Optimal control of unstable water movement in channels of irrigation systems under conditions of discontinuity of water delivery to consumers

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Abstract. The object of research was the site of the South Golodnostepsky main channel, located in the north-east of the Republic of Uzbekistan. Scientific and research work is being carried out in the world aimed at improving and developing methods, quality criteria, developing mathematical models and algorithms for solving problems of optimal control of the operating modes of water facility using modern information systems. We used methods of mathematical modeling and optimal control of systems with distributed parameters, which is considered the main channel. The optimum changes in water levels at times and along the length of the section of the main channel are obtained, and its open gates allow increasing the water level along the length of the section. During \( t = 20.7 \) minutes the water level at the end of the main channel section rises by 1.8 meters.

1. Introduction

Consider an irrigation channel consists of two sections interconnected by a hydraulic structure \([1]\).

As a mathematical model of sections of the irrigation channel, we take the complete system of Saint-Venant equations, which can be written in the following form \([1, 2, 3]\).

\[
\frac{\partial z_i}{\partial t} = f_i \left( z_i, \frac{\partial z_i}{\partial x_i}, q_i \right)
\] (1)
\[
\frac{\partial Q_i}{\partial t} = f_i \left( z_i, Q_i, \frac{\partial z_i}{\partial x_i}, \frac{\partial Q_i}{\partial x_i} \right) i = 1, 2, ..., 0 < x_i < x_j, \; l_i < x_2 < l_j, 0 < t < T
\]

where

\[
f_i \left( z_i, Q_i, \frac{\partial z_i}{\partial x_i}, q_i \right) = -\frac{1}{B_i} \left( \frac{\partial Q_i}{\partial x_i} - q_i \right)
\]

\[
f_i \left( z_i, Q_i, \frac{\partial z_i}{\partial x_i}, \frac{\partial Q_i}{\partial x_i} \right) = g_i \omega_i i_i Q_i^2 \left( \frac{\partial Q_i}{\partial x_i} \right)_i - \frac{2 Q_i}{\omega_i} \frac{\partial Q_i}{\partial x_i} \left( 1 - \frac{Q_i^2}{\omega_i c_i^2} \right) \frac{\partial z_i}{\partial x_i}
\]

where \( Q_i = Q_i( x_i, t ), z_i = z_i( x_i, t ) \) – respectively, the flow rate and the ordinate of the free flow surface of the \( i \) section of the channel, \( B_i = B_i( z_i ) \) – top flow width, \( \omega_i = \omega_i( z_i ) \) – area of the alive sectional, \( c_i = c_i( z_i ) \) – the speed of propagation of small waves, \( K_i = K_i( z_i ) \) – module consumption. The last four values are determined by the morphometric and hydraulic parameters of the channel section. Way inflow (outflow) \( q_i = q_i( x_i, t ) \) – calculated per unit length of the \( i \) channel section is a distributed disturbance. Initial conditions set

\[
Q_i \left( x_i, 0 \right) = Q_{i0} \left( x_i \right), \quad z_i \left( x_i, 0 \right) = z_{i0} \left( x_i \right)
\]

(2)  

Boundary conditions at points \( x_j = 0 \) and \( x_2 = l_2 \) written as follows

\[
Q_i \left( 0, t \right) = g_1 \left( z_1 \left( 0, t \right), u_1 \left( t \right) \right)
\]

(3)  

\[
Q_2 \left( l_2, t \right) = g_2 \left( z_2 \left( l_2, t \right), u_2 \left( t \right), u_3 \left( t \right) \right)
\]

where

\[
g_1 = \mu_i b_i u_i \left( t \right) \sqrt{2g \left( z_1 \left( 0, t \right) - z_1 \left( 0, t \right) \right)}
\]

(4)  

\[
g_2 = \mu_2 b_2 u_2 \left( t \right) \sqrt{2g \left( z_2 \left( l_2, t \right) - z_2 \left( l_2, t \right) \right)}
\]

(5)  

\( u_i = u_i \left( t_i \right), \; i = 1, 2, 3, ..., \) – control functions applied at the boundary points (the height of the openings of the gates), \( b_i, \; i = 1, 2, 3, ..., \) – the width of open valve openings, \( z_i \) – the ordinate of the free surface of the water stream of the upstream of the first shutter.  

Point pairing conditions \( x_j = x_2 = l_j \) are written as follows

\[
Q_i \left( l_j, t \right) = g_i^* \left( z_i \left( l_j, t \right), z_2 \left( l_j, t \right), u_i \left( l_j, t \right), u_2 \left( t \right) \right)
\]

(6)  

\[
Q_2 \left( l_j, t \right) = g_2^* \left( z_2 \left( l_j, t \right), z_2 \left( l_j, t \right), u_2 \left( t \right) \right)
\]

where

\[
g_i^* = \mu_i^* b_i u_i \left( t \right) \sqrt{2g \left( z_i \left( l_j, t \right) - z_i \left( l_j, t \right) \right)} + g_i^e
\]

(7)
\[ g_2^c = \mu_2^c b_2^i u_2^i(t) \sqrt{2g \left( z_2 \left( l_2, t \right) - z_2 \left( l_1, t \right) \right)} \]

where \( u_i^c = u_i^c(t), i = 1, 2, 3, \ldots \) — control functions applied at the connection point of the channel sections (the height of the openings of the gates), \( b_i^c, i = 1, 2, 3, \ldots \) — the width of the openings of the shutters.

2. Methods

For the irrigation channel, a water distribution plan is set for a given period \([0,T]\). In this case, the planned water consumption \( Q_i^*(t) \) for hydraulic structures of lateral and final branches.

The tasks of the optimal distribution of water in the irrigation channel are formulated as follows.

On the period \([0,T]\) required to define such controls \( u_{i_1} \leq u_i(t) \leq u_{i_2}, u_{i_3} \leq u_i(t) \leq u_{i_4}, u_{i_5} \leq u_i(t) \leq u_{i_6}, u_{i_7} \leq u_i(t) \leq u_{i_8}, \) which minimize the linear combination of the integral deviations of the actual flow of water flowing through the hydraulic structures of the lateral and final outlets.

The optimality criterion is written as follows \([4, 5, 6]\)

\[ I = \int_0^T \left[ G_1^c \left[ z_2 \left( l_2, t \right), u_2^c(t) \right] + G_2^c \left[ z_2 \left( l_2, t \right), u_2^c(t) \right] \right] dt + \]
\[ + \int_0^T \left[ G_3^c \left[ z_1 \left( x, T \right), z_1^* \left( x \right) \right] \right] dx_j + \int_0^T \left[ G_4^c \left[ z_2 \left( x, T \right), z_2^* \left( x \right) \right] \right] dx_2 \]

where

\[ G_1^c = \left[ \mu_2^c b_2^c u_2^c(t) \right] \sqrt{2g \left( z_2 \left( l_2, t \right) - z_2 \left( l_1, t \right) \right)} - Q_i^*(t) \]

\[ G_2^c = \left[ \mu_2^c b_2^c u_2^c(t) \right] \sqrt{2g \left( z_2 \left( l_2, t \right) - z_2 \left( l_1, t \right) \right)} - Q_i^*(t) \]

\[ G_i^c = \left[ z_i \left( x, T \right) - z_i^* \left( x \right) \right]^2, \quad i = 3, 4, \ldots \]

\( Q_i^*(t) \) — set water flow rates to side water intakes \( z_i^* \left( x, T \right) \) — set water levels in the channel sections.

Under optimal water supply conditions, the control functions include the Delta functions in x and unit functions in time t, for example,

\[ u_i(t) = \sum_{k=1}^{K} u_{ik} l(t - t_{k1}), \quad u_2(t) = \sum_{k=1}^{K} u_{2k} l(t - t_{k2}) \]

\[ u_3(t) = \sum_{k=1}^{K} u_{3k} l(t - t_{k3}), \quad u_4(t) = \sum_{k=1}^{K} u_{4k} l(t - t_{k4}) \]

where \( u_{ik} \) — discrete values of control actions at time instants \( t_{k1} \).

Introducing Related Functions \( \lambda_i^c(x, t) i = 1, 2, \ldots \), based on the methods of the calculus of variations and Lagrange multipliers \([4, 11]\), without giving intermediate transformations, the explicit form of the expression for the variation of the functional in the vicinity of the optimal trajectory is as follows.
\[
\delta I = \int_0^T \left[ \left( \frac{1}{B_2} \right) \lambda_2^1 - \left( \frac{2Q_2}{\omega_2} \right) \lambda_2^2 \right] \delta u_2 (t) + \frac{\partial G^2}{\partial u_2 (t)} \delta u_2 (t) + \left[ \left( \frac{1}{B_1} \right) \lambda_1^1 - \left( \frac{2Q_1}{\omega_1} \right) \lambda_1^2 \right] \delta u_1 (t) + \frac{\partial G^1}{\partial u_1 (t)} \delta u_1 (t) + \int_{x=0}^{x=x_2} \left[ \left( \frac{\partial g_i}{\partial u_1 (t)} \right) \delta u_1 (t) + \left( \frac{\partial g_i}{\partial u_2 (t)} \right) \delta u_2 (t) \right] dt.
\]

(13)

Conjugate variables \( \lambda_1^1, \lambda_1^2, \lambda_2^1, \lambda_2^2 \) satisfy the following differential equations [5, 6]

\[
\frac{\partial \lambda_i^1}{\partial t} = \left( \frac{B_i}{B_0} \frac{\partial Q_i}{\partial x_i} \right) \lambda_i^1 + \left( \frac{2Q_i}{\omega_i} \frac{\partial Q_i}{\partial x_i} \right) \lambda_i^2 - g_i \omega_i \left( \frac{1}{\omega_i c_i^2} \right) \lambda_i^2 - \left( \frac{g Q_i}{K_i} \frac{K_i}{\omega_i} \right) \lambda_i^2
\]

(14)

where

\[
Q_i = \frac{\partial Q_i}{\partial x_i}, \quad z_i = \frac{\partial z_i}{\partial x_i}, \quad B_i = \frac{\partial B_i}{\partial x_i}, \quad c_i = \frac{\partial c_i}{\partial x_i}, \quad K_i = \frac{\partial K_i}{\partial x_i}.
\]

Boundary conditions for conjugate variables

\[
\left[ \left( \frac{1}{B_1} \right) \lambda_1^1 + \left( \frac{2Q_1}{\omega_1} \right) \lambda_1^2 \right]_{x=0}^{x=x_2} \frac{\partial g_i}{\partial z_i (0,t)} + \left[ \frac{g \omega_i \left( 1 - \frac{Q_i^2}{\omega_i c_i^2} \right) \lambda_1^2}{\partial z_i} \right]_{x=0}^{x=x_2} = 0
\]

(15)
\[
\left( \frac{\partial G_2}{\partial z_2 (l_1, t)} \right)_o - \left[ g \omega_2 \left( 1 - \frac{Q_2^2}{\omega_2 c_2^2} \right) \right] \lambda_2^2 \left( \frac{\partial g_2}{\partial z_2 (l_1, t)} \right)_o = 0
\]

(16)

Pairing conditions

\[
\begin{align*}
\left( \frac{\partial G_c}{\partial z_c (l_1, t)} \right)_o - \left[ g \omega_i \left( 1 - \frac{Q_i^2}{\omega_i c_i^2} \right) \right] \lambda_i^2 \left( \frac{\partial g_c}{\partial z_c (l_1, t)} \right)_o = 0 \\
- \left[ \left( \frac{1}{B_1} \right) \lambda_1^2 \left( \frac{2Q_1}{\omega_1} \right) \left( \frac{\partial g_c^c}{\partial z_c (l_1, t)} \right)_o \right]_{x_1 = l} \\
+ \left[ \left( \frac{1}{B_2} \right) \lambda_2^2 \left( \frac{2Q_2}{\omega_2} \right) \left( \frac{\partial g_c^c}{\partial z_c (l_1, t)} \right)_o \right]_{x_2 = l} = 0
\end{align*}
\]

(17)

\[
\begin{align*}
- \left[ g \omega_2 \left( 1 - \frac{Q_2^2}{\omega_2 c_2^2} \right) \right] \lambda_2^2 \left( \frac{\partial g_2}{\partial z_2 (l_1, t)} \right)_o = 0 \\
- \left[ \left( \frac{1}{B_1} \right) \lambda_1^2 \left( \frac{2Q_1}{\omega_1} \right) \left( \frac{\partial g_2}{\partial z_2 (l_1, t)} \right)_o \right]_{x_1 = l} \\
- \left[ \left( \frac{1}{B_2} \right) \lambda_2^2 \left( \frac{2Q_2}{\omega_2} \right) \left( \frac{\partial g_2}{\partial z_2 (l_1, t)} \right)_o \right]_{x_2 = l} = 0
\end{align*}
\]

(18)

and conditions at the end of the management process

\[
\lambda_2^1 (x_1, T) = 0, \quad \lambda_3^1 (x_2, T) = 0, \\
\lambda_2^2 (x_1, T) = -2 \left[ z_1 (x_1, T) - z^* (x_1) \right], \quad \lambda_3^2 (x_2, T) = -2 \left[ z_2 (x_2, T) - z^* (x_2) \right].
\]

(19)

According to the variation of the optimal control criterion (12), the necessary optimality conditions for control within the constraint region are

\[
\frac{\partial \delta I_j}{\partial u_j (t)} = \left[ \frac{1}{B_1} \right] \lambda_1^2 \left( \frac{2Q_1}{\omega_1} \right) \lambda_i^2 \left( \frac{\partial g_1}{\partial u_j (t)} \right)_o = 0
\]

(20)

\[
\frac{\partial \delta I_j}{\partial u_2 (t)} = \left[ \frac{1}{B_2} \right] \lambda_2^2 \left( \frac{2Q_2}{\omega_2} \right) \lambda_i^2 \left( \frac{\partial g_2}{\partial u_2 (t)} \right)_o = 0
\]

(21)
\[
\frac{\partial \delta I}{\partial \delta u_i(t)}(t) = \left( \frac{\partial G}{\partial u_i(t)} \right)_o - \left[ \left( \frac{1}{B_i} \right)_o \delta_i^1 - \left( \frac{2Q_i}{\omega_i} \right)_o \delta_i^2 \right]_{t \to 0} \left( \frac{\partial g_i^c}{\partial u_i(t)} \right)_o = 0
\]  \(22\)

\[
\frac{\partial \delta I}{\partial \delta u_i(t)}(t) = \left[ \left( \frac{1}{B_i} \right)_o \delta_i^1 + \left( \frac{2Q_i}{\omega_i} \right)_o \delta_i^2 \right]_{x_i \to \delta} \left( \frac{\partial g_i^c}{\partial u_i(t)} \right)_o = 0
\]  \(23\)

In the gradient method, the control sequence has the form [7]
\[
\begin{align*}
    u_i^{m+1}(t) &= u_i^m(t) - \alpha_m \delta u_i^m(t) \\
    u_i^{n+1}(t) &= u_i^n(t) - \alpha_m \delta u_i^n(t) \\
    u_i^{cm+1}(t) &= u_i^{cm}(t) - \alpha_m \delta u_i^{cm}(t) \\
    u_i^{cm+1}(t) &= u_i^{cm}(t) - \alpha_m \delta u_i^{cm}(t)
\end{align*}
\]  \(24\)

Here
\[
\begin{align*}
    \delta u_i^m(t) &= \left[ \left( \frac{1}{B_i} \right)_o \delta_i^1 - \left( \frac{2Q_i}{\omega_i} \right)_o \delta_i^2 \right]_{x_i \to \delta} \left( \frac{\partial g_i^c}{\partial u_i(t)} \right)_o \\
    \delta u_i^n(t) &= \left[ \left( \frac{1}{B_i} \right)_o \delta_i^1 - \left( \frac{2Q_i}{\omega_i} \right)_o \delta_i^2 \right]_{x_i \to \delta} \left( \frac{\partial g_i^c}{\partial u_i(t)} \right)_o \\
    \delta u_i^{cm}(t) &= \left( \frac{\partial G_i}{\partial u_i(t)} \right)_o - \left[ \left( \frac{1}{B_i} \right)_o \delta_i^1 - \left( \frac{2Q_i}{\omega_i} \right)_o \delta_i^2 \right]_{x_i \to \delta} \left( \frac{\partial g_i^c}{\partial u_i(t)} \right)_o \\
    \delta u_i^{cm}(t) &= \left[ \left( \frac{1}{B_i} \right)_o \delta_i^1 + \left( \frac{2Q_i}{\omega_i} \right)_o \delta_i^2 \right]_{x_i \to \delta} \left( \frac{\partial g_i^c}{\partial u_i(t)} \right)_o
\end{align*}
\]  \(25\) \(26\) \(27\) \(28\)

Here \(\alpha_m\) is selected from the interval \(0 < \varepsilon_o \leq \alpha_m \leq 2/(L + 2\varepsilon)\).

In the case of applying the gradient projection method, the control sequence is as follows [8]
\[
\begin{align*}
    u_2^{m+1}(t) &= u_2^m(t) - \alpha_m \delta u_2^m(t) < u_2^{min} \\
    u_2^{m+1}(t) &= u_2^m(t) - \alpha_m \delta u_2^m(t) \quad \text{npu} \quad u_2^{min} \leq u_2^m(t) - \alpha_m \delta u_2^m(t) \leq u_2^{max} \\
    u_2^{m+1}(t) &= u_2^m(t) - \alpha_m \delta u_2^m(t) > u_2^{max} \\
    u_1^{cm+1}(t) &= u_1^{cm}(t) - \alpha_m \delta u_1^{cm}(t) < u_1^{cm} \\
    u_1^{cm+1}(t) &= u_1^{cm}(t) - \alpha_m \delta u_1^{cm}(t) \quad \text{npu} \quad u_1^{cm} \leq u_1^{cm}(t) - \alpha_m \delta u_1^{cm}(t) \leq u_1^{cm} \\
    u_1^{cm+1}(t) &= u_1^{cm}(t) - \alpha_m \delta u_1^{cm}(t) > u_1^{cm}
\end{align*}
\]  \(29\) \(30\)


$$u^{m+1}_2(0) = \begin{cases} 
  u^{m+1}_2 - \alpha_m \delta u^{m}_2(t) < u^{m+1}_2, \\
  u^{m+1}_2 - \alpha_m \delta u^{m}_2(t) \leq u^{m+1}_2, \\
  u^{m+1}_2 > u^{m+1}_2.
\end{cases}$$

Parameter $\alpha_m$ is selected from the interval $0 < \varepsilon_0 < \alpha_m \leq 2/(L + 2\varepsilon)$. Similarly, for the optimal water distribution problem, taking into account the optimal water supply in the channels, one can determine the expression for other methods, for example, the conditional gradient method, the method of possible directions, the conjugate direction method and the Newton method using the full Saint-Venant model of unsteady water movement in the channels of irrigation systems.

3. Results and discussion

Based on the analysis of the methods of the optimal distribution of water between consumers in the conditions of optimum water supply, based on the obtained necessary conditions of optimality, the gradient projection method is the most acceptable.

According to the method of projecting the gradient of the optimal distribution of water between consumers in the conditions of optimal water supply, the improving approximation of control actions for simplified models is determined as follows [9].

For the kinematic wave model

$$u^{m+1}(x,t) = \begin{cases} 
  u^{m}(x,t) - \alpha_m (\lambda)_0 u^{m}(x,t) < u^{m}(x,t), \\
  u^{m}(x,t) - \alpha_m (\lambda)_0 u^{m}(x,t) \leq u^{m}(x,t), \\
  u^{m}(x,t) > u^{m}(x,t).
\end{cases}$$

$$\delta v^{m+1}_1(0) = \begin{cases} 
  \delta v^{m}_1(0) - \alpha_m \lambda(0,t) < \delta v^{m}_1(0), \\
  \delta v^{m}_1(0) - \alpha_m \lambda(0,t) \leq \delta v^{m}_1(0), \\
  \delta v^{m}_1(0) > \delta v^{m}_1(0).
\end{cases}$$

Here $\lambda(x,t)$ – the conjugate variable determined by the solution of the conjugate boundary value problem for the kinematic wave model.

It should be noted that the boundary value problem for the main variables for the complete model is solved in direct time, and the boundary value problem for the conjugate variables is solved in the opposite. These boundary value problems are solved by numerical methods, for example, finite-difference or finite elements [6, 7]. With a good choice of the initial condition and initial controls, the proposed algorithm quickly converges. The initial controls are selected from the solution of the optimal water distribution problem under optimal water supply between consumers obtained using simplified models of unsteady water movement in the channel sections [1], and the initial conditions are determined from the solution of the steady water movement in the channel sections.

The optimal distribution of water in irrigation systems under optimal conditions of water supply is formulated as the problem of optimal control of systems with distributed parameters or as the problem of optimization of systems with distributed parameters [4, 11].

The main optimization methods used in this paper are methods based on the necessary optimality conditions, which are based on determining the gradient or the first derivative of the variation of the control of the functions in question, which are the selected optimality criteria.
Below, we analyze methods that use the necessary optimality conditions for the optimal distribution of water in irrigation systems under optimal water supply conditions.

The gradient minimization method is used for an approximate solution to the problem [10]

\[ I \left( u_m(t) \right) \Rightarrow \min; \quad u_m(t) \in H, \text{ where } H \text{ is a Hilbert space based on the construction of a minimizing sequence, which can be written as} \]

\[ u_{m+1}(t) = u_m(t) - \alpha_m I'(u_m(t)), 0 < t < T, m = 0, 1, 2, \ldots \quad (34) \]

here \( u_0(t) \) – some predetermined initial value of the control action, \( \alpha_m \) – positive value, \( I' \) – functional gradient calculated from a known value \( u_m(t) \). If \( I'(u_m(t)) \neq 0 \), then \( \alpha_m \) – can be chosen so that \( I(u_{m+1}(t)) < I(u_{m+1}(t)) \).

From the condition of double differentiability of the optimality criterion, we have [13, 14]

\[ I(u_{m+1}(t)) - I(u_{m+1}(t)) = \alpha_m \left( -\frac{1}{2} \left\| I'(u_m(t)) \right\|^2 + o(\alpha_m) \right) < 0 \quad (35) \]

For all sufficiently small \( \alpha_m > 0 \), if \( I'(u_m(t)) \neq 0 \), then process (34) stops when the given accuracy of the problem is satisfied and, if necessary, additional studies are conducted of the behavior of the function in the vicinity of the control \( u_m(t) \) to determine whether \( u_m(t) \) it will belong to the optimal control or not. Since for optimal water distribution problems in irrigation systems under optimal water supply conditions, all optimality criteria are convex functions, therefore, the obtained control is close to optimal control.

There are various ways to select a value \( \alpha_m \). We list some of them.

1) \( \alpha_m \) selected from the condition [15, 16]

\[ f_m(\alpha_m) = \inf_{\alpha \geq 0} f_m(\alpha_m) = f_m^*, \quad f_m(\alpha_m) = I(u_m(t) - \alpha_m I'(u_m(t))) \quad (36) \]

This version of the gradient method is commonly called the steepest descent method.

An exact determination of \( \alpha_m \) from (36) is not always possible; therefore, in practice, instead of (36), it uses the condition [17]

\[ f_m^* \leq f_m(\alpha_m) = f_m^* + \delta_m, \quad \delta_m \geq 0, \quad \sum_{t=1}^{\infty} \delta_m \leq \infty \quad (36) \]

or

\[ f_m^* \leq (1 - \lambda_m) f_m(0) + \lambda_m f_m^*, \quad 0 < \lambda \leq \lambda_m \leq 1 \quad (37) \]

The values \( \lambda_m, \delta_m \) here characterize the error in fulfilling the condition (35).

2) assume \( \alpha_m = \alpha > 0 \), then check the condition of monotony \( I(u_{m+1}) < I(u_m) \), and, if necessary, split the value of \( \alpha \), achieving the fulfillment of the condition of monotony.

3) sometimes \( \alpha_m \) determines from the conditions

\[ 0 < \epsilon_0 \leq \alpha_m \leq 2/(L + 2\epsilon) \quad (38) \]

where \( L \) – Lipchitz constant, \( \epsilon, \epsilon_m \) – positive numbers, which are the parameters of the method.

4) possible selection \( \alpha_m \) from the condition
to determine such \( \alpha_m \), \( \alpha_m = \alpha \) is usually set and then it is crushed until the above inequality holds.
5) possible a priori assignment of \( \alpha_m \) values from conditions [18]

\[
\alpha_m > 0, \quad m = 0, 1, 2, \ldots, \quad \sum_{m=0}^\alpha \alpha_m = 0, \quad \sum_{m=0}^\alpha \alpha_m^2 < \infty.
\]

For example, you can take \( \alpha_m = c(k+1)^b \), \( c = const > 0 \), \( \frac{1}{2} < b < 1 \). Such a choice \( \alpha_m \) is simple to implement, but does not guarantee the conditions for the fulfillment of monotonicity and converges slowly.
6) in cases where the value is known in advance \( I_* = \inf I(u_m) > -\infty \), can be accepted [19]

\[
\alpha_m = I(u_m(t)) - I_* / \|I'(u_m(t))\|^2,
\]

In practice, for iterative methods, iterations continue until a condition is met, then there is a criterion for the end of the count. Here it is possible to use such criteria for ending the account as

\[
\|u_{m+1}(t) - u_m(t)\| \leq \varepsilon, \quad u, \quad \|I(u_m(t)) - I(u_{m+1}(t))\| \leq \delta, \quad u, \quad \|I'(u_m(t))\| \leq \gamma,
\]

where \( \varepsilon, \delta, \) and \( \gamma \) – given numbers. Sometimes the number of iterations is specified.
The gradient projection method is used to approximate the solution of problem (1) in the case of imposing a constraint on the control functions, that is [11]

\[
u_m(t) \in U
\]

where \( U \) – convex closed set from Hilbert space \( H \).
The minimizing sequence of solving the optimization problem based on the gradient projection method is constructed according to the rule

\[
u_{m+1}(t) = P_U \left( u_m(t) - \alpha_m I'(u_m(t)) \right), \quad t_0 \leq t \leq T, \quad m = 0, 1, 2, \ldots,
\]

where \( P_U \) – the operator of design in the field of management.
The projection of \( u_m(t) \) onto the \( U \) set is defined as follows

\[
\|u_m(t) - \omega_m(t)\| = \inf \|u_m(t) - \nu_m(t)\|
\]

Expression (45) means that if the calculated control value belongs to the \( U \) set, then the value of the calculated control is taken. If the calculated control value does not belong to the \( U \) set, then close values of the \( U \) set are taken. This method is convenient when there is a formula for the projection of the control onto the set of restrictions \( U \) [20, 21, 22, 23, 24].
The optimal control of unsteady water movement in the channels of irrigation systems was tested on the example of the section of the South Golodnostepsky main channel (SGMC), which is located in the Syrdarya and Jizzakh regions in the north-east of the Republic of Uzbekistan.
The hydraulic and morphometric parameters of the SGMC site are as follows: water flow in the channel section \( Q_0 = 42 \text{m}^3/\text{s} \); depth of water flow in this section of the channel \( H_0 = 3.67 \text{m} \); gravitational constant \( g = 9.8 \text{m/s}^2 \); Shezy coefficient \( \gamma = 1/6 \); channel length \( l = 11.2 \text{ km} \); channel
bottom slope $i = 0.00001$; water flow width at the bottom of the live section of the channel $B_0 = 26.18$m; water flow rate $u_0 = 0.9 \text{m}^2/\text{s}$; coefficient of efficiency $CE = 0.9$.

![Figure 2](image.png)

**Figure 2.** Change in water levels over time and along the length of the main channel section.

From Figure 2 it can be seen that after the gates are opened, the increased flow rate at the beginning of the channel section makes it possible to increase the water level along the length of the specified section of SGMC. During $t = 12,444 \text{ s (20.7 min.)}$ The water level at the end of the section increases by 1.8 m.

The obtained results of numerical and field experiments show that the water level at the end of the channel section is stabilizing, which is necessary for the water intakes from the SGMC located there. A comparison of the results of numerical experiments and field studies carried out in this section of the SGMC shows that the parameters of the water level in them differ slightly, their error is no more than 3-5%.

### 4. Conclusions

Algorithms and a software package have been developed for solving problems of optimal control of the distribution of water in the channels of irrigation systems under conditions of discreteness of water supply to consumers. The developed optimal control of the unsteady water movement in the channels of irrigation systems was tested on the example of the SGMC site, the results of which show that the opening of the gates of the SGMC site increases the flow rate at the beginning of the SGMC site, which allows increasing the water level along the length of the SGMC site. During $t = 20.7$ minutes the water level at the end of the plot is increased by 1.8 meters.

A comparison of the results of numerical and full-scale studies shows that the water level parameters in the SGMC site differ slightly.

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