The peripheral vortex biome of confined quantum fluids and its influence on vortex dipole annihilation

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Received 23 June 2022, revised 12 September 2022
Accepted for publication 7 October 2022
Published 28 October 2022

Abstract
The self-annihilation of a pair of oppositely charged optical vortices (vortex dipole) in a quantum fluid is hindered by nonlinearity and promoted by radial confinement, resulting in rich life-cycle dynamics of such pairs. The competing effects generate a biome of peripheral vortices that can directly interact with the original pair to produce a sequence of surrogation events. Numerical simulation is used to elucidate the role of the vortex biome as a function of nonlinearity strength and the initial spacing between the engineered vortices. The results apply directly to other nonlinear quantum fluids as well and may be useful in the control of complex condensates in which vortex dynamics produce topologically protected phases.

Keywords: optical vortex dipole, BEC, Gross–Pitaevskii equation, harmonically trapped quantum fluid, phase diagram

(Some figures may appear in colour only in the online journal)

1. Introduction
The ability to anticipate and control vortex dynamics in two-dimensional, nonlinear quantum fluids is scientifically and technologically important to disciplines ranging from quantum turbulence \[1\] to the generation of non-Abelian anyons \[2, 3\] and their use in topological quantum computing \[4–8\]. While the motion of individual vortices can now be predicted \[9–11\], it is often the nucleation and annihilation of oppositely charged vortex pairs (vortex dipoles) that dominates processes of interest \[12\].

In Bose–Einstein condensates (BECs), for instance, such processes have been observed around a repulsive Gaussian obstacle \[13\] or precipitated by stirring \[14\], and they have been shown to dominate analogs to quantum turbulence experimentally observed in beams of light \[15\]. A vortex dipole can exist for many seconds in an interacting BEC \[13\] but with a rate of annihilation enhanced by repulsive potential barriers \[16\] and velocities that can be tailored with red-detuned and blue-detuned beams \[17\]. Vortex–antivortex lattices have also been predicted for superfluids \[18\], and the collective dynamics of many-body vortex systems have been studied in both linear \[15\] and nonlinear \[19–21\] quantum fluids.

Vortex dipoles can be theoretically studied using Ginzburg–Landau theory \[22–25\], but it is also possible to elucidate few-body vortex nucleation and annihilation using the Gross–Pitaevskii equation \[26, 27\]. Nonlinear optical fluids are also governed by the Gross–Pitaevskii equation \[28\] and so serve as controllable, classical analogs to quantum fluids. The optical setting offers deterministic programming of the initial vortex positions and shapes, direct readout of evolved states, and room temperature operation.

In the present work, the dynamics and annihilation of vortex pairs are computationally studied in trapped, nonlinear optical quantum fluids. The combination of harmonic trapping and nonlinearity of the medium nonlinearity results in an evolution of nucleation and annihilation events in which the original vortex pair interacts with vortices nucleated at the trap boundary.

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The analysis is framed within the setting of nonlinear optical fluids, but the results apply to any two-dimensional fluid governed by the Gross–Pitaevskii equation. In the limiting regime of linear and weakly nonlinear media, the results are consistent with previous investigations [11, 29–31]. Beyond the perturbative setting though, our numerical simulations reveal qualitatively new behavior in which the role of the boundary vortices is elucidated. A wide range of fluid nonlinearities and initial vortex separations are used to generate a phase diagram delineating settings in which the primary vortices either do or do not annihilate as a function of initial vortex separations and medium nonlinearity.

2. Setting

Consider a linearly polarized, monochromatic electromagnetic wave with an electric field of the form

\[ \mathbf{E}(\mathbf{r}_\perp, z, t) = \psi(\mathbf{r}_\perp, z) e^{i k_0(c t - \omega_0 z)} \mathbf{\hat{r}}_\perp. \]

(1)

The free-space wavenumber is \( k_0 \), \( c_0 \) is the speed of light in vacuum, \( \mathbf{\hat{r}}_\perp \) is the unit vector orientation of the polarization, \( \psi \) is a scalar measure of the electric field, the transverse position vector is \( \mathbf{r}_\perp \), and the beam axial coordinate, \( z \), plays the role of time. In fact, we will refer to changes associated with travel along this axis as time evolution.

For media with a third-order susceptibility and beam profiles for which the paraxial approximation [32] is valid, Maxwell’s equations then imply that the evolution of the scalar field, \( \psi \), is well-approximated by the Gross–Pitaevskii equation:

\[ i \partial_t \psi = \left( -\frac{1}{2 k_0 n_0} \nabla^2 - k_0 \Delta n - k_0 n_2 |\psi|^2 \right) \psi. \]

(2)

Here \( n_0 \) and \( n_2 \) are the index of refraction and nonlinear refractive indices of the medium, and \( \Delta n \) is an external change in the index of refraction,

\[ \Delta n = -\frac{1}{2} \gamma r^2, \]

(3)

that is generated by either a patterned medium or an additional laser beam. The dielectric profile curvature, \( \gamma \), has units of inverse length squared and must be sufficiently small, so that \( \gamma \ll k_0^2 \). This condition allows additional terms involving the dielectric profile to be safely neglected. Optical nonlinearity gives rise to a photon–photon interaction of strength \(-k_0 n_2\) that can be either attractive \((n_2 > 0)\) or repulsive \((n_2 < 0)\). Within this setting, the propagation of light is formally analogous to the dynamics of two-dimensional quantum fluids at the mean-field level [33, 34]. The introduction of a characteristic length of \( 1/(k_0 n_0) \) and characteristic field intensity of \( I_0 = \epsilon_0 c_0 E_0^2 / 2 \) allows the position variables and scalar electric field to be expressed non-dimensionally. The resulting evolution equation is

\[ i \partial_t \psi = \left( -\frac{1}{2} \nabla^2 + \frac{\omega^2}{2} r^2 + \beta |\psi|^2 \right) \psi. \]

(4)

Here \( r^2 = x^2 + y^2 \), and

\[ \beta = -\frac{n_2 I_0}{n_0} \quad \text{and} \quad \omega = \sqrt{\frac{\gamma}{k_0 n_0}} \]

(5)

are the non-dimensional strength of the medium nonlinearity and the non-dimensional trap strength, respectively, and \( n_2 \) is the nonlinear index of refraction in terms of intensity, \( \Delta n = n_2 I_0 \).

This theoretical framework can be applied to experiments on both optical fluids [11, 29, 35] and BECs [36–39], with \( \beta \) on the order of 1000 achievable in both settings.

The 2D Gross–Pitaevskii equation is numerically solved using an unconditionally stable time splitting pseudo-spectral method [39, 40] by discretizing the spatial domain into a 1024×1024 grid. The domain size is ±20 in both \( x \) and \( y \) directions, and the evolution step size is prescribed as \( 10^{-3} \).

The GPE of equation (4) is numerically solved for initial states for which a Gaussian profile is implanted with two diametrically-opposed, oppositely-charged vortices that are offset from the center of the fluid by \( x_0 \):

\[ \psi_0(x, y) = \frac{N}{\sqrt{\pi}} e^{-\frac{1}{2}(x^2+y^2)} v_+(x,y) v_-(x,y). \]

(6)

Here \( N \) is a normalization factor, and the initial profiles of the vortices are

\[ v_\pm(x, y) = \sqrt{1 - \frac{p^2}{(x \pm x_0)^2 + y^2}} \frac{(x \mp x_0) \pm iy}{\sqrt{(x \mp x_0)^2 + y^2}}. \]

(7)

The healing length, \( l \), is chosen to make the vortex structures as stable as possible (equation (3.17) in [41]):

\[ l = \frac{1}{\sqrt{\beta \psi_{bg}}} \]

(8)

Here \( \psi_{bg} \) is the background fluid amplitude at the vortex initial position, \( \{x_0, 0\} \), located at the shoulder of a Gaussian profile. It is given by

\[ \psi_{bg} = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2} y_0^2}. \]

(9)

Note that the repulsive interaction \((\beta > 0)\) in a nonlinear fluid tends to compress vortex footprints which results in periodic fluctuations in the core size of evolving vortices. Such core-size fluctuations cannot be completely eliminated, since the fluid amplitude is not evenly distributed around a vortex moving inside the trapped fluid. However, enforcement of the relationship between \( l \) and \( \beta \) in equation (8) best minimizes the influence of this effect [9].

In the limit of a linear fluid \((\beta = 0)\), the core approaches a linear profile:

\[ v_\pm(x, y) = (x \mp x_0) \pm iy. \]

(10)
vortex moves back and forth on an arc that does not intersect with its
with equations (11) and (12), over a range of initial vortex
vortices reverse their course and repeat the annihilation/nucleation at
intervening interval, re-nucleate at this same spot. Then the two
annihilate at the bottom of their arcs (yellow disks), but after an
understand how energy is distributed as the system evolves.
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The white arrow in panel (b) denotes a strong local, upward
fluid velocity in a necked region, identifiable because the
black phase contour lines are very dense. This calls to mind
energy is conserved throughout such dynamics, and the
potential energy due to the trap (and later nonlinear interaction)
is negligible in comparison with the kinetic energy. Maps
of the magnitude of the linear momentum [13], effectively a
conserved quantity in this setting, can therefore be used to
understand how energy is distributed as the system evolves.

It is an especially useful metric for studying dipole nucleation
and annihilation since linear momentum corresponds to gra-
dients in the field that must be sufficiently large to support a
dipole [42]. Viewed from this perspective, vortex annihilation
is the result of a delocalization of linear momentum. The time-
like interval over which the primary vortices are not present
identifies an interim of such delocalization. It ends when a
subsequent localization of linear momentum causes the vortex
pair to reform. A cycle is thus established in which the linear
momentum localizes and delocalizes within the trap. We have
also verified that suppression of the kinetic energy term in the
GPE will result in vortices that do not move, consistent with
known vortex kinetics [10].

For initial separations \( x_0 > 1/\sqrt{2} \), the two vortices cycli-
cally trace out only a fraction of these semicircular arcs and
never meet.

4. Nonlinear fluids

Nonlinear quantum fluids do not exhibit such simple cyclic
behavior. Some headway can be made using a perturbative
analysis provided the nonlinearity is sufficiently weak [31],
but a complete picture of the dynamics emerges only with a
numerical study over a wider range of nonlinearity strength.
The evolution of vortices in trapped, nonlinear quantum fluids
exhibits a new feature, a seething annulus of vortex nucleation
and annihilation at the periphery of the central portion of the
trap. For the most part, the vortices in this rich biome live
and die far away from the trap center and do not meaningfully
affect the primary vortex pair. However, select members of this
ecosystem play an essential role in the life cycle of the central
pair.

The same initial conditions are applied, but now with the
vortex healing length set by equation (8) and trap frequency set to
\[
\omega = \sqrt{1 + \frac{\beta}{2\pi}}.
\] (13)
The same Gaussian background profile is kept in the initial
state for the strongly nonlinear (large \( \beta \)) case. This is because,
in the optical setting, the Gaussian background profile char-
acterizes the intensity fall off of the laser beam in which
the vortices reside. This minimizes the fluid and vortex core
expansion induced by the repulsion from the finite \( \beta \).
The combination of nonlinearity and beam confinement produces a
peripheral biome of vortices. Figure 2 provides a particularly
clear example of this. Although there are multiple boundary
pairs, here we focus on the way in which nucleation occurs
for the boundary pair closest to the center. The slices of the
fluid phases in panels (b) and (c) capture this nucleation event.
The white arrow in panel (b) denotes a strong local, upward
fluid velocity in a necked region, identifiable because the
black phase contour lines are very dense. This calls to mind
the pulling of an oar through water that results in oppositely
spinning whirlpools on either end, and such settings are known
to generate vortex pairs [43]. The resulting vortices are shown
in panel (c).

3. Linear fluids

First consider trapped vortex dynamics for linear media
(\( \beta = 0 \)), where we will see that no peripheral vortices are
spawned, and analytical trajectories of the primary pair have
been previously derived [31]. This gives a baseline for sub-
sequent consideration of nonlinear effects. The initial state
is given by equations (6) and (10), and the trap strength in
equation (4) is chosen as \( \omega = 1 \) to match the size of the mode.
The evolving state is obtained in closed form with the vortex
trajectories then extracted:
\[
x_v(z) = \pm \sqrt{x_0^2 - (y_v(z))^2},
\] (11)
\[
y_v(z) = \left( x_0 - \frac{1}{x_0} \right) \sin(z).
\] (12)

These analytical trajectories for linear fluid are plotted in
figure 1, over a range of initial vortex separations. For initial separations of \( x_0 < 1/\sqrt{2} \), the two vortices cyclically an-
ihilate and nucleate with an intervening interval during which
no vortices exist. The locations of annihilation and nucleation
during the beam propagation along the \( z \) axis can be obtained
by substituting equation (12) into equation (11) and solving
the equation \( x_v(z) = 0 \). The interval for which no vortices
are present lies between the first and second solutions of this
equation.

Energy is conserved throughout such dynamics, and the
potential energy due to the trap (and later nonlinear interaction)
is negligible in comparison with the kinetic energy. Maps
of the magnitude of the linear momentum [13], effectively a
conserved quantity in this setting, can therefore be used to
understand how energy is distributed as the system evolves.

Figure 1. Analytical trajectories for linear, trapped fluid, generated
with equations (11) and (12), over a range of initial vortex
separations. For initial separations less than \( 1/\sqrt{2} \), the vortices
annihilate at the bottom of their arcs (yellow disks), but after an
intervening interval, re-nucleate at this same spot. Then the two
vortices reverse their course and repeat the annihilation/nucleation at
the top (green disks). For initial separations greater than \( 1/\sqrt{2} \), each
vortex moves back and forth on an arc that does not intersect with its
partner.

The same Gaussian background profile is kept in the initial
state for the strongly nonlinear (large \( \beta \)) case. This is because,
in the optical setting, the Gaussian background profile char-
acterizes the intensity fall off of the laser beam in which
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the pulling of an oar through water that results in oppositely
spinning whirlpools on either end, and such settings are known
to generate vortex pairs [43]. The resulting vortices are shown
in panel (c).
Figure 2. Vortex evolution in a trapped fluid for $\beta = 10$ and $x_0 = 1$. (a) The trajectories of two original vortices (red) and two boundary vortices (blue) during the period from the nucleation of the boundary vortex pair at $z = 0.9$ (blue dot) to the annihilation of the original vortex pair at $z = 1.34$ (red dot); (b) and (c) the fluid phases just before and after the nucleation of the boundary vortex pair, where the black contours are phase lines and the white arrows denote the directions of local fluid velocity—i.e. the negative of the phase gradient; (d) for comparison, the fluid phase for a trapped, linear fluid with $\beta = 0$ and $x_0 = 1$ at $z = 0.9$.

While this explains how vortices form in regions with sufficiently steep phase gradients, it does not explain how such gradients arise. For that, it is useful to consider that, for a linear trapped fluid, all the phase contours are circular, as shown in panel (d). Because of this, boundary vortex pairs will never nucleate in a linear trapped fluid. The addition of nonlinearity, though, induces phase inhomogeneities within the trap setting; even a slight deformation of nonlinear fluid in a trap may result in substantial fluid flow in the boundary annulus. In fact, we find that any nonzero value of nonlinearity, $\beta$, will result in the emergence of a boundary vortex biome.

4.1. Vortex annihilation for nonlinear fluids

The annulus of boundary vortices may indirectly influence the trajectories of the primary vortex pair, but a surprisingly intimate relationship actually exists between the two types of vortices. Vortex dynamics in weakly nonlinear, trapped quantum fluids are shown in figures 3 and 4 for an initial separation of $x_0 = 1$ and nonlinearity strengths of $\beta = 1$ and $\beta = 6$, respectively. The corresponding evolution of amplitude and phase are provided as supplemental movies. Spheres represent vortex position, with diameter inversely proportional to distance from the trap center. This makes the many vortex pairs at the periphery essentially invisible. For this initial separation, the vortex positions are stationary in the linear fluid, as shown in equations (11) and (12), but they annihilate in even weakly nonlinear fluids. Figure 3 shows that the primary vortex pair annihilates, but that a boundary vortex pair is nucleated even before this occurs. These boundary vortices rapidly move into the central region of the trap and take up positions near those initially occupied by the primary pair. The process then repeats, with the surrogate pair annihilating even as a new boundary pair nucleates.

Even if the nonlinearity is weak ($\beta = 1$), we can still see the surrogation effect induced by the peripheral vortex biome, as shown in figures 3(f)–(h). The dynamics shown correspond to the interval between the two black lines of figure 3(a). The primary vortex pair (red) quickly annihilates even as the boundary vortex pair (blue) migrates into the central region to replace the primary pair.
of two vortices are basically stationary at \( x_v(z) = \pm 1 \) and \( y_v(z) = 0 \) (though not strictly stationary since the ellipticity of the vortices can be infinity at some specific values of \( z \)). In the weakly nonlinear case, e.g. \( \beta = 1 \), the abrupt disappearance and re-appearance of the primary vortex pair is modified, with the peripheral vortex biome now inducing rapid surrogation dynamics when \( x_v(z) \neq \pm 1 \) in figure 3(a).

A new time scale becomes apparent as the nonlinearity is increased to \( \beta = 6 \), as shown in figure 4. Surrogate pairs periodically nucleate at the boundary, but the \( y \)-range of the central pair increases with each cycle. This occurs, albeit more slowly, for the case shown in figure 3 as well. An increase in nonlinearity strength causes the primary vortices and their surrogates to carry out circuits that evolve outwards, cycle by cycle, resulting in trajectories that are closer to the boundary region of the trap.

An even larger nonlinearity can be used to determine the ultimate fate of the expanding surrogation cycles. Nascent boundary pair surrogates interact more and more strongly with other boundary vortices, and the distinction is lost between surrogate boundary pairs and other boundary vortices. An example of this is shown in figure 5, for \( \beta = 120 \) and \( x_0 = 0.85 \). Eventually, the primary vortex pair self-annihilates, but the nascent surrogates are waylaid en route to the trap center and are themselves annihilated by another pair of boundary vortices. This process shows that the peripheral vortex biome can directly influence the primary vortices by annihilating with them—i.e. through two annihilation events that each involve one member of the biome pair and one member of the primary pair. In figure 5(a), a vortex pair from the biome enters the central region along the blue trajectory, and it directly annihilates with the primary vortex pair at \( z = z_3 \). A steady state is then reached in which all vortices have been swept out of the central region. The system settles into a structure in which the boundary vortex biome is divided into left and right regimes with a sparsely populated central channel. Figure 5 shows this limiting type of evolution, and the final state structure, for \( \beta = 120 \) and \( x_0 = 0.85 \). The vertical channel of low vortex density that coalesces is the result of very rapid vortex speeds associated with nucleation and annihilation, showing that the sequence identified in panel (a) continues.

The clearance of all vortices from the central region in figures 5(e) and (f) is therefore the result of the scattering of the linear momentum of the primary vortices within the biome. The large number of scattering sources there amounts to an absorbing boundary layer that makes the process effectively
Figure 6. Vortex evolution for $\beta = 150$. (a) The trajectories ($0 < z < 20$) of the two central vortices with initial offset, $x_0 = 0.5$. (b) and (c) Fluid amplitude and phase at $z = 20$ corresponding to the end of the trajectory in panel (a). (d) The trajectories ($0 < z < 20$) of the two central vortices for $x_0 = 0.7$. (e) and (f) Plots of the positions of all vortices during sequential intervals, $0 < z < 10$ and $10 < z < 20$, for $x_0 = 0.7$ with colors corresponding the ‘time’: red ($z = 0$) to cyan ($z = 20$). Sphere size is inversely proportional to distance from the trap center. Evolution runs vertically from bottom to top. (g)–(i) Distributions of the magnitude of linear momentum in the x–y plane at $z = 0.5$, $6.5$, and $20$.

irreversible. The local magnitude of linear momentum is given by

$$p = |\psi|^2 \| \nabla_{\perp} \text{Arg}(\psi) \|.$$  (14)

This linear momentum, corresponds to gradients in the field, is dispersed when the original vortex dipole annihilates. Subsequent reflections of the linear momentum can reproduce the large field gradients required to generate a dipole [42]. For small $z$, figure 5(g), linear momentum is localized around the central region. As $z$ increases, though, the linear momentum is scattered and absorbed by the biome as shown in figure 5(h), and the linear momentum eventually loses the degree of coherence necessary to localize and produce another central vortex pair. As shown in figure 5(i), the ultimate result is that all vortices near the central region are cleared out since the peripheral vortex biome has irreversibly absorbed all the linear momentum.

Analogous dynamics and trends characterize the vortex motion for smaller values of initial vortex separation, $x_0$, although the vortex biome is thinner and the rate of assimilation is slower. This fundamental dynamical character actually hold for all values of $x_0$ provided the nonlinearity is sufficiently small. However, there is an entire region of the $\{x_0, \beta\}$ parameter space for which the central vortices never annihilate, as will now be elucidated.

4.2. No annihilation for highly nonlinear fluids

For sufficiently large values of $\beta$, the influence of nonlinearity overwhelms boundary-vortex effects and annihilation of the central vortex pair is no longer possible. The trajectories of figure 6 are associated with this regime. This is shown in figure 6 with $\beta = 150$. An initial offset of $x_0 = 0.5$ causes the central vortices to simply wobble slightly, and the system reaches a steady state with the associated fluid amplitude and phase plotted in panels (b) and (c), respectively. The positions and circular cross-sections of the vortices are essentially unchanged, and a thick, well-separated biome shield wall exists at the periphery. As the initial separation is increased, the wobble of the central vortices becomes more pronounced. This is shown in panels (d)–(f). The two lobes of central vortex trajectories shown in figure 6 has been observed in an interacting BEC experiment [13].

We have shown, in figure 5, that the peripheral vortex biome tends to clear out all vortices from the central region by absorbing their linear momentum. However, the fluid nonlinearity itself, $\beta$, tends to inhibit annihilation of the original vortex pair since a large $\beta$ significantly reduces the vortex healing length defined in equation (8). This reduces the range over which vortices of opposite charge will experience an
The trap frequency is set to an attractive force. In figure 6, the effect of a sufficiently high nonlinearity reduces vortex healing length and effectively reduces the influence of the vortices on each other, thus making them less inclined to annihilate. The fluid is trapped, though, and boundary effects compete against this by promoting vortex annihilation. The boundary influence is weak for vortices with small separation, so a very large nonlinear effect is required to prevent their annihilation. The nonlinearity required for this is less when the vortex separation is larger since they have a lower mutual attraction [10]. If their initial separation is increased sufficiently, though, the boundary effect becomes prominent, driving the vortex pair toward annihilation. To prevent this, the nonlinearity must be increased. It is this competition that explains the parabolic shape of the phase boundary.

In all regions of the phase diagram, the presence of a biome of boundary vortices plays a role in the dynamics of the central vortex pair.

Below the phase boundary, the original pair can either self-annihilate or annihilate with vortices coming from the peripheral biome, as shown in figures 3–5. The effects of self-attraction and self-annihilation of the original pair are inherited from the linear case in figure 1 and promoted by the peripheral vortex biome. This is immediately followed, though, by a surrogation process in which a boundary pair rushes in to take the place of the original vortices. The process then repeats while, on slower time scale, the surrogate pairs become more widely separated and are more strongly influenced by other biome vortices. At some point, an annihilation in the trap center is not replaced because the nascent surrogates are waylaid and annihilated by other biome vortices. The result is a central region that is swept clear of all vortices and surrounded by a stable, clearly delineated shield biome.

Above the phase boundary, strong fluid nonlinearity tends to suppress the annihilation of the original vortex pair. This is because higher values of $\beta$ reduce the vortex healing length, defined in equation (8), and hence reduces the range at which vortices of opposite charge experience an attractive force. This myopia causes vortices in the biome to no longer be attracted to the central pair. Likewise, the central pair are themselves unaware of each other. As a result, linear momentum is no longer dispersed to and absorbed by the vortex biome, and the

\[ f_{\text{trap}}(x_0) = Ae^{-bx_0} + Ce^{dx_0}. \]  

6. Discussion

The construction of a phase diagram allows the observed trends to be put together into a coherent story. Fluid nonlinearity reduces vortex healing length and effectively reduces the influence of the vortices on each other, thus making them less inclined to annihilate. The fluid is trapped, though, and boundary effects compete against this by promoting vortex annihilation. The boundary influence is weak for vortices with small separation, so a very large nonlinear effect is required to prevent their annihilation. The nonlinearity required for this is less when the vortex separation is larger since they have a lower mutual attraction [10]. If their initial separation is increased sufficiently, though, the boundary effect becomes prominent, driving the vortex pair toward annihilation. To prevent this, the nonlinearity must be increased. It is this competition that explains the parabolic shape of the phase boundary.

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\[ f_{\text{trap}}(x_0) = Ae^{-bx_0} + Ce^{dx_0}. \]  

An optimal fit was found for $A = 997.0, \ b = 7.125, \ C = 0.05667, \text{and} \ d = 10.70$, resulting in the black curve in figure 7. Note that this exponential fit does not based on any physical reasoning but gives a surprisingly reasonable fit to the numerical data.

The symmetry of the phase boundary in figure 7 is because the vortex dipole has an equilibrium fixed point in the Thomas–Fermi regime. So a smaller $x_0$ and a larger $x_0$ will behave approximately in the same way because they correspond to the same trajectory, despite that it might be unstable.

5. Phase diagram

We are now in a position to map out the behavior of a pair of oppositely charged vortices as a function of their initial separation and the strength of fluid nonlinearity. An extensive set of simulations, over a grid of values of $x_0$ and $\beta$, was used to produce the phase diagram shown in figure 7. A parabolic-shaped phase boundary (black solid curve) separates the parameter space into two regions for which original vortices either do or do not annihilate. Green dots correspond to phase boundary points for which no annihilation occurs out to $z = 20$. Their locations do not change when the simulation time is doubled. A grid of neighboring values of $\beta$ was tested to determine that the critical values are accurate to within $\pm 5\%$ of the critical $\beta$.

The cupped-shape phase boundary was then fitted to the sum of an exponential decay and an exponential rise (grey dashed curves) using a nonlinear optimization routine:
primary vortex pair simply exhibits a wobbling orbit about its initial position. This is shown in figures 6(a) and (d). Since it is the global change in field generated by their annihilation that causes nascent surrogate vortices to rush in, the biome does not produce any such candidates and a stable shield wall is maintained.

However, for large initial separation of the primary pair and relatively low nonlinearity, self-annihilation of the primary pair is always followed by its replacement with a surrogate boundary pair that was forming and moving toward the center even as the primary pair was undergoing self-annihilation. Higher values of nonlinearity cause the nascent boundary pair to annihilate with other boundary pairs, and the result is a central trap region swept clear of any vortices at all. Figure 7 shows that the system displays a Bogoliubov dispersion relation in which low-wavenumber features propagate like phonons, and high-wavenumber features propagate like interacting particles. Higher values of $\beta$ have been shown [45] in indium tin oxide using a modestly-powered pulsed laser ($2.5 \times 10^{13}$ W m$^{-2}$, 200 fs) and producing a nonlinearity on the order of $\beta = 25$. It is conceivable that even higher nonlinearities can be achieved in other materials [46] and using higher intensity beams. It is important to reiterate that vortex biome effects should be present for any value of $\beta$ greater than zero.

While the vortex biome and annihilation dynamics described here have not been experimentally demonstrated, work in precision control and readout of nonlinear quantum fluids suggests that such measurements could be achievable in the next few years. Light propagating nonlinearly in a medium is a promising system, but those measurements have so far been done in a bulk medium with no confinement—quite different from the harmonic traps considered in this paper that led to the vortex biome. Laser pre-patterning of the dielectric constant in the material is one possible route toward providing the external trap. Another challenge is taking measurements at different propagation steps through a nonlinear medium [47, 48]. Alternatively, atomic BEC could also host the same vortex dynamics with a straightforward means of trapping, but precision vortex preparation and non-perturbative readout remain significant challenges.

Acknowledgments

The authors acknowledge support from the WM Keck foundation.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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References

[1] Anderson B P 2016 Nature **539** 36

[2] Mawson T, Petersen T C, Slingerland J K and Simula T P 2019 Phys. Rev. Lett. **123** 140404

[3] Kobayashi M, Kawaguchi Y, Nitta M and Ueda M 2009 Phys. Rev. Lett. **103** 115301

[4] Jiang K, Dai X and Wang Z 2019 Phys. Rev. X **9** 011033

[5] Hua C, Halász G B, Dumitrescu E, Brahlek M and Lawrie B 2021 Phys. Rev. B **104** 104510

[6] Posské T, Chiu C-K and Thorwart M 2020 Phys. Rev. Res. **2** 023205

[7] Richman B and Taylor J M 2021 PRX Quantum **2** 030309

[8] Nairnet T R et al 2021 Phys. Rev. B **103** 224526

[9] Zhu C, Siemens M E and Lusk M T 2021 Phys. Rev. A **104** 043306

[10] Andersen J M, Voitiv A A, Siemens M E and Lusk M T 2021 Phys. Rev. A **104** 043306

[11] Andersen J M, Voitiv A A, Siemens M E and Lusk M T 2021 Phys. Rev. Lett. **123** 105301

[12] Ruutu V M, Ruohio J J, Krusius M, Plac¸ais B, Sonin E B and Raman C, Abo-Shaeer J R, Vogels J M, Xu K and Ketterle W 2021 Phys. Rev. Lett. **126** 015006

[13] Neely TW, Samson EC, Bradley AS, Davis MJ and Anderson B P 2014 Phys. Rev. Lett. **112** 043403

[14] Suthar K, Roy A and Angom D 2014 J. Phys. B: At. Mol. Opt. Phys. **47** 135301

[15] Aioi T, Kadokura T, Kishimoto T and Saito H 2011 Phys. Rev. X **1** 021003

[16] Zhang S-C 1993 Phys. Rev. Lett. **67** 1214

[17] Kanai T and Guo W 2021 Phys. Rev. Lett. **127** 095301

[18] Simula T P and Blakie P B 2006 Phys. Rev. Lett. **96** 020404

[19] Seo S W, Ko B, Kim J H and Shin Y 2017 Sci. Rep. **7** 4587

[20] Burlachkov L and Burow S 2021 Phys. Rev. B **103** 024511

[21] Sandella E, Lisboa Filho P N, de Souza Silva C C, Eulálio Cabral L R and Aires Ortiz W 2009 Phys. Rev. B **80** 012506

[22] Pac A R, Carlson J, Wadsworth S and Transtrum M K 2020 Phys. Rev. B **101** 144504

[23] Jeon K-R, Ciccarelli C, Kurebayashi H, Cohen I F, Komori S, Robinson J W A and Bliem and M G 2019 Phys. Rev. B **99** 144503

[24] Gross E P 1955–1965 Il Nuovo Cimento **20** 454

[25] Polkinghorne R E S, Groszek A J and Simula T P 2021 Phys. Rev. A **104** 043602

[26] Gross E P 1955–1965 Il Nuovo Cimento **20** 454

[27] Pitaevskii L P 1961 Sov. Phys. JETP **13** 451

[28] Agrawal G P 2013 Nonlinear Fiber Optics (Amsterdam: Elsevier)

[29] Andersen J M, Voitiv A A, Siemens M E and Lusk M T 2021 Phys. Rev. A **104** 033520

[30] Indebetouw G 1993 J. Mod. Opt. **40** 73

[31] Klein A, Jaksh D, Zhang Y and Bao W 2007 Phys. Rev. A **76** 043602

[32] Lax M, Louisell W H and McKnight W B 1975 Phys. Rev. A **11** 1365

[33] Carusotto I 2014 Proc. R. Soc. A **470** 20140320

[34] Ozawa T et al 2019 Rev. Mod. Phys. **91** 015006

[35] Curtis J E and Grier D G 2001 Phys. Rev. Lett. **90** 133901

[36] Haljan P C, Anderson B P, Coddington I and Cornell E A 2001 Phys. Rev. Lett. **86** 2922

[37] Lin Y J, Jiménez-García K and Spielman I B 2011 Nature **471** 83

[38] Polkinghorne R E S, Groszek A J and Simula T P 2021 Phys. Rev. A **104** 043602

[39] Antoine X and Duboscq R 2015 Comput. Phys. Commun. **193** 2969

[40] Antoine X and Duboscq R 2014 Comput. Phys. Commun. **185** 2969

[41] Barenghi C F and Parker N G 2016 A Primer on Quantum Fluids (Berlin: Springer)

[42] Gorshkov V N, Kononenko A N and Koskin M S 2002 Selected Papers from 5th Int. Conf. Correlation Optics vol 4607 ed O V Angelsky (SPIE, International Society for Optics and Photonics) pp 13–24

[43] Sokolinski M S, Bogatyryova G V and Gorshkov V N 2002 Selected Papers from 5th Int. Conf. Correlation Optics vol 4607 ed O V Angelsky (Chernivtsi, Ukraine: SPIE, International Society for Optics and Photonics) pp 40–68

[44] Fontaine Q, Bienaimé T, Pigeon S, Girod E, Bramati A and Glorieux Q 2018 Phys. Rev. Lett. **121** 183604

[45] Alam M Z, De Leon I D and Boyd R W 2016 Science **352** 795

[46] Polkinghorne R E S, Groszek A J and Simula T P 2021 Phys. Rev. A **104** 043602

[47] Antoine X and Duboscq R 2014 Comput. Phys. Commun. **185** 2969

[48] Antoine X and Duboscq R 2014 Comput. Phys. Commun. **185** 2969

[49] Barenghi C F and Parker N G 2016 A Primer on Quantum Fluids (Berlin: Springer)

[50] Gorshkov V N, Kononenko A N and Koskin M S 2002 Selected Papers from 5th Int. Conf. Correlation Optics vol 4607 ed O V Angelsky (SPIE, International Society for Optics and Photonics) pp 13–24

[51] Fontaine Q, Bienaimé T, Pigeon S, Girod E, Bramati A and Glorieux Q 2018 Phys. Rev. Lett. **121** 183604

[52] Alam M Z, De Leon I D and Boyd R W 2016 Science **352** 795

[53] Boyd R W 2008 Nonlinear Optics 3rd edn (New York: Academic)

[54] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[55] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[56] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[57] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[58] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[59] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[60] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[61] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[62] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[63] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[64] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[65] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[66] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[67] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[68] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484

[69] Vocke D, Roger T, Marino F, Wright C, Maccio F and Faccio D 2015 Optica **2** 484