Convergence towards asymptotic state in 1-D mappings: a scaling investigation

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Abstract

Decay to asymptotic steady state in one-dimensional logistic-like mappings is characterized by considering a phenomenological description supported by numerical simulations and confirmed by a theoretical description. As the control parameter is varied bifurcations in the fixed points appear. We verified at the bifurcation point in both; the transcritical, pitchfork and period-doubling bifurcations, that the decay for the stationary point is characterized via a homogeneous function with three critical exponents depending on the nonlinearity of the mapping. Near the bifurcation the decay to the fixed point is exponential with a relaxation time given by a power law whose slope is independent of the nonlinearity. The formalism is general and can be extended to other dissipative mappings.

Key words: Chaos; Survival probability, parameter space, Shrimps.

1. Introduction

Mappings are often used to characterize the evolution of dynamical systems by using the so called discrete time [1]. The interest in the subject was increased after the investigation of May [2] with direct application to biology [3]. After that a large number of applications involving mappings were considered particularly related to physics [4,5,6], chemistry, biology, engineering, mathematics and many others [7,8,9,10,11,12,13,14,15,16,17,18].

The type of dynamics certainly depends on the control parameter. As it is varied, bifurcations appear changing the dynamics of the steady state [1], and for specific rates eventually leads to chaos [19,20]. Collisions of stable and unstable manifolds yield the destruction of chaotic attractors [4,21]. Many of these dynamical properties are already known and are taken indeed at the asymptotic state. The way the system goes to equilibrium is generally disregarded just by considering the evolution for a large transient.

Our main goal in this Letter is to apply a scaling formalism to explore the evolution towards the equilibrium near three types of bifurcations in a logistic-like mapping: (a) transcritical; (b) pitchfork and (c) period-doubling. Indeed at the bifurcation point the orbit relays to equilibrium in a way described by a homogeneous function with well defined critical exponents [22,23,24]. Such exponents are not universal and depend mostly on the nonlinearity of the mapping and on the type of bifurcation. Near a bifurcation, the relaxation to the equilibrium is exponential, with a relaxation time characterized by a power law [22]. Here, two different procedures are used to obtain the exponents. The first one is mostly phenomenological with scaling hypotheses ending up with a scaling law of the three critical exponents. The second one considers transforming the difference equation into a differential one and solving it with the convenient initial conditions. Our analytical results confirm remarkably well the numerical data obtained via computer simulation.

This piece of work is organized as follows. First the mapping, the equilibrium conditions and a phenomenological approach leading to the scaling law is described. Then we discuss the critical exponents by transforming the difference equation into a differential equation. Moving on we present discussions and extensions to other bifurcation when finally our conclusions are drawn.

2. The mapping and phenomenological properties of the steady state

The mapping we consider is written as

\[ x_{n+1} = R x_n (1 - x_n^\gamma) \]  

where \( \gamma \geq 1 \), \( R \) is a control parameter and \( x \) is a dynamical variable. A typical orbit diagram is shown in Fig. 1 for: (a) \( \gamma = 1 \) (logistic map) and; (b) \( \gamma = 2 \) (cubic map).

The fixed points are obtained by solving \( x_{n+1} = x_n = x^* \) and two cases must be considered: (i) \( \gamma \) is an even number or; (ii) \( \gamma \) is any other value (odd, irrational etc). For case (i) there are three fixed points. One is \( x^* = 0 \), which is stable (asymptotically stable) for \( R \in [0,1) \) and the two others are \( x^* = \pm [1 - 1/R]^{1/\gamma} \), which are stable for \( R \in (1, (2 + \gamma)/\gamma) \). The bifurcation at \( R = 1 \) is called pitchfork [25,26]. For case (ii) there are only two fixed points. One is \( x^* = 0 \), stable for \( R \in [0,1) \) and the other is \( x^* = [1 - 1/R]^{1/\gamma} \), being stable for
$R \in (1, (2 + \gamma)/\gamma)$. Transcritical is the bifurcation at $R = 1$ for this case. Both bifurcations are identified in Fig. 1(a,b). Our goal is to consider the convergence to the fixed point $x^* = 0$ at the bifurcation in $R_c = 1$ and in its neighbouring such that $\mu = R_c - R \approx 0$, with $R \leq R_c$.

The orbit diagram allows to extract more properties. After a transcritical bifurcation, as seen in Fig. 1(a), the period-1 orbit is stable for the range $R \in (1, 3)$ when a period-doubling bifurcation happens. After that a period-doubling sequence is observed, obeying a Feigenbaum scaling [19, 20] until reach the chaos. Similar dynamics, for a different range of $R$ is also observed after a pitchfork bifurcation, as shown in Fig. 1(b).

The natural variable to describe the convergence to the steady state is the distance from the fixed point $x^* = 0$. The convergence to the steady state must also depends on the number of iterations $n$, on the initial condition $x_0$, and of course on the parameter $\mu = R_c - R$. The parameter $\mu = 0$ defines the bifurcation point and the convergence to the fixed point is shown in Fig. 2 for two different values of $\gamma$: (a) $\gamma = 1$ and; (b) $\gamma = 2$ and different initial conditions $x_0$, as labelled in the figure.

We see from Fig. 2 that depending on the initial condition $x_0$, the orbit stays confined in a plateau of constant $x$ and, after reaching a crossover iteration number, $n_x$, the orbit suffers a changeover from a constant regime to a power law decay marked by a critical exponent $\beta$. The length of the plateau also depends on the initial $x_0$. Based on the behaviour observed from Fig. 2 we can suppose that:

1. For a sufficient short $n$, say $n \ll n_x$, the behaviour of $x$ vs. $n$ is given by
   \[ x(n) \propto x_0^\alpha, \quad \text{for} \quad n \ll n_x, \]
   and because $x \propto x_0$, we conclude that the critical exponent $\alpha = 1$.

2. For sufficient large $n$, i.e., $n \gg n_x$, the dynamical variable is described as
   \[ x(n) \propto n^\beta, \quad \text{for} \quad n \gg n_x, \]
   where the exponent $\beta$ is called a decay exponent. The numerical value is not universal and depends on the nonlinearity of the mapping.
3. Finally, the crossover iteration number \( n_c \) is given by
\[
n_c \propto x_0^\gamma,
\]
where \( \gamma \) is a changeover exponent.

The exponents \( \beta \) and \( z \) can be obtained by considering specific plots. After the constant plateau, a power law fitting furnishes \( \beta \). Indeed for \( \gamma = 1 \) (logistic map) we found \( \beta = -0.99981(3) \) while for \( \gamma = 2 \) (cubic map) we obtained \( \beta = -0.49969(5) \). To obtain the exponent \( \gamma \) we must have the behaviour of \( n_c \) vs. \( x_0 \), where \( n_c \) is obtained as the crossing of the constant plateau by the power law decay, as shown in Fig. 3.

The slope obtained for \( \gamma = 1 \), as shown in Fig. 3 is \( z = -1.0002(3) \) while for \( \gamma = 2 \) the exponent obtained is \( z = -2.001(2) \).

The behaviour shown in Fig. 3 together with the three scaling hypotheses allow us to describe the behaviour of \( x \) as an homogeneous function of the variables \( n \) and \( x_0 \), when \( \mu = 0 \), of the type
\[
x(x_0, n) = l a(x_0, \beta n) ,
\]
where \( l \) is a scaling factor, \( a \) and \( b \) are characteristic exponents. Because \( l \) is a scaling factor we choose \( l = x_0^{-1/a} \). Substituting this expression in Eq. (5) we obtain
\[
x(x_0, n) = x_0^{-1/a} x_{1/10}^{(a-b/a)}.
\]
Assuming the term \( x_0^{-1/a} x_{1/10}^{(a-b/a)} \) is constant for \( n \ll n_c \), and comparing Eq. (6) with the first scaling hypothesis we conclude that \( b = -1/\alpha \). Moving on and choosing \( \beta n = 1 \), which leads to \( \beta = n^{-1/b} \) and substituting in Eq. (5) we obtain
\[
x(x_0, n) = n^{-1/b} x_{1/10}^{(a-b/a)}.
\]
Again we suppose the term \( x_{1/10}^{(a-b/a)} \) is constant for \( n \gg n_c \). Comparing then with second scaling hypothesis we end up with \( \beta = -1/b \). Finally we compare the two expressions obtained for the scaling factor. It indeed leads to \( n_c = x_0^{\alpha/\beta} \).

A comparison with third scaling hypothesis allow us to obtain a relation between the three critical exponents \( \alpha, \beta \) and \( \gamma \) therefore converging to the following scaling law
\[
z = \frac{\alpha}{\beta} .
\]

The knowledge of any two exponents allow to find the third one by using Eq. (8). Moreover the exponents can also be used to rescale the variables \( x \) and \( n \) in a convenient way such that \( x \rightarrow x/x_0^\alpha \) and \( n \rightarrow n/x_0^\beta \) and overlap all curves of \( x \) vs. \( n \) onto a single and hence universal curve, as shown in Fig. 4.

Before moving to the next section and consider the dynamics in a more analytical way, let us discuss here the dynamics for \( \mu = 0 \). This characterises the neighbouring of the bifurcation.

The convergence to the steady state is marked by an exponential law of the type (see Refs. [22, 23])
\[
x(n, \mu) \propto e^{-n/\tau} ,
\]
where \( \tau \) is the relaxation time described by
\[
\tau \propto \mu^\delta ,
\]
where \( \delta \) is a relaxation exponent. Figure 5 shows the behaviour of \( \tau \) vs. \( \mu \) for two different values of \( \gamma \).
A power law fitting furnishes the exponent \( \delta \approx -1 \) and is independent on the value of the parameter \( \gamma \). In the next section we describe how to obtain the exponents discussed in this section using an analytical approach.

3. An analytical description to the equilibrium

Let us now discuss a different approach to reach the equilibrium. We start first with case (i), i.e., at the bifurcation point \( R = R_c = 1 \). The equation of the mapping is then written as

\[
x_{n+1} = x_n - \frac{x_n}{x_n}^{\gamma+1} \quad \text{for} \quad x_{n+1} < x_n \quad \text{and} \quad n > 0 .
\]  

Very near the fixed point, we suppose the dynamical variable \( x \) can be considered as a continuous variable. Therefore Eq. (11) is rewritten in a convenient way as (see also Ref. [27] for a recent application in a 2-D mapping)

\[
x_{n+1} - x_n = \frac{x_{n+1} - x_n}{(n+1) - n} = \frac{dx}{dn} \approx -x^{\gamma+1} .
\]  

Grouping the terms properly we obtain the following differential equation

\[
\frac{dx}{x^{\gamma+1}} = -dn .
\]  

Indeed the initial condition \( x_0 \) is defined for \( n = 0 \). Of course for a generic \( n \) we have \( x(n) \). Using these terms as limit of the integrals we end up with

\[
\int_{x_0}^{x(n)} dx = \int_{0}^{n} dn .
\]

After integrating Eq. (14) and organising the terms properly we obtain the following expression

\[
x(n) = x_0 \left[ x_0^{\gamma n + 1} \right]^{1/\gamma} .
\]

Let us now discuss the implications of Eq. (15) for specific ranges of \( n \). We start with the case \( \gamma n < 1 \), which is equivalent to the previous section of \( n < n_c \). For such a case we obtain that \( x(n) \approx x_0 \). A quick comparison with first scaling hypothesis allow us to conclude that the critical exponent \( \alpha = 1 \). Second we consider the situation \( \gamma n > 1 \), corresponding to \( n > n_c \) in the previous section. For such case we obtain that

\[
x(n) \approx n^{-1/\gamma} .
\]

Comparing then this result with scaling hypothesis two of the previous section we conclude \( \beta = -1/\gamma \). The last case is obtained when \( \gamma n = 1 \), which is the case of \( n = n_c \). Then we obtain

\[
n_c \approx x_0^{-\gamma} .
\]

A comparison with third scaling hypothesis gives us that \( z = -\gamma \). With this procedure we obtained all the three critical exponents discussed in the previous section as function of the parameter of the nonlinearity \( \gamma \). Numerical simulations were made for several different values of \( \gamma \) considering either odd, even, irrational and other set of numbers. The numerical findings confirm the validity of both the scaling law as well as the analytical procedure.

The last point to discuss is the case of \( R < R_c \), i.e., immediately before the bifurcation. For this case we can rewrite the mapping as

\[
x_{n+1} - x_n = \frac{x_{n+1} - x_n}{(n+1) - n} \approx \frac{dx}{dn} \approx x(R-1) - Rx^{\gamma+1} .
\]

We have to emphasise that near the steady state \( x \approx 0 \) and considering \( \gamma > 1 \), the term \( x^{\gamma+1} \) goes faster to zero as compared with \( x \). Then the last term of Eq. (18) can be disregarded. With this approach we obtain the following differential equation

\[
\frac{dx}{dn} = -x \mu ,
\]  

Figure 5: Plot of the relaxation to the fixed point as a function of \( \mu \) in the logistic-like map for the exponents: (a) \( \gamma = 1 \) and; (b) \( \gamma = 2 \).
where $\mu = 1 - R$. Considering again that for $n = 0$ the initial condition is $x_0$, we have to integrate the following equation

$$
\int_{x_0}^{x(n)} \frac{dx}{x} = -\mu \int_0^\infty dn'.
$$

(20)

After integration and grouping the terms we obtain

$$
x(n) = x_0 e^{-\mu n}.
$$

(21)

Comparing this result with Eqs. (9) and (10) we conclude that the exponent $\delta = -1$. This finding is in good agreement with the simulations shown in Fig. 5.

### 4. Discussions

Convergence to the steady state at a period-doubling bifurcation was also observed to obey a homogeneous function. Indeed to apply the formalism we have to look at the distance to the fixed point. Such observable can be defined as (following Refs. 22, 23) $y(n) = F^m(F^n(x)) - x^*$ where $F$ stands for the mapping, $m = 2, 4, 6, \ldots$ and $x^*$ is indeed the expression of the fixed point. In the previous sections, the fixed point was located at $x^* = 0$, hence it was convenient to rescale the observables in terms of $x_0$, which was the initial distance from the fixed point. In the period-doubling bifurcation considered here, $x^*$ is not zero anymore, hence we represent the distance from the fixed point as $\epsilon$. Furthermore, the scaling is dependent on $\epsilon$. Figure 6 shows the convergence to the fixed point for $\gamma = 1$ and $R = 3$. The slope obtained for the decay is $\beta = -0.49939(7)$. The crossover exponent was obtained as $z = -2.001(4)$, and the same scaling law as obtained previously is applied here too. The exponents obtained for the convergence towards the fixed point in the period-doubling bifurcation show to be independent of the nonlinearity $\gamma$. The scaling (see the overlap of the curves shown in Fig. 6(a) onto a single and universal plot as shown in Fig. 6(b)) is better seen after the transformations: (i) $(x - x^*) \rightarrow (x - x^*/\epsilon^z)$ where $\epsilon$ stands for the distance of the initial condition to the fixed point $x^*$ and; (ii) $n \rightarrow n/\epsilon^z$. For $\gamma = 1$, the first period-doubling bifurcation happens at $R = 3$, and the fixed point is $x^* = 2/3$. For the second period-doubling, $R = 1 + \sqrt{6}$ and the two fixed points are

$$
x_{1,2}^* = \frac{1}{2} \left[ 1 + \frac{1}{R} \pm \frac{1}{R} \sqrt{R^2 - 2R - 3} \right]
$$

(22)
For $R = 1 + \sqrt{6}$, the two fixed points assume the values $x_1^* = 0.8499377796 \ldots$ and $x_2^* = 0.4399601688 \ldots$. In this case, the critical exponents are $\alpha = 1, z = -2$ and $\beta = -0.498(1)$.

The numerical results obtained for the parameter $R = 1 + \sqrt{6}$ are shown in Fig. [3]. The convergence to the fixed point is plotted in Fig. [3](a) for different values of initial conditions, as shown in the figure while the overlap onto a single and universal plot is shown in Fig. [3](b). The slope of the decay obtained is the same one as obtained for the first period-doubling bifurcation observed at $R = 3$. This result confirms the independence of the critical exponents $\alpha, \beta$ and $z$, on the period-doubling bifurcation, as a function of $\gamma$.

5. Conclusions

To summarize, we have considered the convergence to the steady state in a family of logistic-like mappings in a bifurcation point near a transcritical, pitchfork, period-doubling bifurcation, and around its neighbourhood. At the bifurcation point we used a phenomenological description to prove that decay to the fixed point is described by using a homogeneous function with three critical exponents. The three critical exponents are related between themselves via a scaling law of the type $z = \alpha/\beta$. Near the bifurcation point the convergence to the fixed point is given by an exponential decay and the relaxation time is described by a power law of the type $\tau \propto \mu^\delta$. We found $\delta = -1$ and is independent on the nonlinearity of the mapping. The results obtained by the phenomenological approach is confirmed by an analytical description and is valid for any $\gamma \geq 1$. The exponents obtained for the period doubling bifurcation were $\alpha = 1, \beta \approx -0.5, z = -2$ and are independent of the parameter $\gamma$.

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