BRAIDING FOR THE QUANTUM $gl_2$ AT ROOTS OF UNITY

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Abstract. In the preceding papers [KR1, KR2] we started considering the categories of tangles with flat $G$-connections in their complements, where $G$ is a simple complex algebraic group. The braiding (or the commutativity constraint) in such categories satisfies the holonomy Yang–Baxter equation and it is this property which is essential for our construction of invariants of tangles with flat $G$-connections in their complements. In this paper, to any pair of irreducible modules over the quantized universal enveloping algebra of $gl_2$ at a root of unity, we associate a solution of the holonomy Yang–Baxter equation.

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Introduction

The representation theory of quantum groups at roots of unity which we will use was developed in [DCK] and [DCKP]. These quantum groups are finite dimensional over the central Hopf subalgebra generated by $\ell$-th powers of generators and root elements. This central Hopf subalgebra is usually denoted by $Z_0$.

Quasitriangular Hopf algebras and, more generally, tensor categories can be used for construction of invariants of links and 3-manifolds (see [T1] and references therein). In a similar way tensor categories fibered over a braided group can be used for construction of invariants of tangles with flat connections in their complements [T2, KR2]. Less functorially one can say that such invariants can be constructed from any system of solutions to the holonomy Yang–Baxter equation [KR1].

Let $U_\varepsilon$ be the quantized universal enveloping algebra of $gl_2$ at a root of unity $\varepsilon$ of degree $\ell$. The first step towards understanding the braiding structure for quantized universal enveloping algebras was done in [R, Ga]. In this paper we will show that every pair of irreducible modules over $U_\varepsilon$ defines a solution to the holonomy Yang–Baxter equation for the Lie group $GL_2^\ast$. The braiding on this group is birational and it is given by the factorization mapping.

Moreover, one can show that the category of modules over $U_\varepsilon$, where objects are finite dimensional modules irreducible over $Z_0$, is a tensor category fibered over $GL_2^\ast$. One can also show that this is true for any factorizable Lie algebra related to a simple Lie algebra: the category of finite dimensional models irreducible over $Z_0$ is a tensor category fibered over $Z_0$. These topics will be the subject of a separate publication.

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1. Quantized Universal enveloping algebra of $gl_2$

1.1. The algebra $U_h(gl_2)$.

1.1.1. The algebra $U_h(gl_2)$ over the ring $\mathbb{C}[[h]]$ is generated by elements $H, G, X,$ and $Y$ with the defining relations

\begin{align*}
(1) & \quad [H, G] = 0, [H, X] = 2X, [H, Y] = -2Y, \\
(2) & \quad [G, X] = 2X, [G, Y] = -2Y, \\
(3) & \quad [X, Y] = \frac{e^{\frac{hG}{2}} - e^{-\frac{hG}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}} 
\end{align*}
The Hopf algebra structure on \( U_h(gl_2) \) is defined by the action of the comultiplication on the generators:

\[
\begin{align*}
\Delta H &= H \otimes 1 + 1 \otimes H, \quad \Delta G = G \otimes 1 + 1 \otimes G, \\
\Delta X &= X \otimes e^{\frac{hH}{2}} \otimes 1 + 1 \otimes X, \quad \Delta Y = Y \otimes 1 + e^{-\frac{hG}{2}} \otimes Y 
\end{align*}
\] (4)

Elements \( H, G, X \) and \( Y \) are deformed versions of \( 2e_{11}, \ 2e_{22}, \ e_{12} \) and \( e_{21} \) in \( gl_2 \).

The algebra \( U_h(gl_2)[h^{-1}] \) is isomorphic to the Drinfeld double of the quantized universal enveloping algebra \( U_h(b)[h^{-1}] \subset U_h(sl_2)[h^{-1}] \) (see Appendix A). As the double of a Hopf algebra it is quasitriangular with the universal \( R \)-matrix \([DR]\) given by the canonical element in \( U_h(b) \hat{\otimes} U_h(b) ^\vee \) where \( U_h(b) ^\vee \) is a dual counterpart of \( U_h(b) \) and \( \hat{\otimes} \) is the tensor product completed over formal power series in \( h \):

\[
R = \exp\left(\frac{h}{4} H \otimes G\right) \prod_{n \geq 0} \left(1 - e^{-\frac{h}{2}}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^2 X \otimes Y e^{-nh}\right)^{-1}
\] (6)

This is the element of \( U_h(gl_2)^\otimes[[h]] \) which one should consider as a formal power series in \( h \).

Element \( R \) as the universal \( R \)-matrix satisfies the following identities:

\[
\begin{align*}
R \Delta(a) R^{-1} &= \sigma \cdot \Delta(a) \\
(\Delta \otimes id)(R) &= R_{13} R_{23} \\
(id \otimes \Delta)(R) &= R_{13} R_{12}
\end{align*}
\] (7)

and, in particular, it satisfies the Yang–Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}
\]

1.2. The inner automorphism \( \mathcal{R} \). Define the inner automorphism \( \mathcal{R} : U_h(gl_2)^\otimes[[h]] \to U_h(gl_2)^\otimes[[h]] \) as

\[
\mathcal{R}(x \otimes y) = R(x \otimes y) R^{-1}
\] (8)

It is easy to compute the action of \( \mathcal{R} \) on the generators.

**Theorem 1.** The following identities hold true:

\[
\begin{align*}
\mathcal{R}(1 \otimes e^{\frac{hH}{2}}) &= (1 \otimes e^{\frac{hH}{2}})(1 - e^{\frac{h}{2}}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^2 e^{-\frac{hH}{2}} X \otimes Y e^{\frac{hG}{2}})^{-1}, \\
\mathcal{R}(1 \otimes e^{\frac{hG}{2}}) &= (1 \otimes e^{\frac{hG}{2}})(1 - e^{\frac{h}{2}}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^2 e^{-\frac{hH}{2}} X \otimes Y e^{\frac{hG}{2}})^{-1}, \\
\mathcal{R}(X \otimes 1) &= X \otimes e^{\frac{hG}{2}}, \\
\mathcal{R}(1 \otimes Y) &= e^{-\frac{hH}{2}} \otimes Y
\end{align*}
\] (9)
The proof is immediate by the use of the commutation relations between the generators and the equation

\[ f(zq; q) = (1 - z)f(z; q) \]

for the function \( f(z; q) = (1 - z)^{-1}(1 - zq)^{-1}(1 - zq^2)^{-1} \ldots \).

The action of \( R \) on elements \( 1 \otimes X, Y \otimes 1, e^{\frac{hH}{2}} \otimes 1 \), and \( e^{\frac{hG}{2}} \otimes 1 \) can be derived from the formulae above and from identity (7).

The Yang–Baxter equation for \( R \) implies the Yang–Baxter equation for \( \mathcal{R} \):

\[ \mathcal{R}_{12} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{23} = \mathcal{R}_{23} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{12} \]

2. The algebra \( \mathcal{U} \)

2.1. The algebra. The algebra \( \mathcal{U} \) is generated over \( \mathbb{C}[t, t^{-1}] \) by elements \( K, L, E \) and \( F \) with the following defining relations

\[ KL = LK, \ KE = t^2EK, \ KF = t^{-2}FK, \]

\[ LE = t^2EL, \ LF = t^{-2}FL, \]

\[ EF - FE = (t - t^{-1})(K - L^{-1}) \]

The center of \( \mathcal{U} \) is generated freely by Laurent polynomials in \( KL^{-1} \) and

\[ c = EF + Kt^{-1} + L^{-1}t \]

This is a Hopf algebra with comultiplication

\[ \Delta(K) = K \otimes K, \ \Delta(L) = L \otimes L, \]

\[ \Delta(E) = E \otimes K + 1 \otimes E, \ \Delta(F) = F \otimes 1 + L^{-1} \otimes F . \]

The map \( \phi : \mathcal{U} \rightarrow U_h(gl_2) \) acting on the generators as

\[ \phi(K) = \exp \left( \frac{hH}{2} \right), \ \phi(L) = \exp \left( \frac{hG}{2} \right), \phi(t) = e^{\frac{h}{2}} \]

\[ \phi(E) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})X, \ \phi(F) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})Y \]

extends to a homomorphism of Hopf algebras.
2.2. The outer automorphism $\mathcal{R}$. The algebra $\mathcal{U}$ is not quasitriangular. Instead, there is an outer automorphism of the division ring $\mathcal{U}^\otimes 2$ of $\mathcal{U}^\otimes 2$ which we denote by the same letter $\mathcal{R}$ as the automorphism (8) which acts on the generators as

$$\mathcal{R}(1 \otimes K) = (1 \otimes K)(1 - tK^{-1}E \otimes FL)^{-1}$$
$$\mathcal{R}(1 \otimes L) = (1 \otimes L)(1 - tK^{-1}E \otimes FL)^{-1}$$
$$\mathcal{R}(E \otimes 1) = E \otimes L$$
$$\mathcal{R}(1 \otimes F) = K^{-1} \otimes F$$

Define its action on the elements $K \otimes 1$, $L \otimes 1$, $K \otimes 1$, and $L \otimes 1$ through the condition that

$$\mathcal{R}(\Delta(a)) = \sigma \circ \Delta(a)$$

where $a$ is any of the generators of $\mathcal{U}$.

It is clear that the homomorphism $\phi$ brings the outer automorphism $\mathcal{R}$ to (8). It also satisfies the Yang–Baxter relation (10).

3. The algebra $\mathcal{U}_\varepsilon$

3.1. The algebra. Let $\varepsilon$ be a primitive root of 1 of an odd degree $\ell$ (this is a technical assumption, convenient because then $\varepsilon^{2n}$ runs through all possible $\ell$-th roots of 1 for $n = 1, \ldots, \ell$). Denote by $\mathcal{U}_\varepsilon$ the specialization of $\mathcal{U}$ to $t = \varepsilon$. The following theorem was proven in [DCK] for any simple Lie algebra.

Theorem 2.  

- Elements $E^\ell$, $F^\ell$, $K^{\pm \ell}$, and $L^{\pm \ell}$ are central in $\mathcal{U}_\varepsilon$. Denote by $Z_0$ the central subalgebra in $\mathcal{U}_\varepsilon$ which they generate.
- $Z_0$ is a Hopf subalgebra with the comultiplication
  $$\Delta(K^\ell) = K^\ell \otimes K^\ell, \quad \Delta(L^\ell) = L^\ell \otimes L^\ell,$$
  $$\Delta(E^\ell) = E^\ell \otimes K^\ell + 1 \otimes E^\ell, \quad \Delta(F^\ell) = F^\ell \otimes 1 + L^{-\ell} \otimes F^\ell.$$
- The algebra $\mathcal{U}_\varepsilon$ is a free $Z_0$-module of dimension $\ell^4$.
- The center $Z(\mathcal{U}_\varepsilon)$ is generated by $Z_0$ and by the element (11) modulo the relation
  $$\prod_{j=0}^{\ell-1}(c - K\varepsilon^{j+1} - L^{-1}\varepsilon^{-j-1}) = E^\ell F^\ell$$
- Let $a, b, c, d$ be coordinates on the group $B_+ \times B_-$ such that for $b_{\pm} \in B_{\pm}$ we have:
  $$b_+ = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}, \quad b_- = \begin{pmatrix} d & 0 \\ c & 1 \end{pmatrix}$$
  Then the map $F^\ell \to b$, $E^\ell \to -cd^{-1}$, $K^\ell \to a$, $L^\ell \to d^2$ is an isomorphism of Hopf algebras $Z_0 \to C(B_+ \times B_-)$.
• $\mathcal{U}_\varepsilon$ is semisimple over a Zariski open subvariety in $\text{Spec}(\mathbb{Z}_0) \simeq B_+ \times B_-$. 

3.2. $\mathbb{Z}_0$-irreducible quotients. Let $x \in GL^*_2$ be an irreducible $\mathbb{Z}_0$-character and $I_x \subset \mathcal{U}_\varepsilon$ be the corresponding ideal. The quotient algebra 

$$A_x = \mathcal{U}_\varepsilon / I_x$$

is finite-dimensional of dimension $\ell^4$. There are three natural structures of a left module on $A_x$. For $a \in \mathcal{U}_\varepsilon$ denote by $[a]$ the class of $a$ in $A_x$. Then the three actions are:

- $\pi(a)[b] = [ab]$,
- $\phi(a)[b] = [bS(a)]$,
- $\psi(a)[b] = [bS^{-1}(a)]$.

Assume that $x \in GL^*_2$ is generic, i.e. that $A_x$ is semisimple. Fix an isomorphism of algebras $\phi_x : A_x \simeq \bigoplus_{i=1}^{n} \text{Mat}(k_i)$. For the algebra $\mathcal{U}_\varepsilon$ it is known [DCK, DCKP] that $n = \ell$ and $k_i = \ell$ for all $i = 1, \ldots, \ell$. Define

$$t : A_x \rightarrow \mathbb{C}, \quad t(a) = \sum_{i=1}^{\ell} t_i \text{Tr}(\phi^i_x(a))$$

where $\text{Tr}$ is the matrix trace in $\text{Mat}(k_i)$ and $\phi^i_x : A_x \rightarrow \text{Mat}(k_i)$ is the $i$-th component of $\phi_x$. It is clear that $t(a)$ does not depend on a particular choice of $\phi_x$. Indeed, any other such isomorphism differs from $\phi_x$ by an inner automorphism of $\bigoplus_{i=1}^{n} \text{Mat}(k_i)$. Since trace is cyclically invariant, the value of $t(a)$ for such isomorphism will be the same as for $\phi_x$. Thus, for generic $x$ we have an invariant bilinear form on $A_x$:

$$(a, b) = t(ab),$$

It is non-degenerate if $t_i \neq 0$ for each $i = 1, \ldots, n$.

Fix a scalar product on $A_x$ as above. It gives an isomorphism of vector spaces $A^*_x \simeq A_x$. It is easy to verify that the the pairing between $\mathcal{U}_\varepsilon$-modules $(A_x, \phi)$ and $(A_x, \pi)$ given by the map $e_x : (A_x, \phi) \otimes (A_x, \pi) \rightarrow \mathbb{C}$

$$(13) \quad e_x : a \otimes b \rightarrow t(ab)$$

is $\mathcal{U}_\varepsilon$-invariant with respect to the diagonal action. Indeed, by using Sweedler’s notation $\Delta(c) = \sum e^{(1)} \otimes e^{(2)}$ for the action of the comultiplication on element $c$, we have:

$$e_x(\sum e^{(1)} aS(e^{(1)}) \otimes c^{(2)}b) = \sum e^{(1)} t(aS(e^{(1)})c^{(2)}b) = \varepsilon(c)t(ab)$$

Similarly

$$e_x(\sum e^{(1)} a \otimes bS^{-1}(c^{(2)})) = \sum e^{(1)} t(c^{(1)}abS^{-1}(c^{(2)})) = \varepsilon(c)t(ab)$$

Therefore, the map $e_x : (A_x, \pi) \otimes (A_x, \psi) \rightarrow \mathbb{C}$ defined by (13) is also $\mathcal{U}_\varepsilon$-invariant.
BRAIDING FOR THE QUANTUM $gl_2$

Consider the mapping $i_x : \mathbb{C} \to A_x \otimes A_x$ acting at 1 as

$$i_x(1) \mapsto \sum_i e_i \otimes e^i$$

It is easy to see that it is a morphism of $\mathcal{U}_\varepsilon$-modules $\mathbb{C} \to (A_x, \pi) \otimes (A_x, \phi)$. Indeed, let $a^j_i = t(auv)$, then

$$\sum a \sum_i (a^j_i \otimes (S(a^j_i))^e_k) = \sum a \sum_i (a^j_i) (S(a^j_i))^e_k e^k = \sum a \sum_j (a^j_i) S(a^j_i)^e_k e^k = \varepsilon(a) \sum_i e_i \otimes e^i$$

which implies the first statement. Similarly, the mapping $\mathbb{C} \to (A_x, \psi) \otimes (A_x, \pi)$ is also a morphism of $\mathcal{U}_\varepsilon$-modules.

3.3. The action of $R$ on $Z_0 \otimes Z_0$.

**Theorem 3.** The subspace $Z_0 \otimes Z_0 \subset \mathcal{U}_\varepsilon \otimes \mathcal{U}_\varepsilon$ is invariant with respect to the action of the automorphism $R$.

**Proof.** From the action of $R$ on the generators of $\mathcal{U}_\varepsilon$ and from the relations between generators we have:

$$R(1 \otimes K^\ell) = (1 \otimes K^\ell)(1 + K^{-\ell}E^\ell \otimes F^\ell L^\ell)^{-1}$$

$$R(1 \otimes L^\ell) = (1 \otimes L^\ell)(1 + K^{-\ell}E^\ell \otimes F^\ell L^\ell)^{-1}$$

$$R(E^\ell \otimes 1) = E^\ell \otimes L^\ell$$

$$R(1 \otimes F^\ell) = K^{-\ell} \otimes F^\ell.$$

The comultiplication acts on $\ell$-th powers of the generators as

$$\Delta(K^\ell) = K^\ell \otimes K^\ell, \quad \Delta(L^\ell) = L^\ell \otimes L^\ell$$

$$\Delta(E^\ell) = E^\ell \otimes K^\ell + 1 \otimes E^\ell$$

$$\Delta(F^\ell) = F^\ell \otimes 1 + L^{-\ell} \otimes F^\ell.$$ 

These formulae and the defining property $R(\Delta(a)) = \sigma \circ \Delta(a)$ imply that

$$R(K^\ell \otimes K^\ell) = K^\ell \otimes K^\ell,$$

$$R(L^\ell \otimes L^\ell) = L^\ell \otimes L^\ell,$$

$$R(E^\ell \otimes K^\ell + 1 \otimes E^\ell) = K^\ell \otimes E^\ell + E^\ell \otimes 1,$$

$$R(E^\ell \otimes 1 + L^{-\ell} \otimes F^\ell) = F^\ell \otimes L^{-\ell} + 1 \otimes F^\ell.$$ 

These formulas describe implicitly the action of $R$ on the rest of the generators of $Z_0 \otimes Z_0$. In particular, it is clear that the image is in $Z_0 \otimes Z_0$. \qed
Translating the action of $\mathcal{R}$ on the generators of $Z_0 \otimes Z_0$ through the identification of $Z_0$ and $C(GL^*_2)$ we have the following statement.

**Theorem 4.** The automorphism $\mathcal{R}$ is the pull-back of the birational correspondence $\beta : GL^*_2 \times GL^*_2 \to GL^*_2 \times GL^*_2$ defined as follows:

$$\beta : (x, y) \mapsto (x_L(x, y), x_R(x, y))$$

where $I(x_L(x, y)) = x_-I(y)x_-^{-1}$ and $I(x_R(x, y)) = (x_L(x, y))_+^{-1}I(x)(x_L(x, y))_+$. Here $I : GL^*_2 \to GL_2$ is the factorization mapping $I(x_+, x_-) = x_+x_-^{-1}$.

**Proof.** Let $a_1, b_1, c_1, d_1$ and $a_2, b_2, c_2, d_2$ be coordinates of points $x, y \in GL^*_2$ respectively (as in Theorem 2. Let $\alpha_1, \beta_1, \gamma_1, \delta_1$ and $\alpha_2, \beta_2, \gamma_2, \delta_2$ be coordinates of points $x_R, x_L \in GL^*_2$. By definition, the pull-back of the mapping $\beta$ is $\mathcal{R}$. From the explicit action of $\mathcal{R}$ on the coordinate functions we have

$$\alpha_2 = a_2(1 - c_1b_1b_2d_2/a_1d_1d_2)^{-1}$$

$$\delta_1 = d_1(1 - c_1b_1b_2d_2/a_1d_1d_2)^{-1}$$

$$\gamma_1\delta_1^{-1} = c_1d_1^{-1}d_2$$

$$\beta_2 = b_2a_1^{-1}$$

$$\alpha_1\alpha_2 = a_1a_2$$

$$\gamma_1\delta_1^{-1}\alpha_2 + \gamma_2\delta_2^{-1} = a_1c_1d_1^{-1} + \gamma_2\delta_2$$

$$\beta_1 + \delta_1^{-1} = b_1d_2^{-1} + b_2$$

Now, it remains to prove that this mapping can be written as (19). This a simple linear algebra exercise. \qed

Let $I_x$ be the ideal in $\mathcal{U}_\varepsilon$ corresponding to the irreducible $Z_0$-character $x \in GL^*_2$. Theorems 3 and 4 have an important implication. The mapping $\mathcal{R}$ induces an isomorphism of algebras

$$\mathcal{R}(x, y) : A_x \otimes A_y \to A_{xR(x, y)} \otimes A_{xL(x, y)}$$

This mapping is also an isomorphism of the tensor product of left $\mathcal{U}_\varepsilon$-modules.
4. Braiding for irreducible representations of $\mathcal{U}_c$.

4.1. Restriction of the braiding to irreducible representations. Let $Z_1$ be the central subalgebra generated by the central elements $c$ and $KL^{-1}$. From the definition of $\mathcal{R}$ we have

$$\mathcal{R}(c_1 \otimes c_2) = c_1 \otimes c_2$$

for any $c_{1,2} \in Z_1$. For generic $x$ the algebra $A_x$ is semisimple and its irreducible representations are separated by eigenvalues of $c$. Let $A^i_x$ be one of the irreducible representations of $A_x$.

Since $\mathcal{R}$ acts trivially on $Z_1 \otimes Z_1$, the isomorphism $\mathcal{R}(x, y) : A_x \otimes A_y \rightarrow A_{xR(x,y)} \otimes A_{xL(x,y)}$ restricts to the subalgebra $A^i_x \otimes A^j_y$ and induces an algebra isomorphism

$$\mathcal{R}^{i,j}(x, y) : A^i_x \otimes A^j_y \rightarrow A^i_{xR(x,y)} \otimes A^j_{xL(x,y)}.$$  

Algebras $A^i_x$ are fibers of a bundle of algebras with the fiber $Mat(\ell \times \ell)$. This bundle is trivializable over sufficiently small neighborhood of 1. Fix such trivialization, i.e. for each $x$ in this neighborhood fix an algebra isomorphism $\phi^i_x : A^i_x \simeq Mat(\ell)$. Then the mapping $\mathcal{R}^{i,j}(x, y)$ induces an automorphism of $Mat(\ell)^{\otimes 2}$. Since all automorphisms of a matrix algebra are inner, there exists $R^{i,j}(x, y) \in Mat(\ell)^{\otimes 2}$ such that

$$\left(\phi^i_{xR(x,y)} \otimes \phi^j_{xL(x,y)}\right) \circ \mathcal{R}^{i,j}(x, y) \circ \left((\phi^i_x)^{-1} \otimes (\phi^j_y)^{-1}\right)(A \otimes B) = R^{i,j}(x, y)(A \otimes B)R^{i,j}(x, y)^{-1},$$

for any $A, B \in Mat(\ell)$. The Yang–Baxter relation for $\mathcal{R}$ implies the projective holonomy Yang–Baxter equation for $R^{i,j}(x, y)$,

$$R^{i,j}(x'', y'')_{12}R^{i,k}(x, z'')_{13}R^{i,k}(y, z)_{23} = c^{i,j,k}(x, y, z)R^{i,k}(y', z')_{23}R^{i,k}(x', z)_{13}R^{i,j}(x, y)_{12},$$

where $x, x', x'', y, y'$ etc are $GL_2^+$-colorings of the diagrams of Fig. 1 (see [KR1, KR2]). In terms of the mapping $\beta : (x, y) \mapsto (x_L(x,y), x_R(x,y))$ the arguments in this equation are given by

$$z'' = x_L(y, z), \quad y'' = x_R(y, z), \quad x'' = x_R(x, z''), \quad x' = x_R(x, y), \quad y' = x_L(x, y), \quad z' = x_L(x', z).$$

The function $c^{i,j,k}(x, y, z)$ can be determined from the comparison of determinants of the left and right hand sides of the equation. We will describe a normalization of $R^{i,j}(x, y)$ for which $c^{i,j,k}(x, y, z) = 1$.

4.2. Irreducible representations of $\mathcal{U}_c$. Now we fix the isomorphisms $\phi^i_x : A^i_x \rightarrow Mat(\ell)$ by choosing a specific realization of irreducible representations of $\mathcal{U}_c$. This realization is known as a cyclic representation.

Let $u, v$ and $x$ be variables such that

$$K^\ell = u^\ell v^\ell, \quad L^\ell = u^{-\ell} v^\ell, \quad c = u(x + x^{-1}), \quad E^\ell = y^\ell.$$
The following representation of $U_\varepsilon$ in $\mathbb{C}^\ell$ with the linear basis $\{v_m\}_{m=1}^\ell$ is irreducible:

\begin{align}
K v_m &= u \varepsilon^{2m} v_m, \quad L v_m = u^{-1} \varepsilon^{2m} v_m, \quad E v_m = y v_{m+1} \\
F v_m &= y^{-1} u (x v^{-1} \varepsilon^{-2m+1} - 1)(\varepsilon^{2m-1} - x^{-1}) v_{m-1}
\end{align}

Indeed, since $K$ and $L$ are diagonal in this representation, they are diagonal in any subrepresentation. Taking into account the fact that $E$ and $F$ act as cyclic shift operators in the eigen-basis of $K$ and $L$, one deduces that the only possible subrepresentations are the trivial one and the representation itself. It is easy to see that in this representation

\[ F^\ell = u^\ell (x^\ell v^{-\ell} - 1)(v^\ell - x^{-\ell}) \]

The parameters (22) of the irreducible representation are related to the coordinates $a, b, c, d$ on $GL_2$ by the following formulae

\[ a = u^\ell v^\ell, \quad d = u^{-\ell} v^\ell, \quad c = -y^\ell u^{-\ell} v^\ell, \quad b = y^{-\ell} u^\ell (x^\ell - v^\ell - v^{-\ell} + x^{-\ell}) \]

**Remark 1.** We have the identities

\[ Tr(b_+(b_-)^{-1}) = a + d^{-1} - bcd^{-1} = u^\ell (x^\ell - x^{-\ell}) \]

and

\[ det(b_+(b_-)^{-1}) = ad^{-1} = u^{2\ell} \]

**Proposition 1.** (1) Representations with different branches of $v \rightarrow v^\ell$ and $y \rightarrow y^\ell$ are isomorphic.
(2) Branches of $u \rightarrow u^x$, $x \rightarrow x^x$ parameterize isomorphism classes of irreducible representations for generic values of $a, b, c, d$.

The proof is straightforward.  
Let $A$ and $B$ be $\ell \times \ell$ matrices acting on the standard basis in $\mathbb{C}^\ell$ as follows:
\[
Av_n = \varepsilon^{2n}v_n, \quad Bv_n = v_{n+1}
\]
which satisfy the relations
\[
A^\ell = 1, \quad B^\ell = 1, \quad AB = \varepsilon^2 BA,
\]
Then the representation (23) can be written as the following homomorphism of algebras
\[
K \mapsto uvA, \quad L \mapsto u^{-1}vA, \quad E \mapsto yB,
\]
\[
F \mapsto y^{-1}u(xv^{-1}vA^{-1} - 1)(vA^{-1} - x^{-1})B^{-1}
\]
The collection of these homomorphisms is a trivialization $\phi^{i_i} : A^i \rightarrow Math(\ell)$ over a sufficiently small neighborhood of 1 in $GL_2$.

4.3. The braiding for generic irreducible modules. The action of $R$ on the tensor product of two irreducible representations is the specialization of the formulae from section 2.2.  
Substituting this action into (20) we arrive to the following equations for $R$
\[
R(1 \otimes A)R^{-1} = u_2v_2\tilde{u}_2^{-1}\tilde{v}_2^{-1}(1 \otimes A)
\]
\[
\quad \times (1 - \varepsilon u_1^{-1}v_1^{-1}y_1y_2^{-1}\tilde{v}_2^{-1}A^{-1}B \otimes (x_2v_2^{-1}vA^{-1} - 1)(vA^{-1} - x_2)B^{-1}A)^{-1},
\]
\[
\tilde{y}_2^{-1}\tilde{u}_2R(1 \otimes (\tilde{x}_2\tilde{v}_2^{-1}vA^{-1} - 1)(\tilde{v}_2A^{-1} - \tilde{x}_2)B^{-1})R^{-1}
\]
\[
\quad = u_1^{-1}v_1^{-1}(A^{-1} \otimes y_1^{-1}u_2(x_2v_2^{-1}vA^{-1} - 1)(vA^{-1} - x_2)B^{-1}A),
\]
\[
\tilde{u}_1\tilde{v}_1\tilde{u}_2\tilde{v}_2R(A \otimes A)R^{-1} = u_1v_1u_2v_2(A \otimes A),
\]
\[
\tilde{y}_1R(B \otimes 1)R^{-1} = y_1u_2v_2(B \otimes A),
\]
Here $\tilde{u}_1^2 = u_1^2$ and $\tilde{u}_2^2 = u_2^2$.

For sufficiently small $x, v, \text{ and } z$ such that $z^\ell = (x^\ell v^{-\ell} - 1)(v^\ell - x^\ell)$ define a linear operator $U : \mathbb{C}^\ell \rightarrow \mathbb{C}^\ell$
\[
Uv_n = z \prod_{m=1}^{n}((xv^{-1}v^{-2m+1} - 1)(v^{-2m+1} - x^{-1})^{-1}v_n
\]
Let $P_n$ be the projector on the vector $v_n$ in $\mathbb{C}^\ell$:  
\[
P_nv_m = \delta_{n,m}v_m,
\]
Define an operator $D$ acting in $\mathbb{C}^\ell \otimes \mathbb{C}^\ell$ as
\[
D(v_n \otimes v_m) = \varepsilon^{2nm} \chi_1^{-n} \chi_2^m (v_n \otimes v_m)
\]
Assume that the parameters $\chi_1, \chi_2, \varepsilon^a$ satisfy the following relations
\[
u_1 v_1 u_2 v_2 = \chi_1 \varepsilon^{2a} \tilde{u}_1 \tilde{v}_1 \tilde{u}_2 \tilde{v}_2
\]
\[
y_1 u_2 v_2 \chi_2 = \tilde{y}_1
\]
\[
u_1^{-1} v_1^{-1} z_2 \varepsilon^{-2a} = \tilde{z}_2 \tilde{y}_2^{-1} \tilde{u}_2
\]
and define a power series in $z$ given by the expansion of the function
\[
\Phi(z) = \prod_{m=1}^{\ell} (1 - \varepsilon^{2m} z)^{-\frac{m}{2}}
\]

**Theorem 5.** For generic $x, y \in GL_2^*$ any solution to the equations (27) — (30) is a scalar multiple of
\[
R = D(B^a \otimes U) R_1 (1 \otimes \tilde{U}^{-1})
\]
where $R_1 = \Phi(sB \otimes B^{-1} \varepsilon^{-2})$, $s = u_2 v_2 \tilde{u}_2^{-1} \tilde{v}_2^{-1}$, $D$ is defined above and $U$ and $\tilde{U}$ depend on parameters $z_2, u_2, v_2$, and $\tilde{z}_2, \tilde{u}_2, \tilde{v}_2$, respectively.

**Proof.** The operator $U$ satisfies the identity
\[
U^{-1}(xv^{-1} \varepsilon A^{-1} - 1)(v A \varepsilon^{-1} - x) B^{-1} U = B^{-1}
\]
Write $R$ as
\[
R = D(B^a \otimes U) R_1 (1 \otimes \tilde{U}^{-1})
\]
Here operator $\tilde{U}$ depends on $\tilde{z}_2, \tilde{u}_2, \tilde{v}_2$ and $U$ depends on $z_2, u_2, v_2$. Then, the equations for $R$ imply the following relations for the operator $R_1$:
\[
R_1 (A \otimes A) R_1^{-1} = A \otimes A
\]
\[
R_1 (1 \otimes A) R_1^{-1} = t (1 \otimes A) (1 - s B \otimes B)^{-1}
\]
\[
R_1 (1 \otimes B^{-1}) R_1^{-1} = 1 \otimes B^{-1}
\]
\[
R_1 (B \otimes 1) R_1^{-1} = B \otimes 1
\]
where
\[
s = u_2 v_2 \tilde{u}_2^{-1} \tilde{v}_2^{-1}, \quad t = \varepsilon \chi_2 \chi_3 u_1^{-1} v_1^{-1} y_1 \tilde{y}_2^{-1} \tilde{v}_2^{-1}
\]
Identities (15) — (18) imply that $t^\ell = 1 - s^\ell$.

The function $\Phi$ satisfies the following ″difference equation″:
\[
\Phi(\varepsilon^2 z) = (1 - z^\ell)^4 (1 - \varepsilon^2 z^{-1})^{-1} \Phi(z)
\]
From equations (33), (35), (36) we conclude that $R_1$ is a polynomial in $B \otimes B^{-1}$. Equation (34) determines the coefficients of this polynomial up to an overall scalar factor. Taking into account the difference equation for $\phi$, we conclude that there is one dimensional family of solutions to linear equations (34):

$$R_1 = \text{const} \Phi(sB \otimes B^{-1} \varepsilon^{-2})$$

$\square$

Remark 2. The power series $\Phi(z)$ can be analytically continued to the meromorphic function $\Phi(z)$ defined in a neighborhood of $z = 0$. More precisely $\Phi$ determines the curve $C_\Phi = \{\Phi^\ell \prod_{m=1}^{\ell} (1 - \varepsilon^{2m}z)^{-\frac{m}{\varepsilon}} = 1\}$ in $\mathbb{C} \times \mathbb{C}$. The difference equation is the action of $\mathbb{Z}_\ell$ on the curve $C_1 = \{(\Phi, \varepsilon, \phi) \in \mathbb{C}^3 | \Phi^\ell \prod_{m=1}^{\ell} (1 - \varepsilon^{2m}z)^{-\frac{m}{\varepsilon}} = 1, \phi^\ell + z^\ell = 1\}$. The element $\omega \in \mathbb{Z}_\ell$ acts as $(\Phi, \varepsilon, \phi) \rightarrow (\Phi\phi(1 - \omega z)^{-\ell}, z\omega, \phi)$.

It is not difficult to compute the determinant of the matrix $R$:

$$\det(R) = (1 - s^\ell)^{\frac{(\ell+2)}{2}}$$

Since $R$ is fixed yet only up a scalar function of parameters of the representations, then the element

$$\tilde{R} = (1 - s^\ell)^{\frac{\ell+1}{2}} R$$

intertwines the same representations as $R$ with the property $\det(\tilde{R}) = 1$ and at $s = 0$ the matrix $\tilde{R} = D_{12}B_1U_2\tilde{U}^{-1}_2$ satisfies the Yang–Baxter equation. Therefore the factor $c_{ijk}$ is trivial for the matrix $\tilde{R}$.

Thus, every pair of generic irreducible $U_\varepsilon$-modules defines the following solution to the holonomy Yang–Baxter equation:

$$R^{i,j}(x,y) = (1 - s^\ell)^{\frac{i+j}{2}} D(B^n \otimes U) \Phi(sB \otimes B^{-1} \varepsilon^{-2})(1 \otimes \tilde{U}^{-1})$$

Here $x$ and $y$ are in sufficiently small neighborhood of 1 in $GL_2^*$ and all other ingredients are defined above.

**Appendix A**

1. Let $b \subset \mathfrak{sl}_2$ be the Borel subalgebra and $H$, the generator of the Cartan subalgebra. The algebra $U_h b$ is generated by $H$ and $X$ with the defining relation

$$HX - XH = 2X.$$

This is a Hopf algebra over $\mathbb{C}[[h]]$ with the comultiplication

$$\Delta H = H \otimes 1 + 1 \otimes H,$$

$$\Delta X = X \otimes e^{\frac{h}{2}} + 1 \otimes X.$$
2. Let \((U_h b)^0\) be the Hopf algebra generated by elements \(H^\vee, X^\vee\), completed by formal power series in \(h\) and \(H^\vee\) with the defining relations

\[
[H^\vee, X^\vee] = -\frac{h}{2} X^\vee, \\
\Delta H^\vee = H^\vee \otimes 1 + 1 \otimes H^\vee, \\
\Delta X^\vee = X^\vee \otimes 1 + e^{-2H^\vee} \otimes X^\vee,
\]

3. There is a nondegenerate pairing \(\langle \cdot, \cdot \rangle: U_h b \otimes (U_h b)^0 \to \mathbb{C}[[h]]\) of Hopf algebras with

\[
\langle l, ba \rangle = \langle \Delta l, a \otimes b \rangle \tag{38} \\
\langle lm, a \rangle = \langle l \otimes m, \Delta a \rangle \tag{39}
\]

This pairing is defined on generators as

\[
\langle H^\vee, H \rangle = 1, \; \langle X^\vee, X \rangle = 1 \\
\langle H^\vee, X \rangle = \langle X^\vee, H \rangle = 0.
\]

It is easy to extend this pairing to the whole algebra using (38). The pairing between the monomials is

\[
\langle (H^\vee)^n (X^\vee)^m, H^{n'} X^{m'} \rangle = \delta_{nn'} \delta_{mm'} n! b_m
\]

where

\[
b_m = \frac{1 - e^{-hn}}{1 - e^{-h}} \cdot \frac{1 - e^{-h(n-1)}}{1 - e^{-h}} \cdots \frac{1 - e^{-h}}{1 - e^{-h}}.
\]

Indeed, it is clear that the right-hand side is proportional to \(\delta_{nn'} \delta_{mm'}\). Let us commute the coefficient. Using the Hopf properties (38) of the pairing and the grading arguments we obtain:

\[
\langle (H^\vee)^n (X^\vee)^m, H^n X^m \rangle = \langle (H^\vee)^n \otimes (X^\vee)^m, (\Delta H)^n (\Delta X)^m \rangle \\
= \langle (H^\vee)^n \otimes (X^\vee)^m, H^n \otimes X^m \rangle \\
= \langle (H^\vee)^n, H^n \rangle \langle (X^\vee)^m, X^m \rangle.
\]

For the first factor we have

\[
\langle (H^\vee)^n, H^n \rangle = \langle (H^\vee)^{n-1} \otimes H^\vee, (\Delta H)^n \rangle \\
= \langle (H^\vee)^{n-1} \otimes H^\vee, n H^{n-1} \otimes H \rangle \\
= n \langle (H^\vee)^{n-1}, H^n \rangle
\]

therefore

\[
\langle (H^\vee)^n, H^n \rangle = n!
\]
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For the second factor we have
\[
\langle (X^\vee)^n, X^n \rangle = \langle (X^\vee)^{n-1} \otimes X^\vee, (\Delta X)^n \rangle
\]
\[
= \langle (X^\vee)^{n-1} \otimes X^\vee, \frac{e^{-h_n}}{1-e^{-h}} X^{n-1} \otimes e^{\frac{HH}{2}(n-1)} X \rangle.
\]

Here we used the second term in
\[
(\Delta X)^n = (X \otimes e^{\frac{HH}{2}} + 1 \otimes X)^n
\]
\[
= X^n \otimes e^{\frac{HN}{2}} + (X^{n-1} \otimes e^{\frac{h(n-1)H}{2}} X + X^{n-1} \otimes e^{\frac{H}{2}} X e^{\frac{h(n-2)H}{2}} + \cdots + X^{n-1} \otimes X e^{\frac{h(n-1)H}{2}}) + \cdots
\]
\[
= X^n \otimes e^{\frac{HN}{2}} + (1 + e^{-h} + \cdots + e^{-(n-1)h}) X^{n-1} \otimes e^{\frac{(n-1)H}{2}} X + \cdots
\]

Therefore
\[
\langle (X^\vee)^n, X^n \rangle = \frac{1-e^{-h_n}}{1-e^{-h}} \langle (X^\vee)^{n-1}, X^{n-1} \rangle \cdot \langle X^\vee, e^{\frac{HH(n-1)}{2}} X \rangle.
\]

Taking into account that
\[
\langle X^\vee, e^{\frac{HH}{2}(n-1)} X \rangle = \langle \Delta X^\vee, X \otimes e^{\frac{HH}{2}(n-1)} \rangle = 1
\]

we obtain
\[
\langle (X^\vee)^n, X^n \rangle = b_n.
\]

where $b_n$ is defined above.

4. The double of the Hopf algebra $U_h b$ is the Hopf algebra structure on the space $D(U_h b) = U_h b \otimes (U_h b)^0$ completed in formal power series in $h$ and $H^\vee$ such that the coalgebra structure is the tensor product of coalgebras. The algebra structure is completely determined by the condition that the natural embeddings $U_h b, (U_h b)^0 \hookrightarrow D(U_h b)$ are Hopf algebra homomorphisms and that the canonical element $R = \sum_i e_i \otimes e_i^\prime$ intertwines the coproduct with the opposite coproduct
\[
R \Delta(a) = \Delta^{op}(a) R,
\]
where $\Delta^{op}(a) = \sigma_0 \Delta(a), \sigma(x \otimes y) = y \otimes x$.

Equivalently, the product in $D(U_h b)$ can be defined as
\[
(a \otimes l)(b \otimes m) = \sum ab_{(2)} \otimes l_{(2)} m < b_{(1)}, S_{A^*}^{-1}(l_{(1)}) > < b_{(3)}, l_{(3)} >
\]

Here we use the notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ for the comultiplication in $U_h b$ and $\Delta^{op}(l) = \sum l_{(2)} \otimes l_{(1)}$ for the comultiplication in $(U_h b)^0$ (see [Ma]).

Using either of these definitions, it is easy to show that $D(U_h b)$ is isomorphic to the algebra generated by $H, X, H^\vee, X^\vee$, completed in formal power series in $h$ and $H^\vee$ with the defining
relations
\[
[H,H^\vee] = 0, \ [H,X] = 2X, \ [H^\vee,X] = \frac{h}{2}X
\]
\[
[H,X^\vee] = -2X^\vee, \ [H^\vee,X^\vee] = -\frac{h}{2}X^\vee
\]
\[
[X,X^\vee] = e^{hX} - e^{-2H^\vee}.
\]

The universal \(R\)-matrix is the canonical element
\[
R = \sum e_i \otimes e^i \in U_h b \otimes (U_h b)^0
\]
\[
R = \sum_{n,m \geq 0} \frac{1}{n!m!} H^n X^m \otimes (H^\vee)^n(X^\vee)^m
\]
\[
(\Delta \otimes id)(R) = R_{13} R_{23} \quad (id \otimes \Delta)(R) = R_{13} R_{12}
\]
and, in particular, that \(R\) satisfies the Yang–Baxter equation.

5. The quantum \(gl_2\), or \(U_h gl_2\) is the Hopf algebra generated by \(H,G,X,Y\) with the defining relations and the coproduct described in (1)–(3). It is clear that the mapping \(\varphi : D(U_h b) \to U_h gl_2\) acting on generators as
\[
\varphi(H) = H, \ \varphi(H^\vee) = \frac{h}{4}G, \ \varphi(X) = X, \ \varphi(X^\vee) = Y(e^{h/2} - e^{-h/2})
\]
extends to a Hopf algebra homomorphism. The element (6) is the image of the universal \(R\)-matrix (41) and therefore satisfies (7).

Appendix B

In section 2.2 we defined the action of \(\mathcal{R}\) on the generators \(E \otimes 1, 1 \otimes F, 1 \otimes K,\) and \(1 \otimes L\) explicitly and on other generators implicitly by requiring the intertwining property (12). The action of \(\mathcal{R}\) on element \(F \otimes 1\) and on \(1 \otimes E\) can also be found from the condition that \(\mathcal{R}(c_1 \otimes c_2) = c_1 \otimes c_2\) for central elements \(c_1\) and \(c_2\) and \(\mathcal{R}(K \otimes K) = K \otimes K\). Let us verify this.

First, notice that \(\mathcal{R}(K \otimes K) = K \otimes K\) together with (9) implies that
\[
\mathcal{R}(K \otimes 1) = (1 - tK^{-1}E \otimes FL)(K \otimes 1)
\]
Now let us find the action of $\mathcal{R}$ on $1 \otimes E$:

\begin{equation}
\mathcal{R}(1 \otimes E) = \mathcal{R}(1 \otimes E F)(K \otimes F^{-1}) = \\
\mathcal{R}(1 \otimes (c - t^{-1}K - tL^{-1}))(K \otimes F^{-1}) = \\
1 \otimes cF^{-1} - (t^{-1}\mathcal{R}(1 \otimes K) + t\mathcal{R}(1 \otimes L^{-1}))(K \otimes F^{-1}) = \\
1 \otimes E + t^{-1}K \otimes KF^{-1} + tK \otimes L^{-1}F^{-1} - \\
t^{-1}(1 \otimes K)(1 - tK^{-1}E \otimes FL)^{-1}(K \otimes F^{-1} - t(1 - t^{-1}E \otimes FL)(1 \otimes L^{-1}F^{-1})
\end{equation}

\begin{equation}
= 1 \otimes E + E \otimes K - (1 - t^{-1}K^{-1}E \otimes FL)^{-1}E \otimes KL,
\end{equation}

Similarly, we have:

\begin{equation}
\mathcal{R}(F \otimes 1) = (E^{-1} \otimes L^{-1})\mathcal{R}(EF \otimes 1) = \\
(E^{-1} \otimes L^{-1})\mathcal{R}((c - t^{-1}K - tL^{-1}) \otimes 1) = \\
E^{-1}c \otimes L^{-1} - t^{-1}(E^{-1} \otimes L^{-1})(\mathcal{R}(K \otimes 1) + t\mathcal{R}(L^{-1} \otimes 1)) = \\
F \otimes L^{-1} + t^{-1}E^{-1}K \otimes L^{-1} + tE^{-1}L^{-1} \otimes L^{-1} - \\
t^{-1}(E^{-1} \otimes L^{-1})(1 - tK^{-1}E \otimes FL)(K \otimes 1) - t(E^{-1}L^{-1} \otimes L^{-1})(1 - tK^{-1}E \otimes FL)
\end{equation}

\begin{equation}
= F \otimes L^{-1} + 1 \otimes F - (KL^{-1} \otimes F)(1 - tK^{-1}E \otimes FL)^{-1}.
\end{equation}

Taking into account the action of $\mathcal{R}$ on $K \otimes F$ and $E \otimes L^{-1}$, it is easy to see that the previous formulae are equivalent to

$$\mathcal{R}(E \otimes K + 1 \otimes E) = K \otimes E + E \otimes 1,$$

$$\mathcal{R}(F \otimes 1 + L^{-1} \otimes F) = 1 \otimes F + F \otimes L^{-1}.$$

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