COMPLEX PLANAR HAMILTONIAN SYSTEMS: LINEARIZATION AND DYNAMICS

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(Communicated by Kuo-Chang Chen)

Abstract. Global dynamics of complex planar Hamiltonian polynomial systems is difficult to be characterized. In this paper, for general complex quadratic Hamiltonian systems of one degree of freedom, we obtain some sufficient conditions on the existence of family of invariant tori. We also complete characterization on locally analytic linearizability of complex planar Hamiltonian systems with homogeneous nonlinearity of degrees either 2 or 3 at a nondegenerate singularity, and present their global dynamics. For these classes of systems we also prove existence of families of invariant tori, together with isochronous periodic orbits.

1. Introduction and statement of the main results. Consider the planar polynomial Hamiltonian system (real or complex)

\[ \dot{x} = -H_y(x, y), \quad \dot{y} = H_x(x, y), \]

where the dot denotes the derivative with respect to the time \( t \), and \( H \) is a polynomial in the variables \( x \) and \( y \). The degree of system (1) is the maximum degree of the polynomials \( H_x(x, y) \) and \( H_y(x, y) \). On real planar Hamiltonian systems there are lots of known results. Whereas, on complex planar Hamiltonian systems there are very few known results. Even through, the study on dynamics of complex planar Hamiltonian systems has not only its own theoretical importance but also many practical applications. Here we restrict our interesting to complex planar Hamiltonian systems of degree 2 or 3.

2020 Mathematics Subject Classification. Primary: 37J35, 37C10; Secondary: 37C27, 34C14.

Key words and phrases. Complex Hamiltonian system, Liouvillian integrability, linearization, global dynamics, invariant tori.

The third author is partially supported by NNSF of China grant numbers 11671254, 11871334 and 12071284, and also by Innovation Program of Shanghai Municipal Education Commission.

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For a general quadratic Hamiltonian system
\[ \dot{x} = -h_{11}x - 2h_{02}y - h_{21}x^2 - 2h_{12}xy - 3h_{03}y^2, \]
\[ \dot{y} = 2h_{20}x + h_{11}y + 3h_{30}x^2 + 2h_{21}xy + h_{12}y^2, \]  
with the Hamiltonian
\[ H(x, y) = h_{20}x^2 + h_{11}xy + h_{02}y^2 + h_{30}x^3 + h_{21}x^2y + h_{12}xy^2 + h_{03}y^3, \]  
where \( x, y \in \mathbb{C} \) are complex variables, and \( h_{20}, \ldots, h_{03} \in \mathbb{C} \) are complex coefficients, it is still an open problem to characterize their global dynamics. As our computation, even its singularities, except the origin, have extremely complicated expressions, which cause difficulty in investigating local dynamics of the system.

The next result presents a normal form to the linear part of system (2) at the origin.

**Proposition 1.** Assume that the origin of system (2) is non-degenerate. Then the second order term of its associated Hamiltonian \( H \) can be transformed, via a symplectic linear change of coordinates, to the normal form \( \nu uv \) of system (2) at the origin, where \( \nu = \sqrt{h_{11}^2 - 4h_{20}h_{02}} \neq 0 \) is an eigenvalue of system (2) at the origin.

Proposition 1 is useful in studying local dynamics of system (2) at the origin, and its locally analytic linearizability when the origin is nondegenerate. Recall that a singularity is nondegenerate if its eigenvalues do not vanish.

By Proposition 1, in order to study local and global dynamics of system (2) at the nondegenerate singularity, i.e. the origin, we only need to consider the next Hamiltonian
\[ H(x, y) = \nu xy + ax^3 + bx^2y + cxy^2 + dy^3, \]  
and its associated Hamiltonian system
\[ \dot{x} = -\nu x - bx^2 - 2cxy - 3dy^2, \]
\[ \dot{y} = \nu y + 3ax^2 + 2bxy + cy^2, \]  
where \( a, b, c, d, \nu \) are complex coefficients and \( \nu \neq 0 \).

Our first main result provides existence of invariant tori of system (4).

**Theorem 1.1.** For system (4) with \( \text{Re} \nu = 0 \), the following statements hold.

\( (a) \) The system has a family of invariant tori near the origin, which are fulfilled by periodic orbits.

\( (b) \) If \( a = 0 \) and \( b \neq 0 \), the system has two families of invariant tori, each of which surrounds a singularity. The invariant tori are fulfilled by periodic orbits.

We remark that under some other conditions on system (1.1), for instance \( d = 0 \) and \( c \neq 0 \), we can also get some results similar to those of Theorem 1.1. The details are omitted.

As shown in the following proofs, system (4) can be written in a two degrees of freedom real integrable Hamiltonian system in the Liouvillian sense. As our knowledge, for real four dimensional integrable Hamiltonian systems their dynamics at singularities are characterized only for simple singularities. See for example Arnold [2, p. 35] and Lerman and Umanskiy [20, Proposition 2.1.1] for linear systems, and Lerman and Umanskiy [20, Propositions 4.1.1 and 4.1.2] for nonlinear systems in normal form in the center–center case. Of course, for analytic or \( C^\infty \) smooth normal forms there are lots of results, see example Bolsinov and Fomenko
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[5], Eliasson [14], Ito [18], Rüssmann [27], Vũ Ngoc [29] and Zung [33] et al. Recall that a simple singularity is the one with different eigenvalues, see e.g. [20, 29]. For real four dimensional Hamiltonian systems (not necessary integrable) with non–simple singularity, using the fourth order Birkhoff normal form Arnold et al [3, pages 413–415] characterized the local phase portraits in two parameters.

Next we further study locally analytic linearizability of system (4) at the origin. For analytic (real or complex) differential systems, the study on existence of their locally analytic linearization at a singularity has a long history, which can be traced back to Poincaré. This problem has its own interesting in characterizing local dynamics of analytic differential systems around a singularity.

For a real planar locally analytic differential system having a singularity with a pair of pure imaginary eigenvalues, as shown by Poincaré the existence of a locally analytic linearization of the system at the singularity is equivalent to the existence of an isochronous center at the singularity. For precise, a singularity having a pair of pure imaginary eigenvalues of a planar analytic differential system is an isochronous center if and only if the system is linearizable at this singularity via a near identity analytic transformation. Recall that a center is isochronous if all the periodic orbits around it have the same period. As we know, for a concrete analytic differential system (even a polynomial differential system) it is difficult to classify if it is analytically linearizable at its singularity. As we know, there are a few families of polynomial differential systems, whose isochronous centers were characterized. Loud [24] and Jarque and Villadelprat [19] proved respectively that the centers of real planar polynomial systems of degrees 2 and 4 are not isochronous, and consequently it is not analytically linearizable. Christopher and Devlin [11] gave two algorithms to find conditions for a singularity to be an isochronous center. Chavarriaga et al [7, 8, 9] did a series works on isochronous center. For more information, see also [6, 10, 12, 13, 22, 25] for the progress on characterization of isochronous centers and their associated linearizations. Except the relation with isochronous center, linearization of a differential systems at a singularity has its own significance in the study of dynamics of a given differential system. On global linearization of completely integrable systems, there are also some known results, see Giné and Llibre [17] and Llibre et al [23].

When system (1) is real and has an elementary center at the origin, it can be written in (possibly with an invertible real linear change of coordinates and a time resecaling)

\[
\dot{x} = -y - h_y(x, y), \quad \dot{y} = x + h_x(x, y),
\]

with the Hamiltonian \(H(x, y) = \frac{1}{2}(x^2 + y^2) + h(x, y)\), and \(h(x, y)\) without constant and linear terms. Note that in the complex conjugate coordinates \(u = x + iy, v = x - iy\) with \(i = \sqrt{-1}\), the Hamiltonian \(H(x, y)\) can be expressed as

\[
H = \frac{1}{2}uv + \text{higher order term}.
\]  

Under this consideration Llibre and Romanovski [22] studied linearization of the complex Hamiltonian system (1), whose Hamiltonian is of the form \(H(x, y) = -xy + h(x, y)\) with \(x, y \in \mathbb{C}\) and \(h(x, y)\) a complex polynomial without constant and linear terms. They obtained the necessary and sufficient conditions for the complex planar Hamiltonian system (1) with the Hamiltonian (5) to be linearizable at the origin when \(h\) is quadratic; and some sufficient conditions when \(h\) consists of quadratic and cubic homogeneous polynomials.
Now we state the results of Llibre and Romanovski [22, Theorem 3], which characterizes linearization of the complex planar quadratic Hamiltonian system (1) at its singularity with the Hamiltonian
\[ H = -xy + \frac{1}{3}b_{-1}x^3 + a_{10}x^2y + b_{01}xy^2 + \frac{1}{3}a_{-2}y^3. \]

**Theorem A.** The complex quadratic Hamiltonian system
\[
\begin{align*}
\dot{x} &= x - a_{10}x^2 - 2b_{01}xy - a_{-2}y^2, \\
\dot{y} &= -y + b_{-1}x^2 + 2a_{10}xy + b_{01}y^2,
\end{align*}
\]
is linearizable if and only if \( b_{-1} = a_{10} = 0 \) or \( b_{01} = a_{-2} = 0 \).

The aim of the remaining part of this section is to introduce our characterization on locally analytic linearizability of the mentioned Hamiltonian systems with homogeneous nonlinearity of degrees 2 and 3 respectively, and the classification of the global dynamics of these locally analytically linearizable systems.

Our next result provides the necessary and sufficient conditions for system (2) at the origin to be locally analytically linearizable, where there appears linearization quantities. By definition linearization quantities are the coefficients of the resonant monomials in the Birkhoff normal form of system (2) at the origin. Here we have used the fact that the Birkhoff normal form of a Hamiltonian system of one degree of freedom is a product of its linear part with an analytic function whose constant term is 1. Since the eigenvalues of system (2) are 1 : −1 resonant, the analytic function in the Birkhoff normal form consists of the resonant monomials of the form \((xy)^\ell\) with \( \ell \in \mathbb{N} \). The first, second and third, \ldots, linearization quantities are respectively the coefficients of \( xy, (xy)^2, (xy)^3, \ldots \) in this analytic function.

**Theorem 1.2.** For the Hamiltonian system (2) with the Hamiltonian (3), whose origin is nondegenerate, the following statements hold.

(a) The first three linearization quantities of system (2) at the origin are

\[
\begin{align*}
L_1 &= 10(h_{02}^3h_{03}^2 + h_{02}^3h_{30}^2 - 10h_{11}(h_{20}^2h_{12}h_{03} + h_{02}^3h_{30}h_{21})) \\
&\quad + 2(h_{02}h_{02} + h_{11}^2)(h_{20}(2h_{21}h_{03} + h_{12}^2) + h_{02}(h_{21}^2 + 2h_{30}h_{12})) \\
&\quad - h_{11}(6h_{20}h_{02} + h_{11}^2)(h_{30}h_{03} + h_{21}h_{12}), \\
L_2 &= -189(h_{02}^6h_{30} + h_{02}^6h_{03}) + 378h_{11}(h_{02}^5h_{30}h_{21} + h_{02}^5h_{12}h_{03}) \\
&\quad - 108(3h_{20}h_{02} + h_{11}^2)(h_{02}^4h_{30}h_{12} + h_{20}^2h_{30}h_{03}) \\
&\quad + 27(18h_{20}h_{02} - h_{11}^2)(h_{11}(h_{02}^2h_{30} + h_{02}^2h_{03})h_{30}h_{03}) \\
&\quad + 9(58h_{20}h_{02} + 17h_{11}^2)h_{11}(h_{02}^3h_{30} + h_{02}^3h_{03})h_{21}h_{12} \\
&\quad - 9(2h_{20}h_{02} + 31h_{11}^2)(h_{02}^4h_{30}h_{21} + h_{02}^4h_{12}h_{03}) \\
&\quad + 54h_{20}h_{11}h_{02}(17h_{20}h_{02} + h_{11}^2)(h_{02}h_{30}h_{12} + h_{20}h_{21}h_{03})h_{30}h_{03} \\
&\quad - 18(11h_{20}^2h_{02} + 9h_{20}^2h_{11}h_{02} + h_{11}^4)(h_{02}^2h_{30}h_{12} + h_{20}^2h_{21}h_{03}) \\
&\quad - 81h_{20}^2h_{11}^2(10h_{20}h_{02} + h_{11}^2)h_{02}^3h_{03} \\
&\quad - 27(12h_{20}^2h_{02}^2 + 22h_{20}h_{11}h_{02} - h_{11}^2)(h_{02}^2h_{30}h_{21} + h_{20}^2h_{12}h_{03})h_{30}h_{03} \\
&\quad + 18h_{11}(h_{20}h_{02} + 5h_{11}^2)(h_{02}^3h_{30}h_{21} + h_{02}^3h_{12}h_{03}) \\
&\quad + 6h_{11}(29h_{20}^2h_{02} + 32h_{20}h_{11}h_{02} + 2h_{11}^4)(h_{02}h_{30}h_{12} + h_{20}h_{21}h_{03})h_{21}h_{12}
\end{align*}
\]
Proposition 1 we only need to study the next systems generate singularity. For realizing this aim and avoiding tedious expressions, by systems with homogeneous nonlinearity of degree 3 at the origin, which is a nonde-

Theorem 1.3.
The Hamiltonian system

dynamics of the locally linearizable system (2) at the nondegenerate origin can be

or

cases:

Firstly we consider system (2). Since linearly symplectic change of coordinates

does not alter topological structure of the phase portrait of a polynomial differential

system, by Proposition 1 we consider only system (4). According to Theorem 1.2

does not alter topological structure of the phase portrait of a polynomial differential



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\[ L_4 = 9(h_{20}^3 h_{03}^3 + h_{02}^3 h_{30}^3) + h_{20} h_{02}(h_{21}^2 + 6h_{21} h_{03}) + h_{02}(h_{21} + 6h_{30} h_{12}) \]

modulo nonzero factors.

(b) System (2) is analytically linearizable at the origin if and only if their coefficients belong to \( \{ L_1 = 0 \} \cap \{ L_2 = 0 \} \cap \{ L_3 = 0 \} \).

Next we clarify locally analytic linearizability of complex planar differential systems with homogeneous nonlinearity of degree 3 at the origin, which is a nondegenerate singularity. For realizing this aim and avoiding tedious expressions, by Proposition 1 we only need to study the next systems

\[ \dot{x} = -h_{11} x - h_{31} x^3 - 2h_{22} x^2 y - 3h_{13} x y^2 - 4h_{04} y^3, \]
\[ \dot{y} = h_{11} y + 4h_{40} x^3 + 3h_{31} x^2 y + 2h_{22} x y^2 + h_{13} y^3, \]

with the Hamiltonian

\[ H_3 = h_{11} x y + h_{40} x^4 + h_{31} x^3 y + h_{22} x^2 y^2 + h_{13} x y^3 + h_{04} y^4. \]

Theorem 1.3. The Hamiltonian system (7) with the Hamiltonian (8) and \( h_{11} \neq 0 \) is analytically linearizable at the origin if and only if it is one of the following two cases:

(a) \( \dot{x} = -h_{11} x - 3h_{13} x y^2 - 4h_{04} y^3, \) \( \dot{y} = h_{11} y + h_{13} y^3, \) where either \( h_{13} \neq 0 \) or \( h_{31} = 0 \) and \( h_{40} \neq 0 \),

(b) \( \dot{x} = -h_{11} x - h_{13} x^3, \) \( \dot{y} = h_{11} y + 4h_{40} x^3 + 3h_{31} x^2 y, \) where either \( h_{31} \neq 0 \) or \( h_{31} = 0 \) and \( h_{40} \neq 0 \).

We remark that Theorem 1.3 is an improvement of [22, Theorem 4] for system (7), because our conditions contain theirs as a proper subset, and in this case we obtain the necessary and sufficient conditions for system (7) to be locally analytically linearizable at the origin.

Finally we study global dynamics of the locally linearizable systems (2) and (7).

Firstly we consider system (2). Since linearly symplectic change of coordinates does not alter topological structure of the phase portrait of a polynomial differential system, by Proposition 1 we consider only system (4). According to Theorem 1.2 and [22, Theorem 3], system (4) is linearizable at the origin if and only if \( a = b = 0 \), or \( c = d = 0 \). These two conditions are symmetry for system (4). So the global dynamics of the locally linearizable system (2) at the nondegenerate origin can be reduced to the study of that of system

\[ \dot{x} = -\nu x - 2cxy - 3dy^2, \quad \dot{y} = \nu y + cy^2, \]
with the Hamiltonian \( H = \nu x y + c x y^2 + d y^3 \), where all the variables and coefficients are complex.

We remark that there are some results characterizing the local and global dynamics of one dimensional complex autonomous differential equations, see for instance, Alvarez et al [1] and Garijo et al [15] on configurations of singularities and their properties. But according to our knowledge there are few results on global dynamics of higher dimensional complex autonomous differential equations. The related results are the ones for quaternion autonomous differential equations, see Gasull et al [16] and Zhang [30]. In addition, for real planar differential systems Artés and Llibre [4] completed characterization of the global phase portraits of real quadratic Hamiltonian vector fields. But as stated previously, the problem is still open for complex 2–dimensional quadratic Hamiltonian systems.

We must say that a locally linearizable one degree of freedom complex quadratic Hamiltonian system is not necessary globally linearizable, as shown in the next Theorem 1.4, see also the proof of Theorem 3 of [22].

In what follows we will use the notion ‘pole’. For a complex curve \( y = f(x) \), \( x \in \Omega \subset \mathbb{C} \), \( x_0 \in \Omega \) is a pole of order \( k \) of this curve if \( y = f(x) = f_0 (x-x_0)^k g(x) \) with \( 0 \neq f_0 \in \mathbb{C} \), \( g(x) \) analytic in a neighborhood of \( x_0 \) and \( g(x_0) = 1 \).

Our next result presents the global dynamics of system (9), and consequently of the locally linearizable Hamiltonian system (2).

**Theorem 1.4.** The global dynamics of system (9) are the following.

(a) For \( c = 0 \), system (9) has a unique singularity at the origin, outside which the phase space is foliated by an invariant complex line, and 1–dimensional invariant complex curves which except one all have a pole of order 1.

(a1) If \( \text{Re} \nu = 0 \), all regular orbits are periodic and isochronous, and form a period annulus on each invariant complex curve. Two of the period annuli have their inner boundary at the origin, all the other period annuli have their inner boundaries at the infinity of the pole of the invariant complex curves. Moreover, there is a family of 2–dimensional invariant tori surrounding the invariant complex line.

(a2) If \( \text{Re} \nu \neq 0 \), all regular orbits are heteroclinic, and connect the infinity of an invariant complex curve and either the origin or the infinity of the pole of the complex curve.

(b) For \( c \neq 0 \), system (9) has two finite singularities, saying \( S_1 \) and \( S_2 \), and outside which the phase space is foliated by two invariant complex lines containing one of the singularities and 1–dimensional invariant complex curves. Among which, there are two complex curves with a pole of order one and passing one of the singularities, and all the other invariant complex curves have two poles of order one and do not pass the singularities.

(b1) If \( \text{Re} \nu = 0 \), all regular orbits are periodic and isochronous, and form a period annulus on each of the two invariant complex lines, and form two period annuli limited by a 1–dimensional real curve on each invariant complex curve, each of which either surrounds a singularity or has its inner boundary at the infinity of the pole of the invariant complex curve. Furthermore, there are two families of 2–dimensional invariant tori surrounding respectively one of the two invariant complex lines.

(b2) If \( \text{Re} \nu \neq 0 \), all regular orbits are heteroclinic. On the two invariant complex lines, all the heteroclinic orbits connect a singularity and the
infinity. On each of the invariant complex curves all the heteroclinic orbits except two ones approach positively the infinity of the pole of the complex curve and negatively either one singularity or the infinity of the other pole of the complex curve, depending on the complex curve has either a pole and a singularity or two poles, or reverse.

The two exceptional heteroclinic orbits on each invariant complex curve connect either one of the pair of saddles at the infinity of the complex curve and the infinity of one of the two poles in case two poles exist on the complex curve, or one of the pair of saddles at the infinity of the complex curve and a finite singularity or the infinity of the pole in case one pole and a singularity on the complex curve.

We remark that if \( c = d = 0 \), system (9) is linear, and as illustrated in [2, p. 35] by Arnold its phase space is fulfilled by invariant tori and finite number of invariant lines and planes when Re\( \nu = 0 \). In the proofs of \((a_1)\) and \((b_1)\), we will apply this result by Arnold through local symplectic linearizations at the singularities of system (9).

Now we turn to characterize the global dynamics of the locally analytically linearizable system (7) at the origin. By Theorem 1.3 and the symmetry of the two conditions in the theorem, we only need to study the next system

\[
\dot{x} = -\nu x - 3 c x y^2 - 4 d y^3, \quad \dot{y} = \nu y + c y^3, \tag{10}
\]

with the Hamiltonian

\[
H = \nu x y + c x y^3 + d y^4 \tag{11}
\]

where either \( c \neq 0 \), or \( c = 0 \) and \( d \neq 0 \). We have the next results.

**Theorem 1.5.** For system (10), the global dynamics are the following.

(a) For \( c = 0 \), the phase space \( \mathbb{C}^2 \) of system (10) is foliated by an invariant complex line and a simply connected invariant curve both passing the unique singularity, and a family of invariant complex curves with a pole of order 1.

\((a_1)\) If Re\( \nu = 0 \), all the invariant line and curves are fulfilled by periodic orbits of the same period. Furthermore, system (10) has a family of 2-dimensional invariant tori surrounding the origin.

\((a_2)\) If Re\( \nu \neq 0 \), all the invariant line and curves are fulfilled by heteroclinic orbits, which connect the infinity and either the origin or the pole of the complex curve.

(b) For \( c \neq 0 \), the phase space \( \mathbb{C}^2 \) of system (10) is foliated by three invariant complex lines each of which containing a unique singularity, and a family of invariant complex curves with one containing the origin and having two poles of order 1 and others having three poles of order 1.

\((b_1)\) If Re\( \nu = 0 \), all the invariant lines and curves are fulfilled by periodic orbits of the same period. These periodic orbits form three families of invariant tori, each of which surrounding one of the three invariant lines and limited by two invariant hyperbolic cylinders.

\((b_2)\) If Re\( \nu \neq 0 \), all the invariant lines and curves are fulfilled by heteroclinic orbits. The heteroclinic orbits on each of the three lines connects the unique finite singularity and the infinity. On each of the curves, except the four ones connecting the two pairs of saddles at infinity all the others connect the pole at \( y = 0 \) and one of the other two poles.
The paper is organized as follows. In the next section, we will prove Proposition 1. Section 3 is the proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 will be given in Section 4. And the proof of Theorems 1.4 and 1.5 will be given in Section 5.

2. Proof of Proposition 1. Easy calculation shows that the origin of system (2) has the eigenvalues \( \nu = \pm \sqrt{h_1^2 - 4h_{20}h_{02}}. \) In order that the origin is a non-degenerate singularity, we must have \( h_1^2 - 4h_{20}h_{02} \neq 0. \)

Take the linear symplectic transformation \( x = au + bv, \ y = cu + dv \) with \( ad - bc = 1, \) under which the second order part of \( H(x, y) \) in (3) becomes

\[
H_{2s} = Au^2 + Buv + Cv^2,
\]

where

\[
A = h_{20}a^2 + h_{11}ac + h_{02}c^2,
B = 2h_{02}cd + h_{11}(2bc + 1) + 2h_{20}ab,
C = h_{20}b^2 + h_{11}bd + h_{02}d^2.
\]

If \( h_{20} \neq 0 \), choose \( a = \frac{-h_{11} + \sqrt{h_1^2 - 4h_{20}h_{02}}}{2h_{20}}c, \ b = \frac{-h_{11} - \sqrt{h_1^2 - 4h_{20}h_{02}}}{2h_{20}}d, \)

Then one has \( A = C = 0, \) and

\[
B = -\frac{h_{11} - 4h_{20}h_{02}}{h_{20}}cd, \ ad - bc = \frac{\sqrt{h_1^2 - 4h_{20}h_{02}}}{h_{20}}cd.
\]

This shows that the Hamiltonian \( H(x, y) \) in (3) can be transformed, via a symplectic change of coordinates, to a new one with the second order part being of the form \( -\sqrt{h_1^2 - 4h_{20}h_{02}} uv. \)

If \( h_{02} \neq 0 \), choose

\[
c = \frac{-h_{11} - \sqrt{h_1^2 - 4h_{02}h_{20}}}{2h_{02}}a, \ d = \frac{-h_{11} + \sqrt{h_1^2 - 4h_{02}h_{20}}}{2h_{02}}b.
\]

Then the same argument as in the case \( h_{20} \neq 0 \) holds.

If \( h_{20} = h_{02} = 0 \) then the second order term of the Hamiltonian \( H \) in (3) is already in normal form. It completes the proof of the proposition. \( \square \)

3. Proof of Theorem 1.1. In order to prove the theorem, we need to use the results on the convergence of the transformation sending an analytic Liouvillian integrable Hamiltonian system to its Birkhoff normal form. For doing so, we adopt the real expressions of the complex Hamiltonian system (4). Set \( \gamma = \gamma_1 + i\gamma_2, \ \gamma \in \{x, y, a, b, c, d\}, \) and \( \nu = \alpha + i\beta. \) Then the complex two dimensional system (4) can be equivalently written in a real four dimensional system

\[
\begin{align*}
\dot{x}_1 &= -\alpha x_1 + \beta x_2 - x_1(b_1 x_1 - b_2 x_2) + x_2(b_2 x_1 + b_1 x_2) - 2(c_1 x_1 - c_2 x_2)y_1 + 2(c_2 x_1 + c_1 x_2)y_2 - 3(d_1 y_1 - d_2 y_2)y_1 + 3(d_2 y_1 + d_1 y_2)y_2, \\
\dot{x}_2 &= -\alpha x_2 - \alpha x_2 - x_1(b_1 x_1 + b_2 x_2) + x_2(b_1 x_1 - b_2 x_2) - 2(c_1 x_1 + c_2 x_2)y_1 - 2(c_2 x_1 - c_1 x_2)y_2 - 3(d_1 y_1 + d_2 y_2)y_1 - 3(d_2 y_1 - d_1 y_2)y_2, \\
\dot{y}_1 &= \alpha y_1 - \beta y_2 + 3x_1(a_1 x_1 - a_2 x_2) - 3x_2(a_2 x_1 + a_1 x_2) + 2(b_1 x_1 - b_2 x_2)y_1 - 2(b_2 x_1 + b_1 x_2)y_2 + (c_1 y_1 - c_2 y_2)y_1 - (c_2 y_1 + c_1 y_2)y_2,
\end{align*}
\]
We can check that the Lie bracket of \( X \) algebra is the real and imaginary parts of the Hamiltonian \( H \) is a nilpotent subalgebra Cartan subalgebra generate a Cartan subalgebra in the symplectic Lie algebra \( \text{sp} \) Hamiltonian system in a neighborhood of a nondegenerate isolated singularity of \( H \) can check that \( H \) order 2. This means that system (12) is a Hamiltonian system. Furthermore, we where \( z \) is the transpose of a matrix. Then system (12) can be written in

\[
\dot{z} = J \nabla_z H_R, \quad J = \begin{pmatrix} 0 & -E_2 \\ E_2 & 0 \end{pmatrix},
\]

where \( \nabla_z H_R \) is the gradient of \( H_R \) with respect to \( z \), and \( E_2 \) is the unit matrix of order 2. This means that system (12) is a Hamiltonian system. Furthermore, we can check that \( H_I \) is a first integral of system (13) and it is functionally independent of \( H_R \). So system (13) is completely integrable in the sense of Lieville.

Denote by \( X_R \) the Hamiltonian vector field associated to (13), and \( X_I \) the Hamiltonian vector field defined by \( J \nabla_z H_I \). Recall that the set \( \mathcal{S} \) of smooth vector fields defined on \( \mathbb{R}^4 \) forms a Lie algebra together with a Lie bracket in the standard form, i.e. \([X, Y] = X Y - Y X\) for arbitrary \( X, Y \in \mathcal{S} \). Denote by \( X_{1R} \) and \( X_{1I} \) the linear parts of \( X_R \) and \( X_I \), i.e.

\[
X_{1R} = - (\alpha x_1 - \beta x_2) \frac{\partial}{\partial x_1} + (\beta y_1 + \alpha y_2) \frac{\partial}{\partial y_2},
\]

\[
X_{1I} = - (\beta x_1 + \alpha x_2) \frac{\partial}{\partial x_1} - (\alpha y_1 - \beta y_2) \frac{\partial}{\partial x_2} + (\beta y_1 - \alpha y_2) \frac{\partial}{\partial y_1} + (\alpha x_1 - \beta x_2) \frac{\partial}{\partial y_2}.
\]

We can check that the Lie bracket of \( X_{1R} \) and \( X_{1I} \) vanishes. So \( X_{1R} \) and \( X_{1I} \) generate a Cartan subalgebra in the symplectic Lie algebra \( \text{sp}(\mathbb{R}^4, J) \). Recall that a Cartan subalgebra is a nilpotent subalgebra \( \eta \) of a Lie algebra that is self-normalising (i.e. the Lie bracket \([X, Y] \in \eta \) for all \( X \in \eta \) then \( Y \in \eta \)). A subalgebra of Lie algebra is nilpotent if \( \eta_n = 0 \) for some \( n \in \mathbb{N} \), i.e.

\[
[X_1, [X_2, \cdots [X_n, Y] \cdots ]] = 0 \quad \text{for all } X_1, \ldots, X_n, Y \in \eta.
\]

This shows that the singularity \( S_1 = (0, 0, 0, 0) \) is a nondegenerate common equilibrium of the first integrals \( H_R \) and \( H_I \) (here common equilibrium means that \( \nabla_z H_R(S_1) = \nabla_z H_I(S_1) = 0 \)). In addition, the singularity \( S_1 \) is isolated.

Vey [28] has shown in general that an analytic completely integrable symplectic Hamiltonian system in a neighborhood of a nondegenerate isolated singularity
can be sent to its Birkhoff normal form via a convergent symplectic change of coordinates. This result extends Moser’s one [26], in which Moser proved that the transformation to Birkhoff normal form of a general analytic Hamiltonian system in $\mathbb{C}^2$ is convergent (and so analytic), in principle, only if the non-degenerate equilibrium is of saddle-centre or saddle-focus type. In fact, since our Hamiltonian system (12) is integrable in the sense of Liouville, Zung [33] has shown that system (12) is analytically symplectically equivalent to its Birkhoff normal form. Now by assumption $\text{Re}\nu = 0$, it follows that the linear part of the Hamiltonian vector field $X_R$ has two pairs of pure imaginary eigenvalues. So for the integrable Hamiltonian vector field $X_R$ associated to the Hamiltonian $H_R$, there exist Darboux coordinates $(q_1, q_2, p_1, p_2)$ under which the Hamiltonian $H_R$ can be put in the Birkhoff normal form $F(q_1^2 + p_1^2, q_2^2 + p_2^2)$ with $F(z_1, z_2) = \beta(z_1 + z_2) + \text{higher order terms}$. The Hamiltonian system associated to the Hamiltonian $F$ has the functionally independent first integrals $q_1^2 + p_1^2$ and $q_2^2 + p_2^2$, the intersections of whose level sets in general form a family of two dimensional invariant compact surfaces. By the classical Liouville–Arnold integrability theorem, it follows that these two dimensional invariant surfaces are tori. This proves that the Hamiltonian system associated to $F$ has a family of invariant tori around the origin. Since an invertible symplectic analytic change of coordinates maintains topological structures of phase portraits of an analytic Hamiltonian system, it verifies that the integrable Hamiltonian system $X_R$ has a family of invariant tori near the origin. Consequently, system (4) has a family of invariant tori near the origin. Statement (a) follows.

For proving statement (b), note that under the assumption $a = 0$ and $b \neq 0$, the complex Hamiltonian system (4) has at least two singularities, which are the origin $S^*_1 = (0, 0)$, $S^*_2 = (-\nu b^{-1}, 0)$ and the others: whose number could be 0, 1 and 2 depending on the conditions of the parameters of the system.

As proved before, system (4) has a family of 2-dimensional invariant tori around the singularity $S^*_1$. For the singularity $S^*_2$, we can check that the linearization of system (4) at $S^*_2$ has the eigenvalues $-\nu, \nu$. Using the same arguments as in the proof to $S_1$ together with some calculations, we can show that the singularity $S_2$ of the Hamiltonian vector fields $X_R$ corresponding to $S^*_2$ of system (4) is a nondegenerate isolated singularity, and it is elliptic–elliptic type because of $\text{Re}\nu = 0$. Again with the help of the Vey’s result or of the Zung’s result we get that the Liouvillian integrable Hamiltonian vector field $X_R$ has a family of invariant tori around $S_2$. Hence, the complex Hamiltonian system (4) has a family of invariant tori surrounding $S^*_1$. These last results mean that the complex Hamiltonian system (4) has two families of invariant tori surrounding respectively $S^*_1$ and $S^*_2$. Statement (b) follows.

It completes the proof of the theorem. 

4. Proof of Theorems 1.2 and 1.3. We separate this section in two parts. The first one is on Theorem 1.2, and the second one is on Theorem 1.3.

4.1. Proof of Theorem 1.2. In processing linearization of system (2) at the origin, we first study the situation $h_{20} = h_{02} = 0$. In this circumstance, the nonresonant terms of the normalization sending system (2) to its associated Poincaré–Dulac (Birkhorff in fact) normal form are of the form $x^k y^\ell$ with $k, \ell \in \mathbb{Z}_+$ satisfying $k - \ell \neq 0$, and of course the resonant terms are of the form $(xy)^k$ with $k \in \mathbb{N}$ and $k \neq 0$. Here $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The next proof for $h_{20} = h_{02} = 0$ is similar to that of Theorem 3 of [22]. For completeness we present it here with a different
presentation. The main difference between Birkhorff and Poincaré–Dulac normal forms is that in the former the normalization preserves the Hamiltonian structure of a Hamiltonian system after transformation. For more information on Birkhorff and Poincaré–Dulac normal forms, see for instance [18, 33] and [32], respectively.

Write system (2) with \( h_{20} = h_{02} = 0 \) in the form

\[
\dot{z} = Az + f_2(z), \quad A = \begin{pmatrix} -h_{11} & 0 \\ 0 & h_{11} \end{pmatrix},
\]

where \( z = (x, y)^\tau \) with \( \tau \) the transpose of a matrix, and \( f_2(z) \) is the quadratic homogeneous part of system (2). Take the near identity complex coordinate change \( w = (u, v)^\tau = (x, y)^\tau + h(x, y)^\tau + \sum_{\ell=2}^{\infty} h_\ell(z) \), with \( h_\ell = (p_\ell(x, y), q_\ell(x, y))^\tau \), where \( p_\ell \) and \( q_\ell \) are homogeneous polynomials of degree \( \ell \) in \( x, y \), under which system (14) is transformed to

\[
\dot{w} = Aw.
\]

Then we have the homological equation

\[
\frac{\partial h}{\partial z} Az - Ah = -f_2(z) - \frac{\partial h}{\partial z} f_2(z).
\]

Comparing its homogeneous polynomials of the same degree yields

\[
\frac{\partial h_2}{\partial z} Az - Ah_2 + f_2(z) = 0, \quad \ell = 2, 3, \ldots
\]

Write \( h_\ell = (p_\ell(x, y), q_\ell(x, y))^\tau \), where \( p_\ell \) and \( q_\ell \) are homogeneous polynomials of degree \( \ell \) in \( x, y \) being of the form

\[
p_\ell(x, y) = \sum_{i+j=\ell} p_{ij} x^i y^j, \quad q_\ell(x, y) = \sum_{i+j=\ell} q_{ij} x^i y^j, \quad \ell = 2, 3, \ldots
\]

with their coefficients to be determined.

The coefficients of equation (15) with the expressions of \( p_2 \) and \( q_2 \) form a system of six equations, which have a unique solution in \( p_{20}, p_{11}, p_{02}, q_{20}, q_{11}, q_{02} \).

Using the expressions of \( p_3 \) and \( q_3 \) in (17), the first and second equations of (16) have respectively the coefficients of \( x^2 y \) and \( x y^2 \) being \( 6(h_{21}h_{12} + h_{30}h_{03})/(\alpha + i\beta) \) and \(-6(h_{21}h_{12} + h_{30}h_{03})/(\alpha + i\beta) \) with \( h_{11} = \alpha + i\beta \neq 0 \), which provide the first linearization quantity

\[
L_1^*(h_{30}, h_{21}, h_{12}, h_{03}) := h_{21}h_{12} + h_{30}h_{03}
\]

to system (2) at the origin. The other coefficients of (16) with \( \ell = 3 \) provide a unique solution \( p_{30}, p_{12}, p_{03}, q_{30}, q_{21}, q_{03} \) with \( p_{21} \) and \( p_{12} \) arbitrary. We mention that \( L_1^* \) is \( L_1 \) with \( h_{20} = h_{02} = 0 \) modulo a nonzero factor, where \( L_1 \) is the quantity defined in the theorem.

The coefficients of equation (16) with \( \ell = 4 \) and the expressions of \( p_4 \) and \( q_4 \) form a system of ten equations, which have a unique solution in \( p_{40}, p_{31}, p_{22}, p_{13}, p_{04}, q_{40}, q_{31}, q_{22}, q_{13}, q_{04} \), which depends linearly on either \( p_{21} \) or \( q_{12} \).
Using the expressions of \( p_5 \) and \( q_5 \) in (17), the first and second equations of (16) have respectively the coefficients of \( x^3 y^2 \) and \( x^2 y^3 \) being \( 60 L_2^*(h_{30}, h_{21}, h_{12}, h_{03})/(\alpha + i \beta)^2 \) and \( -60 L_2^*(h_{30}, h_{21}, h_{12}, h_{03})/(\alpha + i \beta)^3 \) provided that \( L_1^* = 0 \), where
\[
L_2^*(h_{30}, h_{21}, h_{12}, h_{03}) := h_{30} h_{12}^3 - 3 h_{21}^2 h_{12}^2 + h_{21}^3 h_{03}.
\]
Hence we obtain the second linearization quantity \( L_2^* \) to system (2) at the origin when \( h_{20} = h_{02} = 0 \). The other coefficients of (16) with \( \ell = 5 \) yield a unique solution \( p_{50}, p_{41}, p_{23}, p_{14}, p_{05}, q_{50}, q_{41}, q_{32}, q_{14}, q_{05} \) depending linearly on either \( p_{21} \) or \( p_{12} \), with \( p_{32} \) and \( p_{23} \) arbitrary.

The coefficients of equation (16) with \( \ell = 6 \) and the expressions of \( p_6 \) and \( q_6 \) form a system of fourteen equations, which have a unique solution in \( p_6 \) and \( q_6 \), which depend linearly on either \( p_{21} \) and \( p_{32} \), or \( q_{12} \) and \( q_{23} \).

Using the expressions of \( p_7 \) and \( q_7 \) in (17), the first and second equations of (16) have respectively the coefficients of \( x^3 y^4 \) and \( x^2 y^5 \) being \(-7560 h_{21}^3 h_{12}^2/(\alpha + i \beta)^5\) provided that \( L_1^* = L_2^* = 0 \). This gives the third linearization quantity
\[
L_3^*(h_{30}, h_{21}, h_{12}, h_{03}) := h_{21}^3 h_{12}^3
\]
to system (2) at the origin when \( h_{20} = h_{02} = 0 \). The other coefficients of (16) with \( \ell = 7 \) yield a unique solution \( p_{70}, p_{61}, p_{52}, p_{43}, p_{34}, p_{25}, p_{16}, p_{07}, q_{70}, q_{61}, q_{52}, q_{43}, q_{34}, q_{25}, q_{16}, q_{07} \) depending linearly on either \( p_{21} \) and \( p_{32} \), or \( q_{12} \) and \( q_{23} \), with \( p_{43} \) and \( p_{34} \) arbitrary.

The above proofs present the first three linearization quantities of system (2) at the origin when \( h_{20} = h_{02} = 0 \). They vanish simultaneously if and only if \( h_{30} = h_{21} = 0 \) or \( h_{12} = h_{03} = 0 \). As shown in the proof of Theorem 3 of [22], we can provide concrete expressions of the symplectic linearization transformation which sends system (2) at the origin to its linear part provided that one of the two conditions holds. This shows that the conditions are also sufficient for system (2) to be linearizable at the origin. This proves the theorem when \( h_{20} = h_{02} = 0 \).

If \( h_{20} \neq 0 \), as in the proof of Proposition 1 we take the symplectic change of coordinates
\[
x = a_0 u + b_0 v, \quad y = c_0 u + d_0 v,
\]
with
\[
a_0 = -\frac{h_{11} + \sqrt{h_{11}^2 - 4 h_{20} h_{02}}}{2}, \quad b_0 = -\frac{h_{11} - \sqrt{h_{11}^2 - 4 h_{20} h_{02}}}{2 h_{20} \sqrt{h_{11}^2 - 4 h_{20} h_{02}}},
\]
\[
c_0 = h_{20}, \quad d_0 = \frac{1}{\sqrt{h_{11}^2 - 4 h_{20} h_{02}}},
\]
which transforms the Hamiltonian (3) to
\[
H_1 = -\nu uv + A_1 u^3 + B_1 u^2 v + C_1 uv^2 + D_1 v^3,
\]
where \( \nu = \sqrt{h_{11}^2 - 4 h_{02} h_{20}} \), and \( A_1, B_1, C_1 \) and \( D_1 \) depend on the coefficients of \( H \), and they have the next expressions
\[
A_1 = \frac{1}{2} (h_{20} h_{12} - h_{20} h_{11} h_{21} - h_{20} h_{02} h_{30} + h_{21}^2 h_{30}) \sqrt{h_{11}^2 - 4 h_{20} h_{02}}
\]
\[+
\]
\[
\frac{1}{2} (2 h_{20}^2 h_{03} - h_{20} h_{11} h_{12} - 2 h_{20} h_{02} h_{21} + 3 h_{20} h_{11} h_{02} h_{30} + h_{20} h_{12}^2 h_{21} - h_{11}^2 h_{30}),
\]
These three quantities written in the expressions of $A$ origin are Hamiltonian system with the associated Hamiltonian (18) to be linearizable at the origin, where $L$ quantities, which are the same as those in (19).

If $h_{20} \neq 0$, from the proof of Proposition 1 we take the symplectic change of coordinates

$$x = a_0 u + b_0 v,$$

$$y = c_0 u + d_0 v,$$

with

$$a_0 = h_{20},$$

$$b_0 = \frac{1}{\sqrt{h_{11}^2 - 4 h_{20} h_{20}}},$$

$$c_0 = -\frac{h_{11} - \sqrt{h_{11}^2 - 4 h_{20} h_{20}}}{2},$$

$$d_0 = \frac{-h_{11} + \sqrt{h_{11}^2 - 4 h_{20} h_{20}}}{2 h_{20} \sqrt{h_{11}^2 - 4 h_{20} h_{20}}},$$

which transforms the Hamiltonian (3) to

$$H_2 = -\nu w + A_2 u^3 + B_2 u^2 v + C_2 uv^2 + D_2 v^3,$$

where $A_2$, $B_2$, $C_2$ and $D_2$ depend on the coefficients of $H$ and their concrete expressions are omitted here. Inserting the expressions of $A_2$, $B_2$, $C_2$, $D_2$ into the formulae of $L_1^*, L_2^*, L_3^*$ by replacing the $h_{30}$, $h_{21}$, $h_{12}$, $h_{03}$, we obtain three corresponding quantities, which are the same as those in (19).

By the above proofs we get the first three linearization quantities, whose variety provides the necessary conditions for system (2) to be linearizable at the origin. Note that an invertible linear change of coordinates does not affect linearization of a polynomial differential system. So as in the previous proof for $h_{20} = h_{02} = 0$, it follows that the necessary conditions are also sufficient for system (2) to be linearizable at the origin in general, i.e. when $h_{20} \neq 0$ or $h_{02} \neq 0$.

It completes the proof of the theorem. $\square$
4.2. Proof of Theorem 1.3. The proof of the necessary part of the theorem follows from the same algorithms as those in subsection 4.1 for proving Theorem 1.3. The detail processes are tedious and will be omitted here. In summary, by those algorithms we could obtain the first four linearization quantities:

\[
\begin{align*}
L_1 &= h_{22}, \\
L_2 &= h_{13}h_{31} + h_{04}h_{40}, \\
L_3 &= h_{04}h_{31}^2 + h_{13}^2h_{40}, \\
L_4 &= h_{13}^2h_{31}^2.
\end{align*}
\]

So the necessary condition for system (7) to be linearizable is that \( L_1 = L_2 = L_3 = L_4 = 0 \). Some easy calculations show that these last conditions are equivalent to one of the following ones:

\[
\begin{align*}
(C_{l1}) & \quad h_{22} = 0, \quad h_{13} = 0, \quad h_{31} \neq 0, \quad \text{and} \quad h_{04} = 0, \\
(C_{l2}) & \quad h_{22} = 0, \quad h_{31} = 0, \quad h_{13} \neq 0, \quad \text{and} \quad h_{40} = 0, \\
(C_{l3}) & \quad h_{22} = 0, \quad h_{13} = 0, \quad h_{31} = 0, \quad \text{and} \quad h_{04} = 0, \\
(C_{l4}) & \quad h_{22} = 0, \quad h_{13} = 0, \quad h_{31} = 0, \quad \text{and} \quad h_{40} = 0.
\end{align*}
\]

Next we prove that these necessary conditions are also sufficient. Under condition \((C_{l1})\), the linearizable symplectic change of variables

\[
\mu = \frac{x}{\sqrt{1 + h_{31}h_{11}x^2}}, \quad \nu = \left(y + \frac{h_{40}}{h_{11}}x^3 + \frac{h_{31}}{h_{11}}x^2y\right)\sqrt{1 + \frac{h_{31}}{h_{11}}x^2},
\]

sends system (7) to its linear part in a neighborhood of the origin. Note that this transformation is not defined in the full complex plane.

Under condition \((C_{l2})\), the linearizable symplectic change of variables

\[
\mu = \left(x + \frac{h_{13}}{h_{11}}y^2 + \frac{h_{04}}{h_{11}}y^3\right)\sqrt{1 + \frac{h_{31}}{h_{11}}y^2}, \quad \nu = \frac{y}{\sqrt{1 + \frac{h_{31}}{h_{11}}y^2}},
\]

sends system (7) to its linear part in a neighborhood of the origin. Note that this transformation is also not defined in the full complex plane.

Under the condition either \((C_{l3})\) or \((C_{l4})\), the linearizable symplectic change of variables is either

\[
\mu = x, \quad \nu = y + \frac{h_{40}}{h_{11}}x^3,
\]

or

\[
\mu = x + \frac{h_{04}}{h_{11}}y^3, \quad \nu = y.
\]

Note that these two transformations are both globally defined in the full complex plane.

These prove that the four conditions \((C_{l1})\), \((C_{l2})\), \((C_{l3})\) and \((C_{l4})\) are the necessary and sufficient conditions for system (7) to be analytically linearizable at the origin. Furthermore, by reorganization it follows that these four conditions are essentially equivalent to one of the two conditions given in the theorem.

It completes the proof of the theorem.

5. Proof of Theorems 1.4 and 1.5. The proofs of these two theorems are separated in two subsections.
5.1. Proof of Theorem 1.4. (a) $c = 0$. System (9) is reduced to
\[ \dot{x} = -\nu x - 3dy^2, \quad \dot{y} = \nu y, \] (20)
with the Hamiltonian $H_0 = \nu xy + dy^3$. Clearly system (20) has a unique singularity, which is at the origin. By the Hamiltonian function $H_0$ it follows that the complex 2–dimensional phase space is foliated by
- the invariant complex line $y = 0$, and
- the invariant complex algebraic curves $x = \frac{1}{\nu} (hy^{-1} - dy^2)$, $h \in \mathbb{C}$.

On the invariant complex line $y = 0$, system (20) is linear, and it has the general solution
\[ x = x_0 e^{-\nu t} = x_0 e^{-\alpha t} (\cos(\beta t) - i \sin(\beta t)), \]
where $\alpha = \text{Re} \nu$ and $\beta = \text{Im} \nu$. So the origin is either a focus if $\text{Re} \nu \text{Im} \nu \neq 0$, or a center if $\text{Re} \nu = 0$, or a star node if $\text{Im} \nu = 0$.

The invariant complex algebraic curve $S_0 := \{ x = -\nu^{-1} dy^2 \}$ is diffeomorphic to the $y$–axis, and restricted to it the dynamics of system (20) is determined by $\dot{y} = \nu y$. So the orbits on $S_0$ are either all periodic when $\text{Re} \nu = 0$, or all heteroclinic connecting the origin and the infinity at the endpoints of the $x$–axis in the Poincaré sphere when $\text{Re} \nu \neq 0$.

The invariant complex algebraic curves $C_h := \{ x = \nu^{-1} (hy^{-1} - dy^2) \}$ with $h \neq 0$ are defined on the complex $y$–axis except at $y = 0$, which is double connected. Each complex curve $C_h$ has two boundaries, which are the infinity of the pole $y = 0$, and the infinity at the end of the complex $x$–axis in the Poincaré sphere. On $C_h$ with $h \neq 0$, any orbit of system (20) is determined by
\[(x(t), y(t)) = (\nu^{-1}h_0^{-1} e^{-\nu t} - \nu^{-1} d_0^2 e^{2\nu t}, y_0 e^{\nu t}),\]
with $0 \neq y_0 \in \mathbb{C}$. Hence, the orbits on $C_h$ are either all periodic when $\text{Re} \nu = 0$, or all heteroclinic connecting the infinity of the pole and the infinity at the end of the $x$–axis in the Poincaré sphere when $\text{Re} \nu \neq 0$.

Note that system (20) is symplectically equivalent to $\dot{u} = -\nu u$, $\dot{v} = \nu v$ via the symplectic change of coordinates $u = x + \nu^{-1} dy^2$, $v = y$. The latter has the two functionally independent first integrals $u\tau + v\sigma$ and $uv$ when $\text{Re} \nu = 0$. It follows that if $\text{Re} \nu = 0$, system (20) has a family of invariant tori, each of which is fulfilled by periodic orbits. Statement (a) follows.

(b) $c \neq 0$. System (9) has two singularities
\[ S_1 = (0, 0), \quad S_2 = (3c^{-2} d\nu, -c^{-1} \nu), \]
whose eigenvalues are both $-\nu$ and $\nu$. By the Hamiltonian $H$ it follows that the complex 2–dimensional phase space is foliated by
- the two invariant complex lines $y = 0$, $y = -c^{-1} \nu$,
- and the invariant complex curves
\[ C_\ell := \left\{ x = \frac{\ell - dy^3}{\nu y + cy^3} \right\}, \quad \ell \in \mathbb{C}. \]

Note that for $\ell = 0$, the level set $H = \ell$ has two branches $L_1 := \{ y = 0 \}$ and $C_1 := \{ x = -dy^2 (\nu + cy)^{-1} \}$, which transversally intersect at the singularity $S_1$. For $\ell = -c^{-3} d\nu^3$ the level set $H = \ell$ has also two branches $L_2 := \{ y = -c^{-1} \nu \}$ and $C_2 := \{ x = -(d\nu^2 - c\nu y + c^2 dy^2) c^{-3} y^{-1} \}$, which transversally intersect at the
singularity $S_2$. For all $\ell \in \mathbb{C} \setminus \{0, -c^{-3}dv^3\}$ the level set $H = \ell$ has a unique branch, which is defined in $\mathbb{C} \setminus \{0, -c^{-1}v\}$ and has the two poles at $y = 0$ and $y = -c^{-1}v$.

Summarizing the above analysis we achieve the next results.

- The complex two dimensional phase space is foliated by two complex invariant lines $L_1$ and $L_2$, the two complex invariant curves $C_1$ and $C_2$ each of which has a unique pole with $C_1$ intersecting $L_1$ at $S_1$ and $C_2$ intersecting $L_2$ at $S_2$, and the complex invariant curves $C_\ell$, $\ell \in \mathbb{C} \setminus \{0, -c^{-3}dv^3\}$, each of which is triple connected and has two poles that provide two inner boundaries of $C_\ell$ located at the infinities of the lines $L_1$ and $L_2$ (or of the two poles).

Restricted to the two invariant complex lines $L_1$ and $L_2$ respectively, system (9) is reduced to

\[
\begin{align*}
\dot{x} &= -\nu x, \\
\dot{x} &= \nu x - 3c^{-2}d\nu^2,
\end{align*}
\]

respectively, which are both linear. Clearly, the singularities of these two complex 1-dimensional equations have the same properties but with different stability when $\text{Re} \nu \neq 0$.

On each invariant complex curve $C_\ell$, $\ell \in \mathbb{C}$, the dynamics of system (9) is determined by the equation

\[
\dot{y} = \nu y + cy^2, \quad (21)
\]

which has two singularities $y = 0$ and $y = -c^{-1}\nu$ with the eigenvalues $\nu$ and $-\nu$, respectively. According to [1, Theorem 2.1], the two singularities are both

- isochronous centers if $\text{Re} \nu = 0$,
- nodes with different stability if $\text{Re} \nu \neq 0$ and $\text{Im} \nu = 0$
- foci with different stability if $\text{Re} \nu \text{Im} \nu \neq 0$.

In addition, equation (21) has a unique singularity at infinity, which is a saddle.

For more precise on dynamics of equation (21), we have the next results.

Set

\[
F_1 = y\bar{y}, \quad F_2 = (y + c^{-1}\nu)(y + c^{1-1}\nu).
\]

Computing the derivatives of $F_1$ and $F_2$ along equation (21) gives

\[
\begin{align*}
\frac{dF_1}{dt} \bigg|_{(21)} &= (\nu + \bar{\nu} + cy + \bar{c}y)F_1, \\
\frac{dF_2}{dt} \bigg|_{(21)} &= (cy + \bar{c}y)F_2.
\end{align*}
\]

These mean that $F_1$ and $F_2$ are Darboux polynomials of equation (21). For more information on the Darboux theory of integrability, see for instance [21, 31, 32].

From the two equalities in (22) one gets that

\[
\frac{d}{dt} \left( \frac{F_1}{F_2} \right) = (\nu + \bar{\nu}) \frac{F_1}{F_2}, \quad \frac{d}{dt} \left( \frac{F_2}{F_1} \right) = -(\nu + \bar{\nu}) \frac{F_2}{F_1}.
\]

These show that $F_2/F_1$ and $F_1/F_2$ are rational first integrals of the system if $\text{Re} \nu = 0$. We note that this claim can also be obtained directly from (22) using the Darboux theory of integrability.

For the next analysis, we study the shape of the level set $F_2/F_1 = r_1$, with $r_1 \in \mathbb{R}$. One can check that if $r_1 \neq 1$, $F_2 - r_1F_1 = 0$ can be equivalently expressed
as
\[(y + \frac{c^{-1}\nu}{1 - r_1}) \left( y + \frac{\nu^{-1} \psi}{1 - r_1} \right) - \frac{r_1 c^{-1} \nu \psi}{(1 - r_1)^2} = 0, \tag{24} \]
which is a circle on the complex line (resp. a circular cylinder in the complex plane), denoted by \(E_{r_1}\), when \(r_1 > 0\) and \(r_1 \neq 1\). Of course, if \(r_1 < 0\) the level set is empty. If \(r_1 = 1\), \(F_2 - r_1 F_1 = 0\) is the real 1–dimensional line \(L: c\nu y + \nu \psi + \nu \psi = 0\) on the complex \(y\)-axis.

**Case 1.** \(\text{Re}\,\nu = 0\), i.e. \(\nu + \nu = 0\). Equation (21) has the rational first integral
\[R_1 = \frac{F_2}{F_1} \quad (\text{resp. } R_2 = \frac{F_1}{F_2}),\]
which is analytic at the singularity \(y = -c^{-1}\nu\) (resp. \(y = 0\)) of equation (21). Using these first integrals we provide an independent proof to the fact that the two singularities of equation (21) are both centers via the classical Poincaré theorem, which states that a nondegenerate monodromy singularity of a real planar analytic differential system is a center if and only if the system has an analytic first integral defined in a neighborhood of the singularity. We remark that the complex one dimensional differential equation (21) can be treated as a real two dimensional differential system.

Lifting the orbits of equation (21) to the complex two dimensional \((x, y)\)-space, together with some further computations, we obtain that for the complex differential system (1), the level set \(R_1 = r_1\), with \(r_1 \in \mathbb{R}\) and \(r_1 \geq 0\), is
- the invariant complex line \(y = -c^{-1}\nu\) if \(r_1 = 0\),
- the invariant complex line \(y = 0\) if \(r_1 = \infty\),
- the invariant real three dimensional hyperplane \(L\) if \(r_1 = 1\),
- the invariant generalized circular cylinder \(E_{r_1}\) given by (24), and surround either the complex line \(y = 0\) if \(r_1 > 1\), or the complex line \(y = -c^{-1}\nu\) if \(0 < r_1 < 1\).

Note that treating \(L\) as an invariant line of equation (21) in the \(y\)-axis, it passes the unique pair of singularities at the infinity (which is a saddle), and it separates the two period annuli in two disjoint regions.

Summarizing the above analysis, one has the next results.
- On each of the two complex invariant lines \(L_1\) and \(L_2\), system (9) is linear and its unique finite singularity is a center.
- The invariant complex curve \(C_1\) of system (9) is fulfilled by two unbounded period annuli limited by a real 1–dimensional invariant line, with one around the singularity \(S_1\) and the other surrounding the infinity of \(L_1\).
- The invariant complex curve \(C_2\) of system (9) is fulfilled also by two unbounded invariant period annuli limited by a real 1–dimensional invariant line, with one around the singularity \(S_2\) and the other surrounding the infinity of \(L_2\).
- The invariant complex curve \(C_\ell, \ell \in \mathbb{C} \setminus \{0, -c^{-3}dv^3\}\), of system (9) is fulfilled by two unbounded invariant period annuli limited by a real 1–dimensional invariant line, which have their inner boundaries approaching the infinities of \(L_1\) and of \(L_2\), respectively.

Note that system (9) can be linearized at the singularity \(S_1\) by the symplectic change of coordinates
\[u = (x + \nu^{-1} cxy + \nu^{-1} dy^2)(1 + \nu^{-1} cy), \quad v = y/(1 + \nu^{-1} cy),\]
and that it can also be linearized at the singularity $S_2$ by the symplectic transformation

$$u = (c^{-1}d - \nu^{-1}d y + \nu^{-2}c^2 x y + \nu^{-2}c y^2) y, \quad v = -\nu c^{-2}(\nu + c y) y^{-1}.$$ 

So we can further get that if $\Re \nu = 0$, system (9) has a family of 2-dimensional invariant tori surrounding the invariant complex line $L_1$, and has also a family of 2-dimensional invariant tori surrounding the invariant complex line $L_2$.

This proves statement $\textbf{(b)}_1$.

\textbf{Case 2.} $\Re \nu \neq 0$. Without loss of generality we assume that $\Re \nu > 0$. By (23) it follows that $R_1 = F_2/F_1$ decreases along any positive orbit, being located outside $F_1 = 0$ and $F_2 = 0$, of system (9). Recall that $R_1 = r_1$, $0 < r_1 \neq 1$, is a real 1-dimensional circle on the complex $y$-axis, and $R_1 = 1$ is the real 1-dimensional line $L$. From (24) and the fact that the singularity at infinity is a unique pair of saddle, we can obtain the next conclusions.

- If a regular orbit of equation (21) always stays in one of the two regions limited by $L$, then it is heteroclinic and connects one of the two finite singularities and one of the pair of saddles at infinity. There are exactly two heteroclinic orbits of this type, because the infinity of the $y$-axis has exactly one pair of singularities of equation (21), which is a saddle.

- If a regular orbit of equation (21) passes through the line $L$, according to (24) it must positively approach $y = -c^{-1}\nu$ and negatively approach $y = 0$, and so it is heteroclinic, too.

Note that each of these heteroclinic orbits of equation (21) can be lifted to an invariant generalized real three dimensional cylinder of system (9), which is heteroclinic to the invariant complex lines $L_2$ and $L_1$.

By these last arguments, we have the next results.

- Restricted to each of the two invariant complex lines $L_1$ and $L_2$, system (9) is a linear focus or a linear node, which has different stability on $L_1$ and $L_2$, respectively.

- On the invariant complex curve $C_1$, all regular orbits of system (9) are heteroclinic ones, which except two are heteroclinic to the singularity $S_1$ and the infinity of the complex line $L_2$, which is at the pole of the complex curve $C_1$. One of the two exceptional ones connects $S_1$ and the infinity (the saddle) of the complex curve $C_1$, and the other connects the infinity of $L_2$ and the infinity (the saddle) of the complex curve $C_1$.

- On the invariant complex curve $C_2$, all regular orbits of system (9) are heteroclinic ones, which except two are heteroclinic to the singularity $S_2$ and the infinity of the complex line $L_1$, which is at the pole of the complex curve $C_2$. One of the two exceptional ones connects $S_2$ and the infinity (the saddle) of the complex curve $C_2$, and the other connects the infinity of $L_1$ and the infinity (the saddle) of the complex curve $C_2$.

- On the invariant complex curve $C_\ell$, $\ell \in \mathbb{C} \setminus \{0, -c^{-3}d^3\}$, all regular orbits of system (9) are heteroclinic, which except two are heteroclinic to the infinities of the invariant complex lines $L_1$ and $L_2$, which are at the two poles. The two exceptional ones connect one of the infinities of $L_1$ and $L_2$ and the infinity (the saddle) of the complex curve $C_\ell$.

This proves statement $(b)$, and consequently the theorem. \qed
5.2. Proof of Theorem 1.5. For the Hamiltonian system (10) with the Hamiltonian (11), we distinguish the two cases $c = 0$ and $d \neq 0$, and $c \neq 0$.

(a) $c = 0$. System (10) is simply of the form
\[ \dot{x} = -\nu x - 4dy^3, \quad \dot{y} = \nu y, \] (25)
whose associated Hamiltonian is $H_0 = \nu xy + dy^4$. Some easy calculations show that
- system (25) has the unique singularity at the origin;
- the phase space $\mathbb{C}^2$ of system (10) is foliated by
  - the invariant complex line $y = 0$, and
  - the invariant complex curves $x = \nu^{-1} (hy^{-1} - dy^3)$, $h \in \mathbb{C}$.

On the invariant complex line $y = 0$, system (25) is simply $\dot{x} = -\nu x$, which has the general solution
\[ x(t) = x_0 e^{-\nu t}, \]
where $x_0 \in \mathbb{C}$ is the initial value. Set $x_0 = \rho_0 e^{\theta i}$ and $\nu = \alpha + i\beta$, we can write the solution $x(t)$ in
\[ x(t) = \rho_0 e^{-\alpha t} e^{(\theta-\beta)it}. \]
This shows that on the invariant line $y = 0$, the origin is a center if $\alpha = 0$ and $\beta \neq 0$, or a star node if $\alpha \neq 0$ and $\beta = 0$, or a focus if $\alpha \beta \nu \neq 0$.

On the family of invariant complex algebraic curves $x = \nu^{-1} (hy^{-1} - dy^3)$, $h \in \mathbb{C}$, we distinguish $h = 0$ and $h \neq 0$. For $h = 0$ the curve $S_0 := \{ x = -\nu^{-1} dy^3 \}$ is diffeomorphic to the $y$-axis, and on it system (25) is simply $\dot{y} = \nu y$, whose dynamics is the same as those on $y = 0$ with reverse stability and rotating direction. For $h \neq 0$ each of the curves $C_h := \{ x = \nu^{-1} (hy^{-1} - dy^3) \}$ has a pole at $y = 0$ and it is diffeomorphic to the $y$-axis outside $y = 0$. Restricted to the invariant complex curve $C_h$, the dynamics of system (25) is determined by $\dot{y} = \nu y$. So according to the previous analysis we get the dynamics on $C_h$ as follows:
- If $\text{Re} \nu = 0$, $C_h$ is fulfilled by periodic orbits.
- If $\text{Re} \nu \neq 0$, all orbits are heteroclinic, and connect the two boundaries of $C_h$, which are the infinity at the pole and the infinity of $C_h$. Moreover, if $\text{Im} \nu \neq 0$ (resp. $= 0$), the orbits are spiral (resp. are radical).

We note that under the conditions of the statement system (25) is globally linearizable through the symplectic change of coordinates $u = -\nu x + dy^2$, $v = \nu^{-1} y$. By the dynamics of linear system $\dot{x} = -\nu x$ and $\dot{y} = \nu y$, one gets that if $\text{Re} \nu = 0$, it follows from Arnold [2, p.35] that the periodic orbits of system (25) form a family of invariant tori. This proves the statement.

(b) $c \neq 0$. Combining the Hamiltonian $H$ together with some easy calculations, we get the next results.
- The singularities are
  \[ S_1 = (0, 0), \]
  \[ S_2 = \left( 2ic^{-3/2} d\nu^{1/2}, -ic^{-1/2} \nu^{1/2} \right), \]
  \[ S_3 = \left( 2ic^{-3/2} d\nu^{1/2}, ic^{-1/2} \nu^{1/2} \right), \]
where $S_1$ has the eigenvalues $-\nu$ and $\nu$, and $S_2$ and $S_3$ have the same eigenvalues $-2\nu$ and $2\nu$.
- The phase space $\mathbb{C}^2$ is foliated by
  - three invariant complex lines $y = 0$, $y = \pm ic^{-1/2} \nu^{1/2}$, and
the invariant complex curves
\[ C_h := \left\{ x = \frac{h - dy^4}{\nu y + cy^3} \right\}, \quad h \in \mathbb{C}. \]

Restricted to the invariant line \( y = 0 \), the system is reduced to the linear one \( \dot{x} = -\nu x \), whose dynamics is well known as shown previously.

Restricted to the invariant lines \( y = \pm ic^{-1/2}\nu^{1/2} \), the system is reduced to \( \dot{x} = 2\nu x \pm 4i\nu^{-3/2}d\nu^{3/3} \). So their dynamics are the same as those on \( y = 0 \) with reverse orientation.

On the invariant complex curves \( C_h, \ h \in \mathbb{C} \), the dynamics of system (10) is determined by
\[ \dot{y} = \nu y + cy^3 = Q_0(y). \]

Now the three singularities of equation (26) correspond to the three invariant complex lines of system (10). For determining the local structure of these singularities we need Theorem 2.1 of [1], which states as follows.

**Lemma 5.1.** For a complex polynomial differential equation \( \dot{z} = f(z) \) of degree \( n \), \( z \in \mathbb{C} \), the following statements hold.

(1) Assume that \( z = z_0 \) is a simple finite singularity of the equation.

   (I) If \( f'(z_0) \in i\mathbb{R} \), then \( z_0 \) is an isochronous center,

   (II) If \( f'(z_0) \in \mathbb{R} \) then \( z_0 \) is a node,

   (III) If \( \text{Re}f'(z_0) \text{Im}f'(z_0) \neq 0 \) then \( z_0 \) is a focus.

(II) If \( z = z_0 \) is a zero of multiplicity \( m \geq 2 \) of \( f \), then it is a union of \( 2(m-1) \) elliptic sectors.

(III) At infinity the equations has exactly \( n - 1 \) couples of singularities, all of them being hyperbolic saddles. Moreover, they are uniformly distributed on the boundary of the Poincaré disk.

Simple calculations show that \( Q_0'(0) = \nu \), and \( Q_0'(\pm ic^{-1/2}\nu^{1/2}) = -2\nu \). So by Lemma 5.1 and \( \nu \neq 0 \) it follows that the three finite singularities of equation (26) are either all isochronous centers if \( \text{Re}\nu = 0 \), or all nodes if \( \text{Im}\nu = 0 \), or all foci if \( \text{Re}\nu\text{Im}\nu \neq 0 \). In case \( \text{Re}\nu \neq 0 \), the singularity \( y = 0 \) has an orientation different from that of \( y = \pm ic^{-1/2}\nu^{1/2} \). If case \( \text{Im}\nu \neq 0 \), the singularity \( y = 0 \) has different stability as that of \( y = \pm ic^{-1/2}\nu^{1/2} \).

By some calculations we get that equation (26) has the first integral
\[ F(t, y) = \frac{e^{-2(\nu + \overline{\nu})t}(y\overline{\nu})^2}{(\nu + cy^2)(\overline{\nu} + c\overline{y}^2)}, \quad (27) \]

where bar denotes the conjugate of a complex number. Now we apply this first integral to further discuss dynamics of equation (26).

(b1) If \( \text{Re}\nu = 0 \), i.e. \( \nu + \overline{\nu} = 0 \), the first integral \( F \) is independent of the time \( t \), and it is real analytic at \( y = 0 \). Moreover \( F^{-1} \) is real analytic at the other two singularities, i.e. \( y = \pm ic^{-1/2}\nu^{1/2} \), it follows from the classical Poincaré theorem that these three singularities are all centers. The Poincaré theorem we used here states that an elementary monodromy singularity of a real planar analytic differential system is a center if and only if the system has an analytic first integral defined in a neighborhood of the singularity.

From the expression of the first integral \( F \), one gets that in the phase space \( \mathbb{C} \), the orbits of equation (26) are described by
\[ \nu\overline{\nu} + \nu c\overline{y}^2 + c\nu y^2 + (c\overline{\nu} - \ell)(y\overline{\nu})^2 = 0, \quad \ell \in \mathbb{R}. \quad (28) \]
If $\ell = c\nu$, equation (28) defines an invariant hyperbola, denoted by $H$, which has the real expression $\nu_2 + 4c_1y_1y_2 + 2c_2(y_1^2 - y_2^2)$, where $\nu_2 = \text{Im}\nu$, $c_1 = \text{Re}c$, and $y_1 = \text{Re}y$ and $y_2 = \text{Im}y$. This invariant hyperbola $H$ separates the three centers in three disjoint regions. This claim follows from the facts that the hyperbola has either the asymptote $y_1 = 0$ and $y_2 = 0$ if $c_2 = 0$, or the asymptote $y_1 = -\left(c_1c_2^{-1} \pm (c_1^2c_2^{-2} + 1)\right)y_2$ if $c_2 \neq 0$, and that the two nonzero singularities $y = \pm \sqrt{-c^{-1}\nu}$ are symmetric with origin, and $W(0) = \nu\tau > 0$ and $W(\pm \sqrt{-c^{-1}\nu}) = -\nu\tau < 0$, where

$$W(y) := \nu\tau + \nu\tau y^2 + cy^2.$$  

For $\ell \neq c\nu$, taking the polar coordinate change of variables $y = re^{i\theta}$, the algebraic equation (28) can be written in

$$(c\nu - \ell)r^4 + (c\nu e^{2i\theta} + c\nu e^{-2i\theta}) r^2 + \nu\tau = 0.$$  

This shows that each invariant algebraic curve defined by (28) with $\ell \neq c\nu$ is periodic with the same period $\pi$.

We claim that these periodic orbits form three families of invariant tori surrounding each of the three invariant lines. Indeed, lifting the hyperbola $H$ to the phase space $\mathbb{C}^2$ one gets an invariant hyperbolic cylinder, which limits the three invariant lines in three disjoint regions. At each of the three finite singularities, system (10) can be linearized by a symplectic change of coordinates. This verifies the existence of the family of invariant tori around each of the three singularities.

(b2) If $\text{Re}\nu \neq 0$, i.e. $\nu + \tau \neq 0$, all three singularities are either all elementary nodes or all elementary foci. All regular orbits are given by

$$(\nu + cy^2)(\tau + c\nu y^2) = \ell e^{-2(\nu+i\tau)t}(y\bar{y})^2, \quad \ell \in \mathbb{R} \setminus \{0\},$$  

which follows from (27), and is equivalently written in

$$(\nu + cy^2)(\tau + c\nu y^2)e^{2(\nu+i\tau)t} = \ell(y\bar{y})^2, \quad \ell \in \mathbb{R} \setminus \{0\}.$$  

By Lemma 5.1 it follows that the infinity of equation (26) has exactly two pairs of hyperbolic saddles. Moreover, with the help of the information on stability of $y = 0$ and $y = \pm \sqrt{-c^{-1}\nu}$ we obtain that

- there are exactly four heteroclinic orbits connecting the two pairs of saddles at the infinity, in which two ones connecting the origin and one pair of singularities; and other two ones connecting $y = \pm \sqrt{-c^{-1}\nu}$ with one of the other pair of singularities.

By (29) and (30), and taking into account the limits $t \to \pm \infty$, one further gets that

- all the other regular orbits are also heteroclinic, and they either positively approach one of the singularities $y = \pm \sqrt{-c^{-1}\nu}$ and negatively to the origin when $\text{Re}\nu > 0$, or inverse when $\text{Re}\nu < 0$.

These last arguments verify statement (b2) and consequently statement (b).

It completes the proof of the theorem. □

Acknowledgments. The authors appreciate the anonymous referee for his/her nice comments and suggestions, which really improve the presentation.
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Received March 2020; revised October 2020.

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