Locally Nilpotent Derivations of Free Algebra of Rank Two

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Abstract. In commutative algebra, if $\delta$ is a locally nilpotent derivation of the polynomial algebra $K[x_1, \ldots, x_d]$ over a field $K$ of characteristic 0 and $w$ is a nonzero element of the kernel of $\delta$, then $\Delta = w\delta$ is also a locally nilpotent derivation with the same kernel as $\delta$. In this paper we prove that the locally nilpotent derivation $\Delta$ of the free associative algebra $K\langle X, Y \rangle$ is determined up to a multiplicative constant by its kernel. We show also that the kernel of $\Delta$ is a free associative algebra and give an explicit set of its free generators.

Key words: free associative algebras; locally nilpotent derivations; algebras of constants

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To the 80th anniversary of Dmitry Fuchs

1 Introduction

Let $K$ be a field of characteristic 0. Locally nilpotent derivations $\delta$ of polynomial algebras $K[x_1, \ldots, x_d]$ and their kernels $\ker(\delta)$ are subjects of active investigation. Traditionally, the kernel of a derivation $\delta$ of $K[x_1, \ldots, x_d]$ is called the algebra of constants of $\delta$ and is denoted by $K[x_1, \ldots, x_d]^\delta$. The algebras of constants of locally nilpotent derivations play an essential role in the study of the automorphism group of $K[x_1, \ldots, x_d]$, including the generation of $\text{Aut}(K[x, y])$ by tame automorphisms, the Jacobian conjecture, in invariant theory, fourteenth Hilbert’s problem and other important topics. See the books by Nowicki [18], van den Essen [29], and Freudenburg [10] for details. In particular, using locally nilpotent derivations, Rentschler [20] gave an easy proof of the theorem of Jung–van der Kulk [11, 30] that all automorphisms of $K[x, y]$ are tame. Another natural proof based on locally nilpotent derivations was given by Makar-Limanov [15], see also the book [6]. The most natural way to define the Nagata automorphism [17]

$$(x, y, z) \rightarrow (x - 2(xz + y^2)y - (xz + y^2)^2z, y + (xz + y^2)z, z)$$

is also in terms of locally nilpotent derivations, see Bass [1] and Smith [25]. The famous Jacobian conjecture is equivalent to several conjectures stated in the language of locally nilpotent derivations, see [29]. Several nice counterexamples to fourteenth Hilbert’s problem are obtained as algebras of constants of locally nilpotent derivations, see the survey and the book by Freudenburg [9, 10] and the survey by Nowicki [19]. On the other hand, the well known theorem of

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Weitzenböck [31] states that if $\delta$ is a nilpotent linear operator acting on the $d$-dimensional vector space $K x_1 \oplus \cdots \oplus K x_d$, then the algebra of constants of the locally nilpotent derivation of $K[x_1, \ldots, x_d]$ which extends $\delta$ is a finitely generated algebra. A modern proof of the theorem is given by Seshadri [22], with further simplification by Tyc [27], see also [18]. Clearly, the algebra of constants $K[x_1, \ldots, x_d]^{\delta}$ coincides with the algebra of invariants of the linear operator

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots.$$ 

If $\delta$ is a locally nilpotent derivation of $K[x_1, \ldots, x_d]$ and $0 \neq w \in K[x_1, \ldots, x_d]^{\delta}$, then $\Delta = w\delta$ is also a locally nilpotent derivation with the same algebra of constants as $\delta$. In particular, starting from the Weitzenböck derivation of $K[x, y, z]$ defined by

$$\delta(x) = -2y, \quad \delta(y) = z, \quad \delta(z) = 0,$$

$w = xz + y^2 \in K[x, y, z]^{\delta}$, and $\Delta = (xz + y^2) \delta$ one obtains the Nagata automorphism as $\exp(\Delta)$. We would like to mention that Shestakov and Umirbaev [23, 24] proved the Nagata conjecture that the Nagata automorphism is wild with methods of noncommutative algebra.

Locally nilpotent derivations of free associative algebras $K\langle X_1, \ldots, X_d \rangle$ have not been studied as intensively as in the commutative case. We shall mention the old result of Falk [8] who described the intersection of the kernels of the formal partial derivatives $\partial / \partial X_j$ of $K\langle X_1, \ldots, X_d \rangle$, and the relations of the formal partial derivatives with theory of algebras with polynomial identity due to Specht [26], see also [6] for further development. Drensky and Gupta [7] studied the kernels of Weitzenböck derivations of $K\langle X_1, \ldots, X_d \rangle$ and established that in all nontrivial cases the kernel is not finitely generated. As in the case of polynomial algebras, the candidate for a wild automorphism, the automorphism of Anick [2, p. 343]

$$(X, Y, Z) \to (X + Z(XZ - ZY), Y + (XZ - ZY)Z, Z)$$

can also be expressed as $\exp(\Delta)$ for the locally nilpotent derivation $\Delta$ of $K\langle X, Y, Z \rangle$ defined by

$$\Delta(X) = Z(XZ - ZY), \quad \Delta(Y) = (XZ - ZY)Z, \quad \Delta(Z) = 0.$$ 

The wildness of the Anick automorphism was established by Umirbaev [28].

In this paper we study locally nilpotent derivations $\Delta$ of the free unitary associative algebra $K\langle X, Y \rangle$ over a field $K$ of characteristic 0. As in the commutative case we shall call the kernel of $\Delta$ the algebra of constants of $\Delta$ and denote it by $K\langle X, Y \rangle^\Delta$. Our main result is that the locally nilpotent derivations of $K\langle X, Y \rangle$ are determined up to a multiplicative constant by their algebras of constants.

It is easy to see that $\Delta$ is of the form $\Delta(U) = 0$, $\Delta(V) = f(U)$, with respect to a suitable system of generators $U, V$ of $K\langle X, Y \rangle$. This follows from the description of Rentschler [20] of the locally nilpotent derivations of $K[x, y]$ and the isomorphism of the automorphism groups of $K[x, y]$ and $K\langle X, Y \rangle$ which is a consequence of the theorem of Jung–van der Kulk [11, 30] and its analogue for the automorphisms of $K\langle X, Y \rangle$ due to Czerniakiewicz [3, 4] and Makar-Limanov [14]. This result is similar to the recent description of locally nilpotent derivations of the free Poisson algebra with two generators given by Makar-Limanov, Turusbekova, and Umirbaev [16].

As a consequence of the result of Lane [13] and Kharchenko [12] the algebra of constants $K\langle X, Y \rangle^\Delta$ of the nontrivial Weitzenböck derivation $\Delta$ of $K\langle X, Y \rangle$ is a free associative algebra. A set of free generators of this algebra was given by Drensky and Gupta [7]. We generalize this result and show that the algebra $K\langle X, Y \rangle^\Delta$ is free for any locally nilpotent derivation $\Delta$ of $K\langle X, Y \rangle$. As in [7] we give an explicit set of free generators of $K\langle X, Y \rangle^\Delta$. See also [5] where it is shown that $K\langle X, Y \rangle^\Delta$ is a free associative algebra for a nontrivial homogeneous derivation (and from which the freeness in our case can be deduced).
2 Preliminaries

For an algebra $R$ over a field $K$ a linear operator $\delta: R \to R$ is called a derivation if it satisfies the Leibniz law $\delta(ab) = \delta(a)b + a\delta(b)$. The kernel of a derivation $\delta$ is denoted by $R^\delta$ and the elements of the kernel are called $\delta$-constants (or just constants when this is not confusing). A derivation $\delta$ is called locally nilpotent if for any $r \in R$ there exists a natural number $n$ (which depends on $r$) for which $\delta^n(r) = 0$. The function

$$\deg(r) = \max\{d \mid \delta^d(r) \neq 0\}, \quad \deg(0) = -\infty,$$

is a degree function with familiar properties:

\begin{align*}
\deg(r_1r_2) &= \deg(r_1) + \deg(r_2), \\
\deg(r_1 + r_2) &= \max(\deg(r_1), \deg(r_2)) \quad \text{when} \quad \deg(r_1) \neq \deg(r_2), \\
\deg(\delta(r)) &= \deg(r) - 1 \quad \text{if} \quad \delta(r) \neq 0.
\end{align*}

The set of all lnds (locally nilpotent derivations) of $R$ is denoted by $\text{LND}(R)$.

The intersection $\bigcap R^\delta$, $\delta \in \text{LND}(R)$, of kernels of all locally nilpotent derivations of $R$ is denoted by $\text{AK}(R)$ (absolute Konstanten of $R$, sometimes denoted as $\text{ML}(R)$).

If $\delta \in \text{LND}(R)$ and characteristic of $K$ is zero then the linear operator

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

is an automorphism of $R$.

In the sequel we fix a field $K$ of characteristic 0 and consider the polynomial algebra $K[x, y]$ and the free associative algebra $K\langle X, Y \rangle$. Let

$$\pi: K\langle X, Y \rangle \to K[x, y]$$

be the natural homomorphism. We denote the elements $U$, $V$, etc. of $K\langle X, Y \rangle$ by upper case symbols and their images under $\pi$ by the same lower case symbols $u$, $v$, etc. Let $C$ be the commutator ideal of $K\langle X, Y \rangle$. It is generated by the commutator

$$T_1 = [Y, X] = YX - XY.$$

By the theorem of Jung–van der Kulk [11, 30], the automorphisms of $K[x, y]$ are tame, i.e., are compositions of affine automorphisms

$$x \to a_1x + a_2y + a_3, \quad y \to b_1x + b_2y + b_3, \quad a_i, b_i \in K, \quad a_1b_2 - a_2b_1 \neq 0,$$

and triangular automorphisms

$$x \to x, \quad y \to y + p(x), \quad p(x) \in K[x].$$

A similar theorem of Czerniakiewicz [3, 4] and Makar-Limanov [14] states that the automorphisms of $K\langle X, Y \rangle$ are also tame. Therefore

$$\Psi(T_1) = cT_1, \quad c \in K^*,$$

for any automorphism $\Psi$ of $K\langle X, Y \rangle$ (indeed, just check that this is true for affine and triangular automorphisms).

The structure of the automorphism groups of $K[x, y]$ and $K\langle X, Y \rangle$ is also known, it is a free product of the subgroups of affine and triangular automorphisms with amalgamation along
their intersection [21]. So we can think that there is a group $H$ isomorphic to $\text{Aut} \ K[x, y]$ and $\text{Aut} \ K\langle X, Y \rangle$ which acts on $K[x, y]$ and $K\langle X, Y \rangle$.

Any automorphism of $K\langle X, Y \rangle$ induces an automorphism of $K[x, y]$ and, since the structure of the group $H$ insures that this is one to one correspondence, any automorphism of $K[x, y]$ can be uniquely lifted to an automorphism of $K\langle X, Y \rangle$.

We shall use below a lexicographic ordering of monomials of $K\langle X, Y \rangle$ defined by $Y \gg X > 1$ and denote by $S$ the leading monomial of $S \in K\langle X, Y \rangle$.

In the sequel we shall show that we can reduce our considerations to the case when the lnd $\Delta$ is such that

$$\Delta(X) = 0, \quad \Delta(Y) = F = f(X),$$

where $0 \neq f(x) \in K[x]$. In this special case we shall define the operator $\Box$ on $K\langle X, Y \rangle$ by

$$\Box(A) = YAF - FAY, \quad A \in K\langle X, Y \rangle,$$

and shall fix the sequence $T_1, T_2, \ldots$, starting with $T_1 = YX - XY$ and then inductively

$$T_{i+1} = \Box^i(T_1).$$

### 3 Description of locally nilpotent derivations

Though the lnds of $K\langle X, Y \rangle$ are similar to the lnds of $K[x, y]$ there are also significant differences.

It is quite clear that $\text{AK}(K[x, y]) = K$ (just observe that the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are locally nilpotent) but we shall show later that $\text{AK}(K\langle X, Y \rangle) = K[T_1]$. The following lemma shows that $\text{AK}(K\langle X, Y \rangle) \supseteq K[T_1]$.

**Lemma 3.1.** $\delta(T_1) = 0$ for any lnd of $K\langle X, Y \rangle$.

**Proof.** If $\delta \in \text{LND}(K\langle X, Y \rangle)$ then $\lambda \delta \in \text{LND}(K\langle X, Y \rangle)$ for any $\lambda \in K$. Take $\Psi_\lambda = \exp(\lambda \delta)$; then $\Phi_\lambda([Y, X]) = c(\lambda)[Y, X]$, where $c(t) \in K[t]$ (recall that $\delta$ is an lnd). On the other hand $\Phi_\lambda \Phi_\mu = \Phi_{\lambda+\mu}$, i.e., $c(s)c(t) = c(s + t)$. Since $c(s) \neq 0$ this is possible only if $c(t) = 1$. Hence $\delta([Y, X]) = 0$. $\blacksquare$

Now we shall prove that lnds of $K\langle X, Y \rangle$ are similar to those of $K[x, y]$.

**Proposition 3.2.** Let $\Delta$ be a locally nilpotent derivation of $K\langle X, Y \rangle$. Then there is a system of generators $U, V$ of $K\langle X, Y \rangle$ and a polynomial $f(U)$ depending on $U$ only, such that $\Delta(U) = 0, \Delta(V) = f(U)$.

**Proof.** Let $\Delta$ be a locally nilpotent derivation of $K\langle X, Y \rangle$. Clearly, $\Delta$ induces a locally nilpotent derivation $\delta$ of $K[x, y]$. By the theorem of Rentschler [20], $K[x, y]$ has a system of generators $u, v$ such that $\delta(u) = 0, \delta(v) = f(u)$ for some $f(u) \in K[u]$.

As was mentioned above this pair of generators can be uniquely lifted to the pair $U, V$ of generators of $K\langle X, Y \rangle$.

Let us consider the automorphisms

$$\Phi = \exp(\Delta) \in \text{Aut} \ K\langle X, Y \rangle = \text{Aut} \ K\langle U, V \rangle$$

and

$$\varphi = \exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots \in \text{Aut} \ K[x, y] = \text{Aut} \ K[u, v].$$
Then
\[ \varphi: \ u \to u, \quad \varphi: \ v \to v + f(u). \]

From the uniqueness mentioned in Section 2
\[ \varphi(u) = u, \quad \varphi(v) = v + f(u), \]
implies \( \Phi(U) = U, \quad \Phi(v) = v + f(U) \). Since \( \Phi = \exp(\Delta) = 1 + \Theta \), where
\[ \Theta = \frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \cdots \]
and \( \Theta^n(S) = 0 \) for \( S \in K\langle X, Y \rangle \) and a sufficiently large \( n \), we have that
\[ \Delta = \log(1 + \Theta) = \Theta - \frac{\Theta^2}{2} + \frac{\Theta^3}{3} - \cdots \]
and \( \Phi \) determines uniquely the lnd \( \Delta \). Hence \( \Delta(U) = 0, \Delta(V) = f(U) \).

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Another difference between the locally nilpotent derivations of \( K[x,y] \) and \( K\langle X,Y \rangle \) is that in the latter case they can be distinguished by their algebras of constants.

Theorem 3.3. Let \( \Delta_1 \) and \( \Delta_2 \) be two non-zero locally nilpotent derivations of \( K\langle X,Y \rangle \). Then \( \Delta_1 \) and \( \Delta_2 \) have the same algebras of constants if and only if \( \Delta_2 = \alpha \Delta_1 \) for a nonzero \( \alpha \in K \).

Proof. Changing the generators of \( K\langle X,Y \rangle \), by Proposition 3.2 we may assume that \( \Delta_1(X) = 0, \Delta_1(Y) = f(X) = F \) for some nonzero \( F = f(X) \in K\langle X,Y \rangle \). Since \( K\langle X,Y \rangle^{\Delta_1} = K\langle X,Y \rangle^{\Delta_2} \) we have that \( \Delta_2(X) = 0 \). By Lemma 3.1
\[ \Delta_2(T_1) = [\Delta_2(Y),X] + [Y,\Delta_2(X)] = [\Delta_2(Y),X] = 0. \]
Therefore \( \Delta_2(Y) = g(X) = G \). A direct computation gives that
\[ T_2 = YT_1F - FT_1Y \in K\langle X,Y \rangle^{\Delta_1}. \]
Hence \( \Delta_2(T_2) = GT_1F - FT_1G = g(X)T_1f(X) - f(X)T_1g(X) = 0 \) which implies that \( g(x) = \alpha f(x) \) for some \( \alpha \in K \). Therefore \( \Delta_2 = \alpha \Delta_1 \). Since \( \Delta_1, \Delta_2 \neq 0 \), we obtain that \( \alpha \neq 0 \).

4 Algebras of constants of derivations of \( K\langle X,Y \rangle \)

By Proposition 3.2, up to a change of the free generators of \( K\langle X,Y \rangle \) every nontrivial locally nilpotent derivation \( \Delta \) of \( K\langle X,Y \rangle \) is of the form
\[ \Delta(X) = 0, \quad \Delta(Y) = f(X), \]
where \( 0 \neq f(x) \in K[x] \). In the sequel we shall fix \( \deg(f) = m \geq 0 \) and \( \Delta \) as defined above.

Proposition 4.1. \( AK(K\langle X,Y \rangle) = K[T_1] \).

Proof. Let us consider derivations
\[ \delta_m: \quad \delta_m(X) = 0, \quad \delta_m(Y) = X^m. \]
Suppose \( \delta_m(P) = 0 \) for all \( m \). We may assume that \( P \) is homogeneous relative to \( X \) and \( Y \). Write \( P = XP_0 + YP_1 \), then
\[ 0 = \delta_m(P) = X\delta_m(P_0) + X^mP_1 + Y\delta_m(P_1). \]
Hence $\delta_m(P_1) = 0$ and we can assume by induction on $\deg_Y$ that $P_1$ belongs to the subalgebra $K\langle X, T_1 \rangle$ of $K\langle X, Y \rangle$ generated by $X$ and $T_1$ and write $P_1 = XP_{10} + T_1P_{11}$. If $P_{11} \neq 0$ then $X^mT_1P_{11}$ cannot be canceled by any monomial of $X\delta_m(P_0)$ if $m$ is sufficiently large. Hence $P_{11} = 0$ and $P_{10} \in K\langle X, T_1 \rangle$. Therefore

$$P = XP_0 + YXP_{10} = XP_0 + T_1P_{10} + XYP_{10} = X(P_0 + YP_{10}) + T_1P_{10}.$$  

Then $\delta_m(P_0 + YP_{10}) = 0$ because $T_1P_{10} \in K\langle X, T_1 \rangle$ and we can assume by induction on $\deg_X$ that $P_0 + YP_{10} \in K\langle X, T_1 \rangle$, i.e., $P \in K\langle X, T_1 \rangle$. Of course

$$\text{AK}(K\langle X, Y \rangle) \subseteq K\langle X, T_1 \rangle \cap K\langle Y, T_1 \rangle = K[T_1]$$

since we can switch $X$ and $Y$.

Consider the operator $\square$ on $K\langle X, Y \rangle$ defined in Section 2. We shall prove in this section that the algebra of constants of $\Delta$ is the minimal algebra $R_F$ which contains $K\langle X, T_1 \rangle$ and is closed under this operator. Since $\square \Delta = \Delta \square$ it is clear that $R_F \subseteq K\langle X, Y \rangle^\Delta$. It is worth observing that the kernel of $\square$ is $K[Y]$ if $\deg_X(F) = 0$ and 0 if $\deg_X(F) > 0$ and that $\deg(\square(A)) = \deg(A)$ (where deg is the degree function induced by $\Delta$) if $\deg_X(F) > 0$. We shall also denote $\square(A)$ by $\{A\}$. This bracketing is a bit unusual since $\square^n(A)$ will be recorded as $\{\ldots \{A\} \ldots\}$ with the same number $n$ of the left and right brackets and there can be more than two terms inside of a pair of brackets, but as in the ordinary bracketing in a configuration of three brackets like this $\{A_1\{A_2\}$ the first bracket cannot match the third bracket, it should be matched by a bracket $\}$ to the right of the third bracket and second and third brackets are matched.

**Theorem 4.2.** Let $L \in K\langle X, Y \rangle$. If $\Delta^n(L) = 0$ then $L$ belongs to the linear span $R_F^n$ of elements $A_1Y_1A_2Y_2 \cdots Y_1A_k$, where $k \leq n$ and each $A_i, 1 \leq i \leq k$, is a monomial from $R_F$, endowed with an arbitrary number of matching pairs of brackets $\{\}$.  

**Proof.** We consider two cases separately.

(a) $m = 0$ (we can assume that $\Delta(Y) = 1$). Consider the sequence of elements $T_1, \ldots, T_i, \ldots$ defined in Section 2 by $T_1 = YX - XY$, $T_{i+1} = \square^i(T_1)$. In this case $T_i = Y^iX$ and any element $S \in K\langle X, Y \rangle$ can be written as $S = \sum_{j=0}^{k} S_jY^j$ where $S_j \in K\langle X, T_1, \ldots, T_i, \ldots \rangle$. Since

$$\Delta(S) = \sum_{j=0}^{k} jS_jY^{j-1}, \Delta^n(S) = 0, \text{ and } \Delta^k(S) \neq 0 \text{ if } S_k \neq 0 \text{ it is clear that } k < n.$$  

(b) $m > 0$. Let us introduce a weight degree function on $K\langle X, Y \rangle$ by $w(X) = 1$, $w(Y) = m$. Then the space $V_N$ spanned by monomials of the weight not exceeding $N$ is mapped by the derivation into itself. We proceed by induction on $w(S)$. If $w(S)$ is sufficiently small, say does not exceed $m$, the claim is obvious. Assume that for the weight less than $N$ the claim is true.

Take an $L$ for which $w(L) = N$ and $L^{(k)} = 0$ (here and further on $L^{(k)}$ denotes $\Delta^k(L)$). We can assume that $L(X, 0) = 0$ and write

$$L = L_mF + \sum_{i=0}^{m-1} L_iYX^i.$$  

Then

$$L^{(k)} = k \sum_{i=0}^{m-1} L_i^{(k-1)}X^iF + \sum_{i=0}^{m-1} L_i^{(k)}YX^i = 0.$$
Hence \( L_i^{(k)} = 0 \) for \( i < m \) and
\[
\left( L_m^{(k)} + k \sum_{i=0}^{m-1} L_i X^i \right)^{(k-1)} = 0.
\]
Therefore \( \hat{L}^{(k)} = 0 \) for \( \hat{L} = L_m F + \sum_{i=0}^{m-1} L_i X^i Y \).

It is sufficient to check the claim for \( \hat{L} \) since \( L - \hat{L} = \sum_{i=0}^{m-1} L_i [Y, X^i] \) satisfies the claim by induction \((w(L_i) < N \text{ and } [Y, X^i] \in R_F)\).

Write \( \hat{L} = L_m F + H_0 Y \). Then \( H_0^{(k)} = 0 \) and \( (L_m + kH_0)^{(k-1)} = 0 \). Hence \( L_m^{(k+1)} = 0 \) and \( \hat{L}^{(k)} = 0 \) for \( \hat{L} = kL_m F - L_m' Y \). It is sufficient to check the claim for \( \hat{L} \) since \( k\hat{L} - \hat{L} = (kH_0 + L_m') Y \) and \( kH_0 + L_m' \) satisfy the claim by induction.

Since \( L_m^{(k+1)} = 0 \) and \( w(L_m) < N \) we can write
\[
L_m = \sum_{\mu} \alpha_{j\mu} Y \alpha_{j_1} Y \cdots Y \alpha_{j_k} + S,
\]
where \( \alpha_{j_\mu} \in R_F \), the summands are endowed with brackets \( \{ \} \), and \( S \) is the sum of terms in which \( Y \) appears less than \( k \) times. We can omit \( S \) since \( kSF - S'Y \in R_F^q \).

Take one of the summands \( \mu_\lambda \) and consider \( \nu_\lambda = k\mu_\lambda F - \mu_\lambda' Y \). Since \( \Delta \) and \( \square \) commute
\[
\nu_\lambda = k\mu_\lambda F - \sum_{i=1}^k \alpha_{j_\mu} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} F \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y,
\]
where each term \( \alpha_{j_\mu} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} F \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y \) has the same bracketing as \( \mu = \mu_\lambda \).

Consider \( P_i = \mu F - \alpha_{j_\mu} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} F \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y \). It is clear that \( P_i^{(k)} = 0 \) so we should check that \( P_i \) can be recorded as a sum of terms containing only \( k-1 \) entries of \( Y \) (we do not count \( Y \)’s appearing in \( \square \)).

Write \( \mu = V_1 Y U_1 \) where \( Y \) is the one which is replaced by \( F \) in \( P_i \) and introduce two operations:
\[
\triangledown_{r,U}(V_1 Y U_1) = V_1 Y U_1 F - V_1 F U_1 Y \quad \text{and} \quad \triangledown_{l,U}(V_1 Y U_1) = F U_1 Y U_1 - Y U_1 F U_1.
\]
We shall write \( \triangledown_r \) and \( \triangledown_l \) when \( U = 1 \), so \( P_i = \triangledown_r(V_1 Y U_1) \).

The operator \( \square \) is defined on all algebra while the operations \( \triangledown_{r,U}, \triangledown_{l,U} \) are defined only on specially recorded elements and their extension does not seem to be canonical.

Assume that \( V_1 Y U_1 = \square(V_2 Y U_2) \). Then we need to simplify \( \triangledown_r(\square(V_2 Y U_2)) \). In order to do this let us compute \([\triangledown_r, \square](V_2 Y U_2)\).

This is a bit tedious but not difficult:
\[
\triangledown_r(\square(V_2 Y U_2)) = [Y (V_2 Y U_2) F - F (V_2 Y U_2) Y] F - [Y (V_2 F U_2) F - F (V_2 F U_2) Y] Y,
\]
\[
\square(\triangledown_r(V_2 Y U_2)) = Y ((V_2 Y U_2) F - (V_2 F U_2) Y] F - F [(V_2 Y U_2) F - (V_2 F U_2) Y] Y.
\]
Hence
\[
[\triangledown_r, \square](V_2 Y U_2) = -F (V_2 Y U_2) Y F + F (V_2 Y U_2) F Y - Y (V_2 F U_2) F Y + Y (V_2 F U_2) Y F
= [Y (V_2 F U_2) F - (V_2 Y U_2)] [Y, F] = -\triangledown_l (V_2 Y U_2) [Y, F].
\]
Therefore
\[ \nabla_r(\square(V_2YU_2)) = \square(\nabla_r(V_2YU_2)) - \nabla_l(V_2YU_2)[Y,F]. \]

Since \( w(V_2YU_2) < w(V_1YU_1) \) we can apply induction.

Assume now that either \( \mu = V \square (V_1YU_1) \) or \( \mu = \square(V_1YU_1)U \). If \( \mu = V \square (V_1YU_1) \) then
\[ \nabla_r(V \square (V_1YU_1)) = V \nabla_r(\square(V_1YU_1)). \]
If \( \mu = \square(V_1YU_1)U \) then \( \nabla_r(\mu) = \nabla_r,U(\square(V_1YU_1)). \)

Now,
\[ [\nabla_r,U,\square](V_1YU_1) = \square[\nabla_r(V_1YU_1)U - \nabla_r,U'(V_1YU_1)] - \nabla_l(V_1YU_1) \square (U) \]

and induction can be applied in these cases as well.

The last case is when \( Y \) does not belong to a bracketed subword. Then \( \mu = V_1YU_1 \) and \( \nabla_r(\mu) = V_1 \square (U_1) \).

The proof is completed.

**Corollary 4.3.** The algebra of constants \( K\langle X,Y \rangle^\Delta \) coincides with the algebra \( R_F \).

**Proof.** As we already mentioned \( R_F \subseteq K\langle X,Y \rangle ^\Delta \) and it is sufficient to show that if \( \Delta(L) = 0 \) for \( L \in K\langle X,Y \rangle \), then \( L \) belongs to \( R_F \). But this is a direct consequence of the case \( n = 1 \) in Theorem 4.2.

Now we are able to establish one of the main properties of the algebra of constants \( K\langle X,Y \rangle ^\Delta \).

**Theorem 4.4.** The algebra of constants \( K\langle X,Y \rangle ^\Delta \) is a free algebra.

**Proof.** By Corollary 4.3 we may work with the algebra \( R_F \) instead with \( K\langle X,Y \rangle ^\Delta \). When \( m = 0 \) we have seen (in the proof of Theorem 4.2) that \( R_1 \) is generated by \( X,T_1,T_2, \ldots \). Since \( T_i = Y^iX \) these elements freely generate \( R_1 \). For \( m > 0 \) producing a generating set is more involved but the freeness can be deduced from a theorem of de W. Jooste [5]. It follows from his theorem that the kernel of the derivation \( \Delta(X) = 0, \Delta(Y) = X^m \) is a free algebra. For this derivation any \( w \)-homogeneous component (recall that \( w(X) = 1, w(Y) = m \)) of a constant is also a constant, hence there is a homogeneous free generating set \( F_1,F_2, \ldots \) of \( R_{X^m} \). There is a bijection \( \pi \) between the elements of \( R_{X^m} \) and \( R_F \) obtained by replacing \( X^m \) in each bracket of an element of \( R_{X^m} \) by \( F = f(X) \). Therefore \( \pi(F_1), \pi(F_2), \ldots \) is a generating set of \( R_F \) which is free since \( w(\pi(F_1) - F_1) < w(F_1) \).

It remains to produce a homogeneous set freely generating \( R_{X^m} \).

**Lemma 4.5.** The algebra \( R_{X^m} \) is generated by \( X \) and bracketed words
\[ T_1^{i_1}X^{j_1} \cdots X^{j_{k-1}}T_1^{i_k}, \]
where \( i_1, i_2, \ldots, i_k > 0, j_1, j_2, \ldots, j_{k-1} < m \), and where the right brackets \{ \} are preceded by \( T_1 \) (i.e., there are no configurations \( X \}).

**Proof.** Denote by \( D \) the subalgebra of \( R_{X^m} \) which is generated by words described in the lemma. Any element of \( R_{X^m} \) can be written as a linear combination of bracketed words \( \mu = X^nT_1^{i_1}X^{j_1} \cdots T_1^{i_k}X^{j_k} \). We shall find an element \( B \in D \) with the same leading monomial \( \overline{B} \) as the leading monomial \( \overline{\mu} \) of \( \mu \) in the lexicographic order defined by \( Y \gg X > 1 \). Clearly this is sufficient for the proof of the lemma.

To find the leading monomial \( \overline{\mu} \) of a bracketed word \( \mu \) we should replace all left brackets \{ by \( Y \) and all right brackets \} by \( X^m \).
If $\varpi$ starts with $X$ then $\mu = X\mu_1$ (as an element of $K\langle X,Y \rangle$) where $\mu_1 \in R_{X^m}$ and we can use induction on weight to claim that there is an element $B_1 \in D$ such that $\overline{\varpi} = \overline{B_1}$ (or even that $\mu_1 \in B$).

If $\mu$ cannot be written as $\sqcup(\nu)$ then $\mu = (\mu_1)(\mu_2)$ where brackets $(\ )$ separate elements of $R_{X^m}$ and $w(\mu_1) < w(\mu)$. Hence we can use induction to claim that $\overline{\varpi} = \overline{B_1}, \overline{\varpi_2} = \overline{B_2}$ where $B_1 \in D$.

If $\mu = \sqcup(\nu)$ then $w(\mu) = w(\nu) + 2m$ and we may assume that $\varpi = \varpi_1$ where $B \in D$. Since $B \in D$ we can write $B = (X^{j_0})(V_1)(X^{j_1}) \cdots (V_k)(X^{j_k})$ where $V_i \in D$ and $(X^j) = X^j$ and $\varpi = YX^{j_0}(V_1)(X^{j_1}) \cdots (V_k)(X^{j_k} + m)$. Inasmuch as $V_i \in D$ we may assume that the first and the last letters in all $V_i$ (as bracketed words) are $T_1$.

If $j_0 > 0$ then $T_1(\prod_{i=1}^{j_0-1}(V_i)(X^{j_i}) \cdots (V_k)(X^{j_k} + m)) = \overline{\varpi}$.

If $j_0 = 0$, $j_s \geq m$ where $s$ is the smallest possible then

$$\left\{(V_1)(X^{j_1}) \cdots (V_s)\right\}(X^{j_s-m}) \cdots (V_k)(X^{j_k+m}) = \overline{\varpi}.$$  

If all $j_s < m$ then $\mu \in D$. \hfill \blacksquare

**Theorem 4.6.** The algebra $D = R_{X^m}$, $m > 0$, is freely generated by $X$, $T_1$ and words $\sqcup(T_1^1X^{j_1} \cdots X^{j_{k-1}}T_1^{j_k})$, where $i_1, i_2, \ldots, i_k > 0$, $j_1, j_2, \ldots, j_{k-1} < m$, and $T_1^1X^{j_1} \cdots X^{j_{k-1}}T_1^{j_k}$ are bracketed words described in Lemma 4.5 (we shall refer to these words as permissible and to $T_1^1X^{j_1} \cdots X^{j_{k-1}}T_1^{j_k}$ without brackets as the root of the corresponding word).

**Proof.** It is sufficient to check that the leading monomial $\overline{\varpi}$ of a permissible word cannot be presented as a product of the leading monomials of permissible words of a smaller weight.

To check this consider the leading monomial $\overline{\varpi} = Y^{b_1} \cdots X^{a_{s-1}}Y^{b_s}X^{a_s}$ of a permissible $\mu$. (Observe that $b_1 > 0$, $a_s = m + 1$ since $\sqcup(\nu) = Y \nu X^m$.)

The number of $T_1$ in the bracketed representation of $\mu \in D$ must be equal to $s$ since in the leading monomial of any word from $D$ a subword $YX$ can appear only as $T_1$. So the number of brackets $\{ \mu \}$ in $\mu$ is $\deg_Y(\overline{\varpi}) - s$. Of course the number of brackets $\{ \mu \}$ is the same.

A subword $Y^{b_1}X^{a_i}$ can appear in $\overline{\varpi}$ only as $\{ \ldots \{ T_1 \} \ldots \} X^{d_i}$ where the number of left brackets is $b_i - 1$, the number of right brackets is the integral part of $a_i - 1$ and $0 \leq d_i < m$ is the remainder of the division of $a_i - 1$ by $m$. Therefore the root and the bracketing of $\mu$ are uniquely determined by $\overline{\varpi}$. But we would have two different bracketings if $\overline{\varpi} = (\overline{\varpi_1})(\overline{\varpi_2})$. This finishes a proof of the theorem. \hfill \blacksquare

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