The genealogy of nearly critical branching processes in varying environment

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Abstract

Building on the spinal decomposition technique in [15] we prove a Yaglom limit law for the rescaled size of a nearly critical branching process in varying environment conditional on survival. In addition, our spinal approach allows us to prove convergence of the genealogical structure of the population at a fixed time horizon – where the sequence of trees are envisioned as a sequence of metric spaces – in the Gromov–Hausdorff–Prohorov (GHP) topology. We characterize the limiting metric space as a time-changed version of the Brownian coalescent point process [48].

Beyond our specific model, we derive several general results allowing to go from spinal decompositions to convergence of random trees in the GHP topology. As a direct application, we show how this type of convergence naturally condenses the limit of several interesting genealogical quantities: the population size, the time to the most-recent common ancestor, the reduced tree and the tree generated by $k$ uniformly sampled individuals.

As in a recent article by the authors [15], we hope that our specific example illustrates a general methodology that could be applied to more complex branching processes.

Keywords and Phrases. branching process in varying environment, Yaglom law, Kolmogorov asymptotics, spinal decomposition, many-to-few, Gromov–Hausdorff–Prohorov topology, coalescent point process

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1 Introduction

Describing the genealogies of different population models as the size of the population gets large is a prominent task in the theory of mathematical population genetics as well as in the study of branching processes, and goes back at least to the two seminal papers of Kingman [35, 36]. As of today there is a vast literature on the genealogies of various type of populations models, see for example [11, 29, 33, 46, 48, 49]. Here, we focus among other topics on the scaling limit of the genealogy of nearly critical branching processes in varying environment conditional on survival, for a precise definition see Section 2.
The scope of this paper is two-fold. On the one hand we prove a Yaglom-type result \[51\] for the size of a nearly critical branching process in varying environment conditioned to survive for a long time, as well as a result on the scaling limit of the genealogy. For this purpose we prove a Kolmogorov-type asymptotics \[37\] for the probability of survival for an unusually long time, which might be interesting in its own right. On the other hand, our objective is to use this specific model to give an illustration of the techniques developed in \[15, 19, 17, 22\]. Our approach relies on new general results on random metric spaces that could be applied widely to branching processes with a critical behaviour. The spinal decomposition method in \[15\] gives an approach to study genealogies by showing convergence of random metric measure spaces in the Gromov-weak topology via a method of moments. To prove convergence of the \(k\)-th moment of a genealogy, a many-to-few formula allows us to reduce this problem to calculating the distribution of a tree with \(k\) leaves, the so-called \(k\)-spine tree. The distribution of the \(k\)-spine tree is obtained through a random change of measure relying on choosing a suitable ansatz for the genealogy of the process. In order to make the \(k\)-spine method work one would require the offspring distribution to have moments of all order, which is far from the optimal conditions. However, using a truncation argument, we are able to work under a second moment condition. Finally, we show how to reinforce the Gromov-weak convergence of the genealogy to a Gromov–Hausdorff–Prohorov convergence. By doing so, we obtain as a direct consequence of our main result the convergence of many interesting genealogical quantities.

In summary, a fraction of this paper will be dedicated to proving general results on random metric spaces. It is only at the end of the paper that this general formalism will be used to treat our specific problem as a (semi-)direct application. Along the way, we will obtain new criteria to obtain a Yaglom law for branching processes in varying environment (see Remark 2.2 below). We hope that this application will help to convince the reader that the abstract formalism introduced in \[15, 19, 17, 22\] (among others) and further developed in this paper is particularly well suited when treating general branching processes near criticality.

1.1 Relevant literature
The genealogical structure of branching processes is a topic that has drawn a lot of attention, especially in the critical case. Let us give a non-exhaustive account of some results related to our work. A first class of results focuses on the so-called reduced process \[13, 52, 47\], that counts the number of individuals having descendants at a fixed focal generation in the future. The limit of the reduced process is often expressed as a (time-inhomogeneous) pure birth process, see \[33\] for results in this direction for branching processes in varying environment. Second, in the same spirit as the theory of exchangeable coalescents, there has been an interest in describing the genealogy obtained by sampling a fixed number \(k\) of individuals uniformly at a given time \[29, 26, 40\]. We also want to point at \[38\] for an explicit construction of the genealogy of splitting trees as a discrete coalescent point process, which plays a central role in \[40\].

In this work, we show how the convergence of the previous two quantities, the reduced process and the genealogy of a uniform sample, as well as the convergence of the population size can be aggregated by viewing the population as a random ultrametric measure space converging in the Gromov–Hausdorff–Prohorov topology. Encoding trees as random metric spaces is a long-standing idea that has proved fruitful both for constant-size population models \[11, 19\] and for branching processes \[2, 9\]. Let us point at the following difference between our
approach and more common convergence results in the Gromov–Hausdorff–Prohorov topology such as [2, 13, 45]. Here, we consider the genealogy of the population at a fixed time horizon, whereas many works [2, 13, 45] are interested in the limit of the whole genealogical tree. Although in the limit one can construct the genealogy at a fixed time (the Brownian coalescent point process) from that of the whole tree (the Brownian continuum random tree), see [48], it is not straightforward to go from the convergence of the latter object to our type of convergence result. Also, from a technical point of view, the convergence of the whole tree is typically obtained by studying a height function, whereas our approach relies on a $k$-spine method.

The spinal decomposition of branching processes is widely used in the theory of branching processes. In particular, the case of a single spine has been proven to be an important tool, see [44, 50, 16]. The classical Yaglom limit and the Kolmogorov estimate for critical Galton–Watson processes is obtained in [44] using this technique. Here, we adapt the one spine decomposition from [44] to prove the Kolmogorov asymptotics for branching processes in varying environment, see Theorem 1. These techniques have been developed further for example in [27, 15] to allow for a $k$-spine construction, which in turn allows us to compute higher moments of the tree by a random change of measure. Such a $k$-spine construction has also been used to extract information about the genealogy of several branching processes [26, 29, 15], and the current work follows a similar path. We also want to point at [18, 25] for a computation of higher order moments for general spatial branching processes, and the derivation of a Yaglom law in this context.

For an introduction to branching processes in varying environment (BPVE) we refer to the monograph of Kersting and Vatutin [34]. The work of Kersting [32] gives a complete classification of critical branching processes in varying environment (among the other cases, supercritical, subcritical and asymptotically degenerate) and proves a Yaglom limit result in this case. Later, Cardona-Tobón and Palau in [7] managed to prove the Yaglom-type result for critical branching processes in varying environment by applying a two-spine decomposition and therefore providing a probabilistic proof of the result obtained by analytical methods in [32]. They make use of the remarkable fact that the distribution of the time to the most recent common ancestor of two individuals can be found explicitly. Recently, in [33] Kersting managed to obtain the asymptotic distribution of the time to the most recent common ancestor of all individuals living at some large time $n$. Our work is a continuation and extension of these results, but does not build strongly on them, since we are dealing with nearly critical processes instead.

In the process of finishing this article we became aware of the works of Harris, Palau and Pardo [24] and of Conchon–Kerjan, Kious and Mailler [8]. In the former work, building on techniques of theirs developed in [27] and [7], the authors give a forward in time construction of the $k$-spine for a critical branching process in varying environment. Subsequently, via a change of measure relating the original branching process in varying environment and the $k$-spine tree they obtain their main result on the asymptotic distribution describing the splitting times for a uniform $k$-sample from the tree at large times, compare Corollary 2.3 (iii). Their results and the spinal decomposition are related to our work and are in the same spirit. However, due to the different points of view and approaches in their work and ours, we believe that these two papers can be seen as complementary to each other. We refer to Remark 2.2 and Remark 2.4 for a more in depth comparison. In the latter work [8], the authors derive the scaling limit of the whole tree structure of a branching process in i.i.d. random environment for the Gromov–Hausdorff–Prohorov topology. Their main result is consistent with ours in the sense that, under their hypothesis, the limit that we find (the Brownian coalescent
point process) corresponds to the reduced tree at a given time of their limit (the Brownian continuum random tree). Nevertheless, as mentioned above, our type of result is not directly implied by the convergence of the whole tree and the techniques are different. Also, our hypothesis can be seen as more general, in the sense that we do not require the environment to have properties of i.i.d. sequences, and consider a sequence of nearly critical environments rather than a single strictly critical environment. In particular, stationary environments (and thus i.i.d. environments) can only lead to the Brownian coalescent point process in the limit, whereas we recover any time-change of a Brownian coalescent point process, including non-binary trees whenever the limiting variance process has jumps. Finally, let us point at [20] which considers a binary branching process in a random environment, modelling the effect of a catalyst that modifies the branching rate of the population. They provide the scaling limit of the genealogy of this branching process, recovering in the limit a time-changed version of a coalescent point process similar to the one arising here. Our work goes way beyond the case of binary branching processes covered in [20], and in a sense extends this result to a much broader universality class.

1.2 Outline

The outline of this paper is as follows. First, in Section 2 we will define precisely the notion of a nearly critical branching process in varying environment and give some examples. We also state our two main results, the Kolmogorov asymptotic (Theorem 1) and the convergence of the genealogy (Theorem 2), and give in Corollary 2.3 straightforward consequences of our Gromov–Hausdorff–Prohorov convergence. In Section 3 we introduce the topologies on metric measure spaces needed in this work, and prove some continuity results for ultrametric spaces. The limiting object for the genealogy, the environmental coalescent point process, is introduced in Section 4, where we also derive some of its properties. The last sections are devoted to the proofs of our results. In Section 5 we recall the spinal decomposition tools for branching processes that we need. The proof of the Kolmogorov asymptotics (Theorem 1) is carried out in Section 6, and that of the convergence of the genealogy (Theorem 2) in Section 7. Lastly, some results of technical nature are given in the Appendix A.

2 Model

Branching processes in varying environment (BPVE for short) are a natural generalisation of classical Galton–Watson processes, which allow for a changing offspring distribution between the generations. Throughout this work we will try to follow the notation of [34].

Let us start by properly defining the notion of a branching process in varying environment. Denote by \( P(\mathbb{N}_0) \) the space of probability measures on \( \mathbb{N}_0 = \{0, 1, 2, \ldots \} \) and let \( v = (f_1, f_2, \ldots) \) be a sequence of probability measures on \( \mathbb{N}_0 \).

**Definition 2.1.** We call a process \((Z_n, n \geq 0)\) a BPVE with environment \( v = (f_1, f_2, \ldots) \), with \( v \in P(\mathbb{N}_0)^\mathbb{N} \), if \((Z_n, n \geq 0)\) has the representation

\[
Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{i,n}, \quad Z_0 = 1,
\]

where for each \( n \geq 1 \) the family \( \{\xi_{i,n}, i \geq 1\} \) consists of independent random variables with distribution \( f_n \).
Note that we enforce the BPVE to consist of a single individual in generation 0. In the subsequent parts it will be useful to identify measures \( f \in \mathcal{P}(\mathbb{N}_0) \) with their generating function

\[
f(s) = \sum_{k=0}^{\infty} s^k f[k], \quad s \in [0, 1],
\]

where \( f[k], k \geq 0, \) denotes the weight of the distribution \( f \) in \( k \). Note that we will use the same symbol \( f \) for the distribution as well as for the corresponding generating function \( f(s) \).

Recall that the first and second factorial moments of a random variable with distribution \( f \) can be expressed as

\[
f'(1) = \sum_{k=1}^{\infty} kf[k], \quad f''(1) = \sum_{k=2}^{\infty} k(k-1)f[k].
\]

In the following we will consider a sequence \( Z^{(N)} = (Z_n^{(N)}, n \geq 0) \) of branching processes in varying environment indexed by \( N \). That is, we consider a sequence of environments \( v^{(N)} = (f_1^{(N)}, f_2^{(N)}, \ldots) \) and study the asymptotic probability for this process to survive \( N \) generations as well as the asymptotic genealogy of the population at a large time.

Due to the fact that we are working with a sequence of branching processes, all the quantities introduced so far as well as the quantities which will be introduced in the subsequent sections depend on \( N \). Additionally, all expectations and probabilities will depend on \( N \). For the convenience of the reader and to avoid overloading the notation, we will drop the superscript \( N \) most of the times. The reader should keep the implicit dependency of \( N \) in mind. Furthermore, whenever there is a sum ranging from 1 to \( \lfloor Nt \rfloor \) we will abuse the notation and simply write \( Nt \) as the upper limit. More generally, whenever no confusion is possible, we simply write \( Nt \) instead of \( \lfloor Nt \rfloor \), as in \( f_{Nt} \) or \( \xi_{Nt} \).

Finally, it is usual to encode the genealogy of a branching process as a random subset of the set of finite words

\[\mathcal{U} = \{\emptyset\} \cup \bigcup_{n \geq 1} \mathbb{N}^n.\]

A word \( u = (u_1, \ldots, u_n) \) is interpreted as an individual in the population living at generation \( |u| := n \). For two individuals \( u, v \in \mathcal{U} \), we also use the notation \( uv \) for their concatenation, and \( u \wedge v \) for their most-recent common ancestor.

We can construct a random tree out of an independent collection \((K_u, u \in \mathcal{U})\) of r.v. with \( K_u \sim f_{|u|+1} \) by setting

\[T = \{u = (u_1, \ldots, u_n) : \forall i \leq n, u_i \leq K_{(u_1, \ldots, u_{i-1})}\}.
\]

This construction is carried out more carefully for BPVEs in [34 Section 1.4]. The variable \( Z_n \) is recovered as the size of the \( n \)-th generation of \( T \),

\[Z_n = \text{Card} T_n, \quad T_n := \{u \in T : |u| = n\}, \quad n \in \mathbb{N}.
\]

### 2.1 Assumptions

In order to prove the main theorems, Theorem 1 and Theorem 2, we will work under the following assumptions on the sequence of environments \( v = (f_1, f_2, \ldots) \), which we name a
nearly critical varying environment. Recall that we do not denote the dependency of \( v \) on \( N \) explicitly. Let us introduce the following quantities,

\[
\mu_k = \prod_{i=1}^{k} f'_i(1), \quad \mu_0 = 1,
\]

which is the expectation of the BPVE at time \( k \), precisely \( \mathbb{E}[Z_k] = \mu_k \). We call a sequence of branching processes in varying environment nearly critical, if

\[
(\mu_{tN}, t \geq 0) \to (e^{X_t}, t \geq 0), \quad \text{as } N \to \infty,
\]

in the Skorohod topology for some càdlàg process \((X_t, t \geq 0)\). (Having in mind the important case where the environment \( v \) is a realization of an i.i.d. sequence, we use the terminology process for \((X_t, t \geq 0)\) although it is a deterministic càdlàg function). The next assumptions ensure that the variance between the generations does not fluctuate too strongly. Precisely, for some càdlàg non-decreasing function \( \sigma^2 : \mathbb{R}_+ \to \mathbb{R}_+ \) and some sequence \( \kappa_N \to \infty \), we assume that for all \( t \in \mathbb{R}_+ \) such that \( \sigma^2 \) is continuous at \( t \),

\[
\frac{1}{\kappa_N} \sum_{k=1}^{Nt} f''_k(1) \to \sigma^2(t), \quad \text{as } N \to \infty.
\]

We also assume that \( \sigma^2 \) and \((X_t, t \geq 0)\) do not jump at the same time. Lastly, we will enforce the following Lindeberg-type condition

\[
\forall \varepsilon > 0, \quad \frac{1}{\kappa_N} \sum_{k=1}^{Nt} \mathbb{E}[\xi_k^2 1_{\xi_k \geq \varepsilon \sqrt{\kappa_N}}] \to 0, \quad \text{as } N \to \infty,
\]

for a generic copy \( \xi_k \) distributed as \( f_k \). It will turn out that assumption \((3')\) is only required to derive the asymptotics of the survival probability (the Kolmogorov asymptotics, see Theorem 1), and that all other results only rely on the following weaker assumption

\[
\forall \varepsilon > 0, \quad \frac{1}{\kappa_N} \sum_{k=1}^{Nt} \mathbb{E}[\xi_k^2 1_{\xi_k \geq \varepsilon \kappa_N}] \to 0, \quad \text{as } N \to \infty,
\]

which prevents any single individual from having a large number of offspring of order \( \kappa_N \). We believe that given \((1)\) and \((2)\) this assumption, which was already used in [6], is nearly optimal. For instance, for Galton–Watson processes \((3')\) is known to be optimal for the process started from a population size of order \( \kappa_N = N \) to converge to a Feller diffusion [21].

**Remark 2.2.** Our definition of near criticality is directly inspired by that for Galton–Watson processes, where one generally imposes that the mean and the variance of the process converge. An alternative notion of criticality was proposed for BPVEs in [32] and was also further used in [33, 7, 24]. It relies on the behaviour of the following two series

\[
\mu_{tN} \sum_{k=1}^{tN} \frac{1}{f'_k(1)^2} \frac{f''_k(1)}{\mu_{k-1}} \sim \kappa_N \cdot \int_{[0,t]} e^{X_1 - X_s} \sigma^2(ds) \to \infty, \quad \text{as } N \to \infty,
\]
and
\[ \sum_{k=1}^{tN} \frac{1}{f_k''(1)} \frac{f_k''(1)}{f_k'(1)^2} \mu_k^{-1} \sim k_N \cdot \int_{[0,t]} e^{-X \sigma^2(s)} \, ds \to \infty, \quad \text{as } N \to \infty, \]
where the asymptotics under assumptions (1) and (2) are obtained in Lemma A.4. Since both series diverge, our BPVE would also be classified as critical in [32]. Assumptions (1) and (2) are not necessary for these series to diverge and can thus be thought of as restrictive in that sense.

It should be noted however that these stronger assumptions allow us to work with sequences of environments and lead to a notion of near criticality, whereas the aforementioned works consider fixed environment. Also, we only need the mild moment condition (3) on the offspring size which we believe to be nearly optimal, whereas [7, 24] require a finite third moment, and [32, 33] a stronger second moment assumption. In particular, if (1) and (2) hold, the latter condition (assumption (B) in [32]) implies that for any sequence \( K_N \to \infty \)
\[
\lim_{N \to \infty} \frac{1}{K_N} \sum_{i=1}^{tN} E[\xi_k^2 1_{\xi_k > K_N}] = 0, \tag{4}
\]
see Lemma A.6. Compare this to the weaker assumptions (3) and (3'). In particular, (4) prevents the limit process \( \sigma^2 \) from jumping, which in turn implies that the limiting tree is binary.

2.2 Examples

Let us give some examples illustrating the conditions (1) and (2) which fit the class of models which we are considering here.

1. Consider a sequence of nearly critical Galton–Watson processes \( Z_n^{(N)}, n \geq 0 \) with offspring distribution \( f^{(N)}(N) \) such that
\[
[f^{(N)}]'(1) = 1 + \frac{\alpha}{N} + o\left(\frac{1}{N}\right) \quad \text{and} \quad [f^{(N)}]''(1) = \sigma^2 + o(1),
\]
with \( \alpha \in \mathbb{R} \) and \( \sigma^2 > 0 \). We then have
\[
(\mu_{sN}, s \geq 0) = \left( \exp \left( \sum_{k=1}^{sN} \log(1 + \alpha/N + o(1/N)) \right), s \geq 0 \right) \to (e^{\alpha s}, s \geq 0),
\]
in the Skorohod topology, and the variance process (2) converges to \( \sigma^2(t) = \sigma^2 t \).

2. In [5] the authors considered a sequence of branching processes in random environment such that the random first and second moment are given by independent copies of
\[
[F^{(N)}]'(1) = 1 + \frac{\alpha}{N} + \frac{1}{\sqrt{N}} \zeta, \quad [F^{(N)}]''(1) = \sigma^2 + o(1),
\]
with \( \alpha \in \mathbb{R}, \sigma^2 > 0 \) and some zero mean random variable \( \zeta \) with finite second moment. Then
\[
(\mu_{sN}, s \geq 0) = \left( \exp \left( \sum_{k=1}^{sN} \log[F_k'(1)]_+(1) \right), s \geq 0 \right) \to (\exp(\alpha's + B_{\rho s}), s \geq 0)
\]
in law for the Skorohod topology, where \((B_s, s \geq 0)\) is a standard Brownian motion and \(\alpha', \rho\) are some constants depending on the moments of \(\zeta\).

3. The setting of the second example is easily adapted to cases where the corresponding random walk converges to a general Lévy process instead.

4. Let \(p \in (0, 1)\). Consider a varying environment where

\[
\forall k \in \frac{N}{2} + [0, N^{1-p}], \quad P(\xi_k = i) = \begin{cases} \frac{1}{Np}, & \text{if } i = Np \\ 1 - \frac{1}{Np}, & \text{if } i = 0. \end{cases}
\]

Elsewhere, suppose that the environment is constant with mean 1 and variance \(\beta\). Then the variance process converges to the linear function \(\beta t\) with a jump of size \(1/2\).

Note that (3’) is fulfilled, whereas (3) is only fulfilled for \(p < 1/2\). By the same token, one can extend the construction to generate limiting varying environments where \(\sigma^2\) is a subordinator and \(X\) an arbitrary independent Lévy process.

### 2.3 Main results

We now discuss our two main results, namely the estimate for the survival probability and the scaling limit of the genealogy of nearly critical BPVEs.

**Theorem 1** (Kolmogorov estimate). Let \((Z_n, n \geq 0)\) be a sequence of BPVEs. For \(t > 0\) such that \(t\) is a continuity point of \(\sigma^2\) and \(\sigma^2\) strictly increasing at \(t\), then as \(N \to \infty\) under the near critical assumptions (1), (2) and (3) we have

\[
P(Z_{tN} > 0) \sim \frac{2}{\kappa_N} \frac{1}{\int_{[0,1]} e^{-X_s \sigma^2(ds)}}.
\]

We note that in the case of critical or nearly critical Galton–Watson processes, Theorem 1 reduces to well known results, see [37] and [47]. The proof of Theorem 1 will be given in Section 6.

Our second main result, Theorem 2, shows that the genealogical structure of the BPVE, when envisioned as a random metric measure space and conditional on survival, converges in the Gromov–Hausdorff–Prohorov (GHP) topology to a limiting continuous tree, the environmental coalescent point process. Stating the result requires some preliminary notation.

Recall the tree construction of the branching process in varying environment from Section 2 and the notation \(T_n\) for the \(n\)-th generation of the process. We define a distance on the vertices at generation \(n\) via

\[
\forall u, v \in T_n, \quad d_n(u, v) = n - |u \land v|,
\]

which corresponds to the time to the most-recent common ancestor of \(u\) and \(v\). Consider the empirical measure

\[
\lambda_n = \sum_{u \in T_n} \delta_u.
\]

Then \([T_n, d_n, \lambda_n]\) is a random metric measure space that encodes the genealogy of generation \(n\) of the BPVE. Formally, we can define a topology on the space of metric measure spaces,
the Gromov–Hausdorff–Prohorov topology \[1\], and \([T_n, d_n, \lambda_n]\) is a random element of this space. The definition of this topology is recalled in Section 3.

In order to state our main result we briefly introduce here the scaling limit of the latter metric measure space, and name it the environmental coalescent point process (CPP). We construct it as a time-change of the more standard Brownian CPP \[48\]. A more direct construction is provided in Section 4.

Consider a Poisson point process \(P\) on \([0, \infty) \times (0, 1]\) with intensity measure \(ds \otimes \frac{1}{z^2} dx\). Define a distance on \(\mathbb{R}^+\) as

\[
\forall x \leq y, \quad d_B(x, y) = \sup\{z \in (0, 1]: (s, z) \in P, x \leq s \leq y\}.
\]

Consider an independent standard exponential random variable \(Z_B\). The Brownian CPP is the random metric measure space \([0, Z_B), d_B, \text{Leb}\], where Leb is the Lebesgue measure on the interval \((0, Z_B)\). It corresponds to the universal scaling limit of genealogies of critical branching processes with finite variance and is illustrated in Figure 1.

The environmental CPP is now defined as a time-changed version of the Brownian CPP, which reflects the heterogeneity of means and variances due to the varying environment. Fix a time \(t > 0\) and define

\[
\forall s \leq t, \quad F(s) := \frac{1}{2\rho_t} \int_{[t-s, t]} e^{X_t-X_u \sigma^2}(du), \quad \bar{\rho}_t := \frac{1}{2} \int_{[0, t]} e^{X_t-X_u \sigma^2}(du). \tag{5}
\]

Let \(F^{-1}\) be the right-continuous inverse of \(F\). The environmental CPP is defined as the random metric measure space

\[
[(0, Z_e), d_{\nu_e}, \text{Leb}] := [(0, Z_B), F^{-1} \circ d_B, \bar{\rho}_t \text{Leb}].
\]

Note that going from the Brownian to the environmental CPP requires to rescale time according to \(F\) but also to rescale mass (that is, population size) by \(\bar{\rho}_t\).
Figure 2: Simulation of the genealogy of a BPVE until generation $N = 500$. All offspring distributions are negative binomials with mean $1 + \alpha/N$, with $\alpha = 1$. Outside of the shaded region, the variance is $\sigma^2 = 2$. In the shaded region, the variance is increased in such a way that the sum of the variances in that region is $cN$, with $c = 2$.

**Theorem 2.** Suppose that (1), (2) and (3) hold and fix $t > 0$ that fulfils the same assumption as in Theorem 1. Conditional on survival at time $tN$ the following holds in distribution for the Gromov–Hausdorff–Prohorov topology

$$
\lim_{N \to \infty} \left[ T_{tN}, \frac{d_{tN}}{N}, \frac{\lambda_{tN}}{\kappa_{tN}} \right] = [ (0, Z_e), d_{\nu_e}, \text{Leb} ],
$$

with $[ (0, Z_e), d_{\nu_e}, \text{Leb} ]$ the environmental CPP.

This result is proved in Section 7. We want to point at the following fact. The limiting genealogy, the environmental CPP, is obtained by time-changing the Brownian CPP according to the function $F$ which might be discontinuous (as in the last example of Section 2.2). As a consequence, although the Brownian CPP is a binary tree, the environmental CPP might have (simultaneous) multiple mergers. These multiple mergers correspond to an accumulation of binary branch points over a time-scale shorter than $N$, due to a region of highly variable environments. This phenomenon is illustrated in Figure 2.

### 2.4 Consequences of Theorem 2

The convergence of the population in the GHP topology might seem daunting at first sight to the reader not familiar with random metric measure spaces. We show in this section how the GHP convergence in Theorem 2 condenses the limit of several genealogical quantities of interest. This requires some preliminary notation for these quantities and their limits.

For each $i \leq tN$, let $Z_{i,tN}$ be the number of individuals at generation $i$ having descendants at generation $tN$ in the BPV individuals chosen uniformly at random from $T_{tN}$ (the set of individuals at generation $tN$) and recall that $d_{tN}(U_i, U_j)$ denotes the time to the most-recent common ancestor of $U_i$ and $U_j$. 
For \( k \in \mathbb{N} \) let \((H_1, \ldots, H_{k-1})\) be a sequence of independent random variables with cumulative distribution function \( F \) defined in (5). We define the array \((H_{i,j}; i, j \leq k)\) via

\[
\forall i \leq j, \quad H_{i,j} = H_{j,i} = \max\{H_i, \ldots, H_{j-1}\},
\]

with the convention that \( \max \emptyset = 0 \). Additionally, for \( \theta \geq 0 \) we introduce \( H^\theta \) with

\[
\forall s \leq t, \quad \mathbb{P}(H^\theta \leq s) = \frac{(1 + \theta)F(s)^k}{1 + \theta F(s)}.
\]

Lastly, we introduce the array \((H^\theta_{i,j}; i, j \leq k)\) in the same manner as in (6). We note that (7) in fact defines a valid probability distribution for any \( \theta > -1 \). However, these values of \( \theta \) are not relevant in our case. Some motivation for this distribution in connection with coalescent point processes is provided in Section 4.2.

**Corollary 2.3.** Let \( t > 0 \) fulfil the same assumptions as in Theorem 1. The following limits hold in distribution as \( N \to \infty \), conditional on \( Z_{Nt} > 0 \).

(i) We have

\[
\frac{Z_{Nt}}{\kappa_N} \to \mathcal{E}(\bar{\rho}_t),
\]

where \( \mathcal{E}(x) \) is an exponentially distributed random variable with mean \( x > 0 \).

(ii) For the reduced process we have

\[
(Z_{sN,tN}, s < t) \to \left(Y \left(\log \frac{\int_{[0,t]} e^{-u\sigma^2} (du)}{\int_{[s,t]} e^{-u\sigma^2} (du)}\right), s < t\right)
\]

in the sense of finite-dimensional distribution at continuity points of \( \sigma^2 \), and where \( (Y(t), t \geq 0) \) is a Yule process. If \( \sigma^2 \) is continuous, the convergence holds in the Skorohod sense.

(iii) Recall that \( d_{tN}(U_i, U_j) \) denotes the time to the most-recent common ancestor of two individuals \( U_i \) and \( U_j \) sampled uniformly from the population at time \( tN \). Then

\[
\left(\frac{d_{tN}(U_i, U_j)}{N}\right)_{i,j} \to (H_{\sigma, \sigma})_{i,j},
\]

where \( \sigma \) is a uniform permutation of \( \{1, \ldots, k\} \) independent of \( (H_{i,j})_{i,j} \), and the law of the latter matrix is given by

\[
\mathbb{E}[\phi((H_{i,j})_{i,j})] = k \int_0^\infty \frac{1}{(1 + \theta)^2} \left(\frac{\theta}{1 + \theta}\right)^{k-1} \mathbb{E}[\phi((H^\theta_{i,j})_{i,j})] d\theta,
\]

where \((H^\theta_{i,j})_{i,j}\) has distribution (7) and \( \phi: \mathbb{R}^{k \times k} \to \mathbb{R}_+ \) is continuous bounded.

In all three items of the previous result, the limiting random variable can be expressed as the image of the environmental CPP under an appropriate functional. The previous result follows from Theorem 2 by showing that these functionals are continuous and computing the law of the corresponding images of the limiting tree.
Finally, let us illustrate our result by specifying the previous corollary to our first example, a nearly critical Galton–Watson process with mean $1 + \alpha/N + o(1/N)$ and variance $\sigma^2 + o(1)$. Theorem I becomes

$$\mathbb{P}(Z_{tN} > 0) \sim \begin{cases} 
\frac{1}{N} \frac{2}{\sigma^2 t} & \alpha = 0 \\
\frac{2\alpha}{N} \frac{1}{\sigma^2 (1 - e^{-\alpha t})} & \alpha \neq 0.
\end{cases}$$

Note that the case $\alpha = 0$ corresponds to the classical Kolmogorov asymptotics [37], whereas the case $\alpha \neq 0$ can be found in [47]. Furthermore, the first point of Corollary 2.3 yields that, conditional on $Z_{tN} > 0$,

$$\frac{Z_{tN}}{N} \to \begin{cases} 
\sigma^2 t & \alpha = 0 \\
\frac{\alpha}{2} \sigma^2 (1 - e^{-\alpha t}) & \alpha \neq 0.
\end{cases}$$

These are well-known asymptotics for nearly critical branching processes, see for instance [47]. In turn, point (iii) of Corollary 2.3 and a small calculation prove that the joint limiting distribution of the split times $s_1, \ldots, s_{k-1} \in [0, t]^{k-1}$ of the subtree spanned by $(U_1, \ldots, U_k)$ has density $f$ given by

$$f(s_1, s_2, \ldots, s_{k-1}) = \begin{cases} 
k \left( \frac{1}{t} \right)^{k-1} \int_0^\infty \frac{\theta^{k-1}}{(1 + \theta)^2} \prod_{i=1}^{k-1} \frac{1}{1 + \theta \frac{\alpha}{t}} d\theta, & \alpha = 0, \\
k(\alpha (e^{\alpha t} - 1))^{k-1} \int_0^\infty \frac{\theta^{k-1}}{(1 + \theta)^2} \prod_{i=1}^{k-1} \frac{e^{\alpha s_i}}{(e^{\alpha s_i} - 1) + e^{\alpha t} - 1} d\theta, & \alpha \neq 0,
\end{cases}$$

which is exactly the limit in [26] Theorem 3 for $\alpha = 0$. Finally, point (ii) shows that the reduced process is asymptotically distributed as the time-changed Yule process,

$$\left( Y \left( \log \frac{1 - e^{-\alpha t}}{e^{-\alpha s} - e^{-\alpha t}} \right), s < t \right),$$

which again is a known result, see [47] Remark 1]. Our work extends these results to the setting of nearly critical branching processes in varying environment.

**Remark 2.4.** Let

$$\forall s \leq t, \quad G_{\bar{N}}(s) = \bar{\rho}_{s}^{(N)} , \quad \bar{\rho}_{s}^{(N)} = \sum_{k=1}^{s+N} \frac{1}{f_k(1)^2} \frac{f_k''(1)}{\mu_{k-1}},$$

The main results in [33] and [24] are respectively that, after making a time-rescaling $u = G_{\bar{N}}(s)$, the reduced process and the law of a $k$ sample from a BPVE converge to the corresponding expressions for the Brownian CPP. Since $G_{\bar{N}}(s) \to G(s) := \frac{\alpha}{\mu}$, (see Lemma A.4), reverting the limiting time-change, that is taking $s = F^{-1}(u)$, yields the environmental CPP (see Proposition 4.2). Thus points (ii) and (iii) of Corollary 2.3 are consistent with the results obtained in [33] and [24]. Note again that we work under different settings, see Remark 2.2.
3 Random metric measure spaces and Gromov topologies

Our main result shows the convergence of the genealogy of the BPVE, viewed as a random metric measure space. This section introduces the topological notions that are required to prove this result, as well as some results regarding ultrametric spaces. We will need to work with two distinct topologies on the space of metric measure spaces: the Gromov-weak topology and the stronger Gromov–Hausdorff–Prohorov (GHP) topology.

The bulk of our work is dedicated to proving convergence in the Gromov-weak topology which is introduced in Section 3.1.1. In this topology, convergence in distribution is strongly connected to convergence of moments of the metric space, which are computed out of samples from the metric space. These moments can be computed efficiently for branching processes using the many-to-few formula of Section 5.2. Applying this method of moments requires the reproduction laws to have moments of any order, whereas our convergence result only requires a finite moment of order two. To overcome this difficulty, we use a truncation procedure similar to that in [26]. Section 3.1.2 contains some perturbation results that are used to prove that the truncated version of the BPVE remains close in the Gromov-weak topology to the original BPVE.

Although proving convergence in distribution in the Gromov-weak topology is made considerably easier by the method of moments, this topology is too weak to ensure the convergence of some natural statistics of genealogies, such as the convergence of the reduced process. This motivates the introduction of a stronger topology in Section 3.2, the Gromov–Hausdorff–Prohorov topology. Once convergence is established for the Gromov-weak topology by computing moments, we will show how to strengthen it to a GHP convergence through a simple tightness argument which was developed in [3]. A similar argument should apply broadly to critical branching processes satisfying a Yaglom law.

3.1 Convergence in the Gromov-weak topology

3.1.1 The Gromov-weak topology

We briefly recall the definition of the Gromov-weak topology and some of its properties that are needed in this work. The reader is referred to [19, 17, 22] for a more complete account.

A metric measure space is a triple \([X, d, \nu]\) such that \((X, d)\) is a complete separable metric space and \(\nu\) is a finite measure on the corresponding Borel \(\sigma\)-field. Given some \(k \geq 1\) and a bounded continuous map \(\phi: [0, \infty)^{k \times k} \to \mathbb{R}\), we define a functional \(\Phi\) on the space of metric measure spaces, called a polynomial, as

\[
\Phi([X, d, \nu]) = \int_{X^k} \phi((d(x_i, x_j))_{i,j \leq k}) \nu(dx_1) \ldots \nu(dx_k).
\]

The Gromov-weak topology is defined as the topology induced by the polynomials, that is, the smallest topology making all polynomials continuous. The Gromov-weak topology allows us to define a random metric measure space as a random variable with values in metric measure spaces, endowed with the Gromov-weak topology and the corresponding Borel \(\sigma\)-field. The moment of a random metric measure space associated to \(\Phi\) is the expectation of the corresponding polynomial.

Recall the tree construction of the BPVE from Section 2 and the notation \(T_n\) for the \(n\)-th generation of the process. Recall that \(d_n\) stands for the tree distance defined as

\[
\forall u, v \in T_n, \quad d_n(u, v) = n - |u \wedge v|
\]
and consider the measure
\[ \lambda_n = \sum_{u \in T_n} \delta_u. \]

Then \([T_n, d_n, \lambda_n] \) is the random metric measure space encoding the genealogy of generation \(n\) of the BPVE.

The appeal of the Gromov-weak topology is well illustrated by the following result, which connects convergence in distribution for the Gromov-weak topology to the convergence of moments, see [9, Lemma 2.8].

**Proposition 3.1** (Method of moments). Consider a sequence \([X_n, d_n, \nu_n], n \geq 1\) of random metric measure spaces such that
\[ \forall \Phi, \quad \mathbb{E}[\Phi([X_n, d_n, \nu_n])] \to \mathbb{E}[\Phi([X, d, \nu])] < \infty, \]
for some limiting space \([X, d, \nu]\). Then if the limiting space further fulfils that
\[ \limsup_{k \to \infty} \frac{\mathbb{E}[\nu(X)^k]^{1/k}}{k} < \infty, \]
the sequence \([X_n, d_n, \nu_n], n \geq 1\) converges to \([X, d, \nu]\) in distribution for the Gromov-weak topology.

**Remark 3.2** (Identification of the limit). Let us compare the previous result with the usual method of moments for real random variables. Let \((X_n)\) be a sequence of real r.v.s and assume that for every \(k \in \mathbb{N}\) there exists some \(m_k\) such that
\[ \mathbb{E}[X_n^k] \to m_k, \quad \text{as} \quad n \to \infty. \]
Subject to an additional technical condition, the method of moments leads to two key outcomes: (1) the \(m_k\)'s coincide with the moments of an underlying random variable \(X_\infty\), and (2) the sequence \((X_n)\) converges in distribution to \(X_\infty\). This stands in contrast to the prior result where (1) is not guaranteed; specifically, the limiting moments are required to coincide with the moments of a limiting metric space. Notably, Proposition 3.1 can be enhanced by relaxing this requirement, as detailed in [15], Section 4. This improvement hinges on extending the Gromov-weak topology to non-separable ultrametric spaces through a de Finetti representation for exchangeable coalescents, as obtained in [14].

### 3.1.2 Truncation in the Gromov-weak topology

Applying the method of moments requires to carry out a truncation of the reproduction laws for all moments to be finite. In order to deduce from the convergence of the truncated BPVE that of the original BPVE, we need to show that the two trees remain close with large probability for an adequate choice of distance between trees. It turns out that one can define a distance on the space of metric measure spaces that induces the Gromov-weak topology [19] [17] [22]. There are several formulations for such a distance, we work with the Gromov–Prohorov distance of [19] [9]. It is defined as
\[ d_{GP}([X, d, \nu], [X', d', \nu']) = \inf_{Z, \phi, \phi'} d_P(\nu \circ \phi^{-1}, \nu' \circ (\phi')^{-1}), \]
where the infimum is taken over all complete separable metric spaces \((Z, d_Z)\) and isometric embeddings \(\phi: X \hookrightarrow Z\) and \(\phi': X' \hookrightarrow Z\), \(\nu \circ \phi^{-1}\) is the pushforward of \(\nu\) through the map \(\phi\), and \(d_p\) is the Prohorov distance on finite measures on \((Z, d_Z)\). The following simple lemma will be the key to control the distance between the truncated and the original tree.

**Lemma 3.3.** Consider a metric measure space \([X, d, \nu]\). Let \(X'\) be a closed subset of \(X\), and let \(d'\) and \(\nu'\) be the restrictions of \(d\) and \(\nu\) to \(X'\). Then

\[
d_{GP}([X, d, \nu], [X', d', \nu']) \leq \nu(X \setminus X').
\]

**Proof.** Since the canonical injection from \(X'\) to \(X\) is an isometry,

\[
d_{GP}([X, d, \nu], [X', d', \nu']) \leq d_p(\nu, \nu') \leq \nu(X \setminus X'),
\]

where \(d_p\) stands for the Prohorov distance on finite measures on \((X, d)\).

The practical interest of the previous result lies in the following corollary.

**Corollary 3.4.** Let \([X_n, d_n, \nu_n], n \geq 1\) be a sequence of random metric measure spaces and, for \(n \geq 1\), let \(X_n' \subseteq X_n\) be a closed subset, and \(d'_n\) and \(\nu'_n\) be the restrictions of \(d_n\) and \(\nu_n\) to \(X_n'\). Suppose that, as \(n \to \infty\),

1. we have \(|\nu_n(X_n) - \nu_n(X_n')| \to 0\) in probability;
2. we have \([X_n', d'_n, \nu'_n] \to [X, d, \nu]\) in distribution for the Gromov-weak topology.

Then \(([X_n, d_n, \nu_n])_n\) also converges to \([X, d, \nu]\) in distribution for the Gromov-weak topology.

**Proof.** By Lemma 3.3

\[
P(d_{GP}([X_n, d_n, \nu_n], [X_n', d'_n, \nu'_n]) \geq \varepsilon) \leq P(\nu_n(X_n) - \nu_n(X_n') \geq \varepsilon) \to 0, \quad \text{as } n \to \infty.
\]

Since \(d_{GP}\) induces the Gromov-weak topology, the convergence of \(([X_n, d_n, \nu_n])_n\) now follows from that of \(([X_n', d'_n, \nu'_n])_n\).

### 3.2 The Gromov–Hausdorff–Prohorov topology

We now turn to the definition of two very related topologies, the Gromov–Hausdorff–Prohorov and the Gromov–Hausdorff-weak topologies. Following [3], we say that a sequence of metric measure spaces \(([X_n, d_n, \nu_n])_{n \geq 1}\) converges in the Gromov–Hausdorff-weak sense to \([X, d, \nu]\) if there exists a complete separable metric space \((Z, d_Z)\) and isometries \(\phi_n: \text{supp } \nu_n \hookrightarrow Z\) and \(\phi: \text{supp } \nu \hookrightarrow Z\) such that

\[
d_H(\phi_n(\text{supp } \nu_n), \phi(\text{supp } \nu)) + d_p(\nu_n \circ \phi_n^{-1}, \nu \circ \phi^{-1}) \to 0, \quad \text{as } n \to \infty,
\]

where \(d_H\) is the Hausdorff distance on \((Z, d_Z)\) and \(d_p\) is the Prohorov distance on finite measures on \((Z, d_Z)\). On the other hand, this sequence converges in the Gromov–Hausdorff–Prohorov sense if there exist isometries \(\phi_n: X_n \hookrightarrow Z\) and \(\phi: X \hookrightarrow Z\) such that

\[
d_H(\phi_n(X_n), \phi(X)) + d_p(\nu_n \circ \phi_n^{-1}, \nu \circ \phi^{-1}) \to 0, \quad \text{as } n \to \infty.
\]
Clearly, a sequence of metric measure spaces converges Gromov–Hausdorff-weakly if and only if the sequence of supports of the measures converges in the Gromov–Hausdorff–Prohorov sense. The two notions of convergence coincide when the spaces have full support, that is, when $\text{supp } \nu = X$.

The Gromov–Hausdorff–Prohorov topology is fairly standard and has been used extensively for deriving scaling limits of trees and other metric spaces, which makes it a natural choice for formulating our main result. However, when trying to reinforce a Gromov-weak convergence to a GHP one, one runs into the problem that the two topologies are not formally defined on the same state space. The GHP topology is defined on the space of strong equivalence classes of metric measure spaces, where two such spaces are said to be strongly equivalent if there exists a bijective isometry between them that preserves the measures, see Definition 2.4 in [1]. On the other hand, the Gromov-weak topology is defined on weak equivalence classes of metric measure spaces, where two such spaces are weakly equivalent if their supports are strongly equivalent in the above sense, see Definition 2.1 in [19]. The Gromov–Hausdorff-weak topology is an adaptation of the GHP topology to weak equivalence classes; it is less standard, but more convenient to use when working with moments and the Gromov-weak convergence. In order to obtain a GHP convergence, we will first prove the Gromov-weak convergence, then reinforce it to a Gromov–Hausdorff-weak one, and finally use the fact that the metric measure spaces considered here have full support to deduce the GHP convergence.

In the rest of this section, we first introduce the notion of reduced process of an ultrametric space and show that this reduced process is a continuous functional of the GHP topology. Then we apply the results in [3] to give a criterion for reinforcing a Gromov-weak convergence to a Gromov–Hausdorff-weak one for ultrametric spaces.

### 3.2.1 Ultrametric spaces and reduced processes

All metric spaces that are considered in this work are constructed as genealogies of population models at a fixed time horizon. They all have the further property of being ultrametric spaces. A metric space $(U, d)$ is called ultrametric if it fulfils the stronger triangle inequality

$$\forall x, y, z \in U, \quad d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$ 

In this section we construct the reduced process associated to a general ultrametric space and prove that the function mapping an ultrametric space to its reduced process is continuous in the GHP topology.

A well-known property of ultrametric spaces is that the set of open balls of a given radius form a partition of the space, that is,

$$x \sim_t y \iff d(x, y) < t$$

is an equivalence relation. We will use the notation $B_{x,t}(U) = \{y \in U : d(x, y) < t\}$ for the open ball of radius $t$ around $x$, and

$$B_t(U) = \{B_{x,t}(U), x \in U\}$$

for the set of balls of radius $t$ of an ultrametric space $(U, d)$. In the genealogical interpretation of an ultrametric space, a ball of radius $t$ corresponds to the set of individuals sharing a
common ancestor at time \( t \) in the past. Thus, we can envision \( B_t(U) \) as the set of ancestral lineages of the population at time \( t \) in the past. This motivates the introduction of the process

\[
\forall t > 0, \quad L_t(U) = \text{Card} \, B_t(U)
\]
counting the number of ancestral lines, which we will refer to as the reduced process of the ultrametric space \((U, d)\). (Note the time reversal compared to the usual definition of a reduced process: time is flowing backward and not forward.)

The result proved in this section shows that the convergence of the reduced process is a consequence of the convergence of the ultrametric spaces in the GHP topology. Actually, this continuity property would hold under the weaker Gromov–Hausdorff topology, defined for instance in Section 4 of [22], but we only state the result for the GHP topology so as to not introduce notions that will not be required later on.

**Proposition 3.5.** Suppose that \( [(U_n, d_n, \nu_n), n \geq 1] \) is a sequence of compact ultrametric measure spaces converging in the Gromov–Hausdorff–Prohorov topology to \([U, d, \nu]\). Then

\[
(L_t(U_n), t > 0) \to (L_t(U), t > 0), \quad \text{as } n \to \infty,
\]
in the sense of pointwise convergence at each continuity point of \((L_t(U), t > 0)\). If \((U, d)\) is binary, that is, \((L_t(U), t > 0)\) only makes jumps of size 1, the convergence also holds in the Skorohod space \(\mathbb{D}([t_0, \infty))\) for all continuity point \(t_0 > 0\) of the reduced process.

**Proof.** Let \((Z, d_Z)\) and \(\phi: U \to Z, \phi_n: U_n \to Z\) be as in (8). Fix \(n \geq 1\) and suppose that

\[
d_H(\phi(U), \phi_n(U_n)) < \varepsilon, \tag{9}
\]
for some \(\varepsilon > 0\). Fix \(t > 2\varepsilon\) and let \((x_i)_{i \leq L_t(U)} \in U\), such that each \(x_i\) belongs to a distinct open ball of radius \(t\) of \(U\). (That is, they are such that \(d(x_i, x_j) \geq t\), for \(i \neq j\).) By (9), for each \(i\) there exists \(y_i \in U_n\) such that \(d_Z(\phi(x_i), \phi_n(y_i)) < \varepsilon\). Moreover,

\[
d_n(y_i, y_j) = d_Z(\phi_n(y_i), \phi_n(y_j)) > d_Z(\phi(x_i), \phi(x_j)) - 2\varepsilon \geq t - 2\varepsilon.
\]
This shows that there at least \(L_t(U_n)\) open balls of \((U_n, d_n)\) of radius \(t - 2\varepsilon\), that is,

\[
L_{t-2\varepsilon}(U_n) \geq L_t(U).
\]
We deduce that (9) implies that

\[
L_{t+2\varepsilon}(U) \leq L_t(U_n) \leq L_{t-2\varepsilon}(U). \tag{10}
\]

Let \(t > 0\) be a continuity point of \((L_n(U), s \geq 0)\). We can find \(\varepsilon > 0\) such that \(L_{t+2\varepsilon}(U) = L_{t-2\varepsilon}(U)\). Equation (10) shows that, for \(n\) large enough,

\[
L_{t+2\varepsilon}(U) = L_t(U_n) = L_{t-2\varepsilon}(U),
\]
hence the first part of the result.

If the process \((L_t(U), t > 0)\) only makes jumps of size 1, the Skorohod convergence follows for instance from [28] Chapter VI, Theorem 2.15. \(\square\)
3.2.2 From Gromov-weak to GHP convergence

In this last section, following [3], we indicate how to reinforce a convergence of ultrametric spaces in the Gromov-weak topology to a convergence in the Gromov–Hausdorff-weak topology. (As discussed earlier, in our context Gromov–Hausdorff-weak convergence will be equivalent to the Gromov–Hausdorff–Prohorov one.)

In the general case of [3], this relies on verifying that the metric measure spaces fulfil the so-called lower mass-bound property, see [3, Definition 3.1]. For ultrametric spaces, since balls form a partition of the space, this property turns out to have a much simpler formulation.

**Proposition 3.6.** Let \([U_n, d_n, \nu_n], n \geq 1\) be a sequence of compact ultrametric measure spaces such that \(\text{supp} \nu_n = U_n\). Suppose that it converges to a limit \([U, d, \nu]\) in distribution in the Gromov-weak topology. If, for all \(t > 0\), the collection of random variables

\[
\max\{\nu_n(B)^{-1} : B \in B_t(U_n)\}, \quad n \geq 1
\]

is tight, then the sequence also converges in distribution in the Gromov–Hausdorff–Prohorov topology.

**Remark 3.7.** In this result, \([U_n, d_n, \nu_n]\) should be interpreted as a random sequence of strong isometry classes such that the corresponding sequence of weak isometry classes converges in distribution in the Gromov-weak topology to \([U, d, \nu]\). The limit of \([U_n, d_n, \nu_n]\) in the GHP topology is the unique strong isometry class with full support whose weak isometry class is \([U, d, \nu]\).

**Proof of Proposition 3.6.** We first prove that the convergence holds in the Gromov–Hausdorff-weak sense, for which it is sufficient to prove that the sequence \([U_n, d_n, \nu_n]\), \(n \geq 1\) is tight in the Gromov–Hausdorff-weak topology. Fix a decreasing sequence \((t_p)p \geq 1\) such that \(t_p \to 0\).

For any \(\varepsilon > 0\) and \(p \geq 1\), using (11) we can find \(\eta_p > 0\) such that

\[
\inf_{n \geq 1} \mathbb{P}(\forall B \in B_{t_p}(U_n), \nu_n(B) \geq \eta_p) \geq 1 - \varepsilon 2^{-p}.
\]

By the convergence of \([U_n, d_n, \nu_n], n \geq 1\), we can also find a compact set (for the Gromov-weak topology) such that \(\mathbb{P}(U_n, d_n, \nu_n) \in K\) \(\geq 1 - \varepsilon\) for all \(n \geq 1\). We claim that

\[
K' := K \cap \bigcap_{p \geq 1} \{[U', d', \nu'] : \forall B \in B_{t_p}(U'), \nu'(B) \geq \eta_p\}
\]

is compact in the Gromov–Hausdorff-weak topology. To see that, note that \(K'\) satisfies the so-called global lower mass-bound property that

\[
\forall t > 0, \quad \inf_{[U', d', \nu'] \in K'} \inf_{B \in B_t(U')} \nu'(B) > 0,
\]

see [3, Definition 3.1]. Since \(K' \subseteq K\), any sequence in \(K'\) admits a subsequence that converges Gromov-weakly and, according to [3, Theorem 6.1], this subsequence also converges Gromov–Hausdorff-weakly. This proves that \(K'\) is Gromov–Hausdorff-weakly compact, and since

\[
\inf_{n \geq 1} \mathbb{P}([U_n, d_n, \nu_n] \in K') \geq 1 - 2\varepsilon,
\]

we have shown that \(([U_n, d_n, \nu_n], n \geq 1\) is tight in the Gromov–Hausdorff-weak topology.
There is a canonical injection \( \iota \) from the set of weak isometry classes to that of strong isometry classes, which is obtained by choosing a representative with full support, see [3, equation (5.6)]. Clearly, it is a homeomorphism between the Gromov–Hausdorff-weak topology and the GHP topology. Therefore, we have proved that
\[
\iota([U_n, d_n, \nu_n]) \to \iota([U, d, \nu]), \quad \text{as } n \to \infty,
\]
in the GHP topology. Since \([U_n, d_n, \nu_n]\) has full support, \(\iota([U_n, d_n, \nu_n]) = [U_n, d_n, \nu_n]\) and the result is proved.

4 The environmental coalescent point process

The limiting genealogy of a nearly critical BPVE is described by a continuous tree called coalescent point process (CPP). This connection is well-known for critical Galton–Watson processes, for which the limiting object is the Brownian CPP constructed in [48]. As our main theorem shows, a similar result holds for varying environments provided that we consider a larger class of limiting CPPs. In Section 2.3 we introduced these objects as time-changes of the Brownian CPP. We provide a more direct construction here, and refer the reader to [48, 41, 10, 39] for more background on CPPs as well as for a more careful construction of the underlying metric measure space.

Fix some height \( t > 0 \) and consider a Poisson point process \( P \) on \([0, \infty) \times (0, t]\) with intensity measure \( ds \otimes \nu(dx) \), for some measure \( \nu \) on \((0, t]\) such that
\[
\forall x > 0, \quad \nu((x, t]) < \infty, \quad \nu((0, t]) = \infty.
\]
Define the distance \( d_\nu \) such that
\[
\forall x \leq y, \quad d_\nu(x, y) = \sup\{z : (s, z) \in P, x \leq s \leq y\}, \quad (12)
\]
see Figure 1 for an illustration. Consider a positive real \( m \) and an independent exponentially distributed random variable \( Z \) with mean \( m \). The CPP at height \( t \) associated to \( \nu \) and \( m \) is the random metric measure space \((0, Z), d_\nu, \text{Leb})\). Its distribution will be denoted by \( \text{CPP}_t(\nu, m) \). Note that the Brownian CPP of Section 2.3 has distribution \( \text{CPP}_1(dx, x^2, 1) \).

Remark 4.1. In the definition of a metric measure space of Section 3.1.1 we require the metric space to be complete and the measure to be defined on the corresponding Borel \( \sigma \)-field. Almost surely \((0, Z), d_\nu\) is not complete, and we always work with its completion without further mention. The completion can be obtained explicitly by adding a countable number of points as described in [42]. Moreover, the Lebesgue measure \( \text{Leb} \) is formally defined on the usual \( \sigma \)-field on the interval \((0, Z)\), which might be different from the Borel \( \sigma \)-field of \((0, Z), d_\nu\). The two \( \sigma \)-fields actually coincide, as can be seen directly by noting that the balls of \( d_\nu \) are intervals of \((0, Z)\), see [14, Lemma 1.4] for a formal proof.

For a nearly critical branching process in varying environment, the rescaled genealogy at time \( tN \) will have \( \text{CPP}_t(\nu_e, \bar{\rho}_t) \) as its limiting distribution, with \( \bar{\rho}_t \) as in (5) and
\[
\forall s \leq t, \quad \nu_e([s, t]) = \left(\frac{1}{2} \int_{[t-s, t]} e^{X_t-X_u} \sigma^2(du)\right)^{-1} - \left(\frac{1}{2} \int_{[0, t]} e^{X_t-X_u} \sigma^2(du)\right)^{-1}. \quad (13)
\]
We will refer to this CPP as the \textit{environmental CPP}. As will be shown in Proposition 4.2, this definition agrees with the definition of the environmental CPP as a time-change of the Brownian CPP given in Section 2.3. Note that the Brownian CPP also corresponds to the environmental CPP at height 1 with \( X_t = 0 \) and \( \sigma^2(t) = \sigma^2t \).

The rest of this section collects some properties of CPPs that are mostly required in the proof of Corollary 2.3. These properties are well-known for the Brownian CPP. The following result will prove convenient to transfer these results from the Brownian CPP to any environmental CPP.

**Proposition 4.2.** Let \([(0, Z_B), d_B, \text{Leb}]\) be the Brownian CPP with distribution \( \text{CPP}_1 \left( \frac{dx}{2x_2}, 1 \right) \). Recall the expression of \( F \),

\[
\forall s \leq t, \quad F(s) = \frac{1}{2\bar{\rho}_t} \int_{[s-t]} e^{X_t-X_u}\sigma^2(du),
\]

and consider \( F^{-1} \) its right-continuous inverse. Then \([(0, Z_B), F^{-1}(d_B), \bar{\rho}_t \text{ Leb}]\) is distributed as \( \text{CPP}_i(\nu_e, \bar{\rho}_t) \), with \( \nu_e \) defined in (13).

**Proof.** Let \( P \) be the Poisson point process on \((0, \infty) \times (0, 1)\) out of which the Brownian CPP is constructed in (12). We have

\[
\forall x < y \leq Z, \quad F^{-1}(d_B(x, y)) = \sup\{F^{-1}(z) : (s, z) \in P, x < s < y\}
\]

\[
= \sup\{z' : (s, z') \in P', x < s < y\},
\]

where \( P' \) is the point process obtained by scaling the second coordinate of \( P \) according to \( F^{-1} \), namely

\[
P' = \sum_{(s, z) \in P} \delta_{(s, F^{-1}(z))}.
\]

By the mapping theorem for Poisson point processes, \( P' \) is again a Poisson point process on \((0, \infty) \times (0, t)\), with intensity \( ds \otimes \nu' \), where \( \nu' \) is the push-forward of \( \frac{1}{2t} dx \) on \((0, 1)\) by \( F^{-1} \). By standard properties of generalized inverses, \( F^{-1}(u) > s \iff u > F(s) \) and

\[
\nu'|(s, t]) = \frac{1}{F(s)} - 1 = \frac{2\bar{\rho}_t}{\int_{[s-t]} e^{X_t-X_u}\sigma^2(du)} - 1 = \bar{\rho}_t \nu_e((s, t])\).
\]

Thus, \([(0, Z_B), F^{-1}(d_B), \bar{\rho}_t \text{ Leb}]\) is the CPP at height \( t \) associated to the measure \( \bar{\rho}_t \nu_e \) and to the random population size \( Z_B \). By a simple scaling argument, it is not hard to see that the distribution of \([(0, Z_B), F^{-1}(d_B), \bar{\rho}_t \text{ Leb}]\) is the same as \([(0, Z_e), d_{\nu_e}, \text{Leb}]\), the CPP constructed from \( \nu_e \) and \( Z_e \). (Note that the total mass needs to be scaled so that the isometry between \([(0, Z_B), F^{-1}(d_B)]\) and \([(0, Z_e), d_{\nu_e}]\) is measure-preserving.) \( \square \)

### 4.1 Moments of the CPP

In order to prove convergence in the Gromov-weak topology we have to know the moments of the environmental CPP. This is provided by the following result.

**Proposition 4.3.** Let \([(0, Z_e), d_{\nu_e}, \text{Leb}]\) be the environmental CPP at time \( t \). Then for any bounded function \( \phi \) with associated polynomial \( \Phi \) we have

\[
E[\Phi((0, Z_e), d_{\nu_e}, \text{Leb})] = \bar{\rho}_t^k k! E[\phi((H_{s, \sigma})_{i, j})],
\]

where \( \sigma \) is an independent and uniform permutation of \([k]\) and \((H_{i,j})_{i,j}\) has distribution \([\text{Leb}]\).
Proof. We will apply [15, Proposition 4]. The CPP construction in [15, Section 4.3] is slightly different from that presented here: the measure ν is defined on the whole real line and Z is defined as the first atom of P above level t. Obviously, the two constructions are equivalent as long as \( E[Z] = 1/\nu((t, \infty)) \). Therefore, using [15, Proposition 4] with \( \nu((t, \infty)) = 1/\bar{\rho}t \), the result is proved if we show that

\[
\forall s \leq t, \quad \mathbb{P}(H \leq s) = \frac{1/\bar{\rho}t}{\nu_e((s,t]) + 1/\bar{\rho}t}
\]

holds. This follows from the simple computation

\[
(\bar{\rho}t\nu_e((s,t]) + 1)^{-1} = \frac{1}{2\bar{\rho}t} \int_{[t-s,t]} e^{X_t - X_u} \sigma^2(du).
\]

4.2 Sampling from the CPP

In this section we are interested in the distribution of the subtree spanned by \( k \) individuals chosen uniformly from an environmental CPP. In the context of the CPP, choosing these individuals amounts to sampling \( k \) points \((U_1, \ldots, U_k)\) uniformly on the interval \((0, Z_e)\), which splits \((0, Z_e)\) into \( k + 1 \) subintervals. The pairwise distances between \((U_1, \ldots, U_k)\) are then connected to the maximum of the point process \( P \) used in the construction of \( d\nu_e \) on these \( k + 1 \) subintervals. Computing the joint distribution of these maxima is made difficult by the fact that the lengths of these subintervals are not independent.

This problem would be much easier if the points were sampled according to an independent Poisson point process with intensity \( \theta/\bar{\rho}dt \). The atoms of such a point process split \((0, Z_e)\) into a geometric number of subintervals with success parameter \( 1/(1 + \theta) \), and the lengths of these intervals are independent exponential random variables with mean \( \bar{\rho}t/(1 + \theta) \). Thus, the distances between any two consecutive atoms are independent, and distributed as the maximum of \( P \) over an interval with exponential length of mean \( \bar{\rho}t/(1 + \theta) \). If \( H^\theta \) denotes a random variable with the same distribution as this maximum, a direct computation shows that

\[
\forall s \leq t, \quad \mathbb{P}(H^\theta \leq s) = \frac{(1 + \theta)F(s)}{1 + \theta F(s)},
\]

where \( F \) is defined in (5), and corresponds to the cumulative distribution function of the maximum of the point process \( P \) over the whole interval \((0, Z_e)\). It turns out that we can express the distribution of the pairwise distances of \( k \) points sampled uniformly from the CPP as a mixture of that for the Poisson sampling. This can be seen as an instance of the Poissonization method developed in [40, 30].

Proposition 4.4. For any \( k \geq 1 \), let \((U_1, \ldots, U_k)\) be uniformly distributed on \((0, Z_e)\), then

\[
\mathbb{E}[\phi((d\nu_e(U_i, U_j)), i, j)] = k \int_0^\infty \frac{1}{(1 + \theta)^2} \left( \frac{\theta}{1 + \theta} \right)^{k-1} \mathbb{E}[\phi((H^\theta_{i,j}), i, j)] d\theta,
\]

where \( \sigma \) is a uniform permutation of \([k]\) and \((H^\theta_{i,j}), i, j\) as in (7).

The key point in our proof is the observation that the exponential distribution can be expressed as a mixture of Gamma distribution.
Lemma 4.5. Fix some \( k \geq 2 \), then
\[
\forall x \geq 0, \quad e^{-x} = k \int_0^\infty \frac{1}{(1 + \theta)^2} \left( \frac{\theta}{1 + \theta} \right)^{k-1} f_{k+1,\theta+1}(x) d\theta,
\]
where \( f_{k,\theta}(x) \) is the density of a Gamma\((k, \theta)\) random variable, that is,
\[
\forall x \geq 0, \quad f_{k,\theta}(x) = \frac{\theta^k x^{k-1}}{(k-1)!} e^{-\theta x}.
\]

Proof. The result follows by a direct computation,
\[
\int_0^\infty \frac{k}{(1 + \theta)^2} \left( \frac{\theta}{1 + \theta} \right)^{k-1} f_{k+1,\theta+1}(x) d\theta = \int_0^\infty \frac{k}{(1 + \theta)^2} \left( \frac{\theta}{1 + \theta} \right)^{k-1}(\theta + 1) + 1 x^k k! e^{-x(\theta+1)} d\theta
\]
\[= \int_0^\infty \frac{\theta^{k-1} x^k}{(k-1)!} e^{-x(\theta+1)} d\theta = e^{-x}. \]
\[\square\]

Remark 4.6. The previous result can be extended to geometric and negative binomial distributions, by noting that these distributions are mixtures of a Poisson distribution by an exponential and Gamma distribution respectively. This can in turn be used to compute the distribution of \( k \) points sampled uniformly from a discrete coalescent point process. In particular this provides an alternative proof to the main result of [40], see Theorem 3.

Proof of Proposition 4.4. Let \([0, Z_e), d_{\nu_e}, \text{Leb}\] have distribution CPP\(_\ell(\nu_e, \bar{\rho}_e)\), let \((V_1, \ldots, V_k)\) be i.i.d. uniform random variables on \((0,1)\) and define \(U_i = V_i Z_e, i \leq k\). Since \(Z_e, d_{\nu_e}, \) and \((V_i)\) are independent, using Lemma 4.5 we can write
\[
\mathbb{E}\left[\phi((d_{\nu_e}(U_i, U_j))_{i,j})\right] = k \int_0^\infty \frac{1}{(1 + \theta)^2} \left( \frac{\theta}{1 + \theta} \right)^{k-1} \mathbb{E}\left[\phi((d_{\nu_e}(V_i Z_e, V_j Z_e))_{i,j})\right] d\theta,
\]
where \(Z_e\) has the Gamma\((k + 1, (\theta + 1)/\bar{\rho}_e)\) distribution and all other random variables are independent. The variables \((U_1, \ldots, U_k)\) split \((0, Z_e)\) into \(k + 1\) subintervals. By well-known properties of Gamma distributions, the lengths of these subintervals are independent exponential random variables with mean \(\bar{\rho}_e/(\theta + 1)\). Let \((U^*_1, \ldots, U^*_k)\) be the order statistics of \((U_1, \ldots, U_k)\), and define
\[
\forall i < k, \quad H^\theta_i = \sup\{s : (s, s) \in P \text{ and } s \in [U^*_i, U^*_i+1]\}.
\]
According to the previous discussion, the random variables \((H^\theta_1, \ldots, H^\theta_{k-1})\) are i.i.d. and distributed as the maximum of the point process \(P\) over an interval with exponential length and mean \(\bar{\rho}_e/(1 + \theta)\). The distribution of this maximum is given by (14). By construction of the CPP distance,
\[
\forall i < j, \quad d_{\nu_e}(U^*_i, U^*_j) = H^\theta_{i,j} := \max\{H^\theta_i, \ldots, H^\theta_{j-1}\}.
\]
Thus,
\[
\mathbb{E}\left[\phi((d_{\nu_e}(U^*_i, U^*_j))_{i,j})\right] = \mathbb{E}\left[\phi((H^\theta_{i,j})_{i,j})\right]
\]
and the result is proved by noting that the unique partition \(\sigma\) of \([k]\) such that
\[
\forall i \leq k, \quad U^*_i = U_{\sigma_i},
\]
is uniform and independent of all other variables.  \[\square\]
4.3 The reduced process of a CPP

Recall the definition of the reduced process \((L_s, 0 < s < t)\) of an ultrametric tree from Section 3.2.1. The process \(L_s\) counts the number of ancestral lineages at time \(s\), starting from the leaves at \(s = 0\) and going toward the root at \(s = t\). We give a characterization of the reduced process associated to the environmental CPP. It is usual to express the reduced process of a branching process as a time-changed pure birth process, also known as a Yule process.

**Proposition 4.7.** Let \((L_s, 0 < s \leq t)\) be the reduced process associated to the environmental CPP \([(0, Z_e), d\nu_e, \text{Leb}]\). Then

\[
(L_{t-s}, s < t) \overset{(d)}{=} \left( Y \left( \log \frac{\int_{[0,t]} e^{-X_u \sigma^2(du)}}{\int_{[s,t]} e^{-X_u \sigma^2(du)}} \right), s < t \right),
\]

for a standard Yule process \((Y(s), s \geq 0)\).

**Proof.** Consider a Brownian CPP, that is the random ultrametric space with distribution \(\text{CPP}_1(\frac{dt}{2}, 1)\), denoted by \([(0, Z_B), d_B, \text{Leb}]\), with associated reduced process \((L'_s, s \leq 1)\). It is well known that

\[
(L'_{1-s}, s < 1) \overset{(d)}{=} \left( Y \left( \log \frac{1}{1-s} \right), s < 1 \right),
\]

where \((Y(u), u \geq 0)\) is a Yule process, see for instance [10, Lemma 5.5] or [47, Theorem 2.3]. By Proposition 4.2, \([(0, Z_B), F^{-1}(d_B), \bar{\rho}_t \text{Leb}]\) is distributed as the environmental CPP \((13)\). Since the identity \(F^{-1}(d(x, y)) < t \iff d(x, y) < F(t)\) holds, the reduced process \((L_s, 0 < s \leq t)\) of \([(0, Z_B), F^{-1}(d_B), \bar{\rho}_t \text{Leb}]\) is

\[
\forall s \leq t, \quad L_s = L'_{F(s)},
\]

and since \(L_{t-s} = L'_{F(t-s)} \overset{(d)}{=} Y(- \log F(t-s))\) the result follows by noting that

\[
\forall s \leq t, \quad \frac{1}{F(t-s)} = \frac{2\bar{\rho}_t}{\int_{[s,t]} e^{X_u \sigma^2(du)}}.
\]

5 Spinal decompositions

5.1 Spinal decomposition and 1-spine

The proof of the Kolmogorov estimate (Theorem 1) uses ideas of [44] adapted to the case of a varying environment. It relies on expressing the distribution of the whole tree structure \(T\) of the branching process in terms of a distinguished lineage (the 1-spine) on which subtrees are grafted. We refer to this type of result as a spinal decomposition, see for instance [50]. A spinal decomposition for BPVEs has already been proposed for instance in [34, Section 1.4.2]. Let us briefly recall it here and introduce some notation.

The spine is a line of individuals, which we think of as a set of marked individuals. We construct a tree \(T\) with one marked individual \(v_n\) at each generation \(n\). Start the population from a single marked individual \(v_0 = \emptyset\). At generation \(n\), let the marked individual have a
number of offspring distributed as a size-biased realization of \( f_{n+1} \). That is, the individual on the spine has a number \( \xi_{n+1}^* \) of children, with
\[
P(\xi_{n+1}^* = k) = \frac{k f_{n+1}[k]}{f'_{n+1}(1)}.
\]

Pick one child of the marked individual uniformly at random and mark it, let all other children be unmarked. The other individuals at generation \( n \) reproduce independently according to the offspring distribution \( f_{n+1} \). We denote by \( Q^* \) the distribution of the resulting tree \( T \) with the set of marked individuals \((\emptyset, v_1, v_2, \ldots)\), and by \( Q \) the distribution of \( T \). (That is, \( Q \) is the projection of \( Q^* \) on the space of trees, with no distinguished vertices.) The importance of the measure \( Q^* \) comes from the following well-known connection between \( Q^* \) and \( P \).

**Lemma 5.1.** Let \((T,(v_0, v_1, \ldots))\) be distributed as \( Q^* \). Conditional on the first \( n \) generations of \( T \), \( v_n \) is uniformly distributed among individuals alive at generation \( n \). Moreover \( Q_n \ll P_n \) and
\[
\forall n \geq 0, \quad \frac{dQ_n}{dP_n} = \frac{Z_n}{\mu_n},
\]
where \( Q_n \) (resp. \( P_n \)) refers to the restriction of \( Q \) (resp. \( P \)) to the first \( n \) generations.

**Proof.** The proof can be found in [34], Lemma 1.2 therein.

### 5.2 The many-to-few formula

Before we introduce the \( k \)-spine tree, let us recall some facts about discrete ultrametric trees that are needed in the proofs and in the construction. First, a tree \( \tau \) is called a (planar) ultrametric tree with height \( n \geq 1 \) if all its leaves lie at generation \( n \), that is,
\[
\forall u \in \tau, \quad d_u = 0 \iff |u| = n,
\]
where \( d_u \) denotes the out-degree of \( u \). An ultrametric tree with \( k \) leaves can always be encoded as a sequence of \( k - 1 \) elements of \( \{1, \ldots, n\} \) giving the depth of the successive coalescence times between the leaves. More precisely, let \( \ell_1, \ldots, \ell_k \) be the leaves of \( \tau \), ordered such that
\[
\ell_1 < \cdots < \ell_k,
\]
where \(<\) is the planar (lexicographical) order of the tree. Then \( \tau \) can be encoded as the sequence
\[
\mathcal{H}(\tau) = (n - |\ell_1 \wedge \ell_2|, \ldots, n - |\ell_{k-1} \wedge \ell_k|),
\]
where \( u \wedge v \) denotes the most recent common ancestor of \( u \) and \( v \). Conversely, given a sequence \((h_1, \ldots, h_{k-1}) \in \{1, \ldots, n\} \) one can find an ultrametric tree \( \tau \) such that \( \mathcal{H}(\tau) = (h_1, \ldots, h_{k-1}) \), that is, \( \mathcal{H} \) is a bijection. The tree \( \mathcal{H}^{-1}(h_1, \ldots, h_{k-1}) \) is obtained through the discrete CPP construction illustrated in Figure 3.

Finally, for an ultrametric tree \( \tau \) we will denote by \( B(\tau) \) the set of branch points, that is,
\[
B(\tau) = \{ u \in \tau : d_u > 1 \}.
\]
Note that there are at most \( k - 1 \) such branch points, and that we have the identity
\[
\sum_{u \in B(\tau)} (d_u - 1) = k - 1.
\]
We now construct the $k$-spine tree $S_k$ using i.i.d. coalescence depths in the CPP construction. More precisely, for each $N$ consider a probability distribution $(p_i, i \leq tN)$ on $\{1, \ldots, tN\}$. (As in the rest of the paper we make the dependency on $N$ implicit). Let $(H_i(N); i \geq 1)$ be a sequence of i.i.d. random variables with distribution $(p_i, i \leq tN)$. The $k$-spine tree $S_k$ is defined as the random ultrametric tree $S_k = H^{-1}(H_1^{(N)}, \ldots, H_{k-1}^{(N)})$. As in (6) introduce

$$\forall i < j, \quad H_{i,j}^{(N)} = H_{j,i}^{(N)} = \max\{H_i^{(N)}, H_{i+1}^{(N)}, \ldots, H_{j-1}^{(N)}\}.$$ 

Following the notation in [15] we will denote the distribution of this array by $Q_{k,N}$. Although $Q_{k,N}$ is formally defined as a measure on $\{0, \ldots, tN\}^{k \times k}$, by construction it corresponds to the matrix of pairwise distances between the leaves of the tree $S_k$. By abuse of notation, when convenient, we also think of $Q_{k,N}$ the law of $S_k$, that is as a measure on ultrametric trees.

**Remark 5.2.** The reader might be confused by the fact that $Q$ denotes the law of the size-biased BPVE tree, whereas $Q_{k,N}$ is the law of a tree $S_k$ with a fixed number $k$ of leaves. One could derive an extension of Lemma 5.1 that connects the law $P$ (biased by its $k$-th factorial moment) to that of a tree with $k$ distinguished lineages obtained by grafting independent subtrees on the spine tree $S_k$, see [15] Theorem 3]. In [15] the distribution of this larger tree is denoted by $\overline{Q}_{k,N}$, and $Q_{k,N}$ corresponds to the law of the tree spanned by the $k$ distinguished lineages. We follow this notation for easier comparison with [15], even though we do not need such a precise result here.

The many-to-few formula relates the factorial moments of a branching process to the distribution of $S_k$. It involves a random bias, that we denote by $\Delta_k$, which has the following
expression in the context of branching processes in varying environment,

\[ \Delta_k \equiv \Delta_k(S_k) = k! \mu^k_{tN} \cdot \prod_{u \in B(S_k)} \frac{1}{(p_{tN - |u|} \mu|u|)^{d_u - 1}} \frac{1}{d_u!} \int_{|u|+1}^{d_u} (1) \frac{d u}{f_{|u|+1}(1)} (1) \]

where \( f_{|u|+1}(1) \) denotes the \( d_u \)-th derivative of the generating function of the offspring distribution in generation \(|u|\). Note that we have included the \( k! \) term in the definition of \( \Delta_k \) compared to [15]. (As in the rest of the paper, we make the dependence of \( \Delta_k \) and \((p_i, i \leq tN)\) on \( N \) implicit, but keep it explicit for \( Q^{k,N} \) and \( H_i^{(N)} \) for later purpose.)

**Proposition 5.3 (Many-to-few).** For any \( k \geq 1 \) and any measurable map \( \phi: [0, \infty)^{k \times k} \rightarrow [0, \infty) \),

\[ E \left[ \sum_{u_1, \ldots, u_k \in T_{tN} \backslash \langle u_i \rangle \text{ distinct}} \phi((d_{tN}(u_i, u_j))_{i,j}) \right] = Q^{k,N} \left[ \Delta_k \cdot \phi((H_{i_{\sigma_i, \sigma_j}}^{(N)})_{i,j}) \right] \]

where, under \( Q^{k,N} \), \( \sigma \) is an independent permutation of \([k]\).

**Proof.** The result follows from the many-to-few formula derived in [15, Proposition 2] for branching processes with a general type space. We interpret time in a branching process with varying environment as a type, an individual \( u \) at generation \( n \) is endowed with type \( X_u = n \). The formula in [15] requires to find an appropriate harmonic function for the process. Since the process \((Z_n/\mu_n, n \geq 0)\) is a non-negative martingale, this readily implies that

\[ h: n \mapsto 1/\mu_n \]

is a harmonic function. The result now follows from [15, Proposition 2] by recalling that \( f_{n+1}(1) \) is the \( k \)-th factorial moment of the offspring distribution of an individual living at generation \( n \). It might help the reader to note that, in the current work, the height of the \( i \)-th branch point is encoded as its distance to the leaves \( H_i \), whereas in [15] it is encoded as the distance to the root \( W_i \), so that \( H_i = [tN] - W_i \). \( \square \)

### 6 Proof of the Kolmogorov estimate

We adapt the proof of Lyons, Pemantle and Peres in [44] for classical Galton–Watson processes to the case of a varying environment and make use of the 1-spine decomposition for BPVEs as introduced in Section 5.1. Our proof of the Kolmogorov estimate will also require an *a priori* uniform upper bound on the probability of survival. We derive it using the shape function \( \varphi_k(s) \) in generation \( k \) which, for \( s \in [0, 1) \), is defined via

\[ \frac{1}{1 - f_k(s)} = \frac{1}{(1 - s)f_k(1)} + \varphi_k(s). \]

Note that \( \varphi_k(s) \) can be extended continuously to \([0, 1]\) by setting

\[ \varphi_k(1) = \frac{f_k''(1)}{2f_k'(1)}. \]

For more details on the shape function we refer to [32].
Lemma 6.1. Let $t > 0$ be as in Theorem [7] and let $q_k = \mathbb{P}(Z_{Nt} = 0 \mid Z_k = 1)$ be the probability of extinction in generation $Nt$ when starting the process in generation $k$. Let $\varepsilon > 0$ be such that $\sigma^2(t) - \sigma^2(t - \varepsilon) > 0$. For all $k \leq (1 - \varepsilon)Nt$,

$$q_k \geq 1 - c\varepsilon/\sqrt{\kappa N},$$

for some constant $c\varepsilon > 0$.

Proof. By Equation (5) of [5] we get

$$\frac{1}{\mathbb{P}(Z_{Nt} > 0 \mid Z_k = 1)} = \frac{\mu_k}{\mu_N} + \sum_{j=k}^{Nt-1} \frac{\mu_k \varphi_{j+1}(q_{j+1})}{\mu_j}.$$ 

(16)

We aim to prove that $q_k \to 1$ with a uniform bound for all $k \leq (1 - \varepsilon)Nt$, which implies that it is sufficient to prove that the r.h.s. in (16) diverges uniformly to $\infty$ for all $k \leq (1 - \varepsilon)Nt$. We have the estimate

$$\sum_{j=(1-\varepsilon)Nt}^{Nt-1} \varphi_{j+1}(0) \geq \frac{1}{2} \sum_{j=(1-\varepsilon)Nt}^{Nt-1} \varphi_{j+1}(0),$$

by Lemma 1 in [32]. The r.h.s above can be bounded from below via

$$\sum_{j=(1-\varepsilon)Nt}^{Nt-1} \varphi_{j+1}(0) \geq c\sqrt{\kappa N}(\sigma^2(t) - \sigma^2(t - \varepsilon)),$$

by Lemma A.5. Therefore, recalling that by [1] $\mu_j$ is uniformly bounded, it follows for all $k \leq (1 - \varepsilon)Nt$ that

$$\frac{1}{1 - q_k} = \frac{1}{\mathbb{P}(Z_{Nt} > 0 \mid Z_k = 1)} \geq \sum_{j=(1-\varepsilon)Nt}^{Nt-1} \frac{\mu_k \varphi_j(0)}{\mu_j} \geq c' \sqrt{\kappa N}$$

which immediately gives

$$q_k \geq 1 - \frac{1}{c' \sqrt{\kappa N}}.$$

Proof of Theorem [7]. In order to ease the notation, we only prove the result for $t = 1$. The result for general $t$ follows by linear scaling. We follow the method of proof for estimating the survival probability of a critical Galton–Watson process designed in [44]. Recall the construction of the spinal measure $Q^*$ in Section 5.1. Let us denote by $A_N$ the event that the spine is the left-most individual at generation $N$. Recalling that the spine is uniformly distributed among individuals at generation $N$ we have

$$Q^*(Y \mathbb{1}_{A_N}) = \mathbb{Q} \left( \frac{Y}{Z_N} \right) = \frac{1}{\mu_N} \mathbb{P}(Y \mathbb{1}_{Z_N>0}),$$

27
for any random variable $Y$ measurable w.r.t. the first $N$ generations of $T$. In particular, taking $Y = Z_N$ yields

$$Q^*(Z_N \mid A_N) = \frac{Q^*(Z_N \mathbb{1}_{A_N})}{Q(1/Z_N)} = \frac{\mu_N}{\mathbb{P}(Z_N > 0)}.$$ 

We are now left with estimating the expectation on the left-hand side.

We need to introduce some notation. Let us denote by $R_{N,i}$ (resp. $L_{N,i}$) the number of individuals at generation $N$ whose most-recent common ancestor with the spine lives at generation $i-1$, and that are to the right (resp. left) of the spine. Let $Z_{N,i} = L_{N,i} + R_{N,i}$ be the total number of individuals that are descended from the offspring of the spine at generation $i$. We have that

$$A_N = \bigcap_{i=1}^{N} A_{N,i} := \bigcap_{i=1}^{N} \{Z_{N,i} = R_{N,i}\},$$

and the $(A_{N,i}; i)$ are independent events. Let us also denote by $R_i$ the total number of individuals living at generation $i$ born to the right of the spine, and $\xi_i$ the total number of children of the spine at generation $i$ (including the spine individual at generation $i$). Let $v_i$ be the index of the spine at generation $i$.

As a preliminary step, we notice that $R_{N,i}$ and $A_{N,i}$ are negatively correlated. Let $Z_j^{(N-i)}$ for each $j \geq 1$ be the size at generation $j$ of a BPVE started from a single individual in generation $i$ and stopped at generation $N$. We get

$$\mathbb{E}\left[R_{N,i}\mathbb{1}_{A_{N,i}}\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^{R_i} Z_j^{(N-i)} \xi_i^{j-R_i-1} \mid R_i, \xi_i^{1}\right]\right]$$

$$= \mathbb{E}\left[R_i \frac{\mu_N}{\mu_i} \xi_i^{j-R_i-1}\right] \leq \mathbb{E}\left[R_i \frac{\mu_N}{\mu_i}\right] \mathbb{E}\left[\xi_i^{j-R_i-1}\right] = \mathbb{E}\left[R_{N,i}\right] \mathbb{E}\left[\mathbb{1}_{A_{N,i}}\right],$$

(17)

where we applied Chebyshev’s sum inequality for the increasing function $k \mapsto k$ and the decreasing one $k \mapsto q_i^{-k}$. Now, let $R'_{N,i}$ be an independent r.v. such that $R'_{N,i}$ has the distribution of $R_{N,i}$ conditional on $A_{N,i}$. Let $A'_{N,i}$ denote the complement of $A_{N,i}$ and introduce

$$R'_{N,i} = R_{N,i} \mathbb{1}_{A_{N,i}} + R'_{N,i} \mathbb{1}_{A'_{N,i}}.$$ 

Clearly, $R_N^* := 1 + \sum_{i=1}^{N} R_{N,i}^*$ has the distribution of $Z_N$ under $Q^*(\cdot \mid A_N)$. The result will follow by first computing the expectation of $R_N := 1 + \sum_{i=1}^{N} R_{N,i}^*$ under $Q^*$, then proving that the $L^1$ distance of $R_N$ and $R_N^*$ goes to 0 as $N \to \infty$.

Let us start by computing the expectation of $R_N$. The expected number of children of the spine individual at generation $i-1$ that are not the spine individual at generation $i$ is $f''_i(1)/f'_i(1)$. Each of these children ends up to the right of the spine with probability 1/2, and grows an independent tree with expected size $\mu_N/\mu_i$. Therefore,

$$\frac{1}{\kappa_N} Q^*[R_N] = \frac{1}{\kappa_N} \left(1 + \sum_{i=1}^{N} \frac{f''_i(1)}{2f'_i(1)} \frac{\mu_N}{\mu_i}\right) \to e^{X_1} \frac{\mu_N}{2} \int_{[0,1]} e^{-s \sigma^2} ds,$$

as $N \to \infty$, where the convergence follows from Lemma A.4.
Now, let us compare \( R_N \) to \( R_N^* \). We first have

\[
\frac{1}{\kappa_N} \mathbb{Q}^* \left[ \frac{1}{\kappa_N} |R_N - R_N^*| \right] \leq \frac{1}{\kappa_N} \sum_{i=1}^{N} \mathbb{Q}^* [R_N,i \mathbb{1}_{A_N,i}] + \mathbb{Q}^*[R_N,i] \mathbb{Q}^*(\bar{A}_N,i).
\]

Recall that \( \mathbb{Q}^*[R^*_N,i] = \mathbb{Q}^*[R^*_N,i \mid A_N,i] \). Since \( R_N,i \) and \( A_N,i \) are negatively correlated (see (17)), we have \( \mathbb{Q}^*[R_N,i \mid A_N,i] \leq \mathbb{Q}^*[R_N,i] \). In turn, \( R_N,i \) and \( \bar{A}_N,i \) are positively correlated so that

\[
\mathbb{Q}^*[R_N,i] \mathbb{Q}^*(\bar{A}_N,i) = \mathbb{Q}^*[R_N,i] \mathbb{Q}^*(\bar{A}_N,i) \leq \mathbb{Q}^*[R_N,i \mathbb{1}_{A_N,i}].
\]

Next,

\[
\mathbb{Q}^* \left[ R_N,i \mathbb{1}_{A_N,i} \right] = \mathbb{Q}^* \left[ R_N,i \mathbb{1}_{A_N,i} \mid \xi^*_i, v_i \right] = \mathbb{Q}^* \left[ R_N,i \mathbb{1}_{A_N,i} \mid \xi^*_i, v_i \right] \mathbb{Q}^*(\bar{A}_N,i \mid \xi^*_i, v_i)
\]

\[
\leq \frac{\mu_N}{\mu_i} \mathbb{Q}^* \left[ r_i \mathbb{Q}^*(\bar{A}_N,i \mid \xi^*_i, v_i) \right] = \frac{\mu_N}{\mu_i} \mathbb{Q}^* \left[ (1 - q^*_i - r_i - 1) \right]
\]

\[
\leq \frac{\mu_N}{\mu_i} \mathbb{Q}^* [(\xi^*_i - 1)(1 - q^*_i - 1)].
\]

Gathering the previous estimates yields

\[
\frac{1}{\kappa_N} \mathbb{Q}^* \left[ |R_N - R_N^*| \right] \leq \frac{2}{\kappa_N} \sum_{i=1}^{N-1} \frac{\mu_N}{\mu_i} \mathbb{Q}^* [(\xi^*_i - 1)(1 - q^*_i - 1)].
\]

Now, for \( \varepsilon, \varepsilon' > 0 \), using Lemma 6.1 and Lemma A.4

\[
\frac{2}{\kappa_N} \sum_{i=1}^{N} \frac{\mu_N}{\mu_i} \mathbb{Q}^* [(\xi^*_i - 1)(1 - q^*_i - 1)]
\]

\[
\leq \frac{2\mu_N}{\kappa_N} \left( \sum_{i=1}^{(1-\varepsilon)N} \frac{1}{\mu_i} \mathbb{Q}^* [\xi^*_i \mathbb{1}_{\xi^*_i \geq \varepsilon' \sqrt{\kappa_N}}] + \sum_{i=1}^{(1-\varepsilon)N} \frac{1}{\mu_i} \mathbb{Q}^* [\xi^*_i - 1](1 - q^*_i - \varepsilon' \sqrt{\kappa_N}) + \sum_{i=(1-\varepsilon)N}^{N} \frac{1}{\mu_i} \mathbb{Q}^* [\xi^*_i - 1] \right)
\]

\[
\leq \frac{2\mu_N}{\kappa_N} \left( \sum_{i=1}^{N} \frac{1}{\mu_i f_i(1)} E[\xi^*_i] \right) + \sum_{i=1}^{N} \frac{1}{\mu_i f_i(1)} \left( 1 - (1 - \frac{\varepsilon}{\sqrt{\kappa_N}}) \right) + \sum_{i=(1-\varepsilon)N}^{N} \frac{f_i(1)}{\mu_i f_i(1)} \rightarrow C(1 - e^{\varepsilon' \varepsilon}) + C(\mu(1 - \mu(1 - \varepsilon)), \text{ as } N \to \infty,
\]

where \( \mu(t) = \int_{[0,1]} e^{-Xs} \sigma^2(ds) \) and \( C \) is a constant independent of \( \varepsilon, \varepsilon' \). Since this limit can be made arbitrarily small by letting \( \varepsilon', \varepsilon \to 0 \), we obtain the desired bound on the \( L^1 \) distance, hence

\[
P( Z_N > 0 ) = \frac{\mu_N}{\mathbb{Q}^*[Z_N \mid A_N]} \sim \frac{\mu_N}{\kappa_N \mathbb{Q}^*[R_N]} \sim \frac{2}{\kappa_N} \int_{[0,1]} e^{-Xs} \sigma^2(ds).
\]

\[\square\]

Remark 6.2. A careful reading of the argument shows that the strong Lindeberg condition \( [3] \) is needed because of our crude estimate for the extinction probability of Lemma 6.1. If we could replace \( \sqrt{\kappa_N} \) by \( \kappa_N \) in the lemma, our Kolmogorov estimate would hold under the weaker condition \( [3] \).
7 Convergence of the BPVE

This last section is dedicated to the proof of Theorem 2. Recall the encoding of generation $tN$ of the BPVE as a random metric measure space $[T_{tN}, d_{tN}, \lambda_{tN}]$ from Section 7.

As alluded to in the introduction, we follow the general proof strategy proposed in an earlier work by two of the authors [15]. We start by proving that the genealogy of the BPVE converges in the Gromov-weak topology. According to Proposition 3.1, proving convergence in distribution for the Gromov-weak topology amounts to computing the limit of the moments of the random metric measure space $[T_{tN}, d_{tN}, \lambda_{tN}]$. Using the many-to-few formula, see Proposition 5.3, these moments can be expressed in terms of the distribution of a finite tree, the $k$-spine tree [27, 26, 15]. This $k$-spine tree is studied in Section 7.1 where we derive the limit of the moments of the BPVE. A small caveat to this approach is that, for the moments to be well-defined, we need to make a truncation of the reproduction laws, which is carried out in Section 7.2. We relate the truncated and the untruncated versions via Corollary 3.4. Once the convergence is established in the Gromov-weak topology, we reinforce it to a convergence in the GHP topology by using Proposition 3.6.

7.1 Moments of the BPVE

Our assumptions ensure that the BPVE has a finite second moment, but moments of higher order are in general not finite. Thus, the method of moments cannot be applied directly. However, we will show in Section 7.2 that, up to making a cutoff in the offspring size, we can safely assume that there exists a sequence $\beta_N = o(\kappa_N)$ such that

$$\forall i \geq 1, k \geq 2, f_i^{(k)}(1) \leq \beta_N^{k-2} f_i''(1),$$

where $f_i^{(k)}$ denotes the $k$-th derivative of $f$. Under this assumption, the moments of a nearly critical BPVE can be computed.

**Proposition 7.1.** Consider a BPVE satisfying (1) and (2), and a fixed $t > 0$ such that $t$ is a continuity point of $\sigma^2$ and $X$, and $\sigma^2$ is strictly increasing in $t$. If the moment growth assumption (18) is fulfilled, then for any bounded continuous $\phi$,

$$\frac{1}{\kappa_N^{-1}} \mathbb{E}\left[ \sum_{u_1, \ldots, u_k \in T_{tN}} \phi\left(\frac{d_{jN}(u_i, u_j)}{N}\right)_{i,j} \right] \to e^{X_t \beta^{k-1} k!} \mathbb{E}\left[ \phi(\sigma_i, \sigma_j)_{i,j} \right],$$

where $\sigma$ is an independent and uniform permutation of $[k]$ and with $((H_{i,j})_{i,j})$ as in (6).

**Proof.** Let $|B_{u_1, \ldots, u_k}|$ denote the number of branch points of the subtree spanned by $(u_1, \ldots, u_k)$. We decompose the expectation with respect to the value of $|B_{u_1, \ldots, u_k}|$.

$$\mathbb{E}\left[ \sum_{u_1, \ldots, u_k \in T_{tN}} \phi\left(\frac{d_{jN}(u_i, u_j)}{N}\right)_{i,j} \right] = \sum_{b=1}^{k-1} \mathbb{E}\left[ \sum_{u_1, \ldots, u_k \in T_{tN}} \mathbb{1}_{|B_{u_1, \ldots, u_k}|=b} \phi\left(\frac{d_{jN}(u_i, u_j)}{N}\right)_{i,j} \right].$$

There are two steps in the proof. First, we prove that, due to our assumption on the moments [15], the contributions of the terms with $b < k - 1$ vanish. Secondly, we compute the limit of the term with only binary branch points and show that we recover the moments of the environmental CPP.
Step 1. Fix $b < k - 1$. We apply the many-to-few formula with branch lengths $H_{i}^{(N)}$ uniformly distributed on $\{1, \ldots, tN\}$. Let $Q^{k,Nt}$ be the corresponding $k$-spine tree distribution. By Proposition 5.3 we have

$$E \left[ \sum_{u_{1}, \ldots, u_{k} \in T_{N} \atop (u_{i}) \text{ distinct}} \mathbb{1}_{B} = b \phi \left( \left( \frac{d_{N}(u_{i}, u_{j})}{N} \right)_{i,j} \right) \right] = Q^{k,Nt} \left[ \Delta_{k} \mathbb{1}_{B} = b \phi \left( \left( \frac{H_{i}^{(N)}}{N} \right)_{i,j} \right) \right],$$

where $|B|$ denotes the number of branch points in the $k$-spine tree. Recall the expression of $\Delta_{k}$ from (15). Using assumption (18) we have

$$\Delta_{k} = (tN)^{k-1}k! \frac{1}{\kappa_{N}} \mu_{N} \prod_{u \in B} f_{|u|}^{(d_{u})}(1) \leq (tN)^{k-1}C \prod_{u \in B} \beta_{k,N}^{d_{u}} f_{|u|+1}^{(d_{u})}(1),$$

for some constant $C$ that depends on the uniform bounds on $\mu_{i}$ and $f_{i}^{(d_{u})}$ from Lemma A.3. Using that $\sum_{u \in B} d_{u} = 1 = k - 1$ leads to

$$\frac{1}{\kappa_{N}} Q^{k,Nt} \left[ \Delta_{k} \mathbb{1}_{B} = b \phi \left( \left( H_{i}^{(N)} \right)_{i,j} \right) \right] \leq \left( \frac{N_{t}}{\kappa_{N}} \right)^{k-1} C \| \phi \|_{\infty} Q^{k,Nt} \left[ \beta_{k,N}^{k-1+b} \prod_{u \in B} f_{|u|+1}^{(d_{u})}(1) \mathbb{1}_{B} = b \right]$$

$$= \beta_{k,N}^{k-1+b} \frac{1}{\kappa_{N}} O \left( \left( \sum_{j=1}^{N_{t}} f_{j}^{(1)}(1) \right)^{b} \right)$$

$$= \left( \frac{\beta_{N}}{\kappa_{N}} \right)^{k-1+b} O \left( \left( \sum_{j=1}^{N_{t}} f_{j}^{(1)}(1) \right)^{b} \right)$$

$$\to 0, \quad \text{as } N \to \infty,$$

where we have used Lemma A.1 in the second line, assumption (2) and that $\beta_{N} = o(\kappa_{N})$ in the last line.

Step 2. To compute the contribution of binary branch points, we also apply the many-to-few formula, but with a different branch length distribution. We assume that $H_{j}^{(N)}$ are independent and identically distributed with

$$\forall 1 \leq i \leq tN, \quad \bar{\rho}_{i} = \mathbb{P}(H^{(N)} = i) := \frac{1}{\bar{\rho}^{(N)}_{i}} \frac{1}{2\kappa_{N}} \frac{1}{\mu_{N}} \frac{f_{i}^{(N)}(1)}{f_{N-i+1}(1)^{2}},$$

where $\bar{\rho}^{(N)}_{i}$ is the renormalization constant making $(\bar{\rho}_{i}, i \leq tN)$ a probability measure:

$$\bar{\rho}^{(N)}_{i} = \frac{\mu_{N}}{2\kappa_{N}} \sum_{j=0}^{tN-1} \frac{1}{\mu_{j+1}} f_{j+1}^{(1)}.$$

To distinguish the distribution of this $k$-spine tree from that derived with uniform branch lengths, let us denote it by $\bar{Q}^{k,Nt}$. We obtain by Proposition 5.3 that

$$E \left[ \sum_{u_{1}, \ldots, u_{k} \in T_{N} \atop (u_{i}) \text{ distinct}} \mathbb{1}_{B} = k-1 \phi \left( \left( \frac{d_{N}(u_{i}, u_{j})}{N} \right)_{i,j} \right) \right] = \bar{Q}^{k,Nt} \left[ \Delta_{k} \mathbb{1}_{B} = k-1 \phi \left( \left( \frac{H_{i}^{(N)}}{N} \right)_{i,j} \right) \right],$$

31
where $\Delta_k$ now takes the simple expression

$$
\Delta_k = \kappa_{N_k}^{k-1}k! \mu_{tN}(\tilde{\rho}^{(N)}_t)^{k-1}.
$$

By Lemma A.4, the distribution $H^{(N)}_{k,N} / N$ under $\bar{Q}^{k,N_t}$ converges to the law with cumulative distribution function $F$ defined in (5). We want to derive the almost sure limit of $\Delta_k$ under $\bar{Q}^{k,N_t}$ and use the bounded convergence theorem to conclude the proof. Since $t$ is a continuity point of $(X_s, s \geq 0)$, by Lemma A.4,

$$
\mu_{tN} \to e^{X_t} \quad \text{and} \quad \tilde{\rho}^{(N)}_t \to \bar{\rho}_t \quad \text{as} \quad N \to \infty.
$$

Finally, by the first part of the proof,

$$
\bar{Q}^{k,N_t} \mathbb{1}_{|B|<k-1} = O \left( \frac{1}{\kappa_{N}} \bar{Q}^{k,N_t} \left[ \Delta_k \mathbb{1}_{|B|<k-1} \right] \right) = O \left( \frac{1}{\kappa_{N}} \mathbb{E} \left[ \sum_{u_1, \ldots, u_k \in T_N (u_i) \text{ distinct}} \mathbb{1}_{|B|<k-1} \right] \right) \to 0.
$$

Altogether this shows that

$$
\frac{1}{\kappa_{N}} \Delta_k \mathbb{1}_{|B|=k-1} \to k! e^{X_t} \bar{\rho}_t^{k-1},
$$

and the bounded convergence theorem yields

$$
\frac{1}{\kappa_{N}} \bar{Q}^{k,N_t} \mathbb{1}_{|B|=k-1} \phi \left( \frac{H^{(N)}_{i,j}}{N} \right) \to k! e^{X_t} \bar{\rho}_t^{k-1} \mathbb{E} \left[ \phi \left( (H_{\sigma_i,\sigma_j})_{i,j} \right) \right],
$$

which ends the proof.

7.2 Truncating the offspring distribution

Let $(\beta_N)_N$ be a sequence of natural numbers. Let us define the truncated offspring size $\tilde{\xi}_i := \xi_i \wedge \beta_N$, where $\xi_i$ denotes a generic copy of the offspring distribution in generation $i$, hence has distribution $f_i$. In the same manner we define $(\tilde{Z}_i, i \geq 0)$ constructed from the truncated variables $(\tilde{\xi}_{i,j}, i \geq 1, j \geq 1)$. In this section, we will need the sequence of natural numbers $(\beta_N, N \geq 1)$ to satisfy $\beta_N = o(\kappa_N)$,

$$
\sum_{j=1}^{N_t} \mathbb{E}[\xi_j \mathbb{1}_{\xi_j > \beta_N}] \to 0 \quad \text{and} \quad \frac{1}{\kappa_N} \sum_{j=1}^{N_t} \mathbb{E}[\xi_j^2 \mathbb{1}_{\xi_j > \beta_N}] \to 0 \quad \text{as} \quad N \to \infty.
$$

We know, thanks to Lemma A.2, that such a sequence exists provided assumption (3) holds. The next result shows that the truncated BPVE satisfies all the required properties to apply the method of moments.

**Lemma 7.2.** Consider a BPVE satisfying assumptions (1), (2) and (3). Consider a sequence $(\beta_N, N \geq 1)$ such that (19) holds and the corresponding truncated r.v. $\tilde{\xi}_j = \xi_j \wedge \beta_N$, with distribution $f_i$. Then
(i) as $N \to \infty$,
\[
\left( \prod_{j=1}^{sN} \tilde{f}_j'(1), s \geq 0 \right) \to (e^{X_s}, s \geq 0),
\]
in the usual Skorohod sense.

(ii) for all $t > 0$ such that $\sigma^2$ is continuous at $t$
\[
\frac{1}{sN} \sum_{j=1}^{tN} \tilde{f}_j''(1) \to \sigma^2(t), \quad \text{as } N \to \infty.
\]

The sequence of truncated environments also fulfills that
\[
\mathbb{E} \left[ \tilde{\xi}^{(k)} \right] \leq \frac{1}{sN} f_j''(1), \quad (20)
\]
for all $k \geq 2$, where $\mathbb{E} \left[ X^{(k)} \right] = \mathbb{E} [X(X-1) \cdots (X-k+1)]$ denotes the $k$-th factorial moment of a random variable $X$.

Proof. To show that (20) holds, note that
\[
\mathbb{E} \left[ \tilde{\xi}^{(k)} \right] = \sum_{i=k}^{\beta_N} i(i-1) \cdots (i-k+1) f_j[k] + \beta_N^{(k)} \mathbb{P} (\xi_j > \beta_N)
\]
\[
\leq \beta_N^{k-2} \left( \sum_{i=k}^{\beta_N} i(i-1) f_j[k] + \beta_N^2 \mathbb{P} (\xi_j > \beta_N) \right) \leq \beta_N^{k-2} f_j''(1).
\]

We now prove (i). By the continuous mapping theorem it is sufficient to prove
\[
\sum_{j=1}^{sN} \log \tilde{f}_j'(1) \to X_s
\]
for the Skorohod topology. Therefore, consider
\[
\sum_{j=1}^{sN} \log \tilde{f}_j'(1) = \sum_{j=1}^{sN} \log \frac{\tilde{f}_j'(1)}{f_j'(1)} + \sum_{j=1}^{sN} \log f_j'(1). \quad (21)
\]

By assumption (1) the second term in (21) converges to $(X_s, s \geq 0)$. Therefore, we only need to prove that the first term converges to 0 uniformly. Note that $\tilde{f}_j'(1) \leq f_j'(1)$, hence
\[
\left| \sum_{j=1}^{sN} \log \frac{\tilde{f}_j'(1)}{f_j'(1)} \right| = \sum_{j=1}^{sN} \log \frac{f_j'(1)}{\tilde{f}_j'(1)} \leq \sum_{j=1}^{sN} \frac{f_j'(1)}{\tilde{f}_j'(1)} - 1
\]
\[
= \sum_{j=1}^{sN} \frac{f_j'(1) - \tilde{f}_j'(1)}{f_j'(1)}.
\]
By [19] we have
\[
\sum_{j=1}^{sN} (f_j'(1) - \tilde{f}_j'(1)) \leq \sum_{j=1}^{sN} E[\xi_j 1_{\xi_j > \beta_N}] \rightarrow 0, \quad \text{as } N \rightarrow \infty. \tag{22}
\]

Point (i) is thus proved if we can lower bound $\tilde{f}_j'(1)$ uniformly in $N$ and $j \leq sN$. Write
\[
\tilde{f}_j'(1) = f_j'(1) - (f_j'(1) - \tilde{f}_j'(1)) \geq f_j'(1) - \sum_{i=1}^{sN} (f_i'(1) - \tilde{f}_i'(1)).
\]
The second term is going to 0 by (22), whereas $f_j'(1)$ is uniformly bounded away from 0 by Lemma A.3, hence we obtain the required bound.

For point (ii), notice
\[
\frac{1}{\kappa_N} \sum_{j=1}^{sN} \tilde{f}_j''(1) = \frac{1}{\kappa_N} \sum_{j=1}^{sN} f_j''(1) + \frac{1}{\kappa_N} \sum_{j=1}^{sN} (\tilde{f}_j''(1) - f_j''(1)),
\]
and by assumption (2), we get the right limit. \qed

**Lemma 7.3.** Consider a sequence of BPVEs verifying (1), and a sequence $(\beta_N, N \geq 1)$ fulfilling (19). Let $\tilde{Z}_{tN}$ be the size of the BPVE at time $tN$ whose offspring law is truncated by $\beta_N$. Then
\[
\mathbb{P}(Z_{tN} - \tilde{Z}_{tN} > \varepsilon \kappa_N | Z_{tN} > 0) \rightarrow 0 \quad \text{as } N \rightarrow \infty.
\]

**Proof.** By Lemma 7.2
\[
\mathbb{E}[Z_{tN} - \tilde{Z}_{tN}] = \prod_{j=1}^{tN} f_j'(1) - \prod_{j=1}^{tN} \tilde{f}_j'(1) \rightarrow 0 \quad \text{as } N \rightarrow \infty.
\]
We obtain
\[
\mathbb{P}(Z_{tN} - \tilde{Z}_{tN} > \varepsilon \kappa_N | Z_{tN} > 0) \leq \frac{\mathbb{E}[Z_{tN} - \tilde{Z}_{tN}]}{\varepsilon \kappa_N \mathbb{P}(Z_{tN} > 0)}.
\]
Due to Theorem 1, the denominator is converging to some constant and the result follows. \qed

### 7.3 Proof of the Gromov-weak convergence

We now prove that the rescaled genealogy converges in the Gromov-weak topology. Recall that $[T_{tN}, d_{tN}, \lambda_{tN}]$ is the random metric measure space representing the genealogy of the population at generation $tN$.

**Theorem 3.** Fix $t > 0$. Under the assumptions of Theorem [1] the following convergence holds in the Gromov-weak topology
\[
\lim_{N \rightarrow \infty} [T_{tN}, d_{tN}, \lambda_{tN}] = [(0, Z_e), d_{\nu}, \text{Leb}],
\]
with $[(0, Z_e), d_{\nu}, \text{Leb}]$ the environmental CPP as defined in (13).

34
Proof. By Lemma 7.3 there exists a sequence $\beta_N = o(\kappa_N)$ fulfilling (18). Consider the corresponding truncated BPVE $(\tilde{Z}_n, n \geq 1)$, and the corresponding metric measure space $[\tilde{T}_{iN}, \tilde{d}_{iN}, \tilde{\lambda}_{iN}]$ giving the genealogy at time $tN$. By Lemma 7.3 we have that for any $\varepsilon > 0$,

$$\mathbb{P}(Z_{tN} - \tilde{Z}_{iN} \geq \varepsilon \kappa_N \mid Z_{tN} > 0) \to 0 \quad \text{as } N \to \infty.$$ 

Applying Corollary 3.4 to $[T_{iN}, \frac{d_{iN}}{\kappa_N}, \frac{\lambda_{iN}}{\kappa_N}]$ and $[\tilde{T}_{iN}, \frac{\tilde{d}_{iN}}{\kappa_N}, \frac{\tilde{\lambda}_{iN}}{\kappa_N}]$ under the measure $\mathbb{P}(\cdot \mid Z_{tN} > 0)$ shows that the result is proved if we can prove that $[\tilde{T}_{iN}, \frac{\tilde{d}_{iN}}{\kappa_N}, \frac{\tilde{\lambda}_{iN}}{\kappa_N}]$ converges to the environmental CPP. Up to replacing $T_{iN}$ by $\tilde{T}_{iN}$, we can therefore assume that the environment fulfills (18).

By Proposition 3.4 we need to show the following convergence for any polynomial $\Phi$,

$$\lim_{N \to \infty} \mathbb{E} \left[ \Phi(T_{iN}, \frac{d_{iN}}{\kappa_N}, \frac{\lambda_{iN}}{\kappa_N}) \right] = \mathbb{E} \left[ \Phi((0, Z_e, d_{\nu_e}, \text{Leb}) \right]. \quad (23)$$

Note that due to Proposition 4.3 the right hand side of (23) translates to

$$\mathbb{E} \left[ \Phi((0, Z_e, d_{\nu_e}, \text{Leb}) \right] = k! \rho^k \mathbb{E} \left[ \phi((H_{\sigma}, \sigma), i,j) \right],$$

where $(H_{\sigma}, \sigma)$ are given by (6) and $\sigma: [k] \to [k]$ is an independent random permutation of $[k]$. Therefore, we show that the left hand side in (23) converges to the above limit.

Using the many-to-few formula, we have shown in Proposition 7.1 that

$$\mathbb{E} \left[ \Phi(T_{iN}, \frac{d_{iN}}{\kappa_N}, \frac{\lambda_{iN}}{\kappa_N}) \mid Z_{tN} > 0 \right] = \frac{1}{\kappa_N \mathbb{P}(Z_{tN} > 0)} \frac{1}{\kappa_N^{k-1}} \mathbb{E} \left[ \sum_{(u_1, \ldots, u_k) \in T_{iN}} \phi((\frac{d_{iN}(u_i, u_j)}{\kappa_N})_{i,j}) \right]$$

$$\sim \frac{1}{\kappa_N \mathbb{P}(Z_{tN} > 0)} \frac{1}{\kappa_N^{k-1}} \mathbb{E} \left[ \sum_{(u_1, \ldots, u_k) \in T_{iN}} \phi((\frac{d_{iN}(u_i, u_j)}{\kappa_N})_{i,j}) \right]$$

$$\sim \tilde{\rho} e^{-X_t} \cdot e^{X_t} \cdot \rho^k \mathbb{E} \left[ \phi((H_{\sigma}, \sigma), i,j) \right],$$

where we applied Theorem 4 to obtain the asymptotics of the survival probability. We recover the moments of the environmental CPP and this completes the proof.

\[\square\]

### 7.4 Completing the proof of Theorem 2

Before completing the proof of the main result, we need the following consequence of the Gromov-weak convergence. Fix $t > 0$ and recall the notation $Z_{i,tN}$ for the number of individuals at generation $i$ having descendants at generation $tN$. For $j \leq Z_{i,tN}$, let us further denote by $[T_{i,N}]_j$ the number of descendants of the $j$-th individual at generation $i$ whose offspring survives until time $tN$.

**Corollary 7.4.** Fix $s < t$. Then conditional on survival at time $tN$,

$$\left( \frac{|T_{i,N}(1)|}{\kappa_N}, \ldots, \frac{|T_{i,N}(Z_{i,N})|}{\kappa_N} \right) \to (E_1, \ldots, E_K), \quad \text{as } N \to \infty$$

in distribution, where $(E_j, j \geq 1)$ are i.i.d. exponential random variables with mean

$$\frac{1}{2} \int_{[s,t]} e^{X_t-X_u} \sigma^2(du),$$

35
and $K$ is an independent geometric random variable on $\mathbb{N}$ with success parameter $\frac{1}{1+c}$ with
\[ c = \frac{\int_{[0,1]} e^{-X_u\sigma^2}(du)}{\int_{[s,t]} e^{-X_u\sigma^2}(du)} . \] (24)

Proof. By Theorem 3 conditional on survival at time $sN$, the population size, rescaled by $\kappa_N$, converges to an exponential distribution with mean $\bar{\rho}_s$. By the branching property, each of these individuals has a probability $P(Z_{sN} > 0 \mid Z_{sN} = 1)$ of leaving some descendants at time $tN$. Using Theorem 1, since by assumption (1), for $u > s$,
\[ \mathbb{E}[Z_{uN} \mid Z_{sN} = 1] \to e^{X_u - X_s}, \]
we have
\[ P(Z_{tN} > 0 \mid Z_{sN} = 1) \sim \frac{2}{\kappa_N \int_{[s,t]} e^{-X_u\sigma^2}(du)} . \]
These two facts readily imply that, conditional on survival at time $sN$, $Z_{sN,tN} \to K'$ in distribution with
\[ \forall k \geq 0, \quad P(K' = k) = \frac{1}{1+c} \left( \frac{c}{1+c} \right)^k \]
with $c$ the constant defined in (24). By the branching property, the number of descendants of each of these $Z_{sN,tN}$ individuals is distributed as $Z_{tN}$ conditional on $\{Z_{sN} = 1, Z_{tN} > 0\}$. Thus, another application of Theorem 3 shows that
\[ \mathbb{E}[\phi(Z_{tN}) \mid Z_{tN} > 0, Z_{sN} = 1] \to \mathbb{E}[\phi(E_1)], \quad \text{as } N \to \infty. \]
The result is proved by noting that further conditioning on $\{Z_{tN} > 0\}$ only amounts to conditioning on $\{K' > 0\}$, which is a geometric random variable shifted by 1, as claimed.

Proof of Theorem 2. We apply Proposition 3.6. We only need to check that (11) holds for the sequence of random metric measure spaces $(\{T_{sN}, d_{sN}, \nu_{sN}\})_{N}$. In the context of the BPVE, the balls of radius $t - s$, with $s < t$, correspond to the ancestral lineages at time $s$. That is, each ball corresponds to one of the $Z_{sN,tN}$ individuals at time $sN$ having descendants at $tN$. Moreover, the mass of a ball is the number of descendants of the corresponding ancestor, rescaled by $\kappa_N$. In the notation of Corollary 7.4, this corresponds to the random variables $(\frac{T_{sN}(i)}{\kappa_N}, i \leq Z_{sN,tN})$. By Corollary 7.4, the number of such balls converges to a geometric random variable, and the mass of the balls converge to i.i.d. exponential random variables. This readily shows that condition (11) is fulfilled, proving the result.

Proof of Corollary 2.3. Both maps
\[ [X, d, \nu] \mapsto \nu(X), \quad [X, d, \nu] \mapsto [X, d, \frac{\nu}{\nu(X)}] \]
are continuous w.r.t. the Gromov-weak topology. Since the BPVE converges in the GHP topology by Theorem 2 this readily shows (i), and since for any functional $\phi$, the expectation $\mathbb{E}[\phi(\frac{d_{N}(U_i, U_j)}{N})]$ is nothing but the moment of order $k$ of $[0, Z_c, d_{N}, Z_c^{-1} \text{Leb}]$ associated to $\phi$, Proposition 4.4 proves (iii). The finite-dimensional part of point (ii) follows from Proposition 3.5. For the Skorohod part, note that if $\sigma^2$ is continuous, the reduced process associated to the corresponding environmental CPP only makes jump of size 1 almost surely, and the result follows again from Proposition 3.5.
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A Appendix

Lemma A.1. For each $N$ let $g_N : \{0, \ldots, N-1\} \to [0, \infty)$ and let $S_k$ be the uniform ultrametric tree with $k$ leaves. For any $k \geq 1$,

$$
\mathbb{E} \left[ \prod_{u \in B(S_k)} g_N(|u|) \mathbb{I}_{|B(S_k)|=b} \right] = O_N \left( \frac{1}{N^{k-1}} \left( \sum_{i=0}^{N-1} g_N(i) \right)^b \right).
$$

Proof. We prove the result by induction on $k$. For $k = 1$ there is no branch point and the claim holds. By the CPP construction, a uniform ultrametric tree $S_{k+1}$ with $k+1$ leaves is obtained by grafting a branch $k+1$ with uniform length to the right of a uniform ultrametric tree $S_k$ with $k$ leaves. There are two options for $S_{k+1}$ to have $b$ branch points. Either $S_k$ had $b-1$ branch points and a new branch point is created. Or $S_k$ has $b$ branch points, and no new branch point is created. This requires that $H_k^{(N)}$ takes the same value as one of the variables $(H_1^{(N)}, \ldots, H_{k-1}^{(N)})$, which occurs with probability at most $(k-1)/N$. Therefore,

$$
\mathbb{E} \left[ \prod_{u \in B(S_{k+1})} g_N(|u|) \mathbb{I}_{|B(S_{k+1})|=b} \mid S_k \right] 
\leq \frac{1}{N} \sum_{i=0}^{N-1} g_N(i) \cdot \prod_{u \in B(S_k)} g_N(|u|) \mathbb{I}_{|B(S_k)|=b-1} + \frac{k-1}{N} \prod_{u \in B(S_k)} g_N(|u|) \mathbb{I}_{|B(S_k)|=b},
$$

and the result follows. \hfill \Box

Lemma A.2. Assume that the environment satisfies (3). Then for any $t > 0$ we can find $\beta_N = o(\kappa_N)$ such that

$$
\frac{1}{\kappa_N} \sum_{j=1}^{N_t} \mathbb{E}[\xi_j^2 \mathbb{I}_{\xi_j > \beta_N}] \to 0, \quad \sum_{j=1}^{N_t} \mathbb{E}[\xi_j \mathbb{I}_{\xi_j > \beta_N}] \to 0, \quad \text{as } N \to \infty. \quad (25)
$$

Proof. We simply adapt the argument of [26 Lemma 22] to our case. By assumption (3'), for any fixed $\varepsilon$

$$
\sum_{j=1}^{tN} \mathbb{E}[\xi_j \mathbb{I}_{\xi_j > \varepsilon \kappa_N}] \leq \frac{1}{\varepsilon \kappa_N} \sum_{j=1}^{tN} \mathbb{E}[\xi_j^2 \mathbb{I}_{\xi_j > \varepsilon \kappa_N}] \to 0, \quad \text{as } N \to \infty.
$$

Now take $\varepsilon_k \to 0$ and chose recursively $N_{k+1} > N_k$ such that

$$
\forall k \geq 1, \forall N \geq N_k, \quad \sum_{j=1}^{tN} \mathbb{E}[\xi_j \mathbb{I}_{\xi_j > \varepsilon_k \kappa_N}] < \varepsilon_k, \quad \text{and} \quad \frac{1}{\kappa_N} \sum_{j=1}^{N_t} \mathbb{E}[\xi_j^2 \mathbb{I}_{\xi_j > \varepsilon_k \kappa_N}] < \varepsilon_k.
$$

Set $\beta_N = \varepsilon_k \kappa_N$ for $N_k \leq N < N_{k+1}$. Clearly $\beta_N = o(\kappa_N)$ and (25) is fulfilled. \hfill \Box
Lemma A.3. Suppose that (1) holds. For any \( t > 0 \) there exist two constants \( c, C > 0 \) depending only on \( t \) such that
\[
\forall N \geq 1, \forall i \leq tN, \quad c \leq f_i'(1) \leq C.
\]
Moreover, for each continuity point \( s \geq 0 \) of \( (X_u, u \geq 0) \) and sequence \( s_N \rightarrow s \),
\[
f_{s_N}(1) \rightarrow 1 \quad \text{as } N \rightarrow \infty.
\]
Proof. We have
\[
\forall t \geq 0, \quad f_{t'}(1) = \frac{\mu_{t'}}{\mu_{t'-1}} = \exp \left( \log \mu_{t'} - \log \mu_{t'-1} \right),
\]
with the convention that \( \mu_{t'-1} = 1 \). By assumption (1), \( (\log \mu_{t'}, t \geq 0) \) converges in the Skorohod topology to \( (X_t, t \geq 0) \). Therefore, at every continuity point \( s \) of \( (X_t, t \geq 0) \), and sequence \( s_N \rightarrow s \), leading to the second statement. Moreover, since \( (\log \mu_{t'}, t \geq 0) \) converges in the Skorohod topology, \( (|\log \mu_{t'}|, t \geq 0) \) is uniformly bounded in \( N \) on compact intervals. Taking the exponential leads to the first part of the claim.

Lemma A.4. For any continuity point \( t > 0 \) of \( \sigma^2 \), and any bounded continuous map \( \phi \), under assumptions (1) and (2),
\[
\frac{1}{\kappa_N} \sum_{i=0}^{tN} \phi\left( \frac{i}{N} \right) f_i''(1) \frac{1}{f_{i+1}(1) \mu_{i+1}} \rightarrow \int_{[0,t]} \phi(s) e^{-X_s} \sigma^2(ds), \quad \text{as } N \rightarrow \infty.
\]
Proof. Let \( X_N \) be distributed as
\[
\forall i \leq tN, \quad P(X_N = i) = \frac{f_i''(1)}{\sum_{j=1}^{tN} f_j''(1)}.
\]
The integral on the left-hand side can be expressed as
\[
\left( \frac{1}{\kappa_N} \sum_{i=1}^{tN} f_i''(1) \right) \mathbb{E}\left[ \frac{\phi\left( \frac{X_N}{N} \right)}{f_{X_N+1}(1) \mu_{X_N+1}} \right].
\]
By assumption (2), the distribution of \( X_N/N \) converges to the distribution with distribution function \( s \mapsto \sigma^2(s)/\sigma^2(t) \) and the first term in the above expression converges to \( \sigma^2(t) \). By Lemma A.3, for any continuity point \( s \) of \( (X_t, t \geq 0) \) and sequence \( s_N \rightarrow s \), \( f_{s_N}(1) \rightarrow 1 \) and by (1) we have \( \mu_{s_N} \rightarrow e^{X_s} \). Therefore using for instance [31, Theorem 4.27] proves that
\[
\left( \frac{1}{\kappa_N} \sum_{i=1}^{tN} f_i''(1) \right) \mathbb{E}\left[ \frac{\phi\left( \frac{X_N}{N} \right)}{f_{X_N+1}(1) \mu_{X_N+1}} \right] \rightarrow \sigma^2(t) \int_{[0,t]} \phi(x) e^{-X_x} \frac{\sigma^2(ds)}{\sigma^2(t)}, \quad \text{as } N \rightarrow \infty,
\]
yielding the claim.

Lemma A.5. Under the assumptions (1), (2) and (3), for any fixed \( t > 0 \), there exists some constant \( c > 0 \) such that
\[
\forall s \leq t, \quad \sum_{j=Ns}^{Nt} \varphi_j(0) \geq c \sqrt{\kappa_N} (\sigma^2(t) - \sigma^2(s)),
\]
where \( \varphi_j \) is the shape function of \( f_j \).
Proof. Recall the definition of the shape function
\[ \varphi_j(0) = \frac{1}{1 - f_j(0)} - \frac{1}{f_j'(1)} = \sum_{k=2}^{\infty} f_j[k](k - 1) \frac{1}{\mathbb{P}(\xi_j > 0)} f_j'(1). \]

Fix some \( \varepsilon > 0 \), we have
\[ \sigma^2(t) - \sigma^2(s) = \lim_{N \to \infty} \frac{1}{\kappa_N} \sum_{j=Ns}^{Nt} f_j''(1) \]
\[ = \lim_{N \to \infty} \frac{1}{\kappa_N} \sum_{j=Ns}^{Nt} \left( \sum_{k=2}^{\varepsilon/\sqrt{\kappa_N}} f_j[k](k - 1) + \sum_{k=\varepsilon/\sqrt{\kappa_N} + 1}^{\infty} f_j[k](k(k-1)) \right). \quad (26) \]

By assumption (\[\ref{assumption3}\]) the second term vanishes. Additionally, we compute for the first term in (26)
\[ \sum_{j=Ns}^{Nt} \sum_{k=2}^{\varepsilon/\sqrt{\kappa_N}} f_j[k](k - 1) \leq \varepsilon/\sqrt{\kappa_N} \sum_{j=Ns}^{Nt} \sum_{k=2}^{\varepsilon/\sqrt{\kappa_N}} f_j[k](k - 1) \mathbb{P}(\xi_j > 0) f_j'(1) \]
\[ \leq \varepsilon/\sqrt{\kappa_N} \sum_{j=Ns}^{Nt} \varphi_j(0)f_j'(1) \]
\[ \leq \varepsilon/\sqrt{\kappa_N} \sup_{1 \leq j \leq Nt} f_j'(1) \sum_{j=Ns}^{Nt} \varphi_j(0). \]

Due to Lemma A.3 \( \sup_{1 \leq j \leq Nt} f_j'(1) \) is bounded, and choosing \( \varepsilon \) sufficiently small yields the claim. \( \square \)

Finally, consider the following assumption. For any \( \varepsilon > 0 \), there exists \( K \) such that
\[ \forall N \geq 1, \quad \forall k \leq tN, \quad \mathbb{E}[\xi_k^2 \mathbb{1}_{\xi_k > K(1 + f_k'(1))}] \leq \varepsilon \mathbb{E}[\xi_k^2 \mathbb{1}_{\xi_k \geq 2}] < \infty. \quad (27) \]

This corresponds to [32, Assumption (B)] and [33, Assumption (*)].

Lemma A.6. Consider a sequence of environments that fulfils (1) and (2). If it further fulfils (27), then the following equivalent uniform integrability properties hold:

(i) For any sequence \( K_N \to \infty \),
\[ \lim_{N \to \infty} \frac{1}{\kappa_N} \sum_{k=1}^{tN} \mathbb{E}[\xi_k^2 \mathbb{1}_{\xi_k > K_N}] = 0. \]

(ii) For any \( \varepsilon > 0 \), there exists \( K \) such that
\[ \lim_{N \to \infty} \frac{1}{\kappa_N} \sum_{k=1}^{tN} \mathbb{E}[\xi_k^2 \mathbb{1}_{\xi_k > K}] \leq \varepsilon. \]

42
Proof. It is easily seen that (i) and (ii) are equivalent. We thus only prove (ii). By Lemma A.3, the expectations $f_k'(1)$ are bounded uniformly in $N$ and $k \leq N t$, so that (27) yields that for any $\varepsilon > 0$ there is $K$ such that

$$\forall N \geq 1, \quad \forall k \leq t N, \quad E[\xi_k^2 \mathbb{1}_{\xi_k > K}] \leq \varepsilon E[\xi_k^2 \mathbb{1}_{\xi_k \geq 2}] \leq \varepsilon E[\xi_k^2].$$

Taking a sum on both sides, and using the convergence of the variance (2) leads to

$$\frac{1}{\kappa N} \sum_{k=1}^{t N} E[\xi_k^2 \mathbb{1}_{\xi_k > K}] \leq \frac{\varepsilon}{\kappa N} \sum_{k=1}^{t N} E[\xi_k^2] \rightarrow \varepsilon \sigma^2(t) \quad \text{as } N \rightarrow \infty$$

yielding point (ii). \qed