Convex Hulls in 2D Complexes with Non-Positive Curvature

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1 Introduction
Convex hulls and algorithms to find them are very well understood in Euclidean spaces [7], but less so in non-Euclidean spaces. We consider the problem of computing the convex hull in a 2D polyhedral cone complex that is non-positively curved, or CAT(0). In such a space, geodesic paths are unique and the convex hull of a set of points \( P \) is the minimal set \( S \) that contains \( P \) and the geodesic path between any two points in \( S \). It is not even known if such convex hulls are closed.

A main example of such a space is the Billera-Holmes-Vogtmann space of phylogenetic trees with 4 leaves [1]. Being able to compute convex hulls in this space would give a method for computing confidence intervals for sets of trees [3].

Our main result is an algorithm to find the convex hull of a finite set of points in the special case where the 2D CAT(0) polyhedral complex has a single vertex. Maftuleac [5] previously solved the special case of a 2D CAT(0) polyhedral surface.

Definitions. A 2D polyhedral complex is a set of polygons ("cells") glued together by isometries along their edges. We consider a space with a single vertex, called the origin. The distance between two points in the complex is the length of the shortest path between them, where the length of a path is the sum of the Euclidean lengths of the pieces of the path in each cell of the complex. For any vertex \( v \) of a 2D complex, we define the link graph, \( G_v \), as follows. The vertices of \( G_v \) correspond to the edges incident to \( v \) in the complex. The edges of \( G_v \) correspond to the cells incident to \( v \) in the complex: if \( r \) and \( s \) are edges of cell \( C \) with \( r \) and \( s \) incident to \( v \), then we add an edge between vertices \( r \) and \( s \) in \( G \) with weight equal to the angle between \( r \) and \( s \) in \( C \). A 2D polyhedral complex \( K \) is CAT(0) if and only if it is simply connected and for every vertex \( v \in K \), every cycle in the link graph \( G_v \) has length at least \( 2\pi \) [2]. In a general CAT(0) space, there is a unique geodesic (locally shortest path) between any two points and this property characterizes CAT(0) complexes [2].

2 Convex Hulls
Let \( P \) be a finite set of points in a CAT(0) complex. We first give examples to show that the following properties of convex hulls in Euclidean space do not carry over to CAT(0) complexes.

1. Any point on the boundary of the convex hull of points \( P \) in 2D is on a shortest path between two points of \( P \).

2. In any dimensional space, the convex hull of three points is 2-dimensional.

The example in Figure 1a shows the shortest paths between pairs of points in \( P \) do not determine the convex hull, and thus property 1 fails. The example in Figure 1b shows that the convex hull of three points in a 3D CAT(0) complex may contain a 3D ball, and thus property 2 fails. We note that Lin et al. [4] showed that for certain CAT(0) cube complexes, including tree space, there exist three points whose convex hull contains a simplex of dimension half that of the maximal cubes.

Our main result is an algorithm (using linear programming) to find convex hulls in any 2D CAT(0) complex with a single vertex \( O \):

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Theorem 1. There is a polynomial time reduction from the problem of finding the convex hull of a finite set of points $P$ in a 2D CAT(0) complex with a single vertex $O$ to linear programming. For the special case of a cube complex this provides a polynomial-time convex hull algorithm.

First our algorithm uses the link graph to test if point $O$ is in the convex hull and to identify the edges of the complex that intersect the convex hull at points other than $O$. These are done in time $O(m(n + m))$ and $O((n + m)^2)$, respectively, where $n$ is the number of cells in the complex and $m$ is the size of $P$. Then we formulate the exact computation of the convex hull as a linear program whose variables represent the boundary points of the convex hull on the edges of the complex. This step requires the real-RAM model of computation because we compute sines of input angles.

Solving the resulting linear program finds the convex hull. There are polynomial-time linear programming algorithms [6], but the run-times depend on the number of bits in the input numbers. When the cells have angles of $90^\circ$ at the origin, such as in tree space, we have a cube complex, sine computations are easy, and our linear program has coefficients with a polynomial number of bits, so our algorithm runs in polynomial time. However, more generally we need the real-RAM model of computation, and we must resort to the simplex method for linear programming [6] which is not known to run in polynomial time.

Our proof of Theorem 1 implies that the convex hull of a finite set of points in a single-vertex 2D CAT(0) complex is a closed set.

References

[1] L. J. Billera, S. P. Holmes, and K. Vogtmann. Geometry of the space of phylogenetic trees. *Advances in Applied Mathematics* 27(4):733–767, 2001.
[2] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Springer, 1999.
[3] S. Holmes. Statistical approach to tests involving phylogenies. *Mathematics of evolution and phylogeny*. *Oxford University Press*, Oxford, UK pp. 91–120, 2005.
[4] B. Lin, B. Sturmfels, X. Tang, and R. Yoshida. Convexity in tree spaces. arXiv:1510.08797, 2015.
[5] D. Maftuleac. Algorithms for distance problems in planar complexes of global nonpositive curvature. *International Journal of Computational Geometry & Applications* 24(01):1–38, 2014.
[6] A. Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998.
[7] R. Seidel. Convex hull computations. *Handbook of Discrete and Computational Geometry (2nd edition)*, pp. 495–512. CRC Press, Inc., 2004.