SUBGROUPS OF THE DIRECT PRODUCT OF GRAPHS OF GROUPS WITH FREE
ABELIAN VERTEX GROUPS

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ABSTRACT. A result of Baumslag and Roseblade states that a finitely presented subgroup of the direct
product of two free groups is virtually a direct product of free groups. In this paper we generalise this
result to the class of cyclic subgroup separable graphs of groups with free abelian vertex groups and
cyclic edge groups. More precisely, we show that a finitely presented subgroup of the direct product
of two groups in this class is virtually $H$-by-(free abelian), where $H$ is the direct product of two groups
in the class. In particular, our result applies to 2-dimensional coherent right-angled Artin groups and
residually finite tubular groups. Furthermore, we show that the multiple conjugacy problem and
the membership problem are decidable for finitely presented subgroups of the direct product of two
2-dimensional coherent RAAGs.

1. Introduction

Since the 60’s it is well known that finitely generated subgroups of the direct product of non-abelian
free groups are very complex and, in particular, most algorithmic problems such as the conjugacy,
the isomorphism and the membership problem are undecidable (see [28, 29]).

In [19], Grunewald showed that the subgroups with this complex behaviour are not finitely
presented. Remarkably, in [3] (see [8, 30, 33] for alternative proofs), Baumslag and Roseblade
clarified the situation and proved that finitely presented subgroups of the direct product of two
free groups have a very tame structure - they are virtually the direct product of two free groups.
The study of finitely presented subgroups of the direct product of arbitrarily many free groups (and
more generally, limit groups over free groups) was conducted in a series of papers that culminated
in [7] where the authors prove that these subgroups also have a tame structure and that the main
algorithmic problems are decidable.

Right-angled Artin groups (RAAGs) are defined by presentations where the relations are commuta-
tion of some pairs of generators and so it extends the class of (direct products of) finitely generated
free groups. In view of the results about subgroups of the direct product of free groups, one
may wonder if finitely presented subgroups of RAAGs have a tame structure and, in particular,
if the main algorithmic problems are decidable in that class. Unfortunately, this is not the case as
Bridson showed in [5] that there is a right-angled Artin group $A$ and a finitely presented subgroup
$S < A \times A$ for which the conjugacy and the membership problems are undecidable.

This work is part of a series that aims at describing the structure of finitely presented subgroups of
the direct product of (limit groups over) coherent RAAGs, that is, the structure of finitely presented
residually coherent RAAGs, and at showing that the main algorithmic problems are decidable for
this class. This programme was carried over for the subclass of RAAGs whose finitely generated
subgroups are again RAAGs in [27].

In this paper, we begin studying the class of 2-dimensional coherent RAAGs. More precisely, we
generalise Baumslag and Roseblade’s result for free groups and we describe the structure of finitely
presented subgroups of the direct product of two 2-dimensional coherent RAAGs:
**Theorem.** Let $S$ be a finitely presented subgroup of the direct product of two 2-dimensional coherent RAAGs. Then, $S$ is virtually $H$-by-(free abelian), where $H$ is the direct product of two subgroups of 2-dimensional coherent RAAGs.

Furthermore, we show that these finitely presented subgroups have a good algorithmic behaviour. Namely, we prove the following:

**Corollary.** Finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs have decidable multiple conjugacy problem and membership problem.

This corollary shows that Bridson’s example of a right-angled Artin group $A$ and an algorithmically bad finitely presented subgroup of $A \times A$ is not 2-dimensional coherent (and we conjecture that $A$ cannot be coherent).

In fact, our results apply to a wider class of groups, the class $\mathcal{A}_t$ which is the $\mathbb{Z}^*$ closure (see Section 2 for the definition) of the class of cyclic subgroup separable graphs of groups with free abelian vertex groups and cyclic edge groups. This class contains 2-dimensional coherent RAAGs and residually finite tubular groups among others. Recall that a tubular group is a finitely generated graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups.

**Theorem 5.1.** Let $S$ be a finitely presented subgroup of $G_1 \times G_2$ where $G_1, G_2 \in \mathcal{A}$. Then, $S$ is virtually $H$-by-(free abelian), where $H$ is the direct product of two groups in $\mathcal{A}$.

Furthermore, $S$ is virtually the kernel of a homomorphism $f$ from $H_1 \times H_2$ to $\mathbb{R}$, for some $H_1, H_2 \in \mathcal{A}$. More precisely, $H$ is equal to $L_1 \times L_2$, where $L_i = S \cap G_i$, $i \in \{1, 2\}$ and either

- $L_1 \times L_2 <_{f_1} S <_{f_1} G_1 \times G_2$, or
- $S$ is virtually the kernel $\ker f$ where $f : H_1 \times H_2 \mapsto \mathbb{Z}$ for some $H_i \in \mathcal{A}$, $i \in \{1, 2\}$, or
- $S$ is virtually the kernel $\ker f$ where $f : H_1 \times H_2 \mapsto \mathbb{Z}^2$ for some $H_i \in \mathcal{A}$, $i \in \{1, 2\}$. In this case, $L_i$ is the free product of finitely generated groups in $\mathcal{A}$ for $i \in \{1, 2\}$.

In [4], Bestvina and Brady examined the finiteness properties of the kernels of homomorphisms from RAAGs to $\mathbb{Z}$ using Morse theory. As a consequence of our main result, we obtain that the finitely presented subgroup $S$ is virtually a kernel of a homomorphism from the direct product of two groups in our class to $\mathbb{Z}^n$, and so we expect that Morse theory can also be useful to further study the finiteness properties of these subgroups.

The abelian factor in the description of $S$ is directly related to the edge groups of the decomposition of the groups in our class. In particular, if we consider graphs of groups in $\mathcal{A}$ with trivial edge groups, we deduce the following theorem and recover the result of Baumslag and Roseblade for direct products of free groups:

**Theorem 5.3.** Let $\mathcal{A}'$ be the subclass of $\mathcal{A}$ containing the groups which have a non-trivial free product decomposition and let $S$ be a finitely presented subgroup of the direct product of two groups in the class $\mathcal{A}'$. Then, $S$ is virtually the direct product of two subgroups in $\mathcal{A}'$.

We require the graphs of groups to be cyclic subgroup separable. Recall that a group $G$ is cyclic subgroup separable if for each cyclic subgroup $C \subset G$ and each $h \in G \setminus C$, there is a finite quotient $Q$ of $G$ and a homomorphism $\pi : G \mapsto Q$ such that $\pi(h) \not\in \pi(C)$. Therefore, cyclic subgroup separability is a residual property that generalises residual finiteness. Although cyclic subgroup separability
is a property of algebraic nature, we believe that it carries consequences on the geometry of the
groups in the class $A$. Indeed, we believe that the groups in the class $A$ are CAT(0) and in fact we
conjecture that they are virtually compact special. More precisely, we propose the following:

**Conjecture 1.1.** Let $G$ be a finitely generated graph of groups with free abelian vertex groups and
with cyclic edge groups. Then, the following are equivalent:

- $G$ is cyclic subgroup separable;
- $G$ has a finite index subgroup with isolated edge groups;
- $G$ is virtually compact special.

Furthermore, if $G$ is freely indecomposable, then $G$ is cyclic subgroup separable if and only if it is
residually finite and non-solvable.

Recall that a subgroup $H < G$ is isolated if, for each $g \in G$ and each $n \in \mathbb{N} \setminus \{0\}$, $g^n \in H$ implies
$g \in H$.

We review some of the results that support the conjecture. In the case of Generalised Baumslag-
Solitar groups, we have that they are cyclic subgroup separable if and only if they are unimodular
if and only if they have a finite index subgroup with isolated edge groups if and only if they are non-solvable and residually finite (see [26, Corollary 7.7]). Unimodular Generalised Baumslag-
Solitar groups are precisely the virtually compact special ones. Therefore, the conjecture holds for
Generalised Baumslag-Solitar groups.

In [20], the authors prove that for tubular groups the properties of being cyclic subgroup separable,
of being residually finite and of having a finite index subgroup with isolated edge groups are equivalent. Furthermore, in [10], Button proves that a tubular group whose underlying graph is a
tree is virtually special.

For the general case, any finitely generated graph of groups such that the underlying graph is
a tree, with free abelian vertex groups and with cyclic edge groups is cyclic subgroup separable
(see [18, Theorem 3.4]). As in the case of tubular groups, we expect these groups to be virtually
special.

If the underlying graph has only one loop, that is, if $G = \langle T, s \mid a^s = b \rangle$ and $T$ is a tree of free abelian
vertex groups and cyclic edge groups, then the group $G$ is cyclic subgroup separable if and only if
it has a finite index subgroup which is a graph of groups with isolated edge groups. Furthermore,
the later is equivalent to either $(a) \cap (b) = \{1\}$ or $a = c^k$ and $b = c^{\pm k}$ for some $c \in G$ and $k \in \mathbb{Z}$ (see
[23, Corollary 3.5]).

In addition, for graphs of groups with free abelian vertex groups and cyclic edge groups, having
isolated edge groups is a sufficient condition in order to be cyclic subgroup separable (see [23,
Theorem 3.7]).

As mentioned above, non-unimodular (Generalised) Baumslag-Solitar groups are not cyclic sub-
group separable, and so they do not belong to our class. We therefore ask the following:

**Question 1.2.** What is the structure of finitely presented subgroups of the direct product of two
non-unimodular (Generalised) Baumslag-Solitar groups?

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1In [20], the authors use the terminology virtually primitive.
Our results apply to 2-dimensional coherent RAAGs but we conjecture that they should extend to all coherent RAAGs and, more generally, to cyclic subgroup separable graphs of groups with free abelian vertex groups. Interestingly enough, in order to generalise the results from 2-dimensional coherent RAAGs to all coherent RAAGs, it would suffice to prove a generalised version of Itô’s theorem (see [21]). Itô’s theorem states that if a group $Q$ is the product $AB$ of two abelian subgroups $A$ and $B$, then $Q$ is metabelian. In our case, we would need to generalise this result to finitely many cosets of a product of two abelian groups, namely, if

$$Q = A_1 A_2 \cup A_1 A_2 a_1 \cup \cdots \cup A_1 A_2 a_r,$$

where $A_1$ and $A_2$ are abelian subgroups, then one would like to conclude that the group $Q$ is virtually metabelian. We did not succeed in proving this general statement and so we formulate it as a question:

**Question 1.3.** Can Itô’s theorem be extended in the presence of cosets? That is, if $Q$ is a group such that

$$Q = A_1 A_2 \cup A_1 A_2 a_1 \cup \cdots \cup A_1 A_2 a_r,$$

where $A_1$ and $A_2$ are abelian subgroups, is $Q$ virtually metabelian?

When one restricts to groups in our class, that is with cyclic edge groups, the abelian subgroups $A_1$ and $A_2$ in the decomposition above are actually cyclic groups. This allows us to further reduce the problem to the case when a Baumslag-Solitar group admits such a decomposition, that is when a Baumslag-Solitar group can be covered by finitely many cosets of the product of two cyclic subgroups. In this case, we use the structure of Baumslag-Solitar groups to show that this is only possible when the Baumslag-Solitar group is abelian.

We also notice that for graphs of groups with free abelian edge groups the condition on cyclic subgroup separability is more subtle than in the case of cyclic edge groups as there are examples of amalgamated free products of free abelian groups that are not residually finite (see [25, Remark 11.6]).

In relation to algorithmic problems, it is known that the isomorphism problem for finitely presented subgroups of the direct product of two free groups is decidable (see [7]), although for finitely presented subgroups of the direct product of finitely many free groups it is still open. In the case of 2-dimensional coherent RAAGs, we formulate it as a question:

**Question 1.4.** Is the isomorphism problem for finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs decidable?

The paper is organized as follows. In Section 2 we introduce the class of groups that we will study and describe some properties of these groups.

In Section 3 we review Miller’s proof for free groups. The idea is to first show that if $S$ is a finitely presented subgroup of the direct product of two free groups $F_1$ and $F_2$, then the subgroups $L_i = S \cap F_i$ are finitely generated, and after that use the fact that non-trivial finitely generated normal subgroups of free groups have finite index to conclude that the direct product $L_1 \times L_2$ has finite index in $S$.

When considering a finitely presented subgroup $S$ of the direct product of two 2-dimensional coherent RAAGs, say $G_1$ and $G_2$, the intersections $S \cap G_1$ and $S \cap G_2$ are not necessarily finitely generated. Furthermore, non-trivial finitely generated normal subgroups of coherent RAAGs
do not need to be of finite index. Indeed, coherent RAAGs fiber, that is, they admit non-trivial epimorphisms onto $\mathbb{Z}$ with finitely generated kernel.

We address these issues in Section 4. We show that although the subgroup $L_i$ may not be finitely generated, a cyclic extension of $L_i$ is (see Proposition 4.5). We also characterise finitely generated normal subgroups $N$ of a group $G$ in $\mathcal{A}$: either $N$ is in the center of the group $G$ or $G/N$ is virtually abelian (see Proposition 4.4).

Finally, in Section 5, the main result is proved. We show that the quotient $Q = G_i/L_i$ is covered by finitely many cosets of the product of two cyclic subgroups and that this covering lifts to a covering of a Baumslag-Solitar group. We use the structure of Baumslag-Solitar groups to deduce that it is free abelian and conclude that $Q$ is virtually free abelian.

2. The class of groups $\mathcal{A}$

Given a class of groups, there is a natural way to construct other groups using operations such as taking free products or adding center.

**Definition 2.1.** Let $C$ be a class of groups. The $Z^*$-closure of $C$, denoted by $Z^*(C)$, is the union of classes $(Z^*(C))_k$ defined recursively as follows. At level 0, the class $(Z^*(C))_0$ is the class $C$. A group $G$ lies in $(Z^*(C))_k$ for $k \geq 1$ if and only if

$$G \cong \mathbb{Z}^m \times (G_1 \times \cdots \times G_n),$$

where $m \in \mathbb{N} \cup \{0\}$ and the groups $G_i$ belong to $(Z^*(C))_{k-1}$ for all $i \in \{1, \ldots, n\}$.

If the class $C$ is $\mathbb{Z}$, then Droms proved in [16] that the $Z^*$-closure of $\mathbb{Z}$ is precisely the family of RAAGs with the property that all their finitely generated subgroups are again RAAGs.

The class of groups $\mathcal{A}$ is defined as the $Z^*$-closure of the class of groups $\mathcal{G}$, defined as follows.

**Definition 2.2** (Class $\mathcal{A}$ and class $\mathcal{G}$). Let $\mathcal{G}$ be the class of cyclic subgroup separable graphs of groups with free abelian vertex groups and cyclic edge groups.

The class of groups $\mathcal{A}$ is the $Z^*$-closure of the class $\mathcal{G}$.

**Definition 2.3.** Let $G$ be a group in the class $\mathcal{G}$. Any splitting of $G$ as a graph of groups with free abelian vertex groups and cyclic edge groups is called a standard splitting of $G$.

Notice that cyclic subgroup separability extends from the class $\mathcal{G}$ to the class $\mathcal{A}$; that is, if $G$ is a group in $\mathcal{A}$, then $G$ is cyclic subgroup separable. Indeed, by assumption, groups in the class $\mathcal{G}$ are cyclic subgroup separable. Furthermore, cyclic subgroup separability is closed under taking free products (see [18] Lemma 2.3) and adding center: $G$ is cyclic subgroup separable if and only if so is $\mathbb{Z}^n \times G$, $n \in \mathbb{N} \cup \{0\}$.

The class of groups $\mathcal{A}$ is closed under taking subgroups: if $G \in \mathcal{A}$ and $H < G$, then $H \in \mathcal{A}$. In particular, subgroups of groups in $\mathcal{A}$, modulo the center, have a non-trivial free product decomposition or split over an infinite cyclic subgroup. More precisely, we have the following

**Lemma 2.4.** Let $G$ be a group of $\mathcal{A}$ and let $H$ be a finitely generated subgroup of $G$. Then, $H \in \mathcal{A}$ and $H \cong \mathbb{Z}^n \times H'$ for $n \in \mathbb{N} \cup \{0\}$, where

- $H'$ is a non-trivial free product of groups in $\mathcal{A}$, and so in particular, it is residually finite, or
• $H'$ belongs to $\mathcal{G}$.

**Proof.** The proof is by induction on the level of the group $G \in \mathcal{A}$. The class $\mathcal{G}$ is closed under subgroups: if $G \in \mathcal{G}$, then it is a cyclic subgroup separable graph of groups with free abelian vertex groups and cyclic edge groups and so is any finitely generated subgroup $H$.

Assume now that the level of $G$ is $k > 0$. Then, $G$ is of the form $\mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n)$ for some $m \in \mathbb{N} \cup \{0\}$ and $G_i \in \mathcal{A}$ of level less than $k$. Then, there is a short exact sequence

$$1 \longrightarrow \mathbb{Z}^m \longrightarrow G \xrightarrow{p} G_1 \ast \cdots \ast G_n \longrightarrow 1.$$

It follows that $H$ is of the form $(H \cap \mathbb{Z}^m) \times p(H)$ and $p(H)$ is a subgroup of $G_1 \ast \cdots \ast G_n$. If $n = 1$, $p(H)$ is a group of $G$ and so the first alternative of the statement holds. If $n > 1$, then the result follows from the Kurosh subgroup theorem and the induction hypothesis. □

The class $\mathcal{A}$ contains interesting families of groups. Of special interest to us are 2-dimensional coherent RAAGs and residually finite tubular groups. We next recall the definitions of these families of groups and show that they indeed belong to the class $\mathcal{A}$.

Recall that given a finite simplicial graph $X$ with vertex set $V(X)$ and edge set $E(X)$, the corresponding **right-angled Artin group** (RAAG), denoted by $G_X$, is given by the following presentation:

$$G_X = \langle V(X) \mid xy = yx \iff (x, y) \in E(X) \rangle.$$

To each right-angled Artin group $G_X$, we can associate a CAT(0) cube complex, called the **Salvetti complex**. The **dimension** of the RAAG $G_X$ is defined as the (topological) dimension of the Salvetti complex which coincides with the number of vertices of the largest complete full subgraph of $X$. In particular, $G_X$ is 2-dimensional if and only if the underlying graph $X$ is triangle-free. See [13] for a general survey of right-angled Artin groups.

Recall that a group is **coherent** if all its finitely generated subgroups are also finitely presented. Carls Droms characterized coherent RAAGs in terms of the defining graph as follows (see [15]): the group $G_X$ is coherent if and only if $X$ does not have induced cycles of length greater than 3. In particular, 2-dimensional coherent RAAGs are defined by graphs which are forests, that is, they are free products of tree groups, where by a **tree group** we mean a right-angled Artin group whose defining graph is a tree.

The RAAG defined by the path with 4 vertices $P_4$ and which, abusing the notation, we also denote by $P_4$ is the group given by the presentation:

$$\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle.$$

Note that $P_4$ admits the following splitting as a graph of groups with free abelian vertex groups and cyclic edge groups:

$$P_4 = \langle a, b \rangle \ast \langle b, c \rangle \ast \langle c, d \rangle,$$

and it is cyclic subgroup separable (see, for instance, [18 Theorem 3.4]). Therefore, $P_4 \in \mathcal{G}$.

The RAAG $P_4$ plays an important role in the theory of 2-dimensional coherent RAAGs as it serves as universe for them. Indeed, in [24] it is shown that any 2-dimensional coherent RAAG embeds in the group $P_4$. In particular, since $P_4$ belongs to $\mathcal{G}$ and the class is closed under subgroups, we deduce that 2-dimensional coherent RAAGs also belong to $\mathcal{G}$.

In fact, all tree groups are examples of **residually finite tubular groups**. Tubular groups are finitely generated groups that split as graphs of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. Despite their simple definition, they have a surprisingly rich source of diverse behaviour. Tubular groups
provide examples of finitely generated 3-manifold groups that are not subgroup separable; of free-by-cyclic groups that do not act properly and semi-simply on a CAT(0) space; of groups that are CAT(0) but not Hopfian, etc (see [20] and references there). In the same paper, the authors give a characterisation of cyclic subgroup separable tubular groups and prove that a tubular group is cyclic subgroup separable if and only if it is residually finite if and only if it is virtually primitive. Recall that a tubular group $G$ is primitive if each edge group is a maximal cyclic subgroup of its vertex groups. From this characterisation we have that residually finite tubular groups belong to the class $\mathcal{A}$.

3. Miller’s proof and counterexamples in tree groups

Baumslag and Roseblade’s result states that given $F_1$ and $F_2$ two finitely generated free groups and $S$ a finitely presented subgroup of $F_1 \times F_2$, then $S$ is free or $S$ is virtually the direct product of two free groups.

We now briefly sketch Miller’s strategy to highlight the relevant properties of free groups that are used in his proof of the aforementioned result. Recall that a subgroup of a direct product is called a subdirect product if its projection to each factor is surjective.

First of all, we can reduce to the case when $S$ is a subdirect product. Indeed, if we consider the projection maps $p_1 : S \mapsto F_1$ and $p_2 : S \mapsto F_2$, $p_i(S)$ is a finitely generated free group for $i \in \{1, 2\}$, so we can assume that the projection maps are surjective.

Let us define $L_i$ to be $S \cap F_i$, $i \in \{1, 2\}$. That is,

$$L_1 = \{ s \in F_1 \mid (s, 1) \in S \} \quad \text{and} \quad L_2 = \{ s \in F_2 \mid (1, s) \in S \}.$$

Observe that there is a short exact sequence

$$1 \longrightarrow L_2 \longrightarrow S \overset{p_1}{\longrightarrow} F_1 \longrightarrow 1.$$

If $L_2$ is trivial, $S$ is isomorphic to $F_1$ and so $S$ is free. A symmetric argument applies if $L_1$ is trivial.

Now assume that $L_1$ and $L_2$ are both non-trivial. Miller then proves, by using Marshall Hall’s theorem for free groups, that for $i \in \{1, 2\}$, $S$ is virtually an HNN extension with associated subgroup $L_i$ and since $S$ is finitely presented, $L_i$ needs to be finitely generated (see [30] Lemma 2, Theorem 1).

From here one deduces that for $i \in \{1, 2\}$, $L_i$ is a non-trivial finitely generated normal subgroup of $F_i$, so $L_i$ has finite index in $F_i$. Hence, since $L_1 \times L_2$ is a subgroup of $S$, $L_1 \times L_2$ has finite index in $S$.

Summarizing, the key points in Miller’s proof are the following ones:

1. Subgroups of free groups are free;
2. if $S$ is finitely presented, the groups $L_1$ and $L_2$ are finitely generated;
3. finitely generated non-trivial normal subgroups of free groups are of finite index.

In the rest of the section, we show that none of the above conditions necessarily hold for tree groups. For that, we will use the right-angled Artin group $P_4$. 7
Firstly, let us give an example to show that non-trivial finitely generated normal subgroups do not need to have finite index in tree groups. Consider the homomorphism \( \varphi: P_4 \rightarrow \mathbb{Z} \) defined as
\[
\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = 1,
\]
and let \( S \) be the kernel of that homomorphism. It can be checked that \( S \) is a free group of rank 3 with basis \( \{ab^{-1}, bc^{-1}, cd^{-1}\} \). In particular, it is a non-trivial finitely generated normal subgroup of \( P_4 \), but \( P_4/S \) is infinite cyclic.

Secondly, subgroups of tree groups do not need to be tree groups or not even RAAGs. In [16], Droms considers the homomorphism \( \alpha: P_4 \rightarrow \mathbb{Z}_2 \) such that
\[
\alpha(a) = \alpha(b) = \alpha(c) = \alpha(d) = 1,
\]
and shows that the kernel of that homomorphism is not a right-angled Artin group.

Finally, let us give an example of a finitely presented subgroup of \( P_4 \times P_4 \) such that \( L_1 \) is not finitely generated. Suppose that
\[
P_4^1 = \langle a, b, c, d \rangle \quad \text{and} \quad P_4^2 = \langle a', b', c', d' \rangle.
\]
Consider the homomorphism \( f: P_4^1 \times P_4^2 \rightarrow \mathbb{Z} \) defined as
\[
f(a) = f(b) = f(d) = 1, \quad f(c) = 0, \quad f(a') = f(b') = f(c') = f(d') = 1.
\]
The kernel of that homomorphism, say \( S \), is finitely presented, but \( S \cap P_4^1 \) is not finitely generated, see [4].

When studying the structure of subgroups of the direct product of \( n \) (limit groups over) free groups, Bridson, Howie, Miller and Short require stronger finiteness conditions (see [8]), namely they consider subgroups of type \( FP_n(Q) \), in order to generalise Baumslag and Roseblade’s result and obtain a structure theorem for these subgroups. In the view of this, one may wonder whether requiring stronger finiteness conditions on the subgroup \( S \) may improve the situation, and for instance, ensure that the subgroups \( L_1 \) and \( L_2 \) are finitely generated. However, this is not the case. In our last example, let us consider the short exact sequence
\[
1 \longrightarrow L_2 \longrightarrow S \overset{P_4}{\longrightarrow} P_4^1 \longrightarrow 1.
\]
The associated Hochschild-Lyndon-Serre spectral sequence converging to \( H_n(S; \mathbb{Z}) \) has
\[
E_2^{pq} = H_p(P_4^1; H_q(L_2; \mathbb{Z})).
\]
Since \( L_2 \) is a free group, \( H_q(L_2; \mathbb{Z}) = 0 \) for all \( q \geq 2 \). It follows that the only terms with \( p + q = n \) that can possibly be nonzero are
\[
H_n(P_4^1, H_0(L_2; \mathbb{Z})) \quad \text{and} \quad H_{n-1}(P_4^1, H_1(L_2; \mathbb{Z})).
\]
Right-angled Artin groups are of type \( FP_\infty \), so both of them are finitely generated abelian groups. Since subgroups and quotients of finitely generated abelian groups are finitely generated, it follows that all the groups on the \( E^\infty \) page that contribute to \( H_n(S; \mathbb{Z}) \) are finitely generated. Therefore, \( H_n(S; \mathbb{Z}) \) is finitely generated for all \( n \geq 0 \).

4. Alternative properties in the class \( \mathcal{G} \)

The aim of this section is to study the class \( \mathcal{G} \) and to see how the properties of free groups used in Miller’s proof generalise for this class. Recall from the previous section that there are three key properties:

1. Subgroups of free groups are free;
2. If \( S \) is finitely presented, the groups \( L_1 \) and \( L_2 \) are finitely generated;
(3) finitely generated non-trivial normal subgroups of free groups are of finite index.

In this section we will prove that these properties generalise to the following ones for groups in \( G \):

1. Subgroups of groups in \( G \) lie in \( G \);
2. if \( S \) is finitely presented, then a cyclic extension of \( L_i \) is finitely generated, \( i \in \{1,2\} \) (see Proposition[4.5]);
3. if \( N \) is a non-trivial finitely generated normal subgroup of a group \( G \in G \) with trivial center, then \( G/N \) is either finite or virtually abelian (see Proposition[4.4]).

Recall that if \( G \) is a graph of groups and \( T \) is the Bass-Serre tree corresponding to a splitting of \( G \), an element \( g \) of a group \( G \) satisfies the weak proper discontinuity condition (or \( g \) is a WPD element) if for each vertex group \( A \), we have that \( A \cap A^g \) is a finite group. In our case, the vertex groups are torsion-free, so the condition that \( A \cap A^g \) is a finite group reduces to \( A \cap A^g = 1 \).

In order to consider actions with non-trivial kernel, we define relative WPD elements.

**Definition 4.1.** Let \( G \) be a group in \( G \), let \( T \) be the Bass-Serre tree corresponding to a standard splitting of \( G \) and let \( K \) be the kernel of the action of \( G \) on \( T \). An element \( h \) of \( G \) is a relative WPD element if for each vertex group \( A \), \( A \cap A^h \) is the semidirect product of \( K \) and a finite group.

Note that if the vertex groups are torsion-free, then a relative WPD element requires \( A \cap A^g = K \). Furthermore, if the action of \( G \) on \( T \) is faithful, i.e. \( K = 1 \), then a relative WPD element is a WPD element.

We first prove the existence of relative WPD elements in groups in the class \( G \).

**Lemma 4.2.** Let \( G \) be a finitely generated group in \( G \), let \( T \) be the Bass-Serre tree corresponding to a standard splitting of \( G \) and let \( K \) be the kernel of the action of \( G \) on \( T \). Then, \( G \) has a relative WPD element. Moreover, there is a finite index subgroup \( G_1 \) of \( G \) that has center \( K \).

**Proof.** Let \( G \in G \) be finitely generated. If \( G \) has a non-trivial free product decomposition, then \( G \) has a WPD element, see [31]. Therefore, we can assume that \( G \) is freely indecomposable, that is, all edge groups are infinite cyclic.

Let \( \Gamma \) be the underlying graph of \( G \). Note that since \( G \) is finitely generated, \( \Gamma \) is a finite graph.

Let \( Y_0 \) be a lift of a maximal tree of \( \Gamma \) in the Bass-Serre tree \( T \) and let \( t_1,\ldots,t_s \) be the stable letters corresponding to the edges in \( \Gamma \setminus Y_0 \). Let \( C \) be the intersection \( \bigcap_{v \in V(Y_0), g \in \{1,t_1,\ldots,t_s\}} G_{v,g} \), where \( G_v \) is the vertex stabiliser of \( v \in V(Y_0) \).

First, assume that \( C \) is infinite cyclic generated by \( c \). Then, since edge groups are cyclic, \( C \) has finite index in each edge subgroup of a vertex group \( G_{v,g}, v \in V(Y_0), g \in \{1,t_1,\ldots,t_s\} \). The vertex groups are abelian, so \( C \) is central in the vertex stabilizers of vertices in \( Y_0 \).

Furthermore, since \( C \) has finite index in each edge group, we have that for each stable letter \( t \in \{t_1,\ldots,t_s\} \), there are \( n = n(t), m = m(t) \in \mathbb{Z} \) such that \( c^n = c^m \). Thus, \( \langle t,c \rangle \) is isomorphic to the Baumslag-Solitar group \( BS(m,n) \). Since \( G \) is cyclic subgroup separable, so is \( \langle t,c \rangle \) and by [26, Corollary 7.7], we have that \( |m| = |n| \).

Therefore, \( t \) normalises the subgroup \( \langle c^m \rangle = \langle c^n \rangle \) for each stable letter \( t \in \{t_1,\ldots,t_s\} \). It follows that there is a power of \( c \), say \( c^k \), that is normalized by \( t \) and commutes with \( t^i \) for all \( i \in \{1,\ldots,s\} \) and it also commutes with the vertex stabilizers of vertices in \( Y_0 \). Without loss of generality, we assume that \( k \) is positive and minimal with these properties.

Therefore, the subgroup \( \langle c^k \rangle \) is normal in \( G \) and \( K = \langle c^k \rangle \). Since \( \langle c^k \rangle \) is the kernel of the action, there is an equivariant epimorphism \( G \mapsto G/\langle c^k \rangle \). Now, \( G/\langle c^k \rangle \) acts on \( T \) acylindrically since it has
finite edge stabilisers. It follows that \( G/\langle c^s \rangle \) has a WPD element and so \( G \) contains a relative WPD element.

Let \( \alpha : G \mapsto \langle t_1, \ldots, t_s \rangle \) be the epimorphism that sends the vertex groups to 1 and let \( \beta \) be the epimorphism \( \langle t_1, \ldots, t_s \rangle \mapsto \mathbb{Z}_2 \) that sends each generator \( t_i \) to the generator of the group of order 2. The kernel \( G' < G \) of the epimorphism \( \beta \circ \alpha \) is an index 2 subgroup which contains precisely the set of words \( w \) that have an even number of letters \( \{t_1, \ldots, t_s, t_1^{-1}, \ldots, t_s^{-1}\} \). As we mentioned, \( (c^k)^i = c^{ik} \) for \( i \in \{1, \ldots, s\} \) and \( c^k \) commutes with the elements of the vertex groups. It follows that if \( w \in G' \), then \( w \) commutes with \( c^k \). Therefore, \( G' \) is an index 2 subgroup of \( G \) with center \( \langle c^k \rangle \).

Assume now that \( C \) is trivial. Let \( g = \prod_{v \in V(Y_0), j \in \{1, \ldots, s\}} a_v^{-1} a_v^j \), where \( a_v \) is an element of the stabiliser \( G_v \) of the vertex \( v \in V(Y_0) \).

If \( h \in G_v \cap G_v^g \), then \( h \) fixes the path from the vertex \( G_v \) to the the vertex \( gG_v \). Since this path contains all the vertices \( G_u^t, u \in V(Y_0), t \in \{1, t_1, \ldots, t_s\} \), it follows that \( h \in C \) and so \( h = 1 \). Therefore, by [31 Theorem 4.17], \( g \) is a WPD element.

We now turn our attention to the study of (finitely generated) normal subgroups of groups in the class \( \mathcal{G} \).

**Lemma 4.3.** Let \( G \) be a finitely generated group in \( \mathcal{G} \), let \( T \) be the Bass-Serre tree corresponding to a standard splitting of \( G \) and let \( K \) be the kernel of the action of \( G \) on \( T \). Suppose that \( N \) is a non-trivial normal subgroup of \( G \). Then, either \( N < K \) or \( N \) contains hyperbolic elements and it acts minimally on \( T \).

**Proof.** We first show that if \( N \) is elliptic, then it is contained in the kernel \( K \) of the action. Indeed, assume that \( N \) is a subgroup of a vertex stabilizer \( A \). By Lemma 4.2 the group \( G \) has a relative WPD element, say \( h \), and so we have that \( A \cap A^h = K \). Since \( N \) is normal, we have that \( N^K = N \) is contained in \( A \cap A^h = K \) and so \( N \) is contained in the kernel of the action.

If \( N \) is not elliptic it contains a hyperbolic element. Hence, the union of the axes of such elements is the unique minimal \( N \)-invariant subtree \( X_0 \) of \( T \) (see [6, Proposition 2.1 (6)]). Since \( N \) is normal in \( G \), the \( N \)-invariant subtree \( X_0 \) is also invariant under the action of \( G \). But \( T \) is minimal as a \( G \)-tree, so \( X_0 = T \). Thus, \( N \) acts minimally on \( T \). □

**Proposition 4.4.** Let \( G \) be a finitely generated group in \( \mathcal{G} \), let \( T \) be the Bass-Serre tree corresponding to a standard splitting of \( G \), and let \( K \) be the kernel of the action of \( G \) on \( T \). Suppose that \( N \) is a non-trivial finitely generated normal subgroup of \( G \). Then, either \( N < K \) or \( G/N \) is virtually cyclic.

Furthermore, if \( G/N \) is virtually \( \mathbb{Z} \), then \( N \) is a free product of free abelian groups whose ranks are bounded above by \( r \), where \( r \) is the maximum of the ranks of the (free abelian) vertex groups of \( G \). In particular, if \( G \) is a residually finite tubular group, \( N \) is a finitely generated non-trivial normal subgroup of \( G \) and \( G/N \) is virtually \( \mathbb{Z} \), we have that \( N \) is a free group.

**Proof.** Suppose that \( N \) is not contained in \( K \). By Lemma 4.3 \( N \) acts minimally on \( T \). In addition, \( N \) is finitely generated, so by [1 Proposition 7.9], \( N \setminus T \) is finite.

Let \( C \) be a cyclic edge stabilizer. Then, \( |N \setminus G/C| \) is finite because the number of edges in \( N \setminus T \) is an upper bound for that number.

From the fact that \( N \) is normal in \( G \), \( N \setminus G/C \) and \( G/NC \) are isomorphic. Therefore, \( |G/NC| \) needs to be finite. Notice that if an edge group is trivial, \( N \) is of finite index in \( G \). Hence, we further assume that \( G \) is freely indecomposable, that is, all the edge groups are infinite cyclic.
The Second Isomorphism Theorem gives us that
\[ NC/N \cong C/(N \cap C). \]

It follows that \( N \cap C \neq \{1\} \) if and only if \( N \) has finite index in \( NC \). Therefore, if \( N \) intersects non-trivially an edge group \( C \), we have that \( N \) also has finite index in \( G \).

We are left to consider the case when \( N \) intersects trivially each edge group in the standard splitting of \( G \). In this case, for each edge group \( C \),
\[ NC/N \cong C. \]

Thus, since \( NC \) has finite index in \( G \), \( G/NC \) is virtually \( \mathbb{Z} \). Furthermore, since \( N < G \) and \( N \) intersects trivially each edge group, \( N \) gets induced a decomposition as a free product of free abelian groups, where the free abelian groups are the intersections of the (conjugates of the) vertex groups of \( N \). Since \( G \) is freely indecomposable, each vertex group has an infinite cyclic edge group as subgroup and since \( N \) does not intersect any edge group, it follows that the intersection of \( N \) with a (conjugate of a) vertex group \( G_v \) of \( G \) has rank at most the rank \( G_v \) minus 1. In the particular case of tubular groups, since all the vertex groups are isomorphic to \( \mathbb{Z}^2 \), we have that the intersection with \( N \) is at most of rank 1 and so \( N \) is a free group. \( \square \)

Our second goal is to find an alternative for the fact that \( L_1 \) and \( L_2 \) are finitely generated in the case of free groups. For the class \( \mathcal{G} \), we prove the following:

**Proposition 4.5.** Let \( G \) be a finitely generated group in \( \mathcal{G} \), let \( T \) be the Bass-Serre tree corresponding to a standard splitting of \( G \) and let \( K \) be the kernel of the action of \( G \) on \( T \). Let \( G' \) be any group and let \( S < G \times G' \) be a finitely presented subdirect product. Define \( L_1 \) and \( L_2 \) to be \( S \cap G \) and \( S \cap G' \), respectively. Assume that \( L_1 \) is non-trivial and it is not contained in \( K \). Then, there is \( y \in G' \) such that \( \langle L_2, y \rangle \) is finitely generated.

**Proof.** As \( L_1 \) is non-trivial and it is not contained in \( K \), by Lemma 4.3 we have that \( L_1 \) contains a hyperbolic isometry, say \( t \in L_1 \).

Since \( G \) is cyclic subgroup separable, it follows from [6, Theorem 3.1] that there is a finite index subgroup \( M \) in \( G \), which is an HNN extension with stable letter \( t \) and associated cyclic subgroup \( \langle c \rangle \). As \( G \) is finitely generated, \( M \) is also finitely generated; suppose that \( M = \langle s_1, \ldots, s_n \rangle \), with \( s_1 = c \) and \( s_2 = t^{-1}s_1t \).

Let us denote \( p_1^{-1}(M) \) by \( M' \). Since \( M \) has finite index in \( G \), \( M' \) has finite index in \( S \) and since \( S \) is finitely presented so is \( M' \). The HNN decomposition of \( M \) induces a decomposition of \( M' \) as an HNN extension. Let us pick \( \hat{s}_i \in S \) such that \( p_1(\hat{s}_i) = s_i \) for \( i \in \{1, \ldots, n\} \). Note that \( t \) is an element in \( S \). We then have that
\[ M' = \langle L_2, \hat{s}_1, \ldots, \hat{s}_n, t | t^{-1}\hat{s}_1t = \hat{s}_2, t^{-1}bt = b, \forall b \in L_2, \mathcal{R}' \rangle, \]
where \( \mathcal{R}' \) is a set of relations in the elements \( L_2 \cup \{\hat{s}_1, \ldots, \hat{s}_n\} \). Recall that \( s_1 = c \), so that \( \hat{s}_1 \) is an element of the form \( cy \) in \( S \).

Since \( M' \) is finitely generated, there are \( a_1, \ldots, a_k \) in \( L_2 \) such that
\[ M' = \langle a_1, \ldots, a_k, \hat{s}_1, \ldots, \hat{s}_n, t | t^{-1}\hat{s}_1t = \hat{s}_2, t^{-1}bt = b, \forall b \in L_2, \mathcal{R}' \rangle. \]

Let \( D \) be the subgroup \( \langle a_1, \ldots, a_k, \hat{s}_1, \ldots, \hat{s}_n \rangle \). Note that \( L_2 \) is a subgroup of \( D \) because \( L_2 < M' \) and \( t \) is an element in \( L_1 \). In conclusion,
\[ M' = \langle D, t | t^{-1}\hat{s}_1t = \hat{s}_2, t^{-1}bt = b, \forall b \in L_2 \rangle. \]
Since $M'$ is finitely presented, by [30, Lemma 2], we deduce that $\langle L_2, s_1 \rangle$ is finitely generated. Finally, since $L_2$ is contained in $G'$ and $s_1 = cy$ with $c \in G$ and $y \in G'$, we have that the image of $\langle L_2, s_1 \rangle$ under the natural projection $\pi: G \times G' \mapsto G'$ is finitely generated, that is, $\langle L_2, y \rangle$ is finitely generated. \qed

Remark 4.6. Let $S$ be a subdirect product of $G_1 \times G_2$ and let us define $L_i$ to be $G_i \cap S$, $i \in \{1, 2\}$. Then,

$$G_1/L_1 \cong S/(L_1 \times L_2) \cong G_2/L_2.$$ 

Indeed, since $S$ surjects onto $G_1$, we have an epimorphism $\pi: S \mapsto G_1/L_1$ with kernel $L_1 \times L_2$.

5. Main result

The main goal of this section is to prove the following:

**Theorem 5.1.** Let $S$ be a finitely presented subgroup of $G_1 \times G_2$ where $G_1, G_2 \in \mathcal{A}$. Then, $S$ is virtually $H$-by-(free abelian), where $H$ is the direct product of two groups in $\mathcal{A}$.

Furthermore, $S$ is virtually the kernel of a homomorphism $f$ from $H_1 \times H_2$ to $\mathbb{R}$, for some $H_1, H_2 \in \mathcal{A}$. More precisely, $H$ is equal to $L_1 \times L_2$, where $L_i = S \cap G_i$, $i \in \{1, 2\}$ and either

- $L_1 \times L_2 < f_i S < f_i G_1 \times G_2$, or
- $S$ is virtually the kernel $\ker f$ where $f: H_1 \times H_2 \mapsto \mathbb{Z}$ for some $H_i \in \mathcal{A}$, $i \in \{1, 2\}$, or
- $S$ is virtually the kernel $\ker f$ where $f: H_1 \times H_2 \mapsto \mathbb{Z}^2$ for some $H_i \in \mathcal{A}$, $i \in \{1, 2\}$. In this case, $L_i$ is the free product of finitely generated groups in $\mathcal{A}$ for $i \in \{1, 2\}$.

Let $G_1$ and $G_2$ be two groups in $\mathcal{A}$ and let $S < G_1 \times G_2$ be a finitely presented subgroup. Let $p_1: G_1 \times G_2 \mapsto G_1$ and $p_2: G_1 \times G_2 \mapsto G_2$ be the two natural projection maps. Then, $S$ is a subdirect product of $p_1(S) \times p_2(S)$. By Lemma 2.4 for $i \in \{1, 2\}$, $p_i(S)$ is either the direct product of a residually finite free product and a free abelian group (type (1)), or it is the direct product of a group of $G$ and a free abelian group (type (2)).

**Remark 5.2.** Suppose that $S$ is a subdirect product of $G_1 \times G_2$ such that $G_i = \mathbb{Z}^{n_i} \times H_i$, $n_i \in \mathbb{N} \cup \{0\}$ and $H_i$ is either a residually finite free product or a group in $G$ for $i \in \{1, 2\}$. That is, $S$ is a subdirect product of

$$(\mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}) \times (H_1 \times H_2).$$

Let $\pi$ be the projection map

$$(\mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}) \times (H_1 \times H_2) \mapsto H_1 \times H_2.$$

Notice that in order to prove Theorem 5.1 it suffices to prove it for $\pi(S)$. Indeed, if $\pi(S)$ is virtually $H$-by-(free abelian), where $H$ is the direct product of two groups in $\mathcal{A}$, then $S$ is also virtually $H$-by-(free abelian).

In order to prove Theorem 5.1, we will distinguish three cases depending on the type of $p_1(S)$ and $p_2(S)$ described in Lemma 2.4.
5.1. The groups $p_1(S)$ and $p_2(S)$ are of type (1). By Remark 5.2 we can assume that the groups are residually finite free products.

**Theorem 5.3.** Let $G_1 \times G_2$ be the direct of two finitely generated residually finite groups that admit a non-trivial free product decomposition. Let $S$ be a finitely presented subdirect product in $G_1 \times G_2$ and define $L_i$ to be $S \cap G_i$, $i \in \{1, 2\}$.

- If $L_1 = 1$, then $S$ is isomorphic to $G_2$.
- If $L_2 = 1$, then $S$ is isomorphic to $G_1$.
- Otherwise, $L_i$ has finite index in $G_i$, $i \in \{1, 2\}$ and so $L_1 \times L_2$ has finite index in $S$. If particular, for $i \in \{1, 2\}$, if $G_i \in \mathcal{A}$, then $L_i \in \mathcal{A}$ and it is finitely generated.

We will first show that the groups $L_1$ and $L_2$ defined in the statement are finitely generated.

**Proposition 5.4.** Let $A \times K$ be the direct product of a group $A$ with $K$, where $K$ is a finitely generated residually finite group that admits a non-trivial free product decomposition. Suppose that $S$ is a subdirect product in $A \times K$ which intersects $K$ non-trivially. Then, $L = S \cap A$ is finitely generated.

**Proof.** By hypothesis, since $S$ intersects $K$ non-trivially, there is a non-trivial element $t$ in $S \cap K$. Since $K$ has a non-trivial free product decomposition, it acts minimally on a tree $T$ with trivial edge stabilisers. Moreover, $K$ is residually finite, so the trivial group is closed in the pro-finite topology. Thus, by [6] Theorem 3.1, there is a finite index subgroup $M$ in $K$ which is a free product of the form $B * \langle t \rangle$. Since $K$ is finitely generated, so is $M$. Let $\{s_1, \ldots, s_n\}$ be a generating set for $M$. For $i \in \{1, \ldots, n\}$, let us pick $s_i \in p_2^{-1}(s_i)$, where $p_2$ is the projection map $S \mapsto K$. Let $S_0 = p_2^{-1}(M)$. Note that $S_0$ is of finite index in $S$ since $M$ is of finite index in $K$. The free product decomposition of $M$ induces a splitting of $S_0$ of the form: $S_0 = \langle L, t, s_1, \ldots, s_n | t^{-1}bt = b, s_i^{-1}bs_i = \phi_i(b), i \in \{1, \ldots, n\}, \forall b \in L \rangle$, where $\phi_i$ is the automorphism of $L$ induced by conjugation by $s_i$.

Now, since $S$ is finitely presented and $S_0$ is of finite index in $S$, we have that $S_0$ is also finitely presented. Suppose that $S_0$ is generated by elements $a_1, \ldots, a_k$ in $L$ together with the elements $t, s_1, \ldots, s_n$. Let $D$ be the group $\langle a_1, \ldots, a_k, s_1, \ldots, s_n \rangle$. Since $L$ is a subgroup of $S_0$ and $t \in K$, $L$ is a subgroup of $D$. Moreover, $S_0 = \langle D, t | t^{-1}bt = b, \forall b \in L \rangle$.

Finally, since $S_0$ is finitely presented, by [30] Lemma 2, we have that $L$ is finitely generated. □

We now address the proof of Theorem 5.3.

**Proof of Theorem 5.3** Note that there are short exact sequences

$$1 \rightarrow L_i \rightarrow S \rightarrow G_j \rightarrow 1,$$

for $i \neq j \in \{1, 2\}$, so if $L_i$ is trivial for some $i \in \{1, 2\}$, $S$ is isomorphic to $G_j$ for $j \neq i \in \{1, 2\}$.

Now suppose that $L_1$ and $L_2$ are non-trivial. By Proposition 5.4, they are finitely generated. Then, for $i \in \{1, 2\}$, $L_i$ is a non-trivial finitely generated normal subgroup of $G_i$, and since by assumption $G_i$ admits a non-trivial free product decomposition, it follows from [2] that $L_i$ has finite index in $G_i$. Therefore, $L_1 \times L_2$ has finite index in $S$. □
5.2. The group \( p_1(S) \) is of type (1) and \( p_2(S) \) is of type (2). By Remark 5.2, it suffices to check the result of Theorem 5.1 for \( G_1 \times G_2 \), where \( G_1 \) is a residually finite free product and \( G_2 \) is a group in \( \mathcal{G} \).

**Theorem 5.5.** Let \( G_1 \) be a finitely generated residually finite group that decomposes as a non-trivial free product and let \( G_2 \) be a finitely generated group in \( \mathcal{G} \). Let \( S \) be a finitely presented subdirect product in \( G_1 \times G_2 \). Then, \( S \) is virtually \( H \)-by-(free abelian), where \( H \) is the direct product of two groups in \( \mathcal{A} \).

More precisely, \( H \) is equal to \( L_1 \times L_2 \), where \( L_i = S \cap G_i \), \( i \in \{1, 2\} \) and either

- \( L_1 \times L_2 < f_i S < f_i G_1 \times G_2 \), or
- \( S \) is virtually the kernel \( \ker f \) where \( f: H_1 \times H_2 \mapsto \mathbb{Z} \) for some \( H_i < f_i G_i \), \( i \in \{1, 2\} \). Furthermore, if \( G_1 \in \mathcal{A} \), then \( L_2 < G_2 \) is finitely generated and \( L_1 \) is either finitely generated or the free product of finitely generated groups in \( \mathcal{A} \).

**Proof.** Let us define \( L_1 \) and \( L_2 \) to be \( S \cap G_1 \) and \( S \cap G_2 \), respectively. There are short exact sequences

\[
1 \longrightarrow L_i \longrightarrow S \longrightarrow G_j \longrightarrow 1,
\]

for \( i \neq j \in \{1, 2\} \), so if \( L_i \) is trivial for some \( i \in \{1, 2\} \), \( S \) is isomorphic to \( G_j \) for \( j \neq i \in \{1, 2\} \).

Now suppose that \( L_1 \) and \( L_2 \) are non-trivial. Let \( T \) be the Bass-Serre tree corresponding to a standard splitting of \( G_2 \) and let \( K = \langle \langle \rangle \rangle \) be the kernel of the action of \( G_2 \) on \( T \). By Lemma 4.2, we have that \( G_2 \) has a relative WPZ element and that there is a subgroup of finite index, say \( K_2 \), with center \( \langle \langle \rangle \rangle \). Then, \( S \cap K_2 \) has finite index in \( S \), so it suffices to show that \( S \cap K_2 \) is virtually \( H \)-by-(free abelian), where \( H \) is the direct product of two groups in \( \mathcal{A} \). That is, we may assume that \( G_2 = K_2 \).

By Lemma 4.3, either \( L_2 < K \) or \( L_2 \) acts minimally on \( T \). If \( L_2 < K \), since there is a short exact sequence

\[
1 \longrightarrow L_2 \longrightarrow S \longrightarrow G_1 \longrightarrow 1,
\]

and \( L_2 \) lies in the center of \( K_2 \), which is cyclic (possibly trivial), we have that \( S \) is \( L_2 \times G_1 \) where \( L_2 \) is cyclic.

We now need to deal with the case when \( L_2 \) is not contained in \( K \). By Proposition 5.4, \( L_2 \) is finitely generated. Moreover, there is \( e \in G_1 \) such that \( (L_1, e) \) is finitely generated (see Proposition 4.5).

First, by [8, Theorem 4.1], \( \langle L_1, e \rangle \) has finite index in \( G_1 \). Therefore, \( G_1/L_1 \) is finite or virtually \( \mathbb{Z} \). Second, \( L_2 \) is a finitely generated normal subgroup of \( G_2 \) which is not contained in \( K \), so from Proposition 4.4 we get that \( G_2/L_2 \) is finite or virtually cyclic. To sum up, since we have that

\[
L_1 \times L_2 < S < G_1 \times G_2,
\]

and \( (G_1 \times G_2)/(L_1 \times L_2) \) is virtually abelian, so is \( S/(L_1 \times L_2) \).

Let us now describe the structure of \( H = L_1 \times L_2 \). Assume that \( H \) is not finitely generated. Since \( H = L_1 \times L_2 \) and \( L_2 \) is finitely generated, it follows that \( L_1 \) is not finitely generated. Then, we have that \( G_1/L_1 \) is virtually \( \mathbb{Z} \). We claim that in this case \( L_1 \) is a free product of finitely generated groups from \( \mathcal{A} \). We prove it by induction on the height of the group \( G_1 \) in the class \( \mathcal{A} \).

Since \( L_1 < G_1 \) and as \( G_1 \) has, by assumption, a free product decomposition, \( L_1 \) is a free product of groups of the form \( L_1 \cap G_{v_i} \) and a free group, where \( G_{v_i} \) is a vertex stabiliser of the standard splitting of \( G_1 \). If \( G_1 \in \mathcal{G} \), then \( G_{v_i} \) is a finitely generated free abelian group and so is the intersection with \( L_1 \). Assume that \( G_1 \) is of height \( k \geq 1 \). In this case, \( G_{v_i} = \mathbb{Z}^n \times (G_{v_2} \times \cdots \times G_{v_n}) \) and \( G_{v_2} \in \mathcal{A} \) are of height less than \( k \), \( i \in \{1, \ldots, n\} \). Since by the induction hypothesis we have that \( L_1 \cap G_{v_i} \) is the free product of finitely generated groups from \( \mathcal{A} \), the same holds for \( L_1 \cap G_{v_i} \).

\[ \square \]
5.3. **The groups** $p_1(S)$ and $p_2(S)$ **are of type** (2). By Remark 5.2 we can assume that the groups belong to $G$, and by the previous cases, that they are freely indecomposable. Therefore, it suffices to prove the following:

**Theorem 5.6.** Let $G_i$ be a freely indecomposable finitely generated group in $G$, for $i \in \{1, 2\}$. Let $S$ be a finitely presented subdirect product of $G_1 \times G_2$. Then, $S$ is virtually $H$-by-(free abelian), where $H$ is the direct product of two groups in $G$.

More precisely, $H$ is equal to $L_1 \times L_2$, where $L_i = S \cap G_i$, $i \in \{1, 2\}$ and either

- $L_1 \times L_2 <_{f_j} S <_{f_j} G_1 \times G_2$, or
- $S$ is virtually the kernel $\ker f$ where $f : H_1 \times H_2 \mapsto \mathbb{Z}$ for some $H_i \in \mathcal{A}$, $i \in \{1, 2\}$, or
- $S$ is virtually the kernel $\ker f$ where $f : H_1 \times H_2 \mapsto \mathbb{Z}^2$ for some $H_i \in \mathcal{A}$, $i \in \{1, 2\}$. In this case, $L_i$ is the free product of finitely generated groups in $\mathcal{A}$ for $i \in \{1, 2\}$.

**Proof.** Let $p_1 : S \mapsto G_1$ and $p_2 : S \mapsto G_2$ be the natural projection maps and let $L_1 = S \cap G_1$ and $L_2 = S \cap G_2$. Let us deal with the case when $L_1$ or $L_2$ is trivial. There are short exact sequences

$$1 \longrightarrow L_i \longrightarrow S \longrightarrow G_j \longrightarrow 1,$$

for $i \neq j \in \{1, 2\}$, so if $L_i$ is trivial for some $i \in \{1, 2\}$, $S$ is isomorphic to $G_j$ for $j \neq i \in \{1, 2\}$.

Now, suppose that $L_1$ and $L_2$ are non-trivial. For $i \in \{1, 2\}$, let $T_i$ be the Bass-Serre tree corresponding to a standard splitting of $G_i$ and let $K_i = \langle \iota_i \rangle$ be the kernel of the action of $G_i$ on $T_i$. By Lemma 4.3 there is a finite index subgroup, say $H_i$, in $G_i$ with center $\langle \iota_i \rangle$ and $G_i$ has a relative WPD element. Since $H_i$ has finite index in $G_i$, we may assume that $G_i = H_i$.

By Lemma 4.3, either $L_i < K_i$ or $L_i$ acts minimally on $T_i$. If $L_i < K_i$ for some $i \in \{1, 2\}$, say $L_1 < K_1$, then $L_1$ is contained in the center of the group $H_1$. Therefore, $S$ is virtually $L_1 \times G_2$, where $L_1$ is cyclic.

We now deal with the case when $L_i$ is not contained in $K_i$, for $i \in \{1, 2\}$. In this case, we show that $G_2/L_2$ is finite or virtually free abelian. A symmetric argument shows that $G_1/L_1$ is finite or virtually free abelian. Therefore, since $L_1 \times L_2 < S < G_1 \times G_2$, $S/(L_1 \times L_2)$ is virtually free abelian.

Let $T = T_2$ be the Bass-Serre tree associated to a standard splitting of $G_2$. By Lemma 4.3, $L_2$ acts minimally on $T$. By Proposition 4.5, there is $e \in G_2$ such that $\langle L_2, e \rangle$ is finitely generated. This group contains $L_2$, so it also acts minimally on $T$. Furthermore, it is finitely generated, so by [1 Proposition 7.9], the graph $\langle L_2, e \rangle \setminus T$ is finite. Then, for every cyclic edge stabilizer $\Gamma_e$, we have that

$$\left|\langle L_2, e \rangle \setminus G_2/\Gamma_e\right|$$

is finite since it is bounded above by the number of edges in the graph $\langle L_2, e \rangle \setminus T$. Then, there are $z_1, \ldots, z_m \in G_2$ such that

$$G_2 = \bigcup_{j \in [1, \ldots, m]} \langle L_2, e \rangle z_j \Gamma_e.$$

Suppose that $C = \langle \iota \rangle$ is a cyclic edge stabilizer and $C < A$ is a free abelian vertex stabilizer.
We now deal with the case when for each vertex stabilizer free abelian. Summarizing, the subgroup and by taking further cosets as we did in the previous case, there are for each distinct natural numbers the element of generality, we assume that . Note that, by the standing assumption , we have that . Therefore, under the standing assumption, we have that for all different powers of and so without loss of generality, we assume that . Indeed, otherwise, would be equal to modulo . Since , we deduce that is congruent to an element of , say , modulo . If we define for and , we have that

\[ G_2 = \bigcup_{j \in \{1, \ldots, m\}, s \in \{0, \ldots, k_2 - k_1 - 1\}} \langle L_2, a' \rangle z_{s,j}(c) = \bigcup_{j \in \{1, \ldots, m\}, s \in \{0, \ldots, k_2 - k_1 - 1\}} \langle L_2, a' \rangle z_{s,j}(c). \]

The next goal is to obtain a decomposition of as a disjoint union of single cosets. The element lies in for all , . Therefore, there are distinct natural numbers and such that

\[ c^l z_{s,j} = l(a')^t s z_{s,j} c^m \]

and

\[ c^{l'} z_{s,j} = l'(a')^s z_{s,j} c^m, \]

for some and . Equating and using the normality of , we deduce that there is such that

\[ l'' c^{-l_2} (a')^{s_2 - s_1} c^l z_{s,j} = z_{s,j} c^{m_1 - m_2}. \]

Denote the element by . Then the previous equation is of the form

\[ l'' a'' z_{s,j} = z_{s,j} c^{m_1 - m_2}, \]

where and . Again, by the standing assumption, we have that and are different and by taking further cosets as we did in the previous case, there are such that

\[ G_2 = \bigcup_{j \in \{1, \ldots, r\}} \langle L_2, a', a'' \rangle f_j. \]

Summarizing, the subgroup has finite index in and so has finite index in . Moreover, by the Second Isomorphism Theorem, is isomorphic to , and since is abelian, so is . Therefore, under the standing assumption, we have that is virtually free abelian.

We now deal with the case when for each vertex stabilizer and each edge stabilizer , for each there are such that

\[ a^n = c^m \quad \text{in} \quad G_2/L_2. \]
In particular, for any two edge stabilizers $\langle c_1 \rangle$ and $\langle c_2 \rangle$, there are $w_1, w_2 \in \mathbb{Z}$ such that $c_1^{w_1} = c_2^{w_2}$ modulo $L_2$.

Recall the double coset splitting of $G_2$,

\[ G_2 = \bigcup_{j \in [1, \ldots, m]} \langle L_2, e \rangle z_j \Gamma_e. \tag{3} \]

Suppose that $\Gamma_e = \langle \gamma_e \rangle$. Note that for each $j \in \{1, \ldots, m\}$,

\[ \langle L_2, e \rangle z_j \langle \gamma_e \rangle = \langle L_2, e \rangle z_j \langle \gamma_e \rangle z_j^{-1} z_j = \langle L_2, e \rangle \langle \gamma_e z_j^{-1} \rangle z_j. \]

Observe that $\langle \gamma_e z_j^{-1} \rangle$ is an edge stabilizer, so by assumption, there are two numbers $n_j, m_j \in \mathbb{Z}$ such that

\[ \langle \gamma_e z_j^{-1} \rangle^{n_j} = \gamma_e^{m_j} \quad \text{in} \quad G_2 / L_2. \]

We again take cosets as in the previous cases to get that

\[ G_2 = \bigcup_{j \in [1, \ldots, m]} \langle L_2, e, \gamma_e \rangle q_j, \tag{4} \]

for some $q_1, \ldots, q_m \in G_2$.

Let us distinguish two cases. First, suppose that $e$ is elliptic. Then, $e \in A$ for a vertex stabilizer $A$. Let $\langle c \rangle$ be an edge stabilizer such that $c \in A$. We are now in the same situation as in the previous case since $e, c \in A$: $L_2A$ has finite index in $G_2$ and so we have again that $G_2 / L_2$ is virtually free abelian.

Finally, we need to deal with the case when $e$ is hyperbolic.

By \[6\] Corollary 3.2, there is $M$ a finite index subgroup of $G_2$ such that $M$ is an HNN extension with stable letter $e$ and amalgamated subgroup $M \cap C$, where $C$ is an edge stabilizer of $T$. If $c$ is a generator of $C$, then $M \cap C = \langle c^r \rangle$ for some $r \in \mathbb{N} \cup \{0\}$. Let us denote $c^r$ by $c_1$. Since $M$ has finite index in $G_2$, $M$ is finitely presented and admits a presentation of the form

\[ \langle b_1, \ldots, b_k, c_1, e, c_2 \mid R, e^{-1} c_1 e = c_2 \rangle, \]

where $R$ is a set of relations in the words $\{b_1, \ldots, b_k, c_1, c_2\}$ and $c_2$ is a power of a generator of another edge stabilizer in $T$. Then, we are in the following situation:

\[ \begin{array}{c}
G_2 \\
\downarrow M \\
L_2 \\
\downarrow M \cap L_2 \\
\downarrow \langle e, c_1 \rangle (M \cap L_2) \\
\downarrow M \cap L_2
\end{array} \]

From the double coset representation \[4\] applied to $c_1$, we have that

\[ G_2 = \bigcup_{j \in [1, \ldots, m]} \langle L_2, e, c_1 \rangle q_j. \tag{5} \]
The group $L_2$ is a subgroup of $ML_2$ and the elements $e, c_1$ belong to $M < ML_2$. Then, there are $z_1, \ldots, z_t \in ML_2$ such that

$$ML_2 = \bigcup_{j \in \{1, \ldots, t\}} \langle L_2, e, c_1 \rangle z_j.$$

Thus, there are $\overline{z}_1, \ldots, \overline{z}_t \in M$ such that

$$M = \bigcup_{j \in \{1, \ldots, t\}} \langle M \cap L_2, e, c_1 \rangle \overline{z}_j.$$

Assume that the elements $\overline{z}_1, \ldots, \overline{z}_t$ belong to $\langle e, c_1 \rangle (M \cap L_2)$ and that $\overline{z}_{t+1}, \ldots, \overline{z}_t$ do not belong to $\langle e, c_1 \rangle (M \cap L_2)$. Then, we have the double coset decomposition of $\langle e, c_1 \rangle (M \cap L_2)$:

$$(6) \quad \langle e, c_1 \rangle (M \cap L_2) = \bigcup_{j \in \{1, \ldots, t\}} \langle M \cap L_2, e, c_1 \rangle \overline{z}_j.$$

By the standing assumption (2), we have that $e^{-1}c_1^e \equiv c_1^q \pmod{L_2}$ for some $m, n \in \mathbb{Z}$. Therefore, the group $\langle e, c_1 \rangle (M \cap L_2)/(M \cap L_2)$ is isomorphic to the quotient of the Baumslag-Solitar group

$$BS(m, n) = \langle x, t \mid t^{-1}x^mt = x^n \rangle.$$

That is, there is $N$ a normal subgroup of $BS(m, n)$ and an isomorphism

$$f: \langle e, c_1 \rangle (M \cap L_2)/(M \cap L_2) \to BS(m, n)/N$$

with $f(eM \cap L_2) = tN, f(c_1M \cap L_2) = xN$. By the decomposition given in (6), there are elements $a_1, \ldots, a_s \in BS(m, n)$ such that

$$(7) \quad BS(m, n) = \bigcup_{j \in \{1, \ldots, s\}} N \langle t, x \rangle a_j.$$

If $m$ is equal to $n$, $c_1^m$ commutes with $e$ modulo $L_2$. Then, from the decomposition given in (5), we have that $G_2/L_2$ is virtually abelian. If $m = -n$, $c_1^m$ commutes with $e^2$ modulo $L_2$. From the decomposition (5) we get that there are elements $r_1, \ldots, r_g \in G_2$ such that

$$G_2 = \bigcup_{j \in \{1, \ldots, g\}} \langle L_2, e^2, c_1^m \rangle r_j,$$

so $G_2/L_2$ is virtually abelian.

Let us deal with the case when $m$ is not equal to $\pm n$. In this case, our aim is to show that there is $q \in \mathbb{N} \setminus \{0\}$ such that $x^q \in N$ and using the isomorphism $f$, we deduce that $c_1^q \in M \cap L_2 < L_2$. Notice that if $c_1^q \in L_2$ for $q \neq 0$, then it follows from the decomposition (5) that $G_2/L_2$ is virtually cyclic.

Note that we can assume that $\gcd(m, n) = 1$. Otherwise, if $\gcd(m, n) = d$, then $m = dm'$ and $n = dn'$ for some $m', n' \in \mathbb{Z}$ with $\gcd(m', n') = 1$ and $e^{-1}(c_1^d)^m e = (c_1^d)^n$ modulo $L_2$.

Let us denote the normal closure of $x$ in $BS(m, n)$ by $\langle \langle x \rangle \rangle$. Let us prove that for each $g \in \langle \langle x \rangle \rangle$ there is $S = S(g) \in \mathbb{N}$ such that $(x^S)^g = x^S$. If $g \in \langle \langle x \rangle \rangle$, then

$$g = (g_0x^{\pm 1}g_0^{-1})(g_1x^{\pm 1}g_1^{-1}) \cdots (g_nx^{\pm 1}g_n^{-1}),$$

for some $g_i \in BS(m, n), i \in \{0, \ldots, n\}$. For each $i \in \{0, \ldots, n\}$, let

$$x_i = \max \left\{ \text{number of } t \text{'s in } g_i, \text{number of } t^{-1} \text{'s in } g_i \right\},$$

and let $S = \max \{x_i \mid i \in \{1, \ldots, n\}\}$. Then,

$$(x^{[\log |g|]})^g = x^{[\log |g|]g^g}.$$
Suppose that \( N \) is not contained in \( \langle x \rangle \), that is, there is an element \( h \in N \) which is not in \( \langle x \rangle \). Since \( BS(m, n) = \langle x \rangle(t) \), we can write \( h = gt^k \) for some \( g \in \langle x \rangle \) and \( k \in \mathbb{Z} \setminus \{0\} \). By the previous paragraph, there is \( S \) such that \( \left(x^{[m]}\right)^g = x^{[m]}[m] \). Let us take \( M \) to be \( \max\{S, |k|\} \).

If \( m \) and \( n \) are greater than 0, then
\[
\left(x^{mMnM} \right)^h = x^{mM-knM} \quad \text{if } k > 0, \quad \left(x^{mMnM} \right)^h = x^{mM-knM} \quad \text{if } k < 0.
\]
So in \( BS(m, n)/N \),
\[
x^{mMnM}N = x^{mM-knM}N \quad \text{or} \quad x^{mMnM}N = x^{mM-knM}N.
\]
That is,
\[
x^{mM-k(n^k-n^f)} \in N \quad \text{or} \quad x^{mM-k(n^k-n^f)} \in N.
\]
Since \( m \) is not equal to \( \pm n \), \( m^k - n^k \neq 0 \) and \( n^k - m^k \neq 0 \). So there is \( q \in \mathbb{N} \) such that \( x^q \in N \).

If \( m < 0 \) and \( n > 0 \),
\[
\left(x^{[m]} \right)^h = (x^{-1})^{mM-knM} \quad \text{if } k > 0, \quad \left(x^{[m]} \right)^h = (x^{-1})^{mM-knM} \quad \text{if } k < 0.
\]
Therefore, as in the previous case,
\[
x^{mM-knM} \in N \quad \text{or} \quad x^{mM-knM} \in N.
\]
Since \( m \) is not equal to \( \pm n \), again we have that there is \( q \in \mathbb{N} \) such that \( x^q \in N \).

Therefore, we are left to consider the case when \( N \) is a subgroup of \( \langle x \rangle \). Let us first show that \( \langle x \rangle/N \) is virtually cyclic. For that, we show that \( \langle N, x \rangle \) has finite index in \( \langle x \rangle \). Let \( g \in \langle x \rangle \).

By the decomposition (7), there is an element \( n \in N \), some \( m, k \in \mathbb{Z} \) and \( j \in \{1, \ldots, s\} \) such that \( g = nt^m a_j x^k \). Observe that \( n^{-1}g^{-1}x^{-k} \) is an element of \( \langle x \rangle \) and so \( t^m a_j \) also belongs to \( \langle x \rangle \). The sum of the powers of \( t \) is 0 in every element of \( \langle x \rangle \), so since \( a_j \) is a fixed element, \( m \) also needs to be a fixed number, say \( k_j \). Therefore, from the above observation and the decomposition (7) we get that

\[
\langle x \rangle = \bigcup N y_i(x),
\]
where \( y_i = t^{k_j}a_j \in \langle x \rangle \) and so \( \langle x \rangle/N \) is virtually cyclic.

Let us denote \( \langle x \rangle \) by \( H \) and its commutator subgroup by \( [H, H] \). Then, we are in the following situation:

\[
\begin{array}{ccc}
\langle x \rangle = H & \text{[H,H]}(N, x) & [H, H]N \\
\{N, x\} & [H, H]N & [H, H] \\
N & & \end{array}
\]

Since \( H/N \) is virtually cyclic, then either \( N \) has finite index in \( \langle N, x \rangle \) or \( \langle N, x \rangle/N \) is cyclic. In the former, we have that \( x^q \in N \). If \( (N, x)/N \) is cyclic, we have that \( [H, H](N, x)/[H, H]N \) is also cyclic and so \( N \) has finite index in \( [H, H]N \).

By Lemma [5.7], we have that \( H/[H, H] \) is isomorphic to \( \mathbb{Z}\left[\frac{1}{mn}\right] \). The group \( [H, H]N/[H, H] \) is a \( \mathbb{Z} \)-submodule of \( \mathbb{Z}\left[\frac{1}{mn}\right] \). If \( [H, H]N/[H, H] \) is trivial, then \( N \subseteq [H, H] \) and as \( H/[H, H] \cong \mathbb{Z}\left[\frac{1}{mn}\right] \) and \( H/N \) is virtually cyclic, it follows that \( mn = \pm 1 \). This case is covered in the case when \( m \) is equal to \( \pm n \). Therefore, we need to deal with the case when \( [H, H]N/[H, H] \) is non-trivial. In this case, there
is a non-trivial element in \([H,H]N/[H,H]\), so \(y\) is of the form \(\frac{d}{(mn)}\) for some \(d \in \mathbb{Z} \setminus \{0\}\) and \(k \in \mathbb{N}\). But since \([H,H]N/[H,H]\) is a \(\mathbb{Z}\)-submodule, then \(d \in [H,H]N/[H,H]\). Therefore, \(x^d \in [H,H]N\). Since \(N\) has finite index in \([H,H]N\), we have that \(x^d \in N\) for some \(q \in \mathbb{N}\). Therefore, in all cases \(G_2/L_2\) is virtually abelian.

Note that the kernel of the natural epimorphism \(f_i : S \rightarrow G_i/L_i\) is \(L_1 \times L_2\), so

\[
G_1/L_1 \cong S/(L_1 \times L_2) \cong G_2/L_2.
\]

We have just proved that \(G_i/L_i\) is either finite, virtually \(\mathbb{Z}\) or virtually \(\mathbb{Z}^2\). If \(G_i/L_i\) is finite, since \(G_i\) is finitely generated, so is \(L_i\).

If \(G_i/L_i\) is virtually \(\mathbb{Z}\), there is \(H\) a finite index subgroup in \(S\) such that \(H/(L_1 \times L_2)\) is \(\mathbb{Z}\) and \((f_1(H) \times f_2(H))/(L_1 \times L_2)\) is \(\mathbb{Z}^2\). Thus \(H\) is normal in \(f_1(H) \times f_2(H)\) and \((f_1(H) \times f_2(H))/H\) is \(\mathbb{Z}\). Since \(f_i(H)\) is a finite index subgroup of \(G_i\), it lies in \(G\) and is finitely generated. In conclusion, \(H\) is a finite index subgroup of \(S\) and it is the kernel of a homomorphism \(f_1(H) \times f_2(H) \rightarrow \mathbb{Z}\), where \(f_i(H) \in G\).

Finally, we deal with the case when \(G_i/L_i\) is virtually \(\mathbb{Z}^2\). Since the group \(L_i\) is a subgroup of \(G_i\), it acts on the Bass-Serre tree \(T_i\), and so it inherits a graph of groups decomposition. We claim that the intersection of \(L_i\) with each edge group is trivial and so the decomposition of \(L_i\) is in fact a non-trivial free product decomposition. Let us prove it for \(i = 2\) being the case \(i = 1\) analogous. If \(\Gamma_e\) is a cyclic edge stabilizer of \(T_2\) such that \(L_2 \cap \Gamma_e \neq 1\), then from (1) we obtain that \(G_2/L_2\) is virtually \(\mathbb{Z}\) contradicting our assumption.

The following lemma is a well-known fact on Baumslag-Solitar groups but we add the proof here for completeness.

**Lemma 5.7.** Let \(BS(m,n) = \langle x, t \mid t^{-1}x^m t = x^n \rangle\) such that \(\text{gcd}(m,n) = 1\) and denote the normal closure of \(x\) in \(BS(m,n)\) by \(\langle \langle x \rangle \rangle\). Then, \(H_1(\langle \langle x \rangle \rangle; \mathbb{Z})\) is isomorphic to \(\mathbb{Z}[\frac{1}{mn}]\).

**Proof.** Let us denote \(\langle \langle x \rangle \rangle\) by \(L\). If \(m \in \{1,-1\}\) or \(n \in \{1,-1\}\), \(L\) is free abelian and isomorphic to \(\mathbb{Z}[\frac{1}{n}]\) (see [14]). Thus, we may assume that \(m, n \in \mathbb{Z} \setminus \{0,1,-1\}\).

Let \(x_i = t^{-i}x t^i\) for \(i \in \mathbb{Z}\). Then, \(x_i^n = x_i^n\) and \(L\) has a decomposition as a two-way infinite amalgamated free product:

\[
\cdots \langle x_{-1}, x_0 \mid x_{-1} = x_0^n \rangle_{\langle \langle x_0 \rangle \rangle} \langle x_0, x_1 \mid x_0^n = x_1^n \rangle_{\langle \langle x_1 \rangle \rangle} \cdots
\]

Let us define the epimorphism \(f : L \rightarrow \mathbb{Z}[\frac{m}{n}, \frac{n}{m}]\) such that \(f(i) = \left(\frac{m}{n}\right)^i\) for all \(i \in \mathbb{Z}\). That is,

\[
f(x_0) = 1, \quad f(x_i) = \left(\frac{m}{n}\right)^i \quad \text{if} \; i > 0, \quad f(x_i) = \left(\frac{n}{m}\right)^{-i} \quad \text{if} \; i < 0.
\]

Let us first show that \(\mathbb{Z}[\frac{m}{n}, \frac{n}{m}]\) and \(\mathbb{Z}[\frac{1}{mn}]\) are isomorphic. Since the greater common divisor of \(m\) and \(n\) is 1, there are \(k_1, k_2 \in \mathbb{Z}\) such that \(1 = mk_1 + nk_2\). Then,

\[
\frac{1}{m} = k_1 + \frac{n}{m}k_2 \quad \text{and} \quad \frac{1}{n} = k_2 + \frac{m}{n}k_1.
\]

Therefore, \(\mathbb{Z}[\frac{m}{n}, \frac{n}{m}] = \mathbb{Z}[\frac{1}{n}, \frac{1}{m}]\) and this is clearly isomorphic to \(\mathbb{Z}[\frac{1}{mn}]\).

Finally, we need to check that \(\ker f\) coincides with the commutator subgroup \(L' = [L,L]\). Since \(L/\ker f\) is abelian, \(L'\) is contained in \(\ker f\), so we need to prove that \(\ker f \subseteq L'\).
Firstly, we show that if \( x_{i_1}^{x_1} \ldots x_{i_k}^{x_k} \in \text{ker } f \), the number of \( x_{i_j} \)'s in the word \( x_{i_1}^{x_1} \ldots x_{i_k}^{x_k} \) is equal to the number of \( x_{i_j}^{-1} \)'s in \( x_{i_1}^{-1} \ldots x_{i_k}^{-1} \), for each \( j \in \{1, \ldots, k\} \). Let us prove it by induction on \( k \). If \( k = 1 \), \( x_{i_1}^{x_1} \) is not an element in ker \( f \). If \( k = 2 \), there are some options for \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) \):

1. \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = \pm \frac{m}{n} i_1 \pm \frac{m}{n} i_2 \),
2. \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = \pm \frac{m}{n} i_1 \pm 1 \),
3. \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = \pm \frac{m}{n} i_1 \pm \frac{m}{n} i_2 \),
4. \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = \pm 1 \pm 1 \),
5. \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = \pm \frac{m}{n} i_1 \pm \frac{m}{n} i_2 \),
6. \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = \pm \frac{m}{n} i_1 \pm 1 \).

In the first case, \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) = 0 \) if and only if \( i_1 = i_2 \) and the powers have opposite sign. The cases (4) and (5) are similar. In cases (2), (3) and (6), \( f(x_{i_1}^{x_1} x_{i_2}^{x_2}) \) is not 0.

Now suppose that the statement holds for \( k = 1 \) and let us check it for \( k \). Let \( x_{i_1}^{x_1} \ldots x_{i_k}^{x_k} \in \text{ker } f \) and suppose that \( i_1, \ldots, i_k \) are the positive values among \( i_1, \ldots, i_k \) and \( i_1, \ldots, i_i \) are the negative ones. Then,

\[
\pm \left( \frac{m}{n} \right)^{i_1} \pm \cdots \pm \left( \frac{m}{n} \right)^{i_k} \pm \left( \frac{n}{m} \right)^{-i_1} \pm \cdots \pm \left( \frac{n}{m} \right)^{-i_k} + k = 0.
\]

By taking \( n^{i_1} \cdots i_k \cdot m^{-i_1} \cdots i_k \) as the denominator, we have

\[
\pm n^{i_1} m^{i_1} \pm n^{i_2} m^{i_2} \pm \cdots \pm n^{i_k} m^{i_k} = 0,
\]

for some \( j_1, l_1 \in \mathbb{N}, i \in \{1, \ldots, k\} \).

Suppose that \( j_1 \leq \cdots \leq j_k \). If \( j_1 \) is equal to \( j_k \), we obtain that \( \pm m^{i_1} \pm \cdots \pm m^{i_k} = 0 \). If \( j_1 < j_k \), there is \( t \in \{1, \ldots, k\} \) such that \( j_1 \leq j_2 \leq \cdots \leq j_t < j_{t+1} = \cdots = j_k \). Then,

\[
\pm n^{i_1} m^{i_1} \pm \cdots \pm n^{i_t} m^{i_t} = n^k (\pm m^{i_1} \pm \cdots \pm m^{i_{t+1}}).
\]

If \( \pm m^{i_1} \pm \cdots \pm m^{i_{t+1}} \) is different from 0, \( n^k \) divides \( \pm m^{i_1} \pm \cdots \pm m^{i_{t+1}} \) which is not possible. Therefore,

\[
\pm m^{i_1} \pm \cdots \pm m^{i_{t+1}} = 0.
\]

Assume that \( l_1 \leq \cdots \leq l_{t+1} \). Then,

\[
m^k (\pm 1 \pm m^{l_{t+1} \cdots l_{t+1}} \pm \cdots \pm m^{l_{t+1} \cdots l_{t+1}}) = 0.
\]

Thus, \( l_{t+1} - l_k = 0 \) and the sign of \( \pm m^{l_{t+1} \cdots l_{t+1}} \) is different from the one of \( \pm 1 \). At this point, the induction hypothesis can be used, so the statement also holds for \( k \).

Recall what we have proved: if \( x_{i_1}^{x_1} \ldots x_{i_k}^{x_k} \in \text{ker } f \), the number of \( x_{i_j} \)'s in the word \( x_{i_1}^{x_1} \ldots x_{i_k}^{x_k} \) is equal to the number of \( x_{i_j}^{-1} \)'s in \( x_{i_1}^{-1} \ldots x_{i_k}^{-1} \), for each \( j \in \{1, \ldots, k\} \). The last step is to show that this implies that \( x_{i_1}^{x_1} \ldots x_{i_k}^{x_k} \) is an element of \( L' \). We prove it by induction on \( k \). The cases \( k \in \{1, 2\} \) are routine, so assume that the statement holds for \( k - 1 \) and let us check it for \( k \). By hypothesis,

\[
x_{i_1}^{x_1} \ldots x_{i_k}^{-1} = x_{i_1}^{x_1} \ldots x_{i_{j_1}^{-1}} x_{i_{j_1}}^{-1} \ldots x_{i_k}^{-1} \text{ or } x_{i_1}^{x_1} \ldots x_{i_k}^{-1} = x_{i_1}^{-1} \ldots x_{i_{j_1}}^{-1} x_{i_{j_1}} x_{i_1}^{-1} \ldots x_{i_k}^{-1}.
\]

If we denote \( x_{i_2}^{x_1} \ldots x_{i_k}^{x_1} \) by \( w \), then we have that either

\[
x_{i_1}^{x_1} \ldots x_{i_k}^{x_1} = x_{i_1} w x_{i_1}^{-1} w^{-1} w x_{i_1}^{-1} \ldots x_{i_k}^{-1} \text{ or } x_{i_1}^{x_1} \ldots x_{i_k}^{x_1} = x_{i_1}^{-1} w x_{i_1}^{-1} w^{-1} w x_{i_1}^{-1} \ldots x_{i_k}^{-1}.
\]
Observe that \( x_{i_{1}}w_{i_{1}}^{-1}w^{-1} \) (or \( x_{i_{1}}^{-1}w_{i_{1}}w^{-1} \)) is an element of \( L' \) and by the inductive hypothesis, \( wx_{i_{1}}^{1}\ldots x_{i_{k}}^{1} \in L' \). In conclusion, \( x_{i_{1}}^{1}\ldots x_{i_{k}}^{1} \in L' \). \( \square \)

### 6. Algorithmic problems

In this section, we study algorithmic problems for finitely presented subdirect products of (some) groups in the class \( \mathcal{A} \). Our approach follows closely the one in [7, Section 7.1, Section 7.2].

Let \( \mathcal{A}' \subset \mathcal{A} \) be the subclass of groups that are \( \text{CAT}(0) \) and have unique roots. Note that, conjecturally, every group in the class \( \mathcal{A} \) has a finite index subgroup that belongs to \( \mathcal{A}' \).

The class \( \mathcal{A}' \) contains all 2-dimensional coherent RAAGs and more generally, all graphs of groups such that the underlying graph is a tree with free abelian vertex groups and cyclic edge groups.

We next prove that the multiple conjugacy problem is decidable for the class of finitely presented subgroups of the direct product of two groups in the class \( \mathcal{A}' \).

The multiple conjugacy problem for a finitely generated group \( G \) asks if there is an algorithm that, given an integer \( l \) and two \( l \)-tuples of elements of \( G \), say \( x = (x_{1}, \ldots, x_{l}) \) and \( y = (y_{1}, \ldots, y_{l}) \), can determine if there exists \( g \in G \) such that \( gx_{i}g^{-1} = y_{i} \) in \( G \), for \( i \in \{1, \ldots, l\} \).

The solution to the multiple conjugacy problem for finitely presented residually free groups described in [7, Section 7.1] has two steps. In the first one, the authors give sufficient conditions for a subgroup of a bicombable group to have decidable multiple conjugacy problem. More precisely, they prove the following:

**Proposition 6.1.** [7, Proposition 7.1] Let \( \Gamma \) be a bicombable group, let \( H < \Gamma \) be a subgroup, and suppose that there exists a subgroup \( L < H \) normal in \( \Gamma \) such that \( \Gamma / L \) is nilpotent. Then \( H \) has a solvable multiple conjugacy problem.

Our main result, Theorem 5.1, states that if \( S \) is a finitely presented subgroup of the direct product of two groups in the class, then \( S \) has a finite index subgroup \( S_{0} \) which is \( H \)-by-(free abelian). The above result is intended to prove that \( S_{0} \) has decidable multiple conjugacy problem. In general, the decidability of the conjugacy problem does not pass from finite index subgroups to the group. However, it does if the group has unique roots.

**Lemma 6.2.** [7, Lemma 7.2] Suppose \( G \) is a group in which roots are unique and \( H < G \) is a subgroup of finite index. If the multiple conjugacy problem for \( H \) is solvable, then the multiple conjugacy problem for \( G \) is solvable.

These two results are the main tools to prove the following:

**Theorem 6.3.** The multiple conjugacy problem is solvable in every finitely presented subgroup \( S \) of the direct product of two groups in \( \mathcal{A}' \) (\( S \) given by a finite presentation and an embedding to \( G_{1} \times G_{2} \) where \( G_{i} \in \mathcal{A}' \) and the embedding is as a neat subdirect product).

**Proof.** Let \( S < G_{1} \times G_{2} \) be a finitely presented neat subdirect product of \( G_{1}, G_{2} \in \mathcal{A}' \). That is, \( L_{i} = S \cap G_{i} \neq \{1\} \) and the projection map \( \pi_{i} : S \mapsto G_{i} \) is an epimorphism for \( i \in \{1, 2\} \).

Now, the group \( G_{1} \times G_{2} \) is \( \text{CAT}(0) \) since by assumption, both \( G_{1} \) and \( G_{2} \) are \( \text{CAT}(0) \). Furthermore, as shown in the proof of Theorem 5.1, we have that \( (G_{1} \times G_{2})/(L_{1} \times L_{2}) \) is virtually abelian and so
there is a finite index subgroup $G$ of $G_1 \times G_2$ such that $G/(L_1 \times L_2)$ is abelian. Furthermore, $G$ is CAT(0) for being a finite index subgroup of a CAT(0) group and so it is bicombable. Then, since $L_1 \times L_2 < G < G$ and $(G \cap S)/(L_1 \times L_2)$ is abelian, it follows from Proposition 6.1 that $G \cap S$ has decidable multiple conjugacy problem. Now $S \cap G$ has finite index in $S$ and since $G_1 \times G_2$ has unique roots, so does $G$. Finally, from Lemma 6.2 we conclude that $S$ has decidable multiple conjugacy problem. □

Notice that in Theorem 6.3 we require the finitely presented group $S$ to be given as a neat subdirect product of groups in the class $\mathcal{A}'$. If we restrict to the family of 2-dimensional coherent RAAGs, then given a finite presentation for $S$, one can effectively determine two 2-dimensional coherent RAAGs $A_1$ and $A_2$, and an embedding $f: S \hookrightarrow A_1 \times A_2$ such that $A_i \cap S \neq 1$ for $i \in \{1, 2\}$ (see [12], and also see [7, Section 7.1] to see how to relate the approaches). Then $\pi_i(f(S)) = G_i$ is a finitely generated subgroup of $A_i$, where $\pi_i: A_1 \times A_2 \twoheadrightarrow A_i$ is the natural projection map, $i \in \{1, 2\}$, and from [22] (see Lemma 6.6 below), given a finite set of generators (which are the image of the generators of $S$), one can effectively describe the presentation of $G_i$, $i \in \{1, 2\}$. Therefore, given a finite presentation of $S$, one can effectively determine $G_1 \times G_2$ so that $S$ is a neat subdirect product of $G_1 \times G_2$. Furthermore, since finitely generated subgroups of coherent RAAGs are CAT(0) (see [11, Corollary 9.5]) and have unique roots (see [17]), we have that $G_1, G_2 \in \mathcal{A}'$. Therefore, from Theorem 6.3 and the discussion above, we deduce the following:

**Corollary 6.4.** The multiple conjugacy problem is decidable for the class of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs.

Let us now focus on the membership problem. Recall that the class $\mathcal{G}$ is defined as the class of cyclic subgroup separable graphs of groups with free abelian vertex groups and cyclic edge groups. We aim to show the following result:

**Theorem 6.5.** If $S$ is a finitely presented subgroup of the direct product of two groups from the class $\mathcal{G}$ (given by a finite presentation and an embedding to $G_1 \times G_2$, $G_i \in \mathcal{G}$) and $H \subseteq S$ is a finitely presentable subgroup (given by a finite generating set of words in the generators of $S$), then the membership problem for $H$ is decidable, i.e., there is an algorithm which, given $g \in S$ (as a word in the generators) will determine whether or not $g \in H$.

In [7, Section 7.2], the authors prove that the membership problem is decidable for finitely presented subgroups of finitely presented residually free groups. The two key ingredients used in the proof are that limit groups have decidable membership problem (in fact, in [34], Wilton proved that limit groups are subgroup separable) and the fact that presentations can be effectively described given a set of generators: if $\Gamma$ is a limit group over a free group, then there is an algorithm that, given a finite set $X \subseteq \Gamma$, will output a finite presentation for the subgroup generated by $X$ (see [7, Lemma 7.5]).

For groups in the class $\mathcal{G}$, these results also hold, namely, we have the following:

**Lemma 6.6.** [22, Corollary 1.3] Let $G \in \mathcal{G}$. Then $G$ has solvable uniform membership problem. Moreover, there is an algorithm which, given a finite subset $X \subseteq G$, constructs a finite presentation for the subgroup $U = \langle X \rangle < G$.

The proof of Theorem 6.5 is now similar to the proof of [7, Theorem K] and we sketch it below.
Proof of Theorem 6.5. Suppose that $S$ is a subgroup of $D = G_1 \times G_2$, where $G_1$ and $G_2$ are in $G$. A solution to the membership problem $H \subseteq D$ provides a solution for $H \subseteq S$. Define $L_i$ to be $H \cap G_i$ and $p_i$ to be the projection map $H \mapsto G_i$ for $i \in \{1, 2\}$ and let $g = (g_1, g_2) \in G_1 \times G_2$.

Suppose that some $L_i$ is trivial, say $L_1$. Then, $H$ is isomorphic to $p_2(H)$. In particular, $p_2(H)$ is finitely presented. By Lemma 6.6, there is an algorithm that determines if $g_2$ lies in $p_2(H)$. If it does not, then $g \notin H$. If it does, we eventually find a word $w$ in the generators of $H$ so that $g^{-1}w$ projects to 1. Since $L_1 = H \cap G_1$, we deduce that $g \in H$ if and only if $g^{-1}w = 1$ and this equality can be checked because the word problem is solvable in residually finite groups and so in particular, in groups from $G$.

It remains to consider the case when $L_1$ and $L_2$ are non-trivial. By Lemma 6.6, we can determine algorithmically if $g_i \in H_i = p_i(H_i)$ for $i \in \{1, 2\}$. If $g_i \notin H_i$ for some $i \in \{1, 2\}$, then $g \notin H$. Otherwise, we replace $G_1 \times G_2$ by $H_1 \times H_2$. We are now reduced to the case when $H$ is a full subdirect product. By Theorem 5.1, $Q = D/L$ is virtually free abelian, where $L = L_1 \times L_2$.

Let $\phi: D \mapsto Q$ be the quotient map. Virtually free abelian groups are subgroup separable, so if $\phi(g) \notin \phi(H)$, there is a finite quotient of $G$ that separates $g$ from $H$. But since $L = \ker \phi \subseteq H$, $\phi(g) \in \phi(H)$ if and only if $g \in H$. This an enumeration of the finite quotients of $D$ provides a procedure for proving that $g \notin H$ in this case. This terminates if $g \notin H$. We run this procedure in parallel with an enumeration of $g^{-1}w$ that will terminate if $g \in H$. \qed

The algorithm described to solve the membership problem is not uniform (it depends on the subgroup). In [9], Bridson and Wilton showed that finitely presented residually free groups are actually subgroup separable providing a uniform algorithm to solve the membership problem. The RAAG $P_4$ is not subgroup separable (see [32]), and so in general, coherent RAAGs are not subgroup separable. Therefore, one can not take this approach to obtain a uniform algorithm to solve the membership problem.

As we discussed above, since given a finitely presented subgroup of the direct product of two 2-dimensional coherent RAAGs, one can effectively describe the RAAGs, we obtain the following:

Corollary 6.7. If $S$ is a finitely presented subgroup of the direct product of two 2-dimensional coherent RAAGs (given by a finite presentation) and $H \subseteq S$ is a finitely presentable subgroup (given by a finite generating set of words in the generators of $S$), then the membership problem for $H$ is decidable, i.e., there is an algorithm which, given $g \in S$ (as a word in the generators) will determine whether or not $g \in H$. 

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