The Art Gallery Problem is $∃R$-complete

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Abstract

We prove that the art gallery problem is equivalent under polynomial time reductions to deciding whether a system of polynomial equations over the real numbers has a solution. The art gallery problem is a classical problem in computational geometry, introduced in 1973 by Victor Klee. Given a simple polygon $P$ and an integer $k$, the goal is to decide if there exists a set $G$ of $k$ guards within $P$ such that every point $p \in P$ is seen by at least one guard $g \in G$. Each guard corresponds to a point in the polygon $P$, and we say that a guard $g$ sees a point $p$ if the line segment $pg$ is contained in $P$.

The art gallery problem has stimulated extensive research in geometry and in algorithms. However, the complexity status of the art gallery problem has not been resolved. It has long been known that the problem is NP-hard, but no one has been able to show that it lies in NP. Recently, the computational geometry community became more aware of the complexity class $∃R$, which has been studied earlier by other communities. The class $∃R$ consists of problems that can be reduced in polynomial time to the problem of deciding whether a system of polynomial equations with integer coefficients and any number of real variables has a solution. It can be easily seen that $NP \subseteq ∃R$. We prove that the art gallery problem is $∃R$-complete, implying that (1) any system of polynomial equations over the real numbers can be encoded as an instance of the art gallery problem, and (2) the art gallery problem is not in the complexity class NP unless $NP = ∃R$. As a corollary of our construction, we prove that for any real algebraic number $α$, there is an instance of the art gallery problem where one of the coordinates of the guards equals $α$ in any guard set of minimum cardinality. That rules out many natural geometric approaches to the problem, as it shows that any approach based on constructing a finite set of candidate points for placing guards has to include points with coordinates being roots of polynomials with arbitrary degree. As an illustration of our techniques, we show that for every compact semi-algebraic set $S \subseteq [0,1]^2$, there exists a polygon with corners at rational coordinates such that for every $p \in [0,1]^2$, there is a set of guards of minimum cardinality containing $p$ if and only if $p \in S$.

In the $∃R$-hardness proof for the art gallery problem, we introduce a new $∃R$-complete problem ETR-INV. We believe that this problem is of independent interest, as it can be used to obtain $∃R$-hardness proofs for other problems. In particular, ETR-INV has been used very recently to prove $∃R$-hardness of other geometric problems.
1 Introduction

The art gallery problem. Given a simple polygon $P$, we say that two points $p, q \in P$ see each other if the line segment $pq$ is contained in $P$. A set of points $G \subseteq P$ is said to guard the polygon $P$ if every point $p \in P$ is seen by at least one guard $g \in G$. Such a set $G$ is called a guard set of $P$, and the points of $G$ are called guards. A guard set of $P$ is optimal if it is a minimum cardinality guard set of $P$.

In the art gallery problem we are given an integer $g$ and a polygon $P$ with corners at rational coordinates, and the goal is to decide if $P$ has a guard set of cardinality $g$. We consider a polygon as a Jordan curve consisting of finitely many line segments and the region that it encloses. The art gallery problem has been introduced in 1973 by Victor Klee, and it has stimulated extensive research in geometry and in algorithms. However, the complexity status of the art gallery problem has stayed unresolved. We are going to prove that the problem is $\exists R$-complete. Below, we give a formal definition of the complexity class $\exists R$.

The complexity class $\exists R$. The first order theory of the reals is a set of all true sentences involving real variables, universal and existential quantifiers, boolean and arithmetic operators, constants 0 and 1, parenthesis, equalities and inequalities, i.e., the alphabet is the set

$$\{X_1, X_2, \ldots, \forall, \exists, \land, \lor, \neg, 0, 1, +, -, \cdot, (, ), =, <, \leq\}.$$  

A formula is called a sentence if it has no free variables, so that each variable present in the formula is bound by a quantifier. Note that within such formulas, one can easily express integer constants (using binary expansion) and powers. Each formula can be converted to a prenex form, which means that it starts with all the quantifiers and is followed by a quantifier-free formula. Such a transformation changes the length of the formula by at most a constant factor.

The existential theory of the reals is a set of all true sentences of the first-order theory of the reals in prenex form with existential quantifiers only, i.e., sentences of the form

$$\left(\exists X_1 \exists X_2 \ldots \exists X_n \right) \Phi(X_1, X_2, \ldots, X_n),$$

where $\Phi$ is a quantifier-free formula of the first-order theory of the reals with variables $X_1, \ldots, X_n$. The problem ETR is the problem of deciding whether a given existential formula of the above form is true. The complexity class $\exists R$ consists of all problems that are reducible to ETR in polynomial time. It is currently known that

$$NP \subseteq \exists R \subseteq PSPACE.$$  

It is not hard see that the problem ETR is NP-hard, for instance by the following reduction from 3SAT. For each boolean variable $x$ in an instance of 3SAT, we introduce a real variable $x'$, and require that $x' \cdot (1 - x') = 0$ in order to ensure that $x' \in \{0, 1\}$. For any clause of the 3SAT formula we construct a function which evaluates to 1 if the corresponding clause is satisfied, and to 0 otherwise. For a clause $C$ of the form $x \lor y \lor \neg z$, the corresponding function $C'$ is $1 - (1 - x')(1 - y')z'$. The conjunction of clauses $C_1 \land \ldots \land C_m$ is then translated to the equation $C'_1 \cdot \ldots \cdot C'_m - 1 = 0$. Clearly, a formula of 3SAT is true if and only if the constructed set of equations has a solution in $\mathbb{R}$. The containment $\exists R \subseteq PSPACE$ is highly non-trivial, and it has first been established by Canny [14].

By the reduction from 3SAT to ETR sketched above we know that a problem of deciding whether a given polynomial equation over $\{0, 1\}$ with integer coefficients has a solution is NP-hard. The problem is also in NP, as a satisfying assignment clearly serves as a witness. Therefore, NP-complete problems are the problems equivalent (under polynomial time reductions) to deciding
whether a given polynomial equation over \{0, 1\} with integer coefficients has a solution. A well-known \(\exists \mathbb{R}\)-complete problem is the problem of deciding whether a single polynomial equation \(Q(x_1, \ldots, x_n) = 0\) with integer coefficients has a solution in \(\mathbb{R}\) \cite{32} Proposition 3.2]. Therefore, the \(\exists \mathbb{R}\)-complete problems are equivalent to deciding whether a given polynomial equation over \(\mathbb{R}\) with integer coefficients has a solution.

Our results and their implications. We prove that solving the art gallery problem is, up to a polynomial time reduction, as hard as deciding whether a system of polynomial equations and inequalities over the real numbers has a solution.

**Theorem 1.** The art gallery problem is \(\exists \mathbb{R}\)-complete, even the restricted variant where we are given a polygon with corners at integer coordinates.

It is a classical result in Galois theory, and has thus been known since the 19th century, that there are polynomial equations of degree five with integer coefficients which have real solutions, but with no solutions expressible by radicals (i.e., solutions that can be expressed using integers, addition, subtraction, multiplication, division, raising to integer powers, and the extraction of \(n\)’th roots). One such example is the equation \(x^5 - x + 1 = 0\) \cite{39}. It is a peculiar fact that using the reduction described in this paper, we are able to transform such an equation into an instance of the art gallery problem where no optimal guard set can be expressed by radicals. More general, we can prove the following.

**Theorem 2.** Given any real algebraic number \(\alpha\), there exists a polygon \(\mathcal{P}\) with corners at rational coordinates such that in any optimal guard set of \(\mathcal{P}\) there is a guard with an \(x\)-coordinate equal \(\alpha\).

Our results justify the difficulty in constructing algorithms for the art gallery problem, and explain the lack of combinatorial algorithms for the problem (see the subsequent summary of related work). In particular, Theorem 2 rules out many algorithmic approaches to solving the art gallery problem. A natural approach to finding a guard set for a given polygon \(\mathcal{P}\) is to create a candidate set for the guards, and select a guard set as a subset of the candidate set. For instance, a candidate set can consist of the corners of \(\mathcal{P}\). The candidate set can then be expanded by considering all lines containing two candidates and adding all intersection points of these lines to the candidate set. This process can be repeated any finite number of times, but only candidates with rational coordinates can be obtained that way, and the candidate set will thus not contain an optimal guard set in general. Algorithms of this kind are discussed for instance by de Rezende et al. \cite{18}. One can get a more refined set of candidates by also considering certain quadratic curves \cite{9}, or more complicated curves. Our results imply that if the algebraic degree of the considered curves is bounded by a constant, such an approach cannot lead to an optimal solution in general, since the coordinates of the candidates will also have algebraic degree bounded by a constant.

A semi-algebraic set is a set of the form \(\{x \in \mathbb{R}^n : \Phi(x)\}\), where \(\Phi\) is a quantifier-free formula of the first-order theory of the reals with \(n\) variables. For many \(\exists \mathbb{R}\)-complete problems, there is a deep connection between their solution spaces and semi-algebraic sets. The most famous result of this kind is Mnev’s universality theorem \cite{34}. It yields that for every semi-algebraic set \(S\) there exists a pseudoline arrangement \(P\) such that the space \(L(P)\) (of all line arrangements topologically equivalent to \(P\)) is homotopy equivalent to \(S\). We show a similar correspondence for the art gallery problem, see Theorem 47 and Theorem 48 in Appendix C. Moreover, we can show the following result.

**Theorem 3** (Picasso Theorem). For any compact semi-algebraic set \(S \subset [0, 1]^2\), there is a polygon \(\mathcal{P}_S\) with corners at rational coordinates such that for any point \(p \in [0, 1]^2\) we have \(p \in S\) if and only if there exists an optimal guard set \(G\) of \(\mathcal{P}_S\) with \(p \in G\).
The name of the last theorem stems from the following imaginative interpretation. We are given any black and white picture (≈ semi-algebraic set), and we construct a special art gallery with this picture drawn at the floor. The theorem says that we can guard the gallery optimally if and only if one of the guards stands on one of the black points of the picture.

Related work. The art gallery problem has been extensively studied, with some books, surveys, and book chapters dedicated to it [35, 45, 19, 37, 17, 31, 19, 36]. The research is stimulated by a large number of possible variants of the problem and related questions that can be studied. The version of the art gallery problem considered in this paper is the classical one, as originally formulated by Victor Klee (see O’Rourke [35]). Other versions of the art gallery problem include restrictions on the positions of the guards, different definitions of visibility, restricted classes of polygons, restricting the part of the polygon that has to be guarded, etc.

The art gallery problem has been studied both from the combinatorial and from the algorithmic perspective. Studies have been made on algorithms performing well in practice on real-world and simulated instances of the problem [12, 18]. Another branch of research investigates approximation algorithms for the art gallery problem and its variants [22, 11, 20].

The first exact algorithm for solving the art gallery problem was published in 2002 in the conference version of a paper by Efrat and Har-Peled [21]. They attribute the result to Micha Sharir. Before that time, the problem was not even known to be decidable. The algorithm computes a formula in the first order theory of the reals corresponding to the art gallery instance, and uses standard algebraic methods, such as the techniques provided by Basu et al. [7], to decide if the formula is true. No algorithm is known that avoids the use of this powerful machinery.

Lee and Lin [28] proved, by constructing a reduction from 3SAT, that the art gallery problem is NP-hard when the guards are restricted to the corners of the polygon (note that this version, called the vertex-guard art gallery problem, is obviously in NP). It has subsequently been shown by Aggarwal [11], see also [34] that this argument can be extended to the case with no restrictions on the guard placement. Various papers showed other hardness results or conditional lower bounds for the art gallery problem and its variations [28, 11, 22, 38, 27, 11, 25].

A problem related to the art gallery problem is the terrain guarding problem. Here, the area above an $x$-monotone polygonal curve $c$ has to be guarded by a minimum number of guards restricted to $c$. Friedrichs et al. [23] showed recently that terrain guarding is NP-complete.

The authors of the present paper [3] provided a simple instance of the art gallery problem with a unique optimal guard set consisting of three guards at points with irrational coordinates. Any guard set using only points with rational coordinates requires at least four guards. This could be an indication that the art gallery problem is actually more difficult than the related problems discussed above. Friedrichs et al. [23] stated that “[…] it is a long-standing open problem for the more general Art Gallery Problem (AGP): For the AGP it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits”. In the present paper we prove that under the assumption $\text{NP} \neq \exists \mathbb{R}$ the art gallery problem is not in NP, and such a representation does not exist.

A growing class of problems turn out to be equivalent (under polynomial time reductions) to deciding whether polynomial equations and inequalities over the reals have a solution. These problems form the family of $\exists \mathbb{R}$-complete problems as it is currently known. Although the name $\exists \mathbb{R}$ has been established not a long time ago, see [31, 43], algorithms on deciding ETR, which forms the core problem of $\exists \mathbb{R}$, have long been studied, see e.g. the book of Basu et al. [6]. The class $\exists \mathbb{R}$ includes problems like the stretchability of pseudoline arrangements [34, 47], recognition of intersection graphs of various objects (e.g. segments [32], unit disks [33], and general convex sets [41]), recog-
nition of point visibility graphs [16], the Steinitz problem for 4-polytopes [40], deciding whether a graph with given edge lengths can be realized by a straight-line drawing [42, 2], deciding whether a graph has a straight line drawing with a given number of edge crossings [10], decision problems related to Nash equilibria [24], and positive semidefinite matrix factorization [46]. We refer the reader to the lecture notes by Matoušek [32] and surveys by Schaefer [41] and Cardinal [15] for more information on the complexity class $\exists \mathbb{R}$.

Overview of the paper and techniques. In Section 2 with details in Appendix A, we show that the art gallery problem is in the complexity class $\exists \mathbb{R}$. For that we present a construction of an ETR-formula $\Phi$ for any instance $(P, g)$ of the art gallery problem such that $\Phi$ has a solution if and only if $P$ has a guard set of size $g$. The idea is to encode guards by pairs of variables and compute a set of witnesses (which depend on the positions of the guards) of polynomial size such that the polygon is guarded if and only if all witnesses are seen by the guards.

The proof that the art gallery problem is $\exists \mathbb{R}$-hard is the main result of the paper, and it consists of two parts. The first part is of an algebraic nature, and in that we introduce a novel $\exists \mathbb{R}$-complete problem which we call ETR-INV. A common way of making a reduction from ETR to some other problem is to build gadgets corresponding to each of the equations $x = 1$, $x + y = z$, and $x \cdot y = z$ for any variables $x, y, z$. Usually, the multiplication gadget is the most involved one. An instance of ETR-INV is a conjunction of formulas of the form $x = 1$, $x + y = z$, and $x \cdot y = 1$, with the requirement that each variable must be in the interval $[1/2, 2]$. In particular, the reduction from ETR-INV requires building a gadget for inversion (i.e., $x \cdot y = 1$), which involves only two variables, instead of a more general gadget for multiplication involving three variables. The formal definition of ETR-INV and the proof that it is $\exists \mathbb{R}$-complete is presented in Section 3 with details in Appendix B. We believe that the problem ETR-INV might be of independent interest, and that it will allow constructing $\exists \mathbb{R}$-hardness proofs for other problems, in particular those for which constructing a multiplication gadget was an obstacle that could not be overcome. ETR-INV has already been used to prove $\exists \mathbb{R}$-completeness of a geometric graph drawing problem with prescribed face areas [20], and to prove $\exists \mathbb{R}$-completeness of completing a partially (straight-line) drawn graph [29].

In Section 4 with details in Appendix C, we describe a polynomial time reduction from ETR-INV to the art gallery problem, which shows that the art gallery problem is $\exists \mathbb{R}$-hard. This reduction constructs an art gallery instance $(P(\Phi), g(\Phi))$ from an ETR-INV instance $\Phi$, such that $P(\Phi)$ has a guard set of size $g(\Phi)$ if and only if the formula $\Phi$ has a solution. We construct the polygon so that it contains $g(\Phi)$ guard segments (which are horizontal line segments within $P$) and stationary guard positions (points within $P$). By introducing pockets we enforce that if $P$ has a guard set of size $g(\Phi)$, then there must be exactly one guard at each guard segment and at each stationary guard position. Each guard segment represents a variable of $\Phi$ (with multiple segments representing the same variable) in the sense that the position of the guard on the segment specifies the value of the variable, the endpoints of a segment corresponding to the values 1/2 and 2.

We develop a technique for copying guard segments, i.e., enforcing that guards at two segments correspond to the same variable. We do that by introducing critical segments within the polygon, which can be seen by guards from two guard segments (but not from other guard segments). Then the requirement that a critical segment is seen introduces dependency between the guards at the corresponding segments. Different critical segments enforce different dependencies, and by enforcing that two guards together see two particular critical segments we ensure that the guards represent the same value. The stationary guards are placed to see the remaining areas of the polygon.

With this technique, we are able to copy two or three segments from an area containing guard
segments corresponding to all variables into a gadget, where we will enforce a dependency between the values of the variables represented by the two or three segments. This is done by constructing a corridor containing two critical segments for each pair of copied segments. The construction is technically demanding, as it requires the critical segments not to be seen from any other segments.

Within the gadgets, we build features that enforce the variables \(x, y, z\) represented by the guards to satisfy one of the conditions \(x + y \geq z\), \(x + y \leq z\), or \(x \cdot y = 1\). The conditions are enforced by a requirement that two or three guards can together see some areas, where for the case of a gadget with three variables the area to be seen is a quadrilateral instead of a line segment.

At last, in Appendix D we prove the Picasso Theorem (Theorem 3).

2 The art gallery problem is in \(\exists \mathbb{R}\)

Theorem 4. The art gallery problem is in the complexity class \(\exists \mathbb{R}\).

Sketch of proof (details in Appendix A). Let \((P, g)\) be an instance of the art gallery problem. Consider the following formula \(\Psi\) that Micha Sharir described to Efrat and Har-Peled [21]

\[
\Psi := \left[ \exists x_1, y_1, \ldots, x_g, y_g \forall p_x, p_y : \text{INSIDE-POLYGON}(p_x, p_y) \implies \bigvee_{i=1}^{g} \text{SEES}(x_i, y_i, p_x, p_y) \right].
\]

For each \(i \in \{1, \ldots, g\}\), the variables \(x_i, y_i\) represent the position of a guard \(g_i := (x_i, y_i)\), and \(p := (p_x, p_y)\) represents an arbitrary point. The predicate \(\text{INSIDE-POLYGON}(p_x, p_y)\) tests if the point \(p\) is contained in the polygon \(P\) and \(\text{SEES}(x_i, y_i, p_x, p_y)\) checks if the guard \(g_i\) can see the point \(p\). Thus, the formula is true if and only if \(P\) has a guard set of cardinality \(g\).

The formula \(\Psi\) is not an existential formula. Our main idea behind obtaining a formula with no universal quantifier is finding a polynomial number of points inside \(P\) which, if all seen by the guards, ensure that all of \(P\) is seen. We denote such a set of points as a witness set.

Consider a set of guards in \(P\), and a set of lines \(\mathcal{L}\) containing either an edge of \(P\), or a guard \(g_i\) for \(i \in \{1, \ldots, g\}\) and a corner \(v\) of \(P\). We can show that the lines in \(\mathcal{L}\) partition the plane into a collection of regions \(\mathcal{A}\) which are convex polygons, each one either contained in \(P\) or contained in the complement of \(P\), and such that for each region \(R \in \mathcal{A}\), each guard \(g_i\) either sees all points of \(R\), or sees no point of \(R\). Thus, it is sufficient to test that for each region \(R \in \mathcal{A}\) which is in \(P\), at least one point in \(R\) is seen by a guard.

Let \(X\) be the set of all intersection points of lines in \(\mathcal{L}\), i.e., the set of all corners of all regions in \(\mathcal{A}\). In the formula \(\Phi\), we generate all points in \(X\). Our witness set is the set of centroids for all triples of points in \(X\). For each such point \(p\) we check that if \(p \in P\), then \(p\) is seen by a guard.

3 The problem ETR-INV

We will prove \(\exists \mathbb{R}\)-hardness of the art gallery problem by a reduction from the problem ETR-INV, which we introduce below. In this section, we sketch the proof that ETR-INV is \(\exists \mathbb{R}\)-complete. The details are in Appendix B.

Definition 5 (ETR-INV). In the problem ETR-INV, we are given a set of real variables \(\{x_1, \ldots, x_n\}\), and a set of equations of the form

\[
x = 1, \quad x + y = z, \quad x \cdot y = 1,
\]

for \(x, y, z \in \{x_1, \ldots, x_n\}\). The goal is to decide whether the system of equations has a solution when each variable is restricted to the range \([1/2, 2]\).
Moreover, if there is no solution to $\Phi$, each guard set of segments contained in the interior of $P$ such that $\Phi$ has a solution if and only if $P$ can be guarded by some number $g := g(\Phi)$ of guards. The number $g$ will follow from the construction. A sketch of the polygon $P$ is shown in Figure 1.

Each variable $x \in X$ is represented by a collection of guard segments, which are horizontal line segments contained in the interior of $P$. Consider one guard segment $s := ab$, where $a$ is to the left of $b$, and assume that $s$ represents the variable $x$ and that there is exactly one guard $p$ placed on $s$. The guard $p$ on $s$ specifies the value of the variable $x$, usually as $\frac{1}{2} + \frac{3|ap|}{2|ab|}$, i.e., as a linear map from $s$ to $[1/2, 2]$. Suppose that $\Phi$ has a solution. Then we will that any optimal guard set $G$ of $P$ has size $g(\Phi)$ and specifies a solution to $\Phi$ in the sense that it satisfies the following two properties.

- Each variable $x \in X$ is specified consistently by the corresponding guards in $G$.
- The guard set $G$ is feasible, i.e., the values of $X$ thus specified give a solution to $\Phi$.

Moreover, if there is no solution to $\Phi$, each guard set of $P$ consists of more than $g(\Phi)$ guards.

### Theorem 6

The problem ETR-INV is $\exists \mathbb{R}$-complete.

**Sketch of proof.** As the conjunction of the equations of ETR-INV, together with the inequalities describing the restricted range $[1/2, 2]$ of the variables, is a quantifier-free formula of the first-order theory of the reals, ETR-INV is in $\exists \mathbb{R}$. To show $\exists \mathbb{R}$-hardness of ETR-INV, we perform a series of polynomial time reductions, starting from an instance $(\exists X_1 \ldots \exists X_k) \Phi(X_1, \ldots, X_k)$ of ETR. We use a result of Schaefer and Štefankovič [44] to reduce $\Phi$ to a polynomial $F$ with integer coefficients, so that the equation $F = 0$ has a solution if and only if $\Phi$ has a solution. Using a standard technique, we transform the equation $F = 0$ into a formula $\Phi'$ which is a conjunction of equations of the form $x = 1, x + y = z,$ and $x \cdot y = z$. It is known that for such a formula $\Phi'$ we either have no solution at all or there exists a solution inside the ball $B(r)$ centered at the origin with radius $r$. As we know some convenient upper bound on $r$, we can use this to scale down all variables to the range $[-1/8, 1/8]$. Thereafter, we shift them to the range $[1/2, 2]$. In the final step, which is our novel contribution, we show how to substitute each equation of the form $x + y = z$ with an equivalent set of equations using only addition and inversion, i.e., equations of the form $x + y = z$ and $x \cdot y = 1$, while ensuring that all variables stay in the range $[1/2, 2]$. Without this last constraint, the result would follow from the proof by Aho et al. [5, Section 8.2] that squaring and taking reciprocals is equivalent to multiplication.

The high-level idea of our construction is as follows. As for $x \notin \{0, 1\}$ we have $\frac{1}{x-1} - \frac{1}{x} = \frac{1}{x^2-x}$, a variable $V_{x^2}$ satisfying $V_{x^2} = x^2$ can be constructed from $x$ using only a sequence of additions and inversions. Similarly, as $(x + y)^2 - x^2 - y^2 = 2xy$, a variable $V_{xy}$ satisfying $V_{xy} = xy$ can be constructed from $x$ and $y$ using a sequence of additions and squaring. By controlling the possible ranges of variables at all stages of the reduction, we can ensure that in the final construction all variables stay in the range $[1/2, 2]$.

### 4 Reduction from ETR-INV to the art gallery problem

**Overview of the construction.** The details of this reduction are in Appendix [C]. Let $\Phi$ be an instance of ETR-INV consisting of $k$ equations with $n$ variables $X$. We show that there exists a polygon $P := P(\Phi)$ with corners at rational coordinates which can be computed in polynomial time such that $\Phi$ has a solution if and only if $P$ can be guarded by some number $g := g(\Phi)$ of guards. The number $g$ will follow from the construction. A sketch of the polygon $P$ is shown in Figure 1.

Each variable $x \in X$ is represented by a collection of guard segments, which are horizontal line segments contained in the interior of $P$. Consider one guard segment $s := ab$, where $a$ is to the left of $b$, and assume that $s$ represents the variable $x$ and that there is exactly one guard $p$ placed on $s$. The guard $p$ on $s$ specifies the value of the variable $x$, usually as $\frac{1}{2} + \frac{3|ap|}{2|ab|}$, i.e., as a linear map from $s$ to $[1/2, 2]$.

Suppose that $\Phi$ has a solution. Then we will that any optimal guard set $G$ of $P$ has size $g(\Phi)$ and specifies a solution to $\Phi$ in the sense that it satisfies the following two properties.

- Each variable $x \in X$ is specified consistently by the corresponding guards in $G$.
- The guard set $G$ is feasible, i.e., the values of $X$ thus specified give a solution to $\Phi$. 

Moreover, if there is no solution to $\Phi$, each guard set of $P$ consists of more than $g(\Phi)$ guards.
The polygon $\mathcal{P}$ is constructed in the following way. The bottom part of the polygon consists of a collection of pockets, containing in total $4n$ collinear and equidistant guard segments $s_1, \ldots, s_{4n}$. We denote the horizontal line containing these guard segments as the base line or $\ell_b$. At the left and at the right side of $\mathcal{P}$, there are some corridors attached, each of which leads into a gadget. The entrances to the corridors at the right side of $\mathcal{P}$ are line segments contained in a vertical line $\ell_r$. Likewise, the entrances to the corridors at the left side of $\mathcal{P}$ are contained in a vertical line $\ell_l$. The gadgets also contain guard segments, and they are used to impose dependencies between the guards in order to ensure that if there is a solution to $\Phi$, then any minimum guard set of $\mathcal{P}$ consists of $g(\Phi)$ guards and specifies a solution to $\Phi$ in the sense defined above. The corridors are used to copy the positions of guards on guard segments on the base line to guards on guard segments inside the gadgets. Each gadget corresponds to a constraint of one of the types $x + y \geq z$, $x + y \leq z$, $x \cdot y = 1$, $x + y \geq 5/2$, and $x + y \leq 5/2$. The first three types of constraints are used to encode the dependencies between the variables in $X$ as specified by $\Phi$, whereas the latter two constraints are used to attain different representations of the same variable, as needed in the inversion gadget. The constraints of type $x = 1$ are enforced by modifying the pocket containing a guard segment corresponding to $x$.

**Creating stationary guard positions and guard segments.** We denote some points in $\mathcal{P}$ as stationary guard positions. A guard placed at a stationary guard position is called a stationary guard. A stationary guard position is the unique point $p \in \mathcal{P}$ such that a guard placed at $p$ can see some set of corners (usually two corners) of $\mathcal{P}$. We will show that not placing a guard at a stationary guard position necessarily leads to a suboptimal solution. We use stationary guards for the purpose of seeing some region on one side of a line segment $\ell$, but no points on the other side of $\ell$. See Figure 2 (left) for an explanation of how such a construction can be made.

We likewise denote some horizontal line segments of $\mathcal{P}$ as guard segments. A guard segment $s$ consists of all points from which a guard can see some set of four corners of $\mathcal{P}$. See Figure 2 (middle) for an example of such a construction. The constraints of type $x = 1$ are enforced by modifying this construction, see Figure 2 (right).

In the full version of the reduction, we will prove that for any guard set of size of at most $g(\Phi)$, there is one guard placed on each stationary guard position and at each guard segment, and there are no guards except for these positions. As explained earlier, guards placed on the guard segments will be used to encode the values of the variables of $\Phi$. 

![Figure 1: A high-level sketch of the construction of the polygon $\mathcal{P}$.](image)
Imposing inequalities by nooks and umbras. The nooks and umbras are our basic tools used to impose dependency between guards placed on two different guard segments, see Figure 3. A nook and an umbra is a quadrilateral $Q$, defined by a pair of guard segments $r_0, r_1$ and a pair of corners $c_0, c_1$ of the polygon. The corners of $Q$ are $c_0, c_1$, and the two intersection points $f_0, f_1$ of rays originating at the endpoints of the guard segments, and containing the corners $c_0$ and $c_1$. For the case of the nook, the line segment $c_0c_1$ is in the interior of $P$, while $f_0f_1$ lies on the boundary of $P$. For the case of the umbra, $c_0c_1$ lies on the boundary of $P$, while $f_0f_1$ lies in the interior of $P$. The critical difference between the nook and an umbra is that the corners $c_0, c_1$ for nooks and umbras are blocking visibility of the line segment $f_0f_1$ from different sides. The line segment $f_0f_1$ is called a critical segment of $Q$. Formal definitions of nooks and umbras can be found in the appendix, see Definition 16 and Definition 17.

We will construct nooks and umbras for pairs of guard segments where we want to enforce dependency between the values of the corresponding variables. When making use of an umbra, we will also create a stationary guard position from which a guard sees the whole quadrilateral $Q$, but nothing on the other side of the critical segment $f_0f_1$. In this way we can enforce that the guards on $r_0$ and $r_1$ together see all of $f_0f_1$, since they need to see an open region on the other side of, and bounded by, $f_0f_1$. See Figure 3 for an example of a construction of both a nook and an umbra for a pair of guard segments.

A nook (or an umbra) $Q$ with the property that the corresponding corners $c_0$ and $c_1$ have the same $y$-coordinate is called a copy-nook (or a copy-umbra, respectively). Consider a pair of guard segments $r_0, r_1$, for which there is both a copy-nook and a copy-umbra. We can show that if there
Figure 4: $Q_1$ is a copy-nook of the segments $r_0 := a_0b_0$ and $r_1 := a_1b_1$ with a critical segment $f_0f_1$, and $Q_2$ is a copy-umbra for the same pair with a critical segment $f_2f_3$. It can be seen that this polygon cannot be guarded by fewer than 3 guards, and any guard set with 3 guards must contain a guard $g_0$ on $r_0$, a guard $g_1$ on $r_1$, and a stationary guard at the point $g_2$. The guards $g_0$ and $g_1$ must specify the same value on $r_0$ and $r_1$, respectively.

is one guard placed on each segment, and the guards together see the critical segments of the nook and the umbrella, then the two guards specify the same value. We use this observation to make one guard segment a copy of another. See Figure 4 where the guard segment $a_1b_1$ is a copy of $a_0b_0$.

**Corridors.** Inside each gadget there are three (or two) guard segments $r_i, r_j, r_l$ (or $r_i, r_j$) corresponding to three (or two) pairwise different guard segments from the base line $s_i, s_j, s_l$ (or $s_i, s_j$). A corridor ensures that the segments $r_i, r_j, r_l$ are copies of the segments $s_i, s_j, s_l$, respectively. To obtain that, we need to construct within the corridor a copy-nook and a copy-umbra for each pair of corresponding segments, see Figure 5 (left) for a simplified illustration, and Figure 5 (right) for more details within the corridor.

**Figure 5:** Left: In this figure, we display a simplified corridor construction. The corners $c_0, c_1$ are corners for three copy-umbras simultaneously for the pairs $(s_i, r_i), (s_j, r_j), (s_l, r_l)$. Each of these pairs also has a small copy-nook in the top of the corridor. The entrances $c_0d_0$ and $c_1d_1$ to the corridor are sufficiently small so that the critical segments of the nook and umbra of each pair of segments $s_\sigma, r_\sigma$ (contained in the small boxes at the top of the figure) are not seen by other guard segments. Right: The complete construction of the corridor.

The lower wall of the corridor is a horizontal edge $c_0, c_1$. The corners $c_0, c_1$ of the corridor are corners of three overlapping copy-umbras for the pairs $(s_i, r_i), (s_j, r_j)$, and $(s_l, r_l)$, respectively, see
the brown areas in Figure 5 (right). We construct the top wall of the corridor so that it creates three copy-nooks for the same pairs. To enforce that for any guard set of size $g(\Phi)$, for each $\sigma \in \{i,j,l\}$ the guard segments $s_\sigma$ and $r_\sigma$ specify the same value, we have to ensure that no guards on guard segments other than $s_\sigma$ and $r_\sigma$ can see the critical segments of the copy-umbra and the copy-nook of the pair $s_\sigma, r_\sigma$. The precise construction of the corridor that ensures this property is highly non-trivial. The high-level idea behind the construction is as follows. By placing the corridor sufficiently far away from the segments on the base line, and by making the corridor entrance (the line segment $c_0d_0$ in the figures) sufficiently small, we obtain that each point in the corridor which is far enough from the corridor entrances can be seen by points from at most one guard segment placed on the base line, and from at most one guard segment placed within the gadget. Stationary guards within the corridor – placed at points $p_i, p_j, p_l$ in Figure 5 (right) – ensure that the remaining area of the corridor is seen.

**Addition gadget.** For any equation of the form $x + y = z$ in $\Phi$, we construct a $\geq$-addition gadget which represents the inequality $x + y \geq z$, and a $\leq$-addition gadget for the inequality $x + y \leq z$. The general idea behind the construction of the gadget imposing the $\geq$-inequality can be seen in Figure 6.

![Figure 6](image.png)

We can place three guard segments in such a way that a certain quadrilateral $\Gamma$ is seen by the guards $g_x, g_y, g_z$ placed on the respective guard segments if and only if the corresponding values $x, y, z$ satisfy the inequality $x + y \geq z$. The area of the gadget except $\Gamma$ can be seen with the help of stationary guards. The actual gadget is more complicated, as the guard segments within the gadget need to be copies (where copying is performed via corridor, as described earlier) of the base line guard segments.

The gadget enforcing the $\leq$-inequality is obtained in a similar way, and the idea behind it can be pictured by a reflection of the picture from Figure 6.

**Inversion gadget.** In the inversion gadget, we have two guard segments corresponding to some variables $x$ and $y$. We construct an umbra $Q_u$ (this time not a copy-umbra) such that the two guards placed on the segments see the whole critical segment of $Q_u$ if and only if the corresponding values satisfy $x \cdot y \leq 1$. We also construct a nook $Q_n$ which enforces the inequality $x \cdot y \geq 1$. Enforcing that the guards together see both $Q_u$ and $Q_n$ yields the equality $x \cdot y = 1$. The difficulty
in the construction was finding rational coordinates of the corners of the nook and the umbra, which seemed only to be possible if the two guard segments were at different heights. This construction also requires changing orientation of one of the guard segments, such that the left endpoint corresponds to the value 2, and the right endpoint corresponds to the value $1/2$.

5 Conclusion

We have shown that the art gallery problem has strong connections to real algebra, in the algorithmic as well as the topological sense. While Micha Sharir showed that the art gallery problem can be solved using algebraic methods, our results justify the lack of exact algorithms which do not use this machinery. Indeed, any algorithm for the art gallery problem must be implicitly capable of deciding feasibility of arbitrary systems of polynomial equations and inequalities in any number of variables.

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A The art gallery problem is in $\exists R$

In this section we will prove that the art gallery problem is in the complexity class $\exists R$. Our proof works also for a more general version of the art gallery problem, where the input polygon can have polygonal holes.

Theorem 7. The art gallery problem is in the complexity class $\exists R$.

Proof. Let $(P, g)$ be an instance of the art gallery problem where the polygon $P$ has $n$ corners, each of which has rational coordinates represented by at most $B$ bits. We show how to construct a quantifier-free formula $\Phi := \Phi(P, g)$ of the first-order theory of the reals such that $\Phi$ is satisfiable if and only if $P$ has a guard set of cardinality $g$. The formula $\Phi$ has length $O(g^2 n^7 B^2) = O(n^{14} B^2)$ and can be computed in polynomial time. It has been our priority to define the formula $\Phi$ so that it is as simple as possible to describe. It might be possible to construct an equivalent but shorter formula.

The description of $\Phi$ is similar to the formula $\Psi$ that Micha Sharir described to Efrat and Har-Peled [21]

$$
\Psi := \left[ \exists x_1, y_1, \ldots, x_k, y_k \forall p_x, p_y : \text{INSIDE-POLYGON}(p_x, p_y) \implies \bigvee_{i=1}^{k} \text{SEES}(x_i, y_i, p_x, p_y) \right].
$$

For each $i \in \{1, \ldots, k\}$, the variables $x_i, y_i$ represent the position of guard $g_i := (x_i, y_i)$, and $p := (p_x, p_y)$ represents an arbitrary point. The predicate $\text{INSIDE-POLYGON}(p_x, p_y)$ tests if the point $p$ is contained in the polygon $P$, and $\text{SEES}(x_i, y_i, p_x, p_y)$ checks if the guard $g_i$ can see the point $p$. Thus, the formula is satisfiable if and only if there is a guard set of cardinality $g$. Note that although the implication “$\implies$” is not allowed in the first order theory of the reals, we can always substitute “$A \implies B$” by “$\neg A \lor B$”.

For the purpose of self-containment, we will briefly repeat the construction of the predicates $\text{INSIDE-POLYGON}(p_x, p_y)$ and $\text{SEES}(x_i, y_i, p_x, p_y)$. The elementary tool is evaluation of the sign of the determinant $\det(\vec{u}, \vec{v})$ of two vectors $\vec{u}, \vec{v}$. Recall that the sign of the expression $\det(\vec{u}, \vec{v})$ determines whether $\vec{v}$ points to the left of $\vec{u}$ (if $\det(\vec{u}, \vec{v}) > 0$), is parallel to $\vec{u}$ (if $\det(\vec{u}, \vec{v}) = 0$), or points to the right of $\vec{u}$ (if $\det(\vec{u}, \vec{v}) < 0$).

We compute a triangulation $T$ of the polygon $P$, e.g., using an algorithm from [17], order the corners of each triangle of $T$ in the counter-clockwise order, and orient each edge of the triangle accordingly. A point is contained inside the polygon if and only if it is contained in one of the triangles of $T$. A point is contained in a triangle if and only if it is on one of the edges or to the left of each edge. Thus the predicate $\text{INSIDE-POLYGON}(p_x, p_y)$ has length $O(nB)$.

A guard $g_i$ sees a point $p$ if and only if no two consecutive edges of $P$ block the visibility. See Figure 7 on why it is not sufficient to check each edge individually. Given a guard $g_i$, a point $p$, and two consecutive edges $e_1, e_2$ of $P$, it can be checked by evaluating a constant number of determinants whether $e_1, e_2$ block the visibility between $g_i$ and $p$. Thus $\text{SEES}(x_i, y_i, p_x, p_y)$ has length $O(nB)$ and consequently $\bigvee_{i=1}^{n} \text{SEES}(x_i, y_i, p_x, p_y)$ has length $O(knB)$.

Note that the formula $\Psi$ is not a formula in ETR because of the universal quantifier. The main idea to get an equivalent formula with no universal quantifier is to find a polynomial number of points inside $P$ which, if all seen by the guards, ensure that all of $P$ is seen. We denote such a set of points as a witness set.

Creating a witness set. We are now ready to describe the witness set that replaces the universal quantifier. Let $L := \{\ell_1, \ldots, \ell_m\}$ be the set of lines containing either an edge of $P$, or a guard
\[ g \in G \text{ and a corner } v \in P^\bullet. \]

The well-defined lines in \( L \) partition the plane into \textit{regions}, which are connected components of \( \mathbb{R}^2 \setminus \bigcup_{\ell \in L} \ell \) (see Figure 7 for an example).

Let \( A \) be the set of these regions. It is easy to verify that \( A \) has the following properties:

- Each region in \( A \) is an open convex polygon.
- Each region in \( A \) is either contained in \( P \) or contained in the complement of \( P \).
- The closure of the union of the regions that are contained in \( P \) equals \( P \).
- For each region \( R \in A \), each guard \( g \in G \) either sees all points of \( R \) or sees no point in \( R \). In particular, if a guard \( g \) sees one point in \( R \), it sees all of \( R \) and its closure.

Thus it is sufficient to test that for each region \( R \in A \) which is in \( P \), at least one point in \( R \) is seen by a guard. For three points \( a, b, c \), define the \textit{centroid} of \( a, b, c \) to be the point \( C(a, b, c) := (a+b+c)/3 \).

If \( a, b, c \) are three different corners of the same region \( R \in A \), then the centroid \( C(a, b, c) \) must lie in the interior of \( R \). Note that each region has at least three corners and thus contains at least one such centroid. Let \( X \) be the set of all intersection points between two non-parallel well-defined lines in \( L \), i.e., \( X \) consists of all corners of all regions in \( A \). In the formula \( \Phi \), we generate all points in \( X \). For any three points \( a, b, c \) in \( X \), we also generate the centroid \( C(a, b, c) \). If the centroid is in \( P \), we check that it is seen by a guard. Since there are \( O((kn)^2) \) centroids of three points in \( X \) and each is tested by a formula of size \( O(knB) \), we get a formula of the aforementioned size.

**Constructing the formula \( \Phi \).** Each line \( \ell_i \) is defined by a pair of points \( \{(p_i, q_i), (p'_i, q'_i)\} \). Let \( \bar{\ell}_i := (p'_i - p_i, q'_i - q_i) \) be a direction vector corresponding to the line. A line \( \ell \) is well-defined if and only if the corresponding vector \( \bar{\ell} \) is non-zero.

The lines \( \ell_i, \ell_j \) are well-defined and non-parallel if and only if \( \det(\bar{\ell}_i, \bar{\ell}_j) \neq 0 \). If two lines \( \ell_i, \ell_j \) are well-defined and non-parallel, their intersection point \( X^{ij} \) is well-defined and it has coordinates

\[
\left( \frac{(p_j q'_j - q_j p'_j)(p'_i - p_i) - (p_i q'_i - q_i p'_i)(p_j - p_j)}{\det(\bar{\ell}_i, \bar{\ell}_j)} \right), \quad \left( \frac{(p_j q'_j - q_j p'_j)(p_i - p'_i) - (p_i q'_i - q_i p'_i)(q'_j - q_j)}{\det(\bar{\ell}_i, \bar{\ell}_j)} \right).
\]

\*Note that if a line is defined as passing through a guard \( g \in G \) and a corner \( v \in P \) such that \( g \) and \( v \) are coincident, the line is not well-defined. Such lines are not considered in the partition into regions described below, but they are included in the set \( L \). Later we will show how to ignore these lines in our formula.
For each pair \((i, j) \in \{1, \ldots, m\}^2\), we add the variables \(x_{ij}, y_{ij}\) to the formula \(\Phi\) and we define \(\text{INTERSECT}(i, j)\) to be the formula
\[
\det(\vec{\ell}_i, \vec{\ell}_j) \neq 0 \implies \left[\det(\vec{\ell}_i, \vec{\ell}_j) \cdot x_{ij} = (p_j q'_j - q_j p'_j)(p'_i - p_i) - (p_i q'_i - q_i p'_i)(p'_j - p_j) \land \right.
\[
\left. \det(\vec{\ell}_i, \vec{\ell}_j) \cdot y_{ij} = (p_j q'_j - q_j p'_j)(q'_i - q_i) - (p_i q'_i - q_i p'_i)(q'_j - q_j) \right].
\]
It follows that if the formula \(\text{INTERSECT}(i, j)\) is true then either
- \(\ell_i\) or \(\ell_j\) is not well-defined or they are both well-defined, but parallel, or
- \(\ell_i\) and \(\ell_j\) are well-defined and non-parallel and the variables \(x_{ij}\) and \(y_{ij}\) are the coordinates of the intersection point \(X_{ij}\) of the lines.

Let \(\Lambda := \{\lambda_1, \ldots, \lambda_{m^6}\} = \{1, \ldots, m\}^6\) be all the tuples of six elements from the set \(\{1, \ldots, m\}\). Each tuple \(\lambda := (a, b, c, d, e, f) \in \Lambda\) corresponds to a centroid of the following three points: the intersection point of the lines \(\ell_a, \ell_b\), the intersection point of the lines \(\ell_c, \ell_d\), and the intersection point of the lines \(\ell_e, \ell_f\). For each tuple \(\lambda\), we proceed as follows. We define the formula \(\text{CENTROID-DEFINED}(\lambda)\) to be
\[
\det(\vec{\ell}_a, \vec{\ell}_b) \neq 0 \land \det(\vec{\ell}_c, \vec{\ell}_d) \neq 0 \land \det(\vec{\ell}_e, \vec{\ell}_f) \neq 0.
\]
We add the variables \(u_\lambda, v_\lambda\) to the formula \(\Phi\), and define the formula \(\text{CENTROID}(\lambda)\) as
\[
3u_\lambda = x_{ab} + x_{cd} + x_{ef} \land 3v_\lambda = y_{ab} + y_{cd} + y_{ef}.
\]
It follows that if the formulas \(\text{CENTROID-DEFINED}(\lambda)\) and \(\text{CENTROID}(\lambda)\) are both true, then the lines in each of the pairs \((\ell_a, \ell_b), (\ell_c, \ell_d), (\ell_e, \ell_f)\) are well-defined and non-parallel, and the variables \(u_\lambda\) and \(v_\lambda\) are the coordinates of the centroid \(C(X_{ab}, X_{cd}, X_{ef})\).

We are now ready to write up our existential formula as
\[
\exists x_1, y_1, \ldots, x_k, y_k \exists x_{11}, y_{11}, x_{12}, y_{12}, \ldots, x_{mm}, y_{mm} \exists u_\lambda_1, v_\lambda_1, \ldots, u_\lambda_{m^6}, v_\lambda_{m^6} : \Phi,
\]
where
\[
\Phi := \left[\bigwedge_{(i, j) \in \{1, \ldots, m\}^2} \text{INTERSECT}(i, j) \right] \land \\
\left[\bigwedge_{\lambda \in \Lambda} \left[\text{CENTROID-DEFINED}(\lambda) \implies \left[\text{CENTROID}(\lambda) \land \left[\text{INSIDE-POLYGON}(u_\lambda, v_\lambda) \implies \bigvee_{i=1}^k \text{SEES}(x_i, y_i, u_\lambda, v_\lambda)\right]\right]\right]\right].
\]
B  The problem ETR-INV

To show that the art gallery problem is ∃R-hard, we will provide a reduction from the problem ETR-INV, which we introduce below. In this section, we will show that ETR-INV is ∃R-complete.

Definition 5 (ETR-INV). In the problem ETR-INV, we are given a set of real variables \(\{x_1, \ldots, x_n\}\), and a set of equations of the form

\[
x = 1, \quad x + y = z, \quad x \cdot y = 1,
\]

for \(x, y, z \in \{x_1, \ldots, x_n\}\). The goal is to decide whether the system of equations has a solution when each variable is restricted to the range \([1/2, 2]\).

If \(\Phi(x)\) is an instance of ETR-INV with variables \(x := (x_1, \ldots, x_n)\), the space of solutions \(S_\Phi := \{x \in [1/2, 2]^n : \Phi(x)\}\) consists of the vectors in \([1/2, 2]^n\) that satisfy all the equations of \(\Phi\). In one sense, the range \([1/2, 2]\) is the simplest possible: The range of course needs to contain 1, since we have equations of the form \(x = 1\). In order to include just one more integer, namely 2, we also need to include \(1/2\) since we have equations of the form \(x \cdot y = 1\).

In order to show that ETR-INV is ∃R-complete, we make use of the following problem.

Definition 8. In the problem ETR\(^c_+,+,\cdot\), where \(c \in \mathbb{R}\), we are given a set of real variables \(\{x_1, \ldots, x_n\}\), and a set of equations of the form

\[
x = c, \quad x + y = z, \quad x \cdot y = z,
\]

for \(x, y, z \in \{x_1, \ldots, x_n\}\). The goal is to decide whether the system of equations has a solution.

A modified version of the problem, where we additionally require that \(x_1, \ldots, x_n \in [a, b]\) for some \(a, b \in \mathbb{R}\), is denoted by ETR\(^c_+,+,\cdot\)\([a, b]\).

We are now ready to prove that ETR-INV is ∃R-complete.

Theorem 6. The problem ETR-INV is ∃R-complete.

Proof. To show that ETR-INV is ∃R-hard, we will perform a series of polynomial time reductions, starting from ETR and subsequently reducing it to the problems ETR\(^1_+,+,\cdot\), ETR\(^{1/8,+,\cdot}\)\([1/2, 2]\), and ending with ETR-INV.

To simplify the notation, while considering a problem ETR\(^c_+,+,\cdot\) or ETR\(^c_+,+,\cdot\)\([a, b]\), we might substitute any variable in an equation by the constant \(c\). For instance, \(x + c = z\) is a shorthand for the equations \(x + y = z\) and \(y = c\), where \(y\) is an additional variable.

Reduction to ETR\(^1_+,+,\cdot\). We will first argue that ETR\(^1_+,+,\cdot\) is ∃R-hard. This seems to be folklore, but we did not find a formal statement. For the sake of self-containment and rigorousness, we present here a short proof based on the following lemma.

Lemma 9 (Schaefer, Stefaňi\v{c} [13]). Let \(\Phi(x)\) be a quantifier-free formula of the first order theory of the reals, where \(x := (x_1, x_2, \ldots, x_n)\) is a vector of variables. We can construct in polynomial time a polynomial \(F : \mathbb{R}^{n+m} \to \mathbb{R}\) of degree 4, for some \(m = O(|\Phi|)\), so that

\[
\{x \in \mathbb{R}^n : \Phi(x)\} = \{x \in \mathbb{R}^n : (\exists y \in \mathbb{R}^m) F(x, y) = 0\}.
\]

The coefficients of \(F\) have bitlength \(O(|\Phi|)\).
Thus it is $\exists\mathbb{R}$-hard to decide if a polynomial has a real root. We reduce this problem to $\text{ETR}^{1,+.}$. Consider a polynomial equation $Q = 0$. First, we generate all variables corresponding to all the coefficients of $Q$, by using only the constant 1, addition and multiplication. For example, a variable corresponding to 20 can be obtained as follows: $V_1 = 1, V_2 = 1 + 1, V_2^2 = V_1 \cdot V_1, V_3 = V_2^2 \cdot V_2, V_4 = V_2^2 + V_2$. We are now left with a polynomial $Q'$ consisting entirely of sums of products of variables, and we keep simplifying $Q'$ as described in the following. Whenever there is an occurrence of a sum $x + y$ or a product $x \cdot y$ of two variables in $Q'$, we introduce a new variable $z$. In the first case, we add the equation $x + y = z$ to $\Phi$ and substitute the term $x + y$ by $z$ in $Q'$. In the latter case, we add the equation $x \cdot y = z$ to $\Phi$ and substitute $x \cdot y$ by $z$ in $Q'$. We finish the construction when $Q'$ has been simplified to consist of a single variable, i.e., $Q' = x$, in which case we add the equation $x + V_1 = V_1$ (corresponding to the equation $Q' = 0$) to $\Phi$. When the process finishes, $\Phi$ yields an instance of $\text{ETR}^{1,+.}$, and the solutions to $\Phi$ are in one-to-one correspondence with the solutions to the original polynomial equation $Q = 0$.

Reduction to $\text{ETR}_{[1/8,1/8]}^{1/8,+.}$. We will now present a reduction from the problem $\text{ETR}^{1,+.}$ to $\text{ETR}_{[1/8,1/8]}^{1/8,+.}$. We use the following result from algebraic geometry, which was stated by Schaefer and Štefankovič [43] in a simplified form. Given an instance $\Phi(x)$ of $\text{ETR}$ over the vector of variables $x := (x_1, \ldots, x_n)$, we define the semi-algebraic set $S_\Phi$ as the solution space

$$S_\Phi := \{x \in \mathbb{R}^n : \Phi(x)\}.$$ 

The complexity $L$ of a semi-algebraic set $S_\Phi$ is defined as the number of symbols appearing in the formula $\Phi$ defining $S_\Phi$ (see [32]).

Corollary 10 (Schaefer and Štefankovič [43]). Let $B$ be the set of points in $\mathbb{R}^n$ at distance at most $2^{L^8 n} = 2^{8^n \log L}$ from the origin. Every non-empty semi-algebraic set $S$ in $\mathbb{R}^n$ of complexity at most $L \geq 4$ contains a point in $B$.

Let $\Phi$ be an instance of $\text{ETR}^{1,+.}$ with $n$ variables $x_1, \ldots, x_n$. We construct an instance $\Phi'$ of $\text{ETR}_{[1/8,1/8]}^{1/8,+}$ such that $\Phi$ has a solution if and only if $\Phi'$ has a solution. Let us fix $k := \lceil 8n \cdot \log L + 3 \rceil$ and $\varepsilon := 2^{-2k}$. In $\Phi'$, we first define a variable $V_\varepsilon$ satisfying $V_\varepsilon = \varepsilon$, using $\Theta(k)$ new variables $V_{1/2^2}, V_{1/2^3}, \ldots, V_{1/2^{2k}}$ and equations

$$V_{1/2^2} + V_{1/2^2} = 1/8, \quad V_{1/2^2} \cdot V_{1/2^2} = V_{1/2^3}, \quad V_{1/2^3} \cdot V_{1/2^3} = V_{1/2^4}, \quad \ldots \quad V_{1/2^{2k-1}} \cdot V_{1/2^{2k-1}} = V_\varepsilon.$$ 

In $\Phi'$, we use the variables $V_{\varepsilon x_1}, \ldots, V_{\varepsilon x_n}$ instead of $x_1, \ldots, x_n$. An equation of $\Phi$ of the form $x = 1$ is transformed to the equation $V_{\varepsilon x} = V_\varepsilon$ in $\Phi'$. An equation of $\Phi$ of the form $x + y = z$ is transformed to the equation $V_{\varepsilon x} + V_{\varepsilon y} = V_{\varepsilon z}$ of $\Phi'$. An equation of $\Phi$ of the form $x \cdot y = z$ is transformed to the following equations of $\Phi'$, where $V_{\varepsilon z}$ is a new variable satisfying

$$V_{\varepsilon x} \cdot V_{\varepsilon y} = V_{\varepsilon z}, \quad V_\varepsilon \cdot V_{\varepsilon z} = V_{\varepsilon z}.$$ 

19
Assume that $\Phi$ is true. Then there exists an assignment of values to the variables $x_1, \ldots, x_n$ of $\Phi$ that satisfies all the equations and where each variable $x_i$ satisfies $|x_i| \in \left[0, 2^{8n \log L}\right]$. Then the assignment $V_{x_i} = \varepsilon x_i$ and (when $V_{x_i}$ appears in $\Phi'$) $V_{x_i} = \varepsilon x_i$ yields a solution to $\Phi'$ with all variables in the range $[-1/8, 1/8]$. On the other hand, if there is a solution to $\Phi'$, an analogous argument yields a corresponding solution to $\Phi$. We have given a reduction from $\text{ETR}^{1,+}$ to $\text{ETR}^{1,+}$.

The length of the formula increases by at most a polylogarithmic factor.

**Reduction to $\text{ETR}^{1,+}$.** We will now show a reduction from $\text{ETR}^{1/8,+}$ to $\text{ETR}^{1/2}$. The reduction is similar to the one in [47]. We substitute each variable $x_i \in [-1/8, 1/8]$ by $V_{x_i} + 7/8$ which will be assumed to have a value of $x_i + 7/8$. Instead of an equation $x = 1/8$ we now have $V_{x_i + 7/8} = 1$. Using addition and the variable equal 1, we can easily get the variables $V_{1/2}, V_{3/2}, V_{3/4}, V_{7/4}, V_{7/8}$ with corresponding values of $1/2, 3/2, 3/4, 7/4$, and $7/8$. Instead of each equation $x + y = z$ we now have equations:

\[
V_{x+7/8} + V_{y+7/8} = V_{(z+7/8)+7/8}, \\
V_{z+7/8} + V_{7/8} = V_{(z+7/8)+7/8}.
\]

As the original variables $x, y, z$ have values in the interval $[-1/8, 1/8]$, the added variables $V_{(z+7/8)+7/8}$ have a value in $[13/8, 15/8]$.

Instead of each equation $x \cdot y = z$ we have the following set of equations

\[
V_{x+7/8} + V_{y+7/8} = V_{x+y+14/8}, \quad (V_{x+y+14/8} \in [12/8, 2]) \\
V_{x+y+7/8} + V_{7/8} = V_{x+y+14/8}, \quad (V_{x+y+7/8} \in [5/8, 9/8]) \\
V_{x+7/8} + V_{i} = V_{x+i+15/8}, \quad (V_{x+i+15/8} \in [14/8, 2]) \\
V_{x+1} + V_{7/8} = V_{x+15/8}, \quad (V_{x+1} \in [7/8, 9/8]) \\
V_{y+7/8} + V_{i} = V_{y+i+15/8}, \quad (V_{y+i+15/8} \in [14/8, 2]) \\
V_{y+1} + V_{7/8} = V_{y+i+15/8}, \quad (V_{y+1} \in [7/8, 9/8]) \\
V_{x+1} \cdot V_{y+1} = V_{xy+x+y+1}, \quad (V_{xy+x+y+1} \in [49/64, 81/64]) \\
V_{xy+x+y+1} + V_{1/2} = V_{xy+x+y+3/2}, \quad (V_{xy+x+y+3/2} \in [81/64, 113/64]) \\
V_{xy+5/8} + V_{x+y+7/8} = V_{xy+x+y+3/2}, \quad (V_{xy+5/8} \in [39/64, 41/64]) \\
V_{xy+5/8} + 1 = V_{xy+13/8}, \quad (V_{xy+13/8} \in [103/64, 105/64]) \\
V_{z+7/8} + V_{3/4} = V_{xy+13/8}.
\]

Each formula $\Phi$ of $\text{ETR}^{1/8,+}$ is transformed to a formula $\Phi'$ of $\text{ETR}^{1/2}$, as explained above. If there is a solution for $\Phi$, there clearly is a solution for $\Phi'$, as all the newly introduced variables have a value within the intervals claimed above. If there is a solution for $\Phi'$, there is also a solution for $\Phi$, as the newly introduced variables $V_{2x+7/4}, V_{x+5/8} \in [1/2, 2]$ ensure that $x \in [-1/8, 1/8]$. The increase in the length of the formula is linear.

Note that the only place where we use multiplication is in the formula $V_{x+1} \cdot V_{y+1} = V_{xy+x+y+1}$, where $V_{x+1}, V_{y+1} \in [7/8, 9/8]$. We will use this fact in the next step of the reduction.

**Reduction to $\text{ETR-INV}$.** We will now show that $\text{ETR}^{1/8,+}$ reduces to $\text{ETR-INV}$. In the first step, we reduce a formula $\Phi$ of $\text{ETR}^{1/8,+}$ to a formula $\Phi'$ of $\text{ETR}^{1/2}$, as described in the step above. We now have to show how to express each equation $x \cdot y = z$ of $\Phi'$ using only the equations allowed in $\text{ETR-INV}$. Note that, as explained above, multiplication is used only

20
for variables $x, y \in [7/8, 9/8]$. Some of the steps in this reduction rely on techniques also used in the proof by Aho et al. [5] Section 8.2] that squaring and taking reciprocals is equivalent to multiplication.

We first show how to define a new variable $V_{x^2}$ satisfying $V_{x^2} = x^2$, where $x \in [7/8, 9/8]$.

$$x + V_{3/4} = V_{x+3/4}, \quad (V_{x+3/4} \in [13/8, 15/8])$$

$$V_{1/(x+3/4)} \cdot V_{x+3/4} = 1, \quad (V_{1/(x+3/4)} \in [8/15, 8/13])$$

$$V_{x-1/4} + 1 = V_{x+3/4}, \quad (V_{x-1/4} \in [5/8, 7/8])$$

$$V_{1/(x-1/4)} \cdot V_{x-1/4} = 1, \quad (V_{1/(x-1/4)} \in [8/7, 8/5])$$

$$V_{1/(x^2+x/2-3/16)} + V_{1/(x+3/4)} = V_{1/(x-1/4)}, \quad (V_{1/(x^2+x/2-3/16)} \in [64/105, 64/65])$$

$$V_{1/(x^2+x/2-3/16)} \cdot V_{x^2+x/2-3/16} = 1, \quad (V_{x^2+x/2-3/16} \in [65/64, 105/64])$$

$$x + V_{7/8} = V_{x+7/8}, \quad (V_{x+7/8} \in [14/8, 2])$$

$$V_{x+1/8} + 3/4 = V_{x+7/8}, \quad (V_{x+1/8} \in [1, 10/8])$$

$$V_{x/2+1/16} + V_{x/2+1/16} = V_{x+1/8}, \quad (V_{x/2+1/16} \in [1/2, 10/16])$$

$$V_{x^2-1/4} + V_{x/2+1/16} = V_{x^2+x/2-3/16}, \quad (V_{x^2-1/4} \in [33/64, 65/64])$$

$$V_{x^2-1/4} + V_{3/4} = V_{x^2+1/2}, \quad (V_{x^2+1/2} \in [81/64, 113/64])$$

$$V_{x^2} + V_{1/2} = V_{x^2+1/2}.$$  

Note that the constructed variables are in the range $[1/2, 2]$. In the following, as shorthand for the construction given above, we allow to use equations of the form $x^2 = y$, for a variable $x$ with a value in $[7/8, 9/8]$. We now describe how to express an equation $x \cdot y = z$, where $x, y \in [7/8, 9/8]$.

$$x + V_{7/8} = V_{x+7/8}, \quad (V_{x+7/8} \in [14/8, 2])$$

$$V_{(x+7/8)/2} + V_{(x+7/8)/2} = V_{x+7/8}, \quad (V_{(x+7/8)/2} \in [14/16, 1])$$

$$y + V_{7/8} = V_{y+7/8}, \quad (V_{y+7/8} \in [14/8, 2])$$

$$V_{(y+7/8)/2} + V_{(y+7/8)/2} = V_{y+7/8}, \quad (V_{(y+7/8)/2} \in [14/16, 1])$$

$$V_{(x+y)/2} + V_{(x+y)/2} = V_{(x+y)/2+7/8}, \quad (V_{(x+y)/2+7/8} \in [14/8, 2])$$

$$V_{(x+y)/2 + V_{7/8}} = V_{(x+y)/2+7/8}, \quad (V_{(x+y)/2+7/8} \in [7/8, 9/8])$$

$$V_{2/(x+y)/2} = V_{((x+y)/2)^2}, \quad (V_{((x+y)/2)^2} \in [49/64, 61/64])$$

$$V_{((x+y)/2)^2} + V_{1/2} = V_{((x+y)/2)^2+1/2}, \quad (V_{((x+y)/2)^2+1/2} \in [81/64, 113/64])$$

$$x^2 = V_{x^2}, \quad (V_{x^2} \in [49/64, 61/64])$$

$$y^2 = V_{y^2}, \quad (V_{y^2} \in [49/64, 61/64])$$

$$V_{x^2} + V_{1/2} = V_{x^2+1/2}, \quad (V_{x^2+1/2} \in [81/64, 113/64])$$

$$V_{x^2/2+1/4} + V_{x^2/2+1/4} = V_{x^2+1/2}, \quad (V_{x^2/2+1/4} \in [81/128, 113/128])$$

$$V_{y^2} + V_{1/2} = V_{y^2+1/2}, \quad (V_{y^2+1/2} \in [81/64, 113/64])$$

$$V_{y^2/2+1/4} + V_{y^2/2+1/4} = V_{y^2+1/2}, \quad (V_{y^2/2+1/4} \in [81/128, 113/128])$$

$$V_{x^2+y^2}/2 + V_{1/2} = V_{(x^2+y^2)/2+1/2}, \quad (V_{(x^2+y^2)/2+1/2} \in [81/64, 113/64])$$

$$V_{(x^2+y^2)/4+1/4} + V_{(x^2+y^2)/4+1/4} = V_{(x^2+y^2)/2+1/2}, \quad (V_{(x^2+y^2)/4+1/4} \in [81/128, 113/128])$$

$$V_{(x^2+y^2)/4+1/4} + V_{xy/2+1/4} = V_{((x+y)/2)^2+1/2}, \quad (V_{xy/2+1/4} \in [81/128, 113/128])$$
\[ V_{xy/2+1/4} + V_{xy/2+1/4} = V_{xy+1/2}; \quad (V_{xy+1/2} \in [81/64, 113/64]) \]
\[ z + V_{1/2} = V_{xy+1/2}. \]

The constructed variables are in a range \([1/2, 2]\).

A formula \(\Phi\) of \(\mbox{ETR}_{1/8, +}^{1/2, \cdot} \) has been first transformed into a formula \(\Phi'\) of \(\mbox{ETR}_{[1/2, 2]}^{1/2, \cdot}\), and subsequently into a formula \(\Phi''\) of \(\mbox{ETR-INV}\). If \(\Phi\) is satisfiable, then both \(\Phi'\) and \(\Phi''\) are satisfiable. If \(\Phi''\) is satisfiable, then both \(\Phi'\) and \(\Phi\) are satisfiable. We get that \(\mbox{ETR-INV}\) is \(\exists \mathbb{R}\)-hard.

As the conjunction of the equations of \(\mbox{ETR-INV}\), together with the inequalities describing the allowed range of the variables within \(\mbox{ETR-INV}\), is a quantifier-free formula of the first-order theory of the reals, \(\mbox{ETR-INV}\) is in \(\exists \mathbb{R}\), which yields that \(\mbox{ETR-INV}\) is \(\exists \mathbb{R}\)-complete.

Lemma 9 and Corollary 10 together with the reductions explained in this section imply the following lemma.

**Lemma 11.** Let \(\Phi\) be an instance of \(\mbox{ETR}\) with variables \(x_1, \ldots, x_n\). Then there exists an instance \(\Psi\) of \(\mbox{ETR-INV}\) with variables \(y_1, \ldots, y_m\), \(m \geq n\), and constants \(c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{Q}\), such that

- there is a solution to \(\Phi\) if and only if there is a solution to \(\Psi\), and
- for any solution \((y_1, \ldots, y_m)\) to \(\Psi\), there exists a solution \((x_1, \ldots, x_n)\) to \(\Phi\) where \(y_1 = c_1 x_1 + d_1, \ldots, y_n = c_n x_n + d_n\).

For the proof of Theorem 48 and Theorem 3, we will need a stronger statement. Recall that Corollary 10 says that there is a large ball which intersects a given semi-algebraic set. The following related result by Basu and Roy [8] says that if the semi-algebraic set is compact, the ball will in fact contain the set.

**Corollary 12** (Basu and Roy [8]). For any compact semi-algebraic set \(S \subseteq \mathbb{R}^n\) with description complexity \(L\) it holds that \(S \subseteq B(0, 2^{2O(L \log L)}\).

From the corollary, we get the following alternative stronger version of Lemma 11 for compact semi-algebraic sets.

**Lemma 13.** Let \(\Phi\) be an instance of \(\mbox{ETR}\) with variables \(x_1, \ldots, x_n\) and a compact set of solutions. Then there exists an instance \(\Psi\) of \(\mbox{ETR-INV}\) with variables \(y_1, \ldots, y_m\), \(m \geq n\), and constants \(c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{Q}\), such that \((y_1, \ldots, y_m)\) is a solution to \(\Psi\) if and only if there exists a solution \((x_1, \ldots, x_n)\) to \(\Phi\) with \(y_1 = c_1 x_1 + d_1, \ldots, y_n = c_n x_n + d_n\).

The only change we need to make in the construction of the \(\mbox{ETR-INV}\) instance \(\Psi\) is that instead of defining \(k := [8n \cdot \log L + 3]\), as when we use Corollary 10 we now define \(k := [C \cdot L \log L + 3]\), where \(C\) is the constant hidden in \(O(L \log L)\) in Corollary 12.
C Reduction from ETR-INV to the art gallery problem

C.1 Notation

Given two different points \( p, q \), the line containing \( p \) and \( q \) is denoted as \( \overrightarrow{pq} \), the ray with the origin at \( p \) and passing through \( q \) is denoted as \( \overrightarrow{pq} \), and the line segment from \( p \) to \( q \) is denoted as \( pq \). For a point \( p \), we let \( x(p) \) and \( y(p) \) denote the \( x \)- and \( y \)-coordinate of \( p \), respectively. Table 1 shows the definitions of some objects and distances frequently used in the description of the construction.

C.2 Overview of the construction

Let \( \Phi \) be an instance of the problem ETR-INV with \( n \) variables \( X := \{x_1, \ldots, x_n\} \) and consisting of \( k \) equations. We show that there exists a polygon \( P := P(\Phi) \) with corners at rational coordinates which can be computed in polynomial time such that \( \Phi \) has a solution if and only if \( P \) can be guarded by some number \( g := g(\Phi) \) of guards. The number \( g \) will follow from the construction. A sketch of the polygon \( P \) is shown in Figure 8.

![Figure 8: A high-level sketch of the construction of the polygon \( P \).](image)

Each variable \( x_i \in X \) is represented by a collection of guard segments, which are horizontal line segments contained in the interior of \( P \). Consider one guard segment \( s := ab \), where \( a \) is to the left of \( b \), and assume that \( s \) represents the variable \( x_i \) and that there is exactly one guard \( p \) placed on \( s \). The guard segment \( s \) can be oriented to the right or to the left. The guard \( p \) on \( s \) specifies the value of the variable \( x_i \) as \( \frac{1}{2} + \frac{3\|ap\|}{2\|ab\|} \) if \( s \) is oriented to the right, and \( \frac{1}{2} + \frac{3\|bp\|}{2\|ab\|} \) if \( s \) is oriented to the left, i.e., the value is a linear map from \( s \) to \([\frac{1}{2}, 2]\).

Suppose that there is a solution to \( \Phi \). We will show that in that case any minimum guard set \( G \) of \( P \) has size \( g(\Phi) \) and specifies a solution to \( \Phi \) in the sense that it satisfies the following two properties.

- Each variable \( x_i \in X \) is specified consistently by \( G \), i.e., there is exactly one guard on each guard segment representing \( x_i \), and all these guards specify the same value of \( x_i \).
- The guard set \( G \) is feasible, i.e., the values of \( X \) thus specified is a solution to \( \Phi \).
| Name                  | Description/value                                                                 |
|-----------------------|-----------------------------------------------------------------------------------|
| $\Phi$                | instance of ETR-INV that we reduce from                                            |
| $X := \{x_1, \ldots, x_n\}$ | set of variables of $\Phi$                                                       |
| $P := P(\Phi)$        | final polygon to be constructed from $\Phi$                                       |
| $g := g(\Phi)$        | number of guards needed to guard $P$ if and only if $\Phi$ has a solution         |
| $k$                   | number of equations in $\Phi$                                                     |
| $n$                   | number of variables in $\Phi$                                                    |
| $N$                   | $4n + k$                                                                          |
| $C$                   | 200000                                                                            |
| $\ell_b$ (base line)  | line that contains $4n$ guard segments at the bottom of $P$                       |
| $s_i := a_i b_i$       | guard segment on $\ell_b$, $i \in \{1, \ldots, 4n\}$, $\|a_i b_i\| = 3/2$        |
| $P_M$ (main part of $P$) | middle part of $P$, without corridors and gadgets                               |
| $\ell_r$, $\ell_l$    | vertical lines bounding $P_M$                                                     |
| corridor              | connection between the main part of $P$ and a gadget                               |
| $c_0 d_0, c_1 d_1$     | corridor entrances, $d_0 := c_0 + (0, \frac{3}{CN^2})$ and $d_1 := c_1 + (0, \frac{1.5}{CN^2})$ |
| $r_i, r_j, r_l, r'_i$  | guard segments within gadgets, of length $\frac{1.5}{CN^2}$                      |
| $a'_\sigma, b'_\sigma, \sigma \in \{i, j, l\}$ | left and right endpoint of $r_\sigma$                                             |
| $\ell_c$              | vertical line through $\frac{a_i + c_0}{2}$                                     |
| $o, o'$               | intersections of rays $\overrightarrow{a_i c_0}$ and $\overrightarrow{b_i c_1}$ with the line $\ell_c$ |
| $\delta$, $\rho$, $\varepsilon$ | $\delta := \frac{13.5}{CN^2}$, $\rho := \frac{\delta}{5} = \frac{1.5}{CN^2}$, $\varepsilon := \frac{\rho}{12} = \frac{1.5}{8CN^2}$ |
| $V$                   | $\frac{a_i + c_0}{2} + (0, 1) + [-38N\rho, +38N\rho] \times [-38N\rho, +38N\rho]$ |
| slab $S(q,v,r)$       | region of all points with distance at most $r$ to the line through $q$ with direction $v$ |
| center of slab        | line in the middle of a slab                                                      |
| $L$-slabs, $R$-slabs   | uncertainty regions for visibility rays, see page 35                             |

Table 1: Parameters, variables, and certain distances that are frequently used are summarized in this table for easy access. Some descriptions are much simplified.
Moreover, if there is no solution to $\Phi$, each guard set of $P$ consists of more than $g(\Phi)$ guards.

The polygon $P$ is constructed in the following way. The bottom part of the polygon consists of a collection of pockets, containing in total $4n$ collinear and equidistant guard segments. We denote the horizontal line containing these guard segments as the base line or $\ell_b$. In order from left to right, we denote the guard segments as $s_1, \ldots, s_{4n}$. The segments $s_1, s_3, \ldots, s_{4n}$ are right-oriented segments representing the variables $x_1, \ldots, x_n$, as are the segments $s_{n+1}, \ldots, s_{2n}$, and $s_{2n+1}, \ldots, s_{3n}$. The segments $s_{3n+1}, \ldots, s_{4n}$ are left-oriented and also represent the variables $x_1, \ldots, x_n$. At the left and at the right side of $P$, there are some corridors attached, each of which leads into a gadget. The entrances to the corridors at the right side of $P$ are line segments contained in a vertical line $\ell_r$. Likewise, the entrances to the corridors at the left side of $P$ are contained in a vertical line $\ell_l$. The gadgets also contain guard segments, and they are used to impose dependencies between the guards in order to ensure that if there is a solution to $\Phi$, then any minimum guard set of $P$ consists of $g(\Phi)$ guards and specifies a solution to $\Phi$ in the sense defined above. The corridors are used to copy the positions of guards on guard segments on the base line to guards on guard segments inside the gadgets. Each gadget corresponds to a constraint of one of the types $x + y \geq z$, $x + y \leq z$, $x \cdot y = 1$, $x + y \geq 5/2$, and $x + y \leq 5/2$. The first three types of constraints are used to encode the dependencies between the variables in $X$ as specified by $\Phi$, whereas the latter two constraints are used to encode the dependencies between the right-oriented and left-oriented guard segments representing a single variable in $X$.

The reason that we need three right-oriented guard segments $s_i, s_{i+n}, s_{i+2n}$ representing each variable is that in the addition gadgets, we need to copy in guards to three right-oriented guard segments, and they are allowed to all represent the same variable. In contrast to this, we need at most one left-oriented guard segment in each gadget. Furthermore, since the right-oriented guard segments appear in three groups ($s_1, \ldots, s_n$, $s_{n+1}, \ldots, s_{2n}$, and $s_{2n+1}, \ldots, s_{3n}$), it is possible for each set $\{i, j, l\} \subseteq \{1, \ldots, n\}$ to choose three right-oriented guard segments $s_i, s_j, s_l$ representing $x_i, x_j, x_l$, respectively, and appearing in any prescribed order on $\ell_b$.

### C.3 Creating a stationary guard position

We denote some points of $P$ as stationary guard positions. A guard placed at a stationary guard position is called a stationary guard. We will often define a stationary guard position as the unique point $p \in P$ such that a guard placed at $p$ can see some two corners $q_1, q_2$ of the polygon $P$.

We will later prove that for any guard set of size at most $g(\Phi)$, there is a guard placed at each stationary guard position. For that, we will need the lemma stated below. For an example of the application of the lemma, see Figure 11. The stationary guard position $g_2$ is the only point from which a guard can see both corners $q_1$ and $q_2$. Applying Lemma 14 with $p := g_2$, $W := \{q_1, q_2\}$, $A := P$ and $M := \{t_1, t_2\}$, we get that there must be a guard placed at $g_2$ in any guard set of size 3. The purpose of the area $A$ is so that we can restrict our arguments to a small area of the polygon.

**Lemma 14.** Let $P$ be a polygon, $A \subseteq P$, and $M$ a set of points in $A$ such that no point in $M$ can be seen from a point in $P \setminus A$, and no two points in $M$ can be seen from the same point in $P$. Suppose that there is a point $p \in A$ and a set of points $W \subset A$ such that

1. no point in $W$ can be seen from a point in $P \setminus A$,
2. the only point in $P$ that sees all points in $W$ is $p$, and
3. no point in $P$ can see a point in $M$ and a point in $W$ simultaneously.

Then any guard set of $P$ has at least $|M| + 1$ guards placed within $A$, and if a guard set with $|M| + 1$ guards placed within $A$ exists, one of its guards is placed at $p$. 

25
Proof. Let \( q \) be a point in \( W \). Since no two points in the set \( \{q\} \cup M \) can be seen from the same point in \( P \), and no point from \( P \setminus A \) can see a point in \( \{q\} \cup M \), at least \( |M| + 1 \) guards are needed within \( A \). Suppose that a guard set with exactly \( |M| + 1 \) guards placed within \( A \) exists. There must be \( |M| \) guards in \( A \) such that each of them can see one point in \( M \) and no point in \( W \). The last guard in \( A \) has to be at the point \( p \) in order to see all points in \( W \).

In the polygon \( P \) we often use stationary guards for the purpose of seeing some region on one side of a line segment \( \ell \), but no points on the other side of \( \ell \). Other guards have the responsibility to see the remaining area. See Figure 9 (left) for an explanation of how a stationary guard position can be constructed.

![Figure 9: Left: The construction of a stationary guard position \( p \) that sees an area in \( P \) below a line segment \( \ell \). The brown areas are the regions of points that see \( q_1 \) and \( q_2 \), and \( p \) is the only point that sees both \( q_1 \) and \( q_2 \). The point \( p \) sees the points in the blue wedge, and the angle of the wedge can be adjusted by choosing the point \( h \) accordingly. Right: The construction of a guard segment \( s \) (the blue segment). In order to see the points \( t_0, t_1, u_0, u_1 \), a guard must be on the horizontal dotted segment. Furthermore, in order to see \( u_0, u_1 \), the guard must be between the vertical dotted segments that contain the endpoints of \( s \). Thus, a guard sees \( t_0, t_1, u_0, u_1 \) if and only if the guard is at \( s \).](image)

**C.4 Creating a guard segment**

In the construction of \( P \) we will denote some horizontal line segments of \( P \) as *guard segments*. We will later prove that for any guard set of size of at most \( g(\Phi) \), there is exactly one guard placed on each guard segment.

We will always define a guard segment \( s \) by providing a collection of four corners of \( P \) such that a guard within \( P \) can see all these four corners if and only if it is placed on the line segment \( s \). See Figure 9 (right) for an example of such a construction. To show that there is a guard placed on a guard segment, we will use the following lemma.

**Lemma 15.** Let \( P \) be a polygon, \( A \subseteq P \), and \( M \) a set of points in \( A \) such that no point in \( M \) can be seen from a point in \( P \setminus A \), and no two points in \( M \) can be seen from the same point in \( P \). Suppose that there is a line segment \( s \) in \( A \), and points \( t_0, t_1, u_0, u_1 \in A \) such that

1. no point in \( \{t_0, t_1, u_0, u_1\} \) can be seen from a point in \( P \setminus A \),
2. a guard in \( P \) sees all of the points \( t_0, t_1, u_0, u_1 \) if and only if the guard is at \( s \), and
3. no point in \( P \) can see a point in \( M \) and one of the points \( t_0, t_1, u_0, u_1 \).

Then any guard set of \( P \) has at least \( |M| + 1 \) guards placed within \( A \), and if a guard set with \( |M| + 1 \) guards placed within \( A \) exists, one of its guards is placed on the line segment \( s \).

**Proof.** Similar to the proof of Lemma 14. \( \square \)
Consider once more the example pictured in Figure 11 where we want to guard the polygon with only three guards. We define two guard segments, $a_0b_0$ and $a_1b_1$. The first one is defined by the corners $t_0, t_1, u_0, u_1$, and the second one by the corners $t_2, t_3, u_2, u_3$. Applying Lemma 15 we get that there must be a guard placed on $a_0b_0$ (we set $A := P$ and $M := \{t_2, q_2\}$) and at $a_1b_1$ (we set $A := P$ and $M := \{t_1, q_2\}$) in any guard set of size 3.

As already explained in Section C.2 guards placed on the guard segments will be used to encode the values of the variables of $\Phi$.

C.5 Imposing inequalities by nooks and umbras

In this section we introduce nooks and umbras, which are our basic tools used to impose dependency between guards placed on two different guard segments. For the following definitions, see Figure 10.

![Figure 10: The brown area $Q$ representing a nook (top), and an umbra (bottom). In the left figure, note that if a guard $p_1$ placed on the segment $a_1b_1$ has to see the whole line segment $f_0f_1$ together with $p_0$, then $p_1$ must be on or to the left of the point $\pi_1^{-1}(e)$, where $e := \pi_0(p_0)$.](image-url)

**Definition 16** (nook and umbra). Let $\mathcal{P}$ be a polygon with guard segments $r_0 := a_0b_0$ and $r_1 := a_1b_1$, where $r_0$ is to the left of $r_1$. Let $c_0, c_1$ be two corners of $\mathcal{P}$, such that $c_0$ is to the left of $c_1$. Suppose that the rays $b_0c_0$ and $b_1c_1$ intersect at a point $f_0$, the lines $a_0c_0$ and $a_1c_1$ intersect at a point $f_1$, and that $Q := c_0c_1f_1f_0$ is a convex quadrilateral contained in $\mathcal{P}$. For each $i \in \{0, 1\}$ define
the function \( \pi_i : r_i \rightarrow f_0f_1 \) such that \( \pi_i(p) \) is the intersection of the ray \( pc_i \) with the line segment \( f_0f_1 \), and suppose that \( \pi_i \) is bijective.

We say that \( Q \) is a nook of the pair of guard segments \( r_0, r_1 \) if for each \( i \in \{0, 1\} \) and every \( p \in r_i \), a guard at \( p \) can see all of the segment \( \pi_i(p)f_{i-1} \) but nothing else of \( f_0f_1 \). We say that \( Q \) is an umbra of the segments \( r_0, r_1 \) if for each \( i \in \{0, 1\} \) and every \( p \in r_i \), a guard at \( p \) can see all of the segment \( \pi_i(p)f_i \) but nothing else of \( f_0f_1 \). The functions \( \pi_0, \pi_1 \) are called projections of the nook or the umbra.

We will construct nooks and umbras for pairs of guard segments where we want to enforce dependency between the values of the corresponding variables. When making use of an umbra, we will also create a stationary guard position from which a guard sees the whole quadrilateral \( Q \), but nothing on the other side of the line segment \( f_0f_1 \). In this way we can enforce that the guards on \( r_0 \) and \( r_1 \) together see all of \( f_0f_1 \), since they need to see an open region on the other side of, and bounded by, \( f_0f_1 \). For the case of a nook, the segment \( f_0f_1 \) will always be on the polygon boundary, and then there will be no stationary guard needed. See Figure 11 for an example of a construction of both a nook and an umbra for a pair of guard segments.

**Definition 17** (critical segment and shadow corners). Consider a nook or an umbra \( Q := c_0c_1f_0f_1 \) of a pair of guard segments \( r_0, r_1 \). The line segment \( f_0f_1 \) is called the critical segment of \( Q \), and the corners \( c_0, c_1 \) are called the shadow corners of \( Q \).

Consider a nook or an umbra of a pair of guard segments \( r_0, r_1 \). Let \( p_0, p_1 \) be the guards placed on the guard segments \( r_0 \) and \( r_1 \), respectively, and assume that \( p_0 \) and \( p_1 \) together see all of the critical segment \( f_0f_1 \). Let \( e := \pi_0(p_0) \). The condition that \( p_0, p_1 \) together see all of \( f_0f_1 \) enforces

\[ \text{Figure 11: Q}_1 \text{ is a copy-nook of the segments } r_0 := a_0b_0 \text{ and } r_1 := a_1b_1 \text{ with a critical segment } f_0f_1, \text{ and } Q_2 \text{ is a copy-umbra for the same pair with a critical segment } f_2f_3. \text{ Lemmas 14 and 15 imply that this polygon cannot be guarded by fewer than 3 guards, and any guard set with 3 guards must contain a guard } g_0 \text{ on } r_0, \text{ a guard } g_1 \text{ on } r_1, \text{ and a stationary guard at the point } g_2. \text{ The guards } g_0 \text{ and } g_1 \text{ must specify the same value on } r_0 \text{ and } r_1, \text{ respectively.}
\]

\[ ^\dagger \text{Our choice of the term "umbra" was inspired by its meaning in astronomy: “the complete or perfect shadow of an opaque body, as a planet, where the direct light from the source of illumination is completely cut off” [1].} \]
dependency between the position of the guard $p_1$ and the point $\pi_1^{-1}(e)$. If $Q$ is a nook, $p_1$ must be in the closed wedge $W$ between the rays $\overrightarrow{cc_0}$ and $\overrightarrow{cc_1}$. If $Q$ is an umbra, $p_1$ must be in $\overline{P \setminus W}$ (the closure of the complement of $W$), i.e., either on or to the right of $\pi_1^{-1}(e)$. This observation will allow us to impose an inequality on the $x$-coordinates of $p_0, p_1$, and thus on the variables corresponding to the guard segments $r_0, r_1$.

### C.6 Copying one variable

**Definition 18.** Let $Q$ be a nook or an umbra of a pair of guard segments $r_0 := a_0b_0$ and $r_1 := a_1b_1$ with the same orientation, such that the shadow corners $c_0$ and $c_1$ have the same $y$-coordinate. We then call $Q$ a copy-nook or a copy-umbra, respectively.

We can show the following result.

**Lemma 19.** Let $Q$ be a copy-nook or a copy-umbra for a pair of guard segments $r_0 := a_0b_0$ and $r_1 := a_1b_1$. Then for every point $e \in f_0f_1$ we have $\frac{||a_0\pi_0^{-1}(e)||}{||a_0b_0||} = \frac{||a_1\pi_1^{-1}(e)||}{||a_1b_1||}$, i.e., the points $\pi_0^{-1}(e)$ and $\pi_1^{-1}(e)$ on the corresponding guard segments $r_0$ and $r_1$ represent the same value.

**Proof.** See Figure 12. Let $\ell := \overrightarrow{c_0c_1}$ be the horizontal line containing the line segment $c_0c_1$, and $\ell'$ a horizontal line passing through $e$. Let $f_2$ be an intersection point of the line $\overrightarrow{f_0f_1}$ with the line $\ell$. Let $a'_0, a'_1, b'_0, b'_1$ be the intersection points of the rays $\overrightarrow{a_0c_0}, \overrightarrow{a_1c_1}, \overrightarrow{b_0c_0}, \overrightarrow{b_1c_1}$, respectively, with the line $\ell'$.

We obtain
\[
\frac{||a_0\pi_0^{-1}(e)||}{||\pi_0^{-1}(e)b_0||} \cdot \frac{||\pi_1^{-1}(e)b_1||}{||a_1\pi_1^{-1}(e)||} = \frac{||ea'_0||}{||b'_0e||} \cdot \frac{||b'_1e||}{||ea'_1||} = \frac{||b'_0||}{||b'_1||} \cdot \frac{||c_0f_2||}{||c_1f_2||} \cdot \frac{||c_1f_2||}{||c_0f_2||} = 1.
\]

The first equality holds as the following pairs of triangles are similar: $a_0\pi_0^{-1}(e)c_0$ and $a'_0ec_0$, $\pi_0^{-1}(e)b_0c_0$ and $eb'_0c_0$, $a_1\pi_1^{-1}(e)c_1$ and $a'_1ec_1$, $\pi_1^{-1}(e)b_1c_1$ and $eb'_1c_1$. The third equality holds as...
Let \( r_0, r_1 \) be a pair of guard segments oriented in the same way for which there is both a copy-nook and a copy-umbra. Suppose that there is exactly one guard \( p_0 \) placed on \( r_0 \) and one guard \( p_1 \) placed on \( r_1 \), and that the guards \( p_0 \) and \( p_1 \) together see both critical segments. Then the guards \( p_0 \) and \( p_1 \) specify the same value.

**Definition 21.** Let \( r_0, r_1 \) be a pair of guard segments for which there is both a copy-nook and a copy-umbra. We say that \( r_1 \) is a copy of \( r_0 \). If there is only a copy-nook or a copy-umbra of the pair \( r_0, r_1 \), we say that \( r_1 \) is a weak copy of \( r_0 \).

It will follow from the construction of the polygon \( \mathcal{P} \) that if there is a solution to \( \Phi \), then for any optimal guard set of \( \mathcal{P} \) and any pair of guard segments \( r_0, r_1 \) such that \( r_1 \) is a copy of \( r_0 \), there is exactly one guard on each segment \( r_0, r_1 \), and the guards together see the whole critical segment of both the copy-nook and the copy-umbra.

### C.7 The overall design of the polygon \( \mathcal{P} \)

Recall the high-level sketch of the polygon \( \mathcal{P} \) in Figure 8. The bottom part of the polygon consists of pockets containing \( 4n \) guard segments \( s_1, \ldots, s_{4n} \). The guard segments are placed on the base line \( \ell_b \), each segment having a width of \( 3/2 \) and contained within a pocket of width \( 13.5 \). Therefore the horizontal space used for the \( 4n \) guard segments on the base line \( \ell_b \) is \( 54n \). The wall of \( \mathcal{P} \) forming the \( 4n \) pockets is denoted the bottom wall. The detailed description of the pockets and of the bottom wall is presented in Section C.8.

Let \( N := 4n + k \). At the right side of \( \mathcal{P} \) and at the left side of \( \mathcal{P} \) there will be at most \( N \) corridors attached, each of which leads into a gadget. The entrances to the corridors are contained in the vertical lines \( \ell_r \) and \( \ell_l \). The corridors are described in Section C.9 and they are placed equidistantly, with a vertical distance of 3 between the entrances of two consecutive corridors along the lines \( \ell_r \) and \( \ell_l \). The gadgets are described in Sections C.10–C.13. The total vertical space occupied by the corridors and the gadgets at each side of \( \mathcal{P} \) is at most \( 3N \).

Consider the sketch of the polygon \( \mathcal{P} \) in Figure 13. Define a constant \( C := 200000 \). Let \( w_l \) and \( w_r \) denote the left and right endpoint of the bottom wall, respectively. The horizontal distance from \( w_l \) to the line \( \ell_l \) is \( CN^2 - 54n + 6 \), as is the horizontal distance from \( w_r \) to \( \ell_r \). The horizontal distance from the left endpoint \( a_1 \) of the leftmost segment to \( \ell_r \), as well as the horizontal distance from the right endpoint \( b_{4n} \) of the rightmost segment to \( \ell_l \), is \( CN^2 \). The vertical distance from \( \ell_b \) up to the entrance of the first corridor is \( CN^2 \). The boundary of \( \mathcal{P} \) contains an edge connecting \( w_l \) to the point \( v_l := w_l + (-CN^2 - 54n + 6)CN^2 - 1 \) on \( \ell_l \), and an edge connecting \( w_r \) to the point \( v_r := w_r + (CN^2 - 54n + 6)CN^2 - 1 \) on \( \ell_r \). Let \( t_l := v_l + (0, 3N) \) and \( t_r := v_r + (0, 3N) \). The main area \( \mathcal{P}_M \) of \( \mathcal{P} \) is the area bounded by the bottom wall of \( \mathcal{P} \) (to be defined in Section C.8) and a polygonal curve defined by the points \( w_l v_l t_l v_r w_r \). The entrances to the corridors are on the segment \( v_l t_l \) in the left side and on the segment \( v_r t_r \) in the right side. The set \( \mathcal{P} \setminus \mathcal{P}_M \) outside of the main area consists of corridors and gadgets.

The reason why we need the distances from the guard segments on the base line \( \ell_b \) to the gadgets to be so large is that we want all the rays from the guard segments on the base line through the corridor entrances on \( \ell \) (\( \ell_l \)) to have nearly the same slopes. That will allow us to describe a general method for copying guard segments from the base line into the gadgets.
Each gadget corresponds to a constraint involving either two or three variables, where each variable corresponds to a guard segment on the base line. Gadgets are connected with the main area $P_M$ via corridors. A corridor does not contain any guard segments, and its aim is enforcing consistency between (two or three) pairs of guard segments, where one segment from each pair is in $P_M$ and the other one is in the gadget. Each corridor has two vertical entrances, the entrance $c_0d_0$ of height $\frac{3}{\sqrt{N^2}}$ connecting it with $P_M$, and the entrance $c_1d_1$ of height $\frac{1.5}{\sqrt{N^2}}$ connecting it with a gadget. The bottom wall of a corridor is a horizontal line segment $c_0c_1$ of length 2. The shape of the upper wall is more complicated, and it depends on the indices of the guard segments involved in the corresponding constraint, and on the height at which the corridor is placed with respect to the base line $\ell_b$.

A gadget can be thought of as a room which is connected with the main area $P_M$ of $P$ via a corridor, i.e., attached to the corners $c_1$ and $d_1$ of the corridor. There are five different kinds of gadgets, each corresponding to a different kind of inequality or equation, and, unlike for the case of corridors, all gadgets of the same type are identical. Each gadget contains one or two guard segments for each variable present in the corresponding formula. All guard segments within a gadget are of length $\frac{1.5}{\sqrt{N^2}}$, and are placed very close to the middle point of the gadget, defined as $m := c_1 + (1, -1)$ for gadgets at the right side of $P$, and $m := c_1 + (-1, -1)$ for gadgets at the left side of $P$.

C.8 Construction of the bottom wall

In this section we present the construction of the bottom wall of $P$. We first describe the overall construction, as shown in Figure 14 and later we introduce small features corresponding to each
equation of the type \( x_i = 1 \) in \( \Phi \).

The bottom wall forms 4\( n \) pockets, each pocket containing one guard segment on the base line \( \ell_b \). Each pocket has a width of 13.5. Each guard segment has a length of 3/2, and the distance between two consecutive segments is 12.

Figure 14: The construction of three consecutive guard segments (blue) on the base line. A pocket corresponding to a single guard segment \( s_i := a_i b_i \) is marked in grey.

Let \( s_1 := a_1 b_1, \ldots, s_{4n} := a_{4n} b_{4n} \) be the guard segments in order from left to right. A pocket for a guard segment \( s_i := a_i b_i \) is constructed as shown in Figure 14 (the grey area in the figure). The left endpoint of the pocket is at the point \( a_i + (-6, 1) \), and the right endpoint of the pocket is at the point \( a_i + (7.5, 1) \). The guard segment is defined by the following points, which are corners of the pocket: \( t_0^i := a_i + (-2, 0), t_1^i := a_i + (3.5, 0), u_0^i := a_i + (0, -5), u_1^i := a_i + (1.5, -5) \). The vertical edges of the pocket are contained in lines \( x = x(a_i) - 1.5 \), \( x = x(a_i) \), \( x = x(a_i) + 1.5 \), and \( x = x(a_i) + 3 \). The horizontal edges of the pocket are contained in lines \( y = 0 \), \( y = -0.5 \), and \( y = -5 \). The remaining edges are constructed so that the points \( t_0^i, t_1^i \) can be seen only from within the pocket, and that any point on \( s_i \) sees the whole pocket.

Consider an equation of the form \( x_i = 1 \) in \( \Phi \). There are four guard segments representing \( x_i \), i.e., the guard segments \( s_i, s_{i+n}, s_{i+2n}, \) and \( s_{i+3n} \), where the first three are right-oriented and the last one is left-oriented. We add two spikes in the construction of the leftmost of these guard segments, i.e., the segment \( s_i \), as shown in Figure 15. The dashed lines in the figure intersect at the point \( g_i \in s_i \), where \( g_i := a_i + (1/2, 0) \). The spike containing \( q_1^i \) enforces the guard to be at the point \( g_i \) or to the right of it, while the spike containing \( q_2^i \) enforces the guard to be at \( g_i \) or to the left of it. Also, the points \( q_1^i \) and \( q_2^i \) are chosen so that they cannot be seen by any points from within the corridors or gadgets. The guard segment is thus reduced to a stationary guard position \( g_i \) corresponding to the value \( x_i = 1 \).

Figure 15: The spikes with corners at \( q_1^i \) and \( q_2^i \) enforce the guard from the guard segment \( s_i \) to be at the point \( g_i \) corresponding to the value of 1.

Note that we only need to add such spikes to the pocket containing \( s_i \), since the construction described in Section C.12 will enforce the guards on all the segments \( s_i, s_{i+n}, s_{i+2n}, \) and \( s_{i+3n} \) to specify the value of \( x_i \) consistently. We have now specified all the details of the main area \( P_M \) of \( P \) (recall the definition of \( P_M \) from Section C.7). The following lemma holds.
Lemma 22. There is a constant $\zeta$ such that for any instance $\Phi$ of ETR-INV, we can construct in polynomial time the bottom wall corresponding to $\Phi$ such that every corner has rational coordinates, with the numerator bounded from above by $\zeta N$ and denominator bounded from above by $\zeta$.

We can prove the following lemma.

Lemma 23. Any guard set $G$ of the polygon $\mathcal{P}$ satisfies the following properties.

- $G$ has at least $4n$ guards placed in $\mathcal{P}_M$.
- If $G$ has exactly $4n$ guards placed in $\mathcal{P}_M$, then it has one guard within each of the $4n$ guard segments $s_1, \ldots, s_{4n}$.
- If $G$ has exactly $4n$ guards placed in the main area of $\mathcal{P}$, then for each variable $x_i \in X$ such that there is an equation $x_i = 1$ in $\Phi$, the guard at the segment $s_i$ is at the position corresponding to the value 1.

Moreover, let $G'$ be any set of points such that (i) $G'$ has a point within each of the $4n$ guard segments of $\mathcal{P}_M$, and (ii) for each variable $x_i \in X$ such that there is an equation $x_i = 1$ in $\Phi$, there is a guard at $s_i$ at the position corresponding to the value of 1. Then $G'$ is a guard set of $\mathcal{P}_M$.

Proof. Recall that each point $t_i^1$, for $i \in \{1, \ldots, 4n\}$, can be seen only from within the pocket corresponding to the guard segment $s_i$. Therefore any guard set requires at least one guard within each of these pockets, i.e., it contains at least $4n$ guards in $\mathcal{P}_M$.

Now, consider any guard segment $s_i$, for $i \in \{1, \ldots, 4n\}$. Let $M := \{t_j^1 : j \in \{1, 2, \ldots, 4n\}, j \neq i\}$. Note also that none of the points $t_i^1, t_i^0, u_0^i, u_1^i$, and also no point from $M$ can be seen by a guard placed within a corridor or a gadget of $\mathcal{P}$, i.e., outside of the main area of $\mathcal{P}$. By Lemma 15, by taking $t_i^0, t_i^1, u_0^i, u_1^i$ as $t_0, t_1, u_0, u_1$ and $A := \mathcal{P}_M$, we obtain that if a guard set $G$ has exactly $4n$ guards placed in $\mathcal{P}_M$, then it has a guard within $s_i$.

For the third property, consider any variable $x_i \in X$ such that there is an equation $x_i = 1$ in $\Phi$. As none of the points $q_1^i$ and $q_2^i$ can be seen from guards within the corridors or gadgets, or from the guards within the other guard segments on the base line, both of them must be seen by the only guard $g_i$ placed on $s_i$. Then, $g_i$ must be placed at the position corresponding to the value 1.

Finally, consider any set $G'$ of points such that (i) $G'$ has a point within each of the $4n$ guard segments of $\mathcal{P}_M$, and (ii) for each variable $x_i \in X$ such that there is an equation $x_i = 1$ in $\Phi$ there is a guard at $s_i$ at the position corresponding to the value of 1. We now show that $G'$ is a guard set of $\mathcal{P}_M$. From the construction of the pockets, a guard within a pocket containing $s_i$ can see the whole pocket (in particular, the guards at positions corresponding to the value of 1 can see all of the added spikes). The guard at $s_i$ also sees all of $\mathcal{P}_M$ which is above the pocket, i.e., all points of $\mathcal{P}_M$ with $x$-coordinates in $[x(a_i) - 6, x(a_i) + 7.5]$. The part of $\mathcal{P}_M$ to the left of the leftmost pocket can be seen by the guard on the leftmost guard segment, and the part of $\mathcal{P}_M$ to the right of the rightmost pocket can be seen by the guard on the rightmost guard segment.

C.9 Construction of a corridor

In this section, we describe the construction of a corridor. Inside each gadget there are three (or two) guard segments $r_i, r_j, r_l$ (or $r_i, r_j$) corresponding to three (or two) pairwise different guard segments from the base line $s_i, s_j, s_l$ (or $s_i, s_j$). We require that for each $\sigma \in \{i, j, l\}$ the guard segments $s_\sigma, r_\sigma$ have the same orientation. For the corridors attached at the right side of $\mathcal{P}$ we assume $i < j < l$, and for the corridors attached at the left side we assume $i > j > l$. We describe here how to construct a corridor that ensures that the segments $r_i, r_j, r_l$ are copies of the
segments $s_i, s_j, s_l$, respectively. This construction requires that the guard segments within the gadget satisfy the conditions of some technical lemmas (see Lemma 24 and 29).

Note that this construction can be generalized for copying an arbitrary subset of guard segments, but since we only need to copy two or three segments, we explain the construction in the setting of three segments. The construction for two segments is analogous but simpler. We first describe how to copy into a gadget at the right side of the polygon $P$ – copying into gadgets at the left side of $P$ can be done in a symmetric way and is described shortly in Section C.9.4.

As described briefly in Section C.7, the lower wall of the corridor of the gadget is a horizontal edge $c_0c_1$ of length 2, where $c_0$ is on the line $ℓ_r$ and $c_1$ is to the right of $c_0$. The upper wall of the corridor is more complicated, and it will be described later. It has the left endpoint at $d_0 := c_0 + (0, \frac{3}{CN^2})$, and the right endpoint $d_1 := c_1 + (0, \frac{1.5}{CN^2})$. The vertical line segments $c_0d_0$ and $c_1d_1$ are called the entrances of the corridor.

C.9.1 Idea of the copying construction

![Diagram of corridor construction](image)

Figure 16: In this figure, we display a simplified corridor construction. The corners $c_0, c_1$ serve as shadow corners for three copy-umbras simultaneously for the pairs $(s_i, r_i), (s_j, r_j), (s_l, r_l)$. Each of these pairs also have a small copy-nook in the top of corridor. The entrances $c_0d_0$ and $c_1d_1$ to the corridor are sufficiently small so that the critical segments of the nook and umbra of each pair of segments $s_σ, r_σ$ (contained in the small boxes at the top of the figure) are not seen by other guard segments.

To ensure that the segments $r_i, r_j, r_l$ are copies of the segments $s_i, s_j, s_l$, we need to construct within the corridor copy-nooks and copy-umbras for the pairs of corresponding segments, see Figure 16 for a simplified illustration. The corners $c_0, c_1$ of the corridor act as shadow corners in three overlapping copy-umbras for the pairs $(s_i, r_i), (s_j, r_j), (s_l, r_l)$, respectively. We construct the chain of $P$ from $d_0$ to $d_1$ bounding the corridor from above so that it creates three copy-nooks for the same pairs. To enforce that for any guard set of size $g(Φ)$, for each $σ ∈ \{i, j, l\}$ the guard segments $s_σ$ and $r_σ$ specify the same value, we have to ensure that no guards on guard segments other than $s_σ$ and $r_σ$ can see the critical segments of the copy-umbra and the copy-nook of the pair $s_σ, r_σ$. For each $σ ∈ \{i, j, l\}$ we also introduce a stationary guard position, so that guards placed at these positions together see all the copy-umbras, but nothing on the other sides of the critical segments of the copy-umbras. We also need to ensure that the guards placed on the stationary guard positions cannot see the critical segments of the copy-umbra and the copy-nooks of other pairs. We then obtain (see Lemma 27) that for any guard set with one guard at each guard segment, and with no guards placed outside of the guard segments and stationary guard positions, the segments $s_σ, r_σ$ specify the value of $x_σ$ consistently.

34
Our construction will ensure that for any \( \sigma \in \{i, j, l\} \), only the guard segment \( s_\sigma \) from the base line, and only the guard segment \( r_\sigma \) from within the gadget can see the critical segments of the corresponding copy-nook and copy-umbra. In particular, we will ensure that the vertical edge of \( P \) directly above the entrance \( c_0d_0 \) blocks visibility from all guard segments \( s_\sigma \) for \( \sigma' \in \{1, \ldots, \sigma - 1\} \), whereas the vertical edge of \( P \) directly below \( c_0d_0 \) blocks visibility from all guard segments \( s_\sigma \) for \( \sigma' \in \{\sigma + 1, \ldots, 4n\} \). An analogous property will be ensured for the gadget guard segments.

The main idea to achieve the above property is to make the entrances \( c_id_i \) of the corridor sufficiently small. However, we cannot place the point \( \sigma \) and the right endpoint of the critical segment of the copy-nook for the pair \( s \) and \( c \) through the points \( f_0, f_1 \), respectively, in the context of Section \( \ref{C.5} \). By placing the corridor sufficiently far away from the segments on the base line, we obtain that the visibility lines from the guard segment endpoints through the points \( c_0, d_0 \) are all almost parallel and can be described by a simple pattern. The same holds for the pair of points \( c_1, d_1 \) and the endpoints of the guard segments \( r_\sigma, \sigma \in \{i, j, l\} \). The pattern enables us to construct the corridor with the desired properties.

In the following, we introduce objects that make it possible to describe the upper corridor wall and prove that the construction works as intended.

### C.9.2 Introducing slabs

In a small area around the point \( \frac{a_0 + c_1}{2} + (0, 1) \), every ray from an endpoint of a base line guard segment through one of the points \( c_0, d_0 \) intersects every ray from an endpoint of a gadget guard segment through one of the points \( c_1, d_1 \). These rays intersect at angles close to \( \pi/2 \), and they form an arrangement consisting of quadrilaterals, creating a nearly-regular pattern. However, the arrangement of rays is not completely regular. We therefore introduce a collection of thin slabs, where each slab contains one of the rays in a small neighbourhood around \( \frac{a_0 + c_1}{2} + (0, 1) \), and such that the slabs form an orthogonal grid with axis \((1, 1)\) and \((-1, 1)\). Thus, the slabs are introduced in order to handle the “uncertainty” and irregularity of the rays.

Given a point \( q \) and a vector \( v \), the slab \( S(q, v, r) \) consists of all points at a distance of at most \( r \) from the line through \( q \) parallel to \( v \). The center of the slab \( S(q, v, r) \) is the line through \( q \) parallel to \( v \).

Let \( r_i := a_i'b_i, r_j := a_j'b_j, r_l := a_l'b_l \). Let \( \ell_c \) be a vertical line passing through the middle of the segment \( c_0c_1 \). Recall that here we describe the construction of a corridor to be attached at the right side of \( P \). Let \( o \) be the intersection point of the ray \( \overrightarrow{a_0c_0} \) with \( \ell_c \), and \( o' \) the intersection point of the ray \( \overrightarrow{b_0c_1} \) with \( \ell_c \). All the points \( o, o', \frac{a_0 + c_1}{2} + (0, 1) \) lie on the vertical line \( \ell_c \).

Let us define vectors \( \alpha := (1, 1), \beta := (-1, 1) \), and introduce a grid of slabs parallel to \( \alpha \) and \( \beta \). Let us fix
\[
\delta := \frac{13.5}{CN^2}, \quad \rho := \frac{\delta}{9} = \frac{1.5}{CN^2}, \quad \text{and} \quad \varepsilon := \frac{\rho}{12} = \frac{1}{8CN^2}.
\]

For each \( \sigma \in \{1, \ldots, 4n\} \) and \( \gamma \in \{0, 1, 2, 3\} \) we define a slab
\[ L^\gamma_\sigma := S(o + (0, (\sigma - 1)\delta + \gamma\rho), \alpha, \varepsilon), \]
which we denote as an \( L \)-slab. Let \( \tau(i) := 2, \tau(j) := 1, \) and \( \tau(l) := 0 \). For each \( \sigma \in \{i, j, l\} \) and \( \gamma \in \{0, 1, 2, 3\} \) we define a slab
\[ R^\gamma_\sigma := S(o' + (0, \tau(\sigma)\delta + \gamma\rho), \beta, \varepsilon), \]
which we denote as an \( R \)-slab.
In the case of gadgets with just two guard segments $r_i, r_j$, we define the point $o'$ as the intersection point of the ray $b'_j c_1$ with $\ell_c$, and we define $\tau(i) := 1$ and $\tau(j) := 0$. Then, the $R$-slabs $R^\gamma_\sigma$ are defined as above for $\sigma \in \{i, j\}$.

See Figure 17 for an illustration of the area where the $L$-slabs intersect the $R$-slabs.

Let $V$ be the square $\frac{c_0 + c_1}{2} + (0, 1) + [-38N\rho, 38N\rho] \times [-38N\rho, 38N\rho]$. We now prove that the area where the $L$-slabs and the $R$-slabs intersect is contained in $V$. Let us denote by $\mathcal{R}$ all the rays with endpoint at one of the guard segments in the main area, going through $c_0$ and $d_0$ and all the rays from the endpoints of gadget guard segments through the points $c_1, d_1$. We also ensure that all rays in $\mathcal{R}$ are inside a predefined slab within the area $V$.

In sections specific to the particular gadgets, we will prove the following lemma. We state the lemma for gadgets with three guard segments $r_i, r_j, r_l$, but it has a natural analogue for gadgets with just two guard segments $r_i, r_j$.

**Lemma 24.** For any gadget to be attached at the right side of the polygon $\mathcal{P}$ and containing the guard segments $r_i := a'_i b'_i, r_j := a'_j b'_j, r_l := a'_l b'_l$ the following holds, where $c_1$ is the bottom-right endpoint of the corridor corresponding to the gadget.

1. The intersection of any $R$-slab with the line $\ell_c$ is contained in $V$.

2. For each $\sigma \in \{i, j, l\}$, it holds that $\overrightarrow{b'_\sigma c_1} \cap V \subset R^0_\sigma$, $\overrightarrow{a'_\sigma c_1} \cap V \subset R^1_\sigma$, $\overrightarrow{b'_\sigma d_1} \cap V \subset R^2_\sigma$, and $\overrightarrow{a'_\sigma d_1} \cap V \subset R^3_\sigma$.

3. There are no stationary guard positions or guard segments different from $r_i, r_j, r_l$ within the gadget from which any point of the corridor can be seen.

Assuming that the above lemma holds, we will prove the following.

**Lemma 25.** For any corridor to be attached at the right side of the polygon $\mathcal{P}$, the following properties are satisfied.

---

Figure 17: The $L$-slabs have slope 1 and the $R$-slabs have slope $-1$. For each guard segment we get 4 equidistant slabs. The width of each slab is $2\varepsilon$. The distance between two slabs from the same group is $\rho$ and the distance between two groups is $\delta$. All intersections are contained in the region denoted by $V$. 
1. The intersection of any $L$-slab with any $R$-slab is contained in $V$.

2. For each $\sigma \in \{1, \ldots, 4n\}$, it holds that $\overrightarrow{a_{\sigma}c_0} \cap V \subset L^0_{\sigma}, \overrightarrow{b_{\sigma}c_0} \cap V \subset L^1_{\sigma}, \overrightarrow{a_{\sigma}d_0} \cap V \subset L^2_{\sigma}$, and $\overrightarrow{b_{\sigma}d_0} \cap V \subset L^3_{\sigma}$.

Proof. We will first show that the intersection of any $L$-slab with the line $\ell_c$ is contained in $V$. Consider Figure 18. Recall that $o$ is the intersection point of the ray $\overrightarrow{a_{\sigma}c_0}$ with $\ell_c$. The horizontal distance between $\ell_c$ and $c_0$ is 1. Let $u$ be the intersection point of the lines $\ell_b$ and $\ell_c$. From the polygon description in Section C.7 we know that $\|a_1u\| = CN^2$, and $\|c_0u\| \in [CN^2, CN^2 + 3N]$. The distance between the point $a_{\sigma}c_0 + \frac{2}{2}u$ and the point $o$ is $\|c_0u\| = \| \frac{C}{2}N + 1 \| [1 + 2N].$ From the definition of the $L$-slabs, the intersection of the $L$-slabs with the vertical line $\ell_c$ is contained in the line segment $(o - (0, 2\varepsilon), o + (0, 4N\delta)) \subseteq (a_{\sigma}c_0 + (0, 1 - 2\varepsilon), c_0 + (0, 1 + 2N\rho + 4N\delta)),$ which is contained in $V$ as $\delta := 9\rho$.

By Property 1 of Lemma 24, any point in the intersection of an $L$-slab and an $R$-slab must have a $y$-coordinate within the range of $V$. As the angles of the slabs are exactly $\pi/4$ and $3\pi/4$, we get that also the $x$-coordinates of the intersection must be within the range of $V$, see also to the right of Figure 18. That gives us Property 1.

![Figure 18: Left: Structure of rays with origins $a_1$ and $b_1$, containing points $c_0$ and $d_0$. Right: Even in the case that a left ray intersects $\ell_c$ at the very top of $V$ and a right ray intersects at the very bottom of $V$, they still have to intersect within $V$.](image)

For Property 2, let us first consider $\sigma = 0$. Let us define $o^{ac}, o^{ad}, o^{bc}, o^{bd}$ as the intersection points of the rays $\overrightarrow{a_1c_0}, \overrightarrow{a_1d_0}, \overrightarrow{b_1c_0}, \overrightarrow{b_1d_0}$ with the line $\ell_c$ (see Figure 18). We have $o = o^{ac}$. The points $o^{ad}, o^{bc}, o^{bd}$ lie above $o$, and we will now estimate the distance between each of them and $o$. We do that as follows.

First, consider the distance $\|oo^{ad}\|$. From below, we have a trivial bound

$$\|oo^{ad}\| \geq \|c_0d_0\| = \frac{3}{C N^2} = 2\rho.$$

From the similarity of triangles $a_1c_0d_0$ and $a_1o^{ad}$, and as the distance between the line $\ell_c$ and the line $\ell_r$ equals 1, we obtain the following upper bound for $\|oo^{ad}\|$

$$\|oo^{ad}\| = \|c_0d_0\| \cdot \frac{\|a_1u\| + 1}{\|a_1u\|} = \frac{3}{C N^2} \cdot \frac{C N^2 + 1}{C N^2} = \frac{3}{C N^2} \cdot \left(1 + \frac{1}{C N^2}\right) \leq 2\rho + \epsilon.$$
Let \( b' \) be a vertical projection of \( b_1 \) on the ray \( \overrightarrow{a_1c_0} \). From similarity of triangles \( c_0\overrightarrow{oo}b \) and \( c_0b'b_1 \), and triangles \( a_1b_1b' \) and \( a_1u_0 \), we get the following equality

\[
\|\overrightarrow{oo}b\| = \|\overrightarrow{b_1b'}\| \cdot \frac{1}{\|\overrightarrow{b_1u}\|} = \|\overrightarrow{u_0u}\| \cdot \frac{1}{\|\overrightarrow{a_1b_1}\|} \cdot \frac{1}{\|\overrightarrow{b_1u}\|} = 3/2 \cdot \frac{\|\overrightarrow{u_0u}\|}{\|\overrightarrow{a_1u}\|\|\overrightarrow{b_1u}\|}.
\]

We instantly get

\[
\|\overrightarrow{oo}b\| \geq 3/2 \cdot \frac{CN^2}{(CN)^2} = \rho.
\]

For an upper bound, we compute

\[
\frac{\|\overrightarrow{a_1u}\|\|\overrightarrow{b_1u}\|}{\|\overrightarrow{u_0u}\|} \leq \frac{CN^2(CN^2 - 3/2)}{CN^2 + 3N} \geq \frac{(CN^2 + 3N)(CN^2 - 4N)}{CN^2 + 3N} = CN^2 - 4N \geq \frac{CN^2}{1 + 1/72},
\]

where the last inequality follows since \( C \geq 73 \cdot 4 = 292. \) That gives us

\[
\|\overrightarrow{oo}b\| \leq 3/2 \cdot \frac{1 + 1/72}{CN^2} = \frac{3/2}{CN^2} + \frac{1/48}{CN^2} = \rho + \frac{\varepsilon}{6}.
\]

In the same way as for \( \|\overrightarrow{oo}b\| \) we obtain the following bounds:

\[
\rho \leq \|\overrightarrow{o\sigma d}b\| \leq \rho + \varepsilon/6.
\]

As \( \|\overrightarrow{oo}b\| = \|\overrightarrow{o\sigma d}b\| + \|\overrightarrow{od\sigma b}\| \), we instantly get

\[
3\rho \leq \|\overrightarrow{oo}b\| \leq 3\rho + \varepsilon/3.
\]

Summarizing this part, we have the following bounds:

\[
\rho \leq \|\overrightarrow{oo}b\| \leq \rho + \varepsilon/3, \quad 2\rho \leq \|\overrightarrow{o\sigma d}b\| \leq 2\rho + \varepsilon/3, \quad 3\rho \leq \|\overrightarrow{oo}b\| \leq 3\rho + \varepsilon/3.
\]

Therefore, the intersection points of the rays \( \overrightarrow{a_1c_0}, \overrightarrow{b_1c_0}, \overrightarrow{a_1d_0}, \overrightarrow{b_1d_0} \) are contained in the required slabs, at a vertical distance of at most \( \varepsilon/3 \) from the centers of the slabs.

Now, consider any \( \sigma \in \{1, \ldots, 4n\} \), \( \tau \in \{a, b\} \) and \( \eta \in \{c, d\} \). Let us denote by \( R \) all the rays with endpoint at one of the guard segments in the main area, going through \( c_0 \) and \( d_0 \) and all the rays from the endpoints of gadget guard segments through the points \( c_1, d_1 \).

Let \( \overrightarrow{\tilde{o}0} \) be the intersection point of the ray \( \overrightarrow{\tau_\sigma \tilde{\eta}0} \) with the line \( \ell_c \) (see Figure 19). We first bound the distance \( \|\overrightarrow{\sigma\tau\eta}\| \). Recall that \( u \) is the intersection point of \( \ell_b \) and \( \ell_c \). Let \( \tau_\sigma \) be the point on the ray \( \overrightarrow{\tau_\sigma \tilde{\eta}0} \) vertically above \( \tau_\sigma \).

As the triangles \( \eta_0\overrightarrow{\sigma\tau\eta} \) and \( \eta_0\tau_\sigma\tau_\sigma \) are similar, the triangles \( \tau_1\tau_\sigma\tau_\sigma \) and \( \tau_1\eta_0 \) are similar, and the distance between the lines \( \ell_c \) and \( \ell_r \) is 1, we get the following equality

\[
\|\overrightarrow{\sigma\tau\eta}\| = \|\tau_\sigma\tau_\sigma\| \cdot \frac{1}{\|\tau_\sigma u\|} = \|\tau_1\tau_\sigma\| \cdot \frac{1}{\|\tau_\sigma u\|} \cdot \frac{1}{\|\tau_\sigma u\|} = 13.5\sigma \cdot \|\tau_1\eta_0\| = 13.5\sigma \cdot \|\tau_1\eta_0\| = 13.5\sigma \cdot \|\tau_1\eta_0\| = 13.5\sigma \cdot \|\tau_1\eta_0\|.
\]

We first bound \( \|\overrightarrow{\sigma\tau\eta}\| \) from above. As \( C \geq 1297 \cdot 58 = 75226 \), we get

\[
\|\tau_1\eta_0\| \geq (CN^2 - 3/2)(CN^2 - 54N) \geq \frac{(CN^2 + 3N)(CN^2 - 58N)}{CN^2 + 3N} = \frac{CN^2 - 58N}{1 + 1/1296N}.
\]

38
and, as $\sigma \leq 4N$, we can bound
\[
\|\hat{o}\| \leq 13.5\sigma \cdot \left(1 + \frac{1}{1296N}\right) \leq \sigma \cdot \frac{13.5}{CN^2} + \frac{1}{24} \cdot \frac{1}{N^2} = \sigma \delta + \frac{\varepsilon}{3}.
\]

To bound $\|o\|$ from below, we compute
\[
\|o\| = 13.5\sigma \cdot \frac{\|u\|}{\|u\|} \geq \frac{13.5}{CN^2} \leq 1 + \frac{1}{1368N} \cdot \frac{1}{N^2} = \frac{1}{3} \cdot 38N\rho = \frac{\varepsilon}{3},
\]
we get that $\hat{o} \cap V$ is contained in the corresponding slab.

Figure 20: The grid of $L$- and $R$-slabs (bounded by blue and red lines) and an approximate shape $\Lambda$ (black) of the upper wall of the corridor copying the guard segments $s_i, s_j, s_l$ to the guard segments $r_i, r_j, r_l$ in the gadget. Here, there are 8 guard segments $s_1, \ldots, s_8$ on the base line, and we have $i = 3$, $j = 6$, and $l = 7$. The blue lines bound the slabs corresponding to the rays originating at the left endpoints of the guard segments (i.e., slabs $L_{i\sigma}^0, L_{i\sigma}^2, R_{j\sigma}^1, R_{j\sigma}^3$), and the red lines bound the slabs corresponding to the rays originating at the right endpoints. The full lines bound the slabs corresponding to the rays passing through $c_0$ or $c_1$ (i.e., slabs $L_{i\sigma}^0, L_{j\sigma}^1, R_{i\sigma}^0, R_{i\sigma}^1$), and the dashed lines bound the slabs corresponding to the rays passing through $d_0$ or $d_1$. The blue points and the red points are the intersections of the rays $\overrightarrow{a_\sigma c_0} \cap \overrightarrow{a_\sigma' c_1}$ and $\overrightarrow{b_\sigma c_0} \cap \overrightarrow{b_\sigma' c_1}$, respectively, for $\sigma \in \{i, j, l\}$. The green segments each contains a critical segment (the part between the blue and the red point) of a copy-umbra for $s_\sigma$ and $r_\sigma$, with shadow corners $c_0, c_1$. 

40
C.9.3 Constructing the corridor using slabs

We are now ready to describe the exact construction of the corridor. As mentioned before, the bottom wall is simply the line segment $c_0c_1$. We first describe the approximate shape of the upper wall, defined by a polygonal curve $\Lambda$ connecting the points $d_1$ and $d_9$. Later we will present how to modify $\Lambda$ into a final polygonal curve $\Lambda'$, which is exactly the upper wall of the corridor.

Note that in the corridor construction here we assume that $i < j < l$. In particular, the $L$-slabs $L_i^\gamma$ are above the $L$-slabs $L_j^\gamma$, which are above $L_l^\gamma$. For the $R$-slabs it is the opposite, i.e., the $R$-slabs $R_i^\gamma$ are below the $R$-slabs $R_j^\gamma$, which are below $R_l^\gamma$.

Figure 20 shows the grid of slabs and a sketch of the curve $\Lambda$ approximating the upper wall (excluding most of the leftmost and rightmost edge of $\Lambda$, with endpoints at $d_9$ and $d_1$, respectively, since they are too long to be pictured together with the middle segments). For $\sigma \in \{i, j, l\}$, let $u_\sigma$ be the intersection point of the rays $a_\sigma d_0$ and $b_\sigma d_1$. Let $v_{ij}$ be the intersection point of the rays $a_j d_0$ and $b_j d_1$, and $v_{jl}$ the intersection point of the rays $a_j d_0$ and $b_l d_1$. The curve $\Lambda$ is then a path defined by the points $d_9 u_i v_{ij} u_j v_{jl} u_l d_9$. By Lemma 25 $\Lambda \cap V$ is contained in the union of the $L$-slabs and the $R$-slabs, as shown in Figure 20. Due to the relative position of the slabs $L_i^\gamma, L_j^\gamma, L_l^\gamma$ and $R_i^\gamma, R_j^\gamma, R_l^\gamma$, as discussed above, the curve $\Lambda$ is $x$-monotone, and the point $v_{ij}$ (resp. $v_{jl}$) has smaller $y$-coordinate than the neighbouring points $u_i, u_j$ (resp. $u_j, u_l$), i.e., the curve $\Lambda$ always has a zig-zag shape resembling the one from Figure 20.

We will now show how to modify $\Lambda$ by adding to the curve some features. The first modification is in order to construct copy-nooks $Q_i^\sigma, Q_j^\sigma, Q_l^\sigma$ for each of the pairs $(s_i, r_i)$, $(s_j, r_j)$, and $(s_l, r_l)$, respectively. Note that the area above $c_0c_1$ and below $\Lambda$ already contains a copy-umbra $Q_{1}^\sigma$ for each pair $s_\sigma, r_\sigma$ for $\sigma \in \{i, j, l\}$ with shadow corners $c_0$ and $c_1$ (as $Q_{1}^\sigma$ is contained in the triangular area bounded above $c_0 c_1$ and below the rays $b_\sigma c_0, a_\sigma c_1$, which, due to Lemmas 25 and 24, is below $\Lambda$). The second reason why we need to modify $\Lambda$ is in order to create stationary guard positions $p_1, p_j, p_l$ that see the areas of the copy-umbra, but nothing above their critical segments. In the following, we explain how to modify the fragment of $\Lambda$ consisting of the leftmost two edges, i.e., the path $v_{ij} u_l d_9$. The construction is presented in Figure 21. We then perform similar modifications for the fragments of $\Lambda$ consisting of the paths $v_{ij} u_j v_{jl}$ and $d_9 u_i v_{ij}$.

First, we show how to construct a copy-nook $Q_i^\sigma$ of $s_l$ and $r_l$ with shadow corners at $\Lambda$. The curve $\Lambda$ will then be modified so that $Q_i^\sigma$ is contained within the corridor. Let $N_l$ be the square consisting of points which are above the slabs $L_l^1$ and $R_l^1$, but not above $L_l^3$ or $R_l^3$. The square $N_l$ is approximately the area which is seen both from the right endpoint $b_l$ of $s_l$, and the left endpoint $a_l$ of $r_l$. Note that $N_l$ contains the point $u_l$ (as $u_l \in L_l^1 \cap R_l^1$, and thus $u_l$ is above $L_l^1$ and $R_l^1$ and below $L_l^3$ and $R_l^3$). The copy-nook $Q_i^\sigma$ for the pair $s_l, r_l$ will be created inside $N_l$ (see Figure 22). Consider the two intersection points of the boundary of $N_l$ with the line segments $v_{ij} u_l$ and $u_l d_0$. Let $y_N$ be the larger of the $y$-coordinates of these two intersection points. The shadow corners of the nook $Q_i^\sigma$ are chosen as intersection points of the horizontal line $y = y_N$ with the line segments $v_{ij} u_l$ and $u_l d_0$, and they are denoted by $z_{11}$ and $z_{10}$, respectively. In this way we ensure that both shadow corners are visible from any point within the segments $s_l$ and $r_l$, and that they define a copy-nook $Q_i^\sigma$ for the pair of segments $s_l, r_l$. Note that the entire nook $Q_i^\sigma$ is contained in $N_l$, since by Lemmas 24 and 25 no point on $s_l$ or $r_l$ can see a point above the slabs $L_l^3$ or $R_l^3$. We now modify the curve $\Lambda$ as follows. Let $P_l$ be a quadrilateral with two corners at $z_{10}$ and $z_{11}$, and such that it contains vertical edges incident to $z_{10}$ and $z_{11}$, and an edge containing the critical segment for $Q_i^\sigma$. We modify $\Lambda$ so that between the points $z_{10}$ and $z_{11}$, it consists of the vertical edges and the topmost edge of $P_l$.

Now, consider the copy-umbra $Q_i^\sigma$ for the pair of segments $s_l, r_l$ with shadow corners $c_0$ and $c_1$.
Let \( f_0 := \overrightarrow{b_0 c_0} \cap \overrightarrow{b_1 c_1} \) and \( f_1 := \overrightarrow{a_0 c_0} \cap \overrightarrow{a_1 c_1} \). Note that the points \( f_0, f_1 \) correspond to the red and blue points in Figures 20 and 21. The segment \( f_0 f_1 \) is the critical segment for \( Q^i \). By Lemmas 24 and 25, \( f_0 \in L^1_i \cap R^1_i \) and \( f_1 \in L^0_i \cap R^1_i \), and the squares \( L^1_i \cap R^0_i, L^0_i \cap R^1_i \) have a sidelength of \( 2\varepsilon \). Therefore the slope of the line \( \overrightarrow{f_0 f_1} \) is in the interval \( \left[ -\frac{2\sqrt{2}\varepsilon}{\sqrt{2\rho}}, \frac{2\sqrt{2}\varepsilon}{\sqrt{2\rho}} \right] = [-1/6, 1/6] \), and this line intersects both line segments \( v_{ij} u_i \) and \( v_i d_0 \). Let \( z'_{i0}, z'_{i1} \) be the intersection points of the line \( \overrightarrow{f_0 f_1} \) with the line segments \( u_i d_0 \) and \( u_i v_{ij} \), respectively. (We similarly define points \( z'_{i0}, z'_{i1} \) on \( d_1 u_i v_{ij} \) as the intersection points with the line containing the critical segment of the umbra \( Q^i \), and \( z'_{i0}, z'_{i1} \) on \( v_{ij} u_i v_{ij} \) as the intersection points with the line containing the critical segment of the umbra \( Q^i_j \).) We introduce a stationary guard position \( p_i \) by creating a pocket which will require modifying the curve \( \Lambda \) again. The pocket is extruding to the right from \( v_{ij} u_i \), following the line \( \overrightarrow{f_0 f_1} \), as pictured in Figure 21. Likewise, it is extruding vertically up from \( v_{ij} \). The pocket contains a stationary guard position \( p_i \) on the line \( \overrightarrow{f_0 f_1} \). Clearly, a guard placed at \( p_i \) sees nothing above the line segment \( f_0 f_1 \). Note that it sees the part of \( Q^i \) to the left of the vertical line through \( v_{ij} \).

For the middle two edges \( v_{ij} u_j v_{ji} \) of \( \Lambda \), we place the stationary guard position \( p_j \) vertically above \( v_{ij} \) so that it sees an area below the critical segment of the umbra \( Q^i_j \) and to the left of the vertical line through \( v_{ij} \). For the rightmost edges \( d_1 u_i v_{ji} \), we place the stationary guard position \( p_i \)
vertically above $d_1$ so that it sees an area below the critical segment of the umbra $Q_i^n$ to the left of the vertical line through $d_1$. Let $\Lambda'$ be the wall obtained by doing the modifications to $\Lambda$ described here, and let $C$ be the corridor, that is, the area bounded by the lower wall $c_0c_1$, the upper wall $\Lambda'$ between $d_0$ and $d_1$, and by the vertical entrance segments $c_0d_0$ and $c_1d_1$. See Figure 23 for a picture of the complete corridor.

Lemma 26. The stationary guard positions $p_i, p_j, p_l$ have the following three properties.

- The three stationary guard positions $p_i, p_j, p_l$ together see all of the corridor except the points above the segments $z'_{i0}z'_{i1}, z'_{j0}z'_{j1}, z'_{l0}z'_{l1}$.

- None of the guards can see anything to the right of the right entrance $c_1d_1$.

- None of the stationary guard positions $p_i, p_j, p_l$ for the pairs $(s_i, r_i), (s_j, r_j), (s_l, r_l)$, respectively, can see any point on the critical segment of the nook or umbra of one of the other pairs.

Proof. See Figure 23. For the first claim, note that the vertical lines through $v_{jl}$ and $v_{ij}$ divide the corridor into three parts. It is now clear that all points in the leftmost part below $z'_{i0}z'_{i1}$ are seen by $p_l$, all points in the middle part below $z'_{j0}z'_{j1}$ are seen by $p_j$, and all points in the rightmost part below $z'_{l0}z'_{l1}$ are seen by $p_i$.

For the second part, observe that the point $p_i$ cannot see any point to the right of the vertical line through $d_1$, and the visibility of $p_j$ and $p_l$ is bounded by vertical lines more to the left.

For the last part, we note that the curve $\Lambda'$ passes through points $v_{jl}$ and $v_{ij}$, blocking visibility between stationary guard positions and critical segments corresponding to different pairs. □

Lemma 27. Suppose that in each of the pairs $(s_i, r_i), (s_j, r_j), (s_l, r_l)$ of guard segments corresponding to a corridor $C$, the two segments have the same orientation. Then $C$ satisfies the following properties.
1. In any guard set $G$ of $\mathcal{P}$ there are at least 3 guards placed within the corridor $C$, and if there are exactly 3 then they are placed at the stationary guard positions $p_i, p_j, p_l$. (The number is 2 instead of 3 if we construct $C$ to copy only two segments.)

2. Let $G$ be any set of points with exactly one guard on each guard segment and each stationary guard position, and with no guards outside of stationary guard positions and guard segments. If all of $C$ is seen by $G$, then for each of the pairs $(s_i, r_i), (s_j, r_j), (s_l, r_l)$ the two guards on the segments specify the same value.

3. For any set of points $G$ which satisfies the properties: (i) there is a guard at each point $p_i, p_j, p_l$ and at each guard segment $s_i, s_j, s_l$ and $r_i, r_j, r_l$, and (ii) the values specified by the pairs of segments $s_i, s_j, s_l$ and $r_i, r_j, r_l$ are consistent, $G$ sees all of $C$.

4. No guard at a stationary guard position or a guard segment outside the gadget can see any point inside the gadget below the line $d_0 c_1$.

5. The vertical distance from $c_0 c_1$ to the topmost point of the corridor is at most 1.4.

Proof. For Property 1, note that the points defining the stationary guards within $C$ can be seen only from within $C$. We can now use Lemma 14 setting $A$ as the corridor area and choosing the points defining the stationary guards to construct the set $M$, to prove the desired property.

For Property 2 consider the set $G$ as described, and let $\sigma \in \{i, j, l\}$. The stationary guard positions $p_i, p_j, p_l$ cannot see any points above the line containing the critical segments of the umbra $Q^\sigma_a$. Lemma 26 and Property 3 of Lemma 24 give us that guards at the guard segments $s_1, s_2, \ldots, s_{4n}, r_i, r_j, r_l$ must see the critical lines of the nook and umbra $Q^\sigma_a, Q^\sigma_a$. We will now show that among all these segments, only guards placed on the segments $s_\sigma, r_\sigma$ are able to do so. From Lemma 25 we get that for any $\sigma' > \sigma$, no guard on the guard segment $s_{\sigma'}$ can see a point in the square $V$ below $L^0_{\sigma'}$. As $L^0_{\sigma'}$ is above $L^3_{\sigma'}$, it is also above the critical segments of the nook $Q^\sigma_a$ and the umbra $Q^\sigma_a$ of the pair $s_\sigma, r_\sigma$. Likewise, for any $\sigma' < \sigma$, no guard on the guard segment $s_{\sigma'}$ can see a point in $V$ above $L^3_{\sigma'}$. As $L^3_{\sigma'}$ is below $L^0_{\sigma'}$, it is also below the critical segments of the pair $s_\sigma, r_\sigma$. A similar argument shows that among the guard segments $r_i, r_j, r_l$, only guards on $r_\sigma$ can see any points on the critical segments for the pair $s_\sigma, r_\sigma$. Therefore the two guards on $s_\sigma, r_\sigma$ must together see both critical segments of that pair, and by Lemma 20 the guards must specify the same value.
For Property $3$, consider a set $G$ satisfying (i) and (ii). By Lemma $26$, the stationary guards can see all of $C$ except of the points which are above the line segments $z_0^i z_1^i$, $z_0^j z_1^j$, $z_0^r z_1^r$ containing the critical segments of the umbras. Consider any $\sigma \in \{i, j, l\}$, and the guard segments $s_\sigma, r_\sigma$. As these guards can see the complete critical segment for the umbra $Q_\sigma^u$, they can see the whole area contained above the critical segment and below $\Lambda$. As they can see the complete critical interval for the nook $Q_\sigma^u$, they can see all of the polygon $P_\sigma$. Therefore, they can see the whole area above the critical interval and below $\Lambda'$.

Property $4$ is a clear consequence of Lemma $26$.

For Property $5$, note that the top wall of the corridor can only extend beyond the square $V$ due to a part of the wall $\Lambda'$ creating a stationary guard position. Recall that the slope of the critical segments of the umbras $Q_i^u, Q_j^u, Q^u_l$ is in the range $[-1/6, 1/6]$. Since $\|c_0c_1\| = 2$, it follows that any point on $\Lambda'$ is at height at most $1 + 38N\rho + 2 \cdot 1/6 < 1.4$ above $c_0c_1$.

\[\square\]

**Lemma 28.** Assume that the endpoints of guard segments corresponding to a corridor $C$ are at rational points, with the nominators and the denominators upper-bounded by $(\zeta CN^2)^{(1)}$. Then, we can construct the corridor $C$ in such a way that each corner of $C$ has rational coordinates, with the nominator and the denominator upper-bounded by $(\zeta CN^2)^{(1)}$. The corridor construction can be done in polynomial time.

**Proof.** Note that the entrances to the corridor are also at rational points, with nominators and denominators upper-bounded by $(\zeta CN^2)^{(1)}$. Therefore, each of the lines defining the polygonal curve $\Lambda$ is defined by two rational points with this property. The same holds for the lines bounding the $L$-slabs and the $R$-slabs.

Consider the construction of a copy-nook $Q_\sigma^u$ within the corridor. The corners of $Q_\sigma^u$ are then at points which are defined as intersection of two lines, where each line is defined by two rational points with nominators and denominators upper-bounded by $(\zeta CN^2)^{(1)}$. Therefore, the corners of the nook, and therefore also the corners of the quadrilateral $P_\sigma$ are also of this form.

The stationary guard positions $p_i, p_j, p_l$ are the intersection points of two lines, again each line defined by two points with the above property. Therefore, the nominators and denominators of the stationary guard positions are also upper-bounded by $(\zeta CN^2)^{(1)}$. The corners of the pockets corresponding to $p_i, p_j, p_l$ can be chosen with much freedom, and therefore they can also be at points satisfying the lemma statement.

\[\square\]

**C.9.4 Corridor construction for gadgets at the left side of $\mathcal{P}$**

For the gadgets attached at the left side of the polygon $\mathcal{P}$, the construction of the corridor is analogous. Now the points $c_0, d_0$ lie on the line $\ell_l$ instead of $\ell_r$, and the points $c_1, d_1$ are to the left of $c_0, d_0$. As we want the points $o (o')$ to correspond to the lowest intersection point of a ray from an endpoint of a guard segment in the base line (in the gadget, respectively) containing the point $c_0 (c_1$, respectively) with $\ell_c$, we redefine these points in the following way. The point $o$ is the intersection point of the ray $b_{1n}c_0$ with $\ell_c$, and $o'$ is the intersection point of the ray $a'l_{c1}$ with $\ell_c$. As we want the slabs $L^\sigma_o$ to contain fragments of rays from the endpoints of the segment $s_\sigma$, we redefine

\[L^\sigma_o := S(o + (0, (4n - \sigma)\delta + \gamma\rho), \beta, \varepsilon).\]

Similarly, for $\sigma \in \{i, j, l\}$, we redefine

\[R^\sigma_o := S(o' + (0, \tau(\sigma)\delta + \gamma\rho), \alpha, \varepsilon),\]

45
where \( \tau: \{i, j, l\} \rightarrow \{0, 1, 2\} \) is as defined for gadgets at the right side of \( \mathcal{P} \).

As now the left endpoints of the gadget guard segments are further away from the line \( \ell_c \) than the right endpoints, each gadget attached to the left side of \( \mathcal{P} \) has to satisfy the following (instead of Lemma 24), which will be proven in sections specific to the particular gadgets.

**Lemma 29.** For any gadget to be attached to the left side of the polygon \( \mathcal{P} \) and containing the guard segments \( r_i := a_i^j b_i^j, r_j := a_j^j b_j^j, r_l := a_l^j b_l^j \) the following holds, where \( c_1 \) is the bottom-left endpoint of the corridor corresponding to the gadget.

1. The intersection of any \( R \)-slab with the line \( \ell_c \) is contained in \( V \).
2. For each \( \sigma \in \{i, j, l\} \), it holds that \( \overrightarrow{a_i^c} c_1 \cap V \subset R_{3\sigma}^0, \overrightarrow{b_i^c} c_1 \cap V \subset R_{3\sigma}^1, \overrightarrow{a_l^c} d_1 \cap V \subset R_{3\sigma}^2, \overrightarrow{b_l^c} d_1 \cap V \subset R_{3\sigma}^3 \).
3. There are no stationary guard positions or guard segments different from \( r_i, r_j, r_l \) within the gadget, from which any point of the corridor can be seen.

For the same reason, instead of Lemma 25 we get the following lemma.

**Lemma 30.** Within any corridor to be attached to the left side of the polygon \( \mathcal{P} \) the following properties are satisfied.

1. The intersection of any \( L \)-slab with any \( R \)-slab is contained in \( V \).
2. For each \( \sigma \in \{1, \ldots, 4\} \), it holds that \( \overrightarrow{b_i^c} c_0 \cap V \subset L_{3\sigma}^0, \overrightarrow{a_i^c} c_0 \cap V \subset L_{3\sigma}^1, \overrightarrow{b_l^c} d_0 \cap V \subset L_{3\sigma}^2, \overrightarrow{a_l^c} d_0 \cap V \subset L_{3\sigma}^3 \).

Due to symmetry of \( \mathcal{P}_M \), the proof of Lemma 30 is the same as the proof of Lemma 25. The corridor construction and the proof of Lemma 27 is then the same as for the corridor attached at the right side of \( \mathcal{P} \). The only difference is that in order to ensure that the \( L \)-slabs \( L_i^j \) are above the \( R \)-slabs \( L_i^j \) which are above \( L_i^j \) (which is required to get a meaningful zig-zag shape of the upper wall of the corridor) we now have to assume that \( i > j > l \) (as the definition of the \( L \)-slabs has changed).

### C.10 The \( \geq \)-addition gadget

In this section we present the construction of the \( \geq \)-addition gadget which represents an inequality \( x_i + x_j \geq x_l \), where \( i, j, l \in \{1, \ldots, n\} \). In Section C.11 we will show how to modify the construction to obtain the \( \leq \)-addition gadget for the inequality \( x_i + x_j \leq x_l \). For any equation of the form \( x_i + x_j = x_l \) in \( \Phi \) we will then add both gadgets to our polygon \( \mathcal{P} \).

#### C.10.1 Idea behind the gadget construction

We first describe the general idea behind the construction of a gadget imposing an inequality \( x_i' + x_j \geq x_l \) for three variables \( x_i', x_j, x_l \). See Figure 24 for a sketch of the construction. Let \( w, v, h > 0 \) be rational values such that \( w > v + 3/2 \). Let \( r_i', r_j, r_l \) be right-oriented guard segments\(^4\) of length 3/2 such that \( r_j \) has its left endpoint at the point \((-w, 0)\), \( r_i' \) has its right endpoint at \((w, 0)\), and \( r_l \) has its left endpoint at \((-2, -h)\). Let \( g_i' := (w - 2 + x_i', 0), g_j := (-w - 1/2 + x_j, 0), \) and \( g_l := (-5/2 + x_l, -h) \) be three guards on \( r_i', r_j, r_l \), respectively, representing the values \( x_i', x_j, x_l \in [1/2, 2] \).

\(^4\)We use \( r_i' \) instead of \( r_i \) here (and \( x_i' \) instead of \( x_i \)), as the guard segment \( r_i' \) specifying the value \( x_i' \) will only be a weak copy of the segment from the base line with a value \( x_i \), i.e., it will hold that \( x_i' \leq x_i \). More details will be provided later.
Let $e_i := (v, h), e_j := (-v, h), e_l := (0, h)$. Let $\Gamma$ be a collection of points $\gamma$ such that the ray $\overrightarrow{\gamma e_i}$ intersects $r_i'$, and the ray $\overrightarrow{\gamma e_j}$ intersects $r_j$. Then $\Gamma$ is a quadrilateral, bounded by the following rays: the rays with origin at the endpoints of $r_i'$ and containing $e_i$, and the rays with origin at the endpoints of $r_j$ and containing $e_j$. Suppose that

- for every point $g_i'$ on $r_i'$ and $\gamma$ in $\Gamma$, the points $\gamma$ and $g_i'$ can see each other if and only if $\gamma$ is on or to the right of the line $\overrightarrow{g_i'e_i}$,
- for every point $g_j$ on $r_j$ and $\gamma$ in $\Gamma$, the points $\gamma$ and $g_j$ can see each other if and only if $\gamma$ is on or to the right of the line $\overrightarrow{g_j e_j}$,
- for every point $g_l$ on $r_l$ and $\gamma$ in $\Gamma$, the points $\gamma$ and $g_l$ can see each other if and only if $\gamma$ is on or to the left of the line $\overrightarrow{g_l e_l}$.

These properties are enforced by polygon edges in Figure 24. Then, we can show the following result.

**Lemma 31.** The guards $g_i', g_j, g_l$ can together see the whole quadrilateral $\Gamma$ if and only if $x_i' + x_j \geq x_l$.

**Proof.** Let $\gamma \in \Gamma$ be the intersection point of the rays $\overrightarrow{g_i'e_i}$ and $\overrightarrow{g_j e_j}$.

Suppose that the guards $g_i', g_j, g_l$ together see the whole quadrilateral $\Gamma$. Since $g_i'$ cannot see the area to the left of the line $\overrightarrow{\gamma g_l}$, and $g_j$ cannot see the area to the left of the line $\overrightarrow{\gamma g_j}$, there are
points arbitrarily close to γ which are not seen by any of the guards \( g'_i, g_j \). Therefore, \( g_l \) has to see γ.

Consider the rays \( \overrightarrow{\gamma e_i}, \overrightarrow{\gamma e_j}, \) and \( \overrightarrow{\gamma e_l} \). Let \( \chi \) be the intersection point of the ray \( \overrightarrow{\gamma e_l} \) with a horizontal line \( y = 0 \), and \( \chi' \) the intersection point of the ray \( \overrightarrow{\gamma e_l} \) with a horizontal line \( y = -h \). Note that the guard \( g_l \) can see the point γ if and only if \( g_l \) is coincident with γ’ or to the left of γ’.

From the similarity of triangles \( g_j g'_i \gamma \) and \( e_j e_l \gamma \) we get that \( \frac{y(\gamma)}{y(\gamma)-h} = \frac{|\chi - g_j|}{v} \), and therefore \( |\chi - g_j| = w + x'_l/2 - x_j/2 - 3/4 \), and \( \chi = (x'_l/2 + x_j/2 - 5/4, 0) \). Let \( O := (0, 0) \) and \( O' := (0, -h) \). From the similarity of triangles \( O\chi e_l \) and \( O'\chi' e_l \) we get that \( \chi' = (x'_l + x_j - 5/2, 0) \). The condition that the guard \( g_l \) is coincident with \( \chi' \) or to the left of \( \chi' \) is equivalent to \(-5/2 + x_l \leq x'_l + x_j - 5/2 \), i.e., \( x'_l + x_j \geq x_l \).

On the other hand, if \( x'_l + x_j \geq x_l \) then the guard \( g_l \) is coincident with \( \chi' \) or to the left of \( \chi' \), and therefore \( g_l \) can see γ. Then the guards \( g'_i, g_j, g_l \) can together see the whole \( \Gamma \).

\[ \square \]

C.10.2 Fragment of the gadget for testing the inequality

We now present the construction of a polygon \( \mathcal{P}_{ineq} \) containing three guard segments \( r'_i, r_j, r_l \) and corners \( e_i, e_j, e_l \) with coordinates as described above, where we set \( w := 26 \), \( v := 10 \), and \( h := 10.5 \), enforcing an inequality on the values corresponding to the guard segments. The main part of the polygon \( \mathcal{P}_{ineq} \) is pictured in Figure \ref{fig:ineq-solution}. The three blue line segments correspond to the guard segments, and the three green dots to stationary guard positions. The stationary guard positions \( g_t, g_m, g_b \) have been chosen as follows. The point \( g_t \) is at the ray with origin at the right endpoint of \( r_j \) and containing \( e_j \), \( g_m \) is at the ray with origin at the right endpoint of \( r_j \) and containing \( e_l \), and \( g_b \) is at the ray with origin at the right endpoint of \( r'_i \) and containing \( e_i \). For each guard segment and each stationary guard position we introduce pockets of the polygon, such that the points defining each guard segment and stationary guard position cannot be seen from any other guard segment or stationary guard position. Two edges in the left of the figure are only shown partially. They end to the left at corners \( c_1 := (-CN^2, CN^2) \) and \( d_1 := (-CN^2, CN^2 + 1.5) \), respectively. These corners are connected by an edge \( c_1 d_1 \) which closes the polygon. We can show the following result.

**Lemma 32.** Consider the polygon \( \mathcal{P}_{ineq} \) from Figure \ref{fig:ineq-solution}. A set of guards \( G \subset \mathcal{P}_{ineq} \) of cardinality at most 6 guards \( \mathcal{P}_{ineq} \) if and only if

- there is exactly one guard placed on each guard segment \( r'_i, r_j, r_l \) and at each stationary guard position \( g_t, g_m, g_b \), and
- the variables \( x'_i, x_j, x_l \) corresponding to the guard segments \( r'_i, r_j, r_l \), respectively, satisfy the inequality \( x'_i + x_j \geq x_l \).

**Proof.** Assume first that the two conditions are satisfied. Observe that the stationary guard positions \( g_t, g_m, g_b \) have been chosen so that guards placed at them cannot see any point in the interior of \( \Gamma \), but together with the guards \( g'_i, g_j, g_l \) placed on \( r'_i, r_j, r_l \) can see all of \( \mathcal{P}_{ineq} \setminus \Gamma \). (For this property to hold, the position of \( g'_i, g_j, g_l \) within the corresponding guard segments does not matter.) Since additionally the inequality \( x'_i + x_j \geq x_l \) is satisfied, then Lemma \ref{lem:ineq-separation} yields that all of \( \Gamma \) is seen, and hence \( G \) guards \( \mathcal{P}_{ineq} \).

Assume now that a set \( G \) of at most 6 guards sees all of \( \mathcal{P}_{ineq} \). Using Lemmas \ref{lem:ineq-separation} and \ref{lem:ineq-separation-2} (where we set \( A := \mathcal{P}_{ineq} \) and choose the points in \( M \) among the following: one point \( t_1 \) defining each stationary guard position, and one point \( q_2 \) defining each guard segment) we can show that the
Figure 25: The main part of a polygon $P_{ineq}$. The quadrilateral $\Gamma$ consists of all the intersection points $\overrightarrow{p_j e_j} \cap \overrightarrow{p_i e_i}$ where $p_j \in r_j$ and $p_i \in r'_i$.

The polygon requires at least 6 guards, and if there are 6 guards then there must be one guard at each stationary guard position, and at each guard segment. Then, as $\Gamma$ is seen by the guards, by Lemma 31 we get that the inequality $x'_i + x_j \geq x_l$ holds.

In the actual $\geq$-addition gadget we modify $P_{ineq}$ in a way described later and scale it down by a factor of $\frac{1}{\sqrt{C N}}$. Then we connect it to the main part of the polygon $P$. The polygon $P_{ineq}$ is attached at the right side of $P$, and the connection between $P_{ineq}$ and the main area of $P$ is via a corridor. The point which corresponds to $O := (0, 0)$ in the polygon $P_{ineq}$ is then at the position $m := c_1 + (1, -1)$ in $P$, where $c_1$ is the bottom-right corner of the corridor.

C.10.3 Copying guard segments to their final position

Let $r'_i, r_j, r_l$ denote the guard segments of $P_{ineq}$. We now show how to enforce dependency between appropriate guard segments from the base line of the polygon $P$ and the guard segments $r'_i, r_j, r_l$.

We ensure that the guard segments $r_j$ and $r_l$ are copies of segments from the base line by connecting the (scaled) polygon $P_{ineq}$ to the polygon $P$ via a corridor. However, the segment $r'_i$ cannot be copied in this way, as the edges of $P_{ineq}$ are blocking visibility. Instead, we introduce an additional guard segment $r_i$ within the gadget, and we copy appropriately chosen segments $s_i, s_j, s_l$ from the base line into $r_i, r_j, r_l$. This part is explained in detail in Section C.10.5. Then, by introducing a copy-nook within the construction of $P_{ineq}$, we ensure that $r'_i$ is a weak copy of $r_i$. This is explained in detail in Section C.10.4. In Section C.10.6 we summarize the properties of the constructed gadget.

The gadget is scaled by a factor of $\frac{1}{\sqrt{C N}}$ before it is attached to the corridor, and the points $c_1, d_1$ of the gadget are coincident with the points defining the right entrance of the corridor, which have
the same names. After this operation, the middle point of the gadget (i.e., the point corresponding to $O := (0, 0)$ in the coordinate system of the gadget) satisfies the equality $m = c_1 + (1, -1)$, as stated in Section C.7.

For the picture of the complete gadget see Figure 26.

![Figure 26: Detailed construction of the $\geq$-addition gadget. Note that the dotted lines show that: no guard on $r_i$ can see any point in $\Gamma$ because of the corner $z$; a guard on $r_i$ can always see both shadow corners of the copy-nook $Q_i$; and no point on $r_i$ sees any point of $Q_i$ because of the corner $e_i$. For each of the segments $r_\sigma$, $\sigma \in \{i, j, l\}$, the rays from points on $r_\sigma$ through the corridor entrance $c_1 d_1$ are between the two grey dashed rays emitting from the endpoints of $r_\sigma$.](image)

C.10.4 Introducing a new guard segment $r_i$

Consider the setting as described in Section C.10.2 and the polygon $P_{\text{ineq}}$ from Figure 25. We explain how to modify $P_{\text{ineq}}$ into a polygon $P'_{\text{ineq}}$, which is a scaled version of our gadget. The main part of $P'_{\text{ineq}}$ is shown in Figure 26, where again the edges to the left with endpoints at $c_1$ and $d_1$ are not fully shown.

The polygon $P'_{\text{ineq}}$ is obtained from $P_{\text{ineq}}$ in the following way. First, we add an additional guard segment $r_i$ of length $3/2$ (i.e., the same as the length of other guard segments in the construction), with its left endpoint at the point $(-20.5, -17)$. This requires introducing a pocket corresponding to the added guard segment. We ensure that $r'_i$ is a weak copy of $r_i$ by creating a copy-nook $Q_i$ for the pair of guard segments $r_i, r'_i$, which cannot be seen from any other guard segment or stationary guard position. The shadow corners of $Q_i$ are $(2.5, 34)$ and $(4.5, 34)$. We have to ensure that after
this modification, the gadget still enforces the desired inequality. In particular, we have to ensure that a guard placed on \( r_i \) cannot see any point in the interior of \( \Gamma \). We can do that by introducing a new corner \( z := (-12.5, -1.5) \) of the polygon that blocks \( r_i \) from seeing \( \Gamma \). Note that \( z \) does not block \( r_i \) from seeing the segment \( c_1d_1 \) (as shown by the dashed grey lines in Figure 26).

**Lemma 33.** A set of guards \( G \subset \mathcal{P}'_{\text{ineq}} \) of cardinality at most 7 guards \( \mathcal{P}'_{\text{ineq}} \) if and only if

- there is exactly one guard placed on each guard segment \( r'_i, r_i, r_j, r_l \) and at each stationary guard position,
- the variables \( x_i, x'_i \) corresponding to the guard segments \( r_i, r'_i \), respectively, satisfy the inequality \( x_i \geq x'_i \), and
- the variables \( x'_i, x_j, x_l \) corresponding to the guard segments \( r'_i, r_j, r_l \), respectively, satisfy the inequality \( x'_i + x_j \geq x_l \).

**Proof.** Assume that \( \mathcal{P}'_{\text{ineq}} \) is guarded by a set \( G \) of at most 7 guards. Similarly as in Lemma 32, we can show that there must be exactly one guard at each guard segment and each stationary guard position. As the copy-nook \( Q_i \) can be seen only by guards placed on \( r_i \) and \( r'_i \), it follows from Lemma 19 that the polygon is guarded by \( G \) only if the variables \( x_i, x'_i \) corresponding to \( r_i, r'_i \), respectively, satisfy the inequality \( x_i \geq x'_i \). As the quadrilateral \( \Gamma \) cannot be seen by a guard from \( r_i \), we get from Lemma 32 that the variables \( x'_i, x_j, x_l \) corresponding to the guard segments \( r'_i, r_j, r_l \), respectively, must satisfy the inequality \( x'_i + x_j \geq x_l \).

Now assume that there is exactly one guard placed on each guard segment \( r'_i, r_i, r_j, r_l \) and at each stationary guard position, the variables \( x_i, x'_i \) satisfy the inequality \( x_i \geq x'_i \), and the variables \( x'_i, x_j, x_l \) satisfy the inequality \( x'_i + x_j \geq x_l \). Then all of \( \Gamma \) is seen by the guards, as is all of the nook \( Q_i \). The remaining area is also seen by the guards, which we can show in the same way as in Lemma 32.

---

### C.10.5 Copying three guard segments via a corridor

We are given a formula from \( \Phi \) of the form \( x_i + x_j = x_l \), where \( i, j, l \in \{1, \ldots, n\} \) and want to construct a gadget imposing an inequality \( x_i + x_j \geq x_l \). We need to show that the values of the required variables can be copied into the guard segments \( r_i, r_j, r_l \) in the gadget described above.

First, we will explain how to choose the segments from the base line to be copied into the gadget. Then, we will show that the \( \geq \)-addition gadget satisfies properties required by Lemma 24 which will ensure that the gadget can be connected to the main area by a corridor.

In order to apply a corridor construction as described in Section C.9 to copy three guard segments from the base line into the gadget, we require the segments in order from left to right on the base line to represent the variables \( x_i, x_j, x_l \). Recall that there are \( n \) variables \( x_1, \ldots, x_n \) in the formula \( \Phi \), but that for any \( \sigma \in \{1, \ldots, n\} \), we use \( x_{\sigma + n}, x_{\sigma + 2n} \) and \( x_{\sigma + 3n} \) as synonyms for \( x_\sigma \). Therefore, the inequality \( x_i + x_j \geq x_l \) is equivalent to \( x_i + x_{n+j} \geq x_{2n+l} \). The guard segments on the base line are \( s_1, \ldots, s_{3n} \), where each \( s_\sigma \) represents the variable \( x_\sigma \). The segments \( s_1, \ldots, s_{3n} \) are right-oriented whereas \( s_{3n+1}, \ldots, s_{4n} \) are left-oriented. (In Section C.12 we explain how to obtain these dependencies between the guard segments.) Therefore, with slight abuse of notation, we redefine \( j := j + n \) and \( l := l + 2n \) so that \( i < j < l < 3n \). Then, the guard segments \( s_i, s_j, s_l \) satisfy our requirements.

We now need to show that our gadget construction satisfies the conditions of Lemma 24. Recall that our gadget is the polygon \( \mathcal{P}'_{\text{ineq}} \) scaled by a factor of \( \frac{1}{CN^r} \). First, we will prove an auxiliary lemma.
Lemma 34. Let $G$ be any gadget to be attached at the right side of the polygon $P$ such that the guard segments $r_1, r_j, r_l$ have a length of $\frac{3}{\sqrt{C_2}}$ and are contained in the box $m + [-\Delta, \Delta] \times [-\Delta, \Delta]$, where $\Delta := \frac{50}{C_2}$, which is the axis-parallel square centered at $m$ with side length $2\Delta$. Suppose furthermore that the left endpoint of $r_j$ is on the line through $a_i' + (\delta, 0)$ parallel to the vector $(-1, 1)$ and the left endpoint of $r_l$ is on the line through $a_i' + (2\delta, 0)$ parallel to the same vector, where $a_i'$ is the left endpoint of $r_i$. Then properties 1 and 2 of Lemma 24 both hold for $G$.

Proof. Assume in this proof without loss of generality that $c_1 := (0, 0)$. Then $m = (1, -1)$. Define $(\omega_{xi}, \omega_{yi}) := CN^2 \cdot (b_i - m)$, $(\omega_{xi}', \omega_{yi}') := CN^2 \cdot (b_i' - m)$, and $(\omega_{xi}, \omega_{yi}) := CN^2 \cdot (b_i - m)$. It follows from the conditions in the lemma that each of these values is in $[-50, 50]$.

Define $d_o'$ for each $\sigma \in \{i, j, l\}$ to be the intersection point of the ray $b_o' \rightarrow c_1$ with $\ell_c$. Recall that the point $o'$ is defined as $o'_l$. We first verify that Property 1 holds, that is, each point $d_o'$ is in the rectangle $V$. We thus need to ensure that $y(d_o') \in [1 - 38N\rho, 1 + 38N\rho]$. The horizontal distance between $\ell_c$ and $c_1$ is 1. Therefore, the vertical distance between $c_0c_1$ and the point $d_o'$ is

$$y(d_o') = \frac{-y(b_{\sigma})}{x(b_{\sigma})} = \frac{CN^2 - \omega_{yi}}{CN^2 + \omega_{yi}}.$$  

We have $y(d_o') \geq \frac{CN^2 - 50}{CN^2 + 50}$. Note that the lower edge of the square $V$ has $y$-coordinate $1 - 38N\rho = 1 - \frac{57}{CN}$. Now, the inequality $\frac{CN^2 - 50}{CN^2 + 50} \geq 1 - \frac{57}{CN}$ is equivalent to $57CN^3 + 2850N \geq 100CN^2$, which is true for all $N \geq 2$.

Similarly, we have that $y(d_o') \leq \frac{CN^2 + 50}{CN^2 - 50}$. The upper edge of $V$ has $y$-coordinate $1 + 38N\rho = 1 + \frac{57}{CN}$. The inequality $\frac{CN^2 + 50}{CN^2 - 50} \leq 1 + \frac{57}{CN}$ is equivalent to $100CN^2 \leq 57CN^3 - 2850N$, which is also true for all $N \geq 2$. Hence, Property 1 holds.

Fix any $\sigma \in \{i, j, l\}$. Recall that the vertical distance between the centers of two consecutive rays from $R_o^0, R_o^1, R_o^2, R_o^3$ is $\rho$. We now show that the following four properties imply that Property 2 holds. Afterwards, we will prove that the four properties are satisfied.

- **a** The vertical distance from $d_o'$ to the center of the slab $R_o^0$ is at most $\varepsilon/4$,
- **b** the distance $d_o''$ between the intersection points of $b_o' \rightarrow c_1$ and $b_o' \rightarrow d_1$ with $\ell_c$ is in the interval $[2\rho - \varepsilon/4, 2\rho + \varepsilon/4]$,
- **c** for any symbol $\mu \in \{c, d\}$ the distance $d_o''$ between the intersection points of $a_o' \rightarrow \mu_1$ and $b_o' \rightarrow \mu_1$ with $\ell_c$ is in $[\rho - \varepsilon/4, \rho + \varepsilon/4]$, and
- **d** the absolute value of the slope of any ray with origin at an endpoint of $r_o$ and passing through a point in $c_1d_1$ is in $\left[1 - \frac{1}{38N\rho} \cdot \varepsilon, 1 + \frac{1}{38N\rho} \cdot \varepsilon\right]$.

The first three properties yield that all rays $b_o' \rightarrow c_1, a_o' \rightarrow c_1, b_o' \rightarrow d_1$, and $a_o' \rightarrow d_1$ intersect the line $\ell_c$ within their corresponding slabs at the vertical distance of at most $\frac{3\varepsilon}{4}$ from the center of the slab. The last property yields that the rays are contained in the corresponding slabs throughout the square $V$.

We now prove Property 2. For $\sigma = l$, the distance is 0 by definition. We thus have to bound the distances $\|o'o'_l\|$ and $\|o'o'_j\|$.

Note that the conditions in the lemma gives that $y(b'_j) = y(b'_l) + x(b'_l) - x(b'_j) - \delta$ and $y(b'_i) = y(b'_j) + x(b'_j) - x(b'_i) - 2\delta$. We have

$$\|o'o'_j\| = y(o'_j) - y(o') = \frac{-y(b'_j)}{x(b'_j)} - \frac{y(b'_j)}{x(b'_j)} = \frac{-y(b'_j) - x(b'_j) + x(b'_j) + \delta}{x(b'_j)} + \frac{y(b'_j)}{x(b'_j)}$$

$$= \frac{-y(b'_j) - x(b'_j) + \delta}{x(b'_j)} + \frac{y(b'_j) + x(b'_j)}{x(b'_j)}$$
A similar argument gives \( \|o'o'\| \in [2\delta - \epsilon/4, 2\delta + \epsilon/4] \).

We will prove Property 2 as follows. We have \( |x(b'_\sigma) - x(m)| \leq \Delta \leq \frac{1}{1500} \) (as \( C > 75000 \)), and

\[
\begin{align*}
d^0_{\sigma} &= \|c_1d_1\| \
&\in \left[ \rho \cdot \frac{2 - \frac{1}{1500}}{1 + \frac{1}{1500}}, \rho \cdot \frac{2 + \frac{1}{1500}}{1 - \frac{1}{1500}} \right] = \left[ \rho \cdot \left( 2 - \frac{2}{1501} \right), \rho \cdot \left( 2 + \frac{3}{1499} \right) \right] \subset \left[ 2\rho - \frac{\epsilon}{4}, 2\rho + \frac{\epsilon}{4} \right].
\end{align*}
\]

For Property 3, denote \( H \) as the distance between the point \( a'_\sigma \) and its vertical projection on the ray \( b'_\sigma \mu_1 \). We have \( \frac{d^0_{\sigma}}{H^0} = \frac{1}{1 + x(b'_\sigma) - x(m)} = \frac{1}{1 + x(b'_\sigma) - x(m) - \rho} \). We have \( H / \rho = \frac{1 + y(m) - y(b'_\sigma) + \|\mu_1c_1\|}{1 + x(b'_\sigma) - x(m) - \rho} \) and \( \rho \leq \frac{1}{50000} \) (as \( C \geq 75000 \)), and therefore

\[
\begin{align*}
d^0_{\sigma} &= \rho \cdot \frac{1 + y(m) - y(b'_\sigma) + \|\mu_1c_1\|}{1 + x(b'_\sigma) - x(m) - \rho} \cdot \frac{1}{1 + x(b'_\sigma) - x(m)} \
&\in \left[ \rho \cdot \frac{1 - \frac{1}{1500}}{1 + \frac{1}{1500}}, \rho \cdot \frac{1 + \frac{1}{1500}}{1 - \frac{1}{1500}} \right] = \left[ \rho \cdot \left( 1 - \frac{4501}{2253001} \right), \rho \cdot \left( 1 + \frac{458897}{22495603} \right) \right] \subset \left[ \frac{\delta}{4} - \frac{\epsilon}{4} + \frac{\epsilon}{4} \right].
\end{align*}
\]

For Property 4, we have to bound the slope of the rays. Note that \( \frac{1}{38N\rho} \cdot \frac{\epsilon}{4} = \frac{1}{1824N} \). Since \( C \geq 50 \cdot 1824 = 91200 \), we get that the slope is at least

\[
\frac{y(c_1) - y(b'_\sigma)}{x(b'_\sigma) - x(c_1)} = \frac{1 + y(m) - y(b'_\sigma)}{1 + x(b'_\sigma) - x(m)} \geq \frac{1 - \Delta}{1 + \Delta} = 1 - \frac{50}{C N^2 + 50} \geq 1 - \frac{50}{C N} \geq 1 - \frac{1}{1824N}.
\]

On the other hand, since \( C \geq 101.5 \cdot 1824 + 50 = 185186 \), we get that the slope is at most

\[
\frac{y(d_1) - y(b'_\sigma)}{x(b'_\sigma) - x(d_1)} \leq \frac{1 + \Delta + \rho}{1 - \Delta} = 1 + \frac{101.5}{C N^2 - 50} \leq 1 + \frac{101.5}{(C - 50)N} \leq 1 + \frac{1}{1824N}.
\]

Since all four properties hold, so does Property 2.

\[\square\]

**Proof of Lemma 24 for the \( \geq \)-addition gadget.** Note that in the \( \geq \)-addition gadget, the segments \( r_1, r_2, r_3 \) have lengths of \( \frac{3\sqrt{2}}{3N} \) and their right endpoints are placed at positions \( m + \left( -\frac{20.5}{C N^2}, -\frac{17.5}{C N^2} \right), m + \left( \frac{-244.5}{C N^2}, 0 \right), m + \left( \frac{-16.5}{C N^2}, \frac{-16.5}{C N^2} \right) \), respectively, where \( m := c_1 + (1, -1) \). As the conditions of Lemma 34 are satisfied, Properties 1 and 2 hold.

Property 3 also holds, as the edge \( e_jd_1 \) blocks all points at stationary guard positions and at the guard segment \( r'_i \) from seeing \( c_1d_1 \).  

\[\square\]
C.10.6 Summary

Lemma 35. Consider the addition gadget together with the corresponding corridor representing an inequality \( x_i + x_j \geq x_l \), as described above. The following properties hold.

- The gadget and the corridor fit into a rectangular box of height 3.
- For any guard set of \( \mathcal{P} \), at least 10 guards have to be placed in the corridor and the gadget.
- Assume that in the main area \( \mathcal{P}_M \), there is exactly one guard at each guard segment, and there are no guards outside of the guard segments. Then 10 guards can be placed in the corridor and the gadget so that the whole corridor and gadget is seen if and only if the values \( x_i, x_j, x_l \) specified by the guards at the guard segments \( s_i, s_j, s_l \) satisfy the inequality \( x_i + x_j \geq x_l \).

Proof. Recall that by Lemma \( \ref{lem:distance} \), the distance from \( c_0c_1 \) to the topmost point in the corridor is at most 1.4. The main part of the gadget is centered around the point \( c_0 + (1, -1) \), and as it is of size \( \Theta(\frac{1}{\sqrt{n}}) \), the vertical space of at most 1.1 below the line segment \( c_0c_1 \) is enough to fit the gadget. From Lemma \( \ref{lem:distance} \) there are at least 3 guards placed within the corridor. From Lemma \( \ref{lem:guard} \), there are at least 7 guards placed within the gadget. That gives us at least 10 guards needed.

Assume that there are exactly 10 guards within the corridor and gadget and that the corridor is completely seen. Then, from Lemma \( \ref{lem:distance} \) and \( \ref{lem:guard} \), there is exactly one guard at each guard segment and each stationary guard position. Then, by Lemma \( \ref{lem:distance} \), the values \( x_i, x_j, x_l \) specified by \( s_i, s_j, s_l \), and the values specified by \( r_i, r_j, r_l \), are the same. By Lemma \( \ref{lem:guard} \), the values \( x_i, x'_i \) corresponding to \( r'_i, r_i \) satisfy \( x_i \geq x'_i \), and we also have \( x_i' + x_j \geq x_l \). That enforces inequality \( x_i + x_j \geq x_l \).

On the other hand, assume that \( x_i + x_j \geq x_l \). We first place a guard at every of the 6 stationary guard positions in the corridor and gadget. By Lemmas \( \ref{lem:distance} \) and \( \ref{lem:guard} \), if we set guards at the 4 guard segments so that the values specified by guards at \( r_i, r_j, r_l \) are \( x_i, x_j, x_l \), and the value \( x_i' \) specified by the guard at \( r'_i \) is the same as \( x_i \), then all of the gadget is guarded.

\( \square \)

C.11 The \( \leq \)-addition gadget

In this section we present construction of a gadget representing an inequality \( x_i + x_j \leq x_l \), where \( i, j, l \in \{1, \ldots, n\} \). The idea of the construction of this gadget is analogous to the construction of the \( \geq \)-addition gadget presented in Section \( \text{C.10} \), and the basic principle is presented in Section \( \text{C.11.1} \).

The principle underlying the \( \geq \)-addition gadget, as explained in Section \( \text{C.10.1} \), required the polygon to have edges blocking the visibility between the segment \( r'_i \) (placed at the right side) and the segments \( r_j, r_l \) (placed at the left side). We managed to get around that by making \( r'_i \) a weak copy of a segment \( r_i \), which in turn was a copy of a segment \( s_i \) on the base line. In contrast to this, the principle underlying the \( \leq \)-addition gadget presented here requires the polygon to have an edge separating \( r'_i \) placed at the left side from the segments \( r_j, r_l \) at the right side. If we were to build a gadget to be placed at the right side of \( \mathcal{P} \), we would have to copy the variables corresponding to both of the segments \( r_j, r_l \) weakly, and the gadget would not enforce the desired inequality. To avoid this problem, we will place the gadget at the left side of \( \mathcal{P} \). Then, we introduce an additional guard segment \( r_i \), and we make \( r'_i \) a weak copy of \( r_i \) using a copy-nook \( Q_i \). As \( r'_i \) is to the left of \( r_i \), the copy-nook \( Q_i \) enforces the inequality \( x_i' \geq x_i \), where \( r'_i, r_i \) represent \( x'_i, x_i \), respectively. The result is that the gadget enforces the desired inequality. The relative placement of the segments \( r_i, r_j, r_l, r'_i \) (in particular, the value of \( w \) from Section \( \text{C.10.2} \)) has to be slightly different than in the construction of \( \geq \)-addition gadget, as it does not seem to be possible to make the gadgets completely symmetrical.
Figure 27: Detailed construction of the $\leq$-addition gadget. As previously, the dotted line shows that no guard on $r_i$ can see any point in $\Gamma$ because of the corner $z$, a guard on $r_i$ can always see both shadow corners of the copy-nook $Q_i$, and no point on $r_l$ sees any point of $Q_i$ because of the corner $e_l$. For each of the segments $r_\sigma$, $\sigma \in \{i,j,l\}$, the rays from points on $r_\sigma$ through the corridor entrance $c_1d_1$ are between the two grey dashed rays emitting from the endpoints of $r_\sigma$.

C.11.1 Idea behind the gadget construction

The idea behind a gadget imposing an inequality $x_i' + x_j \leq x_l$ is similar as for the $\geq$-addition gadget described above. As before, consider rational values $w,v,h > 0$, where $w > v + 3/2$, and let $r_i',r_j,r_l$ be right-oriented guard segments of length $3/2$ such that $r_i'$ has its left endpoint at the point $(-w,0)$, $r_j$ has its right endpoint at $(w,0)$, and $r_l$ has its left endpoint at $(-2,-h)$. Let $g_i' := (-w - 1/2 + x_i,0)$, $g_j := (w - 2 + x_j,0)$, and $g_l := (-5/2 + x_l,-h)$ be three guards on $r_i',r_j,r_l$, respectively, representing the values $x_i',x_j,x_l \in [1/2,2]$.

Suppose that there are corners $e_i := (-v,h), e_j := (v,h), e_l := (0,h)$ of $P$. As before, let $\Gamma$ be a collection of points $\omega$ such that the ray $\vec{\omega e_i}$ intersects $r_i'$, and the ray $\vec{\omega e_j}$ intersects $r_j$. Then $\Gamma$ is a quadrilateral, bounded by the following rays: the rays with origin at the endpoints of $r_i'$ and containing $e_i$, and the rays with origin at the endpoints of $r_j$ and containing $e_j$. Suppose that

- for every point $g_i'$ on $r_i'$ and $\omega$ in $\Gamma$, the points $\omega$ and $g_i'$ can see each other if and only if $\omega$ is on or to the left of the line $g_i'e_i$,
- for every point $g_j$ on $r_j$ and $\omega$ in $\Gamma$, the points $\omega$ and $g_j$ can see each other if and only if $\omega$ is
Lemma 36. The guards $g_i, g_j, g_l$ can together see the whole quadrilateral $\Gamma$ if and only if $x_i + x_j \leq x_l$.

C.11.2 A specification of the gadget

We will present the construction of a gadget with four guard segments $r_i, r_j, r_l, r'_i$, where the segments $r'_i, r_j, r_l$ correspond to the segments in the idea described in Section C.11.1, where this time we set $w := 23.5, v := 10$, and $h := 10.5$. The gadget is shown in Figure 27 and should be attached to the left side of the main area $P_M$ using a left corridor as described in Section C.9.4.

As for the case of the $\geq$-addition gadget, there are three stationary guards which do not see any point within $\Gamma$, but which enforce that the whole area except of $\Gamma$ is seen whenever a guard is placed on each of the guard segments $r_i, r_j, r_l, r'_i$. The guard segment $r_i$ has length $3/2$ (as do $r'_i, r_j, r_l$) and is placed with its left endpoint at the point $(13.75, -21.75)$. We will ensure that $r'_i$ is a weak copy of $r_i$ by creating a copy-nook $Q_i$ for the pair of guard segments $r_i, r'_i$. The nook $Q_i$ has shadow corners $(-1.5, 35)$ and $(0.5, 35)$. To ensure that a guard placed on $r_i$ cannot see any point in the interior of $\Gamma$, we introduce a new corner $z := (8.5, -2.5)$ of the polygon that blocks $r_i$ from seeing $\Gamma$. Two edges to the right in the figure are not fully shown. They end at corners $c_1 := (CN^2, CN^2)$ and $d_1 := (CN^2, CN^2 + 1.5)$.

In order to attach the gadget to the main area $P_M$, we scale down the construction described here by the factor $\frac{1}{CN^2}$ and translate it so that the point which corresponds to $O := (0, 0)$ in the gadget will be placed at position $m := c_1 + (-1, -1)$. (Recall that the left entrance to the corridor, at which we attach the gadget, is the segment $c_1d_1$.)

The following lemma is proved in the same way as Lemma 33.

Lemma 37. Let $P_{rev-ineq}'$ be the polygon obtained from the gadget described above by closing it by adding the edge $c_1d_1$. A set of guards $G \subset P_{rev-ineq}'$ of cardinality at most 7 guards $P_{rev-ineq}'$ if and only if

- there is exactly one guard placed on each guard segment $r'_i, r_i, r_j, r_l$ and at each stationary guard position,
- the variables $x_i, x'_i$ corresponding to the guard segments $r_i, r'_i$, respectively, satisfy the inequality $x_i \leq x'_i$, and
- the variables $x'_i, x_j, x_l$ corresponding to the guard segments $r'_i, r_j, r_l$, respectively, satisfy the inequality $x'_i + x_j \leq x_l$.

C.11.3 Copying three guard segments via a corridor

Here we proceed as in Section C.10.5. We need to show is that the variables $x_i, x_j, x_l$ can be copied into guard segments $r_i, r_j, r_l$ from three guard segments on the base line. Due to the gadget construction, we now require the segment corresponding to the variable $x_l$ to be the leftmost one, and all guard segments have to be right-oriented. With slight abuse of notation, we redefine $i := i + n$ and $j := j + 2n$. Then, the segments $s_i, s_j, s_l$ satisfy our requirements.
As before, to prove that the corridor construction enforces the required dependency between the guards on the base line and guards within the gadget, i.e., for Lemma 27 to work, we need to show that our gadget construction satisfies the conditions of Lemma 29. In particular, we use the following symmetric version of Lemma 34 for gadgets attached to the left side of $P$.

**Lemma 39.** Consider the gadget and the corridor fit into a rectangular box of height 3.

- The gadget and the corridor fit into a rectangular box of height 3.
- For any guard set of $P$, at least 10 guards have to be placed in the corridor and the gadget.
- Assume that in the main area $P_M$, there is exactly one guard at each guard segment, and there are no guards outside of the guard segments. Then 10 guards can be placed in the corridor and the gadget so that the whole corridor and gadget is seen if and only if the values $x_i, x_j, x_l$ specified by the guards at the guard segments $s_i, s_j, s_l$ satisfy the inequality $x_i + x_j \leq x_l$.

### C.11.4 Summary

In the same way as in Lemma 35, we get the following result.

**Lemma 39.** Consider the $\leq$-addition gadget together with the corridor, corresponding to the inequality $x_i + x_j \leq x_l$. The following properties hold.

- The gadget and the corridor fit into a rectangular box of height 3.
- For any guard set of $P$, at least 10 guards have to be placed in the corridor and the gadget.
- Assume that in the main area $P_M$, there is exactly one guard at each guard segment, and there are no guards outside of the guard segments. Then 10 guards can be placed in the corridor and the gadget so that the whole corridor and gadget is seen if and only if the values $x_i, x_j, x_l$ specified by the guards at the guard segments $s_i, s_j, s_l$ satisfy the inequality $x_i + x_j \leq x_l$.

### C.12 The $\geq$- and $\leq$-orientation gadgets

In this section we explain how to enforce consistency between the guard segments on the base line which represent the same variable $x_i$, for $i \in \{1, \ldots, n\}$. Recall that there are four guard segments $s_i, s_{n+i}, s_{2n+i}, s_{3n+i}$ representing the variable $x_i$, and that the first three ones are right-oriented, and the last one is left-oriented.

We will present a gadget enforcing that two guard segments corresponding to the same variable $x_i$ and oriented in different directions specify the variable consistently. We will then use this gadget for the following pairs of guard segments: $(s_i, s_{3n+i}), (s_{n+i}, s_{3n+i})$, and $(s_{2n+i}, s_{3n+i})$.

Consider two guard segments $s_i, s_j$ on the base line, where $s_i$ is right-oriented and $s_j$ is left-oriented, and assume that there is one guard placed on each of these segments. Let $x_i$ and $x_j$ be the values represented by $s_i$ and $s_j$, respectively. Let $x_j^r$ be the value that would be specified by $s_j$ if $s_j$ was right-oriented instead of left-oriented. We have $x_j + x_j^r = 2.5$. Therefore $s_i$ and $s_j$ specify the same value if and only if $x_i + x_j^r = 2.5$.

Performing a simple modification of the $\geq$- and $\leq$-addition gadgets, we obtain the $\geq$- and $\leq$-orientation gadgets, which together enforce the equality $x_i + x_j^r = 2.5$. See Figure 28 for a
detailed picture of the main part of the $\geq$-orientation gadget, which enforces that $x_i + x_j^r \geq 2.5$, or, equivalently, $x_i \geq x_j$. In the $\geq$-addition gadget, we copy three values from the base line into the gadget. Here, we copy only the value of the two segments $s_i, s_j$. Instead of the guard segment $r_l$ inside the gadget, we create a stationary guard position $p$ at the line containing $r_l$ at distance $\frac{1}{2CN}$ to the right of the right endpoint of $r_l$. Then $p$ corresponds to the value of $5/2$ on $r_l$ (ignoring that $p$ lies outside $r_l$).

![Figure 28: Detailed construction of the $\geq$-orientation gadget for $x_i + x_j^r \geq 2.5$, which is a modified version of the $\geq$-addition gadget.](image)

The $\leq$-orientation gadget, which corresponds to the inequality $x_i + x_j^r \leq 5/2$, is obtained by an analogous modification of the $\leq$-addition gadget. Note that in both of these orientation gadgets, we create 4 stationary guard positions in the corridor instead of 6 for the addition gadgets, and the gadget itself contains 4 stationary guards and 3 guard segments, instead of 3 and 4 in the addition gadgets, respectively.

We summarize the properties of the orientation gadgets by the following Lemma, which can be proven in a way analogous to Lemmas 35 and 39.

**Lemma 40.** Consider the $\geq$-orientation gadget (resp. $\leq$-orientation gadget) together with the corresponding corridor for making $r_i, r_j^r$ copies of guard segments $s_i, s_j$ on the base line, where $s_i$ is right-oriented and $s_j$ is left-oriented. The following properties hold.

- The gadget and the corridor fit into a rectangular box of height 3.
- For any guard set of $\mathcal{P}$, at least 9 guards have to be placed in the corridor and the gadget.
- Assume that in the main area $\mathcal{P}_M$, there is exactly one guard at each guard segment, and there are no guards outside of the guard segments. Then 9 guards can be placed in the corridor and

58
C.13 The inversion gadget

In this section we present the construction of the inversion gadget which represent an inequality \( x_i \cdot x_j = 1 \), where \( i, j \in \{1, \ldots, n\} \). We made use of Maple [30] for the construction and verification of this gadget.

C.13.1 Idea behind the gadget construction

We first describe the principle underlying the inversion gadget. Let \( r_i \) and \( r_j \) be two guard segments representing variables \( x_i \) and \( x_j \), respectively. We want to construct an umbra \( Q_u \) such that if the guards at \( r_i \) and \( r_j \) together see the critical segment \( f_0 f_1 \) of \( Q_u \), then one of the inequalities \( x_i x_j \leq 1 \) or \( x_i x_j \geq 1 \) follows. Likewise, we want to construct a nook \( Q_n \) such that if the guards at \( r_i \) and \( r_j \) together see the critical segment \( f_2 f_3 \) of \( Q_n \), then the other of the two inequalities follows, so that in effect, \( x_i x_j = 1 \). This does not seem possible to obtain if both guard segments \( r_i, r_j \) are right-oriented, but as we will see, it is possible when \( r_i \) is right-oriented and \( r_j \) is left-oriented. Furthermore, in order to get rational coordinates of the shadow corners of \( Q_u \) and \( Q_n \), it seems necessary to have \( r_i \) and \( r_j \) at different \( y \)-coordinates.

Some rather complicated fractions are used as the coordinates of the shadow corners of \( Q_u \) and \( Q_n \). In the following, we explain why they need to be somewhat complicated, and how they were found. Let \( a'_i := (1/2, 0) \), \( b'_i := (2, 0) \), \( a'_j := (13.9, 0.1) \), \( b'_j := (15.4, 0.1) \), and suppose that \( r_i := a'_i b'_i \) is right-oriented and \( r_j := a'_j b'_j \) is left-oriented, see Figure 29. Suppose that an umbra \( Q_u \) of \( r_i \) and \( r_j \) has shadow corners \( \xi_0 := (7, h_l) \) and \( \xi_1 := (9, h_r) \). Then the critical segment of \( Q_u \) has endpoints

\[
\begin{align*}
f_0 := & \left( \frac{770h_r - 45 + 128h_l}{64h_l + 50h_r - 5}, \frac{134h_l h_r - 7h_l}{64h_l + 50h_r - 5} \right) \quad \text{and} \\
f_1 := & \left( \frac{1807h_r - 117 + 49h_l}{98h_l + 130h_r - 13}, \frac{268h_l h_r - 17h_l}{98h_l + 130h_r - 13} \right).
\end{align*}
\]

Let \( \pi_i: r_i \to f_0 f_1 \) and \( \pi_j: r_j \to f_0 f_1 \) be the projections associated with \( Q_u \). We want the function \( \pi_j^{-1} \circ \pi_i: r_i \to r_j \) to map a point on \( r_i \) specifying the value \( x_i \) to a point on \( r_j \) specifying the value \( x_j = 1/x_i \). Note that the point \((1, 0)\) on \( r_i \) specifies the value 1 of \( x_i \). Therefore, the point \( \pi_j^{-1}(\pi_i((1, 0))) = (p, 0.1) \), where

\[
p := \frac{34009h_l^2 - 71042h_l h_r + 1147h_l + 34153h_r^2 - 859h_r}{2385h_l^2 - 4980h_l h_r + 80h_l + 2395h_r^2 - 60h_r},
\]

has to specify the value 1 on \( r_j \). The value specified by \((p, 0.1)\) on \( r_j \) is \( 15.9 - p \). Solving the equation \(15.9 - p = 1\) for \( h_r \) gives that

\[
h_r = \frac{632h_l + 7 \pm 613\sqrt{24881h_l^2 - 2186h_l + 49}}{613}.
\]

In order for \( h_l \) and \( h_r \) both to be rational, it is thus required that \( h_l \) is a rational number for which \( 24881h_l^2 - 2186h_l + 49 \) is the square of a rational number. This is in general not – but luckily sometimes – the case. Indeed, if we define \( h_l := \frac{541}{154} \), then \( 24881h_l^2 - 2186h_l + 49 = \left( \frac{8409}{154} \right)^2 \). We used Maple to find such a number \( h_l \) in a suitable range that made it possible to create the gadget.
These considerations lead us to the following construction. Let \( \xi_0 := (7, \frac{541}{184}) \approx (7, 2.94) \) and \( \xi_1 := (9, \frac{259139}{112792}) \approx (9, 2.30) \) and suppose that \( \xi_0, \xi_1 \) are shadow corners of an umbra \( Q_u \) with corners \( \xi_0 f_1 f_0 \) of \( r_i, r_j \). Then \( f_0 := (\frac{499811}{70923}, \frac{38731813}{13049832}) \approx (7.05, 2.97) \) and \( f_1 := (\frac{112379}{15432}, \frac{4355591}{1419744}) \approx (7.28, 3.07) \).

**Lemma 41.** If guards \( p_i, p_j \) on \( r_i, r_j \), respectively, together see \( f_0 f_1 \), then \( x_i x_j \leq 1 \).

**Proof.** Let \( \pi_i : r_i \rightarrow f_0 f_1 \) and \( \pi_j : r_j \rightarrow f_0 f_1 \) be the projections associated with \( Q_u \). Note that since \( p_i \) represents the variable \( x_i \), we must have \( p_i := (x_i, 0) \). Let

\[
e := \pi_i(p_i) = \left( \frac{258288 x_i - 16765}{36994 x_i - 3065}, \frac{20013754 x_i - 1295695}{6806896 x_i - 563960} \right).
\]

Now, \( \pi_j^{-1}(e) = (15.9 - 1/x_i, 1/10) \), which represents the value \( 15.9 - (15.9 - 1/x_i) = 1/x_i \) on \( r_j \). In order to see \( f_0 f_1 \) together with \( p_i \), the guard \( p_j \) has to stand on \( \pi_j^{-1}(e) \) or to the right. This corresponds to \( x_j \) being at most \( 1/x_i \). In other words, if a guard \( p_j \) on \( r_j \) sees \( f_0 f_1 \) together with \( p_i \), then \( x_i x_j \leq 1 \).

We now construct a nook that impose the guards to satisfy the opposite inequality \( x_i x_j \geq 1 \): Let \( \xi_2 := (7, \frac{8805}{732}) \approx (7, 11.79) \) and \( \xi_3 := (9, \frac{4214815}{406976}) \approx (9.914) \) and suppose that \( \xi_2, \xi_3 \) are shadow corners of a nook \( Q_n \) with corners \( \xi_2 \xi_3 f_2 f_3 \) of \( r_i, r_j \). Then \( f_2 := (\frac{182083}{25835}, \frac{231222490}{1947920}) \approx (7.05, 11.90) \) and \( f_3 := (\frac{205139}{190288905}, \frac{50644192}{4148784}) \approx (7.29, 12.31) \).

**Lemma 42.** If guards \( p_i, p_j \) on \( r_i, r_j \), respectively, together see \( f_2 f_3 \), then \( x_i x_j \geq 1 \).

**Proof.** Let \( \hat{\pi}_0 \) and \( \hat{\pi}_1 \) be the associated projections from \( r_i \) and \( r_j \) to \( f_2 f_3 \), respectively. Let

\[
\hat{e} := \hat{\pi}_0(p_0) = \left( \frac{470184 x_i - 29953}{67346 x_i - 5517}, \frac{59702290 x_i - 37933335}{50644192 x_i - 4148784} \right).
\]

Now, \( \hat{\pi}_j^{-1}(\hat{e}) = (15.9 - 1/x_i, 1/10) \), which represents the value \( 15.9 - (15.9 - 1/x_i) = 1/x_i \) on \( r_j \). In order to see \( f_2 f_3 \) together with \( p_i \), the guard \( p_j \) has to stand on \( \hat{\pi}_j^{-1}(\hat{e}) \) or to the left. This corresponds to \( x_j \) being at least \( 1/x_i \) so that \( x_i x_j \geq 1 \).

We thus have the following lemma:

**Lemma 43.** If guards \( p_i \) and \( p_j \) placed on guard segments \( r_i \) and \( r_j \), respectively, see both critical segments \( f_0 f_1 \) and \( f_2 f_3 \), then the corresponding values specified by \( p_i \) and \( p_j \) satisfy \( x_i x_j = 1 \).

### C.13.2 The construction of the gadget

We now explain how to make the complete gadget for the equation \( x_i x_j = 1 \), as shown in Figure 29. We make the wall in the gadget so that it creates the umbra \( Q_u \) and the nook \( Q_n \) as described before. We also create a stationary guard position at the green point in the figure which sees the umbra \( Q_u \), but nothing above the line containing the critical segment of \( Q_u \). Two edges at the left side of the gadget are not fully shown. They end at corners \( c_1 := (-CN^2, CN^2) \) and \( d_1 := (-CN^2, CN^2 + 1.5) \), respectively.

The gadget contains two guard segments \( r_i \) and \( r_j \) representing \( x_i \) and \( x_j \), respectively, and it is required that \( r_i \) is right-oriented and \( r_j \) is left-oriented. Therefore, with slight abuse of notation, we redefine \( j := j + 3n \), so that \( s_i, s_j \) are guard segments on the base line representing \( x_i, x_j \), respectively, where \( s_i \) is right-oriented and \( s_j \) is left-oriented. We then use a corridor as described
Figure 29: The inversion gadget. The nook and umbra (the brown areas) for the pair of guard segments \( r_i, r_j \) impose the inequality \( x_i x_j = 1 \) on the variables \( x_i \) and \( x_j \) represented by \( r_i \) and \( r_j \). The stationary guard position \( g_s \) sees the umbra but nothing above the line \( f_0f_1 \).

in Section C.9 to make \( r_i, r_j \) copies of \( s_i, s_j \), respectively. Recall that the endpoints of the right entrance of the corridor are denoted \( c_1, d_1 \). In order to attach the gadget to the corridor, we first scale it down by the factor \( \frac{1}{CN^2} \) and then translate it so that the points \( c_1, d_1 \) of the gadget coincides with the endpoints of the right entrance of the corridor with the same names. We thus obtain that the point \( O := (0, 0) \) in the gadget becomes the point \( m := c_1 + (1, -1) \) in \( P \).

**Lemma 44.** Let \( P_{in} \) be the polygon obtained from the inversion gadget by closing it by adding the edge \( c_1d_1 \). A set of guards \( G \subseteq P_{in} \) of cardinality at most 3 guards \( P_{in} \) if and only if

- there is exactly one guard placed on each guard segment \( r_i, r_j \) and at the stationary guard position, and
- the variables \( x_i, x_j \) corresponding to the guard segments \( r_i, r_j \), respectively, satisfy the equation \( x_i \cdot x_j = 1 \).

**Proof.** Assume that \( P_{in} \) is guarded by a set \( G \) of at most 3 guards. Similarly as in Lemma \ref{lem:3guards} we can show that there must be exactly one guard at each guard segment and at the stationary guard position. It then follows from Lemma \ref{lem:umbra} that \( x_i \cdot x_j = 1 \).

Now assume that there is exactly one guard placed on each guard segment \( r_i, r_j \) and at the stationary guard position, and that the variables \( x_i, x_j \) represented by the guards at \( r_i, r_j \) satisfy \( x_i \cdot x_j = 1 \). Then all of \( Q_u \) and \( Q_n \) is seen by the guards. The remaining area is clearly also seen by the guards.

**C.13.3 Connecting the gadget with a corridor**

We now need to show that our gadget construction satisfies the conditions of Lemma \ref{lem:connect}.
Proof of Lemma 24 for the inversion gadget. Note that in the inversion gadget, the segments $r_i, r_j$ have lengths of $\frac{3/2}{CN^2}$ and their right endpoints are placed at positions $m + (\frac{2}{CN^2}, 0)$ and $m + (\frac{15/4}{CN^2}, 0)$, respectively, where $m := c_1 + (1, -1)$. As the conditions of Lemma 34 (here used in a simplified version for a gadget with only two guard segments $r_i$ and $r_j$) are satisfied, Properties 1 and 2 hold.

Property 3 also holds, as the stationary guard position in the gadget cannot see the edge $c_1 d_1$. \hfill \square

C.13.4 Summary

Lemma 45. Consider the inversion gadget together with the corresponding corridor representing an inequality $x_i \cdot x_j = 1$, as described below. The following properties hold.

- The gadget and the corridor fit into a rectangular box of height 3.
- For any guard set of $P$, at least 5 guards have to be placed in the corridor and the gadget in total.
- Assume that in the main area $P_M$, there is exactly one guard at each guard segment, and there are no guards outside of the guard segments. Then all of the corridor and the gadget can be guarded by 5 guards (together with the guards on the base line) if and only if the values $x_i, x_j$ specified by the guards at the guard segments $s_i, s_j$ satisfy the inequality $x_i \cdot x_j = 1$.

Proof. The proof for the first property is similar to that for the addition gadget in Lemma 35. From Lemma 27 there are at least 2 guards placed within the corridor. Furthermore, there must be at least 3 guards placed within the gadget by Lemma 44. That gives us at least 5 guards needed.

Assume that there are exactly 5 guards within the corridor and gadget and that the corridor is completely seen. Then, from Lemma 27 and 44, there is exactly one guard at each guard segment and each stationary guard position. Then, by Lemma 27, the values $x_i, x_j$ specified by $s_i, s_j$, and the values specified by $r_i, r_j$, are the same. Lemma 27 gives that the guards at $r_i, r_j$ must see the critical segments of both $Q_n$ and $Q_u$. Then, by Lemma 44, the values $x_i, x_j$ thus satisfy $x_i \cdot x_j = 1$.

On the other hand, assume that $x_i \cdot x_j = 1$. We first place a guard at every of the 3 stationary guard positions in the corridor and gadget. By Lemmas 27 and 44 if we set guards at the 2 guard segments so that the values specified by guards at $r_i, r_j$ are $x_i, x_j$, then all of the gadget is guarded. \hfill \square

C.14 Putting it all together

Let $\Phi$ be an ETR-INV formula with $n$ variables, $k_1$ equations of the form $x_i + x_j = x_l$, and $k_2$ equations of the form $x_i \cdot x_j = 1$. We have already explained how to construct the polygon $P(\Phi)$, but we shall here give a brief summary of the process. We start by constructing the main area with $4n$ guard segments. We modify the pockets corresponding to the variables $x_i$ for which $\Phi$ contains equation $x_i = 1$, as described in Section C.8. To enforce dependency between the base line guard segments corresponding to the same variable, we construct $3n \geq$-orientation gadgets (attached at the right side of the polygon) and $3n \leq$-orientation gadgets (attached at the left side), as described in Section C.12. For each equality of the form $x_i + x_j = x_l$ in $\Phi$, we construct a corresponding $\geq$-addition gadget (attached at the right side), and a $\leq$-addition gadget (attached at the left side), as described in Sections C.10 and C.11, respectively. For each equality of the form $x_i \cdot x_j = 1$ in $\Phi$ we construct a corresponding inversion gadget (attached at the right side), as described in
Section C.13 The total number of gadgets at each side of \( P \) is therefore at most \( 3n + k_1 + k_2 \leq N \), as stated in Section C.7.

Without loss of generality, we assume that the \( y \)-coordinate of the base line of \( P(\Phi) \) is 0. We set \( g(\Phi) := 58n + 20k_1 + 5k_2 \). Then \( (P(\Phi), g(\Phi)) \) is an instance of the art gallery problem, and the following theorem holds.

**Theorem 46.** Let \( \Phi \) be an instance of ETR-INV. The polygon \( P(\Phi) \) has corners at rational coordinates, which can be computed in polynomial time. Moreover, there exist constants \( d_1, \ldots, d_n \in \mathbb{Q} \) such that for any \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( x \) is a solution to \( \Phi \) if and only if there exists a guard set \( G \) of cardinality \( g(\Phi) \) containing guards at all the positions \( (x_1 + d_1, 0) \ldots, (x_n + d_n, 0) \).

**Proof.** Consider a guard set \( G \) of the polygon \( P := P(\Phi) \). By Lemma 23, \( G \) has at least 4n guards placed in \( P_M \), and if the number of guards within \( P_M \) equals 4n, then there must be exactly one guard at each guard segment. Lemma 40 implies that within each of the \( 6n \) orientation gadgets of \( P \) together with the corresponding corridors, there are at least 9 guards, giving at least \( 54n \) guards in total. Similarly, from Lemmas 35 and 39 we obtain that there must be at least 10 guards placed within each \( \geq \)-addition gadget and each \( \leq \)-addition gadget plus the corresponding corridors, giving at least \( 20k_1 \) guards. By Lemma 45, there are at least 5 guards within each inversion gadget and the corresponding corridor, giving at least \( 5k_2 \) guards in total. Therefore, \( G \) has at least \( 58n + 20k_1 + 5k_2 \) guards, which is equal to \( g(\Phi) \).

If a guard set of size \( g(\Phi) \) exists, then there are exactly 4n guards in \( P_M \), 9 guards within each orientation gadget, 10 guards within each \( \geq \)-addition gadget and each \( \leq \)-addition gadget, and 5 guards within each inversion gadget. The same lemmas give us then that there is exactly one guard at each guard segment and each stationary guard position, and no guards away from the guard segments or the stationary guard positions. Also, the variables \( x_1, \ldots, x_n \) specified by the guard segments \( s_1, \ldots, s_n \) are a solution to \( \Phi \).

On the other hand, if there exists a solution to \( \Phi \), then we get a guard set of size \( g(\Phi) \) by placing the guards accordingly. It is thus clear that the solutions to \( \Phi \) correspond to the optimal guard sets of \( P \), as stated in the theorem.

Due to Lemmas 22 and 28 we get that the corners of \( P_M \) and the corridor corners are all rational, with the nominators and denominators polynomially bounded. Next, consider all the corners of the gadgets. Each gadget is first described as a polygon with coordinates where nominators and denominators are both of size \( \Theta(1) \) (as this construction is fixed and it does not depend on the formula \( \Phi \); and we can easily choose the corners so that they are all at rational coordinates), and this polygon is subsequently scaled down by a factor of \( \frac{1}{CN^2} \) and attached at a corridor entrance, which also has polynomially bounded nominators and denominators. Thus, the coordinates of the corners in the gadgets have polynomially bounded complexity and can be computed in polynomial time.

We can now prove the main theorem of the paper.

**Theorem 1.** The art gallery problem is \( \exists \mathbb{R} \)-complete, even the restricted variant where we are given a polygon with corners at integer coordinates.

**Proof.** By Theorem 4, the art gallery problem is in the complexity class \( \exists \mathbb{R} \). From Theorem 6 we know that the problem ETR-INV is \( \exists \mathbb{R} \)-complete. We presented a polynomial time construction of an instance \( (P(\Phi), g(\Phi)) \) of the art gallery problem from an instance \( \Phi \) of ETR. Theorem 46 gives that it is \( \exists \mathbb{R} \)-hard to solve the art gallery problem when the coordinates of the polygon corners are given by rational numbers. Note that the number of corners of \( P \) is proportional to the input length \( |\Phi| \). By Theorem 46 there is a polynomial \( |\Phi|^{m} \) which is a bound on every denominator.
of a coordinate of a corner in \( P \). The product \( \Pi \) of denominators of all coordinates of corners of \( P \) thus has size at most \( |\Phi|^n \, O(|\Phi|) \). It follows that we can express \( \Pi \) by \( O(m|\Phi| \log |\Phi|) \) bits. By multiplying every coordinate of a coordinate of a corner in \( P \) an instance \( g \) exponentially large in the size of \( \Phi \). Also note that the number set, so is \( n \) than the number \( \Phi \), \( S \) be obtained from \( P \), \( S \) is a guard set of size \( g \), and

\- for any guard set \( G \) of \( P \) of size \( g \), there exists a solution \((x_1, \ldots, x_n) \in \mathbb{R}^n\) to \( \Phi \) such that \( G \) contains guards at positions \((c_1 x_1 + d_1, 0), \ldots, (c_n x_n + d_n, 0)\).

Note that Theorem 47 only gives a correspondence between solutions to \( \Phi \) and the art gallery problem in one direction, namely from guard sets of \( P \) to solutions to \( \Phi \). This is inherently unavoidable for two reasons. First, the solution space for \( \Phi \) is in general unbounded, whereas the guards are restricted to \( P \). Second, the set of guard sets that guard \( P \) is closed in the following sense. Consider a sequence of guard sets \( G^i, G^2, \ldots \), each consisting of \( g \) guards. Each guard set \( G^i \) is considered as a point in \( \mathbb{R}^{2g} \), where the coordinates \( 2j \) and \( 2j + 1 \) are the coordinates of the \( j \)th guard, \( j \in \{1, \ldots, g\} \). Suppose that \( G^i \) converges to \( G^* \in \mathbb{R}^{2g} \), i.e., \( \|G^i - G^*\| \to 0 \) as \( i \to \infty \). Then the limit \( G^* \) is clearly also a guard set of \( P \), so the guard sets with \( g \) guards is a closed subset of \( \mathbb{R}^{2g} \). By restricting ourselves to compact semi-algebraic sets \( S \), we get a one-to-one correspondence between guard sets of \( P \) and points in \( S \).

**Theorem 48** (Compact Semi-Algebraic Sets). Let \( S \) be a compact semi-algebraic set in \( \mathbb{R}^n \). Then there is an instance \((P, g)\) of the art gallery problem and constants \( c_1, d_1, \ldots, c_n, d_n \in \mathbb{Q} \) such that the following holds. For all \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \), the point \( x \) is in \( S \) if and only if \( P \) has a guard set of size \( g \) containing guards at \((c_1 x_1 + d_1, 0), \ldots, (c_n x_n + d_n, 0)\).

**Proof.** By Lemma 13 there is an instance \( \Phi \) of ETR-INV with solution set \( S' \), such that \( S \) can be obtained from \( S' \) by removing some coordinates from the points in \( S' \) and scale and shift the remaining coordinates. The statement now follows from Theorem 46.

We note that both in Theorem 47 and 48 the constants \( c_1, d_1, \ldots, c_n, d_n \in \mathbb{Q} \) might be doubly exponentially large in the size of \( \Phi \). Also note that the number \( g := g(\Phi) \) of guards will be larger than the number \( n \) of variables in \( \Phi \). Finally, we want to point out that if \((p_1, \ldots, p_g)\) is a guard set, so is \((p_g, \ldots, p_1)\), as the order does not play any role. In contrast to this, if \((x_1, \ldots, x_n)\) satisfies \( \Phi \), then we usually would not expect the same to be the case for \((x_n, \ldots, x_1)\).

We can now prove Theorem 2 restated below.

**Theorem 2.** Given any real algebraic number \( \alpha \), there exists a polygon \( P \) with corners at rational coordinates such that in any optimal guard set of \( P \) there is a guard with an \( x \)-coordinate equal \( \alpha \).

**Proof.** Let \( P(x) \) be a polynomial of degree more than 0 in one variable \( x \) such that the equation \( P(x) = 0 \) has \( \alpha \) as a solution. The equation might have other solutions as well, but we can choose integers \( p_1, p_2, q_1, q_2 \) such that \( \alpha \) is the only solution in the interval \([p_1/q_1, p_2/q_2]\). Then the formula \( P(x) = 0 \land p_1 \leq q_1 x \land q_2 x \leq p_2 \) is an instance of the problem ETR with a unique solution \( x = \alpha \). Now, by Theorem 17 there exists a polygon \( P \) and rational constants \( c, d \) such that in any optimal guard set of \( P \), one guard has coordinates \((c \alpha + d, 0)\). By subtracting \( d \) from the \( x \)-coordinate of all corners of \( P \) and then dividing all coordinates by \( c \), we get a polygon \( P' \) such that any optimal guard set of \( P' \) has a guard at the point \((\alpha, 0)\).
D  Picasso Theorem

This section is devoted to the proof of Theorem 3:

**Theorem 3 (Picasso Theorem).** For any compact semi-algebraic set \( S \subset [0, 1]^2 \), there is a polygon \( \mathcal{P}_S \) with corners at rational coordinates such that for any point \( p \in [0, 1]^2 \) we have \( p \in S \) if and only if there exists an optimal guard set \( G \) of \( \mathcal{P}_S \) with \( p \in G \).

In order to construct the polygon \( \mathcal{P}_S \), we make use of the construction described in Appendix C together with two additional gadgets, the Picasso gadget and the scaling gadget.

D.1  The idea behind the construction

We start with an informal description of the ideas behind the construction and then prove the theorem.

![Figure 30: Left: A half-nook \( Q \) for \( a_0b_0 \) and \( T \). The guards at \( s \) and \( p \) together see the critical segment \( f_0f_1 \). Right: A half-umbra.](image)

It is easy to restrict a guard by a half-plane by a construction similar to that of nooks and umbras. The idea is, as in the construction of nooks and umbras, that two guards have to see a critical segment, but now only one of the guards is not restricted to a guard segment, but instead to a square. We use the terms half-nooks and half-umbras to denote such constructions, see Figure 30. We give a more formal description a little later.

The next geometric observation is that one can describe a point uniquely as the intersection of three closed half-planes:

**Observation 49.** Let \( \ell_i, \ell_j, \ell_l \) be three non-vertical lines through a point \( p \). If \( h_i \) and \( h_l \) are half-planes bounded from above by \( \ell_i \) and \( \ell_l \), respectively, and \( h_j \) is bounded from below by \( \ell_j \), then \( h_i \cap h_j \cap h_l = p \).

This motivates us to design a gadget where a guard is restricted to one square \( T \) while forming two half-nooks with guards on guard segments \( r_j, r_l \), respectively, and one half-umbra with a guard on guard segment \( r_j \). The two half-nooks define two half-planes bounded from above and the half-umbra defines a half-plane bounded from below, see Figure 31. Any position of a point \( p \) gives rise to three guard positions \( g_i, g_j, g_l \) on \( r_i, r_j, r_l \), respectively, such that \( p \) is in the boundary of each of the corresponding half-planes, and \( p \) is the unique intersection point of the half-planes. We denote by \( x_i(p), x_j(p), x_l(p) \) the values that the three guards \( g_i, g_j, g_l \) represent on their respective
segments. Using rational functions, one can express how \( x_i(p), x_j(p), \) and \( x_l(p) \) depends on the coordinates of \( p \).

![Figure 31: The point \( p \) is inside the unit square \([0, 1]^2\). The intersection of the three half-planes equals \( p \). The two half-nooks define two half-planes bounded from above and the half-umbra defines a half-plane bounded from below.]

We now explain how the Picasso gadget will be used. Let a quantifier-free formula \( \Phi \) of the first-order theory of the reals be given that has exactly two free variables \( x, y \), and suppose that the set \( S := \{(x, y) \in \mathbb{R}^2 : \Phi(x, y)\} \) is a closed subset of \([0, 1]^2\). Let \( \Psi \) be a quantifier-free formula of the first-order theory of the reals with five free variables \( x, y, x_i, x_j, x_l \) such that when \( p = (x, y) \in [0, 1]^2 \), the tuple \((x, y, x_i, x_j, x_l)\) satisfies \( \Psi \) if and only if \( x_\sigma = x_\sigma(p) \) for each \( \sigma \in \{i, j, l\} \). Such a formula exists since \( p \mapsto x_\sigma(p) \) is a rational function of \( p \). We consider the formula \( \Phi' := \Phi \land \Psi \). Assume for the ease of presentation that \( \Phi' \) is an instance of ETR-IN\( V \). This is in general not the case, but we will later explain how to get around that issue (this is where the other gadget introduced in this section, the scaling gadget, will be used). We construct the polygon \( P := P(\Phi') \) as described in Section [C]. Then a point \( p = (x, y) \in [0, 1]^2 \) is in \( S \) if and only if there is an optimal guard set of \( P \) such that for each \( \sigma \in \{i, j, l\} \), the guards representing \( x_\sigma(p) \) specify the value \( x_\sigma(p) \). We get the polygon \( P_S \) from Theorem [C] by adding the Picasso gadget to \( P \) and copy the three variables \( x_i, x_j, x_l \) to the guard segments \( r_i, r_j, r_l \) in the gadget. By shifting and scaling the polygon, we may assume that the square \( T \) in the Picasso gadget is the unit square \([0, 1]^2\). It follows that \( p \in S \) if and only if there is an optimal guard set of \( P_S \) in which a guard is placed at \( p \), and this proves the theorem.

### D.2 Half-nooks and half-umbras

Half-nooks and half-umbras are defined in the same way as nooks and umbras with the only difference that one of the guard segments is replaced by a square \( T \). See Figure [30]. Here we define only the case that the square is on the right of the guard segment. In the case that the square is to the left, half-nooks and half-umbras are defined in an analogous way.

**Definition 50** (half-nook and half-umbra). Let \( P \) be a polygon containing a guard segments \( r := a_0b_0 \) and an axis-parallel square \( T \), where \( r \) is to the left of \( T \). Let \( abcd \) be the corners of \( T \) in clockwise order, where \( a \) is the topmost left corner. Let \( c_0, c_1 \) be two corners of \( P \), such that \( c_0 \) is
to the left of \(c_1\). Suppose that the rays \(\overrightarrow{b_0c_0}\) and \(\overrightarrow{bc_1}\) intersect at a point \(f_0\), the rays \(\overrightarrow{a_0c_0}\) and \(\overrightarrow{ac_1}\) intersect at a point \(f_1\), and that \(Q := c_0c_1f_1f_0\) is a convex quadrilateral contained in \(\mathcal{P}\). We define the function \(\pi_0 : r \to f_0f_1\) such that \(\pi_0(p)\) is the intersection of the ray \(\overrightarrow{pc_0}\) with the line segment \(f_0f_1\). Analogously, we define \(\pi_1 : T \to f_0f_1\) such that \(\pi_1(p)\) is the intersection of the ray \(\overrightarrow{pc_1}\) with the line segment \(f_0f_1\). We suppose that \(\pi_0\) is bijective and \(\pi_1\) is surjective.

We say that \(Q\) is a half-nook for \(r\) and \(T\) if for every \(p \in r\), a guard at \(p\) can see all of the segment \(\pi_0(p)f_1\) but nothing else of \(f_0f_1\) and for every \(p \in T\), a guard at \(p\) can see all of the segment \(\pi_1(p)f_0\) but nothing else of \(f_0f_1\).

We say that \(Q\) is a half-umbra for \(r\) and \(T\) if for every \(p \in r\), a guard at \(p\) can see all of the segment \(\pi_0(p)f_0\) but nothing else of \(f_0f_1\) and for every \(p \in T\), a guard at \(p\) can see all of the segment \(\pi_1(p)f_1\) but nothing else of \(f_0f_1\).

The functions \(\pi_0, \pi_1\) are called projections of the half-nook or the half-umbra.

**Observation 51.** Let \(Q\) be a half-nook (half-umbra) for a guard segment \(r\) and a square \(T\), and let \(s \in r\) and \(p \in T\). Then it holds that \(p\) and \(s\) together see the half-nook (half-umbra) if and only if \(p\) is on or below (above) the line \(\pi_0(s)c_1\).

**D.3 The Picasso gadget**

We are now ready to give an explicit description of the Picasso-gadget, see Figure 32. Two edges at the left side of the gadget are not fully shown. They end at corners \(c_1 := (-CN^2, CN^2)\) and \(d_1 := (-CN^2, CN^2 + 1.5)\), respectively, where \(c_1\) is the right shadow corner of the corridor as in Section C.9. The gadget contains three guard segments \(r_i := a_i'b'_i, \ r_j := a_j'b'_j\), and \(r_l := a_l'b'_l\), each of width 1.5, defined by the left endpoints

\[
\begin{align*}
a_i' := (-31, -7), & \quad a_j' := (0, -24.5), \quad a_l' := (4, -15).
\end{align*}
\]

The gadget contains an axis-parallel square \(T\), in which a guard will “realize” the set \(S\). The side length of \(T\) is 2 and the upper left corner is \((-18, -24)\). Let \(Q_i, Q_j, Q_l\) be a half-nook, half-umbra, half-nook of \(r_i, r_j, r_l\) and \(T\), respectively, with shadow corners:

\[
\begin{align*}
Q_i : (-23, 20) & \quad \text{and} \quad (-21, 20), & \quad Q_j : (-9, -22) & \quad \text{and} \quad (-6.5, -22), & \quad Q_l : (2, -10) & \quad \text{and} \quad (4, -10).
\end{align*}
\]

For \(\sigma \in \{i, j, l\}\), let \(\pi_{\sigma 0}\) and \(\pi_{\sigma 2}\) be the projections of \(Q_{\sigma}\). Furthermore, let \(x_{\sigma} : T \to [1/2, 2]\) be defined so that \(x_{\sigma}(p)\) is the value represented by \(\pi_{\sigma 0}^{-1}(\pi_{\sigma 2}(p))\). There is a stationary guard position \(g_s\) that sees \(Q_j\), but nothing above the critical segment of \(Q_j\).

We then use a corridor as described in Section C.9 to copy in the values of three variables \(x_i, x_j, x_l\) at \(r_i, r_j, r_l\). Recall that the endpoints of the right entrance of the corridor are denoted \(c_1, d_1\). In order to attach the gadget to the corridor, we first scale it down by the factor \(\frac{1}{CN^2}\) and then translate it so that the points \(c_1, d_1\) of the gadget coincides with the endpoints of the right entrance of the corridor with the same names. We thus obtain that the point \(O := (0, 0)\) in the gadget becomes the point \(m := c_1 + (1, -1)\) in \(\mathcal{P}_S\).

**Lemma 52.** Let \(\mathcal{P}_{Pic}'\) be the polygon obtained from the Picasso gadget by closing it by adding the edge \(c_1d_1\). Consider a set of guards \(G \subseteq \mathcal{P}_{Pic}'\) of size at most 5. If \(G\) guards \(\mathcal{P}_{Pic}'\), then there is exactly one guard placed on each guard segment \(r_i, r_j, r_l\), a guard at \(g_s\), and a guard in \(T\). Consider a point \(p \in T\) and suppose that \(G\) contains \(x_{\sigma}(p)\) for each \(\sigma \in \{i, j, l\}\). Then \(G\) guards \(\mathcal{P}_{Pic}'\) if and only if \(G\) also contains \(g_s\) and \(p\).
Proof. Let $G$ be a set of guards of size at most 5. Similarly as in Lemma 32 we can show that if $P_{Pic}'$ is guarded by $G$, there must be exactly one guard in $T$, at one each guard segment, and one at $g_s$. It is straightforward to check that if $G$ contains $p \in T$, $g_s$, and $x_\sigma(p)$ for each $\sigma \in \{i,j,l\}$, then $G$ guards $P_{Pic}'$. On the other hand, if there is a point $p \in T$ such that $G$ contains guards $g_s$ and $x_\sigma(p)$ for each $\sigma \in \{i,j,l\}$, then Observations 49 and 51 imply that $G$ must contain $p$ in order to guard $P_{Pic}'$.

We now need to show that our gadget construction satisfies the conditions of Lemma 24.

Proof of Lemma 24 for the Picasso gadget. We make use of Lemma 34. First check that the length of every guard segment is $\frac{3}{2CN}$ as required. Note that all guard segments are contained in the square $m + [-\Delta, \Delta] \times [-\Delta, \Delta]$, with $\Delta := \frac{50}{CN}$ and $m := c_1 + (1,-1)$. Furthermore, $a'_j$ is on the line through $a'_i + (\delta,0)$ and direction $(1,-1)$, and $a'_i$ is on the line through $a'_i + (2\delta,0)$ and direction $(1,-1)$. Thus Property 1 and 2 of Lemma 24 are met. It remains to show that no stationary guard or the guard in the square $T$ can see into the corridor. That is clear from the construction. 

Figure 32: The Picasso-gadget.
D.4 The scaling gadget

Before we can move to the final step of proving Theorem 3, we need another tool, which is a gadget to scale variables. This will be needed to reverse the scaling that occurs as we apply the transformation described in Appendix B. The gadget is shown in Figure 33. Two edges at the left side of the gadget are not fully shown. They end at corners $c_1 := (−CN^2, CN^2)$ and $d_1 := (−CN^2, CN^2 + 1.5)$, respectively, where $c_1$ is the right shadow corner of the corridor as in Section C.9. The gadget contains two guard segments $r_i := a'_i b'_i$ and $r_j := a'_j b'_j$, each of width 1.5, defined by the left endpoints $a'_i := (0, 0)$ and $a'_j := (13.5, 0)$. Let $ab$ be a segment on $r_i$, where $a$ is to the left of $b$. The scaling gadget contains a copy-nook $Q_n$ of $ab$ and $r_j$ with shadow corners $(5.5, 3)$ and $(9.5, 3)$. Likewise, it contains a copy-umbra $Q_u$ of $ab$ and $r_j$ with shadow corners $(5.5, 11)$ and $(9.5, 11)$. It is important here to note that in general, $ab$ is not the full segment $r_i$, but just a subset. This enforces that the value represented by $a$ of $r_i$ will be copied to $1/2$ on $r_j$ and the value represented by $b$ will be copied to $2$. There is a stationary guard position $g_s$ that sees $Q_u$, but nothing above the critical segment of $Q_u$.

Lemma 53. Let $P'_sca$ be the polygon obtained from a scaling gadget by closing it by adding the edge $c_1 d_1$. A set of guards $G \subset P'_sca$ of cardinality at most 3 guards $P'_sca$ if and only if

- there is exactly one guard placed on each guard segment $r_i, r_j$ and at $g_s$, and

- the guards $p_i, p_j$ at $x_i, x_j$, respectively, satisfy that $p_i \in ab$ and $\frac{||ap_i||}{||ab||} = \frac{||a'_jp_i||}{||a'_jb'_j||}$.

Proof. Assume that $P'_sca$ is guarded by a set $G$ of at most 3 guards. Similarly as in Lemma 32 we can show that there must be exactly one guard at each of $r_i, r_j$ and at $g_s$. Since the guards $p_i, p_j$ must together see the critical segments of $Q_n$ and $Q_u$, it follows that $p_i \in ab$ and $\frac{||ap_i||}{||ab||} = \frac{||a'_jp_i||}{||a'_jb'_j||}$.
Now assume that there is exactly one guard placed on each guard segment \( r_i, r_j \) and at \( g_s \), and that \( p_i \in ab \) and \( \|\mathbf{ap}_i\| = \|\mathbf{ap}_j\| \). Then all of \( Q_u \) and \( Q_n \) is seen by the guards. The remaining area is clearly also seen by the guards.

We then use a corridor as described in Section C.3 to copy in the values of two variables \( x_i, x_j \) at \( r_i, r_j \). Recall that the endpoints of the right entrance of the corridor are denoted \( c_1, d_1 \). In order to attach the gadget to the corridor, we first scale it down by the factor \( \frac{1}{\sqrt{CN}} \) and then translate it so that the points \( c_1, d_1 \) of the gadget coincides with the endpoints of the right entrance of the corridor with the same names. We thus obtain that the point \( a'_i \) in the gadget becomes the point \( m := c_1 + (1, -1) \) in \( \mathcal{P}_S \). We need to prove Lemma 24 in order to guarantee that the corridor copies the guard positions appropriately.

**Lemma 24 for the Scaling-Gadget.** We make use of Lemma 34. First check that the length of \( r_i \) and \( r_j \) is \( \frac{1}{cN} \) as required. Both guard segments are contained in the square \( m + [-\Delta, \Delta] \times [-\Delta, \Delta] \), with \( \Delta := \frac{\sqrt{50}}{cN} \) and \( m := c_1 + (1, -1) \). Furthermore, the left endpoint of \( a'_i \) is on the point \( a'_i + (\delta, 0) \). Thus Property 1 and 2 of Lemma 24 are met. It remains to observe that \( g_s \) cannot see into the corridor.

**D.5 Proof of the Picasso Theorem**

Using the scaling gadget and the Picasso gadget, we are now ready to describe an art gallery \( \mathcal{P}_S \) as described in Theorem 3.

**Proof of Theorem 3.** Let a quantifier-free formula \( \Phi \) of the first-order theory of the reals be given that has exactly two free variables \( x, y \), and suppose that the set of solutions \( S := \{(x, y) \in \mathbb{R}^2 : \Phi(x, y)\} \) is a closed subset of \([0, 1]^2\). The theorem is trivially true of \( S = \emptyset \), so assume that \( S \) is non-empty. Recall that there is an axis-parallel square \( T \) in the Picasso gadget and three guard segments \( r_1, r_2, r_3 \) and corresponding half-nooks \( Q_1, Q_2, Q_3 \) and a half-umbra \( Q_4 \). We identify each point in \( T \) with the corresponding point in \([0, 1]^2\) under the natural linear bijection, so that the upper left corners correspond, etc. Thus, with slight abuse of notation, consider \( S \) as a subset of \( T \). Let \( g_s \) be a point on \( r_\sigma, \sigma \in \{i, j, l\} \), and \( p \) a point in \( T \). In order to see the critical segment of \( Q_s \) together, the point \( g_s \) restricts \( p \) to a half-plane. The point \( \pi_0^{-1}(\pi_1(p)) \) is the point on \( r_\sigma \) such that \( p \) is on the boundary of that half-plane. The point \( \pi_0^{-1}(\pi_1(p)) \) specifies the number \( x_\sigma(p) \in [1/2, 2] \). The function \( x_\sigma : T \longrightarrow [1/2, 2] \) is a rational function of the \( x \)- and \( y \)-coordinate of \( p \). Therefore, there is a quantifier-free formula of the first-order theory of the reals \( \Psi \) with five free variables \( x, y, x_i, x_j, x_l \) such that when \( p = (x, y) \in T \), the tuple \( (x, y, x_i, x_j, x_l) \) satisfies \( \Psi \) if and only if \( x_\sigma = x_\sigma(p) \) for each \( \sigma \in \{i, j, l\} \).

We consider the formula \( \Phi' := \Phi \land \Psi \), which has a compact set of solutions in \( \mathbb{R}^5 \).

We now apply Corollary 48 to obtain an instance \( \Phi'' \) of ETR-INV in which the variables of \( \Phi' \) appear scaled down and shifted. Consider the variable \( x_\sigma \) in \( \Phi' \), which has domain \([1/2, 2] \). In \( \Phi'' \), there is a variable \( x_\sigma' \) with corresponding domain \([a, b] \subseteq [1/2, 2] \). Thus, \( x_\sigma \) has been scaled down by a factor of \( \frac{b-a}{3/2} \) and shifted in order to get \( x_\sigma' \). Let the remaining variables in \( \Phi'' \) be \( y'_1, \ldots, y'_k \), so that the complete set of variables is \( \{x_\sigma', x_1', x_j', y'_1, \ldots, y'_k\} \). We now make another instance \( \Phi''' \) of ETR-INV, which is identical to \( \Phi'' \), except that \( \Phi''' \) has three extra variables \( x_i, x_j, x_l \) appearing in no equations. Let \( (\mathcal{P}, g) \) be the instance of the art gallery problem as described in Theorem 46 for the instance \( \Phi''' \) of ETR-INV. Thus, in the main area of \( \mathcal{P} \), there are guard segments representing \( x_i, x_j, x_l \), but they are not copied into any gadget. We might need to use a slightly larger value of \( N \) than defined in Section C.7 in order to have vertical space for 4 more gadgets to the right. For each \( \sigma \in \{i, j, l\} \), consider the variable \( x_\sigma' \) with domain \([a, b] \subseteq [1/2, 2] \), as defined above. We
construct an art gallery $\mathcal{P}'$ from $\mathcal{P}$ by adding a scaling gadget into which we copy $x'_\sigma$ and $x_\sigma$, such that the domain $[a,b]$ of $x'_\sigma$ is scaled up to $[1/2,2]$ of $x_\sigma$. Finally, we also add the Picasso gadget to $\mathcal{P}'$ and copy in $x_i, x_j, x_l$. By shifting and scaling $\mathcal{P}'$, we obtain that the square $T$ in the Picasso gadget coincides with $[0,1]^2$. It now follows from Lemma 53 and Lemma 52 that for each $p \in [0,1]^2$, there is an optimal guard set of $\mathcal{P}_S$ containing $p$ if and only if $p \in S$. \hfill \qed