FINITE ABELIAN SUBGROUPS IN THE GROUPS OF BIRATIONAL AND
BIMEROMORPHIC SELFMAPS

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Abstract. Let $X$ be a complex projective variety. Suppose that the group of birational automorphisms of $X$ contains finite subgroups isomorphic to $(\mathbb{Z}/N\mathbb{Z})^r$ for $r$ fixed and $N$ arbitrarily large. We show that $r$ does not exceed $2 \dim(X)$. Moreover, the equality holds if and only if $X$ is birational to an abelian variety. We also show that an analogous result holds for groups of bimeromorphic automorphisms of compact Kähler spaces under some additional assumptions.

1. Introduction

In the present paper we study finite abelian subgroups in the groups of birational automorphisms of projective algebraic varieties (over a field of zero characteristic), or in the groups of bimeromorphic automorphisms of compact Kähler spaces. The starting point for us is the following recent theorem by I. Mundet i Riera [22, Theorem 1.9].

Theorem 1.1. Let $X$ be a connected compact Kähler manifold. Suppose that there exists $r \in \mathbb{N}$ such that for arbitrarily large positive integers $N$ the group $\text{Aut}(X)$ contains a subgroup isomorphic to $(\mathbb{Z}/N\mathbb{Z})^r$. Then $\text{Aut}(X)$ contains a subgroup isomorphic to a compact real torus of dimension $r$. In addition, $r \leq 2 \dim(X)$, and if $r = 2 \dim(X)$ then $X$ is biholomorphic to a compact complex torus.

The maximal number $r$ satisfying the assumptions of Theorem 1.1 is called in [22] the (holomorphic) discrete degree of symmetry of $X$. More generally, in [22] I. Mundet i Riera defines and studies this invariant for continuous group actions on topological manifolds. In some cases, the discrete degree of symmetry can be compared to the maximal dimension of a torus acting effectively on a manifold [22, Theorem 1.7]. In connection with Theorem 1.1 I. Mundet i Riera also asks whether the same bound on $r$ holds also for birational automorphism groups. In fact, this invariant has implicitly appeared in the study of $p$-subgroups of birational automorphism groups. For instance, in [34, Theorem 2.9] J. Xu proved the following result for non-uniruled algebraic varieties.

Theorem 1.2. Let $X$ be a non-uniruled algebraic variety over an algebraically closed field of characteristic zero. There exists a constant $b(X)$ such that the group $\text{Bir}(X)$ contains an element of order greater than $b(X)$ if and only if $X$ is birational to a variety $X'$ which admits an effective action of an abelian variety.

A remarkable result of Yu. Prokhorov and C. Shramov [25, Theorem 1.10], together with C. Birkar’s solution of the BAB conjecture [3, Theorem 1.1], provides a stronger bound for rationally connected varieties.

Theorem 1.3. Let $X$ be a rationally connected algebraic variety of dimension $n$ over an algebraically closed field of characteristic zero. There exists a constant $L = L(n)$ such that, for any prime number $p > L(n)$, each finite $p$-subgroup $G \subset \text{Bir}(X)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$ for some $r \leq n$.

By a result of J. Xu [35] the constant $L$ in the above theorem can be taken to be $n + 1$. Moreover, J. Xu proved a rationality criterion for rationally connected varieties admitting an action of $(\mathbb{Z}/p\mathbb{Z})^r$ in terms of $r$ and $p$ (see [34, Theorem 4.5]).

Theorem 1.4. Let $X$ be a rationally connected algebraic variety of dimension $n$ over an algebraically closed field of characteristic zero. Then there exists a constant $R(n)$ such that if $\text{Bir}(X)$ contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ for some $p > R(n)$ then $X$ is rational.

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Informally speaking, these results suggest that existence of finite abelian subgroups in $\text{Bir}(X)$ of unbounded orders should imply existence of algebraic groups (of positive dimension depending on $r$) acting on $X$ by birational automorphisms, at least if the ranks $r$ of the finite abelian groups are close to maximal. For smaller values of $r$ the relation between finite abelian and algebraic subgroups of $\text{Bir}(X)$ is more delicate. For instance, there exists a sequence of finite cyclic subgroups of $\text{Cr}_2(\mathbb{C}) = \text{Bir}(\mathbb{P}^2_\mathbb{C})$ which generate a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z}$ but are not contained in any torus in the Cremona group \cite{33}. Existence of finite abelian subgroups of unbounded orders in the group $\text{Bir}(X)$, where $X$ is a non-rationally connected threefold, is a difficult open problem (see \cite{26} Question 4.8). Another related open problem (cf. \cite{34} Conjecture 1.7) is a conjectural description of projective varieties with non-Jordan groups of birational automorphisms. The first examples of such varieties were constructed in \cite{30}; a complete description exists in dimension 3 by \cite{26} Theorem 1.8 and \cite{34} Theorem 1.6.

We should also mention a “toroidalization principle” recently studied by J. Moraga in his works on Kawamata log terminal singularities \cite{19, 20, 21}. In particular, he showed that existence of “large” finite abelian groups of rank $n$ acting on a projective Fano type variety of dimension $n$ implies that $X$ is birational to a log Calabi–Yau toric pair \cite{19} Theorem 2). In \cite{21} Theorem 1) a general result on toroidalization for finite group actions on klt singularities is proved. The case of cyclic group actions on Fano type surfaces is studied in \cite{20}.

The aim of this paper is to initiate a systematic study of an invariant similar to the discrete degree of symmetry for groups of birational (and bimeromorphic) automorphisms. Our main result is a generalization of Theorem 1.1 to groups of birational automorphisms.

**Theorem 1.5.** Let $X$ be a projective algebraic variety over an algebraically closed field of zero characteristic. Suppose that there exists an unbounded sequence $\{N_i\}_{i \in \mathbb{N}}$ of positive integers such that the group $\text{Bir}(X)$ contains subgroups isomorphic to $\mathbb{Z}/N_i\mathbb{Z}$ for some fixed $r$. Then $r \leq 2 \dim(X)$, and in case of equality $X$ is birational to an abelian variety.

Compared to Theorem 1.2 we consider also uniruled varieties; moreover, we do not assume that the orders $N_i$ of generators of the finite groups are prime. The main idea of the proof is to consider the action of $\text{Bir}(X)$ on the maximal rationally connected (MRC) fibration of $X$ (see Definition 2.15 below); this idea is already present in J. Xu’s work (see \cite{34} Proposition 2.12). Combining it with some technical results from our paper \cite{10}, we prove an analogous result for groups of bimeromorphic selfmaps of compact Kähler spaces. We have to assume the existence of quasi-minimal models (see Definition 3.11 below) for the (non-uniruled) base of the MRC fibration of $X$ and the relation between finite abelian and algebraic subgroups of $\text{Bir}(X)$ of positive integers; this idea is already present in J. Xu’s work \cite{34} Proposition 2.12). Combining it with some technical results from our paper \cite{10}, we prove an analogous result for groups of bimeromorphic selfmaps of compact Kähler spaces. We have to assume the existence of quasi-minimal models (see Definition 3.11 below) for the (non-uniruled) base of the MRC fibration of $X$ and the relation between finite abelian and algebraic subgroups of $\text{Bir}(X)$ of positive integers.

**Theorem 1.6.** Let $X$ be a compact Kähler space. Assume that the base $B$ of the MRC fibration of $X$ admits a quasi-minimal model. Suppose that there exists an unbounded sequence $\{N_i\}_{i \in \mathbb{N}}$ of positive integers such that the group $\text{Bim}(X)$ contains subgroups isomorphic to $\mathbb{Z}/N_i\mathbb{Z}$ for some fixed $r$. Then we have $r \leq 2 \dim(X)$ and in case of equality $X$ is bimeromorphic to a compact complex torus.

Let us outline the structure of the paper. In Section 2 we gather some technical results. Section 3 is devoted to the proof of our main theorem. First, in subsection 3.1 we use techniques from our previous paper \cite{10} to generalize Theorem 1.1 to pseudoautomorphisms of compact Kähler spaces with rational singularities (see Theorem 3.8). Then, in subsection 3.2 we prove the main theorem for non-uniruled projective varieties (Theorem 3.14), following the ideas from \cite{34} Section 2). In subsection 3.3 we use the results of Prokhorov and Shramov from \cite{25} to derive the bound on $r$ for abelian groups acting on rationally connected varieties. Finally, in subsection 3.4 we derive Theorem 1.5 from Theorems 3.14 and 3.18 using the maximal rationally connected fibration of $X$. We also prove Theorem 1.6 in this section.

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In what follows we consider irreducible and reduced compact complex spaces, see [12] Chapter 1 for a general reference on complex analytic spaces. A complex manifold is a nonsingular complex space. We consider only compact Kähler manifolds. For the definition of a singular compact Kähler space see Definition 3.1; this definition follows the one in [14].

2.2. Structure of abelian subgroups. In this subsection we collect a few technical statements about subgroups of finite abelian groups.

Definition 2.1. Let $G$ be a finite abelian group. The rank $r(G)$ is defined as the minimal size of a generating set of $G$. An elementary abelian group of rank $r$ is an abelian group isomorphic to $(\mathbb{Z}/N\mathbb{Z})^r$.

Lemma 2.2. Let $G \simeq (\mathbb{Z}/N\mathbb{Z})^r$ be a finite abelian group. Let $H \subseteq G$ be a subgroup of index $I_H \leq N - 1$.

Then there exists an elementary subgroup $H' \subseteq H$ such that $H' \simeq (\mathbb{Z}/N'\mathbb{Z})^r$ for some $N' \geq N/I_H$.

Proof. Let $H \subseteq G$ be a subgroup of index $I_H \leq N - 1$. Then $H$ is a finite abelian group of order at least $N(r - 1) + 1$. The orders of generators of $H$ do not exceed $N$, so the rank of $H$ is at least $r$. By the structure theorem for finite abelian groups we have

$$H \simeq \bigoplus_{1 \leq i \leq r} \mathbb{Z}/N_i \mathbb{Z}$$

where $N_i | N_{i+1}$ for all $i \in \{1, \ldots, r-1\}$. Next, from the equality

$$|H| = N'/I_H = N_1 \cdots N_r$$

we have $N_1 \geq N/I_H$. Now it suffices to take the elementary subgroup $H' = (\mathbb{Z}/N_i \mathbb{Z})^r \subseteq H$.

Lemma 2.3. Let $G \simeq (\mathbb{Z}/N\mathbb{Z})^r$ be a finite abelian group and let $H \subseteq G$ be a subgroup. There exist a set of generators $\{b_1, \ldots, b_r\}$ for $H$ and a set of generators $\{a'_1, \ldots, a'_r\}$ for $G$ such that the embedding $H \rightarrow G$ can be written as the direct sum of homomorphisms

$$\mathbb{Z}/N_i \mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z}.$$ 

Proof. We choose a set of generators $a_1, \ldots, a_r$ for $\mathbb{Z}^r$ such that their images under the projection

$$\mathbb{Z}^r \rightarrow \mathbb{Z}^r/(N\mathbb{Z})^r$$

give an isomorphism $(\mathbb{Z}/N\mathbb{Z})^r \simeq G$. Let us denote by $\tilde{H}$ the preimage of $H$ under the map $\mathbb{Z}^r \rightarrow G$. Then $\tilde{H}$ is a free abelian group of rank $r$ containing the subgroup $(N\mathbb{Z})^r$. Let $A$ be a presentation matrix for $\tilde{H}$. Using, for example, [2, Theorem (4.3)] we find that there exists the Smith normal form

$$A' = QAP^{-1}$$

where $Q, P \in \text{GL}_r(\mathbb{Z})$ and the matrix $A'$ is diagonal with entries $N_1, \ldots, N_r$ such that $N_i$ divides $N_{i+1}$ for any $i \in \{1, \ldots, r-1\}$. So there exists a basis $a'_1, \ldots, a'_r$ of $\mathbb{Z}^r$ such that $\tilde{H}$ is generated by

$$b_1 = a'_1 N_1, \ldots, b_r = a'_r N_r.$$ 

Since multiplication by invertible matrices preserves the sublattice $(N\mathbb{Z})^r \subset \mathbb{Z}^r$, we can take the images of $b_1, \ldots, b_r$ under the projection $\mathbb{Z}^r \rightarrow \mathbb{Z}^r/(n\mathbb{Z})^r \simeq G$ as generators for $H$.

For the discussion that follows it will be convenient to introduce the following definition.

Definition 2.4. Let $\{G_i\}_{i \in \mathbb{N}}$ be a sequence of finite groups. We define the asymptotic rank of the sequence $\{G_i\}$ to be the minimal number $r$ such that the following condition is satisfied. There exists a constant $L$ such that, for infinitely many indices $i \in \mathbb{N}$, we can find an abelian subgroup $H_i \subseteq G_i$ such that

- $H_i$ is generated by $r$ elements;
- the orders of the subgroups $H_i$ are unbounded as $i$ tends to infinity;
- the index of $H_i$ in $G_i$ does not exceed $L$.

Example 2.5. If the orders of the groups $G_i$ are bounded by a constant then the asymptotic rank of the sequence $\{G_i\}$ is equal to zero. The asymptotic rank of the sequence $G_i = (\mathbb{Z}/i\mathbb{Z})^r \times (\mathbb{Z}/2\mathbb{Z})^10$ is equal to $r$. The asymptotic rank of the sequence $G_r = (\mathbb{Z}/r\mathbb{Z})^r$ is infinite.
Remark 2.7. The motivation for the above definition comes from the study of Jordan groups. Recall from [23] Definition 2.1 that a group $G$ is Jordan if there exists a constant $J(G) \in \mathbb{N}$ such that for every finite subgroup $H \subset G$ there exists a normal abelian subgroup $A \lhd H$ of index at most $J(G)$. Suppose that the group $G$ is Jordan and that the orders of finite subgroups of $G$ are unbounded. Then there exists a sequence $\{G_i\}_{i \in \mathbb{N}}$ of finite subgroups of $G$ satisfying the assumptions in Definition 2.3 for some $r$ and $L = J(G)$ (the Jordan constant of $G$). The maximum value of $r$ over all such sequences of finite subgroups of $G$ is a natural invariant of the group $G$.

Remark 2.8. Let $\{G_i\}$ be a sequence of finite groups of asymptotic rank $r$. Then we can take a sequence of abelian subgroups $H_i \subset G_i$ as in Definition 2.4, then the asymptotic rank of $\{H_i\}$ is equal to the asymptotic rank of $\{G_i\}$. More generally, if $\{G'_i \subset G_i\}$ is a sequence of subgroups of uniformly bounded index, then the asymptotic ranks of the sequences $\{G'_i\}$ and $\{G_i\}$ are equal. Therefore, by Lemma 2.2, it suffices to consider sequences of elementary abelian groups.

For a sequence of finite abelian groups, the asymptotic rank can be computed using the direct sum decomposition provided by the structure theorem.

**Proposition 2.9.** Let $\{G_i\}_{i \in \mathbb{N}}$ be a sequence of finite abelian groups. Suppose that for every $i \in \mathbb{N}$ the rank of $G_i$ is $r$. Consider the decomposition

$$G_i \simeq \bigoplus_{1 \leq k \leq r} \mathbb{Z}/N_{i,k}\mathbb{Z},$$

where $N_{i,k} | N_{i,k+1}$ for every $k \in \{1, \ldots, r-1\}$. Then the asymptotic rank of the sequence $\{G_i\}$ is equal to

$$r - \max\{ k \mid \text{the sequence } \{N_{i,k}\}, i \in \mathbb{N} \text{ is bounded as } i \to \infty \}.$$

**Proof.** We set

$$k_{\max} = \max\{ k \mid \text{the sequence } \{N_{i,k}\}, i \in \mathbb{N} \text{ is bounded as } i \to \infty \}.$$

Considering the sequence of subgroups

$$H_i = \bigoplus_{k_{\max} + 1 \leq k \leq r} \mathbb{Z}/N_{i,k}\mathbb{Z} \subset G_i,$$

we find that the asymptotic rank of the sequence $\{G_i\}$ is at most $r - k_{\max}$. We denote by $L$ a constant such that $\|G_i\|/|H_i| \leq L$ for all $i \in \mathbb{N}$.

Suppose that the asymptotic rank $r'$ of $\{G_i\}$ is smaller than $r - k_{\max}$. Then there exists a sequence of subgroups $\{H'_i \subset G_i\}$ such that for every $i \in \mathbb{N}$ the group $H'_i$ is generated by $r' < r - k_{\max}$ elements, and the indices $|G_i|/|H'_i|$ are bounded as $i \to \infty$. We have

$$|H'_i| = |H'_i/(H_i \cap H'_i)| \cdot |H_i \cap H'_i| \leq L \cdot |H_i \cap H'_i|.$$  

Hence

$$\frac{|G_i|}{|H'_i|} = \frac{|G_i|/|H'_i|}{|H_i|/|H'_i|} \geq \frac{|G_i|}{L \cdot |H_i|} \cdot \left(\frac{|H_i|}{|H_i \cap H'_i|}\right).$$

On the other hand, since the subgroup $H_i \cap H'_i$ is generated by $r' < r - k_{\max}$ elements, the indices $|H_i|/|H_i \cap H'_i|$ are unbounded as $i \to \infty$. This contradiction shows that the asymptotic rank of $\{G_i\}$ is equal to $r - k_{\max}$. \qed

Another convenient way to express the asymptotic rank of a sequence of finite abelian groups is given in the corollary below.

**Corollary 2.10.** Let $\{G_i\}$ be a sequence of finite abelian groups. Suppose that for every $i \in \mathbb{N}$, the group $G_i$ can be generated by $r$ elements. Then the asymptotic rank of $\{G_i\}$ is equal to

$$\max\{ r \mid G_i \supset (\mathbb{Z}/M_i\mathbb{Z})^r \text{ for an infinite number of } i \in \mathbb{N} \text{ and } M_i \to \infty \}.$$
Proof. By Proposition 2.9, the asymptotic rank of the sequence \( \{G_i\} \) is \( r - k_{\text{max}} \), where
\[
k_{\text{max}} = \max \{k \mid \text{the sequence } \{N_{i,k}\}, i \in \mathbb{N} \text{ is bounded as } i \to \infty \}.
\]
Consider the sequence of subgroups
\[
H_i = \bigoplus_{k_{\text{max}} + 1 \leq k \leq r} \mathbb{Z}/N_{i,k}\mathbb{Z} \subset G_i.
\]
Then each \( H_i \) contains a subgroup isomorphic to \((\mathbb{Z}/M_i\mathbb{Z})^{r - k_{\text{max}}}\) where \( M_i = N_{i,k_{\text{max}} + 1} \). Suppose that for infinitely many \( i \in \mathbb{N} \) we can find subgroups \( H'_i \subset G_i \) such that \( H'_i \simeq (\mathbb{Z}/M'_i\mathbb{Z})^r \) for \( i \to \infty \). Consider the images of \( H'_i \) under the quotient homomorphisms \( G_i \to G_i/H_i \). Since the indices \( |G_i|/|H_i| \) are bounded by a constant \( L \) independent of \( i \in \mathbb{N} \), the number \( s \) does not exceed \( r - k_{\text{max}} \).

We deduce the following important subadditivity property for asymptotic ranks.

Lemma 2.11. Let \( \{G_i\} \) be a sequence of abelian groups. Consider a sequence of subgroups \( G'_i \subset G_i \) for \( i \in \mathbb{N} \), and denote the quotients by \( G''_i \). Suppose that the asymptotic rank of the sequence \( \{G'_i\} \) is at most \( r' \) and that the asymptotic rank of the sequence \( \{G''_i\} \) is at most \( r'' \). Then the asymptotic rank \( r \) of the sequence \( \{G_i\} \) is at most \( r' + r'' \).

Proof. By Remark 2.8 we may assume that \( G_i \simeq (\mathbb{Z}/N_i\mathbb{Z})^r \) are elementary abelian subgroups for some \( N_i \) that tend to infinity. By Lemma 2.3 we may choose compatible systems of generators in \( G'_i \) and \( G_i \) for every \( i \in \mathbb{N} \). Let us denote by
\[
N'_{i,1} | N'_{i,2} | \cdots | N'_{i,r}
\]
the divisors in the decomposition of \( G'_i \) given by the structure theorem. Then the quotient groups \( G''_i \) are isomorphic to
\[
\bigoplus_{1 \leq j \leq r} \mathbb{Z}/N'_{i,j}\mathbb{Z}.
\]
By the assumption, the asymptotic rank of the sequence \( \{G'_i\} \) is at most \( r' \). By Proposition 2.10,
\[
r - \max \{j \mid \text{the sequence } \{N_{i,j}\} \text{ is bounded as } i \to \infty \} \leq r'.
\]
Similarly, since the asymptotic rank of the sequence \( \{G''_i\} \) is at most \( r'' \), we have, by Proposition 2.10,
\[
r - \min \{j \mid \text{the sequence } \{N'_{i,j}\} \text{ is bounded as } i \to \infty \} \leq r''.
\]
However, since \( N_i \) tend to infinity, the sequences \( \{N'_{i,j}\} \) and \( \{N_{i}/N'_{i,j}\} \) for a fixed \( j \in \{1, \ldots, r\} \) cannot be bounded simultaneously. Therefore, adding the above inequalities we obtain
\[
r \leq r' + r'',
\]
the result required. \( \Box \)

2.3. The MRC fibration. In this subsection we briefly recall the construction of the maximal rationally connected (MRC) fibration for a compact Kähler manifold \( X \). In this generality, the existence of the MRC fibration was established in [8]. We refer to [8] for details, including the definition of the cycle space (Barlet space) \( \mathcal{C}(X) \) for a compact complex space \( X \). For a purely algebraic proof of this result in the case of projective algebraic varieties, see [7, Théorème 2.3] or [17].

Definition 2.12. A covering family of cycles on a complex space \( X \) is a complex subspace \( S \subset \mathcal{C}(X) \) such that
- \( S \) is a countable union of compact irreducible complex subspaces;
- For \( s \in S \), a general point the cycle \( Z_s \) is irreducible and reduced;
- \( X \) is a union of \( \text{Supp}(Z_s) \) for \( s \in S \).

A covering family of cycles induces an equivalence relation \( R(S) \) on points of \( X \). Namely, two points \( x, y \in X \) are equivalent if and only if they are contained in a connected union of a finite number of cycles parameterized by \( S \). The following theorem (see [8, Theorem 1.1] for the proof) shows the existence of meromorphic reduction maps for covering families of cycles. Recall that a fibration is a dominant meromorphic map of normal complex spaces with connected fibers. A typical fiber of a fibration is a fiber over a point in the complement to a proper analytic subset in the base.
Theorem 2.13. Let $X$ be a normal connected complex space. Let $S \subset \mathcal{C}(X)$ be a covering family of cycles on $X$. Denote by $R(S)$ the equivalence relation on $X$ induced by $S$. Then there exists a meromorphic fibration $q_S : X \to B_S$ such that a typical fiber of $q_S$ is an equivalence class for $R(S)$.

An important result, proved independently in [9] and [18], is the compactness of the irreducible components of $\mathcal{C}(X)$ in the Kähler case.

Theorem 2.14. Let $X$ be a compact Kähler manifold. Then each irreducible component of the cycle space $\mathcal{C}(X)$ is compact.

As a consequence of Theorem 2.14 for a compact Kähler manifold one can define the following natural meromorphic fibration.

Definition 2.15. Let $X$ be a compact Kähler manifold and let $S$ be a family of all rational curves on $X$. The fibration $f : X \to B$ corresponding to $S$ by Theorem 2.13 is called the maximal rationally connected (MRC) fibration of $X$.

Obviously, if $X$ is not covered by rational curves then $f$ is birational. A crucial property of the MRC fibration is that its base $B$ is not covered by rational curves. This statement was shown in [11, Corollary 1.4] for $X$ an algebraic variety. The same argument generalizes to the Kähler case (see, for instance, [13, Remark 3.2] or [27, Proposition 3.8]).

Theorem 2.16. Let $X$ be a compact Kähler manifold. Consider the MRC fibration $f : X \to B$.

Then the base $B$ is not uniruled.

Moreover, the smooth fibers of the MRC fibration of a compact Kähler manifold $X$ are in fact projective, see [27, Theorem 3.9] for the proof.

Proposition 2.17. Let $X$ be a rationally connected compact Kähler manifold. Then $X$ is projective.

2.4. Finite group actions. We will need a well-known result (see e.g. [24, Lemma 3.1]) on existence of regularizations of birational actions of finite groups.

Proposition 2.18. Let $X$ be a normal projective variety and let $G \subset \text{Bir}(X)$ be a finite group. Then there exists a smooth projective variety $\overline{X}$ with a regular action of $G$ and a $G$-equivariant birational map $\varphi : \overline{X} \to X$.

Proof. Replacing $X$ by an affine open subset, we may assume that the action of $G$ on $X$ is regular. Then by [31, Theorem 3], there exists a $G$-equivariant projective completion $\varphi : \overline{X} \to X$.

Replacing $\overline{X}$ by a $G$-equivariant resolution of singularities of $\overline{X}$ (see e.g. [31]) we may assume $\overline{X}$ to be smooth.

The following proposition shows that actions of finite groups by automorphisms can be linearized in the fixed points. For the proof in the complex analytic setup we refer to [11, p. 38].

Proposition 2.19. Let $G$ be a finite group acting on a compact complex space $X$ by biholomorphic automorphisms with a fixed point $p \in X$. Then the induced action of $G$ on the tangent space $T_p(X)$ is faithful.

3. Main results

3.1. Groups of pseudoautomorphisms. In this section we extend Theorem 1.14 to automorphism groups of singular compact Kähler spaces. For a complex space $X$ we denote the subsets of its singular and non-singular points by $X_{\text{sing}}$ and $X_{\text{ns}}$, respectively.

Definition 3.1. Let $X$ be an irreducible and reduced complex space. A Kähler form on $X$ is a closed positive real $(1,1)$-form $\omega$ on $X_{\text{ns}}$ satisfying the following condition: for any $x \in X_{\text{sing}}$ there exists an open neighborhood $x \in U \subset X$ with a closed embedding $i_U : U \subset V$ into an open subset $V \subset \mathbb{C}^N$ such that

$$\omega|_{U \cap X_{\text{ns}}} = i\partial\overline{\partial}f|_{U \cap X_{\text{ns}}}$$
for a smooth strictly plurisubharmonic function $f: V \to \mathbb{C}$. An irreducible and reduced complex space $X$ is Kähler if there exists a Kähler form on $X$.

**Remark 3.2.** Below we consider only those singular Kähler spaces that are normal and have rational singularities. In particular, minimal and quasi-minimal compact Kähler spaces (or complex projective varieties) satisfy these conditions.

**Remark 3.3.** If $X$ is a singular Kähler space, one can always find a resolution of singularities $\varphi: X' \to X$ where $X'$ is a compact Kähler manifold [14, Remark 2.3]. The MRC fibration for $X$ can be defined as the MRC fibration of (any) compact Kähler manifold $X'$ bimeromorphic to $X$.

We need the following simple lemma (cf. [32, Lemma 9.11]).

**Lemma 3.4.** Let $X$ be a normal compact Kähler space. Suppose that there exists a bimeromorphic morphism

$$\varphi: T \to X,$$

where $T$ is a compact complex torus. Then $\varphi$ is an isomorphism.

**Proof.** Let $E$ be an irreducible component of the exceptional locus of $\varphi$ of dimension $c > 0$. Consider a Kähler class $\omega$ on $X$. By the projection formula,

$$(\varphi^*\omega)^c \cdot E = \omega^c \cdot (\varphi^* E) = 0.$$ 

On the other hand, we can choose a general translation $\tau: T \to T$ such that the image of $\tau^*(E)$ under the map $\varphi$ is not contained in the singular locus of $X$. Therefore,

$$(\varphi^*\omega)^c \cdot \tau^* E = \omega^c \cdot (\varphi^*\tau^* E) > 0.$$ 

However, since $\tau$ is an automorphism of $T$, we have $(\varphi^*\omega)^c \cdot E = (\varphi^*\omega)^c \cdot (\tau^* E)$. This contradiction shows that the exceptional locus of $\varphi$ is empty, so $\varphi$ is an isomorphism. □

We also state another result by I. Mundet i Riera (see [22, Theorem 1.10]). Theorem 1.9 is immediate from this result.

**Theorem 3.5.** Let $G$ be a Lie group with finitely many connected components. For every natural number $r$, the following properties are equivalent:

- the group $G$ contains subgroups of the form $(\mathbb{Z}/N\mathbb{Z})^r$ for arbitrarily large positive integers $N$;
- the group $G$ contains a subgroup isomorphic to a compact real torus $(S^1)^r$ of real dimension $r$.

We can deduce the following corollary from Theorem 3.5 by an argument similar to the proof of [22, Theorem 1.9].

**Corollary 3.6.** Let $X$ be a (possibly singular) normal compact Kähler space. For every natural number $r$, the following properties are equivalent:

- the group $\text{Aut}(X)$ contains finite abelian subgroups of the form $(\mathbb{Z}/N\mathbb{Z})^r$ for arbitrarily large positive integers $N$;
- the group $\text{Aut}(X)$ contains a subgroup isomorphic to $(S^1)^r$.

In addition, if $r \leq 2 \dim(X)$, and if $r = 2 \dim(X)$ then $X$ is biholomorphic to a compact complex torus.

**Proof.** The group of connected components $\text{Aut}(X)/\text{Aut}^0(X)$ has bounded finite subgroups (see [15, Lemma 3.1]). So, by Lemma 2.2 we may assume that the finite abelian subgroups in question lie in $\text{Aut}^0(X)$. By a well-known theorem of S. Bochner and H. Montgomery, the group $\text{Aut}^0(X)$ is a connected complex Lie group acting holomorphically on $X$ (see e. g. [11, Theorem on p. 40] for a modern proof). Now the first statement of the corollary follows from Theorem 3.5.

Suppose that there is an effective action of $(S^1)^r$ on $X$ by holomorphic automorphisms. Then by the results of [4], we can take a $(S^1)^r$-equivariant resolution of singularities $\varphi: X' \to X$, where $X'$ is a compact Kähler manifold. Applying Theorem 1.1 to $X'$, we get the estimate

$$r \leq 2 \dim(X') = 2 \dim(X).$$

Now, if $r = 2 \dim(X')$, then $X'$ is biholomorphic to a compact complex torus. By Lemma 3.4 $\varphi$ is an isomorphism, and so $X$ is nonsingular and biholomorphic to a compact complex torus. □
The next step is to extend the above result to groups of pseudoautomorphisms of singular compact Kähler spaces. Recall that a bimeromorphic map \( f : X \to X \) is a pseudoautomorphism if both \( f \) and \( f^{-1} \) do not contract divisors. The group of pseudoautomorphisms of \( X \) is denoted by \( \text{Psaut}(X) \).

For convenience of the reader, we reproduce here the following result (see [10, Corollary 4.6]).

**Proposition 3.7.** Let \( X \) be a normal compact Kähler space with rational singularities. Let \( f : X \to X \) be a pseudoautomorphism. Suppose that there exists a Kähler class \( \omega \) such that \( f_* \omega \) is also a Kähler class. Then \( f \) is a biholomorphic automorphism of \( X \).

Now using this proposition we can easily generalize Corollary 3.6 to the group \( \text{Psaut}(X) \).

**Theorem 3.8.** Let \( X \) be a normal compact Kähler space with rational singularities. Suppose that there exists \( r \in \mathbb{N} \) such that the group \( \text{Psaut}(X) \) contains finite abelian subgroups isomorphic to \((\mathbb{Z}/N\mathbb{Z})^r\) for arbitrarily large \( N \). Then \( r \leq 2 \dim(X) \) and \( \text{Psaut}(X) \) contains a subgroup isomorphic to a compact real torus \((S^1)^r\). In addition, if \( r = 2 \dim(X) \), then \( X \) is biholomorphic to a compact complex torus.

**Proof.** As in the proof of [10, Theorem 4.5], we consider the action of \( \text{Psaut}(X) \) on \( H^2(X, \mathbb{Q}) \) by push-forward. We have an exact sequence of groups

\[
1 \to \text{Psaut}(X)_r \to \text{Psaut}(X) \to \text{Psaut}(X)/\text{Psaut}(X)_r \to 1,
\]

where we set

\[
\text{Psaut}(X)_r = \{ f \in \text{Psaut}(X) \mid f_*|_{H^2(X, \mathbb{Q})} = \text{Id} \}.
\]

Note that the quotient group \( \text{Psaut}(X)/\text{Psaut}(X)_r \) embeds into \( \text{GL}(H^2(X, \mathbb{Q})) \), therefore, by Minkowski’s theorem (see e.g. [30, Theorem 1]) the orders of finite subgroups of \( \text{Psaut}(X)/\text{Psaut}(X)_r \) are bounded by a constant \( M(X) \) depending on \( h^2(X, \mathbb{Q}) \) only. Hence the group \( \text{Psaut}(X)_r \) contains a sequence of finite abelian subgroups of asymptotic rank \( r \); in addition, by Lemma 2.2, we may assume that these subgroups are of the form \((\mathbb{Z}/N_i\mathbb{Z})^r\), where \( N_i \) tend to infinity. The group \( \text{Psaut}(X)_r \) acts trivially on \( H^2(X, \mathbb{R}) = H^2(X, \mathbb{Q}) \otimes \mathbb{R} \) and, in particular, it preserves every Kähler class on \( X \). Thus by Proposition 3.7, the group \( \text{Psaut}(X)_r \) is contained in \( \text{Aut}(X) \). The theorem now follows from Corollary 3.6. \( \square \)

### 3.2. Non-uniruled varieties and complex spaces.

In this subsection we use Theorem 3.8 to derive a slightly more general version of Theorem 1 from the Introduction.

To define minimal and quasi-minimal models of compact Kähler spaces, we need to introduce notions of nefness and modified nefness in the non-projective context (see [3, 14] for more details).

**Definition 3.9.** Let \( X \) be a normal compact Kähler space with rational singularities. We say that a class \( \alpha \in H^{1,1}(X, \mathbb{R}) \) is

- nef if it belongs to the closure of the cone of Kähler classes;
- modified nef if it belongs to the closure of the cone generated by classes of the form \( \mu_* \omega \) where \( \mu : Y \to X \) is an arbitrary bimeromorphic morphism from a smooth compact Kähler manifold \( Y \) and \( \omega \) is a Kähler class on \( Y \).

**Definition 3.10.** A compact Kähler space (or a projective variety) \( X \) with terminal \( \mathbb{Q} \)-factorial singularities is called

- minimal (or a minimal model) if the canonical class \( K_X \) is nef;
- quasi-minimal (or a quasi-minimal model) if \( K_X \) is modified nef.

Note that a minimal model is also quasi-minimal. Existence of quasi-minimal models for non-uniruled projective varieties was shown in [24, Lemma 4.4].

**Proposition 3.11.** Let \( X \) be a non-uniruled projective variety. Then there exists a quasi-minimal model of \( X \), that is, a quasi-minimal variety \( X' \) birational to \( X \).

In the case of non-uniruled compact Kähler spaces of dimension 3, minimal models exist by [14, Theorem 1.1].

**Theorem 3.12.** Let \( X \) be a compact Kähler space of dimension 3. Then there exists a minimal compact Kähler space \( X' \) bimeromorphic to \( X \).

The reason to consider quasi-minimal models is the following description of their bimeromorphic (or birational) automorphisms. The case when \( X \) is a projective variety was settled in [24, Corollary 4.7]; for the general case of compact Kähler spaces see [10, Proposition 4.2].
Proposition 3.13. Let $X$ be a quasi-minimal compact Kähler space. Let $f : X \dasharrow X$ be a bimeromorphic map. Then $f$ is a pseudoautomorphism.

Now we can prove Theorem 3.8 for a non-uniruled projective variety $X$ over an algebraically closed field $k$ of zero characteristic. Without loss of generality we may assume that $k = \mathbb{C}$.

Theorem 3.14. Let $X$ be a non-uniruled projective variety over the field of complex numbers. Suppose that there exists $r \in \mathbb{N}$ such that the group $\text{Bir}(X)$ contains finite abelian subgroups isomorphic to $(\mathbb{Z}/N\mathbb{Z})^r$ for arbitrarily large positive integers $N$. Then

$$r \leq 2 \dim(X),$$

and the group $\text{Bir}(X)$ contains a subgroup isomorphic to an abelian variety of dimension $\lfloor r/2 \rfloor$. In the case $r = 2 \dim(X)$ the variety $X$ is birational to an abelian variety.

Proof. Since $X$ is non-uniruled, by Proposition 3.11 there exists a quasi-minimal projective variety $X'$ birational to $X$. By Proposition 3.13 we have $\text{Bir}(X) \simeq \text{Bir}(X') = \text{Psaut}(X')$. The upper bound $r \leq 2 \dim(X)$ now follows from Theorem 3.8. Since $X$ is not covered by rational curves, the compact real torus $(S^1)^r$ in the connected component $\text{Aut}^0(X)$ can only be contained in an abelian variety of complex dimension at least $\lfloor r/2 \rfloor$.

An analogous result holds for a compact Kähler space $X$, under the assumption that a quasi-minimal model of $X$ exists. By Theorem 3.13 this condition holds if $\dim(X) \leq 3$.

Proposition 3.15. Let $X$ be a non-uniruled compact Kähler space admitting a quasi-minimal model. Let, for some $r \in \mathbb{N}$, the group $\text{Bim}(X)$ contain finite abelian subgroups isomorphic to $(\mathbb{Z}/N\mathbb{Z})^r$ for arbitrarily large $N$. Then $r \leq 2 \dim(X)$, and the group $\text{Bim}(X)$ contains a subgroup isomorphic to a compact complex torus of dimension $\lfloor r/2 \rfloor$. In addition, if $r = 2 \dim(X)$, then $X$ is bimeromorphic to a compact complex torus.

Proof. By the assumption, there exists a quasi-minimal compact Kähler space $X'$ bimeromorphic to $X$. By Proposition 3.13 $\text{Bim}(X) \simeq \text{Bim}(X') = \text{Psaut}(X')$. Now the required result is secured by Theorem 3.8.

3.3. Rationally connected varieties. We recall an important result on boundedness for finite groups acting on rationally connected algebraic varieties.

Proposition 3.16. Let $X$ be a rationally connected algebraic variety of dimension $n$ over an algebraically closed field $k$ of zero characteristic. Then there exists a constant $J(n)$, depending on $n$ only, such that for any finite subgroup $G \subseteq \text{Aut}(X)$ there exists a subgroup $H \subseteq G$ of index at most $J(n)$ acting on $X$ with a fixed point.

This result is immediate from [26, Theorem 4.2] and [3, Theorem 1.1].

As a result, we have the following upper bound for ranks of finite abelian subgroups in the group $\text{Bir}(X)$, where $X$ is a geometrically rationally connected algebraic variety over any field of zero characteristic.

Corollary 3.17. Let $X$ be a geometrically integral and geometrically rationally connected algebraic variety of dimension $n$ over an arbitrary field $k$ of zero characteristic. There exists a constant $M = M(n)$ such that, for any finite subgroup $G \subseteq \text{Bir}(X)$, there exists an abelian subgroup $H \subseteq G$ of index at most $M(n)$ and such the rank of $H$ does not exceed $n$.

Proof. We pass to the algebraic closure $\overline{k}$ of $k$ and replace $X$ by $X \times_k \overline{k}$, Let $\varphi : X' \dasharrow X$ be a smooth birational regularization of the action of $G$, which exists by Proposition 2.13. Note that $X'$ is rationally connected as well. Therefore by Proposition 3.16 there exists a constant $J'(n)$ such that $G$ contains a subgroup $H$ of index at most $J'(n)$ acting on $X'$ with a fixed point. By Proposition 2.19 the group $H$ embeds into $\text{GL}_n(\overline{k})$. Therefore by Jordan’s theorem (see [15] or [29]) there exists a constant $J''(n)$ such that the group $H$ contains an abelian subgroup $A \subseteq H$ of index at most $J''(n)$. Then $A \subseteq G$ is a subgroup of index at most

$$M(n) = J'(n) \cdot J''(n),$$

moreover, since $A$ is linear, it is generated by at most $n$ elements.
Let us now show that any sequence of finite abelian subgroups in Bir($X$) has asymptotic rank at most $n$ (see Corollary 2.11).

**Theorem 3.18.** Let $X$ be a geometrically integral and geometrically rationally connected algebraic variety of dimension $n$ over an arbitrary field $k$ of zero characteristic. Suppose that there exists an unbounded sequence $\{N_i\}_{i \in \mathbb{N}}$ of positive integers such that the group Bir($X$) contains a subgroup $G_i \cong (\mathbb{Z}/N_i \mathbb{Z})^r$ for some fixed $r \in \mathbb{N}$. Then $r \leq n$.

**Proof.** By Proposition 3, there exists a constant $M(n)$ such that, for each $i \in \mathbb{N}$, there exists an abelian subgroup $H_i \subseteq G_i$ of index $\leq M(n)$. Hence the asymptotic rank of $\{G_i\}$ is equal to that of $\{H_i\}$, which is at most $n$, because all finite abelian groups $H_i$ are of rank $\leq n$. This proves the theorem. \(\square\)

### 3.4. The general case.

Now we can prove Theorems 1.5 and 1.6 from the Introduction.

**Proof of Theorem 1.5.** Passing to a resolution of singularities, we may assume that $X$ is smooth. If $X$ is not uniruled then the result follows from Theorem 3.14. Suppose that $X$ is uniruled and consider its MRC fibration $f: X \dashrightarrow B$, where $\dim(B) < \dim(X)$. Then for every $i \in \mathbb{N}$ we have the exact sequence

$$1 \rightarrow G_i' \rightarrow G_i \rightarrow G''_i \rightarrow 1,$$

where the action of $G''_i$ is fiberwise with respect to $f$ (that is, every element $g \in G''_i$ maps a point in a fiber of $f$ where $g$ is defined to a point in the same fiber) and $G''_i$ acts faithfully on the base $B$.

Let $X_\eta$ be the scheme-theoretic generic fiber of $f$. Then for every $i \in \mathbb{N}$ we have $G_i' \subseteq \text{Bir}(X_\eta)$. We denote $n' = \dim(X_\eta)$. Then by Theorem 3.18 the asymptotic rank of the sequence $\{G_i'\}$ is at most $n'$. Since $B$ is not uniruled by Proposition 2.16 we apply Theorem 3.14 and obtain that the asymptotic rank of the sequence $\{G''_i\}$ is at most $2\dim(B)$. Therefore by Lemma 2.11 the asymptotic rank $r$ of the sequence $\{G_i\}$ is at most $n' + 2\dim(B)$. In particular,

$$r \leq 2\dim(B) + n' = \dim(X) + \dim(B) < 2n,$$

the result required. \(\square\)

It is also possible to describe projective varieties such that Bir($X$) contains a sequence of finite abelian groups of submaximal asymptotic rank.

**Corollary 3.19.** Let $X$ be a projective variety over an algebraically closed field of zero characteristic. Suppose that there exists an unbounded sequence $\{N_i\}_{i \in \mathbb{N}}$ of positive integers such that the group Bir($X$) contains subgroups $G_i$ isomorphic to $(\mathbb{Z}/N_i \mathbb{Z})^r$ for $r = 2\dim(X) - 1$. Then $X$ is birational either to

- an abelian variety $A$;
- the product $\mathbb{P}^1 \times A$ where $A$ is an abelian variety of dimension $\dim(X) - 1$.

**Proof.** For $\dim(X) = 1$ the result is obvious, since $X$ is then isomorphic to a rational or elliptic curve. Suppose from now on that $\dim(X) > 1$. If $X$ is not uniruled, then by Theorem 3.14 there exists a birational model $X'$ of $X$ and a faithful action of an abelian variety of dimension

$$\frac{2\dim(X') - 1}{2} = \dim(X') = \dim(X)$$

on $X'$, so that by Theorem 1.5 $X$ is birational to an abelian variety.

Suppose now that $X$ is uniruled and consider the MRC fibration $f: X \dashrightarrow B$. Since $\dim(X) > 1$, we have $2\dim(X) - 1 > \dim(X)$ and therefore by Theorem 3.14 $X$ cannot be rationally connected, that is, $\dim(B) > 0$. Let $X_\eta$ be the general fiber of $f$. Let also $\{G_i'\}$ be the sequence of subgroups acting fiberwise with respect to $f$, and denote by $\{G''_i\}$ the sequence of quotient groups. By Theorem 3.18 the asymptotic rank of $\{G_i'\}$ does not exceed $\dim(X_\eta) \leq \dim(X)$. Therefore the asymptotic rank of $\{G''_i\}$ is at least

$$2\dim(X) - 1 - \dim(X_\eta) \geq 2\dim(B) > 0.$$ 

Now by Theorem 3.14 the non-uniruled variety $B$ is birational to an abelian variety $A$; moreover, $A$ has maximal possible dimension, equal to $\dim(X) - 1$. Since the asymptotic rank of $\{G''_i\}$ is equal to 1, it follows by [6, Theorem 4.14] that $X_\eta \cong \mathbb{P}^1_{k(B)}$, and so $X$ is birational to a product $\mathbb{P}^1 \times A$. \(\square\)
Before proceeding with the proof of Theorem 1.6, we need the following technical lemma (see [28, Lemma 3.1]) for the proof. Recall that a very typical fiber of a dominant meromorphic map \( \alpha: X \to Y \) is a fiber \( X_t = \alpha^{-1}(t) \) over a point \( t \in Y \) in the complement to at most countable union of proper analytic subspaces of \( Y \). By \( \text{Bim}(X)_{\alpha} \) we denote the subgroup of elements of \( \text{Bim}(X) \) acting fiberwise with respect to \( \alpha \).

**Lemma 3.20.** Let \( \alpha: X \to Y \) be a dominant meromorphic map of compact complex manifolds. Then there exist a constant \( I = I(\alpha) \) with the following property. Let \( \{G_i\}_{i \in \mathbb{N}} \) be a sequence of finite subgroups of \( \text{Bim}(X)_{\alpha} \). Then there exists a reduced fiber \( F \) of \( \alpha \) and its irreducible component \( F' \) of dimension \( \dim(X) - \dim(Y) \) such that for every \( i \in \mathbb{N} \) the group \( G_i \) contains a subgroup of index at most \( I \), which is isomorphic to a subgroup of \( \text{Bim}(F') \). Moreover, if \( \dim(Y) > 0 \) the fiber \( F \) can be chosen to be very typical.

Now we can apply the same line of reasoning to the case of compact Kähler spaces, applying Lemma 3.20 to the MRC fibration of \( X \).

**Proof of Theorem 1.6.** Passing to a resolution of singularities, we may assume that \( X \) is smooth. If \( X \) is not uniruled then the result follows from Proposition 3.15.

Suppose that \( X \) is uniruled. Then we consider the MRC fibration \( f: X \to B \) with \( B \) non-uniruled and \( \dim(B) < \dim(X) \). If \( \dim(B) = 0 \) then \( X \) is rationally connected and hence projective by Proposition 2.17; this case follows from Theorem 3.18. Assume from now on that \( \dim(B) > 0 \). Then for every \( i \in \mathbb{N} \) there exists an exact sequence of groups

\[
1 \to G_i' \to G_i \to G_i'' \to 1,
\]

where the action of \( G_i'' \) is fiberwise with respect to \( f \) and \( G_i'' \) acts faithfully on \( B \). Since the set of finite groups \( \{G_i\}_{i \in \mathbb{N}} \) is countable, by Lemma 3.20 we may assume that

\[
G_i' \subset \text{Bim}(X_t),
\]

where \( X_t \) is a very typical (in particular, smooth) fiber of \( f \). Note that by Proposition 2.17, smooth fibers of \( f \) are projective. Now by Theorem 3.18 the asymptotic rank of the sequence \( \{G_i'\} \) is at most \( \dim(X_t) \). Moreover, by the assumptions on \( B \) and by Theorem 3.14 the asymptotic rank of the sequence \( \{G_i''\} \) is at most \( 2 \dim(B) \). By Lemma 2.11 the asymptotic rank \( r \) of the sequence \( \{G_i\} \) is at most \( 2 \dim(B) + \dim(X_t) \); in particular,

\[
r \leq \dim(X_t) + 2 \dim(B) < 2 \dim(X),
\]

as desired. \( \square \)

**Remark 3.21.** To prove Theorem 1.6 in full generality, it suffices to prove that “large” finite abelian subgroups of \( \text{Bim}(X) \) can be pseudo-regularized on a compact Kähler manifold \( X' \) birational to \( X \). By considering the algebraic reduction (see [32, Definition 3.3]) of a compact Kähler space \( X \), it suffices to resolve the above problem in the case when \( X \) has algebraic dimension 0. In particular, if \( X \) has no divisors (like a general compact complex torus) this statement is clear, since \( \text{Bim}(X) = \text{Psaut}(X) \) in this case.

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