THE CARTIER CORE MAP FOR CARTIER ALGEBRAS

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ABSTRACT. Let $R$ be a commutative Noetherian $F$-finite ring of prime characteristic and let $\mathcal{D}$ be a Cartier algebra. We define a self-map on the Frobenius split locus of the pair $(R, \mathcal{D})$ by sending a point $P$ to the splitting prime of $(R_P, \mathcal{D}_P)$. We prove this map is continuous, containment preserving, and fixes the $\mathcal{D}$-compatible ideals. We show this map can be extended to arbitrary ideals $J$, where in the Frobenius split case it gives the largest $\mathcal{D}$-compatible ideal contained in $J$. Finally, we apply Glassbrenner’s criterion to prove that the prime uniformly $F$-compatible ideals of a Stanley-Reisner rings are the sums of its minimal primes.

1. Introduction

Frobenius splitting is an important tool in characteristic $p$ commutative algebra and algebraic geometry. Locally, Frobenius splitting (or $F$-purity) is a restriction on singularities, analogous to log canonicity for complex singularities. Strong $F$-regularity is a strengthening of Frobenius splitting, analogous to how Kawamata log terminality is a strengthening of log canonicity. There is much interest in understanding the world of Frobenius split objects that are not strongly $F$-regular.

For a local ring, Aberbach and Enescu introduced the splitting prime as a way to measure the difference between Frobenius splitting and strong $F$-regularity—the elements in the splitting prime are obstructions to strong $F$-regularity, and in particular, the splitting prime of a domain is zero precisely for strongly $F$-regular rings [1]. Aberbach and Enescu’s splitting prime can also be described as the largest uniformly $F$-compatible ideal in the sense of Schwede [22]. In this paper, we will be working with a generalization of the splitting prime following two different directions.

Frobenius splitting and strong $F$-regularity have been generalized to further settings, including pairs $(X, \Delta)$ consisting of a $\mathbb{Q}$-divisor $\Delta$ on a smooth variety $X$ of characteristic $p$; pairs $(R, a^t)$ where $R$ is a ring of characteristic $p$, $a$ is an ideal, and the formal exponent $t$ is a positive real number; as well as more general settings [12, 19, 21, 25]. The Cartier algebra (see Definition 2.1) gives a unified and more general approach, allowing us to talk about Frobenius splitting and strong $F$-regularity for an arbitrary subalgebra of the Cartier algebra [23]. An important problem is to understand the extent to which strong $F$-regularity fails for an arbitrary Frobenius split subalgebra of the Cartier algebra [4, 22].

Fix a Frobenius split pair $(R, \mathcal{D})$, where $\mathcal{D}$ is a Cartier subalgebra (see Definition 2.4). One main theme of this paper is to consider splitting primes via the perspective of the Cartier core map

$$C_{\mathcal{D}} : \text{Spec } R \to \text{Spec } R$$
which assigns to each prime \( P \in \text{Spec} R \) the splitting prime \( C_D(P) \) corresponding to the pair \((R_P, D_P)\). Alternatively, if \( R \) is not Frobenius split one can define \( C_D \) on the Frobenius split locus, \( U_D \), of \((R, D)\). In this context, we show the following main result.

**Theorem A** (Theorem 3.9). Let \( R \) be an \( F \)-finite Noetherian ring of characteristic \( p \), and let \( D \) be a Cartier subalgebra. Then the Cartier core map
\[
U_D \to \text{Spec } R \quad P \mapsto C_D(P)
\]
is a continuous containment preserving map on the \( F \)-pure locus \( U_D \) of the pair \((R, D)\) which fixes the \( D \)-compatible ideals. The image of \( C_D \) is the set of prime \( D \)-compatible ideals and is always finite. The image is the set of minimal primes of \( R \) precisely when the pair \((R, D)\) is strongly \( F \)-regular.

In another direction, for an arbitrary (not necessarily Frobenius split) pair \((R, D)\), we can associate to any ideal \( J \) of \( R \) the largest \( D \)-compatible ideal \( C_D(J) \) contained in \( J \). We call this ideal the Cartier core of \( J \), following Badilla-Céspedes, who considered this earlier for the non-pair setting \([3]\). We will prove some basic facts about the Cartier core, many of which mirror results of Schwede in the triples setting and of Badilla-Céspedes in the setting where \( D \) is the full Cartier algebra.

Returning to the non-pair setting, we also give the following explicit formula for the Cartier core of an arbitrary ideal in a quotient of a regular ring, making use of a criterion for strong \( F \)-regularity due to Glassbrenner \([11]\).

**Theorem B** (Theorem 4.6). Let \( S \) be a regular \( F \)-finite ring, let \( I \subseteq J \) be ideals of \( S \), and let \( R = S/I \). Then
\[
C_R(J/I) = \left( \bigcap_{e > 0} J^{[p^e]} :_S (I^{[p^e]} :_S I) \right) / I.
\]

This presentation of the Cartier core allows us to prove that the Cartier core map commutes with basic operations such as localizing, adjoining a variable, and in the case of quotients of polynomial rings, with homogenization (see Lemma 3.7, Proposition 4.8, Lemma 4.10). As an application of these techniques, we give an exact description of the Cartier core map in the case of Stanley-Reisner rings.

**Theorem C** (Theorem 5.1, Corollary 5.2). Let \( R \) be a Stanley-Reisner ring over a field that has prime characteristic and is \( F \)-finite. Let \( Q \) be any prime ideal of \( R \). Then
\[
C_R(Q) = \sum_{P \subseteq \text{Min}(R)} P.
\]
In particular, the image of the Cartier core map on primes, i.e., the set of generic points of \( F \)-pure centers of \( R \), is the set of sums of minimal primes. Further, if \( J \) is any ideal, then
\[
C_R(J) = \sum_{Q \subseteq \text{Min}(R)} \left( \bigcap_{P \subseteq Q} P \right) \cap J.
\]
This theorem extends existing work on computing certain specific uniformly $F$-compatible ideals and $D$-compatible ideals, including the splitting prime and test ideals, for Stanley-Reisner rings \cite{1,3,8,26}.

**Assumptions.** All rings in this paper (other than the Cartier algebras) are commutative, Noetherian, and unital. Furthermore, all such rings are of prime characteristic $p$ and are $F$-finite.

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## 2. Background

For a ring $R$ of prime characteristic $p$, the Frobenius endomorphism is the ring map $F : R \to R$ where $F(r) = r^p$. To distinguish the copies of $R$, we will write $F_s R$ for the codomain. As a ring, this Frobenius pushforward $F_s R = \{ F_s r \mid r \in R \}$ is exactly the same as $R$, just with this formal symbol $F_s$ prepended everywhere. For example, multiplication is $(F_s r)(F_s s) = F_s(rs)$.

The benefit of this notation is that it clarifies the formal symbol $F_s$ with prepended everywhere. For example, multiplication is $(F_s r)(F_s s) = F_s(rs)$. We can iterate the Frobenius, writing $F^e_s R \to F^e_s R$, where $F^e(r) = F^e_s(r^p)$ and $rF^e_s s = F^e_s(r^p s)$.

We will utilize this $R$-module structure on $F^e_s R$, but first we need a cohesive way to consider the certain maps in $\text{Hom}_R(F^e_s R, R)$. First, given any map $\psi \in \text{Hom}_R(F^d_R, R)$, we write $F^e_s \psi : F^{e+d}_s R \to F^e_s R$ for the Frobenius pushforward of the map, where

\[ (F^e_s \psi)(F^{e+d}_s r) = (F^e_s \psi)(F^e_s(F^d_s r)) = F^e_s(\psi(F^d_s r)). \]

Now we define a (non-commutative) multiplication on the abelian group $\bigoplus_s \text{Hom}_R(F^e_s R, R)$ as follows. Given maps $\phi \in \text{Hom}_R(F^e_s R, R)$ and $\psi \in \text{Hom}_R(F^d_s R, R)$, we define their product as

\[ \phi \cdot \psi = \phi \circ F^e_s \psi. \]

More concretely, for any $r \in R$ we have

\[ (\phi \cdot \psi)(F^{e+d}_s r) = \phi \left( F^e_s(\psi(F^d_s r)) \right). \]

**Definition 2.1.** The (full) Cartier algebra on $R$ is the graded non-commutative ring

\[ \mathcal{C}_R = \bigoplus_{e \geq 0} \text{Hom}_R(F^e_s R, R), \]

where multiplication is as defined in Equation (1).

Note that we are writing $F^0_0 R$ to mean $R$ as an $R$-module, so that $(\mathcal{C}_R)_0 = \text{Hom}_R(R, R) \cong R$. We will often write the “multiplication by $c$” map as simply $c$, and its pushforward as $F^e_s c$, so that $(F^e_s c)(F^e_s r) = F^e_s (c r)$. However, this copy of $R$ is rarely central in $\mathcal{C}_R$, because for $\phi$ of degree $e$, we have $r \cdot \phi = \phi \cdot r^e$. In particular, $R$ is central only if $R = \mathbb{F}_p$. 

Definition 2.2. A Cartier subalgebra $\mathcal{D}$ is a graded subalgebra of $\mathcal{C}_R$ such that $\mathcal{D}_0 = R$. In particular, $\mathcal{D}$ has the form $\mathcal{D} = \bigoplus_e \mathcal{D}_e$ where $\mathcal{D}_e \subseteq \text{Hom}_R(F_e^e R, R)$ for all $e \geq 0$.

Example 2.3 ([23 Rmk. 3.10]). Let $(R, a^t)$ be a pair where $a$ is an ideal and the formal exponent $t$ is a positive real number. Then the corresponding Cartier subalgebra $\mathcal{C}_{a^t}$ has 

$$\mathcal{C}_{a^t} = \text{Hom}_R(F_e^e R, R) \cdot a^{t(p^e - 1)}.$$ 

Definition 2.4. Let $R$ be a ring of prime characteristic, and let $\mathcal{D}$ be a Cartier subalgebra on $R$.

- The pair $(R, \mathcal{D})$ is $F$-finite if $R$ is $F$-finite, i.e., $F_* R$ is a finite $R$-module. Every ring $R$ in this paper will be $F$-finite.

- The pair $(R, \mathcal{D})$ is Frobenius split or (sharply) $F$-pure if there exists some $e > 0$ and some $\phi \in \mathcal{D}_e$ with $\phi(F_e^e 1) = 1$.

- If $c$ is an element of $R$, then the pair $(R, \mathcal{D})$ is eventually Frobenius split along $c$, or $F$-pure along $c$ if there exists some $e > 0$ and some $\phi \in \mathcal{D}_e$ with $\phi(F_e^e c) = 1$.

- The pair $(R, \mathcal{D})$ is strongly $F$-regular if it is eventually Frobenius split along every $c$ which is not in any minimal prime of $R$.

We will follow the example of Blickle, Schwede, and Tucker and omit the adjective “sharp” when discussing $F$-purity of pairs [4 Def. 2.7]. Observe that if $\phi \in \mathcal{D}_e$ is a splitting of $F_e$, then there is a splitting in any multiple of the degree, given by $\phi^n \in \mathcal{D}_{en}$.

Since we will consider only pairs $(R, \mathcal{D})$ where $R$ is Noetherian and $F$-finite, this means that for any ring $S$ such that $R \to S$ is flat, we have by [18 Thm. 7.11],

$$S \otimes_R \text{Hom}_R(F_e^e R, R) \cong \text{Hom}_S(S \otimes_R F_e^e R, S).$$

In the case that $S$ is a localization of $R$ we further know that $S$ commutes with the Frobenius, that is, for any multiplicative set $W$,

$$W^{-1} R \otimes_R \text{Hom}_R(F_e^e R, R) \cong \text{Hom}_{W^{-1} R}(F_e^e(W^{-1} R), W^{-1} R).$$

We will use this isomorphism freely: if $u \otimes \phi$ is a pure tensor in $W^{-1} R \otimes_R \text{Hom}_R(F_e^e R, R)$, we will identify this with the map in $\text{Hom}_{W^{-1} R}(F_e^e(W^{-1} R), W^{-1} R)$ which sends $F_e^e(\frac{z}{u})$ to $r\phi(F_e^e(su^{p^e - 1}))$. This identification is easier to understand if we first rewrite $F_e^e(\frac{z}{u})$ as

$$F_e^e\left(\frac{sw^{p^e - 1}}{w^{p^e}}\right) = \frac{1}{u} \cdot F_e^e(su^{p^e - 1}) = \frac{F_e^e(su^{p^e - 1})}{u}.$$ 

Thus we have a natural containment $W^{-1} R \otimes_R \mathcal{D}_e \subseteq (\mathcal{C}_{W^{-1} R})_e$. We can therefore construct a new Cartier subalgebra $W^{-1} \mathcal{D}$ on $W^{-1} R$ using this isomorphism, so that

$$(W^{-1} \mathcal{D})_e = W^{-1} R \otimes \mathcal{D}_e.$$ 

When we are localizing at a prime ideal $P$, we write this Cartier subalgebra as $\mathcal{D}_P$.

Now that we have the setup to discuss localizations of Cartier subalgebras, we can state and prove the following standard result on the Frobenius split locus in the setting of Cartier algebra pairs.
Theorem 2.5. Let $R$ be an $F$-finite ring, and $D$ a Cartier subalgebra. Then the set of primes $P$ of $R$ at which $(R_P, D_P)$ is $F$-pure is open. Further, the pair $(R, D)$ is $F$-pure if and only if the localized pair $(R_P, D_P)$ is $F$-pure for all primes $P$.

Proof. For any $e$, we get a module map $\Psi_e : D_e \rightarrow R$ via evaluation at $F^e1$. The pair $(R, D)$ is $F$-pure exactly when this map is surjective for some $e > 0$, or equivalently, when there exists an $e > 0$ such that $R/\text{im } \Psi_e = 0$. The localization $(\Psi_e)_P$ corresponds to the evaluation map $(D_P)_e \rightarrow R_P$, so the pair $(R_P, D_P)$ is not $F$-pure if and only if $R_P/\text{im}(\Psi_e)_P \neq 0$ for all $e$. Thus the non-$F$-pure locus is precisely the closed set $\bigcap_{e>0} V(\text{im } \Psi_e)$.

For the second statement, if $(R, D)$ is $F$-pure, then there exists some $e > 0$ and $\phi \in D_e$ with $\phi(F^e1) = 1$. By definition, the localization $\phi_P : F^e(R_P) \rightarrow R_P$ is in $(D_P)_e$, and so $(R_P, D_P)$ is also $F$-pure.

Conversely, if each $(R_P, D_P)$ is $F$-pure, then the complements of the sets $V(\text{im } \Psi_e)$ give an open cover of Spec $R$. Since Spec $R$ is compact, only finitely many are needed, say, the complements of $V(\text{im } \Psi_{e_1}), \ldots, V(\text{im } \Psi_{e_t})$. Then taking $e = e_1 \cdots e_t$ to be the product of these indices, we must have that $(R_P, D_P)$ has a splitting in $D_e$ for every prime $P$. Thus the map $\Psi_e$ is surjective at every prime, and therefore is surjective.

This proof in fact shows that for any $c$, the set of primes $P$ such that $(R_P, D_P)$ is not eventually Frobenius split along $c$ is closed. Further, it also shows that $(R, D)$ is eventually Frobenius split along $c$ if and only if $(R_P, D_P)$ is for every prime ideal $P$. In particular, this shows that just like in the classical case, $(R, D)$ is strongly $F$-regular if and only if every $(R_P, D_P)$ is as well.

3. The Cartier core map

Fix a pair $(R, D)$ where $R$ is an $F$-finite and Frobenius split ring and where $D$ is a Cartier subalgebra. In this section we will define an explicit continuous map

$$C_D : \text{Spec } R \rightarrow \text{Spec } R$$

that has some especially nice properties. The image of our map is the set of $D$-compatible primes of Spec $R$, which in the case $D = C_R$ is the set of (generic points of) $F$-pure centers. If $R$ is not Frobenius split, we can instead define $C_D$ on the open locus of Frobenius split points.

More generally, the map $C_D$ can be viewed as an endomorphism defined on the set of all ideals of $R$ (not necessarily proper), which is especially interesting on the class of radical ideals in a Frobenius split ring.

Definition 3.1. Let $R$ be an $F$-finite ring of prime characteristic. Let $J$ be an ideal of $R$. Let $D \subseteq C_R$ be a Cartier subalgebra. Then the Cartier core of $J$ in $R$ with respect to $D$ is

$$C_D(J) = \{ r \in R \mid \phi(F^e r) \in J \quad \forall e > 0, \quad \forall \phi \in D_e \} .$$

We will write $C_R(J)$ to mean the Cartier core with respect to the full Cartier algebra $C_R$, and just $C(J)$ when the ring and Cartier subalgebra are clear from context. In the case that $D = C_R$, the Cartier core $C_R(J)$ is also denoted (e.g., in [3]) as $\mathcal{P}(J)$.
Note that for an $D$ we can also express the Frobenius pushforward of the $e$-th Cartier contraction as

$$F_e^*(A_{D_e}(J)) = \bigcap_{\phi \in D_e} \phi^{-1}(J).$$

When $D = C_R$, the $e$-th Cartier contraction $A_{D_e}(J)$ is sometimes denoted by $J_e$.

To motivate the definition of the Cartier core, note that the condition $J \subseteq C_D(J)$, i.e., that $\phi(F_e^*(J)) \subseteq J$ for all $e$ and for all $\phi \in D_e$, is precisely the condition that $J$ is $D$-compatible.

The Cartier core was defined for the case $C_R = D$ by Badilla-Céspedes [3, Def. 4.12] as a generalization of Aberbach and Enescu’s splitting prime [1] and of Brenner, Jeffries, and Núñez Betancourt’s differential core [5]. Here we generalize this definition to the context of pairs, similar to Blickle, Schwede, and Tucker’s generalization of the splitting prime to the context of pairs [4].

Further, as the next two results show, the Cartier core of a prime ideal $P$ carries information about the localization $(R_P, D_P)$.

**Proposition 3.3** ([4, Prop. 2.12]). Let $(R, D)$ be an $F$-finite pair and let $P$ be a prime ideal of $R$. Then $r \notin C_D(P)$ if and only if the pair $(R_P, D_P)$ is $F$-pure along $r/1$. In particular, $(R_P, D_P)$ is $F$-pure if and only if $C_D(P)$ is proper.

**Proof.** Since $D_P = D \otimes R_P$, saying $\phi(F_e^*(r)) \in P$ for some $\phi \in D_e \subseteq \text{Hom}_R(F_e^*R, R)$ is equivalent to saying $\phi(F_e^*(r/1)) \in PR_P$, viewing $\phi \in (D_P)_e \subseteq \text{Hom}_{R_P}(F_e^*(R_P), R_P)$.

The pair $(R_P, D_P)$ is $F$-pure if and only if there is some $\phi \in D_P$ such that $\phi(F_e^*(1))$ is a unit, i.e., not in $PR_P$, which by the above is equivalent to having $1 \notin C_D(P)$.

**Proposition 3.4** (Cf. [4, Thm. 2.11,Prop. 2.12]). Let $(R, D)$ be an $F$-finite pair and let $P$ be a prime ideal of $R$. Then the pair $(R_P, D_P)$ is strongly $F$-regular if and only if $C_D(P)$ is contained in some minimal prime of $R$.

**Proof.** The pair $(R_P, D_P)$ is strongly $F$-regular if and only if $(R_P, D_P)$ is $F$-pure along every non-zero divisor, i.e., $C_D(P)$ is contained in the union of the minimal primes of $R$. Since
$C_D(P)$ is an ideal, prime avoidance says this is equivalent to having $C_D(P)$ contained in some minimal prime of $R$. \hfill \Box$

Now that we have provided some motivation for the Cartier core construction, we will discuss some of its nice properties.

**Proposition 3.5.** Let $(R,D)$ be an $F$-finite pair. If $J_1 \subseteq J_2$ in $R$, then $C_D(J_1) \subseteq C_D(J_2)$.

**Proof.** For every $e$, $A_{D_e}(J_1) \subseteq A_{D_e}(J_2)$, since if $\phi(F^*_e r) \in J_1$ for some $\phi \in D_e$, we also have $\phi(F^*_s r) \in J_2$. Taking the intersection over all $e$ gives our result. \hfill \Box

**Proposition 3.6** (Cf. [3, Prop 4.6]). Let $\{J_\alpha\}$ be an arbitrary collection of ideals in an $F$-finite ring $R$, and let $D$ be a Cartier subalgebra. Then

$$C_D\left(\bigcap_\alpha J_\alpha\right) = \bigcap_\alpha C_D(J_\alpha).$$

**Proof.** We see that

$$C_D\left(\bigcap_\alpha J_\alpha\right) = \left\{ r \in R \mid \phi(F^*_e r) \in \bigcap_\alpha J_\alpha \forall e, \forall \phi \in D_e \right\}$$

$$= \bigcap_\alpha \left\{ r \in R \mid \phi(F^*_e r) \in J_\alpha \forall e, \forall \phi \in D_e \right\}$$

$$= \bigcap_\alpha C_D(J_\alpha).$$ \hfill \Box

In particular, the set of Cartier cores with respect to $D$ is closed under arbitrary intersection. We will see in Proposition 3.15 that this set is also closed under arbitrary sum for $F$-pure pairs.

Our next goal is to show that the Cartier core construction commutes with localization. To do so, we need the following lemma.

**Lemma 3.7.** Let $(R,D)$ be an $F$-finite pair, let $Q$ be a $P$-primary ideal of $R$, and let $W$ be a multiplicative set avoiding $P$, so that $W \cap P = \emptyset$. Then

$$C_{W^{-1}D}(QW^{-1}R) \cap R = C_D(Q).$$

**Proof.** By our discussion in Section 2, $W^{-1}D_e$ is generated by the maps $\frac{\phi}{w} : F_*(W^{-1}R) \to R$ for $\phi \in D_e$ and $w \in W$, where $\frac{\phi}{w}(F_*(\frac{s}{u})) = \frac{\phi(F^*_s(\frac{s}{u}))}{wu}$. We will start by showing that $\frac{s}{t} \in A_{W^{-1}D_e}(QW^{-1}R)$ if and only if $s \in A_{D_e}(Q)$.

By definition, $\frac{s}{t} \in A_{W^{-1}D_e}(QW^{-1}R)$ if and only if $\psi(F^*_s(\frac{s}{t})) \in QW^{-1}R$ for all $\psi \in W^{-1}D_e$. This is equivalent to having

$$\frac{\phi(F^*_s(s))}{w} \in QW^{-1}R$$

for all $\phi \in D_e$ and all $w \in W$. This means that we can write $\frac{\phi(F^*_s(s))}{w} = \frac{j}{u}$ for some $j \in Q$, $u \in W$, i.e., there exists $v \in W$ such that $vu\phi(F^*_s(s)) = vuj$. The latter is in $Q$, but $vu \notin P$. \hfill \Box
so by \( P \)-primaryness of \( Q \) we must then have \( \phi(F^*_s s) \in Q \). This holds for all \( \phi \) exactly when \( s \in A_{D_\infty}(Q) \).

Now we have shown our first claim, which implies \( A_D(Q) = A_{W^{-1}D_\infty}(Q) \cap R \). Intersecting both sides over all \( e > 0 \), we see
\[
C_{W^{-1}D}(Q) \cap R = C_D(Q).
\]

\[\square\]

**Theorem 3.8.** Let \((R,D)\) be an \( F \)-finite pair, let \( J \) be an ideal of \( R \), and let \( W \) be a multiplicative set avoiding every prime in \( \text{Ass}(J) \). Then
\[
C_{W^{-1}D}(JW^{-1}R) \cap R = C_D(J) \quad \text{and} \quad C_D(J)W^{-1}R = C_{W^{-1}D}(JW^{-1}R).
\]

**Proof.** Write \( J = Q_1 \cap \cdots \cap Q_t \) a minimal primary decomposition of \( J \) with corresponding primes \( P_i = \sqrt{Q_i} \). Then since intersection commutes with applying \( C_D \) and with contraction,
\[
C_{W^{-1}D}(J) \cap R = \bigcap_{i=1}^t \left( C_{W^{-1}D}(Q_i) \cap R \right).
\]
By Lemma 3.7 since \( W \cap P_i = \emptyset \) we have \( C_{W^{-1}D}(Q_i) \cap R = C_D(Q_i) \) and so
\[
C_{W^{-1}D}(J) \cap R = \bigcap_{i=1}^t C_D(Q_i) = C_D(J).
\]
For the second equality, we note
\[
C_{W^{-1}D}(J) = (C_{W^{-1}D}(J) \cap R)W^{-1}R = C_D(J)W^{-1}R
\]
since contracting then extending to a localization preserves ideals. \(\square\)

Now that we have established the preliminary results for arbitrary ideals, we move to considering prime ideals. Our main results of the rest of this section can be summarized in the following theorem.

**Theorem 3.9.** Let \( R \) be an \( F \)-finite Noetherian ring, and let \( D \) be a Cartier subalgebra. Then the Cartier core construction with respect to \( D \) induces a well-defined, continuous, and containment preserving map on the \( F \)-pure locus of the pair \((R,D)\) which fixes \( D \)-compatible ideals. The image of the map is the set of \( D \)-compatible ideals in \( U_D \) and is always finite. The image is the set of minimal primes of \( R \) precisely when the pair \((R,D)\) is strongly \( F \)-regular.

**Proof.** We have already seen in Proposition 3.5 that the Cartier core is containment preserving, even without restricting to primes. Corollary 3.12 will show that the map \( C : U_D \to U_D \) is well-defined. Theorem 3.23 will show that this map is continuous, and Proposition 3.21 discusses the finiteness of the image. Theorem 3.19 will show that the image is precisely the set of \( F \)-pure \( D \)-compatible ideals, which combined with Proposition 3.16 shows that all the \( D \)-compatible ideals in \( U_D \) are fixed.

The one statement that doesn’t have a stand-alone proof elsewhere is the last one. \((R,D)\) is strongly \( F \)-regular if and only if each \((R_P,D_P)\) is strongly \( F \)-regular. By Proposition 3.4 this occurs exactly when each \( C_D(P) \) is contained in a minimal prime of \( R \). But since \( C_D(P) \) is prime, this is equivalent to having \( C_D(P) \) be a minimal prime. \(\square\)
It is known that the splitting prime, which in our notation is $C_R(m)$ for $(R, m)$ local, is indeed prime \[\text{Prop. 2.12}\], even in the case of an arbitrary Cartier subalgebra \[\text{Prop. 2.12}\]. After localizing, the same proof works here, which we repeat for the reader’s convenience.

**Proposition 3.10** (\[\text{Prop. 2.12}\]). If $P$ is prime and $C_D(P)$ is proper, then $C_D(P)$ is prime.

**Proof.** Suppose $c_0, c_1 \notin C_D(P)$. Then we will show $c_0 c_1 \notin C_D(P)$. Our assumption means that $(R_P, D_P)$ is $F$-pure along each $c_i$, i.e., there exists an $e_i$ and $\psi_i \in (D_P)_{e_i}$ such that $\psi_i(F_*^{e_i}c_i) = 1$. Then applying the map $\psi_1 \circ F_*^{e_1} \psi_0 \circ F_*^{e_0+e_1}(c_1^{e_0-1})$ to $F_*^{e_0+e_1}(c_0 c_1)$, where we are writing $F_*^{e_0+e_1}(c_1^{e_0-1})$ to mean multiplication by this ring element, we get

$$
F_*^{e_0+e_1}(R_P) \xrightarrow{F_*^{e_0+e_1}(c_1^{e_0-1})} F_*^{e_0+e_1}(R_P) \xrightarrow{F_*^{e_1} \psi_0} F_*^{e_1}(R_P) \xrightarrow{\psi_1} R_P
$$

$$
F_*^{e_0+e_1}(c_0 c_1) \xrightarrow{F_*^{e_0+e_1}(c_0 c_1)} F_*^{e_0+e_1}(c_0 c_1) = F_*^{e_1}(c_1 F_*(c_0)) \xrightarrow{F_*^{e_1} \psi_1} F_*^{e_1} c_1 \xrightarrow{1} 1.
$$

Rewriting this map as $\psi_1 \circ F_*^{e_1} \psi_0 \circ F_*^{e_0+e_1}(c_1^{e_0-1}) = \psi_1 \cdot \psi_0 \cdot c_1^{e_0-1}$, we see that it is in $(D_P)_{e_0+e_1}$, and thus that that $(R_P, D_P)$ is also $F$-pure along $c_0 c_1$, as desired. \[\square\]

**Proposition 3.11.** Let $R$ be a characteristic $p$, $F$-finite ring, and let $D$ be a Cartier subalgebra. If $Q$ is a $P$-primary ideal of $R$ and $C_D(P)$ is proper, then $C_D(Q) \subseteq Q$.

**Proof.** Since $C_D(P)$ is proper, there is some $e > 0$ and $\psi \in D_e$ with $\psi(F_*^e 1) \notin P$. Consider $r \notin Q$ and the map $\psi \circ (F_*^e r^{p^e-1}) = \psi \cdot r^{p^e-1}$ in $D_e$. Then by $P$-primariness,

$$
r \psi(F_*^e) = \psi(F_*^e r^{p^e}) = (\psi \cdot r^{p^e-1})(F_*^e r) \notin Q,
$$

and so $r \notin C_D(Q)$ as desired. \[\square\]

**Corollary 3.12.** Let $(R, D)$ be an $F$-finite pair, with $F$-pure locus $U_D$. Then the Cartier core construction induces a well-defined map $C_D : U_D \to U_D$.

**Proof.** Let $P$ be a prime ideal in $U_D$. Then $(R_P, D_P)$ is Frobenius split, so Proposition \[3.3\] gives that $C_D$ is proper, and thus prime by Proposition \[3.10\]. This gives a map $C_D : U_D \to \text{Spec } R$.

Then Proposition \[3.11\] says $C_D(P) \subseteq P$. Since the $F$-pure locus is open, this means $C_D(P)$ must also be in the $F$-pure locus. \[\square\]

**Corollary 3.13** (Cf. \[22\] Cor. 4.8]). Suppose the pair $(R, D)$ is $F$-finite and $F$-pure. If $P$ is a minimal prime of $R$, then $C_D(P) = P$.

**Proof.** Since $(R, D)$ is $F$-pure, $C_D(P) \subseteq P$. Since $C_D(P)$ is prime by Proposition \[3.10\] and $P$ is minimal, we must have that $C_D(P) = P$. \[\square\]
Corollary 3.14 (Cf. [3] Prop. 4.5). If the pair \((R, \mathcal{D})\) is \(F\)-finite and \(F\)-pure, then for any ideal \(J\) we have \(C_D(J) \subseteq J\).

Proof. Write \(J = Q_1 \cap \cdots \cap Q_t\), where the \(Q_i\) give a primary decomposition of \(J\). Then by Proposition 3.6,
\[
C_D(J) = C_D(Q_1) \cap \cdots \cap C_D(Q_t).
\]
Since \((R, \mathcal{D})\) is Frobenius split, for every prime \(P\) the pair \((R_P, \mathcal{D}_P)\) is also Frobenius split, and thus has \(C_D(P)\) proper by Proposition 3.3. By Proposition 3.11, each \(C_D(Q_i) \subseteq Q_i\). Intersecting, we get that \(C_D(J) \subseteq J\) as desired. \(\square\)

Proposition 3.15 (Cf. [22] Lemma 3.5]). Let \((R, \mathcal{D})\) be an \(F\)-finite, \(F\)-pure pair, and let \(\{J_\alpha\}_{\alpha \in \mathcal{A}}\) be a collection of ideals with \(C_D(J_\alpha) = J_\alpha\) for all \(\alpha \in \mathcal{A}\). Then we have
\[
C_D\left(\sum_\alpha J_\alpha\right) = \sum_\alpha C_D(J_\alpha).
\]
Proof. Since \(J_\beta \subseteq \sum J_\alpha\), we have \(C_D(J_\beta) \subseteq C_D\left(\sum J_\alpha\right)\) for all \(\beta \in \mathcal{A}\) by Proposition 3.5, and so
\[
\sum_\alpha C_D(J_\alpha) \subseteq C_D\left(\sum_\alpha J_\alpha\right).
\]
For the reverse containment, we use our assumption that \(C_D(J_\alpha) = J_\alpha\) and Corollary 3.14 to see that
\[
C_D\left(\sum J_\alpha\right) = C_D\left(\sum C_D(J_\alpha)\right) \subseteq \sum C_D(J_\alpha)
\]
which is our desired opposite inclusion. \(\square\)

Proposition 3.16. If the pair \((R, \mathcal{D})\) is \(F\)-finite and \(F\)-pure, then for any ideal \(J\) in \(R\),
\[
C_D(J) = C_D\left(C_D(J)\right).
\]
Proof. By Corollary 3.14, we know that \(C_D(J) \subseteq J\). Then \(C_D(C_D(J)) \subseteq C_D(J)\) by Proposition 3.5, so it suffices to show the other direction.

Consider \(f \notin C_D(C_D(J))\). Thus there exists \(\epsilon > 0\) and \(\phi \in \mathcal{D}_\epsilon\) with \(\phi(F_\epsilon^e f) \notin C_D(J)\). Then there must also exist \(\epsilon'\) and \(\phi' \in \mathcal{D}_{\epsilon'}\) with \(\phi'(F_{\epsilon'}^{e'} \phi(F_\epsilon^ef)) \notin J\). This term can be rewritten as \((\phi' \cdot \phi)(F_{\epsilon'}^{e'+\epsilon}(f)) = \phi'\left(F_{\epsilon'}^e(\phi(F_\epsilon^ef))\right)\), and so \(f \notin C_D(J)\). \(\square\)

Remark 3.17. If the pair \((R, \mathcal{D})\) is \(F\)-finite and \(F\)-pure, then combining Corollary 3.14, Proposition 3.5, and Proposition 3.16, shows that \(C_D\) is a relative interior operation on ideals of \(R\), in the sense of Epstein, R.G., and Vassilev [21] Def. 2.2].

The following result is known when \(\mathcal{D} = C_R\) [3], and for triples \((R, \Delta, \alpha^i)\) [22]. The proof in the Cartier algebra setting proceeds the same as Badilla-Césedes’ proof, with a little care needed for the exponents used.

Proposition 3.18 (Cf. [3] Rmk. 4.14], [22] Cor. 3.3]). If the pair \((R, \mathcal{D})\) is \(F\)-finite and \(F\)-pure, then for any ideal \(J\), the Cartier core \(C_D(J)\) is radical.
Proof. Suppose $r \in \sqrt{C_D(J)}$. Then there exists some $n$ so that $r^{q^n} \in C_D(J)$. Since the pair is $F$-pure, there also exists some $\psi \in D_{d}$ so that $\psi(F^{q^n}_1) = 1$. Take $e = nd$, so that there is $\phi \in D_{e}$ with $\phi(F^{q^n}_e 1) = 1$, and so that Proposition 3.16 gives $r^{q^n} \in C_D(J) = C_D(C_D(J))$. Then

$$\phi(F^{q^n}_*(r^{q^n})) = r\phi(F^{q^n}_*1) = r \in C_D(J).$$

The hypothesis that $(R, D)$ be $F$-pure is necessary. Consider $R = k[x]/(x^2)$ where $k$ is an $F$-finite field, and let $D = C_R$. This ring $R$ is non-reduced, so can't be $F$-pure. For any ideal $J \subset k[x]$, use $J$ to denote the image of $J$ in $R$. Now using the presentation from Theorem 4.6 we compute

$$A_e((x^2)) = \frac{(x^2)^{[p^e]}}{(x^2)^{[p^e]} : k[x]} \left(\frac{(x^2)^{[p^e]} : k[x]}{(x^2)}\right) = \frac{(x^{2p^e})}{k[x]} \frac{(x^{2p^e} - 2)}{(x^2)}. $$

Intersecting over all $e$, we see that $C_R((x^2)) = (x^2)$, a non-radical ideal.

Theorem 3.19 (Cf. [3] Prop. 4.9, Thm. 4.10]). If the pair $(R, D)$ is $F$-finite and $F$-pure, then the set of Cartier cores with respect to $D$, i.e., the set \{C_D(J) \mid J an ideal of R\}, is precisely the set of $D$-compatible ideals.

Proof. An ideal $J$ is $D$-compatible precisely if $\phi(F^e_*(J)) \subseteq J$ for all $e$ and for all $\phi \in D_{e}$, and thus by construction $J$ is $D$-compatible if and only if $J \subseteq C_D(J)$. By Corollary 3.14, if the pair $(R, D)$ is $F$-pure then this is equivalent to having $J = C_D(J)$. This shows that every $D$-compatible ideal is a Cartier core.

Conversely, the Cartier core $C_D(J)$ is $D$-compatible since by Proposition 3.16 we have $C_D(J) = C_{D}(C_D(J))$. \qed

Corollary 3.20 (Cf. [3] Prop. 4.11]). If the pair $(R, D)$ is $F$-finite and $F$-pure and $J$ is an ideal of $R$, then $C_D(J)$ is the largest $D$-compatible ideal contained in $J$.

Proof. $C_D(J)$ is $D$-compatible by the previous result. If another $D$-compatible ideal $J'$ has $C_D(J) \subseteq J' \subseteq J$, then by Proposition 3.5 we have $C_D(C_D(J)) \subseteq C_D(J') \subseteq C_D(J)$, and by Proposition 3.16 we in fact have $C_D(J') = J' = C_D(J)$. \qed

The following result, originally due to Schwede [20] Cor. 5.10] and to Kumar and Mehta [16 Thm. 1.1], captures another nice property of the Cartier core map. Recent work of Datta and Tucker [6] Prop. 3.4.1] provides an alternate proof that uses similar language to the rest of this paper.

Proposition 3.21 ([6] Prop. 3.4.1]). If $(R, D)$ is an $F$-finite, $F$-pure pair, then there are only finitely many Cartier cores with respect to $D$, i.e., there are only finitely many $D$-compatible ideals.

Remark 3.22. If additionally $R$ is local, one can in fact get concrete bounds on the number of $D$-compatible ideals. Using Theorem 4.2 of [24] or the argument from Remark 3.4 of [15], the number of prime Cartier cores with respect to $D$ of coheight $d$ is bounded above by $\binom{n}{d}$, where $n$ is the embedding dimension of $R$. 


Theorem 3.23. Let \((R, \mathcal{D})\) be an \(F\)-finite pair, and let \(\mathcal{U}_\mathcal{D}\) denote the \(F\)-pure locus of \((R, \mathcal{D})\). Then the map \(C_\mathcal{D}: \mathcal{U}_\mathcal{D} \to \mathcal{U}_\mathcal{D}\) is continuous under the Zariski topology.

Proof. We will show that the inverse image of the closed set \(V = \mathbb{V}(J) \cap \mathcal{U}_\mathcal{D}\) is also closed, where \(J\) is an ideal of \(R\). Let \(K\) be the intersection of all Cartier cores containing \(J\) which come from primes, so that

\[
K = \bigcap_{P \in \mathcal{U}_\mathcal{D}, C_\mathcal{D}(P) \in \mathbb{V}(J)} C_\mathcal{D}(P).
\]

Since the set of Cartier cores with respect to \(\mathcal{D}\) is closed under infinite intersection by Proposition 3.6, \(K = C_\mathcal{D}(K)\) is also a Cartier core. We claim that \(C_\mathcal{D}^{-1}(V) = \mathbb{V}(K) \cap \mathcal{U}_\mathcal{D}\).

Suppose \(P \in C_\mathcal{D}^{-1}(V)\). Then since \(P \in \mathcal{U}_\mathcal{D}\), we have \(C_\mathcal{D}(P) \subseteq P\) by Proposition 3.11. Since \(C_\mathcal{D}(P) \in \mathbb{V}(J)\), we have \(K \subseteq C_\mathcal{D}(P)\) by construction. Thus \(K \subseteq P\) and so \(C_\mathcal{D}^{-1}(V) \subseteq \mathbb{V}(K) \cap \mathcal{U}_\mathcal{D}\).

Conversely, if \(P \in \mathbb{V}(K) \cap \mathcal{U}_\mathcal{D}\), then \(K \subseteq P\) and by Proposition 3.5

\[
J \subseteq K = C_\mathcal{D}(K) \subseteq C_\mathcal{D}(P).
\]

Thus \(\mathbb{V}(K) \cap \mathcal{U}_\mathcal{D} \subseteq C_\mathcal{D}^{-1}(V)\). □

4. Quotients of Regular Rings

Now that we have seen some abstract properties of the Cartier core map, \(C_\mathcal{D}\), we shift our focus to actually computing it. In this section we give a concrete description of the Cartier core in the case when \(R\) is presented as a quotient of a regular ring and \(\mathcal{D}\) is the full Cartier algebra \(\mathcal{C}_R\). We will then use this concrete description to show that the Cartier core commutes with adjoining a variable and with homogenization (in the case that our regular ring is a polynomial ring).

One reason to focus on this case is that regularity of \(S\) forces \(F^e_* S\) and \(\text{Hom}_S(F^e_* S, S)\) to be well-behaved, as the following result of Kunz and result of Fedder illustrate.

Theorem 4.1 ([17, Cor. 2.7]). If \(R\) is a Noetherian ring of prime characteristic, then \(R\) is regular if and only if \(F^e_* R\) is a flat \(R\)-module.

Lemma 4.2 ([10, Lemma 1.6]). If \(S\) is an \(F\)-finite regular local ring, then \(\text{Hom}_S(F^e_* S, S)\) is a free rank one \(F^e_* S\) module.

Further, Glassbrenner, building on work of Fedder, gives us the following description of the \(R\)-module structure on maps in the local case.

Lemma 4.3 (Fedder’s Lemma [11, Lemma 2.1]). Let \(S\) be an \(F\)-finite regular local ring and let \(R = S/I\) for some ideal \(I\). Then

\[
\text{Hom}_R(F^e_* R, R) \cong F^e_* \left( \frac{I^{[p^e]}}{I^{[p^e]}} : I \right)
\]

as \(R\)-modules.
This description of \( \text{Hom}_R(F^e_s R, R) \) is the core of Fedder’s criterion and of Glassbrenner’s criterion.

**Proposition 4.4** ([10, Prop. 1.7]). Let \((S, \mathfrak{m})\) be an \(F\)-finite regular local ring of prime characteristic \(p\), and let \(I\) be an ideal of \(S\). Then \(R\) is \(F\)-pure if and only if \((I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}\).

**Lemma 4.5** ([11 Lemma 2.2]). Let \((S, \mathfrak{m})\) be an \(F\)-finite regular local ring of prime characteristic \(p\). Let \(I\) be an ideal of \(S\). Then the map \(S/I \to F^e_s(S/I)\), where \(1 \mapsto F^e_s c\), splits as an \((S/I)\)-module map exactly when \(c \not\in \mathfrak{m}^{[p]} : (I^{[p]} : I)\).

Since the Cartier core of \(J\) is composed precisely of the elements which cannot be split in this manner, the technique of this lemma naturally leads to the following result. For the following, we use \(\overline{J}\) to denote the image of an ideal \(J\) in a quotient ring, and similarly \(\mathfrak{c}\) to denote the image of an element \(c\).

**Theorem 4.6.** Let \(S\) be a regular \(F\)-finite ring, let \(I \subseteq J\) be ideals of \(S\), and let \(R = S/I\). Fix \(e \geq 1\), \(c \in S\). Then there exists some \(\phi \in \text{Hom}_R(F^e_s R, R)\) with \(\phi(\mathfrak{c}) \notin \overline{J}\) if and only if \(c \not\in J^{[p]} : (I^{[p]} : I)\). In particular,

\[
A_{e; R}(J) = J^{[p]} :_S (I^{[p]} :_S I) \quad \text{and} \quad C_{R}(J) = \bigcap_e J^{[p]} :_S (I^{[p]} :_S I).
\]

**Proof.** The representations of \(A_{e}\) and \(C_{R}\) follow directly from the first statement, so it suffices to prove that \(c \not\in J^{[p]} : (I^{[p]} : I)\) if and only if there is some \(\phi \in \text{Hom}_R(F^e_s R, R)\) with \(\phi(\mathfrak{c}) \notin \overline{J}\).

For our fixed \(e\), let \(E : \text{Hom}_R(F^e_s R, R) \to R\) be the “evaluation at \(c\)” map, so that \(E(\phi) = \phi(F^e_s c)\). Our goal is to show \(\text{im}(E) \subseteq \overline{J}\) if and only if \(c \in J^{[p]} : (I^{[p]} : I)\). By the discussion in Section 2, we can view the localization of \(E\) as a map \(\text{Hom}_{R_p}(F^e_s(R_p), R_p) \to R_p\) so that \((\text{im} E)_p \cong \text{im}(E_p)\). Since localization also commutes with Frobenius and with ideal colon, we can without loss of generality assume that \((S, \mathfrak{m})\) is local.

Let \(\Psi\) be a generator of \(\text{Hom}_S(F^e_s S, S)\) as an \(F^e_s S\) module. By **Lemma 4.3**, the maps \(\phi \in \text{Hom}_R(F^e_s R, R)\) are exactly those maps induced by something of the form \(\Psi \circ F^e_s(s)\) where \(s \in I^{[p]} : I\). Thus

\[
\phi(F^e_s(\mathfrak{c})) = (\Psi \circ F^e_s)(F^e_s c) = \Psi(F^e_s(sc))
\]

and so there exists \(\phi\) with \(\phi(F^e_s(\mathfrak{c})) \notin \overline{J}\) if and only if there exists \(s \in I^{[p]} : I\) with \(\Psi(F^e_s(sc)) \notin J\), i.e., if and only if

\[
\Psi \left( F^e_s(c(I^{[p]} : I)) \right) = (F^e_s c \cdot \Psi)(I^{[p]} : I) \not\subseteq J.
\]

Using [10 Lemma 1.6], this occurs if and only if

\[
F^e_s(c) \notin (J F^e_s) : (F^e_s(I^{[p]} : I)) = F^e_s(J^{[p]}) : F^e_s(I^{[p]} : I).
\]

Since \(S\) is regular, the flat Frobenius commutes with colon and is injective, thus this is equivalent to

\[
c \not\in J^{[p]} : (I^{[p]} : I).
\]

\(\square\)
We will frequently move between considering $C_R(J)$ in $R$ and its lift $\bigcap_{e \geq 0} J^{[p^e]} : (I^{[p^e]} : I)$ in $S$, which we will denote as either $\tilde{C}_R(J)$ or $\tilde{C}_R(J)$. Similarly, we will denote the lift of $A_{e;R}(J)$ as $\tilde{A}_{e;R}(J)$ or $\tilde{A}_{e;R}(J)$.

We now prove results which let us connect Cartier cores of related ideals computed in different, related rings.

**Lemma 4.7.** Let $S_1 \to S_2$ be a flat map of regular $F$-finite rings. Consider ideals $I \subseteq J_1$ in $S_1$, and ideal $J_2$ in $S_2$ contracting to $J_1$. Let $R_1 = S_1/I$ and $R_2 = S_2/IS_2$. Then

$$C_{R_1}(J_1)R_2 \subseteq C_{R_2}(J_2).$$

**Proof.** Finite intersections always commute with flat base change. Thus for any sequence of ideals $\{K_{e}\}_{e \in \mathbb{N}}$ and for any $n$,

$$\left(\bigcap_{e=1}^{\infty} K_{e}\right) S_2 \subseteq \left(\bigcap_{e=1}^{n} K_{e}\right) S_2 = \bigcap_{e=1}^{n} (K_eS_2)$$

and in particular we must have $(\bigcap_{e=1}^{\infty} K_{e}) S_2 \subseteq \bigcap_{e=1}^{\infty} (K_eS_2)$. Colon commutes with flat base change when the ideals are finitely generated [18, Thm. 7.4]. Thus

$$\left(\bigcap_{e \geq 1} J_1^{[p^e]} : (I^{[p^e]} : I)\right) S_2 \subseteq \bigcap_{e \geq 1} ((J_1S_2)^{[p^e]} : ((IS_2)^{[p^e]} : IS_2)) \subseteq \bigcap_{e \geq 1} (J_2^{[p^e]} : ((IS_2)^{[p^e]} : IS_2)),$$

which by using Theorem 4.6 to pass to the quotient gives

$$C_{R_1}(J_1)R_2 \subseteq C_{R_2}(J_2).$$

In the case of a general flat map, even a general faithfully flat map, containment is the best we can do. For example, consider $S_1 = k[x^p]$ and $S_2 = k[x]$ where $k$ is a perfect field. The inclusion of $S_1$ into $S_2$ is faithfully flat since it corresponds to the Frobenius on the regular ring $k[x]$. Now consider $I = J_1 = \langle x^p \rangle \subset S_1$ and $J_2 = \langle x \rangle \subset S_2$. Then $R_1 = S_1/I \cong K$ which is Frobenius split, so $C_{R_1}(J_1) = \tilde{J}_1$. But $R_2 = S_2/IS_2 = k[x]/\langle x^p \rangle$ is not reduced, thus cannot be Frobenius split. Since $\tilde{J}_2$ is a prime ideal, this means $C_{R_2}(\tilde{J}_2) = R_2$.

However, it turns out that in the case of adjoining a variable, we can get a stronger result.

**Proposition 4.8.** Let $R$ be a quotient of a regular $F$-finite ring, let $J$ be an ideal of $R$, and let $J'$ be an ideal of $R[x]$ such that $JR[x] \subseteq J' \subseteq JR[x] + \langle x \rangle$. Then

$$C_R(J)R[x] = C_{R[x]}(J') \quad \text{and} \quad C_{R[x]}(J') \cap R = C_R(J).$$

**Proof.** By Proposition 3.5

$$C_{R[x]}(JR[x]) \subseteq C_{R[x]}(J') \subseteq C_{R[x]}(JR[x] + \langle x \rangle).$$

Our first step will be to show

$$C_{R[x]}(JR[x]) \supseteq C_{R[x]}(JR[x] + \langle x \rangle),$$

which will then give us $C_{R[x]}(JR[x]) = C_{R[x]}(J') = C_{R[x]}(JR[x] + \langle x \rangle)$.

To do so, note that by assumption we can write $R = S/I$ where $S$ is a regular $F$-finite ring, and so we can also write $R[x] = S[x]/IS[x]$. We use $\sim$ to denote lifting an ideal from $R$ or
denote the minimal homogenization of $f$, as appropriate. Consider $S[x]$ to be $\mathbb{N}$-graded by $x$. Since $JR[x] + \langle x \rangle$, the lift of $JR[x] + \langle x \rangle$ to $S[x]$, is homogeneous, as is $IS[x]$, our lift of the Cartier core

$$\tilde{C}_{R[x]}(J[x] + \langle x \rangle) = \bigcap_{e>0} JR[x] + \langle x \rangle : (IS[x]^q : IS[x])$$

is also homogeneous. Consider some homogeneous $g$ in this lift of the Cartier core. Ideal colon commutes with flat maps, and $S \rightarrow S[x]$ and the Frobenius are both flat. Thus for every $q = p^e$ we have

$$IS[x]^q : IS[x] = (I^q : I)S[x].$$

Since $g \in \tilde{A}_e(J)$, we must have $g(I^q : I) \subseteq (JR[x] + \langle x \rangle)^q$. However, any element of $(JR[x] + \langle x \rangle)^q$ of degree less than $q$ must be expressible in terms of elements of $JR[x]^q$. In particular, if $q > \deg g$ then $g(I^q : I) \subseteq JR[x]^q$. Thus for $e \gg 0$, we have

$$g \in JR[x]^q : (IS[x]^q : IS[x]) = \tilde{A}_{e,R[x]}(JR[x]).$$

By [3] Prop. 4.15, since $C_{R[x]}(JR[x]) = \bigcap_{e>0} A_{e,R[x]}(JR[x])$, this tells us that

$$C_{R[x]}(JR[x] + \langle x \rangle) \subseteq C_{R[x]}(JR[x])$$

as desired.

Now we have shown $C_{R[x]}(JR[x]) = C_{R[x]}(J')$, and it suffices to show $C_R(J)R[x] = C_{R[x]}(J[x])$. To do so, we will show that adjoining a variable commutes with infinite intersection. Consider an arbitrary ideal $K = \bigcap_{\alpha} K_\alpha$ in $S$. As a set, each $K_\alpha S[x]$ is polynomials with coefficients in $K_\alpha$, and so the polynomials in $\bigcap_{\alpha} K_\alpha S[x]$ are those with coefficients in $K_\alpha$ for every $\alpha$, which is precisely $KS[x]$, as desired.

This lets us repeat the argument in Lemma 4.7 but with equalities, and thus

$$C_R(J)R[x] = C_{R[x]}(J[x]) = C_{R[x]}(J')$$

as desired. The contraction result then follows directly from the fact that adjoining a variable is faithfully flat, so that

$$C_R(J) = C_R(J)R[x] \cap R = C_{R[x]}(J') \cap R.$$  

If $R$ is a quotient of a polynomial ring by a homogeneous ideal, we can also look at how the Cartier core behaves under homogenization. More concretely, take $R = S/I$ for $S = k[x_1, \ldots, x_d]$ and $I$ a homogeneous ideal of $S$, so that $R$ is $\mathbb{N}$-graded. If $f \in R$, we let $f^h$ denote the minimal homogenization of $f$ in $R[t]$, so that

$$f^h = t^{\deg f} f \left( \frac{x_1}{t}, \ldots, \frac{x_d}{t} \right).$$

If $J$ is an ideal of $R$, we define its homogenization in $R[t]$ to be $J^h = \langle f^h | f \in J \rangle$.

For any degree-preserving lift of $f$ to $S$, there is a corresponding lift of $f^h$ to $S[t]$ so that the lift of the homogenization is the homogenization of the lift. This means we can freely consider a given homogenization to live either in $R[t]$ or in $S[t]$. Further, the ideals $(\tilde{J}^h)$ and $(\tilde{J})^h$ are the same: $(\tilde{J}^h)$ is generated by the lifts of the homogenizations of elements of $J$, and $(\tilde{J})^h$ is generated by homogenizations of lifts of elements of $J$. 


There is also a corresponding dehomogenization map $\delta : R[t] \to R$ defined by $\delta(t) = 1$, which ensures that $\delta(f^h) = f$.

We recall the following straightforward facts about homogenization.

**Lemma 4.9.** Let $R$ be a quotient of a polynomial ring by a homogeneous ideal. Let $I, J$ be ideals of $R$, and $\{ I_\alpha \}$ a family of ideals. Let $f$ be an element of $R$. Then the following statements all hold.

- $f \in I$ if and only if $f^h \in I^h$.
- $(I : J)^h = I^h : J^h$ and $(\bigcap I_\alpha)^h = \bigcap (I_\alpha^h)$.
- $(I^h)^{[p^e]} = (I^{[p^e]})^h$.

**Proof.** For the first two bullets, see Problems 3.15 and 3.17 in [7]. For the third bullet, use Proposition 3.15 of [7] and Theorem 6.2 of [13].

Using these facts, we will prove the following lemma.

**Lemma 4.10.** Let $R = S/I$ where $S$ is a polynomial ring over an $F$-finite field and $I$ is a homogeneous ideal. Let $J$ be an ideal of $R$. Then

$$(C_R(J))^h = C_{R[t]}(J^h) \quad \text{and} \quad C_R(J) = \delta(C_{R[t]}(J^h))$$

**Proof.** If we lift to $S[t]$ using Theorem 4.6 and the above discussion on lifting and homogenization, then

$$(\widetilde{C_R(J)})^h = \left( \widetilde{C_R(J)} \right)^h$$

$$= \left( \bigcap_{e > 0} (\widetilde{J}^{[q]} : (I^{[q]} : I)^h) \right)^h$$

$$= \bigcap_{e > 0} \left( (\widetilde{J}^h)^{[q]} : ((I^h)^{[q]} : I^h) \right)$$

$$= \bigcap_{e > 0} \left( (\widetilde{J}^h)^{[q]} : (I^{[q]} : I) \right)$$

$$= \widetilde{C}_{R[t]}(J^h)$$

and so contracting back to $R[t]$ via Theorem 4.6

$$(C_R(J))^h = C_{R[t]}(J^h).$$

The last statement follows directly from dehomogenizing each side of the equation. □
5. **Stanley-Reisner Rings**

A ring $R$ is a *Stanley-Reisner ring* if it can be written as $R = S/I$, where $S$ is a polynomial ring and $I$ is a square-free monomial ideal. The following theorem gives a complete description of the Cartier core map for Spec $R$ where $R$ is a Stanley-Reisner ring.

**Theorem 5.1.** Let $R$ be a Stanley-Reisner ring over a field that has prime characteristic and is $F$-finite. Let $Q$ be any prime ideal. Then

$$C_R(Q) = \sum_{P \in \text{Min}(R) \atop P \subseteq Q} P.$$ 

In particular, the set of prime Cartier cores of $R$, i.e., the set of generic points of $F$-pure centers of $R$, is the set of sums of minimal primes.

This theorem extends some earlier results. Aberbach and Enescu showed that the splitting prime of a Stanley-Reisner ring, which is its largest proper uniformly $F$-compatible ideal, is the sum of the minimal primes [1, Prop 4.10]. For the reader’s convenience, we will reprove this in our proof of Theorem 5.1. At the other extreme, Vassilev showed that the test ideal of a Stanley-Reisner ring, which is its smallest non-zero uniformly $F$-compatible ideal, is $\bigcap_{e=1}^{t} \cap_{j \neq i} P_j$ where $P_1, \ldots, P_t$ are the minimal primes of $R$ [20, Thm. 3.7]. In a related but different direction, for a specific choice of $\phi : F^e_* R \to R$, Enescu and Ilioaea showed that the $\phi$-compatible primes of $R$ are precisely the prime monomial ideals which contain a minimal prime of $R$. They used this to give a combinatorial description the test ideal of the pair $(R, \phi)$ [8, Prop. 3.9, Prop. 3.10].

**Proof of Theorem 5.1.** Our proof will proceed as follows: First we will reduce to the case where every minimal prime is contained in $Q$. Then we will homogenize and trap $Q^h$ between a sum of minimal primes and the homogeneous maximal ideal, and use Proposition 3.5 and the convenient form of monomial primes to get our desired equality.

Let $\bar{Q}$ denote the lift of any ideal to $S$, let $I' = \bigcap_{P \in \text{Min}(R), \ P \subseteq Q} \bar{P}$ be the intersection of the minimal primes contained in $Q$, and let $R' = S/I'$. Then

$$R_Q \cong S_{\bar{Q}}/I_{\bar{Q}} = S_{I'/Q} \cong R_{Q'},$$

and so by Lemma 3.7,

$$C_R(Q)R_Q = C_{R_Q}(Q) = C_{R_Q}(Q) = C_{R'}(Q)R_{Q'}. $$

Stanley-Reisner rings are $F$-pure [14, Prop. 5.8], and so $C_R(Q) \subseteq Q$ by Corollary 3.14, and thus when we lift back to $S$ using Theorem 4.6, we see

$$\bigcap_{e>0} \bar{Q}^{[ep]} : (I^{[ep]} : I') = \bigcap_{e>0} \bar{Q}^{[ep]} : (I'^{[ep]} : I').$$
Thus we can use $I'$ as our new $I$, and so we can assume $P \subseteq Q$ for all minimal primes $P$.

Relabel the variables so that $\sum_{P \in \text{Min}(R)} P = \langle x_1, \ldots, x_c \rangle$ and define $A = k[x_1, \ldots, x_c]/I$, so that $R = A[x_{c+1}, \ldots, x_d]$. Now we homogenize, so $Q^h \subseteq m$ where $m$ is the homogeneous maximal ideal in $S[t]$. Then Proposition 3.5 tells us

$$C_{R[t]} \left( \sum_{P \in \text{Min}(R)} P^h \right) \subseteq C_{R[t]}(Q^h) \subseteq C_{R[t]}(m).$$

Each minimal prime $P$ of $R$ remains a minimal prime of $R[t]$ after homogenizing, so Corollary 3.13 says $C_{R[t]}(P^h) = P^h$, and Proposition 3.15 then says that their sum is also preserved by the Cartier core map. Applying Proposition 4.8 to $m$, we get

$$\langle x_1, \ldots, x_c \rangle R[t] = C_A(P_1 + \cdots + P_t)A[x_{c+1}, \ldots, x_d, t] = C_{R[t]}(m).$$

Thus

$$\langle x_1, \ldots, x_c \rangle = C_{R[t]} \left( \sum_{P \in \text{Min}(R)} P^h \right) \subseteq C_{R[t]}(Q^h) \subseteq C_{R[t]}(m) = \langle x_1, \ldots, x_c \rangle$$

and by Lemma 4.10,

$$(C_{R}(Q))^h = C_{R[t]}(Q^h) = \langle x_1, \ldots, x_c \rangle.$$

Dehomogenizing the homogenization always gives back the original ideal, and so

$$C_{R}(Q) = \langle x_1, \ldots, x_c \rangle = \sum_{P \in \text{Min}(R)} P.$$

For the last statement of the theorem, note that since each minimal prime of $R$ corresponds to an ideal of $S$ which is generated by variables, any sum of minimal primes is also prime, and thus is fixed by the Cartier core map. \hfill \Box

Since taking the Cartier core commutes with intersection, Theorem 5.1 immediately gives a formula for the Cartier core of any radical ideal in terms of the Cartier cores of its minimal primes. The following corollary instead gives a formula for the Cartier core of an arbitrary ideal which is more analogous to the previous one.

**Corollary 5.2.** Let $R$ be a Stanley-Reisner ring over a field that has prime characteristic and is $F$-finite. Let $J$ be any ideal. Then

$$C_{R}(J) = \sum_{Q \subseteq \text{Min}(R)} \left( \bigcap_{P \in Q} P \right).$$

**Proof.** First, we will show our desired formula gives an ideal contained in $C_{R}(J)$. The restriction on the $P$’s appearing ensures that the resulting ideal is contained in $J$. Further, each $P$ appearing is minimal, so $P = C_{R}(P)$ by Corollary 3.13. Using Propositions 3.6...
and \(3.15\) (our intersection and sum results) to apply \(C_R\) to the formula and then using Proposition \(3.5\) to preserve the containment gives

\[
\sum_{Q \subset \text{Min}(R)} \left( \bigcap_{P \in Q} P \right) = C_R \left( \sum_{Q \subset \text{Min}(R)} \left( \bigcap_{P \in Q} P \right) \right) \subseteq C_R(J).
\]

To show equality, we will show that summing over a specific smaller subset in fact already yields \(C_R(J)\), and so the larger sum above must yield \(C_R(J)\) as well. Since \(R\) is Frobenius split, \(C_R(J)\) is radical by Proposition \(3.18\) so we can write \(C_R(J) = \bigcap_{i=1}^n Q_i\) as the intersection of its minimal primes. By the same argument as in the proof of Theorem \(3.23\) applying Proposition \(3.10\), Corollary \(3.14\) and Proposition \(3.16\) shows that \(C_R(Q_i) = Q_i\) for each \(i\).

Now since the Cartier core commutes with intersection, we use Theorem \(5.1\) to see

\[
C_R(J) = \bigcap_i C_R(Q_i) = \bigcap_i \left( \sum_{P \in \text{Min}(R)} P \right).
\]

Writing \(R = S/I\) as a quotient of a polynomial ring and lifting back up to \(S\), this says that the lift of \(C_R(J)\) is an intersection of sums of monomial ideals. Sum and intersection of monomial ideals commute \([7\) Problem 1.17], and so passing back to the quotient gives

\[
C_R(J) = \sum_{P_1, \ldots, P_n \in \text{Min}(R) \atop P_i \subset Q_i} \left( \bigcap_{i=1}^n P_i \right).
\]

For each possibility for \(P_1, \ldots, P_n\) in the above sum, we have \(\bigcap_{i=1}^n P_i \subset C_R(J) \subset J\), and so this sum is a subset of our desired formula, which thus must be equal to \(C_R(J)\) as well. \(\square\)

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