RATIONAL FACTORS OF $D$-VARIETIES OF DIMENSION 3 WITH REAL ANOSOV FLOW

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Abstract. In this article, we prove that, under some suitable assumptions, the generic type of a real absolutely irreducible $D$-variety of dimension 3 with a mixing Anosov real-analytic flow satisfies exactly one of the two following behaviors: either it is minimal and disintegrated, or it is interalgebraic over $\mathbb{R}$ with the third power of the generic type of a disintegrated $D$-curve.

For that matter, we prove a conjecture of my thesis on the algebraic dynamical properties of an algebraically presented mixing Anosov flow — namely that, under the same suitable assumptions, a $D$-variety of dimension 3 with a mixing Anosov real-analytic flow does not admit non-trivial rational factors.

Then, the first statement follows from the second — using the Trichotomy Theorem in the theory $\text{DCF}_0$ — and the property of orthogonality to the constants for the aforementioned $D$-varieties.

1. In the end of my thesis, I formulated a conjecture on the algebraic dynamical properties of the geodesic flow of an algebraically presented Riemannian manifold of dimension 2 with negative curvature. The main motivation was that — using techniques from geometric stability theory in a differentially closed fields and the results of [Jao16] — I proved in the third chapter of my thesis [Jao17a], that this conjecture has strong consequences for the nature of the algebraic relations for geodesic motions in negative curvature. The purpose of this note is to prove this conjecture in the more general setting of mixing Anosov flows:

**Theorem A.** Let $(X,v)$ be an absolutely irreducible $D$-variety of dimension 3 over $\mathbb{R}$. Assume that the real-analytification $X(\mathbb{R})^{an}$ of $X$ admits a compact (non-empty) connected component $C_\mathbb{R}$ contained in the regular locus of $X$.

If the real analytic flow $(C_\mathbb{R},(\phi_t|_{C_\mathbb{R}})_{t \in \mathbb{R}})$ is a mixing Anosov flow, then $(X,v)$ does not admit any non-trivial rational factor.

Note that since $C_\mathbb{R}$ is contained in the regular locus of $X$, it is Zariski-dense in $X$ if and only if it is non-empty. Hence — similarly to the criterion of orthogonality to the constants of [Jao16] — one needs only to describe the dynamics of the vector field $v$ on a Zariski-dense connected component of $X(\mathbb{R})^{an}$ contained in the regular locus of $X$, in order to apply Theorem A.

A first consequence of the compactness and smoothness assumptions on the connected component $C_\mathbb{R}$ is that the real-analytic flow $(\phi_t|_{C_\mathbb{R}})_{t \in \mathbb{R}}$ of the vector field $v$ restricted to $C_\mathbb{R}$ is well defined at any time $t \in \mathbb{R}$. Then, we require in Theorem A that this real-analytic flow is a (necessarily compact and of dimension 3) mixing Anosov flow.

The Anosov flows are uniformly hyperbolic flows defined by Anosov in [Ano69]. It is well-known (see for instance [Ehe73]) that the geodesic flow of a compact Riemannian manifold with negative curvature is an Anosov flow. For an Anosov flow $(M,(\phi_t)_{t \in \mathbb{R}})$, the various notions of mixing — topologically weakly-mixing, topologically mixing and the mixing properties relatively to an equilibrium measure— collapse into a single one (see for example [Cont14]). We will simply say that $(M,(\phi_t)_{t \in \mathbb{R}})$ is a mixing Anosov flow to mean that one of the previous properties is satisfied.

One of the most important classes of mixing Anosov flows is that of geodesic flows of compact Riemannian manifolds with negative curvature (see, for instance, [Dal99]). In particular, Theorem A
can be applied to study the geodesic flow of an algebraically presented compact Riemannian manifold with negative curvature, as stated in my thesis.

Let’s explain in more details the content of the conclusion of Theorem A. A rational factor of \((X, v)\) is a dominant rational map \(\phi : (X, v) \rightarrow (Y, w)\) towards another \(D\)-variety \((Y, w)\), that is a \(\mathbb{R}\)-rational map \(\phi : X \rightarrow Y\) such that \(d\phi(v) = w\). Such a rational factor is called trivial if either \(Y\) is a point or \(\dim(Y) = \dim(X)\) (which happens if and only if \(\phi\) is generically finite).

The main heuristic on which Theorem A is based on is the property that mixing Anosov flows should give rise to minimal differential equations. Here — in the same way as simple groups are those who can not be split into simpler (in a naive sense) parts — a differential equation is minimal if its resolution can not be reduced to the resolution of simpler differential equations. Hence, for an absolutely irreducible \(D\)-variety \((X, v)\), a non-trivial rational factor is a very concrete instance of non-minimality.

Conversely, the property for a \(D\)-variety of admitting no non-trivial rational factors is a special instance concerning the theory \(\text{DCF}_0\) of the same notion investigated in \([\text{MPL}]\) in the more general setting of stable theories. In particular, they noticed that this property — that, in contrast with the stronger notion of minimality, avoid the consideration of base-changes by non constant differential fields — implies a weaker minimality property called semi-minimality.

2. In the next paragraph, we explain the main application of Theorem A that we have in mind — in the formalism of geometric stability theory and then in a more concrete geometric form — for autonomous differential equation of order 3 with a mixing Anosov real-analytic flow. As well as on Theorem A, the next theorem also relies on Hrushovski-Sokolovic classification of minimal types interpretable in the theory \(\text{DCF}_0\) (cf. \([\text{HS96}]\)) and on the results of \([\text{Jao16}]\) regarding the property of orthogonality to the constants:

**Theorem B.** Let \((X, v)\) be an absolutely irreducible \(D\)-variety of dimension 3 over \(\mathbb{R}\). Assume that the real-analytification \(X(\mathbb{R})^\text{an}\) of \(X\) admits a compact (non-empty) connected component \(C_\mathbb{R}\) contained in the regular locus of \(X\).

If the real-analytic flow \((C_\mathbb{R}, (\phi_t)_{t \in \mathbb{R}})\) is a mixing Anosov flow, then exactly one of the two following cases holds:

(i) Either the generic type of \((X, v)\) is minimal and disintegrated.

(ii) Or there exists a strictly disintegrated type \(r\) of order 1 over \(\mathbb{R}\) such that the generic type of \((X, v)\) and \(r^{(3)}\) are interalgebraic over \(\mathbb{R}\).

A self-contained exposition of Theorem B (assuming that Theorem A holds) can be found in the third chapter of my thesis (see, for instance, \([\text{Jao17a}]\) Theorem 5.3.1). We include a sketch of proof with references to my thesis when needed.

**Sketch of proof.** Let \((X, v)\) be an absolutely irreducible \(D\)-variety of dimension 3 over \(\mathbb{R}\) satisfying the assumptions of Theorem B. Assume that the real-analytic flow \((C_\mathbb{R}, (\phi_t)_{t \in \mathbb{R}})\) is a mixing Anosov flow and denote by \(p \in S(\mathbb{R})\) the generic type (in the theory of differentially closed field) of the \(D\)-variety \((X, v)\).

Now, Theorem A ensures that \((X, v)\) does not admit non-trivial rational factors. As explained in the first paragraph, this property implies that its generic type \(p\) is semi-minimal (see \([\text{Jao17a}]\) Proposition 4.1.8]). Hence, there exist a differential fields extension \((\mathbb{R}, 0) \subset (K, \delta)\) and a minimal type \(r \in S(B)\) such that \(p\) is almost internal to the set of \(\mathbb{R}\)-conjugates of the minimal type \(r\).

Moreover, since \((C_\mathbb{R}, (\phi_t)_{t \in \mathbb{R}})\) is a mixing flow, Theorem B of \([\text{Jao10}]\) implies that the type \(p\) is orthogonal to the constants. It follows that \(r\) is a minimal type, orthogonal to the constants, and non-orthogonal to a type defined over a constant differential field (namely, the type \(p\)). Under these assumptions, Hrushovski-Sokolovic Theorem of \([\text{HS96}]\) ensures that the minimal type \(r\) has to be disintegrated (see \([\text{Jao17a}]\) Theorem 4.1.6]).
To conclude, one uses standard arguments from (one-based) geometric stability theory to ensure that one can avoid extending parameters to witness almost-internality for $p$ in $r$: Since $r$ is disintegrated, there exists a minimal type $r' \in S(\mathbb{R})$, not orthogonal to $r$ such that $p$ is interalgebraic (over $\mathbb{R}$) with a power $r'^{(n)}$ of $r'$. Comparing the order of types, we get:

$$3 = \text{ord}(p) = \text{ord}(r) n.$$  

Since $3$ is prime, there are only two possibilities:

- either $n = 1$: the type $p$ is interalgebraic with $r$ and hence minimal (since we already assumed that it is stationary).
- or $n = 3$ and $\text{ord}(r') = 1$. In that case, unpublished results of Hrushovski imply that the type $r'$ has to be $\omega$-categorical (see also [FM16]) and therefore non-orthogonal to a strictly disintegrated type $r''$. Since $p$ and $r''(3)$ are interalgebraic over $\mathbb{R}$, this concludes the proof. □

In [Jao17b], we proved that the hypotheses of Theorem [B] are satisfied for unbounded families of real $D$-varieties — namely the geodesic $D$-varieties of real pseudo-Riemannian varieties, whose real-analytification is compact with negative curvature. Consequently, Theorem [B] provides unbounded families of types satisfying the alternative in the conclusion of Theorem [B]. When I started working on geodesic flows with negative curvature, I expected a stronger version of Theorem [B] to hold, namely that the case (ii) never occurs and that the generic type of a $D$-variety satisfying the hypothesis of Theorem [B] is always minimal and disintegrated. This strengthening of Theorem [B] would be very interesting and is still open.

3. We now formulate a geometric version of Theorem [B] not involving any reference to model-theory. Fix a $D$-variety $(X,v)$ over some constant differential field $(k,0)$. For $n \geq 2$, denote by $\mathcal{I}^\text{gen}_n$ the set of closed invariant subvarieties of $(X,v)^n$ which project generically on all the factors. We say that the $D$-variety $(X,v)$ is generically disintegrated if for every $n \geq 3$, every member $Z \in \mathcal{I}^\text{gen}_n$ can be written as an irreducible component (which projects generically on each factors) of:

$$\bigcap_{1 \leq i \neq j \leq n} \pi_{i,j}^{-1}(Z_{i,j}).$$

where $\pi_{i,j} : X^n \rightarrow X^2$ is the projection on the $i^{\text{th}}$ and $j^{\text{th}}$ coordinates and $Z_{i,j} \in \mathcal{I}^\text{gen}_2$ for every $i \neq j$. With this terminology in place, Theorem [B] can be formulated as follows:

**Corollary C.** Let $(X,v)$ be an absolutely irreducible $D$-variety of dimension 3 over $\mathbb{R}$. Assume that the real-analytification $X(\mathbb{R})^{\text{an}}$ of $X$ admits a compact (non-empty) connected component $C_K$ contained in the regular locus of $X$.

If the real-analytic flow $(C_K,\phi_t)_{t \in \mathbb{R}}$ is a mixing Anosov flow, then the $D$-variety $(X,v)$ is generically disintegrated. Moreover, one of the following two cases holds:

(i) Either $\mathcal{I}^\text{gen}_2$ contains only generically finite-to-finite correspondences and the generic type of $(X,v)$ is minimal.

(ii) Or $\mathcal{I}^\text{gen}_2$ contains at least a closed subvariety of $X \times X$ of codimension 1. Moreover, in that case, $\mathcal{I}^\text{gen}_2$ is a finite set.

In this very concrete geometric form, Corollary C reduces the classification of the closed invariant subvarieties (projecting generically on all factors) of $(X,v)^n$ for $n \geq 3$ to the classification of closed invariant subvarieties of $(X,v) \times (X,v)$. Hence, stronger forms of Theorem [B]—such as the minimality property discussed above— are related to the description of the set $\mathcal{I}^\text{gen}_2$ (and of its "non-generic" counterpart $\mathcal{I}_2$) of invariant closed subvarieties of $(X,v) \times (X,v)$ projecting generically on both factors. In particular, such a description for mixing Anosov flows would allow a complete description of the algebraic relations for an algebraically presented mixing Anosov flow.

Moreover, the minimal differential equation of order 3 studied by Freitag and Scanlon in [FS14] shows that in the first case, one can not expect $\mathcal{I}^\text{gen}_2$ to be finite. In the second case, this is a consequence of $\omega$-categoricity for disintegrated $D$-curves.
4. A major difficulty in the proof of Theorem [A] is to understand the relative position of (the real-analytification of) the indeterminacy locus of a rational factor \( \phi : (X, v) \rightarrow (Y, w) \) and the vector field \( v \). In [Jao16], this issue was solved for rational integrals — that is when the vector field \( w \) is everywhere zero. Indeed, we proved that the indeterminacy locus of a rational integral of \((X, v)\) has to be a closed invariant subvariety (see [Jao16, Lemma 3.3.5]).

For a more general vector \( w \), it is probably unreasonable to expect such a strong statement to hold. Indeed, such a statement would be a very strong instance of quantifier elimination for \( D \)-varieties which is known not to hold in general for \( D \)-varieties (see for instance [Pil06]). For a more geometric perspective, this is also reminiscent of the absence of projective models for varieties endowed with vector fields.

In this article, we address this issue by considering (possibly singular) foliations on \( X \)—which in contrast with vector fields and rational factors — are known to extend to any smooth projective model of \( X \). Starting from a rational factor \( \phi : (X, v) \rightarrow (Y, w) \), we consider the relative tangent vector bundle \( T_\phi \) of rank \( \dim(X) - \dim(Y) \) on the biggest open subset \( U \subset X \) where \( \phi \) is smooth. Then, using foliation formalism, we extend this (non-singular) foliation to a (possibly singular) foliation \( F_\phi \) on \( X \). In contrast with the indeterminacy locus of \( \phi \), the singular locus of the foliation \( F_\phi \) will satisfy nice properties of invariance.

This article is organized as follows: In the first section, we recall various classical facts about Lie-derivative in the setting of algebraic geometry. We try to emphasize the connection with differential algebra by formulating most of the results at the level of \( D \)-schemes.

The second section is devoted to the notion of (possibly singular) algebraic foliations on a smooth algebraic variety. In fact, apart from the definition itself, we will only be using the extension principle (mentioned above) for algebraic foliations on smooth varieties. We refer to [Bru15] in dimension 2 and [CP06] in higher dimension for more advanced geometrical results.

In the third section, we define the notion of an invariant foliation on a smooth \( D \)-variety defined over some constant differential field. Then, we explain how to reduce the study of rational factors of a \( D \)-variety to the study of its invariant foliations. This fairly general statement, which constitutes the core of our strategy to the proof of Theorem [A] might be useful in other contexts than the one of Anosov flows.

In the last section, we prove Theorem [A]. We start by giving a classification of invariant continuous foliation on a compact Anosov flow of dimension 3. Apart from the results from the previous section, the main ingredient is then a theorem of Plante in [Pla72] that ensures that no invariant continuous foliation appearing in the aforementioned classification can come from a rational factor.

5. The results of this article have been worked out during and slightly after my PhD in Orsay University. They are based on the many interesting ideas and suggestions of my PhD supervisors: Jean Benoît Bost and Martin Hils. I would also like to thank them for their numerous comments on earlier versions of this article. I am also very grateful to Rahim Moosa for many useful discussions and comments while I was writing this article as a post-doctoral fellow in Waterloo University.

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1. Lie derivative

In this section, we recall some classical facts on the Lie derivative in the setting of algebraic and analytic geometry.

The Lie derivative of a vector field (or more generally of a tensor field) with respect to another vector field is a standard notion of differential calculus. Surprisingly, it seems that this notion has not been considered before in the setting of differential algebra à la Buium [Bui93].

On the other hand, the effectiveness of the Lie derivative in order to study “non-integrability properties” of a differential equation given by a vector field on a manifold already appears in the work of Morales-Ruiz and Ramis (cf. [MR99] and [MRR01]) by means of the so-called variational equation (see also [Aud08, Part 3.1, p.46]). Moreover, the notion of Lie-derivative lies — through Frobenius Integrability Theorem — at the heart of the theory of algebraic foliations of dimension $\geq 2$, that we will study in the second section of this article.

1.1. Definition. We fix $k$ a field of characteristic 0. Recall that if $A$ is a $k$-algebra then the $A$-module $\text{Der}_{k}(A) = \text{Hom}(\Omega_{A/k}^{1}, A)$ is endowed with a Lie-bracket by the formula:

$$[\delta_{1}, \delta_{2}] = \delta_{1} \circ \delta_{2} - \delta_{2} \circ \delta_{1}. \tag{1}$$

The Lie-bracket is compatible with localization. Indeed, if $A$ is a $k$-algebra and $S$ is a multiplicative system of $A$ then the natural isomorphism of $S^{-1}A$-modules:

$$S^{-1}\text{Der}_{k}(A) \cong \text{Der}_{k}(S^{-1}A)$$

is also an isomorphism of $k$-Lie algebras.

Definition 1.1.1. Let $(X, \mathcal{O}_{X})$ be a $k$-scheme. The Lie-bracket defined by the formula (1) on each affine open subset defines a Lie-bracket on the coherent sheaf $\Theta_{X/k} = \text{Der}_{k}(\mathcal{O}_{X})$ of derivations of the structural sheaf of $X$.

Similarly, if $(M, \mathcal{O}_{M})$ is a (real or complex) analytic space, then the formula (1) defines a Lie-bracket on the coherent sheaf $\Theta_{M} = \text{Der}(\mathcal{O}_{M})$.

In both cases, under an additional smoothness assumption on $X$ (respectively on $M$), the sheaf $\Theta_{X/k}$ is a locally free on $X$ and is, indeed, the sheaf of sections of the vector bundle $T_{X/k}$ — or in other words, the sheaf of vector fields on $X$.

Lemma 1.1.2. Let $k$ be either the field of real or complex numbers and $(X, \mathcal{O}_{X})$ be a $k$-scheme. The (algebraic) Lie-bracket on $X$ satisfies the obvious compatibility relation with the analytic one on $X(k)^{an}$:

$$[v, w]^{an} = [v^{an}, w^{an}]$$
where $-^{\text{an}}$ denotes either the real or the complex analytification and the Lie-bracket on the right-hand side is the (real or complex) analytic one.

**Proof.** This can easily be derived from the standard properties of the analytification functor. It is also a direct consequence of the formula (2) of the next paragraph in both analytic and étale coordinates. □

**Definition 1.1.3.** Let $(X, \delta_X)$ be a $D$-scheme over a constant differential field $(k, 0)$ and $v$ a vector field on $X$. The **Lie-derivative** of $\delta_X$, denoted $\mathcal{L}_{\delta_X} : \Theta_{X/k} \to \Theta_{X/k}$ is the $k$-linear morphism defined by:

$$\mathcal{L}_{\delta_X}(\delta) = [\delta_X|U, \delta]$$

for every local section $\delta \in \Theta_{X/k}(U)$.

**Lemma 1.1.4.** Let $(X, \delta_X)$ be a $D$-scheme over a constant differential field $(k, 0)$, let $\delta, \delta' \in \Theta_{X/k}(U)$ be two local sections defined on the same open set $U$ and $a \in \mathcal{O}_X(U)$. We have:

$$\begin{cases} \mathcal{L}_{\delta_X}(\delta + \delta') = \mathcal{L}_{\delta_X}(\delta) + \mathcal{L}_{\delta_X}(\delta') \\ \mathcal{L}_{\delta_X}(a \delta) = \delta_X(a) \delta + a \mathcal{L}_{\delta_X}(\delta). \end{cases}$$

**Proof.** These two properties follow immediately from the formula (1). □

Before studying, more generally, the $k$-linear operator on a coherent sheaf satisfying the properties of Lemma 1.1.4, we compute, in the next paragraph, the Lie-bracket of two vector fields on a smooth variety in local coordinates (analytic coordinates in the analytic case and étale coordinates in the algebraic one).

1.2. **Computation in analytic and étale coordinates.** Every analytic manifold $(M, \mathcal{O}_M)$ can be covered by analytic charts. More precisely, there exists a covering of $M$ by open subsets endowed with analytic coordinates $x_1, \ldots, x_n$ (meaning that the map $x = (x_1, \ldots, x_n) : U \to k^n$ is an analytic isomorphism onto its image).

In order to prove local properties of the Lie-derivative, we will sometimes work locally inside these coordinates. Instead of analytic coordinates, we will use étale coordinates when working with algebraic varieties.

**Definition 1.2.1.** Let $U$ be a smooth variety over some field $k$ of dimension $n$. A **system** $(x_1, \ldots, x_n)$ of étale coordinates on $U$ is an étale morphism $x = (x_1, \ldots, x_n) : U \to \mathbb{A}^n$.

In other words, it is a $n$-tuple $(x_1, \ldots, x_n)$ of regular functions on $U$ such that the section $dx_1 \wedge \cdots \wedge dx_n$ of the canonical line bundle $\Omega^n_{U/k} = \wedge^n \Omega^1_{U/k}$ does not vanish on $U$.

**Remark 1.2.2.** Note that, if $(x_1, \ldots, x_n) : U \to \mathbb{A}^n$ is a system of étale coordinates then, by definition, $\{dx_1, \ldots, dx_n\}$ define a trivialization of $\Omega^n_{U/k}$. It follows that the dual basis of vector fields $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ define a trivialization of the tangent bundle of $X$.

**Lemma 1.2.3.** Let $X$ be a smooth algebraic variety over some field $k$ of characteristic $0$. There exists a covering of $X$ by Zariski-open subsets $(U; x_1, \ldots, x_n)$ endowed with étale coordinates. □

**Example 1.2.4.** Let $X$ be a smooth algebraic variety over a field $k$ of characteristic $0$. Consider an open set $U \subset X$ and $(x_1, \ldots, x_n)$ a system of étale coordinates on $U$. Since the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ define a trivialization of the tangent bundle of $X$, we may write

$$v_{|U} = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \quad \text{and} \quad w_{|U} = \sum_{i=1}^n w_i \frac{\partial}{\partial x_i}$$

for some functions $v_i, w_i \in \mathcal{O}_X(U)$. 


Lemma 1.2.5. With the notation above, the Lie-bracket of the vector fields \( v \) and \( w \) is given by:

\[
[v, w]_U = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} v_j \frac{\partial w_i}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.
\]

Proof. The lemma follows from Lemma 1.1.4 applied to both \( \mathcal{L}_v \) and \( \mathcal{L}_w \). \( \square \)

Note that when one works with (a subfield of) the field of complex numbers, one may use analytic coordinates instead of étale coordinates and the formula (2) also holds in this analytic setting.

1.3. Lie-derivative and \( D \)-coherent sheaves. The notion of \( D \)-coherent sheaves over some \( D \)-scheme \( (X, \delta_X) \) formalizes the notion of a "linear differential equation over \( (X, \delta_X) \)". This is a straightforward generalization to \( D \)-schemes of the notion of \( \delta \)-module over differential fields, that appears for example in [PZ03, Section 3].

Definition 1.3.1. Let \( (X, \delta_X) \) be a \( D \)-scheme over some constant differential field \( (k, 0) \). A \( D \)-coherent sheaf over \( (X, \delta_X) \) is a pair \((E, \nabla)\) where \( E \) is a coherent sheaf over \( X \) and \( \nabla : E \rightarrow E \) is a \( k \)-linear sheaf morphism satisfying the Leibniz-rule with respect to scalar multiplication:

\[
\nabla(a.m) = \delta_X(a).m + a.\nabla(m)
\]

for every local sections \( a \in \mathcal{O}_X(U) \) and \( m \in E(U) \) on some open subset \( U \) of \( X \).

If \((E, \nabla_E)\) and \((F, \nabla_F)\) are both \( D \)-coherent sheaves over \( (X, \delta_X) \), then a morphism of \( D \)-coherent sheaves over \( (X, \delta_X) \) is a morphism \( f : E \rightarrow F \) of coherent sheaves over \( X \) such that

\[
f \circ \nabla_E = \nabla_F \circ f.
\]

Remark 1.3.2. The notion of \( D \)-coherent sheaf is closely related to the more usual notion of a coherent sheaf \( E \) endowed with a connexion \( \nabla \). Recall that if \( X \) is a scheme over a field \( k \), a connexion \( \nabla \) on a coherent sheaf \( E \) over \( X \) is \( k \)-bilinear morphism:

\[
\nabla : E \times \Theta_{X/k} \rightarrow E
\]

which satisfies the Leibniz rule with respect to scalar multiplication on \( E \) and is \( \mathcal{O}_X \)-linear with respect to scalar multiplication on \( \Theta_{X/k} \).

Lemma 1.3.3. Let \( (X, \delta_X) \) be a \( D \)-scheme over some constant differential field \( (k, 0) \) and \((E, \nabla)\) a coherent sheaf endowed with a connexion on \( X \). Then \((E, \nabla_{\delta_X})\) is a \( D \)-coherent sheaf.

Proof. By definition, the morphism \( \nabla_{\delta_X} \) is \( k \)-linear and satisfies the Leibniz rule. \( \square \)

In particular, we get the following example:

Example 1.3.4. Let \( (X, \delta_X) \) be a \( D \)-scheme over some constant differential field \( (k, 0) \) and \( E = \mathcal{O}_X \epsilon_1 \oplus \cdots \oplus \mathcal{O}_X \epsilon_n \) be a free sheaf of rank \( n \) over \( X \). Define the \( k \)-linear map \( \nabla_0 : E \rightarrow E \) by the formula:

\[
\nabla_0(\sum_{i=1}^{n} f_i \epsilon_i) = \sum_{i=1}^{n} \delta_X(f_i) \epsilon_i.
\]

Then \((E, \nabla_0)\) is a \( D \)-coherent sheaf over \( (X, \delta_X) \).

Lemma 1.3.5. Let \( (X, \delta_X) \) be a \( D \)-scheme over some constant differential field \( (k, 0) \) and let \( E \) be a coherent sheaf on \( X \). If \((E, \nabla)\) and \((E, \nabla')\) are both \( D \)-coherent sheaves then:

\[
\nabla - \nabla' \in \operatorname{End}_{\mathcal{O}_X}(E).
\]
Proof. For a local function \( a \in \mathcal{O}_X(U) \) and a local section \( \sigma \in \mathcal{O}_X(U) \), we have:

\[
(\nabla - \nabla')(a, \sigma) = a(\nabla(\sigma) - \nabla'(\sigma)) + \delta_X(a).\sigma - \delta_X(a)\sigma = a(\nabla - \nabla')\sigma.
\]

It follows that \( \nabla - \nabla' \) is \( \mathcal{O}_X \)-linear. \( \square \)

**Example 1.3.6.** Let \((X, \delta_X)\) be a smooth \( D \)-variety over a constant differential field \((k, 0)\) and \((\mathcal{E}, \nabla)\) a locally free \( D \)-coherent sheaf.

Consider an open set \( U \subset X \) for which \( \mathcal{E}|_U = \mathcal{O}_U \epsilon_1 \oplus \cdots \oplus \mathcal{O}_U \epsilon_n \) is free. Using Example 1.3.4 and Lemma 1.3.5, there are functions \( a_{i,j} \in \mathcal{O}_X(U) \) for \( i, j = 1, \ldots, n \) such that:

\[
(3) \quad \nabla(\sum_{i=1}^n f_i \epsilon_i) = \nabla_0(\sum_{i=1}^n f_i \epsilon_i) + A(f_1, \ldots, f_n)
\]

where \( A = (a_{i,j}) \) is the \( n \times n \)-matrix with coefficients \( a_{i,j}(x) \in \mathcal{O}_X(U) \).

Conversely, for every matrix \( A = (a_{i,j})_{i,j \leq n} \) with coefficients \( a_{i,j}(x) \in \mathcal{O}_X(U) \), \( \nabla = \nabla_0 + A \) defines a \( D \)-coherent sheaf on \( \Theta_{U/k} \).

**Example 1.3.7.** Let \((X, \delta_X)\) be a \( D \)-scheme over some constant differential field \((k, 0)\). By Lemma 1.1.4, the pair \((\Theta_X/k, \delta_X)\) is a \( D \)-coherent sheaf over \((X, \delta_X)\).

Assume now that \( X \) is a smooth variety. The derivation \( \delta_X \) is associated to a vector field \( v \) on \( X \). Consider \( U \subset X \) an open subset and \((x_1, \ldots, x_n)\) a system of étale coordinates on \( U \). Combining the formulas (3) and (2), we get:

\[
L_v(w) = \nabla_0(w) - A.w \text{ where } a_{i,j} = \frac{\partial v_i}{\partial x_j} \in \mathcal{O}_X(U).
\]

Since the coefficients \( a_{i,j} \) of the matrix \( A \) do not depend linearly on the vector field \( v \), this kind of \( D \)-coherent sheaves never come by Lemma 1.3.3 from a connexion on \( X \).

### 1.4. Categories of \( D \)-coherent sheaves over \( D \)-schemes.

**Lemma 1.4.1.** Let \((X, \delta_X)\) be a \( D \)-scheme over some constant differential field \((k, 0)\). The category of \( D \)-coherent sheaves over \((X, \delta_X)\) is an Abelian category.

**Proof.** If \( f : (\mathcal{E}, \nabla_\mathcal{E}) \to (\mathcal{F}, \nabla_\mathcal{F}) \) is a morphism of \( D \)-coherent sheaves, it is easy to check that the kernel and the image of \( f \) are respectively \( D \)-subcoherent sheaves of \((\mathcal{E}, \nabla_\mathcal{E})\) and \((\mathcal{F}, \nabla_\mathcal{F})\) respectively.

Moreover, if \( \mathcal{G} \) is a \( D \)-coherent sheave of \((\mathcal{E}, \nabla_\mathcal{E})\), then there exists a unique \( D \)-coherent sheaf structure \((\mathcal{E}/\mathcal{G}, \nabla_{\mathcal{E}/\mathcal{G}})\) which makes the canonical projection into a morphism of \( D \)-coherent sheaves.

Using the forgetful functor to the category of coherent sheaves on \( X \), which is an Abelian category, it is easy to check that the axioms of an Abelian category are satisfied. \( \square \)

**Definition 1.4.2.** Let \( \phi : (X, \delta_X) \to (Y, \delta_Y) \) be a morphism of \( D \)-schemes over some constant differential field \((k, 0)\) and \((\mathcal{E}, \nabla_\mathcal{E})\) a \( D \)-coherent sheaf over \((Y, \delta_Y)\).

The pull-back of \((\mathcal{E}, \nabla_\mathcal{E})\) by \( \phi \), denoted \( \phi^*(\mathcal{E}, \nabla_\mathcal{E})\), is the coherent sheaf \( \phi^*\mathcal{E} \) over \( X \) endowed with the derivation:

\[
\phi^*\nabla_\mathcal{E} = \phi^*\nabla_\mathcal{E} \otimes 1 + \text{Id} \otimes \delta_X.
\]

**Example 1.4.3.** Let \( \phi : (X, \delta_X) \to (Y, \delta_Y) \) be a morphism of \( D \)-schemes over some constant differential field \((k, 0)\) and \((\mathcal{E}, \nabla_\mathcal{E})\) a locally free \( D \)-coherent sheaf over \((Y, \delta_Y)\).

Consider an open subset \( U \subset Y \) such that the restriction \( \mathcal{E}|_U = \mathcal{O}_U \epsilon_1 \oplus \cdots \oplus \mathcal{O}_U \epsilon_n \) of \( \mathcal{E} \) to \( U \) is free. Using formula (3), we can write:

\[
\nabla|_U = \nabla_0 + A
\]

where \( A = (a_{i,j}) \) is an \( n \times n \) matrix with coefficients \( a_{i,j} \in \mathcal{O}_Y(U) \).

Set \( V = \phi^{-1}(U) \). Note that the restriction \( \phi^*\mathcal{E}|_V = \mathcal{O}_V \phi^*\epsilon_1 \oplus \cdots \oplus \mathcal{O}_V \phi^*\epsilon_n \) of \( \phi^*\mathcal{E} \) to \( V \) is also free.
Lemma 1.4.4. With the notations above, the $D$-coherent structure on $\phi^*E$ is given by:

$$\phi^* \nabla_E = \nabla_0 + A^\phi$$

where $A^\phi$ is the matrix with coefficients $a_{i,j} \circ \phi \in \mathcal{O}_X(V)$.

1.5. Main Proposition.

Proposition 1.5.1. Let $\phi : (X, v) \rightarrow (Y, w)$ be a morphism of smooth $D$-varieties over some constant differential field $(k, 0)$. The derivative of $\phi$ defines a morphism of $D$-coherent sheaves over $X$:

$$d\phi : (\Theta_{X/k}, \mathcal{L}_v) \rightarrow \phi^*(\Theta_{Y/k}, \mathcal{L}_w)$$

Proof. It is sufficient to work locally in the Zariski topology. Consider étale coordinates $(x_1, \ldots, x_p) : U \rightarrow \mathbb{A}^p$ and $(y_1, \ldots, y_q) : V \rightarrow \mathbb{A}^q$ be étale coordinates on $Y$ such that $\phi : U \rightarrow V$.

We want to show that $d\phi \circ \mathcal{L}_v = \phi^* \mathcal{L}_w \circ d\phi$. Using Example 1.3.7 and Lemma 1.4.4, we can write:

$$\left\{ \begin{array}{l}
\mathcal{L}_v = \nabla_0 - A \text{ where } a_{i,j} = \frac{\partial \phi}{\partial x_j} \in \mathcal{O}_X(U) \\
\phi^* \mathcal{L}_w = \nabla'_0 - B^\phi \text{ where } b_{i,j} = \frac{\partial \phi}{\partial y_j} \in \mathcal{O}_Y(V).
\end{array} \right.$$ 

Using this notations, the previous equality translates into:

$$B^\phi . d\phi = d\phi . A + (\nabla_0 . d\phi - d\phi . \nabla_0)$$

which is an identity between two matrices of size $p \times q$ with coefficients in $\mathcal{O}_X(U)$.

Now, since $\phi$ is a morphism of $D$-varieties, we have $d\phi(v) = \phi^* w$, which — after denoting $\phi_j = y_j \circ \phi$, the coordinate function of $\phi$ — translates in these coordinates by:

$$\sum_{k=1}^p \frac{\partial \phi_j}{\partial x_k} v_k = w_j \circ \phi.$$

For $1 \leq i \leq p$ and $1 \leq j \leq q$, the chain rule for derivation as well as the Leibniz rule imply that:

$$(B^\phi . d\phi)_{i,j} = \frac{\partial}{\partial x_i} (w_j \circ \phi) = \frac{\partial}{\partial x_i} \left( \sum_{k=1}^p \frac{\partial \phi_j}{\partial x_k} v_k \right) = (d\phi . A)_{i,j} + \sum_{k=1}^p v_k \frac{\partial^2 \phi_j}{\partial x_k x_i}.$$

Moreover, since $\nabla_0 (\frac{\partial}{\partial x_i}) = 0$, we have that:

$$(\nabla'_0 . d\phi - d\phi . \nabla_0)_{i,j} = (\nabla'_0 . d\phi)_{i,j} = v(\frac{\partial \phi_j}{\partial x_i}) = \sum_{k=1}^p v_k \frac{\partial^2 \phi_j}{\partial x_k x_i}.$$

This concludes the proof of the proposition. □

We now gather two corollaries of Proposition 1.5.1 which deal respectively with two different geometric situations.

Corollary 1.5.2. Let $\phi : (X, v) \rightarrow (Y, w)$ be a dominant morphism of smooth $D$-varieties over a constant differential field $(k, 0)$. The coherent subsheaf $\Theta_{X/Y} = \text{Ker}(d\phi)$ is invariant under the Lie-derivative $\mathcal{L}_v$ of $v$.

Corollary 1.5.3. Let $(X, v)$ be a smooth $D$-variety and $Y \subset X$ a closed smooth invariant submanifold. We have an exact sequence of sheaves over $Y$:

$$0 \rightarrow \Theta_{Y/k} \rightarrow i^* \Theta_{X/k} \rightarrow \mathcal{N}_{X/Y} \rightarrow 0$$

where $\mathcal{N}_{X/Y}$ denotes the normal bundle of $Y$ in $X$. The Lie-derivative $\mathcal{L}_v$ induces a well-defined $D$-coherent sheaf structure on $\mathcal{N}_{X/Y}$. 
1.6. Cauchy formula on an analytic manifold.

Construction 1.6.1. Let \((A, \delta_A)\) be a differential ring of characteristic 0. Consider the differential ring \((A((t)), \frac{\partial}{\partial t})\) of formal power series over \(A\) endowed with the derivation sending \(t\) to 0 and the morphism of differential rings \(\phi^* : (A, \delta_A) \longrightarrow (A((t)), \frac{\partial}{\partial t})\) given by the expansion in power series:

\[
a \mapsto \sum_{k=0}^{\infty} \frac{\delta^k_A(a)}{k!} t^k.
\]

We denote by \(-|_{t=0} : A((t)) \longrightarrow A\) the morphism of evaluation at \(t = 0\) and we denote by \(\delta_A\), the unique extension of \(\delta_A\) to \(A((t))\) satisfying \(\delta_A(t) = 0\). Note that with this derivation, the morphism of rings \(-|_{t=0}\) becomes a morphism of differential rings:

Lemma 1.6.2. With the notation above for every \(f \in A\), \(\phi^*(f)\) may be described as the unique solution of the differential equation:

\[
\begin{align*}
\phi^*(f)_{|_{t=0}} &= f \\
\frac{\partial}{\partial t} \phi^*(f) &= \delta_A(\phi^*(f))
\end{align*}
\]

Remark 1.6.3. Consider \(M\) an analytic manifold, \(v\) an analytic vector field on \(M\) and \(a \in M\). The vector field \(v\) induces a derivation \(\delta_v\) on \(A = \mathcal{O}_{M,a}\). In [Jao16, Lemme 3.1.21], we proved — using Cauchy integral formula the bound the norm of \(\delta^k_v(f)\) — that in that case, the morphism \(\phi^*\) may be factored as:

\[
\phi^* : A \longrightarrow \mathcal{O}_{M \times \mathbb{C}, (a,0)} \subset A((t)).
\]

As noted in [Jao16], one easily checks that if \(\phi_a\) denotes the local analytic flow of the vector field at \(v\), this implies that:

\[
f \circ \phi_a = \phi^*(f) = \sum_{k=0}^{\infty} \frac{\delta^k_A(f)}{k!} t^k.
\]

This property was then used to translate — for a closed submanifold of \(M\) — invariance properties with respect to the vector field in terms of invariance properties with respect to the local analytic flow (see [Jao16, Proposition 3.1.20]).

Construction 1.6.4. Let \((A, \delta_A)\) be a differential ring of characteristic 0 and \((M, \nabla_M)\) be a \(D\)-module over \(A\). One may define a morphism of \(A\)-modules \(M \longrightarrow M \otimes A((t))\) by the formula:

\[
m \mapsto \sum_{k=0}^{\infty} \frac{\nabla^k_M(m)}{k!} \otimes t^k.
\]

When \((M, \nabla_M)\) is given by the Lie-derivative of a vector field \(v\) on an analytic manifold \(M\), we will use once again Cauchy integral formula to bound the sequence \(\left(\sum_{k=0}^{\infty} \frac{\nabla^k_M(m)}{k!}\right)_{k \in \mathbb{N}}\):

Let \(M\) be a smooth analytic manifold and let \(v\) be a vector field on \(M\). For every point \(a \in M\), the Lie-derivative of vector field \(v\) defines a derivation \(\nabla_v : \Theta_{M,a} \longrightarrow \Theta_{M,a}\). We denote by \(\mathcal{O}_{M,a}\{t\}\) the ring of local analytic functions on \(\mathbb{C} \times M\). Pull-back along the local flow \(\phi_a\) of \(v\) defines a morphism of \(\mathcal{O}_{M,a}\)-modules:

\[
\phi^*_a : \Theta_{a,M} \longrightarrow \Theta_{a,M} \otimes \mathcal{O}_{M,a}\{t\}
\]

We extend the derivation \(\mathcal{L}_v\) on \(\Theta_{a,M}\) to a derivation on \(\Theta_{a,M} \otimes \mathcal{O}_{M,a}\{t\}\) still denoted \(\mathcal{L}_v\) by setting \(\mathcal{L}_v(t) = 0\).

Lemma 1.6.5 (Cauchy formula for the Lie derivative). With the notation above for every vector fields \(w\) in a neighborhood of \(p\), we have:

\[
\begin{align*}
\phi^*_a(w)_{|_{t=0}} &= w \\
\frac{\partial}{\partial t} \phi^*_a(w) &= \mathcal{L}_v(\phi^*_a(w))
\end{align*}
\]
Moreover, we have:

$$\phi^*_a w = \sum_{n=0}^{\infty} \frac{L^n_v(w)}{n!} t^n.$$ 

Proof. The relation between the Lie-derivative of a vector field $v$ and the pull-back by the local flow of the vector field $v$, given in the first part is well-known and holds more generally on smooth manifolds (see any textbook of differential geometry).

For the second part, one needs to check that the right hand-side converges normally to an analytic function in a neighborhood of $p$. By formal derivation of power series, it follows easily that the left-hand side satisfies the differential equation given by the first part and therefore must be equal to the local flow $\phi^*_a w$. Similar formulas already appear in the work of Cauchy.

Given a vector field $w$ on $M$, we need a uniform bound for $L^n_v(w)$ in a neighborhood of $p$, in order to prove normal convergence. Since $M$ is smooth and that we work locally on $M$, we may assume that $M = \mathbb{C}^n$ and $p = 0$.

Fix $r_0 < R_0$ are the radii of two complex polydisks of $p$ and $n$ a natural number. By applying Cauchy integral formula to the holomorphic coordinates of $w$, there exists a constant $C > 0$ such that for all $\epsilon > 0$, all radii $r > 0$ and all vector fields $w$, we have:

$$||L_v(w)||_{\infty, r-\epsilon} \leq \frac{C}{\epsilon} ||w||_{\infty, r}.$$ 

where $||-||_{\infty, r}$ denotes the supremum norm on the polydisk with radius $r$. It then follows from $n$ successive applications of the previous inequality to the radii $r_k = r_0 + (R_0 - r_0) \frac{k}{n}$ that:

$$||L^n_v(w)||_{\infty, r_0} \leq \left( \frac{C}{n} \right)^n ||w||_{\infty, r_0}$$

The normal convergence of the left-hand side follows from this inequality. \hfill $\square$

2. Foliations on a smooth algebraic variety

In this section, we recall the standard definition of an algebraic foliation — in the setting of algebraic varieties over a field $k$ of characteristic 0 — that we will use in this article.

Let $X$ be a smooth irreducible algebraic variety over $k$. Intuitively, a foliation $\mathcal{F}$ on $X$ is the data of a subspace $T_{\mathcal{F}, x}$ of the tangent space $TX$ at $x$ for every point $x$ of $X$ that depends algebraically on the point $x \in X$ and such that the sheaf of sections of $\mathcal{F}$ is stable under Lie-bracket.

The dimension of the subspace $F_\eta$ at the generic point $\eta$ of $X$ is called the rank of the foliation. A singularity of the foliation $\mathcal{F}$ is simply a point $x$ of $X$ where the dimension of the fibre $F_x$ is less than the rank of the foliation.

More restrictively, we will require the foliations to satisfy an additional assumption of saturation. On the one hand, the involutivity property — that is the stability under Lie-bracket — ensures the local analytic integrability of the foliation, outside of the singular locus. On the other hand, the saturation hypothesis ensures that the singular locus of $\mathcal{F}$ is “small”, namely of codimension at least 2 in $X$.

The main motivation for these additional requirement is the extension result (Proposition 2.3.1) for any algebraic foliation $\mathcal{F}$ on a quasi-projective smooth variety $X$ to a foliation $\overline{\mathcal{F}}$ on any projective closure.

2.1. Algebraic foliations and their singular locus. Let $X$ be an irreducible and smooth variety over a field $k$ of characteristic 0. The coherent sheaf $\Theta_{X/k}$ is a locally free sheaf of rank $n = \dim(X)$.

Definition 2.1.1. A foliation $\mathcal{F}$ on $X$ is a sub-sheaf $T_\mathcal{F}$ of the tangent bundle $\Theta_{X/k}$ which satisfies:

(i) The sub-sheaf $T_\mathcal{F}$ is involutive, that is, stable under the Lie-bracket.

(ii) The sub-sheaf $T_\mathcal{F}$ is saturated, that is, the quotient $\Theta_{X/k}/T_\mathcal{F}$ does not have torsion.
The coherent sheaf \( T_F \) is the sheaf of vector fields tangent to the foliation \( F \). The rank of the foliation \( F \) is the (generic) rank of the coherent sheaf \( T_F \). Alternatively, we say that \( F \) is a \( k \)-foliation to mean that \( F \) is a foliation with rank \( k \).

**Remark 2.1.2.** We first comment on the two main assumptions in Definition 2.1.1, namely involutivity and saturation.

1. In this article, we are mainly interested in smooth algebraic varieties defined over the field of real or complex numbers. In that case, the involutivity assumption is crucial to ensure local analytic integrability of the algebraic foliation under study (see section 2.5). However, for a coherent sub-sheaf \( L \) of \( \Theta_{X/k} \) of rank 1, the condition of involutivity is automatically satisfied.

2. The property of saturation for \( T_F \) (inside a locally free sheaf) implies that the coherent sheaf \( T_F \) is reflexive (namely, isomorphic to its bidual). This property, which is extensively studied in [Har80], is weaker than being locally free but stronger than being torsion-free.

In particular, a 1-foliation is always defined by an invertible sheaf \( T_F \) whereas for a 2-foliation on a smooth algebraic variety \( X \) of dimension 3, it is always defined by a coherent sheaf \( T_F \) which is locally free outside a finite set of points (see [Har80]) of \( X \). Since the main result of this article deals with \( D \)-variety of dimension 3, these two examples are the most important ones for the proof of Theorem A.

**Lemma 2.1.3.** Let \( X \) be a smooth algebraic variety over \( k \) and let \( T_F \subset \Theta_{X/k} \) be a coherent sub-sheaf of the tangent sheaf of \( X \). The following properties are equivalent:

(i) \( \Theta_{X/k}/T_F \) does not have torsion.

(ii) There exists an open set \( U \subset X \) such that \( \text{codim}_X(X \setminus U) \geq 2 \) and

\[
0 \longrightarrow T_F|_U \longrightarrow \Theta_{U/k} \longrightarrow \Theta_{U/k}/T_F|_U \longrightarrow 0
\]

is an exact sequence of locally free-sheaves on \( U \).

**Proof.** Indeed, fix \( x \in X \) of codimension 1. Since \( X \) is a smooth algebraic variety, the local ring \( \mathcal{O}_{X,x} \) is a principal local ring. It follows that the exact sequence of torsion-free \( \mathcal{O}_{X,x} \)-modules:

\[
0 \longrightarrow T_{F,x} \longrightarrow \Theta_{X/k,x} \longrightarrow \Theta_{X/k,x}/T_{F,x} \longrightarrow 0
\]

is in fact an exact sequence of free \( \mathcal{O}_{X,x} \)-module which therefore have to split. Since this is true for every point \( x \in X \) of codimension 1, the lemma follows.

**Definition 2.1.4.** Let \( F \) be a foliation on \( X \). The singular locus of \( F \), denoted \( \text{Sing}(F) \), is the set of points \( x \in X \) such that \( \Theta_{X/k}/F \) is not locally-free in a neighborhood of \( x \), i.e.

\[
\text{Sing}(F) = \{ x \in X \mid (\Theta_{X/k}/F)_x \text{ is not a } \mathcal{O}_{X,x} \text{ - free module} \}.
\]

**Definition 2.1.5.** Let \( F \) be a foliation on \( X \). The foliation \( F \) on \( X \) is called non-singular if \( \text{Sing}(F) = \emptyset \).

**Proposition 2.1.6.** Let \( F \) be a foliation on \( X \). The singular locus of \( F \) is a closed subset of \( X \) of codimension \( \geq 2 \).

**Proof.** The fact that the singular locus of a foliation is closed follows from the general properties of morphisms of coherent sheaves. This also follows from the computations of Example 2.1.7. Moreover, the previous lemma shows that its codimension is greater than 2.

**Example 2.1.7.** Let \( F \) be a foliation on \( X \) and \( U \subset X \) an open subset endowed with a system of étale coordinates \( x_1, \ldots, x_n : U \to \mathbb{A}_k^n \). Recall that the tangent sheaf \( \Theta_{U/k} \) may be identified with the free sheaf \( \mathcal{O}_{U/k}^r \) with basis \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \).

Since the coherent sub-sheaf \( F|_U \) is finitely generated, it can be described as the coherent sheaf on \( U \) spanned by some vector fields \( v_1, \ldots, v_r \) written as:
$$v_i = \sum_{i=1}^{n} f_{ij} \frac{\partial}{\partial x_i}. $$

We can now explicit the system of equations describing the singular locus of $\mathcal{F}$ restricted to $U$ in terms of these datas:

$$\text{Sing}(\mathcal{F}) \cap U = \{ p \in U \mid \text{rank}(f_{ij}(p)) \text{ is not maximal} \}.$$

Hence, if $\mathcal{F}$ is a foliation of rank $p$, the system of equations describing the singular locus of $\mathcal{F}$ is given by the vanishing of the $p \times p$ minors of the matrix $(f_{ij})_{1 \leq i \leq r, 1 \leq j \leq n}.$

2.2. **Analytification of an algebraic foliation.** Assume that $k$ is the field of real or complex number. We have the following counterpart for Definition 2.1.1 in the analytic setting:

**Definition 2.2.1.** Let $M$ be a real or complex analytic manifold. An analytic foliation on $M$ is a coherent subsheaf $\mathcal{E}$ of the tangent sheaf $\Theta_M$ of $M$ which is both involutive and saturated.

Similarly to the algebraic case, for an analytic foliation $\mathcal{E}$ on an analytic manifold $M$, one can define its rank and its singular locus $\text{Sing}(\mathcal{E})$. The singular locus of $\mathcal{E}$ is a closed analytic subspace of $M$ of codimension $\geq 2$.

**Lemma 2.2.2.** Let $X$ be a complex (resp. a real) smooth algebraic variety and $\mathcal{F}$ a foliation on $X$. Through the complex-analitcification (resp. real-analyticalization) functor $-^{an}$, $\mathcal{F}$ defines an analytic foliation $\mathcal{F}^{an}$ on $X^{an}$.

Moreover, the rank of $\mathcal{F}^{an}$ is the rank of $\mathcal{F}$ and the singular locus of $\mathcal{F}^{an}$ is the analytification of the singular locus of $\mathcal{F}$.

**Remark 2.2.3.** When we will work in the analytic setting, we will mainly work with non-singular foliation. The only exception to that rule is the proof of Proposition 3.2.2 where we work with the local analytic flow of a vector field to prove invariance properties for its singular locus.

Once this has been established, we will simply throw away the singular locus $\text{Sing}(\mathcal{F})$ of the foliation $\mathcal{F}$ at the level of scheme before applying the analytification functor.

2.3. **Saturation of an algebraic foliation on a open subset.**

**Proposition 2.3.1** (Saturation). Let $X$ be a smooth algebraic variety over $k$ and let $U$ be a dense open subset. Any algebraic foliation $\mathcal{F}$ on $U$ extends uniquely to an algebraic foliation on $X$.

**Proof.** Let $\mathcal{F}$ be an algebraic foliation on $U$. There exists a coherent subsheaf $\mathcal{G}$ of $\Theta_{X/k}$ such that $\mathcal{G}|_U = \mathcal{F}$ (namely, the sheaf $\mathcal{G}$ of vector fields $\mathcal{G}|_U = \mathcal{F}$, which, as a simple verification shows, is quasi-coherent, hence coherent).

Let $\mathcal{G}$ be the saturation of $\mathcal{G}$ in $\Theta_{X/k}$ (see [Har80]). By definition, $\mathcal{F}$ is a saturated subsheaf of $\Theta_{X/k}$, whose restriction to $U$ is $\mathcal{F}$. Therefore, it suffices to check that $\mathcal{F}$ is involutive.

Let $v \in \mathcal{F}(V)$ be a local section of $\mathcal{F}$. The Lie-bracket with $v$ defines a morphism of $\mathcal{O}_X$-modules:

$$[-, v] : \mathcal{F}|_V \longrightarrow (\Theta_{X/k}/\mathcal{F})|_V$$

Since the algebraic foliation $\mathcal{F}$ is involutive, this morphism is zero on $U \cap V$. Consequently, the image of this morphism is a coherent sheaf whose support is a proper closed subvariety of $X$, hence a torsion sheaf. Since the coherent sheaf $\Theta_{X/k}/\mathcal{F}$ has no torsion, this morphism is zero.

The uniqueness is a direct consequence of the saturation hypothesis, since two saturated subsheaves of $\Theta_{X/k}$ which have the same generic fibre are equal (see [Har80]).

Proposition 2.3.1 is useful to construct algebraic foliations on smooth algebraic variety $X$: it shows that one only needs to construct them on a dense open subset.

**Construction 2.3.2.** Let $\phi : X \longrightarrow Y$ be a rational dominant morphism of smooth irreducible varieties over a field $k$. Denote by $n$ the dimension of $X$ and $m$ the dimension of $Y$. Let $U$ be the biggest open set where $\phi$ is defined and smooth. The restriction of the differential $d\phi : \Theta_{X/k} \longrightarrow \phi^*\Theta_{Y/k}$ of $\phi$ to $U$ has constant rank $n - m$. 
Definition 2.4.1. Let $v$ be a non-zero rational vector field on $X$ of rank $(n-m)$. The condition of coprimeness simply expresses that $v$ and $w$ define the same foliation on an open set of $X$. In particular, in that setting, the notion of closed invariant subvarieties naturally extends to closed subvarieties of $X$. We illustrate the previous remark by computing the foliation on $\mathbb{P}^2$ associated to a polynomial vector field on $\mathbb{A}^2$.

Example 2.4.4. Let $k$ be a field of characteristic 0. We consider polynomial vector fields $v$ on the plane $\mathbb{A}^2$:

$$v(x, y) = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$$

where $a(x, y)$ and $b(x, y)$ are two coprime polynomials with coefficients in $k$, with the same degree $n$. The condition of coprimeness simply expresses that $v$ defines a saturated subsheaf of $\mathbb{A}^2$. We now describe the foliation $\mathcal{F}_v$ tangent to $v$ on $\mathbb{P}^2$ in the chart $(s, t)$ where:

$$s = \frac{1}{x} \quad \text{and} \quad t = \frac{y}{x}.$$
Now, if we denote by $w(s,t)$ the vector field on the right-hand side, the vector fields $w(s,t)$ and $v(s,t)$ have the same singularities outside of the line $x = 0$ and $s = 0$. Hence, the vector field $w(s,t)$ defines a saturated sub-sheaf of $\Theta_{\mathbb{A}^2/k}$ if and only if the vector field $w(s,t)$ does not vanish on the line at infinity. This condition can be rewritten as:

$$P(t) := -t.a(0, t) + b(0, t) \neq 0 \text{ or } Q(x, y) = x^n P(y/x) = x.b_n(x, y) - y.a_n(x, y) \neq 0.$$ 

where $a_n(x, y)$ and $b_n(x, y)$ are respectively the homogeneous parts of degree $n$ of $a(x, y)$ and $b(x, y)$ respectively.

We have proven the following:

**Lemma 2.4.5.** Let $v(x, y) = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ be a polynomial vector field on the plane where $a(x, y)$ and $b(x, y)$ are two coprime polynomials with degree $n$. Assume moreover that:

$$Q(x, y) = x.b_n(x, y) - y.a_n(x, y) \neq 0$$

Then the line at infinity is invariant for the foliation $\mathcal{F}_v$ and the singularities $\text{Sing}(\mathcal{F}_v)_{\infty}$ of $\mathcal{F}_v$ on the line at infinity are given (in homogeneous coordinates) as the roots of $Q(x, y)$.

**Remark 2.4.6.** This simple lemma admits strong consequences for the structure of closed invariant varieties for the $D$-variety $(\mathbb{A}^2, v)$ for a vector field $v$ satisfying the assumption of the lemma. For example, if $L_{\infty}$ denotes the line at infinity then every closed invariant curve $C$ of $(\mathbb{A}^2, v)$ satisfies:

$$C_{\infty} = \overline{C \cap L_{\infty}} \subset \text{Sing}(\mathcal{F}_v)_{\infty}.$$ 

This property heavily constrains the possible closed invariant curves for $(\mathbb{A}^2, v)$. For example, with refinements of this idea in [CMS09], Coutinho and Menasché describe an algorithm to compute polynomial vectors fields on the plane without closed invariant curves.

### 2.5. Analytic leaves of a non-singular analytic foliation.

**Theorem 2.5.1** (Frobenius Integrability Theorem, [IY08, Theorem 2.9, Part I]). Let $M$ be a complex analytic manifold of dimension $n$, $\mathcal{F}$ be a non-singular analytic $r$-foliation on $M$ and $x \in M$.

There exist a neighborhood $U$ of $x$ in $M$, open subsets $V_1 \subset \mathbb{C}^r$, $V_2 \subset \mathbb{C}^{n-r}$ and an analytic isomorphism $\phi : U \longrightarrow V_1 \times V_2$ such that for all $t \in V_2$, the tangent space of $L_t = \phi^{-1}(V_1 \times \{t\})$ at every point $a \in L_t$ is $F_a$.

**Remark 2.5.2.** Let $M$ be a complex analytic manifold of dimension $n$, $\mathcal{F}$ be a non-singular analytic $r$-foliation on $M$ and $x \in M$.

The germ at $x$ of the analytic subvariety $L_y = \phi^{-1}(V_1 \times \{y\})$ where $y = \phi_2(x)$ does not depend on the chosen trivialisation $\phi$ given by Theorem 2.5.1 and is called the germ of the leaf of $\mathcal{F}$ through $x$.

**Definition 2.5.3.** Let $M$ be a complex analytic manifold of dimension $n$ and $\mathcal{F}$ be a non-singular analytic $r$-foliation on $M$. A leaf of $\mathcal{F}$ is a maximal immersed analytic subvariety $N$ of $M$, which is the germ of a leaf at any point of $N$.

**Corollary 2.5.4.** Let $M$ be a complex analytic manifold of dimension $n$ and $\mathcal{F}$ be a non-singular analytic $r$-foliation on $M$. The leaves of $\mathcal{F}$ define a partition of $M$ into immersed analytic subvariety of dimension the rank of $\mathcal{F}$.

**Proof.** Indeed, by maximality, two distinct leaves of $\mathcal{F}$ never intersect. Moreover, by Theorem 2.5.1 any point $x \in M$ is contained in a leaf. \hfill $\square$

**Definition 2.5.5.** Let $X$ be an algebraic manifold and $\mathcal{F}$ a possibly singular foliation of $X$. Denote by $U$ the complementary of the singular locus $\text{Sing}(F)$ of $\mathcal{F}$.

A leaf $\mathcal{L}$ of the non-singular foliation $\mathcal{F}_{\mid U}$ is called algebraic if there exists a closed algebraic subvariety $Z$ of $X$ such that

$$Z(\mathbb{C}) \cap U(\mathbb{C}) = \mathcal{L}.$$
Example 2.5.6. Let $X$ be a smooth complex algebraic variety. By [CP06], there exists at least one rational vector field $v$ (in fact, most of them), such that the foliation $\mathcal{F}_v$ defined in Example 2.4.1 does not admit any algebraic leaf.

However, by definition, the foliation associated to a rational factor $\mathcal{F}_\phi$ in Example 2.3.2 there exists a non-empty open set $U$ such that any leaf that encounters $U$ is algebraic.

2.6. Continuous foliations on an analytic manifold. Let $M$ be a (real or complex) analytic manifold and let $E \subset TM$ be a continuous distribution on $M$, that is a continuous sub-bundle $E$ of $TM$. Under such a weak regularity assumption, the property of involutivity formulated in terms of the Lie-derivative is not available anymore to distinguish foliations (satisfying local-integrability properties) from other distributions.

Let’s start by recalling the definition of an Anosov flow, which evidences two important continuous distributions.

**Definition 2.6.1.** Let $(M, g)$ be a Riemannian manifold of dimension $\geq 3$ and $v$ a $C^\infty$-vector field on $M$ with a complete flow $(\phi_t)_{t \in \mathbb{R}}$. The flow $(M, (\phi_t)_{t \in \mathbb{R}})$ is called an Anosov flow if the vector field $v$ does not vanish and there exists a splitting of the tangent bundle $TM$ into continuous sub-bundles

\[(4) \quad TM = E^{ss} \oplus \mathbb{R} v \oplus E^{su}\]

satisfying:

(i) The sub-bundles $E^{ss}$ and $E^{su}$ are non-trivial bundles which are $(d\phi_t)_{t \in \mathbb{R}}$-invariant.

(ii) There exist $C, C' > 0$ and $0 < \lambda < 1$ such that for all $u \in E^{ss}$,

\[||d\phi_t(u)|| \leq C |t| \lambda^t ||u|| \text{ and } ||d\phi_{-t}(u)|| \geq C' \lambda^{-t} ||u|| \text{ for all } t > 0.\]

(iii) There exist $C, C' > 0$ and $0 < \lambda < 1$ such that for all $w \in E^{su}$,

\[||d\phi_t(w)|| \geq C \lambda^{-t} ||w|| \text{ and } ||d\phi_{-t}(w)|| \leq C' \lambda^t ||w|| \text{ for all } t > 0.\]

where the norm on the tangent bundle $TM$ is given by the Riemannian metric on $M$.

The distributions $E^{ss}$ and $E^{su}$ are called respectively strongly stable and strongly unstable distributions. Note that if $M$ is compact then any two Riemannian metrics on $M$ are equivalent. Hence, for a vector $v$ on a smooth compact manifold $M$, the property that expresses that its real-analytic flow is Anosov is independent of the choice of a Riemannian metric on $M$. In particular, for compact manifold $M$, these two distributions are uniquely determined by the vector field $v$.

**Remark 2.6.2.** The work of Hasselblat in [Has94] (that goes back to Anosov in the case of diffeomorphisms) imply that these distributions do not, in general, satisfy stronger regularity properties (such as being $C^2$). Consequently, when working with Anosov flows, we will need to work at this degree of regularity.

The procedure that allows that — which produces, after real analyticity, from an algebraic foliation $\mathcal{F}$, a continuous foliation outside of the singular locus — will be essential for our purposes.

**Definition 2.6.3.** Let $M$ be a analytic manifold of dimension $n$. A continuous (non-singular) foliation $\mathcal{F}$ on $M$ of rank $p$ is a continuous atlas $\{(U_i, \phi_i) \mid i \in I\}$ on $M$ such that:

(i) The image $\phi_i(U_i) = V^1_i \times V^2_i \subset \mathbb{R}^p \times \mathbb{R}^{n-p}$ can be written as a product of two connected open subsets of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively.

(ii) The transition maps $\psi_{i,j}$ are “locally triangular” with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^{n-p}$ i.e.

\[\psi_{i,j}(x,y) = (h_{i,j}(x,y), g_{i,j}(y)) \quad \text{where } x \in \mathbb{R}^p \text{ and } y \in \mathbb{R}^{n-p}.\]

for some functions $h_{i,j}$ and $g_{i,j}$ defined on the appropriate open subspaces of $\mathbb{R}^n$.

The leaves of the foliation $\mathcal{F}$ are then the equivalence classes for the equivalence relation $\sim$ generated by $xRy$ if:

(iii) $\exists i \in I$, $x, y \in U_i$ and the last $(n-p)$ coordinates of $\psi_i(y)$ and $\psi_i(x)$ are equal.
In particular, the leaves \( \{ L_\alpha, \alpha \in A \} \) of the foliation \( \mathcal{F} \) define a partition of \( M \). The compatibility between this definition and Definition 2.2.1 in the analytic case is ensured by Frobenius Integrability Theorem.

**Theorem 2.6.4** (Hadamard-Perron Theorem – [Has02, Section 2.2.i]). Let \( (M, (\phi_t))_{t \in \mathbb{R}} \) be an Anosov flow as above. The stable and unstable distributions \( E^s \) and \( E^u \) are integrable by continuous foliations with \( C^1 \) leaves.

### 3. Foliations invariant by a vector field

In this section, we study foliations in the setting of smooth \( D \)-variety \((X, v)\) over some constant differential field \((k, 0)\). In the same way that one defines invariant closed subscheme of \((X, v)\), we define **invariant foliations** by means of the Lie-derivative along the vector field \( v \).

The invariant foliations of a smooth irreducible \( D \)-variety \((X, v)\) provide a effective tool to study rational factors — namely rational dominant morphisms \( \phi : (X, v) \rightarrow (Y, w) \) of \( D \)-varieties — of the \( D \)-variety \((X, v)\). Indeed, we prove that any such rational factor induces through its tangent sheaf, an invariant foliation on \((X, v)\) (see Proposition 3.1.5).

We also prove that the singular locus of an invariant foliation is in fact invariant (as a closed invariant subvariety). We get this result by using the local complex analytic flow of the vector field \( v \) instead of the vector field itself (see Corollary 3.2.2).

When working dynamically in the fourth section, this convenient result will allow us to work far away from the singularities of the foliation. A similar invariance result, regarding the indeterminacy locus of a rational integral, was also needed in the proof of the criterion of orthogonality to the constants in \([Jao16]\).

#### 3.1. Definition.

**Definition 3.1.1.** Let \( X \) be an algebraic variety over \( k \) and \( v \) a vector field on \( X \). We say that a coherent subsheaf \( \mathcal{F} \) of \( \Theta_{X/k} \) is **\( v \)-invariant** if \( \mathcal{F} \) is a \( D \)-coherent subsheaf of \((\Theta_{X/k}, [v, -])\).

In other words, a coherent subsheaf \( \mathcal{F} \) of \( \Theta_{X/k} \) is **\( v \)-invariant** if

\[
[v, \mathcal{F}(U)] \subset \mathcal{F}(U)
\]

for every open subspace \( U \subset X \).

**Remark 3.1.2.** Let \((\mathcal{E}, \nabla_\mathcal{E})\) be a \( D \)-coherent sheaf over a \( D \)-scheme \((X, \delta_X)\). For every coherent subsheaf \( \mathcal{F} \subset \mathcal{E} \), the derivation \( \nabla_\mathcal{E} \) induces a morphism of \( \mathcal{O}_X \)-sheaves:

\[
\phi_\mathcal{F} : \mathcal{F} \rightarrow \mathcal{E}/\mathcal{F}.
\]

Moreover, \( \mathcal{F} \) is a \( D \)-coherent subsheaf of \( \mathcal{E} \) if and only if this map is the null morphism.

Now, consider the case of a linear differential equation defined over some constant differential field.

**Example 3.1.3.** If \( x_1, \ldots, x_n \) are coordinates on \( \mathbb{A}^n \), then the vector fields \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \) define a trivialization of the tangent bundle \( T_{\mathbb{A}^n/k} \). We will work in this trivialization.

Consider a linear vector field \( v(x) = A.x \) for some matrix \( A \in \mathcal{M}_n(k) \) where \((k, 0)\) is a constant differential field. Let \( E \subset \mathbb{A}^n \) be a \( A \)-invariant linear subspace. Then, the sheaf of sections \( \mathcal{E} \) of \( E \times \mathbb{A}^n \) is \( v \)-invariant as a subsheaf of \( \Theta_{\mathbb{A}^n/k} \).

In particular, if \( k \) is algebraically closed and the eigenvalues of \( A \) are simple then the tangent bundle

\[
\Theta_{\mathbb{A}^n/k} = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_n}.
\]

splits as a direct sum of \( 1 \)-dimensional \( v \)-invariant coherent subsheaves.

**Lemma 3.1.4.** Let \( X \) be an irreducible algebraic variety over \( k \), let \( \mathcal{F} \subset \Theta_{X/k} \) be a coherent saturated subsheaf and let \( U \) be a non-empty open subset of \( X \).

The subsheaf \( \mathcal{F} \) if \( v \)-invariant if and only if \( \mathcal{F}|_U \) is \( v|_U \)-invariant.

**Proof.** The direct implication is obvious. Conversely, assume that \( \mathcal{F}|_U \) is \( v|_U \)-invariant. Using Remark 3.1.2 we can consider the morphism of \( \mathcal{O}_X \)-coherent sheaves defined by the Lie-bracket with \( v \):
By assumption, this morphism is the null morphism on $U$. Since $\Theta_{X/k}/F$ does not have torsion, this implies that it is the null morphism. □

The principal examples of invariant foliations that we will encounter in these notes will be derived from rational factors by the following proposition.

**Proposition 3.1.5.** Let $\phi : (X, v) \to (Y, w)$ be a rational morphism of $D$-varieties over $(k, 0)$. The tangent foliation $F_\phi$ of $\phi$ is $v$-invariant.

**Proof.** By Lemma 3.1.4, we may assume that $X$ and $Y$ are smooth and that $\phi$ is regular and smooth. In particular, we have an exact sequence of locally free sheaves over $X$:

$$0 \to F_\phi \to \Theta_{X/k} \xrightarrow{d\phi} \phi^* \Theta_{Y/k} \to 0$$

Moreover, since $\phi$ is a morphism of $D$-varieties, Lemma 1.5.1 ensures that $d\phi$ is a morphism of $D$-coherent sheaves over $(X, v)$:

$$d\phi : (\Theta_{X/k}, [v, -]) \to \phi^* (\Theta_{Y/k}, [w, -])$$

From Lemma 1.4.1, we conclude that its kernel $F_\phi$ is a $D$-subcoherent sheaf of $\Theta_{X/k}$.

3.2. **Analytification of an invariant coherent sheaf.**

**Proposition 3.2.1.** Let $F$ be a coherent analytic subsheaf of $\Theta_M$ and $v$ an analytic vector field on $M$. The following are equivalent:

(i) The subsheaf $F$ is $v$-invariant.

(ii) For every open set $U \subset M$ and $t$ such that the flow of $v$ is defined at $t$ for every $x \in U$,

$$\phi_t^*(F(U)) \subset F(\phi^{-t}(U)).$$

**Proof.** Using the semi-group properties of flows, the property (ii) is equivalent to:

(ii)' For every $x \in M$ and $t \in \mathbb{C}$ sufficiently small

$$\phi_t^*(F_x) \subset F_{\phi^{-t}(x)}.$$  

Denote by $\pi : \Theta_M \to \Theta_M/F$, the canonical projection.

Let $w \in F_x$ be a local section of $F$. Consider $\epsilon > 0$ such that $\phi_t^* w$ is defined and bounded on $B(x, \epsilon)$ for $t \in \mathbb{C}$ sufficiently small.

Now, the space of bounded sections of $\Theta_M$ and $\Theta_M/F$ on $B(x, \epsilon)$ is a Banach space.

Using Cauchy formulas (Lemma 1.6.5) we know that $t \mapsto \phi_t^* w$ is the solution of the differential equation:

$$\begin{cases}
\phi_t^*(w)|_{t=0} = w \\
\frac{d}{dt} \phi_t^*(w) = L_v(\phi_t^*(w))
\end{cases}$$

Therefore, composing with the linear map $\pi$, the path $t \mapsto \pi \circ \phi_t^* (w)$ is a solution of the differential equation:

$$\begin{cases}
\pi \circ \phi_t^*(w)|_{t=0} = 0 \\
\frac{d}{dt} (\pi \circ \phi_t^*(w)) = \pi \circ L_v(\phi_t^*(w))
\end{cases}$$

In the Banach space of bounded sections of $\Theta_M/F$ on $B(x, \epsilon)$, we get that

$$\pi \circ \phi_t^* w = 0 \text{ for } t \text{ sufficiently small if and only if } \pi \circ L_v(w) = 0$$

which exactly means the equivalence between (i) and (ii). □

**Corollary 3.2.2.** Let $v$ be a vector field on $M$ and $F$ an analytic foliation such that $F$ is $v$-invariant. Then, $\text{Sing}(F)$ is a closed invariant analytic subspace of $(M, v).$
Proof. By definition the singular locus of $\mathcal{F}$ is described by:

$$\text{Sing}(\mathcal{F}) = \{ x \in M \mid \Theta_{M,x}/\mathcal{F}_x \text{ is not a free } \mathcal{O}_{M,x} \text{ submodule of rank 1} \}$$

For every point $p \in M$ and $t \in \mathbb{C}$ sufficiently small, the local flow $\phi_t$ at $p$ is a local diffeomorphism. It defines a ring-isomorphism between $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,\phi_t(x)}$ and its differential defines an isomorphism of modules:

$$d\phi_t : \Theta_{M,x} \rightarrow \Theta_{M,\phi_t(x)}$$

Since $\mathcal{F}$ is $v$-invariant, by Proposition 3.2.1, we have that $\mathcal{F}_{\phi_t(x)} = \phi_t(\mathcal{F}_x)$. Therefore, $d\phi_t$ defines an isomorphism of exact sequences:

$$
\begin{array}{cccc}
0 & \rightarrow & \mathcal{F}_x & \rightarrow & \Theta_{M,x} & \rightarrow & \Theta_{M,x}/\mathcal{F}_x & \rightarrow & 0 \\
& & \downarrow{d\phi_t} & & \downarrow{d\phi_t} & & \downarrow{d\phi_t} & & \downarrow{d\phi_t} & \\
0 & \rightarrow & \mathcal{F}_{\phi_t(x)} & \rightarrow & \Theta_{M,\phi_t(x)} & \rightarrow & \Theta_{M,\phi_t(x)}/\mathcal{F}_{\phi_t(x)} & \rightarrow & 0
\end{array}
$$

Therefore, the first exact sequence is a direct product if and only if the second one is. We conclude that for $t$ sufficiently small, the germ of $\text{Sing}(\mathcal{F})$ at $p$ is invariant by the local flow at $p$. Since this is true through any point of $\text{Sing}(\mathcal{F})$, we conclude that $\text{Sing}(\mathcal{F})$ is a closed invariant analytic subspace of $M$. \hfill \Box

**Corollary 3.2.3** (Preparatory Lemma). Let $(X,v)$ be an absolutely irreducible real $D$-variety such that $M = X(\mathbb{R})^n$ is regular, compact, and Zariski-dense in $X$. Suppose that $(X,v)$ admits an invariant saturated coherent subsheaf $\mathcal{F}$ of $\Theta_{X/k}$. Then

- The singular locus $Z = \text{Sing}(\mathcal{F})$ is invariant.
- On the dense invariant open subset $U = X(\mathbb{R})^n \setminus Z(\mathbb{R})$, the real analytification of $\mathcal{F}$ defines a continuous vector subbundle $F$ of $T_M$ such that

$$\forall t \in \mathbb{R}, \forall x \in U, d\phi_t(F_x) = F_{\phi_t(x)}.$$ 

**Proof.** Using Corollary 3.2.2, the singular locus is invariant. Therefore, the restriction of the flow of $v$ to $U = X(\mathbb{R})^n \setminus Z(\mathbb{R})$ is complete. We then apply Proposition 3.2.1. \hfill \Box

4. Rational factors of mixing Anosov flows of dimension 3

The aim of this section is to prove Theorem A. If $(X,v)$ be a $D$-variety defined over $(\mathbb{R},0)$, we already reduced in the previous sections the understanding of the rational factors of $(X,v)$ to the understanding of the invariant algebraic foliations on $(X,v)$.

When the real-analytic flow $(M, (\phi_t)_{t \in \mathbb{R}})$ of $(X,v)$ is an Anosov flow, there exists, by definition, a splitting of the tangent bundle of $M$ into continuous invariant (non-singular) foliations:

$$T_M = W^{su} \oplus \mathbb{R}v \oplus W^{ss}$$

where $W^{ss}$ (resp. $W^{su}$) is called the strongly stable foliation (resp. strongly unstable foliation).

In dimension 3, these three invariant foliations have rank 1 and therefore, can not be split again into invariant foliations of smaller rank. Looking at the periodic orbits of $(M, (\phi_t)_{t \in \mathbb{R}})$, we are able to describe all continuous invariant foliations of $(M, (\phi_t)_{t \in \mathbb{R}})$ from the strongly stable foliation $W^{ss}$, the strongly unstable one $W^{su}$ and the direction of the flow $\mathbb{R}v$ (Proposition 4.1.1 and Proposition 4.1.2).

Using this explicit description of invariant continuous foliations and a result of Plante in [Pla72], we conclude that none of these foliations comes from a rational factor.
4.1. Continuous invariant subbundles of a mixing Anosov flow.

**Proposition 4.1.1.** Let \((M, (\phi_t)_{t \in \mathbb{R}})\) be a compact and connected Anosov flow of dimension 3 and \(\Sigma\) a proper closed invariant subset. The \((d\phi_t)_{t \in \mathbb{R}}\)-invariant continuous line subbundles defined on \(U = M \setminus \Sigma\) are exactly:

- the (non-singular) foliation \(F\) tangent to the flow.
- the strongly stable and strongly unstable foliations \(W^{ss}\) and \(W^{su}\), associated to the Anosov structure.

**Proof.** By definition of an Anosov flow, these three continuous foliations are \((d\phi_t)_{t \in \mathbb{R}}\)-invariant. We denote by \(\sigma_1, \ldots, \sigma_2, \sigma_3 : M \to \text{Gr}_1(TM)\) the three (continuous) sections of the 1-Grassmanian bundle of \(TM\) defining these 1-foliations.

Conversely let \(L \subset TM\) be a continuous line subbundle defined on \(U \subset M\) and denote by \(\sigma : M \to \text{Gr}_1(TM)\) the associated section.

Since \((M, (\phi_t)_{t \in \mathbb{R}})\) is an Anosov flow, the periodic points of \((\phi_t)_{t \in \mathbb{R}}\) are dense in \(U\).

Let \(p \in M\) be a periodic points of \((\phi_t)_{t \in \mathbb{R}}\) of period \(T > 0\). Then, the fibre \(L_p \subset T_pM\) is a stable line of the linear map:

\[
(d\phi_T)|_{T_pM} : T_pM \to T_pM
\]

Since \((M, (\phi_t)_{t \in \mathbb{R}})\) is an Anosov flow, this linear map has exactly three eigenvalues and the associated eigenspaces are precisely \(W^{pu}_p\), \(W^{ss}_p\) and \(F_p\).

We have proven that, on a dense set, \(\sigma\) agrees with one of the sections \(\sigma_1, \sigma_2, \sigma_3\). For \(i \leq 3\), denote by \(F_i\) the closed subset of \(M\) where \(\sigma\) agrees with \(\sigma_i\). Since \(F_1 \cup F_2 \cup F_3\) is closed and contain a dense set, we have that \(F_1 \cup F_2 \cup F_3 = M\). Moreover, they are disjoint since two distinct \(\sigma_i\) have distinct values at every point of \(M\). Since \(M\) is connected, \(M = F_i\) for some \(i \leq 3\). This implies that \(L\) is either the stable foliation, or the unstable foliation or the direction of the flow itself. \(\square\)

**Proposition 4.1.2.** Let \((M, (\phi_t)_{t \in \mathbb{R}})\) be a compact and connected Anosov flow of dimension 3 and \(\Sigma\) a proper closed invariant subset. The \((d\phi_t)_{t \in \mathbb{R}}\)-invariant continuous subbundles of rank 2 defined on \(U = M \setminus \Sigma\) are exactly:

- the stable and unstable foliations \(W^s = F \oplus W^{ss}\) and \(W^u = F \oplus W^{su}\) associated to the Anosov structure, where \(F\) is the direction tangent to the flow.
- the direct sum of the strongly stable and strongly unstable foliations \(W^{ss} \oplus W^{su}\).

**Proof.** Since these three examples are direct sum of \((d\phi_t)_{t \in \mathbb{R}}\)-invariant line bundles, there are also \((d\phi_t)_{t \in \mathbb{R}}\)-invariant.

Conversely let \(P \subset TM\) be a continuous plane subbundle defined on \(U \subset M\). Since \((M, (\phi_t)_{t \in \mathbb{R}})\) is an Anosov flow, the periodic points of \((\phi_t)_{t \in \mathbb{R}}\) are dense in \(U\).

Let \(p \in M\) be a periodic points of \((\phi_t)_{t \in \mathbb{R}}\) of period \(T > 0\). Then, the fibre \(L_p \subset T_pM\) is a stable plane of the linear map:

\[
(d\phi_T)|_{T_pM} : T_pM \to T_pM
\]

But these stable planes are exactly \(W^s_p\), \(W^u_p\) and \((W^{ss} \oplus W^{su})_p\). One concludes in the same way as Proposition 4.1.1. \(\square\)

**Theorem 4.1.3** ([Pla72, Theorem 1.3]). Let \((M, (\phi_t)_{t \in \mathbb{R}})\) be a mixing Anosov flow.

Every leaf of the strongly stable foliation \(W^{ss}\) and every leaf of the strongly unstable foliation \(W^{su}\) is dense in \(M\).

We will use the Theorem 4.1.3 combined with the two preceding propositions in the following form:

**Corollary 4.1.4.** Let \((M, (\phi_t)_{t \in \mathbb{R}})\) be a compact and connected mixing Anosov flow of dimension 3, \(\Sigma\) a proper closed invariant subset and \(F\) a continuous foliation on \(M \setminus \Sigma\) with positive rank.

Suppose that the foliation \(F\) is \((d\phi_t)_{t \in \mathbb{R}}\)-invariant and distinct from the foliation tangent to the flow. Then, every leaf of \(F\) is dense in \(M\).
Proof. Denote by $r$ the rank of the foliation. If $r = 3$, it is true since $\Sigma$ has empty interior (the flow is mixing so topologically transitive). For $r = 1, 2$, we do a case by case inspection:

- If $r = 1$, then by Proposition 4.1.1, $F$ is either the strongly stable foliation or the strongly unstable one. In both cases, Theorem 4.1.3 implies that every leaf of $F$ is dense.
- If $r = 2$, then by Proposition 4.1.2, $F$ is either the stable foliation or the strongly unstable one, or the direct sum of the strongly stable and the strongly unstable ones. In those three cases, the foliation contains either the strongly stable foliation or the strongly unstable one. Using Theorem 4.1.3, we conclude that every leaf of $F$ is dense in $M$.

\[\Box\]

4.2. Proof of the Theorem \[\textit{A}\]

Theorem 4.2.1. Let $(X, v)$ be an absolutely irreducible $D$-variety of dimension 3 over $\mathbb{R}$. Assume that the real-analytification $X(\mathbb{R})^{an}$ of $X$ admits a compact (non-empty) connected component $C_{\mathbb{R}}$ contained in the regular locus of $X$.

If the real analytic flow $(C_{\mathbb{R}}, (\phi_{t})_{t \in \mathbb{R}})$ is a mixing Anosov flow, then $(X, v)$ does not admit any non-trivial rational factor.

Proof. Let $(X, v)$ be a real $D$-variety satisfying the hypothesis of Theorem 4.2.1. Suppose that $(X, v)$ admits a non-trivial rational factor $\pi : (X, v) \to (Y, w)$. Since we already now that $(X, v)$ has no non-trivial rational integral (see [Jao17b]), we may assume that $w \neq 0$. By Proposition 3.1.3, the tangent foliation $F_{\pi}$ is $v$-invariant and does not contain the foliation generated by $v$, since $w \neq 0$.

Now, by corollary 2.1.4, the singular locus $Z$ of $F_{\pi}$ is a closed invariant subvariety. On the open set $U = X \setminus Z$, $F_{\pi}$ is a non-singular foliation. Consequently, outside of the closed invariant set $\Sigma = Z(\mathbb{R})$, $F = F^{an}_{\pi}$ is a non-singular foliation on $X(\mathbb{R})^{an} \setminus \Sigma$.

- Since $F_{\pi}$ is $v$-invariant, the continuous foliation is a $(d\phi_{t})_{t \in \mathbb{R}}$-invariant subbundle of $T_{M}$.
- It is distinct from the foliation generated by $v$, since $w \neq 0$.

By Corollary 2.1.4, we conclude that every leaf of $F$ is dense in $C_{\mathbb{R}}$. This contradicts the fact that $F_{\pi}$ is a foliation tangent to the algebraic fibration $\pi$ (see Example 2.5.6).

\[\Box\]

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