Quantum computational webs

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We introduce the notion of quantum computational webs: These are quantum states universal for measurement-based computation which can be built up from a collection of simple primitives. The primitive elements—reminiscent of building blocks in a construction kit—are (i) one-dimensional states ("computational quantum wires") with the power to process one logical qubit and (ii) suitable couplings which connect the wires to a computationally universal "web". All elements are preparable by nearest-neighbor interactions in a single pass, as are accessible in a number of physical architectures. We provide a complete classification of qubit wires—this being the first instance where a physically well-motivated class of universal resources can be fully understood. Finally, we sketch possible realizations in superlattices, and explore the power of coupling mechanisms based on Ising or exchange-interactions.

It is an intriguing fact that one can perform universal quantum computation just by performing local measurements on certain quantum many-body systems [1–8]. Despite enormous interest in this phenomenon, our understanding of which quantum systems offer a quantum computational speed-up and which do not is still rudimentary. Indeed, for years the only states known to be universal for quantum computation by measurements were the cluster state and very close relatives [1, 2, 9]. This was unsatisfactory both from a fundamental point of view and for experimentalists aiming to tailor resource states to their physical systems in the lab. In Refs. [7, 8] a framework for the construction of new schemes for MBQC was introduced (further applied e.g. in Refs. [11, 12]). There, it was shown that many of the singular properties of the cluster are not necessary for a computational speed-up—hence weakening the requirements for MBQC. This newly found flexibility notwithstanding, it has been established that universality is a rare property among quantum many-body states [10]. Therefore, it would be very desirable to obtain a full classification of the relatively few states which are universal. While the unqualified problem still seems daunting, we show in this work that under reasonable, physically motivated constraints, a complete understanding is possible.

The basic idea is to break up resource states into smaller primitives, which are more amenable to analysis. Indeed, most known states universal for MBQC come in two versions: (i) states on a 1-D chain of qubits, which have the ability to transport and process one logical qubit worth of quantum information [1, 7, 8, 11, 12], and (ii) 2-D versions, obtained by suitably entangling several 1-D strands. We will refer to such 1-D states as quantum computational wires. They form the measurement-based equivalent of a single qubit. Likewise, the couplings used to form truly universal 2-D resources (referred to as quantum computational webs) are the analogues of entangling unitaries in the gate model. Splitting the analysis of universal states into wires and couplings has two advantages: (i) the primitives are far easier to understand than the compound state they give rise to and (ii) in a manner reminiscent of a construction kit, wires and couplings may be freely combined to form diverse sets of universal resources (c.f. Fig. 1).

**Full classification of qubit wires.** For most of what follows we focus on qubit systems, for which we can provide a full theory. We impose the physically reasonable requirement that wires can be build up from product states by means of nearest-neighbor interactions \( U = e^{-iH_{\text{spin}}(1,2)} \) in a single translationally invariant pass. The physical realizations we have in mind here are atoms in an optical lattice as in an "atomic sorting device" [12], settings exploiting optical superlattices [11, 14], or other architectures such as ones involving interacting quantum dots [15] or instances of networks [17]. More specifically, by a qubit computational wire we mean

(i) a family of pure states \( |\phi_n\rangle \) of a \( n \)-qubit spin chain,

(ii) preparable from a product state \( |0, \ldots, 0\rangle \) by the sequential action of a unitary gate \( U \)

\[
|\psi_n\rangle = U^{(n,n-1)} \cdots U^{(3,2)} U^{(2,1)} |0, \ldots, 0\rangle.
\]  

(iii) In the limit of large \( n \), the entanglement between the left and the right half of the chain (in the sense of an "area law") approaches one ebit.

These axioms may seem surprisingly weak: earlier, we loosely characterized computational wires as states with the power to "transport and process one logical qubit". It is one central result of this work that any state fulfilling (i)–(iii) is automatically useful for information processing. Below, we...
will explain and prove the following complete classification of qubit wires up to local basis changes:

**Observation 1** (Classification of qubit wires). A wire is specified by an

(a) “always-on operation” $W \in SU(2)$, acting on correlation space (see below) after every step, independent of the basis chosen or the measurement outcome, and a

(b) “by-product angle” $\phi$, specifying how sensitive the resource is to the inherent randomness of measurements.

To make sense of this statement, first note that any $|\psi_n\rangle$ has a matrix product state (MPS) representation [7, 8]:

$$|\psi_n\rangle = \sum_{x_1, \ldots, x_n} \langle x_n | A[x_{n-1}] \ldots A[x_1] | 0 \rangle | x_1, \ldots, x_n \rangle,$$

where $x_i \in \{0, 1\}$, and $A[0], A[1]$ are $2 \times 2$-matrices. (Eq. [2] follows from Eq. [1] by setting $A[x_{i,j}] = \langle i, x | U[0, j] \rangle$.) The auxiliary two-dimensional vector space the matrices $A[0/1]$ act on is called correlation space. We recall very briefly the basic idea of Refs. [7, 8]. Let $|\phi^{(i)}\rangle = c_0^{(i)} |0\rangle + c_1^{(i)} |1\rangle$ be a local state vector and set $A[\phi^{(i)}] = c_0^{(i)} A[0] + c_1^{(i)} A[1]$. Then

$$((|\phi^{(1)}\rangle \otimes \cdots \otimes |\phi^{(n)}\rangle)|\psi_n\rangle = (\phi_n|A[\phi_n] \cdots A[\phi_1]|0\rangle).$$

Hence, a local measurement with outcome corresponding to $|\phi_i\rangle$ is connected with the action of the operator $A[\phi_i]$ on correlation space. MBQC can be understood completely in terms of this relation between local measurements and logical computations on correlation space [7, 8]. With these notions, the precise statement of Obs. [1] is that any wire allows for an MPS representation with matrices

$$B[0] = 2^{-1/2} W, \quad B[1] = 2^{-1/2} WS(\phi),$$

where $S(\phi) = \text{diag}(e^{-i\phi/2}, e^{i\phi/2})$, see Fig 2(a). (That is to say, any matrix arising from Eq. (2) can be brought into this form by a suitable rescaling and conjugation, see below).

Obs. [1] goes a long way towards understanding the structure of qubit wires. Assume that we measure site by site in the computational basis. By Eq. [3], at every step the same “always-on” operation $W$ will be applied to the correlation space, irrespective of the measurement outcome. Some tribute must be paid to the random nature of quantum measurements. It comes in the form of the by-product operation $S(\phi)$, acting on the correlation system in case the “wrong” measurement outcome (“$1$”), instead of “$0$”) is obtained. It is remarkable that this penalty is described by a single parameter: the by-product angle $\phi$ [20].

**Examples of qubit wires.** The paradigmatic qubit wire is the cluster state. Here, $W = H$, the Hadamard gate, and $\phi = \pi$, the highest possible value [21, 22]. We can thus put two well-known properties of the cluster into a more general context: (i) in every step a Hadamard gate $H$ is applied to the logical qubit and (ii) a “wrong” measurement outcome causes the application of an extra $S(\pi) \simeq \sigma_z$ gate on correlation space.

Another interesting new resource where the role of the by-product angle can be highlighted is the T-resource, named after the common notation $T = S(\pi/2)$ for a phase gate. Here, we take $W = H$ (as for the cluster), but the by-product angle is just $\phi = \pi/2$ (so that a measurement in the computational basis gives rise to either $H$ or $HT$). This qubit wire has non-maximal entropy of entanglement of a single wire w.r.t. the rest of the lattice. The intuitive explanation is that $T$ is “close” to the identity, so the state of the correlation system (and hence the rest of the chain) does not strongly depend on the outcomes of local measurements on any given site.

The proof of Obs. [1] will make repeated use of the theory of MPS’s [18] and of qubit channels [19]. Any MPS can be represented with matrices s.t. $A[0] A[0] + A[1] A[1] = 1$ [18]. The matrices give rise to a trace-preserving channel $\rho \rightarrow \mathbb{E}(\rho) = \sum_x A[x] \rho A[x]^\dagger$. Assuming that $\mathbb{E}$ has a spectral gap [22] the half-chains share one ebit of entanglement iff $\mathbb{E}$ is unital [18]. In this case, it follows easily from Ref. [19] that $\mathbb{E}(\rho) = p_0 U_0 \rho U_0^\dagger + p_1 U_1 \rho U_1^\dagger$, with suitable $U_i \in SU(2)$. From the basic theory of quantum channels, we know that there is a unitary $V \in SU(2)$ such that $p_1 U_i = \sum_j V_{i,j} A[j].$ That being nothing but the transformation rule for MPS representations under local basis change, we conclude that there is a basis in which $|\psi_n\rangle$ is represented with matrices $A'[i] = p_i^{1/2} U_i$. Next, an MPS does not change if both matrices are conjugated by the same operator $X$. There is an $X \in SU(2)$ such that $X U_0 X^\dagger = e^{i\alpha} S(\phi)$ for $\alpha, \phi \in \mathbb{R}$. Setting $W = X U_0 X^\dagger$ and $B[i] = X A[i] X^\dagger$ implies $B[0] = p_0^{1/2} W, B[1] = p_1^{1/2} e^{-i\alpha} W S(\phi)$. Performing the local basis change $|1\rangle \rightarrow e^{i\alpha}|1\rangle$ if necessary, we may assume that $\alpha = 0$. The fact that $p_0, p_1$ can be chosen to be $1/2$ will be explained below in a more general context. Conversely, any MPS with matrices as in Eq. [3] is a qubit wire. A translationally invariant preparation scheme can easily be derived by inverting the construction below Eq. (2).

**Computation with qubit wires.** So far we have shown that one can implement some unitary operation in a quantum wire, i.e. transport quantum information. In order to process it, one must have some freedom to choose which operation to apply. It will turn out—rather surprisingly—that two coincidences conspire to make any qubit quantum wire useful for that purpose. To that end, consider the one-parameter family of bases

$$|0_\theta\rangle = \sin(\theta) |0\rangle + \cos(\theta) |1\rangle, \quad |1_\theta\rangle = \cos(\theta) |0\rangle - \sin(\theta) |1\rangle.$$
One may check directly that the operations $A[0] \propto W(\sin \theta \mathbb{I} + \cos \theta S(\phi))$ are unitary up to scaling. The two unexpected coincidences are: (i) for any quantum wire, there is a continuous family of projections which give rise to unitary evolution and (ii) the set these projections includes entire bases—so that measuring in these bases corresponds to a unitary logical computation regardless of the outcome.

**Observation 2** (Unitary evolution). For any computational wire, a measurement in any basis from the one-parameter set $\{(0)_b, |1\rangle_b\}$ induces a unitary evolution in correlation space.

Let us investigate the realizable unitaries. Clearly, $A[0]$ has the form $WU(\theta, \phi)$, where $U(\theta, \phi)$ is a diagonal matrix with eigenvalues $\lambda_{\pm} = \sin(\theta) \pm e^{\pm i \phi}/2$. Let $\delta = \arctan(\lambda_+)$ and $p = |\lambda_+|^2$. Then $U(\theta, \phi) = \sqrt{p} S(-2\delta)$ and basic MPS theory yields that the corresponding measurement outcome is obtained with probability $p$. For fixed $\phi$, the set of phase gates $S(-2\delta)$ thus realizable forms an ellipse, see Fig. 2(c), in the complex plane with parametrization

\[
(\text{re}\lambda_+(\theta, \phi), \text{im}\lambda_+(\theta, \phi))^T = \begin{pmatrix}
\cos \phi/2 & \sin \theta \\
0 & \cos \theta
\end{pmatrix}.
\]

**Observation 3** (Phase gates). In any computational wire, an arbitrary phase gate $S(\delta)$ can be implemented in a single step.

Leaving aside the issue of randomness for a moment, we see that one can realize any unitary of the form $U = WS(\delta_n)WS(\delta_{n-1})\ldots WS(\delta_1)$ for some $n$. Invoking assumption 22, every $U \in SU(2)$ is of that form.

**Observation 4** (Universal rotations). Except from a set of measure zero, all computational qubit wires allow for the implementation of any unitary $U \in SU(2)$ in correlation space.

**Local properties.** From MPS theory 18 one finds that the reduced state of a single site far away from the boundary is given by $\rho = \sum_{i,j} \text{tr}(A[i]^\dagger A[j]) |i\rangle\langle j|/2$. Explicitly:

\[
\rho = \begin{pmatrix}
1 & \cos \phi/2 \\
0 & 1
\end{pmatrix} /2.
\]

Interestingly, we see that the always-on operation $W$ does not affect the local properties of the state. One can hence conclude (see Fig. 2(b)):

**Observation 5** (Small entanglement in wires). Computational wires with arbitrarily low local entanglement exist.

**Compensating randomness.** In the above classification, we required from a “qubit wire” to allow for “transporting and processing one logical qubit”. We yet also need to clarify how to deal with the inherent randomness of quantum measurements. If the always-on term $W$ and the by-product operator $S(\phi)$ generate a finite group $B$, there is a simple and efficient possibility to cope with randomness, introduced in Ref. [7]. Suppose we would like to implement $WS(\delta)$, but instead obtain a measurement outcome which causes $WS(\delta')$ to be realized. Now, by measuring several consecutive sites in the computational basis, we effectively implement a random walk on the finite group $B$ in correlation space. This random walk will visit any element of $B$ after a finite expected number of steps. We will hence obtain $W^{-1}B$ after several steps, yielding a total evolution of $W^{-1}WS(\delta') = S(\delta')$. Then, one tries to implement $S(-\delta' + \delta)$, which is possible by Obs. 24. It remains to be shown how logical information in the correlation system can be prepared and read out. As for preparation, note that $A[2^{-1/2}(0) - e^{i\phi/2}1|] \propto |1\rangle1\rangle$ has rank one. Hence, if after a local measurement the outcome corresponding to $2^{-1/2}(0) - e^{i\phi/2}1|1\rangle$ is obtained, the correlation system will be in $1\rangle1\rangle$, irrespective of its previous state—so preparation is possible. A read-out scheme can be devised along these lines.

**Observation 6** (Preparation and readout). For any qubit wire, one can efficiently prepare the correlation system in a known state and read out the latter by local measurements.

**Ising coupling.** All wires introduced so far can be coupled to form a 2-D state, universal for quantum computation. Remarkably, there are several coupling schemes, which work equally well for all 1-D states so far introduced. Space limitations require us to describe only one and be somewhat sketchy (however, all central points are explained: see Ref. 24 for further details). The coupling scheme, depicted in Fig. 3, is based on a setting where $\{1, 2, 3\}$ and $\{5, 6, 7\}$ belong to any wire and 4 has been prepared in $2^{-1/2}(0) + 1|1\rangle$. One now entangles sites $\{2, 4\}$ and sites $\{4, 6\}$ via Ising interactions in a suitable bases. More concretely, one performs a controlled-$\sigma_z$ gate $(CZ(4, 2))$ between site 2 and site 4 and then applies

\[
W(6)CZ(4, 6)(W(6))^\dagger, W = \begin{pmatrix} 1 & 1 \\ e^{i\phi/2} & -e^{-i\phi/2} \end{pmatrix} 2^{-1/2}
\]

between systems 4 and 6. To decouple the wires, just measure 4 in the computational basis. In case of the $|0\rangle$-outcome, we have un-done the coupling; a $|1\rangle$-outcome brings us back to the original state, up to the action of $\sigma_z$ on site 2 and $W\sigma_z W^\dagger$ on 6. To perform an entangling gate, one measures 6 in the $\sigma_z$-basis and 4 in the $\sigma_x$-basis, getting outcomes $x_4, z_6 \in \{0, 1\}$ respectively. Let us assume that $x_4 = 0$ and $z_6$ is even. Choose $\gamma, \varepsilon$ such that $e^{i\varepsilon/2} \sin \gamma = 1/2(1 - e^{i\phi})$, and let $\delta$ be the solution to $|\sin \delta| = |\sin \delta \sin \gamma + \cos \delta \cos \gamma e^{i\phi/2}|$ (which always exists). Finally, measure site 2 in the basis $|\psi_0\rangle = e^{-i\varepsilon} \sin \delta |0\rangle + \cos \delta |1\rangle$, $|\psi_1\rangle = -e^{-i\varepsilon} \cos \delta |0\rangle + \sin \delta |1\rangle$. A lengthy—but by these definitions fully specified—calculation shows that if we get the $|\psi_0\rangle$ outcome, then one implements the unitary entangling operation

\[
V = W[0]0\langle 0 | \otimes (\cos \delta) A[1] \\
+ W[1]1\langle 1 | \otimes (\sin \delta \sin \gamma) A[0] + \cos \delta \cos \gamma A[1])
\]

between the upper and lower correlation spaces. The orthogonal outcome and the case of odd $x_4 + z_6$ may be treated analogously.

**Observation 7** (Ising-type coupling). Arbitrary qubit wires can be coupled with suitable phase gates.

We use the remaining paragraphs to give an outlook on further results and ideas.

**Exchange interaction coupling.** Using the ideas presented above, one may check that cluster wires can be coupled together using an exchange interaction Hamiltonian: $\hat{H}_{ex} =$
for time $t$ one shifts the superlattice, so that neighboring pairs which sites of each double-well interact with $H$ | right site of each double-well occupied by a single particle, where the potential forms a string of double-wells, with the Bose-Hubbard optical superlattices [11, 14], subject to an Widening our scope beyond qubits, we look at bosons in Observation 8

ings”, see Figs. 1(b). The coupling operation used to obtain a cellular automaton | A coupling of cluster wires based on an exchange interaction. We also state the

\[ \Psi^- | \Psi^- \rangle, \text{ where } | \Psi^- \rangle = 2^{-1/2} ((0,1) - |1,0\rangle) \]. The topology used here is a “hexagonal lattice with additional space-\(s\)”, see Figs. 1(b). The coupling operation used to obtain a universal resource is given by $U = e^{i\pi/2H_{2\pi}}$ [25].

**Observation 8 (Exchange interaction coupling).** An exchange interaction Hamiltonian can be used to couple cluster wires.

**Bose-Hubbard-type and continuous-variable wires.** Widening our scope beyond qubits, we look at bosons in optical superlattices [11–14], subject to an Bose-Hubbard interaction (compare also Ref. [15]). Consider the situation where the potential forms a string of double-wells, with the right site of each double-well occupied by a single particle, $|\Psi(t = 0)\rangle = |0,1,\ldots,0,1\rangle$. In a first step, one lets the two sites of each double-well interact with $H = a_L^\dagger a_R + a_R^\dagger a_L$ for time $t = \pi/4$, leading to pairs in the state $|0,1\rangle + i|1,0\rangle$. Secondly—in the fashion of a quantum cellular automaton—one shifts the superlattice, so that neighboring pairs which have not previously interacted are subjected to the Hamiltonian above. One obtains a globally entangled state with up to three excitations per site and entropy of entanglement between half-chains of up to $E(\rho) = 1.725$. Assuming the power to perform tilted measurements in the particle number basis (or making use of suitable internal degrees of freedom) it is easily checked that this “Bose-Hubbard wire” allows for the transport of one logical qubit, and arbitrary rotations along one axis. This is an example of a primitive where the local Hilbert space dimension is in principle infinite. Further steps towards continuous-variable (CV) schemes could be done by considering correlation spaces where only a subspace of superpositions of finitely many coherent states is occupied, such that the correlation space is still finite-dimensional. The framework established here forms a starting point to study such CV computational schemes.

**Observation 9 (Bose-Hubbard wires).** Suitable states preparable by Bose-Hubbard interactions in superlattices allow for transport of one logical qubit.

**Summary.** We have introduced a toolbox of primitives for constructing new quantum computational schemes. For the qubit case, we provide a full classification. The results constitute a further step towards the goal of understanding what is ultimately needed for quantum computation and what degree of freedom there is in designing computational schemes.

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[19] M. B. Ruskai, S. Szarek, and E. Werner, Lin. Alg. Appl. 347, 159 (2002).
[20] Note that for the definition of a qubit wire as such, we do not require being able to compensate randomness of outcomes by exploiting a finite group structure of the by-product operators.
[21] Note that our definition differs from the conventional one by the action of a local Hadamard gate on every site.
[22] Away from (and independently of) the boundaries, an MPS is completely specified by the matrices Eq. [3] iff the map $E$ has a spectral gap [18]. This is true iff $W$ is neither diagonal nor equal to $\sigma_x$. We will always implicitly assume this generic situation.
[23] We can now prove the earlier claim that in the normal form Eq. [3] the weights of the two matrices may be chosen to be equal. That follows from the fact that there are two perpendicular vectors intersecting the ellipse at the same length $\sqrt{p}$. More generally, the method sketched above may be implemented as soon as there is some basis $\{|0_0\}, |1_0\rangle \}$ s.t. $A|0_0\rangle, A|1_0\rangle$ generate a finite group (up to scalars). It can be
shown that whenever one such basis exists, there is a one-parameter set of bases with the same property [25]. This gives rise to continuous families of wires in which randomness can be compensated by the same method.

[25] D. Gross and J. Eisert, in preparation.