Preliminarily group classification of a class of 2D nonlinear heat equations

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Abstract
A preliminary group classification of the class 2D nonlinear heat equations $u_t = f(x, y, u, u_x, u_y)(u_{xx} + u_{yy})$, where $f$ is arbitrary smooth function of the variables $x, y, u, u_x$ and $u_y$ using Lie method, is given. The paper is one of the few applications of an algebraic approach to the problem of group classification: the method of preliminary group classification.

Key words: 2D Nonlinear heat equation, Optimal system, Preliminarily group classification.

1 Introduction
It is well known that the symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie. The first paper on this subject is [1], where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity). He computed the maximal invariance group of the one-dimensional heat conductivity equation and utilized this symmetry to construct its explicit solutions. Saying it the modern way, he performed symmetry reduction of the heat equation. Nowadays symmetry reduction is one of the most powerful tools for solving nonlinear partial differential equations (PDEs). Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [2] developed the method of partially invariant solutions. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study. In an attempt to study nonlinear effects Saied and Hussain [3] gave some new similarity solutions of the (1+1)-nonlinear heat equation. Later Clarkson and Mansfield [4] studied classical and nonclassical symmetries of the (1+1)-heat equation and gave new reductions for the linear heat equation and a catalogue of closed-form solutions for a special choice of the function $f(x, y, u, u_x, u_y)$ that appears in their model. In higher dimensions Servo [5] gave some conditional symmetries for a nonlinear heat equation while Goard et al. [6] studied the nonlinear heat equation in the degenerate case. Nonlinear heat equations in one or higher dimensions are also studied in literature by using both symmetry as well as other methods [7,8]. There are a number of papers to study (1+1)-nonlinear heat equations from the point of view of Lie symmetries method. The (2+1)-dimensional nonlinear heat equations

$$u_t = f(u)(u_{xx} + u_{yy}), \quad (1.1)$$

are investigated in [9] and in present paper we studied

$$u_t = f(x, y, u, u_x, u_y)(u_{xx} + u_{yy}), \quad (1.2)$$

Similarity techniques are applied in [10,11,12,13] for (2+1)-dimensional wave equations.

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2 Symmetry Methods

Let a partial differential equation contains \( p \) dependent variables and \( q \) independent variables. The one-parameter Lie group of transformations

\[
x_i \longmapsto x_i + \epsilon^i \xi(x, u) + O(\epsilon^2); \quad u_\alpha \longmapsto u_\alpha + \epsilon \varphi^\alpha(x, u) + O(\epsilon^2),
\]
where \( i = 1, \ldots, p \) and \( \alpha = 1, \ldots, q \). The action of the Lie group can be recovered from that of its associated infinitesimal generators. We consider general vector field

\[
X = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^q \varphi^\alpha(x, u) \frac{\partial}{\partial u_\alpha},
\]

on the space of independent and dependent variables. The symmetry generator associated with (2.4) given by

\[
\phi(\xi(x, y, t, u)) = 0 \text{ whenever } (2.10)
\]

that its coefficients are obtained with following formulas

\[
\varphi^x = D_x \varphi - u_x D_x \xi^1 - u_y D_x \xi^2 - u_t D_x \xi^3,
\]

\[
\varphi^t = D_t \varphi - u_x D_t \xi^1 - u_y D_t \xi^2 - u_t D_t \xi^3,
\]

\[
\varphi^{yt} = D_y \varphi^y - u_x D_y \xi^1 - u_y D_y \xi^2 - u_t D_y \xi^3,
\]

\[
\varphi^{yt} = D_y \varphi^t - u_x D_y \xi^1 - u_y D_y \xi^2 - u_t D_y \xi^3.
\]

where the operators \( D_x, D_y \) and \( D_t \) denote the total derivatives with respect to \( x, y \) and \( t \):

\[
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{xt} \frac{\partial}{\partial u_t} + \ldots
\]

\[
D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yt} \frac{\partial}{\partial u_t} + \ldots
\]

\[
D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + \ldots
\]

By theorem 6.5. in [14], \( X^{(2)}[u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy})]_{|0} = 0 \) whenever

\[
u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) = 0.
\]

Since

\[
X^{(2)}[u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy})] = \varphi^t - (f_x \xi^1 + f_y \xi^2 + f_u \varphi + f_u \varphi^x + f_{uu} \varphi^y)(u_{xx} + u_{yy}) - f(x, y, u, u_x, u_y)(\varphi^{xx} + \varphi^{yy}),
\]

therefore we obtain the following determining function:

\[
\varphi^t - (f_x \xi^1 + f_y \xi^2 + f_u \varphi + f_u \varphi^x + f_{uu} \varphi^y)(u_{xx} + u_{yy}) - f(x, y, u, u_x, u_y)(\varphi^{xx} + \varphi^{yy}) = 0.
\]

In the case of arbitrary \( f \) it follows

\[
\xi^1 = \xi^2 = \varphi = 0,
\]
\[ \xi^1 = \xi^2 = \varphi = 0, \quad \xi^3 = C. \]  

(2.12)

Therefore, for arbitrary \( f(x, y, u, u_x, u_y) \) Eq. (1.1) admits the one-dimensional Lie algebra \( \mathfrak{g}_1 \), with the basis

\[ X_1 = \frac{\partial}{\partial t}. \]  

(2.13)

\( \mathfrak{g}_1 \) is called the principle Lie algebra for Eq. (1.1). So, the remaining part of the group classification is to specify the coefficient \( f \) such that Eq. (1.1) admits an extension of the principal algebra \( \mathfrak{g}_1 \). Usually, the group classification is obtained by inspecting the determining equation. But in our case the complete solution of the determining equation (2.10) is a wasteful venture. Therefore, we don’t solve the determining equation but, instead we obtain a partial group classification of Eq. (1.1) via the so-called method of preliminary group classification. This method was suggested in \cite{10} and applied when an equivalence group is generated by a finite-dimensional Lie algebra \( \mathfrak{g}_e \). The essential part of the method is the classification of all nonsimilar subalgebras of \( \mathfrak{g}_e \). Actually, the application of the method is simple and effective when the classification is based on finite-dimensional equivalence algebra \( \mathfrak{g}_e \).

### 3 Equivalence transformations

An equivalence transformation is a nondegenerate change of the variables \( t, x, y, u \) taking any equation of the form (1.1) into an equation of the same form, generally speaking, with different \( f(x, y, u, u_x, u_y) \). The set of all equivalence transformations forms an equivalence group \( \mathcal{E} \). We shall find a continuous subgroup \( \mathcal{E}_C \) of it making use of the infinitesimal method.

We consider an operator of the group \( \mathcal{E}_C \) in the form

\[ Y = \xi^1(x, y, t, u)\frac{\partial}{\partial x} + \xi^2(x, y, t, u)\frac{\partial}{\partial y} + \xi^3(x, y, t, u)\frac{\partial}{\partial t} + \varphi(x, y, t, u)\frac{\partial}{\partial u} + \mu(x, y, t, u, u_x, u_y, f)\frac{\partial}{\partial f}, \]  

(3.14)

from the invariance conditions of Eq. (1.1) written as the system:

\[
\begin{align*}
  u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) &= 0, \\
  f_t &= f_{u_t} = 0,
\end{align*}
\]  

(3.15)

where \( u \) and \( f \) are considered as differential variables: \( u \) on the space \( (x, y, t) \) and \( f \) on the extended space \( (x, y, t, u, u_x, u_y) \).

The invariance conditions of the system (3.15) are

\[
\begin{align*}
  Y^{(2)}(u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy})) &= 0, \\
  Y^{(2)}(f_t) &= Y^{(2)}(f_{u_t}) = 0,
\end{align*}
\]  

(3.16)

where \( Y^{(2)} \) is the prolongation of the operator (3.14):

\[ Y^{(2)} = Y + \varphi^x \frac{\partial}{\partial u_x} + \varphi^y \frac{\partial}{\partial u_y} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xy} \frac{\partial}{\partial u_{xy}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{yy} \frac{\partial}{\partial u_{yy}} \]

\[
+ \varphi^{yt} \frac{\partial}{\partial u_{yt}} + \varphi^{ut} \frac{\partial}{\partial u_{ut}} + \mu^t \frac{\partial}{\partial f_t} + \mu^{uu} \frac{\partial}{\partial f_{u_t}}.
\]  

(3.17)

The coefficients \( \varphi^x, \varphi^y, \varphi^t, \varphi^{xx}, \varphi^{xy}, \varphi^{xt}, \varphi^{yy}, \varphi^{yt}, \varphi^{ut}, \mu^t, \mu^{uu} \) are given in (2.7) and the other coefficients of (3.17) are obtained by applying the prolongation procedure to differential variables \( f \) with independent variables \( (x, y, t, u, u_x, u_y, u_t) \). We have

\[
\begin{align*}
  \mu^t &= \tilde{D}_t(\mu) - f_x \tilde{D}_x(\xi^1) - f_y \tilde{D}_y(\xi^2) - f_u \tilde{D}_u(\varphi) - f_{u_x} \tilde{D}_{u_x}(\varphi) - f_{u_y} \tilde{D}_{u_y}(\varphi), \\
  \mu^{uu} &= \tilde{D}_{u_t}(\mu) - f_x \tilde{D}_{u_t}(\xi^1) - f_y \tilde{D}_{u_t}(\xi^2) - f_u \tilde{D}_{u_t}(\varphi) - f_{u_x} \tilde{D}_{u_t}(\varphi) - f_{u_y} \tilde{D}_{u_t}(\varphi),
\end{align*}
\]  

(3.18)  

(3.19)
where
\[
\tilde{D}_t = \frac{\partial}{\partial t}, \quad \tilde{D}_{u_t} = \frac{\partial}{\partial u_t}.
\] (3.20)

So, we have the following prolongation formulas:
\[
\begin{align*}
\mu_t & = \mu_t - f_x \xi_1^t - f_y \xi_2^t - f_u \varphi_t - f_{u_u} (\varphi^u)_t - f_{u_y} (\varphi^y)_t, \\
\mu_{u_t} & = \mu_{u_t} - f_{u_x} (\varphi^x)_{u_t} - f_{u_y} (\varphi^y)_{u_t},
\end{align*}
\] (3.21)

By the invariance conditions (3.16) give rise to
\[
\begin{align*}
\mu_t & = \mu_{u_t} = 0, \\
\xi_1^t & = \xi_2^t = \varphi_t = 0 \\
(\varphi^x)_t & = (\varphi^x)_{u_t} = (\varphi^y)_t = (\varphi^y)_{u_t} = 0
\end{align*}
\] (3.22)

Moreover with substituting (3.17) into (3.16) we obtain
\[
\varphi_t - f(x, y, u, u_x, u_y)(\varphi^{xx} + \varphi^{yy}) - \mu(u_{xx} + u_{yy}) = 0.
\] (3.24)

We are left with a polynomial equation involving the various derivatives of \(u(x, y, t)\) whose coefficients are certain derivatives of \(\xi_1, \xi_2, \xi_3\) and \(\varphi\). Since \(\xi_1, \xi_2, \xi_3, \varphi\) only depend on \(x, y, t, u\) we can equate the individual coefficients to zero, leading to the complete set of determining equations:
\[
\begin{align*}
\xi_1 & = \xi_1(x, y) \\
\xi_2 & = \xi_2(y) \\
\xi_3 & = \xi_3(t) \\
\varphi_{uu} & = 0 \\
2\varphi_{xu} & = \xi_1 + \xi_1^t \\
\varphi_{yy} & = \xi_2 + \xi_2^t \\
\varphi_u & = \xi_1^t = \xi_2^t \\
\mu & = (\xi_1^t - \xi_2^t)f \\
\varphi_{tt} & = f(\varphi^{xx} + \varphi^{yy})
\end{align*}
\] (3.25-3.34)

so, we find that
\[
\begin{align*}
\xi_1(x) & = c_1 x + c_2 y + c_3, \\
\xi_2(t) & = c_1 y + c_4, \\
\xi_3(t) & = a(t), \\
\varphi(x, y, u) & = c_1 u + \beta(x, y), \\
\mu & = (c_1 - a'(t))f,
\end{align*}
\] (3.34)

with constants \(c_1, c_2, c_3\) and \(c_4\), also we have \(\beta_{xx} = -\beta_{yy}\).

We summarize: The class of Eq. (1.2) has an infinite continuous group of equivalence transformations generated by the following infinitesimal operators:
\[
Y = (c_1 x + c_2 y + c_3) \frac{\partial}{\partial x} + (c_1 y + c_4) \frac{\partial}{\partial y} + a(t) \frac{\partial}{\partial t} + (c_1 u + \beta(x, y)) \frac{\partial}{\partial u} + (c_1 - a'(t)) f \frac{\partial}{\partial f}.
\] (3.35)
Table 1
Commutation relations satisfied by infinitesimal generators in (4.38)

|   | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
|---|---|---|---|---|---|---|
| $Y_1$ | 0 | 0 | $-Y_3$ | $-Y_4$ | 0 | $-Y_6$ |
| $Y_2$ | 0 | 0 | 0 | $-Y_3$ | 0 | 0 |
| $Y_3$ | $Y_3$ | 0 | 0 | 0 | 0 | 0 |
| $Y_4$ | $Y_4$ | $Y_4$ | 0 | 0 | 0 | 0 |
| $Y_5$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $Y_6$ | $Y_6$ | 0 | 0 | 0 | 0 | 0 |

Table 2
Adjoint relations satisfied by infinitesimal generators in (4.38)

|   | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
|---|---|---|---|---|---|---|
| $Y_1$ | $Y_1$ | $Y_2$ | $e^tY_3$ | $e^tY_4$ | $Y_5$ | $e^tY_6$ |
| $Y_2$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4 + sY_3$ | $Y_5$ | $Y_6$ |
| $Y_3$ | $Y_1 - sY_3$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
| $Y_4$ | $Y_1 - sY_4$ | $Y_2 - sY_3$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
| $Y_5$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
| $Y_6$ | $Y_1 - sY_6$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |

Therefore the symmetry algebra of the Burgers’ equation (1.2) is spanned by the vector fields

\[
Y_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \quad Y_2 = y \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial t}, \quad Y_4 = \frac{\partial}{\partial y}, \quad Y_5 = \frac{\partial}{\partial t} - f \frac{\partial}{\partial f}, \quad Y_6 = \frac{\partial}{\partial u}.
\] (3.36)

Moreover, in the group of equivalence transformations there are included also discrete transformations, i.e., reflections

\[
t \rightarrow -t, \quad x \rightarrow -x, \quad u \rightarrow -u, \quad f \rightarrow -f.
\] (3.37)

4 Preliminary group classification

One can observe in many applications of group analysis that most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $g_E$. We call these extensions $\mathcal{E}$-extensions of the principal Lie algebra. The classification of all nonequivalent equations (with respect to a given equivalence group $G_E$) admitting $\mathcal{E}$-extensions of the principal Lie algebra is called a preliminary group classification. Here, $G_E$ is not necessarily the largest equivalence group but, it can be any subgroup of the group of all equivalence transformations.

So, we can take any finite-dimensional subalgebra (desirable as large as possible) of an infinite-dimensional algebra with basis (3.30) and use it for a preliminary group classification. We select the subalgebra $g_6$ spanned on the following operators:

\[
Y_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \quad Y_2 = y \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial t}, \quad Y_4 = \frac{\partial}{\partial y}, \quad Y_5 = \frac{\partial}{\partial t} - f \frac{\partial}{\partial f}, \quad Y_6 = \frac{\partial}{\partial u}.
\] (4.38)
The adjoint action is given by the Lie series

$$\ \text{Ad}(\exp(s Y_j)) Y_j = Y_j - s [Y_i, Y_j] + \frac{s^2}{2} [Y_i, [Y_i, Y_j]] - \cdots, \quad (4.39)$$

where $s$ is a parameter and $i, j = 1, \cdots, 6$. The adjoint representations of $g_6$ is listed in Table 2, it consists the separate adjoint actions of each element of $g_6$ on all other elements.

**Theorem 4.1.** An optimal system of one-dimensional Lie subalgebras of general Burgers’ equation (1.2) is provided by those generated by

1) $Y^1 = Y_1 = x \partial_x + y \partial_y + t \partial_t + u \partial_u + f \partial_f$,  
2) $Y^2 = Y_2 = y \partial_x$,  
3) $Y^3 = Y_3 = -\partial_y$,  
4) $Y^4 = Y_1 + Y_5 = x \partial_x + y \partial_y + (t+1) \partial_t + u \partial_u$,  
5) $Y^5 = Y_1 - Y_2 = (x-y) \partial_x + y \partial_y + t \partial_t + u \partial_u + f \partial_f$,  
6) $Y^6 = Y_2 - Y_4 = y \partial_x - \partial_y$,  
7) $Y^7 = -Y_4 + Y_6 = -\partial_y + \partial_u$,  
8) $Y^8 = -Y_4 - Y_6 = -\partial_y - \partial_u$,  
9) $Y^9 = Y_2 + Y_5 = y \partial_x + \partial_t - f \partial_f$,  
10) $Y^{10} = Y_2 - Y_3 = y \partial_x - \partial_t + f \partial_f$,  
11) $Y^{11} = Y_2 + Y_6 = y \partial_x + \partial_u$,  
12) $Y^{12} = Y_2 - Y_6 = y \partial_x - \partial_u$,  
13) $Y^{13} = Y_1 + Y_2 = (x+y) \partial_x + y \partial_y - t \partial_t + u \partial_u + f \partial_f$,  
14) $Y^{14} = -Y_4 + Y_5 + Y_6 = -\partial_y + \partial_t + \partial_u - f \partial_f$,  
15) $Y^{15} = Y_2 - Y_1 - Y_5 + Y_6 = y \partial_x - \partial_y - \partial_t + \partial_u + f \partial_f$,  
16) $Y^{16} = Y_2 - Y_4 + Y_5 = y \partial_x - \partial_y + \partial_u$,  
17) $Y^{17} = Y_2 - Y_4 + Y_5 - Y_6 = y \partial_x - \partial_y + \partial_t - \partial_u - f \partial_f$,  
18) $Y^{18} = Y_2 - Y_4 - Y_6 = y \partial_x - \partial_y - \partial_u$,  
19) $Y^{19} = Y_1 + Y_2 + Y_5 = (x+y) \partial_x + (t+1) \partial_t + u \partial_u$,  
20) $Y^{20} = Y_2 + Y_5 + Y_6 = y \partial_x + \partial_t + \partial_u - f \partial_f$,  
21) $Y^{21} = Y_2 + Y_5 - Y_6 = y \partial_x + \partial_t - \partial_u - f \partial_f$,  
22) $Y^{22} = Y_2 - Y_5 - Y_6 = y \partial_x - \partial_t - \partial_u + f \partial_f$,  
23) $Y^{23} = Y_2 - Y_6 + Y_6 = y \partial_x - \partial_t + \partial_u + f \partial_f$,  
24) $Y^{24} = -Y_4 + Y_5 - Y_6 = -\partial_y - \partial_t - \partial_u + f \partial_f$,  
25) $Y^{25} = -Y_4 - Y_5 + Y_6 = -\partial_y - \partial_t + \partial_u + f \partial_f$,  
26) $Y^{26} = -Y_4 + Y_5 - Y_6 = -\partial_y + \partial_t - \partial_u - f \partial_f$,  
27) $Y^{27} = Y_2 - Y_4 + Y_5 + Y_6 = y \partial_x - \partial_y + \partial_t + \partial_u - f \partial_f$,  
28) $Y^{28} = Y_1 + Y_2 - Y_5 = (x+y) \partial_x + y \partial_y + (t-1) \partial_t + u \partial_u + 2 f \partial_f$,  
29) $Y^{29} = Y_1 - Y_2 - Y_5 = (x-y) \partial_x + y \partial_y + (t-1) \partial_t + u \partial_u + 2 f \partial_f$,  
30) $Y^{31} = Y_1 - Y_2 + Y_5 = (x-y) \partial_x + y \partial_y + (t+1) \partial_t + u \partial_u$,  
31) $Y^{31} = Y_1 - Y_5 = x \partial_x + y \partial_y + (t+1) \partial_t + u \partial_u + 2 f \partial_f$,  
32) $Y^{32} = Y_2 - Y_4 - Y_5 - Y_6 = y \partial_x - \partial_y - \partial_t - \partial_u + f \partial_f$,
Proof. Let \( \mathfrak{g}_6 \) is the symmetry algebra of Eq. (1.2) with adjoint representation determined in Table 2 and

\[
Y = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 + a_5 Y_5 + a_6 Y_6,
\]

(4.40)
is a nonzero vector field of \( \mathfrak{g}_6 \). We will simplify as many of the coefficients \( a_i; i = 1, \ldots, 6 \), as possible through proper adjoint applications on \( Y \). We follow our aim in the below easy cases:

Case 1:
At first, assume that \( a_1 \neq 0 \). Scaling \( Y \) if necessary, we can assume that \( a_1 = 1 \) and so we get

\[
Y = Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 + a_5 Y_5 + a_6 Y_6.
\]

(4.41)

Using the table of adjoint (Table 2), if we act on \( Y \) with \( \text{Ad}(\exp(a_3 Y_3)) \), the coefficient of \( Y_3 \) can be vanished:

\[
Y' = Y_1 + a_2 Y_2 + a_4 Y_4 + a_5 Y_5 + a_6 Y_6.
\]

(4.42)

Then we apply \( \text{Ad}(\exp(a_4 Y_4)) \) on \( Y' \) to cancel the coefficient of \( Y_4 \):

\[
Y'' = Y_1 + a_2 Y_2 + a_5 Y_5 + a_6 Y_6.
\]

(4.43)

At last, we apply \( \text{Ad}(\exp(a_6 Y_6)) \) on \( Y'' \) to cancel the coefficient of \( Y_6 \):

\[
Y''' = Y_1 + a_2 Y_2 + a_5 Y_5.
\]

(4.44)

Case 1a:
If \( a_2, a_5 \neq 0 \) then we can make the coefficient of \( Y_2 \) and \( Y_5 \) either +1 or −1. Thus any one-dimensional subalgebra generated by \( Y \) with \( a_2, a_5 \neq 0 \) is equivalent to one generated by \( Y_1 \pm Y_2 \pm Y_5 \) which introduce parts 19), 28), 29) and 30) of the theorem.

Case 1b:
For \( a_2 = 0, a_5 \neq 0 \) we can see that each one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \( Y_1 \pm Y_5 \) which introduce parts 4) and 31) of the theorem.

Case 1c:
For \( a_2 \neq 0, a_5 = 0 \), each one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \( Y_1 \pm Y_2 \) which introduce parts 5) and 13) of the theorem.

Case 1d:
For \( a_2 = 0, a_5 = 0 \), each one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \( Y_1 \) which introduce parts 1) of the theorem.

Case 2:
The remaining one-dimensional subalgebras are spanned by vector fields of the form \( Y \) with \( a_1 = 0 \).

Case 2a:
If \( a_4 \neq 0 \) then by scaling \( Y \), we can assume that \( a_4 = -1 \). Now by the action of \( \text{Ad}(\exp(a_3 Y_3)) \) on \( Y \), we can cancel the coefficient of \( Y_3 \):

\[
\overline{Y} = a_2 Y_2 - Y_4 + a_5 Y_5 + a_6 Y_6.
\]

(4.45)

Let \( a_2 \neq 0 \) then by scaling \( Y \), we can assume that \( a_2 = 1 \), and we have

\[
\overline{Y}' = Y_2 - Y_4 + a_5 Y_5 + a_6 Y_6.
\]

(4.46)

Case 2a-1:
Suppose \( a_5 = a_6 = 0 \), then the one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \( Y_2 - Y_4 \) which introduce parts 6).

Case 2a-2:
Suppose \( a_5 = 0, a_6 \neq 0 \), all of the one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \( Y_2 - Y_4 \pm Y_6 \) which introduce parts 16) and 18).

Case 2a-3:
Suppose \( a_5 \neq 0, a_6 \neq 0 \), all of the one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \( Y_2 - Y_4 \pm Y_5 \pm Y_6 \) which introduce parts 15), 17), 27), and 32).
Now if \( a_2 = 0 \), we have

\[
\bar{Y}'' = -Y_4 + a_5 Y_5 + a_6 Y_6.
\] (4.47)

**Case 2a-4:**
Suppose \( a_5 = a_6 = 0 \), then the one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \(-Y_4 \) which introduce parts 3).

**Case 2a-5:**
Suppose \( a_5 = 0, a_6 \neq 0 \), all of the one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \(-Y_4 \pm Y_6 \) which introduce parts 7) and 8).

**Case 2a-6:**
Suppose \( a_5 \neq 0, a_6 \neq 0 \), all of the one-dimensional subalgebra generated by \( Y \) is equivalent to one generated by \(-Y_4 \pm Y_5 \pm Y_6 \) which introduce parts 14), 24), 25) and 26).

**Case 2b:**
Let \( a_4 = 0 \) then \( Y \) is in the form

\[
\bar{Y} = a_2 Y_2 + a_5 Y_5 + a_6 Y_6.
\] (4.48)

Suppose that \( a_2 \neq 0 \) then if necessary we can let it equal to 1 and we obtain

\[
\bar{Y}' = Y_2 + a_5 Y_5 + a_6 Y_6.
\] (4.49)

**Case 2b-1:**
Let \( a_5 = a_6 = 0 \), then \( Y_2 \) is remained and find 2) section of the theorem.

**Case 2b-2:**
If \( a_5 \neq 0, a_6 \neq 0 \), then \( \bar{Y}' \) is equal to \( Y_2 \pm Y_5 \pm Y_6 \). Hence this case suggests part 20), 21), 22) and 23).

**Case 2b-3:**
If \( a_5 \neq 0, a_6 = 0 \), then \( \bar{Y}' = Y_2 \pm Y_5 \). Hence this case suggests part 9) and 10).

**Case 2b-4:**
If \( a_5 = 0, a_6 \neq 0 \), then \( Y_2 \pm Y_6 \) is obtained. So, this case suggests part 11) and 12).

There is not any more possible case for studying and the proof is complete. \( \square \)

The coefficients \( f \) of Eq. (1.2) depend on the variables \( x, y, u, u_x, u_y, f \). Therefore, we take their optimal system’s projections on the space \((x, y, u, u_x, u_y, f)\). We have

1) \( Z^1 = Y^1 = x \partial_x + y \partial_y + u \partial_u + f \partial_f \),
2) \( Z^2 = Y^2 = y \partial_x \),
3) \( Z^3 = Y^3 = - \partial_y \),
4) \( Z^4 = Y^4 = (x + y) \partial_x + y \partial_y + u \partial_u + f \partial_f \),
5) \( Z^5 = Y^5 = x \partial_x + y \partial_y + u \partial_u \),
6) \( Z^6 = Y^6 = x \partial_x + y \partial_y + 2 f \partial_f \),
7) \( Z^7 = Y^7 = (x - y) \partial_x + y \partial_y + u \partial_u + f \partial_f \),
8) \( Z^8 = Y^8 = y \partial_x - \partial_y \),
9) \( Z^9 = Y^9 = - \partial_y + \partial_u \),
10) \( Z^{10} = Y^{10} = - \partial_y - \partial_u \),
11) \( Z^{11} = Y^{11} = y \partial_x - f \partial_f \),
12) \( Z^{12} = Y^{12} = y \partial_x + f \partial_f \),
13) \( Z^{13} = Y^{13} = x \partial_x - y \partial_y \),
14) \( Z^{14} = Y^{14} = - \partial_x - \partial_y \),
15) \( Z^{15} = Y^{15} = x \partial_x + y \partial_y + f \partial_f \),
16) \( Z^{16} = Y^{16} = x \partial_x + y \partial_y - f \partial_f \),
17) \( Z^{17} = Y^{17} = (x - y) \partial_x + y \partial_y + u \partial_u + 2 f \partial_f \),
18) \( Z^{18} = Y^{18} = (x + y) \partial_x + u \partial_u \),
19) \( Z^{19} = Y^{19} = y \partial_x + \partial_u - f \partial_f \),
20) \( Z^{20} = Y^{20} = y \partial_x - \partial_u - f \partial_f \),
21) \( Z^{21} = Y^{21} = y \partial_x - \partial_u + f \partial_f \),
22) \( Z^{22} = Y^{22} = y \partial_x + \partial_u + f \partial_f \),
23) \( Z^{23} = Y^{23} = y \partial_x - \partial_y + \partial_u \),
24) \( Z^{24} = Y^{24} = y \partial_x - \partial_y - \partial_u \),
25) \( Z^{25} = Y^{25} = y \partial_x + \partial_u - f \partial_f \),
26) \( Z^{26} = Y^{26} = y \partial_x + \partial_u + f \partial_f \),
27) \( Z^{27} = Y^{27} = - \partial_y + \partial_u + f \partial_f \),
28) \( Z^{28} = Y^{28} = - \partial_y - \partial_u - f \partial_f \).
13) \( Z^{13} = Y^{13} = y\partial_x + \partial_u \),
14) \( Z^{14} = Y^{14} = y\partial_x - \partial_u \),
15) \( Z^{15} = Y^{15} = (x - y)\partial_x + y\partial_y + u\partial_u \),
16) \( Z^{16} = Y^{16} = (x + y)\partial_x + y\partial_y + u\partial_u + 2f\partial_f \),
29) \( Z^{29} = Y^{29} = y\partial_x - \partial_y + \partial_u - f\partial_f \),
30) \( Z^{30} = Y^{30} = y\partial_x - \partial_y - \partial_u + f\partial_f \),
31) \( Z^{31} = Y^{31} = y\partial_x - \partial_y + \partial_u + f\partial_f \),
32) \( Z^{32} = Y^{32} = y\partial_x - \partial_y - \partial_u - f\partial_f \).

**Proposition 4.2.** Let \( \mathfrak{g}_m := \langle Y_1, \ldots, Y_m \rangle \), be an \( m \)-dimensional algebra. Denote by \( Y^i (i = 1, \ldots, r, 0 < r \leq m, r \in \mathbb{N}) \) an optimal system of one-dimensional subalgebras of \( \mathfrak{g}_m \) and by \( Z^i (i = 1, \ldots, t, 0 < t \leq r, t \in \mathbb{N}) \) the projections of \( Y^i \), i.e., \( Z^i = \text{pr}(Y^i) \). If equations

\[ f = \Phi(x, y, u, u_x, u_y) \]  \hspace{1cm} (4.52)

are invariant with respect to the optimal system \( Z^i \) then the equation

\[ u_t = \Phi(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) \] \hspace{1cm} (4.53)

admits the operators \( X^i = \text{projection of } Y^i \) on \( t, x, y, u, u_x, u_y \).

**Proposition 4.3.** Let Eq. (4.53) and the equation

\[ u_t = \Phi'(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) \] \hspace{1cm} (4.54)

be constructed according to Proposition 4.2. via optimal systems \( Z^i \) and \( Z'^i \), respectively. If the subalgebras spanned on the optimal systems \( Z^i \) and \( Z'^i \), respectively, are similar in \( \mathfrak{g}_m \), then Eqs. (4.53) and (4.54) are equivalent with respect to the equivalence group \( G_m \), generated by \( \mathfrak{g}_m \).

Now we apply Proposition 4.2. and Proposition 4.3. to the optimal system (3.16) and obtain all nonequivalent Eq. (1.2) admitting \( \mathcal{E} \)-extensions of the principal Lie algebra \( \mathfrak{g} \), by one dimension, i.e., equations of the form (1.2) such that they admit, together with the one basic operators (4.55) of \( \mathfrak{g} \), also a second operator \( X^{(2)} \). For every case, when this extension occurs, we indicate the corresponding coefficients \( f, g \) and the additional operator \( X^{(2)} \).

We perform the algorithm passing from operators \( Z^i (i = 1, \ldots, 32) \) to \( f \) and \( X^{(2)} \) via the following example.

Let consider the vector field

\[ Z^{32} = y\partial_x - \partial_y - \partial_u - f\partial_f \],

then the characteristic equation corresponding to \( Z^6 \) is

\[ \frac{dx}{y} = \frac{dy}{-1} = \frac{du}{-1} = \frac{df}{-f} \] \hspace{1cm} (4.56)

and can be taken in the form

\[ I_1 = u + \frac{x}{y}, \hspace{1cm} I_2 = e^{\frac{x}{y}}f. \] \hspace{1cm} (4.57)

From the invariance equations we can write

\[ I_2 = \Phi(I_1), \] \hspace{1cm} (4.58)

it follows that

\[ f = e^{-\frac{x}{y}}\Phi(\lambda), \] \hspace{1cm} (4.59)

where \( \lambda = I_1 \).
From Proposition 4.2. applied to the operator $Z^6$ we obtain the additional operator $X^{(2)}$

$$y\partial_x - \partial_y + \partial_t - \partial_u. \quad (4.60)$$

After similar calculations applied to all operators (3.16) we obtain the following result (Table 3) for the preliminary group classification of Eq. (1.2) admitting an extension $g_3$ of the principal Lie algebra $g_1$.

5 Conclusion

In this paper, following the classical Lie method, the preliminary group classification for the class of heat equation (1.2) and investigated the algebraic structure of the symmetry groups for this equation, is obtained. The classification is obtained by constructing an optimal system with the aid of Propositions 4.2. and 4.3.. The result of the work is summarized in Table 3. Of course it is also possible to obtain the corresponding reduced equations for all the cases in the classification reported in Table 3.
Table 3
The result of the classification

| N | Z | Invariant | Equation | Additional operator $X^{(2)}$ |
|---|---|----------|---------|-------------------------------|
| 1 | $Z^1$ | $\frac{u}{x}$ | $u_t = x\Phi(u_{xx} + u_{yy})$ | $x\partial_x + y\partial_y + t\partial_t + u\partial_u$ |
| 2 | $Z^2$ | u | $u_t = \Phi(u_{xx} + u_{yy})$ | $y\partial_x$ |
| 3 | $Z^3$ | u | $u_t = \Phi(u_{xx} + u_{yy})$ | $-\partial_y$ |
| 4 | $Z^4$ | $\frac{u}{x+y}$ | $u_t = y\Phi(u_{xx} + u_{yy})$ | $(x+y)\partial_x + y\partial_y + u\partial_u$ |
| 5 | $Z^5$ | $\frac{u}{x}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $x\partial_x + y\partial_y + (t+1)\partial_t + u\partial_u$ |
| 6 | $Z^6$ | $\frac{u}{y}$ | $u_t \equiv x\Phi(u_{xx} + u_{yy})$ | $x\partial_x + y\partial_y + (t-1)\partial_t + u\partial_u$ |
| 7 | $Z^7$ | $\frac{u}{x-y}$ | $u_t \equiv (x-y)\Phi(u_{xx} + u_{yy})$ | $(x-y)\partial_x + y\partial_y + t\partial_t + u\partial_u$ |
| 8 | $Z^8$ | u | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_y$ |
| 9 | $Z^9$ | u | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $-\partial_y + \partial_u$ |
| 10 | $Z^{10}$ | x | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $-\partial_y - \partial_u$ |
| 11 | $Z^{11}$ | u | $u_t = e^{-\frac{u}{x}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x + \partial_t$ |
| 12 | $Z^{12}$ | u | $u_t = e^{-\frac{u}{y}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t$ |
| 13 | $Z^{13}$ | $u - \frac{v}{y}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $y\partial_x + \partial_u$ |
| 14 | $Z^{14}$ | $u + \frac{v}{y}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_u$ |
| 15 | $Z^{15}$ | $\frac{u}{x+y}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $(x+y)\partial_x + y\partial_y + (t+1)\partial_t + u\partial_u$ |
| 16 | $Z^{16}$ | $\frac{u}{x+y}$ | $u_t \equiv (x+y)^2\Phi(u_{xx} + u_{yy})$ | $(x+y)\partial_x + y\partial_y + (t-1)\partial_t + u\partial_u$ |
| 17 | $Z^{17}$ | $\frac{u}{x+y}$ | $u_t \equiv (x-y)^2\Phi(u_{xx} + u_{yy})$ | $(x-y)\partial_x + y\partial_y + (t-1)\partial_t + u\partial_u$ |
| 18 | $Z^{18}$ | $\frac{u}{x+y}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $(x+y)\partial_x + (t+1)\partial_t + u\partial_u$ |
| 19 | $Z^{19}$ | $u - \frac{v}{y}$ | $u_t = e^{-\frac{u}{x}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x + \partial_t + \partial_u$ |
| 20 | $Z^{20}$ | $u + \frac{v}{y}$ | $u_t = e^{-\frac{u}{x}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x + \partial_t - \partial_u$ |
| 21 | $Z^{21}$ | $u - \frac{v}{y}$ | $u_t = e^{-\frac{u}{x}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t - \partial_u$ |
| 22 | $Z^{22}$ | $u + \frac{v}{y}$ | $u_t = e^{-\frac{u}{x}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t + \partial_u$ |
| 23 | $Z^{23}$ | $u - \frac{v}{y}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t + \partial_u$ |
| 24 | $Z^{24}$ | $u + \frac{v}{y}$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t - \partial_u$ |
| 25 | $Z^{25}$ | $u + y$ | $u_t \equiv \Phi(u_{xx} + u_{yy})$ | $-\partial_y + \partial_t + \partial_u$ |
| 26 | $Z^{26}$ | $u - y$ | $u_t \equiv e^{-y}\Phi(u_{xx} + u_{yy})$ | $-\partial_y - \partial_t - \partial_u$ |
| 27 | $Z^{27}$ | $u + y$ | $u_t \equiv e^{-y}\Phi(u_{xx} + u_{yy})$ | $-\partial_y - \partial_t + \partial_u$ |
| 28 | $Z^{28}$ | $u - y$ | $u_t \equiv e^{-y}\Phi(u_{xx} + u_{yy})$ | $-\partial_y + \partial_t - \partial_u$ |
| 29 | $Z^{29}$ | $u + \frac{\bar{v}}{y}$ | $u_t = e^{-\bar{v}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t + \partial_u$ |
| 30 | $Z^{30}$ | $u + \frac{\bar{v}}{y}$ | $u_t = e^{-\bar{v}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t - \partial_u$ |
| 31 | $Z^{31}$ | $u - \frac{\bar{v}}{y}$ | $u_t = e^{-\bar{v}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t + \partial_u$ |
| 32 | $Z^{32}$ | $u + \frac{\bar{v}}{y}$ | $u_t = e^{-\bar{v}}\Phi(u_{xx} + u_{yy})$ | $y\partial_x - \partial_t - \partial_u$ |

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