Maximal subgroups of amalgams of finite inverse semigroups.

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Abstract

We use the description of the Schützenberger automata for amalgams of finite inverse semigroups given by Cherubini, Meakin, Piochi in [5] to obtain structural results for such amalgams. Schützenberger automata, in the case of amalgams of finite inverse semigroups, are automata with special structure possessing finite subgraphs, that contain all essential information about the automaton. Using this crucial fact, and the Bass-Serre theory, we show that the maximal subgroups of an amalgamated free-product are either isomorphic to certain subgroups of the original semigroups or can be described as fundamental groups of particular finite graphs of groups build from the maximal subgroups of the original semigroups.


1 Introduction

If $S_1$ and $S_2$ are semigroups (groups) such that $S_1 \cap S_2 = U$ is a non-empty subsemigroup (subgroup) of both $S_1$ and $S_2$ then $[S_1, S_2; U]$ is called an amalgam of semigroups (groups). The amalgamated free-product $S_1 *_U S_2$ associated with this amalgam in the category of semigroups (groups) is defined by the usual universal diagram.

The amalgam $[S_1, S_2; U]$ is said to be strongly embeddable in a semigroup (group) $S$ if there are injective homomorphisms $\phi_i : S_i \to S$ such that $\phi_1|U = \phi_2|U$ and $S_1\phi_1 \cap S_2\phi_2 = U\phi_1 = U\phi_2$. It is well known that every amalgam of groups embeds in a group while semigroup amalgams do not necessarily embed in any semigroup \([14]\). On the other hand, every amalgam of inverse semigroups (in the category of inverse semigroups) embeds in an inverse semigroup, and hence in the corresponding amalgamated free product in the category of inverse semigroups \([11]\).

An inverse semigroup is a semigroup $S$ with the property that for each element $a \in S$ there is a unique element $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$, $a^{-1}$ is called the inverse of $a$. A consequence of the definition is that the set of the idempotents $E(S)$ is a semilattice. One may also define a natural partial order on $S$ putting $a \leq b$ if and only if $a = eb$ for some $e \in E(S)$.

Inverse semigroups may be regarded as semigroups of partial one-to-one transformations, so they arise very naturally in several areas of mathematics and more recently also in computer science, mainly since the inverse of an element can be seen as the “undo with a trace” of the action represented by that element. We refer the reader to the book of Petrich \([18]\) for basic results and notation about inverse semigroups and to the more recent books of Lawson \([15]\) and Paterson \([17]\) for many references to the connections between inverse semigroups and other branches of mathematics.

The free object on a set $X$ in the category of inverse semigroup is denoted by $FIS(X)$. It is the quotient of the free semigroup $(X \cup X^{-1})^+$ by the least congruence $\nu$ that makes the resulting quotient semigroup inverse (see \([18]\)). The inverse semigroup $S$ presented by a set $X$ of generators and a set $T$ of relations is denoted by $S = Inv(X; T)$. This is the quotient of the free semigroup $(X \cup X^{-1})^+$ by the least congruence $\tau$ that contains $\nu$ and the relations in $T$.

The structure of $FIS(X)$ was studied via graphical methods by Munn \([16]\). Munn’s work was greatly extended by Stephen \([24]\) who introduced the notion of Schützenberger graphs associated with presentations of inverse semigroups. These graphs were widely used in the study algorithmic
problems and the structure of several classes of inverse semigroups (see, for instance [3, 4, 5, 6, 7, 8, 10, 12, 13, 20, 21, 22, 23]). In particular Haataja, Margolis and Meakin were the first to show that Bass-Serre theory may be applied to study the structure of maximal subgroups, to obtain results for amalgams of inverse semigroups where $U$ contained all idempotents of $S_1$ and $S_2$. Their construction was extended by Bennett [3] and Jajcayová [12] to respectively study the maximal subgroups of a special class of amalgams and HNN-extensions of inverse semigroups.

In [5], the word problem for amalgams of finite inverse semigroups was shown to be decidable by constructing an automaton that is a good approximation of the Schützenberger automaton. Here, we make use of this construction to study the structure of maximal subgroups in such amalgams. This is done along the lines of Bennett’s study of maximal subgroups of lower bounded amalgams, but as amalgams of finite inverse semigroups are not necessarily lower bounded, Schützenberger automata of amalgams of finite inverse semigroups differ from Bennett’s automata, mainly by the fact the Schützenberger graphs of the two original semigroups do not appear as subgraphs of the resulting Schützenberger automaton of the amalgam. This leads to several important technical differences in the treatment and in the results.

The paper is organized as follows: in Section 2 we recall basic definitions and relevant results concerning Schützenberger automata of inverse semigroups, and the structure and properties of Schützenberger automata of amalgams of finite inverse semigroups in particular. In Section 3 we prove that the automorphism groups of the Schützenberger graphs of amalgams (isomorphic to the maximal subgroups) are isomorphic to the automorphism groups of particular subgraphs with special properties. We study these subgraphs and their properties in Section 4. In Section 5 we give a brief review of the Bass-Serre theory of groups acting on graphs. Finally, merging this theory with the previous results, in Section 6 we give a complete description of the maximal subgroups in an amalgam of finite inverse semigroups.

2 Preliminaries

In this section we review definitions and results concerning Schützenberger automata of inverse semigroups, and briefly describe the construction of Schützenberger graphs of amalgams of finite inverse semigroups. We refer the reader to [2, 5, 18, 24] for more details.

An inverse word graph over an alphabet $X$ is a strongly connected la-
labelled digraph whose edges are labelled over $X \cup X^{-1}$, where $X^{-1}$ is the set of formal inverses of elements in $X$, so that for each edge $e$ labelled by $x \in X$ there is an edge labelled by $x^{-1}$ in the reverse direction. A finite sequence of edges $e_i = (\alpha_i, a_i, \beta_i)$, $1 \leq i \leq n$, with $a_i \in X \cup X^{-1}$ and $\beta_i \in \alpha_i + 1$ for all $i$ with $1 \leq i < n$, is an $\alpha_1 - \beta_n$ path of $\Gamma$ labelled by $a_1 a_2 \ldots a_n \in (X \cup X^{-1})^+$. An inverse automaton over $X$ is a triple $A = (\alpha, \Gamma, \beta)$ where $\Gamma$ is an inverse word graph over $X$ with set of vertices $V(\Gamma)$ and $\alpha, \beta \in V(\Gamma)$ are two special vertices called the initial and final state of $A$. The language $L[A]$ recognized by $A$ is the set of labels of all $\alpha - \beta$ paths of $\Gamma$.

Morphisms between inverse word graphs are graph morphisms that preserve labelling of edges and are referred to as $V$-homomorphisms in [24]. In this paper we simply refer to them as homomorphisms, and in the case a morphism is surjective, injective or bijective, we refer to it as an epimorphism, monomorphism or isomorphism, respectively. The group of all automorphisms of graph $\Gamma$ is denoted by $\text{Aut}(\Gamma)$. If $\Gamma$ is an inverse word graph over $X$ and $\rho$ is an equivalence relation on the set of vertices of $\Gamma$, the corresponding quotient graph $\Gamma/\rho$ is called a $V$-quotient of $\Gamma$ (see [24] for details). There is a least equivalence relation on the vertices of an inverse automaton $\Gamma$ such that the corresponding $V$-quotient is deterministic. A deterministic $V$-quotient of $\Gamma$ is called a $DV$-quotient. There is a natural homomorphism from $\Gamma$ onto a $V$-quotient of $\Gamma$. The notions of morphism, $V$-quotient and $DV$-quotient of inverse graphs extend analogously to inverse automata. (see [24]).

Let $S = \text{Inv}\langle X; T \rangle \simeq (X \cup X^{-1})^+ / \tau$ be an inverse semigroup. The Schützenberger graph $\text{SG}(X; T; w)$ for a word $w \in (X \cup X^{-1})^+$ relative to the presentation $\langle X | T \rangle$ has the $R$-class of $w\tau$ in $S$ as the set of vertices and its edges consist of all the triples $(s, x, t)$ with $x \in X \cup X^{-1}$, and $s \cdot x\tau = t$. We view this edge as being directed from $s$ to $t$. The graph $\text{SG}(X; T; w)$ is a deterministic inverse word graph over $X$. The structure of Schützenberger graphs is strictly connected with the Green’s relations on $S$. In particular the following results by Stephen will be important for our purposes.

**Proposition 1.** Let $S = \text{Inv}(X; T)$ be an inverse semigroup and let $e, f \in E(S)$. Then

1. $eDf$ if and only if there exists a $V$-isomorphism $\phi : \text{SG}(X; T; e) \rightarrow \text{SG}(X; T; f)$ [24, Theorem 3.4 (a)].

2. The $H$-class of $e$ and $\text{Aut}(\text{SG}(X; T; e))$ are isomorphic groups [24].
The second statement identifies the group of symmetries of \( ST(X, T; e) \) with the maximal subgroup of \( S \) having \( e \) as unity. This fact is fundamental since it gives us a geometric interpretation of these maximal subgroups and it will be implicitly used throughout the paper. The automaton \( A(X, T; w) \) whose underlying graph is \( ST(X, T; w) \) with the vertex \( w^{-1}τ \) as the initial state and the vertex \( wτ \) as the terminal state, is called the Schützenberger automaton of \( w ∈ (X ∪ X^{-1})^+ \) relative to the presentation \( ⟨X | T⟩ \).

In [24] Stephen provides an iterative (but in general not effective) procedure to build \( A(X, T; w) \) via two operations, the elementary determination and the elementary expansion. We briefly recall such operations. Let \( Γ \) be an inverse word graph over \( Y \), an elementary determination consists of folding two edges starting from the same vertex and labeled by the same letter of the alphabet \( Y ∪ Y^{-1} \). The elementary expansion applied to \( Γ \) relative to a presentation \( ⟨Y | T⟩ \) consists in adding a path \( ⟨ν_1, r, ν_2⟩ \) to \( Γ \) wherever \( r \) is a path in \( Γ \) and \( ⟨r, t⟩ \) ∈ \( T \cup T^{-1} \). An inverse word graph is called closed with respect to \( ⟨Y | T⟩ \) if it is a deterministic word graph where no expansion relative to \( ⟨Y | T⟩ \) can be performed. An inverse automaton is closed with respect to \( ⟨Y | T⟩ \) if its underlying graph is so.

Let \( S_i = Inv⟨X_i | R_i⟩ = (X_i ∪ X_i^{-1})^+/η_i, i = 1, 2 \), where the \( X_i \) are disjoint alphabets. For an amalgam \( [S_1, S_2, U] \) we view the natural image of \( u ∈ U \) in \( S_i \) under the embedding of \( U \) as a word in the alphabet \( X_i \) and \( ⟨X_1 ∪ X_2 | R_1 ∪ R_2 ∪ W⟩ \) with \( W = \{(ϕ_1(u), ϕ_2(u))| u ∈ U\} \) is a presentation of \( S_1 ∗_U S_2 \). We put \( X = X_1 ∪ X_2 \) and \( R = R_1 ∪ R_2 \) and we call \( ⟨X | R ∪ W⟩ \) the standard presentation of \( S_1 ∗_U S_2 \) with respect to the presentations of \( S_1 \) and \( S_2 \), for short the standard presentation of \( S_1 ∗_U S_2 ≃ (X ∪ X^{-1})^+/τ \). In the sequel we will use the superscript notations \( D^U, D^{S_i}, D^{S_1 ∗_U S_2} \) to discriminate the \( D \)-classes in \( U, S_i, (S_1 ∗_U S_2) \), respectively. We keep this convention for all the Green’s relations as well as for their classes. For instance, for the maximal subgroup in \( S_1 \) of an idempotent \( e \) in \( S_1 \) we use the symbol \( H_e^{S_1} \), instead if we consider the maximal subgroup in the free-product with amalgamation we use the notation \( H_e^{S_1 ∗_U S_2} \). We adhere to this notation throughout the remain of the paper and we always assume that \( S_1 \) and \( S_2 \) are finite inverse semigroups.

Let \( Γ \) be an inverse word graph labeled over \( X = X_1 ∪ X_2 \) with \( X_1 ∩ X_2 = \emptyset \), an edge of \( Γ \) that is labeled from \( X_i ∪ X_i^{-1} \) (for some \( i ∈ \{1, 2\} \)) is said to be colored \( i \). A subgraph of \( Γ \) is called monochromatic if all its edges have the same color. A lobe of \( Γ \) is defined to be a maximal monochromatic connected subgraph of \( Γ \). The coloring of edges extends to coloring of lobes.
Two lobes are said to be adjacent if they share common vertices, called intersection vertices. If \( \nu \in V(\Gamma) \) is an intersection vertex, then it is common to two unique lobes, which we denote by \( \Delta_1(\nu) \) and \( \Delta_2(\nu) \), colored 1 and 2, respectively. We define the lobe graph of \( \Gamma \) to be the graph whose vertices are the lobes of \( \Gamma \) and whose edges correspond to adjacency of lobes. 

We remark that a nontrivial inverse word graph \( \Delta \) colored \( i \) and closed relative to \( \langle X_i | R_i \rangle \) contains all the paths \((\nu_1, v', \nu_2)\) with \( v' \in (X_i \cup X_i^{-1})^+ \) such that \( v' \eta_i = \nu \eta_i \), provided that \((\nu_1, v, \nu_2)\) is a path of \( \Delta \). Hence we often say that there is a path \((\nu_1, s, \nu_2)\) with \( s \in S_i \) in \( \Delta \) whenever \( \{ (\nu_1, v, \nu_2) | v \eta_i = s \} \neq \emptyset \). Similarly we say that \((\nu_1, u, \nu_2)\) with \( u \in U \) is a path of \( \Delta \) to mean that \((\nu_1, \phi_i(u), \nu_2)\) is a path of \( \Delta \). For all \( \nu \in V(\Delta) \) we denote by \( L_U(\nu, \Delta) \) the set of all the elements \( u \in U \) such that \((\nu, u, \nu)\) is a loop based at \( \nu \) in \( \Delta \).

Thus following the notation of [5], that slightly modifies the analogous one introduced in [2], we say that an inverse word graph \( \Gamma \) on \( X \) is an opuntoid graph if it satisfies the following properties.

- \( \Gamma \) is a deterministic inverse word graph;
- the lobe graph is a tree, denoted \( T(\Gamma) \);
- each lobe of \( \Gamma \) is a finite closed DV-quotient of a Schützenberger graph relative to \( \langle X_i | R_i \rangle \), \( i \in \{1, 2\} \);
- (loop equality property) for each intersection vertex \( \nu \) of \( \Gamma \), \( L_U(\nu, \Delta_1(\nu)) = L_U(\nu, \Delta_2(\nu)) \);
- (assimilation property) for each intersection vertex \( \nu \) of \( \Gamma \), and for each \( \nu' \in V(\Delta_1(\nu)) \cap V(\Delta_2(\nu)) \) with \( \nu' \neq \nu \) there is \( u \in U \) such that \((\nu, u, \nu')\) is a path in both \( \Delta_1(\nu) \) and \( \Delta_2(\nu) \). Moreover \((\nu, u', \nu')\) with \( u' \in U \) is path in \( \Delta_1(\nu) \) if and only if \((\nu, u', \nu')\) is a path in \( \Delta_2(\nu) \).

We remark that assimilation property implies that if \( \nu \) is an intersection vertex of \( \Gamma \) and \((\nu, u', \nu')\) is a path of \( \Gamma \), then \( \Delta_i(\nu) = \Delta_i(\nu') \) with \( i \in \{1, 2\} \). This property is referred as the related pair separation property in [2]. A subopuntoid subgraph \( \Theta \) is an inverse word subgraph of an opuntoid graph \( \Gamma \) such that if \( \Delta \) is a lobe of \( \Theta \), then \( \Delta \) is also a lobe of \( \Gamma \). If \( \phi : \Gamma \to \Gamma' \) is a homomorphism between the two opuntoid graphs \( \Gamma, \Gamma' \) and \( \Theta \) is a subopuntoid subgraph of \( \Gamma \), we often denote \( \phi|_{\Theta} \) the restriction of \( \phi \) to \( \Theta \). Opuntoid automata are automata whose underlying graphs are so. Since \( S_1, S_2 \) are finite, for each lobe \( \Delta \) of an opuntoid graph \( \Gamma \) colored \( i \in \{1, 2\} \) and for each \( \nu \in V(\Delta) \), there is a minimum idempotent, denoted by \( e_i(\nu) \),
labeling a loop based at \( \nu \). Moreover if \( L_U(\nu, \Delta) \neq \emptyset \) then there is also a minimum idempotent belonging to \( L_U(\nu, \Delta) \) which is denoted by \( f(e_i(\nu)) \).

In [5] the Schützenberger automaton \( A(X, R \cup W; w) \) is built by means of a sequence of constructions. The first three constructions are iteratively applied finitely many times to the linear automata \( (\alpha, \text{lin}(w)), (\beta) \). These constructions are applied on the lobes and the intersection vertices, they do not increase the number of lobes of \( \text{lin}(w) \) and after applying the fourth one, a finite opuntoid automaton \( \text{Core}(w) = (\alpha, \Gamma_0, \beta) \) called \( \text{Core automaton} \) or briefly \( \text{Core} \) of \( w \) is obtained. \( \text{Core}(w) \) contains all the information to build the Schützenberger automaton of \( w \) and results to be a subopuntoid automaton of the Schützenberger automaton of \( w \). It is closed with respect to \( \langle X|R \rangle \) but it is not in general closed relative to \( \langle X|R \cup W \rangle \) and approximates \( A(X, R \cup W; w) \) in the sense of Stephen [24].

Then a last construction, called Construction 5, is applied in general infinitely many times to \( \text{Core}(w) \) to obtain \( A(X, R \cup W; w) \). Let \( \Gamma \) be a non-complete opuntoid graph and let \( \Delta \) be a lobe colored \( i \in \{1, 2\} \). Then \( \nu \in V(\Gamma) \) is a \( \text{bud} \) of \( \Gamma \) (see [2]) if it is not an intersection vertex and \( L_U(\nu, \Delta) \neq \emptyset \).

**Construction 5** (see [5])

- Let \( \Gamma \) be a non-complete opuntoid graph and let \( \nu \in V(\Delta) \) be a bud belonging to a lobe \( \Delta \) colored by some \( i \in \{1, 2\} \). Then \( \nu \in V(\Gamma) \) is a bud of \( \Gamma \) (see [2]) if it is not an intersection vertex and \( L_U(\nu, \Delta) \neq \emptyset \). Let \( f = f(e_i(\nu)) \) and let \((x, \Lambda, x) = A(X_{3-i}, R_{3-i}; f)\). Consider the set of vertices (called as \( \text{net} \)):

\[
N(x, \Lambda) = \{ y \in V(\Lambda) : (x, u, y) \text{ is a path in } \Lambda \text{ for some } u \in L_U(\nu, \Delta) \}
\]

and let \( \rho \subseteq V(\Lambda) \times V(\Lambda) \) be the least equivalence relation that identifies each vertex of \( N(x, \Lambda) \) with \( x \) and such that \( \Lambda/\rho \) is deterministic and put \( \Delta' = \Lambda/\rho \). Consider the inverse word automaton \( B = (\nu, \Gamma, \nu) \times (x \rho, \Delta', x \rho) \) (see [24]), then for all \( u \in U \) such that \((\nu, u, y) \) and \((\nu, u, y') \) are paths of \( \Delta \) and \( \Delta' \) respectively, consider the equivalence relation \( \kappa \) on \( V(B) \) that identifies \( y \) and \( y' \) and call \( \Gamma \) the underlying graph of \( B/\kappa \).

Lemma 10 of [5] states that \( \Gamma \) is an opuntoid graph with one more lobe than \( \Gamma \) and the graph \( \Gamma \) is unchanged by this construction.

Let \( \Delta \) and \( \Delta' \) be two adjacent lobes of an opuntoid graph \( \Gamma \). Following [2] [19], we say that \( \Delta' \) directly feeds off \( \Delta \), in symbols \( \Delta \mapsto \Delta' \), if \( \Delta' \) can be obtained from \( \Delta \) by applying Construction 5 at some intersection vertex.
Moreover for each pair of lobes $\Delta, \Delta'$ of $\Gamma$ we say that $\Delta'$ feeds off $\Delta$, $\Delta \mapsto^* \Delta'$, if $\Delta$ and $\Delta'$ are related in the transitive closure of $\mapsto$. We remark that the definition of feeding off in our case differs from the one given in [2] because in Bennett’s case no quotient of the lobe $\Lambda$ is needed.

For each finite opuntoid graph $\Gamma$ consider the sequence

$$\Gamma = \Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \ldots \Gamma_m \hookrightarrow \ldots$$

where $\Gamma_{m+1}$ is obtained from $\Gamma_m$ applying Construction 5 at some bud of $V(\Gamma_m)$ and consider the directed limit $\lim_i \Gamma_i = \bigcup_{k>0} \Gamma_k$. Then $\lim_i \Gamma_i$ is a closed automaton with respect to $\langle X|R \cup W \rangle$ (see [2]); this operation is called in the sequel the closure of $\Gamma$ and it is denoted by $cl_{R,W}(\Gamma)$. The uniqueness of the closure of an opuntoid graph follows from the work of Stephen [24, 25]. Note that the closure of the underlying graph $\Gamma_0$ of $Core(w)$ is the Schützenberger graph $S\Gamma(X,R \cup W;w)$. This graph is in general an infinite graph, however a direct consequence of the finiteness of $S_1, S_2$ is that there are finitely many different types of lobes up to isomorphisms.

We say that a lobe $\Delta'$ of an opuntoid graph $\Gamma$ that is adjacent to precisely one other lobe $\Delta$ of $\Gamma$ is called a parasite (see also [3]) of $\Gamma$ if $\Delta'$ feeds off $\Delta$.

A subopuntoid subgraph $\Gamma'$ of an opuntoid graph $\Gamma$ is called a host of $\Gamma$ if:

- its lobe tree is finite,
- it is parasite-free,
- every lobe of $\Gamma$ not belonging to $\Gamma'$ feeds off some lobe of $\Gamma'$.

A host of an opuntoid automaton is a host of its underlying graph.

It is straightforward to see that a host $\Theta$ of an opuntoid graph $\Gamma$ is a minimal subopuntoid subgraph of $\Gamma$ such that $cl_{R,W}(\Theta) \supseteq \Gamma$. Moreover we have the following proposition whose statement and proof are formally equal to the ones of [2, Lemma 6.2] even if the definitions differ in some technical details

**Proposition 2.** Let $\Gamma$ be an opuntoid graph. Then a host of $\Gamma$ is a maximal parasite-free subopuntoid subgraph of $\Gamma$. If $\Gamma$ has more than one host, then every host is a lobe and the unique reduced lobe path connecting any two hosts consists entirely of lobes that are hosts.

Obviously an opuntoid with finite-lobe graph has always a host, and since the underlying graph $\Gamma_0$ of $Core(w)$ has finitely many lobes and

\[ \nu \in V(\Delta) \cap V(\Delta'). \]
\[ S \Gamma(X, R; w) = cl_{R \cup W}(\Gamma_0), \]
by Lemma 6.1 of [2] we derive that the Schützenberger graph \( S \Gamma(X, R; w) \) posses always a host contained in \( \Gamma_0 \).
Moreover the union of all hosts of an opuntoid graph \( \Gamma \) that has a host is a subopuntoid subgraph of \( \Gamma \), denoted by \( Host(\Gamma) \).

The closure of an opuntoid automaton \((\alpha, \Gamma, \beta)\) is \((\alpha', cl_{R \cup W}(\Gamma), \beta')\) where \( \alpha', \beta' \) are the natural images of \( \alpha \) and \( \beta \) respectively in \( cl_{R \cup W}(\Gamma) \). Then we have the following description of Schützenberger automata that slightly extends Theorem 3 in [5] (see also [19])

**Proposition 3.** Let \( S = S_1 \ast_U S_2 \) be an amalgamated free-product of finite inverse semigroups \( S_1 \) and \( S_2 \) amalgamating a common inverse subgroup \( U \), where \( \langle X_i | R_i \rangle \) are presentations of \( S_i \) for \( i = 1, 2 \). Let \( X = X_1 \cup X_2 \), \( R = R_1 \cup R_2 \) and \( W \) be the set of all pairs \((\phi_1(u), \phi_2(u))\) for \( u \in U \). Then the Schützenberger automata relative to \( \langle X | R \cup W \rangle \) are complete opuntoid automata possessing a host.

It is important for the sequel to point out the differences between the definitions of opuntoid graphs, feeding off relation and Schützenberger automata given in Bennett’s papers [2, 3] and the ones presented here. According to Bennett, lobes of opuntoid graphs are Schützenberger graphs relative to \( \langle X_i | R_i \rangle \) for some \( i \in \{ 1, 2 \} \), instead here lobes are in general \( DV\)-quotients of such graphs. The lower bound equality property of opuntoid graphs according to Bennett is here replaced by the loop equality property that, as remarked in [5] p.10, coincides with lower bound equality property in the case that lobes are Schützenberger graphs. In [2] a lobe \( \Delta' \) colored \( i \) directly feeds off \( \Delta \) in a vertex \( \nu \) if \( \Delta' = S \Gamma(X_i, R_i; f(\varepsilon_{3-i}(\nu))) \) while in our case some \( DV\)-quotient is in general needed to guarantee the loop equality property in \( \nu \). Lastly, Schützenberger automata of lower bounded amalgams are completely characterized as complete opuntoid automata with hosts, Schützenberger automata of amalgams of finite semigroups are only described in such a way because complete opuntoid automata with hosts are not necessarily Schützenberger automata.

### 3 Automorphisms of opuntoid graphs

We shall adopt the same notion as in the previous section for the presentations of \( S_1, S_2 \) and \( S_1 \ast_U S_2 \) and all the considered opuntoid graphs shall be assumed to be determined by these presentations. The main result proved here, shows that there is a group isomorphism between the automorphism group of a complete opuntoid graph \( \Gamma \) and the automorphism group of the...
subopuntoid $Host(\Gamma)$ formed by the union of all the hosts of $\Gamma$.

Also in our case we can state Lemma 6 and Lemma 7 of [3], namely the proofs of these lemmas in [3] only use the facts that the lobe graphs are trees, homomorphisms of graphs preserve labels (and so colors) of edges and automorphisms of deterministic inverse word graphs that agree on a vertex are equal.

**Lemma 1.** Let $\Gamma$, $\Gamma'$ be opuntoid graphs and let $\varphi : \Gamma \to \Gamma'$ be a homomorphism. Then $\varphi$ is an isomorphism if and only if it induces an automorphism of lobe trees and maps lobes isomorphically onto lobes.

Let $\Gamma$ be an opuntoid automaton with finitely many lobes. It is straightforward to check that the automorphism group of $\Gamma$ embeds in the automorphism group of some lobe of $\Gamma$ by using the fact that the automorphism group of a finite tree fixes a vertex or an edge (see [1, Subsection 27.1.3]).

Let $\Gamma$ be an opuntoid graph and let $\Theta$ be a subopuntoid subgraph of $\Gamma$. For any lobe $\Delta$ not belonging to $\Theta$ the notation $\Theta^\Delta$ means $\Theta \cup \Delta$. We have the following lemma.

**Lemma 2.** Let $\Gamma, \Gamma'$ be two complete opuntoid graphs and let $\Theta, \Theta'$ be subopuntoid subgraphs containing a host of $\Gamma$ and $\Gamma'$ respectively. Let $\nu$ be a bud of $\Theta$ and let $\Delta_i(\nu)$ be a lobe of $\Theta$ for some $i \in \{1, 2\}$. Let $\varphi : \Theta \to \Theta'$ be an isomorphism and let $\nu' = \varphi(\nu)$. Then $\varphi$ can be extended to an isomorphism from $\Theta \cup \Delta_{3-i}(\nu)$ onto $\Theta' \cup \Delta_{3-i}(\nu')$.

**Proof.** Since $\nu$ is a bud of $\Theta$ and $\Delta_i(\nu)$ is a lobe of $\Theta$, the lobe $\Delta_{3-i}(\nu)$ of $\Gamma$ does not belong to $\Theta$. Moreover since $\Theta$ contains a host of $\Gamma$, then $\Delta_i(\nu) \mapsto \Delta_{3-i}(\nu)$. Let $f = f(e_i(\nu))$ be the minimum idempotent in $U$ labelling a loop based at $\nu$ in $\Delta_i(\nu)$. Since $\phi$ preserves the labels, then $f = f(e_i(\varphi(\nu)))$. Since $\Gamma'$ is complete, $\varphi(\nu)$ is not a bud, hence it is an intersection vertex. Moreover, by Lemma $\prod \Delta_i(\varphi(\nu))$ is a lobe of $\Theta'$. If also $\Delta_{3-i}(\varphi(\nu))$ is a lobe of $\Theta'$ then again by Lemma $\prod \Delta_i(\nu) = \varphi^{-1}(\Delta_i(\varphi(\nu))$ and $\varphi^{-1}(\Delta_{3-i}(\varphi(\nu)))$ are adjacent lobes in $\Theta$ with intersection vertex $\nu$, hence $\nu$ is not a bud of $\Theta$, a contradiction. So $\Delta_{3-i}(\varphi(\nu))$ is not a lobe of $\Theta'$ and $\Delta_i(\varphi(\nu)) \to \Delta_{3-i}(\varphi(\nu))$ because $\Theta'$ contains a host. Since $f(e_i(\varphi(\nu))) = f(e_i(\nu)) = f$ and $L_\nu(\nu, \Delta_i(\nu)) = L_\nu(\varphi(\nu), \Delta_i(\varphi(\nu)))$, by definition of direct feed off, $(\nu, \Delta_{3-i}(\nu), \nu)$ and $(\varphi(\nu), \Delta_{3-i}(\varphi(\nu)), \varphi(\nu))$ are isomorphic to the same $DV$-quotient of the Schützenberger automaton $A(X_{3-i}, R_{3-i}; f)$ and so the lobes $\Delta_{3-i}(\nu)$ and $\Delta_{3-i}(\varphi(\nu))$ are isomorphic under an isomorphism $\psi$ such that $\psi(\nu) = \varphi(\nu)$. So it is straightforward to see that $\varphi$ can be
extended to an isomorphism from \( \Theta \cup \Delta_{3-i}(\nu) \) onto \( \Theta' \cup \Delta_{3-i}(\varphi(\nu)) \) whose restriction to \( \Delta_{3-i}(\nu) \) is \( \psi \).

From this lemma we deduce the following property.

**Proposition 4.** Let \( \Gamma, \Gamma' \) be two complete opuntoid graphs and let \( \Theta, \Theta' \) be subopuntoid subgraphs containing a host of \( \Gamma, \Gamma' \) respectively. Let \( \varphi : \Theta \to \Theta' \) be an isomorphism. Then \( \varphi \) can be extended to an isomorphism \( \varphi^* : \Gamma \to \Gamma' \).

**Proof.** Let \( P \) be the set of the pairs \( (\phi, \Gamma) \) where \( \Gamma \) is a subopuntoid subgraph of \( \Gamma \) containing \( \Theta \) and \( \phi \) is a graph monomorphism of \( \Gamma \) into \( \Gamma' \) such that \( \phi|_{\Theta} = \varphi \). Obviously \( (\varphi, \Theta) \in P \). Let \( \leq \) be the natural partial order on \( P \) defined by \( (\phi_1, \Gamma_1) \leq (\phi_2, \Gamma_2) \) if \( \Gamma_1 \) is a subopuntoid subgraph of \( \Gamma_2 \) and \( \phi_2|_{\Gamma_1} = \phi_1 \). By Hausdorff maximality lemma there is a maximal chain \( \Omega = \{ (\phi_\alpha, \Gamma_\alpha) \}_{\alpha \in I} \). Consider the pair \( (\hat{\phi}, \hat{\Gamma}) \) where \( \hat{\Gamma} = \bigcup_\alpha \Gamma_\alpha \) and \( \hat{\phi}(\nu) = \phi_\alpha(\nu) \) for \( \nu \in V(\Gamma_\alpha) \). It is easy to show that the element \( (\hat{\phi}, \hat{\Gamma}) \) belongs to \( P \) and in particular it is a maximal element of the chain \( \Omega \).

We claim \( \hat{\Gamma} = \Gamma \). Indeed suppose that, contrary to our claim, \( \hat{\Gamma} \neq \Gamma \). Then \( \hat{\Gamma} \) has a bud \( \nu \) and so only one of the two lobes \( \Delta_1(\nu), \Delta_2(\nu) \) of \( \Gamma \) is in \( \hat{\Gamma} \). Assume without loss of generality that it is \( \Delta_1(\nu) \). Then by Lemma 2 the monomorphism \( \hat{\phi} \) can be extended to an isomorphism from \( \hat{\Gamma} \cup \Delta_2(\nu) \) onto a subopuntoid graph of \( \Gamma' \) against the maximality of \( (\hat{\phi}, \hat{\Gamma}) \) whence \( \hat{\Gamma} = \Gamma \) and \( \hat{\phi} \) is an isomorphism between the opuntoid graph \( \Gamma \) onto the subopuntoid subgraph \( \hat{\phi}(\Gamma) \subseteq \Gamma' \). Suppose \( \hat{\phi}(\Gamma) \neq \Gamma' \), then \( \hat{\phi}(\Gamma) \) has a bud \( \mu \) and again only one of the two lobes \( \Delta_1(\mu), \Delta_2(\mu) \) of \( \Gamma' \) belongs to \( \hat{\phi}(\Gamma) \). Repeating the above argument on the isomorphism \( \hat{\phi}^{-1} : \hat{\phi}(\Gamma) \to \Gamma \), we get \( \hat{\phi}(\Gamma) = \Gamma' \).

We are now in position to prove the following proposition.

**Proposition 5.** Let \( \Gamma \) be a complete opuntoid graph which poses a host. Then the automorphism group of \( \Gamma \) is isomorphic to the automorphism group of the union of all hosts of \( \Gamma \).

**Proof.** Let \( Host(\Gamma) \) be the union of all hosts of \( \Gamma \). We know from Proposition 4 that each \( \phi \in Aut(Host(\Gamma)) \) can be extended to an automorphism \( \varphi \in Aut(\Gamma) \). We prove that \( \varphi \) preserves the feed off relation. Assume that \( \Delta \to \Delta' \) and let \( \nu \in V(\Delta) \cap V(\Delta') \). Then obviously \( \varphi(\nu) \in V(\varphi(\Delta)) \cap V(\varphi(\Delta')) \), moreover, if \( i \in \{1, 2\} \) is the color of \( \Delta \), then \( f(e_i(\nu)) = f(e_i(\varphi(\nu))) \) and \( L_U(\nu, \Delta) = L_U(\varphi(\nu), \varphi(\Delta)) \). Let \( f = f(e_i(\nu)) \), since \( \Delta \to \Delta' \) then

\[
(v, \Delta', \nu) \simeq A(X_{3-i}, R_{3-i}; f)/\rho
\]
where \( \rho \) is the least equivalence relation that identifies the initial vertex \( \nu \) of \( A(X_{3-i}, R_{3-i}; f(e_i(\nu))) \) with the final vertices \( y \) of all the paths \((\nu, u, y)\) with \( u \in L_U(\nu, \Delta) \) and makes the quotient deterministic. Thus

\[
(\varphi(\nu), \varphi(\Delta'), \varphi(\nu)) \simeq A(X_{3-i}, R_{3-i}; f)/\rho
\]

and so by the definition of direct feed off, \( \varphi(\Delta) \mapsto \varphi(\Delta') \). Therefore \( \varphi \) sends hosts into hosts and so \( \varphi|_{Host(\Gamma)} \) belongs to \( Aut(Host(\Gamma)) \). It is straightforward to check that the map \( \chi \) defined by \( \chi(\varphi) = \varphi|_{Host(\Gamma)} \) is a group isomorphism from \( Aut(\Gamma) \) onto \( Aut(Host(\Gamma)) \).

\[\square\]

4 Union of hosts of Schützenberger graphs

In this section we consider the union of hosts of a Schützenberger graph of the free amalgamated product \( S_1 \ast_U S_2 \) of the finite inverse semigroups \( S_1, S_2 \). First we characterize the Schützenberger graphs of the free amalgamated product \( S_1 \ast_U S_2 \) of the finite inverse semigroups \( S_1, S_2 \) with more than one host. We need to recall some results from [20].

**Proposition 6.** [20, Proposition 10]

Let \( \Gamma \) be an opuntoid graph. Let \( \Delta, \Delta' \) be two lobes of \( \Gamma \) colored respectively by \( i, 3 - i \) for some \( i = 1, 2 \) with \( \Delta \mapsto \Delta' \). Let \( \nu \in V(\Delta) \cap V(\Delta') \) be an intersection vertex of \( \Gamma \). Then \( f = f(e_i(\nu)) = e_{3-i}(\nu) \in E(U) \). Conversely if \( \Delta \simeq STi(X_{3-i}, R_{3-i}; f) \) is a lobe of \( \Gamma \) and \( f \in E(U) \), then \( \Delta' \simeq STi(X_{3-i}, R_{3-i}; f) \) is a lobe of \( \Gamma \) and \( \Delta' \mapsto \Delta \).

**Proposition 7.** [20, Theorem 23 and Proposition 18]

Let \( \Delta, \Delta' \) be two lobes of \( STi(X, R \cup W; w) \) colored respectively by \( i, 3 - i \) for some \( i = 1, 2 \) with \( \Delta \mapsto \Delta' \). Let \( \nu \in V(\Delta) \cap V(\Delta') \), \( f \in E(U) \) such that \( (\nu, \Delta', \nu) \simeq (x, STi(X_{3-i}, R_{3-i}; f)/\rho, x) \) where \( \rho \) is the least equivalence relation on \( STi(X_{3-i}, R_{3-i}; f) \) which identifies the net \( N(x, STi(X_{3-i}, R_{3-i}; f)) \) and makes \( STi(X_{3-i}, R_{3-i}; f)/\rho \) deterministic. Then

\[
(\nu, STi(X, R \cup W; f), \nu) \simeq (x, STi(X, R \cup W; f)/\Xi, x)
\]

where

\[
\Xi \subseteq V(STi(X, R \cup W; f)) \times V(STi(X, R \cup W; f))
\]

is defined by: \( q \Xi q' \) if there are \( y, y' \in N(x, STi(X_{3-i}, R_{3-i}; f)) \) and \( t \in (X \cup X^{-1})^* \) such that \( (y, t, q) \) and \( (y', t, q') \) are paths in \( STi(X, R \cup W; f) \). Moreover the following lifting property for \( \Xi \) holds: if \( (p h, q \Xi) \) is a path in \( STi(X, R \cup W; f)/\Xi \) then for each \( p \in p \Xi \) there is a path \( (p h, q') \) in \( STi(X, R \cup W; f) \) with \( q' \in q \Xi \).
We have the following characterization.

**Theorem 1.** Let \([S_1, S_2; U]\) be an amalgam of finite inverse semigroups, let \(w \in (X \cup X^{-1})^+\). The following are equivalent:

1. \(\text{ST}(X, R \cup W; w)\) has more than one host.

2. Each host of \(\text{ST}(X, R \cup W; w)\) is the Schützenberger graph of some idempotent of \(U\) relative to the presentation \(\langle X_i|R_i \rangle\) of \(S_i\) for some \(i \in \{1, 2\}\).

3. \(ww^{-1}D^1 \ast wS_2 f\) for some idempotent \(f \in E(U)\).

**Proof.**

1) \(\Rightarrow\) 3). Assume that \(\text{ST}(X, R \cup W; w)\) has more than one host. Then by Proposition 2 there are (at least) two adjacent lobes of \(\text{ST}(X, R \cup W; w)\) which are hosts. Let \(\nu\) be an intersection vertex between these adjacent hosts \(\Delta_i = \Delta_i(\nu), i \in \{1, 2\}\) and let \(f = f(e_1(\nu))\). By definition of host \(c_{R_i,W}(\Delta_2) = \text{ST}(X, R \cup W; w)\). Moreover \(\Delta_1 \Rightarrow \Delta_2\), so by Proposition 7 \(\nu, \text{ST}(X, R \cup W; w), \nu \simeq (x\Xi, \text{ST}(X, R \cup W; f)/\Xi, x\Xi)\). Let \(e\) be an idempotent labelling a loop based at \(\nu \in \text{ST}(X, R \cup W; w)\) then again by Proposition 7 \(e\) labels also a loop based at \(x \in \text{ST}(X, R \cup W; f)\), hence \(e \geq f\). So \(f\) is the minimum idempotent labelling a loop based at \(\nu \in \text{ST}(X, R \cup W; w)\) whence \(\text{ST}(X, R \cup W; w) = \text{ST}(X, R \cup W; ww^{-1}) \simeq \text{ST}(X, R \cup W; f)\), then by Proposition 4 \(ww^{-1}D^1 \ast wS_2 f\).

3) \(\Rightarrow\) 2). Put \(\Delta_i = \text{ST}(X_i, R_i; f)\) for \(i = 1, 2\). By Proposition 4 \(\text{ST}(X, R \cup W; ww^{-1}) \simeq \text{ST}(X, R \cup W; f)\). Obviously \(\text{ST}(X, R \cup W; f)\) is obtained by iterated applications of Construction 5 to \(\Delta_1\), then \(\Delta_1\) is a host of \(\text{ST}(X, R \cup W; w)\). Now let \(\Delta\) be any host of \(\text{ST}(X, R \cup W; w)\) and assume that it is colored \(j \in \{1, 2\}\). We prove that \(\Delta\) is a Schützenberger graph of some idempotent of \(U\) by induction on the length \(n\) of the reduced lobe path connecting \(\Delta_1\) to \(\Delta\). If \(n = 0\) the statement is trivially true. So let \(P : \Delta_1, \Delta_2, \ldots, \Delta_n = \Delta\) be the reduced lobe path connecting \(\Delta_1\) with \(\Delta\). Since \(\Delta_1\) and \(\Delta_n = \Delta\) are hosts, by Proposition 2 \(\Delta_{n-1}\) is a host and by induction hypothesis it is a Schützenberger graph of some idempotent of \(U\). Let \(\nu \in V(\Delta_{n-1}) \cap V(\Delta_n)\). Since \(\Delta_{n-1} \Rightarrow \Delta_n\), then by Proposition 6 \(e_{3-j}(\nu) = f(e_j(\nu)) \in E(U)\). Hence, since \(\Delta_{n-1}\) is a Schützenberger graph, \((\nu, \Delta_{n-1}, \nu) \simeq A(X_{3-j}, R_{3-j}; f(e_j(\nu)))\) and so by Proposition 6 \(\Delta_{n} \simeq \text{ST}(X_j, R_j; f(e_j(\nu)))\).

2) \(\Rightarrow\) 1). Let \(\Delta\) be a host colored \(i\) of \(\text{ST}(X, R \cup W; w)\). Then \(\Delta \simeq \text{ST}(X_i, R_i; f)\) for some \(f \in E(U)\). Then \(f = e_i(\nu) = f(e_i(\nu))\) for some \(\nu \in V(\Delta)\). Applying Construction 5 at \(\nu\) one gets a new lobe \(\Delta'\) such that \(\Delta' \Rightarrow \Delta\) by Proposition 6. Let \(\Lambda\) be any lobe of \(\text{ST}(X, R \cup W; w)\). Since \(\Delta\)
is a host, then $\Lambda$ feeds off $\Delta$ that in turns directly feeds off $\Delta'$. So $\Lambda$ feeds off $\Delta'$ and $\Delta'$ is a host.

In the sequel for a Schützenberger graph $S\Gamma(X, R \cup W; w)$ we denote $\text{Host}(S\Gamma(w))$ the union of all its hosts. We characterize Schützenberger graphs $S\Gamma(X, R \cup W; e)$, such that $\text{Host}(S\Gamma(e))$ is an infinite graph. Since a host has finitely many finite lobes, these opuntoids need to have infinitely many hosts, hence by Theorem 1 we necessarily have $eD^{S_1 S_2} f$ for some idempotent $f \in E(U)$. By the above Theorem in such case all hosts are lobes which are Schützenberger graphs of some idempotents of $U$ relative to the presentation $(X_i | R_i)$. Let

**Definition 1.** Let $\Delta, \Delta'$ be two lobes of an opuntoid graph such that $\Delta' = \phi(\Delta)$ for some isomorphism $\phi$. Let $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ be the reduced lobe path connecting $\Delta$ to $\Delta'$ and let $\nu_1 \in V(\Delta_0) \cap V(\Delta_1)$. The isomorphism $\phi$ is called a shift-isomorphism (and $\Delta, \Delta'$ are called shift-isomorphic by $\phi$) if $\phi(\nu_1) \notin V(\Delta_{n-1}) \cap V(\Delta_n)$.

The lobes $\Delta, \Delta'$ are called successive isomorphic lobes if no $\Delta_i$ ($0 < i < n$) is isomorphic to $\Delta_0$.

We have the following lemma.

**Lemma 3.** Let $e \in E(S_1 * U S_2)$ with $S_1, S_2$ finite inverse semigroups and let $eD^{S_1 * U S_2} f$ for some idempotent $f \in E(U)$. Let $\Delta, \Delta'$ be two distinct lobes of $\text{Host}(S\Gamma(e))$ colored $i$ such that $\Delta' = \phi(\Delta)$ for some $\phi \in \text{Aut}(\text{Host}(S\Gamma(e)))$.

Let $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ be the reduced lobe path connecting $\Delta$ to $\Delta'$. Then either for some $j$ with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$ $\phi|_{\Delta_j}$ is a shift-isomorphism or, for all $j$ with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, $\phi(\Delta_j) = \Delta_{n-j}$.

**Proof.** By Proposition 2 and Theorem 1 each $\Delta_h$, $0 \leq h \leq n$, is a host which is the Schützenberger graph of some idempotent of $U$ relative to the presentation $(X_i | R_i)$ for some $i = 1, 2$. We prove the statement by induction on $n \geq 2$. The base of induction is trivial. If $\phi|_{\Delta_0}$ is a shift-isomorphism of $\Delta_0$ onto $\Delta_n$ the statement is trivially true. So assume that $\phi|_{\Delta_0}$ is not a shift-isomorphism. Let $\nu_1 \in V(\Delta_0) \cap V(\Delta_1)$, then $\phi(\nu_1) \in V(\Delta_{n-1}) \cap V(\Delta_n)$. Moreover by Proposition 3 we get $f_1 = e_i(\nu_1) \in E(U)$. Hence $e_i(\phi(\nu_1)) = f_1 \in E(U)$, and so $(\phi(\nu_1), \Delta_{n-1}, \phi(\nu_1)) \simeq (\nu_1, \Delta_1, \nu_1)$, $\phi(\Delta_1) = \Delta_{n-1}$. Since the reduced lobe path from $\Delta_1$ to $\Delta_{n-1}$ has length $n - 1$ the statement holds by induction hypothesis.

**Proposition 8.** Let $e \in E(S_1 * U S_2)$ with $S_1, S_2$ finite inverse semigroups and let $eD^{S_1 * U S_2} f$ for some idempotent $f \in E(U)$. Then the following are equivalent

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1. \textit{Host}(\textit{ST}(e)) is infinite;

2. \textit{Host}(\textit{ST}(e)) has infinitely many lobes;

3. There are two isomorphic hosts of \textit{ST}(e) which are not successive isomorphic lobes;

4. There is a shift-isomorphism between two hosts of \textit{ST}(e).

\textit{Proof.} The equivalence between 1) and 2) is trivial.

1) $\Rightarrow$ 3). By Theorem 4 each lobe of \textit{Host}(\textit{ST}(e)) is a Schützenberger graph of some idempotent of $U$ relative to the presentation $\langle X_i | R_i \rangle$ for some $i \in \{1, 2\}$. Since $S_1, S_2$ are finite, there are finitely many Schützenberger graphs of idempotents of $U$ relative to the presentations $\langle X_i | R_i \rangle$ with $i \in \{1, 2\}$. Since all the lobes are finite, each lobe has finitely many adjacent lobes and so the degree of each vertex of the lobe tree $T(\textit{Host}(\textit{ST}(e)))$ is finite. Therefore there is an infinite reduced lobe path in $T(\textit{Host}(\textit{ST}(e)))$ in which there are at least three isomorphic lobes, whence there are two isomorphic hosts which are not successive.

3) $\Rightarrow$ 4). Suppose that \textit{ST}(e) has two isomorphic hosts $\Delta$ and $\Delta'$ which are not successive isomorphic lobes. Thus, in the reduced lobe path $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ connecting them, there is a lobe $\Delta_h$ with $1 \leq h \leq n-1$ isomorphic to both $\Delta$ and $\Delta'$. We can assume without loss of generality that no $\Delta_j$ with $1 \leq j \leq n-1, j \neq h$ is isomorphic to $\Delta$. Since isomorphic lobes have the same color then $n \geq 4$ is even and let $t = n/2$. Let $\phi$ be the isomorphism sending $\Delta$ onto $\Delta'$. By Propositions 4 and 5 $\phi$ can be extended to an automorphism $\overline{\phi} \in \text{Aut}(\textit{Host}(\textit{ST}(e)))$. Assume that for each lobe $\Lambda$ of \textit{Host}(\textit{ST}(e)) $\overline{\phi}|_{\Lambda}$ is not a shift-isomorphism of $\Lambda$ onto some host $\Lambda'$. Then by Lemma 3 for all $j$ with $0 \leq j < t$, $\overline{\phi}(\Delta_j) = \Delta_{2t-j}$. Then $t = h$, otherwise both $\Delta_h$ and $\Delta_{n-h}$ would be isomorphic to $\Delta$, hence $\overline{\phi}|_{\Delta_h} \in \text{Aut}(\Delta_h)$. Let $\nu_t \in V(\Delta_{t-1}) \cap V(\Delta_t)$. Then $\overline{\phi}(\nu_t) \in V(\Delta_t) \cap V(\Delta_{t+1})$. Now let $\psi : \Delta \to \Delta_t$ be an isomorphism. If $\psi$ is a shift-isomorphism then we are done, otherwise if $\nu_1 \in V(\Delta_0) \cap V(\Delta_1)$, then $\psi(\nu_1) \in V(\Delta_t) \cap V(\Delta_{t-1})$. Using the fact that $\psi$ preserves labeling, it is easy to see that $\psi$ is actually a bijection between the two sets $V(\Delta_0) \cap V(\Delta_1)$ and $V(\Delta_t) \cap V(\Delta_{t-1})$. Thus let $\nu'$ be the vertex of $V(\Delta_0) \cap V(\Delta_1)$ such that $\psi(\nu') = \nu_t \in V(\Delta_{t-1}) \cap V(\Delta_t)$. Hence $\overline{\phi}(\psi(\nu')) = \overline{\phi}(\nu_t) \in V(\Delta_t) \cap V(\Delta_{t+1})$, i.e. $\overline{\phi}(\psi(\nu')) \notin V(\Delta_{t-1}) \cap V(\Delta_t)$. Hence the map $\psi \cdot \overline{\phi} : \Delta \to \Delta_t$ (defined by $\psi \cdot \overline{\phi}(\nu) = \overline{\phi}(\psi(\nu))$) is a shift-isomorphism from $\Delta$ to $\Delta_t$.

4) $\Rightarrow$ 2). Assume by contradiction that \textit{Host}(\textit{ST}(e)) has two shift-isomorphic lobes, and finitely many lobes. Let $\Delta, \Delta'$ be two shift-isomorphic lobes by $\phi$
in \( \text{Host}(\Gamma(e)) \) such that the reduced lobe path \( \Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta' \) from \( \Delta \) to \( \Delta' \) is of maximal length. By Proposition 2 and Theorem 4, each \( \Delta_j, 0 \leq j \leq n \) is a host and the Schützenberger graph of some idempotent of \( \Gamma(e) \). Moreover by Proposition 3, \( \nu_j \), \( \nu_j \) are isomorphic under a shift-isomorphism. Then the maximal length among the reduced lobe paths connecting two hosts which are isomorphic under a shift-isomorphism is infinite if and only if \( \Delta_0 = \Delta \). By Proposition 4, \( \nu_j \) the isomorphism \( \phi : \Delta \to \Delta' \) can be extended to an automorphism \( \phi \in \text{Aut}(\text{Host}(\Gamma(e))) \).

We prove by induction on \( h \) that, for all \( h \) with \( 0 \leq h \leq n \), \( \overline{\phi} \) maps the subopuntoid subgraph \( \Theta = \bigcup_{0 \leq j \leq h} \Delta_j \) of \( \text{Host}(\Gamma(e)) \) onto a subopuntoid subgraph of \( \text{Host}(\Gamma(e)) \) whose lobes, except eventually \( \Delta_n \), are all different from the lobes of \( \Theta \).

Moreover \( \phi \) is a shift-isomorphism between \( \Delta_h \) and \( \Delta' \). The base of induction is trivial. So let \( \Theta_{h-1} = \bigcup_{0 \leq j \leq h-1} \Delta_j \) and put \( \Delta_n + j = \overline{\phi}(\Delta_j) \) for all \( 0 \leq j \leq h-1 \). By Lemma 1, \( \overline{\phi}(\Theta_{h-1}) = \bigcup_{0 \leq j \leq h-1} \Delta_n + j \) and for all \( j \) with \( 0 \leq j \leq h-2 \), \( \Delta_n + j \) is adjacent to \( \Delta_n + j + 1 \). Moreover by induction hypothesis \( \Theta_{h-1} \) and \( \overline{\phi}(\Theta_{h-1}) \) have disjoint sets of lobes and \( \overline{\phi} \) is a shift-isomorphism of \( \Delta_{h-1} \) onto \( \Delta_{n+h-1} \). Let \( \nu_h \) be an intersection vertex between \( \Delta_{h-1} \) and \( \Delta_h \) and put \( \Delta_n + j = \overline{\phi}(\Delta_j) \) for all \( 0 \leq j \leq h-1 \). By Lemma 2, \( \overline{\phi}(\Theta_{h-1}) = \bigcup_{0 \leq j \leq h-1} \Delta_n + j \) and for all \( j \) with \( 0 \leq j \leq h-2 \), \( \Delta_n + j \) is adjacent to \( \Delta_n + j + 1 \). Moreover by induction hypothesis \( \Theta_{h-1} \) and \( \overline{\phi}(\Theta_{h-1}) \) have disjoint sets of lobes and \( \overline{\phi} \) is a shift-isomorphism between \( \Delta_h \) and \( \Delta_n + h \). In particular for \( h = n \), \( \overline{\phi} \) is a shift-isomorphism of \( \Delta_n \) onto \( \Delta_{2n} \), and \( \overline{\phi}^2 \) is a shift-isomorphism of \( \Delta_0 \) onto \( \Delta_{2n} \), against the assumption that the reduced lobe path connecting \( \Delta = \Delta_0 \) to \( \Delta' = \Delta_n \) is a path of maximal length among the reduced lobe paths connecting two hosts which are isomorphic under a shift-isomorphism. Then \( \text{Host}(\Gamma(e)) \) has infinitely many lobes.

From the above proposition we derive the following corollary.

**Corollary 1.** Let \( e \in E(S_1 * U S_2) \) with \( S_1, S_2 \) finite inverse semigroups and let \( e \Delta S_1 * U S_2 f \) for some idempotent \( f \in E(U) \). Then \( \text{Host}(\Gamma(e)) \) is infinite if and only if in \( \text{Host}(\Gamma(e)) \) there is a reduced lobe path \( P : \Delta_0, \ldots, \Delta_{t-1}, \Delta_t, \Delta_{t+1}, \ldots, \Delta_{2t} \) with \( \Delta_0 \simeq \Delta_t \simeq \Delta_{2t} \).

**Proof.** By Proposition 8 if \( \text{Host}(\Gamma(e)) \) is infinite then there is a shift-isomorphism between two lobes of \( \text{Host}(\Gamma(e)) \). Let \( \Delta \) and \( \Delta' \) be such lobes and then let \( \phi : \Delta \to \Delta' \) be a shift-isomorphism. If \( P \) is the reduced lobe path from \( \Delta \) to \( \Delta' \), by the same argument of the proof of implication 4) \( \Rightarrow \) 2) of Proposition 8, \( P \cup \phi(P) \) is a reduced lobe path in \( \text{Host}(\Gamma(e)) \) satisfying the statement. Conversely, let \( P : \Delta_0, \ldots, \Delta_{t-1}, \Delta_t, \Delta_{t+1}, \ldots, \Delta_{2t} \)
with $\Delta_0 \simeq \Delta_t \simeq \Delta_{2t}$ be a reduced lobe path of $\text{Host}(S\Gamma(e))$, then in $P$ there are two non successive isomorphic lobes and so $\text{Host}(S\Gamma(e))$ is infinite by Proposition 8.

5 Review of the Bass-Serre theory

For the sake of completeness, we shortly review the Bass-Serre theory of groups acting on graphs. We refer the reader interested in more details to [9] [23]. Let $G$ be a group, and let $X = (\text{Vert}(X), \text{Edge}(X))$ be a graph with initial and terminal vertex maps $\alpha, \omega : \text{Edge}(x) \to \text{Vert}(X)$. If the action $\cdot$ of $G$ on the set $\text{Vert}(X) \cup \text{Edge}(X)$ satisfies for all $g \in G, y \in \text{Edge}(X)$ the conditions: $\alpha(g \cdot y) = g \cdot \alpha(y), \omega(g \cdot y) = g \cdot \omega(y)$ and $g \cdot \overline{y} = g \cdot y$ (where $\overline{y}$ denotes the opposite edge of $y$), we say that $G$ acts on the graph $X$.

The action of $G$ on the graph $X$ is without inversions if $g \cdot y \neq \overline{y}$ for all $y \in \text{Edge}(X), g \in G$. If $G$ acts on a graph $X$ without inversions, the quotient graph of the action of $G$ on $X$ is the graph $G \backslash X$ whose vertex and edge sets are respectively the set $\{G \cdot v|v \in \text{Vert}(X)\}$ of the orbits of $G$ of the vertices of $X$ and the set $\{G \cdot y|y \in \text{Edge}(X)\}$ of the orbits of $G$ of the edges of $X$, with the incidence relation defined by $\alpha(G \cdot y) = G \cdot \alpha(y), \omega(G \cdot y) = G \cdot \omega(y)$ and $G \cdot y = G \cdot \overline{y}$ for all $y \in \text{Edge}(X)$. The map $v \to G \cdot v, y \to G \cdot y$, for all $v \in \text{Vert}(X), y \in \text{Edge}(X)$ is a map of the graph $X$ onto $G \backslash X$ and when $X$ is a connected graph the subtrees of $G \backslash X$ lift to subtrees of $X$ [23].

Let $X$ be a connected non-empty graph and let $\mathcal{G}$ be a mapping assigning to each $v \in \text{Vert}(X)$ and to each $y \in \text{Edge}(X)$ a group $G_v$ and $G_y$ so that $G_y = G_\overline{y}$. Assume that for each $y \in \text{Edge}(X)$ there are two group monomorphisms $\sigma_y : G_y \to G_{\alpha(y)} : \tau_y : G_y \to G_{\omega(y)}$ such that $\sigma_y = \tau_\overline{y}$. Then $X$ with the group assignment $\mathcal{G}$ is called a graph of groups $(\mathcal{G}(-), X)$.

Let $T$ be any maximal subtree of $X$, the fundamental group $\pi(\mathcal{G}(-), X, T)$ of $(\mathcal{G}(-), X)$ with respect to $T$ is generated by the disjoint union of vertex groups $G_v, v \in \text{Vert}(X)$ and by the edges $\text{Edge}(X)$, subject to the relations $\{\overline{y} = y^{-1}, y^{-1}\sigma_y(a)y = \tau_y(a)|y \in \text{Edge}(X), a \in G_y\} \cup \{y = 1|y \in T\}$. The fundamental group of a graph of groups is, up to isomorphisms, independent on the choice of $T$ and so it will be denoted by $\pi(\mathcal{G}(-), X)$. From the structure theorem of Bass-Serre theory we know that

**Proposition 9.** Let $(\mathcal{G}(-), X)$ be a graph of groups. Then each vertex group $G_v$ and edge group $G_y$ is embedded in the fundamental group $\pi(\mathcal{G}(-), X)$.

Two graphs of groups $(\mathcal{G}(-), X), (\mathcal{H}(-), Y)$ are said to be isomorphic if there is a graph isomorphism $\Phi : X \to Y$ together with a collection of group
isomorphisms $\Phi_v : G_v \to H_{\Phi(v)}$, $\Phi_y : G_y \to H_{\Phi(y)}$ satisfying the conditions $\Phi_y = \Phi_\sigma$, $\Phi_\sigma(y) = \sigma_\Phi(y)$ and $\Phi_\omega(y) \tau_y = \tau_{\Phi(y)} \Phi(y)$. It is easy to see that isomorphic graphs of groups have isomorphic fundamental groups.

A graph of groups $(\mathcal{H}(-), X)$ is conjugate to $(\mathcal{G}(-), X)$ if it has the same group assignments of $(\mathcal{G}(-), X)$, the same embedding $\sigma_y$ and whose embedding $\tau_y$ are the ones of $(\mathcal{G}(-), X)$ followed by a conjugation by an element of $G_\omega(y)$. A graph of groups $(\mathcal{H}(-), Y)$ is conjugate isomorphic to the graph of groups $(\mathcal{G}(-), X)$ if it is isomorphic to a conjugate of $(\mathcal{G}(-), X)$. Two conjugate isomorphic graphs of groups have the same fundamental groups.

Now we outline how to construct a graph of groups starting from the action of a group $G$ on a connected non-empty graph $X$. Let $Y = G \setminus X$ and let $A$ be an orientation of the edges of $Y$, i.e a subset of edges of $Y$ containing exactly one edge for each pair of opposite edges in $Y$. Let $T$ be a maximal subtree of $Y$ and $T'$ its lifting to $X$, for each $v \in Vert(T) = Vert(Y), y \in Edge(T)$, we denote by $j(v), j(y)$ the lift of $v$ and $y$ in $T'$. For each $v \in Vert(T)$ and $y \in Edge(T)$ we put $G_v = Stab_G(j(v))$ and $G_y = Stab_G(j(y))$. Since $Stab_G(j(y)) \subseteq Stab_G(j(\alpha(y)))$ and $Stab_G(j(y)) \subseteq Stab_G(j(\omega(y)))$, then for each $y \in Edge(T)$ the monomorphisms $\sigma_y, \tau_y$ are the inclusions. Now let $y \in (\text{Edge}(Y) - \text{Edge}(T)) \cap A$ and let $x = j(y) \in \text{Edge}(X)$ an edge mapping in $y$ such that $\alpha(x) = j(\alpha(x))$. Again we put $G_y = Stab_G(j(y))$ and $\sigma_y$ is the inclusion of $G_y$ in $G_\alpha(x)$. Moreover since $\omega(x)$ and $j(\omega(y))$ belong to the same vertex orbit of $G$ there exists $g_y \in G$ mapping $\omega(x)$ to $j(\omega(y))$ and the two groups $Stab_G(\omega(x)), Stab_G(j(\omega(y)))$ are conjugate in $G$ by $g_y$. So we set $\tau_y = g_y \cdot \iota \cdot g_y^{-1}$ where $\iota$ is the inclusion of $Stab_G(j(y))$ in $Stab(\omega(x))$. Then we put for each edge $y \in \text{Edge}(Y) - A G_y = G_\sigma = Stab_G(\gamma), \sigma_y = \tau_\sigma, \tau_y = \sigma_\tau$, completely constructing a graph of groups $(\mathcal{G}(-), Y)$. Its fundamental group $\pi(\mathcal{G}(-), Y)$ is homomorphic on $G$ by the group homomorphism $\Phi$ defined by the inclusions $G_v \to G$ and the mapping $\Phi(y) = g_y$ where $g_y$ is the element of $G$ mapping $\omega(x)$ on $j(\omega(y))$. The structure theorem of Bass-Serre theory says

**Theorem 2.** Let $G$ be a group acting without inversions on a connected graph $X$. Then $X$ is tree if and only if $\Phi : \pi(\mathcal{G}(-), G \setminus X) \to G$ is an isomorphism.

6 Maximal subgroups

Let $e \in E(S_1 \ast_U S_2)$, the lobe graph $\mathcal{T}_e = \mathcal{T}(\text{Host}(\text{ST}(e)))$ can be seen as a directed tree $(\text{Vert}(\mathcal{T}_e), \text{Edge}(\mathcal{T}_e), \alpha, \omega)$ where $\text{Vert}(\mathcal{T}_e)$ is the set of lobes of union of all hots $\text{Host}(\text{ST}(e))$. $\text{Edge}(\mathcal{T}_e)$ is formed by the pairs of adjacent
lobes of $\text{Host}(S\Gamma(e))$, oriented so that, for each $y = (\Delta, \Delta') \in \text{Edge}(T_e)$, $\alpha(y)$ is the lobe $\Delta$ colored 1 and $\omega(y)$ is the lobe $\Delta'$ colored 2. Lemma 1 and Proposition 5 prove that the group $H_{\text{e}}^{S_1 * U S_2} \simeq \text{Aut}(S\Gamma(e))$ acts on $T_e$ without inversions. Thus we can build a graph of groups $(G(-), G \setminus T_e)$ starting from the action of the group $G = H_{\text{e}}^{S_1 * U S_2}$ on the connected non-empty graph $T_e$ as described in the previous section. As in Section 5, the quotient graph $G \setminus T_e$ will be denoted by $Y$. Theorem 2 gives us immediately the following:

**Corollary 2.** Let $e \in E(S_1 * U S_2)$ be an idempotent in the amalgamated free product of two finite inverse semigroups $S_1, S_2$. Let $Y = H_{\text{e}}^{S_1 * U S_2} \setminus T_e$. Then

$$H_{\text{e}}^{S_1 * U S_2} \simeq \pi(G(-), Y).$$

To study the structure of maximal subgroups in more details, we analyze the two following different situations:

- **Case 1:** $e$ is an “old idempotent”: $e$ is $D^{S_1 * U S_2}$-related to some idempotent of $S_1$ or $S_2$.

- **Case 2:** $e$ is a “new idempotent”: $e$ is not $D^{S_1 * U S_2}$-related to any idempotent of $S_1$ or $S_2$.

### 6.1 Case 1

Although in general the lobes of Schützenberger graphs of elements of amalgams of finite inverse semigroups are only DV-quotients, if we restrict our attention to hosts of Schützenberger graphs of original idempotents the situation is nicer:

**Theorem 3.** Let $e$ be an idempotent in $S_1$ or $S_2$. With the above notations, let $Y = H_{\text{e}}^{S_1 * U S_2} \setminus T_e$. Then

1. each $\Delta \in \text{Vert}(Y)$ is a Schützenberger graph $S\Gamma(X_i, R_i; e_i(\nu))$ for some $\nu \in \text{Vert}(\Delta)$;
2. $Y$ is finite;
3. $(\Delta_1, \Delta_2) \in \text{Edge}(Y)$ if and only if the following conditions hold
   - $e$ is $D^{S_1 * U S_2}$-related to some idempotent of $U$;
   - $\Delta_i(\nu) \simeq S\Gamma(X_i, R_i; f)$ for some $f \in E(U)$;
there is a lobe $\Delta'_2$ of $\text{Host}(ST(e))$ such that $(\Delta_1, \Delta'_2) \in \text{Edge}(\mathcal{T}_e)$, $\psi(\Delta'_2) = \Delta_2$, for some automorphism $\psi \in \text{Aut}(\text{Host}(ST(e)))$, and $e_i(\nu') = f$ for intersection vertex $\nu'$ between $\Delta_1$ and $\Delta'_2$.

Proof. Assume that $e$ is an idempotent of $S_1$. If we start from a word $u \in (X_1 \cup X_1^{-1})^*$ equivalent to $e$ in $S_1$, then it is clear that the underlying graph $\Delta_0$ of $\text{Core}(u)$ is isomorphic to $ST(X_1, R_1; u)$. Thus $\Delta_0$ is a host of $\Gamma = ST(X, R \cup W; e)$. If $e$ is not $D^{S_1 \ast U S_2}$-related to any idempotent of $U$ then it is the unique host, otherwise $\Gamma$ has more than one host, each host is a lobe and a Schützenberger graph of some idempotent of $U$ relative to the presentation $\langle X_i | R_i \rangle$, for some $i \in \{ 1, 2 \}$ by Theorem 1 so statement 1 is proved.

If $e$ is not $D^{S_1 \ast U S_2}$-related to any idempotent of $U$ then $Y$ is trivially finite. So assume that $e$ is $D^{S_1 \ast U S_2}$-related to some idempotent of $U$. Since by Theorem 1 all hosts of $ST(e)$ are Schützenberger graphs of idempotents of $U$, there are at most $| U |$ different hosts up to isomorphisms. From Propositions 1 2 it follows that two isomorphic hosts lie in the same $H^{S_1 \ast U S_2}_e$-orbit, and so statement 2 holds.

Let us prove statement 3. The “if” part is trivial. We prove the “only if” part. Let $(\Delta_1, \Delta_2) \in \text{Edge}(Y)$, then $\Gamma$ has more than one host, hence by Theorem 1 $e$ is $D^{S_1 \ast U S_2}$-related to some idempotent of $U$. Moreover there is $(\Delta'_1, \Delta'_2) \in \text{Edge}(\mathcal{T}_e)$ which lies in the same $H^{S_1 \ast U S_2}_e$-orbit of $(\Delta_1, \Delta_2)$. The lobes $\Delta'_1, \Delta'_2$ are adjacent, feed off each other and share at least a common vertex $q$. By Theorem 1 $\Delta'_i \simeq ST(X_i, R_i; e_i(q))$ with $e_i(q) \in E(U)$ for $i = 1, 2$. Moreover there is an automorphism $\varphi$ of $\text{Host}(ST(e))$ such that $\varphi(\Delta'_1) = \Delta_1$, then $\Delta_1$ and $\varphi(\Delta'_2)$ are adjacent lobes that share the vertex $\varphi(q) = \nu'$. Hence $e_i(\nu') = e_i(q) \in E(U)$. $\varphi(\Delta'_2) \simeq ST(X_2, R_2; e_2(\nu'))$ is a host and $(\varphi(\Delta'_2)) \in \text{Edge}(\mathcal{T}_e)$ lies in the same $H^{S_1 \ast U S_2}_e$-orbit of $(\Delta_1, \Delta_2)$. Then there is an automorphism $\psi$ of $\text{Host}(ST(e))$ such that $\psi(\varphi(\Delta'_2)) = \Delta_2$, $\psi(\Delta_1) = \Delta_1$ and $\psi(\nu') = \nu$, hence $e_1(\nu) \in E(U)$ and $\Delta_1 \simeq ST(X_1, R_i; f)$ with $f = e_1(\nu)$.

As an immediate consequence of the previous proof, we get that the graph of groups, in the case when $e$ is an original idempotent but it is not $D$-related in the amalgam to any idempotent in $U$, consists of just a single vertex. This vertex corresponds to the Schützenberger graph of some $g$ in $S_i$ and we get by, Proposition 1 the following description of maximal subgroups in this situation.

**Corollary 3.** Let $e$ be $D$-related in $S_1 \ast U S_2$ to some idempotent $g$ of $S_i \setminus U$, $i = 1, 2$, then the maximal subgroup $H^{S_1 \ast U S_2}_e \simeq H^S_{g^i}$. 
Let \( e \) be \( D \)-related in \( S_1 \ast \cup S_2 \) to some idempotent of \( U \), then by Theorem 1 the Schützenberger graph \( ST(X, R \cup W; e) \) has more than one host and each host is a lobe and a Schützenberger graph of some idempotent of \( U \) relative to the presentation \( \langle X_i | R_i \rangle \) of \( S_i \), for some \( i \in \{1, 2\} \). Moreover, by Lemma 3 the graph \( Y \) is finite and in this case the graph of groups \( (\mathcal{H}_e(-), Z_e) \) built according to Bennett’s construction can be used to describe the structure of the maximal subgroup of \( S_1 \ast \cup S_2 \) containing \( e \). Namely, we can prove, along the same line of the proof of Theorem 2 in [24] that the graph of \( \mathcal{H}(\cdot, Y) \), built starting from the action of the group \( G = H_e^{S_1 \ast \cup S_2} \) on the connected non-empty graph \( T_e \) as described in the Section 3 is conjugate isomorphic to the graph of groups \( (\mathcal{H}(\cdot, Z_e)) \). However, the graph of groups \( (\mathcal{G}(\cdot, Y)) \) gives us more information, thanks also to Theorem 3 from [24] from which we can obtain a better description of the associated groups which are stabilizers of vertices and edges in \( T_e \). Indeed, by Theorem 3 vertices of \( T_e \) are Schützenberger graphs of idempotents belonging to \( U \). Thus we immediately derive from Proposition 1 and Propositions 4, 5 that the stabilizers of the vertices appearing in the graph of groups \( (\mathcal{G}(\cdot, Y)) \) are maximal subgroups of idempotents of \( U \) in the original semigroups \( S_1 \), \( S_2 \) (depending on the color of the lobe). The next proposition gives us a description of the stabilizer of an edge in \( T_e \), but first we need the following lemma.

**Lemma 4.** Let \( (\nu, \Delta, \nu) = \mathcal{A}(X_k, R_k, f) \) for some \( k \in \{1, 2\} \) and \( f \in U \). Let \( I(\nu, \Delta) = \{ y \in V(\Delta) : (\nu, u, y) \text{ is a path in } \Delta \text{ for some } u \in U \} \). Then

\[
H^U_I \simeq \{ \varphi \in \text{Aut}(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta) \}
\]

**Proof.** Theorem 3.5 of [24] shows that \( H^U_I \simeq \text{Aut}(\Delta) \) by the isomorphism \( m \mapsto \phi_m \) defined by \( \phi_m(v) = m^{-1}v \). Since \( \phi_k \) is an embedding of \( H^U_I \) into \( H^U_I \), then \( H^U_I \) also embeds into \( \text{Aut}(\Delta) \). We claim that the map \( u \mapsto \psi_{\phi_k(u)} \) defined by and \( \psi_{\phi_k(u)}(v) = \phi_k(u^{-1})v \) with \( v \in V(\Delta) \), \( u \in H^U_I \) is an isomorphism from \( H^U_I \) onto \( \{ \varphi \in \text{Aut}(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta) \} \). To show that \( \psi_{\phi_k(u)}(\nu) \in \{ \varphi \in \text{Aut}(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta) \} \) it is enough to prove that \( \psi_{\phi_k(u)}(\nu) \in I(\nu, \Delta) \) since each element of \( I(\nu, \Delta) \) is connected to \( \nu \) by some element of \( U \) and \( \psi_{\phi_k(u)} \in \text{Aut}(\Delta) \). Since \( f \) is the unity of \( H^U_I \) we get \( u = fu\bar{f} \), moreover since \( \nu = \phi_k(f) \) we get:

\[
\psi_{\phi_k(u)}(\nu) = \phi_k(fu^{-1}f)\phi_k(f) = \phi_k(f)\phi_k(fu^{-1}f) = \nu\phi_k(fu^{-1}f)
\]

so by [24] \( fu^{-1}f \) labels a path connecting \( \nu \) to \( \psi_{\phi_k(u)}(\nu) \), whence \( \psi_{\phi_k(u)}(\nu) \in I(\nu, \Delta) \). It remains to show that \( u \mapsto \psi_{\phi_k(u)} \) is surjective. Let \( \psi \in \{ \varphi \in \text{Aut}(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta) \} \) then there is some \( u \in U \) which labels a
path in $\Delta$ connecting $\nu$ to $\psi(\nu)$. Since $\psi$ is an automorphism also $fuf$ labels a path connecting $\nu$ to $\psi(\nu)$. Note that the element $fu^{-1}f \in H^U_f$ and $\nu\phi_k(fuf) = \psi(\nu)$. Thus, consider the automorphism $\psi_{\phi_k(fu^{-1}f)}$, then:

$$\psi_{\phi_k(fu^{-1}f)}(\nu) = \psi_{\phi_k(fu^{-1}f)}(\phi_k(f)) = \phi_k(f)\phi_k(fuf) = \nu\phi_k(fuf) = \psi(\nu)$$

and so $\psi_{\phi_k(fu^{-1}f)} = \psi$ since they coincide on a vertex. □

**Proposition 10.** Let $\Delta_1, \Delta_2$ be two adjacent lobes of $\text{Host}(S\Gamma(e))$ and let $f = f(e_1(\nu)) = f(e_2(\nu))$ for some intersection vertex $\nu \in V(\Delta_1) \cap V(\Delta_2)$. If we let $e = (\Delta_1, \Delta_2)$, then $\text{Stab}_G(e) \simeq H^U_f \ (G = H^{S_1*U*S_2}_e)$.

**Proof.** Using Propositions 4 and 5 it is straightforward to check that $\text{Stab}_G(e)$ is isomorphic to $\{ \varphi \in \text{Aut}(\Delta_1) : \varphi(V(\Delta_1) \cap V(\Delta_2)) \subseteq V(\Delta_1) \cap V(\Delta_2) \}$. By the assimilation property we have

$$V(\Delta_1) \cap V(\Delta_2) = \{ y \in V(\Delta_1) : (\nu, u, y) \text{ is a path in } \Delta_1 \text{ for some } u \in U \}.$$

and so $\text{Stab}_G(e) \simeq H^U_f$ by Lemma 3. □

For the clarity of the presentation we record the previous facts in the following theorem.

**Theorem 4.** With the above notations, if $e$ is $D^{S_1*U*S_2}$-related to some idempotent of $U$, then

$$H^{S_1*U*S_2}_e \simeq \pi(G(-), Y)$$

where $Y$ is finite. Moreover, the group $G_v,$ $v \in \text{Vert}(Y)$, is a maximal subgroups in $S_1$ or $S_2$ of some idempotents of $U$, while $G_e,$ $e \in \text{Edge}(Y)$, is a maximal subgroup in $U$.

Since $Y$ is finite, from [9 page 14] follows that, $H^{S_1*U*S_2}_e$ is built by iteratively performing an amalgamated free-product for each edge belonging to the maximal subtree $T$ of $Y$, followed by HNN-extensions for each edge not in $T$. Therefore the next natural steps is to characterize whether $Y$ is a tree or not. This clearly gives us a characterization which reveals whether the construction of $H^{S_1*U*S_2}_e$ involves just iterated group amalgams, or it also involves HNN-extensions. First we characterize the case when $H^{S_1*U*S_2}_e$ is finite.

**Proposition 11.** Let $e \in E(S_1*U*S_2)$ with $eD^{S_1*U*S_2}f$ for some $f \in E(U)$. Then $H^{S_1*U*S_2}_e$ is finite if and only if $H^{S_1*U*S_2}_e \simeq H^{S_k}_g$, for some $g \in E(U)$, $k \in \{1, 2\}$.
Proof. By Proposition \[5\] and Propositions \[4\] and \[5\] $H_e^{S_1\ast U\ast S_2}$ is finite if and only if $Host(S\Gamma(f))$ is finite. If $Host(S\Gamma(e))$ is finite, since the automorphism group of a finite tree fixes a vertex or an edge (see [1, Subsection 27.1.3]), then it is straightforward to check that in this case each automorphism $\phi$ of $Host(S\Gamma(e))$ has to fix a lobe $\Delta = S\Gamma(X_k, R_k; g)$, for some $k \in \{1, 2\}$, $g \in E(U)$. Thus $Aut(Host(S\Gamma(e))) \simeq Aut(\Delta)$, whence $H_e^{S_1\ast U\ast S_2} \simeq H_{g}^{S_k}$. The converse is trivial.

For infinite maximal subgroups we have the following characterization.

**Theorem 5.** Let $e \in E(S_1\ast U\ast S_2)$ with $eD^{S_1\ast U\ast S_2}f$ for some $f \in E(U)$. Then the following are equivalent:

1. $H_e^{S_1\ast U\ast S_2}$ is infinite;
2. there is a sequence $f_1, f_2, ..., f_{2t-2}$ of idempotents of $U$ for some $t > 1$ such that:
   - $fD^{S_1\ast U\ast S_2}f_1$,
   - for each $1 \leq i < 2t-2$, $f_{i-1}$ and $f_i$ are not $D^U$-related,
   - there is some $k \in \{1, 2\}$ such that $f_1D^{S_k}f_1D^{S_k}f_{2t-2}$, and for each $1 \leq i < 2t-2$ even $f_{i-1}D^{S_3-k}f_iD^{S_k}f_{i+1}$, and $f_{2t-3}D^{S_3-k}f_{2t-2}$
3. $Y = H_e^{S_1\ast U\ast S_2}\setminus \mathcal{T}_e$ is not a tree.

Proof. Again, by Proposition \[5\] and Propositions \[4\] and \[5\] $H_e^{S_1\ast U\ast S_2}$ is infinite if and only if $Host(S\Gamma(f))$ is infinite. Moreover by Corollary \[1\] $Host(S\Gamma(f))$ is infinite if and only if there is a reduced lobe path

$$P : \Delta_1, \ldots, \Delta_t, \ldots, \Delta_{2t-1}$$

with $\Delta_1 \simeq \Delta_t \simeq \Delta_{2t-1}$. We prove the equivalence 1 $\iff$ 2 by showing that this geometric characterization is equivalent to the algebraic conditions described in the statement 2.

1) $\Rightarrow$ 2) Take any intersection vertex $\nu_i$ of $V(\Delta_i) \cap V(\Delta_{i+1})$ for $1 \leq i \leq 2t-2$ of $P$. Assume without loss of generality that the color of $\Delta_1$ is 1, by Proposition \[5\] we have a sequence

$$e_1(\nu_1) = e_2(\nu_1), e_2(\nu_2) = e_1(\nu_2), \ldots, e_2(\nu_{2t-2}) = e_1(\nu_{2t-2})$$

of idempotents of $U$ with $e_1(\nu_l)D^{S_1\ast U\ast S_2}f$. Put $f_i = e_1(\nu_i)$. Since $\Delta_1 \simeq \Delta_t \simeq \Delta_{2t-1}$, then $f_1D^1f_1D^1f_{2t-2}$. Moreover it is straightforward to check that this sequence satisfies also the other conditions of statement 2.
2) \(\Rightarrow\) 1) Assuming without loss of generality \(k = 1\), then \(\Delta_1 = \text{ST}(X_1, R_1; f_1)\) is a host of \(\Gamma = \text{ST}(X, R \cup W; f) \simeq \text{ST}(X, R \cup W; e)\). Let \(\nu_1 \in V(\Delta_1)\) such that \(e_1(\nu_1) = f_1\). Since \(\Gamma\) is complete \(\nu_1\) is an intersection vertex, so let \(\Delta_2\) be the lobe of \(\Gamma\) that shares the vertex \(\nu_1\) with \(\Delta_1\). Then \(\Delta_2 = \text{ST}(X_2, R_2; f_1)\) is a host by Proposition \(\square\). Since \(f_1 \mathcal{D} S_2 f_2\) and \(f_2\) is not \(\mathcal{D} U\) related to \(f_1\), then there is a vertex \(\nu_2 \in V(\Delta_2)\) which is not connected to \(\nu_1\) by any path labeled by an element in \(U\) and \(e_2(\nu_2) = f_2\). Thus \(\nu_2\) does not belong to the intersection vertices of \(\Delta_1, \Delta_2\), and so there is a lobe \(\Delta_3\), different from \(\Delta_1\), such that \(\nu_2 \in V(\Delta_2) \cap V(\Delta_3)\) and \(\Delta_3 \simeq \text{ST}(X_1, R_1; f_2)\) is a host by Proposition \(\square\). Using now \(f_2 \mathcal{D} S_1 f_3\) and the fact that \(f_2\) and \(f_3\) are not \(\mathcal{D} U\)-related we get that there is a vertex \(\nu_3\) with \(e_1(\nu_3) = f_3\) which is not an intersection vertex between \(\Delta_2, \Delta_3\). Continuing in this way we build a reduced lobe path \(P : \Delta_1, \ldots, \Delta_t, \ldots, \Delta_{2t-1}\) such that \(\nu_i \in V(\Delta_1) \cap V(\Delta_{i+1})\) for \(1 \leq i \leq 2t - 2\). Continuing this way we build a reduced lobe path \(P : \Delta_1, \ldots, \Delta_{2t-1}\) such that \(\Delta_1\) is sent onto \(\Delta_{2t-1}\) by \(\varphi\) and the automorphism on the lobe graph induced by \(\varphi\) does not map the edge \((\Delta_1, \Delta_2)\) into the edge \((\Delta_{2t-1}, \Delta_{2t-2})\). Therefore, these two edges do not belong to the same \(H_{\mathcal{S} f_1 \ast \mathcal{U} S_2}\)-orbit, so in \(Y\) there is a non-trivial loop \(P'\) induced by \(P\).

3) \(\Rightarrow\) 1) A reduced loop \(P'\) in \(Y\) lifts to a reduced lobe path \(P : \Delta_1, \ldots, \Delta_{2t-1}\) in \(\text{Host}(\text{ST}(f))\) for some \(t > 1\) and with \(\Delta_1, \Delta_{2t-1}\) belonging to the same \(H_{\mathcal{S} f_1 \ast \mathcal{U} S_2}\)-orbit. Hence there is an automorphism \(\varphi \in \text{Aut}(\text{Host}(\text{ST}(f)))\) which sends \(\Delta_1\) onto \(\Delta_{2t-1}\). Furthermore, any automorphism does not send the edge \((\Delta_1, \Delta_2)\) into the edge \((\Delta_{2t-1}, \Delta_{2t-2})\), otherwise \(P'\) would no be reduced. Hence, \(\varphi|_{\Delta_1} : \Delta_1 \rightarrow \Delta_{2t-1}\) is a shift-isomorphism. Therefore, by Proposition \(\square\), \(\text{Host}(\text{ST}(f))\) is infinite.

From the above theorem we obtain that \(Y\) is a tree if and only if \(H_{\mathcal{S} f_1 \ast \mathcal{U} S_2}\) is finite. This is equivalent to the fact that the only case when \(H_{\mathcal{S} f_1 \ast \mathcal{U} S_2}\) is isomorphic to iterated amalgams of groups is when \(H_{\mathcal{S} f_1 \ast \mathcal{U} S_2}\) is finite.

**Remark 1.** We recall that an amalgam \([S_1, S_2; U]\) respects the \(\mathcal{J}\)-order if for each \(e_1, e_2 \in E(U)\), \(e_1 \mathcal{J} S_2 e_2\) implies \(e_1 \mathcal{J} S_{2-k} e_2\) for each \(k \in \{1, 2\}\). With such condition, if \(e\) is \(\mathcal{J} S_1 \ast S_2\)-related to some idempotent \(f \in U\), then, using an argument similar to the one in proof of Theorem \(\square\), it is straightforward to check that each host is isomorphic to either \(\text{ST}(X_1, R_1; f)\) or \(\text{ST}(X_2, R_2; f)\). Therefore \(H_{\mathcal{S} f_1 \ast \mathcal{U} S_2}\) is finite, \(|Y| = 2\), and

\[
H_{\mathcal{S} f_1 \ast \mathcal{U} S_2} \simeq H_f^{S_1} \ast H_f^{S_2}.
\]
6.2 Case 2

Let \( e \) be not \( D^{S_1, S_2} \)-related to any idempotent of \( S_1 \) or \( S_2 \). Then \( S(\Gamma(X, R \cup W; e)) \) has a unique host that is a subopuntoid subgraph of the underlying graph of \( Core(f) \). Thus \( H^{S_1, S_2} \) stabilizes some lobes \( \Delta \) of the host. Since this lobe is finite for any \( \nu \in V(\Delta) \) there is a minimum idempotent, namely \( e = e_k(\nu) \), labelling a loop based at \( \nu \). Thus, by [5, Lemma 2] \( (\nu, \Delta, \nu) \) is a DV-quotient of the Schützenberger automaton \( A(X_k, R_k; e) = (\alpha, \Sigma, \alpha) \) called in [5] the maximum determinizing Schützenberger automaton of \( (\nu, \Delta, \nu) \). Denoting by \( \pi : (\alpha, \Sigma, \alpha) \to (\nu, \Delta, \nu) \) the natural homomorphism induced by this quotient we show that we can lift an automorphism \( \phi \) of \( \Delta \) to an automorphism \( \varphi \) of \( \Sigma \) for which the following diagram commutes:

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi} & \Sigma \\
\pi \downarrow & & \downarrow \pi \\
\Delta & \xrightarrow{\phi} & \Delta
\end{array}
\]

**Theorem 6.** Let \( (\nu, \Delta, \nu) \) be a closed inverse automaton relative to the presentation \( \langle X_k | R_k \rangle \) for some \( k \in \{1, 2\} \). With the above notation let \( (\alpha, \Sigma, \alpha) \) be the maximum determinizing Schützenberger automaton of \( (\nu, \Delta, \nu) \) where \( \pi(\alpha) = \nu \). Then every automorphism \( \phi \in Aut(\Delta) \) can be lifted to an automorphism \( \varphi \in Aut(\Sigma) \) such that \( \varphi \circ \pi = \pi \circ \phi \). Moreover there is a group epimorphism from the subgroup \( H := \{ \varphi \in Aut(\Sigma) : \exists \phi \in Aut(\Delta), \varphi \circ \pi = \pi \circ \phi \} \) of \( Aut(\Sigma) \) onto \( Aut(\Delta) \) with kernel \( N = H \cap S \) where \( S = \{ \varphi \in Aut(\Sigma) : \varphi(\pi^{-1}(\nu)) \subseteq \pi^{-1}(\nu) \} \).

**Proof.** Let \( \phi \in Aut(\Delta) \), let \( \nu' = \phi(\nu) \). Since \( \phi \) is labelling preserving, then \( e_k(\nu') = e_k(\nu) = e \). Thus there is a word \( w \in (X \cup X^{-1})^* \) labelling a path \( (\nu, w, \nu') \) in \( \Delta \) such that \( ww^{-1} = w^{-1}w = e \). Since \( (\alpha, \Sigma, \alpha) = A(X_k, R_k; e) \) there is also a path \( (\alpha, w, \alpha') \) for some \( \alpha' \in V(\Sigma) \) and by the minimality of \( e_k(\nu') \) we get \( e_k(\alpha') = e \). Therefore \( (\alpha', \Sigma, \alpha') \) and \( (\alpha, \Sigma, \alpha) \) are Schützenberger automata that accept the same language, hence by Proposition [1] there is an automorphism \( \varphi \in Aut(\Sigma) \) such that \( \varphi(\alpha) = \alpha' \). We prove that \( \varphi \) is the automorphism satisfying the lifting property \( \varphi \circ \pi = \pi \circ \phi \).

For this purpose let \( \nu \) be a vertex of \( \Sigma \) and let \( r \in (X \cup X^{-1})^+ \) be a word labelling a path \( (\alpha, r, \nu) \), so applying the automorphism \( \varphi \) this path goes to \( (\alpha', r, \nu') \) with \( \nu' = \varphi(\nu) \). Consider \( \pi(\nu) \) then clearly \( (\nu, r, \pi(\nu)) \) is a path in \( \Delta \) thus the image of this path by \( \phi \) is \( (\nu', r, \phi(\pi(\nu))) \), hence \( (\nu, wr, \phi(\pi(\nu))) \) is also a path in \( \Delta \). Consider now \( \pi(\nu') \), since \( (\alpha, wr, \nu') \) is a path in \( \Sigma \) then
(ν, wr, π(ν')) is also a path in Δ, whence by the determinism of Δ we get 
φ(π(ν)) = π(ν') = π(φ(ν)).
Let 
H := {φ ∈ Aut(Σ) : ∃φ ∈ Aut(Δ), φ ◦ π = π ◦ φ}. It is straightforward checking that 
H is a subgroup of Aut(Δ).
For any φ ∈ H, the relation π−1 ◦ φ ◦ π ⊆ V(Δ) × V(Δ) is a function,
since by definition of H there is a 
φ such that φ ◦ π = π ◦ φ and so, taking into account that π is surjective, then for any left inverse π−1 we have

π−1 ◦ (φ ◦ π) = π−1 ◦ (π ◦ φ) = (π−1 ◦ π) ◦ φ = 1Δ ◦ φ = φ

So there is a map λ : H → Aut(Δ) defined by

λ(φ1 ◦ φ2) = π−1 ◦ (φ1 ◦ φ2) ◦ π =
= (π−1 ◦ φ1) ◦ (π ◦ φ2) = (π−1 ◦ φ1 ◦ π) ◦ φ2 =
= λ(φ1) ◦ φ2 = λ(φ1) ◦ λ(φ2)

The last statement is a routine calculus which involves only the definitions of H and S.

Note that without the finiteness condition of the inverse semigroup S, in general it is not possible to define the maximum determinizing Schützenberger automaton of a closed inverse word automaton relative to the presentation ⟨X|R⟩ of the inverse semigroup S. It is also quite easy to produce an example where it is not possible to lift an automorphism of a closed DV-quotient Δ of a Schützenberger automaton Σ to an automorphism of Σ (see [19]). Moreover the subgroup H in the previous theorem is in general a proper subgroup of Aut(Σ). To prove this fact it is enough to consider the dihedral group 
D4 = GP(r, s|r^2, s^2, (rs)^4) which is clearly a finite inverse semigroup with only one Schützenberger graph which is the Cayley graph of D4. If in the Cayley graph of D4 we identify the identity e with s and then we determinize, we obtain an inverse word graph Δ with Aut(Δ) ∼= GP(σ|σ^2). It is easy to show that σ can be lifted up to the automorphism (sr)^2, however the automorphism (sr) is not in H.

The next theorem covers the last case.

**Theorem 7.** Let e ∈ E(S1 *U S2) with S1, S2 finite inverse semigroups and suppose that e is not D(S1 *U S2)-related to any idempotent of S1 or S2.
Therefore $H_{S_1 \ast U S_2}^S$ is a homomorphic image of some subgroup of the maximal subgroup $H_{g}^{S_k}$ of $S_k$ for some $k \in \{1, 2\}$ and $g \in E(S_k)$.

Proof. We already remarked that in this case $H_{e}^{S_1 \ast U S_2}$ is isomorphic to the automorphism group of some lobe $\Delta$ of Host$(E(e))$. By Theorem $\Box$. $Aut(\Delta)$ is an homomorphic image of $Aut(\Sigma)$ where $\Sigma = ST(X_k, R_k; g)$ for some $g \in E(S_k)$. Therefore $H_{e}^{S_1 \ast U S_2}$ is a homomorphic image of $Aut(\Sigma) \simeq H_{g}^{S_k}$. In particular $H_{e}^{S_1 \ast U S_2} \simeq H_{g}^{S_k}/N$ where $N$ is the normal subgroup described in Theorem $\Box$.

Remark 2. We note that when $S_1$ and $S_2$ are $E$-unitary then no quotient has to be performed in the construction of the Schützenberger graph of some word with respect to the standard presentation of the amalgam. Then, if $e$ is not $D^{S_1 \ast U S_2}$-related to any idempotent of $S_1$ or $S_2$, $H_{e}^{S_1 \ast U S_2}$ is isomorphic to the maximal subgroup $H_{g}^{S_k}$ of $S_k$ for some $k \in \{1, 2\}$ and $g \in E(S_k)$.

7 Conclusion

We have completely determined the structure of the maximal subgroups of the amalgamated free-product of an amalgam of finite inverse semigroups. All these groups are finitely presented, and we sketch the proof that their presentations are effectively computable. For more details of the proof see the final chapter of [19].

Theorem 8. Let $e \in E(S_1 \ast U S_2)$ with $S_1, S_2$ finite inverse semigroups, then there is an algorithm to compute a presentation of $H_{e}^{S_1 \ast U S_2}$.

Proof. If the host is unique then $H_{e}^{S_1 \ast U S_2}$ is the automorphism group of a lobe $\Delta$ of the host. The host is finite, so such a lobe can be determined as well as a presentation of $Aut(\Delta)$. If $ST(e)$ has more than one host, it is enough to find a maximal subtree of $Y$ and then to compute the presentation of $H_{e}^{S_1 \ast U S_2}$. Starting from any host in $Core(u)$ for some word $u$ equivalent to $e$ in $S_1 \ast U S_2$, we can build a maximal subtree of $Y$ recursively adding at each step adjacent hosts which are non-isomorphic to the ones previously chosen. It is straightforward to check that when we obtain an opuntoid graph $\Theta$ for which each adjacent host is isomorphic to a lobe occurring in $\Theta$, then by Propositions $\Box$ and $\Box$ all the lobes of $\Theta$ are representatives of all the orbits. ☐

We end the section considering the case when $S_1, S_2$ are combinatorial. Note that a finite inverse semigroup which is combinatorial is a semilattice.
Thus, in our case any Schützenberger automaton is formed by at most two adjacent lobes. Hence, as an easy consequence of the above results, we have the following

**Corollary 4.** Let $S_1, S_2; U$ be an amalgam of finite inverse semigroups, then $S_1 *_U S_2$ is combinatorial if and only if $S_1$ and $S_2$ are both combinatorial.

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