NUMERICAL SIMULATION OF FLUID-STRUCTURE INTERACTION PROBLEMS WITH HYPERELASTIC MODELS I: A PARTITIONED APPROACH

ULRICH LANGER AND HUIDONG YANG

ABSTRACT. In this work, we consider fluid-structure interaction simulation with nonlinear hyperelastic models in the solid part. We use a partitioned approach to deal with the coupled nonlinear fluid-structure interaction problems. We focus on handling the nonlinearity of the fluid and structure sub-problems, the near-incompressibility of materials, the stabilization of employed finite element discretization, and the robustness and efficiency of Krylov subspace and algebraic multigrid methods for the linearized algebraic equations.

1. Introduction

Recently, some nonlinear hyperelastic models have been adopted in the fluid-structure interaction (FSI) simulation; see, e.g., [52, 37, 80, 64, 10]. In contrast to linear, isotropic and elastic models subjected to large displacements and small strains in the FSI simulation (see, e.g., [60, 19]), the nonlinear hyperelastic models are often used to describe materials undergoing finite deformation and very large strains, that are characterized by the anisotropic effects, material coefficient jumps across distinct layers, and near-incompressibility constraints; see, e.g., the arterial layers modeling in [36, 31] among others. As phrased in [10], when incorporating the above characteristics is not feasible, some simplified isotropic solid models may be used to predict wall deformation under the action of the fluid forces. In this work, we use a model of modified hyperelastic Mooney-Rivlin material (see, e.g., [12]), and a model of thick-walled artery with the media and adventitia layer (see [36]). For this two-layer thick-walled artery, we assume the angles between collagen fibers and the circumferential direction in the media and adventitia are specified for a healthy young arterial wall. In [8], a novel Laplace-Dirichlet Rule-Based (LDRB) algorithm to assign fiber orientation to heart models has been presented. Besides, there are other important issues concerning the arterial modeling, e.g., how to obtain arterial wall prestress and to incorporate it into the solid
momentum balance equation (see [37]) or how to estimate the element-based zero-stress state for arterial FSI computations in order to obtain correct reference geometry for the finite hyperelastic models (see [61]), and how to derive adequate boundary conditions for the solid domain due to the external tissue support surrounding the arterial wall (see, e.g., [52, 51]). These issues are beyond the scope of this work, we thus refer readers to the corresponding references.

The material coefficient jumps of the two-layer thick-walled artery prescribed in [36] are not so high, that is less problematic for our numerical simulation. Therefore, we concentrate on the near-incompressibility of the materials with very large bulk modulus $\kappa$. This brings difficulties to develop stable discretization and efficient iterative solution methods. To enforce incompressibility, in [34], a displacement-dilatation-pressure three-field Hu-Washizu variational approach combined with an augmented Lagrangian optimization technique has been employed. In [2], a finite element tearing and interconnecting (FETI, see, e.g., [55]) method is used to simulate biological tissues characterized by anisotropic and nonlinear materials. However a pure displacement formulation hired therein may lead to locking phenomena.

In this work, we use a displacement-pressure two-filed variational approach to overcome the locking phenomena. On the one hand, the stabilized equal order mixed finite element method in hyperelastic finite deformation using a residual-based stabilized formulation is well understood, as described in [47, 49] for two different types of Neo-Hooke material models. An application of this approach to the nonlinear inverse problem for hyperelastic tissues has been presented in [32]. This mixed displacement-pressure method was also used in the FSI simulation of a patient-specific atherosclerotic coronary artery modeled as hyperelastic (Fung) material; see [65]. For the linear elastic model, the error estimates of the mixed pressure-displacement formulation have been analyzed in [30]. The application to the FSI simulation using such a mixed formulation has been reported in [77].

On the other hand, the linear system of algebraic equations arising from the finite element discretization of the mixed hyperelastic formulation (after linearization using Newton’s method [21]) has a very nice form suitable for a (nearly) optimal algebraic multigrid (AMG) method. This AMG method has been utilized to solve the discretized Oseen and linear elastic equations in the FSI simulation; see [77], where a stabilized coarsening strategy and an efficient Braess-Sarazin smoother (see the original and approximate versions in [13] and [81], respectively) have been applied to both fluid and structure sub-problems.
The number of iterations for the AMG method to converge is independent of the problem size and the incompressibility. The original contribution of such an AMG method for the fluid problem (Oseen equations) can be found in, e.g., [71, 72]. Therein, in order to keep inf - sup stability on coarse levels (since meshes on coarse levels are not available), a special coarsening strategy with a proper scaling technique is discussed. The two-grid convergence of such an AMG method for the Stokes equations using a Petrov-Galerkin type coarse grid operator is shown in [50].

For the hyperelastic problems, unfortunately a direct application of such an AMG method fails. It is found from our numerical experiments, that the Braess-Sarazin smoother does not damp efficiently high-frequency errors. We therefore consider another class of commonly used smoothers based on corrections of many small local problems, that are projected from the global problem on properly chosen local patches, in a Gauss-Seidel manner. For the geometrical multigrid (GMG) methods in computational fluid mechanics, the so-called Vanka smoother based on a symmetrical coupled Gauss-Seidel (SCGS) was tested in [68], where a finite-difference formulation using staggered locations of the velocity and pressure variables is employed. An additive version of such a multiplicative Vanka-type smoother under the finite element discretization has been analyzed in [59]. The Vanka smoother has been widely used in computational fluid dynamics; see, e.g., [42, 41], where some stable finite element pairs with higher order velocity spaces are used. However, according to [71], the Vanka smoother deteriorates rapidly in the AMG method for solving the 3D Oseen equations discretized with the stabilized $P_1 - P_1$ pair. This is also observed in our numerical experiments to solve the linearized Navier-Stokes (NS) equations in a Newton iteration, especially for high Reynolds number.

The situation is different in the hyperelastic models. We apply this smoother to the multigrid method for our hyperelastic problems. Using a proper relaxation, the smoother works well under sufficient smoothing steps. In [76], it was observed, the Vanka smoother in the GMG methods works quite robustly for the $Q_1 - Q_1$ discretized finite elastic problem using a hyperelastic Neo-Hooke material in 2D. Compared with the Braess-Sarazin smoother, in each Vanka smoothing step, there appear many small local sub-problems, that are usually tackled by direct solvers. Thus altogether it leads to a complexity approximately proportional to $Nn^3$, where $N$ denotes the number of patches, and $n$ the average local sub-problem size.
We point out, that the AMG method we used in this work is not aiming to compete with the FETI method in, e.g., [2], since parallelization has been applied there. However, the (nearly) optimal AMG method may be used to solve sub-domain problems in the FETI framework. As a comparison, we also consider some Krylov subspace methods combined with efficient preconditioners, that are derived from the block $LU$ factorization of the mixed algebraic equations (see [69]). The iteration numbers of these Krylov subspace methods slightly increase when the mesh is refined, that usually leads to suboptimal solvers.

The reminder of this paper is organized in the following way. In Section 2 we describe the preliminary of this work. The complete coupled FSI system is formulated in Section 3. Section 4 deals with the temporal and spatial discretization of the fluid and structure sub-problems, and their linearization using Newton’s method. In Section 5 solution methods for the nonlinear FSI coupled problem, and the nonlinear fluid and structure sub-problems are discussed. Some numerical experiments are presented in Section 6. Finally, the conclusions are drawn in Section 7.

2. Preliminary

2.1. A model problem. As an illustration in Fig. 1 a fluid-structure interaction (FSI) driven by an inflow condition is considered, which is decomposed into fluid and structure sub-problems, and an interaction in between. Considering different application fields, we include both isotropic and anisotropic hyperelastic models. In the isotropic case, the material holds the same properties through the layer; while in the anisotropic case, the collagenous matrix of the material varies across the layers.

2.2. Geometrical description and arbitrary Lagrangian Euler mapping. The computational FSI domain $\Omega^f \subset \mathbb{R}^3$ at time $t$ is composed of the fluid sub-domain $\Omega^f_t \subset \mathbb{R}^3$ and structure sub-domain $\Omega^s_t \subset \mathbb{R}^3$: $\Omega^t = \Omega^f_t \cup \Omega^s_t$, $\Omega^s_t \cap \Omega^f_t = \emptyset$; see a schematic representation in Fig. 2. At time $t = 0$, $\Omega^0$ is referred to as a reference (initial) configuration: $\Omega^0 = \overline{\Omega^f_0} \cup \overline{\Omega^s_0}$, $\overline{\Omega^s_0} \cap \overline{\Omega^f_0} = \emptyset$. For the fluid sub-problem, $\Gamma^f_{in}$ and $\Gamma^f_{out}$ are parts of boundary $\partial \Omega^f_t$ where certain Neumann forces are applied. In an analogous manner, $\Gamma^s_d$ and $\Gamma^s_n$ are employed to denote parts of boundary $\partial \Omega^s_t$, where structure Dirichlet and Neumann conditions are applied, respectively. The intersection $\Gamma^t$ between two sub-domains at time $t$ is called the FSI interface: $\Gamma^t = \partial \Omega^f_t \cap \partial \Omega^s_t$. 
Again at $t = 0$, $\Gamma^0$ is the reference (initial) interface: $\Gamma^0 = \partial \Omega^0_f \cap \partial \Omega^0_s$, where certain interface conditions are fulfilled.

In order to track particle motion of the structure body $\Omega^0_s$, we adopt the Lagrangian mapping $\mathcal{L}'(\cdot) : \mathcal{L}'(x_0) = x_0 + \hat{d}_s(x_0, t)$ for all $x_0 \in \Omega^0_s$ and $t \in (0, T)$, with the structure displacement $\hat{d}_s(\cdot, \cdot) : \Omega^0_s \times (0, T] \mapsto \mathbb{R}^3$. An arbitrary Lagrangian Euler (ALE, see, e.g., [27, 22, 40]) mapping $\mathcal{A}'(\cdot)$ is used to capture the movement of the fluid sub-domain $\Omega^0_f$: $\mathcal{A}'(x_0) = x_0 + \hat{d}_f(x_0, t)$ for all $x_0 \in \Omega^0_f$ and $t \in (0, T)$, where the fluid
displacement \( \hat{d}_f(\cdot, \cdot) : \Omega_f^0 \times (0, T] \mapsto \mathbb{R}^3 \), only follows particle motion of the fluid body on the interface \( \Gamma^0 \), but is arbitrarily extended into the domain \( \Omega_f^0 \). A classical option for the ALE mapping is the so-called harmonic extension:

\[
\begin{align*}
(1a) & \quad - \Delta \hat{d}_f = 0 \quad \text{in} \quad \Omega_f^0, \\
(1b) & \quad \hat{d}_f = \hat{d}_s \quad \text{on} \quad \Gamma^0
\end{align*}
\]

with homogeneous Dirichlet boundary conditions \( \hat{d}_f = 0 \) on \( \Gamma_{in} \cup \Gamma_{out} \). The fluid sub-domain velocity \( w_f : \Omega_f^t \mapsto \mathbb{R}^3 \) at time \( t \) is then given by

\[
(2) \quad w_f = \frac{\partial \hat{d}_f}{\partial t} \circ \mathcal{A}^{-1}.
\]

In our situation, the simple harmonic extension \([1]\) provides satisfying results. Other more advanced mesh motion techniques under the ALE framework can be found in, e.g., \([74]\).

### 2.3. Nonlinear hyperelastic modeling

In order to formulate the hyperelastic sub-problem in Section 3.2, we first introduce the following useful notations in nonlinear continuum mechanics, that can be found in, e.g., \([35, 12, 54, 75]\). Let \( F \) denote the deformation gradient tensor given by \( F = \partial \mathcal{L}_t / \partial x_0 = I + \nabla \hat{d}_s \) and \( J \) the determinant given by \( J = \det F \). We then define the right Cauchy-Green tensor \( C \) as \( C = F^T F \) and the three principal invariants are respectively given by

\[
(3) \quad I_1 = C : I, \quad I_2 = \frac{1}{2} (I_1^2 - C : C), \quad I_3 = \det(C).
\]

For a hyperelastic material, the second Piola-Kirchoff tensor \( S \) is defined as

\[
(4) \quad S = 2 \frac{\partial \Psi}{\partial C},
\]

where \( \Psi \) represents the energy functional depending on the invariants. The energy functional is chosen according to material properties, and the second Piola-Kirchoff tensor is then computed accordingly.

#### 2.3.1. A modified model of Mooney-Rivlin material

We first consider a modified model of Mooney-Rivlin material, for which the energy functional is given by

\[
(5) \quad \Psi = \frac{c_{10}}{2} (J_1 - 3) + \frac{c_{01}}{2} (J_2 - 3) + \frac{1}{2} \kappa (J - 1)^2,
\]

where the first two invariants are respectively given by

\[
(6) \quad J_1 = I_1 I_3^{-1/3}, \quad J_2 = I_2 I_3^{-2/3}.
\]
Material parameters $c_{10} > 0$ and $c_{01} > 0$ are related to shear modulus $\mu^l$ for consistency with linear elasticity in the limit of small strains: $\mu^l = 2(c_{10} + c_{01})$. The bulk modulus $\kappa > 0$ indicates material compressibility: the larger the bulk modulus is, the more incompressible the material becomes. In this case, the second Piola-Kirchoff tensor is given by

$$S = c_{10} \frac{\partial J_1}{\partial C} + c_{01} \frac{\partial J_2}{\partial C} + 2\kappa(J - 1) \frac{\partial J}{\partial C}. \tag{7}$$

In order to overcome the usual locking phenomena with large bulk modulus, we introduce the pressure $\hat{p}_s := \hat{p}_s(x, t) = -\kappa(J - 1) : \Omega_s^0 \times (0, T] \mapsto \mathbb{R}^3$ for all $x \in \Omega_s^0$ at time $t$. Using the notation

$$S' = c_{10} \frac{\partial J_1}{\partial C} + c_{01} \frac{\partial J_2}{\partial C},$$

then the second Piola-Kirchoff tensor is rewritten as

$$S = S' - \hat{p}_s JC^{-1}, \tag{8}$$

where we use $\partial J/\partial C = JC^{-1}/2$. The Mooney-Rivlin model is often used to model elastic response of rubber-like materials (see, e.g., [12]). We use this model in the FSI simulation as an isotropic hyperelastic case for test purpose; see the one-layered model in Fig. [1].

2.3.2. A two-layer thick-walled artery. In many applications, simple isotropic hyperelastic models might be insufficient to describe the mechanical response of materials like arterial tissues; see, e.g., [36, 31, 2]. When studying mechanical response of a healthy young arterial (with no pathological intimal changes) in hemodynamics, we usually model the artery as a two-layer thick-walled tube consisting of the media and the adventitia layers. The media layer has a common FSI interface with the fluid. The energy functional of such an arterial model is prescribed by:

$$\Psi = \Psi_{iso} + \Psi_{aniso} + \frac{1}{2} \kappa(J - 1)^2, \tag{9}$$

where the isotropic response in each layer and the strain energy stored in the collagen fibers are respectively prescribed by the classical neo-Hookean model and an exponential function (see [36, 7]):

$$\Psi_{iso} = \frac{c_{10}}{2} (J_1 - 3), \quad \Psi_{aniso} = \frac{k_1}{2k_2} \sum_{i=4,6} (\exp(k_2(J_i - 1)^2) - 1). \tag{10}$$

Here $c_{10} > 0$ is a stress-like material parameter and $\kappa > 0$ the bulk modulus as in the modified Mooney-Rivlin material, $k_1 > 0$ a stress-like material parameter, and $k_2 > 0$ a dimensionless parameter. The
invariants $J_4 > 1$ and $J_6 > 1$ (active in extension) are given by:

$$J_4 = I_3^{-1/3}A_1 : C, \quad J_6 = I_3^{-1/3}A_2 : C,$$

where the tensors $A_1$ and $A_2$ characterizing the media and adventitia structure are prescribed by

$$A_1 = a_{01} \otimes a_{01}, \quad A_2 = a_{02} \otimes a_{02}$$

with the direction vectors $a_{01}$ and $a_{02}$ that are specified by the angle $\alpha$ between the collagen fibers and the circumferential direction in the media or adventitia. In particular, they have the forms

$$a_{01} = (0, \cos \alpha, \sin \alpha)^T, \quad a_{01} = (0, \cos \alpha, -\sin \alpha)^T.$$

Then, the second Piola-Kirchoff tensor $S$ for this hyperelastic model is computed as follows:

$$S = c_{10} \frac{\partial J_i}{\partial C} + k_1 \sum_{i=4,6} (\exp(k_2(J_i - 1)^2)(J_i - 1) \frac{\partial J_i}{\partial C}) + 2\kappa(J-1) \frac{\partial J}{\partial C}.$$

Using the pressure $\hat{p}_s$ as defined for the modified Mooney-Rivlin material and the notation

$$S' = c_{10} \frac{\partial J_i}{\partial C} + k_1 \sum_{i=4,6} (\exp(k_2(J_i - 1)^2)(J_i - 1) \frac{\partial J_i}{\partial C}),$$

the second Piola-Kirchoff tensor is reformulated in the following form

$$S = S' - \hat{p}_sJC^{-1}.$$

The material parameters $c_{10}$, $k_1$, $k_2$, and the angle $\alpha$ in the model take different values in the media and adventitia, i.e., they have jumps across the layers. In particular, the constants $k_1$ and $k_2$ are associated with the anisotropic contribution of collagen to the response; see an illustration in Fig. 14 of [36], from which we also adopt the material and geometrical data of a carotid artery from a rabbit as a benchmark in our numerical experiments. Therefore, we consider this model as an anisotropic hyperelastic model in the FSI simulation; see the illustration of two-layered model in Fig. 1.

2.3.3. A short remark on hyperelastic modeling. As stated in [36], although the material parameters are constants independent of the geometry, opening angle or fiber angle, the internal pressure/radius response does depend on these quantities. For simplicity, we use the same assumption as therein that the fiber angles are the same in the load-free configuration. Unfortunately, in patient-specific arterial FSI computations, the image-based arterial geometry is usually not stress-free. In order to overcome this difficulty, in, e.g., [37], the prestress is...
FSI SIMULATION WITH HYPERELASTIC MODELS

directly integrated into the momentum equation, or in, e.g., [61], the authors developed a methodology to find a zero-stress configuration for the artery, that is used as a reference configuration for the hyperelastic models. Another interesting issue is related to proper descriptions of boundary conditions for the hyperelastic model since the artery is surrounded by tissues. In [52, 51, 11], the viscoelastic support conditions along the artery wall are proposed to model the effect of external tissues. These important issues are not the focus of this work. However, the solution methods we are proposing are not only restrict to these two hyperelastic models used in this work. Indeed it is easy to integrate the above effects into our methodology.

3. The coupled system

3.1. The fluid sub-problem. The fluid sub-problem in a deformable domain $\Omega^t_f \subset \mathbb{R}^3$ is governed by the incompressible NS equations under the ALE framework: Find the fluid velocity $u := u(x,t) : \Omega^t_f \mapsto \mathbb{R}^3$ and pressure $p_f := p_f(x,t) : \Omega^t_f \mapsto \mathbb{R}$ for all $x \in \Omega^t_f$ at time $t$ such that

\begin{align}
(16a) \quad \rho_f \partial_t u|_{A^t} + \rho_f ((u - w_f) \cdot \nabla)u - \nabla \cdot \sigma_f(u, p_f) &= 0 \quad \text{in } \Omega^t_f, \\
(16b) \quad \nabla \cdot u &= 0 \quad \text{in } \Omega^t_f
\end{align}

with Neumann boundary conditions

\begin{align}
(17a) \quad \sigma_f(u, p_f)n_f &= g_{in} \quad \text{on } \Gamma^t_{in}, \\
(17b) \quad \sigma_f(u, p_f)n_f &= 0 \quad \text{on } \Gamma^t_{out},
\end{align}

where $\rho_f$ denotes the fluid density, $n_f$ the outerward unit normal vector to $\Gamma^t_{in/out}$, $\sigma_f(u, p_f) := 2\mu \varepsilon(u) - \frac{p_f}{\rho_f} I$ the Cauchy stress tensor, $\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$ the strain rate tensor, $\mu$ the dynamic viscosity, $g_{in}$ the given data for the inflow boundary condition, and

\begin{equation}
(18) \quad \partial_t u|_{A^t} := \partial_t u + (w_f \cdot \nabla)u
\end{equation}

the ALE time derivative with the fluid sub-domain velocity $w_f$ defined by (2).

3.2. The structure sub-problem. For finite deformation, the structure sub-problem under the Lagrangian framework in a reference domain $\Omega^0_s \subset \mathbb{R}^3$ is governed by the following equations: Find the structure displacement $\hat{d}_s$ and pressure $\hat{p}_s$ for all $x \in \Omega^0_s$ at time $t$ such that

\begin{align}
(19a) \quad \rho_s \partial_t \hat{d}_s - \nabla \cdot (FS) &= 0 \quad \text{in } \Omega^0_s, \\
(19b) \quad \hat{p}_s &= -\kappa(J - 1) \quad \text{in } \Omega^0_s
\end{align}
with Dirichlet and Neumann boundary conditions

\begin{align}
\dot{d}_s &= 0 \quad \text{on } \Gamma_d^0, \\
FS\hat{n}_s &= 0 \quad \text{on } \Gamma_n^0,
\end{align}

where \( \rho_s \) denotes the structure density, \( \hat{n}_s \) the outward unit normal vector to \( \Gamma_n^0 \) of the initial configuration, and the second Piola-Kirchhoff tensor \( S \) is given by (8) or (15) according to the choice of the hyperelastic models.

3.3. Interface equations. The interface equations on \( \Gamma^0 \) are respectively described by the classical no-slip condition and equivalence of surface tractions:

\begin{align}
(21a) \quad u \circ \mathcal{A}^t &= \partial_t \hat{d}_s \quad \text{on } \Gamma^0, \\
(21b) \quad \sigma_f (u, p_f) n_f \circ \mathcal{A}^t + FS\hat{n}_s &= 0 \quad \text{on } \Gamma^0.
\end{align}

The fully coupled FSI problem consists of (1), (16)-(21), and proper initial conditions: \( u(x, 0) = 0 \) for all \( x \in \Omega^0_f \), \( d_s(x, 0) = \partial_t \hat{d}_s(x, 0) = 0 \) for all \( x \in \Omega^0_s \).

4. Temporal and spatial discretization and linearization

4.1. Temporal discretization. In order to solve the FSI problem, we first make temporal discretization and seek the FSI solution at each time level. Thus we subdivide the time period \( (0, T] \) into \( N \) equidistant intervals and the time step size is given by \( \Delta t = T/N \). We aim to find the FSI solution at each time level \( t^n = n\Delta t, \quad n = 1, \ldots, N \). At time level \( t^0 \), the FSI solution is known by the initial conditions. For simplicity of notations, a function \( f \) at time level \( t^n \) is denoted by \( f^n := f(\cdot, t^n) \).

4.1.1. Temporal discretization for the fluid sub-problem. An fully implicit Euler scheme for the fluid sub-problem is adopted:

\begin{align}
\partial_t u(x, t^n)|_{\mathcal{A}^n} &\approx (u^n - u^{n-1} \circ \mathcal{A}^{t^{n-1}} \circ (\mathcal{A}^{t^n})^{-1}) / \Delta t. \tag{22}
\end{align}

At each nonlinear FSI iteration, the fluid sub-domain \( \Omega^0_f \) at time level \( t^n \) is extrapolated by the sub-domain from the previous iteration. The fluid domain velocity \( w^f \) in the convection term is then computed using the first order Euler scheme:

\begin{align}
w^n_f &= (\hat{d}^f_n - \hat{d}^{f-1}_n) \circ (\mathcal{A}^{t^n})^{-1} / \Delta t, \tag{23}
\end{align}

where \( \hat{d}^f_n \) is computed by solving the harmonic extension problem (1) with obtained interface displacement from previous nonlinear iteration, and \( \hat{d}^{f-1}_n \) is the fluid artificial domain displacement from previous time level \( t^{n-1} \).
By the ALE property, the fluid velocity \( u^n_{\Gamma t} \), on the interface is then given by the fluid domain velocity, i.e., \( u^n_{\Gamma t} = u^n_{\Gamma} \). In addition, since we use a fully implicit scheme, the nonlinearity in the fluid convection term \( u^n \cdot \nabla u^n \) needs to be tackled by Newton’s method.

4.1.2. **Temporal discretization for the structure sub-problem.** To discretize the structure sub-problem in time, a first order Newmark-\( \beta \) scheme is used; see [53]. For simplicity of notations, let \( \dot{\hat{d}}^n_s := \partial_t \hat{d}_s(x, t^n) \) and \( \ddot{\hat{d}}^n_s := \partial_{tt} \hat{d}_s(x, t^n) \). Then the Newmark-\( \beta \) scheme is given by

\[
\ddot{\hat{d}}^n_s \approx \frac{1}{\beta \Delta t^2} (\hat{d}^n_s - \hat{d}^{n-1}_s) - \frac{1}{\beta \Delta t} \dot{\hat{d}}^{n-1}_s - \frac{1 - 2\beta}{2\beta} \ddot{\hat{d}}^{n-1}_s,
\]

\[
\dot{\hat{d}}^n_s \approx \dot{\hat{d}}^{n-1}_s + \gamma \Delta t \ddot{\hat{d}}^{n-1}_s + (1 - \gamma) \Delta t \ddot{\hat{d}}^{n-1}_s,
\]

with constants \( 0 < \beta \leq 1 \) and \( 0 \leq \gamma \leq 1 \). As observed in [78] for linear elastic models using the mixed displacement and pressure formulation, the conventional choice of \( 2\beta = \gamma = 1 \) in the numerical experiments leads to temporal oscillation of the pressure field, and the choice of \( \beta > 0.5 \) leads to stable time discretization. Thus in our numerical experiments, we have chosen \( \beta = 0.625 \) and \( \gamma = 1 \) in order to avoid the pressure oscillation in time.

4.2. **Time semi-discretized weak formulations.**

4.2.1. **Function spaces.** We aim to find weak FSI solution on proper function spaces. Let \( H^1(\Omega^n_f), H^1(\Omega^n_s), L^2(\Omega^n_f) \) and \( L^2(\Omega^n_s) \) denote the standard Sobolev and Lebesgue spaces (see, e.g., [1]) on \( \Omega^n_f \) and \( \Omega^n_s \), respectively. The function spaces for the fluid velocity and pressure are respectively given by \( V^n_f := \{ v_f : v_f \circ A^t \in H^1(\Omega^n_f)^3 \} \) and \( Q^n_f := \{ q_f : q_f \circ A^t \in L^2(\Omega^n_f) \} \); the function spaces \( V^n_s \) and \( Q^n_s \) for the structure displacement and pressure shall be appropriately chosen according to the nonlinearities; see, e.g., [6, 18].

4.2.2. **The time semi-discretized fluid weak formulation.** After time discretization, the weak formulation of the fluid sub-problem reads: Find \( u^n \in V^n_{f,D} := \{ v \in V^n_f : v = g_d \in \mathbb{R}^3 \text{ on } \Gamma^n \} \) and \( p^n_f \in Q^n_f \) such that for all \( v \in V^n_{f,\partial} := \{ v \in V^n_f : v = 0 \text{ on } \Gamma^n \} \) and \( q \in Q^n_f \)

\[
\tilde{R}^1_f((u^n, p^n_f), v) = 0,
\]

\[
\tilde{R}^2_f((u^n, p^n_f), q) = 0,
\]
where the residuals \( \tilde{R}_1^f(\cdot, \cdot) \) and \( \tilde{R}_2^f(\cdot, \cdot) \) on the function spaces \( V_f^n \) and \( Q_f^n \) are given by

\[
\begin{align*}
\tilde{R}_1^f((u, p), v) &= \left( \frac{\rho_f}{\Delta t} u + \rho_f (\nabla (u - w^n) \cdot \nabla) u, v \right)_{\Omega_f^n} + 2\mu (\varepsilon(u), \varepsilon(v))_{\Omega_f^n} \\
&\quad - (p, \nabla \cdot v)_{\Omega_f^n} - (\frac{\rho_f}{\Delta t} u^{n-1}, v)_{\Omega_f^n} - \langle g_n, v \rangle_{\Gamma^n}, \\
\tilde{R}_2^f((u, p), q) &= -(q, \nabla \cdot u)_{\Omega_f^n},
\end{align*}
\]

Here we assume a Dirichlet boundary condition \( g_d \) is prescribed on the interface \( \Gamma^n \).

### 4.2.3. The time semi-discretized structure weak formulation.

In an analogous way, the time semi-discretized weak formulation of the structure sub-problem reads: Find \( d_n^s \in V_{s,0} := \{ v \in V_s : v = 0 \text{ on } \Gamma^0_d \} \) and \( p_n^s \in Q_s \) for all \( v \in V_{s,0} \) and \( q \in Q_s \) such that

\[
\begin{align*}
\tilde{R}_1^s((d, p^s), v) &= 0, \\
\tilde{R}_2^s((d, p^s), q) &= 0,
\end{align*}
\]

where the residuals \( \tilde{R}_1^s(\cdot, \cdot) \) and \( \tilde{R}_2^s(\cdot, \cdot) \) on the function spaces \( V_s \) and \( Q_s \) are given by

\[
\begin{align*}
\tilde{R}_1^s((d, p^s), v) &= \left( \frac{\rho_s}{\beta \Delta t^2} d, v \right)_{\Omega^s_0} + (S', (F^T \nabla v))_{\Omega^s_0} - (pJF^{-T}, \nabla v)_{\Omega^s_0} \\
&\quad - \langle g_n, v \rangle_{\Gamma^0} - (r_s, v)_{\Omega^s_0}, \\
\tilde{R}_2^s((d, p^s), q) &= -(J - 1, q)_{\Omega^s_0} - \frac{1}{\kappa} (p, q)_{\Omega^s_0},
\end{align*}
\]

where \( r_s = \frac{\rho_s}{\beta \Delta t^2} d^{n-1} + \frac{\rho_s}{\beta \Delta t^s} \hat{d}_{s}^{n-1} + \frac{\rho_s (1-2\beta)}{2\beta} \ddot{d}_{s}^{n-1} \). Here we assume a Neumann data \( g_n \in \mathbb{R}^3 \) on the interface \( \Gamma^0 \) is known.

### 4.3. Stabilized finite element methods.

Finite element discretization requires a triangulation of the computational FSI domain. For this purpose, we use Netgen \([58]\) to generate a tetrahedral mesh with conforming grids on the FSI interface. In addition it resolves different structure layers with prescribed material parameters; see an illustration in Fig. 4.3.

Equal order linear finite element spaces for the velocity and pressure fields in the fluid sub-problem, and the displacement and pressure fields in the structure sub-problem suffer from instability. Therefore some stabilization techniques are employed and adapted.
4.3.1. Stabilization for the fluid sub-problem. Let $\mathcal{T}_{f,h}$ be the triangulation of the fluid sub-domain $\Omega_f^h$, that is obtained by the ALE mapping on the triangulation $\mathcal{T}_{f,h}^0$ of the reference fluid sub-domain $\Omega_f^0$. The finite element spaces of the fluid velocity and pressure are given by $V_{f,h} := \{ v_f \in C^0(\Omega_f^h)^3 : v_f|_T = P_1(T)^3, \forall T \in \mathcal{T}_{f,h} \}$ and $Q_{f,h} := \{ q_f \in C^0(\Omega_f^0) : q_f|_T = P_1(T), \forall T \in \mathcal{T}_{f,h} \}$, respectively, where $P_1(T)$ denotes the linear polynomial defined on a tetrahedral element.

As is well known, the discrete solution may contain unphysical oscillations, e.g., spurious pressure modes, when equal order finite element spaces are used in the fluid problem, that do not fulfill the inf−sup or LBB (Ladyshenskaya-Babuška-Brezzi) stability condition; see, e.g., [3, 14, 15]. In addition, the instability may occur in advection dominated regions of the domain. To suppress instability, we employ
a unified stabilization technique, streamline-upwind-/ and pressure-stabilizing/Petrov-Galerkin (SUPG/PSPG) method, developed in [16, 39, 38, 62, 23, 63], that is able to enhance velocity field stability and meanwhile allows the use of equal order finite element spaces for both the velocity and pressure fields. This stabilization technique was adapted to the fluid problem in the FSI simulation; see, e.g., [73, 20, 9]. In [79], it was used for the stabilized finite element discretization of the Oseen-type equations on hybrid meshes. Applications of more recently developed stabilization techniques to the fluid problems in the FSI simulation can be found in, e.g., [29, 28].

The fully discretized and stabilized fluid sub-problem reads: Find $u_h \in V_{f,h} \cap V_{t,n}^{f,d}$ and $p_{f,h} \in Q_{f,h}$ such that for all $v_h \in V_{f,h} \cap V_{t,n}^{f,0} =: V_{f,h}$ and $q_h \in Q_{f,h}$

\begin{align}
R_1^f((u^n_h, p^n_{f,h}), v_h) &= 0, \\
R_2^f((u^n_h, p^n_{f,h}), q_h) &= 0,
\end{align}

where $R_1^f(\cdot, \cdot) = \tilde{R}_1^f(\cdot, \cdot) + W_1^f(\cdot, \cdot)$ and $R_2^f(\cdot, \cdot) = \tilde{R}_2^f(\cdot, \cdot) + W_2^f(\cdot, \cdot)$. Here we include $W_1^f(\cdot, \cdot)$ and $W_2^f(\cdot, \cdot)$ for corresponding stabilization terms coming from the SUPG/PSPG method.

4.3.2. Stabilization for the structure sub-problem. Let $T_{s,h}$ be the triangulation of the structure sub-domain $\Omega^0_s$; let $V_{s,h} := \{v_s \in C^0(\Omega^0_s)^3 : v_s|_T = P_1(T)^3, \forall T \in T_{s,h}\} \cap V_{s,0}$ and $Q_{s,h} := \{q_s \in C^0(\Omega^0_s) : q_s|_T = P_1(T), \forall T \in T_{s,h}\}$ represent the finite element spaces of the structure displacement and pressure, respectively.

Thanks to the absence of convection term in the hyperelastic model, we only face instability due to equal order interpolation spaces for both displacement and pressure fields. The PSPG method introduced in [38] is used to circumvent the inf − sup stability condition; see the application in, e.g., [17] to two different types of Neo-Hooke material models, and [32] to a soft tissue model. We adapt this technique to hyperelastic models used in this work. The stabilized formulation is obtained by augmenting the Galerkin discretization with the following introduced least-square finite element stabilization term:

\begin{align}
W_s((d_{s,h}, p_{s,h}, q_h) &= \sum_{T \in T_{s,h}} \tau(r_{s,h} = \frac{\rho_s}{\beta \Delta t^2} d_{s,h} - \nabla \cdot (F S), F^{-T} \nabla q_h)_{T})
\end{align}

for $d_{s,h} \in V_{s,h}$ and $p_{s,h}, q_h \in Q_{s,h}$, where $r_{s,h} = \frac{\rho_s}{\beta \Delta t^2} d_{s,h}^{n-1} + \frac{\rho_s}{\beta \Delta t^2} \ddot{d}_{s,h}^{n-1} + \frac{\rho_s(1-\gamma)}{2\beta \Delta t} \dot{d}_{s,h}^{n-1}$ corresponds to the discretization of the right hand side of the structure momentum equation. The stabilized parameter $\tau$ is
given by $\tau = \frac{h_T^2}{4\mu}$, where $h_T$ is the characteristic element length for the tetrahedral element $T$. The parameter $\mu$ is given by $\mu^l = 2(c_{10} + c_{01})$ and $\mu^l = 2(c_{10} + \frac{k_1}{2k_2})$ for the modified Mooney-Rivlin material and two-layer thick-walled artery, respectively. Since we are working with linear finite elements, the derivative of $S'$ cancel out. Then the above stabilization term (28) is simplified as follows:

\[
W_s(d_{s,h}, p_{s,h}, q_h) = \sum_{T \in T_{s,h}} \tau (r_{s,h} - \frac{p_s}{\beta \Delta t^2} d_{s,h} - \nabla \cdot (p_{s,h}JF^{-T}), F^{-T}\nabla q_h)T
\]

The fully discretized and stabilized structure sub-problem reads: Find $d_{s,h} \in V_{s,h}$ and $p_{s,h} \in Q_{s,h}$ such that for all $v_h \in V_{s,h}$ and $q_h \in Q_{s,h}$

\[
R^1_s(d^n_{s,h}, p^n_{s,h}, v_h) = 0,
\]

\[
R^2_s(d^n_{s,h}, p^n_{s,h}, q_h) = 0,
\]

where $R^1_s(\cdot, \cdot)$ and $R^2_s(\cdot, \cdot)$ correspond to an additional term derived from the PSPG method.

4.4. **Newton’s Method.** It is obvious both (27) and (30) are nonlinear systems of stabilized finite element equations. To solve these nonlinear equations, we employ Newton’s method, that requires Jacobian computations of the nonlinear equations at the current state variables. The equations are linearized using standard concept of directional derivatives in continuum mechanics; see, e.g., [12, 35, 54]. Since both (27) and (30) have similar forms, let $w_k$ denote the fluid velocity or structure displacement, and $p_k$ be the fluid or structure pressure at $k$th iteration. For simplicity of notations, we use unified notations $R^1((w_k, p_k), v)$ and $R^2((w_k, p_k), q)$ to represent the residuals at the $k$th iteration $(k \geq 0)$ in the fluid sub-problem (27) or the residuals in the structure sub-problem (30). At each Newton iteration, we need to find corrections $\delta_w$ and $\delta_p$ of the following linearized equations as the mixed formulation below:

\[
\frac{\partial R^1((w_k, p_k), v)}{\partial w} \cdot \delta_w + \frac{\partial R^1((w_k, p_k), v)}{\partial p} \cdot \delta_p = -R^1((w_k, p_k), v),
\]

\[
\frac{\partial R^2((w_k, p_k), q)}{\partial w} \cdot \delta_w + \frac{\partial R^2((w_k, p_k), q)}{\partial p} \cdot \delta_p = -R^2((w_k, p_k), q),
\]
for all \( v \in V_{f,h}^{0}/V_{s,h} \) and \( q \in Q_{f,h}/Q_{s,h} \). The solution is then updated by \( w_{k+1} = w_{k} + \delta_w \) and \( p_{k+1} = p_{k} + \delta_p \).

To express (31) as a system of linear algebraic equations, we introduce two finite elements basis \( \{ \phi^i \} \) and \( \{ \varphi^i \} \), \( i = 1, \ldots, m \), for the fluid velocity/structure displacement, and fluid/structure pressure, respectively. We choose the standard hat function on tetrahedral elements as the basis function for both the fluid velocity/structure displacement and the fluid/structure pressure. The finite element functions \( w_k \) and \( p_k \) at Newton iteration \( k \) can be represented as linear combinations of the basis

\[
(32) \quad w_k = \sum_{i=1}^{3m} w_k^i \phi^i \quad \text{and} \quad p_k = \sum_{i=1}^{m} p_k^i \varphi^i,
\]

where \( w_k^i \) and \( p_k^i \) are the degrees of freedom, and \( m \) is the number of vertices of the mesh. In an analogous manner, the corrections \( \delta_w \) and \( \delta_p \) at Newton iteration \( k \) are then expressed as linear combinations of the basis

\[
(33) \quad \delta_w = \sum_{i=1}^{3m} \Delta w_k^i \phi^i \quad \text{and} \quad \delta_p = \sum_{i=1}^{m} \Delta p_k^i \varphi^i,
\]

where \( \Delta w_k = (\Delta w_k^1, \ldots, \Delta w_k^{3m})^T \) and \( \Delta p_k = (\Delta p_k^1, \ldots, \Delta p_k^m)^T \) are the coefficient vectors of Newton increments. The global linear algebraic system at \( k \)th iteration is then written in the following block structure:

\[
(34) \quad K_k \begin{bmatrix} \Delta w_k \\ \Delta p_k \end{bmatrix} = \begin{bmatrix} A & B_1^T \\ B_2 & -C \end{bmatrix} \begin{bmatrix} \Delta w_k \\ \Delta p_k \end{bmatrix} = \begin{bmatrix} r_{1,k} \\ r_{2,k} \end{bmatrix},
\]

where \( K_k := K(w_k, p_k) \) is the computed Jacobian matrix containing four block matrices \( A, B_1, B_2 \) and \( C \). For the fluid sub-problem, we have \( A \neq A^T, B_1 \neq B_2 \), and \( C = C^T \). For the structure sub-problem, we have \( A = A^T, B_1 \neq B_2, C = C^T \). In any case, we have to deal with an unsymmetric algebraic equation with \( K_k \neq K_k^T \).

Then, Newton’s method consists of the following steps: given an initial guess \( w_0 \) and \( p_0 \), for \( k \geq 0 \),

1. assemble and solve the linearized system (34) up to a relative residual error reduction factor \( \varepsilon_1 \),
2. update the solutions by \( \tilde{w}_{k+1} = w_k + \delta_w \) and \( \tilde{p}_{k+1} = p_k + \delta_p \), and go to step 1. until

\[
(35) \quad \| (\tilde{w}_{k+1} - w_k, \tilde{p}_{k+1} - p_k) \|_{L_2} \leq \varepsilon_2 \| (\tilde{w}_1 - w_0, \tilde{p}_1 - p_0) \|_{L_2}
\]

is fulfilled.
The Newton iteration is always terminated if the stopping criterion (35) is fulfilled with \( \varepsilon_2 = 10^{-8} \). In the first part of our numerical studies presented in Section 6, we choose \( \varepsilon_1 := 10^{-8} \) as stopping criterion for the solvers of the linearized problems, whereas in the second part, i.e. in Subsection 6.5, we use an adaptive error control on inner and outer iterations in order to improve the overall efficiency.

More precisely, when employing the AMG method as the inner linear solver, with the matrix-vector notations, the above nonlinear stopping criterion (35) can be reformulated as follows.

Let \( v_k = [w_k^T, p_k^T]^T \) denote the solution at the \( k \)th Newton iteration for the nonlinear system arising from the finite element discretization of (27) or (30):

\[
F(v) = 0,
\]

where \( v = [w^T, p^T]^T \) denotes the exact solution we are aiming to find. Then under appropriate assumption, see, e.g., [21], we have the following quadratic convergence rate of the Newton iteration

\[
\|v - v_{k+1}\| \leq C\|v - v_k\|^2,
\]

where the norm \( \| \cdot \| \) has to be chosen accordingly, and the constant \( C > 0 \) may depend on the mesh size on the discrete level and the choice of norms. The correction step of the Newton iteration is given by

\[
v_{k+1} = v_k + \Delta v_k
\]

with the exact correction

\[
\Delta v_k := (F'(v_k))^{-1}(-F(v_k)) = K_{k}^{-1} r_k.
\]

Let \( \mathcal{M}_k \) denote the AMG iteration matrix at the \( k \)th Newton iteration. By applying AMG iterations to the linearized system (34) with an initial guess 0, we then have the following relation

\[
\Delta \tilde{v}_k = C_k^{-1} r_k
\]

with the approximated solution \( \Delta \tilde{v}_k := [\Delta \tilde{w}_k^T, \Delta \tilde{p}_k^T]^T \). The iteration matrix \( C_k^{-1} \) of the AMG method is given by

\[
C_k^{-1} = (I - \mathcal{M}_k^{(k)})K_k^{-1}
\]

with the function \( l(k) \) indicating the number of AMG iterations for solving the linearized system. Since we have only approximate solutions \( \tilde{v}_k^{k+1} \) by applying AMG iterations, we obtain

\[
\tilde{v}_{k+1} = v_k + \Delta \tilde{v}_k.
\]
Now we want to estimate the difference \( v - \tilde{v}_{k+1} \) in the proper norm in order to control the relation between the inner and outer accuracy. This can be done in the following way

\[
\| v - \tilde{v}_{k+1} \| = \| v - v_{k+1} + v_{k+1} - \tilde{v}_{k+1} \| \\
\leq \| v - v_{k+1} \| + \| v_{k+1} - \tilde{v}_{k+1} \| \\
\leq C \|v - v_k\|^2 + \| \Delta v_k - \Delta \tilde{v}_k \|
\]  
(43)

\[
= C \|v - v_k\|^2 + \| K_k^{-1} r_k - C_k^{-1} r_k \|
\]

\[
= C \|v - v_k\|^2 + \| \mathcal{M}_k^{(k)} K_k^{-1} r_k \|
\]

\[
\leq C \|v - v_k\|^2 + \| \mathcal{M}_k \| \| (K_k^{-1} r_k) \|
\]

Now using the relation that \( K_k^{-1} r_k = \Delta v_k = v_{k+1} - v_k \) and the quadratic convergence of the Newton iteration, we have

\[
\| K_k^{-1} r_k \| = \| v_{k+1} - v_k \| = \| v - v_{k+1} \| + \| v - v_k \| \\
\leq C \|v - v_k\|^2 + \| v - v_k \|
\]
(44)

Combining (43) and (44), we arrive at

\[
\| v - \tilde{v}_{k+1} \| \leq C \|v - v_k\|^2 + \| \mathcal{M}_k \| \| (K_k^{-1} r_k) \|
\]
(45)

Now it is easy to see, that in order to avoid the deterioration of the quadratic convergence rate of the Newton iteration, the following relation has to be satisfied:

\[
\| \mathcal{M}_k \| \leq c \|v - v_k\|
\]
(46)

with a constant \( c \).

However, the quantities in the above estimates are not computable since we have no exact solution \( v \) available. Thus, in practice, for the outer iteration, we use the \( L_2 \) norm of the increment in (35) that can be reformulated as

\[
\|(\tilde{w}_{k+1} - w_k, \tilde{p}_{k+1} - p_k)\|_{L_2} = \| \Delta \tilde{v}_k \|_M = \| C^{-1} r_k \|_M \\
= (MC_k^{-1} r_k, C^{-1} r_k)^{1/2}_{L_2} \\
= (C_k^{-T} M C_k^{-1} r_k, r_k)^{1/2}_{L_2} \\
= \|r_k\| c^{-T} M C_k^{-1} \\
\leq \varepsilon \|r_0\| c^{-T} M C_0^{-1},
\]

where \( M = \text{diag}[M_1, M_2] \) denotes the mass matrix with the velocity/displacement mass matrix \( M_1 \) and pressure mass matrix \( M_2 \) on the
diagonal. From this point of view, (35) is nothing but the relative residual error reduction in the \(C_k^{-T}MC_k^{-1}\)-energy norm with the symmetric and positive definite matrix \(C_k^{-T}MC_k^{-1}\).

For the inner iteration, we use the relative error reduction of the AMG iterations. On the one hand, we have

\[
\|\Delta v_k - \Delta \tilde{v}_k\|_{K_k^T M K_k} = \|K_k^{-1}r_k - C_k^{-1}r_k\|_{K_k^T M K_k} \\
\quad = \|K_k^{-1}r_k - (I - M_k^{(k)}) K_k^{-1}r_k\|_{K_k^T M K_k} \\
\quad = \|M_k^{(k)} K_k^{-1} r_k\|_{K_k^T M K_k} \\
\quad \leq \|M_k\|_{K_k^T M K_k} \|K_k^{-1} r_k\|_{K_k^T M K_k} \\
\quad = \|M_k\|_{K_k^T M K_k} \|r_k\|_M.
\]

On the other hand, we have

\[
\|\Delta v_k - \Delta \tilde{v}_k\|_{K_k^T M K_k} = (K_k^T M K_k(\Delta v_k - \Delta \tilde{v}_k), (\Delta v_k - \Delta \tilde{v}_k))^{1/2} \\
\quad = (M K_k(\Delta v_k - \Delta \tilde{v}_k), K_k(\Delta v_k - \Delta \tilde{v}_k))^{1/2} \\
\quad = (M r_{AMG}, r_{AMG})^{1/2} \\
\quad = \|r_{AMG}\|_M.
\]

Thus, for the relative residual norm of the linearized system, we have the following estimate

\[
\|r_{AMG}\|_M \leq \|M_k\|_{K_k^T M K_k} \|r_k\|_M,
\]

where \(r_{AMG} := r_k - K_k \Delta \tilde{v}_k\) denotes the residual of the linearized system in the AMG iterations. Since on the discrete level, all the norms are equivalent (up to some scaling), we have chosen the error reduction factor of the inner iteration such that

\[
\|M_k^{(k)}\|_{K_k^T M K_k} \leq \|M_k\|_{K_k^T M K_k} \leq \varepsilon_{2,k-1}
\]

is fulfilled, where \(\varepsilon_{2,k-1}\) indicates the error reduction factor at the previous \((k-1)\)th Newton iteration. The relative residual error reduction of the inner iterations will then have the following form

\[
\|r_{AMG}\|_M \leq \varepsilon_{2,k-1}^2 =: \varepsilon_{1,k}.
\]

For more details see [44, 45, 43, 33].

In our numerical examples in Section 6 we have always \(\|(w_1 - u_0, p_1 - p_0)\|_{L_2} \approx 100\). Thus we simply require to stop the iteration if \(\|(w_{k+1} - w_k, p_{k+1} - p_k)\|_{L_2} \leq \varepsilon_2 = 100 \cdot \tilde{\varepsilon}_2\).
5. Solution methods

5.1. A Dirichlet-Neumann FSI iteration. To solve the nonlinear FSI coupling in this work, we use a classical Dirichlet-Neumann (DN) FSI iteration (see, e.g., [48]), that is very easy and efficient to implement. Each DN FSI iteration consists of the following three steps:

1. solve the harmonic extension problem with prescribed fluid domain displacement on $\Gamma^0$ from the solution of the structure sub-problem, compute the current fluid domain $\Omega^t_f$ and the fluid domain velocity $w^t$,
2. with the prescribed fluid velocity condition on $\Gamma^v$ and the computed fluid sub-domain velocity $w^v$, and in the computed current fluid domain $\Omega^t_f$, solve the nonlinear fluid sub-problem (27) using Newton’s method,
3. compute the coupling force on $\Gamma^0$ from the solutions of the fluid sub-problem, solve the structure sub-problem (30) using Newton’s method with the computed force.

Repeat the above three steps until the stopping criterion $|r^v_{\Gamma,k}|/\sqrt{n} < \varepsilon$ is reached; see [48]. Here we set $\varepsilon = 10^{-8}$ in the numerical experiments, $n$ is the number of vertices on the interface $\Gamma^0$ and $r^v_{\Gamma,k}$ the interface residual introduced by $r^v_{\Gamma,k} = \tilde{d}^v_{\Gamma,k} - d^v_{\Gamma,k-1}$, where $d^v_{\Gamma,k-1}$ denotes the previously computed interface displacement on $\Gamma^0$, and $\tilde{d}^v_{\Gamma,k}$ the currently computed structure displacement restricted to $\Gamma^0$.

5.2. Aitken relaxation. As is well known, the above simple DN FSI iteration may lead to very slow convergence due to the added-mass effect (see, e.g., [17]). Using the Aitken relaxation we are able to guarantee and accelerate the convergence of the DN iteration. The relaxation step after each DN cycle reads:

$$d^v_{\Gamma,k} = \omega_{k-1}d^v_{\Gamma,k} + (1 - \omega_{k-1})d^v_{\Gamma,k-1},$$

where the acceleration parameter $\omega_k$ is defined by a recursion (see [48])

$$\omega_k = -\omega_{k-1} \frac{(r^v_{\Gamma,k})^T (r^v_{\Gamma,k+1} - r^v_{\Gamma,k})}{|r^v_{\Gamma,k+1} - r^v_{\Gamma,k}|^2}.$$

The sequence $d^v_{\Gamma,k}$, $k = 1, 2, \ldots$ converges to a point $d^v_{\Gamma}$ that is a fixed-point of the following FSI interface equation:

$$d^v_{\Gamma} = S^{-1}(F(d^v_{\Gamma})),$$

where $F$ denotes the fluid Dirichlet-to-Neumann mapping from the fluid velocity to the fluid force on $\Gamma^0$ by solving (27), and $S^{-1}$ the
Neumann-to-Dirichlet mapping from the structure forces to the structure displacement on $\Gamma^0$ by solving (30). Using the above mapping notations, the fixed-point FSI iteration with relaxation is given by

$$
 d^n_{\Gamma,k} = \omega_{k-1} S^{-1}(F(d^n_{\Gamma,k-1})) + (1 - \omega_{k-1})d^n_{\Gamma,k-1}.
$$

For more details, we refer to [48].

5.3. A short remark on other FSI iterations. In case of a semi-implicit coupling, where we use a first order extrapolation to approximate the current fluid domain and semi-implicit treatment for the fluid convection term, at each time step the Robin-Neumann (RN) preconditioned Krylov subspace method has been used in [4, 5, 77]. This method can be viewed as a generalization of the DN FSI iteration (in the semi-implicit coupling setting), that contains a more sophisticated preconditioner with weighted contributions from both fluid and structure sides. Proper choices of involved weighting parameters, combined with Krylov subspace acceleration, lead to very fast convergence of the FSI iterations. In a fully implicit scheme, we may run a two-level approach, that includes nested iterations. The outer iteration handles the geometrical and fluid convective nonlinearity using first order extrapolation, and the inner iteration runs the RN preconditioned Krylov subspace method. In this work, we stick to the above simple and efficient DN FSI iterations with Aitken relaxation.

5.4. Preconditioned Krylov subspace methods. The remaining cost in the DN FSI iteration is to solve the fluid and structure subproblems. At each DN iteration, we need to solve the highly nonlinear fluid and structure sub-problems using Newton’s method. Each Newton iteration requires an efficient solver for the linearized system (34). From now on, for simplicity, instead of $K_k$ we use the notation $K$ to indicate the linearized Jacobian. We first consider some Krylov subspace methods with efficient preconditioners for the linear system of algebraic equations, that we use in this work:

$$
 P^{-1}K \begin{bmatrix} \Delta w_k \\ \Delta p_k \end{bmatrix} = P^{-1} \begin{bmatrix} r_{1,k} \\ r_{2,k} \end{bmatrix},
$$

where $P$ is a properly chosen preconditioner.

An efficient preconditioner for the linear system (34) is developed according to the following $LU$ factorization of the matrices in the linear system of the form (34) (see, e.g., [69, 57]):

$$
 \begin{bmatrix} I & 0 \\ B_2A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} I & A^{-1}B_1^T \\ 0 & I \end{bmatrix} =: LDU,
$$
where \( S = B_2 A^{-1} B_1^T + C \) is the (negative) Schur complement. The inverses of \( A \) and \( S \) are very expensive in general. Thus we construct efficient preconditioner based on the following modified \( LU \) factorization.

\[
\begin{bmatrix}
I & 0 \\
B_2 \bar{A}^{-1} & I
\end{bmatrix}
\begin{bmatrix}
\bar{A} & 0 \\
0 & -\bar{S}
\end{bmatrix}
\begin{bmatrix}
I & \bar{A}^{-1} B_1^T \\
0 & I
\end{bmatrix} =: \bar{L} \bar{D} \bar{U},
\]

where \( \bar{A}^{-1} \) and \( \bar{S}^{-1} \) are proper approximations for \( A^{-1} \) and \( S^{-1} \), that are easy to invert.

5.4.1. A preconditioner for the fluid sub-problem. To solve the fluid sub-problem, we use a Generalized Conjugate Residual (GCR) method (see, e.g., [57]) with a preconditioner \( P_R = \bar{D} \bar{U} \). The inverse of the preconditioner is given by

\[
P_R^{-1} = \begin{bmatrix} \bar{A}^{-1} & \bar{A}^{-1} B_1^T \bar{S}^{-1} \\ 0 & -\bar{S}^{-1} \end{bmatrix},
\]

where \( \bar{A}^{-1} \) is realized by applying AMG cycles (see, e.g., [56, 46]), and \( \bar{S}^{-1} \) is computed by the operator splitting technique (see, e.g., [66, 26]). The method starts with the splitting of the velocity matrix \( A \)

\[
A = \frac{\rho_f}{\Delta t} M + \mu D + \rho_f C,
\]

where \( M \) stands for velocity mass matrix, \( D \) the diffusion matrix, \( C \) the convection matrix. Using the properties of corresponding operators, the Schur complement is then approximated by the following addition

\[
(B_2 A^{-1} B_1^T)^{-1} \approx \frac{\rho_f}{\Delta t} (B_2 M^{-1} B_1^T)^{-1} + \mu (B_2 D^{-1} B_1^T)^{-1} + \rho_f (B_2 C^{-1} B_1^T)^{-1}
\]

\[
\approx \frac{\rho_f}{\Delta t} D_p^{-1} + \mu M_p^{-1} + \rho_f M_p^{-1} C_p D_p^{-1} =: \bar{S}^{-1},
\]

where \( D_p \) and \( C_p \) are the stiffness matrices associated with the finite element discretization of the Laplacian operator and the scalar convection operator in the pressure space, respectively, \( M_p \) the mass matrix in the pressure space. In actual calculations, \( D_p \) is inverted by AMG method, and \( M_p \) is replaced by \( \text{diag}(M_p) \), i.e., the diagonal of \( M_p \). These operations are relatively cheap to realize. In [25], this method was also used to solve the linearized NS equations using the \( P_2 - P_1 \) mixed velocity and pressure formulation.

It is easy to see, that the operation of \( P_R^{-1} \) applied to a vector consists of two steps: solve the pressure Schur complement equation with \( \bar{S} \), and then solve the velocity equation with \( \bar{A} \).
5.4.2. A preconditioner for the structure sub-problem. To solve the structure sub-problem, we use the Biconjugate Gradient Stabilized (BiCGStab) method (see [67]) with a preconditioner $P_L = \tilde{L}\tilde{D}$. The inverse of the preconditioner is given by

\begin{equation}
P_L^{-1} = \begin{bmatrix}
\tilde{A}^{-1} & 0 \\
\tilde{S}^{-1}B\tilde{A}^{-1} & -\tilde{S}^{-1}
\end{bmatrix},
\end{equation}

where $\tilde{A}^{-1}$ is performed by applying AMG cycles to $A$, and the Schur complement is approximated by

\begin{equation}
S \approx \tilde{S} = (\frac{1}{\theta} + \frac{1}{\kappa})\text{diag}(M_p),
\end{equation}

where $M_p$ is the mass matrix in the structure pressure space. This preconditioner has demonstrated the robustness with respect to near-incompressibility. In our numerical experiments for both hyperelastic models, we use $\theta = O(c_{10})$. A good choice of $\theta$ is usually adjusted from the numerical tests on a coarse mesh. Once chosen, the value is held fixed for finer meshes. See [24] for the application of this method using $P_2 - P_1$ mixed displacement and pressure formulation in the modified Mooney-Rivlin hyperelastic model.

It is obvious to see, that the operation of $P_L^{-1}$ applied to a vector consists of two steps: solve the velocity equation with $\tilde{A}$ and then solve the pressure equation with the approximated Schur complement $\tilde{S}$.

5.5. AMG methods for the mixed problems. To improve performance of the linear iterative solvers, a class of (nearly) optimal AMG methods for the $P_1 - P_1$ mixed formulation are considered. In [71, 72, 79, 77], the robust coarsening strategy is discussed for stabilized $P_1 - P_1$ discretization of the Oseen equations and nearly incompressible linear elastic equations, that guarantees the discrete inf–sup conditions on all coarse levels in an algebraic manner. In this work, we will focus on proper choices of smoothers in the AMG methods for both fluid and structure sub-problems. As observed there exists no black-box solution, that can be applied to both sub-problems. In the AMG method, we need both pre- and post-smoothing steps, that are applied to all levels. For simplicity of presentation, we only discuss a smoothing step on one particular level. The smoothing step on a coarse level is a straightforward application to the corresponding linear system obtained by the stabilized Galerkin projection.
5.5.1. The Braess-Sarazin Smoother. The Braess-Sarazin consists of three steps of the inexact symmetric Uzawa algorithm:

\[
\Delta w_k = \Delta w_k + \tilde{A}^{-1}(r_{1,k} - A\Delta w_k - B_1^T \Delta p_k),
\]

\[
\Delta p_k = \Delta p_k - \tilde{S}^{-1}(r_{2,k} - B_2 \Delta w_k + C \Delta p_k),
\]

with \(\Delta w_k := r_{p,k}\)

which corresponds to a Richardson iteration applied to (34):

\[
\begin{bmatrix}
\Delta w_k \\
\Delta p_k
\end{bmatrix} = \begin{bmatrix}
\Delta w_k \\
\Delta p_k
\end{bmatrix} + P_F^{-1} \left( \begin{bmatrix}
r_{1,k} \\
r_{2,k}
\end{bmatrix} - K \begin{bmatrix}
\Delta w_k \\
\Delta p_k
\end{bmatrix} \right)
\]

with the full preconditioner \(P_F\) given by

\[
P_F = \begin{bmatrix}
\tilde{A} & B_1^T \\
B_2 & B_2 \tilde{A}^{-1} B_1^T - \tilde{S}
\end{bmatrix}.
\]

An essential issue is how to choose preconditioners \(\tilde{A}\) and \(\tilde{S}\) in order to fulfill the smoothing property. We use \(\tilde{A} = 2D\), where \(D\) denotes the diagonal of \(A\) (a relaxed Jacobi iteration). The original Braess-Sarazin smoother (see [13]) needs to solve the pressure correction equation exactly with the Schur-complement \(\tilde{S} = B_2 \tilde{A}^{-1} B_1 + C\). A relaxed version (see [81]) only solves this equation approximately, using e.g., an inner AMG method with starting value 0. As a price to pay, in each smoothing step of the outer AMG method, we first construct a Schur complement \(\tilde{S}\) on each level using the Galerkin projected matrices \(B_2\), \(\tilde{A}^{-1}\), \(B_1\) and \(C\). Then on each level of the inner AMG method, we need to construct a Schur complement using standard Galerkin projection applied to the previously constructed Schur complement. In [71, 72, 79, 77], the Braess-Sarazin smoother demonstrates robustness and efficiency in the AMG methods for solving the Oseen and linear elastic equations. However, as observed in this work, this smoother fails in the AMG method for solving the hyperelastic equations.

5.5.2. The multiplicative Vanka smoother. We finally consider the multiplicative Vanka smoother. In each smoothing step, we need to solve a sequence of local problems on properly chosen patches \(P_i\), \(i = 1, \ldots, n\), where \(n\) is the number of pressure degrees of freedom on that level. Each patch \(P_i\) contains a pressure degree of freedom, and the connected velocity degrees of freedom (given by the connectivity of matrix \(B_2\)). The local problem on \(P_i\) is then constructed by a canonical projection of the global matrix \(K\) on the patch \(P_i\), that has a form similar
to the global problem \((34)\):
\[
K^i \begin{bmatrix}
\Delta w^i_k \\
\Delta p^i_k
\end{bmatrix} = \begin{bmatrix}
A^i & (B_1^T)^i \\
B_2^i & -C^i
\end{bmatrix} \begin{bmatrix}
\Delta w^i_k \\
\Delta p^i_k
\end{bmatrix} = \begin{bmatrix}
\Delta w^i_k \\
\Delta p^i_k
\end{bmatrix} = \begin{bmatrix}
r^i_{1,k} \\
r^i_{2,k}
\end{bmatrix},
\]
where the current local residual \([(r^i_{1,k})^T; (r^i_{2,k})^T]^T\) is assembled by incorporating updated solutions from all previously treated local problems, which corresponds to a Gauss-Seidel manner. Thanks to small size of each local problem, direct solvers are applicable efficiently. In order to achieve smoothing property, a relaxation is usually necessary in the solution updating step:
\[
\begin{bmatrix}
\Delta w_k \\
\Delta p_k
\end{bmatrix} = \begin{bmatrix}
\Delta w_k \\
\Delta p_k
\end{bmatrix} + \omega P^i_k \begin{bmatrix}
\Delta w^i_k \\
\Delta p^i_k
\end{bmatrix},
\]
where \(P^i_k\) is the canonical embedding from the local patch \(P_i\) to the global, and \(\omega \in [0.5, 0.9]\) is the relaxation parameter used in our numerical experiments. We fix \(\omega = 0.78\) and \(\omega = 0.86\) for the structure sub-problems modeled by the modified hyperelastic Mooney-Rivlin material and the two-layer thick-walled artery, respectively. Such a Vanka smoother and its variants have been used in the GMG methods for solving both the fluid and hyperelastic problems in 2D or 3D, see, e.g., \([68, 59, 42, 41, 76]\). However, in the AMG method for solving the linearized NS equations, the smoothing property deteriorates rapidly as observed in our numerical experiments, that leads to an invalidity of the multigrid convergence; see also the failure report of the AMG method for the 3D Oseen equations in \([70]\). For solving the linearized hyperelastic equations used in this work, the Vanka smoother works quite well with sufficiently large number of smoothing steps.

6. Numerical experiments

6.1. Geometrical and material parameters. For the hyperelastic model of two-layer thick-walled artery from a rabbit, the geometry and material parameters are chosen according to Fig. 14 in \([36]\). The thickness of the media and adventitia is set to 0.26 mm and 0.13 mm, respectively. The angles between the collagen fibers and the circumferential direction in the media and adventitia are set to 29.0° and 62.0°, respectively. The radius of the artery is set to 1.43 mm. The length of the artery is set to 18 mm.

The shear stress-like material parameters \(c_{10}\) are set to 3 kPa and 0.3 kPa, \(k_2\) are set to 2.3632 kPa and 0.562 kPa, the dimensionless parameters \(k_2\) are set to 0.8393 and 0.7112, for the media and adventitia, respectively. The bulk modulus is set to \(10^5\) kPa for both two layers,
which corresponds to (nearly) incompressible material. The density on the reference configuration is set to 1.2 mg/mm$^3$.

For the modified hyperelastic model of Mooney-Rivlin material, we adopt the same geometrical configuration as in the artery. We set material parameters $c_{10}$ and $c_{01}$ to 3 kPa and 0.3 kPa, respectively. The bulk modulus is set to $10^5$ kPa. The density on the reference configuration is set to 1.2 mg/mm$^3$.

For the fluid model, we set the density to 1 mg/mm$^3$, the dynamic viscosity to 0.035 Poise.

The fluid and structure are at the rest in the beginning. The structure body is fixed at two ends. We give a pressure pulse (Neumann data) for the fluid at the inlet $\Gamma_{in}$ of the fluid domain: $(0, 0, 1.332)$ kPa for time $t \leq 1$ ms, and set $(0, 0, 0)$ kPa after 1 ms in the case of hyperelastic Mooney-Rivlin material and $(0, 0, 1.332)$ kPa for time $t \leq 0.125$ ms, and set $(0, 0, 0)$ kPa after 0.125 ms in the case of two-layer thick-walled artery. On the outlet $\Gamma_{out}$, a doing-nothing condition (0 Neumann boundary conditions) is applied. Time step size is set 0.125 ms for both models.

6.2. **Coarse and fine meshes.** On the fluid coarse mesh, we have 8120 tetrahedral elements, 2259 vertices and 9036 degrees of freedom. On the structure coarse mesh, we have 30472 tetrahedral elements, 6524 vertices and 26096 degrees of freedom. On the fluid fine mesh, we have 64960 tetrahedral elements, 14249 vertices, and 56996 degrees of freedom. On the structure fine mesh, we have 243776 tetrahedral elements (131072 for the media and 112704 for the adventitia), 46356 vertices, and 185424 degrees of freedom. We use the automatic mesh generator Netgen \cite{58} to generate the FSI meshes, that have conforming mesh grids on the FSI interface and resolve different layers of hyperelastic models so that different material parameters on layers are assigned accordingly.

6.3. **Nonlinear DN FSI iterations.** We first demonstrate the performance of nonlinear DN FSI solvers for the Mooney-Rivlin material and two-layer thick-walled artery in Fig. 4 and Fig. 5, respectively. We depict the value of the Atiken relaxation parameter (vertical lines) with respect to each DN iteration (horizontal lines) in the left plot, and the convergence history (vertical lines) with respect to the DN iteration (horizontal lines) in the right plot. Since at each time step, the nonlinear DN solver behaves in an analogous manner, we only plot the results at the second time step.

As observed from Fig. 4 and Fig. 5, the DN FSI solver for both nonlinear hyperelastic models runs robustly and efficiently. The Atiken
parameters are updated dynamically according to (53) at each DN iteration, and are in the range of (0, 1). The iteration numbers stay in the same range on both coarse and fine meshes for each structure model.

In Fig. 6, we show the number of nonlinear DN FSI iterations at different time steps (up to the 72th time step), using the model of the modified Mooney-Rivlin material and the model of the two-layer thick-walled artery. As we see from the numerical results, the iteration
numbers of the DN iteration stay in a similar range with different time steps.

![Graph showing number of nonlinear DN FSI iterations at different time steps using the model of the modified Mooney-Rivlin material ('+-+') and the model of the two-layer thick-walled artery ('.-.*'). The horizontal line indicates the time steps, the vertical line indicates the number of DN iterations.]

Figure 6. Number of nonlinear DN FSI iterations at different time steps using the model of the modified Mooney-Rivlin material ('+-+') and the model of the two-layer thick-walled artery ('.-.*'). The horizontal line indicates the time steps, the vertical line indicates the number of DN iterations.

For an illustration of the FSI simulation results, we visualize the structure deformation, the fluid velocity for the Mooney-Rivlin material and the two-layer thick-walled artery at time 8 ms in Fig. [7]

6.4. Solution methods for sub-problems. We first present numerical results concerning Newton’s method combined with the Krylov subspace methods: GCR and BiCGStab for the fluid and structure sub-problems, respectively. After that numerical results of Newton’s method combined with the AMG methods for both sub-problems are shown. For simplicity of presentation, we only present the convergence history of Newton’s method at the second DN FSI nonlinear iteration at the first time step. The iteration numbers of GCR and BiCGStab methods are recorded for each Newton iteration. We observe similar behavior of both Newton’s method and Krylov subspace methods at other time steps. On the left plots of the following figures, the horizontal lines represent the iteration number of Newton’s method, the vertical lines represent the corresponding errors. On the right plots of
the following figures, the horizontal lines represent the iteration number of Newton’s method, the vertical lines represent the iteration number of linear solvers needed at each Newton iteration.

6.4.1. *Krylov subspace methods.* The convergence history of Newton’s method to solve the NS equations on the coarse and fine meshes and the iteration numbers of the GCR method to solve the linearized NS equations are recorded in Fig. 8.

As observed from the numerical results, we obtain quadratic convergence rate for Newton’s method. The GCR method combined with the
preconditioner (59) demonstrates the robustness with respect to the mesh refinement, and the material parameters.

For the structure sub-problem modeled by the modified hyperelastic Mooney-Rivlin material, the convergence history of Newton’s method and the iteration numbers of the BiCGStab method to solve the linearized hyperelastic equations are recorded in Fig. 9.

![Convergence history of Newton's method](image1)

![Iteration numbers of BiCGStab solvers](image2)

Figure 9. Convergence history of Newton’s method (left) and iteration numbers of BiCGStab solvers (right) for the structure sub-problem modeled by modified hyperelastic Mooney-Rivlin material on coarse and fine meshes.

As observed from the numerical results, we obtain quadratic convergence rate for Newton’s method. The BiCGStab method combined with the preconditioner (62) shows medium number of iteration numbers on both coarse and fine meshes. We observe increasing number of iterations with mesh refinement. However, the method is robust with respect to the material parameters.

For the structure sub-problem modeled by the two-layer thick-walled artery, the convergence history of Newton’s method and the iteration numbers of the BiCGStab method to solve the linearized hyperelastic equations are recorded in Fig. 10.

![Convergence history of Newton's method](image3)

![Iteration numbers of BiCGStab solvers](image4)

As observed from the numerical results, we obtain the quadratic convergence of Newton’s method on the coarse mesh. On the fine mesh, the quadratic convergence degenerates at the fourth step, that is because the BiCGStab method at this step does not solve the linearized hyperelastic equations accurately. As we see from the right plot in Fig. 10, we stop the linear solver at 200 steps if the relative residual error does not reach the factor $10^{-8}$. This is cured later on by the more robust AMG method. As in the previous hyperelastic model, we observe the
medium number of iteration numbers on both coarse and fine meshes, and increasing number of iterations with mesh refinement.

6.4.2. The AMG methods. In order to run the AMG methods, we first construct the hierarchy of matrices on all levels in pure algebraic way (distinct from the GMG methods, where a hierarchy of nested meshes are provided). We use the coarsening strategy in [46]. On the coarse mesh, we arrive at three and four levels in the AMG methods for the fluid and structure sub-problems, respectively. For the fluid sub-problem on the coarse mesh, we have 9034, 1684 and 252 degrees of freedom on three levels after coarsening. For the structure sub-problem, we have 26096, 4012, 648 and 88 degrees of freedom on four levels. For the fluid sub-problem on the fine mesh, we have 56996, 9034, 1684 and 252 degrees of freedom on four levels after coarsening. For the structure sub-problem, we have 185424, 26096, 4012, 648 and 88 degrees of freedom on five levels.

As we see from Fig. 11, the number of degrees of freedom is reduced by a factor about 8 from the fine to coarse levels.

For the fluid sub-problem, the convergence of Newton’s method and the iteration numbers of the AMG method to solve the linearized NS equations using different number of Braess-Sarazin smoothing steps are plotted in Fig. 12.

To sum from Fig. 12, a quadratic convergence rate of Newton’s method is observed. The number of AMG iterations is drastically decreased compared to the GCR method combined with the preconditioner. In addition, the iteration numbers are less effected by the mesh refinement. By doubling the pre- and post-smoothing steps (from 4 to
Figure 11. Number of degrees of freedom for the fluid and structure sub-problems on AMG levels using the coarse (left) and fine (right) meshes.

Figure 12. Convergence history of Newton’s method (left) and iteration numbers of AMG method (right) for the fluid sub-problem on coarse and fine meshes, using 4 and 8 pre- and post-smoothing steps, respectively.

8), we observe the reduced AMG iteration numbers by almost a factor 2.

For the structure sub-problem modeled by the modified hyperelastic Mooney-Rivlin material, the convergence of Newton’s method and the iteration numbers of the AMG method to solve the linearized NS equations using different number of Vanka smoothing steps are plotted in Fig. [13].

As expected, a quadratic convergence rate of Newton’s method is observed from the numerical results. The number of AMG iterations is drastically decreased compared to the preconditioned BiCGStab method.
In addition, the iteration numbers are less affected by the mesh refinement. By doubling the pre- and post-smoothing steps (from 6 to 12), we observe the reduced AMG iteration numbers by almost a factor 2.

For the structure sub-problem modeled by the two-layer thick-walled artery, the convergence of Newton’s method and the iteration numbers of the AMG method to solve the linearized NS equations using different number of Vanka smoothing steps are plotted in Fig. 14.
As seen from the numerical results, a quadratic convergence rate of Newton’s method is recovered. The number of AMG iterations is drastically decreased compared to the preconditioned BiCGStab method. In addition, the iteration numbers are less effected by the mesh refinement. By doubling the pre- and post-smoothing steps (from 6 to 12), we observe the reduced AMG iteration numbers by almost a factor 2.

6.5. **Adaptive error control.** For test purpose, we use the adaptive strategy in Section 4.4 to control the error $\varepsilon_2$ of the outer Newton iterations and the error $\varepsilon_{1,k}$ of the inner AMG iterations. We present the numerical results only for the second nonlinear DN FSI iteration at the first time step. For other DN iterations and time steps, we observe similar results.

For the fluid sub-problem, we use the AMG method with Braess-Sarazin smoother (8 pre- and post-smoothing steps). The control of the outer and inner iterations is prescribed in Table 1 and 2 for the coarse and fine meshes, respectively.

| Newton-It | $\varepsilon_2$ | $\varepsilon_{1,k}$ | #AMG-It |
|-----------|----------------|---------------------|---------|
| $k = 1$   | 9.2e+01       | 4.1e-02             | 1       |
| $k = 2$   | 6.2e-03       | 1.8e-03             | 1       |
| $k = 3$   | 6.6e-06       | 1.6e-05             | 2       |
| $k = 4$   | 3.1e-10       | 1.6e-11             | 5       |

**Table 1.** Adaptive error control on the outer (Newton: $\varepsilon_2$) and inner (AMG: $\varepsilon_{1,k}$) iterations for the fluid sub-problem on the coarse mesh.

| Newton-It | $\varepsilon_2$ | $\varepsilon_{1,k}$ | #AMG-It |
|-----------|----------------|---------------------|---------|
| $k = 1$   | 9.4e+01       | 7.1e-02             | 1       |
| $k = 2$   | 8.6e-03       | 1.2e-02             | 1       |
| $k = 3$   | 4.4e-05       | 4.6e-06             | 3       |
| $k = 4$   | 7.9e-10       | 1.5e-10             | 6       |

**Table 2.** Adaptive error control on the outer (Newton: $\varepsilon_2$) and inner (AMG: $\varepsilon_{1,k}$) iterations for the fluid sub-problem on the fine mesh.

It is easy to observe from the numerical results in Table 1 and 2, the adaptive error control requires fewer inner AMG iterations (#AMG-It) without deterioration of Newton convergence rate of the outer iteration.
For the structure sub-problem, we use the AMG method with Vanka smoother (12 pre- and post-smoothing steps). The control of the outer and inner iterations for the modified model of Mooney-Rivlin material is shown in Table 3 and 4 for the coarse and fine meshes, respectively. The control of the outer and inner iterations for the two-layer thick-walled arterial model is presented in Table 5 and 6 for the coarse and fine meshes, respectively.

| Newton-It | $\varepsilon_2$ | $\varepsilon_{1,k}$ | #AMG-It |
|-----------|-----------------|--------------------|---------|
| $k = 1$   | 2.4e+01         | 9.2e-07            | 1       |
| $k = 2$   | 1.5e-02         | 4.0e-06            | 1       |
| $k = 3$   | 1.9e-06         | 1.1e-05            | 2       |
| $k = 4$   | 2.6e-11         | 3.3e-13            | 5       |

Table 3. Adaptive error control on the outer (Newton: $\varepsilon_2$) and inner (AMG: $\varepsilon_{1,k}$) iterations for the structure sub-problem on the coarse mesh using the modified model of Mooney-Rivlin material.

| Newton-It | $\varepsilon_2$ | $\varepsilon_{1,k}$ | #AMG-It |
|-----------|-----------------|--------------------|---------|
| $k = 1$   | 2.5e+01         | 6.4e-03            | 1       |
| $k = 2$   | 1.8e-01         | 1.9e-02            | 1       |
| $k = 3$   | 2.8e-03         | 2.8e-02            | 1       |
| $k = 4$   | 1.1e-04         | 1.4e-06            | 4       |
| $k = 5$   | 1.5e-09         | 1.8e-09            | 7       |

Table 4. Adaptive error control on the outer (Newton: $\varepsilon_2$) and inner (AMG: $\varepsilon_{1,k}$) iterations for the structure sub-problem on the fine mesh using the modified model of Mooney-Rivlin material.

From the numerical results in Table 3-6, we observe the satisfying convergence rate of the outer iterations with the adaptive error control, that requires fewer inner AMG iterations (#AMG-It). The number of Newton iterations is very close to what we obtained by using the inner AMG method with the fixed error $\varepsilon_1 = 10^{-8}$.

7. Conclusions

We test a partitioned approach for the FSI simulation using two hyperelastic models with near-incompressibility constrains. The numerical results show the robustness and efficiency of the partitioned
approach combined with Newton’s method to tackle the nonlinear fluid and structure sub-problems. Stabilized finite element methods, efficient Krylov subspace and AMG methods are used to handle the fluid and structure sub-problems, that demonstrate the feasibility of our methodology to handle such highly nonlinear systems. The AMG method shows more robustness than the preconditioned Kyrlov subspace method. The adaptive error control for the nonlinear problems requires fewer inner AMG iterations without deterioration of the convergence rate of the outer Newton iterations.

8. Acknowledgement

We would like to thank Dr. Christoph Augustin and Prof. Gerhard A. Holzapfel for the discussion on modeling of biological tissues. Special thanks go to Prof. Jean-Frédéric Gerbeau and Prof. Johan Hoffman for the discussion on FSI modeling.
References

[1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, vol. 140 of Pure and Applied Mathematics, Academic Press, Amsterdam, Boston, second ed., 2003.

[2] C. Augustin and O. Steinbach, FETI methods for the simulation of biological tissues, in Domain Decomposition Methods in Science and Engineering XX, R. Bank, M. Holst, O. Widlund, and J. Xu, eds., vol. 91 of Lecture Notes in Computational Science and Engineering, Springer Berlin Heidelberg, 2013, pp. 503–510.

[3] I. Babuška, The finite element method with Lagrangian multipliers, Num. Math., 20 (1973), pp. 179–192.

[4] S. Badia, F. Nobile, and C. Vergara, Fluid-structure partitioned procedures based on Robin transmission conditions, Journal of Computational Physics, 227 (2008), pp. 7027–7051.

[5] ———, Robin-Robin preconditioned Krylov methods for fluid-structure interaction problems, Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 1768–2784.

[6] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Archive for Rational Mechanics and Analysis, 63 (1976), pp. 337–403.

[7] D. Balzani, P. Neff, J. Schröder, and G. Holzapfel, A polyconvex framework for soft biological tissues. Adjustment to experimental data, International Journal of Solids and Structures, 43 (2006), pp. 6052 – 6070.

[8] J. Bayer, R. Blake, G. Plank, and N. Trayanova, A novel rule-based algorithm for assigning myocardial fiber orientation to computational heart models, Annals of Biomedical Engineering, 40 (2012), pp. 2243–2254.

[9] Y. Bazilevs, V. Calo, T. Hughes, and Y. Zhang, Isogeometric fluid-structure interaction: theory, algorithms, and computations, Computational Mechanics, 43 (2008), pp. 3–37.

[10] Y. Bazilevs, M.-C. Hsu, Y. Zhang, W. Wang, T. Kvamsdal, S. Hentschel, and J. Isaksen, Computational vascular fluid-structure interaction: methodology and application to cerebral aneurysms, Biomechanics and Modeling in Mechanobiology, 9 (2010), pp. 481–498.

[11] C. Bertoglio, P. Moireau, and J.-F. Gerbeau, Sequential parameter estimation for fluid-structure problems: Application to hemodynamics, International Journal for Numerical Methods in Biomedical Engineering, 28 (2012), pp. 434–455.

[12] J. Bonet and R. Wood, Nonlinear Continuum Mechanics for Finite Element Analysis, Cambridge University Press, 2008.

[13] D. Braess and R. Sarazin, An efficient smoother for the stokes problem, Applied Numerical Mathematics, 23 (1997), pp. 3 – 19.

[14] F. Brezzi, On the existence uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, RAIRO, 2 (1974), pp. 129–151.

[15] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer series in computational mathematics, Springer, New York, 1991.

[16] A. N. Brooks and T. J. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Computer Methods in Applied Mechanics and Engineering, 32 (1982), pp. 199 – 259.
38 ULRICH LANGER AND HUIDONG YANG

[17] P. CAUSIN, J. F. GERBEAU, AND F. NOBILE, Added-mass effect in the design of partitioned algorithm for fluid-structure problems, Comp. Meth. Appl. Mech. Eng., 194 (2005), pp. 4506–4527.

[18] P. G. CIARLET, Mathematical Elasticity Volume I: Three-Dimensional Elasticity, vol. 20 of Studies in Mathematics and Its Applications, North-Holland, New York, 1988.

[19] P. CROSETTO, P. REYMOND, S. DEPARIS, D. KONTAXAKIS, N. STERGIOPULOS, AND A. QUARTERONI, Fluid-structure interaction simulation of aortic blood flow, Computers & Fluids, 43 (2011), pp. 46 – 57.

[20] W. DETTMER AND D. PERIĆ, A computational framework for fluid-structure interaction: Finite element formulation and applications, Computer Methods in Applied Mechanics and Engineering, 195 (2006), pp. 5754 – 5779.

[21] P. DEUFLHARD, Newton Methods for Nonlinear Problems, vol. 35 of Springer series in computational mathematics, Springer, Heidelberg, 2005.

[22] J. DONEA, A. HUERTA, J. PONTHOT, AND A. FERRAN, Arbitrary Lagrangian-Eulerian methods, in The Encyclopedia of Computational Mechanics, E. Stein, R. Borst, and T. Hughes, eds., vol. 1, Wiley& Sons, Ltd, 2004, pp. 413–437.

[23] J.-J. DROUX AND T. J. HUGHES, A boundary integral modification of the Galerkin least squares formulation for the Stokes problem, Computer Methods in Applied Mechanics and Engineering, 113 (1994), pp. 173 – 182.

[24] A. EL MALIKI, M. FORTIN, N. TARDIEU, AND A. FORTIN, Iterative solvers for 3D linear and nonlinear elasticity problems: Displacement and mixed formulations, International Journal for Numerical Methods in Engineering, 83 (2010), pp. 1780–1802.

[25] A. EL MALIKI AND R. GUNETTE, Efficient preconditioning techniques for finite-element quadratic discretization arising from linearized incompressible Navier-Stokes equations, International Journal for Numerical Methods in Fluids, 63 (2010), pp. 1394–1420.

[26] H. ELMAN, D. SILVESTER, AND A. J. WATHEN, Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics, Oxford University Press, 1 ed., 2005.

[27] L. FORMAGGIA AND F. NOBILE, A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements, East-West Journal of Numerical Mathematics, 7 (1999), pp. 105–132.

[28] C. FÖRSTER, Robust methods for fluid-structure interaction with stabilised finite elements, PhD thesis, University Stuttgart, 2007.

[29] C. FÖRSTER, W. WALL, AND E. RAMM, Stabilized finite element formulation for incompressible flow on distorted meshes, International Journal for Numerical Methods in Fluids, 60 (2009), pp. 1103–1126.

[30] L. P. FRANCA AND R. STENBERG, Error analysis of some Galerkin least squares methods for the elasticity equations, SIAM Journal on Numerical Analysis, 28 (1991), pp. 1680–1697.

[31] T. C. GASSER, R. W. OGDEN, AND G. A. HOLZAPFEL, Hyperelastic modelling of arterial layers with distributed collagen fibre orientations, Journal of The Royal Society, 3 (2006), pp. 15–35.
[32] S. Goenezen, P. Barbone, and A. A. Oberai, Solution of the nonlinear elasticity imaging inverse problem: The incompressible case, Computer Methods in Applied Mechanics and Engineering, 200 (2011), pp. 1406 – 1420.

[33] G. Haase and U. Langer, Modern Methods in Scientific Computing and Applications, vol. 75 of NATO Science Series II. Mathematics, Physics and Chemistry, Kluwer Academic Press, Dordrecht, 2002, ch. Multigrid Methods: From Geometrical to Algebraic Versions, pp. 103–154.

[34] G. Holzapfel, T. Gasser, and M. Stadler, A structural model for the viscoelastic behavior of arterial walls: Continuum formulation and finite element analysis, European Journal of Mechanics - A/Solids, 21 (2002), pp. 441 – 463.

[35] G. A. Holzapfel, Nonlinear Solid Mechanics: A Continuum Approach for Engineering, Wiley, 2000.

[36] G. A. Holzapfel, T. C. Gasser, and R. W. Ogden, A new constitutive framework for arterial wall mechanics and a comparative study of material models, Journal of elasticity and the physical science of solids, 61 (2000), pp. 1–48.

[37] M. Hsu and Y. Bazilevs, Blood vessel tissue prestress modeling for vascular fluid-structure interaction simulation, Finite Elements in Analysis and Design, 47 (2011), pp. 593 – 599.

[38] T. J. Hughes, L. P. Franca, and M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the babuska-brezzi condition: a stable Petrov-Galerkin formulation of the stokes problem accommodating equal-order interpolations, Computer Methods in Applied Mechanics and Engineering, 59 (1986), pp. 85 – 99.

[39] T. J. Hughes, L. P. Franca, and G. M. Hulbert, A new finite element formulation for computational fluid dynamics: VIII. The galerkin/least-squares method for advective-diffusive equations, Computer Methods in Applied Mechanics and Engineering, 73 (1989), pp. 173 – 189.

[40] T. J. Hughes, W. K. Liu, and T. K. Zimmermann, Lagrangian-eulerian finite element formulation for incompressible viscous flows, Computer Methods in Applied Mechanics and Engineering, 29 (1981), pp. 329 – 349.

[41] V. John, On the efficiency of linearization schemes and coupled multigrid methods in the simulation of a 3D flow around a cylinder, International Journal for Numerical Methods in Fluids, 50 (2006), pp. 845–862.

[42] V. John and G. Matthies, Higher-order finite element discretizations in a benchmark problem for incompressible flows, International Journal for Numerical Methods in Fluids, 37 (2001), pp. 885–903.

[43] M. Jung and U. Langer, Application of multilevel methods to practical problems, Surveys on Mathematics for Industry, 1 (1991), pp. 217–257.

[44] ——, Methode der finiten Elemente für Ingenieure, Verlag Springer Vieweg, 2013.

[45] M. Jung, U. Langer, A. Meyer, W. Queck, and M. Schneider, Multigrid preconditioners and their applications, in Third Multigrid Seminar, Biesenthal 1988, G. Telschow, ed., no. Report R-MATH-03/89, Berlin, 1989, Karl–Weierstrass–Institut, pp. 11–52.
[46] F. Kickinger, Algebraic multigrid for discrete elliptic second-order problems, in Multigrid Methods V. Proceedings of the 5th European Multigrid conference (ed. by W. Hackbush), Lecture Notes in Computational Sciences and Engineering, vol. 3, Springer, 1998, pp. 157–172.

[47] O. Klaas, A. Maniatty, and M. S. Shephard, A stabilized mixed finite element method for finite elasticity: Formulation for linear displacement and pressure interpolation, Computer Methods in Applied Mechanics and Engineering, 180 (1999), pp. 65–79.

[48] U. Küttler and W. A. Wall, Fixed-point fluid-structure interaction solvers with dynamic relaxation, Comput. Mech., 43 (2008), pp. 61–72.

[49] A. M. Maniatty, Y. Liu, O. Klaas, and M. S. Shephard, Higher order stabilized finite element method for hyperelastic finite deformation, Computer Methods in Applied Mechanics and Engineering, 191 (2002), pp. 1491 – 1503.

[50] B. Metsch, Algebraic Multigrid (AMG) for Saddle Point Systems, PhD thesis, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2013.

[51] P. Moireau, C. Bertoglio, N. Xiao, C. Figueroa, C. Taylor, D. Chapelle, and J.-F. Gerbeau, Sequential identification of boundary support parameters in a fluid-structure vascular model using patient image data, Biomechanics and Modeling in Mechanobiology, 12 (2013), pp. 475–496.

[52] P. Moireau, N. Xiao, M. Astorino, C. Figueroa, D. Chapelle, C. Taylor, and J.-F. Gerbeau, External tissue support and fluid-structure simulation in blood flows, Biomechanics and Modeling in Mechanobiology, 11 (2012), pp. 1–18.

[53] N. Newmark, A method of computation for structural dynamics, Journal of Engineering Mechanics, 85 (EM3) (1959), pp. 67–94.

[54] R. Ogden, Non-linear Elastic Deformations, Dover Publications, 1997.

[55] C. Pechstein, Finite and boundary element tearing and interconnecting solvers for multiscale problems, vol. 90 of Lecture Notes in Computational Science and Engineering, Springer, Heidelberg, 2013.

[56] J. W. Ruge and K. Stüben, Algebraic multigrid (AMG), in Multigrid Methods, vol. 3 of Frontiers in Applied Mathematics, SIAM, Philadelphia, 1987, pp. 73–130.

[57] Y. Saad, Iterative Methods for Sparse Linear Systems, Society for Industrial and Applied Mathematics, 2 ed., 2003.

[58] J. Schöberl, NETGEN - An advancing front 2D/3D-mesh generator based on abstract rules, Computing and Visualization in Science, 1 (1997), pp. 41–52.

[59] J. Schöberl and W. Zulehner, On Schwarz-type smoothers for saddle point problems, Numerische Mathematik, 95 (2003), pp. 377–399.

[60] C. M. Scotti, A. D. Shkolnik, S. C. Muluk, and E. A. Finol, Fluid-structure interaction in abdominal aortic aneurysms: effects of asymmetry and wall thickness, Biomedical Engineering OnLine, 4 (2005), pp. 1–22.

[61] K. Takizawa, H. Takagi, T. E. Tezduyar, and R. Torii, Estimation of element-based zero-stress state for arterial fsi computations, Computational Mechanics, (2013), pp. 1–16.

[62] T. Tezduyar, S. Mittal, S. Ray, and R. Shih, Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity-pressure elements, Computer Methods in Applied Mechanics and Engineering, 95 (1992), pp. 221 – 242.
[63] T. E. Tezduyar and Y. Osawa, *Finite element stabilization parameters computed from element matrices and vectors*, Computer Methods in Applied Mechanics and Engineering, 190 (2000), pp. 411–430.

[64] R. Torii, M. Oshima, T. Kobayashi, K. Takagi, and T. E. Tezduyar, *Fluid-structure interaction modeling of a patient-specific cerebral aneurysm: influence of structural modeling*, Computational Mechanics, 43 (2008), pp. 151–159.

[65] R. Torii, M. Oshima, T. Kobayashi, K. Takagi, and T. E. Tezduyar, *Influence of wall thickness on fluid-structure interaction computations of cerebral aneurysms*, International Journal for Numerical Methods in Biomedical Engineering, 26 (2010), pp. 336–347.

[66] S. Turek, *Efficient Solvers for Incompressible Flow Problems*, Springer, Berlin, first ed., 1999.

[67] H. van der Vorst, *BI-CGSTAB: a fast and smoothly converging variant of BI-CG for the solution of nonsymmetric linear systems*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 631–644.

[68] S. Vanka, *Block-implicit multigrid solution of navier-stokes equations in primitive variables*, Journal of Computational Physics, 65 (1986), pp. 138–158.

[69] P. S. Vassilevski, *Multilevel Block Factorization Preconditioners*, Springer, 2008.

[70] M. Wabro, *Algebraic Multigrid Methods for the Numerical Solution of the Incompressible Navier-Stokes Equations*, PhD thesis, Johannes Kepler University Linz, 2003.

[71] M. Wabro, *Coupled algebraic multigrid methods for the Oseen problem*, Computing and Visualization in Science, 7 (2004), pp. 141–151.

[72] M. Wabro, *AMGe—coarsening strategies and application to the Oseen equations*, SIAM Journal on Scientific Computing, 27 (2006), pp. 2077–2097.

[73] W. A. Wall, *Fluid-Struktur-Interaktion mit stabilisierten Finiten Elementen*, PhD thesis, Universität Stuttgart, 1999.

[74] T. Wick, *Fluid-structure interactions using different mesh motion techniques*, Computers & Structures, 89 (2011), pp. 1456 – 1467.

[75] P. Wiggers, *Computational Contact Mechanics*, Springer, 2006.

[76] H. Wobker and S. Turek, *Numerical studies of Vanka-type smoothers in computational solid mechanics*, Advances in Applied Mathematics and Mechanics, 1 (2009), pp. 29–55.

[77] H. Yang, *Partitioned solvers for the fluid-structure interaction problems with a nearly incompressible elasticity model*, Computing and Visualization in Science, 14 (2011), pp. 227–247.

[78] H. Yang, *Improved robustness of the fluid-structure interaction simulation with a nearly incompressible elasticity model*, Tech. Rep. 2012-19, Johann Radon Institute for Computational and Applied Mathematics (RICAM), 2012.

[79] H. Yang and W. Zulehner, *Numerical simulation of fluid-structure interaction problems on hybrid meshes with algebraic multigrid methods*, Journal of Computational and Applied Mathematics, 235 (2011), pp. 5367 – 5379.

[80] Q. Zhang and B. Zhu, *An integrated coupling framework for highly nonlinear fluid-structure problems*, Computers & Fluids, 60 (2012), pp. 36 – 48.

[81] W. Zulehner, *A class of smoothers for saddle point problems*, Computing, 65 (2000), pp. 227–246.
Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences and Institute of Computational Mathematics, Johannes Kepler University, Altenberger Strasse 69, A-4040 Linz, Austria

E-mail address: ulanger@numa.uni-linz.ac.at
URL: http://www.numa.uni-linz.ac.at/~ulanger/

Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Strasse 69, A-4040 Linz, Austria

E-mail address: huidong.yang@oeaw.ac.at
URL: http://people.ricam.oeaw.ac.at/h.yang/