Nonlinearity and linearity, friends or enemies?
Algebraic Analyzation of Science:)

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Abstract

Certain operator-valued functions and new generating structures (instead of generating functionals) are proposed for the analysis of equations for n-point information (n-pi). Some remarks are made concerning the intertwining of linearity and nonlinearity, and functions defined on non-numerical objects.

1 Introduction

Let us start from two equations:

$$y = ax + b$$  \hspace{1cm} (1)

and

$$y^2 + x^2 = r^2$$  \hspace{1cm} (2)

To describe them, six different symbols were used. In the first case: $y, a, x, +, b$ and in the second: $y, 2, +, x, =, r$. In the first case we have a linear object - a straight line, in the second, we have a nonlinear object - a circle. On these grounds, it is difficult to decide which object is easier or more complicated to handle. In fact, we have here used certain convention which allows to us simplify equations, which should be written as follows:

$$y = a \cdot x + b$$  \hspace{1cm} (3)

$$(y \cdot y)^2 + (x \cdot x)^2 = (r \cdot r)^2$$  \hspace{1cm} (4)

In this case first equation needs 7 different symbols and second equation needs 9! In the most simple case

$$y = x$$  \hspace{1cm} (5)

and
\[(y \cdot y)^2 + (x \cdot x)^2 = 1\]  

(6)

we have 3 and 8 different symbols to describe the particular straight line and circle. If we wanted to have the same generality in a circle as in the case of a straight line, then we would have to enter additional 2 parameters (a total of 10). These simple examples show that the description of nonlinearity includes some additional complexity which are not in the linear models.

Other, well known examples, are homogeneous and non homogeneous linear differential equations encountered in physics or engineering, to which one can find relatively easy specific, and in many cases general solutions or to use effective methods of approximations. These may be arguments for trying to look for linear models, even if their original versions are nonlinear. It turns out that this can be done at the expense of introducing additional, infinite number of variables (e.g. correlation functions), having hope that in this way the properties of linear systems will be effectively used. Unfortunately, an infinite number of variables needed to linearize the original nonlinear problem leads at least to two issues: We have too many solutions which can not be related to reasonable physical conditions, see [?], and we have difficulties in a precise definition of a number of terms appearing in equations for n-point informations (n-pi).

These two issues were addressed in greater or lesser degree in the previous author’s papers. In the present work we will focus on a definition of certain operator-valued functions and we introduce, for the n-pi, instead of generating functionals - the new generating structures leading to an algebraization of physics.

In this paper we will address these two problems taking into account linear properties of appropriate entities. At this point we would like to note in passing that talking about linearity of considered formulas, or equations, we usually mean that they depend on the first power of certain set of dependent variables. In this way the nonlinearity of the original theory appears into linearized theory in different ways and in fact we are speaking about the relative linearity.

In the case of formula (3), the absolute linearity would mean that

\[y = a + x + b\]  

(7)

which goes on to describe a straight line but inclined at an angle of 45° and otherwise translated with respect to the coordinate system. This example shows that when building models of various phenomena a linearity request should be used with a sense and we are rather using the relative linearity, which de facto means coexistence of non-linearity with linearity.

2 Vector-valued functions (v-vf)

In mathematics we are talking about linear spaces and linear mappings (v-vf) as indeed one of the latter is closely related. Given a vector space its elements are present as linear combinations of basis vectors B. The numbers used in
these combinations are components of vectors. We can view this by means of (relative) linear mapping \( f \):

\[
\{ f(\varrho; B) \}_{\varrho \in \mathbb{R}^n} \iff V^n
\]

where \( V^n \) is \( n \)-D linear vector space. In fact, the linear mapping

\[
f(\alpha' \varrho' + \alpha'' \varrho''; B) = \alpha' f(\varrho'; B) + \alpha'' f(\varrho''; B)
\]

represents isomorphically \( n \)-D linear space created with vectors \((\varrho_1, ..., \varrho_n)\):

\[
f(\varrho; B) \equiv \sum_{i=1}^{n} \varrho_i \vec{e}_i \iff (\varrho_1, ..., \varrho_n)
\]

If a new base, \( B' \), is chosen, then we should have:

\[
f(\varrho; B) = f(\varrho'; B')
\]

From what we have previously said it results that for the description of the linear vector spaces one can use \( v \)-vf depending on variables \( \varrho \) and ’parameters’ \( B \).

In the case of

\[
f = f(\varrho; B(P))
\]

we have base \( B \) depending on the point \( P \) and relation has a local character; to every point \( P \) a linear space of vectors is related. In fact, we are dealing here with hidden nonlinearity and, like in the linearized theories, the non-linearity does not permit to forget about yourself. Similar intertwining of linearity and non-linearity exists in the case of non-linear manifolds to which, at each point, the tangent space is introduced, see also [13].

**Symmetry**

If a certain symmetry takes place, like the permutation symmetry:

\[
\varrho = S \varrho; \quad S = S^*
\]

then we have:

\[
f(\varrho; B) = f(S \varrho; B) = f(\varrho; SB) = f(S \varrho; SB)
\]

This equality indicates that one can use a base richer than in the absence of symmetry which may lead to left or right invertibility of useful set of operators which make possible to introduce appropriate projectors.

For a linear transformation \( A \)

\[
f(\varrho'; B) = f(A \varrho; B) = f(\varrho; B') \iff f' = Af
\]
where as an exercise see: $B' = ?$

In all these formulas the base $B$ can be finite or even uncountably dimensional as in the case of Fock space used for description of linearized equations, see [1]. In the latter case, a linear function $f$ depends on the uncountable number of parameters $B$.

3 Operator-valued functions

In many areas of science, however, non-linear functions are used which, although they depend on one or more parameters are really difficult to define a meaningful, because of the fact that their arguments (variables) are operators. The functional calculus is defined sometimes as a branch of mathematics about inserting operators into functions to get in result meaningful, or, at least formally correct, new operators, see, e.g., [8], [2, 3], [4, 5]. In this section we try to identify the operator associated with the function

$$f(x) = a \frac{x}{1 - x} \quad (16)$$

using a slightly generalized functional calculus. First, we will try to determine the operator

$$f(\hat{M}) = \frac{\hat{M}}{I - \lambda_2 \hat{M}} = ? \quad (17)$$

where $\hat{M}$ is a right invertible operator. In other words, there is an operator $\hat{M}^{-1}$ such that

$$\hat{M}\hat{M}^{-1} = \hat{I} \quad (18)$$

where $\hat{I}$ is the unit operator in a considered linear space $F$, see [6] and App.3. Then in $F$ there is projector

$$\hat{P} = \hat{I} - \hat{M}^{-1} \hat{M} \equiv \hat{I} - \hat{Q} \quad (19)$$

projecting on the null space of the operator $\hat{M}$:

$$\hat{M}\hat{P} = 0, \quad \hat{M}\hat{Q} = \hat{M} \quad (20)$$

We would like to specify the operator (17) in such way that the following equality would take place:

$$\lambda_1' \hat{M} \frac{1}{I - \lambda_2 \hat{M}} + \lambda_1'' \frac{1}{I - \lambda_2 \hat{M}} \hat{M} = (\lambda_1' + \lambda_1'') \hat{B} \quad (21)$$

where $\hat{B}$ is an operator given in a moment. The property (21) is weaker than the assumption that the two operators standing on the l.h.s. of Eq. (21) are identical.

First, we have to spacific the formal operator
\[
\frac{1}{\hat{I} - \lambda_2 \hat{M}} \equiv \hat{Y} = \left( \hat{I} - \lambda_2 \hat{M} \right)^{-1}_R
\] (22)

which we will treat as a right inverse operator to the operator \( \hat{I} - \lambda_2 \hat{M} \):

\[
\left( \hat{I} - \lambda_2 \hat{M} \right) \hat{Y} = \hat{I}
\] (23)

Multiplying this equation by the right inverse \( \lambda^{-1}_2 \hat{M}^{-1}_R \) we get equivalent equation:

\[
\left( \lambda^{-1}_2 \hat{M}^{-1}_R - \hat{Q} \right) \hat{Y} = \lambda^{-1}_2 \hat{M}^{-1}_R
\]

hence we get the following equation for \( \hat{Y} \)

\[
\left( \hat{I} - \lambda^{-1}_2 \hat{M}^{-1}_R \right) \hat{Y} = \hat{P} \hat{Y} - \lambda^{-1}_2 \hat{M}^{-1}_R
\] (24)

in which the projection \( \hat{P} \hat{Y} \) of the right inverse operator \( \hat{Y} \) is an arbitrary element. Assuming, for a sake of simplicity that \( \hat{I} - \lambda^{-1}_2 \hat{M}^{-1}_R \) is both side invertible operator, we get:

\[
\hat{Y} = \left( \hat{I} - \lambda^{-1}_2 \hat{M}^{-1}_R \right)^{-1} \left( \hat{P} \hat{Y} - \lambda^{-1}_2 \hat{M}^{-1}_R \right) \equiv \frac{1}{\hat{I} - \lambda_2 \hat{M}} \hat{Y} = \hat{P} \hat{Y} - \lambda^{-1}_2 \hat{M}^{-1}_R
\] (25)

This formula shows all the uncertainty of the expression \( \frac{1}{\hat{I} - \lambda_2 \hat{M}} \). Now it is easy to show that if

\[
\hat{B} = \hat{B} \hat{Q}
\] (26)

and if the arbitrary term

\[
\hat{P} \hat{Y} = 0
\] (27)

then Eq.(24) is satisfied. In this case the operator \( \hat{M} \)

\[
f(\hat{M}) = \frac{\hat{M}}{\hat{I} - \lambda_2 \hat{M}} = \hat{B} = \hat{B} \hat{Q}
\]

\[
\left\{ \hat{M} \left( \hat{I} - \lambda^{-1}_2 \hat{M}^{-1}_R \right) - \lambda^{-1}_2 \hat{M}^{-1}_R \right\} \hat{Q}
\]

\[
= \lambda^{-1}_2 \left\{ \left( \lambda^{-1}_2 \hat{M}^{-1}_R - \hat{I} \right) - \lambda^{-1}_2 \hat{M}^{-1}_R \right\} \hat{Q}
\]

\[
= 2 \lambda^{-1}_2 \left( \lambda^{-1}_2 \hat{M}^{-1}_R - \hat{I} \right) \hat{Q}
\] (28)

where the projector \( \hat{Q} = \hat{M}^{-1}_R \hat{M} \). Here it is worth noting that the property \( \hat{B} = \hat{B} \hat{Q} \), which underlies formula (25), is consistent with the formal expression for the function \( f \), for which \( f \simeq \hat{M} \), for \( \lambda_2 \simeq 0 \). Do not we will get it, if
not to force a linear relationship, for (21), with the parameters $\lambda'_1$ and $\lambda''_1$. It is interesting, however, that $B$ has one additional, symmetry property, which does not have a formal prototype, namely that $B = QB$. Could it be a clue in defining the operator valued functions?

Derived formula depends however on the choice of operator $\hat{M}^{-1}_R$. Thus, additional conditions are required in order to reduce its ambiguity, see [8], for example, we can demand that $\hat{M}^{-1}_R$ is the same type as $\hat{M}$, (e.g. local), see [1], Sec.4.

In the derivation of the above formula we was influenced mainly by features (21) and (26) which can be only formally justified. What does it really mean? It means so much that if the formula (17) made sense, it would be those properties that we want to take over the (inherited) already correctly defined formula (28). There is another aspect here which is not insignificant when considering the equations for n-point correlation functions, or more general, for the n-point information (n-pi), namely that the formula which is correctly specified, in many cases is the sum of the diagonal and lower triangular operators, see previous author papers. This means that it does not lead to additional links (branches) with higher n-pi. Moreover, since the formal (ill defined) formula (17), for small $\lambda_2$, describes formally in many cases polynomial interaction, we can consider the correctly defined non-polynomial formula (28) as a candidate for description of such polynomial interaction. Taking into account that closed equations for n-pi can be obtained by means of highly complicated nonlinear interactions, which approximate much simpler, polynomial interactions, see [10] we are inclined to say that the non-linearity and linearity are more friends than enemies. But that last sentence would suggest(?) that perhaps more effective is the search for simple equations than seeking for simple interactions!

4 Linearity fetish

The popular belief is that the linearity means more simplicity and effectiveness of the systems and phenomena description. The basic concepts of mechanics as radius vector, force, momentum, angular momentum - linearly depend on the variables defining them. Cartesian reference systems include the concept of linearity for both themselves and the relations binding them (Galilean transformation). It is widely believed that it is easier to solve linear systems of equations than nonlinear systems. It is surprising that it is not always the case, even when a solution is presented in the form of formal, functional integrals (see, e.g., quantum or stochastical field theories) because in that case the functional integral prevents to obtain the effective final result. Moreover, some people including me believe that the functional integrals encountered in the field theory are generally not computable, see [20]; remarks about computability.

It turns out that the linear equations satisfied by the n-pi generated by these functional integrals, branch out to infinity. This means that it can not be possible to write out a reasonable, closed set of equations, for a defined, finite set of n-pi. It is called the closure problem which is usually related to a
nonlinearity of the original (before linearization) equations.

5 A new paradigm?

These observations seem to indicate that in the existing approaches leading to an impasse - titled the closure problem - linearity and non-linearity are closely related. We propose to move away from the primacy of the detailed description of the dynamic components of the system and replacing it with the primacy of the less detailed descriptions. In this description, in the first place, will be placed on n-pi

\[ < \varphi(\tilde{x}_1) \cdots \varphi(\tilde{x}_n) > \]

with \( n=1,2,\ldots \), taking a less detailed description of the system than description supplied by the 'field' \( \varphi \), see [11]. A detailed description will be something secondary and should result from the first in which rather global properties of the system are taken into account. Thinking of this kind provides a not so old discoveries in astronomy and still threatening economic crises. Climate change also seem to suggest a different paradigm. Moving away from the detailed description is the basis of abstraction and it allows to cope with of extremely high complexity of the considered system. But the problem is to use it in a more fundamental way. In this approach, concepts such as - local, global - will play at least equivalent role. But how to accomplish this?

6 A ‘new’ approach. Free Fock space?

Motto:

'Our experience hitherto justifies us in trusting that nature is the realization of the simplest that is mathematically conceivable' Albert Einstein

In the proposed new approach, we start with the equation on the generating vector \( |V> \) for the functions \( V(\tilde{x}_{(n)}) \); \( n = 1,2,\ldots,\infty \):

\[ |V> = \sum_{n=1} \int d\tilde{x}_{(n)} V(\tilde{x}_{(n)})|\tilde{x}_{(n)}> + |0>_{info} \]

The n-point functions \( V(\tilde{x}_{(n)}) \), which we call the n-point information (n-pi) about the system, will have different interpretation in classical and quantum physics, see, e.g., [?]|\tilde{x}_{(n)}>, for \( n=1,2,\ldots \), are linearly independent orthonormal vectors, the vector \( |0>_{info} \) describes so called the local information vacuum, see [1].

The generating vector \( |V> \) satisfies the following linear equation:

\[ \left( \hat{L} + \lambda \hat{N} + \hat{G} \right) |V> = |0>_{info} = \hat{P}_0 |0>_{info} \]
The operator $\hat{L}$ is a right invertible operator, which in the case of classical (e.g. statistical) field theory is a diagonal operator:

$$P_n \hat{L} = \hat{L} P_n$$

with respect to the projectors $\hat{P}_n$; $n = 1, 2, ..., \infty$, where the projector $\hat{P}_0$ projects on the $n$-th term in the expansion (30), see App1. The projector $\hat{P}_0$ projects on the subspace of the linear space $F$ constituted by means of vectors (30). The subspace $\hat{P}_0 F$ does not contain any local information about the system. In the papers [1] and [2] we call a vector belonging to subspace $\hat{P}_0 F$ - the local information vacuum. It is surprising that both in the classical and quantum description of systems they lead to additional nonperturbative corrections.

In the case of quantum field theory the operator $\hat{L}$ is an invertible or right invertible diagonal, plus - a lower triangular operator - related to the commutation relations of the canonical conjugate operator variables, with respect to the same set of projectors $\hat{P}_n$.

In the case of polynomial nonlinearity, the operator $\hat{N}$ is an upper triangular operator in a classical as well as in quantum field theory:

$$\hat{P}_n \hat{N} = \sum_{n<m} \hat{P}_n \hat{N} \hat{P}_m$$

see [1].

The operator $\hat{G}$, in the both cases, is a left invertible operator, which is lower triangular operator:

$$\hat{P}_n \hat{G} = \sum_{m<n} \hat{P}_n \hat{G} \hat{P}_m$$

All these operators are linear operators acting in the linear space $F$ constituted from the vectors (30). If linearly independent orthonormal vectors

$$|\tilde{x}_{(n)} > = \tilde{\eta}^* (\tilde{x}_1) \cdots \tilde{\eta}^* (\tilde{x}_n) |0 >$$

see Sec.7, then we call space $F$ the free Fock space (FFS).

**Why the free Fock space (FFS) F?**

Because our experience hitherto justifies us in trusting that in such a space constructed by means of operators satisfying Cuntz relations, see, e.g., [1], it is easier to find the inverse operations to multiple operators which occur in the equations for the generating vectors as Eq.[31] see previous author papers. In some sense, we have similarity to the difference which exists in the construction of the inverse matrices by Euler’s and Gauss methods: The effectiveness of Gauss type methods, in our opinion, is due to the fact that they use effectively linearity of matrices themselves. In fact, this is a FFS task.

But that’s not all. It turns out that in this space there are operators which leads to a closed equation for n-pi, see [1] [10] [4], and so on, that for small values of the so called minor coupling constant at least formally approximate operators used in the usual (not free) Fock space.
It is not excluded that in this way the entangled together problems of non-linearity and closing are significantly overcome.

In FFS it is also possible to introduce vectors describing local and global information which allows the use of a unique language to describe phenomena belonging to different fields of human activity such as physics or economics, complex systems, etc.

7 New generating structures describing classical and quantum physics; noncommutative rings and algebraization of physics

In fact, we can avoid the introduction of the vector space of generating vectors (30) by considering only (generating) operators acting in FFS. For this purpose, instead of vectors (30) with the base vectors \( |\tilde{x}(n)\rangle = \hat{\eta}^*(\tilde{x}_1) \cdots \hat{\eta}^*(\tilde{x}_n) |0\rangle \), we introduce the lower triangular operator

\[
\hat{V}_0 \equiv |V <0| = \sum_{n=1} \int d\tilde{x}(n) V(\tilde{x}(n))\hat{\eta}^*(\tilde{x}_1) \cdots \hat{\eta}^*(\tilde{x}_n) \hat{P}_0 + \hat{P}_0 \quad (36)
\]

with \( \tilde{x}(n) \equiv \tilde{x}_1, ..., \tilde{x}_n \) and \( \hat{P}_0 = |0 > <0| \), where operators \( \hat{\eta} \) satisfy the Cuntz relations

\[
\hat{\eta}(\tilde{y})\hat{\eta}^*(\tilde{x}) = \delta(\tilde{y} - \tilde{x}) \cdot \hat{I} \quad (37)
\]

and vectors \( |0 >, <0| \) describe local information vacuum, see [1]. The star means an involution of the operator \( \hat{\eta} \), \( (\hat{\eta}^*)^* = \hat{\eta} \), and the projector

\[
\hat{P}_0 \sim \hat{P}_{info} \quad (38)
\]

We also assume that, for arbitrary 'vectors' \( \tilde{x} \), the projector \( \hat{P}_0 = \hat{P}_0^* \) and operators \( \hat{\eta} \) have the following properties:

\[
\hat{\eta}(\tilde{x})\hat{P}_0 = \hat{P}_0\hat{\eta}^*(\tilde{x}) = 0 \quad (39)
\]

From the above, we have

\[
\hat{P}_0\hat{\eta}(\tilde{y}_1) \cdots \hat{\eta}(\tilde{y}_n)\hat{V}_0 = V(\tilde{y}(n))\hat{P}_0 \quad (40)
\]

We also have

\[
\hat{V}_0 = \hat{V}_0\hat{P}_0, \quad \hat{P}_0\hat{V}_0 = \hat{P}_0 \quad (41)
\]

Operators \( \hat{V}_0 \) satisfy very similar equations as Eq.31

\[
\left( \hat{L} + \lambda \hat{N} + \hat{G} \right) \hat{V}_0 = \hat{P}_{info} \sim \hat{P}_0 \quad (42)
\]

One can introduce a more general generating operators than the operators (36), with diagonal, lower and upper triangular elements,
\[
\hat{V} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int d\tilde{x}(m)d\tilde{y}(n)V_{m,n}(\tilde{x}(m), \tilde{y}(n))\hat{\eta}^*(\tilde{x}_1) \cdots \hat{\eta}^*(\tilde{x}_m)\hat{P}_0\hat{\eta}(\tilde{y}_1) \cdots \hat{\eta}(\tilde{y}_n) + \hat{P}_{mfo}
\]

while we agree that the subscript zero means that the variable does not exist in the given expression.

We have:

\[
\hat{P}_0\hat{\eta}(\tilde{x}_1) \cdots \hat{\eta}(\tilde{x}_m)\hat{V}\hat{\eta}^*(\tilde{y}_1) \cdots \hat{\eta}^*(\tilde{y}_n)\hat{P}_0 = V_{m,n}(\tilde{x}(m), \tilde{y}(n))\hat{P}_0
\]

for \( k = 0, ..., n \) and \( n = 0, 1, ..., \infty \).

We postulate, for the operators \( \hat{V} \), the following equation:

\[
\left( \hat{L} + \lambda\hat{N} + \hat{G} \right)\hat{V} = \hat{\Phi}
\]

with a 'source' operator \( \hat{\Phi} \). Imposing on \( \hat{V} \) the condition:

\[
\hat{P}_0\hat{V} = \hat{P}_0
\]

and on the source term \( \hat{\Phi} \) the condition:

\[
\hat{\Phi}\hat{P}_0 = \hat{P}_{mfo}
\]

we see that the component \( \hat{V}\hat{P}_0 = \hat{V}_0 \) of the generating operator \( \hat{V} \) satisfies exactly the same Eq. 42 as the generating operator \( \hat{V}_0 \).

An algebraization of equations introduced here leads to description of considered equations in which the sought entities and entities used to describe equations - belong to the same category of notions, for example, they are operators. This allows for raising new questions. To see this let us assume that the generating operator \( \hat{V} \) satisfies a more general equation

\[
\hat{A}\hat{V} = \hat{\Phi}
\]

with given operators \( \hat{A}, \hat{\Phi} \). Now, in addition to Eq. 46 we postulate that

\[
\hat{\Phi}\hat{P}_0 = \hat{\Phi}_0
\]

which may be different from Eq. 47.

Like in the case of Eq. 44 let us assume that the operator \( \hat{A} \) is a right invertible. This means that a solution can be expressed as

\[
\hat{V} = \hat{A}_R^{-1}\hat{\Phi} + \hat{P}_A\hat{V}
\]

with an arbitrary projection \( \hat{P}_A\hat{V} \), where \( \hat{P}_A = \hat{I} - \hat{A}_R^{-1}\hat{A} \) is a projector on the null space of the operator \( \hat{A} \) and \( \hat{A}_R^{-1} \) is its a right inverse. Now we can see what we would get if the generating operator \( \hat{V} \) was also a right invertible with \( \hat{V}_R^{-1} \) as its right inverse: \( \hat{V}\hat{V}_R^{-1} = \hat{I} \)? From Eq. 46, we get
\[ i = \hat{A}_R^{-1} \hat{\Phi} \hat{V}_R^{-1} + \hat{P}_A \]  
\hspace{1cm} (51)

and this would mean that product of operators
\[ \hat{A}_R^{-1} \hat{\Phi} \hat{V}_R^{-1} = i - \hat{P}_A \equiv \hat{Q}_A \]  
\hspace{1cm} (52)

which is a projector, would not depend on the arbitrary source operator \( \hat{\Phi} \). But on the above limitation one can look in a more positive way, namely, that in the case of a more fundamental theory the 'sources' (including in this name the currents) and interactions described by operators \( \hat{\Phi} \) and \( \hat{A} \) are somehow related to each other. In fact, Eq. (52) like original Eq. (48) relates three entities: \( \hat{A}_R^{-1}, \hat{\Phi} \) and \( \hat{V}_R^{-1} \). But in the case of (52) this relation is a more restrictive and indicating a certain 'entanglement' or unification of them.

Multiplying Eq. (52) by \( \hat{A}_R \) we get equation
\[ \hat{\Phi} \hat{V}_R^{-1} = \hat{A} \hat{Q}_A = \hat{A} \]  
\hspace{1cm} (53)

which can be regarded as an equation for a right inverse \( \hat{V}_R^{-1} \). Having calculated \( \hat{V}_R^{-1} \), we can calculate the operator \( \hat{V}^\ast \) by means of the equation:
\[ \left( \hat{V}_R^{-1} \right)^\ast \hat{V}^\ast = i \]  
\hspace{1cm} (54)

and the generating operator \( \hat{V} = \left( \hat{V}^\ast \right)^\ast \). Hence, finally,
\[ |V> = \hat{V}|0> \]  
\hspace{1cm} (55)

It shows how algebraization of equations allows for a new approach to old problems.

In the case of transformed Eq. (41) and after its symmetrization, see, e.g., [1],

\[ \left( i + \lambda \left( \hat{I} + \hat{S} \hat{L}_R^{-1} \hat{G} \right)^{-1} \hat{S} \hat{L}_R^{-1} \hat{N} \right) |V> = \]  
\[ \left( i + \hat{S} \hat{L}_R^{-1} \hat{G} \right)^{-1} \left( \hat{S} \hat{L}_R^{-1} |0>_{\text{inf}} + \hat{S} \hat{P}_L |V> \right) \]  
\hspace{1cm} (56)

we can consider the operator equation

\[ \left( i + \lambda \left( \hat{I} + \hat{S} \hat{L}_R^{-1} \hat{G} \right)^{-1} \hat{S} \hat{L}_R^{-1} \hat{N} \right) \hat{V} = \left( i + \hat{S} \hat{L}_R^{-1} \hat{G} \right)^{-1} \left( \hat{S} \hat{L}_R^{-1} \hat{P}_{\text{inf}} + \hat{S} \hat{P}_L \hat{V} \right) \]  
\hspace{1cm} (57)

This means that in Eq. (48) the operator
\[ \hat{A} = \left( i + \lambda \left( \hat{I} + \hat{S} \hat{L}_R^{-1} \hat{G} \right)^{-1} \hat{S} \hat{L}_R^{-1} \hat{N} \right) \]  
\hspace{1cm} (58)

and the operator
\[ \hat{\Phi} = \left( \hat{I} + \hat{S}\hat{L}_R^{-1}\hat{G} \right)^{-1} \left( \hat{S}\hat{L}_R^{-1}\hat{P}_{mfo} + \hat{S}\hat{P}_L\hat{V} \right) \] (59)

8 Final remarks and comments on symmetrization of calculations; too much symmetry in science?

The operator-valued functions of the right invertible operators incorporating three properties of the formal formula as:

i. linearity, see Eq.21,
ii. commutativity with respect to the \( \hat{M} \) operator, see again Eq.21, and
iii. projecton properties, see Eq.28

where constructed.

To construct such operator-valued functions, we did not take into account spectral properties of used operators like in the usual approach of the functional calculus, but we have used the most primitive methods of defining such functions, namely, to present them in the form of infinite power series. The specificity of the submitted approach is that in many cases, for interesting projections, only a finite number of terms of such series gives contributions, see previous author papers. In this sense a new approach, illustrated by the motto to Sec.6, is possible:) An important element of this paper is also a new generating structure (operator) for the n-pi \( V(\tilde{x}_{(n)}) \) that allows to describe systems and considered equations by means of the noncommutative ring with the unity, see, e.g., [12]. In this way, the obstacle has been removed associated with the use of the basic concepts of physics, namely - vector spaces in which are not defined vector products. In the proposed approach has been abolished demarcation line between the description of equations (operators) and the description of physical systems (vectors). For both objects, we use the elements of noncommutative ring with unity.

Usually, the division on the operators and vectors is justified by the demand that an action of the operator on the vector should gives the vector. Such demand is automatically realized by the ring in which vectors are represented by the one column matrices, first one. If, however, we resign from that demand then the vectors can be substituted by the operators, e.g. the diagonal matrices. In many cases such matrices representing vectors can be inverted which is a useful property in many solving procedures.

Similar reasoning lies behind the idea of replacing the generating functions or functionals by the generating vectors. In this case, it was possible due to the fact that the generating functions or functionals do not have to be convergent. As a result, obtained equations admit more general representations.

And one more thing related to the paper title: the considered generating structures depend in the nonlinear way on the auxiliary field operators \( \hat{\eta} \) and \( \hat{\eta}^* \), see, for example, formula (36). One can introduce the equivalent generating
structures which linearly depend on the infinite set of auxiliary n-point functions $g(x_{(n)})$, see [1]. This leads, as we think, to the more complex formulas, especially in defining the operator-valued functions.

As far as the algebraization idea of physics and science in general, we would like to note that it is much less appreciated by the scientific community than the geometrization idea. In fact, we think that there is too much symmetry in science which is reflected in the assumptions on the generating functions or functionals - the entities having only auxiliary character. The mere transfer of symmetry of physical quantities on the generating structures leads in general to the divergent power series which we call the formal power series. Excess of symmetry is often masked in a natural way as a result of differences in the laws of nature (equations) from the initial or/and boundary conditions. We speak then about spontaneous symmetry breaking. You must also be aware of the fact that the very existence of the reference frame disturbs the symmetry of the described system, for instance the Universe. See [16]. See also [21].

Each symmetry is associated with some limitations. So if the auxiliary entities will unnecessarily inherit restrictions that apply to the physical entities then we will needlessly deprive ourselves effective calculations. Since the multiscale, complex systems mostly deal with permutation symmetry, it is worth recalling the Cayley's theorem that every group is isomorphic to a group of permutations. See also Klein's Erlangen program of relation of symmetry with geometry.

That's what we're talking about is similar to reductionism in science, which uses a quasi-invariance (quasi-isolation) of the system under study with respect to changes in the environment. In this analogy, the environment would be a generating vector or operator. Reduction is symmetry, see [16].

We believe that algebraization of description of the multiscale and complex systems will significantly improve the process of computing. It will also allow for a broader look at the different areas of mathematics and physics, see [15, 14, 13], and especially [18]. See, however, [16]. See also [17], where algebraic approach to quantum field theory is criticized, but there it was mainly concerned with the problem of renormalization.

At the end of work I would like to draw attention to the fact that the description of physical systems based on moving away from the details and its algebraization leading to the noncommutative rings is similar to the way that leads to free probability and noncommutative geometry, see [18]. The difference lies in the fact that approach proposed here is realized in a more transparent manner.

We would also like to draw the reader's attention to another aspect of the generalization of this and previous works. In Eq.[31] the term associated with the linear nature of the phenomenon, the so-called kinematic term, is described by the operator $\hat{L}$, which is the right invertible operator. In the simplest case, this would be a derivative of the first or higher orders. Maybe this is the reason why sometimes right invertible operators are called the derivatives. To these operators are related the basic physical quantities such as velocity, acceleration, the existence of free waves and the existence of physically interesting solutions
to considered equations. Interesting is also the fact that these are mostly diagonal plus lower triangular operators. Nonlinear phenomena are described mostly by the upper triangular operators, and external fields in which systems are submerged, are described by the lower triangular, left invertible operators. By the lower triangular operators are also described quantum properties of systems, see [9]. For this reason, it seems interesting definition of operator-valued functions considered in Sec.3 which is not leaving the above class of operators at least, for polynomial nonlinearity. Does this mean a greater unification of linear and non-linear, or, classical and quantum phenomena? That is the question.

App.1 Projectors $\hat{P}_n$.

These are projectors projecting on the n-th terms of the expansion (60). They are:

$$\hat{P}_n = \int \hat{\eta}^* (\bar{x}_1) \cdots \hat{\eta}^* (\bar{x}_n) \hat{P}_0 \hat{\eta} (\bar{x}_n) \cdots \hat{\eta} (\bar{x}_1) d\bar{x}_{(n)}$$ (60)

for n=1,2,.... They can be expressed in another form as:

$$\hat{P}_n = \int \hat{\eta}^* (\bar{x}_1) \cdots \hat{\eta}^* (\bar{x}_n) \left( \hat{I} - \int \hat{\eta}^* (\bar{x}) \hat{\eta} (\bar{x}) d\bar{x} \right) \hat{\eta} (\bar{x}_n) \cdots \hat{\eta} (\bar{x}_1) d\bar{x}_{(n)}$$ (61)

in which the vacuum projector $\hat{P}_0$ does not appear. The name of $\hat{P}_0$ comes from interpretation of functions $V(\bar{x}_{(n)})$ as the n-p-i about the field $\varphi$, see (29). We have got, of course, that the unit operator $\hat{I}$,

$$\hat{I} = \sum_{n=1}^{\infty} \hat{P}_n + \hat{P}_0$$ (62)

App.2 Other projectors

The operator

$$\hat{R} = \sum_{n=1}^{\infty} \int d\bar{x}_{(n)} \hat{\eta}^* (\bar{x}_1) \cdots \hat{\eta}^* (\bar{x}_n) \hat{P}_0 + \hat{P}_0 \equiv \sum_{n=1}^{\infty} \hat{R}_n + \hat{P}_0$$ (63)

and operators $\hat{R}_n$ are very lower triangular operators with respect to projectors $\hat{P}_n$. We can see that

$$\hat{R}_n^* \hat{R}_n = \hat{P}_0 vol^n$$ (64)

and

$$\hat{R}_n \hat{R}_n^* \cdot \hat{R}_n \hat{R}_n^* = \hat{R}_n \hat{R}_n^* \cdot vol^n$$ (65)
In other words, the diagonal products $\hat{R}_n\hat{R}_n^*$ behave like pseudo-projectors which for the unit volume are projectors. In fact they are projectors after division by $\text{vol}^{n/2}$. In contrast to the orthogonal projectors $\hat{P}_n$:

$$\hat{P}_m\hat{P}_n = \delta_{mn}\hat{P}_n$$ (66)

projectors $\hat{R}_n$ are not orthogonal.

**App.3 Algebraic analysis?**

In this as well as in the previous papers we are using certain results of algebraic analysis, see [6]. Since the same name stands for two different branches of mathematics, see [19] to form an opinion on this terminological confusion.

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