are interested in the possible occurrence of a quantum phase transition to a photon condensate, and we therefore wish to make general statements about the ground state \( |\Psi\rangle \) of \( \hat{H}_A \), in the 2D thermodynamic limit \( N \rightarrow \infty, S \rightarrow \infty \), with constant \( n_{2D} = N/S \). In this limit, we can safely assume that \( |\Psi\rangle \) does not contain light-matter entanglement, i.e. we can take \( |\Psi\rangle = |\psi\rangle |\Phi\rangle \), where \( |\psi\rangle \) and \( |\Phi\rangle \) are matter and light states. The effective Hamiltonian for the photonic degrees of freedom is \( \hat{H}_{\text{ph}}[\psi] \equiv \langle \psi | \hat{H}_A | \psi \rangle \).

The order parameter for 2D photon condensation is \( \delta_{q_{l},\sigma,n_z} \equiv \langle \Phi | \hat{a}_{q_{l},\sigma,n_z} | \Phi \rangle \), which, at the putative QCP, is small.

Since the diamagnetic term in Eq. (74) is quadratic in \( \delta_{q_{l},\sigma,n_z} \), close to the QCP we can approximate the matter content in the diamagnetic term with its value in the absence of light-matter interactions. By further assuming, as in the 3D case, that the ground state of the 2DES in the absence of light-matter interactions is homogenous and isotropic, i.e. that \( \langle \psi | \hat{n}(q) | \psi \rangle = N \delta_{q_{l},a} \), the effective photon Hamiltonian can be written as
\[ \hat{H}_{\text{ph}}[\psi] = \langle \psi | \hat{H}_2 \text{D} | \psi \rangle + \hat{H}_\text{ph} + \hat{H}_p + \hat{H}_d, \]  
(76)
where the paramagnetic contribution is given by
\[
\hat{H}_p = \sum_{n_z} \sum_{q_i,\sigma} (-1)^{n_z} \frac{e}{\epsilon} A^{(2D)}_{q_i,n_z} \left[ \hat{a}_{q_i,\sigma,n_z} \hat{u}_{q_i,\sigma} \cdot \hat{j}_p(-q_i) \right] + \hat{a}_{q_i,\sigma,n_z}^{\dagger} \hat{u}_{q_i,\sigma} \cdot \hat{j}_p(q_i)
\]
(77)
and the diamagnetic one by
\[
\hat{H}_d = \sum_{n_z,n'_z} \sum_{q_i,\sigma} (-1)^{n_z+n'_z} \frac{2e^2}{\epsilon^2} A^{(2D)}_{q_i,n_z} A^{(2D)}_{q_i,n'_z} \times \left( \hat{a}_{q_i,\sigma,n_z}^{\dagger} + \hat{a}_{q_i,\sigma,n'_z}^{\dagger} \right)(\hat{a}_{q_i,\sigma,n_z}^{\dagger} \hat{a}_{q_i,\sigma,n'_z}^{\dagger} + \hat{a}_{q_i,\sigma,n_z} + \hat{a}_{q_i,\sigma,n'_z}) .
\]
(78)

In Eq. (77) we have introduced
\[
\hat{j}_p(q_i) \equiv \langle \psi | \hat{j}_p(q_i) | \psi \rangle .
\]
(79)

For future use, we also introduce \( J_{q_i,\sigma} = \hat{u}_{q_i,\sigma} \cdot \hat{j}_p(q_i) \).

As we have seen in Sect. II A, point v), in order to calculate the energy functional, it is sufficient to evaluate the expectation value of the effective Hamiltonian \( \hat{H}^{\text{eff}}[\psi] \) on a trial photonic wavefunction of the form \( |\alpha'\rangle \equiv \otimes_{q_i,\sigma} |\alpha_{q_i,\sigma}\rangle \), namely on a tensor product of coherent states of the \( \hat{a}_{q_i,\sigma} \) operators, i.e. \( \hat{a}_{q_i,\sigma} |\alpha'\rangle = \alpha_{q_i,\sigma} |\alpha'\rangle \). This procedure corresponds to replacing the photonic operators in Eq. (9) with \( c \)-numbers, \( \hat{a}_{q_i,\sigma,n_z} \rightarrow \alpha_{q_i,\sigma,n_z} \). Carrying out this procedure we find:

\[
E[\{\alpha_{q_i,\sigma,n_z}\}, \psi] = \langle \psi | \hat{H}_{2D} | \psi \rangle + \sum_{n_z} \sum_{q_i,\sigma} (-1)^{n_z} \sqrt{2D} \left[ \alpha_{q_i,\sigma,n_z} J_{q_i,\sigma} + \text{c.c.} \right] - \frac{e^2}{\epsilon^2} \sum_{n_z} \sum_{q_i,\sigma} \sum_{n'_z} \frac{A^{(2D)}_{q_i,n_z} A^{(2D)}_{q_i,n'_z}}{m \omega_{q_i,n_z} \omega_{q_i,n'_z}} \times \left( \alpha_{q_i,n_z}^{\dagger} + \alpha_{q_i,n'_z} \right) \left( \alpha_{q_i,n_z}^{\dagger} \alpha_{q_i,n'_z} + \alpha_{q_i,n_z}^{\dagger} + \alpha_{q_i,n'_z} \right) + \sum_{q_i,\sigma,n_z} \hbar \omega_{q_i,n_z} \alpha_{q_i,\sigma,n_z}^{\dagger} \alpha_{q_i,\sigma,n_z}^{\dagger} \alpha_{q_i,\sigma,n_z} \alpha_{q_i,\sigma,n_z}^{\dagger} .
\]
(80)

where \( D \equiv \pi^2 \hbar^2 / (L_x S \epsilon) \). (As discussed in Sect. II, if one is interested in finding the exact photonic eigenstate, a different and much more cumbersome root need to be followed. This is described at length in Appendix C. The end result, from the point of view of energy differences, is identical to the one that one obtains using Eq. (80).) Note that all the modes with even \( n_z \) are completely decoupled from matter degrees of freedom. For these modes, the minimum of the energy functional is trivially obtained at \( \alpha_{q_i,n_z} = 0 \). Hence, we can completely disregard even values of \( n_z \); from now on, the index \( n_z \) will take only odd values.

It turns out to be useful to express the energy functional \( E[\{\alpha_{q_i,\sigma,n_z}\}, \psi] \) in terms of \( \{z_{q_i,\sigma,n_z}\} = \{z_{q_i,\sigma,n_z}, y_{q_i,\sigma,n_z}\} \) where \( x_{q_i,\sigma,n_z} = \langle \alpha_{q_i,\sigma,n_z}, \alpha_{q_i,\sigma,n_z}^{\dagger} \rangle / 2 \) and \( y_{q_i,\sigma,n_z} = \langle \alpha_{q_i,\sigma,n_z} - \alpha_{q_i,\sigma,n_z}^{\dagger} \rangle / 2i \).

\[
\hbar \omega_{q_i,n_z} x_{q_i,\sigma,n_z} + \frac{2N}{m} \sum_{n'_z} \frac{g(n_z-1)/2(q_i)g(n'_z-1)/2(q_i) x_{q_i,\sigma,n'_z}}{2} = -g(n_z-1)/2(q_i) J_{q_i,\sigma} \,.
\]
(82)

where \( n_z \) is odd.

The first equation is trivially solved by \( y_{q_i,\sigma,n_z} = 0 \). From Eq. (82), we find that the optimal value of \( \{x_{q_i,\sigma,n_z}\} \) is the solution of a linear system in terms of \( J_{q_i,\sigma} \), and it is non-trivial (i.e. \( x_{q_i,\sigma,n_z} \neq 0 \)) only if \( J_{q_i,\sigma} \) takes a finite value. Using the stiffness theorem\(^{14}\), one has, up to second order in \( J_{q_i,\sigma} \),
Similarly to the 3D case, we now express the response function \( \chi_{uq_{||},\sigma} \hat{J}_p(q_{||}),uq'_{||},\sigma \hat{J}_q(-q_{||}')(0) \) in terms of the physical current-current response tensor, which contains a paramagnetic as well as a diamagnetic contribution:

\[
\chi_{i,k}(q_{||},0) = \frac{n_{2D}}{m} \delta_{i,k} + \chi_{p,i}(q_{||}) \hat{J}_p(-q_{||})(0) .
\]  

(85)

Since we are considering a homogeneous and isotropic system, the rank-2 tensor \( \chi_{i,k}(q_{||},0) \) can be decomposed in terms of the longitudinal, \( \chi^{L}_{i,k}(q_{||},0) \), and transverse, \( \chi^{T}_{i,k}(q_{||},0) \), current-current response functions:

\[
\chi^{L}_{i,k}(q_{||},0) = \chi^{L}_{i}(q_{||},0) \frac{q_{i,k}}{q_{||}} + \chi^{T}_{i}(q_{||},0) \left( \delta_{i,k} - \frac{q_{i,k}q_{i,k}}{q_{||}^2} \right) .
\]  

(86)

Using Eqs. (85)-(86) in Eq. (84), we finally find:

\[
\chi_{uq_{||},\sigma} \hat{J}_p(q_{||}),uq'_{||},\sigma \hat{J}_q(-q_{||}')(0) = \left[ \chi^{L}_{i}(q_{||},0) - \frac{n_{2D}}{m} \right] .
\]  

(87)

We now calculate the energy difference between a generic phase with \( \{ z_{q_{||},\sigma} , \psi \} \) and the normal phase with \( \{ z_{q_{||},\sigma} = 0, \psi \} \):

\[
E[\{ z_{q_{||},\sigma} , \psi \}] - E[\{ z_{q_{||},\sigma} = 0, \psi \}] = \sum_{q_{||},\sigma} \left\{ \frac{1}{2S} \left[ \frac{n_{2D}}{m} - \chi^{L}_{i}(q_{||},0) \right] \langle \hat{J}_{q_{||},\sigma} \hat{J}_{-q_{||},\sigma} \rangle + \sum_{\text{odd } n_z} \left[ \frac{\hbar \omega_{n_z} (x_{q_{||},\sigma,n_z} x_{-q_{||},\sigma,n_z} + y_{q_{||},\sigma,n_z} y_{-q_{||},\sigma,n_z})}{m} + \frac{2N}{m} \sum_{\text{odd } n_z'} g(n_{z'}-1/2(q_{||})g(n_{z'}-1/2(q_{||})x_{q_{||},\sigma,n_z} x_{-q_{||},\sigma,n_z} + 2J_{q_{||},\sigma} g(n_{z'}-1/2(q_{||})x_{q_{||},\sigma,n_z} x_{-q_{||},\sigma,n_z} ) \right) \right\} .
\]  

(88)

Minimizing this quantity with respect to \( \hat{J}_{q_{||},\sigma} \), we obtain the following result:

\[
\hat{J}_{q_{||},\sigma} = 2S \left[ \chi^{L}_{i}(q_{||},0) - \frac{n_{2D}}{m} \right] \times \sum_{\text{odd } n_z} g(n_{z'}-1/2(q_{||})x_{q_{||},\sigma,n_z} .
\]  

(89)

Replacing Eq. (89) in Eq. (88), we find that the energy difference, minimized with respect to the matter wave-function and denoted by \( E[\{ z_{q_{||},\sigma} \}] \equiv \min_{\psi} \{ E[\{ z_{q_{||},\sigma} , \psi \}] - E[\{ z_{q_{||},\sigma} = 0, \psi \}] \} \) taking the following quadratic form:

\[
E[\{ z_{q_{||},\sigma} \}] = \sum_{q_{||},\sigma} \sum_{\text{odd } n_z} \left[ \frac{\hbar \omega_{n_z} (x_{q_{||},\sigma,n_z} x_{-q_{||},\sigma,n_z} + y_{q_{||},\sigma,n_z} y_{-q_{||},\sigma,n_z})}{m} + \frac{2S}{m} \chi^{L}_{i}(q_{||},0) \sum_{\text{odd } n_z'} g(n_{z'}-1/2(q_{||})x_{q_{||},\sigma,n_z} x_{-q_{||},\sigma,n_z} ) \right] .
\]  

(90)

which can be written compactly as

\[
E[\{ z_{q_{||},\sigma} \}] = \sum_{q_{||},\sigma} z^{L}_{q_{||},\sigma} \mathcal{M}_{q_{||}} z_{q_{||},\sigma} .
\]  

(91)

Here, \( \mathcal{M}_{q_{||}} \) is a symmetric matrix, which is independent of the polarization \( \sigma \). For photon condensation to occur we need the photon condensate phase to be energetically favored with respect to the normal phase. This occurs, at a given \( q_{||} \) and \( \sigma \), if at least one eigenvalue \( \lambda_{q_{||},n} \) of \( \mathcal{M}_{q_{||}} \) is negative. For each \( q_{||} \), the determinant \( \Delta_{q_{||}} = \text{Det}(\mathcal{M}_{q_{||}}) \) of the quadratic form in Eq. (91) can be written as:

\[
\Delta_{q_{||}} = \left[ 1 + \chi^{L}_{i}(q_{||},0) \frac{2\pi e^2}{\ell_{c}^2} \tanh \left( \frac{q_{||} L_{z}}{2} \right) \right] \times \prod_{\text{odd } n_z} \left( \hbar \omega_{n_z} \right) .
\]  

(92)

Using the relation \( \Delta_{q_{||}} = \prod_{n} \lambda_{q_{||},n} \) between eigenvalues and determinant, and noting that the second line in Eq. (92) is positive definite, we conclude that, in order to have at least one negative eigenvalue, the following inequality needs to be satisfied:

\[
- \chi^{L}_{i}(q_{||},0) \frac{2\pi e^2}{\ell_{c}^2} \tanh \left( \frac{q_{||} L_{z}}{2} \right) > 1 .
\]  

(93)

Let us consider first the case of zero photon momentum, \( q_{||} = 0 \). In this case, the condition (93) for the occurrence of the photon condensation reduces to

\[
- \chi^{L}_{i}(0,0) \frac{\pi e^2 L_{z}}{c^2} = 0 .
\]  

(94)

As discussed in Sect. II, in systems with no long-range order, \( \lim_{q_{||} \to 0} \chi^{L}_{i}(q_{||},0) = 0 \). Such diamagnetic sum-rule then yields an absurd (0 > 1), expressing the no-go
theorem for the occurrence of photon condensation in a spatially-uniform cavity field.

As in the 3D case, we now introduce the 2D non-local orbital susceptibility

\[ \chi_{\text{orb}}(q_\parallel) = -\frac{e^2}{c^2} \frac{\chi_{\text{orb}}^4(q_\parallel, 0)}{q_\parallel^2}. \]  

(95)

Introducing this definition in Eq. (93), we finally obtain the condition for the occurrence of photon condensation in a 2DES:

\[ \chi_{\text{orb}}(q_\parallel) > \frac{1}{2 \pi q_\parallel \tanh(q_\parallel L_z/2)}. \]  

(96)

This is the most important result of this Section.

As in the 3D case discussed in Sect. II, the criterion in Eq. (96) emphasizes that the route towards the discovery of photon condensate states relies entirely on the knowledge of the orbital magnetic response function \( \chi_{\text{orb}} \) of ESs.

A. Discussion

We invite the reader to compare the 2D criterion (96) with the 3D one in Eq. (33). The two criteria display a dramatic qualitative difference. While in the 3D case photon condensation can occur also in the quasi-homogeneous \( q \to 0 \) limit (provided that Eq. (33) is satisfied in that limit), in the 2D case the right-hand side of Eq. (96) diverges as \( 1/q_\parallel^2 \) in the \( q_\parallel \to 0 \) limit. On the other hand, the left-hand side is usually finite in the same limit. In order to satisfy the inequality, we need to hunt for 2DESs whose OMS is not only positive (orbital paramagnets), but also diverging in the quasi-homogeneous \( q_\parallel \to 0 \) limit.

In 1991, Vignale demonstrated\(^{63} \) that when the Fermi energy is sufficiently close to a saddle point of the band structure, non-interacting 2DESs is a periodic potential are such that

\[ \lim_{q_\parallel \to 0} \chi_{\text{orb}}(q_\parallel) = +\infty. \]  

(97)

The divergence is due to a diverging density of states at the saddle point. The positive sign is an exquisite quantum effect, which is easy to understand. Near a saddle point the semiclassical approximation breaks down, and tunneling from one quasi-classical trajectory to the neighboring one occurs. Due to tunneling, electrons rotate around the saddle point in a direction opposite to the classical direction of rotation and the induced magnetic moment is reversed. We emphasize that the positive sign (i.e. paramagnetic character of the response) for non-interacting electrons is surprising, in view of the fact that non-interacting parabolic-band ESs are characterized by a negative OMS (Landau diamagnetism).

More recently, the OMS of the 2DES in graphene has received some attention. In the massless Dirac fermion continuum model, the 2DES in graphene is strongly diamagnetic,\(^{64} \) \( \chi_{\text{OMS}} \propto -\delta(E_F) \), when the Fermi energy lies at the Dirac point and electron-electron interactions are neglected. On the other hand, the lattice contribution\(^{65} \) to the OMS beyond the massless Dirac fermion continuum model is positive for a wide range of Fermi energies and diverges at the saddle point, in agreement with Ref. 63. Electron-electron interactions display the same tendency and, in the massless Dirac fermion continuum model, turn the 2DES in graphene into an orbital paramagnet\(^{66} \) when the Fermi energy is away from the Dirac point.

The OMS of multi-band systems with a pair of Dirac points interpolating between honeycomb and dice lattices has been studied by Raoux et al.\(^{67} \). Orbital paramagnetic behavior, stemming from a topological Berry phase changing continuously from \( \pi \) (graphene) to 0 (dice), has been found in this work even at Dirac crossings. A novel geometric contribution to the OMS has been shown to give rise to very strong orbital paramagnetism in models with flat bands\(^{68} \). It is therefore very natural to expect the same behavior also in twisted bilayer graphene close to the magic angle\(^{69} \).

Other instances of orbital paramagnetic behavior have been found in normal metals in proximity to a superconductor\(^{70} \) and, much more recently, in a non-interacting 2DES in the presence of Rashba spin-orbit coupling and a perpendicular static magnetic field\(^{71} \). In particular, in their model, Nataf et al.\(^{72} \) showed that Eq. (97) is satisfied every time that two Landau levels with opposite helicity cross.

V. SUMMARY AND CONCLUSIONS

In summary, we have derived criterions for the occurrence of “superradiant” (i.e. photon condensate) states in electrons system coupled to a spatially-varying electromagnetic field. In three spatial dimensions, the criterion, reported in Eq. (33), is identical to the Condon criterion for the occurrence of magnetic domains.

The Zeeman coupling of the electronic spin degrees of freedom to the cavity field leads to the criterion in Eq. (67) and implies that in a real material one needs to know both orbital and spin non-local response functions to make quantitative predictions on the occurrence of a photon condensate phase.

Finally, the condition for the occurrence of photon condensates in 2D systems embedded in quasi-2D cavities is reported in Eq. (96) and poses severe bounds on the observability of this phenomenon. We have indeed shown that in order to satisfy this criterion in the quasi-homogeneous limit, one needs to hunt for materials with a divergent orbital paramagnetic character. A few possibilities have been discussed in Sect. IV A.

The prediction of the possible coexistence in strongly
correlated materials of exotic orders and photon condensate states requires accurate microscopic theories of the non-local orbital and spin response functions that take into account the role of electron-electron interactions.

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As this manuscript was being finalized for publication, we learned about related work by Guerci et al., where a similar criterion for the occurrence of a superradiant phase transition in a cavity with a single mode was obtained. It is a great pleasure to thank Daniele Guerci, Pascal Simon, and Christophe Mora for sharing their results with us prior to publication.

Appendix A: Disentangling light and matter

In this Appendix, we show that, in the thermodynamic $N \to \infty$ limit, it is permissible to assume a factorized ground state of the form

$$|\Psi\rangle = |\psi\rangle |\Phi\rangle \ .$$

(A1)

We begin by defining the electron-photon Hamiltonian $\hat{H}_{el-ph} \equiv \hat{H}_{el-ph}^{(1)} + \hat{H}_{el-ph}^{(2)}$, where

$$\hat{H}_{el-ph}^{(1)} \equiv \sum_{i=1}^{N} \frac{e}{mc} \hat{A}(\hat{r}_i) \cdot \hat{p}_i$$

(A2)

and

$$\hat{H}_{el-ph}^{(2)} \equiv \sum_{i=1}^{N} \frac{e^2}{2mc^2} \hat{A}^2(\hat{r}_i) \ .$$

(A3)

The photon Hamiltonian $\hat{H}_{ph}$ has been defined in the main text. Let us split the matter Hamiltonian into the sum of kinetic and potential terms, i.e. we write $\hat{H} \equiv \hat{H}_K + \hat{H}_V$, where:

$$\hat{H}_K \equiv \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m}$$

(A4)

and

$$\hat{H}_V \equiv \frac{1}{2} \sum_{i \neq j} v(\hat{r}_i - \hat{r}_j) \ .$$

(A5)

In order to guarantee the correct thermodynamic limit, $\hat{H}_{el-ph}, \hat{H}_{ph}, \hat{H}$ must scale extensively with $N$. This implies that photonic and electronic operators must in turn scale properly with $N$ in this limit. Let us discuss this fact explicitly. We begin by considering the photon Hamiltonian $\hat{H}_{ph}$. We denote by the symbol $N_{\text{modes}}$ the number of “non-negligible” modes, i.e. modes that cannot be neglected in the thermodynamic limit. The photon term $\hat{H}_{ph}$ can have an extensive scaling with $N$ in two different cases:

- $N_{\text{modes}}$ is an intensive quantity (i.e. $N_{\text{modes}}$ does not scale with $N$). In this case, the operator $\hat{a}_{q_0, \sigma}$ characterized by a given $q_0$ acquires a macroscopic occupation $\hat{a}_{q_0, \sigma} \sim \sqrt{N}$,

- $N_{\text{modes}}$ is an extensive quantity, while the occupation number of each mode $\hat{a}_{q_0, \sigma}^\dagger \hat{a}_{q_0, \sigma}$ is not macroscopic, i.e. $\hat{a}_{q, \sigma} \sim \sqrt{N/N_{\text{modes}}} \sim 1$. In the following we are going to prove that this case it is not relevant for the occurrence of photon condensation. Heuristically, it is known that bosons condense populating a single energetically favored mode rather than assume a uniformly distributed equilibrium distribution, which is rather typical of fermions.
Let us prove, by inspection, that this case is not relevant for photon condensation. The paramagnetic electron-photon interaction $\hat{H}_{\text{el-ph}}^{(1)}$ scales like:

$$\hat{H}_{\text{el-ph}}^{(1)} \sim \sum_q A_q \hat{a}_{q,\sigma} \hat{j}_p(q).$$

(A6)

In the case of interest $A_q \hat{a}_{q,\sigma} \sim 1/\sqrt{N_{\text{modes}}}$, while $\sum_q \hat{j}_p(q)$ is extensive in $N$, giving a vanishing paramagnetic contribution in the thermodynamic limit $\hat{H}_{\text{el-ph}}^{(1)} \sim N/\sqrt{N_{\text{modes}}} \sim \sqrt{N}$. The paramagnetic electron-photon interaction $\hat{H}_{\text{el-ph}}^{(1)}$ is the energetic term responsible for lowering the energy of the photon condensate phase giving rise to the phase transition. The fact that this term can be asymptotically neglected means that in this case the phase transition can be excluded.

Since we are interested in photon condensation we consider only the former case, i.e. only a finite number of modes acquire a macroscopic occupation number, assuming that $N_{\text{modes}}$ is an intensive quantity. Provided that such scaling of the photonic operator is assumed, Hamiltonians in Eqs.(A2,A3) are extensive. Let us now focus on electronic operators. Being a sum of $N$ independent terms, $\hat{H}_K$ (A4) is explicitly extensive. Conversely, $\hat{H}_V$ in Eq.(A5) contains Coulomb interactions, which are composed by $N^2$ terms. Nevertheless, it is possible to show that, due to the ground state equilibrium condition, i.e. charge neutrality, $\hat{H}_V$ scales with $N$.

Below, we therefore work with the rescaled operators $\hat{H}/N$, $\hat{H}_{\text{ph}}/N$, and $\hat{H}_{\text{el-ph}}/N$ which are well defined in the thermodynamic limit, $N \to \infty$.

In order to prove Eq. (A1) we will show that in the limit $N \to \infty$

$$\left[ \frac{\hat{H}}{N}, \frac{\hat{H}_{\text{el-ph}}}{N} \right] \to 0,$$

(A7)

and

$$\left[ \frac{\hat{H}_{\text{ph}}}{N}, \frac{\hat{H}_{\text{el-ph}}}{N} \right] \to 0.$$

(A8)

The left-hand side of Eq. (A7) for the kinetic Hamiltonian reads as following:

$$\left[ \frac{\hat{H}_K}{N}, \frac{\hat{H}_{\text{el-ph}}^{(1)}}{N} \right] = \sum_{i=1}^N \frac{\hbar}{2cm^2N^2} \left\{ \hat{p}_i \cdot q - \sum_{q,\sigma} A_q (\hat{a}_{q,\sigma} e^{iq \cdot \hat{r}_i} - \hat{a}^\dagger_{q,\sigma} e^{-iq \cdot \hat{r}_i}) u_{q,\sigma} \cdot \hat{p}_i + \sum_{q,\sigma} A_q (\hat{a}_{q,\sigma} e^{iq \cdot \hat{r}_i} - \hat{a}^\dagger_{q,\sigma} e^{-iq \cdot \hat{r}_i}) \hat{p}_i \cdot q u_{q,\sigma} \cdot \hat{p}_i \right\}. $$

(A9)

This quantity vanish as $1/N$, since $\sum_{i=1}^N$ is the only extensive factor, while terms like $\sum_q A_q \hat{a}_{q,\sigma}$ are intensive in $N$. This commutator is at least linear in $q$ consistently with the fact that this term does not appear in the correspondent proof for a uniform vector field.

The left-hand side of Eq. (A7) for the potential Hamiltonian reads as following:

$$\left[ \frac{\hat{H}_V}{N}, \frac{\hat{H}_{\text{el-ph}}^{(1)}}{N} \right] = \frac{1}{N^2} \frac{1}{2} \sum_{i \neq j} v(\hat{r}_i - \hat{r}_j), \sum_{j=1}^N \frac{e}{mc} \hat{A}(\hat{r}_j) \cdot \hat{p}_j \right] .$$

(A10)

Using that $[f(\hat{r}_i), \hat{p}_j] = \delta_{i,j} i \hbar \nabla_{\hat{r}_i} f(\hat{r}_i)$ and introducing the Coulomb force $\hat{F}_{i,j}^C = -\nabla_{\hat{r}_i} v(\hat{r}_i - \hat{r}_j)/2$ we get:

$$\left[ \frac{\hat{H}_V}{N}, \frac{\hat{H}_{\text{el-ph}}^{(1)}}{N} \right] = - \sum_{i=1}^N \frac{i e \hat{A}(\hat{r}_i)}{mcN^2} \sum_{j \neq i} \hat{F}_{i,j}^C.$$

(A11)

$\hat{F}_{i,j}^C \equiv \sum_{j \neq i} \hat{F}_{i,j}^C$ is the total force acting on the $i$–th particle. Even though this sum formally contains $N$ terms, we are going to prove that, close to the equilibrium condition, the total force acting on the $i$–th particle is at most an extensive quantity. Heuristically, this come from the fact that at equilibrium, the total force acting on the $i$–th particle is zero by definition. Since we are not interested in the factorization for all states, but only for the ground state $|\Psi\rangle$ we project the commutator on the ground state, considering the quantity $\langle \Psi | [\hat{H}_V/N, \hat{H}_{\text{el-ph}}^{(1)}/N] | \Psi \rangle$. Even though $\sum_{j \neq i} \hat{F}_{i,j}^C$ contains $N^2$ terms, this quantity, evaluated on the ground state, is extensive at most, since it is the spatial derivative of $\hat{H}_V$, which is imposed to be extensive to guarantee the correct thermodynamic limit.
\[
\hat{H} \equiv \frac{\hat{H}_{el}^{(2)} - \hat{H}_{el-ph}^{(2)}}{N} = \sum_{i=1}^{N} \frac{e^2}{2mc^2} \sum_{q,\sigma} \left\{ \hat{p}_i \cdot q \hat{A}(\hat{r}_i) \cdot u_{q,\sigma} \hat{A}_q \left( \hat{a}_{q,\sigma} e^{iq \cdot r_i} - \hat{a}_{q,\sigma}^\dagger e^{-iq \cdot r_i} \right) + \right. \\
\left. + \sum_{q,\sigma} \hat{A}(\hat{r}_i) \cdot u_{q,\sigma} \hat{A}_q \left( \hat{a}_{q,\sigma} e^{iq \cdot r_i} - \hat{a}_{q,\sigma}^\dagger e^{-iq \cdot r_i} \right) \hat{p}_i \cdot q \right\}.
\]

Again, this quantity scales like $1/N$, since we assume that $\sum_{i=1}^{N}$ is the only extensive factor, while terms like $\sum_{q} A_q \hat{a}_{q,\sigma}$ and $\hat{A}(\hat{r}_i)$ are intensive in $N$.

In order to derive Eq. (A8) it is convenient to recast the light matter interaction as a function of the paramagnetic current $\hat{j}_p(r)$ and the density $\hat{n}(r)$ which are defined as following:

\[
\hat{n}(r) = \sum_{i=1}^{N} \delta(\hat{r}_i - r), \\
\hat{j}_p(r) = \frac{1}{2m} \sum_{i=1}^{N} \left[ \hat{p}_i \delta(\hat{r}_i - r) + \delta(\hat{r}_i - r) \hat{p}_i \right].
\]

Exploiting this definition we can recast Eq. (A15) as:

\[
\hat{H}_{el-ph}^{(1)} = \frac{\hat{e}}{c} \sum_{q} \hat{j}_p(r) \cdot A(r), \\
\hat{H}_{el-ph}^{(2)} = \frac{e^2}{2mc^2} \int dr \hat{n}(r) \hat{A}^2(r).
\]

Exploiting the commutator $[\hat{a}_{q,\sigma}, \hat{a}_{q',\sigma'}^\dagger] = \delta_{q,q'} \delta_{\sigma,\sigma'}$, we can rewrite the left-hand side of Eq. (A8) as:

\[
\left[ \frac{\hat{H}_{ph}}{N}, \frac{\hat{H}_{el-ph}^{(1)}}{N} \right] = \sum_{q,\sigma} \hbar \omega_q \left\{ \frac{e}{c} \int dr \hat{j}_p(r) \cdot u_{q,\sigma} \hat{A}_q \left( \hat{a}_{q,\sigma} e^{iq \cdot r} - \hat{a}_{q,\sigma}^\dagger e^{-iq \cdot r} \right) \right\},
\]

\[
\left[ \frac{\hat{H}_{ph}}{N}, \frac{\hat{H}_{el-ph}^{(2)}}{N} \right] = \sum_{q,\sigma} \hbar \omega_q \left\{ \frac{e}{2mc} \int dr \hat{n}(r) \hat{A}(r) \cdot u_{q,\sigma} \hat{A}_q \left( \hat{a}_{q,\sigma} e^{iq \cdot r} - \hat{a}_{q,\sigma}^\dagger e^{-iq \cdot r} \right) \right\}.
\]

Appendix B: Disentangling light and matter in the Zeeman coupling case

In this Appendix, we show that, in the thermodynamic $N \to \infty$ limit, it is allowed to assume a factorized ground state of the form

\[
|\Psi\rangle = |\psi\rangle |\Phi\rangle,
\]

also when a Zeeman electron-photon interaction is taken into account.

We begin by defining the electron-photon Hamiltonian $\hat{H}_{el-ph}$.

\[
\hat{H}_{el-ph} = \frac{g \mu_B}{2} \sum_{i=1}^{N} \hat{A}_i \cdot \mathbf{B}(\mathbf{r}_i),
\]

where coefficients are defined in the main text. The electron Hamiltonian $\hat{H}$ and the photon Hamiltonian $\hat{H}_{ph}$ have been defined in the main text. We remind the explicit form of the magnetic field, $\mathbf{B}(\mathbf{r}) = \sum_{q,\sigma} A_q \mathbf{u}_{T,q,\sigma} \left( \hat{a}_{q,\sigma} e^{iq \cdot r} - \hat{a}_{q,\sigma}^\dagger e^{-iq \cdot r} \right)$. Again, in order to assure thermodynamic consistency we assume that a finite number of relevant $q$ (i.e. a
number which does not scale with \( N \) acquires a macroscopic occupation \( \hat{a}_{q_0,\sigma} \sim \sqrt{N} \). Since the electron Hamiltonian does not depends on the spin \( \vec{\sigma} \), we have \([\hat{H}/N, \hat{H}_{\text{el-ph}}]/N = 0\).

Hence, in order to prove Eq. (B1) we will prove that, the limit \( N \to \infty \)

\[
\left[ \frac{\hat{H}_{\text{ph}}}{N}, \frac{\hat{H}_{\text{el-ph}}}{N} \right] \to 0. \tag{B3}
\]

In order to derive Eq. (B3) it is convenient to recast electron-photon Hamiltonian \( \hat{H}_{\text{el-ph}} \) as a function of the spin density \( \hat{S}(r) \) which is defined as following:

\[
\hat{S}(r) = \sum_{i=1}^{N} \hat{\sigma}_i \delta(\vec{r}_i - \vec{r}). \tag{B4}
\]

Using this definition we can recast Eq. (B2) as:

\[
\hat{H}_{\text{el-ph}} = \frac{g_{\mu B}}{2} \int d\vec{r} \hat{S}(r) \cdot \vec{B}(r). \tag{B5}
\]

Exploiting the commutator \([\hat{a}_{q,\sigma}, \hat{a}_{q',\sigma'}^\dagger] = \delta_{q,q'} \delta_{\sigma,\sigma'}\), we can rewrite the left-hand side of Eq. (B3) as:

\[
\left[ \frac{\hat{H}_{\text{ph}}}{N}, \frac{\hat{H}_{\text{el-ph}}}{N} \right] = -\sum_{q,\sigma} \frac{i \hbar q \sigma g_{\mu B}}{2N^2} \left\{ \int d\vec{r} \hat{S}(r) \cdot \vec{u}_{\pi,\sigma} A_q \left( \hat{a}_{q,\sigma} e^{iq \vec{r}} + \hat{a}_{q,\sigma}^\dagger e^{-iq \vec{r}} \right) \right\}. \tag{B6}
\]

Again, this quantities scale like \( 1/N \), since \( \int d\vec{r} \hat{S}(r) \sim N \), while \( \sum_{q,\sigma} \sim 1 \) and \( A_q \hat{a}_{q,\sigma} \sim 1 \).

### Appendix C: Proof of Eq. (80)

The Hamiltonian in Eq. (76) is a quadratic form of the photonic fields. We now carry out a suitable transformation, switching from the bosonic operators \( \hat{a}_{q_1,\sigma, n_z} \) and \( \hat{a}_{q_1,\sigma, n_z}^\dagger \) with odd \( n_z \) to new bosonic operators \( \hat{b}_{q_1,\sigma, j} \) and \( \hat{b}_{q_1,\sigma, j}^\dagger \) with integer \( j \). Bosonic operators \( \hat{a}_{q_1,\sigma, n_z} \) and \( \hat{a}_{q_1,\sigma, n_z}^\dagger \) with even mode index \( n_z \) are decoupled from matter degrees of freedom. The Bogoliubov transformation reads as following:

\[
\hat{b}_{q_1,\sigma, j} = \sum_{\ell} [X_{j,\ell}(q_{||}) \hat{a}_{q_1,\sigma, 2\ell+1} + Y_{j,\ell}(q_{||}) \hat{a}_{q_1,\sigma, 2\ell+1}^\dagger], \tag{C1}
\]

with \( \ell, j \) integers. Applying the Hermitian conjugation to the expression above and replacing \( q_{||} \to -q_{||} \), one has

\[
\hat{b}_{q_1,\sigma, j}^\dagger = \sum_{\ell} [Y_{j,\ell}^*(q_{||}) \hat{a}_{q_1,\sigma, 2\ell+1} + X_{j,\ell}(q_{||}) \hat{a}_{q_1,\sigma, 2\ell+1}^\dagger]. \tag{C2}
\]

For every \( q_{||}, \sigma \), we can therefore write the Bogoliubov transformation in the following compact form

\[
\begin{bmatrix} \{ \hat{b}_{q_1,\sigma, j} \} \\ \{ \hat{b}_{q_1,\sigma, j}^\dagger \} \end{bmatrix} = \begin{bmatrix} X(q_{||}) & Y(q_{||}) \\ Y^*(q_{||}) & X^*(-q_{||}) \end{bmatrix} \begin{bmatrix} \{ \hat{a}_{q_1,\sigma, 2\ell+1} \} \\ \{ \hat{a}_{q_1,\sigma, 2\ell+1}^\dagger \} \end{bmatrix}. \tag{C3}
\]

It acts only on the photon modes with odd mode index and has a trivial dependence on the polarization \( \sigma \). For this reason, we have omitted polarization labels from the Bogoliubov transformation matrices \( X(q_{||}) \) and \( Y(q_{||}) \).

We would like to find \( X(q_{||}) \) and \( Y(q_{||}) \) such that:

\[
\hat{H}_{\text{ph}} + \hat{H}_{\text{d}} = \sum_{q_1,\sigma} \sum_{\text{even } n_z} \hbar \omega_{q_1,n_z} \left( \hat{a}_{q_1,\sigma, n_z}^\dagger \hat{a}_{q_1,\sigma, n_z} + \frac{1}{2} \right) + \sum_j \hbar \Omega_{q_{||},j} \left( \hat{b}_{q_{||},\sigma,j}^\dagger \hat{b}_{q_{||},\sigma,j} + \frac{1}{2} \right), \tag{C4}
\]

with suitable \( \Omega_{q_{||},j} \). Notice that, differently from the main text, we have restored the vacuum contribution. If (C4) holds true, one has

\[
[\hat{H}_{\text{ph}} + \hat{H}_{\text{d}}, \hat{b}_{q_{||},\sigma,j}] = -\hbar \Omega_{q_{||},j} \hat{b}_{q_{||},\sigma,j} \tag{C5}.
\]
Using Eq. (C1) we can write Eq. (C5) as

\[
\sum_{\ell} \left[ \hat{H}_{ph} + \hat{H}_d, X_{j,\ell}(q_\parallel) \hat{a}_{q_\parallel, \sigma, 2\ell+1} + Y_{j,\ell}(q_\parallel) \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1} \right] \\
= -\hbar \Omega_{q_\parallel,j} \sum_{\ell} X_{j,\ell}(q_\parallel) \hat{a}_{q_\parallel, \sigma, 2\ell+1} + Y_{j,\ell}(q_\parallel) \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1},
\]

which is equivalent to

\[
h \Omega_{q_\parallel,j} \sum_{\ell} X_{j,\ell}(q_\parallel) \hat{a}_{q_\parallel, \sigma, 2\ell+1} + Y_{j,\ell}(q_\parallel) \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1} \\
= \sum_k X_{j,k}(q_\parallel)[h \omega_{q_\parallel, 2k+1} \hat{a}_{q_\parallel, \sigma, 2k+1} + \frac{N}{m} \sum_{\ell} g_k(q_\parallel) g_\ell(q_\parallel) (\hat{a}_{q_\parallel, \sigma, 2\ell+1} + \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1})] \\
- Y_{j,k}(q_\parallel)[h \omega_{q_\parallel, 2\ell+1} \hat{a}^\dagger_{q_\parallel, \sigma, 2\ell+1} + \frac{N}{m} \sum_{\ell} g_k(q_\parallel) g_\ell(q_\parallel) (\hat{a}_{q_\parallel, \sigma, 2\ell+1} + \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1})],
\]

where \( g_j(q_\parallel) = (-1)^j \sqrt{2D/\omega_{q_\parallel, 2j+1}} \), the expression above can be written compactly as

\[
[K_{q_\parallel} - h \Omega_{q_\parallel,j} \mathbb{1}_{2N_{\text{max}}}] \psi_j(q_\parallel) = 0,
\]

where we introduced a cutoff for the number of modes \( N_{\text{max}} \) in order to deal with finite-size matrices. The vector \( \psi_j(q_\parallel) \) reads

\[
\psi_j(q_\parallel) = [X_{j,0}(q_\parallel) \hat{a}_{q_\parallel, \sigma, 1}, \cdots, X_{j,N_{\text{max}}-1}(q_\parallel) \hat{a}_{q_\parallel, \sigma, 2N_{\text{max}}-1}, Y_{j,0}(q_\parallel) \hat{a}^\dagger_{-q_\parallel, \sigma, 1}, \cdots, Y_{j,N_{\text{max}}-1}(q_\parallel) \hat{a}^\dagger_{-q_\parallel, \sigma, 2N_{\text{max}}-1}]^T.
\]

The solutions of the linear algebra problem posed by Eq. (C8) can be founded setting determinant of the matrix \([K_{q_\parallel} - h \Omega_{q_\parallel,j} \mathbb{1}_{2N_{\text{max}}}]\) equal to zero,

\[
\det[K_{q_\parallel} - h \Omega_{q_\parallel,j} \mathbb{1}_{2N_{\text{max}}}] = 0,
\]

The calculation of this determinant is a purely mathematical formality and details are given in Appendix D.

Once the limit \( N_{\text{max}} \to \infty \) is taken, from Eq. (D17) we have that the eigenvalues of the matrix \( K_{q_\parallel} \) are the roots of the transcendental equation below

\[
1 + \frac{n_{2D} 2 \pi c^2}{m} \frac{2 \pi c^2}{c^2} \tan \left( \frac{L_s \sqrt{\epsilon \Omega_{q_\parallel,j}^2/c^2 - q_\parallel^2/2}}{\sqrt{\epsilon \Omega_{q_\parallel,j}^2/c^2 - q_\parallel^2}} \right) = 0,
\]

where \( n_{2D} = N/S \). Since \( K_{q_\parallel} = K_{-q_\parallel} \), one has \( \Omega_{q_\parallel,j} = \Omega_{-q_\parallel,j} \) and \( \psi_j(q_\parallel) = \psi_j(-q_\parallel) \) (i.e. \( X(q_\parallel) = X(-q_\parallel) \) and \( Y(q_\parallel) = Y(-q_\parallel) \)).

Similarly to the previous case, we calculate the following commutator

\[
[K_{q_\parallel} - \mathbb{1}_{N_{\text{max}}}, \hat{b}_{-q_\parallel, \sigma, j}^\dagger] = h \Omega_{-q_\parallel,j} \hat{b}_{-q_\parallel, \sigma, j},
\]

and replacing Eq. (C2), one has

\[
h \Omega_{-q_\parallel,j} \sum_{\ell} Y^*_{j,\ell}(-q_\parallel) \hat{a}_{q_\parallel, \sigma, 2\ell+1} + X^*_{j,\ell}(-q_\parallel) \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1} \\
= \sum_k X^*_{j,k}(-q_\parallel)[h \omega_{q_\parallel, 2k+1} \hat{a}^\dagger_{-q_\parallel, \sigma, 2k+1} + \frac{N}{m} \sum_{\ell} g_k(q_\parallel) g_\ell(q_\parallel) (\hat{a}_{q_\parallel, \sigma, 2\ell+1} + \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1})] \\
- Y^*_{j,k}(-q_\parallel)[h \omega_{q_\parallel, 2\ell+1} \hat{a}_{q_\parallel, \sigma, 2\ell+1} + \frac{N}{m} \sum_{\ell} g_k(q_\parallel) g_\ell(q_\parallel) (\hat{a}_{q_\parallel, \sigma, 2\ell+1} + \hat{a}^\dagger_{-q_\parallel, \sigma, 2\ell+1})],
\]
the expression above can be written compactly as
\[
[K\alpha_{\perp} - \hbar\Omega_{q_{\perp}j}\mathbb{1}_{2N_{\text{max}}}](\nu_{\perp}^j(-q_{\perp})) = 0 ,
\] (C14)
where \(\Omega_{q_{\perp}j} = \Omega_{q_{\perp}j}\). Since this eigenvalue problem is identical to Eq. (C8), one has and \(\nu_{\perp}^j(-q_{\perp}) = \nu_j(q_{\parallel})\) (i.e. \(X(q_{\parallel}) = X^*(-q_{\parallel})\) and \(Y(q_{\parallel}) = Y^*(-q_{\parallel})\)).

Because of the symmetries of Eqs. (C8) and (C14) (i.e. \(X(q_{\parallel}) = X^*(-q_{\parallel}) = X(-q_{\parallel})\) and \(Y(q_{\parallel}) = Y^*(-q_{\parallel}) = Y(-q_{\parallel})\)), we can write
\[
\left[ \begin{array}{c} \{\hat{b}_{q_{\perp}j},\alpha,2\ell+1\} \\ \{\hat{a}_{-q_{\perp}j},\alpha,2\ell+1\} \end{array} \right] = \left[ \begin{array}{cc} X(q_{\parallel}) & Y(q_{\parallel}) \\ Y(q_{\parallel}) & X(q_{\parallel}) \end{array} \right] \left[ \begin{array}{c} \{\hat{b}_{q_{\perp}j},\alpha,2\ell+1\} \\ \{\hat{a}_{-q_{\perp}j},\alpha,2\ell+1\} \end{array} \right] ,
\] (C15)
imposing the bosonic commutation rules \(\{\hat{b}_{q_{\perp}j},\alpha,2\ell+1,\hat{b}_{q_{\perp}j},\alpha,2\ell+1\} = \delta_{q_{\perp}j,\alpha,2\ell+1,\alpha,2\ell+1}\) and \(\{\hat{b}_{q_{\perp}j},\alpha,2\ell+1,\hat{b}_{q_{\perp}j},\alpha,2\ell+1\} = 0\) we obtain the following properties
\[
X(q_{\parallel})X^*(q_{\parallel}) - Y(q_{\parallel})Y^*(q_{\parallel}) = \mathbb{1} ,
\] (C16)
and
\[
X(q_{\parallel})Y^*(q_{\parallel}) - Y(q_{\parallel})X^*(q_{\parallel}) = 0 .
\] (C17)
By using the properties above, it is easy to obtain the inverse Bogoliubov transformation
\[
\left[ \begin{array}{c} \{\hat{b}_{q_{\perp}j},\alpha,2\ell+1\} \\ \{\hat{a}_{-q_{\perp}j},\alpha,2\ell+1\} \end{array} \right] = \left[ \begin{array}{cc} X(q_{\parallel}) & -Y(q_{\parallel}) \\ -Y(q_{\parallel}) & X(q_{\parallel}) \end{array} \right] \left[ \begin{array}{c} \{\hat{b}_{q_{\perp}j},\alpha,2\ell+1\} \\ \{\hat{a}_{-q_{\perp}j},\alpha,2\ell+1\} \end{array} \right] .
\] (C18)
In terms of the new bosonic operators \(\hat{b}_{q_{\perp}j},\alpha,2\ell+1\), \(\hat{b}_{q_{\perp}j},\alpha,2\ell+1\) the effective Hamiltonian reads as following:
\[
\hat{H}_{\text{ph}}^{\text{eff}}[\psi] = \langle \psi | \hat{H}_{2D} | \psi \rangle + \sum_{q_{\perp},\alpha} \left\{ \sum_{\text{even } n} \hbar\omega_{q_{\perp}n} \left( \hat{a}_{q_{\perp}n,\alpha,2\ell+1} \right)^\dagger \hat{a}_{q_{\perp}n,\alpha,2\ell+1} + \frac{1}{2} \right\} 
+ \sum_{j} \hbar\Omega_{q_{\perp}j}(\hat{b}_{q_{\perp}j,\alpha,2\ell+1} + \frac{1}{2}) + \text{J}_{q_{\perp}j,\sigma,\sigma'} \sum_{j,\ell} g_{\ell}(q_{\parallel}) (\hat{b}_{q_{\perp}j,\alpha,2\ell+1} + \hat{a}_{-q_{\perp}j,\alpha,2\ell+1})(X_{j,\ell}(q_{\parallel}) - Y_{j,\ell}(q_{\parallel})) \right] .
\] (C19)
The previous Hamiltonian can be written as an explicitly Hermitian form
\[
\hat{H}_{\text{ph}}^{\text{eff}}[\psi] = \langle \psi | \hat{H}_{2D} | \psi \rangle + \sum_{q_{\perp},\alpha} \left\{ \sum_{\text{even } n} \hbar\omega_{q_{\perp}n} \left( \hat{a}_{q_{\perp}n,\alpha,2\ell+1} \right)^\dagger \hat{a}_{q_{\perp}n,\alpha,2\ell+1} + \frac{1}{2} \right\} 
+ \sum_{j} \hbar\Omega_{q_{\perp}j}(\hat{b}_{q_{\perp}j,\alpha,2\ell+1} \hat{b}_{q_{\perp}j,\alpha,2\ell+1} + \frac{1}{2}) + \text{J}_{q_{\perp}j,\sigma,\sigma'} \sum_{j,\ell} g_{\ell}(q_{\parallel}) (X_{j,\ell}(q_{\parallel}) - Y_{j,\ell}(q_{\parallel})) + \text{H.c.} \right] .
\] (C20)
In the Hamiltonian above, the even photon modes are independent of the light-matter interaction, while the odd photon modes are renormalized by the diamagnetic term and expressed as a sum of displaced harmonic oscillators, the ground state \(|\Phi\rangle\) of \(\hat{H}_{\text{ph}}^{\text{eff}}[\psi]\), for every matter state \(|\psi\rangle\), is a tensor product \(|\Phi\rangle \equiv \otimes_{q_{\perp},\sigma} |\beta_{q_{\perp}j,\sigma,\sigma} \rangle\) of coherent states of the \(\hat{b}_{q_{\perp}j,\sigma,\sigma} \) operators, i.e. \(\hat{b}_{q_{\perp}j,\ell,\sigma'} |\Phi\rangle = \beta_{q_{\perp}j,\ell,\sigma'} |\Phi\rangle\).

We now introduce the following energy functional, obtained by taking the expectation value of \(\hat{H}_{\text{ph}}^{\text{eff}}[\psi]\) over \(|\Phi\rangle\):
\[
E[\{\beta_{q_{\perp}j,\sigma,\sigma}\}, \psi] \equiv \langle \Phi | \hat{H}_{A} | \psi \rangle = \langle \Phi | \hat{H}_{\text{ph}}^{\text{eff}}[\psi] | \Phi \rangle ;
\] (C21)
\[
E[\{\beta_{q_{\perp}j,\sigma,\sigma}\}, \psi] = \langle \psi | \hat{H}_{2D} | \psi \rangle + \sum_{q_{\perp},\alpha} \left\{ \sum_{\text{even } n} \hbar\omega_{q_{\perp}n} \left( \hat{a}_{q_{\perp}n,\alpha,2\ell+1} \right)^\dagger \hat{a}_{q_{\perp}n,\alpha,2\ell+1} + \frac{1}{2} \right\} 
+ \text{J}_{q_{\perp}j,\sigma,\sigma'} \sum_{j,\ell} g_{\ell}(q_{\parallel}) (X_{j,\ell}(q_{\parallel}) - Y_{j,\ell}(q_{\parallel})) + \text{H.c.} \right] .
\] (C20)

Note that the order parameter \(\alpha_{q_{\perp}j,\alpha,2\ell+1,\sigma}_{\sigma,\sigma,\sigma'}\), introduced above is linearly-dependent on \(\beta_{q_{\perp}j,\sigma,\sigma}\), i.e.
\[
\left[ \begin{array}{c} \{\alpha_{q_{\perp}j,\alpha,2\ell+1,\sigma}_{\sigma,\sigma,\sigma'}\} \\ \{\alpha_{-q_{\perp}j,\alpha,2\ell+1,\sigma}_{\sigma,\sigma,\sigma'}\} \end{array} \right] = \left[ \begin{array}{cc} X(q_{\parallel}) & -Y(q_{\parallel}) \\ -Y(q_{\parallel}) & X(q_{\parallel}) \end{array} \right] \left[ \begin{array}{c} \{\beta_{q_{\perp}j,\sigma,\sigma}\} \\ \{\beta_{-q_{\perp}j,\sigma,\sigma}\} \end{array} \right] .
\] (C22)
By using the linear relation above, we can express the energy functional \(E[\{\beta_{q_{\perp}j,\sigma,\sigma}\}, \psi]\) in terms of \(\{\alpha_{q_{\perp}j,\sigma,\sigma}\}, \psi\), Eq. (80) of the main text.
Appendix D: Proof of Eq. (C10)

It is useful to write the matrix $K_{\|} - \hbar \Omega_{\|,j} \mathbb{1}_{2N_{\text{max}}}$ defined in Eq. (C8) in the block form:

$$K_{\|} - \hbar \Omega_{\|,j} \mathbb{1}_{2N_{\text{max}}} = \begin{bmatrix} Q(q_{\|}) + V(q_{\|}) - \hbar \Omega_{q_{\|},j} \mathbb{1}_{N_{\text{max}}} & -V(q_{\|}) \\ V(q_{\|}) & -Q(q_{\|}) - V(q_{\|}) - \hbar \Omega_{q_{\|},j} \mathbb{1}_{N_{\text{max}}} \end{bmatrix},$$

(D1)

where

$$Q_{k,\ell}(q_{\|}) = \hbar \omega_{q_{\|},2\ell+1} \delta_{k,\ell},$$

(D2)

and

$$V_{k,\ell}(q_{\|}) = \frac{N}{m} g_{k}(q_{\|}) g_{\ell}(q_{\|}).$$

(D3)

After algebraic manipulations, one has

$$K_{\|} - \hbar \Omega_{\|,j} \mathbb{1}_{2N_{\text{max}}} = \begin{bmatrix} \mathbb{1}_{2N_{\text{max}}} + \mathcal{W}(q_{\|}) & \mathcal{W}(-q_{\|}) & \mathcal{W}(q_{\|}) \\ \mathcal{W}(-q_{\|}) & \mathcal{W}(-q_{\|}) & \mathcal{W}(q_{\|}) \end{bmatrix},$$

(D4)

$$\mathcal{W}(q_{\|}) = \begin{bmatrix} \mathcal{W}_{-}(q_{\|}) & \mathcal{W}_{+}(q_{\|}) \\ \mathcal{W}_{-}(q_{\|}) & \mathcal{W}_{+}(q_{\|}) \end{bmatrix},$$

(D5)

where

$$\mathcal{W}_{\pm}(q_{\|}) = \left( \pm \hbar \Omega_{q_{\|},\pm} \mathbb{1}_{N_{\text{max}}} + Q(q_{\|}) \right)^{-1} V(q_{\|}).$$

(D6)

Using the expression above, one can write the following determinant as

$$\det[K_{\|} - \hbar \Omega_{\|,j} \mathbb{1}_{2N_{\text{max}}} = \prod_{\ell} (\hbar \omega_{q_{\|},2\ell+1})^{2} \det[\mathbb{1}_{2N_{\text{max}}} + \mathcal{W}(q_{\|})].$$

(D7)

Now, we focus on $\det[\mathbb{1}_{2N_{\text{max}}} + \mathcal{W}(q_{\|})]$. So, by using the following well-known algebraic property

$$\det[\mathbb{1}_{2N_{\text{max}}} + \mathcal{W}(q_{\|})] = \exp\{\operatorname{Tr}[\ln(\mathbb{1}_{2N_{\text{max}}} + \mathcal{W}(q_{\|})]\},$$

(D8)

the trace expressed in the right-hand side can be written as

$$\operatorname{Tr}[\ln(\mathbb{1}_{2N_{\text{max}}} + \mathcal{W}(q_{\|})] = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \operatorname{Tr}[\mathcal{W}^{j}(q_{\|})],$$

(D9)

Since for block matrices the following property holds,

$$\operatorname{Tr}\begin{bmatrix} A & B \\ A & B \end{bmatrix} = \operatorname{Tr}\{(A + B)(C + D)\},$$

(D10)

we can prove that

$$\operatorname{Tr}[\mathcal{W}^{j}(q_{\|})] = \operatorname{Tr}[(\mathcal{W}_{+}(q_{\|}) + \mathcal{W}_{-}(q_{\|}))^{j}].$$

(D11)

Furthermore, it is possible to show that

$$\operatorname{rank}[\mathcal{W}_{+}(q_{\|}) + \mathcal{W}_{-}(q_{\|})] = 1,$$

(D12)

this can be proved by inspection, showing that all the columns of the matrix $\mathcal{W}_{+}(q_{\|}) + \mathcal{W}_{-}(q_{\|})$ can be obtained, for example, multiplying the first column for a suitable constant.
Hence, the matrix $W_+(q ||) + W_-(q ||)$ has only one non-zero eigenvalue and the following relation holds true,

$$\text{Tr}[W^j(q ||)] = \text{Tr}[(W_+(q ||) + W_-(q ||))^j] = \text{Tr}[(W_+(q ||) + W_-(q ||))^{j-1}] \text{Tr}[W(q ||)] = \text{Tr}[W(q ||)]^j. \quad (D13)$$

By replacing the result above in Eq. (D9), one has

$$\text{Tr}[\ln(\mathbb{I}_{2N_{\text{max}}} + W(q ||))] = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \text{Tr}[W(q ||)]^j = \ln(1 + \text{Tr}[W(q ||)]), \quad (D14)$$

where

$$\text{Tr}[W(q ||)] = \text{Tr}[(W_+(q ||) + W_+(q ||))] = \sum_{\ell=0}^{N_{\text{max}}-1} \frac{2\omega_{q_1,2\ell+1}N_{q_1}^2(q ||)}{m\hbar(\omega_{q_1,2\ell+1}^2 - \Omega^2)} = \sum_{\ell=0}^{N_{\text{max}}-1} \frac{4DN}{m\hbar(\omega_{q_1,2\ell+1}^2 - \Omega_{q_1,j}^2)}. \quad (D15)$$

Replacing Eq. (D15) in Eq. (D8), one finds

$$\text{Det}[\mathbb{I}_{2N_{\text{max}}} + W_{q ||}] = 1 + \sum_{\ell=0}^{N_{\text{max}}-1} \frac{4DN}{m\hbar(\omega_{q_1,2\ell+1}^2 - \Omega_{q_1,j}^2)} = 1 + \frac{n_{2D}}{m} \frac{2\pi e^2}{c^2} \tan\left(\frac{L_z}{\epsilon_l \Omega^2/c^2 - q ||^2/2}\right), \quad (D16)$$

where we have used $\sum_{\ell}[(2\ell + 1)^2 - x^2]^{-1} = \pi \tan(\pi x/2)/(4x)$ and the limit $N_{\text{max}} \to \infty$ has been taken. Finally, one can write

$$\text{Det}[K_{q ||} - \hbar \Omega_{q_1,j} \mathbb{I}_{2N_{\text{max}}}]] = \prod_{\ell} [(-\hbar \Omega_{q_1,j})^2 - (\hbar \omega_{q_1,2\ell+1})^2]^{1/n_{2D}} \frac{2\pi e^2}{c^2} \tan\left(\frac{L_z}{\epsilon_l \Omega_{q_1,j}^2/c^2 - q ||^2/2}\right). \quad (D17)$$

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Strictly speaking, in the (unscreened) jellium model the dielectric constant $\varepsilon_0 = \varepsilon(r_0)$, which will be introduced below. Nowhere in our proof we will assume this specific form of $\varepsilon(r)$.