Time Asymmetric Quantum Theory —
I Modifying an Axiom of Quantum Physics

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Abstract

A slight modification of one axiom of quantum theory changes a reversible theory into a time asymmetric theory. Whereas the standard Hilbert space axiom does not distinguish mathematically between the space of states (in-states of scattering theory) and the space of observables (out—“states” of scattering theory) the new axiom associates states and observables to two different Hardy subspaces which are dense in the same Hilbert space and analytic in the lower and upper complex energy plane, respectively. As a consequence of this new axiom the dynamical equations (Schrödinger or Heisenberg) integrate to a semi-group evolution. Extending this new Hardy space axiom to a relativistic theory provides a relativistic theory of resonance scattering and decay with Born probabilities that fulfill Einstein causality and the exponential decay law.

1 Introduction—Time Asymmetry

Time asymmetry, irreversibility, time reversal non-invariance are different concepts and they are (probably) not (all) related to each other [1], [2], [3]. These concepts are usually called arrows-of-time. Time asymmetry comes from time asymmetric boundary conditions of time symmetric equations; its most prominent consequence is causality. The radiation arrow of time is its example from classical physics. Irreversibility is usually associated with probability or entropy increase and called the thermodynamic arrow of time in classical physics. In quantum mechanics or quantum statistical mechanics entropy increase is associated [4] with the effect of the environment or of the measurement apparatus upon the physical system.

The possibility of connecting time asymmetry for the solution of the Maxwell equations to probability was discussed in the Einstein-Ritz arguments [5], where Einstein maintains that time asymmetric boundary conditions for the Maxwell equations are not needed and the radiation arrow of time is a consequence of probability, whereas Ritz insists that the initial-boundary conditions are the basis of irreversibility. Peierls [6] argued that the implied boundary conditions of Boltzmann’s Stosszahl Ansatz are the origin of irreversibility.

Time-reversal non-invariance is a non-invariance of the dynamical equations and of the Hamiltonian with respect to the anti-unitary time-reversal operator [7], and is thus an entirely different concept and can

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(probably) not be related to the above arrows-of-time.

In this article we do not want to discuss possible connections between the arrows of time but are only concerned with time asymmetry, i.e. the arrow of time due to boundary conditions.

In classical theories one can have time symmetric dynamical equations with time asymmetric boundary conditions.

These time asymmetric boundary conditions come in pairs: given one time asymmetric boundary condition, its time reversed boundary condition can also be formulated mathematically. An example is the retarded and advanced solutions in classical electrodynamics (Maxwell equations). Nature chooses the retarded solutions; this is the radiation arrow of time: Radiation must be emitted first by the source, before it can be detected by the receiver. Another example is the big bang and big crunch solution of general relativity (Einstein equations). The universe expands and the big bang gives us a means of defining the cosmic time and its origin $t = t_0 = 0$. This is called the cosmological arrow of time.

Quantum physics also has arrows of time. In terms of experimental arrangements one can formulate it in close analogy to the radiation arrow of time as the preparation-registration arrow of time [8]: A state must be prepared first by a preparation apparatus (e.g., accelerator) before an observable can be detected in it by a registration apparatus (detector). The radiative decay of an excited state of an atom or a nucleus or a relativistic particle is the quantum analogue of the classical radiation arrow of time.

Though the correspondence of the arrows of time for classical electromagnetic radiation and quantum radiative decays is obvious, there does not exist such a correspondence between the respective theories, since standard quantum theory in Hilbert space (von Neumann(1931)) [9] does not allow time-asymmetric boundary conditions for the Schrödinger equation or the Heisenberg equation. The Hilbert space axiom, inevitably (by the Stone-von Neumann theorem, [9], [10]) leads to unitary and therewith reversible time evolution. In the heuristic scattering theory one circumvents this problem by using retarded (advanced) Green’s functions [11] or purely outgoing boundary conditions [12], which — very much like the retarded (advanced) potentials of classical electrodynamics — contain time asymmetry. With the Green’s functions one has admitted distributions into the theory and is outside the Hilbert space. It is the conflict between the Hilbert space mathematics and the time asymmetry of (resonance) scattering described by these heuristic methods, which lead to such puzzles like violation of causality or the impossibility of an exponential decay law.

The axioms of quantum theory are not to be understood as mathematical axioms from which everything can be derived without using further judgment or creativity. An approach of this kind does not appear possible in physics. The axioms, or basic assumptions of physics are to be considered as a concise way of formulating the quintessence of many experimental facts. As such one could modify them. One could also leave the old axioms intact and allow their consequences to partially disagree with reality; then the axioms
provide an approximate (fuzzy) theory, and since every theoretical description can only be an approximate
description, these approximate methods could be adequate in many respects. But if the discrepancy between
the consequences of the set of axioms and the observations becomes too pronounced, one would do well to
make minor changes in the axioms of a theory and devise a modified theory with wider applicability and
greater accuracy.

We want to change just one of the axioms: The Hilbert space axiom of traditional quantum mechanics
will be modified (infinitesimally) into an axiom using two (dense) Hardy subspaces of the same Hilbert Space.
The motivation that led to this modification was to obtain a consistent and exact mathematical theory of
resonance scattering and exponential decay.

2 Axioms of Quantum Theory

In quantum mechanics one speaks of states and of observables.

(AI) States are mathematically described by state operators (denoted by \( \rho, W \)) or, if they are pure states,
by vectors \( \phi \). Observables are also described by operators \( A = (A^\dagger) \), \( \Lambda, P = (P^2) \); but if they are
projections \( P = |\psi\rangle\langle\psi| \) they can also be described by vectors \( \psi \) (up to a phase factor). The vectors
\( \psi, \phi \in \Phi \), are elements of a linear space (pre-Hilbert space) with scalar product \( \langle \psi | \phi \rangle = (\psi, \phi) \).

The operators \( A, \Lambda \in A \), are often assumed to be elements of an algebra \( A \) of linear operators in \( \Phi \).

In the usual practice of quantum mechanics, the space \( \Phi \), though often called a Hilbert space, is treated
as a space in which the convergence of infinite sequences is not a problem, i.e. as a pre-Hilbert space. One
“kind” of quantum physical systems is associated to a space \( \Phi \). Any vector of \( \Phi \) can represent a state or an
observable. The state operator is usually normalized, \( \text{Tr} W = 1 \).

Though mathematically one does not distinguish between vectors that describe states and vectors that
describe observables, in experiments, states and observables are defined by different devices:

(Al) A State \( W, \phi \) is prepared by a preparation apparatus (e.g., accelerator) and an Observable \( \Lambda, \psi \) is
registered (or detected) by a registration apparatus (detector).

Most treatments of the foundations of quantum mechanics agree on ascribing a separate fundamental importance
to states and to observables. The observed quantities, i.e. the experimental numbers, are interpreted
as probabilities to measure an observable \( \Lambda \) in a state \( W \) at a time \( t \).

(AIII) The probabilities \( P_W(\Lambda(t)) \) are calculated in the theory as Born probabilities:

\[
P_W(t,\Lambda_0) = \text{Tr}(\Lambda_0 W(t)) \quad (\text{Schrödinger picture}) \tag{2.1S}
\]

\[
P_{W_0}(\Lambda(t)) = \text{Tr}(\Lambda(t)W_0) \quad (\text{Heisenberg picture}). \tag{2.1H}
\]
For the special case that $\Lambda = |\psi^\pm\rangle\langle\psi^\pm|$ and $W = |\phi^+\rangle\langle\phi^+|$ this is:

$$\mathcal{P}_{\phi^+}(|\psi^-(t)\rangle) = |\langle\psi^-|\phi^+(t)\rangle|^2 \quad \text{(Schrödinger picture)}$$

$$= |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{(Heisenberg picture)}.$$  \hspace{1cm} (2.1'S)

The probabilities $\mathcal{P}_W(\Lambda)$ are measured as ratios of large numbers of detector counts

$$\mathcal{P}_W(\Lambda(t))_{\text{exp}} \approx N_{\Lambda(t)}/N.$$  \hspace{1cm} (A_{IV})

The time evolution in quantum mechanics is given by the Hamiltonian operator $H$ and described by the following dynamical equations.

In the Schrödinger picture the observables are kept time independent, and for the state operator $W(t)$, or in the special case $W = |\phi^+\rangle\langle\phi^+|$, for the state vector $\phi^+(t)$ the dynamical equations are

$$\frac{\partial W(t)}{\partial t} = \frac{i}{\hbar}[W(t), H]$$

or

$$ih\frac{\partial \phi^+(t)}{\partial t} = H\phi^+(t), \quad \phi^+(t = 0) = \phi_0^+.$$  \hspace{1cm} (2.2S)

In the Heisenberg picture the state is kept time independent and for the observable $\Lambda(t)$, or in the special case $\Lambda = |\psi^-\rangle\langle\psi^-|$, for the observable vector $\psi^-(t)$ the dynamical equations are

$$\frac{\partial \Lambda(t)}{\partial t} = \frac{i}{\hbar}[\Lambda(t), H]$$

or

$$ih\frac{\partial \psi^-(t)}{\partial t} = -H\psi^-(t), \quad \psi^-(t = 0) = \psi_0^-.$$  \hspace{1cm} (2.2H)

For the sake of simplicity we shall here mainly treat the special case described by state vectors $\phi^+$ and observable vectors $\psi^-$. And we have used the notation of scattering theory, $\phi^+$ for the interaction incorporating (“exact”) in-states and $\psi^-$ for the interaction incorporating out-observables, since we have the theory of scattering and decay in mind.

Before we turn to the boundary value conditions imposed on the solutions of the dynamical differential equations (2.3S) or (2.3H) we want to discuss the calculational methods used for Born probabilities.

The trace and the scalar product in (2.1) is calculated in the following way:

$$\text{Tr}(AW) = \sum_n^n \langle n | AW | n \rangle$$

or

$$\text{Tr}(AW) = \int dE \langle E | AW | E \rangle.$$  \hspace{1cm} (2.4a)

or

$$\text{Tr}(AW) = \int dE \langle E | AW | E \rangle.$$  \hspace{1cm} (2.4b)
For states \( W = |\phi\rangle\langle\phi| \) and observables \( \Lambda = |\psi\rangle\langle\psi| \) this is given by
\[
|\langle\psi|\phi\rangle|^2 = \left| \sum_{n}^{N} \langle \psi|n\rangle\langle n|\phi\rangle \right|^2
\]
(2.4c)
or
\[
|\langle\psi|\phi\rangle|^2 = \left| \int dE \langle \psi|E\rangle\langle E|\phi\rangle \right|^2 .
\]
(2.4d)

In order to write and calculate these formulas one needs the *basis vector expansions*, i.e. the existence of a complete basis system of eigenvectors \( |n\rangle \) or \( |E\rangle \). The basis vector expansions are generalizations of the expansion of a vector \( \vec{x} \) in the 3-dimensional Euclidean space:
\[
\vec{x} = \sum_{i=1}^{3} \vec{e}_{i}x^{i} .
\]
(2.5a)

The generalization of (2.5a) used in (2.4) and (2.5c) is to an \( N \) dimensional (complex) linear scalar product space. For \( N = \text{finite} \) this eigenvector expansion is well established for all self-adjoint, normal and unitary operators. This means for every \( \phi \) in a finite dimensional space \( \Phi \) one has
\[
\phi = \sum_{n=1}^{N} |n\rangle\langle n|\phi\rangle , \quad N = \text{finite}
\]
(2.5b)

where \( |n\rangle \) are eigenvectors of any self-adjoint (or normal or unitary) operator, usually representing a prominent quantum mechanical observable, (e.g., the Hamiltonian \( H \)):
\[
H|n\rangle = E_{n}|n\rangle .
\]
(2.5c)

In case \( N = \infty \), i.e., for the infinite dimensional (complex) Hilbert space, not all operators important in quantum mechanics have a discrete set of eigenvalues (discrete spectrum), so that (2.5b) with (2.5c) holds only for some observables. But for all (so far) known quantum systems and all observables there are spaces \( \Phi \) for which a second, continuously infinite generalization of (2.5b) holds. This means, for every \( \phi \in \Phi \) Dirac’s continuous eigenvector expansion holds:
\[
\phi = \int_{E_{0}=0}^{\infty} dE |E\rangle\langle E|\phi\rangle .
\]
(2.5d)

Here \( |E\rangle \) is a generalized eigenvector or eigenket
\[
H^{\times}|E\rangle = E|E\rangle , \quad E_{0} \leq E \leq \infty .
\]
(2.5e)

This eigenvalue equation precisely means
\[
\langle \phi|H^{\times}|E\rangle \equiv \langle H\phi|E\rangle = E\langle \phi|E\rangle \quad \text{for all vectors } \phi \in \Phi \subset H ,
\]
(2.5f)

where \( \Phi \) is a dense subspace of \( \mathcal{H} \), but \( |E\rangle \) is not an element of \( \Phi \) or \( \mathcal{H} \), but \( |E\rangle \in \Phi^{\times} \), the space of continuous antilinear functionals on \( \Phi \). Dirac had omitted the \( \times \) in (2.5c). \( H^{\times} \) is defined by the first equality (2.5f) as a unique extension of \( H^\dagger \), the Hilbert space adjoint operator of \( H \), to the space \( \Phi^{\times} \supset \mathcal{H} \).
The meaning of the integral in (2.5d) and of the eigenkets in (2.5e), (2.5f) depends upon the choice of the space \( \Phi \) of vectors \( \{ \phi \} \). Usually one chooses
\[
\phi, \psi \in \Phi , \quad \text{where } \Phi \text{ is the abstract Schwartz space. (2.6a)}
\]
This means that the energy wave function
\[
\phi(E) \equiv \langle E|\phi \rangle = \langle \phi|E \rangle \in \mathcal{S} , \quad (2.6b)
\]
which corresponds by (2.5a) to the vector \( \phi \):
\[
\Phi \ni \phi \leftrightarrow \phi(E) \in \mathcal{S} , \quad (2.6c)
\]
is a Schwartz function (infinitely differentiable, rapidly decreasing function of \( E \)).

The spaces \( \Phi \subset \mathcal{H} \) together with the space of antilinear continuous functionals on \( \Phi \) form a Rigged Hilbert Space (RHS) [13]
\[
\Phi \subset \mathcal{H} \subset \Phi^* \quad (2.7a)
\]
which is realized by the RHS of Schwartz functions
\[
\mathcal{S} \subset L^2(\mathbb{R}_+,dE) \subset \mathcal{S}^* , \quad (2.7b)
\]
where \( \mathcal{S}^* \) is the space of tempered distributions (\( \mathbb{R}_+ \) denotes the positive real semiaxis). This means that the RHS’s (2.7a) and (2.7b) are equivalent. The ordinary Dirac kets are usually defined as \( |E\rangle \in \Phi^* \), i.e., functionals on the Schwartz space \( \Phi \) which fulfill the eigenvalue relation (2.5c) or (2.5d).

The above basic assumptions of quantum mechanics \((A_1) \cdots (A_{IV})\) are part of the standard mathematical theory in Hilbert space [9], including the calculational rules (2.4), except that the kets \(|E\rangle\) cannot be defined with the Hilbert space only. But using for the integrals in (2.4b) and (2.4d) Lebesgue (rather than Riemann) integrals (and admitting for \( \phi(E) = \langle E|\phi \rangle, \psi(E) = \langle \psi|E \rangle \) all the elements of \( L^2(\mathbb{R}_+,dE) \) not just the smooth functions) one can also use (2.4b), (2.4d) in Hilbert space quantum mechanics.

Thus one can add to \((A_1) \cdots (A_{IV})\) the Hilbert space axiom:

\((A_{oldV})\) The set of states \( \{ \phi^+ \} \) is equal to the set of observable \( \{ \psi^- \} \) and both are equal to the whole Hilbert space \( \mathcal{H} \):
\[
\{ \phi^+ \} \equiv \{ \psi^- \} \equiv \mathcal{H} . \quad (2.8)
\]

\((A_1)\) to \((A_{oldV})\) are the basic assumptions of conventional quantum mechanics.

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1. The function \( \phi(E) \) corresponds to the vector \( \phi \) in the same way as the coordinates \( x^i \) corresponds to the vector \( \vec{x} \) in (2.5a). One calls this correspondence (2.6c) of the abstract linear topological space \( \Phi \) by the function space \( \mathcal{S} \) a “realization” of \( \Phi \) by \( \mathcal{S} \).

2. There exist bicontinuous operators from (2.7a) onto (2.7b) and vice versa.
A slightly different version of axiom \((A^\text{old}_V)\) uses the Rigged Hilbert space of \((2.7)\). This version includes the Dirac kets \(|E\rangle\) and the Dirac formalism, i.e., the basis vector expansion \((2.5d)\) and an algebra of observables in \((A_1)\), which is an algebra of continuous (bounded) operators in \(\Phi\) (and therefore defined everywhere in \(\Phi\)). This version of \((A^\text{old}_V)\) states

\[
(A^\text{old}_V)
\]

\[
\{\phi^+\} \equiv \{\psi^-\} \equiv \Phi \subset \mathcal{H} \subset \Phi^* \tag{2.8'}
\]

where \(\Phi\) is the Schwartz space. The Dirac ket \(|E\rangle\) is an element of the space \(\Phi^*\).

Using the axioms \((A_1) \cdots (A_V)\) and \((A^\text{old}_V)\) one avoids infinite energy states and similar pathologies of the Hilbert space and one does not require Lebesgue integrals for \((2.4b)\), \((2.4d)\). But one does not describe different physics than with \((A_1) \cdots (A^\text{old}_V)\).

An immediate consequence of \((A^\text{old}_V)\) is that the solutions of the dynamical equations \((A_IV)\) with the boundary conditions \(\psi^- \in \mathcal{H}, \phi^+ \in \mathcal{H}\) are for the Schrödinger equation \((2.3S)\) given by

\[
\phi^+(t) = U^\dagger(t)\phi^+_0 = e^{-iHt}\phi^+_0 , \quad -\infty < t < \infty \tag{2.9}
\]

(\text{or for the general state } W)

\[
W(t) = e^{-iHt}W_0e^{iHt} , \quad -\infty < t < \infty \tag{2.9b}
\]

and for the Heisenberg equation \((2.3H)\) they are given by

\[
\psi^-(t) = U(t)\psi^-_0 = e^{iHt}\psi^-_0 , \quad -\infty < t < \infty \tag{2.10}
\]

(\text{or for the general observable } \Lambda)

\[
\Lambda(t) = e^{iHt}\Lambda_0e^{-iHt} , \quad -\infty < t < \infty \tag{2.10b}
\]

This is a mathematical consequence of the Hilbert space boundary condition \((A^\text{old}_V)\) for the dynamical equations \((2.2)\) and \((2.3)\) and follows from the Stone-von Neumann theorem \([10]\) for the Hilbert space. These results mean that the time evolution is given by the unitary group \((2.9)\) or \((2.10)\) and is thus time symmetric. The state \(\phi^+\) (in the Schrödinger picture) can evolve forward and backward in time, and the observable \(\psi^-\) (in the Heisenberg picture) can evolve forward and backward and consequently the Born probabilities \((2.1)\) can be predicted for all positive and negative values of \(t\). The same follows from the Schwartz space axiom \((A^\text{old}_V)\) \([14]\).

Quantum theory in Hilbert space is time symmetric. This is not so bad for the description of spectra and structure of quantum physical systems, whose states are (or are considered as) stationary. But this is particularly detrimental for the description of decay processes and resonance scattering, which are intrinsically irreversible processes. There is no consistent theoretical description for decaying states and resonances.
in Hilbert space quantum mechanics. There are only Weisskopf-Wigner methods of which “there does not exist...a rigorous theory to which these various methods can be considered as approximations” \[15\]. Therefore we suggest an alternative hypothesis in place of the Hilbert space axiom $(A_\text{old}^V)$ (and $(A_\text{old}^V')$). This new axiom $(A_V)$ will be stated using $(A_\text{old}^V)$ (the RHS in analogue of $(A_\text{old}^V)$). Therefore it will include the Dirac formalism from the start, i.e., the kets, the basis vector expansion, and an algebra of operators without the need to worry about domain questions for operators. We first shall formulate the axiom $(A_V)$ and then will discuss and motivate it.

$(A_V)$ The set of states defined physically by preparation apparatuses (accelerator) (e.g., the in-states $\phi^+$ of a scattering experiment) are mathematically described by

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times .$$

(2.11)

The set of observables defined by registration apparatuses (detector) (e.g., the out-observables usually called out-states of a scattering experiment) are mathematically described by

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times .$$

(2.12)

The space $\Phi_-$ and $\Phi_+$ are different (dense) Hardy subspaces of the same Hilbert space $\mathcal{H}$.

Though observables and states are defined physically as different entities, the axiom $(A_\text{old}^V)$ identifies them mathematically, i.e., $\{\phi^+\} = \{\psi^-\}$. The same is true for $(A_\text{old}^V')$ (which in scattering theory is called asymptotic completeness). The new hypothesis $(A_V)$ distinguishes also mathematically between states and observables by assigning them to different dense subspaces of the Hilbert space $\mathcal{H}$, the Hardy spaces $\Phi_-$ and $\Phi_+$, respectively. We shall explain the mathematical properties of Hardy Spaces a little better in the following section, and first describe here some consequences.

The solutions of the dynamical equation (2.3S) with the new boundary condition (2.11) are for the states $\phi^+ \in \Phi_-$, given by

$$\phi^+(t) = e^{-iHt}\phi^+ \equiv U_-^\dagger(t)\phi^+ ; \quad 0 \leq t < \infty .$$

(2.13)

The solutions of the dynamical equation (2.3H) with the new boundary condition (2.12) are for the observables $\psi^- \in \Phi_+$, given by

$$\psi^-(t) = e^{iHt}\psi^- \equiv U_+(t)\psi^- ; \quad 0 \leq t < \infty .$$

(2.14)

Thus, in place of the unitary group solution (2.9), (2.10) which one obtains from the dynamical equations (2.2), (2.3) one obtains under the new boundary conditions (2.11), (2.12) the semigroup solution (2.13), (2.14).

3Precisely, the semigroup generator $H = H_+$ in (2.14) is the restriction of the self-adjoint operator $H$ to the (dense in $\mathcal{H}$) subspace $\Phi_+$ and the generator $H = H_+$ in (2.13) is the restriction of $H$ to $\Phi_-$. The same notation is used for the $U(t)$. We often omit the subscripts of the operators which are the same as the subscripts of the spaces.
Thus we see that as a consequence of the change of boundary conditions from \((A_0^{\text{old}})\) to \((A_V)\), keeping all other axioms of quantum mechanics including the dynamical equations the same, we obtain a completely new situation. For \((A_0^{\text{old}})\) (and the same for \((A_V^{\text{old}})\)) we obtain the reversible time evolution given by the unitary group \(U(t) = e^{iHt}\) (or \(U^\dagger(t) = e^{-iHt}\)) with \(-\infty < t < \infty\). For \((A_V)\) we have only a semigroup time evolution \(0 \leq t < \infty\), which cannot be reversed to negative time.

This singles out a particular time \(t_0\), the mathematical semigroup time \(t_0 = 0\). To interpret this time \(t_0\) we calculate the Born probability of the observable \(|\psi^-(t)\rangle\langle\psi^-|\) in the state \(\phi^+\), using the Heisenberg picture. From (2.14) follows:

\[
P(t) = |(\psi^-(t), \phi^+)|^2 = |(e^{iHt}\psi^-, \phi^+)|^2, \quad \text{for } t \geq 0 \text{ only.} \tag{2.15}
\]

The same result one obtains using the Schrödinger picture for the probability of the observable \(|\psi^-\rangle\langle\psi^-|\) in the state \(\phi^+(t)\). The prediction \(t \geq 0 = t_0\) means that the probability for an observable \(\psi^-(t)\) in a state \(\phi^+\) makes sense only for times \(t \geq t_0 = 0\). This is a mathematical consequence of the new hypothesis \((A_V)\).

Whereas a group like (2.9), (2.10) does not have a distinguished time \(t_0\) since \(-\infty < t < \infty\), the semigroups (2.13) and (2.14) introduce a distinguished time \(t_0 = 0\). We interpret this \(t_0\) as the time at which the state has been prepared and at which the registration of an observable in this state can start. The existence of such a time \(t_0\) in quantum mechanics is fairly obvious, because, as stated in Section 1, a state must be prepared first (by \(t_0\)) before an observable can be detected in it. The detailed interpretation of \(t_0\) and its determinations in each particular process can, however, be quite intricate as will be discussed below and in a subsequent paper [16].

The conclusion of this section is that we can have two systems of axioms. They differ from each other in the one axiom that specifies the boundary conditions for the solutions of the dynamical equations. All other axioms agree with each other and are the same as formulated or practiced in the traditional quantum mechanics. Using the set of axioms \((A_I), \ldots, (A_{IV})\) and \((A_V^{\text{old}})\) one has the conventional, time symmetric quantum mechanics with reversible time evolution, using the set of axioms \((A_I), \ldots, (A_{IV})\) and \((A_V)\) one obtains a time asymmetric quantum theory. It is in particular the choice of the Hardy spaces for the axiom \((A_V)\) that leads to time asymmetry.

3 Similarities and Differences with the Traditional Practices

After having decided that the states \(\phi^+\) and the observables \(\psi^-\) have their own spaces, \(\phi^+ \in \Phi_-\) and \(\psi^- \in \Phi_+\) respectively, each of them must have their own linear basis vector expansion. We denote their...
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respective basis kets as $|E, b^+\rangle \in \Phi_+^\times$ and $|E, b^-\rangle \in \Phi_-^\times$, where the $|E, b^\pm\rangle$ are generalized (in the sense of (2.5f)) eigenvectors of the exact Hamiltonian $H = H_0 + V$,

$$H|E, b^\pm\rangle = E|E, b^\pm\rangle, \quad E_0 \leq E < \infty$$  \hspace{1cm} (3.1)

the $b$ in (3.1) are degeneracy labels (e.g., the angular momentum quantum numbers $j, j_3$). The basis vector expansions of the in-states $\phi^+ \in \Phi_-$ and the out-observables $\psi^- \in \Phi_+$ are given by:

$$\Phi_- \ni \psi^- = \sum_b \int_0^\infty dE \langle E, b^-|\langle -E, b^-|\psi^- \rangle, \quad (3.2)$$

$$\Phi_+ \ni \phi^+ = \sum_b \int_0^\infty dE \langle E, b^+|\langle +E, b^+|\phi^+ \rangle. \quad (3.3)$$

These are analogous to the Dirac basis vector expansion (2.5d) and mathematically they are the Nuclear Spectral Theorem [13] for the two RHS's (2.11), (2.12). The energy wave functions $\langle +E|\phi^+ \rangle$ and $\langle -E|\psi^- \rangle$ describe the apparatus that prepares the state $\phi^+$ and the apparatus that detects the observable $\psi^-$, respectively:

$$\phi^+(E) \equiv \langle +E|\phi^+ \rangle = \langle E|\phi^{\text{in}} \rangle \quad \text{is given by the energy distribution of the prepared incident beam,} \hspace{1cm} (3.4)$$

$$\psi^-(E) \equiv \langle -E|\psi^- \rangle = \langle E|\psi^{\text{out}} \rangle \quad \text{is given by the energy resolution of the detector and measures the} \hspace{1cm} (3.5)$$

energy distribution of the detected observables (out-states).

We have thus two sets of basis vectors $|E, b^\pm\rangle = |E^\pm\rangle \in \Phi_+^\times$ corresponding to the two RHS (2.11) and (2.12). The analogy of (3.2) and (3.3) with the conventional scattering theory suggests that the $|E, b^\pm\rangle$ correspond to the in- and out-plane wave “states” $|E^+\rangle$ and $|E^-\rangle$. These plane wave states are conventionally specified as solutions of the Lippmann-Schwinger equations, which are used to describe a pair of time asymmetric boundary conditions in a heuristic way:

$$|E^\pm\rangle = |E\rangle + \frac{1}{E - H \pm i\varepsilon} V|E\rangle = \Omega^\pm|E\rangle, \quad (3.6)$$

where $(H - V)|E\rangle = E|E\rangle$.

We shall therefore also call our $|E, b^\pm\rangle$, which are mathematically defined as the functionals on the Hardy spaces $\Phi_\pm$, Lippmann-Schwinger kets. They are more general than the ordinary Dirac kets which are defined as functionals on Schwartz spaces.

The Schwartz energy wave functions $\langle E|\phi \rangle \in S$ are smooth (infinitely differentiable), rapidly decreasing functions on the real positive energy axis. From the $\pm i\varepsilon$ in energy, $|E^\pm\rangle = \lim_{\varepsilon \to +0} |E \pm i\varepsilon\rangle$, of the Lippmann-Schwinger equation (3.6) we can conclude that the energy wave functions

$$\phi^+(E) = \langle +E|\phi^+ \rangle = \langle \phi^+|E^+ \rangle$$  \hspace{1cm} (3.7)

out-states, cf. [16] section 3) and their basis vectors $|E, b^\pm\rangle$, the plane wave solutions of the Lippmann-Schwinger equation (3.6). The labels $-$ and $+$ of the spaces refer to the standard notation that mathematicians use for Hardy spaces of the lower and upper complex semiplane. Since we want to retain the physicists and the mathematicians conventions we have to accept the mismatch in the notation of (2.11) and (2.12) where the physics of prepared in-states and detected out-observables is mapped to the Hardy spaces of the lower and upper complex energy plane, respectively.
must be analytic in the lower half plane, and the energy wave functions

\[ \psi^-(E) = \langle -E|\psi^- \rangle = \overline{\langle \psi^-|E^- \rangle} \]  

(3.8)

must be analytic in the upper half plane, at least in an infinitesimal strip below and above the real axis, respectively.

From this we conjecture that the energy wave functions (3.4) and (3.5) should not only be smooth functions on the real axis but they should only be those smooth functions that can be analytically continued into the lower (for (3.7)) and upper (for (3.8)) complex energy semiplanes. We also would want them to vanish rapidly enough when one goes to the infinite semicircle. This is essentially the definition of Hardy functions on the lower and upper semiplane (for a definition see [17]). To make this into a precise hypothesis we postulate:  

\[ \langle +E - i\varepsilon|\phi^+ \rangle = \overline{\langle \phi^+|E + i\varepsilon^+ \rangle} \in \mathcal{H}^2_+ \cap S||_{R^+}, \]  

(3.9)

\[ \langle -E + i\varepsilon|\psi^- \rangle = \overline{\langle \psi^-|E - i\varepsilon^- \rangle} \in \mathcal{H}^2_+ \cap S||_{R^+}, \]  

(3.10)

where \( \mathcal{H}^2_+ \cap S||_{R^+} \) is the space of smooth (\( \in S \)) Hardy functions on the lower/upper complex \( E \)-plane on the second sheet of the Riemann surface for the \( S \)-matrix. That we choose the second sheet of the \( S \)-matrix is related to the analyticity property of the \( S \)-matrix. This postulate (3.9), (3.10) is the new axiom (\( AV \)) because \( \Phi_\mp \) are defined as the abstract linear topological spaces which are realized by the function spaces \( \mathcal{H}^2_+ \cap S||_{R^+} \). This means, in analogy to (2.7a), (2.7b), that the following triplets are equivalent:

\[ \phi^\pm \in \Phi_\mp \subset \mathcal{H} \subset \Phi_\mp^* \iff \langle \pm E|\phi^\pm \rangle \in \mathcal{H}^2_+ \cap S||_{R^+} \subset L^2(R^+) \subset (\mathcal{H}^2_+ \cap S||_{R^+})^*. \]  

(3.11)

The important property of the Hardy space triplets is that they are indeed Rigged Hilbert spaces [18], so that the basis vector expansions (3.2) and (3.3) are fulfilled as the nuclear spectral theorem for the RHS’s in (3.11).

With the arguments that led from (3.6) to (3.9), (3.10) we have given a heuristic justification of the Hardy space axiom (\( AV \)). A more compelling argument (which exceeds the scope of this paper) is that only with the Hardy space properties can one obtain a theory that relates Breit-Wigner resonances to exponential decay [19], [20].

From (\( AV \)) follows the time asymmetry (2.13) (2.14) and (2.15). [19, sect. 5.6] The important mathematical theorem behind this time asymmetry is the Paley-Weiner Theorem [19, Appendix A] for Hardy functions. The choice of the Schwartz space \( S \) in (3.11) is not crucial for time asymmetry. Therefore \( S \) could be—and will be in the relativistic case—replaced by some other suitable spaces.

With the Hardy Rigged Hilbert Spaces (3.11) we have a wealth of new mathematical objects which are not contained in \( \mathcal{H} \) or in the Schwartz RHS (2.7). In addition to apparatus prepared states \( \phi^+ \in \Phi_+ \) with

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6A consequence of (3.10) is that \( \langle \psi^-|E - i\varepsilon^- \rangle \in \mathcal{H}^2_- \cap S||_{R_-} \) and similar for (5.9).
3 SIMILARITIES AND DIFFERENCES WITH THE TRADITIONAL PRACTICES

smooth, analytic in $\mathbb{C}_-$, wave function $\phi^+(E) = \langle +E|\phi^+ \rangle$, describing the energy distribution $|\phi^+(E)|^2$ of accelerator beam, and in addition to the detected observables $\psi^- \in \Phi_+$ with smooth, analytic in $\mathbb{C}_+$ wave functions $\psi^-(E) = \langle -E|\psi^- \rangle$ describing the energy resolution of the detector one has in RHS’s and generalized vectors (continuous antilinear functionals on $\Phi_{\pm}$) or kets $F^\mp \in \Phi_{\pm}^\times$.

An example of these kets are the Lippmann-Schwinger scattering states which are the generalized eigenvectors $(3.1)$ of the exact Hamiltonian $H$ with real eigenvalues given by the scattering energies. In addition to these eigenkets with real eigenvalues $7$, there are many more functionals $F^\mp \in \Phi_{\pm}^\times$. In particular, there are generalized eigenvalues of the self-adjoint Hamiltonian $H$ with complex eigenvalue $z \in \mathbb{C}_\pm$. One special example of these are the exponentially decaying state vectors $\psi^G = |E_R - i\Gamma/2, b^-\rangle \sqrt{2\pi\Gamma} \in \Phi_+^\times$ which are eigenkets of the self-adjoint Hamiltonian $H$ with complex eigenvalue,

$$H^\times \psi^G = (E_R - i\Gamma/2)\psi^G, \quad (3.12)$$

where the generalized eigenvalue $E_R - i\Gamma/2$ is the position of the resonance pole of the $S$-matrix in the complex energy plane, second Riemann sheet. These generalized eigenvectors we call Gamow vectors or Gamow kets. They describe exponentially decaying resonance states given by first order poles of the $S$-matrix; their generalization to relativistic physics is the subject of the two subsequent papers $[16]$ $[27]$.

There are many more special vectors in $\Phi_{\pm}^\times$ (and in $\Phi_{\mp}^\times$) than the Lippmann-Schwinger kets $(3.1)$ and the Gamow kets $(3.12)$. Some of these, the Gamow-Jordan kets associated to higher than first order poles of the $S$-matrix, will be presented in the Appendix $[28]$ $[29]$. The states associated to higher order $S$-matrix poles have the following curious features: 1.) they have—contrary to the standard opinion $[31]$, $[32]$—an exponential time evolution $[29]$, and 2.) they are not described by state vectors but by density operators that cannot be reduced further into “pure” states given by dyadic products $[29]$. Other examples of decaying mixed states are associated to cuts in the lower half-plane, their asymmetric time evolution is described by the extended Liouvillian $[33]$. We want to restrict our discussions in this introductory paper on time asymmetric quantum theory to the simplest and experimentally well-documented case of a first order pole resonance and to the Gamow states $(3.12)$.

Using the bra-ket (i.e., the functional $F^-(\psi^-) = \langle \psi^-|F^- \rangle$ for all $\psi^- \in \Phi_+$) the time evolution of the generalized vector $F^-(t) \in \Phi_+^\times$ can be defined as

$$\langle \psi^-|F^- \rangle = \langle e^{iHt}\psi^-|F^- \rangle = \langle \psi^-|e^{-iH^\times t}\psi^- \rangle \quad (3.13)$$

$7$They are boundary values of kets with complex eigenvalues in the upper and lower complex half plane, respectively.
since \( (2.14) \) holds only for \( t \geq 0 \). \( F^-(t) \) is again a solution of the Schrödinger equation

\[
i \frac{\partial}{\partial t} F^-(t) = H^\times F^-(t) \quad \text{but with boundary condition} \quad F^-(0) = F^- \in \Phi_+^x.
\]  

(3.14)

Thus \( F^-(t) \) represents a generalized state with semigroup time evolution

\[
F^-(t) = U^\times(t)F^- = e^{-iH^\times t}F^- \quad \text{for} \quad t \geq 0.
\]

(3.15)

In analogy to the Born probabilities we would want to interpret the matrix elements \( \langle \psi^-(t) | \psi^-(t) \rangle \) as something like probability amplitudes (as done for Dirac ket, where \( |\langle \psi|E\rangle|^2 \) is the probability density for \( E \)). This is a generalization of \( (2.1') \).

The probability to measure an observable \( |\psi^-(t)\rangle \langle \psi^-(t) | \) (or \( \Lambda^- (t) \)) in the generalized state \( F^- \in \Phi_+^x \) is given by

\[
\mathcal{P}_{F^-}(t) = |\langle \psi^-(t) | F^- \rangle|^2 = |\langle \psi^- | F^-(t) \rangle|^2
\]

(3.16)

and it is defined for \( t \geq 0 \) only.

As mentioned in Section 1, the semigroup time \( t = 0 \) will be interpreted as any arbitrary but finite time \( t_0 \) at which the generalized state described by \( F^- \) has been prepared. The quantum system represented by the generalized state vector \( F^- \) is an ensemble of individual micro-particles prepared or created under identical conditions. The generalized state \( F^- \) starts its dynamical evolution generated by the Hamiltonian \( H^\times \supset H \) at the semigroup time \( t = 0 \). An experiment in quantum physics is done on an ensemble of individual micro systems prepared under identical conditions; this does not mean that they are prepared at the same time \( t_0 \) in the life of the experimentalist. In most cases, the semigroup time \( t = 0 \) means, in fact, a collection of times \( t_0^{(1)}, t_0^{(2)}, \ldots, t_0^{(i)}, \ldots \) and how these times are measured will be discussed in the subsequent paper \[10\]. All these times \( t_0^{(i)} \) are represented by the mathematical semigroup time \( t = 0 \) of the state \( F^-(t) \) that describes the ensemble of microsystems.

If we choose for \( F^- \) the Gamow vector \( \psi^G \) we obtain from \( \[9\] [12] \)

\[
\psi^G(t) = e^{-iH^\times t} \psi^G = e^{-iE_{R\ast}t} e^{-\frac{1}{2}it} \psi^G; \quad t \geq 0.
\]

(3.17)

\[8\] Precisely, the semigroup generator \( H = H_+ \) in \( \Phi_+ \) is the restriction of the self-adjoint operator \( H \) to \( \{ \psi \} \) of the (dense in \( \mathcal{H} \)) subspace \( \Phi_+ \) and the generator \( H = H_- \) in \( \Phi_- \) is the restriction of \( H \) to \( \Phi_- \). For simplicity of notation we have omitted the subscripts for the operators. But in order to make our results precise, we have to be more accurate in our notation and specify the space on which the operators act. The semigroup \( (e^{iH^\times t})^\times \) acts on \( \Phi_+^x \) and has the generator \( H_+^\times = e^{-iH^\times t} \).

Analogously, the semigroup \( (e^{-iH^- t})^\times = e^{iH_-^\times t} \) acts on \( \Phi_-^x \) and has the generator \( H_-^\times = e^{-iH^\times t} \). The operators \( H, H^\dagger \) are generators of the unitary group \( \Phi_+ \) and \( \Phi_- \) in \( \Phi_+^x \) (where the differentiation \( H = \frac{1}{i} \frac{dU(t)}{dt} \) at \( t = 0 \) is defined with respect to the topology \( \Phi_+ \)). The operators \( H_+, H_- \) are generators of the semigroup \( \Phi_+^x \) and of the semigroup \( \Phi_-^x \), respectively. This means that the differentiation \( H_+ = \frac{1}{i} \frac{dU_+}{dt} \) and \( H_-^\dagger = -\frac{1}{i} \frac{dU_-^\dagger}{dt} \) is defined with respect to the topology in \( \Phi_+ \) and \( \Phi_-^x \), respectively. That the restrictions and extensions of the generators of the group \( \Phi_+ \) are the generators of the semigroup in \( \Phi_+ \) and of its conjugate in \( \Phi_-^x \) is highly non-trivial.
For the probability rate to detect the decay products \( \psi^- \in \Phi_+ \) in state \( \psi^G \in \Phi_+^\times \) at time \( t \) we obtain then

\[
P_{\psi^G(t)} = |\langle \psi^- (t) | \psi^G \rangle|^2 = e^{-\Gamma t} |\langle \psi^- | \psi^G \rangle|^2 \quad \text{for } t \geq 0 \text{ only.} \tag{3.18}
\]

The time asymmetry \( t \geq 0 \) means that no probability is predicted for times \( t \) before the quantum system was prepared at \( t = 0 \), as is in agreement with all observations (causality). Similar arguments also apply to the Lippmann-Schwinger kets \( F^- = |E^- \rangle \in \Phi_+^X \), for which this time asymmetry has been unrecognized:

\[
\langle \psi^- (t) | E^- \rangle = \langle e^{iHt} \psi^- | E^- \rangle = \langle \psi^- | e^{-iH^X t} | E^- \rangle = e^{-iEt} \langle \psi^- | E^- \rangle \quad \text{for } t \geq 0 \text{ only} \tag{3.19}
\]

or as a functional equation in the space \( \Phi_+^X \):

\[
e^{-iH^X t} | E^- \rangle = e^{-iEt} | E^- \rangle \quad \text{for } t \geq 0 \text{ only.} \tag{3.20}
\]

This time asymmetry is the difference between the Lippmann-Schwinger kets and the ordinary Dirac kets, defined as functionals on the Schwartz space \( |E \rangle \in \Phi^X \). The Schwartz space kets fulfill:

\[
e^{-iH^X t} | E \rangle = e^{-iEt} | E \rangle \quad \text{for all } - \infty < t < \infty, \tag{3.21}
\]

precisely

\[
\langle e^{iHt} \psi | E \rangle = e^{-iEt} \langle \psi | E \rangle \quad \text{for all } \psi \in \Phi. \tag{3.22}
\]

And this has always been assumed for the Dirac kets, even when they were not precisely defined as functionals. On the dual of the Schwartz space \( \Phi^X \) the extension \( H^X \supset H^\dagger \) of the self-adjoint \( H \) generates a group, on the dual of the Hardy space \( \Phi_+^X \) the extension \( H_+^X \supset H_+^\dagger \) of the self-adjoint \( H \) generates a semigroup.

It is important to realize that the popular Lippmann-Schwinger kets \( |E^\pm \rangle \) cannot fulfill the following two conditions simultaneously:

1) Be boundary values of “analytic kets” in the complex half-plane,\(^9\) as indicted by the \( i\epsilon \) in (3.6).

2) Fulfill the unitary group evolution (3.21).

One of these conditions has to go. For the ordinary Dirac kets (functionals on the Schwartz space) one keeps (3.21); then one cannot have the analyticity required for the in- or out- boundary condition. For the Lippmann-Schwinger kets one keeps the time asymmetric boundary conditions, because that was the purpose for introducing them; then they cannot fulfill (3.21). But, — after turning them into mathematically well defined objects by specifying the spaces \( \Phi^\pm \) on which they are eigenfunctionals—they fulfill (3.14).

\(^9\)This means that the functions \( \langle E^+ | \phi^+ \rangle \) and \( \langle E^- | \psi^- \rangle = \langle \psi^- | E^- \rangle \) are analytic functions in the lower complex halfplane \( \langle E^- | \psi^- \rangle \) analytic in the upper half-plane.)
We have modified the system of traditional axioms of quantum mechanics slightly by exchanging one of its axioms, the Hilbert space axiom \((A_{\text{old}})\), for the Hardy space axiom \((A_V)\). This new axiom \((A_V)\) distinguishes between observables and states and describes them by two different (dense) subspaces of the same Hilbert space. The idea of using two different spaces for two kinds of “states” has been mentioned before in footnote 14 of the historical paper \[21\]. Feynman distinguishes between the “state at times \(t' < t_0\) defined by the preparation” (our prepared states \(\{\phi^+\}\)) and the “state characteristic of the experiment” at times \(t'' > t_0\) (our detected observables \(\{\psi^-\}\)). The possibility, that \(\phi^+\) and \(\psi^-\) are from two different spaces, he mentions in footnote 14 attributing it to H. Snyder, but does not consider it any further. Here we have implemented this possibility by choosing for \(\{\phi^+\}\) and \(\{\psi^-\}\) the two different Hardy spaces \(\Phi_-\) and \(\Phi_+\) which are related by the Paley-Wiener theorem \[17\] to \(t' < t_0 = 0\) and \(t'' > t_0 = 0\), respectively.

The new axiom \((A_V)\), \(\{\phi^+\} \subset \Phi_-\), means that the energy distribution in the accelerator beam \(|\phi^+(E)|^2\) is described by a smooth rapidly decreasing function \(\phi^+(E)\) that, additionally, can be analytically continued into the lower half complex energy plane. Similarly, \(\{\psi^-\} \subset \Phi_+\) means that the energy resolution of the detector \(|\psi^-(E)|^2\) is described by a smooth rapidly decreasing function \(\psi^-(E)\) that can be analytically continued into the upper half complex energy plane.

In contrast, the Hilbert space axiom \((A_{\text{old}})\) states that \(\phi^+, \psi^- \in \mathcal{H}\), which means that the energy distributions \(|\psi^-(E)|^2\) and \(|\phi^+(E)|^2\) are both described by Lebesgue square-integrable functions \(\psi^-(E)\) and \(\phi^+(E)\). The modified version \((A_{\text{old}})\) of this axiom (which is just a refinement of \((A_{\text{old}})\) justifying many of the calculational tools that physicists use) means that the energy distribution functions \(\psi^-(E)\) and \(\phi^+(E)\) can be given by any smooth rapidly decreasing (Schwartz space) functions. In both cases, \((A_{\text{old}})\) and \((A_{\text{old}})\), it does not matter whether the function can be analytically continued into the complex energy plane or not. Thus the Hardy space hypothesis differs from the old axiom \((A_{\text{old}})\) only by the additional requirement that the wave function of the states \(\phi^+(E)\) can be analytically continued into the lower complex energy half-plane and the wave functions of the observables \(\psi^-(E)\) can be analytically continued into the upper complex energy half-plane.

Observing whether or not an energy wave function can be analytically continued to complex energies using the energy resolution of the apparatus does not appear possible. Thus the two axioms \((A_{\text{old}})\) and \((A_V)\) cannot be distinguished from each other by direct observations.

However, the differences in the consequences of \((A_{\text{old}})\) and \((A_V)\) are enormous. It is remarkable that such minor, and experimentally imperceptible changes in the mathematics (topology) of the axioms can lead to such enormous changes in the consequences of the mathematical theory.

The consequences of the Hardy space axiom \((A_V)\) are:
1. a consistent mathematical theory of resonance scattering and decay for which the lifetime-width relation
\[ \tau = \frac{\hbar}{\Gamma} \]
is an exact property of the Gamow state which is the new idealized physical notion provided by the Hardy space \([14]\), and

2. time asymmetry and causality versus time symmetry and problems with causality for \((A^\text{old}_V)\).

The dominant opinion about quantum mechanics—supported by the Axiom \((A^\text{old}_V)\)—is that the time evolution (for isolated systems) is reversible. It is described by the unitary group \((2.9)\) with the reverse evolution of \(U(t)\) given by \(U^{-1}(t) = U(-t)\). But the idea of time asymmetry in quantum mechanics has a long history and is connected with many distinguished names. It probably goes back to R. Peierls and his school (1938), who formulated it in terms of purely outgoing boundary conditions \([12]\) \([22]\) for the Schrödinger equation \((2.3S)\). The irreversible nature of quantum decay has been mentioned in \([23]\) and in a few monographs \([2]\) \([24]\); T. D. Lee called it the “impossibility of constructing time reversed quantum solutions for a microphysical system.” That the irreversibility should be intrinsic, rather than caused by the external effects of a quantum reservoir or the environment was emphasized by Prigogine and his school \([25]\). Gell-Mann and Hartle \([26]\) call it a fundamental arrow of time and refer to Feynman \([21]\) when they use time asymmetry in order to avoid inconsistencies for the probabilities of histories in their quantum mechanics of cosmology.

The axioms of non-relativistic time-asymmetric quantum mechanics can be extended to a relativistic theory by an appropriate extension of the Hardy space axiom into the relativistic domain. There the hypothesis \((A_V)\) will lead to new predictions. This relativistic theory of resonance scattering and decay, in which the time evolution semigroup is generalized to the causal Poincaré semigroup, is the subject of subsequent papers \([10]\), \([27]\).

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Appendix A Gamow Jordan Vectors

Though the operator \(H^\dagger\) in Hilbert space is a self-adjoint operator its (unique) extension \(H^\times \ni H^\dagger\) in the space \(\Phi^+_\times \ni \mathcal{H}\) is not. Self-adjoint operators in \(\mathcal{H}\) and in finite dimensional subspaces thereof are always diagonalizable. But \(H^\times\) is not self-adjoint or normal on all finite dimensional subspaces of \(\Phi^+_\times\). Therefore there may be subspaces on which \(H^\times\) is not diagonalizable. This does not happen if the \(S\)-matrix has only first order pole singularities, but it will happen if the \(S\)-matrix has a second or in general \(r\)th (finite) order pole \([28]\), \([29]\). In the same way as one derives the Gamow vectors \([8,13]\) from the first order pole, one derives
from the \( r \)th order \( S \)-matrix pole at \( z_r = E_r - i\Gamma/2 \) on the second (or higher) Riemann sheet \( r \) Gamow vectors of order \( k = 0, 1, 2, \ldots, (r - 1) \):

\[
|z_r^{-}^{(0)}, |z_r^{-}^{(1)}, \ldots, |z_r^{-}^{(r-1)} .
\]  

(A1)

The \( k \)th order Gamow vectors \( |z_r^{-}^{(k)} , k = 0, 1, 2, \ldots, r - 1 \) are Jordan vectors of degree \( k + 1 \).\( ^{30} \) They fulfill the “generalized” eigenvector equation

\[
(H^\times - z_R)^{k+1}|z_r^{-}^{(k)} = 0 .
\]

(A2)

and fulfill in detail

\[
egin{align*}
H^\times |z_r^{-}^{(0)} &= z_R |z_r^{-}^{(0)} \\
H^\times |z_r^{-}^{(0)} &= z_R |z_r^{-}^{(1)} + \Gamma |z_r^{-}^{(0)} \\
\vdots \\
H^\times |z_r^{-}^{(r-1)} &= z_R |z_r^{-}^{(r-1)} + \Gamma |z_r^{-}^{(r-2)} .
\end{align*}
\]

(A3)

This means \( |z_r^{-}^{(k)} \in \Phi^+ \) and the \( r \)th order \( S \)-matrix pole is associated to an \( r \) dimensional subspace \( M_{z_r} \subset \Phi^+ \), spanned by the \( |z_r^{-}^{(k)} , k = 0, 1, 2, \ldots, (r - 1) \), i.e., to the set of all \( F_{z_r}^{-} \in M_{z_r} \subset \Phi^+ \) is

\[
F_{z_r}^{-} = \sum_{k=0}^{r-1} |z_r^{-}^{(k)} c_k , \quad c_k \in \mathbb{C} .
\]

(A4)

On \( M_{z_r} \subset \Phi^+ \) the Hamiltonian \( H^\times \) (i.e., the extension of the self-adjoint operator \( H^\dagger \) to \( \Phi^+ \)) is not diagonalizable, but can only be brought into the normal form of a Jordan block \( ^{30} \). This means that \( H^\times \) restricted to the subspace \( M_{z_r} \) is a Jordan operator of degree \( r \) (in the standard notation it is rather the operator \( \dagger H^\times \) which is called a Jordan operator).

These equations \( ^{30} \) are, like the eigenvector equation for Dirac \( ^{25,32} \) and for Gamow vectors \( ^{31,32} \) (Gamow vectors of order 0 are Jordan vectors of degree 1), understood as generalized eigenvector equations \( ^{24,51} \) over the space \( \Phi^+ \), that means as

\[
\langle H\psi^- |z_r^{-}^{(k)} \rangle \equiv \langle \psi^- |H^\times |z_r^{-}^{(k)} \\
= z_r \langle \psi^- |z_r^{-}^{(k)} + \Gamma \langle \psi^- |z_r^{-}^{(k-1)} \quad \text{for all } \psi^- \in \Phi^+ .
\]

(A5)

Therefore the \( |z_r^{-}^{(r)} \) are generalized vectors in two respects, firstly they are functionals on the space \( \Phi^+ \) and secondly they are generalized eigenvectors as expressed by \( ^{30,31} \). Therefore we call these vectors Gamow-Jordan vectors. The matrix representation of the operator \( H^\times \) is given by a matrix that has in the diagonal complex eigenvalues for the ordinary Gamow kets and Jordan blocks for the Gamow-Jordan kets.
For instance if the $S$-matrix has two first order poles at $z = z_{Rk} = E_{Rk} - i\Gamma_{Rk}/2$, $k = 1, 2$, and one second order pole at $z = z_2 = E_2 - i\Gamma_2$ then the matrix of the Hamiltonian $H$ is given by

$$
\frac{\langle H\psi^- | z_2^r \rangle}{\langle H\psi^- | z_2^- \rangle} = \begin{pmatrix}
\frac{\langle\psi^-|H^r|z_2^r \rangle}{\langle\psi^-|H^0|z_2^r \rangle} & \frac{\langle\psi^-|H^1|z_2^r \rangle}{\langle\psi^-|H^0|z_2^r \rangle} & \ldots & \frac{\langle\psi^-|H^{r-1}|z_2^r \rangle}{\langle\psi^-|H^0|z_2^r \rangle} \\
0 & \Gamma_2 & z_{R_1} & \ldots & \frac{\langle\psi^-|H^{r-1}|z_2^r \rangle}{\langle\psi^-|H^0|z_2^r \rangle} \\
\Gamma_2 & z_{R_1} & z_{R_2} & \ldots & \frac{\langle\psi^-|H^{r-1}|z_2^r \rangle}{\langle\psi^-|H^0|z_2^r \rangle} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
$$

(A6)

where $(E)$ denotes the continuously infinite matrix with diagonal elements $E : E_0 \leq E < \infty$ corresponding to (3.1). Each $z_{R_i}$ corresponds to (3.12) and the $2 \times 2$ matrix in the top right corner is the Jordan block corresponding to (3.1) with $r = 2$.

The state associated to the pole term of the $S$-matrix for the $r$-th order pole can no longer be described by a state vector, like the bound states by $|E_n\rangle$ with real discrete eigenvalues, or the 1st order resonance states (Gamow states) by $|z_{R_1}\rangle$ with complex eigenvalue $z_{R_1}$ of the self-adjoint Hamiltonian $H$. Instead, the state associated with the $r$-th order $S$-matrix pole term is described by a non-diagonalizable density operator or state operator

$$
W_{PT} = 2\pi\Gamma \sum_{n=0}^{r-1} \binom{r}{n+1} (-i)^n W^{(n)}
$$

(A7)

where the operators $W^{(n)}$ are defined as

$$
W^{(n)} = \sum_{k=0}^{n} |z_r^-(k)\langle n-k|z_r^-|W^{(n)} = |\psi^G\rangle\langle\psi^G|.
$$

(A8)

The pole term of the $r$th order $S$-matrix term is associated with a sum (A7) of the operators $W^{(n)}$. The operators $W^{(n)}$ represent components of this state $W_{PT}$ which are in a certain way, “irreducible” (as expressed by its property (A9) below).

In the case $r = 1$ (ordinary first order resonance pole) the operator (A7) becomes

$$
W_{PT} = 2\pi|z_1^-\rangle\langle z_1^-| = 2\pi\Gamma W^{(0)} = |\psi^G\rangle\langle\psi^G|.
$$

(A9)

This is the operator description of the generalized state whose vector description is given by $\psi^G$ of (3.12).

For the case $r = 2$ (second order pole at $z_R$) we have two irreducible components:

$$
W^{(0)} = |z_2^-\rangle\langle z_2^-| = |\psi_0\rangle\langle\psi_0|,
$$

(A10)

and

$$
W^{(1)} = \left(|z_2^-\rangle\langle z_2^-| + |z_2^+\rangle\langle z_2^+|\right).
$$

(A11)

The state associated with the $n$th order pole is a mixed state $W_{PT}$ all of whose components $W^{(n)}$, except for the zeroth component $W^{(0)}$ cannot be reduced further into something like “pure” states given by dyadic
products like (A9) which could be described by a vector $\psi^G = |z^{-}\rangle \sqrt{2\pi \Gamma}$. The operator $W_{PT}$ associated to the 2nd order pole term is

$$W_{PT} = 2\pi \Gamma \left( 2W^{(0)} - iW^{(1)} \right)$$

$$= 2\pi \Gamma \left( 2|z_2^{-}\rangle \langle z_2^{(0)}| - i \left( |z_2^{-}\rangle \langle z_2^{(1)}| + |z_2^{-}\rangle \langle z_2^{(1)}| \right) \right).$$

(A12)

The generalized vectors $|z_r^{(k)}\rangle$, $k = 0, \ldots, r - 1$ have very complicated time evolution given by

$$e^{-iH^x t} |z_r^{(k)}\rangle = e^{-iz_r t} \sum_{\nu=0}^{k} \frac{\Gamma^\nu}{\nu!} (-it)^\nu |z_r^{(\nu)}\rangle, \quad t \geq 0.$$  

(A13)

These are representations of the time transformation semigroup which (for $r > 1$) are not one dimensional. The existence of this kind of representation for the causal spacetime translation group has already been mentioned in reference [31]. For the special case of a double pole, $r = 2$, $k = 0, 1$, the formula (A13) for the zeroth order Gamow vector is

$$e^{-iH^x t} |z_r^{-}\rangle^{(0)} = e^{-iz_r t} e^{(-\Gamma/2)t} |z_r^{-}\rangle^{(0)}, \quad t \geq 0,$$

(A14)

and for the first order Gamow vector it is

$$e^{-iH^x t} |z_r^{-}\rangle^{(1)} = e^{-iz_r t} \left( |z_r^{-}\rangle^{(1)} + (-it)\Gamma |z_r^{-}\rangle^{(0)} \right), \quad t \geq 0.$$  

(A15)

That vectors associated with double poles of the $S$-matrix have in addition to the exponential a strong linear dependence of magnitude $\Gamma$ as in (A14) and (A13) has been known for a long time and was the reason for dismissing double poles as viable resonance states, since the strong linear time dependence (A15) of a deviation from the exponential law had never been observed for decaying states. However, since the state associated with the $PT$ of the $S$-matrix is not a vector state but given by the complicated density operator (A7), (A8) the relevant property is the time evolution of the state operator $W_{PT}$ in (A7). For the first order pole this is given according to (A9) by the operator equivalent of (3.17) and (3.18). Writing (3.17) in terms of the state operator gives

$$W^G(t) = e^{-iH^x t} |\psi^G\rangle \langle \psi^G| e^{iHt}$$

$$= e^{-iz_r t} |\psi^G\rangle \langle \psi^G| e^{iz_r t}$$

$$= e^{-i(E_r-i(\Gamma/2))t} |\psi^G\rangle \langle \psi^G| e^{i(E_r+i(\Gamma/2))}$$

$$= e^{-\Gamma t} W^G(0).$$

(A16)

The operator $W^G$ represents the microsystem that affects the detector. The vectors $\psi^{-} \in \Phi^+$ represent observables defined by the detector (registration apparatus). The probability that the microsystem affects the detector at $t > 0$ (later than the time $t = 0$ at which the microsystem was created) is according to (2.1H)
\[
\text{Tr} \left( |\psi^-(t)\rangle \langle \psi^-(t)| W^G \right) = \langle \psi^-(t) | W^G | \psi^-(t) \rangle \\
= \langle e^{-iHt} \psi^- | W^G | e^{iHt} \psi^- \rangle \\
= \langle \psi^- | e^{-iH \times t} W^G e^{iHt} | \psi^- \rangle \\
= \langle \psi^- | W^G(t) | \psi^- \rangle \\
= e^{-\Gamma t} \langle \psi^- | W^G | \psi^- \rangle 
\]
which is (A17).

We now apply the time evolution operator \( e^{-iH \times t} \) to the state operator \( W^{(n)} \) of (A8), \( n = 0, 1, \ldots, (r-1) \) and then the \( W_{PT} \) of (A7), which is the state operator associated to the \( S \) matrix pole term of order \( r \) (any finite integer):

\[
W^{(n)}(t) = e^{-iH \times t} W^{(n)} e^{iHt} \\
= \sum_{k=0}^{n} e^{-iH \times t} | z^{- \uparrow \downarrow}^k (k-n) \rangle \langle - z_r | e^{iHt} .
\]

(A18)

Using (A13) and its conjugate

\[
| z^{- \uparrow \downarrow}^k - z_r | e^{iHt} = e^{i z_r t} \sum_{\nu=0}^{k} \Gamma^{\nu}_{\nu!} (it)^{\nu} \langle (k-\nu) \uparrow \downarrow - z_r |
\]

(A13)

one obtains after a complicated calculation [29] a very simple expression

\[
W^{(n)}(t) = e^{-iH \times t} W^{(n)} e^{iHt} = e^{-\Gamma t} \sum_{k=0}^{n} | z^{- \uparrow \downarrow}^k (k-n) \rangle \langle - z | e^{-\Gamma t} W^{(n)}(0) , \ t \geq 0 .
\]

(A19)

The complicated state operator \( W^{(n)} \) has thus a very simple exponential time evolution. Considering the complicated time evolution of (A13) and (A13) the simple result (A19) looks like a miracle.

This result means that the complicated non-reducible (“mixed”) microphysical state operator \( W^{(n)} \) defined by (A8) has a simple purely exponential semigroup time evolution, like the zeroth order Gamow state (A16) and thus leads to the exponential law for the probabilities, as in (A17). This operator is probably the only operator formed by the dyadic products \( | z^{- \uparrow \downarrow}^m (m-\ell) \rangle \langle - z_r | \) with \( m, \ell = 0, 1, \ldots, n \), which has purely exponential time evolution. Thus \( W^{(n)} \) of equation (A8) is distinguished from all other operators in \( M^{(n)} \).

The microphysical decaying state operator associated with the \( r \)-th order pole of the unitary \( S \)-matrix is according to its definition (A7) a sum of the \( W^{(n)} \). Because of the simple form (A19) (independence of the time evolution of \( n \)) this sum has again a simple and exponential time evolution

\[
W_{PT}(t) \equiv e^{-iH \times t} W_{PT} e^{iHt} = e^{-\Gamma t} W_{PT} ; \quad t \geq 0 .
\]

(A20)

Thus we have seen that the state operator (A7) which is the operator associated to the \( r \)-th order pole of the \( S \)-matrix, describes a non-reducible “mixed” microphysical decaying state which obeys an exact exponential decay law.
Summarizing the Appendix, Jordan blocks arise naturally from higher order $S$-matrix poles. They represent a self-adjoint Hamiltonian by a complex matrix in a finite dimensional subspace like the two dimensional Jordan block in $\text{(A6)}$. This finite dimensional subspace is contained in the dual $\Phi^+_\mathfrak{C}$ of the Rigged Hilbert space of Hardy type $\Phi^+$. Although higher order $S$-matrix poles are not excluded by any theoretical argument, there has been so far very little experimental evidence for their existence. It was always believed on the basis of $\text{(A15)}$ that states associated with higher order poles must have polynomial time dependence and therewith deviations from the exponential time dependence. A deviation from the exponential law of the magnitude as predicted by $\text{(A15)}$ (i.e., of magnitude $\Gamma$) is excluded experimentally. However, since $\text{(A19)}$ and $\text{(A20)}$ show that all non-reducible state operators associated to the higher order $S$-matrix pole evolve purely exponential in time, there remains no experimental evidence against their existence. These mathematically beautiful objects may therefore have some future applications in physics.

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