Generation of small-scale structures in the developed turbulence

K.P. Zybin, V.A. Sirota, A.S. Ilyin, A.V. Gurevich

Abstract

The Navier-Stokes equation for incompressible liquid is considered in the limit of infinitely large Reynolds number. It is assumed that the flow instability leads to generation of steady-state large-scale pulsations. The excitation and evolution of the small-scale turbulence is investigated. It is shown that the developed small-scale pulsations are intermittent. The maximal amplitude of the vorticity fluctuations is reached along the vortex filaments. Basing on the obtained solution, the pair correlation function in the limit $r \to 0$ is calculated. It is shown that the function obeys the Kolmogorov law $r^{2/3}$.

1 Introduction

In the turbulent flow, in addition to the average velocity of the flow, random velocity pulsations are excited. These pulsations could be presented as a sum of different scales random movements. The large-scale pulsations with scale $L$ of the same order as the characteristic parameters of the flow play a leading role (for example, in the tube of radius $R$ the scale is $L \sim R/5$). Large-scale pulsations have the highest amplitudes.

The small-scale pulsations with scales $l << L$ are excited also. They have much smaller velocity amplitudes, and they could be considered as a fine structure set to the main large-scale movement. The small-scale pulsations contain only a small part of the whole turbulent kinetic energy (see Landau, Lifchitz [1], Monin, Yaglom [2]).

If the viscosity of the liquid $\nu$ is small enough, and the Reynolds number $R_u$ correspondingly is large, then the spectrum of the small-scale pulsations becomes very wide. This type of turbulence is called developed. Let $\lambda_0$ be the maximal scale where the viscosity is still significant; then the range of scales $\lambda_0 << l << L$ is called the inertial interval. The pulsations developed inside the inertial interval in different scales are determined by nonlinear processes only, since the viscosity $\nu$ is negligible. Therefore, it is possible to study the inertial interval of the turbulence in the limit $\nu \to 0$, $\lambda_0 \to 0$, $R_u \to \infty$ (Frisch [3]).

In the turbulent flow velocities are random. So, the correlation functions could be used to describe them. Let us consider the isotropic turbulence. The pair correlation function

$$K(r) = \langle [v(\rho) - v(\rho + r)]^2 \rangle,$$
determines the relation between the values of velocity in two near points \( \rho \) and \( \rho + \mathbf{r} \). Since the turbulent pulsations are isotropic, the correlation function depends on the distance \( r \) between the points only. The pair correlation function measured in numerous experiments has the universal form:

\[
K(r) = Cr^{2/3}
\]  

(1)

The distance is restricted by the condition \( r << L \), i.e. the experimental result (1) refers to the inertial interval only. The Fourier-transform \( S(k) \) of the correlation function (1) has been also investigated experimentally. These investigations give

\[
S(k) = C_f k^{-5/3},
\]

(2)

where \( k \) is the wave vector. The spectrum (2) is called the five-thirds law. The limit \( r \to 0 \) corresponds to \( k \to \infty \). In the developed turbulence the five-thirds law is observed inside a wide range of wave numbers – up to three-four orders of magnitude [2], [3].

We emphasize that the experimental measurements of the correlation function in the small-scale region were initiated by the theoretical predictions. A.N. Kolmogorov in his fundamental works in 1941 [4] derived the expressions (1), (2) for the velocity correlation function in the homogeneous and isotropic turbulence. \(^1\) The Kolmogorov’s theory is phenomenological. Its basic conception is the uniform dissipation of energy in the turbulent liquid. There is a stationary flux of energy in the Fourier space: the energy is generated in large-scale pulsations, and flows uniformly through the whole inertial interval of scales. In this process, the flux of the energy is conserved. The dissipation occurs only outside the inertial interval, at the smallest “dissipative” scales \( \leq \lambda_0 \). Relaying on this physical model, using the relations of similarity and dimensions and the general properties of hydrodynamics equations, the correlation function was found.

This fundamental Kolmogorov’s result was later confirmed in numerous experiments. It stimulated a huge amount of theoretical, mathematical and (in the recent time) numerical investigations. In these works the theory of turbulence was widely developed (see the monographs [2], [3], [6] – [9] and literature therein). Recently, the methods of field theory and solid-state physics have been used [10], [11]. However, the attempts to obtain the expression for the correlation function directly from the Navier-Stokes equation without any additional assumptions have not been successful up to now (see [3] for more details).

Another approach to the problem is based on the physical ideas of the leading role of singularity in the small-scale structures of developed turbulence [3], [13]. However, despite significant efforts in this direction, neither the correlation function has been derived from the Navier-Stokes equation, nor even the existence of singular solutions has been proved.

Thus, the problem of derivation of the fundamental Kolmogorov’s result directly from hydrodynamic equations is not solved yet (see the monographs [2], [3]).

In this paper we propose a new approach to the problem. It allows to find the structure of the small-scale turbulence and the pair correlation function.

From the hydrodynamic equations written in the Lagrangian reference frame, we derive the equation describing the joint probability density of vorticity \( \mathbf{\omega} = \nabla \times \mathbf{v} \) and its time

\(^1\)The law (2), which is the direct consequence of (1), was written in an explicit form in the papers by A.M. Obukhov [5].
derivative. We show that moments of the vorticity distribution grow unrestrictedly in time. Then we find an asymptotic solution at infinitely large time. Basing on it, we obtain the spatial distribution of the vorticity where it is large. These are vortex filaments. They give the main contribution to the pair correlation function.

The paper is organized as follows.

In Section 2 the equations of motion of incompressible liquid are considered. Their decomposition in the vicinity of trajectory of an arbitrary lagrangian particle is written. It is shown that local vorticity growth is determined by anisotropic part of large-scale pulsations of pressure.

In Section 3, supposing the randomness of large-scale pulsations of pressure, the equation for probability density of vorticity and its time derivative is obtained. We show that even moments of the vorticity distribution grow exponentially, the higher moments growing faster than the lower ones. This is the manifestation of intermittency of hydrodynamic turbulence in small scales.

In Section 4 we find the large time asymptotic solution for the joint probability density of vorticity and its time derivative.

In Section 5, on the ground of the obtained asymptotic solution, the spatial structures contributing mainly to the asymptote of the probability density are investigated. We show that these are the vortex filament structures which determine the pair correlation function of turbulent pulsations (1) as $r \to 0$.

In Conclusion we formulate and discuss the main results of the paper.

## 2 The statement of the problem

Let us consider the Navier-Stokes equation for incompressible liquid. It is known that at the scales larger than the viscous scale $\lambda_0$ it takes the form of the Euler equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla p}{\rho} = 0; \quad \nabla \cdot \mathbf{v} = 0 \quad (3)$$

Here $\mathbf{v}$ is the velocity of the flow, $p$ is the pressure. The density $\rho$ is taken unity below. The second equation expresses the incompressibility of the liquid. The equations (3) describe the processes on the scales inside the inertial interval (see [3]). From (3) one can find the relation connecting the pressure with the flow velocity:

$$-\Delta p = \nabla_i v_j \cdot \nabla_j v_i$$

To investigate the local properties of the turbulent flow we pass on to a coordinate system co-moving to some element of the liquid with coordinates $\xi(t)$:

$$\mathbf{r}' = \mathbf{r} - \xi(t), \quad \mathbf{v}' = \mathbf{v} - \dot{\xi}; \quad \ddot{\xi} = -\nabla \dot{p} \bigg|_{\mathbf{r} = \xi(t)} \quad (4)$$

Dot means the time derivative. After such change of variables the equations (3) take the form

$$\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}' \cdot \nabla) \mathbf{v}' + \nabla \dot{P} = 0, \quad \nabla \dot{P} = \nabla \dot{p} + \dddot{\xi}; \quad \nabla \cdot \mathbf{v}' = 0$$

3
Since the reference frame is chosen to be co-moving, at the point \( r = \xi(t) \) we have

\[
\nabla P(r' = 0) = 0, \quad \mathbf{v}'(r' = 0) = 0
\]

Expanding the velocity \( \mathbf{v}' \) and the pressure \( P \) into a Taylor series in the vicinity of the co-moving point we have the main term:

\[
v'_i = \left( \frac{1}{2} \varepsilon_{ikj} x_k + b_{ij} \right) r'^j
\]

\[
P = \frac{1}{2} \rho_{ik} r'^i r'^k, \quad \rho_{ik} = \nabla_i \nabla_k P
\]

Here the tensor \( \frac{\partial v'_i}{\partial r'^j} \bigg|_{r'^j=0} \) is decomposed into a sum of symmetric \( b_{ij} \) and antisymmetric \( \frac{1}{2} \varepsilon_{ikj} x_k \) parts. Note that \( b_{ii} = 0 \) since \( \nabla \mathbf{v}' = 0; \rho_{ij} \) is symmetric. It is easy to check that the vector \( x_i \) defined by the asymmetric part of \( \partial v'_i/\partial r'^j \) is the vorticity \( \mathbf{\omega} = \nabla \times \mathbf{v} \) of the flux at the point \( \xi \):

\[
\omega_i \bigg|_{r' = 0} = x_i.
\]

Combining (5), (6) and (4), we obtain

\[
\left[ \left( \dot{b}_{ij} + \frac{1}{4} (x_i x_j - x^2) \delta_{ij} + b_{ik} b_{kj} + \rho_{ij} \right) + \frac{1}{2} (\varepsilon_{ikj} \dot{x}_k - \varepsilon_{jkn} x_k b_{in} + \varepsilon_{ikn} x_k b_{jn}) \right] r'^j = 0
\]

The first term in the square brackets is symmetric, the second term is antisymmetric. Since the equation should hold for any \( r' \), both terms are equal to zero. Actually, multiplying the expression in the brackets by \( \varepsilon_{ijn} \) and taking \( b_{ii} = 0 \) into account, we find for \( x_i \) and \( b_{ij} \):

\[
\dot{b}_{ij} + \frac{1}{4} (x_i x_j - x^2) \delta_{ij} + b_{ik} b_{kj} + \rho_{ij} = 0 \quad (7)
\]

\[
\dot{x}_n = b_{nk} x_k \quad (8)
\]

Taking the time derivative of (8), we obtain finally the equations set for the three components \( x_i \):

\[
\ddot{x}_n = -\rho_{nk} x_k \quad (9)
\]

Let us clarify now the physical meaning of the symmetric part of velocity \( b_{ik} \). Namely, let us express it in terms of space distribution of the vorticity \( \mathbf{\omega}(\mathbf{r}) \). We shall see that in the completely isotropic flow \( b_{ij} = 0 \). Hence in accordance with (8), the vorticity at the lagrangian point does not change. In the real flow there are two regions where the isotropy may be broken: either local, on account of small-scale pulsations of pressure in the vicinity of the point under consideration; or global, the remote areas close to the boundary of the system, at the scales of the order \( R \). We shall show that \( b_{ij} \) is determined just by the global break of isotropy.

Since \( \nabla \cdot \mathbf{v} = 0 \), there exists a vector potential \( \mathbf{A} \):

\[
\mathbf{v} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0.
\]
Then

\[ \Delta A = -\omega. \quad (10) \]

To separate the singularity accurately, let us expand \( A(r) \) and \( \omega(r) \) into a series on spherical harmonics:

\[
A = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_{lm}(r) Y_{lm}(\theta, \varphi), \quad \omega = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \omega_{lm}(r) Y_{lm}(\theta, \varphi).
\]

The solution of the Poisson equation (10) is:

\[
A_{lm}(r) = \frac{r^{l-1}}{2l + 1} \int_{0}^{r} \omega_{lm}(r') r'^{l+2} dr' + \frac{r^l}{2l + 1} \int_{r}^{\infty} \omega_{lm}(r') r'^{l-1} dr'
\]

(11)

The integration limits are chosen to provide the convergence of the integrals at \( r \to 0 \) and \( r \to \infty \). Note that for analytic function \( \omega_{lm}(r) \propto r^l \) as \( r \to 0 \).

To evaluate \( b_{ij} \), we need to determine the limit

\[
\nabla_i v_i = \epsilon_{ijn} \nabla_j A_n \quad \text{as} \quad r \to 0.
\]

Only quadratic (in the coordinates \( r_i \)) part of \( A \) contributes in it. This quadratic part consists of two terms proportional to the zeroth \( A_{00} \) and the second \( A_{2m} \) spherical harmonics. Hence, we are interested in the two harmonics only.

The zeroth harmonic \( A_{00} \) gives the local contribution corresponding to the antisymmetric part of the velocity tensor: \( x = \omega_{00}|_{r=0} = -\Delta A_{00}|_{r=0} \). However, in the symmetric tensor \( b_{ij} \) its quadratic component \( A_{00} \propto r^2 \) is cancelled: it is just the fact that gives \( b_{ij} = 0 \) in isotropic medium. So, only the second harmonic \( A_{2m} \) remains. Since \( \omega_{2m}(r) \sim r^2 \) as \( r \to 0 \), we see that the first integral in (11) behaves like \( r^4 \), and the second one - like \( r^2 \) as \( r \to 0 \). Hence, the contribution of the ”local” item is negligibly small, and the symmetric part of the velocity tensor \( b_{ij} \) is determined by ”global”, large-scale properties of the whole flow. ²

Returning to the rectangular coordinates and taking the derivative, we obtain

\[
b_{ij} = \epsilon_{jnk} \int \frac{\omega_n(r')}{r'^3} \left( \delta_{ik} - 3 \frac{r'^i r'^k}{r'^2} \right) dr' + (i \leftrightarrow j)
\]

According to our analysis, the integrand has no singularity at \( r = 0 \); the integral accumulates at the scales of the order of \( R \), where the isotropy breaks.

The analogous argumentation shows that the pulsations of pressure \( \rho_{ik} \) (see (7)) could also be presented as a sum of local and large-scale pulsations; the local part of the tensor is \( x_i x_k - \delta_{ik}x^2 \). From (9) it follows that this tensor does not affect the local vorticity of the flow. Hence, the local dynamics of vorticity (9) depends on the large-scale pulsations of the pressure \( \rho_{nk} \) only.

Thus we obtain the first main property of the turbulent flow: the local vorticity along the streamline in homogeneous and isotropic flow is determined by anisotropic part of large-scale pulsations of the pressure.

²Note that in two-dimensional flow such a division into local and large-scale components is not possible. The zeroth cylindrical harmonic \( A_0 \) diverges logarithmically, and as a result the ”local” component should influence on the large-scale component.
3 Probability density equation

Since we interested in statistical properties of the flow, let us introduce the probabilistic description. We consider now the vorticity $\mathbf{\omega}(t)$ as a random quantity. Its change still obeys to (9). Instead of one equation of the second order, let us consider a system of two first-order equations:

$$\begin{align*}
\dot{x}_i &= y_i \\
\dot{y}_i &= -\rho_{ij} x_j
\end{align*}$$

(12)

Here $x_i \equiv \omega_i$ and $y_i \equiv \dot{\omega}_i$. We introduce a joint probability density $f(t, x, y) = \langle \delta(x - x(t))\delta(y - y(t)) \rangle$.

(13)

Here $x(t), y(t)$ are the solutions of (12) at the given realization of $\rho_{ij}$ and initial conditions; the average is taken over the ensemble of all possible realizations.

The aim of the paper is to study a steady-state turbulent flow in the inertial interval of scales, i.e. at scales $l$ and time $t$ satisfying to the conditions

$$l << L, \quad t >> \tau_c$$

(14)

Here $L$ and $\tau_c$ are the characteristic space and time correlation scales of the large-scale vortices. These large-scale vortices depend on the specific geometry of the installation and on the boundary conditions. According to the experimental data [14], the large-scale velocity pulsations are random and Gaussian. Thus, in the equations (9) or (12) for the local vorticity, the matrix $\rho_{ij}(t)$ describing the large-scale fluctuations of pressure could be taken Gaussian and, because of (14), delta-correlated in time. These propositions would be discussed below.

Note that, as it follows from (9), the ”random” behavior of vorticity (or velocity) is caused by the randomness of the large-scale flow and the corresponding matrix $\rho_{kk}(t)$.

The Gaussian random process is described by a pair correlation function

$$\langle \rho_{ij}(t)\rho_{kl}(t') \rangle = D_{ijkl} \delta(t - t')$$

(15)

Using (12) and taking time derivative of the probability density function, we obtain

$$\frac{\partial f}{\partial t} + y_k \frac{\partial f}{\partial x_k} = x_p \frac{\partial}{\partial y_k} \langle \rho_{kp}\delta(x - x(t))\delta(y - y(t)) \rangle$$

(16)

Let $R(x, y, \rho)$ be a functional of $\rho$. To find the correlation function $\langle \rho_{kp}R(x, y, \rho) \rangle$ we use the standard averaging technics for delta-correlated random process (see the monograph by Klyackin, [15]):

$$\langle z_k R[z] \rangle = \sum_{k'} \int dt' \langle z_k(t) z_{k'}(t') \rangle \left\langle \frac{\delta R[z, t]}{\delta z_{k'}(t')} \right\rangle$$

Taking (15) into account, we get:

$$\langle \rho_{kp}R(x, y, \rho) \rangle = \sum_{k'p'} D_{kpkp'} \left\langle \frac{\delta R[x, y, t]}{\delta \rho_{k'p'}(t)} \right\rangle$$

(17)
To evaluate the variational derivative (17), we use the equations of motion (12); it follows
\[
\frac{\delta y_k(t)}{\delta \rho_{k'p'}(t')} \bigg|_{t=t'} = -\delta_{kk'} x_{p'}(t), \quad \frac{\delta x_k(t)}{\delta \rho_{k'p'}(t')} \bigg|_{t=t'} = 0
\]
Combining this with (16), we obtain the Fokker-Planck equation for the function \( f(t, x, y) \):
\[
\frac{\partial f}{\partial t} + y_k \frac{\partial f}{\partial x_k} = D_{ijkl} x_j x_l \frac{\partial^2 f}{\partial y_i \partial y_k}
\]  
(18)
The matrix \( \rho_{ij}(t) \) is symmetric (6). Hence, in the homogeneous and isotropic medium the general form of the matrix \( D_{ijkl} \) is
\[
D_{ijkl} = D \delta(t - t') (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + \Gamma \delta_{ij} \delta_{kl})
\]  
(19)
The constants \( D \) and \( \Gamma \) depend on the large-scale flow. In addition to the isotropy and homogeneity, it is natural to suppose statistical independence of different components \( \rho_{ik} \). In this case one has \( \Gamma = 0 \). However, the values \( D \) and \( \Gamma \) appear to be unimportant. The parameter \( D \) in the equation vanishes as a result of time normalization. As we shall see below, the resulting properties of the turbulence depend only weakly on the parameter \( \Gamma \) (the positive definiteness leads to a restriction \( \Gamma > -2 \)).
Substituting (19) into (18), we obtain finally:
\[
\frac{\partial f}{\partial t} + y_k \frac{\partial f}{\partial x_k} = \left[ x_i \frac{\partial^2 f}{\partial y^2} + \gamma \left( x_k \frac{\partial}{\partial y_k} \right)^2 f \right]
\]  
(20)
Here \( \gamma = 1 + \Gamma \), time \( t \) is normalized by \( D^{1/3} \). The value \( D^{-1/3} \) is the characteristic time of probability density change. As it was shown above, in the completely isotropic turbulence \( D = 0 \). Taking into account small anisotropy, we have \( D^{-1/3} > > \tau_c \). This allows to use the delta-correlation approximation in derivation of the equation (20).
We now itemize the main properties of the equation (20):
1. All momenta of values \( x_k \) and \( y_j \) of the order \( n \) are connected by a system of the first order linear differential equations.
2. Even momenta grow exponentially. Independently of the initial conditions, the function \( f \) at large values \( t \) depends on the modules \( x, y \) and the cosine of the angle between the vectors \( \mu = (x, y)/xy \) only.
3. The higher even momenta grow faster than the lower ones.
To illustrate these statements, consider the momenta of the second and the fourth order. Integrating the equation (20) in \( x \) and in \( y \), we obtain for the second-order momenta:
\[
\frac{d}{dt} < x_i x_j > = < x_i y_j + x_j y_i >
\]
\[
\frac{d}{dt} < x_i y_j + x_j y_i > = 2 < y_i y_j >
\]
\[
\frac{d}{dt} < y_i y_j > = 2 \delta_{ij} < x^2 > + 2 \gamma < x_i x_j >
\]  
(21)
Let us consider the invariant momenta of the second order, i.e. \(< x^2 >\), \(< y^2 >\) and \(< x \cdot y >\). Their evolution is determined by characteristic equation

\[ \lambda^3 - 4\Gamma - 16 = 0. \]

Asymptotically as \(t \to \infty\) we obtain

\[ < x^2 > \propto < y^2 > \propto < x \cdot y > \propto \exp(\Lambda_2 t) \]

Here \(\Lambda_2 = (16 + 4\Gamma)^{1/3}\).

For the other momenta \((i \neq j)\) one has

\[ \lambda_2^3 = 4 + 4\Gamma \]

We see that \(\Lambda_2 > \lambda_2\). Hence, at large time the invariant momenta are much larger than the others. In other words, the probability density function at large time depends on three variables \(x, y, \mu = x_i y_i / (xy)\) only.

The characteristic equation for invariant momenta of the fourth order takes the form:

\[ \lambda^6 - (84\Gamma + 224)\lambda^3 - 1280 = 0. \]

For example, for \(< x^4 >\) one has \(< x^4 > \propto \exp(\Lambda_4 t)\), where

\[ \Lambda_4 = \frac{1}{8} \left[ \left( (84\Gamma + 224)^2 + 5120 \right)^{1/2} + (84\Gamma + 224) \right]^{1/3} \]

One can check up that for the values \(\Gamma > -0.9\) holds \(\Lambda_4 > 2\Lambda_2\). Hence, as \(t \to \infty\) one has \(< x^4 > \gg < x^2 >\).

The obtained relations demonstrate that the higher momenta of the vorticity module grow exponentially in time. This property is called intermittency. It reveals itself in the instability of small-scale flow. Physically this instability means that under the influence of large-scale random pulsations a drop of incompressible liquid stretches out. This leads to generation of vortex filaments. These filaments provide the basis of the small-scale turbulence (a simple physical example demonstrating the process of filament growing is considered in Appendix).

It will be shown later that the domain of parameters \(y \gg x\) plays an especially important role. In this domain the suggestion of gaussian random process is not needed. Actually, from (12) it follows that the change of \(y\) during the correlation time is \(\Delta y \sim x\). Hence,

\[ \frac{\Delta y}{y} \approx \frac{x}{y} \ll 1 \quad \text{if} \quad y \gg x \quad (22) \]

In this case the fluctuations of the probability function are very small: \(\delta f \ll < f >\), and one can obtain the equation (20) using the perturbation theory. As a result, the equation has the Fokker-Planck form.
4 Asymptotic form of the probability density function

As it was shown in the previous section, the probability density function \( f(x, y, \mu) \) at large time depends on three variables only: \( f(x, y, \mu) = f(x, y, \mu) \), where \( \mu = (x, y)/xy \). Besides, the equation (20) and the initial conditions to the probability function allows integrating over three other variables.

As a result, the equation (20) takes the form:

\[
\frac{\partial f}{\partial t} + \frac{y}{x^2} \frac{\partial}{\partial x} (\mu x^2 f) + \frac{y}{x \partial \mu} \left( (1 - \mu^2) f \right) = \frac{x^2}{y^2} \frac{\partial}{\partial y} \left( y^2 \frac{\partial f}{\partial y} \right) + \frac{x^2}{y^2} \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial f}{\partial \mu} \right) + \gamma \left( \mu x \frac{\partial}{\partial y} + \frac{x}{y} (1 - \mu^2) \frac{\partial}{\partial \mu} \right)^2 f
\]

The function \( f \) must satisfy the normalization condition

\[
\int f \, dx \, dy = \int f x^2 y^2 dx dy d\mu = 1,
\]

and two conditions of zero flux from the boundaries \( x = 0 \) and \( y = 0 \). Let us specify the meaning of these conditions. For that we return to the equation (20). It has the divergent form \( \partial f/\partial t = \nabla \alpha J_\alpha, \alpha = 1,..,6 \). The flux density \( J_\alpha \) in a 6-dimensional space \((x, y)\) is

\[
J = \left\{ -y f, x^2 \frac{\partial f}{\partial y} + \gamma x \left( x \cdot \frac{\partial f}{\partial y} \right) \right\}
\]

The no-flow boundary condition at \( y = 0 \) means that the integral of \( J \) over the 5-dimensional surface \(|y| = \epsilon\) vanishes as \( \epsilon \to 0 \). After integrating in all angles \( d\Omega_x d\Omega_y = 4\pi \cdot 2\pi d\mu \), in terms of variables \( x, y, \mu \) we have

\[
\int \left( x^2 \frac{\partial f}{\partial y} + \gamma x \left( x \cdot \frac{\partial f}{\partial y} \right) \right) \cdot \frac{y}{y^2} x^2 dx dy = 8\pi \int (1 + \gamma \mu^2) \frac{\partial f}{\partial y} x^2 y^2 dx dy \mu \to 0 \quad (24)
\]

Similarly, the no-flow condition at \( x = 0 \) leads to

\[
\int \mu y f x^2 y^2 d\mu dy \to 0 \quad x \to 0 \quad (25)
\]

The expressions (24) and (25) are the boundary conditions for the equation (23).

We now return to (23). We search for a stationary solution as \( t \to \infty \). Choosing a new variables \( x, z = y^3/(3x^3) \), \( \mu \) let us present the function \( f(x, y, \mu) \) in the form

\[
f(x, y, \mu) = \sum_\alpha x^{-2} x^{-\alpha} F(z, \mu; \alpha)
\]

The set of eigenvalues \( \alpha \) is to be found by solution of (23) with boundary conditions (24), (25).

The equation (23) then takes the form

\[
\frac{2}{z} \frac{\partial^2 F}{\partial z^2} + \left( \frac{4}{3} + \mu z \right) \frac{\partial F}{\partial z} + \frac{\alpha}{3} \mu F - \frac{1}{3} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) F \right] + \frac{1}{9z} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial F}{\partial \mu} \right]
\]
+ γ \left[ z \left( \frac{\mu}{z} \frac{\partial}{\partial z} + \frac{1 - \mu^2}{3z} \frac{\partial}{\partial \mu} \right)^2 F + \frac{2}{3} \mu \left( \frac{\mu}{z} \frac{\partial}{\partial z} + \frac{1 - \mu^2}{3z} \frac{\partial}{\partial \mu} \right) F \right] = 0

Integrate (27) on variable $\mu$ and define functions $\overline{p}(z)$, $\overline{p}^2(z)$:

$$\overline{p}(z) = \frac{\int_{-1}^{1} \mu F d\mu}{\int_{-1}^{1} F d\mu} = \frac{F_1}{3F_0}, \quad \overline{p}^2(z) = \frac{\int_{-1}^{1} \mu^2 F d\mu}{\int_{-1}^{1} F d\mu} = \frac{2F_2}{15F_0} + \frac{1}{3} \tag{28}$$

Here $F_k(z)$ are coefficients in Legendre expansion of the function $F$. The equation (27) takes the form

$$z(1 + \gamma \overline{p}^2) F_{0zz} + \left( \frac{4}{3} + \overline{p}z + \gamma \left( 2z\overline{p}^2 + \frac{7\overline{p}^2 - 1}{3} \right) \right) F_{0z}$$

$$+ \left( \frac{\alpha}{3} \overline{p} + z\overline{p}z + \gamma \left( z\overline{p}^2 + \frac{7\overline{p}^2 - 1}{3} \right) \right) F_0 = 0 \tag{29}$$

Substituting $F_0(z) = w(z) \exp \left( - \int_{0}^{z} \frac{\overline{p}(\mu) d\mu}{1 + \gamma \mu^2} \right)$, we get

$$(1 + \gamma \overline{p}^2) zw_{zz} + \left( \frac{4}{3} + \gamma \frac{7\overline{p}^2 - 1}{3} + 2\gamma z\overline{p}^2 - \overline{p}z \right) w_z + \left( \frac{\alpha}{3} - \frac{4}{3(1 + \gamma \overline{p}^2)} \right) \overline{p}w$$

$$+ \gamma \left( z\overline{p}^2 + \frac{7\overline{p}^2 - 1}{3} + \frac{3\overline{p}^2}{1 + \gamma \overline{p}^2} - \frac{z\overline{p}^2}{1 + \gamma \overline{p}^2} + \overline{p} + \frac{7\overline{p}^2 - 1}{3(1 + \gamma \overline{p}^2)} \right) w = 0 \tag{30}$$

The solutions of (30) could be presented as a series $w = z^{s} \sum_{n=0}^{\infty} c_n z^n$. It converges on the domain $0 < z < \infty$ (if $\overline{p}(z)$ and $\overline{p}^2(z)$ have no singularity).

In order to find $s$ and $c_n$, let us consider the asymptote $z \to \infty$. Expanding (29) into Legendre series, we get

$$zF_{0zz} + \frac{4}{3} F_{0z} + \frac{1}{3} \left[ zF_{1z} + \frac{\alpha}{3} F_1 \right] = 0$$

$$zF_{1zz} + \frac{4}{3} F_{1z} + \left( \frac{\partial}{\partial z} + \frac{\alpha}{3} \right) \left( F_0 + \frac{2}{5} F_2 \right) + \frac{2}{3} F_0 - \frac{2}{15} F_2 = \frac{2}{9} F_1$$

$$\ldots$$

$$zF_m'' + \frac{4}{3} F_m'' + \left( \frac{\partial}{\partial z} + \frac{\alpha}{3} \right) \left( \frac{m}{2m-1} F_{m-1} + \frac{m+1}{2m+3} F_{m+1} \right)$$

$$+ \frac{1}{3} \left( \frac{m(m+1)}{2m-1} F_{m-1} - \frac{m(m+1)}{2m+3} F_{m+1} \right) = \frac{m(m+1)}{9z} F_m$$

$$+ \gamma \left( \frac{m^2(m-1)(m-2)}{(2m-1)(2m-3)} F_{m-2} + \frac{m(m+1)}{2} \frac{(m+1)^2(m-2)}{(2m-1)(2m+3)} F_{m+1} - \frac{m(m+1)}{9z} F_m \right)$$

$${}^3$$Zero is a regular critical point of (30), since $\overline{p}(z)$ and $\overline{p}^2(z)$ are unbounded, and $1 + \gamma > 0$.

$${}^4$$As it has been mentioned in the end of previous section, in this limit the equation for probability function has the Fokker-Planck form independently of statistical properties of the large-scale random process.
\[
\gamma \frac{m(m-1)}{(2m-1)(2m-3)} F''_{m-2} + \frac{2m(m+1)-1}{(2m-1)(2m+3)} F''_m + \frac{(m+2)(m+1)}{(2m+3)(2m+5)} F''_{m+2} \\
+ \frac{\gamma}{3} \left( -\frac{m(m-1)(2m-5)}{(2m-1)(2m-3)} F''_{m-2} + \frac{8m(m+1)-4}{(2m-1)(2m+3)} F'_m + \frac{(m+1)(m+2)(2m+7)}{(2m+3)(2m+5)} F'_{m+2} \right) = 0
\]

As \( z \to \infty \), we neglect the terms proportional to \( \left( \frac{F}{z^2} \right) \). The resulting equations set has the solution

\[
F_m = (2m+1)F_0(z)
\]  

(31)

where \( F_0(z) \) satisfies (29) for \( \mu = \mu^2 = 1 \). (Actually, the coefficients (31) are the Legendre coefficients of the function \( 2\delta(1-\mu)F_0(z;\alpha) \).) Combining (31) with the definition of \( \mu(z) \), \( \mu^2(z) \) (28), we get

\[
\mu(z) = 1 - O\left( \frac{1}{z} \right) \quad \mu^2(z) = 1 - O\left( \frac{1}{z} \right)
\]

The equation (30) takes the form

\[
(1 + \gamma)zw_{zz} + \left( \frac{4}{3} + 2\gamma - z \right) w_z + \left( \frac{\alpha}{3} - \frac{4}{3(1+\gamma)} \right) w + 2\gamma \left( \frac{1}{9z} - \frac{1}{(1+\gamma)} \right) w = 0
\]

This is equivalent to Kummer degenerate hypergeometric equation [16]. The solutions of this equation are

\[
w_1(z) = z^{-\frac{4+\alpha}{1+\gamma}} M \left( a, b; \frac{z}{1+\gamma} \right)
\]

(32)

\[
w_2(z) = z^{-1/3} M \left( 1+a-b, 2-b; \frac{z}{1+\gamma} \right)
\]

where \( M \) is the Kummer function

\[
M(a, b, \zeta) = 1 + \frac{a}{b} \frac{z}{1+\gamma} + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!(1+\gamma)} + ...
\]

the parameters \( a \) and \( b \) are

\[
a = \frac{4-\alpha}{3} \quad b = \frac{2+\gamma}{3(1+\gamma)}
\]

We have found the general solution. Let us now check the boundary conditions (24), (25). The solution \( w_2 \) gives \( F_0 \sim z^{-1/3} \) as \( z \to 0 \). The correspondent \( \int y^{2f} d\mu \) does not vanish as \( y \to 0 \). This contradicts to the boundary condition (24). Hence, the solution of our problem is \( w_1 \), since it satisfies (24).

The Kummer functions behave like \( M(a, b, z) \sim e^z z^{a-b} \) as \( z \to \infty \) if \( a \) is not negative integer. Therefore, for the corresponding values \( \alpha = 4 - 3a \) and for small \( x \) we have \( F(z) \sim z^{-\alpha/3} \), \( f x^2 \sim y^{-\alpha} \). This means that the no-flux condition on the boundary \( x = 0 \) (25) is not satisfied. Hence, to satisfy both boundary conditions (24) and (25) one should take the solution \( w_1 \) with the values \( \alpha \) that correspond to “discrete" spectrum

\[
\alpha = 4 + 3n, \quad n = 1, 2, 3,...
\]

(33)

\( n = 0 \) is excluded since it does not satisfy the normalization condition for \( f \).
For these values $\alpha$ the series $M$ contains a finite number of terms, the leading term being $\sim z^n$.

The solution (32) together with (33) and (31) gives asymptotic behavior of the function $f$ as $t \to \infty$. We stress that, according to (26), (33), the full probability density function $f(x, y, \mu)$ in the leading asymptotic term behaves like

$$f(x, y, \mu) \sim x^{-9} F(z, \mu; 7)$$

as $x \to \infty$. Below, we will need integrability of the function $F$ only.

5 Spatial distribution of vorticity. Singularity of vorticity and pair correlation function.

In the previous section we found an asymptotic solution for probability distribution of $\omega, \dot{\omega}$. It is important that the solution has power fall as $x \to \infty$. This means a significant probability of large-amplitude fluctuations of $|\omega|$. This is the manifestation of intermittency in the turbulence: in some spatial domains the value $|\omega|$ is much larger than its average. The question is how the flow in this domains should look like to provide the obtained asymptote.

For this purpose, let us define the probability density function for the module of vorticity based on the combined probability density of $\omega, \dot{\omega}$, (13),(20):

$$P(x, t) = \int f(t, x, y) x^2 d\mu = \langle \delta(x - |x(t)|) \rangle$$

(34)

On the other hand, we independently define a probability density $P_1(x, t)$ as space average of some realization of the turbulent flow:

$$P_1(x, t) = \frac{1}{V} \int \delta(x - X(t, r)) dr.$$  

(35)

Here $X(t, r) = |\omega(t, r)|$ is the vorticity module at time $t$ and at the point $r$, $V$ is the volume of space occupied by the flow.

The first expression for the probability density $P(x, t)$ is the ensemble average along a trajectory of liquid particle $\xi(t)$, and the second expression for $P_1(x, t)$ is the space average.

Owing to ergodicity (i.e. the equality of ensemble and space averages) we obtain

$$P(x, t) = P_1(x, t)$$

(36)

If the function $P(x, t)$ is known, then it is possible to derive space distribution of the vorticity module $X(t, r)$ using (36) and (35). Since we are interested in possible singularities and their surrounding, let us consider the limit $t \to \infty$ and next $x \to \infty$.

Suppose that the singularity is reached at some surface. Taking the point of origin on the surface and the axis $z$ perpendicular to it, we find

$$P_1(x, t) = \frac{1}{V} \int \delta(x - X(t, z)) d\sigma dz = \frac{1}{\left|X_z\right|_t}$$

(34)
Here \(d\sigma\) is the element of the surface area. The simplest example of surface where the vorticity grows unrestrictedly large is the tangential break of flow velocity. Indeed, in the case the velocity is \(V_0\) on one side of the contact surface and \(-V_0\) on the other side. Hence, the vorticity is concentrated on the surface.

Let us now consider the most interesting case: the maximum of \(X(t, r)\) is reached along a vortex line. Then, choosing the cylindrical variables \(z, r, \phi\) with \(z\) axis oriented along the line, we obtain

\[
P_1(x, t) = \frac{1}{V} \int \delta(x - X(t, r)) r dr d\phi dz = \frac{r}{|X'|_{X(t,r)=x}}
\]

In the case of point-like maximum, using spherical coordinates \(r, \theta, \phi\), we would obtain

\[
P_1(x, t) = \frac{1}{V} \int \delta(x - X(t, r)) r^2 dr d\theta d\phi = \frac{r^2}{|X'|_{X(t,r)=x}}
\]

Taking the limit \(t \to \infty\), with account of (36) we find the spatial distribution of vorticity module in the vicinity of singularity:

\[
X'(z)P(X) = 1 - \text{singular surface}
\]

\[
X'(r)P(X) = r - \text{singular line}
\]

\[
X'(r)P(X) = r^2 - \text{singular point}
\] (37)

In Section 4 we found the asymptotic expression for probability density function \(F_0 = \int F(z, \mu, x) d\mu\) (32), (33). Integrating it with respect to \(z\), we obtain the function \(P(x)\):

\[
P(x) = \sum_{\alpha} p_\alpha x^{3-\alpha} \quad \alpha = 4 + 3n \quad n = 1, 2, 3, ...
\] (38)

Combining (38) with (37) and integrating (37) in the vicinity of the singular point, we get in the leading asymptotic term

\[
X(z) \sim z^{-\frac{1}{3}} - \text{singularity of vortex surface}
\] (39)

\[
X(r) \sim r^{-\frac{2}{3}} - \text{singularity of vortex line}
\]

\[
X(r) \sim r^{-1} - \text{singular point}
\] (40)

We see that in the case of singular point, \(X(r)\) diverges less than any singularity specified by the Laplace equation. This means that the singularity of such kind cannot exist as the isolated point.

The most divergence in (40) is given by the singularity along the filament. We shall show that the dependence \(X(r) \sim r^{-\frac{2}{3}}\) corresponds to the Kolmogorov law. Consider the correlator of transverse velocities

\[
K(r) = \langle (v_{\perp}(r, t) - v_{\perp}(0, t))^2 \rangle
\] (41)
Here $v_\perp$ is the velocity component perpendicular to the current line $\xi(t)$. According to the definition (4), the expression $v_\perp(r, t) - v_\perp(0, t)$ is identically equal to $\omega \times r$ in the vicinity of the vorticity’s singularity. Hence, as $r \to 0$ we get

$$K(r_\perp) \propto \omega^2 r_\perp^2 \propto r_\perp^{2/3}$$  \hspace{1cm} (42)

Since the direction of the line is arbitrary, the turbulent pulsations change it. From the isotropy it naturally follows that the space average over the main scale of the turbulence is equivalent to the average over the angles. The expression (42) then transforms to

$$K(r) \propto r^{2/3}$$  \hspace{1cm} (43)

So, the main expression (1), (43) follows naturally from our consideration. It means that Kolmogorov’s correlation function (43) is determined by the system of vortex filaments. The input of regular part of the velocity to the correlation function ($\propto r^2$) is negligible.

### 6 Conclusion

In this paper we investigated the small-scale structure generation in the developed turbulence. Here we summarize and discuss briefly the main results.

1. Vortex structures

We considered the Navier-Stokes equation as $\nu \to 0$. We derived the equations describing the growth of small-scale pulsations along the Lagrangian trajectory under the action of the large-scale turbulence. We showed that the small-scale part of vorticity grew exponentially in time. This growth led to formation of a system of filaments and surfaces where the vorticity grew intensively. We derived the characteristic parameters of the vortex structure growth in time.

2. Singularity

We showed that in the non-dissipative limit $\nu \to 0$ the absolute value of the vorticity $\omega$ tended to infinity along the vortex filaments as $t \to \infty$. Notice that constructing the probability density function (20) we linearized the hydrodynamic equations near the Lagrangian trajectory (9). We obtained the exponential growth of vorticity which may be cut either by nonlinear corrections or by viscosity of the flow. Let us discuss now the nonlinear corrections. The feedback effect of the small-scale pulsations could be estimated by comparison of the energy density of small-scale pulsations with the energy of large-scale ones. The energy density of the main pulsations on the scale $L$ is

$$E_0 = \frac{1}{2} \rho U^2$$

here $U$ is the velocity of the large-scale pulsations. From (39) we estimate the velocity of the filament having width $r_0$: $v_n \sim U (r_0/L)^{1/3}$. Taking into account the part of volume occupied by the filaments, we obtain the relation between the energy densities:

$$\frac{E_n}{E_0} \sim N_f \left(\frac{v_n}{U}\right)^2 \left(\frac{r_0}{L}\right)^2 \sim N_f \left(\frac{r_0}{L}\right)^{8/3} \ll 1.$$
Here $N_f$ is the ratio of the number of filaments to the number of large-scale vortex in the volume unit.

Thus, we see that the feedback effect of small-scale pulsations on the large-scale ones is insignificant. Hence, the singularity should be cut off by viscosity. The situation is quite analogous to that in supersonic hydrodynamics: the singularities (strong and weak discontinuities) in Euler flow are cut off by viscosity.

3. Correlation function

We found the solution of the equation describing the vorticity distribution in the vicinity of the vortex filaments. The solution had the form $|v_\perp| \propto r_\perp^{1/3}$ in the plane perpendicular to the vortex filament. This was the solution that determined the form of the pair correlation function in small scales. Thus, we found the velocity correlation function (1), (43) in the steady-state turbulent flow directly from the Navier-Stokes equation in non-viscous limit $\nu \to 0$.

4. Intermittency

According to the Kolmogorov-Richardson assumption the energy flux in turbulent flow cascades from larger scales to smaller ones and dissipates at the smallest scales uniformly in space and time. Landau pointed out that this assumption was controversial (see [3] §6.4). Gurvich in 1960 [18], and later the other researchers discovered experimentally a very strong time and space inhomogeneity of velocity and energy flux. This property of turbulence is called intermittency. Variety of approaches to this effect was considered by many authors (see monographs [2],[3] and citations therein).

Let us list the intermittency features that follow from the presented theory.

1) The vorticity distribution in space is very inhomogeneous. Near the vortex axis it could possess the value many times exceeding its average.

2) Even moments of the correlation functions should grow with number of the moment.

3) The energy dissipation in the developed turbulent flow is localized near the axes of vortex filaments and the vortex surfaces. It is distributed very inhomogeneously in space and time due to the vortex structures motion. Besides, the strong nonuniform dissipation is the most pronounced manifestation of the intermittency [19].

Note that the filaments give the maximum degree of singularity as $t \to \infty$ and are responsible for the form of the pair correlation function. However, the surface-type singularities may affect the dissipation process, since they could occupy a significant part of the volume of the flow.

The authors are grateful to V.L. Ginzburg for the attention to this work, and to A.S. Gurvich, V.S. Lvov, E.A. Kuznetsov, S.M. Apenko, V.V. Losyakov and M.O. Ptitsyn for useful discussions.

This research was partially supported by the RAS Presidium Program "Mathematical methods in nonlinear dynamics".

Appendix

Let us consider an axially symmetric flow. The hydrodynamic equations in the cylindric
coordinates take the form

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\phi^2}{r} = - \frac{\partial p}{\partial r} \quad (A.1)
\]

\[
\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_\phi v_r}{r} = 0 \quad (A.2)
\]

\[
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = - \frac{\partial p}{\partial z} \quad (A.3)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{\partial v_z}{\partial z} = 0 \quad (A.4)
\]

Here \(v_r, v_\phi, v_z\) are the radial, azimuthal and parallel to the cylinder’s axis velocity components, respectively.

We search a solution of the system (A.1) – (A.4) in the form

\[
v_\phi = \omega r, \quad v_r = ar, \quad v_z = bz \quad (A.5)
\]

Then the pressure should be

\[
p(r, z, t) = \frac{P_1(t)}{2} r^2 + \frac{P_2(t)}{2} z^2
\]

From (A.4) follows a relation between \(a\) and \(b\):

\[
2a + b = 0, \quad (A.6)
\]

This relation expresses the volume conservation in the liquid. Indeed, let us consider a cylindrical drop with radius \(R(t)\) and length \(Z(t)\). Then from (A.5) follows

\[
\dot{R} = a(t) R, \quad \dot{Z} = b(t) Z,
\]

hence

\[
R(t) = R_0 \exp \left( \int_0^t a(t_1) dt_1 \right), \quad Z(t) = Z_0 \exp \left( \int_0^t b(t_1) dt_1 \right)
\]

The cylinder volume at arbitrary time \(t\) is

\[
\pi R(t)^2 Z(t) = \pi R_0^2 Z_0 \exp \left( \int_0^t (2a(t_1) + b(t_1)) dt_1 \right) = \pi R_0^2 Z_0.
\]

We see that volume conserves. For example, if \(b > 0\) then the cylinder stretches, and its transversal radius decreases.

Combining (A.5) with (A.1) – (A.4), we obtain a system of ordinary differential equations:

\[
\dot{a} + a^2 - \omega^2 = -P_1 \quad (A.7)
\]

\[
\dot{\omega} + 2a\omega = 0
\]

\[
\dot{b} + b^2 = -P_2
\]
Note that the system allows one arbitrary function of time. Actually, with account of (A.6) we have four equations and five unknown functions: $a, b, \omega, P_1, P_2$. Without loss of generality one can choose $P_2(t)$ as such arbitrary function. We also note that the change of vorticity $\omega(t)$ is connected unambiguously with change of the ”cylinder length” $Z(t)$: $\omega(t) = \omega_0 Z(t)/Z_0$.

Differentiating the second equation of the system (A.7) and substituting other equations, we get

$$\ddot{\omega} = -P_2(t) \omega$$

This equation is a particular case of (9). We assume that $P_2(t)$ is rather complicated ”random” function and its time average is zero. Then the time intervals when $P_2(t) > 0$ and $P_2(t) < 0$ are equally probable. However, at $P_2(t) > 0$ the function $\omega(t)$ oscillates, the oscillation amplitude changing weakly. To the contrary, at $P_2(t) < 0$ the function $\omega(t)$ grows exponentially. It is clear that in the average the value $\omega$ grows. Since $\omega$ and $Z$ are proportional, such growth means ”stretching out” the cylinder.
References

[1] L.D.Landau, E.M.Lifshitz "Hydrodynamics" Pergamon Press 1975, ch.3.
[2] A.S.,Monin A.M.Yaglom "Statistical Fluid Mechanics" vol.1 Ed.J.Lumley, MIT Press, Cambridge, MA, 1971; vol.2 1975.
[3] U.Frisch, ”Turbulence. The Legacy of A.N.Kolmogorov” Cambridge University Press, 1995
[4] A.N.Kolmogorov, Doklady Academy of Science USSR, 30, 9 - 13, 1941; 31, 583 - 540, 1941; 32, 16 - 18, 1941 (in Russian)
[5] A.M.Obukhov, Doklady Academy of Science USSR, 32, 22-24, 1941; (in Russian)
[6] V.E.Zakharov, V.S.L’vov, G.Falkovich, ”Kolmogorow spectra of turbulence”, Springer, Berlin, 1992
[7] M.J.Vishik, A.F.Fursikov ”Mathematical problems of statistical hydrodynamics” Kluwer, Dordrecht, 1988
[8] C.Foias, O.Manley, R.Rosa and R.Temam ”Navier-Stokes equations and Turbulence” Cambridge Univ. Press 2001
[9] P.G.Saffman ”Vortex dynamics” Cambridge Univ.Press, Cambridge, 1992
[10] V.S.L’vov, I.Procaccia ”Analytic calculation of anomalous exponents in turbulence: Using the fusion rules to flush out a small parameter” Phys.Rev.E, v.62, N 6, 8037 - 8057, (2000).
[11] V.Yakhot ”Probability density in strong turbulence” arXiv:physics/0512102 v3 (2005).
[12] V.I.Belincher, V.S.L’vov, Sov.Phys. JETP 66, 349 (1977).
[13] E.A.Kuznetsov and V.P.Ruban, JETP, 91, 775-785 (2000).
[14] A.Noullez, G.Wallace, W.Lempert, R.Miles, U.Frisch J.Fluid. Mech., 339; 287 - 307, (1997)
[15] V.I.Klyatskin ”Dynamics of Stochastic Systems” Fizmatlit, 2003
[16] M.Abramowitz I.Stegun ”Handbook of Mathematical Functions” National Bureau of Standards, 1964
[17] A.S.Gurvich, V.V.Pachomov, A.M.Cheremuchin, Radiofizika, v.7, 76-80, (1971).
[18] A.S.Gurvich, Izvestiya Academy of Sci USSR, geofizika, 7, 1042-1055, 1960
[19] C.M.Menevean, K.R.Sreenivasan, ”The multifractal nature of turbulent energy dissipation”, J.Fluid Mech. 224, 429 - 484, 1991