Quasiclassical QCD Pomeron

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Abstract

The Regge behaviour of the scattering amplitudes in perturbative QCD is governed in the generalized leading logarithmic approximation by the contribution of the color–singlet compound states of Reggeized gluons. The interaction between Reggeons is described by the effective hamiltonian, which in the multi–color limit turns out to be identical to the hamiltonian of the completely integrable one–dimensional XXX Heisenberg magnet of non-compact spin $s = 0$. The spectrum of the color singlet Reggeon compound states, – perturbative Pomerons and Odderons, is expressed by means of the Bethe Ansatz in terms of the fundamental $Q$–function, which satisfies the Baxter equation for the XXX Heisenberg magnet. The exact solution of the Baxter equation is known only in the simplest case of the compound state of two Reggeons, the BFKL Pomeron. For higher Reggeon states the method is developed which allows to find its general solution as an asymptotic series in powers of the inverse conformal weight of the Reggeon states. The quantization conditions for the conserved charges for interacting Reggeons are established and an agreement with the results of numerical solutions is observed. The asymptotic approximation of the energy of the Reggeon states is defined based on the properties of the asymptotic series, and the intercept of the three–Reggeon states, perturbative Odderon, is estimated.

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1. Introduction

Recently, a lot of attention has been renewed to the old problem of understanding QCD Pomeron. The interest was partially inspired by new exciting experimental results, which confirmed a growth of the structure function of deep inelastic scattering, $F_2(x,Q^2)$, at small values of the Bjorken variable $x$ and increasing with energy of the total hadronic cross-sections, $\sigma_{\text{tot}}(s)$. Both phenomena allow us to test QCD in the extreme limit of high energies and fixed transferred momenta, the limit in which the famous Regge model emerges [1]. Being one of the most popular subjects in 70’s, the Regge theory interprets the increasing of physical cross-sections with the energy by introducing the concept of the Regge poles. The Regge poles can be classified into different families according to their quantum numbers and their contribution to the amplitude of elastic scattering of hadrons $h_1$ and $h_2$ in the Regge limit, $s \gg -t$, has the form $A_{h_1h_2}(s, t) \sim s^{\alpha(t)}$, with $\alpha = \alpha(t)$ being universal Regge pole trajectory independent on particular choice of the scattered hadrons [1].

Among all possible families of the Regge poles there is a special one, with the quantum numbers of vacuum, the so-called Pomeron. The Pomeron provides the leading contribution to the higher energy behaviour of the total hadronic cross-section [1]

$$\sigma_{h_1h_2}^{\text{tot}}(s) = s^{-1} \text{Im} A_{h_1h_2}(s, t = 0) \sim s^{\alpha_{\text{IP}}(0)}$$

with $\alpha_{\text{IP}}(0)$ being the Pomeron intercept. Replacing one of incoming hadrons by virtual photon with invariant mass $-Q^2$ we may find a similar expression for the asymptotics of the structure function of deep inelastic scattering $\gamma^*(Q) + h \to X$ in the Regge limit, $x = Q^2/s \ll 1$ and $Q^2 = \text{fixed}$. There is however an important difference between these two processes from point of view of underlying QCD dynamics. The Regge behavior of the total hadronic cross-sections is essentially nonperturbative phenomenon and it is governed by the “soft” QCD Pomeron [2] with the intercept $\alpha_{\text{IP}}(0) \approx 1.08$. At the same time, the deep inelastic scattering can be analyzed for large enough $Q^2$ within the framework of perturbative QCD by using the operator product expansion, or factorization, and the small $-x$ asymptotics of the structure function is controlled by the “hard” Pomeron [2], $F_2(x,Q^2) \sim x^{1-\alpha_{\text{IP}}(0)}$, with larger intercept $\alpha_{\text{IP}}(0) \approx 1.38$.

Although the Regge model is extremely successful in describing a wealth of different data [1, 2] it is still unclear whether it can be justified from the first principles of QCD. Since we do not have a complete understanding of nonperturbative QCD we have to restrict ourselves to processes involving “hard” Pomeron. As an example of such processes, one may consider the scattering of two onium states each consisting of a heavy quark-antiquark pair [3, 4]. The first attempts to describe the “hard” Pomeron in QCD revealed the remarkable property of gluon reggeization [3, 5]. It was found from the analysis of the Feynman diagram in the leading logarithmic approximation (LLA), $\alpha_s \ln s \sim 1$ and $\alpha_s \ll 1$, that interacting with each other gluons form a new collective excitation, Reggeon, which according to its contribution to the scattering amplitude can be interpreted as a Regge pole with quantum numbers of a gluon. As a result, QCD can be replaced in the Regge limit by an effective field theory [6] in which Reggeons play a role of new elementary fields while Pomerons appear as compound states of interacting Reggeons. The simplest example of such state built from only two Reggeons, Balitsky–Fadin–Kuraev–Lipatov Pomeron, was found many years ago [3] and its intercept was calculated in the LLA as

$$\alpha_{\text{BFKL}} = 1 + \frac{\alpha_s \cdot N}{\pi} 4 \ln 2$$

(1.1)
where $\alpha_s$ is the QCD coupling constant and $N$ is the number of quark colors. Apart from the BFKL Pomeron there is an infinite number of another states with vacuum quantum numbers, which are built from an arbitrary number of Reggeons $n > 2$.

To identify higher $n$ Reggeon compound states we have to go beyond the LLA in calculating the scattering amplitudes. One possible way of doing this has been proposed in [7, 8] and is called the generalized leading logarithmic approximation. In this approximation, the amplitude of the onium-onium scattering, $A(s, t)$, is given in the Regge limit by the sum of diagrams describing the propagation of an arbitrary number $n$ of interacting Reggeons in the $t$–channel

$$A(s, t) = \sum_{n=2}^\infty \alpha_s^{n-2} A_n(s, t).$$

The Reggeon interaction has interesting properties in the generalized LLA. If one introduces a fictitious time in the $t$–channel, then at the same moment of “time” the interaction occurs only between two Reggeons (pair–wise interaction) [7, 8, 9]. It conserves the number of Reggeons (elastic scattering) but changes color and two-dimensional transverse momenta of Reggeons. As a result, the diagrams contributing to the amplitude $A_n(s, t)$ have a conserved number $n$ of Reggeons in the $t$–channel and have a form of generalized ladder diagrams [7, 8, 9]. These diagrams can be effectively resumed using the Bethe–Salpeter approach [8, 9] and the resulting expression for the scattering amplitude $A_n(s, t)$ can be represented as a sum of contributions of the $n$ Reggeon compound states propagating in the $t$–channel

$$A_n(s, t) = is \sum_{\{q\}} s^{E_{n,\{q\}}} \beta_{n \rightarrow h_1}^{\{q\}}(t) \beta_{n \rightarrow h_2}^{\{q\}}(t).$$

Here, $E_{n,\{q\}}$ is the energy of the $n$ Reggeon compound state $|\chi_{n,\{q\}}\rangle$, which satisfies the $(2+1)$–dimensional Schrodinger equation, the so–called Bartels–Kwiecinski–Praszalowicz equation [8, 9]

$$\mathcal{H}_n|\chi_{n,\{q\}}\rangle = E_{n,\{q\}}|\chi_{n,\{q\}}\rangle$$

with $\{q\}$ being the set of quantum numbers parameterizing all possible solutions. The hamiltonian $\mathcal{H}_n$ corresponds to the elastic pair–wise interaction of $n$ Reggeons. It acts on 2–dimensional transverse momenta of Reggeons and describes the evolution of the $n$ Reggeon compound state $|\chi_{n,\{q\}}\rangle$ in the $t$–channel. The residue factors in (1.3) measure the coupling of the Reggeon compound states to the hadronic state $|\Phi_h\rangle$ and they are defined as

$$\beta_{n \rightarrow h}^{\{q\}}(t) = \langle \Phi_h(t)|\chi_{n,\{q\}}\rangle.$$

For short–distance hadronic states like onium $\beta_{n \rightarrow h}^{\{q\}}(t)$ can be calculated in perturbative QCD [3, 8, 4]. Expression (1.3) for the scattering amplitude is consistent with the Regge theory predictions provided that we interpret the Regge poles as the $n$ Reggeon compound states and identify the intercept as their maximal energy

$$\alpha(t) - 1 = \max_{\{q\}} E_{n,\{q\}}.$$

Combining relations (1.2) and (1.3) together we obtain the Regge asymptotics of the scattering amplitude in the generalized LLA as an infinite sum over all possible $n$ Reggeon compound states propagating in the $t$–channel. For $n = 2$ the first term in the sum (1.2) corresponds to
the BFKL Pomeron. The next term, \( n = 3 \), is associated with three Reggeon compound states which belong to the Odderon family of the Regge poles [10]. Although they have a zero color charge, in contrast with the BFKL Pomeron their charge conjugation is negative, \( C = -1 \). As a result, they cannot couple to the hadronic states with \( C = 1 \) like virtual photon in the deep inelastic scattering, but for the hadronic states with \( C = -1 \) like proton they contribute to the total cross sections \( \sigma_{pp}^{tot} \) and \( \sigma_{pp}^{tot} \). Moreover, their contribution is responsible for the increasing with the energy of the difference [10], \( \Delta \sigma = \sigma_{pp}^{tot} - \sigma_{pp}^{tot} \). The Regge behaviour of \( \Delta \sigma \) is controlled by the intercept of the Odderon [11, 12], which despite of a lot of efforts is unknown yet.

As it follows from (1.2), the contributions of the \( n \) Reggeon states to the scattering amplitude is suppressed by powers of \( \alpha_s \) with respect to that of the BFKL Pomeron. Nevertheless, they have to be taken into account in order to preserve unitarity of the \( S \)–matrix of QCD [7, 8]. To derive the higher Reggeon compound states we have to solve the \((2+1)\)–Schrodinger equation (1.4) for \( n \) interacting Reggeons. For \( n = 2 \) it was done in [3], but for \( n \geq 3 \) the problem becomes extremely complicated partially due to interaction between color charges of Reggeons. The latter interaction can be drastically simplified by taking the multi–color limit [13], \( N \to \infty \) and \( \alpha_s N = \text{fixed} \), and passing by means of Fourier transformation from two-dimensional transverse momenta of Reggeons, \( k_\perp = (k_1, k_2) \), to two-dimensional impact parameter space, \( b_\perp = (b_1, b_2) \).

After these transformations, the Reggeon hamiltonian takes the following form [14]

\[
\mathcal{H}_n = \frac{\alpha_s N}{4\pi} \left( H_n + \bar{H}_n \right) + \mathcal{O}(N^{-2}).
\]  

(1.7)

The hamiltonians \( H_n \) and \( \bar{H}_n \) act on holomorphic and antiholomorphic coordinates of the Reggeons in the impact parameter space, \( z = b_1 + i b_2 \) and \( \bar{z} = b_1 - i b_2 \), respectively, and describe nearest–neighbour interaction between \( n \) Reggeons

\[
H_n = \sum_{k=1}^{n} H(z_k, z_{k+1}), \quad \bar{H}_n = \sum_{k=1}^{n} H(\bar{z}_k, \bar{z}_{k+1}), \quad [H_n, \bar{H}_n] = 0,
\]

where \( z_k \) and \( \bar{z}_k \) are holomorphic and antiholomorphic coordinates of the \( k \)–th Reggeon and periodic boundary conditions \( z_{n+1} = z_1 \) and \( \bar{z}_{n+1} = \bar{z}_1 \) are imposed. Here, \( H(z_1, z_2) + H(\bar{z}_1, \bar{z}_2) \) is the interaction hamiltonian of two Reggeons with coordinates \((z_1, \bar{z}_1)\) and \((z_2, \bar{z}_2)\) in the impact parameter space, the so–called BFKL kernel [5, 14],

\[
H(z_1, z_2) = -\psi(-J_{12}) - \psi(1 + J_{12}) + 2\psi(1)
\]  

(1.8)

where \( \psi(x) = \frac{d}{dx} \ln \Gamma(x) \) and the operator \( J_{12} \) is defined as a solution of the equation

\[
J_{12}(1 + J_{12}) = -(z_1 - z_2)^2 \partial_1 \partial_2
\]

with \( \partial_k = \partial/\partial z_k \). The expression for \( H(\bar{z}_1, \bar{z}_2) \) is similar to (1.8). Thus, in the multi–color limit, the 2–dimensional Reggeon hamiltonian \( \mathcal{H}_n \) describing the pair–wise interaction of \( n \) Reggeons turns out to be equivalent to the sum (1.7) of two 1–dimensional mutually commuting hamiltonians \( H_n \) and \( \bar{H}_n \). This allows us to reduce the original \((2+1)\)–dimensional problem (1.4) to the system of two \((1+1)\)–dimensional Schrodinger equations [14]

\[
H_n |\varphi_{n,\{q\}}\rangle = \varepsilon_{n,\{q\}} |\varphi_{n,\{q\}}\rangle, \quad \bar{H}_n |\bar{\varphi}_{n,\{q\}}\rangle = \bar{\varepsilon}_{n,\{q\}} |\bar{\varphi}_{n,\{q\}}\rangle
\]  

(1.9)
where the wave functions $\varphi_{n,\{q\}}$ and $\bar{\varphi}_{n,\{q\}}$ depend only on holomorphic and antiholomorphic coordinates of the Reggeons, respectively. Then, in the multi–color limit the spectrum of the $n$ Reggeon states can be found as

$$E_{n,\{q\}} = \frac{\alpha_s N}{4\pi} \left( \epsilon_{n,\{q\}} + \bar{\epsilon}_{n,\{q\}} \right),$$

$$|\chi_{n,\{q\}}\rangle = |\varphi_{n,\{q\}}\rangle |\bar{\varphi}_{n,\{q\}}\rangle.$$

Although equations (1.9) are much simpler than the original Schrödinger equation (1.4) it is not obvious that they can be solved exactly for an arbitrary number of Reggeons. A significant progress has been achieved recently [15, 16] after it was realized that the system (1.9) of $(1 + 1)$–dimensional Schrödinger equations has interesting interpretation in terms of one–dimensional lattice models [17].

Let us consider the one-dimensional lattice with periodic boundary conditions and the number of sites, $n$, equal to the number of Reggeons. We parameterize $n$ sites by holomorphic coordinates $z_k$ ($k = 1, ..., n$) and introduce the interaction between nearest neighbors on the lattice with the holomorphic Reggeon Hamiltonian $H_n$, (1.8). Thus defined “holomorphic” lattice model is described by the same Schrödinger equation as in (1.9) and it obeys the following remarkable property [15, 16]. It turns out to be equivalent to the celebrated XXX Heisenberg magnet of spin $s = 0$ corresponding to the principal series representation of the noncompact conformal $SL(2,\mathbb{C})$ group. The same is true, of course, for the antiholomorphic Schrödinger equation in (1.9). Therefore, the $n$ Reggeon compound states in multi–color QCD share all their properties with the eigenstates of the XXX Heisenberg magnet defined on the periodic one–dimensional lattice with $n$ sites and their intercept (1.6) is closely related to the ground state energy of the magnet.\footnote{Indeed, from point of view of lattice models it is more natural to change a sign of the Reggeon Hamiltonian, $H_n$, and interpret $(-\max E_n)$ in (1.6) as a ground state energy of the XXX magnet.} This result opens a possibility to apply the powerful methods of exactly solvable models [18, 19, 20, 21] to the solution of the Regge problem in QCD. The first step has been undertaken in [15, 22] where the generalized Bethe ansatz has been developed for diagonalization of the Reggeon Hamiltonians. The expressions for the $n$ Reggeon wave functions and their corresponding energies were found in terms of the fundamental $Q$–function which satisfies the Baxter equation for the XXX Heisenberg magnet of spin $s = -1$. The solution of the Baxter equation was found in the simplest case of $n = 2$ Reggeon state, or BFKL Pomeron. For higher Reggeon states the problem becomes much more complicated and it is still open [23]. In the present paper we continue the study of the Baxter equation initiated in [15, 22] and develop a method which allows us to find its general solutions in the form of asymptotic expansion similar to the well–known quasiclassical approximation in quantum mechanics.

The paper is organized as follows. In Section 2 we summarize the Bethe ansatz solution for $n$ Reggeon compound states and introduce the Baxter equation. We interpret the Baxter equation as a discrete one–dimensional Schrodinger equation and identify inverse conformal weight of the Reggeon states as a small parameter which plays a role of the Planck constant. The quasiclassical expansion of the solution of the Baxter equation in powers of this parameter is performed in Section 3. Similar to the situation in quantum mechanics it leads to the asymptotic series for the energy of Reggeon states whose properties are studied in Section 4. The asymptotic expansions for the Reggeon quantum numbers are derived in Section 5. In Section 6 we use the obtained results to estimate the energy of higher Reggeon states. Summary and concluding remarks are given in Section 7. The analytical properties of the energy of the Reggeon states are considered in
Appendix A. Relation between Reggeon states and conformal operators is discussed in Appendix B. Some useful properties of the asymptotic series are described in Appendix C.

2. Generalized Bethe Ansatz

The fact that the system of Schrodinger equations (1.9) is completely integrable implies that there exists a family of “hidden” holomorphic and antiholomorphic conserved charges, \{\(q\)\} and \{\(\bar{q}\)\}, which commute with the Reggeon Hamiltonian and among themselves. Their explicit form can be found using the quantum inverse scattering method as \[16, 15\]

\[
q_k = \sum_{n \geq i_1 > i_2 > \ldots > i_k \geq 1} i^k z_{i_1 i_2} z_{i_2 i_3} \ldots z_{i_k i_1} \partial_{i_1} \partial_{i_2} \ldots \partial_{i_k} \tag{2.1}
\]

with \(z_{ij} \equiv z_i - z_j\) and the expression for \(\bar{q}_k\) is similar. The appearance of these operators is closely related to the invariance of the Reggeon Hamiltonian (1.7) under conformal \(SL(2, \mathbb{C})\) transformations \[5\]

\[
z \to \frac{az + b}{cz + d}, \quad \bar{z} \to \frac{a\bar{z} + \bar{b}}{c\bar{z} + d} \tag{2.2}
\]

where \(ac - bd = \bar{a}c - \bar{b}d = 1\). Indeed, we recognize \(q_2\) and \(\bar{q}_2\) as the quadratic Casimir operators of the \(SL(2, \mathbb{C})\) group while the remaining conserved charges \(\{q_k, \bar{q}_k\}, k = 3, \ldots, n\) can be interpreted as higher Casimir operators. The Reggeon compound states belong to the principal series representation of the \(SL(2, \mathbb{C})\) group and under the conformal transformations (2.2) they are transformed as quasiprimary fields with conformal weights \((h, \bar{h})\) \[24\].

The \(n\) Reggeon states diagonalize the operators \(\{q, \bar{q}\}\) and the eigenvalues of the conserved charges \(q_2, q_3, \ldots, q_n\) play a role of their additional quantum numbers. In particular, the eigenvalues of the quadratic Casimir operators are related to the conformal weights of the \(n\) Reggeon state as

\[
q_2 = -h(h - 1), \quad \bar{q}_2 = -\bar{h}(\bar{h} - 1), \quad \bar{q}_2 = q_2^*.
\]

where the possible values of \(h\) and \(\bar{h}\) can be parameterized by integer \(m\) and real \(\nu\)

\[
h = \frac{1 + m}{2} + i\nu, \quad \bar{h} = \frac{1 - m}{2} + i\nu, \quad m = \mathbb{Z}, \quad \nu = \mathbb{R}. \tag{2.3}
\]

As to remaining charges, their possible values also become quantized. The explicit form of the corresponding quantization conditions is more complicated and will be discussed in Sect. 5.

To find the explicit form of the eigenstates and eigenvalues of the \(n\) Reggeon compound states corresponding to a given set of quantum numbers \(\{q, \bar{q}\}\) we apply the generalized Bethe ansatz developed in \[15, 22\]. The Bethe ansatz for Reggeon states in multi–color QCD is based on the solution of the Baxter equation

\[
\Lambda(\lambda)Q(\lambda) = (\lambda + i)^n Q(\lambda + i) + (\lambda - i)^n Q(\lambda - i). \tag{2.4}
\]

Here, \(Q(\lambda)\) is a real function of the spectral parameter \(\lambda\), \(\Lambda(\lambda)\) is the eigenvalue of the so–called auxiliary transfer matrix for the XXX Heisenberg magnet of spin \(s = 0\)

\[
\Lambda(\lambda) = 2\lambda^n + q_2\lambda^{n-2} + \ldots + q_n \tag{2.5}
\]
and $n$ is the number of Reggeized gluons or, equivalently, the number of sites of the one-dimensional spin chain. For a fixed $n$ it is convenient to introduce the function

$$\tilde{Q}(\lambda) = \lambda^n Q(\lambda)$$

and rewrite the Baxter equation (2.4) as

$$\tilde{Q}(\lambda + i) + \tilde{Q}(\lambda - i) - 2\tilde{Q}(\lambda) = \left( -\frac{h(h-1)}{\lambda^2} + \frac{q_3}{\lambda^3} + \ldots + \frac{q_n}{\lambda^n} \right) \tilde{Q}(\lambda).$$

(2.7)

Once we know the function $\tilde{Q}(\lambda)$, the energy $E_n$ of the $n$-Reggeon compound state can be evaluated using the relation

$$E_n = \frac{\alpha_s N}{2\pi} \text{Re} \varepsilon_n(h, q_3, \ldots, q_n)$$

(2.8)

where the holomorphic energy $\varepsilon_n$ is defined as

$$\varepsilon_n(h, q_3, \ldots, q_n) = i \frac{d}{d\lambda} \log \frac{\tilde{Q}(\lambda + i)}{\tilde{Q}(\lambda - i)} \bigg|_{\lambda=0}.$$  

(2.9)

The expression for the wave function of the $n$ Reggeon states in terms of the function $\tilde{Q}$ can be found in [15, 22] and it will not be discussed in the present paper.

The Baxter equation (2.7) has the following properties [22]. We notice that for $q_n = 0$ it is effectively reduced to a similar equation for the states with $n-1$ Reggeized gluons. The corresponding solution, $Q(\lambda)$, gives rise to the degenerate unnormalizable $n$ Reggeon states with the energy

$$\varepsilon_n(h, q_3, \ldots, q_{n-1}, 0) = \varepsilon_{n-1}(h, q_3, \ldots, q_{n-1}).$$

(2.10)

These states should be excluded from the spectrum of the $n$ Reggeon hamiltonian and solving the Baxter equation for $n$ Reggeized gluons we have to satisfy the condition

$$q_n \neq 0.$$  

(2.11)

As a function of the quantum numbers, the holomorphic energy obeys the relations

$$\varepsilon_n(h, q_3, \ldots, q_n) = \varepsilon_n(1-h, q_3, \ldots, q_n) = \varepsilon_n(h, -q_3, \ldots, (-)^n q_n),$$

(2.12)

which follow from the symmetry of the Baxter equation (2.7) under the replacement $h \rightarrow 1-h$ or $\lambda \rightarrow -\lambda$ and $q_k \rightarrow (-)^k q_k$. This relation means that the spectrum of the Reggeon hamiltonian is degenerate with respect to quantum numbers $h, q_3, \ldots, q_n$. Then, assuming that the ground state of the XXX Heisenberg magnet of spin $s = 0$ is not degenerate we can identify the quantum numbers corresponding to the maximal value of the Reggeon energy as [22]

$$h \bigg|_{\text{max}} = \frac{1}{2}.$$  

(2.13)

and for the states with only even number of Reggeons, $n = 2k$,

$$q_3 \bigg|_{\text{max}} = q_5 \bigg|_{\text{max}} = \ldots = q_{2k-1} \bigg|_{\text{max}} = 0.$$  

7
For the states with odd number of Reggeons the latter condition is not consistent with (2.11).

We notice that the conformal weight $h$ enters as a parameter into the Baxter equation (2.4) and, in general, one is interesting to find the solutions of (2.4) only for its special values (2.3). Moreover, since the equation (2.7) is invariant under the replacement $h \to 1 - h$ we may restrict ourselves to the region
\[ \text{Re } h \geq 1/2, \tag{2.14} \]
or equivalently $m \geq 0$ in (2.3).

Our strategy in solving the Baxter equation will be the following [15, 22]. We will first try to solve (2.4) for integer positive values of the conformal weight $h = \mathbb{Z}_+$ and then analytically continue the resulting expression for the energy (2.9) to all possible values (2.3) including the most interesting one (2.13). It is clear that this procedure is ambiguous since for any integer $Z$ one could multiply the energy by a factor $\exp(i\pi Z h)$. However, according to the Carlson’s theorem [1], the holomorphic energy is uniquely defined in the region (2.3) by its values at the integer positive values of the conformal weight $h$ provided that the function $\varepsilon_n$ is regular in (2.3) and its asymptotics at infinity is $O(e^{c|h|})$ with $c < \pi$. As we show in Appendix A, these two conditions are indeed satisfied.

The Baxter equation (2.4) has two linear independent solutions and in order to select only one of them we impose the additional condition on the function $Q(\lambda)$ for $h = \mathbb{Z}_+$
\[ Q(\lambda) \xrightarrow{\lambda \to \infty} \lambda^{h-n}. \tag{2.15} \]

For integer positive conformal weight, $h \geq n$, the solution, $Q(\lambda)$, of the Baxter equation (2.4) under the additional condition (2.15) is given by a polynomial of degree $h - n$ in the spectral parameter $\lambda$. This implies in turn that $\tilde{Q}(\lambda)$ is a polynomial of degree $n$ in $\lambda$, which can be expressed in terms of its roots $\lambda_1, \ldots, \lambda_n$ as follows [15, 22]
\[ \tilde{Q}(\lambda) = \prod_{i=1}^{h} (\lambda - \lambda_i) = \exp \left( \sum_{i=1}^{h} \ln(\lambda - \lambda_i) \right) \xrightarrow{\lambda \to \infty} \lambda^{h}. \tag{2.16} \]

Substituting (2.16) into (2.7) and putting $\lambda = \lambda_k$ we obtain that the roots satisfy the Bethe equation for the XXX spin chain of spin $s = 0$
\[ 1 = \prod_{j \neq k}^{n} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \quad k = 1, \ldots, n. \tag{2.17} \]

There is, however, one additional important condition on the roots, which follows from the definition (2.6). Comparing (2.16) with (2.6) we find that $\lambda_1, \ldots, \lambda_{h-n}$ are roots of $Q(\lambda)$ and $\lambda_{h-n+1} = \ldots = \lambda_h = 0$. This means that among all the solutions of the Bethe equation (2.17) we have to select only those which have the $n-$time degenerate root $\lambda_k = 0$.

The explicit solution of the Baxter equation (2.4) is known only for $n = 2$ [15, 22]
\[ Q_{n=2}(\lambda; h) = i^h h(1 - h) \binom{1 + h, 2 - h, 1 - i\lambda}{2, 2, 1} \tag{2.18} \]
and it can be identified as a continuous symmetric Hahn orthogonal polynomial [25]. Substituting the solution (2.18) into (2.9) we obtain the holomorphic energy of $n = 2$ Reggeon states as [22]
\[ \varepsilon_2(h) = -4 [\psi(h) - \psi(1)] \tag{2.19} \]
where \( \psi \)-function was defined in (1.8). We substitute \( \varepsilon_2(h) \) into (2.8) and analytically continue the result from integer \( h \) to all possible complex values (2.3)

\[
E_2(h) = -2\frac{\alpha_sN}{\pi} \Re \left[ \psi\left(1 + \frac{|m|}{2} + i\nu \right) - \psi(1) \right].
\]

This relation coincides with the well-known expression [3] for the energy of the \( n = 2 \) Reggeon compound state, the BFKL Pomeron. The maximum value of the energy, \( E_2^{\text{max}} = \frac{\alpha_sN}{\pi} 4 \ln 2 \), is achieved at \( h = 1/2 \) and it is in agreement with (1.1), (1.6) and (2.13).

Solving the Baxter equation (2.4) for \( n \geq 3 \) one may try to look for the solution as a linear combination of the \( n = 2 \) solutions \([22]\),

\[
Q_n = \sum_k C_k(h, q_3, ..., q_n) Q_{n=2}(\lambda; k).
\]

Using the properties of \( Q_{n=2} \) as orthogonal polynomials we obtain that the coefficients \( C_k(h, q_3, ..., q_n) \) satisfy the multi-term recurrent relations. Most importantly, the quantum numbers \( q_3, ..., q_n \) corresponding to the polynomial solutions of the Baxter equation turned out to be quantized and their possible values as well as the values of the roots \( \lambda_k \) were found to be real \([22]\).

\[
\text{Im} \lambda_k = \text{Im} q_k = 0.
\] (2.20)

Although the explicit form of the recurrent relations for \( \{C_k\} \) is known \([22]\) and it is not difficult to solve them numerically for lowest values of the conformal weight \( h \) and then find the quantized charges \( \{q_k\} \) and the energy \( \varepsilon_n \), the analytical expression for the solution \( Q(\lambda) \) similar to (2.18) is still missing. At the same time, the results of numerical solution of the Baxter equation for \( n = 3 \) and \( n = 4 \) presented in [22] indicate that the analytical solution should exist. In this paper we will find such kind of solution for an arbitrary \( n \) in the special limit, \( h \gg 1 \). We will show that it is possible to develop an asymptotic expansion for the solutions of the Baxter equation, as well as for the energy of the \( n \) Reggeon states, in powers of \( 1/h \).

### 2.1. The origin of the quasiclassical approximation

Let us rescale in (2.7) the spectral parameter as \( \lambda \to \lambda h \) in order to get rid of a large factor \( h(h - 1) \) in the r.h.s. of the Baxter equation (2.7) as \( h \to \infty \). Then, we introduce new functions \( f(\lambda) \) and \( \Phi(\lambda) \)

\[
f(\lambda) = \exp(h\Phi(\lambda)) = e^{-h \ln h} \tilde{Q}(h\lambda),
\] (2.21)

where according to the definition (2.16)

\[
\Phi(\lambda) = \frac{1}{h} \sum_{i=1}^{n} \ln \left( \lambda - \frac{\lambda_i}{h} \right).
\] (2.22)

Substituting (2.21) into (2.9) we find the following expression for the holomorphic energy of the \( n \) Reggeon state in terms of the function \( \Phi(\lambda) \)

\[
\varepsilon_n = -i \left[ \Phi'\left(\frac{i}{h}\right) - \Phi'\left(-\frac{i}{h}\right) \right],
\] (2.23)

where prime denotes a derivative with respect to the spectral parameter \( \lambda \).

The main reason for considering the function \( \Phi(\lambda) \) is that in the limit \( h \to \infty \) we expect the scaling \( \lambda_i = \mathcal{O}(h) \) and \( \Phi(\lambda) \sim h^0 \), which allows us to expand \( \Phi(\lambda) \) in powers of \( 1/h \) as

\[
\Phi(\lambda) = \Phi_0(\lambda) + \frac{1}{h} \Phi_1(\lambda) + \frac{1}{h^2} \Phi_2(\lambda) + ... \] (2.24)
with all functions $\Phi_i(\lambda)$ being $h$–independent.

The substitution of the relation (2.21) into (2.7) yields the following equation for $f(\lambda)$

$$f\left(\lambda + \frac{i}{h}\right) + f\left(\lambda - \frac{i}{h}\right) - 2f(\lambda) = -V(\lambda)f(\lambda),$$

(2.25)

where the notation was introduced for the "potential"

$$V(\lambda) = \frac{1}{\lambda^2} \left(1 - \frac{1}{h}\right) - \frac{1}{\lambda^3 h^3} - \cdots - \frac{1}{\lambda^n h^n}.$$  

(2.26)

Taking a naive limit $h \to \infty$ in (2.25) we find that $f(\lambda)$ satisfies the one–dimensional Schrodinger equation, $-\frac{1}{2\hbar^2} \frac{d^2}{d\lambda^2} f = 2m(E - V(\lambda))f$, for the wave function of a particle with mass $m = 1/2$ and energy $E = 0$ moving in the singular potential $V(\lambda)$. In this equation $1/h$ plays a role of the Planck constant $\hbar$ and the relation (2.21) has a form of the WKB ansatz, $f(\lambda) = \exp\left(\frac{1}{h}\Phi(\lambda)\right)$.

This suggests that $h \to \infty$ limit of the Baxter equation (2.25) should be closely related to the quasiclassical approximation $\hbar \to 0$ in quantum mechanics.

We would like to notice that there is an intriguing relation between the Bethe Ansatz solution of our model and that of the one–dimensional Toda chain [26]. In both cases after the separation of variables one has to solve the Baxter equation which has the form of one–dimensional discretized Schrödinger equation (2.25) for a particle in an external potential. The only difference between models is in the form of the potential and in different boundary conditions which one imposes on the solution of the Baxter equation.

The condition for all terms in the expression for the potential (2.26) to have the same behaviour at large $h$ implies the following scaling of the holomorphic quantum numbers

$$\frac{q_3}{h^3} = \hat{q}_3 + \frac{1}{h} q_3^{(1)} + \frac{1}{h^2} q_3^{(2)} + \mathcal{O}\left(\frac{1}{h^3}\right), \quad \ldots, \quad \frac{q_n}{h^n} = \hat{q}_n + \frac{1}{h} q_n^{(1)} + \frac{1}{h^2} q_n^{(2)} + \mathcal{O}\left(\frac{1}{h^3}\right).$$

(2.27)

with $\hat{q}_3 \equiv q_3^{(0)}$, ..., $\hat{q}_n \equiv q_n^{(0)}$ and all coefficients $q_k^{(i)}$ being $h$–independent. This leads to the decomposition of the potential $V(\lambda)$ similar to that in (2.24)

$$V(\lambda) = V_0(\lambda) + \frac{1}{h} V_1(\lambda) + \frac{1}{h^2} V_2(\lambda) + \ldots$$

(2.28)

where

$$V_0(\lambda) = \lambda^{-2} - \hat{q}_3 \lambda^{-3} - \ldots - \hat{q}_n \lambda^{-n},$$

(2.29)

$$V_1(\lambda) = -\lambda^{-2} - q_3^{(1)} \lambda^{-3} - \ldots - q_n^{(1)} \lambda^{-n},$$

$$V_2(\lambda) = -q_3^{(2)} \lambda^{-3} - \ldots - q_n^{(2)} \lambda^{-n}, \quad \ldots$$

Finally, to solve (2.25) and find the functions $\Phi_i(\lambda)$ we have to substitute (2.24) into (2.21) and (2.25), take into account the decomposition (2.28) and equate the coefficients in front of different powers of $1/h$ in the both sides of the discrete Schrodinger equation (2.25).

### 3. Leading large–$h$ approximation

As the conformal weight $h$ increases the number of roots of the solution of the Baxter equation (2.16) and (2.22) also increases. According to (2.20), the roots are real and in the limit $h \to \infty$
it is convenient to introduce their distribution density on the real axis as

\[ \rho(\sigma) = \frac{1}{h} \sum_{i=1}^{h} \delta \left( \sigma - \frac{\lambda_i}{h} \right) \]  

(3.1)

with the normalization condition

\[ \int_{-\infty}^{\infty} d\sigma \ \rho(\sigma) = 1. \]  

(3.2)

In the large–\( h \) limit \( \rho(\sigma) \) becomes a smooth positive definite function of a real parameter \( \sigma \). It vanishes outside some (not necessary connected) finite interval \( \mathcal{S} \) on the real axis

\[ \rho(\sigma) \neq 0, \quad \text{for} \quad \sigma \in \mathcal{S} = [\sigma_1, \sigma_2] \cup [\sigma_3, \sigma_4] \cup \ldots \cup [\sigma_{2k-1}, \sigma_{2k}] \]  

(3.3)

with the number \( k \) depending on the “potential” \( V(\lambda) \), or equivalently on the quantum numbers \( q_3, \ldots, q_n \). The explicit form of \( \rho(\sigma) \) can be obtained from the Baxter equation (2.25) in the limit \( h \to \infty \) as follows. Using (2.22) and (3.1) we get

\[ \Phi(\lambda) = \int_{\mathcal{S}} d\sigma \ \rho(\sigma) \ln(\lambda - \sigma) \]

and after differentiation of the both sides with respect to \( \lambda \)

\[ \Phi'(\lambda) = \int_{\mathcal{S}} d\sigma \ \frac{\rho(\sigma)}{\lambda - \sigma} \lambda \to \infty \sim \frac{1}{\lambda}, \]  

(3.4)

where integration is performed along the finite interval (3.3). The last relation implies that \( \Phi'(\lambda) \) is an analytical function of the spectral parameter \( \lambda \) in the complex \( \lambda \)–plane with the cut along the finite interval \( \mathcal{S} \) and singularity at \( \lambda = \infty \). Taking discontinuity of \( \Phi'(\lambda) \) across the cut we can reconstruct the distribution density of roots as

\[ \rho(\sigma) = \frac{i}{2\pi} [\Phi'(\sigma + i0) - \Phi'(\sigma - i0)] . \]  

(3.5)

Similar to (2.24), the distribution density can be expanded in powers of \( 1/h \)

\[ \rho(\sigma) = \rho_0(\sigma) + \frac{1}{h} \rho_1(\sigma) + \frac{1}{h^2} \rho_2(\sigma) + \ldots. \]  

(3.6)

Here, the functions \( \rho_k(\sigma) \) are defined using (3.5) as a discontinuity of the functions \( \Phi_k(\lambda) \) and they satisfy the normalization conditions, \( \int_{\mathcal{S}} d\sigma \rho_0(\sigma) = 1 \) and \( \int_{\mathcal{S}} d\sigma \rho_k(\sigma) = 0 \) for \( k \geq 1 \), which follow from (3.2).

To find the solution of the Baxter equation (2.25) in the leading large–\( h \) limit we use the identity

\[ \frac{f(\lambda \pm i/h)}{f(\lambda)} \equiv \exp(h[\Phi(\lambda \pm i/h) - \Phi(\lambda)]) = \exp(\pm i\Phi_0'(\lambda) + \mathcal{O}(1/h)) \]  

(3.7)

and keep the leading \( 1/h \)–terms in the both side of (2.25) to get

\[ 4\sin^2 \left( \frac{1}{2} \Phi_0'(\lambda) \right) = V_0(\lambda). \]  

(3.8)
The equation has two different solutions for $\Phi'_0(\lambda)$ but only one of them satisfies the additional condition (3.4) as $\lambda \to \infty$,

$$\Phi'_0(\lambda) = -2i \ln \left( \sqrt{1 - \frac{1}{4}V_0(\lambda)} + i\sqrt{\frac{1}{4}V_0(\lambda)} \right). \quad (3.9)$$

Notice that $\Phi'_0(\lambda)$ can be defined from (3.8) up to a constant $4\pi Z Z$ which does not contribute, however, neither to the energy (2.23), nor to the distribution density (3.5).

Let us consider the properties of $\Phi'_0(\lambda)$ in the complex $\lambda$–plane. Taking into account that the potential $V_0(\lambda)$ is a real function of $\lambda$, we find from (3.9) that $\Phi'_0(\lambda)$ is not analytical in the regions of $\lambda$, in which the arguments of the square roots change their sign, that is for $V_0(\lambda) > 4$ or $V_0(\lambda) < 0$, and in compact form

$$|V_0(\lambda) - 2| > 2, \quad \lambda \in \mathcal{S}. \quad (3.10)$$

Solving this inequality we find the interval $\mathcal{S}$ on the real axis across which the function $\Phi'_0(\lambda)$ has a discontinuity and, as a consequence of (3.5), the distribution density of roots (3.3) is different from zero. The same result can be easily deduced from (3.8) as follows. In order to have roots on $\mathcal{S}$, the function $f(\lambda)$ should be oscillating on this interval or equivalently $\Phi_0(\lambda)$ should have an imaginary part for $\lambda \in \mathcal{S}$. If $\lambda$ is outside $\mathcal{S}$, then the function $\Phi_0(\lambda)$ is real and it follows from (3.8) that $0 \leq V_0(\lambda) \leq 4$.

Once we solved (3.10) and obtained the explicit form of $\mathcal{S}$, we substitute (3.9) into (3.5), evaluate the discontinuity of $\Phi'_0(\lambda)$ across the interval $\mathcal{S}$ and get the distribution density as

$$\rho_0(\sigma) = \frac{2}{\pi} \ln \left( \sqrt{1 - \frac{1}{4}V_0(\sigma)} + \sqrt{-\frac{1}{4}V_0(\sigma)} \right), \quad \text{for} \quad V_0(\sigma) < 0 \quad (3.11)$$

and

$$\rho_0(\sigma) = \frac{2}{\pi} \ln \left( \sqrt{\frac{1}{4}V_0(\sigma) - 1} + \sqrt{\frac{1}{4}V_0(\sigma)} \right), \quad \text{for} \quad V_0(\sigma) > 4. \quad (3.12)$$

These expressions define the distribution density of roots in the leading large $-h$ limit for an arbitrary number of Reggeons, $n \geq 2$.

### 3.1. Critical values of quantum numbers

Let us consider the special case $n = 2$, in which we can compare (2.21) and (3.9) with the exact solution (2.18) of the Baxter equation. For $n = 2$ we substitute $V_0(\lambda) = \lambda^{-2}$ into (3.10) and find the explicit form of $\mathcal{S}$ as

$$\rho_0(\sigma) \neq 0, \quad \text{for} \quad \mathcal{S} = [-\frac{1}{2}, \frac{1}{2}].$$

Then, it follows from the definition (3.1) that the zeros of $Q_{n=2}(\lambda; h)$ are located on the interval $-h/2 < \lambda_i < h/2$. Substituting $V_0(\lambda)$ into (3.12) we obtain the distribution density of zeros of the $n = 2$ solutions of the Baxter equation in the leading large $-h$ limit as

$$\rho_0(\sigma) = \frac{1}{\pi} \ln \frac{1 + \sqrt{1 - 4\sigma^2}}{1 - \sqrt{1 - 4\sigma^2}}, \quad \sigma \in \mathcal{S} = [-\frac{1}{2}, \frac{1}{2}]. \quad (3.13)$$
One can check that this expression satisfies the normalization condition (3.2). To test (3.13) we use the explicit expression (2.18) for $Q_{n=2}(\lambda; h)$ at $h = 50$, find numerical values of all 48 normalized roots, $\{\lambda_k/h\}$, and compare on fig. 1 the histogram of their distribution with (3.13).

Let us generalize our analysis and consider the possible solutions of (3.10) for $n = 3$ Reggeon states. For $n = 3$ the potential (2.29) is given by $V_0(\lambda) = \lambda^{-2} - \hat{q}_3 \lambda^{-3}$ and, depending on the value of the rescaled quantum number $\hat{q}_3$, there are two different solutions of (3.10) shown on fig. 2. For $\hat{q}_3^2 < (q_3^*)^2$ the equation $V_0(\lambda) = 4$ has three real roots $\lambda = \sigma_1, \sigma_3, \sigma_4$, while for

Figure 1: The histogram of the normalized roots of the solution of the $n = 2$ Baxter equation for $h = 50$ versus the distribution density $\rho(\sigma)$ in the leading large-$h$ limit.

Figure 2: Possible forms of the potential $V_0(\lambda)$ entering into $n = 3$ Baxter equation which lead to two different solutions of (3.10): (a) $\hat{q}_3 = 0.18 < q_3^*$ and (b) $\hat{q}_3 = 0.25 > q_3^*$. The shadow area represents the support $\mathcal{S}$. 

13
\( \hat{q}_3^2 \geq (q^*_3)^2 \) there is only one real root, \( \lambda = \sigma_1 \). The “critical” value of the quantum number

\[
(q^*_3)^2 = \frac{1}{27}
\]
can be found as a solution of the system of equations, \( V'_0(\lambda^*) = 0 \) and \( V_0(\lambda^*) = 4 \). Notice that in both cases, \( \hat{q}_3^2 < (q^*_3)^2 \) and \( \hat{q}_3^2 \geq (q^*_3)^2 \), the equation \( V_0(\lambda) = 0 \) has only one solution \( \sigma_2 = \hat{q}_3 \).

Then, taking for simplicity \( \hat{q}_3 > 0 \) we identify the finite support \( S \) for the distribution density of roots at \( n = 3 \) as

\[
S = [\sigma_1, \sigma_2] \cup [\sigma_3, \sigma_4], \quad \text{for} \quad 0 < \hat{q}_3 \leq \frac{1}{\sqrt{27}}
\]
with \( \sigma_1 < 0 < \sigma_2 = \hat{q}_3 < \sigma_3 < \frac{3}{2} \hat{q}_3 < \sigma_4 \) and

\[
S = [\sigma_1, \sigma_2], \quad \text{for} \quad \hat{q}_3 > \frac{1}{\sqrt{27}}
\]
with \( \sigma_1 < 0 < \sigma_2 = \hat{q}_3 \). The generalization of these relations to the case \( \hat{q}_3 < 0 \) is obvious.

The distribution densities corresponding to (3.14) and (3.15) are shown on fig. 3(a) and (b), respectively.

![Figure 3: Two different forms of the distribution density for the \( n = 3 \) Baxter equation: (a) \( \hat{q}_3 < q^*_3 \) and (b) \( \hat{q}_3 \geq q^*_3 \), corresponding to the potentials of fig. 2 (a) and (b), respectively.](image)

It is important to note that, in contrast with the previous case, for \( n = 3 \) there exists a “critical” value of the quantum number \( \hat{q}_3 = \pm \frac{1}{\sqrt{27}} \), at which the gap in the distribution of the roots vanishes and (3.14) is replaced by (3.15), or equivalently fig. 3(a) is replaced by fig. 3(b).

One might expect that this value separates two different “phases” of the Baxter equation and its solution should have different properties for \( q^2_3 \leq \frac{1}{27} \) and \( q^2_3 > \frac{1}{27} \). To show that this is really the case we recall that we are interesting in polynomial solutions of the Baxter equation (2.16) under the additional condition that they should have \( n \)-time degenerate root \( \lambda_k = 0 \). Till now we did not satisfy this condition and “good” solutions could appear in our analysis together with “unwanted” ones. In order to get rid of them we have to impose additional constraints on the solutions of the Baxter equation in the large \( h \) limit. Our assumption is that for \( n = 3 \) this can be achieved by restricting the possible values of the quantum number \( q_3 \) to lie in the region

\[
-\frac{1}{\sqrt{27}} \leq \hat{q}_3 \leq \frac{1}{\sqrt{27}}.
\]
This condition is equivalent to the statement that for the $n = 3$ Reggeon compound states the distribution density of roots, $\rho(\sigma)$, should have the support $S$ given by (3.14) and consisting of two connected intervals (see fig. 3(a)). To confirm our assumption (3.16) we present on fig. 4 the results (see eq. (6.46) and fig. 8 in [22]) of the numerical solution of the quantization conditions for $q_3/h^3$, corresponding to $n = 3$ and integer conformal weight $3 \leq h \leq 40$. We find that all numerically found quantized values of $q_3/h^3$ are in perfect agreement with (3.16). They lie inside the strip defined by (3.16) and for large $h$ their maximal/minimal values asymptotically approach the critical values $q_3/h^3 = \pm \frac{1}{\sqrt{27}}$. Due to the symmetry $q_3 \rightarrow -q_3$ of the Baxter equation, (2.12), the quantized $q_3$ are distributed symmetrically with respect to the axis $q_3 = 0$. Moreover, looking on fig. 4 it is difficult do not notice that the possible values of $q_3$ for different $h$ try to form a family of one-parametric curves in the two-dimensional $(h, q_3)$ plane which can be labeled by integer $k = 0, 1, \ldots$. An example of such a curve is shown on fig. 4(a). The curves start at the points with $h = k + 3$ and $q_3$ taking the absolute minimal value, cross the axis $q_3 = 0$ and for $h \to \infty$ they asymptotically approach the maximal value inside the strip (3.16). This suggests that all possible values of the quantum number $q_3$ can be parameterized in the following form

$$\frac{q_3}{h^3} = a_0(k) + \frac{1}{h}a_1(k) + \frac{1}{h^2}a_2(k) + O(h^{-3}), \quad k = 0, 1, \ldots$$

(3.17)

where $a_j(k)$ are some unknown coefficients depending on integer $k$. Comparing this ansatz with (2.27) we find that $a_0(k) = \hat{q}_3$ and all remaining coefficients $a_1(k), a_2(k), \ldots$ correspond to higher $1/h$-terms in the expansion (2.27). To test (3.17) we have to evaluate the nonleading $1/h$ corrections to the quantum number $q_3/h^3$ in (2.27). This will be done in Sect. 5.

Figure 4: The distribution of quantized $q_3$ for different values of the conformal weight $3 \leq h \leq 40$: (a) The results of the numerical solution of the $n = 3$ Baxter equation. The dashed line represents the upper and lower limits for $q_3$ in (3.16). The solid line indicates one of the curves to which quantized $q_3$ belong. (b) One-parametric families of curves defined in (5.15) and (5.16).

Let us generalize the condition (3.16) for the Baxter equation at $n > 3$. For an arbitrary $n$ the solutions of (3.10) have the following properties. Since $V_0(\lambda) \sim 1/\lambda^2 \rightarrow 0_+$ as $\lambda \rightarrow \infty$ the interval $S$ does not contain infinity and has a finite size. Then, the potential is singular at the
origin, \( V_0(\lambda) \sim -\hat{q}_n/\lambda^n \to \infty \) as \( \lambda \to 0 \), and, according to (3.10), the point \( \sigma = 0 \) is always inside \( \mathcal{S} \). In general, depending on the value of the quantum numbers \( \hat{q}_3, \ldots, \hat{q}_n \), the interval \( \mathcal{S} \) may consist of up to \( n - 1 \) connected intervals. We require that the number of connected intervals inside \( \mathcal{S} \) should have \( n - 2 \) and \( n \) real roots, respectively. The solution of these constraints leads to the quantization conditions on \( \hat{q}_3, \ldots, \hat{q}_n \) similar to (3.16). To satisfy them the quantized values of the conserved charges should lie inside the domain in the \( n - 2 \) dimensional space of all possible \( \hat{q}_3, \ldots, \hat{q}_n \). The boundary of the domain defines the “critical” \( n - 3 \) dimensional hypersurface

\[
\Sigma_n(q_3^*, \ldots, q_n^*) = 0
\]  

(3.18)

which separates different phases of the Baxter equation. For \( n = 3 \) we have \( \Sigma_3(q_3^*) = (q_3^*)^2 - 27 \) and the quantized \( q_3 \) belong to the interval (3.16). Let us find the explicit form of the quantization conditions for \( n = 4 \).

For \( n = 4 \) the potential is given by \( V_0(\lambda) = \lambda^{-2} - \hat{q}_3 \lambda^{-3} - \hat{q}_4 \lambda^{-4} \) and in order for the equation \( V_0(\lambda) = 0 \) to have two real roots we have to require that \( \hat{q}_4 \geq -\frac{1}{4} \hat{q}_3^2 \). Under this condition the function \( V_0(\lambda) \) has two extrema — two local maximums for \( \hat{q}_4 > 0 \) and one maximum and one minimum for \( \hat{q}_4 < 0 \). Their positions, \( \lambda = \lambda_{\text{max}} \) or \( \lambda = \lambda_{\text{min}} \), can be found as solutions of the equation \( \frac{dV_0}{d\lambda} = 0 \). Then, one can easily check that the second condition on \( V_0(\lambda) \), namely that \( V_0(\lambda) = 4 \) has four real solutions, becomes equivalent to the requirement \( V_0(\lambda_{\text{max}}) \geq 4 \). After some algebra we find that the second constraint on \( \hat{q}_3 \) and \( \hat{q}_4 \) can be represented together with the first one in the following form

\[
-4\hat{q}_4 \leq \hat{q}_3^2 \leq \frac{8\hat{q}_4 \left(\sqrt{48\hat{q}_4 + 1} - 2\right)^2}{9 \sqrt{48\hat{q}_4 + 1} - 1}.
\]  

(3.19)

These inequalities define the two-dimensional domain on the \( (\hat{q}_3, \hat{q}_4) \)–plane shown on fig. 5. Using the results of numerical solutions of the \( n = 4 \) Baxter equation [22] we plot on fig. 5 the numerical values of quantized \( (q_3/h^3, q_4/h^4) \) corresponding to the conformal weight \( h = 10 \) (see fig. 10 in [22]). We observe a complete agreement of the numerical results with (3.19). Moreover, the quantized values of \( q_3 \) and \( q_4 \) are distributed on fig. 5 in a regular way, but similar to the previous case \( n = 3 \), in order to describe their “fine structure” we need to know nonleading \( 1/h \) corrections to both quantum numbers.

The one–dimensional boundary of the domain (3.19) defines the “critical” values of the quantum numbers. The range of quantized charges \( \hat{q}_3 \) and \( \hat{q}_4 \) can be found from (3.19) as

\[
-\frac{1}{4} \leq \hat{q}_3 \leq \frac{1}{4}, \quad -\frac{1}{64} \leq \hat{q}_4 \leq \frac{1}{16}.
\]

We notice that taking \( \hat{q}_4 = 0 \) in (3.19) we obtain the relation \( \hat{q}_3^2 \leq 1/27 \), which we identify as the condition (3.16) on \( \hat{q}_3 \) for \( n = 3 \). In the same manner, the point \( \hat{q}_3 = \hat{q}_4 = 0 \) on fig. 5 corresponds to the possible range of the quantum numbers for the \( n = 2 \) Baxter equation. Continuing this

\[2\]This condition has its analog in the Toda chain [27, 28]. To separate the variables and reduce the Bethe Ansatz to the solution of the Baxter equation, one performs the canonical transformation and introduces the set of \( n - 1 \) generalized coordinates \( x_1, \ldots, x_{n-1} \) [29]. Then, the \( n - 1 \) closed intervals inside \( \mathcal{S} \) correspond to the regions of the classical motion of \( x_1, \ldots, x_{n-1} \).
analogy, we may conclude that the $n - 2$ dimensional domain of the quantized charges $\hat{q}_3, \ldots, \hat{q}_n$ corresponding to the $n$ Reggeon states lies on an intersection of the hyperplane $\hat{q}_{n+1} = 0$ with the $n - 1$ dimensional domain of $\hat{q}_3, \ldots, \hat{q}_n, \hat{q}_{n+1}$ corresponding to the $(n + 1)$ Reggeon states. The same property can be expressed in terms of (3.18) as $\Sigma_n(q^*_3, \ldots, q^*_n) = \Sigma_{n+1}(q^*_3, \ldots, q^*_n, q^*_{n+1} = 0)$.

### 3.2. Divergences of quasiclassical approximation

Let us use the quasiclassical expansion (2.24) of the solution of the Baxter equation to evaluate the holomorphic energy (2.23). In order to apply (2.23) we need to know the behaviour of $\Phi'(\lambda)$ at $\lambda = \pm i/h$. In the limit $h \to \infty$, the points $\lambda = \pm i/h$ move toward the origin from different sides of the imaginary axis and the energy (2.23) is different from zero because the point $\lambda = 0$ belongs to the interval $S$. Using (3.9) and (2.29) we find the asymptotic behaviour of the function $\Phi'_0(\lambda)$ for $\lambda \to 0$ at the both sides of the cut as

$$\Phi'_0(\lambda \pm i0) \xrightarrow{\lambda \to 0} \mp i \ln \hat{q}_n \lambda^n + \ldots$$

and applying (3.5) we conclude that the density of roots is logarithmically divergent at the origin (see figs. 1 and 3)

$$\rho(\sigma) \xrightarrow{\sigma \to 0} \frac{1}{\pi} \ln \frac{\hat{q}_n}{\sigma^n} + \mathcal{O}(\sigma).$$

Substituting (3.20) into (2.23) we obtain the following expression for the holomorphic energy

$$\varepsilon_n = -2 \ln(\hat{q}_n h^n) \approx -2 \ln q_n, \quad \text{as } h \to \infty.$$  \hfill (3.22)

Thus, the holomorphic energy of the $n$ Reggeon compound states scales in the limit $h \to \infty$ as a logarithm of the “higher” quantum number $q_n$.\(^3\) We stress that (3.22) was found in the

\(^3\)A somewhat similar result was obtained in [14].
leading large—\( h \) limit and in order to justify this expression we have to estimate the level at which next–to–leading \( 1/\hbar \)–corrections to the energy appear. Naively one might expect that nonleading corrections to \( \varepsilon_n \) will arise at the \( O(1/\hbar) \) level. However, as we will show in a moment, this is not true due to singular behaviour of the potential \( V_0(\lambda) \) at \( \lambda = 0 \) and, as a consequence, the qusicalcassical expansion of the energy does not work properly. Indeed, beyond the leading large—\( h \) approximation the energy (2.23) gets contribution from \( \Phi_1, \Phi_2, \ldots \), which in the quasiclassical approximation are proportional to the derivatives of the function \( \Phi_0(\lambda) \) (see eq. (4.1) below). In particular, \( \frac{d\Phi_k}{d\lambda} \sim \frac{d^{k+1}\Phi_0}{d\lambda^{k+1}} \sim \lambda^{-k} \) as \( \lambda \to 0 \) and the contribution of \( \Phi_k \) to the holomorphic energy can be estimated using (2.23) as \( \varepsilon_n \sim \lambda^{-k} h^{-k} = O(h^k) \) for \( \lambda = \pm i/\hbar \) and it is of the same order as the leading order contribution (3.22). Thus, in order to get an expansion of the holomorphic energy in powers of \( 1/\hbar \) it is not enough to restrict ourselves by the leading order result (3.22).

4. Beyond the leading order

To find the solution of the Baxter equation (2.25) beyond the leading \( 1/\hbar \) order we have to keep nonleading terms \( \Phi_1, \Phi_2, \ldots \) in (2.24) and expand further the r.h.s. of (3.7) in powers of \( 1/\hbar \). Substituting the expansion for \( f(\lambda \pm i\hbar)/f(\lambda) \) into the Baxter equation (2.25) and comparing the coefficients in front of powers of \( 1/\hbar \) in the both sides of the equation we obtain the following system of equations

\[
\begin{align*}
\Phi_1'(\lambda) &= -\frac{1}{2} \Phi_0'' \cot \Phi_0' + \frac{V_1(\lambda)}{2 \sin \Phi_0'} \\
\Phi_2'(\lambda) &= \frac{1}{6} \Phi_0''' + \frac{1}{2} \Phi_0'' \Phi_1' + \cot \Phi_0' \left( \frac{1}{8} (\Phi_0'')^2 - \frac{1}{2} (\Phi_1')^2 - \frac{1}{2} \Phi_1'' \right) + \frac{V_2(\lambda)}{2 \sin \Phi_0'}, \quad \ldots \quad (4.1)
\end{align*}
\]

where prime denotes a derivative with respect to the spectral parameter \( \lambda \), the function \( \Phi_0(\lambda) \) was defined in (3.8) and (3.9) and the potentials \( V_1, V_2, \ldots \) were introduced in (2.28) and (2.29).

Integrating these relations with the boundary conditions \( \Phi_k(\infty) = 0 \) one can get the functions \( \Phi_1(\lambda), \Phi_2(\lambda), \ldots \) and then reconstruct the solution of the Baxter equation using (2.21) and (2.24).

Since we would like to use the solution of (4.1) to find the energy of the \( n \) Reggeon states (2.23) it is of most interest for us to consider the behaviour of \( \Phi_k(\lambda) \) at small \( \lambda = O(h^{-1}) \). However, as it follows from (4.1) and (3.20), the expansion for the function \( \Phi(\lambda) \) in powers of \( 1/\hbar \) is not well defined for small \( \lambda \) due to singular behaviour of the potential at the origin, \( V(\lambda) \sim 1/\lambda^a \) as \( \lambda \to 0 \). Indeed, as was anticipated at the end of previous section, the functions \( \Phi_k(\lambda) \) are proportional to the higher derivatives of \( \Phi_0(\lambda) \). For small \( \lambda \) one can use (3.20) to find from (4.1) the following general form of their asymptotic behaviour as \( \lambda \to 0 \)

\[
\begin{align*}
\Phi_0(\lambda) &= a_{-1} \ln(i\lambda) + a_0 + a_1 i\lambda + a_2 (i\lambda)^2 + O(\lambda^3) \\
\Phi_1'(\lambda) &= b_0/(i\lambda) + b_1 + b_2 i\lambda + O(\lambda^2) \\
\Phi_2'(\lambda) &= c_0/(i\lambda)^2 + c_1/(i\lambda) + c_2 + O(\lambda), \quad \ldots \quad (4.2)
\end{align*}
\]

where the coefficients depend on whether \( \lambda \) approaches the origin from above or below the cut \( S \). To calculate the holomorphic energy (2.23) we substitute \( \lambda = \pm i/\hbar \) into (4.2) and evaluate
\( \Phi'(\lambda) \) using (2.24). In particular, \( \Phi'(-i/h) \) has the following expansion

\[
\Phi'(-i/h) = -a_1 \ln h + (a_0 + b_0 + c_0 + \ldots) + (a_1 + b_1 + c_1 + \ldots)/h + (a_2 + b_2 + c_2 + \ldots)/h^2 + O(1/h^3)
\]

(4.3)

where dots denote the contributions of the functions \( \Phi_k \) with \( k \geq 3 \). We conclude that different terms in the large-\( h \) expansion (2.24) give the contributions to \( \Phi'(\pm i/h) \) of the same order in \( 1/h \). As a consequence, in order to calculate the holomorphic energy \( \varepsilon_n \) we have to find an effective way to resum the contributions of \( \Phi_k'(\lambda) \) at small \( \lambda = O(1/h) \) to all orders \( k \).

### 4.1. Leading order resummation

Let us try to simplify the system of equations (4.1) in the limit of small \( \lambda \). As was shown in Sect. 3, the function \( \Phi'_0(\lambda) \) has a cut along the interval \( \mathcal{S} \) of the real axis and for \( \lambda \rightarrow 0 \) we have to consider separately two cases when \( \lambda \) approaches the origin from above and below the cut, \( \text{Im} \lambda > 0 \) and \( \text{Im} \lambda < 0 \), respectively. Using (3.20) we obtain the following identities

\[
\exp(i\Phi'_0(\lambda)) \xrightarrow{\lambda \rightarrow 0} \begin{cases} 
\frac{2\lambda}{\lambda_0} (1 + O(\lambda)) , & \text{for } \text{Im} \lambda > 0 \\
\frac{2\lambda}{\lambda_0} (1 + O(\lambda)) , & \text{for } \text{Im} \lambda < 0
\end{cases}
\]

(4.4)

Without lack of generality we may concentrate on the first case, \( \text{Im} \lambda > 0 \), and apply the relation (4.4) in the form \( \exp(i\Phi'_0) \gg \exp(-i\Phi'_0) \) to simplify the trigonometric functions entering into the system (4.1) as

\[
\sin(\Phi'_0) = \frac{1}{2i} \exp(i\Phi'_0)(1 + O(\lambda^{2n})), \quad \cot(\Phi'_0) = i + O(\lambda^{2n}).
\]

(4.5)

Then, in the limit \( \lambda \rightarrow 0 \) and \( \text{Im} \lambda > 0 \) the system of equations (4.1) is replaced by

\[
\Phi'_1(\lambda) = -\frac{i}{2} \Phi''_0 + iV_1(\lambda)e^{-i\Phi'_0} + O(\lambda^{2n-1}),
\]

\[
\Phi'_2(\lambda) = \frac{1}{6} \Phi''''_0 + \frac{1}{2} \Phi'''_0\Phi'_1 + \frac{i}{8} (\Phi''_0)^2 - \frac{i}{2} (\Phi'_1)^2 - \frac{i}{2} \Phi''_1 + iV_2(\lambda)e^{-i\Phi'_0} + O(\lambda^{2n-2}), ...
\]

(4.6)

where \( \Phi'_0 \) can be defined from (3.8) as \( \exp(i\Phi'_0) = 2 - V_0(\lambda) + O(\lambda^{2n}) \). One easily checks that performed approximation introduces ambiguities into the definition of the functions \( \Phi'_k(\lambda) \) at the level of \( O(\lambda^{2n-k}) \) corrections, \( \delta\Phi'_1 \sim \lambda^{2n-1}, \delta\Phi'_2 \sim \lambda^{2n-2}, ... \). Therefore substituting these relations into (2.23) and (2.24) and putting \( \lambda = \pm i/h \) one can find the expression for the holomorphic energy with the following accuracy

\[
\delta\varepsilon_n \sim \delta\Phi'_0 + \frac{1}{h} \delta\Phi'_1 + \frac{1}{h^2} \delta\Phi'_2 \sim \frac{1}{h^{2n}}.
\]

This means that using the approximation (4.5) and (4.6) one can define the holomorphic energy up to \( O(h^{-2n}) \) terms.

The reason why we are interesting in the approximation (4.5) is that instead of solving the infinite set of equations (4.6) for \( \Phi'_k \) one can find a closed expression for the function \( \Phi'(\lambda) \). Our observation is that for small \( \lambda \) the Baxter equation (2.25) can be effectively replaced by a new equation

\[
\frac{f(\lambda + i/h)}{f(\lambda)} = 2 - V(\lambda), \quad \text{for } \lambda \rightarrow 0 \text{ and } \text{Im} \lambda > 0
\]

(4.7)
and
\[
\frac{f(\lambda - i/h)}{f(\lambda)} = 2 - V(\lambda), \quad \text{for } \lambda \to 0 \text{ and } \text{Im} \lambda < 0,
\] (4.8)
which reproduces the relations (4.6) and which can be solved exactly. A simplest way to show
this is to use (4.4) and (3.7) and to take into account that for small \(\lambda\) in the Baxter equation
(2.25) we have \(f(\lambda - i/h) \ll f(\lambda + i/h)\) for \(\text{Im} \lambda > 0\) and \(f(\lambda - i/h) \gg f(\lambda + i/h)\) for \(\text{Im} \lambda < 0\).

Let us consider the relation (4.7) and use the ansatz (2.21) to rewrite it as
\[
\exp \left[ h \left( e^{i\partial/h} - 1 \right) \Phi(\lambda) \right] = 2 - V(\lambda).
\] where \(\partial \equiv \frac{\partial}{\partial \lambda}\). Its solution has a form
\[
i \Phi'(\lambda) = \frac{\frac{i}{h} \partial}{\exp(\frac{i}{h} \partial) - 1} \ln (2 - V(\lambda)), \quad \text{for } \text{Im} \lambda > 0.
\] (4.9)

As a check, we substitute \(\Phi = \sum_{k \geq 0} h^{-k} \Phi_k\) and \(V = \sum_{k \geq 0} h^{-k} V_k\) into the both sides of (4.9), compare the coefficients in front of powers of \(1/h\) and reproduce (4.6). An analog of (4.9) for \(\text{Im} \lambda < 0\) can be obtained from (4.8) as
\[
i \Phi'(\lambda) = \frac{\frac{i}{h} \partial}{\exp(-\frac{i}{h} \partial) - 1} \ln (2 - V(\lambda)), \quad \text{for } \text{Im} \lambda < 0.
\] (4.10)

Substituting (4.9) and (4.10) into (2.23) and putting \(\lambda = \pm i/h\) we derive after some algebra the following expression for the holomorphic energy of the \(n\) Reggeon state
\[
\epsilon_n = -\frac{\partial}{\exp(\partial) - 1} \ln \left( \left[ 2 - V\left( \frac{i\lambda}{h} \right) \right] \left[ 2 - V\left( -\frac{i\lambda}{h} \right) \right] \right) \bigg|_{\lambda=1} + \mathcal{O}(h^{-2n})
\] (4.11)
and taking into account that the potential (2.28) is a real function of \(\lambda\)
\[
\epsilon_n = -2 \Re \frac{\partial}{\exp(\partial) - 1} \ln \left( 2 - V\left( \frac{i\lambda}{h} \right) \right) \bigg|_{\lambda=1} + \mathcal{O}(h^{-2n}).
\] (4.12)

Here, the last term was added to indicate explicitly that the expression for the holomorphic energy was found within the approximation (4.5) and it is valid only up to \(\mathcal{O}(h^{-2n})\) terms. This implies in particular, that expanding the operator
\[
\frac{\partial}{\exp(\partial) - 1} = \sum_{k=0}^{\infty} \frac{1}{k!} B_k (\partial)^k
\]
with \(B_k\) being Bernoulli numbers, we may keep in (4.12) only \(2n - 1\) first terms.

The expression for the energy (4.12) is real. Another interesting property of (4.12) is that
although it has a rather formal form it can be explicitly evaluated for an arbitrary potential
\(V(\lambda)\). The calculation is based on the relation
\[
\frac{\partial}{\exp(\partial) - 1} \ln(a \lambda) = \ln(a \lambda) - \frac{1}{2 \lambda} - \frac{1}{12 \lambda^2} + \frac{1}{120 \lambda^4} + \ldots = \psi(\lambda) + \ln a
\] (4.13)
\footnote{Since in both cases the points \(\lambda \pm i/h\) should lie on the same side from the real axis, \(\lambda\) cannot approach the origin too close, \(|\text{Im} \lambda| \geq 1/h|\).}
which is valid for an arbitrary \( a \) and which follows from the definition of the \( \psi \)-function given in (1.8) and from identity \( (\ln \lambda)' = (e^\delta - 1)\psi(\lambda) = \psi(\lambda + 1) - \psi(\lambda) = 1/\lambda \). Let us consider the equation \( V(\lambda) = 2 \) with the potential defined in (2.26). It has \( n \) roots which we denote as \( \delta_1, \ldots, \delta_n \)
\[
2 - V(\lambda) = 2 - \frac{h(h-1)}{(h\lambda)^2} + \frac{q_3}{(h\lambda)^3} + \ldots + \frac{q_n}{(h\lambda)^n} = \frac{2}{\lambda^n} \prod_{i=1}^{n} (\lambda - \delta_i) . \tag{4.14}
\]
Substituting this relation into (4.11) and applying the identity (4.13) we find
\[
\varepsilon_n = -2\ln 2 - \sum_{i=1}^{n} [\psi(1 + ih\delta_i) + \psi(1 - ih\delta_i) - 2\psi(1)] + \mathcal{O}(h^{-2n}) . \tag{4.15}
\]
Notice that the roots \( \delta_i \) depend on \( h \) and in the limit \( h \to \infty \) they scale as \( \delta \sim h^0 \). For example, for \( n = 2 \) one can easily obtain from (4.14) their explicit form as \( \delta_{1,2}^2 = \frac{2\pi}{n\gamma_e} \).

It is interesting to note that the parameters \( \delta_i \) have a simple interpretation in terms of the auxiliary transfer matrix \( \Lambda(\lambda) \) defined in (2.5). Comparing the definitions (2.5) and (2.26) we conclude that \( h\delta_1, \ldots, h\delta_n \) entering into (4.15) can be determined as zeros of the auxiliary transfer matrix
\[
\Lambda(h\delta_k) = (h\delta_k)^n(2 - V(\delta_k)) = 0 , \quad k = 1, \ldots, n .
\]
Using the well–known asymptotic expansion (4.13) of the \( \psi \)-function we get from (4.15) the expansion of the energy in powers of \( 1/h \)
\[
\varepsilon_n = -\ln q_n^2 - 2n\psi(1) - 4\sum_{k=1}^{n-1} \frac{1}{2k} \frac{(2k-1)!}{(2\pi)^{2k}} \sum_{i=1}^{n} \delta_i^{-2k} + \mathcal{O}(h^{-2n}) \tag{4.16}
\]
where \( \psi(1) = -\gamma_e = -0.577216 \) is the Euler constant and \( \zeta(2k) \) is the Riemann zeta function. The comparison of (4.16) and (3.22) shows that, in accordance with our expectations (4.3), nonleading \( 1/h \)—corrections to the solution of the Baxter equation generate \( \mathcal{O}(h^0) \) contribution to the energy, \( 2n\psi(1) \), as well as an asymptotic series in \( 1/h \). We should stress that applying (4.16) we have to take also into account the \( h \)–dependence of the roots \( \delta_k \) and expand \( \sum_{i=1}^{n} \delta_i^{-2k} \) in powers of \( 1/h \). Although (4.16) looks like an even function of \( 1/h \) the latter expansion will give rise to odd powers of \( 1/h \).

Let us apply (4.16) for \( n = 2 \). Using the explicit form of the roots, \( \delta_i^{-2} = \frac{2\pi}{n\gamma_e} \), together with \( q_2 = -h(h-1) \) we obtain from (4.16)
\[
\varepsilon_2(h) = -2\ln(h(h-1)) - 4\gamma_e - \frac{2}{3h(h-1)} + \mathcal{O}(h^{-4}) = -4\ln h - 4\gamma_e + \frac{2}{h} + \frac{1}{3h^2} + \mathcal{O}(h^{-4}) . \tag{4.17}
\]
Comparing this expression with the exact result (2.19) we verify that up to \( \mathcal{O}(h^{-4}) \) corrections both expressions do coincide.

Expression (4.16) defines only \( 2n - 1 \) first terms of the infinite asymptotic series for \( \varepsilon_n \) and in order to apply (4.16) it is very important to understand how accurately these few terms extrapolate the exact result for \( \varepsilon_n \) for an arbitrary value of the conformal weight, \( 1/2 \leq h < \infty \). The answer to this question is based on the properties of the asymptotic series [30, 31, 32] and it will be discussed in Sect. 6. To anticipate our conclusions, we substitute the most interesting value of the conformal weight, (2.13), into (4.17) and (2.19) and obtain the following numerical values
\[
\varepsilon_2^{\text{app}}(1/2) = 5.797059 , \quad \varepsilon_2(1/2) = 8\ln 2 = 5.545177 . \tag{4.18}
\]
Remarkably enough, the approximate result turns out to be only 4% bigger the exact expression, which determines the intercept of the BFKL Pomeron (1.1).
4.2. Improved approximation

The approximation performed in the previous section has a strong limitation. It does not allow us to evaluate the holomorphic energy with the accuracy better that $\mathcal{O}(h^{-2n})$. Since we do not know in advance how many terms in the $1/h$-expansion of the energy we will need in order to approach the physically interesting value $\varepsilon_n(h = 1/2)$ it is desirable to have an improved scheme, which would allow us to predict higher $1/h$ corrections to the energy.

There is a simple way how one can improve the approximation. Let us start with the Baxter equation (2.25) and rewrite it in the following form

$$\frac{f(\lambda + i/h)}{f(\lambda)} = 2 - V(\lambda) - \left(\frac{f(\lambda)}{f(\lambda - i/h)}\right)^{-1}.$$  \hspace{1cm} (4.19)

As we have seen in Sect. 4.1, in the limit of small $\lambda$ and $\text{Im}\lambda > 0$ the ratio $f(\lambda + i/h)/f(\lambda)$ behaves as $\sim -V(\lambda) \sim q_n \lambda^{-n}$. This fact allows us to neglect the last term in the r.h.s. of the Baxter equation (4.19) and reproduce (4.7). The next step will be to consider the same term as a small perturbation and iterate the Baxter equation (4.19) as

$$\frac{f(\lambda + i/h)}{f(\lambda)} = 2 - V(\lambda) - \frac{1}{2 - V(\lambda - i/h) - \left(\frac{f(\lambda - i/h)}{f(\lambda - 2i/h)}\right)^{-1}}.$$  \hspace{1cm} (4.20)

Repeating this procedure $k$-times we obtain

$$\frac{f(\lambda + i/h)}{f(\lambda)} = 2 - V(\lambda) - \left[2 - V(\lambda - \frac{i}{h}) - \left[2 - V(\lambda - \frac{2i}{h}) - \ldots - \left[2 - V(\lambda - \frac{ki}{h}) - \left(\frac{f(\lambda - ik/h)}{f(\lambda - (k+1)i/h)}\right)^{-1}\right]^{-1}\right]^{-1}\right]^{-1}.$$  \hspace{1cm} (4.20)

Here, the ratio $(f(\lambda - ik/h)/f(\lambda - i(k+1)/h))^{-1}$ is much smaller than $V(\lambda - ik/h)$ at small $\lambda \sim 1/h$ provided that $\text{Im}\lambda > 0$ and the arguments of $f$ still lie in the upper half plane. Otherwise, for $\text{Im}\lambda < 0$, the same ratio becomes large and we are not allowed to treat it as a small parameter. This means, that for fixed number of iterations, we have to replace $\text{Im}\lambda > 0$ by a stronger condition $\text{Im}\lambda > (k+1)/h$. In particular, we cannot continue the iterations infinitely and replace the r.h.s. of (4.20) by continued fraction since this will push the allowed region of $\lambda$ away from the origin.

Let us consider (4.20) in the case when $\lambda = \mathcal{O}(1/h)$ and $\text{Im}\lambda > (k+1)/h$. Then, using the relations $V(\lambda - ik/h) = \mathcal{O}(h^n)$ and $f(\lambda - ik/h)/f(\lambda - i(k+1)/h) \sim -V(\lambda - i(k+1)/h) = \mathcal{O}(h^n)$ we can find that neglecting $(f(\lambda - ik/h)/f(\lambda - i(k+1)/h))^{-1}$ in (4.20) we modify the r.h.s. of the Baxter equation (4.20) at the level of $\mathcal{O}(h^{-(2k+1)n})$ terms. Performing this transformation and introducing notation for the "iterated" potential

$$V_k(\lambda) = 2 - V(\lambda) - \left[2 - V(\lambda - \frac{i}{h}) - \left[2 - V(\lambda - \frac{2i}{h}) - \ldots - \left[2 - V(\lambda - \frac{ki}{h})^{-1}\right]^{-1}\right]^{-1}\right]^{-1}.$$  \hspace{1cm} (4.21)

we can rewrite the Baxter equation (4.20) in the following form

$$\frac{f(\lambda + i/h)}{f(\lambda)} = V_k(\lambda) + \mathcal{O}(h^{-(2k+1)n}),$$
where $\text{Im} \lambda > (k + 1)/h$ and $\lambda = \mathcal{O}(1/h)$. For $k = 0$ this equation is equivalent to (4.7) and similar to (4.9) its solution can be represented for an arbitrary $k$ as

$$i\Phi'(\lambda) = \frac{\frac{i}{h} \partial}{\exp\left(\frac{i}{h} \partial\right) - 1} \ln \left(\mathcal{V}_k(\lambda) + \mathcal{O}(h^{-(2k+1)n})\right) = \frac{\frac{i}{h} \partial}{\exp\left(\frac{i}{h} \partial\right) - 1} \ln \mathcal{V}_k(\lambda) + \mathcal{O}(h^{-(2k+2)n})$$

(4.22)

where we took into account that $\mathcal{V}_k(\lambda) = \mathcal{O}(h^n)$ for $\lambda = \mathcal{O}(1/h)$. We recall that (4.22) was derived under the additional conditions $\text{Im} \lambda > (k + 1)/h$ and $\lambda = \mathcal{O}(1/h)$. To find the expression for $i\Phi'(\lambda)$ in the lower half–plane, $\text{Im} \lambda < 0$, we have to iterate the Baxter equation (4.19) considering $f(\lambda + \frac{i}{h})/f(\lambda)$ as a small parameter and follow the same steps which led to (4.10). The final expression is similar to (4.10) with $2 - V(\lambda)$ replaced by complex conjugated iterated potential $\mathcal{V}_k^*(\lambda)$ and it is defined for $\text{Im} \lambda < -(k + 1)/h$ and $\lambda = \mathcal{O}(1/h)$.

Although (4.22) was found for $\text{Im} \lambda > (k + 1)/h$ we may analytically continue the result to $\lambda = i/h$ and substitute it into (2.23) to get the following expression for the holomorphic energy

$$\varepsilon_n = -\frac{\partial}{\exp(\partial) - 1} \ln \left(\mathcal{V}_k\left(i\frac{\lambda}{h}\right)\mathcal{V}_k^*\left(-i\frac{\lambda}{h}\right)\right)\bigg|_{\lambda=1} + \mathcal{O}(h^{-2(k+1)n})$$

and using the fact that $V(\lambda)$ is a real function of $\lambda$

$$\varepsilon_n = -2 \text{Re} \frac{\partial}{\exp(\partial) - 1} \ln \mathcal{V}_k\left(i\frac{\lambda}{h}\right)\bigg|_{\lambda=1} + \mathcal{O}(h^{-2(k+1)n}) .$$

(4.23)

Here, $k$ is a positive integer which enters into the definition (4.21) of the potential $\mathcal{V}_k$. We conclude from (4.23) that the accuracy of the approximation is controlled by the number of iterations $k$. Increasing this number in the definition (4.21) of the potential $\mathcal{V}_k$ and using (4.23) we can obtain the expansion of the holomorphic energy $\varepsilon_n$ in powers of $1/h$ with an arbitrary accuracy. For $k = 0$ the expressions for the energy (4.23) and (4.12) coincide. Each new iteration, $k \to k + 1$, adds $2n$ additional terms to the expansion of $\varepsilon_n$.

One can apply (4.23) in a two different ways. The simplest one is to expand $\ln \mathcal{V}_k(i\lambda/h)$ and $\partial/(\exp(\partial) - 1)$ in powers of $1/h$ and $\partial$, respectively, and keep only first $2(k+1)n - 1$ terms. The second way is based on the identity (4.13). Similar to (4.14) one has to decompose $\mathcal{V}_k(\lambda)$ into simple factors and obtain the expression for $\varepsilon_n$ as a sum of $\psi$–functions.

5. **Fine structure of quantum numbers**

The expressions for the energy, (4.23) and (4.21), depend on the potential $V(\lambda)$. Applying (4.23) one finds that the higher terms in the asymptotic expansion of the holomorphic energy become sensitive to the nonleading $1/h$ corrections to the quantum numbers $q_k/h^k$ entering into the definition (2.28) and (2.29) of $V(\lambda)$. The results of the numerical solution of the Baxter equation indicate that the nonleading corrections to the quantum numbers are organized in such a way that for different values of the conformal weight $h$ the quantized values of $q_3, \ldots, q_n$ belong to the family of one–parametric curves for $n = 3$ (see fig. 4), two-parametric surfaces for $n = 4$ and, in general, to $n - 2$ parametric $(n - 2)$–dimensional hypersurfaces for an arbitrary $n$. There is a simple way how one can understand this fine structure within the quasiclassical approximation.

Let us recall that in the large $h$ limit, the distribution density of roots of the polynomial solutions of the Baxter equation has the support $S$ consisting of $n - 1$ connected intervals. The
total number of roots is equal to the conformal weight \( h \) and, as a consequence, the distribution density (3.1) satisfies the normalization condition (3.2). All \( h \) roots take real values and they are distributed on the interval \( S = [\sigma_1, \sigma_2] \cup ... \cup [\sigma_{2n-3}, \sigma_{2n-2}] \). Let us denote by \( N_k \) the number of roots belonging to the interval \([\sigma_{2k-1}, \sigma_{2k}]\) with \( k = 1, ..., n - 1 \). We remember that one of the intervals, say \([\sigma_{2n-3}, \sigma_{2n-2}]\), necessary contains the origin \( \sigma = 0 \) and the corresponding root of the polynomial solution should be \( n \)-time degenerate. This leads to the following conditions on \( N_k \)

\[
\sum_{k=1}^{n-1} N_k = h, \quad 0 \leq N_1, N_2, ..., N_{n-2} \leq h - n, \quad n \leq N_{n-1} \leq h.
\]

We recognize that for a given \( n \) and \( h \) among all integer numbers \( N_1, ..., N_{n-1} \) there are only \( n - 2 \) independent. It is natural to expect that these are the sets of the integer numbers \( \{N_k\} \), which parameterize \((n - 2)\)-dimensional hypersurfaces describing the distribution of the quantized \( q_3, ..., q_n \) for different values of \( h \).

To show this we use the definition (3.1) of the distribution density and express the numbers \( N_k \) as

\[
\int_{\sigma_{2k-1}}^{\sigma_{2k}} d\sigma \rho(\sigma) = \frac{N_k}{h}.
\]

In the large \( h \) limit, we substitute (3.6) into this relation and obtain expansion of its l.h.s. in powers of \( 1/h \). At the same time, depending on the value of \( N_k \), the r.h.s. contains only \( \sim h^0 \) and \( \sim h^{-1} \) terms. Then, from the comparison of the both sides we get

\[
\int_{\sigma_{2k-1}}^{\sigma_{2k}} d\sigma \left( \rho_0(\sigma) + \frac{1}{h} \rho_1(\sigma) \right) = \frac{N_k}{h}, \quad \int_{\sigma_{2k-1}}^{\sigma_{2k}} d\sigma \rho_j(\sigma) = 0,
\]

where \( k = 1, ..., n - 1 \) numerates the intervals inside \( S \) and \( j = 2, 3, ... \) refers to the level of nonleading \( 1/h \)-terms in the expansion of the distribution density (3.6). We notice that for small values of \( N_k \), such that \( N_k = \mathcal{O}(h^0) \), the first equation in (5.1) can be further split into two independent relations

\[
\int_{\sigma_{2k-1}}^{\sigma_{2k}} d\sigma \rho_0(\sigma) = 0, \quad \int_{\sigma_{2k-1}}^{\sigma_{2k}} d\sigma \rho_1(\sigma) = N_k, \quad \text{for } N_k = \mathcal{O}(h^0).
\]

Finally, using the definition (3.5) of the distributions \( \rho_j \) as discontinuity of the function \( \Phi'_j \) across the cut, we rewrite the integral of \( \rho_j \) over the cut \([\sigma_{2k-1}, \sigma_{2k}]\) as a contour integral of \( \Phi'_j \) around the cut

\[
\int_{\sigma_{2k-1}}^{\sigma_{2k}} d\sigma \rho_j(\sigma) = \oint_{C_k} \frac{d\lambda}{2\pi i} \Phi'_j(\lambda)
\]

where the contour \( C_k \) encircles the interval \([\sigma_{2k-1}, \sigma_{2k}]\) in the complex \( \lambda \)-plane in the anticlockwise direction. Using the relations (4.1) and replacing the potentials \( V_j(\lambda) \) by their definitions (2.28) and (2.29), we can express the r.h.s. of (5.3) in terms of quantized \( q^{(j)}, ..., q^{(n)} \). Combining (5.1), (5.2) and (5.3) together we obtain an infinite set of equations, in which we treat the numbers \( N_1, ..., N_{n-1} \) as parameters and \( q^{(j)}_3, ..., q^{(j)}_n \) as unknown variables. For a given set of integers \( N_1, ..., N_{n-1} \) their solution will give us the expressions for \( q^{(j)}_3, ..., q^{(j)}_n \) which we will substitute into (2.27) to find the asymptotic expansion of the quantum numbers \( q_3, ..., q_n \). This result can be expressed in the following form

\[
h = \sum_{j=1}^{n-1} N_j, \quad q_k = q_k(h; N_1, N_2, ..., N_{n-2}), \quad N_1, ..., N_{n-2} \geq 0, \quad N_{n-1} \geq n
\]
where \( k = 3, \ldots, n \). If we relax the condition for \( N_k \) to be integer, then these parametric relations define the \( n - 2 \) dimensional domain in the space of quantum numbers \( h, q_3, \ldots, q_n \). The interpretation of (5.4) within the framework of two-dimensional conformal field theories is proposed in Appendix B.

5.1. Nonleading corrections at \( n = 3 \)

Let us find the explicit form of the function \( q_k = q_k(h; N_1, \ldots, N_{n-2}) \) for the \( n = 3 \) Reggeon states. For \( n = 3 \) the distribution density of roots on \( S = [\sigma_1, \sigma_2] \cup [\sigma_3, \sigma_4] \) is described by two integer numbers \( 0 \leq N_1 \leq h - 3 \) and \( N_2 = h - N_1 \geq 3 \). According to our notations, \( N_2 \geq 3 \) counts the number of roots on the interval \([\sigma_3, \sigma_4]\) including the 3-time degenerate root \( \lambda_k = 0 \). For integer conformal weight \( h = N_1 + N_2 \) one can form only \( h - 2 \) possible sets of integers \((N_1, N_2) = (N_1, h - N_1)\) with \( N_1 = 0, \ldots, h - 3 \). This means that in agreement with the numerical results shown on fig. 4, for fixed conformal weight \( h \geq 3 \) there are only \( h - 2 \) possible quantized values of \( q_3 \).

Let us analyze the system of equations (5.1) in the special limit \( N_1 = O(h^0) \), in which (5.2) holds. The first equation in (5.2) involves the leading distribution density \( \rho_0(\sigma) \), which according to (3.12) and (3.11) is a positive definite smooth function on \( S \) (see fig. 3(a)). Therefore, in order to satisfy \( \int_{\sigma_1}^{\sigma_2} d\sigma \rho_0(\sigma) = 0 \) we have to require that the interval of the integration is shrinking to a point, \( \sigma_1 = \sigma_2 \). As we have shown in Sect. 3.1, this is the same condition which defines the critical value of the quantum numbers. We conclude that the solution of the first equation in (5.2) is that the leading quantum numbers \( \hat{q}_3, \ldots, \hat{q}_n \) should take their critical values \( \hat{q}_k = q_k^0 \), or equivalently belong to the critical hypersurface \( \Sigma_n \), (3.18). For \( n = 3 \) this amounts to say that

\[
\hat{q}_3 \equiv q_3^{(0)} = \pm \frac{1}{\sqrt{27}}. \tag{5.5}
\]

Let us consider the second equation in (5.2). It contains the integral of the first non-leading correction to the distribution density, \( \rho_1(\sigma) \), over the vanishing interval. In order for the integral to be different from zero, the function \( \rho_1(\sigma) \) should have singularities on \([\sigma_1, \sigma_2]\). To show this we introduce a small parameter \( \varepsilon \),

\[
\sigma_2 - \sigma_1 = \varepsilon, \quad V_0(\sigma_1) = V_0(\sigma_2) = 4 \tag{5.6}
\]

and approach the limit \( \sigma_2 - \sigma_1 \to 0 \) as \( \varepsilon \to 0 \). Here, the second relation follows from (3.10) (see also fig. 2(a)) and the potential \( V_0 \) for \( n = 3 \) was defined in (2.29). Choosing for simplicity \( \hat{q}_3 > 0 \), we use (5.6) to obtain the following expansions at small \( \varepsilon \)

\[
\sigma_{1,2} = \frac{1}{\sqrt{12}} \pm \frac{\varepsilon}{2}, \quad \hat{q}_3 = \frac{1}{\sqrt{27}} - \frac{\sqrt{3}}{2} \varepsilon^2, \quad V_0(\sigma) = 4 + 144x(1 - x)\varepsilon^2, \tag{5.7}
\]

where \( \sigma = \sigma_1 + \varepsilon x \) and \( 0 \leq x \leq 1 \). Substituting these expressions into (3.8) and (4.1) we find the asymptotic behaviour of \( \Phi_0(\lambda) \) and \( \Phi_1(\lambda) \) at the vicinity of the infinitesimal interval \([\sigma_1, \sigma_2]\) as

\[
\Phi_0(\lambda) = 12\varepsilon \sqrt{z(z-1)} + O(\varepsilon^2), \quad \Phi_1(\lambda) = -\frac{1}{4\varepsilon} \left( \frac{1}{z} + \frac{1}{z-1} \right) + \frac{V_1(\sqrt{12})}{24\varepsilon \sqrt{z(z-1)}} + O(\varepsilon^0) \tag{5.8}
\]
where \( \lambda = \sigma_1 + z\varepsilon \), \( z \) has an infinitesimal imaginary part and \( 0 \leq \text{Re} \, z \leq 1 \). Taking discontinuity (3.5) across the cut \( 0 \leq z \leq 1 \) we obtain the first nonleading correction to the distribution density

\[
\rho_1(\sigma_1 + x\varepsilon) = \frac{1}{\varepsilon} \left[ -\frac{1}{4}(\delta(x) + \delta(x - 1)) + \frac{1}{24\pi} \frac{V_1(\sqrt{12})}{\sqrt{x(1 - x)}} \right] + \mathcal{O}(\varepsilon^0). \tag{5.9}
\]

In accordance with our expectations it is divergent for \( \varepsilon \to 0 \). Applying the identity

\[
\int_{\sigma_1}^{\sigma_1} d\sigma \, \rho_j(\sigma) = \varepsilon \int_0^1 dx \, \rho_j(\sigma_1 + x\varepsilon) = \varepsilon \oint_{C_1} \frac{dz}{2\pi i} \Phi_j'(\sigma_1 + \varepsilon z) \tag{5.10}
\]

with \( C_1 \) encircling the interval \([0, 1]\) in complex \( z \)-plane, we may use (5.9) to evaluate the integral in (5.10). However, a more effective way to obtain the same result is to consider the contour integral of \( \Phi_1'(\sigma_1 + \varepsilon z) \) in (5.10) and realize, using (5.8), that the contour \( C_1 \) can be deformed away from the interval \([0, 1]\). Then, the contour integral in (5.10) is given by a residue of \( \Phi_j'(\sigma_1 + \varepsilon z) \) at \( z = \infty \) (with \( \varepsilon z = \text{fixed}! \)) and its substitution into the second equation (5.2) yields

\[
V_1\left(\frac{1}{\sqrt{12}}\right) = 24\left(N_1 + \frac{1}{2}\right).
\]

Finally, using the definition (2.29) of the potential \( V_1(\lambda) \) we obtain the first nonleading correction to the quantum number \( q_3 \) as

\[
q_3^{(1)} = -\frac{N_1 + 1}{\sqrt{3}}. \tag{5.11}
\]

where \( N_1 = \mathcal{O}(\hbar^0) \) is a positive integer parameterizing all possible solutions of the quantization conditions (5.1) and (5.2).

Let us repeat similar calculation and find the next correction, \( q_3^{(2)} \). We start with the equation (4.1) for \( \Phi_2 \) and use the asymptotics (5.8) to estimate the leading singularity of \( \Phi_2'(\sigma_1 + \varepsilon z) \) in the limit \( \varepsilon \to 0 \) as \( \sim \varepsilon^{-3} \). This seems to imply that the integral over \( \rho_2(\sigma) \) in (5.10) should be divergent at \( \varepsilon \to 0 \). However, despite of the fact that all these singular terms potentially contribute to \( \rho_2(\sigma) \), the integral \( \int_{\sigma_1}^{\sigma_1} d\sigma \rho_2(\sigma) \) gets a nonzero contribution only from \( \mathcal{O}(\varepsilon^{-1}) \) term and is finite. To understand this property we consider the contour integral of \( \Phi_2'(\sigma_1 + \varepsilon z) \) in (5.10) and take into account, using (5.8), that to any finite order of the expansion of the functions \( \Phi_j'(\sigma_1 + \varepsilon z) \) \((j = 0, 1, \ldots) \) in powers of \( \varepsilon \) their singularities are located on the interval \([0, 1]\) and at \( z = \infty \) in the complex \( z \)-plane. As a result, the contour integral in (5.10) is given by the residue of the function \( \Phi_j'(\sigma_1 + \varepsilon z) \) at infinity \( z = \infty \) but with \( \varepsilon z = \text{fixed} \). One can find from (5.8) that in the limit \( z = \infty \) and \( \varepsilon z = \text{fixed} \) the functions \( \Phi_j'(\sigma_1 + \varepsilon z) \) have the following asymptotic behaviour

\[
\Phi_j' \sim \sum_k (\varepsilon z)^{-k} f_k
\]

with some coefficients \( f_k \). All \( (\varepsilon z)^{-k} \)-terms with \( k \neq 1 \) in the expansion of \( \Phi_2'(\sigma_1 + \varepsilon z) \) have a zero residue at infinity and the contour integral (5.10) gets a nonzero contribution only from \( \mathcal{O}((\varepsilon z)^{-1}) \) term. Carefully expanding (4.1) in powers of \( \varepsilon \) we identify after some algebra the proper terms in \( \Phi_2'(\sigma_1 + \varepsilon z) \) as

\[
\Phi_2'(\sigma_1 + \varepsilon z) \sim \varepsilon^{-1}(\varepsilon z)^{-1} \left( 2(q_3^{(1)})^2 - \frac{1}{\sqrt{3}} q_3^{(1)} + \frac{5}{9} + \frac{1}{24} V_2\left(\frac{1}{\sqrt{12}}\right) \right) + \ldots \tag{5.12}
\]

where dots denote terms with another (smaller or larger) powers of \( 1/(\varepsilon z) \). Here, the potential \( V_2(\lambda) = -\lambda^{-3} q_3^{(2)} \) was defined in (2.29) and \( q_3^{(2)} \) is the next nonleading correction to the quantum number \( q_3 \). The quantization condition (5.1) for \( \rho_2 \) has a form \( \int_{\sigma_1}^{\sigma_1} d\sigma \rho_2(\sigma) = \oint dz \Phi_2' = 0 \) and
it requires that the residue of (5.12) at \( z = \infty \) should be equal to zero. As a result, the second correction to the quantum number \( q_3 \) can be found from (5.12) as
\[
q_3^{(2)} = \frac{1}{\sqrt{3}} \left( 2(q_3^{(1)})^2 - \frac{1}{\sqrt{3}}q_3^{(1)} - \frac{5}{9} \right),
\]
and after substitution of (5.11) the explicit dependence of \( q_3^{(2)} \) on integer number \( N_1 \) is
\[
q_3^{(2)} = \frac{\sqrt{3}}{27} \left( 6N_1^2 + 15N_1 + 4 \right). \quad (5.13)
\]
Continuing the same procedure it is straightforward to obtain the next nonleading corrections to \( q_3 \) from the quantization conditions (5.1). Here, we present the results of our computations using Maple V Symbolic Computation System
\[
q_3^{(3)} = -\frac{\sqrt{3}}{81} \left( 2N_1^3 + 12N_1^2 + 32N_1 + 8 \right)
\]
\[
q_3^{(4)} = \frac{\sqrt{3}}{2187} \left( 30N_1^4 + 60N_1^3 - 942N_1^2 - 972N_1 - 688 \right)
\]
\[
q_3^{(5)} = \frac{\sqrt{3}}{6561} \left( 114N_1^5 + 330N_1^4 - 5478N_1^3 - 10050N_1^2 - 17012N_1 - 7360 \right) \quad (5.14)
\]
\[
q_3^{(6)} = \frac{\sqrt{3}}{59049} \left( 1344N_1^6 + 5058N_1^5 - 96210N_1^4 - 254382N_1^3 - 709062N_1^2 - 668508N_1 - 261472 \right)
\]
\[
q_3^{(7)} = \frac{\sqrt{3}}{531441} \left( 16560N_1^7 + 76104N_1^6 - 1708338N_1^5 - 5887110N_1^4 - 23283058N_1^3 - 34667562N_1^2 - 30609348N_1 - 10272032 \right)
\]
Finally, combining (5.5), (5.11), (5.13) and (5.14) together we obtain the first eight terms of the asymptotic expansion (2.27) of the quantized \( q_3 \) in powers of \( h \). We notice that (5.11), (5.13) and (5.14) were found for positive \( q_3^{(0)} \). To get the corresponding expressions for negative \( q_3^{(0)} \) we may use the symmetry of the Baxter equation (2.12) under \( q_3 \to -q_3 \) to change a sign of all nonleading coefficients \( q_3^{(i)} \) in (5.11), (5.13) and (5.14).

### 5.2. Comparison with numerical calculations

Substituting (5.14) into (2.27) we find that the resulting expression for quantized \( q_3 = q_3(h; N_1) \) has a parametric form (5.4) with \( h = N_1 + N_2, \ N_1 \geq 0 \) and \( N_2 \geq 3 \). Choosing different values of the integers \( N_1 \) and \( N_2 \) we get the spectrum of quantized \( q_3 \) and \( h \) which should be compared with the numerical results shown on fig. 4(a). To perform the comparison we remove the condition for \( N_1 \) and \( N_2 = h - N_1 \) to be integer and consider the dependence of \( q_3 \) on \( N_1 \) and \( h \) in two different cases. In the first case, to which we will refer as to \( A \), we take \( N_1 \) to be zero or positive integer and \( N_2 \geq 3 \) to be positive real. This gives us two families of curves
\[
A_\pm: \quad q_3 = \pm q_3(h; N_1), \quad N_1 = \mathbb{Z}_+, \ h \geq 3 \quad (5.15)
\]
where \( \pm \) refers to the symmetry \( q_3 \to -q_3 \) and \( h \) is a continuous. In the second case, \( B \), we choose \( N_2 \) to be positive integer and \( N_1 = h - N_2 \) to be positive real,
\[
B_\pm: \quad q_3 = \pm q_3(h; h - N_2), \quad N_2 = 3 + \mathbb{Z}_+, \ h \geq N_2. \quad (5.16)
\]
As a result, we obtain the families $A_+^3$ and $B_+^3$ of the one-parametric curves in the $(h, q_3/h^3)$-plane shown on fig. 4(b). The curves from $A_+^3$ and $B_+^3$ and from $A_-^3$ and $B_-^3$ lie above and below the axis $q_3 = 0$, respectively. The points in which the curves from different families cross each other correspond to integer values of both $N_1$ and $N_2$ and their coordinates define the quantized values of $h$ and $q_3$. Trying to compare the expressions (5.5), (5.11), (5.13) and (5.14) for $q_3$ with the numerical results we should keep in mind that we calculated only first few terms of the asymptotic expansion of the functions $q_3 = q_3(h; N_1)$ for small values of integer $N_1$. Nevertheless, a careful examination of fig. 4(b) allows us to speculate about possible behaviour of the function $q_3$. For fixed $N_1$, the function $q_3(h; N_1)$ from $A_+^3$ increases as the conformal weight $h$ grows and for large $h$ it approaches the maximal value. For different $N_1$ the following hierarchy holds

$$\frac{h^3}{\sqrt{27}} > q_3(h; 0) > q_3(h; 1) > ... > 0 \quad (5.17)$$

As conformal weight $h$ decreases, the nonleading terms in the asymptotic expansion of $q_3(h; N_1)$ become important. Using the first 8 terms in the expansion of $q_3$ we find that the function $q_3(h; N_1)$ vanishes as $h$ approaches the value $h = 2N_1 + 3$ (see fig. 4(b))

$$q_3(h; N_1) \approx 0, \quad \text{for } h = 2N_1 + 3 = Z_+^3 \quad (5.18)$$

We recall, however, that the quantum number $q_3 = 0$ has a special meaning as corresponding to the degenerate solutions of the Baxter equation (2.11). Notice that equality in (5.18) is approximate because the integer $N_1$ takes the value, $N_1 = O(h)$, beyond the approximation (5.2), $N_1 = O(h^0)$, under which the corrections to quantized $q_3$ have been found in (5.14). Therefore, using our expressions for $q_3(h; N_1)$ we are not allowed to approach $q_3 = 0$ very closely. That is the reason why all curves from $A_+^3$ and $B_+^3$ should be terminated at the vicinity of the axis $q_3 = 0$. To describe the region near $q_3 = 0$ we have to improve the asymptotic approximation for $q_3 = q_3(h; N_1)$ in contrast with $A_+^3$, the curve $q_3 = q_3(h; h - N_2)$ from $B_+^3$ is decreasing functions of $h$ for fixed $N_2$. They take maximal value $q_3(h; 0)$ at $h = N_2$ and for $N_2 = 3, 4, ...$ all these points belong to the curve $q_3(h; 0)$ from $A_+^3$. As $h$ grows the function $q_3(h; h - N_2)$ decreases and it crosses subsequently another curves from $A_+^3$ with $N_1 = 1, 2, ..., N_2 - 3$. At $h = 2N_2 - 3 = 2N_1 + 3$ it takes a zero value in accordance with (5.18).

Although we cannot approach small values of the quantized $q_3$ within the approximation (5.2), a careful examination of fig. 4(b) allows us to speculate about possible behaviour of the function $q_3(h; N_1)$ below the axis $q_3 = 0$. The comparison of figs. 4(a) and (b) suggests that close to the axis $q_3 = 0$ the different curves from $A_+^3$ can be considered as a continuation of the curves from $B_+^3$ to the upper half-plane $q_3 > 0$. One of such possible curves is shown on fig. 4(a). Together with (5.15) and (5.16) this assumption corresponds to the following property of the function $q_3(h; N_1)$

$$q_3(h; N_1) \approx -q_3(h; h - 3 - N_1), \quad 0 \leq N_1 \leq h - 3 \quad (5.19)$$

This relation is consistent with (5.18). It allows us now to identify the curve on fig. 4(a) as $q_3 = q_3(h; 6)$. The value of the integer, $N_1 = 6$, can be fixed using (5.18) from the position of zero $q_3 = 0$. 

28
5.3. Quantization conditions for higher $n$

Let us generalize the analysis of the quantization conditions (5.1) to the higher $n$ Reggeon states. As we have shown in Sect. 5.1, the possible values of quantized $q_3$, ..., $q_n$ are parameterized by integers $N_1$, ..., $N_{n-1}$. According to the definition (5.4), their values satisfy either $N_j = O(h^0)$, or $N_j = O(h)$. Let us solve the quantization conditions (5.1) in the special case

$$N_1, N_2, ..., N_{n-2} = O(h^0), \quad N_{n-1} = O(h), \quad (5.20)$$

when almost all roots of the solution of the Baxter equation belong to a single interval. As we will see in a moment, the advantage of (5.20) is that for small $N_1$, ..., $N_{n-2}$ one is able to find analytical expressions for quantized $q_3$, ..., $q_n$. This does not mean however that the quantization conditions (5.1) can not be solved for the values of integers different from (5.20). In the latter case the equations for quantized $q_3$, ..., $q_n$ become more complicated and their solution is more involved.

Using (5.3) we represent the quantization conditions (5.2) in the following form

$$\oint_{C_j} \frac{d\lambda}{2\pi i} \Phi_0(\lambda) = 0, \quad \oint_{C_j} \frac{d\lambda}{2\pi i} \Phi_1(\lambda) = N_j, \quad j = 1, ..., n-2, \quad (5.21)$$

where the functions $\Phi_0'$ and $\Phi_1'$ were defined in (3.8) and (4.1). As was explained in Sect. 5.1, for each $j = 1, ..., n-2$ the solution of the first condition (5.21) is that the quantum numbers $q_3^{(o)}$, ..., $q_n^{(o)}$ take their critical values, that is they belong to the critical $n-2$ dimensional hypersurface (3.18). The system of $n-2$ equations, (5.21), fixes their position on $\Sigma_n$ “almost” uniquely. Namely, the solution of (5.21) defines some points on $\Sigma_n$, which we identify as quantized values of $q_3^{(o)}$, ..., $q_n^{(o)}$. For $n = 3$ we found $q_3^{(o)} = \pm 1/\sqrt{27}$, but for $n = 4$ these points can be identified as “corners” of the generalized triangle on fig. 5:

$$(q_3^{(o)} = 0, \quad q_4^{(o)} = \frac{1}{16}) \quad \text{and} \quad (q_3^{(o)} = \pm \frac{1}{4}, \quad q_4^{(o)} = -\frac{1}{64}). \quad (5.22)$$

To solve the second equation in (5.21) and find the next corrections, $q_3^{(1)}$, ..., $q_n^{(1)}$, we have to define the limit in which $q_3^{(o)}$, ..., $q_n^{(o)}$ approach their quantized values. Similar to the situation for $n = 3$, we introduce small parameters $\varepsilon_1$, ..., $\varepsilon_{n-2}$, which measure the size of the infinitesimal intervals, $\varepsilon_j = \sigma_{2j} - \sigma_{2j-1}$. None of these intervals contains the origin $\sigma = 0$ and their boundaries satisfy the relations $V_0(\sigma_{2j}) = V_0(\sigma_{2j-1}) = 0$ or $V_0(\sigma_{2j}) = V_0(\sigma_{2j-1}) = 4$. Let us concentrate on the latter case. Then, the potential $V_0(\lambda)$ has a local maximum on the interval $[\sigma_{2j}, \sigma_{2j-1}]$ at $\sigma = \sigma_j^*(\varepsilon_j)$ and

$$\sigma_{2j,2j-1} = \sigma_j^*(\varepsilon_j) \pm \frac{\varepsilon_j^2}{2} + O(\varepsilon_j^2).$$

The expansion of $V_0(\lambda)$ around $\lambda = \sigma_j^*$ looks like

$$V_0(\lambda) = V_0(\sigma_j^*(\varepsilon_j)) + \frac{1}{2}V_0''(\sigma_j^*(\varepsilon_j))(\lambda - \sigma_j^*)^2 + \ldots.$$

Substituting $\lambda = \sigma_{2j-1} + \varepsilon_jx$ and taking into account that $V_0(\lambda) = 4$ for $x = 0$ and $x = 1$, we obtain the expansion of the potential in the limit $\varepsilon_j \to 0$ as

$$V_0(\sigma_{2j-1} + \varepsilon_jx) - V_0(\sigma_j^*) = \frac{\varepsilon_j^2}{2}V_0''(\sigma_j^*)x(x-1) + O(\varepsilon_j^2). \quad (5.23)$$
Here, \( \sigma_j^* = \sigma_j^*(0) \) is the position of the maximum of \( V_0(\lambda) \) in the limit \( \varepsilon_j = \sigma_{2j} - \sigma_{2j-1} \to 0 \). It can be found as a solution of the equations

\[
V_0(\sigma_j^*) = 4, \quad V_0'(\sigma_j^*) = 0 \quad \text{and} \quad V_0''(\sigma_j^*) < 0.
\] (5.24)

Repeating analysis for \( V_0(\sigma_{2j}) = V_0(\sigma_{2j-1}) = 0 \) one can show that the same expansion (5.23) holds provided that we define \( \sigma_j^* \) as a local minimum of the potential,

\[
V_0(\sigma_j^*) = 0, \quad V_0'(\sigma_j^*) = 0 \quad \text{and} \quad V_0''(\sigma_j^*) > 0.
\] (5.25)

One can check that for \( n = 3 \) the relation (5.23) coincides with (5.7). Substitution of the expansion (5.23) into (3.8) and (4.1) yields

\[
\Phi'_0(\sigma_{2j-1} + \varepsilon_j z) = \frac{\varepsilon_j}{2} \sqrt{2|V_0''(\sigma_j^*)|z(z-1)} + \mathcal{O}(\varepsilon_j^2)
\]
\[
\Phi'_1(\sigma_{2j-1} + \varepsilon_j z) = \frac{1}{\varepsilon_j} \left[ -\frac{1}{4} (z + \frac{1}{z}) - \frac{1}{\sqrt{z(z-1)}} \frac{V_1(\sigma_j^*)}{\sqrt{2|V_0''(\sigma_j^*)|}} \right] + \mathcal{O}(\varepsilon_j^0),
\]

where the potential \( V_1(\lambda) \) was defined in (2.29) and \( j = 1, \ldots, n-2 \) numerates the intervals inside \( S \). The evaluation of the contour integral of \( \Phi'_1(\lambda) \) in (5.21) is similar to that in (5.10). It is given by the residue of \( \Phi'_1(\sigma_{2j-1} + \varepsilon_j z) \) at \( z \to \infty \) and \( \varepsilon_j z = \text{fixed} \). Finally, the solution of the second equation (5.21) has the form

\[
V_1(\sigma_j^*) = \sqrt{2|V_0''(\sigma_j^*)|} \left( N_j + \frac{1}{2} \right), \quad j = 1, \ldots, n-2.
\]

Using the definition (2.29) of \( V_1(\lambda) \) we obtain from this relation the system of \( (n-2) \) equations for quantized \( q_3^{(i)}, \ldots, q_n^{(i)} \)

\[
\frac{q_3^{(i)}}{\sigma_j^*} + \frac{q_4^{(i)}}{(\sigma_j^*)^2} + \ldots + \frac{q_n^{(i)}}{(\sigma_j^*)^n} = -1 - \sqrt{2(\sigma_j^*)^4|V_0''(\sigma_j^*)|} \left( N_j + \frac{1}{2} \right).
\] (5.26)

Here, \( \sigma_j^* \) is defined as a solution of (5.24) or (5.25). As an example, we consider the system of equation (5.26) for the \( n = 4 \) Reggeon compound states. Using two different sets (5.22) of the quantum numbers \( (q_3^{(o)}, q_4^{(o)}) \) we find the solution of (5.26) in the following form

\[
q_3^{(i)} = -\frac{1}{2} (N_1 - N_2), \quad q_4^{(i)} = -\frac{1}{8} - \frac{\sqrt{2}}{8} (N_1 + N_2 + 1)
\]

for the first set in (5.22) and

\[
q_3^{(i)} = \pm \frac{1}{4} \left( N_1 - 2\sqrt{2}N_2 - 1 - \sqrt{2} \right), \quad q_4^{(i)} = \frac{\sqrt{2}}{32} - \frac{1}{16} \left( N_1 - \sqrt{2}N_2 \right)
\]

for the second set in (5.22). Here, \( 0 \leq N_1, N_2 \leq h - 4 \) are integers parameterizing all possible solutions for \( q_3 \) and \( q_4 \).

30
6. Asymptotic expansions for the energy of the Reggeon states

In previous sections we developed the method for evaluation of the energy and conserved charges of the \( n \) Reggeon states in the quasiclassical approximation, \( h \to \infty \). The resulting expressions have a form of the asymptotic expansion in powers of \( 1/h \). Although they were found for large \( h \), it is of most importance to calculate their values for the conformal weight \( h = 1/2 \), which corresponds to the maximal energy of the Reggeon state, (2.13), and which defines the intercept of the QCD Pomerons and Odderons (1.6). The natural question appears whether it is meaningful to substitute \( h = 1/2 \) into the large \( h \) asymptotic expansion and estimate the approximate value of the energy. To test this idea we start with \( n = 2 \) Reggeon states and apply (4.16) and (4.23) to find the first few terms of the expansion of \( \varepsilon_2(h) \) in \( 1/h \)

\[
\varepsilon_2(h) = -4 \ln(he^{\gamma_E}) + \frac{2}{h} + \frac{1}{30h^2} + \frac{1}{63h^6} - \frac{1}{60h^8} + \frac{1}{33h^{10}} - \frac{691}{8190h^{12}} + \frac{1}{3h^{14}} + O(h^{-16}).
\]  

(6.1)

One can check that this series coincides with the asymptotic expansion of the exact result (2.19). The series (6.1) is a sign–alternating and its coefficients rapidly grow in higher orders. It is obvious that for \( h = 1/2 \) the series (6.1) becomes divergent while the correct answer for the energy \( \varepsilon_2(1/2) \) is known to be finite (4.18). On the other hand, as follows from (4.17) and (4.18), the sum of the first four terms of the divergent series (6.1) gives for \( h = 1/2 \) the result which is close to the exact answer. These two properties are remarkable features of the asymptotic series [30, 31, 32], which we will explore below to find the energy of the Reggeon states. Our goal will be to identify the partial sums which give the best asymptotic approximation to \( \varepsilon_n \). However, any asymptotic approximation should be accompanied by bounds for the corresponding error terms and in the case of the asymptotic series their determination becomes extremely nontrivial [30, 31, 32]. In what follows, instead of giving the rigorous proof we apply a “practical” version of the asymptotic approximations [32] based on the well–known examples and briefly described in Appendix C.

6.1. Euler transformation for \( n = 2 \) Reggeon states

Let us apply the “practical” asymptotic approximation to the asymptotic series (4.16) and (4.23) for the energy of the \( n = 2 \) Reggeon state and then compare the approximant \( \varepsilon_2^{\text{app}}(h) \) with the exact result (2.19) for \( \varepsilon_2(h) \). To start the analysis, we substitute \( h = 1/2 \) into (6.1) and examine the first few terms of the series

\[
\varepsilon_2(1/2) = 0.46 + 4 + 1.33 - 0.53 + 1.02 - 4.27 + 31.03 - 345.58 + 5461.33 + \ldots.
\]  

(6.2)

We find that apart from the first term their absolute value decreases until the \( n = 4 \) term is reached and then it rapidly increases. This suggests to truncate the series (6.2) and approximate \( \varepsilon_2(1/2) \) by a partial sum of the first 4 terms. For our purposes, however, it is convenient to exclude the 4–th term from the partial sum because after this the approximant has a simple interpretation. We notice that the 4–th term in (6.1) and (6.2) has a sign opposite to all preceding terms and its inclusion diminishes the partial sum. Therefore, the sum of the first 3 terms corresponds to the local maximum, \( S_{\text{max}} \), of the partial sum \( S_n \) as a function of a discrete variable \( n \),

\[
\varepsilon_2^{\text{app}}(h) = S_{\text{max}} = \max_{n=1,2,\ldots} S_n.
\]  

(6.3)
It is this interpretation of the approximant which will be used below.

Thus defined asymptotic approximation \( \varepsilon_{2 \text{app}}(h) \) to (6.1) coincides with (4.17) and it is in a good agreement with the exact result (4.18). We notice that (4.17) provides us asymptotic approximation to \( \varepsilon_2(h) \) not only for \( h = 1/2 \) but for larger values of the conformal weight \( h \) as well. The accuracy of the approximation (6.3) and (4.17) is not clear yet. However, the fact that the asymptotic series (6.1) and (6.2) are sign-alternating allows us to estimate the optimum remainder term [31] (see Appendix C for details). If we assume that the first three terms in (6.1) and (6.2) correctly describe the asymptotics of \( \varepsilon_2(h) \), then the absolute value of the optimum remainder term, \( \delta \varepsilon_2(h) = \varepsilon_2(h) - \varepsilon_{2 \text{app}}(h) \), is smaller than the last term included into the approximant and the first term excluded from the approximant. In application to the series (6.2) this property allows us to estimate the remainder as \(-0.53 < \delta \varepsilon_2(1/2) < 0\), which is in agreement with the numerical results (4.18).

Let us apply the Euler transformation [30, 32] (for definition see Appendix C) to find better approximation to the maximal energy \( \varepsilon_2(1/2) \). To this end, we take the asymptotic series (6.1) and reexpand it around the point \( h + x_0 \) up to \( O((h + x_0)^{-16}) \) terms. Although the original series (6.1) contains only even powers of \( 1/h \) in higher orders, the transformed series has all powers of \( 1/(h + x_0) \). Let us denote the partial sum of the first \( n \) terms of the transformed series as \( S_n(h, x_0) \). According to the “practical” asymptotical approximation (6.3), to find the best approximation to \( \varepsilon_2(h) \) we have to examine the behaviour of \( S_n(h, x_0) \) as a function of \( n \), identify the lowest value of \( n = n_0 \) at which \( S_n \) has a maximum and estimate the best approximation to \( \varepsilon_2(h) \) as \( S_{\text{max}} = S_{n_0}(h, x_0) \). To understand the role of the parameter \( x_0 \) in this procedure we fix the conformal weight as \( h = 1/2 \) and evaluate \( S_n(1/2, x_0) \) for \( n = 1, ..., 15 \) and two different values of \( x_0 \): \( x_0 = 0.25 \) and \( x_0 = 0.5 \). The results of our calculations are shown on fig. 6. We notice that in both cases the partial sum \( S_n \) grows with \( n \) until it approaches the local maximum, \( S_{\text{max}} \). Then, the region of stability follows after which \( S_n \) starts to oscillate around \( S_{\text{max}} \) and the

\[\begin{array}{c}
\text{(a)}
\end{array}\]

\[\begin{array}{c}
\text{(b)}
\end{array}\]

**Figure 6:** The partial sums \( S_n \) of the Euler transformed asymptotic series for the energy, (6.1), corresponding to \( h = 1/2 \) and two different values of the parameter of transformation: (a) \( x_0 = 0.25 \) and (b) \( x_0 = 0.5 \). The solid line denotes the exact expression for the energy \( \varepsilon_2(1/2) = 8 \ln 2 \), the dotted line indicates the position of the local maximum \( S_{\text{max}} \).
size of the fluctuations rapidly grows with $n$. It is important to realize that the position, $n$, of the maximum $S_n = S_{\text{max}}$ depends on the parameter $x_0$. For $x_0 = 0.25$ we have $S_{\text{max}} = 5.489602$ at $n = 5$ and for $x_0 = 0.5$ it is $S_{\text{max}} = 5.539171$ at $n = 10$. The dependence of $S_{\text{max}}(h, x_0)$ on $x_0$ for $h = 1/2$ is shown on fig. 7(a). The cusps correspond to the “critical” values of $x_0$, at which the number of terms contributing to $S_{\text{max}}$ increases by 1. As parameter $x_0$ increases the number of terms contributing to $S_{\text{max}}$ also increases and, hence, the approximation with which one can evaluate $S_{\text{max}}$ becomes better. However, for large $x_0$ the position of the local maximum of $S_n$ is shifted toward larger values of $n$ and in order to apply the Euler transformation we have to add higher $1/h$-terms to the asymptotic expansion of the energy (6.1). Namely, the expression (6.1) defines the asymptotic expansion of the energy up to $O(h^{-16})$ terms and under the Euler transformation the maximal value of the $n$ corresponding to $S_{\text{max}} = S_n$ cannot exceed $n = 15$. One can show that this condition is still satisfied for $x_0 = 0.75$ and the corresponding value of the partial sum gives the approximate value of the energy 

$$\varepsilon_2^{\text{app}}(1/2) = 5.545150$$

which is remarkably close to the exact answer (4.18). One might expect that the accuracy of the asymptotic approximation should be better for larger values of the conformal weight, $h > 1/2$. Indeed, taking the parameter of the Euler transformation as $x_0 = 0.75$ and repeating the calculation of $S_{\text{max}}$ for different values of $h$ we obtain the results for $\varepsilon_2^{\text{app}}(h)$ and $\delta\varepsilon_2(h)$ summarized in Table 1. We conclude that the “practical” version of the asymptotic approximation combined with the Euler transformation works perfectly well for the energy of the $n = 2$ Reggeon states. Although we are not able to find the exact value of the energy we have the method which allows us to find the approximation, $\varepsilon_2^{\text{app}}(h)$, and improve its accuracy by varying the value of the parameter of the Euler transformation, $x_0$. This gives us a hope that the same procedure can be applied to the evaluation of the energy of the higher Reggeon states.

Figure 7: The dependence of the asymptotic approximation to the energy, $S_{\text{max}}(h = 1/2, x_0)$, on the parameter of the Euler transformation $x_0$ for (a) $n = 2$ and (b) $n = 3$ Reggeon compound state. The dotted line represents the exact value of the energy $\varepsilon_2(h = 1/2) = 8 \ln 2$. 

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33
Substituting the asymptotic expansion (2.27) and (5.14) of the quantized
\(\epsilon(4.23)\) and (4.21) we find the first 8 terms of the expansion of
the energy
\[ \epsilon(h,q) = \epsilon(0) + \frac{1}{h} \epsilon_{1}(h) + \frac{1}{h^{2}} \epsilon_{2}(h) + \ldots \]
integer numbers
\(q\) the same numbers...
... makes the definition of the asymptotic expansion of the energy very problematic.
This implies, in particular, that similar to the distribution of the quantized
\(q_{3}, \ldots, q_{n}\). Their values are quantized and depend on the conformal weight \(h\) as well as on the
set of integer numbers defined in (5.4). The latter dependence was studied in Sect. 5 and the
expressions for the quantum numbers were obtained in the form of asymptotic series in \(1/h\). To find
the asymptotic expansion of the energy of the higher Reggeon states we have to supplement
the relation (4.23) by analogous expressions (5.4) for the quantum numbers \(q_{3}, ..., q_{n}\). Moreover,
as we have shown in Sect. 4, the quasiclassical expansion of the energy in powers of \(1/h\) becomes
divergent. The lowest order terms in the expansion of \(\epsilon_{n}\) in powers of \(1/h\) may depend, in general,
on an arbitrary higher order terms in the expansion of the quantum numbers \(q_{3}, ..., q_{n}\) and this
makes the definition of the asymptotic expansion of the energy very problematic.

Luckily enough, a close examination of the expression (4.23) shows that the \(\mathcal{O}(h^{-k})\)–term in
the expansion of the energy \(\epsilon_{n}\) depends on the finite number of nonleading terms in the expansion
(2.27) of the quantum numbers: \(q_{n}^{(h)}, q_{n-1}^{(k-2)}, ..., q_{3}^{(k-2(n-3))}\). Therefore, having an expression for the
quantum numbers up to \(\mathcal{O}(h^{-k})\) terms we will able to derive from (4.23) the expansion of the
energy with the same accuracy. To find the asymptotic series for \(q_{3}, ..., q_{n}\) we have to solve the
quantization conditions (5.1) and (5.2) as was described in Sect. 5. Their solution is well–defined
for integer values of the conformal weight \(h \geq n\) and it was parameterized in (5.4) by the set of
integer numbers \(N_{1}, ..., N_{n-1}\). As a result, the energy of the \(n\) Reggeon states also depends on
the same numbers
\[ \epsilon_{n} = \epsilon_{n}(h,q_{3}, ..., q_{n}) = \epsilon_{n}(h,N_{1}, ..., N_{n-2}) . \]  
This implies, in particular, that similar to the distribution of the quantized \(q_{3}, ..., q_{n}\), all possible
values of the energy belong to the families of curves in the \((h, \epsilon_{n})\)–plane labelled by integers \(N_{1}, ...
,..., N_{n-2}\).

Let us find the explicit form of the function \(\epsilon_{3} = \epsilon_{3}(h; N_{1})\) for the \(n = 3\) Reggeon states.
Substituting the asymptotic expansion (2.27) and (5.14) of the quantized \(q_{3} = q_{3}(h; N_{1})\) into
(4.23) and (4.21) we find the first 8 terms of the expansion of \(\epsilon_{3}\) in powers of \(1/h\) as
\[ \epsilon_{3}(h; N_{1}) = -6 \ln \left( \frac{h \epsilon_{3}}{\sqrt{3}} \right) + \frac{1}{h} \epsilon_{3}^{(1)} + \ldots + \frac{1}{h^{7}} \epsilon_{3}^{(7)} + \mathcal{O}(h^{-8}) , \]  
where the coefficients are given by
\[ \epsilon_{3}^{(1)} = 6 N_{1} + 6 \]
\[ \epsilon_{3}^{(2)} = 5 N_{1}^{2} + 8 N_{1} + \frac{11}{6} \]

| \(h\)   | 0.5  | 1    | 1.5  | 2    | 2.5  | 3    | 3.5  | 4    |
|-------|------|------|------|------|------|------|------|------|
| \(\epsilon_{2}^{pp}\) | 5.5452 | \(\sim 10^{-5}\) | -2.4548 | -4.0000 | -5.1215 | -6.0000 | -6.7215 | -7.3333 |
| \(\delta \epsilon_{2}\) | \(\sim 10^{-6}\) | \(\sim 10^{-5}\) | \(\sim 10^{-6}\) | \(\sim 10^{-7}\) | \(\sim 10^{-8}\) | \(\sim 10^{-9}\) | \(\sim 10^{-9}\) |

Table 1: The asymptotic approximation to \(\epsilon_{2}(h)\) based on the Euler transformation for \(x_{0} = 0.75\).
The expression (6.5) should be compared with the numerical results for the energy of the \( n = 3 \) Reggeon states presented on fig. 8(a) (see also fig. 3 in [22]). Let us perform the comparison following the same procedure as we have used in Sect. 5.2 for quantized \( q_3 = q_3(h; N_1) \). Similar to (5.15) and (5.16) the function (6.5) defines two different families of the curves in the \((h, \varepsilon_3)\)-plane shown on fig. 8(b),

\[
A : \quad \varepsilon_3 = \varepsilon_3(h; N_1), \quad h \geq 2N_1 + 3; \quad B : \quad \varepsilon_3 = \varepsilon_3(h; h - N_1 - 3), \quad 2N_1 + 3 \geq h \geq N_1 + 3,
\]

with \( N_1 \geq 0 \) being an integer. Notice that due to the invariance (2.12) under replacement \( q_3 \rightarrow -q_3 \) the energy \( \varepsilon_3(h; N_1) \) takes the same value for the curves from \( A_+ \) and \( A_- \). The quantized values of the energy correspond to the points on fig. 8(b), at which the curves from \( A \) and \( B \) cross each other. There is one-to-one correspondence between different curves on figs. 4(b) and 8(b), such that each function \( q_3 = q_3(h; N_1) \) is mapped into the corresponding function \( \varepsilon_3 = \varepsilon_3(h; N_1) \). As an example, the solid line on fig. 4(a) corresponding to \( q_3 = q_3(h; 6) \) induces the curve \( \varepsilon_3 = \varepsilon_3(h; 6) \) shown on fig. 8(a). In the same manner, the axis \( q_3 = 0 \) on fig. 4(a) is mapped into the dashed line on fig. 8(a) describing the degenerate values of the energy \( \varepsilon_3(h, q_3 = 0) = \varepsilon_2(h) \). Similar to the distribution of the quantized \( q_3 \) on fig. 4, we observe a good agreement on fig. 8 between numerical results for the energy and analytical expression (6.5) everywhere except of the points close to the “degenerate” curve \( \varepsilon_3 = \varepsilon_3(h, q_3 = 0) \). Nevertheless, as one can see from fig. 8(a), the function \( \varepsilon_3(h, q_3) \) is a smooth function of \( h \) at the vicinity of \( q_3 = 0 \) and according to (5.19) it satisfies the following relation

\[
\varepsilon_3(h; N_1) \approx \varepsilon_3(h; h - N_1 - 3),
\]

which allows us to consider (6.6) as definition of the unique function \( \varepsilon_3 = \varepsilon_3(h; N_1) \) for \( N_1 + 3 \leq h < \infty \). Solving the Baxter equation, we are interesting to find the maximal value of the energy of the \( n \) Reggeon states, (1.6) and (2.8). As was shown in [22] and as it can be easily seen from fig. 8(a), for positive integer conformal weight, \( h \geq n \), the energy \( \varepsilon_n \) cannot exceed the upper bound

\[
\varepsilon_n(h, q_3, \ldots, q_n) \leq -4(\psi(h) - \psi(1)), \quad h \geq n
\]

given by the energy of \( n = 2 \) Reggeon state with the conformal weight \( h \). As \( h \) decreases the maximal value of the energy \( \max_{q_3, \ldots, q_n} \epsilon_n(h, q_3, \ldots, q_n) \) grows until \( h \) approaches the value \( h = n \), beyond which the Baxter equation does not have polynomial solutions. On the other hand, we
Figure 8: Holomorphic energy of the \( n = 3 \) Reggeon compound states: (a) the results of the numerical solution; (b) two different families of curves defined in (6.6). The solid line on (a) corresponds to \( \varepsilon_3 = \varepsilon_3(h, q_3) \) with the function \( q_3 = q_3(h; 6) \) shown on fig. 4(a). The dashed line on (a) represents the energy of the \( n = 2 \) Reggeon states (2.19) and the dotted line on (a) is described by the function \( \varepsilon_3(h; 0) \).

expect from (2.13) that the absolute maximum of energy \( \varepsilon_n \) is achieved for \( h = 1/2 \) and not for \( h = n \). This means that in order to approach physically interesting values of the energy we have to penetrate through the \( h = n \) barrier to the region of smaller values of the conformal weight \( h \). In what follows we restrict consideration to the \( n = 3 \) Reggeon states while generalization to the higher Reggeon states is straightforward.

To perform analytical continuation of the energy \( \varepsilon_3(h, q_3) \) for \( h < 3 \) we use fig. 8(a) to observe the following properties of the function \( \varepsilon_3(h; N_1) \). Let us consider the flow of the quantized values of the energy along the curve \( \varepsilon_3 = \varepsilon_3(h; N_1) \) for fixed \( N_1 \). The energy increases along \( \varepsilon_3(h; N_1) \) as conformal weight decreases from \( h = \infty \) to \( h = 2N_1 + 3 \). At the point \( h = 2N_1 + 3 \) it takes the degenerate value \( \varepsilon_3(h; N_1) = \varepsilon_3(h) \) corresponding to \( q_3(h; N_1) = 0 \) in (5.18). For smaller values of the conformal weight, \( h < 2N_1 + 3 \), the curve \( \varepsilon_3 = \varepsilon_3(h; N_1) \) is “reflected” from the upper limit \( \varepsilon_3 = \varepsilon_2(h) \) and for \( h = N_1 + 3 \) it approaches the end–point, \( \varepsilon_3 = \varepsilon_3(h; N_1) = \varepsilon_3(h; 0) \), and is terminated.

We notice from fig. 8 that any two curves \( \varepsilon_3(h; N_1) \) and \( \varepsilon_3(h; N_1') \), corresponding to different values of integers \( N_1 \) and \( N_1' \), cross each other at only one point. Its position can be easily found from (6.7) as

\[
\varepsilon_3(h; N_1) = \varepsilon_3(h; N_1'), \quad \text{for} \quad h = N_1 + N_1' + 3.
\]

It is clear, that if the value of one of the functions is bigger for \( h < N_1 + N_1' + 3 \), then for \( h > N_1 + N_1' + 3 \) the relation between them becomes opposite. As a result, the functions satisfy the following hierarchy

\[
-4(\psi(h) - \psi(1)) \geq \varepsilon_3(h; N_1) > \varepsilon_3(h; N_1'), \quad \text{for} \quad h > N_1 + N_1' + 3, \quad N_1 > N_1'.
\]
These inequalities allow us to identify the maximal value of the energy corresponding to the integer conformal weight $h \geq 3$. We find that for the values of the conformal weight inside the interval $2k + 2 \leq h < 2k + 4$, with $k$ positive integer, among all the functions $\varepsilon_3 = \varepsilon_3(h; N_1)$ the one with $N_1 = k = \left[\frac{h-2}{2}\right]$ takes a maximal value

$$\max_{N_1} \varepsilon_3(h; N_1) = \varepsilon_3\left(h; \left[\frac{h-2}{2}\right]\right).$$

Therefore, as $h$ decreases from $h = \infty$ to $h = 4$ the maximum of the energy “jumps” from one curve $\varepsilon_3(h; N_1)$ to the next one, $N_1 \rightarrow N_1 - 1$, and finally for $h < 4$ it follows the function $\varepsilon_3(h; 0)$.

For $h \geq 3$ the quantized values of the energy are confined to the region on fig. 8 which is restricted from above by the function $\varepsilon_3\left(h; \left[\frac{h-2}{2}\right]\right)$ and by the function $\varepsilon_3(h; 0)$ from below,

$$\min_{N_1} \varepsilon_3(h; N_1) = \varepsilon_3(h; 0).$$

As conformal weight $h$ decreases, the upper and lower bounds for the energy, (6.9) and (6.10), respectively, move toward each other leaving no phase space for quantized values of energy $\varepsilon_3$. At $h < 4$ both boundaries coincide and the energy is given by

$$\varepsilon_3 = \varepsilon_3(h; 0), \quad \text{for} \quad h < 4.$$  (6.11)

For $h = 3$ it takes the value $\varepsilon_3 = \varepsilon_3(3; 0) = -6$ corresponding to the $n = 3$ degenerate Reggeon state with $q_3(3; 0) = -0$. It is of the most importance now to understand the behaviour of the function $\varepsilon_3(h; 0)$ beyond the barrier, $h < 3$, where the Baxter equation does not have polynomial solutions. We conclude from (6.9) and (6.10) that once the conformal weight approaches the barrier $h = 3$, among all functions $\varepsilon_3(h; N_1)$ describing quantized values of energy only one, (6.11), with $N_1 = 0$ survives. Therefore, performing analytical continuation to $h < 3$ we may restrict ourselves to the function $\varepsilon_3(h; 0)$.

6.3. Euler transformation for $n = 3$

To define $\varepsilon_3(h; 0)$ for $h \leq 3$ we use its asymptotic expansion in powers of $1/h$ and apply the Euler transformation to evaluate the best asymptotic approximation. The asymptotic expansion for $q_3(h; 0)$ can be found from (2.27) and (5.14) up to $O(h^{-10})$ terms as

$$q_3(h; 0) = \frac{\sqrt{3}}{9} h^3 \left(1 - \frac{3}{h} + \frac{4}{3 h^2} - \frac{8}{9 h^3} - \frac{688}{243 h^4} - \frac{7360}{729 h^5} \right.$$

$$- \frac{261472}{6561 h^6} - \frac{10272032}{59049 h^7} - \frac{447738400}{531441 h^8} - \frac{724437856}{1594323 h^9} + O(h^{-10}) \right).$$

As was explained in Sect. 6.2, substituting (6.12) into (4.23) we can obtain the expansion of the energy with the same accuracy

$$\varepsilon_3(h; 0) = -6 \ln \left(\frac{h e^7 e}{\sqrt{3}}\right) + \frac{6}{h} + \frac{11}{6 h^2} - \frac{56}{9 h^3} - \frac{190883}{4860 h^4} - \frac{135976}{729 h^5}$$

$$- \frac{80780333}{91854 h^6} - \frac{259171832}{59049 h^7} - \frac{50622906961}{21257640 h^8} - \frac{226978281368}{1594323 h^9} + O(h^{-10}).$$

\footnote{In comparison with (5.14) we calculated and added to $q_3(h; 0)$ two additional nonleading terms.}
Let us compare this expression with an analogous expansion (6.1) of the energy of the \( n = 2 \) Reggeon states. We notice that in both cases the absolute value of the coefficients in front of powers of \( 1/h \) rapidly increases in higher orders revealing the asymptotic nature of the series. However, in contrast with (6.1) the series for \( \varepsilon_3(h; 0) \) is not a sign–alternating. Starting from \( \mathcal{O}(h^{-3}) \) all terms in the series (6.13) are negative and this makes its properties different from (6.1). As one can see from fig. 8(b), the expression (6.13) is in a good agreement with the numerical results for the energy for large conformal weight \( h \geq 5 \). For small \( h \) the absolute value of the higher order terms in (6.13) starts to grow and the series (6.13) becomes divergent. In particular, substituting \( h = 1/2 \) into (6.13) we obtain the series

\[
\varepsilon_3(1/2; 0) = 3.99 + 12 + 7.33 - 49.78 - 628.42 - 5968.77 - 56284.34 - 561804.51 - 6096389.64 - 72891678.83 + \ldots,
\]

which should be compared with an analogous series in (6.2).

To evaluate the energy \( \varepsilon_3(h; 0) \) for \( h < 5 \) we have to define the asymptotic approximation to the series (6.13). Let us denote the partial sum of the first \( n \) terms of the series (6.13) as \( S_n \). Then, it follows from (6.13) and (6.14) that the partial sum \( S_n \) grows as \( n \) increases, at \( n = 3 \) it takes a maximal value, \( S_{\text{max}} \), and for \( n \geq 4 \) it rapidly decreases. It is clear that \( S_{\text{max}} \) overestimates the energy while the partial sum \( S_n \) for large \( n \) underestimates it. This means that the remainder term, as a function of a discrete \( n \), first decreases for small \( n \) until it approaches a minimal negative value corresponding to \( S_n = S_{\text{max}} \) and then it rapidly grows to \( \infty \). To our practical needs, we identify the asymptotic approximation to the energy as

\[
\varepsilon_3^{\text{app}}(h; 0) = S_{\text{max}} = \max_{n=1, 2, \ldots} S_n.
\]  

Under this choice, the remainder term \( \delta \varepsilon_3 = \varepsilon_3 - \varepsilon_3^{\text{app}} \) is always negative and the approximant (6.15) defines an upper limit for the energy \( \varepsilon_3(h; 0) \). For the series (6.13) and (6.14) the expression for \( S_{\text{max}} \) is given by the sum of the first three terms. To increase the number of terms contributing to \( S_{\text{max}} \) and, as a consequence, to improve the accuracy of the asymptotic approximation (6.15) we apply the Euler transformation to the series (6.13). The behaviour of the partial sum, \( S_n(h = 1/2, x_0) \), as a function of \( n \) is shown on figs. 9(a) and (b) for two different values of the parameter of transformation, \( x_0 = 4 \) and \( x_0 = 7.85 \), respectively. Changing \( x_0 \) we find from fig. 9 that as \( x_0 \) increases, the growth of \( S_n \) with \( n \) becomes less steep and the value of \( S_{\text{max}} \) slowly decreases. Since \( S_{\text{max}} \) defines the upper bound for the energy \( \varepsilon_3(h; 0) \), the latter property implies that the accuracy of the asymptotic approximation (6.15) becomes better. The dependence of \( S_{\text{max}}(h; x_0) \) on \( x_0 \) for \( h = 1/2 \) is shown on fig. 7(b). It suggests that approximant \( S_{\text{max}} \), being a decreasing function of \( x_0 \), asymptotically approaches the energy \( \varepsilon_3(h; 0) \) from above for large enough \( x_0 \).

Increasing the value of \( x_0 \) we have to check that the number of terms contributing to \( S_{\text{max}} \) is less than 10, that is the number of terms in the expansion (6.13). Otherwise, additional higher \( 1/h \)–terms should be taken into account in the asymptotic expansion (6.13). We found that the maximal value of \( x_0 \) for the series (6.13) which satisfies this condition is close to \( x_0 = 7.85 \). Applying the Euler transformation to the series (6.13) and taking \( x_0 = 7.85 \) we obtain the asymptotic approximation to the energy of the \( n = 3 \) Reggeon states, \( \varepsilon_3^{\text{app}}(h; 0) \), shown on fig. 10.

Let us summarize the properties of the function \( \varepsilon_3^{\text{app}}(h; 0) \). According to the definition (6.15), this function approximates the energy of the \( n = 3 \) Reggeon states \( \varepsilon_3 = \varepsilon_3(h; 0) \) for an arbitrary value of the conformal weight \( 1/2 \leq h < \infty \). Although its accuracy was not defined yet, we know
Figure 9: Partial sums $S_n$ of the Euler transformed series for the energy of $n = 3$ Reggeon states, (6.13), corresponding to $h = 1/2$ and two values of the parameter of transformation: (a) $x_0 = 4$ and (b) $x_0 = 7.85$. The dotted line indicates the position of the asymptotic approximation $S_{\text{max}}$. In advance that the asymptotic approximation (6.15) provides upper bound for the energy. For large values of the conformal weight, $h > 5$, the expression (6.13) is in a good agreement with the numerical results for the energy and the asymptotic approximation $\varepsilon^\text{app}_3(h;0)$ makes it even better. For smaller values of $h$ the expansion (6.13) becomes divergent and it is approximated by the function $\varepsilon^\text{app}_3(h;0)$. To estimate the accuracy of the approximation we use the numerical results for the energy, $\varepsilon_3(h;0)$, and compare them with the corresponding values of $\varepsilon^\text{app}_3(h;0)$ to find the remainder terms $\delta\varepsilon_3 = \varepsilon_3 - \varepsilon^\text{app}_3$. The numerical data are summarized in Table 2. As was expected, the remainder terms are negative.

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
h & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\varepsilon^\text{app}_3 & -5.0809 & -7.1050 & -8.6696 & -9.9333 & -10.9880 & -11.8907 & -12.6783 & -13.3763 \\
\delta\varepsilon_3 & -0.9191 & -0.3950 & -0.1637 & -0.0667 & -0.0273 & -0.0114 & -0.0050 & -0.0022 \\
\hline
\end{array}
$$

Table 2: The asymptotic approximation to $\varepsilon_3(h;0)$ based on the Euler transformation for $x_0 = 7.85$.

Performing analytical continuation of the energy to $h < 3$ we use the symmetry (2.12) of the Baxter equation under the replacement $h \to 1 - h$ to restrict the value of the conformal weight to the fundamental domain, $1/2 \leq h < \infty$. We notice from fig. 10 that $\varepsilon^\text{app}_3(h;0)$ is a decreasing function of $h$. This implies that, in accordance with (2.13), the maximal value of the energy of the $n = 3$ Reggeon state corresponds to $h = 1/2$ and it is given by

$$
\max_h \varepsilon^\text{app}_3(h;0) = \varepsilon^\text{app}_3(1/2;0) = 4.826208.
$$

We stress that this expression should be considered as upper bound for the energy $\varepsilon_3(1/2;0)$. The function $\varepsilon^\text{app}_3(h;0)$ depends on the parameter of the Euler transformation $x_0$. For $x_0 = 7.85$ we find from fig. 10 that $\varepsilon^\text{app}_3(h;0)$ crosses $\varepsilon_2(h) = -4(\psi(h) - \psi(1))$ at two points, $h_+ = 4.38$ and $h_- = 0.56$. For fixed $h$, the value of $\varepsilon^\text{app}_3(h;0)$ decreases as $x_0$ increases (see fig. 7(b)) and the distance between $h_+$ and $h_-$ becomes smaller. One might expect that the distance $h_+ - h_-$
controls the accuracy of the asymptotic approximation and in the limit $h_+ - h_- \to 0$ the function $\varepsilon_3^{\text{app}}(h; 0)$ coincides with the exact expression for the energy, $\varepsilon_3(h; 0)$. We recall that for integer $N_1 \geq 1$ the function $\varepsilon_3(h; N_1)$ crosses $\varepsilon_2(h)$ at only one point $h = 2N_1 + 3$. If one assumes that the same property holds for $N_1 = 0$, then $h_+ = h_- = 3$ and $\varepsilon_3(h; 0) \leq -4(\psi(h) - \psi(1))$. For $h = 1/2$ this leads to the relation
\[
\varepsilon_3(1/2; 0) < 8 \ln 2 ,
\]
which is in agreement with (6.16). Finally, taking into account (6.16) we find from (1.6) and (2.8) the asymptotic approximation of the intercept of the Odderon as
\[
\alpha_{\text{Odderon}}^{\text{app}} = 1 + \frac{\alpha_s N}{\pi} 2.413104 .
\]
This expression is smaller than the intercept of the BFKL Pomeron, (1.1), and it approximates $\alpha_{\text{Odderon}}$ from above. The estimate (6.18) is also in agreement with the lower bound for the Odderon intercept proposed in [11].

One may try to repeat the analysis and apply the Euler transformation to the asymptotic series (6.12) to find the approximation to the quantum number $q_3(1/2; 0)$ corresponding to the maximal energy of the $n = 3$ Reggeon states. We find however that since all terms in the power series (6.12) except of the $O(h^{-2})$ term have the same sign, the Euler transformation does not improve its convergency properties and we are not able to construct the convergent sequence of the partial sums similar to (6.15) for the energy. Moreover, as we will show in the next section, the coefficients in (6.12) grow as factorials and the asymptotic series for $q_3(h; 0)$ is not Borel

---

**Figure 10:** Holomorphic energy of the $n = 3$ Reggeon compound states: (a) Dots represent numerical results for the energy $\varepsilon_3 = \varepsilon_3(h; 0)$. Dotted line corresponds to the asymptotic series (6.13). Thick line on (a) and (b), $\varepsilon_3 = \varepsilon_3^{\text{app}}(h; 0)$, was found after the Euler transformation of the series (6.13) with the parameter $x_0 = 7.85$. Thin line on (b) is the energy of the $n = 2$ Reggeon states, $\varepsilon_2(h)$, defined in (2.19).
summable. Unfortunately, there is no regular way for approximating non Borel summable series like (6.12) and in order to calculate \( q_3(1/2; 0) \) one has to develop another summation method.

### 6.4. Borel summability

As was found in the previous sections, the asymptotic expansion of the energy of the \( n \)-Reggeon states has the following general form

\[
\varepsilon_n(h, \{q\}) = -2 \ln(h^n \hat{q}_n e^{n\gamma_E}) + \sum_{k \geq 1} \frac{f_k}{h^k}
\]  

(6.19)

where \( \hat{q}_2 = 1, \hat{q}_3 = \frac{1}{\sqrt{27}}, \ldots \) and \( f_k \) are rapidly growing coefficients. It is important now to understand whether the asymptotic series (6.19) defines uniquely the energy. To this end we perform the Borel transformation of the power series entering into (6.19)

\[
\varepsilon_n(h, \{q\}) = -2 \ln(h^n \hat{q}_n e^{n\gamma_E}) + \int_0^\infty dt \, B_{\varepsilon_n}(t) e^{-ht}.
\]  

(6.20)

Here, the function \( B_{\varepsilon_n}(t) \) is defined as

\[
B_{\varepsilon_n}(t) = \sum_{k \geq 0} \frac{f_{k+1}}{k!} t^k = \sum_{k \geq 0} A_{k,\{q\}} e^{-kt} - \frac{2n}{t}
\]  

(6.21)

where in the last relation we used (A.9) and (A.11) and the additional term, \(-2n/t\), was added to provide regularity of \( B_{\varepsilon_n} \) as \( t \to 0 \). The properties of the function \( B_{\varepsilon_n}(t) \) depend on the large order behaviour of the coefficients \( f_k \). Let us introduce the following ansatz [33]

\[
f_k \sim c k! a^{-k} b^k (1 + O(1/k)) \quad \text{as } k \to \infty,
\]  

(6.22)

which being substituted into (6.21) leads to the singularity of the function \( B_{\varepsilon_n}(t) \) at \( t = a \)

\[
B_{\varepsilon_n}(t) \sim \frac{1}{(t-a)^{b+2}}.
\]  

(6.23)

To verify (6.22) one has to compare \( f_k \) with their numerical values in the asymptotic expansions (6.1) and (6.13) and extract the values of the parameters \( a, b \) and \( c \).

For the \( n = 2 \) Reggeon states one uses the last three terms in the expansion (6.1) to find the following values: \( a = \pm 6.2746 i, b = -1.0201 \) and \( c = -8.2657 \), where the sign of \( a \) can not fixed since the expansion (6.1) contains only even powers of \( h \) in higher orders. These values are very close to the exact expressions

\[
a = \pm 2\pi i, \quad b = -1, \quad c = -8,
\]  

(6.24)

which follow from the well–known asymptotic expansion of the \( \psi \)–function in (2.19). Moreover, the function \( B_{\varepsilon_n}(t) \) can be evaluated explicitly using (6.21) and (A.10) as

\[
B_{\varepsilon_2}(t) = \frac{4}{1 - e^{-t}} - \frac{4}{t}. 
\]

We check that, in accordance with (6.23) and (6.24), the nearest to the origin singularity of this function is located at \( t = \pm 2\pi i \). At the same time, the function \( B_{\varepsilon_2}(t) \) does not have singularities
on the integration path in (6.20) and as a consequence the asymptotic series (6.20) for the energy of the \( n = 2 \) Reggeon states is Borel summable and it can be approximated using the well–known methods [33].

Let us check the large order behaviour of the energy of the \( n = 3 \) Reggeon states (6.13) using the ansatz (6.22). We find that for the set of parameters

\[
a = 1.8098, \quad b = 1.8195, \quad c = -1.5000
\]

the relation (6.22) reproduces the coefficients \( f_4, \ldots, f_9 \) with 8% accuracy. This result indicates that the function \( B_{\varepsilon_3}(t) \) has singularity (6.23) at the point \( t = 1.8098 \) which belongs to the integration path in (6.20). Therefore the Borel transformation does not exist and the asymptotic series for the energy of the \( n = 3 \) Reggeon states is not Borel summable. Different prescriptions for integrating the singularity of \( B_{\varepsilon_3}(t) \) lead to the results for the energy which differ in \( \sim \exp(-ha) \) terms. Thus, the asymptotic series (6.13) does not define the energy uniquely but rather represents one member of an infinite class of functions having the same asymptotic expansion. To avoid ambiguity and fix a particular member of the class we have to provide additional information about analytical properties of the energy. This requires solution of Baxter equation beyond the large \( h \) approximation.

We notice that the asymptotic expansion of \( q_3(h; 0) \) defined in (6.12) exhibits the large order behaviour similar to (6.13). Indeed, rewriting (6.12) as

\[
q_3(h; 0) = \sqrt{3} h^3 \left( 1 + \sum_{k \geq 1} g_k h^k \right)
\]

we find that the coefficients \( g_k \) \((k = 4, \ldots, 9)\) satisfy the asymptotic behaviour (6.22) with 2% accuracy provided that

\[
a = 1.8174, \quad b = 0.8019, \quad c = -0.4650.
\]

As a consequence, the asymptotic series for \( q_3(h; 0) \) suffers from the same problems as that for \( \varepsilon_3(h; 0) \). We notice that the expressions (6.25) and (6.27) suggest the following relation between asymptotic expansions for the \( n = 3 \) Reggeon states: \( g_k k/f_k \approx 0.28 \).

7. Conclusions

The spectrum of the color singlet \( n \) Reggeon compound states – perturbative Pomerons and Odderon, is expressed by means of the Bethe Ansatz in terms of the solution of the Baxter equation for the XXX Heisenberg magnet. In this paper we developed the method which allows us to find the solution of the Baxter equation for an arbitrary number of the Reggeized gluons.

The method is based on the observation that in the limit of large integer conformal weight \( h \) of the Reggeon states, the Baxter equation takes a form of a discrete one–dimensional Schrodinger equation for a particle in the external potential \( V(\lambda) \), which depends on the conserved charges \( q_3, \ldots, q_n \) of the Reggeon states and is singular at the origin \( \lambda \to 0 \). The inverse conformal weight, \( 1/h \), plays a role of the Planck constant and this fact allows us to apply the well–known quasiclassical expansion and obtain the solution of the Baxter equation, \( \Phi(\lambda) \), as an asymptotic series in \( 1/h \). We found that the derivative of the solution, \( \Phi'(\lambda) \), has a discontinuity across a
finite disconnected interval \( S \) on the real axis in the complex \( \lambda \)-plane, whose position and the number of connected intervals inside \( S \) depend on the potential or equivalently on the value of the charges \( q_3, ..., q_n \). The quantization conditions for the charges can be reformulated as certain constraints on \( S \). Namely, for the \( n \) Reggeon compound states the interval of nonanalyticity \( S \) should consist of \( n - 1 \) connected intervals. The corresponding quantization conditions have solutions for charges which can be parameterized by the set of integer numbers \( N_1, ..., N_{n-1} \). We have shown that the value of quantized \( q_3, ..., q_n \) can be obtained as a series in \( 1/h \) and the expressions for the lowest order coefficients were found. We observed that for \( n = 3 \) and \( n = 4 \) Reggeon states they are in a complete agreement with the results of the numerical solutions of the Baxter equation.

The energy of the \( n \) Reggeon states depends on the conformal weight \( h \) and quantized charges \( q_3, ..., q_n \). Using the quasiclassical solution of the Baxter equation we obtained the expansion of the energy in powers of \( 1/h \) for \( h \geq n \) and then analytically continued the result for small values of conformal weight, \( 1/2 \leq h < n \). For large integer \( h \) it agrees with the results of the numerical solutions while for small \( h \) the asymptotic expansion of the energy becomes divergent and it should be replaced by the asymptotic approximation. Using the properties of the asymptotic series and applying the Euler transformation we defined the asymptotic approximation of the energy for small \( h \) and checked its validity in the special case of the \( n = 2 \) Reggeon states by comparing with the exact expression for the energy. As first nontrivial application, we derived the asymptotic approximation of the energy of the \( n = 3 \) Reggeon state, perturbative Odderon, and estimated its intercept. The generalization of the obtained results to the higher Reggeon states is straightforward.

This work was supported in part by the National Science Foundation under grant PHY 9309888.

Appendix A. Analytical properties of the energy

Let us show that the energy of the \( n \)-Reggeon compound states satisfies the conditions of the Carlson’s theorem and therefore it has the unique analytical continuation in the region (2.14).

We start with the simplest case of the \( n = 2 \) Reggeon states, in which the holomorphic energy depends only on the conformal weight \( h \) and it is given by (2.19). The function \( \varepsilon_2(h) \) has poles at the origin \( h = 0 \) and at negative integer \( h \)

\[
\varepsilon_2(h) \xrightarrow{h \to -k} \frac{4}{h+k}, \quad k = 0, 1, 2, ... \tag{A.1}
\]

and its asymptotics at infinity is

\[
\varepsilon_2(h) \xrightarrow{h \to \infty} -4 \ln h. \tag{A.2}
\]

Thus, the function \( \varepsilon_2(h) \) has the unique continuation from integer to complex \( h \) defined in (2.3).

Let us generalize the relations (A.1) and (A.2) to the \( n \)-Reggeon states. Their holomorphic energy \( \varepsilon_n \) was obtained in Sect. 4 in the form of the asymptotic series in \( 1/h \) whereas the analytical expression for \( \varepsilon_n \) similar to (2.19) is not available yet. In the large \( h \) limit the asymptotic behaviour of \( \varepsilon_n \) was found in (3.22) as

\[
\varepsilon_n(h) \xrightarrow{h \to \infty} -2n \ln h, \tag{A.3}
\]
and it is in agreement with (A.2) for $n = 2$. Trying to satisfy the second condition of the Carlson’s theorem we find that having only a few first terms, (4.16) and (4.23), of the asymptotic series for $\varepsilon_n$ we are not able to identify all singularities of $\varepsilon_n$ in the complex $h-$plane.

There is however another way to study the singularities of the energy of the Reggeon compound states based on their relation with the anomalous dimensions of composite higher twist operators entering into the operator product expansion of the structure function of deep inelastic photon–hadron scattering at small values of the Bjorken variable $x$ [5]. Let us consider the structure function at small $x$ in the generalized leading logarithmic approximation [22]. We choose for simplicity the hadron to be a perturbative onium state built from two heavy quarks and created via the decay of the photon with invariant mass $m$. The structure function $F(x, Q^2)$ is defined as an imaginary part of the amplitude of the forward photon–onium scattering with the center-of-mass energy $s = Q^2(1 - x)/x$ and photon virtuality $-Q^2$. Let us consider the asymptotics of the structure function in the double scaling limit of large $Q^2/m^2$, and small $x \approx Q^2/s$.

In the limit $Q^2 \gg m^2$, the structure function can be expanded in powers of $1/Q^2$ using the operator product expansion (OPE). In the leading $\ln Q-$approximation, $\alpha_s \ll 1$ and $\alpha_s \ln(Q^2/m^2) \sim 1$, this expansion looks like

$$F_\omega(Q^2) \equiv \int_0^1 dx \ x^{\omega - 1} F(x, Q^2) = \frac{1}{Q^2} \sum_{k=1}^{\infty} C_\omega^{(k)} \left( \frac{m^2}{Q^2} \right)^{k-1+\gamma_\omega^{(k)}} ,$$

where the $k-$th term is associated with the contribution of twist–$2k$ operators. Their matrix elements have the anomalous dimensions $\gamma_\omega^{(k)} = O(\alpha_s)$ which have a perturbative expansion in powers of $\alpha_s$.

In the small $x$ limit, or equivalently in the limit of large energy $s \approx Q^2/x \gg Q^2$, the structure function has the Regge behaviour which it is governed by the compound Reggeon states propagating in the $t-$channel

$$F_\omega(Q^2) = \sum_{n=2}^{\infty} \alpha_s^{n-2} F_{\omega,n}(Q^2) ,$$

where the contribution of the $n-$Reggeon states can be found using (1.3) for $s \approx Q^2/x$ and $t = 0$ as

$$F_{\omega,n}(Q^2) = \sum_{\{q\}} \frac{1}{\omega - E_{n,\{q\}}} \beta_{n \to \gamma^*(Q^2)}^{\{q\}}(0) \left( \beta_{n \to h(m^2)}^{\{q\}}(0) \right)^* .$$

Here the summation is performed over all quantum numbers $h, q_3, ..., q_n$ corresponding to the $n$ Reggeon compound state with the energy $E_{n,\{q\}}$. The residue factors have been defined in (1.5) and in the case of the forward scattering, $t = 0$, they depend only on the invariant masses of scattering particles, $Q^2$ and $m^2$. In the generalized leading logarithmic approximation, one may calculate the residue factors for perturbative states of virtual photon, $\gamma^*(Q^2)$, and onium, $h(m^2)$, in the Born approximation and neglect $\alpha_s$ corrections. As a result, $\beta_{n \to \gamma^*}^{\{q\}}$ and $\beta_{n \to h}^{\{q\}}$ do not have anomalous dimension and their scaling dimensions are equal to the sum of the scaling dimensions of the $n$ Reggeon state, $h + \bar{h} = 1 + 2i\nu$, and the scaling dimensions of photon and onium states

$$\beta_{n \to \gamma^*(Q^2)}^{\{q\}}(0) = C_{n \to \gamma^*}^{\{q\}} Q^{-1 + 2i\nu} , \quad \beta_{n \to h(m^2)}^{\{q\}}(0) = C_{n \to h}^{\{q\}} m^{-1 + 2i\nu} .$$
where the dimensionless coefficients depend on the quantum numbers of the Reggeon states and scattering particles. Substituting these relations into (A.5) we obtain
\[
F_{\omega,n}(Q^2) = \frac{1}{Q^2} \sum_{q_3,\ldots,q_n} \int_{-\infty}^{\infty} d\nu \sum_{m \geq 0} \omega - \frac{\alpha_s N}{4\pi} \left[ \varepsilon_n \left( \frac{1+m}{2} + i\nu; \{q\} \right) + \varepsilon_n \left( \frac{1+m}{2} - i\nu; \{q\} \right) \right] \left( \frac{m}{Q} \right)^{-1-2\nu}.
\] (A.6)

Here, we extracted the sum over quantized values (2.3) of the conformal weight, that is summation over discrete \(m\) and integration over continuous \(\nu\), from the sum over all quantum numbers in (A.5).

Let us consider (A.6) in the limit \(Q^2 \gg m^2\), in which one should be able to reproduce \(1/Q^2\) expansion (A.4). For \(Q^2 \gg m^2\) one can enclose the integration contour over \(\nu\) into the lower half-plane, \(\text{Im} \nu < 0\), and calculate the integral over \(\nu\) in (A.6) by taking the residue at the values of \(\nu\) which satisfy the relation
\[
\frac{4\pi \omega}{\alpha_s N} = \varepsilon_n \left( \frac{1+m}{2} + i\nu; \{q\} \right) + \varepsilon_n \left( \frac{1+m}{2} - i\nu; \{q\} \right).
\] (A.7)
Solving this equation one can find the values of \(i\nu\) which determine the power of \(m/Q\) in the \(1/Q\)–expansion of the structure function (A.4), or equivalently define the scaling dimensions of the composite operators entering into the OPE. The comparison of (A.6) with (A.4) requires that the solutions of (A.7) should have the following form for \(\text{Im} \nu < 0\):
\[
i\nu = -\left( \frac{1}{2} + Z \right) + \mathcal{O}(\alpha_s), \quad Z = 0, 1, \ldots.
\] (A.8)

Let us now take into account that in order for the contribution of the pole in \(\nu\) to (A.6) to be nonvanishing, the residue factors, or equivalently the coefficients \(C_{n \to \gamma}^{(q)}\) and \(C_{n \to h}^{(q)}\), should be different from zero. This condition imposes selection rules on the quantum numbers \(h, q_3, \ldots, q_n\) of the Reggeon states. In particular, for the residue factors to be scalar, the conformal spin of the Reggeon state, \(h - \bar{h} = m\), should be equal to the conformal spin of the photon and onium states [5]: \(m = 0\) or \(m = 2\).

Then, substituting (A.8) into (A.7) and taking the limit \(\alpha_s \to 0\) we find that the holomorphic energy \(\varepsilon_n = \varepsilon_n(h; \{q\})\) has simple poles at the origin and at the integer negative values of the conformal spin
\[
\varepsilon_n(h; \{q\}) \sim \frac{A_{k,\{q\}}}{h+k}, \quad k = 0, 1, 2, \ldots.
\]
Moreover, using the asymptotic behaviour (A.3) and calculating the discontinuity of the energy at the negative \(h\) one can write the dispersion relation for the function \(\varepsilon_n(h; \{q\})\) in the complex \(h\)–plane which leads to
\[
\varepsilon_n^{\text{pole}}(h; \{q\}) = \sum_{k=0}^{\infty} \frac{A_{k,\{q\}}}{h+k} + C
\] (A.9)
with \(C\) some infinite \(h\)–independent subtraction constant. One can check that this relation holds for the energy of the \(n = 2\) Reggeon states (2.19) provided that
\[
A_k = 4
\] (A.10)
and the constant \(C\) can be found using the condition \(\varepsilon_2(1) = 0\). The asymptotics of the energy (A.3) implies the following relation:
\[
A_{k,\{q\}} \sim 2n, \quad \text{as } k \to \infty.
\] (A.11)
Thus, the consistency of the small $x$ asymptotics and the large $Q^2$ expansion of the structure function allows us to identify singularities of the holomorphic energy of the $n$–Reggeon states and, as a consequence, uniquely define $\varepsilon_n$ in the half-plane $\text{Re} \, h \geq 1/2$ by its values at the integer positive $h$.

### Appendix B. Conformal operators

The $n$ Reggeon compound states corresponding to the polynomial solutions of the Baxter equation have a simple interpretation in terms of the so–called conformal operators [34]–[39]. Let us consider the holomorphic wave function of the Reggeon states defined in (1.9). As was explained in Sect. 1, the same wave function diagonalizes the hamiltonian of the XXX Heisenberg magnet of spin $s = 0$. The explicit expression for $\varphi_n(\{z_k\}; z_0)$ in terms of the solution of the Baxter equation was found in [15, 22] and it is based on the correspondence between eigenstates of the XXX Heisenberg magnets of spins $s = 0$ and $s = -1$

$$\varphi_n(\{z_k\}; z_0) = (z_{12}z_{23}...z_{n1}) \varphi_n^{(s=1)}(\{z_k\}; z_0) \tag{B.1}$$

where $z_k$ are the holomorphic coordinates of the Reggeons and $z_0$ is the holomorphic coordinate of the center–of–mass of the Reggeon compound state. Invariance of the Reggeon hamiltonian under the conformal transformations (2.2) leads to the following constraints

$$\left( \sum_{k=1}^{n} \partial_k + \partial_0 \right) \varphi_n^{(s=1)} = 0, \quad \left( \sum_{k=1}^{n} z_k \partial_k + z_0 \partial_0 + h + n \right) \varphi_n^{(s=1)} = 0, \quad \left( \sum_{k=1}^{n} z_k^2 \partial_k + 2z_k + z_0^2 \partial_0 + 2hz_0 \right) \varphi_n^{(s=1)} = 0. \tag{B.2}$$

The expression for $\varphi_n(\{z_k\}; z_0)$ proposed in [15, 22] satisfies these relations. However, instead of using the Bethe Ansatz solution for $\varphi_n(\{z_k\}; z_0)$ we would like to interpret (B.2) as conformal Ward identities for $(n+1)$–point correlation functions in some two–dimensional conformal field theory [5, 14]

$$\varphi_n^{(s=1)}(\{z_k\}; z_0) = \langle \phi(z_1)...\phi(z_n)O_h(z_0) \rangle. \tag{B.3}$$

Here, the field $\phi(z_k)$ describes the Reggeon with the holomorphic coordinate $z_k$ and the operator $O_h(z_0)$ extrapolates the compound $n$ Reggeon state. To satisfy (B.2), the fields $\phi(z_k)$ and $O_h(z_0)$ should be quasiprimary operators with conformal weights 1 and $h$, respectively [24]. The conformal Ward identities fix two– and three–point correlation functions up to a normalization constant as

$$\langle \phi(z_1)\phi(z_2) \rangle = \text{const.} \times z_{12}^{-2}, \quad \langle \phi(z_1)\phi(z_2)O_h(z_0) \rangle = \text{const.} \times z_{10}^{-h}z_{20}^{-h}z_{12}^{-2}.$$

Substituting the last relation into (B.3) and (B.1) one can find the holomorphic wave function of the $n = 2$ Reggeon compound state [3].

For integer $h \geq n$ one can construct $O_h(z_0)$ as a composite operator built from $n$ Reggeon fields $\phi$ and their derivatives. The corresponding construction has been developed in QCD many years ago [35, 36, 37, 39] within the framework of the so–called “collinear” conformal group and it
can be trivially generalized to the “transverse” conformal group which is a symmetry group of the Reggeon states. The properties of the resulting conformal operators can be summarized as follows. For positive integer conformal weight \( h \geq n \), one can define the basis of the conformal operators and then represent all possible quasiprimary operators \( O_h(z) \) as their linear combination. For fixed number of the Reggeons, \( n \), the conformal basis consists of the operators \( O_{l_1,\ldots,l_{n-1}}(z) \) labelled by integers \( l_1, \ldots, l_{n-1} \) satisfying the following conditions

\[
l_1, \ldots, l_{n-1} \geq 0, \quad \sum_{k=1}^{n-1} l_k = h - n.
\]

The explicit form of the operators \( O_{l_1,\ldots,l_{n-1}}(z) \) for \( z = 0 \) is [37, 39]

\[
O_{l_1,\ldots,l_{n-1}}(0) = \partial^{h-n} P_{l_1,\ldots,l_{n-1}}\left( \frac{\partial}{\partial z}, \ldots, \frac{\partial}{\partial z} \right) \phi_1(0) \ldots \phi_{n-1}(0) \phi_n(0), \quad (B.4)
\]

where \( \partial = \partial_1 + \ldots + \partial_n \), the derivative \( \partial_k \) acts on the free field \( \phi_k \equiv \phi \) and \( P_{l_1,\ldots,l_{n-1}}(\alpha_1, \ldots, \alpha_{n-1}) \) is the so-called conformal polynomial in \( \alpha_1, \ldots, \alpha_{n-1} \). For different values of \( l_1, \ldots, l_{n-1} \) the functions \( P_{l_1,\ldots,l_{n-1}}(\alpha_1, \ldots, \alpha_{n-1}) \) form the system of biorthogonal polynomials on the simplex \( \alpha_1 + \ldots + \alpha_{n-1} + \alpha_n = 1 \) with the weight function and the measure

\[
w(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = (2n-1)! \alpha_1 \ldots \alpha_{n-1} \alpha_n, \quad \int D\alpha = \int_0^1 d\alpha_1 \ldots d\alpha_n \delta\left( \sum_{k=1}^{n} \alpha_k - 1 \right).
\]

In the special case of the \( n = 2 \) Reggeon state, thus defined conformal polynomial coincides with the Gegenbauer polynomial [34, 35, 36], \( P_l(\alpha) = C_l^{3/2}(2\alpha - 1) \), and the conformal operator interpolating the BFKL Pomeron for integer conformal weight \( h \geq 2 \) has a form

\[
O_h^{(BFKL)}(0) = (\partial_1 + \partial_2)^{h-2} C_{h-2}^{3/2} \left( \frac{\partial_1 - \partial_2}{\partial_1 + \partial_2} \right) \phi_1(0) \phi_2(0).
\]

For \( n = 3 \) Reggeon states, the conformal operators (B.4) are parameterized by two integers, \( l_1, l_2 \geq 0 \) and \( l_1 + l_2 = h - 3 \). The conformal polynomials, \( P_{l_1,l_2} \), are given by the Appell polynomials [36] and the basis of conformal operators looks like

\[
O_{l_1,l_2}(0) = \partial^{h-3} J_{l_1,l_2} \left( 5, 2, 2; \frac{\partial_1}{\partial}, \frac{\partial_2}{\partial} \right) \phi_1(0) \phi_2(0) \phi_3(0), \quad (B.5)
\]

with \( \partial = \partial_1 + \partial_2 + \partial_3 \). We recall that an arbitrary conformal operator \( O_h(0) \) is a linear combination of the basis operators \( O_{l_1,l_2}(0) \). However, for the conformal operator \( O_h(0) \) to interpolate the \( n = 3 \) Reggeon state, the correlation function (B.3) has to satisfy the additional condition of being the eigenstate of the conserved charge \( q_3 \) defined in (2.1). Substituting the operators (B.5) into (B.3) and (B.1) one can define the basis of the wave functions \( \varphi_{l_1,l_2} \), in which the operator \( q_3 \) can be represented as a finite-dimensional matrix \( (q_3)_{l_1,l_2,k_1',k_2'} \). The eigenvalues of this matrix give the quantized values of the charge \( q_3 \) and the corresponding eigenstates define the decomposition of the interpolating operator over the basis of conformal operators (B.5). It is clear that quantized values of \( q_3 \) can be parameterized in the same form as the diagonal elements of the matrix

\[
q_3 = q_3(l_1,l_2), \quad l_1, l_2 \geq 0 \quad l_1 + l_2 = h - 3.
\]
Comparing this relation with (5.4) we find that they are identical provided that we identify integers as \( N_1 = l_1 \) and \( N_2 = l_2 + 3 \). It can be easily seen that the same correspondence holds for the higher Reggeon states, \( N_1 = l_1, \ldots, N_{n-2} = l_{n-2} \) and \( N_{n-1} = l_{n-1} + n \).

Appendix C. “Practical” asymptotical approximation

Suppose \( f(z) \) is a function of \( z \) having the asymptotic expansion

\[
f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots. \tag{C.1}
\]

Let us assume that for a given \( z \) the absolute values of the first \( n \) successive terms of the series decrease and then they increase starting from \((n+1)\)-th term. Then, the asymptotic expansion is expecting to give the best approximation to \( f(z) \) when it is truncated at about \( n = n(z) \) terms [32]. The corresponding remainder term

\[
R_n(z) \equiv f(z) - \sum_{k=0}^{n-1} \frac{a_k}{z^{k+1}}
\]

is called the optimum remainder term. The accuracy of the approximation is controlled by the value of \( R_n(z) \).

Euler transformation.

There is a simple method, the so-called Euler’s transformation, which allows us to increase the accuracy of the asymptotic approximation [30, 32]. Let us perform the identical transformation of the series (C.1) by expanding its each term in a series of inverse powers of \( z + x_0 \) as

\[
f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{b_k}{(z + x_0)^{k+1}}, \quad b_k = \sum_{j=0}^{k} \binom{k}{j} x_0^j a_{k-j}
\]

with \( x_0 \) being a parameter of the transformation. It turns out that under proper choice of the parameter \( x_0 \) the sum of the first \( n' = n'(z, x_0) \) terms of a new series converges more rapidly and the corresponding optimum remainder term \( R_{n'} \) becomes smaller than \( R_n \) for the original series.

Estimation of the remainder terms.

Suppose that \( S_n = \sum_{k=0}^{n} a_n \) is the partial sum of sign–alternating series \( S_\infty = \sum_{k=0}^{\infty} a_n \) and \( R_n \) is the corresponding remainder term. Then, \( S_n \) overestimate the \( S_\infty \) when the last term included is positive, \( a_n > 0 \), and underestimate it when \( a_n < 0 \). This means that the remainder term \( R_n \) has a sign opposite to \( a_n \) and it follows from the relation \( R_n - R_{n+1} = a_{n+1} \) that

\[
|R_n| < |a_n|, \quad |R_n| < |a_{n+1}|.
\]

Thus, the absolute value of the optimum remainder term is smaller than the last term included into the approximant and the first term excluded from the approximant [31].
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