Let $k$ be a $p$-adic field. Some time ago, D. Harbater [9] proved that any finite group $G$ may be realized as a regular Galois group over the rational function field in one variable $k(t)$, namely there exists a finite field extension $F/k(t)$, Galois with group $G$, such that $F$ is a regular extension of $k$ (i.e. $k$ is algebraically closed in $F$). Moreover, one may arrange that a given $k$-place of $k(t)$ be totally split in $F$. Harbater proved this theorem for $k$ an arbitrary complete valued field. Rather formal arguments ([10, §4.5]; §2 hereafter) then imply that the theorem holds over any ‘large’ field $k$. This in turn is a special case of a result of Pop [15], hence will be referred to as the Harbater/Pop theorem. We refer to [10], [16], [6] for precise references to the literature (work of Dèbes, Deschamps, Fried, Haran, Harbater, Jarden, Liu, Pop, Serre, and Völklein).

Most proofs (see [10], [19, 8.4.4, p. 93] and Liu’s contribution to [16]; see however [15]) first use direct arguments to establish the theorem when $G$ is a cyclic group (here the nature of the ground field is irrelevant), then proceed by patching, using either formal or rigid geometry, together with GAGA theorems.

In the present paper, where I take the case of algebraically closed fields for granted, I show how a technique recently developed by Kollár [12] may be used to give a quite different proof of the Harbater/Pop theorem, when the ‘large’ field $k$ has characteristic zero. This proof actually gives more than the original result (see comment after statement of Theorem 1).

Before I formally state the main result, let us recall what a ‘large’ field is. Let $k$ be a field and let $k((y))$ be the quotient field of the ring $k[[y]]$ of formal power series in one variable. Following F. Pop, we shall say that $k$ is ‘large’ if it satisfies one of the three equivalent properties ([15, Prop. 1.1]):

(i) It is existentially closed in $k((y))$: any $k$-variety with a $k((y))$-point has a $k$-point.

(ii) On a smooth integral $k$-variety with a $k$-point, $k$-points are Zariski dense.

(iii) On a smooth integral $k$-curve with a $k$-point, $k$-points are Zariski dense.
(Such a field is clearly infinite. By going over to the completion at a smooth $k$-point of a curve, one sees that (i) implies (iii). That (iii) implies (ii) is easy (consider a regular system of parameters). In characteristic zero, one may use resolution of singularities to show that (ii) implies (i).)

Known examples of ‘large’ fields $k$ are fraction fields of a henselian discrete valuation ring, such as a $p$-adic field or a field of the shape $k = F((x))$ for $F$ some field.

Other well-known examples are real closed fields. That these are ‘large’ is a special instance of the following fact, which seems to have escaped the attention of specialists: any field $F$, all finite field extensions of which are of degree a power of a fixed prime $p$, is a ‘large’ field. To see this, one only needs to observe that on a regular, projective, connected curve $C$ over a field $F$, given any nonempty open set $U$, any zero-cycle (divisor) $z$ on $C$ is rationally equivalent to a zero-cycle $z_1$ whose support is contained in $U$ (a semi-local Dedekind ring is a principal ideal domain); the degree (over $F$) of $z$ and $z_1$ clearly coincide. Applying this to an $F$-point of $C$, one produces a zero-cycle $\sum n_i P_i$ ($n_i \in \mathbb{Z}$, $P_i$ closed points) with support in $U$, such that the degree $\sum n_i [F(P_i) : F] = 1$. For $F$ as above, this forces one of the degrees $[F(P_i) : F]$ to be one.

Other known examples are the fields of totally real algebraic numbers and of totally $p$-adic algebraic numbers (that these fields are ‘large’ is a very special case of a theorem of Moret-Bailly [14, Thm. 1.3]). The property trivially holds for so-called pseudo algebraically closed fields, such as infinite algebraic extensions of a finite field.

**Theorem 1.** Let $G$ be a finite group. Let $k$ be a ‘large’ field of characteristic zero. Let $\mathcal{E} = \text{Spec}(K)$ be a $G$-torsor over $\text{Spec}(k)$. Then there exist an open set $U$ of the affine line $\mathbb{A}^1_k$ containing a $k$-point $O$ and a $G$-torsor $V \to U$ such that the following two properties hold:

(i) The fibre of $V \to U$ over $O$ is isomorphic to $\mathcal{E}$ (as a $G$-torsor over $\text{Spec}(k)$);

(ii) The smooth $k$-curve $V$ is geometrically connected.

The ring $K$ is a finite separable extension of $k$; it need not be a field. In loose terms: given a Galois extension $K/k$ with group $G$, one may realize $G$ as the Galois group of a ‘regular’ extension of $k(t)$, in such a way that over a suitable $k$-place of $k(t)$, the extension specializes to $K/k$.

When the $G$-torsor $\mathcal{E}/\text{Spec}(k)$ is trivial, i.e. $\mathcal{E} = \coprod_{g \in G} \text{Spec}(k)$, we recover the result of Harbater and Pop. The question whether $\mathcal{E}$ may be chosen arbitrary had been investigated for special groups by several authors (see [6]). For arbitrary groups, Débes proves a weaker result ([6, Thm. 3.1]) when $k$ is
‘large’, and he proves the theorem in the case where \( k \) is a pseudo algebraically closed field ([6, Thm. 3.2]).

Using general results from [EGA IV_3], we immediately obtain a series of concrete corollaries. These will be detailed in Section 2. In the case of a split \( \mathcal{E}/k \), most of them had already been obtained, with somewhat different proofs.

After the paper was submitted, I was asked whether in Theorem 1 one may impose arbitrary \( G \)-torsors as fibres of \( V \to U \) at more than one \( k \)-point of \( U \subset A^1_k \). The answer is in general in the negative, as shown in the appendix.

Let us say a few words on the tools used in this article. In a series of papers which appeared in 1992, Kollár, Miyaoka and Mori developed a technique which enables them, under some assumptions, to smooth a tree of rational curves into a single rational curve ([13, Thm. (2.1)]; see also [11, Chap. II. 7, pp. 154–158] and [5, §4.2]). That work was over an algebraically closed field. In his recent paper [12], Kollár extends the technique over ‘large’ fields (e.g. local fields). Under certain assumptions, he manages to deform a set of conjugate \( P^1 \)'s into a single \( P^1 \) defined over the ground field. From this he gets the finiteness of the set of \( R \)-equivalence classes on \( k \)-points of a geometrically rationally connected variety defined over a local field \( k \). That the key lemma of [12] precisely holds for ‘large’ fields provided the incentive for the present paper.

The proof I give for Theorem 1 starts from the classical fact that a finite group \( G \) is a Galois group over \( k(t) \) when \( k \) is algebraically closed of characteristic zero. It then uses a natural versal model for a \( G \)-torsor, and applies the deformation result of [12] to (a smooth compactification of) the base space of this \( G \)-torsor. The proof uses the existence of such a smooth compactification, but it avoids any consideration of the divisor at infinity: there is no discussion of inertia groups at all.

The idea of using a versal model of a \( G \)-torsor, originally due to E. Noether, has come up a number of times in the literature, notably in work of E. Fischer, D. Saltman [17], F. A. Bogomolov [1]; see [20] and [21] for further references.

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1. Proof of Theorem 1

In this section, we shall assume that the ground field \( k \) (which is of characteristic zero) is uncountable. The proof in the countable case will be given in Section 2.

Let \( \overline{k} \) be an algebraic closure of \( k \). Given a \( k \)-scheme \( Z \), let us write \( \overline{Z} = Z \times_k \overline{k} \).
(1) Let $G$ be a finite group and $E/\text{Spec}(k)$ a $G$-torsor. Let us fix an embedding of $G$ into some general linear group $\text{GL}_n$. Here $G$ is viewed as a constant (split) $k$-group scheme, $\text{GL}_n$ is the linear group over $k$ and $i: G \to \text{GL}_n$ is a homomorphism of $k$-group schemes. Let $U = \text{GL}_n/G$ be the affine $k$-variety of ‘left classes’. This is the affine $k$-scheme whose ring is the ring of invariants for $G$ acting on the ring $k[\text{GL}_n]$. The projection map $\text{GL}_n \to U$ makes $\text{GL}_n$ into a right $G$-torsor $V$ over $U$. The left action of $\text{GL}_n$ on itself induces a left action of $\text{GL}_n$ on $U = \text{GL}_n/G$ and the projection $V \to U$ is equivariant for these (left) actions.

Let us recall basic facts from noncommutative étale cohomology. Given any smooth affine $k$-group scheme $H$, and any commutative $k$-algebra $A$, we denote by $H^1_\text{ét}(A, H)$ the pointed cohomology set which classifies (étale) (right) $H \times_k A$-torsors over $\text{Spec}(A)$ (up to nonunique isomorphism). Such torsors will simply be called $H$-torsors over $A$. For any such $A$, there is an “exact sequence”

$$V(A) \to U(A) \to H^1_\text{ét}(A, G) \to H^1_\text{ét}(A, \text{GL}_n).$$

Let us detail this sequence. The map $V(A) \to U(A)$ is the obvious one; it respects the (left) action of $\text{GL}_n(A)$ on both sets. The right $G$-torsor $V \to U$ defines an element $\xi \in H^1_\text{ét}(U, G)$. To an element $\rho \in U(A) = \text{Hom}_k(\text{Spec}(A), U)$, the map $U(A) \to H^1_\text{ét}(A, G)$ associates the class $\rho^*(\xi) \in H^1_\text{ét}(A, G)$ of the pullback $\rho^*(V \to U)$, which is a $G$-torsor over $A$. Two points $x, y \in U(A)$ have the same image in $H^1_\text{ét}(A, G)$ if and only if there exists $\alpha \in \text{GL}_n(A)$ such that $\alpha x = y$. By Grothendieck’s version of Hilbert’s Theorem 90, the set $H^1_\text{ét}(A, \text{GL}_n)$ classifies projective modules of rank $n$ over $A$. Thus if $A$ is semilocal, or if $A$ is a Dedekind ring with trivial class group, then $H^1_\text{ét}(A, \text{GL}_n)$ is reduced to one element, and for any right $G$-torsor $\mathcal{T}$ over $A$ there exists an element $\rho \in U(A)$ such that $\mathcal{T}$ and $\rho^*(V \to U)$ are isomorphic $G$-torsors over $A$. In particular, there exists a $k$-point $P \in U(k)$ such that the fibre $V_P$ of $V$ above $P$ is a $G$-torsor isomorphic to the given $E/k$. We shall fix such a $k$-point $P$.

(2) By classical results (see [19, Chap. 6]), we know that $G$ is a ‘regular’ Galois group over $\bar{k}(t)$. In other words there exist a nonempty open set $W$ of the affine line $\mathbb{A}^1_k = \text{Spec}(\bar{k}[t])$ and a $G$-torsor over $W$ whose underlying variety is integral. Let $A$ be the semi-local ring of $\bar{k}[t]$ at $t = 0$ and $t = 1$, and let $S = \text{Spec}(A)$. Let us abuse notation and call 0, respectively 1, the points of $S$ defined by $t = 0$, respectively $t = 1$. Changing coordinates and semi-localizing produces a $G$-torsor $\mathcal{T}$ over $S$ such that $\mathcal{T}$ is an integral scheme.

By (1), there exists a nonconstant $\bar{k}$-morphism $\rho: S \to \bar{U}$ such that the pull-back of the $G$-torsor $\bar{V} \to \bar{U}$ under $\rho$ is isomorphic to the $G$-torsor $\mathcal{T}/S$. Given any $\alpha \in \text{GL}_n(A)$, the $G$-torsor $(\alpha, \rho)^*(\bar{V} \to \bar{U})$ is $G$-isomorphic to the $G$-torsor $\mathcal{T}$. In particular, it is an integral scheme.
(3) The action of $\text{GL}_n(\mathbb{K})$ on $\overline{U}(\mathbb{K})$ is transitive; hence the obvious action of $\text{GL}_n(\mathbb{K}) \times \text{GL}_n(\mathbb{K})$ on $\overline{U}(\mathbb{K}) \times \overline{U}(\mathbb{K})$ is also transitive. Reduction of $A$ modulo $t$ and modulo $t-1$ induces a surjective homomorphism $\text{GL}_n(A) \rightarrow \text{GL}_n(\mathbb{K}) \times \text{GL}_n(\mathbb{K})$. Thus given two points $M, N \in \overline{U}(\mathbb{K})$, there exists $\alpha \in \text{GL}_n(A)$ such that $\alpha.\rho \in \overline{U}(A)$ sends the point $t=0$ to $M$ and the point $t=1$ to $N$.

Remark. One should compare the present general position argument with ‘Kuyk’s lemma’ (see [20, Lemma 4.5]).

(4) Since $\text{char}(k)=0$, by Hironaka’s theorem, there exist smooth, projective, geometrically integral $k$-varieties $X_1$ and $X$, with $V$ open in $X_1$ and $U$ open in $X$, together with a $k$-morphism $p : X_1 \rightarrow X$ extending the map $V \rightarrow U$ and inducing a $k$-isomorphism $V \simeq p^{-1}(U)$.

(5) According to a theorem of Kollár, Miyaoka and Mori ([13]; [11, Thm. II. 3.11, p. 118]), to the point $\mathcal{P} \in \overline{U}(\mathbb{K}) \subset \overline{X}(\mathbb{K})$ one may associate countably many proper subvarieties $V_i (i \in I)$ of the smooth projective variety $\overline{X}$ such that if $f : \mathbb{P}^1_{\mathbb{K}} \rightarrow \overline{X}$ is a nonconstant morphism, $f(0) = \mathcal{P}$ and the image of $f$ is not contained in the union of the $V_i$’s, then $f$ is free over $0 \in \mathbb{P}^1_{\mathbb{K}}$. By definition (see [11, II. 3.1, p. 113]), this means that the coherent cohomology group $H^1(\mathbb{P}^1_{\mathbb{K}}, f^*T_{\overline{X}}(-2))$ vanishes (here $T_{\overline{X}}$ denotes the tangent bundle of $\overline{X}$), which amounts to the hypothesis that in Grothendieck’s decomposition of the vector bundle $f^*T_{\overline{X}}$ over $\mathbb{P}^1_{\mathbb{K}}$ as a sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(n_j)$, we have $n_j > 0$ for each $j$ (this is the ampleness property for the vector bundle $f^*T_{\overline{X}}$ on $\mathbb{P}^1_{\mathbb{K}}$, see [11, II.3.8, p. 116]).

Since $k$ is uncountable, there exists a point $Q \in \overline{U}(\mathbb{K})$, $Q \neq \mathcal{P}$, which does not lie on any of the $V_i$’s (proof: use a generically finite projection to projective space and induct on dimension). By (3), there exists $\alpha \in \text{GL}_n(A)$ such that $\alpha.\rho \in \overline{U}(A)$ sends the point $t=0$ to $\mathcal{P}$ and the point $t=1$ to $Q$. Since $X/k$ is proper, the morphism $\alpha.\rho : S \rightarrow \overline{U}$ extends to a (nonconstant) morphism $f : \mathbb{P}^1_{\mathbb{K}} \rightarrow \overline{X}$. The image of $f$ contains $\mathcal{P}$ and is not contained in the union of the $V_i$’s, since this image contains $Q$. By the quoted theorem ([11, II.3.11]), we conclude:

(5.1) The vector bundle $f^*T_{\overline{X}}$ on $\mathbb{P}^1_{\mathbb{K}}$ is ample.

On the other hand, we have:

(5.2) The underlying variety of the $G$-torsor $f^*(\mathbb{V} \rightarrow \overline{U})$ over $f^{-1}(\overline{U})$ is integral.

Indeed, this follows from the same statement for the restriction of this $G$-torsor over $S = \text{Spec}(A) \subset f^{-1}(\overline{U})$, which was pointed out at the end of (2).
(6) We have now reached the situation studied in [12]. Starting from \( f : \mathbf{P}^1_k \to \overline{X} \) such that \( f(0) = \overline{T} \) and \( f^* T_{\overline{X}} \) is ample, Kollár ([12, 3.2], I change notation) produces, over the ground field \( k \), a smooth integral \( k \)-curve \( C \) with a \( k \)-point \( O \), a smooth geometrically integral \( k \)-surface \( Z \) proper over \( C \), together with a \( k \)-morphism \( h : Z \to X \), with the following properties:

(6.a) The projection \( Z \to C \) admits a \( k \)-section \( \sigma : C \to Z \) which by \( h \) is mapped to \( P \in X \).

(6.b) The geometric fibre \( Z_{\overline{T}} \) of \( Z \to C \) at the point \( O \) is a comb \( D + \sum_{i \in I} C_i \) on \( \mathbb{Z} \) (here \( I \) is a nonempty finite set, the \( C_i \)'s are the teeth of the comb, see [11, II.7.7, p. 156]), each component of which is a nonsingular curve of genus zero; the map \( \overline{h} : Z \to \overline{X} \) sends \( D \) to \( \overline{T} \) and induces on \( C_i \) a conjugate of \( f : \mathbf{P}^1_k \to \overline{X} \).

(6.c) Over any closed point \( M \) of \( C \) different from \( O \), the fibre \( Z_M \) of \( Z \to C \) is \( k(M) \)-isomorphic to the projective line \( \mathbf{P}^1_{k(M)} \); the fibre is a smooth, geometrically irreducible, projective curve of genus zero over the residue field \( k(M) \), and it contains the \( k(M) \)-rational point \( \sigma(M) \).

(7) Since the map \( \overline{h} : Z_{\overline{T}} \to \overline{X} \) is not constant (because its restriction to any \( C_i \) is not constant), the closed set \( h^{-1}(P) \subset Z \) is a proper closed set. Thus, after shrinking \( C \), we may assume: for no \( M \in C \) is \( h \) constant on the fibre \( Z_M \) (note that on any fibre \( Z_M \), \( h \) assumes the value \( h(\sigma(M)) = P \times_k k(M) \)).

Let \( \Omega \subset Z \) be the inverse image of \( U \) under \( h \). Note that \( \Omega \) contains \( \sigma(C) \), hence the composite map \( \Omega \subset Z \to C \) is surjective. Let \( \Omega_1 \to \Omega \) be the inverse image of the \( G \)-torsor \( V \to U \) under \( h : \Omega \to U \). Let \( M \) be a closed point in \( C \). We shall show: For all but finitely many \( M \in C \), the total space of the induced \( G \)-torsor \( \Omega_{1,M} \to \Omega_M \subset Z_M \cong \mathbf{P}^1_{k(M)} \) is a smooth geometrically integral \( k(M) \)-variety.

To prove this, it is enough to prove the corresponding statement over \( \overline{k} \). For the rest of the proof of (7), to simplify notation, let us set \( k = \overline{k} \). Points \( M \) will be \( \overline{k} \)-rational points on \( C \). For \( M \neq O \), the (nonempty) variety \( \Omega_M \) is smooth and connected and the variety \( \Omega_{1,M} \) is a finite étale cover of \( \Omega_M \), hence is smooth. To prove that a given \( \Omega_{1,M} , M \neq O \), is integral, it is thus enough to show that it is connected.

The inverse image in \( \Omega_1 \) of \( D \cap \Omega \) is a disjoint union of copies \( D_g \) \( (g \in G) \) of \( D \cap \Omega \), each with multiplicity one; by (5.2) and (6.b), for a given \( i \in I \) the inverse image in \( \Omega_1 \) of each \( C_i \cap \Omega \) is a (smooth) connected curve, which meets each \( D_g \) \( (g \in G) \), since \( C_i \) meets \( D \) (see (6.b)). Thus \( \Omega_{1,O} \), which is the inverse image of \( D + \sum_{i \in I} C_i \), is a reduced connected divisor on \( \Omega_1 \).
That $\Omega_{1,M}$ is connected for all but finitely many $M \in C$ now follows from the general lemma (where $X$ and $Y$ have nothing to do with the previous $Y$ and $X$), to be applied to $X = \Omega_1$ and $Y = \Omega$:

**Lemma.** Let $C$ be a smooth, connected curve over an algebraically closed field $k$, and let $O \in C(k)$. Let $X, Y, C$ be smooth varieties over $k$, equipped with faithfully flat $k$-morphisms $X \to Y$ and $Y \to C$. Assume that the generic fibre of $Y \to C$ is smooth and geometrically integral. Assume that $X \to Y$ is finite and étale. Assume moreover that the inverse image of $O$ under the composite map $X \to Y \to C$ is a connected divisor on $X$ and is not a multiple divisor. Then there exists a finite set $S$ of points of $C$ such that for $M \in C, M \notin S$, the inverse image $X_M$ of $M$ under the composite map $X \to Y \to C$ is a smooth connected variety.

**Proof.** Note first that $X$ is connected. Indeed if it was not connected, the finite étale cover $X \to Y$ would break up into a disjoint union of finite étale (hence faithfully flat) covers $X_i \to Y$, and the fibre of $X \to Y \to C$ over $O$ would not be connected. Thus $X$ is connected; since it is smooth, it is integral. Let $D$ be the normalization of $C$ in the function field of $X$. This is a smooth integral curve, and the map $D \to C$ is flat and finite. Since $X$ is normal, the map $X \to C$ factors through $D$. The finite (étale) map $X \to Y$ factors through the scheme $Y \times_C D$. The scheme $Y \times_C D$ is integral, because $C$ is its own normalization in $Y$, since we have assumed that the generic fibre of $Y \to C$ is geometrically integral. The finite map of integral varieties $X \to Y \times_C D$ is dominant, hence surjective as a morphism of schemes (it need not be flat). In particular, it is surjective on $k$-points (recall $k = \overline{k}$). The projection map $Y \times_C D \to D$ is faithfully flat, since it is obtained by base change from the faithfully flat map $Y \to C$. In particular, $Y \times_C D \to D$ is surjective on $k$-points. We conclude that $X \to D$ is surjective on $k$-points. But then the scheme-theoretic inverse image of $O \in C$ under the map $D \to C$ must consist of one reduced point, since the inverse image of $O$ under the composite map $X \to D \to C$ is a connected divisor which is not multiple. Since $D \to C$ is finite and flat, this implies that $D \to C$ is an isomorphism. Thus the function field of $C$ is algebraically closed in the function field of $X$, hence the generic fibre of $X \to C$ is a smooth geometrically integral variety. By [EGA IV$_3$, (9.7.7)] this implies the same statement for all fibres of $X \to C$ away from a proper closed subset of $C$. □

(8) We finally make use of the hypothesis that the field $k$ is ‘large.’ Since the curve $C$ has a $k$-rational point, namely $O$, this hypothesis implies that there exists a $k$-point $M$ on $C$ away from the finitely many points excluded in (7), such that the map $\mathbb{P}^1_k \to X$ induced by $h$ on the fibre $Z_M \simeq \mathbb{P}^1_k$
does what we want: the inverse image of the $G$-torsor $V \to U$ under the map $h : h^{-1}(U) \cap \mathbf{P}^1 \to U$ is a $G$-torsor over the open set $h^{-1}(U) \subset \mathbf{P}^1$, whose fibre at $\sigma(M) \in h^{-1}(U)(k) \subset \mathbf{P}^1(k)$ is isomorphic to the fibre of $V \to U$ at $P$, hence is isomorphic to $E$ (by the very choice of $P$, see (1)), and whose total space is a geometrically integral $k$-variety (see (7)).

2. Corollaries

**Theorem 2.** Let $O$ be a $\mathbb{Q}$-point of the projective line $\mathbf{P}^1_{\mathbb{Q}}$. Let $G$ be a finite group and let $E = \text{Spec}(K) \to \text{Spec}(\mathbb{Q})$ be a $G$-torsor. There exist a smooth, geometrically integral curve $Y/\mathbb{Q}$ whose smooth compactification has a $\mathbb{Q}$-point, an open set $U \subset \mathbf{P}^1 \times_{\mathbb{Q}} Y$ containing $O \times_{\mathbb{Q}} Y$, and a $G$-torsor $V \to U$ (an étale Galois cover with group $G$), whose restriction to $O \times_{\mathbb{Q}} Y$ is the $G$-torsor $E \times_{\mathbb{Q}} Y$, and such that the fibre of the composite map $V \to U \to Y$ at any geometric point of $Y$ is nonempty and connected (hence integral).

**Proof.** Let $G \hookrightarrow \text{GL}_{n,\mathbb{Q}}$ be an embedding. The varieties $U, V, X, X_1$ which appear in the proof of Theorem 1 may all be defined over $\mathbb{Q}$. We also have $P \in U(\mathbb{Q}) \subset X(\mathbb{Q})$.

For any field $F$ with $\mathbb{Q} \subset F$, let us in this proof say that an $F$-morphism $f : \mathbf{P}^1 \to X_F$ is good if $f(O) = P_F$ and the inverse image of $V_F \to U_F$ under $f$ (restricted to $f^{-1}(U_F)$) is a geometrically integral $F$-variety. Let $Z = \text{Hom}_{\mathbb{Q}}(\mathbf{P}^1, X, O \to P)$ (notation as in [11, II.1.4, p. 94]). This is a countable union of $\mathbb{Q}$-varieties $Z_d$ ($d$ for degree of the image of $\mathbf{P}^1$, in a fixed projective embedding of $X$). An $F$-point of $Z$ will be called good if the corresponding $F$-morphism $f : \mathbf{P}^1 \to X_F$ is good. Given arbitrary field extensions $\mathbb{Q} \subset E_1 \subset E_2$, a point in $Z(E_1)$ is good if and only if its image in $Z(E_2)$ is good.

The field $\mathbb{Q}(\!(x)\!)$ is uncountable. By Theorem 1 over such a field, as proved in Section 1, there exists a good $\mathbb{Q}(\!(x)\!)$-point on $Z$, hence on $Z_d$ for some $d$. Let $Y \subset Z_d$ be the scheme-theoretic closure of the image of the corresponding morphism $\text{Spec}(\mathbb{Q}(\!(x)\!)) \to Z_d$. The $\mathbb{Q}$-variety $Y$ is geometrically integral. We have the field embeddings $\mathbb{Q} \subset \mathbb{Q}(Y) \subset \mathbb{Q}(\!(x)\!)$. Thus on the one hand the generic point of $Y$ is a good $\mathbb{Q}(Y)$-point of $Z$; on the other hand any $\mathbb{Q}$-compactification of $Y$ has a $\mathbb{Q}$-point. Indeed, for any such compactification $Y_c$, the map $\text{Spec}(\mathbb{Q}(\!(x)\!)) \to Y$ extends to a $\mathbb{Q}$-morphism $\text{Spec}(\mathbb{Q}(\![x]\!)) \to Y_c$; the image of $x = 0$ is a $\mathbb{Q}$-point of $Y_c$.

Replacing $Y$ by a nonempty open set, one may ensure ([EGA IV$_3$, (8.8.2)]) that the corresponding good $\mathbb{Q}(Y)$-morphism $\mathbf{P}^1_{\mathbb{Q}(Y)} \to X_{\mathbb{Q}(Y)}$ extends to a $Y$-morphism $\varphi : \mathbf{P}^1 \times_{\mathbb{Q}} Y \to X \times_{\mathbb{Q}} Y$ which sends $O \times_{\mathbb{Q}} Y$ to $P \times_{\mathbb{Q}} Y$.  


Let $\Omega = \varphi^{-1}(U \times \mathbb{Q}^1 Y) \subset P^1 \times \mathbb{Q}^1 Y$ and let $\Omega_1 \to \Omega$ be the $G$-torsor which is the inverse image of the $G$-torsor $V \times \mathbb{Q}^1 Y \to U \times \mathbb{Q}^1 Y$ under $\varphi$. Upon replacing $Y$ by a nonempty open set (this is actually not necessary), the restriction of this $G$-torsor over $O \times \mathbb{Q}^1 Y \subset \Omega$ is isomorphic to $E \times \mathbb{Q}^1 Y$ (indeed, this is true over the generic point of $Y$). We have the maps $\Omega_1 \to \Omega \to Y$. The first map is finite étale of constant rank, the second one is smooth and surjective. Thus the composite map $\Omega_1 \to Y$ is smooth. Since the generic point of $Y$ corresponds to a good point of $Z$, the generic fibre $\Omega_1, \mathbb{Q}^1 Y$ is geometrically integral over $\mathbb{Q}(Y)$. Upon replacing $Y$ by a nonempty open set ([EGA IV 3, (9.7.7)(iv)]), we therefore have that all geometric fibres of the map $\Omega_1 \to Y$ are smooth and geometrically integral. In particular for any field $F$ with $\mathbb{Q} \subset F$ and any $F$-point of $Y$, the morphism $\varphi_F : P^1_F \to X_F$ induced by $\varphi$ is good.

On a smooth projective model $Y_c$ of $Y$ over $\mathbb{Q}$, there exists a $\mathbb{Q}$-point $R$. By considering a regular system of parameters at $R$ one produces a geometrically integral $\mathbb{Q}$-curve $C \subset Y_c$, smooth at $R$, and which meets $Y$. One now replaces $Y$ by $Y \cap C$. This completes the proof of Theorem 2.

**Remarks and corollaries.**

(1) Note that $Y$ in Theorem 2 need not have a $\mathbb{Q}$-point. But for any field $k$ containing $\mathbb{Q}$ such that $Y(k) \neq \emptyset$, $G$ is a ‘regular’ Galois group over the rational field $k(t)$, with the added information that the fibre at the point $t = 0$ is isomorphic to the torsor $E \times \mathbb{Q} k$. This applies in particular to any ‘large’ field of characteristic zero, thus completing the proof of Theorem 1 for fields which are countable.

(2) One should compare Theorem 2 with the contribution of Deschamps in [16], and the proof given here with that given in [7, 4.2].

(3) One amusing corollary is that for any finite group $G$, there exists a finite set of number fields $k_i$ such that the greatest common denominator of the degrees $[k_i : \mathbb{Q}]$ is equal to one, and such that $G$ is a ‘regular’ Galois group over each $k_i(t)$, hence in particular a Galois group over each $k_i$. The proof is simple: on the smooth compactification $Y_c$ of the curve $Y$, there exists a $\mathbb{Q}$-point, call it $M$. If we let $S \subset Y_c$ be the complement of $Y$ in $Y_c$, there exists a zero-cycle $\sum_{i \in I} n_i P_i$ (here the $n_i$ are integers, $P_i$ is a closed point and $I$ is finite) on $Y_c$ which is rationally equivalent to $M$, hence of degree one, and whose support is foreign to $S$, i.e. whose support is contained in $Y$. Let $k_i$ be the residue field at the closed point $P_i$. Then $\sum_{i \in I} n_i[k_i : \mathbb{Q}] = 1$ and $Y(k_i) \neq \emptyset$ for each $i$, hence the claim.

One could say that, for any group $G$, the inverse Galois group problem over $\mathbb{Q}$ acquires a positive answer when passing from rational points to ‘zero-cycles of degree one’.
This could have been noticed earlier. For any prime $p$, let $K_p$ be the fixed field of a pro-$p$-Sylow subgroup of the absolute Galois group of $\mathbb{Q}$. As proved in the introduction of this paper, $K_p$ is a ‘large’ field. By Theorem 1 (or, for that matter, the Harbater/Pop theorem), $G$ is a regular Galois group over $K_p(t)$. There exists a finite subextension $L_p/\mathbb{Q}$ of $K_p/\mathbb{Q}$, such that $G$ is a regular Galois group over $L_p(t)$. By Hilbert’s irreducibility theorem, $G$ is a Galois group over the number field $L_p$, whose degree $[L_p : \mathbb{Q}]$ is prime to $p$.

(4) Starting from the statement of Theorem 2 and writing a model of the whole situation over an open set of the ring of integers (same references to [EGA IV.3] as above), one easily deduces the following result, which is a special case of a theorem of Fried and Völklein: For a given finite group $G$, for almost all primes $p$ (‘almost all’ depending on $G$), $G$ is a ‘regular’ Galois group over $\mathbb{F}_p(t)$ (see [10] and [7, 3.9] for references; in [7] a model-theoretic argument is given). Simply note that if $Y/Z$ is a smooth integral model of the smooth, geometrically integral curve $Y/\mathbb{Q}$, then by classical estimates (Weil) we have $Y(\mathbb{F}_p) \neq \emptyset$ for almost all primes $p$. Here again, the present proof enables us to get more: if we start off with a given $G$-torsor $E$ over a nonempty open set of $\text{Spec}(\mathbb{Z})$, we may satisfy the additional requirement that for almost all primes $p$ the ‘regular’ Galois extension over $\mathbb{F}_p(t)$ be unramified at $t = 0$, the fibre being isomorphic to $E \times_{\mathbb{Z}} \mathbb{F}_p$.

Appendix

In this appendix, where for simplicity I assume all fields to be of characteristic zero, I address the question:

Let $k$ be a field, $G$ a finite group, $n \geq 1$ an integer. Let $E_1, \ldots, E_n$ be $G$-torsors over $k$. Can one find an open set $U \subset \mathbb{A}^1_k$, a $G$-torsor $V \to U$ and $n$ points $P_1, \ldots, P_n \in U(k)$ such that for each $i$, the fibre $V_{P_i}$ is isomorphic to $E_i$ as a $G$-torsor over $k$?

Here are two cases where the answer is in the affirmative:

(i) $G$ is an abelian group, its 2-primary subgroup is of exponent $2^r$, the cyclotomic field extension $k(\mu_{2^r})/k$ is cyclic, and $n$ is arbitrary. This is a special case of [3, Thm. 7.9] (various versions of this statement exist in the literature; see [17], [20]).

(ii) $G$ is arbitrary, $k$ is ‘large’ and $n = 1$: this is Theorem 1 of the present paper (with the additional piece of information that $V$ may be chosen geometrically integral).

In this appendix, I show by examples that for $n \geq 2$ and $k$ ‘large’ the answer to the above question is in general in the negative.
In the first part of the appendix, written in April 1999, I consider the case left open in (i) above. I give an example with $G = \mathbb{Z}/8$ and $k$ the 2-adic field $\mathbb{Q}_2$. As may be expected, this example is closely related to Wang’s counterexample to Grunwald’s theorem.

In the second part of the appendix, written in November 1999, for an arbitrary prime $p$, I give examples with $G$ a $p$-group and $k$ a suitable ‘large’ field. That part builds upon work of Saltman [18].

Background and references for the first part of the appendix (algebraic tori, quasi-trivial and flasque tori, groups of multiplicative type, $R$-equivalence) will be found in [2], [3], and [21]. For $G$ a commutative algebraic group over a field $k$, the étale cohomology group $H^1_{\text{ét}}(k, G)$ may be identified with a Galois cohomology group, and will be simply denoted $H^1(k, G)$.

**Proposition A.1.** Let $k$ be a field and $A$ be a finite abelian group. One may embed the constant $k$-group scheme $A$ into a commutative diagram of exact sequences of $k$-groups of multiplicative type:

$$
\begin{array}{cccccc}
1 & \to & A & \to & P_1 & \to & T & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & = \\
1 & \to & F & \to & P_2 & \to & T & \to & 1 \\
\end{array}
$$

where $T$ is a $k$-torus, $F$ is a flasque $k$-torus and $P_1$ and $P_2$ are quasi-trivial $k$-tori.

**Proof.** By the well-known duality $M \mapsto \hat{M} = \text{Hom}_{k-\text{gr}}(M, \mathbb{G}_{m,k})$ between $k$-groups of multiplicative type and finitely generated Galois modules over $k$, it is enough to prove the dual result. There exist exact sequences of finitely generated Galois modules

$$0 \to \hat{T} \to \hat{P}_1 \to \hat{A} \to 0$$

and

$$0 \to \hat{P} \to \hat{F} \to \hat{A} \to 0$$

with $\hat{P}_1$ and $\hat{P}$ permutation modules, and $\hat{F}$ a flasque module (for the second sequence, see [3, (0.6.2)]). The pull-back of the first sequence under the map $\hat{F} \to \hat{A}$ is an exact sequence

$$0 \to \hat{T} \to \hat{P}_2 \to \hat{F} \to 0$$

where the module $\hat{P}_2$ is an extension of the permutation module $\hat{P}_1$ by the permutation module $\hat{P}$, hence is itself a permutation module. Taking duals yields the proposition. $\Box$
For a quasi-trivial $k$-torus $P$, Hilbert's Theorem 90 implies $H^1(k, P) = 0$. Passing over to Galois cohomology in the diagram of Proposition A.1, we get the commutative diagram of exact sequences

\[
P_1(k) \rightarrow T(k) \rightarrow H^1(k, A) \rightarrow 0
\]
\[
P_2(k) \rightarrow T(k) \rightarrow H^1(k, F) \rightarrow 0.
\]

From this diagram it immediately follows that the map $H^1(k, A) \rightarrow H^1(k, F)$ is onto.

Let us recall the following basic fact from [2]: the map $T(k) \rightarrow H^1(k, F)$ induces an isomorphism $T(k)/R \cong H^1(k, F)$. Here $R$ denotes $R$-equivalence ([2, §4]) on the set of $k$-points of the $k$-torus $T$.

**Proposition A.2.** With notation as above, assume that there exists $\xi \neq 0 \in H^1(k, F)$. Let $\eta \in H^1(k, A)$ denote a lift of $\xi$ under the surjective map $H^1(k, A) \rightarrow H^1(k, F)$. Then there do not exist an open set $U \subset \mathbb{A}^1_k$ and an $A$-torsor $X \rightarrow U$ with the following properties: there exist points $M, N \in U(k)$ such that the fibre of $X \rightarrow U$ at $M$ is trivial while the fibre of $X \rightarrow U$ at $N$ has class $\eta \in H^1(k, A)$.

**Proof.** Let us assume there exist such $U, M, N$. Since $P_1$ is a quasi-trivial $k$-torus, for any $k$-scheme $V$ the étale cohomology group $H^1_{\text{ét}}(V, P_1)$ is isomorphic to a sum of groups $\text{Pic}(V \times_k K_i)$, where the $K_i/k$ are finite separable field extensions of $k$. For $U \subset \mathbb{A}^1_k$, we thus have $H^1_{\text{ét}}(U, P_1) = 0$. Hence the map $T(U) \rightarrow H^1_{\text{ét}}(U, A)$ associated to the upper exact sequence in the diagram of Proposition A.1 is onto. There thus exists a $k$-morphism $\varphi : U \rightarrow T$ such that $\varphi^*(P_1) \rightarrow T$ is isomorphic to the $A$-torsor $X \rightarrow U$. The map $T(k) \rightarrow H^1(k, A)$ sends $\varphi(M)$ to 0, and it sends $\varphi(N)$ to $\eta$. Thus the map $T(k) \rightarrow H^1(k, F)$ sends $\varphi(M)$ to 0, and it sends $\varphi(N)$ to $\xi \neq 0$. Now since $U$ is an open set of $\mathbb{A}^1_k$, the points $\varphi(M) \in T(k)$ and $\varphi(N) \in T(k)$ are $R$-equivalent: their images under the map $T(k) \rightarrow H^1(k, F)$ should coincide. This contradiction establishes our contention. \qed

We still need to exhibit one case where the hypotheses of Proposition A.2 are fulfilled. Let $k$ be a field, let $A = \mathbb{Z}/8$ and let $T$ and $F$ be two $k$-tori as in Proposition A.1. Suppose the cyclotomic field extension $k(\mu_8)/k$ has degree 4. Its Galois group is then $\mathbb{Z}/2 \times \mathbb{Z}/2$. In that case, we have $H^1(k, F) = \mathbb{Z}/2$ ([21, §7.4, p. 79]). If $k$ is a $p$-adic field, then the finite abelian groups $H^1(k, S)$ and $H^1(k, S)$ are dual (Tate-Nakayama). Let $k$ be the 2-adic field $\mathbb{Q}_2$. The field extension $\mathbb{Q}_2(\mu_8)/\mathbb{Q}_2$ has degree 4; we thus have $H^1(\mathbb{Q}_2, F) \neq 0$.\n
This completes the construction of the announced example, but one can be more explicit. Let $k = \mathbb{Q}_2$. As a class $\eta \neq 0 \in H^1(k, \mathbb{Z}/8)$, let us take the class of the degree 8 unramified field extension $E$ of $k = \mathbb{Q}_2$. Let us write the commutative diagram in Proposition A.1 over $\mathbb{Q}$. One may then write the ensuing commutative diagram over $\mathbb{Q}$ and over $\mathbb{Q}_2$, in a compatible manner. Let $M \in T(k)$ be any point with image $\eta$ in $H^1(k, \mathbb{Z}/8)$. Suppose the image of $\eta$ in $H^1(k, F)$ is trivial. Then $M$ comes from a $k$-point of $P_2$. But then the point $M$ lies in the closure of $T(\mathbb{Q})$ in $T(\mathbb{Q}_2)$, since $P_2/\mathbb{Q}$ is a quasi-trivial torus, hence $\mathbb{Q}$-isomorphic to an open set of some affine space over $\mathbb{Q}$. One can then find a $\mathbb{Q}$-point $N$ of $T$ such that the fibre of $P_1 \to T$ at $N$ is a Galois extension $F/\mathbb{Q}$ with group $\mathbb{Z}/8$ and such that $F \otimes_{\mathbb{Q}} \mathbb{Q}_2 \simeq E$ (as Galois extensions of $\mathbb{Q}_2$ with group $\mathbb{Z}/8$). But there is no such extension (Wang’s well-known counterexample to Grunwald’s theorem, see [17] and [20]). Thus the image of $\eta$ in $H^1(k, F)$ is nontrivial.

Let us now turn to other types of examples.

**Proposition A.3.** Let $p$ be a prime number. There exist a $p$-group $G$, a ‘large’ field $k$, and $G$-torsors $\mathcal{E}_1$ and $\mathcal{E}_2$ over $k$ with the following property: given any $G$-torsor $f : V \to U$ over an open set $U$ of $\mathbb{A}_k^1$, there do not exist $k$-points $P, Q \in U(k)$ such that the $G$-torsor $V_P$ is isomorphic to $\mathcal{E}_1$ and the $G$-torsor $V_Q$ is isomorphic to $\mathcal{E}_2$.

**Proof.** Saltman’s work [18] (extended by Bogomolov [1], see [21, §7.6 and §7.7]) produces finite $p$-groups $G$ together with faithful (finite dimensional) linear representations $W$ of $G$ over the complex field $\mathbb{C}$, such that the unramified Brauer group $\text{Br}_{nr}(F)$ of $F = \mathbb{C}(W)^G$ is a nontrivial ($p$-primary) group. Here by $\mathbb{C}(W)$ we denote the fraction field of the symmetric algebra on $W$. The unramified Brauer group of $F$ is the subgroup of the Brauer group $\text{Br}(F)$ consisting of classes which are unramified with respect to any (rank one) discrete valuation on $F$. As is well-known, the group $\text{Br}_{nr}(\mathbb{C}(W)^G)$ does not depend on the particular faithful (finite dimensional) linear representation of $G$.

Let us fix one such $p$-group $G$. As in the beginning of Section 1, let us fix a homomorphic embedding $G \to \text{GL}_n = \text{GL}_n, \mathbb{C}$. We may take for $W$ the vector space of $\mathbb{C}$-points of $M_n$ (the ring scheme of $n$ by $n$ matrices over $\mathbb{C}$), with the action induced by left multiplication. Let $U = \text{GL}_n/G$ and $V = \text{GL}_n \subset M_n$. Projection $V \to U$ makes $V$ into a $G$-torsor, whose properties are described at the beginning of Section 1.

By Hironaka’s theorem, there exists a smooth projective variety $X/\mathbb{C}$ containing $U$ as a dense open set. The function field $\mathbb{C}(X)$ of $X$ is $F$. By results of Grothendieck, the natural map from the étale Brauer group $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ to $\text{Br}(F)$ is one-to-one, and it induces an isomorphism $\text{Br}(X) \simeq \text{Br}_{nr}(F)$ (see [4]). Let $A \in \text{Br}(X) \subset \text{Br}(F)$ be a nontrivial element. Let $X_F$
be the smooth, projective $F$-variety $X_F = X \times_C F$. This contains the open set $U_F = U \times_C F$. On the other hand, the natural field embedding $C \subset F$ induces an inclusion $X(C) \subset X_F(F)$ of the set of $C$-rational points of $X$ into the set of $F$-rational points of $X_F$, and similarly $U(C) \subset U_F(F)$. Let $P \in U_F(F)$ be an arbitrary point in that subset. On the other hand, the generic point $\text{Spec}(F) \to X$ of $X$ gives rise (via the diagonal map) to an $F$-rational point $Q$ of $Y$. Let $A_F \in \text{Br}(X_F)$ be the inverse image of $A$ under the projection map $X_F \to X$. Let us evaluate $A_F$ on the $F$-rational points $P$ and $Q$. We have $A_F(P) = 0 \in \text{Br}(F)$ because $A_F(P)$ comes from $\text{Br}(C)$. We have $A_F(Q) \neq 0 \in \text{Br}(F)$ because $A_F(Q)$ is none other than the image of $A \in \text{Br}(X)$ under the embedding $\text{Br}(X) \hookrightarrow \text{Br}(F)$. Let $k$ be a field, $F \subset k$, such that the induced map $\text{Br}(F) \to \text{Br}(k)$ is one-to-one. Changing the base field from $F$ to $k$, we obtain rational points which we still denote $P, Q$ in $X_k(k)$, such that $A_k(P) = 0$ and $A_k(Q) \neq 0$ in $\text{Br}(k)$. The points $P, Q$ both lie in $U_k = U \times_C k$. Let $E_1 = V_P$, respectively $E_2 = V_Q$, be the $G$-torsors over $k$ defined as the fibre of the $G$-torsor $V \to U$ at $P$, respectively $Q$. Suppose there exist a $G$-torsor $Z \to Y$ over an open set $Y \subset A^1_k$ and two $k$-points $p, q \in Y(k)$ such that the fibre $Z_p$, respectively $Z_q$, is a $G$-torsor over $k$ isomorphic to $E_1$, respectively $E_2$. By the general properties of the $G$-torsor $V_k \to U_k$ (see beginning of §1) and the fact that $\text{Pic}(Y) = 0$, there exists a $k$-morphism $r : Y \to U_k$ such that the inverse image of the $G$-torsor $V_k \to U_k$ under $r$ is isomorphic to the $G$-torsor $Z \to Y$. Let $P_1 = r(p) \in U(k)$ and $Q_1 = r(q) \in U(k)$. Then $V_P$ and $V_{P_1}$ are isomorphic as $G$-torsors over $k$, and similarly $V_Q$ and $V_{Q_1}$. The general properties of the $G$-torsor $V \to U$ then imply that there exist $g, h \in \text{GL}_n(k)$ such that $gP_1 = P$ and $hQ_1 = Q$. Since $\text{GL}_n$ is an open set of an affine space over $k$, this implies that the $k$-points $P_1$ and $P$ of $U_k(k) \subset X_k(k)$ are $R$-equivalent. Similarly, $Q_1$ and $Q$ are $R$-equivalent. Clearly, $P_1$ and $Q_1$ are $R$-equivalent. Thus $P$ and $Q$ are $R$-equivalent on the projective $k$-variety $X_k$. By Prop. 16 of [2] (p. 213) this implies $A_k(P) = A_k(Q)$. But then we cannot have $A_k(P) = 0$ and $A_k(Q) \neq 0$.

To complete the proof of Proposition A.3, it remains to notice that the field $k = F((t))$ of formal power series in one variable is a ‘large’ overfield of $F$ for which the map $\text{Br}(F) \to \text{Br}(k)$ is one-to-one. \hfill \Box

Whether examples as in Proposition A.3 may be exhibited over a $p$-adic field remains to be seen.

C.N.R.S., U.M.R. 8628, Université de Paris-Sud, Orsay, France
E-mail address: colliot@math.u-psud.fr
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