Smooth Beginning of the Universe

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Abstract

The breaking down of the equivalence principle, when discussed in the context of Sikorski’s differential space theory, leads to the definition of the so-called differentially singular boundary (d-boundary) and to the concept of differential space with singularity associated with a given space-time differential manifold. This enables us to define the time orientability, the beginning of the cosmological time and the smooth evolution for the flat FRW world model with the initial singularity. The simplest smoothly evolved models are studied. It is shown, that the cosmological matter causing such an evolution can be of three different types. One of them is the fluid with dark energy properties, the second the fluid with attraction properties, and the third a mixture of the other two. Among all investigated smoothly evolved solutions, models qualitatively consistent with the observational data of type Ia supernovae have been found.

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1 Introduction

It is generally believed that the Universe had a beginning and everything indicates that it indeed had the beginning. According to contemporary ideas the Planck era is the beginning of the Universe. During the flow of cosmic time the known Universe

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emerges from the Planck era. Certainly it is not an immediate process and one can imagine that individual space-time structures emerge from the Planck era gradually. One can expect that the simplest structures such as a set structure or a topology on this set can emerge first. The appearance of the manifold structure is the key event of the process. Then the equivalence principle, in its non metric form, appears and makes possible a creation of ”higher” structures such as, for example, the Lorentzian structure. From this perspective it is interesting to ask how the space-time geometry is ”nested” in theories more general than the differential geometry since the emerging process of gravitation has to be associated with a theory which is more general than the geometry of space-time manifold.

In this paper the Sikorski’s differential spaces theory (see Appendix A) is applied to the discussion of the breaking down process of the equivalence principle. Within this theory the non-metric version of the equivalence principle is one of the axioms of the space-time differential manifold (d-manifold) definition. Let us consider points at which this axiom is not satisfied. The set of this points forms the so-called differentially singular boundary (d-boundary). This issue is discussed in Section 2. The construction of this type of boundary is also described.

In Section 3, the construction from Section 2 is used for building the differential space (d-space) with d-boundary for the flat FRW model. The issue of the prolongation of cosmic time and time-orientability to the d-boundary for this model is discussed in Section 4. In Section 5, we describe the concept of the smooth evolution of this model starting from the initial singularity. We find the simplest smoothly evolved flat FRW models in Section 6. Some of them (see Section 7) have the kinematical properties which are qualitatively agreeing with the recent cosmological observations [1, 2]. Namely, these models begin their evolution with the decelerate expansion which changes into the accelerated one.

The discussion of properties of the cosmological fluid which causes that type of smooth evolution for the simplest cosmological models is carried out in Section 7. Behaviour of any smoothly evolving flat model in a neighbourhood of the initial singularity is investigated in Section 8. Finally, in Section 9, we summarize the main results of this paper.

2 Sikorski’s differential spaces and singularities

Space-time is a 4-dimensional, Lorentzian and time-orientable d-manifold of the class $C^\infty$ [3]. This definition is a mathematical synthesis of what is known about gravitational field. It implicitly includes the equivalence principle [4, 5, 6].
The equivalence principle, in its non-metric version, is implicit in the axiom stating that space-time is a d-manifold which means that it is locally homeomorphic to an open set in \( \mathbb{R}^n \), where \( n = 4 \). Differential properties of these homeomorphisms (maps) are determined by the atlas axioms through the assumption, that the composition of two maps \( \varphi_1 \circ \varphi_2^{-1} \), as a mapping between open subsets of \( \mathbb{R}^n \), is a diffeomorphism of the class \( C^k \), \( k \in \mathbb{N} \). In the present paper it is assumed that these diffeomorphisms are smooth. It is worth to notice, that the non-metric version of the equivalence principle is encoded into two levels of the classical d-manifold definition, namely in the assumption of the existence of local homeomorphisms to \( \mathbb{R}^n \) and in the axioms of atlases.

One can look at space-time, or more generally at a d-manifold, from the viewpoint of theories that are more general than classical differential geometry. In these theories d-manifolds are special examples of more general objects. Examples of theories of this type are: the theory of sub-cartesian spaces by Aronszajn and Marshall, the theory of Mostow’s spaces or Chen’s theory. In the present paper we study the initial singularity problem and the problem of the beginning of cosmological time with help of Sikorski’s differential spaces (d-spaces for brevity). In this theory, the generalization level does not lead to an excessive abstraction and therefore the d-space’s theory may have applications in physics.

Generally speaking, d-space is a pair \((M, \mathcal{M})\), where \( M \) is any set, and \( \mathcal{M} \) is a family of functions \( \varphi: M \rightarrow \mathbb{R} \) (see Appendix A). This family satisfies the following conditions: a) \( \mathcal{M} \) is closed with respect to superposition with smooth functions from \( C^\infty(\mathbb{R}^n, \mathbb{R}) \), and b) \( \mathcal{M} \) is closed with respect to localization. The precise sense of these notions is not important at the moment. The d-space \((M, \mathcal{M})\) is also a topological space \((M, \tau_M)\), where \( \tau_M \) denotes the induced topology on \( M \) given by the family \( \mathcal{M} \). The topology \( \tau_M \) is the weakest topology in which all functions from \( \mathcal{M} \) are continuous. With in the theory of d-spaces the d-manifold definition assumes the form

**Definition 2.1** Non empty d-space \((M, \mathcal{M})\) is n-dimensional d-manifold, if the following condition is satisfied

\[(\star) \text{ for every } p \in M, \text{ there are neighbourhoods } U_p \in \tau_M \text{ and } O \in \tau_{\mathbb{R}^n}, \text{ such that d-subspaces } (U_p, \mathcal{M}|_{U_p}) \text{ and } (O, \mathcal{E}^{(n)}|_O) \text{ of d-spaces } (M, \mathcal{M}) \text{ and } (\mathbb{R}^n, \mathcal{E}^{(n)}) \text{ respectively are diffeomorphic, where } \mathcal{E}^{(n)} := C^\infty(\mathbb{R}^n, \mathbb{R}).\]

Briefly speaking, d-space \((M, \mathcal{M})\) is a n-dimensional d-manifold if it is locally diffeomorphic to open subspaces of \( \mathbb{R}^n \) which are treated as d-spaces (see Definitions A.8
Diffeomorphisms appearing in condition (⋆) are generalized maps. Unlike in the classical definition, these maps are automatically diffeomorphisms. Definition 2.1 is equivalent to the classical d-manifold definition [12].

In the d-spaces theory, the non-metric version of the equivalence principle is “localized” in the single axiom (⋆). This enables us to give the following definition of space-time

**Definition 2.2** A space-time is a non empty d-space \((M, \mathcal{M})\) such that

a) \((M, \mathcal{M})\) satisfies the non-metric equivalence principle as expressed by the condition (⋆) for \(n=4\),

b) a Lorentzian metric form \(g\) is defined on \(M\),

c) Lorentzian d-manifold \((M, \mathcal{M}, g)\) is time orientable.

If the non-metric equivalence principle in the classical definition of space-time is thrown out, the time-orientability, the Lorentzian metric structure, the classical differential structure and the topological manifold structure are automatically destroyed. Only the topological space structure survives.

In the case space-time Definition 2.2 the situation is different. Throwing out the non-metric equivalence principle in the form of axiom (⋆) leaves a rich and workable structure on which one can define generalized counterparts of classical notions such as orientability, Riemannian and pseudo-Riemannian metrics, tensors, etc [12, 13, 14, 15, 16, 17, 18, 19, 20].

Looking at space-time manifolds from the d-spaces perspective, one can imagine a situation when a background of our considerations is a sufficiently “broad” d-space \((M, \mathcal{M})\) which is not a 4-dimensional d-manifold. Let us additionally suppose that in \((M, \mathcal{M})\) there exists a subset \(B \subset M\) having a structure of a 4-dimensional d-manifold satisfying the condition (⋆) for \(n = 4\), regarded as a d-subspace of \((M, \mathcal{M})\). Naturally, at each point \(p \in B\) the non-metric equivalence principle is satisfied. The points not belonging to \(B\) can considered as singular points because at these points the four-dimensional non-metric equivalence principle is violated. Among such singular points in \(M\) the most interesting are accumulation points of the set \(B\) in the topological space \((M, \tau_M)\). One can call a set of all singular accumulation points of \(B\) the singular boundary of \((B, \mathcal{M}_B)\) space-time d-manifold. Singular points in \(M\) not belonging to the singular boundary can be excluded from considerations as unattainable from the interior \(B\) of the space-time d-manifold.

**Definition 2.3** Let \((M, \mathcal{M})\) and \((M_0, \mathcal{M}_0)\) be Sikorski’s d-spaces, such that
a) \((M_0, \mathcal{M}_0)\) is a \(n\)-dimensional \(d\)-manifold,
b) \((M, \mathcal{M})\) is not a \(n\)-dimensional \(d\)-manifold,
c) \(M_0 \subset M\) and \(M_0\) is a dense set in the topological space \((M, \tau_M)\), 
d) \((M_0, \mathcal{M}_0)\) is a \(d\)-subspace of the \(d\)-space \((M, \mathcal{M})\).

The set \(\partial_d M_0 := M - M_0\) is said to be differentially-singular boundary (\(d\)-boundary for brevity) of \(d\)-manifold \((M_0, \mathcal{M}_0)\), if for every \(p \in \partial_d M_0\) and for every neighbourhood \(U_p \in \tau_M\), the \(d\)-subspace \((U_p, \mathcal{M}_{U_p})\) is not diffeomorphic to any \(d\)-subspace \((V, C^\infty(\mathbb{R}^n)_{V})\) of the \(d\)-space \((\mathbb{R}^n, C^\infty(\mathbb{R}^n))\), where \(V \in \tau_{\mathbb{R}^n}\). Then, the \(d\)-space \((M, \mathcal{M})\) is said to be \(d\)-space with differentially singular boundary (or \(d\)-space with \(d\)-boundary for brevity) associated with the \(d\)-manifold \((M_0, \mathcal{M}_0)\).

In the following we shall restrict our considerations to the case of Definition 2.3 for \(n = 4\). The \(d\)-boundary definition for a space-time’s \(d\)-space \((M_0, \mathcal{M}_0)\) depends on the choice of the \(d\)-space \((M, \mathcal{M})\). Therefore, a reasonable method of the \(d\)-space \((M, \mathcal{M})\) construction is necessary. Fortunately, the idea described above suggests such a construction. Namely, one chooses a sufficiently ”broad” \(d\)-space ”well” surrounding the whole investigated space-time \(M_0\) and then one carries out the process of determination of all accumulation points for \(M_0\). These points form the \(d\)-boundary \(\partial_d M_0\) of space-time. Next, one defines the set \(M\) as \(M := M_0 \cup \partial_d M_0\). By treating \(M\) as a \(d\)-subspace of the sufficiently ”broad” \(d\)-space, the \(d\)-space with the \(d\)-boundary \((M, \mathcal{M})\) is uniquely determined. One can easily check that the pair of \(d\)-spaces \((M, \mathcal{M})\) and \((M_0, \mathcal{M}_0)\), where \((M, \mathcal{M})\) is determined by above described method, satisfies Definition 2.3.

The choice of a sufficiently ”broad” \(d\)-space ”well” surrounded the whole investigated space-time is almost obvious. As it is well known [21], every space-time can be globally and isometrically embedded in a sufficiently dimensional pseudo-Euclidean space \(E^{p,q}\), where \(p, q\) depends on space-time model. The space \(E^{p,q}\) is also the \(d\)-space \((\mathbb{R}^{p+q}, C^\infty(\mathbb{R}^{p+q}, \mathbb{R}))\) which ”well” surrounds the studied space-time \(d\)-manifold \((M_0, \mathcal{M}_0)\).

Such a choice of the surrounding \(d\)-space is well motivated since all causal properties of the studied space-time are taken into account, even if we do not refer to them directly. This is because of the isometricity of the embedding. Furthermore, the concrete form of the isometrical embedding provides us with a practical method of constructing the \(d\)-structure \(\mathcal{M}\) for the the \(d\)-space \((M, \mathcal{M})\). Namely, \(\mathcal{M} = C^\infty(\mathbb{R}^{p+q}, \mathbb{R})_M\).
To test this method of constructing \((M, \mathcal{M})\) we shall apply it to the flat FRW cosmological model.

3 A differential space for the flat FRW model with singularity

Let us consider the flat FRW model with the metric

\[
g = -dt^2 + a_0^2(t)(dx^2 + dy^2 + dz^2),
\]

where \((t, x, y, z) \in W^0 := D_a^0 \times \mathbb{R}^3\) and \(D_a^0\) is a domain of the scale factor. The scale factor \(a_0: D_a^0 \ni t \rightarrow a_0(t) \in \mathbb{R}^+, a_0 \in C^\infty(D_a^0)\), is an even real function of the cosmological time \(t\). A domain of the scale factor \(D_a^0 \subset \mathbb{R}^+\), is an open and connected set. For convenience, let us assume that \(t = 0\) is an accumulation point of the set \(D_a^0\) and \(0 \notin D_a^0\). In addition, the model has an initial singularity at \(t = 0\) i.e. \(\lim_{t \rightarrow 0^+} a_0(t) = 0\). In the next parts of the paper, the symbol \(a\) is reserved for the following function:

\[
a: D_a \ni t \rightarrow a(t) = a_0(t) \text{ for } t \in D_a, a(t) = 0 \text{ for } t = 0 \text{ where } D_a := D_a^0 \cup \{0\}.
\]

A construction of a d-space with d-boundary for the flat FRW model pass in analogical way to the construction of a differential space for the string-generated space time with a conical singularity described in [22].

Every 4-dimensional Lorentzian manifold can be isometrically embedded in a pseudo-Euclidean space [21]. In particular, the manifold \((W^0, g)\) of model [1] can be isometrically embedded in \((\mathbb{R}^5, \eta^{(5)})\) by means of the following mapping

\[
F^0: W^0 \longrightarrow F^0(W^0) \subset \mathbb{R}^5
\]

\[
F_1^0(t, x, y, z) = \frac{1}{2}a_0(t)(x^2 + y^2 + z^2 + 1) + \frac{1}{2} \int_0^t \frac{d\tau}{a_0(\tau)},
F_2^0(t, x, y, z) = \frac{1}{2}a_0(t)(x^2 + y^2 + z^2 - 1) + \frac{1}{2} \int_0^t \frac{d\tau}{a_0(\tau)},
F_3^0(t, x, y, z) = a_0(t) x, F_4^0(t, x, y, z) = a_0(t) y, F_5^0(t, x, y, z) = a_0(t) z,
\]

where \(\eta^{(5)} = \text{diag}(-1, 1, 1, 1, 1)\).

As every manifold, space-time \((W^0, g)\) is also a differential space \((W^0, \mathcal{W}^0)\), where the differential structure \(\mathcal{W}^0 := \mathcal{E}^{(4)}_{W^0}\) is a family of local \(\mathcal{E}^{(4)}\)-functions on \(W^0 \subset \mathbb{R}^4\), where \(\mathcal{E}^{(4)} = C^\infty(\mathbb{R}^4)\). On the other hand, the set \(F^0(W^0)\) can be equipped, in a natural way, with a differential structure treated as a differential subspace of the \((\mathbb{R}^5, \mathcal{E}^{(5)})\), where \(\mathcal{E}^{(5)} = C^\infty(\mathbb{R}^5)\). Then the family \(\mathcal{E}^{(5)}_{F^0(W^0)}\) of local \(\mathcal{E}^{(5)}\)-functions
on $F^0(W^0)$ is a differential structure and the pair $(F^0(W^0), \mathcal{E}^{(5)}_{F^0(W^0)})$ is a differential space. In addition, if the integral $\int_0^t d\tau/\dot{a}(\tau)$ is convergent for every $t \in D_0$, then the mapping $F^0$ is a diffeomorphism of the differential spaces $(F^0(W^0), \mathcal{E}^{(5)}_{F^0(W^0)})$ and $(W^0, \mathcal{W}^0)$ \cite{23, 22}.

The process of attaching the initial singularity depends on the completion of the set $F^0(W^0)$ by means of points from the surrounding space $\mathbb{R}^5$ in the way controlled by the isometry $F^0$. It enables us to define the d-space with d-boundary (with the initial singularity) for the flat FRW model.

Let the mapping $F: W \to \mathbb{R}^5$ denotes a prolongation of $F^0$ to the set $W := D_a \times \mathbb{R}^3$. The values of $F$ are given by formul\ae\ \cite{2} changing the symbol $F^0$ onto $F$. Then for every $x, y, z \in \mathbb{R}$ the value of $F$ at the initial moment $t = 0$ is $F(0, x, y, z) = (0, 0, 0, 0, 0)$, since $a(0) = 0$ by assumption. Therefore, the cosmological initial singularity (d-boundary) distinguished by the embedding procedure is represented by the single point: $\partial_d F^0(W^0) = \{(0, 0, 0, 0, 0)\} \subset \mathbb{R}^5$.

The pseudo-Euclidean space $(\mathbb{R}^5, \eta^{(5)})$ is a differential space $(\mathbb{R}^5, \mathcal{E}^{(5)})$. The differential structure $\mathcal{E}^{(5)}$ of this space is finitely generated. The projections on the axes of the Cartesian system $\pi_i: \mathbb{R}^5 \to \mathbb{R}$, $\pi_i(z_1, z_2, ..., z_5) = z_i$, $i = 1, 2, ..., 5$, are generators of this structure and therefore: $\mathcal{E}^{(5)} = \text{Gen}(\pi_1, \pi_2, ..., \pi_5)$. As is well known \cite{15, 22}, every subset $A$ of a support $M$ of a differential space $(M, \mathcal{C})$ is a differential subspace with the following differential structure: $\text{Gen}(\mathcal{C}|_A)$. For finitely generated differential spaces $(M, \mathcal{C})$, every differential subspace with a support $A \subset M$ is also finitely generated. Generators of the differential structure on $A$ are generators of the differential structure $\mathcal{C}$ restricted to $A$.

The differential space $(F(W), \mathcal{E}^{(5)}_{F(W)})$ is a differential subspace of the differential space $(\mathbb{R}^5, \mathcal{E}^{(5)})$ and represents the d-space with d-boundary for model \cite{1}. Since $\mathcal{E}^{(5)}$ is finitely generated, the d-structure $\mathcal{E}^{(5)}_{F(W)}$, induced on $F(W)$, is also finitely generated and

$$\mathcal{E}^{(5)}_{F(W)} = \text{Gen}(\pi_1|_{F(W)}, \pi_2|_{F(W)}, ..., \pi_5|_{F(W)}).$$

The d-space $(F(W), \mathcal{E}^{(5)}_{F(W)})$ is not convenient for further discussion because it depends on the embedding procedure. Let us define a d-space $(W, \mathcal{W})$ diffeomorphic to $(F(W), \mathcal{E}^{(5)}_{F(W)})$ which will enable us to apply the Sikorski's geometry in a form similar to the standard differential geometry.

Let us consider an auxiliary d-space $(W, \mathcal{W})$, $\mathcal{W} = \text{Gen}(\beta_1, \beta_2, ..., \beta_5)$, where

$$\beta_i: W \to \mathbb{R}, \quad \beta_i(t, x, y, z) = \pi_i \circ F(t, x, y, z), \quad i = 1, 2, ..., 5.$$
This space is not diffeomorphic to the d-space with d-boundary \((F(W), \mathcal{E}^{(5)}_{F(W)})\), since the function \(F: W \to \mathbb{R}^5\) is not one to one. Additionally, the generators \(\beta_i\) do not distinguish the following points \(p \in \partial W := W - W^0\). Therefore, the d-space \((W, \mathcal{W})\) is not a Hausdorff space. With the help of this space one can build a d-space with d-boundary \((\bar{W}, \bar{\mathcal{W}})\) diffeomorphic to \((F(W), \mathcal{E}^{(5)}_{F(W)})\).

Let \(\varrho_H\) be the following equivalence relation
\[
\forall p, q : \quad p \varrho_H q \iff \forall \beta \in \mathcal{W} : \beta(p) = \beta(q).
\]
The quotient space \(\bar{W} = W/\varrho_H\) can be equipped with a d-structure \(\bar{\mathcal{W}} := W/\varrho_H\) coinduced from \(W\)
\[
\bar{\mathcal{W}} := \text{Gen}(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_5),
\]
where
\[
\bar{\beta}_i: \bar{W} \to \mathbb{R}, \quad \bar{\beta}_i([p]) := \beta_i(p), \quad i = 1, 2, ..., 5.
\]
The symbol \([p]\) denotes the equivalence class of a point \(p \in W\) with respect to \(\varrho_H\).

The pair \((\bar{W}, \bar{\mathcal{W}})\) is a d-space \([24, 23, 22]\).

Let us define the following mapping
\[
\bar{F}: \bar{W} \to F(W), \quad \bar{F}([p]) := F(p).
\]

**Theorem 3.1** The d-space \((\bar{W}, \bar{\mathcal{W}})\) is diffeomorphic to the d-space with d-boundary \((F(W), \mathcal{E}^{(5)}_{F(W)})\). \(\square\)

**Proof.** The mapping \(\bar{F}: \bar{W} \to F(W)\) is a bijection. In addition, \(\bar{W}\) is by construction a Hausdorff topological space with the topology given by the generators \(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_5\) \([12, 13, ?]\). The mapping
\[
\bar{F}([p]) = (\bar{\beta}_1(p), \bar{\beta}_2(p), ..., \bar{\beta}_5(p)) = (\bar{\beta}_1([p]), \bar{\beta}_2([p]), ..., \bar{\beta}_5([p]))
\]
is a diffeomorphism of the d-space \((\bar{W}, \bar{\mathcal{W}})\) onto its image \((\bar{F}(\bar{W}), \mathcal{E}^{(5)}_{\bar{F}(\bar{W})}) = (F(W), \mathcal{E}^{(5)}_{F(W)})\) \([25]\).

According to Definition \([2, 3]\) \((\bar{W}, \bar{\mathcal{W}})\) is the d-space with d-boundary for the flat FRW d-manifold \((W^0, \mathcal{W}^0)\), where the d-boundary is represented by the set \(\partial_d W^0 := \bar{W} - W^0\).

The d-space with d-boundary \((\bar{W}, \bar{\mathcal{W}})\) has been constructed with help of the d-space \((W, \mathcal{W})\) and the relation \(\varrho_H\). Generally speaking, d-spaces of the type of \((M, \mathcal{C})\) and \((M/\varrho_H, \mathcal{C}/\varrho_H)\) have a lot of common features because of the isomorphism of the algebras \(\mathcal{C}\ i\ \mathcal{C}/\varrho_H\). In particular, moduli of smooth vector fields \(\textbf{X}(M)\) and \(\textbf{X}(M/\varrho_H)\)
are isomorphic \cite{23, 22}. This property will, in the next parts of the paper, enable us to work with the help of the more convenient d-space \((\overline{W}, \overline{W})\) instead of \((\overline{W}, \overline{W})\).

The d-structure \(\mathcal{W}\) of the d-space \((\overline{W}, \overline{W})\) is finitely generated by means of functions \(\beta_i, \ i = 1, 2, ..., 5\). However, in the next parts of the paper, we use a different, but equivalent, system of generators

\[
\alpha_1 := \beta_1 - \beta_2, \quad \alpha_2 := \beta_1 + \beta_2, \quad \alpha_i := \beta_i, \quad i = 3, 4, 5. \tag{3}
\]

Then the d-structure \(\mathcal{W}\) has the form

\[
\mathcal{W} = \text{Gen}(\alpha_1, \alpha_2, ..., \alpha_5).
\]

4 Time orientability

The flat FRW model is a time orientable Lorentzian manifold \(M\). By definition, there is a timelike directional field generated by a nowhere vanishing timelike vector field \(X\). If \(X\) generates the directional field then the field \(\lambda X\) generates it also, where \(\lambda\) is a nowhere vanishing scalar field on \(M\). The field \(X\) caries a part of information included in the casual structure of \(M\), which enables us to define the direction of the stream of time and the succession of events \cite{20}.

The manifold structure and the casual structures of space-time are broken down at the initial singularity. In the hierarchy of space-time structures the Sikorski’s d-structure is placed below the casual structure \cite{15}. Therefore, the d-space with d-boundary \((\overline{W}, \overline{W})\) of the flat FRW model is timeless independently of the fact that one of coordinates is called time and the moment \(t = 0\) is named the beginning of time. In this situation one cannot say that the cosmological singularity (d-boundary) is an initial or final state of the cosmic evolution. It is necessary to introduce a notion which would be a substitute of time orientability.

Let \((W^0, W^0)\) and \((W, W)\) be the pair of d-spaces described in the section \cite{3}. For convenience we can consider the d-space \((W, W)\) instead of the d-space with d-boundary \((\overline{W}, \overline{W})\) according to the remark after Theorem \cite{3, 11}. In this representation the set of not Hausdorff separated points \(\partial W := W - W^0\) is a counterpart of the d-boundary \(\partial_d W^0\) for the flat FRW d-manifold.

Let in addition, \(X^0 : W^0 \rightarrow TW^0\) be a timelike and smooth vector field without critical points, tangent to the manifold \((W^0, W^0)\), fixing the time-orientability on the manifold \((W^0, g)\).

**Definition 4.1** The d-space \((W, W)\) is said to be time oriented by means of a vector field \(X\) if
a) there is a nonzero vector field \( X: W \to TW \) tangent to \((W, W)\) given by the formula

\[
\forall \alpha \in W : \quad X(p)(\alpha) := \begin{cases} 
X^0(p)(\alpha|_{W^0}) & \text{for } p \in W^0 \\
\lim_{q \to p} X^0(q)(\alpha|_{W^0}) & \text{for } p \in \partial W, \quad q \in W^0
\end{cases}
\]

b) and there is a function \( \lambda \in W, \lambda(q) > 0 \) (or \( \lambda(q) < 0 \)) for \( q \in W^0 \subset W \) such that the vector field \( V := \lambda X \) is smooth on \((W, W)\).

A coordinate defined by means of \( X \) is called time and the moment \( t = 0 \) the beginning of time \( t \). We also say that the d-space \((W, W)\) is oriented with respect to time \( t \).

In the flat FRW model (11) the cosmological time \( t \) is a time variable. The vector field of the form

\[
X^0: W^0 \to TW^0, \quad X^0(p)(\alpha^0) := \frac{\partial \alpha^0(p)}{\partial t}, \quad (4)
\]

where \( \alpha^0 \in W^0, p \in W^0 \), establishes the time orientation on \((W^0, g)\). The vector field is smooth on the manifold \((W^0, W^0)\) since derivation \( \tilde{X}^0: W^0 \to \mathbb{R}^n, \tilde{X}^0(\alpha^0)(p) := X^0(p)(\alpha^0) \) satisfies the condition \( \tilde{X}^0(W^0) \subset W^0 \) (Definition A.12).

In the next parts of the paper, cosmological models for which the vector field \( X^0 \) can be extended on \((W, W)\) are discussed. Then if remaining conditions of Definition 4.1 are satisfied, \((W, W)\) is a d-space with boundary \( \partial W \) of the flat FRW d-manifold which is time oriented with respect to the vector field \( X \).

Lemma 4.1 The mapping \( X: W \to TW, X(p)(\alpha) := \frac{\partial \alpha(p)}{\partial t} \) is a vector field tangent to \((W, W)\) if and only if for every \( t \in D_a, \dot{a}(t) \) is finite and \( \dot{a}(t) \neq 0 \). ■

Proof. A vector field is tangent to \((W, W)\) if its value \( X(p)(\alpha) \) is finite for every \( \alpha \in W \) and \( p \in W \). It is enough to check this property on the generators \( \alpha_1, \alpha_2, ..., \alpha_5 \) since the d-space \((W, W)\) is finitely generated. Straightforward calculations show that the value of the field \( X(p)(\alpha_2) \) is finite for \( p \in W \) iff \( \dot{a}(t) \) is finite and \( \dot{a}(t) \neq 0 \) for \( t \in D_a \). Then, the value of \( X \) on the remaining generators is always finite. □
Lemma 4.2 If \( V := \lambda X, \lambda \in \mathcal{W} \), is a smooth vector field tangent to the d-space \((W, \mathcal{W})\) and for every \( p \in W^0 \) the value of the function \( \lambda(p) \neq 0 \) then \( \lambda(p) = 0 \) for \( p \in \partial W \). In other words, the smooth vector field \( V \) has a critical point at the boundary \( \partial W \).

Proof. Every smooth function \( \gamma \in \mathcal{W} \) is a local \( \mathcal{W} \)-function on \( W \) (see Definition A.1). This means that for every \( p \in W \) there is \( U_p \in \tau_W \) and \( f_p \in C^\infty(\mathbb{R}^5, \mathbb{R}) \) such that \( \gamma(q) = f_p(\alpha_1(q), \alpha_2(q), ..., \alpha_5(q)) \) for \( q \in U_p \). Therefore, the value of \( \gamma \) in \( p = (0, x, y, z) \in \partial W \), where \( x, y, z \) are any, is a constant function of \( x, y, z \):

\[
\gamma(0, x, y, z) = f_p(\alpha_1(p), \alpha_2(p), ..., \alpha_5(p)) = f_p(0, 0, 0, 0) = \text{const}.
\]

Now, let us suppose that \( \lambda(p) \neq 0 \) for \( p \in \partial W \) also. Smoothness conditions for the field \( V \) have the form: \( \hat{V}(\alpha_i) \in \mathcal{W} \), \( i = 1, 2, ..., 5 \). In particular, the conditions \( \hat{V}(\alpha_1) \in \mathcal{W} \) and \( \hat{V}(\alpha_2) \in \mathcal{W} \) lead to \( \dot{a} \in \mathcal{W} \) and \( \dot{a} \neq 0 \) for \( p \in W \). Then, for example, the function \( \eta := \hat{V}(\alpha_3)/\lambda \dot{a} \), \( \eta(p) = x \), is a smooth function \( \eta \in \mathcal{W} \) since \( \lambda \in \mathcal{W} \) and \( \lambda(q) \neq 0 \) for \( q \in W \). This is a contradiction since this function is not a constant function on \( \partial W \) and therefore it is not generated by means of \( \alpha_i, i = 1, 2, ..., 5 \). □

5 A smooth evolution with respect to cosmological time

The vector field \( X \) defined by formula (5) is not smooth on \((W, \mathcal{W})\). Its value on a smooth function is not necessarily smooth. This means, that there are functions \( \alpha \in \mathcal{W} \) such that \( \hat{X}(\alpha) \notin \mathcal{W} \), where the derivation \( \hat{X}: \mathcal{W} \to \mathbb{R}^W \) is given by the formula \( \hat{X}(\alpha)(p) := X(p)(\alpha) \). But also it can happen that, for a class of smooth functions, values of \( X \) on functions from this class are smooth.

According to the definition of \( X \), its restriction \( X^0 := X|_{W^0} \) is a smooth vector field on \((W^0, \mathcal{W}^0)\), and therefore its value \( \dot{a}_0^0 := X^0(\alpha_1^0) = \dot{a}_0 \) on the smooth function \( \alpha_1^0 := \alpha_{1|W^0} \) is smooth: \( \dot{a}_0^0 \in \mathcal{W}^0 \). The generator \( \alpha_1 = \dot{a} \) is by definition a smooth function on \((W, \mathcal{W})\). According to the earlier argumentation, the value of \( X \) on the function \( \alpha_1 \), \( \dot{a}_1 := X(\alpha_1) = \dot{a} \), is not necessarily smooth in the Sikorski’s sense. But, from the physical point of view, it is natural that the velocity of the expansion of the universe is a smooth function even at the beginning of time \( t = 0 \) since a moment later, it is smooth \( (\dot{a}_1^0 \in \mathcal{W}^0) \) both in the classical sense and Sikorski’s sense. Let us distinguish a class of cosmological models with such a property.
Definition 5.1 An evolution of cosmological model \( \mathcal{M} \) is said to be smooth from the beginning of cosmological time if \( \dot{\alpha}_1 := \dot{X}(\alpha_1) = \dot{a} \in \mathcal{W} \) and \( \dot{a}(t) \neq 0 \) for \( t \in D_a \). We shall also say that the cosmological model is smoothly evolving.

Theorem 5.1 If the flat FRW model is smoothly evolving, then the d-space \((W, W)\) with boundary \( \partial W \) of this model is time oriented with respect to the cosmological time \( t \). A smooth vector field \( \hat{V} \) defining time orientability on \((W, \mathcal{W})\) has the form
\[
\hat{V}: \mathcal{W} \rightarrow \mathcal{W}, \quad \hat{V} := \alpha_1^2 \frac{\partial}{\partial t}.
\]

Proof. Proof consists of verification whether the following inclusion is satisfied
\[
\hat{V}(\alpha_i) = \alpha_1^2 \frac{\partial \alpha_i}{\partial t} \in \mathcal{W}, \quad i = 1, 2, ..., 5.
\]

\[\square\]

6 The simplest smoothly evolving models

Lemma 6.1 A function \( \varphi: \mathcal{W} \rightarrow \mathbb{R} \) which depends on the single coordinate \( t \) only is smooth \( \varphi \in \mathcal{W} \) if and only if, for every \( p \in \mathcal{W} \), there is a neighbourhood \( U_p \in \tau_\mathcal{W} \) and a function \( f \in \mathcal{E}^{(2)} \) such that \( \varphi|_{U_p} = f(\alpha_1, \xi)|_{U_p} \), where
\[
\xi := \alpha_1 \alpha_2 - \alpha_3^2 - \alpha_4^2 - \alpha_5^2, \quad \xi(t, x, y, z) = a(t) \int_0^t \frac{d\tau}{a(\tau)}.
\]

Proof. Let a function \( \varphi \) be a smooth one. The d-structure \( \mathcal{W} \) is finitely generated. Therefore for every \( p \in \mathcal{W} \) there are a neighbourhood \( U_p \in \tau_\mathcal{W} \) and a function \( f^{(5)} \in \mathcal{E}^{(5)} \) such that \( \varphi|_{U_p} = f^{(5)}(\alpha_1, \alpha_2, ..., \alpha_5)|_{U_p} \). Since \( \varphi \) is a constant function with respect to \( x, y, z \), it has to be generated by means of such combinations of the generators admissible in the algebra \( \mathcal{W} \) that a resulting function depends on \( t \) only. The evident form of the generators leads to conclusion that such combinations have the following form \( \alpha_1 \) and \( \xi := \alpha_1 \alpha_2 - \alpha_3^2 - \alpha_4^2 - \alpha_5^2 \). Therefore \( f^{(5)} \) has the following structure
\[
f^{(5)}(\alpha_1, \alpha_2, ..., \alpha_5) = f(\alpha_1, \alpha_1 \alpha_2 - \alpha_3^2 - \alpha_4^2 - \alpha_5^2) = f(\alpha_1, \xi), \quad f \in \mathcal{E}^{(2)}.
\]

The implication in the opposite direction is obvious. \[\square\]

According to Lemma (6.1), if \( \dot{\alpha}_1 \in \mathcal{W} \) then, in a neighbourhood \( U_p \) of \( p \in \mathcal{W} \), the function \( \dot{\alpha}_1 \) is of the form \( \dot{\alpha}_1|_{U_p} = f(\dot{\alpha}_1, \xi)|_{U_p} \). Especially interesting points in \( \mathcal{W} \) are those not Hausdorff separated points from the boundary \( p_* \in \partial \mathcal{W} \), \( p_* = (0, x, y, z) \). For a given \( f \), from the whole class of all possible neighbourhoods \( U_{p_*} \) such that \( \dot{\alpha}_1|_{U_{p_*}} = f(\dot{\alpha}_1, \xi)|_{U_{p_*}} \) one can select the maximal neighbourhood in the sense of inclusion. This means in practice that the maximal neighbourhood has the
form $U_p = D_a \times \mathbb{R}^3$, where the domain $D_a$ of the scale factor is a subset of the set
\{ $t \in [0, \infty) : 0 \leq a(t) < \infty, 0 < \dot{a}(t) < \infty$ \} such that $0 \in D_a$. In the neighbourhood $U_p$, the smoothness condition for $\dot{a}_1$ (see Definition 5.1) has the form

$$\dot{a}(t) = f(a(t), a(t) \int_0^t \frac{d\tau}{\dot{a}(\tau)}),$$

(7)

where

$$f(0, 0) > 0, \quad f \in \mathcal{E}^{(2)}.$$  

(8)

Formula (7) is an equation for $a(t)$ with an initial condition $a(0) = 0$. The function $f$, is in a principle, arbitrary. The only restriction on $f$ is condition (8) which is a consequence of Lemmas 4.1, 6.1 and the physical assumption that the real universe expands from the initial singularity. The simplest choice is the following function

$$f(z_1, z_2) := \beta + \gamma_1 z_1 + \gamma_2 z_2, \quad \beta > 0, \quad \gamma_1, \gamma_2 \in \mathbb{R}.$$  

Now, the smoothness equation (7) for $a(t)$ has the form

$$\dot{a}(t) = \beta + \gamma_1 a(t) + \gamma_2 a(t) \int_0^t \frac{d\tau}{\dot{a}(\tau)}.$$  

(9)

Solutions of (9) have to satisfy the following conditions

$$a(0) = 0, \quad \dot{a}(0) = \beta > 0, \quad a(t) > 0, \quad \dot{a}(t) > 0 \text{ for } t > 0.$$  

(10)

Proposition 6.1 When $\gamma_2 = 0$ then solutions of the smoothness equation (7) satisfying conditions (10) have the form

$$a(t) = \beta t, \quad t \in [0, \infty), \quad \text{for } \gamma_1 = 0,$$

(11)

and

$$a(t) = \frac{\beta}{\gamma_1} (e^{\gamma_1 t} - 1), \quad t \in [0, \infty), \quad \text{for } \gamma_1 \neq 0.$$  

(12)

Proof. Obvious calculus.  

Solution (11) represents the well known model of the universe which expands with the constant velocity $\dot{a} = \beta$ and which is a solution of the Friedman’s equations with the following equation of state $p = -\rho/3$. For $\gamma_1 > 0$ solution (12) is the universe model which is asymptotically ($t \to \infty$) the de-Sitter model. The model
expands from the very beginning with a positive acceleration. The parameter $\gamma_1$ can be asymptotically interpreted as a cosmological constant. When $\gamma_1 < 0$ cosmological model (12) describes an expanding universe, and the expansion slows down from the very beginning. For great $t$, the size of universe fixes on the level $a(t) \approx \lim_{t\to \infty} a(t) = \beta/|\gamma_1|$ and $\dot{a}$ and $\ddot{a}$ tend to zero when $t \to \infty$. Such a universe asymptotically becomes the Minkowski space-time.

In the case $\gamma_2 \neq 0$, let us introduce the following auxiliary symbols

$$\gamma_1 := \tilde{\gamma}_1 \tilde{\gamma}/\sqrt{3}, \quad \gamma_2 := \text{sgn}(\gamma_2) \beta \tilde{\gamma}_2^2/3, \quad K := \sqrt{3}H/\tilde{\gamma}_2, \quad \tilde{\gamma}_2 > 0,$$

$$\bar{a}(K) := \frac{\tilde{\gamma}_2 a(K)}{\beta \sqrt{3}}, \quad \bar{t}(K) := \tilde{\gamma}_2 t(K)/\sqrt{3}, \quad (13)$$

where $H(t) := \dot{a}(t)/a(t)$.

**Proposition 6.2** If $\gamma_2 > 0$ then solutions of smoothness equation (9) have the form

$$\bar{a}(K) = (K - \tilde{\gamma}_1 - \text{arccoth}K)^{-1}, \quad \bar{t}(K) = \int_{K}^{\infty} \frac{\bar{a}(y)ydy}{y^2 - 1}, \quad (14)$$

where $K \in (K_f, \infty)$, and $K_f$ is a solution of the following equation

$$K_f - \text{arccoth}K_f = \tilde{\gamma}_1, \quad K_f \in (1, \infty). \quad (15)$$

Proof. Solution of an elementary differential equation. □

**Proposition 6.3** If $\gamma_2 > 0$ then

1. if $\tilde{\gamma}_1 \geq 0$, acceleration $\ddot{a}(t) > 0$ for $t \in (0, \infty),$

2. if $\tilde{\gamma}_1 < 0$, acceleration $\ddot{a}(t) < 0$ for $t \in (0, t_*)$, $\ddot{a}(t_*) = 0$ and $\ddot{a}(t) > 0$ for $t \in (t_*, \infty)$, where $t_* := t(K_*)$ and $K_*$ is a solution of the following equation

$$\tilde{\gamma}_1 + \frac{K_*}{K_*^2 - 1} + \text{arccoth}K_* = 0, \quad K_* \in (K_f, \infty). \quad (16)$$

Proof. By obvious calculation. □

In the case $\gamma_2 > 0$, there are two essentially different scenarios of a smooth evolution with respect to the cosmological time $t$:
a) If $\tilde{\gamma}_1 \geq 0$ the model accelerates from the very beginning and expands indefinitely.

b) In the case $\tilde{\gamma}_1 < 0$, initially the expansion slows down, but at the moment $t_s$ the unlimited and infinitely long accelerated expansion is initiated (see Figure 1).

Figure 1: Scale factor $\tilde{a}(\tilde{t})$ for smooth solutions with $\gamma_2 > 0$. When $\tilde{\gamma}_1 \geq 0$ models expand with acceleration. For $\tilde{\gamma}_1 < 0$ solutions initially expand with a negative acceleration but at a moment, denoted by a circle on the graph, an accelerated expansion is initiated. The black point denotes the initial singularity.

**Proposition 6.4** If $\gamma_2 < 0$, solutions of smoothness equation (9) have the following form

$$
\tilde{a}(K) = (K - \tilde{\gamma}_1 - \arctan K + \pi/2)^{-1}, \quad \tilde{t}(K) = \int_K^{\infty} \frac{\tilde{a}(y)ydy}{y^2 + 1}, \quad (17)
$$

where $K \in (K_f, \infty)$ and $K_f$ is a solution of the following equation

$$
K_f - \arctan K_f + \pi/2 = \tilde{\gamma}_1, \quad K_f \in \mathbb{R}. \quad (18)
$$

Proof. Solution of an elementary differential equation. □

**Proposition 6.5** If $\gamma_2 < 0$ and $\tilde{\gamma}_1 < \pi/2$, then cosmological model (11) has a final curvature singularity at $t_s < \infty$. The set $[0, t_s)$ is a domain of the scale factor $a(t)$, where $t_s := t(0)$ and the function $t(K)$ is given by formulae (17) and (13).
Proof. Proposition is the result of a fact that some of components of the curvature tensor are indefinite at $t_s$, because $\dddot{a}(t) \to -\infty$ when $t \to t_s^{-}$.

According to Propositions 6.2 and 6.4 solutions of smoothness equation (9) are defined in the domain $(K_f, \infty)$. But in the case of solutions discussed in the Proposition 6.5 there appears an additional restriction for the domains of $a(K)$ and $t(K)$ coming from the geometrical interpretation of $a(K)$ as the scale factor for a flat FRW model. The final curvature singularity ends the evolution of the model.

**Proposition 6.6** If $\gamma_2 < 0$ then

1. if $\bar{\gamma}_1 \geq \pi/2$, acceleration $\dddot{a}(t) > 0$ for $t \in (0, \infty)$,

2. if $0 < \bar{\gamma}_1 < \pi/2$, the scale factor $a(t)$ has the inflexion point at $t_*$ ($\dddot{a}(t_*) = 0$), $\dddot{a}(t) > 0$ for $t \in [0, t_*)$ and $\dddot{a}(t) < 0$ for $t \in (t_*, t_s)$,

3. if $\bar{\gamma}_1 \leq 0$, acceleration $\dddot{a}(t) < 0$ for $t \in [0, t_s)$,

where $t_* := t(K_*)$. The quantity $K_*$ is a solution of the following equation

$$
\frac{K_*}{K_*^2 + 1} - \arctan K_* + \pi/2 = \bar{\gamma}_1.
$$

Proof. By properties of the function given by formulae (17).

If $\gamma_2 < 0$, the smooth evolution of the universe with respect to the cosmological time $t$ can proceed on three different ways:

a) If $\bar{\gamma}_1 \geq \pi/2$, a smooth accelerated evolution starts from the initial singularity. The acceleration goes on continuously during an infinite period of time.

b) For $\bar{\gamma}_1 \in (0, \pi/2)$, these smoothly evolving universes initially accelerate but the acceleration is slowing down so as to change, at $t = t_*$, into a deceleration. Smooth evolution ends at the final curvature singularity within the finite period of time $[0, t_s)$. These models have two singularities: the initial and final one.

c) Models with $\bar{\gamma}_1 \leq 0$ start their evolution in the Big-Bang and decelerate. The rate of expansion is strongly slowing down and these models end their evolution at curvature singularities in a finite time $t_s$. Models of this class have also two singularities.
Figure 2: Scale factors $\tilde{a}(\tilde{t})$ for smoothly evolving models with $\gamma_2 < 0$. Curves on the graph with $\tilde{\gamma}_1 \geq \pi/2$ represent accelerated solutions. For models with $\tilde{\gamma}_1 \in (0, \pi/2)$ initially accelerating expansion is slowing down. At the moment $t_*$ (circles on the graph) the acceleration changes into a strong deceleration. When $\tilde{\gamma}_1 \leq 0$ solutions expand with negative acceleration. The black points on the graph denote initial and final curvature singularities.

7 Primordial matter and dark energy

Let us assume that solutions of the smoothness equation represent cosmological models. Then the Friedman equations

$$p = -2\ddot{a}/a - \dot{a}^2/a^2, \quad \rho = 3\dot{a}^2/a^2,$$

(20)

can serve as a definition of a pressure $\tilde{p}$ and energy density $\tilde{\rho}$ of a kind of cosmological fluid which causes the smooth evolution of models, where $p = \kappa \tilde{p}$ and $\rho = \kappa \tilde{\rho}$. In the present paper this fluid is called the cosmological primordial fluid, or primordial fluid for brevity.

**Proposition 7.1** If $\gamma_1 \in \mathbb{R}$ and $\gamma_2 = 0$ then the equation of state for the primordial fluid has the form

$$p = -\frac{1}{3} \rho - \frac{2\gamma_1}{\sqrt{3}} \sqrt{\rho}.$$

(21)

In addition
a) the energy density \( \varrho \) is a decreasing function of cosmological time and \( \lim_{t \to 0^+} \varrho(t) = \infty \),

b) at the initial moment \( p(0) = \lim_{t \to 0^+} p(t) = -\infty \), and the remaining details of the dependence \( p(t) \) are shown in Figure 3.

c) \[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma_2 &= 0 & \varrho & p & \text{SEC} & \text{WEC} \\
\hline
\gamma_1 > 0 & \lim_{t \to \infty} \varrho(t) = 3\gamma_1^2 & \lim_{t \to \infty} p(t) = -3\gamma_1^2 & \text{no} & \text{yes} \\
\gamma_1 \leq 0 & \lim_{t \to \infty} \varrho(t) = 0 & \lim_{t \to \infty} p(t) = 0 & \text{yes} & \text{yes} \\
\hline
\end{array}
\]

Proof. An elementary calculus. □

The abbreviations SEC and WEC on the above and next tables denote the strong and week energy conditions and the statements below are answers to the question of whether the strong or the week energy conditions are satisfied.

Figure 3: Dependence \( p(t) \) for \( \gamma_2 = 0 \). In the case \( \gamma_1 < 0 \) the pressure has a single positive maximum.

In the case \( \gamma_2 \neq 0 \) it is convenient to introduce the following abbreviations

\[
\tilde{p} := p\gamma_2^{-2}, \quad \tilde{\varrho} := \varrho\gamma_2^{-2}.
\]
Proposition 7.2 If $\gamma_2 > 0$ and $\tilde{\gamma}_1 \in \mathbb{R}$ then the equation of state of the primordial fluid has the form

$$\tilde{p} = -\frac{2}{3} - \frac{1}{3} \tilde{\rho} - \frac{2}{3} (\tilde{\gamma}_1 + \text{arccoth} \sqrt{\tilde{\rho}}) (\sqrt{\tilde{\rho}} - 1/\sqrt{\tilde{\rho}}).$$

(23)

In addition

a) energy density $\tilde{\rho}$ is an increasing function of time and $\lim_{t \to 0^+} \tilde{\rho}(t) = +\infty$, $\lim_{t \to \infty} \tilde{\rho}(t) = K_f^2$ where $K_f$ is a solution of equation (15),

b) pressure at the beginning and end of the evolution is $\lim_{t \to 0^+} \tilde{p}(t) = -\infty$, $\lim_{t \to \infty} \tilde{p}(t) = -K_f^2$, and the remaining details of dependence $\tilde{p}(t)$ are shown in Figure 4,

c) the weak energy condition is satisfied during the whole evolution,

d) for $\tilde{\gamma}_1 \geq 0$ the strong energy condition is broken down,

e) if $\tilde{\gamma}_1 < 0$, the strong energy condition is satisfied for $t \in (0, t_*)$. For remaining $t > t_*$ this condition is broken down. The moment $t_*$ is defined in Proposition 6.3

Proof. An elementary calculus.

Figure 4: For $\tilde{\gamma}_1 \geq -1.35$, pressure is a decreasing function of cosmological time. For remaining $\tilde{\gamma}_1$, function $\tilde{p}(\tilde{t})$ has both the maximum and minimum. The minimum is not well visible on the graph.
Proposition 7.3 If $\gamma_2 < 0$ and $\tilde{\gamma}_1 \in \mathbb{R}$ then the equation of state of the primordial fluid has the following form

$$\tilde{p} = \frac{2}{3} - \frac{1}{3} \tilde{\varrho} - \frac{2}{3} (\tilde{\gamma}_1 - \text{arccot} \sqrt{\tilde{\varrho}})(\sqrt{\tilde{\varrho}} + 1/\sqrt{\tilde{\varrho}}).$$

(24)

Additionally

a) energy density $\tilde{\varrho}$ is a decreasing function of the cosmological time, and $\lim_{t \to 0^+} \tilde{\varrho}(t) = +\infty$,

b) initial pressure is $\tilde{p}(0) := \lim_{t \to 0^+} \tilde{p}(t) = -\infty$, and the remaining details of dependence $\tilde{p}(t)$ are shown in the Figure 5,

c) $\gamma_2 < 0$ $\tilde{\gamma}_1 \leq 0$ $\lim_{t \to t_*} \tilde{\varrho}(t) = 0$ $\lim_{t \to t_*} \tilde{p}(t) = +\infty$ SEC yes WEC yes

d) if $0 < \tilde{\gamma}_1 < \pi/2$, the cosmological fluid violates the strong energy condition for $t \in (0, t_*)$. In the remaining range $t \in (t_*, t_s)$, the SEC is satisfied.

Quantities $K_f$, $t_*$ and $t_s$ are defined in Propositions 6.4, 6.5 and 6.6.

Proof. Elementary calculations. □

The simplest solution, $\gamma_1 = \gamma_2 = 0$, of smoothness equation (9) represents a model filled with the primordial fluid with the equation of state $p = -1/3\varrho$. In the present paper, this fluid is called a $\gamma_0$-matter. In the case of the following parameters system $\{\gamma_1 \neq 0, \gamma_2 = 0\}$, the primordial fluid consists of the $\gamma_0$-matter enriched by a material ingredient connected with the generator $\alpha_1(t) = a(t)$ in formula (9). This enriched primordial fluid we call a $\gamma_1$-matter when $\gamma_1 < 0$, or a $\gamma_1$-energy when $\gamma_1 > 0$. Similarly, in the case of the following parameter system $\{\gamma_1 = 0, \gamma_2 \neq 0\}$, the primordial fluid composed of the $\gamma_0$-matter and a matter connected with the generator $\alpha_2$, through the function $\xi$ in formula (9), we call a $\gamma_2$-matter when $\gamma_2 < 0$, or a $\gamma_2$-energy when $\gamma_2 > 0$.

Taking into account Proposition 7.4 $\gamma_1$-energy has the properties of a dark energy. This energy causes the expansion to accelerate. During the evolution acceleration grows to infinity. After an infinitely long evolution pressure and energy density reach
the finite values \( p_f = -3\tilde{\gamma}_1^2 \) and \( \rho_f = 3\tilde{\gamma}_1^2 \) respectively. Let us notice that then the following equation of state for the cosmological constant is satisfied, \( p_f = -\rho_f \). The \( \gamma_1 \)-energy reaches this property at last stage of the evolution.

In contrast to \( \gamma_1 \)-energy, \( \gamma_1 \)-matter satisfies the strong energy condition during the whole evolution. It has an attraction property. Therefore the expansion of the cosmological model is slowing down in such a manner that at the last stages of the evolution the model becomes static. The \( \gamma_1 \)-matter changes its properties during the evolution. Initially, it has a negative pressure. But later it transforms itself into a kind of matter with a positive pressure. At the last stages of the evolution the pressure and the energy density of \( \gamma_1 \)-matter become zero: \( p_f = 0 \) and \( \rho_f = 0 \). After an infinite period of time since the Big-Bang this cosmological model becomes, in an asymptotic sense, the Minkowski space-time.

A model of the universe filled with \( \gamma_2 \)-energy monotonically accelerates. Initially, the jostling property of \( \gamma_2 \)-energy has a small influence on the expansion but its inflationary power is disclosed at the last stages of evolution of the model. The initially infinite energy density strongly decreases and at the end of the evolution is on the level of \( \rho_f = \tilde{\gamma}_2^2 K_{0f}^2 \), where \( K_{0f} \approx 1.19 \) is a solution of equation (15) for \( \tilde{\gamma}_1 = 0 \). A negative pressure rapidly grows from \(-\infty\) to a finite level of \( p_f = -\tilde{\gamma}_2^2 K_{0f}^2 \). At the end of the evolution the equation of state is as for the cosmological constant: \( p_f = -\rho_f \). During the whole evolution the strong energy condition is violated. \( \gamma_2 \)-energy can be interpreted as a dark energy of different kind then \( \gamma_1 \)-energy. Details

Figure 5: The graph shows the great qualitative differences in the \( \tilde{p}(\tilde{t}) \) dependence for various ranges of \( \tilde{\gamma}_1 \).
can be found in Figures 1, 4 and in Proposition 7.2.

$\gamma_2$-matter has a strong attraction property and therefore the expansion of a model with such a fluid is rapidly slowing down. Acceleration quickly decreases from 0 to $-\infty$ in a finite period. At the end of the evolution the model is stopped $\dot{a}(t_f) = 0$, and its scale factor reaches the maximal, finite value. The final curvature singularity ends the evolution. Properties of $\gamma_2$-matter are changing during the evolution. Pressure rapidly increases from $-\infty$ to $+\infty$ in the finite period $(0, t_f)$. Simultaneously, the energy density decreases from $+\infty$ to zero independently of the fact that the scale factor is finite ($a(t_f) < \infty$) at $t_f$. This kind of matter has very interesting properties at the end of the evolution: it has a slight energy density but simultaneously a huge positive pressure. Details can be found in Figures 2, 5 and in Proposition 7.3.

A mixture of $\gamma_1$-energy and $\gamma_2$-energy ($\gamma_1, \gamma_2 > 0$). During the whole period of evolution of this model the mixture has the properties of a dark energy. Both components interact with each other causing increased acceleration. In the last stages of the infinitely long evolution, the equation of state for the mixture has the form of equation of state for the cosmological constant $\rho_f = -p_f = \tilde{\gamma}_2 K_f^2$, where $K_f$ is a solution of equation (15).

A mixture of $\gamma_1$-matter and $\gamma_2$-matter ($\gamma_1 \leq 0, \gamma_2 < 0$). This kind of the primordial fluid satisfies the strong energy condition during the whole finite period of evolution. The mixture is a fluid with interacting components. At the final singularity $\gamma_2$-matter absorbs, in a sense, $\gamma_1$-matter and finally the mixture vanishes, $\dot{a}_f = 0$ at an infinite pressure. An admixture of $\gamma_1$-matter into $\gamma_2$-matter shortens the life time of the cosmological model.

A mixture of $\gamma_1$-matter and $\gamma_2$-energy ($\gamma_1 \leq 0, \gamma_2 > 0$). Components of the mixture are interacting fluids. The beginning of the model evolution is dominated by $\gamma_1$-matter. The universe expands with a negative acceleration and the mixture satisfies the strong energy condition. But the influence of $\gamma_2$-energy is still rising. For $t > t_*$ the evolution is dominated by $\gamma_2$-energy. Since $t = t_*$, the mixture has properties of a dark energy and changes the further evolution into accelerated expansion. During final stages of the evolution the equation of state has the following form $\rho_f = -p_f = \tilde{\gamma}_2 K_f^2$, where $K_f$ is a solution of equation (15).

The evolutionary behaviour of the model is extremely interesting because such an evolution is qualitatively consistent with the observational data of Ia type supernovae [1, 2]. Preliminary quantitative investigations of the consistency of the discussed model have been carried out with the help of the $\tilde{H}_0(z)$ dependence published in [27]. Results of the best-fit procedure depend on $\tilde{H}_0$. For $70.6 \leq \tilde{H}_0 \leq 77.8 \text{ km s}^{-1}\text{Mpc}^{-1}$ [28] the best-fit parameters are in the range $\tilde{\gamma}_1 \in [-1.829, -1.075]$, $\tilde{\gamma}_2 \in [2.82, 4.011] \times 10^{-4}\text{Mpc}^{-1}$ and $\chi^2_{\text{min}} \in [8.66, 9.76]$ (see also Figure 6). Values of parameters $\tilde{\gamma}_1$ and
\( \gamma_2 \) enable us to find: an age of universe \( \bar{t}_0 \in [13.561, 14.241] \times 10^9 \text{y} \), the moment of the acceleration beginning \( \bar{t}_* \in [7.427, 8.471] \times 10^9 \text{y} \), the Hubble constant in the acceleration moment \( \bar{H}_* \in [106.31, 108.86] \times \text{km s}^{-1}\text{Mpc}^{-1} \) and redshift \( z_* \in [0.648, 0.743] \), where

\[
z(K) := \frac{a_0 \tilde{\gamma}_2}{\sqrt{3\beta \bar{a}(K)}} - 1, \quad \bar{t}_0 = c^{-1} t(K_0), \quad \bar{t}_* = c^{-1} t(K_*), \quad K_0 := \frac{\sqrt{3}}{c \gamma_2} \bar{H}_0.
\]

Quantities \( K, \bar{a}(K), t(K) \) are given by formulas (13) and (14), and \( K_* \) is a solution of equation (16). In the present Section bar over quantities denotes that we use the systems of units in which \( c \neq 1 \).

![Figure 6: Comparison of the theoretical \( \bar{H}(z) \) dependence for the smoothly evolved model (\( \gamma_1 < 0 \) and \( \gamma_2 > 0 \)) with \( \bar{H}_{obs}(z) \) for \( \bar{H}_0 = 72.0 \text{ km s}^{-1}\text{Mpc}^{-1} \). The solid line represents the smoothly evolved model with the best-fit parameter values \( \tilde{\gamma}_1 = -1.241, \quad \tilde{\gamma}_2 = 0.000312\text{Mpc}^{-1} \) and \( \chi^2_{\text{min}} = 8.79 \). In this case \( \bar{t}_0 = 14.103 \times 10^9 \text{y}, \quad \bar{t}_* = 8.241 \times 10^9 \text{y}, \quad \bar{H}_* = 106.87 \text{ km s}^{-1}\text{Mpc}^{-1} \) and \( z_* = 0.668 \). The dashed line represents the best-fit of the \( \Lambda \text{CDM} \) model.](image)

*Figure 6: Comparison of the theoretical \( \bar{H}(z) \) dependence for the smoothly evolved model (\( \gamma_1 < 0 \) and \( \gamma_2 > 0 \)) with \( \bar{H}_{obs}(z) \) for \( \bar{H}_0 = 72.0 \text{ km s}^{-1}\text{Mpc}^{-1} \). The solid line represents the smoothly evolved model with the best-fit parameter values \( \tilde{\gamma}_1 = -1.241, \quad \tilde{\gamma}_2 = 0.000312\text{Mpc}^{-1} \) and \( \chi^2_{\text{min}} = 8.79 \). In this case \( \bar{t}_0 = 14.103 \times 10^9 \text{y}, \quad \bar{t}_* = 8.241 \times 10^9 \text{y}, \quad \bar{H}_* = 106.87 \text{ km s}^{-1}\text{Mpc}^{-1} \) and \( z_* = 0.668 \). The dashed line represents the best-fit of the \( \Lambda \text{CDM} \) model.*

*A mixture of \( \gamma_1 \)-energy and \( \gamma_2 \)-matter (\( \gamma_1 > 0, \quad \gamma_2 < 0 \)).* For \( t \in [0, t_*] \) the mixture has properties of a dark energy and therefore this model accelerates from the very
beginning. But later, $\gamma_2$-matter component begins to play a bigger and bigger role. The further evolution depends on the value of the parameter $\gamma_1$.

When $\tilde{\gamma}_1 \in (0, \pi/2)$, the repulsive properties of $\gamma_1$-energy are not able to dominate the evolution and the acceleration is gradually stopped because of a greater and greater attractive influence of $\gamma_2$-matter. The moment $t_*$, defined in the Proposition 6.6 is the end of acceleration. Starting from this moment, the expansion is strongly slowing down till the final singularity. The behaviour of pressure is interesting, Figure 5, for $t > t_*$. After $t_*$ the mixture behaves as a dust ($\tilde{p} \approx 0$). But later, pressure rapidly increases to infinity at the final singularity. Simultaneously, because of expansion, the energy density decreases to zero.

When $\tilde{\gamma}_1 \geq \pi/2$, repulsive influence of $\gamma_1$-energy is dominating during the whole period the infinitely long evolution. The mixture acts like a dark energy causing acceleration, independently of the $\gamma_2$-matter presence. As for previously considered accelerating models, the equation of state, in the last stages of the evolution, is as the one for the cosmological constant, i.e., $\varrho_f = -p_f = \tilde{\gamma}_2^2 K_f^2$, where $K_f$ is a solution of equation (18).

8 Smoothly evolved models in a neighbourhood of singularity

The function $f$ appearing in the smoothness equation (7) can be expanded into a series in a neighbourhood of the point $(0,0)$:

$$f(z_1, z_2) = \beta + \partial_1 f(0,0) \cdot z_1 + \partial_2 f(0,0) \cdot z_2 + ....$$

Then the smoothness equation assumes the following form

$$\dot{a}(t) = \beta + \partial_1 f(0,0) \cdot a(t) + \partial_2 f(0,0) \cdot a(t) + \frac{d\tau}{\dot{a}(\tau)} + ....$$

For sufficiently small $t$, or equivalently in a small neighbourhood of the initial singularity, one can omit higher powers of the expansion and consider the smoothness equation in the linear approximation. If one assumes that $\gamma_1 := \partial_1 f(0,0)$ and $\gamma_2 := \partial_2 f(0,0)$, the above equation, in the linear approximation, is identical with the smoothness equation (9) for the simplest smoothly evolved models (Section 6).

The above observation leads to the conclusion that properties of the solutions studied in Sections 6 and 7 for small $t$, are typical for every smoothly evolving flat FRW model in a small neighbourhood of the initial singularity. In particular, every smoothly evolving model during the initial stages of its evolution is filled with a cosmological fluid which is $\gamma_i$-matter or $\gamma_i$-energy, $i = 1, 2$, or one of mixtures...
described in Section 7. These fields satisfy the following approximate equation of state
\[
p = -\frac{\rho}{3} - \frac{2\gamma_1}{\sqrt{3}} + \frac{2\gamma_1 \tilde{\gamma}_2}{9} + \frac{4}{45} \tilde{\gamma}_2^2 + \ldots,
\]
where \(\gamma_1 \in \mathbb{R}, \tilde{\gamma}_2 \geq 0, \varepsilon = \text{sign}(\gamma_2)\).

9 Summary and Discussion

a) The discussion of the equivalence principle and its breaking down has led to the formulation of the d-boundary notion which is, roughly speaking, an "optimal" set of points at which the equivalence principle is broken. A space-time d-manifold with the attached d-boundary is not only a topological space but also an object with a rich geometrical structure called differential space (see Appendix A). The effective construction of both d-boundary and d-space with d-boundary for any space-time d-manifold has also been described (see Section 2).

b) For every flat FRW cosmological model with the initial singularity the d-space with d-boundary \((\bar{W}, \bar{W})\) has been constructed. In this case the d-boundary is a single point (see Section 3).

c) In the d-space formalism it is possible to extend the concept of time orientability to the d-boundary of the FRW models. In this way, the intuitive understanding of the beginning of the cosmological time obtains a precise mathematical form (see Section 4). However, not every flat FRW model with the initial singularity has the well defined beginning of the cosmological time.

d) In the whole class of flat FRW models with the well defined beginning of the cosmological time we distinguish the large subclass of the so-called smoothly evolved models (see Definition 5.1). The condition defining this subclass is called the smoothness equation. The most practical form of this equation is given by formula (7).

e) The simplest flat FRW models with the well defined beginning of the cosmological time have been found in the explicit form. This set of models can be divided into six qualitatively different classes (Lemmas 6.1, 6.2 and 6.4). The most important classes are

- solutions with parameters \(\tilde{\gamma}_1 < \pi/2\) and \(\gamma_2 < 0\) which have two curvature singularities: the initial singularity and the final singularity (see Figure 2),

- the subfamily with \(\tilde{\gamma}_1 < 0\) and \(\gamma_2 > 0\), being qualitatively consistent with the observational \(\bar{H}_{obs}(z)\) data from [27]. The quantitative consistency with the data...
is discussed in Section 7 and is shown in Figure 6. The level of the consistency is similar to that for the ΛCDM model.

f) The Friedman equation without cosmological constant, in application to the simplest solutions, may serve as a definition of energy density $\bar{\rho}$ and pressure $\bar{p}$ of a cosmological fluid (primordial fluid) which causes the smooth evolution. This strategy enables us to find main phenomenological properties (in particular, the equation of state) of this primordial fluid (see Lemmas 7.1, 7.2 and 7.3).

g) In Section 7 we present an interpretation of the primordial fluid as a mixture of more elementary interacting primordial fluids $\gamma_i$-matter and $\gamma_i$-energy, $i = 1, 2$.

h) From the analysis of smoothness equation (7) in a neighbourhood of the initial singularity, i.e., for small $t$, one can conclude (see Section 8) that in earlier stages of the evolution every smoothly evolving flat FRW model is filled with a primordial cosmological fluid with properties characteristic for particular solutions found in this paper (Sections 6 and 7).

i) It is very surprising that without any assumption concerning physical nature of the cosmological fluid, among the simplest solutions of the smoothness equation (7), it is possible to find models consistent with the observed evolution of the Universe (see Figure 6). The existence of the well defined beginning of the cosmological time was the only requirement which has been assumed. Primordial mixture of fluids $\gamma_1$-matter and $\gamma_2$-energy is a consequence of this simple assumption. However, we cannot expect that every detail of the cosmic evolution can be determined in this way. The material ingredients such as radiation, dust and dark matter should also be taken into account.

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A Sikorski’s differential spaces

Let $\mathcal{C}$ be a non empty family of real functions defined on a set $M$. The family $\mathcal{C}$ generates on $M$ a topology denoted by the symbol $\tau_{\mathcal{C}}$. It is the weakest topology on $M$ in which every function from $\mathcal{C}$ is continuous.

Let $A \subset M$ be a subset of $M$ and let the symbol $\mathcal{C}\lvert A$ denotes the set of all functions belonging to $\mathcal{C}$ restricted to $A$. On $A$ one can define the induced topology $\tau_{\mathcal{C}} \cap A = \tau_{\mathcal{C}\lvert A}$. The topological space $(A, \tau_{\mathcal{C}\lvert A})$ is a topological subspace of $(M, \tau_{\mathcal{C}})$. 

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Next, we introduce two key notions in the d-spaces theory: a) the closure of \( C \) with respect to localization and b) the closure of \( C \) with respect to superposition with smooth functions from \( \mathcal{E}^{(m)} := C^\infty(\mathbb{R}^m, \mathbb{R}) \), \( m = 0, 1, 2, \ldots \).

**Definition A.1** A function \( \gamma : A \to \mathbb{R} \) is said to be a local \( C \)-function on a subset \( A \subset M \) if, for every \( p \in A \), there is a neighbourhood \( U_p \in \mathcal{T}_{C|A} \) and a function \( \phi \in \mathcal{C} \) such that \( \gamma|U_p = \phi|U_p \). The set of all local \( C \)-functions on \( A \) is denoted by \( \mathcal{C}_A \).

It is easily to check that in general \( \mathcal{C}|_A \subset \mathcal{C}_A \). In particular \( \mathcal{C} \subset \mathcal{C}_M \).

**Definition A.2** A family \( \mathcal{C} \) of real functions on a set \( M \) is said to be closed with respect to localization if \( \mathcal{C} = \mathcal{C}_M \).

**Definition A.3** A family of functions \( \mathcal{C} \) is closed with respect to superposition with smooth functions from \( \mathcal{E}^{(m)} \), \( m = 0, 1, 2, \ldots \), if for every function \( \omega \in \mathcal{E}^{(m)} \) and for every system of \( m \) functions \( \varphi_1, \varphi_2, \ldots, \varphi_m \in \mathcal{C} \), the superposition \( \omega(\varphi_1, \varphi_2, \ldots, \varphi_m) \) is a function from \( \mathcal{C} \); \( \omega(\varphi_1, \varphi_2, \ldots, \varphi_m) \in \mathcal{C} \).

The above described system of concepts make possible to define an object (a d-space) which is a commutative generalisation of the d-manifold concept.

**Definition A.4** A pair \( (M, \mathcal{C}) \), where \( M \) is a set of points and \( \mathcal{C} \) a family of real functions on \( M \), is said to be a differential space (d-space for brevity) if

1. \( \mathcal{C} \) is closed with respect to localization, \( \mathcal{C} = \mathcal{C}_M \),
2. \( \mathcal{C} \) is closed with respect to superposition with smooth functions from \( \mathcal{E}^{(m)} \).

The family \( \mathcal{C} \) is called a differential structure on \( M \) (d-structure for brevity) and the set \( M \) a support of the d-structure \( \mathcal{C} \). Functions \( \varphi \in \mathcal{C} \) are called smooth functions.

Every d-space is simultaneously a topological space with the topology \( \mathcal{T}_C \) given by the d-structure \( \mathcal{C} \) in the standard way. The d-structure itself, with the usual multiplication, is a commutative algebra. The notion of smoothness, given by the condition \( \varphi \in \mathcal{C} \), is an abstract generalization of the smoothness notion for functions defined on \( \mathbb{R}^n \). Differential structure, by definition, is a set of all smooth functions on \( M \). There are no other smooth functions on \( M \). This class of smooth functions may consists of functions which are not smooth in the traditional sense. This is a great advantage of the d-spaces theory. The simplest example of a d-space is the n-dimensional Euclidean d-space \((\mathbb{R}^n, \mathcal{E}^{(n)})\), where \( \mathcal{E}^{(n)} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \).
There exists a procedure to construct a d-structure with the help of a chosen set of real functions on $M$. Let us denote it as $C_0$. The method consists in adding to a given $C_0$ missing functions so as to satisfy the axioms of the closure with respect to superposition with smooth functions and the closure with respect to localization. The closure with respect to superposition with smooth real functions on $\mathbb{R}^n$, is denoted by mathematicians by $\text{sc}(C_0)$ and the closure with respect to localization is denoted by $(C_0)_M$ (see Definition A.1) or $\text{lc}(C_0)$.

It is easy to check that

**Lemma A.1** Let $C_0$ be a set of real functions defined on a set $M$. The family of functions $C := \text{lc}(\text{sc}(C_0)) = (\text{sc}(C_0))_M$ is the smallest, in the sense of inclusion, d-structure on $M$ containing $C_0$.

Sometimes one uses the following abbreviation: $C = \text{Gen}(C_0) := \text{lc}(\text{sc}(C_0)) = (\text{sc}(C_0))_M$.

**Definition A.5** The set $C_0$ in Lemma A.1 is said to be a set of generators. Functions $\varphi \in C_0$ are called generators of the d-structure $C := \text{sc}(C_0)$. If $C_0$ is finite then the d-structure $C$ is called finitely generated.

The method of constructing a d-structure with the help of generators is the great advantage of the Sikorski’s theory, especially in the case of finitely generated d-spaces such as, for example, the d-space with singularity associated with the flat FRW world model.

**Definition A.6** If $(M, C)$ is a d-space and $A \subset M$ then the d-space $(A, C_A)$ is said to be a differential subspace (d-subspace) of the d-space $(M, C)$.

The above definition enables us to determine a d-structure for any subset $A$ of $M$. It is enough to “localize” every function from the d-structure $C$ to $A$. In the case of a finitely generated d-spaces $(M, C)$, a simpler situation occurs. Then $C = \text{Gen}(C_0)$, $C_0 := \{\beta_1, \beta_2, ..., \beta_n\}$, $n \in \mathbb{N}$, where $\beta_1, \beta_2, ..., \beta_n$ are given functions. The d-structure $C_A$ is given in terms of generators $\tilde{C}_0 = C_0|A$ which is a set of restrictions of the set $C_0$ to $A$. Then $C_A = \text{Gen}(\tilde{C}_0)$.

**Definition A.7** Let $(M, C)$ and $(N, D)$ be a d-spaces.

1. A mapping $f: M \rightarrow N$ is said to be smooth if $\forall \beta \in D: \beta \circ f \in C$.

2. A mapping $f: M \rightarrow N$ is said to be a diffeomorphism from a d-space $(M, C)$ to a d-space $(N, D)$, if it is a bijection from $M$ to $N$ and both mappings $f: M \rightarrow N$ and $f^{-1}: N \rightarrow M$ are smooth.

In this case, we say that $(M, C)$ and $(N, D)$ are diffeomorphic.
A smooth mapping $f$ transforms smooth functions on $N$ onto smooth functions on $M$. The notion of d-spaces diffeomorphism is the key notion from the point of view of the present paper. If there is a diffeomorphism $f$ between d-spaces $(M, C)$ and $(N, D)$ then these d-spaces, from the viewpoint the d-spaces theory, are equivalent.

**Definition A.8** Let $(M, C)$ and $(N, D)$ be d-spaces. A d-space $(M, C)$ is said to be locally diffeomorphic to the d-space $(N, D)$ if for every $p \in M$ there is $U_p \in \tau_C$ and a mapping $f_p : U_p \rightarrow f_p(U_p) \in \tau_D$ such that $f_p$ is a diffeomorphism between the d-subspaces $(U_p, C_{U_p})$ and $(f_p(U_p), D_{f_p(U_p)})$ of the d-spaces $(M, C)$ and $(N, D)$, respectively.

**Definition A.9** A d-space $(M, C)$ is said to be an $n$-dimensional d-manifold, if it is locally diffeomorphic to the d-space $(\mathbb{R}^n, E(n))$.

Applying Definition A.8 to Definition A.9 leads to condition $(\star)$ in Definition 2.1. Local diffeomorphisms $f_p$ are obviously maps. A set of maps forms an atlas. It turns out that Definition A.9 is equivalent to the classical definition of d-manifold [12].

**Theorem A.1** Let $(M, C)$ be a finitely generated d-space with the structure $C$ generated by a finite set of functions: $C_0 := \{\beta_1, \beta_2, \ldots, \beta_n\}$, $C = \text{Gen}(C_0)$. Then the mapping $F : M \rightarrow \mathbb{R}^n$, $F(p) := (\beta_1(p), \beta_2(p), \ldots, \beta_n(p))$ is a diffeomorphism of the d-space $(M, C)$ onto the d-subspace $(F(M), E^{(n)}_{F(M)})$ of the d-space $(\mathbb{R}^n, E(n))$.

Proof, see [25].

The d-subspace $(F(M), E^{(n)}_{F(M)})$ of $(\mathbb{R}^n, E(n))$ is an image of the d-space $(M, C)$ in the mapping $F$. Theorem A.1 is called the theorem on a diffeomorphism onto the image and it is important for the construction of the d-space for the flat FRW model with the initial singularity.

**Definition A.10** A mapping $v : C \rightarrow \mathbb{R}$ is said to be a tangent vector to a d-space $(M, C)$ at a point $p \in M$ if

1. $\forall \alpha, \beta \in C \forall a, b \in \mathbb{R} : v(a\alpha + b\beta) = av(\alpha) + bv(\beta),$
2. $\forall \alpha, \beta \in C : v(\alpha\beta) = v(\alpha)\beta(p) + \alpha(p)v(\beta), \ p \in M.$

The set of all tangent vectors to $(M, C)$ at $p \in M$ is said to be a tangent vector space to $(M, C)$ at $p \in M$ and is denoted by $T_pM$. The symbol $TM$ denotes the following disjoint sum:

$$TM := \bigcup_{p \in M} T_pM.$$
**Definition A.11** Let \((M, C)\) be a \(d\)-space. The mapping \(X: M \rightarrow TM\) such that \(\forall p \in M, \ X(p) \in T_p M\) is said to be a vector field tangent to \((M, C)\).

With help of a vector field \(X: M \rightarrow TM\) one can define the following mapping

\[
\hat{X}: C \rightarrow \mathbb{R}^M, \quad \hat{X}(\alpha)(p) := X(p)(\alpha),
\]

where \(\alpha \in C\) and \(p \in M\). The mapping is linear and satisfies the Leibnitz rule \(\forall \alpha, \beta \in C : \hat{X}(\alpha \beta) = \hat{X}(\alpha) \beta + \alpha \hat{X}(\beta)\). Therefore, it is a derivation and a global alternative for the definition of a vector field (A.11).

**Definition A.12** A vector field tangent to \((M, C)\) is said to be smooth if the mapping \(\hat{X}\) satisfies the condition: \(\hat{X}(C) \subset C\).

**Definition A.13** Let \(f: M \rightarrow N\) be a smooth mapping. The mapping \(f_{\ast p}: T_p M \rightarrow T_{f(p)} N\), given by the formula

\[
\forall v \in T_p M, \beta \in D : \quad f_{\ast p}(v)(\beta) := v(\beta \circ f),
\]

is said to be differential of the mapping \(f\) at the point \(p \in M\).

Let us define the following mapping \(id_A: A \rightarrow M, \ id_A(p) = p\), where \(A \subset M\).

**Definition A.14** A vector field \(Y: M \rightarrow TM\) on \((M, C)\) is said to be tangent to a \(d\)-subspace \((A, C_A)\), if there is a vector field \(X: A \rightarrow TA\) on \((A, C_A)\) such that

\[
\forall p \in A : Y(p) = id_{A \ast p}(X).
\]

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