GENERATING THE MAPPING CLASS GROUPS BY TORSIONS

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Abstract. Let $S_g$ be the closed oriented surface of genus $g$ and let $\text{Mod}(S_g)$ be the mapping class group. When the genus is at least 3, $\text{Mod}(S_g)$ can be generated by torsion elements. We prove the follow results. For $g \geq 4$, $\text{Mod}(S_g)$ can be generated by 4 torsion elements. Three generators are involutions and the forth one is an order 3 element. $\text{Mod}(S_3)$ can be generated by 5 torsion elements. Four generators are involutions and the fifth one is an order 3 element.

1. Introduction

Let $S_g$ be the closed oriented surface of genus $g$. The mapping class group $\text{Mod}(S_g)$ is defined by $\text{Homeo}^+(S_g)/\text{Homeo}_0(S_g)$, the group of homotopy classes of oriented-preserving homeomorphisms of $S_g$.

The study of the generating set of the mapping class group of an oriented closed surface begun in the 1930s. The first generating set of the mapping class group was found by Dehn ([2]). This generating set consist of $2g(g-1)$ Dehn twists for genus $g \geq 3$. About a quarter of a century later, Lickorish found a generating set consisting of $3g-1$ Dehn twists for $g \geq 1$ ([8]). The numbers of Dehn twists in the generating set was improved to $2g+1$ by Humphries ([4]). In fact this is the minimal number of the generators if we require that all of the generators are Dehn twists. This is also proved by Humphries.

If we consider generators other than Dehn twists, it is possible to find smaller generating sets. Wajnryb found that the mapping class groups $\text{Mod}(S_g)$ can be generated by two elements ([12]), each of which is some product of Dehn twists. One element in Wajnrb’s generating set has finite order. Korkmaz in [7] gave another generating set consisting of two elements. One of Korkmaz’s generators is a Dehn twist and the other one is a torsion element of order $4g+2$.

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The mapping class group $\text{Mod}(S_g)$ can be generated only by torsion elements. By an \textit{involution} in a group we simply mean any element of order 2. Luo proved that for $g \geq 3$, $\text{Mod}(S_g)$ is generated by finitely many involutions ([9]). In Luo’s result, the number of the involutions grows linearly with $g$. Brendle and Farb found a universal upper bound for the number of involutions that generate $\text{Mod}(S_g)$. They proved that for $g \geq 3$, $\text{Mod}(S_g)$ can be generated by 7 involutions ([1]). Kassabov improved the number of involution generators to 4 if $g \geq 7$, 5 if $g \geq 5$ and 6 if $g \geq 3$ ([6]).

For the extended mapping class group, Stukow proved that $\text{Mod}^\pm(S_g)$ can be generated by three orientation reversing elements of order 2 ([11]).

If we consider torsion elements of higher orders, not only involutions, Brendle and Farb in [1] also found a generating set consisting of 3 torsion elements. The minimal generating set of torsion elements was found by Korkmaz. He showed that $\text{Mod}(S_g)$ can be generated by two torsion elements, each of which is of order $4g + 2$ ([7]). For generating set of smaller order, Monden found that for $g \geq 3$, $\text{Mod}(S_g)$ is generated by 3 elements of order 3, or generated by 4 elements of order 4 ([10]).

Since the group generated by two involution is a dihedral group, Farb and Margalit in [3], Kassabov in [6] and Monden in [10] mentioned that the following problem is open:

\textbf{Problem 1.1.} Whether or not $\text{Mod}(S_g)$ can be generated by three involutions?

In this paper, we give generating sets consisting of 3 involutions and a torsion of order 3:

\textbf{Theorem 1.2.} For $g \geq 4$, $\text{Mod}(S_g)$ can be generated by 4 torsion elements, among which 3 elements are involutions and the forth one is an order 3 torsion. For $g = 3$, $\text{Mod}(S_3)$ can be generated by 5 torsion elements, among which 4 elements are involutions and the fifth one is an order 3 torsion.

\textbf{Remark 1.3.} When $g < 7$, the number of involutions in Kassabov’s result is at least 5. We give a smaller generating set in these cases. For $g \geq 4$ cases, our result uses the generating sets of the same form.

\textbf{Remark 1.4.} Monden’s result use only 3 torsions of order 3. Korkmaz’s result use only 2 torsions of order $4g + 2$. Our advantage is the use of involutions.

\section{Preliminaries}

We use the convention of functional notation, namely, elements of the mapping class group are applied right to left. The composition $fh$ means that $h$ is applied first.
Let $c$ be the isotopy class of a simple closed curve on $S_g$. Then the (left-hand) Dehn twist $T_c$ about $c$ is the homotopy class of the homeomorphism obtained by cutting $S_g$ along $c$, twisting one of the sides by $2\pi$ to the left and gluing two sides of $c$ back to each other. See Fig. 1.

![Fig. 1](image)

We recall the following results (see, for instance, [3]):

**Lemma 2.1** (Conjugacy relation). For any $f \in \text{Mod}(S_g)$ and any isotopy class $c$ of simple closed curves in $S_g$, we have:

$$T_{f(c)} = f T_c f^{-1}.$$  

**Lemma 2.2** (Dehn twists along disjoint curves commute). Let $a, b$ be two simple closed curves on $S_g$. If $a$ is disjoint from $b$, then

$$T_a T_b = T_b T_a.$$  

**Lemma 2.3** (Braid relation). Let $a, b$ be two simple closed curves on $S_g$. If the geometric intersection number of $a$ and $b$ is one, then

$$T_a T_b T_a = T_b T_a T_b.$$  

Let $c_1, c_2, \ldots, c_t$ be simple closed curves on $S_g$. If $i(c_j, c_{j+1}) = 1$ for all $1 \leq j \leq k - 1$ and $i(c_j, c_k) = 0$ for $|j - k| > 1$, then we call $c_1, c_2, \ldots, c_t$ a chain. Take a regular neighbourhood of their union. The boundary of this neighbourhood consist of one or two simple closed curves, depending on whether $t$ is even or odd. In the odd case, denote the boundary curve by $d$. In the even case, denote the boundary curves by $d_1$ and $d_2$. We have the following chain relation:

**Lemma 2.4** (Chain relation).

$$(T_{c_1} T_{c_2} \cdots T_{c_t})^{2t+2} = T_d$$ for even $t$, 

$$(T_{c_1} T_{c_2} \cdots T_{c_t})^{t+1} = T_{d_1} T_{d_2}$$ for odd $t$.

Dehn discovered the lantern relation and Johnson rediscovered it ([3]). This is a very useful relation in the theory of mapping class groups.

**Lemma 2.5** (Lantern relation). Let $a, b, c, d, x, y, z$ be the curves showed in Figure 2 on a genus zero surface with four boundaries. Then
Lickorish’s result on the set of generators is the following:

**Lemma 2.6 (Lickorish’s generators).** The mapping class group $\text{Mod}(S_g)$ is generated by the set of Dehn twists $\{T_{a_i} (1 \leq i \leq g), T_{b_i} (1 \leq i \leq g), T_{a_i} (1 \leq i \leq g - 1)\}$ (See Fig. 3).

3. **Main results**

Now we are ready to prove the main results of this paper.

**Theorem 3.1.** For $g \geq 4$, $\text{Mod}(S_g)$ can be generated by 4 torsion elements, among which 3 elements are involutions and the forth one is an order 3
torsion. For $g = 3$, $\text{Mod}(S_3)$ can be generated by 5 torsion elements, among which 4 elements are involutions and the fifth one is an order 3 torsion.

Proof. According to Lickorish’s theorem, we only need to construct a set of torsion elements such that each Dehn twist of the Lickorish curves \(\{T_{a_i} (1 \leq i \leq g), T_{b_i} (1 \leq i \leq g), T_{c_i} (1 \leq i \leq g - 1)\}\) can be generated by these torsion generators. By the conjugacy relation of Dehn twists, if the group \(G\) generated by some torsions has the following two properties, then it is $\text{Mod}(S_g)$:

property 1, under the action of \(G\), \(a_i's, b_i's, c_i's\) are in the same orbit;
property 2, some \(T_{a_i}\) is in \(G\).

There are involutions \(f_1, f_2\) on the surface such that \(f_1\) and \(f_2\) are \(\pi\)-rotations and \(f_2f_1\) is an order \(g\) element, hence \(f_2f_1\) permutes \(a_i's, b_i's, c_i's\) respectively (See Fig. 3).

![Fig. 3](image_url)

Meanwhile, in the lantern relation (see the previous figure 2), since each curve in \(\{a, b, c, d\}\) is disjoint from \(x, y, z\), and each curve in \(\{a, b, c, d\}\) is disjoint from each other, the Dehn twistss along them are commutative. The lantern relation can be rewritten as
\[ T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1}). \]

Locally, there is a \((2\pi/3)\)-rotation \(h\) on the genus zero surface with four boundary, mapping the curve \(d\) to itself, sending \(a\) to \(b\), \(b\) to \(c\), \(c\) to \(a\), \(x\) to \(y\), \(y\) to \(z\) and \(z\) to \(x\). Hence

\[ T_d = (T_x T_a^{-1})(h T_x T_a^{-1} h^{-1})(h^2 T_x T_a^{-1} h^{-2}). \]

Compare figure 4 to figure 2, there is a genus-zero-and-four-boundary subsurface bounded by \(\{a_1, c_1, c_2, a_3\}\). \(a_2\) separates \(\{a_1, c_1\}\) from \(\{c_2, a_3\}\). Hence the subgroup \(G \leq \text{Mod}(S_g)\) generated by \(\{f_1, f_2, T_{a_1} f_2 T_{a_1}^{-1}, f_3\}\) includes all the \(T_{a_i}\)'s and \(T_{c_i}\)'s. It remains to show how we get \(T_{b_i}\)'s by torsion elements. We only need to construct a torsion sending some \(a_i\) to \(b_i\).

In the case of genus \(g \geq 4\), we make the global order 3 homeomorphism \(f_3\) sending some \(a_i\) to \(b_i\) as follows. See figure 5.

\[ h \text{ is an order 3 local homeomorphism on the subsurface with four boundaries. Each point on this subsurface (including the point on the boundary) has period 3 except the fix point at the center. The complement of the such a subsurface is a genus } g - 3 \text{ subsurface with also four boundaries } a_1, a_3, c_1, c_2. \]
On the complement, we need to construct a periodic map with the following properties: (1) \(c_1\) is mapped to itself; (2) \(a_1, a_3, c_2\) are permuted cyclicly; (3) each point on the complement subsurface (including the point on the boundary) has period 3 except the fix points; (4) some \(a_i\) is sent to \(b_i\).

We cut the complement along curves \(e_4, \ldots, e_{g-1}\) into \(g-3\) pieces, each of which is of genus 1. See figure 5. When \(g \geq 5\), such pieces can be divided into 3 classes: class (i): has only one boundary \(e_{g-1}\); class (ii): has 5 boundaries \(e_4, a_1, a_3, c_2, c_1\); class (iii): has 2 boundaries \(e_{i-1}, e_i\). When \(g = 4\), the complement has only one piece with 4 boundaries \(a_1, a_3, c_2, c_1\).

To construct the periodic maps of order 3 on such pieces, sending some meridian \(a_i\) to the longitude \(b_i\), take the genus 1 surfaces as the quotient space of the hexagon (perhaps with holes) gluing the opposite sides. See figure 6. The \(2\pi/3\)-rotation of the plane is the obvious homeomorphism on the quotient space we want.

Now at the beginning, we have the order 3 homeomorphism \(h\) on the lantern. Then we can extend the order 3 homeomorphism to the adjacent piece, inducing an order 3 homeomorphism on the rest component of the boundary of the adjacent piece. Then piece by piece, we make a global order 3 homeomorphism \(f_3\) of the genus \(g\) surface. All the \(a_i's, b_i's, c_i's\) is in the same orbit under the action of the group \(G\) generated by \:\{\(f_1, f_2, T_{a_1}f_2T_{a_1}^{-1}, f_3\}\). \(T_{a_1}\) is in \(G\). Hence \(G = \text{Mod}(S_g)\).

In the case of genus \(g = 3\), the complement of the lantern bounded by \(a_1, a_3, c_2, c_1\) is a surface of genus zero has four boundaries. The global order 3 homeomorphism \(f_3\) on complement of the lantern can only be the same form as \(h\). \(f_3\) cannot send some \(a_i\) to some \(b_j\). We need one more involution to send some \(a_i\) to some \(b_j\). Now \(a_3\) and \(b_3\) are two non-separating curves on the surface. So there is a homeomorphism \(\sigma = T_{a_3}T_{b_3}\) sending \(a_3\) to \(b_3\) and fixing \(a_1, b_1, a_2, b_2\). Then \(\sigma^{-1}f_1\sigma\) is an involution, sending \(a_3\) to \(b_3\). The group generated by \:\{\(f_1, f_2, T_{a_1}f_2T_{a_1}^{-1}, f_3, \sigma^{-1}f_1\sigma\}\) is \(\text{Mod}(S_3)\).

\(\square\)
REFERENCES

[1] T. E. Brendle and B. Farb, Every mapping class group is generated by 3 torsion elements and by 6 involutions. J. Algebra 278 (2004), 187C198.
[2] M. Dehn, Papers on group theory and topology (Springer-Verlag, New York, 1987) (Die Gruppe der Abbildungsklassen, Acta Math. Vol. 69 (1938), 135C206).
[3] B. Farb and D. Marglait, A Primer on Mapping Class Groups. Princeton Math. Ser., (Princeton University Press, 2012), 623C658.
[4] S. P. Humphries, Generators for the mapping class group. Topology of low-dimensional manifolds. Proc. Second Sussex Conf. Chelwood Gate 1977 Lecture Notes in Math. 722 (Springer, 1979), 44C47.
[5] D. Johnson, The structure of Torelli group I: A finite set of generators for I. Ann. of Math. 118 (1983), 423C442.
[6] M. Kassabov, Generating Mapping Class Groups by Involutions. arXiv:math.GT/0311455 v1 25 Nov 2003.
[7] M. Korkmaz, Generating the surface mapping class group by two elements. Trans. Amer. Math. Soc. 357 (2005), 3299C3310.
[8] W. B. R. Lickorish, A finite set of generators for the homeotopy group of a 2-manifold. Proc. Camb. Phils. Soc. 60 (1964), 769C778.
[9] F. Luo, Torsion Elements in the Mapping Class Group of a Surface. arXiv:math.GT/0004048 v1 8 Apr 2000.
[10] N. Monden, Generating the mapping class group by torsion elements of small order. Mathematical Proceedings of the Cambridge Philosophical Society, 154 (2013), pp 41-62, doi:10.1017/S0305004112000357
[11] M. Stukow, The extended mapping class group is generated by 3 symmetries. C. R. Math. Acad. Sci. Paris 338 (2004), no. 5, 403C406.
[12] B. Wajnryb, Mapping class group of a surface is generated by two elements. Topology, 35(2), 377-383, 1996.

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