Continuous anti-forcing spectra of cata-condensed hexagonal systems

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Abstract

The anti-forcing number of a perfect matching $M$ of a graph $G$ is the minimal number of edges not in $M$ whose removal make $M$ as a unique perfect matching of the resulting graph. The anti-forcing spectrum of $G$ is the set of anti-forcing numbers of all perfect matchings of $G$. In this paper we prove that the anti-forcing spectrum of any cata-condensed hexagonal system is continuous, that is, it is an integer interval.

Key words: Perfect matching; Anti-forcing number; Anti-forcing spectrum; Hexagonal system

1 Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A perfect matching or 1-factor of $G$ is a set of disjoint edges which covers all vertices of $G$. Harary et al. \cite{10} proposed the forcing number of a perfect matching $M$ of a graph $G$. The roots of this concept can be found in an earlier paper by Klein and Randić \cite{13}. There, the forcing number has been called the innate degree of freedom of a Kekulé structure. The forcing number of a perfect matching $M$ of a graph $G$ is equal to the smallest cardinality of some subset $S$ of $M$ such that $M$ is completely determined by this subset (i.e., $S$ is not contained in other perfect matchings of $G$). The minimum (resp. maximum) forcing number of $G$ is the minimum (resp. maximum) value over

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forcing numbers of all perfect matchings of \( G \). The set of forcing numbers of all perfect matchings of \( G \) is called the *forcing spectrum* of \( G \) \([1]\). The sum of forcing numbers of all perfect matchings of \( G \) is called the *degree of freedom* of \( G \), which is relative to Clar’s resonance-theoretic ideals \([5]\). For more results on the matching forcing problem, we refer the reader to \([2,11,12,14,16–20,23,24,26,27]\).

In 2007, Vukićević and Trinajstić \([21]\) introduced the *anti-forcing number* that is opposite to the forcing number. The anti-forcing number of a graph \( G \) is the smallest number of edges whose removal result in a subgraph of \( G \) with a unique perfect matching. After this initial report, several papers appeared on this topic \([4,7,8,22,30]\).

Recently, Lei, Yeh and Zhang \([15]\) define the *anti-forcing number of a perfect matching* \( M \) of a graph \( G \) as the minimal number of edges not in \( M \) whose removal to make \( M \) as a single perfect matching of the resulting graph, denoted by \( af(G,M) \). By this definition, the anti-forcing number of a graph \( G \) is the smallest anti-forcing number over all perfect matchings of \( G \). Hence the anti-forcing number of \( G \) is the *minimum anti-forcing number* of \( G \), denoted by \( af(G) \). Naturally, the *maximum anti-forcing number* of \( G \) is defined as the largest anti-forcing number over all perfect matchings of \( G \), denoted by \( Af(G) \). They \([15]\) also defined the *anti-forcing spectrum* of \( G \) as the set of anti-forcing numbers of all perfect matchings of \( G \), and denoted by \( \text{Spec}_{af}(G) \). If \( \text{Spec}_{af}(G) \) is an integer interval, then the anti-forcing spectrum of \( G \) is called to be *continuous*.

Let \( M \) be a perfect matching of a graph \( G \). A cycle \( C \) of \( G \) is called an *\( M \)-alternating cycle* if the edges of \( C \) appear alternately in \( M \) and \( E(G) \setminus M \). If \( C \) is an \( M \)-alternating cycle of \( G \), then the symmetric difference \( M \triangle C := (M \setminus C) \cup (C \setminus M) \) is another perfect matching of \( G \).

A set \( A \) of \( M \)-alternating cycles of a graph \( G \) is called a *compatible \( M \)-alternating set* if any two members of \( A \) either are disjoint or intersect only at edges in \( M \). Let \( c'(M) \) denote the cardinality of a maximum compatible \( M \)-alternating set of \( G \). For a planar bipartite graph \( G \) with a perfect matching \( M \), the following minimax theorem reveals the relationship between \( af(G,M) \) and \( c'(M) \).

**Theorem 1.1** \([15]\). Let \( G \) be a planar bipartite graph with a perfect matching \( M \). Then \( af(G,M) = c'(M) \).

A *hexagonal system* (or *benzenoid system*) \([6]\) is a finite 2-connected planar bipartite graph in which each interior face is surrounded by a regular hexagon of side length one. Hexagonal systems are of great important for theoretical chemistry since they are the molecular graphs of benzenoid hydrocarbons.

Let \( H \) be a hexagonal system with a perfect matching \( M \). A set of \( M \)-alternating hexagons (the intersection is allowed) of \( H \) is called an *\( M \)-alternating set*. A *Fries set* of \( H \) is a maximum alternating set over all perfect matchings of \( H \). The size of a Fries
set of $H$ is called the Fries number of $H$ and denoted by $Fries(H)$ \cite{9}. It is obvious that an $M$-alternating set of $H$ is also a compatible $M$-alternating set. By Theorem \ref{thm1} $Af(H) \geq Fries(H)$. The following theorem implies that the equality holds.

**Theorem 1.2** \cite{15}. Let $H$ be a hexagonal system with a perfect matching. Then $Af(H) = Fries(H)$.

In this paper we consider the anti-forcing spectra of cata-condensed hexagonal systems. In the next section, we introduce some graph-theoretic terms relevant to our subject and give some useful lemmas. In Section 3, we prove that the anti-forcing spectrum of any cata-condensed hexagonal system is continuous. It is quite different from the case for forcing spectrum. In fact, the forcing spectra of some cata-condensed hexagonal systems have gaps (see \cite{15,26}).

## 2 Preliminaries and lemmas

The inner dual graph $H^*$ of a hexagonal system $H$ is a graph whose vertices correspond to hexagons of $H$, and two such vertices are adjacent by an edge of $H^*$ if and only if they correspond to two adjacent hexagons (i.e., these two hexagons have one common edge). Then $H$ is cata-condensed if and only if $H^*$ is a tree \cite{3}.

We can see that edges of a cata-condensed hexagonal system $H$ can be classified into boundary edges (edges are on the boundary of $H$) and shared edges (edges are shared by two hexagons of $H$), and all vertices of $H$ are on the boundary (i.e., $H$ has no inner vertices). A subgraph $G'$ of a graph $G$ is nice if $G - V(G')$ (the graph obtained by deleting vertices of $V(G')$ and their incident edges from $G$) has a perfect matching. It is well known that every cata-condensed hexagonal system $H$ has perfect matchings and every cycle in it is nice \cite{5}, so each hexagon of $H$ can be $M$-alternating with respect to some perfect matching $M$ of $H$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{A cata-condensed hexagonal system with one branched hexagon and three kinks.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{The linear chain with five hexagons.}
\end{figure}
A hexagon $s$ of a cata-condensed hexagonal system $H$ has one, two, or three neighboring hexagons. $s$ is called *terminal* if it has one neighboring hexagon, and *branched* if it has three neighboring hexagons. $s$ has exactly two neighboring hexagons is a *kink* if $s$ possesses two adjacent vertices of degree 2, is *linear* otherwise. An illustration is given in Fig. 1. A cata-condensed hexagonal system with no branched hexagons is called a *hexagonal chain*. A hexagonal chain with no kinks is called a *linear chain*, an example is shown in Fig. 2.

A linear chain $B$ contained in a cata-condensed hexagonal system $H$ is called *maximal* if $B$ is not contained in other linear chains of $H$. For example, see Fig. 1. $B_1$ and $B_2$ are two maximal linear chains.

Let $B$ be a maximal linear chain of a cata-condensed hexagonal system $H$. We draw a straight line $l$ passing through the two centers of the two terminal hexagons of $B$. Let $E$ be the set of those edges which intersecting $l$. By the Lemma 2.1 in [28], the following lemma is immediate.

**Lemma 2.1.** Let $M$ be any perfect matching of $H$. Then $|M \cap E| = 1$.

Let $\mathcal{A}$ be a compatible $M$-alternating set with respect to a perfect matching $M$ of a planar bipartite graph $G$. Two cycles $C_1$ and $C_2$ of $\mathcal{A}$ are *crossing* if they share an edge $e$ in $M$ and the four edges adjacent to $e$ alternate in $C_1$ and $C_2$ (i.e., $C_1$ enters into $C_2$ from one side and leaves from the other side via $e$). $\mathcal{A}$ is *non-crossing* if any two cycles in $\mathcal{A}$ are not crossing. Lei, Yeh and Zhang [15] proved that any compatible $M$-alternating set can be improved to be a non-crossing compatible $M$-alternating set with the same cardinality. Let $H$ be a cata-condensed hexagonal system with a perfect matching $M$. For a cycle $C$ of $H$, let $h(C)$ denote the number of hexagons in the interior of $C$. Then we can choose a maximum non-crossing compatible $M$-alternating set $\mathcal{A}$ such that $|\mathcal{A}| = af(H, M)$ and $h(\mathcal{A}) = \sum_{C \in \mathcal{A}} h(C)$ ($h(\mathcal{A})$ is called $h$-*index* of $\mathcal{A}$ [15]) is as small as possible. By using those notations, we give the following lemma.

**Lemma 2.2.** $\mathcal{A}$ contains all $M$-alternating hexagons of $H$, and any two non-hexagon cycles of $\mathcal{A}$ have at most one common edge in $M$ and are interior disjoint (i.e., have no common areas).

**Proof.** Let $s$ be any $M$-alternating hexagon of $H$. Suppose $s \notin \mathcal{A}$. By the maximality of $|\mathcal{A}|$, $\mathcal{A} \cup \{s\}$ is not a compatible $M$-alternating set. So there is a cycle $C \in \mathcal{A}$ which is not compatible with $s$. Since any two $M$-alternating hexagons must be compatible, $C$ is not a hexagon of $H$. $s$ is in the interior of $C$ since $H$ is cata-condensed. We claim that $(\mathcal{A} \setminus \{C\}) \cup \{s\}$ is a compatible $M$-alternating set. Otherwise there is a cycle $C' \in \mathcal{A} \setminus \{C\}$ such that $C'$ and $s$ are not compatible. Hence $s$ is in the interior of $C'$. It implies that $C$ and $C'$ are either not compatible or crossing, a contradiction.
Therefore, \((A \setminus \{C\}) \cup \{s\}\) is a maximum non-crossing compatible \(M\)-alternating set with smaller \(h\)-index, a contradiction. Hence \(s \in A\).

Let \(C_1\) and \(C_2\) be two non-hexagon cycles in \(A\). First we prove that \(C_1\) and \(C_2\) are interior disjoint. If not, without loss of generality, we may suppose \(C_1\) is contained in the interior of \(C_2\) since \(C_1\) and \(C_2\) are not crossing. Then there is an \(M\)-alternating hexagon \(s\) in the interior of \(C_1\) since \(C_1\) is \(M\)-alternating [25, 29]. By the above discussion, we have that \(s \in A\) and \(s\) is compatible with both \(C_1\) and \(C_2\). Since \(H\) is cata-condensed, \(C_1\) must pass through the vertices of \(s\) and the three edges of \(s\) in \(M\), but not the other three edges of \(s\). Hence \(C_1\) must pass through the six edges going out of \(s\). Similarly, \(C_2\) must pass through the six edges going out of \(s\). It implies that \(C_1\) and \(C_2\) are not compatible, a contradiction.

Next, we prove that \(C_1\) and \(C_2\) have at most one common edge in \(M\). If \(C_1\) and \(C_2\) have at least two common edges in \(M\), then there must generate inner vertices in \(H\), a contradiction.

**Lemma 2.3.** Let \(H\) be a cata-condensed hexagonal system with at least two hexagons. Then \(af(H) < F\text{ries}(H)\).

**Proof.** By Theorem 1.2, \(Af(H) = F\text{ries}(H)\). It is sufficient to prove that \(af(H) < Af(H)\). Let \(s\) be a terminal hexagon of \(H\). Then \(s\) has a neighboring hexagon \(s'\) since \(H\) has at least two hexagons. Let \(e\) be the sheared edge of \(s\) and \(s'\). Since \(s'\) is nice, there is a perfect matching \(M\) of \(H\) such that \(s'\) is \(M\)-alternating and \(e \in M\). Note that \(s\) is also \(M\)-alternating. So \(F = M \Delta s\) is a perfect matching of \(H\) such that \(s\) is \(F\)-alternating. Let \(A\) be a maximum non-crossing compatible \(F\)-alternating set with smallest \(h\)-index. By Lemma 2.2 we have that \(s \in A\). We can see that no cycle of \(A\) passing through the three edges of \(s'\) not in \(M\). So \(A \cup \{s'\}\) is a compatible \(M\)-alternating set. By Theorem 1.1 we have that \(af(H) \leq |A| < |A \cup \{s'\}| \leq af(H, M) \leq Af(H)\). 

### 3 Continuous anti-forcing spectra

Let \(a\) and \(b\) be two integer numbers, and \(a \leq b\). In the following, we use \([a, b]\) to denote the integer interval from \(a\) to \(b\).

**Theorem 3.1.** Let \(H\) be a cata-condensed hexagonal system. Then anti-forcing spectrum of \(H\) is continuous.

**Proof.** We proceed by induction on the number \(n\) of hexagons of \(H\). If \(H\) is a single hexagon, then \(\text{Spec}_{af}(H) = \{1\}\). Suppose \(n \geq 2\). Take a maximal linear chain \(B\) in \(H\) such that one end hexagon of \(B\) is a terminal hexagon of \(H\). Let \(h_0, h_1, \ldots, h_r\) \((r \geq 1)\) be hexagons of \(B\) in turn, and \(h_r\) be terminal (see Fig. 3).
If $h_0$ is also a terminal hexagon of $H$, then $H = B$ is a linear chain with $n > 1$ hexagons. We can check that $\text{Spec}_{af}(H) = [1, 2]$.

If $h_0$ is not terminal, then $h_0$ is a kink or branched hexagon of $H$. Let $B'$ be the linear chain obtained from $B$ by removing hexagon $h_0$ and $H'$ the cata-condensed hexagonal system obtained from $H$ by removing the hexagons of $B'$ (see Fig. 3). Then $H'$ has less than $n$ hexagons. By the induction hypothesis, the anti-forcing spectrum of $H'$ is continuous. By Theorem 1.2, $Af(H') = \text{Fries}(H')$. Let $af(H') = a'$. Then $\text{Spec}_{af}(H') = [a', \text{Fries}(H')]$.

**Claim 1.** $[a' + 1, \text{Fries}(H')] \subseteq \text{Spec}_{af}(H)$.

Since $h_0$ is not terminal, $H'$ has at least two hexagons. By Lemma 2.3, $a' + 1 \leq \text{Fries}(H')$. For any $i \in [a' + 1, \text{Fries}(H')]$, we want to prove $i \in \text{Spec}_{af}(H)$. Since $i - 1 \in [a', \text{Fries}(H') - 1]$, by the induction hypothesis, there is a perfect matching $M'$ of $H'$ such that $af(H', M') = |A'| = i - 1 \geq 1$, where $A'$ is a maximum non-crossing compatible $M'$-alternating set of $H'$ with smallest $h$-index. Note that $M = M' \cup \{e_1, f_2, \ldots, f_r, g_1, g_2, \ldots, g_r\}$ is a perfect matching of $H$ and $d_i \notin M$ for each $1 \leq i \leq r$. By Lemma 2.4, either $e_4 \in M'$ or $e_1 \in M'$.

If $e_1 \in M'$, then $h_1$ is $M'$-alternating. By Lemma 2.2, $h_1 \in A$, where $A$ is a maximum non-crossing compatible $M$-alternating set of $H$ with smallest $h$-index. We can see that $A \setminus \{h_1\}$ is a compatible $M'$-alternating set of $H'$, and $|A| = |A \setminus \{h_1\}| + 1 \leq |A'| + 1 = i$. On the other hand, $A' \cup \{h_1\}$ is a compatible $M$-alternating set of $H$, so $i = |A'| + 1 = |A' \cup \{h_1\}| \leq |A|$, i.e., $|A| = i$. By Theorem 1.1, $af(H, M) = i \in \text{Spec}_{af}(H)$.

From now on, we suppose $e_4 \in M'$. Then $\{e_2, e_6\} \subseteq M'$. So $h_0$ is $M'$-alternating and $M$-alternating. By Lemma 2.2, $h_0 \in A'$ and $h_0 \in A$. See Fig. 3, $H - e_2 - e_4 - e_6$ consists of three disjoint sub-catacondensed hexagonal systems: $H_1$, $H_2$ and $B'$ (the former two may be single edges). Note that there is a possible non-hexagon cycle $Q$ in $A$ which containing $h_0$.

If such $Q$ exists, then $Q$ must pass through $g_1$ and $f_1$ since $Q$ and $h_0$ are compatible.
So $A \setminus \{Q\}$ is a compatible $M'$-alternating set of $H'$, we have $|A \setminus \{Q\}| \leq |A'| = i - 1$. If $|A'| = |A \setminus \{Q\}|$, then $af(H, M) = |A| = i \in \text{Spec}_{af}(H)$. If $|A'| > |A \setminus \{Q\}|$, then $|A'| \geq |A|$. On the other hand, $A'$ is also a compatible $M$-alternating set, so $|A'| \leq |A|$. We have $|A'| = |A| = i - 1$. Note that $Q$ does not pass through $e_3$ and $e_5$. Let $M_1 = M \triangle h_0 \triangle h_1 \triangle \ldots \triangle h_j$ ($j = 0, 1, \ldots, r$), $Q_1 = (E(Q) \cap E(H_1)) \cup \{e_5\}$ and $Q_2 = (E(Q) \cap E(H_2)) \cup \{e_3\}$. Then $Q_1$ and $Q_2$ both are $M_0$-alternating cycles. We can see that $(A \setminus \{Q\}) \cup \{Q_1, Q_2, h_1\}$ is a compatible $M_0$-alternating set with cardinality $i+1$, so $c'(M_0) \geq i+1$. On the other hand, $c'(M_0)$ is at most $|A| + 2 = i + 1$. So $c'(M_0) = i + 1$.

Note that $h_r$ is the unique $M_r$-alternating hexagon contained in $B_i$ by Theorem 1.3 and Lemma 2.2. We have $af(H, M_r) = c'(M_r) = c'(M_0) - 1 = i \in \text{Spec}_{af}(H)$.

If such cycle $Q$ does not exist, then there is no cycle in $A$ which passing through the edges going out of $h_0$ since $h_0 \notin A$. So $A$ is also a maximum compatible $M'$-alternating set on $H'$, and $|A'| = |A| = i - 1$. Note that any cycle of $A \setminus \{h_0\}$ is completely contained in $H_1$ or $H_2$. Let $i_1$ (resp. $i_2$) be the number of cycles of $A$ which are completely contained in $H_1$ (resp. $H_2$). Then $|A| = i_1 + i_2 + 1 = i - 1$. Since $M$ and $M_0$ only differ on $h_0$ and $e_5 \in M_0$ (resp. $e_3 \in M_0$), the size of maximum compatible $M_0$-alternating set on $H_1$ (resp. $H_2$) is $i_1$ or $i_1 + 1$ (resp. $i_2$ or $i_2 + 1$). Let $A_0$ be a maximum compatible non-crossing $M_0$-alternating set of $H$ with minimal $h$-index. Note that $h_0$ and $h_1$ both are $M_0$-alternating. By Lemma 2.2, $h_0 \in A_0$ and $h_1 \in A_0$. It implies that cycles in $A_0 \setminus \{h_0, h_1\}$ are completely contained in $H_1$ or $H_2$. Hence $i = i_1 + i_2 + 2 \leq |A_0| \leq i_1 + i_2 + 4 = i + 2$. If $|A_0| = i$, then $af(H, M_0) = i \in \text{Spec}_{af}(H)$. If $|A_0| = i + 1$, then $(A_0 \setminus \{h_0, h_1\}) \cup \{h_r\}$ is a maximum compatible $M_r$-alternating set with size $i$ since $M_0$ and $M_r$ only differ on $B_i$. So $af(H, M_r) = i \in \text{Spec}_{af}(H)$.

If $|A_0| = i + 2$, then $H_1$ (resp. $H_2$) contains exactly $i_1 + 1$ (resp. $i_2 + 1$) cycles of $A_0$. Let $M_0'$ (resp. $M_0''$) be the restriction of $M_0$ to $H_1$ (resp. $H_2$). Then $af(H_1, M_0') = i_1 + 1$ (resp. $af(H_2, M_0'') = i_2 + 1$). Hence $af(H_1) \leq i_1 + 1$ (resp. $af(H_2) \leq i_2 + 1$). If $af(H_1) < i_1 + 1$ (resp. $af(H_2) < i_2 + 1$), then $i_1 \in \text{Spec}_{af}(H_1)$ (resp. $i_2 \in \text{Spec}_{af}(H_2)$) by the induction hypothesis. So there is a perfect matching $F_1$ (resp. $F_2$) of $H_1$ (resp. $H_2$) such that $c'(F_1) = i_1$ (resp. $c'(F_2) = i_2$). We can see that $M_1' = F_1 \cup (M_r \cap (E(H_1) \cup E(B'))) \cup (E(B'))$ (resp. $M_2' = F_2 \cup (M_r \cap (E(H_1) \cup E(B'))) \cup (E(B'))$) is a perfect matching of $H$ such that $af(H, M_1') = c'(M_1') = i_1 + i_2 + 1 + 1 = i \in \text{Spec}_{af}(H)$ (resp. $af(H, M_2') = c'(M_2') = i_2 + i_1 + 1 + 1 = i \in \text{Spec}_{af}(H)$).

Now suppose $af(H_1) = i_1 + 1$ and $af(H_2) = i_2 + 1$, $F'$ is a perfect matching of $H'$ with $af(H', F') = a'$. Note that $F = F' \cup \{f_1, f_2, \ldots, f_r, g_1, g_2, \ldots, g_e\}$ is a perfect matching of $H$. By Lemma 2.2, either $e_1 \in F'$ or $e_4 \in F'$. We assert that $e_4 \in F'$. Otherwise $e_1 \notin F'$, then the restrictions $F_1'$ and $F_2'$ of $F'$ to $H_1$ and $H_2$ are perfect matchings of $H_1$ and $H_2$ respectively. So $a' \geq c'(F_1') + c'(F_2') \geq af(H_1) + af(H_2) = i_1 + 1 + i_2 + 1 = i \in [a' + 1, \text{Fries}(H')], a$ contradiction. Since $e_4 \notin F'$,
\{e_2, e_6\} \subseteq F'$. So $h_0$ is $F'$-alternating, and $F' \triangle h_0$ is a perfect matching of $H'$. Since the restrictions of $F' \triangle h_0$ to $H_1$ and $H_2$ are perfect matchings of $H_1$ and $H_2$ respectively, $c'(F' \triangle h_0) \geq af(H_1) + af(H_2) + 1 = i + 1 \geq a' + 2$. On the other hand, by Lemma 2.2 we have that $c'(F' \triangle h_0) \leq a' + 2$. So $c'(F' \triangle h_0) = a' + 2$ and $a' + 1 = i$. Let $\mathcal{A}^*$ be a maximum non-crossing compatible $F' \triangle h_0$-alternating set of $H'$ with minimal $h$-index. Then $|\mathcal{A}^*| = c'(F' \triangle h_0) = a' + 2$. Note that $c'(F') = a'$, but $c'(F' \triangle h_0) = a' + 2$, it implies that there must be an $F' \triangle h_0$-alternating cycle $C_1$ (resp. $C_2$) of $\mathcal{A}^*$ passing through $e_5$ (resp. $e_3$) and contained in $H_1$ (resp. $H_2$). We can see that $D = (C_1 - e_3) \cup (C_2 - e_3) \cup \{e_2, e_6, g_1, f_1, f_1', d_1\}$ is an $F'$-alternating cycle containing $h_0$, and $D$ is compatible with each cycle of $\mathcal{A}^* \setminus \{C_1, C_2\}$. So $\{D\} \cup (\mathcal{A}^* \setminus \{C_1, C_2\})$ is a compatible $F$-alternating set with size $i$. By theorem 1.1 $af(H, F) \geq i$. On the other hand, by Theorem 1.1 and Lemma 2.2 we have $i = a' + 1 = c'(F') + 1 \geq c'(F) = af(H, F)$. Therefore, $af(H, F) = i \in \text{Spec}_{af}(H)$.

By the arbitrariness of $i$, we proved that $[a' + 1, \text{Fries}(H')] \subseteq \text{Spec}_{af}(H)$.

Claim 2. $a' \leq af(H)$.

Let $af(H) = a$, $M$ be a perfect matching of $H$ with $af(H, M) = a$. By Lemma 2.1 just one edge of $\{e_4, e_1, d_1, d_2, \ldots, d_r\}$ belongs to $M$. If $e_4 \in M$ or $e_1 \in M$, then the restriction $M'$ of $M$ to $H'$ is a perfect matching of $H'$. So $a' \leq c'(M') \leq c'(M) = a$. If $d_i \in M$ ($1 \leq i \leq r$), then $h_i$ is $M$-alternating. Let $M_i = M \triangle h_i \triangle h_{i-1} \triangle \cdots \triangle h_1$. Then $M_i$ is a perfect matching of $H$ and $e_1 \in M_i$. Note that $M$ and $M_i$ only differ on $B'$. By Lemma 2.2 $c'(M_i) \leq c'(M) + 1 = a + 1$. Since $e_1 \in M_i$, the restriction $M_i'$ of $M_i$ to $H'$ is a perfect matching of $H'$. So $a' \leq c'(M'_i)$. Let $\mathcal{A}_i'$ be a maximum non-crossing compatible $M_i'$-alternating set with minimal $h$-indices in $H'$. Then $\mathcal{A}_i' \cup \{h_1\}$ is a compatible $M_i$-alternating set of $H$ since $h_1$ is $M_i$-alternating. Hence $|\mathcal{A}_i' \cup \{h_1\}| \leq c'(M_i)$. Further, $a' + 1 \leq c'(M'_i) + 1 = |\mathcal{A}_i'| + 1 \leq c'(M_i) \leq a + 1$, i.e., $a' \leq a$.

Claim 3. $\text{Fries}(H) \leq \text{Fries}(H') + 2$.

Let $M$ be a perfect matching of $H$ and $h'(M)$ denote the the number of $M$-alternating hexagons of $H$. Suppose $h'(M) = \text{Fries}(H)$. By Lemma 2.1 only one of $\{e_4, e_1, d_1, d_2, \ldots, d_r\}$ belongs to $M$. If $e_1 \in M$ or $e_4 \in M$, then the restriction $M'$ of $M$ to $H'$ is a perfect matching of $H'$. Note that $B'$ contains at most one $M$-alternating hexagon. So $h'(M) \leq h'(M') + 1 \leq \text{Fries}(H') + 1$. If $d_i \in M$ ($1 \leq i \leq r$), then $h_i$ is $M$-alternating. Let $M_i = M \triangle h_i \triangle h_{i-1} \triangle \cdots \triangle h_1$. Then $M_i$ is a perfect matching of $H$ such that the restriction $M_i'$ of $M_i$ to $H'$ is a perfect matching of $H'$. Since $h_1$ is the unique $M_i$-alternating hexagon contained in $B'$, $h'(M_i) = h'(M') + 1$. Note that $h'(M_i) = h'(M)$ or $h'(M_i) = h'(M) - 1$ since $B$ is a linear chain. If $h'(M_i) = h'(M)$, then $h'(M) = h'(M'_i) + 1 \leq \text{Fries}(H') + 1$. If $h'(M_i) = h'(M) - 1$, then $h'(M) = h'(M'_i) + 2 \leq \text{Fries}(H') + 2$.

Claim 4. If $\text{Fries}(H) = \text{Fries}(H') + 2$, then $\text{Fries}(H') + 1 \in \text{Spec}_{af}(H)$.
The proof of Claim 3 implies that if $\text{Fries}(H) = \text{Fries}(H') + 2$, then there is a perfect matching $M$ of $H$ with $h'(M) = \text{Fries}(H)$ such that just two adjacent $M$-alternating hexagons $h_i$ and $h_{i+1}$ ($1 \leq i < r$) contained in $B'$. Let $M^* = M \triangle h_{i+1} \triangle h_{i+2} \triangle \cdots \triangle h_r$. Then $M^*$ is a perfect matching of $H$ and $h_r$ is the unique $M^*$-alternating hexagon contained in $B'$. By Lemma 2.2, $c'(M^*) = h'(M) - 1 = \text{Fries}(H') + 1$. By Theorem 1.1, $af(H, M^*) = \text{Fries}(H') + 1 \in \text{Spec}_{af}(H)$.

We can see that Claims 1, 2, 3 and 4 imply that there is no gap in the anti-forcing spectrum of $H$. □

According to Theorems 1.2 and 3.1 the following corollary is immediate.

**Corollary 3.2.** Let $H$ be a cata-condensed hexagonal system. Then $	ext{Spec}_{af}(H) = [af(H), \text{Fries}(H)]$.

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