Deformations of Bundles and the Standard Model

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We modify a recently proposed heterotic model hep-th/0703210, giving three next-generations of standard model fermions, to get rid of an additional $U(1)$ factor in the gauge group. The method employs a stable $SU(5)$ bundle on a Calabi-Yau three-fold admitting a free involution. The bundle has to be built as a deformation of the direct sum of a stable $SU(4)$ bundle and the trivial line bundle.

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In this note, which constitutes an addendum to [1], we propose a method to construct a model of the $E_8 \times E_8$ heterotic string giving in four dimensions the gauge group and chiral matter content of the standard model. For this we embed an $SU(5)$ bundle in the first $E_8$ leading to a GUT gauge group $SU(5)$, which is afterwards broken by a Wilson line to the standard model gauge group. The non-simply connected Calabi-Yau threefold is obtained by modding a simply connected cover Calabi-Yau space $X$ by a free involution. Therefore we search on $X$ for an invariant $SU(5)$ bundle of net-generation number $\pm 6$ (for other constructions along these lines cf. [2], [3], [4] ). Models of this kind were recently constructed in [1] (cf. also [5]) but had an additional $U(1)$ in the unbroken gauge group due to the specific form of the bundle

$$V_5 = V_4 \otimes \mathcal{O}_X(-\pi^* \beta) \oplus \mathcal{O}_X(4\pi^* \beta)$$

(0.1)

This is a polystable bundle and has structure group $SU(4) \times U(1)_A$ (on the Lie algebra level) of $V_5$ and therefore $SU(5) \times U(1)_A$ as unbroken gauge group.

All the conditions, stability of the bundle, invariance under the involution, solution of the anomaly cancellation equation by having an effective five-brane class and finally the phenomenologically net-generation number were therefore essentially solved already on the level of $V_4$. The bundle $V_4$ alone would give an unbroken gauge group $SO(10)$, which cannot be broken to the standard model gauge group by just turning on a $Z_2$ Wilson line corresponding to $\pi_1(X/Z_2)$. Therefore in [1] $V_4$ had to be enhanced to an $SU(5)$ bundle by adding a line bundle (the combined conditions of stability and five-brane effectivity make a non-trivial extension for $V_5$ problematical as explored in [1]). Then the structure (0.1) caused the additional $U(1)_A$ in the gauge group.

So in a $(4 + 1)$-decomposition of the rank 5 structure group one has

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in SU(5)$$

where the $U(1)_A$ is embedded as

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

. Our goal is to turn on the off-diagonal block elements to get a full, irreducible $SU(5)$, i.e.

$$\begin{pmatrix} a & * \\ * & d \end{pmatrix}$$

. The possibility to do this will be measured (in first order) be two $Ext^1$-groups, corresponding to each of the off-diagonal blocks, respectively. The process of turning on these off-diagonal terms means that the bundle $V' = V_4 \oplus \mathcal{O}_X$ (for simplicity we set $\beta = 0$) is deformed to a more generic bundle $V$; conversely $V'$ occurs as a specialization or
degeneration \( V \rightarrow V' \) where the off-diagonal terms again go to zero and one gets the reducible object. More precisely, we say a vector bundle \( V' \) deforms to a stable bundle \( V \) if there is a connected curve \( C \) and a vector bundle \( \mathcal{V} \) over \( C \times X \) such that \( V' \cong \mathcal{V}_{\{0\} \times X} \) for some point \( 0 \in C \) and \( \mathcal{V}_{\{t\} \times X} \cong V \) for some other point \( t \in C \)

\[
\begin{array}{c|c|c}
\mathcal{V} & V & \rightarrow & V' \\
\hline
C & t & 0
\end{array}
\]  

(0.2)

This process will thereby cause the following changes in the structure group \( G \) of the bundles and the unbroken gauge group \( H \) of the four-dimensional low-energy observer who sees the commutator in \( E_8 \) of \( G \)

\[
\begin{array}{c|c|c}
G & SU(5) & \rightarrow & SU(4) \times U(1)_A \\
\hline
H & SU(5) & \rightarrow & SU(5) \times U(1)_A
\end{array}
\]  

(0.3)

For a generic choice of parameters with \( \alpha \beta \neq 0 \) (where \( \alpha \in H^{11}(B) \) is a further twist class inherent in the construction, cf. [1] and below) one would find that the \( U(1)_A \), which occurs in the structure group and in the gauge group, is anomalous and thereby gets massive by the Green-Schwarz mechanism. However the group theory of the \( ad \ E_8 \) decomposition relevant here tells us that to secure the absence of exotic matter multiplets one has just to impose the condition \( \alpha \beta = 0 \) (cf. [1]). Thereby the \( U(1)_A \) remains non-anomalous and remains in the light spectrum. It is this problem for which the present paper shows a way out by embedding the reducible bundle in a family of proper irreducible \( SU(5) \) bundles where the \( U(1)_A \) is therefore again massive on the compactification scale (as all other elements of \( E_8 \) which are broken for the four-dimensional low-energy observer by the specific gauge background turned on on \( X \)); for the specialization \( * \rightarrow 0 \), where the off-diagonal elements are turned off again, one would then get a restauration of four-dimensional gauge symmetry as this corresponds to \( m_{U(1)_A} \rightarrow 0 \), i.e., at this special point on the boundary of the bundle moduli space the \( U(1)_A \) returns into the light spectrum.

So, to get rid of this additional \( U(1)_A \) factor we will construct a stable holomorphic \( SU(5) \) bundle \( V_5 \) by deforming the complex structure of the given polystable \( SU(5) \)
bundle \( V' = V_4 \oplus \mathcal{O}_X \) (as said, for simplicity we work from now on with \( \beta = 0 \)). \( V_4 \) is a stable \( SU(4) \) bundle, and \( \mathcal{O}_X \) is the trivial one-dimensional line bundle; therefore \( V' \) is a polystable bundle and solves the Donaldson-Uhlenbeck-Yau (DUY) equations. If such a deformation to a stable holomorphic bundle exist, the theorems of [6], [7] guarantee that \( V_5 \) is a solution of the DUY equations, i.e., the equations of motion of the heterotic string.

In [10] (Corollary B.3) it has been shown that the direct sum of two stable vector bundles (say \( V, W \)) of the same slope \( \mu(V) = \int c_1(V)J^2/rk(V) \) deforms to a stable vector bundle if the sum has unobstructed deformations and both spaces \( H^1(X, \text{Hom}(V, W)) \) and \( H^1(X, \text{Hom}(W, V)) \) do not vanish. Applied to our case we therefore have to show that \( H^1(X, \text{Hom}(V_4, \mathcal{O}_X)) \) and \( H^1(X, \text{Hom}(\mathcal{O}_X, V_4)) \) do not vanish and that \( V_4 \oplus \mathcal{O}_X \) has unobstructed deformations.

\( H^1(X, \text{End}(V')) \) is the space of all first-order deformations of \( V' \). The obstruction to extending a first order deformation to second (or higher) order lives in \( H^2(X, \text{End}_0(V')) \). Thus if \( H^2(X, \text{End}_0(V')) = 0 \) we can always lift to higher order, i.e., the deformations would be unobstructed. For instance, the tangent bundle \( TX \) of a \( K3 \) surface has unobstructed deformations since \( H^2(X, \text{End}_0(TX)) \cong H^0(X, \text{End}_0(TX))^* = 0 \). As we are on a Calabi-Yau threefold, Serre duality shows that the dimensions of \( H^1(X, \text{End}(V')) \) and \( H^2(X, \text{End}(V')) \) are equal, so there are as many obstructions as deformations.

The vanishing of \( H^2(X, \text{End}(V')) \) is, however, only a sufficient condition for the existence of a global (in contrast to first-order) deformation. So in principle it is still possible to have global deformations although the obstruction space is non-vanishing. That this hope is not in vain is born out by the example of \( X \) being the quintic in \( \mathbb{P}^4 \) and \( V' = TX \oplus \mathcal{O}_X \) (this example was considered first in [12], [13]), where nevertheless knows that a global deformation exists [10], [11].

The tangent space \( H^1(X, \text{End}(V')) \) to the deformations decomposes as follows (using the fact that \( H^1(X, \mathcal{O}_X) = 0 \))

\[
H^1(X, \text{End}(V')) \cong H^1(X, \text{End}(V_4)) \oplus H^1(X, \text{Hom}(V_4, \mathcal{O}_X)) \oplus H^1(X, \text{Hom}(\mathcal{O}_X, V_4))
\]

where the last two terms \( \text{Ext}^1(V_4, \mathcal{O}_X) = H^1(X, V_4^*) \) and \( \text{Ext}^1(\mathcal{O}_X, V_4) = H^1(X, V_4) \)
parametrize non-trivial extensions

\[
0 \to \mathcal{O}_X \to W \to V_4 \to 0 \\
0 \to V_4 \to W' \to \mathcal{O}_X \to 0
\] (0.4)

We note first that the index theorem gives (by stability of \(V_4\) and \(\mu(V_4) = 0\) we have \(H^i(X, V_4) = 0\) for \(i = 0, 3\); note that also \(V^*\) is stable and has \(\mu(V^*) = 0\))

\[
\dim H^1(X, V_4) - \dim H^1(X, V_4^*) = -\frac{1}{2} c_3(V_4)
\] (0.6)

which for the physical relevant \(V_4\) is non-zero, so at least one of the two off-diagonal spaces is already non-vanishing. To actually prove that \(H^1(X, V_4)\) and \(H^1(X, V_4^*)\) are both non-vanishing we recall first the explicit construction of \(V_4\) form [1]. Note that whereas in [1] we considered the case \(x > 0, \beta \neq 0\), we consider here the case \(x < 0\) (this simplifies some arguments below) and \(\beta = 0\).

As in [1] the rank four vector bundle \(V_4\) will be constructed as an extension

\[
0 \to \pi^* E_1 \otimes \mathcal{O}_X(-D) \to V_4 \to \pi^* E_2 \otimes \mathcal{O}_X(D) \to 0
\] (0.7)

where \(E_i\) are stable bundles on \(B\) and \(D = x \Sigma + \pi^* \alpha\) is a divisor in \(X\) (here \(X\) is a Calabi-Yau threefold elliptically fibered with two sections \(\sigma_i\) over \(B = \mathbb{P}^1 \times \mathbb{P}^1\), cf. [1]; furthermore \(\Sigma = \sigma_1 + \sigma_2\) and \(F\) will denote the fiber). The argument for stability of \(V_4\) runs exactly parallel to the one given in [1]. To prove stability of \(V_4\) we first note that given zero slope stable vector bundles \(E_i\) on \(B\), one can prove that the pullback bundles \(\pi^* E_i\) are stable on \(X\) [8], [9] for a suitable Kähler class \(J\). Now for the zero slope bundle \(V_4\) constructed as an extension (with \(\pi^* E_i\) stable) we have two immediate conditions which are necessary for stability: first that \(\mu(\pi^* E_1 \otimes \mathcal{O}_X(-D)) < 0\) and second that \(\pi^* E_2 \otimes \mathcal{O}_X(D)\) of \(\mu(\pi^* E_2 \otimes \mathcal{O}_X(D)) > 0\) is not a subbundle of \(V_4\), i.e., the extension is non-split. The first condition reduces to

\[
D J^2 = 2x(h-z)^2 c_1^2 + 2z(2h-z) \alpha c_1 > 0
\] (0.8)

(where \(c_i := \pi^* c_i(B)\)) this implies in our case \(x < 0\) the condition

\[
\alpha c_1 > 0
\] (0.9)

The non-split condition can be expressed as \(Ext^1(\pi^* E_2 \otimes \mathcal{O}_X(D), \pi^* E_1 \otimes \mathcal{O}_X(-D)) \neq 0\) where \(\mathcal{E} = E_1 \otimes E_2\). As in [1] applying the Leray spectral
sequence to $\pi: X \to B$ yields as sufficient condition for $H^1(X, \mathcal{E} \otimes \mathcal{O}_X(-2D)) \neq 0$ the following condition (for $x < 0$)

$$\chi(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha)) = 4 + 8\alpha^2 - 4\alpha c_1 - 2(k_1 + k_2) < 0 \quad (0.10)$$

Finally, it remains to determine the range in the Kähler cone where $V_4$ is stable, i.e., that for any coherent subsheaf $F$ of rank $0 < r < 4$ we have $\mu(F) < 0$. Solving as in [1] the corresponding inequalities we find for the general Kähler class $J = z\Sigma + h\pi^*c_1$ where $z, h \in \mathbb{R}$ with $0 < z < h$ the range (for $x < 0$; here $\zeta := h - z$)

$$\frac{-xc_1^2}{(\alpha - xc_1)c_1} h^2 < h^2 - \zeta^2 < \frac{-xc_1^2}{(\alpha - xc_1)c_1 - 1} h^2 \quad (0.11)$$

In summary, we find $V_4$ is stable if (0.10) and (0.11) are satisfied (the latter just fixes an appropriate range of $z$).

Let us now derive sufficient conditions for $H^1(X, V_4^*) \neq 0$ and $H^1(X, V_4^*)' \neq 0$. The Leray spectral sequence gives the following exact sequence

$$0 \to H^1(B, \pi_* V_4^*) \to H^1(X, V_4^*) \to$$

Thus it suffices to show that $H^1(B, \pi_* V_4^*) \neq 0$. For this we apply $\pi_*$ to the defining exact sequence of $V_4^*$ and find the exact sequence

$$0 \to E_2^* \otimes \mathcal{O}_B(-\alpha) \otimes \pi_* \mathcal{O}_X(-x\Sigma) \to \pi_* V_4^* \to E_1^* \otimes \mathcal{O}_B(\alpha) \otimes \pi_* \mathcal{O}_X(x\Sigma) \to$$

For $x < 0$ one has $\pi_* \mathcal{O}_X(x\Sigma) = 0$ and finds $E_2^* \otimes \mathcal{O}_B(-\alpha) \otimes \pi_* \mathcal{O}_X(-x\Sigma) \cong \pi_* V_4^*$. It follows

$$H^1(B, \pi_* V_4^*) = H^1(B, E_2^* \otimes \mathcal{O}_B(-\alpha) \otimes \pi_* \mathcal{O}_X(-x\Sigma)) \quad (0.13)$$

As $\pi_* \mathcal{O}_X(-x\Sigma) = \mathcal{O}_B \oplus \ldots$ it will be sufficient to show that $H^1(B, E_2^* \otimes \mathcal{O}_B(-\alpha)) \neq 0$. The index theorem gives $\chi(B, E_2^* \otimes \mathcal{O}_B(-\alpha)) = 2 + \alpha^2 - \alpha c_1 - k_2$ from which we conclude that

$$2 + \alpha^2 - \alpha c_1 - k_2 < 0 \implies H^1(B, E_2^* \otimes \mathcal{O}_B(-\alpha)) \neq 0 \implies H^1(X, V^*) \neq 0 \quad (0.14)$$

The same reasoning applied for $H^1(X, V)$ yields

$$2 + \alpha^2 - \alpha c_1 - k_1 < 0 \implies H^1(B, E_1^* \otimes \mathcal{O}_B(-\alpha)) \neq 0 \implies H^1(X, V) \neq 0 \quad (0.15)$$
Let us now determine the physical constraints we have to impose on $V_5$ in order to get a viable standard model compactification of the heterotic string. What concerns the invariance of the deformed bundle one can, as in [1] app. B, argue for the existence of invariant elements in the two non-trivial extension spaces (to solve both conditions simultaneously one can use reflection twists $v \rightarrow -v$ in both fiber vector spaces of $V_4$ and $\mathcal{O}_X$). Further one has still to make sure that the deformability to first order, which we have checked, extends to a full global construction, which we assume can be done.

Note further that the characteristic classes of $V'$ are invariant under deformations. Therefore we have $c(V_5) = c(V')$ and a direct computation yields

\begin{align}
  c_2(V_5) &= -2x(2\alpha - xc_1)\Sigma - 2\alpha^2 + k_1 + k_2 \\
  \frac{c_3(V_5)}{2} &= 2x(k_1 - k_2)
\end{align}

(0.16) (0.17)

Further one has to satisfy the heterotic anomaly condition $c_2(X) - c_2(V_5) = [W] = w_B \Sigma + a_f F$ where $W$ is a space-time filling fivebrane wrapping a holomorphic curve of $X$. This leads to the condition that $[W]$ is an effective curve class in $X$, which in turn can be expressed by the two conditions $w_B \geq 0$ and $a_f \geq 0$. Inserting the expressions for $c_2(X) = 6c_1\Sigma + 5c_1^2 + c_2$ and $c_2(V)$ gives the conditions

\begin{align}
  w_B &= (6 - 2x^2)c_1 + 4x\alpha \geq 0 \\
  a_f &= 44 + 2\alpha^2 - k_1 - k_2 \geq 0
\end{align}

(0.18) (0.19)

Finally, the physical net-generation number of chiral fermions, downstairs on $X/\mathbb{Z}_2$, is given by

\begin{align}
  N_{gen}^{phys} = x(k_1 - k_2)
\end{align}

(0.20)

To summarize, we get the following list of constraints (besides $x < 0$)

\begin{align}
  \alpha c_1 &> 0 \\
  2 + \alpha^2 - \alpha c_1 - k_i &< 0, \quad \text{where } i = 1, 2 \\
  2 + 4\alpha^2 - 2\alpha c_1 - (k_1 + k_2) &< 0 \\
  (6 - 2x^2)c_1 + 4x\alpha &\geq 0 \\
  44 + 2\alpha^2 - (k_1 + k_2) &\geq 0 \\
  x(k_1 - k_2) &= \pm 3
\end{align}

(0.21) (0.22) (0.23) (0.24) (0.25) (0.26)
(and \(k_i \geq 8\) for \(h = \frac{1}{2}\), cf. [1], app. B). One realizes that (0.24) entails \(x = -1\) and so
\[
\alpha \leq c_1
\] (0.27)
One finds that the following \(\alpha\)’s are possible (the entries in \((p, q)\) refer to the multiples of the two generators in \(B = P^1 \times P^1\)):
\[
\alpha = (-1, 2), (1, 1), (1, 0), (0, 2), (1, 2), (2, 2)
\] (0.28)
besides interchanging the entries. For instance, one finds then for \(\alpha = (1, 1)\) that \(k_1 = 8 + i\), \(k_2 = 11 + i\) where \(i = 0, \ldots, 14\) or \(\alpha = (1, 0)\) and \(i = 0, \ldots, 12\) (besides interchanging the \(k_i\)).

References

[1] B. Andreas and G. Curio, *Extension bundles and the standard model*, hep-th/0703210.
[2] R. Donagi, B. Ovrut, T. Pantev and D. Waldram, *Standard-model bundles on non-simply connected Calabi-Yau threefolds*, hep-th/0008008, JEHP 0108 (2001) 053.
[3] V. Buchard and R. Donagi, *An SU(5) heterotic standard model*, hep-th/0512149, Phys.
Lett. B633 (2006) 783.
[4] V. Bouchard, M. Cvetic and R. Donagi, *Tri-linear couplings in an heterotic minimal supersymmetric standard model*, hep-th/0602096, Nucl. Phys. B745 (2006) 62.
[5] R. Blumenhagen, G. Honecker and T. Weigand, *Loop-corrected compactifications of the heterotic string with line bundles*, hep-th/0504232, JHEP 0506 (2005) 020.
[6] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. 39 (1986), pp. 257-293., Frontiers of the mathematical sciences: 1985 (New York, 1985).
[7] S. Donaldson, Proc. London Math. Soc. 3, 1 (1985).
[8] B. Andreas and G. Curio, *Stable bundle extensions on elliptic Calabi-Yau threefolds*, math.ag/0611762.
[9] B. Andreas and G. Curio, *Heterotic models without fivebranes*, hep-th/0611309.
[10] D. Huybrechts, *The tangent bundle of a Calabi-Yau manifold - Deformations and restrictions to rational curves*, Comm. Math. Phys. 171 (1995), 139.
[11] R. Donagi, R. Reinbacher and S.T. Yau, *Yukawa Couplings on Quintic Threefolds*, hepth/0605203.
[12] E. Witten, *New Issues in Manifolds of SU(3) Holonomy,"* Nucl. Phys. B 268, 79 (1986).
[13] L. Witten and E. Witten, *Large Radius Expansion of Superstring Compactifications*, Nucl. Phys. B 281, 109 (1987).