CONSTRUCTION AND EXAMPLES OF HIGHER-DIMENSIONAL MODULAR CALABI-YAU MANIFOLDS

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Abstract. We construct several examples of higher-dimensional Calabi-Yau manifolds and prove their modularity.

1. Introduction

As a consequence of Wiles’ proof of the Taniyama-Shimura-Weil conjecture [20] there has been considerable interest in the modularity of Calabi-Yau manifolds in recent years.

The case of dimension 2 was first considered by Shioda and Inose [17] who studied $K3$ surfaces with maximal Picard number, so-called singular $K3$ surfaces. They showed that these surfaces can be defined over number fields and computed their Hasse-Weil zeta-function. In the case of a singular $K3$ surface the transcendental lattice is 2-dimensional. If the surface is defined over $\mathbb{Q}$, then Livn´e [11] showed that the corresponding 2-dimensional Galois representation is related to a weight 3 modular form.

In dimension 3 rigid Calabi-Yau manifolds are simplest in the sense that they have 2-dimensional middle cohomology. By a variant of the Fontaine-Mazur conjecture [7], also asked by Yui (see e. g. [21] for a recent account), one expects that the middle cohomology of a rigid Calabi-Yau threefold defined over $\mathbb{Q}$, gives rise to an $L$-series, which is that of a weight 4 modular form. After numerous examples by various authors were exhibited, Dieulefait and Manoharmayum [6] proved the modularity conjecture for rigid Calabi-Yau threefolds under mild conditions on the primes of bad reduction. Examples and results about non-rigid modular Calabi-Yau threefolds can be found e.g. in [8], [9]. For a very recent survey, including lists of practically all known examples we refer the reader to the book by Meyer [13].

However, practically no examples seem to be known in higher dimension, and it is the aim of this paper to fill this gap. The first type of
examples we give, arises inductively from the Kummer construction described in Proposition 2.1 in section 2. The manifolds obtained in this way are resolutions of quotients of products of Calabi-Yau manifolds by a group of the form $\mathbb{Z}_n^2$. With this method one can construct several examples of modular Calabi-Yau manifolds (in any dimension). The middle cohomology (if the dimension is odd), resp. the transcendental lattice (if the dimension is even) is a tensor product of the middle cohomologies of modular Calabi-Yau manifolds of lower dimension. In some cases this tensor product (or, more precisely, its semi-simplification) splits into 2-dimensional modular pieces.

In order to obtain higher dimensional Calabi-Yau manifolds with small (e. g. 2-dimensional) middle cohomology, one has to refine the Kummer construction by taking quotients with respect to bigger groups. We consider suitable actions of the groups $G = \mathbb{Z}_n^3$ or $\mathbb{Z}_n^4$ and discuss this in particular in the case of quotients of the form $(E \times \ldots \times E)/G$, where $E$ is an elliptic curve with extra automorphisms (see Sections 3 and 4). We show that these quotients have a smooth Calabi-Yau model, whose middle cohomology (if the dimension is odd), resp. the transcendental lattice (if the dimension is even) is 2-dimensional. Moreover we show modularity and determine the corresponding cusp forms.

The final example, which we discuss, goes back to Ahlgren [2]. He considers a 5-dimensional affine variety $X$ which is a double cover of 5-space branched along 12 hyperplanes and relates the number of points of $X(\mathbb{F}_p)$ to the cusp form $g_6(q) = \eta(q^2)^{12}$ of weight 6 and level 4. We prove in Theorem 5.1 that $X$ has a smooth projective model which is a 5-dimensional Calabi-Yau manifold with $b_1 = b_3 = 0$ and $b_5 = 2$, whose $L$-series of the middle cohomology is that of the weight form $g_6$.

Acknowledgements. We are grateful to the DFG for support under grant Hu 337/5-2 in the frame of the Schwerpunktprogramm SPP 1084 “Globale Methoden in der komplexen Geometrie”. This grant supported the stay of the first named author at the University of Hannover, who would like to thank this University for kind hospitality and excellent working conditions. We would also like to thank M. Schütt for numerous discussions.

2. The Kummer construction

We start by generalizing the Kummer construction, which has been used to construct Calabi-Yau threefolds as quotients of the product of a $K3$ surface with an involution and an elliptic curve modulo the diagonal involution. To begin with, let $Y$ be a projective manifold of dimension $n$ with $H^q(\mathcal{O}_Y) = 0$ for $q > 0$ and let $D \in | -2K_Y|$ be a
smooth divisor. The line bundle $-K_Y$ defines a double covering

$$
\pi : X \longrightarrow Y
$$

branched along the divisor $D$, and

$$
K_X = \pi^*(K_Y + (-K_Y)) = 0.
$$

Moreover, since

$$
\pi^*(\mathcal{O}_X) = \mathcal{O}_Y \oplus K_Y
$$

it follows that for $0 < q < n$:

$$
H^q(\mathcal{O}_X) \cong H^q(\mathcal{O}_Y) \oplus H^q(K_Y) \cong H^q(\mathcal{O}_Y) \oplus H^{n-q}(\mathcal{O}_Y) = 0
$$

and, therefore, the variety $X$ is a Calabi-Yau manifold.

Now assume that we have a pair $Y_i, i = 1, 2$ of algebraic manifolds of dimension $n_i$, together with smooth divisors $D_i \in |-2K_{Y_i}|$. Moreover, assume that $H^q(\mathcal{O}_{Y_i}) = 0$ for $i = 1, 2$ and $q > 0$ and let $X_i$ be the double covers described above. By construction, the product $X_1 \times X_2$ admits an action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

**Proposition 2.1.** Under the above assumptions the quotient of the product $X_1 \times X_2$ by the diagonal involution admits a crepant resolution $X$, which is a (smooth) Calabi-Yau manifold. Moreover, there is a double cover $X \rightarrow Y$, branched along a smooth divisor $D$ with $H^q(\mathcal{O}_Y) = 0$ for $q > 0$.

**Proof.** The resolution may be described as follows: we start with the blow-up

$$
\sigma : Y \longrightarrow Y_1 \times Y_2
$$

of $Y_1 \times Y_2$ along $D_1 \times D_2$. Denote the exceptional divisor by $E$ and let

$$
D = \sigma^*(D_1 \times Y_2 + Y_1 \times D_2) - 2E
$$

be the strict transform of $D_1 \times Y_2 \cup Y_1 \times D_2$. Since $D_1 \times Y_2$ and $Y_1 \times D_2$ intersect transversally along $D_1 \times D_2$, the divisor $D$ is smooth and isomorphic to the disjoint union of $D_1 \times Y_2$ and $Y_1 \times D_2$.

Moreover $D = \sigma^*(D_1 \times Y_2 + Y_1 \times D_2) - 2E \sim \sigma^*(\pi_1^*(-2K_{Y_1}) + \pi_2^*(-2K_{Y_2})) - 2E = \sigma^*(-2K_{Y_1} \times Y_2) - 2E \sim -2K_Y$. Since $Y$ and $Y_1 \times Y_2$ are smooth birational projective manifolds $H^q(\mathcal{O}_Y) \cong H^q(\mathcal{O}_{Y_1 \times Y_2}) = 0$ for $q > 0$ (by the K"unneth decomposition), and hence the double cover $X$ of $Y$ branched along $D$ is a Calabi-Yau manifold.

Clearly, $X$ is birational to the quotient of $X_1 \times X_2$ by the action of the diagonal in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. More precisely, the fixed point set of the diagonal involution is the inverse image $B$ of $D_1 \times D_2$. The quotient by the diagonal involution has transversal $A_1$-singularities along the image of $B$. Let $Z$ be the blow-up of $X_1 \times X_2$ along $B$. The involution
lifts to $Z$ with fixed point set equal to the exceptional divisor $\tilde{B}$, which is ruled over $B$. The quotient of $Z$ by this involution is isomorphic to $X$, i.e., we have a commutative diagram of the form

$$
\begin{array}{c}
X_1 \times X_2 / \mathbb{Z}_2 \leftarrow Z / \mathbb{Z}_2 = X \\
\downarrow \quad \downarrow \pi \\
Y_1 \times Y_2 \leftarrow Y
\end{array}
$$

where the horizontal lines are inverse maps to blow-ups and the vertical lines are branched double covers. □

The above proposition allows us to use the covering $X \rightarrow Y$ inductively, and thus to construct higher-dimensional Calabi-Yau manifolds.

The Euler characteristic of $X$ depends not only on the Euler characteristics of $X_1$ and $X_2$, but also on the involution. By standard topological arguments we obtain

$$
e(X) = \frac{1}{2} e(X_1) e(X_2) + \frac{3}{2} e(D_1) e(D_2)
$$

and

$$
e(D) = \frac{1}{2} e(X_1) e(D_2) + \frac{1}{2} e(D_1) e(X_2) + e(D_1) e(D_2).
$$

In the special case, when $Y_2$ is an elliptic curve, branched over 4 points in $\mathbb{P}^1$, we have

$$
e(X) = 6 e(D_1), \quad e(D) = 2 e(X_1) + 4 e(D_1).
$$

**Example 2.1.** The case where $X_1$ is a $K3$-surfaces with an involution whose quotient is rational and $X_2$ is an elliptic curve was studied independently by Borcea [4] and Voisin [19] in the context of mirror symmetry. Already in this case we have several possibilities leading to different Euler numbers. Namely, $e(D_1)$ is an even integer ranging from $-18$ (for a smooth plane sextic) to 20 (10 lines coming from the resolution of six lines in $\mathbb{P}^2$ with four triple points). If $X_2$ is an elliptic curve we get Calabi-Yau 3-folds with Euler numbers equal to $-108, -96, -84, -72, -60, -48, -36, -24, -12, 0, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120$.

The Hodge numbers of $X$ cannot be computed in a similarly straight-forward way. If we know the Hodge numbers of $X_1$ and $X_2$, we can compute the Hodge numbers of $X_1 \times X_2$. The involution will kill the skew-symmetric part of the Hodge groups and preserve the symmetric part. But we also have to take into account the contribution to the cohomology coming from the blow-up of $B$ and describe the action of the involution on it.
Proposition 2.2. Let $X_1, \ldots, X_n$ be Calabi-Yau manifolds with involutions as above. The quotient of the product $X_1 \times \cdots \times X_n$ by the action of \{$(m_1, \ldots, m_n) \in \mathbb{Z}^n_2 | m_1 + \cdots + m_n = 0$\} $\cong \mathbb{Z}^{n-1}_2$ has a crepant resolution of singularities which is a Calabi-Yau manifold.

Proof. We shall proceed by induction on $n$. The case $n = 2$ follows from Proposition 2.1. Since the resulting Calabi-Yau manifold has again an involution, we can iterate the procedure. For a sequence of Calabi-Yau manifolds $X_i$ with involution we have the following factorization

$$(X_1 \times \cdots \times X_n)/\mathbb{Z}^{n-1}_2 \cong ((X_1 \times \cdots \times X_n)/\mathbb{Z}^{n-2}_2) / (\mathbb{Z}^{n-1}_2/\mathbb{Z}^{n-2}_2)$$

where $\mathbb{Z}^{n-2}_2$ denotes the group \{$(m_1, \ldots, m_n) \in \mathbb{Z}^n_2 | m_1 + \cdots + m_{n-1} = m_n = 0$\}. Consequently

$$(X_1 \times \cdots \times X_n)/\mathbb{Z}^{n-1}_2 \cong (((X_1 \times \cdots \times X_{n-1})/\mathbb{Z}^{n-2}_2) \times X_n)/\mathbb{Z}_2,$$

which proves the proposition. \hfill $\Box$

Corollary 2.3. Let $E_i; i = 1, \ldots, n$ be elliptic curves. The quotient $E_1 \times \cdots \times E_n$ by the action of $\mathbb{Z}^{n-1}_2$ has a smooth model $X^n$ which is a Calabi-Yau manifold with Euler characteristic $e(X^n) = \frac{1}{2}(6^n + 3(−2)^n)$.

We would like to remark that quotients of the form $(E_1 \times E_2 \times E_3)/\mathbb{Z}^2_n$ were first considered by Borcea [3], who also proved that the resulting Calabi-Yau threefolds have CM if and only if the factors $E_i$ have CM.

Lemma 2.4. If $n$ is odd, then

$$H^n(X^n) \cong H^n(E_1 \times \cdots \times E_n)^{\mathbb{Z}^{n-1}_2} \cong H^1(E_1) \otimes \cdots \otimes H^1(E_n).$$

For $n$ even the (invariant) submotive $H^n(E_1 \times \cdots \times E_n)^{\mathbb{Z}^{n-1}_2}$ of $H^n(X^n)$ is isomorphic to the direct sum of a submotive generated by cycles of products of $n/2$ fibres and a submotive $I(X^n) \cong H^1(E_1) \otimes \cdots \otimes H^1(E_n)$. The motive $I(X^n)$ contains the transcendental submotive, i.e., the orthogonal complement to the algebraic cycles of $X^n$.

Proof. We first consider the invariant part of the middle cohomology of $E_1 \times \cdots \times E_n$. Any tensor product $\otimes j H^{i_j}(E_j)$ which contributes to this must have $\sum_j i_j = n$. Now assume that at least one $i_j = 1$. Then we must have that all $i_j = 1$, since otherwise one can find some $e \in \mathbb{Z}^{n-1}_2$ which acts by $−1$ on $\otimes j H^{i_j}(E_j)$. If $n$ is odd, then $\sum_j i_j = n$ can only occur if at least one, and hence, by the above argument, all $i_j = 1$. We finally remark that $X^n$ is of the form $Z_n/\mathbb{Z}^{n-1}_2$ where $Z_n$ arises from the product $E_1 \times \cdots \times E_n$ by blowing-up rational submanifolds. This only contributes to the even cohomology and this contribution is spanned by algebraic cycles. \hfill $\Box$
This discussion easily implies the

**Proposition 2.5.** Assume that the $E_i$ are defined over $\mathbb{Q}$ with the involution given as $x \mapsto -x$ and let $L(X^n, s)$, resp. $L(I(X^n), s)$ be the L-series associated to the Galois action on $H^n(X^n)$ for $n$ odd and the submotive $I(X^n)$ for $n$ even. Then

$$L(X^n, s) = L(g_{E_1} \otimes \cdots \otimes g_{E_n}, s),$$

resp. $L(I(X^n), s) = L(g_{E_1} \otimes \cdots \otimes g_{E_n}, s)$

where the $g_{E_i}$ are the cusp forms associated to $E_i$.

**Proof.** The only statement, which requires a proof, is that $X_n$ is defined over $\mathbb{Q}$. But this is clear, since the factors $E_i$, the involutions and the locus which is blown up are all defined over $\mathbb{Q}$. □

Here we consider $g_{E_1} \otimes \cdots \otimes g_{E_n}$ as the tensor product of Galois-modules. For the analytic properties of (some) tensor products see [10].

**Remark 2.6.** For a generic choice of elliptic curves $I(X^n)$ equals the transcendental submotive of $X^n$, whereas in special cases it may be strictly bigger. For instance, if the factors $E_{2i-1}$ and $E_{2i}$ ($i = 1, \ldots, n/2$) are isogeneous, then $I(X^n)$ contains the product of the graphs of isogenies. If, moreover, the $E_i$’s have complex multiplication then $I(X^n)$ contains also the product of graphs of complex multiplications. Note that this is in agreement with the appearance of the factors $L(s - \frac{d}{2})$ and $L(\chi_{-d}, s - \frac{n}{2})$ in the L-series given below.

We now specialize the situation even further and assume that all $E_i$ are isomorphic to an elliptic curve $E$ with complex multiplication in $\mathbb{Q}(\sqrt{-d})$. If $n$ is odd, then

$$L(X, s) = L(g_{n+1}, s)^{\binom{n}{2}} L(g_{n-1}, s - 1)^{\binom{n}{1}} \cdots L(g_2, s - \frac{n-1}{2})^{\binom{n}{(n-1)/2}}$$

and if $n$ is even, then

$$L(I(X), s) = L(g_{n+1}, s)^{\binom{n}{2}} L(g_{n-1}, s - 1)^{\binom{n}{1}} \cdots L(g_3, s - \frac{n-2}{2})^{\binom{n}{(n-2)/2}} \times L(\chi_{-d}, s - \frac{n}{2})^{\binom{n}{(n/2)}} L(s - \frac{n}{2})^{\frac{1}{2} \binom{n}{(n/2)}}.$$

Here $\zeta(\chi_{-d}, s)$ is the Dirichlet L-function defined by the character associated to the number field $K = \mathbb{Q}(\sqrt{-d})$, i.e $\chi_{-d}(p) = (\frac{-d}{p})$ and $g_k$ is the cusp form corresponding to the $(k-1)$st power of the Grössencharakter $\psi$ of the elliptic curve $E$ [13]. The cusp form $g_k$ has weight $k$ and complex multiplication in the same field as $E$. The Fourier coefficient $a_n(g_k)$ is given by the sum of the values of the Grössencharakter $\psi^{k-1}$ at the ideals in the ring $\mathcal{O}_K$ of integers in $K$ of norm $n$, relatively prime to the conductor of $E$. For a prime $p$ which is inert in $\mathcal{O}_K$, we get...
$a_p = 0$, because there is no ideal in $\mathcal{O}_K$ with norm $p$. For a split prime $p$ we have $p = \alpha_p \bar{\alpha}_p$ for some $\alpha_p \in \mathcal{O}_K$, which is determined by $E$. Then $a_p(g_k) = \alpha_p^{k-1} + \bar{\alpha}_p^{k-1}$, more explicitly, we have $a_p(g_3) = a_p^2 - 2p$, $a_p(g_4) = \alpha_p^3 - 3p\alpha_p$, $a_p(g_5) = \alpha_p^4 - 4p\alpha_p^2 + 2p^2$ and so on.

In terms of the associated Galois representations, the connection between the forms $g_k$ and $g_2$ can be described as follows. Consider the representation associated to $g_2$ and let $(\alpha_p, \bar{\alpha}_p)$ be the eigenvalues of Frob$_p$ for primes $p$ with $\chi_{-d}(p) = 1$. If $\chi_{-d}(p) = -1$, then the corresponding eigenvalues are $(ip^{\frac{k}{2}}, -ip^{\frac{k}{2}})$. The eigenvalues of $g_k$ are then $(\alpha_p^{k-1}, \bar{\alpha}_p^{k-1})$ for $\chi_{-d}(p) = 1$ and $(p^{\frac{k-1}{2}}, -p^{\frac{k-1}{2}})$ for $k$ odd and $\chi_{-d}(p) = -1$, resp. $(ip^{\frac{k-1}{2}}, -ip^{\frac{k-1}{2}})$ for $k$ even and $\chi_{-d}(p) = -1$.

We want to conclude this section by discussing one further example of our Kummer construction. As the first factor we choose the rigid Calabi-Yau 3-fold $X_3$, constructed as a resolution of singularities of the double covering of $\mathbb{P}^3$ branched along the following arrangement of eight planes

$$xt(x - z - t)(x - z + t)y(y + z - t)(y + z + t)(y + 2z) = 0.$$  

For a discussion of the properties of this (and other) double octics see [5] and [13, Octic Arr. No. 19]. As the second factor we take the $K3$ surface $S$ which is obtained as a desingularization of the double sextic branched along the following arrangement of six lines

$$xy(x + y + z)(x + y - z)(x - y + z)(x - y - z) = 0.$$  

Both $X_3$ and $S$ come with natural involutions which allow us to apply Proposition 2.1. In this way we obtain a smooth Calabi-Yau fivefold $X_5$, which is the quotient of a blow-up $\widetilde{X_3} \times \widetilde{S}$ of $X_3 \times S$ by an involution. So the Hodge groups of $X_5$ are the invariant part of the Hodge groups of $\widetilde{X_3} \times \widetilde{S}$. Since we blow up products of lines and blown-up planes, the odd-dimensional cohomology groups of $\widetilde{X_3} \times \widetilde{S}$ and $X_3 \times S$ are the same.

Now, the odd-dimensional cohomology groups of $S$ vanish, whereas the only odd-dimensional cohomology of $X_3$ is $H^3(X_3) = H^{3,0} \oplus H^{0,3}$, which is anti–invariant. The anti–invariant part of the cohomology of $S$ is $H^{2,0} \oplus H^{0,2} \cong T(S) \otimes_{\mathbb{Z}} \mathbb{C}$, where $T(S)$ is the transcendental lattice. Consequently, $b_1(X_5) = b_3(X_3) = 0$ and $b_5(X_4) = 4$ and moreover

$$H^5(X_5) \cong H^5(X_3) \otimes T(S).$$  

Recall that (see [11] and [13, p. 57])

$$L(T(S), s) \cong L(g_3, s), \quad L(X_3, s) \cong L(g_4, s)$$  

where \( g_3 \) and \( g_4 \) are the unique weight 3, resp. weight 4 Hecke eigenforms of level 16 and 32 with complex multiplication by \( i \). As usual \( \cong \) denotes equality up to a finite number of Euler factors. In concrete terms

\[
g_3(q) = \eta(q^4)^6 = q - 6q^5 + 9q^9 + 10q^{13} - 30q^{17} + \ldots
\]

and

\[
g_4(q) = q + 22q^5 - 27q^9 - 18q^{13} - 94q^{17} + 359q^{25} + \ldots
\]

where \( \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \) is the Dedekind \( \eta \)-function. Both of these forms can be derived from the unique weight 2 level 32 newform

\[
g_2(q) = \eta(q^8)^2 \eta(q^4)^2 = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} + \ldots
\]

by taking the second, resp. third power of the Grössencharakter of \( \mathbb{Q}[i] \) given as \( \psi((\alpha)) = \alpha \) for \( \alpha \in \mathbb{Z}[i], \alpha \equiv 1 \mod 2 + 2i \).

Hence we obtain that the L-series of \( X_5 \) is the product of the L-series associated to \( X_3 \) and \( S \) and we also find that it factors as

\[
L(X_5, s) \cong L(g_4 \otimes g_3, s) \cong L(g_6, s)L(g_2, s - 2)
\]

where \( g_2 \) is as above and \( g_6 \) is a level 32 cusp form of weight 6, namely

\[
g_6(q) = q - 82q^5 - 243q^9 - 1194q^{13} + 2242q^{17} + 3599q^{25} + \ldots
\]

which can be derived from \( g_2 \) by taking the fifth power of the Grössencharakter. Obviously, we can iterate this procedure to obtain modular Calabi-Yau manifolds of higher dimension (with increasingly complex middle cohomology).

3. **Calabi-Yau manifolds with an endomorphism of order 3**

We shall construct for any positive integer \( n \) a Calabi-Yau \( n \)-fold \( X_n \) with an endomorphism of order 3 such that \( \dim H^n(X_n) = 2 \) for \( n \) odd and \( \dim T(X_n) = 2 \) for \( n \) even, where \( T(X_n) \subset H^n(X_n) \) is the transcendental part. Moreover, we shall show that the (semi-simplifications of) the Galois representation on \( H^n(X_n) \) (resp. \( T(X_n) \)) and the Galois representation associated to a suitable cusp form with CM by \( \sqrt{-3} \) are isomorphic.

Fix the primitive third root of unity \( \zeta = e^{2\pi i/3} \). Let \( X_1 \) and \( X_2 \) be two Calabi-Yau manifolds admitting \( \mathbb{Z}_3 \)-actions, which do not preserve the canonical form. Moreover, assume that the fixed point set of the action on \( X_1 \) is a smooth divisor, whereas on \( X_2 \) it is a disjoint union of a smooth divisor and a smooth codimension two submanifold. Fix an automorphism \( \eta_1 \) of \( X_1 \) such that \( \eta_1^* \omega_{X_1} = \zeta \omega_{X_1} \) and an automorphism \( \eta_2 \) of \( X_2 \) such that \( \eta_2^* \omega_{X_2} = \zeta^2 \omega_{X_2} \) such that they act on \( X_1 \) and \( X_2 \) as described above. Then \( \eta_1 \) is given locally near the branch-divisor.
on $X_1$ as $(\zeta, 1, 1\ldots)$, whereas $\eta_2$ is given locally either as $(\zeta^2, 1, 1\ldots)$ near the branch divisor on $X_2$ or as $(\zeta, \zeta, 1\ldots)$ near the codimension 2 fixed locus.

On $X_1 \times X_2$ we have an action of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, and we consider the action of $\mathbb{Z}_3$ on $X_1 \times X_2$, given by the automorphism $\eta = \eta_1 \times \eta_2$.

**Proposition 3.1.** Under the above assumptions the quotient variety $X_1 \times X_2 / \mathbb{Z}_3$ has a resolution of singularities $X$, which is a Calabi-Yau manifold. The manifold $X$ admits a $\mathbb{Z}_3$-action which satisfies the same assumptions as for $X_2$.

**Proof.** The singularities of $X_1 \times X_2 / \mathbb{Z}_3$ correspond to the fixed locus of $\eta$, which is the cartesian product of the fixed point sets of $\eta_1$ and $\eta_2$. Consequently, we get two kind of singularities: a singular codimension two stratum $W_1$, which is a transversal $A_2$-singularity, and a codimension three stratum $W_2$, which is a transversal cone over a triple Veronese surface. Both types of singularities admit a crepant resolution (described explicitly below), and we denote the resulting manifold by $X$. Since the canonical form on $X_1 \times X_2$ is $\eta$-invariant it descends to the quotient, and thus to the crepant resolution. Consequently, we get $\omega_X \cong \mathcal{O}_X$.

Denote by $W_1$ (resp. $W_2$) the union of the codimension two (resp. three) strata of the fixed point set of $\eta$ and consider the blow-up $Z_1$ of $X_1 \times X_2$ along $W_1 \cup W_2$. Then $\eta$ lifts to $Z_1$ and the fixed point set is a codimension two subvariety lying over $W_1$ and a divisor over $W_2$. Let $Z_2$ be the blow-up of $Z_1$ along the codimension two fixed submanifold. Again, the action of $\mathbb{Z}_3$ lifts to $Z_2$ and the fixed point set is a divisor. So the quotient $Z$ of $Z_2$ by the action of $\mathbb{Z}_3$ is a smooth manifold and it is a blow-up of $X$. (In terms of the $A_2$-singularity, the difference between $Z$ and $X$ is, that we blow up the point of intersection of the two $(-2)$-curves which come from the resolution of the $A_2$-singularity.)

Now observe that

$$H^0(Z_2, \Omega_{Z_2}^q) = H^0(X_1 \times X_2, \Omega_{X_1 \times X_2}^q) = 0$$

for $q \neq 0, n_1, n_2, n_1 + n_2$ and hence, by taking the invariant part with respect to the action of $\eta$, we obtain that $H^0(Z, \Omega_Z^p) = H^0(X, \Omega_X^p) = 0$ for $p \neq 0, n_1 + n_2$. This proves that $X$ is a (smooth) Calabi-Yau manifold.

The action of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ on $X_1 \times X_2$ induces an action of $\mathbb{Z}_3$ on $X$ generated by the induced action of $\text{id} \times \eta_2$ to $X$. We shall study this action in local coordinates. For the transversal $A_2$-singularity we can find local coordinates on $X_1 \times X_2$ in which the action is given as $(\zeta, \zeta^2)$. Note that for simplicity we shall omit the coordinates on which $\eta$ acts trivially.
The quotient map is given by \((x_1, x_2) \mapsto (u_1, u_2, u_3) = (x_1^3, x_2^3, x_1x_2)\), and the image has equation \(u_3^3 = u_1u_2\). The resolution of singularities is given by blowing up the submanifold \(u_1 = u_2 = u_3 = 0\). In suitable charts on the blown up surface, the quotient map is then given by
\[
(y_1, y_2) \mapsto (y_1, y_1^2y_2^3, y_1y_2) \\
(y_1, y_2) \mapsto (y_1^3, y_2^2, y_1y_2) \\
(y_1, y_2) \mapsto (y_1^2y_2, y_1y_2^2, y_1y_2).
\]
The map from \(X_1 \times X_2\) to the resolution of the quotient in local analytic terms is given by \((x_1^3, \frac{x_2}{x_1^2}), (\frac{x_2}{x_1^2}, x_2^3), (\frac{x_2}{x_1^2}, \frac{x_3}{x_2})\), depending on the charts we work in. The action of \(\text{id} \times \eta_2\) is given on \(X_1 \times X_2\) as \((1, \zeta^2)\), so it lifts to \(X\) as \((1, \zeta, \zeta)\), \((\zeta^2, 1)\) or \((\zeta, \zeta)\) respectively, depending on the charts we consider.

For the cone over the Veronese triple embedding the resolution is given by the so-called canonical resolution \(\mathbb{R}^{16.10, p. 199}\). If the \(\text{id} \times \eta_2\) is given on \(X_1 \times X_2\) as \((1, \zeta^2)\), so it lifts to \(X\) as \((1, \zeta, \zeta)\), \((\zeta^2, 1)\) and \((\zeta, \zeta)\) respectively.

In all cases \(X\) satisfies the assumptions made for \(X_2\).

\[\square\]

**Remark 3.2.** We can now use the Calabi-Yau manifold \(X\) together with the \(\mathbb{Z}_3\)-action on it to repeat this constructions inductively.

We consider an elliptic curve defined over \(\mathbb{Q}\) with an automorphism of order 3, which we can, without loss of generality, assume to be in Weierstrass form \(y^2 = x^3 - D\). The automorphism \(\eta\) is given by \(x \mapsto \zeta x\).

**Theorem 3.3.** Let \(E\) be the elliptic curve with an automorphism \(\eta\) of order 3 and let \(\bar{X}_n\) be the quotient of \(E^n\) by the action of the group
\[\{(\eta^{a_1} \times \cdots \times \eta^{a_n}) \in \text{End}(E^n) : a_1 + \cdots + a_n \equiv 0 \mod 3\} .\]

Then \(\bar{X}_n\) has a smooth model \(X_n\), which is a Calabi-Yau manifold and \(\dim(H^n(X_n)) = 2\), if \(n\) is odd, resp. \(\dim(T(X_n)) = 2\), if \(n\) is even, where \(T(X_n)\) is the transcendental part of the cohomology.

Moreover, \(X_n\) is defined over \(\mathbb{Q}\) and \(L(H^n(X_n), s) \cong L(g_{n+1}, s)\), resp. \(L(T(X_n)) \cong L(g_{n+1}, s)\), where \(g_{n+1}\) is the weight \(n + 1\) cusp form with complex multiplication in \(\mathbb{Q}(\sqrt{-3})\), associated to the \(n\)-th power of the Grössencharakter of \(E\).

**Proof.** The claim about \(X_n\) being a Calabi-Yau manifold follows by repeated application of Proposition \(\text{3.1}\). To compute the middle cohomology, resp. its transcendental part, we first notice that it is enough
to compute the invariant part of the cohomology of $E^n$. This follows, since the divisors which we introduce by blowing up, are linear spaces blown-up in some subspaces, so their cohomology is generated by algebraic cycles. The subspace $\otimes H^{10}(E) \oplus \otimes H^{01}(E)$ is always invariant. If $n$ is odd, then, by an argument similar to the one we used in the proof of Proposition 2.4 this is the only contribution to the invariant part of $H^n(E^n)$. If $n$ is even we have, in addition, summands of the form $H^{i1}(E) \otimes \ldots \otimes H^{in}(E)$, where $i_k = 0$ or 2 and $\sum i_k = n$, which are also generated by algebraic cycles.

Now we turn to the arithmetic statements. We first note that $X_n$ is defined over $\mathbb{Q}$, since it is defined over $\mathbb{Q}(\sqrt{-3})$ and invariant under the Galois group. Since we blow up in submanifolds defined over $\mathbb{Q}$, the resolution $X_n$ is also defined over $\mathbb{Q}$.

The endomorphism $\eta$ induces endomorphisms $\eta_p : E(\bar{\mathbb{F}}_p) \rightarrow E(\bar{\mathbb{F}}_p)$ (for $p \neq 3, p \nmid D$), also of order 3. The induced endomorphisms $\eta_p$ have three fixed points, and hence the Lefschetz fixed point formula implies $\text{tr} \eta_p^* = -1$. Now, if $l \equiv 1 \mod 6$, then $\mathbb{Q}_l$ contains a primitive root of unity $\rho_l$ and the eigenvalues of $\eta_p^*$ are powers of $\rho_l$ which sum up to $-1$, and are, therefore, equal to $\rho_l$ and $\rho_l^2$. Denote by $v_1, v_2 \in H^1_{\text{et}}(E_p)$ the corresponding eigenvectors. It is easy to see that the subspace of $H^1_{\text{et}}(E_p)^{\otimes n}$ invariant under the action of $\mathbb{Z}_l^n$ is generated by $v_1^{\otimes n} = v_1 \otimes \cdots \otimes v_1$ and $v_2^{\otimes n} = v_2 \otimes \cdots \otimes v_2$, so we need to compute the images of the tensor power of Frobenius on $v_1^{\otimes n}$ and $v_2^{\otimes n}$. To this end, we shall need to compute the action of Frobenius $\text{Frob}_p^*$ in the base $v_1, v_2$.

We shall consider the cases $p \equiv 1, 5 \mod 6$ separately. For $p \equiv 1 \mod 6$ the Frobenius map $\text{Frob}_p^*$ commutes with $\eta_p^*$, so it acts as $v_1 \mapsto \alpha_p v_1$ and $v_2 \mapsto \alpha_p v_2$, where $\alpha_p$ and $\bar{\alpha}_p$ are the eigenvalues of $\text{Frob}_p^*$. Consequently, the eigenvalues of Frobenius on the invariant part of $H^1_{\text{et}}(E_p)^{\otimes n}$ equal $\alpha_p^2$ and $\bar{\alpha}_p^2$.

For $p \equiv 5 \mod 6$ we have $\text{Frob}_p^* \circ \eta_p^* = (\eta_p^*)^{-1} \circ \text{Frob}_p^*$, which easily implies that in the base $v_1, v_2$ Frobenius is given by the matrix $
abla \begin{pmatrix} 0 & \lambda \\ -\frac{\lambda^2}{\lambda} & 0 \end{pmatrix}$. Consequently, the action of Frobenius on the invariant subspace of $H^1_{\text{et}}(E_p)^{\otimes n}$ equals $\begin{pmatrix} 0 & \lambda^n \\ (-\frac{\lambda^2}{\lambda})^n & 0 \end{pmatrix}$, with eigenvalues equal to $\pm p^{n/2}$ for $n$ even, and $\pm ip^{n/2}$ for $n$ odd.

Taking all the cases together, we see that the Galois representation on $H^n(X^n)$ for $n$ odd, resp. $T(X^n)$ for $n$ even, has the same eigenvalues as the representation associated to the cusp form $g_{n+1}$ associated to the $n$-th power of the Grössencharakter of the elliptic curve $E$. $\square$
Remark 3.4. In the case where $E$ is given by the equation $y^2 = x^3 - 1/4$ the form $g_2$ is the unique weight 2 newform of level 27, namely

$$g_2(q) = \eta(q^9)^2 \eta(q^3)^2 = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \ldots$$

In this case

$$g_3(q) = q + 4q^4 - 13q^7 - q^{13} + 16q^{16} + 11q^{19} + 25q^{25} + \ldots$$

and

$$g_4(q) = \eta(q^3)^8 = q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} + \ldots$$

which are the unique level 27 and 9 forms of weight 3 and 4. Cusp forms $g_k$ correspond to powers of the Grössencharakter of the field $\mathbb{Q}(\sqrt{-3})$ given by $\psi(\alpha) = \alpha$ for $\alpha \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}], \alpha \equiv 1 \mod 3.$ For other models of $E$ one obtains appropriate twists of these forms.

4. Calabi-Yau manifolds with an endomorphism of order 4

In this section we shall construct a similar example as in the previous section, but with an endomorphism of order 4. Let $X_1$ and $X_2$ be two Calabi-Yau manifolds admitting $\mathbb{Z}_4$-actions $\eta_1$ and $\eta_2$. Assume that the fixed point set of $\eta_1$ is a divisor, near which the action has a linearization of the form $(1, i)$, whereas the fixed point set of $\eta_2$ is a disjoint union of submanifolds of codimension one, two or three, near which the action has a linearization as $(-i, 1, \ldots)$, $(-1, i, 1, \ldots)$ and $(i, i, i, 1 \ldots)$ respectively.

On $X_1 \times X_2$ we have an action of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, and we consider the action of $\mathbb{Z}_4$ on $X_1 \times X_2$ given by the automorphism $\eta = \eta_1 \times \eta_2$.

Proposition 4.1. Under the above assumptions the quotient $X_1 \times X_2/\mathbb{Z}_4$ has a resolution of singularities $X$, which is a Calabi-Yau manifold. The manifold $X$ admits a $\mathbb{Z}_4$-action which satisfies the same assumptions as for $X_2$.

Proof. We shall show that the quotient admits a crepant resolution of singularities. We shall consider the three cases, depending on the codimension of the component of the fix-point set of $\eta_2$, separately.

Near the fixed divisor of $\eta_2$ the action of $\eta$ on $X_1 \times X_2$ is locally given by $(i, -i)$ (as in similar proofs before, we omit the variables on which $\eta$ acts trivially). Consequently the quotient is a transversal $A_3$ singularity along the singular subvariety, which can be resolved by blowing-up twice. In local coordinates, the map from $X_1 \times X_2$ inverse to the resolution is given in affine charts as $(x^4, \frac{y}{x^3}), (y^4, \frac{x}{y^3}), (\frac{x^4}{y^3}, \frac{y^4}{x^3}), (\frac{x^4}{y^3}, \frac{y^4}{x^3})$ or $(\frac{x^2}{y^3}, xy).$ The action of $id \times \eta_2$ on $X_1 \times X_2$ has a linearization $(1, -i)$, so...
it lifts to the resolution as $(1, -i), (1, -i), (i, -1), (i, -1)$ or $(-1, -i)$. In all the cases, except the last one, the action is exactly as we assume for $X_2$, but in the last case the fixed point of the action is $(0, 0)$, which does not belong to the domain of the map.

Now consider the singularity corresponding to a codimension two fixed stratum of $\eta_2$. Then the action on $X_1 \times X_2$ has a local linearization of the form $(i, i, -1)$. We first divide by the square of $\eta$, which is an involution with fixed point set of codimension two resulting in transversal $A_1$-singularities. These we resolve by blowing-up the singular locus. The action of $\mathbb{Z}_4$ lifts to this resolution again as an involution with a codimension two fixed point set, leading once more to transversal $A_1$-singularities, which we resolve with a single blow-up. Simple computations show, that in terms of local coordinates, the map from $X_1 \times X_2$ inverse to the resolution looks in local coordinates like $(x^4, z^2, y, \frac{y}{z})$, $(x^2, x^2 z, y, \frac{y}{x})$, $(x, x^2 y^2, z, \frac{z}{x y})$, $(x, y^2, z, \frac{z}{x})$, $(z^2, \frac{z^2}{x}, x y z)$, $(y^4, \frac{y^4}{y}, x y z)$, $(z^2, \frac{y^2}{x}, \frac{z}{y})$, or $(\frac{x^2}{y}, y^2 z, \frac{z}{x})$. The action of $\text{id} \times \eta_2$ on $X_1 \times X_2$ is linearized by $(1, i, -1)$, so it lifts to the resolution as $(1, -1, i), (1, -1, i), (-1, 1, -i), (-i, 1, -i), (1, 1, -i), (1, 1, i)$ or $(1, 1, -i)$. The lifting satisfies the assumption made for $X_2$ in all except the cases 3, 4, 5 and 6, when the fixed points do not lie in the domain of the map.

The last case is the fixed point stratum of $\eta_2$ of codimension 3, so the action on $X_1 \times X_2$ has a local linearization of the form $(i, i, i, i)$. Here again, it is easier to resolve in one step. On the quotient we get a transversal cone over the Veronese fourfold embedding of $\mathbb{P}^3$. The crepant resolution is given by the so-called canonical resolution [18 (16.16, p. 199)] for which the inverse map is given as $(x^4, \frac{x}{z}, \frac{x}{y}, \frac{1}{x})$, $(x^2, \frac{x}{z}, \frac{z^4}{x}, \frac{z}{y})$, $(\frac{z^2}{x}, \frac{y^2}{x}, \frac{z}{x})$, and so the action of $\text{id} \times \eta_2$ lifts as $(1, i, i, i), (-i, 1, 1, 1), (-i, 1, 1, 1)$ and $(-i, 1, 1, 1)$ respectively, which completes the proof.

\[\{\eta^{a_1} \times \cdots \times \eta^{a_n}\} \in \text{End}(E^n) : a_1 + \cdots + a_n \equiv 0 \mod 4\].
Then $X_n$ has a smooth model $X_n$, which is a Calabi-Yau manifold and $\dim(H^n(X_n)) = 2$ if $n$ is odd, resp. $\dim(T(X_n)) = 2$ if $n$ is even, where $T(X_n)$ is the transcendental part of the cohomology.

Moreover, $X_n$ is defined over $\mathbb{Q}$ and $L(H^n(X_n), s) \cong L(g_{n+1}, s)$, resp. $L(T(X_n)) \cong L(g_{n+1}, s)$, where $g_{n+1}$ is a weight $n + 1$ cusp form with complex multiplication in $\mathbb{Q}(i)$.

Proof. The existence of a crepant resolution follows from repeated application of Proposition 4.1, and the remaining statements can be proved exactly in the same way as in the proof of Theorem 3.3. □

5. The example of Ahlgren

Let $\bar{X}$ be the double cover of $\mathbb{P}^5$ branched along the union of the twelve hyperplanes

$$x(x - u)(x - v)y(y - u)(y - v)z(z - u)(z - v)t(t - u)(t - v) = 0.$$  

This is a projective closure of the fivefold studied by Ahlgren [2]. He proved that the number of points defined over $\mathbb{F}_p$ on the affine part $(u = 1)$ of this variety equals

$$N(p) = p^5 + 2p^3 - 4p^2 - 9p - 1 - a_p,$$

where $a_p$ is the $p$-th Fourier coefficient of the unique normalized weight 6 and level 4 cusp form (which is equal to $\eta^{12}(q^2)$).

Our goal here is to prove the following

Theorem 5.1. The variety $\bar{X}$ has a smooth model $X$ (defined over $\mathbb{Q}$), which is a Calabi-Yau fivefold with Betti numbers $b_1(X) = b_3(X) = 0$, $b_5(X) = 2$. More precisely, $h^{50} = h^{05} = 1$, $h^{14} = h^{23} = h^{32} = h^{41} = 0$. The (semi-simplifications of the) Galois representation of the action of Frobenius on $H^5(X)$ and the Galois representation corresponding to the unique normalized cusp form of level 4 and weight 6 (which is $\eta^{12}(q^2)$), are isomorphic.

Before we can give the proof we need some preparations.

The variety $\bar{X}$ is a double cover of a degree twelve arrangement, in the sense of Definition 5.4 (see subsection 5.1 at the end of this section, where we collect the necessary statements). In Proposition 5.6 we describe a procedure, how to resolve singularities of such a double cover, and our goal here is to check that the arrangement satisfies the assumptions of that proposition.

We shall distinguish the singularities by their multiplicity and dimension and denote the resulting classes by $T_k$. Let $N_k$ be the number of singularities of type $T_k$ that contain a given singularity. Then the situation can be summed up by table 1. We see that $T_2, T_4, T_5, T_7, T_8, T_9$
are near pencil, whereas $T_1, T_3, T_6, T_{10}$ and $T_{11}$ satisfy $\left\lfloor \frac{m(C)}{2} \right\rfloor = n - d(C) - 1$. Hence $\bar{X}$ has a crepant resolution of singularities $X$, which is a smooth Calabi-Yau variety.

Studying the singularities in the above table, we see that the only prime of bad reduction is 2 (due to taking the double cover). The exterior powers of the matrix of coefficients of the arrangement of hyperplanes have coefficients equal to 0, ±1, so the reduction modulo an odd prime has the same number and type of singularities as in characteristic 0. Consequently, the same blow-ups as in characteristic 0 give a resolution of singularities.

To prove modularity of $X$, we have to study the number of points of $X_p$ in $\mathbb{F}_p$. In principle, it should be possible to give an explicit formula, as was done in the analogous situation in dimension 3. However, in this case there are many more different types of singularities and so the computations would be very long and tedious. For our purpose it is enough to have the following information on the “shape” of that number.

**Proposition 5.2.** For any odd prime $p$ we have

$$\#X(\mathbb{F}_p) = 1 + \sum_{i=1}^{4} \sum_{j=1}^{b_{2i}} \left( \frac{a_{i,j}}{p} \right) p^i + p^5 - a_p,$$

where the $a_{i,j}$ are square-free non-zero integers.
Sketch of the proof of Proposition 5.2. Using Ahlgren’s result we only have to take into account the effect of adding the hypersurface at infinity and of all blow-ups. All these varieties are resolutions of certain double covers branched along divisors of small degrees. Using the projection formula for finite maps, it is not difficult to show that the Hodge spaces contributing to the odd cohomology groups are all zero. The even cohomology groups are spanned by algebraic cycles, which project (under the double covering) onto cycles defined over $\mathbb{Q}$. Consequently, the even cohomology groups can be generated by cycles which are either defined over $\mathbb{Q}$ or over some quadratic extension. Fix a non-invariant irreducible algebraic subvariety $Z$ and denote by $Z'$ its image under the involution defined by the double cover. Clearly $Z + Z'$ is defined over $\mathbb{Q}$. Assume that $Z$ (and hence also $Z'$) is defined over a quadratic extension $\mathbb{Q}(\sqrt{a})$. Recall that $p$ is an odd prime, and hence a prime of good reduction. Over $\overline{\mathbb{F}}_p$ the sum $(Z + Z')_p$ splits into a sum of two cycles $Z_p$ and $Z'_p$. Frobenius maps the class $Z_p$ to the class of $p^i Z_p$ or $p^i Z'_p$ ($i = 5 - \dim Z$) depending on whether $a$ is a square in $\mathbb{F}_p$ or not. Consequently the class of the cycle $Z_p - Z'_p$ is an eigenvector with eigenvalue $(\frac{a}{p}) p^i$, where $(\frac{a}{p})$ is the Legendre symbol. Using the Lefschetz fixed-point formula we obtain the proposition.

Proof of Theorem 5.1. For every $1 \leq i \leq 4$ and $1 \leq j \leq b_{2i}$ we consider the one-dimensional Galois representation $\rho_{i,j}$ with eigenvalues $(\frac{a_{i,j}}{p}) p^i$ and define $\tilde{\rho}$ to be the direct sum of all $\rho_{i,j}$. So $\tilde{\rho}$ is the Galois representation associated to the algebraic cycles. Let $\tilde{\rho}_i$ be the Galois action on the $i$-th cohomology and denote by $\rho$ the direct sum of $\rho_{2i}$, $i = 1, \ldots, 4$. Finally, denote by $\rho$ the Galois representation associated to the unique cusp form of level 4 and weight 6. By Proposition 5.2 we can write the number of points of $X(\mathbb{F}_p)$ as

$$1 + p^5 + \text{tr}(\tilde{\rho}_p) - \text{tr} \rho_p.$$

By the Lefschetz fixed point formula this is equal to

$$1 + p^5 + \text{tr}(\tilde{\rho}_p) - \text{tr}(\rho_{1,p}) - \text{tr}(\rho_{3,p}) - \text{tr}(\rho_{5,p}) - \text{tr}(\rho_{7,p}) - \text{tr}(\rho_{9,p}).$$

Comparing the above two formulas and clearing the signs we get

$$\text{tr}(\tilde{\rho}_p) + \text{tr}(\rho_p) = \text{tr}(\tilde{\rho}_p) + \text{tr}(\rho_{1,p}) + \text{tr}(\rho_{3,p}) + \text{tr}(\rho_{5,p}) + \text{tr}(\rho_{7,p}) + \text{tr}(\rho_{9,p})$$

So the representations

$$\tilde{\rho} \oplus \rho$$

and

$$\tilde{\rho} \oplus \rho_1 \oplus \rho_3 \oplus \rho_5 \oplus \rho_7 \oplus \rho_9$$
have equal traces for any odd prime, and consequently they have isomorphic semi-simplifications (see [16, Lemma p. 1-11]). Semi-simplification preserves the eigenvalues. By construction and the Weil conjectures the representation $\bar{\rho} \oplus \rho$ has no eigenvalue with absolute value equal to $p^{1/2}$ or $p^{3/2}$, and only two eigenvalues with absolute value equal to $p^{5/2}$. So $H^1(X) = H^3(X) = H^7(X) = H^9(X) = 0$ and the Galois representations $\rho$ and $\rho_5$ have equal eigenvalues and hence isomorphic semi-simplifications. □

Remark 5.3. The Ahlgren variety is birational to the quotient of the fourfold fiber product of the Legendre family, resp. the extremal rational elliptic surface with three singular fibers of Kodaira types $I_2, I_2, I_2^*$ (which in [14] is denoted by $X_{222}$) by the group $\mathbb{Z}_3$. In each fiber this is the construction described in Section 2 so it is fibered by Calabi-Yau 4-folds.

5.1. Resolution of singularities of double arrangements. In this subsection we shall describe in detail the procedure which we use to resolve the singularities of Ahlgren’s fivefold. Let $Y$ be an $n$-dimensional smooth projective manifold.

Definition 5.4. A sum $D = \bigcup_{i=1}^N D_i$ of smooth hypersurfaces $D_i$ in $Y$ is called an arrangement if for each subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, N\}$ the (ideal-theoretic) intersection $C_{i_1, \ldots, i_r} = D_{i_1} \cap \cdots \cap D_{i_r}$ is smooth.

The following lemma is obvious from the definitions.

Lemma 5.5. Let $D = D_1 \cup \cdots \cup D_N \subset Y$ be an arrangement. Then

1. If $\dim(D_{i_1} \cap \cdots \cap D_{i_r}) = n-r$ for some $\{i_1, \ldots, i_r\} \subset \{1, \ldots, N\}$, then $D_{i_1} \cdots D_{i_r}$ intersect transversally.

2. For any $\{i_1, \ldots, i_r\} \subset \{1, \ldots, N\}$ the tangent space to the intersection $D_{i_1} \cap \cdots \cap D_{i_r}$ (at any point) equals the intersection of the tangent spaces to the divisors $D_i$.

We now consider the decomposition of the singular locus of $D$ by multiplicities. For this we take the set $\mathcal{S}$ of all components $C$ of intersections $D_{i_1} \cap \cdots \cap D_{i_r}$, where $r \geq 2$ and $\{i_1, \ldots, i_r\} \subset \{1, \ldots, N\}$. To each element $C \in \mathcal{S}$ we assign its multiplicity $m(C) = \text{mult}_C D = \# \{i : C \subset D_i\}$ and dimension $d(C) = \dim C$. An element $C \in \mathcal{S}$ will be called near-pencil if it is contained in an element $C' \in \mathcal{S}$ with $d(C) = d(C') - 1$ and $m(C) = m(C') + 1$ (i.e. $C$ is cut-out from $C'$ by a single hypersurface).

If the arrangement $D \subset Y$ is even (as an element of the Picard group $\text{Pic}(Y)$), then there exists a double cover $\pi : X \to Y$ of $Y$ branched
along \( D \). Such a double cover is uniquely determined by fixing a line bundle \( \mathcal{L} \) on \( Y \) with \( \mathcal{O}(D) \cong \mathcal{L}^2 \).

**Proposition 5.6.** Assume that for every singular variety \( C \in S \) either \( C \) is near-pencil or \( \left\lfloor \frac{m(C)}{2} \right\rfloor = n - d(C) - 1 \) then \( X \) admits a projective crepant resolution of singularities.

**Proof.** Let \( C \in S \) be of dimension \( d(C) = d \) and multiplicity \( m(C) = m \). By the definition of an arrangement, this is a smooth subvariety of \( Y \) and we consider the blow-up \( \sigma : \tilde{Y} \rightarrow Y \) of \( Y \) along \( C \) with exceptional divisor \( E \). Recall that \( C \), and hence \( E \), are irreducible by the definition of \( S \). The pullback \( \sigma^*D \) of \( D \) to \( \tilde{Y} \) is even in the Picard group of \( \tilde{Y} \), but it is in general not reduced. We define \( D^* \) as the unique reduced and even divisor satisfying \( \tilde{D} \leq D^* \leq \sigma^*D \), where \( \tilde{D} \) is the strict transform of the branch locus as the new branch locus, whereas when the multiplicity is odd we add the exceptional divisor. Equivalently \( D^* = \sigma^*D - 2\left\lfloor \frac{m}{2} \right\rfloor E \).

We have \( K_{\tilde{Y}} + \frac{1}{2}D^* = \sigma^*(K_Y + \frac{1}{2}D) + (n - d - 1 - \left\lfloor \frac{m}{2} \right\rfloor)E \), and so \( K_{\tilde{Y}} + \frac{1}{2}D^* = \sigma^*(K_Y + \frac{1}{2}D) \) exactly when \( \left\lfloor \frac{m}{2} \right\rfloor = n - d - 1 \). We shall call a blow-up for which this equality holds admissible.

Assume now that \( C \in S \) is a minimal element (with respect to inclusion) among those components which are not near pencil. Then, by assumption, the blow-up \( \sigma \) along \( C \) is admissible. We want to show that \( D^* \) is again an arrangement satisfying the assumptions of the theorem. Let \( D_1, \ldots, D_k \) be the components of \( D \) that contain \( C \). Let us pick some other components \( D_{k+1}, \ldots, D_{k+p} \) and denote by \( C_1 \) the intersection \( C_1 = C \cap D_{k+1} \cap \cdots \cap D_{k+p} \). As the problem is local we can assume that \( C_1 \) is irreducible. Our aim is to show that the intersection \( \tilde{D}_1 \cap \cdots \cap \tilde{D}_l \cap D_{k+1} \cap \cdots \cap D_{k+p} \) is smooth.

The intersection consists of two parts, namely the strict transform of the intersection and the intersection of the exceptional loci. The dimension of the former is less than or equal to \( \dim C_1 + \codim C - 1 - l \), and so its codimension is greater than or equal to \( \codim C_1 - \codim C + 1 + l \). Since all the intersections of \( C \) with \( D_{k+j} \) are near pencil we obtain that \( \codim C_1 - \codim C = p \) and that the codimension of the intersection of the exceptional loci is greater than \( p + l \), and hence this
is not a component of the intersection $\tilde{D}_1 \cap \cdots \cap \tilde{D}_l \cap \tilde{D}_{k+1} \cap \cdots \cap \tilde{D}_{k+p}$. Consequently, the intersection $\tilde{D}_1 \cap \cdots \cap \tilde{D}_l \cap \tilde{D}_{k+1} \cap \cdots \cap \tilde{D}_{k+p}$ equals the strict transform of the intersection $D_1 \cap \cdots \cap D_l \cap D_{k+1} \cap \cdots \cap D_{k+p}$, and hence is smooth. To conclude that $D^*$ is an arrangement in the case of $m$ odd, we also have to take the exceptional divisor of the blow-up into account. But this is transversal to any strict transform.

To show that the arrangement $D^*$ satisfies the assumption of the proposition, we observe that in the case of $m$ even the exceptional varieties for $D^*$ are blow-ups of the exceptional varieties for $D$, with the same multiplicities and dimensions. In the case of $m$ odd, we have to add the intersections with the exceptional divisors, but these are near-pencil singularities.

A resolution of singularities of $X$ can now be obtained by blowing-up all the components $C \in S$ which are not near-pencil, starting from the smallest dimension. Since every blow-up decreases the number of not near-pencil elements, the process will terminate. As the intersection of two hyperplanes cannot be near-pencil, the components of the final branch locus must be disjoint, and hence we get a resolution of singularities. Finally, since all blow-ups are admissible, the resulting resolution is crepant.

Denote by $\sigma : \tilde{Y} \to Y$ the composition of all inverse maps to the blow-ups, and by $\pi : X \to Y$ (resp. $\tilde{\pi} : \tilde{X} \to \tilde{Y}$) the double cover of $Y$ (resp. of $\tilde{Y}$) branched along the divisor $D$ (resp. along $\tilde{D}$). Then there exists a unique map $\tilde{\sigma} : \tilde{X} \to X$ making the following diagram commutative

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\sigma}} & X \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\tilde{Y} & \xrightarrow{\sigma} & Y
\end{array}
\]

So the constructed crepant resolution of $X$ is given by a proper, birational morphism. \hfill \Box

Clearly, the resolution is in general not unique, but depends on the order of the blow-ups.

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