Towards Tight Lower Bounds for Scheduling Problems

Abbas Bazzi\(^1\) and Ashkan Norouzi-Fard\(^2\)

\(^1\) School of Computer and Communication Sciences, EPFL. abbas.bazzi@epfl.ch
\(^2\) School of Computer and Communication Sciences, EPFL. ashkan.norouzifard@epfl.ch

Abstract. We show a close connection between structural hardness for \(k\)-partite graphs and tight inapproximability results for scheduling problems with precedence constraints. Assuming a natural but nontrivial generalisation of the bipartite structural hardness result of \([1]\), we obtain a hardness of \(2 - \epsilon\) for the problem of minimising the makespan for scheduling precedence-constrained jobs with preemption on identical parallel machines. This matches the best approximation guarantee for this problem \([6,4]\). Assuming the same hypothesis, we also obtain a super constant inapproximability result for the problem of scheduling precedence-constrained jobs on related parallel machines, making progress towards settling an open question in both lists of ten open questions by Williamson and Shmoys \([17]\), and by Schuurman and Woeginger \([14]\).

The study of structural hardness of \(k\)-partite graphs is of independent interest, as it captures the intrinsic hardness for a large family of scheduling problems. Other than the ones already mentioned, this generalisation also implies tight inapproximability to the problem of minimising the weighted completion time for precedence-constrained jobs on a single machine, and the problem of minimising the makespan of precedence-constrained jobs on identical parallel machine, and hence unifying the results of Bansal and Khot \([11]\) and Svensson \([15]\), respectively.

Keywords: hardness of approximation, scheduling problems, unique game conjecture

1 Introduction

The study of scheduling problems is motivated by the natural need to efficiently allocate limited resources over the course of time. While some scheduling problems can be solved to optimality in polynomial time, others turn out to be NP-hard. This difference in computational complexity can be altered by many factors, from the machines model that we adopt, to the requirements imposed on the jobs, as well as the optimality criterion of a feasible schedule. For instance, if we are interested in minimising the completion time of the latest job in a schedule (known as the maximum makespan), then the scheduling problem is NP-hard
to approximate within a factor of $3/2 - \epsilon$, for any $\epsilon > 0$, if the machines are unrelated, whereas it admits a Polynomial Time Approximation Scheme (PTAS) for the case of identical parallel machines \[8\]. Adopting a model in between the two, in which the machines run at different speeds, but do so uniformly for all jobs (known as uniform parallel machines), also leads to a PTAS for the scheduling problem \[9\].

Although this somehow suggests a similarity in the complexity of scheduling problems between identical parallel machines and uniform parallel machines, our hopes for comparably performing algorithms seem to be shattered as soon as we add precedence requirements among the jobs. On the one hand, we know how to obtain a 2-approximation algorithm for the problem where the parallel machines are identical \[6, 4\] (denoted as $P|\text{prec} | C_{\text{max}}$ in the language of \[7\]), whereas on the other hand the best approximation algorithm known to date for the uniform parallel machines case (denoted as $Q|\text{prec} | C_{\text{max}}$), gives a $\log(m)$-approximation guarantee \[3, 2\], $m$ being the number of machines. In fact obtaining a constant factor approximation algorithm for the latter, or ruling out any such result is a major open problem in the area of scheduling algorithms. Perhaps as a testament to that, is the fact that it is listed by Williamson and Shmoys \[17\] as Open Problem 8, and by Schuurman and Woeginger \[14\] as Open Problem 1.

Moreover, our understanding of scheduling problems even on the same model of machines does not seem to be complete either. On the positive side, it is easy to see that the maximum makespan of any feasible schedule for $P|\text{prec} | C_{\text{max}}$ is at least $\max\{L, n/m\}$, where $L$ is the length of the longest chain of precedence constraints in our instance, and $n$ and $m$ are the number of jobs and machines respectively. The same lower bound still holds when we allow preemption, i.e., the scheduling problem $P|\text{prec, pmtn}| C_{\text{max}}$. Given that both 2-approximation algorithms of \[6\] and \[4\] rely in their analysis on the aforementioned lower bound, then they also yield a 2-approximation algorithm for $P|\text{prec, pmtn}| C_{\text{max}}$. However, on the negative side, our understanding for $P|\text{prec, pmtn}| C_{\text{max}}$ is much less complete. For instance, we know that it is NP-hard to approximate $P|\text{prec} | C_{\text{max}}$ within any constant factor strictly better than $4/3$ \[10\], and assuming (a variant of) the UNIQUE GAMES Conjecture, the latter lower bound is improved to 2 \[15\]. However for $P|\text{prec, pmtn}| C_{\text{max}}$, only NP-hardness is known. It is important to note here that the hard instances yielding the $(2 - \epsilon)$ hardness for $P|\text{prec} | C_{\text{max}}$ are easy instances for $P|\text{prec, pmtn}| C_{\text{max}}$. Informally speaking, the hard instances for $P|\text{prec} | C_{\text{max}}$ can be thought of as $k$-partite graphs, where each partition has $n + 1$ vertices that correspond to $n + 1$ jobs, and the edges from a layer to the layer above it emulate the precedence constraints. The goal is to schedule these $(n + 1)k$ jobs on $n$ machines. If the $k$-partite graph is complete, then any feasible schedule has a makespan of at least $2k$, whereas if the graph was a collection of perfect matchings between each two consecutive layers, then there exists a schedule whose makespan is $k + 1$\footnote{In fact, the gap is between $k$-partite graphs that have nice structural properties in the completeness case, and behave like node expanders in the soundness case.} However, if we allow preemption, then
it is easy to see that even if the $k$-partite graph is complete, one can nonetheless find a feasible schedule whose makespan is $k + 1$.

The effort of closing the inapproximability gap between the best approximation guarantee and the best known hardness result for some scheduling problems was successful in recent years; two of the results that are of particular interest for us are \cite{1} and \cite{15}. Namely, Bansal and Khot studied in \cite{1} the scheduling problem $1|\text{prec}|\sum w_j C_j$, the problem of scheduling precedence constrained jobs on a single machine, with the goal of minimizing the weighted sum of completion time, and proved tight inapproximability results for it, assuming a variant of the \textsc{unique games} Conjecture. Similarly, Svensson proved in \cite{15} a hardness of $2 - \epsilon$ for $P|\text{prec}, pmtn|C_{\max}$, assuming the same conjecture. In fact, both papers relied on a structural hardness result for bipartite graphs, first introduced in \cite{1}, by reducing a bipartite graph to a scheduling instance which leads to the desired hardness factor.

Our results We propose a natural but non-trivial generalisation of the structural hardness result of \cite{1} from bipartite to $k$-partite graphs, that captures the intrinsic hardness of a large family of scheduling problems. Concretely, this generalisation yields

1. A super constant hardness for $Q|\text{prec} \mid C_{\max}$, making progress towards resolving an open question by \cite{17,14}
2. A hardness of $2 - \epsilon$ for $P|\text{prec}, pmtn|C_{\max}$, even for the case where the processing time of each job is 1, denote by $P|\text{prec}, pmtn, p_j = 1|C_{\max}$, and hence closing the gap for this problem.

Also, the results of \cite{1} and \cite{15} will still hold for $1|\text{prec} \mid \sum w_j C_j$ and $P|\text{prec} \mid C_{\max}$, respectively, under the same assumption.

On the one hand, our generalisation rules out any constant factor polynomial time approximation algorithm for the scheduling problem $Q|\text{prec} \mid C_{\max}$. On the other hand, one may speculate that the preemption flexibility when added to the scheduling problem $P|\text{prec} \mid C_{\max}$ may render this problem easier, especially that the hard instances of the latter problem become easy when preemption is allowed. Contrary to such speculations, our generalisation to $k$-partite graphs enables us to prove that it is NP-hard to approximate the scheduling problem $P|\text{prec}, pmtn, p_j = 1|C_{\max}$ within any factor strictly better than 2. Formally, we prove the following:

**Theorem 1.** Assuming Hypothesis\cite{1}, it is NP-hard to approximate the scheduling problems $P|\text{prec}, pmtn, p_j = 1|C_{\max}$ within any constant factor strictly better than 2, and $Q|\text{prec} \mid C_{\max}$ within any constant factor.

This suggests that the intrinsic hardness of a large family of scheduling problems seems to be captured by structural hardness results for $k$-partite graphs. For the case of $k = 2$, our hypothesis coincides with the structure bipartite hardness result of \cite{1}, and yields the following result:
Theorem 2. Assuming a variant of the unique games Conjecture, it is NP-hard to approximate the scheduling problem $P|\text{prec, pmtn}, p_j = 1|C_{\text{max}}$ within any constant factor strictly less than $3/2$.

In fact, the $3/2$ lower bound holds even if we only assume that $1|\text{prec} | \sum_j w_j C_j$ is NP-hard to approximate within any factor strictly better than 2, by noting the connection between the latter and a certain bipartite ordering problem. This connection was observed and used by Svensson [15] to prove tight hardness of approximation lower bounds for $P|\text{prec} | C_{\text{max}}$, and this yields a somehow stronger statement; even if the UNIQUE GAMES Conjecture turns out to be false, $1|\text{prec} | \sum_j w_j C_j$ might still be hard to approximate to within a factor of $2 - \epsilon$, and our result for $P|\text{prec, pmtn}, p_j = 1|C_{\text{max}}$ will still hold as well. Formally,

Corollary 1. For any $\epsilon > 0$, and $\eta \geq \eta(\epsilon)$, where $\eta(\epsilon)$ tends to 0 as $\epsilon$ tends to 0, if $1|\text{prec} | \sum_j w_j C_j$ has no $(2 - \epsilon)$-approximation algorithm, then $P|\text{prec, pmtn}, p_j = 1|C_{\text{max}}$ has no $(3/2 - \eta)$-approximation algorithm.

Although we believe that Hypothesis 5 holds, the proof is still eluding us. Nonetheless, understanding the structure of $k$-partite graphs seems to be a very promising direction to understanding the inapproximability of scheduling problems, due to its manifold implications on the latter problems. As mentioned earlier, a similar structure for bipartite graphs was proved assuming a variant of the UNIQUE GAMES Conjecture in [1] (see Theorem 4), and we show in Section B how to extend it to $k$-partite graphs, while maintaining a somehow similar structure. However the resulting structure does not suffice for our purposes, i.e., does not satisfy the requirement for Hypothesis 5. Informally speaking, a bipartite graph corresponding to the completeness case of Theorem 4, despite having a nice structure, contains some noisy components that we cannot fully control. This follows from the fact that these graphs are derived from UNIQUE GAMES PCP-like tests, where the resulting noise is either intrinsic to the UNIQUE GAMES instance (i.e., from the non-perfect completeness of the UNIQUE GAMES instance), or artificially added by the test. Although we can overcome the latter, the former prohibits us from replicating the structure of the bipartite graph to get a $k$-partite graph with an equally nice structure.

Further Related Work The scheduling problem $P|\text{prec, pmtn}, p_j = 1|C_{\text{max}}$ was first shown to be NP-hard by Ullman [16]. However, if we drop the precedence rule, the problem can be solved to optimality in polynomial time [11]. Similarly, if the precedence constraint graph is a tree [12,13,5] or the number of machines is 2 [12,13,16], the problem also becomes solvable in polynomial time. Yet, for an arbitrary precedence constraints structure, it remains open whether the problem is polynomial time solvable when the number of machines is a constant greater than or equal to 3 [17]. A closely related problem to $P|\text{prec, pmtn} | C_{\text{max}}$ is $P|\text{prec} | C_{\text{max}}$, in which preemption is not allowed. In fact the best 2-approximation algorithms known to date for $P|\text{prec, pmtn} | C_{\text{max}}$ were originally designed to approximate $P|\text{prec} | C_{\text{max}}$ [6,4], by noting the common lower bound for a makespan to any feasible schedule for both problems. As mentioned earlier, [10] and [15]
prove a $4/3 - \epsilon$ NP-hardness, and $2 - \epsilon$ UGC-hardness respectively for $P|\text{prec}|C_{\text{max}}$, for any $\epsilon > 0$. However, to this date, only NP-hardness is known for the $P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}$ scheduling problem. Although one may speculate that allowing preemption might enable us to get better approximation guarantees, no substantial progress has been made in this direction since [6] and [4].

One can easily see that the scheduling problem $P|\text{prec}|C_{\text{max}}$ is a special case of $Q|\text{prec}|C_{\text{max}}$, since it corresponds to the case where the speed of every machine is equal to 1, and hence the $(4/3 - \epsilon)$ NP-hardness of [10] and the $(2 - \epsilon)$ UGC-hardness of [15] also apply to $Q|\text{prec}|C_{\text{max}}$. Nonetheless, no constant factor approximation for this problem is known; a log($m$)-approximation algorithm was designed by Chudak and Shmoys [3], and Chekuri and Bender [2] independently, where $m$ is the number of machines.

Outline  We start in Section 2 by defining the UNIQUE GAMES problem, along with the variant of the UNIQUE GAMES Conjecture introduced in [1]. We then state in Section 3 the structural hardness result for bipartite graphs proved in [1], and propose our new hypothesis for k-partite graphs (Hypothesis 5) that will play an essential role in the hardness proofs of Section 4. Namely, we use it in Section 4.1 to prove a super constant inapproximability result for the scheduling problem $Q|\text{prec}|C_{\text{max}}$, and $2 - \epsilon$ inapproximability for $P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}$. The reduction for the latter problem can be seen as replicating a certain scheduling instance $k - 1$ times, and hence we note that if we settle for one copy of the instance, we can prove an inapproximability of $3/2$, assuming the variant of the UNIQUE GAMES Conjecture of [1]. In Section 5, we prove a structural hardness result for k-partite graphs which is similar to Hypothesis 5 but not sufficient for our scheduling problems of interest. We note in Section 4 that the integrality gap instances for the natural Linear Programming (LP) relaxation for $P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}$ have a very similar structure to the instances yielding the hardness result.

2 Preliminaries

In this section, we start by introducing the UNIQUE GAMES problem, along with a variant of Khot’s UNIQUE GAMES Conjecture as it appears in [1], and then we formally define the scheduling problems of interest.

Definition 1. A UNIQUE GAMES instance $\mathcal{U}(G = (V, W, E), [R], \Pi)$ is defined by a bipartite graph $G = (V, W, E)$ with bipartitions $V$ and $W$ respectively, and edge set $E$. Every edge $(v, w) \in E$ is associated with a bijection map $\pi_{v,w} \in \Pi$ such that $\pi_{v,w} : [R] \mapsto [R]$, where $[R]$ is the label set. The goal of this problem is to find a labeling $\Lambda : V \cup W \mapsto [R]$ that maximises the number of satisfied edges in $E$, where an edge $(u, v) \in E$ is satisfied by $\Lambda$ if $\pi_{v,w}(\Lambda(w)) = \Lambda(v)$.

Bansal and Khot [1] proposed the variant of the UNIQUE GAMES Conjecture in Hypothesis 3 and used it to (implicitly) prove the structural hardness result for bipartite graphs in Theorem 4.
Theorem 4. [Section 7.2 in [1]] For arbitrarily small constants \( \eta, \zeta, \delta > 0 \), there exists an integer \( R = R(\eta, \zeta, \delta) \) such that for a unique games instance \( \mathcal{U}(G = (V, W, E), [R], \Pi) \), it is NP-hard to distinguish between:

- (YES Case: ) There are sets \( V' \subseteq V, W' \subseteq W \) such that \( |V'| \geq (1 - \eta)|V| \) and \( |W'| \geq (1 - \eta)|W| \), and a labeling \( \Lambda : V \cup W \mapsto [R] \) such that all the edges between the sets \( (V', W') \) are satisfied.
- (NO Case: ) No labeling to \( \mathcal{U} \) satisfies even a \( \zeta \) fraction of edges. Moreover, the instance satisfies the following expansion property. For every \( S \subseteq V, T \subseteq W, |S| = \delta|V|, |T| = \delta|W| \), there is an edge between \( S \) and \( T \).

Hypothesis 3 [Variant of the UGC[1]] For arbitrarily small constants \( \eta, \zeta, \delta > 0 \), there exists an integer \( R = R(\eta, \zeta, \delta) \) such that for a unique games instance \( \mathcal{U}(G = (V, W, E), [R], \Pi) \), it is NP-hard to distinguish between:

- (YES Case: ) There are sets \( V' \subseteq V, W' \subseteq W \) such that \( |V'| \geq (1 - \eta)|V| \) and \( |W'| \geq (1 - \eta)|W| \), and a labeling \( \Lambda : V \cup W \mapsto [R] \) such that all the edges between the sets \( (V', W') \) are satisfied.
- (NO Case: ) No labeling to \( \mathcal{U} \) satisfies even a \( \zeta \) fraction of edges. Moreover, the instance satisfies the following expansion property. For every \( S \subseteq V, T \subseteq W, |S| = \delta|V|, |T| = \delta|W| \), there is an edge between \( S \) and \( T \).

Theorem 4. [Section 7.2 in [1]] For every \( \epsilon, \delta > 0 \), and positive integer \( Q \), the following problem is NP-hard assuming Hypothesis 3 given an \( n \)-by-\( n \) bipartite graph \( G = (V, W, E) \), distinguish between the following two cases:

- YES Case: \( V \) can be partitioned into \( V_0, \ldots, V_{Q-1} \) and \( W \) can be partitioned into \( W_0, \ldots, W_{Q-1} \), such that

  • There is no edge between \( V_i \) and \( W_j \) for all \( 0 \leq j < i < Q \).
  • \( |V_0| \geq \left( \frac{1 - \epsilon}{Q} \right) n \) and \( |W_i| \geq \left( \frac{1 - \epsilon}{Q} \right) n \), for all \( i \in [Q] \).
- NO Case: For any \( S \subseteq V, T \subseteq W, |S| = \delta n, |T| = \delta n \), there is an edge between \( S \) and \( T \).

In the scheduling problems that we consider, we are given a set \( \mathcal{M} \) of machines and a set \( \mathcal{J} \) of jobs with precedence constraints, and the goal is find a feasible schedule in a way to minimise the makespan, i.e., the maximum completion time.

We will be interested in the following two variants of this general setting:

- \( \text{P|prec, pmtn |C_{max}} \): In this model, the machines are assumed to be parallel and identical, i.e., the processing time of a job \( J \in \mathcal{J} \) is the same on any machine \( M \in \mathcal{M} \) (\( p_{i,j} = p_j \) for all \( M_i \in \mathcal{M} \)). Furthermore, preemption is allowed, and hence the processing of a job can be paused and resumed at later stages, not necessarily on the same machine.

- \( \text{Q|prec |C_{max}} \): In this model, the machines are assumed to be parallel and uniform, i.e., each machine \( M_i \in \mathcal{M} \) has a speed \( s_i \), and the time it takes to process job \( J_j \in \mathcal{J} \) on this machine is \( p_j / s_i \).

Before we proceed we give the following notations that will come in handy in the remaining sections of the paper. For a positive integer \( Q \), \( [Q] \) denotes the set \( \{0, 1, \ldots, Q-1\} \). In a scheduling context, we say that a job \( J_i \) is a predecessor of a job \( J_j \), and write it \( J_i \prec J_j \), if in any feasible schedule, \( J_j \) cannot start executing before the completion of job \( J_i \). Similarly, for two sets of jobs \( \mathcal{J_i} \) and \( \mathcal{J_j} \), \( \mathcal{J_i} \prec \mathcal{J_j} \) is equivalent to saying that all the jobs in \( \mathcal{J_i} \) are successors of all the jobs in \( \mathcal{J_j} \).

### 3 Structured \( k \)-partite Problem

We propose in this section a natural but nontrivial generalisation of Theorem 4 to \( k \)-partite graphs. Assuming hardness of this problem, we can get the following hardness of approximation results:
1. It is NP-hard to approximate \( Q \prec \{ C_{max} \} \) within any constant factor.
2. It is NP-hard to approximate \( P \prec \{ \text{pmtn, } p_j = 1 \} \prec \{ C_{max} \} \) within a \( 2 - \epsilon \) factor.
3. It is NP-hard to approximate \( 1 \prec \{ \sum_j w_j C_j \} \) within a \( 2 - \epsilon \) factor.
4. It is NP-hard to approximate \( P \prec \{ \sum_j w_j C_j \} \) within a \( 2 - \epsilon \) factor.

The first and second result are presented in Section 4.1 and 4.2, respectively. Moreover, one can see that the reduction presented in [1] for the scheduling problem \( 1 \prec \{ \sum_j w_j C_j \} \) holds using the hypothesis for the case that \( k = 2 \). The same holds for the reduction in [19] for the scheduling problem \( P \prec \{ \sum_j w_j C_j \} \). This suggests that this structured hardness result for \( k \)-partite graphs somehow unifies a large family of scheduling problems, and captures their common intrinsic hard structure.

**Hypothesis 5** [\( k \)-partite Problem] For every \( \epsilon, \delta > 0 \), and constant integers \( k, Q > 1 \), the following problem is NP-hard: given a \( k \)-partite graph \( G = (V_1, ..., V_k, E_1, ..., E_{k-1}) \) with \( |V_i| = n \) for all \( 1 \leq i \leq k \) and \( E_i \) being the set of edges between \( V_i \) and \( V_{i+1} \) for all \( 1 \leq i < k \), distinguish between following two cases:

- **YES Case**: every \( V_i \) can be partitioned into \( V_{i,0}, ..., V_{i,Q-1} \), such that
  - There is no edge between \( V_{i,j_1} \) and \( V_{i-1,j_2} \) for all \( 1 < i \leq k \), \( j_1 < j_2 \in [Q] \).
  - \( |V_{i,j}| \geq \frac{(1-\epsilon)}{Q} n \) for all \( 1 \leq i \leq k \), \( j \in [Q] \).
- **NO Case**: For any \( 1 < i \leq k \) and any two sets \( S \subseteq V_{i-1}, T \subseteq V_i \), \( |S| = \delta n \), \( |T| = \delta n \), there is an edge between \( S \) and \( T \).

This says that if the \( k \)-partite graph \( G = (V_1, ..., V_k, E_1, ..., E_{k-1}) \) satisfies the YES Case, then for every \( 1 \leq i \leq k - 1 \), the induced subgraph \( G = (V_i, V_{i+1}, E_i) \) behaves like the YES Case of Theorem 4, and otherwise, every such induced subgraph corresponds to the NO case. Moreover, if we think of \( G \) as a directed graph such that the edges are oriented from \( V_i \) to \( V_{i-1} \), then all the partitions in the YES case are consistent in the sense that a vertex \( v \in V_{i,j} \) can only have paths to vertices \( v' \in V_{i',j'} \) if \( i' < i \leq k \) and \( j' \leq j \leq Q - 1 \).

We can prove that assuming the previously stated variant of the UNIQUE GAMES Conjecture, Hypothesis 5 holds for \( k = 2 \). Also we can extend Theorem 4 to a \( k \)-partite graph using a perfect matching approach which results in the following theorem. We delegate its proof to Appendix B.

**Theorem 6.** For every \( \epsilon, \delta > 0 \), and constant integers \( k, Q > 1 \), the following problem is NP-hard: given a \( k \)-partite graph \( G = (V_1, ..., V_k, E_1, ..., E_{k-1}) \) with \( |V_i| = n \) and \( E_i \) being the set of edges between \( V_i \) and \( V_{i+1} \) , distinguish between following two cases:

- **YES Case**: every \( V_i \) can be partitioned into \( V_{i,0}, ..., V_{i,Q-1}, V_{i,\text{err}} \), such that
  - There is no edge between \( V_{i,j_1} \) and \( V_{i-1,j_2} \) for all \( 1 < i \leq k \), \( j_1 \neq j_2 \in [Q] \).
  - \( |V_{i,j}| \geq \frac{(1-\epsilon)}{Q} n \) for all \( 1 \leq i \leq k \), \( j \in [Q] \).
- **NO Case**: For any \( 1 < i \leq k \) and any two sets \( S \subseteq V_i, T \subseteq V_{i-1} \), \( |S| = \delta n \), \( |T| = \delta n \), there is an edge between \( S \) and \( T \).
Note that in the YES Case, the induced subgraphs on \( \{ V_{i,j} \} \) for \( 1 \leq i \leq k, \ 0 \leq j \leq Q - 1 \), have the perfect structure that we need for our reductions to scheduling problems. However, we do not get the required structure between the noise partitions (i.e., \( \{ V_{i,err} \} \) for \( 1 \leq i \leq k \)), which will prohibit us from getting the desired gap between the YES and NO Cases when performing a reduction from this graph to our scheduling instances of interest. The structure of the noise that we want is that the vertices in the noise partition are only connected to the vertices in the noise partition of the next layer.

4 Lower Bounds for Scheduling Problems

In this section, we show that, assuming Hypothesis 5, there is no constant factor approximation algorithm for the scheduling problem \( Q|\text{prec}|C_{\text{max}} \), and there is no \( c \)-approximation algorithm for the scheduling problem \( P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}} \), for any constant \( c \) strictly better than 2. We also show that, assuming a special case of Hypothesis 5 i.e., \( k = 2 \) which is equivalent to (a variant) of UNIQUE GAMES Conjecture (Hypothesis 3), there is no approximation algorithm better than \( 3/2 - \epsilon \) for \( P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}} \), for any \( \epsilon > 0 \).

4.1 \( Q|\text{prec}|C_{\text{max}} \)

In this section, we reduce a given \( k \)-partite graph \( G \) to an instance \( I(k) \) of the scheduling problem \( Q|\text{prec}|C_{\text{max}} \), and show that if \( G \) corresponds to the YES Case of Hypothesis 5, then the maximum makespan of \( I(k) \) is roughly \( n \), whereas a graph corresponding to the NO Case leads to a scheduling instance whose makespan is roughly the number of vertices in the graph, i.e., \( nk \). Formally, we prove the following theorem.

**Theorem 7.** Assuming Hypothesis 5 it is NP-hard to approximate the scheduling problem \( Q|\text{prec}|C_{\text{max}} \) within any constant factor.

**Reduction** We present a reduction from a \( k \)-partite graph \( G = (V_1, \ldots, V_k, E_1, \ldots, E_{k-1}) \) to an instance \( I(k) \) of the scheduling problem \( Q|\text{prec}|C_{\text{max}} \). The reduction is parametrised by a constant \( k \), a constant \( Q \gg k \) such that \( Q \) divides \( n \), and a large enough value \( m \gg nk \).

- For each vertex in \( v \in V_i \), let \( J_{v,i} \) be a set of \( m^{2(k-i)} \) jobs with processing time \( m^{i-1} \), for every \( 1 \leq i \leq k \).
- For each edge \( e = (v,w) \in E_i \), we have \( J_{v,i} \preceq J_{w,i+1} \), for \( 1 \leq i < k \).
- For each \( 1 \leq i \leq k \) we create a set \( M_i \) of \( m^{2(k-i)} \) machines with speed \( m^{i-1} \).

**Completeness** We show that if the given \( k \)-partite graph satisfies the properties of the YES Case, then there exist a schedule with makespan \( (1 + \epsilon_1)n \) for some small \( \epsilon_1 > 0 \). Towards this end, assume that the given \( k \)-partite graph satisfies
on the machines in can schedule all the jobs in a set \( \tilde{J} \) satisfied. The partitioning of the vertices naturally induces a partitioning \( \{\tilde{J}_{i,j}\} \) for the jobs for \( 1 \leq i \leq k \) and \( 0 \leq j \leq Q - 1 \) in the following way:

\[
\tilde{J}_{i,j} = \bigcup_{v \in V_{v,i}} J_v, i
\]

Consider the schedule where for each \( 1 \leq i \leq k \), all the jobs in a set \( \tilde{J}_{i,0}, \ldots, \tilde{J}_{i,Q-1} \) are scheduled on the machines in \( \mathcal{M}_i \). Moreover, we start the jobs in \( \tilde{J}_{i,j} \) after finishing the jobs in both \( \tilde{J}_{i-1,j} \) and \( \tilde{J}_{i,j-1} \) (if such sets exist). In other words, we schedule the jobs as follows (see Figure 1):

- For each \( 1 \leq i \leq k \), we first schedule the jobs in \( \tilde{J}_{i,0} \), then those in \( \tilde{J}_{i,1} \) and so on up until \( \tilde{J}_{i,Q-1} \). The scheduling of the jobs on machines in \( \mathcal{M}_0 \) starts at time 0 in the previously defined order.
- For each \( 2 \leq i \leq k \), we start the scheduling of jobs \( \tilde{J}_{i,0} \) right after the completion of the jobs in \( \tilde{J}_{i-1,0} \).
- To respect the remaining precedence requirements, we start scheduling the jobs in \( \tilde{J}_{i,j} \) right after the execution of jobs in \( \tilde{J}_{i,j-1} \) and as soon as the jobs in \( \tilde{J}_{i-1,j} \) have finished executing, for \( 2 \leq i \leq k \) and \( 1 \leq j \leq Q - 1 \).

By the aforementioned construction of the schedule, we know that the precedence constraints are satisfied, and hence the schedule is feasible. That is, since we are in YES Case, we know that vertices in \( V_{i,j} \) might only have edges to the vertices in \( V_{i,j} \) for all \( 1 \leq i' < i \leq k \) and \( 1 \leq j' \leq j < Q \), which means that the precedence constraints may only be from the jobs in \( \tilde{J}_{i,j} \) to jobs in \( \tilde{J}_{i,j} \) for all \( 1 \leq i' < i \leq k \) and \( 0 \leq j' \leq j < Q \). Therefore the precedence constraints are satisfied.

Moreover, we know that there are at most \( m^{2(k-i)}n(1 + \epsilon)/Q \) jobs of length \( m^{i-1} \) in \( \tilde{J}_{i,j} \), and \( m^{2(k-i)} \) machines with speed \( m^{i-1} \) in each \( \mathcal{M}_i \) for all \( 1 \leq i \leq k \), \( j \in [Q] \). This gives that it takes \( (1 + \epsilon)n/Q \) time to schedule all the jobs in \( \tilde{J}_{i,j} \) on the machines in \( \mathcal{M}_i \) for all \( 1 \leq i \leq k \), \( j \in [Q] \), which in turn implies that we can schedule all the jobs in a set \( \tilde{J}_{i,j} \) between time \( (i + j - 1)(1 + \epsilon)n/Q \) and \( (i + j)(1 + \epsilon)n/Q \). This gives that the makespan is at most \( (k + Q)(1 + \epsilon)n/Q \) which is equal to \( (1 + \epsilon_1)n \), by the assumption that \( Q \gg k \).

**Soundness** We shall now show that if the \( k \)-partite graph \( G \) corresponds to the NO Case of Hypothesis 5, then any feasible schedule for \( \mathcal{I}(k) \) must have a makespan of at least \( cnk \), where \( c := (1 - 2\delta)(1 - k^2/m) \) can be made arbitrary close to one.

**Lemma 1.** In a feasible schedule \( \sigma \) for \( \mathcal{I}(k) \) such that the makespan of \( \sigma \) is at most \( nk \), the following is true: for every \( 1 \leq i \leq k \), at least a \( (1 - k^2/m) \) fraction of the jobs in \( \mathcal{L}_i = \bigcup_{v \in V_i} J_v, i \) are scheduled on machines in \( \mathcal{M}_i \).
Proof. We first show that no job in $L_i$ can be scheduled on machines in $M_j$, for all $1 \leq j < i \leq k$. This is true, because any job $J \in J_i$ has a processing time of $m^{i-1}$, whereas the speed of any machine $M \in M_j$ is $m^{j-1}$ by construction, and hence scheduling the job $J$ on the machine $M$ would require $m^{i-1}/m^{j-1} \geq m$ time steps. But since $m \gg nk$, this contradicts the assumption that the makespan is at most $nk$.

We now show that at most $k^2/m$ fraction of the jobs in $L_i$ can be scheduled on the machines in $M_j$ for $1 \leq i < j \leq k$. Fix any such pair $i$ and $j$, and assume that all the machines in $M_j$ process the jobs in $L_i$ during all the $T \leq nk$ time steps of the schedule. This accounts for a total $T(m^{2(k-j)}m^{i-1}) = m^{2k-j-i}nk$ jobs processed from $L_i$, which constitutes at most $\frac{m^{2k-j-i}nk}{nm^{i-1}nk} \leq \frac{k}{m}$ fraction of the total number of jobs in $L_i$.

Let $\sigma$ be a schedule whose makespan is at most $nk$, and fix $\gamma > k^2/m$ to be a small constant. From Lemma we know that for every $1 \leq i \leq k$, at least an $(1-\gamma)$ fraction of the jobs in $L_i$ is scheduled on machines in $M_i$. From the structure of the graph in the NO Case of the $k$-partite Problem, we know that we cannot start more than $\delta$ fraction of the jobs in $L_i$ before finishing $(1-\delta)$ fraction of the jobs in $L_{i-1}$, for all $2 \leq i \leq k$. Hence the maximum makespan of any such schedule $\sigma$ is at least $(1-2\delta)(1-\gamma)nk$. See figure 4.

4.2 $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$

We present in this section a reduction from a $k$-partite graph to an instance of the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$, and prove a tight inapproximability result for the latter, assuming Hypothesis 5. Formally, we prove the following result:

**Theorem 8.** Assuming Hypothesis 5, it is NP-hard to approximate the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$ within any constant factor strictly better than $2$.

To prove this, we first reduce a $k$-partite graph $G = (V_1, ..., V_k, E_1, ..., E_{k-1})$ to a scheduling instance $\tilde{I}(k)$, and then show that

1. If $G$ satisfies the YES Case of Hypothesis 5 then $\tilde{I}(k)$ has a feasible schedule whose makespan is roughly $kQ/2$.
2. If $G$ satisfies the NO Case of Hypothesis 5 then any schedule for $\tilde{I}(k)$ must have a makespan of roughly $kQ$.

**Reduction** The reduction has three parameters: an odd integer $k$, an integer $Q$ such that $Q \gg k$ and $n$ divides $Q$, and a real $\epsilon \gg 1/Q^2 > 0$.

Given a $k$-partite graph $G = (V_1, ..., V_k, E_1, ..., E_{k-1})$, we construct an instance $\tilde{I}(k)$ of the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$ as follows:

- For each vertex $v \in V_{2i-1}$ and every $1 \leq i \leq (k+1)/2$, we create a set $J_{2i-1,v}$ of $Qn - (Q-1)$ jobs.
For each vertex \(v \in V_2\) and every \(1 \leq i < (k + 1)/2\), we create a chain of length \(Q - 1\) of jobs, i.e., a set \(J_{2i,v}\) of \(Q - 1\) jobs
\[
J_{2i,v} = \{J^1_{2i,v}, J^2_{2i,v}, \ldots, J^{Q-1}_{2i,v}\}
\]
where we have \(J^l_{2i,v} \prec J^{l+1}_{2i,v}\) for all \(l \in \{1, 2, \ldots, Q - 2\}\).

For each edge \(e = (v, w) \in E_{2i-1}\) and every \(1 \leq i < (k + 1)/2\), we have \(J_{2i-1,v} \prec J_{2i,w}\).

For each edge \(e = (v, w) \in E_{2i}\) and every \(1 \leq i < (k + 1)/2\), we have \(J^{Q-1}_{2i,v} \prec J_{2i+1,w}\).

Finally the number of machines is \((1 + Q\epsilon)n^2\).

Theorem 8 now follows from the following lemma, whose proof can be found in Appendix A.

**Lemma 2.** Scheduling instance \(\tilde{I}(k)\) has the following two properties.

1. If \(G\) satisfies the YES Case of Hypothesis \([3]\), then \(\tilde{I}(k)\) has a feasible schedule whose makespan is \((1 + \epsilon)kQ/2\), where \(\epsilon\) can be arbitrary close to zero.
2. If \(G\) satisfies the NO Case of Hypothesis \([3]\), then any feasible schedule for \(\tilde{I}(k)\) must have a makespan of \((1 - \epsilon)kQ\), where \(\epsilon\) can be arbitrary close to zero.

Although not formally defined, one can devise a similar reduction for the case of \(k = 2\), and prove a \(3/2\)-inapproximability result for \(P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}\), assuming the variant of the UNIQUE GAMES Conjecture in \([1]\). We illustrate this in Appendix C and prove the following result:

**Theorem 9.** For any \(\epsilon > 0\), it is NP-hard to approximate \(P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}\) within a factor of \(3/2 - \epsilon\), assuming \((a\ variant\ of)\ the\ UNIQUE\ GAMES\ Conjecture.\)

5 Discussion

We proposed in this paper a natural but nontrivial generalisation of Theorem 4 that seems to capture the hardness of a large family of scheduling problems with precedence constraints. It is interesting to investigate whether this generalisation also illustrates potential intrinsic hardness of other scheduling problems, for which the gap between the best known approximation algorithm and the best known hardness result persists.

On the other hand, a natural direction would be to prove Hypothesis \([5]\), we show in Section \([3]\) how to prove a less-structured version of it using the bipartite graph resulting from the variant of the UNIQUE GAMES Conjecture in \([1]\). One can also tweak the dictatorship \(T_{\epsilon,t}\) of \([1]\), to yield a \(k\)-partite graph instead of a bipartite one. However, composing this test with a UNIQUE GAMES instance adds a noisy component to our \(k\)-partite graph, that we do not know how to control, since it is due to the non-perfect completeness of the UNIQUE GAMES instance. One can also try to impose \((a\ variant\ of)\ this\ dictatorship\ test\ on\ \textit{d-to-1\ GAMES}\ instances,\ and\ perhaps\ prove\ the\ hypothesis\ assuming\ the\ \textit{d-to-1\ Conjecture,}\ although\ we\ expect\ the\ size\ of\ the\ partitions\ to\ deteriorate\ as\ \(k\)\ increases.\)
Acknowledgments

The authors are grateful to Ola Svensson for inspiring discussions and valuable comments that influenced this work. We also wish to thank Hyung Chan An, Laurent Feuilloley, Christos Kalaitzis and the anonymous reviewers for several useful comments on the exposition.

References

1. N. Bansal and S. Khot. Optimal long code test with one free bit. In Proc. FOCS 2009, FOCS ’09, pages 453–462, Washington, DC, USA, 2009. IEEE Computer Society.
2. C. Chekuri and M. Bender. An efficient approximation algorithm for minimizing makespan on uniformly related machines. Journal of Algorithms, 41(2):212–224, 2001.
3. F. A. Chudak and D. B. Shmoys. Approximation algorithms for precedence-constrained scheduling problems on parallel machines that run at different speeds. Journal of Algorithms, 30(2):323–343, 1999.
4. D. Gangal and A. Ranade. Precedence constrained scheduling in (2-7/(3p+1)) optimal. Journal of Computer and System Sciences, 74(7):1139–1146, 2008.
5. T. F. Gonzalez and D. B. Johnson. A new algorithm for preemptive scheduling of trees. Journal of the ACM (JACM), 27(2):287–312, 1980.
6. R. L. Graham. Bounds for certain multiprocessing anomalies. Bell System Technical Journal, 45(9):1563–1581, 1966.
7. R. L. Graham, E. L. Lawler, J. K. Lenstra, and A. H. G. Rinnooy Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. Annals of discrete mathematics, 5(2):287–326, 1979.
8. D. S. Hochbaum and D. B. Shmoys. Using dual approximation algorithms for scheduling problems theoretical and practical results. Journal of the ACM (JACM), 34(1):144–162, 1987.
9. D. S. Hochbaum and D. B. Shmoys. A polynomial approximation scheme for scheduling on uniform processors: Using the dual approximation approach. SIAM journal on computing, 17(3):539–551, 1988.
10. J. K. Lenstra and A. R. Kan. Computational complexity of discrete optimization problems. Annals of Discrete Mathematics, 4:121–140, 1979.
11. R. McNaughton. Scheduling with deadlines and loss functions. Management Science, 6(1):1–12, 1959.
12. R. R. Muntz and E. G. Coffman Jr. Optimal preemptive scheduling on two-processor systems. Computers, IEEE Transactions on, 100(11):1014–1020, 1969.
13. R. R. Muntz and E. G. Coffman Jr. Preemptive scheduling of real-time tasks on multiprocessor systems. Journal of the ACM (JACM), 17(2):324–338, 1970.
14. P. Schuurman and G. J. Woeginger. Polynomial time approximation algorithms for machine scheduling: Ten open problems. Journal of Scheduling, 2(5):203–213, 1999.
15. O. Svensson. Hardness of precedence constrained scheduling on identical machines. SIAM Journal on Computing, 40(5):1258–1274, 2011.
16. J. D. Ullman. Complexity of sequencing problems. Computer and Job-Shop Scheduling Theory, EG Co man, Jr.(ed.), 1976.
17. D. P. Williamson and D. B. Shmoys. The design of approximation algorithms. Cambridge University Press, 2011.
A Proof of Lemma 2

In this section, we prove Lemma 2 that is we show that the reduction in Section 4.2 from a k-partite graph $G$ to a scheduling instance $I(k)$ yields a hardness of $2 - \epsilon$ for the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$, for any $\epsilon > 0$. This follows from combining Lemmas 3 and 4.

Lemma 3 (Completeness). If the given $k$-partite graph $G$ satisfies the properties of the YES case of Hypothesis 5, then there exists a valid schedule for $\tilde{I}(k)$ with maximum makespan $(1 + \epsilon')kQ/2$, where $\epsilon'$ can be made arbitrarily close to zero.

Proof. Assume that $G$ satisfies the properties of the YES Case of Hypothesis 5 and let $\{V_{s,\ell}\}$ for $s \in \{1, \ldots, k\}, \ell \in [Q]$ denote the good partitioning of the vertices of $G$. We use this partitioning to derive a partitioning $\{S_{i,j}\}$ for the jobs in the scheduling instance $\tilde{I}(k)$ for $1 \leq i \leq (k - 1)Q/2 - 1$, $j \in [Q]$, where a set of jobs $S_{i,j}$ can be either big or small.

The intuition behind this big/small distinction is that a job $J$ is in a big set if it is part of the $Qn - (Q-1)$ copies of a vertex $v \in V_{2i-1}$ for $1 \leq i \leq (k+1)/2$, and in a small set otherwise.

These sets can now be formally defined as follows:

Big sets: $S_{Q(i-1)+1,j} := \bigcup_{v \in V_{2i-1},v} J_{2i-1,v} \forall 1 \leq i \leq \frac{k+1}{2}$, $j \in [Q]$

Small sets: $S_{Q(i-1)+1+l,j} := \bigcup_{v \in V_{2i},v} J_{2i,v} \forall 1 \leq i < \frac{k+1}{2}$, $j \in [Q], l \in [Q-1]$

We first provide a brief overview of the schedule before defining it formally. Since $S_{1,0}$ is the set of the jobs corresponding to the vertices in $V_{1,0}$, scheduling all the jobs in $S_{1,0}$ in the first time step enables us to start the jobs at the first layer of the chain corresponding to vertices in $V_{2,0}$ (i.e., $S_{2,0}$). Therefore in the next time step we can schedule the jobs corresponding to the vertices in $V_{1,1}$, (i.e. $S_{1,1}$) and $S_{2,0}$. This further enables us to continue to schedule the jobs in the second layer of the chain corresponding to the vertices in $V_{2,0}$ (i.e., $S_{2,0}$), the jobs at the first layer of the chain corresponding to vertices in $V_{2,1}$ (i.e., $S_{2,1}$), and the jobs corresponding to the vertices in $V_{1,2}$ (i.e., $S_{1,2}$). We can keep going the same way, until we have scheduled all the jobs. Since the number of partitions of each vertex set $V_t$ is $Q$, and length of each of our chains is $Q - 1$, we can see that in the suggested schedule, we are scheduling in each time step at most $Q$ sets, out of which exactly one is big, and none of the precedence constraints are violated (see Figure 2).

Formally speaking, let $T_t$ be the union of $S_{i,j}$ such that $t = i + j - 1$, where $1 \leq i \leq (k - 1)Q/2 + 1$ and $j \in [Q]$, hence each $T_t$ consist of at most $Q$ sets of the jobs in which exactly one of them is a big set and at most $Q - 1$ of them are
small sets. Therefore, for \( t \in [(k + 1)Q/2] \) we have

\[
|\mathcal{T}_t| \leq |V_{2i-1,j}| \cdot (Qn - (Q - 1)) + |V_{2i,j}| \cdot (Q - 1) \\
\leq (\frac{1}{Q} + \epsilon)n \cdot (Qn - (Q - 1)) + (\frac{1}{Q} + \epsilon)n \cdot (Q - 1) \\
\leq (\frac{1}{Q} + \epsilon)n \cdot (Qn) \leq (1 + Q\epsilon)n^2
\]

One can easily see that all the jobs in a set \( \mathcal{T}_t \) can be scheduled in a single time step since the number of machines is \((1 + Q\epsilon)n^2\). Hence consider the following schedule: for each \( \mathcal{J}_t \in [((k + 1)/2)], \) schedule all the jobs in \( \mathcal{T}_t \) between time \( t \) and \( t + 1 \). We claim that this schedule does not violate any precedence constraint. This is true because we first schedule the predecessors of the job, and then the job in the following steps. Formally, if \( J_1 < J_2 \) with \( J_1 \in \mathcal{T}_{t_1} \) and \( J_2 \in \mathcal{T}_{t_2} \), then \( t_1 < t_2 \). The structure of such schedule is depicted in Figure 3.

**Lemma 4 (Soundness).** If the given \( k \)-partite graph \( G \) satisfies the properties of the No Case of Hypothesis 3 then any feasible schedule for \( \mathcal{J}(k) \) has a maximum makespan of at least \((1 - \epsilon')kQ\), where \( \epsilon' \) can be made arbitrary close to zero.

**Proof.** Assume that \( G \) satisfies the NO Case of Hypothesis 5 and consider the following partitioning of the jobs:

**Big partitions:** \( \mathcal{S}_{Q(i-1)+1} := \bigcup_{v \in V_{2i-1}} \mathcal{J}_{2i-1,v} \quad \forall 1 \leq i \leq (k + 1)/2 \)

**Small partitions:** \( \mathcal{S}_{Q(i-1)+1+l} := \bigcup_{v \in V_{2i}} \mathcal{J}_{2i,v} \quad \forall 1 \leq i < (k + 1)/2, l \in [Q - 1] \)

Note that \( \{\mathcal{S}\} \) partitions the jobs into \((k - 1)Q/2 + 1\) partitions such that the size of a big partition is \( n(nQ - c) \geq n(n - 1)Q \) and the size of a small partition is \( n \). Let \( f_i \) be the first time that a \((1 - \delta)\) fraction of the jobs in \( \mathcal{S}_i \) is completely executed, and let \( s_i \) be the first time that more than \( \delta \) fraction of the jobs in \( \mathcal{S}_i \) is started. Because of the expansion property of the NO Case, we can not start more that \( \delta \) fraction of the jobs in the second partition, before finishing at least \( 1 - \delta \) fraction of the jobs in the first partition. This implies that \( f_1 \leq s_2 \). Similarly, \( f_1 + 1 \leq s_3 \) and \( f_1 + Q - 2 \leq s_Q \). The same inequalities hold for any big partition and the small partitions following it. This means that, beside \( \delta \) fraction of the jobs in the \( i \)-th and \((i + 1)\)-th big partitions, the rest of the jobs in the \((i + 1)\)-th big partition start \( Q - 1 \) steps after finishing the jobs in the \( i \)-th big partition. Also we need at least \( \frac{(1 - \delta)n(n - 1)Q}{(1 + Q\epsilon)n^2} = (1 - \epsilon_1)Q \) time to finish \( 1 - \delta \) fraction of the jobs in a big partition. This gives that the makespan is at least:

\[
(1 - \epsilon_1)(k + 1)Q/2 + (k - 1)(Q - 1)/2 \geq (1 - \epsilon_2)kQ
\]

where \( \epsilon_2 = \epsilon_2(Q, k, \epsilon, \delta) \), which can be made small enough for an appropriate choice of \( Q, k, \epsilon \) and \( \delta \).
B The Perfect Matching Approach

In this section, we prove Theorem 10 by presenting a direct reduction from a bipartite graph of Theorem 10 to a $k$-partite graph. It is also proved in [11] that Theorem 10 holds assuming a variant of the UNIQUE GAMES Conjecture, and note that the former implies Theorem 4.

**Theorem 10.** For every $\epsilon, \delta > 0$, and positive integer $Q$, the following problem is NP-hard assuming Hypothesis $\frac{3}{2}$ given an $n$-by-$n$ bipartite graph $G = (V, W, E)$, distinguish between the following two cases:

- **YES Case:** We can partition $V$ into disjoint sets $V_0, V_1, \ldots, V_{Q-1}, V_{err}$ with $|V_0| = |V_1| = \cdots = |V_{Q-1}| = \frac{1-\delta}{Q}$, and $W$ into disjoint sets $W_0, W_1, \ldots, W_{Q-1}, W_{err}$ with $|W_0| = |W_1| = \cdots = |W_{Q-1}| = \frac{1-\delta}{Q}$ such that for every $i \in [Q]$ and any vertex $w \in W_i$, $w$ only have edges to vertices in $V_i \cup V_{err}$.
- **NO Case:** For any $S \subseteq V$, $T \subseteq W$, $|S| = \delta n$, $|T| = \delta n$, there is an edge between $S$ and $T$.

**Reduction** We present a reduction from an $n$-by-$n$ bipartite graph $G = (V, W, E)$ to a $k$-partite graph $G_k = (U_1, \ldots, U_k, E_1, \ldots, E_{k-1})$. From the expansion property of the No Case in Theorem 10, we get that the size of the maximum matching is at least $(1-\delta)n$. Therefore we can assume that the graph $G$ has a matching of size at least $(1-\delta)n$. We find a maximum matching $M$ and remove all the other vertices from $G$. Let the resulting graph be $G' = (V', W', E')$, where $|V'| = |W'| = n' \geq n(1-\delta)$, $V' = \{v_0, \ldots, v_{n'}\}$ and $W' = \{w_0, \ldots, w_{n'}\}$. Also assume w.l.o.g. that $v_i$ is matched to $w_i$ in the matching $M$ for all $i \in [n']$.

**Observation 11** Assume that $G$ satisfies the YES Case of Theorem 10, and let $\{V_i\}, \{W_i\}$ for $i \in [Q]$ and $V_{err}, W_{err}$ be the good partitioning. We use the latter to define a new partitioning $\{V'_i\}, \{W'_i\}$ for $i \in [Q]$ , and $V'_{err}, W'_{err}$ as follows:

\[
\begin{align*}
V'_{err} &:= (V_{err} \cup \{v_i, w_i \in W_{err}\}) \cap V' \\
W'_{err} &:= (W_{err} \cup \{w_i, v_i \in V_{err}\}) \cap W'
\end{align*}
\]

then the following two observations hold:

- For all $i \in [Q]$, $|V'_i| \geq (1-\delta - 2\epsilon)n$, $|W'_i| \geq (1-\delta - 2\epsilon)n$.
- For any vertex $w \in W'_i$, $w$ only have edges to vertices in $V'_i \cup V'_{err}$.

**Observation 12** Assume that $G$ satisfies the NO Case of Theorem 10, then $G'$ satisfies the NO Case as well, i.e., for any two sets $S \subseteq V'$, $T \subseteq W'$, $|S| = \delta n$, $|T| = \delta n$, there is an edge between $S$ and $T$.

We are now ready to construct the $k$-partite graph $G_k$ from $G'$.

- Let $U_i = \{u_{i,0}, u_{i,1}, \ldots, u_{i,n'-1}\}$ be a set of vertices of size $n'$ for all $i \in \{1, \ldots, k\}$.
- For any edge $e = (v_i, v_j) \in E'$ add edge $(u_{i,l}, u_{l+1,j})$ to $E_l$, for all $l \in \{1, \ldots, k-1\}$.
Completeness We show that if the given bipartite $G$ satisfies the properties of the YES Case of Theorem 10 then the YES Case of Theorem 6 holds. Hence assume that we are in the YES Case and let $\{V_i\}$ for $i \in [Q]$ denote the good partitioning and $\{V'_i\}$ denote the partitioning derived from it as described in Observation 11. For all $l = \{1, \ldots, k\}$ and $i \in [Q]$ let

$$U_{l,i} = \{u_{l,j} | v_j \in V'_i\}$$

$$U_{1,\text{err}} = \{u_{1,j} | v_j \in V'_{\text{err}}\}$$

It follows from Observation 11 and the fact that we have the same set of edges in all the layers, that the new partitioning has the properties of the YES Case of Theorem 6.

Soundness We show that if the given bipartite $G$ satisfies the properties of the NO Case of Theorem 10, then the YES Case of Theorem 6 holds. To that end, assume that we are in the NO Case, therefore the given bipartite graph satisfy that for any $S \subseteq V$, $T \subseteq W$, $|S| = |T| = \delta n$, there is an edge between $S$ and $T$. From Observation 12 we get that the same expansion property holds for $G'$, i.e. for any $S \subseteq V$, $T \subseteq W$, $|S| = |T| = \delta n$, there is an edge between $S$ and $T$. Moreover, we have the same set of the edges in all the layers, so we get that each layer has the expansion property.

C Hardness of Approximation

In this section, we prove Theorem 9 of Section 4.2. For the sake of presentation, we restate Theorem 4 as it is a key component in the reduction. In other words, we prove that assuming the variant of the UNIQUE GAMES Conjecture in [1], it is NP-hard to approximate the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$ within any constant factor strictly better than $3/2$. To do so, we present a reduction from a bipartite graph of Theorem 13 to a scheduling instance $\tilde{I}$ such that:

- If $G$ satisfies the YES Case of Theorem 13 then $\tilde{I}$ has a schedule whose makespan is roughly $2Q$.
- If $G$ satisfies the NO Case of Theorem 13 then every schedule for $\tilde{I}$ must have a makespan of at least $3Q - 1 - \epsilon Q$.

**Theorem 13.** For every $\epsilon, \delta > 0$, and positive integer $Q$, given an $n$ by $n$ bipartite graph $G = (V, W, E)$ such that, assuming a variant of UNIQUE GAMES Conjecture, it is NP-hard to distinguish between the following two cases:

- **YES Case:** We can partition $V$ into disjoints sets $V_0, V_1, \ldots, V_{Q-1}, V_{\text{err}}$ with $|V_0| = |V_1| = \cdots = |V_{Q-1}| = \frac{1-\epsilon}{Q}$, and $W$ into disjoint sets $W_0, W_1, \ldots, W_{Q-1}, W_{\text{err}}$ with $|W_0| = |W_1| = \cdots = |W_{Q-1}| = \frac{1-\epsilon}{Q}$ such that for every $i \in [Q]$ and any vertex $w \in W_i$, $w$ only have edges to vertices in $V_i \cup V_{\text{err}}$.
- **NO Case:** For any $S \subseteq V$, $T \subseteq W$, $|S| = \delta |V|$, $|T| = \delta |W|$, there is an edge between $S$ and $T$. 


Reduction We present a reduction from an $n \times n$ bipartite graph $G = (V, W, E)$ to a scheduling instance $\tilde{I}$, for some integer $Q$ that is the constant of Theorem 10:

- For each vertex $v \in V$, we create a set $J_v$ of $Qn$ jobs each of size 1, and let $J_V := \bigcup_{v \in V} J_v$.
- For each vertex $w \in W$, we create a set $J_w$ of $Q(n + 1) - 1$ jobs $J_w = \{J^1_w, J^2_w, \ldots, J^{Q-1}_w\} \cup J^Q_w$ where $J^Q_w$ is the set of the last $Qn$ jobs, and the first $Q - 1$ jobs are the chain jobs. We also define $J_W := \bigcup_{w \in W} J^Q_w$.
- For each edge $e = (v, w) \in E$, we have a precedence constraint between $J_v \prec J^1_w$ for all $J_v \in J_v$.
- For each $w \in W$, we have the following precedence constraints:
  \[
  J^1_w \prec J^i_w \quad \forall i \in \{1, 2, \ldots, Q - 2\}
  \\
  J^{Q-1}_w \prec J^Q_w
  \]

In total, the number of jobs and precedence constraints is polynomial in $n$ since

\[
\text{number of the jobs} \leq Qn^2 + n(Q(n + 1) - 1) = 2Qn^2 + Qn - n
\]

For a subset $S$ of jobs in our scheduling instance $\tilde{I}$, we denote by $\Psi(S) \subseteq V \cup W$ the set of their representative vertices in the starting graph $G$. Similarly, for a subset $S \subseteq V \cup W$, $\Psi^{-1}(S) \subseteq J_V \cup J_W$ is the set of all jobs, except for chain jobs, corresponding to vertices in $S$, i.e.,

\[
\Psi^{-1}(S) = \left( \bigcup_{v \in \tilde{S} \setminus W} J_v \right) \cup \left( \bigcup_{w \in \tilde{S} \setminus V} J^Q_w \right)
\]

A subset $S$ of jobs with $S = \Psi(S)$ is said to be complete if $S = \Psi^{-1}(S)$.

W.l.o.g. assume that $Q$ divides $n$. Finally the number of machines is $n^2$.

Before proceeding with the proof of Theorem 9 we record the following easy observations:

**Observation 14** If for some $w \in W$, there exist a feasible schedule $\sigma$ in which a job $J \in J^Q_w$ starts before time $T$, then the set $A \subseteq J_V$ of all its predecessors in $J_V$ must have finished executing in $\sigma$ prior to time $T - Q$. Moreover $A$ is complete, i.e., $A = J_V$.

**Observation 15** For any subset $A \subseteq J_V \cup J_W$, we have that

\[
|\Psi(A)| \geq \frac{|A|}{nQ}
\]

where the bound is met with equality if $A$ is complete.
Completeness  Let $V_0, V_1, \ldots, V_{Q-1}, V_{err}, W_0, W_1, \ldots, W_{Q-1}, W_{err}$ be the partitions as in the YES Case of Theorem 13. For ease of notation, we merge $V_{err}$ with $V_0$, and $W_{err}$ with $W_{Q-1}$, i.e., $V_0 = V_0 \cup V_{err}$ and $W_{Q-1} = W_{Q-1} \cup W_{err}$.

Note that this implies that for all $i \in [Q]$, any vertex $w \in W_i$ is only connected to vertices in $V_j$ where $j \leq i$ also:

$$|W_i|, |V_i| \leq \left(\frac{1 - \epsilon}{Q} + \epsilon\right) n \quad \forall i \in [Q] \quad (1)$$

For a subset $S \subseteq V \cup W$, we denote by $J_S$ the set of jobs corresponding to vertices in $S$, i.e., $J_S = \bigcup_{u \in S} J_u$. Also, for an index $i \in \{1, 2, \ldots, 2Q\}$, we define a job set $T_i$ as follows:

$$T_i = \begin{cases} S_i \cup J_{V_{i-1}} & 0 \leq i < Q \\ S_i \cup J_{W_{i-Q-1}} & Q \leq i < 2Q \end{cases}$$

where

$$S_i = \{ J_{W_{k-1}} : 1 \leq \ell < Q, \ k \in [Q], \ \text{and} \ k + \ell = i \}$$

The intuition behind partitioning the jobs into $S$ and $T$ follows from the same reasoning of the completeness proof of Appendix A. Observe here that using the structure of the graph, we get that if there exists two jobs $J, J'$ such that $J' \prec J$ and $J \in J_{W_k}$, then $J'$ can only be in one of the following two sets:

$$J' \in J_{V_k} \ \text{s.t.} \ k' \leq k \quad \text{or} \quad J' \in J_{W_{k'}} \ \text{s.t.} \ k' = k, \ell' < \ell$$

This then implies that a schedule $\sigma$ in which we first schedule $T_1$ then $T_2$, and so on up to $T_{2Q}$ is indeed a valid schedule. Now using equation (1) and the construction of our scheduling instance $\tilde{I}$, we get that

$$|T_i| \leq Qn^2 \left(\frac{1 - \epsilon}{Q} + \epsilon\right) + nQ \left(\frac{1 - \epsilon}{Q} + \epsilon\right) \leq n(n + 1)(1 + \epsilon Q)$$

Hence the total makespan of $\sigma$ is at most

$$\sum_{i=1}^{2Q} \frac{|T_i|}{n^2} = 2Q \left(1 + \epsilon Q + O\left(\frac{1}{n}\right)\right)$$

which tends to $2Q + \epsilon'$ for large values of $n$.

Soundness  Assume towards contradiction that there exists a schedule for $\tilde{I}$ with a maximum makespan less than $t := 3Q - 1 - 2\epsilon Q$, and let $A$ be the set of jobs in $J_W$ that started executing by or before time $s := 2Q - 1 - \epsilon Q$, and denote by $B$ the set of their predecessors in $J_V$. Note that $B$ is complete by Observation 14. Now since $t - s = Q - \epsilon Q$, we get that $|A| \geq \epsilon Q n^2$, and hence,
by Observation 15, $|A| := |\Psi(A)| \geq \frac{\epsilon Qn^2}{Qn} = \epsilon n$. Applying Observation 14 one more time, we get that all the jobs in $B$ must have finished executing in $\sigma$ by time $Q - \epsilon Q$, and hence $|B| \leq Qn^2(1 - \epsilon)$. Using the fact that $B$ is complete, we get that $|B| := |\Psi(B)| \leq \frac{Qn^2(1 - \epsilon)}{Qn} = (1 - \epsilon)n$, which contradicts with the NO Case of Theorem 4.

It is important to note here that we can settle for a weaker structure of the graph corresponding to the completeness case of Theorem 4. In fact, we can use a graph resulting from Theorem 2 in [15], and yet get a hardness of $3/2 - \epsilon$. This will then yield this somehow stronger statement:

**Theorem 16.** For any $\epsilon > 0$, and $\eta \geq \eta(\epsilon)$, where $\eta(\epsilon)$ tends to 0 as $\epsilon$ tends to 0, if $1|\text{prec} \mid \sum_j w_j C_j$ has no $(2 - \epsilon)$-approximation algorithm, then $P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}$ has no $(3/2 - \eta)$-approximation algorithm.

### D Linear Programming Formulation for $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$

In this section, we will be interested in a feasibility Linear Program, that we denote by $[LP]$, for the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$. For a makespan guess $T$, $[LP]$ has a set of indicator variables $\{x_{j,t}\}$ for $j \in \{1, 2, \ldots, n\}$ and $t \in \{1, 2, \ldots, T\}$. A variable $x_{j,t}$ is intended to be the fraction of the job $J_j$ scheduled between time $t - 1$ and $t$. The optimal makespan $T^*$ is then obtained by doing a binary search and checking at each step if $[LP]$ is feasible:

$$\sum_{j=1}^{n} x_{j,t} \leq m \quad \forall t \in \{1, 2, \ldots, T\} \quad (2)$$

$$\sum_{t=1}^{T} x_{j,t} = 1 \quad \forall j \in \{1, 2, \ldots, n\} \quad (3)$$

$$\sum_{t=1}^{t'} x_{\ell,t} + \sum_{t=t'+1}^{T} x_{k,t} \geq 1 \quad \forall J_{\ell} < J_k, \forall t' \in \{1, 2, \ldots, T\} \quad (4)$$

$$x_{j,t} \geq 0 \quad \forall j \in \{1, 2, \ldots, n\}, \forall t \in \{1, 2, \ldots, T\} \quad (5)$$

To see that $[LP]$ is a valid relaxation for the scheduling problem $P|\text{prec, pmtn, } p_j = 1|C_{\text{max}}$, note that constraint (3) guarantees that the number of jobs processed at each time unit is at most the number of machines, and constraint (4) says that in any feasible schedule, all the jobs must be assigned. Also any schedule that satisfies the precedence requirements must satisfy constraint (4).

#### D.1 Integrality Gap

In order to show that $[LP]$ has an integrality gap of 2, we start by constructing a family of integrality gap instances of $3/2$ and gradually increase this gap to 2. The reason is that the $3/2$ case captures the intrinsic hardness of the problem, and we show how to use it as basic building block for the construction of the target integrality gap instance of 2.
Basic Building Block  We start by constructing a $P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}$ scheduling instance $I(d)$ parametrised by a large constant $d \geq 2$, that shows that the integrality gap of $[\text{LP}]$ is $3/2$, and constitutes our main building block for the next reduction. Let $m$ be the number of machines, and $n = 2dm - (d - 1)$ the number of jobs. The instance $I(d)$ is then constructed as follows:

- The first $dm - (d - 1)$ jobs $J_1, \ldots, J_{dm - (d - 1)}$ have no predecessors [Layer 1].
- A chain of $(d - 1)$ jobs $J_{dm - (d - 1) + 1}, \ldots, J_{dm}$ such that $J_{dm - (d - 1) + 1}$ is the successor of all the jobs in the Layer 1, and $J_{k-1} \prec J_k$ for $k \in \{dm - (d - 1) + 2, \ldots, dm\}$ [Layer 2].
- The last $dm - (d - 1)$ jobs $J_{dm + 1}, \ldots, J_{2dm - (d - 1)}$ are successors of $J_{dm}$ [Layer 3].

We first show that $I(d)$ is an integrality gap of $3/2$ for $[\text{LP}]$. This basically follows from the following lemma:

**Lemma 5.** Any feasible schedule for $I(d)$ has a makespan of at least $3d - 2$, however $[\text{LP}]$ has a feasible solution $\{x_{j,t}\}$, for $t \in \{1, 2, \ldots, 2d\}$ and $j = 1, \ldots, 2dm - (d - 1)$ of value $2d$. Moreover, for $t = d + 1 + \ell$, $\ell \in \{1, 2, \ldots, d - 1\}$, the machines in the feasible LP solution can still execute a load of $\frac{\ell}{d}$, i.e., $m - \sum_{j \in J} x_{j,t} \geq \frac{\ell}{d}$.

**Proof.** Consider the following fractional solution:

[Layer 1] \[ x_{1,1} = 1 \quad \text{and} \quad x_{j,t} = \frac{1}{d} \quad \forall j \in \{2, 3, \ldots, dm - (d - 1)\}, \forall t \in \{1, 2, \ldots, d\} \]

[Layer 2] \[ x_{dm - (d - 1) + 1, t+1} = x_{dm - (d - 1) + 2, t+2} = \cdots = x_{dm, t+d-1} = \frac{1}{d} \quad \forall \ell \in \{1, 2, \ldots, d\} \]

[Layer 3] \[ x_{j,t} = \frac{1}{d} \quad \forall j \in \{dm + 1, dm + 2, \ldots, 2dm - (d - 1)\}, \forall t \in \{d + 1, d + 2, \ldots, 2d\} \]

One can easily verify that each job $J$ is completely scheduled, i.e., $\sum_{t=1}^{2d} x_{j,t} = 1$. Moreover, the workload at each time step is at most $m$. To see this, we consider the following three types of time steps:

1. For $t = 1$, the workload is
   \[ 1 + \frac{1}{d} \times (dm - (d - 1) - 1) = m \]

2. For $t = 2, \ldots, d$, the workload is
   \[ \frac{1}{d} \times ((dm - (d - 1) - 1) + t - 1) \leq m \]
3. For \( t = d + 1, \ldots, 2d \), the workload is

\[
\frac{1}{d} ((dm - (d - 1)) + (2d - t)) = m - \frac{t - (d + 1)}{d}
\]

Note that in this feasible solution, we have that for \( t = d + 1 + i, i \in \{1, 2, \ldots, d - 1\} \), the machines can still execute a load of \( \frac{1}{d} \).

We have thus far verified that \( \{x_{j,t}\} \) satisfies the constraints (2) and (3) of \( \text{[LP]} \). Hence it remains to check (4). Except for job \( J_1 \), any two jobs \( J_k \) and \( J_\ell \), such that \( J_k \) is a direct predecessor of \( J_\ell \), satisfy the following properties by construction: If \( t_k = \min \{t : x_{k,t} > 0\} \), then

1. \( \max \{t : x_{k,t} > 0\} = t_k + d - 1 \).
2. \( \min \{t : x_{k,t} > 0\} = t_k + 1 \).
3. \( \max \{t : x_{\ell,t} > 0\} = t_k + d \).

Hence for any such jobs \( J_k \) and \( J_\ell \), and for any \( t \in \{t_k, t_k + 1, \ldots, t_k + d\} \) we get

\[
\sum_{t=1}^{t-1} x_{k,t} + \sum_{t=t+1}^{2d} x_{k,t} = \sum_{t=t_k}^{t-1} x_{k,t} + \sum_{t=t_k+1}^{t_k+d} x_{k,t} \quad \text{and} \quad \sum_{t=t_k+1}^{t+1} x_{\ell,t} = \frac{t - 1 - t_k + 1}{d} + \frac{t_k + d - (t + 1) + 1}{d} = 1
\]

Similarly, for \( t \in \{1, 2, \ldots, t_k - 1\} \) (respectively \( t \in \{t_k + d + 1, \ldots, 2d\} \)), the second (and respectively first) summation will be 1.

On the other hand, one can see that we should schedule all the jobs in Layer 1 in order to start with the first job in Layer 2. Similarly, due to the chain-like structure of Layer 2, it requires \( d - 1 \) times steps to be scheduled, before any job in Layer 3 can start executing. Hence the makespan of any feasible schedule is at least

\[
\frac{dm - (d - 1)}{m} + (d - 1) + \frac{dm - (d - 1)}{m} = 3d - \frac{2(d - 1)}{m} = 1 \geq 3d - 2.
\]

**Final Instance** We now construct our final integrality gap instance \( \mathcal{I}(k, d) \), using the basic building block \( \mathcal{I}(d) \). This is basically done by replicating the structure of \( \mathcal{I}(d) \), and arguing that any feasible schedule for \( \mathcal{I}(k, d) \) must have a makespan of roughly \( 2kd \), whereas we can extend the the LP solution of Lemma 5 for the instance \( \mathcal{I}(d) \), to a feasible LP solution for \( \mathcal{I}(k, d) \) of value \( (k + 1)d \). A key point that we use here is that the structure of the LP solution of Lemma 5 enables us to schedule a fraction of the chain jobs of a layer, while executing the non-chain jobs of the previous layer. We now proceed to prove that the integrality gap of \( \text{[LP]} \) is 2, by constructing a family \( \mathcal{I}(k, d) \) of scheduling instances, using the basic building block \( \mathcal{I}(d) \).

**Theorem 17.** \( \text{[LP]} \) has an integrality gap of 2.
Proof. Consider the following family of instances \( \mathcal{I}(k, d) \) for constant integers \( k \) and \( d \), constructed as follows:

- We have \( k + 1 \) layers \( \{ \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{k+1} \} \) similar to Layer 1 in \( \mathcal{I}(d) \), and \( k \) layers \( \{ \mathcal{L}_1^2, \mathcal{L}_2^2, \ldots, \mathcal{L}_k^2 \} \) similar to Layer 2. i.e., \( \mathcal{L}_1 \) has \( dm - (d - 1) \) jobs \( J_{i,1}, \ldots, J_{i, dm - (d - 1)} \) for all \( i \in \{1, \ldots, k + 1\} \) and \( \mathcal{L}_i^2 \) has \( (d - 1) \) jobs \( J_{i,1}^2, \ldots, J_{i, (d-1)}^2 \) for all \( i \in \{1, \ldots, k + 1\} \).

- For \( i \in \{1, 2, \ldots, k\} \):
  - Connect \( \mathcal{L}_i^1 \) to \( \mathcal{L}_i^2 \) in the same way that Layer 1 is connected to Layer 2 in \( \mathcal{I}(d) \), that is, the job \( J_{i,1}^1 \in \mathcal{L}_i^1 \) is a successor for all the jobs in \( \mathcal{L}_i^1 \).
  - Connect \( \mathcal{L}_i^2 \) to \( \mathcal{L}_{i+1}^1 \) in the same way that Layer 2 is connected to Layer 3 in \( \mathcal{I}(d) \), that is, all the jobs in \( \mathcal{L}_{i+1}^1 \) are successors for the job \( J_{i, (d-1)}^2 \in \mathcal{L}_i^2 \).

Notice that for \( k = 1 \), the scheduling instance \( \mathcal{I}(1, d) \) is the same as the previously defined instance \( \mathcal{I}(d) \). In any feasible schedule, we need to first schedule the jobs in \( \mathcal{L}_1^1 \), then those in \( \mathcal{L}_1^2 \), then \( \mathcal{L}_2^1 \), and so on, until \( \mathcal{L}_{k+1}^1 \). Hence the makespan of any such schedule is at least

\[
(k + 1) \frac{dm - (d - 1)}{m} + k(d - 1) > 2kd + d - k - 1.
\]

We now show that [LP] has a feasible solution of value \((k + 1)d\). Let \( \{x_{j,t}\} \) for \( t = 1, \ldots, 2d \) and \( j = 1, \ldots, 2dm - (d - 1) \) be the feasible solution of value \( 2d \) obtained in Lemma 5. It would be easier to think of \( \{x_{j,t}\} \) as \( \{x_{j,t}^1\} \cup \{x_{j,t}^2\} \cup \{x_{j,t}^3\} \) where for \( i = 1, 2, 3 \), \( \{x_{j,t}^i\} \) is the set of LP variables corresponding to variables in Layer \( i \) in \( \mathcal{I}(d) \). We now construct a feasible solution \( \{y_{j,t}\} \) for \( \mathcal{I}(k, d) \). We similarly think of \( \{y_{j,t}\} \) as \( \{y_{j,t}^1\} \cup \{y_{j,t}^2\} \), where \( y_{j,t}^1 \) is the set of LP variables corresponding to jobs in \( \mathcal{L}_1^1 \) for some \( 1 \leq \ell \leq k + 1 \). The set \( \{y_{j,t}^2\} \) for \( \mathcal{I}(k, d) \) can then be readily constructed as follows:

- for \( J_{i,1}^1 \in \mathcal{L}_1^1 \), \( y_{j,t}^1 = x_{j,t}^1 \) for \( t \leq 2d \), and 0 otherwise.
- for \( J_{i,1}^2 \in \mathcal{L}_i^2 \), \( y_{j,t+(i-1)d}^2 = x_{j,t}^2 \) for \( i = 1, 2, \ldots, k \), and \( t \leq 2d \), and 0 otherwise.
- for \( J_{i,1}^1 \in \mathcal{L}_i^1 \), \( y_{j,t+(i-1)d}^1 = x_{j,t}^1 \) for \( i = 2, 3, \ldots, k + 1 \), and \( t \leq 2d \), and 0 otherwise.

Using Lemma 5, we get that for \( t = d + 1 + i \), \( i \in \{1, 2, \ldots, d - 1\} \), the machines in the feasible LP solution can still execute a load of \( \frac{d}{2} \), and hence invoking the same analysis of Lemma 5 with the aforementioned observation for every two consecutive layers of jobs, we get that \( \{y_{j,t}\} \) is a feasible solution for [LP] of value \((k + 1)d\).
Fig. 1: Structure of the Soundness Versus Completeness of $Q|\text{prec}|C_{\text{max}}$ assuming Hypothesis 5.

The schedule on the left corresponds to the case where the graph represents the NO Case of Hypothesis 5; note that most of the machines are idle but for a small fraction of times. The schedule on the right corresponds to the case where the graph represents the YES Case; the schedule is almost packed. This case also illustrates the ordering of the jobs within each machine according to the partitioning of the jobs in the $k$-partite graph.
Fig. 2: Example of the construction of sets \( \{S\} \) for \( P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}} \) in the YES Case of Hypothesis 5 for \( k = 3 \) and \( Q = 4 \), along with their respective finishing time in the defined schedule. The boxes in the figure represent sets of jobs, and the sets that are grouped together have no precedence constraints within each others. Hence a feasible schedule is to schedule each group during the same time step. These groups corresponds to the sets \( T_i \) of Appendix A.
Fig. 3: Structure of Soundness versus Completeness of $P|\text{prec}, \text{pmtn}, p_j = 1|C_{\text{max}}$. Assuming Hypothesis \(\text{5}\)  

The schedule on the left corresponds to the case where the starting graph represents the NO Case of Hypothesis \(\text{5}\). Note the most of the machines are idle most of the time in this case. The schedule on the right corresponds to the case where the starting graph represents the YES Case of the hypothesis; Note that all the machines are packed almost all the time. This case also illustrates our partitioning of the jobs in sets \(\{T_i\}\), where \(T_i = \bigcup_{i,j:i+j-1=m} S_{i,j}\).