Separating Geometric Thickness from Book Thickness

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Abstract

We show that geometric thickness and book thickness are not asymptotically equivalent: for every $t$, there exists a graph with geometric thickness two and book thickness $\geq t$.

1 Introduction

Graph drawing [2] concerns itself with the geometric layout of abstract graphs, with applications including information visualization and VLSI design. The graphs arising from these applications are frequently impossible to arrange in the plane without edge crossings; one way of dealing with this problem is to partition the edges of the graph into a number of planar layers, each of which might be drawn using a different color or placed within a different physical layer of a VLSI circuit. This leads naturally to the concept of thickness, of which there are several variants:

- The thickness of a graph $G$, denoted $\theta(G)$, is the minimum number of planar subgraphs into which the edges of $G$ can be partitioned. Equivalently, it is the minimum number of layers in a planar drawing of $G$, such that each edge belongs to a single layer, no two edges in the same layer cross, and edges are allowed to be drawn as arbitrary curves [5].

- The geometric thickness of a graph $G$, denoted $\bar{\theta}(G)$, is the minimum number of layers in a planar drawing of $G$, such that each edge belongs to a single layer, no two edges in the same layer cross, and edges must be drawn as straight line segments. This parameter was introduced under the name “real linear thickness” by Kainen [5], and further studied by Dillencourt et al. [3].

- The book thickness of a graph $G$, denoted $bt(G)$, can be defined as the minimum number of layers in a planar drawing of $G$, such that each edge belongs to a single layer, no two edges in the same layer cross, and edges must be drawn as straight line segments, with the further restriction that the vertices of $G$ must be placed in convex position [1].

It is not difficult to define further variants; for instance, Wood [7] considers layouts in which each edge is drawn with at most one bend, at which it may change layers. For more results on thickness, see the survey of Mutzel et al. [6].

Clearly, from these definitions, $\theta(G) \leq \bar{\theta}(G) \leq bt(G)$, and these inequalities have been shown to be strict for certain graphs [3]. However, it was not known whether the geometric thickness is asymptotically equivalent to either of the other two parameters; that is, whether $bt(G) = \mathcal{O}(\theta(G))$ or whether $\bar{\theta}(G) = \mathcal{O}(\theta(G))$. In this paper, we answer the first of these two questions in the negative, by exhibiting a family of graphs for which $\bar{\theta} = 2$ but for which $bt = \omega(1)$.

Our construction is closely related to the layout method of Wood [2] however rather than allowing bends in the edges of our drawings we build them into the input by subdividing the edges of a complete graph. Our

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construction demonstrates also that in Wood’s one-bend layout model, every graph can be drawn in one or two layers, unless we introduce further restrictions such as Wood’s that the layout stay within a small area relative to the vertex separation.

2 Bounded geometric thickness

We define the graph $G_k$ by subdividing every edge of the complete $k$-vertex graph $K_k$ into a path of two edges. Equivalently, the vertices of $G_k$ can be viewed as corresponding to the singleton and doubleton subsets of a $k$-element set, with an edge between every two subsets having an inclusion relation. Thus, $G_k$ has $k + \binom{k}{2}$ vertices and $2\binom{k}{2}$ edges. Figure 1 depicts $G_8$. As the figure hints, all graphs $G_k$ can be drawn with small geometric thickness:

**Theorem 1** For $k \geq 5$, $\bar{\theta}(G_k) = 2$.

**Proof:** Since $K_k$ and its subdivisions are nonplanar for $k \geq 5$, it is clear that $\bar{\theta}(G_k) \geq 2$, so it remains to demonstrate the existence of a two-layer drawing. We let $v_i$, $0 \leq i < k$, denote the vertices of the complete graph $K_k$ from which $G_k$ is formed, and place vertex $v_i$ at the point with coordinates $(i, i + 1)$. To place the two-edge path between $v_i$ and $v_j$, $i < j$, we use an edge on the first layer from $v_i$ to $(j + 1, i)$ and an edge on the second layer from that point to $v_j$. All edges within a given row of the first layer, or within a given column of the second layer, have a common endpoint, so there can be no crossings within either layer. □

3 Unbounded book thickness

To show that $G_k$ does not have bounded book thickness, we need a standard result of Ramsey Theory [4], which we state in the form we need:
Lemma 1  For every pair of positive integers $c$ and $\ell$ there is an integer $R_c(\ell)$ with the following property: If the edges of the complete graph $K_{R_c(\ell)}$ are partitioned into $c$ graphs, then at least one of the graphs contains a complete $\ell$-vertex subgraph.

Theorem 2  For every positive integer $t$ there exists a $k$ such that $bt(G_k) \geq t$.

Proof: Let $k = R_c(5)$, where $c = \binom{t-1}{2}$. We then show that $bt(G_k) \geq t$. Suppose to the contrary that $G_k$ has a book embedding with $t - 1$ layers. We use this drawing to partition the edges of $K_k$ into $c$ subgraphs: an edge $v_iv_j$ is assigned to a subgraph according to the unordered pair of layers used by the path connecting $v_i$ to $v_j$ in $G_k$. If any of the two-edge paths in $G_k$ uses only a single layer, the corresponding edge of $K_k$ may be placed arbitrarily into any of the $t - 2$ subgraphs involving that layer.

By Lemma 1, we can find a copy of $K_5$ in one of the subgraphs. This copy corresponds to a graph $G_5$, drawn with a book embedding of only two layers. However, $G_5$ is nonplanar, so its book thickness is at least three, a contradiction. \qed

4 Discussion

We have shown that, for certain graphs $G_k$, the book thickness grows arbitrarily while the geometric thickness is bounded. It remains open how strongly separated these quantities are. Our proof uses Ramsey theory, so yields only weak lower bounds on the book thickness of $G_k$. The best upper bound we have been able to achieve is $bt(G_k) = O(\sqrt{k})$, by partitioning the vertices of $K_k$ into blocks, using one layer per vertex within each block to connect the vertices to reordered transfer points adjacent to the block, and using one layer per block to connect the transfer points to vertices in the other blocks. We conjecture that this upper bound is tight.

The question of whether geometric thickness and thickness are asymptotically equivalent is very interesting, and remains open.

References

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