CESÁRO CONDITION FOR CURVES IN THE FLAT PSEUDO-HERMITIAN MANIFOLDS

YEN-CHANG HUANG

Abstract. By considering the three dimensional Heisenberg group $H_1$ as a flat model of pseudo-hermitian manifolds, the authors in [8] derived the Frenet-Serret formulas for curves in $H_1$. In this notes we show three applications of the Frenet-Serret formulas. The first is the Cesáro immobility condition, which provides the criterion of curves being contained in a given rotationally symmetric surface. Secondly, we show that any horizontally regular curve is a Bertrand curve, and give all characterizations of those curves. The final application is a classification of curves depending on whether the position vector of the curve lies on the planes spanned by any pair of its unit tangent, normal, and binormal vectors.

1. Introduction

Given a regular curve $r$ parametrized by arc length in the three dimensional Euclidean space $\mathbb{R}^3$, it is well known that the famous Frenet–Serret formulas

\[
\begin{align*}
    t' &= \kappa n, \\
    n' &= -\kappa t + \tau b, \\
    b' &= -\tau n,
\end{align*}
\]

where $t, n, b$ are the unit tangent, unit normal, and unit binormal vector respectively, and $\kappa, \tau$ are respectively the curvature and the torsion for $r$. Several fundamental applications related to the formula reveal geometric information about curves in $\mathbb{R}^3$. For instance, curves in $\mathbb{R}^3$ can be characterized by their position vectors: if the position vector of the curve lies in its osculating plane at each point, then it is a plane curve; if its position vector lies in its normal plane, it lies on a sphere. See [19, Proposition 4.7, 4.8] for more detail. After that, Chen [3] characterizes the curves when the position vectors lie on their rectifying planes and hitherto a classification of curves by position vectors has been established.

In this notes, we show three applications of the Frenet-Serret formulas derived by the author with the coauthors in [8] (see (4) below) for curves in the three dimensional Heisenberg group $H_1$. First, Cesáro immobility condition (10) in $H_1$ will be derived in Section 3. In $\mathbb{R}^3$, the condition provides a necessary and sufficient condition for curves being on the standard spheres in terms of the corresponding curvature and torsion of the curve. See the original book [2] (Chapter IV, 4, equation (8) and Chapter X, 1, equation (5)) by Ernesto Cesáro and Remark 3.2 in Section 3 for the precise formula. The importance of the condition is that it provides a relation between the position vector of the given curve and its geometric quantities which are invariant in the sense of rigid motions [16]. It also can be applied to kinematic geometry of linkages from the planar, spherical, to spatial curves.

2010 Mathematics Subject Classification. Primary: 53A04, 32V30, Secondary: 53C23, 53C17.

Key words and phrases. Cesáro condition, Bertrand curves, Mannheim curves, Heisenberg groups.

This work was funded by Ministry of Science and Technology (MOST), Taiwan, with grant Number: 110-2115-M-024 -002 -MY2.
See [22, Sec. 1.2]. In the paper, we obtain several necessary and sufficient conditions for curves with p-curvatures $\kappa$ and contact normality $\tau$ (defined in [2]) being on the rotationally symmetric surfaces in $\mathbb{H}_1$ (Theorem 3.3 and Corollary 3.4). In particular, we characterize the rotationally symmetric surfaces on which there exists at least one curve with constant p-curvature (Corollary 3.6) and show that there is no any horizontal curve (i.e. $\tau = 0$) on the standard sphere in $\mathbb{H}_1$ (Example 3.7).

The second application is the existence of Bertrand mates in $\mathbb{H}_1$. A curve $r$ in $\mathbb{R}^3$ is a Bertrand curve, by definition, if there exists another curve $\bar{r}$ such that both curves own the same principal normals and the curves $r, \bar{r}$ are called the Bertrand mates. One of the important properties for Bertrand curves is that a curve $r$ is Bertrand if and only if there exist a linear relation $A\kappa + B\tau = 1$ for some nonzero constants $A, B$, where $\kappa, \tau$ are the curvature and the torsion of $r$ respectively [11, page 27]. We refer the reader to [12,13,15,18] for further references when considering the Bertrand curves in different ambient spaces. However, applying the similar concept to define Bertrand-like curves in $\mathbb{H}_1$, we observe that there is no constraint for the existence of Bertrand-like curves (see Theorem 4.1): any horizontally regular curve $r$ in $\mathbb{H}_1$ has the corresponding Bertrand mate $\bar{r}$ and all Bertrand mates can be specified by the frame $\{t, n, b\}$ of the given curve $r$. We point out that although $\mathbb{H}_1$ is also a three dimensional Lie group, our result is different from that in [24] by Okuyucu-Gök-Yayli-Ekmekci, where the authors give a necessary and sufficient condition (the same linear relation as in $\mathbb{R}^3$) for the existence of Bertrand curves. The frames of the curves considered in [24] are not invariant under pseudo-hermitian transformations as we considered in $\mathbb{H}_1$.

Besides the Bertrand mates which involve two curves with equal normal vectors, we also discuss other interactions between the moving frames $\{t, n, b\}$ and $\{t, n, b\}$ of two curves $\bar{r}$ and $r$, respectively. But the possible situations in $\mathbb{H}_1$ are simpler. In fact, we do not particularly consider whether the condition $t = \bar{t}$ holds or not, because the almost complex structure $J$ ensures that $n = \bar{n}$ if and only if $t = \bar{t}$; neither to consider whether $b = \bar{b}$ since it occurs globally in $\mathbb{H}_1$. For the mixed types, we only have to make the assumption $\bar{n} = gt$ (equivalently, $-\bar{t} = gn$ ) for some nonzero function $g$, and ignore the possibilities for $t = gb$ and $n = gb$.

The last application characterizes the curves $r$ in $\mathbb{H}_1$ based on the relation between the position vectors of $r$ and some specific planes containing $r$. Those planes are spanned by any two of the members in the moving frame $\{t, n, b\}$ of $r$, which give some geometric information about the curve. When the planes are osculating or normal planes in $\mathbb{R}^3$, the results have been mentioned at the very beginning of this section. When the curve lies on its rectifying plane, Bang-Yen Chen [3] gave some characterizations about them. We ask if curves in $\mathbb{H}_1$ can be classified by the similar notions of osculating, normal, and rectifying planes where the position vectors of the curve lie on. In Theorem 5.1 we show that the curves contained in the planes $\text{span}\{t, n\}, \text{span}\{t, b\}$, and $\text{span}\{n, b\}$ respectively can only be one of lines, plenary curves, or circular helices.

The paper is organized as follows. In Section 2 we give some fundamental background about curves in $\mathbb{H}_1$ with the p-curvatures and contact normalities related to our results. In Section 3 we derive the Cesáro immobility condition and show the necessary and sufficient condition for curves in the rotationally symmetric surfaces. In Section 4 we prove the existence of Bertrand curves with their properties. In Section 5 we will prove a classification theorem for curves, which is based on the relation between position vectors of the curve and the corresponding planes where the curve is contained.
2. Preliminary

In 2017, Chiu-Huang-Lai [8] studied the group $PSH(1)$ of pseudo-hermitian transformations in the Heisenberg groups $\mathbb{H}_1$ and obtained that $PSH(1) = U(1) \ltimes T(1)$, the semidirect product of the unitary group $U(1)$ and the group $T(1)$ of left translations in $\mathbb{H}_1$. For any points $p = (x, y, z), q = (a, b, c) \in \mathbb{H}_1$, the element $L_p$ in $T(1)$ is a left translation defined by

$$L_p(q) = (a + x, b + y, c + z + ya - xb),$$

and the standard left-invariant unit vectors at the point $p$ are defined by

$$\hat{e}_1(p) = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} := (1, 0, y),$$

$$\hat{e}_2(p) = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} := (0, 1, -x),$$

$$T(p) = \frac{\partial}{\partial z} := (0, 0, 1).$$

The space $\mathbb{H}_1$ is considered as a flat pseudo-hermitian manifolds (zero Webster torsion) with the almost complex structure $J$ defined in $\mathbb{H}_1$ satisfying

$$Je_1(p) = \hat{e}_2(p), \ Je_2(p) = -\hat{e}_1(p), \ JT = 0.$$

There also exists the standard contact structure $\xi$ in $\mathbb{H}_1$ with the contact planes at $p$ defined by $\xi_p := \text{span}\{\hat{e}_1(p), \hat{e}_2(p)\}$ for all $p \in \mathbb{H}_1$. In [8], a regular curve $r : s \in [a, b] \subset \mathbb{R} \to \mathbb{H}_1$ is horizontally regular if the orthogonal decomposition of the velocity vector $r'(s) = r'_\xi(s) + r'_{\xi^\perp}(s)$ has the nonzero contact part $r'_\xi(s)$ for all $s \in [a, b]$, where $r'_\xi(s) \in \xi_r(s)$ and $r'_{\xi^\perp}(s) \in \text{span}\{T\}$. Also, $r$ can always be reparametrized by the horizontal arc-length $s$ such that the length of the contact part is equal to one, namely, $|r_\xi(s)| = 1$ for all $s \in [a, b]$, where the length is in the sense of Levi-metric. For this reason, we always assume that the curve is parametrized by horizontal arc-length throughout the present paper if the context is clear.

We also introduced two important geometric invariants: the $p$-curvature $\kappa$ and the contact normality $\tau$ (also called $T$-variation in [7]) defined by

$$\kappa(s) := \langle \frac{dr'(s)}{ds}, Jr'(s) \rangle,$$

$$\tau(s) := \langle r'(s), T \rangle.$$

and prove the fundamental theorem of curves in $\mathbb{H}_1$ [8, Theorem 1.2]. That is, given two $C^1$-functions $\kappa$ and $\tau$, there exists uniquely the horizontally regular curve $r(s)$ with horizontal arc-length $s$ such that its $p$-curvature and contact normality are $\kappa$ and $\tau$, respectively. In addition, any two curves with the same $\kappa$ and $\tau$ differ from an pseudo-hermitian transformation in $PSH(1)$. The result has also been generalized to the Heisenberg groups $\mathbb{H}_n$ of higher dimensions for $n \geq 2$ in [6]. We emphasis that both $\kappa$ and $\tau$ are invariant under pseudo-hermitian transformations $PSH(1)$ [5, Section 4] and the geometric interpretation are as follows: $\kappa(s)$ is exactly the Euclidean curvature of the projection of $r$ onto the $xy$-plane, and $\tau(s)$ measures how far the curve is from being horizontal. By [9], a curve $r$ is horizontal if and only if $\tau \equiv 0$, equivalently, the tangents of $r$ are pointwise contained in the corresponding contact planes. When the curve $r(u) = (x(u), y(u), z(u))$ is parametrized by arbitrary parameter $u$ (not necessarily the horizontal arc-length parameter), the
p-curvature and the contact normality are respectively given by

$$\kappa(u) := \frac{x'y'' - x''y'}{(x')^2 + (y')^2}^{1/2}(u),$$

(3)

$$\tau(u) := \frac{x'y' - x'y + z'}{(x')^2 + (y')^2}^{1/2}(u).$$

By taking Élie Cartan’s method of moving frames, for any horizontally regular curve \( r(s) \) in \( \mathbb{H}_1 \) parametrized by horizontal arc-length \( s \) we obtain the Frenet-Serret formulas \([8, p 12]\)

$$\begin{cases}
    r' &= t + \kappa n, \\
    t' &= \kappa n, \\
    n' &= -\kappa t - b, \\
    b' &= 0,
\end{cases}$$

(4)

where

$$t = t(s) := r_\xi(s),$$

$$n = n(s) := Jr_\xi(s),$$

$$b = b(s) := T(s).$$

In this formula we only consider the projection of the usual tangent vector \( r' \) of \( r \) onto the contact plane \( \xi_\alpha \) as the “unit tangent vector” \( t \) because, in contrast to \( r' \), the contact part \( r''_\xi \) can be represented by the linear combination of the standard basis \( e_1, e_2 \), which are the left invariant vector fields in \( \mathbb{H}_1 \).

To our purpose, we may write \( t \) and \( n \) in the Euclidean coordinates for any horizontally regular curve \( r \in \mathbb{H}_1 \) parametrized by horizontal arc-length \( s \). Indeed, if \( r = (x, y, z) \), then

$$r' = (x', y', z') = x'e_1 + y'e_2 + (-x'y + xy' + z')T,$$

(5)

Thus,

$$t = x'e_1 + y'e_2 = (x', y', x'y - xy'),$$

(6)

$$n = Jt = -y'e_1 + x'e_2 = (-y', x', -yy' - xx').$$

(7)

3. Cesáro conditions

Let \( r : [a, b] \to \mathbb{H}_1 \) be a horizontally regular curve with horizontal arc-length \( s \) in \( \mathbb{H}_1 \). Consider the curve \( \bar{r} \) defined by

$$\bar{r}(s) = r(s) + u_1(s)t(s) + u_2(s)n(s) + u_3(s)b(s),$$

(8)

where \( u_1, u_2, \) and \( u_3 \) are functions of \( C^2 \)-class to be determined. Take the derivative with respect to \( s \) and use the Frenet frame formula \([11]\) to have

$$\bar{r}' = r' + u_1't + u_2'n + u_3'b + u_1t' + u_2n'$$

$$= (1 + u_1' - \kappa u_2)t + (u_2' + \kappa u_1)n + (\tau + u_3' - u_2)b$$

$$:= \alpha_1t + \alpha_2n + \alpha_3b.$$

Notice that the derivatives \( u_1', u_2', \) and \( u_3' \) in the first equation of \([11]\) are the components of the relative velocity of the associate curve \( \bar{r} \) with respect to the original curve \( r \) in terms of the orthonormal basis \( \{t, n, b\} \), while \( \alpha_1, \alpha_2, \alpha_3 \) are that of the absolute velocity of \( \bar{r} \) with respect to the origin \( O \) in \( \mathbb{H}_1 \) in terms of the orthonormal basis \( \{t, n, b\} \). Hence, the condition \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \)
is equivalent to that the absolute velocity \( \mathbf{r}' \) is zero, namely, \( \mathbf{r} \) is a fixed point with respect to \( O \) whenever \( \mathbf{r} \) travels anywhere in \( \mathbb{H}_1 \). Therefore, we have reached the Cesáro immobility condition

\[
\begin{cases}
  u_1' = \kappa u_2 - 1, \\
  u_2' = -\kappa u_1, \\
  u_3' = u_2 - \tau.
\end{cases}
\]

Remark 3.1. The components of the position vector of the curve \( \mathbf{r}(s) \) in \( \mathbb{H}_1 \) is exactly same as that of \( \mathbf{r} \) in the tangent space \( T_{\mathbf{r}(s)}\mathbb{H}_1 \) at \( \mathbf{r}(s) \). Indeed, by (11)

\[
\mathbf{r}(s) = \left( x(s), y(s), z(s) \right)
\]

Moreover, if the standard basis \( \mathbf{e}_1(\mathbf{r}(s)) \) keeps the angle \( \theta(s) \) with the unit tangent \( \mathbf{t} \) at \( \mathbf{r}(s) \), then

\[
\mathbf{e}_1 = \cos \theta \mathbf{t} + \sin \theta \mathbf{n} \quad \text{and} \quad \mathbf{e}_2 = -\sin \theta \mathbf{t} + \cos \theta \mathbf{n}.
\]

Substitute \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) into (11), one has that

\[
|\mathbf{r}(s)|_{\mathbb{R}^3} := \sqrt{x^2 + y^2 + z^2} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2},
\]

namely, the Euclidean distance of \( \mathbf{r} \) to the origin in \( \mathbb{H}_1 \) can be represented by the components \( \bar{x}, \bar{y}, \) and \( z \) under the frame \( \{\mathbf{t}, \mathbf{n}, \mathbf{b}\} \) on the curve \( \mathbf{r} \). Also, note that the (Euclidean) distance of \( \mathbf{r} \) to the \( z \)-axis equals \( \sqrt{x^2 + y^2} = \sqrt{\bar{x}^2 + \bar{y}^2} \) and the (Euclidean) height of any point on \( \mathbf{r} \) is equal to the coefficient \( z \).

Remark 3.2. Since \( \{\mathbf{t}, \mathbf{n}, \mathbf{b}\} \) is an orthonormal basis along \( \mathbf{r} \), we may write \( \mathbf{r} = \bar{u}_1 \mathbf{t} + \bar{u}_2 \mathbf{n} + \bar{u}_3 \mathbf{b} \) for some functions \( \bar{u}_i \). The expression of \( \mathbf{r} \) is equivalent to (5) when we set \( \bar{u}_1 = -u_1, \bar{u}_2 = -u_2, \bar{u}_3 = -u_3 \) and \( \mathbf{r} = \mathbf{0} \). By (9), we conclude that any functions \( \bar{u}_i \) such that \( -\bar{u}_1 \) satisfy the Cesáro conditions (11) if and only if \( \bar{u}_i \) are the coefficients of some curve \( \mathbf{r} \) with respect to its basis \( \{\mathbf{t}, \mathbf{n}, \mathbf{b}\} \) defined by (5). Notice that using the same argument as in (9), we can derive the Cesáro condition in the Euclidean space \( \mathbb{R}^3 \)

\[
\begin{cases}
  u_1' = \kappa u_2 - 1, \\
  u_2' = -\kappa u_1 + \tau u_3, \\
  u_3' = -\tau u_2.
\end{cases}
\]

By comparing this formula and (10) in \( \mathbb{H}_1 \), we make a key observation that there is no \( u_3 \) in the first two equations of (10), and this helps us to find the exact solutions of \( u_i \) (see (21), (22), (23) below), equivalently, to construct the curve \( \mathbf{r} \) from its coefficients \( u_i \) (see the proof of Theorem 3.3).

Now we seek solving \( u_i, i = 1, 2, 3 \), satisfying the Cesáro condition in terms of the geometric invariants \( \kappa \) and \( \tau \) of the curve \( \mathbf{r} \). When \( \kappa = 0 \), the Cesáro condition implies that \( u_1 = -s + c_1, u_2 = c_2, \) and \( u_3 = c_2 - \tau \) for some constant \( c_1, c_2 \). When \( \kappa \neq 0 \), by using the first two equations of (10), one has two second order linear ordinary differential equations (ODEs) of nonhomogeneous type:

\[
\begin{align*}
  u_1'' - \frac{\kappa'}{\kappa} u_1' + k^2 u_1 - \frac{\kappa'}{\kappa} &= 0, \\
  u_2'' - \frac{\kappa'}{\kappa} u_2' + k^2 u_2 - \kappa &= 0.
\end{align*}
\]

To solve the ODEs, let us recall the following result.

**Theorem 3.3.** (page 130) A nonhomogeneous linear differential equation of the second order is given by

\[
f_2(s)y''(s) + f_1(s)y'(s) + f_0(s)y(s) = g(s).
\]
If \( y_1 = y_1(s) \) and \( y_2 = y_2(s) \) are two nontrivial linearly independent (namely, \( \frac{y_1}{y_2} \neq \text{constant} \)) solutions of the corresponding homogeneous equation with \( \omega \equiv 0 \) in (15), then the general solution for (15) can be found from the formula
\[
y = C_1 y_1 + C_2 y_2 + y_2 \int y_1 \frac{g}{f_2} \, ds - y_1 \int y_2 \frac{g}{f_2} \, ds,
\]
where \( W := y_1(y_2)' - y_2(y_1)' \).

For the ODE (13), we first find two linearly independent solutions for the homogeneous equation
\[
u'' - \frac{\kappa'}{\kappa} u' + \kappa^2 u = 0.
\]
This equation is a special case of the form
\[
y'' - f'y' + a^2 e^{2f} y = 0
\]
if we set the function \( f = \ln |\kappa| \) and the constant \( a = 1 \). Notice that since \( \kappa \neq 0 \) and solving ODE is a local computation, without loss of generality we may assume that \( \kappa > 0 \). According to [23, page 218, Equation 63]), the general solution for (18) is \( y = C_1 \sin(\int f \, ds) + C_2 \cos(\int f \, ds) \) for any constants \( C_1, C_2 \), and so the homogeneous equation (17) has the general solution
\[
u_1 = \int C_1 \sin(\int \kappa) + C_2 \cos(\int \kappa).
\]
Suppose \( u_1 = C_3 \sin(\int \kappa) + C_4 \cos(\int \kappa) \) is another solution for (17) with
\[
C_2 C_3 \neq C_1 C_4.
\]
Then \( u_1, v_1 \) are linearly independent solutions for (17). By Theorem 3.3 it is clear that \( W = \kappa (C_2 C_3 - C_1 C_4) \neq 0 \), and hence the general solution for the nonhomogeneous equation (13) is
\[
u_1 = (C_1 C_5 + C_3 C_6) \sin \theta + (C_2 C_5 + C_4 C_6) \cos \theta
\]
\[
+ (C_3 \sin \theta + C_4 \cos \theta) \int (C_1 \sin \theta + C_2 \cos \theta) \frac{\kappa'}{\kappa^2 \Delta} \, ds
\]
\[
- (C_1 \sin \theta + C_2 \cos \theta) \int (C_3 \sin \theta + C_4 \cos \theta) \frac{\kappa'}{\kappa^2 \Delta} \, ds,
\]
where \( \theta = \theta(s) := \int^s \kappa(t) \, dt \), \( C_5, C_6 \) are some constants, and \( \Delta := C_2 C_3 - C_1 C_4 \).

Similarly, by using (14) and repeating above process (or using the first equation in (10)), we have
\[
u_2 = (C_1 C_5 + C_4 C_6) \cos \theta - (C_2 C_5 + C_4 C_6) \sin \theta
\]
\[
+ (C_3 \cos \theta - C_4 \sin \theta) \int (C_1 \sin \theta + C_2 \cos \theta) \frac{\kappa'}{\kappa^2 \Delta} \, ds
\]
\[
- (C_1 \cos \theta - C_2 \sin \theta) \int (C_3 \sin \theta + C_4 \cos \theta) \frac{\kappa'}{\kappa^2 \Delta} \, ds + \frac{1}{\kappa}.
\]
Finally, \( u_3 \) can be obtained by the third equation of (10), namely,
\[
u_3 = \int (u_2 - \tau) \, ds + \text{const.},
\]
and we have obtained all solutions \( u_1, u_2, u_3 \) in terms of the geometric invariants \( \kappa \) and \( \tau \) of the curve \( r \).
Next we provide several applications for the Cesàro condition \( (10) \). At first, we prove the necessary and sufficient condition for the curve \( r \) on the rotationally symmetric surface.

**Theorem 3.4.** Let \( \Sigma : X(s, t) = (g(s) \cos t, g(s) \sin t, f(s)) \), \( 0 \leq t \leq 2\pi \), \( g(s) \geq 0 \), \( s \in [0, s_0] \), be a rotationally symmetric surface about the \( z \)-axis and \( r \) be a horizontally regular curve parametrized by horizontally arc-length with \( p \)-curvature \( \kappa \neq 0 \) and the contact normality \( \tau \) in \( \mathbb{R}_1 \). Suppose \( u_1, u_2, u_3 \) are defined in terms of \( \kappa \) and \( \tau \) as in \( (21), (22), \) and \( (23) \) respectively with any constants \( C_1, \ldots, C_6 \) satisfying \( (20) \). Then \( r \) is on \( \Sigma \) if and only if

\[
\begin{align*}
(u_1)^2 + (u_2)^2 &= g^2, \\
u_3 &= -f.
\end{align*}
\]

**Proof.** Suppose that \( r \) is on the surface \( \Sigma \). We can express the curve \( r = \tilde{u}_1 t + \tilde{u}_2 n + \tilde{u}_3 b \) in terms of the orthonormal frames for some functions \( \tilde{u}_i \), \( i = 1, 2, 3 \). By Remark 3.1 the Euclidean distance of \( r \) to the \( z \)-axis implies that \( (\tilde{u}_1)^2 + (\tilde{u}_2)^2 = g^2 \), and the height of \( r \) implies that \( \tilde{u}_3 = f \). By setting \( u_i = -\tilde{u}_i \), the identities \( (24) \) immediately hold. It suffices to prove that \( u_i \)'s satisfy \( (21), (22), \) and \( (23) \). Indeed, since \( -\tilde{u}_i \)'s satisfy the Cesàro condition by Remark 3.2, it is clear that the functions \( u_1, u_2 \) satisfy \( (13), (14) \), respectively, and so all \( u_i \) are of the forms as shown in \( (21), (22), \) and \( (23) \), and the procedure of solving ODEs in the previous paragraph.

On the other hand, if \( r \) is any curve in \( \mathbb{R}_1 \) defined by \( r = \tilde{u}_1 t + \tilde{u}_2 n + \tilde{u}_3 b \) for some functions \( \tilde{u}_i \) and \( (24) \) holds, then by Remark 3.2 \( \tilde{u}_i \)'s satisfy the Cesàro conditions and so they must be the form of \( (21), (22), (23) \), namely, \( -\tilde{u}_i = u_i \) for all \( i \). Also, by the assumption and Remark 3.1 \( \sqrt{u_1^2 + u_2^2} = \sqrt{(-\tilde{u}_1)^2 + (-\tilde{u}_2)^2} = g \) is the distance of \( r \) to the \( z \)-axis. The height of any point on \( r \) is uniquely determined by \( \tilde{u}_3 = -u_3 = f \), and hence the curve \( r \) is on the surface \( \Sigma \). \( \Box \)

We also prove that the conditions \( (24) \) in the previous theorem can be written in terms of the geometric invariants \( \kappa \) and \( \tau \) of the curve and then we have the following result.

**Corollary 3.5.** Suppose \( \Sigma \) is a rotationally symmetric surface about the \( z \)-axis defined as in Theorem 3.4 and \( r \) is a horizontally regular curve with \( \kappa \neq 0 \) and any \( \tau \). If \( r \) is on \( \Sigma \), then the functions \( f, g, \kappa, \tau \) satisfy

\[
\begin{align*}
f' - \tau &= \frac{\frac{1}{2}(g^2)'' - 1}{\kappa}, \\
n'' - \tau' &= \frac{-\kappa (g^2)'}{2}.
\end{align*}
\]

Conversely, if the functions \( f, g, \kappa, \tau \) satisfy \( (25) \) and \( (26) \), then the curve \( r \) is on the surface of revolution obtained from \( \Sigma \) by a constant stretch along the radial direction up to a Euclidean translation.

**Proof.** If \( r \) is on \( \Sigma \), then the functions \( u_i \), \( i = 1, 2, 3 \), defined by \( (21), (22), \) and \( (23) \) in terms of its \( \kappa \) and \( \tau \) satisfy \( (24) \) by Theorem 3.4 and \( u_i \)'s also satisfy the Cesàro conditions. Differentiate \( u_1^2 + u_2^2 = g^2 \) and use the first two equations of \( (10) \) to have \( u_1 = \frac{1}{2}(g^2)' \). Use \( (10) \) again to get

\[
(-f)' = u_3' = u_2 - \tau = \frac{u_2'}{\kappa} - \tau = \frac{(-\kappa (g^2)')'}{\kappa} - \tau,
\]

and so \( (25) \) holds. Use the second equation of \( (10) \), we have \((-f)'' = (u_2 - \tau)' = -\kappa u_1 - \tau' = \frac{\kappa (g^2)''}{\kappa} - \tau' \) and the result follows.

Conversely, let \( u_1 = -\frac{1}{2}(g^2)' \), \( u_2 = -f' + \tau \), and \( u_3 = -f \). Then by \( (25), (26) \), we have that \( u_2 = \frac{-\frac{1}{2}(g^2)''}{\kappa} \), and \( u_1, u_2, u_3 \) satisfy the Cesàro conditions \( (10) \). The first two equations of \( (10) \)
imply that $u_1 u_1' + u_2 u_2' = -u_1 = \frac{1}{2}(g^2)'$ and so $(u_1)^2 + (u_2)^2 = g^2 + c_1$ for some constant $c_1$. If $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are the orthonormal frames on $\mathbf{r}$ as defined in (5), then by the Frenet-Serret formulas (11) one has $\mathbf{0} = (1 + u_1' - \kappa u_2)\mathbf{t} + (u_2' + \kappa u_1)\mathbf{n} + (\tau - u_3' - u_2)\mathbf{b} = \mathbf{r}' + (u_1 \mathbf{t} + u_2 \mathbf{n} + u_3 \mathbf{b})'$. Thus, the curve $\mathbf{r}$ can be represented by $\mathbf{r} = -u_1 \mathbf{t} - u_2 \mathbf{n} - u_3 \mathbf{b} + \mathbf{c}$ for some constant vector $\mathbf{c}$. Let $\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{c}$ be the curve obtained from $\mathbf{r}$ by a Euclidean translation and $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ be the corresponding orthonormal frame on $\tilde{\mathbf{r}}$. By the definition (5), we have that $\mathbf{t} = \tilde{\mathbf{t}}, \mathbf{n} = \tilde{\mathbf{n}},$ and $\mathbf{b} = \tilde{\mathbf{b}},$ so the expression of $\tilde{\mathbf{r}}$ in terms $\mathbf{t}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}$ has the same coefficients as that of $\mathbf{r}$, namely,

$$
\tilde{\mathbf{r}} = \hat{u}_1 \mathbf{t} + \hat{u}_2 \tilde{\mathbf{n}} + \hat{u}_3 \tilde{\mathbf{b}} = -u_1 \mathbf{t} - u_2 \mathbf{n} - u_3 \mathbf{b} = \mathbf{r} - \mathbf{c}.
$$

By Remark (3.1), the Euclidean distance of $\tilde{\mathbf{r}}$ to the $z$-axis equals

$$\sqrt{(\hat{u}_1)^2 + (\hat{u}_2)^2} = \sqrt{(-u_1)^2 + (-u_2)^2} = \sqrt{g^2 + c_1}
$$

and the height of any point on $\tilde{\mathbf{r}}$ is $\hat{u}_3 = -u_3 = f$. Thus, the curve $\tilde{\mathbf{r}}$ is on the rotationally symmetric surface obtained from $\Sigma$ by a constant stretch along the radial direction. We conclude that the curve $\mathbf{r}$ is on the same surface up to a Euclidean translation. \hfill $\square$

We also characterize the surface of revolution $\Sigma$ in $\mathbb{H}_1$ on which there exits a horizontally regular curve with constant $p$-curvature.

**Corollary 3.6.** Given the rotationally symmetric surface $\Sigma$ defined as in Theorem 3.4

1. If there exists a horizontally regular curve $\mathbf{r}$ on $\Sigma$ with nonzero constant $\kappa$ on $\Sigma$, then $\Sigma$ is generated by the functions $f, g$ satisfying

$$
\begin{aligned}
g &= \pm \left[ \frac{1}{\kappa}(-C_1 \cos(\kappa s) + C_2 \sin(\kappa s) + C_3) \right]^{1/2}, \\
f &= (\int \tau) + \frac{1}{2\kappa}(C_1 \sin(\kappa s) + C_2 \cos(\kappa s)) - \frac{s}{\kappa} + C_3,
\end{aligned}
$$

for any constants $C_1, C_2, C_3$.

2. If there exists a horizontally regular curve $\mathbf{r}$ with constant $\tau$ and nonzero $\kappa$ (not necessarily constant) on $\Sigma$, then the generating function $f$ and $g$ satisfying $g^2 = -2 f u_1$ and $f = f(\sqrt{w^2 + 1} - \kappa s)$, where $u_1$ is defined by (21).

**Proof.** (1) If $\mathbf{r}$ is on the surface $\Sigma$ and $\kappa$ is a nonzero constant, then $g$ satisfies $(g^2)^{\prime\prime} = -\kappa^2(g^2)'$ by (25) (26). Set $G = (g^2)'$ and then one gets the differential equation $G'' = -\kappa^2 G$. The general solution for this equation is $G = C_1 \sin(\kappa s) + C_2 \cos(\kappa s)$ for any constants $C_1, C_2$. Thus, $g$ can be recovered from $G$ by integration as shown in (27). Finally, by integrating (25) we have the function $f$.

(2) Since the differentiation of (25) is (26), the function $\frac{-1}{2}(g^2)'$ satisfies the differential equation (12), namely, $\frac{-1}{2}(g^2)'$ is of the form as $u_1$ in (21), and we get the result for $g^2$. The result for $f$ is obvious by using (25). \hfill $\square$

**Example 3.7.** When $\Sigma$ is the standard sphere $S^2_R$ centered at the origin with radius $R > 0$, then $g(s) = R \sin(s)$ and $f(s) = R \cos(s)$ for $(s, t) \in [0, \pi] \times [0, 2\pi]$. The necessary condition (25) for $\mathbf{r}$ being on the sphere is $2R^2 \sin^2 s - \kappa R \sin(s) - R^2 + 1 - \kappa \tau = 0$. If $\tau = 0$, then $\kappa = \frac{2R^2 \sin^2 s - R^2 + 1}{R \sin s}$ for any $s \in (0, \pi)$. In addition, by (26) we also have that $\kappa = \frac{-2f''}{(g^2)'} = \frac{1}{R \sin s}$. Both conditions hold for $\kappa$. \hfill $\square$
if and only if \( \sin^2 s = \frac{1}{2} \), namely, some isolated points of the curve \( r \). Therefore, we conclude that there is no horizontal curve (i.e. \( \tau = 0 \)) on \( S^2_R \).

**Example 3.8.** The Pansu sphere \( P_\lambda \), for any \( \lambda > 0 \), in \( \mathbb{H}_1 \) is one of the important rotationally symmetric compact surfaces in \( \mathbb{H}_1 \) \([20]\). Several researchers study the properties of \( P_\lambda \) with different approaches. For instance, \([14, 21]\) by the sub-Riemannian point of view; \([5, 9]\) by the pseudo-hermitian perspective. Recently, Cheng \([4]\) published a survey paper about the Pansu spheres and related problems of submanifolds in the CR manifolds. More precisely, \( P_\lambda \) is defined by the union of the graphs of the functions \( F \) and \(-F\) defined by

\[
F(x, y) = \frac{1}{2\lambda^2} \left( \lambda \sqrt{x^2 + y^2} \sqrt{1 - \lambda^2(x^2 + y^2)} + \cos^{-1}(\lambda \sqrt{x^2 + y^2}) \right), \quad \sqrt{x^2 + y^2} \leq \frac{1}{\lambda}.
\]

The Pansu sphere can also be obtained by rotating the curve \( r(s) = (x(s), y(s), z(s)) \), \( s \in [0, \pi] \) about the \( z \)-axis, where

\[
\begin{align*}
x(s) &= \frac{1}{2\lambda} \sin(2\lambda s), \\
y(s) &= -\frac{1}{2\lambda} \cos(2\lambda s) + \frac{1}{2\lambda}, \\
z(s) &= \frac{1}{4\lambda^2} \sin(2\lambda s) - \frac{s}{2\lambda} + \frac{\pi}{4\lambda^2}.
\end{align*}
\]

The curve \( r \) is parametrized by horizontal arc-length \( s \) and joins the ”north and south poles” at \((0, 0, \pm \frac{\pi}{2\lambda}) \) in \( \mathbb{H}_1 \). It is a geodesic (in the sense of pseudo-hermitian manifolds) with constant \( p \)-curvature \( \kappa = 2\lambda \), and so it is horizontal. According to the first result in Corollary 3.6 the Pansu sphere can be generated by some functions \( g(s) \) and \( f(s) \) with suitable constants \( C_1, C_2, C_3 \) as shown in (27). Indeed, if we set \( C_1 = \frac{1}{\lambda}, C_2 = 0, C_3 = \frac{1}{\lambda} \), and define \( g(s) = \frac{1}{\lambda} \cos(-2\lambda s) \) and \( f(s) = \frac{1}{4\lambda^2} (\sin(-2\lambda s) - 2\lambda s) + \frac{1}{\lambda} \), one can easily check that the set

\[
\{ (g(s) \cos(t), g(s) \sin(t), f(s)) \mid s \in [0, 2\pi], t \in [-\frac{\pi}{\lambda}, 0] \}
\]

is exactly same as the Pansu sphere defined by (25). Moreover, both \( f, g \) satisfy (25) for the case \( \kappa = 2\lambda \) and \( \tau = 0 \). Also, by using the half-angle formula for sine and cosine, we have

\[
g(s) = \frac{1}{\lambda} \cos(-\lambda s) = \left[ \frac{1 + \cos(-2\lambda s)}{2\lambda^2} \right]^{1/2},
\]

and

\[
f(s) = \frac{1}{4\lambda^2} (\sin(-2\lambda s) - 2\lambda s) + \frac{1}{\lambda} = \frac{-1}{2\kappa} \frac{1}{\lambda} \sin(\kappa s) - \frac{s}{\kappa} + \frac{1}{\lambda} = \frac{1}{2\kappa} \frac{C_1 \sin(\kappa s)}{\kappa} - \frac{s}{\kappa} + C_3.
\]

Both above show that the functions \( g, f \) can be written in the forms as shown in (27).
4. Bertrand mates

In this section we will provide the criterions for any pair of horizontally regular curves in $\mathbb{H}_1$ being Bertrand mates. Let $r(s)$ and $\bar{r}(\bar{s})$ in $\mathbb{H}_1$ be horizontally regular curves with horizontal arc-length $s$ and $\bar{s}$ respectively, and $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the corresponding orthonormal basis as mentioned in the previous section. The notations $r' := \frac{dr(s)}{ds}$ and $\bar{r}' := \frac{d\bar{r}(\bar{s})}{d\bar{s}}$ are the derivatives of $r$ and $\bar{r}$ with respect to the parameters $s$ and $\bar{s}$, respectively. We will present some observation about Bertrand-like curves in $\mathbb{H}_1$ and then give the formal definition for Bertrand mates. We may consider $\bar{s} = \bar{s}(s)$, a function of $s$, and assume that $\bar{t}(\bar{s}) = g(s)n(s)$ for some nonzero function $g(s)$. First, we get $\bar{t}(\bar{s}) = -J\bar{n}(\bar{s}) = -J(g(s)n(s)) = g(s)t(s)$. Secondly, by taking the derivative with respect to $s$ on the assumption and using the Frenet frame formula (4), on one hand we have
\[ \bar{n} \frac{d\bar{s}}{ds} = g' n + g(-\kappa t - b); \]
on the other hand,
\[ \bar{n}' \frac{d\bar{s}}{ds} = (-\bar{\kappa} t - b) \frac{d\bar{s}}{ds} = (-\bar{\kappa} g t - b) \frac{d\bar{s}}{ds}. \]

Compare with the coefficients of $t, n, b$ above to get
\[
\begin{align*}
\frac{d\bar{s}}{ds} &= g \equiv \text{const.} \neq 0, \\
\bar{\kappa} &= \kappa.
\end{align*}
\]

The result, $g$ is a nonzero constant, suggests that when studying the Bertrand-like curves in $\mathbb{H}_1$ we may assume that $\bar{n} = gn$ for some constant $g$. Therefore, we set $g = 1$ for simplicity and introduce the following definition.

**Definition.** A horizontally regular curve $r(s)$ in $\mathbb{H}_1$ is a Bertrand curve if there exists an associated horizontally regular curve $\bar{r}(\bar{s})$ such that the corresponding normal vectors $n(s)$ and $\bar{n}(\bar{s})$ satisfying $\bar{n}(\bar{s}) = n(s)$ for all $s, \bar{s}$ in the their intervals. If $r$ is a Bertrand curve, we call $\bar{r}$ a Bertrand mate of $r$.

Note that in general the parameter $s$ of $r$ may not necessarily be the arc-length parameter for $\bar{r}$; however, if $r$ and $\bar{r}$ are Bertrand mates, then $\frac{d\bar{s}}{ds} = 1$ by (29) and our setting $g = 1$.

The following theorem gives the existence and nonuniqueness for Bertrand curves in $\mathbb{H}_1$: (1) any horizontal regular curve (without any constraint) is a Bertrand curve. (2) A Bertrand curve may have more than one Bertrand mates.

**Theorem 4.1.** Any horizontally regular curve in $\mathbb{H}_1$ is a Bertrand curve. In particular, for a given horizontally regular curve $r$ in $\mathbb{H}_1$ with the geometric invariants $\kappa$ and $\tau$, we can characterize the associated Bertrand mates $\bar{r}$ by the value of $\kappa$:

1. When $\kappa(s) \equiv 0$, then $\bar{r}$ is a Bertrand mate of $r$ with the geometric invariants $\bar{\kappa}$ and $\bar{\tau}$ if and only if $\bar{r}(s) = r(s) + c_1 t(s) + c_2 n(s) + g(s)b(s)$ for any constants $c_1, c_2$ and some function $g(s)$ with $\bar{\kappa}(\bar{s}) \equiv 0$ and $\bar{\tau}(\bar{s}) = \tau(s) - c_2 + g'(s)$.

2. When $\kappa(s) \neq 0$, then the curve $\bar{r}$ is a Bertrand mate of $r$ with $\bar{\kappa}$ and $\bar{\tau}$ if and only if $\bar{r}(s) = r(s) + u_1 t(s) + u_2 n(s) + u_3 b(s)$, where
\[ u_1(s) = c_1 \sin(\int^s \kappa) + c_2 \cos(\int^s \kappa), \quad u_2(s) = c_1 \cos(\int^s \kappa) - c_2 \sin(\int^s \kappa), \]
and

\[ u_3(s) = \int_s^\tau (u_2 - \tau + \bar{\tau}) \]

for some constants \( c_1, c_2 \).

**Proof.** (1) If \( \kappa(s) \equiv 0 \) and \( \bar{\mathbf{r}}(s) = \mathbf{r}(s) + c_1 \mathbf{t}(s) + c_2 \mathbf{n}(s) + g(s) \mathbf{b}(s) \) for some constants \( c_1, c_2 \) and some function \( g(s) \). Substitute \( \kappa = 0 \), \( u_1 = c_1 \), \( u_2 = c_2 \), and \( u_3 = g \) into (31), one has

\[ \frac{d\bar{\mathbf{r}}}{ds} = \mathbf{t} + (\tau + g' - c_2) \mathbf{b}. \]

Thus, \( \bar{\mathbf{r}} \) has the horizontal arc-length \( s \), \( \bar{\mathbf{t}} = \mathbf{t} \) (and so \( \bar{\mathbf{n}} = \mathbf{n} \)), \( \bar{\tau} = \tau + g' - c_2 \), and hence \( \bar{\mathbf{r}} \) is a Bertrand mate of \( \mathbf{r} \). Using the Frenet formula (4) to have \( \bar{\kappa} \bar{\mathbf{n}} = \bar{\mathbf{t}} = \kappa \mathbf{n} \), which implies that \( \bar{\kappa} \equiv \kappa \equiv 0 \). In this case, the projections of both curves \( \mathbf{r}, \bar{\mathbf{r}} \) onto the \( xy \)-plane are lines.

Conversely, let \( \bar{\mathbf{r}}(\bar{s}(s)) \) be a Bertrand mate of \( \mathbf{r} \) with horizontal arc-length parameter \( \bar{s} \). Since \( \bar{\mathbf{n}}(\bar{s}) = \mathbf{n}(s) \), taking the derivative with respect to the parameter \( s \) to get

\[ \frac{d\bar{s}}{ds} = \mathbf{n}' \frac{ds}{d\bar{s}} = \mathbf{n}' = -\kappa \mathbf{t} - \mathbf{b} = -\mathbf{b}. \]

Here we have used \( \frac{ds}{d\bar{s}} = 1 \) in the second equality. Thus, \( \bar{\kappa} \equiv 0 \). Writing

\[ \bar{\mathbf{r}}(\bar{s}) = \mathbf{r}(s) + u_1(s) \mathbf{t}(s) + u_2(s) \mathbf{n}(s) + u_3(s) \mathbf{b}(s) \]

for some functions \( u_1, u_2, u_3 \) to be determined, and using (33) with the assumption \( \kappa = 0 \), we have

\[ (\bar{\mathbf{t}} + \bar{\tau} \mathbf{b}) \frac{d\bar{s}}{ds} = \mathbf{r}' \frac{d\bar{s}}{d\bar{s}} = (1 + u_1') \mathbf{t} + u_2' \mathbf{n} + (\tau + u_3' - u_2) \mathbf{b}. \]

We conclude that \( u_1' = u_2' \equiv 0 \) and \( \bar{\tau} = \tau + u_3' - u_2 \). By setting \( u_1 = c_1, u_2 = c_2 \), and \( u_3(s) = g(s) = \int (\tau - \tau + c_2) \) for some constants \( c_1, c_2 \), the result follows.

(2) Let \( \bar{\mathbf{r}}(\bar{s}) \) be a Bertrand mate of \( \mathbf{r} \) and be defined by (33). By (32), the horizontal arc-length parameters \( \bar{s} \) and \( s \) satisfy \( \frac{d\bar{s}}{ds} = 1 \). Using the similar argument as in (34), one has that

\[ (\bar{\mathbf{t}} + \bar{\tau} \mathbf{b}) \frac{d\bar{s}}{ds} = \mathbf{r}' \frac{d\bar{s}}{d\bar{s}} = (1 + u_1' - \kappa u_2) \mathbf{t} + (u_2' + \kappa u_1) \mathbf{n} + (\tau + u_3' - u_2) \mathbf{b}, \]

and so the functions \( u_1, u_2, u_3 \) satisfy the conditions

\[ \begin{aligned} u_1' &= \kappa u_2, \\ u_2' &= -\kappa u_1, \\ \bar{\tau} &= \tau + u_3' - u_2. \end{aligned} \]

Since \( \kappa \neq 0 \), the first two equations in (36) imply that both \( u_1, u_2 \) satisfy the differential equation (17), and hence the general solutions \( u_1, u_2 \) for the O.D.Es are as shown in (30) (according to (19)). Finally, by the third equation of (36), the function \( u_3 \) is uniquely determined (up to a constant) by the integral of \( u_2, \tau, \bar{\tau} \) as in (31), and the result follows.

Conversely, if \( \bar{\mathbf{r}} := \mathbf{r} + u_1 \mathbf{t} + u_2 \mathbf{n} + u_3 \mathbf{b} \), where \( u_1, u_2, u_3 \) are defined as in (30) (31). Take the derivative with respect to \( s \) on \( \bar{\mathbf{r}} \) and then we get (35). Since \( u_1, u_2, u_3 \) satisfy (36), the equation (35) implies that \( \frac{ds}{d\bar{s}} = 1 \) and \( \bar{\mathbf{t}} = \mathbf{t} \), and hence \( \bar{\mathbf{n}} = \mathbf{n} \). We conclude that \( \bar{\mathbf{r}} \) is a Bertrand mate of \( \mathbf{r} \).

Notice that in this case, \( \bar{\kappa} = \kappa \) and \( \bar{\tau} \) satisfies (31). \( \square \)

We also have the following result which gives the same result as that of Bertrand curves in the Euclidean spaces.
Corollary 4.2. If \( \bar{r} \) and \( r \) are two Bertrand mates in \( \mathbb{H}_1 \), then the distance between two curves \( \text{dist}(\bar{r}(s), r(s)) = \text{const.} \) for any \( s \) in the interval.

Proof. By Theorem 4.1, the distance between \( r \) and \( \bar{r} \) in both cases of \( \kappa = 0 \) and \( \kappa \neq 0 \) is given by \( \text{dist}(\bar{r}(s), r(s)) = \sqrt{c_1^2 + c_2^2} \). \( \square \)

For any curves \( r \) and \( \bar{r} \), one can study all possible relations between the unit vectors in the corresponding orthonormal frames \( \{t, n, b\} \) and \( \{\bar{t}, \bar{n}, \bar{b}\} \). According to the orthogonality and orientation for the frames, we only have to consider the three possibilities: \( \bar{n} = n, \bar{b} = b, \text{ or } \bar{t} = n \). Bertrand mates belongs to the first case. For the second, there does not exist any pair of curves satisfying the condition since the vector \( \bar{b} = b \) is always perpendicular to \( n \). However, there indeed exist two curves \( \bar{r}, r \) in the Euclidean spaces \( \mathbb{R}^n \) satisfying that \( \bar{n}(s) = b(s) \) (such curves are called Mannheim partner curves). Several properties have been derived for those curves. See [1, 10, 17] and the references therein. Thus, the notion for Mannheim partner curves does not work in \( \mathbb{H}_1 \). In the rest of the section, we will find if there exists any pair of curves satisfying \( \bar{t}(s) = g(s)n(s) \), and the following theorem shows that the answer is also negative.

Theorem 4.3. Given a horizontally regular curve \( r \) in \( \mathbb{H}_1 \). There does not exist any curve \( \bar{r} \) satisfying \( \bar{t}(s) = g(s)n(s) \) or \( \bar{n}(s) = -g(s)t(s) \) for any function \( g(s) \).

Proof. Notice that the conditions \( t = gn \) and \( n = -gt \) are equivalent, and hence we only show the nonexistence for the first case. If \( r \) and \( \bar{r} \) are the curves satisfying \( \bar{t} = gn \). By taking the derivative with respect to \( s \) and use the Frenet formula (4), we have

\[
\kappa(-gt) \frac{ds}{ds} = \bar{n} = \frac{d\bar{t}}{ds} = \frac{d(gn)}{ds} = g'n + g(-\kappa t - b).
\]

Simplify to have \( g(\kappa - \kappa)t + g'n - gb = 0 \). Since \( \{t, n, b\} \) is the set of linearly independent vectors, \( g \equiv 0 \), and we conclude that there does not exist the curve \( r \) satisfying the condition \( \bar{t} = gn \). \( \square \)

5. A classification of curves in \( \mathbb{H}_1 \)

The last application of (3) will be presented here. Suppose the left-hand side of (3) is zero. If any point on the curve \( r(s) \) lies on the planes: \( \text{span}\{t((s), n(s))\} \), \( \text{span}\{t(s), b(s)\} \), and \( \text{span}\{n(s), b(s)\} \) respectively, for all \( s \), we have the following results.

Theorem 5.1. Let \( r(s) \) be a horizontally regular curve parametrized by horizontal arc-length \( s \).

1. Suppose \( r(s) \in \text{span}\{t(s), n(s)\} \) for all \( s \).
   - (a) If \( \kappa(s) \equiv 0 \), then \( r \) is a line on the \( xy \)-plane.
   - (b) If \( \kappa(s) \neq 0 \), then \( r \) is a planar curve but not a line on the \( xy \)-plane.

2. If \( r(s) \in \text{span}\{t(s), b(s)\} \) for all \( s \), then \( r \) is a curve on the vertical plane (perpendicular to the \( xy \)-plane).

3. If \( r(s) \in \text{span}\{n(s), b(s)\} \) for all \( s \), then \( r \) is a circular helix about the \( z \)-axis.

Proof. (1) Since \( r(s) \in \text{span}\{t(s), n(s)\} \), we assume that \( r = u_1 t + u_2 n \) for some functions \( u_1, u_2 \).

By setting \( \dot{r} = 0 \), the functions \( u_1, u_2, u_3 \), in the Cesáro condition (10) shall be replaced by \( -u_1, -u_2, \) and \( u_3 = 0 \), respectively, namely,

\[
\begin{align*}
1 - u_1' + \kappa u_2 &= 0, \\
u_2' + \kappa u_1 &= 0, \\
t + u_2 &= 0.
\end{align*}
\]

(37)
Case (a): If $\kappa \equiv 0$, then by (37) we have $u_1 = s + c_1$ and $u_2 = -\tau = c_2$ for some constants $c_1, c_2$. Thus, the curve must be of the form $r(s) = (s + c_1)t(s) - \tau n(s)$. Represent the curve componentwise, $r = (x, y, z)$, and use (38), we have

$$\begin{cases} x &= (s + c_1)x' + \tau y', \\ y &= (s + c_1)y' - \tau x', \\ z &= (s + c_1)(xy - xy') + \tau(yy' + xx'). \end{cases}$$

Take the derivative in the first two equations to have $(s + c_1)y'' = \tau x''$ and $-\tau y'' = (s + c_1)x''$. Thus, one gets either $(s + c_1) : \tau = -\tau : (s + c_1)$ or $x'' = y'' = 0$. But it is impossible to be the former, otherwise $r$ degenerates to a point. The latter implies that $x, y$ both are linear in $x$, namely, $x = as + b$ and $y = cs + d$ for some constants $a, b, c, d$. Again, substituting $x$ and $y$ into the first two equations to obtain $b = ac_1 + \tau c$ and $d = cc_1 - \tau a$. Then substituting $b, c$ into the third equations with a simple computation implies that $\tau = 0$. In consequence, $r$ is a line on the $xy$-plane.

Case (b): If $\kappa \neq 0$, then by $f = \frac{\tau}{\kappa}$, then the third equation of (37) implies that $u_2 = -\tau$ and $u_1 = \frac{\tau}{\kappa} = f$, and hence $r = ft - \tau n$. When writing $r$ in terms of the coordinates (using (3) (7)), one gets

$$\begin{align*}
(38) & \quad x = f x' + \tau y', \\
(39) & \quad y = f y' - \tau x', \\
(40) & \quad z = f(xy - xy') + \tau(yy' + xx').
\end{align*}$$

On one hand, multiply (38) and (39) by $x'$ and $y'$, respectively, and sum them up to have $xx' + yy' = f((x')^2 + (y')^2)$. On the other hand, multiply (38) and (39) by $y'$, $-x'$ and sum them together to have $xy' - yx' = \tau((x')^2 + (y')^2)$. As a result by (40) we have $z = 0$, i.e., the curve $r$ lies on the $xy$-plane. Finally, since $\kappa \neq 0$, $r$ is never a line.

(2) If $r \in \text{span}\{t, b\}$, by setting $\mathbf{r} = 0$, $u_2 \equiv 0$ in (8), and replacing $u_1, u_3$ by $-u_1, -u_3$ in (10), the similar argument as the previous case (1) implies that $u'_1 = 1, u_1 = 0$, and $u'_3 = \tau$. Thus, we have $\kappa = 0$ and $r = u_1 t + u_3 b = (s + c_1)t + (\int \tau) b$ for some constant $c_1$. By using (6) again, the components of the curve $r(s) = (x(s), y(s), z(s))$ satisfy $x = (s + c_1)x'$, $y = (s + c_1)y'$, and $z = (s + c_1)(xy - xy') + \int \tau$. Solve the differential equations to have

$$x(s) = c_2(s + c_1), \quad y(s) = c_3(s + c_1), \quad z(s) = \int \tau$$

for some constants $c_1, c_2, c_3$. Therefore, the curve $r$ lies on the vertical plane $\{c_3 x = c_2 y\}$ as the result.

(3) By setting $r = u_2 n + u_3 b$ and using the same argument, one gets $1 + \kappa u_2 = 0$, $u'_2 = 0$, and $u'_3 - u_2 - \tau = 0$, and so $u_2 = c_1, \kappa = -\frac{1}{c_1}$, $u_3 = c_1 s + c_2 + \int \tau$ for some constants $c_1 \neq 0$ and $c_2$. Then the curve $r = c_1 n + (c_1 s + c_2 + \int \tau) b$ with $\kappa \neq 0$. Write $r$ in terms of its components we have

$$\begin{align*}
x &= -c_1 y', \\
y &= c_1 x', \\
z &= -c_1(yy' + xx') + c_1 s + c_2 + \int \tau.
\end{align*}$$

The first two equations above imply that

$$\begin{align*}
x(s) &= c_3 \sin\left(\frac{s}{c_1}\right) + c_4 \cos\left(\frac{s}{c_1}\right), \\
y(s) &= c_3 \cos\left(\frac{s}{c_1}\right) - c_4 \sin\left(\frac{s}{c_1}\right),
\end{align*}$$

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for any constant $c_3, c_4$. Note that $x^2 + y^2 = (c_3)^2 + (c_4)^2 = \text{const.}$ This means that $r$ lies on a cylinder with axis $\{x = y = 0\}$. Substitute $x$ and $y$ into the third equation to have $z(s) = c_1 s + c_2 + \int^s \tau$. In consequence, $r$ is a circular helix on the cylinder $\{x^2 + y^2 = \text{const.}\}$.

□

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YEN-CHANG HUANG, DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL UNIVERSITY OF TAINAN, TAINAN, TAIWAN

Email address: ychuang@mail.nutn.edu.tw