Exactly solvable Gaussian and non-Gaussian mean-field games and collective swarms dynamics

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Abstract
The collective behaviour of stochastic multi-agents swarms driven by Gaussian and non-Gaussian environments is analytically discussed in a mean-field approach. We first exogenously implement long range mutual interactions rules with strengths that are modulated by the real-time distance separating each agent with the swarm barycentre. Depending on the form of this barycentric modulation, a transition between drastically collective behaviours can be unveiled. A behavioural bifurcation threshold due to the tradeoff between the desynchronisation effects of the stochastic environment and the synchronising interactions is analytically calculated. For strong enough interactions, the emergence of a swarm soliton propagating wave is observable. Alternatively, weaker interactions cannot overcome the environmental noise and evanescent diffusive waves result. In a second and complementary approach, we show that the emergent solitons can alternatively be interpreted as being the optimal equilibrium of mean-field games (MFG) models with ad-hoc running cost functions which are here exactly determined. The MFG’s equilibria resulting from the optimisation of individual utility functions are solitons that are therefore endogenously generated. Hence for the classes of models here proposed, an explicit correspondence between exogenous and endogenous interaction rules ultimately producing similar collective effects can be explicitly constructed. For both Gaussian and non-Gaussian environments our exact results unveil new classes of exactly solvable mean-field games dynamics.

Keywords: stochastic multi-agents dynamics, Brownian motion, piecewise deterministic dynamics, nonlinear Fokker-Planck equation, Burger’s equation, nonlinear two-velocities Boltzmann equation, barycentric interactions, dynamic programming, Hamilton-Bellman-Jacobi equations, mean-field games, behavioural phase transitions, soliton waves.

1 Introduction
Yoshiki Kuramoto proposed for the first time in 1975 a fully analytic study describing the collective behaviour of a swarm of interacting Brownian phase oscillators [1]. In this now

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paradigmatic multi-agent model, each phase oscillator evolving on a one-dimensional compact state space (circle) interacts with all its neighbours (long-range type interactions). The dynamics is described by a set of coupled stochastic differential equations (SDE) driven by independent White Gaussian Noise (WGN). In the very large population limit, it is legitimate to use of a mean-field (MF) approach enabling to summarise the swarm behaviour into an oscillator density measure. Thanks to the underlying WGN the density measure evolves according to a nonlinear deterministic Fokker-Planck equation (FPE) which, in the stationary regime, can be analytically discussed. The trade-off between the desynchronising tendency due to the random environment and the synchronising effect due to the mutual interaction leads to bifurcation threshold separating two drastically different swarm behaviours. Namely for low coupling to noise ratio, the collective motion is diffusive and fully disorganised, for large coupling to noise ratio however, a collective synchronised swarm emerges. While for the basic Kuramoto model, the mutual interactions rule leading to the collective behaviour is exogenously given, a recent approach [2] shows how the Kuramoto’s corporative behaviour may alternatively be viewed as being an equilibrium of a mean-field games (MFG) formalism pioneered in [3, 4, 5, 6, 7] with ad-hoc running cost functions. In this MFG approach each agent (i.e. phase oscillator) minimises an individual cost function which depends on the whole swarm and it is this collective minimisation procedure which leads to a global equilibrium which coincides with the Kuramoto’s synchronised phase. Adopting the alternative MFG point of view, one may interpret the collective behaviour as being the result of an endogenous rule (each agent possesses its own objective function to minimise). We hope that our present models offer an additional elementary explicitly solvable illustration of the recent mathematically oriented literature devoted to MFG dynamics [8, 9, 10, 11,12].

The central goal in this paper is to construct fully solvable classes of scalar multi-agents models with state space \( \mathbb{R} \) instead of the compact Kuramoto’s circular state space. The underlying stochastic environment is either the Brownian motion process or a two-states Markov chain in continuous time (also known as the telegraphic noise) which introduces correlations into the dynamics. To separately discuss the resulting dynamics, our presentation proceeds via two main sections. In section 2 the agents’ dynamics are driven by Brownian environment sources and Propostion 1 shows how a soliton propagating wave emerges from an exogenously specified algorithm referred as the avoid to be the laggard (ABL) rule. Depending on the strength of a barycentric factor which modulates the actual influence of agents depending of their locations relative to the swarm’s barycentre, a behavioural transition, of the Kuramoto’s type arises and the exact bifurcation threshold can be calculated. In Proposition 2, we construct associated MFG’s with equilibria that are determined by solving a nonlinear Schrödinger equation. This yields soliton waves similar to those obtained in Proposition 1. A similar presentation architecture adopted in section 3 where the corresponding Propositions 3 and 4 are obtained in presence of telegraphic noise environments. Proposition 4, unveils a new class of analytically solvable MFG (i.e. not belonging to the linear drift with quadratic costs optimal control dynamics).

2 Nonlinear diffusive dynamics

Let us consider a set of \( N \) scalar interacting diffusion processes \( X_{k,t} \in \mathbb{R} \) with time \( t \in \mathbb{R}^+ \):

\[
dX_{k,t} = I(X_t, X_{k,t})dt + \sigma dB_{k,t}, \quad k = 1, 2, \ldots, N,
\]
with $\mathbf{X}_t := (X_{1,t}, X_{2,t}, \ldots, X_{N,t})$, $\sigma \in \mathbb{R}^+$ and $dB_{k,t}$ are $N$ independent standard Brownian motions [13]. The drift $\mathcal{I}(\mathbf{X}_t, X_{k,t})$ defines a mutual-interaction kernel exogenously implementing the algorithm:

**Avoid being a laggard algorithm (ABL)**.

i) For $k = 1, 2, \ldots, N$ and in real time, agent $A_k$ observes the positions $X_{j,t}$ of his fellows $A_j$ for $j \neq k$ and $j = 1, 2, \ldots, N$.

ii) For $k = 1, 2, \ldots, N$ agent $A_k$ accounts the number $n_k(t)$ of her leaders $A_j$ for which $X_{j,t} \geq X_{k,t}$ and for $j \neq k$.

iii) For $k = 1, 2, \ldots, N$ agent $A_k$ implements her instantaneous drift according to the rule:

$$\mathcal{I}(\mathbf{X}_t, X_{k,t}) = \frac{n_k(t)}{N}. \quad (2)$$

In view of Eq. (2), $A_k$ effectively avoids to remain a swarm’s laggard since the more leaders she finds, the higher is her incentive to increase a drift velocity.

In the sequel, we will systematically focus attention on large populations (i.e. $N \to \infty$) enabling us define an empirical agents population density $\rho(x,t) \in [0,1]$ as

$$\rho(x,t) = \frac{1}{N} \sum_{j=1}^{N} \delta(X_{j,t} - x). \quad (3)$$

Since the agents population is homogeneous, (i.e. $\mathcal{I}_k(\cdot) \equiv \mathcal{I}(\cdot)$), we may randomly select one representative (i.e. index independent) fellow $A$ located at $X_t \in \mathbb{R}$. For $A$, the ABL rule is formally implemented as:

$$dX_t = \left[ \int_{X_t}^{\infty} \rho(y,t) dy \right] dt + \sigma dB_t. \quad (4)$$

**Remark 1 (mean-field dynamics).** Eq. (4) implements infinite range interactions since agent $A$ has to take into account the locations of the whole swarm population (except herself) to determine her own drift. This basically realises the mean-field approach of the swarm’s dynamics.

The probabilistic properties of the trajectories solving the (Markovian) stochastic differential equation (SDE) Eq. (4) can be found by solving the associated nonlinear Fokker-Planck equation (FPE) [13]:

$$\partial_t \rho(x,t) = \sigma^2 \frac{1}{2} \partial_{xx} \rho(x,t) - \partial_x \left( \rho(x,t) \left[ \int_{x}^{\infty} \rho(y,t) dy \right] \right). \quad (5)$$

Let us now generalise the ABL rule Eq. (4) by further introducing an (infinitely differentiable) barycentric weighting function $G[\mathbf{X}_t - \langle X(t) \rangle]$ : $\mathbb{R} \to \mathbb{R}^+$. Accordingly, Eq. (5) will be now generalised as:

$$\left\{ \begin{array}{l}
\partial_t \rho(x,t) = \frac{\sigma^2}{2} \partial_{xx} \rho(x,t) - \partial_x \left\{ \rho(x,t) \int_{x}^{\infty} (G[y - \langle X(t) \rangle] \rho(y,t)dy) \right\}, \\
\langle X(t) \rangle = \int_{\mathbb{R}} x \rho(x,t) dx, \\
\int_{\mathbb{R}} \rho(x,t) dx = 1,
\end{array} \right. \quad (6)$$
For agent $A$, the weight $G(\cdot)$ modulates the relative influence of the leaders depending on their relative remoteness to the swarm barycentre. For $z$-increasing $G(z)$ we effectively describe situations where leaders are more influential than close neighbours. Conversely, decreasing $G(z)$ describe dynamics where agents are mostly influenced by their neighbour-fellows. It is important to point out that in both situations the interactions remain of long range type. For the nonlinear swarm dynamics expressed in Eq.(6), we now can establish:

**Proposition 1.**

Assuming the class of kernel functions $G_{\eta,\sigma}(x) := A(\eta,\sigma) \cosh(x)^{\eta}$ with parameters $\sigma \in \mathbb{R}^+, \eta \in ]-2, \infty]$ and the pre-factor $A(\eta,\sigma) = \frac{\sigma^2(2+\eta)}{2N(\eta)}$, Eq.(6) is solved by the normalised soliton propagating waves:

$$
\begin{align*}
\rho(x,t) &= N(\eta) \cosh^{-(2+\eta)}(x - \omega t), \quad (2 + \eta) > 0, \\
\omega &= N(\eta) A(\eta,\sigma), \\
N(\eta)^{-1} &= B(1/2, 1 + \eta/2) = \frac{\sqrt{\pi}((1+\eta/2))}{1((3/2+\eta/2))},
\end{align*}
$$

(7)

where $B(1/2, 1 + \eta/2) := \frac{\sqrt{\pi}((1+\eta/2))}{1((3/2+\eta/2))}$ is the beta function [14].

**Proof of proposition 1.**

Introduce the change of variables $x \mapsto \xi = (x - \omega t)$ and $t \mapsto \tau$ and assume the $\tau$-independent stationary evolution $\rho(\xi,\tau) = \rho(\xi) = N(\eta) \cosh^{-(2+\eta)}(\xi)$. Accordingly, we have:

$$
\begin{align*}
\langle X(t) \rangle &= \int_{\mathbb{R}} x \rho(x,t) dx = \int_{\mathbb{R}} \langle \xi + \omega t \rangle \rho(\xi) d\xi = \omega t, \\
\partial_t \mapsto -\omega \partial_\xi + \partial_\tau \quad \text{and} \quad \partial_x \mapsto \partial_\xi, \\
\int_{\mathbb{R}} [G_{\eta,\sigma}(y - \langle X(t) \rangle)] \rho(y,t) dy &\mapsto \int_{\mathbb{R}} G_{\eta,\sigma} = \int_{\mathbb{R}} A(\sigma,\eta) N(\eta) \cosh^{-2}(\xi) d\xi.
\end{align*}
$$

It follows that Eq.(6) can be rewritten as:

$$
\partial_\xi \left\{ \frac{\sigma^2}{2} \partial_\xi \rho(\xi) + \omega \rho(\xi) - \rho(\xi) \int_{\mathbb{R}} A(\sigma,\eta) N(\eta) \cosh^{-2}(\xi) d\xi \right\} = 0.
$$

Integrating once the last equation with respect to $\xi$ (with vanishing integration constant in order to fulfil the normalisation constraint, one has to impose $\lim_{|\xi| \to \infty} \rho(\xi) = 0$) and dividing by $\rho(\xi) > 0$, we straightforwardly obtain:

$$
\frac{\sigma^2}{2} \partial_\xi \log[\rho(\xi)] = -\omega + \int_{\mathbb{R}} A(\sigma,\eta) N(\eta) \cosh^{-2}(\xi) d\xi.
$$

Plugging $\rho(\xi) = N(\eta) \cosh^{-(2+\eta)}(\xi)$ into the last equation and using $\int_{\mathbb{R}} \cosh^{-2}(\xi) d\xi = 1 - \tanh(\xi)$, the last expression reads:

$$
-\frac{\sigma^2}{2} (2 + \eta) \tanh(\xi) = -\omega + A(\sigma,\eta) N(\eta)[1 - \tanh(\xi)].
$$

By direct identification, we see that we need to fulfil:
\[
\begin{align*}
A(\sigma, \eta) &= \frac{\sigma^2 (2 + \eta)}{2 N(\eta)}, \\
\omega &= A(\sigma, \eta) N(\eta).
\end{align*}
\] (8)

The normalisation factor \( N(\eta) \) imposes \((2 + \eta) > 0\) and, invoking [14], we have:

\[
N(\eta)^{-1} = \int_{\mathbb{R}} \cosh^{-(2+\eta)}(x) dx = B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right).
\]

End of the proof

Remark 1. It is worth to observe that a normalised soliton cannot be generated for weights \( G(z) \propto \cosh^{\eta}(z) \) for \( \eta < -2 \). This can be heuristically understood by the fact that decreasing \( \eta \), reduces the cooperative influence of remote leaders. This weakens the possibility to sustain a collective evolution. Hence \( \eta = 2 \) is the bifurcation value separating two drastically different swarm evolution, namely for \( \eta > -2 \), one observes the emergence of a co-operative soliton which cannot be sustained when \( \eta \leq -2 \).

2.1 Corresponding mean-field game dynamics

Consider the diffusive dynamics:

\[
dX_t = [b + u(X_t)] dt + \frac{\sigma^2}{2} dB_t,
\] (9)

with \( b \) a constant and \( u(X_t) \) a control function dependent on the whole population of agents. In parallel, we now introduce a MF running cost function \( \mathcal{L}[u, \rho(x, t)] \) in the form [14]:

\[
\mathcal{L}[u, \rho(x, t)] := \frac{1}{2\mu} [u(x)]^2 - g \rho(x, t)^a.
\] (10)

with \( g, \mu, a \in \mathbb{R}^+ \) For a time horizon \( T \), we introduce for \( t \in [0, T] \) a cost functional:

\[
\mathcal{J}[X(\cdot), u(\cdot)] = \mathbb{E} \left\{ \int_0^T \mathcal{L}[u(X_s), \rho(X_s, s)] ds \right\} + C_T(X_T),
\] (11)

where \( C_T(X_T) \) stands for a final cost. Minimisation of the cost given by Eq. (11) leads to a set of nonlinear coupled pde’s which have to be simultaneously solved forward/backward in time [15]:

\[
\begin{align*}
\partial_t \rho(x, t) &= \partial_x \left[ \left( \frac{1}{\mu} \partial_x u(x, t) - b \right) \rho(x, t) \right] + \frac{\sigma^2}{2} \partial_{xx} \rho(x, t), \quad \text{(Fokker Planck)}, \\
\partial_t u(x, t) + b \partial_x u(x, t) - \frac{1}{2\mu} [\partial_x u(x, t)]^2 + \frac{\sigma^2}{2} \partial_{xx} u(x, t) &= g \rho(x, t)^a, \quad \text{(Hamilton Bellman Jacobi)}.
\end{align*}
\] (12)

Assume now that we deal with sufficiently large time horizons \( T \) so that for the range of times \( 0 \ll t \ll T \), an ergodic regime [12, 15, 16] can be reached. In this stationary regime, we approximately have:

\[
\frac{u(x, t)}{t} \simeq \epsilon \quad \text{for } 0 \ll t \ll T,
\] (13)
with \( \epsilon \) an a priori unknown constant. In this time range, the initial conditions and the final cost barely affect the solution of Eq. (12) and we can establish:

**Proposition 2.**

Given the parameters \( b, \sigma \) in Eq. (10) and \( a, \mu \) in Eq. (10) and for \( g = \frac{\mu \sigma^4 (a+1) [B \left( \frac{1}{2}, \frac{1}{a} \right)]^{2a}}{2a^a} \) in Eq. (10), the probability density \( \rho(x,t) \) associated with ergodic regime of the MFG dynamics defined by Eq. (12) reads:

\[
\rho(x,t) = \frac{B \left( \frac{1}{2}, \frac{1}{a} \right)}{[\cosh(x - bt)]^{1/a}},
\]

**Proof of Proposition 2**

We first introduce a couple of auxiliary scalar fields \([\Phi(x,t), \Psi(x,t)]\) defined by:

\[
\begin{align*}
\Phi(x,t) &= e^{-\left[ \frac{u(x,t)-\epsilon t}{\mu \sigma^2} \right]}, \\
\Psi(x,t) &= e^{+\left[ \frac{u(x,t)-\epsilon t}{\mu \sigma^2} \right]} \rho(x,t)
\end{align*}
\]

and hence \( \rho(x,t) = \Phi(x,t)\Psi(x,t) \). In terms of \([\Phi(x,t), \Psi(x,t)]\), Eq. (12) can be rewritten as, (see Appendix):

\[
\begin{align*}
\epsilon \Phi(x,t) - \mu \sigma^2 \partial_t \Phi(x,t) &= +\mu \sigma^2 b \partial_x \Phi(x,t) + \frac{\mu \sigma^4}{2} \partial_{xx} \Phi(x,t) + g [\Phi(x,t)\Psi(x,t)]^a \Phi(x,t), \\
\epsilon \Psi(x,t) + \mu \sigma^2 \partial_t \Psi(x,t) &= -\mu \sigma^2 b \partial_x \Psi(x,t) + \frac{\mu \sigma^4}{2} \partial_{xx} \Psi(y,t) + g [\Phi(x,t)\Psi(x,t)]^a \Psi(x,t).
\end{align*}
\]

Introduce the Galilean frame of coordinates \((\tau, \xi)\) defined by:

\[
t \mapsto \tau, \quad x \mapsto \xi = x - bt \quad \Rightarrow \quad \partial_t \mapsto \partial_{\tau} - b \partial_{\xi}, \quad \partial_x \mapsto \partial_{\xi}
\]

implying that Eq. (16) takes the form:

\[
\begin{align*}
\epsilon \Phi(\xi,\tau) - \mu \sigma^2 \partial_{\tau} \Phi(\xi,\tau) &= \frac{\mu \sigma^4}{2} \partial_{\xi\xi} \Phi(\xi,\tau) + g [\Phi(\xi,\tau)\Psi(\xi,\tau)]^a \Phi(\xi,\tau), \\
\epsilon \Psi(\xi,\tau) + \mu \sigma^2 \partial_{\tau} \Psi(\xi,\tau) &= \frac{\mu \sigma^4}{2} \partial_{\xi\xi} \Psi(\xi,\tau) + g [\Phi(\xi,\tau)\Psi(\xi,\tau)]^a \Psi(\xi,\tau).
\end{align*}
\]

In the stationary regime reached when \( \partial_{\tau} \Phi(\xi,\tau) = \partial_{\tau} \Psi(\xi,\tau) = 0 \), the resulting nonlinear ODE’s for \( \Psi(\xi) \) and \( \Phi(\xi) \) coincide and are formally similar to a nonlinear Schrödinger equation [15]. Imposing therefore that \( \Psi(\xi) = \Phi(\xi) \) and integrating the \( \tau \)-independent version of Eq. (18) by separation of variables, we obtain:

\[
\int \frac{d\Phi(\xi)}{\sqrt{A_1(\epsilon)\Phi^2(\xi) - A_2(\mu)\Phi^{2a+2}(\xi)}} = \xi, \quad A_1(\epsilon) = \frac{2\epsilon}{\mu \sigma^4} \quad \text{and} \quad A_2(\mu) = \frac{2g}{(a+1)\mu \sigma^4}.
\]

Using the identity \( \cosh^2(x) - 1 = \sinh^2(x) \), it is straightforward to verify that
provided we have:

\[
\begin{align*}
A_1(\epsilon) a^2 &= 1 \quad \Rightarrow \quad \epsilon = \frac{1}{2} \mu \sigma^4 a^2, \\
A_2(g) a^2 &\mathcal{N}(a)^{2a} = 1, \quad \Rightarrow \quad g = \frac{\mu \sigma^{4(a+1)} |\mathcal{N}(a)|^{-2a}}{2\pi},
\end{align*}
\]

the normalised solution of Eq. (19) reads as:

\[
\Phi(\xi) = \Phi(x - bt) = \frac{\mathcal{N}(a)}{\cosh^{1/a}(x - bt)},
\]

(20)

with the normalisation factor \(\mathcal{N}(a)\) is given by [13]:

\[
\mathcal{N}(a)^{-1} = \int_{\mathbb{R}} \frac{d\xi}{[\cosh(\xi)]^{1/a}} = B(1/2, 1/2a).
\]

End of the proof

**Remark 2.** By identifying \(\omega = b\) and \(a^{-1} = (2 + \eta) > 0\), we see that both solitons arising in Eqs.(7) and (20) are identical. Therefore, one can directly assert that the exogenously given ABL algorithm generates a collective behaviour which coincides with the optimal ergodic equilibrium of the MFG with running cost function \(\mathcal{L}[u, \rho(x, t)]\) given by Eq.(10). Large \(a\) parameters in Eq.(10) describe MFG situations where interaction costs are confined to close neighbours and hence relatively small corresponding parameters \(\eta\). Conversely small \(a\)'s lead to MFG with widely spread interactions to which corresponds large \(\eta\)'s in the exogenously defined ABL rule.

### 3 Piecewise evolution dynamics

In this section, the exposition delivered in section 2 will be repeated for environments driven by two states Markov chains in continuous time (i.e. telegraphic process) [17]. Instead of the diffusion dynamics given by Eq.(1), we shall here consider a set of discrete two velocity Boltzman’s equation, (i.e. also known as the Ruijgrok-Wu dynamics (RW) [18]) with random Poisson switchings between the two velocity states. Specifically, one considers a set of \(N\) agents evolving on \(\mathbb{R}\) evolving with velocities either \(-1\) or \(+1\). Driven by a couple of Poisson processes with rates \(u_\pm \geq 0\), the agents’ velocities spontaneously switch from \(-1\) to \(+1\) (respectively \(+1\) to \(-1\)) velocity states. In addition to the spontaneous switchings, a Boltzman nonlinear collision term implies that when a pair of particles with velocities \((-1, +1)\) collide, it emerges with a given rate a \((+1, +1)\) pair. In the sequel, we shall assume that the \((-1, +1)\) collision rate is itself modulated by the instantaneous configuration of the swarm of particles. For \(x \in \mathbb{R}\) and time \(t \in \mathbb{R}^+\), \(P(x, t)dx\) (respectively \(Q(x, t)dx\)) stand for the proportion of agents with velocities \(+1\) (respectively \(-1\)) located in \([x, x + dx]\). Instead of the Fokker-Planck Eq.(3), the corresponding dynamics now reads as a generalised version of the velocity Boltzman’s equation:
For the dynamics defined by Eq. (21) with the choices

\[
\dot{P}(x,t) + \partial_x P(x,t) = -u_+ P(x,t) + u_- Q(x,t) + \Omega_t[P(x,t),Q(x,t)],
\]

\[
\dot{Q}(x,t) - \partial_x P(x,t) = +u_+ P(x,t) - u_- Q(x,t) - \Omega_t[P(x,t),Q(x,t)],
\]

\[
\Omega_t[P(x,t),Q(x,t)] = \alpha P(x,t) \int_x^\infty G[y - \langle X(t) \rangle] Q(y,t) dy + \beta Q(x,t) \int_x^\infty G[y - \langle X(t) \rangle] P(y,t) dy,
\]

(21)

with \(u_-, u_+, \alpha, \beta \in \mathbb{R}^+\) and initial conditions \(P_0(x)\) and \(Q_0(x)\). Associated with Eq. (21), we have the normalisation constraint and the definition:

\[
\int_{\mathbb{R}} [P(x,t) + Q(x,t)] dx \equiv 1, \quad \text{(normalisation of the total probability mass)}
\]

\[
\langle X(t) \rangle := \int_{\mathbb{R}} x[P(x,t) + Q(x,t)] dx, \quad \text{(barycenter location, of the total probability mass)}
\]

(22)

For the nonlinear dynamics Eq. (21), we will now establish:

**Proposition 3**

For the dynamics defined by Eq. (21) with the choices \(\omega \in [0,1], \eta \in \mathbb{R} \geq -2, +\infty\), the couple of positive switching rates \(u_+ > 0, u_- > 0\) solving:

\[
\frac{u_+}{(1 - \omega)} - \frac{u_-}{(1 + \omega)} = 2 + \eta
\]

(23)

and with the barycentre weight function:

\[
G(x) \equiv G_{a,b,\eta}(x) = \frac{(2 + \eta)}{2(a + b)} B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right) \cosh(\eta x),
\]

Eq. (21) is solved by the soliton waves:

\[
P(x - \omega t) = \frac{Q(x - \omega t)}{1 + \omega} = \frac{1}{2} B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right) [\cosh(x - \omega t)]^{-(2 + \eta)}.
\]

(24)

**Proof of Proposition 3**

We introduce the change of variables \(t \mapsto \tau\) and \(x \mapsto \xi = (x - \omega t)\) and we focus on the stationary regime \(\partial_\tau P(\xi, \tau) = \partial_\tau Q(\xi, \tau) = 0\). We assume the symmetry \(P(\xi) = P(-\xi)\) and \(Q(\xi) = Q(-\xi)\) and so Eq. (21) implies:

\[
\langle X(t) \rangle = \int_{\mathbb{R}} (\xi + \omega t) [P(\xi) + Q(\xi)] d\xi = \omega t.
\]

In the stationary regime, Eq. (21) can be rewritten as:

\[
(1 - \omega) \partial_\xi P(\xi) = (1 + \omega) \partial_\xi Q(\xi) = \alpha P(\xi) \int_\xi^\infty G_{a,b,\eta}(y) Q(y) dy + \beta Q(\xi) \int_\xi^\infty G_{a,b,\eta}(y) P(y) dy - u_+ P(\xi) + u_- Q(\xi).
\]

(25)

Introducing the rescaling factors:
\[
\begin{align*}
P(\xi) := (1 + \omega) \hat{P}(\xi) \quad \text{and} \quad Q(\xi) := (1 - \omega) \hat{Q}(\xi), \\
u_+ := \hat{u}_+(1 - \omega) \quad \text{and} \quad u_- := \hat{u}_-(1 + \omega),
\end{align*}
\]
we can rewrite Eq. (26) as:
\[
\partial_t \hat{P}(\xi) = \partial_{\xi} \hat{Q}(\xi) = -\hat{u}_+ \hat{P}(\xi) + \hat{u}_- \hat{Q}(\xi) + a \hat{P}(\xi) \int_{\xi}^{\infty} G(a, b, \eta)(y) \hat{Q}(y) dy + b \hat{Q}(\xi) \int_{\xi}^{\infty} G(a, b, \eta)(y) \hat{P}(y) dy.
\]
In view of Eqs. (24) and (27), we now assume that \( G_{a,b,\eta}(\xi) = A(a, b, \eta) \cosh(\xi) \) and \( \hat{P}(\xi) = \hat{Q}(\xi) = N(\eta) \cosh^{-2+\eta}(\xi) \). and a direct substitution into Eq. (27) yields:
\[
-\hat{N}(\eta)(2 + \eta) [\cosh(\xi)]^{-(\eta+3)} \sinh(\xi) = [\hat{u}_- - \hat{u}_+] \hat{N}(\eta) [\cosh(\xi)]^{-(2+\eta)} + 
\hat{N}^2(\eta)(a + b) [\cosh(\xi)]^{-(2+\eta)} \int_{\xi}^{\infty} \frac{A(a, b, \eta)}{[\cosh(\xi)]^{2+\eta}} d\xi.
\]
By direct identification, one concludes that one has to fulfil:
\[
\begin{align*}
2 + \eta &= A(a, b, \eta)(a + b) \hat{N}(\eta) \quad \Rightarrow \quad A(a, b, \eta) = \frac{2+\eta}{(a+b) \hat{N}(\eta)}, \\
[\hat{u}_- - \hat{u}_+] + A_{a,b,\eta}(a + b) \hat{N}(\eta) &= 0 \quad \Rightarrow \quad [\hat{u}_+ - \hat{u}_-] = 2 + \eta.
\end{align*}
\]
Finally, the first line in Eq. (26) implies:
\[
2\mathcal{N}^{-1}(\eta) = \int_R (1 + \omega) \frac{d\xi}{[\cosh(\xi)]^{2+\eta}} + \int_R (1 - \omega) \frac{d\xi}{[\cosh(\xi)]^{2+\eta}} = B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right).
\]
\textbf{End of the Proof}

\textbf{Remark 3.} Here again we observe that a normalised soliton cannot be generated for modulation kernels \( G(z) \propto \cosh^\eta(z) \) for \( \eta < -2 \) and therefore, similarly to section 2, one concludes that \( \eta = 2 \) is the bifurcation threshold separating two drastically swarm propagation modes.

### 3.1 Corresponding mean-field game dynamics for piecewise deterministic evolutions

Following the development followed in section 2 where Brownian motion environments drive the dynamics, we now construct a MFG dynamics with telegraphic noise driving which leads to ergodic regimes similar to the soliton found in Proposition 3. To this aim, consider the controllable piecewise evolution dynamics:
\[
\begin{align*}
\partial_t P(x, t) + \partial_x P(x, t) &= -u_+(x, t) P(x, t) + u_-(x, t) Q(x, t), \\
\partial_t Q(x, t) - \partial_x Q(x, t) &= +u_+(x, t) P(x, t) - u_-(x, t) Q(x, t),
\end{align*}
\]
which differs from Eq. (24) by the fact that the Poisson switching rates \( u_-(x, t) \) and \( u_+(x, t) \) are now explicitly \( (x, t) \)-dependent. For a time horizon \( t \in [0, T] \), let us introduce a couple of cost functions \( \mathcal{J}_\pm \) in the form discussed in [19]:
\[
\begin{align*}
&\mathcal{J}_\pm [X(\cdot), u_\pm(\cdot)] = \mathbb{E} \left\{ \int_0^T \{ \mathcal{L}(u_\pm(X(s), s) + \mathcal{W}[P(\cdot), s), Q(\cdot), X(s), s)] ds \right\} + C_{\pm,T}(X(T)) \\
\mathcal{L}(u_\pm(x, t), t) &:= u_\pm(x, t) \ln [u_\pm(x, t)] - u_\pm(x, t) + 1
\end{align*}
\]
where \( C_T(X(T)) \) stands for a final cost. In Eq. (31), the running cost \( W[P(x, t), Q(x, t)] \) depends only on the probability densities \( P(x, t) \) and \( Q(x, t) \). This functional structure conveys to the dynamics its MFG character. The objective is now to minimise the global costs \( J_\pm [\cdot, u_\pm (\cdot)] \) by optimally adjusting the switching rates \( u_\pm (x, t) \). Invoking the dynamic programming (DP) principle, we may now derive the associated Hamilton-Belman-Jacobi (HBJ) equation and for the resulting couple of value functions \( V_\pm (x, t) \), we obtain:

\[
\begin{align*}
\partial_t V_+(x, t) + \partial_x V_+(x, t) + \min_{u_+} \{ L(u_+(x, t), t) + u_+ [V_-(x, t) - V_+(x, t)] \} + W(P(x, t), Q(x, t)) &= 0, \\
\partial_t V_-(x, t) - \partial_x V_-(x, t) + \min_{u_-} \{ L(u_-(x, t), t) + u_- [V_+(x, t) - V_-(x, t)] \} + W(P(x, t), Q(x, t)) &= 0.
\end{align*}
\]

Performing the required minimisations, Eq. (32) becomes [19]:

\[
\begin{align*}
\partial_t V_+(x, t) + \partial_x V_+(x, t) + [1 - e^{V_+(x, t) - V_-(x, t)}] + W(P(x, t), Q(x, t)) &= 0, \\
\partial_t V_-(x, t) - \partial_x V_-(x, t) + [1 - e^{V_-(x, t) - V_+(x, t)}] + W(P(x, t), Q(x, t)) &= 0.
\end{align*}
\]

and the optimal switching rates \( u_+^*(x, t) \) and \( u_-^*(x, t) \) are given by:

\[
\begin{align*}
u_+^*(x, t) &= e^{V_+(x, t) - V_-(x, t)}, \\
u_-^*(x, t) &= e^{V_-(x, t) - V_+(x, t)}.
\end{align*}
\]

**Proposition 4.**

Given \( \omega \in [0, 1] \) and with the cost in Eq. (37) defined as:

\[
\begin{align*}
W(P, Q) &= g(q, \omega) |PQ|^{q}, & q > 0, \\
g(q, \omega) &= \frac{(q+1)(1+\omega^2)^{(2q-2)}}{4q(2q-2)^q} \left[ B \left( \frac{1}{2}, \frac{1}{2q} \right) \right]^{2q},
\end{align*}
\]

the probability densities \( P(x, t) \) and \( Q(x, t) \) solving the ergodic regime of the MFG Eq. (35) are given by the soliton waves:

\[
\begin{align*}
(1 - \omega)P(x - \omega t) &= (1 + \omega)Q(x + \omega t) = \frac{\hat{N}(q)}{\cosh(x - \omega t)}^{1/\gamma}, \\
\left[ \hat{N}(q) \right]^{-1} &= \frac{(1 - \omega^2)}{2} B \left( \frac{1}{2}, \frac{1}{2q} \right).
\end{align*}
\]

**Proof of Proposition 4.**

Let us introduce the following transformations:

\[
\begin{align*}
\varphi_A &= e^{-\epsilon t - V_+} \quad \text{and} \quad \Gamma_A = Pe^{\epsilon t + V_+} \Rightarrow P = \Gamma_A \varphi_A, \\
\varphi_B &= e^{-\epsilon t - V_-} \quad \text{and} \quad \Gamma_B = Qe^{\epsilon t + V_-} \Rightarrow Q = \Gamma_B \varphi_B,
\end{align*}
\]

with \( \epsilon \in \mathbb{R}^+ \). From Eqs. (35) and (37), we have:

\[
\begin{align*}
u_+^* &= \frac{\varphi_B}{\varphi_A}, \quad \text{and} \quad u_-^* = \frac{\varphi_A}{\varphi_B}.
\end{align*}
\]

Using Eq. (37) and substituting the definitions of \( \varphi_A \) and \( \varphi_B \) into Eq. (32), we obtain:

\[
\begin{align*}
\partial_t [\varphi_A] + \partial_x [\varphi_A] - \varphi_A + \varphi_B - |W - \epsilon| \varphi_A &= 0, \\
\partial_t [\varphi_B] - \partial_x [\varphi_B] + \varphi_A - \varphi_B - |W - \epsilon| \varphi_B &= 0.
\end{align*}
\]

\(^1\)In the sequel, for simplicity of the notation, we shall omit to repeat the ubiquitous \((x, t)\) argument.
Introducing \(u^*_\xi(x,t)\) and \(u^*_\tau(x,t)\) given by Eq. (31) into Eq. (30) and using once more Eq. (37), we end with:

\[
\begin{aligned}
\begin{cases}
\partial_t[\Gamma_A] + \partial_x[\Gamma_A] + \Gamma_A - \Gamma_B + [W - \epsilon] \Gamma_A = 0, & \text{iii)} \\
\partial_t[\Gamma_B] - \partial_x[\Gamma_B] - \Gamma_A + \Gamma_B + [W - \epsilon] \Gamma_B = 0. & \text{iv)}
\end{cases}
\end{aligned}
\]  
(40)

In view of Eqs. (39) and (40) and the set of definitions introduced in Eq. (37), we can derive:

\[
\begin{aligned}
\begin{cases}
[i] \times \Gamma_A & + [iii] \times \varphi_A \Rightarrow \partial_t P + \partial_x P + \{\Gamma_A \varphi_B - \varphi_A \Gamma_B\} = 0, \\
[ii] \times \Gamma_B & + [iv] \times \varphi_B \Rightarrow \partial_t Q - \partial_x Q - \{\Gamma_A \varphi_B - \varphi_A \Gamma_B\} = 0.
\end{cases}
\end{aligned}
\]  
(41)

Performing the change of variables:

\[
\begin{aligned}
& t \mapsto \tau, \quad x \mapsto \xi = (x - \omega t), \\
& \partial_t \mapsto \partial_\tau - \omega \partial_\xi, \quad \partial_x \mapsto \partial_\tau,
\end{aligned}
\]  
(42)

enables to rewrite Eq. (11) as:

\[
\begin{aligned}
\frac{1}{(1 + \omega)} & \partial_\tau \hat{P} + \partial_\xi \hat{P} + \{\Gamma_A \varphi_B - \varphi_A \Gamma_B\} = 0, \\
\frac{1}{(1 + \omega)} & \partial_\tau \hat{Q} - \partial_\xi \hat{Q} - \{\Gamma_A \varphi_B - \varphi_A \Gamma_B\} = 0.
\end{aligned}
\]  
(43)

Let us now focus on stationary regimes for which \(\partial_\tau \hat{P} = \partial_\tau \hat{Q} = 0\). Using Eq. (43) and since normalisation imposes that \(\lim_{\xi \to \infty} \hat{P}(\xi) = 0\) and \(\lim_{\xi \to \infty} \hat{Q}(\xi) = 0\), we have \(\hat{P}(\xi) = \hat{Q}(\xi)\) and therefore:

\[
\hat{P}(\xi) = (1 - \omega)P(\xi) = (1 - \omega)\varphi_A(\xi)\Gamma_A(\xi) = \hat{Q}(\xi) = (1 + \omega)Q(\xi) = (1 + \omega)\varphi_B(\xi)\Gamma_B(\xi),
\]  
(44)

Since \(\hat{P}(\xi) = \hat{Q}(\xi)\), the first two lines in Eq. (43) imply that we can write:

\[
\partial_\xi \hat{P} + \partial_\xi \{\Gamma_A \varphi_B - \varphi_A \Gamma_B\} = 0.
\]  
(45)

In the stationary regime, Eqs. (39) and (40) enable to write straightforwardly the following combinations:

\[
\begin{aligned}
\varphi_B \partial_\xi \Gamma_A & = - \varphi_B \Gamma_A \frac{\Gamma_B}{(1 + \omega)} - \varphi_B \Gamma_B \frac{\Gamma_A}{(1 + \omega)} - \varphi_B \Gamma_B (W - \epsilon) \frac{\Gamma_A}{(1 + \omega)}, \\
\Gamma_A \partial_\xi \varphi_B & = + \varphi_A \Gamma_B \frac{\Gamma_A}{(1 + \omega)} - \varphi_A \Gamma_A \frac{\Gamma_B}{(1 + \omega)} - \varphi_A \Gamma_A (W - \epsilon) \frac{\Gamma_B}{(1 + \omega)}, \\
\Gamma_B \partial_\xi \varphi_A & = + \varphi_A \Gamma_B \frac{\Gamma_B}{(1 + \omega)} - \varphi_A \Gamma_B \frac{\Gamma_A}{(1 + \omega)} + \varphi_A \Gamma_B (W - \epsilon) \frac{\Gamma_B}{(1 + \omega)}, \\
\varphi_A \partial_\xi \Gamma_B & = - \varphi_B \Gamma_A \frac{\Gamma_B}{(1 + \omega)} + \varphi_B \Gamma_A \frac{\Gamma_B}{(1 + \omega)} + \varphi_B \Gamma_B (W - \epsilon) \frac{\Gamma_B}{(1 + \omega)}.
\end{aligned}
\]

This enables to write:

\[
\partial_x (\Gamma_B \varphi_A - \Gamma_A \varphi_B) = - \frac{2}{(1 - \omega)} \left\{ (\hat{P} + \hat{Q}) + (\epsilon - 1 - W) (\varphi_A \Gamma_B + \varphi_B \Gamma_A) \right\}.
\]  
(46)

Consistent with Eqs. (37) and (44), we now assume that:

\[
\varphi_B = \varphi_A (1 - \omega) \quad \text{and} \quad \Gamma_B = \Gamma_A \frac{1}{(1 + \omega)} \Rightarrow (\varphi_A \Gamma_B + \varphi_B \Gamma_A) = \left( \frac{2 - \omega^2}{1 - \omega^2} \right) \hat{P}.
\]  
(47)
Since $\dot{P}(\xi) = \dot{Q}(\xi)$, Eqs. (45), (46) and (47) imply:

$$\partial_{\xi} \dot{P}(\xi) = \frac{2}{(1-\omega^2)} \left[ 2 + \left( \epsilon - 1 - \mathcal{W}(\dot{P}, \dot{Q}) \right) \left( \frac{2}{1-\omega^2} \right) \right] \dot{P}(\xi)$$

(48)

We now focus on the set of running cost functions:

$$\mathcal{W}(P, Q) = \mathcal{W}(PQ) = \mathcal{W}\left( \frac{\dot{P}Q}{1-\omega^2} \right) := g \left[ \dot{P}(\xi) \right]^{2q}, \quad g, q \in \mathbb{R}^+.$$  

(49)

It is immediate to realise that Eq. (48) exhibits the standard form of the nonlinear Schrödinger equation:

$$\partial_{\xi} \dot{P}(\xi) = 2 \left[ \frac{2(2-\omega^2)-\omega^2}{(1-\omega^2)^2} \right] \dot{P}(\xi) - g \left( \frac{2-\omega^2}{1-\omega^2} \right) \left[ \dot{P}(\xi) \right]^{2q+1}.$$ 

(50)

Provided appropriate constants $g, q, \epsilon, \omega$ are chosen, Eq. (50) can be integrated to yield a soliton which coincides with the one found in Eq. (24). To see this, multiply both sides of Eq. (48) by $\dot{P}$ and integrate once with respect to $\xi$ (with zero integration constant), we obtain:

$$\left( \partial_{\xi} \dot{P}(\xi) \right)^2 = 2 \left[ \frac{2(2-\omega^2)-2}{(1-\omega^2)^2} \right] \dot{P}(\xi)^2 - g \left( \frac{2-\omega^2}{q+1(1-\omega^2)^2} \right) \dot{P}(\xi)^{2q+2} = 0.$$  

(51)

Using once again the separation of variable technique, Eq. (51) leads to:

$$\begin{cases} 
\int \frac{d\dot{P}(\xi)}{\sqrt{A_1(\epsilon, \omega)P^2(\xi) - A_2(g, \omega, q)\dot{P}(\xi)^{2q+2}}} = \xi, \\
A_1(\epsilon, \omega) := 2 \frac{[(2-\omega^2)-\omega^2]}{(1-\omega^2)^2}, \\
A_2(g, \omega, q) := \frac{g(2-\omega^2)}{(q+1)(1-\omega^2)^2}. 
\end{cases}$$  

(52)

Now we can verify that the Ansatz $\dot{P}(\xi) = \frac{\mathcal{N}(q)}{\left( \cosh(\xi) \right)^{1/q}}$ with $q > 0$ solves Eq. (52) provided we impose:

$$\begin{cases} 
1 = A_1(\epsilon, \omega)q^2 \quad \Rightarrow \quad \epsilon = \frac{1}{2-\omega^2} \left( \frac{1-\omega^2}{2q} + \omega^2 \right) \\
1 = A_2(g, \omega, q)q^2 \left( \mathcal{N}(q) \right)^{2q}. 
\end{cases}$$  

(53)

So given the couple constants $g > 0$ and $\omega \in [0, 1]$, we choose $g$ to satisfy the last line of Eq. (53) and then $\epsilon$ follows. Since $\dot{P}(\xi) = \dot{Q}(\xi)$, the normalisation factor $\mathcal{N}(q)$ is given by:

$$\int_{\mathbb{R}} [P(\xi) + Q(\xi)] \, d\xi = 1 \Rightarrow \left[ \mathcal{N}(q) \right]^{-1} = \frac{2}{(1-\omega^2)} \int_{\mathbb{R}} \frac{d\xi}{\left( \cosh(\xi) \right)^{1/q}} = \frac{2}{(1-\omega^2)} B \left( \frac{1}{2}, \frac{1}{2q} \right).$$

End of the proof

Remark 4. Fixing $\omega$ as given by Eq. (23) and choosing $1/q = (2 + \eta) > 0$, once more one observes that the solitons given in Eqs. (24) and (36) coincide. Hence under telegraphic noise environments, the soliton generated by exogenous interaction rule Eq. (6) coincide, in the ergodic regime, with the optimal solution of a MFG with an appropriate choice of the running cost function.

Remark 5. As shown in [17] (see Chapter 9), the WGN can be derived from the telegrapher’s process by an ad-hoc rescaling of the switching rates and the jumps sizes. Exploiting this observation, several contributions [20, 21, 22] show how the parabolic Burgers’ equation describing the diffusive dynamics of section 2 coincides, via an an ad-hoc limiting procedure, with the hyperbolic
discrete velocity Boltzmann equation which describes the piecewise deterministic dynamics used of section 3. In other words the RW dynamics [17] is a generalisation of the Burgers’s equation. Along the same lines, the assertions made in our present Propositions 3 and 4 are generalisations of the assertions made in Propositions 1 and 2.

4 Conclusion

To a stationary collective motions sustained by exogenously defined interactions’ rules, it corresponds a mean-field games (MFG) with optimal stationary equilibria yielding the same collective evolution. This parallel is analytically exemplified here in situations where the driving stochastic environment is either a Brownian motion or a two-states Markov chains in continuous time. For long-range interacting scalar agents each evolving on the real line, we explicitly show the existence of bifurcation threshold separating two drastically different swarm propagation modes: one either observes a stable soliton or a diffusive evanescent wave. As in the Kuramoto’s dynamics for agents evolving on a compact state space, the transition is due to the competition between a synchronisation effect due to the mutual interactions and the desynchronisation due to the random environment. Since our proposed models are exactly solvable, they hopefully enrich the yet rather scarce collection of fully solvable models relevant for multi-agents dynamics.

Appendix

This goal of this appendix is to derive the second line in Eq. (16). Using the fact that $\rho = \Phi \Psi$ and $u = -\mu \sigma^2 \ln \Phi - \epsilon t$, the FPE Eq. (12) reads:

$$ (\partial_t \Phi) \Psi + \Phi (\partial_t \Psi) = \partial_x \left\{ \frac{1}{\mu} \left[ -\mu \sigma^2 \frac{\partial_x \Phi}{\Phi} \right] \Phi \Psi - b \Phi \Psi \right\} + \frac{\sigma^2}{2} \Psi \partial_{xx} \Phi + \sigma^2 (\partial_x \Phi)(\partial_x \Psi) + \frac{\sigma^2}{2} \Phi \partial_{xx} \Psi $$

or equivalently:

$$ (\partial_t \Phi) \Psi + \Phi (\partial_t \Psi) = \partial_x \left\{ -b \Phi \Psi \right\} - \frac{\sigma^2}{2} \Psi \partial_{xx} \Phi + \frac{\sigma^2}{2} \Phi \partial_{xx} \Psi $$

This can be rewritten as:

$$ \Psi \left\{ \partial_t \Phi + b \partial_x \Phi + \frac{\sigma^2}{2} \partial_{xx} \Phi \right\} = -\Phi \left\{ \partial_t \Psi + b \partial_x \Psi - \frac{\sigma^2}{2} \partial_{xx} \Psi \right\}. $$

(54)

The first line in Eq. (16) implies:

$$ \left\{ \partial_t \Phi + b \partial_x \Phi + \frac{\sigma^2}{2} \partial_{xx} \Phi \right\} = \frac{1}{\mu \sigma^2} \left\{ \epsilon \Phi - g [\Phi \Psi]^a \Phi \right\}. $$

Hence Eq. (54) takes the form:

$$ \Psi \frac{1}{\mu \sigma^2} \left\{ \epsilon \Phi - g [\Phi \Psi]^a \Phi \right\} = -\Phi \left\{ \partial_t \Psi + b \partial_x \Psi - \frac{\sigma^2}{2} \partial_{xx} \Psi \right\}. $$

or dividing by $-\frac{\Phi}{\mu \sigma^2}$, we obtain:

$$ -\epsilon \Psi + g [\Phi \Psi]^a \Psi = \mu \sigma^2 \left\{ \partial_t \Psi + b \partial_x \Psi - \frac{\sigma^2}{2} \partial_{xx} \Psi \right\} $$

and hence the second line in Eq. (16) follows:

$$ \epsilon \Psi + \mu \sigma^2 \partial_t \Psi = -\mu \sigma^2 b \partial_x \Psi + \frac{\mu \sigma^2}{2} \partial_{xx} \Psi + g [\Phi \Psi]^a \Psi. $$

2All arguments of the scalar fields $\Phi(x, t)$ and $\Psi(x, t)$ are omitted on this Appendix.
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