HILBERT-KUNZ DENSITY FUNCTION FOR GRADED DOMAINS

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Abstract. We prove the existence of HK density function for a pair \((R, I)\), where \(R\) is a \(\mathbb{N}\)-graded domain of finite type over a perfect field and \(I \subset R\) is a graded ideal of finite colength. This generalizes our earlier result where one proves the existence of such a function for a pair \((R, I)\), where, in addition \(R\) is standard graded.

As one of the consequences we show that if \(G\) is a finite group scheme acting linearly on a polynomial ring \(R\) of dimension \(d\) then the HK density function \(f_{R^G, \mathfrak{m}_G}\) of the pair \((R^G, \mathfrak{m}_G)\), is a piecewise polynomial function of degree \(d - 1\).

We also compute the HK density functions for \((R^G, \mathfrak{m}_G)\), where \(G \subset SL_2(k)\) is a finite group acting linearly on the ring \(k[X, Y]\).

1. Introduction

In this paper a pair \((R, I)\) is a graded pair if \(R\) is an \(\mathbb{N}\)-graded domain of dimension \(d \geq 2\) and finite type over a perfect field \(k\) of characteristic \(p > 0\), and \(I\) is a graded ideal of finite colength. The main result here is to prove the existence of the HK density function for such a pair.

The notion of HK density function was introduced in [T] for the purpose of studying the Hilbert-Kunz multiplicity (or HK multiplicity) \(e_{HK}(R, I)\). Recall that the notion of HK multiplicity \(e_{HK}(R, I)\) was introduced by P. Monsky [M] for an arbitrary Noetherian ring \(R\) (in characteristic \(p > 0\)) and an ideal \(I \subset R\) of finite colength. In the same paper he showed that it is positive real number given by

\[
e_{HK}(R, I) = \lim_{n \to \infty} \frac{\ell(R/I[q^n])}{q^n}.
\]

The HK density function behaves well (when it exists) for various operations like tensor products, Segre products etc. Moreover it is a limit of a uniformly converging sequence (which could be suitably renormalized to study a given specific property).

In [T], we proved the existence of HK density function for a standard graded pair \((R, I)\), where by a standard graded pair we mean a graded pair, where \(R\) is a standard graded ring (that is, \(R\) is generated by \(R_1\) as a \(k\)-algebra) in addition.

Theorem 1.1 [T]. Let \((R, I)\) be a standard graded pair. Then for a finitely generated graded module \(M\) over \(R\) there is a sequence \(\{g_n(M_R, I) : [0, \infty) \to [0, \infty]\}_n\) of compactly supported continuous and piecewise linear functions such that

1. the sequence \(\{g_n(M_R, I)\}_n\) is uniformly convergent. Moreover
2. the HK density function \(f_{M_R, I} : [0, \infty) \to [0, \infty)\) defined as \(f_{M_R, I}(x) = \lim_{n \to \infty} g_n(M_R, I)(x)\) is a compactly supported continuous function, and

\[
e_{HK}(M, I) = \int_0^\infty f_{M_R, I}(x)dx.
\]

Here, for a finitely generated graded \(R\)-module \(M\), \(\{g_n(M_R, I) : [0, \infty) \to [0, \infty]\}_n\) denotes the sequence of functions given as follows:
For \( x \geq 0 \), if \( x = (1 - t)\frac{|xq|}{q} + (t)\frac{|xq+1|}{q} \), for some \( t \in [0,1) \) then we define
\[
g_n(M_R, I)(x) = \frac{1}{q^{d-1}} \left( (1 - t)\ell(M/I[|q|]M)_{|xq|} + (t)\ell(M/I[|q|]M)_{|xq+1|} \right).
\]

In this paper we generalize the above result to the case of \textit{graded pair} \((R, I)\), where \( R \) need not be standard graded. (There are many interesting \( \mathbb{N} \)-graded rings which are not standard graded, for example, the ring of invariants and the positive affine semigroup rings, in particular affine toric rings).

To do this we need to generalize the notion of \( g_n(M_R, I) \) (see Definition 2.2) which coincides with the above notion of \( g_n(M_R, I) \) whenever \( \gcd \{n \mid R_n \neq 0\} = 1 \).

More precisely we prove the following

**Theorem 1.1.** (Main Theorem). If \( M \) is a finitely generated graded \( R \)-module, where \((R, I)\) is a graded pair then there is a sequence \( \{g_n(M_R, I) : [0, \infty) \to [0, \infty)\}_n \) of compactly supported continuous and piecewise linear functions such that

1. \( \{g_n(M_R, I)\}_{n \in \mathbb{N}} \) is a uniformly convergent sequence of compactly supported functions.
2. If \( f_{M_R, I} : [0, \infty) \to [0, \infty) \) given by \( x \to \lim_{n \to \infty} g_n(M_R, I)(x) \) then \( f_{M_R, I} \) is a compactly supported continuous function such that
   
   (a) \( f_{M_R, I} = (\text{rank } M) f_{R, I} \) and (b) \( e_{HK}(M, I) = \int_0^\infty f_{M_R, I}(x)dx \).

We recall some key aspects of the proof in the situation of standard graded pair.

If \( R \) is a standard graded domain (which need not be normal) as in [T] with \( I \) generated by homogeneous generators \( f_1, \ldots, f_s \) of degrees \( d_1, \ldots, d_s \) then there exists a very ample invertible sheaf \( \mathcal{O}_X(D) \) on \( X \) (associated to a Cartier divisor \( D \)) such that there is a graded inclusion \( R \to \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)) \) which is an isomorphism in all graded degrees \( m \gg 0 \). This gives us a short exact sequence of \( \mathcal{O}_X \)-modules
\[
0 \to V \to \bigoplus_i \mathcal{O}_X((1 - d_i)D) \xrightarrow{\phi} \mathcal{O}_X(D) \to 0,
\]
where \( \phi(\sum_i a_i) = \sum_i a_i f_i \). Since \( \mathcal{O}_X(D) \) (in fact every \( \mathcal{O}_X(mD) \)) is invertible the sequence
\[
0 \to \bigoplus_i \mathcal{O}_X((q - q_d)D) \xrightarrow{\phi_{q_d}} \mathcal{O}_X(qD) \to 0.
\]

Since \( \mathcal{O}_X(mD) \otimes \mathcal{O}_X(nD) \simeq \mathcal{O}_X((m+n)D) \) tensoring (1.2) by \( \mathcal{O}_X(mD) \) we get the exact sequence
\[
0 \to F^{n*}V \otimes \mathcal{O}_X(mD) \to \bigoplus_i \mathcal{O}_X((m + q - q_d)D) \xrightarrow{\phi_{m,q}} \mathcal{O}_X((m + q)D) \to 0.
\]

Now, for every \( x \geq 0, \lfloor xq \rfloor = m + q \), for some integer \( m \). Hence we may define step functions
\[
f_n(R, I)(x) := f_n(R, I)(\frac{m + q}{q}) = \frac{1}{q^{d-1}}\ell(R/I[|q|]_{m+q})
\]
\[
= \frac{1}{q^{d-1}} [h^0(X, \mathcal{O}_X(m + q)D) - \oplus_i h^0(X, \mathcal{O}_X(m + q - q_d)D) + h^0(X, F^{n*}V \otimes \mathcal{O}_X(mD))].
\]
The sequence $g_n(R,I)$ is obtained from $f_n(R,I)$ in an obvious way.

In particular the computations depend on the cohomologies of the Frobenius pull-backs of the locally free sheaves $V$ and $\mathcal{O}_X(D)$ and their twists (by the line bundles $\mathcal{O}_X(mD)$).

On the other hand if $R = \oplus_{m \geq 0} R_m$ is an arbitrary normal graded domain then by the theorem of Demazure (see Theorem 1.1 below), there is a $\mathbb{Q}$-divisor $D$ such that $R_m = H^0(X, \mathcal{O}_X(mD))$, for all $m$. But $\mathcal{O}_X(D)$ need not be invertible and the multiplication map $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(nD) \to \mathcal{O}_X((m+n)D)$ need not be an isomorphism, in general. In particular the sequence (1.1) need not be locally split exact and $V$ may not be locally free. Hence a version of (1.3) cannot be derived from a single sequence like (1.2), and therefore $\text{Ker} \phi_{m,q}$ does not come from ‘twists of’ a single sheaf (unlike in the standard graded situation, where $\text{Ker} \phi_{m,q} = F^n V \otimes \mathcal{O}_X(mD)$, for all $m$ and $q$).

However $\mathcal{O}_X(mD)$, associated to such a $\mathbb{Q}$-divisor, does have some special properties which we exploit, for example $\mathcal{O}_X(mD)$ is a reflexive sheaf of $\mathcal{O}_X$-modules, hence invertible outside the singular locus of $X$. As a result though one does not have a direct relation between the sequences (1.3) (as $m$ and $q$ vary), we are able to relate their cohomologies by estimates

$$|h^0(X, \text{Ker} \phi_{mp+nq,qp}) - p^{d-1} h^0(X, \text{Ker} \phi_{m,q})| = O(m + q)^{d-2}, \quad 0 \leq n_1 < p.$$ 

In particular the fact (Theorem 3.1) that each $R_m$ is the space of sections of the divisor $mD$ allows us to give a simpler proof (than in [T]) for this more general setting (a graded pair).

However in [T] we prove the existence of the HK density function $f_{M_R,I}$ directly (and without the assumption that $R$ is a domain). Here we prove the Main Theorem when $R$ is a domain, and the proof is in three steps: We prove the theorem when $(M_R,I) = (R,I)$ and where $R$ is a normal domain such that $\text{gcd} \{ m > 0 \mid R_m \neq 0 \} = 1$. This is the main part. Then we extend the result for the pair $(R,I)$, where $R$ is a general graded domain. Then we further extend this to graded modules over such pairs.

We can extend the result to the case, when $R$ may not be a domain, by defining

$$f_{R,I} := \sum_{p \in \Lambda} \Lambda(M_P) f_{R/P,(I+P)/P},$$

where $\Lambda = \{ p \in \text{Spec} R \mid \dim R = \dim R/P \}$. This is clearly an additive function and hence can be extended canonically to the notion of $f_{M_R,I}$. In particular $\int f_{M_R,I}(x) dx = e_{HK}(M_R,I)$ as $e_{HK}(-)$ is an additive function.

However, if $\text{gcd} \{ m \mid R_m \neq 0 \} = n_0$, say, the equality

$$f_{R,I}(x) = \lim_{n \to \infty} 1/q^{d-1} \ell(R/I[q])_{\lfloor xq \rfloor n_0}$$

may not hold any longer unless $\text{gcd} \{ m \mid (R/P)_m \neq 0 \} = n_0$, for all $P \in \Lambda$.

As a consequence of our Main Theorem (Theorem 1.1) we get the following

**Corollary 1.2.** Let $(S,I)$ be a graded pair of dimension $d > 1$. Suppose there is a graded ring $R$ with a degree preserving map $S \subset R$ such that $R$ is $S$-finite, and $\text{proj dim}_R (R/I) < \infty$.

Then the HK density function $f_{S,I}$ is a piecewise polynomial function of degree $d-1$, explicitly given in terms of the graded Betti numbers of the resolution of $IR$.

In particular, if $R = k[X_1, \ldots, X_d]$ is a polynomial ring and $G$ is a finite group (scheme) acting linearly on $R$ then for any graded pair $(R^G,I)$, where $R^G$ is the ring of invariants, the function $f_{R^G,I}$ is a piecewise polynomial of degree $d-1$. 

Then the following three conditions are equivalent:

Lemma 2.4. Let $\text{supp}$

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Definition 2.2. Let $\mu \circ \oplus$

Proof. $g$

Lemma 2.3. Each $g_n(M_R, I)$ is a compactly supported continuous function. Moreover, for a given pair $(M_R, I)$ there is a constant $\tilde{m}$ (independent of $n$) such that

\[
\text{supp } g_n(M_R, I) \subseteq [0, \tilde{m}], \quad \text{for all } n \geq 1.
\]

In particular $\text{supp } f_n(M_R, I) \subseteq [0, \tilde{m}]$, for all $n \geq 1$.

Proof. We choose the integers $s$, $l$, $m_\mu$ and $n_\nu$ as follows: Let $\mu(I) = s$. Let $J = \oplus_{m \geq 0} R_m$ with a set of homogeneous generators $h_1, \ldots, h_\mu$ of degrees, say, $m_1 \leq \cdots \leq m_\mu$, respectively. Let $l$ be an integer such that $J^l \subseteq I$. Let $M$ be generated by homogeneous elements $g_1, \ldots, g_\nu$ of degrees $n_1 \leq \cdots \leq n_\nu$.

Since $R_m = h_1 R_{m-m_1} + \cdots + h_\mu R_{m-m_\mu}$ and $M_m = g_1 R_{m-n_1} + \cdots + g_\nu R_{m-n_\nu}$,

\[
m - n_\nu \geq (m_\mu)lsq \implies M_m \subseteq J^{lsq}M \subseteq I^{lsq}M \subseteq I^{[0]}M.
\]

Hence $(M/I^{[0]}M)_m = 0$, for all $m \geq n_s + (m_\mu)lsq$. \hfill \Box

The following is a well known result.

Lemma 2.4. Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian graded domain such that $R_0$ is a field. Then the following three conditions are equivalent:

1. $\gcd \{ n > 0 \mid R_n \neq 0 \} = n_0$.
2. $n_0 > 0$ is the least integer with the property: there is $m_1 > 0$ such that $R_{mn_0} \neq 0$, for all $m \geq m_1$. 

2. Preliminaries

Notations 2.1. By a graded pair $(R, I)$ we mean that $R = \oplus_{m \geq 0} R_m$ is a Noetherian graded domain of dimension $d \geq 2$, and of finite type over a perfect field $k = R_0$ of characteristic $p > 0$, and $I \subset R$ is a graded ideal such that $\ell(R/I) < \infty$.

Let $(R, I)$ be a graded pair, and let $M$ be a finitely generated graded $R$-module. We extend the definition of $g_n(M_R, I)$ (given in the introduction) as follows.

Definition 2.2. Let $n_0 = \gcd \{ n \mid R_n \neq 0 \}$. Then $f_n(M_R, I) : [0, \infty) \rightarrow [0, \infty)$ is the step function given by

\[
f_n(M_R, I)(x) = \frac{1}{q^{-1}} \left( \ell(M/I^{[0]}M)[xq]_{n_0} + \cdots + \ell(M/I^{[0]}M)[xq]_{n_0+n_0-1} \right).
\]

If $x = (1-t)\frac{xq}{q} + (t)\frac{xq+1}{q}$, for some $t \in [0, 1)$ then the function $g_n(M_R, I) : [0, \infty) \rightarrow [0, \infty)$ is given by

\[
g_n(M_R, I)(x) = (1-t)f_n(M_R, I)(x) + (t)f_n(M_R, I)(x + \frac{1}{q}).
\]

In particular, each $g_n(R, I)$ is continuous, and the uniform convergence of the sequence $\{g_n(M_R, I)\}_n$ is equivalent to the uniform convergence of the sequence $\{f_n(M_R, I)\}_n$.

We also make the following observation that the functions $g_n(M_R, I)$ and $f_n(M_R, I)$ are compactly supported with a bound on the support which is independent of $n$.

Lemma 2.3. Each $g_n(M_R, I)$ is a compactly supported continuous function. Moreover, for a given pair $(M_R, I)$ there is a constant $\tilde{m}$ (independent of $n$) such that $\text{supp } g_n(M_R, I) \subseteq [0, \tilde{m}]$, for all $n \geq 1$.

In particular $\text{supp } f_n(M_R, I) \subseteq [0, \tilde{m}]$, for all $n \geq 1$.

Proof. We choose the integers $s$, $l$, $m_\mu$ and $n_\nu$ as follows: Let $\mu(I) = s$. Let $J = \oplus_{m \geq 0} R_m$ with a set of homogeneous generators $h_1, \ldots, h_\mu$ of degrees, say, $m_1 \leq \cdots \leq m_\mu$, respectively. Let $l$ be an integer such that $J^l \subseteq I$. Let $M$ be generated by homogeneous elements $g_1, \ldots, g_\nu$ of degrees $n_1 \leq \cdots \leq n_\nu$.

Since $R_m = h_1 R_{m-m_1} + \cdots + h_\mu R_{m-m_\mu}$ and $M_m = g_1 R_{m-n_1} + \cdots + g_\nu R_{m-n_\nu}$,

\[
m - n_\nu \geq (m_\mu)lsq \implies M_m \subseteq J^{lsq}M \subseteq I^{lsq}M \subseteq I^{[0]}M.
\]

Hence $(M/I^{[0]}M)_m = 0$, for all $m \geq n_s + (m_\mu)lsq$. \hfill \Box

The following is a well known result.

Lemma 2.4. Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian graded domain such that $R_0$ is a field. Then the following three conditions are equivalent:

1. $\gcd \{ n > 0 \mid R_n \neq 0 \} = n_0$.
2. $n_0 > 0$ is the least integer with the property: there is $m_1 > 0$ such that $R_{mn_0} \neq 0$, for all $m \geq m_1$. 

(3) $n_0 > 0$ is the least integer such that the quotient field of $R$ has an homogeneous element of degree $n_0$.

Proof. Left as an exercise for the reader. \qed

3. The HK density functions for normal graded domains

In this section we prove the existence of the HK density function (in Proposition 3.8) for a graded pair $(R, I)$, where, in addition, $R$ is a normal domain. We will make use of a technical lemma (Lemma 3.7), which we will prove in Section 5.

For such a ring $R$ we will be use the following result of Demazure [D].

**Theorem 3.1.** (Demazure). Let $R = \oplus_{n \geq 0} R_n$ be a normal graded domain of finite type over a field $k$. Suppose there is an homogeneous element $T$ of degree 1 in the quotient field of $R$. Then for $X = \text{Proj } R$, there exists a unique Weil $\mathbb{Q}$-divisor $D$ in $W\text{div}(X, \mathbb{Q})$ such that $R_n = H^0(X, \mathcal{O}_X(nD)).T^n$, for every $n \geq 0$.

We recall some general facts about $\mathbb{Q}$-divisors.

**Notations 3.2.** Let $X$ be a normal projective variety over a perfect field $k$ (in our case $X = \text{Proj } R$, where $R$ is a normal graded domain).

The set $W\text{div}(X)$ is the set of Weil divisors, where a Weil divisor is a formal sum of codimension 1 integral subschemes (prime divisors) of $X$. The set

$$\text{Div}(X, \mathbb{Q}) = W\text{div}(X, \mathbb{Q}) = W\text{div}(X) \otimes_\mathbb{Z} \mathbb{Q},$$

is the set of formal linear combinations of codimension one integral subschemes of $X$ with coefficients in $\mathbb{Q}$ (called $\mathbb{Q}$-divisors). Let $K(X)$ denote the function field of $X$. For $D \in W\text{div}(X, \mathbb{Q})$ the $\mathcal{O}_X$-sheaf $\mathcal{O}_X(D)$ is the sheaf whose space of sections on an open set $U \subset X$ is given by

$$H^0(U, \mathcal{O}_X(D)) = \{f \in K(X) \mid \text{div}(f) \mid_U + D \mid_U \geq 0\},$$

where $\text{div}(f) = \sum_i v_{D_i}(f)D_i$ and $v_{D_i} : K(X) \mapsto \mathbb{Z} \cup \{\infty\}$ is the discrete valuation of $K(X)$ corresponding to the prime divisor $D_i$.

In particular, if $D = \sum_i a_iD_i \in W\text{div}(X, \mathbb{Q})$ is a formal sum of prime divisors $D_i$, where $a_i \in \mathbb{Q}$ then $\mathcal{O}_X(D) = \mathcal{O}_X([D])$, where $[D] = \oplus_i [a_i]D_i$.

For the following basic theory of reflexive sheaves we refer to [H1] (one can also look up the notes by [S] on his homepage).

**Definition 3.3.** A coherent sheaf $\mathcal{F}$ on $X$ is reflexive if the natural map of $\mathcal{O}_X$-modules $\alpha : \mathcal{F} \mapsto (\mathcal{F}^\wedge)^\wedge$ is an isomorphism, where $\mathcal{F}^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. A rank one reflexive sheaf is invertible on the regular locus of $X$. In fact

$$\{\mathcal{O}_X(D) \mid D \in W\text{div}(X)\} = \{\text{rank 1 reflexive subsheaves of } K(X)\}$$

and (even if $R$ is not normal)

$$\{\text{the Cartier divisors of } X\} = \{\text{invertible (hence reflexive) subsheaves of } K(X)\}.$$

Hence if $D$ is a Cartier divisor then $D = \sum a_iD_i$, where $a_i \in \mathbb{Z}$ and hence $D = [D]$.

As we discussed earlier, in case $R$ is standard graded there is a Cartier divisor $D$ such that for $m >> 0$, $R_m = H^0(X, \mathcal{O}_X(mD))$. On the other hand if $R$ is graded normal domain then (by the above theorem of Demazure) there exists a $\mathbb{Q}$-divisor $D$ (which need not be Cartier, but some positive integer multiple of $D$ is a Cartier divisor) such that $R_m = H^0(X, \mathcal{O}_X(mD))$, for all $m \geq 0$. 
We recall (in Lemma 3.4) some relevant properties of $R$ and $\mathcal{O}_X(nD)$ (see [D]).

By Lemma 2.4, the existence of an homogeneous element $T$ of degree 1 in the quotient field of $R$ is equivalent to the condition that $R_m \neq 0$ for all $m >> 0$ which is equivalent to saying that gcd $\{m > 0 \mid R_m \neq 0\} = 1$.

**Lemma 3.4.** For $R$ and $D$ as in Theorem 3.1 let $h_1, \ldots, h_\mu$ denote a set of homogeneous generators of $R$ as an $R_0$-algebra, of degrees $m_1, \ldots, m_\mu$ respectively, and let $l_1 = \text{lcm} (m_1, \ldots, m_\mu)$. Then

(a) for $n \in l_1 \mathbb{N}$, the sheaf $\mathcal{O}_X(nD)$ is a line bundle on $X$. In particular the canonical multiplication map

$$\mathcal{O}_X(nD) \otimes \mathcal{O}_X(iD) \rightarrow \mathcal{O}_X((n + i)D)$$

is an isomorphism, for all $i$.

(b) For $r = l_1 \mu$ the line bundle $\mathcal{O}_X(rD)$ is very ample on $X$.

**Proof.** (a): The variety $X$ has the affine open cover $\{D_+(h_i)\}_i$, where

$$\mathcal{O}_X(nD) \mid_{D_+(h_i)} = \{f/h_i^m \mid \deg(f) - m \deg(h_i) = n, \ f \in R_{\deg(f)}\} = h_i^{n/m_i} \mathcal{O}_X \mid_{D_+(h_i)}\text{.}$$

is generated by the element $h_i^{n/m_i} \in H^0(D_+(h_i), \mathcal{O}_X \mid_{D_+(h_i)})$, for all $i$.

Since $\mathcal{O}_X(nD)$ is a Cartier divisor $\mathcal{O}_X(nD + [iD]) = \mathcal{O}_X([i(n + i)D])$.

(b): If $R^{(r)} := \oplus_{m \geq 0} R_{rm}$ and $R^{(r)}_{rm} := R_{rm}$. Then $R^{(r)}$ is a standard graded ring, as for $m \geq 1$

$$R^{(r)}_m = R_{rm} \subseteq R^{(m-1)}_{li} R_{1i} \mu \subseteq R^{m-1}_{li} R_{1i} \mu = (R^{(r)}_1)^m.$$ 

Since $X = \text{Proj} R^{(r)}$, the sections of $\mathcal{O}_X(rD)$ give a closed immersion of $X$ into $\mathbb{P}^h_k$ where $h = h^0(X, \mathcal{O}_X(rD)) - 1$.

In the rest of the section we have the following notations.

**Notations 3.5.** The pair $(R, I)$ is a fixed graded pair, where $R$ is a normal graded domain and gcd $\{n \mid R_n \neq 0\} = 1$.

We fix a homogeneous element $T$ of degree 1 in the quotient field of $R$.

Let $D \in \text{Div}(X, \mathbb{Q})$ be the divisor, as in Theorem 3.1, so that $R = R(X, D) = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)). T^n$.

We fix $r \in \mathbb{N}$, so that $\mathcal{O}_X(rD)$ is a very ample divisor on $X$ and $R_m \neq 0$, for all $m \geq r$.

For $I$, we fix a set of homogeneous generators $f_1, \ldots, f_s$ of degrees $d_1, \ldots, d_s$ respectively.

For the sake of abbreviation we adopt the following Notations.

Let $\mathcal{O}_n = \mathcal{O}_X(nD)$.

Let $\mathcal{L} = \mathcal{O}_r$ be the very ample line bundle on $X$.

Let $m_{q,d} = m + q - q d_i$ where $q = p^n$, for some $n \geq 1$.

Let $(m_{q,d}) = \lceil \frac{m+q}{r} \rceil - q d_i$.

Since gcd $\{m > 0 \mid R_m \neq 0\} = 1$, the definition of the sequences $\{f_n(R, I)\}_n$ and $\{g_n(R, I)\}_n$ is same as in the case of standard graded pair (given in [T]).

**Definition 3.6.** For the pair $(R, I)$, the function $\{f_n(R, I) : [0, \infty) \rightarrow [0, \infty)\}_n \in \mathbb{N}$ is given by

$$f_n(R, I)(x) = 1/q^{d-1} \ell (R/I^{[q]})_{xq} \text{, where } q = p^n$$

where $\ell (R/I^{[q]})_{xq}$
and, for $x = (1 - t)m/q + t(m + 1)/q$, where $t \in [0, 1)$,
\[ g_n(R, I)(x) = (1 - t)f_n(R, I)(m/q) + t)f_n(R, I)(m + 1/q). \]

For given $m \in \mathbb{N}$ and $q = p^n$, we consider the following short exact sequence of $\mathcal{O}_X$-modules

\[(3.1) \quad 0 \to \mathcal{F}_{m,q} \to \bigoplus_{i=1}^s \mathcal{O}_{m_q,d_i} \xrightarrow{\varphi_{m,q}} \mathcal{O}_{m+q} \to 0, \]

where $\varphi_{m,q}(a_1, \ldots, a_s) = \sum_i a_i f_i^q$.

Then
\[ f_n(R, I)(m/q) = \frac{1}{q^i} \ell(R/I[q])_{m+q} = \frac{1}{q^i} \left[ \ell(R_{m+q}) - \sum_{i=1}^s \ell(f_i^q R_{m+q-d_i}) \right] \]
\[ = \frac{1}{q^i} \left[ h^0(X, \mathcal{O}_{m+q}) - \oplus_i h^0(X, \mathcal{O}_{m_q,d_i}) + h^0(X, \mathcal{F}_{m,q}) \right]. \]

To compare $f_n$ and $f_{n+1}$, we use the following crucial technical result, which will be proved below in Section 5.

**Lemma 3.7.** (Main Lemma). For $\mathcal{F}_{m,q}$ as in (3.1), there exists a constant $C$ such that, for all $m \geq 0$ and $q = p^n$ and $0 \leq n_1 < p$,
\[ |h^0(X, \mathcal{F}_{m+p+1, qp}) - p^{d-1}h^0(X, \mathcal{F}_{m,q})| \leq C(mp + qp)^{d-2}, \]
and
\[ |h^0(X, \mathcal{O}_{m+p+1}) - p^{d-1}h^0(X, \mathcal{O}_{m_q})| \leq C(mp + qp)^{d-2}, \]
where $m_q = m_q, m = (m_q)_{d_i}$, for $1 \leq i \leq s$, or $m_q = m + q$.

The following proposition proves the existence of the HK density function for normal graded domains.

**Proposition 3.8.** If $R = \bigoplus_{n \geq 0} R_n$ is a normal graded domain and $\gcd \{n \mid R_n \neq 0\} = 1$ then for a graded pair $(R, I)$ the sequence $\{f_n(R, I)\}_{n}$ is uniformly convergent.

**Proof.** For brevity, in the rest of the proof, we denote $f_n(R, I)$ by $f_n$.

Let $x \geq 1$. For $q = p^n$ and $m + q \leq x \leq m + q + 1$,
\[ f_n(x) = 1/q^{d-1} \ell(R/I[q])_{m+q} = 1/q^{d-1} \ell(R/I[q])_{m+q}. \]

Therefore there is $n_1$ such that $0 \leq n_1 < p$ and
\[ f_{n+1}(x) = 1/(q^p)^{d-1} \ell(R/I[q^p])_{mp+qp+n_1}. \]

For $\mathcal{O}_m = \mathcal{O}_X(mD)$ consider the short exact sequences of $\mathcal{O}_X$-modules (as in (3.1))
\[ 0 \to \mathcal{F}_{m,q} \to \bigoplus_{i=1}^s \mathcal{O}_{m+q-d_i} \xrightarrow{\varphi_{m,q}} \mathcal{O}_{m+q} \to 0, \]
\[ 0 \to \mathcal{F}_{mp+1, qp} \to \bigoplus_{i=1}^s \mathcal{O}_{mp+qp+n_1-d_i} \xrightarrow{\varphi_{mp+1, qp}} \mathcal{O}_{mp+qp+n_1} \to 0. \]

Therefore
\[ f_n(x) = \frac{p^{d-1}[h^0(X, \mathcal{O}_{m+q}) - \sum_i h^0(X, \mathcal{O}_{m+q-qd_i}) + h^0(X, \mathcal{F}_{m,q})]}{(qp)^{d-1}}, \]
\[ f_{n+1}(x) = \frac{[h^0(X, \mathcal{O}_{mp+1, qp}) - \sum_i h^0(X, \mathcal{O}_{mp+1, qp-n_1-d_i}) + h^0(X, \mathcal{F}_{mp+1, qp})]}{(qp)^{d-1}}. \]
By the Main Lemma 3.7 there is a constant $C_0 > 0$ such that

$$|f_n(x) - f_{n+1}(x)| \leq C_0(mp + qp)^{d-2}/(qp)^{d-1}.$$

Since $\text{supp } f_n \subseteq [0, \tilde{m}]$, where $\tilde{m}$ is as in Lemma 2.3 we can assume $(m + q)/q \leq \tilde{m}$ and hence there is a constant $C_1$ such that

$$|f_n(x) - f_{n+1}(x)| \leq C_1/qp, \text{ for all } x \geq 1.$$

We can further choose $C_1$ such that the above inequality also holds for all $0 \leq x \leq 1$.

Because if $0 \leq x < 1$ then

$$f_n(x) = \ell(R)_{[q]}/q^{d-1} = P([xq])/(q^{d-1}) \text{ for all } q = p^n > 0,$$

where $P(x) \in \mathbb{Q}[X]$ is the Hilbert polynomial of $R$ hence of degree $d - 1$. This proves the proposition. \hfill $\square$

4. The Main Theorem

In this section we will prove that the Main Theorem holds for the general graded pairs $(R, I)$. (Here we still assume the Main Lemma 3.7 which will be proved in the next section).

Throughout this section $(R, I)$ is a graded pair and $\text{gcd } m | R_m \neq 0 = n_0$.

For a finitely generated graded $R$-module, the functions $f_n(M_R, I)$ and $g_n(M_R, I)$ are as in Definition 2.2.

**Remark 4.1.** If $S \rightarrow R$ is a degree preserving module-finite map of graded domains, where $(S, I)$ is a graded pair and $n_0 = \text{gcd } n | R_n \neq 0$ and $m_0 = \text{gcd } n | S_n \neq 0$, then the HK density function of $(R, IR)$ as a module over itself is $(q = p^n)$

$$f_{R, IR}(x) := \lim_{n \rightarrow \infty} f_n(R, IR)(x) = \frac{1}{q^{d-1}} \left( \ell(R/I[q] R)_{[xq]n_0} \right).$$

Whereas the HK density function of $(R, IR)$ as a module over $S$ is

$$f_{R, S, I}(x) := \lim_{n \rightarrow \infty} f_n(R_S, I)(x) = \frac{1}{q^{d-1}} \left( \ell(R/I[q] R)_{[xq]m_0} + \cdots + \ell(R/I[q] R)_{[xq]m_0+m_0-1} \right).$$

The existence of both the limits is shown in the following Theorem 4.1.

We use the following lemma to reduce the problem of convergence of $f_n(M_R, I)$ to the problem of convergence of $\{f_n(S, IS)\}$, where $S$ is the normalization of $R$ in $Q(R)$.

**Lemma 4.2.** If $\text{gcd } m > 0 | R_m \neq 0 = 1$ and $N$, $N'$ are finitely generated graded $R$-modules with the exact sequence of graded $R$-linear maps

$$0 \rightarrow N \overset{\phi}{\rightarrow} N' \rightarrow Q'' \rightarrow 0$$

such that the supp dim $Q'' < d$ and the map $\phi$ is of degree 0.

Then the sequence $\{f_n(N, R, I)\}_n$ is uniformly convergent if and only if $\{f_n(N'_R, I)\}_n$ is so. Moreover in that case

$$\lim_{n \rightarrow \infty} f_n(N, R, I) = \lim_{n \rightarrow \infty} f_n(N'_R, I).$$

**Proof.** Here if $M$ is a graded $R$-module then the function $f_n(M, IR) : [0, \infty) \rightarrow [0, \infty)$ is given by $x \rightarrow \ell(M/I[q] M)_{[xq]}/q^{d-1}$, where $q = p^n$.

Let $f$ have homogeneous generators $f_1, \ldots, f_s$ of degree $d_1, \ldots, d_s$ respectively. Then for any graded $R$-module $M$, we define

$$\Phi_M : \oplus_{i}^s M(-qd_i) \rightarrow M \text{ given by } (m_1, \ldots, m_s) \rightarrow \sum_i f_i^q m_i$$
This proves the lemma.

Now the snake lemma applied to (4.1) gives the following exact sequence of graded $R$-modules

$$\rightarrow \text{Ker } \Phi_{Q''} \rightarrow \text{Coker } \Phi_N \rightarrow \text{Coker } \Phi_{N'} \rightarrow \text{Coker } \Phi_{Q''} \rightarrow 0,$$

where $f_n(N'_R, I)(\frac{m+q}{q}) = \ell(\text{Coker } \Phi_{N'})_{m+q}$ and $f_n(N_R, I)(\frac{m+q}{q}) = \ell(\text{Coker } \Phi_N)_{m+q}$.

Let $C_{Q''}$ be constant such that, for all $m > 0$, $\ell(Q''_m) \leq C_{Q''}m^{d-2}$ (such a constant exists by the hypothesis on support dimensions).

$$|\ell(\text{Coker } \Phi_{N'})_{m+q} - \ell(\text{Coker } \Phi_N)_{m+q}| \leq 2C_{Q''}(m + q)^{d-2}.$$

Now, for $x \geq 1$ we have $m + q \leq xq < m + q + 1$ for some $m \geq 0$ and so we have

$$|f_n(N_R, I)(x) - f_n(N'_R, I)(x)| \leq 2C_{Q''}x_0^{d-2}/q,$$

where by Lemma 2.3, we may fix an $x_0$ such that $\text{supp } f_n(N_R, I)$ and $\text{supp } f_n(N'_R, I)$ are subsets of $[0, x_0]$, for all $n \geq 1$.

If $0 \leq x < 1$ then $m \leq xq < m + 1$, for some $m < q$. It is easy to check that in this case

$$|f_n(N_R, I)(x) - f_n(N'_R, I)(x)| = 2C_{Q''}m^{d-2}/q^{d-1} \leq 2C_{Q''}/q.$$

This proves the lemma. \qed

Now we are ready to prove the Main Theorem.

**Proof of the Main Theorem** [17]. Let $\gcd \{n \mid R_n \neq 0\} = n_0$. Let $S = R^{(n_0)}$, where the $n$th degree component of $R^{(n_0)}$ is $R_{n n_0}$. Then $S$ is a graded domain, where $\gcd \{n \mid S_n \neq 0\} = 1$. (Note that $S = R$ as rings, but the grading is changed.)

Note that assertion (2) (b) follows from assertion (1).

It is easy to prove that the uniform convergence of $\{g_n(M_R, I)\}_n$ is equivalent to the uniform convergence of $\{f_n(M_R, I)\}_n$.

We first prove the theorem for $M = R$, where it is sufficient to prove the uniform convergence of $\{f_n(R, I)\}_n$. By definition, $f_n(R, I) = f_n(S, I)$, for all $n$.

Let $\tilde{S} = \oplus_n \tilde{S}_n$ denote the normalization of $S$ in its quotient field then the inclusion map $S \rightarrow \tilde{S}$ is a module finite graded map of degree 0, and we have the short exact sequence of graded $S$-modules

$$0 \rightarrow S \rightarrow \tilde{S} \rightarrow Q'' \rightarrow 0,$$

where support dim $Q'' \leq d - 1$.

By Proposition 3.8, the sequence $\{f_n(\tilde{S}, IS)\}$ is uniformly convergent. But $f_n(\tilde{S}_S, I) = f_n(S, IS)$. Hence the uniform convergence of $\{f_n(R, I)\}_n$ follows by Lemma 4.2.

We now consider the general case of a finite graded module $M$.

Let $M = \oplus_n M_n$, where $M_n = M_{n n_0} + \cdots + M_{n n_0 + n_0 - 1}$ denotes the degree $n$ component of $M$. If $M$ is generated by homogeneous elements $g_1, \ldots, g_\nu$ as an $R$-module then $\overline{M}$ is generated by $g_1, \ldots, g_\nu$ as an $S$-module. Hence $\overline{M}$ is a finitely generated graded $S$-module. Also, for all $n \geq 1$, $f_n(M_R, I) = f_n(M_S, IS)$ and $\text{rank}_R M = \text{rank}_S \overline{M}$.
Claim. There exists an exact sequence of graded $S$-modules

$$0 \rightarrow \bigoplus_{n=1}^n S(-a) \xrightarrow{\phi} \overline{M} \rightarrow Q'' \rightarrow 0,$$

where $\phi$ is a graded map of degree 0 and $\dim(Q'') \leq d - 1$.

Proof of the claim: For the multiplicatively closed set $T = S \setminus \{0\}$, the $T^{-1}S$-module $T^{-1}\overline{M}$ is free of finite rank, say $n_1$ and is generated by a finite set of homogeneous elements. Hence we can choose homogeneous elements $m_1, \ldots, m_{n_1}$ in $\overline{M}$ of degrees $d_1, \ldots, d_{n_1}$ respectively such that the $m_i$'s give a basis for $T^{-1}\overline{M}$.

Since $\gcd\{n \mid S_n \neq 0\} = 1$, we have $m_0$ such that $S_{m_0} \neq 0$, for all $m \geq m_0$. Let $a > 0$ such that $a \geq \max\{m_0 + d_i, m_0\}$ and let $s_i \in S_{a - d_i} \setminus \{0\}$. Then $s_1 m_1, \ldots, s_{n_1} m_{n_1} \in \overline{M}$ are homogeneous elements (each of degree $a$) and generate $T^{-1}\overline{M}$ as $T^{-1}S$-module. Hence we have a generically isomorphic map $\bigoplus_{n=1}^n S(-a) \rightarrow \overline{M}$ of graded $S$-modules of degree 0. The map is injective as $S$ is a domain. This proves the claim.

Now the theorem (1) and (2) (a) follows from Lemma 4.2.

Note that assertion (2) (b) follows from assertion (1).

As we remarked earlier (Remark 4.1), for a finite map $S \rightarrow R$ as in Notations 4.1 the two HK density functions, for the pair $(R, IR)$, namely $f_{R,IR}$ and $f_{RS, I}$ need not be the same functions but can be recovered from each other as follows.

Lemma 4.3. Let $S \rightarrow R$ be the module-finite map as in Notations 4.1. Let $m_0 = \gcd\{n > 0 \mid S_n \neq 0\}$ and $n_0 = \gcd\{n > 0 \mid R_n \neq 0\}$ then

$$(l_0) f_{R,IR}(x_{l_0}) = f_{RS,I}(x) = (\text{rank}_SR) f_{S,I}(x), \quad \text{for all} \quad x \in R_{\geq 0},$$

where $l_0 = m_0/n_0$ is an integer. Hence $f_{R,IR} \equiv f_{RS,I}$ if $m_0 = n_0$.

Proof. We first prove that $n_0$ divides $m_0$. Otherwise $m_0 = n_0 l_0 + n_1$, where $0 < n_1 < n_0$. Now if $x \in Q(S)$ is an homogeneous element of degree $m_0$ and $y \in Q(R)$ is an homogeneous element of degree $n_0$ then $(x)(y^{-l_0})$ is an homogeneous element of degree $n_1$ in $Q(R)$. By Lemma 2.3, this contradicts the hypothesis that $\gcd\{n > 0 \mid R_n \neq 0\} = n_0$.

Now, replacing $R$ by $R^{(n_0)}$ and $S$ by $S^{(n_0)}$ we can assume $n_0 = 1$ and $m_0 = l_0$.

By definition

$$f_{RS,I}(x) = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \left( \ell(R/I[q]R)_{[xq]l_0} + \cdots + \ell(R/I[q]R)_{[xq]l_0 + l_0 - 1} \right),$$

and

$$f_{R,IR}(x_{l_0}) = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \left( \ell(R/I[q]R)_{[xql_0]} \right).$$

For all $x \geq 0$ and $q = p^n$, we have $|[xql_0] - [xq]l_0| \leq l_0$. Let $m_1$ be such that $R_{m_1} \neq 0$, for $m \geq m_1$. Then for each $0 \leq l_i \leq 2l_0$, we have generically isomorphic graded maps $R(-l_i) \rightarrow R(m_1)$ and $R \rightarrow R(m_1)$ of degree 0. Now by Lemma 4.2 there is a constant $C_{l_0}$ such that

$$\frac{1}{q^{d-1}} \ell(R/I[q])_{[xql_0]} - \ell(R/I[q])_{[xql_0] + l_i} \leq C_{l_0}, \quad \text{for all} \quad x, \, q \text{ and } 0 \leq l_i \leq 2l_0$$

which implies

$$f_{R,I}(x_{l_0}) = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \ell(R/I[q])_{[xql_0]} = \lim_{n \rightarrow \infty} \frac{1}{q^{d-1}} \ell(R/I[q])_{[xql_0] + l_i}.$$
5. Proof of the Main Lemma

Here we prove that the Main technical Lemma 3.7 which will complete the proof of the Main Theorem.

Throughout this section we follow the Notations 3.5.

As we mentioned earlier, the sequence (3.1) need not be locally split exact as the sheaf $\mathcal{O}_{m+q}$ is not invertible in general. Hence we consider the following locally split (as $\mathcal{O}_{m+q/r} \simeq \mathcal{L}^{[m+q/r]}$ is invertible) exact sequence of $\mathcal{O}_X$-modules

\[
0 \to \mathcal{G}_{m,q} \to \bigoplus_{i=1}^{s} \mathcal{O}_{(m,q,d_i)} \overset{\varphi_{m,q}}{\to} \mathcal{L}^{[m+q/r]} \to 0,
\]

where $\varphi_{m,q}(a_1, \ldots, a_s) = \sum_i a_i f_i^{q/r}$ and where $\mathcal{G}_{m,q} = \mathcal{F}_{[m+q/r]r-q,g}$ (note however that $\mathcal{G}_{m,q}$ may not be locally free). In case $m+q$ is divisible by $r$, the sequence (5.1) is same as the sequence

\[
0 \to \mathcal{F}_{m,q} \to \bigoplus_{i=1}^{s} \mathcal{O}_{m,q,d_i} \overset{\varphi_{m,q}}{\to} \mathcal{O}_{m+q} \to 0,
\]

as in (3.1).

In Lemma 5.3 we show that the length of the cohomology of $\mathcal{F}_{m,q}$ (or of $\mathcal{O}_{m+q}$) differs from the length of the cohomology of $\mathcal{G}_{m,q}$ ($\mathcal{L}^{[m+q/r]}$ respectively) by a function of order $O(m+q)^{d-2}$, where $\dim X = d-1$. Hence it will be sufficient to prove the Main Lemma 5.7 for $\mathcal{G}_{m,q}$ instead of $\mathcal{F}_{m,q}$.

In the rest of the section we use the following

**Terminology**

We fix a pair $(R, I)$, the line bundle $\mathcal{L}$, the integer $r$ along with a choice of generators $f_1, \ldots, f_s$ of $I$ as in Notations 3.5.

Given $L$, where $L$ might be a number, a set, a map or a coherent sheaf, $C_L$ denotes a constant which depends only on $L$ (with the fixed data $(R, I)$, $\mathcal{L}$ etc. as above). By $\supp \dim \mathcal{F}$, we mean the dimension of the support of $\mathcal{F}$.

Here supp dim $\mathcal{O}_n = \supp \dim X = d - 1 \geq 1$.

**Lemma 5.1.** For a given coherent sheaf $\mathcal{N}$ of $\mathcal{O}_X$-modules with $\supp \dim \mathcal{N} < d - 1$, there is a constant $C_N$ such that

1. $h^j(X, \mathcal{N} \otimes \mathcal{L}^m) \leq C_N(|m|)^{d-2}$, for every $j \geq 0$ and $m \in \mathbb{Z}$.
2. $h^j(X, \mathcal{O}_m \otimes \mathcal{N}) \leq C_N(|m|)^{d-2}$, for all $j \geq 0$ and for all $m \in \mathbb{Z}$.

In particular, for $m \geq 0$ and $0 \leq j \leq d - 1$,

3. there exists $C$ such that $h^j(X, \mathcal{O}_{m,q} \otimes \mathcal{N}) \leq C_N(m+q)^{d-2}$, where, for $1 \leq i \leq s$, $m_q = m_{q,d_i}$, $m_q = (m_{q,d_i})$ or $m_q = m + q$ (as in Notations 3.5), and

4. $h^j(X, \mathcal{G}_{m,q} \otimes \mathcal{N}) \leq C_N(m+q)^{d-2}$, for all $m, j \geq 0$ and $q$.

**Proof.** (1) By the Serre vanishing theorem ([HI]) $h^j(X, \mathcal{N} \otimes \mathcal{L}^m) = 0$, for $j > 0$ and $m > 0$. Also, for $m >> 0$, $h^0(X, \mathcal{N} \otimes \mathcal{L}^m)$ is a polynomial of degree equal to $\dim \mathcal{N} < d - 1$. Hence the assertion (1) follows by induction on $\dim \mathcal{N}$ and the Serre’s duality ([HI]).

(2) By Lemma 3.4 we have $\mathcal{O}_m = \mathcal{L}^{[m/r]} \otimes \mathcal{O}_{r_1}$, where $r_1 = m - [m/r]r < r$. Since the support of $\mathcal{O}_m \otimes \mathcal{N} = \text{the support of } \mathcal{N}$, the assertion (2) follow from the fact that $\mathcal{O}_{m,q} \otimes \mathcal{N}$ belongs to the finite set $\{\mathcal{O}_0 \otimes \mathcal{N}, \mathcal{O}_1 \otimes \mathcal{N}, \ldots, \mathcal{O}_{r-1} \otimes \mathcal{N}\}$ of coherent sheaves of $\mathcal{O}_X$-modules.

The assertion (3) follows from (2) as $|m+q-d_iq| \leq d_i(m+q)$.

Since the sequence (5.1) is locally split exact, the induced sequence

\[
0 \to \mathcal{G}_{m,q} \otimes \mathcal{N} \to \bigoplus_{i=1}^{s} \mathcal{O}(m,q,d_i) \otimes \mathcal{N} \overset{\varphi_{m,q} \otimes \mathcal{N}}{\to} \mathcal{L}^{[m+q/r]} \otimes \mathcal{N} \to 0
\]
is exact. Now the assertion (4) follows from (2) and (3).

Lemma 5.2. (1) Let \( S = \{ \psi_j : \mathcal{E}_j \rightarrow \mathcal{F}_j \mid 1 \leq j \leq s \} \) be a finite set of \( \mathcal{O}_X \)-linear maps, where \( \mathcal{E}_j \) and \( \mathcal{F}_j \) are coherent sheaves of \( \mathcal{O}_X \)-modules. For \( m \in \mathbb{Z} \), let

\[
\psi_j(m) := \text{Id}_{\mathcal{L}^m} \otimes \psi_j : \mathcal{L}^m \otimes \mathcal{E}_j \rightarrow \mathcal{L}^m \otimes \mathcal{F}_j
\]

be the canonically induced maps. Assume that \( \text{supp dim} (\ker \psi_j) \) and \( \text{supp dim} (\text{coker} \ psi_j) \) are each \( < d - 1 \). Then there exists a constant \( C_S \) such that

\[
h^i(X, \ker \psi_j(m)) \leq C_S m^{d-2} \quad \text{and} \quad h^i(X, \text{coker} \ psi_j(m)) \leq C_S m^{d-2}, \quad \text{for all} \quad i \geq 0.
\]

(2) Moreover if \( \{ 0 \rightarrow \mathcal{N}_m' \rightarrow \mathcal{M}_m' \xrightarrow{\phi_m} \mathcal{M}_m \rightarrow \mathcal{N}_m \rightarrow 0 \}_{m \in \mathbb{Z}} \) denote a family of exact sequences of \( \mathcal{O}_X \)-modules and \( C_1 \) and \( C_2 \) are constants such that

\[
h^i(X, \mathcal{N}_m') \leq C_1 (n_m)^{d-2} \quad \text{and} \quad h^i(X, \mathcal{N}_m) \leq C_2 (n_m)^{d-2}, \quad \text{for all} \quad i \geq 0,
\]

then

\[
|h^0(X, \mathcal{M}_m') - h^0(X, \mathcal{M}_m)| \leq (C_1 + C_2) (n_m)^{d-2}.
\]

Proof. We note that, for any \( m \in \mathbb{Z} \),

\[
\ker \psi_j(m) \simeq \mathcal{L}^m \otimes \ker \psi_j \quad \text{and} \quad \text{coker} \ psi_j(m) \simeq \mathcal{L}^m \otimes \text{coker} \ psi_j,
\]

where \( \ker \psi_j \) and \( \text{coker} \ psi_j \) are in a fixed family of finite number of coherent sheaves of \( \mathcal{O}_X \)-modules. Hence the first assertion follows by Lemma 5.1.

The second assertion follows by splitting the exact sequence into two canonical two short exact sequences

\[
0 \rightarrow \mathcal{N}_m' \rightarrow \mathcal{M}_m' \rightarrow \text{Im}(\phi_m) \rightarrow 0,
\]
\[
0 \rightarrow \text{Im}(\phi_m) \rightarrow \mathcal{M}_m \rightarrow \mathcal{N}_m \rightarrow 0.
\]

\( \square \)

Lemma 5.3. For all \( m \) and \( q = p^n \),

(1) there is a constant \( C \) such that

\[
|h^0(X, \mathcal{G}_{m,q}) - h^0(X, \mathcal{F}_{m,q})| \leq C (m + q)^{d-2}.
\]

(2) For given integer \( l_0 \), there exists a constant \( C_{l_0} \) such that for every \( 0 \leq l \leq l_0 \),

\[
|h^0(X, \mathcal{G}_{m,q}) - h^0(X, \mathcal{G}_{m+l,q})| \leq C_{l_0} (m + q)^{d-2},
\]

\[
|h^0(X, \mathcal{O}_{m,q}) - h^0(X, \mathcal{O}_{m+q,l})| \leq C_{l_0} (m + q)^{d-2},
\]

where, for \( 1 \leq j \leq s \), \( m_q = m_{q,d_j} \) or \( m_q = (m_{q,d_j}) \), or \( m_q = m + q \).

Proof. Claim (A). For a given \( \bar{r} \in \mathbb{Z} \), if \( H^0(X, \mathcal{O}_{\bar{r}}) \neq \{ 0 \} \) then there exists a constant \( C_{\bar{r}} \) such that, for \( i \geq 0 \) and \( m \geq 0 \)

\[
|h^0(X, \mathcal{F}_{m,q}) - h^0(X, \mathcal{F}_{m+\bar{r},q})| \leq C_{\bar{r}} (m + q)^{d-2}.
\]

Proof of the claim: An element \( h \in H^0(X, \mathcal{O}_{\bar{r}}) \setminus \{ 0 \} \) gives an injective map \( \Phi_h : \mathcal{O}_m \rightarrow \mathcal{O}_{m+\bar{r}} \), for all \( m \). In particular we have the following canonical diagram of sheaves of \( \mathcal{O}_X \)-modules:
1. If \( (1) \), it follows from the proof of Assertion (2).

Assertion \( \varphi \) and the map \( \text{Assertion (2)} \) of the lemma.

Case \( C \) there exists \( x \) such that \( O \) for \( i \) such that \( m \) and \( l \) such that \( i \) is the multiplication map by \( h \).

Similarly we prove the lemma for \( G \).

The map \( \oplus_{i} \Phi_{h} : \oplus_{i} O_{m+q-qd_{i}} \longrightarrow \oplus_{i} O_{m+r+q-qd_{i}} \) is same as

\[
\oplus_{i}(\text{Id}_{\mathcal{L}} \otimes \phi_{i}) : \oplus_{i}(\mathcal{L}^{[m_{q,d_{i}}/r]} \otimes \mathcal{E}_{i}) \longrightarrow \oplus_{i}(\mathcal{L}^{[m_{q,d_{i}}/r]} \otimes \mathcal{F}_{i})
\]

and where

\[
\mathcal{E}_{i} = \mathcal{O}_{m+q-qd_{i}-[m_{q,d_{i}}/r]r} \quad \text{and} \quad \mathcal{F}_{i} = \mathcal{O}_{m+r+q-qd_{i}-[m_{q,d_{i}}/r]r}
\]

and the map \( \phi_{i} : \mathcal{E}_{i} \longrightarrow \mathcal{F}_{i} \) is the multiplication map by \( h \).

Also the map \( \Phi_{h} : \mathcal{O}_{m+q} \longrightarrow \mathcal{O}_{m+r+q} \) is

\[
\text{Id}_{\mathcal{L}} \otimes \phi_{0} : \mathcal{L}^{[m+q/r]} \otimes \mathcal{E}_{0} \longrightarrow \mathcal{L}^{[m+q/r]} \otimes \mathcal{F}_{0},
\]

where

\[
\mathcal{E}_{0} = \mathcal{O}_{m+q-[m+q/r]r} \quad \text{and} \quad \mathcal{F}_{0} = \mathcal{O}_{m+r+q-[m+q/r]r}
\]

and the map \( \phi_{0} : \mathcal{E}_{0} \longrightarrow \mathcal{F}_{0} \) is given by the multiplication by \( h \). Note that \( \mathcal{E}_{i} \in \{\mathcal{O}_{0}, \ldots, \mathcal{O}_{r-1}\} \) and \( \mathcal{F}_{i} \in \{\mathcal{O}_{r}, \ldots, \mathcal{O}_{r+r-1}\} \) and supp dim (coker \( \phi_{i} \)) < \( d - 1 \). Hence the claim follows by Lemmas [5.1] and [5.2] and the short exact sequence

\[
0 \longrightarrow \text{coker} \Phi_{h} \longrightarrow \text{coker} (\oplus_{i} \Phi_{h}) \longrightarrow \text{coker} \Phi_{h} \longrightarrow 0.
\]

Assertion (2). It is enough to prove the Assertion (2) for \( \mathcal{F}_{m,q} \) instead of \( \mathcal{G}_{m,q} \). Since there exists \( x_{1} \in \text{H}^{0}(X, \mathcal{O}_{2r}) \setminus \{0\} \), the above claim implies that we have a constant \( C_{2r} \) such that

\[
|h^{0}(X, \mathcal{F}_{m,q}) - h^{0}(X, \mathcal{F}_{m+2r,q})| \leq C_{2r}(m + q)^{d-2}, \quad \text{for} \quad i \geq 0.
\]

Case 1. If \( l \leq r \) then there exists \( x_{2} \in \text{H}^{0}(X, \mathcal{O}_{2r-l}) \setminus \{0\} \), and therefore we have a constant \( C_{2r-l} \) such that, for \( i \geq 0 \),

\[
|h^{0}(X, \mathcal{F}_{m+l,q}) - h^{0}(X, \mathcal{F}_{m+2r,q})| \leq C_{2r-l}(m + q)^{d-2}.
\]

Case 2. If \( l \geq r \) then we can choose \( x_{3} \in \text{H}^{0}(X, \mathcal{O}_{l}) \setminus \{0\} \) and therefore get a constant \( C_{l} \) such that

\[
|h^{0}(X, \mathcal{F}_{m,q}) - h^{0}(X, \mathcal{F}_{m+l,q})| \leq C_{l}(m + q)^{d-2},
\]

for \( i \geq 0 \). Since, for given \( 0 \leq l \leq l_{0} \), there are finitely many choices of such \( C_{l} \), we get Assertion (2) of the lemma.

Similarly we prove the lemma for \( \mathcal{O}_{m,q} \).

Assertion (1). It follows from the proof of Assertion (2). \( \square \)

5.1. The Main Lemma for \( \mathcal{G}_{m,q} \). Here we compare \( h^{0}(X, \mathcal{G}_{mp,qp}) \) (or \( h^{0}(X, (\mathcal{O}_{mp,qp})) \) with \( h^{0}(X, \oplus^{p_{d-1}} \mathcal{G}_{m,q}) \) (or \( h^{0}(X, \oplus^{p_{d-1}} (\mathcal{O}_{m,q})) \) respectively) in Lemma [5.4] and in Lemma [5.5]. Since the sequence [5.1] is locally split exact, it remains exact for the functor \((-) \otimes \mathcal{M}) \), for any sheaf of \( \mathcal{O}_{X} \)-modules \( \mathcal{M} \). In particular we construct below a generically isomorphic map \( F^{*} \mathcal{G}_{m,q} \longrightarrow \mathcal{G}_{m',q'p} \), provided \( |mp - m'| \) bounded by constant for all \( m \) and \( m' \).
Lemma 5.4. There is a constant $C_0$ such that
\[ |h^0(X,(F^*G_{m,q})) - h^0(X,G_{mp,qp})| \leq C_0(mp + qp)^{d-2}, \]
\[ |h^0(X,(F^*G_{mq})) - h^0(X,G_{mq,p})| \leq C_0(mp + qp)^{d-2}, \]
where $m_q = m_{q,d_j}$, $m_q = (m_{q,d_j})$, for $1 \leq j \leq s$, or $m_q = m + q$ and where $m \geq 0$ and $q = p^n$.

Proof. Claim. For given $n$ there is a generically isomorphic map $\psi_n : F^nO_n \rightarrow O_{np}$.

Proof of the claim: By notation $O_n = O_X(nD)$, where $D$ is a $\mathbb{Q}$-Weil divisor. Let $D = \sum a_iD_i$, where $a_i \in \mathbb{Q}$ and $D_i$ are prime divisors. Then
\[ [npD] = \sum [a_i n]pD + \sum m_iD_i = p[nD] + \sum m_iD_i, \]
where $0 \leq m_i \leq p$ are integers. Let $\mathcal{M} = O_X([nD])$ then $\mathcal{M} \xrightarrow{f_n} O_{np}$ is an inclusion such that supp dim coker $f_n < d - 1$.

On the other hand, we can define the map $\phi_n : F^*O_n \rightarrow \mathcal{M}$ as follows: For the Frobenius map $F : X \rightarrow X$ let $F^nO_n = F^{-1}O_n \otimes_{F^{-1}O_X} O_{X_1}$ and let $\{D_+(f)\}_f$ denote the affine open cover of $X$, where $f \in R$ is an homogeneous element of $R$. Then the map $\phi_n |_{D_+(f)}$ is given by
\[ v/f^i \otimes u/f^i \rightarrow (v/f^i)^p \cdot u/f^i, \text{ if } v/f^i \in F^{-1}O_n \text{ and } u/f^i \in O_{X_1}. \]

The map $\phi_n$ is isomorphism on the regular locus $X_{reg}$ of $X$ as $O_n |_{X_{reg}}$ is invertible. In particular $\psi_n = \bar{f}_n \cdot \phi_n$ is generically an isomorphism. This proves the claim.

Now the map $\psi = \bigoplus_i \psi_{(m_q,d_i)} : \bigoplus_i F^*O_{(m_q,d_i)} \rightarrow \bigoplus_i O_{(m_q,d_i)}$ is generically an isomorphism, and $\phi$ is an isomorphism such that $\phi \circ F^*\phi_{m,q} = \phi_{m',qp} \circ \phi$. This gives us a map $F^*G_{m,q} \rightarrow G_{m',qp}$ such that the following diagram commutes
\[ \begin{array}{ccccccc}
0 & \rightarrow & G_{m',qp} & \rightarrow & \bigoplus_i F^*O_{(m_q,d_i)} & \xrightarrow{\phi_{m',qp}} & \mathcal{L}^{[m+q/r]}p & \rightarrow & 0 \\
\uparrow{f_{m,q}} & & \uparrow{\psi} & & \uparrow{\phi} & & \uparrow{\phi} & & \\
0 & \rightarrow & F^*G_{m,q} & \rightarrow & \bigoplus_i F^*O_{(m_q,d_i)} & \xrightarrow{F^*\phi_{m,q}} & F^*\mathcal{L}^{[m+q/r]} & \rightarrow & 0,
\end{array} \]
where $m' = \lceil (m + q)/r \rceil r p - q p$. Therefore $m' = mp - r_1p$, for some $0 \leq r_1 < r$.

Note that the map $\psi_{(m,q,d_i)} : F^*O_{(m_q,d_i)} \rightarrow O_{(m_q,d_i)}$ is the same as the map
\[ \psi_{[m,q,d_i/r]} \otimes \psi_{i,j} : F^*\mathcal{L}^{[m,q,d_i/r]} \otimes F^*O_{i,j} \rightarrow \mathcal{L}^{[m,q,d_i/r]}p \otimes O_{i,j}, \]
where the map $\psi_{[m,q,d_i/r]} : F^*\mathcal{L}^{[m,q,d_i/r]} \rightarrow \mathcal{L}^{[m,q,d_i/r]}p$ is an isomorphism and the generically isomorphic map $\psi_{i,j} \in \{\psi_j : F^*O_j \rightarrow O_{ip} \mid -r \leq j \leq r\}$.

Similarly for any $m \in \mathbb{Z}$, the map $\psi_m : F^*O_m \rightarrow O_{mp}$ is same as the map
\[ \psi_{[m/r]} \otimes \psi_i : F^*\mathcal{L}^{[m/r]} \otimes F^*O_i \rightarrow \mathcal{L}^{[m/r]}p \otimes \mathcal{O}(ip), \text{ where } 0 \leq i < r \]
and where $\psi_{[m/r]}$ is an isomorphism. Since the map $\phi$ is an isomorphism, we have ker $f_{m,q} = \ker \psi$ and coker $f_{m,q} = \coker \psi$ and each have supp dim $< d - 1$. Hence the lemma follows by Lemma 5.2 \qed
Lemma 5.5. There is a constant $C_1$ such that
\[
|p^{d-1}h^0(X, G_{m,q}) - h^0(X, F^*G_{m,q})| \leq C_1(mp + qp)^{d-2}
\]
where $m_q = m_{q,d_i}$, $m_q = (m_{q,d_i})$, for $1 \leq j \leq s$, or $m_q = m + q$, and where $m \geq 0$ and $q = p^n$.

Proof. Recall $X = \text{Proj } R = \text{Proj } R^r$, where $R^r$ is a standard graded domain. Therefore by Lemma 2.9 in [T], there is an integer $m_2 \in \mathbb{N}$ (it will be a multiple of $r$) such that we have a short exact sequence of sheaves of $\mathcal{O}$-modules
\[
0 \rightarrow \bigoplus_{j=1}^{d-1} \mathcal{O}_X(-m_2D) \xrightarrow{\eta} F_*\mathcal{O}_X \rightarrow Q'' \rightarrow 0,
\]
where support dimension $Q''$ is $< d - 1$.

Let $M_1 = \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_2}$ and $M = F_*\mathcal{O}_X$. Then the short exact sequences $0 \rightarrow M_1 \xrightarrow{\eta} M \rightarrow Q'' \rightarrow 0$ and (5.1) give the following commutative diagram of canonical maps
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_{q,d_i}} \otimes Q'' & \rightarrow & \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_{q,d_i}} \otimes Q'' & \rightarrow & L^{m+q/r} \otimes Q'' & \rightarrow & 0 \\
\uparrow & & \uparrow \psi_{m_{q,d_i}} & & \uparrow \psi_{m_{q,d_i}} & & \uparrow \psi_{m_{q,d_i}} & & \uparrow \psi_{m_{q,d_i}} \\
0 & \rightarrow & \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_{q,d_i}} \otimes M & \rightarrow & \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_{q,d_i}} \otimes M & \rightarrow & L^{m+q/r} \otimes M & \rightarrow & 0 \\
\uparrow & & \uparrow \phi_{m_{q,d_i}} & & \uparrow \phi_{m_{q,d_i}} & & \uparrow \phi_{m_{q,d_i}} & & \uparrow \phi_{m_{q,d_i}} \\
0 & \rightarrow & \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_{q,d_i}} \otimes M_1 & \rightarrow & \bigoplus_{j=1}^{d-1} \mathcal{O}_{m_{q,d_i}} \otimes M_1 & \rightarrow & L^{m+q/r} \otimes M_1 & \rightarrow & 0 \\
\uparrow \ker(f_{m_{q,d_i}}) & & \uparrow \ker(f_{m_{q,d_i}}) & & \uparrow \ker(f_{m_{q,d_i}}) & & \uparrow \ker(f_{m_{q,d_i}}) & & \uparrow \ker(f_{m_{q,d_i}}) \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

Since $\mathcal{O}_{m_{q,d_i}} = L^{m+q/r} \otimes \mathcal{E}_i$, where $\mathcal{E}_i = \mathcal{O}_{m_{q,d_i}} - \lfloor m_{q,d_i}/r \rfloor \in \{\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{2r}\}$ the map $\phi_{m_{q,d_i}}$ is the same as the map
\[
\bigoplus_{j=1}^{d-1} \text{Id}_{\mathcal{L}} \otimes \psi_i : \bigoplus_{j=1}^{d-1} \mathcal{L}^{m_{q,d_i}/r} \otimes \mathcal{E}_i \otimes M_1 \rightarrow \bigoplus_{j=1}^{d-1} \mathcal{L}^{m_{q,d_i}/r} \otimes \mathcal{E}_i \otimes M,
\]
where the map $\psi_i = \text{Id}_{\mathcal{E}_i} \otimes \eta : \mathcal{E}_i \otimes M_1 \rightarrow \mathcal{E}_i \otimes M$ is generically isomorphic. Therefore
\[
\text{supp dim of } \left( \ker(f_{m_{q,d_i}}) = \ker(f_{m_{q,d_i}}) = \bigoplus_{j=1}^{d-1} \mathcal{L}^{m_{q,d_i}/r} \otimes \ker(\psi_i) \right) < d - 1.
\]

Now the long exact sequence
\[
0 \rightarrow \ker(f_{m_{q,d_i}}) \rightarrow \mathcal{G}_{m,q} \otimes M_1 \rightarrow \mathcal{G}_{m,q} \otimes M \rightarrow \mathcal{G}_{m,q} \otimes Q'' \rightarrow 0
\]
gives (Lemma 5.2 (2))
\[
| h^0(X, \mathcal{G}_{m,q} \otimes M_1) - h^0(\mathcal{G}_{m,q} \otimes F_*\mathcal{O}_X) | = C_1(m + q)^{d-2},
\]
for some constant $C_1$. On the other hand, as $F$ is a finite map, for any coherent sheaf $M$ of $\mathcal{O}_X$-modules the projection formula $F_* (F^* \mathcal{G}_{m,q} \otimes M) = \mathcal{G}_{m,q} \otimes F_* M$ holds.

This implies
\[
h^0(X, \mathcal{G}_{m,q} \otimes F_*\mathcal{O}_X) = h^0(X, F_* (F^* \mathcal{G}_{m,q})) = h^i(X, F^* \mathcal{G}_{m,q})
\]
Now the lemma follows by (5.3) and (5.4).
The second assertion follows by the same line of arguments. □

Proof of Main Lemma 3.7 It follows from Lemma 5.4, Lemma 5.5 and Lemma 5.3 (1).

6. HK density functions for Segre products of graded rings

Here we show that the HK density function is multiplicative. Let \((R, I)\) and \((S, J)\) be two pairs, where \(R = \oplus_{n \geq 0} R_n\) and \(S = \oplus_{n \geq 0} S_n\) are graded domains of dimension \(d_1 \geq 2\) and \(d_2 \geq 2\) respectively, over a perfect field \(k\), and \(I \subset R\) and \(J \subset S\) are graded ideals of finite colengths.

Moreover let \(F_R : [0, \infty) \rightarrow [0, \infty)\) and \(F_S : [0, \infty) \rightarrow [0, \infty)\) be the Hilbert-Samuel density functions given by

\[
F_R(x) = e_0(R)x^{d_1-1}/(d_1 - 1)! , \quad \text{and} \quad F_S(x) = e_0(S)x^{d_2-1}/(d_2 - 1)!,
\]

where, for a graded ring \(R\), \(e_0(R)\) is the Hilbert-Samuel multiplicity of \(R\) with respect to its graded maximal ideal.

In [\(T\)], we had proved that the HK density function is multiplicative for Segre products of standard graded rings. In Theorem 6.1 and Corollary 6.3 we show that this property extends to graded domains.

**Theorem 6.1.** For the pairs \((R, I)\) and \((S, J)\) as above if \(\gcd \{ m \mid R_m \neq 0 \} = 1\) and \(\gcd \{ m \mid S_m \neq 0 \} = 1\). Then the HK density function of the pair \((R\#S, I\#J)\), where \(R\#S = \oplus_{n \geq 0} R_n \otimes_k S_n\) is the Segre product of \(R\) and \(S\), is given by

\[
\epsilon_{HK}(R\#S, I\#J) = \frac{e_0(R)}{(d_1 - 1)!} \int_0^\infty x^{d_1-1} f_{S,J}(x) dx + \frac{e_0(S)}{(d_2 - 1)!} \int_0^\infty x^{d_2-1} f_{R,I}(x) dx
\]

and also

\[
\epsilon_{HK}(R\#S, I\#J) = \int_0^\infty f_{R,I}(x) f_{S,J}(x) dx.
\]

**Proof.** Note that \(R\#S\) is a graded integral domain with \(\gcd \{ n \mid (R\#S)_n \neq 0 \} = 1\). Therefore

\[
(q^{d_1-1} q^{d_2-1}) f_n(R\#S, I\#J)(m/q) = \ell(R\#S/(I\#J)[q])_m
\]

\[
= \ell(R_m) \ell(S_m) - \left[ \ell(R_m) - \ell(R/I[q])_m \right] \left[ \ell(S_m) - \ell(S/J[q])_m \right].
\]

Hence

\[
f_n(R\#S, I\#J)(x) = f_n(S, J) q^{\ell(S)[xq]} q^{d_1-1} + f_n(R, I) q^{\ell(R)[xq]} q^{d_2-1} - f_n(R, I) f_n(S, J).
\]

Since \(\{ f_n(R\#S, I\#J) \}_n\), \(\{ f_n(R, I) \}_n\) and \(\{ f_n(S, J) \}_n\) are uniformly convergent sequences with bounded supports, taking limit as \(n \to \infty\) we get,

\[
f_{R\#S, I\#J}(x) = F_R(x) f_{S,J}(x) + F_S(x) f_{R,I}(x) - f_{R,I}(x) f_{S,J}(x) \quad \text{for all} \quad x \geq 0.
\]

The rest of the proof follows as \(F_{R\#S}(x) = F_R(x) F_S(x)\). □

**Notations 6.2.** For a graded domain \(R = \oplus_{n \geq 0} R_n\) with a graded ideal \(I = \oplus_{n \geq 1} I_n\),
we denote \(R^{(m)} = \oplus_{n \geq 0} R_{nm}\) with degree \(n\) component \(= R_{nm}\)
and \(I^{(m)} = I \cap R^{(m)} = \oplus_{n \geq 1} I_{nm}\).
Corollary 6.3. If \( \gcd \{ m \mid R_m \neq 0 \} = n_1 \) and \( \gcd \{ m \mid S_m \neq 0 \} = n_2 \). Then
\[
F_{R^\#} - f_{R^\#,I^\#} = \left[ F_{R^{(l)}} - f_{R^{(l)},I^{(l)}} \right] F_{S^{(l)}} - f_{S^{(l)},J^{(l)}}
\]
and
\[
e_{HK}(R^\#, I^\#) = \frac{e_0(R^{(l)})}{(a_1 - 1)!} \int_0^\infty x^{d_1 - 1}f_{S^{(l)},J^{(l)}}(x)dx + \frac{e_0(S^{(l)})}{(a_2 - 1)!} \int_0^\infty x^{d_2 - 1}f_{R^{(l)},I^{(l)}}(x)dx
\]
\[
- \int_0^\infty f_{R^{(l)},I^{(l)}}(x)f_{S^{(l)},J^{(l)}}(x)dx,
\]
where \( l = \text{lcm}(n_1, n_2) \).

Proof. Since \( R^\# = R^{(l)}^\# S^{(l)} \) and \( I^\# = I^{(l)}^\# J^{(l)} \) and \( \gcd \{ m \mid R_m^{(l)} \neq 0 \} = \gcd \{ m \mid S_m^{(l)} \neq 0 \} = 1 \), the corollary follows from the above theorem.

7. Applications and examples

The coefficients of the HK function for a pair \((R, I)\) (given as \( HK(q) = 1/q^d \ell(R/I^{[q]}) \)) have a nice geometric description (see \[K\]) provided \( \text{proj dim}_R(R/I) < \infty \). Here we prove that the HK density function too has a nice description in the case of such graded pairs.

Proposition 7.1. Let \((R, I)\) be a graded pair such that \( \text{proj dim}_R(R/I) < \infty \) then the HK density function \( f_{R,I} \) is a piecewise polynomial function of degree \( d - 1 \), where \( f_{R,I} \) (and hence \( e_{HK}(R, I) \)) is given in terms of the graded Betti numbers of the minimal graded \( R \)-resolution of \( R/I \).

Proof. Consider the minimal graded resolution of \( R/I \) over the graded ring \( R \)
\[
0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{d,j}} \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{d-1,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}} \longrightarrow R \longrightarrow R/I \longrightarrow 0.
\]
Since the functor of Frobenius is exact on the category of modules of finite type and finite projective dimension (a corollary of the acyclicity lemma by Peskine-Szpiro [PS]), we have a long exact sequence
\[
0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-qj)^{\beta_{d,j}} \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-qj)^{\beta_{d-1,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-qj)^{\beta_{1,j}} \longrightarrow R \longrightarrow R/I \longrightarrow 0.
\]
Let \( \tilde{e}_0 = e_0(R, m)/(d - 1)! \) and let
\[
\mathbb{B}(j) = \beta_{0j} - \beta_{1j} + \beta_{2j} + \cdots + (-1)^d \beta_{dj}.
\]
Note that \( \beta_{00} = 1 \) and \( \beta_{0,j} = 0 \) for \( j \neq 0 \). For \( j < 1, \mathbb{B}(j) = 0 \). If \( l \) be the largest integer such that \( \beta_{il} \neq 0 \) for some \( i \). Then
\[
\ell(R/I^{[l]})_m = \ell(R_m) + \mathbb{B}(1)\ell(R_{m-q}) + \mathbb{B}(2)\ell(R_{m-q}) + \cdots + \mathbb{B}(l)\ell(R_{m-lq})
\]
and therefore
\[
f_{R,I}(x) = \begin{cases} 0 & x \leq 1 \\ \tilde{e}_0 \left[ x^{d-1} + \mathbb{B}(1)(x - 1)^{d-1} \right] & 1 \leq x \leq 2 \\ \vdots & \\ \tilde{e}_0 \left[ x^{d-1} + \mathbb{B}(1)(x - 1)^{d-1} + \cdots + \mathbb{B}(i)(x - i)^{d-1} \right] & i \leq x \leq (i + 1) \\ \tilde{e}_0 \left[ x^{d-1} + \mathbb{B}(1)(x - 1)^{d-1} + \cdots + \mathbb{B}(l - 1)(x - l + 1)^{d-1} \right] & l - 1 \leq x \leq l \\ 0 & x \leq l \end{cases}
\]
Note that \( f_{R,I} \) is a compactly supported function which implies that the polynomial
\[
x^{d-1} + \mathbb{B}(1)(x - 1)^{d-1} + \cdots + \mathbb{B}(l)(x - l)^{d-1} = 0.
\]
Hence $\text{supp}(f_{R,I}) \subseteq [0, l]$.
Moreover
\[ e_{HK}(R, I) = \frac{c_0}{d!} \left[ B(0)^d + B(1)(l - 1)^d + \cdots + B(i)(l - i)^d + \cdots + B(l - 1) \right] \]

Proof of Corollary 7.2. Let $m_0 = \gcd\{n > 0 \mid S_n \neq 0\}$ and $n_0 = \gcd\{n > 0 \mid R_n \neq 0\}$ and $l_0 = m_0/n_0$. Then, by Theorem 1.1 and Lemma 4.3 for $x \geq 0$,
\[ f_{S,I}(x) = \frac{1}{\text{rank}_S R} f_{R,S,I}(x) = \frac{l_0}{\text{rank}_S R} f_{R,IR}(x|0) \quad \text{and} \quad e_{HK}(S, I) = \frac{e_{HK}(R, IR)}{\text{rank}_S R}. \]
Hence the corollary follows from Proposition 7.1. □

Remark 7.2. If $(R, I)$ is a graded pair such that $R/IR$ has the finite pure resolution
\[ 0 \longrightarrow \oplus \beta^0 R(-j_0) \longrightarrow \cdots \longrightarrow \oplus \beta^2 R(-j_2) \longrightarrow \oplus \beta^1 R(-j_1) \longrightarrow R \longrightarrow R/IR \longrightarrow 0 \]
then $j_1 < j_2 < \cdots < j_d$ and
\[ B(1) = \cdots = B(j_1 - 1) = 0 \quad \text{and} \quad B(j_1) = -\beta_1 \]
\[ B(j_{n-1} + 1) = \cdots = B(j_{n} - 1) = 0 \quad \text{and} \quad B(j_{n}) = (-1)^n \beta_n. \]
Hence
\[ f_{R,I}(x) = \begin{cases} \bar{e}_0 \left[ x^{d-1} \right] & 0 \leq x \leq j_1 \\ \bar{e}_0 \left[ x^{d-1} - \beta_1(x - j_1)^{d-1} \right] & j_1 \leq x \leq j_2 \\ \bar{e}_0 \left[ x^{d-1} - \beta_1(x - j_1)^{d-1} + \cdots + (-1)^{d-1} \beta_{d-1}(x - j_{d-1})^{d-1} \right] & j_{d-1} \leq x \leq j_d \\ 0 & j_d \leq x. \end{cases} \]
Here the maximum support of $f_{R,I} = \alpha(R, I) = j_d$, as $\beta_d \neq 0$.

7.1. Some concrete examples. We recall the Hilbert-Burch theorem (see [BH]).

Theorem 7.3. Let $\psi : R^n \longrightarrow R^{n+1}$ be a $R$-linear map, where $R$ is a Noetherian ring. Let $I = I_n(\psi)$ be the ideal generated by $n + 1$ elements consisting of $n \times n$ minors of the matrix given by $\psi$.

Then grade $I_n(\psi) \geq 2$ implies the ideal $I$ has the resolution of the form
\[ 0 \longrightarrow R^n \stackrel{\psi}{\longrightarrow} R^{n+1} \longrightarrow I \longrightarrow 0. \]

In the following examples we compute the HK density function $f_{S,I}$, where $S = R^G$ is the ring of invariants in $R = k[x_1, x_2]$ with $G \in \{A_n, D_n, E_6, E_7, E_8\}$ and $I \subset S$ is its graded maximal ideal. This also recovers the computations of $e_H(R^G, I)$ given in Theorem 5.1 of [WY]. For this it is enough to construct the minimal graded resolution of $IR$ as a $R$-module.

Note that in all the following cases $R^G = k[h_1, h_2, h_3] \subset k[x_1, x_2]$, where $h_1, h_2, h_3$ are explicit homogeneous polynomials in $x_1, x_2$ (see Chap X, page 225 of [MBD]). Using the Hilbert-Burch theorem, we will construct a $R$-resolution for $IR = (h_1, h_2, h_3)R$, which is of the following type:
\[ (7.1) \]
\[ 0 \longrightarrow R(-l_1) \oplus R(-l_2) \stackrel{\psi}{\longrightarrow} R(-\deg h_1) \oplus R(-\deg h_2) \oplus R(-\deg h_3) \stackrel{\phi}{\longrightarrow} IR \longrightarrow 0, \]
where $\phi$ is given by the matrix $[h_1, h_2, h_3]$. In the forthcoming set of examples we define the map $\psi$ by giving a $3 \times 2$ matrix in $R$ (this will also determine the values $l_1, l_2$). Since grade $IR = 2$, to prove that (7.1) is exact, it only remains to check that $I_2(\psi) = IR$ which can be done easily.

**Example 7.4.** Let $G = A_n$ then $|G| = n \geq 2$ and char $k = p \geq 2$ and $(p, n) = 1$.

$$R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(x_1^2 + x_1x_2 + x_2^2)}$$

where $h_1 = x_1x_2$, $h_2 = x_1^2$ and $h_3 = x_2^2$. The map $\psi$ is given by the matrix

$$\begin{bmatrix}
x_1^{n-1} & -x_2 & 0 \\
x_2^{n-1} & 0 & -x_1
\end{bmatrix}.$$ 

Then the sequence

$$0 \longrightarrow R(-n - 1) \oplus R(-n) \xrightarrow{\psi} R(-2) \oplus R(-n) \oplus R(-n) \longrightarrow IR \xrightarrow{\phi} 0$$

is the minimal resolution for $IR$ as $I_2(\psi) = I$. Here $\mathbb{H}(2) = -1$, $\mathbb{B}(n) = -2$ and $\mathbb{B}(n + 1) = 2$. If $n$ is even then the HK density function $f_{S,I}$ is given by

$$f_{S,I}(x) = \frac{4x}{(n + 1)} \quad \text{if} \quad 0 \leq x \leq 1$$

$$= \frac{4}{(n + 1)} \quad \text{if} \quad 1 \leq x \leq n/2$$

$$= \frac{2(2 - 4x + 2n)}{n + 1} \quad \text{if} \quad n/2 \leq x \leq (n + 1)/2$$

If $n$ is odd then the HK density function $f_{S,I}$ is given by

$$f_{S,I}(x) = \frac{x}{(n + 1)} \quad \text{if} \quad 0 \leq x \leq 2$$

$$= \frac{2}{(n + 1)} \quad \text{if} \quad 2 \leq x \leq n$$

$$= \frac{2 - 2x + 2n}{n + 1} \quad \text{if} \quad n \leq x \leq n + 1$$

**Example 7.5.** Let $G = D_n$ the dihedral group then $|G| = 4n$ and char $k = p \geq 3$ and $(p, n) = 1$.

$$R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(x_1^2 + x_2^2 + x_2^3 + x_1^3)}$$

where

$$h_1 = -2x_1x_2^2, \quad h_2 = x_1^{2n} + (-1)^nx_2^{2n}, \quad h_3 = x_1x_2(x_1^{2n} - (-1)^nx_2^{2n}).$$

We assume $n$ is even. Let map $\psi$ is given by the matrix

$$\begin{bmatrix}
-2x_1^{n-1} & x_1x_2^2 & x_2 \\
-2x_2^{n-1} & -x_1^2x_2 & x_1
\end{bmatrix}$$

Then the sequence

$$0 \longrightarrow R(-2n - 3) \oplus R(-2n - 3) \xrightarrow{\psi} R(-4) \oplus R(-2n) \oplus R(-2n - 2) \xrightarrow{\phi} IR \longrightarrow 0$$

is the minimal resolution for $IR$ as $I_2(\psi) = IR$.

Here $\mathbb{H}(4) = -1$, $\mathbb{B}(2n) = -1$ and $\mathbb{B}(2n + 2) = -1$. If $n$ is even then the HK density function $f_{S,I}$ is given by

$$f_{S,I}(x) = \frac{x}{n - 2} \quad \text{if} \quad 0 \leq x \leq 2$$

$$= \frac{2}{n - 2} \quad \text{if} \quad 2 \leq x \leq n$$

$$= \frac{(n + 2 - x)/n - 2} \quad \text{if} \quad n \leq x \leq n + 1$$

$$= \frac{(2n + 3 - 2x)/n - 2} \quad \text{if} \quad n + 1 \leq x \leq n + 3/2$$

and hence $e_{HK}(R^G, I) = 2 - 1/4n$. 
Example 7.6. Let \( G = E_6 \) the tetrahedral group then \(|G| = 24\) and \( \text{char } k = p \geq 5 \).

\[ R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(6ax_1^2 - x_2^3 + x_3^3)}, \text{ where } a = 2\sqrt{-3} \]

\[ h_1 = x_1^5 x_2 - x_1 x_2^5, \quad h_2 = x_1^4 + ax_1^2 x_2^2 + x_2^4, \quad h_3 = x_1^3 - ax_1^2 x_2 + x_2^3. \]

Let \( \psi \) be given by the matrix

\[ \begin{bmatrix} x_1 & -(a/2)x_1^2 x_2 - x_3^2 & (a/2)x_1^2 x_2 - x_3^2 \\ x_2 & x_1^2 + (a/2)x_1 x_2^2 & x_1^2 - (a/2)x_1 x_2^2 \end{bmatrix} \]

Then \( I_2(\psi) = (ah_1, h_2, h_3)R \). If \( a \neq 0 \) in \( k \) then the canonical sequence

\[ 0 \rightarrow R(-7) \oplus R(-7) \xrightarrow{\psi} R(-6) \oplus R(-4) \oplus R(-4) \xrightarrow{\phi} IR \rightarrow 0 \]

is the minimal \( R \)-resolution of \( IR \).

Here \( \mathcal{B}(4) = -2, \mathcal{B}(6) = -1 \) and \( \mathcal{B}(7) = 2 \) and the HK density function \( f_{S,I} \) is given by

\[ f_{S,I}(x) = \begin{cases} x/6 & \text{if } 0 \leq x \leq 2 \\ (4-x)/6 & \text{if } 2 < x \leq 3 \\ (7-2x)/6 & \text{if } 3 < x \leq 7/2 \\ 0 & \text{otherwise} \end{cases} \]

Example 7.7. Let \( G = E_7 \) octahedral group then \(|G| = 24\) and \( \text{char } k \geq 5 \) and

\[ R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(108x_1^4 - x_2^3 + x_3^2)}, \text{ where } a = x_1^2 - x_2^3 + x_3^4 \]

\[ h_1 = x_1^5 x_2 - x_1 x_2^5, \quad h_2 = x_1^8 + 14x_1^4 x_2^4 + x_2^8, \quad h_3 = x_1^{12} - 33(x_1^8 x_2^4) - 33(x_1^4 x_2^8) + x_2^{12}. \]

Let \( \psi \) be given by the matrix

\[ \begin{bmatrix} -7x_1^4 x_2^3 - x_2^7 & x_1^5 & x_1 \\ 7x_1^3 x_2^4 + x_1^7 & x_1^5 & x_2 \end{bmatrix} \]

If \( \text{char } k > 3 \) then \( (h_1, h_2, h_3)R = I_2(\psi) \) and hence the minimal \( R \)-resolution for \( IR \) is given by

\[ 0 \rightarrow R(-13) \oplus R(-13) \xrightarrow{\psi} R(-6) \oplus R(-8) \oplus R(-12) \xrightarrow{\phi} IR \rightarrow 0. \]

Here \( \mathcal{B}(6) = -1, \mathcal{B}(8) = -1, \mathcal{B}(12) = -1 \) and \( \mathcal{B}(13) = 2 \). Hence the HK density of \((S,I)\) is given by

\[ f_{S,I}(x) = \begin{cases} x/48 & \text{if } 0 \leq x \leq 6 \\ 6/48 & \text{if } 6 \leq x \leq 8 \\ (14-x)/48 & \text{if } 8 \leq x \leq 12 \\ (2x-2x)/48 & \text{if } 12 \leq x \leq 13 \\ 0 & \text{otherwise} \end{cases} \]

and hence \( e_{\text{HK}}(R^G, I) = 2 - (1/24) \).

Example 7.8. Let \( G = E_8 \) the icosahedral group then \(|G| = 120\) and \( \text{char } k \geq 7 \). Now

\[ R^G = k[h_1, h_2, h_3] = \frac{k[x_1, x_2, x_3]}{(x_2^3 + x_3^3 - 1728x_1^5)}, \]

where
\[ h_1 = x_1x_2(x_1^{10} + 11x_1^5x_2^5 - x_2^{10}) \]

\[ h_2 = x_1^{30} + x_2^{30} + 522(x_1^{25}x_2^5 - x_2^{25}x_1^5) - 10005(x_1^{20}x_2^{10} + x_1^{10}x_2^{20}) \]

\[ h_3 = -x_1^{20} - x_2^{20} + 228(x_1^{15}x_2^5 - x_2^{15}x_1^5) - 494(x_1^{10}x_2^{10}) \]

Let \( \psi \) be given by the matrix

\[
\begin{bmatrix}
  x_1 & f_2 & f_3 \\
  x_2 & g_2 & g_3
\end{bmatrix}
\]

where

\[ f_2 = -x^{11} - (11/2)x_1^6x_2^5. \quad f_3 = x_2^{19} + ax_1^{14}x_2 + (b/2)x_1^{10}x_2^{10} \]

\[ g_2 = -x_1^{11} + (11/2)x_1^6x_2^5 \quad g_3 = -x_1^{19} + ax_1^{14}x_2^5 - (b/2)x_1^{10}x_2^{10} \]

and where \( a = 228 \) and \( b = 494 \).

In particular \( (h_1, h_2, h_3)R = I_2(\psi) \).

Hence the minimal \( R \)-resolution for \( IR \) is given by

\[
0 \longrightarrow R(-31) \oplus R(-31) \xrightarrow{\psi} R(-12) \oplus R(-30) \oplus R(-20) \longrightarrow IR \longrightarrow 0.
\]

\[ \mathbb{B}(12) = -1, \mathbb{B}(20) = -1, \mathbb{B}(30) = -1 \text{ and } \mathbb{B}(31) = 2. \]

Hence the HK density of \((S, I)\) is given by

\[
\begin{align*}
    f_{S, I}(x) & = x/30 \quad \text{if} \quad 0 \leq x \leq 6 \\
    & = 6/30 \quad \text{if} \quad 6 \leq x \leq 10 \\
    & = (16 - x)/30 \quad \text{if} \quad 10 \leq x \leq 15 \\
    & = (31 - 2x)/30 \quad \text{if} \quad 15 \leq x \leq 31/2 \\
    & = 0 \quad \text{otherwise}
\end{align*}
\]

and hence \( e_{HK}(R^G, I) = 2 - (1/120) \).

**Remark 7.9.** If \((R, I)\) is a two dimensional graded pair then its HK density function \( f_{R, I} \) is an explicit piecewise linear polynomial with rational coefficients and rational break points (the proof follows from the same arguments as in [TW]):

Let \( f_1, \ldots, f_s \) be a set of homogeneous generators of \( I \) of degrees \( d_1, \ldots, d_s \). Let \( \widetilde{S} \) denote the normalization of \( R'^0 = \oplus n \geq 0 R_{n0} \), where \( \gcd \{ m \mid R_m \neq 0 \} = n_0 \). Let \( X = \text{Proj}(\widetilde{S}) \), then for the \( \mathbb{Q} \)-Weil divisor \( D \) (which is Cartier in this case) corresponding to the normal ring \( \widetilde{S} \) (as in Theorem 3.1) the sheaf \( \mathcal{O}_n = \mathcal{O}_X(D) \) is invertible. Hence the sequence \((3.1)\) is

\[
0 \longrightarrow F^n V \otimes \mathcal{O}_m \longrightarrow \oplus_i \mathcal{O}_{m+q-d_i} \xrightarrow{\phi_{m,q}} \mathcal{O}_{m+q} \longrightarrow 0,
\]

where

\[
0 \longrightarrow V \longrightarrow \oplus_i \mathcal{O}_{1-d_i} \xrightarrow{\phi} \mathcal{O}_1 \longrightarrow 0,
\]

where \( \phi(x_1, \ldots, x_s) = \sum x_if_i \). This gives

\[
f_{R, I}(x) = f_{V, \mathcal{O}_1}(x) - f_{\mathbb{E}, \mathcal{O}_{1-d_i}, \mathcal{O}_1}(x), \quad \text{for} \quad x \geq 0,
\]
where, for a vector bundle $E$ on $X$ with strong HN data $(\{a_1, \ldots, a_{l+1}\}, \{r_1, \ldots, r_{l+1}\})$ and $d = \deg \mathcal{O}_1$, the function $f_{E,\mathcal{O}_1}$ denotes the HK density function of $E$ with respect to $\mathcal{O}_1$ and is given by

$$x < 1 - a_1/d \implies f_{E,\mathcal{O}_1}(x) = -\left[ \sum_{i=1}^{l+1} a_i r_i + d(x-1)r_i \right]$$

$$1 - a_i/d \leq x < 1 - a_{i+1}/d \implies f_{E,\mathcal{O}_1}(x) = -\left[ \sum_{k=i+1}^{l+1} a_k r_k + d(x-1)r_k \right].$$

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