Faster Adaptive Momentum-Based Federated Methods for Distributed Composition Optimization

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Abstract

Federated Learning is a popular distributed learning paradigm in machine learning. Meanwhile, composition optimization is an effective hierarchical learning model, which appears in many machine learning applications such as meta learning and robust learning. More recently, although a few federated composition optimization algorithms have been proposed, they still suffer from high sample and communication complexities. In the paper, thus, we propose a class of faster federated compositional optimization algorithms (i.e., MFCGD and AdaMFCGD) to solve the nonconvex distributed composition problems, which builds on the momentum-based variance reduced and local-SGD techniques. In particular, our adaptive algorithm (i.e., AdaMFCGD) uses a unified adaptive matrix to flexibly incorporate various adaptive learning rates. Moreover, we provide a solid theoretical analysis for our algorithms under non-i.i.d. setting, and prove our algorithms obtain a lower sample and communication complexities simultaneously than the existing federated compositional algorithms. Specifically, our algorithms obtain lower sample complexity of $\tilde{O}(\epsilon^{-3})$ with lower communication complexity of $\tilde{O}(\epsilon^{-2})$ in finding an $\epsilon$-stationary solution. We conduct the numerical experiments on robust federated learning and distributed meta learning tasks to demonstrate the efficiency of our algorithms.

1 Introduction

Composition optimization is an effective hierarchical model, which is widely used to many applications such as reinforcement learning [Wang et al., 2017b, Huo et al., 2018], meta learning [Wang et al., 2021], risk management [Huo et al., 2018] and deep AUC maximization [Yuan et al., 2022]. In the paper, we study the following distributed composition optimization problem:

$$\min_{x \in \mathbb{R}^d} F(x) := \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\xi^m \sim D^m} \left[ f^m \left( \mathbb{E}_{\zeta^m \sim S^m} \left[ g^m(x; \xi^m) \right] ; \xi^m \right) \right],$$

(1)

where $y^m = g^m(x) = \mathbb{E}_{\xi^m \sim S^m} \left[ g^m(x; \xi^m) \right]$ and $f^m(y^m) = \mathbb{E}_{\xi^m \sim D^m} \left[ f^m(y^m; \xi^m) \right]$ for any $m \in [M]$ denote the inner and outer objective functions respectively in $m$-th client. Here $\xi^m$ and $\zeta^m$ for any $m \in [M]$ are independent random variables follow unknown distributions $D^m$ and $S^m$ respectively. For any $m, j \in [M]$ possibly $D^m \neq D^j$, $S^m \neq S^j$ and $D^m \neq D^j$. Applications of the problem (1)

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Table 1: Sample and Communication complexities comparison of the representative federated compositional optimization algorithms in finding an $\epsilon$-stationary solution of the distributed composition optimization problem (1), i.e., $\mathbb{E}\|\nabla F(x)\| \leq \epsilon$ or its equivalent variants. ALR denotes adaptive learning rate.

| Algorithm       | Reference         | Sample Complexity | Communication Complexity | ALR |
|-----------------|-------------------|-------------------|--------------------------|-----|
| ComFedL         | Huang et al. [2021a] | $O(\epsilon^{-8})$ | $O(\epsilon^{-8})$       |     |
| LocalMOML       | Wang et al. [2021]  | $O(\epsilon^{-8})$ | $O(\epsilon^{-8})$       |     |
| FEDNEST         | Tarzanagh et al. [2022] | $O(\epsilon^{-4})$ | $O(\epsilon^{-4})$       |     |
| Local-SCGDM     | Gao et al. [2022]   | $O(\epsilon^{-4})$ | $O(\epsilon^{-4})$       |     |
| MFCGD           | Ours               | $\tilde{O}(\epsilon^{-3})$ | $\tilde{O}(\epsilon^{-2})$ | ✓   |
| AdaMFCGD        | Ours               | $\tilde{O}(\epsilon^{-3})$ | $\tilde{O}(\epsilon^{-2})$ | ✓   |

Involves many machine learning problems with a compositional structure, which include model-agnostic meta learning [Tutunov et al., 2020, Chen et al., 2020b, Wang et al., 2021], reinforcement learning [Wang et al., 2017b, Huo et al., 2018] and sparse additive models [Wang et al., 2017a]. In the following, we give two specific applications that can be formulated as the distributed composition optimization problem (1).

1). Task-Distributed Meta Learning. Meta Learning is to learn some properties in the optimal model to improve model performances with more experiences, i.e., learning to learn [Andrychowicz et al., 2016]. Model-Agnostic Meta Learning (MAML) [Finn et al., 2017] is a class of popular meta learning methods, which is to find a common initialization that can adapt to a desired model for a set of new tasks after taking several gradient descent steps. In the paper, we consider a class of task-distributed MAMLs, where a set of tasks $\{T_m\}_{m=1}^M$ are drawn from a certain task distribution and each task is assigned in each client. Specifically, we solve the following task-distributed MAML problem:

$$
\min_{x \in \mathbb{R}^d} \frac{1}{M} \sum_{m=1}^M f^m (x - \eta \nabla f^m(x)),
$$

(2)

where $f^m(x) = \mathbb{E}_{\xi^m \sim \mathcal{D}^m}[f(x; \xi^m)]$, and random variable $\xi^m$ follows the unknown distribution $\mathcal{D}^m$, and $\eta > 0$ is a learning rate. Let $f^m(g^m) = f^m(g^m(x))$ and $g^m = g^m(x) = x - \eta \nabla f^m(x)$, the above problem (2) is a special case of the above composition problem (1).

2). Distributionally Robust Federated Learning. Federated learning (FL) [McMahan et al., 2017, Kairouz et al., 2019] is a distributed and privacy preserving machine learning method to learn a global model collaboratively from decentralized data distributed over a network of devices. To tackle the data heterogeneity from different devices, some robust FL algorithms [Mohri et al., 2019, Reisizadeh et al., 2018, Deng et al., 2020b] have been studied. In the paper, as in [Huang et al., 2021a], we consider solving the following distributed composition problem to reach distributionally robust FL, defined as

$$
\min_{x \in \mathbb{R}^d} \frac{1}{M} \sum_{m=1}^M f \left( \mathbb{E}[g^m(x; \xi^m)] \right),
$$

(3)

where $g^m(x) = \mathbb{E}[g^m(x; \xi^m)]$ denotes the loss function in the $m$-th client, and $f(\cdot)$ is a monotonically increasing function. Clearly, the problem (3) is a special case of the above problem (1).
Although recently many compositional gradient algorithms have been proposed to solve the composition problems, few distributed algorithms focus on solving the distributed composition optimization problems. More recently, Huang et al. [2021a], Wang et al. [2021], Gao et al. [2022], Tarzanagh et al. [2022] proposed some federated compositional gradient algorithms for the distributed stochastic composition problems. However, few adaptive algorithm focuses on the composition optimization problems under the distributed setting. Meanwhile, these existing federated composition optimization methods suffer from large sample complexity and communication complexity (Please see Table 1). Then there exists a natural question:

Could we develop faster and adaptive federated learning methods to solve the distributed composition optimization problem (1)?

In the paper, we provide an affirmative answer to the above question and propose a class of faster momentum-based federated compositional gradient descent algorithms (i.e., MFCGD and AdaMFCGD) to solve the problem (1), which builds on the local Stochastic Gradient Descent (SGD) and momentum-based variance reduced techniques to obtain a lower sample and communication complexities simultaneously. Our main contributions are as follows:

1. We propose a class of faster momentum-based federated compositional gradient descent algorithms (i.e., MFCGD and AdaMFCGD) to solve the nonconvex distributed composition problems, which builds on the momentum-based variance reduced and local-SGD techniques. In particular, our adaptive algorithm (i.e., AdaMFCGD) uses a unified adaptive matrix to flexibly incorporate various adaptive learning rates.

2. We provide a solid convergence analysis framework for our algorithms under non-i.i.d. setting, and prove that our algorithms obtain simultaneously lower sample complexity of $\tilde{O}(\epsilon^{-3})$ and lower communication complexity of $\tilde{O}(\epsilon^{-2})$ than the existing federated composition methods for finding an $\epsilon$-stationary solution (Please see Table 1).

3. Experimental results demonstrate efficiency of our algorithms on the robust federated learning and task-distributed meta learning.

2 Related Works

In this section, we overview some representative composition optimization, federated optimization and adaptive optimization methods, respectively.

2.1 Composition Optimization

Composition optimization has been widely applied to many applications such as reinforcement learning [Wang et al., 2017b], model-agnostic meta Learning [Tutunov et al., 2020] and risk management [Huo et al., 2018]. Recently, many compositional gradient-based methods have recently been proposed to solve these composition optimization problems. For example, stochastic compositional gradient methods [Wang et al., 2017a, b] and Ghadimi et al. [2020] have been proposed to solve these problems. Subsequently, some variance-reduced compositional algorithms [Huo et al.].
2018, Lin et al. 2018, Zhang and Xiao. 2019 have been proposed for composition optimization. Tutunov et al. 2020, Chen et al. 2020b presented a class of momentum-based compositional gradient methods for stochastic composition optimization. More recently, Jiang et al. 2022 proposed a class of efficient momentum-based variance reduced methods for non-convex stochastic composition optimization. Huang and Gao 2022 studied the stochastic composition optimization on Riemannian manifolds.

For the distributed setting, Huang et al. 2021a firstly studied federated learning algorithm for the general distributed composition optimization. Meanwhile, Wang et al. 2021 studied personalized federated learning algorithm based on the composition optimization. Subsequently, Gao et al. 2022, Tarzanagh et al. 2022 proposed some accelerated federated learning algorithms for the distributed composition optimization.

2.2 Federated Optimization

Federated Learning (FL) is a popular distributed machine learning framework for collaboratively training the global model without sharing the local data, and is widely used in many applications such as healthcare informatics Xu et al. 2021 and automatic diagnosis of COVID-19 Yang et al. 2021. McMahan et al. 2017 first studied FL and proposed the FedAvg algorithm for FL based on local-SGD algorithms Stich 2019, where each client conducts multiple steps of SGD with its local data and then sends the learned model to the server for averaging. Subsequently, Li et al. 2019, Karimireddy et al. 2019, Deng and Mahdavi. 2021 have studied the convergence properties of the local-SGD and FedAvg algorithms or their variations. To accelerate the vanilla local-SGD and FedAvg algorithms, various accelerated FL algorithms Yuan and Ma. 2020, Karimireddy et al. 2020, Khanduri et al. 2021, Chen et al. 2020a have been developed and studied. For example, Karimireddy et al. 2020 proposed a stochastic controlled averaging algorithm for FL by adopting the variance-reduced technique of SARA/N Nguyen et al. 2017/SPIDER Fang et al. 2018. Subsequently, Khanduri et al. 2021 proposed a faster federated algorithm based on momentum-based variance reduced technique of STORM Cutkosky and Orabona. 2019 and ProxHSGD Tran-Dinh et al. 2022, which obtains lower sample and communication complexities simultaneously.

To solve the data heterogeneity in FL, Mohri et al. 2019, Deng et al. 2020b proposed some effective robust FL algorithms by learning the worst-case loss based on the minimax optimization problems. To further incorporate personalization in FL, some personalized federated learning models Fallah et al. 2020, Deng et al. 2020a, Li et al. 2021 have been developed and studied. For example, Li et al. 2021 proposed an effective and efficient personalized FL algorithm (i.e., Ditto) by learning a regularized local model for each client.

2.3 Adaptive Optimization Methods

Adaptive optimization methods Duchi et al. 2011, Kingma and Ba. 2014 are a class of efficient optimization methods due to using adaptive learning rates in machine learning, and they have been widely studied in machine learning community. For example, AdaGrad Duchi et al. 2011 is the first adaptive gradient method. Adam Kingma and Ba. 2014 is a popular variation of AdaGrad algorithm based on the momentum technique, which is the default optimization algorithm for training large-scale machine learning models. Meanwhile, some variants of Adam algorithm Reddi et al. 2019, Chen et al. 2019 have been proposed to obtain a convergence guarantee under the nonconvex setting. To further improve the performance of Adam algorithm, recently some new
its variants such as AdamW \cite{Loshchilov2018} have been developed. More recently, some accelerated adaptive gradient methods \cite{Cutkosky2019, Huang2021} have been proposed based on the momentum-based variance reduced techniques. In parallel, some adaptive gradient methods \cite{Reddi2020, Chen2020} are proposed for distributed optimization. For example, \cite{Reddi2020} proposed a class of adaptive federated algorithms for FL by using adaptive learning rates at the server side.

3 Preliminaries

3.1 Notations

Let $[M]$ denote the set \{1, 2, $\cdots$, $M$\}. $\|\cdot\|$ denotes the $\ell_2$ norm for vectors and Frobenius norm for matrices. \langle x, y \rangle$ denotes the inner product of two vectors $x$ and $y$. For vectors $x$ and $y$, $x^r$ ($r > 0$) denotes the element-wise power operation, $x/y$ denotes the element-wise division and $\max(x, y)$ denotes the element-wise maximum. $I_d$ denotes a $d$-dimensional identity matrix. $A \succ 0$ denotes that $A$ is a positive definite matrix. $a_t = O(b_t)$ denotes that $a_t \leq cb_t$ for some constant $c > 0$. The notation $\tilde{\mathcal{O}}(\cdot)$ hides logarithmic terms. $\Pi_{C}[x] = \arg \min_{\|w\| \leq C} \|x - w\|^2$ denote a projection onto the ball with radius $C > 0$.

3.2 Classic Federated Learning

The classic Federated Learning (FL) solves the following distributed optimization problem:

$$\min_{x \in \mathbb{R}^d} \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\xi_m \sim \mathcal{D}^m}[\ell_m(x; \xi^m)],$$

(4)

where $\ell_m(x; \xi^m)$ is the loss function on $m$-th device, and $\mathcal{D}^m$ denotes the data distribution on $m$-th device. In FL, the data distributions $\{\mathcal{D}^m\}_{m=1}^{M}$ generally are different, i.e., for any $m, j \in [M]$ possibly $\mathcal{D}^m \neq \mathcal{D}^j$. The goal of FL is to learn a global variable $x$ based on these heterogeneous data from different data distributions.

3.3 Federated Composition Optimization

In the paper, we studied Federated Composition Optimization (FCO) defined as:

$$\min_{x \in \mathbb{R}^d} F(x) := \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\xi_m \sim \mathcal{D}^m} \left[ f^m \left( \mathbb{E}_{\xi_m \sim \mathcal{S}^m} \left[ g^m(x; \xi^m) \right] ; \xi^m \right) \right],$$

(5)

where $y^m = g^m(x) = \mathbb{E}_{\xi_m \sim \mathcal{S}^m} \left[ g^m(x; \xi^m) \right]$ and $f^m(y^m) = \mathbb{E}_{\xi_m \sim \mathcal{D}^m} \left[ f^m(y^m; \xi^m) \right]$ for any $m \in [M]$ denote the inner and outer objective functions respectively in $m$-th client. Here $\xi^m$ and $\zeta^m$ for any $m \in [M]$ are independent random variables follow unknown distributions $\mathcal{D}^m$ and $\mathcal{S}^m$ respectively. For any $m, j \in [M]$ possibly $\mathcal{D}^m \neq \mathcal{D}^j$, $\mathcal{S}^m \neq \mathcal{S}^j$ and $\mathcal{D}^m \neq \mathcal{S}^j$. Due to the existence of composition objective function and double unknown distributions, FCO has more challenges than classic FL.
Algorithm 1 MFCGD and AdaMFCGD Algorithms

1: Input: $T, q$, tuning parameters $\{\gamma, \eta_t, \alpha_t, \beta_t, \vartheta_t\}$ and initial input $x_1 \in \mathbb{R}^d$;
2: initialize: Set $x_1^m = x_1$ for $m \in [M]$, and draw $2q$ independent samples $\{\xi_{i,j}^m\}^q_{j=1}$ and $(\xi_{i,j}^m)_{j=1}^q$, and then compute $h_1^m = \frac{1}{q} \sum_{j=1}^q g^m(x_1^m; \xi_{i,j}^m)$, $u_1^m = \frac{1}{q} \sum_{j=1}^q \nabla g^m(x_1^m; \xi_{i,j}^m)$ and $v_1^m = \frac{1}{q} \sum_{j=1}^q \nabla f(h_1^m; \xi_{i,j}^m)$ for all $m \in [M]$; Generate adaptive matrix $A_1 \in \mathbb{R}^{d \times d}$.
3: for $t = 1$ to $T$ do
4:   if $\bmod(t, q) = 0$ then
5:     $\bar{w}_t = \frac{1}{M} \sum_{m=1}^M w_t^m$ and $\bar{x}_t = \frac{1}{M} \sum_{m=1}^M x_t^m$;
6:     Generate the adaptive matrix $A_t \in \mathbb{R}^{d \times d}$;
7:     Compute $a_t = \vartheta_t a_{t-1} + (1 - \vartheta_t) a_t^2$; $A_t = \text{diag}(\sqrt{a_t} + \rho)$;
8:     Choose uniformly random from $\{\bar{x}_t\}_{t=1}^T$.
9:   else
10:      for each client $m \in [M]$ (in parallel) do
11:       $w_t^m = (u_t^m)^T v_t^m$;
12:       $x_t^m = \arg \min_{x \in \mathbb{R}^d} \left\{ \langle x, w_t^m \rangle + \frac{1}{2q \gamma} (x - x_t^m)^T A_t (x - x_t^m) \right\}$;
13:       $A_{t+1} = A_t$;
14:      end if
15:   end for
16:   for each client $m \in [M]$ (in parallel) do
17:       Draw two independent samples $\xi_{t+1}^m$ and $\zeta_{t+1}^m$;
18:       $h_{t+1}^m = g^m(x_{t+1}^m; \xi_{t+1}^m) + (1 - \alpha_{t+1}) (h_t^m - g^m(x_t^m; \xi_{t+1}^m))$;
19:       $u_{t+1}^m = \Pi_{C_q} \left[ \nabla g^m(x_{t+1}^m; \xi_{t+1}^m) + (1 - \beta_{t+1}) (u_t^m - \nabla g^m(x_t^m; \xi_{t+1}^m)) \right]$;
20:       $v_{t+1}^m = \Pi_{C_f} \left[ \nabla f(h_{t+1}^m; \zeta_{t+1}^m) + (1 - \vartheta_{t+1}) (v_t^m - \nabla f(h_t^m; \zeta_{t+1}^m)) \right]$;
21:       $w_{t+1}^m = (u_{t+1}^m)^T v_{t+1}^m$;
22: end for
23: end for
24: Output: Chosen uniformly random from $\{\bar{x}_t\}_{t=1}^T$.

4 Federated Compositional Gradient Descent Algorithms

In this section, we propose a class of faster momentum-based federated compositional gradient descent algorithms (i.e., MFCGD and AdaMFCGD) to solve the problem (I), which builds on the local-SGD and momentum-based variance reduced techniques. Specifically, the local-SGD technique reduce the communication complexity and the momentum-based variance reduced technique reduce the sample complexity without relying on large batches. Meanwhile, our AdaMFCGD algorithm uses the unified adaptive matrix to flexibly incorporate various adaptive learning rates in updating variables. Specifically, Algorithm (I) provides a procedure framework of our MFCGD and AdaMFCGD algorithms.

In Algorithm (I) when $\bmod(t, q) = 0$ (i.e., synchronization step), the server receives the local variables $\{x_{t+1}^m\}_{m=1}^M$ and local gradients $\{w_t^m\}_{m=1}^M$ from the clients, and then averages them to obtain the averaged variables $\{\bar{x}_t\}$ and averaged gradients $\{\bar{w}_t\}$. Based on these averaged gradients $\{\bar{w}_t\}$,
we can generate some adaptive matrices \( \{A_t\}_{t \geq 1} \) (i.e., adaptive learning rates). **Note that** for our non-adaptive MFCDG algorithm, we only set \( A_t = I_d \) for all \( t \geq 1 \) in Algorithm 1. Besides one example given at the line 6 of Algorithm 1, we can also generate many other adaptive matrices. For example, we can generate adaptive matrix \( A_t \) as the norm-type of Adam, defined as

\[
a_t = \vartheta_t a_{t-1} + (1 - \vartheta_t) \| \tilde{w}_t \|, \quad A_t = \text{diag}(a_t + \rho),
\]

where \( 0 < \vartheta \leq 1 \). Note that we can directly choose \( \alpha_t, \beta_t \) or \( \vartheta_t \) instead of \( \vartheta_t \) to reduce the number of tuning parameters in our algorithm. Next, based on these adaptive matrices, we can update the variable \( x \) in the server, then sent it to each client.

When \( \mod(t, q) \neq 0 \) (i.e., asynchronous step), the clients update the variables \( \{\hat{x}_{t+1}\} \) and the generated adaptive matrices \( \{A_t\} \) from the server. Then the clients use the momentum-based variance reduced technique of STORM [Cutkosky and Orabona, 2019] and Prox-HSGD [Tran-Dinh et al., 2022] to update the stochastic gradients based on local data: for \( m \in [M] \)

\[
h^m_{t+1} = g^m(x^m_t; \zeta^m_{t+1}) + (1 - \alpha_t)(h^m_t - g^m(x^m_t; \zeta^m_{t+1})) \tag{7}
\]

\[
u^m_{t+1} = \Pi_{C_g} \left[ \nabla g^m(x^m_{t+1}; \zeta^m_{t+1}) + (1 - \beta_{t+1})(u^m_t - \nabla g^m(x^m_{t}; \zeta^m_{t+1})) \right] \tag{8}
\]

\[
u^m_{t+1} = \Pi_{C_f} \left[ \nabla f^m(h^m_{t+1}; \zeta^m_{t+1}) + (1 - \vartheta_{t+1})(v^m_t - \nabla f(h^m_{t}; \zeta^m_{t+1})) \right], \tag{9}
\]

where \( \alpha_t \in (0, 1), \beta_t \in (0, 1) \) and \( \vartheta_t \in (0, 1) \). Here the projection functions \( \Pi_{C_g}[ \cdot ] \) and \( \Pi_{C_f}[ \cdot ] \) ensure that the estimated stochastic gradients \( u^m_{t+1} \) and \( v^m_{t+1} \) are bounded, i.e., \( \|u^m_{t+1}\| \leq C_g \) and \( \|v^m_{t+1}\| \leq C_f \) for any \( t \geq 1 \). Based on the estimated stochastic gradients and adaptive matrices, the clients update the variables \( \{x^m_{t+1}\}_{m=1}^M \), defined as

\[
x^m_{t+1} = x^m_t - \gamma_t A_t^{-1} w^m_t = \arg \min_{x \in \mathbb{R}^d} \left\{ \langle x, w^m_t \rangle + \frac{1}{2\eta_t} \|x - x^m_t\|^2 A_t(x - x^m_t) \right\}, \tag{10}
\]

where \( \gamma > 0 \) and \( \eta_t > 0 \). In our algorithms, all clients use the same adaptive matrix generated from the server as in [Chen et al., 2020]. **Note that** the existing adaptive FL algorithms such as local-AMSGrad [Chen et al., 2020] only builds on some specific adaptive learning rates such as AMSGrad [Reddi et al., 2019]. However, our algorithms can use the unified adaptive matrix to flexibly incorporate various adaptive learning rates.

## 5 Convergence Analysis

In this section, we study the convergence properties of our MFCDG and AdaMFCGD algorithms under some mild assumptions. All related proofs are provided in the Appendix. We first review some useful lemmas and assumptions.

**Assumption 1. (Lipschitz Gradients)** For any \( m \in [M] \), there exist constants \( L_f \) and \( L_g \) for \( \nabla f^m(y; \zeta^m) \), \( \nabla g^m(x; \zeta^m) \) respectively satisfying

\[
\| \nabla g^m(x_1, \zeta^m) - \nabla g^m(x_2, \zeta^m) \| \leq L_g \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^d,
\]

\[
\| \nabla f^m(y_1; \zeta^m) - \nabla f^m(y_2; \zeta^m) \| \leq L_f \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^p.
\]

**Assumption 2. (Bounded Gradients)** For any \( m \in [M] \), gradient \( \nabla g^m(x; \zeta^m) \) and Jacobian matrix \( \nabla f^m(y; \zeta^m) \) have the upper bounds \( C_g \) and \( C_f \) respectively, i.e.,

\[
\| \nabla g^m(x; \zeta^m) \| \leq C_g, \quad \| \nabla f^m(y; \zeta^m) \| \leq C_f, \quad \forall x \in \mathbb{R}^d, \ y \in \mathbb{R}^p.
\]
**Assumption 3.** (Bounded Variances) For any \( m \in [M] \), functions \( f^m(y; \xi^m) \) and \( g^m(x; \zeta^m) \) and its gradients are unbiased and the bounded variances, i.e., we have \( \mathbb{E}[g^m(x; \zeta^m)] = g^m(x) \), \( \mathbb{E}[\nabla f^m(y; \xi^m)] = \nabla f^m(y) \) and

\[
\mathbb{E}||g^m(x; \zeta^m) - g^m(x)||^2 \leq \sigma^2, \quad \mathbb{E}||\nabla g^m(x; \zeta^m) - \nabla g^m(x)||^2 \leq \sigma^2;
\]

\[
\mathbb{E}||\nabla f^m(y; \xi^m) - \nabla f^m(y)||^2 \leq \sigma^2, \quad \forall x \in \mathbb{R}^d, \; y \in \mathbb{R}^p
\]

where \( \sigma > 0 \).

**Assumption 4.** \( F(x) \) has a lower bound, i.e., \( F^* = \inf_{x \in \mathbb{R}^d} F(x) \).

**Assumption 5.** In our algorithms, the adaptive matrix \( A_t \) for all \( t \geq 1 \) satisfies \( A_t \geq \rho I_d \), where \( \rho > 0 \) is an appropriate positive number.

**Assumption 6.** For any \( m, j \in [M], x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^p \), we have \( ||\nabla f^m(y) - \nabla f^j(y)|| \leq \delta_f \), \( ||\nabla g^m(x) - \nabla g^j(x)|| \leq \delta_g \), where \( \delta_f > 0 \) and \( \delta_g > 0 \) are constants.

Assumptions 1-4 ensure the smoothness of functions \( f^m(y; \xi^m) \), \( g^m(x; \zeta^m) \) for any \( m \in [M] \). Assumption 5 ensures the bounded gradients (or Jacobian matrix) of functions \( f^m(y; \xi^m) \) and \( g^m(x; \zeta^m) \) for any \( m \in [M] \). Assumption 6 ensures the bounded variances of stochastic gradient or value of functions \( f^m(y; \xi^m) \) and \( g^m(x; \zeta^m) \) for any \( m \in [M] \). Assumption 7 guarantees the feasibility of the problem (1). Assumptions 1-4 have been commonly used in the convergence analysis of the stochastic composition algorithms [Wang et al., 2017, 2019]. Assumption 5 has been commonly used in the existing adaptive methods [Huang et al., 2021]. Assumption 6 is the standard condition constrained the data heterogeneity in non-i.i.d FL setting [Li et al., 2019]. In fact, we can obtain the part results of Assumption 6 based on Assumptions 1-2. For example, we have

\[
||\nabla f^m(y) - \nabla f^j(y)|| = ||\nabla f^m(y) - \nabla f^m(y; \xi^m) + \nabla f^m(y; \xi^m) - \nabla f^j(y; \xi^j) + \nabla f^j(y; \xi^j) - \nabla f^j(y)|| \\
\leq ||\nabla f^m(y) - \nabla f^m(y; \xi^m)|| + ||\nabla f^m(y; \xi^m)|| + ||\nabla f^j(y; \xi^j)|| + ||\nabla f^j(y; \xi^j) - \nabla f^j(y)|| \\
\leq 2\sigma + 2C_f,
\]  

(11)

where the last inequality holds by Assumptions 1-2. Similarly, we have \( ||\nabla g^m(x) - \nabla g^j(x)|| \leq 2\sigma + 2C_g \) based on Assumptions 1-2.

**Lemma 1.** Given the above Assumptions 1-2, the function \( F(x) = \frac{1}{M} \sum_{m=1}^{M} f^m(g^m(x)) \) is \( L \)-smooth, i.e., for any \( x_1, x_2 \in \mathbb{R}^d \), we have

\[
||\nabla F(x_1) - \nabla F(x_2)||^2 \leq L^2 ||x_1 - x_2||^2,
\]

(12)

where \( L = \sqrt{2C_f^2 L_g^2 + 2C_g^4 L_f^2} \).

**Lemma 2.** Assume the gradient estimator \( \{\bar{w}_t\}_{t=1}^{T} \) generated from Algorithm 1 where \( w_t = \frac{1}{M} \sum_{m=1}^{M} w_t^m \), we have

\[
||\bar{w}_t - \nabla F(x_t)||^2 \leq \frac{1}{M} \sum_{m=1}^{M} \left( 2C_f^2 ||w_t^m - \nabla g^m(x_t)||^2 + 4C_g^2 ||w_t^m - \nabla f^m(h_t^m)||^2 \\
+ 4C_g^2 L_f^2 ||h_t^m - g^m(x_t)||^2 \right).
\]

(13)
5.1 Convergence Properties of AdaMFCGD Algorithm

In this subsection, we provide the convergence properties of our AdaMFCGD algorithm.

**Theorem 1.** Assume the sequence \( \{\tilde{x}_t\}_{t=1}^T \) be generated from AdaMFCGD algorithm. Under the above Assumptions, and let \( \eta_k = \frac{1}{(n+t)^{1/3}} \) for all \( t \geq 0 \), \( \alpha_{t+1} = c_1 \eta_k^2 \), \( \beta_{t+1} = c_2 \eta_k^2 \), \( \theta_{t+1} = c_3 \eta_k^2 \), \( n \geq \max(2k, k^3, (c_1k)^3, (c_2k)^3, (c_3k)^3, \frac{(2k^3+4k\mathcal{L}_gC_g)^2}{24}) \), \( k > 0 \), \( c_1 \geq \frac{2}{3k^2} + B \), \( c_2 \geq \frac{2}{3k^2} + 5C_g^2 \), \( c_3 \geq \frac{2}{3k^2} + 5C_g^2 \), \( \frac{\mu(\gamma+2)\mathcal{L}_g^2}{12\mathcal{L}_g^2+2\mathcal{L}_g^2C_g^2} \leq \gamma \leq \min \left( \frac{3\eta_k\mathcal{L}_gC_g}{4\mathcal{L}_g^2+3\mathcal{L}_g^2C_g^2}, \frac{\mu^2}{2k^2} \right) \), \( B \geq 20C_g^2\mathcal{L}_g^2 + \frac{c_1^2C_g^2\mathcal{L}_g^2}{24} + \Theta_k\gamma (c_1^2+3c_2^2) + \Theta_k\gamma (c_1^2+3c_2^2) \), \( \Theta = \left( 5C_g^2\mathcal{L}_g^2 + \frac{c_1^2C_g^2\mathcal{L}_g^2}{24} + \Theta_k\gamma (c_1^2+3c_2^2) \right) \), and \( \Theta = \left( 2C_g^2 + \mathcal{L}_g\gamma \right) \), we have

\[
\frac{1}{T} \sum_{t=1}^T \mathbb{E} \| \nabla F(\tilde{x}_t) \| \leq \left( \frac{\sqrt{2Gn^{1/6}}}{T^{1/2}} + \frac{\sqrt{2G}}{T^{1/3}} \right) \frac{1}{T} \sum_{t=1}^T \mathbb{E} \| A_t \|^2,
\]

where \( C_g^2 = \max(C_f^2, C_g^2) \), \( L_g^2 = L_f^2C_g^2 + L_g^2 \), \( G = \frac{4(F(\tilde{x}_1) - F^*)}{k\rho \gamma} + \frac{12n^{1/6}L_f^2}{k^2\rho \gamma^2} + 4k^2 \left( \frac{\delta^2}{4\gamma^2L_g^2} + \frac{(c_1^2+c_2^2+c_3^2)\gamma^2}{4\gamma^2L_g^2} \right) \) \( n + T \) and \( \delta^2 = 2c_1^2L_f^2\sigma^2 + c_3^2\sigma^2 + 4c_1^2\delta_2^2 + 4c_2^2L_g^2\delta^2 + c_2^2\sigma^2 + 3c_2^2\delta_2^2 \).

**Remark 1.** Under the above Assumption, we have \( \frac{1}{T} \sum_{t=1}^M \left( \nabla g^m(\tilde{x}_t) \right)^T \nabla f^m(g^m(\tilde{x}_t)) \| \leq C_fC_g \). When the adaptive matrix \( A_t \) is generated from line 6 of Algorithm 1, we have \( \frac{1}{T} \sum_{t=1}^T \mathbb{E} \| A_t \|^2 \leq 2(C_f^2C_g^2 + \rho) \). Without loss of generality, let \( k = O(1) \), \( \rho = O(1) \), \( c_1 = O(1) \), \( c_2 = O(1) \), \( c_3 = O(1) \) and \( n = O(q^3) \), we have and \( G = O(1) \). Let \( q = T^{1/3} \) and

\[
\frac{1}{T} \sum_{t=1}^T \mathbb{E} \| \nabla F(\tilde{x}_t) \| \leq \bar{O} \left( \frac{\sqrt{q}}{\sqrt{T}} + \frac{1}{T^{1/3}} \right) = \bar{O} \left( \frac{1}{T^{1/3}} \right) \leq \epsilon,
\]

then we have \( T = \bar{O}(\epsilon^{-3}) \). Since our AdaMFCGD algorithm requires 2 samples at each iteration except for the first iteration requires 2q samples, it has a sample complexity of 2q+2T = \( \bar{O}(q^{-3}) \). Thus, our AdaMFCGD algorithm requires \( \bar{O}(\epsilon^{-3}) \) sample (or gradient) complexity and \( \frac{T}{q} = T^{2/3} = \bar{O}(\epsilon^{-2}) \) communication complexity to find an \( \epsilon \)-stationary point of the distributed composition problem 1.

**Remark 2.** From Theorem 1, our AdaMFCGD algorithm simultaneously have lower sample and communication complexities than the existing federated compositional optimization algorithms (Please see Table 1). Moreover, our AdaMFCGD algorithm simultaneously have lower sample and communication complexities than the existing adaptive single-level FL algorithms such as the local-AMSGrad algorithm that needs sample complexity of \( O(\epsilon^{-4}) \) and communication complexity of \( O(\epsilon^{-3}) \) for finding an \( \epsilon \)-stationary point of the distributed single-level optimization problem, i.e., the above problem 1 with \( g^m(x) = x \) for all \( m \in [M] \).

5.2 Convergence Properties of MFCGD Algorithm

In this subsection, we provide the convergence properties of our non-adaptive MFCGD algorithm, i.e., set \( A_t = I_d \) for all \( t \geq 1 \).
Theorem 2. Assume the sequence \( \{\bar{x}_t\}_{t=1}^T \) be generated from MFCGD algorithm, i.e., \( A_t = I_d \) for all \( t \geq 1 \) in Algorithm\[\text{I}\]. Under the above Assumptions, and let \( \eta_t = \frac{k}{(n+T)^{1/4}} \) for all \( t \geq 0 \), \( \alpha_{t+1} = c_1\eta_t^2 \), \( \beta_{t+1} = c_2\eta_t^2 \), \( \theta_{t+1} = c_3\eta_t^2 \), \( n \geq \max (2,k^3,(c_1 k)^3,(c_2 k)^3,(c_3 k)^3, (24k^2 q L_{fg} C_{fg})^3) \), \( k > 0 \), \( c_1 \geq \frac{k}{12} + B \), \( c_2 \geq \frac{k}{12} + 5C^2 \), \( c_3^3 + c_2^2 \leq \frac{(24)^3 q^3 (L_{fg})^3 C_{fg}^3}{c} \), \( \gamma \leq \min \left( \frac{64 q L_{fg} C_{fg}}{c (L_{fg} + 2c L_{fg})}, \frac{c^3}{5} \right) \), \( g \leq 20C^2_q + \frac{c^2 C^2_q L^2_{fg}}{304 q^2 L_{fg} C_{fg}^2} \), \( \Theta = \left( \frac{5C^2_q L^2_{fg}}{2} + \frac{c^2 C^2_q L^2_{fg}}{304 q^2 L_{fg} C_{fg}^2} \right) \) and communication complexity of \( \Theta \). The proof of Theorem 2 can totally follow the proofs of the above Theorem\[\text{I}\] with the parameter \( \rho = 1 \). Without loss of generality, let \( k = O(1) \), \( c_1 = O(1) \), \( c_2 = O(1) \), \( c_3 = O(1) \) and \( n = O(q^3) \), we have and \( G = \tilde{O}(1) \). Let \( q = T^{1/3} \) and

\[
\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|
abla F(\bar{x}_t)\|] \leq \tilde{O}\left( \frac{\sqrt{q}}{T^{1/3}} + \frac{1}{T^{1/3}} \right) = \tilde{O} \left( \frac{1}{T^{1/3}} \right) \leq \epsilon,
\]

then we have \( T = \tilde{O}(\epsilon^{-3}) \). Since our MFCGD algorithm requires 2 samples at each iteration expect for the first iteration requires 2q samples, it has a sample complexity of \( 2q + 2T = \tilde{O}(\epsilon^{-3}) \). As the above AdaMFCGD algorithm, our MFCGD algorithm also obtain lower sample complexity of \( \tilde{O}(\epsilon^{-3}) \) and communication complexity of \( \tilde{O}(\epsilon^{-2}) \) in finding an \( \epsilon \)-stationary point of the problem \( \Pi \).

6 Numerical Experiments

In this section, we apply some numerical experiments to demonstrate the efficiency of our MFCGD and AdaMFCGD algorithms on robust federated learning and distributed meta learning tasks. In the experiments, we compare our algorithms with the existing algorithms in Table\[\text{II}\] for solving distributed composition optimization problems.

6.1 Robust Federated Learning

6.2 Task-Distributed Meta Learning

7 Conclusions

In the paper, we proposed a class of faster momentum-based federated compositional gradient descent algorithms (i.e., MFCGD and AdaMFCGD) to solve the nonconvex distributed composition problems. Our adaptive algorithm (i.e., AdaMFCGD) uses a unified adaptive matrix to flexibly
incorporate various adaptive learning rates. Moreover, we established a solid convergence analysis framework for our algorithms, and proved that our methods obtain lower sample and communication complexities simultaneously than the existing federated composition optimization methods.

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Appendix

In this section, we provide the detailed convergence analysis of our algorithms.

We first introduce some useful notations: 
\[
\bar{w}_t = \frac{1}{M} \sum_{m=1}^{M} w^m_t, \quad \bar{x}_t = \frac{1}{M} \sum_{m=1}^{M} x^m_t,
\]
\[
F(x) = \frac{1}{M} \sum_{m=1}^{M} f^m(g^m(x)), \quad \nabla F(x) = \frac{1}{M} \sum_{m=1}^{M} (\nabla g^m(x))^T \nabla f^m(g^m(x)).
\]

Next, we review and provide some useful lemmas.

Lemma 3. Given \( M \) vectors \( \{u^m\}_{m=1}^{M} \), the following inequalities satisfy:
\[
\|u^m + u^j\|^2 \leq (1 + c) \|u^m\|^2 + (1 + \frac{1}{c}) \|u^j\|^2 \quad \text{for any } c > 0,
\]
and
\[
\sum_{m=1}^{M} \|u^m\|^2 \leq M \sum_{m=1}^{M} \|u^m\|^2.
\]

Lemma 4. Given a finite sequence \( \{u^m\}_{m=1}^{M} \), and \( \bar{u} = \frac{1}{M} \sum_{m=1}^{M} u^m \), the following inequality satisfies
\[
\sum_{m=1}^{M} \|u^m - \bar{u}\|^2 \leq \sum_{m=1}^{M} \|u^m\|^2.
\]

Given a \( \rho \)-strongly convex function \( \varphi(x) \), we define a prox-function (Bregman distance) \cite{Censor and Lent 1981, Censor and Zenios 1992} associated with \( \varphi(x) \) as follows:
\[
D(z, x) = \varphi(z) - \left[ \varphi(x) + (\nabla \varphi(x), z - x) \right]. \tag{18}
\]

Then we define a generalized projection problem as in \cite{Ghadimi et al. 2016}:
\[
x^+ = \arg \min_{x \in \mathcal{X}} \{ \langle z, w \rangle + \frac{1}{\gamma} D(z, x) + h(z) \}, \tag{19}
\]
where \( \mathcal{X} \subseteq \mathbb{R}^d \), \( w \in \mathbb{R}^d \) and \( \gamma > 0 \). In the paper, we consider \( h(x) = 0 \). Meanwhile, we also define a generalized projected gradient (a.k.a., gradient mapping):
\[
G_{\mathcal{X}}(x, w, \gamma) = \frac{x - x^+}{\gamma}. \tag{20}
\]

Lemma 5. (Lemma 1 in \cite{Ghadimi et al. 2016}) Let \( x^+ \) be given in (19). Then, for any \( x \in \mathcal{X}, w \in \mathbb{R}^d \) and \( \gamma > 0 \), we have
\[
\langle w, G_{\mathcal{X}}(x, w, \gamma) \rangle \geq \rho \|G_{\mathcal{X}}(x, w, \gamma)\|^2 + \frac{1}{\gamma} [h(x^+) - h(x)], \tag{21}
\]
where \( \rho > 0 \) depends on \( \rho \)-strongly convex function \( \varphi(x) \).

When \( h(x) = 0 \), in the above lemma 5 we have
\[
\langle w, G_{\mathcal{X}}(x, w, \gamma) \rangle \geq \rho \|G_{\mathcal{X}}(x, w, \gamma)\|^2. \tag{22}
\]

Lemma 6. (Restatement of Lemma 1) Given the above Assumptions \cite{Jaggi et al. 2012} the function \( F(x) \) is \( L \)-smooth, i.e., for any \( x_1, x_2 \in \mathbb{R}^d \), we have
\[
\|\nabla F(x_1) - \nabla F(x_2)\|^2 \leq L^2 \|x_1 - x_2\|^2, \tag{23}
\]
where \( L = \sqrt{2C_f^2 L_0^2 + 2C_g^2 L_f^2} \).
Proof. Based on Assumption\[12\] the deterministic functions $f^m(y) = \mathbb{E}[f^m(y; \xi^m)]$ and its gradients also satisfy the Lipschitz gradients and bounded gradients. For example, for any $y_1, y_2 \in \mathbb{R}^n$

$$\|
abla f^m(y_1) - \nabla f^m(y_2)\| = \mathbb{E}[\|
abla f^m(y_1; \xi^m) - \nabla f^m(y_2; \xi^m)\|]$$

$$\leq \mathbb{E}[\|
abla f^m(y_1; \xi^m) - \nabla f^m(y_2; \xi^m)\|] \leq L_f \|y_1 - y_2\|,$$

where the first inequality holds by Jensen’s inequality, and the last inequality holds by Assumption\[4\].

Since $F(x) = \frac{1}{M} \sum_{m=1}^M f^m(g^m(x))$, we have

$$\|
abla F(x_1) - \nabla F(x_2)\|^2$$

$$= \left\| \frac{1}{M} \sum_{m=1}^M (\nabla g^m(x_1))^T \nabla f^m(g^m(x_1)) - \frac{1}{M} \sum_{m=1}^M (\nabla g^m(x_2))^T \nabla f^m(g^m(x_2)) \right\|^2$$

$$\leq \frac{1}{M} \sum_{m=1}^M \left\| (\nabla g^m(x_1))^T \nabla f^m(g^m(x_1)) - (\nabla g^m(x_2))^T \nabla f^m(g^m(x_2)) \right\|^2$$

$$= \frac{1}{M} \sum_{m=1}^M \left\| (\nabla g^m(x_1))^T \nabla f^m(g^m(x_1)) - (\nabla g^m(x_2))^T \nabla f^m(g^m(x_1)) + (\nabla g^m(x_2))^T \nabla f^m(g^m(x_1)) 
- (\nabla g^m(x_2))^T \nabla f^m(g^m(x_2)) \right\|^2$$

$$\leq \frac{1}{M} \sum_{m=1}^M 2C_f^2 \left\| \nabla g^m(x_1) - \nabla g^m(x_2) \right\|^2$$

$$+ \frac{1}{M} \sum_{m=1}^M 2C_f^2 \left\| \nabla f^m(g^m(x_1)) - \nabla f^m(g^m(x_2)) \right\|^2$$

$$\leq 2C_f^2 L_f^2 \|x_1 - x_2\|^2 + 2C_f^2 \|x_1 - x_2\|^2 = (2C_f^2 L_f^2 + 2C_f^2) \|x_1 - x_2\|^2,$$

where the second last and the last inequalities hold by Assumptions\[12\].

Lemma 7. (Restatement of Lemma 2) Assume the gradient estimator \(\{\tilde{w}_t\}_{t=1}^T\) generated from Algorithm\[4\] where $w_t = \frac{1}{M} \sum_{m=1}^M u^m_t$, we have

$$\|
abla \tilde{w} - \nabla F(\tilde{x}_t)\|^2 \leq \frac{1}{M} \sum_{m=1}^M \left( 2C_f^2 \|u^m_t - \nabla g^m(\tilde{x}_t)\|^2 + 4C_f^2 \|v^m_t - \nabla f^m(h^m_t)\|^2 + 4C_f^2 L_f^2 \|h^m_t - g^m(\tilde{x}_t)\|^2 \right).$$

Proof. Since $\tilde{w}_t = \frac{1}{M} \sum_{m=1}^M (u^m_t)^T v^m_t$, we have

$$\|
abla \tilde{w} - \nabla F(\tilde{x}_t)\|^2$$

$$= \left\| \frac{1}{M} \sum_{m=1}^M (u^m_t)^T v^m_t - \frac{1}{M} \sum_{m=1}^M (\nabla g^m(\tilde{x}_t))^T \nabla f^m(g^m(\tilde{x}_t)) \right\|^2$$

$$= \left\| \frac{1}{M} \sum_{m=1}^M (u^m_t)^T v^m_t - \frac{1}{M} \sum_{m=1}^M (\nabla g^m(\tilde{x}_t))^T v^m_t + \frac{1}{M} \sum_{m=1}^M (\nabla g^m(\tilde{x}_t))^T v^m_t - \frac{1}{M} \sum_{m=1}^M (\nabla g^m(\tilde{x}_t))^T \nabla f^m(g^m(\tilde{x}_t)) \right\|^2$$

$$\leq \frac{1}{M} \sum_{m=1}^M 2C_f^2 \|u^m_t - \nabla g^m(\tilde{x}_t)\|^2$$

$$+ \frac{1}{M} \sum_{m=1}^M 2C_f^2 \|v^m_t - \nabla f^m(h^m_t)\|^2$$

$$= 2C_f^2 \sum_{m=1}^M \|u^m_t - \nabla g^m(\tilde{x}_t)\|^2$$

$$+ 2C_f^2 \sum_{m=1}^M \|v^m_t - \nabla f^m(h^m_t)\|^2$$

$$\leq 2C_f^2 \sum_{m=1}^M \|u^m_t - \nabla g^m(\tilde{x}_t)\|^2$$

$$+ 4C_f^2 \sum_{m=1}^M \|v^m_t - \nabla f^m(h^m_t)\|^2$$

$$+ 4C_f^2 \sum_{m=1}^M \|h^m_t - g^m(\tilde{x}_t)\|^2,$$

where the last inequality holds by Assumption\[4\].

\[\square\]
where the first inequality is due to Assumptions 1 and the above Lemma 3.

**Lemma 8.** Suppose that the sequence \( \{\tilde{x}_t\}_{t=1}^T \) generated from Algorithm 1 where \( \tilde{x}_t = \frac{1}{M} \sum_{m=1}^{M} x_t^{m} \).

Let \( 0 < \gamma \leq \frac{\epsilon}{L^2} \), then we have

\[
F(\tilde{x}_{t+1}) \leq F(\tilde{x}_t) + \frac{1}{M} \sum_{m=1}^{M} \left( \frac{2C^2\eta t}{\rho} \|u_t^m - \nabla g^m(\tilde{x}_t)\|^2 + \frac{4C^2\eta t}{\rho} \|v_t^m - \nabla f^m(h_t^m)\|^2 \right)
+ \frac{4C^2\eta t^2}{\rho} \|h_t^m - g^m(\tilde{x}_t)\|^2 - \frac{\rho}{2\eta} \|x_t - \tilde{x}_t\|^2.
\]

(28)

**Proof.** According to the above Lemma 3, the function \( F(x) \) is \( L \)-smooth. Thus we have

\[
F(\tilde{x}_{t+1}) \leq F(\tilde{x}_t) + \langle \nabla F(\tilde{x}_t), \tilde{x}_{t+1} - \tilde{x}_t \rangle + \frac{L}{2} \|\tilde{x}_{t+1} - \tilde{x}_t\|^2
\]

(29)

\[
= F(\tilde{x}_t) + \langle \tilde{w}_t, \tilde{x}_{t+1} - \tilde{x}_t \rangle + \langle \nabla F(\tilde{x}_t) - \tilde{w}_t, \tilde{x}_{t+1} - \tilde{x}_t \rangle + \frac{L}{2} \|\tilde{x}_{t+1} - \tilde{x}_t\|^2.
\]

According to Assumption 3, i.e., \( A_t \succeq \rho I_d \) for any \( t \geq 1 \), the mirror function \( \varphi_t(x) = \frac{1}{2}x^T A_t x \) is \( \rho \)-strongly convex, then we can define a Bregman distance as in [Ghadimi et al. 2016],

\[
D_t(x, \tilde{x}_t) = \varphi_t(x) - \left[ \varphi_t(\tilde{x}_t) + \langle \nabla \varphi_t(\tilde{x}_t), x - \tilde{x}_t \rangle \right] = \frac{1}{2} \|x - \tilde{x}_t\|^2 A_t(x - \tilde{x}_t).
\]

(30)

When \( t = s_t = q[t/q] + 1 \), according to the line 7 of Algorithm 1 we have \( \tilde{x}_{t+1} = \arg \min_{x \in \mathbb{R}^d} \left\{ \langle \tilde{w}_t, x \rangle + \frac{1}{2\eta t} (x - \tilde{x}_t)^T A_t (x - \tilde{x}_t) \right\} \). By using Lemma 1 in [Ghadimi et al. 2016], to the problem \( \tilde{x}_{t+1} = \arg \min_{x \in \mathbb{R}^d} \left\{ \langle \tilde{w}_t, x \rangle + \frac{1}{2\eta t} (x - \tilde{x}_t)^T A_t (x - \tilde{x}_t) \right\} \), we can obtain

\[
\langle \tilde{w}_t, \frac{1}{\eta t} (x - \tilde{x}_{t+1}) \rangle \geq \rho \| \frac{1}{\eta t} (x - \tilde{x}_{t+1}) \|^2.
\]

(31)

When \( t \in (s_t, s_t + q) \), according to the line 11 of Algorithm 1 we have \( x_t^m = \arg \min_{x \in \mathbb{R}^d} \left\{ \langle w_t^m, x \rangle + \frac{1}{2\eta t} (x - x_t^m)^T A_t (x - x_t^m) \right\} \). Similarly, we have

\[
\langle w_t^m, \frac{1}{\eta t} (x_t^m - x_{t+1}^m) \rangle \geq \rho \| \frac{1}{\eta t} (x_t^m - x_{t+1}^m) \|^2.
\]

(32)

Then we have

\[
\frac{1}{M} \sum_{m=1}^{M} \langle w_t^m, \frac{1}{\eta t} (x_t^m - x_{t+1}^m) \rangle \geq \rho \frac{1}{M} \sum_{m=1}^{M} \| \frac{1}{\eta t} (x_t^m - x_{t+1}^m) \|^2
\]

\[
\geq \rho \| \frac{1}{\eta t} \sum_{m=1}^{M} (x_t^m - x_{t+1}^m) \|^2 = \rho \| \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \|^2.
\]

(33)

Thus we have

\[
\langle \tilde{w}_t, \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \rangle \geq \rho \| \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \|^2.
\]

(34)

Since \( \tilde{w}_t = \frac{1}{M} \sum_{m=1}^{M} w_t^m \), averaging the above inequality (33) from \( m = 1 \) to \( M \), we can obtain

\[
\langle \tilde{w}_t, \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \rangle = \frac{1}{M} \sum_{m=1}^{M} \langle w_t^m, \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \rangle \geq \rho \frac{1}{M} \sum_{m=1}^{M} \| \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \|^2 = \rho \| \frac{1}{\eta t} (\tilde{x}_t - \tilde{x}_{t+1}) \|^2.
\]

(35)
Then we have for any \( t \in [s_t, s_t + q] \),

\[
T_1 = \langle \bar{w}_t, \bar{x}_{t+1} - \bar{x}_t \rangle \leq -\frac{\rho}{\eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2. \tag{36}
\]

Since \( s_t = q[t/q] + 1 \) and all \( t \in [s_t, s_t + q) \), clearly, we have, for all \( t \geq 1 \)

\[
T_1 = \langle \bar{w}_t, \bar{x}_{t+1} - \bar{x}_t \rangle \leq -\frac{\rho}{\eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2. \tag{37}
\]

Next, consider the bound of the term \( T_2 \), we have

\[
T_2 = \langle \nabla F(\bar{x}_t) - \bar{w}_t, \bar{x}_{t+1} - \bar{x}_t \rangle
\leq \| \nabla F(\bar{x}_t) - \bar{w}_t \| \| \bar{x}_{t+1} - \bar{x}_t \|
\leq \frac{\eta \gamma}{\rho} \| \nabla F(\bar{x}_t) - \bar{w}_t \|^2 + \frac{\rho}{4 \eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2, \tag{38}
\]

where the first inequality is due to the Cauchy-Schwarz inequality and the last is due to Young’s inequality.

By combining the above inequalities (36), (37), with (38), we obtain

\[
F(\bar{x}_{t+1}) \leq F(\bar{x}_t) + \langle \nabla F(\bar{x}_t) - \bar{w}_t, \bar{x}_{t+1} - \bar{x}_t \rangle + \| \bar{x}_{t+1} - \bar{x}_t \|^2
\leq F(\bar{x}_t) + \frac{\eta \gamma}{\rho} \| \nabla F(\bar{x}_t) - \bar{w}_t \|^2 + \frac{\rho}{4 \eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2 - \frac{\rho}{\eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2 + \frac{L}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2
\leq F(\bar{x}_t) + \frac{\eta \gamma}{\rho} \| \nabla F(\bar{x}_t) - \bar{w}_t \|^2 - \frac{\rho}{2 \eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2 - \frac{\rho}{2 \eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2
\leq F(\bar{x}_t) + \frac{1}{M} \sum_{m=1}^{M} \frac{2C_\rho^2 \eta \gamma}{\rho} \| u_{t+1}^m - \nabla g^m(\bar{x}_t) \|^2 + \frac{4C_\rho^2 \eta \gamma}{\rho} \| v_{t+1}^m - \nabla f^m(h_{t+1}^m) \|^2
\leq F(\bar{x}_t) + \frac{4C_\rho^2 L_\rho^2 \eta \gamma}{\rho} \| h_{t+1}^m - g^m(\bar{x}_t) \|^2 - \frac{\rho}{2 \eta \gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2, \tag{39}
\]

where the second last inequality is due to \( 0 < \gamma \leq \frac{\rho}{2 \eta \gamma} \), and the last inequality holds by Lemma[1].

Lemma 9. Under the above assumptions, and assume the stochastic gradient estimators \( \{h_{t+1}^m, u_{t+1}^m, v_{t+1}^m\}^T \) be generated from Algorithm[1] we have, for any \( m \in [M] \)

\[
E[|h_{t+1}^m - g^m(x_{t+1}^m)|^2] \leq (1 - \alpha_{t+1})E[|h_{t+1}^m - g^m(x_{t+1}^m)|^2 + 2\alpha_{t+1}^2 \sigma^2 + 2C_\rho^2 E[|x_{t+1}^m - x_t^m|^2] , \tag{40}
\]

\[
E[|u_{t+1}^m - \nabla g^m(x_{t+1}^m)|^2] \leq (1 - \beta_{t+1})E[|u_{t+1}^m - \nabla g^m(x_{t+1}^m)|^2 + 2\beta_{t+1}^2 \sigma^2 + 2L_\rho^2 E[|x_{t+1}^m - x_t^m|^2] , \tag{41}
\]

\[
E[|v_{t+1}^m - \nabla f^m(h_{t+1}^m)|^2] \leq (1 - \theta_{t+1})E[|v_{t+1}^m - \nabla f^m(h_{t+1}^m)|^2 + 4L_\rho^2 E[|x_{t+1}^m - x_t^m|^2 + 2\theta_{t+1}^2 \sigma^2 + 8\alpha_{t+1}^2 L_\rho^2 E[|h_{t+1}^m - g^m(x_{t+1}^m)|^2 + 8L_\rho^2 \alpha_{t+1}^2 \sigma^2, \tag{42}
\]
Proof. Without loss of generality, we only prove the above inequality \((\ref{eq:1})\), and it is similar to the other inequalities. Since \(v_{t+1}^m = \Pi_C \left[ \nabla f^m(h_{t+1}^m; \xi_{t+1}^m) + (1 - \rho_{t+1})(v_t^m - \nabla f^m(h_t^m; \xi_t^m)) \right] \), we have

\[
\mathbb{E}[v_{t+1}^m - \nabla f^m(h_{t+1}^m)]^2 \\
\leq \mathbb{E}[\Pi_C \left[ \nabla f^m(h_{t+1}^m; \xi_{t+1}^m) + (1 - \rho_{t+1})(v_t^m - \nabla f^m(h_t^m; \xi_t^m)) \right]] - \mathbb{E}[\Pi_C \left[ \nabla f^m(h_{t+1}^m) \right]]^2 \\
\leq \mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m)]^2 \\
= \mathbb{E}[(1 - \rho_{t+1})(v_t^m - \nabla f^m(h_t^m; \xi_t^m)) - \rho_{t+1}(\nabla f^m(h_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m))] \\
+ (1 - \rho_{t+1})\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m) + \nabla f^m(h_{t+1}^m)]^2 \\
= (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + \mathbb{E}[(1 - \rho_{t+1})\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m)]^2 \\
- (1 - \rho_{t+1})^2\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m) + \nabla f^m(h_{t+1}^m)]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m)]^2 \\
+ 2(1 - \rho_{t+1})^2\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m) + \nabla f^m(h_{t+1}^m)]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\sigma^2 + 2(1 - \rho_{t+1})^2\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m)]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\sigma^2 + 2(1 - \rho_{t+1})^2\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m)]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\sigma^2 + 2(1 - \rho_{t+1})^2\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m)]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\sigma^2 + 2(1 - \rho_{t+1})^2\mathbb{E}[\nabla f^m(h_{t+1}^m; \xi_{t+1}^m) - \nabla f^m(h_{t+1}^m; \xi_{t+1}^m)]^2, \tag{43}
\]

where the third equality holds by the following fact:

\[
\mathbb{E}[(\xi_{t+1}^m - \mathbb{E}[\xi_{t+1}^m])^2] = 0,
\]

and the second last inequality holds by the inequality \(\mathbb{E}[\xi - \mathbb{E}[\xi]]^2 \leq \mathbb{E}[\xi]^2\) and Assumption \((\ref{assumption:2})\), the last inequality is due to Assumption \((\ref{assumption:1})\).

Since \(h_{t+1}^m = g^m(x_{t+1}; \zeta_{t+1}^m) + (1 - \alpha_{t+1})(h_t^m - g^m(x_t; \zeta_{t+1}^m))\), we have

\[
\mathbb{E}[h_{t+1}^m - h_t^m]^2 \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 2\alpha_{t+1}\mathbb{E}[h_t^m - g^m(x_t; \zeta_{t+1}^m)]^2 \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 2\alpha_{t+1}\mathbb{E}[h_t^m - g^m(x_t; \zeta_{t+1}^m)]^2 \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 2\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2] \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 4\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2] \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 4\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2] \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 4\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2] \\
\leq 2\mathbb{E}[g^m(x_{t+1}; \zeta_{t+1}^m) - g^m(x_t; \zeta_{t+1}^m)]^2 + 4\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2] + 4\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2], \tag{44}
\]

where the second inequality holds by Assumption \((\ref{assumption:1})\) and Assumption \((\ref{assumption:2})\).

Combining the above inequalities \((\ref{eq:1})\) with \((\ref{eq:2})\), we have

\[
\mathbb{E}[v_{t+1}^m - \nabla f^m(h_{t+1}^m)]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\sigma^2 + 2(1 - \rho_{t+1})^2\mathbb{E}[h_{t+1}^m - h_t^m]^2 \\
\leq (1 - \rho_{t+1})^2\mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 2\rho_{t+1}\sigma^2 + 2L_f^2\mathbb{E}[\|h_{t+1}^m - h_t^m\|^2] \\
+ 8\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2] + 8L_f^2\alpha_{t+1}\mathbb{E}[\|h_t^m - g^m(x_t; \zeta_{t+1}^m)\|^2],
\]

where the last inequality holds by \(0 < \rho_{t+1} \leq 1\).
Lemma 10. Based on the above Assumptions 1-2 and 6, we have

\[
\begin{align*}
\sum_{m=1}^{M} \mathbb{E}\|f_i^m(h_t^m) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(h_t^j)\|^2 &\leq 8L_f^2 \sum_{m=1}^{M} \mathbb{E}\|h_t^m - g^m(\bar{x}_t)\|^2 + 4M\delta_f^2 + 4ML^2\delta_g^2, \\
\sum_{m=1}^{M} \mathbb{E}\|g^m(x_t^m) - \frac{1}{M} \sum_{j=1}^{M} g_j^i(x_t^j)\|^2 &\leq 6L_g^2 \sum_{m=1}^{M} \mathbb{E}\|x_t^m - \bar{x}_t\|^2 + 3M\delta_g^2, \\
\sum_{m=1}^{M} \mathbb{E}\|g^m(x_t^m) - \frac{1}{M} \sum_{j=1}^{M} g_j^i(x_t^j)\|^2 &\leq 6C_g^2 \sum_{m=1}^{M} \mathbb{E}\|x_t^m - \bar{x}_t\|^2 + 3M\delta_g^2.
\end{align*}
\]

Proof. Consider the term \(\sum_{m=1}^{M} \mathbb{E}\|f_i^m(h_t^m) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(h_t^j)\|^2\), we have

\[
\begin{align*}
&\sum_{m=1}^{M} \mathbb{E}\|f_i^m(h_t^m) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(h_t^j)\|^2 \\
&\quad = \sum_{m=1}^{M} \mathbb{E}\|f_i^m(h_t^m) - f_i^m(g^m(\bar{x}_t)) + f_i^m(g^m(\bar{x}_t)) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(g^m(\bar{x}_t)) + \frac{1}{M} \sum_{j=1}^{M} f_j^i(g^m(\bar{x}_t))\|^2 \\
&\quad \quad \quad - \frac{1}{M} \sum_{j=1}^{M} f_j^i(g^m(\bar{x}_t)) + \frac{1}{M} \sum_{j=1}^{M} f_j^i(g^m(\bar{x}_t)) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(h_t^j)^2 \\
&\quad \leq \sum_{m=1}^{M} 4\mathbb{E}\|f_i^m(h_t^m) - f_i^m(g^m(\bar{x}_t))\|^2 + \sum_{m=1}^{M} 4\mathbb{E}\|f_i^m(g^m(\bar{x}_t)) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(g^m(\bar{x}_t))\|^2 \\
&\quad \quad \quad + \sum_{m=1}^{M} 4\mathbb{E}\|f_i^m(\bar{x}_t)\|^2 - \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\|f_j^i(\bar{x}_t)\|^2 + \sum_{m=1}^{M} 4\mathbb{E}\|f_i^m(g^m(\bar{x}_t)) - \frac{1}{M} \sum_{j=1}^{M} f_j^i(g^m(\bar{x}_t))\|^2 \\
&\quad \leq 4L_f^2 \sum_{m=1}^{M} \mathbb{E}\|h_t^m - g^m(\bar{x}_t)\|^2 + 4 \sum_{m=1}^{M} \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\|f_i^m(g^m(\bar{x}_t)) - f_j^i(g^m(\bar{x}_t))\|^2 \\
&\quad \quad \quad + 4L_f^2 \sum_{m=1}^{M} \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\|g^m(\bar{x}_t) - g^m(\bar{x}_t)\|^2 + 4L_f^2 \sum_{m=1}^{M} \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\|g^m(\bar{x}_t) - h_t^m\|^2 \\
&\quad \leq 8L_f^2 \sum_{m=1}^{M} \mathbb{E}\|h_t^m - g^m(\bar{x}_t)\|^2 + 4M\delta_f^2 + 4ML^2\delta_g^2,
\end{align*}
\]

where the last inequality holds by Assumption 5.
Next, we have
\[
\sum_{m=1}^{M} \mathbb{E}\|g^m(x_t^m) - \frac{1}{M} \sum_{j=1}^{M} g^j(x_t^j)\|^2 \\
= \sum_{m=1}^{M} \mathbb{E}\|g^m(x_t^m) - \nabla g^m(\bar{x}_t) + \nabla g^m(\bar{x}_t) - \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(\bar{x}_t) + \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(\bar{x}_t) - \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(x_t^j)\|^2 \\
\leq \sum_{m=1}^{M} 3\mathbb{E}\|\nabla g^m(x_t^m) - \nabla g^m(\bar{x}_t)\|^2 + \frac{M}{3} \sum_{m=1}^{M} \mathbb{E}\|\nabla g^m(\bar{x}_t) - \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(\bar{x}_t)\|^2 \\
+ \sum_{m=1}^{M} 3\mathbb{E}\|\frac{1}{M} \sum_{j=1}^{M} \nabla g^j(\bar{x}_t) - \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(x_t^j)\|^2 \\
\leq 3L_g^2 \sum_{m=1}^{M} \mathbb{E}\|x_t^m - \bar{x}_t\|^2 + \frac{3}{M} \sum_{m=1}^{M} \mathbb{E}\|\nabla g^m(\bar{x}_t) - \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(\bar{x}_t)\|^2 + \frac{3}{M} \sum_{m=1}^{M} \mathbb{E}\|\frac{1}{M} \sum_{j=1}^{M} \nabla g^j(\bar{x}_t) - \frac{1}{M} \sum_{j=1}^{M} \nabla g^j(x_t^j)\|^2 \\
\leq 6L_g^2 \sum_{m=1}^{M} \mathbb{E}\|x_t^m - \bar{x}_t\|^2 + 3M\delta_g^2, \tag{46}
\]
where the last inequality is due to the above Assumption \[4\].

Similarly, we can obtain
\[
\sum_{m=1}^{M} \mathbb{E}\|g^m(x_t^m) - \frac{1}{M} \sum_{j=1}^{M} g^j(x_t^j)\|^2 \leq 6C_g^2 \sum_{m=1}^{M} \mathbb{E}\|x_t^m - \bar{x}_t\|^2 + 3M\delta_g^2. \tag{47}
\]

Lemma 11. Suppose the iterates \(\{x_t^m\}_{t=1}^{T}\), for all \(m \in [M]\) generated from Algorithm \[1\] satisfy:
\[
\sum_{m=1}^{M} \mathbb{E}\|x_t^m - \bar{x}_t\|^2 \leq (q - 1) \sum_{l=s_t}^{t-1} \gamma^2 \eta_t^2 \sum_{m=1}^{M} \mathbb{E}\|d_t^m - \tilde{d}_t\|^2, \tag{48}
\]
where \(\bar{x}_t = \frac{1}{M} \sum_{m=1}^{M} x_t^m\), \(d_t^m = \frac{x_{t+1}^m - x_t^m}{\eta_t}\), and \(\tilde{d}_t = \frac{\bar{x}_{t+1} - \bar{x}_t}{\eta_t}\).

Proof. According to the lines 7 and 11 of Algorithm \[1\] we have
\[
x_{t+1}^m = x_t^m - \gamma \eta_t A_t^{-1} w_t^m = \arg \min_{x \in \mathbb{R}^d} \{ (w_t^m, x) + \frac{1}{2\eta_t \gamma} (x - x_t^m)^T A_t (x - x_t^m) \},
\]
\[
\bar{x}_{t+1} = \bar{x}_t - \gamma \eta_t A_t^{-1} \bar{w}_t = \arg \min_{x \in \mathbb{R}^d} \{ (\bar{w}_t, x) + \frac{1}{2\eta_t \gamma} (x - \bar{x}_t)^T A_t (x - \bar{x}_t) \},
\]
and then we define the gradient mappings as in the above \[20\]: \(d_t^m = \frac{x_{t+1}^m - x_t^m}{\eta_t}\) and \(\tilde{d}_t = \frac{\bar{x}_{t+1} - \bar{x}_t}{\eta_t}\).

From the line 7 of Algorithm \[1\] when \(t = s_t = q|t/q| + 1\), we have \(x_t^m = \bar{x}_t = \frac{1}{M} \sum_{m=1}^{M} x_t^m\) for any \(m \in [M]\), so the above inequality in the lemma holds trivially.

When \(t \in (s_t, s_t + q)\), we have
\[
x_t^m = x_{s_t}^m - \sum_{l=s_t}^{t-1} \gamma \eta_t d_t^m, \quad \text{and} \quad \bar{x}_t = \bar{x}_{s_t} - \sum_{l=s_t}^{t-1} \gamma \eta_t \tilde{d}_t.
\]

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Thus we have
\[ \sum_{m=1}^{M} \mathbb{E}||x_t^m - \bar{x}_t||^2 = \sum_{m=1}^{M} \mathbb{E}||\tilde{x}_m^m - \bar{x}_t - \left( \sum_{l=1}^{t-1} \gamma_l \eta_l \tilde{d}_l^m - \sum_{l=t}^{t-1} \gamma_l \tilde{d}_l \right) ||^2 \]
\[ = \sum_{m=1}^{M} \mathbb{E} \left( \left( \sum_{l=1}^{t-1} \gamma_l \eta_l \tilde{d}_l^m - \sum_{l=t}^{t-1} \gamma_l \tilde{d}_l \right) \right)^2 \leq (q-1) \sum_{l=1}^{t-1} \gamma_l \eta_l \sum_{m=1}^{M} \mathbb{E}||d_l^m - \tilde{d}_l||^2, \]
where the above inequality is due to \( t - s_t \leq q - 1 \).

Lemma 12. Let \( C_f^2 = \max(C_f^1, C_f^2) \), \( L_f^g = L_f^2C_f^2 + L_f^2 \) and \( \eta_t \leq \frac{5}{\delta^2 \gamma_t} \) for all \( t \geq 0 \). Further let \( \alpha_{t+1} = c_1 \eta_t^2 \), \( \beta_{t+1} = c_2 \eta_t^2 \) and \( \varrho_{t+1} = c_3 \eta_t^2 \), \( c_1, c_2, c_3 > 0 \) and \( c_1^2 + c_2^2 \leq \frac{(24)4^2 \gamma_t^2 \gamma_t^s C_f^2}{9 \eta_t^3} \). Set \( s_t = q(t/q) + 1 \) and \( t \in [s_t, s_t + q - 1] \), we have
\[ \sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{m=1}^{M} \mathbb{E}||d_t^m - \tilde{d}_t||^2 \]
\[ \leq \frac{6M}{5} \sum_{t=s_t}^{s_t+q-1} \eta_t \mathbb{E}||\tilde{d}_t||^2 + \rho^2 \left( c_1^2 + c_2^2 \right) \sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{m=1}^{M} \mathbb{E}||h_t^m - g^m(\bar{x}_t)||^2 + \frac{3M \delta^2}{5 \gamma_t^2} \sum_{t=s_t}^{s_t+q-1} \eta_t^3, \]
where \( \delta^2 = 2c_1^2 L_f^2 \gamma_t^2 + c_3^2 \gamma_t^2 + 4c_1^2 \delta_t^2 + 4\gamma_t^2 L_f^2 \gamma_t^2 + 3c_1^2 \delta_t^2 \).

Proof. According to the lines 7 and 11 of Algorithm [1] we have
\[ x_{t+1}^m = \arg \min_{x \in \mathbb{R}^n} \left\{ \langle w_t^m, x \rangle + \frac{1}{2 \eta_t^2} (x - x_t^m)^T A_t(x - x_t^m) \right\}, \]
\[ \bar{x}_{t+1} = \arg \min_{x \in \mathbb{R}^n} \left\{ \langle \bar{w}_t, x \rangle + \frac{1}{2 \eta_t^2} (x - \bar{x}_t)^T A_t(x - \bar{x}_t) \right\}, \]
and then we define the gradient mappings as in the above [20]: \( d_t^m = \frac{x_{t+1}^m - x_t^m}{\eta_t^2} = A_t^{-1} w_t^m \) and \( \tilde{d}_t = \frac{x_t - \bar{x}_{t+1}}{\eta_t \gamma_t} \).

Then we have
\[ \sum_{m=1}^{M} \mathbb{E}||d_t^m - \tilde{d}_t||^2 = \sum_{m=1}^{M} \mathbb{E}||A_t^{-1} (w_t^m - \bar{w}_t)||^2 \leq \frac{1}{\rho^2} \sum_{m=1}^{M} \mathbb{E}||w_t^m - \bar{w}_t||^2 \]
\[ \leq \frac{1}{\rho^2} \sum_{m=1}^{M} \left( 3C_f^2 E||v_t^m - \bar{v}_t||^2 + 3C_f^2 E||u_t^m - \bar{u}_t||^2 + 3||v_t^m - \bar{v}_t||^2 \right), \]
where the first inequality holds by Assumption [3] i.e., \( A_t \succeq \rho I_d \) for all \( t \geq 1 \), and the last inequality holds
by \(\|u_t^m\|^2 \leq C_d^2\) and \(\|\tilde{v}_t\|^2 \leq C_f^2\). Consider the term \(\|(\bar{u}_t)^T \bar{v}_t - \frac{1}{M} \sum_{m=1}^M (u_{t}^m)^T v_t^m\|^2\), we have

\[
\|(\bar{u}_t)^T \bar{v}_t - \frac{1}{M} \sum_{m=1}^M (u_{t}^m)^T v_t^m\|^2 = \|(\bar{u}_t)^T \bar{v}_t - \frac{1}{M} \sum_{m=1}^M (u_{t}^m)^T v_t^m\|^2 \\
\leq \frac{1}{M} \sum_{m=1}^M \|(\bar{u}_t)^T \bar{v}_t - (u_{t}^m)^T v_t^m\|^2 \\
= \frac{1}{M} \sum_{m=1}^M \|(\bar{u}_t)^T \bar{v}_t - (u_{t}^m)^T v_t^m + (\bar{u}_t)^T v_t^m - (u_{t}^m)^T v_t^m\|^2 \\
\leq \frac{1}{M} \sum_{m=1}^M \left( 2C_d^2 \|v_t^m - \bar{v}_t\|^2 + 2C_f^2 \|u_{t}^m - \bar{u}_t\|^2 \right).
\]

By combining the above inequalities (50) and (51), we have

\[
\sum_{m=1}^M \mathbb{E}[\|d_t^m - \tilde{d}_t\|^2] \leq \frac{9C_d^2}{\rho^2} \sum_{m=1}^M \mathbb{E}[\|v_t^m - \bar{v}_t\|^2] + \frac{9C_f^2}{\rho^2} \sum_{m=1}^M \mathbb{E}[\|u_{t}^m - \bar{u}_t\|^2].
\]

Let \(t = s_t = q[t/q] + 1\). When \(t = s_t\), we have \(v_t^m = \bar{v}_t\) and \(u_{t}^m = \bar{u}_t\) for any \(m \in [M]\), so we have \(\sum_{m=1}^M \mathbb{E}[\|v_t^m - \bar{v}_t\|^2] = 0\) and \(\sum_{m=1}^M \mathbb{E}[\|u_{t}^m - \bar{u}_t\|^2] = 0\). According to the above inequality (52), when \(t = s_t\), we have \(\sum_{m=1}^M \mathbb{E}[\|d_t^m - \tilde{d}_t\|^2] = 0\). Clearly, the about inequality (52) in the lemma holds trivially.

When \(t \in (s_t, s_t + q)\), we first consider the term \(\sum_{m=1}^M \mathbb{E}[\|v_t^m - \bar{v}_t\|^2]\) as follows:

\[
\sum_{m=1}^M \mathbb{E}[\|v_t^m - \bar{v}_t\|^2] \\
= \sum_{m=1}^M \mathbb{E}[\|v_t^m - \frac{1}{M} \sum_{m=1}^M v_t^m\|^2] \\
= \sum_{m=1}^M \mathbb{E}[\Pi_{C_f} [\nabla f^m(h_t^m; \xi_t^m) + (1 - \varrho_t)(v_{t-1}^m - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m))] - \frac{1}{M} \sum_{m=1}^M \Pi_{C_f} [\nabla f^m(h_t^m; \xi_t^m) + (1 - \varrho_t)(v_{t-1}^m - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m))]\|^2 \\
\leq \sum_{m=1}^M \mathbb{E}[\|\nabla f^m(h_t^m; \xi_t^m) + (1 - \varrho_t)(v_{t-1}^m - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m))\|^2 - (1 + \nu)(1 - \varrho_t)^2 \sum_{m=1}^M \mathbb{E}[\|v_{t-1}^m - \bar{v}_{t-1}\|^2] + (1 + \frac{1}{\nu}) \sum_{m=1}^M \mathbb{E}[\|\nabla f^m(h_t^m; \xi_t^m)\|^2 - (1 + \varrho_t)(\nabla f^m(h_t^m; \xi_t^m) - \frac{1}{M} \sum_{m=1}^M \nabla f^m(h_{t-1}^m; \xi_{t-1}^m))\|^2].
\]
Then, we consider the last term of (53):

\[
\sum_{m=1}^{M} \mathbb{E} \left[ \| \nabla f^m(h_t^m; \xi_t^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_t^m; \xi_t^m) - (1 - \vartheta_t) (\nabla f^m(h_{t-1}^m; \xi_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m; \xi_{t-1}^m)) \right]^2
\]

\[
= \sum_{m=1}^{M} \mathbb{E} \left[ \| \nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} (\nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m)) \right]^2 \\
+ \vartheta_t \left( \nabla f^m(h_t^m; \xi_t^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \right)^2 \\
\leq 2 \sum_{m=1}^{M} \mathbb{E} \left[ \| \nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \right]^2 + 2 \vartheta_t^2 \sum_{m=1}^{M} \mathbb{E} \left[ \| \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \right]^2 \\
\leq 2L_f^2 \sum_{m=1}^{M} \mathbb{E} \left[ \| h_t^m - h_{t-1}^m \right]^2 - 2 \vartheta_t^2 \sum_{m=1}^{M} \mathbb{E} \left[ \| \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \right]^2 \\
\leq 2M \sigma^2 + 16L_f^2 \sum_{m=1}^{M} \mathbb{E} \left[ \| h_{t-1}^m - g^m(\bar{x}_{t-1}) \right]^2 + 8M \delta_f^2 + 8ML_f^2 \delta^2,
\]

where the last inequality holds by the above Lemma [10].

Consider the term \( \sum_{m=1}^{M} \| \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \|_2^2 \), we have

\[
\sum_{m=1}^{M} \| \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \|_2^2 \\
= \sum_{m=1}^{M} \| \nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} (\nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m)) \|_2^2 \\
+ \| \nabla f^m(h_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m) \|_2^2 \\
\leq 2 \sum_{m=1}^{M} \| \nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} (\nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m)) \|_2^2 \\
+ 2 \sum_{m=1}^{M} \| \nabla f^m(h_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m) \|_2^2 \\
\leq 2 \sum_{m=1}^{M} \| \nabla f^m(h_t^m; \xi_t^m) - \nabla f^m(h_{t-1}^m; \xi_{t-1}^m) \|_2^2 + 2 \sum_{m=1}^{M} \| \nabla f^m(h_{t-1}^m) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m) \|_2^2 \\
\leq 2M \sigma^2 + 16L_f^2 \sum_{m=1}^{M} \mathbb{E} \left[ \| h_{t-1}^m - g^m(\bar{x}_{t-1}) \right|^2 + 8M \delta_f^2 + 8ML_f^2 \delta^2,
\]

where the last inequality holds by the above Lemma [10].
Since \( h_t^m = g^m(x_t^m; \zeta_t^m) + (1 - \alpha_t)(h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)) \), we have

\[
\begin{align*}
\mathbb{E}\|h_t^m - h_{t-1}^m\|^2 &= \mathbb{E}\|g^m(x_t^m; \zeta_t^m) - g^m(x_{t-1}^m; \zeta_{t-1}^m) - \alpha_t(h_t^m - g^m(x_t^m; \zeta_t^m))\|^2 \\
&\leq 2\mathbb{E}\|g^m(x_t^m; \zeta_t^m) - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 + 2\alpha_t^2\mathbb{E}\|h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 \\
&\leq 2C_g^2\|x_t^m - x_{t-1}^m\|^2 + 2\alpha_t^2\mathbb{E}\|h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 \\
&= 2C_g^2\|x_t^m - x_{t-1}^m\|^2 + 2\alpha_t^2\mathbb{E}\|h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 + 4\alpha_t^2\mathbb{E}\|g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 \\
&\leq 2C_g^2\|x_t^m - x_{t-1}^m\|^2 + 4\alpha_t^2\mathbb{E}\|h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 + 4\alpha_t^2\sigma^2 \\
&= 2C_g^2\|x_t^m - x_{t-1}^m\|^2 + 4\alpha_t^2\mathbb{E}\|h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 + 4\alpha_t^2\sigma^2 \\
&\leq 2C_g^2\|x_t^m - x_{t-1}^m\|^2 + 8\alpha_t^2\mathbb{E}\|h_{t-1}^m - g^m(x_{t-1}^m; \zeta_{t-1}^m)\|^2 + 8\alpha_t^2C_g^2\|x_{t-1}^m - x_{t-1}^m\|^2 + 4\alpha_t^2\sigma^2,
\end{align*}
\]

where the second inequality holds by Assumption 3.

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By combining the above inequalities (53), (54), (55), and (56), we have

\[
\sum_{m=1}^{M} \mathbb{E} \| v_{t}^m - \bar{v}_{t} \|^2 \\
\leq (1 + \nu)(1 - \varrho_t)^2 \sum_{m=1}^{M} \mathbb{E} \| v_{t-1}^m - \bar{v}_{t-1} \| \|^2 + (1 + \frac{1}{\nu}) \sum_{m=1}^{M} \mathbb{E} \| \nabla f^m(h_{t}^m; \xi^m_t) 
- \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t}^m; \xi^m_t) - (1 - \varrho_t)\left( \nabla f^m(h_{t-1}^m; \xi^m_t) - \frac{1}{M} \sum_{m=1}^{M} \nabla f^m(h_{t-1}^m; \xi^m_t) \right) \|^2 \\
\leq (1 + \nu)(1 - \varrho_t)^2 \sum_{m=1}^{M} \mathbb{E} \| v_{t-1}^m - \bar{v}_{t-1} \|^2 + 4L_h^2 C_g^2 \sum_{m=1}^{M} \mathbb{E} \| x_{t}^m - x_{t-1}^m \|^2 + 16\alpha_t^2 M_{f} \sum_{m=1}^{M} \mathbb{E}\| h_{t-1}^m - g^m(\bar{x}_{t-1}) \|^2 \\
+ 16\alpha_t^2 L_{f}^2 C_g^2 \sum_{m=1}^{M} \mathbb{E}\| \bar{x}_{t-1} - x_{t-1}^m \|^2 + 8ML_{f}^2 \alpha_t \sigma^2 \\
+ 4M_{f}^2 \varrho_t^2 \sigma^2 + 32\gamma^2 L_{f}^2 \sum_{m=1}^{M} \mathbb{E}\| h_{t-1}^m - g^m(\bar{x}_{t-1}) \|^2 + 16M_{g}^2 \varrho_t^2 \delta^2 + 16ML_{f}^2 C_g^2 \delta^2 \]

(57)

\[
\leq (1 + \nu)(1 - \varrho_t)^2 \sum_{m=1}^{M} \mathbb{E} \| v_{t-1}^m - \bar{v}_{t-1} \|^2 + 4L_h^2 C_g^2 \sum_{m=1}^{M} \mathbb{E} \| x_{t}^m - x_{t-1}^m \|^2 + 8L_h^2 C_g^2 \sum_{m=1}^{M} \mathbb{E}\| \bar{x}_{t-1} - d_{t-1} \|^2 \\
+ 16\alpha_t^2 L_{f}^2 M_{f} \sum_{m=1}^{M} \mathbb{E}\| h_{t-1}^m - g^m(\bar{x}_{t-1}) \|^2 + 16M_{g}^2 \varrho_t^2 \delta^2 + 16ML_{f}^2 C_g^2 \delta^2 \]

(58)

where the second last inequality holds by the above Lemma \text{[11]} and the above inequality \text{(52)}, and the last inequality holds by \text{$d_{t-1}^m = \frac{x_{t-1}^m - g^m(\bar{x}_{t-1})}{\| x_{t-1}^m - g^m(\bar{x}_{t-1}) \|}$}, and the above inequality \text{(52)}. 

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Next, we consider the term $\sum_{m=1}^{M} E\|u_{t}^{m} - \tilde{u}_{t}\|^2$ as follows:

$$
\sum_{m=1}^{M} E\|u_{t}^{m} - \tilde{u}_{t}\|^2 = \sum_{m=1}^{M} E\|u_{t}^{m}\|^2 - \frac{1}{M} \sum_{m=1}^{M} E\|u_{t}^{m}\|^2 \\
= \sum_{m=1}^{M} E[|C_{g}\| (\nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})) - \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})] + (1 - \beta_{t}) (u_{t-1}^{m} - \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m}))
$$

$$
\leq \sum_{m=1}^{M} E\|\nabla g^{m}(x_{t}^{m}; \zeta_{t}^{m}) - (1 - \beta_{t}) (\nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2}
$$

(59)

Then, we consider the last term of (59):

$$
\sum_{m=1}^{M} E\|\nabla g^{m}(x_{t}^{m}; \zeta_{t}^{m}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t}^{m}; \zeta_{t}^{m})\|^{2}
$$

$$
= \sum_{m=1}^{M} E\|\nabla g^{m}(x_{t}^{m}; \zeta_{t}^{m}) - \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2} - \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2}
$$

$$
+ \beta_{t} (\nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2}
$$

$$
\leq 2 \sum_{m=1}^{M} E\|\nabla g^{m}(x_{t}^{m}; \zeta_{t}^{m}) - \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2} + 2 \beta_{t}^{2} \sum_{m=1}^{M} E\|\nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2}
$$

(60)

$$
\leq 2L_{g} \sum_{m=1}^{M} E\|x_{t}^{m} - x_{t-1}^{m}\|^{2} + 2 \beta_{t}^{2} \sum_{m=1}^{M} E\|\nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^{2}
$$

(61)

where the second last inequality is due to Young inequality and the above Lemma 4.
Consider the term $\sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}; \zeta^m_t) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^m(x^m_{t-1}; \zeta^m_t) \|^2$, we have

$$\sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}; \zeta^m_t) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^m(x^m_{t-1}; \zeta^m_t) \|^2 = \sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}; \zeta^m_t) - \nabla g^m(x^m_{t-1}) - \frac{1}{M} \sum_{m=1}^{M} (\nabla g^m(x^m_{t-1}; \zeta^m_t) - \nabla g^m(x^m_{t-1})) \|^2$$

$$+ \nabla g^m(x^m_{t-1}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^m(x^m_{t-1}) \|^2 \leq 2 \sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}; \zeta^m_t) - \nabla g^m(x^m_{t-1}) - \frac{1}{M} \sum_{m=1}^{M} (\nabla g^m(x^m_{t-1}; \zeta^m_t) - \nabla g^m(x^m_{t-1})) \|^2$$

$$+ 2 \sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^m(x^m_{t-1}) \|^2 \leq 2 \sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}; \zeta^m_t) - \nabla g^m(x^m_{t-1}) \| + 2 \sum_{m=1}^{M} \| \nabla g^m(x^m_{t-1}) - \frac{1}{M} \sum_{m=1}^{M} \nabla g^m(x^m_{t-1}) \|^2$$

$$\leq 2M \sigma^2 + 12L_g^2 \sum_{m=1}^{M} \mathbb{E} \| x^m_{t-1} - \bar{x}_{t-1} \|^2 + 6M \delta_g^2,$$ (62)

where the last inequality holds by the above Lemma 10.
where the last inequality holds by the above inequality (52).

By combining the above inequalities (59), (61) and (62), we have

$$
\sum_{m=1}^{M} \mathbb{E}\|u_{t}^{m} - \bar{u}_{t}\|^2
\leq (1 + \nu)(1 - \beta_t)^2 \sum_{m=1}^{M} \mathbb{E}\|u_{t-1}^{m} - \bar{u}_{t-1}\|^2 + (1 + \frac{1}{\nu}) \sum_{m=1}^{M} \mathbb{E}\|\nabla g^{m}(x_{t}^{m}; \zeta_{t}^{m})
- \frac{1}{M} \sum_{m=1}^{M} \nabla g^{m}(x_{t-1}^{m}; \zeta_{t}^{m})\|^2
$$

$$
\leq (1 + \nu)(1 - \beta_t)^2 \sum_{m=1}^{M} \mathbb{E}\|u_{t-1}^{m} - \bar{u}_{t-1}\|^2 + (1 + \frac{1}{\nu}) \left(2L^2 \sum_{m=1}^{M} \mathbb{E}\|x_{t}^{m} - x_{t-1}^{m}\|^2
+ 4M\sigma^2 \beta_t^2 + 24L^2_\rho \beta_t^2 \sum_{m=1}^{M} \mathbb{E}\|x_{t-1}^{m} - \bar{x}_{t-1}\|^2 + 12M\delta^2 \beta_t^2\right)
$$

$$
\leq (1 + \nu)(1 - \beta_t)^2 \sum_{m=1}^{M} \mathbb{E}\|u_{t-1}^{m} - \bar{u}_{t-1}\|^2 + (1 + \frac{1}{\nu}) \left(4L^2_\rho \eta_t^{-1} \gamma^2 \sum_{m=1}^{M} \mathbb{E}\|d_t^{m} - \bar{d}_{t-1}\|^2 + 4L^2_\rho \eta_t^{-1} \gamma^2 \sum_{m=1}^{M} \mathbb{E}\|\bar{d}_{t-1}\|^2
+ 4M\sigma^2 \beta_t^2 + 24(q - 1)L^2_\rho \beta_t^2 \sum_{t=1}^{T-2} \gamma^2 \eta_t^{-1} \sum_{m=1}^{M} \mathbb{E}\|d_t^{m} - \bar{d}_{t}\|^2 + 12M\delta^2 \beta_t^2\right)
$$

$$
\leq (1 + \nu)(1 - \beta_t)^2 \sum_{m=1}^{M} \mathbb{E}\|u_{t-1}^{m} - \bar{u}_{t-1}\|^2 + (1 + \frac{1}{\nu}) \left(\frac{36C^2_\rho L^2_\rho \eta_t^{-1} \gamma^2}{\rho^2} \sum_{m=1}^{M} \mathbb{E}\|u_{t-1}^{m} - \bar{u}_{t-1}\|^2
+ \frac{36C^2_\rho L^2_\rho \eta_t^{-1} \gamma^2}{\rho^2} \sum_{m=1}^{M} \mathbb{E}\|u_{t-1}^{m} - \bar{u}_{t-1}\|^2
+ 4L^2_\rho \eta_t^{-1} \gamma^2 \sum_{m=1}^{M} \mathbb{E}\|\bar{d}_{t-1}\|^2 + 4M\sigma^2 \beta_t^2 + 12M\delta^2 \beta_t^2
+ 24(q - 1)L^2_\rho \beta_t^2 \sum_{t=1}^{T-2} \gamma^2 \eta_t^{-1} \sum_{m=1}^{M} \mathbb{E}\|d_t^{m} - \bar{d}_{t}\|^2 + 12M\delta^2 \beta_t^2\right),
$$

(64)

where the last inequality holds by the above inequality (64).
Similarly, since \( \eta \) is bounded, we have

\[
\sum_{m=1}^{M} (E\|u_t^m - \bar{u}_t\|^2 + E\|v_t^m - \bar{v}_t\|^2)
\]

\[
\leq (1 + \nu)(1 - \beta_t)^2 \sum_{m=1}^{M} E\|u_t^m - \bar{u}_{t-1}\|^2 + (1 + \frac{1}{\nu})(\frac{36C^2_a L^2_a \eta^2_t - 1 \gamma^2}{\rho^2} \sum_{m=1}^{M} E\|v_t^m - \bar{v}_{t-1}\|^2
\]

\[
+ \frac{36C^2_a L^2_a \eta^2_t - 1 \gamma^2}{\rho^2} \sum_{m=1}^{M} \sum_{l=1}^{M} \gamma^2 \eta^2\left(\frac{9C^2_a}{\rho^2} \sum_{m=1}^{M} E\|v_t^m - \bar{v}_t\|^2 + \frac{9C^2_a}{\rho^2} \sum_{m=1}^{M} E\|u_t^m - \bar{u}_t\|^2\right)
\]

\[
+ 24(q - 1)\delta_t^2 \sum_{m=1}^{M} E\|u_t^m - \bar{u}_{t-1}\|^2 + 8L^2_a C^2_a \gamma^2 \sum_{m=1}^{M} E\|d_{t-1}\|^2
\]

\[
+ 16(q - 1)\delta_t^2 \sum_{m=1}^{M} E\|h_{t-1}^m - g_{t-1}(\bar{x}_{t-1})\|^2 + 8ML^2_a \alpha^2 \delta_t^2 + 4M \delta_t^2 \sigma_t^2 + 16M \delta_t^2 \delta_t^2
\]

\[
\leq \max\left((1 + \nu)(1 - \beta_t)^2 + (1 + \frac{1}{\nu})\frac{72C^2_a (L^2_a C^2_a + L^2_a) \eta^2_t - 1 \gamma^2}{\rho^2}, 1 + \nu\right) (1 - \beta_t)^2 + (1 + \frac{1}{\nu})\frac{72C^2_a (C^2_a L^2_a + L^2_a) \eta^2_t - 1 \gamma^2}{\rho^2}
\]

\[
\cdot \sum_{m=1}^{M} (E\|u_t^m - \bar{u}_{t-1}\|^2 + E\|v_t^m - \bar{v}_{t-1}\|^2) + 8(1 + \frac{1}{\nu})(L^2_a C^2_a + L^2_a) \eta^2_t - 1 \gamma^2 \sum_{m=1}^{M} E\|d_{t-1}\|^2
\]

\[
+ 24(1 + \frac{1}{\nu})(q - 1)(L^2_a C^2_a \alpha_t^2 + L^2_a \delta_t^2) \sum_{m=1}^{M} \gamma^2 \eta^2\left(\frac{9C^2_a}{\rho^2} \sum_{m=1}^{M} E\|v_t^m - \bar{v}_t\|^2 + \frac{9C^2_a}{\rho^2} \sum_{m=1}^{M} E\|u_t^m - \bar{u}_t\|^2\right)
\]

\[
+ 32(1 + \frac{1}{\nu})(\delta_t^2 + \alpha_t^2) L^2_t \sum_{m=1}^{M} E\|h_{t-1}^m - g_{t-1}(\bar{x}_{t-1})\|^2 + (1 + \frac{1}{\nu})\left(8ML^2_a \alpha_t^2 \sigma_t^2 + 4M \delta_t^2 \sigma_t^2 + 16M \delta_t^2 \delta_t^2
\]

\[
+ 16ML^2_a \delta_t^2 \sigma_t^2 + 4M \sigma_t^2 \delta_t^2 + 12M \delta_t^2 \delta_t^2\right).
\]

Let \( C^2_{\bar{f}_g} = \max(C^2_f, C^2_a) \), \( L^2_{\bar{f}_g} = L^2_f C^2_a + L^2_g \), \( \nu = \frac{\alpha^2}{\bar{g}} \) and \( \eta \leq \frac{\rho}{24\sqrt{q}L^2_f \bar{f}_{\bar{f}_g}} \) for all \( t \geq 0 \). Since \( \beta_t \in (0, 1) \) for all \( t \geq 0 \), we have

\[
(1 + \nu)(1 - \beta_t)^2 + (1 + \frac{1}{\nu})\frac{72C^2_a (L^2_a C^2_a + L^2_a) \eta^2_t - 1 \gamma^2}{\rho^2}
\]

\[
\leq 1 + \frac{1}{q} (1 + q)\frac{72C^2_a (L^2_a C^2_a + L^2_a) \gamma^2}{\rho^2} \frac{\rho^2}{576\gamma^2 q^2 L^2_f C^2_{\bar{f}_g}}
\]

\[
\leq 1 + \frac{1}{q} + \frac{1 + q}{8q^2} \leq 1 + \frac{5}{4q}
\]

(67)

Similarly, since \( \bar{g}_t \in (0, 1) \) for all \( t \geq 0 \), we have

\[
(1 + \nu)(1 - \bar{g}_t)^2 + (1 + \frac{1}{\nu})\frac{72C^2_a (L^2_a C^2_a + L^2_a) \eta^2_t - 1 \gamma^2}{\rho^2} \leq 1 + \frac{5}{4\bar{g}_t}
\]

(68)
Based on the above inequality (65) and the parameters, then we have

\[
\sum_{m=1}^{\mathcal{M}} (\mathbb{E}\|u_t^m - \bar{u}_t\|^2 + \mathbb{E}\|v_t^m - \bar{v}_t\|^2) \leq (1 + \frac{5}{4q}) \sum_{m=1}^{\mathcal{M}} (\mathbb{E}\|u_t^m - \bar{u}_{t-1}\|^2 + \mathbb{E}\|v_t^m - \bar{v}_{t-1}\|^2) + 8(q+1) L_f^2 \eta_{t-1}^2 \gamma^2 \sum_{m=1}^{\mathcal{M}} \mathbb{E}\|\tilde{d}_{t-1}\|^2
\]

\[
+ 216(q^2 - 1) C_{f}^2 \beta_f \gamma^2 (\alpha_t^2 + \beta_t^2) \sum_{l=t}^{t-1} \eta_t^2 \sum_{m=1}^{\mathcal{M}} (\mathbb{E}\|v_t^m - \bar{v}_t\|^2 + \mathbb{E}\|u_t^m - \bar{u}_t\|^2)
\]

\[
+ 32(1 + q)(\alpha_t^2 + \beta_t^2) L_f^2 \sum_{m=1}^{\mathcal{M}} \mathbb{E}\|h_{t-1}^m - g^m(\bar{x}_{t-1})\|^2
\]

\[
+ 4M(q+1) \left( 2L_f^2 \alpha_t^2 \sigma^2 + \alpha_t^2 \beta_t^2 + G_t^2 \beta_t^2 + 4L_f^2 \beta_t^2 \sigma_t^2 + \sigma^2 \beta_t^2 + 3 \delta^2_t \beta_t^2 \right)
\]

\[
\leq (1 + \frac{5}{4q}) \sum_{m=1}^{\mathcal{M}} (\mathbb{E}\|u_t^m - \bar{u}_{t-1}\|^2 + \mathbb{E}\|v_t^m - \bar{v}_{t-1}\|^2) + \frac{\rho^2}{36q C_{f}^2} \sum_{m=1}^{\mathcal{M}} \mathbb{E}\|\tilde{d}_{t-1}\|^2
\]

\[
+ \frac{3(c_1^2 + c_2^2)}{8} \eta_{t-1} \sum_{l=t}^{t-1} \eta_t^2 \sum_{m=1}^{\mathcal{M}} (\mathbb{E}\|v_t^m - \bar{v}_t\|^2 + \mathbb{E}\|u_t^m - \bar{u}_t\|^2) + \frac{\rho^2 (c_1^2 + c_2^2)}{9q \gamma^2 C_{f}^2} \sum_{m=1}^{\mathcal{M}} \mathbb{E}\|h_{t-1}^m - g^m(\bar{x}_{t-1})\|^2
\]

\[
+ \frac{M \nu}{3 \gamma L_f C_{f}^3} \left( 2c_1^2 \gamma L_f \sigma^2 + c_2^2 \sigma^2 + 4c_2^2 \beta_t^2 + 4c_2^2 \gamma L_f^2 \delta_t^2 + c_2^2 \sigma^2 + 3c_2^2 \delta_t^2 \right) \eta_{t-1}^2,
\]

where the first inequality holds by the above inequality (59) and \( \nu = \frac{1}{q} \), and the last inequality holds by \( \alpha_t = c_1 \eta_{t-1}^2, \beta_t = c_2 \eta_{t-1}^2, \gamma_t = c_3 \eta_{t-1}^2 \) and \( \eta_t \leq \frac{\rho \eta_{t-1}}{2k \gamma L_f C_{f}^2} \) for all \( t \geq 0 \), and \( \frac{c_2^2}{c_1^2} \leq \frac{1}{c_2^2} \).
According to the above inequality (63), we have

\[
\sum_{m=1}^{M} \left( \mathbb{E}\|u_t^m - \bar{u}_t\|^2 + \mathbb{E}\|v_t^m - \bar{v}_t\|^2 \right) 
\leq \frac{\rho^2}{36qC_{f,g}^2} \sum_{s=1}^{t-1} \left( 1 + \frac{5}{4q} \right)^{t-1-s} \sum_{m=1}^{M} \mathbb{E}\|\tilde{d}_s\|^2 
+ \frac{3(c_f^2 + c_g^2)}{8} \sum_{s=1}^{t-1} \left( 1 + \frac{5}{4q} \right)^{t-1-s} \eta_s^2 \sum_{t=1}^{q} M_{\eta_s^2} \left( \mathbb{E}\|e_t^m - \bar{v}_t\|^2 + \mathbb{E}\|u_t^m - \bar{u}_t\|^2 \right) 
+ \frac{\rho^2(c_f^2 + c_g^2)}{9q^7C_{f,g}^2} \sum_{s=1}^{t-1} \left( 1 + \frac{5}{4q} \right)^{t-1-s} \eta_s^2 \sum_{m=1}^{M} \mathbb{E}\|h_t^{m-1} - g^m(\bar{x}_{s-1})\|^2 
+ \frac{M\rho}{3\gamma L_f gC_{f,g}} \left( 2c_f^2 L_f^2 \sigma^2 + c_g^2 \sigma^2 + 4c_f^2 \delta_f^2 + 4c_g^2 L_f^2 \delta_g^2 + c^2 \sigma^2 + 3c^2 \delta_g^2 \right) \sum_{s=1}^{t-1} \left( 1 + \frac{5}{4q} \right)^{t-1-s} \eta_s^3 
\leq \frac{\rho^2}{9qC_{f,g}^2} \sum_{s=1}^{t-1} \left( 1 + \frac{5}{4q} \right)^{t-1-s} \mathbb{E}\|\tilde{d}_s\|^2 
+ \frac{3(c_f^2 + c_g^2)}{2} \sum_{s=1}^{t-1} \eta_s^2 \sum_{m=1}^{M} \mathbb{E}\|e_t^m - \bar{v}_t\|^2 + \mathbb{E}\|u_t^m - \bar{u}_t\|^2 
+ \frac{4\rho^2(c_f^2 + c_g^2)}{9q^7C_{f,g}^2} \sum_{s=1}^{t-1} \eta_s^2 \sum_{m=1}^{M} \mathbb{E}\|h_t^{m-1} - g^m(\bar{x}_{s-1})\|^2 
+ \frac{4M\rho}{3\gamma L_f gC_{f,g}} \left( 2c_f^2 L_f^2 \sigma^2 + c_g^2 \sigma^2 + 4c_f^2 \delta_f^2 + 4c_g^2 L_f^2 \delta_g^2 + c^2 \sigma^2 + 3c^2 \delta_g^2 \right) \sum_{s=1}^{t-1} \eta_s^3,
\]

(70)

where the second inequality holds by \((1 + \frac{5}{4q})^{t-1-s} \leq (1 + \frac{5}{4q})^{t-1} \leq e^{5/4} \leq 4\) and the last inequality holds by \(\eta_s \leq \frac{1}{24\gamma q L_f gC_{f,g}}\) for all \(t \geq 0\).

By multiplying both sides of (70) by \(\eta_t\) and summing over \(t = s_t + q - 1\), we have

\[
\sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{m=1}^{M} \left( \mathbb{E}\|u_t^m - \bar{u}_t\|^2 + \mathbb{E}\|v_t^m - \bar{v}_t\|^2 \right) 
\leq \frac{M\rho^2}{9C_{f,g}^2} \left( \frac{3}{243} + \frac{4\rho^2(c_f^2 + c_g^2)}{24 + 16q^7g^2 L_f^4 C_{f,g}^4} \right) \sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{m=1}^{M} \mathbb{E}\|e_t^m - \bar{v}_t\|^2 + \mathbb{E}\|u_t^m - \bar{u}_t\|^2 
+ \frac{\rho^2(c_f^2 + c_g^2)}{24 + 54q^7g^2 L_f^2 C_{f,g}^2} \sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{m=1}^{M} \mathbb{E}\|h_t^{m-1} - g^m(\bar{x}_t)\|^2 
+ \frac{M\rho^2}{18g^2 L_f^2 C_{f,g}^2} \left( 2c_f^2 L_f^2 \sigma^2 + c_g^2 \sigma^2 + 4c_f^2 \delta_f^2 + 4c_g^2 L_f^2 \delta_g^2 + c^2 \sigma^2 + 3c^2 \delta_g^2 \right) \sum_{t=s_t}^{s_t+q-1} \eta_t^3,
\]

(71)
Given $c_1^2 + c_2^2 \leq \frac{(24^4 \gamma^4 L_f^4 C_f^4)}{\eta_\rho^2}$, we have $\eta_\rho \leq 1 - \frac{\rho^4 (c_1^2 + c_2^2)}{2^{12} \cdot 16 \gamma^4 L_f^4 C_f^4}$, we have

$$\sum_{t = s_1}^{s_1 + q - 1} \eta_t \sum_{m = 1}^{M} (E\|u_t^m - \tilde{u}_t\|^2 + E\|v_t^m - \tilde{v}_t\|^2)$$

$$\leq \frac{2M \rho^2}{15C_f^2} \sum_{t = s_1}^{s_1 + q - 1} \eta_t \|\tilde{d}_t\|^2 + \frac{\rho^4 (c_1^2 + c_2^2)}{1080 \gamma^4 L_f^4 C_f^4} \sum_{t = s_1}^{s_1 + q - 1} \eta_t \sum_{m = 1}^{M} E\|h_{m}^t - g^m(\tilde{x}_t)\|^2$$

$$+ \frac{M \rho^2}{15\gamma^4 L_f^4 C_f^4} \left(2c_1^2 L_f^2 \sigma^2 + c_3^2 \delta_f^2 + 4c_1^2 L_f^2 \sigma^2 + 4c_3^2 \delta_f^2 + c_2^2 \sigma^2 + 3c_2^2 \delta_f^2\right) \sum_{t = s_1}^{s_1 + q - 1} \eta_t^3.$$  \hfill (72)

According to the above inequality [92] and $C_f^2 = \max(C_f^2, C_f^2)$, we have

$$\sum_{m = 1}^{M} E\|d_t^m - \tilde{d}_t\|^2 \leq \frac{9C_f^2 M}{\rho^2} \sum_{m = 1}^{M} E\|u_t^m - \tilde{u}_t\|^2 + \frac{9C_f^2 M}{\rho^2} \sum_{m = 1}^{M} E\|u_t^m - \tilde{u}_t\|^2 \leq \frac{9C_f^2 M}{\rho^2} \sum_{m = 1}^{M} (E\|u_t^m - \tilde{u}_t\|^2 + E\|v_t^m - \tilde{v}_t\|^2).$$  \hfill (73)

Thus we have

$$\sum_{t = s_1}^{s_1 + q - 1} \eta_t \sum_{m = 1}^{M} E\|d_t^m - \tilde{d}_t\|^2$$

$$\leq \frac{9C_f^2 M}{\rho^2} \sum_{t = s_1}^{s_1 + q - 1} \eta_t \sum_{m = 1}^{M} (E\|u_t^m - \tilde{u}_t\|^2 + E\|v_t^m - \tilde{v}_t\|^2)$$

$$\leq \frac{6M}{5} \sum_{t = s_1}^{s_1 + q - 1} \eta_t E\|\tilde{d}_t\|^2 + \frac{\rho^2 (c_1^2 + c_2^2)}{120 \gamma^4 L_f^4 C_f^4} \left(\sum_{t = s_1}^{s_1 + q - 1} \eta_t \sum_{m = 1}^{M} E\|h_{m}^t - g^m(\tilde{x}_t)\|^2\right)$$

$$+ \frac{3M}{5\gamma^4 L_f^4 C_f^4} \left(2c_1^2 L_f^2 \sigma^2 + c_3^2 \delta_f^2 + 4c_1^2 L_f^2 \sigma^2 + 4c_3^2 L_f^2 \delta_f^2 + c_2^2 \sigma^2 + 3c_2^2 \delta_f^2\right) \sum_{t = s_1}^{s_1 + q - 1} \eta_t^3.$$  \hfill (74)

**Theorem 3.** (Restatement of Theorem 1) Assume the sequence $\{\tilde{x}_t\}_{t=1}^{T}$ be generated from AdaMFGCD algorithm. Under the above Assumptions, and let $\eta_t = \frac{k}{(n + t)^{1/2}}$ for all $t \geq 0$, $\alpha_{t+1} = c_1 \eta_t^2$, $\beta_{t+1} = c_2 \eta_t^2$, $\alpha_{t+1} = c_3 \eta_t^2$, $n \geq \max(2, k^2, (c_1 k)^3, (c_2 k)^3, (c_3 k)^3, (24k^2 \delta_f L_f^4 C_f^4)^{1/3})$, $k > 0$, $c_1 \geq \frac{3}{32} \frac{\rho^2}{\eta_\rho^2} + B$, $c_2 \geq \frac{5}{32} \frac{\rho^2}{\eta_\rho^2} + 5C_f^2$, $c_1^2 + c_2^2 \leq \frac{(24)^4 \gamma^4 L_f^4 C_f^4}{\eta_\rho^2}$, $c_3 \geq \frac{1}{32} \frac{\rho^2}{\eta_\rho^2} - 5C_f^2$, $\eta_\rho \leq \gamma \leq \min(\frac{3\rho^2 L_f C_f}{3}, \frac{1}{4(C_f^4 + L_f^2 + 5C_f^2)^{1/3}})$, $B \geq 20C_f^2 L_f^2 + \frac{c_2^2 L_f^2 L_f^4 C_f^4}{30\eta_\rho^2 \gamma^2 L_f^4 C_f^4} + \Theta \left(5C_f^2 L_f^2 + \frac{c_2^2 L_f^2 C_f^4}{864 \eta_\rho^2 \gamma^2 L_f^4 C_f^4}\right)$

\[ \frac{(24)^4 \gamma^4 L_f^4 C_f^4}{\eta_\rho^2} \leq \frac{6M}{5\gamma^4 L_f^4 C_f^4} \left(2c_1^2 L_f^2 \sigma^2 + c_3^2 \delta_f^2 + 4c_1^2 L_f^2 \sigma^2 + 4c_3^2 L_f^2 \delta_f^2 + c_2^2 \sigma^2 + 3c_2^2 \delta_f^2\right) \sum_{t = s_1}^{s_1 + q - 1} \eta_t^3 \]
Proof. Since \( \eta_t = \frac{k}{(n + t)^{3/2}} \) on \( t \) is decreasing and \( n \geq k^3 \), we have \( \eta_t \leq \eta_0 = \frac{k}{(n)^{3/2}} \leq 1 \) and \( \gamma \leq \frac{1}{2k^3} \leq \frac{\rho}{2L_{n0}} \) for any \( t \geq 0 \). Since \( \eta_t \leq \frac{\rho}{23g \sqrt{L_{fg} C_{fg}}} \) for all \( t \geq 0 \), we have \( \frac{k}{n^{3/2}} \leq \tilde{\eta}_0 \leq \frac{\rho}{23g \sqrt{L_{fg} C_{fg}}} \), then we have \( n \geq \frac{(2k^3 \sqrt{L_{fg} C_{fg}})^3}{3} \). Due to \( 0 < \eta_t \leq 1 \) and \( n \geq (c_1 k)^3 \), we have \( \alpha_{t+1} \leq c_1 \eta_t^2 \leq c_1 \eta_t \leq \frac{c_1 k^3}{n^{3/2}} \leq 1 \). Similarly, due to \( n \geq (c_2 k)^3 \) and \( n \geq (c_3 k)^3 \), we have \( \beta_{t+1} \leq c_2 \eta_t^2 \) and \( \beta_{t+1} \leq 1 \).

According to Lemma [3] for any \( m \in [M] \), we have

\[
\frac{1}{\eta_t} E[|h_{t+1}^m - g^m(x_{t+1}^m)|^2 - \frac{1}{\eta_{t-1}} E[|h_t^m - g^m(x_t^m)|^2] \leq \left(1 - \frac{\alpha_{t+1}}{\eta_t} - \frac{1}{\eta_{t-1}} \right) E[|h_t^m - g^m(x_t^m)|^2 + 2L^2 g E[|x_{t+1}^m - x_t^m|^2 + 2\alpha_{t+1}^2] \right)
\]

\[
\leq \left(1 - \frac{\beta_{t+1}}{\eta_t} - \frac{1}{\eta_{t-1}} \right) E[|u_t^m - \nabla g^m(x_t^m)|^2 + 2L^2 g E[|x_{t+1}^m - x_t^m|^2 + 2\beta_{t+1}^2] \right)
\]

where the second equality is due to \( \alpha_{t+1} = c_1 \eta_t^2 \). Similarly, since \( \beta_{t+1} = c_2 \eta_t^2 \), we have

\[
\frac{1}{\eta_t} E[|v_{t+1}^m - \nabla f^m(h_{t+1}^m)|^2 - \frac{1}{\eta_{t-1}} E[|v_t^m - \nabla f^m(h_t^m)|^2] \leq \left(1 - \frac{\beta_{t+1}}{\eta_t} - \frac{1}{\eta_{t-1}} \right) E[|v_t^m - \nabla f^m(h_t^m)|^2 + 2L^2 f E[|x_{t+1}^m - x_t^m|^2 + 2\beta_{t+1}^2] \right)
\]

where the second equality is due to \( \beta_{t+1} = c_2 \eta_t^2 \).

By \( \eta_t = \frac{k}{(n + t)^{3/2}} \), we have

\[
\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} = \frac{1}{k} \left( (n + t)^{3/2} - (n + t - 1)^{3/2} \right) \leq \frac{1}{3k(n + t - 1)^{2/3}} \leq \frac{1}{3k(n + t)^{2/3}} \leq \frac{k}{3k t^{2/3}} \eta_t \leq \frac{2}{3k^3} \eta_t.
\]

where the first inequality holds by the concavity of function \( f(x) = x^{1/3} \), i.e., \( (x + y)^{1/3} \leq x^{1/3} + \frac{y}{3x^{2/3}} \); the second inequality is due to \( n \geq 2 \), and the last inequality is due to \( 0 < \eta_t \leq 1 \).
Let $c_1 \geq \frac{4}{3\text{L}} + B$, for any $m \in [M]$, we have

$$\frac{1}{\eta t} || h_{t+1}^m - g^m(x_t^m) ||^2 - \frac{1}{\eta t - 1} || h_t^m - g^m(x_t^m) ||^2 \leq -B \eta t || h_t^m - g^m(x_t^m) ||^2 + 2C_0^2 \eta t || x_{t+1}^m - x_t^m ||^2 + 2\alpha_{t+1}^2 \sigma^2$$

$$= -B \eta t || h_t^m - g^m(x_t^m) ||^2 + 2C_0^2 \eta t \gamma^2 \overline{E} || d_t - \bar{d}_t ||^2 + 2\alpha_{t+1}^2 \sigma^2$$

$$\leq -B \eta t || h_t^m - g^m(x_t^m) ||^2 + 2C_0^2 \eta t \gamma^2 \overline{E} || d_t - \bar{d}_t ||^2 + 2\alpha_{t+1}^2 \sigma^2$$

$$\leq -B \frac{\eta t}{2} || h_t^m - g^m(x_t^m) ||^2 + BC_0^2 \eta t || x_t^m - \bar{x}_t^m ||^2 + 2C_0^2 \eta t \gamma^2 \overline{E} || d_t - \bar{d}_t ||^2 + 2\alpha_{t+1}^2 \sigma^2,$$

where the last inequality holds by $-|| h_t^m - g^m(x_t^m) ||^2 \leq -\frac{1}{2} || h_t^m - g^m(x_t^m) ||^2 + || g^m(x_t^m) - g^m(\bar{x}_t) ||^2 \leq -\frac{1}{2} || h_t^m - g^m(x_t^m) ||^2 + 2\alpha_{t+1}^2 \sigma^2$.

Let $c_2 \geq \frac{2}{3\text{L}} + 5C_0^2$, for any $m \in [M]$, we have

$$\frac{1}{\eta t} || u_{t+1}^m - \nabla g^m(x_{t+1}^m) ||^2 - \frac{1}{\eta t - 1} || u_t^m - \nabla g^m(x_t^m) ||^2 \leq -5C_0^2 \eta t || u_t^m - \nabla g^m(x_t^m) ||^2 + 2L_0^2 \eta t || x_{t+1}^m - x_t^m ||^2 + 2\gamma_{t+1}^2 \sigma^2$$

$$= -5C_0^2 \eta t || u_t^m - \nabla g^m(x_t^m) ||^2 + 2L_0^2 \eta t \gamma^2 \overline{E} || d_t - \bar{d}_t ||^2 + 2\gamma_{t+1}^2 \sigma^2$$

$$\leq -5C_0^2 \eta t || u_t^m - \nabla g^m(x_t^m) ||^2 + 2L_0^2 \eta t \gamma^2 \overline{E} || d_t - \bar{d}_t ||^2 + 2\gamma_{t+1}^2 \sigma^2$$

$$\leq -5C_0^2 \eta t || u_t^m - \nabla g^m(x_t^m) ||^2 + 2L_0^2 \eta t \gamma^2 \overline{E} || d_t - \bar{d}_t ||^2 + 2\gamma_{t+1}^2 \sigma^2.$$
\[
\sum_{t=s_t}^{s_{t+\varphi-1}} \eta_t \sum_{m=1}^{M} \mathbb{E}[d_t^m - \tilde{d}_t^m]^2 \\
\leq \frac{6M}{5} \sum_{t=s_t}^{s_{t+\varphi-1}} \eta_t \mathbb{E}[\tilde{d}_t^m]^2 + \frac{\rho^2 (c_f^2 + c_g^2)}{120 \eta^2 \gamma^4 C_f^2 L_{f_g}^2 C_g^2} \sum_{t=s_t}^{s_{t+\varphi-1}} \eta_t \sum_{m=1}^{M} \mathbb{E}[h_t^m - g^m(\bar{x})]^2 + \frac{3M \delta^2}{\gamma^4 L_{f_g}^2} \sum_{t=s_t}^{s_{t+\varphi-1}} \eta_t^2, \quad (84)
\]

Next, we define a potential function, for any \( t \geq 1 \)
\[
\Omega_t = \mathbb{E}\left[ F(\bar{x}_t) + \frac{\gamma}{\rho \eta_t} \sum_{m=1}^{M} (\|h_t^m - g^m(x_t^m)\|^2 + \|u_t^m - \nabla g^m(x_t^m)\|^2 + \|v_t^m - \nabla f^m(h_t^m)\|^2) \right].
\]

Then we have
\[
\Omega_{t+1} - \Omega_t
\]
\[
= F(\bar{x}_{t+1}) - F(\bar{x}_t) + \frac{\gamma}{M \rho} \sum_{m=1}^{M} \left( \frac{1}{\eta_t} \mathbb{E}[h_{t+1}^m - g^m(x_{t+1}^m)]^2 - \frac{1}{\eta_t} \mathbb{E}[h_t^m - g^m(x_t^m)]^2 + \frac{1}{\eta_t} \mathbb{E}[u_t^m - \nabla g^m(x_t^m)]^2 \right)
\]
\[
- \frac{1}{\eta_t} \mathbb{E}[u_t^m - \nabla g^m(x_t^m)]^2 + \frac{1}{\eta_t} \mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 - \frac{1}{\eta_t} \mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2
\]
\[
\leq \frac{1}{M} \sum_{m=1}^{M} \left( \frac{2C_f^2 \eta_t}{\rho} \mathbb{E}[u_t^m - \nabla g^m(\bar{x})]^2 + \frac{4C_f^2 \eta_t \gamma}{\rho} \mathbb{E}[u_t^m - \nabla f^m(h_t^m)]^2 + \frac{4C_f^2 \eta_t \gamma}{\rho} \mathbb{E}[h_t^m - g^m(\bar{x})]^2 \right) - \frac{\rho \eta_t}{2} \|\tilde{d}_t^m\|^2
\]
\[
+ \frac{\gamma}{M \rho} \sum_{m=1}^{M} \left( - \frac{B}{2} \mathbb{E}[h_t^m - g^m(x_t^m)]^2 + BC_g^2 \eta_t \mathbb{E}[x_t^m - \bar{x}]^2 + 4C_f^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 4C_f^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 2\beta_{t+1}^2 \sigma^2
\]
\[
- \frac{5C_f^2 \eta_t}{2} \mathbb{E}[u_t^m - \nabla g^m(\bar{x})]^2 + 5C_f^2 \eta_t \mathbb{E}[x_t^m - \bar{x}]^2 + 4L_f^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 4L_f^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 2\beta_{t+1}^2 \sigma^2
\]
\[
- 5C_f^2 \eta_t \mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 + 8L_f^2 C_g^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 8L_f^2 C_g^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 2\beta_{t+1}^2 \sigma^2
\]
\[
+ \frac{e_{g}^2 L_f^2}{864 \eta^2 \gamma^3 L_{f_g}^2 C_f^2} \eta_t \mathbb{E}[x_t^m - \bar{x}]^2 + 8L_f^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2 + 8L_f^2 \eta_t \gamma^2 \mathbb{E}\|d_t^m - \tilde{d}_t^m\|^2
\]
\[
\leq \frac{1}{M} \sum_{m=1}^{M} \left( - \frac{C_f^2 \gamma}{2 \rho} \mathbb{E}[u_t^m - \nabla g^m(\bar{x})]^2 - \frac{C_f^2 \gamma}{\rho} \mathbb{E}[v_t^m - \nabla f^m(h_t^m)]^2 \right)
\]
\[
- \frac{\gamma}{\rho} \left( - \frac{B}{2} - 4C_f^2 L_f^2 - \frac{C_f^2 \eta_t}{864 \eta^2 \gamma^3 L_{f_g}^2 C_f^2} \mathbb{E}[h_t^m - g^m(\bar{x})]^2 \right) - \left( \frac{\rho \eta_t}{2} - \frac{4C_f^2 \eta_t \gamma^3}{\rho} - \frac{4L_f^2 \eta_t \gamma^3}{\rho} - \frac{8L_f^2 C_g^2 \eta_t \gamma^2}{\rho} \right) \|\tilde{d}_t^m\|^2
\]
\[
+ \frac{\gamma}{M \rho} \left( BC_f^2 + 5C_f^2 L_f^2 + \frac{C_f^2 \eta_t}{864 \eta^2 \gamma^3 L_{f_g}^2 C_f^2} \eta_t (q - 1) \sum_{m=1}^{M} \mathbb{E}[d_t^m - \tilde{d}_t^m]^2 \right)
\]
\[
+ \frac{\gamma}{M \rho} \left( 4C_f^2 \eta_t \gamma^2 + 4L_f^2 \eta_t \gamma^2 + 8L_f^2 C_g^2 \eta_t \gamma^2 \right) \sum_{m=1}^{M} \mathbb{E}[d_t^m - \tilde{d}_t^m]^2 + \frac{2\sigma^2 \gamma}{\rho} (\alpha_{t+1}^2 + \beta_{t+1}^2 + \beta_{t+1}^2), \quad (85)
\]

where the first inequality holds by the above inequalities (80), (81), (82) and (83), and the last inequality is due to Lemma \[11\].
Let $s_t = q\lceil t/q \rceil + 1$, summing the above inequality over $t = s_t$ to $s_t + q - 1$, we have

$$\sum_{t=s_t}^{s_t+q-1} (\Omega_{t+1} - \Omega_t) \leq \sum_{t=s_t}^{s_t+q-1} \frac{1}{M} \sum_{m=1}^{M} \left( - \frac{C_2^2 \gamma}{2\rho} \eta_t \|u_t^m - \nabla g^m(\bar{x}_t)\|^2 - \frac{C_2^2 \gamma}{\rho} \eta_t \|v_t^m - \nabla f^m(h_t^m)\|^2 
- \frac{\gamma}{\rho} \left( B - 4C_2^2 L_f^2 - \frac{c_2^2 C_2^2 L_f^2}{864 q^3 \gamma^3 L_f^3 C_f^2 g} \right) \eta_t \|h_t^m - g^m(\bar{x}_t)\|^2 \right) 
- \sum_{t=s_1}^{s_t+q-1} \left( \frac{\rho \gamma \eta_t}{2} - \frac{4C_2^2 \eta_t^2 \gamma^3}{\rho} - 4L_2^4 \eta_t^2 \gamma^3 \right) \|\tilde{d}_t\|^2 
+ \frac{\gamma}{M \rho} \left( BC_2^2 + 5C_2^2 L_g^2 + \frac{c_2^2 C_2^2 L_f^2}{864 q^3 \gamma^3 L_f^3 C_f^2 g} \right) \sum_{t=s_t}^{s_t+q-1} \eta_t (q-1) \sum_{t=s_1}^{t-1} \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} E \|d_t^m - \bar{d}_t\|^2 
+ \frac{\gamma}{M \rho} \left( BC_2^2 + 5C_2^2 L_g^2 + \frac{c_2^2 C_2^2 L_f^2}{864 q^3 \gamma^3 L_f^3 C_f^2 g} \right) \sum_{t=s_t}^{s_t+q-1} \eta_t (q-1) \sum_{t=s_1}^{t-1} \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} E \|d_t^m - \bar{d}_t\|^2 
+ \frac{\gamma}{M \rho} \left( BC_2^2 + 5C_2^2 L_g^2 + \frac{c_2^2 C_2^2 L_f^2}{864 q^3 \gamma^3 L_f^3 C_f^2 g} \right) \sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{t=s_1}^{t-1} \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} E \|d_t^m - \bar{d}_t\|^2 
+ \frac{\gamma}{M \rho} \left( BC_2^2 + 5C_2^2 L_g^2 + \frac{c_2^2 C_2^2 L_f^2}{864 q^3 \gamma^3 L_f^3 C_f^2 g} \right) \sum_{t=s_t}^{s_t+q-1} \eta_t \sum_{t=s_1}^{t-1} \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} E \|d_t^m - \bar{d}_t\|^2 
+ \frac{\gamma^2}{12 q L_f g C_f^2} \left( c_1^2 + c_2^2 + c_3^2 \right) \sum_{t=s_t}^{s_t+q-1} \eta_t^2 \sum_{t=s_1}^{t-1} \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} \eta_t^2 \sum_{m=1}^{M} E \|d_t^m - \bar{d}_t\|^2 \right)
$$

where the second inequality is due to $\eta_t \leq \frac{\rho}{24 \sqrt{2} \gamma C_f C_g} \frac{C_f}{C_g}$ for all $t \geq 0$.

Let $\gamma^2 \geq \frac{\rho^2 C_f^2}{24 \sqrt{2} \gamma C_f C_g}$, we have

$$\frac{\rho^2 C_f^2}{(24)^2 L_f^2 g C_f^2} \frac{\rho^2 (c_1^2 + c_2^2 + c_3^2)}{120 q^2 \gamma^3 C_f^2 L_f^2 g C_f^2} \leq \frac{1}{4}$$

(87)

Set $\Theta = \left( 5C_2^2 L_g^2 + \frac{c_2^2 C_2^2 L_f^2}{864 q^3 \gamma^3 L_f^3 C_f^2 g} \right) \frac{\rho^2}{(24)^2 L_f^2 g C_f^2} + \frac{\gamma^2}{12 q L_f g C_f^2} \left( C_f^2 L_g^2 + 2L_f^2 C_f^2 \right)$ Based on the above Lemma
then we have

\[
\sum_{t+s_1+q-1}^{s_1+q-1} (\Omega_{t+1} - \Omega_t)
\leq \sum_{t+s_1+q-1}^{s_1+q-1} \frac{1}{M} \sum_{m=1}^{M} \left( - \frac{C_2^2 \gamma^2}{2\rho} \eta_t \| u_t^m - \nabla g^m(x_t) \|^2 - \frac{C_2^2 \gamma}{\rho} \eta_t \| v_t^m \|^2 - \nabla f^m(h_t^m) \| ^2 \\
- \frac{\gamma B}{\rho} - 4C_2^2 L_f^2 - \frac{c_7^2 C_2^2 L_f^2}{864 \eta_t^2 \gamma^3 L_{f_g}^2 C_f^2} \eta_t \| h_t^m - g^m(x_t) \|^2 \right) - \gamma \rho \eta_t - 4C_2^2 \eta_t^2 \gamma^3 - \frac{4L_f^2 \eta_t^2 \gamma^3}{\rho} - \frac{8L_f^2 C_2 \eta_t^2 \gamma^3}{\rho} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
+ \gamma \rho M (BC_2^2 + 5C_2^2 L_f^2 + \frac{c_7^2 C_2^2 L_f^2}{864 \eta_t^2 \gamma^3 L_{f_g}^2 C_f^2} (24)^2 L_{f_g} C_f^2) \eta_t \| \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
+ \frac{\gamma}{\rho M} 6qL_{f_g} C_f \left( c_1^2 + c_2^2 + c_3^2 \right) \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
\leq \sum_{t+s_1+q-1}^{s_1+q-1} \frac{1}{M} \sum_{m=1}^{M} \left( - \frac{C_2^2 \gamma^2}{2\rho} \eta_t \| u_t^m - \nabla g^m(x_t) \|^2 - \frac{C_2^2 \gamma}{\rho} \eta_t \| v_t^m \|^2 - \nabla f^m(h_t^m) \| ^2 \\
- \frac{\gamma B}{\rho} - 4C_2^2 L_f^2 - \frac{c_7^2 C_2^2 L_f^2}{864 \eta_t^2 \gamma^3 L_{f_g}^2 C_f^2} \eta_t \| h_t^m - g^m(x_t) \|^2 \right) - \gamma \rho \eta_t - 4C_2^2 \eta_t^2 \gamma^3 - \frac{4L_f^2 \eta_t^2 \gamma^3}{\rho} - \frac{8L_f^2 C_2 \eta_t^2 \gamma^3}{\rho} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
+ \left( \Theta + \frac{BC_2^2 \rho^2}{(24)^2 L_{f_g}^2 C_f^2} \right) \frac{3\gamma^2}{5} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 + \frac{\sigma^2}{12qL_{f_g} C_f} \left( c_1^2 + c_2^2 + c_3^2 \right) \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
\leq \sum_{t+s_1+q-1}^{s_1+q-1} \frac{1}{M} \sum_{m=1}^{M} \left( - \frac{C_2^2 \gamma^2}{2\rho} \eta_t \| u_t^m - \nabla g^m(x_t) \|^2 - \frac{C_2^2 \gamma}{\rho} \eta_t \| v_t^m \|^2 - \nabla f^m(h_t^m) \| ^2 - \frac{\gamma C_2^2 L_f^2}{\rho} \eta_t \| h_t^m - g^m(x_t) \|^2 \right) - \gamma \rho \eta_t - 4C_2^2 \eta_t^2 \gamma^3 - \frac{4L_f^2 \eta_t^2 \gamma^3}{\rho} - \frac{8L_f^2 C_2 \eta_t^2 \gamma^3}{\rho} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
+ \left( \Theta + \frac{BC_2^2 \rho^2}{(24)^2 L_{f_g}^2 C_f^2} \right) \frac{3\gamma^2}{5} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 + \frac{\sigma^2}{12qL_{f_g} C_f} \left( c_1^2 + c_2^2 + c_3^2 \right) \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \right. \\
\left. \leq \sum_{t+s_1+q-1}^{s_1+q-1} \frac{1}{M} \sum_{m=1}^{M} \left( - \frac{C_2^2 \gamma^2}{2\rho} \eta_t \| u_t^m - \nabla g^m(x_t) \|^2 - \frac{C_2^2 \gamma}{\rho} \eta_t \| v_t^m \|^2 - \nabla f^m(h_t^m) \| ^2 - \frac{\gamma C_2^2 L_f^2}{\rho} \eta_t \| h_t^m - g^m(x_t) \|^2 \right) - \gamma \rho \eta_t - 4C_2^2 \eta_t^2 \gamma^3 - \frac{4L_f^2 \eta_t^2 \gamma^3}{\rho} - \frac{8L_f^2 C_2 \eta_t^2 \gamma^3}{\rho} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \\
+ \left( \Theta + \frac{BC_2^2 \rho^2}{(24)^2 L_{f_g}^2 C_f^2} \right) \frac{3\gamma^2}{5} \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 + \frac{\sigma^2}{12qL_{f_g} C_f} \left( c_1^2 + c_2^2 + c_3^2 \right) \sum_{t+s_1+q-1}^{s_1+q-1} \eta_t \| d_t \|^2 \right)
\end{align*}

where the second inequality holds by the above inequality (88), and the last inequality holds by \( B \geq 20C_2^2 L_f^2 + \frac{c_7^2 C_2^2 L_f^2}{864 \eta_t^2 \gamma^3 L_{f_g}^2 C_f^2} + \frac{6\rho \gamma (c_1^2 + c_2^2)}{30 \rho \eta_t^2 \gamma^3 L_{f_g}^2 C_f^2}, \gamma \leq \frac{3\gamma \rho \eta_t \gamma C_f}{4(c_1^2 + c_2^2 + 2L_f^2 C_f^2)} \) (i.e., the following inequality (89)) and \( \Theta + \frac{BC_2^2 \rho^2}{(24)^2 L_{f_g}^2 C_f^2} \leq \frac{5\gamma^2}{5} \).
Since $\eta_t \leq \frac{\rho}{\eta_t |\bar{x}_t|^2}$ and $\gamma \leq \frac{3\rho L_f C_{fg}}{4(C_d^2 + L_d^2 + 2L_f^2 C_d^2)}$, we have

$$\gamma^2 \leq \frac{\rho^2}{32(C_d^2 + L_d^2 + 2L_f^2 C_d^2)} \frac{24q\gamma L_f C_{fg} \rho^2}{\rho} \leq \frac{\rho^2}{32\eta_t(C_d^2 + L_d^2 + 2L_f^2 C_d^2)}.$$  \hspace{1cm} (89)

Summing the above inequality 88 from $t = 1$ to $T$, then we have

$$\sum_{t=1}^{T} (\Omega_{t+1} - \Omega_t) \leq \sum_{t=1}^{T} \frac{1}{1 - \frac{C_d^2 \eta_t \gamma}{C_d^2 \eta_t \gamma + 2L_f^2 C_d^2}} - \sum_{t=1}^{T} \frac{C_d^2 \eta_t \gamma}{C_d^2 \eta_t \gamma + 2L_f^2 C_d^2} \sum_{t=1}^{T} \frac{\rho^2}{\rho} \frac{24q\gamma L_f C_{fg} \rho^2}{\rho} \leq \frac{\rho^2}{32\eta_t(C_d^2 + L_d^2 + 2L_f^2 C_d^2)}.$$  \hspace{1cm} (89)

Since $h^{\omega}_m = \frac{1}{q} \sum_{j=1}^{q} g^{\omega}(x_1^{\omega}; \xi^{\omega}_{1,j})$, $u^{\omega}_m = \frac{1}{q} \sum_{j=1}^{q} \nabla g^{\omega}(x_1^{\omega}; \xi^{\omega}_{1,j})$ and $v^{\omega}_m = \frac{1}{q} \sum_{j=1}^{q} \nabla f(h^{\omega}_m; \xi^{\omega}_{1,j})$ for all $m \in [M]$, we have

$$\Omega_1 = \mathbb{E} \left[ F(\bar{x}_1) + \frac{1}{\rho \eta_0} \sum_{m=1}^{M} \left( \gamma \sum_{m=1}^{M} \left( \|h^{\omega}_m - g^{\omega}(x_1^{\omega})\|^2 + \|u^{\omega}_m - \nabla g^{\omega}(h^{\omega}_m)\|^2 + \|v^{\omega}_m - \nabla f(h^{\omega}_m)\|^2 \right) \right]$$

$$\leq F(\bar{x}_1) + \frac{3\gamma^2}{\eta_0}.$$  \hspace{1cm} (91)

where the last inequality holds by Assumption 2. Since $\eta_t = \frac{\rho}{k + t}$ is decreasing, i.e., $\eta_t^{-1} \geq \eta_t^{-1}$ for any $0 \leq t \leq T$, we have

$$\frac{1}{T} \sum_{t=1}^{T} \left[ \frac{1}{M} \sum_{m=1}^{M} \frac{1}{\rho^2} \left( 2C_d^2 \|u^{\omega}_m - \nabla g^{\omega}(\bar{x}_t)\|^2 + 4C_d^2 \|v^{\omega}_m - \nabla f^{\omega}(h^{\omega}_m)\|^2 + 4C_d^2 L_f^2 \|h^{\omega}_m - g^{\omega}(\bar{x}_t)\|^2 \right) + \|\bar{d}_t\|^2 \right]$$

$$\leq \frac{3\gamma^2}{\eta_0}.$$  \hspace{1cm} (92)

where the second inequality holds by the above inequality (91). Let $G = \frac{4(F(\bar{x}_1) - F^*)}{k \rho \gamma} + \frac{12n^{1/3} \sigma^2}{k^2 \rho^2} + \frac{4k^2 \sigma^2}{3 \rho \gamma L_f C_{fg}} + \frac{\rho^2}{32\eta_t(C_d^2 + L_d^2 + 2L_f^2 C_d^2)}$.
According to Cauchy-Schwarz inequality, we have
\[ 4k^2 \left( \frac{k^2}{4} + \frac{t^2}{3} \right) \ln(n + T) \]. According to the above Lemma \[\text{[1]}\] then we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \frac{1}{\rho^2} \left| \tilde{w}_t - \nabla F(\bar{x}_t) \right|^2 + \| \tilde{d}_t \|^2 \right] \\
\leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^{M} \left( \frac{1}{N} \sum_{n=1}^{N} \left( 2C_{t}^2 \| u_{tn}^m - \nabla g^m(\bar{x}_t) \|^2 + 4C_{t}^2 \| v_{tn}^m - \nabla f^m(h_t^m) \|^2 + 4C_{t}^2 \| h_t^m - g^m(\bar{x}_t) \|^2 \right) + \| \tilde{d}_t \|^2 \right] \\
\leq \frac{G}{T} (n + T)^{1/3}, \quad (93)
\]
where the first inequality holds by Lemma \[\text{[2]}\] and the last inequality holds by \[\text{[92]}\].

Since \( \tilde{d}_t = \frac{\sqrt{\frac{k^2}{4} + \frac{t^2}{3}}}{\sqrt{\frac{k^2}{4} + \frac{t^2}{3}}} = A_t^{-1} \tilde{w}_t \), and let \( \mathcal{G}_t = \frac{1}{\rho} \| \tilde{w}_t - \nabla F(\bar{x}_t) \| + \| \tilde{d}_t \| \), we have
\[
\mathcal{G}_t = \frac{1}{\rho} \| \tilde{w}_t - \nabla F(\bar{x}_t) \| + \| \tilde{d}_t \| \\
= \frac{1}{\rho} \| A_t^{-1} \tilde{w}_t \| + \| \tilde{d}_t \| \\
\geq \frac{1}{\rho} \| A_t^{-1} \tilde{w}_t \| + \frac{1}{\rho} \| \tilde{w}_t - \nabla F(\bar{x}_t) \| \\
\geq \frac{1}{\rho} \| A_t^{-1} \tilde{w}_t \| + \| \tilde{d}_t \| \\
\geq \frac{1}{\rho} \| A_t^{-1} \| \| \nabla F(\bar{x}_t) \| - \| \tilde{d}_t \| \\
\geq \frac{1}{\rho} \| A_t^{-1} \| \| \nabla F(\bar{x}_t) \|, \quad (94)
\]
where the inequality \( (i) \) holds by \( \| A_t \| \geq \rho \) for all \( t \geq 1 \) due to Assumption \[\text{[8]}\]. Then we have
\[
\| \nabla F(\bar{x}_t) \| \leq \| A_t \| \mathcal{G}_t. \quad (95)
\]

According to Cauchy-Schwarz inequality, we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla F(\bar{x}_t) \| \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \mathcal{G}_t \| \| A_t \| \leq \sqrt{\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \mathcal{G}_t \|^2} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| A_t \|^2}. \quad (96)
\]

According to the above inequality \[\text{[93]}\], we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla F(\bar{x}_t) \| \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \frac{2}{\rho^2} \| \tilde{w}_t - \nabla F(\bar{x}_t) \|^2 + 2 \| \tilde{d}_t \|^2 \right] \\
\leq \frac{2G}{T} (n + T)^{1/3}. \quad (97)
\]
Combining the above inequalities \[\text{[95]}\] with \[\text{[97]}\], we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla F(\bar{x}_t) \| \leq \sqrt{\frac{\sqrt{2Gt^{1/2}}}{T^{1/2}} + \frac{\sqrt{2Gt^{1/2}}}{T^{1/2}}} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| A_t \|^2}. \quad (98)
\]