An Ergodic Theorem for Quantum Counting Processes

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I. INTRODUCTION

Modern research on quantum-mechanical counting processes, be it numerical simulations \cite{Car} or experimental investigations \cite{MYK}, usually starts from the tacit assumption that for the study of statistical properties of the counting records it does not make a difference whether a large number of experiments is performed or a single very long one. This assumption amounts to ergodicity of these records. In several recent discussions, e.g. \cite{BESW,NaS,PlK,Cre,DCM}, investigators have addressed the question of its validity. A partial result was obtained by Cresser \cite{Cre}, who proved ergodicity in the $L^2$-sense and to first order in the detection current. In this paper we establish ergodicity in the full sense (Theorem 3), in particular to all orders in the detection current and with probability 1 (Theorem 4). Theorem 5 formulates ergodicity in terms of multi-time coincidences.

For the description of detection records we employ the rigorous formulation of Davies and Srinivas \cite{Dav,SrD}, which has set the tone for later investigations \cite{Car,Mr,WiM,GaZ}.

II. COUNTING PROCESSES ACCORDING TO DAVIES AND SRINIVAS

We consider an open quantum system under continuous observation by use of a finite number $k$ of detectors. The state of the system is described by a density matrix $\rho$ on a Hilbert space, obeying a Master equation $\dot{\rho} = L\rho$, where $L$ is a generator of Lindblad form $[\text{Lin}]$. Normalisation is expressed by the relation

$$\text{tr} L(\rho) = 0 \quad \text{for all } \rho$$ \hspace{1cm} (2.1)

A counting process connected to this quantum evolution is based on an unraveling of the generator

$$L = L_0 + \sum_{i=1}^{k} J_i$$ \hspace{1cm} (2.2)

which is interpreted as follows. The reaction of the detectors to the system consists of clicks at random times. The evolution $\rho \mapsto e^{t L_0} \rho$ denotes the change of the state of the system under the condition that during a time interval of length $t$ no clicks are recorded. The operator $\rho \mapsto J_i(\rho)$ on the state space describes the change of state conditioned on the occurrence of a click of detector $i$. For computational convenience we assume these operators to be bounded. So, if $\rho$ describes the state of the system at time 0, and if, during the time interval $[0,t]$, clicks are recorded at times $t_1, t_2, \ldots, t_n$ of detectors $i_1, i_2, \ldots, i_n$ respectively, and none more, then, up to normalisation, the state at time $t$ is given by

$$e^{(t-t_n) L_0} J_{i_n} e^{(t_{n}-t_{n-1}) L_0} \ldots e^{(t_2-t_1) L_0} J_{i_2} e^{t_1 L_0} \rho$$ \hspace{1cm} (2.3)

The probability density $f((t_1, i_1), \ldots, (t_n, i_n))$ for these clicks to occur is equal to the trace of (2.3).

We imagine the experiment to continue indefinitely. The observation process will then produce an infinite detection record $((t_1, i_1), (t_2, i_2), (t_3, i_3), \ldots)$, where we assume that $0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots$, and $\lim_{n \to \infty} t_n = \infty$ (i.e., the clicks do not accumulate).

Let $\Omega$ denote the space of all such detection records. By an event we mean some property of the record, which we identify with the set $E \subset \Omega$ of all records with this property. The events decidable at or before time $t$ are those of the form $\{\varepsilon \in \Omega \mid \varepsilon(t) \in E\}$, where $E \in \Sigma$, and $\varepsilon$ has been observed up to $t$. We say the event $E$ has occurred if the set $\{\varepsilon \in \Omega \mid \varepsilon(t) \in E\}$ is non-empty, and if a positive time $t$ is an event in $\Sigma_t$ and $\rho$ denotes a state, then we define

$$M_t(E)(\rho) := \sum_{n=0}^{\infty} \sum_{i_1=1}^{k} \cdots \sum_{i_{n-1}=1}^{k} \int_0^{t_1} \cdots \int_0^{t_{n-1}} \int_0^{t_n} L_0 J_{i_n} e^{(t_n-t_{n-1}) L_0} \ldots e^{(t_2-t_1) L_0} J_{i_1} e^{t_1 L_0} \rho$$

$$\times dt_1 dt_2 \cdots dt_n$$ \hspace{1cm} (2.4)

Here $1_E$ denotes the indicator function of the event $E$ and $M_t(E)$ is the effect on the quantum system of the occurrence of $E \in \Sigma_t$. Then
The resulting equation extends to all $F \in \Sigma$, in particular it holds for $F = E$:

$$
\mathbb{P}_\varnothing(E) = \mathbb{P}_\varnothing(E)^2.
$$

It follows that $\mathbb{P}_\varnothing(E)$ is equal to 0 or 1. \hfill \blacksquare

Let us denote the expectation $\int_{\Omega} f(\omega) d\mathbb{P}_\varnothing(\omega)$ of an integrable function $f$ on $\Omega$ by $\mathbb{E}_\varnothing(f)$.

**Theorem 3.** If the evolution $(T_t)_{t \geq 0}$ converges in the mean to $\rho$, then for all integrable functions $h$ on $\Omega$ and all initial states $\varnothing$ we have, almost surely with respect to $\mathbb{P}_\varnothing$,

$$
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau h(\sigma_t(\omega)) \, dt = \mathbb{E}_\varnothing(h) \, .
$$

**Proof.** By Lemma 1 and Theorem 2, $\mathbb{P}_\varnothing$ is stationary and ergodic. Hence, by Birkhoff’s individual ergodic theorem, the limit on the left exists almost surely with respect to $\mathbb{P}_\varnothing$, and is equal to the constant $\mathbb{E}_\varnothing(h)$. Since the set $F$ of points $\omega \in \Omega$ for which (3.3) holds, is time-invariant, we have $\mathbb{P}_\varnothing(F) = \mathbb{P}_\varnothing(F) = 1$ for all states $\varnothing$ by (3.4). \hfill \blacksquare
IV. APPLICATIONS

The main result of the present ergodic theory for quantum counting processes, Theorem 3, can be made considerably more concrete by applying it to detection currents and multi-time coincidences, showing bunching or antibunching.

For simplicity we consider only one detector, which responds to a point event at time $s$ by producing a current $\gamma(t-s)$ at time $t$. (This will be zero for $t<s$.) The total detection current is given by

$$I_t(\omega) := \sum_{s \in \omega} \gamma(t-s).$$

Let $\overline{P}_\sigma$ be the unique stationary extension of $P_\sigma$ to negative times on the configuration space $\overline{\Omega}$ of the full real line. We shall denote expectation with respect to this measure by $E_{\overline{P}_\sigma}$.

**Theorem 4.** Let the quantum evolution $(T_t)_{t \geq 0}$ converge in the mean to a state $\rho$ and let the detector response function $\gamma : \mathbb{R} \to [0,\infty)$ be bounded and integrable. Then for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ and all initial states $\vartheta$ we have, almost surely with respect to $P_\rho$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T I_{t_1+t} \cdots I_{t_n+t} d\omega = E_{\overline{P}_\sigma}(I_{t_1} \cdots I_{t_n}).$$

For $n=2$ this theorem implies a quantum-mechanical version of the Wiener-Khinchin theorem. In the proof we shall make use of the non-exclusive probability density of the stationary process $[vKa,GaZ,CrE]$,\n
$$g_n(t_1, t_2, \ldots, t_n) := \text{tr}(J T_{t_n-t_{n-1}} J \cdots J T_{t_2-t_1} J (\rho)).$$

The functions $g_n$ are related to the probability density $f^\dagger$ from (2.3) of the counting process (where $t \geq t_n$), by

$$g_n(t_1, t_2, \ldots, t_n) = f_n^\dagger(t_1, t_2, \ldots, t_n)$$

$$+ \sum_{m=1}^\infty \int_0^T \cdots \int_0^T f_{m+n}^\dagger(t_1, t_2, \ldots, t_n) \cup \{s_1, \ldots, s_n\})$$

$$\times ds_1 \cdots ds_m = \int_{\Omega_t} f^\dagger\{(t_1, t_2, \ldots, t_n) \cup \omega\} d\omega; \quad (4.1)$$

where $\Omega_t$ is the set of finite subsets of $[0, t]$, which can be identified with the time-ordered points in $\{\emptyset\} \cup \bigcup_{m=1}^\infty [0, t]^m$. By $d\omega$ we mean $ds_1 ds_2 \cdots ds_m$ if $\omega = \{s_1, s_2, \ldots, s_m\}$ with $s_1 \leq s_2 \leq \cdots \leq s_m$.

**Proof of Theorem 4.** First we note that Theorem 3 also holds if $\Omega$, $P_\rho$, and $E_\rho$ are replaced by $\overline{\Omega}$, $\overline{P}_\sigma$, and $E_{\overline{P}_\sigma}$ respectively, as introduced above, and $\sigma_t$ by the left shift of $\omega \subset \mathbb{R}$. Then we have $I_{t_1+t} = I_{\sigma_t}(\omega)$. Now fix $n \in \mathbb{N}$ and $0 \leq t_1 \leq \ldots \leq t_n$. Let $h : \Omega \to \mathbb{R}$ be given by

$$h(\omega) := I_{t_1}(\omega) I_{t_2}(\omega) \cdots I_{t_n}(\omega).$$

It follows that $h \circ \sigma_t = I_{t_1+t} I_{t_2+t} \cdots I_{t_n+t}$, and the statement to be proved follows from Theorem 3, provided that $h$ is integrable. In the Appendix we shall show that this is indeed the case. \hfill $\Box$

As our second application we shall show that the non-exclusive probability densities $g_n$ have a straightforward pathwise interpretation: they are equal to the frequency of multi-time coincidences on almost every detection record. For this, let $N_{[a,b]}(\omega) := \#(\omega \cap [a,b])$ denote the number of clicks detected during the time interval $[a,b]$.

**Theorem 5.** Let $(T_t)_{t \geq 0}$ converge in the mean to the equilibrium state $\rho$. Then for all $n \in \mathbb{N}$, all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, all $\varepsilon$ between 0 and $\min \{ t_j+\varepsilon - t_j \}$, and all initial states $\vartheta$ we have, almost surely with respect to $P_\rho$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \prod_{j=1}^n N_{[t_j+t,t_{j+i}\varepsilon]}(\omega) \right) dt$$

$$= \int_{t_n}^{t_n+\varepsilon} \cdots \int_{t_1}^{t_1+\varepsilon} g(s_1, \ldots, s_n) ds_1 \cdots ds_n. \quad (4.2)$$

**Proof.** Fix $n \in \mathbb{N}$ and a sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ of times. Let $K : \Omega \to \{0,1\}$ be the function that maps $\omega \in \Omega$ to 1 if $\omega$ contains exactly $n$ points, one in each of the intervals $[t_1, t_1+\varepsilon], \ldots, [t_n, t_n+\varepsilon]$, and to 0 otherwise. Then we obtain for $t \geq t_n+\varepsilon$, using set notation and the integral-sum lemma from [LiM],

$$\int_{t_n}^{t_n+\varepsilon} \cdots \int_{t_1}^{t_1+\varepsilon} g(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$= \int_{\Omega_t} K(\omega) g(\omega) d\omega \quad (4.1) \quad \int_{\Omega_t} \int_{\Omega_t} K(\omega) f^\dagger(\omega) d\omega. \quad (4.3)$$

A short calculation shows that

$$\sum_{\alpha \subset \omega} K(\omega) = \prod_{j=1}^n N_{[t_j,t_j+\varepsilon]}(\omega). \quad (4.4)$$

Since $0 \leq g_n(s_1, s_2, \ldots, s_n) \leq \|J\|^n$, the integral (4.3) is convergent, hence the product on the r.h.s. of (4.4) is integrable as a function of $\omega$. Application of Theorem 3 to this product now yields the statement. \hfill $\Box$

V. DISCRETE TIME

There is an obvious analogue of our main result (Theorem 3) in discrete time [MaK]. A Kraus measurement [Kra] is given by a decomposition of a completely positive operator $T$ on state space as

$$T \rho = \sum_{i=1}^k a_i \rho a_i^\dagger,$$
where $\rho \mapsto a_i \rho a_i^*$ describes the state change of the density matrix $\rho$ when the measurement gives the outcome $i$. Thus for initial state $\vartheta$ the probability of finding the sequence of outcomes $i_1, i_2, \ldots, i_m$ by repeated Kraus measurement is given by

$$\text{tr} \left( a_{i_m} \cdots a_{i_1} \vartheta a_{i_1}^* \cdots a_{i_m}^* \right).$$

As in continuous time, this yields a probability measure $P_\vartheta$ on the space of detection records $\Omega \equiv \{1, 2, \ldots, k\}^N$. Again, if $(T^n)_{n \in \mathbb{N}}$ converges in the mean to some state $\rho$, then the only time invariant events in $\Omega$ have measure 0 or 1 for all $P_\vartheta$. In particular, $P_\rho$ is ergodic.

**APPENDIX:**

We shall show that, in the situation of Theorem 4, $h := I_{t_1} \cdots I_{t_n}$ is an integrable function on $\Omega$ provided that the jump operator $J$ is bounded and the detector response function $\gamma : \mathbb{R} \to [0, \infty)$ is bounded and integrable.

Let $M := \max(1, \|\gamma\|_\infty)$. Fix $n \in \mathbb{N}$ and a sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ of times. Let

$$\varphi(t) := \sum_{j=1}^n (t_j - t).$$

Then $\varphi$ is also integrable, with $\|\varphi\|_1 = n\|\gamma\|_1$. For $k \in \mathbb{N}$, let $\mathcal{J}_{n,k}$ denote the set of all surjections $\{1, \ldots, n\} \to \{1, \ldots, k\}$. Then we may write for any $\omega \in \Omega$,

$$I_{t_1}(\omega) I_{t_2}(\omega) \cdots I_{t_n}(\omega) = \sum_{s_1 \in \omega} \cdots \sum_{s_n \in \omega} \gamma(t_1 - s_1) \cdots \gamma(t_n - s_n) = \sum_{k=1}^n \sum_{j \in \mathcal{J}_{n,k}} \sum_{s_1 < a_j(1) \leq s_2 < \cdots < s_k} \gamma(t_1 - a_j(1)) \cdots \gamma(t_n - a_j(n))$$

$$\leq \sum_{k=1}^n \#(\mathcal{J}_{n,k}) \sum_{\alpha \subseteq \omega} \|\gamma\|^{n-k} \int_{\prod_{s \in \alpha} \varphi(s)}.$$

Using set notation and the integral-sum lemma \cite{LiM}, we conclude that, for all $t \geq 0$ and $u \geq t_n + t$,

$$\mathbb{E}_\rho((I_{t_1} I_{t_2} \cdots I_{t_n}) \circ \sigma_t)/M^n n^{n+1} \leq \int_{\Omega_n} \sum_{\alpha \subseteq \omega} \left( \prod_{s \in \alpha} \varphi(s-t) \right) f^u(\omega) \, d\omega$$

$$\leq \int_{\Omega_n} \left( \prod_{s \in \alpha} \varphi(s-t) \right) g(\alpha) \, d\alpha$$

$$\leq \sum_{m=0}^\infty \|J\|^m m! \int_{[0,u]^m} \varphi(s_1 - t) \cdots \varphi(s_m - t) \, ds_1 \cdots ds_m$$

$$\leq \exp \left( \|J\| \int_0^u \varphi(s-t) \, ds \right) \leq e^{n\|J\|\|\gamma\|_1}.$$

Therefore, since the r.h.s. does not depend on $t$,

$$\mathbb{E}_\rho(I_{t_1} \cdots I_{t_n}) = \lim_{t \to \infty} \mathbb{E}_\rho((I_{t_1} \cdots I_{t_n}) \circ \sigma_t) < \infty.$$

\[\square\]

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