On Reversing Operator Choi–Davis–Jensen Inequality

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Abstract
In this paper, we first provide a better estimate of the second inequality in Hermite–Hadamard inequality. Next, we study the reverse of the celebrated Davis–Choi–Jensen’s inequality. Our results are employed to establish a new bound for the operator Kantorovich inequality.

Keywords
Hermite-Hadamard inequality · Davis–Choi–Jensen inequality · Convex function · Self-adjoint operator · Positive operator

Mathematics Subject Classification
Primary: 47A12; 47A30 · Secondary: 26D15

1 Introduction and Preliminaries
A very interesting inequality for convex functions that has been widely studied in the literature is due to Hermite and Hadamard. It provides a two-sided estimate of the mean value of a convex function. The Hermite-Hadamard inequality (for briefly, H–H inequality) has several applications in the nonlinear analysis and geometry of Banach spaces, see (Barnett et al. 2006). Over the past decades, several interesting generalizations, specific cases, and formulations of this remarkable inequality have been obtained for different frameworks. In fact, it provides a necessary and sufficient condition for a function $f$ to be convex. Many well-known inequalities can be obtained using the concept of convex functions. For details, interested readers can refer to Andrica and Rassias (2019), Mitrnović and Lacković (1995) and Mitrovic et al. (1993). This remarkable result of Hermit and Hadamard is as follows:

Theorem A Let $f : J \to \mathbb{R}$ be a convex function, where $a, b \in J$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(z)dz \leq \frac{f(a) + f(b)}{2}.$$  \hspace{1cm} (1.1)

A history of this inequality can be found in Mitrnović and Lacković (1995). An overview of the generalities and various developments can be found in Niculescu and Persson (2003).

In this paper, we first provide a better estimate of the second inequality in H–H inequality (1.1).

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces, and $B(\mathcal{H})$ denotes the algebra of all bounded (linear) operators on $\mathcal{H}$. Recall that an operator $A$ on $\mathcal{H}$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. We write $A > 0$ if $A$ is positive and invertible. For self-adjoint operators $A$ and $B$, we write $A \geq B$ if $A - B$ is positive, i.e., $\langle Ax, x \rangle \geq \langle Bx, x \rangle$ for all $x \in \mathcal{H}$. We call it the usual order. In particular, for some scalars $m$ and $M$, we write $m \leq A \leq M$ if $m\langle x, x \rangle \leq \langle Ax, x \rangle \leq M\langle x, x \rangle$, $\forall x \in \mathcal{H}$.

We extensively use the continuous functional calculus for self-adjoint operators, e.g., see (Furuta et al. 2005, p. 3).
Definition 1.1 A continuous function \( f \) defined on the interval \( J \) is called an operator convex function if
\[
f((1 - v)A + vB) \leq (1 - v)f(A) + vf(B)
\]
for every \( 0 < v < 1 \) and for every pair of bounded self-adjoint operators \( A \) and \( B \) whose spectra are both in \( J \).

Definition 1.2 A linear map \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) is positive if \( \Phi(A) \) is positive for all positive \( A \) in \( \mathcal{B}(\mathcal{H}) \). It is said to be unital (or, normalized) if \( \Phi(1_\mathcal{H}) = 1_\mathcal{K} \).

We recall the Davis–Choi–Jensen inequality (Choi 1974; Davis 1957) for operator convex functions, which is regarded as a noncommutative version of Jensen’s inequality:

Theorem B Let \( A \in \mathcal{B}(\mathcal{H}) \) be a self-adjoint operator with the spectra contained in the interval \( J \) and \( \Phi \) be a positive unital linear map from \( \mathcal{B}(\mathcal{H}) \) to \( \mathcal{B}(\mathcal{K}) \). If \( f \) is operator convex on an interval \( J \), then
\[
f(\Phi(A)) \leq \Phi(f(A)).
\]
(1.2)

Though in the case of convex function the inequality (1.2) does not hold in general, we have the following estimate from (Mićić et al. 2000, Remark 4.14).

Theorem C Let \( A \in \mathcal{B}(\mathcal{H}) \) be a self-adjoint operator with \( Sp(A) \subseteq [m, M] \) for some scalars \( m, M \) with \( m < M \) and \( \Phi \) be a positive unital linear map from \( \mathcal{B}(\mathcal{H}) \) to \( \mathcal{B}(\mathcal{K}) \). If \( f \) is non-negative convex function, then
\[
\frac{1}{x} f(\Phi(A)) \leq f(\Phi(A)) \leq x f(\Phi(A)),
\]
where \( x \) is defined by
\[
x = \max_{m \leq t \leq M} \left\{ \frac{1}{f(t)} \left( \frac{m - t}{M - m} f(m) + \frac{t - m}{M - m} f(M) \right) \right\}.
\]

For other generalizations of inequality, we refer the interested readers to Mićić et al. (2018a, b); Moradi et al. (2019); Sababheh et al. (2020). One of our main aims in this article is to improve the first inequality in the above. For this purpose, we use some ideas from (Moradi et al. 2018, Theorem 3.4).

In this paper, we first provide a better estimate of the second inequality in Hermite-Hadamard inequality. Our results are employed to establish a new bound for the operator Kantorovich inequality. In particular, we show that
\[
\Phi(A^{-1}) \leq \Phi\left( \int_0^1 \left( \frac{M1_\mathcal{H} - A}{M - m} m^{-t} + \frac{A - m1_\mathcal{H}}{M - m} M^{-t} \right) \frac{dt}{4Mm} \right)
\]
(1.4)

where \( m1_\mathcal{H} \leq A \leq M1_\mathcal{H} \) and \( \Phi \) is a positive linear map.

2 Main Results

First we start our work by providing a better estimate for \( H-H \) inequality (1.1).

Theorem 2.1 Let \( f : J \to (0, \infty) \) be a continuous function on the interval \( J \) such that \( f' \) is convex for all \( 0 < t < 1 \). Then for any \( a, b \in J \),
\[
f\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(z)dz
\]
\[
\leq \int_a^b \left( \frac{z - a}{b - a} f'(a) + \frac{b - z}{b - a} f'(b) \right) \frac{dz}{2} \leq \frac{f(a) + f(b)}{2}.
\]

Proof Since \( f' \) is a convex function, we have for any \( x, y \in J \) and \( v \in [0, 1] \)
\[
f'(1 - v)x + vy \leq (1 - v)f'(x) + vf'(y).
\]
(2.1)

Raising both sides of (2.1) to the power of \( 1/t \), we get
\[
f((1 - v)x + vy) = (f'((1 - v)x + vy))^\frac{1}{t}
\]
\[
\leq ((1 - v)f'(x) + vf'(y))^\frac{1}{t}
\]
\[
\leq (1 - v)f'(x)^\frac{1}{t} + v(f'(y))^\frac{1}{t}
\]
\[
= (1 - v)f'(x) + vf'(y).
\]
Consequently,
\[
f((1 - v)x + vy) \leq ((1 - v)f'(x) + vf'(y))^\frac{1}{t}
\]
\[
\leq (1 - v)f'(x) + vf'(y),
\]
(2.2)

which shows the convexity of the function \( f \).

Suppose \( z \in [a, b] \). If we substitute \( x = a, \ y = b, \) and \( 1 - v = (b - z)/(b - a) \) in (2.2), we get
\[
f(z) \leq \left( \frac{b - z}{b - a} f'(a) + \frac{z - a}{b - a} f'(b) \right)^\frac{1}{t}
\]
(2.3)

Since \( z \in [a, b] \), it follows that \( b + a - z \in [a, b] \). Now,
applying the inequality (2.3) to the variable \( b + a - z \), we get
\[
f(b + a - z) \leq \left( \frac{z-a}{b-a} f'(a) + \frac{b-z}{b-a} f'(b) \right)^{\frac{1}{2}} \tag{2.4}
\]
By adding inequalities (2.3) and (2.4), we infer that
\[
f(b + a - z) + f(z) \leq \left( \frac{z-a}{b-a} f'(a) + \frac{b-z}{b-a} f'(b) \right)^{\frac{1}{2}} + \left( \frac{b-z}{b-a} f'(a) + \frac{z-a}{b-a} f'(b) \right)^{\frac{1}{2}}
\]
which, in turn, leads to
\[
\frac{f(a+b)}{2} \leq \frac{f(a+b-z) + f(z)}{2}
\]
and
\[
\frac{f(a+b)}{2} \leq \left( \frac{z-a}{b-a} f'(a) + \frac{b-z}{b-a} f'(b) \right)^{\frac{1}{2}} + \left( \frac{b-z}{b-a} f'(a) + \frac{z-a}{b-a} f'(b) \right)^{\frac{1}{2}}
\]
Now, the result follows by integrating the inequality (2.5) over \( z \in [a, b] \), and using the fact that \( \int_a^b f(z) \, dz = \int_a^b f(a+b-z) \, dz \). \( \square \)

**Remark 2.2** It follows from the proof of Theorem 2.1 that
\[
f(a+b-z) \leq \left( \frac{z-a}{b-a} f'(a) + \frac{b-z}{b-a} f'(b) \right)^{\frac{1}{2}}
\]
This inequality improves (Mercer 2003, Lemma 1.3).

We introduce two notations that will be used in the sequel:
\[
a_f \equiv \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad b_f \equiv \frac{M f(m) - m f(M)}{M - m}.
\]

The following result gives us a refinement of the first inequality in inequality (1.3).

**Theorem 2.3** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a self-adjoint operator with the spectra contained in the interval \([m, M]\) with \( m < M \), and let \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a positive unital linear map. If \( f : [m, M] \to (0, \infty) \) is a continuous function such that \( f' \in (0, 1) \) is convex, then for a given real number \( \alpha \)
\[
\Phi(f(A)) \leq \Phi \left( \left( \frac{M f_1 - A}{M - m} f'(m) + \frac{A - m f_1}{M - m} f'(M) \right)^{\frac{1}{2}} \right)
\]
holds for \( \beta = \max_{m \leq t \leq M} \{ a_f t + b_f \} \).

**Proof** We first observe that the assumptions imply that for any \( m \leq z \leq M \),
\[
f(z) \leq \left( \frac{M - z}{M - m} f'(m) + \frac{z - m}{M - m} f'(M) \right)^{\frac{1}{2}}
\]
Applying the continuous functional calculus for the operator \( A \) whose spectrum is contained in the interval \([m, M]\),
\[
f(A) \leq \left( \frac{M f_1 - A}{M - m} f'(m) + \frac{A - m f_1}{M - m} f'(M) \right)^{\frac{1}{2}}
\]
Since \( \Phi \) is order preserving, we have
\[
\Phi(f(A)) \leq \Phi \left( \left( \frac{M f_1 - A}{M - m} f'(m) + \frac{A - m f_1}{M - m} f'(M) \right)^{\frac{1}{2}} \right)
\]
The above inequality can also be written as
\[
\Phi(f(A)) \leq \Phi \left( \left( \frac{M f_1 - A}{M - m} f'(m) + \frac{A - m f_1}{M - m} f'(M) \right)^{\frac{1}{2}} \right)
\]
Therefore,
Let the hypothesis of Theorem 2.3 be satisfied. Then
\[\Phi(f(A)) - \Phi(f(A)) \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} f'(m) + \frac{A - m\mathbf{1}_H}{M - m} f'(M)\right)^{1/2}\]

\[\leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} f'(m) + \frac{A - m\mathbf{1}_H}{M - m} f'(M)\right)\]

\[\leq \max_{m \leq t \leq M} \left\{a_t f + b_f - \Phi(f(A))\right\} \mathbf{1}_H.

Consequently,
\[\Phi(f(A)) \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} f'(m) + \frac{A - m\mathbf{1}_H}{M - m} f'(M)\right)^{1/2}\]

\[\leq \max_{m \leq t \leq M} \left\{a_t f + b_f - \Phi(f(A))\right\} \mathbf{1}_H + \Phi(f(A))
\]

and this concludes the proof. \(\square\)

We have Corollary 2.4 if we put \(\alpha = 1\) in Theorem 2.3 and Corollary 2.5 if we choose \(\alpha\) such that \(\beta = 0\) in Theorem 2.3.

**Corollary 2.4** Let the hypothesis of Theorem 2.3 be satisfied. Then
\[\Phi(f(A)) \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} f'(m) + \frac{A - m\mathbf{1}_H}{M - m} f'(M)\right)^{1/2}\]

\[\leq \beta \mathbf{1}_H + f(\Phi(A))\]

where
\[\beta = \max_{m \leq t \leq M} \left\{\frac{M - t}{M + m} f(m) + \frac{t - m}{M + m} f(M) - f(t)\right\}.

**Corollary 2.5** Let the hypothesis of Theorem 2.3 be satisfied. Then
\[\Phi(f(A)) \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} f'(m) + \frac{A - m\mathbf{1}_H}{M - m} f'(M)\right)^{1/2}\]

\[\leq \alpha \Phi(f(A))\]

where
\[\alpha = \max_{m \leq t \leq M} \left\{\frac{1}{f(t)} \left(\frac{M - t}{M + m} f(m) + \frac{t - m}{M + m} f(M)\right)\right\}.

Let \(0 < t < 1\) and let \(1 < \frac{1}{2} \leq r\). Consider the function \(f(t) = r^t\). Then we have the following two corollaries.

**Corollary 2.6** Let \(A \in \mathcal{B}(\mathcal{H})\) be a self-adjoint operator with the spectra contained in the interval \([m, M]\) with \(0 < m < M\), and let \(\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})\) be a positive unital linear map. Then for any \(1 < \frac{1}{2} \leq r\), \(t \in (0, 1)\)
\[\Phi(A') \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} - \alpha \Phi(f(\Phi(A)))\right)^{1/2}\]

\[\leq K(m, M, r) \Phi'(A),\]

where the generalized Kantorovich constant \(K(m, M, r)\) ((Furuta et al. 2005, Definition 2.2)) is defined by
\[K(m, M, r) = \frac{(mM' - Mm')}{(r - 1)(M - m)} \left(\frac{r - 1}{r} \frac{M' - m'}{mM' - Mm'}\right)^{1/2},\]

(2.7)

In particular,
\[\Phi(A') \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} - \alpha \Phi(f(\Phi(A)))\right)^{1/2}\]

\[\leq K(m, M, r) \Phi'(A),\]

for any \(r \geq 2\).

The above inequalities also hold when \(r < 0\).

**Remark 2.7** It follows from Corollary 2.6 that
\[\Phi(A^{-1}) \leq \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} - \alpha \Phi(f(\Phi(A)))\right)^{1/2}\]

\[\leq \frac{(M + m)^2}{4Mm} \Phi^{-1}(A),\]

(2.8)

since \(K(m, M, -1) = \frac{(M + m)^2}{4Mm}\). Integrating the inequality (10) over \(t \in [0, 1]\), we find that
\[\Phi(A^{-1}) \leq \frac{1}{0} \Phi\left(\frac{M\mathbf{1}_H - A}{M - m} - \alpha \Phi(f(\Phi(A)))\right)^{1/2}\]

\[\leq \left(\frac{M + m)^2}{4Mm}\right) \Phi^{-1}(A).\]

Since the mapping \(\Phi\) is linear and continuous, then
\[\Phi(A^{-1}) \leq \Phi\left(\frac{1}{0} \left(\frac{M\mathbf{1}_H - A}{M - m} - \alpha \Phi(f(\Phi(A)))\right)^{1/2}\right)\]

\[\leq \left(\frac{M + m)^2}{4Mm}\right) \Phi^{-1}(A).\]

(2.9)

Observe that the inequality (2.9) gives a refinement of the operator Kantorovich inequality (Marshall and Olkin 1990).
Corollary 2.8  Let the hypothesis of Corollary 2.5 be satisfied. Then
\[ \Phi(A') \leq \Phi\left( \frac{M_{1\mathcal{H}} - A}{M - m} \left( m^{r'} + A - m1_{\mathcal{H}} \right) \right) \]
\[ \leq C(m, M, r)1_{\mathcal{K}} + \Phi'(A), \]
where the Kantorovich constant for the difference \( C(m, M, r) \) (Furuta et al. 2005, Theorem 2.58) is defined by
\[ C(m, M, r) = \frac{\text{Mm}^r - \text{mM}^r}{M - m} + (r - 1) \left( \frac{\text{M}^r - \text{m}^r}{r(M - m)} \right)^2. \]

In particular,
\[ \Phi(A') \leq \Phi\left( \frac{M_{1\mathcal{H}} - A}{M - m} \right) \leq C(m, M, r)1_{\mathcal{K}} + \Phi'(A) \]
for any \( r \geq 2 \).

The above inequalities also hold when \( r < 0 \).

Example 2.9  Letting \( t = 1/2 \) and \( \Phi(T) = \frac{1}{2} \text{tr}(T) \). Consider \( A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \). Then, of course, we can choose \( m = 1.35 \) and \( M = 3.8 \). Simple calculations show that
\[ \Phi(A^{-1}) = 0.5, \]
\[ \Phi\left( \frac{M_{1\mathcal{H}} - A}{M - m} \right) \approx 0.51, \]
\[ \frac{(M + m)^2}{4Mm} \Phi^{-1}(A) \approx 0.517. \]

Consequently,
\[ \Phi(A^{-1}) \leq \Phi\left( \frac{M_{1\mathcal{H}} - A}{M - m} \right) \]
\[ \leq \frac{(M + m)^2}{4Mm} \Phi^{-1}(A). \]

Remind that the function \( f(t) = t^{r'} \) for \( r > 1 \) is not operator monotone on \([0, \infty)\). In the sense that \( A \preceq B \) does not always ensure \( A^{r'} \preceq B^{r'} \). Related to this problem, Furuta (Furuta 1998) proved: Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be two positive operators such that their spectrums contained in the interval \( [m, M] \), for some scalars \( 0 < m < M \). If \( A \preceq B \), then \( A^{r'} \preceq K(m, M, r)B^{r'} \) for \( r \geq 1 \).

Next, we present a better estimate than Furuta inequality (2.11). To this end, we recall the following operator version of Jensen’s inequality which is shown by Mond and Pečarić in (Furuta et al. 2005, Theorem 1.2).

Lemma 2.10  Let \( A \in \mathcal{B}(\mathcal{H}) \) be a self-adjoint operator with the spectrums contained in the interval \( J \) and let \( x \in \mathcal{H} \). If \( f : J \to \mathbb{R} \) is convex function, then \( f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle \).

Theorem 2.11  Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be two self-adjoint operators with the spectrums contained in the interval \( [m, M] \) with \( m < M \), and let \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a positive unital linear map. If \( f : [m, M] \to (0, \infty) \) is a continuous increasing function such that \( f^t \) is \( f \) for a given positive real number \( x \)
\[ f(A) \leq \left( \frac{M_{1\mathcal{H}} - A}{M - m} f^t(m) + \frac{A - m1_{\mathcal{H}}}{M - m} f^t(M) \right) \]
\[ \leq \beta 1_{\mathcal{H}} + \gamma f(B), \]
holds for \( \beta = \max_{m < t < M} \{ a f(t) + b f(t) \} \).

Proof  Our assumption implies
\[ f(A) \leq \left( \frac{M_{1\mathcal{H}} - A}{M - m} f^t(m) + \frac{A - m1_{\mathcal{H}}}{M - m} f^t(M) \right) \]
\[ \leq a f(A) + b f(1_{\mathcal{H}}), \]
thanks to (2.6). Then, for any unit vector \( x \in \mathcal{H} \)
\[ \langle f(A)x, x \rangle \leq \left( \frac{M_{1\mathcal{H}} - A}{M - m} f^t(m) + \frac{A - m1_{\mathcal{H}}}{M - m} f^t(M) \right) \]
\[ \leq a \langle Ax, x \rangle + b f(1_{\mathcal{H}}). \]

Therefore,
\[ \langle f(A)x, x \rangle - \gamma f(\langle Bx, x \rangle) \]
\[ \leq \left( \frac{M_{1\mathcal{H}} - A}{M - m} f^t(m) + \frac{A - m1_{\mathcal{H}}}{M - m} f^t(M) \right) \]
\[ \frac{(M + m)^2}{4Mm} \Phi^{-1}(A). \]
\begin{align*}
(f(A)x, x) & \leq \left\langle \left( \frac{M1_H - A}{M-m} f'(m) + \frac{A - m1_H}{M-m} f'(M) \right)^\frac{1}{2}, x, x \right\rangle \\
& \leq \beta + 2f(\langle Bx, x \rangle) \\
& \leq \beta + \lambda (f(B)x, x) \\
& = \langle (\beta 1_H + 2f(B))x, x \rangle.
\end{align*}

This completes the proof. \qed

As a direct consequence of Theorem 2.11, we have:

**Corollary 2.12** Let $A, B \in B(\mathcal{H})$ be two self-adjoint operators with the spectra contained in the interval $[m, M]$ with $0 < m < M$. If $A \leq B$, then for any $1 < \frac{1}{t} \leq r$, $t \in (0, 1)$

\begin{align*}
A' & \leq \left( \frac{M1_H - A}{M-m} m^\alpha + \frac{A - m1_H}{M-m} M^\alpha \right)^\frac{1}{2} \\
& \leq K(m, M, r)B' \\
\end{align*}

and

\begin{align*}
A' & \leq \left( \frac{M1_H - A}{M-m} m^\alpha + \frac{A - m1_H}{M-m} M^\alpha \right)^\frac{1}{2} \\
& \leq C(m, M, r)1_H + B' \\
\end{align*}

where $K(m, M, r)$ and $C(m, M, r)$ are defined as in (2.7) and (2.10), respectively.

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