Wave Particle Duality in Asymmetric Beam Interference

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It is well known that in a two-slit interference experiment, acquiring which-path information about the particle, leads to a degrading of the interference. It is argued that path-information has a meaning only when one can unambiguously tell which slit the particle went through. Using this idea, two duality relations are derived for the general case where the two paths may not be equally probable, and the two slits may be of unequal widths. These duality relations, which are inequalities in general, saturate for all pure states. Earlier known results are recovered in suitable limit.

I. INTRODUCTION

Wave-particle duality is an intriguing aspect of nature, which was first conceptualized by Neils Bohr in his principle of complementarity [1]. It has been in debate right from the beginning when Einstein’s raised objections against it, proposing his famous recoiling slit experiment [2], and continues to be so even today [3]. The two-slit interference experiment has become a cornerstone for investigating such issues.

Bohr had emphasized that the wave and particle natures are mutually exclusive, revealing one, completely hides the other. Wooters and Zurek began by asking what happens if one probes the wave and particles natures at the same time, in a two-slit experiment [4]. They found that it is indeed possible to partially reveal both the natures. This idea was later put on firm mathematical ground by Englert, in the form of a duality relation which puts a bound on how much of each nature can be revealed simultaneously: [5]

D^2 + V^2 \leq 1, \tag{1}

where D is path distinguishability, a measure of the particle nature, and V the visibility of interference, a measure of wave nature. Wave and particle natures are so fundamental to quantum objects that many prefer to call them quantons [6, 7]. A different kind of duality relations are also studied where one tries to predict the path information of the quanton based on the asymmetry of the two beams, without using any path detector [8, 9].

Contemporary thinking is that, in a two-slit interference experiment, if one is able to tell which of the two slits the quanton went through, one has revealed the particle nature of the quanton. On the other hand, if one obtains an interference pattern, one has revealed the wave aspect of the quanton. The duality relation derived by Englert was for a symmetric two-slit experiment, in which the quanton is equally likely to go through both the slits. However, there can be situations where the setup is not symmetric, i.e., the state of the incident quanton is such that the probabilities to go through the two slits are not equal. Another possibility is that the two slits may not be of equal widths. This asymmetric case has not been probed in as much detail as the symmetric case [5]. There have been other studies on asymmetric two-path interference [10, 11], but none provides a tight duality relation for this case. The aim of this paper is to obtain a general duality relation which also holds for asymmetric two-slit interference experiments.

II. DEALING WITH ASYMMETRY

We assume that the state of the quanton that emerges from the double-slit is given by the unnormalized state

|ψ⟩ = \sqrt{p_1}|ψ_1⟩ + \sqrt{p_2}|ψ_2⟩, \tag{2}

where p_1, p_2 quantify the asymmetry of the incoming wave. In addition, we would like to take into account the effect of asymmetric slits, namely the situation where the two slits may have different widths. As different slit widths will also contribute to the probabilities of passing through the two slits, one should assume that the states |ψ_1⟩, |ψ_2⟩, corresponding to the quanton coming out of slit 1 and 2, respectively, are not normalized. One can see that for the case p_1 = p_2 = 1/2, the probability of the quanton to pass through slit 1 and 2 is proportional to ⟨ψ_1|ψ_1⟩ and ⟨ψ_2|ψ_2⟩, respectively.

Given that the incoming quanton state is symmetric, the quanton is more likely to pass through the wider slit. The details of the effect of asymmetry due to the slits will be specified while choosing the form of |ψ_1⟩, |ψ_2⟩. Needless to say, the state |ψ⟩ as a whole should be normalized.
III. GETTING WHICH-WAY INFORMATION

An experiment to find out which of the two paths a quanton has followed, would require some kind of detector which can retrieve and store information on which path a particular quanton took. We assume a fully quantum detector with states corresponding to the quanton taking one path or the other. Without going into the details of what this path detector should be like, we just use von Neumann’s criterion for a quantum measurement [12]. According to von Neumann’s criterion, the basic requirement for a path detector to perform a which-path measurement is that it should interact with the quanton in such a way that its two states should get correlated with the two paths of the quanton. If the state of the quanton that emerges from the asymmetric double-slit is given by (2), and the path-detector is in an initial state $|d_0⟩$, the interaction between the two should be such that it evolves to the following:

$$\sqrt{p_1}|ψ_1⟩ + \sqrt{p_2}|ψ_2⟩ \rightarrow \sqrt{p_1}|ψ_1⟩|d_1⟩ + \sqrt{p_2}|ψ_2⟩|d_2⟩$$

(3)

The quanton goes and registers on the screen, and the path-detector is left with the experimenter. If the experimenter finds that the state of the path-detector is $|d_1⟩$, she can conclude that the quanton went through slit 1, else if the state of the path-detector is $|d_2⟩$, it would imply that the quanton went through slit 2. The interaction between the quanton and the path-detector is designed by the experimenter, and thus the states $|d_1⟩$, $|d_2⟩$ are known. What is not known is, which of the two states one would get, for particular instance of quanton going through the double-slit.

The problem of finding which path the quanton followed then reduces to telling whether the state of the path-detector is $|d_1⟩$ or $|d_2⟩$. To solve this problem, Englert took the approach of calculating the optimum “likelihood for guessing the way (which of the two ways the quanton went) right”. We take a somewhat different route. We believe that for any given instance of quanton passing through the double-slit, one can claim to have which-path knowledge only if one can tell for sure which of the two paths the quanton took. What we mean is, there should be no guessing involved. One should have an unambiguous answer to the question which path the quanton took. In the path-detection scenario discussed above, this would mean one should be able unambiguously tell which of the two states $|d_1⟩$ or $|d_2⟩$, is the given state of the path-detector. If $|d_1⟩$ and $|d_2⟩$ are orthogonal, one can measure an observable of the path-detector which has $|d_1⟩$ and $|d_2⟩$ as its two eigenstates, with distinct eigenvalues. Looking at the eigenvalue of the measurement, one would know for sure that the path-detector was (say) $|d_2⟩$, and hence the quanton went through the lower slit. However, there are situations in which $|d_1⟩$ and $|d_2⟩$ are not orthogonal. There exists a method which allows for unambiguously distinguishing between two non-orthogonal states, and goes by the name of unambiguous quantum state discrimination (UQSD) [13,16]. A downside of this method is that there will be occasions where the method will fail to provide an answer, but the experimenter will know that it has failed. Thus, on the occasions on which UQSD succeeds, it can unambiguously distinguish between the two non-orthogonal states. The measurement method can be tuned to minimize the failure probability, and thus maximizing the probability of unambiguously distinguishing between the two states.

IV. UNAMBIGUOUS PATH DISCRIMINATION

The UQSD approach has been successfully used in defining a new distinguishability of paths, $D_Q$, as the maximum probability of unambiguously distinguishing between the available quanton paths. This resulted in new duality relations for the symmetric two-slit interference [17], three-slit interference [17], and n-slit interference [18, 19]. Here we use it to study wave-particle duality in the case of interference involving asymmetric paths.

We begin at the instance the quanton emerges from the double-slit. The state of the quanton has to be a superposition of two localized parts, in front of slit 1 and 2, respectively. We assume the quanton is traveling along the positive y-axis, and the double-slit is the in the x-z plane, at $y = 0$ (see FIG. 1). For the purpose of interference, the motion along the y-axis is unimportant. It is the spread of the two emerging wave-packets along the x-axis and the overlap, which gives rise to interference. We neglect the dynamics along y-axis, and assume that the quanton is traveling along y-axis with an average momentum $p_0$, and that motion only serves to translate the quanton from the slit to the screen with time. For calculational simplicity, we assume the parts of the state emerging from the double-slit to be Gaussian wave-packets, localized in front of the two slits, namely at positions $x = x_0$ and $x = -x_0$. The state of the quanton at time $t = 0$, is given by

$$⟨x|ψ(0)⟩ = A \left( \sqrt{p_1}e^{-\frac{(x-x_0)^2}{2\epsilon^2}} + \sqrt{p_2}e^{-\frac{(x+x_0)^2}{2\epsilon^2}} \right)$$

(4)

where $A = \left( \frac{2}{\pi\epsilon(p_1+p_2)} \right)^{1/4}$, $d = 2x_0$ is the separation between the slits and $\epsilon$ and $\xi\epsilon$ are the widths of the two Gaussians, and may loosely be considered the widths of the two slits. At the instant of emerging from the double-slit, the quanton interacts with a path-detector, and the combined state of the two should have the following form (as argued earlier):

$$⟨x|ψ(0)⟩ = A \left( \sqrt{p_1}e^{-\frac{(x-x_0)^2}{\epsilon^2}} |d_1⟩ + \sqrt{p_2}e^{-\frac{(x+x_0)^2}{\epsilon^2}} |d_2⟩ \right)$$

(5)

where $|d_1⟩$, $|d_2⟩$ are the two states of the path-detector. The states $|d_1⟩$, $|d_2⟩$ are chosen to be normalized, although they are not orthogonal in general. It may
be mentioned that choosing the probability amplitudes \( \sqrt{p_1}, \sqrt{p_2} \) to be real and positive is not a loss of generality as \( |d_1\rangle, |d_2\rangle \) may contain phases.

Now, the idea is to find out how much path information of the quanton can be retrieved from the path detector, in principle, given the state \( \langle 5 \rangle \). We would like to stress the point that a particular method of probing the path-detector may yield a certain amount of path information, but we are interested in best possible value that can be obtained in principle. UQSD works for the situation where the two states, \( |d_1\rangle, |d_2\rangle \) occur randomly with different probability. If one is given one of the states and asked to tell which of the two it is, UQSD allows one to give the best possible answer. To use this method for the problem at hand, we should ascertain the probabilities with which \( |d_1\rangle, |d_2\rangle \) occur. Looking at \( \langle 5 \rangle \) one may naively jump to the conclusion that the probabilities in question are \( p_1 \) and \( p_2 \). However, the different widths of the two slits would also contribute to the probability of the quanton passing through slit \( k \) resulting the path-detector state \( |d_k\rangle \). The probability amplitude for this possibility is given by \( c_k = \sqrt{\langle \psi_1|d_k\rangle \langle \psi_0|d_k\rangle} \), where \( k = 1, 2 \). Using the Gaussian form given in \( \langle 5 \rangle \), these probability amplitudes turn out to be

\[
    c_1 = \frac{\sqrt{p_1}}{\sqrt{p_1 + \xi p_2}}, \quad c_2 = \frac{\sqrt{p_2}}{\sqrt{p_1 + \xi p_2}}.
\]

As far as measurements on the path-detector are concerned, it can be assumed to randomly found in the state \( |d_1\rangle \) with probability \( c_1^2 \), and in the state \( |d_2\rangle \) with probability \( c_2^2 \). In addition, without loss of generality, we assume that \( c_1 \geq c_2 \).

In order to use UQSD, we assume that the Hilbert space of the path-detector is not two dimensional, but three dimensional, described by an orthonormal basis of states \( |q_1\rangle, |q_2\rangle, |q_3\rangle \). The reason for doing so will become clear in the following analysis. The basis is chosen in such a way that the detector states \( |d_1\rangle, |d_2\rangle \) can be represented as \( \langle 16 \rangle \)

\[
    |d_1\rangle = \alpha |q_1\rangle + \beta |q_3\rangle, \\
    |d_2\rangle = \gamma |q_2\rangle + \delta |q_3\rangle,
\]

where \( \alpha \) and \( \gamma \) are real, and \( \beta, \delta \) satisfy

\[
    |\beta|/|\alpha| \geq \frac{|\langle d_1|d_2\rangle|}{|\langle d_1|d_2\rangle|} \\
    |\beta|^2 = \max \{|\langle d_1|d_2\rangle|c_2/c_1, |\langle d_1|d_2\rangle|^2\}
\]

In the expanded Hilbert space, one can now measure an operator (say)

\[
    A = |q_1\rangle\langle q_1| + 2|q_2\rangle\langle q_2| + 3|q_3\rangle|q_3\rangle.
\]

It is straightforward to see that getting eigenvalue 1 means the state was \( |d_1\rangle \), getting eigenvalue 2 means the state was \( |d_2\rangle \). However, there is also a finite probability of getting eigenvalue 3, in which case one cannot tell if the state was \( |d_1\rangle \) or \( |d_2\rangle \). One would like to minimize the probability of getting the eigenvalue 3, or failure of the state discrimination. It can be shown that chosen values of \( \beta, \delta \) in \( \langle 5 \rangle \) are such that they minimize the probability of failure, and maximize the probability of successfully distinguishing between \( |d_1\rangle \) and \( |d_2\rangle \). We will return to these in more detail later.

\[
    \langle d_1|d_2\rangle = |\langle d_1|d_2\rangle|e^{i\phi}. \quad \text{The probability of the quanton to}
\]

V. INTERFERENCE AND FRINGE VISIBILITY

We now analyze what happens when the quanton reaches the screen. We assume that the quanton takes a time \( t \) to travel along y-axis from the double-slit to the screen, a distance \( D \) (see FIG. 1). The time evolution depends on what is the nature of our quanton. It could be a photon traveling with the speed of light, or it could be a particle of mass \( m \) under free time-evolution. One can write the time evolution of the state in a universal form

\[
    |\psi(t)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |k\rangle \langle \psi(0)| e^{-i\omega_k t} dk
\]

where \( |k\rangle \) the momentum states. For photons \( \omega_k = ck \) and for massive particles \( \omega_k = \sqrt{p^2/c^2 + \hbar^2} \). The state of the quanton, after a time \( t \) (after traveling a distance \( D \) from the double-slit to the screen), can be worked out to be \( \langle 20 \rangle \)

\[
    \langle x|\psi(t)\rangle = B \left( \sqrt{\frac{\pi}{\hbar^2}} e^{-i\omega_k^2 x^2/\hbar} |d_1\rangle + \sqrt{\frac{\pi}{\hbar^2}} e^{-i\omega_k^2 x^2/\hbar} |d_2\rangle \right),
\]

where \( \Gamma = 2\pi/k = \lambda D/\pi \), if one defines \( \lambda = h/p_0 \), and \( B = \sqrt{\frac{2\hbar}{\pi\hbar^2 + \pi\hbar^2}} \). It can be shown that if the quanton is a photon, one gets the same expression with \( \Gamma = \lambda D/\pi \), where \( \lambda \) is the wavelength of the photon. Let us assume a phase factor associated with the detector states: \( \langle d_1|d_2\rangle = |\langle d_1|d_2\rangle|e^{i\phi} \). The probability of the quanton to
arrive at a position $x$ on the screen is then given by

$$\langle x|\psi(t)\rangle^2 = B^2 \left[ \frac{p_1}{\sqrt{c^2+1}} e^{-\frac{2ix^2\alpha_0^2}{c^2+1}} + \xi p_2 e^{-\frac{2ix^2\alpha_1^2}{c^2+1}} + \frac{\sqrt{\xi p_2}}{\sqrt{c^2+1}} \langle d_1|d_2\rangle |e^{-\frac{ix(\alpha_0-\alpha_1)}{c^2+1} - \theta} \right].$$

(12)

In the Fraunhofer limit $\lambda D \gg \epsilon^2$, which implies $\Gamma^2 \gg \epsilon^4$, the above can be simplified to

$$\langle x|\psi(t)\rangle^2 = B^2 \left[ \xi p_2 e^{-\frac{2ix^2\alpha_0^2}{c^2+1}} + \frac{\sqrt{\xi p_2}}{\sqrt{c^2+1}} \langle d_1|d_2\rangle e^{-\frac{ix(\alpha_0-\alpha_1)}{c^2+1} - \theta} \right].$$

(13)

Eqn. (12) represents a two-slit interference pattern, with a fringe width $w = \lambda D / \epsilon$. We assume that the intensity at position $x$ is given by $I(x) \propto |\langle x|\psi(t)\rangle|^2$. The maxima and minima of intensity occur at the values of $x$ where the cosine term is 1 and -1, respectively.

The visibility of the interference pattern is just the contrast in intensities of neighbouring fringes

$$V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}},$$

(14)

where $I_{\text{max}}$ and $I_{\text{min}}$ represent the maximum and minimum intensity in neighbouring fringes. The interference in (13), ignoring the effect of a finite slit-width $\epsilon$, yields the following ideal fringe visibility:

$$V = \frac{2\sqrt{p_1 p_2 \xi}}{p_1 + \xi p_2} |\langle d_1|d_2\rangle|.$$

(15)

If $|d_1\rangle = |d_2\rangle$, which means that the path-detector is effectively absent, the fringe visibility reduces to $V_0 = \frac{2\sqrt{p_1 p_2 \xi}}{p_1 + \xi p_2}$, and is called the a priori fringe visibility. This means that even if no which-path information is extracted, the visibility will be less than 1 if either the incoming state is asymmetric, or the slits are of unequal widths.

VI. DISTINGUISHABILITY & DUALITY RELATIONS

Coming back to the issue of getting unambiguous path information about the quanton, we note that (13) implies two cases: (a) $|\beta|^2 = |\langle d_1|d_2\rangle|^2 c_2/c_1 \geq |\langle d_1|d_2\rangle|^2$ and (b) $|\beta|^2 = |\langle d_1|d_2\rangle|^2 > |\langle d_1|d_2\rangle|^2 c_2/c_1$. These should be discussed separately. We define the distinguishability of two paths, $D_Q$, as the maximum probability with which the two paths can be unambiguously distinguished. To get distinguishability, we first use (17) to rewrite (11) as

$$|\psi(t)\rangle = c_1 |\psi_1(t)\rangle |d_1\rangle + c_2 |\psi_2(t)\rangle |d_2\rangle = c_1 a |\psi_1(t)\rangle |q_1\rangle + c_2 a \gamma |\psi_2(t)\rangle |q_2\rangle + (a \beta |\psi_1(t)\rangle + a \delta |\psi_2(t)\rangle) |q_3\rangle$$

(16)

where $|\psi_1(t)\rangle$, $|\psi_2(t)\rangle$ represent the wave-packets appearing in (11). From (16) one can see that the unambiguous path discrimination fails when one gets the state $|q_3\rangle$ while measuring the operator $A$. The probability of failure is just $|\langle q_3|\psi(t)\rangle|^2$ which turns out to be $c_1^2 |\beta|^2 + c_2^2 |\delta|^2$, using the orthogonality of $|\psi_1(t)\rangle$, $|\psi_2(t)\rangle$. Subtracting that from 1, gives the optimal probability of unambiguously distinguishing between the two paths. Thus we can write

$$D_Q = 1 - |\langle q_3|\psi(t)\rangle|^2,$$

(17)

which is our general expression for path distinguishability. The distinguishability may also be calculated from the successful discrimination as

$$D_Q = |\langle q_1|\psi(t)\rangle|^2 + |\langle q_2|\psi(t)\rangle|^2,$$

(18)

which would just be $c_1^2 a^2 + c_2^2 \gamma^2$.

A. Case: $|\langle d_1|d_2\rangle| c_2/c_1 \geq |\langle d_1|d_2\rangle|^2$

This is the case when the orthogonality of $|d_1\rangle$, $|d_2\rangle$ is on the stronger side, and the asymmetry is not extreme. In this case the values of $a$, $\gamma$, for optimal success, are given by

$$a = \sqrt{1 - |\langle d_1|d_2\rangle|^2 c_2/c_1}, \quad \gamma = \sqrt{1 - |\langle d_1|d_2\rangle|^2 c_1/c_2}.$$

(19)

Using (18), the distinguishability has the form

$$D_Q = 1 - 2c_1 c_2 |\langle d_1|d_2\rangle|.$$

(20)

Using (19) and (15) we arrive at the following relation

$$V = \frac{2\sqrt{p_1 p_2 \xi}}{p_1 + \xi p_2} (1 - D_Q)$$

(21)

Using (6), the above equation reduces to a very simple duality relation

$$D_Q + V = 1.$$

(22)

This duality relation generalizes Englert’s relation (1) to the case of asymmetric incoming quanton state, and is an equality, not an inequality for any pure state. The relation (22) implies that if one is able to unambiguously distinguish between the two paths with a probability $P$ by any method, that $P$ cannot exceed $D_Q$, and the fringe visibility cannot exceed $1 - D_Q$. 

If the state of the incoming quanton happens to be symmetric, i.e., \( p_1 = p_2 = 1/2 \), and the two slits are of same width, i.e., \( \xi = 1 \), we can define a new distinguishability \( D \) as

\[
D^2 = D_Q(2 - D_Q) = 1 - |\langle d_1 | d_2 \rangle|^2,
\]

which is precisely Englert’s distinguishability \([5]\). Using \([15]\) we can write

\[
D^2 + V^2 = 1,
\]

which is just the saturated form of Englert’s duality relation \([1]\). So for the symmetric case, \([22]\) is essentially the same as \([1]\).

### B. Case \(|\langle d_1 | d_2 \rangle|^2 > |\langle d_1 | d_2 \rangle|c_2/c_1\)

This is the case when the orthogonality of \( |d_1\rangle, |d_2\rangle \) is on the lower side, and the asymmetry is large. In this case the values of the constants are as follows \([16]\)

\[
\alpha = \sqrt{1 - |\langle d_1 | d_2 \rangle|^2}, \quad \beta = |\langle d_1 | d_2 \rangle|, \quad \gamma = 0, \quad |\delta| = 1.
\]

The expression for distinguishability can be obtained by using \([18]\):

\[
D_Q = c_1^2(1 - |\langle d_1 | d_2 \rangle|^2).
\]

Combining \([26]\) and \([15]\), we can write

\[
\frac{D_Q}{c_1^2} + \frac{V^2(p_1 + \xi p_2)^2}{4p_1p_2}\xi = 1,
\]

which can be rewritten as a new duality relation for this specific case:

\[
\frac{D_Q}{c_1^2(1 + \mathcal{P}_0)} + \frac{V^2}{\mathcal{V}_0^2} = 1,
\]

where \( \mathcal{V}_0 \) is the \textit{a priori} fringe visibility, and \( \mathcal{P}_0 \) is the \textit{a priori} path-predictability defined as \( \mathcal{P}_0 = |\xi|/\sqrt{|\xi|^2 + |\xi|^2} \). As a consistency check, we consider the case where \( |d_1\rangle, |d_2\rangle \) are identical, and hence \( D_Q \) given by \([26]\) is zero. Here the visibility is reduced to the \textit{a priori} fringe visibility, as it should when there is no path-detection. Another special case is when \( p_1 = 1 \), in which case \( V \) becomes zero, and \([28]\) gives \( D_Q = 1 \). Notice that varying the widths of the slits affects the \textit{a priori} fringe visibility and the \textit{a priori} path-predictability, but the equality \([28]\) continues to hold.

One might wonder if it is possible to have a single duality relation for both the cases. To address this question, we denote the distinguishability in the first case, i.e. \([20]\), by \( D_{Q1} \) and that in the second case \([26]\), by \( D_{Q2} \). Then, in the region \(|\langle d_1 | d_2 \rangle|^2 > |\langle d_1 | d_2 \rangle|^2/c_2/c_1\), one can show that

\[
D_{Q1} - D_{Q2} = c_1^2(\langle d_1 | d_2 \rangle| - c_2/c_1)^2,
\]

which means that \( D_{Q2} \leq D_{Q1} \). This implies that the following inequality holds in all regions

\[
D_Q + V \leq 1,
\]

but it cannot be saturated in the region \(|\langle d_1 | d_2 \rangle|^2 > |\langle d_1 | d_2 \rangle|^2/c_2/c_1\).

So we see that one cannot have a tight single duality relation for all asymmetric two-slit experiments. Depending on the asymmetry and the orthogonality of the path detector states, the duality relation has two distinct forms, \([22]\) and \([28]\).

### C. The general case (pure/mixed)

Till now we have been looking at the case where quanton and the path detector are in a pure state. However, there are effects of decoherence due to which there can be some loss of coherence, and it may become necessary to treat the quanton and path detector combine as a mixed state. In such a situation, the state of the quanton and path detector will be represented by a mixed state density matrix. The treatment of path-distinguishability will remain unchanged. For example, the path distinguishability given by \([17]\) will now be represented as \( D_Q = 1 - \text{Trace}[\rho(t)|q_3\rangle\langle q_3|] \), and that given by \([18]\) will be represented as \( D_Q = \text{Trace}[\rho(t)|q_1\rangle\langle q_1|] + \text{Trace}[\rho(t)|q_2\rangle\langle q_2|] \).

Fringe visibility is a measure of quantum coherence in the system, and any mixedness will degrade the interference. This statement can be put on a strong footing as follows. Recently a new measure of quantum coherence was introduced, which, in a normalized form, can be written as \( C = \frac{1}{n-1} \sum \rho_{ii} \). In our context, \( \rho_{ij} \) are the elements of the density matrix of the quanton, \( i, j \) corresponding to the two paths, and \( n \) is the dimensionality of the Hilbert space (in our case \( n = 2 \) corresponding to the two paths). Using the final state of the quanton plus path detector as \( \rho(t) = c_1|\psi_1(t)\rangle\langle d_1| + c_2|\psi_2(t)\rangle\langle d_2| \), we first trace over the path detector states to get a reduced density matrix. The coherence \( C \) can then be evaluated, and turns out to be

\[
C = 2c_1c_2|\langle d_1 | d_2 \rangle|.
\]

We see that for the two-slit interference, coherence is the same as visibility. It has been shown that any incoherent operation on the system will lead to a decrease in coherence \( C \). In our case it means, any mixedness introduced in the system will lead to a decrease in the visibility \( V \).

Consequently, the visibility will now be less than the maximum allowed by the amount of path information.
that has been acquired by the path detector. For the case $|d_1|d_2| \leq c_2/c_1$, it means $V \leq 1 - D_Q$. Thus the duality relation becomes the inequality
\[ D_Q + V \leq 1. \tag{32} \]
Similarly, for the case $|d_1|d_2| > c_2/c_1$ too, the duality relation becomes
\[ \frac{D_Q}{2(1 + P_0)} + \frac{V^2}{V_0^2} \leq 1. \tag{33} \]
The inequalities (32) and (33) quantify wave-particle duality for an asymmetric two-slit interference. They are saturated for any pure state.

D. Particle or wave?

The thought experiment in the preceding discussion, with an expanded Hilbert space, was introduced to get an upper bound on the probability with which the two paths can be unambiguously distinguished. However, if one were to actually carry out this experiment with an observable $A$ of the path detector giving three measured values, an interesting possibility emerges. Suppose each quanton is detected on the screen in coincidence with measurement of the observable $A$. Once the path detector is in place, the interference does not depend on what observable of the path detector we choose to measure. Everytime we get the measured value 1, we know the quanton went through slit 1, and everytime we get the value 2, we know that the particular quanton went through slit 2. In these two situations, the quanton behaves like a particle, choosing one of the two available paths. However, when the measurement of $A$ yields the value 3, we conclude that the quanton went through both the slits at the same time, behaving like a spread-out wave. In fact, this can be experimentally verified by separating the detected quantons into two groups, one where $A$ gave value 1 or 2, and two where $A$ gave value 3. The first group of quantons will show no interference, since path information for each of them is stored in the path detector. The second group of quantons will show full interference.

The state of the quantons, for which measurement of $A$ gives value 3, can be written using (16) as
\[
\langle q_3|\psi(t)\rangle = c_1\alpha|\psi_1(t)\rangle\langle q_3|q_1\rangle + c_2\gamma|\psi_2(t)\rangle\langle q_3|q_2\rangle \\
+ (c_1\beta|\psi_1(t)\rangle + c_2\delta|\psi_2(t)\rangle)\langle q_3|q_3\rangle \\
= c_1\beta|\psi_1(t)\rangle + c_2\delta|\psi_2(t)\rangle \tag{34} \]

It is obvious that the above state will produce interference. This leads us to conclude that in a two-slit experiment with an imperfect path detector in place, each quanton can be thought of as randomly choosing to behave like a particle or a wave. This behaviour, obviously, is forced by the presence of the path detector, in agreement with the philosophy behind Bohr’s principle of complementarity [1].

VI. CONCLUSION

In conclusion, we have analyzed the issue of wave-particle duality in a two-slit experiment. For symmetric beams and equal slit widths, wave-particle duality can be captured by the well-known inequality [1], which was derived using the ideas of minimum error discrimination of states [5]. The same relation can be derived by defining the distinguishability using the ideas of UQSD [17]. This latter method has proved to be very useful in describing wave-particle duality in multi-slit interference [18,19]. For two-slit experiments where the two beams are asymmetric, and the slits may be of unequal widths, a result as strong as (1) was lacking. We have used this new approach to study wave-particle duality in this asymmetric case.

We argue that in a two-slit experiment, getting path information should mean, being able to tell unambiguously for each quanton, which of the two slits it went through. Using this premise, we use a thought experiment to get which path information about the quantons using UQSD. We define the path distinguishability as the maximum probability with which one can unambiguously tell which slit the quanton went through, in principle. Using it we derive two duality relations for interference where the two paths may not be equally probable or the two slits may not be of equal widths. The two duality relations correspond to two difference ranges of asymmetry. Unlike the well studied symmetric case, a single tight duality relation is not possible for the asymmetric case. Additionally, if the thought experiment is actually performed, one can tell for each quanton if it went through slit 1 or slit 2 like a particle or through both the slits like a wave.

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