Exponential Contraction in Wasserstein Distance for Diffusion Semigroups with Irregular Drifts

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December 31, 2018

Abstract
The exponential contraction in standard Wasserstein distance
\[ W_q(\mu_1 P_t, \mu_2 P_t) \leq ce^{-\kappa t} W_q(\mu_1, \mu_2), \quad t \geq 0, q \geq 1 \]
is proved for the transition semigroup \( P_t \) associated with stochastic differential equations driven by multiplicative noise. Our first result is obtained by using the coupling by reflection and a new auxiliary function. Conditions on coefficients are explicit and can be applied to equations with less regular drifts. To treat equations more singular drift, we assume the diffusion semigroup possesses a nice part which generates an ultracontractive semigroup. A Zvonkin's transformation is given by the ultracontractivity. Our second conclusion is obtained by combining our first result and the Zvonkin’s transformation.

AMS Subject Classification (2010): 60H10
Keywords: Wasserstein distance; coupling by reflection; Zvonkin’s transformation; diffusion processes

1 Introduction
In this paper, we shall consider diffusion processes in \( \mathbb{R}^d \) given by the stochastic differential equations (SDEs for short)
\[ dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad (1.1) \]
where \( \{W_t\}_{t \geq 0} \) is a \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \), \( b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) are measurable. We denote by \( P_t \) the transition semigroup associated with \( X_t \) and \( \mu P_t \) by the distribution of \( X_t \) with initial distribution \( \mu \). To consider the long-time behaviour of \( X_t \) or \( P_t \), many aspects and approaches have been developed. Denote by \( \mathcal{P}(\mathbb{R}^d) \) all the probability measures on \( \mathbb{R}^d \). The \( L^q \)-Wasserstein distance \( (q \geq 1) \) defined as follows for all \( \mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d) \) is used to
measure the evolution of \( \{\mu_t \}_{t \geq 0, \mu \in \mathcal{P}(\mathbb{R}^d)} \) with time \( t \):

\[
W_q(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^q \pi(x, y) \right)^{\frac{2}{q}}
\]

where \( \mathcal{C}(\mu_1, \mu_2) \) consists all couplings of \( \mu_1 \) and \( \mu_2 \); \( | \cdot | \) is the Euclidean norm. The Wasserstein distance has various characterizations and plays an important role in the study on partial differential equations, optimal transportation problem etc. For more discussions, one can consult \[3, 16, 17, 21, 22, 23, 25\] and references there in.

One typical phenomenon of the system \( \{\mu_t \}_{t \geq 0, \mu \in \mathcal{P}(\mathbb{R}^d)} \) associated with some important models is the following exponential contraction in the Wasserstein distance

\[
W_q(\mu_1 P_t, \mu_2 P_t) \leq c e^{-\kappa t} W_q(\mu_1, \mu_2), \quad t \geq 0, q \geq 1. \tag{1.2} \]

For instance, if \( \sigma(\cdot) \equiv 1 \) and \( b_i \) is time independent with the following uniformly dissipative condition holds

\[
(b(x) - b(y), x - y) \leq -\kappa |x - y|^2, \quad x, y \in \mathbb{R}^d, \tag{1.3}
\]

then \eqref{1.2} holds with \( c = 1 \) and the same positive constant \( \kappa \), see e.g. \[2\]. For some SDEs without uniformly dissipative drift, \[9\] showed the following inequality holds

\[
W_q(\delta_x P_t, \delta_y P_t) \leq c_1 e^{-\frac{\kappa q}{q-1}} \left( |x - y| + |x - y|^\frac{1}{q} \right)
\]

for some positive constant \( c_1, c_2 \). Their work is inspired by \[7\], where \eqref{1.2} with \( q = 1 \) was proved for equations without dissipative drift. It is clear that the inequality in \[9\] can not implies \eqref{1.2}, since \( |x - y| < |x - y|^\frac{1}{q} \) for \( |x - y| < 1 \) and \( \frac{a^q}{q} \rightarrow 0 \) as \( q \rightarrow \infty \).

In the Riemannian setting, \eqref{1.2} with \( c = 1 \) (\( \kappa \) may be negative) is equivalent to that the curvature of the generator of the diffusion is bounded below, see e.g. \[26\]. In some negative curvature cases, it has been showed in \[27\] that \eqref{1.2} holds with \( \kappa > 0 \) and \( c > 1 \) for any \( q \geq 1 \) under the assumptions of ultracontractivity of \( P_t \) and the curvature is bounded below. Hence, one aim of this paper, we shall give explicit conditions on coefficients of \eqref{1.1} to imply \eqref{1.2}. Following from \[7\] and \[9\], the coupling by reflection introduced by \[15\] and developed by \[4, 26\] is adopted.

In \[9\], the authors used an auxiliary function motivated by Chen-Wang’s variational formula for principal eigenvalue, see also \[5\]. Here, we shall give another construction of the auxiliary function, which is in the same spirit of the one given in \[6\]. In \eqref{1.2}, it is crucial that the constant \( \kappa \) is independent of \( q \). Since our auxiliary function is explicit and calculable, we can give a clear dependency on \( q \) of the parameters in the auxiliary function, which allows us to give a \( q \)-independent right hand side of \eqref{1.2} by choosing suitable parameters.

On the other hand, to use the auxiliary function, some uniformly conditions are needed for the coefficients of SDEs. More precisely, for instance, the right hans side of \eqref{1.3} depends only on \( |x - y| \), so

\[
\sup_{|x - y| = r} \langle b(x) - b(y), x - y \rangle < \infty, \quad r > 0.
\]

This excludes some drift without this property, i.e. the left hand side of the above inequality is explosion. For a concrete example, one can check the following drift

\[
b(x) = -\frac{x}{|x|} \left( \log \frac{1}{|x|} \wedge 1 \right)^\frac{1}{q} 1_{|x| \neq 0} - |x|^2 x. \tag{1.4}
\]
To allow more singular drift, we shall use the Zvonkin’s transformation. Indeed, reconsider the stander the Wasserstein distance $W_q$, the Euclidean norm $| \cdot |$ can be replaced by other distance in $\mathbb{R}^d$ to obtain a Wasserstein-type distance, see e.g. [7,11,27]. One way is to replace $|x - y|$ in $W_1(\mu_1, \mu_2)$ by a concave and strictly increasing function $\psi$ of $|x - y|$ with $\psi(0) = 0$. Another way is to change the topology of $\mathbb{R}^d$ by using a diffeomorphism on $\mathbb{R}^d$, as in the Riemannian setting. From this point of view, the Zvonkin’s transformation provides the second alternative way. A semigroup $P_t$ with the exponential $W_q$-contraction property usually has a nice dissipative part. However, the Zvonkin’s transformation used in [30] is rooted in the Laplace operator without low order term. To utilise the dissipative unbounded first order term, we develop the method used in [27] and introduce a Zvonkin’s transformation based on the ultracontractivity of the nice part of the semigroup. This provides an alternative way to consider the Zvonkin’s transformation. Combining this transformation and the previous study, we get the exponential $W_q$-contraction property.

This paper is structured as follows. In Section 2, we study the exponential $W_q$-contraction property by the reflection coupling. Section 3 is devoted to the study of Zvonkin’s transformation based on the ultracontractivity and the exponential $W_q$-contraction for SDEs with singular drift.

## 2 Main results by the coupling by reflection

In this section, we consider the exponential convergence in $L^q$-Wasserstein distance for diffusion processes by using the coupling by reflection. We assume that the equation (1.1) has a unique weak solution (or equivalently the martingale solution, see [12 Proposition IV-2.1]) and the solution is pathwise continuous. Then the strong Markov property holds, see for instance [12 Theorem IV-5.1]. Since we shall study the convergence in $L^q$-Wasserstein distance, we assume in addition that the solution $\{X_t\}_{t \geq 0}$ has finite moments of all orders.

If $\sigma_t \sigma_t^* \geq \sigma_0^2$ with $\sigma_0 \in \mathbb{R}^+$, then we can follows [19,26] and set $\bar{\sigma}_t = \sqrt{\sigma_t \sigma_t^* - \sigma_0^2}$ and consider the following equation

$$dX_t = b_t(X_t)dt + \bar{\sigma}_t(X_t)dW_t^{(1)} + \sigma_0 dW_t^{(2)}, \tag{2.1}$$

where $\{W_t^{(1)}\}_{t \geq 0}$ and $\{W_t^{(2)}\}_{t \geq 0}$ are two independent $d$-dimensional Brownian motion, which are independent of $\{W_t\}_{t \geq 0}$. It is clear that the martingale solution of (2.1) is a martingale solution of (1.1). By the weak uniqueness, we shall consider (2.1) instead.

The reflection coupling will be used to obtain the convergence in the Wasserstein distance. A coupling process $\{(X_t, Y_t)\}_{t \geq 0}$ will be considered with $X_t$ given by (2.1) and $Y_t$ given by the following equation

$$dY_t = b_t(Y_t)dt + \bar{\sigma}_t(Y_t)dW_t^{(1)} + \sigma_0 \left( I - 2 \sigma_0^{-1}(X_t - Y_t) \otimes \sigma_0^{-1}(X_t - Y_t) \right) dW_t^{(2)}, \tag{2.2}$$

where $\tau = \inf\{t \geq 0 \mid X_t = Y_t\}$ is the coupling time. Indeed, the stopping time $\tau$ is the first time that $(X_t, Y_t)$ hits the diagonal $D := \{(x, y) \in \mathbb{R}^{2d} \mid x = y\}$, and the process $\{(X_t, Y_t)\}_{t \geq 0}$ can be realised as a diffusion process on $\mathbb{R}^{2d}$ by solving the...
system \( (2.1)-(2.2) \) (without the coupling time \( \tau \)) until it hits the diagonal for the first time then setting \( X_t = Y_t \) for \( t \geq \tau \). Moreover, the process
\[
\left\{ \int_0^s \left( I - 2\frac{\sigma_0^{-1}(X_t - Y_t) \otimes \sigma_0^{-1}(X_t - Y_t)}{|\sigma_0^{-1}(X_t - Y_t)|^2} \mathbb{1}_{[t \leq \tau]} \right) \, dW_t^{(2)} \right\}_{s \geq 0}
\]
is a Brownian motion. Thus \( Y_t \) is also the weak solution of \( (2.1) \). Consequently, \( Y_t \) has the same law of \( X_t \). We define
\[
L_t f(x,y) = \langle b_t(x), \nabla_1 f(x,y) \rangle + \langle b_t(y), \nabla_2 f(x,y) \rangle + \frac{1}{2} \text{tr} \left( \sigma_t(x) \sigma_t^*(x) \nabla_1^2 f(x,y) \right) + \frac{1}{2} \text{tr} \left( \sigma_t(y) \sigma_t^*(y) \nabla_2^2 f(x,y) \right) + \frac{2\sigma_0^2}{|x-y|^2}, \quad f \in C_0^2(\mathbb{R}^{2d} \setminus D)
\]
where \( \nabla_1 \) and \( \nabla_2 \) are gradients w.r.t. the first component and the second component of \( f(x,y) \) respectively. The operator \( L_t \) is called the coupling operator of reflection coupling, see [4]. We assume that there is unique solution for the martingale solution of \( L_t \) up to the time \( \tau \), i.e. for all \( (x,y) \in \mathbb{R}^{2d} \setminus D \) there exists probability measure \( \mathbb{P}^{x,y} \) such that for all \( f \in C_0^2(\mathbb{R}^{2d} \setminus D) \)
\[
f(X_t,Y_t) - \int_0^t L_s f(X_s,Y_s) \, ds
\]
is a \( \mathbb{P}^{x,y} \) martingale.

**Theorem 2.1.** Assume that \((1.1)\) has a unique weak solution with finite moments of all orders and the martingale problem of \( L_t \) has a unique solution up to the time \( \tau \). Suppose that
\[
\sigma_t(x) \sigma_t^*(x) \geq \sigma_0^2, \quad x \in \mathbb{R}^d
\]
with \( \sigma_0 \in \mathbb{R}^+ \), and there exist \( \theta \geq 0 \) such that for \( q \geq 1 \), there are a positive constant \( \bar{K}_2 \) and a positive function \( \bar{K}_1 \) continuous on \((0,\infty)\) satisfying
\[
\lim_{v \to +\infty} \bar{K}_1(v) v^{-1} = 0, \quad \int_0^t v^{-1} \bar{K}_1^{\frac{1}{q-1}}(v) \, dv < \infty, \quad t > 0, \quad (2.3) \tag{K-1}
\]
such that
\[
\langle b_t(x) - b_t(y), x - y \rangle + \frac{1}{2} |\dot{\sigma}_t(x) - \dot{\sigma}_t(y)|_{HS}^2 + \frac{(2q-3)(\dot{\sigma}_t^*(x) - \dot{\sigma}_t^*(y))(x-y)^2}{2|x-y|^2} \leq \bar{K}_1 |x-y| - \bar{K}_2 |x-y|^{2+\theta}, \quad (2.4) \tag{cond-b-si}
\]
where \( \dot{\sigma}_t = \sqrt{\sigma_t \sigma_t^* - \sigma_0^2} \). Then there exist positive constants \( c_1, c_2 \) and \( \bar{C} \) such that
\[
W_q(\dot{\delta}_x P_t, \dot{\delta}_y P_t) \leq \left( \frac{c_2}{c_1} \right)^{\frac{1}{\theta}} \left( 1 + c_1^{\frac{\theta}{\theta}} \left( 1 - e^{-\frac{c_1}{a}} \right) |x-y|^\theta \right)^{\frac{1}{\theta}} e^{-\frac{c_1}{a} |x-y|}, \quad (2.5) \tag{W-p-con}
\]
In addition to the above assumptions, if \( \bar{K}_2 \) is independent of \( q \) and \( \bar{K}_1 \) also satisfies
\[
\sup_{q \geq 1} \int_0^t v^{-1} \bar{K}_1^{\frac{1}{q-1}}(v) \, dv < \infty; \quad \bar{K}_1(v) \leq C(2q-1)^m v^{\alpha}, \quad v \geq 1, \quad q \geq 1 \quad (2.6) \tag{tld-K}
\]
with $q$-independent constants $C \geq 0$, $\alpha \in [0,2+\theta)$ and $m \geq 0$, then there exists $c_0 > 0$ and $\kappa > 0$ such that

$$W_q(\delta_x P_t, \delta_y P_t) \leq c_0 \left(1 + \left(1 - e^{-\kappa \theta t}\right) |x - y|^\beta\right)^{-\frac{1}{\theta}} e^{-\kappa t |x - y|}, \; q \in [1, \infty].$$

We remark here that $\bar{K}_1$ and $\bar{K}_2$ may depend on $q$. As a direct consequence, we present the following corollary. The proof follows from Theorem 2.1 and [14, Lemma 3.3], and we omit it.

**Corollary 2.2.** Under all the assumptions of Theorem 2.1, then $W_q$-exponential contraction holds

$$W_q(\mu_1 P_t, \mu_2 P_t) \leq e^{-\alpha t} W_q(\mu_1, \mu_2), \; t \geq 0, \mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d), \; q \in [1, \infty].$$

The following examples are presented to illustrate our results. For the diffusion coefficient $\sigma$, we assume that there exist $C > 0$ and $\alpha_1 \in (0, 1]$ such that

$$|\sigma_t(x) - \sigma_t(y)| \leq C |x - y| \wedge |x - y|^{\alpha_1}, \; x, y \in \mathbb{R}^d.$$

Then it follows from [19] Lemma 3.3 and [1] Theorem 1 that

$$|\tilde{\sigma}_t(x) - \tilde{\sigma}_t(y)|_{HS} \leq \frac{1}{2 \sigma_0} \left|\sqrt{\sigma_t \sigma_t^*} - \sqrt{\sigma_t \sigma_t^*}(y)\right|_{HS} \leq \frac{1}{\sqrt{2 \sigma_0}} |\sigma_t^*(x) - \sigma_t^*(y)|_{HS} \leq \frac{C}{\sqrt{2 \sigma_0}} |x - y| \wedge |x - y|^{\alpha_1}.$$  

The following example has been studied in [9] and [28].

**Example 2.3.** If $b_t$ satisfies

$$\langle b_t(x) - b_t(y), x - y \rangle \leq \left((K_1 + K_2) 1_{|x - y| \leq r_0} - K_2\right) |x - y|^2, \; x, y \in \mathbb{R}^d,$$

then (2.4) holds with $\bar{K}_2 = K_2$ and

$$K_1(v) = C_1 \left(K_1 + K_2 + \left(q - \frac{3}{2}\right)^+\right) v^2 \wedge v^{2\alpha_1}, \; v > 0$$

with some positive $C_1$ independent of $q$. Since

$$\sup_{q \geq 1} \left\{C_1 \left(K_1 + K_2 + \left(q - \frac{3}{2}\right)^+\right)\right\} < \infty,$$

it is easy to check that (2.6) holds.

Next example was studied in [9] for equations driven by the additive noise. The following example can also be applied to the finite dimensional case studied in [27].

**Example 2.4.** If $b_t = b_t^{[1]} + b_t^{[2]}$ with (2.4) holds for $b_t^{[1]}$ with $\bar{K}_1 \equiv 0$ and $\bar{K}_2$ independent of $q$; $b_t^{[2]}$ is bounded measurable. Then (2.4) holds with $\bar{K}_2$

$$K_1(v) = C_1 \left(v + q \nu^2 \wedge v^{2\alpha_1}\right), \; v > 0$$

with some $C_1 > 0$ independent of $q$. Then (2.6) holds.
Finally, we finish this section by a similar result to [9, Proposition 1.6]. The proof can completely follow from [9, Proposition 1.6], so we omit it.

**Proposition 2.5.** If there is \( r_0 > 0 \) and \( c > 0 \) such that for all \( |x - y| = r_0 \) it holds that
\[
\frac{2(b_t(x) - b_t(y), x - y) + |\sigma_t(x) - \sigma_t(y)|_{HS}^2}{|x - y|} \leq -c,
\]
then there are \( c_1 > 0 \) and \( \delta > 0 \) such that
\[
\sup_{|x - y| = r} \frac{2(b_t(x) - b_t(y), x - y) + |\sigma_t(x) - \sigma_t(y)|_{HS}^2}{|x - y|} \leq -c_1 r + \delta.
\]

### 2.1 Proofs

Before the proof of the theorem, we give a lemma on the auxiliary function which will be used in the proof of **Theorem 2.1**. For each \( p \geq 1 \), let
\[
p_1(v) = pv^{\frac{\beta - 2}{\beta}} \tilde{K}_1^\frac{1}{\beta}(v) - p\tilde{K}_2 v^{1 + \frac{\beta}{p}} + 2\sigma_0^2 p(p - 1)v^{\frac{p - 2}{p}}
\]
\[
p_0(v) = 4\sigma_0^2 \tilde{K}_2 v^{\frac{2p - 2}{p}}, \quad v > 0.
\]

It follows from (2.3) that
\[
\lim_{v \to +\infty} v^{\frac{p - 2}{p}} \tilde{K}_1^\frac{1}{\beta}(v) = \lim_{v \to +\infty} \tilde{K}_1^\frac{1}{\beta}(v) = 0,
\]
which, together with \( \lim_{v \to +\infty} p_0(v)v^{\frac{2p}{p} - 2} = 0 \), implies that there exist \( v_0 > 0 \) and \( \tilde{c}_1 > 0 \) such that
\[
-\left(1 - \frac{1}{2p}\right) \frac{p_0(v)}{v} + p_1(v) \leq -\tilde{c}_1 v^{1 + \frac{\beta}{p}}, \quad v \geq v_0. \tag{2.8}\]

Let
\[
q(v) = \left(-p_1(v) + \left(1 - \frac{1}{2p}\right) \frac{p_0(v)}{v}\right) 1_{[v_0, \infty)} + \tilde{K}_2 v 1_{(0,v_0)},
\]
where \( \tilde{K}_2 \) is a positive constant such that \( p_1(v) + q(v) > 0 \) for \( v \in (0,v_0) \), and let
\[
\psi(r) = \int_0^r \exp \left(-\int_{v_0}^u \frac{p_1(v) + q(v)}{p_0(v)} dv\right) du. \tag{2.9}\]

The function \( \psi \) is the auxiliary function we shall use in the proof of **Theorem 2.1** which has the following properties.

**Lemma 2.6.** Assume \( \tilde{K}_1 \) satisfies (2.3) with \( q = \frac{\beta + 1}{2} \). Then

1. The function \( \psi \) is well defined and satisfies \( \psi'' < 0 \) and
\[
p_1(v)\psi'(v) + p_0(v)\psi''(v) = -q(v)\psi'(v), \quad v > 0.
\]

2. There exists \( \tilde{C} > 0 \) such that
\[
q(v)\psi'(v) \geq \tilde{C}\psi(v) \left(1 + \psi^{\frac{2p}{p+1}}(v)\right), \quad v > 0.
\]
(3) There exist positive constants $c_1$ and $c_2$ such

$$c_1 v^{\frac{p+1}{p}} \leq \psi(v) \leq c_2 v^{\frac{p+1}{p}}.$$ 

If $\bar{K}_1$ satisfies (2.4) with $q = \frac{p+1}{2}$ in addition and $\bar{K}_2$ is independent of $p$, then $c_1$, $c_2$ and $\bar{C}$ can be chosen such that

$$c_1 \geq 2, \quad \sup_{p \geq 1} \frac{c_2}{c_1} < \infty, \quad \inf_{p \geq 1} \frac{\bar{C}}{p+1} > 0.$$ 

**Proof of Theorem 2.1:**

By Itô’s formula,

$$d |X_t - Y_t|^2 = 2(b_t(X_t) - b_t(Y_t), X_t - Y_t) dt + |\tilde{\sigma}_t(X_t) - \tilde{\sigma}_t(Y_t)|^2_{HS} dt$$

$$+ 4 \left| \left( \frac{\partial}{\partial X} \psi \right)(X_t) \right|^2_{HS} dt$$

$$+ 2 \left\langle X_t - Y_t, \tilde{\sigma}_t(X_t) - \tilde{\sigma}_t(Y_t) \right\rangle dW_t^{(1)}$$

$$+ 4 \left\langle X_t - Y_t, \frac{\partial}{\partial X} \psi \right\rangle_{HS} dW_t^{(2)}, \quad t < \tau.$$ 

Consequently, for $p := 2q - 1$, we have

$$d |X_t - Y_t|^p = \frac{p}{2} \left( |X_t - Y_t|^2 \right)^{\frac{p-1}{2}} d |X_t - Y_t|^2$$

$$+ \frac{p}{4} \left( \frac{p}{2} - 1 \right) |2 \left( \tilde{\sigma}_t^* (X_t) - \tilde{\sigma}_t^* (Y_t) \right) (X_t - Y_t)|^2 \left| X_t - Y_t \right|^{2(\frac{p}{2} - 2)} dt$$

$$+ \frac{p}{4} \left( \frac{p}{2} - 1 \right) \left( \frac{4 |X_t - Y_t|^2}{|\sigma_0^{-1}(X_t - Y_t)|^2} \right) ^2 \left| X_t - Y_t \right|^{2 \left(\frac{p}{2} - 2\right)} dt$$

$$\leq \left\{ \frac{p}{2} \left| X_t - Y_t \right|^{p-2} \left( 2(b_t(X_t) - b_t(Y_t), X_t - Y_t) + |\tilde{\sigma}_t(X_t) - \tilde{\sigma}_t(Y_t)|^2_{HS} \right) \right\} dt$$

$$+ \frac{p(p-2)}{2} \left( \tilde{\sigma}_t^* (X_t) - \tilde{\sigma}_t^* (Y_t) \right) (X_t - Y_t)^2 \left| X_t - Y_t \right|^{p-4} dt$$

$$+ 2p(p-1) |\sigma_0^2| X_t - Y_t |^{p-2} dt + dM_t, \quad t < \tau$$

with

$$dM_t = p |X_t - Y_t|^p \left( \tilde{\sigma}_t(X_t) - \tilde{\sigma}_t(Y_t) \right) dW_t^{(1)}$$

$$+ 2p |X_t - Y_t|^{p-2} \left( \tilde{\sigma}_t(X_t) - \tilde{\sigma}_t(Y_t) \right) \left( X_t - Y_t \right) dW_t^{(2)}.$$ 

Setting $V_t = |X_t - Y_t|^p$, it follows from (2.4), (1) and (2) of Lemma 2.6 that

$$d\psi(V_t) = \psi'(V_t) dV_t + \psi''(V_t) d\psi(V_t)$$

$$= p_1(V_t) \psi'(V_t) dt + \psi'(V_t) dM_t$$

$$+ \left( p_2 V_t^2 \psi''(V_t) |(\tilde{\sigma}_t^* (X_t) - \tilde{\sigma}_t^* (Y_t) \right) (X_t - Y_t)^2 + p_0(V_t) \right) \psi''(V_t) dt$$

$$\leq p_1(V_t) \psi'(V_t) dt + p_0(V_t) \psi''(V_t) dt + \psi'(V_t) dM_t$$

$$= -q(V_t) \psi'(V_t) dt + \psi'(V_t) dM_t$$
Let $y$. It can be checked that $U$ where $U$

Then in fact, if there is $U$ for $t > 0$, we have 

Next, we shall estimate $\mathbb{E} \psi(V_t)$. Let $\tau_n = \{t \geq 0 \mid |X_t - Y_t| \notin [\frac{1}{n}, n]\}$. Then $\tau_n \uparrow \tau$.

It follows from (2.10) that

$$
\mathbb{E} \psi(t \wedge \tau_n) \leq \mathbb{E} \psi(s \wedge \tau_n) - \tilde{C}_3 \mathbb{E} \int_{s \wedge \tau_n}^{t \wedge \tau_n} \psi(V_r) \left(1 + \psi_{\frac{2\theta}{p+1}}(V_r)\right) dr
$$

$$
\leq \mathbb{E} \psi(s \wedge \tau_n) - \tilde{C}_3 \int_{s}^{t} \left(\mathbb{E} \psi(V_r \wedge \tau_n) + (\mathbb{E} \psi(V_r \wedge \tau_n))^{1+\frac{2\theta}{p+1}}\right) dr
$$

$$
= \mathbb{E} \psi(s \wedge \tau_n) - \tilde{C}_3 \int_{s}^{t} \mathbb{E} \psi(V_r \wedge \tau_n) \left(1 + (\mathbb{E} \psi(V_r \wedge \tau_n))^{\frac{2\theta}{p+1}}\right) dr.
$$

Let $y_t = \mathbb{E} \psi(t \wedge \tau_n)$. Then $y_t$ satisfies

$$
y_t \leq y_s - \tilde{C}_3 \int_{s}^{t} y_r(1 + y_r^{\frac{2\theta}{p+1}}) dr, \ y_0 = \psi(V_0).
$$

Set

$$
U_\theta(t) = \left(1 + (1 - e^{-\frac{2\tilde{C}_3 \theta t}{p+1}})\psi_{\frac{2\theta}{p+1}}(V_0)\right)^{-\frac{p+1}{2\theta}} e^{-\tilde{C}_3 \theta t} \psi(V_0),
$$

where $U_0(t)$ (i.e. $\theta = 0$) is defined as follows

$$
U_0(t) = e^{-2\tilde{C}_3 \theta t} \psi(V_0) = \lim_{\theta \rightarrow 0^+} U_\theta(t).
$$

It can be checked that $U_\theta(t)$ satisfies

$$
U_\theta(t) = U_\theta(s) - \tilde{C}_3 \int_{s}^{t} U_\theta(r) \left(1 + U_\theta^{\frac{2\theta}{p+1}}(r)\right) dr, \ U_\theta(0) = \psi(V_0) = y_0.
$$

Then

$$
y_t \leq U_\theta(t), \ t \geq 0. \quad (2.11)
$$

In fact, if there is $t_0 > 0$ such that $y_0 > U_\theta(t_0)$, then by the continuity of $y_t$ and $U_\theta(t)$, we can set $t_1 = \sup\{s \leq t_0 \mid y_s \leq U_\theta(s)\}$. Thus $y_{t_1} = U_\theta(t_1)$ and $y_r > U_\theta(r)$ for $r \in (t_1, t_0)$. Consequently,

$$
y_t \leq y_{t_1} - \int_{t_1}^{t} y_r(1 + y_r^{\frac{2\theta}{p+1}}) dr
$$

$$
< U_\theta(t_1) - \int_{t_1}^{t} U_\theta(r) \left(1 + U_\theta^{\frac{2\theta}{p+1}}(r)\right) dr
$$

$$
= U_\theta(t),
$$

which leads to a contradiction.

Letting $n \rightarrow \infty$, it follows form (2.11) that $\mathbb{E} \psi(V_{t \wedge \tau}) \leq U_\theta(t)$. Since $V_t = 0$ for $t \geq \tau$, we have

$$
\mathbb{E} \psi(V_{t \wedge \tau}) = \mathbb{E} \psi(V_t) 1_{[t < \tau]} + \mathbb{E} \psi(V_\tau) 1_{[t \geq \tau]} = \mathbb{E} \psi(V_t).
$$
Thus $E\psi(V_t) \leq U_\theta(t)$. Combining this with (3) of Lemma 2.6

$$E|X_t - Y_t|^{\frac{p+1}{2}} \leq \frac{1}{c_1} E\psi(V_t) \leq \frac{1}{c_1} U_\theta(t)$$

$$\leq \frac{1}{c_1} \left( 1 + \left( 1 - e^{-\frac{2\tilde{C}}{p+1}} \right) \psi^{\frac{p}{p+1}}(V_0) \right)^{-\frac{p}{p+1}} e^{-\tilde{C}t \psi(V_0)}$$

$$\leq \frac{c_2}{c_1} \left( 1 + e^{-\frac{2\tilde{C}}{p+1}} \right)^{-\frac{p}{p+1}} \left( 1 - e^{-\frac{2\tilde{C}}{p+1}} \right) |x - y|^{\theta} e^{-\tilde{C}t |x - y|^{\frac{p+1}{2}}}.$$ 

Therefore

$$W_{\frac{p+1}{2}}(\delta_x P_t, \delta_y P_t) \leq \left( E|X_t - Y_t|^{\frac{p+1}{2}} \right)^{\frac{2}{p+1}}$$

$$\leq \left( \frac{c_2}{c_1} \right)^{\frac{2}{p+1}} \left( 1 + e^{-\frac{2\tilde{C}}{p+1}} \right)^{-\frac{p}{p+1}} \left( 1 - e^{-\frac{2\tilde{C}}{p+1}} \right) |x - y|^{\theta} e^{-\tilde{C}t |x - y|}.$$

Recalling that $p = 2q - 1$, we get (2.5).

It follows from (2.6) and (3) of Lemma 2.6 that

$$c_2 \theta^{p+1} \geq \frac{2}{p+1} \sup_{p \geq 1} \left( \frac{c_2}{c_1} \right)^{\frac{2}{p+1}}, \quad \inf_{p \geq 1} \frac{\tilde{C}}{p+1} > 0.$$

Let

$$\kappa = 2 \inf_{p \geq 1} \frac{\tilde{C}}{p+1}, \quad c_0 = \sup_{p \geq 1} \left( \frac{c_2}{c_1} \right)^{\frac{2}{p+1}}.$$

Then the proof is completed.

\[ \square \]

**Proof of Lemma 2.6.**

For $0 < u \leq v_0$,

$$\int_{v_0}^u \frac{p_1(v) + q(v)}{p_0(v)} dv = \int_{v_0}^u \left( \frac{K_1^\frac{1}{p} v^{\frac{p}{2}}}{p^2 \alpha_0^2 v^2} + \frac{p - 1}{2p} v_0 v^{-1} + \frac{\tilde{K}_2}{\sigma_0^2 \beta^2} v_0^{-1+\frac{p}{2}} \right) dv, \quad (2.12)$$  

which, together with (2.3), implies that there exists positive constants $C_1$ and $C_2$ such that

$$\log u^{\frac{p-1}{2p}} - C_1 \leq \int_{v_0}^u \frac{p_1(v) + q(v)}{p_0(v)} dv \leq \log u^{\frac{p-1}{2p}} - C_2, \quad u \in (0, v_0]. \quad (2.13)$$  

Then

$$\frac{2pe^{\psi(r)}}{\psi^{\frac{p}{2p}}} \leq \psi'(r) \leq \frac{2pe^{\psi(r)}}{\psi^{\frac{p}{2p}}} \leq \frac{2pe^{\psi(r)}}{\psi^{\frac{p}{2p}}}, \quad r \in (0, v_0], \quad (2.14)$$  

with

$$e^{\psi(r)} \leq \psi'(r) \leq e^{\psi(r)} \leq \psi'(r), \quad r \in (0, v_0]. \quad (2.15)$$
For \( u \geq v_0 \),
\[
\int_{v_0}^{u} \frac{p_1(v) + q(v)}{p_0(v)} \, dv = \int_{v_0}^{u} \left( \frac{1}{2} - \frac{1}{2p} \right) \frac{1}{v} \, dv = \log u^{\frac{p-1}{2p}} - \log v_0^{\frac{p-1}{2p}}.
\]

Then
\[
\psi(r) = \int_0^r \exp \left( - \int_{v_0}^{u} \frac{p_1(v) + q(v)}{p_0(v)} \, dv \right) \, du + \int_{v_0}^{r} \exp \left( - \int_{v_0}^{u} \frac{p_1(v) + q(v)}{p_0(v)} \, dv \right) \, du
= \psi(v_0) + \int_{v_0}^{r} \frac{2p v_0^{\frac{p-1}{2p}}}{p+1} \left( r^{\frac{1}{2p}} - v_0^{\frac{1}{2p}} \right), \quad r \geq v_0,
\]
with
\[
\psi'(r) = \exp \left( - \int_{v_0}^{u} \frac{p_1(v) + q(v)}{p_0(v)} \, dv \right) = v_0^{\frac{p}{r^{\frac{1}{2p}}}}, \quad r \geq v_0.
\]

Hence, \( \psi \) is well defined. It is easy to check that (1) holds.

By the definition of \( q \) and (2.8), it is easy to see that
\[
q(v) \geq \tilde{K}_2 r \mathbbm{1}_{(0,v_0)}(v) + \tilde{c}_1 v^{1+\frac{\theta}{p}} \mathbbm{1}_{[v_0,\infty)}(v) > 0,
\]
Combining this with (2.15) and (2.17), there exist positive constants \( \tilde{C}_1, \tilde{C}_2 \) and \( \tilde{C}_3 \) such that
\[
- q(v) \psi'(v) \leq - \left( \tilde{K}_2 v \mathbbm{1}_{(0,v_0)}(v) + \tilde{c}_1 v^{1+\frac{\theta}{p}} \mathbbm{1}_{[v_0,\infty)}(v) \right) \psi'(v)
\leq - \tilde{C}_1 v^{\frac{p+1}{2p}} \mathbbm{1}_{(0,v_0)} - \tilde{C}_2 v^{\frac{p+1}{2p}+\frac{\theta}{p}} \mathbbm{1}_{[v_0,\infty)}
\leq - \tilde{C}_3 \psi(v) \left( 1 + \psi^{\frac{2p}{p+1}}(v) \right), \quad v > 0.
\]

Hence, (2) holds.

It follows from (2.14) and (2.16) that there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
\tilde{c}_1 r^{\frac{p+1}{2p}} \leq \psi(r) \leq \tilde{c}_2 r^{\frac{p+1}{2p}}, \quad r > 0.
\]

Then the first claim of (3) holds. Next, we shall prove the second claim.

By (2.12) and (2.13), we can choose
\[
C_2 = \log v_0^{\frac{p-1}{2p}}, \quad C_1 = \log v_0^{\frac{p-1}{2p}} + \frac{1}{p^2 \sigma_0^2} \int_{v_0}^{v_0} v^{-1} \tilde{K}_1(i) \, dv + \tilde{K}_2 \sigma_0^{\frac{2}{p}} v_0^{\frac{p}{2}}.
\]

Then
\[
e^{C_2} = v_0^{\frac{p}{2p}}, \quad e^{C_1} = v_0^{\frac{p}{2p}} \exp \left\{ \frac{1}{p^2 \sigma_0^2} \int_{v_0}^{v_0} v^{-1} \tilde{K}_1(i) \, dv + \tilde{K}_2 \sigma_0^{\frac{2}{p}} v_0^{\frac{p}{2}} \right\}.
\]

Consequently, by (2.16) and (2.14), we have
\[
c_2 = \sup_{r \geq v_0} \left\{ \frac{2p}{p+1} v_0^{\frac{p-1}{2p}} + r^{\frac{p-1}{2p}} \left( \psi(v_0) - \frac{2p}{p+1} v_0 \right) \right\} \vee \left( \frac{2p}{p+1} e^{C_1} \right).
\]
Thus we set
\[ v^p \leq \left\{ \frac{2p}{p+1} \frac{v_{0\text{p}}}{v_{0}} + v_{0} \frac{p+1}{p+1} \left( \psi(v_{0}) - \frac{2p}{p+1} v_{0} \right) \right\} \vee \left( \frac{2p}{p+1} c_{1} \right) \]
\[ = \left( \frac{2p}{p+1} \frac{v_{0\text{p}}}{v_{0}} \psi(v_{0}) \right) \vee \left( \frac{2p}{p+1} c_{1} \right) \]
\[ \leq \left( v_{0} \frac{p+1}{p+1} \frac{p+1}{p+1} \right) \vee \left( \frac{2p}{p+1} c_{1} \right) \]
\[ \leq \frac{2p}{p+1} v_{0} \exp \left\{ K_{2} v_{\text{p}}^{2} \frac{\tilde{K}_{2} v_{\text{p}}^{0}}{2\sigma_{0p}} + \frac{1}{p^{2}\sigma_{0}^{2}} \int_{0}^{v_{0}} v^{-1} \tilde{K}_{1}^{2}(v)dv \right\} , \]
and
\[ c_{1} = \inf_{r \geq v_{0}} \left\{ \frac{2p}{p+1} v_{0}^{p+1} + r \frac{p+1}{p+1} \left( \psi(v_{0}) - \frac{2p}{p+1} v_{0} \right) \right\} \wedge \frac{2pe^{C_{2}}}{p+1} \]

Next, we shall choose suitable $v_{0}$ and $\tilde{K}_{2}$. It follows from (2.6) with $q = \frac{p+1}{2}$ that $K_{1}(v) \leq Cp^{m}v^{\alpha}, \quad v \geq 1$. Then
\[ q(v) = -2\sigma_{0}^{2}p(p-1)v^{p-2} + \frac{p-2}{p} K_{1}^{\frac{1}{p}} - pK_{2}v^{1+\frac{\theta}{p}} + 2\sigma_{0}^{2}p(p-1)v^{\frac{p-2}{p}} \]
\[ = pv^{\frac{2+\theta}{p}} \left( -\tilde{K}_{2} + \left( \frac{K_{1}(v)}{v^{2+\theta}} \right)^{\frac{1}{p}} \right) \]
\[ \leq pv^{\frac{2+\theta}{p}} \left( -\tilde{K}_{2} + \left( mCv^{\alpha-2-\theta} \right)^{\frac{1}{p}} \right) , \quad v \geq 1. \]

Thus we set
\[ v_{0} = \left( Cp^{m} \right)^{\frac{1}{2+\theta-\alpha}} \left( \frac{2}{\tilde{K}_{2}} \right) \left( \frac{1}{2+\theta-\alpha} \right) \vee 1. \]

Then
\[ \tilde{c}_{1} \geq \frac{p\tilde{K}_{2}}{2} \quad \text{(2.21)} \]
and
\[ \sup_{p \geq 1} v_{0}^{p} = \sup_{p \geq 1} \left\{ \left( Cp^{m} \right)^{\frac{1}{2+\theta-\alpha}} \left( \frac{2}{\tilde{K}_{2}} \right) \left( \frac{1}{2+\theta-\alpha} \right) \vee 1 \right\} = \left( \frac{2Ce^{\frac{m}{\tilde{K}_{2}}} \vee 1}{\tilde{K}_{2}^{2}} \vee 1 \right)^{\frac{1}{2+\theta-\alpha}}. \quad \text{(2.22)} \]

Set $\tilde{K}_{2} = p\tilde{K}_{2} v_{0}^{\frac{2}{p}} + 1$. Then $p_{1}(v) + \tilde{K}_{2}v > 0$ for all $v \in (0, v_{0})$, and
\[ \frac{K_{2}}{\sigma_{0}^{2}p} v_{0}^{\frac{2}{p}} = \frac{K_{2} v_{0}^{p} + v_{0}^{2}}{\sigma_{0}^{p}}. \]
With \( v_0 \) and \( K_2 \) chosen above, we have
\[
C_1 - C_2 = \frac{1}{p^2\sigma_0^2} \int_0^{v_0} v^{-1} \frac{1}{2} K_1^\frac{1}{p}(v)dv + \frac{\hat{K}_2}{2\sigma_0^2} v_0^\frac{2}{p} \\
\leq \frac{1}{p^2\sigma_0^2} \int_0^{1} v^{-1} \frac{1}{2} K_1^\frac{1}{p}(v)dv + \frac{C_{\sigma p m - 1}}{\alpha^2\sigma_0^2} \left( \frac{v_0^\frac{2}{p}}{\sigma_0^2} - 1 \right) + \frac{\hat{K}_2 v_0^p}{\sigma_0^2} + \frac{v_0^p}{\sigma_0^2p},
\]
which along with (2.22) imply that
\[
\sup_{p \geq 1} e^{C_1 - C_2} < 0.
\]  (2.23) inf-C2-C1

Consequently,
\[
\sup_{p \geq 1} \bar{c}_2 \leq \sup_{p \geq 1} e^{C_1 - C_2} < \infty.
\]

It follows from (2.15) and (2.17) that \( \hat{C}_1 \) and \( \hat{C}_2 \) in (2.19) satisfy
\[
\hat{C}_1 = \hat{K}_2 e^{C_2}, \quad \hat{C}_2 = \bar{c}_1 v_0^{\frac{p-1}{2p}}.
\]

Then (2.14), (2.19), (2.20) and \( p \geq 1 \) yield that
\[
\hat{C} \geq \frac{\hat{C}_1 (p + 1)e^{-C_1}}{2p} \wedge \left( \frac{\hat{C}_2 e^{\frac{1}{2p} - \frac{p-1}{2p}}}{2p} \right) \wedge \hat{K}_2 (p + 1)e^{C_2 - C_1} \wedge \frac{\bar{c}_1 v_0^{\frac{p-1}{2p}} - \frac{\bar{c}_1}{2p} (1 + 2\theta)}{v_0^{\frac{p-1}{2p}}(e^{C_1 - C_2})^{1 + \frac{2\theta}{p+1}}} \geq \left\{ \left( \frac{\hat{K}_2}{2(e^{C_1 - C_2})} \right) \wedge \left( \frac{1 + \hat{K}_2}{8(e^{C_1 - C_2})^{1 + \theta}} \left( \frac{2C e^{\frac{m}{\theta}}}{\hat{K}_2} \right)^{\frac{1}{1 + \theta - \alpha}} \right) \right\} (p + 1).
\]

Combining this with (2.22) and (2.23), we get that \( \inf_{p \geq 1} \frac{\hat{C}}{p+1} > 0 \). Since \( v_0 \geq 1 \) and \( p \geq 1 \), it is clear that \( \bar{c}_1 \geq 2 \).

\[
3 \quad \text{Main results by the Zvonkin’s transformation}
\]

In this section, we shall consider the same problem for equations with some more singular drift term. To cancel the singularity of the drift term, we shall consider the equation in the following form
\[
dX_t = Z(X_t)dt + b_t(X_t)dt + \sigma(X_t)dW_t,
\]  (3.1) main-equ

where \( b_t(\cdot) \) is the singular part of the drift term.

Zvonkin’s transformation will be adapted, and a family of diffeomorphisms \( \{\Phi_t : \mathbb{R}^d \to \mathbb{R}^d\}_{t \geq 0} \) would be given such that \( \mathbb{P}\text{-a.s.} \)
\[
\Phi_t(X_t) = \Phi_0(X_0) + \int_0^t Z(X_s)ds + \lambda \int_0^t \phi_s(X_s)ds
\]
Remark 3.1. The equations (3.3) and (3.4) are $\mathbb{R}^d$-valued, where by $P_{s,t}^Z b_t$ we mean that $P_{s,t}^Z b_t(x) \in \mathbb{R}^d$ with
\[
\langle P_{s,t}^Z b_t(x), v \rangle := \langle P_{s,t}^Z(b_t, v) \rangle(x), \quad v \in \mathbb{R}^d,
\]
and $P_{s,t}^Z \nabla b_t \phi_t$ are understood in the similar way. Thus (3.4) can be written in the following form
\[
\langle \phi_s(x), v \rangle = \int_s^\infty e^{-\lambda(t-s)} P_{s,t}^Z \{ \nabla b_t \phi_t + \langle b_t, v \rangle \} (x) dt, \quad v \in \mathbb{R}^d.
\]

For $Z(x) = Ax$ and $b_t$ is Dini continuous and bounded, the existence of $\phi_t$ has been proved in [27]. To consider unbounded and singular $b_t$, we shall use regular $\sigma$ and more dissipative $Z$:

(A1) The drift $Z \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ has continuous second order derivative. There are $\beta > 0$, $K_2 > 0$, $K_4 \geq 0$ and non-negative constants $K_1, K_3$ such that for all $x, v \in \mathbb{R}^d$
\[
|\nabla Z(x)| \leq K_1|x|^{\beta} + K_3, \quad \langle \nabla Z(x)v, v \rangle \leq \left(-K_2|x|^{\beta} + K_4\right)|v|^2; \quad \text{(3.5) Inequ-nnZ}
\]
and there are nonnegative constants $K_5, K_6$ such that
\[
|\nabla^2 Z(x)| \leq K_5|x|^{(\beta-1)^+} + K_6. \quad \text{(3.6) Inequ-nn2Z}
\]

(A2) The diffusion term $\sigma \in C^2(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ has bounded continuous first and second order derivatives; there is positive constant $\sigma_0$ such that
\[
\sigma \sigma^*(x) \geq \sigma_0^2, x \in \mathbb{R}^d.
\]
The condition (3.5) implies that there is a constant $c > 0$ such that $c - Z$ is a monotone map on $\mathbb{R}^d$, see Lemma 3.5 below. (A1) and (A2) can be used to study the regularity of $P_t^Z$, which will be given in subsection 3.2.

Our main result in this section reads as follows.

**Theorem 3.1.** Assume (A1), (A2) with $\sigma$ is bounded on $\mathbb{R}^d$ in addition. Then $P_t^Z$ has a unique invariant measure $\mu$ such that $\mu(e^{\delta |.|^p})$ for some $\delta > 0$ and $P_t^Z$ is ultracontractive:

$$||P_t^Z||_{L^2(\mu) \to L^\infty(\mu)} \leq \exp(c(1 + t^{-\frac{\beta + 2}{\beta}})), \ t > 0,$$

with some positive constants $c$. If there exist $\zeta > 0$ and $p > \frac{2(\beta + 2)}{\beta}$ such that

$$\sup_{t \geq 0} \mu(e^{\zeta |b_t|^p}) < \infty.$$

Then (3.1) has a unique strong solution. For any $q \geq 1$ there exist positive constants $c_0$ and $\kappa$ such that

$$W_q(\delta_x P_t, \delta_y P_t) \leq c_0 \left(1 + \left(1 - e^{-\kappa t}\right)^{\frac{q}{2}} e^{-\kappa t}|x - y|\right), \ q \in [0, \infty].$$

The proof of this theorem is given in subsection 3.3. Applying this theorem to (1.4) with $Z(x) = -|x|^2 x$ and $b_t(x) = -\frac{x}{|x|} \left(\log \frac{1}{|x|}\right)^{\frac{1}{2}} \mathbf{1}_{\{x \neq 0\}}$, then (3.7) and (3.8) hold with $\beta = 2$ and $p = 5$ with $\zeta < d$.

### 3.1 A study of the integral equation

We first study an $\mathbb{R}^d$-valued integral equation which is slight general than (3.4):

$$\phi_s = \int_s^\infty e^{-\lambda(t-s)} P^Z_{s,t} \{\nabla g_t \phi_t(x) + f_t(x)\} \, dt.$$  

In this subsection, we consider (3.9) under the following general hypothesises on $\mathcal{L}^Z$.

(H1) The differential operator $\mathcal{L}^Z$ generates a unique Markov semigroup $P_t^Z$ with a unique invariant probability measure $\mu$.

(H2) There exists $\gamma > 0$ and $c > 0$ such that (3.7) holds.

(H3) For $f \in L^2(\mu), \ P^Z_{s,t}f \in C^1_b(\mathbb{R}^d)$ with the following gradient estimate

$$|\nabla P^Z_{s,t}f(x)| \leq \frac{ce^{\delta(t-s)}}{\sqrt{t-s}} \left(P^Z_{s,t}f^2\right)^{\frac{1}{2}}(x), \ x \in \mathbb{R}^d$$

holds for some positive constants $c$ and $\delta$.

Let $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ be the set of all Borel measurable functions from $\mathbb{R}^d$ to $\mathbb{R}^d$. We shall use the following notations:

$$|v| = \left(\sum_{i=1}^d v_i^2\right)^{\frac{1}{2}}, \ v \in \mathbb{R}^d; \ ||f||_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|;$$
Lemma 3.2. Assume (H1)–(H3). Suppose there exist \( \zeta > 0 \), \( p \geq 2 \) and \( p > 2\gamma \) such that
\[
\sup_{t \geq 0} \left( \mu(e^{\zeta|g_t|^p}) \vee \mu(e^{\zeta|\nu|^p}) \right) < \infty.
\] (3.11) g-f-exp

Then there is \( \lambda_0 > 0 \) such that for any \( \lambda > \lambda_0 \) the following inequalities hold for the solution of (3.9)
\[
\sup_{t \geq 0} ||\nabla \phi_t||_\infty \leq \frac{C\Theta_1(f)}{1 - C\Theta_1(g)},
\]
\[
\sup_{t \geq 0} ||\phi_t||_\infty \leq C \left( \frac{\Theta_2(g)\Theta_1(f)}{1 - C\Theta_1(g)} + \Theta_2(f) \right),
\]
where \( C_\zeta \) is a positive constant different from line to line; \( \Theta_1(\cdot), \Theta_2(\cdot) \) are defined as follows
\[
\Theta_1(g) = \zeta^{(-\frac{1}{p})} \left( \lambda^2 + \frac{1}{2} \right) \left( 1 - \frac{\gamma}{p} \right) + \frac{1}{\lambda} \sup_{t \geq 0} \left( 1 + \left( \log \mu(e^{\zeta|g_t|^p}) \right)^{\frac{1}{p}} \right),
\]
\[
\Theta_2(g) = \zeta^{(-\frac{1}{p})} \left( \lambda^2 - 1 \right) \left( 1 - \frac{\gamma}{p} \right) + \frac{1}{\lambda} \sup_{t \geq 0} \left( 1 + \left( \log \mu(e^{\zeta|g_t|^p}) \right)^{\frac{1}{p}} \right).
\]

Consequently, if \( f_t \equiv 0, t \geq 0 \), then \( \phi \equiv 0 \).

Proof. By the equation (3.9),
\[
||\phi_s||_\infty \leq \int_s^\infty e^{-\lambda (t-s)} ||P^Z_{s,t} (\nabla g_t \phi_t + f_t) ||_\infty dt.
\]
Since
\[
||P^Z_{s,t} (\nabla g_t \phi_t)||_\infty \leq ||\nabla \phi_t||_\infty ||P^Z_{s,t} g_t||_\infty
\]
\[
\leq \zeta^{(-\frac{1}{p})} ||\nabla \phi_t||_\infty \left( ||\log P^Z_{s,t} e^{\frac{1}{p} |g_t|^p} ||_\infty \right)^{\frac{1}{p}}
\]
\[
\leq \zeta^{(-\frac{1}{p})} ||\nabla \phi_t||_\infty \left( \log \left( e^{c_1(t-s)-\gamma c_2 \left( \mu(e^{\zeta|\nu|^p}) \right)^{\frac{1}{p}}} \right) \right)^{\frac{1}{p}}
\]
\[
\leq C\zeta^{(-\frac{1}{p})} ||\nabla \phi_t||_\infty \left( (t-s) \gamma^{\frac{1}{p}} + 1 + \left( \log \mu(e^{\zeta|\nu|^p}) \right)^{\frac{1}{p}} \right)
\] (3.12) P-nn-g

and
\[
||P^Z_{s,t} f_t||_\infty \leq 2^p \zeta^{(-\frac{1}{p})} \left( \log ||P^Z_{s,t} e^{\frac{1}{p} |f_t|^p} ||_\infty \right)^{\frac{1}{p}}
\]
we get the following inequality

\[
\sup_{s \geq 0} ||\phi_s|| \leq C \zeta^{-\frac{1}{p}} \left( (t-s)^{-\frac{2}{p}} + 1 + \left( \log \mu(e^{\zeta|f_1|^p}) \right)^{\frac{1}{p}} \right),
\]

Thus for

\[
P_{s,t} \left( \sup_{s \geq 0} \left| \nabla \phi_s \right| \right) \leq \sup_{s \geq 0} \left| P_{s,t} \phi_s \right| \]

Similarly,

\[
\| \nabla P_{s,t} \{ \nabla g \phi_t \} \| \leq \frac{C}{\sqrt{t-s}} \left\| \left( P_{s,t} |g| \right) \| \nabla \phi_t \|^2 \right\|_{\infty}
\]

and

\[
\sup_{s \geq 0} \left| \nabla \phi_s \right| \leq C \zeta^{-\frac{1}{p}} \left( (t-s)^{-\frac{2}{p}} + 1 + \left( \log \mu(e^{\zeta|f_1|^p}) \right)^{\frac{1}{p}} \right) \]

Thus for \( \lambda > 2\delta \), it follows from (3.11) that

\[
\zeta^{-\frac{1}{p}} \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \left( (t-s)^{-\frac{2}{p}} + 1 + \left( \log \mu(e^{\zeta|f_1|^p}) \right)^{\frac{1}{p}} \right) dt
\]

\[
\leq \zeta^{-\frac{1}{p}} \left\{ (\lambda - \delta) \frac{2}{p} - \frac{1}{p} \right\} + \frac{1}{(\lambda - \delta)^{\frac{1}{p}}} \sup_{t \geq 0} \left( 1 + \left( \log \mu(e^{\zeta|f_1|^p}) \right)^{\frac{1}{p}} \right)
\]

\[
= 2 \Theta_1(g) < \infty.
\]

It is clear that there exists \( \lambda_0 > 0 \) such that for \( \lambda > \lambda_0 \), \( 1 - 2C\Theta_1(g) > 0 \). Then, letting \( C_1(\lambda) = (1 - 2C\Theta_1(g))^{-1} \), we get that

\[
\sup_{s \geq 0} \left| \nabla \phi_s \right| \leq C_1(\lambda) \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \left| \left( P_{s,t} |f_1|^\delta \right)^{\frac{1}{\delta}} \right| \| P_{s,t} \phi_s \|_{\infty} dt.
\]

\[
\leq CC_1(\lambda) \zeta^{-\frac{1}{p}} \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \left( (t-s)^{-\frac{2}{p}} + 1 + \left( \log \mu(e^{\zeta|f_1|^p}) \right)^{\frac{1}{p}} \right) dt
\]

\[
\leq C(1 - 2C\Theta_1(g))^{-1} \Theta_1(f).
\]
Substituting this into (3.13),
\[
\sup_{s \geq 0} \|\phi_s\|_\infty \leq \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \|P_{s,t}|f_t|\|_\infty \, dt \\
+ C(1 - C\Theta_1(g))^{-1}\Theta_1(f) \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \frac{\|P_{s,t}|f_t|^2\|_\infty}{\sqrt{t-s}} \, dt \\
\leq C\zeta^{-\frac{1}{2}} \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \left((t-s)^{-\frac{\gamma}{2}} + 1 + \left(\log \mu(e^{\zeta |f_t|^p})\right)^\frac{1}{p}\right) \, dt \\
+ C\zeta^{-\frac{1}{2}} \Theta_1(f) \sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)} \frac{1 - C\Theta_1(g)}{\sqrt{t-s}} \left((t-s)^{-\frac{\gamma}{2}} + 1 + \left(\log \mu(e^{\zeta |g_t|^p})\right)^\frac{1}{p}\right) \, dt \\
\leq C(\Theta_2(f) + (1 - C\Theta_1(g))^{-1}\Theta_1(f)\Theta_2(g)). \quad (3.16) \tag{Ineq-ph2}
\]

According to the first inequality in (3.15) and (3.16), if \(f_t \equiv 0\), then
\[
\sup_{s \geq 0}(\|\phi_s\|_\infty + \|\nabla \phi_s\|_\infty) = 0.
\]

\[\square\]

**Remark 3.2.** (1) According to this lemma, it is easy to see that
\[
\lim_{\lambda \to +\infty} \left(\sup_{t \geq 0} \|\nabla \phi_t\|_\infty + \sup_{t \geq 0} \|\phi_t\|_\infty\right) = 0.
\]

(2) If \(\sup_{t \geq 0}(\|f_t\|_\infty + \|g_t\|_\infty) < \infty\), then it is not necessary to assume that \(P_{s,t}^Z\) is ultracontractive. In this case, there exists \(\lambda_0 > 0\) such that
\[
\sup_{t \geq 0}(\|\nabla \phi_t\|_\infty + \|\phi_t\|_\infty) \leq C(\lambda, \sup_{t \geq 0} \|f_t\|_\infty, \sup_{t \geq 0} \|g_t\|_\infty), \lambda > \lambda_0
\]
with
\[
\lim_{\lambda \to +\infty} C(\lambda, \sup_{t \geq 0} \|f_t\|_\infty, \sup_{t \geq 0} \|g_t\|_\infty) = 0.
\]

In the next lemma, we shall study the dependency of the solutions to (3.9) on \(f\) and \(g\). Let \(f_t^{[m]} = P_{s,t}^Z f_t\) and \(g_t^{[m]} = P_{s,t}^Z g_t\), \(m \in \mathbb{N}\), and let \(\phi^{[m]}\) be a solution of (3.9) with \(f\) and \(g\) replaced by \(f_t^{[m]}\) and \(g_t^{[m]}\).

**Lemma 3.3.** Under the same assumptions of Lemma [Y-Z] we have

(1) The functions \(f_t^{[m]}\) and \(g_t^{[m]}\) are in \(L^\infty([0, \infty), C_b^1(\mathbb{R}^d, \mathbb{R}^d))\), and \(f_t^{[m]}, g_t^{[m]} \in C_b([0, \infty), C_b^1(\mathbb{R}^d, \mathbb{R}^d))\) if \(f, g \in C([0, \infty), L^2(\mu)).\)

(2) There is \(\lambda_0 > 0\) such that for any \(\lambda > \lambda_0\), there exist a subsequence of \(\phi^{[m]}\), denoting by \(\{\phi_t^{[m_k]}\}_{k \geq 0}\), such that for any \(R > 0\)
\[
\lim_{k \to \infty} \sup_{t \in [0, R]} \left(\|\phi_t^{[m_k]} - \phi_t\|_\infty + \|\nabla (\phi_t^{[m_k]} - \phi_t)\|_\infty\right) = 0.
\]

(3) For any \(n \in \mathbb{N}\), let
\[
f_t^{[m,n]} = n \int_t^{t+1/n} f_r^{[m]} \, dr, \quad g_t^{[m,n]} = n \int_t^{t+1/n} g_r^{[m]} \, dr,
\]
and let \( \phi^{[m,n]} \) be the solution of the equation \((3.19)\) with \( f_t \) and \( g_t \) replaced by \( f_t^{[m,n]} \) and \( g_t^{[m,n]} \). Then \( f^{[m,n]} \) and \( g^{[m,n]} \) are in \( C_b([0, \infty), C_b^1(\mathbb{R}^d)) \); given \( \lambda > \lambda_0 \) and the subsequence \( \{m_k\}_{k \geq 1} \) in \((2)\), for each \( R > 0 \), there is \( \{n(m_k)\}_{k \geq 1} \) such that

\[
\lim_{k \to \infty} \sup_{t \in [0, R]} \left( \left\| \nabla (\phi_t^{m_k,n(m_k)} - \phi_t) \right\|_\infty + \left\| \phi_t - \phi_t^{m_k,n(m_k)} \right\|_\infty \right) = 0. \tag{3.17}
\]

**Proof.** (1) Letting

\[
f_{n,t} = P_{Z/m}^f f_t 1_{|f_t| \leq n}, \quad g_{n,t} = P_{Z/m}^g g_t 1_{|g_t| \leq n},
\]

it follows from the ultracontractivity of \( P_{1/m}^Z \) and \((3.10)\) that

\[
\sup_{t \geq 0, n \in \mathbb{N}} \left( \left\| \nabla f_{n,t}^{[m]} \right\|_\infty + \left\| f_{n,t}^{[m]} \right\|_\infty \right) \leq C_m \sup_{t \geq 0, n \in \mathbb{N}} \left( \left\| P_{Z}^f f_t 1_{|f_t| \leq n} \right\|_\infty \right) \leq C_m \sup_{t \geq 0} \left( \mu \left( |f_t|^2 \right) \right)^{\frac{1}{\lambda}}, \tag{3.18}
\]

and

\[
\sup_{t \geq 0} \left( \left\| \nabla (f_{n,t}^{[m]} - f_t^{[m]}) \right\|_\infty + \left\| f_{n,t}^{[m]} - f_t^{[m]} \right\|_\infty \right) \leq C_m \sup_{t \geq 0, n \in \mathbb{N}} \left( \left\| P_{Z}^f f_t 1_{|f_t| \geq n} \right\|_\infty \right) \leq C_m \sup_{t \geq 0} \left( \mu \left( |f_t|^2 \right) \right)^{\frac{1}{\lambda}}. \tag{3.19}
\]

It follows from \((3.11)\) that \( \{ |f_t|^2 \}_{t \geq 0} \) is uniformly integrable, which along with \((3.19)\) yields that

\[
\lim_{n \to \infty} \sup_{t \geq 0} \left( \left\| \nabla (f_{n,t}^{[m]} - f_t^{[m]}) \right\|_\infty + \left\| f_{n,t}^{[m]} - f_t^{[m]} \right\|_\infty \right) = 0.
\]

Taking into account \((3.18)\), we have proved that \( f^{[m]} \in L^\infty([0, \infty), C_b^1(\mathbb{R}^d, \mathbb{R}^d)) \) and the same conclusion holds for \( g^{[m]} \).

On the other hand, we have

\[
\left\| \nabla (f_{t_1}^{[m]} - f_{t_2}^{[m]}) \right\|_\infty + \left\| f_{t_1}^{[m]} - f_{t_2}^{[m]} \right\|_\infty \leq C_m \left\| \left( P_{1/m}^Z f_{t_1} - f_{t_2} \right)^2 \right\|_\infty \leq C_m \left( \mu(|f_{t_1} - f_{t_2}|^2) \right)^{\frac{1}{2}}.
\]

Then \( f_t^{[m]}, g_t^{[m]} \in C_b([0, \infty), C_b^1(\mathbb{R}^d, \mathbb{R}^d)) \) if \( f, g \in C([0, \infty), L^2(\mu)) \).

(2) By the equations of \( \phi_s \) and \( \phi_s^{[m]} \),

\[
\left\| \phi_s - \phi_s^{[m]} \right\|_\infty \leq \int_s^\infty e^{-\lambda(t-s)} \left\| P_{s,t}^Z (|\nabla \phi_t| g_t - g_t^{[m]}) \right\| dt.
\]
Substituting this into (3.20) and (3.21), there exists positive constant $C_{\zeta,f,g}$ depending on $\zeta, f, g$ such that

\[
\left\| \phi_s - \phi_s^{[m]} \right\|_\infty \leq \left\| \nabla \phi_t \right\|_\infty \int_0^\infty e^{-\lambda r} \left\| P_r^Z \left( (g_{s+r} - g_{s+r}^{[m]}) \right) \right\|_\infty \, dr \\
+ C_{\zeta,f,g} \int_0^\infty e^{-\lambda r} \left( r^{-\frac{2}{p} + 1} \right) \left\| \nabla (\phi_{r+s} - \phi_{r+s}^{[m]}) \right\|_\infty \, dr \\
+ \int_0^\infty e^{-\lambda r} \left\| P_r^Z [f_{s+r} - f_{s+r}^{[m]}] \right\|_\infty \, dr. \tag{3.23} \text{add-phem} \]

and

\[
\left\| \nabla (\phi_s - \phi_s^{[m]}) \right\|_\infty \leq \left\| \nabla \phi_t \right\|_\infty \int_0^\infty e^{-\lambda r} \left\| P_r^Z \left( (g_{s+r} - g_{s+r}^{[m]}) \right) \right\|_\infty \, dr \\
+ C_{\zeta,f,g} \int_0^\infty e^{-\lambda r} \left( r^{-\frac{2}{p} + 1} \right) \left\| \nabla (\phi_{r+s} - \phi_{r+s}^{[m]}) \right\|_\infty \, dr.
\]

It follows from the ultracontractivity of $P_r^Z$ and the strong continuous of $P_{1/m}^Z$ that

\[
\sup_{m \in \mathbb{N}, s \geq 0} \left( \left\| \left( P_r^Z [f_{r+s}]^{[m]} \right)^{\frac{1}{2}} \right\|_\infty + \left\| \left( P_r^Z [g_{r+s}]^{[m]} \right)^{\frac{1}{2}} \right\|_\infty \right) \\
\leq \sup_{m \in \mathbb{N}, s \geq 0} \left( \left\| \left( P_{r+1/m}^Z [f_{r+s}]^{[m]} \right)^{\frac{1}{2}} \right\|_\infty + \left\| \left( P_{r+1/m}^Z [g_{r+s}]^{[m]} \right)^{\frac{1}{2}} \right\|_\infty \right) \\
\leq C_{\zeta,f,g} \left\{ \left( \log \mu(e^{\zeta |g|^p}) \right)^{\frac{1}{p}} + \left( \log \mu(e^{\zeta |f|^p}) \right)^{\frac{1}{p}} \right\}. \tag{3.22} \text{sup-g-f-m} \]

Substituting this into (3.20) and (3.21), there exists positive constant $C_{\zeta,f,g}$ depending on $\zeta, f, g$ such that

\[
\left\| \phi_s - \phi_s^{[m]} \right\|_\infty \leq \left\| \nabla \phi_t \right\|_\infty \int_0^\infty e^{-\lambda r} \left\| P_r^Z \left( (g_{s+r} - g_{s+r}^{[m]}) \right) \right\|_\infty \, dr \\
+ C_{\zeta,f,g} \int_0^\infty e^{-\lambda r} \left( r^{-\frac{2}{p} + 1} + 1 \right) \left\| \nabla (\phi_{r+s} - \phi_{r+s}^{[m]}) \right\|_\infty \, dr \\
+ \int_0^\infty e^{-\lambda r} \left\| P_r^Z [f_{s+r} - f_{s+r}^{[m]}] \right\|_\infty \, dr.
\]

and

\[
\left\| \nabla (\phi_s - \phi_s^{[m]}) \right\|_\infty \leq \left\| \nabla \phi_t \right\|_\infty \int_0^\infty e^{-\lambda r} \left\| P_r^Z \left( (g_{s+r} - g_{s+r}^{[m]}) \right) \right\|_\infty \, dr \\
+ C_{\zeta,f,g} \int_0^\infty e^{-\lambda r} \left( r^{-\frac{2}{p} + 1} + 1 \right) \left\| \nabla (\phi_{r+s} - \phi_{r+s}^{[m]}) \right\|_\infty \, dr.
\]
+ \int_{0}^{\infty} \frac{e^{-(\lambda-\delta)t}}{\sqrt{t}} \left\| \left( P_{r}^{Z} \mid f_{s+r} - f_{s+r}^{[m]} \right)^{2} \right\| dt \quad (3.24)

Because \frac{2}{p} + \frac{1}{q} < 1,

\left\| \lim_{m \to \infty} P_{r}^{Z} \left( \left| f_{r+s}^{[m]} - f_{r+s} \right|^{2} + \left| g_{r+s} - g_{r+s}^{[m]} \right|^{2} \right) \right\|_{\infty} \leq \lim_{m \to \infty} C_{r} \mu \left( \left| f_{r+s}^{[m]} - f_{r+s} \right|^{2} + \left| g_{r+s} - g_{r+s}^{[m]} \right|^{2} \right) = 0,

and

\sup_{m, \ell} \left( \left\| \phi_{t}^{[m]} \right\|_{\infty} + \left\| \nabla \phi_{t}^{[m]} \right\|_{\infty} \right) < \infty, \quad (3.25)

which follows from Lemma 3.2. We can apply Fatou’s lemma to (3.23) and (3.24) and get that

\sup_{s \geq 0} \lim_{m \to \infty} \left\| \nabla \left( \phi_{t} - \phi_{t}^{[m]} \right) \right\|_{\infty} \leq \frac{\tilde{C} (\zeta, g, f)}{(\lambda - \delta)^{\frac{1}{2} + \frac{2}{p}}} \sup_{s \geq 0} \lim_{m \to \infty} \left\| \nabla \left( \phi_{t} - \phi_{t}^{[m]} \right) \right\|_{\infty}.

Hence there is \lambda_{0} > 0 such that \tilde{C} (\zeta, g, f) (\lambda - \delta)^{-\frac{1}{2} + \frac{2}{p}} < 1 holds for all \lambda > \lambda_{0}, consequently,

\lim_{m \to \infty} \left( \left\| \nabla \left( \phi_{t} - \phi_{t}^{[m]} \right) \right\|_{\infty} + \left\| \phi_{t} - \phi_{t}^{[m]} \right\|_{\infty} \right) = 0. \quad (3.26)

Next, we shall prove that \{ \phi_{[m]}, \phi \}_{m \in \mathbb{N}} is equicontinuous in \( C_{b}^{1}(\mathbb{R}^{d}, \mathbb{R}^{d}) \). Then \{ \phi_{[m]} - \phi \}_{m \geq 0} is equicontinuous. The Arzel-Ascoli theorem will imply that for any \( R > 0 \) there is a subsequence \{ \phi_{[m]} \}_{j \geq 0} such that \left\| \phi_{[m]} - \phi_{[j]} \right\|_{C_{b}^{1}(\mathbb{R}^{d}, \mathbb{R}^{d})} \) converges uniformly on \( t \in [0, R] \). Taking into account (3.26), the limit must be zero indeed. To conclude the claim (2), one only need to use Cantor’s diagonal argument. We shall consider the integrations as follows

\[ H_{s}^{[m]} := \int_{s}^{\infty} e^{\lambda(t-s)} P_{s}^{Z} h_{t}^{[m]} dt, \quad \nabla H_{s}^{[m]} := \int_{s}^{\infty} e^{\lambda(t-s)} \nabla P_{s}^{Z} h_{t}^{[m]} dt. \]

According to Remark 3.11, we can assume \( h_{t}^{[m]} \) is a real value function on \( \mathbb{R}^{d} \) indexed by \( m \in \mathbb{N} \). We only estimate \( \nabla H_{s}^{[m]} \) and set \( h_{t}^{[m]} = \nabla g_{[m]}(\phi_{t}) \) with \( \mathbb{R}^{d} \)-valued functions \( g \) and \( \mathbb{R} \)-valued function \( \phi \). The claims hold for \( h_{t}^{[m]} \) replaced by \( f_{t}^{[m]} \) and also hold for \( H^{[m]} \).

For any \( s_{1}, s_{2} \in [0, \infty) \) and \( s_{1} < s_{2} \),

\[ \left\| \nabla H_{s_{1}}^{[m]} - \nabla H_{s_{2}}^{[m]} \right\|_{\infty} \leq \left\| \int_{s_{1}}^{s_{2}} e^{-\lambda(t-s_{1})} \nabla P_{s_{1}}^{Z} h_{t}^{[m]} dt \right\|_{\infty}. \]
For $I_1$, the ultracontractivity of $P_t^Z$ and (3.10) yield

$$\sup_{m \geq 1} \left\| \int_{s_1}^{s_2} e^{-(\lambda - \delta)u} \nabla P_{u+s_1}^Z h_t^{[m]} \, du \right\|_\infty \leq \int_0^{s_2-s_1} e^{-(\lambda - \delta)u} \sup_{m \geq 0} \left\| P_{u+s_1}^Z |h_t^{[m]}|^2 \right\|_\infty \, du \leq C \zeta^{-\frac{1}{\beta}} \sup_{m \geq 1, t \geq 0} \left\| \nabla \phi_t^{[m]} \right\|_\infty \int_0^{s_2-s_1} e^{-(\lambda - \delta)u} \left\| P_{u+s_1}^Z |g_t^{[m]}|^2 \right\|_\infty \, du,$$

which implies that there is $C > 0$ independent of $m$, $s_1$ and $s_2$ such that $I_1 \leq C |s_1 - s_2|^{-\frac{\beta}{\alpha}}$.

For $I_2$, it follows from the ultracontractivity of $P_t^Z$ that there is a positive constant $C_{\zeta, f, g}$ independent of $m$, $s_1$ and $s_2$ such that

$$\sup_{m \geq 1} \int_{s_1}^{s_2} e^{-(\lambda - \delta)(t-s_2)} \left\| P_{s_2}^Z |g_t^{[m]}|^2 \right\|_\infty \, dt \leq C_{\zeta, f, g} \left( e^{-\lambda(s_2-s_1)} - 1 \right).$$

For $I_3$,

$$I_3 \leq \int_0^{\infty} e^{-\lambda u} \left\| \nabla P_{u+s_2-s_1}^Z h_{u+s_2}^{[m]} - \nabla P_{u}^Z h_{u+s_2}^{[m]} \right\|_\infty \, du \leq \int_0^{\infty} e^{-\lambda u} \left\| \nabla P_{u/3}^Z (P_{u/3}^Z (P_{s_2-s_1}^Z - I) P_{u/3}^Z h_{u+s_2}^{[m]}) \right\|_\infty \, du \leq \sqrt{3} \int_0^{\infty} e^{-\lambda u} \left\| P_{u/3}^Z (P_{s_2-s_1}^Z - I) P_{u/3}^Z h_{u+s_2}^{[m]} \right\|_\infty \, du.$$

Just as in Lemma 3.2, it can be proved that

$$\int_0^{\infty} e^{-\lambda u} \sup_{v \geq 0, s_2 \geq 0} \left\| P_{u/3}^Z (P_v^Z - I) P_{u/3}^Z h_{u+s_2}^{[m]} \right\|_\infty \, du < \infty.$$

On the other hand, the ultracontractivity of $P_{u/3}^Z$ implies that $P_{u/3}^Z$ is a compact operator in $L^2(\mu)$. Then the set $P_{u/3}^Z B_1$ is a relative compact set in $L^2(\mu)$, where $B_1$ is the unit ball in $L^2(\mu)$. Consequently, for all $\epsilon > 0$, there exist $\{f_k\}_{k=1}^{N(\epsilon)}$ with
\[\mu(|f_k|^2) \leq 1\] such that
\[
P_{u/3}B_1 \subset \bigcup_{k=1}^{n(\epsilon)} \left\{ f \in L^2(\mu) \mid \|P_{u/3}f_k - f\|_{L^2(\mu)} < \epsilon \right\}.
\]
Consequently, for any \(\mu(f^2) \leq 1\), there is \(f_k\) such that
\[
\|(P_v^Z - I) P_{u/3}f\|_{L^2(\mu)} \leq \|(P_v^Z - I) P_{u/3}f_k\|_{L^2(\mu)} + \epsilon.
\]
It follows from the strong continuity of \(P_v^Z\) that
\[
\lim_{v \to 0^+} \sup_{1 \leq k \leq n(\epsilon)} \|(P_v^Z - I) P_{u/3}f_k\|_{L^2(\mu)} = 0.
\]
Then
\[
\lim_{v \to 0^+} \sup_{\|f\|_{L^2(\mu)} \leq 1} \|(P_v^Z - I) P_{u/3}f\|_{L^2(\mu)} \leq \epsilon,
\]
which yields
\[
\lim_{v \to 0^+} \|(P_v^Z - I) P_{u/3}\| = 0. \tag{3.27} \]
Hence
\[
\lim_{v \to 0^+} \sup_{s_2 \geq 0, m \geq 1} \left| \left| P_{u/3} (P_v^Z - I) P_{u/3} h_{u+m+s_2}^m \right| \right|_{\infty}
\leq C_u \lim_{v \to 0^+} \sup_{s_2 \geq 0, m \geq 1} \left| \left| (P_v^Z - I) P_{u/3} h_{u+m+s_2}^m \right| \right|_{L^2(\mu)}
\leq C_u \lim_{v \to 0^+} \left| \left| (P_v^Z - I) P_{u/3} \right| \right|_{L^2(\mu)} \sup_{s_2 \geq 0, m \geq 1} \left| \left| P_{1/m} h_{u+s_2} \right| \right|_{L^2(\mu)}
\leq C_u \lim_{v \to 0^+} \left| \left| (P_v^Z - I) P_{u/3} \right| \right|_{L^2(\mu)} \sup_{s_2 \geq 0, m \geq 1} \left( \left| \left| P_{1/m} g_{u+s_2} \right| \right|_{L^2(\mu)} \left| \left| \nabla \phi_{u+s_2}^m \right| \right|_{\infty} \right)
\leq C_u \lim_{v \to 0^+} \left| \left| (P_v^Z - I) P_{u/3} \right| \right|_{L^2(\mu)} \sup_{t \geq 0, m \in \mathbb{N}} \left( \left| \left| g_t \right| \right|_{L^2(\mu)} \left| \left| \nabla \phi_t^m \right| \right|_{\infty} \right)
= 0.
\]
Applying the dominated convergence theorem, we have \(\lim_{v \to 0^+} \sup_{s_2 \geq 0, m \geq 1} I_3 = 0\), which yields the equicontinuous of \(I_3\).

(3) Fixing some \(m \in \mathbb{N}\), it is clear by (1) that
\[
\sup_{t \geq 0} \left( \left| \left| f_t^m \right| \right|_{C_b^1(\mathbb{R}^d)} + \left| \left| g_t^m \right| \right|_{C_b^1(\mathbb{R}^d)} \right) < \infty.
\]
Then following from the Bochner differentiation theorem (see e.g. Theorem 3.8.5)) that
\[
\lim_{n \to \infty} \|f_t^{m,n} - f_t^m\|_{C_b^1(\mathbb{R}^d)} \leq \lim_{n \to \infty} n \int_t^{t+1/n} \|f_r^m - f_t^m\|_{C_b^1(\mathbb{R}^d)} dr
\leq C_m \lim_{n \to \infty} n \int_t^{t+1/n} \|f_r - f_t\|_{L^2(\mu)} dr
\]
Lemma 3.4. Let \( \lambda \) be the existence and uniqueness of solutions to (3.9) follows. We have that \( \Pi(u) \) admits a unique solution.

Proof. Suppose there is some \( \beta > 0 \) such that for any \( R > 0 \),

\[
\lim_{k \to \infty} \sup_{t \in [0,R]} \left( \left\| \nabla \phi_t^{[m,n]} \right\|_\infty + \left\| \phi_t^{[m,n]} - \phi_t^{[m]} \right\|_\infty \right) = 0.
\]

Therefore, for any \( R > 0 \), there is a subsequence \( [m_k, n_k] \) such that (3.17) holds.

Let \( \Pi \) be a map on \( C_b([0,\infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)) \) defined as follows

\[
\Pi(u)_s = \int_s^\infty e^{-\lambda(t-s)} P_s^Z \{ \nabla g_t u_t + f_t \} \, dt, \quad u \in C_b([0,\infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)).
\]

We shall prove that \( \Pi \) is a contraction mapping in \( C_b([0,\infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)) \), then (3.9) admits a unique solution.

Lemma 3.4. Under the same condition of Lemma 3.3, the equation (3.9) has a unique solution in \( C_b([0,\infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)) \).

Proof. By (A3), we have \( P_s^Z \{ \nabla g_t u_t + f_t \} \in C^1_b(\mathbb{R}^d, \mathbb{R}^d) \), which along with (3.12) and (3.14) imply that \( \Pi(u)_s \in C^1_b(\mathbb{R}^d, \mathbb{R}^d) \). Following the proof of (2) in Lemma 3.3 we have that \( \Pi(u) \) is uniformly continuous in \( C^1_b(\mathbb{R}^d, \mathbb{R}^d) \). According to (3.12) and (3.14) again, there exists \( \lambda_0 > 0 \) such that for \( \lambda > \lambda_0 \) there is \( C(\lambda, \zeta) < 1 \) such that

\[
\sup_{s \geq 0} \| \Pi(u^{[1]})_s - \Pi(u^{[2]})_s \|_\infty \leq C(\lambda, \zeta) \sup_{s \geq 0} \| \nabla u^{[1]} - \nabla u^{[2]} \|_\infty
\]

where \( u^{[1]}, u^{[2]} \in C_b([0,\infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)) \). Therefore, the mapping \( \Pi \) is contractive and the existence and uniqueness of solutions to (3.9) follows.

3.2 Bismut formula and the regularity of \( P_t^Z \)

We shall study the regularity of \( P_t^Z \) under the assumptions of (A1) and (A2). We first give a general results on dissipative maps with perturbation.

Lemma 3.5. Suppose there is some \( \beta \geq 0 \) such that there are \( K_2 > 0, K_4 \geq 0 \) \( (K_4 < K_2 \text{ if } \beta = 0) \) and non-negative constants \( K_1, K_3 \) such that

\[
|\nabla Z(x)| \leq K_1 |x|^\beta + K_3, \quad \langle \nabla_v Z(x), v \rangle \leq -K_2 |x|^\beta |v|^2 + K_4 |v|^2.
\]
Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism with
$$
\Phi(x) = x + \phi(x), \quad x \in \mathbb{R}^d,
$$
and let $T = (I + \nabla \phi)^{-1} \nabla \phi$. If
$$
\sup_{x \in \mathbb{R}^d} ||T(x)|| < \infty, \quad K_1 \sup_{x \in \mathbb{R}^d} ||T(x)|| < K_2, \quad |\Phi(x)| \leq c(1 + |x|), \quad x \in \mathbb{R}^d,
$$
for some $c > 0$. Then there is $K > 0$ and $C \geq 0$ such that
$$
(Z \circ \Phi^{-1}(x) - Z \circ \Phi^{-1}(y), x - y) \leq -K|x - y|^\beta + C|x - y|^2. \quad (3.28)
$$
In particular, (A1) implies that $(3.28)$ holds.

**Proof.** Notice that
$$
\nabla \Phi^{-1} = (\nabla \Phi)^{-1} = (I + \nabla \phi)^{-1}
= (I + \nabla \phi)^{-1} - I + I
= I + (I + \nabla \phi)^{-1} \nabla \phi,
$$
where $(\nabla \phi)_{ij} = \partial_j \phi_i$. Then by the chain rule,
$$
(Z \circ \Phi^{-1}(x) - Z \circ \Phi^{-1}(y), x - y)
= \int_0^1 \langle (\nabla Z \circ \Phi^{-1} \cdot \nabla \Phi^{-1})(y + r(x - y))(x - y), x - y \rangle \, dr
= \int_0^1 \langle (\nabla Z \circ \Phi^{-1})(y + r(x - y))(x - y), x - y \rangle \, dr
+ \int_0^1 \langle (\nabla Z \circ \Phi^{-1} \cdot T)(y + r(x - y))(x - y), x - y \rangle \, dr
\leq -K_2|x - y|^2 \int_0^1 |\Phi^{-1}(y + r(x - y))|^{\beta} \, dr + \left( K_4 + K_3 \sup_{x \in \mathbb{R}^d} ||T(x)|| \right) |x - y|^2
+ K_1|x - y|^2 \sup_{x \in \mathbb{R}^d} ||T(x)|| \int_0^1 |\Phi^{-1}(y + r(x - y))|^{\beta} \, dr
\leq -C_\beta \left( K_2 - K_1 \sup_{x \in \mathbb{R}^d} ||T(x)|| \right) |x - y|^2 \int_0^1 |y + r(x - y)|^{\beta} \, dr
+ \left( C_\beta + K_4 + K_3 \sup_{x \in \mathbb{R}^d} ||T(x)|| \right) |x - y|^2. \quad (3.29)
$$
Following from the basis inequality:
$$
|a^q - b^q| \leq |a - b|^q, \quad q \in [0, 1], a \geq 0, b \geq 0,
$$
if $\beta \leq 2$, then
$$
|y^\frac{\beta}{2} - r^\frac{\beta}{2}|x - y|^\frac{\beta}{2} \leq ||y - r||x - y||^\frac{\beta}{2} \leq |y + r(x - y)|^\frac{\beta}{2} \quad r \in [0, 1].$$
Consequently,
\[ |y|^{\beta} - 2r^{\frac{\beta}{2}} |y|^{\frac{\beta}{2}} |x - y|^{\frac{\beta}{2}} + r^{\beta} |x - y|^\beta \leq |y + r(x - y)|^\beta \]
and
\[ \int_0^1 |y + r(x - y)|^\beta \, dr \geq \int_0^1 \left( |y|^{\beta} - 2r^{\frac{\beta}{2}} |y|^{\frac{\beta}{2}} |x - y|^{\frac{\beta}{2}} + r^{\beta} |x - y|^\beta \right) \, dr \]
\[ = |y|^{\beta} - \frac{4}{\beta + 2} |y| |x - y| + \frac{1}{\beta + 1} |x - y|^\beta \]
\[ \geq \frac{\beta^2}{(\beta + 1)(\beta + 2)^2} |x - y|^2. \] (3.30) \text{Ineq-mon1}

On the other hand, if \( \beta > 2 \), then the Jessen inequality yields
\[ \int_0^1 |y + r(x - y)|^\beta \, dr \geq \left( \int_0^1 |y + r(x - y)|^2 \, dr \right)^{\frac{\beta}{2}} \geq 12^{\frac{-\frac{\beta}{2}}{2}} |x - y|^\beta. \] (3.31) \text{Ineq-mon2}

Substituting (3.30) and (3.31) into (3.29), we complete the proof.

If \( \Phi(x) = x = \Phi^{-1}(x) \), then \( \phi \equiv 0 \), and it is easy to get (3.28).

\[ \square \]

Let \( Y_t \) be the solution of the following equation
\[ dY_t = Z(Y_t)dt + \sigma(Y_t)dW_t, Y_0 = y. \] (3.32) \text{equ-Y}

It follows from \cite{10} Theorem 3.1.1 and (3.5) that the equation
\[ d\eta_{s,t} = \nabla_{\eta_{s,t}}Z(Y_{s,t})dt + \nabla_{\eta_{s,t}}\sigma(Y_{s,t})dW_t, \eta_{s,s} = v \in \mathbb{R}^d, t \geq s. \] (3.33) \text{equ-et}

has a unique non-explosive strong solution. Moreover, for all \( p > 0 \), there is a \( c_p > 0 \) which depends on \( p \) such that
\[ \mathbb{E} \sup_{r \in [s,t]} |\eta_{r,t}|^p \leq 2e^{c_p(t-s)}\mathbb{E}|\eta_s|^p. \] (3.34) \text{Inequ-et-sup}

It can be proved that the derivative of \( Y_t \) w.r.t. the initial value along \( v \in \mathbb{R}^d \) exists and satisfies (3.33). We shall denote by the derivative \( \nabla_v Y_t \). Let \( \{h_{s,t}\}_{t \geq s} \) be an adapted process defined as follows
\[ h_s = 0; \quad h'_{s,t} = w_s(t)\sigma_s^{-1}(Y_{s,t})\nabla_v Y_{s,t} \equiv w_t(t)\sigma_s^*(\sigma_s^*)^{-1}(Y_{s,t})\nabla_v Y_{s,t}, \]
where \( w_s(\cdot) \) is a positive continuous function on \( [s, \infty) \). Because of (A2) and (3.34), for any \( T > 0 \) and \( p > 0 \), we have that
\[ \mathbb{E} \sup_{t \in [s,T]} |h'_{s,t}|^p \leq \sigma_0^{-1} \sup_{t \in [0,T]} w_s(t)\mathbb{E} \sup_{t \in [0,T]} |\nabla_v Y_t|^p \]
\[ \leq 2\sigma_0^{-1} \sup_{t \in [s,T]} w_s(t) e^{c_p(T-s)}|v|^p. \] (3.35) \text{Inequ-h'-sup}

It is clear that \( h \) is in the Cameron-Martin space of the Wiener process. Next, we shall prove that \( Y_t \) is Malliavin differentiable along \( h \).
Lemma 3.6. Assume (A1) and (A2). Let h be defined as above. Then $Y_{s,t}$ is differentiable w.r.t. initial value, and is Malliavin differentiable along $h$. The corresponding derivatives processes, denoted by $\nabla_s Y_{s,t}$ and $D_h Y_{s,t}$ respectively, satisfy

$$dD_h Y_{s,t} = \nabla_{D_h Y_{s,t}} Z(Y_{s,t}) dt + \nabla_{D_h Y_{s,t}} \sigma(Y_{s,t}) dW_t + \sigma(Y_{s,t}) h_s' dt, \quad D_h Y_{s,s} = 0. \quad (3.36)$$

Moreover, the second order derivative of $Y_t$ w.r.t. the initial value exists and satisfies

$$\mathbb{E} \sup_{|\eta|,|\theta| \leq 1} \sup_{t \in [0,T]} |\nabla_u \nabla_v Y_t|^{2p} \leq C_p (T \wedge 1) e^{C_p T}, \quad T \geq 0 \quad (3.37)$$

Proof. We just prove for $s = 0$, and for $s > 0$ the proof is similar. Let $Y^\varepsilon$ be the solution of the following equation:

$$Y^\varepsilon_t = Y_0 + \int_0^t Z(Y^\varepsilon_s) ds + \int_0^t \sigma(Y^\varepsilon_s) dW_s + \varepsilon \int_0^t \sigma(Y^\varepsilon_s) h_s' ds.$$

Then

$$d|Y^\varepsilon_t - Y_t|^2 = 2 \langle Z(Y^\varepsilon_t) - Z(Y_t), Y^\varepsilon_t - Y_t \rangle dt + \langle \sigma(Y^\varepsilon_t) - \sigma(Y_t) \rangle H ds dt$$
$$+ 2 \langle Y^\varepsilon_t - Y_t, \sigma(Y^\varepsilon_t) - \sigma(Y_t) \rangle dW_t + 2 \varepsilon \langle \sigma(Y^\varepsilon_t) h_t', Y^\varepsilon_t - Y_t \rangle dt$$

$$\leq C |Y^\varepsilon_t - Y_t|^2 dt + 2 \varepsilon \langle \sigma(Y^\varepsilon_t) h_t', Y^\varepsilon_t - Y_t \rangle dt - K |Y^\varepsilon_t - Y_t|^3 dt$$
$$+ 2 \langle Y^\varepsilon_t - Y_t, \sigma(Y^\varepsilon_t) - \sigma(Y_t) \rangle dW_t,$$

for some positive constants $K$ and $C$. For any $p \geq 1$, there is a positive constant $C_p > 0$ such that

$$d|Y^\varepsilon_t - Y_t|^{2p} \leq C_p |Y^\varepsilon_t - Y_t|^{2p} dt + p |Y^\varepsilon_t - Y_t|^{2p-1} |\sigma(Y^\varepsilon_t)| |h_t'| dt$$
$$+ 2 p |Y^\varepsilon_t - Y_t|^{2p-2} (Y^\varepsilon_t - Y_t, \sigma(Y^\varepsilon_t) - \sigma(Y_t)) dW_t$$
$$\leq C_p |Y^\varepsilon_t - Y_t|^{2p} dt + \varepsilon^{2p} (1 + |Y^\varepsilon_t|)^{2p} |h_t'|^{2p} dt$$
$$+ 2 p |Y^\varepsilon_t - Y_t|^{2p-2} (Y^\varepsilon_t - Y_t, \sigma(Y^\varepsilon_t) - \sigma(Y_t)) dW_t.$$

By B-D-G inequality, we have

$$\mathbb{E} \sup_{t \in [0,s]} |Y^\varepsilon_t - Y_t|^{2p} \leq C_p \mathbb{E} \int_0^s |Y^\varepsilon_t - Y_t|^{2p} dt + \varepsilon^{2p} \mathbb{E} (1 + |Y^\varepsilon_t|)^{2p} |h_t'|^{2p} dt.$$

Then Gronwall’s inequality yields that

$$\mathbb{E} \sup_{t \in [0,T]} |Y^\varepsilon_t - Y_t|^{2p} \leq \varepsilon^{2p} \int_0^T \mathbb{E} (1 + |Y^\varepsilon_t|)^{2p} |h_t'|^{2p} dte^{C_p T}, \quad T > 0. \quad (3.38)$$

Inequ-Y-Y

For $Y^\varepsilon_t$ with $\varepsilon \in [0,1]$, we have

$$d|Y^\varepsilon_t|^2 \leq - \tilde{K}_1 |Y^\varepsilon_t|^{\beta+2} dt + 2 \varepsilon \langle \sigma(Y^\varepsilon_t) h_t', Y^\varepsilon_t \rangle dW_t$$
$$+ 2 p |Y^\varepsilon_t|^{2p} dt + 2 \varepsilon \langle \sigma(Y^\varepsilon_t) h_t', Y^\varepsilon_t \rangle dW_t$$
$$\leq - \tilde{K}_1 |Y^\varepsilon_t|^{\beta+2} dt + 2 \varepsilon \langle \sigma(Y^\varepsilon_t) h_t', Y^\varepsilon_t \rangle dW_t$$
$$+ 2 C |Y^\varepsilon_t| (1 + |Y^\varepsilon_t|) |h_t'| dt + 2 \langle \sigma(Y^\varepsilon_t) h_t', Y^\varepsilon_t \rangle dW_t,$$
for some positive constants $\tilde{K}_1, \tilde{K}_2, C > 0$. Then for any $p \geq 1$,
\[
d|Y_t^\epsilon|^{2p} \leq - c_p |Y_t^\epsilon|^\beta + 2p dt + \tilde{c}_p (1 + |Y_t^\epsilon|^{2p}) dt \\
+ 2p |Y_t^\epsilon|^{2p-1} (1 + |Y_t^\epsilon|) |h_t^\epsilon| dt + 2p |Y_t^\epsilon|^{2p-2} (Y_t^\epsilon, \sigma(Y_t^\epsilon) dW_t) \\
\leq - c_p |Y_t^\epsilon|^\beta + 2p dt + \tilde{c}_p (1 + |Y_t^\epsilon|^{2p}) dt + (|h_t^\epsilon|^{2p} + |h_t^\epsilon|^{\frac{4+\beta}{2}}) dt \\
+ 2p |Y_t^\epsilon|^{2p-2} (Y_t^\epsilon, \sigma(Y_t^\epsilon) dW_t),
\]

where $c_p$ and $\tilde{c}_p$ maybe different positive constants different from line to line. Applying B-D-G inequality and Gronwall’s inequality, we get that for any $s > 0, p \geq 1$ there is $C_p > 0$ such that
\[
\sup_{t \in [0,1]} \left( \mathbb{E} \sup_{\epsilon \in [0,1]} |Y_t^\epsilon|^{2p} + \int_0^s |Y_t^\epsilon|^{\beta + 2p} dt \right) \leq C_p \left( 1 + s + |Y_0^\epsilon|^{2p} + \mathbb{E} \int_0^s |h_t^\epsilon|^{\frac{4+\beta}{2}} dt \right) e^{C_p s}.
\]
Taking into account (3.35), it follows that there is $C_{p, \beta} > 0$ such that
\[
\sup_{t \in [0,1]} \left( \mathbb{E} \sup_{\epsilon \in [0,1]} |Y_t^\epsilon|^{2p} + \int_0^s |Y_t^\epsilon|^{\beta + 2p} dt \right) \\
\leq C_p \left( \sup_{r \in [0,s]} \int_0^{\frac{2p+\beta}{\beta}} (r) \left( 1 + |Y_0^\epsilon|^{2p} \right) e^{C_{p, \beta} s}, s \geq 0, \right)
\]
which, together with (3.35), implies that
\[
\mathbb{E} \sup_{t \in [0,s]} |Y_t^\epsilon - Y_t|^2 \leq e^{2p C_p} \left( \sup_{r \in [0,s]} \int_0^{\frac{2p+2+\beta}{\beta}} (r) \right) e^{C_{p, \beta} s}, s \geq 0.
\]

Let $D_h Y_t$ be a solution of the following equation
\[
dD_h Y_t = \nabla_{D_h Y_t} Z(Y_t) dt + \nabla_{D_h Y_t} \sigma(Y_t) dW_t + \sigma(Y_t) h_t^\epsilon dt, \quad D_h Y_0 = 0,
\]
and let
\[
V_t^\epsilon = \frac{Y_t^\epsilon - Y_t}{\epsilon} = D_h Y_t, \quad U_t^\epsilon = \frac{Y_t^\epsilon - Y_t}{\epsilon}.
\]
Then
\[
dV_t^\epsilon = \int_0^1 \left( \nabla_{U_t^\epsilon} Z(Y_t + \theta (Y_t^\epsilon - Y_t)) - \nabla_{U_t^\epsilon} Z(Y_t) \right) \frac{dt}{\epsilon} \\
+ \nabla_{V_t^\epsilon} Z(Y_t) dt + \nabla_{V_t^\epsilon} \sigma(Y_t) dW_t \\
+ \int_0^1 \left( \nabla_{U_t^\epsilon} \sigma(Y_t + \theta (Y_t^\epsilon - Y_t)) - \nabla_{U_t^\epsilon} \sigma(Y_t) \right) \frac{d\theta}{\epsilon} dW_t \\
+ (\sigma(Y_t^\epsilon) - \sigma(Y_t)) h_t^\epsilon dt.
\]

By Itô’s formula, (A1) and (A2), for any $p \geq 1$, there are positive constants $c_p$ and $\tilde{c}_p$ such that
\[
d|V_t^\epsilon|^{2p} \leq - c_p |V_t^\epsilon|^\beta + 2p |V_t^\epsilon|^{2p} dt + \tilde{c}_p |V_t^\epsilon|^{2p} dt \\
+ \int_0^1 \|\nabla \sigma(Y_t + \theta \epsilon U_t^\epsilon) - \nabla \sigma(Y_t)\|^{2p} d\theta |U_t^\epsilon|^{2p} dt.
\]
\[
\begin{align*}
+ \int_0^1 |\nabla Z(Y_t + \epsilon U_t^I) - \nabla Z(Y_t)|^{2p} \, d\theta |U_t^I|^{2p} \, dt \\
+ 2p |V_t^I|^{2p-2} \left( V_t^I, \int_0^1 (\nabla U_t^I \sigma(Y_t + \epsilon U_t^I) - \nabla U_t^I \sigma(Y_t)) \right) \, d\theta dW_t \\
+ 2p |V_t^I|^{2p-2} \left( V_t^I, \nabla Y_t \sigma(Y_t) \right) dW_t.
\end{align*}
\]

It follows from the B-D-G inequality and Gronwall’s inequality that for any \( T > 0 \)
\[
\mathbb{E} \sup_{t \in [0,T]} |V_t^I|^{2p} + \int_0^T |Y_t|^{3+2|V_t^I|^{2p-2} dt \\
\leq e^{C_p(T + 1)} \int_0^T \int_0^1 \|\nabla \sigma(Y_t + \theta \epsilon U_t^I) - \nabla \sigma(Y_t)|^{2p} \, d\theta |U_t^I|^{2p} dt \\
+ \int_0^T \int_0^1 |\nabla Z(Y_t + \epsilon U_t^I) - \nabla Z(Y_t)|^{2p} \, d\theta |U_t^I|^{2p} dt.
\]

It follows from (3.40), (3.39) and the dominated convergence theorem that
\[
\lim_{\epsilon \to 0^+} \mathbb{E} \sup_{t \in [0,T]} |V_t^I|^{2p} = 0.
\]

Similarly, following from (A1), (A2), (3.34) and (3.39), we can prove that \( \nabla_u Y_t^y \) is differentiable w.r.t. \( y \), i.e. \( Y_t \) is twice differentiable w.r.t. the initial value. Moreover, \( \nabla_u \nabla_t Y_t \) satisfies
\[
d\nabla_u \nabla_t Y_t = \nabla_u \nabla_t Y_t Z(Y_t) dt + \nabla_u \nabla_t Y_t \nabla_u \nabla_t Y_t \sigma(Y_t) dW_t \\
+ \nabla_u \nabla_t Y_t Z(Y_t) dt + \nabla_u \nabla_t Y_t \sigma(Y_t) dW_t, \quad \nabla_u \nabla_t Y_t = 0,
\]
and for any \( p > 0 \), there is \( C_p > 0 \) such that
\[
\mathbb{E} \sup_{t \in [0,T]} |\nabla_u \nabla_t Y_t|^{2p} \leq C_p(T + 1) e^{C_p T} |u|^{2p} |v|^{2p}, \quad T \geq 0. \tag{3.41}
\]

Since \( \nabla \nabla Z(\cdot) \) and \( \nabla \nabla \sigma(\cdot) \) are continuous, we have that
\[
\lim_{y \to x} \mathbb{E} \sup_{t \in [0,T]} |\nabla_u \nabla_t Y_t^y - \nabla_u \nabla_t Y_t^x|^2 = 0, \quad T > 0. \tag{3.42}
\]

Next, we shall introduce a lemma on the gradient estimate of \( P_{s,t}^Z \) for \( t > s \). The method to prove this lemma is due to [27, Lemma 2.1] essentially.

**Lemma 3.7.** Let \( t > s \). Then for any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) and \( q \in (1, \infty] \),
\[
\nabla_u \nabla_t P_{s,t}^Z f(y) = \frac{1}{t-s} \mathbb{E} f(Y_s^y) \int_s^t \langle \hat{\sigma}^{-1}(Y_r^y) \nabla_u Y_r^y, dW_r \rangle, \tag{3.43}
\]
Bismut
\[
|\nabla_u \nabla_t P_{s,t}^Z f| (y) \leq e^{c_q(t-s)} \frac{1}{t-s} (P_{s,t} f^q)^{\frac{1}{q}} (y) |u| \|v|, \tag{3.44}
\]
grad-2-est
and
\[
|\nabla P_{s,t}^Z f(x) - \nabla P_{s,t}^Z f(y)| \leq e^{c_q(t-s)} \left( \frac{|x-y|}{t-s} \wedge \frac{1}{\sqrt{t-s}} \right) \left\| (P_{s,t} f^q)^{\frac{1}{q}} \right\|_{\infty}. \tag{3.45}
\]
nn-P-x-y

If \( P_{s,t} \) is ultracontractive, then for any \( q \in (1, \infty] \), the operator \( P_{s,t}^Z \) is a bounded operator from \( L^q(\mu) \) to \( C_b^2(\mathbb{R}^d) \), and (3.33) hold for any \( f \in L^q(\mu) \).
Proof. For any $t > s \geq 0$ and $v \in \mathbb{R}^d$, set $h_{s,r} = \frac{1}{t-s} \hat{s}^{-1}(Y_{s,r}) \nabla v Y_{s,r}$. Then \{D_{h_{s,r}} \}_{r \in [s,t]} and \{\int_{s}^{r} \nabla v Y_{s,r} \}_{r \in [s,t]} satisfy
\[
d\xi_{s,r} = \nabla \xi_{s,r} Z(Y_{s,r}) dr + \nabla \xi_{s,r} \sigma(Y_{s,r}) dW_{r} + \frac{1}{t-s} \nabla v Y_{s,r} dr, \ r \in [s,t].
\]
Thus for $f \in C_{b}^{1}(\mathbb{R}^d)$
\[
\nabla_{v} P_{s,t}^{Z} f(y) = \mathbb{E}(\nabla f(Y_{s,t}^{y}), \nabla v Y_{s,t}^{y}) = \mathbb{E} \langle \nabla f(Y_{s,t}^{y}), D_{h_{s,t}} Y_{s,t}^{y} \rangle = \mathbb{E} D_{h}(f(Y_{s,t}^{y})) = \frac{1}{t-s} \mathbb{E} f(Y_{s,t}^{y}) \int_{s}^{t} \langle \hat{s}^{-1}(Y_{s,r}) \nabla v Y_{s,r}^{y}, dW_{r} \rangle.
\]
It is clear that $\nabla_{v} P_{s,t}^{Z} f(\cdot)$ is continuous on $\mathbb{R}^d$. Hence $P_{s,t}^{Z} f \in C_{b}^{1}(\mathbb{R}^d)$ and
\[
\| \nabla_{v} P_{s,t}^{Z} f \|_{\infty} < C_{1-s,\sigma_{0}} \|f\|_{\infty} |v|,
\]
which implies that there exists $P_{s,t}^{Z} f \in C_{b}^{1}(\mathbb{R}^d)$ and the formula (3.43) holds for all $f \in C_{b}(\mathbb{R}^d)$. Moreover,
\[
P_{s,t}^{Z} f(y + v) - P_{s,t}^{Z} f(y) = \int_{0}^{1} \nabla_{v} P_{s,t}^{Z} f(y + uv) du = \int_{0}^{1} \mathbb{E} f(Y_{s,t}^{y+uv}) \int_{s}^{t} \langle \hat{s}^{-1}(Y_{s,r}^{y+uv}) \nabla v Y_{s,r}^{y+uv}, dW_{r} \rangle du. \tag{3.46}
\]
Since a bounded Borel measurable function can be approximated by a bounded continuous function pointwise, by the dominated convergence theorem, (3.46) holds for all $f \in \mathcal{B}_{b}(\mathbb{R}^d)$, which yields that $P_{s,t}^{Z} f \in C_{b}(\mathbb{R}^d)$. Since $t$ is arbitrary, for any $0 < \epsilon < t - s$ and $f \in \mathcal{B}_{b}(\mathbb{R}^d)$, $P_{s,t-\epsilon}^{Z} f \in C_{b}(\mathbb{R}^d)$. Then
\[
\nabla_{v} P_{s,t-\epsilon}^{Z} (P_{t-\epsilon}^{Z} f)(y) = \nabla_{v} P_{s,t-\epsilon}^{Z} (P_{t-\epsilon}^{Z} f)(y) = \frac{1}{t-s-\epsilon} \mathbb{E} P_{t-\epsilon}^{Z} f(Y_{s,t-\epsilon}^{y}) \int_{s}^{t-\epsilon} \langle \hat{s}^{-1}(Y_{s,r}^{y}) \nabla v Y_{s,r}^{y}, dW_{r} \rangle
\]
\[
= \frac{1}{t-s-\epsilon} \mathbb{E} \left\{ \mathbb{E} f(Y_{s,t-\epsilon}^{y}) \langle \hat{s}^{-1}(Y_{s,r}^{y}) \nabla v Y_{s,r}^{y}, dW_{r} \rangle \right\}
= \frac{1}{t-s-\epsilon} \mathbb{E} f(Y_{s,t-\epsilon}^{y}) \int_{s}^{t-\epsilon} \langle \hat{s}^{-1}(Y_{s,r}^{y}) \nabla v Y_{s,r}^{y}, dW_{r} \rangle. \tag{3.47}
\]
Letting $\epsilon \to 0^+$, it easy to get (3.43) for all $f \in \mathcal{B}_{b}(\mathbb{R}^d)$ and
\[
\| \nabla_{v} P_{s,t}^{Z} f \|_{\infty} \leq \frac{C_{b}}{\sqrt{t-s}} \left\| (P_{s,t}^{Z} f)_{-}^{\frac{1}{q}} \right\|_{\infty} |v|, \tag{3.48}
\]
Moreover, it follows from the second equality of (3.47) that $\nabla_{v} P_{s,t}^{Z} f(\cdot) \in C_{b}^{1}(\mathbb{R}^d)$.

Next, we shall prove that $P_{s,t}^{Z} f \in C_{b}^{2}(\mathbb{R}^d)$. By the semigroup property,
\[
\nabla_{v} P_{s,t}^{Z} f(y) = \frac{2}{t-s} \mathbb{E} P_{s,t}^{Z} f(Y_{s,t-\epsilon}^{y}) \int_{s}^{t-\epsilon} \langle \hat{s}^{-1}(Y_{s,r}^{y}) \nabla v Y_{s,r}^{y}, dW_{r} \rangle, f \in \mathcal{B}_{b}(\mathbb{R}^d).
\]
Lemma 3.8. Assume (A1) and (A2). Suppose that \( f \) and \( g \) are in \( C_b([0, \infty), C_b^1(\mathbb{R}^d, \mathbb{R}^d)) \). Then there exists \( \lambda_0 > 0 \) such that for \( \lambda > \lambda_0 \), the solution to (3.41) satisfies (3.43).
Proof. We first prove that there is $\lambda_0$ such that for $\lambda > \lambda_0$ and $\delta \in (1, 2)$

$$
\int_s^\infty e^{-\lambda(t-s)}|\nabla^2 P_{s,t} \{\nabla_{g_t} \phi_t + f_t\}|^2(y) dt \leq C_{\lambda, \delta}(1 + |y|)^\delta < \infty. \tag{3.50}
$$

By [27, Lemma 2.2] and (3.45), for $\lambda > c_p$, there is $C(\lambda) > 0$ such that for all $h_t$ with $\sup_{t \geq 0} \|h_t\|_\infty < \infty$

$$
sup_{s \geq 0} \int_s^\infty e^{-\lambda(t-s)}|\nabla P^Z_{s,t} h_t(x) - \nabla P^Z_{s,t} h_t(y)| dt \leq C(\lambda) \left( \sup_{t \geq 0} \|h_t\|_\infty \right) |x - y| \log \left( e + \frac{1}{|x - y|} \right). \tag{3.51}
$$

It follows by (3.9), Lemma 3.2, (2) of Remark 3.2 and (3.51) that there is some $C > 0$ such that

$$
sup_{t \geq 0} |\nabla \phi_t(x) - \nabla \phi_t(y)| \leq C |x - y| \log \left( e + \frac{1}{|x - y|} \right). \tag{3.52}
$$

Taking into account $g_t \in C_b([0, \infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d))$ and $\sup_{t \geq 0} \|\nabla \phi_t\|_\infty < \infty$, we get that

$$
|\nabla g_t \phi_t(x) - \nabla g_t \phi_t(y)| \leq C \left( |x - y| \left( 1 + \log(e + |x - y|^{-1}) \right) \right) \wedge 1
$$

$$
\leq \tilde{C} \left\{ \left( |x - y| \log(e^2 + |x - y|^{-1}) \right) \wedge 1 \right\}, \quad x, y \in \mathbb{R}^d.
$$

Let $\psi = r \log^2(e + \frac{1}{\sqrt{r}})$, which is increasing and concave, and let $y^y_{s,t}$ be the solution of the following equation

$$
y^y_{s,t} = y + \int_s^t Z(y^y_{s,r}) dr, \quad t \geq s.
$$

Then it follows from (3.44) with $p = 2$ that

$$
|\nabla^2 P^Z_{s,t} \nabla g_t \phi_t(y)| = |\nabla^2 P^Z_{s,t} \{\nabla g_t \phi_t(\cdot) - \nabla g_t \phi_t(y)\}(x)|
$$

$$
\leq \frac{e^{c_2(t-s)}}{t-s} \left( \frac{P^Z_{s,t} |\nabla g_t \phi_t(\cdot) - \nabla g_t \phi_t(y^y_{s,t})|^2(y)}{t-s} \right)^{\frac{1}{2}}
$$

$$
= \frac{e^{c_2(t-s)}}{t-s} \left( \frac{\mathbb{E} \left| \nabla g_t \phi_t(Y^y_{s,t}) - \nabla g_t \phi_t(y^y_{s,t}) \right|^2}{t-s} \right)^{\frac{1}{2}}
$$

$$
\leq \frac{C e^{c_2(t-s)}}{t-s} \left( \frac{\mathbb{E} \psi(|Y^y_{s,t} - y^y_{s,t}|^2)}{t-s} \wedge 1 \right)
$$

$$
\leq \frac{C e^{c_2(t-s)}}{t-s} \left( \frac{\psi(\mathbb{E}|Y^y_{s,t} - y^y_{s,t}|^2)}{t-s} \wedge 1 \right). \tag{3.53}
$$

It is easy to get that

$$
\mathbb{E}|Y^y_{s,t} - y^y_{s,t}|^2 \leq C_\beta(1 + |y|^2) \left( (t-s) + (t-s)^2 \right).
$$

Substituting this into (3.53), we get that

$$
|\nabla^2 P^Z_{s,t} \nabla g_t \phi_t(y)| \leq \frac{C}{t-s} \left( \sqrt{\psi((t-s) + (t-s)^2)} \wedge 1 \right) \left( 1 + |y| \right)
$$
where \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are positive constants independent of \( y \). Similarly, there is positive \( C \) independent of \( y \) such that

\[
|\nabla^2 P^Z_{s,t}f_t(y)| \leq \frac{C_2e^{C_2(t-s)}}{t-s} \sqrt{\psi(t-s)}(1+|y|), \quad y \in \mathbb{R}^d.
\] (3.54)  

Hence, (3.50) follows from the definition of \( \psi \) and any large enough \( \lambda \).

Next, we shall prove that \( \phi_t \) satisfies (3.3). Let \( h_t = \nabla_y \phi_t + f_t \). Due to Remark 3.1, we can assume that \( h_t \) is \( \mathbb{R} \)-valued.

We first consider the left derivative of \( \phi \). For \( v > 0 \),

\[
\frac{\phi_{s-v} - \phi_s}{v} = \frac{1}{v} \int_s^s e^{-\lambda(t-s+v)} P^Z_{s-v,t} f_t dt + \int_s^\infty e^{-\lambda(s+v)} \left( P^Z_{s-v,t} f_t - \frac{P^Z_{s,t} f_t}{v} \right) dt
\]

\[=: I_1 + I_2 + I_3.\]

For \( I_1 \), by the time homogeneous property, the boundedness and continuity of \( h \) and the pathwise continuity of \( X \), we have that

\[
\lim_{v \to 0^+} \frac{1}{v} \int_s^s e^{-\lambda(t-s+v)} P^Z_{s-v,t} h_t dt = \lim_{v \to 0^+} \frac{1}{v} \int_s^s e^{-\lambda(t-s+v)} P^Z_{t-s-v} h_t dt
\]

\[= \lim_{v \to 0^+} \frac{1}{v} \int_s^s e^{-\lambda(t-s+v)} \mathbb{E} h_t(X_{t-s+v}) dt
\]

\[= h_s(x).\]

For \( I_2 \), by the boundedness and continuity of \( h \), it follows from the dominated convergence theorem that

\[
\lim_{v \to 0^+} \int_s^\infty \frac{e^{-\lambda u} - 1}{v} e^{-\lambda(t-s)} P^Z_{s-v,t} h_t dt
\]

\[= \lim_{v \to 0^+} \int_s^\infty \frac{e^{-\lambda u} - 1}{v} e^{-\lambda(t-s)} \mathbb{E} h_t(X_{t-s+v}) dt
\]

\[= -\lambda \int_s^\infty e^{-\lambda(t-s)} \mathbb{E} h_t(X_{t-s}) dt
\]

\[= -\lambda \int_s^\infty e^{-\lambda(t-s)} P^Z_{s,t} h_t dt.
\]

For \( I_3 \), since \( P^Z_{t-s} h_t \in C^2(\mathbb{R}^d) \), it follows from the semigroup property and Itô’s formula that

\[
\frac{(P^Z_{s-v,t} h_t - P^Z_{s,t} h_t)}{v} = \frac{P^Z_{v} - I}{v} P^Z_{t-s} h_t = \int_0^v \mathbb{E} LP^Z_{t-s} h_t(Y_r) dr.
\] (3.55)  

By (3.50), and the growth condition of \( \sigma \) and \( Z \), for \( 1 < \delta < 2 \), we have

\[
\int_s^\infty e^{-\lambda(t-s)} \mathbb{E} \left( \frac{1}{v} \int_0^v |(LP^Z_{t-s} h_t)(Y_r)| dr \right) \delta dt
\]
respectively. For the last term, the dominated convergence theorem can be applied and we get that

\[
\int_s^\infty \exp(-\lambda(t-s)) \left( \frac{1}{v} \int_0^v \mathbb{E} \left| (LP_{t-s} h_t) (Y_r) \right|^\delta \right) \, dt \\
\leq \frac{C}{v} \int_0^v \mathbb{E} \left( 1 + |Y_r| \right) \delta \left( \int_s^\infty \exp(-\lambda(t-s)) |\nabla^2 P_{t-s} h_t(Y_r)|^\delta \right) \, dr \\
+ \frac{C}{v} \int_0^v \mathbb{E} \left( 1 + |Y_r| \right) \delta \left( \int_s^\infty \exp(-\lambda(t-s)) |\nabla P_{t-s} h_t(Y_r)|^\delta \right) \, dr \\
\leq \frac{C \lambda \delta}{v} \int_0^v \mathbb{E} \left( 1 + |Y_r| \right) \delta(\beta+1)^2 \, dr < \infty.
\]

Besides, it holds that

\[
\lim_{v \to 0^+} \frac{1}{v} \int_0^v (LP_{t-s} h_t) (Y_r) \, dr = LP_{t-s} h_t(Y_0).
\]

Then, the dominated convergence theorem can be applied and we get that

\[
\lim_{v \to 0^+} I_3 = \lim_{v \to 0^+} \int_s^\infty e^{-\lambda(t-s)} \left( \frac{P_{s+tv} h_t - P_{s,t} h_t}{v} \right) \, dt \\
= \int_s^\infty e^{-\lambda(t-s)} LP_{s,t} h_t \, dt \\
= L \int_s^\infty e^{-\lambda(t-s)} P_{s,t} h_t \, dt,
\]

where we used (3.50) and the dominated convergence theorem in the last equality. Hence,

\[
\partial_s^{-} \phi_s = -L \phi_s - h_s + \lambda \phi_s.
\]

On the other hand,

\[
\frac{\phi_{s+v} - \phi_s}{v} = \frac{1}{v} \int_s^{s+v} e^{-\lambda(t-s)} P_{s+tv} h_t \, dt + \int_s^\infty \frac{e^{-\lambda(t-s)} - 1}{v} e^{-\lambda(t-s)} \frac{P_{s+tv} h_t - P_{s,t} h_t}{v} \, dt \\
+ \int_s^\infty e^{-\lambda(t-s)} \left( \frac{P_{s+tv} h_t - P_{s,t} h_t}{v} \right) \, dt, \quad v > 0.
\]

The convergences of the first term and the second term are similar to $I_1$ and $I_2$ respectively. For the last term,

\[
\int_s^\infty e^{-\lambda(t-s)} \frac{(P_{s+tv} h_t - I)}{v} P_{s+tv} h_t \, dt \\
= \int_s^\infty e^{-\lambda(t-s)} \frac{(P_{s+tv} h_t - P_v h_t)}{v} P_{s+tv} h_t \, dt \\
= e^{-\lambda v} \int_s^\infty e^{-\lambda(t-s)} \frac{(P_{s+tv} h_t - I)}{v} P_{t-s} h_t \, dt \\
= e^{-\lambda v} \int_s^\infty e^{-\lambda(t-s)} \mathbb{E} \left( \frac{1}{v} \int_0^v LP_{t-s} h_t(Y_r) \, dr \right) \, dt.
\]

By (3.53), (3.54) and (3.55), $LP_{t-s} h_t(Y_r)$ is continuous in $y$ uniformly for $v$, i.e.

\[
\lim_{\tilde{y} \to y} \sup_v |LP_{t-s} h_t(Y_r) - LP_{t-s} h_t(Y_r)| = 0.
\]
Then
\[
\lim_{v \to 0^+} \frac{1}{v} \int_0^v L P_{t-s} h_{t+v}(Y_r) dr = L P_{t-s} h_t(Y_0).
\]
Combining this with (3.50), as in the case of the left derivative, the dominated convergence theorem yields
\[
\lim_{v \to 0^+} \int_s^{s+v} e^{-\lambda(t-s)} \left( \frac{P_Z^v I}{v} P_{s+v} h_t dt \right)
= \int_s^{s+v} e^{-\lambda(t-s)} \left( \lim_{v \to 0^+} \frac{1}{v} \int_0^v L P_{t-s} h_{t+v}(Y_r) dr \right) dt
= \int_s^{s+v} e^{-\lambda(t-s)} L P_{t-s} h_t dt
= \int_s^{s+v} e^{-\lambda(t-s)} P_Z^{s+v} h_t dt.
\]
Therefore, replacing \( h_t \) by \( \nabla g_t \phi_t + f_t \), we prove that \( \phi \) satisfies (3.3).

\[ \square \]

3.3 Proof of Theorem 3.1

To prove this theorem, we begin with a lemma on Krylov-Khasminskii type estimate for the solution of (3.1) based on the ultracontractivity of \( P_t^Z \).

\textbf{Lemma 3.9.} Assume (H1) and (H2). Let \( Y_t \) be the solution of (3.32). If there exist \( \zeta > 0 \) and \( p \geq 1 \) and \( p > \gamma \) such that
\[
\int_0^t \left( \log \mu \left( e^{\zeta |f_s|^p} \right) \right)^{\frac{1}{p}} ds < \infty, \quad t > 0
\]
then there is a \( C_{\zeta,p,\gamma} > 0 \) such that for all \( t > 0 \) and \( 0 \leq t_0 \leq t_1 \leq t \)
\[
\mathbb{E} \left\{ \int_{t_0}^{t_1} f_s(Y_s) ds \mid \mathcal{F}_{t_0} \right\}
\leq C_{\zeta,p,\gamma} \left( (t_1 - t_0)^{1-\frac{1}{p}} + (t_1 - t_0) + \int_{t_0}^{t_1} \left( \log \mu \left( e^{\zeta |f_s|^p} \right) \right)^{\frac{1}{p}} ds \right),
\]
consequently,
\[
\mathbb{E} \exp \left\{ r \int_0^t |f_s(Y_s)| ds \right\} < \infty, \quad t > 0, r > 0.
\]

\textbf{Proof.} It follows from (H2) that
\[
\mathbb{E} \left\{ \int_{t_0}^{t_1} f_s(Y_s) ds \mid \mathcal{F}_{t_0} \right\} = \int_{t_0}^{t_1} \mathbb{E} \left\{ f_s(Y_s) \mid \mathcal{F}_{t_0} \right\} ds \leq \int_{t_0}^{t_1} P_{t_0,s} f_s(Y_{t_0}) ds
\leq (2\zeta^{-1})^\frac{1}{p} \int_{t_0}^{t_1} \left( \log P_{t_0,s} e^{\zeta |f_s|^p(Y_{t_0})} \right)^{\frac{1}{p}} ds
\leq (2\zeta^{-1})^\frac{1}{p} \int_{t_0}^{t_1} \left( \log \left( \|P_{t_0,s}^Z\|_{L^2(\mu) \to L^\infty(\mu)} \left( \mu \left( e^{\zeta |f_s|^p} \right) \right)^{\frac{1}{p}} \right) \right)^{\frac{1}{p}} ds
\]
Exponential contraction in Wasserstein distance

\[ \leq C_{\zeta,p} \int_{t_0}^{t_1} \left( (s - t_0)^{-\frac{\gamma}{p}} + \left( \log \mu \left( e^{\zeta |b_s|^p} \right) \right)^{\frac{1}{p}} \right) \, ds, \]

which implies our first claim. For the second claim, it follows from the standard proof of Khasminskii type estimate, see [20, Lemma 1.2.1]

Applying Lemma 3.9 to \(|b_t|^2|\), we have the following corollary, which can be used to get the existence of weak solutions to (3.1).

**Corollary 3.10.** Assume (H1) and (H2). Let \(Y_t\) be the solution of (3.32). If there exist \(\zeta > 0\) and \(p \geq 2\) and \(p > 2\gamma\) such that

\[ \int_{t_0}^{t} \left( \log \mu \left( e^{\zeta |b_s|^p} \right) \right)^{\frac{1}{p}} \, ds < \infty, \quad t > 0, \]

then

\[ E \exp \left\{ r \int_{t_0}^{t} |b_s(Y_s)|^2 \, ds \right\} < \infty, \quad t > 0, \quad r > 0. \]

**Proof of Theorem 3.1:**

(1) We prove the first claim. The proof is due to [18, Corollary 2.5]. Applying Itô’s formula to the solution of (3.32), it follows from (A1) and Lemma 3.5 that there exist positive constants \(K\) and \(c\) such that

\[ d|Y_s|^2 \leq \left( -K|Y_s|^2 + c|Y_s| \right) \, ds + 2(Y_s, \sigma_s(Y_s)dW_s), \quad s \geq 0, \]

which implies that

\[ \frac{1}{t} \int_{0}^{t} P_s^Z |\cdot|^2 \, ds = \frac{1}{t} \int_{0}^{t} \mathbb{E}|Y_s|^2 \, ds \leq C, \quad t \geq 0, \]

holds for some \(C > 0\). Since \(|\cdot|^2\) is a compact function on \(\mathbb{R}^d\), the Krylov-Bogolyubov method yields that \(P_t^Z\) has an invariant measure. Then \(P_t^Z\) can be extended to \(L^1(\mu)\) to be a \(C_0\)-semigroup. By the interpolation, \(P_t^Z\) can be extended to all \(L^q(\mu)\) with \(q \in [1, \infty)\) to be a \(C_0\)-semigroup.

By [19, Lemma 3.3] and [1] Theorem 1,

\[ \left| \sqrt{\sigma(x)\sigma^*(x)} - \sigma_0 - \sqrt{\sigma(y)\sigma^*(y)} - \sigma_0^2 \right|_{HS} \leq \frac{1}{2\sigma_0} \left| \sqrt{\sigma(x)\sigma^*(x)} - \sqrt{\sigma(y)\sigma^*(y)} \right|_{HS} \]

\[ \leq \frac{1}{\sqrt{2\sigma_0}} |\sigma^*(x) - \sigma^*(y)|_{HS} \]

\[ \leq C|x - y|. \]

Taking into account (3.28) with \(\Phi(x) \equiv x\), it is easy to see that exponential contraction in Wasserstein distances holds for \(P_t^Z\), which implies the uniqueness of the invariant probability measure.

By Itô’s formula and that \(\sigma\) is bounded, there exist positive \(K_\beta\), \(c_{\beta,\sigma}\) and \(\bar{c}_{\beta,\sigma}\) such that

\[ \exp \left\{ \delta |Y_t|^{\beta+2} \right\} \leq \left( -\delta K_{\beta} |Y_t|^{2\beta+2} + \delta^2 c_{\beta,\sigma} |Y_t|^{2\beta+2} + \delta \bar{c}_{\beta,\sigma} |Y_t|^{\beta} \right) e^{\delta |Y_t|^{\beta+2}} \, dt \]

For $0 < \delta < \frac{K_\beta}{c_{\beta,\sigma}}$, we have $-\delta K_\beta + \delta^2 c_{\beta,\sigma} < 0$. Then for any $\delta \in (0, \frac{K_\beta}{c_{\beta,\sigma}})$, there exist positive constants $C_1$ and $C_2$ such that for all $t > 0$

$$d e^{\delta |Y_t|^{\beta+2}} \leq \left( -C_1 |Y_t|^{2\beta+2} + C_2 \right) e^{\delta |Y_t|^{\beta+2}} dt + \delta (\beta + 2) |Y_t|^{\beta+2} e^{\delta |Y_t|^{\beta+2}} (Y_t, \sigma(Y_t) dW_t).$$

Let $\tilde{r}_n = \{ t \geq 0 \mid |Y_t| \geq n \}$. Then

$$E e^{\delta |Y_{t\wedge \tilde{r}_n}|^{\beta+2}} - E e^{\delta |Y_{t\wedge \tilde{r}_n}|^{\beta+2}} 
\leq \int_0^t \left( C_2 - C_1 \delta \frac{2\beta+2}{\beta+2} \left( \log E e^{\delta |Y_{t\wedge \tilde{r}_n}|^{\beta+2}} \right)^{\frac{\beta+2}{\beta+2}} \right) \| Y_{t\wedge \tilde{r}_n} \|^{\beta+2} dr.$$ 

Consider the following equation:

$$\frac{du_t}{dt} = \left( C_2 - C_1 \delta \frac{2\beta+2}{\beta+2} (\log u_t) \frac{2\beta+2}{\beta+2} \right) u_t, \quad u_0 = e^{\delta |X_0|^{\beta+2}}. \quad (3.56)$$

Let $c_0 = \exp \left\{ \delta \left( C_2 C_1^{-1} \right) \frac{2\beta+2}{\beta+2} \right\}$. Then $c_0$ is the unique zero of the function

$$u \rightarrow C_2 - C_1 \delta \frac{2\beta+2}{\beta+2} (\log u) \frac{2\beta+2}{\beta+2}.$$ 

If there exists $t_0 > 0$ such that $u_{t_0} = c_0$, then $u_t \equiv c_0$. Otherwise, there exists $t_1 > t_0$ such that $u_{t_1} > c_0$ (for $u_{t_0} < c_0$, the discussion is similar). Let $s_0 = \sup \{ t < t_1 \mid u_t = c_0 \}$. Then we have $t_0 \leq s_0$, $u_{s_0} = c_0$ and $u_t > c_0$ for $t \in (s_0, t_1)$. By (3.56), $u_t < 0$ for $t \in (s_0, t_1)$. However, it follows from the mean value theorem that there exists $s_1 \in (s_0, t_1)$ such that

$$u'_{s_1} = \frac{u_{t_1} - u_{s_0}}{t_1 - s_0} > 0,$$

which leads to a contradiction. Hence, if $u_t < c_0$, then $u_s < c_0$ for $s \in [0, t]$; if $u_t > c_0$, then $u_s > c_0$ for $s \in [0, t]$. Moreover, if $u_0 > c_0$, then it follows from (3.56) that

$$u_t \leq \exp \left\{ \left( \frac{\beta + 2}{\beta C_2} \right)^{\frac{2\beta+2}{\beta+2}} t^{-\frac{2\beta+2}{\beta+2}} \right\}.$$ 

Hence

$$u_t \leq \exp \left\{ \frac{\beta + 2}{\beta C_2} t^{-\frac{2\beta+2}{\beta+2}} \right\} \forall c_0 \leq \exp \left\{ C_{\beta,\sigma,\delta} \left( 1 + t^{-\frac{2\beta+2}{\beta+2}} \right) \right\},$$

for some positive constant $C_{\beta,\sigma,\delta}$ independent of $X_0$. As a consequence of the comparison theorem of ordinary differential equation, we have

$$E e^{\delta |Y_{t\wedge \tilde{r}_n}|^{\beta+2}} \leq u_t \leq \exp \left\{ C_{\beta,\sigma,\delta} \left( 1 + t^{-\frac{2\beta+2}{\beta+2}} \right) \right\}, \quad n \in \mathbb{N}, t > 0.$$ 

Letting $n \rightarrow \infty$, we get that

$$P_t^Z e^{\delta |Y_t|^{\beta+2}} = E e^{\delta |Y_t|^{\beta+2}} \leq \exp \left\{ C_{\beta,\sigma,\delta} \left( 1 + t^{-\frac{2\beta+2}{\beta+2}} \right) \right\}, \quad t > 0.$$
Then
\[ \mu \left( e^{|\cdot|^{\beta+2}} \right) = \mu \left( P_t^Z e^{|\cdot|^{\beta+2}} \right) = \mu \left( E e^{|Y_t|^{\beta+2}} \right) \leq \exp \left\{ C_{\beta, \sigma, \delta} \left( 1 + t^{-\frac{\beta+2}{\sigma}} \right) \right\} < \infty, \]
and for any \( r > 0 \) and \( t > 0 \), we have that
\[ P_t^Z e^{r|\cdot|^2} \leq P_t^Z e^{C_{\beta, \sigma, \delta} + r|\cdot|^2} \leq \exp \left\{ C_{\beta, \sigma, \delta} + C_{\beta, \sigma, \delta} \left( 1 + t^{-\frac{\beta+2}{\sigma}} \right) \right\} < \infty. \]

Therefore, it follows from [24, Corollary 1.3] and [18, Corollary 2.5] that (3.7) holds for \( P_t^Z \).

(2) It follows from Corollary 3.10 and (A2) that
\[ \mathbb{E} \exp \left\{ r \int_0^t (\tilde{\sigma}^{-1} b_s)^2(Y_s) ds \right\} \leq \mathbb{E} \exp \left\{ \frac{r}{\sigma^2} \int_0^t |b_s|^2(Y_s) ds \right\} < \infty, \quad r > 0. \]
Then \( \{ R_t \}_{t \geq 0} \) defined as follows is a martingale
\[ R_t = \exp \left\{ - \int_0^t \langle \tilde{\sigma}^{-1} b_s(Y_s), dW_s \rangle - \frac{1}{2} \int_0^t |\tilde{\sigma}^{-1} b_s(Y_s)|^2 ds \right\}, \quad t \geq 0, \]
and \( \mathbb{E} R_t^q < \infty \) for any \( q > 0 \). Thus (3.11) has a weak solution. By the Girsanov theorem and the uniqueness in law for the solutions to (3.32), which follows from the strong uniqueness, we have that (3.11) has a unique weak solution and the law of \( \{ X_s \}_{s \in [0,t]} \) under \( P \) equals to the law of \( \{ Y_s \}_{s \in [0,t]} \) under \( R_t P \).

Next, we shall prove that there exists a family of diffeomorphisms \( \{ \Phi_t \}_{t \geq 0} \) satisfying (3.2).

By (A1) and (A2), Lemma 3.7 holds. Then Lemma 3.7 (H1) and (3.8) imply that the assumption of Lemma 3.2 holds for the equation (3.4). Consequently, we can choose large enough \( \lambda \) such that
\[ \sup_{t \geq 0} (\| \phi_t \|_\infty + \| \nabla \phi_t \|_\infty) < \frac{1}{2} \frac{K_2}{(K_1 - K_2)^+}, \quad \text{(3.57) add-p-nnp} \]
which implies that \( \Phi_t(x) \) is a diffeomorphism and
\[ \frac{1}{2} |x - y| \leq |\Phi_t(x) - \Phi_t(y)| \leq \frac{3}{2} |x - y|, \quad t \geq 0. \quad \text{(3.58) diff-mor} \]

Applying Lemma 3.4 then (3.4) has a unique solution in \( C_b([0, \infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)) \). For given \( t > 0 \), let \( \{ b^{\lfloor m_k, n(m_k) \rfloor} \}_{k \geq 1} \) be a sequence in \( C_b([0, \infty), C^1_b(\mathbb{R}^d, \mathbb{R}^d)) \) defined as in Lemma 3.3 and let \( \phi^{[k]} \) be the solution of (3.4) with \( \tilde{b} \) replaced by \( b^{\lfloor m_k, n(m_k) \rfloor} \). Then (3.17) holds on \([0, t]\). For the sake of simplicity, we shall denote by \( \{ b^{[k]} \}_{k \geq 1} \) the sequence \( \{ b^{\lfloor m_k, n(m_k) \rfloor} \}_{k \geq 1} \). It follows from Lemma 3.8 that \( \phi^{[k]} \) satisfies the parabolic equation (3.8). Then by applying Itô’s formula to \( \Phi^{[k]}_t(X_r) \equiv X_r + \phi^{[k]}_t(X_r), \ r \in [0, t], \) we have
\[ \Phi^{[k]}_t(X_r) = \int_0^r Z(X_s) ds + \int_0^r \left( b_s(X_s) - b^{[k]}_s(X_s) \right) ds + \int_0^r \nabla b_s - b^{[k]}_s \phi^{[k]}_s(X_s) ds \]
\[ + \int_0^r \lambda \phi^{[k]}_s(X_s) ds + \int_0^r \left( I + \nabla \phi^{[k]}_s(X_s) \right) \sigma(X_s) dW_s, \]
where \((\nabla \phi_s^{[k]})(\cdot)\sigma(\cdot)\) is a matrix with
\[
\left(\nabla \phi_s^{[k]}(\cdot)\sigma(\cdot)\right)_{ij} = \sum_{i=1}^{d} \partial_i(\phi_s^{[k]})t(\cdot)\sigma_{ij}(\cdot).
\]

Next, we shall let \(k \to \infty\) to get (3.24). Firstly,
\[
\mathbb{E} \int_0^t |b_s(X_s) - b_s^{[k]}(X_s)|ds = \mathbb{E} R_t \int_0^t |b_s(Y_s) - b_s^{[k]}(Y_s)|ds \\
\leq \sqrt{\mathbb{E} R_t^2} \int_0^t \left( P_s^Z |b_s - b_s^{[k]}|^2 \right)^{1/2} ds.
\]
By the ultracontractivity of \(P_s^Z\),
\[
\int_0^t P_s^Z |b_s - b_s^{[k]}|^2 ds \leq C_\zeta \int_0^t \left( s^{-2/p} + (s + m_k^{-1})^{-2/p} + 1 \right) ds \\
+ t \sup_{s \geq 0} \left( \log \left( \mu(e^{\zeta |b_s|^p}) \right) \right)^{2/p} < \infty
\]
and
\[
\lim_{k \to \infty} P_s^Z |b_s - b_s^{[k]}|^2 \\
\leq \lim_{k \to \infty} C_n \mu \left( n(m_k) \int_s^{s+1/m_k} |b_s - P_s^Z b_r|^2 dr \right) \\
\leq \lim_{k \to \infty} C_n n(m_k) \int_s^{s+1/m_k} \left( \mu \left( |b_s - P_s^Z b_r|^2 \right) + \mu (|b_r - b_s|^2) \right) dr \\
= 0, \quad \text{a.e.-ds.}
\]

Then the dominated convergence theorem yields
\[
\lim_{k \to \infty} \mathbb{E} \int_0^T |b_t(X_t) - b_t^{[k]}(X_t)|dt \leq \sqrt{\mathbb{E} R_t^2} \lim_{k \to \infty} \int_0^t \left( P_s^Z |b_s - b_s^{[k]}|^2 \right)^{1/2} ds = 0. \quad (3.59)
\]
Secondly, it follows from Lemma 3.2 and Lemma 3.3 that
\[
\sup_{s \geq 0,k} \left( \left\| \phi_s^{[k]} \right\|_{\infty} + \left\| \nabla \phi_s^{[k]} \right\|_{\infty} \right) < \infty
\]
and
\[
\lim_{k \to \infty} \sup_{s \in [0,t]} \|\phi_s^{[k]} - \phi_s\|_{C^1_b(\mathbb{R}^d)} = 0.
\]

Then
\[
\lim_{k \to \infty} \mathbb{E} \sup_{s \in [0,t]} |\Phi_s^{[k]}(X_s) - \Phi(X_s)| = \mathbb{E} \sup_{s \in [0,t]} \|\phi_s^{[k]} - \phi\|_{\infty} = 0,
\]
\[
\lim_{k \to \infty} \int_0^t \mathbb{E} |\phi_s^{[k]} - \phi_s|(X_s)ds \leq \lim_{k \to \infty} \sup_{s \in [0,t]} \|\phi_s^{[k]} - \phi_s\|_{\infty} = 0,
\]
and
\[
\lim_{k \to \infty} \int_0^T \mathbb{E} \left| \left( I + \nabla \phi_{\delta_t}^k \right) \sigma(X_t) - \left( I + \nabla \phi_t(X_t) \right) \sigma(X_t) \right|^2 dt \\
\leq \lim_{k \to \infty} \sup_{s \in [0,t]} \left\| \nabla \phi_{\delta_t}^k(s) - \nabla \phi_s(s) \right\|_\infty^2 \int_0^t \mathbb{E} |\sigma(X_s)|^2 ds \\
\leq C \sqrt{\mathbb{E} R_i^2} \lim_{k \to \infty} \sup_{s \in [0,t]} \left\| \nabla \phi_{\delta_t}^k(s) - \nabla \phi_s(s) \right\|_\infty^2 \int_0^t \left( \mathbb{E} (1 + |Y_s|^4) \right)^{\frac{1}{2}} ds \\
= 0.
\]

Finally, it follows from (3.59) that
\[
\lim_{k \to \infty} \int_0^T \mathbb{E} \left| \nabla b_{\delta_t} - b_t \phi_{\delta_t}^k \right| (X_t) \right| dt \leq \sup_{s \in [0,t]} \left\| \nabla \phi_{\delta_t}^k \right\|_\infty \lim_{k \to \infty} \int_0^T \mathbb{E} \left| (b_{\delta_t} - b_t \phi_{\delta_t}^k) \right| (X_s) ds \\
= 0.
\]

Letting $k \to +\infty$, it follows from the pathwise continuity of $X_t$ that we prove (3.2).

(3) Let $\tilde{X}_t = \Phi_t(X_t)$. Then $\tilde{X}_t$ satisfies
\[
\tilde{X}_t = \tilde{X}_0 + \int_0^t Z \circ \Phi_s^{-1}(\tilde{X}_s) ds + \lambda \int_0^t \phi_s \circ \Phi_s^{-1}(\tilde{X}_s) ds \\
+ \int_0^t (\nabla \Phi_s \sigma) \circ \Phi_s^{-1}(\tilde{X}_s) dW_s, \quad t \geq 0.
\] (3.60)

It follows from (3.57) that
\[
|\Phi_t(x)| \leq 1 + |x|, \quad K_1 \nabla \phi_t \|_\infty < K_2,
\]
which implies by Lemma 3.30 that there exists positive constants $K$ and $C$ such that
\[
\langle Z \circ \Phi_s^{-1}(x) - Z \circ \Phi_s^{-1}(y), x - y \rangle \leq -K|x - y|^\beta + C|x - y|^2.
\]

By (3.52) and the boundedness of $\sigma$, there exist $C > 0$ such that
\[
\left\| (\nabla \Phi_s \sigma) \circ \Phi_s^{-1}(x) - (\nabla \Phi_s \sigma) \circ \Phi_s^{-1}(y) \right\|^2_{HS} \leq C|x - y|^2 \log^2 \left( e + \frac{1}{|x - y|} \right).
\]

Then it follows from [3] Theorem B] that the pathwise uniqueness holds for solutions to (3.60). Hence (3.1) has a unique strong solution.

Finally, we shall prove the $W_{p_r}$-convergence. Set $\tilde{\sigma}_s(x) = (\nabla \Phi_s \sigma) \circ \Phi_s^{-1}(x)$. Then it follows from (3.57) that
\[
\tilde{\sigma}_s \tilde{\sigma}_s^* \geq (I + \nabla \phi_s) \sigma_s \sigma_s^* (I + (\nabla \phi_s)^*) \geq \sigma^2_0 \frac{2}{4},
\]
which implies by [19] Lemma 3.3] and [11] Theorem 1] that
\[
\left\| \sqrt{\tilde{\sigma}_s \tilde{\sigma}_s^*} (x) - \frac{\sigma_0^2}{4} - \sqrt{\tilde{\sigma}_s \tilde{\sigma}_s^*} (y) - \frac{\sigma_0^2}{4} \right\|_{HS} \leq \frac{1}{\sigma_0} \left\| \sqrt{\tilde{\sigma}_s \tilde{\sigma}_s^*} (x) - \sqrt{\tilde{\sigma}_s \tilde{\sigma}_s^*} (y) \right\|_{HS}
\]
\[ \frac{\sqrt{2}}{\sigma_0} |\tilde{\sigma}_s^*(x) - \tilde{\sigma}_s^*(y)|_{HS} \leq C|x - y| \log \left( e + \frac{1}{|x - y|} \right). \]

Let \( \bar{\sigma}_s = \sqrt{\tilde{\sigma}_s^2 - \frac{\sigma_0^2}{4}} \). Then

\[
\begin{align*}
\langle Z \circ \Phi_s^{-1}(x) - Z \circ \Phi_s^{-1}(y), x - y \rangle + 
\frac{1}{2} |\tilde{\sigma}_s(x) - \tilde{\sigma}_s(y)|^2_{HS} + 
\frac{(2q - 3)}{2|x - y|^2} |(\tilde{\sigma}_s^*(x) - \tilde{\sigma}_s^*(y))(x - y)|^2 
\end{align*}
\]

\[ \leq -K|x - y|^{3+2} + C_\lambda|x - y|^2 + C_\lambda(2q - 3)^+|x - y|^2 \log^2 \left( e + \frac{1}{|x - y|} \right), \]

which implies that (2.4) holds with \( \bar{K}_2 = K \) and

\[ \bar{K}_1(v) = C_\lambda v^2 + C_\lambda(2q - 3)^+v^2 \log^2 \left( e + \frac{1}{v} \right). \]

It is easy to check that (2.3) and (2.6) hold. Hence (2.7) holds for the transition semigroup \( \bar{P}_t \) associated with \( \bar{X}_t \). Since (1.58),

\[ W_q(\delta_x P_t, \delta_y P_t) \leq 2W_q \left( \delta_{\Phi_0(x)} \bar{P}_t, \delta_{\Phi_0(y)} \bar{P}_t \right). \]

Therefore, the conclusion of this theorem follows.

**References**

[1] H. Araki and S. Yamagami, An Inequality for Hilbert-Schmidt Norm. Comm. Phys. Math. 81 (1981), 89–96.

[2] F. Bolley, I. Gentil, and A. Guillin, Convergence to equilibrium in Wasserstein distance for Fokker–Planck equations, J. Funct. Anal. 263 (2012), 2430–2457.

[3] M. F. Chen, From Markov Chains to Non-Equilibrium Particle Systems, Second edition. World Scientific Publishing Co., Inc., River Edge, NJ, (2004).

[4] M.-F. Chen and S.-F. Li, Coupling methods for multidimensional diffusion processes, Ann. Probab. 17 (1989), 151–177.

[5] M. F. Chen and F. Y. Wang, Estimation of spectral gap for elliptic operators, Trans. Amer. Math. Soc. 249 (1997), 1239–1267.

[6] L.-J. Cheng and S.-Q. Zhang, Weak Poincar Inequality for Convolution Probability Measures, arXiv:1407.4910v2.

[7] A. Eberle, Reflection couplings and contraction rates for diffusions, Probab. Theory Related Fields, (2016) 166, 851–886.

[8] S. Fang and T. Zhang. A study of a class of stochastic differential equations with non-Lipschitzian coefficients. Probab. Theory Relat. Fields 132 (2005), 356–390.
[9] D. Luo and J. Wang, Exponential convergence in $L^p$-Wasserstein distance for diffusion processes without uniformly dissipative drift. *Math. Nachr.* **289** (2016), 1909–1926.

[10] W. Liu and M. Röckner. Stochastic Partial Differential Equations: An Introduction. Springer: Switzerland. 2015.

[11] E. Hille and R. S. Phillips. Functional Analysis and Semi-Groups. American Mathematical Society, 1996.

[12] N. Ikeda and S. Watanabe. Stochastic differential equations and diffusion processes.

[13] R. Z. Khasminskii. On Positive Solutions of the Equation $\Delta u + Vu = 0$. *Theory Probab. Appl.*, **4(3)**, 309–318.

[14] K. Kuwada, Duality on gradient estimates and Wasserstein controls. *J. Funct. Anal.* **258** (2010), 3758–3774.

[15] T. Lindvall and L. Rogers, Coupling of multidimensional diffusions by reflection, *Ann. Probab.* **14** (1986), 860–872.

[16] F. Otto, The geometry of dissipative evolution equations: the porous medium equation, *Comm. Part. Diff. Equat.* **26** (2001), 101–174.

[17] M.-K. von Renesse, K.-T. Sturm, Transport inequalities, gradient estimates, entropy, and Ricci curvature, *Comm. Pure Appl. Math.* **58** (2005), 923–940.

[18] M. Röckner and F.-Y. Wang, Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds. *Forum Math.* **15** (2003), 893–921.

[19] E. Priola, F.-Y. Wang, Gradient estimates for diffusion semigroups with singular coefficients, *J. Funct. Anal.* **236** (2006) 244–264.

[20] A.-S. Sznitman, Brownian motion, obstacles, and random media. Springer-Verlag Berlin Heidelberg. (1998)

[21] C. Villani, Topics in Optimal Transportation, Amer. Math. Soc. 2003.

[22] C. Villani, Optimal Transpot, Old and New, Springer, 2009.

[23] F.-Y. Wang, Functional inequalities, Markov semigroups and spectral theory. Science Press. (2004)

[24] F.-Y. Wang, Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds, *Ann. Probab.* **39** (2011), 1449–1467.

[25] F.-Y. Wang, Analysis for Diffusion Processes on Riemannian Manifolds, Springer, 2014

[26] F.-Y. Wang, Asymptotic couplings by reflection and applications for nonlinear monotone SPDES, *Nonlinear Analysis* **117** (2015), 169–188.
[27] F.-Y. Wang, Gradient estimates and applications for SDEs in Hilbert space with multiplicative noise and Dini continuous drift. *J. Differential Equations* **260** (2016), 2792–2829.

[28] F.-Y. Wang, Exponential Contraction in Wasserstein Distances for Diffusion Semigroups with Negative Curvature, [arXiv:1603.05749](https://arxiv.org/abs/1603.05749) 2016.

[29] F.-Y. Wang, Estimates for invariant probability measures of degenerate SPDEs with singular and path-dependent drifts, *Probab. Theory Relat. Fields* (2018), https://doi.org/10.1007/s00440-017-0827-4

[30] L. Xie and X. Zhang, Ergodicity of stochastic differential equations with jumps and singular coefficients, [arXiv:1705.07402](https://arxiv.org/abs/1705.07402) 2017.