AN EXPPLICIT THEORY OF $\pi_{1,\text{crys}}^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$

V : The point of view of $\lim_{N \to \infty} \pi_{1,\text{DR}}^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$

V-1 : The Frobenius extended to $\pi_{1,\text{DR}}^{\text{un}}(\mathbb{P}^1 - \{0, \mu_p^a N, \infty\})$

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Abstract. Let $p$ a prime number. For all $N \in \mathbb{N}^*$ prime to $p$, let $k_N$ be a finite field of characteristic $p$ containing a primitive $N$-th root of unity. Let $X_{k_N,N} = \mathbb{P}^1 - (\{0, \infty\} \cup \mu_N) / k_N$. This work is an explicit theory of the crystalline pro-unipotent fundamental groupoid $(\pi_{1,\text{crys}}^{\text{un}})^r$ of $X_{k_N,N}$. In the parts I to IV, we have considered each possible value of $N$ separately. The purpose of part V is to study the role of the morphisms relating $\pi_{1,\text{crys}}^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$ and $\pi_{1,\text{DR}}^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$ when $N_1$ divides $N_2$. In V-1, we specify this question to the theme of part I, the computation of the Frobenius. For any $N \in \mathbb{N}^*$, let $K_N = \mathbb{Q}_p(\xi_N)$ where $\xi_N \in \mathbb{Q}_p$ is a primitive $N$-th root of unity, and $X_{K_N,N} = \mathbb{P}^1 - (\{0, \infty\} \cup \mu_N) / K_N$. For $N$ prime to $p$, we are used to view the Frobenius of $\pi_{1,\text{crys}}^{\text{un}}(X_{k_N,N})$ as a structure on $\pi_{1,\text{DR}}^{\text{un}}(X_{K_N,N})$.

In V-1, we show that the Frobenius of $\pi_{1,\text{DR}}^{\text{un}}(X_{K_N,N})$, iterated $a \in \mathbb{N}^*$ times, can be extended canonically as a structure of $\pi_{1,\text{DR}}^{\text{un}}(X_{K_{N^a},N^a})$. This allows to define generalizations of adjoint $p$-adic multiple zeta values associated with roots of unity of order $p^a N$, and several related objects. This also gives a canonical framework to relate to each other the direct method of computation of the Frobenius of I-1 and the indirect methods of computation of the Frobenius of I-2 and I-3.

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1. Introduction

1.1. Let \( p \) be a prime number. For each \( N \in \mathbb{N}^* \) prime to \( p \), let \( q_N \in \mathbb{P}^N \) such that the finite field \( k_N = \mathbb{F}_{q_N} \) contains a primitive \( N \)-th root of unity. Let \( X_{k_N,N} \) be the variety \( \mathbb{P}^1 - \{(0, \infty) \cup \mu_N \} \) over \( k_N \).

The purpose of this work is to construct an explicit theory of the crystalline pro-unipotent fundamental groupoid \( \pi_1^{\text{un,crys}} \) of \( X_{k_N,N} \), in the sense of [De], [CL], [S1], [S2], with a particular focus on the \( p \)-adic multiple zeta values at \( N \)-th roots of unity \((pMZV_{\mu_N})'s\) where 'at roots of unity' is omitted when \( N = 1 \) \((pMZV's)\), which are images of \( p \)-adic periods by a reduction map, are defined via the Frobenius and characterize the Frobenius. The complex multiple zeta values at roots of unity \((MZV_{\mu_N})'s\) are Betti-De Rham periods of \( \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}) \), and are the following numbers

\[
\zeta(\xi_{N_1}, \ldots, \xi_{N_d}) = \sum_{0 < m_1 < \ldots < m_d} \frac{(\xi_{N_1}^{m_1} \cdots \xi_{N_d}^{m_d})^{\alpha}}{m_1 \cdots m_d} \in \mathbb{C}
\]

with \( N \in \mathbb{N}^*, \xi_N \) a primitive \( N \)-th root of unity in \( \mathbb{Q}^p \subset \mathbb{C}, n_1, \ldots, n_d \in \mathbb{N}^*, \) and \( j_1, \ldots, j_d \in \{1, \ldots, N\} \), such that \((\xi_{N_1}^{j_1}, \ldots, \xi_{N_d}^{j_d}) \neq (1, 1) \).

Let now \( \xi_N \) be a primitive \( N \)-th root of unity in \( \mathbb{Q}^p \) and \( K_N = \mathbb{Q}_p(\xi_N) \subset \mathbb{Q}_p \). Let \( \alpha \in \pm \mathbb{N}^* \cup \{\pm \infty\} \).

One has for each \( \alpha \) a family of numbers called \( p \)-adic multiple zeta values at roots of unity, denoted by \( \zeta_{p,\alpha}(\xi_N^{j_1}, \ldots, \xi_N^{j_d}) \in K_N \), for all \( n_d, \ldots, n_1 \in \mathbb{N}^*, j_1, \ldots, j_d \in \mathbb{Z}/N\mathbb{Z} \). They have been defined in [DeG], [F1], [F2], [Y], for certain particular values of \( \alpha \), then a different convention involving the inverse of the Frobenius was adopted in [U1], [U2], and the definition was finally generalized to all values of \( \alpha \) in [J I-1] and [J I-3]. They characterize the Frobenius of \( \pi_1^{\text{un,crys}}(X_{k_N,N}) \) at base-points \((-\tilde{\Gamma}_1, \tilde{\mu}_0)\) iterated \( \alpha \) times.

By extending the conjecture in [DeG] §5.28, for each \( \alpha \) and \( N \), the ideal of the algebraic relations satisfied by the numbers \( \zeta_{p,\alpha} \) is conjecturally generated by the algebraic relations satisfied by their complex analogues (1) and the vanishing of the \( p \)-adic analogy of \( 2\pi, \pi \), which implies in particular that, for all \( s \in \mathbb{N}^* \), we have \( \zeta_{p,\alpha}(2m) = 0 \) (where, when \( N = 1 \), \((n_d, \ldots, n_1) = (1, \ldots, 1)\)).

For all \( N \in \mathbb{N}^* \), let \( X_{K_N,N} = \mathbb{P}^1 - \{0, \mu_N, \infty\} / K_N \). When \( N \) is prime to \( p \), following [De], §13, we view \( \pi_1^{\text{un,crys}}(X_{k_N,N}) \) as \( \pi_1^{\text{un,DR}}(X_{k_N,N}) \) equipped with the Frobenius. We have computed the Frobenius, in particular \( pMZV_{\mu_N} \)'s, in part I [J I-1] [J I-2] [J I-3]. We have used our explicit computation to study explicitly \( pMZV_{\mu_N} \)'s in part II [J II-1] [J II-2] [J II-3], part III [J III-1] [J III-2], part IV [J IV-1] [J IV-2]. This part V is the last one of this theory.

1.2. The initial motivation for this part V is the following. Until now, we have considered the groupoids \( \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}) \) for each \( N \) separately. If \( N_1, N_2 \in \mathbb{N}^* \) are such that \( N_1|N_2 \), we have a morphism of groupoids

\[
\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_{N_2}, \infty\})|_{B_{N_1,N_2}} \to \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_{N_1}, \infty\})|_{B_{N_1,N_2}}
\]

where \( |_{B_{N_1,N_2}} \) means the restriction to the set \( B_{N_1,N_2} \) of couples of base-points which are shared by \( \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_{N_2}, \infty\}) \) and \( \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_{N_1}, \infty\}) \). If \( N_1 \) and \( N_2 \) are prime to \( p \), for convenient base-points \( x, y \) over \( \mathbb{F}_p \), we have a morphism

\[
\pi_1^{\text{un,crys}}(\mathbb{P}^1 - \{0, \mu_{N_2}, \infty\}, y, x) \to \pi_1^{\text{un,crys}}(\mathbb{P}^1 - \{0, \mu_{N_1}, \infty\}, y, x)
\]

which means that, if \( \tilde{x} \) and \( \tilde{y} \) are convenient lifts of \( x \) and \( y \) over \( W(\mathbb{F}_p) \), we have a morphism of pro-affine schemes between the De Rham realizations \( \pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{N_2}, \infty\}, \tilde{y}, \tilde{x}) \to \pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{N_1}, \infty\}, \tilde{y}, \tilde{x}) \), which is compatible with the Frobenius. In particular, the notation \( \zeta_{p,\alpha} \) above is consistent : two equal sequences \((\xi_{N_1}^{j_1}, \ldots, \xi_{N_d}^{j_d}) = (\xi_{N_2}^{j_1}, \ldots, \xi_{N_d}^{j_d})\) give the same number. The morphisms (2) form a projective system when \( (N_1, N_2) \) varies. Moreover, each of the morphisms (2) has a natural
section, and these sections form an inductive system when \((N_1, N_2)\) varies. It seems reasonable to expect that these objects have a certain role to play in this theory. The subject of this part V is to realize this expectation. In this V-1, we specialize this problem to the topic of part I: what we want is to bring together the computation of the Frobenius and the morphisms (2).

1.3. Let us see what is at stake by the question above. First, the computation of the Frobenius and the morphisms (3), as well as their sections, actually commute. Indeed, the morphisms (3) actually commute with the canonical presentations of the groupoids \(\pi_1^{\text{un, DR}}(X_{K,N})\), such that the groupoid \(\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})\) can be described functorially in terms of its fiber at \(\omega_{\text{DR}}\), and the Hopf algebra of global functions on the affine group scheme \(\pi_1^{\text{un, DR}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}, \omega_{\text{DR}})\) is the shuffle Hopf algebra over the alphabet \(\{e_0, e_{\xi_1}, \ldots, e_{\xi_N}\}\), generated as a \(\mathbb{Q}\)-vector space by words over \(\{e_0, e_{\xi_1}, \ldots, e_{\xi_N}\}\); this alphabet represents a basis of \(H^{1, \text{DR}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})\), and the canonical choice of basis, which is used all the time, is given by the differential forms \(\omega_0(z) = \frac{dz}{z}\) and \(\omega_j(z) = \frac{dz}{z - \xi_N}\), \(j = 1, \ldots, N\). Since, when \(N_1|N_2\), we have an inclusion \(\{\omega_0, \omega_1, \ldots, \omega_{N_1}\} \subset \{\omega_0, \omega_1, \ldots, \omega_{N_2}\}\), and since the Frobenius is functorial, the computation of the Frobenius of \(\pi_1^{\text{un, DR}}(X_{K,N_1},N_1)\) can be viewed canonically as a computation within \(\pi_1^{\text{un, DR}}(X_{K,N_2},N_2)\) and, conversely, the computation of the Frobenius of \(\pi_1^{\text{un, DR}}(X_{K,N_2},N_2)\) restricts to the computation of the Frobenius of \(\pi_1^{\text{un, DR}}(X_{K,N_1},N_1)\). This shows that a part of the question to bring together the computation of the Frobenius and the morphisms (2) is trivial.

Let us now consider \(\pi_1^{\text{un, DR}}(X_{K,N_1},N_1|N')\) with \(N'\) non-prime to \(p\), and divisible by \(N\) which is prime to \(p\), say \(N = N'p^{v_p(N')}\). Note that since \(K_N\) is ramified, it would make no sense to apply to \(\pi_1^{\text{un, DR}}(X_{K,N_1},N_1|N')\) the construction of the Frobenius structure of [De], §11, and that the reduction modulo \(p\) of \(X_{K,N_1},N_1\) is equal to the one of \(X_{K,N},N\). One has a morphism

\[
\pi_1^{\text{un, DR}}(X_{K,N_1},N_1|N')|_{B_{N_1},N'} \rightarrow \pi_1^{\text{un, DR}}(X_{K,N},N)|_{B_{N'},N'}
\]

Let us consider the Frobenius of \(\pi_1^{\text{un, DR}}(X_{K,N})\) iterated \(\alpha \in \mathbb{N}^* \cup -\mathbb{N}^*\) times, with \(N' = Np^{[\alpha]}\). There is a basic reason why our study of \(\pi_1^{\text{un, crys}}(X_{K,N})\) is somewhat related to the morphism (4). Let \(W_N = W(k_N)\) be the ring of Witt vectors of \(k_N\), and let \(X_{W_N,N} = \mathbb{P}^1 - \{0, \mu_N, \infty\}/W_N\), which has the global lift of Frobenius \(z \mapsto z^p\). For \(\alpha \in \mathbb{N}^*\), the map \(z \mapsto z^p\) sends the groupoid \(\pi_1^{\text{un, DR}}(X_{K,N})\), with its connection \(\nabla_{\text{KZ}}^N\) to a variant on \(X_{W_N,N} = X_{W_N,N}|_{\text{Spec}(W_N)} \times_{\text{Spec}(W_N)} \text{Spec}(K_N)\) where \(X_{W_N,N}\) is equal to the pullback of \(X_{W_N,N}\) by the \(\alpha\)-th iteration of the Frobenius automorphism \(\sigma : W_N \rightarrow W_N\) of \(W_N\). Whereas \(\nabla_{\text{KZ}}^N\) is the map \(f \mapsto f^{-1}(df - \sum_{i=0}^{N-1} e_{z_i,N}dz, z = z_{i,N})\), with \(z_{0,N} = 0\) and \(z_{i,N} = \xi_i\), \(i = 1, \ldots, N\), the variant \(\nabla_{\text{KZ}}^{p,\alpha}\) is \(f \mapsto f^{-1}(df - \sum_{i=0}^{N} e_{z_i,N}p^i dz)\) since, the \(z_i,N\)‘s being roots of unity, we have \(\sigma^\alpha(z_{i,N}) = z_{i,N}^p\) for all \(i \in \{1, \ldots, N\}\). The Frobenius of \(\pi_1^{\text{un, DR}}(X_{K,N})\) is a natural isomorphism between \((\pi_1^{\text{un, DR}}(X_{K,N}),\nabla_{\text{KZ}}^N)\) and \((\pi_1^{\text{un, DR}}(X_{K,N}^{p,\alpha}),\nabla_{\text{KZ}}^{p,\alpha})\). By writing :

\[
\frac{dz}{z^\rho} = \sum_{\rho \in \rho^{p,\alpha}(\mathbb{Z})} z_{i,N}dz = \sum_{\rho \in \rho^{p,\alpha}(\mathbb{Z})} \left[ \sum_{l \in L_{i,N} < 1} \frac{z_{i,N}dz}{z^\rho} \right] < 1
\]

we see that the Frobenius of \(\pi_1^{\text{un, DR}}(X_{K,N})\) can be viewed as an isomorphism between two quotients of the groupoid \(\pi_1^{\text{un, DR}}(X_{K,N},\rho^{p,\alpha})\) with its connection \(\nabla_{\text{KZ}}^{p,\alpha}\).

This suggests to study, more generally, how \(\pi_1^{\text{un, DR}}(X_{K,N^{[\alpha]},\rho^{[\alpha]}})\) is connected to \(\pi_1^{\text{un, crys}}(X_{K,N})\) where \(\alpha \in \mathbb{N}^* \cup -\mathbb{N}^*\) is the number of iterations of the Frobenius and \(N \in \mathbb{N}^*\) is prime to \(p\).

We are going to adopt a radical form of this question: reformulate, or enrich, the Frobenius structure of
\((\pi^\text{un,DR}_{1}(\mathbb{P}^1 - \{0,\mu_N,\infty\}), \nabla^K_\mathbb{KZ})\), iterated \(\alpha\) times, with its explicit formulas from part I, into a structure on \((\pi^\text{un,DR}_{1}(\mathbb{P}^1 - \{0,\mu_{p^\alpha N},\infty\}), \nabla^K_{p^\alpha N})\), with explicit formulas within \((\pi^\text{un,DR}_{1}(\mathbb{P}^1 - \{0,\mu_{p^\alpha N},\infty\}), \nabla^K_{p^\alpha N})\).

1.4. Given the question above, let us review how we have computed the Frobenius in part I [J I-1], [J I-2], [J I-3]. The results of the computation were expressed in terms of weighted multiple harmonic sums at roots of unity: for \(n, d \in \mathbb{N}^*, s_d, \ldots, s_1 \in \mathbb{N}^*, j_1, \ldots, j_{d+1} \in \mathbb{Z}/N\mathbb{Z},\)

\begin{equation}
\text{har}_m \left( \frac{\zeta^{j_{d+1}}_N, \ldots, \zeta^{j_1}_N}{n_d, \ldots, n_1} \right) = m^{n_d + \cdots + n_1} \sum_{0 < m_1 < \cdots < m_d < m} \frac{\zeta^{j_{d+1}}_N}{m_1^{n_1 + \cdots + n_d}} \cdots \frac{\zeta^{j_1}_N}{m_d^{n_d}} \left( \frac{1}{\zeta^{j_{d+1}}_N} \right)^n \tag{6}
\end{equation}

These are, up to a multiplicative factor in \(m^2\), the regularized iterated integrals of \((7)\) are overconvergent analytic functions on the rigid analytic space \((P^1, \mathbb{C}^*) = \bigcup_N \{z \mid |z - \xi_N|_p < 1\})/K_N. The sequences of coefficients of the power series expansions of the iterated integrals of the forms \((7)\) are, up to a simple multiplicative factor, the "multiple harmonic sums with congruences" modulo powers of \(p\), at roots of unity of order \(N\) prime to \(p\): namely, for any \(n \in \mathbb{N}^*,\) for any index as in \((6), a \in \mathbb{N}^*\) and \(I \subset \{1, \ldots, d\} :\)

\begin{equation}
\text{har}_{n_d, \ldots, n_1} \left( \frac{\zeta^{j_{d+1}}_N, \ldots, \zeta^{j_1}_N}{n_d, \ldots, n_1} \right) = n^{n_d + \cdots + n_1} \sum_{0 < n_1 < \cdots < n_d < n, \text{mod } p^a} \frac{\zeta^{j_{d+1}}_N}{n_1^{s_1 + \cdots + s_{d}}} \cdots \frac{\zeta^{j_1}_N}{n_d^{s_d}} \left( \frac{1}{\zeta^{j_{d+1}}_N} \right)^n \tag{8}
\end{equation}

In [J I-2] and [J I-3], we have made a distinction between three frameworks of computations:

- the "framework \(f^1_0\)" which involves the scheme \(\pi^\text{un,DR}_{1}(X_{K,N}, \mathbb{Z}, -I_0, I_0)\) and operations on it.
- the "framework \(f^c_0\)" which involves multiple harmonic sums viewed as coefficients of power series expansions of hyperlogarithms and operations on \(\pi^\text{un,DR}_{1}(X_{K,N})\).
- the "framework \(\Sigma\)" which involves multiple harmonic sums viewed as elementary finite iterated sums via their formula \((6).\)

In [J I-2] and [J I-3], we have shown using [J I-1] that the differential equation of the Frobenius had a simplification in a certain "limit", in which it was equivalent to a relation between weighted multiple harmonic sums and adjoint \(p\)-adic multiple zeta values at roots of unity (abbreviated Ad \(p\text{MZV}_{\mu_N}\)'s, and denoted by \(\zeta^\text{Ad}_{\mu_N}(w)\)), which are variants of \(p\text{MZV}_{\mu_N}\)'s equivalent to them up to a polynomial base-change defined over \(\mathbb{Q}\). We called harmonic Frobenius (standing for "incarnation of the Frobenius which is natural from the point of view of multiple harmonic sums"), the corresponding "limit" of the Frobenius. The harmonic Frobenius is sufficient to reconstruct the whole of the Frobenius.

We have computed indirectly the harmonic Frobenius, i.e. we have expressed it in two different ways, one of them explicit in terms of multiple harmonic sums in the framework \(\Sigma\), the other one in terms of Ad \(p\text{MZV}_{\mu_N}\)'s in the framework \(f^c_0\) or \(f^1_0\) and we showed that the two expressions can be identified.

The results were expressed by the following objects. We have defined "\(p\)-adic harmonic Ihara actions", \(c^\text{har}_1, c^\text{har}_2, c^\text{har}_3, c^\text{har}_4\). These are continuous group actions, related to the Ihara bracket on Lie \(\pi^\text{un,DR}_{1}(X_K, \mathbb{Z}, I_0)\), of a complete topological subgroup of \(\pi^\text{un,DR}_{1}(X_K, \mathbb{Z}, I_0)(K)\) equipped with an appropriate topology, on a complete space containing as an element the generating sequence of all multiple harmonic sums. We have
defined *maps of comparison between series and integrals* $\text{comp}^{\Sigma-\Sigma}$ and $\text{comp}^{\Sigma-\Sigma}$ relating the harmonic Ihara actions to each other. We also have defined in [J I-3] two maps $\text{iter}^I_{\bar{\alpha}}$ and $\text{iter}^N_{\bar{\alpha}}$ of "iterations of the harmonic Frobenius". We have written fully explicit formulas for all these objects. The explicit formulas for $p\text{MZV}_{\mu,N}'$'s amount then to say that the generating series of $\text{Ad} p\text{MZV}_{\mu,N}'$'s is the image of the one of prime weighted multiple harmonic sums by the map $\text{comp}^{\Sigma-\Sigma}$.

The flat sections of the connection $\nabla_{KZ}^{\mu,N}$ on $\pi_1^{\text{un}DR}(X_{K_{\mu,N},p^aN})$ are the iterated integrals of the differential forms $\frac{dx}{x}$ and $\frac{dz}{z}$, with $p$ a $p^a$-th root of unity and $\xi$ an $N$-th root of unity. They are called hyperlogarithms as in the case of $\pi_1^{\text{un}DR}(X_{K_{N},N})$. The coefficients of their power series expansions at 0 are up to a multiplicative factor in $m^2$, the weighted multiple harmonic sums at roots of unity of order $p^aN$:

$$
\text{har}_m \left( \begin{array}{c} \xi^{j_1}_{1}, \ldots, \xi^{j_{n_d}}_{n_d}, \ldots, \xi^{j_1}_{n_1} \\ n_d, \ldots, n_1 \end{array} \right) = m^{n_d+\ldots+n_1} \sum_{0<m_1<\ldots<m_d<m} \left( \begin{array}{c} \xi^{\rho_1}_{m_1}, \ldots, \xi^{\rho_d}_{m_d} \\ \xi^{\rho_1}_{m_1}, \ldots, \xi^{\rho_d}_{m_d} \end{array} \right) \prod_{i=1}^d m_i \left( \frac{1}{\xi^{j_i}_{m_i}} \right)^{m_i}
$$

with $\alpha \in \mathbb{N}^*$ and $N \in \mathbb{N}^*$ prime to $p$, and $\rho_{p^a}$ a primitive $p^a$-th root of unity in $K_{p^a,N}$, $m \in \mathbb{N}^*$, $(\xi^{j_1}_{n_1}, \ldots, \xi^{j_{n_d}}_{n_d})$ as in (6) and $j_1, \ldots, j_{n_d+1} \in \mathbb{Z}/p^a\mathbb{Z}$.

1.5. We are going to show that there exists a canonical relation between $\pi_1^{\text{un}DR}(X_{K_{\mu,p^aN},p^aN})$ and $\pi_1^{\text{un}DR}(X_{\mu,N})$, defined over $K_{p^a,N}$, which extends the Frobenius of $\pi_1^{\text{un}DR}(X_{K_{N},N})$ (which is an isomorphism between $\pi_1^{\text{un}DR}(X_{K_{N},N})$ and $\pi_1^{\text{un}DR}(X_{\mu,N})$). The first immediate application is to define a notion of adjoint $p$-adic multiple zeta values at $p^aN$-th roots of unity (Ad $p\text{MZV}_{\mu,p^aN}$’s). We are also going to show that all the tools of computation of the Frobenius of part I, reviewed in §1.4, can be generalized to this framework, giving a computation of the Frobenius extended to $\pi_1^{\text{un}DR}(X_{K_{\mu,p^aN},p^aN})$, and in particular of (Ad $p\text{MZV}_{\mu,p^aN}$’s).

**Theorem-Definition V-1.a : the Frobenius extended to $\pi_1^{\text{un}DR}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\})$ ; adjoint $p$-adic multiple zeta values at $p^aN$-th roots of unity**

There exist a unique element of $\mathbb{A}^1(U_{\mathbb{N}}) \otimes_{K_N} K_{p^aN} \langle (e_{\rho_{p^a\xi}})_{j=1,\ldots,p^a} \rangle K_{p^aN} \langle (e_{\rho_{p^a\xi}})_{j=1,\ldots,p^a} \rangle$, denoted by $(L_{p,\mu,p^{aN}}, \psi^{\mu,p^{aN}}_{\xi}, \ldots, \psi^{\mu,p^{aN}}_{\xi})$, which satisfies the following extension of the differential equation of the Frobenius

$$
dL_{p,\mu,p^{aN}} = \left(p^a\omega_0(z)e_0 + \sum_{\rho_{p^a\xi}} \omega_{\rho_{p^a\xi}}(z)e_{\rho_{p^a\xi}} \right) L_{p,\mu,p^{aN}} - L_{p,\mu,p^{aN}} \left(p^a\omega_0(z)e_0 + \sum_{j=1}^{N} \omega_{\rho_{p^a\xi}}(z^{p^a})\psi^{\mu,p^{aN}}_{\xi} \right)
$$

and $L_{p,\mu,p^{aN}}(0) = 1$ and a certain regularity property of $L_{p,\mu,p^{aN}}$ (see §4.3). Moreover, $(L_{p,\mu,p^{aN}}, \psi^{\mu,p^{aN}}_{\xi}, \ldots, \psi^{\mu,p^{aN}}_{\xi})$ satisfies an extension of the bounds of valuation and the formulas of the direct computation proved in [J I-1]. For all words $w$, we denote by

$$
\zeta_{p,\mu,w}(w) = \psi^{\mu,p^{aN}}_{\xi}[w]
$$

and call these numbers the adjoint $p$-adic multiple zeta values at $p^aN$-th roots unity (Ad $p\text{MZV}_{\mu,p^aN}$’s). This also enables to define (non-adjoint) $p$-adic multiple zeta values at $p^aN$-th roots unity ($p\text{MZV}_{\mu,p^{aN}}$’s) denoted by $\zeta_{p,\mu,w}(w)$.

**Theorem-Definition V-1.b : the harmonic Frobenius, harmonic Ihara actions, iterations of the harmonic Frobenius and comparison maps extended to $\pi_1^{\text{un}DR}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\})$**
There exist
\(f[0][0]_N : K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \times \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \to \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle)\)
\(f[0][1]_N : K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \times \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \to \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle)\)
whose restrictions to \(K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \times \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle)\) are equal to the harmonic Ihara actions defined in part I, given by explicit formulas.

b) comparison maps between sums and integrals at \(p^N\)-th roots of unity:
\(\text{comp}^{\Sigma-f} : K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \times \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \to \text{Map}(N, K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle)\)
whose restrictions to \(K[N] \langle \{e[0]\rangle) \) are the comparison maps defined in part I, and given by explicit formulas.

c) maps of iteration of the harmonic Frobenius at \(p^N\)-th roots of unity:
\(\text{iter}^0_{\text{har}} : K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \to K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle)\)
\(\text{iter}^1_{\text{har}} : K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle) \to K[p][N] \langle \{e[0][0]_j\}_{j=1}^m \rangle)\)

Theorem 1.5.
\(\sum z^{\langle A\rangle} \not\equiv \psi[0]_{\text{har}} \; \text{mod } \langle \{e[0][0]_j\}_{j=1}^m \rangle)\)

and at words \(w\) such that \(\frac{w}{\alpha[0]} > \text{depth}(w)\), with \(\text{iter}^1_{\text{har}} \not\equiv \psi[0]_{\text{har}} \; \text{mod } \langle \{e[0][0]_j\}_{j=1}^m \rangle)\) and \(\text{iter}^1_{\text{har}} \not\equiv \psi[0]_{\text{har}} \; \text{mod } \langle \{e[0][0]_j\}_{j=1}^m \rangle)\).

These results are a new illustration of the interest of considering all the iterates of the Frobenius rather than only the Frobenius : by considering the Frobenius iterated \(\alpha\) times and extending it canonically, we can define generalizations of adjoint \(p\)-adic multiple zeta values at roots of unity of order \(p^N\) : thus by making \(N\) and \(\alpha\) vary, we attain any possible order in \(N^+\).

1.6. We are actually going to prove a slightly more general result than Theorem-Definition V.1.a and Theorem-Definition V.1.b.

We consider the differential forms \(\frac{dz}{z}\) and \(\frac{dz}{z-\xi[0]}\), \(j = 1, \ldots, N\). Our main observation will be the following : the whole of part I remains true if we replace \(\frac{dz}{z-\xi[0]} = -\xi[0]^{-j} \sum_{m \geq 1} \xi[0]^{-jm} z^m dz\) by, more generally,
\(-\xi[0]^{-j} \sum_{m \geq 1} f(m) \text{ mod } p^\alpha \xi[0]^{-jm} z^m, \) with \(f\) is any function \(\mathbb{Z}/p^\alpha \mathbb{Z} \to \mathbb{C}_p\) with values having not too big \(p\)-adic norm, let us say with values in \(O[\mathbb{C}_p]\). If we choose \(f : m \mapsto \rho[\rho]_p^m\), with \(\rho[\rho]_p\) a primitive \(p^\alpha\)-th root of
unity, we obtain the canonical basis of \( H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\}) \) which was implicit above. However, any choice of basis would work, and there is another choice which is interesting for us: \( f: m \mapsto 1_{m \equiv m_0 \mod p^a}, \) for \( m_0 \in \mathbb{Z}/p^a\mathbb{Z} \) (where \( 1 \) denotes the characteristic function). Thus, we will also prove:

**Theorem-Definition V-1.c:** compatibility with the change of basis of \( H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\}) \) and adjoint \( p^a \)-adic multiple zeta values at \( N \)-th roots of unity with congruences modulo \( p^a \)

i) In Theorem-Definitions V-1.a and V-1.b, let us replace the canonical basis of \( H^{1,\text{DR}}(X_{K,\mu_{p^aN},p^aN}), \) which is implicit, by any other basis obtained in the way above from the canonical basis of \( H^{1,\text{DR}}(X_{K,N}) \) and a basis of \( \text{Map}(\mathbb{Z}/p^a\mathbb{Z}, \mathcal{O}_{\mathbb{C}_p}) \). Then, the Theorem-Definitions V-1.a and V-1.b remain valid.

ii) Let two different bases \( B \) and \( B' \) of \( H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\}) \) as in i) such that the matrix decomposing \( B \) in \( B' \) has coefficients in \( \mathcal{O}_{\mathbb{C}_p} \).

Then the automorphism of \( H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\}) \) sending \( B' \) to \( B \) induces functorially a morphism between the generalizations of regularized iterated integrals, harmonic Ihara actions, iterations of the harmonic Frobenius associated respectively with \( B \) and \( B' \) by i).

iii) Let us choose the basis of \( H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\}) \) arising from the maps, \( f: m \mapsto 1_{m \equiv m_0 \mod p^a}, \) for \( m_0 \in \mathbb{Z}/p^a\mathbb{Z} \). The analogs of the objects defined in Theorem-Definitions V-1.a and V-1.b associated with this basis are called as in Theorem-Definitions V-1.a and V-1.b except that "at \( \mu \)" is replaced by "at \( \mu_{p^aN} \)."

1.7. Extending to \( \pi_{1,\text{un,DR}}^{(X_{K,\mu_{p^aN},p^aN})} \) the Frobenius of \( \pi_{1,\text{un,DR}}^{(X_{K,N})} \) breaks a dissymmetry between the indirect method and the direct method to compute it: in the direct method [J I-1], we had to use iterated integrals of the set of differential forms \( \{ d\frac{z}{x}, \frac{dz}{z - \zeta_N}, j = 1, \ldots, N, \frac{dz}{z^p - (\zeta_N)^{j'}} j' = 1, \ldots, N \} \), which involves at the same time \( \pi_{1,\text{un,DR}}^{(X_{K,N},N)}, \nabla_{\mathbb{K}^Z} \) and its pull back \( \pi_{1,\text{un,DR}}^{(X_{K,N},N)}(p^{\alpha}, \nabla_{\mathbb{K}^Z}(p^{\alpha})) \); whereas, in the indirect methods [J I-2] and [J I-3], \( \pi_{1,\text{un,DR}}^{(X_{K,N},N)}, \nabla_{\mathbb{K}^Z} \) and \( \pi_{1,\text{un,DR}}^{(X_{K,N},N)}(p^{\alpha}), \nabla_{\mathbb{K}^Z}(p^{\alpha}) \) were separated from each other, appearing in different terms of the equations.

In the new context, \( \{ d\frac{z}{x}, \frac{dz}{z - \zeta_N}, j = 1, \ldots, N, \frac{dz}{z^p - (\zeta_N)^{j'}} j' = 1, \ldots, N \} \) generates, both in the indirect and direct methods of computation, a subspace of the space of differential forms under consideration. Thus this context seems to be a natural framework for relating the direct and the indirect methods of computation of the Frobenius.

**Proposition V-1.d : Descent from \( \pi_{1,\text{un,DR}}^{(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\})} \) to \( \pi_{1,\text{un,DR}}^{(\mathbb{P}^1 - \{0, \mu N, \infty\})} \)** (see §9 for a more precise statement) The regularization, the harmonic Ihara actions, the iterations of the harmonic Frobenius and the comparison maps, when restricted to the subspace of iterated integrals of \( \{ p^{\alpha} d\frac{z}{x}, p^{\alpha} \frac{dz}{z - \zeta_N}, j = 1, \ldots, N, \frac{dz}{z^p - (\zeta_N)^{j'}} j' = 1, \ldots, N \} \), have coefficients defined within \( \pi_{1,\text{un,DR}}^{(X_{K,N},N)} \), i.e. convergent infinite summations of linear combinations of prime weighted multiple harmonic sums of \( \pi_{1,\text{un,DR}}^{(X_{K,N},N)} \).

**Proposition V-1.e : the regularization and the iteration of the harmonic Frobenius** (see §9 for a more precise statement) The regularization of iterated integrals on \( X_{K,\mu_{p^aN},p^aN} \) defined in Theorem-Definition V.1.a can be computed by using the iteration of the harmonic Frobenius defined in Theorem-Definition V.1.b.

**Conclusion :** Via Theorem-Definition V-1.c, Proposition V-1.e and Proposition V-1.d give a connection between the formulas of the direct [J I-1] and indirect [J I-2] [J I-3] computations of the Frobenius, as well as a meaning of this connection in terms of a descent of the Frobenius extended to \( \pi_{1,\text{un,DR}}^{(\mathbb{P}^1 - \{0, \mu_{p^aN}, \infty\})} \).

The present version of this paper is preliminary. In the next version, we will write explicit formulas.
for all the maps involved (they are not significantly different from the formulas for the maps of part I) and we will write commutative diagrams interconnecting them. This will finish to state the compatibility between the direct computation of [J I-1] and the indirect computations of [J I-2] and [J I-3].

Finally, these facts also have an algebraic meaning and will be related, for example, to the fact that the spaces of multiple harmonic sums with congruences of (8) are stable by the double shuffle relations. This will be explained in [J V-2].

1.8. The previous considerations also suggest to retrieve the maximal versions of the results of part I, relatively to curves \(\mathbb{P}^1\) - punctures (one can also ask for the maximality relatively to the terms of multiple harmonic sums : in the framework \(\Sigma\), this is treated, in part I and this paper, by remarking that it is possible to replace in most computations the maps \(n_i \mapsto \frac{1}{n_i}\) by certain locally analytic group homomorphisms \(K^\times \to K^\times\).

We propose an element of answer in the Appendix A. Let us consider a more general \(\pi_1^{un,DR}(\mathbb{P}^1 - \{0 = z_0, z_1, \ldots, z_r, \infty\})\) over a complete normed field \(K\) of characteristic 0, with \(z_1, \ldots, z_r\) of norm 1. It is still equipped with the similar connection \(\nabla_{KZ} : f \mapsto f^{-1}(df - \sum_{r=0}^{r} \frac{dz}{z - z_i})\). The flat sections of \(\nabla_{KZ}\), called hyperlogarithms, have the power series expansions of their flat sections expressed in terms of multiple harmonic sums : in the framework \(\Sigma\), this is treated, in part I and this paper, by remarking that it is possible to replace in most computations the maps \(n_i \mapsto \frac{1}{n_i}\) by certain locally analytic group homomorphisms \(K^\times \to K^\times\).

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\[
\log_m \left( \frac{z_{j_d+1}}{n_d}, \ldots, \frac{z_{j_1}}{n_1} \right) = mn_d + \cdots + n_1 \sum_{0 < m_1 < \cdots < m_d < m} \frac{z_{j_d} m_1}{n_1} \cdots \frac{z_{j_1} m_d}{n_d} \frac{1}{z_{j_d+1}^m} \in K
\]

For the next results, we do not claim a crystalline interpretation similar to the previous ones (let us recall that \(z \mapsto z^p\) stabilizes sets of roots of unity but not a generic subset \(z_1, \ldots, z_r\) of \(K\)).

In Appendix A, we construct a generalization of the \(\Sigma\)-harmonic Ihara action and the equations that it satisfies to this setting, provided certain hypothesis on \(z_1, \ldots, z_r\). By viewing \(z_1, \ldots, z_r\) as variables, this leads to a notion of \(p\)-adic pseudo adjoint multiple polylogarithms.

Our strategy consists in writing (if \(K = \mathbb{C}_p\) or more generally if this is possible) \(z_i = \omega(z_i) + \epsilon_i\) with \(\omega(z_i)\) a root of unity of order prime to \(p\) and \(|\epsilon_i|_p < 1\), and writing a power series expansions with respect to \(\epsilon_i\)'s ; the computation works when \(|\epsilon_i|_p < p^{-\infty}\).

We could use the strategy of Appendix A to generalize the rest of this paper to \(\pi_1^{un,DR}(\mathbb{P}^1 - \{0 = z_0, z_1, \ldots, z_r, \infty\})\).

The qualitative difference between the results for \(\pi_1^{un,DR}(\mathbb{P}^1 - \{0, \mu_{p^rN}, \infty\})\) and \(\pi_1^{un,DR}(\mathbb{P}^1 - \{0 = z_0, z_1, \ldots, z_r, \infty\})\) illustrate the particularity of the case of \(\pi_1^{un,DR}(\mathbb{P}^1 - \{0, \mu_{p^rN}, \infty\})\) and the meaning of the results for that case.

1.9. Related work.

- Aside from our papers [J I-1], [J I-2], [J I-3], two notes of announcement [J N1] and [J N2] which announce some parts of [J I-1] and [J I-2], and the present paper, the question of computing of the Frobenius of \(\pi_1^{un,cryst}(X_{K,N})\) or very closely related questions appear, to our knowledge, in the work of Deligne [De], §19.6, Besser and de Jeu [BD-J], Unver [U1], [U2], [U3], [U4], Yamashita [Y] §3, Dan-Cohen and Chatzistamatiou [D-C].

- Our results give new proofs, explicit formulas and generalizations to the Proposition 2.9 in [U4]. The main result of [U4] (Theorem 1.1) is proved by our paper [J I-1] and generalized in this paper. One can join Proposition 2.5 and Proposition 2.9 of [U4] into a statement giving an explicit basis of a certain space of multiple harmonic sums with congruences (which we interpret as a space of iterated integrals over \(X_{K,p^rN,N}\)) and a decomposition of each element of the space in the basis (we interpret the elements of the basis as iterated integrals over \(X_{K,N,N}\)). The results of [U1], [U2], [U3] are included in those of [U4].

- The \(p\)-adic iterated integrals on certain subspaces of \(\mathbb{P}^1\) of the differential forms \(\frac{dz}{z}\) and \(\frac{dz}{z-p}\), where
Proposition 2.2. (Follows from [De], §12) The group scheme $\text{Spec}(\mathbb{F})$ the shuffle product 
iii) the counit $\varepsilon$ is canonically isomorphic to

The Hopf algebra obtained in this way is called the shuffle Hopf algebra over the alphabet

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2. Review on $\pi_{1}^{\text{uni,DR}}$ and $\pi_{1}^{\text{uni,crys}}$

We review $\pi_{1}^{\text{uni,DR}}(X_K)$ where $X_K$ is of the form $\mathbb{P}^1 - \{0, z_1, \ldots, z_r, \infty\}$ over a (complete normed) field $K$ of characteristic zero, and then $\pi_{1}^{\text{uni,crys}}(X_{K,N})$, viewed as $\pi_{1}^{\text{uni,DR}}(X_{K,N,N})$ equipped with the Frobenius, where $N$, $k_N$, $K_N$ are as in §1.1. For more details, in particular about the definitions, see [J-I-II] §2.

2.1. $\pi_{1}^{\text{uni,DR}}(\mathbb{P}^1 - \{0, z_1, \ldots, z_r, \infty\} / K)$. Let $K$ be any field of characteristic zero, and $X_K = \mathbb{P}^1 - \{0, z_1, z_2, \ldots, z_r \} / K$, where $r \in \mathbb{N}$ and $z_1, z_2, \ldots, z_r \in K$; let $z_0 = 0$, and $e_{\{z_0, \ldots, z_r\}}$ the alphabet $\{e_0, e_{z_1}, \ldots, e_{z_r}\}$.

Proposition-Definition 2.1. Let $\mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}}$ be the $\mathbb{Q}$-vector space $\mathbb{Q}(e_{\{z_0,\ldots, z_r\}}) = \mathbb{Q}(e_0, e_{z_1}, \ldots, e_{z_r})$, freely generated by words on $e_{\{z_0, \ldots, z_r\}}$ including the empty word. The following operations i) to iv) make it into a graded Hopf algebra over $\mathbb{Q}$, where the grading is the number of letters (called the weight) of words:

i) the shuffle product $\varpi: \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}} \otimes \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}} \rightarrow \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}}$ defined by, for all words:

$$\varpi(e_{z_{i_1}}, \ldots, e_{z_{i_1}+1}) = \sum_{\sigma \text{ permutation of } \{1, \ldots, l+1\}} e_{z_{\sigma^{-1}(1)}} \cdot \cdots \cdot e_{z_{\sigma^{-1}(l+1)}}$$

s.t. $\sigma(1) < \cdots < \sigma(l+1)$

ii) the deconcatenation coproduct $\Delta_{\text{dec}}: \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}} \rightarrow \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}} \otimes \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}}$, defined by, for all words:

$$\Delta_{\text{dec}}(e_{z_{i_1}}, \ldots, e_{z_{i_1}+1}) = \sum_{l'=0}^{l} e_{z_{i_1}} \cdots e_{z_{i_1}+l'} \otimes e_{z_{i_1}+l'} \cdots e_{z_{i_1}+1}$$

iii) the counit $\varepsilon: \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}} \rightarrow \mathbb{Q}$ sending all non-empty words to 0.

iv) the antipode $S: \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}} \rightarrow \mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}}$, defined by, for all words:

$$S(e_{z_{i_1}}, \ldots, e_{z_{i_1}+1}) = (-1)^l e_{z_{i_1}}, \ldots, e_{z_{i_1}+1}$$

The Hopf algebra obtained in this way is called the shuffle Hopf algebra over the alphabet $\{e_{z_0}, \ldots, e_{z_r}\}$.

Proposition 2.2. (Follows from [De], §12) The group scheme $\text{Spec}(\mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}})$ is pro-unipotent, and is canonically isomorphic to $\pi_{1}^{\text{uni,DR}}(X_K, \omega_{\text{DR}})$, where $\omega_{\text{DR}}$ is the canonical base-point of $\pi_{1}^{\text{uni,DR}}(X_K)$ in the sense of [De], §12.

If $A$ is a ring, let $A(e_{\{z_0, \ldots, z_r\}})$ be the non-commutative $A$-algebra of power series over the alphabet $e_{\{z_0, \ldots, z_r\}}$. Let us denote by $\mathcal{W}(e_{\{z_0, \ldots, z_r\}})$ the set of words over $e_{\{z_0, \ldots, z_r\}}$; the coefficients of the decomposition of an element $f \in A(e_{\{z_0, \ldots, z_r\}})$ in the basis $\mathcal{W}(e_{\{z_0, \ldots, z_r\}})$ are denoted by $f[\text{word}]$, as follows

$$f = \sum_{w \in \mathcal{W}(e_{\{z_0, \ldots, z_r\}})} f[w]w \tag{11}$$

1Following a common abuse of notation, we denote in the same way $e_{z_i}$ in $\mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}}$, and $e_{z_j}$ in $A(e_{\{z_0, \ldots, z_r\}})$, although, as we are going to see, $A(e_{\{z_0, \ldots, z_r\}})$ can be equipped in a natural way with the structure of the dual of the topological Hopf algebra of $\mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}}$ and we will view it as containing the group of points $\text{Spec}(\mathcal{O}^{m,e}_{\{z_0, \ldots, z_r\}})(A)$.
Proposition-Definition 2.3. (see for instance [U1], §4) The dual \((O^{\infty,(\zeta_0,...,\zeta_r)})^\vee\) of the topological Hopf algebra \(O^{\infty,(\zeta_0,...,\zeta_r)}\) is \(\mathcal{Q}((e_{(\zeta_0,...,\zeta_r)}))\), the completion of the universal enveloping algebra of the complete free Lie algebra over the variables \(e_{\zeta_0}, e_{\zeta_1}, ..., e_{\zeta_r}\), equipped with its canonical structure of topological Hopf algebra. For any \(\mathcal{Q}\)-algebra \(A\), the group of points \(\text{Spec}(O^{\infty,(\zeta_0,...,\zeta_r)})(A)\) consists of the group-like elements of \((O^{\infty,(\zeta_0,...,\zeta_r)})^\vee \otimes \mathcal{Q} A\) and is:

\[
\{ f \in A((e_{(\zeta_0,...,\zeta_r)})) \mid \forall w, w' \in W(e_{(\zeta_0,...,\zeta_r)}), f[w \mathbin{\bullet} w'] = f[w]f[w'], \text{ and } f[0] = 1 \}
\]

and the Lie algebra of \((O^{\infty,(\zeta_0,...,\zeta_r)})^\vee \otimes \mathcal{Q} A\) consists thus of the primitive elements of \((O^{\infty,(\zeta_0,...,\zeta_r)})^\vee \otimes \mathcal{Q} A\) and is:

\[
\{ f \in A((e_{(\zeta_0,...,\zeta_r)})) \mid \forall w, w' \in W(e_{(\zeta_0,...,\zeta_r)}), f[w \mathbin{\bullet} w'] = 0 \}
\]

One says that the elements \(f\) in (12) satisfy the shuffle equation, and that the elements \(f\) in (13) satisfy the shuffle equation modulo products.

Proposition-Definition 2.4. (follows from [De], §7.30 and §12)

i) The canonical connection of \(\pi^{\text{un,DR}}_1(X_K)\), called the Knizhnik-Zamolodchikov (or KZ) connection, is given, in the sense of [De], §7.30, by

\[
\nabla_X^{KZ} : f \mapsto f^{-1}(df - e_0f dz - \sum_{i=1}^r e_i f dz_{\alpha i})
\]

Its horizontal sections have their coefficients defined by iterated integrals of \(dz_i\) and \(dz_{\alpha i}\), \(i = 1,\ldots, r\), called hyperlogarithms.

ii) Assume now that \(K\) is a completed normed field and that \(z_1,\ldots, z_r\) have norm 1. The hyperlogarithms are the functions given by the following power series expansion, for \(|z|_K < 1\) : for \(d \in \mathbb{N}^*, s_1,\ldots, s_d \in \mathbb{N}^*, j_1,\ldots, j_d \in \{1,\ldots, r\},

\[
\text{Li}(\frac{z_{j_1}}{s_1}, \ldots, \frac{z_{j_d}}{s_d}) (z) = \sum_{0 < n_1 < \ldots < n_d} \frac{(\frac{z_{j_1}}{s_1})^{n_1} \ldots (\frac{z_{j_d}}{s_d})^{n_d}}{n_1! \ldots n_d!} K
\]

In particular, \(e_{\alpha i}\) is the residue of \(\nabla_X^{KZ}\) at \(z_i\); for certain purposes, it will be natural to view \(\mathbb{Q}(e_{(\zeta_0,...,\zeta_r)})\) as the \(\mathbb{Q}\)-vector space freely generated by words on the larger alphabet \(\{e_0, e_{\zeta_1}, \ldots, e_{\zeta_r}, e_{\infty}\}\), mounded out by the sum of all the residues \(e_0 + e_{\zeta_1} + \ldots + e_{\zeta_r} + e_{\infty}\).

Definition 2.5. (Following [DeG], §5) Let \(\tau\) be the action of \(\mathbb{G}_m(K)\) on \(K((e_{(0)}),\mu_{P})\), that maps \((\lambda, f) \in \mathbb{G}_m(K) \times K((e_{(0)}),\mu_{P})\) to \(\sum_{w \in W(e_{(0)}),\mu_{P}} \lambda^\text{weight}(w) f[w]w\), where \(f\) is written as in equation (1).

2.2. \(\pi^{\text{un,crys}}_1(X_{k,N})\). The crystalline pro-unipotent fundamental groupoid of \(X_{k,N}\) can be defined as the fundamental groupoid associated to the Titanaki category the unipotent \(F\)-isocrystals over \(X_{k,N}\), following Chiarellotto and Le Stum [CL] ; a variant using logarithmic geometry is due to Shiho [S1], [S2] ; one has then a theorem of comparison relating this object to \(\pi^{\text{un,DR}}_1(X_{k,N})\) [CL] [S1] [S2] ; alternatively, following Deligne [De], one can define directly a Frobenius structure on \(\pi^{\text{un,DR}}_1(X_{k,N})\), and call \(\pi^{\text{un,crys}}_1(X_{k,N})\) the data of \(\pi^{\text{un,DR}}_1(X_{k,N})\) plus the Frobenius. We review this more elementary point of view in this paragraph.

Let us fix a prime \(p\), and \(N \in \mathbb{N}^*\) prime to \(p\). We now go back to the notations of the introduction, and assume that \(K = K_N\), \(r = N\), and \((z_1,\ldots, z_r) = (\xi_N^1,\ldots, \xi_N^N)\) where \(\xi_N\) is a primitive \(N\)-th root of unity in \(K_N\). Following [J I-1] and [J I-3], for each \(\alpha \in \mathbb{N}^*\), we adopt the convention that the Frobenius iterated \(\alpha\) times is \(\tau(p^{\alpha})\phi^\alpha\) where \(\phi\) is in the sense of [De], §13, and, for each \(\alpha \in -\mathbb{N}^*\), the Frobenius iterated \(\alpha\) times is \(F_\alpha\), where \(F_\alpha = \phi^{-1}\) is in the sense of [De], §11.

We have \(O(\pi^{\text{un,DR}}_1(X_K, \omega_{\text{DR}}(X_K)))\), resp. \(O(\pi^{\text{un,DR}}_1(X_K^{(p^{\alpha})}, \omega_{\text{DR}}(X_K^{(p^{\alpha})})))\) is the shuffle Hopf algebra over the alphabet \(\{e_0, e_{z_1}, ..., e_{z_r}\}\) resp. \(\{e_0, e_{z^1_{\alpha}}, ..., e_{z^r_{\alpha}}\}\). We denote by \(w \mapsto w^{(p^{\alpha})}\) the isomorphism between them that sends \(e_0 \mapsto e_0^{(p^{\alpha})}\) and \(e_{z_i} \mapsto e_{z^i_{\alpha}}\), \(i \in \{1,\ldots, N\}\).
For any $\alpha \in \mathbb{N}^* \cup -\mathbb{N}^*$, the Frobenius is determined by its values at the canonical paths $j_1 x$ in the sense of [De], §12; they are determined by the couple $(Li^\dagger_{p,\alpha}, \Phi_{p,\alpha})$ made of the non-commutative generating series of overconvergent $p$-adic hyperlogarithms ($p$HL's) and the non-commutative generating series of $p$MZV-$\mu$’s, defined as follows:

**Definition 2.6.** i) (see §1.1 and [J I-1] for references) Let $U_N$ be the rigid analytic space $\mathbb{P}^{1,\text{an}} - \cup_{i=1}^{N+1} \{ z \mid |z - \xi_N|_p < 1 \} / K_N$ and $\mathfrak{A}^i(U_N)$ the $K_N$-algebra of overconvergent rigid analytic functions on it.

For $\alpha \in \mathbb{N}^*$, let $Li^\dagger_{p,\alpha}$, resp. $Li^\dagger_{p,-\alpha} \in \pi_1^{\text{un,DR}}(X_K, \vec{0})(\mathfrak{A}_i(U_N))$, be the map $z \mapsto (p^\alpha)^{\text{weight}} \delta_{z}^{\alpha}(1_{\mathfrak{A}_i})$, resp. $z \mapsto F_{s}^{\alpha}(z_{\mathfrak{A}_i})$, the arguments of $\mathfrak{A}_i(U_N)$, are called overconvergent $p$-adic hyperlogarithms.

ii) (see §1.1 for references) We denote by $\Phi_{p,\alpha}(z) = \tau(p^\alpha)\phi^\alpha(\frac{\pi}{1 - p^\alpha}) \in \pi_1^{\text{un,DR}}(X_{K_N,N}, \vec{1}, \vec{0})(K_N)$ if $\alpha > 0$, and $\Phi_{p,\alpha}^{-}(z) = F_{s}^{\alpha}(z_{\mathfrak{A}_i}) \in \pi_1^{\text{un,DR}}(X_{K_N,N}, \vec{1}, \vec{0})(K_N)$ if $\alpha < 0$.

For $w = e_{0}^{s_0-1}e_{i_1}^{s_1} \cdots e_{i_t}^{s_t} = (z_{i_0}, \ldots, z_{i_t}, s_{j_0}, \ldots, s_{j_l})$, with $d \in \mathbb{N}^*$, and $s_1, \ldots, s_d \in \mathbb{N}^*$, and $i_1, \ldots, i_t \in \{1, \ldots, N\}$, one calls $p$-adic multiple zeta values at roots of unity the numbers $\zeta_{p,\alpha}(w) = \Phi_{p,\alpha}(w)$ if $\alpha > 0$, and $\zeta_{p,\alpha}^{-}(w) = \Phi_{p,\alpha}^{-}(w)$ if $\alpha < 0$.

iii) For all objects $\ast$ above, and $\alpha = \frac{\log(q_\alpha)}{\log(p)}$, let $*_\alpha \ast = *_{p,\alpha}$.

iii) (Furusho [FI] for $N = 1$, Yamashita [Y] for any $N$). We fix a determination $\log_p$ of the $p$-adic logarithm. Let $Li_{KZ}^\dagger_{p,N}$ resp. $Li_{KZ}^{-}$ be the unique Coleman function on $X_K$, resp. $X_K^{-}$, which is a horizontal section of $\nabla_{KZ}$ and has the asymptotic behaviour $Li_{KZ}^\dagger_{p,N}(z) \sim z \to 0 e^{\alpha_0 \log_p(z)}$, resp. $Li_{KZ}^{-}(z) \sim z \to 0 e^{\alpha_0 \log_p(z)}$.

Below, we assume $\alpha > 0$. The properties for $\alpha < 0$ are similar. For all $j = 1, \ldots, N$, we denote by $z_j = \xi_N$, by $\Phi_{p,\alpha}^{(\xi_N)}(x)$ the image of $\Phi_{p,\alpha}$ by the automorphism $(x \mapsto z_j x)$, and $\omega_{z_j}(x) = \frac{dx}{x-z_j}$. We denote by $z_0 = 0$. The $Li_{KZ}$'s are related to multiple harmonic sums of equation (6) as follows:

**Fact 2.7.** For $z \in \mathbb{C}_p$ such that $|z|_p < 1$, and for each word $w = e_{j_0}^{s_0-1}e_{j_1}^{s_1} \cdots e_{j_t}^{s_t} = (j_0, \ldots, j_t)$, with $j_0, \ldots, j_t \in \{1, \ldots, N\}$, $n_d, \ldots, n_1 \in \mathbb{N}^*$, we have

$$Li_{KZ}^\dagger_{p,N}(w)(z) = (-1)^d \sum_{0 \leq m_1 \leq \ldots \leq m_d} \left( \frac{\xi_N^{m_1}}{\xi_N^{m_1}} \right)^{\sum_{l=1}^d \left( \frac{\xi_N^{m_l}}{\xi_N^{m_l-1}} \right) m_{d-l}} \frac{z \xi_N^{m_d}}{m_1 \cdots m_d}$$

$$Li_{KZ}^{-}(w)(z) = (-1)^d \sum_{0 \leq m_1 \leq \ldots \leq m_d} \left( \frac{\xi_N^{m_1}}{\xi_N^{m_1}} \right)^{\sum_{l=1}^d \left( \frac{\xi_N^{m_l}}{\xi_N^{m_l-1}} \right) m_{d-l}} \frac{z \xi_N^{m_d}}{m_1 \cdots m_d}$$

**Proposition 2.8.** The couple $(Li^\dagger_{p,\alpha}, \Phi_{p,\alpha})$ is determined by:

$$dLi^\dagger_{p,\alpha} = \left( \sum_{j=0}^{N} \omega_{z_j}(z)e_{z_j} \right) Li^\dagger_{p,\alpha} - Li^\dagger_{p,\alpha} \left( \sum_{j=0}^{N} \omega_{z_j}(z^{p^\alpha}) \Phi_{p,\alpha}^{-1} e_{z_j} \Phi_{p,\alpha} \right)$$

i.e.

$$Li^\dagger_{p,\alpha}(z)(e_0, e_{z_1}, \ldots, e_{z_N}) \times Li_{KZ}^{-}(z^{p^\alpha})(e_0, \Phi_{p,\alpha}^{-1} e_{z_1} \Phi_{p,\alpha}, \ldots, \Phi_{p,\alpha}^{-1} e_{z_N} \Phi_{p,\alpha}) = Li_{KZ}^{-}(z)(p^{p^\alpha}e_0, p^{p^\alpha}e_{z_1}, \ldots, p^{p^\alpha}e_{z_N})$$
and
\begin{equation}
\epsilon_0 + \sum_{i=1}^{N} (\Phi^{(i)}_{p,\alpha})^{-1}e_z, \Phi^{(i)}_{p,\alpha} + \text{Li}_{p,\alpha}(\infty)^{-1}(\epsilon_0 + \sum_{i=1}^{N} e_z) \text{Li}_{p,\alpha}(\infty) = 0
\end{equation}

3. Setting for computations

3.1. Bases of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$ and presentations of $\pi^{\text{un,DR}}_1(X_{K_{p^a}, \mu_p N})$; two particular examples.

3.1.1. Bases of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu N, \infty\})$, bases of $\text{Map}(\mathbb{Z}/p^n \mathbb{Z}, K_{p^a})$ and bases of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$.

**Definition 3.1.** Let $\omega_0(z) = \frac{dz}{z}$, $\omega_{\xi_N}(z) = \frac{dz}{z - \xi_N}$, where $\xi_N = \frac{1}{\xi_N} \sum_{m \geq 0} (\xi_N - m)^{-1} m z^m$, $j = 1, \ldots, N$ be the canonical basis of $\Omega^1(\mathbb{P}^1 - \{0, \mu N, \infty\})$. Let $b = (f_1, \ldots, f_r)$ be a set of maps $\text{Map}(\mathbb{Z}/p^a \mathbb{Z}, K_{p^a})$. Then we consider the sequence of differential forms
\begin{equation}
\left( \frac{1}{\xi_N} \sum_{m \geq 0} f_i(m \mod p^a)(\xi_N - m)^{-1} m z^m \right)_{i=1,\ldots,r}
\end{equation}

**Remark 3.2.** We could also generalize this definition and our computations by replacing the canonical basis of $\Omega^1(\mathbb{P}^1 - \{0, \mu N, \infty\})$ by any basis. This would give ultimately variants of $\text{pMZV}_{\mu N}$'s, for example expressed by multiple harmonic sums involving congruences modulo $N$. However, this object is less natural for our current purposes. We could also choose $N$ different $b$'s.

We are interested in the cases where $r = p^a$ and $b$ is a basis of $\text{Map}(\mathbb{Z}/p^a \mathbb{Z}, K_{p^a})$, and (17) join with $\omega_0$ define a basis of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$. We will be interested in two examples:

**Example 3.3.** i) With $r = p^a$ and $f_i : m \mapsto \rho_p^m$, $i = 1, \ldots, r$, where $\rho_p$ is a primitive $p^a$-th root of unity, we obtain the canonical basis $B_{\text{can}}$ of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$:
\begin{equation}
\left\{ \frac{dz}{z}, \frac{dz}{z - \rho_p^i \xi_N}, i = 1, \ldots, p^a, j = 1, \ldots, N \right\}
\end{equation}

ii) With $r = p^a$ and $f_i : m \mapsto 1_{m \equiv i \mod p^a}$, we obtain another basis $B_{\text{cong}}$ of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$, the 'canonical basis with congruences modulo $p^a$':
\begin{equation}
\left\{ \frac{dz}{z}, \frac{dz'}{z' - \rho_p^i \xi_N}, r = 1, \ldots, p^a, j = 1, \ldots, N \right\}
\end{equation}

Of course, we are interested in the second example partly because we have the lift of Frobenius $z \mapsto z^{p^a}$ on $\mathbb{P}^1 - \{0, \mu N, \infty\}$.

3.1.2. Presentation of $(\pi^{\text{un,DR}}_1(X_{K_{p^a}, \mu_p N}), \nabla^{\text{un,DR}}_{KZ}_{p^aN})$ from the canonical basis with congruences of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$. In this paragraph, we formalize the presentation of $\pi^{\text{un,DR}}_1(X_{K_{p^a}, \mu_p N}, \nabla^{\text{un,DR}}_{KZ}_{p^aN})$ defined by the canonical basis with congruences modulo $p^a$ of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_p N, \infty\})$ (Example 3.3, ii)).

**Definition 3.4.** Let $e_{0,\mu_p N \mod p^a}$ be the alphabet formed by $e_0$ and the formal variables $e_{r \mod p^a}$, $j = 1, \ldots, N$, $r = 0, \ldots, p^a - 1$.

**Definition 3.5.**

i) Let $\mathcal{O}^{\text{un,DR}_{0,\mu_p N \mod p^a}}$ be the the shuffle Hopf algebra over the alphabet $\{e_0\} \cup \{e_{r \mod p^a}, r = 0, \ldots, p^a - 1, j = 1, \ldots, N\}$.

ii) Let $\text{Spec}(\mathcal{O}^{\text{un,DR}_{0,\mu_p N \mod p^a}})$ is a pro-unipotent algebraic group over $\mathbb{Z}$.

iii) Let also $K_N(\langle e_0, e_{r \mod p^a} \rangle)$ be the non-commutative algebra of formal power series over the variables $e_0$ and $e_{r \mod p^a}$, $r = 0, \ldots, p^a - 1, j = 1, \ldots, N$, with coefficients
in \( K_N \). It is naturally equipped with the structure of topological Hopf algebra dual to \( \mathcal{O}_K \cdot e_0 \to K_N \times \mathbb{Z}/p^\infty \mathbb{Z} \).

iv) Let \( \nabla_{\mu}^{\mod \ p^\alpha} : K_{p^\alpha N}[[z]](\langle e_{0,ij}^{\mod \ p^\alpha} \rangle) \to K_{p^\alpha N}[[z]]\frac{dz}{z}(\langle e_{0,ij}^{\mod \ p^\alpha} \rangle) \), defined by

\[
L \mapsto dL - \left( \frac{dz}{z} e_0 + \sum_{r=0}^{p^\alpha-1} \sum_{i=1}^N e_i^{\mod \ p^\alpha} \frac{z^r dz}{z^{p^\alpha} - z^{p^\alpha_i}} \right) L
\]

**Definition 3.6.** We have defined the canonical presentation with congruences modulo \( p^\alpha \) of \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \).

The presentation reviewed in \( \S 2.1 \) is the canonical presentation of \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \).

**Remark 3.7.** The previous objects are defined over \( Y_{K_{p^\alpha N}, N}^{\mod \ p^\alpha} = \text{Spec}(K_N[z, \frac{1}{z^{p^\alpha_i} - z^{p^\alpha_{i+1}}}, \ldots, \frac{1}{z^{p^\alpha_i} - z^{p^\alpha_{i+p^\alpha-1}}}] \), but the singularities of \( \frac{1}{z^{p^\alpha_i} - z^{p^\alpha_{i+p^\alpha-1}}} \) are logarithmic only over \( K_{p^\alpha N} \) and we have \( Y_{K_{p^\alpha N}, N}^{\mod \ p^\alpha} \times_{\text{Spec}(K_N)} \text{Spec}(K_{p^\alpha N}) = X_{K_{p^\alpha N}, p^\alpha N} \).

Of course, there is an adaptation of these constructions for other choices of bases of \( \Omega^1(X_{K_{p^\alpha N}, p^\alpha N}) \).

3.1.3. **Base-change relating the two presentations of** \( (\pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}), \nabla_{\mu}^{\mod \ p^\alpha}) \).

**Proposition 3.8.** i) The correspondence \( e_i^{\mod \ p^\alpha} = \frac{1}{p^\alpha} \sum_{r=0}^{p^\alpha-1} (\frac{1}{p^\alpha})^{p^\alpha-r} e_i^{\mod \ p^\alpha} \) defines a \( K_{p^\alpha N} \)-linear automorphism \( K_{p^\alpha N}e_0 \oplus \bigoplus_{i=0}^N K_{p^\alpha N} e_i^{\mod \ p^\alpha} \to K_{p^\alpha N}e_0 \oplus \bigoplus_{i=0}^N K_{p^\alpha N} e_i^{\mod \ p^\alpha} \), whose inverse is \( e_i^{\mod \ p^\alpha} \to \sum_{r=0}^{p^\alpha-1} (\frac{1}{p^\alpha})^{p^\alpha-r} e_i^{\mod \ p^\alpha} \).

ii) The dual of this isomorphism is characterized by the commutativity of the diagram

\[
\begin{array}{ccc}
K_{p^\alpha N} [[z]] (\langle e_{0,ij}^{\mod \ p^\alpha} \rangle) & \xrightarrow{\nabla_{\mu}^{\mod \ p^\alpha}} & K_{p^\alpha N} [[z]] (\langle e_{0,ij}^{\mod \ p^\alpha} \rangle) \\
\downarrow & & \downarrow \\
K_{p^\alpha N} [[z]] (\langle e_{0,ij}^{\mod \ p^\alpha} \rangle) & \xrightarrow{\nabla_{\mu}^{\mod \ p^\alpha}} & K_{p^\alpha N} [[z]] (\langle e_{0,ij}^{\mod \ p^\alpha} \rangle)
\end{array}
\]

where the horizontal arrows arise from the canonical isomorphism \( \Pi_N^{\mod \ p^\alpha} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(K_{p^\alpha N}) \simeq \Pi_{p^\alpha N} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(K_{p^\alpha N}) \) defined by this linear isomorphism.

**Proof.** Let \( Z \) be a formal variable, \( \rho \in \mu(p^\alpha(K_{p^\alpha N})) \) and \( \xi \in \mu_N(K_{p^\alpha N}) \). By writing the power series expansion of \( (1 - \rho^{-1} \xi^{-1} z)^{-1} \), we get

\[
\frac{1}{Z^{1-\xi}} = \sum_{n=0}^{\infty} \left( \frac{1}{\rho} \right)^n \frac{\xi^n}{z^{n+1}}.
\]

Conversely, we have

\[
Z^{-1-\xi} = \sum_{n=0}^{\infty} \left( \frac{1}{\rho} \right)^n \frac{\xi^n}{z^{n+1}}.
\]

Indeed, \( Z^{-1-\xi} \) is not \( p^\infty \)-convergent and, for all \( t \in \mathbb{N}^* \) and \( \eta \in K - \{0\} \),

\[
\frac{1}{Z^{(\xi-\eta)}} = \frac{1}{\rho} \frac{\xi^{-t}}{z^{t+1}} + \sum_{t=1}^{\infty} \frac{1}{\rho^t} \frac{\xi^{t-1}}{z^{t+1}}.
\]

whence

\[
Z^{-1-\xi} = \frac{1}{p^\alpha} \sum_{r=0}^{p^\alpha-1} \sum_{i=1}^N e_i^{\mod \ p^\alpha} \frac{z^r dz}{z^{p^\alpha} - z^{p^\alpha_i}}
\]

and \( \sum_{r=0}^{p^\alpha-1} \sum_{i=1}^N e_i^{\mod \ p^\alpha} = 0 \) for each \( t \in \{1, \ldots, p^\alpha-1\} \).

\[ \square \]

3.1.4. **Comment on viewing** \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) **with its presentation with congruences modulo** \( p^\alpha \) **as an extension of** \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \).

**Remark 3.9.** Whereas \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) with its canonical presentation admits \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, N}) \) as a natural quotient and subobject, \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) with its presentation with congruences modulo \( p^\alpha \) \((\S 3.1.2)\) admits the pull-back by Frobenius \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) as a natural quotient and sub-object.

Thus we may view \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) with its presentation with congruences modulo \( p^\alpha \) as a generalization of \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) to \( X_{K_{p^\alpha N}, p^\alpha N} \). However, the Frobenius that we are going to construct will not be an isomorphism relating two different presentations \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) but rather a relation between \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \) and \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, N}) \) having coefficients in \( K_{p^\alpha N} \). It would be possible to extend our construction as a relation between two different presentations of \( \pi_1^{\un, \DR}(X_{K_{p^\alpha N}, p^\alpha N}) \), but this would be more artificial and less interesting arithmetically.
The view of $\pi^{\text{un,DR}}_1(X_{K^p,N}, p^N)$ with its presentation with congruences modulo $p^n$ as an extension of $\pi^{\text{un,DR}}_1(X_{K,N})$ leads however to the following observations.

**Remark 3.10.** The Proposition 3.8 and the natural passage from the i) of Example 3.3 to the ii) of Example 3.3 are better expressed by the map:

$$Eul_{p^N} : \begin{array}{c} K_{p^N}[z] \cong K_{p^N}[y][p^\infty] \\ S(z) \mapsto (S(y)^{10 \mod p^n}, \ldots, S(y)^{p^n-1 \mod p^n}) \end{array}$$

characterized by the equation $S(z) = \sum_{r=0}^{p^n-1} z^r S(r \mod p^n)(z^{p^n})$.

$Eul_{p^N}$ extends in a natural way to an isomorphism $K_{p^N}[z] \cong K_{p^N}[y][p^\infty]$ dy and further as an isomorphism $K_{p^N}[z] \cong K_{p^N}[y][p^\infty] \otimes K_{p^N} dy$, characterized by $z^{p^n} \mapsto y$ and the commutation with the differentials $d$, given by $\frac{dz}{z} \mapsto \frac{1}{p^n} \frac{dy}{y}$ and the following equation: for $L \in K_{p^N}[z]$, for $z^{p^n}$.

$$dS(z) = \sum_{r=0}^{p^n-1} z^r d \left( S(r \mod p^n)(z^{p^n}) \right) + \sum_{r=1}^{p^n-1} rz^r S(r \mod p^n)(z^{p^n}) d(z^{p^n})$$

**Remark 3.11.** Using $Eul_{p^N}$, $\nabla_{KZ}^{p^\infty}$ as a connection on a trivial bundle over $\mathbb{P}^1 - \{0, \mu, \infty\}$ defined over $K_N$, but which is not pro-unipotent. We have:

$$\frac{dz}{z} \epsilon_0 + \sum_{r=0}^{p^n-1} \sum_{j=1}^{N} e^{r} \xi_N \mod p^n \frac{z^r dz}{z^{p^n} - \xi^{p^n}_N} = \frac{d(z^{p^n})}{p^n z^{p^n} \epsilon_0} + \sum_{r=0}^{p^n-1} \sum_{j=1}^{N} e^{r} \xi_N \mod p^n \frac{z^{r+1} d(z^{p^n})}{p^n z^{p^n} z^{p^n} - \xi_N^{p^n}}$$

Thus, by substituting equations (19) and (20) inside the expression $\nabla_{KZ}^{p^\infty}$ is defined to be $L = 0$ where $\nabla_{KZ}^{p^\infty}$ is isomorphic to $\nabla_{KZ}^{(L_0, \ldots, L_0)}$ defined by, for $r = 0, \ldots, p^n-1$,

$$(\nabla_{KZ}^{L_r}) \epsilon_0 = \frac{L_r}{p^n} \frac{dy}{y} \epsilon_0 - r + \sum_{0 \leq r_1 + r_2 = p^n-1 \mod p^n}^{N-1} \sum_{j=0}^{N-1} e^{r_1} \xi_N \mod p^n \frac{dy}{y} (\epsilon_0 - r + r_2)$$

That bundle keeps a weight filtration, but the flat sections of its weight-graded quotients are built out of the functions $y \mapsto y^{r_2}$, $r = 0, \ldots, p^n - 1$.

**3.2. Multiple harmonic sums at roots of unity attached to elements of $\text{Map}(\mathbb{Z}/p^s \mathbb{Z}, K_{p^N})$.**

We extend the framework established in part I to deal with multiple harmonic sums as elementary $p$-adic functions.

**3.2.1. Definition.** We suppose chosen a basis $B = (f_1, \ldots, f_{p^n})$ of $\text{Map}(\mathbb{Z}/p^s \mathbb{Z}, K_{p^N})$ such that the operation defined in §3.1 defines a basis of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu, \infty\})$. We suppose that $f_1, \ldots, f_{p^n}$ take values in $O_{K_{p^N}}$.

**Definition 3.12.** For $m \in \mathbb{N}^*$, $d \in \mathbb{N}^*$, $j_1, \ldots, j_d \in \{1, \ldots, p\}$, $n_1, \ldots, n_d \in \mathbb{N}^*$, we call weighted multiple harmonic sums at $N$-th roots of unity with the numbers

$$\text{har}_m \left( \begin{array}{c} f_{i_1}, \ldots, f_{i_l} \\ \xi_{i_1}, \ldots, \xi_{i_l} \end{array} \right) = m^{n_d + \ldots + n_1} \sum_{0 < m_1 < \ldots < m_d < m} f_{i_1}(m_1) \frac{(\xi_{i_1}^{m_1})^{m_1}}{\xi_{i_1}^{m_1}} \ldots f_{i_d}(m_d) \frac{(\xi_{i_d}^{m_d})^{m_d}}{\xi_{i_d}^{m_d}} \frac{1}{\xi_{i_d}^{m_d}} \in K_{p^N}$$
Remark 3.13. The slightly more general variant
\[ \text{har}_m \left( \frac{f_{i_{d+1}, \ldots, f_{i_1}}}{\xi_N^{i_{d+1}, \ldots, \xi_N^{i_1}}} \right) = m^{p^{d+\ldots+n_1}} \sum_{0<m_1<\ldots<m_d<m} f_{i_1}(m_1) \frac{\epsilon_{\xi_N^{i_1}}} {\xi_N^{m_1}} \cdots f_{i_d}(m_d) \frac{\epsilon_{\xi_N^{i_{d+1}}}^{m_d}} {\xi_N^{m_1} \cdots m_d} f_{i_{d+1}}(m) \frac{\epsilon_{\xi_N^{i_{d+1}}}^{m}} {\xi_N^{m_1} \cdots m_d} \]
incorporates implicitly the isomorphism \( \text{Eucl}_{ρ^N} \) appearing in §3.1.4 but we will not use it.

Remark 3.14. Let \( G_{p, α, N} \) be the subgroup of \( \text{Hom}_N(K^*_pN, K^*_pN) \) made of elements \( α \) such that, for all \( ϵ \in K \) such that \( |ϵ|_p \leq \frac{1}{1} \), we have an absolutely convergent expansion \( ϵ(1 + ϵ) = \sum_{l \geq 0} ϵ^{(li)}(1) ε^l \)
with \( ϵ^{(li)}(1) \in K^N \). In particular, the elements of \( G_{p, α, N} \) are locally analytic maps. Moreover, \( G_{p, α, N} \) contains the elements \( m_i \mapsto \frac{1}{m_i}, s_i \in \mathbb{N}^* \).

Then, as in part I, most of the computations of the next paragraphs remain true if we replace the maps \( m_i \mapsto \frac{1}{m_i}, n_i \in \mathbb{N}^* \) by \( χ_i(m_i) \), where \( χ_1, \ldots, χ_d \in G_{p, α, N} \).

This generalization seems to attain the maximal version of our statements concerning multiple harmonic sums in the \( Σ \) framework.

The notations concerning generating sequences of multiple harmonic sums in this sense are delayed to the next paragraphs, because they depend on the choice of framework: \( f^1_1, f^*_2<1 \) or \( Σ \).

3.2.2. Bounds of valuation of prime weighted multiple harmonic sums. These extend a definition and a lemma of part I.

Definition 3.15. We call prime weighted multiple harmonic sums the weighted multiple harmonic sums as in Definition 3.12 with \( m \) equal to a power of \( p \).

Lemma 3.16. For any word \( w \), for any \( α \in \mathbb{N}^* \), we have \( v_p(\text{har}_p^\alpha(w)) \geq \text{weight}(w) \), and, with \( w = \left( \frac{f_{i_{d+1}, \ldots, f_{i_1}}}{\xi_N^{i_{d+1}, \ldots, \xi_N^{i_1}}} \right), w' = \left( \frac{f'_{i_{d+1}, \ldots, f'_{i_1}}}{\xi_N^{i_{d+1}, \ldots, \xi_N^{i_1}}} \right) \) where \( f'_i : m \mapsto f_i(p^{a_i}m) \),
\[ \text{p-weight}(w) \text{har}_p^\alpha(w) \equiv \text{har}_p^{\alpha}(w') \mod p \]

Proof. Same proof with the case of \( \mathbb{P}^1 - \{0, μN, ∞\} : \text{har}_p^\alpha(w) \) viewed as an iterated sum indexed by \( 0 < n_1 < \ldots < n_d < p^{p^c} \) can be split into its subsum indexed by \( p^{a_1}n_1, \ldots, p^{a_d}n_d \), which is equal to \( \text{har}_p(w) \) and the subsum indexed by the complement of \( \{p^{a_1}n_1, \ldots, p^{a_d}n_d\} \), whose valuation is \( \geq \text{weight}(w) + 1 \), since \( |z|_K = \ldots = |z^r+1|_K = 1 \).

3.3. Algebraic and topological structures on \( π_3^{\text{un}, \text{DR}}(X_{K^*_pN, p^N}) \) with a different presentation.

3.3.1. Notations. We suppose chosen a basis \( B = (f_1, \ldots, f_{p^c}) \) of the space of maps \( \mathbb{Z}/p^cZ \rightarrow K^*_pN \) which defines by Definition 3.1 a basis \( B \) of \( H^1_{\text{DR}}[\mathbb{P}^1 - \{0, μp^cN, ∞\}] \).

Definition 3.17. i) Let \( e_{0, ϵ_0, ϵ_j} \) be the alphabet \( \{ε_0, (ε_\xi^{j,i}, f_i), j = 1, \ldots, N, i = 1, \ldots, p^c\} \).
ii) The weight of a word over \( e_{0, ϵ_0, ϵ_j}^B \) is its number of letters. The depth of such a word is its number of letters different from \( \epsilon_0 \).
Let \( \mathcal{W}(e_{0, ϵ_0, ϵ_j}^B) \) be the set of words over \( e_{0, ϵ_0, ϵ_j}^B \).
iii) Let \( Ω^B_0, Ω^B_{0, ϵ_0, ϵ_j}^B \) be the shuffle Hopf algebra over the alphabet \( e_{0, ϵ_0, ϵ_j}^B \).
iv) Let \( K^*_pN(\{e_{0, ϵ_0, ϵ_j}^B\}) \) be the non-commutative \( K^*_pN \)-algebra of power series over the variables \( \{ε_0, (ε_\xi^{j,i}, f_i), j = 1, \ldots, N, i = 1, \ldots, p^c\} \).

The changes of basis above preserve the weight and the depth. We maintain in this case the notation concerning the coefficients of formal power series, namely \( f = \sum w \text{word } f[w] w \) for \( f \in K^*_pN(\{e_{0, ϵ_0, ϵ_j}^B\}) \).
3.3.2. *Extension of the adjoint Ihara product.* What follows is an extension of the notion of "adjoint Ihara product" defined in [J I-2]. We assume that the alphabet $\epsilon_{0,\mu_N}$ is identified to a set of $K_{p^\alpha,N}$-linear combinations of the letters of the alphabet $e_{0,\mu_P,N}$.

**Definition 3.18.** For $g_{\xi_1}, \ldots, g_{\xi_N} \in K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$ and $f \in K_N(\langle e_{0,\mu_P,N} \rangle)$, let

$$ (g_{\xi_1}, \ldots, g_{\xi_N}) \circ^B_{\text{Lie}} f = f(e_0, g_{\xi_1}, \ldots, g_{\xi_N}) $$

3.3.3. **Topologies on $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$**. We view an element of $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$ as a function $\mathcal{W}(e_{0,\mu_P,N}^B) \to K_{p^\alpha,N}$.

**Definition 3.19.**

i) Let $\mathcal{N}_{\Lambda, D} : K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle) \to \mathbb{R}_+[[\Lambda, D]]$ be the map

$$ f \mapsto \sum_{(n,d) \in \mathbb{N}^2} \max_{w \text{ of weight } n \text{ and depth } d} |f[w]|_p \Lambda^n D^d $$

ii) The $\mathcal{N}_{\Lambda, D}$-topology on $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$ is the topology of simple convergence of functions $\mathcal{W}(e_{0,\mu_P,N}^B) \to K_{p^\alpha,N}$. It makes $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$ into a topological $K_{p^\alpha,N}$-algebra.

**Definition 3.20.** i) Let $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)_b \subset K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$ be the subset of elements $f$ such that, for all $d$, the supremum $\sup |f[w]|_p$ over words $w$ of depth $d$ is finite.

ii) Let $\mathcal{N}_D : K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)_b \to \mathbb{R}_+[[D]]$ be the map

$$ f \mapsto \sum_{d \in \mathbb{N}} \max_{w \text{ of depth } d} |f[w]|_p D^d $$

iii) The $\mathcal{N}_D$-topology on $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)_b$ is the topology of convergence of functions $\mathcal{W}(e_{0,\mu_P,N}^B) \to K_{p^\alpha,N}$ which is uniform on $\{w \text{ word of depth } d\}$ for each $d$. It makes $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)_b$ into a topological $K_{p^\alpha,N}$-algebra.

**Definition 3.21.** Let $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)_S \subset K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)$ ($S$ stands for summable) be the subspace of elements $f$ such that we have, for all $d$,

$$ \max_{w \text{ of weight } n \text{ and depth } d} |f[w]|_p \to n \to \infty 0 $$

One can equip it with the $\mathcal{N}_D$-topology.

**Lemma 3.22.** If we have two basis $B$ and $B'$ such that the decomposition of $B$ in $B'$ and the decomposition of $B$ in $B'$ have coefficients of norm $\leq 1$, then one has an isomorphism of topological algebras $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle) \simeq K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^{B'} \rangle)$, and $K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^B \rangle)_b \simeq K_{p^\alpha,N}(\langle e_{0,\mu_P,N}^{B'} \rangle)_b$.

4. The differential equation of the Frobenius extended to $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{p^\alpha,N}, \infty\})$

We suppose chosen $B = (f_1, \ldots, f_{p^\alpha})$ a basis of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^\alpha,N}, \infty\})$ of the type of §3.1 and we work in the framework of §3. We show that the differential equation characterizing the Frobenius $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{p^\alpha,N}, \infty\})$ can be canonically extended as a structure on $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{p^\alpha,N}, \infty\})$.

4.1. Regularization of algebraic iterated integrals over $X_{K_{p^\alpha,N}}$. Let again $\rho_{p^\alpha}$ be a primitive $p^\alpha$-th root of unity. We are going to define regularized the iterated integrals of $\omega_0 = \frac{dz}{z}$, $\omega_{p^\alpha, \xi_N} = \frac{dz}{z - \rho_{p^\alpha, \xi_N} \rho_{p^\alpha}^j}$, $i = 0, \ldots, p^\alpha - 1$, $j = 1, \ldots, N$ as the image of the regularization of part I and the formulas for it by the isomorphisms of §3.2.

We are going to write the definition and formulas in a more direct way, by adapting a part of [J I-1], §4.

Below, $\mathbb{Z}_p^{(N)} = \lim_{N \to \infty} \mathbb{Z}/Np^\alpha \mathbb{Z} = \Pi_{0 \leq m \leq N-1}(m + N\mathbb{Z}_p)$. The next definition is the variant of Definition 4.2.1 and Definition 4.2.2 in [J I-1] with coefficients in $K_{p^\alpha,N}$:
4.2. The differential equation of the Frobenius of \( \pi_{1,\mu}^{\text{DR}}(\mathbb{P}^1 - \{0, \mu, \infty\}) \) extended to \( \pi_{1,\mu}^{\text{DR}}(\mathbb{P}^1 - \{0, \mu, \infty, \infty\}) \). We imitate the differential equation of the Frobenius with the regularization defined above.

**Problem 4.3.** Do there exist, \( (L, \Theta_{\xi_0^1}, \ldots, \Theta_{\xi_N^1}) \), elements of \( \mathcal{A}^1(U_N) \otimes_{K_N} K_{p^\infty,N} \langle (e_0, \{e_{\xi_0^j, f_i}, i = 1, \ldots, p^\infty\}) \rangle \times K_{p^\infty,N} \langle (e_0, \{e_{\xi_N^j, f_i}, i = 1, \ldots, p^\infty\}) \rangle \) which satisfies

i) the following extension of the differential equation of the Frobenius

\[
(22) \quad dL = \left( p^\alpha \omega_0(z) e_0 + \sum_{j=1}^{N} \omega_{\xi_0^j, f_i}(z) e^B_{\xi_0^j, f_i} \right) L = L \left( p^\alpha \omega_0(z) e_0 + \sum_{j=1}^{N} \omega_{\xi_N^j}(z^p) \Theta_{\xi_N^j} \right) 
\]

ii) \( L(0) = 1 \)

iii) For any word \( w \) over \( e_{\xi_0, \mu, \infty} \), \( L[w] \) is in the space \( \text{LAE}_{S_{\xi_0, \xi_N}}^{\text{reg}}(\mathbb{Z}^N - \{0\}, K_{p^\infty,N}) \) defined in §3.2.

**Remark 4.4.** For any solution above the image of \( L \) by the canonical projection \( \mathcal{A}^1(U_N) \langle (e_{\xi_0, \mu, \infty}) \rangle \to \mathcal{A}^1(U_N) \langle (e_{\xi_0, \mu, \infty}) \rangle \) is \( \text{Li}^1_{p, \mu}(\mu_N) \) reviewed in §2.2: this is because, by [De], §11, the Frobenius of \( \pi_{1,\mu}^{\text{DR}}(X_{K_N,N}) \) is uniquely characterized as the unique isomorphism \( \pi_{1,\mu}^{\text{DR}}(X_{K_N,N}) \to \pi_{1,\mu}^{\text{DR}}(X_{K_Z,N}) \) which is horizontal with respect to \( \langle \nabla_{\mu,Z}, \nabla_{\mu,Z} \rangle \).

**Proposition 4.5.** The Problem 4.3 has a unique solution.

**Proof.** Let \( f \) be the canonical integration operator from 0 to a formal variable \( z \), which send sequences of differential forms to an element in \( K_{p^\infty,N}[[z]] \). By induction on the weight, we have to show that, for any weight \( n \), the problem has a unique solution up to weight \( n - 1 \) and, for any word \( w \) of weight \( n \), there is a unique way to define \( \Theta_{\xi_0^1}, \ldots, \Theta_{\xi_N^1} \) in weight \( n \) such that the following iterated integral, necessarily
equal to $L[w]$, is regular:

\begin{equation}
(23) \quad \int \omega_0 L[\partial_e (w)] + \sum_{j=1}^{N} \omega_{\xi_j} f_j(z) \int \nu L[\partial_{\xi_j} f_j(w)] - \int p^\omega \omega_0 L[\partial_e (w)] - \sum_{j=1}^{N} \omega_{\xi_j} \nu \left( z^{p^\omega} \right) \Theta_{\xi_j} [w] - \sum_{w' \in \nu(w') \geq 1} L[w'] \Theta_{\xi_j} [w']
\end{equation}

The singularity at 0 vanishes by the condition $L(0) = 1$. The regularity of (23) amounts to say that the right-hand side of (23) is equal to its image by the regularization of §4.1.

In (23), only the term $- \sum_{j=1}^{N} \omega_{\xi_j} \nu \left( z^{p^\omega} \right) C_{\xi_j}$ characterized in terms of the polar coefficients of the iterated integral, we see that the regularity of (23) amounts to a unique choice of $(\psi_{\xi_j} [w], \ldots, \psi_{\xi_j} [w])$, and thus a unique choice of $L[w]$. □

4.3. Definitions.

**Definition 4.6.** Let $(\text{Li}_{\mu_\alpha N}^1, \psi_{\mu_\alpha N}, \ldots, \psi_{\mu_{p_\alpha N}})$ be the unique solution to Problem 4.3.

**Definition 4.7.** Let adjoint $p$-adic multiple zeta values at $p^\alpha N$-th roots of unity be the coefficients of $\psi_1$, namely:

$\psi_{\mu_{\alpha N}}^\text{Ad}(w) = \psi_1[w]$

**Definition 4.8.** The Frobenius of $\pi^\text{un,DR}_1(\mathbb{P}^1 \setminus (\{0, \mu_\alpha N, \infty\})$ extended to $\pi^\text{un,DR}_1(\mathbb{P}^1 \setminus (\{0, \mu_\alpha N, \infty\}$ is the map $K_{p^\alpha N}[[z]]/(\langle e_{0, \mu_\alpha N} \rangle) \to K_{p^\alpha N}[[z]]/(\langle e_{0, \mu_\alpha N} \rangle)$ which sends $f \mapsto \text{Li}_{\mu_\alpha N}^1(\mu_\alpha N)^{f_0}(e_0, e_{\xi_N}, \ldots, e_{\xi_N})$.

**Definition 4.9.** The Frobenius of $\pi^\text{un,DR}_1(\mathbb{P}^1 \setminus (\{0, \mu_\alpha N, \infty\}, -1_0)$ is the map $f \mapsto f(e_0, e_{\xi_N}, \ldots, e_{\xi_N})$, $K_{p^\alpha N}((e_{0, \mu_\alpha N})) \to K_{p^\alpha N}((e_{0, \mu_\alpha N}))$.

We reformulate the definition in a more meaningful way. Let us choose an infinite sequence $\rho_D = (f_i)_i \in \mathbb{N}$ of elements of $B = (f_1, \ldots, f_{p^\alpha})$.

We associate to such a sequence a linear injection

$\Omega^\text{un,DR}_{e_0, e_{\xi_N}} \to \Omega^\text{un,DR}_{e_0, e_{\xi_N}}$

We have $\cup \rho_D \Omega_D^\text{un,DR}_{e_0, e_{\xi_N}} = \Omega^\text{un,DR}_{e_0, e_{\xi_N}}$. However, $\rho_D \Omega_D^\text{un,DR}_{e_0, e_{\xi_N}}$ is not stabilized by the shuffle product. The definition of the Frobenius of $X_{K_{p^\alpha N}, p^\alpha N}$ can be interpreted as follows: in the differential equation of the Frobenius of $X_{K, N}$, let us replace the factor $\text{Li}_{p}^k$, viewed as a function on $\Omega^\text{un,DR}_{e_0, e_{\xi_N}}$, by $\text{Li} \circ i_D$ : there is a unique way to transform $(\text{Li}_{p}^k, \text{Ad}_{\rho_{\alpha N}}(e_1))$ into variants in order to obtain an equation having regular solutions. This regular equation is the restriction to $i_D \rho_{\alpha N}$ of $\Omega^\text{un,DR}_{e_0, e_{\xi_N}}$ differential equation (22). In a summary :

i) for each sequence $\rho_D$, we have a "transformed copy" of the Frobenius of $\pi^\text{un,DR}_1(X_{K_{p^\alpha N}, p^\alpha N})$ inside $\pi^\text{un,DR}_1(X_{K_{p^\alpha N}, p^\alpha N})$, which acts on iterated integrals of $X_{K, N}$ transformed by $i_D$.

ii) the Frobenius extend to $\pi^\text{un,DR}_1(X_{K_{p^\alpha N}, p^\alpha N})$ in the previous sense is just an infinite sequence of "transformed copies" of the Frobenius of $\pi^\text{un,DR}_1(X_{K_{p^\alpha N}, p^\alpha N})$, with transformed coefficients.

We can now sketch the definition of (non-adjoint) $p$-adic multiple zeta values at $p^\alpha N$-th roots of unity : - by formalizing the machinery of "transformed copies" described above and applying it to multiple harmonic sums and the explicit formulas for $\zeta_{p, \alpha}$ found in [J-I-1], we get their definition
Alternatively, consider the expression of the coefficients of $\Phi_{p,\alpha}$ in terms of those of $\Phi_{p,\alpha}^{-1}e_1\Phi_{p,\alpha}$ obtained in [J I-1]. There is a natural way to extend it to power series over the alphabet $e_{B_{0,\mu_{p,N}}}$. This leads to a definition $\mathcal{C}_{p,\alpha}$; however, one has to check that it does not depend on choices.

- In principle, the equality between all the possible definitions follows from the uniqueness of the solution to Problem 4.3.

The $p$-adic multiple zeta values at $p^\alpha N$-th roots of unity should be closely related to the 'twisted $p$-adic multiple $L$-values' defined in Theorem 3.38 of [FKMT]; note that in [FKMT], $p$-adic multiple $L$-value means $p$-adic multiple zeta values at root of unity of order prime to $p$, following a terminology of Yamashita [Y], and not values of the $p$-adic multiple $L$-functions introduced in [FKMT].

4.4. Combinatorics and bounds of valuation for $\text{Li}_{p,\alpha}^{1,\mu_{p,N}}$. We sketch a generalization to $\mathbb{P}^1 - \{0, \mu_{p,N}, \infty\}$ of the solution of the differential equation of the Frobenius explained in [J I-1]. The differential equation (22) amounts to

$$\text{Li}_{p,\alpha}^{1,\mu_{p,N},B} = \text{Li}_{p}^{KZ_{\mu_{p,N}}}(e_0, e_{\xi_{\alpha,j}}, \zeta_{\alpha}) \times \text{Li}_{p}^{KZ_{\mu_{p,N}}}(e_0, \psi_{\xi_{\alpha,j}}, \ldots, \psi_{\xi_{\alpha,N}})(z^{p^\alpha})^{-1}$$

Let us view $L$ as a linear map $O_{\mathbb{P}^1, e_{\mu_{p,N}}} \rightarrow K_{p,N}$.

- In [J I-1], we defined a 'regularization by the Frobenius' $\text{reg}_{\text{Frob},\alpha}$ and wrote a formula for it:

This can be extended immediately into a map $\text{reg}_{\psi_{\xi_{\alpha,j}}}, \ldots, \psi_{\xi_{\alpha,N}}$; the formula for it involves a convolution product of functions over two different shuffle Hopf algebras, and a certain coproduct related to the Goncharov motivic coproduct.

- Then, the regularity condition of $\text{Li}_{p,\alpha}^{1,\mu_{p,N},B}$ amounts to the equation $\int \text{reg}_{\psi_{\xi_{\alpha,j}}, \ldots, \psi_{\xi_{\alpha,N}}} = \int \text{reg}_{\phi_{\xi_{\alpha,j}}, \ldots, \phi_{\xi_{\alpha,N}}}$. The bounds of valuations of the coefficients of $\text{Li}_{p,\alpha}^{1,\mu_{p,N}}$ and $\mathcal{C}_{p,\alpha}$ of the Appendix to Theorem I-1 remain true, with the notions of depth and weight of words over $e_{B_{0,\mu_{p,N}}}$ defined in §3.1.

5. The $f_0^{z<1}$-harmonic Ihara action extended to $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{p,N}, \infty\})$

Again we suppose given a basis of $H^{1,\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p,N}, \infty\})$ of the type of §3.1, and we work in the framework of §3. Using §4, we extend the $f_0^{z<1}$-harmonic Ihara action built in [J I-2] into a structure on $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu_{p,N}, \infty\})$.

5.1. The main technical lemma. We extend to $\mathbb{P}^1 - \{0, \mu_{p,N}, \infty\}$ the main preliminary lemma of this theory:

**Lemma 5.1.** For any word $w$ over $e_{B_{0,\mu_{p,N}}}$, we have $\text{Li}_{p,\alpha}^{1,\mu_{p,N},B}[e_{0,L}^L w] \rightarrow_{l \rightarrow \infty} 0$.

**Proof.** Similar to the proof of Lemma 3.2.1 in [J I-2], as a direct consequence of the generalization, explained in §4.4, of the bounds of valuation of $\text{Li}_{p,\alpha}^{1,\mu_{p,N}}$ in [J I-1]. More generally, the result is true if we replace the sequence $(e_{w_{0}}^L w)_{i \in \mathbb{N}}$ by any sequence of words $(w_{i})_{i \in \mathbb{N}}$ such that $\text{weight}(w_{i}) \rightarrow_{l \rightarrow \infty} \infty$ and $\text{lim sup depth}(w_{i}) < \infty$. $\square$

5.2. Notations for generating series of multiple harmonic sums in the framework $f_0^{z<1}$. We take notations for the non-commutative generating series of multiple harmonic sums in the framework $f_0^{z<1}$.

**Definition 5.2.** Let $K_{p,N}(\langle e_{B_{0,\mu_{p,N}}}^L \rangle_{0 \leq l \leq \delta_{z<1}}) \subset K_{p,N}(\langle e_{B_{0,\mu_{p,N}}}^L \rangle_{0 \leq l})$ be the vector subspace consisting of the elements $f \in K_{p,N}(\langle e_{B_{0,\mu_{p,N}}}^L \rangle_{0 \leq l})$ such that, for all words $w$ on $e_{B_{0,\mu_{p,N}}}$, the sequence $(f[e_{L}^L w])_{i \in \mathbb{N}}$ is constant.
Definition 5.3. Let \((K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{har}}^{f_0<\zeta})\) be the subset of elements defined by the condition \(\sup_{w} |f[w]|K \to 0\) of depth \(d\).

Definition 5.4. i) For all \(n \in \mathbb{N}^*, \) let \(\text{har}^{\mu^{\alpha} N,B}_{p^\alpha N} = (\text{har}_{n}(w))_{w \in \mathcal{W}_N^{\text{har}}} \in K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle^{\Sigma}_{\text{har}}\).

ii) For all \(I \subset \mathbb{N}, \) let \(\text{har}^{\mu^{\alpha} N,B}_{p^\alpha N} = (\text{har}_{n}^{\mu^{\alpha} N,B})_{n \in I} \in \text{Map}(I, K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle^{\Sigma}_{\text{har}}).

5.3. The \(f_0^{\zeta<\zeta}<\zeta\)-harmonic Ihara action and of the \(f_0^{\zeta<\zeta}-\text{harmonic Frobenius generalized to} \ p^\alpha - \{0, \mu^{\alpha} N, \infty\}.

Definition 5.5.
i) Let \(K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{lim}} \subset K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle\) be the vector subspace consisting of the elements \(f \in K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle\) such that, for all words \(w \in \varepsilon_{0,\mu^{\alpha} N}, \) the sequence \((f[e_0^l w])_{l \in \mathbb{N}}\) has a limit in \(K_{p^\alpha N}\) when \(l \to \infty\).

ii) We have a map that we call "limit":

\[
\lim : K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{lim}} \to K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{har}}^{f_0<\zeta}
\]

defined by, for all words \(w,\)

\[
(\lim f) = \sum_{w \text{ word}} \left( \lim_{l \to \infty} f[e_0^l w] \right) w
\]

Lemma 5.6. The map \(\text{lim}\) of Definition 5.5 is continuous for restriction of the \(\mathcal{N}_D\)-topology on the source and target.

Proof. Clear.

Proposition-Definition 5.7. The map :

\[
\circ_{\text{har}}^{f_0<\zeta} : K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{lim}} \times \text{Map}(\mathbb{N}, K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{har}}^{f_0<\zeta}) \to \text{Map}(\mathbb{N}, K_{p^\alpha N} \langle \langle \varepsilon_{0,\mu^{\alpha} N} \rangle \rangle_{\text{har}}^{f_0<\zeta})
\]

defined by the equation

\[
(g_{\xi_1}^1, \ldots, g_{\xi_N}^N) \circ_{\text{har}}^{f_0<\zeta} (n \mapsto h_n) = (n \mapsto \lim \left( \tau(n)(g_{\xi_1}^1, \ldots, g_{\xi_N}^N) \circ_{\text{har}}^{f_0<\zeta} h_n \right))
\]

is well-defined. We call it the \(f_0^{\zeta<\zeta}\)-harmonic Ihara action of \(X_{K_{p^\alpha N}, p^\alpha N}.

Proposition 5.8. We have

\[
\text{har}^{\mu^{\alpha} N} = e_1^{p^\alpha N} \circ_{\text{har}}^{f_0<\zeta} \text{har}^{\mu^{\alpha} N}_N
\]

Proof. In the extended differential equation of the Frobenius rewritten as (24), we take the coefficient of degree \(p^\alpha n\) in the power series expansion at \(0\) (\(n \in \mathbb{N}^*\)), apply \(\tau(n)\) (where \(\tau\) is defined in Definition 2.5), we specify the equality for a word \(e_0^\prime \tilde{e}_{b_{i_d+1}} e_0^{n_d-1} \tilde{e}_{b_{i_d}} \ldots e_0^{n_1-1} \tilde{e}_{b_{i_1}},\) and compute the limit \(l \to \infty\) by applying the Lemma 5.1.

6. The \(\Sigma\)-harmonic Ihara action extended to \(\pi_{1}^{\text{un,DR}}(\mathbb{P}^1 - \{0, \mu^{\alpha} N, \infty\})\)

6.1. Basic equations of multiple harmonic sums. We adapt a few (simple) basic properties of multiple harmonic sums written in [J 1-2], §4 to this more general case. The first property is the “formula of shifting”.

Lemma 6.1. Let \(m \in \mathbb{N}\) and \(\delta \in \mathbb{N}\); we have by definition

\[
\text{har}_{\delta,m+\delta} \left( {f_1}_{\delta_{N}^{\zeta}}, \ldots, {f_1}_{\delta_{N}^{\zeta}} \right) = \varepsilon^{m_1 + \ldots + m_1} \prod_{0 < m_1 < \ldots < m_d < m_1 + \ldots + m_d + \delta = 1} \left( \frac{\xi_{\delta_{N}^{\zeta}}^{d+1}}{\xi_{\delta_{N}^{\zeta}}^{d}} \right)^{m_1} f_{\delta_{N}^{\zeta}}^{(m_d + \delta)}
\]

Assume moreover that \(m = p^\alpha,\) and \(\delta = up^\alpha, u \in \mathbb{N}\) (in particular for all \(m' \in [1, m - 1]\), we have \(v_p(m') < v_p(\delta)\). Then, we have:
(25) \( \text{har}_{s+m \Sigma} \left( \frac{f_{id_1}, \ldots, f_{id_t}}{\xi_N^{i_1}, \ldots, \xi_N^{i_1}} \right) = \sum_{t_1, \ldots, t_d \geq 0} \left( \prod_{j=1}^{d} \delta_j \left( -n_j \right) \right) \text{har}_m \left( \frac{f_{id_1}, \ldots, f_{id_t}}{\xi_N^{i_1}, \ldots, \xi_N^{i_1}} \right) \)

Proof. Clear. \( \square \)

The second property is the formula of splitting. \( \text{har}_m \) refers to the non-weighted multiple harmonic sums: \( \text{har}_m(w) = m^{\text{weight}(w)} \text{har}_m(w) \). See [J I-2], §4 and §5 for more details on the notations.

Lemma 6.2. Let \( N \in \mathbb{N}^* \), \( \{t_1, \ldots, t_r\} \subset [1, N - 1] \) with \( t_1 < \ldots < t_r \). We also denote by \( t_0 = 0 \) and \( t_{r+1} = N \).

(26) \( \text{har}_m \left( \frac{f_{id_1}, \ldots, f_{id_t}}{n_d, \ldots, n_{1}} \right) = \sum_{S=\{t_1, \ldots, t_{r}\} \subset \{t_1, \ldots, t_{r}\}} \prod_{j=1}^{r} \frac{f_{i(j)}(m_{i(j)} \mod p^{\alpha_j})}{m_{i(j)}} \prod_{i=0}^{r} \text{har}_{t_i, t_{i+1}} \left( \frac{f_{id_1}, \ldots, f_{id_t}}{n_d, \ldots, n_{1}} \right) \text{restricted to } S_i \)

where the condition \((*)\) is that \( S_0 \Pi \ldots \Pi S_r \) is an increasing connected partition of \( \{1, \ldots, d\} - f(S) \), such that \( S_i \Pi \ldots \Pi S_{i+1} \) is an increasing connected partition of \( f(t_{i+1}), f(t_{i+1}) \)\{.

Proof. Clear. \( \square \)

6.2. Notations for generating series of multiple harmonic sums in the framework \( \Sigma \). In the framework \( \Sigma \), we need to take into account both the generating series of multiple harmonic sums and the generating series of localized multiple harmonic sums.

Definition 6.3. i) Let \( \mathcal{W}_{\Sigma, \mu, \nu}^{m, B, \text{loc}} \) be the set of words of the form \( \left( \frac{f_{id_1}, \ldots, f_{id_t}}{\xi_N^{i_1}, \ldots, \xi_N^{i_1}} \right) \), as in equation (10) called \( \Sigma \)-harmonic words. Let \( \mathcal{W}_{\Sigma, \mu, \nu}^{m, B} \) be the larger set of similar words with \( n_d, \ldots, n_{1} \in \mathbb{Z} \), called localized \( \Sigma \)-harmonic words.

ii) Let \( K^{\nu, \nu} (\langle e_{\{|\mu, \nu\}}^{\nu, \nu} \rangle)^{\Sigma} \) be the set of words of the form \( \left( \frac{f_{id_1}, \ldots, f_{id_t}}{\xi_N^{i_1}, \ldots, \xi_N^{i_1}} \right) \), as in equation (10) called \( \Sigma \)-harmonic words. Let \( \mathcal{W}_{\Sigma, \mu, \nu}^{m, B, \text{loc}} \) be the larger set of similar words with \( n_d, \ldots, n_{1} \in \mathbb{Z} \), called localized \( \Sigma \)-harmonic words.

iii) The coefficient of an element \( f \in K^{\nu, \nu} (\langle e_{\{|\mu, \nu\}}^{\nu, \nu} \rangle)^{\Sigma} \) in front of a word \( w \in \mathcal{W}_{\Sigma, \mu, \nu}^{m, B, \text{loc}} \) is denoted by \( f[w] \) (as for the elements of \( K(\langle e_{\{2, \ldots, \nu\}} \rangle)^{\Sigma} \) in $\S 2.1$). Same notation for the localized variant.

iv) Let \( (K^{\nu, \nu} (\langle e_{\{|\mu, \nu\}}^{\nu, \nu} \rangle)^{\Sigma} \) be the set of elements \( f \) defined by the condition \( \sup_{w \in \mathcal{W}_{\Sigma, \mu, \nu}^{m, B, \text{loc}}} \left| f[w] - \right|_{p}^{n \rightarrow \infty} 0 \).

Definition 6.4. i) For all \( n \in \mathbb{N}^* \), let \( \text{har}_{n}^{m, \mu, \nu} = (\text{har}_n(w))_{w \in \mathcal{W}_{\Sigma, \mu, \nu}^{m, B, \text{loc}}} \) be the \( \Sigma \)-harmonic words. Let \( \text{har}_{n}^{m, \mu, \nu} = (\text{har}_n(w))_{w \in \mathcal{W}_{\Sigma, \mu, \nu}^{m, B, \text{loc}}} \) be the \( \Sigma \)-harmonic words.

ii) For all \( I \subset \mathbb{N}^* \), let \( \Sigma_{\mu, \nu}^{m, B} = (\text{har}_n^{m, \mu, \nu})_{n \in I} \in \text{Map}(I, K^{\nu, \nu} (\langle e_{\{|\mu, \nu\}}^{\nu, \nu} \rangle)^{\Sigma} \) be the \( \Sigma \)-harmonic words. Let \( \text{har}_{n}^{m, \mu, \nu} = (\text{har}_n^{m, \mu, \nu})_{n \in I} \in \text{Map}(I, K^{\nu, \nu} (\langle e_{\{|\mu, \nu\}}^{\nu, \nu} \rangle)^{\Sigma} \) be the \( \Sigma \)-harmonic words.

Definition 6.5. For any map \( \varphi : \{\xi_1^{N}, \ldots, \xi_N^{N}\} \rightarrow \{\xi_1^{N}, \ldots, \xi_N^{N}\} \), we define \( \varphi(\text{har}_n^{m, \nu}) \) as the element of \( K^{\nu, \nu} (\langle e_{\nu}^{\nu, \nu} \rangle)^{\Sigma} \) whose coefficient at a word \( \left( \frac{\xi_1^{N}, \ldots, \xi_N^{N}}{n_d, \ldots, n_{1}} \right) \) is \( \text{har}_n(\varphi(\xi_1^{N}, \ldots, \xi_N^{N})) \).
6.3. The localized $\Sigma$-harmonic Ihara action at $p^n N$-th roots of unity.

**Proposition-Definition 6.6.** There exists a natural explicit map

\[ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} : (Kp^n N \langle \langle e^B_{0, \mu p^n N} \rangle \rangle_{\Sigma}) S \times \text{Map}(\mathbb{N}, Kp^n N \langle \langle e_{0, \mu p^n N} \rangle \rangle_{\Sigma}) \to \text{Map}(\mathbb{N}, Kp^n N \langle \langle e^B_{0, \mu p^n N} \rangle \rangle) \]

which satisfies the equation

\[ \text{har}^{\mu p^n N, B}_\Sigma = \text{har}^{\mu p^n N, B}_\Sigma \circ (z \mapsto z^{p^n}) \circ \text{har}^{\mu, \Sigma, \text{loc}}_N \]

**Proof.** The proof is the same with the proposition-definition of \((\varphi_{\Sigma \mu p^n N, B}^{\Sigma})_{\text{loc}}\) for \(X_{K, N}\) in [J I-2], §5. We start with \(\text{har}^{\mu p} (w)\), for a word \(w\), written as an iterated sum indexed by \(0 < m_1 < \ldots < m_d < m\), and we introduce the Euclidean division of \(m_i\) by \(p^n\) : \(m_i = p^n u_i + r_i\); we write, for \(n_i \in \mathbb{N}^\ast\), \(m_i^{n_i} = r_i^{n_i} (\frac{p^n}{p^n} + 1)^{-n_i} = r_i^{n_i} \sum_{l \geq 0} (-n_i^l) \left( \frac{p^n}{p^n} \right)^l i\). This gives the result, via the formula of shifting (Lemma 6.1) and the formula of splitting (Lemma 6.2).

If \(n \in \mathbb{N}^\ast\) the Euclidean division by \(p^n\) defined by \(n = p^n u + r\), we have, for all \(i = 1, \ldots, p^n\) and \(j = 1, \ldots, N\), \(f_i(n \mod p^n) \xi^j_N = (f_i(r) \xi^j_N) (\xi^j_N)^n\) : we see that only \(\xi^j_N\) appears in the part of the computation which depends on \(u\); this is why the computation works in the same way with \(X_{K, N}\).

6.4. Conclusion. In [J I-2], §5, we have constructed a map of "elimination of the localization" : this is a natural map

\[ \text{elim} : \text{Map}(\mathbb{N}, Kp^n N \langle \langle e^B_{0, \mu p^n N} \rangle \rangle) \to \text{Map}(\mathbb{N}, Kp^n N \langle \langle e^B_{0, \mu p^n N} \rangle \rangle) \]

which satisfies

\[ \text{elim} (\text{har}^\mu) = \text{har}^\mu_{\text{loc}} \]

(The terminology refers to the dual of elim, which indeed suppresses the localization.)

**Proposition-Definition 6.7.** Let

\[ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} \circ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} = \varphi_{\Sigma \mu p^n N, B}^{\Sigma} \circ (\text{id} \times \text{elim}) \]

We call it the \(\Sigma\)-harmonic Ihara action of \(X_{K, p^n N, \mu.p^n N}\). We have :

\[ \text{har}^{\mu p^n N, B}_N = \text{har}^{\mu p^n N, B}_N \circ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} (z \mapsto z^{p^n}) \circ \text{har}^{\mu, \Sigma, \text{loc}}_N \]

**Proof.** Consequence of §6.3 and of the property of elim reviewed above.

**Remark 6.8.** It can be sometimes more natural to consider the dual of \(\varphi_{\Sigma \mu p^n N, B}^{\Sigma}\). We call it the \(\Sigma\)-harmonic Ihara coaction of \(X_{K, p^n N, \mu.p^n N}\); same terminology for the duals of the other harmonic Ihara actions.

**Definition 6.9.** i) Let \(\text{comp}^{\Sigma \mu p^n N, B} \circ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} : (Kp^n N \langle \langle e^B_{0, \mu p^n N} \rangle \rangle_{\Sigma}) S \to Kp^n N \langle \langle e^B_{0, \mu p^n N} \rangle \rangle S\) be defined by : \(\text{comp}^{\Sigma \mu p^n N, B} \circ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} \circ (z \mapsto z^{p^n}) \circ \text{har}^{\mu, \Sigma, \text{loc}}_N\)

of the equation (31).

ii) Let \(\text{comp}^{\Sigma \mu p^n N, B} \circ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} \circ (z \mapsto z^{p^n}) \circ \text{har}^{\mu, \Sigma, \text{loc}}_N\)

We call them the comparison maps between sums and integrals.

**Proposition 6.10.** We have \(\text{comp}^{\Sigma \mu p^n N, B} \circ \varphi_{\Sigma \mu p^n N, B}^{\Sigma} = \text{id}\).

**Proof.** This follows from the proof of the analog statement in [J I-2], §5 : it is indeed a consequence of a property of the coefficients \(B\) of the elimination of the localization.
Lemma 7.2. We call it the There exists a unique map as a map However, because of the equation involving multiple harmonic sums above, we can also view it as a map Let us adopt momentaneously this point of view.

Proof. In [J I-3], the definition and basic properties of of multiple harmonic sums at they require nothing crystalline. They only require the Ihara product, which is defined on Let .

Definition 7.1. Let comp be the submodule of elements such that, for all words the series is convergent ( stands for summable). It contains in particular and its image by . We extend a map of comparison defined in §6.

Definition 7.1. Let comp where the sum is over the words of the form over the words is convergent. It contains in particular and its image by . We extend a map of comparison defined in §6.

Lemma 7.2. The image of is equal to the subset of the elements such that we have for all and for all words .

Proof. Indeed, each element of is in and equal to its own image by .

Proposition-Definition 7.3. There exists a unique map making the following diagram commutative:

We call it the harmonic Ihara action of X and denote it by .
Proof. Let us consider the coefficient $\text{comp}^{\Sigma - f \cdot \mu_{p^\infty}, B}(\text{Ad}_e(e_1) \circ \text{Ad}_f(e_1))[w]$ at any word $w$, viewed as a function of $g$ and $f$. By the formula for the dual of the adjoint Ihara action, one can see that its dependence on $f$ factorizes in a natural way through $\text{comp}^{\Sigma - f \cdot \mu_{p^\infty}, B}(f)$. This defines $\phi_{\text{har}}^f$ and one can check that this map is unique.

The $f^\dagger_0$-harmonic Ihara action is thus $\text{comp}^{\Sigma - f \cdot \mu_{p^\infty}, B}(\text{Ad}_e(e_1) \circ \text{Ad}_f(e_1))(\text{Ihara product}))$ (where the push-forward refers to the set which is acted upon), characterized by the equation:

\[
(27) \quad \text{comp}^{\Sigma - f}(\text{Ad}_g(e_1) \circ \text{Ad}_f(e_1)) = g \circ \phi_{\text{har}}^f \text{comp}^{\Sigma - f}(\text{Ad}_e(e_1))
\]

One can also write an extension of where $\phi_{\text{har}}^f$ is replaced by $\phi_{\text{Lie}}^f$ defined in §3.3.2.

8. The Iteration of the Harmonic Frobenius Extended to $\Sigma^\text{un,DR}_{1}(\mathbb{P}^1 - \{0, \mu_{p^\infty}, \infty\})$

We take again the setting of §3, a basis $B$ of $H^1_{\text{DR}}(\mathbb{P}^1 - \{0, \mu_{p^\infty}, \infty\})$ being chosen.

8.1. Introduction. We have studied in [J I-3] how the Frobenius iterated $\alpha$ times depends on $\alpha$ viewed as a $p$-adic integer. Unlike for $X_{K,N}$, in the present case, the coefficients of the multiple sums which we consider depend on $p$ and $\alpha$, via the functions $f_1, \ldots, f_{p^\infty}$. It is thus less easy than in the usual case to study the limit $\alpha \to \infty$.

In [J I-3], given formal variables $a$, $\Lambda$, we have defined a map:

\[
(\text{iter}^{\Sigma}_{\text{har}})_{\Lambda}^a : (K_N(\langle e_0, e_{\mu_N} \rangle)_{\text{har}}^{\Sigma})_{\Lambda} \to \langle e_0, e_{\mu_N} \rangle_{\text{har}}^\Sigma
\]

such that the map $(\text{iter}^{\Sigma}_{\text{har}})_{\Lambda}^a$ defined as the composition of $(\text{iter}^{\Sigma}_{\text{har}})_{\Lambda}^a \circ (\text{iter}^{\Sigma}_{\text{har}})_{\Lambda}^a$ such that $h = (\alpha_0, \tilde{\alpha}) \in (\mathbb{N}^*)^2$ such that $\alpha_0 | \tilde{\alpha}$.

8.2. Generalizations to $X_{X_{\mu_{p^\infty}}, \mu_{p^\infty}, N}$. The proof of the part of Theorem-Definition V-1.a concerning the maps of iteration of the harmonic Frobenius follows from going back to the proofs in [J I-3], and making again the same observations: the computations can be adapted to our more general setting because the maps $f_1, \ldots, f_{p^\infty}$, applied to elements $m_1 < \ldots < m_d \in \mathbb{N}$, depend only on the remainders of $m_1, \ldots, m_d$ modulo $q^\alpha = p^\alpha$ and have coefficients of norm $\leq 1$.

9. Application to the Relation Between Direct and Indirect Methods to Compute the Frobenius

We sketch why the regularization of iterated integrals defined in §4 can be computed in terms of the iteration of the harmonic Frobenius defined in §8, and that the restriction of these maps to a certain subspaces has coefficients expressed in terms of $\Sigma^\text{un,DR}_{1}(X_{K,N})$. This gives a connection between the formulas of the direct computation of the Frobenius [J I-1] and the ones of the indirect computation of the Frobenius [J I-2] [J I-3] and an interpretation of this connection in terms of a descent of the Frobenius extended to $\Sigma^\text{un,DR}_{1}(X_{K_{p^\infty}, \mu_{p^\infty}, N})$.

9.1. A subspace and descent to $\Sigma^\text{un,DR}_{1}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$. We sketch the proof of Proposition V-1.d stated in §1.7.

Definition 9.1. Let $e_{0, \mu_N, j, \mu_{p^\infty}}$, be the set of letters formed by $e_0, e_{j, \mu_N}, j = 1, \ldots, N$, and $e_{(\mu_{p^\infty})^{p^\alpha}}$, $j \in \{1, \ldots, N\}, r \in \{0, \ldots, p^\alpha - 1\}$. 

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By associating \( e_{z_i} \) to \( \frac{dz_i}{z_i} \), and \( e'_{z_i} \mod p^\alpha \) to \( \frac{d^{\alpha} e_{z_i}^{p-1}}{z_i^{p^\alpha - 1} e'_{z_i}} \), we get a specific type of multiple harmonic sums.

**Definition 9.2.** We now assume \( X_K = X_{KN} \) and \( K = K_N \), with \( N \in \mathbb{N}^* \) prime to \( p \). The weighted multiple harmonic sums at roots of unity with congruences the numbers, for \( d \in \mathbb{N}^*, n_d, \ldots, n_1 \in \mathbb{N}^*, j_1, \ldots, j_d+1 \in \{1, \ldots, N\} \), and \( I, I' \subset \{1, \ldots, d\}, n \in \mathbb{N}^*, \)

\[
(29) \quad \text{har}_{m,I} \mod p^\alpha \left( \frac{z_{j_d+1}, \ldots, z_{j_1}}{n_d, \ldots, n_1} \right) = \sum_{0=m_0<\ldots<m_d<m, \text{ for } i \in I,n_{i-1} \equiv n_i \mod p^\alpha} \left( \frac{1}{z_{j_d+1}} \right) \prod_{j=1}^{m} \left( \frac{\left( \frac{z_{j_d+1}}{z_{j_1}} \right)^{m_j}}{m_j^{n_j}} \right)
\]

We can now sketch the proof of Proposition V-1.d.

**Proof.** (sketch) The result is obtained by redoing the constructions of the previous paragraphs with this particular type of multiple harmonic sums. It comes from how the conditions of the type \( m_i \equiv m_{i-1} \mod p^\alpha \) are expressed in terms of the reduction of \( n_i \) and \( m_{i-1} \mod p^\alpha \) is equivalent to \( r_i = r_{i-1} \) where \( r_i \) and \( r_{i-1} \) are the remainders of, respectively, \( m_i \) and \( m_{i-1} \mod p^\alpha \).

9.2. **Iteration of the harmonic Frobenius at \( p^\alpha N \)-th roots of unity and regularization at \( p^\alpha N \)-th roots of unity.** We sketch the proof of Proposition V-1.e stated in §1.7.

The idea behind it is to play on the ambiguity of the parameter \( q^\alpha \) in \( \text{har}_{p^\alpha} \):
- \( q^\alpha \) is the upper bound of the domain of summation of \( \text{har}_{p^\alpha} \) viewed as an iterated sum ; the question of studying the maps \( m \in \mathbb{N}^* \mapsto \text{har}_{m}(w) \in K \) with \( m \) and \( \text{har}_{m}(w) \) viewed as \( p \)-adic numbers has appeared a lot in part I, via the study of the regularized version \( m \in \mathbb{N}^* \mapsto \text{har}_{m,p^\alpha}^{\pm}(w) \in K \), generalized here in §4.
- \( q^\alpha \) also expresses via \( \hat{\alpha} \) the number of iterations of the Frobenius ; the question of studying the iterated Frobenius as a function of the number of iterations has appeared in [JI-3], and this has been generalized here in §8.

**Proof.** (sketch) The Proposition follows from the uniqueness of some power series expansions.

9.3. **Interpretation in terms of the computation of the Frobenius.** The strategy of computation of the Frobenius in [JI-1] has three steps :

i) **Formal algebraic solution to the differential equation of the Frobenius (14).**

By an immediate induction on the weight, equation (14) amounts to say that each function \( Li^{p,\alpha}_{I,\mathbb{N}}[w] \) is a \( K_N \)-linear combination of iterated integrals of the differential forms (7), where the coefficients of the linear combination are numbers \( \Phi_{p,\alpha}[w'] \) with weight \( (w') < \text{weight}(w) \).

We write a close formula for this property ; it involves the convolution product on the shuffle Hopf algebra as well as some combinatorics of another Hopf algebra related to a motivic Hopf algebra. We show that, in this close formula, each iterated integral of (7) can be replaced by a regularized version. We also show that the close formula, which is initially inductive with respect to the weight, can actually be made inductive with respect to the depth, which is more efficient algorithmically.

ii) **Explicit computation and bounds of valuation for each regularized iterated integrals of (7).**

We find a formula for these functions which is inductive with respect to the depth. We show that they lie in a very small subspace of \( \mathfrak{N}^1(U_N) \), defined in terms of \( p \)-adically and weight-adically convergent infinite sums of prime weighted multiple harmonic sums (the prime weighted multiple harmonic sums are the weighted multiple harmonic sums \( \text{har}_{\alpha}(w) \) such that \( n = p^\alpha \) where \( \alpha \) is the number of iterations of the Frobenius). We prove at the same time some bounds of evaluations of the values of these functions.

iii) **Reformulation of equation (16).**
We rewrite equation (16) as a way to obtain the numbers $\zeta_{p,\alpha}(w)$ as $\mathbb{Q}$-linear combinations of the numbers $\text{Li}_{p,\alpha}^\dagger[w]$ which is, first, compatible with the depth filtration, and secondly, which involves rational coefficients having not too big $p$-adic norms.

Bringing together steps i), ii), iii), we deduce, first, an explicit formula for $(\text{Li}_{p,\alpha}^\dagger, \zeta_{p,\alpha})$ written through an induction on the depth, and, secondly, bounds of valuations for the functions $\text{Li}_{p,\alpha}^\dagger[w]$ (Here we are referring to the canonical structure of the algebra on the rigid analytic functions on $U_N$ as a complete normed $K_N$-algebra) and for the numbers $\zeta_{p,\alpha}(w)$.

We see that the step of regularization can be done purely in terms of prime weighted multiple harmonic sums ; conversely, certain coefficients of multiple harmonic sums are actually equal to certain coefficients of iterated integrals.

Remark 9.3. This point of view has also the advantage of providing a canonical regularization. In I-1, §4, the regularization of $p$-adic iterated integrals depended on choices.

Remark 9.4. This point of view also gives an interpretation of the formulas with parameters sketched in [I I-1], §6.3, c).

**APPENDIX A.** $\pi_1^{un,DR}(\mathbb{P}^1 - \{z_0, z_1, \ldots, z_r, \infty\})$ AND $p$-ADIC PSEUDO ADJOINT MULTIPLE POLYLOGARITHMS

Let $X_K$ be $\mathbb{P}^1 - \{0, z_1, \ldots, z_r, \infty\}$ over a complete ultrametric normed field $K \supset \mathbb{Q}_p$ with $z_1, \ldots, z_r \in K$ of norm 1. We partially generalize to $\pi_1^{un,DR}(X_K)$ the operations on multiple harmonic sums defined in part I and the previous paragraphs.

**A.1. Setting.** The weighted multiple harmonic sums under consideration are those of equation (10). Their localized variants are obtained by allowing the parameters $n_1, \ldots, n_d$ in (10) to be any elements of $\mathbb{Z}$. They can be generalized as functions depending on locally analytic group homomorphisms $K^* \to K^*$, as in Remark 3.14.

Let us formalize the indices of multiple harmonic sums :

**Definition A.1.** Let $\mathcal{W}_{\Sigma, X_K}^{\Sigma, X_K}$ be the set of words of the form $(\frac{z_{j_d+1} \ldots z_{j_1}}{n_d, \ldots, n_1})$, as in equation (10) called $\Sigma$-harmonic words.

Let $\mathcal{W}_{\Sigma, X_K}^{\Sigma, X_K, \text{loc}}$ be the larger set of sequences $(\frac{z_{j_d+1} \ldots z_{j_1}}{n_d, \ldots, n_1})$, with $n_d, \ldots, n_1 \in \mathbb{Z}$, called localized $\Sigma$-harmonic words.

We say that $(\frac{z_{j_d+1} \ldots z_{j_1}}{n_d, \ldots, n_1})$ has weight $n_d + \ldots + n_1$ and depth $d$.

Let us formalize the spaces containing the generating sequences of multiple harmonic sums, and subspaces of 'summable' elements :

**Definition A.2.** i) Let $K((e_{z_0}, \ldots, e_{z_r}))_{\Sigma, X_K}^{\Sigma, X_K} = \prod_{w \in \mathcal{W}_{\Sigma, X_K}^{\Sigma, X_K}} K.w$

Let $K((e_0^{\pm 1}, e_{z_1}, \ldots, e_{z_r}))_{\Sigma, X_K}^{\Sigma, X_K, \text{loc}} = \prod_{w \in \mathcal{W}_{\Sigma, X_K}^{\Sigma, X_K, \text{loc}}} K.w$

ii) The coefficient of an element $f \in K((e_{z_0}, \ldots, e_{z_r}))_{\Sigma, X_K}^{\Sigma, X_K}$ in front of a word $w \in \mathcal{W}_{\Sigma, X_K}^{\Sigma, X_K}$ is denoted by $f[w]$ (as for the elements of $K((e_{z_0}, \ldots, e_{z_r}))_{\Sigma, X_K}^{\Sigma, X_K}$ in §2.1). Same notation for $f \in (K((e_0^{\pm 1}, e_{z_1}, \ldots, e_{z_r}))_{\Sigma, X_K}^{\Sigma, X_K, \text{loc}}$.

iii) Let $(K((e_{z_0}, \ldots, e_{z_r}))_{\Sigma, X_K}^{\Sigma, X_K})_{\Sigma, X_K}$ be the subset of elements defined by the condition $\sup_{\text{depth } d} |f[w]|_K \to s \to \infty 0$.

Let us formalize the generating sequences of multiple harmonic sums :
Definition A.3. i) For all $n \in \mathbb{N}^*$, let $\text{har}_n^{\Sigma_K} = (\text{har}_n(w))_{w \in \mathbb{W}_{\text{har}}} \in K(\langle e_{z_0}, \ldots, e_{z_r} \rangle)_{\text{har}}^{\Sigma_K}$ and $\text{har}_n^{\Sigma_K, \text{loc}} = (\text{har}_n(w))_{w \in \mathbb{W}_{\text{har,loc}}} \in K(\langle e_{z_0}^{\pm 1}, e_{z_1}, \ldots, e_{z_r} \rangle)_{\text{har}}^{\Sigma_K}$.

ii) For all $I \subset \mathbb{N}$, let $\text{har}_I^{\Sigma_K} = (\text{har}_I(X))_{X \in I}$ viewed as a map $I \to K(\langle e_{z_0}, \ldots, e_{z_r} \rangle)_{\text{har}}^{\Sigma_K}$, and $\text{har}_I^{\Sigma_K, \text{loc}} = (\text{har}_I^{\Sigma_K, \text{loc}})_{n \in I}$ viewed as a map $I \to K(\langle e_{z_0}^{\pm 1}, e_{z_1}, \ldots, e_{z_r} \rangle)_{\text{har}}^{\Sigma_K}$.

A.2. The localized $\Sigma$-harmonic Ihara action. Generalizing the localized $\Sigma$-harmonic Ihara action to this setting is simple. Below, the notation $(z \mapsto z^{p^n})$, refers to Definition 6.5.

Proposition-Definition A.4. There exists a natural explicit map, generalizing the localized $\Sigma$-harmonic Ihara action of [J I-2] §6 and §6,

$$c_{\text{har}}^{\Sigma, X_K}_{\text{loc}} : (K(\langle e_{z_0}, \ldots, e_{z_r} \rangle)_{\text{har}})_{\Sigma} \times \text{Map}(\mathbb{N}, K(\langle e_{0}^{\pm 1}, e_{z_1}, \ldots, e_{z_r} \rangle)_{\text{har}}) \to \text{Map}(\mathbb{N}, K(\langle e_{z_0}, \ldots, e_{z_r} \rangle))$$

such that we have

$$\text{har}_{p^N}^{\Sigma_K} = c_{\text{har}}^{\Sigma, X_K}_{\text{loc}}(z \mapsto z^{p^n})_{\ast}(\text{har}_{N}^{X_K, \text{loc}}).$$

Proof. This is the same computation with in §6 and [J I-2] §5. □

A.3. The elimination of the localization. Unlike the previous step, the step of elimination of the localization is more subtle in the generic case than in the case of roots of unity. There is, first, a naive way to write it:

Proposition-Definition A.5. There exists a natural explicit map, generalizing the elimination of the localization in [J I-2] §6 and §6,

$$\text{elim}^{X_K} : \text{Map}(\mathbb{N}, K(\langle e_0, e_{z_1}, \ldots, e_{z_r} \rangle)) \to \text{Map}(\mathbb{N}, K(\langle e_{0}^{\pm 1}, e_{z_1}, \ldots, e_{z_r} \rangle))$$

such that we have

$$\text{elim}^{X_K}(\text{har}_{N}^{X_K}) = \text{har}_{N}^{X_K, \text{loc}}.$$

Moreover, elim commutes with the map $(z \mapsto z^{p^n})_{\ast}$, and more generally with all the maps of the type of Definition 6.5. One has an explicit formula for $\text{elim}^{X_K}$ involving the analogs of the coefficients $B$ defined in [J I-2], but depending on $z_1, \ldots, z_r$, instead of roots of unity.

Proof. This is the same computation with in §6 and [J I-2] §5. □

In §6, in order to define $c_{\text{har}}^{\Sigma, X_K}$ for $\mathbb{P}^1 \setminus \{0, \mu_{p^N}, \infty\}$, we had to apply $\text{elim}$ only in the case of roots of unity of order prime to $p$. This is not the case anymore here; and we do not have satisfactory bounds of valuations of the coefficients $B$ in our more general case. In the case of roots of unity of order prime to $p$ we had the following fact leading to satisfactory bounds of valuations of the coefficients $B$: for each product $z$ of elements of $\{z_1, \ldots, z_r\}$, either $z = 1$ or $|x_1|_p = \frac{1}{|x_2|_p} = 1$. This does not remain true if $\{z_1, \ldots, z_r\}$ is not included in the set of roots of unity of order prime to $p$: at least one $\frac{1}{|x_2|_p}$ is bigger. This prevents us from defining $c_{\text{har}}^{\Sigma, X_K}$ as $(c_{\text{har}}^{\Sigma, X_K})_{\text{loc}} \circ (\text{id} \times \text{elim}^{X_K})$ as we did in [J I-2] for $\mathbb{P}^1 \setminus \{0, \mu_{p^N}, \infty\}$ and in §6 for $\mathbb{P}^1 \setminus \{0, \mu_{p^N}, \infty\}$; it would involve divergent series. Thus, we will express differently the elimination of the localization.

For any $x \in \mathbb{C}_p$ such that $|x|_p = 1$, let $\omega(x)$ be the unique root of unity of order prime to $p$ in $\mathbb{C}_p$ such that $|x - \omega(x)|_p < 1$. This defines a morphism of multiplicative groups from $O_{\mathbb{C}_p}$ to $\bigcup_{p \not| N} \mu_N(\mathbb{C}_p)$.

Let us assume that either $K$ is a sub-topological field of $\mathbb{C}_p$, or $K$ contains $\mathbb{C}_p$ as a sub-topological field and that $z_1, \ldots, z_r$ are chosen such that for all $j \in \{1, \ldots, r\}$, there exists a (necessarily unique) root of unity in $\mathbb{C}_p$ of order prime to $p$, which we denote again by $\omega(z_j)$, such that $|z_j - \omega(z_j)|_p < 1$. We have a morphism of multiplicative groups from the subgroup of $K^\times$ generated by the $z_j$'s to $\bigcup_{p \not| N} \mu_N(\mathbb{C}_p)$. We will then write

$$z_j = \omega(z_j) + \epsilon_j$$
and expand the result in terms of powers of $\epsilon_j$'s. We will first replace $\epsilon_1, \ldots, \epsilon_r$ by formal variables $E_1, \ldots, E_r$ in order to neutralize the possible problems of convergence. We let for convenience $y_i = \frac{1}{z_i}$ for $i = 1, \ldots, d$. The multiple harmonic sums under consideration become:

**Definition A.6.** Let, for $n_d, \ldots, n_1 \in \mathbb{Z},$ and $j_{d+1}, \ldots, j_1 \in \{1, \ldots, r\}$,

$$\text{har}_{m}^{E_d, \ldots, E_1} \left( \frac{z_{j_{d+1}}, \ldots, z_{j_1}}{n_d, \ldots, n_1} \right) = m_{n_d+\ldots+n_1} \sum_{0<m_1<\ldots<m_d<m} \left( \frac{\omega(y_{i_1})+E_{i_1}}{\omega(y_{i_2})+E_{i_2}} \right)^{m_1} \cdots \left( \frac{\omega(z_{j_{d+1}})+E_{j_{d+1}}}{\omega(z_{j_d})+E_{j_d}} \right)^{m_d} \left( \omega(y_{j_{d+1}})+E_{j_{d+1}} \right)^{m} \in \mathbb{Q}_p^{\text{ unr}}[E_1, \ldots, E_r]$$

Let $\text{har}_{m, \text{loc}}^{E_d, \ldots, E_1}$ be their generating sequence and let $\text{har}_{m, \text{loc}}^{E_d, \ldots, E_1}$ be the map $m \mapsto \text{har}_{m, \text{loc}}^{E_d, \ldots, E_1}$. Same notations without loc for the restrictions to $n_d, \ldots, n_1 \geq 1$.

(30) has an expansion as a polynomial of the $E_j$'s (the sum below is finite):

**Lemma A.7.** We have an expression of the form:

$$\text{har}_{m}^{E_d, \ldots, E_1} \left( \frac{z_{j_{d+1}}, \ldots, z_{j_1}}{n_d, \ldots, n_1} \right) = \sum_{l_1, \ldots, l_d \geq 0} \prod_{r=1}^{d} \frac{E_{j_r}}{l_r!} \left( \frac{\omega(y_{i_r})}{\omega(y_{i_{r+1}})} \right)^{l_r} \times \left( \sum_{0<m_1<\ldots<m_d<m} \left( \frac{1}{l_1} \right)^{m_1-1} \omega(y_{i_1})^{m_1-l_1} \omega(y_{i_2})^{m_2-l_2} \omega(y_{i_{d+1}})^{m_d-l_d} \frac{\omega(y_{j_{d+1}})^{m}}{\omega(y_{j_d})^{m_d}} \right)$$

which, by rewriting the binomial coefficients, is of the following form, with $P_{l_1, \ldots, l_d} \in \mathbb{Z}[M_1, \ldots, M_d]$

$$\sum_{l_1, \ldots, l_d \geq 0} \left( \prod_{r=1}^{d} \frac{E_{j_r}}{\omega(y_{i_r})^{l_r}l_r!} \right) \sum_{0<m_1<\ldots<m_d<m} P_{l_1, \ldots, l_d} (m_1, \ldots, m_d) \frac{\omega(y_{i_1})^{m_1} \omega(y_{i_2})^{m_2-m_1} \omega(y_{i_{d+1}})^{m_d-m_d}}{m_1! \cdots m_d!}$$

We obtain the result by writing each $P_{l_1, \ldots, l_d}$ as a sum of monomials. □

We deduce another way to write the elimination of the localization:

**Proposition-Definition A.8.** There exists an explicit map (the source and target of $\text{elim}^{E_1, \ldots, E_r}$ are defined as in A.1)

$$\text{elim}^{E_1, \ldots, E_r} : \mathbb{Q}_p^{\text{ unr}}[E_1, \ldots, E_r]/\langle \pi_0, \pi_{j \in \mu} \rangle \to \mathbb{Q}_p^{\text{ unr}}[E_1, \ldots, E_r]/\langle \pi_0^{1+1} \cup \mu \rangle \text{ har, loc}$$

such that we have

$$\text{elim}^{E_1, \ldots, E_r} (\text{har}^{E_1, \ldots, E_r}) = \text{har}_{\text{loc}}^{E_1, \ldots, E_r}$$

**Proof.** This follows from the result of elimination of the localization for multiple harmonic sums at roots of unity of order prime to $p$ ([J I-2], §5), and the previous lemma, noting that the coefficients in the previous lemma are in $\mathbb{Z}$, and thus do not affect the convergence of the series. □

It is also possible to define this map in a more universal way which is not relative to $X_K$ but involves one variable $E_\xi$ per root of unity $\xi$ of order prime to $p$.

**A.4. The $\Sigma$-harmonic Ihara action and related objects.** In this paragraph, we assume that $|z_j - \omega(z_j)|_p < p^{-\frac{1}{\mu}}$ for all $j$. We can now generalize the $\Sigma$-harmonic Ihara action.
Proposition-Definition A.9. If $|e_j|_p < p^{-\frac{1}{p-1}}$ for all $j$, then
\[
\sigma^{\Sigma,X_K}_{\text{har}} = (\sigma^{\Sigma,X_K}_{\text{har}})_{\text{loc}} \circ (\text{id} \times \text{elim} \varepsilon_1, \ldots, \varepsilon_r)
\]
is well-defined and we have
\[
\text{har}^{X_K}_{\rho^\alpha_N} = \text{har}^{X_K}_{\rho^\alpha} \sigma^{\Sigma,X_K}_{\text{har}} (z \mapsto z^\rho)(\text{har}_{\rho^\alpha_N})
\]
where $N$, prime to $p$, is the lcm of the orders of $\omega(z)$'s as roots of unity.

Proof. The well-definedness is a consequence of A.3, and the equation (31) is a direct consequence of A.2, A.3 and the fact that power series $\sum_{l \geq 0} \frac{x^l}{l!}$ is convergent in $\mathbb{Q}_p$, for $x \in \mathbb{Q}_p$ such that $v_p(x) > \frac{1}{p-1}$. \qed

Remark A.10.

i) In this point of view, the coefficient of $(\varepsilon_1 \ldots \varepsilon_r)^0$ looks like a "regularization" (in a quite unusual sense: regularization "by roots of unity of order prime to $p$") of the $\Sigma$-harmonic Ihara action.

ii) The results of §6, concerning $X_{K_{p^\alpha,N}}$, give the particular case where $\varepsilon_1 = \ldots = \varepsilon_d = 0$.

iii) Thus in fine, this result is a sort of analytic interpretation, where "analytic" refers to functions of $\varepsilon_1, \ldots, \varepsilon_d$, of the definition of the $\Sigma$-harmonic Ihara action for $X_{K_{p^\alpha,N}}$ in §6.

The coefficients of $\sigma^{\Sigma,X_K}_{\text{har}}$ are not iterated integrals a priori: but, by analogy with the case of roots of unity, we will call them pseudo-iterated integrals, and define:

Definition A.11. i) Let $\text{comp}^{\Sigma}\sigma(X_K) : (K(\langle \varepsilon_{z_1}, \ldots, \varepsilon_{z_r}\rangle)_{\text{har}})_S \to (K(\langle \varepsilon_{z_1}, \ldots, \varepsilon_{z_r}\rangle)_S$ be defined by:
\[
\text{comp}^{\Sigma}\sigma(X_K)[e_0^{d_1} z_{j_{d_1+1}}^{n_{d_1}} \ldots z_{j_1}^{n_1}]
\]
is the coefficient of $n! \text{har}_n(\emptyset)$ in the coefficient of
\[
(z_{j_{d_1+1}}, \ldots, z_{j_1})
\]
of the equation (31).

ii) Let $\text{comp}^{\Sigma}\sigma(X_K) : (K(\langle \varepsilon_{z_1}, \ldots, \varepsilon_{z_r}\rangle)_{\text{har}})_S \to (K(\langle \varepsilon_{z_1}, \ldots, \varepsilon_{z_r}\rangle)_S$ be the map defined by
\[
\text{comp}^{\Sigma}\sigma(X_K)(f)(\frac{z_{j_{d_1+1}}, \ldots, z_{j_1}}{n_{d_1}, \ldots, n_1}) = \sum_{l \geq 0} f(e_0^{d_1} z_{j_{d_1+1}}^{n_{d_1}} \ldots z_{j_1})^{n_1}
\]
We call them the maps of comparison between sums and pseudo-integrals.

Proposition A.12. We have $\text{comp}^{\Sigma}\sigma \circ \text{comp}^{\Sigma} = \text{id}$.

Proof. Follows from the Proposition-Definition A.9 and the proof for roots of unity in §6 and in [J I-2] §5. \qed

The following definition generalizes the definition of adjoint $p$-adic multiple zeta values at roots of unity and $\Sigma$-harmonic Frobenius [J I-2], and their generalizations to $X_{K_{p^\alpha,N}}$ in §5.

Definition A.13. i) Let $\text{PseudoLi}_{p^\alpha,N}^{Ad}(X_K) = (\text{comp}^{\Sigma}\sigma)(X_K) (\text{har}^{X_K}_{\rho^\alpha} \in K(\langle \varepsilon_{z_1}, \varepsilon_{z_2}, \ldots, \varepsilon_{z_r}\rangle)$. We call $p$-adic pseudo adjoint hyperlogarithms the coefficients of $\text{PseudoLi}_{p^\alpha,N}^{Ad}(X_K)$.

ii) We call $\Sigma$-harmonic pseudo-Frobenius the map
\[
K(\langle \varepsilon_{z_1}, \ldots, \varepsilon_{z_r}\rangle)_{\text{har}} \to K(\langle \varepsilon_{z_1}, \ldots, \varepsilon_{z_r}\rangle)_{\text{har}}
\]
\[
h \mapsto \text{PseudoLi}_{p^\alpha,N}^{Ad}(X_K)(\text{har}_n^\rho h)
\]
The next statement generalizes the expansion of multiple harmonic sums at roots of unity in terms of $p$-adic multiple zeta values at roots of unity (Corollary 1-2.a in [J I-2]) and its generalization to $X_{K_{p^\alpha,N}}$ in §5.

Proposition A.14. We have $\text{har}^{X_K}_{\rho^\alpha} = \text{comp}^{\Sigma}\sigma \text{PseudoLi}_{p^\alpha,N}^{Ad}(X_K)$, i.e., for all words,
\[
\text{har}^{X_K}_{\rho^\alpha} (\frac{z_{j_{d_1+1}}, \ldots, z_{j_1}}{n_{d_1}, \ldots, n_1}) = \sum_{l \geq 0} \text{PseudoLi}_{p^\alpha,N}^{Ad}(\frac{z_{j_{d_1+1}}, \ldots, z_{j_1}}{l, n_{d_1}, \ldots, n_1})(X_K)
\]

Proof. By Proposition A.12. \qed
A.5. \( z_1, \ldots, z_r \) viewed as variables: \( p \)-adic pseudo adjoint multiple polylogarithms. Our point of view of replacing \( \epsilon_1, \ldots, \epsilon_r \) by formal parameters and viewing them as perturbations makes better sense if we allow \( \epsilon_1, \ldots, \epsilon_r \) to vary. According to Goncharov [G], the functions obtained from hyperlogarithms (in the sense of Proposition-Definition 2.4) by making \( z_1, \ldots, z_r \) into variables are called multiple polylogarithms. We are going to express how the \( p \)-adic pseudo adjoint hyperlogarithms of §A.2 depend on \( z_1, \ldots, z_r \). Let \( U_K \) be the set of \( z \in K \) such that there exists a (necessarily unique) root of unity \( \eta \) of order prime to \( p \) such that \( |z - \eta|_K < p^{-1} \). For \( r \in \mathbb{N}^* \), \( U_K^r \) is equipped with the product topology.

**Proposition-Definition A.15.** The coefficients of the following map are locally analytic functions (of \((y_1, \ldots, y_r) = (\frac{1}{z_1}, \ldots, \frac{1}{z_r})\)):

\[
PseudoLi^{\text{Ad}}_{p,\alpha}: \; \; U_K^r \rightarrow \; \; K\langle\langle e_{z_1}, \ldots, e_{z_r}\rangle\rangle_{\text{har}}^\Sigma (z_1, \ldots, z_r) \mapsto \text{PseudoLi}^{\text{Ad}}_{p,\alpha}(\mathbb{P}^1 - \{0, z_1, \ldots, z_r, \infty\}/K)
\]

We call its coefficients the \( p \)-adic pseudo adjoint multiple polylogarithms. (Pseudo Ad \( p \)MPL’s)

**Proof.** They are convergent infinite sums of products of locally constant functions by rational functions having poles only at \( z_i = 0 \). □

The values of Pseudo Ad \( p \)MPL’s at tuples of roots of unity are the Ad \( p \)MZV \( \mu_{p, N} \)’s defined in this paper; their values at tuples of roots of unity of order prime to \( p \) are the Ad \( p \)MZV \( \mu_{N} \)’s.
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