Knowledge by Direct Measurement versus Inference from Steering

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If Alice and Bob start out with an entangled state $|\Psi\rangle_{AB}$, Bob may update his state to $|\varphi\rangle$ either by performing a suitable measurement himself, or by receiving the information that a measurement by Alice has steered that state. While Bob’s update on his state is identical, his update on Alice’s state differs: if Bob has performed the measurement, he has steered the state $|\chi_{\rightarrow}(\varphi)\rangle_A$ of Alice; if Alice has made the measurement, to steer $|\varphi\rangle_B$ on Bob she must have found a different state $|\chi_{\leftarrow}(\varphi)\rangle_A$. Based on this observation, a consequence of the well-known “Hardy’s ladder”, we show that information from direct measurement must trump inference from steering. The erroneous belief that both paths should lead to identical conclusions can be traced to the usual prejudice that measurements should reveal a pre-existing state of affairs. We also prove a technical result on Hardy’s ladder: the minimum overlap between the steered and the steering state is $2^{\sqrt{p_0 p_{n-1}}/(p_0 + p_{n-1})}$, where $p_0$ and $p_{n-1}$ are the smallest (non-zero) and the largest Schmidt coefficients of $|\Psi\rangle_{AB}$.

I. MEASUREMENT AND STEERING IN A BIPARTITE SETTING

A. A question

A source produces the state $|\Psi\rangle_{AB}$ and sends one subsystem to Alice’s lab, the other to Bob’s lab. Bob leaves his assistants to take care of the task and goes to his office to perform some administrative duty. Shortly later, Bobby comes from the lab to inform Bob that, in some given rounds of the experiment, the state in their lab had been updated to $|\varphi\rangle_B$. Those particles, that have been kept in a quantum memory, are ready for use in subsequent tasks. Can Bob update also his knowledge of Alice’s state in those same rounds?

B. Formalisation

We write the initial state as

$$|\Psi\rangle_{AB} = \sum_k \sqrt{p_k} |k\rangle_A \otimes |k\rangle_B. \quad (1)$$

and Bob’s subsequent update as

$$|\varphi\rangle_B = \sum_k \beta_k |k\rangle_B. \quad (2)$$

The probability $P_\beta = \sum_k p_k |\beta_k|^2$ of Bob finding this state is strictly positive, unless $\beta_k \neq 0$ only for indices $k$ such that $p_k = 0$.

We consider now two ways in which Bob’s update may have come about. Suppose first that the measurement was done in Bob’s lab. It is a measurement in a basis that comprises $|\varphi\rangle$ and which, in that particular round, happened to yield that result. In this situation, Bob updates Alice’s state to be the $\varphi$-steered state

$$|\chi_{\rightarrow}(\varphi)\rangle_A = \sum_k \beta_k \sqrt{p_k} \sqrt{P_\beta} |k\rangle_A. \quad (3)$$

As second case, suppose that the measurement was done in Alice’s lab and Alice has informed Bobby that she has steered his state to $|\varphi\rangle_B$. This means that, in that round, Alice’s measurement had yielded the $\varphi$-steering state

$$|\chi_{\rightarrow}(\varphi)\rangle_A = \sum_k \alpha_k |k\rangle \text{ with } \alpha_k \sqrt{p_k} \sqrt{P_\alpha} = \beta_k \quad (4)$$

where $P_\alpha = \sum_k p_k |\alpha_k|^2$ is the probability of that outcome.

At this point, it is obvious for anyone familiar with Hardy’s ladder \[1\] that the two states $|\chi_{\rightarrow}(\varphi)\rangle_A$ and $|\chi_{\leftarrow}(\varphi)\rangle_A$ are generally different. Indeed,

$$\langle\chi_{\rightarrow}(\varphi)|\chi_{\leftarrow}(\varphi)\rangle = \frac{P_\alpha}{P_\beta} = \frac{\sum_k p_k |\alpha_k|^2}{(\sum_k p_k^2 |\alpha_k|^2)^{1/2}} \quad (5)$$

is equal to 1 only in either of two cases: first, if all the $p_k$ are equal, i.e. if $|\Psi\rangle_{AB}$ is maximally entangled; second, if $\alpha_k = \delta_{k,k'}$ for a given $k'$, which implies also $\beta_k = \delta_{k,k'}$, and means that the measurement (be it done by Alice or by Bob) is made in the Schmidt basis.

Also, the $\varphi$-steered and the $\varphi$-steering states are never orthogonal for a given $|\Psi\rangle_{AB}$. To the best of our knowledge, the minimum of the scalar product \[6\] was never reported for states of arbitrary dimensions. Using techniques from convex fractional optimisation (Appendix A), we find

$$\min_{\chi_{\rightarrow}(\varphi)} \langle\chi_{\rightarrow}(\varphi)|\chi_{\leftarrow}(\varphi)\rangle = \frac{2\sqrt{p_0 p_{n-1}}}{p_0 + p_{n-1}} \quad (6)$$

where $p_0$ and $p_{n-1}$ are, respectively, the smallest and the largest non-zero Schmidt coefficients of $|\Psi\rangle_{AB}$. If $p_0$ and $p_{n-1}$ are not degenerate, the minimum is achieved for

$$|\varphi\rangle_B = \frac{1}{\sqrt{2}} \left( |0\rangle + e^{i\lambda} |n - 1\rangle \right) \quad (7)$$

with arbitrary $\lambda \in \mathbb{R}$. This state steers

$$|\chi_{\rightarrow}(\varphi)\rangle_A = \sqrt{\frac{p_0}{p_0 + p_{n-1}}} |0\rangle + e^{-i\lambda} \sqrt{\frac{p_{n-1}}{p_0 + p_{n-1}}} |n - 1\rangle$$
and is steered by

\[ |\chi\rightarrow(\varphi)\rangle_A = \sqrt{\frac{p_{n-1}}{p_0 + p_{n-1}}} |0\rangle + e^{-i\lambda} \sqrt{\frac{p_0}{p_0 + p_{n-1}}} |n - 1\rangle. \]

II. MEASUREMENT TRUMPS STEERING

A. Narrative of the previous observation

Bob knows that, in the \( n \) rounds under consideration, the state in his lab was \( |\varphi\rangle_B \). Because of no-signaling, he can’t know what Alice has done, if anything. At this stage, he can update his knowledge of Alice’s state to the steered state \( |\chi\rightarrow(\varphi)\rangle_A \) in the sense that whatever Alice reports later won’t be of zero probability given that state, the fact that \( \square \) is never zero being a special instance of this. However, if later Alice informs Bob that the measurement was done in her lab, and that Bob’s knowledge came from Alice updating him [?], then Bob must update Alice’s state to the steered state \( |\chi\rightarrow(\varphi)\rangle_A \). This update will lead to more accurate predictions.

For a classical mind, there is something troubling in what we have written. After all, we have allowed Alice to perform one out of only two operations: either do nothing, or measure and tell Bob what is the state she steered on his side. That these two operations don’t lead to the same state of knowledge means that Alice’s measurement creates a state of affairs that could not have been known to Bob in advance (and that’s why he has to update again his knowledge). This is certainly counter-intuitive but not exactly new: it has been a tenet of quantum theory since the early days and was conclusively demonstrated by Bell’s theorem.

B. A variation, the same message

The point may be reinforced by considering another set of rounds and possibly other measurements. We look at what happens when Bob’s information comes from both sources: \( |\varphi\rangle_B \) was found as a result of measurements in his lab, and he is informed by Alice that she has steered his state to a possibly different state \( |\varphi'\rangle_B \). Bob’s should then update his knowledge to the outcomes of the two measurements, namely \( |\varphi\rangle_B \) for himself and \( |\chi\rightarrow(\varphi')\rangle_A \) for Alice. To see it, consider one possible chronology (thanks to no-signaling, timing does not matter). Bob measures first and gets \( |\varphi\rangle_B \): he infers that he must have steered \( |\chi\rightarrow(\varphi)\rangle_A \) on Alice’s side. Alice’s message later informs him that she has made her own measurement and found \( |\chi\rightarrow(\varphi')\rangle_A \). The only states \( |\varphi'\rangle_B \) for which the story is impossible on Alice’s side are those such that \( \langle\chi\rightarrow(\varphi)|\varphi'\rangle = 0 \). But \( \langle\chi\rightarrow(\varphi)|\chi\rightarrow(\varphi')\rangle = \sum_k \beta_k^* \beta'_{k'} \), the impossible states are those that are orthogonal to \( |\varphi\rangle_B \), i.e. those for which the story is impossible on Bob’s side too.

C. What happens if steering trumps measurement

Failing to give direct measurement priority over inferences from steering may lead to absurd situations, the following échange de politesses being an extreme one. Bob measures his system and finds \( |\psi_0\rangle = \sum_k b_k |k\rangle \). He then informs Alice that he has steered her state to \( |\psi_1\rangle \propto \sum_k c_k b_k |k\rangle \), where for simplicity of notation we denote \( c_k = \sqrt{p_k} \), assume \( a_k \in \mathbb{R} \), and omit normalisation. So far so good; but now Alice replies back as if she had performed the measurement: if she has the state \( |\psi_1\rangle \), then by steering Bob must have \( |\psi_2\rangle \propto \sum_k c_k^2 b_k |k\rangle \). Bob accepts Alice’s inference on his system, then believes that he has done the measurement and infers Alice that her state must be \( |\psi_3\rangle \propto \sum_k c_k^2 b_k |k\rangle \); and so on. As soon as \( b_k \neq \delta_{k,k'} \), the iteration’s convergence is dominated by the largest Schmidt coefficient \( p_{\text{max}} \) of \( |\Psi\rangle_{AB} \): both the even (Bob’s) and the odd (Alice’s) sequence converge to \( |\psi_\infty\rangle = \sum_k b_k |k\rangle \) where the sum is on the indices \( k \) such that \( p_k = p_{\text{max}} \). The fact that Alice and Bob converge to an agreement may be desirable for peace but not for knowledge, since \( |\psi_\infty\rangle \) has nothing to do with what they actually had in their labs (unless the initial state is maximally entangled, in which case all the \( |\psi_m\rangle \) are equal).

D. Relation to Frauchiger & Renner’s Thought Experiment

The tension between updates from steering and updates from measurements may be detected in the argument put forward by Frauchiger and Renner (FR) \[2\], which is indeed a discussion of knowledge and certainty regarding measurements via the inference of various agents about each other’s states. In this discussion, we mention some notations of that paper without explaining all of them.

In the FR thought experiment, the structure of the quantum state is

\[ |\Psi\rangle_{AB} = \sqrt{\frac{1}{3}} \left( |0\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \right) \]

where \( |0\rangle_A = |\text{tails}\rangle_R \otimes |\bar{R}\rangle_L \), \( |1\rangle_A = |\text{heads}\rangle_R \otimes |\bar{R}\rangle_L \), \( |0\rangle_B = |\uparrow\rangle_S \otimes |\bar{R}\rangle_L \) and \( |1\rangle_B = |\downarrow\rangle_S \otimes |\bar{R}\rangle_L \).

The instances of steering and inference in question involve the time steps denoted \( n : 11 \rightarrow n : 01 \) and \( n : 01 \rightarrow n : 31 \) in Table 3 of the paper. Bob’s measurement updated his state to \( |1\rangle_B \): through steering, he infers that Alice’s state is \( |0\rangle_A \). When Alice is informed of this, by steering she would infer that Bob holds \( \sqrt{3/2} (|0\rangle_B + |1\rangle_B) \). Suppose she communicates her inference to Bob, and Bob buys this update rather than keeping the knowledge coming from his own measurement. Then, the paper’s reasoning regarding the other two agents follow, namely: if \( \bar{w} = \text{ok} \) then it is certain that \( w = \text{fail} \); which translates to \( P(\bar{w} = \text{ok} | w = \text{ok}) = 0 \), against the quantum prediction \( P(\bar{w} = \text{ok} | w = \text{ok}) = 1/2 \). In other words, the
FR argument exploits the minimal version of the échange de politesse discussed before, where the dialogue stops at $|\psi_2\rangle$.

Also, as we argued above, the equivalence between updating from measurement and updating from steering could be assumed if measurement just reveals a pre-existing state of affairs. Thus, the similarity between the FR argument and Hardy’s paradox in nonlocality may not be a mere mathematical incident, but likely stems from the same prejudice.

III. CONCLUSIONS

We have shown that information from measurement must trump that from steering when updating an agent’s knowledge on another agent’s state. If this rule is not followed, paradoxical situation may appear. The fact that this rule is not trivial originates from the same prejudice that leads to formulating the local hidden variable assumption, namely, that measurements should just reveal a pre-existing state of affairs.

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Appendix A: Minimisation of the scalar product

In this appendix, for simplicity of notation we assume $\alpha_k \in \mathbb{R}$, without loss of generality. Also, the notation $x^*$ indicates the solution of an optimisation, rather than complex conjugation as in the main text.

Theorem 1. Let $\mathbf{p} \in \mathbb{R}^n$ be a probability distribution such that $0 < p_k \leq p_{k+1}$ for any $k$. The following non-convex fractional optimization problem:

$$\mathbf{a}^* := \arg\min_{\mathbf{a} \in \mathbb{R}^n} \frac{\sum_{k} p_k a_k^2}{\sum_{k} p_k^2 a_k^2},$$

(A1)

is solved by any $\mathbf{a}^*$ such that

$$\sum_{k \in K_{\min}} (\alpha_k^*)^2 = \frac{p_{n-1}}{p_0 + p_{n-1}},$$

$$\sum_{k \in K_{\max}} (\alpha_k^*)^2 = \frac{p_0}{p_0 + p_{n-1}},$$

where $K_{\min}$ and $K_{\max}$ are the sets of indexes $k$’s such that $p_k < p_j$, $\forall k \in K_{\max}$, $\forall j \notin K_{\min}$, $p_k > p_j$, $\forall k \in K_{\min}$, $\forall j \notin K_{\max}$, and $\alpha_k^* = 0$ for any other $k$. In particular, if $|K_{\min}| = |K_{\max}| = 1$, one has

$$\alpha_0^* = \pm \sqrt{\frac{p_{n-1}}{p_0 + p_{n-1}}},$$

$$\alpha_{n-1}^* = \pm \sqrt{\frac{p_0}{p_0 + p_{n-1}}}.$$

The figure of merit in Eq. (A1) evaluates to

$$\frac{\sum_{k} p_k (\alpha_k^*)^2}{\sqrt{\sum_{k} p_k^2 (\alpha_k^*)^2}} = 2 \frac{\sqrt{p_0 p_{n-1}}}{p_0 + p_{n-1}}.$$

Proof. First, notice that for any set $\mathcal{K}$ of indexes $k$’s such that $p_k = p_j$ for any $k, j \in \mathcal{K}$, by direct computation one has that the figure of merit in Eq. (A1) evaluates the same for any $\mathbf{a}$ and $\mathbf{b}$ such that $\alpha_k = \beta_k$ for any $k \notin \mathcal{K}$, and

$$\sum_{k \in \mathcal{K}} (\alpha_k^*)^2 = \sum_{k \in \mathcal{K}} (\beta_k^*)^2.$$

Hence, without loss of generality, we assume $p_k < p_{k+1}$ for any $k$.

The remaining of the proof is lengthy and will be split into the following three lemmas.

Lemma 1. Let $\mathbf{p} \in \mathbb{R}^n$ be a probability distribution such that $0 < p_k < p_{k+1}$ for any $k$. The optimization problem in Eq. (A1) is equivalent to the following optimization problem, linear in $\mathbf{a}$:

$$(\mathbf{a}^*, s^*) := \arg\min_{\mathbf{a} \geq 0, s > 0, \sum_{k} a_k = 1} s \sum_{k} p_k a_k, \quad (A2)$$

where $\alpha_k = \pm \sqrt{a_k}$ for any $k$.

Proof. Equation (A1) is an instance of fractional programming. The numerator and the denominator of the figure of merit are convex functions, however the constraint is not a convex set. Hence, Eq. (A1) is not an instance of convex fractional programming. However, Eq. (A1) can be recast as a convex fractional programming by means of the following substitution. By setting $\alpha_k := \alpha_k^2$ for any $k$ one has that $\alpha_k^* = \pm \sqrt{a_k^*}$ for any $k$, where

$$\mathbf{a}^* := \arg\min_{\mathbf{a} \geq 0} \frac{\sum_{k} p_k a_k^2}{\sum_{k} p_k^2 a_k}, \quad (A3)$$

Equation (A3) is now an instance of convex fractional programming. A subset of convex fractional programming for which special results hold (see Case 1 at the end of page 3 of Ref. [4] or Proposition 7 of Ref. [5]) is the case in which the denominator of the figure of merit is affine, which is not the case in Eq. (A3). However, this can be amended by another simple transformation.
Since the figure of merit is non-negative on the domain of optimization, taking its square one has
\[ a^* := \arg\min_{a \geq 0} \left( \frac{\sum_b p_b a_b}{\sum_b p_b^2 a_b} \right)^2. \] (A4)

Notice that the numerator and the denominator of the figure of merit are a convex and a linear function, respectively, and that the optimization is over a convex set. Hence, Eq. (A4) is an instance of convex fractional programming with affine denominator. It was shown (see Case 1 at the end of page 3 of Ref. [4] or Proposition 7 of Ref. [1]) that by setting \( a_k = b_k/t \) for any \( k \) one has \( a_k^* = b_k^*/t^* \) for any \( k \), where
\[ (b^*, t^*) := \arg\min_{b \geq 0, t > 0} t \left( \sum_k b_k \frac{b_k}{t} \right)^2. \] (A5)

Notice that Eq. (A5) represents an optimization problem convex in variable \( b \). Yet another simple transformation recasts Eq. (A5) as an optimization problem linear in \( t \)
\[ (b^*, t^*) := \arg\min_{b \geq 0, t > 0} \sqrt{t} \sum_k p_k b_k t. \] (A6)

Finally, to recast Eq. (A6) as the optimization problem in Eq. (A2), we set \( s := \sqrt{t} \) and substitute back \( a_k = b_k/t \) for any \( k \).

Lemma 2. The optimization problem in Eq. (A2) is equivalent to the following scalar optimization problem:
\[ (k_0^*, k_1^*, s^*, a^*) := \arg\min_{k_0, k_1, s \geq 0, a \geq 0} s \left( p_{k_0} a + p_{k_1} (1 - a) \right) \] (A7)
\[ a^* \text{ where } a_k^* = a^* \text{ and } a_{k_1}^* = 1 - a^* \text{, and } a_{k_0}^* = 0 \text{ for any } k \neq k_0^*, k_1^*. \]

Proof. Since \( s \) and \( p_k \)'s appear in terms of the same degree in the figure of merit and in the constraints in Eq. (A2), by setting \( q_k := s^* p_k \) for any \( k \), where \( s^* = \sqrt{t^*} \), one has
\[ a^* := \arg\min_{a \geq 0} \sum_k q_k a_k, \] (A8)
which is a linear optimization problem whose constraint is a polytope given by the intersection of the probability simplex with the hyperplane \( \sum_k q_k^2 a_k = 1 \).

The (possibly local) extrema of the figure of merit in Eq. (A8) under its constraints are either in the bulk of such a polytope or on its boundary, that is, when at least one of the elements of \( a \) is zero. In the former case, extrema can be found by looking for extrema when the inequality constraint \( a \geq 0 \) is relaxed, and selecting those that lie inside the polytope. In the latter case, extrema can be found by setting one element of \( a \) to zero, and proceeding as in the previous case. Proceeding recursively, one needs to look for extrema when any possible subset of the elements of \( a \) are set to zero. In the following, we use the technique of Lagrange multipliers to show that any such an extremum has at most two non-null elements.

By introducing Lagrange multipliers \( \lambda \) and \( \mu \), for any set \( K \) subset of the set of all possible indexes \( k \)'s, that is \( K \subseteq [0, n - 1] \), one can write the following auxiliary function
\[ L(K) := \sum_{k \in K} (q_k + \lambda + \mu q_k^2) a_k. \]

Hence, a necessary condition for \( a \) to be a (possibly local) extrema of the figure of merit in Eq. (A8) over its constraint is that
\[ \frac{\partial}{\partial a_k} L(K) = 0, \quad \forall k, \] (A9)
for at least one set \( K \subseteq [0, n - 1] \). By explicit computation one has
\[ \frac{\partial}{\partial a_k} L(K) = \begin{cases} (q_k + \lambda + \mu q_k^2), & \text{if } k \in K, \\ 0, & \text{otherwise}. \end{cases} \]

Since all \( q_k \)'s are different, the system in Eq. (A9) contains \( |K| \) linearly independent equations in variables \( \lambda \) and \( \mu \), hence such a system admits solutions if and only if \( |K| \leq 2 \). Hence the statement follows.

Lemma 3. The optimization problem in Eq. (A7) is solved by \( k_0^* = 0, k_1^* = n - 1 \), and
\[ a^* = \frac{p_{k_1}^*}{p_{k_0}^* + p_{k_1}^*}. \]

Proof. Let us first solve the problem in \( a \) for any given \( k_0, k_1, \) and \( s \). Form the constraint, by direct computation one has
\[ a^* = \frac{1 - s^2 p_{k_1}^2}{s^2 (p_{k_0}^2 - p_{k_1}^2)}. \] (A10)

Let us now solve the problem in \( s \) for any given \( k_0 \) and \( k_1 \). By substituting Eq. (A10) into Eq. (A7) one immediately has
\[ (k_{0}^*, k_{1}^*, s^*) := \arg\min_{k_{0}, k_{1}, s \geq 0} \frac{1}{p_{k_0} + p_{k_1}} \left( \frac{1}{s} + sp_{k_0} p_{k_1} \right). \] (A11)
By explicit computation, the figure of merit is a convex function in $s$ and is thus minimized in $s$ by taking the zero of its first derivative. Hence, by explicit computation one has

$$s^* = \frac{1}{\sqrt{p_k_0 p_k_1}}$$  \hspace{1cm} (A12)

Let us finally solve the problem in $k_0$ and $k_1$. Upon replacing Eq. (A12) into Eq. (A11) one has

$$(k_0^*, k_1^*) := \arg\min_{k_0, k_1} \frac{\sqrt{p_k_0 p_k_1}}{p_k_0 + p_k_1}.$$  \hspace{1cm} (A13)

Without loss of generality, let us take $k_0 < k_1$, and hence $p_k_0 < p_k_1$. Since $k_0$ and $k_1$ are discrete variables, one cannot directly apply optimization techniques based on differential methods. However, it follows by direct computation that, upon defining $r := p_k_0 / p_k_1$, the figure of merit in Eq. (A13) can be written as

$$\sqrt{p_k_0 p_k_1} \frac{1}{p_k_0 + p_k_1} = \frac{\sqrt{r}}{r + 1}.$$  \hspace{1cm} (A14)

Hence, the figure of merit in Eq. (A13) depends on $p_k_0$ and $p_k_1$ only through their ratio $r$.

We can now apply differential methods to variable $r$. By explicit computation, one has that the first derivative of Eq. (A14) in $r$ is positive in the range $0 \leq r < 1$ that we are considering since $p_k_0 < p_k_1$. Hence, the figure of merit in Eq. (A13) is monotonically increasing in $r$, and is thus minimized by the minimal $r$. By definition of $r$, this is achieved by $k_0^* = 0$ and $k_1^* = n - 1$. \hfill \Box

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