Translated points for prequantization spaces over monotone toric manifolds

Brian Tervil
University of Haifa

Abstract

We prove a version of Sandon’s conjecture on the number of translated points of contactomorphisms for the case of a prequantization bundle over a closed monotone toric manifold. Namely we show that any contactomorphism of this prequantization bundle lying in the identity component of the contactomorphism group possesses at least \( N \) translated points, where \( N \) is the minimal Chern number of the toric manifold. The proof relies on the theory of generating functions coupled with equivariant cohomology, whereby we adapt Givental’s approach to the Arnold conjecture for rational symplectic toric manifolds to the context of prequantization bundles.

1 Introduction and result

1.1 The main result

A major driving force in symplectic topology is the celebrated Arnold conjecture \cite{Arn65}:

The number of fixed points of a Hamiltonian symplectomorphism of a closed symplectic manifold is at least the number of critical points of a smooth function.

Its importance lies in the fact that in general, diffeomorphisms and even symplectomorphisms have far fewer fixed points. Its has now been proved in full generality in a homological version using Floer homology (see for instance \cite{Flo89}, \cite{HS95}, \cite{Ono95}, \cite{LT98}, \cite{Ono99}): on any closed symplectic manifold \((M, \omega)\), \textit{non-degenerate} Hamiltonian symplectomorphisms have at least \( \dim H^*(M, \mathbb{Q}) \) fixed points. There are also results about general, not necessarily non-degenerate Hamiltonian symplectomorphisms, see Oh \cite{Oh90}, Schwarz \cite{Sch98}, and Givental \cite{Giv95}. The present paper is closer in spirit to the latter results.

The analog of the Arnold conjecture in contact topology was introduced by S. Sandon \cite{San11}, via the notion of \textit{translated points}. Recall that a \textbf{contact manifold} is a pair \((V, \xi)\), where \( V \) is an odd-dimensional manifold, and \( \xi \) is a maximally non-integrable hyperplane field, called a \textbf{contact structure}. A \textbf{contactomorphism} of \((V, \xi)\) is a diffeomorphism preserving \( \xi \). Since contact manifolds are odd-dimensional, in general contactomorphisms have no fixed points. Sandon introduced the notion of translated points as the contact analog of a fixed point. To define it, assume that \( \xi \) is the kernel of a \textbf{contact form} \( \alpha \), which is a 1-form such that \( d\alpha|_\xi \) is non-degenerate. The \textbf{Reeb vector field} \( R_\alpha \) of \( \alpha \) is defined by

\[
\alpha(R_\alpha) = 1 \quad \text{and} \quad \iota_{R_\alpha} d\alpha = 0,
\]

and its flow is denoted \( \{\phi^t_\alpha\}_{t \in [0,1]} \). Given a contactomorphism \( \phi \), a point \( x \in V \) is called an \textbf{\( \alpha \)-translated point} if \( x \) and \( \phi(x) \) belong to the same Reeb orbit and if moreover \( \phi \) preserves the contact form \( \alpha \) at \( x \):

\[
\exists s \in \mathbb{R} \quad \text{such that} \quad \phi(x) = \phi^s_\alpha (x) \quad \text{and} \quad (\phi^* \alpha)_x = \alpha_x.
\]

We let \( \text{Cont}(V, \xi) \) be the group of contactomorphisms of \((V, \xi)\) and \( \text{Cont}_0(V, \xi) \) its identity component.
Conjecture (San13 conjecture 1.2). Let \((V, \xi)\) be a closed contact manifold, and \(\phi \in \text{Cont}_0(V, \xi)\). For any choice of contact form \(\alpha\), the number of \(\alpha\)-translated points of \(\phi\) is at least the number of critical points of a function.

A symplectic manifold \((M, \omega)\) is called monotone if the cohomology class of its symplectic form is positively proportional to the first Chern class \(c\) of \(M\). We denote \(N_M\) the minimal Chern number of \(M\), that is, the positive generator of \((c, H_2(M, \mathbb{Z})) \subset \mathbb{Z}\). A toric manifold \((M^{2d}, \omega, T)\) is a symplectic manifold endowed with an effective Hamiltonian action of a torus \(T\) of dimension \(d\).

A prequantization space over a symplectic manifold endowed with an effective Hamiltonian action of a torus \(M\) is a principal \(T\)-bundle \(\pi : (V, \alpha) \to (M, \omega)\) such that \(\pi^*\omega = d\alpha\), and the Reeb vector field \(R_\alpha\) induces the free \(S^1\)-action on \(V\), where \(S^1 = \mathbb{R}/\mathbb{Z}\), \(h > 0\) being the minimal period of a closed Reeb orbit.

**Theorem 1.1.1.** Let \((M, \omega)\) be a closed monotone toric manifold different from \(\mathbb{C}P^n\). There exists a prequantization space \((V, \xi = \ker \alpha)\) over \((M, \omega)\) such that any \(\phi \in \text{Cont}_0(V, \xi)\) has at least \(N_M\) \(\alpha\)-translated points.

Sandon’s conjecture was previously established for numerous prequantization spaces over \(\mathbb{C}P^n\): on the standard contact sphere \(S^{2n+1}\) and real projective space \(\mathbb{R}P^{2n+1}\) in \([\text{San13}]\), and later for all lens spaces in \([\text{GKPS17}]\). Note that \(\mathbb{C}P^n\) is monotone, however we will see that in this setting our arguments do not apply. The existence of translated points was also obtained in other contexts, for instance in \([\text{AM13}], \text{She17},\) and \([\text{MN18}]\).

Our approach to Theorem 1.1.1 is based on the theory of generating functions and equivariant cohomology, as developed by A. Givental \([\text{Giv95}]\). In the next section, we will give an overview of the constructions and of the proof of Theorem 1.1.1.

### 1.2 Overview of the paper and proof of the theorem

Generating functions were extensively used in the eighties and nineties by numerous authors (see for instance \([\text{Sik84}, \text{Giv90}, \text{Giv95}, \text{The95}, \text{The98}]\)). They provide a powerful tool when the manifold can be obtained somehow from a symplectic vector space. In \([\text{Giv95}]\), Givental used this approach along with equivariant cohomology to establish the Arnold conjecture for rational toric manifolds. A symplectic manifold \((M, \omega)\) is called rational if the cohomology class \([\omega]\) of its symplectic form lies in the image of the natural homomorphism \(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\). Note that a symplectic manifold is the base of a prequantization space if and only if it is rational, and that a monotone symplectic manifold is in particular rational.

We explain here the main lines of our construction. It holds for certain prequantization spaces over general rational toric manifolds. Given a closed toric manifold \((M, \omega)\), we view \(M\) as a symplectic reduction of \(\mathbb{C}^n\) by the Hamiltonian action of a torus \(\mathbb{K} \subset \mathbb{T}^n := (S^1)^n\). Based on a procedure of Borman-Zapolsky \([\text{BZ15}]\), we associate to \((M, \omega)\) a prequantization space \((V, \alpha)\), obtained by contact reduction of a contact sphere in \(\mathbb{C}^n\) by the action of a codimension 1 subtorus \(\mathbb{K}_0 \subset \mathbb{K}\). Given a contact isotopy \(\{\phi^t\}_{t \in [0, 1]}\) on \(V\), we then lift it up to an \(\mathbb{R}_{>0}\) and \(\mathbb{K}_0\)-equivariant Hamiltonian isotopy \(\{\Phi^t\}_{t \in [0, 1]}\) on \(\mathbb{C}^n\). This procedure constitutes sections 3.3 and 3.4.

In sections 3.3 - 3.6, we mainly follow \([\text{Giv95}]\). We first decompose the isotopy \(\{\Phi^t\}_{t \in [0, 1]}\) into \(2N_1\) Hamiltonian symplectomorphisms \(\Phi^1 = \Phi_{2N_1} \circ \ldots \circ \Phi_1\) which are close to \(I_{d\alpha^n}\). Taking the torus action into account, we come to the definition of a family of generating functions

\[
\mathcal{F}_N : \mathbb{C}^{2nN} \times \Lambda_N \to \mathbb{R}, \quad \mathcal{F}_N^{(N)} := \mathcal{F}_N(\cdot, \lambda), \quad N \geq N_1,
\]

parametrized by compact subsets \(\Lambda_N \subset \text{Lie}(\mathbb{K})\), which exhaust the Lie algebra \(\text{Lie}(\mathbb{K})\) of \(\mathbb{K}\) when \(N \to \infty\). Each \(\mathcal{F}_N^{(N)}\) is \(\mathbb{K}_0\)-invariant and homogeneous of degree 2 with respect to \(\mathbb{R}_{>0}\). Moreover,
certain critical points of $\mathcal{F}^{(N)}_{\lambda}$ correspond to translated points of the time 1 map $\phi^1$ on $V$. Denote $S_N \subset \mathbb{C}^{2nN}$ the unit sphere. For any real number $\nu \in \mathbb{R}$, we consider sets:

$$F_N(\nu) := \{(x, \lambda) \in S_N \times \Lambda_N : \mathcal{F}_N(x, \lambda) \leq 0 \text{ and } p(\lambda) = \nu\}, \quad \partial F_N(\nu) := F_N(\nu) \cap (S_N \times \partial \Lambda_N), \quad (1)$$

where $p = [\omega]$ denotes the cohomology class of the symplectic form $\omega$. Without loss of generality, we assume that $p$ is primitive, which means in particular that

$$p = \frac{c}{N_M}.$$ 

The torus $\mathbb{K}_0$ has $\mathfrak{t}_0 := \ker p \subset \text{Lie}(\mathbb{K})$ as its Lie algebra. When $\nu$ is generic, we construct natural homomorphisms of $\mathbb{K}_0$-equivariant cohomology groups over the field $\mathbb{C}$:

$$H^{*+2nN}_{\mathbb{K}_0}(F_N^- (\nu), \partial F_N^- (\nu)) \to H^{*+2nN'}_{\mathbb{K}_0}(F_N^- (\nu), \partial F_N^- (\nu)), \quad N \leq N'.$$

In the limit $N \to \infty$, we obtain a cohomology group

$$H^*_{\mathbb{K}_0}(F^- (\nu)) := \lim_{N \to \infty} H^{*+2nN}_{\mathbb{K}_0}(F_N^- (\nu), \partial F_N^- (\nu))$$

naturally associated with the Hamiltonian isotopy $\{\Phi^t\}_{t \in [0,1]}$. Similar sets as in (1) can be defined in the whole space $\mathbb{C}^{2nN} \times \Lambda_N$, and contract onto $\Lambda_N \cap p^{-1}(\nu)$. By the same limit process as above, these give rise to a cohomology group which is independent of $\nu$ and the isotopy $\{\Phi^t\}_{t \in [0,1]}$: We denote it $\mathcal{R}_0$. There is a natural homomorphism called augmentation map

$$\mathcal{R}_0 \to H^*_{\mathbb{K}_0}(F^- (\nu)).$$

Moreover, the group $H^*_{\mathbb{K}_0}(F^- (\nu))$ is endowed with the Novikov action of $H_2(M, \mathbb{Z})$:

$$m, H^*_{\mathbb{K}_0}(F^- (\nu)) = H^{*+2c(m)}_{\mathbb{K}_0}(F^- (\nu + p(m))), \quad m \in H_2(M, \mathbb{Z}),$$

where $c$ is the first Chern class of $M$. Let us denote $\mathcal{J}^*_{\mathbb{K}_0}(F^- (\nu))$ the kernel of the augmentation map, and describe the restriction of the Novikov action to it. Let $\mathfrak{k}$ denote the Lie algebra of the torus $\mathbb{K}$, $\mathfrak{k}_\mathbb{Z} := \ker(\exp : \mathfrak{k} \to \mathbb{K})$ the kernel of the exponential map, and $\iota : \mathfrak{k} \hookrightarrow \mathbb{R}^n$ the inclusion of $\mathfrak{k}$ into $\mathbb{R}^n \simeq \text{Lie}((S^1)^n)$. Under the natural isomorphisms

$$H_2(M, \mathbb{Z}) \simeq \mathfrak{k}_\mathbb{Z} \quad \text{and} \quad H^2(M, \mathbb{Z}) \simeq \mathfrak{k}_\mathbb{Z}^*, \quad (2)$$

the first Chern class writes

$$c(m) = \sum_{j=1}^n m_j, \quad \iota(m) = (m_1, ..., m_n).$$

We denote $H^*_{\mathbb{T}_n}(\text{pt})$ and $H^*_{\mathbb{K}_0}(\text{pt})$ the singular cohomology of the classifying spaces $B\mathbb{T}_n$ and $B\mathbb{K}_0$ respectively. The cohomology groups $\mathcal{R}_0$, $H^*_{\mathbb{K}_0}(F^- (\nu))$, and $\mathcal{J}^*_{\mathbb{K}_0}(F^- (\nu))$ are endowed with the structure of $H^*_{\mathbb{K}_0}(\text{pt})$-modules, and therefore with the structure of $H^*_{\mathbb{T}_n}(\text{pt})$-modules under the natural homomorphism

$$H^*_{\mathbb{T}_n}(\text{pt}) \to H^*_{\mathbb{K}_0}(\text{pt}).$$

The cohomology $H^*_{\mathbb{T}_n}(\text{pt})$ is naturally isomorphic to the ring $\mathbb{C}[u_1, ..., u_n]$ of regular functions on $\mathbb{C}^n$, and $H^*_{\mathbb{K}_0}(\text{pt})$ is a quotient of this ring, namely the ring of regular functions on $\mathfrak{t}_0 \otimes \mathbb{C}$.

From now one, we assume that $M \neq \mathbb{C}P^n$. Note that by compactness of $M$, the above homomorphism sends any monomial $u_i$ to a positive degree element in $H^*_{\mathbb{K}_0}(\text{pt})$.

A key novelty in this paper is the use of a Gysin exact sequence for equivariant cohomology (section [3], relating our cohomology groups $H^*_{\mathbb{K}_0}(F^- (\nu))$ to the ones constructed by Givental in [Giv95].
Using the sequence, we will view $\mathcal{J}_{K_0}^+(F^-(\nu))$ as a submodule of the ring of regular functions on the intersection $(\mathfrak{k}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$, where $(\mathbb{C}^\times)^n$ is the complex torus. Let

$$\mathbb{C}[u_1, ..., u_n, u_1^{-1}, ..., u_n^{-1}]$$

be the ring of regular functions on $(\mathbb{C}^\times)^n$. The Novikov action on $\mathcal{J}_{K_0}^+(F^-(\nu))$ is given by the multiplication by $u_1^{m_1}...u_n^{m_n}$:

$$m_* \mathcal{J}_{K_0}^+(F^-(\nu)) = u_1^{m_1}...u_n^{m_n} \mathcal{J}_{K_0}^+(F^-(\nu)) \simeq \mathcal{J}_{K_0}^{*+2c(m)}(F^-(\nu + p(m))). \quad (3)$$

Let $\nu_0 < \nu_1$ be generic. There is a natural homomorphism

$$\mathcal{H}_{K_0}^+(F^-(\nu_1)) \to \mathcal{H}_{K_0}^+(F^-(\nu_0)).$$

There is a contact analog of the action spectrum: given a contactomorphism $\phi$ on a contact manifold $(V, \xi)$, the translated spectrum (or simply spectrum) is the set

$$\text{Spec}(\phi) := \{s \in \mathbb{R} : \exists x \in V, \; x \text{ is a translated point of } \phi \text{ with } \phi(x) = \phi_\alpha^s(x)\}. \quad (4)$$

Note that on a prequantization space, the spectrum of any contactomorphism is periodic of period $\hbar$. In the sequel we assume that $\hbar = 1$, which is equivalent to $p$ being a primitive integral vector in $\mathfrak{k}_\mathbb{Z}$.

Let us denote $\phi := \phi^t$ the time 1 map of the original contact isotopy on $V$. The cohomology groups and the translated points of $\phi$ are related by the two following analogs of [Giv95] Propositions 6.2, 6.3, which are proved in section 5.6.

**Proposition 1.2.1.** Suppose that $[\nu_0, \nu_1] \cap \text{Spec}(\phi_\hbar) = \emptyset$. Then the homomorphism above is an isomorphism

$$\mathcal{H}_{K_0}^+(F^-(\nu_1)) \simeq \mathcal{H}_{K_0}^+(F^-(\nu_0)).$$

**Proposition 1.2.2.** Suppose that the segment $[\nu_0, \nu_1]$ contains only one value $\nu \in \text{Spec}(\phi_\hbar)$, and that all the associated translated points are isolated. Let $\nu \in \mathcal{H}_{K_0}^+(\text{pt})$ be an element of positive degree, and $q \in \mathcal{R}_0$. Suppose that $q \in \mathcal{J}_{K_0}^+(\nu_0)$. Then $v_\nu \in \mathcal{J}_{K_0}^+(\nu_1)$.

We now assume that $(M, \omega)$ is monotone. We will see in section 6 that, in this situation, the kernel $\mathcal{J}_{K_0}^+(F^-(\nu))$ admits in some sense elements of minimal degree:

**Proposition 1.2.3.** There exists $q \in \mathcal{H}_{K_0}^+(\nu)$, such that $q \notin \mathcal{J}_{K_0}^+(F^-(\nu))$, but $u_i q \in \mathcal{J}_{K_0}^+(F^-(\nu))$ for any $i = 1, ..., n$.

**Remark 1.2.1.** Note that in the above proposition the monotonicity assumption cannot be lifted, since in general, it can happen that $\mathcal{J}_{K_0}^+(F^-(\nu)) \neq \mathcal{H}_{K_0}^+(\nu)$, and so the proof of Theorem 1.1.1 does not work in this case. Note also that if $M$ was the complex projective space $\mathbb{C}P^n$, the ring of regular functions on the intersection $(\mathfrak{k}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$ would be trivial, since we would have $\mathfrak{k}_0 = \{0\}$, and in particular $\mathcal{J}_{K_0}^+(F^-(\nu)) = \{0\}$, so the above statement would not hold.

The above discussion allows us to prove our main result, Theorem 1.1.1.

**Proof.** (Theorem 1.1.1) We adapt the proof of [Giv95] Theorem 1.1] to the contact setting. Consider the prequantization space $(V, \alpha)$ constructed by the procedure described above. Let $\{\phi_t\}_{t \in [0, 1]}$ be a contact isotopy on $V$. For any generic $\nu \in \mathbb{R}$, we have a cohomology group $\mathcal{H}_{K_0}^+(F^-(\nu))$, and the kernel $\mathcal{J}_{K_0}^+(F^-(\nu))$ of the augmentation map satisfies proposition 1.2.3. Let $\nu \notin \text{Spec}(\phi)$. Since $\text{Spec}(\phi)$ is periodic of period 1, it is enough to count the number of elements of $\text{Spec}(\phi)$ between $\nu$ and $\nu + 1$. To every such element corresponds at least one $\alpha$-translated point of $\phi$ on $V$. Therefore, to one $\alpha$-translated point of $\phi$ there correspond $l$ elements of the spectrum between $\nu$ and $\nu + l$, where $l \in \mathbb{Z}_{>0}$. Let us assume that all the $\alpha$-translated points of $\phi$ are isolated (otherwise the statement is trivial).
Let \( m \in \mathfrak{f}_Z \setminus \{0\} \) such that \( \nu(m) = (m_1, \ldots, m_n) \in \mathbb{R}_0^n \). Suppose that the number \( \# \) of elements in \( \text{Spec}(\phi) \) between \( \nu \) and \( \nu + p(m) \) is strictly less than \( c(m) = \sum_{j=1}^{n} m_j \). Let \( q \in H_{\nu}^{(2)}(\nu) \) be such that \( q \notin J_{\nu}^{(2)}(F^-) \), but \( u_i q \in J_{\nu}^{(2)}(F^-(\nu)) \) for any \( i = 1, \ldots, n \). Since \( \# < c(m) \), propositions \([1.2.1] \) and \([1.2.2] \) imply that \( u_0^{m_1} \cdots u_n^{m_n} q \in J_{\nu}^{r+2c(m)}(\nu + p(m)) \). This is precisely the Novikov action \( (3) \) of \( m \), which is an isomorphism
\[
J_{\nu}^{(2)}(\nu) \xrightarrow{u_0^{m_1} \cdots u_n^{m_n}} J_{\nu}^{r+2c(m)}(\nu + p(m)).
\]
In particular, \( u_0^{m_1} \cdots u_n^{m_n} q \notin J_{\nu}^{r+2c(m)}(\nu + p(m)) \), which is a contradiction. This means that the number \( \# \) of elements of \( \text{Spec}(\phi) \) between \( \nu \) and \( \nu + p(m) \) is not less that \( c(m) \), that is the number of \( \alpha \)-translated points of \( \phi \) is not less than \( N_M \).

**Acknowledgements.** I would like to warmly thank my advisor Frol Zapolsky for his support throughout this project. His professionalism and patience were crucial in every step of this work. I would also like to thank Leonid Polterovich and Michael Entov for listening to a preliminary version of this work and their interest, and Alexander Givental for his generous help regarding the paper \([Giv95] \). Finally, I would like to thank the participants of the workshop "Floer homology and contact topology" held at the University of Haifa in September 2017, for their questions and attention during my talk about this project. I was supported by grant number 1825/14 from the Israel Science Foundation.

## 2 Preliminaries

### 2.1 Fronts and deformation of sublevel sets

We recall here several technical results stated in \([Giv95] \) section 3. Let \( F : X \times \Lambda \to \mathbb{R} \) be a family of functions \( F_{\lambda} \) on a compact manifold \( X \), where \( \Lambda \) is a parametrizing manifold, and \( F \in C^{1,1} \) at all points \((x, \lambda)\) such that \( x \) is a critical point of \( F_{\lambda} \) with critical value zero. We consider restrictions of \( F \) to submanifolds \( \Gamma \subset \Lambda \) with boundary \( \partial \Gamma \), and look at the following sets
\[
F_{\Gamma}^- := \{(x, \lambda) \in X \times \Lambda | \lambda \in \Gamma, f(x, \lambda) \leq 0\}, \quad F_{\Gamma}^+ := \{(x, \lambda) \in X \times \Lambda | \lambda \in \Gamma, f(x, \lambda) \geq 0\}.
\]
Assume that 0 is a regular value of \( F \). The **front of \( F \)** is defined as
\[
L := \{\lambda \in \Lambda : 0 \text{ is a singular value of } F_{\lambda}\}.
\]
It is the singular locus of the projection
\[
F^{-1}(0) \to \Lambda,
\]
and since \( F \in C^{\infty} \) at all critical points with critical value zero, it is of zero measure. It is actually a singular hypersurface in \( \Lambda \), provided at every point with tangent hyperplanes (there can be more than one hyperplane at each point). The construction goes as follows: let \( N \) be a smooth neighborhood in \( X \times \Gamma \) of the set of critical points of \( F \) with critical value 0. The zero set \( F_{\nu}^{-1}(0) \subset X \times \Lambda \) gives rise naturally to a Legendrian submanifold
\[
\mathcal{L} := \{(x, \lambda, T_{(x,\lambda)}F_{\nu}^{-1}(0)), (x, \lambda) \in X \times \Lambda \} \subset PT^*(X \times \Lambda)
\]
in the space \( PT^*(X \times \Lambda) \) of contact elements. Denote by \( \mathcal{P} \) the space of vertical contact elements: a hyperplane \( H_{(x, \lambda)} \) is in \( \mathcal{P} \) if and only if \( T_x X \times \{\lambda\} \subset H_{(x, \lambda)} \). Then the intersection \( \hat{\mathcal{L}} := \mathcal{L} \cap \mathcal{P} \) projects to a Legendrian submanifold \( \hat{\mathcal{L}} \subset PT^*(\Lambda) \) in the space of contact elements \( PT^*(\Lambda) \). The front is then defined as the projection of \( \hat{\mathcal{L}} \) to the base \( \Lambda \), whereas a tangent hyperplane to \( \lambda \in L \) is an element of \( \hat{\mathcal{L}} \cap PT^*_\lambda \Lambda \).
Proposition 2.1.1 ([Giv95] proposition 3.1). A submanifold $\Gamma \subset \Lambda$ is transversal to $L$ if and only if the hypersurface $F^{-1}(0)$ is transversal to $X \times \Gamma$.

Proof. $X \times \Gamma$ is tangent to $F^{-1}(0)$ at a point $(x, \lambda) \in X \times \Lambda$ if and only if $T_{(x,\lambda)}(X \times \Gamma) \subset T_{(x,\lambda)}F^{-1}(0)$, if and only if $T_{(x,\lambda)}(X \times \Gamma) \subset \hat{\mathcal{L}}$, if and only if $T_{\Gamma} \subset \hat{\mathcal{L}} \cap PT^*(\Lambda)$. □

Corollary 2.1.1 ([Giv95] proposition 3.2). Let $\Gamma_t := \rho^{-1}(t)$ be non-singular levels of some smooth map $\rho : \Gamma \to \mathbb{R}^m$. Then almost all $\Gamma_t$ are transversal to $L$.

The two following propositions can be proved using standard gradient flow deformations.

Proposition 2.1.2. Let $F : X \times \Lambda \times [0,1] \to \mathbb{R}$ be a $C^{1,1}$ family of functions $F_s : X \times \Lambda \to \mathbb{R}$ satisfying similar properties as above. Suppose that for any $s \in [0,1]$, $\Gamma \subset \Lambda$ and $\partial \Gamma$ are transversal to $L_s$. Then there exists a $C^{0,1}$ isotopy $I : X \times \Lambda \times [0,1] \to X \times \Lambda$ such that

$$I_s(F_{\Gamma,0}^\pm, \partial F_{\partial \Gamma,0}^\pm) = (F_{\Gamma,s}^\pm, \partial F_{\partial \Gamma,s}^\pm),$$

where we have denoted by $F_{\Gamma,s}^\pm, F_{\partial \Gamma,s}^\pm$ the sublevel sets defined for $F_s$ as in [9]. If moreover each $F_s$ is invariant under the action of a compact Lie group $G$ on $X$, the isotopy can be made $G$-equivariant.

Proposition 2.1.3. Suppose that we have a $C^{1,1}$ family $\{\Gamma_s\}_{s \in [0,1]}$ of submanifolds $\Gamma_s \subset \Lambda$, such that for any $s \in [0,1]$, $\Gamma_s$ and $\partial \Gamma_s$ are transversal to $L$. Then there exists a $C^{0,1}$ isotopy $I : X \times \Lambda \times [0,1] \to X \times \Lambda$ such that

$$I_s(F_{\Gamma,0}^\pm, \partial F_{\partial \Gamma,0}^\pm) = (F_{\Gamma,s}^\pm, \partial F_{\partial \Gamma,s}^\pm).$$

If moreover $F$ is invariant under the action of a compact Lie group $G$ on $X$, the isotopy can be made $G$-equivariant.

Remark 2.1.1. The above results remain true if $\Gamma$ is a stratified manifold. In this setting, one must improve the notion of transversality: $\Gamma$ is transversal to $L$ if each of its strata is.

2.2 Equivariant cohomology and conical spaces

We state here several facts from equivariant cohomology, fix some notations, and describe a simple identification which holds for conical spaces. Let $X$ be a topological space provided with the action of a compact Lie group $G$. Equivariant cohomology $H_G(X)$ is defined as the singular cohomology $H^*(X_G)$ of the quotient

$$X_G := (X \times EG)/G,$$

where $EG \to BG$ is the universal principal $G$-bundle over the classifying space $BG$. The canonical projection

$$(X \times EG)/BG \to EG/G$$

provides $H_G(X)$ with the structure of a module over $H^*(BG)$, which plays the role of the coefficient ring $H_G^*(pt)$ in equivariant cohomology theory. If $G$ is the $n$-dimensional torus $\mathbb{T}^n = (S^1)^n$, the coefficient ring $H_G^*(pt)$ is naturally isomorphic to a polynomial algebra in $n$ variables $u = (u_1, ..., u_n)$ of degree 2. More precisely, there is a natural algebra isomorphism, called the Chern-Weil isomorphism

$$\chi : H_G^*(pt) \simeq \text{Sym}^*(\mathbb{R}^{n*}),$$

between the cohomology of the classifying space $B\mathbb{T}^n$ and the symmetric algebra of $\mathbb{R}^{n*}$.

We will consider singular cohomology with complex coefficients, so that we will use the notation $H^*(Y) := H^*(Y, \mathbb{C})$. If $(u_1, ..., u_n)$ denotes the standard basis of $\mathbb{R}^{n*}$, the Chern-Weil isomorphism writes

$$H_G^*(pt) \simeq \mathbb{C}[u_1, ..., u_n].$$

Let $A \subset X$ be a $G$-invariant subspace of $X$. Denote by $pr : X \times EG \to X_G$ the canonical projection. We introduce the following notations:

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Suppose that $X$ and $A$ are closed conical subspaces of $\mathbb{R}^m$, that is $tX \subset X$ and $tA \subset A$ for any $t > 0$, and that $G$ is a torus. Then the natural retraction of $\mathbb{R}^m$ onto the closure $\overline{B}$ of its unit ball $B$ induces a $G$-equivariant retraction of $X$ (resp. $A$) onto $X \cap \overline{B}$ (resp. $A \cap \overline{B}$), and $X \setminus (X \cap B)$ (resp. $A \setminus (A \cap B)$) onto $X \setminus S$ (resp. $A \setminus S$), where $S = \partial \overline{B}$. Moreover, the inclusion $C_G(X, X \setminus X \cap B) \hookrightarrow C^*_G(X)$ is a $G$-equivariant homotopy equivalence, since any compact $K \subset X$ is included in an intersection $X \setminus \overline{B}$, where $\overline{B}$ is a ball centered at 0, and $X \setminus \overline{B}$ is $G$-equivariantly homotopic to $X \cap B$, since $X$ is conical and $G$ commutes with $\mathbb{R}_{>0}$. The same argument applies when replacing $X$ with $A$. Putting all these simple facts together yields a natural isomorphism

$$H^*(C^*_G(X \setminus S, A \setminus S)) \simeq H^*(C^*_G(X, A)/C^*_G(X, A)).$$

3 The constructions

3.1 The prequantization space

In this section, we describe a construction of a prequantization space over a rational toric symplectic manifold $(M^d, \omega, T)$. The action of $T$ is induced by a momentum map $M \rightarrow t^*$, where $t^*$ is the dual to the Lie algebra $t$ of $T$. The image $\Delta$ of the momentum map is called the moment polytope. If $\Delta$ has $n$ facets, it is given by

$$\Delta = \{x \in t^* | \langle x, v_j \rangle + a_j \geq 0 \text{ for } j = 1, \ldots, n\},$$

where the conormals $v_j$ are primitive vectors in the integer lattice $t_\mathbb{Z} := \ker(\exp : t \rightarrow T)$, and $a := (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n$. The polytope $\Delta$ is compact and smooth, that is each $k$-codimensional face of $\Delta$ is the intersection of exactly $k$ facets, and the $k$ associated conormals $\{v_{i_1}, \ldots, v_{i_k}\}$ can be extended to an integer basis for the lattice $t_\mathbb{Z}$. We mainly follow [BZ15] (see remark 3.1.2).

3.1.1 Delzant’s construction of symplectic toric manifolds

Let us first recall Delzant’s construction of toric manifolds [De88]. The standard Hamiltonian action of the torus $T^m := \mathbb{R}^m/\mathbb{Z}^m$ on $(\mathbb{C}^n, \omega_{\text{std}} := \sum_{j=1}^n dx_j \wedge dy_j)$ by rotation in each coordinate is induced by the momentum map

$$P : \mathbb{C}^n \rightarrow \mathbb{R}^m, \quad \langle \lambda, P(z) \rangle = \pi \sum_{j=1}^n \lambda_j |z_j|^2, \quad \text{and} \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^m. \quad (6)$$
Consider the following surjective linear map:
\[ \beta_\Delta : \mathbb{R}^n \to t, \quad \epsilon_j \mapsto v_j \quad \text{for} \quad j = 1, \ldots, n, \]
where \((\epsilon_1, \ldots, \epsilon_n)\) is the standard basis of \(\mathbb{R}^n\), and \(v_j \in t_\mathbb{Z}\) are the conormals. Since \(\Delta\) is compact and smooth, the map \(\beta_\Delta\) satisfies \(\beta_\Delta(\mathbb{Z}^n) = t_\mathbb{Z}\), and therefore induces a homomorphism \([\beta_\Delta] : \mathbb{T}^n \to \mathbb{T}\).

We define the connected subtorus
\[ K \subset \mathbb{T}^n \]
as the kernel of \([\beta_\Delta]\). It has Lie algebra
\[ \mathfrak{k} := \ker(\beta_\Delta : \mathbb{R}^n \to t), \]
and if \(\iota : \mathfrak{k} \hookrightarrow \mathbb{R}^n\) denotes the inclusion, the momentum map for the action of \(K\) on \(\mathbb{C}^n\) is given by
\[ P_K := \iota^* \circ P : \mathbb{C}^n \to \mathfrak{k}^*. \]
The torus \(K\) acts freely on the regular level set
\[ P_K^{-1}(p), \quad \text{where} \quad p := \iota^*(a) \in \mathfrak{k}^* \setminus \{0\}, \]
and if \(X_\lambda(z) = 2i\pi(\lambda_1 z_1, \ldots, \lambda_n z_n) \in \mathbb{C}^n = T_z\mathbb{C}^n\) denotes the Hamiltonian vector field for the function \(\langle \lambda, P \rangle : \mathbb{C}^n \to \mathbb{R}\), and \(\alpha_{\text{std}} = \frac{1}{i} \sum_{j=1}^{d} (x_j dy_j - y_j dx_j)\) is the standard 1-form on \(\mathbb{C}^n\) where \(d\alpha_{\text{std}} = \omega_{\text{std}}\), one has
\[ \alpha_{\text{std}}(X_\lambda) = \langle \lambda, P \rangle \quad \text{and} \quad \iota_{X_\lambda} d\alpha_{\text{std}} = \iota_{X_\lambda} \omega_{\text{std}} = -d\langle \lambda, P \rangle. \]
In particular, \((L_{X_\lambda} \omega_{\text{std}})|_{P_K^{-1}(p)} = 0\). Therefore, symplectic reduction gives rise to a symplectic manifold \((M_\Delta, \omega_\Delta)\), where
\[ M_\Delta := P_K^{-1}(p)/K,\quad \text{and the symplectic form} \ \omega_\Delta \ \text{is induced by} \ \omega_{\text{std}}|_{P_K^{-1}(p)}. \]
Finally, Delzant’s theorem [Del88] shows that \((M_\Delta, \omega_\Delta, \mathbb{T}^n/K)\) and \((M, \omega, \mathbb{T})\) are equivariantly symplectomorphic as toric manifolds.

**Remark 3.1.1.** This construction provides natural isomorphisms
\[ H_2(M, \mathbb{R}) \cong \mathfrak{k} \quad \text{and} \quad H^2(M, \mathbb{R}) \cong \mathfrak{k}^*, \]
which restrict to the isomorphisms (2).

### 3.1.2 The contact sphere

Our construction of generating functions is based on a lifting procedure from a contact sphere to its symplectization (see section 3.2.2). It is important that this sphere is \(K\)-invariant and contains the sublevel set \(P_K^{-1}(p)\). Its existence is ensured by compactness of the toric manifold. We refer to [And12] for details on the properties of toric manifolds.

The image of the momentum map \(P\) from equation (6) is the first orthant \(\mathbb{R}^n_{\geq 0}\), and the polytope \(\Delta\) is identified with
\[ (\iota^*)^{-1}(p) \cap \mathbb{R}^n_{\geq 0}. \]
Moreover, compactness of \(M_\Delta\) is equivalent to that of \(\Delta\), which is ensured by the condition
\[ \ker \iota^* \cap \mathbb{R}^n_{\geq 0} = \{0\}. \]
Since \(\ker \iota^*\) is equal to the annihilator of \(\iota(\mathfrak{k})\) in \(\mathbb{R}^n_{\geq 0}\), this implies in particular that \(\iota(\mathfrak{k}) \cap \mathbb{R}^n_{\geq 0} \neq \{0\}\). Moreover, \(p\) lies in the image of \(\mathbb{R}^n_{\geq 0}\), and therefore it is positive on \(\iota(\mathfrak{k}) \cap \mathbb{R}^n_{\geq 0}\). Fix \(b \in \mathfrak{k}\) such that \(\iota(b) = (b_1, \ldots, b_n) \in \mathbb{R}^n_{\geq 0}\). We define the contact sphere by
\[ S_p := \{ z \in \mathbb{C}^n : \sum_{j=1}^{n} b_j \pi |z_j|^2 = p(b) \}, \quad \alpha_p := \alpha_{\text{std}}|_{S_p}. \quad (8) \]
3.1.3 The prequantization space

From now on we assume that the toric manifold \((M, T, \omega)\) is rational. Notice that since \(\beta_{\Delta}(\mathbb{Z}^n) = \mathfrak{t}_\mathbb{Z}\), the inclusion \(\iota\) satisfies
\[
\iota(\mathfrak{t}_\mathbb{Z}) \subset \mathbb{Z}^n.
\]
Thus, the rationality condition is equivalent to
\[
\iota(\mathfrak{t}_\mathbb{Z}) \subset \mathbb{Z}^n.
\]
Let \(\mathfrak{t}_0\) denote the kernel of \(p : \mathfrak{t} \to \mathbb{R}\). Then \(\mathfrak{t}_0 \subset \mathbb{Z}\) is a sublattice of \(\mathfrak{t}_\mathbb{Z}\). This means that \(\exp(\mathfrak{t}_0)\) is a subtorus \(\mathbb{K}_0 \subset \mathbb{K}\) of codimension 1 which, in particular, acts freely on the regular level set \(P_{\mathbb{K}}^{-1}(p)\). Let \(j : \mathfrak{t}_0 \to \mathfrak{t}\) denote the inclusion of Lie algebras. Then \(P_{\mathbb{K}}^{-1}(p)\) is the zero-level of the momentum map
\[
\mu : S_p \to \mathfrak{t}_0^*, \quad \mu(z) := j^* \circ P_{\mathbb{K}}(z)
\]
associated with the action of \(\mathbb{K}_0\) on the contact sphere \((S_p, \alpha_p)\). Therefore, contact reduction yields a contact manifold
\[
(V := P_{\mathbb{K}}^{-1}(p)/\mathbb{K}_0, \xi := \ker \alpha), \quad \rho^* \alpha = \alpha_p,
\]
where \(\rho : P_{\mathbb{K}}^{-1}(p) \to V\) denotes the canonical projection (see [Gei08] for more on contact reductions). For the circle \(S^1 := \mathbb{K}/\mathbb{K}_0\), the projection
\[
\pi : (V, \alpha) \to (M_{\Delta}, \omega_{\Delta})
\]
defines a principal \(S^1\)-bundle, and satisfies \(\pi^* \omega_{\Delta} = \alpha\), since \(\omega_{\text{std}} = \alpha_{\text{std}}\). Finally, one can choose an equivariant symplectomorphism \((M_{\Delta}, \omega_{\Delta}, \mathbb{T}^n/\mathbb{K}) \simeq (M, \omega, \mathbb{T})\), and obtain a prequantization space over the toric manifold \((M, \omega, T)\).

Remark 3.1.2. Note that the construction of \(V\) is independent of the sphere \(S_p\). Indeed, equation (7) shows that the infinitesimal action of \(\mathbb{K}_0\) is tangent to \(\ker \alpha_{\text{std}}\) along \(P_{\mathbb{K}}^{-1}(p)\), and therefore the contact form \(\alpha\) is well defined.

3.2 Lifting contact isotopies

In this section we explain a procedure for lifting a contact isotopy of \(V\) to a Hamiltonian isotopy of \(\mathbb{C}^n\). Recall that for any contact manifold \((V, \xi)\) and any choice of contact form \(\alpha\) such that \(\xi = \ker \alpha\), any contact Hamiltonian \(h : V \times [0, 1] \to \mathbb{R}\) gives rise to a unique time-dependent vector field satisfying
\[
\alpha(X^t_h) = h_t \quad \text{and} \quad \alpha(X^t_h), \quad -dh_t + dh_t(R_{\alpha}) = h_t := h(\cdot, t).
\]
The vector field \(\{X^t_h\}_{t \in [0, 1]}\) preserves \(\xi\) and integrates into a contact isotopy denoted \(\{\phi^t_h\}_{t \in [0, 1]}\). This establishes a bijection, depending on the contact form \(\alpha\), between smooth time-dependent functions \(h : V \times [0, 1] \to \mathbb{R}\) and the identity component Con_{0}(V, \xi) of the group of contactomorphisms.

3.2.1 Lift to the contact sphere

Following [BZ14] (definition 1.6), we say that a closed submanifold \(Y \subset V\) transverse to \(\xi\) is strictly coisotropic with respect to \(\alpha\) if is is coisotropic, that is the subbundle \(TY \cap \xi\) of the symplectic vector bundle \((\xi_{|Y}, \alpha)\) is coisotropic:
\[
\{X \in \xi_{|Y} | \iota_X \alpha = 0 \text{ on } T_y Y \cap \xi_{|Y}\} \subset T_y Y \cap \xi_{|Y}, \quad \text{for all} \quad y \in Y,
\]
and additionally \(R_{\alpha} \in T_y Y\) for all \(y \in Y\), that is the Reeb field is tangent to \(Y\).
Consider the setting

$$(S_p, \alpha_p) \supset (P_{K}^{-1}(p), \alpha_p|P_{K}^{-1}(p)) \xrightarrow{\rho} (V, \xi, \alpha).$$

Then $P_{K}^{-1}(p)$ is strictly coisotropic with respect to $\alpha_p$. Let $h : V \times [0, 1] \to \mathbb{R}$ be a time-dependent contact Hamiltonian on $V$, and let $\phi_h := \phi_h^t$ denote the time-1 map of the contact isotopy $\{\phi_h^t\}_{t \in [0, 1]}$ generated by $h$. We first lift $h_t := h(, t)$ to a $\mathbb{K}_0$-invariant function

$$\tilde{h}_t : P_{K}^{-1}(p) \to \mathbb{R}, \quad \tilde{h}_t := \rho^* h_t,$$

and then extend $\tilde{h}$ somehow to a $\mathbb{K}_0$-invariant contact Hamiltonian $\tilde{h} : S_p \times [0, 1] \to \mathbb{R}$, so that $\tilde{h}_{t|P_{K}^{-1}(p)} = h_t$. By [BZ15] (lemma 3.1 and 3.2), the contact isotopy $\{\phi_h^t\}_{t \in [0, 1]}$ generated by $\tilde{h}$ and the Reeb flow $\{\phi_{\alpha_p}^t\}_{t \in \mathbb{R}}$ of $\alpha_p$ preserve $P_{K}^{-1}(p)$, and project to the contact isotopy $\{\phi_h^t\}_{t \in [0, 1]}$ and the Reeb flow $\{\phi_{\alpha_p}^t\}_{t \in \mathbb{R}}$ of $\alpha$ respectively. More precisely, the following holds:

$$\rho \circ \phi_{\alpha_p}^t|P_{K}^{-1}(p) = \phi_h^t \circ \rho : P_{K}^{-1}(p) \to V \quad \text{for all } t \in \mathbb{R};$$

$$\rho \circ \phi_h^t|P_{K}^{-1}(p) = \phi_h^t \circ \rho : P_{K}^{-1}(p) \to V \quad \text{for all } t \in [0, 1].$$

Let $q \in V$ be an $\alpha$-translated point of $\phi_h$, that is

$$\phi_h(q) = \phi_h^s(q) \quad \text{for some } s \in \mathbb{R}, \text{ and } (\phi_h^s \alpha)_q = \alpha_q.$$

Then $q$ is a **discriminant point** of $\phi_h^s \circ \phi_h$, that is an $\alpha$-translated point which is also fixed. For any $z \in P_{K}^{-1}(p)$ such that $\rho(z) = q$, the equations above show moreover that

$$\rho(\phi_h^s \circ \phi_h^t(z)) = \rho(z).$$

In other words, $\phi_h^s \circ \phi_h^t(z)$ and $z$ are on the same $\mathbb{K}_0$-orbit in $P_{K}^{-1}(p)$. Since the Reeb orbits of $\alpha$ generate the circle $\mathbb{K}/\mathbb{K}_0$, there exists $\lambda \in \mathfrak{k}$ such that $p(\lambda) = -s$ and $\exp(\lambda) \circ \phi_h^t(z) = z$. Moreover,

$$(\phi_h^\lambda \alpha_p)z = (\phi_h^s \rho^* \alpha)_z = ((\rho \circ \phi_h^t)^* \alpha)_z = ((\phi_h \circ \rho)^* \alpha)_z = (\rho^* \phi_h^t \alpha)_z = (\rho^* \alpha)_z = (\alpha_p)_z,$$

and $\mathbb{K}$ acts by $\alpha_p$-preserving transformations (see equation (7)). In other words, $z$ is a discriminant point of $\exp(\lambda) \circ \phi_h^t$.

### 3.2.2 Lift to a symplectic vector space

A convenient way to pass from the contact to the symplectic setting consists of associating to a contact manifold its symplectization. We briefly recall this procedure, and apply it to lift contactomorphisms of the contact sphere $(S_p, \alpha_p)$ to symplectomorphisms of $(\mathbb{C}^n, \omega_{\text{std}})$.

Let $(N, \xi = \ker \alpha)$ be a cooriented contact manifold. Its **symplectization** is the symplectic manifold

$$SN := N \times \mathbb{R}, \text{ with symplectic form } d(e^r \alpha),$$

where $r$ is a coordinate on $\mathbb{R}$. Let $\phi$ be a contactomorphism of $N$. Then $\phi$ lifts up to a symplectomorphism

$$\Phi : SN \to SN$$

$$(x, r) \to (\phi(x), r - g(x)),$$

where $g : N \to \mathbb{R}$ is the function satisfying $\phi^* \alpha = e^{g} \alpha$. In particular, a discriminant point $q$ of $\phi$ corresponds to a $\mathbb{R}$-line $\{(q, r) \in SN, r \in \mathbb{R}\}$ of fixed points of $\Phi$. If $\phi \equiv \phi_h$ is the time-1 map of a contact isotopy $\{\phi_h^t\}_{t \in [0, 1]}$ generated by a contact Hamiltonian $h : N \times [0, 1] \to \mathbb{R}$, $\Phi$ is the time-1 map

$$\Phi := \Phi_H$$

$$(x, r) := e^t h_t(x), \quad t \in [0, 1].$$
Suppose now that \( N \subset \mathbb{C}^n \) is a star-shaped hypersurface, that is the image of a map 
\[
S^{2n-1} \to \mathbb{C}^n
\]
\[
z \to f(z) := (f_1(z)z_1, \ldots, f_n(z)z_n),
\]
where \( f := (f_1, \ldots, f_n) \in (C^\infty(S^{2n-1}, \mathbb{R}_{>0}))^n \). One can show that the standard Liouville form \( \alpha_{\text{std}} \) on \( \mathbb{C}^n \) restricts to a contact form on \( N \), and the symplectization \((SN, d(\epsilon^r \omega_{\text{std}}))\) is symplectomorphic to \((\mathbb{C}^n \setminus \{0\}, \omega_{\text{std}})\), via the symplectomorphism 
\[
\Psi : N \times \mathbb{R} \to \mathbb{C}^n \setminus \{0\},
\]
\[
(x, r) \to e^{\frac{r}{\sqrt{2}}i}x
\]
We then have 
\[
\Psi \Phi_t \Psi^{-1} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\},
\]
\[
z \to \frac{|z|}{|pr(z)|} \phi(pr(z))
\]
where 
\[
pr : \mathbb{C}^n \setminus \{0\} \to N
\]
\[
z \to f(\frac{z}{|z|}) \frac{z}{|z|}
\]
is the radial projection, and \( |z| := \sqrt{\sum_{j=1}^n |z_j|^2} \). For the time-1 map \( \phi_h \) of a contact isotopy \( \{\phi_h^t\}_{t \in [0,1]} \), the Hamiltonian of \( \mathbb{C}^n \setminus \{0\} \) generating the Hamiltonian isotopy \( \{\Psi h \Psi^{-1}\}_{t \in [0,1]} \) is of the form 
\[
\tilde{H}_t(z) := \frac{|z|^2}{|pr(z)|^2} h_t(pr(z)).
\]
It is homogeneous of degree 2 with respect to the action of \( \mathbb{R}_{>0} \) on \( \mathbb{C}^n \setminus \{0\} \), that is 
\[
\tilde{H}_t(rz) = r^2 \tilde{H}_t(z), \text{ for any } r > 0.
\]
The Hamiltonian isotopy \( \{\Phi_{\tilde{H}}^t\}_{t \in [0,1]} = \{\Psi h \Psi^{-1}\}_{t \in [0,1]} \) is therefore \( \mathbb{R}_{>0} \)-equivariant, and we can extend \( \tilde{H}_t \) and \( \Phi_{\tilde{H}}^t \) continuously to \( \mathbb{C}^n \) by 
\[
\tilde{H}_t(0) = 0
\]
\[
\Phi_{\tilde{H}}^t(0) = 0.
\]
Back to our setting, the contact sphere \((S_p, \alpha_p)\) is a star-shaped hypersurface of \( \mathbb{C}^n \), with 
\[
f_i(z) := \sqrt{\frac{p(h)}{\pi b_i}}, \text{ for all } i = 1, \ldots, 2n.
\]
The action of \( K \) on \( S_p \) lifts to the linear action of \( K \) on \( \mathbb{C}^n \), and \( pr \) is \( K \)-equivariant. Consider a contact Hamiltonian \( h : V \times [0,1] \to \mathbb{R} \), and its \( K_0 \)-invariant lift \( \tilde{h} : S_p \times [0,1] \to \mathbb{R} \). The Hamiltonian \( \tilde{h} \) lifting \( h \) is \( K_0 \)-invariant, and therefore the Hamiltonian isotopy \( \{\Phi_{\tilde{h}}^t\}_{t \in [0,1]} \) is \( K_0 \)-equivariant. We call \( \tilde{h} \) a Hamiltonian lift of \( h \).

Let \( q \in V \) be an \( \alpha \)-translated point of \( \phi_h \). Any \( z \in P^{-1}(p) \) such that \( \rho(z) = q \) is a discriminant point of \( \exp(\lambda) \circ \phi_{\tilde{h}}^r \), for some \( \lambda \in \mathfrak{k} \). On \( \mathbb{C}^n \), it becomes a ray of fixed points of \( \exp(\lambda) \circ \Phi_{\tilde{h}} \):
\[
\exp(\lambda) \circ \Phi_{\tilde{h}}(rz) = rz, \text{ for any } r > 0.
\]

**Remark 3.2.1.** Notice that \( \lambda \) is not unique, since \( \exp(\lambda + \lambda_0) = \exp(\lambda) \), for any \( \lambda_0 \in \mathfrak{k}^\mathbb{Z} \).

Conversely, let \( z \in \mathbb{C}^n \) be a fixed point of \( \exp(\lambda) \circ \Phi_{\tilde{h}} \) such that \( rz \in P^{-1}(p) \), for some \( r > 0 \). Then \( rz \in S_p \) is a discriminant point of \( \exp(\lambda) \circ \phi_{\tilde{h}}^r \), and therefore \( \rho(rz) \) is a discriminant point of 
\[
\phi_{\alpha}^{\rho(\lambda)} \circ \phi_h.
\]
In other words, \( \rho(pr(z)) \) is an \( \alpha \)-translated point of \( \phi_h \).
3.3 The generating families

In this section, we introduce the generating families from which we will derive the cohomology groups. We first recall the general construction, and then apply it to the lifts of the previous section. Finally, we add the torus action to this construction, and come to the notion of generating families. We closely follow [Giv95], except that in our case, the generating functions are $K_0$-invariant, and their critical points correspond to translated points on $V$.

3.3.1 General construction

Let $H : \mathbb{C}^n \times [0, 1] \to \mathbb{R}$ be a time-dependent Hamiltonian, and $\{\phi^t_H\}_{t \in [0,1]}$ be the Hamiltonian isotopy generated by $H$. Dividing the interval $[0,1]$ into an even number of parts, say $2N$, we decompose the time-1 map $\phi_H$ as follows:

$$\phi_H = \phi_{2N} \circ ... \circ \phi_1,$$

where $\phi_j := \phi_{2N}^j \circ (\phi_{2N}^{j+1})^{-1}$. If $N$ is big enough so that, for any $z$, $-1$ is not an eigen value of $d_z\phi_H$, the graph

$$Gr_{\phi_j} := \{(z, \phi_j(z)) : z \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n$$

projects diffeomorphically to the diagonal

$$\Delta := \{(z, z) : z \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n,$$

where $\mathbb{C}^n$ denotes the symplectic vector space $(\mathbb{C}^n, -\omega_{std})$. The linear symplectomorphism

$$\Psi : \mathbb{C}^n \times \mathbb{C}^n \to (T^*\mathbb{C}^n, -d(pdq))$$

$$(z, w) \mapsto \left(\frac{z + w}{2}, i(z - w)\right)$$

sends the diagonal $\Delta$ to the zero-section $0_{\mathbb{C}^n} \subset T^*\mathbb{C}^n$, and therefore $\Psi(Gr_{\phi_j})$ is the graph of closed 1-form. This form is exact (either because $H^1(\mathbb{C}^n, \mathbb{R}) = \{0\}$ or $\phi_j$ is Hamiltonian):

$$\Psi(Gr_{\phi_j}) = Gr_{dH_j}.$$

The function $H_j$ is a generating function for the Lagrangian submanifold $\Psi(Gr_{\phi_j})$. In particular, the critical points of $H_j$ are, through $\Psi$, in one-to-one correspondence with the intersection points of $Gr_{\phi_j}$ with $\Delta$, that is, with the fixed points of $\phi_j$.

Consider now the Lagrangian product

$$Gr_{\phi_1} \times ... \times Gr_{\phi_{2N}} \subset (\mathbb{C}^n \times \mathbb{C}^n)^{2N}.$$

Applying the above to each component and the identification $(T^*\mathbb{C}^n)^{2N} = T^*\mathbb{C}^{2nN}$, we can write

$$\prod_{j=1}^{2N} \Psi(Gr_{\phi_j}) := \Psi(Gr_{\phi_1}) \times ... \times \Psi(Gr_{\phi_{2N}}) = Gr_{d\mathcal{H}},$$

were

$$\mathcal{H} : \mathbb{C}^{2nN} \to \mathbb{R}$$

$$(x_1, ..., x_{2N}) \mapsto \sum_{j=1}^{2N} H_j(x_j).$$

The critical points of $\mathcal{H}$ are in one-to-one correspondence with the fixed points of the product

$$\prod_{j=1}^{2N} \phi_j := \phi_1 \times ... \times \phi_{2N} : (\mathbb{C}^n)^{2N} \to (\mathbb{C}^n)^{2N}.$$
Yet, they do not correspond to fixed points of $\phi_h$, which are rather in one-to-one correspondence with the solutions of the equation

$$(z_2, \ldots, z_{2N}, z_1) = (\phi_1(z_1), \ldots, \phi_N(z_{2N})).$$

The graph $Gr_q \subset (\mathbb{C}^n)^{2N} \times (\mathbb{C}^n)^{2N} \simeq (\mathbb{C}^n \times \mathbb{C}^n)^{2N}$ of the "twisted" cyclic shift

$$q : (\mathbb{C}^n)^{2N} \to (\mathbb{C}^n)^{2N}, \quad (z_1, \ldots, z_{2N}) \mapsto (z_2, \ldots, z_{2N}, -z_1)$$

corresponds, through $\prod_{j=1}^{2N} \Psi$, to a Lagrangian subvector space of $T^*\mathbb{C}^{2nN}$, which has a generating quadratic form. Since we have decomposed $\phi_H$ into an even number of parts and have a factor $-1$ in the map $q$ above, the graph $Gr_q$ has trivial intersections with both the multi-diagonal and multi-antidiagonal

$$((\pm \Delta)^N := \{(z_1, \pm z_1, \ldots, z_{2N}, \pm z_{2N}) : z_j \in \mathbb{C}^n\},$$

These are sent, through $\prod_{j=1}^{2N} \Psi$, to the zero-section $0_{\mathbb{C}^{2nN}}$ and the fiber $\{0\} \times \mathbb{C}^{2nN} \subset T^*\mathbb{C}^{2nN}$ respectively. Therefore, in $T^*\mathbb{C}^{2nN}$, we can write

$$\prod_{j=1}^{2N} \Psi(Gr_q) = Gr_dQ,$$

where $Q : \mathbb{C}^{2nN} \to \mathbb{R}$ is a non-degenerate quadratic form. The intersection points of $Gr_q$ with $\prod_{j=1}^{2N} Gr_{\phi_j}$ are in one-to-one correspondence with the fixed points of the Hamiltonian symplectomorphism

$$-Id_{\mathbb{C}^n} \circ \phi_h.$$

Consider the function

$$\mathcal{F}^{(N)} : \mathbb{C}^{2nN} \to \mathbb{R}, \quad \mathcal{F}^{(N)} := Q - H.$$

The critical points of $\mathcal{F}^{(N)}$ are in one-to-one correspondence with the fixed points of $-Id_{\mathbb{C}^n} \circ \phi_h$. The function $\mathcal{F}^{(N)}$ is called the generating function of the decomposition $\phi_h = \phi_N \circ \ldots \circ \phi_1$.

3.3.2 Generating functions for the Hamiltonian lifts

Let $h : V \times [0, 1] \to \mathbb{R}$ be a contact Hamiltonian, and $\tilde{H} : \mathbb{C}^n \times [0, 1] \to \mathbb{R}$ a Hamiltonian lift of $h$. We apply the construction of the previous section to the Hamiltonian symplectomorphism $\exp(\lambda) \circ \Phi_{\tilde{H}}$, $\lambda \in \mathfrak{g}$ from section 3.2.2. Consider a decomposition

$$\Phi_{\tilde{H}} = \Phi_{2N_1} \circ \ldots \circ \Phi_1$$

of the Hamiltonian symplectomorphism $\Phi_{\tilde{H}}$ into $2N_1$ small parts $\Phi_j := \Phi_{\frac{\lambda}{2N_1}} \circ \Phi_{\frac{\lambda}{2N_1}}$, and similarly, a decomposition

$$\exp(\lambda) = \exp\left(\frac{\lambda}{2N_2}\right) \circ \ldots \circ \exp\left(\frac{\lambda}{2N_2}\right)$$

of the Hamiltonian symplectomorphism $\exp(\lambda)$. We denote by

$$\mathcal{F}^{(N)} := Q - H_{\lambda} : \mathbb{C}^{2nN} \to \mathbb{R}, \quad N = N_1 + N_2,$$
the generating function associated with the decomposition
\[
\exp(\lambda) \circ \Phi_{\tilde{H}} = \exp\left(\frac{\lambda}{2N_2}\right) \circ \ldots \exp\left(\frac{\lambda}{2N_2}\right) \circ \Phi_{2N_1} \circ \ldots \circ \Phi_1.
\]

The function \( H_\lambda \) is of the form
\[
H_\lambda(x_1, \ldots, x_{2N}) := \sum_{j=1}^{2N_1} H_j(x_j) + \sum_{j=2N_1+1}^{2N_2} T_\lambda(x_j), \quad x_j \in \mathbb{C}^n,
\]
where \( H_j \) and \( T_\lambda \) are the generating functions of \( \Phi_j \) and \( \exp(\frac{\lambda}{2N_2}) \) respectively. Moreover, a direct computation shows that
\[
T_\lambda(x_j) = \sum_{k=1}^{n} \tan(\frac{\pi k}{2N_2}) |q_j^k|^2, \quad x_j = (q_j^1, \ldots, q_j^n) \in \mathbb{C}^n \quad \text{and} \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{k} \subset \mathbb{R}^n.
\]

Let us list several properties of the function \( F_\lambda^{(N)} \):

1. the Hamiltonian lift \( \Phi_{\tilde{H}} \) is \( C^\infty \) on \( \mathbb{C}^n \setminus \{0\} \), and is \( C^2 \) at 0 only if it is quadratic. However, it is homogeneous of degree 2, and therefore it is \( C^1 \) on \( \mathbb{C}^n \) with Lipschitz differential near 0. Therefore, for any \( \lambda \in \mathfrak{k} \), \( F_\lambda^{(N)} \) is \( C^1 \) on \( \mathbb{C}^{2nN} \) with Lipschitz differential, and smooth on \( (\mathbb{C}^n \setminus \{0\})^{2N} \);

2. for any \( \lambda \in \mathfrak{k} \), \( F_\lambda^{(N)} \) is homogeneous of degree 2 and \( \mathbb{K}_0 \)-invariant. In particular, the critical points of \( F_\lambda^{(N)} \) appear as rays of \( \mathbb{K}_0 \)-orbits in \( \mathbb{C}^{2nN} \), and have critical value 0;

3. the function \( F_\lambda^{(N)} \) is well-defined as long as \( \iota(\lambda) \in [-N_2, N_2]^n \subset \mathbb{R}^n \);

4. the family \( \lambda \mapsto F_\lambda^{(N)} \) decreases in positive directions: for any \( s \in \mathfrak{k} \) such that \( \iota(\lambda + s) \in \mathbb{C}^n \) and \( \iota(s) \in (\mathbb{R}_{>0})^n \), we have
\[
F_{\lambda+s}^{(N)} \leq F_{\lambda}^{(N)};
\]

5. for any \( \lambda \in \mathfrak{k} \), the critical points of \( F_\lambda^{(N)} \) are in one-to-one correspondence with the fixed points of the Hamiltonian symplectomorphism
\[
-Id_{\mathbb{C}^n} \circ \exp(\lambda) \circ \Phi_{\tilde{H}}.
\]

The symplectomorphism \( -Id_{\mathbb{C}^n} \) is Hamiltonian \( \mathbb{K} \)-equivariant. It preserves the sphere \( S_p \) and the contact form \( \alpha_p \), and the level set \( P_{\mathbb{K}}^{-1}(p) \). Therefore, it projects to an \( \alpha \)-preserving contactomorphism \( g \in \text{Cont}_0(V, \xi) \). In particular, any estimate for the number of translated points of all the compositions of \( g \) with contactomorphisms in \( \text{Cont}_0(V, \xi) \) will give rise to the same estimate for the contactomorphisms in \( \text{Cont}_0(V, \xi) \) themselves. In other words, the twist by \( -Id_{\mathbb{C}^n} \) does not affect the estimation on the number of translated points.

### 3.3.3 Adding the torus action

Recall that we are looking for fixed points of the composition
\[
\exp(\lambda) \circ \Phi_{\tilde{H}}
\]
for all values of \( \lambda \), and therefore we shall consider \( \mathfrak{f} \) as the space of Lagrange multipliers. Consider the following function
\[
F_N: \mathbb{C}^{2nN} \times \Lambda_N \to \mathbb{R}, \quad F_N(x, \lambda) := F_\lambda^{(N)}(x),
\]
where \( \Lambda_N \) is a closed submanifold of \( \mathfrak{t} \) with boundary \( \partial \Lambda_N \) such that \( \iota(\Lambda_N) \subset ([-N_2, N_2]^n \setminus \Lambda_N)^n \). We denote by \( S_N \) the unit sphere \( S^{4nN-1} \subset \mathbb{C}^{2nN} \). A critical ray of \( \mathcal{F}_\Lambda^{(N)} \) corresponds to a critical point on \( S_N \). We denote by \( F_N \) the restriction of \( \mathcal{F}_\Lambda \) to \( S_N \times \Lambda_N \). We call \( F_N \) (resp. \( \mathcal{F}_\Lambda \)) the generating family (resp. homogeneous generating family) associated with the decomposition \( \Phi_1 = \Phi_2 \circ \ldots \circ \Phi_1 \).

In the sequel, we fix this decomposition, and for any \( N > N_1 \), we take \( \Lambda_N \) as cube in \( \mathfrak{t} \) centered at the origin, of fixed size growing linearly with \( N \), and such that \( \cup \Lambda_N = \mathfrak{t} \).

\begin{remark}
Restricting the generating families to \( \Lambda_N \) will serve us in the sequel, when we investigate equivariant cohomology groups of sets relative to the boundary \( \partial \Lambda_N \). When studying regularity and critical point sets, we will keep in mind that these functions are actually defined on \( \mathfrak{t} \cap ([-N_2, N_2]^n \setminus \Lambda_N)^n \).

We are looking for critical points of the functions \( F_\Lambda^{(N)} \), which lie in rays that, through \( \Psi \), intersect \( P_K^{-1}(p) \). Such critical points have critical value 0 and moreover, the zero-set \( F_N^{-1}(0) \subset S_N \times \Lambda_N \) is \( \mathbb{K}_0 \)-invariant. Consider the function
\[
\hat{p}_N : F_N^{-1}(0) \rightarrow \mathbb{R},
(\lambda, z) \mapsto \lambda(z).
\]
It is \( \mathbb{K}_0 \)-invariant, and we have the following contact analogue of [Giv95], proposition 4.3.

\begin{proposition}
0 is a regular value of \( F_N \), and critical \( \mathbb{K}_0 \)-orbits of \( \hat{p}_N \) correspond to translated points of \( \phi_h \).
\end{proposition}

\begin{proof}
Notice first that by homogeneity of \( F_N \), we have \( d_x F_\Lambda^{(N)} = d_x F_\Lambda^{(N)} \) for any \( x \in (F_\Lambda^{(N)})^{-1}(0) \).

Let \( (x, \lambda) \in F_N^{-1}(0) \) be a critical point of \( F_N \). Then \( x \in S_N \) is a critical point of \( F_\Lambda^{(N)} \). On \( \mathbb{C}^n \), it corresponds to a fixed point \( z \) of the decomposition
\[
-1d_{C^n} \circ \exp(\frac{\lambda}{2N_2}) \circ \ldots \circ \exp(\frac{\lambda}{2N_2}) \circ \Phi_2 \circ \ldots \circ \Phi_1.
\]

Let us denote the corresponding discrete trajectory in \( \mathbb{C}^{2nN} \) by \( (z_1, \ldots, z_{2N} = -z_1) \), that is, \( z_1 = z \), and \( z_j \) is obtained by applying the \( (j - 1) \)-th symplectomorphism of the above decomposition to \( z_{j-1} \).

Choose coordinates on \( \mathfrak{t} = \mathbb{R}^k \), and write \( \lambda = (\lambda_1, \ldots, \lambda_k) \). By the Hamilton-Jacobi equation, the derivative of \( F_N \) in \( \lambda \) is given by minus the Hamiltonian associated with the infinitesimal action of \( \exp(\lambda) \) on the \( N \) last coordinates of \( x \). Through \( \Psi \), this means that
\[
\frac{\partial F_N}{\partial \lambda_j}(x, \lambda) = -P^j_K(z_{N+1}) - \ldots - P^j_K(z_{2N}),
\]
where \( P_K = (P_K^1, \ldots, P_K^k) : \mathbb{C}^n \rightarrow \mathbb{R}^k \). On the other hand, we have \((z_{2N_1+i} = \exp(\frac{\lambda}{2N_2})z_{2N_1+i-1}, \) for any \( i = 1, \ldots, 2N_2 \). Since \( P_K^j \) is \( \mathbb{K}_0 \)-invariant, we obtain
\[
\frac{\partial F_N}{\partial \lambda_j}(x, \lambda) = -NP^j_K(z_{2N_1+1}).
\]

If \( (x, \lambda) \) is a critical point of \( F_N \), then \( P^j_K(z_{2N_1+i}) = 0 \), for any \( j = 1, \ldots, k \). In particular, this means that \( P(z_{2N_1+i}) \in \ker \iota^* \), and by compactness of the toric manifold \( M \), this is possible only if \( z_{2N_1+1} = 0 \).

By homogeneity of the symplectomorphisms in the decomposition above, this implies that \( z = 0 \), and thus \( x = 0 \), which is impossible on \( S_N \). Thus 0 is a regular value of \( F_N \).

By the method of Lagrange multipliers, the critical points of \( \hat{p}_N \) are the points \( (x, \lambda) \in F_N^{-1}(0) \) such that \( x \) is a critical point of \( F_\Lambda^{(N)} \), and \( \frac{\partial F_N}{\partial \lambda_j}(x, \lambda) \) is proportional to \( p_j \), where \( p = (p_1, \ldots, p_k) \in \mathbb{R}^k \).

By the discussion above, this means that \( x \) corresponds to a discrete trajectory \( (z_1, \ldots, z_{2N} = -z_1) \) satisfying
\[
P^j_K(z_{2N_1+i}) \sim p_j.
\]

But then, \( P^j_K(z_{2N}) = P^j_K(z) \sim p_j \). In other words, \( z \) lies in a ray which intersects \( P_K^{-1}(p) \).
\end{proof}
3.4 From a decomposition to another

We describe here how the generating families are related when \( N \) grows. We closely follow \[\text{Giv95}\]. Notice that the front of \( F_N \) depends only on \( N \) and the time 1-map \( \Phi_H \). We begin by an observation. Let \( \Phi_H = \Phi_{2N'} \circ \ldots \circ \Phi_1 \) be the decomposition of a Hamiltonian isotopy \( \{ \Phi_H \}_{t \in [0,1]} \), such that the first \( 2(N' - N) \) parts consist of a loop \[
\text{Id}_{C^n} = \Phi_{2(N' - N)} \circ \ldots \circ \Phi_1.
\]

We relate the generating function \( F^{(N')} \) associated with the whole decomposition
\[
- \text{Id}_{C^n} \circ \Phi_H = - \text{Id}_{C^n} \circ \Phi_{2N'} \circ \ldots \circ \Phi_1,
\]
and the generating functions \( F^{(N)} \) and \( G^{(N' - N)} \) of the parts
\[
- \text{Id}_{C^n} \circ \Phi_{2N'} \circ \ldots \circ \Phi_{2(N' - N) + 1}, \quad - \text{Id}_{C^n} \circ \Phi_{2(N' - N)} \circ \ldots \circ \Phi_1,
\]
in the following way. Consider the following deformation of the graph \( Gr_q \) from section 3.3.1
\[
Q_\epsilon := \{(z_1, w_1, \ldots, z_{2N'}, w_{2N'}): (z_j, w_j) \in C^n \times C^n, \quad w_j = z_{j+1} \text{ for } j \notin \{2(N' - N), 2N'\} \}.
\]

Then \( Q_\epsilon \) is a Lagrangian subspace of \( (C^n \times C^n)^{2N'} \), which is transversal both to the multi-diagonal \( \Delta^{2N'} \) and the multi-antidiagonal \( -(\Delta)^{2N'} \). Thus, it corresponds in \( T^*C^{2N'} \) to a non-degenerate quadratic form \( Q_\epsilon \), and we can define the generating function
\[
F^{(N')}_{\epsilon} : Q_\epsilon \to H,
\]
the critical points of which correspond to the points in the intersection
\[
\prod_{j=1}^{2N'} \Psi(Gr_{\phi_j}) \cap Q_\epsilon.
\]

For \( \epsilon = 0 \), \( Q_0 \) is the product \( Q_1 \times Q_2 \), where \( Q_1 \) and \( Q_2 \) correspond to the twisted cyclic shifts in \( (C^n)^{2(N' - N)} \) and \( (C^n)^{2N} \) respectively. Since \( H \) is the sum of the generating functions associated with the small Hamiltonian parts \( \Phi_j \), we get
\[
F^{(N')}_0 = F^{(N)} \oplus G^{(N' - N)} : C^{2nN} \times C^{2n(N' - N)} \to \mathbb{R},
\]
and moreover, \( F^{(N')}_0 \) admits a critical point if and only if \( F^{(N)} \) does, if and only if \( F^{(N')} \) does. For \( \epsilon = 1 \), we have \( Q_\epsilon = Gr_q \), so that
\[
F^{(N')}_1 = F^{(N')}.
\]

Finally, for \( \epsilon \in [0,1[ \), notice that
\[
(z_1, w_1, \ldots, z_{2N'}, w_{2N'}) \in \prod_{j=1}^{2N'} \Psi(Gr_{\phi_j}) \cap Q_\epsilon \iff \left( \frac{1}{\epsilon} z_1, \frac{1}{\epsilon} w_1, \ldots, \frac{1}{\epsilon} z_{2(N' - N)}, \frac{1}{\epsilon} w_{2(N' - N)}, z_{2(N' - N) + 1}, w_{2(N' - N) + 1}, \ldots, z_{2N'}, w_{2N'} \right) \in \prod_{j=1}^{2N'} \Psi(Gr_{\phi_j}) \cap Q,
\]
that is, critical points of \( F^{N'}_\epsilon \) are in one-to-one correspondence with critical points of \( F^{N'} \), for any \( \epsilon \in [0,1] \).
The observation now, is that given two decompositions

\[ \Phi_H^1 = \Phi_{2N} \circ \ldots \circ \Phi_1, \quad \Phi_H^1 = \Phi_{H'}^1 \]

\[ \Phi_{H'}^1 = \Phi_{2N'} \circ \ldots \circ \Phi_1' \]

of the same Hamiltonian isotopy \( \{ \Phi_H^t \}_{t \in [0,1]} = \{ \Phi_{H'}^t \}_{t \in [0,1]} \) on \( \mathbb{C}^n \) with, say, \( N' > N \), one can always use a reparametrization \( \tilde{H}_t^s \) of \( H_t' \), such that \( \tilde{H}_t^s = H_t', \) and \( \tilde{H}_t^s \) generated the decomposition

\[ \Phi_{\tilde{H}_s}^1 = \Phi_{2N} \circ \ldots \circ \Phi_1 \circ Id_{\mathbb{C}^n} \circ \ldots \circ Id_{\mathbb{C}^n}. \]

In particular, consider the homogeneous generating family \( \mathcal{F}_N \) associated with the decomposition

\[ \Phi_{\tilde{H}}^1 = \Phi_{N_1} \circ \ldots \circ \Phi_1, \]

where \( \tilde{H} \) is a Hamiltonian lift of \( h \). Let us denote by \( g^{(K)}_0 : \mathbb{C}^{2nK} \rightarrow \mathbb{R} \) the generating function associated with the decomposition

\[ \exp\left( \frac{\lambda}{2K} \right) \circ \ldots \circ \exp\left( \frac{\lambda}{2K} \right), \]

as defined in 3.3.1. Note that the front of the restriction \( \mathcal{F}_{N+K}|_{\Lambda_N} \) is the front of \( F_N : L_{N+K} \cap \Lambda_N = L_N \).

**Proposition 3.4.1.** There exists a fiberwise \( \mathbb{K}_0 \)-invariant smooth homotopy between the restricted homogeneous family

\[ \mathcal{F}_{N+K}|_{\Lambda_N} \]

and the fiberwise direct sum

\[ \mathcal{F}_N \oplus_{\Lambda_N} g^{(K)}_0 : \mathbb{C}^{2n(N+K)} \times \Lambda_N \rightarrow \mathbb{R}, \]

in a way that the front of the associated generating families on \( S_{N+K} \times \Lambda_N \) remains unchanged during the deformation.

**Proof.** Recall that \( \mathcal{F}_N^{(N+K)} \) and \( \mathcal{F}_N^{(N)} \) are the generating functions associated respectively with the decompositions

\[ \exp(\lambda) \circ \Phi_{\tilde{H}} = \exp\left( \frac{\lambda}{2(N_2 + K)} \right) \circ \ldots \circ \exp\left( \frac{\lambda}{2(N_2 + K)} \right) \circ \Phi_{2N_1} \circ \ldots \circ \Phi_1 \]

\[ \exp(\lambda) \circ \Phi_{\tilde{H}} = \exp\left( \frac{\lambda}{2N_2} \right) \circ \ldots \circ \exp\left( \frac{\lambda}{2N_2} \right) \circ \Phi_{2N_1} \circ \ldots \circ \Phi_1. \]

Up to reparametrization of the \( 2(N_2 + K) \) last factors, the first decomposition becomes

\[ \exp(\lambda) \circ \Phi_{\tilde{H}} = \exp\left( \frac{\lambda}{2N_2} \right) \circ \ldots \circ \exp\left( \frac{\lambda}{2N_2} \right) \circ Id_{\mathbb{C}^n} \circ \ldots \circ Id_{\mathbb{C}^n} \circ \Phi_{2N_1} \circ \ldots \circ \Phi_1 \]

\[ = \exp\left( \frac{\lambda}{2N_2} \right) \circ \ldots \circ \exp\left( \frac{\lambda}{2N_2} \right) \circ \Phi_{2N_1} \circ \ldots \circ \Phi_1 \circ Id_{\mathbb{C}^n} \circ \ldots \circ Id_{\mathbb{C}^n}. \]

which, along with the discussion above, yields the result. \( \square \)
Remark 3.4.1. Notice that the result above is independent of the reparametrization, since any two such reparametrizations are always homotopic. Moreover, one can show by a similar argument that the front of the generating family $F_N$ remains unchanged if one modifies the decomposition of $\Phi_{\bar{H}}$ into $2N_2$ parts. In other words, it depends only on $N = N_1 + N_2$ and the time 1 maps $\Phi_{\bar{H}}$.

3.5 Sublevel sets and transversality

In this section, we introduce the sublevel sets of the generating families which will be used to define the cohomology groups. We consider the following sets

$$\mathcal{F}_N^+: = \{F_N \geq 0 \text{ (resp. } \leq 0)\}, \quad \mathcal{F}_N^- : = \{F_N \geq 0 \text{ (resp. } \leq 0)\},$$

For any $\nu \in \mathbb{R}$, we denote by $\Gamma_N(\nu)$ the intersection $\Lambda_N \cap p^{-1}(\nu)$, and define

$$\mathcal{F}_N^+(\nu): = \mathcal{F}_N^+ \cap (\mathbb{C}^{2nN} \times \Gamma_N(\nu)), \quad \partial \mathcal{F}_N^+(\nu): = \mathcal{F}_N^+ \cap (\mathbb{C}^{2nN} \times \partial \Gamma_N(\nu)),$$

$$\mathcal{F}_N^-(\nu): = \mathcal{F}_N^- \cap (S_N \times \Gamma_N(\nu)), \quad \partial \mathcal{F}_N^-(\nu): = \mathcal{F}_N^- \cap (S_N \times \partial \Gamma_N(\nu)),$$

where

$$\partial \Gamma_N(\nu): = \Gamma_N(\nu) \cap \partial \Lambda_N.$$ 

Let $L_N$ denote the front of $F_N$:

$$L_N : = \{\lambda \in \Lambda_N : 0 \text{ is a singular value of } F^{(N)}_\lambda\}.$$ 

Recall that the homogeneous generating function $F^{(N)}_\lambda$ is smooth on $(\mathbb{C}^n \setminus \{0\})^{2N}$. If $x$ is a critical point of $F^{(N)}_\lambda$, it corresponds, through $\Psi$, to a solution of the equation

$$(z_2, \ldots, z_{2N}, -z_1) = (\gamma_1(z_1), \ldots, z_{2N}, \gamma_2(z_{2N}))$$

where $\gamma_j = \Phi_j$ if $j \leq 2N_1$, and $\gamma_j = \exp\left(\frac{\lambda}{2N_2}\right)$ otherwise. If $x$ lies on the coordinate cross, there exists $j$ such that $\gamma_j(z_j) = -z_j$. Since $\gamma_j$ is close to $Id_{\mathbb{C}^n}$, this can happen only if $z_j = 0$. Since $\gamma_j(0) = 0$ for all $j$, this implies that $x = 0$. Therefore, $F^{(N)}_\lambda$ is smooth at any non-zero critical points. In particular, $L_N$ is of zero-measure.

The relation between the front $L_N$ and the spectrum $Spec(\phi_h)$ (equation (1)) shall be understood as follows: the front is made of all the elements $\lambda$ such that the Hamiltonian diffeomorphism

$$-Id_{\mathbb{C}^n} \circ \exp(\lambda) \circ \Phi_{\bar{H}} : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

admits a fixed point, whereas the spectrum is made of real numbers for which an $\alpha$-translated point of $\phi_h$ appears on $V$. In other words, meeting the front corresponds to the notion of fixed points on $\mathbb{C}^n$, whereas by propositions 2.1.1 and 3.1.1, being tangent to the front corresponds to the notion of fixed point on $F^{-1}_\lambda(p)$, that is, to that of translated points on $V$.

Note however that $\nu \notin Spec(\phi_h)$ does not mean that the boundary $\partial \Gamma_N(\nu)$ is transversal to $L_N$. By corollary 2.1.1 the set of $\nu$ such that $\Gamma_N(\nu)$ and $\partial \Gamma_N(\nu)$ are transverse to the front $L_N$ is of full measure. We will say that $\nu \in \mathbb{R}$ is generic if $\Gamma_N(\nu)$ and $\partial \Gamma_N(\nu)$ are transversal to $L_N$ for all $N$ (we use here that a countable intersection of sets of full measure is of full measure). Note that this notion depends on $\Phi_{\bar{H}}$ and $\{\Lambda_N\}_N$.

We have the following "homogeneous" version of proposition 5.1 in [Giv95].

Proposition 3.5.1. Let $\mathcal{F} : \mathbb{C}^M \rightarrow \mathbb{R}$ be a homogeneous of degree 2 function, and $\hat{\mathcal{F}} : \mathbb{C}^{M+1} \rightarrow \mathbb{R}$ be its suspension

$$\hat{\mathcal{F}}(x, z) : = \mathcal{F}(x) + |z|^2.$$
We denote by $\mathcal{F}^\pm$ and $\hat{\mathcal{F}}^\pm$ the sets
\[ \mathcal{F}^\pm := \{ x \in \mathbb{C}^M : \mathcal{F}(x) \geq 0 \text{ (resp.} \leq 0) \}, \quad \hat{\mathcal{F}}^\pm := \{ (x, z) \in \mathbb{C}^{M+1} : \hat{\mathcal{F}}(x) \geq 0 \text{ (resp.} \leq 0) \}. \]

Then there exist natural homotopy equivalences
\[ \hat{\mathcal{F}}^- \simeq \mathcal{F}^-, \quad \hat{\mathcal{F}}^+ \simeq \mathcal{F}^+ \times \mathbb{C}. \]

Moreover, if $\mathcal{F}$ is invariant relatively to a $S^1$-action on $\mathbb{C}^M$, then the above homotopy equivalences can be made equivariant with respect to the diagonal action on $\mathbb{C}^{M+1}$. If $\mathcal{F}$ depends continuously on additional parameters, then the homotopy equivalences depend continuously on them.

**Proof.** We first treat the case where $z \in \mathbb{R}_{\geq 0}$. The meridional contraction from the North pole $P = (0, 1)$ of the unit sphere $S^{2M+1}$ preserves $\hat{\mathcal{F}}^- \cap (S^{2M-1} \times \{0\})$, and therefore $\hat{\mathcal{F}}^- \cap S^{2M+1}$ retracts onto $\hat{\mathcal{F}}^- \cap (S^{2M-1} \times \{0\})$. Extending this contraction homogeneously provides a deformation retraction from $\hat{\mathcal{F}}^-$ to $\mathcal{F}^-$ as well:
\[ \hat{\mathcal{F}}^- \simeq \mathcal{F}^- \]

Moreover, each meridional arc from the North pole to $\mathcal{F}^+ \cap S^{2M-1}$ lies in $\hat{\mathcal{F}}^+ \cap S^{2M+1}$, and therefore the same contraction provides a homotopy equivalence of pairs:
\[ (\hat{\mathcal{F}}^+, \mathcal{F}^+) \simeq (\mathbb{C}^M \times \mathbb{R}_{\geq 0}, \mathcal{F}^+). \]

Now, the pairs $(\mathbb{C}^M \times \mathbb{R}_{\geq 0}, \mathcal{F}^+)$ and $(\mathcal{F}^+ \times \mathbb{R}_{\geq 0}, \mathcal{F}^+)$ are naturally homotopy equivalent, and therefore we have
\[ (\hat{\mathcal{F}}^+, \mathcal{F}^+) \simeq (\mathcal{F}^+ \times \mathbb{R}_{\geq 0}, \mathcal{F}^+). \]

For the general case, we view $\mathbb{C}^M \times \mathbb{C}$ as the quotient $\mathbb{C}^M \times \mathbb{R}_{\geq 0} \times S^1 / \sim$, where $\mathbb{C}^M \times \{0\} \times S^1 \sim \mathbb{C}^M$. Then $\mathcal{F}^\pm$ and $\hat{\mathcal{F}}^\pm$ are just given by their restrictions to $\mathbb{C}^M \times \mathbb{R}_{\geq 0}$ multiplied by $S^1$, under the relevant identifications. We obtain
\[ \hat{\mathcal{F}}^- \simeq \mathcal{F}^-, \quad \hat{\mathcal{F}}^+ \simeq \mathcal{F}^+ \times \mathbb{C}. \]

Since all the homotopies from above are carried out in a canonical way, they respect group actions and parametric dependence.

Recall the notation $\mathcal{G}_\lambda^{(N)}$ for the generating function of the decomposition
\[ \exp \left( \frac{\lambda}{2N} \right) \circ \cdots \circ \exp \left( \frac{\lambda}{2N} \right), \]

as defined in [3.3.1]. We look at the associated homogeneous generating family parametrized by a point (we do not make $\lambda$ vary):
\[ \mathcal{G}_{\lambda,N} : \mathbb{C}^{2nN} \to \mathbb{R}, \quad \mathcal{G}_{\lambda,N}(x) = \mathcal{G}_\lambda^{(N)}(x). \]

It is a quadratic form, and we will use the notation $\mathcal{G}_{\lambda,N}^\pm$ to denote its non-negative and non-positive eigen spaces, after the following diagonalization.

**Lemma 3.5.1.** There exists an equivariant linear isomorphism
\[ \mathbb{C}^{2nN} \simeq \bigoplus_{j,k} \mathbb{C}v_j^k, \quad j = 1, \ldots, n, \quad k = 1, \ldots, 2N, \]

where $v_j^k$ are vectors in $\mathbb{C}^{2nN}$, such that:

1. for any $k$, the maximal torus $\mathbb{T}$ acts on the complex line generated by $v_j^k$ via its standard characters $\chi_j : (e^{i\theta_1}, \ldots, e^{i\theta_n}) \mapsto e^{i\theta_j}$. In particular, the direct sum $\bigoplus_k \mathbb{C}v_j^k$ is $\mathbb{T}$-invariant;
2. for any \( \lambda \), the quadratic form \( \mathcal{G}_{\lambda}^{(N)} \) is diagonal in the basis \( \{v_j^k\}_{j,k} \), and we have

\[
\dim(\mathcal{G}_{\lambda}^{(N)}) = \sum_{j=1}^{n} 2(N + \lfloor \lambda_j + \frac{1}{2} \rfloor), \quad \lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_N,
\]

where \( \lfloor . \rfloor \) denotes the integer part.

**Proof.** Recall that for any \( \lambda \in \mathfrak{t} \subset \mathbb{R}^n \), the quadratic form \( \mathcal{G}_{\lambda}^{(N)} \) is given by

\[
\mathcal{G}_{\lambda}^{(N)} = Q - H_{\lambda}.
\]

We have \( H_{\lambda} = -\sum_{i=1}^{n} \frac{\tan(\frac{\pi}{2N})}{2} |q_i|^2 + ... + |q_{2N}^i|^2 \), where \( q = (q_1^1, ..., q_n^1, ..., q_2^N, ..., q_{2N}^n) \in \mathbb{C}^{2nN} \), and moreover, a direct calculation shows that the non-degenerate quadratic form \( Q \) writes \( Q(q) = \langle Cq, q \rangle \), where

\[
C = i(Id - A)(Id + A)^{-1}, \quad A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -1 & 0 & & & \end{bmatrix} \in M_{2N \times 2N}(\mathbb{C}^n).
\]

For any \( k = -N, ..., N - 1 \), and \( j = 1, ..., n \), we define the following vector:

\[
X_j^k = (e_j, e^{i(2k+1)n}, e_{j}, e^{i(2N-1)}e_{j}) \in (\mathbb{C}^n)^{2N},
\]

where \( e_j = (0, 0, ..., 0, 1, 0, ..., 0) \) is the \( j \)-th standard vector in \( \mathbb{C}^n \). The reader may check that the following hold:

- we have \( i(A - Id)X_j^k = \tan(\frac{2k+1}{4N})X_j^k \) and therefore the family \( \{v_j^k := (A + Id)(X_j^k)\}_{j,k} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}^{2nN} \) made of eigen vectors of \( C \), with eigen values \( -\tan(\frac{2k+1}{4N}) \);

- the maximal torus \( T \) acts on the complex line generated by \( v_j^k \) via its character \( \chi_j \). In particular, it is \( T \)-invariant;

Let’s put \( V_j = \mathop{\oplus}_{k=1}^{2N} C v_j^k \). Since \( Q \) is \( T \)-invariant, each two lines \( C v_i^k \) and \( C v_j^l \) are orthogonal whenever \( i \neq j \). It remains to diagonalize the restrictions \( Q|_{V_j} \). The action of \( T \) on the basis \( \{v_j^1, ..., v_j^{2N}\} \) is given by \( \chi_j \). Since it is diagonal and linear, we can find a new basis (still denoted by \( \{v_j^1, ..., v_j^{2N}\} \)) for \( V_j \), on which \( T \) acts by the character \( \chi_j \), and such that \( C_j|_{V_j} \) is diagonal. It remains to concatenate these bases for \( j = 1, ..., n \).

The spectrum of \( Q \) is equal to

\[
\text{Spec}(Q) = \{-\tan(\frac{\pi(2k+1)}{4N}) : -N \leq k \leq N - 1\},
\]

and therefore the spectrum of \( \mathcal{G}_{\lambda}^{(N)} \) is given by

\[
\text{Spec}(\mathcal{G}_{\lambda}^{(N)}) = \{\tan(\frac{\pi(2k+1)}{4N}) - \tan(\frac{\pi\lambda_j}{2N}) : -N \leq k \leq N - 1, j = 1, ..., n\}.
\]

We have

\[
\tan(\frac{\pi(2k+1)}{4N}) - \tan(\frac{\pi\lambda_j}{2N}) < 0 \iff k < \lambda_j - \frac{1}{2}.
\]

There are \( N + \lfloor \lambda_j + \frac{1}{2} \rfloor \) such \( k \)'s, and therefore the real dimension of \( \mathcal{G}_{\lambda,N} \) is:

\[
\dim(\mathcal{G}_{\lambda,N}) = \sum_{j=1}^{n} 2(N + \lfloor \lambda_j + \frac{1}{2} \rfloor),
\]

as claimed. \( \square \)
This change of basis is canonical, that is, it depends only on $N$ and the quadratic form $Q$. Consider the fiberwise direct sum $F_N \oplus_{\Lambda_N} G_0^{(K)}$. The non-degenerate quadratic form $G_0^{(K)}$ has $2nK$ negative eigen values, and is diagonal in the basis of the above proposition. Applying proposition 3.5.1 multiple times, we get

**Corollary 3.5.1.** There exists a fiberwise $\mathbb{K}_0$-equivariant homotopy equivalence

$$\{F_N \oplus_{\Lambda_N} G_0^{(K)} \leq 0\} \simeq F_N^- \times G_0^{-K} \simeq F_N^- \times \mathbb{C}^{nK}.$$

Applying proposition 3.4.1 along with proposition 2.1.2, we get

**Proposition 3.5.2.** If $\nu$ is generic, there exists a fiberwise $\mathbb{K}_0$-equivariant homotopy equivalence

$$\big( F_{N+K|\Gamma_N(\nu)}, F_{N+K|\partial\Gamma_N(\nu)}^- \big) \simeq \big( F_{N}^- (\nu) \times \mathbb{C}^{nK}, \partial F_{N}^- (\nu) \times \mathbb{C}^{nK} \big).$$

### 3.6 The cohomology groups

In this section we study the equivariant cohomology of the sublevel sets of the previous section, and come to the definition of our limit of cohomology groups. Consider the homogeneous generating family

$$F_N : \mathbb{C}^{2nN} \times \Lambda_N \to \mathbb{R},$$

associated with a decomposition of the Hamiltonian symplectomorphism $\Phi_{\tilde{H}}$, where $\tilde{H}$ is the lift of a contact Hamiltonian $h$, and fix a generic $\nu$. We look at the $\mathbb{K}_0$-equivariant cohomology groups

$$H_{\mathbb{K}_0}^* (F_N^- (\nu), \partial F_{N}^- (\nu)).$$

We consider the following short exact sequence

$$0 \to C_{\mathbb{K}_0, c}^* (F_N^- (\nu), \partial F_{N}^- (\nu)) \to C_{\mathbb{K}_0}^* (F_N^- (\nu), F_N^- (\nu)) \to C_{\mathbb{K}_0}^* (F_{N}^- (\nu), F_{N}^- (\nu)) / C_{\mathbb{K}_0, c}^* (F_N^- (\nu), \partial F_{N}^- (\nu)) \to 0,$$

and identify the cohomology of the third term with that of the complex $C_{\mathbb{K}_0}^* (F_{N}^- (\nu), \partial F_{N}^- (\nu))$, that is with $H_{\mathbb{K}_0}^* (F_{N}^- (\nu), \partial F_{N}^- (\nu))$. Consider the $\mathbb{C}^{nK}$-bundle

$$(F_{N}^- (\nu) \times \mathbb{C}^{nK})_{\mathbb{K}_0} \to F_{N}^-(\nu)_{\mathbb{K}_0}.$$

The Thom isomorphism of this bundle

$$H_{\mathbb{K}_0}^* (F_{N}^- (\nu), \partial F_{N}^- (\nu)) \simeq H_{\mathbb{K}_0, cv}^{*+2nK} (F_{N}^- (\nu) \times \mathbb{C}^{nK}, \partial F_{N}^- (\nu) \times \mathbb{C}^{nK}),$$

and the natural homomorphism

$$H_{\mathbb{K}_0, cv}^{*+2nK} (F_{N}^- (\nu) \times \mathbb{C}^{nK}, \partial F_{N}^- (\nu) \times \mathbb{C}^{nK}) \to H_{\mathbb{K}_0}^* (F_{N}^- (\nu) \times \mathbb{C}^{nK}, \partial F_{N}^- (\nu) \times \mathbb{C}^{nK})$$

preserve compact supports, and therefore they induces a homomorphism

$$H_{\mathbb{K}_0}^* (F_{N}^- (\nu), \partial F_{N}^- (\nu)) \to H_{\mathbb{K}_0}^* (F_{N}^- (\nu) \times \mathbb{C}^{nK}, \partial F_{N}^- (\nu) \times \mathbb{C}^{nK}).$$

Along with proposition 3.5.2, we obtain a homomorphism

$$H_{\mathbb{K}_0}^* (F_{N}^- (\nu), \partial F_{N}^- (\nu)) \to H_{\mathbb{K}_0}^* (F_{N}^- (\nu) \times \mathbb{C}^{nK}, \partial F_{N}^- (\nu) \times \mathbb{C}^{nK}).$$

We now apply the equivariant version of the excision formula to the triple

$$\big( F_{N+K}(\nu), F_{N+K|\Gamma_N+K(\nu)}^- (\nu), F_{N+K|\partial\Gamma_N(\nu)}^-(\nu) \big).$$
where $\Gamma_N(\nu)$ denotes the interior of $\Gamma_N(\nu)$ (note here the importance of the fact that transversality is an open condition). It induces an isomorphism

$$H^*_\mathbb{K}_0(\mathcal{F}_{N+K}\nu,\mathcal{F}_{N+K}\partial\Gamma_N(\nu)) \cong H^*_\mathbb{K}_0(\mathcal{F}_{N+K}\nu,\mathcal{F}_{N+K}\nu,\mathcal{F}_{N+K}\nu,\Gamma_N(\nu)),$$

which preserves compact supports. Moreover, there is an inclusion of pairs

$$(\mathcal{F}_{N+K}\nu,\partial\mathcal{F}_{N+K}\nu) \subset (\mathcal{F}_{N+K}\nu,\mathcal{F}_{N+K}\nu,\mathcal{F}_{N+K}\nu,\Gamma_N(\nu)).$$

Putting all these maps together, we obtain a homomorphism

$$H^*_\mathbb{K}_0(F_N(\nu),F_N(\nu)) \rightarrow H^*_\mathbb{K}_0(F_{N+K}(\nu),\partial F_{N+K}(\nu)).$$

We will take a limit in $N \rightarrow \infty$, and therefore it will be convenient to shift the grading by $2nN$. Thus we have built a homomorphism:

$$f^N_{K} : H^{*+2nN}(F_N(\nu),\partial F^{-}(\nu)) \rightarrow H^{*+2n(N+K)}(F_{N+K}(\nu),\partial F_{N+K}^{-}(\nu)).$$

Notice that all the maps involved in the construction of $f^N_{K}$ are natural in cohomology: they involve topological inclusions, excision, the Thom isomorphism, and the deformation of proposition 3.5.2. In particular, we have the following cocycle condition

$$f^{N+K}_{K} \circ f^{N}_{K} = f^{N+K+K'}_{K}.$$

**Definition 3.6.1.** We define the **cohomology of $\tilde{H}$ of level $\nu$** as the limit

$$H^*_\mathbb{K}_0(F^{-}(\nu)) := \lim_{N \rightarrow \infty} H^{*+2nN}(F_N(\nu),\partial F^{-}(\nu)).$$

**Remark 3.6.1.** This definition is independent of the choice of a sequence $\{\Lambda_N\}_N$. Indeed, for any other sequence $\{\Lambda'_N\}_N$, and for any $N$, there exists $N'$ such that $\Lambda_N \subseteq \Lambda'_N$. Applying the Thom isomorphism and excision, one can then build a map from one limit to the other, and similarly in the other direction. Then, one concludes by naturality of the maps involved.

Note that we have a natural inclusion of pairs $(F_N(\nu),\partial F^{-}(\nu)) \subset (\mathcal{F}_{N}(\nu),\partial \mathcal{F}_{N}^{-}(\nu))$, and the latter is equivariantly homotopic to the pair $(\Gamma_N(\nu),\partial \Gamma_N(\nu))$. In particular, there is a natural homomorphism

$$H^*_\mathbb{K}_0(\Gamma_N(\nu),\partial \Gamma_N(\nu)) \rightarrow H^*_\mathbb{K}_0(F_N^-)(\nu,\partial F_N^-(\nu)).$$

We denote $H^*_\mathbb{K}_0(\nu)$ the limit

$$H^*_\mathbb{K}_0(\nu) := \lim_{N \rightarrow \infty} H^{*+2nN}(\Gamma_N(\nu),\partial \Gamma_N(\nu)),$$

and call the induced homomorphism

$$H^*_\mathbb{K}_0(\nu) \rightarrow H^*_\mathbb{K}_0(F^-)(\nu),$$

the **augmentation map**. Note that the groups $H^*_\mathbb{K}_0(F^-)(\nu)$ and $H^*_\mathbb{K}_0(F^-)(\nu)$ inherit from there finite parts the structure of $H^*_\mathbb{K}_0(pt)$-modules, and that the augmentation map is a module homomorphism. We denote by $J^*_\mathbb{K}_0(F^-)(\nu)$ its kernel.

Recall that under the isomorphism $H^2(M,\mathbb{Z}) \cong \mathfrak{t}_Z^*$, the first Chern class $c$ of $M$ is given by $c(m) = \sum_{j=1}^{n} m_j$. Let $\iota(m) = (m_1,\ldots,m_n)$. Notice that since $\nu$ is generic and the front $L_N$ of $F_N$ is $\mathfrak{t}_Z$-invariant (remark 3.2.1), $\Gamma_N(\nu) + m$ and $\partial \Gamma_N(\nu) + m$ are transversal to $L_N$, for any $N$. Thus we
can apply proposition 3.5.2 replacing 0 with $m \in \mathbb{Z}$, and along with the Thom isomorphism and the excision we obtain a homomorphism in equivariant cohomology

$$H_{K_0}^{*+2nN}(F_N^-(\nu), \partial F_N^-(\nu)) \to H_{K_0}^{*+2n(N+K)+2c(m)}(F_{N+K}^-(\nu + p(m)), \partial F_{N+K}^-(\nu + p(m))).$$

In the limit, we obtain

$$\mathcal{H}_{K_0}^*(F^-) \to \mathcal{H}_{K_0}^{*+2c(m)}(F^-(\nu + p(m))).$$

It is an isomorphism (with inverse given by applying proposition 3.5.2 replacing 0 with $-m$), which reflects the Novikov action of $H_2(M, \mathbb{Z})$. The latter induces an isomorphism between the kernels

$$\mathcal{J}_K^*(F^-) \simeq \mathcal{J}_{K_0}^{*+2c(m)}(F^-(\nu + p(m))). \quad (9)$$

Notice that the torus $K_0$ acts trivially on the pair $(\Gamma_N, \partial \Gamma_N)$, and therefore the cohomology group $H^*_{K_0}(\Gamma_N, \partial \Gamma_N)$ is a free $H^*_0$ (pt) module of rank 1 generated by the fundamental cocycle of the sphere $\Gamma_N(pt)$. Under this identification, the homomorphism

$$H^*_{K_0}(\Gamma_N, \partial \Gamma_N) \to H^*_{K_0}(\Gamma_{N+K}, \partial \Gamma_{N+K})$$

is simply the multiplication by the Euler class of the bundle

$$(\mathbb{C}^{nK+c(m)})_{K_0} \to B\mathbb{K}_0.$$

Recall the Chern-Weil isomorphism

$$H^*_\mathbb{C}(pt) \simeq \mathbb{C}[u_1, \ldots, u_n].$$

Under the surjective ring homomorphism

$$H^*_\mathbb{C}(pt) \to H^*_\mathbb{K}_0(pt),$$

the Euler class of this bundle is given by $u_1^{K+m_1} \cdots u_n^{K+m_n}$. In the limit, this implies that the isomorphism (9) is the multiplication by $u_1^{m_1} \cdots u_n^{m_n}$ (in proposition 4.2 will see that $\mathcal{H}_K^*(\nu)$ is the ring of regular functions on $(\mathfrak{t}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$, which will in particular clarify why this multiplication is well-defined).

We now turn to the proofs of propositions 1.2.1 and 1.2.2.

**Proof.** (proposition 1.2.1) Assume that $[\nu_0, \nu_1] \cap \text{Spec}(\phi_t) = \emptyset$. For any $\nu \in [\nu_0, \nu_1]$ and any $N$, $\Gamma_N(\nu)$ is transversal to the front $L_N$. Moreover, the set of $\nu$ such that $\partial \Gamma_N(\nu)$ is transversal to $L_N$ for all $N$ is of full measure. Let $\nu \in [\nu_0, \nu_1]$ be in this set. Since transversality is an open condition, there exists $\epsilon > 0$ such that, by proposition 2.1.3, there is a $\mathbb{K}_0$-equivariant homotopy equivalence

$$(F_N^-(\nu + \epsilon), \partial F_N^-(\nu + \epsilon)) \simeq (F_N^-(\nu - \epsilon), \partial F_N^-(\nu - \epsilon)).$$

In the limit $N \to \infty$ of $\mathbb{K}_0$-equivariant cohomology groups, this yields an isomorphism

$$\mathcal{H}_K^*(F_N^-(\nu + \epsilon), \partial F_N^-(\nu + \epsilon)) \simeq \mathcal{H}_K^*(F_N^-(\nu - \epsilon), \partial F_N^-(\nu - \epsilon)).$$

Choosing a finite subcovering of $[\nu_0, \nu_1]$ by such segments $[\nu - \epsilon, \nu + \epsilon]$ yields an isomorphism

$$\mathcal{H}_K^*(F^-(\nu_1)) \simeq \mathcal{H}_K^*(F^-(\nu_0)),$$

as claimed. \qed
Proof. (proposition 1.2.2) We rephrase the statement as follows: suppose that $q_1 \in H^*_{\mathbb{K}_0}(F^-(\nu_1))$ and $q_0 \in H^*_{\mathbb{K}_0}(F^-(\nu_0))$ are the images of $q$ under the augmentation maps:

$$q \in \mathcal{R}_0 \mapsto H^*_{\mathbb{K}_0}(F^-(\nu_1)) \ni q_1 \quad \text{and} \quad H^*_{\mathbb{K}_0}(F^-(\nu_0)) \ni q_0.$$

Then $q_0 = 0$ implies $v \alpha = 0$.

Without loss of generality, we can assume that $\nu_0 = \nu - \epsilon$, $\nu_1 = \nu + \epsilon$, and that there is only one $\mathbb{K}_0$-orbit of fixed points associated with $\nu$. Using proposition 2.1.3, one can show that for any $N$, the pair $(F_N^-(\nu_0), \partial F_N^-(\nu_0))$ is embedded into $(F_N^-(\nu_1), \partial F_N^-(\nu_1))$ as the complement of a neighborhood to the $\mathbb{K}_0$-orbit of critical points of $\hat{p}_N$. There exists a non-zero representative $\hat{q} \in H^*_{\mathbb{K}_0}(F_N^-(\nu_1), F_N^-(\nu_1))$ of $q_1$ (otherwise $q_1 = 0$ and the statement is trivial), which vanishes when restricted to $(F_N^-(\nu_0), \partial F_N^-(\nu_0))$. In particular, $\hat{q}$ is the image of some element $\alpha \in H^*_{\mathbb{K}_0}(F_N^-(\nu_1), F_N^-(\nu_1))$ under the long exact sequence:

$$\ldots \to H^*_{\mathbb{K}_0}(F_N^-(\nu_1), F_N^-(\nu_0)) \xrightarrow{f} H^*_{\mathbb{K}_0}(F_N^-(\nu_1), \partial F_N^-(\nu_1)) \to H^*_ {\mathbb{K}_0}(F_N(\nu_0), \partial F_N^-(\nu_0)) \to \ldots$$

The torus $\mathbb{K}_0$ acts freely in a neighborhood of the $\mathbb{K}_0$-orbit of critical points of $\hat{p}_N$ (since $M$ is compact), hence the equivariant cohomology $H^*_ {\mathbb{K}_0}(F_N^-(\nu_1), F_N^-(\nu_0))$ is simply the singular cohomology $H^*(F_N^-(\nu_1)/\mathbb{K}_0, F_N^-(\nu_0)/\mathbb{K}_0)$. Now, the coefficient ring $H^*_ {\mathbb{K}_0}(pt)$ acts by 0 on the usual cohomology, and thus

$$v \alpha = 0 \quad \text{and} \quad v \hat{q} = f(v \alpha) = 0.$$

$\square$

4 The Gysin sequence

Let $G_N : \mathbb{C}^{2nN} \times \Lambda_N \to \mathbb{R}$ be the homogeneous generating family associated with the Hamiltonian lift 0 of the trivial contact Hamiltonian on $V$. The homogeneous generating functions $G_N^{(N)}$ generate the action of the torus $\mathbb{K}$, and therefore they are $\mathbb{K}$-invariant. This means that one can consider either the $\mathbb{K}_0$-equivariant cohomology groups $H^*_ {\mathbb{K}_0}(G_N^-(\nu), \partial G_N^-(\nu))$, or the $\mathbb{K}$-equivariant cohomology groups $H^*_ {\mathbb{K}}(G_N^-(\nu), \partial G_N^-(\nu))$. We write:

$$H^*_K(G^-(\nu)) := \lim_{N \to \infty} H^{*+2nN}_K(G_N^-(\nu), \partial G_N^-(\nu))$$

$$H^*_K(\nu) := \lim_{N \to \infty} H^{*+2nN}_K(\Gamma_N(\nu), \partial \Gamma_N(\nu)).$$

In [Giv95], Givental showed that the augmentation map

$$H^*_K(\nu) \to H^*_K(G^-(\nu))$$

has trivial cokernel, and he described its kernel in terms of Newton diagrams associated with the level $p^{-1}(\nu)$ (see [Giv95 Proposition 5.4]). Note that here the augmentation map is a homomorphism of $H^*_K(pt)$-modules.

We write $H^*_K(pt) \simeq \mathbb{C}[u]$, where $u = (u_1, \ldots, u_n)$. If $I$ (resp. $I_0$) denotes the ideal of $\mathbb{C}[u]$ generated by polynomials vanishing on the complexified Lie algebra $\mathfrak{k} \otimes \mathbb{C} \subset \mathbb{C}^n$ (resp. $\mathfrak{k}_0 \otimes \mathbb{C} \subset \mathbb{C}^n$), there are natural isomorphisms

$$H^*_K(pt) \simeq \mathbb{C}[u]/I \quad \text{and} \quad H^*_K(\nu) \simeq \mathbb{C}[u]/I_0.$$
In other words, $H^*_{\mathbb{K}}(pt)$ (resp. $H^*_{\mathbb{K}_0}(pt)$) is the ring of regular functions on the complexified Lie algebra $\mathfrak{k} \otimes \mathbb{C}$ (resp. $\mathfrak{k}_0 \otimes \mathbb{C}$). Let us denote by $\mathcal{R}$ and $\mathcal{R}_0$ the rings of regular functions on the intersections $(\mathfrak{k} \otimes \mathbb{C}) \cap (\mathbb{C}^*)^n$ and $(\mathfrak{k}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^*)^n$ respectively. We have:

$$\mathcal{R} := \mathbb{C}[u, u^{-1}]/I_0 \mathbb{C}[u, u^{-1}] \quad \text{and} \quad \mathcal{R}_0 := \mathbb{C}[u, u^{-1}]/I_0 \mathbb{C}[u, u^{-1}].$$

Let $J^*(\nu)$ denote the $\mathbb{C}[u]$-submodule of $\mathbb{C}[u, u^{-1}]$ generated by monomials whose degrees lie in the intersection $\mathfrak{k}_0 \cap p^{-1}(\nu)$:

$$J^*(\nu) := \langle u^{(m)} | m \in \mathfrak{k}_0, p(m) \geq \nu \rangle, \quad (m) = (m_1, ..., m_n), \quad (10)$$

and let $J^*_G(\nu)$ denote its projection to $\mathcal{R}$. Givental proved the following:

**Proposition 4.1** ([Giv95] proposition 5.4). There are isomorphisms

$$\mathcal{H}^*_G(\nu) \simeq \mathcal{R} \quad \text{and} \quad \mathcal{H}^*_G(G^-(\nu)) \simeq \mathcal{R}/J^*_G(\nu).$$

Let $J^*_G(\nu)$ be the projection of $J^*(\nu)$ to $\mathcal{R}_0$. We claim that it is the kernel of the augmentation homomorphism

$$\mathcal{H}^*_G(\nu) \to \mathcal{H}^*_G(G^-(\nu)).$$

Let us denote it by $J^*_G(G^-(\nu))$ for now. For any $N$, the $\mathbb{K}$-equivariant and $\mathbb{K}_0$-equivariant cohomology groups above are related by the following Gysin sequence:

$$\cdots \to H^{2n+2N}_G(G_N^-(\nu), \partial G_N^- (\nu)) \xrightarrow{p} H^{2n+2N+2}_G(G_N^-(\nu), \partial G_N^- (\nu)) \xrightarrow{\pi_N}$$

$$H^{2n+2N+2}_G(G_N^-(\nu), \partial G_N^- (\nu)) \xrightarrow{f} H^{2n+2N+1}_G(G_N^-(\nu), \partial G_N^- (\nu)) \xrightarrow{p} \cdots$$

Here $p$ stands for the multiplication, induced in relative cohomology, by the Euler class of the $S^1$-bundle $\pi_N : (G_N^-(\nu))_{\mathbb{K}_0} \to (G_N^- (\nu))_\mathbb{K}$. Indeed, we have $p \in H^*(B(\mathbb{K}/\mathbb{K}_0)) \simeq \mathbb{C}[t/t_0] \simeq \mathbb{C}[p]$, and we still denote $p$ its pull back by the classifying map of the bundle $\pi_N$. The other two maps involved are the pull-back $\pi_N^*$ and the integration $\int$ along the fibers. They commute with the limit in $N \to \infty$, and the multiplication by the Euler class is functorial. Since directs limits of exact sequences are exact sequences, we obtain a Gysin sequence

$$\cdots \to \mathcal{H}^*_G(G^- (\nu)) \to \mathcal{H}^{2n+2}_G(G^- (\nu)) \to \mathcal{H}^{2n+2}_G(G^- (\nu)) \to \mathcal{H}^{2n+1}_G(G^- (\nu)) \to \cdots.$$  

The same applies for the equivariant cohomology groups of the pair $(\Gamma_N(\nu), \partial \Gamma_N(\nu))$, for which the Gysin sequence in the limit is given by

$$\cdots \to \mathcal{H}^*_G(\nu) \to \mathcal{H}^{2n+2}_G(\nu) \to \mathcal{H}^{2n+2}_G(\nu) \to \mathcal{H}^{2n+1}_G(\nu) \to \cdots \quad (11)$$

**Proposition 4.2.** There is a natural isomorphism

$$\mathcal{H}^*_G(\Gamma(\nu)) \simeq \mathcal{R}_0.$$

**Proof.** Since the intersection $(\mathfrak{k} \otimes \mathbb{C}) \cap (\mathbb{C}^*)^n$ is an irreducible variety, there are no zero divisors in $\mathcal{R}$. In particular, the multiplication by $p$ is injective, and the Gysin sequence is a short exact sequence

$$0 \to \mathcal{R} \xrightarrow{p} \mathcal{R} \to \mathcal{H}^*_G(\nu) \to 0.$$  

The isomorphism is then obtained by the very definition of $\mathfrak{k}_0 = ker p$. □

**Remark 4.1.** Notice that if $M = \mathbb{C}P^n$, $\mathfrak{k}_0 = \{0\}$, and therefore $I_0 = \mathbb{C}[u, u^{-1}]$. This means that $\mathcal{R}_0 = \{0\}$. Moreover, the Euler class $p$ in this case is the generator $v$ of $\mathcal{R} \simeq \mathbb{C}[v, v^{-1}]$, and thus it is invertible. Therefore the Gysin sequence gives $\mathcal{H}^*_G(G^-(\nu)) = \{0\}$.  

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We have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & J^*_K(\nu) & \rightarrow & H^*_K(\nu) & \rightarrow & H^*_K(G^-) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & J^*_K(G^-) & \rightarrow & H^*_K(G^-) & \rightarrow & H^*_K(G^-) & \rightarrow & 0 \\
\end{array}
\]

where the vertical right and middle sequences are Gysin sequences, and the horizontal sequences come from the augmentation maps. Note that the left vertical sequence is not necessarily exact at the middle term. From the discussion above, the diagram is clear, except for the surjectivity of the map \(f\). From the two bottom rows of the diagram and the snake lemma, we have the following short exact sequence:

\[
0 \rightarrow \ker f \rightarrow \ker g \rightarrow \ker h \rightarrow \text{coker } f \rightarrow 0.
\]

Since the homomorphism \(\ker g \rightarrow \ker h\) is simply the surjection \(p \colon H^*_K(\nu) \rightarrow H^*_K(G^-)\), \(f\) is onto. In particular, the kernel \(J^*_K(G^-(\nu))\) is the image of \(J^*_K(\nu)\) under the map \(g\). In other words,

\[
J^*_K(G^-(\nu)) \simeq J^*_K(\nu).
\]

5 Elements of minimal degree in the monotone case

In this section we prove proposition 1.2.3. The proof of theorem 1.1.1 is adapted from [Giv95], and relies on a strong algebraic property of the kernel \(J^*_K(F^-)\). More precisely, Givental showed that in the \(\mathbb{K}\)-equivariant setting, the kernel \(J^*_K(F^-)\) of the augmentation map

\[
\mathcal{R} \simeq H^*_K(\nu) \rightarrow H^*_K(F^-)
\]

admits, in some sense, elements of minimal degree ([Giv95, Corollary 1.3]). It appears that this is not always true in the \(\mathbb{K}_0\)-equivariant case. For instance, for the product \(\mathbb{C}P^1 \times \mathbb{C}P^1\) endowed with the non-monotone symplectic form whose cohomology class is, say, \(p = (1, 2) \in \mathbb{Z}^2\), the kernel \(J^*_K(G^-)\) of the augmentation map

\[
\mathcal{R}_0 \rightarrow H^*_K(G^-(\nu))
\]

is the whole ring \(\mathcal{R}_0\).

Suppose that \((M, \omega)\) is monotone, and let \(p \in H^2(M, \mathbb{R}) \simeq \mathfrak{t}^*\) be the cohomology class of \(\omega\). Suppose that \(p\) is integral and primitive, and let \(N_M\) denote the minimal Chern number. Then we have

\[
p = \frac{c}{N_M},
\]

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Consider the kernel of the augmentation map $\mathcal{R}_0 \to \mathcal{H}_{K_0}^\ast (F^-(\nu))$. Let $(V, \alpha)$ be the prequantization space over $(M, \omega)$ constructed in section \textbf{3.1.3} $h$ be a contact Hamiltonian on $V$, and $\overline{h}$ a lift of $h$ to the sphere $S_p$ from equation \textbf{(8)}. Given two constants $c_-, c_+ \in \mathbb{R}$ such that $c_- < \overline{h} < c_+$, we denote by $H_-, H_+$ be the quadratic Hamiltonians on $\mathbb{C}^n$ generating respectively the Hamiltonian symplectomorphisms

\[ \Phi_{H_-}(z) = \exp(c_- \frac{b}{p(b)})z \quad \text{and} \quad \Phi_{H_+}(z) = \exp(c_+ \frac{b}{p(b)})z. \]

Let $\tilde{H}$ denote the Hamiltonian lift of $h$ obtained from $\overline{h}$. We have naturally

\[ H_- < \tilde{H} < H_+, \]

and for $\nu_- := \nu + c_- < \nu < \nu_+ := \nu + c_+$, this yields inclusions of pairs

\[ (G_N^-(\nu_-), \partial G_N^-(\nu_-)) \subset (F_N^-(\nu), \partial F_N^-(\nu)) \subset (G_N^-(\nu_+), \partial G_N^-(\nu_+)), \]

and therefore homomorphisms

\[ H_{K_0}^\ast (G_N^-(\nu_+), \partial G_N^-(\nu_+)) \to H_{K_0}^\ast (F_N^-(\nu), \partial F_N^-(\nu)) \to H_{K_0}^\ast (G_N^-(\nu_-), \partial G_N^-(\nu_-)). \]

If $\nu_-, \nu$ and $\nu_+$ are generic, we obtain homomorphisms in the limit $N \to \infty$:

\[ \mathcal{H}_{K_0}^\ast (G - (\nu_+)) \to \mathcal{H}_{K_0}^\ast (F^-(\nu)) \to \mathcal{H}_{K_0}^\ast (G^-(\nu_-)), \]

and thus inclusions of kernels

\[ \mathcal{J}_{K_0}^\ast (\nu_+) \subset \mathcal{J}_{K_0}^\ast (F^-(\nu)) \subset \mathcal{J}_{K_0}^\ast (\nu_-). \]

Note that as in \textbf{[Giv95]}, for any $\nu \in \mathbb{R}$, the $\mathbb{C}[u]$-module $J^\ast(\nu)$ from equation \textbf{(11)} lies between two modules of the form

\[ J_{r_\pm} := \langle u^{(m)} \mid m \in \mathbb{Z}, p(m) \geq r_\pm \rangle, \quad r_- < r_. \]

In particular, one can find $r_- < r_+$ such that we have inclusions

\[ J_{r_+} \subset J^\ast(\nu_+) \subset J^\ast(\nu_-) \subset J_{r_-}, \]

which lead to

\[ \mathcal{J}_{r_+}^0 \subset \mathcal{J}_{K_0}^\ast (F^-(\nu)) \subset \mathcal{J}_{r_-}^0, \]

where $\mathcal{J}_{r_\pm}^0$ denote the images of $J_{r_\pm}$ by the quotient map $\mathbb{C}[u, u^{-1}] \to \mathbb{C}[u, u^{-1}]/I_0\mathbb{C}[u, u^{-1}] \simeq \mathcal{R}_0$.

\textit{Proof.} (proposition \textbf{1.2.3}). \textbf{Step 1.} Consider the projection

\[ \text{pr} : \mathbb{C}[u] \subset \mathbb{C}[u, u^{-1}] \to \mathbb{C}[u, u^{-1}]/I_0\mathbb{C}[u, u^{-1}] \simeq \mathcal{R}_0. \]

For $r \in \mathbb{R}$, the preimage $\text{pr}^{-1}(\mathcal{J}_r^0)$ is the intersection of $\mathbb{C}[u]$ with the preimage of $\mathcal{J}_r^0$ by the quotient map $\mathbb{C}[u, u^{-1}] \to \mathcal{R}_0$, which equals $J_r + I_0\mathbb{C}[u, u^{-1}]$. Therefore, we have

\[ \text{pr}^{-1}(\mathcal{J}_r^0) = \mathbb{C}[u] \cap (J_r + \mathbb{C}[u, u^{-1}]I_0) \supset (\mathbb{C}[u] \cap J_r) + I_0. \]

\textbf{Step 2.} For $m \in \mathbb{Z}$, we saw that $p(m) = \frac{1}{N_M} \sum_{i=1}^{n} m_i$. Consider the $\mathbb{C}[u]$-module

\[ J_r = \langle u^{(m)} \mid m \in \mathbb{Z}, p(m) \geq r \rangle. \]
Then $J_r$ consists of polynomials whose monomials are of the form $u^{(m)+m'}$, where $m \in \mathbb{F}_Z$ is such that $p(m) = r$, and $m' = (m_1', \ldots, m_n') \in \mathbb{Z}_{\geq 0}^n$. In particular, we have

$$\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m'_i \geq \sum_{i=1}^{n} m_i = N_M p(m) = r N_M.$$ 

Therefore, letting $\mathbb{C}[u, u^{-1}] \geq d$ denote the submodule generated by monomials of total degree at least $d$, we obtain

$$J_r \subset \mathbb{C}[u, u^{-1}] \geq d.$$ 

**Step 3.** The quotient map $\mathbb{C}[u, u^{-1}] \to \mathbb{C}[u, u^{-1}]/I_0 \mathbb{C}[u, u^{-1}]$ is the restriction map from the ring of regular functions on the complex torus $(\mathbb{C}^\times)^n$ to the ring of regular functions on the intersection $(\mathfrak{t}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$. If $f$ is a homogeneous regular function of degree $d$ on $(\mathbb{C}^\times)^n$, then $\mathbb{C}^\times$ acts on $f$ by $\mu . f = \mu^d f$, for any $\mu \in \mathbb{C}^\times$. This characterizes entirely the degree of $f$. Moreover, $\mathbb{C}^\times$ acts on the ring of regular functions on $(\mathfrak{t}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$ in the same way, and the restriction is equivariant with respect to this action. This means that $f$ restricts to a regular function on $(\mathfrak{t}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$ which is of same degree, or equals 0. Thus, if $\mathcal{R}_0 \geq d$ denotes the ring of regular functions of degree at least $d$ on $(\mathfrak{t}_0 \otimes \mathbb{C}) \cap (\mathbb{C}^\times)^n$, we have

$$\mathcal{J}_r^0 \subset \mathcal{R}_0 \geq r N_M.$$ 

**Step 4.** It is clear from the definition of $J_r$ that for any $b \in \mathbb{F}_Z$, we have $u^b J_r = J_{r + r_0}$, where $r_0 = p(b)$. In particular

$$u^b \mathcal{J}_r^0 = \mathcal{J}_{r + r_0}^0.$$ 

Pick $b \in \mathbb{F}_Z$ so that $(r_+ + r_0) N_M \geq 1$. Then

$$u^b \mathcal{J}_r^0 = \mathcal{J}_{r_+ + r_0}^0 \subset \mathcal{R}_0 \geq (r_+ + r_0) N_M \subset \mathcal{R}_0 \geq 1.$$ 

In particular $1 \notin u^b \mathcal{J}_{r_0}^0$, which means that $1 \notin \mathbb{C}[u]$ is not mapped to $u^b J_{r_0}$ by the projection $\text{pr} : \mathbb{C}[u] \to \mathcal{R}_0$, and thus is also not mapped to $u^b \mathcal{J}_{r_0}^0 (F^-(\nu)) \subset u^b \mathcal{J}_{r_0}^0$.

**Step 5.** Let

$$A := \{ u^a \in \mathbb{C}[u] \mid \text{pr}(u^a) \notin u^b \mathcal{J}_{r_0}^0 (F^-(\nu)) \}.$$ 

In the previous step we saw that $1 = u^0 \notin A$, so $A \neq \emptyset$. We claim that the maximal degree $\sum_{i=1}^{n} a_i$ of an element of $A$ is bounded from above. Since $u^b \mathcal{J}_r^0 \subset u^b \mathcal{J}_{r_0}^0 (F^-(\nu))$, we see that $\text{pr}(u^a) \notin u^b \mathcal{J}_{r_0}^0 (F^-(\nu))$ implies $\text{pr}(u^a) \notin u^b \mathcal{J}_{r_0}^0$, and thus $u^a \notin \mathbb{C}[u] \cap J_r + I_0$. Therefore, $A$ lies in the complement in $\mathbb{C}[u]$ of the ideal $\mathbb{C}[u] \cap J_r + I_0$. By [Giv95, Proposition 1.2], the zero set $Z(\mathbb{C}[u] \cap J_r + I)$ has at most one point, the origin. Since $I_0 \supset I$, we have

$$Z(\mathbb{C}[u] \cap J_r + I_0) \subset Z(\mathbb{C}[u] \cap J_r + I) \subset \{0\}.$$ 

By the Nullstellensatz, this implies that for every $i$, there exists $m_i \geq 0$ such that $u^{m_i} \in \mathbb{C}[u] \cap J_r + I_0$, and it is easy to see that every monomial of total degree $\geq \sum_{i=1}^{n} m_i$ must then also belong to the ideal. The conclusion is that $\mathbb{C}[u] \cap J_r + I_0$ contains all monomials of sufficiently high degree, and as a result the maximal degree of a monomial $u^a \in A$ is bounded from above.

**Conclusion.** Let $u^a \in A$ have maximal degree, which means that $u_i u^a \notin A$ for any $i = 1, \ldots, n$. This implies that $u^a \notin u^b \mathcal{J}_{r_0}^0 (F^-(\nu))$, while $u_i u^a \in u^b \mathcal{J}_{r_0}^0 (F^-(\nu))$. Therefore $q = u^{a-b} \in \mathcal{R}_0 \setminus \mathcal{J}_{r_0}^0 (F^-(\nu))$, but $u_i q \in \mathcal{J}_{r_0}^0 (F^-(\nu))$ for any $i = 1, \ldots, n$, as claimed.

□

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