Around evaluations of biset functors.

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Abstract

It is known that there is a deep and complex relation between the category of biset functors and the category of modules over the double Burnside algebra. The evaluation functor carries a lot of informations of the category of biset functors into the category of modules over the double Burnside algebra. Curiously, the converse also holds. Our purpose here, is to study in more details this relation. Unfortunately, we are immediately stuck by the old problem of vanishing of simple biset functors. In this article, we remark that this is in fact crucial for the understanding of the double Burnside algebra. In order to avoid this difficulty we look at finite groups for which there are no non-trivial vanishing of simple functors. We call them non-vanishing groups. This family contains all the abelian groups, but also infinitely many non abelian groups. We show that for a non-vanishing group, there is an equivalence between the category of modules over the double Burnside algebras and a category of biset functors. Then, we deduce results about the highest-weight structure, and the self-injective property of the double Burnside algebra. We also revisit Barker’s Theorem on the semi-simplicity of the category of biset functors.

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1 Introduction

Let $k$ be a field and $G$ be a finite group. The double Burnside ring $B(G,G)$ is the Grothendieck ring of the category of all finite $G$-$G$-bisets. Extending scalars to $k$, we have an algebra $kB(G,G)$ which is called the double Burnside algebra of the finite group $G$. This algebra, and some of its subalgebras are crucial objects in recent developments in representation theory of finite groups, fusion systems and homotopy theory. We refer to the Introduction of [7], for explicit motivations. The last fifteen years, it has been studied by several mathematicians. In [3], Robert Boltje and Susanne Danz were particularly interested by the subalgebras consisting of left-free bisets or bi-free bisets in characteristic zero. They introduced ghost-rings and mark morphisms for these algebras. They deduced a lot of informations about these two algebras in characteristic 0. In [4], Boltje and Burkhard Külshammer found the central primitive idempotents of these two algebras. However, the question of understanding the central primitive idempotents of the double Burnside algebra is still open, even in characteristic zero. In fact, it is a
general remark. It is particularly difficult to generalize the results of the left-free double Burnside algebra to the whole double Burnside algebra. According to Jacques Thévenaz, there is a ‘quantic gap’ between them.

It is known that the double Burnside algebra is semisimple if and only if the group $G$ is cyclic and the characteristic of the field is suitable (zero e.g.). Unfortunately, it is basically the only known result about the ring structure of the double Burnside algebra.

It was recently shown by Serge Bouc, Radu Stancu and Jacques Thévenaz in [7], that one can deduce a lot of informations about the double Burnside algebra via evaluating biset functors. More surprisingly, they showed that the converse is also true: one can deduce a lot about biset functors by just looking at the double Burnside algebras using adjoints of the evaluation functor, but also more sophisticated tools. In the present article, we continue to develop and use this philosophy. Loosely speaking, the main difference is that we work with an intermediate category of biset functors, which we called category of biset functors over a fixed finite group.

Immediately, we are stuck with the problem of evaluating simple biset functors. It is almost obvious that the evaluation at $G$ of a simple biset functor $S$ is either 0 or a simple module over the endomorphism algebra of $G$. Unfortunately, it is a notoriously difficult combinatorial problem to understand when there is a vanishing. In this article, we observe that this problem is not only annoying, it is also of crucial importance if one wants to understand the double Burnside algebra of a finite group $G$. In particular, we show that the problems of Section 9 of [7] are nothing but problems of vanishing of some simple functors. Moreover, it may give some explanations on the ‘quantic gap’ between the left-free double Burnside algebra and the whole double Burnside algebra.

In order to avoid this difficulty we consider finite groups such that there are no non-trivial vanishing of simple functors (see Definition 4.1 for more details). The first important result, is that it is possible to reformulate this condition involving the simple functors in a condition involving a composition of bisets. This is what we called the generating relation. We give various equivalent interpretations of this relation. One in terms of vanishing of biset functors, another in terms of a composition of bisets and finally one in terms of representable functors in the biset category. Each of these interpretations will be useful at some point in the article. One of them shows that the non-vanishing condition is so strong that there is in fact an equivalence of categories between the category of modules over the double Burnside algebra and the category of biset functors over $G$.

This gives us the motivation to understand the generating relation and more generally, the family of non-vanishing groups. We show that the abelian groups and the so-called self-dual groups are in this family. Unfortunately we did not succeed to classify all the non-vanishing groups. Understanding this family is in theory much easier than understanding the vanishing problem for the simple biset functors, as we just need to understand the so-called generating relation. However, it appears to be difficult to characterize this last problem in group theoretical terms. Curiously, in some example it is even easier to check that the simple biset functors do not vanish at $G$ than to check this composition of bisets. At the end of Section 5, we give an example of non-vanishing
group in characteristic 0. It can be seen that the non-vanishing property of this group is connected to some non-trivial facts about the ordinary representations of $GL(3, 2)$. In particular, this example is so pathologic that it seems very unlikely to end up with a classification of this family of groups. However, over a field of characteristic 3, this group is vanishing. In particular, this very pathologic example is not in the family of the finite groups that are non-vanishing over any field (or over the ring of integers). This last family, may be easier to classify and is also important. Indeed, a finite group $G$ is in this family if and only if the category of modules over the double Burnside ring $\mathbb{Z}B(G, G)$ is equivalent to the category of $\mathbb{Z}$-biset functors over $G$.

In Section 6, we assume the field to be of characteristic zero. We show that the double Burnside algebra of a non-vanishing group is quasi-hereditary. This result is nothing but an application of the equivalence discussed above and the famous Theorem of Peter Webb about the highest-weight structure of the category of biset functors. Quasi-hereditary algebras, and highest-weight categories come historically from the theory of representations of the complex semi-simple Lie algebras. A lot of important notions for Lie algebras admit a generalization, or an axiomatization, to quasi-hereditary algebras. In particular, there is a notion of exact Borel subalgebra of a given quasi-hereditary algebra. Theses subalgebras seem to be of first importance in recent development of the representation theory of algebras. They are particularly important for the so-called theory of boscs. It is known, and highly non-trivial, that for any quasi-hereditary $A$, there is a Morita equivalent algebra $A'$ which admits an exact Borel subalgebra. For biset functors, it turns out that this notion of Borel subalgebras is connected with the so-called deflation functors. That is, biset functors without the inflation morphisms. We show that the category of deflation functors is an exact Borel subcategory of the category of biset functors. In order to prove this result, we observe that the induction functor between the category of deflation functors and the full category of biset functors is exact without any assumption on the field or on the ring of coefficients. At first, the author was doubtful about this result that seems 'too nice to be true'. However, as pointed out by Serge Bouc to the author, it is also a direct corollary of the description of this induction by Rosalie Chevalley in her PhD thesis (See Chapter 5 of [11]).

For the double Burnside algebra, the situation is as usual more complicated. This algebra may be an example of quasi-hereditary algebra which does not admit an exact Borel subalgebra. First, we show that over a field of characteristic zero, the left-free double Burnside algebra is a quasi-hereditary algebra. This result may be of independent interest. It may also not be new, but I was not able to find it in the literature. Unfortunately, the left-free double Burnside algebra is not in general a Borel subalgebra of the double Burnside algebra. The main reason is that there are less simple modules over the left-free algebra than over the whole double Burnside algebra. This is another illustration of the 'quantic gap' between left-free double Burnside algebra and the whole double Burnside algebra. It is also interesting to note that for the category of biset functors, the situation is much nicer!

In the section 7, we use the generating relation to revisit Barker’s Theorem about the semi-simple property of the category of biset functors. We show, that understanding
the semi-simplicity of the double Burnside algebra is equivalent to understand the semi-
simplicity of the category of biset functors. We also deduce a useful characterization of
this semi-simplicity. It is well-known that a group algebra is semi-simple if and only if
the trivial module is projective. We prove that for biset functors and modules over the
double Burnside algebra, we have a similar result for a suitable notion of trivial object.

Over a field of characteristic zero, the double Burnside algebra may be quasi-hereditary.
The example of the group $A_5$ shows that this is not always the case. For $A_5$, one can
see that the double Burnside algebra has infinite global dimension. More precisely one
can show that it has a block which is self-injective. For that reason, we feel natural to
wonder if the double Burnside algebra is a self-injective algebra (or a symmetric algebra).
Using the characterization of the semi-simplicity by the trivial object, we show that the
double Burnside algebra of a finite group is never a self-injective algebra except when it
is semi-simple.

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References

2 Review on bisets and biset functors

In this short section, we fixe our notations and for the convenience of the reader, we
recall some well known facts about biset functors that are crucial for the present article.
We refer to the first chapters of [6] for more details.

Let $G$ be a finite group. We denote by $s_G$ the set of the subgroups of $G$ and by $[s_G]$ a
set of representatives of the conjugacy classes of the subgroups of $G$. As always, if
g $\in G$ and $H$ is a subgroup of $G$, then we denote by $^gH$ the conjugate of $H$ by $g$.

Let $G$ and $H$ be two finite groups. Let $L$ be a subgroup of $G \times H$. There are four
important subgroups associated to $L$:

\[
p_1(L) = \{ g \in G : \exists h \in H, (g, h) \in L \},
\]
\[
p_2(L) = \{ h \in H : \exists g \in G, (g, h) \in L \},
\]
\[
k_1(L) = \{ g \in G : (g, 1) \in L \},
\]
\[
k_2(L) = \{ h \in H : (1, h) \in L \}.
\]

It is clear that $k_i(L) \leq p_i(L)$ for $i = 1, 2$ and that \( (k_1(L) \times k_2(L)) \leq L \). Moreover, there are canonical isomorphisms

\[
p_1(L)/k_1(L) \cong L/\left( k_1(L) \times k_2(L) \right) \cong p_2(L)/k_2(L).
\]

The quotient $L/\left( k_1(L) \times k_2(L) \right)$ will be denoted by $q(L)$.

Let $G$, $H$ and $K$ be three finite groups. Let $L$ be a subgroup of $G \times H$ and $M$ be a subgroup of $H \times K$. Then $L \ast M$ is the subgroup of $G \times K$ defined by:

\[
L \ast M := \{ (g, k) \in G \times K : \exists h \in H, (g, h) \in L \text{ and } (h, k) \in M \}.
\]

Let $G$ and $H$ be two finite groups. A $G$-$H$-biset is a set endowed with a left action of $G$ and a right action of $H$ which commute. In other terms, a $G$-$H$-biset is nothing but a $G \times H^{op}$-set. If $L$ is a subgroup of $G \times H$, the quotient $(G \times H)/L$ is naturally a $G$-$H$-biset for the action given by

\[
a \cdot (g, h)L \cdot b = (ag, b^{-1}h)L, \forall a, g \in G, \forall b, h \in H.
\]

The **double Burnside** group $B(G, H)$ is the Grothendieck group of the category of finite $G$-$H$-bisets. The set of $[(G \times H)/L]$ where $L \in [s_{G \times H}]$ is called the canonical basis of $B(G, H)$. Here, $[X]$ denotes the isomorphism class of the $G$-$H$-biset $X$.

Let $G$, $H$ and $K$ be three finite groups. Let $U$ be a $G$-$H$-biset and $V$ be an $H$-$K$-biset. Then, we denote by $U \times_H V$ the set of $H$-orbits of $U \times V$ where $H$ acts diagonally on the cartesian product. That is $h \cdot (u, v) = (uh^{-1}, hv)$ for $h \in H$ and $(u, v) \in U \times V$. Extending this by bilinearity, we have a bilinear map from $B(G, H) \times B(H, K)$ to $B(G, K)$. This product will be understood as a composition, so we will sometimes use the notation $\circ$ instead of $\times_H$.

There is a Mackey formula for the composition of the transitive bisets. Let $G$, $H$ and $K$ be three finite groups. Let $L$ be a subgroup of $G \times H$ and $M$ be a subgroup of $H \times K$. Then, by Lemma 2.3.24 of [3], we have

\[
((G \times H)/L) \times_H ((H \times K)/M) \cong \bigsqcup_{h \in [p_2(L)]/p_1(M)} (G \times K)/(L \ast (h, 1)M) \tag{1}
\]

Let $R$ be a commutative ring with 1. For $G$ and $H$ two finite groups, we denote by $RB(G, H)$ the $R$-module $R \otimes \mathbb{Z} B(G, H)$. We still denote by $\times_H$ its $R$-bilinear extension.
Definition 2.1. The biset category $RC$ over $R$ is the category where:

- The objects are the finite groups.
- If $G$ and $H$ are two finite groups, then $\text{Hom}_C(G, H) = RB(H, G)$.
- If $G$, $H$, and $K$ are finite groups, the composition is the product
  $$- \times_H - : RB(G, H) \times RB(H, K) \rightarrow RB(G, K).$$
- If $G$ is a finite group, the identity morphism is $[(G \times G)/\Delta(G)]$ where $\Delta(G)$ is the diagonal subgroup of $G$.

A biset functor over $R$ is an $R$-linear functor from $RC$ to $R$-Mod.

Here, we choose to apply an ‘op’-functor on the set of morphisms. This is just for convenience and this will allow us to work with covariant functors instead of contravariant functors.

The category of biset has a very important property. Every morphism can be written as sum of transitive morphisms. Moreover every transitive morphism can be factorised and written as a composition of 5 particular morphisms which are called elementary by Bouc. This is Lemma 2.3.26 of [4]. If $H$ is a subgroup of $G$, then we set $\text{Ind}^G_H := gG_H$ and $\text{Res}^G_H = H_GG$, where the action of $G$ and $H$ are given by the multiplication. If $N$ is a normal subgroup of $G$, then we set $\text{Inf}^G_{G/N} = g(G/N)G/N$ and $\text{Def}^G_{G/N} = G/(G/N)G$ where $G/N$ acts via multiplication and $G$ via the canonical projection onto $G/N$. If $\alpha : H \rightarrow H'$ is a group isomorphism, then we set $\text{Iso}(\alpha) = H'H_H'$ where $H$ acts via the morphism $f$. Then, we have Bouc’s Butterfly Lemma (Lemma 2.3.26 [6]). Let $G$ and $H$ be two finite groups. Let $L$ be a subgroup of $G \times H$. Then,

$$\frac{(G \times H)}{L} \cong \text{Ind}^G_{p_1(L)} \circ \text{Inf}^p_{p_1(L)/k_1(L)} \circ \text{Iso}(\alpha) \circ \text{Def}^p_{p_2(L)/k_2(\Delta)} \circ \text{Res}^H_{p_2(L)},$$

where $\alpha$ is the canonical isomorphism between $p_2(L)/k_2(\Delta)$ and $p_1(L)/k_1(L)$.

We are particularly interested by subcategories of the biset category in which the morphisms will keep this type of decomposition. These categories are called admissible by Bouc.

Definition 2.2 (Definition 4.1.3 [4]). A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called admissible if it contains group isomorphisms and if it satisfies the following conditions:

- If $G$ and $H$ are objects of $\mathcal{D}$, then there is a subset $S(H, G)$ of the set of subgroups of $H \times G$, invariant under $(H \times G)$-conjugation, such that $\text{Hom}_D(G, H)$ is the submodule of $kB(H, G)$ generated by the elements $[(H \times G)/L]$, for $L \in S(H, G)$.
- If $G$ and $H$ are groups of $\mathcal{D}$, and if $L \in S(H, G)$, then $q(L)$ is also an object of $\mathcal{D}$. Moreover, $\text{Def}^p_{p_2(L)/k_2(\Delta)} \circ \text{Res}^G_{p_2(L)}$ and $\text{Ind}^H_{p_1(L)} \circ \text{Inf}^p_{p_1(L)/k_1(L)}$ are morphisms in $\mathcal{D}$.
If $\mathcal{D}$ is an admissible biset category and $H, K \in \mathcal{D}$, we denoted by $\mathcal{D}(K, H)$ the set $\text{Hom}_{\mathcal{D}}(H, K)$.

In this article we are interested by two types of admissible biset categories: replete biset categories and their subcategories consisting of the left-free bisets. More precisely: let $G$ be a finite group. A section of $G$ is a pair $(B, A)$ such that $A \trianglelefteq B \triangleleft G$. The quotient $B/A$ is then called a subquotient of $G$. If $H$ is isomorphic to a subquotient of $G$, then we use the notation $H \subseteq G$. If it is isomorphic to a strict subquotient, we use $H \lhd G$. We denote by $\Sigma(G)$ the set of all subquotients of $G$. Moreover if $\mathcal{D}$ is a class of finite groups, we said that $\mathcal{D}$ is closed under taking subquotients if $H$ is isomorphic to $B/A$ where $(B, A)$ is a section of a group $G$ in $\mathcal{D}$, then $H$ is also in $\mathcal{D}$.

**Definition 2.3.** Let $R$ be a commutative ring. A subcategory of $\mathcal{R}$ is called replete if it is a full subcategory of $\mathcal{R}$ whose class of object is closed under taking subquotients.

In the present article we will work with replete biset category or sometimes subcategories of replete biset category where we only take the left-free bisets as morphisms.

If $\mathcal{D}$ is an admissible biset category we denote by $\mathcal{F}_{\mathcal{D}, R}$ the category of $R$-linear functors from $\mathcal{D}$ to $R$-$\text{Mod}$, and we call it the category of $R$-biset functors over $\mathcal{D}$.

**Definition 2.4.** Let $R$ be a commutative ring and $G$ a finite group. The category $\mathcal{F}_{\Sigma(G), R}$ is called the category of $k$-biset functors over $G$ and simply denoted by $\mathcal{F}_{G, R}$.

**Definition 2.5.** Let $R$ be a commutative ring with 1 and $G$ be a finite group. The double Burnside algebra of $G$ is the endomorphism algebra of $G$ in $R\mathcal{C}$. In other words, the double Burnside algebra of $G$ is the module $RB(G, G)$ with the product induced by the composition of $G$-$G$-bisets.

Let $\mathcal{D}$ be an admissible biset category. As the category $\mathcal{R}$-$\text{Mod}$ is abelian, it is well known that the category of biset functors over $\mathcal{D}$ is an abelian category. The abelian structure is point-wise. In other words, the abelian structure is defined on the evaluations of the functors.

By Yoneda’s Lemma, if $G$ is an object of $\mathcal{D}$, the representable functor (also called the Yoneda functor) $Y_G := \text{Hom}_{\mathcal{D}}(G, -) = \mathcal{D}(-, G)$ is a projective object of $\mathcal{F}_{\mathcal{D}, R}$. Moreover, every object of $\mathcal{F}_{\mathcal{D}, R}$ is quotient of a direct sum of Yoneda functors. So, the category $\mathcal{F}_{\mathcal{D}, R}$ has enough projective objects. Using a duality argument one can show that this category has also enough injective objects. See Corollary 3.2.13 of [6] for more details. We are also particularly interested by the family of simple functors which can be describe (see the next paragraph) using the so-called evaluation functor.

Let $G \in \mathcal{D}$. If $F$ is a biset functor over $\mathcal{D}$, then its value $F(G)$ has a natural structure of module over the endomorphism algebra $\mathcal{D}(G, G)$ of $G$. More precisely, we have a functor $ev_G : \mathcal{F}_{\mathcal{D}, R} \to \mathcal{D}(G, G)$-$\text{Mod}$. Since the abelian structure of $\mathcal{F}_{\mathcal{D}, R}$ is defined on the evaluations, this functor is clearly exact. By usual arguments, it has a left and a right adjoint. A left adjoint is denoted $L_{G,-}$ and can be defined as followed. Let $H \in \mathcal{D}$. Then, the right-multiplication by the elements of $\mathcal{D}(G, G)$ induces a structure of right $\mathcal{D}(G, G)$-module on $\mathcal{D}(H, G)$. Let $V$ be a $\mathcal{D}(G, G)$-module.
Then, we set \( L_{G,V}(H) := \mathcal{D}(H,G) \otimes_{\mathcal{D}(G,G)} V \). It is now straightforward to check that \( L_{G,V} := \mathcal{D}(-,G) \otimes_{\mathcal{D}(G,G)} V \) is a biset functor over \( \mathcal{D} \) and that \( V \mapsto L_{G,V} \) is a functor from \( \mathcal{D}(G,G) \)-Mod to \( \mathcal{F}_{D,R} \) (for more details and proofs see Section 3.3 of [6]).

If \( V \) is a simple \( \mathcal{D}(G,G) \)-module, then \( L_{G,V} \) has a unique simple quotient. Moreover it is easy to see that any simple functor appears as such quotient. However, this construction is not completely satisfying as one simple functor may be realised by many different such pairs \((G,V)\). One can avoid this problem by considering minimal groups and simple modules over a quotient of the double Burnside algebra.

By Proposition 4.3.2 of [6], the quotient of the algebra \( \mathcal{D}(G,G) \) by the ideal \( I_{\mathcal{D}(G,G)} \) consisting of all morphisms factorizing through groups strictly smaller than \( G \) is isomorphic to the algebra of outer automorphisms of the group \( H \), denoted by \( R\text{Out}(H) \).

**Theorem 2.6.** Let \( k \) be a field. Let \( \mathcal{D} \) be an admissible biset category. The set of isomorphism classes of simple objects of \( \mathcal{F}_{D,k} \) is in bijection with the set of isomorphism classes of pairs \((H,V)\) where \( H \) runs through the objects of \( \mathcal{D} \) and \( V \) through the simple \( k\text{Out}(H) \)-modules.

**Proof.** See Theorem 4.3.10 of [6]. \( \square \)

If \( (H,V) \) is a pair consisting of a finite group and a simple \( k\text{Out}(H) \)-module, then since \( k\text{Out}(H) \) is a quotient of \( \mathcal{D}(H,H) \), one can see \( V \) as a simple \( \mathcal{D}(H,H) \)-module by inflation. Then \( S_{H,V} \) is nothing but the quotient \( L_{H,V}/J_{H,V} \), where \( J_{H,V} \) is the unique maximal subfunctor of \( L_{H,V} \). If \( K \in \mathcal{D} \), then by Remark 4.2.6 of [6], we have

\[
J_{H,V}(K) = \left\{ \sum_{i=1}^{n} \phi_i \otimes v_i \in L_{H,V}(K) : \forall \psi \in \mathcal{D}(H,K), \sum_{i=1}^{n} (\psi \circ \phi_i) \cdot v_i = 0 \right\}.
\] (3)

Note that this subfunctor has the property of vanishing at the group \( H \).

Let us recall that a biset functor is **finitely generated** if it is a quotient of a finite direct sum of Yoneda functors. In particular, the simple functors and the Yoneda functors are finitely generated. As in the case of modules over a ring, the choice axiom implies the existence of maximal subfunctors of a finitely generated functor. As always, we define the radical of a biset functor \( F \) as the intersection of all maximal subfunctors, and we denote it by \( \text{Rad}(F) \). Over a field, the category of finitely generated biset functors is a Krull-Schmidt category. In particular, every simple functor \( S_{H,V} \) has a projective cover in \( \mathcal{F}_{D,k} \). We denote by \( P_{H,V} \) a projective cover of \( S_{H,V} \).

In this article, we will also need the following family of biset functors. Let \( H \) and \( K \) be two objects of \( \mathcal{D} \). Then

\[
\bigoplus_{X \in \mathcal{D}} \mathcal{D}(K,X)\mathcal{D}(X,H),
\]

can be viewed as a submodule of \( \mathcal{D}(K,H) \) via composition of morphisms. We denote by \( I_{\mathcal{D}}(K,H) \) this submodule and by \( k\mathcal{D}(K,H) \) the quotient \( k\mathcal{D}(K,H)/I_{\mathcal{D}}(K,H) \). This is a natural right \( k\text{Out}(H) \)-module. If \( V \) is a \( k\text{Out}(H) \)-module, then we denote by \( \Delta_{H,V}^{\mathcal{D}} \) the functor

\[
\Delta_{H,V}^{\mathcal{D}} := K \mapsto \mathcal{D}(K,H) \otimes_{k\text{Out}(H)} V.
\]
When the context is clear enough, we will simply denote it by $\Delta_{H,V}$. These functors were introduced first, in a different form, by Peter Webb in [22]. Webb proved that under suitable hypothesis they are the standard functors in the highest-weight structure of the category of biset functors. Since we think that this is a fundamental result, we will keep this idea in mind by calling them the standard biset functors. Note that the module $V$ is not supposed to be simple here.

Similarly, we let $\nabla_{H,V}^D$ be the quotient of $D(H,K)$ by the $R$-submodule consisting of all morphisms factorizing through groups strictly smaller than $H$. Let $V$ be a $k\Out(H)$-module. We denote by $\nabla_{H,V}^D$ the functor

$$\nabla_{H,V}^D: K \mapsto \Hom_{k\Out(H)} \left( \nabla_{H,V}^D(\mathcal{D}(H,K), V) \right).$$

These functors are called the co-standard biset functors. If $H \in \mathcal{D}$, then we have a functor from $F_{\mathcal{D},k}$ to $k\Out(H)$-Mod defined by

$$F \mapsto F(H) = \bigcap_{U \in \mathcal{D}(K,H)} \ker(F(U)).$$

Lemma 2.7. Let $\mathcal{D}$ be an admissible biset category. Let $H \in \mathcal{D}$. The functor sending a $k\Out(H)$-module $V$ to $\Delta_{H,V}$ is a left adjoint to the functor sending a biset functor $F$ to $F(H)$.

Proof. This is straightforward.

3 Evaluations of biset functors

Let $k$ be a field and $\mathcal{D}$ be an admissible biset category. Let $G \in \mathcal{D}$. Then, we have an evaluation functor from $F_{\mathcal{D},k}$ to $\mathcal{D}(G,G)$-Mod. It is well known that this evaluation functor carries a lot of informations of the category of biset functors into the category of $\mathcal{D}(G,G)$-modules. More recently, Bouc, Stancu and Thévenaz showed in [7] and [8] that the converse is also true. For example, we have:

Proposition 3.1. Let $\mathcal{D}$ be an admissible biset category. Let $S$ be a simple biset functor over $\mathcal{D}$. Let $G$ be a finite group such that $S(G) \neq 0$. Let $F$ be a biset functor over $\mathcal{D}$. Then the following are equivalent.

1. $S$ is isomorphic to a subquotient of $F$.
2. The simple $\mathcal{D}(G,G)$-module $S(G)$ is isomorphic to a subquotient of $F(G)$.

Proof. Proposition 3.5 of [14] for the case of a replete biset category. However, the proof is formal, so it can be applied to any admissible biset category.

However, the evaluation at $G$ is not always compatible with algebraic operations. For example, it does not commute with taking the radical. Indeed, as explain in Section 9 of [7], the evaluation at a group $G$ of the radical of a biset functor is not always the radical of the evaluation. In [19], we observed that this phenomenon is connected with the vanishing property of the simple biset functors.
Lemma 3.2. Let $\mathcal{D}$ be an admissible biset category. Let $F \in \mathcal{F}_{\mathcal{D}, k}$ be a finitely generated biset functor. Let $G \in \mathcal{D}$.

1. $\text{Rad}(F(G)) \subset [\text{Rad}(F)](G)$.

2. If the simple quotients of $F$ do not vanish at $G$, then $\text{Rad}(F(G)) = [\text{Rad}(F)](G)$.

Let $P$ be an indecomposable projective biset functor. If the simple quotient of $P$ does not vanish at $G$, then $P(G)$ is a projective indecomposable $\mathcal{D}(G, G)$-module.

Proof. Let $M$ be a maximal subfunctor of $F$. Then, $M(G)$ is a maximal submodule of $F(G)$ if the simple quotient $F/M$ does not vanish at $G$ and $M(G) = F(G)$ otherwise. For the second part, if $N$ is a maximal submodule of $F(G)$, let $\mathcal{N}$ be the subfunctor of $F$ generated by $N$. There is a maximal subfunctor $M$ of $F$ such that $\mathcal{N} \subseteq M \subset F$. We have $\mathcal{N}(G) = N \subseteq M(G) \subset F(G)$. By maximality, $M(G) = N$. The result follows.

If $P$ is a projective indecomposable functor, it has a simple top $S$. By hypothesis, the simple functor $S$ does not vanish at $G$. By Yoneda’s Lemma, there is a non-zero morphism between the representable functor $Y_G$ and $S$. Since $S$ is simple, this morphism is surjective. Moreover, the functor $P$ is a projective cover of $S$, so $P$ is isomorphic to a direct summand of $Y_G$. This implies that $P(G)$ is a direct summand of $\mathcal{D}(G, G)$. In particular, the evaluation $P(G)$ is a projective $\mathcal{D}(G, G)$-module. Moreover, the module $P(G)$ has a unique simple quotient $S(G)$, so it is indecomposable.

Remark 3.3. The results of Section 9 of [7] can by illuminated by this Lemma.

As corollary, we also have the following useful result.

Corollary 3.4. Let $G$ be a finite group. Let $k$ be a field. Then, the Burnside module $kB(G)$ is an indecomposable projective $kB(G, G)$-module. It is a projective cover of the simple $kB(G, G)$-module $S_{1,k}(G)$.

Proof. Since $kB$ is a representable functor in $\mathcal{F}_{\Sigma(G), k}$, it is projective. Moreover, we have $kB = L_{1,k}$. By the arguments detailed below Definition 2.5 this functor has a simple top which is isomorphic to $S_{1,k}$. Since the trivial group 1 is a quotient of the group $G$, one can write $\text{Id}_{G/G} = \text{Def}^G_{G/G} \circ \text{Inf}^G_{G/G}$. By definition, $S_{1,k}(G/G) \cong k$ and since the identity of $S_{1,k}(G/G)$ factorizes through $S_{1,k}(G)$, the last cannot be zero. So, the simple functor $S_{1,k}$ does not vanish at $G$. The result follows from Lemma 3.2.

Let $k$ be a field. As explained above, the simple biset functors are parametrized by the pairs $(H, V)$ where $H$ runs through the isomorphism classes of objects of $\mathcal{D}$ and $V$ runs through the isomorphism classes of simple $k\text{Out}(H)$-modules. Via evaluation at $G$, we have a classification of the simple $k\mathcal{D}(G, G)$-modules and their projective cover.

Corollary 3.5. Let $k$ be a field. Let $\mathcal{D}$ be an admissible biset category. The set of the $S_{H,V}(G)$ for $(H, V) \in \Lambda$ such that $S_{H,V}(G) \neq 0$ is a complete set of representatives of simple $\mathcal{D}(G, G)$-modules. Moreover, $P_{H,V}(G)$ is a projective cover of $S_{H,V}(G)$.
Proof. The first part is Corollary 3.3 which has to be adapted to the case of admissible biset categories, but one more time, this is straightforward. The second part is an obvious consequence of Lemma 3.2.

Finally we have another family of indecomposable biset functors.

**Corollary 3.6.** Let $k$ be a field. Let $\mathcal{D}$ be an admissible biset category. Let $G$ be a finite group. The $\mathcal{D}(G,G)$-modules $\Delta_{H,V}(G)$ are indecomposable for $(H,V) \in \Lambda$ such that $S_{H,V}(G) \neq 0$.

**Proof.** We know that $\Delta_{H,V}$ is a quotient of $P_{H,V}$. Since the evaluation functor is exact, we have that $\Delta_{H,V}(G)$ is a quotient of the indecomposable projective module $P_{H,V}(G)$. The result follows.

Obviously, this classification is not completely satisfying, as it is well known that understanding which simple biset functors vanish at $G$ is an extremely hard combinatorial problem (see [8] for a recent survey).

4 The generating relation

Since it seems very difficult to understand when a simple functor vanishes at a finite group $G$, we try to avoid this by considering finite groups such that there is no non-trivial vanishing of simple functors.

**Definition 4.1.** Let $k$ be a field. Let $G$ be a finite group. The group $G$ is a non-vanishing group over $k$ if none of the simple functors of $\mathcal{F}_{G,k}$ vanishes at $G$.

**Remark 4.2.** This is clearly equivalent to the fact that $S_{H,V}(G) \neq 0$, for every simple functor $S_{H,V}$ of $\mathcal{F}_{C,k}$ such that $H$ is isomorphic to a subquotient of $G$.

We use the notation $G$ is a NV$_{k}$-group if it is non-vanishing over a field $k$.

Let $H$ and $G$ be two finite groups. The composition in the biset category:

$$kB(H,G) \times kB(G,H) \to kB(H,H)$$

$$(H_U, G_W) \mapsto U \times_G W.$$

induces a morphism of $kB(H,H)$-modules. We will abusively denote by $kB(H,G)B(G,H)$ the image of this composition in $kB(H,H)$. It is the submodule of $kB(H,H)$ consisting of the linear combinations of elements of the form $W \times_G U$ for $W \in RB(H,G)$ and $U \in RB(G,H)$. Using this composition, we have an intrinsic understanding of the non-vanishing at $G$ of all the simple functors $S$ such that $S(H) \neq 0$.

**Proposition 4.3.** Let $k$ be a field. Let $G$ and $H$ be two finite groups of $\mathcal{D}$. The following are equivalent.

1. $S_{H,V}(G) \neq 0$ for every $kB(H,H)$-simple module $V$. 

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2. There is an isomorphism of $k\mathcal{B}(H, H)$-modules between $k\mathcal{B}(H, G)\mathcal{B}(G, H)$ and $k\mathcal{B}(H, H)$.

3. There exists $n \in \mathbb{N}^*$ and for $1 \leq i \leq n$ there are $U_i \in k\mathcal{B}(H, G)$ and $W_i \in k\mathcal{B}(G, H)$ such that
   \[
id_H = \sum_{i=1}^{n} U_i \times_G W_i.
   \]

Remark 4.4. It is important to remark that the family of the simple functors in the first point is not the family consisting of the simple functors with minimal group $H$. Indeed, here we consider all the simple functors indexed by $(H, V)$ where $V$ is a simple $k\mathcal{B}(H, H)$-module and not only a simple $k\text{Out}(H)$-module.

Proof. It is clear that 2 and 3 are equivalent. We show that 1 is equivalent to 2.

Let us assume 2. Since $V \cong S_{H,V}(H) \neq 0$, then the identity of $S_{H,V}(H)$ is non zero. By hypothesis $Id_H \in k\mathcal{B}(H, G)\mathcal{B}(G, H)$. So, this morphism factorizes through $S_{H,V}(G)$ which must be non zero.

Conversely, let $V$ be a simple $k\mathcal{B}(H, H)$-module. Since $S_{H,V}(G) \neq 0$, then $L_{H,V}(G) \neq J_{H,V}(G)$. By the formula (3), this means that there is an element $\sum i \phi_i \otimes v_i \in k\mathcal{B}(G, H) \otimes_{k\mathcal{B}(H, H)} V$ and an element $\psi \in k\mathcal{B}(H, G)$ such that $\sum (\psi \phi_i) \cdot v_i \neq 0$. So the action of the element $\phi \times_G (\sum \phi_i)$ is non zero on $V$. Since $V$ is a simple module, we have:
   \[
k\mathcal{B}(H, G)\mathcal{B}(G, H) \cdot V = V.
   \]

Since this holds for every simple module $V$, we have that $k\mathcal{B}(H, G)\mathcal{B}(G, H)$ is not contained in any maximal submodule of $k\mathcal{B}(H, H)$. So it must be equal to $k\mathcal{B}(H, H)$.

Definition 4.5. Let $k$ be a field. Let $G$ and $H$ be two finite groups. We say that $H$ is $k$-generated by $G$, if we have $k\mathcal{B}(H, G)\mathcal{B}(G, H) = k\mathcal{B}(H, H)$. In this case we use the notation $H \vdash_k G$. If the context is clear enough, we simply use the notation $H \vdash G$.

As immediate Corollary, we have an intrinsic definition of non-vanishing groups.

Proposition 4.6. Let $G$ be a finite group and $k$ be a field. Then, the following are equivalent.

- The group $G$ is $\text{NV}_k$.
- Every subquotient $H$ of $G$ is $k$-generated by $G$.

Proof. It is an easy consequence of Lemma 4.3.
Lemma 4.7. Let $k$ be a field. Let $\mathcal{D}$ be a full subcategory of the biset category. Let $G$ and $H$ be two groups of $\mathcal{D}$. Then the following are equivalent.

1. $H \vdash_k G$.
2. There exists $n \in \mathbb{N}^\ast$ such that $Y_H$ is a direct summand of $(Y_G)^m$.

Proof. It is an easy application of Yoneda’s Lemma. More precisely, if $H \vdash_k G$, then there exists $n \in \mathbb{N}^\ast$ and for $1 \leq i \leq n$ there are $U_i \in kB(H,G)$ and $W_i \in kB(G,H)$ such that

$$id_H = \sum_{i=1}^{n} U_i \times_G W_i.$$  

By Yoneda’s Lemma the biset $U_i$ induces via right multiplication a morphism $\phi_i$ between $Y_H$ and $Y_G$ and the biset $V_i$ induces via right multiplication a morphism $\psi_i : Y_G \to Y_H$. Moreover, the morphism $\sum_{i=1}^{n} \psi_i \circ \phi_i$ corresponds to the identity of $kB(H,H)$ via the isomorphism of Yoneda’s Lemma, so it is the identity of $Y_H$.

Conversely, if there exists $n \in \mathbb{N}$ such that $Y_H$ is a direct summand of $(Y_G)^n$, then for $i \in \{1, \ldots, n\}$ there are morphisms $\phi_i : Y_H \to Y_G$ and $\psi_i : Y_G \to Y_H$ such that:

$$id_{Y_H} = \sum_{i=1}^{n} \psi_i \circ \phi_i.$$  

Using Yoneda’s Lemma one more time, we see that $H \vdash_k G$. \qed

Now, it is not difficult to prove that the non-vanishing groups are exactly the finite groups $G$ such that the evaluation at $G$ induces an equivalence of categories between $\mathcal{F}_{G,k}$ and $kB(G,G)$-Mod.

Theorem 4.8. Let $G$ be a finite group. Let $k$ be a field. Then the following are equivalent.

1. $G$ is a NV$_k$-group.
2. $ev_G : \mathcal{F}_{G,k} \to kB(G,G)$-Mod is an equivalence of categories.

Proof. If the evaluation at $G$ is an equivalence of categories, it cannot kill a simple functor. So $G$ is a non-vanishing group.

Conversely, Lemma 4.7 implies that the representable functor $Y_G$ is a pro-generator of $\mathcal{F}_{G,k}$. By Morita’s Theorem, the functor $\text{Hom}_{\mathcal{F}_{G,k}}(RB_G, -)$ will be an equivalence of categories between $\mathcal{F}_{G,k}$ and $\text{End}_{\mathcal{F}_{G,k}}(RB_G)$. Moreover, Yoneda’s Lemma identifies this functor with the evaluation at $G$ and $\text{End}_{\mathcal{F}_{G,k}}(RB_G)$ with $kB(G,G)$.

Alternatively, we give a direct proof of the result without using Morita’s Theorem. The functor $L_G,$ is a left adjoint to the evaluation at $G$. Let $V$ be a $kB(G,G)$-module. The value at $V$ of the co-unit of the adjunction is the canonical isomorphism

$$V \cong kB(G,G) \otimes_{kB(G,G)} V.$$  

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Let $F$ be a $k$-biset functor over $G$. The value at $F$ of the unit is the following morphism. Let $X$ be a subquotient of $G$, then we have:

$$
\epsilon_F(X) : kB(X,G) \otimes_{kB(G,G)} F(G) \to F(X)
W \otimes f \mapsto F(W)(f).
$$

If $G$ is a NV$_k$-group, then by Proposition 4.3 there is an integer $n$ and some morphisms $U_1, \cdots, U_n \in kB(X,G)$, and $W_1, \cdots, W_n \in kB(G,X)$ such that

$$id_X = \sum_{i=1}^n U_i \times_G W_i.
$$

We let $\delta_F(X) : F(X) \to kB(X,G) \otimes_{kB(G,G)} F(G)$ to be the morphism defined by

$$\delta_F(X)(x) := \sum_{i=1}^n U_i \otimes F(W_i)(x) \text{ for } x \in F(X).$$

It is easy to check that $\delta_F(X)$ is an inverse isomorphism of $\epsilon_F(X)$.

Remark 4.9. The arguments developed here are much more general than the case of a replete biset category.

1. For example one can take for the category $\mathcal{D}$ a category of so-called left-free, or bi-free, biset functors. At least, over a field of characteristic zero it seems that there are a lot of non-vanishing groups for the left-free case. However, since the category $\mathcal{D}$ contains less morphisms, there are a lot of vanishing of simple functors when the characteristic of the field is non-zero, particularly for the bi-free case.

2. One the other hand, the opposite phenomenon can appear. In the context of correspondence functors recently developed by Bouc and Thévenaz (see [10] for lots of details ) every object of their category is generated by a larger object. So there are no non-trivial vanishing. As consequence, in this context if $X$ is a finite set, let $\mathcal{D}$ be the full subcategory of the category of correspondences consisting of all the sets smaller the $X$. Then, the evaluation at $X$ induces an equivalence between the category of $k$-linear functors from $\mathcal{D}$ to $k$-Mod and the category of modules over the algebra of all relations on $X$. This, together with results on the simple correspondences functors implies, and strengthen Theorem 4.1 of [9].

3. Another example of such equivalence appears in the context of Mackey functors. There are various possible definitions of Mackey functors, for example they can be defined as modules over the Mackey algebras as well as particular bivariant functors over the category of $G$-sets. Unfortunately the equivalence between the different definitions is rather technical. For a recent survey on theses equivalences see Section 2 of [18] or the first Sections of [21]. With Lindner’s definition (see [15] ) a Mackey functor is nothing but an additive functor from a category of spans of $G$-sets to the category of abelian groups. Every finite $G$-set $X$ is generated by the $G$-set $\Omega_G := \bigsqcup_{H \leq G} G/H$ in the sense of Definition 4.5. So the evaluation at $\Omega_G$ induces an equivalence of categories between the category of Mackey functors and
the category of modules over $B(\Omega^2_G)$ the algebra of endomorphisms of $\Omega_G$. This last algebra is known as the Mackey algebra introduced by Thévenaz and Webb in [21]. A similar result holds for cohomological Mackey functors and cohomological Mackey algebras (see Section 2 of [17] for more details about the different definitions of cohomological Mackey functors.)

The following result is now obvious but not uninteresting.

**Corollary 4.10.** Let $k$ be a field and $G$ a be NV$_k$-group. Let $F \in \mathcal{F}_{G,k}$ be a biset functor over $G$. Then $F \cong L_{G,F(G)}$.

**Proof.** Since $G$ is a non-vanishing group, the evaluation at $G$ is an equivalence of categories from $\mathcal{F}_{G,k}$ to $kB(G,G)$-Mod. Any quasi-inverse equivalence is isomorphic to the left adjoint of the evaluation $L_{G,-}$. \[\square\]

**Remark 4.11.** It is clear that this result is not valid if $G$ is a vanishing group. Indeed, let $S$ be a simple functor such that $S(G) = 0$. The functor $S$ is a non-zero functor, so it cannot be isomorphic to $L_{G,S(G)} = 0$. Still, we will use a weak degeneration of this result for the proof of Theorem 7.3.

More generally, if $G$ is a vanishing group, we have a situation of recollement:

**Proposition 4.12.** Let $k$ be a field. Let $\mathcal{D}$ be a replete biset category. Let $G \in \mathcal{D}$. We denote by $\mathcal{K}(G)$ the full subcategory of $\mathcal{F}_{\mathcal{D},k}$ consisting of the functors $F$ such that $F(G) = 0$. Then, we have a situation of recollement:

$$
\mathcal{K}(G) \leftarrow \mathcal{F}_{\mathcal{D},k} \rightarrow kB(G,G)\text{-Mod}.
$$

In particular, $kB(G,G)$-Mod $\cong \mathcal{F}_{\mathcal{D},k}/\mathcal{K}(G)$.

**Proof.** We give the different functors between these categories. The result will follow from straightforward verifications of the Axioms of [12]. The functor between $\mathcal{F}_{\mathcal{D},k}$ and $kB(G,G)$-Mod is the evaluation at $G$. It has a left adjoint $L_{G,-}$ and a right adjoint $L_{G,-}^0$ (see 3.3.5 of [6]). The functor between $\mathcal{K}(G)$ and $\mathcal{F}_{\mathcal{D},k}$ is the embedding functor. It has a left adjoint which sends a functor $F$ to its largest quotient which belongs in $\mathcal{K}(G)$. The right adjoint is the functor sending $F$ to its largest subfunctor belonging in $\mathcal{K}(G)$. \[\square\]

### 5 Some non-vanishing groups

In this section, we investigate basic properties of non-vanishing groups. In the first part we give an infinite list of non-vanishing groups. The groups of this list have the particularity of being non-vanishing over any field. Moreover, they are nilpotent (note that there are a lot of nilpotent groups that are vanishing).

Unfortunately we do not succeed to classify the non-vanishing groups. Our problem is in theory much easier than the problem of understanding the vanishing problem of simple
functors. Indeed we ask that none of the simple functors vanishes at \( G \). In particular, this is equivalent to a problem involving composition of bisets (see Proposition \ref{prop:vanishing}). However, it seems that it is still difficult to understand this problem. At the end of this section we construct a non-trivial example of non-vanishing group over a field of characteristic zero. We will see that the non-vanishing property of this group comes from some non trivial results about the representations of \( GL(3,2) \). In particular this example is so pathologic that it seems very improbable to find a classification. Moreover, this example shows that the family of non vanishing groups contains some non-nilpotent groups and is not closed under taking subgroups and subquotients.

Let us start this investigation by looking at the relation \( \rhd \).

**Lemma 5.1.** Let \( k \) be a field.

- The relation \( \rhd_k \) is reflexive, and transitive.
- If \( G \) and \( H \) are two groups such that \( H \rhd_k G \) then \( H \) is isomorphic to a subquotient of \( G \).

**Proof.** Let \( G, H \) and \( K \) be three finite groups. It is clear that \( G \rhd_k G \). Now, let us assume that \( K \rhd_k H \) and \( H \rhd_k G \). One can use the equivalent assertions of Lemma \ref{lem:relation} in order to show that \( K \rhd_k G \). Alternatively, one can use the equivalent characterisation of Lemma \ref{lem:relation}. That is \( K \rhd_k H \) if and only if there is an integer \( n \) such that \( Y_K \mid (Y_H)^n \) in the category of biset functors.

For the second point, if \( H \rhd_k G \), then \( kB(H,G)B(G,H) = kB(H,H) \). Let us denote by \( I(H,H) \) the submodule of \( kB(H,H) \) consisting of all the \( H-H \)-bisets factorising strictly below \( H \). Then \( kOut(H) \cong kB(H,G)B(G,H)/I(H,H) \). In particular, the last quotient is non zero. Moreover, by Bouc’s butterfly lemma (See formula (2)) any element in \( kB(H,G) \) is a \( k \)-linear combination of transitive bisets which factors through subquotients of \( H \) and \( G \). So if \( H \) is not a subquotient of \( G \), then every \( H-G \)-biset factorizes through a proper subquotient of \( H \), and hence is zero in the quotient. \( \square \)

**Lemma 5.2.** Let \( k \) be a field. If \( G \) is an abelian group, then \( G \) is a NV\(_k\)-group.

**Proof.** This is Proposition 3.2 of \cite{ref}. Since the argument is both crucial and easy we recall it. We have to prove that for every subquotient \( H \) of \( G \) and for every simple \( kOut(H) \)-module, we have \( S_{H,V}(G) \neq 0 \). Since the group \( G \) is abelian, every subquotient is isomorphic to a quotient of \( G \). Now if \( H = G/N \) for a normal subgroup \( N \) of \( G \), we have \( id_H = Def_{G/N}^G \inf_{G/N}^G \).

More generally, this argument can be used in order to prove the following result.

**Corollary 5.3.** Let \( G \) be a finite group such that every subgroup is isomorphic to a quotient of \( G \). Then \( G \) is a NV\(_k\)-group for every field \( k \).

**Proof.** By hypothesis, every subgroup of \( G \) is isomorphic to a quotient of \( G \), so it is generated by \( G \). Let’s assume that for every \( H \) subgroup of \( G \) we have \( H \rhd_k G \). Let \( H = B/A \) be a subquotient of \( G \). Then \( H \) is generated by \( B \), and by hypothesis \( B \)}
is generated by $G$. So by transitivity of the relation $\vdash_k$, the group $H$ is generated by $G$.

Such a finite group is called a $s$-self dual group and these groups have been completely classified in [1]. This classification involves two families of finite $p$-groups. Let $p$ be a prime number, then we denote by $X_{p^3}$ an extra-special $p$-group of exponent $p$. Let $n$ be an integer, then we denote by $M_p(n,n)$ the finite group $<a, b \mid a^{p^n} = b^{p^n} = 1, bab^{-1} = a^{1+p^{n-1}}>$.

The following theorem summarizes the classification in [1].

**Theorem 5.4.** Let $G$ be a finite group.

1. The group $G$ is $s$-self dual if and only if $G$ is nilpotent and all Sylow subgroups of $G$ are $s$-self dual.

2. Let $p$ be a prime number. Let $P$ be a finite $p$-group. Then $P$ is $s$-self dual if and only if $P$ is:

   • $P$ is abelian.
   • $P \cong X_{p^3} \times M$ where $M$ is an abelian $p$-group with $\exp(M) \leq p$ when $p$ is an odd prime number.
   • $P \cong M_p(n,n) \times M$ where $M$ is an abelian $p$-group with $\exp(M) < p^n$.

**Proof.** Theorem 7.3 together with Theorem 7.1 of [1].

As consequence the list of non-vanishing groups includes all the abelian groups and this particular family of nilpotent groups. However, in order to exhibit this list, we only used the trivial fact that if $H$ is isomorphic to a quotient of $G$, then $S_{H,V}(G) \neq 0$ for any simple $k\text{Out}(H)$-modules. As it can be seen in Section 3 of [8] there are some less trivial non-vanishing properties involving the geometry of the sections in a finite group. So we cannot hope to have construct all the non-vanishing groups at this stage.

There is a formula for the dimension of the evaluation at a finite group $G$ of the simple functor $S_{H,V}$ in Theorem 7.1 of [7]. This formula involves the computation of a bilinear form which is construct by using the character of the simple $k\text{Out}(H)$-module $V$. In particular, the vanishing or the non-vanishing at $G$ of this simple functors should depend on the simple module $V$ and on the field $k$. Nevertheless, if we are interested in the non vanishing at $G$ of all the simple functors having $H$ as minimal group, then all the sufficient conditions that we know only involve the finite group $G$. More precisely in all these cases $id_H$ is written as product of two transitive bisets (see Section 3 of [8] for more details). In particular, these results are true for any field (even over the ring of integers).

So it seems natural to ask the following two questions:

**Question 5.5.** Let $k$ be a field. Let $G$ be a NV$_k$-group. Let $H$ be a subquotient of $G$. Is it always possible to find an element $U \in kB(H,G)$ and an element $V \in kB(G,H)$ such that $id_H = U \times_G V$?
This question is not hopeless since the corresponding question has a positive answer for the category of finite sets with correspondences (See Lemma 4.1 of [10]).

**Question 5.6.** Let \( k \) be a field and \( G \) be a finite group. If \( G \) is non-vanishing over the field \( k \), is it non-vanishing over any field?

We are going to give a negative answer to these two naive questions. More precisely, we give an example of a non-vanishing group \( G \) over a field of characteristic zero with a subquotient \( H \) such that \( \text{id}_H \) is not a product of two elements. Moreover, this example shows that the structure of the simple modules for the group algebra of the outer automorphism of the subquotients of \( G \) is also involved in the vanishing or non-vanishing property of \( G \). In particular, this group will not be non-vanishing over any field. We start by a useful general lemma about non-vanishing groups.

**Lemma 5.7.** Let \( G \) be a finite group. Let \( k \subset K \) be a field extension. Then \( G \) is NV\(_k\) if and only if \( G \) is NV\(_K\).

**Proof.** By Lemma [4,3], the group \( G \) is NV\(_k\) if and only if \( kB(H,G)B(G,H) = kB(H,H) \) for every \( H \subseteq G \). That is, if and only if the \( kB(H,H) \)-modules \( kB(H,G)B(G,H) \) and \( kB(H,H) \) are isomorphic for every \( H \subseteq G \). Since \( K \otimes_k kB(H,H) = KB(H,H) \) and \( K \otimes_k kB(H,G)B(G,H) = KB(H,G)B(G,H) \), the result follows from Noether Deuring’s Theorem.

We will also use the following result in order to compute the dimension of some evaluations of some simple functors.

**Theorem 5.8 (Bouc).** Let \( k \) be a field of characteristic 0. Let \( p \) be a prime number and \( P \) be a finite \( p \)-group which is not 1 or \( C_p \times C_p \). Let \( G \) be a finite group. Then, \( \dim_k S_{P,k}(G) \) is the number of conjugacy classes of sections \((T,S)\) of \( G \) such that \( T/S \cong P \) and \( T \) is the direct product of a \( p \)-group and a cyclic group.

**Proof.** See the main Theorem of [5].

**Lemma 5.9.** Let \( k \) be a field of characteristic 0. Let \( G = A_4 \times C_2 \) and \( H = C_2 \times C_2 \times C_2 \). Then \( \text{id}_H \) cannot be written as a product of an element of \( kB(H,G) \) and an element of \( kB(H,H) \).

**Proof.** Let us first remark that the fact that \( \text{id}_H \) is the product of an element of \( kB(H,G) \) and an element of \( kB(H,H) \) is equivalent to the fact that \( Y_H \) is a direct summand of \( Y_G \) in the category of biset functors. Moreover, by standard arguments, if \( X \) is a finite group, then the decomposition of the Yoneda functor \( Y_X \) as direct sum of indecomposable projective is given by:

\[
Y_X \cong \bigoplus_{(K,W) \in \Lambda} P_{K,W}^{n_{K,W}(X)},
\]

where \( P_{K,W} \) is a projective cover of \( S_{K,W} \) and

\[
n_{K,W}(X) = \dim_k S_{K,W}(X)/\dim_k \text{End}(W).
\]
So \( Y_H \) is a direct summand of \( Y_G \) if and only if for every simple functor \( S_{K,W} \) we have:

\[
\dim_k S_{K,W}(H) \leq \dim_k S_{K,W}(G).
\]

Now, we claim that we have \( \dim_k S_{C_2,k}(G) = 14 \) and \( \dim_k S_{C_2,k}(H) = 35 \) so \( Y_H \) is not a direct summand of \( Y_G \). It remains to prove the claim. These computations can be, in theory, done by using the arguments of Paragraph 7 of [7]. However, the bilinear forms will be so huge that it is not reasonable to give a full proof of this fact. Instead, we use Theorem 5.8. Since there are 35 sections \( C_2 \) in \( C_2 \times C_2 \times C_2 \), we have that \( \dim_k S_{C_2,k}(H) = 35 \). It is not difficult to check that there are 15 conjugacy classes of sections \( C_2 \) in \( A_4 \times C_2 \): 3 sections \( (C_2,1) \), 1 section \( (C_6,C_3) \), 7 sections \( (C_2 \times C_2,C_2) \), 3 sections \( ((C_2)^3,(C_2)^2) \) and 1 section \( (A_4 \times C_2,A_4) \). However, since \( A_4 \times C_2 \) is not the direct product of a 2-group and a cyclic group, we have to discount the conjugacy class of the section \( (A_4 \times C_2,A_4) \). Finally, we have \( \dim_k S_{C_2,k}(G) = 14 \).  

The next step is to check that \( A_4 \times C_2 \) is a non-vanishing group over a field of characteristic zero. Actually, we prove that it is non-vanishing over any field of characteristic different from 3.

**Lemma 5.10.** Let \( G = A_4 \times C_2 \) and let \( k \) be a field. The group \( G \) is a \( NV_k \)-group if and only if \( \text{char}(k) \neq 3 \).

**Proof.** Using the fact that the relation \( \vdash := \vdash_k \) is transitive, it is enough to check that \( H \vdash G \) for every subgroup \( H \) of \( G \). Up to isomorphism, the subgroups of \( G \) are: 1, \( C_2, C_3, C_2 \times C_2, C_6, C_2 \times C_2 \times C_2, A_4 \) and \( A_4 \times C_2 \). For our purpose it is enough to look at subgroups which are not isomorphic to a quotient of \( G \). It remains \( C_2 \times C_2 \) and \( H := C_2 \times C_2 \times C_2 \). Moreover, the group \( C_2 \times C_2 \) is a quotient of \( H \), so by transitivity of \( \vdash \), it is enough to check that \( H \vdash G \). So we want to check that

\[
\text{id}_H \in kB(H,G)kB(G,H),
\]

but in this case the identity is not a product of bisets, and it is rather technical to check this without the help of a computer. However, this is equivalent to check that for every simple \( k\text{Out}(H) \)-module \( V \) we have \( S_{H,V}(G) \neq 0 \). Since \( \Sigma_H(G) = \{(H,1)\} \), by Proposition 7.1 of [8], we have:

\[
S_{H,V}(G) \cong \text{Tr}^{G/H}_1(V),
\]

where \( G/H \) acts on \( V \) via conjugation.

Since \( G/H \) is a cyclic group of order 3, when \( k \) is a field of characteristic 3, then \( S_{H,k}(G) = 0 \) and \( G \) is a vanishing group. Now let us suppose that \( k \) is a field of characteristic \( p \neq 3 \). By Lemma 5.7, we can assume the field to be algebraically closed. Let \( \Gamma \) be the subgroup of \( GL_3(\mathbb{F}_2) \) consisting of \( \text{conjug} \), for \( g \in G/H \). Let \( V \) be a simple \( k\text{GL}_3(\mathbb{F}_2) \)-module. Since \( \Gamma \) is a group of order 3, the group algebra \( k\Gamma \) is semisimple. So \( \text{Res}_{\Gamma}^{GL_3(\mathbb{F}_2)}(V) \) is a semisimple module. It is a direct sum of the trivial \( k\Gamma \)-module and the two other simple modules of dimension 1. The space of fixed point of either
of the two non trivial simple modules is 0. In conclusion $V^\Gamma \neq \{0\}$ if and only if the trivial $k\Gamma$-module is a direct summand of $\text{Res}^{GL_3(\mathbb{F}_2)} V$. Since the trace map is surjective, $S_{H,V}(G) \neq 0$ if and only if $k \mid \text{Res}^{GL_3(\mathbb{F}_2)} V$. We did not find a structural reason for this fact, and as far as we known, it is maybe just a coincidence for the group $GL_3(\mathbb{F}_2)$ and its subgroup of order 3.

There are basically three cases: $p = 0$, $p = 2$ and $p = 7$. This can be done by looking at the ordinary character table for $p = 0$ and for $p = 2$ and $p = 7$, one can look at the Brauer character tables of $GL_3(\mathbb{F}_2)$. For our purpose, we only need to have the values of the characters on the elements of order 1 and 3. All the elements of order 3 are conjugate in $G$. We let $x$ to be an element of order 3. We have the following tables, where the characters are written horizontally.

| (1) | 1 | 3 | 3 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|
| (x) | 1 | 0 | 0 | 0 | 1 | -1 |

For the prime $p = 2$ we have the following table:

| (1) | 1 | 3 | 3 | 8 |
|-----|---|---|---|---|
| (x) | 1 | 0 | 0 | -1 |

And finally, for $p = 7$, we have:

| (1) | 1 | 3 | 5 | 7 |
|-----|---|---|---|---|
| (x) | 1 | 0 | -1 | 1 |

In the three cases, it is easy to check that the trivial module appears at least ones in $\text{Res}^{GL(3,2)} V$ for every simple $kGL(3,2)$-module $V$.

As corollary, of this example, we have:

**Corollary 5.11.** Let $k$ be a field. The family of $\text{NV}_k$-groups is not closed under taking subgroups or taking quotients. Moreover, it contains non nilpotent groups.

**Proof.** $A_4$ is a subgroup and a quotient of $A_4 \times C_2$. We just have to show that $A_4$ is a vanishing group for every field. Let $C_2 \times C_2$ be the subgroup of order 4 of $A_4$. It is easy to see that $\Sigma_{C_2 \times C_2}(A_4) = \{(C_2 \times C_2, 1)\}$. So if $V$ is a simple $k\text{Out}(C_2 \times C_2)$-module, by Proposition 7.1 of [8], we have:

$$S_{H,V}(G) \cong \text{Tr}_{1}^{N_{G}(T,S)}(V).$$

Here $\text{Out}(C_2 \times C_2) \cong S_3$ and $N_{G}(T,S)$ is the subgroup of $S_3$ of order 3.

1. If $\text{char}(k) = 0$ or 2. Let $V$ be the simple $kS_3$-module of dimension 2. It is easy to check that $\text{Tr}_{1}^{C_3}(V) = 0$.

2. If $\text{char}(k) = 3$, then $\text{Tr}_{1}^{C_3}(k) = 0$.  

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So in every cases, the group $G$ is a vanishing group.

Finally, we summarise the known results over the non-vanishing groups.

**Theorem 5.12.**

1. The abelian groups and, more generally, the $s$-self dual groups are non-vanishing for any field.

2. There are other examples of non-vanishing groups over a field of characteristic 0. However, it is not known whether there are other examples of groups that are non-vanishing for any field.

3. The family of non-vanishing group over a field $k$ is not closed under taking subgroups or quotients.

### 6 Around highest-weight structure.

We recall the famous Theorem of Peter Webb about the Highest-weight structure of the category of biset functors.

**Theorem 6.1** (Webb). Let $\mathcal{D}$ be an admissible category in the sense of Definition 4.1.3 of [6]. Let $k$ be a field such that $\text{char}(k) \nmid |\text{Out}(H)|$ for every $H \in \mathcal{D}$. If $\mathcal{D}$ has only finitely many isomorphism classes of objects, then the category $\mathcal{F}_{\mathcal{D},k}$ is a highest-weight category.

**Proof.** This is a reformulation of Theorem 7.2 of [22]. The standard objects are given by the functors $\Delta_{H,V}$ where $S_{H,V}$ is a simple functor of $\mathcal{D}$. Our context is slightly more general than the one of Webb. Indeed, his theorem is stated in terms of globally defined Mackey functors. These correspond to some particular admissible biset categories. Nevertheless, it is straightforward to check that his result can be extended to our more general situation. Moreover, one can avoid the counting arguments of Theorem 6.3 of [22] by a systematic use of this functorial definition of the standard objects.

The representation theory of quasi-hereditary algebra is philosophically close to the representation theory of semi-simple Lie algebras. One very important feature of semi-simple Lie algebra is the existence of so-called Borel subalgebras. This notion has been generalized to arbitrary quasi-hereditary algebras by König in [14] and seems to be very important in recent development of the theory. The key result, which is highly non trivial, is that for every quasi-hereditary algebra $A$, there is an algebra $A'$ that is Morita equivalent to $A$ such that $A'$ has an exact Borel subalgebra (See Corollary 1.3 of [13]).

In the rest of the section, we describe the category of modules over an exact Borel subalgebra of the biset functor category.

**Definition 6.2** (Definition 2.2 of [13]). Let $(A, \leq)$ be a quasi-hereditary algebra with $n$ simple modules. Then, a subalgebra $B \subseteq A$ is called an exact Borel subalgebra if:

1. The algebra $B$ has also $n$ simple modules denoted by $L_B(i)$ for $i \in \{1, \cdots, n\}$, and $(B, \leq)$ is a quasi-hereditary algebra with simple standard modules.
2. The induction functor $A \otimes_B -$ is exact.

3. There is an isomorphism $A \otimes_B L_B(i) \cong \Delta_A(i)$. Here $\Delta_A(i)$ denotes the standard module with weight $i$ of $A$.

Note that the second point of the definition is nothing but an analogue of the PBW-Theorem.

The category of biset functors is equivalent to a category of modules over an algebra $A$ (without 1 in general). But, in this paper we are more interested by the category of biset functors and not very interested by the choice of an underline algebra. So, if there is a subcategory of $\mathcal{F}_{D,k}$ which is equivalent to the category of modules over an exact Borel subalgebra of $A$, I will abusively say that this category is an exact Borel subcategory.

If $\mathcal{D}$ is a replete biset category, we denote by $\mathcal{D}_0$ the following admissible biset category. The object of $\mathcal{D}_0$ are the object of $\mathcal{D}$. Now, if $H$ and $K$ are two groups of $\mathcal{D}$, then $\text{Hom}_{\mathcal{D}_0}(H, K)$ is given by the left-free double Burnside algebra. That is, we have forgotten the inflation in the five elementary bisets. The biset functors over $\mathcal{D}_0$ are sometimes called deflation functors.

**Lemma 6.3.** Let $\mathcal{D}$ be a replete biset category and let $\mathcal{D}_0$ as above. Let $H \in \mathcal{D}$ and $F \in \mathcal{F}_{\mathcal{D},k}$. We write $\text{Res}_{\mathcal{D}_0}^\mathcal{D}$ the forgetful functor from $\mathcal{F}_{\mathcal{D},k}$ to $\mathcal{F}_{\mathcal{D}_0,k}$. Then, we have an isomorphism, natural in $F$, of $k\text{Out}(H)$-module

$$\text{Res}_{\mathcal{D}_0}^\mathcal{D} F(H) \cong F(H).$$

**Proof.** Let $F \in \mathcal{F}_{\mathcal{D},k}$. Then, by definition we have:

$$F(H) = \bigcap_{K \leq H} \text{Ker}(F(U)).$$

We claim that we have:

$$\text{Res}_{\mathcal{D}_0}^\mathcal{D} F(H) = \bigcap_{(B,A) \in \Sigma(H)} \text{Ker} \left( F(\text{Def}_{B/A} \circ \text{Res}_B^H) \right). \quad (4)$$

It is clear that $F(H)$ is a subset of the right hand side term. Conversely, let $x$ be an element of the right hand side. Let $K$ be a strict subquotient of $H$ and $U \in kB(K,H)$. Then $U$ is a linear combination of transitive bisets $(K \times H)/L$. Moreover, by Bouc’s Butterfly decomposition we have:

$$(K \times H)/L \cong \text{Ind}_D^K \circ \text{Inf}_{D/C}^D \circ \text{Iso}(\alpha) \circ \text{Def}_{B/A}^B \circ \text{Res}_B^H,$$

where $(B,A)$ is a particular section of $H$ and $(D,C)$ is a particular section of $K$. It is clear that $B/A$ is a strict subquotient of $H$ as it is isomorphic to a subquotient of $K$. Now $\text{Def}_{B/A}^B \circ \text{Res}_B^H$ kills $x$ by assumption, so $(K \times H)/L$ kills $x$. This proves that $x \in \text{Ker}(F(U))$, and the equality (4) holds. The same result holds for $\text{Res}_{\mathcal{D}_0}^\mathcal{D} F$, as the butterfly decomposition is still correct in the category $\mathcal{D}_0$. The only difference is that there are no inflation in this decomposition. \qed
**Theorem 6.4.** Let $\mathcal{D}$ be a replete biset category with only finitely many isomorphism classes of objects. Let $k$ be a field such that $\text{char}(k) \nmid |\text{Out}(H)|$ for $H \in \mathcal{D}$. Then, the category $\mathcal{F}_{\mathcal{D}_0,k}$ is an exact Borel subcategory of $\mathcal{F}_{\mathcal{D},k}$.

**Proof.** By Webb’s Theorem 6.1 under these hypothesis the category $\mathcal{F}_{\mathcal{D}_0,k}$ is a highest-weight category. Moreover, by Proposition 9.1 of [22], all the standard functors $\Delta_{\mathcal{D}_0}^D$ are simple. Let $V$ be an arbitrary $k\text{Out}(H)$-module. Let $F$ be any functor in $\mathcal{F}_{\mathcal{D},k}$. Then, using successive adjunctions and the isomorphism of Lemma 6.3 we have:

$$\text{Hom}_{\mathcal{F}_{\mathcal{D},k}} \left( l\text{Ind}_{\mathcal{D}_0}^D (\Delta_{\mathcal{D}_0}^D), F \right) \cong \text{Hom}_{\mathcal{F}_{\mathcal{D}_0,k}} \left( \Delta_{\mathcal{D}_0}^D, \text{Res}_{\mathcal{D}_0}^D F \right),$$

$$\cong \text{Hom}_{k\text{Out}(H)} \left( V, \text{Res}_{\mathcal{D}_0}^D(F)(H) \right),$$

$$\cong \text{Hom}_{k\text{Out}(H)} \left( V, F(H) \right),$$

$$\cong \text{Hom}_{\mathcal{F}_{\mathcal{D},k}} \left( \Delta_{\mathcal{D}}^D, F \right).$$

Since this holds for any functor $F$, we have that $l\text{Ind}_{\mathcal{D}_0}^D$ sends the standard functors for the category $\mathcal{D}_0$ to the standard functors in $\mathcal{F}_{\mathcal{D},k}$.

It remains to check that the functor $l\text{Ind}_{\mathcal{D}_0}^D$ is exact. For that we use the description of this particular induction due to Rosalie Chevalley. See Section 5.1 of [11] for more details. Let $F$ be a deflation functor, then she proved (under slightly stronger, but not important for the proof, hypothesis on the field and the category $\mathcal{D}$) that

$$(l\text{Ind}_{\mathcal{D}_0}^D F)(G) = \left( \bigoplus_{H \leq G} F(N_G(H)/H) \right)_{\text{G}} \cong \bigoplus_{[H \leq G]} F(N_G(H)/H)_{N_G(H)}.$$

Here $[H \leq G]$ denotes a set of representatives of conjugacy classes of conjugacy classes of subgroups of $G$, and the group $N_G(H)$ acts on $F(N_G(H)/H)$ via conjugation. However, the action of $N_G(H)$ over $F(N_G(H)/H)$ is trivial. So, we have:

$$(l\text{Ind}_{\mathcal{D}_0}^D F)(G) \cong \bigoplus_{[H \leq G]} F(N_G(H)/H).$$

In particular, it becomes clear that the functor $l\text{Ind}_{\mathcal{D}_0}^D$ is exact. \hfill \Box

For the double Burnside algebra, the situation is more complicated.

**Theorem 6.5.** Let $G$ be a finite group. Let $k$ be a field such that $\text{char}(k) \nmid |\text{Out}(H)|$ for $H \leq G$.

1. The left-free double Burnside algebra $kB_0(G,G)$ is a quasi-hereditary algebra with simple standard modules.

2. If $G$ is a non-vanishing group, then the double Burnside algebra is quasi-hereditary. However, the left-free double Burnside algebra $kB_0(G,G)$ is not always an exact Borel subalgebra.
Proof. 1. We denote by $\Sigma(G)_0$ the subcategory of the biset category consisting of the subquotients of $G$ and where the morphisms are given by the left-free double Burnside algebra. In general the category of modules over $kB_0(G, G)$ is not equivalent to the category $\mathcal{F}_{\Sigma(G)_0,k}$. Still, by Webb’s Theorem 6.4, the category $\mathcal{F}_{\Sigma(G)_0,k}$ is a highest-weight category with simple standard functors. Let $\Lambda$ be the set of $S_{H,V}(G)$ where $S_{H,V}$ runs a set of representatives of simple functors of $\mathcal{F}_{\Sigma(G)_0,k}$ such that $S_{H,V}(G) \neq 0$. Then, by the arguments of Lemma 3.2, the set $\Lambda$ is a complete set of representatives of simple modules over the left-free double Burnside algebra. Moreover, if $P_{H,V}$ is a projective cover of $S_{H,V}$, then $P_{H,V}(G)$ is a projective cover of $S_{H,V}(G)$. The standard modules will be the evaluation at $G$ of the standard functors $\Delta_{H,V}$ such that $S_{H,V}(G) \neq 0$. By hypothesis, there are subfunctors $0 = M_0 \subset M_1 \subset M_2 \subset \cdots M_n = P_{H,V}$ such that $M_i/M_{i-1} \cong \Delta_{H_i,V_i}$ where $V_i$ is a simple $k\text{Out}(H_i)$-module and $H_i$ is a strict subquotient of $H$. Since the evaluation functor is exact, we have a filtration of $P_{H,V}(G)$. Moreover, $M_i(G)/M_{i-1}(G) \cong \Delta_{H_i,V_i}(G) = \overline{S}_{H_i,V_i}(G)$. This shows that the sequence of the $M_i(G)$ is not a sequence of strict submodules. We let $N_0 \subset N_1 \subset \cdots N_k = P_{H,V}(G)$ to be the filtration obtained by removing the multiple terms. Then by construction $N_i/N_{i-1} \cong \Delta_{H_i,V_i}(G)$ for some simple module $S_{H_i,V_i}(G)$ such that $H_j$ is a strict subquotient of $H$. This implies that every projective indecomposable module is filtered by the standard modules. Moreover the standard modules that appear in a filtration of the projective indecomposable module $P_{H,V}(G)$ are indexed by groups $K$ such that $K$ is isomorphic to a strict subquotient of $H$. The standard modules are simple, so the left-free double Burnside algebra is a quasi-hereditary algebra with simple standard modules.

2. See Theorem 2.1 of [19] for a proof without using the equivalence of Theorem 4.8.

In this case the evaluation at $G$ is an equivalence of categories between $\mathcal{F}_{G,k}$ and $kB(G,G)\text{-Mod}$. The result follows from the fact that under these hypothesis, the category $\mathcal{F}_{G,k}$ is a highest-weight category. If $G$ is a non-vanishing group, this result does not implies that the left-free double Burnside algebra is an exact Borel subalgebra of $kB(G,G)$. Indeed the last one may have more simple objects. For example if $G$ is a s-self dual group which is not self dual. That is a group such that every subgroup is isomorphic to a quotient but there is a quotient $H$ which is not isomorphic to a subgroup. This is clearly a non-vanishing group. The left-free bisets of $B_0(G, H)$ are linear combinations of transitive bisets of the form

$$\text{Ind}_C^G \circ \text{Iso}(f) \circ \text{Def}_B^{B/A} \circ \text{Res}_H^f,$$

where $(B, A)$ is a section of $H$, $C$ is a subgroup of $G$ and $f$ is an isomorphism from $B/A$ to $C$. Since $H$ is not isomorphic to a subgroup of $G$, then $B/A$ has to be a strict subquotient of $H$. In particular the quotient of $kB_0(G, H)$ by the ideal consisting of the morphism factorizing strictly below $H$ is zero. This implies that for any simple $k\text{Out}(H)$-module $V$, we have $\Delta_{H,V}^{\Sigma(G)_0}(G) = 0$. But by Theorem 6.4.
the simple deflation functor $S_{H,V}^{\Sigma(G)}$ is isomorphic to the corresponding standard functor. In particular, we have $S_{H,V}^{\Sigma(G)}(G) = 0$.

As corollaries we have:

**Corollary 6.6.** Let $G$ be a finite group. Let $k$ be a field such that $|\text{Out}(H)|$ is invertible in $k$ for every subquotient $H$ of $G$. Then, the global dimension of the left-free double Burnside algebra $kB_0(G,G)$ is finite.

For the double Burnside algebra, we need more hypothesis.

**Corollary 6.7.** Let $G$ be a finite group. Let $k$ be a field such that $|\text{Out}(H)|$ is invertible in $k$ for every subquotient $H$ of $G$. Then, if $G$ is a NV$_k$-group we have:

1. The global dimension of $kB(G,G)$ is finite.
2. The Cartan matrix of $kB(G,G)$ has determinant 1.

The fact that the double Burnside algebra of a non-vanishing group is quasi-hereditary is a direct consequence of the fact that the category of biset functors is a highest-weight category. However, if the group $G$ is a vanishing group, one can wonder if the double Burnside algebra is still quasi-hereditary. In particular, it may be possible to find an better order on the set of simple modules over the double Burnside algebra in order to avoid the vanishing problems. In [19], we showed that the global dimension of $CB(A_5,A_5)$ is infinite. In particular, this shows that such a better ordering does not exist for the double Burnside algebra for $A_5$. We have.

**Proposition 6.8.** The double Burnside algebra $CB(A_5,A_5)$ is not quasi-hereditary.

**Proof.** Proposition 3.3 of [19].

### 7 Semi-simplicity revisited.

In this section we revisit the semi-simple property of the double Burnside algebra and the category of biset functors. By results of Barker and Bouc, we know precisely when both objects are semi-simple. It is clear that the result of Barker on the category of biset functors implies the result of Bouc about the double Burnside algebra. Curiously, Barker’s result can be reformulated as follows. Let $\mathcal{D}$ be a replete biset category and $k$ be a field. Then, the category $\mathcal{F}_{\mathcal{D},k}$ is semi-simple if and only if the endomorphism algebra of every object of $\mathcal{D}$ is semi-simple. In general, it is easy to find a category where all the endomorphism algebras of objects are semi-simple but the its category of representations is not semi-simple. Here we show that this phenomenon is related to the generating relation. As corollary, this gives a rather simple proof of Barker’s Theorem. We also give a useful characterization of the semi-simple property in terms of the so-called trivial object. More precisely, we show that similarly to the case of group
algebras, these categories are semi-simple if and only if the trivial object is projective. This characterization will be used in the last Section of this article.

We start this section by recalling when the double Burnside algebra is semi-simple.

**Theorem 7.1** (Bouc). Let $k$ be a field and $G$ be a finite group. The double Burnside algebra is semi-simple if and only if $G$ is cyclic and $\text{char}(k) \nmid \phi(|G|)$.

**Proof.** Proposition 6.1.7 of [6].

We need the following Lemma.

**Lemma 7.2.** Let $k$ be a field. Let $D$ be a replete biset category. Let $H$ and $K$ be two groups such that $H$ is $k$-generated by $K$. Let $V$ be a $kB(H,H)$-module. Then,

$L_{K,LH,V(K)} \cong L_{H,V}$.

**Proof.** The co-unit of the adjunction between the evaluation at $K$ and $L_{K,-}$ gives a morphism from $L_{K,LH,V(K)}$ to $L_{H,V}$. This morphism $\phi$ is defined on a group $G$ by

$\phi_G\left(W \otimes (U \otimes v)\right) = (W \times_K U) \otimes v,$

for $W \in kB(G,K)$, $U \in kB(K,H)$ and $v \in V$. Since $H$ is generated by $K$, there are $U_i \in kB(H,K)$ and $W_i \in kB(K,H)$ for $i = 1, \cdots, n$ such that $id_H = \sum_{i=1}^n U_i \times_K W_i$.

We define $\psi_G : kB(G,H) \otimes_k V \to L_{K,LH,V(K)}$ by

$\psi_G(U \otimes v) = \sum_{i=1}^n (U \times_H U_i) \otimes (W_i \otimes v),$

where $U \in kB(G,H)$ and $v \in V$. If $\alpha \in kB(H,H)$, we have:

$\psi_G\left((U \times_H \alpha) \otimes v\right) = \sum_{i=1}^n (U \times_H \alpha \times_H U_i) \otimes (W_i \otimes v),$

$= \sum_{i=1}^n \left( U \times_H \left( \sum_{j=1}^n U_j \times_K W_j \alpha \times_H U_i \right) \otimes (W_i \otimes v) \right),$

$= \sum_{i,j=1}^n (U \times_H U_j) \otimes \left( (W_j \times_H \alpha \times_H U_i \times_K W_i) \otimes v \right),$

$= \sum_{j=1}^n (U \times_H U_j) \otimes \left( W_j \times_H \alpha \times_H \left( \sum_{i=1}^n U_i \times_K W_i \right) \otimes v \right),$

$= \psi_G\left(U \otimes (\alpha \times_H v)\right).$

So $\psi$ can be factorized as morphism from $kB(G,H) \otimes_k kB(H,H) V$ to $L_{K,LH,V(K)}$. Moreover, it is clear that $\phi_G$ and $\psi_G$ are two inverse isomorphisms. \qed
For the category of biset functors, we have the following Theorem. Barker’s theorem (Theorem 1 of [2]) is exactly the equivalence of 1. and 3. with slightly stronger hypothesis on the characteristic of the field $k$.

**Theorem 7.3** (Barker). Let $k$ be a field. Let $\mathcal{D}$ be a replete biset category. Then, the following are equivalent.

1. The category $\mathcal{F}_{D,k}$ is semi-simple.
2. For every group $H \in \mathcal{D}$, the algebra $kB(H,H)$ is a semi-simple algebra.
3. Every group $H$ of $\mathcal{D}$ is cyclic and $\text{char}(k) \nmid \phi(|H|)$.

**Proof.** Let $G \in \mathcal{D}$. If $\mathcal{F}_{D,k}$ is semi-simple, then the Yoneda functor $Y_G$ is direct sum of simple functors. So its endomorphism algebra is semi-simple. Moreover, by Yoneda’s Lemma this last algebra is nothing but the double Burnside algebra of the group $G$. So 1 implies 2. And 3 is nothing but a reformulation of 2 using Bouc’s Theorem 7.1.

Now, we will prove that 3 implies 1. Let $H \in \mathcal{D}$. Let $V$ be a simple $k\text{Out}(H)$-module. Then by inflation, the module $V$ is also a simple $kB(H,H)$-module. Since the last algebra is semi-simple, the module $V$ is a projective indecomposable $kB(H,H)$-module. Since $L_{H,-}$ is a left adjoint to the exact functor $ev_H$, it sends projective modules to projective functors. Moreover, it sends indecomposable modules to indecomposable functors. So the functor $L_{H,V}$ is a projective indecomposable functor with simple quotient $S_{H,V}$. In other words, this functor is a projective cover of $S_{H,V}$. It has a unique maximal subfunctor $J_{H,V}$ and this functor has the property of vanishing at $H$.

• First let us assume that $\mathcal{D}$ is the full subcategory of the biset category consisting of all the cyclic groups. Let $K$ be a group of $\mathcal{D}$. Let $M$ be a lcm of the groups $H$ and $K$ (a minimal cyclic group such that $H$ and $K$ are subgroups of $M$). Then, the group $M$ is in the category $\mathcal{D}$. Since $M$ is abelian, the groups $H$ and $K$ are both isomorphic to a quotient of $M$. In other words, the groups $H$ and $K$ are $k$-generated by $M$. So, by Lemma 7.2 we have an isomorphism

$$\phi : L_{H,V} \cong L_{M,L_{H,V}(M)}.$$

In particular, $\phi$ maps the maximal subfunctor $J_{H,V}$ to a maximal subfunctor of $L_{M,L_{H,V}(M)}$. But as explain in Lemma 3.2 the $kB(M,M)$-module $L_{H,V}(M)$ is a projective indecomposable module. Moreover, by hypothesis $M$ is a cyclic group such that $\phi(|M|)$ is invertible in $k$, so the double Burnside algebra $kB(M,M)$ is semi-simple and $L_{H,V}(M)$ is a simple module. In particular, the functor $L_{M,L_{H,V}(M)}$ has a unique maximal subfunctor $J_{M,L_{H,V}(M)}$ which has the property of vanishing at $M$. In conclusion, we have:

$$J_{H,V}(M) = 0.$$

The group $K$ is also isomorphic to a quotient of $M$, so $id_K$ factorizes through $M$. In particular, the identity of $J_{H,V}(K)$ factorizes through $J_{H,V}(M) = 0$. This implies
that $J_{H,V}(K) = 0$. In conclusion, we proved that every projective indecomposable functor in $\mathcal{F}_{D,k}$ is simple. Any functor in $\mathcal{F}_{D,k}$ is a quotient of a direct sum of projective functors. So, a quotient of a direct sum of simple functors. By usual arguments, this implies that every biset functor over $D$ is semi-simple.

- If $H$ and $K$ are two cyclic groups of $D$, then a lcm of $H$ and $K$ may not be in the category $\mathcal{F}_{D,k}$ and $\mathcal{F}_{\text{Cyc},k}$ where $\text{Cyc}$ is the full subcategory of the biset category consisting of all the cyclic groups. We refer to Section 3.3 of [3] for more details. We use the fact that the projective indecomposable functors of $\mathcal{F}_{D,k}$ are exactly the restriction of the projective indecomposable functors of $\mathcal{F}_{\text{Cyc},k}$ indexed by the groups of $D$. The arguments of the previous point ($\star$) shows that any projective indecomposable functor of $\mathcal{F}_{\text{Cyc},k}$ is simple. Since the restriction functor sends the simple functors indexed by groups of $D$ to simple functors, the result follows.

$\star$ In order to apply the previous point we need to show that if $H$ and $K$ are groups in $D$ and $M$ is a lcm of $H$ and $K$, then the double Burnside algebra of $M$ is semi-simple. By assumption, we now that $\phi(|H|)$ and $\phi(|K|)$ are invertible in $k$ this implies that $\phi(|M|) = \phi(\text{lcm}(|H|,|K|))$ is invertible in $k$. In particular, the algebra $kB(M,M)$ is indeed semi-simple.

In general, when the category of biset functor is not semi-simple, it is possible that some simple functors are also projective. However, the simple functor indexed by the trivial group 1 is always as far from being projective as possible. For that reason we call it the trivial functor. Then, we prove that exactly as for the group algebra case, a category of biset functor is semi-simple if and only if the trivial object is projective.

**Definition 7.4.** Let $k$ be a field.

- Let $D$ be a replete biset category. The simple functor $S_{1,k}$ of $\mathcal{F}_{D,k}$ is called the trivial functor.
- Let $G$ be a finite group. The simple $kB(G,G)$-module $S_{1,k}(G)$ is called the trivial module.

First, we need a technical lemma about the Burnside module.

**Lemma 7.5.** Let $k$ be a field. Let $G$ be a cyclic $p$-group such that $\text{char}(k) \mid |\text{Out}(G)|$. Then $kB(G)$ is not a simple $kB(G,G)$-module.

**Proof.** If $\text{char}(k) \neq p$, then Bouc has already classified the composition factors of the $kB(G,G)$-module $kB(G)$ in Paragraph 5.6.9 of [3]. However, Bouc used the idempotents of the Burnside ring $kB(G)$ in co-prime characteristic. So his method cannot be generalized to case where the characteristic of the field is $p$. Let us look more carefully at the action of $kB(G,G)$ on $kB(G)$. Let $H$ be a subgroup of $G \times G$ and $L$ be a subgroup of
It is clear that action is given by:

\[ \text{G} \text{ is given by the Mackey formula (see formula (1)).} \]

Since \( \text{G} \) is a commutative group, the action is given by:

\[ ((G \times G)/H) \cdot G/L = \|p_2(H)\backslash G/L\| G/(H \bullet L). \]  

(5)

Here \( \|p_2(H)\backslash G/L\| \) is the size of a set of representatives of the double cosets \( p_2(H)\backslash G/L \) and \( H \bullet L \) is the subgroup of \( \text{G} \) defined by:

\[ H \bullet L = \{ g \in \text{G} ; \ \exists \ l \in L \text{ with } (g, l) \in H \}. \]

It is clear that \( k_1(H) \lesssim H \bullet L \lesssim p_1(H) \).

1. If \( \text{char}(k) \mid (p - 1) \), then \( p = 1 \) in \( k \). The action of a transitive \( G\bullet G\)-biset \( (G \times G)/H \) on a transitive \( G\)-set \( G/L \) is given by \((G \times G)/H \cdot G/L = G/(H \bullet L)\). Let us consider \( N(G) \) the subspace of \( k\text{B}(G) \) defined by:

\[ N(G) = \{ \sum_{L \in \text{G}} \lambda_L G/L \in k\text{B}(G) ; \ \sum_{L \in \text{G}} \lambda_L = 0 \in k \}. \]

It is a \( k \)-vector space of codimension 1. Moreover, it is a non zero proper \( k\text{B}(G,G)\)-submodule of \( k\text{B}(G) \).

2. If \( n > 1 \) and \( \text{char}(k) = p \), then \( p^2 \mid |G| \) and \( \text{dim}_k k\text{B}(G) \geq 3 \). Formula (5) becomes:

\[ ((G \times G)/H) \cdot G/L = \begin{cases} 0 & \text{if } L \neq G \text{ and } p_2(H) \neq G, \\ G/(H \bullet L) & \text{otherwise.} \end{cases} \]

Let us consider the subspace \( N'(G) \) of \( k\text{B}(G) \) defined by:

\[ N'(G) := \{ \sum_{L \in \text{G}} \lambda_L G/L ; \ \lambda_G = 0 \text{ and } \sum_{L \in \text{G}} \lambda_L = 0 \}. \]

It is a \( k \)-vector space of codimension 2 and we claim that it is also a \( k\text{B}(G,G)\)-submodule of \( k\text{B}(G) \). Indeed if \( H \) is a subgroup of \( G \times G \) such that \( p_2(H) \neq G \), then the action of \((G \times G)/H \) on \( N'(G) \) is zero. Let \( H \) be a subgroup of \( G \) such that \( p_2(H) = G \). Then the action of \((G \times G)/H \) on a transitive \( G\)-set \( G/L \) is given by:

\[ ((G \times G)/H) \cdot G/L = G/(H \bullet L). \]

We need to check if \( H \bullet L \) can be equal to the group \( G \). Since \( H \bullet L \) is a subgroup of \( p_1(H) \), we can assume that \( p_1(H) = G \). The map which sends \( g \in G/(H \bullet L) \) to an element \( l(g) \in L \) such that \( (g, l(g)) \in H \) induces an isomorphism of groups:

\[ (H \bullet L)/k_1(H) \cong L/(k_2(H) \cap L). \]

• Let \( H \lesssim G \) be such that \( p_1(H) = p_2(H) = G \) and \( k_2(H) = G \). Since \( p_1(H)/k_1(H) \cong p_2(H)/k_2(H) \), this condition implies that \( k_1(H) = G \). Since \( k_1(H) \lesssim H \bullet L \), then \( H \bullet L \cong G \) for every subgroup \( L \) of \( G \). The space \( N'(G) \) is therefore stable by the action of such a transitive \( G\bullet G \) biset.
• If \( k_2(H) = k_1(H) < G \). As \( G \) is a cyclic \( p \)-group, then either \( L \leq k_2(H) \) or \( k_2(H) \leq L \). If \( L \leq k_2(H) \) then \( k_2(H) \cap L = L \), therefore \( H \cdot L = k_1(H) \). If \( k_2(H) = L \) then \( k_2(H) \cap L = k_2(H) \). So we have \( (H \cdot L)/k_1(H) = L/k_2(H) \), since \( k_2(H) = k_1(H) \), we have \( |H \cdot L| = |L| \). In both cases we cannot have \( H \cdot L = G \) and \( N^0(G) \) is a \( kB(G,G) \)-module.

As corollary, we have the following useful reformulation of the Theorems of Barker and Bouc about the semisimplicity of the category of biset functors and category of modules over the double Burnside algebra.

**Theorem 7.6.** Let \( k \) be a field.

1. Let \( \mathcal{D} \) be a replete biset category. Then, the category \( \mathcal{F}_{\mathcal{D},k} \) is semisimple if and only if the simple functor \( S_{1,k} \) is projective.

2. Let \( G \) be a finite group. Then, the double Burnside algebra \( kB(G,G) \) is a semisimple algebra if and only if the simple module \( S_{1,k}(G) \) is projective.

**Proof.** By Theorem 7.3, we know that the category \( \mathcal{F}_{\mathcal{D},k} \) is semisimple if and only if every group in \( \mathcal{D} \) is cyclic and for every \( H \in \mathcal{D} \), \( \text{char}(k) \) does not divide \( |\text{Out}(H)| \). For the double Burnside algebra, by Theorem 7.1 we know that \( kB(G,G) \) is semisimple if and only if \( G \) is cyclic and \( \text{char}(k) \) does not divide the order of \( |\text{Out}(G)| \). So in both cases, it remains to see that if the category is not semisimple, then the trivial functor (resp. module) is not projective. Or equivalently that its projective cover \( kB \) is not simple.

• If \( G \) is not cyclic then \( kB(G) \) is an indecomposable non simple module. Indeed by the proof of Proposition 6.1 of [6], the kernel of the linearization functor is a non zero proper submodule of \( kB(G) \). As consequence, if the category \( \mathcal{D} \) contains a non cyclic group, the functor \( kB \) is non simple.

• If \( G \) is cyclic, then by Theorem 4.8 the evaluation at \( G \) induces an equivalence of categories between \( \mathcal{F}_{G,k} \) and \( kB(G,G)\)-Mod. The group \( G \) is a direct product of cyclic groups of prime power order, \( G = P_1 \times \cdots \times P_r \). Let us assume that \( \text{char}(k) = p \) divides \( |\text{Out}(G)| \), then \( p \mid |\text{Out}(P_s)| \) for some \( s \in \{1,\cdots,r\} \). By Lemma 7.5 the \( kB(P_s,P_s) \)-module \( kB(P_s) \) is not simple. This implies that the functor \( kB \) is not simple in \( \mathcal{F}_{G,k} \) and by using one more times the equivalence of Theorem 4.8 we have that \( kB(G) \) is not simple.

• If \( \mathcal{D} \) is a replete category containing a cyclic group \( G \) such that \( \text{char}(k) \mid |\text{Out}(G)| \), then by the previous point, the \( kB(G,G) \)-module \( kB(G) \) is not simple, so the functor \( kB \) is not simple.
8 Self-injective property of the double Burnside algebra.

As explain in Section 6 if the group $G$ is a non-vanishing group and if $k$ is a field of characteristic zero, then the double Burnside algebra is quasi-hereditary. However, some easy computations show that over a field of positive characteristic the double Burnside algebra may have infinite global dimension. Moreover, in the case of $A_5$, there is a self-injective block isomorphic to $\mathbb{C}[X]/(X^2)$ in $\mathbb{C}B(A_5,A_5)$. As consequence, we wonder under which hypothesis on the field or the group, the double Burnside algebra is a self-injective algebra.

If $D$ is a replete biset category containing only finitely many isomorphism classes of objects, then by Morita’s Theorem the category $\mathcal{F}_{D,k}$ is equivalent to the category of modules over a finite dimensional algebra. In particular, we will say that the category $\mathcal{F}_{D,k}$ is self-injective if the corresponding finite dimensional algebra is self-injective. Then we have to following result.

**Proposition 8.1.** Let $k$ be a field and $D$ be a replete biset category. Then $\mathcal{F}_{D,k}$ is self-injective if and only if it is semisimple.

**Proof.** We only need to prove that if this category is self-injective, it is semisimple. If the category of biset functors over $D$ is self-injective, the application sending the top of a projective indecomposable functor to its simple socle induces a bijection on the set of isomorphism classes of simple functors. This bijection is called the Nakayama’s permutation (see Lemma 1.10.31 of [23] for more details). In particular, the simple functor $S_{1,k}$ must be in the socle of a projective indecomposable functor.

Let $P_{H,V}$ be a projective cover of the simple functor $S_{H,V}$. By Theorem 6.3 of [22], there is a filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = P_{H,V},$$

such that $P_i/P_{i-1} \cong \Delta_{H_i,U_i}$, where $H_i \in D$ and $U_i$ is a direct summand of a permutation $k\text{Out}(H_i)$-module. So we have:

$$\text{Soc}(P_{H,V}) \subseteq \bigoplus \text{Soc}(\Delta_{H_i,U_i}),$$

where the $\Delta$’s runs through the standard quotients of $P_{H,V}$. In particular, if the simple functor $S_{1,k}$ is in the socle of $P_{H,V}$, then it is in the socle of some of its standard factors $\Delta_{H_i,U_i}$. Also, for a finite group $K$, if $\Delta_{H_i,U_i}(K) \neq 0$, then $H_i$ is a subquotient of $K$. So if $S_{1,k}$ is composition factor of $\Delta_{H_i,U_i}$, then $H_i = 1$ and $U_i \cong \oplus k$. Moreover, $\Delta_{1,k} = L_{1,k} = kB$. As consequence, the simple functor $S_{1,k}$ only appears at the top of $\Delta_{1,k}$. So if the simple functor $S_{1,k}$ is in the socle of $P_{H,V}$ then it is in the socle of $\Delta_{1,k}$. This is the case if and only if $\Delta_{1,k}$ is simple. By Theorem 7.6 this is the case if and only if $\mathcal{F}_{D,k}$ is semisimple.

Now, we state the result for the double Burnside algebras.

**Theorem 8.2.** Let $k$ be a field. Let $G$ be a finite group. Then the double Burnside algebra $kB(G,G)$ is self-injective if and only if it is semisimple.
Proof. Since the group $G$ is a quotient of $G$, the evaluation at $G$ of $S_{1,k}$ is a non-zero simple module. Moreover, by Corollary 3.3, the simple $kB(G,G)$-modules are the non-zero evaluation of the simple functors. And if $S_{H,V}(G) \neq 0$, then $P_{H,V}(G)$ is a projective cover of this simple module. By Theorem 6.3 of [22], the simple functor $P_{H,V}$ has a filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = P_{H,V},$$

such that $P_i/P_{i-1} \cong \Delta_{H_i,U_i}$, where $H_i \in \mathcal{D}$ and $U_i$ is a direct summand of a permutation $k\text{Out}(H_i)$-module. Since the evaluation at $G$ is an exact functor, the $kB(G,G)$-module has a (weak form of) filtration:

$$0 = P_0(G) \subseteq P_1(G) \subseteq \cdots \subseteq P_n(G) = P_{H,V}(G),$$

and the quotients of this filtration are:

$$P_i(G)/P_{i-1}(G) \cong \left(P_i/P_{i-1}\right)(G) \cong \Delta_{H_i,U_i}(G).$$

Note that some of these factors may be zero, but not all of them since $S_{H,V}(G) \neq 0$ by hypothesis. Moreover, we have that $\text{Soc}(P_{H,V}(G)) \subseteq \bigoplus \left(\text{Soc}\Delta_{H_i,U_i}(G)\right)$.

Let us assume that $kB(G,G)$ is a self-injective algebra. Then the simple module $S_{1,k}(G)$ must be in the socle of a projective indecomposable module $P_{H,V}(G)$. So $S_{1,k}(G)$ is in the socle of some $\Delta_{H_i,U_i}(G)$. Moreover, by Proposition 8.1, $S_{1,k}(G)$ is composition factor of $\Delta_{H_i,U_i}(G)$ if and only if $S_{1,k}$ is composition factor of $\Delta_{H_i,U_i}$ in the category of biset functors. So by the Proof of proposition 8.1 we have $H_i = 1$ and $U_i \cong \oplus k$. So $S_{1,k}(G)$ must be in the socle of $\Delta_{1,k}(G) = kB(G)$. By Theorem 7.6, this implies that $kB(G,G)$ is a semisimple algebra.

Remark 8.3. As corollary, we have that the double Burnside algebra of a finite group $G$ over a field $k$ is symmetric if and only if it is a semisimple algebra. In [20], the Author studied the symmetry of the Mackey algebra. The main tool was a central linear map on the Mackey algebra which comes from the monoidal structure of the category of modules over the Mackey algebra, that is the category of Mackey functors. There are lot of points in common between the theory of biset functors and the theory of Mackey functors. In particular, the category of biset functors is also a closed symmetric monoidal category under suitable hypothesis on the category $\mathcal{D}$. (see Chapter 8 of [6]). The trace map of this monoidal structure (see Section 4 of [16] for more details about the trace of a monoidal category) is a map which goes from the endomorphism ring of a finitely generated projective biset functor to the endomorphism ring of the Burnside functor. By taking a representable functor $Y_G$, we have a central linear map:

$$\text{tr} : kB(G,G) \to \text{End}_{F_{G,k}}(Y_G) \cong kB(1) \cong k.$$  

One can compute this trace and shows that if $U$ is a $G$-$G$-biset then $\text{tr}(U) = |U/G| \in k$, that is the number of $G$-orbits in $U$ where $G$ acts diagonally on $U$. Unfortunately, this map cannot help to the comprehension of the symmetry of $kB(G,G)$ since the bilinear form $(U,V) \mapsto \text{tr}(U \times_G V)$ is always degenerate if $G \neq 1$.  

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