Meromorphic Painlevé III transcendents and the Joukowski correspondence

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We study a twistor correspondence based on the Joukowski map reduced from one for stationary-axisymmetric self-dual Yang–Mills and adapt it to the Painlevé III equation. A natural condition on the geometry (axis-simplicity) leads to solutions that are meromorphic at the fixed singularity at the origin. We show that it also implies a quantization condition for the parameter in the equation. From the point of view of generalized monodromy data, the condition is equivalent to triviality of the Stokes matrices and half-integral exponents of formal monodromy. We obtain canonically defined representations in terms of a Birkhoff factorization whose entries are related to the data at the origin and the Painlevé constants.

Keywords: Painlevé III; Joukowski; meromorphic.

1. Introduction

The self-dual Yang–Mills equations provide a paradigm of complete integrability by virtue of their twistor correspondence [1]. This expresses local solutions in terms of essentially free holomorphic data on an auxiliary complex manifold and twistor space. Symmetry reductions lead to many of the most basic integrable systems and their integrability can be understood via the reduction of this twistor correspondence [2]. When the self-dual Yang–Mills equations are stationary and axisymmetric, the reduced twistor correspondence is based on a parametrized family of Joukowski (or Zhukovski) transformations [3]. When the gauge group is SL(2), interesting reductions of this type include the Ernst equations of general relativity, and when a further symmetry is imposed, the third and sixth Painlevé equations.

This article started as an exploration of the connections between this Joukowski correspondence and the Quantum Spectral Curve of Gromov et al. [4]. This is based on the same geometry and produces quantum field theoretic anomalous dimensions as functions of just one variable, the coupling constant, via quantum rather than classical integrability. Such objects are naturally expected to be holomorphic near zero coupling. In [2], based on [5], solutions of the stationary axisymmetric self-dual Yang–Mills equations were studied that had well-defined meromorphic behaviour at the axis, termed axis-simple. The radial coordinate plays the role of the coupling constant in the quantum spectral curve, and so anomalous dimension should relate to this axis-simple class given that they should be in fact holomorphic near the origin. There are additional ingredients in the computation of anomalous dimensions, not straightforwardly reflected in these constructions. The Painlevé III equation, however, appears to be a close analogue.1

This was our motivation for the exploration given here for the Painlevé III equation in this axis-simple case. These results seem to be of interest in their own right and so this article provides a separate study

1 Especially to the BPS limits considered in [6, 7].
of these meromorphic solutions to the Painlevé III equation. We find a quantization of parameters that arises from the axis-simple condition and see a simplified Riemann–Hilbert problem for constructing such solutions. From the point of view of isomonodromic deformations [8–10], we show that the axis-simple condition is equivalent to the triviality of the Stokes matrices together with half-integral exponents of formal monodomy.

This article is structured as follows. In the remainder of this Introduction, we introduce the $P_{III}$ equation and state our main results. In Section 2, we review the reduced twistor correspondence for stationary axisymmetric self-dual Yang–Mills equations which gives solutions with meromorphic behaviour at the origin.

In Section 3, we adapt the construction to Painlevé III and show that this leads to a quantization of the parameters and prove our main theorem. We go on to characterize the axis-simple condition in terms of generalized monodromy data. In Section 4, we reconstruct some explicit solutions that reproduce the classical transcendental solutions of Painlevé III meromorphic at $\rho = 0$.

1.1 The Painlevé III equation

The Painlevé equations are second order ordinary differential equations whose only movable singularities, (i.e. the singularities of their solutions whose locations depend on the initial conditions) are poles. The equations have received much attention in mathematical physics over the years [11–17]. They can all be viewed as symmetry reductions of the SL$(2, \mathbb{C})$ SDYM equations by certain three dimensional abelian subgroups of the conformal group [2, 18]. Painlevé III, $P_{III}$, can in particular be obtained from the stationary axisymmetric equations by imposing an additional translational symmetry along the axis [19].

The third Painlevé equation is a family of equations parametrized by four complex parameters $(\alpha, \beta, \gamma, \delta)$:

$$f'' = \left(\frac{f'}{f}\right)^2 - \frac{f'}{\rho} + \frac{1}{\rho} (\alpha f^2 + \beta) + \gamma f^3 + \frac{\delta}{f}. \tag{1}$$

For all $(\alpha, \beta, \gamma, \delta)$, the equation is a meromorphic ODE with a simple pole at its fixed singularity at $\rho = 0$.

It is customary (see [17, 20]) to distinguish four different classes:

$$P_{III}(D_6) = \{ (\alpha, \beta, \gamma, \delta) \mid \gamma \delta \neq 0 \},$$
$$P_{III}(D_7) = \{ (\alpha, \beta, \gamma, \delta) \mid \gamma = 0, \alpha \delta \neq 0, \text{ or } \delta = 0, \beta \gamma \neq 0 \}$$
$$P_{III}(D_8) = \{ (\alpha, \beta, \gamma, \delta) \mid \gamma = \delta = 0, \alpha \beta \neq 0 \},$$
$$P_{III}(Q) = \{ (\alpha, \beta, \gamma, \delta) \mid \alpha = \gamma = 0 \text{ or } \beta = \delta = 0 \}. \tag{2}$$

Other commonly used parameters are

$$\alpha = -8n, \quad \beta = 8(m - k), \quad \gamma = 16l^2, \quad \delta = -16k^2, \tag{3}$$
in terms of which the above families become
\[ P_{III}(D_6) = \{(l, k, m, n) \mid kl \neq 0\}, \]
\[ P_{III}(D_7) = \{(l, k, m, n) \mid l = 0, kn \neq 0, \text{ or } k = 0, lm \neq 0\}, \]
\[ P_{III}(D_8) = \{(l, k, m, n) \mid l = k = 0, n \neq 0, m \neq k\}, \]
\[ P_{III}(Q) = \{(l, k, m, n) \mid n = l = 0 \text{ or } m = k = 0\}. \]
(4)

Rescaling \( f \) and \( \rho \) reduces the number of essential free parameters in each class to 2, 1, 0, 1, respectively.

The most familiar case is the scaling reduction of the Sinh-Gordon equation which can be expressed as
either \( D_6 \) with \( \alpha = \beta = 0 \) or \( D_8 \).

1.2 Monodromy preserving deformations

It is well-known that the Painlevé III equation describes the isomonodromic deformations of a \( 2 \times 2 \) linear system of equations
\[ \frac{dY(\lambda)}{d\lambda} = \left( \frac{A_{0,1}}{\lambda^2} + \frac{A_{0,0}}{\lambda} - A_{\infty,0} - A_{\infty,1}\lambda \right) Y(\lambda) \]
with Poincaré rank 1 at the irregular singularities \( \lambda = 0 \) and \( \lambda = \infty \) [8, 10, 21]. We briefly review
the generalized monodromy data and establish some notation. For simplicity, in what follows we only
consider the case where all matrices \( A_{i,j} \) appearing in the above equation are diagonalizable. Near the
singularities the system has formal solutions
\[ Y(\lambda)^{(0,\infty)}(0,\infty) \sim G(0,\infty) \hat{Y}(\lambda)^{(0,\infty)}(\lambda)e^{T(0,\infty)(\lambda)}. \]
(6)

Here
\[ \hat{Y}(\lambda)^{(0)} = 1 + y_1^{(0)} \lambda + y_2^{(0)} \lambda^2 + \ldots \]
(7)
and
\[ \hat{Y}(\lambda)^{(\infty)} = 1 + y_1^{(\infty)} \lambda^{-1} + y_2^{(\infty)} \lambda^{-2} + \ldots \]
(8)
are power series around the irregular singularities that converge in wedges centred on them, the so-called Stokes sectors \( S_j^{(0,\infty)} \) whose details will not be needed here. The \( G^{(0,\infty)} \)s are constant matrices that
diagonalize \( A_{0,1} \) and \( A_{\infty,1}, \) and
\[ T^{(0)} = -T_{-1}^{(0)} \lambda^{-1} + T_0^{(0)} \log(\lambda) \]
(9)
\[ T^{(\infty)} = -T_{-1}^{(\infty)} \lambda + T_0^{(\infty)} \log \left( \frac{1}{\lambda} \right), \]
(10)
where the \( T_{-1}^{(0,\infty)} \)s are diagonal. For each sector \( S_j^{(0,\infty)} \) labelled by \( j \), there are convergent solutions \( Y_j^{(0,\infty)}(\lambda) \) for
which equation (6) are asymptotic expansions
\[ Y_j^{(0,\infty)}(\lambda) \sim G^{(0,\infty)} \hat{Y}(\lambda)e^{T^{(0,\infty)}(\lambda)} \text{ in } S_j^{(0,\infty)}. \]
(11)
The solutions in overlapping sectors can differ by the constant Stokes matrices,

$$s_j^{(0,\infty)} := Y_j^{(0,\infty)}(\lambda)^{-1} Y_j^{(0,\infty)}(\lambda),$$  \hspace{1cm} (12)

and solutions near 0 and \(\infty\) are related by the connection matrix \(C\)

$$C := Y_1^{(\infty)} \left( Y_1^{(0)} \right)^{-1}.$$  \hspace{1cm} (13)

We associate to the linear system (5) the generalized monodromy data \(\mathcal{M}\) consisting of

- Stokes matrixes \(s_j^{(0,\infty)}\), \(j = 1, 2\)
- Connection matrix \(C\)
- ‘Exponents of formal monodromy’ \(T_0^{(0,\infty)}\).

These data are preserved by deformations of \(T_n^{(0,\infty)}\) iff a non-linear equation on the matrices \(A\) of (5) in the parameter \(\rho\) is satisfied. These equations can be reduced to (1), i.e. Painlevé III, [8]. This has proved to be a powerful tool in the study of solutions to the Painlevé III equation.

Although our starting point will be twistor-theoretic, we will prove that our solutions to Painlevé III are characterized as those whose associated monodromy data \(\mathcal{M}\) has trivial Stokes matrices and \(\mathbb{Z}_2\)-monodromy.

1.3 The axis-simple condition and meromorphicity

In Section 3, we will show that the axis-simple condition on \(P_{III}\) leads to the following Riemann–Hilbert problem:

**Theorem** A solution to the \(D_6 P_{III}\) equation is axis-simple iff it arises from the Riemann–Hilbert problem on the Riemann sphere parametrized by \(\lambda\) given by the following matrix

$$P = \begin{pmatrix}
(\rho \lambda)^\bar{p} & 0
\end{pmatrix}
\begin{pmatrix}
0 & (\rho \lambda)^{-\bar{p}}
\end{pmatrix}
\begin{pmatrix}
\left( \frac{\lambda}{p} \right)^p & 0
\end{pmatrix}
\begin{pmatrix}
\left( \frac{\lambda}{\bar{p}} \right)^{-p}
\end{pmatrix},$$  \hspace{1cm} (14)

where \((c_0, c_1, \tilde{c}_0, \tilde{c}_1, a, \tilde{a})\) are constants with \(c_0 \tilde{c}_0 - c_1 \tilde{c}_1 = 1\) and \(p, \bar{p} \in \mathbb{Z}\) or \(2\), \(p, \bar{p} \in \mathbb{Z} + 1/2\), and \(u = \rho/2(\lambda + 1/\lambda)\). The data are gauge equivalent under pre- and post-multiplying by diagonal constant unit determinant matrices so that there is only one essential parameter in the \(c_0, c_1, \tilde{c}_0, \tilde{c}_1\). The data are related to the standard constants by

$$k^2 = \frac{1}{16} a^2, \quad \ell^2 = \tilde{a}^2, \quad m = -\frac{a}{2} p, \quad n = -2\tilde{a} \bar{p},$$

\[\text{In this half-integral case, we need to observe that there is no overall square-root singularity in } P \text{ as that in the first factor cancels that in the last.}\]
In particular, we have the quantization condition

\[ \frac{1}{2} \frac{m}{k} \frac{1}{2} \frac{n}{l} \in \mathbb{Z} \text{ or } \mathbb{Z} + 1/2. \]

A similar Riemann–Hilbert problem for the type \( Q \) Painlevé III is spelled out in the proof, whereas \( D_7 \) and \( D_8 \) are ruled out by our condition.

We have from Theorem 1 that, because the Riemann–Hilbert problem extends smoothly to \( \rho = 0 \), we can have at worst poles in the solutions at \( \rho = 0 \) when the Riemann–Hilbert problem jumps there so we have

**Corollary 1** The axis-simple solutions presented by the Riemann–Hilbert problem in Theorem 1 are meromorphic at \( \rho = 0 \).

Finally, we show that the axis-simple condition can be understood from the point of view of generalized monodromy data by

**Theorem 2** The axis-simple condition is equivalent to imposing triviality of the Stokes matrices and half-integer exponents of formal monodromy. (We take half-integral to include integer values also.)

The proof of Theorem 1 follows by imposing the axis-simple condition on twistor space as given in [3] to reduce the twistor data to a normal form that gives rise to the above Riemann–Hilbert problem. Corollary 1 follows from [3] and the proof of the Painlevé property in [22]. In this article, we give an explicit proof for a special case. The axis-simple condition itself is reviewed in the next section.

### 2. The Joukowski correspondence

In the context of the stationary axisymmetric systems, we introduce spatial coordinates \((\rho, z)\) with \(\rho\) being the radial distance from the \(z\)-axis. We will work on a region\(^3\) \( U \) in the \((\rho, z)\)-plane that we will take to be connected, simply connected, containing some piece of the axis \( \rho = 0 \) and invariant under \( \rho \to -\rho \).

This requirement that the domain \( U \) contains a piece of the axis \( \rho = 0 \) is the key feature of the *axis-simple case*. The corresponding solutions coming from the twistor correspondence will be allowed to have poles on the axis, but as explained below Theorem 3 only as a consequence of so-called jumping lines, and therefore it will be meromorphic, but not branching. See reference [3] for further discussion.

#### 2.1 The underlying geometry

We are defining the *Joukowski correspondence* to be the reduced twistor correspondence introduced in [3] based on [5]. It is a symmetry reduction of the twistor correspondence between points in (complexified)

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\(^3\) Although the solutions that we are interested in are initially defined only for real \((\rho, z)\), we will be able to take \( U \) to be a region in the complexification as solutions will generically be analytic in our context being solutions to elliptic equations.
Minkowski space-time and lines in $\mathbb{CP}^3$ under a time translation and spatial rotation. It can be summarized in the double fibration

$$U \times \mathbb{CP}^1 \ni (\rho, z, \lambda) \quad (\rho, z) \in U \quad \Upsilon(U) \ni u.$$ (15)

The map $p$ forgets $\lambda$ whereas $q$ projects according to the Joukowski transformation

$$u = \frac{\rho}{2} \left( \frac{\lambda}{\lambda} + 1 \right) + iz.$$ (16)

This gives a family of maps from $\lambda \in \mathbb{CP}^1 \to u \in \mathbb{CP}^1$ that depends on $(\rho, z)$. At fixed $(\rho, z)$, the map $\lambda \to u$ is 2 : 1, branching at $u = \pm \rho + iz$. The unit circle $|\lambda| = 1$ is mapped to the slit $[-\rho, \rho] + iz$.

**Definition 1** We define $\Upsilon(U)$ to be the space of connected leaves in $U \times \mathbb{CP}^1$ on which $u$ is constant.

Naïvely one might expect $\Upsilon(U)$ to be the $u$-Riemann sphere and clearly there is a map $u : \Upsilon(U) \to \mathbb{CP}^1$ defined by $u$. However, at a fixed value $u = u_0$, we will obtain two points when $u = u_0$ has two components in $U \times \mathbb{CP}^1$, although only one when the set $u = u_0$ in $U \times \mathbb{CP}^1$ is connected. Given $u = u_0$, (16) gives two choices of $\lambda$ at each point in $U$ dropping to one on the branching loci $u_0 = \pm \rho + iz$. Thus the criteria for $u = u_0$ in $U \times \mathbb{CP}^1$ to have just one component rather than two is that the branching loci $u_0 = \pm \rho + iz$ should lie in $U$.

Identifying $U$ with a region in the $u$-Riemann sphere, we see that we have a covering $\Upsilon(U) \to \mathbb{CP}^1$ that is 1 : 1 for $u \in U$, and 2 : 1 on $\mathbb{CP}^1 - U$. Thus

**Proposition 1** [3] $\Upsilon(U)$ is the non-Haudorff Riemann surface obtained by gluing two copies of $\mathbb{CP}^1$ together using the identity map on the open set $U \subset \mathbb{CP}^1$.

In practice, we will only be concerned with the example in which $U = \mathbb{C}$. Then $\Upsilon(U)$ is essentially the Riemann sphere, but with two points at $\infty$. It is obtained by gluing two copies of the Riemann sphere $\mathbb{CP}^1$ together for finite values of $u$.

Points $(\rho, z) \in U$ correspond to surjective maps $L_{(\rho, z)} : \mathbb{CP}^1 \to \Upsilon(U)$ given by $L_{(\rho, z)}(\lambda) = q(\rho, z, \lambda)$.

### 2.2 The reduced Ward construction

The main result that we will use follows [3] based on Ward [1, 5] concerning solutions to the stationary axisymmetric SDYM equations. These can be expressed in the form of Yang’s equation

$$\partial_\rho (\rho J^{-1} \partial_\rho J) + \partial_z (\rho J^{-1} \partial_z J) = 0.$$ (17)

where $J(\rho, z)$, the $J$-matrix, takes values in $GL(N, \mathbb{C})$ in the first instance, but it can be adapted to any real or complex Lie group. For a unitary group, for example, it can be taken to be Hermitian.
Theorem 3 There is a 1:1 correspondence between solutions to the stationary axisymmetric self-dual Yang–Mills equations on $U$ with gauge group $SL(N, \mathbb{C})$ and holomorphic vector bundles $E \to \mathbb{T}(U)$ with structure group $SL(N, \mathbb{C})$ such that for any fixed $\rho + iz \in U$, the restriction of $E$ to the line $L_{(\rho,z)} = q \circ p^{-1}(\rho + iz) \subset \mathbb{T}(U)$ is trivial.

Remark: Although that last restriction might seem very strong, if true at one value of $\rho + iz$, it will be true for $\rho + iz$ on a dense open subset of $U$; the points at which it fails correspond precisely to $J$ becoming meromorphic rather than simply holomorphic on $U$. The points $(\rho, z)$ where $J$ is meromorphic correspond to jumping lines $L_{(\rho,z)}$ for the bundle $E$ where it is no longer trivial, but a direct sum of non-trivial line bundles as allowed by Grothendieck’s theorem.

We give an outline of the proof to provide ingredients that will also be needed later. A key role is played by the vector fields

\[
V_1 = \partial_\rho + i\lambda \partial_z + \frac{1}{\rho} \partial_\lambda \\
V_2 = i\partial_z + \lambda \partial_\rho - \frac{1}{\rho} \lambda^2 \partial_\lambda.
\] (18)

They satisfy $V_1 u = V_2 u = 0$ so their integral surfaces define the leaves of constant $u$ in $U \times \mathbb{CP}^1$.

We can take the bundle $E \to \mathbb{T}(U)$ to be defined in the Čech fashion by patching functions $P(u)_{ij}$ defined on overlaps $U_i \cap U_j$ of some open cover $\{U_i\}$ of $\mathbb{T}(U)$. The pull-back $q^*E$ of $E$ to the Riemann sphere $L_{(\rho,z)}$ in $U \times \mathbb{CP}^1$ has patching functions $P(\rho/2(\lambda + 1/\lambda) + iz)_{ij}$. Since the bundle is assumed to be trivial on $L_{(\rho,z)}$, for fixed $(\rho, z)$ we can find $G_i$ such that

\[
P(\rho/2(\lambda + 1/\lambda) + iz)_{ij} = G_i(\rho, z, \lambda)G_j^{-1}(\rho, z, \lambda),
\] (19)

where $G_i$ is holomorphic in $\lambda$ on $q^{-1}(U_i)$. We can normalize the solutions $G_i$ to (19) by requiring $G_0(\rho, z, 0) = 1$ where $q^{-1}(U_0)$ contains $\lambda = 0$ and with this, the $G_i$ are unique. It follows from $V_j P_{ij} = V_2 P_{ij} = 0$ that $G_i^{-1}V_i G_i = G_j^{-1}V_j G_j$ by differentiation of (19) so that this expression is global on the $\lambda$-Riemann sphere, but with a simple pole at $\lambda = \infty$. It follows that we can define the Lax pair by

\[
L_1 := G_i^{-1}V_i \circ G_i = V_i + i\lambda (J^{-1} \partial_\lambda J) \\
L_2 := G_i^{-1}V_2 \circ G_i = V_2 + \lambda (J^{-1} \partial_\rho J),
\] (20)

where the $J$-matrix is defined by

\[
J(\rho, \lambda) = G_\infty(\rho, z, \infty)G_0(\rho, z, 0)^{-1},
\]

\[\text{4} \text{ Up to the caveat that at the cost of factoring out a determinant, the condition on the structure group can be relaxed. See the remark at the end of this section.}\]
where \( U_\infty, U_0 \) denotes the sets whose preimage under \( q \) contains \( \lambda = \infty, 0 \), respectively. The \( L_1 \) and \( L_2 \) simultaneously annihilate \( G_i^{-1} \) by construction. Thus, the Lax pair is compatible

\[
[L_1, L_2] = 0. \tag{21}
\]

This is equivalent to Yang’s equation form of the axisymmetric self-dual Yang–Mills equations (17).

The significance of the theorem is that holomorphic vector bundles on \( \mathbb{T}(U) \) can be described in terms of essentially free data \( P(U) \) although subject to certain consistency conditions. However, the trivialization problem for the bundle is still complicated as there are potentially many sets in an open cover, and the presentation is still subject to a large degree of redundancy also corresponding to changes of frames on the open sets. We will see that it can be put into a standard normal form in interesting situations.

2.3 The normal form in the axis-simple case

In this section, we introduce a normal form for bundles on twistor space satisfying the axis-simple condition. The task of constructing solutions will then be reduced to the Riemann–Hilbert problem equation (19) based on the patching data \( P \) in this normal form. This section is essentially a review of Section 5.4 of [3] and concerns axis-simple stationary axisymmetric solutions to the self-dual Yang–Mills equations. In Section 3.2, we reduce the description to one for Painlevé III.

**Proposition 2** In the axis-simple case, a vector bundle over \( \mathbb{T}(U) \) of rank \( N \) and structure group \( SL(N, \mathbb{C}) \) is characterized completely by a set of \( 2N - 2 \) integers \( (p_1, \ldots, p_N, \tilde{p}_1, \ldots, \tilde{p}_N) \), \( \sum_i p_i = \sum_i \tilde{p}_i = 0 \) and a holomorphic matrix \( P_U(u) \) on \( U \) with values in \( SL(N, \mathbb{C}) \).

The reconstruction of the \( J \)-matrix at \( (\rho, z) \) arises from a Riemann–Hilbert problem on the \( L(\rho, z) \) Riemann sphere parametrized by \( \lambda \) given by

\[
\tilde{P}(\lambda, \rho, z) = G_\infty(\rho, z, \lambda) G_0(\rho, z, \lambda)^{-1}. \tag{22}
\]

where

\[
\tilde{P} = \begin{pmatrix}
(\lambda, \rho)^{\tilde{p}_1} & 0 & \cdots & 0 \\
0 & (\lambda, \rho)^{\tilde{p}_2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & (\lambda, \rho)^{\tilde{p}_N}
\end{pmatrix} P_U(u) \begin{pmatrix}
(\lambda/\rho)^{p_1} & 0 & \cdots & 0 \\
0 & (\lambda/\rho)^{p_2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & (\lambda/\rho)^{p_N}
\end{pmatrix}. \tag{23}
\]

We then have \( J(\rho, z) = G_\infty(\rho, z, \infty) G_0(\rho, z, 0)^{-1} \).

The key simplification in the axis-simple case is that, as mentioned, \( \mathbb{T}(U) \) consists of two copies of the Riemann sphere glued together over \( U \). We use the Čech description in terms of patching matrices to describe bundles \( E \rightarrow \mathbb{T}(U) \). Let the rank of \( E \) be \( N \), and denote the two Riemann spheres by \( \mathbb{CP}_0^1 \) (that contains \( \lambda = 0 \) in its pre-image) and \( \mathbb{CP}_\infty^1 \) that contains \( \lambda = \infty \). We cover \( \mathbb{T}(U) \) with four open sets: \( U_0 \) a copy of \( U \) in \( \mathbb{CP}_0^1 \), \( V_0 \) a neighbourhood of \( u = \infty \in \mathbb{CP}_0^1 \) intersecting \( U_0 \) in an annular region and two analogous open sets \( U_\infty \) and \( V_\infty \) covering \( \mathbb{CP}_\infty^1 \). A standard theorem due to Birkhoff and Grothendieck gives that the restriction of \( E \) to either sphere must be a direct sum of line bundles:

\[
E|_{\mathbb{CP}_0^1} \cong \bigoplus_{i=1}^N \mathcal{O}(p_i), \quad E|_{\mathbb{CP}_\infty^1} \cong \bigoplus_{i=1}^N \mathcal{O}(\tilde{p}_i), \quad p_i \in \mathbb{Z}, \quad \tilde{p}_i \in \mathbb{Z},
\]
where the $p_i$'s and $\tilde{p}_i$'s are the Chern classes of the line bundles. Since $(U_0, V_0)$ and $(U_\infty, V_\infty)$ are standard covers of the spheres, it follows that we can choose holomorphic frames on the four sets such that the transition matrices on $U_0 \cap V_0$ and $U_\infty \cap V_\infty$ are of the form $\text{diag}(u^p_1, \ldots u^p_N)$ and $\text{diag}(u^{\tilde{p}}_1, \ldots u^{\tilde{p}}_N)$, respectively. Assuming the structure group to be $\text{SL}(N, \mathbb{C})$, we also have the constraints $\sum_{i=1}^N p_i = \sum_{i=1}^N \tilde{p}_i = 0$. Lastly, the frames on $U_0$ and $U_\infty$ must be patched by an undetermined $N \times N$ matrix of unit determinant, which we call $P_U$:

$$P_U : U_0 \cap U_\infty = U \to \text{SL}(N, \mathbb{C}).$$

Thus, a vector bundle over $\mathbb{T}(U)$ of rank $N$ and structure group $\text{SL}(N, \mathbb{C})$ is characterized as described in the proposition above in the axis-simple case.

We can reduce the four set cover Riemann–Hilbert problem of (19) to a two-set one on $L(\rho, z)$ in terms of $\lambda$ as follows. Because the patching matrices on $U_0 \cap V_0$ and $U_\infty \cap V_\infty$ are particularly simple, after pulling them back to $L(\rho, z)$, i.e. after the substitution $u = \rho / (\lambda + 1 / \lambda) + iz$, we can factorize $u$ and hence its powers by

$$u = \frac{\rho \lambda}{2} \times \left( 1 + \frac{1}{\lambda^2} + \frac{2iz}{\rho \lambda} \right) \quad \text{near } \lambda = \infty$$

and

$$u = \frac{\rho}{2 \lambda} \times \left( 1 + \lambda^2 + \frac{2iz \lambda}{\rho} \right) \quad \text{near } \lambda = 0.$$

We use the second factors to find frames of the pullback of $E$ to $L(\rho, z)$ on the two sets $\tilde{U}_0 = \{ |\lambda| < 2 \}$ and $\tilde{U}_\infty = \{ |\lambda| > 1/2 \}$. In these frames, there is now just the one patching matrix on $\tilde{U}_0 \cap \tilde{U}_\infty$ as given in (23).

The last step to find the global frame required in (19) is to solve a Riemann–Hilbert problem: given an invertible holomorphic matrix $\tilde{P}(\lambda) \in \text{SL}(N, \mathbb{C})$ defined in a neighbourhood of $|\lambda| = 1$ (an element of the loop group $\text{LSL}(N, \mathbb{C})$), we look for $\text{SL}(N, \mathbb{C})$-valued functions $G_\infty(\lambda)$ and $G_0(\lambda)$ holomorphic in $\lambda$ for $|\lambda| > 1 - \epsilon$ and $|\lambda| < 1 + \epsilon$ for $\epsilon \in \mathbb{R}^+$ such that

$$\tilde{P}(\lambda) = G_\infty(\lambda) G_0(\lambda)^{-1}.$$

According to Birkhoff’s theorem, for generic loop group elements $P(\lambda)$ solutions exist\textsuperscript{5} and are unique up to multiplication by a constant matrix $C$,

$$G_\infty \mapsto G_\infty C, \quad G_0 \mapsto G_0 C.$$ (27)

In our context, we have a family of $P(\lambda)$'s parametrized by $(\rho, z)$, and $C$ can then depend on $(\rho, z)$. Although the transformation leaves $J$ invariant, we remark that it actually corresponds to a gauge transformation of the SDYM connection associated with $J$.

\textsuperscript{5} More precisely, ‘generic’ here means that the element is in the identity component of $\text{LSL}(N, \mathbb{C})$ endowed with uniform convergence topology. The ‘non-generic’ elements have an additional diagonal contribution, which corresponds precisely to the diagonal transition matrices in Birkhoff–Groethendieck’s theorem. Thus, ‘generically’ any holomorphic bundle on $\mathbb{CP}^1$ is trivial.
Remark: we will relax the $\text{SL}(N, \mathbb{C})$ condition on the bundles mildly so that $\sum_{i=1}^{N} p_i + \tilde{p}_i = 0$. This condition leads to $\text{SL}(N, \mathbb{C})$ solutions to the Ernst equation up to a determinant factor consisting of powers of $\rho$ that can be removed.

Although Birkhoff’s factorization theorem gives a generic existence theorem for solutions to the Riemann–Hilbert problem, to obtain explicit solutions, we need to make some further restrictions on the data as will be described in Section 4.

3. Meromorphic Painlevé III transcendants

In this section, we adapt the above construction to Painlevé III and derive Theorem 1. First, we study the Lax pair for $\text{PIII}$ and then go on to characterize the axis-simple holomorphic vector bundles on $\mathbb{T}(U)$ that are invariant under the action of the translational Killing vector $\partial_z$. These are the bundles that yield $\text{PIII}$ solutions. We finally map the free parameters entering the vector bundles to the constants $k, l, m, n$ parametrizing the type of $\text{PIII}$.

3.1 The Lax pair and isomonodromy

The Lax pair for $\text{PIII}$ arises from that for the stationary axisymmetric Yang–Mills equations eq.(20) when the fields are independent of $z$. However, we have tacitly made a gauge choice when writing equation (20), that does not allow $z$-independent fields and so is unsuitable for deriving $\text{PIII}$. We therefore write the Lax pair in a general gauge as

$$L_1 = \partial_\rho + \frac{\lambda}{\rho} \partial_\lambda - A - \lambda B$$
$$L_2 = \lambda \partial_\rho - \frac{\lambda^2}{\rho} \partial_\lambda - C - \lambda D,$$

where the $\text{sl}(2, \mathbb{C})$-valued functions $A, B, C, D$ depend only on $\rho$. We then require as a compatibility condition that the Lax pair commutes up to a linear combinations of itself. After making the gauge choice $A = 0$, this implies

$$\partial_\rho C = 0,$$
$$\rho \partial_\rho B = [\rho D, B],$$
$$\partial_\rho (\rho D) = \rho [B, C].$$

The derivation of $\text{PIII}$ from this system is given in full detail in\textsuperscript{6} [2, p. 103]. We wish to relate the Painlevé transcendent $f$ and the constant parameters $(\alpha, \beta, \gamma, \delta)$ to the matrices $A, B, C, D$ entering the Lax pair.

\textsuperscript{6} The system in [2] is given by $\partial_\rho P = 0, \partial_\rho Q = 2[Q, R], \partial_\rho R = 2\rho [Q, P]$ and comparing with equation (29) for $A, B, C, D$, we find $R = -\frac{1}{2} \rho D, \quad Q = B, \quad P = -\frac{1}{2} C$. 


Comparing with \(^7\) [2], we find that the constants are given by

\[
\begin{align*}
    k^2 &= \frac{1}{32} \text{tr}(C^2), \\
    m &= \frac{\rho}{8} \text{tr}(CD), \\
    l^2 &= \frac{1}{2} \text{tr}(B^2), \\
    n &= -\frac{\rho}{2} \text{tr}(BD),
\end{align*}
\]

(30)

where \(k, l, m, n\) are the parameters introduced in (3). The transcendent reads

\[
f = \begin{cases} \dfrac{-1}{2} \dfrac{D_{12}}{B_{12}} & \text{if } k \neq 0 \\ \dfrac{-1}{2} \dfrac{D_{21}}{B_{21}} & \text{if } k = 0, \end{cases}
\]

(31)

where the subscripts indicate the respective entries of the matrices in a frame in which \(C\) is diagonalized when \(k \neq 0\), or is strictly upper triangular for \(k = 0\).

The monodromy operator:

eliminating \(\partial \rho\) from (28) we obtain

\[
\partial \lambda - \mathcal{A}(\lambda) := \partial \lambda + \rho \dfrac{D}{2\lambda} + \rho \dfrac{C}{2\lambda^2} - \rho \dfrac{B}{2}.
\]

(32)

This defines a holomorphic flat connection on the Riemann sphere with double poles at \(\lambda = 0\) and \(\infty\) that defines the isomonodromy problem associated with the linear system equation (32). The compatibility with (28) means that the generalized monodromy of the operator is independent of \(\rho\).

We will use this flat connection to express \(B, C, D\), in terms of the geometric data representing solutions to \(P_{III}\).

3.2 Characterization of invariant bundles

By differentiation of the incidence relation \(u = \frac{\rho}{2}(\lambda + 1/\lambda) + iz\), it is easy to see that the symmetry \(\partial_z\) acts by \(\partial_u\) on \(\mathbb{T}(U)\). This has \(u = \infty\) as a fixed point, so we cannot simply quotient the space by this action to construct invariant bundles as pullbacks from a quotient. Instead, we must characterize vector bundles \(E \rightarrow \mathbb{T}(U)\) that carry a global holomorphic lift \(L_{\partial_u}\) of \(\partial_u\). In the following, we study axis-simple \(GL(2, \mathbb{C})\) bundles, and so without loss of generality

\[
E\big|_{CP^1} \cong O(p) \oplus O(q), \quad E\big|_{CP^1} \cong O(\bar{p}) \oplus O(\bar{q}).
\]

Since the following discussion applies equally to both spheres, for this first part we will drop the subscripts \([0, \infty]\), and refer generically to ‘the sphere’. We work locally on the sphere’s copy of \(U\) and assume that a frame has been chosen so that the matrix patching \(U\) with a neighbourhood \(V\) of \(\infty \in \mathbb{C}P^1\) is of standard form,

\[
P_{UV} = \text{diag}(u^{-p}, u^{-q}).
\]

(33)

\(^7\) The constants of motion are given in terms of \(P, Q\) and \(R\) by \(k^2 = \frac{1}{2} \text{tr}(P^2) = \frac{1}{32} \text{tr}(C^2), l^2 = \frac{1}{2} \text{tr}(Q^2) = \frac{1}{2} \text{tr}(B^2), m = \text{tr}(PR) = \frac{\rho}{8} \text{tr}(CD), n = \text{tr}(QR) = \frac{\rho}{2} \text{tr}(BD)\).
Generic case.
Assume that \( p > q \). The action of \( \partial_u \) on \( E \) is expressed in terms of a Lie derivative \( L_{\partial_u} \). Locally on \( U \), we have

\[
L_{\partial_u} = \partial_u + \theta_U, \quad \text{where} \quad \theta_U = \begin{pmatrix} a & d \\ c & b \end{pmatrix},
\]

and \( \theta_U \) must be holomorphic on \( U \). On \( V \), we will therefore have

\[
L_{\partial_u} = P^{-1}_{UV}(\partial_u + \theta_U)P_{UV} = \partial_u + \theta_V = \partial_u + P^{-1}_{UV}(\partial_u P_{UV}) + P^{-1}_{UV} \theta_u P_{UV},
\]

so

\[
\theta_V = \begin{pmatrix} a - \frac{p}{u} du^{p-q} \\ ct^{p-q} \\ b - \frac{q}{u} \\
\end{pmatrix},
\]

and this must be holomorphic near \( \infty \). Since \( p > q \) this implies \( d = 0 \), \( a \) and \( b \) are constants, and \( c \) is a polynomial in \( u \) of degree at most \( p - q \), i.e.

\[
d = 0, \quad a, b \in \mathbb{C}, \quad c(u) = \sum_{k=0}^{p-q} c_k u^k.
\]

The general form of \( L_{\partial_u} \) can be restricted further by making use of the residual gauge transformations that preserve the patching matrix equation (33). These are given by

\[
G = \begin{pmatrix} e & 0 \\ f \\ g \end{pmatrix},
\]

where again \( k, m \) are constants, whereas \( l \) is a polynomial of degree \( p - q \). Applying such a gauge transformation to \( \theta \) does not change \( a \) or \( d \), but gives

\[
c' = \frac{ce + (b - a)f + \partial_u l}{g}.
\]

We assume for the moment \( a \neq b \), and also include this (besides the assumption on the Chern classes) as a condition for the generic case. In this generic case, \( l \) can be chosen to cancel \( c' \) precisely. Thus, without loss of generality, we can take

\[
L_{\partial_u} = \partial_u + \theta = \partial_u + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.
\]

Note that we did not make use of \( e, g \) so that we still have a remaining diagonal gauge freedom

\[
G = \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix}.
\]
Having put the Lie derivative in standard form on both spheres, we now study how they are related to each other on $U$ and reintroduce subscripts $\{0, \infty\}$ to distinguish between them. The crucial point is that the patching matrix $P_U$ must send the Lie derivative defined on $U_0$ to that defined on $U_\infty$. This implies that

$$\partial_u P_U = P_U \theta_\infty - \theta_0 P_U, \quad (41)$$

which is a first order matrix ODE for $P_U$. The solution is easily written in terms of exponentials

$$P_U = \exp(-\theta_\infty u) \exp(\theta_0 u), \quad (42)$$

where $C$ is an arbitrary invertible matrix with constant entries, say $c_0, c_1, \tilde{c}_0, \tilde{c}_1$. With $\theta_0$ and $\theta_\infty$ as in equation (39),

$$P_U = \left( e^{(a-\tilde{a})u}c_0 \quad e^{(b-\tilde{b})u}\tilde{c}_1 \right). \quad (43)$$

The solution still contains some redundancy, because of the residual diagonal gauge freedom equation (40). This means that we are free to multiply $P_U$ from the left and from the right by two different constant diagonal matrices. As a consequence, out of the four $c$s, only one is essential.

This completes the characterization of $\partial_u$ invariant vector bundles in the generic case. It follows from the previous section that this data leads to the patching matrix

$$P = \left( \begin{array}{cc} (\rho \lambda)^\hat{p} & 0 \\ 0 & (\rho \lambda)^\hat{q} \end{array} \right) \left( \begin{array}{cc} e^{(a-\tilde{a})u}c_0 \quad e^{(b-\tilde{b})u}\tilde{c}_1 \end{array} \right) \left( \begin{array}{cc} (\hat{\xi})^p & 0 \\ 0 & (\hat{\xi})^q \end{array} \right), \quad (44)$$

**Reduction to SL(2, $\mathbb{C}$).**

For Painlevé equations, we only need SL(2, $\mathbb{C}$)-bundles. First of all, as explained above, in this case we must have $p + q = \hat{p} + \hat{q}$. We can shift $p + q$ to zero by multiplying by a multiple of the identity, perhaps at the cost of introducing a half-integer value for $p$ and $q$ when $p + q$ is odd (and hence also of $\hat{p}$ and $\hat{q}$) leading to the condition

$$p, \hat{p} \in \mathbb{Z} \text{ or } \mathbb{Z} + 1/2.$$  

Since elements of $\mathfrak{sl}(2, \mathbb{C})$ are traceless, we have $a = -b$, $\tilde{a} = -\tilde{b}$. The presentation of the bundle becomes

$$P = \left( \begin{array}{cc} (\rho \lambda)^\hat{p} & 0 \\ 0 & (\rho \lambda)^{-\hat{p}} \end{array} \right) \left( \begin{array}{cc} e^{(a-\tilde{a})u}c_0 \quad e^{(a-\tilde{a})u}\tilde{c}_1 \end{array} \right) \left( \begin{array}{cc} (\hat{\xi})^p & 0 \\ 0 & (\hat{\xi})^{-p} \end{array} \right), \quad (45)$$

where $c_0\tilde{c}_0 - c_1\tilde{c}_1 = 1$ and the unit determinant diagonal gauge transformations reduce the $c$s to one essential parameter. This is equation (14) in the theorem above. Lastly, notice that the degenerate case can only arise when $a = 0$ or $\tilde{a} = 0$. 

Non-generic cases.

We relax the generic restrictions that we have put above. First assume \( a = b \) in \( \theta_0 \). Sticking to the SL\((2, \mathbb{C})\) case, we must have \( a = b = 0 \). Now in equation (38), we cannot cancel the leading term of the polynomial \( \tilde{c} \). Thus we arrive at the standard form

\[
L_{\partial u} = \partial_u + \left( \begin{array}{cc} 0 & 0 \\ cu^2p & 0 \end{array} \right),
\]

(46)

where now \( c \) is a constant. There are no substantial differences in the subsequent discussion. The ODE equation (41) can be solved in terms of exponentials, but we need to take into account that one of the \( \theta \)s depends on \( u \). Assuming e.g. that the degenerate case holds for \( \theta_0 \), the solution becomes

\[
P_U = \exp (-\theta_{\infty}u) \cdot C \cdot \exp (\Theta_0(u)),
\]

(47)

where \( \Theta_0(u) \) is the primitive of \( \theta_0 \),

\[
\Theta_0(u) = \left( \begin{array}{cc} 0 & 0 \\ cu^2p+1 & 0 \end{array} \right).
\]

(48)

The exponential of \( \Theta_0(u) \) is

\[
\exp(\Theta_0(u)) = \left( \begin{array}{cc} 1 & 0 \\ cu^2p+1 & 1 \end{array} \right).
\]

(49)

When \( p = q \) (and so \( p = 0 \) in the SL\((2, \mathbb{C})\) case) and/or \( \tilde{p} = \tilde{q} \) (and similarly \( \tilde{p} = 0 \) restricting to SL\((2, \mathbb{C})\)), dropping subscripts for a moment, all entries of \( \theta \) must be constant from the argument above. Gauge transformations are now constant matrices and simply conjugate \( \theta \). Generically, \( \theta \) is diagonalizable, and so we can still put \( \theta \) in the standard form of equation (39) by means of a gauge transformation. Otherwise, we can put the matrix into strictly lower triangular form, and treat the nilpotent case as above.

3.3 Identification of the parameters

We now compare the data in the normal forms above to the Painlevé constants in the definition of \( P_{III} \), namely the complex numbers \( k, l, m, n \), and to the initial conditions. we prove:

**Lemma 1** The Painlevé parameters are given in terms of the data for invariant bundles given in the previous subsection by

- In the generic case \( (a \neq 0, \tilde{a} \neq 0, p \neq 0, \tilde{p} \neq 0) \)

\[
k^2 = \frac{1}{16} a^2, \quad l^2 = \tilde{a}^2, \quad m = -\frac{a}{2p}, \quad n = -2\tilde{a}\tilde{p},
\]

and the Painlevé equation is of the type \( D_6 \).
If \( p = 0 \), then \( m = 0 \) with no restriction on the other parameters, which are given by the same formulae as above. Similarly if \( \tilde{p} = 0 \), then \( n = 0 \), with the other parameters given as above.

If \( a = 0 \), then \( k = m = 0 \). Similarly, if \( \tilde{a} = 0 \), then \( l = m = 0 \). Therefore, the Painlevé equation is of type \( Q \).

We see in particular that the \( D_7 \) and \( D_8 \) cases do not occur. The only free parameter that can correspond to initial conditions is encoded in \( c_0, c_1, \tilde{c}_0, \tilde{c}_1 \) up to diagonal gauge transformations and subject to

\[
c_0\tilde{c}_0 - c_1\tilde{c}_1 = 1.
\]

We prove the lemma by identifying our parameters in terms of the invariants of the matrices in the isomonodromy operator \( A(\lambda) \) in equation (32). We start by sketching how equation (32) arises from the construction from the twistor data.\(^8\)

The basic idea is that \( duL_\partial u \) defines a flat holomorphic connection \( \nabla \) on the bundle \( E \to T(U) \) with a double pole at \( u = \infty \) (since \( du \) has a double pole and \( \theta \) does not vanish). This pulls back to give the isomonodromy operator on the pullback \( q^*E \) of \( E \) to \( U \times \mathbb{C}P^1 \) along the \( \mathbb{C}P^1 \) factor.

The isomonodromy operator is defined on \( q^*E \) along \( L_\rho, z \) and given in (32) as

\[
\nabla f = d\lambda (\partial_\lambda - A(\lambda)) f. \tag{50}
\]

For \( P_{III} \) this has double poles at \( \lambda = 0, \infty \). The Painlevé constants \( k, l, m, n \) are the invariants of \( A \) at these poles defined by (32) and (30).

We can obtain \( \nabla \) near \( \lambda = 0, \infty \) from our formulae for \( duL_\partial u \) above. First of all, we must use the formula (35) valid near \( u = \infty \) either on \( \mathbb{C}P^1_0 \) or \( \mathbb{C}P^1_\infty \) for \( L_\partial u \). We work to start with in the generic \( SL(2, \mathbb{C}) \) case and so we will have

\[
\theta_{0V} = \begin{pmatrix} a - \frac{p}{u} & 0 \\
0 & -a + \frac{p}{u} \end{pmatrix}, \quad \theta_{\infty V} = \begin{pmatrix} \tilde{a} - \frac{\tilde{p}}{u} & 0 \\
0 & -\tilde{a} + \frac{\tilde{p}}{u} \end{pmatrix}, \tag{51}
\]

where we have put the extra 0 or \( \infty \) subscript on \( \theta_V \) to denote the version of (35) appropriate to \( \mathbb{C}P^1_0 \) or \( \mathbb{C}P^1_\infty \).

The operator in (32) is a holomorphic gauge transformation of \( duL_\partial u \) near \( u = \infty \) obtained from the solution to the Riemann–Hilbert problems (19) which we reduced to (22) by means of (25) and (24). The combined effect is a gauge transformation \( G_0 \) that is holomorphic near \( \lambda = 0 \) (and \( G_\infty \) near \( \lambda = \infty \)). Thus focussing first near \( \lambda = 0 \)

\[
du(\partial_u + \theta_{0V}) = G_0^{-1} \nabla G_0 = d\lambda \partial_\lambda + d\lambda (G_0^{-1} \partial_\lambda G_0 + G_0^{-1} A G_0). \tag{52}
\]

Because \( G_0 \) is holomorphic near \( \lambda = 0 \), the singular terms transform homogeneously. Using \( u = \rho/2\lambda + O(1) \) and \( du = -\rho d\lambda/2\lambda^2 \) near \( \lambda = 0 \),

\[
G_0 A(\lambda) G_0^{-1} = -\frac{\rho}{2\lambda^2} \begin{pmatrix} a - \frac{2\rho p}{\rho} & 0 \\
0 & -a + \frac{2\rho p}{\rho} \end{pmatrix} + O(1)
\]

\(^8\) For more details on this point, see [2, p. 232].
Thus, we have
\[
\text{tr} \left( \mathcal{A}(\lambda)^2 \right) = \frac{\rho^2 a^2}{2\lambda^4} - \frac{2\rho a p}{\lambda^3} + O \left( \frac{1}{\lambda^2} \right).
\] (53)

Given that
\[
G_0 \mathcal{A}(\lambda) G_0^{-1} = G_0 \frac{\rho}{2} \left( \frac{D}{\lambda} + \frac{C}{\lambda^2} \right) G_0^{-1} + O(1),
\]
we can then compare the fourth-order and third-order poles, and read off the desired constants of motion:
\[
k^2 = \frac{1}{32} \text{tr} \left( C^2 \right) = \frac{a^2}{16}, \quad m = \frac{\rho}{8} \text{tr} \left( DC \right) = -\frac{ap}{2}.
\] (54)

Lower order singularities do not yield isomonodromy invariants.

Similarly, working near \( \lambda = \infty \) we obtain
\[
l^2 = \frac{1}{2} \text{tr} \left( B^2 \right) = \tilde{a}^2, \quad n = \frac{\rho}{2} \text{tr} \left( BD \right) = -2\tilde{a}\tilde{p}.
\] (55)

We have thus mapped the constants of motion to the geometric data in the generic case, and the only free parameter left, one of \( c_0, c_1, \tilde{c}_0, \tilde{c}_1 \) encodes therefore the initial conditions.

The degenerate cases can be treated similarly, using the respective standard form for the Lie derivative given for example in equation (46) giving the result stated in the lemma.

3.4 Behaviour as \( \rho \to 0 \)

Although Corollary 1, or equivalently meromorphicity at \( \rho = 0 \) of the solutions encoded in Theorem 1, follows from the general result of [22] together with the axis-simple condition, in this subsection, we investigate this explicitly in a particular case of \( D_6 \), namely \( p = -\tilde{p} \) in equation (14). We also rescale \( c_0, c_1, \tilde{c}_0, \tilde{c}_1 \) so that \( c_0 = \tilde{c}_0 = 1, c_1 = -1/\tilde{c}_1 \). We show that singularities at the origin are simple poles and that the residue is fixed to be \(-\alpha/\gamma\). This result is consistent with a standard Frobenius analysis.9

Our argument is similar to the one in [3, p. 94], and it goes in two steps. First, we perform a splitting of the patching matrix in equation (14) in the \( \rho \to 0 \) limit, and thus obtain an expression for the \( J \)-matrix in this limit. Second, we express the system equation (29) in terms of the \( J \)-matrix, so that we can see how the Painlevé transcendent is given in terms of its components.

---

9 In order to show this, we multiply equation (1) through by \( f \), and insert \( f = \sum_{k \geq 0} a_k \rho^{k+c} \). Assuming \( c \leq -2 \), we see that the lowest powers in \( \rho \) give the constraints
\[
\gamma a_0^4 \rho^{4c} = 0, \quad a_0^3 \rho^{3c-1} = 0,
\] (56)
which, provided both \( \gamma \) and \( a_0 \) are not zero, implies that \( a_0 = 0 \). If \( c = -1 \), the lowest term gives
\[
-(a_0^3 + \gamma a_0^4) \rho^{-4} = 0,
\] (57)
which gives
\[
a_0 = -\alpha/\gamma.
\] (58)
Thus, unless \( \alpha = \gamma = 0 \), the transcendent has at most a simple pole at \( \rho = 0 \).
The first step is to find $G_0(\rho, z, \lambda)$ defined for $\lambda \neq \infty$ and $G_\infty(\rho, z, \lambda)$ so that $P = G_\infty G_0^{-1}$ which we will do in a series in $\rho$. We assume $p = -\bar{p}$ in equation (14), and for notational simplicity we define $\alpha = a - \bar{a}$, $\beta = a + \bar{a}$. Setting $u = \frac{\rho}{2} (\lambda + 1/\lambda) + iz$ to split $e^{e^{iaz}} = e^{(iz+\rho/2\lambda)} e^{e^{iaz}/2}$, we have

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \rho \bar{\rho} & 0 \\ 0 & \rho^{-\bar{\rho}} \end{pmatrix} \begin{pmatrix} e^{au} & \lambda^{2\hat{p}} e^{\beta u} \\ -\lambda^{-\hat{p}} e^{-\beta u} & e^{-au} \end{pmatrix} \begin{pmatrix} \rho \bar{\rho} & 0 \\ 0 & \rho^{-\bar{\rho}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \rho \bar{\rho} e^{\alpha (i+\rho/2\lambda)} & 0 \\ 0 & \rho^{-\bar{\rho}} e^{-\alpha (i+\rho/2\lambda)} \end{pmatrix} \times \begin{pmatrix} (1/2) c_1 \lambda^{\hat{p}} e^{\beta u - \alpha (1/\lambda - \lambda)/2 + ai\bar{z}} \\ c_1 \lambda^{2\hat{p}} e^{(1/\lambda - \lambda)/2 - ai\bar{z}} \end{pmatrix} \times \begin{pmatrix} \rho \bar{\rho} e^{\alpha p\lambda/2} & 0 \\ 0 & \rho^{-\bar{\rho}} e^{-\alpha p\lambda/2} \end{pmatrix}. \quad (59)$$

Without further loss of generality, we assume $\bar{p} \geq 0$, and we focus on the matrix in the middle of the RHS, which we call $\tilde{P}$. Its off diagonal entries can be expanded in $\rho$ and if we do so up to $\rho^{2\bar{p}}$

$$\tilde{P} = \begin{pmatrix} 1 & g \\ -\hat{g} & 1 \end{pmatrix} + \mathcal{O} \left( \rho^{2\bar{p}+1} \right), \quad (60)$$

we find that $g$ is a polynomial in $\lambda$

$$g = c_1 e^{i(\beta - a)z} \lambda^{2\hat{p}} \sum_{i=0}^{2\bar{p}} \rho^{i} ((a + \beta)\lambda + (a - \beta)/\lambda) \frac{1}{2i!} = \sum_{i=0}^{2\bar{p}} g_i \lambda^i,$$

where

$$g_0 = c_1 \frac{(\alpha - \beta)\rho^{2\hat{p}} e^{i(\beta - a)z}}{2^{\hat{p}} (2\bar{p})!}$$

is generically non-vanishing. There is a similar formula for $\hat{g}$ as a polynomial in $1/\lambda$, $\hat{g} = \sum_{i=0}^{2\bar{p}} \hat{g}_i \lambda^{-i}$, with $\hat{g}_0$ generically non-vanishing. Notice further that from their definitions

$$g \hat{g} = 1 + \mathcal{O} \left( \rho^{2\bar{p}+1} \right). \quad (61)$$

We can now easily split the matrix $\tilde{P}$ up to order $\mathcal{O} \left( \rho^{2\bar{p}+1} \right)$,

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ -\hat{g} & 1 + g \hat{g} \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} + \mathcal{O} \left( \rho^{2\bar{p}+1} \right)$$

$$= \begin{pmatrix} 1 & 0 \\ -\hat{g} & 2 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} + \mathcal{O} \left( \rho^{2\bar{p}+1} \right). \quad (62)$$

Therefore,

$$J(\rho, z) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ia\hat{p}z - \lambda \rho} g_0 + \mathcal{O} \left( \rho^{2\bar{p}+1} \right) \\ e^{ia\hat{p}z} g_0 + \mathcal{O} \left( \rho^{2\bar{p}+1} \right) \end{pmatrix} \rho^{-\bar{p} \hat{g} (2 - \hat{g} g_0)} + \mathcal{O} \left( \rho \right) \quad (63)$$
As \( g_0 \) and \( \hat{g}_0 \) are proportional to \( \rho^{2\tilde{\nu}} \), it is clear that the entries of \( J, J^{-1} \) and \( \partial_\rho J \) are all meromorphic when \( \rho \rightarrow 0 \).

We can now proceed to the second step. Recall that in a particular gauge the Lax pair can be written as in equation (20),

\[
L_1 = \partial_\rho + i\lambda \partial_z + \frac{1}{\rho} \lambda \partial_\lambda + i\lambda J^{-1} \partial_\rho J, \tag{64}
\]

\[
L_2 = \partial_z - i\lambda \partial_\rho + \frac{i}{\rho} \lambda^2 \partial_\lambda - i\lambda J^{-1} \partial_\rho J. \tag{65}
\]

In order to get to equation (29), we first need to perform a gauge transformation and go to a \( z \)-independent frame. From equation (45), it is clear that

\[
J^{-1} \partial_z J = J^{-1} \left( \frac{-\tilde{a}i}{0} \ 0 \frac{\tilde{a}i}{0} \right) J + \left( \frac{a}{0} \ 0 \ -a \right). \tag{66}
\]

Therefore,

\[
\left( \begin{array}{cc} e^{ai z} & 0 \\ 0 & e^{-ai z} \end{array} \right) L_1 \left( \begin{array}{cc} e^{-ai z} & 0 \\ 0 & e^{ai z} \end{array} \right) = \partial_\rho + \frac{1}{\rho} \lambda \partial_\lambda + i\lambda J^{-1} \left( \frac{-\tilde{a}i}{0} \ 0 \frac{\tilde{a}i}{0} \right) J|_{z=0}. \tag{67}
\]

The same gauge transformation gives

\[
\left( \begin{array}{cc} ie^{ai z} & 0 \\ 0 & e^{-ai z} \end{array} \right) L_2 \left( \begin{array}{cc} e^{-ai z} & 0 \\ 0 & e^{ai z} \end{array} \right) = \lambda \partial_\rho - \frac{\lambda^2}{\rho} \partial_\lambda + \lambda J^{-1} \partial_\rho J|_{z=0} + \left( \frac{\tilde{a}}{0} \ 0 \ -\tilde{a} \right). \tag{68}
\]

It follows that

\[
B = J^{-1} \left( \frac{-\tilde{a}}{0} \ 0 \frac{\tilde{a}}{0} \right) J|_{z=0}
\]

\[
C = \left( \frac{-\tilde{a}}{\tilde{a}} \ 0 \ 0 \right)
\]

\[
D = -J^{-1} \partial_\rho J|_{z=0}. \tag{69}
\]

In particular, using the knowledge gained in the first step, we can conclude that the entries \( D_{12}, D_{21}, B_{12}, B_{21} \) are meromorphic in \( \rho \) as \( \rho \rightarrow 0 \). In more detail, from equation (31), provided \( \tilde{p} > 0 \), we have

\[
D_{12} = -4\tilde{p} \frac{\tilde{a}}{2^{2\tilde{\nu}} \tilde{p}!} \frac{1}{\rho} + O(\rho^0) \quad B_{12} = -2\tilde{a}^2 \frac{\tilde{p}}{2^{2\tilde{\nu}} \tilde{p}!} \frac{1}{\rho} + O(\rho).
\]

\[
f = -\frac{1}{2} \frac{D_{12}}{B_{12}} = -\frac{\tilde{p}}{\tilde{a}} \frac{1}{\rho} + O(\rho^0) = \frac{1}{2} \frac{n}{\gamma} \frac{1}{\rho} + O(\rho^0) = -\frac{\alpha}{\gamma} \frac{1}{\rho} + O(\rho), \tag{71}
\]

in agreement with the result of the Frobenius analysis equation (58). If \( \tilde{p} = 0 \), we find instead

\[
f = 0 + O(\rho^{-1}),
\]

which is also consistent with the Frobenius analysis.
3.5 Characterization of monodromy data

We now prove Theorem 2 that states that the solutions parametrized in Theorem 1 are the solutions whose
monodromy data $M$ has trivial Stokes matrices and half-integral exponents of formal monodromy. First,
assume that we are given the Riemann–Hilbert problem of Theorem 1. We restrict to the case $\mathcal{D}_6$, as $Q$
can be treated in a similar way. The patching matrix equation (14) can be written as

$$P = \begin{pmatrix}
(\rho \lambda)^p & 0 & 0 & e^{au} & e^{-au} & c_0 & 0 & 0 & 0 & 0 & (\lambda^p)^p & 0 \\
0 & (\rho \lambda)^{-p} & e^{au} & e^{-au} & c_1 & 0 & \tilde{c}_1 & 0 & 0 & 0 & 0 & (\lambda^{-p})^p
\end{pmatrix}. \tag{72}
$$

This patching matrix is meant to relate two frames $\hat{F}^0$ and $\hat{F}^\infty$ holomorphic near $\lambda = 0$ and $\lambda = \infty$
respectively. In order to eliminate the diagonal matrices, redefine the frames so that

$$\hat{F}^0 \mapsto \hat{F}^0 \begin{pmatrix}
\tilde{e}^{au}(\lambda^p)^{-p} & 0 \\
0 & e^{au}(\lambda^{-p})^p
\end{pmatrix} \tag{73}
$$

near $\lambda = 0$, and similarly for the frame near $\lambda = \infty$. Near $\lambda = 0$ we have

$$\hat{F}^0 \begin{pmatrix}
\tilde{e}^{au}(\lambda^p)^{-p} & 0 \\
0 & e^{au}(\lambda^{-p})^p
\end{pmatrix} \sim \hat{Y}^0 \exp \left( \frac{a \rho \lambda}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} + p \log \lambda \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \right). \tag{74}
$$

where we reabsorbed $\lambda$-independent factors of $\hat{F}^0$ in $\hat{Y}^0$. This is of the form of equation (6), and since
the asymptotic expansion for solutions of the linear system equation (5) is unique it must be the same
as equation (5). By construction, $\hat{Y}^0$ is holomorphic near $\lambda = 0$ and therefore the Stokes matrices are
trivial. We conclude that

$$C := \begin{pmatrix}
c_0 & \tilde{c}_1 \\
c_1 & \tilde{c}_0
\end{pmatrix} \tag{75}
$$

is by definition the connection matrix. We also see directly that the exponents formal monodromy are $\pm p$
which are half-integral (or of course integral) by our earlier discussion. Our constructions therefore maps
to monodromy data $M$ with trivial Stokes matrices and half-integer exponents of formal monodromy.

4. The Ward ansatz

The Ward ansatz [5] constructs non-trivial examples of solutions by taking the data to be upper triangular.
One can then solve the Riemann–Hilbert problem explicitly, or at least reduce the procedure to solving
linear equations. What is remarkable is that the solutions $J$ that are obtained cannot be reduced to being
upper triangular when the diagonal entries have non-trivial winding number. Reducing to the $\mathrm{SL}(2, \mathbb{C})$
case, we take patching matrices $P$ of the form

$$P = \begin{pmatrix}
(\rho \lambda)^r e^{\sigma(u)} & \rho^s \lambda^r \gamma(u) \\
0 & \rho^{-s} \lambda^{-r} e^{-\sigma(u)}
\end{pmatrix}. \tag{76}
$$

Here $\sigma$ and $\gamma$ are holomorphic functions of $u$, and with respect to equation (23) we have set $r = p + \tilde{p}$,
$s = p - \tilde{p}$. We must assume $r \geq 0$ in order that there is not a line sub-bundle of positive degree (which
would contradict triviality of the bundle on a line).
The original work is [23]. Details closer to our approach can be found for example in [24, p. 398] et seq. The computation of the $J$-matrix based on the procedure outlined therein leads to the following theorem, which generalizes Proposition A.1 of [25].

**Theorem 4** Let $A = A(\rho, z)$ and $A_\rho = \partial_\rho A$, $A_z = \partial_z A$ satisfying

$$
\left(\frac{1}{\rho} + \partial_\rho\right)A_\rho + \partial_z A_z = 0,
$$

and let $\Delta_k = \Delta_k(\rho, z)$ solve

$$
\left(\partial_\rho - k\right)\Delta_k = -i(\partial_z + A_z)\Delta_{k+1},
$$

$$
\left(\partial_\rho + A_\rho + \frac{k+1}{\rho}\right)\Delta_{k+1} = -i\partial_\rho \Delta_k.
$$

Define

$$
\tau_s^r \equiv \det\left(r_s^r\right), \quad r_s^r \equiv \begin{pmatrix}
\Delta_{s+1} & \cdots & \Delta_s \\
\Delta_{s+2} & \cdots & \Delta_{s+1} \\
\vdots & \ddots & \vdots \\
\Delta_s & \cdots & \Delta_{s+r-1}
\end{pmatrix},
$$

Then, provided $\tau_s^r \neq 0$,

$$
J(\rho, z) = \frac{1}{\tau_s^r} \begin{pmatrix}
\rho^r \tau_{s+1}^r & \rho^r \tau_s^r \\
\rho^{-r} \tau_{s+1}^r & \rho^{-r} \tau_s^r
\end{pmatrix}
$$

is a solution of the stationary axisymmetric SDYM equation, equation (17).

$A$ is an abelian analogue of $\log J$, (so that $A_\rho$, $A_z$ are abelian counterparts of $\partial_\rho J$ and $\partial_z J$ in equation (17)). It arises from viewing $e^{\sigma(u)}$ as a $GL(1, \mathbb{C})$-patching matrix and by following the steps in the proof of Theorem 1. In the abelian case, equation (19) amounts to a splitting

$$
\sigma(\rho/2(\lambda + 1/\lambda) + iz) = \sigma_\infty(\lambda, \rho, z) - \sigma_0(\lambda, \rho, z)
$$

where $\sigma_0(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$, whereas $\sigma_\infty(\lambda)$ is holomorphic in a neighbourhood of $\lambda = \infty$. As $V_1, V_2$ annihilate $\sigma$ restricted to a line, it follows that $V_1 \sigma_0 = V_1 \sigma_\infty$, $V_2 \sigma_0 = V_2 \sigma_\infty$, and by the Liouville-type argument that both expressions are at most linear in $\lambda$. Using the freedom in equation (81) we can remove the constant terms. Thus

$$
V_1 \sigma_0 = iA_\rho \lambda, \quad V_2 \sigma_0 = -iA_\rho \lambda.
$$

and so

$$
(V_1 + iA_\rho \lambda)e^{-\sigma_0} = (V_2 - iA_\rho \lambda)e^{-\sigma_0} = 0.
$$
Consequently

\[ [V_1 + iA_z \lambda, V_2 - iA_\rho \lambda] = 0, \tag{84} \]

which is equivalent to equation (77).

The \( \Delta_k \)'s arise from the following Laurent expansion in \( \lambda \) in an annulus surrounding \( |\lambda| = 1 \)

\[
\exp(-\sigma_\infty - \sigma_0) \gamma(\lambda, \rho, z) = \sum_{i \in \mathbb{Z}} \Delta_i(\rho, z) \lambda^i. \tag{85}
\]

Equation (78) is then a direct consequence of the fact that the vector fields in equation (18) annihilate \( \gamma(u) \).

More geometrically, define the line bundles \( L \rightarrow \mathbb{T}(U) \) by its transition function \( e^{\sigma(u)} \), and \( O(r) \rightarrow \mathbb{C} \mathbb{P}^1 \) by \( \lambda^{-n} \) so that it has Chern class \( n \). Then the patching matrix in equation (76) represents the bundle \( E \) as an extension of \( L(-r) := L \otimes O(-r) \) by its dual. Thus \( E \) fits into the following short exact sequence on \( \mathbb{T}(U) \)

\[
0 \rightarrow L(-r) \rightarrow E \rightarrow L^{-1}(r) \rightarrow 0. \tag{86}
\]

The Penrose–Ward transform identifies the line bundle \( L \) with the stationary-axisymmetric self-dual Maxwell field fields with components \( A_z \) and \( A_\rho \) on the reduced space-time coordinatized by \( (\rho, z) \). On the other hand, the off diagonal entry in \( \lambda^n \gamma(u) \) can be seen as an element of \( H^1(T(U), L^2(-2r)) \). The Penrose transform realizes such cohomology classes as massless field of helicity \( r - 1 \) coupled to the Maxwell field. Such a field has \( 2n + 1 \) components and these are the coefficients \( \Delta_k \) for \( |k| \leq r - 1 \), and the charged massless field equations in this stationary axisymmetric context are (78).

**Painlevé III example.**

As an example, we study the case in which \( c_0 = \tilde{c}_0 = 1, c_1 = 0 \) in equation (45). Defining \( r = p + \tilde{p}, s = -p + \tilde{p} \), we obtain

\[
\begin{pmatrix}
\rho^+ \lambda^r e^{(a-\bar{a})u} & \rho^+ \lambda^r e^{(-a-\bar{a})u \tilde{c}_1} \\
0 & \rho^{-s} \lambda^{-r} e^{(-a+\bar{a})u} \\
\end{pmatrix}, \tag{87}
\]

where

\[
u = \frac{\rho}{2} \left( \lambda + \frac{1}{\lambda} \right) + iz. \tag{88}\]

Based on the previous discussion, we first need to split \( (a - \bar{a})u \) as

\[
\sigma_\infty = (a - \bar{a}) \frac{\rho}{2\lambda}, \quad \sigma_0 = -(a - \bar{a}) \left( \frac{\rho}{2} \lambda + iz \right). \tag{89}\]

In order to perform the Laurent expansion equation (85), we recall the well-known identity

\[
e^{\frac{\rho}{2} (\lambda + \frac{1}{\lambda})} = \sum_{i=-\infty}^{\infty} I_{\lambda i}(\rho) \lambda^i, \tag{90}\]
where the $I_\nu(\rho)$s are modified Bessel functions of the first kind. We therefore rearrange the LHS of equation (85) as follows,

$$\tilde{c}_1 e^{-\sigma_-} e^{(-\tilde{a} - a)u} = \tilde{c}_1 \exp \left( -\rho \sqrt{\tilde{a} a} \left( \tilde{\lambda} + \frac{1}{\tilde{\lambda}} \right) \right) e^{-2\tilde{a}iz},$$  (91)

where

$$\tilde{\lambda} = \sqrt{\tilde{a}/a}.$$  (92)

It then follows directly from equation (90) that

$$\tilde{c}_1 e^{-\sigma_-} e^{(-\tilde{a} - a)u} = \sum_{j \in \mathbb{Z}} \tilde{c}_1 e^{-2\tilde{a}iz} I_{-j} \left( -2\rho \sqrt{\tilde{a} a} \right) \left( \frac{a}{\tilde{a}} \right)^{j/2} \lambda^j,$$  (93)

and thus

$$\Delta_j = \tilde{c}_1 e^{-2\tilde{a}iz} I_{-j} \left( -2\rho \sqrt{\tilde{a} a} \right) \left( \frac{a}{\tilde{a}} \right)^{j/2}.$$  (94)

In terms of the constants $m, n, k, l$ (picking the square-roots $l = \tilde{a}, -4k = a$)

$$\Delta_j = \tilde{c}_1 e^{-2\tilde{a}iz} I_{-j} \left( -\rho \sqrt{-16kl} \right) \left( -4k/l \right)^{j/2}.$$  (95)

and

$$s = \frac{1}{2} \left( \frac{m}{k} - \frac{n}{l} \right), \quad r = \frac{1}{2} \left( \frac{m}{k} + \frac{n}{l} \right).$$  (96)

In the context of the Painlevé equations, solutions involving special functions (Bessel functions in the case of $PIII$) are called classical transcendental solutions. The classical transcendental solutions were classified in [25] and given in terms of $J$-matrices of precisely the form of equation (80), see in particular theorem 4.2 of [25]. These solutions coincide with ours up to a redefinition of the constants.10

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10 The classical transcendental solutions presented in [25] are essentially given by determinants as in (79) with entries

$$\phi_j = (-2\eta_0)^{-j} e^{-\eta_0 z' + \eta_\infty v'} \tilde{w}^{-v+1-j} \rho^{-(v+1-j)} \psi_{v+1-j},$$  (97)

where

$$\psi_v = \begin{cases} c_1 J_v + c_2 Y_v, & 4\eta_0 \eta_\infty = +1 \\ c_1 I_v + c_2 I_{-v}, & 4\eta_0 \eta_\infty = -1, \end{cases}$$  (98)

and

$$v + 1 = \frac{1}{2} \left( \frac{n}{l} + \frac{m}{k} \right), \quad \eta_\infty = 2l, \quad \eta_0 = -2k.$$  (99)
Our solutions reproduce all classical transcendental solutions meromorphic at $\rho = 0$.

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The coordinate $\tilde{w}$ is related to $\rho$ by $\tilde{w} = \rho e^{i\phi}$, where $\phi$ is the angular variable in the space-time cylindrical polar coordinates, and a combination of $z'$ and $\tilde{z}'$ gives our parameter $z$. The factor $\tilde{w}^{v+1-j} \rho^{-(v+1-j)}$, a power of $e^{i\phi}$, factors out from the $J$-matrix as well as exponentials containing $z$ or $\tilde{z}$, and none contributes to the transcendental. With a rescaling of the transcendent, we can set $4\eta_0\eta_\infty = 16kl = \pm 1$ and it can then be checked that the solutions of [25] meromorphic at $\rho = 0$ agree with the ones we have obtained.

This can also be checked from the point of view of the Bäcklund transformations of $P_{III}$. This determines the classical transcendental solutions as particular loci in the parameter space of the constants ($\alpha, \beta, \gamma, \delta$), see for example [26]. The action of the Bäcklund transformations on the twistor data are given in [3].
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