Radially symmetric thin plate splines interpolating a circular contour map

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Abstract

Profiles of radially symmetric thin plate spline surfaces minimizing the Beppo Levi energy over a compact annulus $R_1 \leq r \leq R_2$ have been studied by Rabut via reproducing kernel methods. Motivated by our recent construction of Beppo Levi polyspline surfaces, we focus here on minimizing the radial energy over the full semi-axis $0 < r < \infty$. Using a $L$-spline approach, we find two types of minimizing profiles: one is the limit of Rabut’s solution as $R_1 \to 0$ and $R_2 \to \infty$ (identified as a ‘non-singular’ $L$-spline), the other has a second-derivative singularity and matches an extra data value at 0. For both profiles and $p \in [2, \infty]$, we establish the $L^p$-approximation order $3/2 + 1/p$ in the radial energy space. We also include numerical examples and obtain a novel representation of the minimizers in terms of dilates of a basis function.

Keywords: thin plate spline, radially symmetric function, $L$-spline, Beppo Levi polyspline, approximation order

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1 Introduction

A well-known tool for scattered data interpolation, the thin plate spline surface is defined as the unique minimizer of the squared seminorm

$$
\|F\|_{BL}^2 := \int_{\mathbb{R}^2} \left( |F_{xx}|^2 + 2 |F_{xy}|^2 + |F_{yy}|^2 \right) dx \, dy,
$$

among all admissible $F$ taking prescribed values at given points (not all collinear) in the plane. The notation $BL$ stands for the Beppo Levi energy space of admissible continuous functions $F$ with generalized second-order derivatives in $L^2(\mathbb{R}^2)$. Originally discovered and utilized in aerospace engineering (Sabin [22], Harder and Desmarais [12]), thin plate splines became a research topic in approximation theory with the work of Duchon [11].
A different type of thin plate spline surfaces, matching continuous curves prescribed along concentric circles, has recently been proposed by Bejancu [6], using the *polyspline* method developed by Kounchev [18]. The new surfaces, called *Beppo Levi polysplines on annuli*, are minimizers of the polar coordinate version of (1), namely

\[
\|f\|_{BL}^2 := \int_0^\infty \int_{-\pi}^{\pi} \left\{ |f_{rr}|^2 + 2 \left| \frac{f_\theta}{r} - \frac{f_{r\theta}}{r} \right|^2 + \left| \frac{f_\theta}{r^2} + \frac{f_r}{r} \right|^2 \right\} r \, d\theta \, dr,
\]

(2)

where \(f(r,\theta) := F(r \cos \theta, r \sin \theta)\) is the polar form of \(F \in C^2(\mathbb{R}^2 \setminus \{0\}) \cap BL\). Related Beppo Levi polyspline surfaces interpolating continuous periodic data along parallel lines or hyperplanes have also been studied in [4, 5, 7].

The construction of Beppo Levi polysplines on annuli is based on their Fourier series representation in \(\theta\), with amplitude coefficients depending on \(r\).

In this paper, we focus on the special case of radially symmetric Beppo Levi polysplines, which arise as zero-frequency amplitudes (non-zero frequencies are treated in [8]). Note that, for radial functions \(f(r,\theta) \equiv f(r)\), the energy functional (2) takes the following form (up to a constant factor):

\[
\int_0^\infty \left( r \left| \frac{d^2 f}{dr^2} \right|^2 + \frac{1}{r} \left| \frac{df}{dr} \right|^2 \right) \, dr.
\]

(3)

Therefore, minimizing (3) subject to \(f\) taking prescribed values at the set of positive radii \(r_1 < \ldots < r_n\) is equivalent to determining the profile \(f\) of a radially symmetric thin plate spline surface interpolating a circular contour map at the concentric circles \(r = r_j\), \(1 \leq j \leq n\).

For \(0 < R_1 < R_2 < \infty\), Rabut [21] has previously studied the related problem of minimizing \(\int_{R_1}^{R_2} (r |f''|^2 + \frac{1}{r} |f'|^2) \, dr\) in place of (3), subject to the interpolation conditions at \(r_1, \ldots, r_n\). Rabut obtained a closed form solution to this problem via the method of abstract splines and reproducing kernels.

Our approach (section 2) exploits the fact that an interpolating minimizer of (3) is actually a univariate L-spline over \((0, \infty)\), with 4-dimensional pieces joining-up \(C^2\)-continuously at the knots \(r_1, \ldots, r_n\). The key observation is that, after imposing the interpolation constraints and the condition that (3) be finite, there remains an extra degree of freedom on the interval \((0, r_1)\). This enables us to obtain two types of minimizers: one is the limit of Rabut’s profile as \(R_1 \to 0\) and \(R_2 \to \infty\) (generating a surface which is biharmonic, hence non-singular, at 0), the other has a second-derivative singularity and matches an extra data value at 0. We study the main properties of these profiles in section 3, including their linear representation over \((0, \infty)\) in terms of dilates of a basis function.

For both minimizing profiles and \(p \in [2, \infty]\), in section 4 we establish a \(L^p\)-error bound of order \(O(h^{3/2+1/p})\), where \(h\) is the maximum distance between consecutive radii. This is further applied to derive, in [8], a \(L^2\)-convergence result for transfinite surface interpolation with biharmonic Beppo Levi polysplines on annuli. We also prove that the exponent 3/2 in the above approximation order cannot be increased in general for data functions from the...
radial energy space. In section 5, we illustrate the numerical accuracy of the
interpolatory L-spline profiles and point out a connection with the compactly
supported radial basis functions recently studied by Johnson [15].

2 Preliminaries

2.1 Admissible profiles

Let $AC_{\text{loc}}$ be the vector space of functions $f : (0, \infty) \to \mathbb{C}$ which are absolutely
continuous on any interval $[a, b]$, where $0 < a < b < \infty$. Throughout this paper,
$\Lambda_0$ denotes the energy space of complex-valued functions $f \in C^1(0, \infty)$, such
that $f' \in AC_{\text{loc}}$ and the integral \((\ref{eq:3})\) is finite. For any $f, g \in \Lambda_0$, we define the
semi-inner product

$$
\langle f, g \rangle_0 := \int_0^\infty \left( r f''(r) \overline{g''(r)} + \frac{1}{r} f'(r) \overline{g'(r)} \right) dr.
$$

The induced squared seminorm $\|f\|_0^2 := \langle f, f \rangle_0$ on $\Lambda_0$ equals the radial Beppo Levi energy \((\ref{eq:3})\).

**Lemma 1** If $f \in \Lambda_0$, then $f$ can be extended by continuity at 0. Also, the
relation $f'(r) = O(1)$ holds as $r \to 0$ and as $r \to \infty$.

**Proof.** Using the Leibniz-Newton formula $f(r) = f(s) + \int_s^r f'(t) dt$ for $r, s > 0$, as
well as the Cauchy-Schwarz estimate

$$
|f(r) - f(s)| = \left| \int_s^r t^{1/2} \left[ t^{-1/2} f'(t) \right] dt \right|
\leq \left| \int_s^r t dt \right|^{1/2} \left| \int_s^r \left| t^{-1/2} f'(t) \right|^2 dt \right|^{1/2}
\leq 2^{-1/2} r^{1/2} - s^{1/2} \|f\|_0,
$$

it follows that $f$ is Lipschitz of order 1/2 on any bounded subinterval of $(0, \infty)$.
In particular, $f$ is uniformly continuous on $(0, 1)$, hence it can be extended by
continuity at 0. Next, since $f' \in AC_{\text{loc}}$, we have, for each $r > 0,

$$
r^{-1} f'(r) - f'(1) = \int_1^r \left[ t^{-1} f'(t) \right]' dt
= \int_1^r t^{-3/2} \left[ t^{1/2} f''(t) - t^{-1/2} f'(t) \right] dt.
$$

Therefore the estimate

$$
\left| \int_1^r t^{-3/2} \left[ t^{1/2} f''(t) \right] dt \right| + \left| \int_1^r t^{-3/2} \left[ t^{-1/2} f'(t) \right] dt \right|
\leq \left| \int_1^r t^{-3} dt \right|^{1/2} \left\{ \left| \int_1^r t^{1/2} f''(t) \right|^2 dt \right|^{1/2} + \left| \int_1^r t^{-1/2} f'(t) \right|^2 dt \right|^{1/2}
\leq 2^{1/2} \left| 1 - r^{-2} \right|^{1/2} \|f\|_0,
$$

3
implies the existence of a constant $C_f \geq 0$ such that
\[ |f'(r)| \leq C_f \left( r + |1 - r^2|^{1/2} \right), \quad \forall r > 0, \]
hence $f'(r) = O(1)$ as $r \to 0$. The boundedness of $f'$ at $\infty$ follows similarly from
\[
rf'(r) - f'(1) = \int_1^r [tf'(t)]' \, dt = \int_1^r t^{1/2} \left[ t^{1/2} f''(t) + t^{-1/2} f'(t) \right] \, dt
\]
and the Cauchy-Schwarz estimates:
\[
\left| \int_1^r t^{1/2} \left[ t^{1/2} f''(t) \right] \, dt \right| + \left| \int_1^r t^{-1/2} f'(t) \, dt \right| \leq \left| \int_1^r t^{1/2} \left[ f''(t) \right]^2 \, dt \right|^{1/2} + \left| \int_1^r \left[ t^{-1/2} f'(t) \right]^2 \, dt \right|^{1/2} \leq 2^{-1/2} |r^2 - 1|^{1/2} \|f\|_0.
\]
The proof is complete. ■

2.2 \textit{L}-spline framework

Our \textit{L}-spline approach is motivated by the necessary conditions satisfied by a minimizer of (3). First, note that the self-adjoint Euler-Lagrange differential operator associated to the integrand of (3) is
\[
L_0 := \frac{d^2}{dr^2} \left( r \frac{d^2}{dr^2} \right) - \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \right)
= r \frac{d^4}{dr^4} + 2 \frac{d^3}{dr^3} - \frac{1}{r} \frac{d^2}{dr^2} + \frac{1}{r^2} \frac{d}{dr}
= r \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right)^2.
\]
The substitution $r = e^v$, $\frac{d}{dr} = r \frac{d}{dv}$, changes $L_0$ into a differential operator with constant coefficients in variable $v$. Factoring this operator and reverting to variable $r$, we obtain
\[
L_0 = \frac{1}{r^3} \left( r \frac{d}{dr} - 2 \right)^2 \left( r \frac{d}{dr} \right)^2,
\]
and we identify the null space of $L_0$ as the 4-dimensional vector space
\[
\text{Ker } L_0 = \text{span } \{ r^2, r^2 \ln r, 1, \ln r \}. \quad (6)
\]
Hence usual calculus of variations suggests that a minimizer of \([3]\) subject to prescribed values at \(r_1 < \ldots < r_n\) should be in \(\text{Ker} L_0\) on each of the open intervals \((0, r_1), (r_1, r_2), \ldots, (r_n, \infty)\).

Second, as observed in \([2]\), the convergence of the radial Beppo Levi energy integral \([3]\) at 0 and \(\infty\) implies that a minimizer should in fact belong to span \(\{r^2, r^2 \ln r, 1\}\) for \(r \in (0, r_1)\), and to span \(\{1, \ln r\}\) for \(r \in (r_n, \infty)\). These two spans are null spaces of the following left and right ‘boundary’ operators:

\[
G_0 := \frac{d^3}{dr^3} - \frac{1}{r} \frac{d^2}{dr^2} + \frac{1}{r^2} \frac{d}{dr} = \frac{1}{r^3} \left( r \frac{d}{dr} - 2 \right)^2 \left( r \frac{d}{dr} \right),
\]

\[
R_0 := \frac{1}{r} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] = \frac{1}{r^3} \left( r \frac{d}{dr} \right)^2.
\]

Let \(\rho\) be the given set of positive knots \(r_1 < \ldots < r_n\) and, in addition, let \(r_0 := 0, r_{n+1} := \infty\). The above considerations support the following definition.

**Definition 1** (a) A function \(\eta : [0, \infty) \to \mathbb{C}\) is called a Beppo Levi \(L_0\)-spline on \(\rho\) if the following conditions hold:

(i) \(L_0 \eta(r) = 0, \forall r \in (r_j, r_{j+1}), \forall j \in \{1, \ldots, n - 1\}\);

(ii) \(G_0 \eta(r) = 0, \forall r \in (0, r_1)\), and \(R_0 \eta(r) = 0, \forall r > r_n\);

(iii) \(\eta\) is \(C^2\)-continuous at each knot \(r_1, \ldots, r_n\).

We denote by \(S_0(\rho) \subset \Lambda_0\) the class of all Beppo Levi \(L_0\)-splines on \(\rho\).

(b) A Beppo Levi \(L_0\)-spline \(\eta \in S_0(\rho)\) that satisfies \(\eta \in \text{span} \{r^2, 1\}\) for \(r \in (0, r_1)\) is called non-singular.

**Remark 1** On any interval of positive real numbers, Kounchev \([18, p. 104]\) characterized the null space of \(L_0\) as an extended complete Chebyshev (ECT) system in the sense of Karlin and Ziegler \([16]\), due to the representation

\[
L_0 = D_4 D_3 D_2 D_1 = \frac{d}{dr} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} \frac{d}{dr},
\]

where \(D_1 = \frac{d}{dr}, D_2 = D_4 = \left( \frac{d}{dr} (r) \right)\), \(D_3 = \left( \frac{d}{dr} \right)^2 \left( \frac{1}{r} \right)\). Following Schumaker \([24, p. 398]\), we note that \(L_0\) also admits the factorization

\[
L_0 = L^* L,
\]

where \(L := \frac{1}{\sqrt{r}} \frac{d}{dr} r \frac{d}{dr} = r^{3/2} R_0\) and \(L^*\) denotes the formal adjoint of \(L\). This type of factorization was employed in the early studies of Ahlberg, Nilson, and Walsh \([1]\) and Schultz and Varga \([23]\) to define ‘generalized splines’ and ‘\(L\)-splines’ as functions that are piecewise in the null space of \(L^* L\) and satisfy certain continuity conditions. Our definition adopts the subsequent terminology of Lucas \([20]\) and Jerome and Pierce \([13]\), according to which ‘\(L\)-splines’ are piecewise in the null space of a general self-adjoint differential operator \(L\) with variable coefficients.
Remark 2 The Beppo Levi boundary conditions (ii) of our definition do not agree with the usual ‘natural’ (or ‘type II’) conditions from L-spline literature [20, 13, 24], the latter being formulated in terms of one and the same differential operator outside the interpolation domain or at the endpoints of this domain.

3 Interpolation with Beppo Levi \(L_0\)-splines

Due to the new type of boundary conditions, the main properties of interpolation with Beppo Levi \(L_0\)-splines are not direct consequences of classical L-spline theory, but will be established in this section based on our first theorem below.

3.1 A fundamental orthogonality result

Theorem 1 Let \(\eta \in S_0(\rho)\) and \(\psi \in \Lambda_0\) such that

\[
\psi(r_j) = 0, \quad \forall j \in \{1, \ldots, n\}.
\]

If, in addition, we assume that either \(\psi\) satisfies

\[
\psi(0) = 0,
\]

or \(\eta\) is non-singular, then

\[
\langle \eta, \psi \rangle_0 = 0.
\]

Proof. Recalling the notation \(r_0 := 0, r_{n+1} = \infty\), (4) implies

\[
\langle \eta, \psi \rangle_0 = \sum_{j=1}^{n+1} \int_{r_{j-1}}^{r_j} \left[ r \eta''(r) \bar{\psi}''(r) + \frac{1}{r} \eta'(r) \bar{\psi}'(r) \right] dr.
\]

Since \(\psi' \in AC_{loc}\), we may apply integration by parts to the first term of each integral to obtain

\[
\langle \eta, \psi \rangle_0 = \sum_{j=1}^{n+1} \left[ r \eta''(r) \bar{\psi}'(r) \right]_{r_{j-1}}^{r_j} - \sum_{j=1}^{n+1} \int_{r_{j-1}}^{r_j} \bar{\psi}'(r) \left[ r \eta'''(r) + \eta''(r) - \frac{1}{r} \eta'(r) \right] dr.
\]

Note that the first sum of the above right-hand side is telescopic due to the continuity of \(\eta''\). Hence, we only need to evaluate the boundary terms of this sum corresponding to \(r := r_0 = 0\) and \(r := r_{n+1} = \infty\). We invoke the specific form of \(\eta \in S_0(\rho)\) on the extreme sub-intervals: \(\eta \in \text{span}\{r^2, r^2 \ln r, 1\}\) for \(r \in (0, r_1)\), and \(\eta \in \text{span}\{1, \ln r\}\) for \(r \in (r_n, \infty)\). By Lemma 1, it follows that

\[
r \eta''(r) \bar{\psi}'(r) = \begin{cases} \mathcal{O}(r |\ln r|), & \text{as } r \to 0, \\ \mathcal{O}(r^{-1}), & \text{as } r \to \infty. \end{cases}
\]
Therefore the boundary terms vanish in the limit at 0 and \( \infty \).

On the other hand, the fact that \( \eta \in \ker L_0 \) on each sub-interval implies the existence, for each \( j \in \{1, \ldots, n+1\} \), of a constant \( b_j \) such that

\[
    r \eta'''(r) + \eta''(r) - \frac{1}{r} \eta'(r) = b_j, \quad \forall r \in (r_{j-1}, r_j), \; \forall j \in \{1, \ldots, n+1\},
\]

where \( b_{n+1} = 0 \), since \( \eta(r) \in \text{span} \{1, \ln r\} \) for \( r \in (r_n, \infty) \). Hence

\[
    \langle \eta, \psi \rangle_0 = -\sum_{j=1}^{n} b_j \int_{r_{j-1}}^{r_j} \psi'(r) \, dr = -\sum_{j=1}^{n} b_j \left[ \psi(r) \right]_{r_{j-1}}^{r_j} = b_1 \left[ \psi(0) \right],
\]

the last equality being due to the vanishing condition (7) of \( \psi \) at the positive knots. Therefore (6) implies \( \langle \eta, \psi \rangle_0 = 0 \), as required. The same conclusion applies assuming that \( \eta \) is non-singular, since \( b_1 = 0 \) in that case. \( \blacksquare \)

### 3.2 Existence, uniqueness, variational characterization

Our \( L \)-spline formulation shows that the extra degree of freedom of any \( \eta \in S_0(\rho) \) on the leftmost interval \((0, r_1)\) allows one more restriction to be imposed on \( \eta \), apart from the interpolation conditions at the knots. The next result obtains existence and uniqueness for two types of interpolatory profiles from \( S_0(\rho) \): one matching an extra data value at 0, the other being non-singular.

**Theorem 2** Let \( \alpha, \nu_1, \nu_2, \ldots, \nu_n \) be arbitrary real values.

(a) There exists a unique Beppo Levi \( L_0 \)-spline \( \sigma^A \in S_0(\rho) \), such that

\[
\begin{align*}
\sigma^A(r_j) &= \nu_j, \quad \forall j \in \{1, \ldots, n\}, \\
\sigma^A(0) &= \alpha.
\end{align*}
\]

(b) There exists a unique non-singular Beppo Levi \( L_0 \)-spline \( \sigma^B \in S_0(\rho) \), such that

\[
\sigma^B(r_j) = \nu_j, \quad \forall j \in \{1, \ldots, n\}.
\]

**Proof.** For convenience, we first prove part (b). It is sufficient to establish the existence of a unique function \( \tilde{\sigma} : [r_1, r_n] \rightarrow \mathbb{C} \) with the properties: i) \( \tilde{\sigma} \in \ker L_0 \) on \((r_{j-1}, r_j)\) for \( j \in \{2, \ldots, n\} \); ii) \( \tilde{\sigma} \in C^2(r_1, r_n) \); iii) the interpolation conditions (11) hold for \( \tilde{\sigma} \) in place of \( \sigma^B \); and iv) \( \tilde{\sigma} \) satisfies the endpoint conditions:

\[
\begin{align*}
\left[ \frac{d}{dr} \left( \frac{d}{dr} - 2 \right) \tilde{\sigma}(r) \right]_{r \rightarrow r_1^+} &= 0, \\
\left[ \frac{d}{dr} \tilde{\sigma}(r) \right]_{r \rightarrow r_n^-} &= 0.
\end{align*}
\]

Indeed, one can uniquely determine the constants \( c_m \) for \( m \in \{1, 2, 3, 4\} \) such that the function \( \sigma^B \) defined by

\[
\sigma^B(r) := \begin{cases} 
    c_1 r^2 + c_2, & \text{if } 0 < r < r_1, \\
    \tilde{\sigma}(r), & \text{if } r_1 \leq r \leq r_n, \\
    c_3 + c_4 \ln r, & \text{if } r_n < r,
\end{cases}
\]



is continuous and has a continuous first derivative at $r_1$ and $r_n$. This also ensures that $rac{d^2}{dx^2} \sigma^B$ is continuous at $r_1$ and $r_n$, due to (12) and the fact that \((r_1 \frac{d}{dx}) (r_n \frac{d}{dx} - 2) \sigma^B (r) = 0, \forall 0 < r < r_1,\) and \(R_0 \sigma^B (r) = 0, \forall r > r_n.\) Hence, \(\sigma^B\) verifies the conditions required by the conclusion. Conversely, the above properties of \(\tilde{\sigma}\) hold with necessity for the restriction to \([r_1, r_N]\) of any non-singular Beppo Levi \(L_0\)-spline \(\sigma^B \in S_0 (\rho)\) that satisfies (11).

Note that a function \(\tilde{\sigma}\) as described in the previous paragraph is determined by four coefficients on each subinterval \((r_{j-1}, r_j), j \in \{2, \ldots, n\}\). These coefficients are required to satisfy the homogeneous linear equations given by three \(C^2\)-continuity conditions at each interior knot \(r_2, \ldots, r_{n-1}\) and two end-point conditions (12), as well as the \(n\) interpolation conditions (11). Hence this \(4(n-1) \times 4(n-1)\) system of linear equations becomes homogeneous if we assume zero interpolation values: \(\nu_j = 0, j \in \{1, \ldots, n\}\). Let \(\tilde{\sigma}\) be determined by an arbitrary solution of this homogeneous system and let \(\sigma^B \in S_0 (\rho)\) be the unique extension of \(\tilde{\sigma}\) to a non-singular Beppo Levi \(L_0\)-spline as in the previous paragraph. Choosing \(\eta := \psi := \sigma^B\) in Theorem 1, the orthogonality relation (9) implies \(\|\sigma^B\|_0 = 0\), hence \(\sigma^B\) is a constant function on \((0, \infty)\). Since \(\sigma^B (r_j) = 0, j \in \{1, \ldots, n\}\), we obtain \(\sigma^B \equiv 0\), which shows that the homogeneous linear system admits only the trivial solution. Therefore the \(4(n-1) \times 4(n-1)\) non-homogeneous system corresponding to arbitrary interpolation values has a unique solution, as required.

Part (a) of the theorem is similarly reduced to the existence of a unique function \(\sigma\), this time defined on \([0, r_n]\), with the properties: i) \(\sigma \in \ker L_0\) on \((r_{j-1}, r_j)\) for \(j \in \{2, \ldots, n\}\), while \(\sigma \in \text{span}\{r^2, r^2 \ln r, 1\}\) on \([0, r_1]\); ii) \(\sigma \in C^2(0, r_n)\); iii) \(\sigma\) satisfies (10) (in place of \(\sigma^A\)); iv) \(\sigma\) satisfies the second of the endpoint conditions (12). All these constraints amount to a system of \(4n-1\) linear equations for as many coefficients. Then arguments similar to those of the last paragraph show that the corresponding homogeneous linear system has only the trivial solution, which completes the proof. □

**Remark 3** Existence and uniqueness results for \(L_0\)-spline interpolation have previously been obtained by Kounchev [17, 18] under different boundary conditions. Specifically, Kounchev uses a clamped condition at the right-end point \(r_n\) (i.e., the first-derivative is prescribed at \(r_n\)), while, at the left-end point \(r_1\), his \(L_0\)-spline either is clamped or it satisfies the non-singularity condition stated in part (b) of Definition 1.

The next result shows that our \(L_0\)-spline interpolants are indeed profiles of radially symmetric thin plate spline surfaces minimizing the radial Beppo Levi seminorm (5).

**Theorem 3** Given arbitrary real values \(\alpha, \nu_1, \nu_2, \ldots, \nu_n\), let \(\sigma^A\) and \(\sigma^B\) denote the unique Beppo Levi \(L_0\)-splines obtained in Theorem 2. Then \(\|\sigma^A\|_0 < \|f\|_0\) whenever \(f \in \Lambda_0\) satisfies (10) in place of \(\sigma^A\) and \(f \neq \sigma^A\). Also, if \(f \in \Lambda_0\), \(f \neq \sigma^B\), and \(f\) satisfies conditions (11) in place of \(\sigma^B\), then \(\|\sigma^B\|_0 < \|f\|_0\).
Proof. Assume that \( f \in \Lambda_0 \), \( f \) satisfies the same interpolation conditions as \( \sigma^A \), and let \( \eta := \sigma^A, \psi := f - \sigma^A \). Since \( \psi \) satisfies (7) and (8), by Theorem 1 we have
\[
\langle \sigma^A, f - \sigma^A \rangle_0 = 0,
\]
which implies the first integral relation
\[
\|f\|^2_0 = \|\sigma^A\|^2_0 + \|f - \sigma^A\|^2_0.
\] (13)
Therefore \( \|f\|_0 \geq \|\sigma^A\|_0 \), with equality only if \( \|f - \sigma^A\|_0 = 0 \), which is equivalent to \( (f - \sigma^A)'(r) = 0 \), for all \( r \in (0, \infty) \). Since \( f - \sigma^A \) takes zero values at the knots \( r_1, \ldots, r_n \), this implies \( f \equiv \sigma^A \), which proves the first part of the theorem. The second part follows similarly, based on the corresponding relation (13) with \( \sigma^B \) replacing \( \sigma^A \).

As stated in the Introduction, for \( 0 < R_1 < R_2 < \infty \), Rabut obtained a reproducing kernel solution [21, (3.7)] to the related problem of minimizing
\[
\int_{R_1}^{R_2} (r |f''|^2 + \frac{1}{2} |f'|^2) \, dr
\]
in place of (3), subject to interpolation conditions at \( r_1, \ldots, r_n \) inside the compact interval \([R_1, R_2] \).

Corollary 1 Our non-singular Beppo Levi \( L_0 \)-spline interpolant \( \sigma^B \) coincides with the limit, as \( R_1 \to 0 \) and \( R_2 \to \infty \), of Rabut’s minimizer over \([R_1, R_2] \).

Proof. Let \( s_{R_1, R_2} \) denote the expression [21, (3.7)] of Rabut’s minimizer, where \( B := \{r_1, \ldots, r_n\} \) and we regard the coefficients \( \{\lambda_b\}_{b \in B} \) of this expression as undetermined constants that satisfy \( \sum_{b \in B} \lambda_b = 0 \). Then, letting \( R_1 \to 0 \) and \( R_2 \to \infty \) and using formulas [21, (3.9) & (3.10)] for the remaining two coefficients \( \alpha \) and \( \beta \) of \( s_{R_1, R_2} \), we denote by \( s_{0, \infty} \) the resulting limit expression of \( s_{R_1, R_2} \). Since, by [21, (3.1)], \( H_1 \) is a \( C^2 \)-continuous function of \( r \), it is a straightforward matter to verify that \( s_{0, \infty} \) is in fact a non-singular Beppo Levi \( L_0 \)-spline in the sense of our Definition 1. Given that \( s_{0, \infty} \) satisfies the same interpolation conditions as \( \sigma^B \), Theorem 4 implies \( s_{0, \infty} \equiv \sigma^B \).

Remark 4 Unlike \( \sigma^B \), Rabut’s minimizer \( s_{R_1, R_2} \) satisfies the ‘natural’ boundary conditions from classical calculus of variations. Indeed, expression [21, (2.9)] and the last two lines of [21, (3.4)] show that the end conditions for \( s_{R_1, R_2} \) are formulated by means of identical differential operators at both endpoints \( R_1 \) and \( R_2 \), in agreement with the ‘natural’/’type II’ conditions of \( L \)-spline literature [20] or [13, Theorem 4].

Although the related Beppo Levi \( L \)-splines studied in [8] employ mutually adjoint boundary operators on the extreme subintervals \((0, r_1)\) and \((r_n, \infty)\), it can be verified that our non-singular full-space minimizer \( \sigma^B \) does not share this property. Note that adjoint boundary conditions were first identified in the context of univariate interpolation by full-space Matérn kernels [5, 7] (for a recent treatment of Matérn kernels on a compact interval, see [10]).
3.3 Linear representation with dilates of a basis function

We now obtain a representation of the two types of interpolatory Beppo Levi \(L_0\)-splines in terms of the following basis function:

\[
\varphi_0 (r) := \begin{cases} 
  r^2 - r^2 \ln r, & 0 \leq r \leq 1, \\
  1 + \ln r, & 1 < r.
\end{cases}
\] (14)

Note that \(\varphi_0\) is \(C^2\)-continuous at \(r = 1\), \(G_0 \varphi_0 (r) = 0\) for \(0 < r < 1\), \(R_0 \varphi_0 (r) = 0\) for \(1 < r\), and \(\varphi_0 (0) = 0\).

The expression stated in the next theorem will be used in section 5 to evaluate our Beppo Levi \(L_0\)-spline interpolants on a fine mesh and estimate the numerical accuracy of this approximation procedure. Similar representations for a related family of \(L\)-splines [6, Lemma 3] have played a crucial role in the construction of Beppo Levi polyspline surfaces which interpolate smooth curves prescribed on concentric circles.

**Theorem 4** Let \(\sigma\) be one of the Beppo Levi \(L_0\)-splines \(\sigma^A\) or \(\sigma^B\) satisfying the interpolation conditions (10) or (11) of Theorem 2 for given values \(\alpha, \nu_1, \ldots, \nu_n\). There exist unique coefficients \(c, a_1, \ldots, a_n\), such that, for all \(r \geq 0\), we have the representation

\[
\sigma (r) = c + \sum_{k=1}^{n} a_k \varphi_0 \left( \frac{r}{r_k} \right),
\] (15)

where, if \(\sigma = \sigma^B\), the coefficients also satisfy:

\[
\sum_{k=1}^{n} a_k r_k^{-2} = 0.
\] (16)

**Proof.** Letting \(\sigma_1 (r)\) denote the right-hand side of (15), the noted properties of \(\varphi_0\) imply \(\sigma_1 \in S_0 (\rho)\) and \(c = \sigma_1 (0)\). Also, \(\sigma_1\) is non-singular if and only if (16) holds. Therefore, invoking Theorem 2 we have to prove the existence of unique coefficients such that either \(\sigma_1\) satisfies (10) in place of \(\sigma^A\), or (11) and (16) hold with \(\sigma_1\) in place of \(\sigma^B\). Since these equations are linear, it is sufficient to assume that all data values \(\alpha, \nu_1, \ldots, \nu_n\) are zero and prove that each of the two resulting homogeneous systems admits only the trivial solution.

Indeed, for \(\alpha = 0\), the last equation of (10) implies \(c = 0\), hence the homogeneous system associated to (10) reduces to

\[
\sum_{k=1}^{n} a_k \varphi_0 \left( \frac{r_j}{r_k} \right) = 0, \quad \forall j \in \{1, \ldots, n\}.
\] (17)

To solve this system, we consider the auxiliary function

\[
\psi_0 (t) := e^{-t} \varphi_0 (e^t) = e^{-|t|} [1 + |t|], \quad t \in \mathbb{R},
\]
well-known in geostatistics as a ‘Matérn function’. It has positive Fourier transform
\[ \hat{\psi}_0(\tau) = \int_{-\infty}^{\infty} e^{-it\tau} \psi_0(t) \, dt = \frac{1}{(1 + \tau^2)^{\frac{\nu}{2}}}, \quad \tau \in \mathbb{R}, \]
implying that \( \psi_0 \) is a positive definite function. This means that the \( n \times n \) matrix \( (\psi_0(t_j - t_k))_{j,k=1}^{n} \) is positive definite, hence nonsingular, for any choice of distinct real values \( t_1, \ldots, t_n \). Therefore the homogeneous system
\[ \sum_{k=1}^{n} b_k e^{-(t_j-t_k)} \varphi_0(e^{t_j-t_k}) = 0, \quad \forall j \in \{1, \ldots, n\}, \tag{18} \]
admits only the trivial solution. But, after simplifying its \( j \)-th equation by \( e^{-t_j} \), (18) becomes (17) by letting \( t_k := \ln r_k, b_k := \alpha_k e^{-t_k}, \forall k \in \{1, \ldots, n\} \).

Now, assume that \( \sigma_1 \) satisfies the homogeneous system corresponding to (11) and (16). It follows that
\[ \sigma_1(r) = c + r^2 \sum_{k=1}^{n} \alpha_k \ln r_k / r_k^2, \quad \forall r \in (0, r_1). \]

Since \( \sigma_1 \in \mathcal{S}_0(\rho) \), Theorem 2, part (b), implies that \( \sigma_1 \) is identically zero on each subinterval, in particular \( c = 0 \). This leads to the same system (17) solved in the previous paragraph.

Remark 5 Although each dilation \( \cdot / r_k \) can be regarded as a ‘translation’ in the multiplicative topological group \((0, \infty)\), our representation (15) does not belong to the framework of interpolation via Hankel translates recently considered by Arteaga & Marrero.

4 Convergence orders

In this section, we establish \( L^p \)-error bounds for the interpolatory profiles studied in the previous section. The method of our proofs is based on the error analysis \([1, 23]\) developed for generalized splines and \( L \)-splines.

Theorem 5 For \( n \geq 2 \) and \( r_0 = 0 < r_1 < \ldots < r_n < \infty = r_{n+1} \), let \( \rho := \{r_1, \ldots, r_n\} \) and \( h := \max_{1 \leq j \leq n-1} (r_{j+1} - r_j) \). Given \( f \in \Lambda_0 \), let \( \alpha := f(0) \) and \( \nu_j := f(r_j) \) for \( 1 \leq j \leq n \). If \( \sigma \) denotes one of the corresponding Beppo Levi \( L_0 \)-spline interpolants \( \sigma^A \) or \( \sigma^B \) obtained in Theorem 2 then, for \( l \in \{0, 1\} \),
\[ \left\| f^{(l)} - \sigma^{(l)} \right\|_{L^\infty[r_1, r_n]} \leq \frac{1}{2^{l-1} \sqrt{r_1}} h^{3/2-l} \|f\|_0. \tag{19} \]

Proof. Let \( \psi := f - \sigma \), so \( \psi(r_j) = 0 \) for \( 1 \leq j \leq n \). Since \( \psi \in C^1(0, \infty) \), Rolle’s theorem implies that, for each \( j \in \{1, \ldots, n-1\} \), there exists \( t_j \in (r_j, r_{j+1}) \), such that
\[ \psi'(t_j) = 0. \]
Let $r^*, t^* \in [r_1, r_n]$ satisfy
\[
|\psi(r^*)| = \|\psi\|_{L^\infty[r_1, r_n]}, \quad |\psi'(t^*)| = \|\psi'\|_{L^\infty[r_1, r_n]},
\]
(20)
and choose $k \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, n - 1\}$, such that
\[
|r^* - r_k| \leq h/2, \quad |t^* - t_m| \leq h.
\]
(21)
Since $\psi'$ is locally absolutely continuous, it follows that $\psi''$ is locally integrable and we have the Leibniz-Newton formulae:
\[
\psi(r^*) = \psi(r^*) - \psi(r_k) = \int_{r_k}^{r^*} \psi'(r) \, dr,
\]
\[
\psi'(t^*) = \psi'(t^*) - \psi'(t_m) = \int_{t_m}^{t^*} \psi''(r) \, dr.
\]
(22)
Therefore the first line of (22) and (21) imply the estimate
\[
|\psi(r^*)| \leq \left| \int_{r_k}^{r^*} |\psi'(r)| \, dr \right| \leq \frac{h}{2} \|\psi'\|_{L^\infty[r_1, r_n]},
\]
(23)
while, using Cauchy-Schwarz and (21) in the second line of (22), we obtain
\[
|\psi'(t^*)| \leq \left| \int_{t_m}^{t^*} |\psi''(r)| \, dr \right| \leq \left( \int_{t_m}^{t^*} r^{-1/2} \left| \psi''(r) \right|^2 \, dr \right)^{1/2} \left( \int_{r_1}^{\infty} r \, |\psi'(r)|^2 \, dr \right)^{1/2}
\]
\[
\leq r_1^{-1/2} h^{1/2} \|\psi\|_0.
\]
(24)
On the other hand, the first integral relation (13), valid for both $\sigma^A$ and $\sigma^B$, implies
\[
\|\psi\|_0 = \|f - \sigma\|_0 \leq \|f\|_0.
\]
(25)
The conclusion follows from (20) and (23)–(25). \hfill \blacksquare

**Theorem 6** Under the hypotheses of Theorem 5 for $l \in \{0, 1\}$, we have
\[
\left\| f^{(l)} - \sigma^{(l)} \right\|_{L^2[r_1, r_n]} \leq \frac{1}{2^{1-l} \sqrt{r_1}} h^{2-l} \|f\|_0.
\]
(26)
**Proof.** We employ the notations from the proof of Theorem 5 in particular $\psi := f - \sigma$. For $j \in \{1, \ldots, n - 1\}$, we use the following univariate version of the well-known Friedrichs inequality:
\[
\int_{r_j}^{r_{j+1}} |\psi(r)|^2 \, dr \leq \frac{h^2}{4} \int_{r_j}^{r_{j+1}} |\psi'(r)|^2 \, dr.
\]
(27)
Indeed, if \( m_j := \frac{r_j + r_{j+1}}{2} \) and \( r_j \leq r \leq m_j \), then Leibniz-Newton formula, \( \psi(r_j) = 0 \), and Cauchy-Schwarz imply

\[
|\psi(r)|^2 = \left| \int_{r_j}^{r} \psi'(u) \, du \right|^2 \leq (r - r_j) \int_{r_j}^{r} |\psi'(u)|^2 \, du \leq \frac{h}{2} \int_{r_j}^{m_j} |\psi'(u)|^2 \, du,
\]

hence, by integration,

\[
\int_{r_j}^{m_j} |\psi(r)|^2 \, dr \leq \frac{h^2}{4} \int_{r_j}^{m_j} |\psi'(r)|^2 \, dr.
\]

Now (27) is obtained by adding this to a similar inequality that holds on the interval \([m_j, r_{j+1}]\). Summing both sides of (27) over \( j \) and taking square roots, we obtain

\[
\|\psi\|_{L^2[r_1, r_n]} \leq \frac{h}{2} \|\psi\|_{L^2[r_1, r_n]}.
\]  

(28)

Next, note that \( \psi'' \) is square integrable over \([r_1, r_n]\), due to

\[
\|\psi''\|_{L^2[r_1, r_n]}^2 = \int_{r_1}^{r_n} \frac{1}{r} \left( \frac{\psi''(r)}{r} \right)^2 \, dr \\
\leq \frac{1}{r_1} \int_{r_1}^{r_n} r \left( \frac{\psi''(r)}{r} \right)^2 \, dr \\
\leq \frac{1}{r_1} \|\psi''\|_0^2 \leq \frac{1}{r_1} \|f\|_0^2.
\]

(29)

Hence, for \( j \in \{1, \ldots, n-1\} \) and \( r_j \leq r \leq r_{j+1} \), since \( \psi' \) is locally absolutely continuous and \( \psi'(t_j) = 0 \), where \( r_j < t_j < r_{j+1} \), we have

\[
|\psi'(r)|^2 = \left| \int_{t_j}^{r} \psi''(u) \, du \right|^2 \leq |r - t_j| \int_{t_j}^{r} |\psi''(u)|^2 \, du \leq h \int_{r_j}^{r_{j+1}} |\psi''(u)|^2 \, du,
\]

which implies, via integration, the following analog of (27):

\[
\int_{r_j}^{r_{j+1}} |\psi'(r)|^2 \, dr \leq h^2 \int_{r_j}^{r_{j+1}} |\psi''(r)|^2 \, dr.
\]

Summing once more over \( j \) and using (29), we obtain

\[
\|\psi''\|_{L^2[r_1, r_n]}^2 \leq h^2 \|\psi''\|_{L^2[r_1, r_n]}^2 \leq \frac{1}{r_1} h^2 \|f\|_0^2,
\]

which establishes (26) via (28).

**Remark 6** The result of Theorem 6 is used in [8], along with similar convergence results for the related class of \( L^2 \)-splines studied there, to establish an \( L^2 \)-error bound for transfinite surface interpolation with biharmonic Beppo Levi polysplines on annuli.
Corollary 2 Under the hypotheses of Theorem 3, if \( p \in [2, \infty) \) and \( l \in \{0, 1\} \), then
\[
\left\| f(t) - \sigma(t) \right\|_{L^p[r_1, r_n]} \leq \frac{1}{2^{1-1/2}n^{3/2+1/p}} h^{3/2+1/p-l} \|f\|_{0}.
\] (30)

Proof. We employ a classical result on ‘interpolation between \( L^p \)-spaces’ (see [9, p. 175]): if \( 1 \leq p_0 < p < p_1 \leq \infty \) and \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) for some \( \theta \in (0, 1) \), then
\[
\|\psi\|_{L^p} \leq \|\psi\|^{1-\theta}_{L^{p_0}} \|\psi\|^\theta_{L^{p_1}}, \quad \forall \psi \in L^{p_0} \cap L^{p_1}.
\]
Letting \( p_0 = 2 \), \( p_1 = \infty \), it follows that \( 2 (1 - \theta) + \frac{\theta}{2} = \frac{3}{2} + \frac{1}{2} \), hence (30) is a consequence of (19) and (26). Alternatively, a direct proof of (30) for \( p \in (2, \infty) \) can be obtained via Jensen’s inequality, as in [24, Chapter 6].

Remark 7 For \( 1 \leq p < 2 \), the usual embedding of \( L^2[r_1, r_n] \) into \( L^p[r_1, r_n] \) shows that the \( L^2 \)-approximation order \( O(h^{2-l}) \) of (20) also applies to the corresponding \( L^p \)-norm of the error.

The final theorem of this section shows that the exponent 3/2 appearing in Theorem 3 is asymptotically sharp as \( h \to 0 \), in the sense that it cannot be increased for the class \( \Lambda_0 \) of data functions. A similar result holds for the approximation order obtained in Theorem 4 but is omitted, for brevity. We need the following lemma, whose proof is deferred to the Appendix.

Lemma 2 Let \( (X, \|\cdot\|) \) be a normed vector space of dimension at least \( d + 1 \) and denote by \( \mathcal{H} \subset X^d \) the set of all \( d \)-tuples \( (v_1, \ldots, v_d) \) of \( d \) linearly independent vectors in \( X \). For a given vector \( x \in X \) and any \( (v_1, \ldots, v_d) \in \mathcal{H} \), let \( \delta_x(v_1, \ldots, v_d) \) be the distance from \( x \) to the sub-space of all linear combinations of \( v_1, \ldots, v_d \), i.e.
\[
\delta_x(v_1, \ldots, v_d) = \inf_{v \in \text{span}(v_1, \ldots, v_d)} \|x - v\|.
\]
Then \( \delta_x \) is a continuous function of \( (v_1, \ldots, v_d) \) on \( \mathcal{H} \), where \( \mathcal{H} \) is endowed with the product topology induced from \( X^d \).

Theorem 7 For each \( n \geq 2 \), let \( h = 1/(n-1) \) and let \( \rho_h \) be the set of uniformly spaced knots \( r_j = 1 + (j-1)h \), \( j \in \{1, \ldots, n\} \), hence \( [r_1, r_n] = [1, 2] \). Then, for any \( \varepsilon > 0 \) arbitrarily small, there exists \( f_\varepsilon \in \Lambda_0 \) and a constant \( M > 0 \), such that
\[
M h^{3/2+\varepsilon} \leq \|f_\varepsilon - \sigma_h\|_{L^\infty[1,2]}
\]
holds for all sufficiently small \( h > 0 \) and all \( \sigma_h \in S_0(\rho_h) \).

Proof. We adapt the arguments of [23, Theorem 11], using the fact that the null space \( \text{Ker} L_0 \) described by (6) is a finite-dimensional vector space. Specifically, let \( \mu > 0 \) and, for each \( h \in [0, 1] \), define
\[
\alpha(h) = \alpha(h, \mu) = \inf_{\eta \in \text{Ker} L_0} \|t^\mu - \eta(1+th)\|_{L^\infty[0,1]}.
\]
We claim that \( \alpha \) is a continuous function of \( h \) on \([0, 1]\). To see this, first observe that, for any \( \eta \in \text{Ker} \ L_0 \) and \( h_0 \in [0, 1] \), we have

\[
\| \eta (1 + h) - \eta (1 + h_0) \|_{L^\infty [0, 1]} \to 0, \quad \text{as } h \to h_0,
\]

which is a consequence of the uniform continuity of any such function \( \eta \) on \([1, 2]\). Then use this observation for each of the four basis functions of \([6]\) and apply Lemma \(2\) with \( X := C \ [0, 1] \), \( d := 4 \), to deduce \( \alpha (h) \to \alpha (h_0) \) as \( h \to h_0 \).

Also, note that \( \alpha (h) > 0, \forall h \in [0, 1] \). Indeed, this inequality is easily verified if \( h = 0 \), while, if \( h \in (0, 1] \), it follows from the fact that \( (r - 1)^\mu \notin \text{Ker} \ L_0 \), for \( r \in [1, 1 + h] \). Therefore, letting

\[
\min_{0 \leq h \leq 1} \alpha (h, \mu) =: A (\mu),
\]

we have \( A (\mu) > 0 \) and

\[
\inf_{\eta \in \text{Ker} \ L_0} \| (th)^\mu - \eta (1 + th) \|_{L^\infty [0, 1]} = h^{\mu} \inf_{\eta \in \text{Ker} \ L_0} \| t^\mu - h^{-\mu} \eta (1 + th) \|_{L^\infty [0, 1]} = h^{\mu} \inf_{\zeta \in \text{Ker} \ L_0} \| t^\mu - \zeta (1 + th) \|_{L^\infty [0, 1]} \geq h^{\mu} A (\mu).
\]

Next, let \( g \) be a \( C^\infty \)-smooth function with compact support within \((0, \infty)\), such that \( g (r) = 1, \forall r \in [1, 2] \), and, for \( \varepsilon > 0 \), define

\[
f_\varepsilon (r) := g (r) |r - 1|^{\frac{3}{2} + \varepsilon}, \quad r \in (0, \infty).
\]

It is then easily verified that \( f_\varepsilon \in \Lambda_0 \) for each \( \varepsilon > 0 \). Making the change of variables \( r - 1 = th, \ t \in [0, 1] \), and using \((31)\) with \( \mu = \frac{3}{2} + \varepsilon \), we obtain, for any \( \sigma_h \in S_0 (\rho_h) \),

\[
\| f_\varepsilon - \sigma_h \|_{L^\infty [1, 2]} \geq \| f_\varepsilon - \sigma_h \|_{L^\infty [1, 1 + h]} \geq \inf_{\eta \in \text{Ker} \ L_0} \| f_\varepsilon - \eta \|_{L^\infty [1, 1 + h]} = \inf_{\eta \in \text{Ker} \ L_0} \| (r - 1)^{\frac{3}{2} + \varepsilon} - \eta (r) \|_{L^\infty [1, 1 + h]} \geq \inf_{\eta \in \text{Ker} \ L_0} \| (th)^{\frac{3}{2} + \varepsilon} - \eta (1 + th) \|_{L^\infty [0, 1]} \geq h^{\frac{3}{2} + \varepsilon} A \left( \frac{3}{2} + \varepsilon \right),
\]

hence the conclusion of the theorem holds with \( M := A \left( \frac{3}{2} + \varepsilon \right) \).

## 5 Examples and conclusion

### 5.1 Numerical accuracy of \( L_0 \)-spline interpolation

For a data profile \( f \in \Lambda_0 \), we denote by \( \sigma^A \) and \( \sigma^B \) the Beppo Levi \( L_0 \)-spline interpolants described in Theorem \(5\). To test the numerical accuracy of these two profiles, we partition the interval \([1, 2]\) into \( n - 1 \) equal subintervals of mesh-size \( h := \frac{1}{n - 1} \) by letting \( r_j := 1 + h (j - 1), \ j \in \{1, \ldots, n\} \), hence \( r_1 = 1 \) and \( r_n = 2 \).

Given \( n \) and \( f \), we use Matlab to compute the coefficients of representation \((15)\) for \( \sigma^A \) and \( \sigma^B \). Specifically, if \( \sigma := \sigma^A \), we let \( c := f (0) \) and find \( \{a_k\}_{k=1}^n \). 

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from the system
\[ \sum_{k=1}^{n} a_k \varphi_0 \left( \frac{r_j}{r_k} \right) = f(r_j) - f(0), \quad \forall j \in \{1, \ldots, n\}, \]
while the coefficients of \( \sigma := \sigma^B \) are determined from the corresponding system
\[ c + \sum_{k=1}^{n} a_k \varphi_0 \left( \frac{r_j}{r_k} \right) = f(r_j), \quad \forall j \in \{1, \ldots, n\}, \]
\[ \sum_{k=1}^{n} a_k r_k^{-2} = 0. \]

Next, we use expression \([15]\) to evaluate \( \sigma^{A,B} \) numerically on a ten-times finer grid, i.e. at 9 equi-spaced points of each subinterval \([r_j, r_{j+1}], j \in \{1, \ldots, n-1\}\). Our numerical approximation of the uniform error \( \|f - \sigma^{A,B}\|_{L^\infty[1,2]} \) is then computed as
\[ E_h^{A,B} := \max \left\{ \left| (f - \sigma^{A,B}) \left( 1 + h (j-1) + \frac{hl}{10} \right) \right| : j = 1, \ldots, n-1; \ l = 1, \ldots, 9 \right\}. \]

This procedure is employed successively for \( n - 1 := 2^m, m \in \{4, 5, \ldots, 10\} \). Since the values of \( E_h^{A,B} \) are expected to decay proportionally with \( h^\kappa \), for a positive constant \( \kappa \), we also compute the approximations
\[ \kappa_h^{A,B} := \log_2 \left( \frac{E_h^{A,B}}{E_{h/2}^{A,B}} \right) \approx \kappa. \]

Tables 1–3 display the numerical results for three data profiles \( f \).

| \( n - 1 \) | \( E_h^A \) | \( \kappa_h^A \) | \( E_h^B \) | \( \kappa_h^B \) |
|---|---|---|---|---|
| 16 | \( 9.8230 \times 10^{-4} \) | 1.5369 | \( 1.0307 \times 10^{-3} \) | 1.5709 |
| 32 | \( 3.3851 \times 10^{-4} \) | 1.5670 | \( 3.4691 \times 10^{-4} \) | 1.5839 |
| 64 | \( 1.1425 \times 10^{-4} \) | 1.5827 | \( 1.1572 \times 10^{-4} \) | 1.5919 |
| 128 | \( 3.8144 \times 10^{-5} \) | 1.5913 | \( 3.8390 \times 10^{-5} \) | 1.5959 |
| 256 | \( 1.2659 \times 10^{-5} \) | 1.5956 | \( 1.2700 \times 10^{-5} \) | 1.5979 |
| 512 | \( 4.1886 \times 10^{-6} \) | 1.5978 | \( 4.1954 \times 10^{-6} \) | 1.5990 |
| 1024 | \( 1.3838 \times 10^{-6} \) | \( - \) | \( 1.3849 \times 10^{-6} \) | \( - \) |

Table 1. \( f(r) := f_\varepsilon(r) \) for \( \varepsilon = 0.1 \)
Table 2. $f(r) = r$

| n − 1 | $E_h^A$   | $\kappa_h^A$ | $E_h^B$   | $\kappa_h^B$ |
|------|--------|--------|--------|--------|
| 16   | $1.7625 \times 10^{-4}$ | 1.9414 | $1.8257 \times 10^{-4}$ | 1.9665 |
| 32   | $4.5890 \times 10^{-5}$ | 1.9701 | $4.6715 \times 10^{-5}$ | 1.9829 |
| 64   | $1.1713 \times 10^{-5}$ | 1.9849 | $1.1818 \times 10^{-5}$ | 1.9913 |
| 128  | $2.9590 \times 10^{-6}$ | 1.9924 | $2.9724 \times 10^{-6}$ | 1.9956 |
| 256  | $7.4366 \times 10^{-7}$ | 1.9962 | $7.4534 \times 10^{-7}$ | 1.9978 |
| 512  | $1.8641 \times 10^{-7}$ | 1.9981 | $1.8662 \times 10^{-7}$ | 1.9989 |
| 1024 | $4.6663 \times 10^{-8}$ | $-$ | $4.6690 \times 10^{-8}$ | $-$ |

Table 3. $f(r) = \cos 3r$

| n − 1 | $E_h^A$   | $\kappa_h^A$ | $E_h^B$   | $\kappa_h^B$ |
|------|--------|--------|--------|--------|
| 16   | $1.5906 \times 10^{-3}$ | 2.0093 | $1.7113 \times 10^{-3}$ | 1.9712 |
| 32   | $3.9510 \times 10^{-4}$ | 2.0035 | $4.3646 \times 10^{-4}$ | 1.9840 |
| 64   | $9.8536 \times 10^{-5}$ | 2.0014 | $1.1033 \times 10^{-4}$ | 1.9916 |
| 128  | $2.4609 \times 10^{-5}$ | 2.0006 | $2.7744 \times 10^{-5}$ | 1.9957 |
| 256  | $6.1496 \times 10^{-6}$ | 2.0003 | $6.9566 \times 10^{-6}$ | 1.9978 |
| 512  | $1.5371 \times 10^{-6}$ | 2.0002 | $1.7418 \times 10^{-6}$ | 1.9989 |
| 1024 | $3.8422 \times 10^{-7}$ | $-$ | $4.3577 \times 10^{-7}$ | $-$ |

Table 1 refers to the data function $f_\varepsilon$ defined by (32) for $\varepsilon = 0.1$. In accord with Theorems 5 and 7, the computed values $\kappa_h^{A,B}$ indicate a clear tendency to converge numerically to the approximation order $\kappa = 1.6$. It can be verified that similar numerical results with $\kappa = \frac{3}{2} + \varepsilon$ hold for other positive values of $\varepsilon$, showing a direct link between the smoothness/singularity of the data function and the approximation order $\kappa$.

On the other hand, Tables 2 & 3 refer to $C^\infty$ data functions. Along with similar results that can be observed for other smooth data functions, the two tables suggest the conjecture that, for Beppo Levi $L_0$-spline interpolation to $C^\infty$-smooth data profiles $f$, the uniform norm of the error over the interval $[r_1, r_n]$ decays with (saturation) order $O(h^2)$, as $h \to 0$. The resolution of this conjecture is likely to require different techniques than those employed to establish Theorem 5 and remains a topic for further research. Note that the $L^\infty$-approximation order 2 was also conjectured by Johnson [14] for the related problem of thin plate spline interpolation to scattered data in any planar domain with a smooth boundary.

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1It can be assumed that these data profiles belong to $\Lambda_0$ after suitable multiplication by a smooth mollifier which takes the constant value 1 on $[0, 2]$, and which vanishes identically on an interval $[a, \infty)$ with $a > 2$. 
5.2 Two compactly supported $L_0$-splines

A remarkable example of a Beppo Levi $L_0$-spline has recently been obtained independently by Johnson [15] in the context of radial basis function (RBF) methods for multivariable scattered data interpolation. Specifically, Johnson’s profile $\eta_2$, constructed as part of a family of compactly supported and piecewise polyharmonic RBFs, can be expressed as

$$\eta_2(r) = \frac{4}{3} \left[ \ln 2 - \varphi_0(r) + \varphi_0 \left( \frac{r}{2} \right) \right], \quad r \geq 0,$$

where $\varphi_0$ is defined by (14). Letting $\rho = \{1, 2\}$, it follows that $\eta_2 \in S_0(\rho)$ and $\eta_2$ is supported on the interval $[0, 2]$. As stipulated in [15, Definition 3.9], the ‘singular coefficient’ of $\eta_2$, i.e. the coefficient of $r^2 \ln r$ in the linear representation of $\eta_2$ as a member of Ker $L_0$ over the interval $(0, 1)$, equals 1.

Note that $k = 2$ is the smallest integer $k$ for which there exists a Beppo Levi $L_0$-spline profile in $S_0(\rho)$ with compact support on $[0, k]$, where $\rho = \{1, 2, \ldots, k\}$. Alternatively, if the same existence problem is considered for a non-singular Beppo Levi $L_0$-spline (i.e., having a zero singular coefficient), then $k = 3$ is the smallest integer for which this problem admits a solution. More precisely, for $\rho = \{1, 2, 3\}$, there exists, up to a constant factor, a unique $\beta \in S_0(\rho)$, which is non-singular and compactly supported on $[0, 3]$. It is given by the formula

$$\beta(r) = \frac{1}{5} \left[ 27 \ln 3 - 32 \ln 2 + 5 \varphi_0(r) - 32 \varphi_0 \left( \frac{r}{2} \right) + 27 \varphi_0 \left( \frac{r}{3} \right) \right], \quad r \geq 0.$$

Although not revealed by their plots in Figure 1, there is a fundamental property that recommends $\eta_2$ against $\beta$ for use in RBF interpolation applications. Namely, if $F_2 \eta_2$ denotes the profile of the Fourier transform of the bivariate radially symmetric extension of $\eta_2$, it is proved in [15] that $F_2 \eta_2$ satisfies a Sobolev regularity condition at $\infty$. In particular, this implies $(F_2 \eta_2)(t) > 0$ for $t \geq 0$, 

![Figure 1: Solid line: plot of $\eta_2$. Dash line: plot of $\beta$.](image)
hence \(\eta_2\) is positive definite on \(\mathbb{R}^2\). In contrast, it is verified numerically that 
\((F_2\beta) (t)\) takes both positive and negative values for \(t \geq 0\), since, using \[15\] and 
the Bessel coefficient \(J_0\), one can express \((F_2\beta) (t)\), up to a constant factor, as
\((5J_0 (t) - 8J_0 (2t) + 3J_0 (3t)) t^{-4}\). The reader is referred to the monograph \[25\] for the role of positive definiteness and Sobolev regularity in RBF interpolation.

5.3 Conclusion

We employed a \(L\)-spline approach to the problem of minimizing the radial version \[3\] of the Beppo Levi energy integral over the full semi-axis \((0, \infty)\), subject to interpolation conditions. This treatment led us to the identification of singular/non-singular solution profiles, which were expressed as linear combinations of dilates of a basis function. For \(p \geq 2\) and data functions from the radial energy space, our analysis proved the exact \(L^p\)-approximation power \(3/2 + 1/p\) of the \(L\)-spline profiles. This result is further used in the sequel paper \[8\] to derive a \(L^2\)-error bound for surface interpolation through curves prescribed on concentric circles.

Additional research is needed to establish the improved approximation order observed in our numerical experiments for \(C^\infty\) data functions. Also, the close connections of Beppo Levi \(L\)-splines with RBF and polyspline surfaces motivate the future extension of this work to higher order radially symmetric piecewise polyharmonic surfaces.

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6 Appendix: Proof of Lemma 2

Let \((v_1, \ldots, v_d) \in H\), \(\delta \equiv \delta_x(v_1, \ldots, v_d)\), and, for each \(j \in \{1, \ldots, d\}\), consider a sequence \(\{v_{j,n}\}_{n=1}^\infty\) of vectors in \(X\), such that \(\|v_{j,n} - v_j\| \to 0\) as \(n \to \infty\), and \((v_{1,n}, \ldots, v_{d,n}) \in H, \forall n \geq 1\). We have to show that, for every \(\epsilon > 0\), the following inequalities hold for all \(n\) sufficiently large:

\[
\delta - \epsilon \leq \delta_x(v_{1,n}, \ldots, v_{d,n}) \leq \delta + \epsilon. \tag{33}
\]

Letting \(a_1, \ldots, a_d\) be scalars for which

\[
\|x - \sum_{j=1}^{d} a_j v_j\| = \delta,
\]

and setting \(v := \sum_{j=1}^{d} a_j v_{j,n}\), we have

\[
\|x - v\| \leq \left\|x - \sum_{j=1}^{d} a_j v_j\right\| + \sum_{j=1}^{d} |a_j| \|v_j - v_{j,n}\|.
\]

Hence \(\|x - v\| \leq \delta + \epsilon\) for \(n\) sufficiently large, which implies the right-side inequality of (33).

For the remaining inequality, we argue by contradiction and assume that there exists \(\epsilon > 0\) such that, after selecting and re-indexing a sub-sequence, we have \(\delta_x(v_{1,n}, \ldots, v_{d,n}) < \delta - \epsilon\), for all \(n \geq 1\). Hence, there exist scalar sequences \(\{b_{j,n}\}_{n=1}^\infty, j \in \{1, \ldots, d\}\), such that, if \(w_n := \sum_{j=1}^{d} b_{j,n} v_{j,n}\), then

\[
\|x - w_n\| < \delta - \epsilon, \quad \forall n \geq 1. \tag{34}
\]

Next, we invoke a well-known result [19, Lemma 2.4-1], which guarantees the existence of a constant \(K > 0\), such that

\[
\|c_1 v_1 + \ldots + c_d v_d\| \geq K
\]

for all scalars \(c_1, \ldots, c_d\) with \(\sum_{j=1}^{d} |c_j| = 1\). Also, let \(n_0\) be a positive integer for which \(\|(v_j - v_{j,n})\| \leq \frac{K}{2}, \forall n \geq n_0, \forall j \in \{1, \ldots, d\}\). Then, for all scalars \(c_1, \ldots, c_d\) as above and all \(n \geq n_0\), we have

\[
\left\|\sum_{j=1}^{d} c_j v_j\right\| - \left\|\sum_{j=1}^{d} c_j v_{j,n}\right\| \leq \sum_{j=1}^{d} |c_j| \|(v_j - v_{j,n})\| \leq \frac{K}{2},
\]

hence

\[
\left\|\sum_{j=1}^{d} c_j v_{j,n}\right\| \geq \left\|\sum_{j=1}^{d} c_j v_j\right\| - \frac{K}{2} \geq \frac{K}{2}.
\]

For all \(n \geq n_0\), we obtain, via (34),

\[
\frac{K}{2} \sum_{j=1}^{d} |b_{j,n}| \leq \left\|\sum_{j=1}^{d} b_{j,n} v_{j,n}\right\| = \|w_n\| \leq \|x\| + \delta - \epsilon,
\]

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and therefore each sequence \( \{b_{j,n}\}_{n=1}^{\infty}, \ j \in \{1, \ldots, d\} \), is bounded. Selecting convergent sub-sequences, re-indexed such that \( \lim_{n \to \infty} b_{j,n} =: b_j \), and passing to the limit in (34), we deduce the existence of a vector \( w := \sum_{j=1}^{d} b_j v_j \), such that \( \|x - w\| \leq \delta - \epsilon \), which contradicts the definition of \( \delta \). The lemma is proved. ■