From the AKNS system to the matrix Schrödinger equation with vanishing potentials: Direct and inverse problems

Francesco Demontis | Cornelis van der Mee

Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale, Cagliari, Italy

Correspondence
Francesco Demontis, Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy.
Email: fdemontis@unica.it

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1 INTRODUCTION

For nearly 50 years nonlinear Schrödinger (NLS) equations have served as the basic models for describing surface waves on deep waters,1–3 signals along optical fibers,4–6 particle states in Bose–Einstein condensates,7–9 plasma oscillations,10 and magnetic spin waves.11,12 NLS equations with solutions decaying at infinity have been studied in detail by means of the inverse scattering transform,2,13–16 where a time invariant canonical transformation converts the initial-value problem of the matrix NLS equation into the (elementary) time evolution of the scattering data of the so-called AKNS system.

In this article, we present an apparently new method for solving matrix NLS equations inspired by the Miura transformation,2,17 which traditionally allows one to derive Korteweg-de Vries (KdV)
solutions from modified Korteweg-de Vries (mKdV) solutions (but not necessarily vice versa). By a similar transformation, the direct and inverse scattering theory of the focusing \( n = m_1 + m_2 \) AKNS system can be related to the direct and inverse scattering theory of a matrix Schrödinger equation whose \( n \times n \) matrix potential \( Q \) satisfies the adjoint symmetry relation

\[
Q^\dagger = \sigma_3 Q \sigma_3 = \begin{pmatrix} 0_{m_1 \times m_1} & q \\ -q^\dagger & 0_{m_2 \times m_2} \end{pmatrix},
\]

where \( I_m \) is the identity matrix of order \( m \), \( \sigma_3 = I_{m_1} \oplus (-I_{m_2}) \), and the dagger denotes the matrix conjugate transpose. The traditional application of the matrix Schrödinger equation to quantum graphs, quantum wires, and quantum mechanical scattering of particles with internal structure\(^{18-36}\) has led to the almost exclusive development of matrix Schrödinger scattering theory for self-adjoint potentials for which \( Q^\dagger = Q \). After the seminal papers on the scalar Schrödinger direct and inverse scattering theory by Faddeev\(^{37}\), Deift and Trubowitz\(^{38}\) and others (see Ref. 39), in the matrix case the direct and inverse half-line theory has been studied in detail by Agranovich and Marchenko\(^{40}\) and by Aktosun and Weder\(^{41,42}\) and the direct and inverse full-line theory by Wadati and Kamijo.\(^{43}\) Martinez and Olmedilla\(^{44}\) confined themselves to full line direct scattering theory, trace identities, and conservation laws. The most general small energy asymptotics of the scattering data, crucial to developing the direct and inverse scattering theory rigorously, is due to Klaus\(^{45}\) in the scalar case and to Aktosun et al.\(^{46}\) in the matrix case.

Energy losses in quantum graphs, quantum wires, and particles with internal structure naturally lead to matrix Schrödinger potentials whose imaginary part \([Q - Q^\dagger]/2i\) has constant sign.

In this article, we thus require a modified direct and inverse scattering theory when solving the matrix NLS equation by using the matrix Schrödinger equation for potentials satisfying

\[
Q^\dagger = \sigma_3 Q \sigma_3.
\]

This makes it important to discuss the direct and inverse scattering theory of matrix Schrödinger equations with potentials \( Q \) satisfying (2).

In this article, we depart from the direct and inverse scattering theory of the focusing AKNS system\(^{13,15,16}\)

\[
v_x = (-ik \sigma_3 + Q)v,
\]

where \( v = v(x, k) \) is a vector function with \( n = m_1 + m_2 \) components and the potential \( Q = \begin{pmatrix} 0_{m_1 \times m_1} & q_l \\ q_r & 0_{m_2 \times m_2} \end{pmatrix} \) anticommutes with \( \sigma_3 = I_{m_1} \oplus (-I_{m_2}) \) and satisfies \( Q^\dagger = -Q \). Letting \( L = i\sigma_3[(d/dx)I_n - Q] \) stand for the AKNS Hamiltonian, we easily verify that \( L^2 \) is the matrix Schrödinger Hamiltonian given by

\[
L^2 v = -\sigma_3[(d/dx)I_n - Q]\sigma_3[(d/dx)I_n - Q]v
= -[(d/dx)I_n + Q][(d/dx)I_n - Q]v
= -v_{xx} + Q^2 v - Qv_x + (Qv)_x = -v_{xx} + Qv,
\]

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= -v_{xx} + Q^2 v - Qv_x + (Qv)_x = -v_{xx} + Qv,
\]
where
\[
Q = Q^2 + Q_x = \begin{pmatrix} q_l q_r & [q_l]_x \\ [q_r]_x & q_r q_l \end{pmatrix}
\] (5)

is a matrix Schrödinger potential obtained from \(Q\) by the Miura transform (5). Assuming that the entries of \(Q\) and \(Q_x\) belong to the weighted \(L^1\)-space \(L^1(\mathbb{R}; (1 + |x|)dx)\), we arrive from (3) at the matrix Schrödinger equation
\[
-v_{xx} + Qv = k^2 v,
\] (6)

where \(Q\) is a so-called Faddeev class potential (i.e., its entries belong to \(L^1(\mathbb{R}; (1 + |x|)dx)\)) satisfying the adjoint symmetry relation (2). The above calculation leading to (5) has appeared before in Ref. 44.

It is well known\(^{1,13,15,16}\) that for a suitable time evolution of the AKNS scattering data, the potential \(Q = Q(x; t)\) satisfies the matrix NLSequation
\[
i Q_t + Q_{xx} - 2Q^3 = 0_{n \times n},
\] (7)

where \(Q\) anticommutes with \(\sigma_3\). Direct AKNS scattering theory involves Jost solutions and certain scattering coefficients, which are analytic in \(k\) in either the upper half complex plane \(\mathbb{C}^+\) or the lower half complex plane \(\mathbb{C}^-\), with continuous extensions up to the real \(k\)-line.\(^{13,15,16}\) On the other hand, traditionally direct matrix Schrödinger scattering theory involves Jost solutions and certain scattering coefficients, which are continuous in \(k \in \mathbb{C}^+ \cup \mathbb{R}\) and analytic in \(k \in \mathbb{C}^+\) (see Refs. 40–42 for the half-line theory and Refs. 43, 44, 46 for the full-line theory). In fact, the values \(\pm k\) of the AKNS spectral variable can be made to correspond to the matrix Schrödinger spectral variable \(k\), which greatly simplifies direct and inverse scattering theory. It is therefore natural to wonder which nonlinear evolution equation \(Q(x; t)\) is satisfied when \(Q(x; t)\) is a matrix NLS solution. We are thus lead to a nonlocal integrable equation for \(Q(x; t)\) to be derived from a Lax pair.

The advantages of the present method for solving focusing NLS equations are especially striking when dealing with focusing NLS equations where the potential \(Q(x, t)\) has (distinct) nontrivial \(x \to \pm \infty\) limits of equal spectral norm \(\mu > 0\), a topic we wish to relegate to a different article.\(^{47}\) Here, the accompanying scattering theory relies heavily on the conformal mapping \(\lambda = \sqrt{k^2 + \mu^2}\) cut along the imaginary segment \([-i \mu, i \mu]\), which has the inconvenience of sometimes being 2-to-1 (see, e.g., Refs. 48, 49 and references therein). However, the Miura transform \(Q = Q^2 + \mu^2 I_n + Q_x\) transforms the AKNS system (3) into the matrix Schrödinger equation
\[
-v_{xx} + Qv = \lambda^2 v
\]

in the spectral variable \(\lambda \in \mathbb{C}^+ \cup \mathbb{R}\) with “vanishing” potential \(Q\), which greatly simplifies the discussion of Jost solutions, scattering coefficients, Riemann-Hilbert problems, and Marchenko equations (see Ref. 47).

An important objective of this article is to derive the multisoliton solutions of the focusing NLS system by applying the Miura transform (5) to the matrix Schrödinger equation, which is then solved explicitly by the so-called matrix triplet method (see Ref. 50 and references therein). These solutions are obtained by solving the corresponding Marchenko integral equations. Once obtained, these solutions are verified by direct substitution into the nonlocal integrable equation satisfied by \(Q(x; t)\).

Let us discuss the contents of this article. In Section 2 we derive the nonlocal integrable equation for \(Q\) from a Lax pair. Next, in Sections 3 and 4, we develop the direct and inverse scattering
theory of the matrix Schrödinger equation (6) for Faddeev class potentials $Q$ satisfying (2). In particular, we introduce the Jost solutions and the scattering coefficients, write them as Fourier transforms of $L^1$-functions, and derive the Marchenko integral equations to solve the inverse scattering problem. The rather technical proof of the adjoint symmetry relations of the Marchenko integral kernels is deferred to Appendix B. The relationship between our Jost solutions and scattering coefficients and those prevailing in AKNS scattering theory will be worked out in Section 5. We then go on to derive the time evolution of the scattering data in Section 6. In Section 7, we apply the so-called matrix triplet method to derive the multisoliton solutions of the nonlocal equation (14) and the matrix NLS equation (7). Three appendices have been attached. In Appendix A, we derive the nonlocal integrable equation for $Q$ directly from the NLS equation for $Q$ without using Lax pairs. In Appendix B, we derive the adjoint symmetry relations for the Marchenko integral kernels. Finally, in Appendix C, we prove by direct substitution that the multisoliton solutions satisfy the nonlocal integrable equation and lead to matrix NLS solutions.

We adopt blackboard boldface symbols for many of the quantities pertaining to the AKNS system, thus deviating from the notational system adopted in Ref. 16. Boldface symbols are reserved for many of the quantities pertaining to the matrix Schrödinger equation. We also deviate from the praxis of Refs. 13, 16 in allowing right and left to correspond to the real line endpoints of transmission, both in the AKNS case and in the (matrix) Schrödinger case. Thus, we deviate from the right versus left conventions adopted in Refs. 13, 16.

## 2 | LAX PAIR FOR THE NEW INTEGRABLE MODEL

As to be indicated shortly, the matrix NLS system is governed by a Lax pair \{$L, A$\} of linear operators\(^{2,16,51}\) where \(L u = k u\) is the AKNS eigenvalue problem and \(u_t = A u\) describes the time evolution. Then, the matrix NLS system is equivalent to the zero curvature condition

$$L_t + LA - AL = 0,$$

where \(0\) denotes the zero operator on a suitable function space. We now observe that \(\{L^2, A\}\) is also a Lax pair of linear operators, where \(L^2u = k^2 u\) is the matrix Schrödinger eigenvalue problem and \(u_t = A u\) describes the time evolution. Then, the accompanying nonlinear evolution equation is equivalent to the zero curvature condition

$$(L^2)_t + L^2 A - AL^2 = 0.$$ (9)

Putting

$$L = i\sigma_3(\partial_x I_n - Q),$$ (10a)

$$A = i\sigma_3(2\partial_x^2 I_n - 2Q\partial_x - B_0),$$ (10b)

where \(n = m_1 + m_2\) and \(B_0\) is to be determined, we compute

$$0 = -(L_t + LA - AL)$$

$$= i\sigma_3 Q_t + \sigma_3(\partial_x I_n - Q)\sigma_3(2\partial_x^2 I_n - 2Q\partial_x - B_0)$$
\[-\sigma_3 (2\delta_x^2 I_n - 2Q\delta_x - B_0)\sigma_3 (\delta_x I_n - Q)\]
\[= i\sigma_3 Q_t + (\delta_x I_n + Q)(2\delta_x^2 I_n - 2Q\delta_x - B_0)\]
\[= - (2\delta_x^2 I_n + 2Q\delta_x - \sigma_3 B_0 \sigma_3)(\delta_x I_n - Q)\]
\[= i\sigma_3 Q_t + 2\delta_x^2 I_n - 2Q\delta_x - 2Q\delta_x^2 - (B_0)x - B_0 \delta_x + 2Q\delta_x^2 - 2Q^2 \delta_x - QB_0 - 2\delta_x^2 I_n\]
\[+ 2Q_{xx} + 4Q_x \delta_x + 2Q\delta_x^2 - 2Q\delta_x^2 + 2QQ_x + 2Q^2 \delta_x + \sigma_3 B_0 \sigma_3 \delta_x - \sigma_3 B_0 \sigma_3 \]
\[= i\sigma_3 Q_t - 2(-Q_x + \frac{1}{2}[B_0 - \sigma_3 B_0 \sigma_3])\delta_x\]
\[-(B_0)x - QB_0 + 2Q_{xx} + 2QQ_x - \sigma_3 B_0 \sigma_3 Q.\]  
(11)

To cancel the $\delta_x$ terms, $B_0$ should be the sum of $Q_x$ and a matrix commuting with $\sigma_3$. Thus, taking $B_0 = Q^2 + Q_x = Q$, we get the matrix NLS equation
\[i\sigma_3 Q_t + Q_{xx} - 2Q^3 = 0_{n \times n}.\]  
(12)

Putting $\mathcal{L} = L^2 = -\delta_x^2 + Q$, we compute
\[i\sigma_3 [\mathcal{L}_t + \mathcal{L} A - A \mathcal{L}] = i\sigma_3 Q_t\]
\[-(\delta_x^2 + \sigma_3 Q \sigma_3)[2\delta_x^2 - 2Q\delta_x - Q] + [2\delta_x^2 - 2Q\delta_x - Q](-\delta_x^2 + Q)\]
\[= i\sigma_3 Q_t + 4(-Q_x + \frac{1}{2}[Q - \sigma_3 Q \sigma_3])\delta_x^2\]
\[+ 2(-Q_{xx} + Q_x + \sigma_3 Q \sigma_3 Q - QQ)\delta_x\]
\[+ Q_{xx} + \sigma_3 Q \sigma_3 Q - 2QQ_x - Q^2.\]  
(13)

Then, (5) and $\sigma_3 Q = -Q \sigma_3$ imply that the terms in the third member involving $\delta_x^2$ and $\delta_x$ vanish. Therefore, we have arrived at the nonlinear evolution equation
\[i\sigma_3 Q_t + Q_{xx} - Q^2 + \sigma_3 Q \sigma_3 Q - 2QQ_x = 0_{n \times n},\]  
(14)

where
\[Q(x; t) = - \int_x^\infty dy \frac{1}{2} (Q - \sigma_3 Q \sigma_3) = \int_{-\infty}^x dy \frac{1}{2} (Q - \sigma_3 Q \sigma_3).\]  
(15)

A proof of the integrability of (14) based directly on (5) and (7) will be given in Appendix A. Finally, using (5) in (14) and singling out the block off-diagonal component we easily get the derivative of (12) with respect to $x$ and hence (12) itself; the block diagonal component yields the commutator of (12) and $Q$.

3 | DIRECT SCATTERING PROBLEM

In this section, we introduce the Jost solutions and scattering coefficients for the matrix Schrödinger equation (6) with Faddeev class potential $Q$ satisfying (2). For the scalar Schrödinger equation with real Faddeev class potential, the direct scattering theory is well
For self-adjoint potentials, the matrix theory on the half-line is discussed at length in Refs. 40–42, whereas important aspects of the full-line theory can be found in Refs. 43, 44, 46. Rather than discussing a complete full-line theory, we emphasize the modifications caused by the adjoint symmetry (2).

### 3.1 Jost solutions of the matrix Schrödinger equation

Let us define the Jost solution from the left $F_l(x, k)$ and the Jost solution from the right $F_r(x, k)$ as those solutions of the matrix Schrödinger equation (6) that satisfy the asymptotic conditions

\begin{equation}
F_l(x, k) = e^{ikx}[I_n + o(1)], \quad x \to +\infty,
\end{equation}

\begin{equation}
F_r(x, k) = e^{-ikx}[I_n + o(1)], \quad x \to -\infty,
\end{equation}

where $n = m_1 + m_2$. Calling $m_l(x, k) = e^{-ikx}F_l(x, k)$ and $m_r(x, k) = e^{ikx}F_r(x, k)$ Faddeev functions, we easily define them as the unique solutions of the Volterra integral equations

\begin{equation}
m_l(x, k) = I_n + \int_x^{\infty} dy \frac{e^{2ik(y-x)}-1}{2ik} Q(y)m_l(y, k),
\end{equation}

\begin{equation}
m_r(x, k) = I_n + \int_{-\infty}^x dy \frac{e^{2ik(x-y)}-1}{2ik} Q(y)m_r(y, k).
\end{equation}

Then, for each $x \in \mathbb{R}$, $m_l(x, k)$ and $m_r(x, k)$ are continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $k \in \mathbb{C}^+$, and tend to $I_n$ as $k \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. For $0 \neq k \in \mathbb{R}$, we can reshuffle (17) and arrive at the asymptotic relations

\begin{equation}
F_l(x, k) = e^{ikx}A_l(k) + e^{-ikx}B_l(k) + o(1), \quad x \to -\infty,
\end{equation}

\begin{equation}
F_r(x, k) = e^{-ikx}A_r(k) + e^{ikx}B_r(k) + o(1), \quad x \to +\infty,
\end{equation}

where

\begin{equation}
A_{r,l}(k) = I_n - \frac{1}{2ik} \int_{-\infty}^{\infty} dy \frac{e^{2iky}Q(y)m_{r,l}(y, k)}{Q(y)}.
\end{equation}

\begin{equation}
B_{r,l}(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{-2iky} Q(y)m_{r,l}(y, k).
\end{equation}

Then, $A_{r,l}(k)$ is continuous in $0 \neq k \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $k \in \mathbb{C}^+$, and tends to $I_n$ as $k \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$, while $2ik[I_n - A_{r,l}(k)]$ has the finite limit $-\Delta_{r,l} = \int_{-\infty}^{\infty} dy Q(y)m_{r,l}(y, k)$ as $k \to 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$. By the same token, $B_{r,l}(k)$ is continuous in $0 \neq k \in \mathbb{R}$, vanishes as $k \to \pm \infty$, and satisfies $2ikB_{r,l}(k) \to -\Delta_{r,l}$ as $k \to 0$ along the real $k$-axis.

Putting

\begin{equation}
F_l(x, k) = \begin{pmatrix} F_l(x, -k) & F_l(x, k) \\ F'_l(x, -k) & F'_l(x, k) \end{pmatrix}, \quad F_r(x, k) = \begin{pmatrix} F_r(x, k) & F_r(x, -k) \\ F'_r(x, k) & F'_r(x, -k) \end{pmatrix},
\end{equation}
where the prime denotes differentiation with respect to $x$, we obtain by equating the $x \to \pm \infty$ asymptotics (16) and (18)

$$F_r(x, k) = F_l(x, k) \begin{pmatrix} A_r(k) & B_r(-k) \\ B_r(k) & A_r(-k) \end{pmatrix}, \quad (21a)$$

$$F_l(x, k) = F_r(x, k) \begin{pmatrix} A_l(-k) & B_l(k) \\ B_l(-k) & A_l(k) \end{pmatrix}, \quad (21b)$$

where $0 \neq k \in \mathbb{R}$. Using that $F_{r,l}(x, k)$ satisfies the linear first-order system

$$\begin{pmatrix} V \\ V' \end{pmatrix}' = \begin{pmatrix} 0_{n \times n} & I_n \\ Q(x) - k^2 I_n & 0_{n \times n} \end{pmatrix} \begin{pmatrix} V \\ V' \end{pmatrix}, \quad (22)$$

with traceless system matrix, we see that, for $0 \neq k \in \mathbb{R}$, $F_{r,l}(x, k)$ has a determinant not depending on $x \in \mathbb{R}$. Using (16) we easily verify that $\det F_{r,l}(x, k) = (2ik)^n$ for $0 \neq k \in \mathbb{R}$.

Putting $\Phi(x, k) = \begin{pmatrix} F_r(x, k) & F_l(x, k) \\ F'_r(x, k) & F'_l(x, k) \end{pmatrix}$

$$= F_r(x, k) \begin{pmatrix} I_n \\ 0_{n \times n} \end{pmatrix} B_l(k) = F_l(x, k) \begin{pmatrix} A_r(k) \\ B_r(k) \end{pmatrix} 0_{n \times n} I_n \), \quad (23)$$

we easily see that, for $0 \neq k \in \mathbb{R}$, $\det A_r(k) = \det A_l(k)$. By analytic continuation, we get $\det A_r(k) = \det A_l(k)$ for $0 \neq k \in \mathbb{C}^+ \cup \mathbb{R}$.

Letting $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ denote the second Pauli matrix, we define

$$\sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0_{n \times n} & -i \sigma_3 \\ \sigma_3 & 0_{n \times n} \end{pmatrix}, \quad (24)$$

as the Kronecker product of $\sigma_2$ and $\sigma_3$ (cf. Ref. 52).

**Proposition 1.** For $0 \neq k \in \mathbb{C}^+ \cup \mathbb{R}$, let $V(x, k)$ and $W(x, k)$ be two size compatible matrix solutions of the linear first-order system (22). Then,

$$W(x, -k^*)^\dagger (\sigma_2 \otimes \sigma_3) V(x, k) \quad (25)$$

is independent of $x \in \mathbb{R}$. Similarly, for $0 \neq k \in \mathbb{R}$ the matrix

$$W(x, k)^\dagger (\sigma_2 \otimes \sigma_3) V(x, k) \quad (26)$$

is independent of $x \in \mathbb{R}$.

**Proof.** It is easily verified that

$$(\sigma_2 \otimes \sigma_3) \begin{pmatrix} 0_{n \times n} & I_n \\ Q(x) - k^2 I_n & 0_{n \times n} \end{pmatrix} (\sigma_2 \otimes \sigma_3) = -\begin{pmatrix} 0_{n \times n} & I_n \\ Q(x) - (-k^*)^2 I_n & 0_{n \times n} \end{pmatrix}^\dagger. \quad (27)$$
Then,
\[
\frac{\partial}{\partial x} \left[ W(x, -k^*)^\dagger (\sigma_2 \otimes \sigma_3) V(x, k) \right] = W(x, -k^*)^\dagger \left( \begin{array}{c|c} 0_{n \times n} & I_n \\ \hline Q(x) - (-k^*)^2 I_n & 0_{n \times n} \end{array} \right)^\dagger (\sigma_2 \otimes \sigma_3) V(x, k)
\]
\[
+ W(x, -k^*)^\dagger (\sigma_2 \otimes \sigma_3) \left( \begin{array}{c|c} 0_{n \times n} & I_n \\ \hline Q(x) - k^2 I_n & 0_{n \times n} \end{array} \right) V(x, k) = 0_{2n \times 2n},
\]
(28)
as claimed. The second part follows in the same way.

Let us now apply either part of Proposition 1 to derive identities for the $A$ and $B$ coefficients by equating the asymptotics as $x \to +\infty$ to the asymptotics as $x \to -\infty$. Using the second part for $V = W = \Phi$, we get
\[
A_{r,l}(k)^\dagger \sigma_3 A_{r,l}(k) - B_{r,l}(k)^\dagger \sigma_3 B_{r,l}(k) = \sigma_3,
\]
(29a)
\[
B_{r,l}(k)^\dagger = -\sigma_3 B_{l,r}(k) \sigma_3,
\]
(29b)
where $0 \neq k \in \mathbb{R}$. Using the second part for $V = W = F_{r,l}$, we get
\[
A_{r,l}(k)^\dagger \sigma_3 B_{r,l}(-k) = B_{r,l}(k)^\dagger \sigma_3 A_{r,l}(-k),
\]
(30)
where $0 \neq k \in \mathbb{R}$. Using the second part for $V = F_l$ and $W = F_r$ we obtain
\[
A_r(k)^\dagger = \sigma_3 A_l(-k) \sigma_3, \quad B_r(k)^\dagger = -\sigma_3 B_l(k) \sigma_3,
\]
(31)
where $0 \neq k \in \mathbb{R}$. Finally, using the first part for $V = W = \Phi$, we get
\[
A_{r,l}(-k^*)^\dagger = \sigma_3 A_{l,r}(k) \sigma_3,
\]
(32)
where $0 \neq k \in \mathbb{C}^+ \cup \mathbb{R}$.

Introducing the reflection coefficients
\[
R_{r,l}(k) = B_{r,l}(k) A_{r,l}(k)^{-1} = -A_{l,r}(k)^{-1} B_{l,r}(-k)
\]
(33)
and the transmission coefficients $A_{r,l}(k)^{-1}$, we obtain the Riemann–Hilbert problem
\[
\begin{pmatrix} F_l(x, -k) & F_r(x, -k) \end{pmatrix} = \begin{pmatrix} F_r(x, k) & F_l(x, k) \end{pmatrix} \begin{pmatrix} A_r(k)^{-1} & -R_l(k) \\ -R_r(k) & A_l(k)^{-1} \end{pmatrix},
\]
(34)
where the matrix $S(k)$ containing the $A$ and $R$ quantities is called the scattering matrix and the discussion of the nonsingularity of $A_{r,l}(k)$ will be presented shortly. Then it is easily verified that
\[
R_{r,l}(k)^\dagger = \sigma_3 R_{l,r}(-k) \sigma_3
\]
(35)
and
\[
S(k)^\dagger (\sigma_3 \oplus \sigma_3) S(k) = \sigma_3 \oplus \sigma_3,
\]
(36)
provided $0 \neq k \in \mathbb{R}$ and $\det A_{r,l}(k) \neq 0$. 
Above we have defined $\Delta_{r,l}$ as follows:

$$\Delta_{r,l} = \lim_{k \to 0} 2ikA_{r,l}(k) = - \lim_{k \to 0\pm} 2ikB_{r,l}(k),$$  \tag{37}$$

where the first limit may be taken from the upper half-plane. Then, the equality $\det A_r(k) = \det A_l(k)$ implies that the matrices $\Delta_{r,l}$ have the same determinant. If $\Delta_{r,l}$ is nonsingular, we say that we are in the \textit{generic case}; if instead $\Delta_{r,l}$ is singular, we are in the \textit{exceptional case}. We are said to be in the \textit{superexceptional case} if $\Delta_{r,l} = 0_{n \times n}$ and $A_{r,l}(k)$ tends to a nonsingular matrix, $A_{r,l}(0)$ say, as $k \to 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$.

Throughout this article, we assume the absence of \textit{spectral singularities}, that is, the absence of nonzero real $k$ for which $\det A_{r,l}(k) = 0$. Under this condition the reflection coefficients $R_{r,l}(k)$ are continuous in $0 \neq k \in \mathbb{R}$. For self-adjoint potentials (where $\sigma_3 = I_n$), the identities (29a) imply for each $\xi \in \mathbb{C}^n$

$$\|A_{r,l}(k)\xi\|^2 = \|\xi\|^2 + \|B_{r,l}(k)\xi\|^2, \quad 0 \neq k \in \mathbb{R},$$  \tag{38}$$

which implies the nonsingularity of $A_{r,l}(k)$ for $0 \neq k \in \mathbb{R}$. Thus, there are no spectral singularities for self-adjoint potentials, which generalizes a well-known result for $n = 1$. For general $\sigma_3$-self-adjoint potentials, there may very well be spectral singularities (see Ref. 53 for focusing AKNS examples). Recently, Bilman and Miller\textsuperscript{54} showed that the inverse scattering transform with full account of spectral singularities leads to rogue wave solutions of the focusing NLS equation with nonvanishing boundary conditions.

### 3.2 Triangular representations and Wiener algebras

The Jost solutions allow the triangular representations

$$F_l(x,k) = e^{ikx}I_n + \int_x^\infty dy \ e^{iky}K(x,y),$$  \tag{39a}$$

$$F_r(x,k) = e^{-ikx}I_n + \int_{-\infty}^x dy \ e^{-iky}J(x,y),$$  \tag{39b}$$

where for every $x \in \mathbb{R}$

$$\int_x^\infty dy \ |K(x,y)|| + \int_{-\infty}^x dy \ |J(x,y)| < +\infty.$$  \tag{40}$$

To derive (39), we introduce the auxiliary matrix functions $K(x,y)$ and $J(x,y)$ as the unique solutions of the Volterra integral equations

$$K(x,y) = \frac{1}{2} \int_{\frac{1}{2}[x+y]}^\infty dz \ Q(z) + \frac{1}{2} \int_x^\infty dz \ Q(z) \int_{\max(z,x+y-z)}^{z+y-x} d\hat{y} \ K(\hat{z},\hat{y}),$$  \tag{41a}$$

$$= \frac{1}{2} \int_{\frac{1}{2}[x+y]}^\infty dz \ Q(z) + \frac{1}{2} \int_x^y dz \int_{x+\frac{1}{2}(y-z)}^{\infty} dw \ Q(w)K(w, w + z - x),$$  \tag{41a}$$
\[ J(x, y) = \frac{1}{2} \int_{-\infty}^{\frac{1}{2}[x+y]} dz Q(z) + \frac{1}{2} \int_{-\infty}^{x} dz Q(z) \int_{z+y-x}^{\min(z, x+y-z)} d\hat{y} J(z, \hat{y}) \]

\[ = \frac{1}{2} \int_{-\infty}^{\frac{1}{2}[x+y]} dz Q(z) + \frac{1}{2} \int_{y}^{x} dz \int_{z}^{\min(z, x-y-z)} \hat{Q}(w) J(w, w+z-x). \] (41b)

Equations (41) follow from the Volterra integral equations (12) by Fourier transformation. Solving (41) by iteration we easily derive (39) and (40). Moreover,

\[ K(x, x) = \frac{1}{2} \int_{\infty}^{x} dy Q(y), \quad J(x, x) = \frac{1}{2} \int_{-\infty}^{x} dy Q(y). \] (42)

Let us now introduce some necessary terminology and well-known results on Fourier transforms of \( L^1 \)-functions. By the (continuous) Wiener algebra \( \mathcal{W} \), we mean the complex vector space of constants plus Fourier transforms of \( L^1 \)-functions

\[ \mathcal{W} = \{ c + \hat{h} : c \in \mathbb{C}, h \in L^1(\mathbb{R}) \} \] (43)

endowed with the norm \( |c| + \|h\|_1 \). Here, we define the Fourier transform as follows: \( (Fh)(k) = \hat{h}(k) = \int_{-\infty}^{\infty} dy e^{iky} h(y) \). The invertible elements of the commutative Banach algebra \( \mathcal{W} \) with unit element are exactly those \( c + \hat{h} \in \mathcal{W} \) for which \( c \neq 0 \) and \( c + \hat{h}(k) \neq 0 \) for each \( k \in \mathbb{R} \).

The algebra \( \mathcal{W} \) has the two closed subalgebras \( \mathcal{W}^+ \) and \( \mathcal{W}^- \) consisting of those \( c + \hat{h} \in \mathcal{W} \) for which \( h \) is supported on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), respectively. The invertible elements of \( \mathcal{W}^\pm \) are exactly those \( c + \hat{h} \in \mathcal{W}^\pm \) for which \( c \neq 0 \) and \( c + \hat{h}(k) \neq 0 \) for each \( k \in \mathbb{C}^\pm \cup \mathbb{R} \) \( 55 \). Letting \( \mathcal{W}_0^\pm \) and \( \mathcal{W}_0 \) stand for the (nonunital) closed subalgebras of \( \mathcal{W}_0^\pm \) and \( \mathcal{W} \) consisting of those \( c + \hat{h} \) for which \( c = 0 \), we obtain the direct sum decompositions

\[ \mathcal{W} = \mathbb{C} \oplus \mathcal{W}_0^+ \oplus \mathcal{W}_0^-, \quad \mathcal{W}_0 = \mathcal{W}_0^+ \oplus \mathcal{W}_0^- \] (44)

By \( \Pi_\pm \) we now denote the (bounded) projections of \( \mathcal{W} \) onto \( \mathcal{W}_0^\pm \) along \( \mathbb{C} \oplus \mathcal{W}_0^\mp \). Then, \( \Pi_+ \) and \( \Pi_- \) are complementary projections. In fact,

\[ (\Pi_\pm f)(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\zeta \frac{f(\zeta)}{\zeta - (k \pm i0^+)} \] (45)

where \( f \in \mathcal{W}_0 \cap L^p(\mathbb{R}) \) for some \( p \in (1, +\infty) \). These direct sum decompositions can be coupled by the Fourier transform to the natural direct sum decomposition of \( L^1(\mathbb{R}) \) as follows:

\[ L^1(\mathbb{R}) = L^1(\mathbb{R}^-) \oplus L^1(\mathbb{R}^+) \]

\[ \downarrow F \quad \downarrow F \quad \downarrow F \quad \downarrow F \]

\[ \mathcal{W}_0 \quad = \quad \mathcal{W}_0^- \oplus \mathcal{W}_0^+ \] (46)

Throughout this article, we denote the vector spaces of \( n \times m \) matrices with entries in \( \mathcal{W}, \mathcal{W}^\pm, \) and \( \mathcal{W}_0^\pm \) by \( \mathcal{W}^{n\times m}, \mathcal{W}^\pm^{n\times m}, \) and \( \mathcal{W}_0^\pm^{n\times m} \), respectively. We write \( \mathcal{L}^1(\mathbb{R})^{n\times m} \) and \( \mathcal{L}^1(\mathbb{R}^\pm)^{n\times m} \) for the vector spaces of \( n \times m \) matrices with entries in \( L^1(\mathbb{R}) \) and \( L^1(\mathbb{R}^\pm) \), respectively. Using any submultiplicative matrix norm, we can turn all of these vector spaces into Banach spaces. It is then clear that \( \mathcal{W}^{n\times m} \) and \( \mathcal{W}^\pm^{n\times m} \) are noncommutative Banach algebras with unit element and
\( \mathcal{W}_0^{n \times n} \) are (nonunital) noncommutative Banach algebras. The projections \( \Pi^\pm \) can be extended in a natural way to act on matrices of Wiener algebra elements. Hence, according to (39) and (40), for each \( x \in \mathbb{R} \), the Faddeev functions \( m_{r,l}(x, \cdot) \in \mathcal{W}_+^{n \times n} \).

For \( n = 1 \), the following result is most easily proved using the Gelfand theory of commutative Banach algebras.\(^{55}\) The extension to arbitrary \( n \) is elementary, because the determinant and the cofactor matrix of an element of \( \mathcal{W}^{n \times n} \) belong to \( \mathcal{W} \) and \( \mathcal{W}_{0}^{n \times n} \), respectively.

**Theorem 1.** If for some complex \( n \times n \) matrix \( H_\infty \) and some \( H \in L^1(\mathbb{R})^{n \times n} \) the Fourier transform \( H_\infty + \int_{-\infty}^{\infty} dz e^{ikz} H(z) \) is a nonsingular matrix for every \( k \in \mathbb{R} \) and if \( \det H_\infty \neq 0 \), then there exists \( K \in L^1(\mathbb{R})^{n \times n} \) such that

\[
\left[ H_\infty + \int_{-\infty}^{\infty} dz e^{ikz} H(z) \right]^{-1} = (H_\infty)^{-1} + \int_{-\infty}^{\infty} dz e^{ikz} K(z)
\]

for every \( k \in \mathbb{R} \).

Using that \( m_{r,l}(x, \cdot) \in \mathcal{W}_+^{n \times n} \), we easily prove with the help of (19) that \( 2ik[I_n - A_{r,l}(k)] \) belongs to \( \mathcal{W}_+^{n \times n} \) and \( 2ikB_{r,l}(k) \) belongs to \( \mathcal{W}^{n \times n} \). Assuming the absence of spectral singularities and to be in the generic case, we easily prove that the reflection coefficients \( R_{r,l}(k) \) belong to \( \mathcal{W}_0^{n \times n} \) and the transmission coefficients \( A_{r,l}(k)^{-1} \) to \( \mathcal{W}_+^{n \times n} \). Indeed,

\[
\frac{k}{k+1} A_{r,l}(k) = \frac{k}{k+i} I_n - \frac{2ik[I_n - A_{r,l}(k)]}{2i(k+i)}
\]

belongs to \( \mathcal{W}_+^{n \times n} \) and therefore, by Theorem 1 and \( k/(k+i) \in \mathcal{W} \),

\[
A_{r,l}(k)^{-1} = \frac{k}{k+i} \left[ \frac{k}{k+i} A_{r,l}(k) \right]^{-1}
\]

belongs to \( \mathcal{W}^{n \times n} \). Analogously, because

\[
\frac{k}{k+i} B_{r,l}(k) = \frac{2ikB_{r,l}(k)}{2i(k+i)}
\]

belongs to \( \mathcal{W}^{n \times n} \), we easily see that

\[
R_{r,l}(k) = \left[ \frac{k}{k+i} B_{r,l}(k) \right] \left[ \frac{k}{k+i} A_{r,l}(k) \right]^{-1}
\]

belongs to \( \mathcal{W}_0^{n \times n} \).

The exceptional case requires more attention. Assuming that the potential \( Q \in L^1(\mathbb{R}; (1 + |x|^2) dx) \), we can differentiate (17) with respect to \( k \) and solve the resulting integral equations by iteration. Next, with the help of (19), we obtain

\[
A_{r,l}(k) + B_{r,l}(k) = I_n + \int_{-\infty}^{\infty} dy \frac{e^{\mp 2iky} - 1}{2ik} Q(y)m_{r,l}(y,k).
\]
This expression satisfies
\[
\lim_{k \to 0^\pm} [A_{r,l}(k) + B_{r,l}(k)] = I_n \mp \int_{-\infty}^{\infty} dy yQ(y)\hat{m}_{r,l}(y,0) = I_n \mp E_{r,l},
\] (53)
provided \( Q \in L^1(\mathbb{R}; (1 + |x|)^2 dx) \). Taking the limit as \( k \to 0^\pm \) we get
\[
\lim_{k \to 0^\pm} \frac{2ik A_{r,l}(k) - \Delta_{r,l}}{2ik} = I_n - \frac{1}{2i} \int_{-\infty}^{\infty} dy Q(y)\hat{m}_{r,l}(y,0)
= I_n - \frac{1}{2i} G_{r,l},
\] (54a)
\[
\lim_{k \to 0^\pm} \frac{2ikB_{r,l}(k) + \Delta_{r,l}}{2ik} = \int_{-\infty}^{\infty} dy Q(y)[\mp y\hat{m}_{r,l}(y,0) + \frac{1}{2i}\dot{\hat{m}}_{r,l}(y,0)]
= \mp E_{r,l} + \frac{1}{2i} G_{r,l},
\] (54b)
where the overdot indicates differentiation with respect to \( k \).

In the superexceptional case, where \( \Delta_{r,l} = 0_{n \times n} \), it is clear that \( A_{r,l} \in \mathcal{W}_{n \times n}^+ \), provided \( Q \in L^1(\mathbb{R}; (1 + |x|)^2 dx) \). Assuming the absence of spectral singularities, using the nonsingularity of \( A_{r,l}(0) \), and applying Theorem 1, we get the transmission coefficients \( A_{r,l}(k)^{-1} \) to belong to \( \mathcal{W}_{n \times n}^0 \). Because \( A_{r,l}(k) + B_{r,l}(k) \) belongs to \( \mathcal{W}_{n \times n}^0 \), we conclude that
\[
R_{r,l}(k) = [A_{r,l}(k) + B_{r,l}(k)]A_{r,l}(k)^{-1} - I_n \in \mathcal{W}_{0 \times 0}^{n \times n}.
\] (55)

At present it is not known if, under the absence of spectral singularities, the reflection and transmission coefficients belong to \( \mathcal{W}_{n \times n}^{n \times n} \) in any other exceptional case and for general \( Q \in L^1(\mathbb{R}; (1 + |x|)^2 dx) \). Under the condition \( Q \in L^1(\mathbb{R}; (1 + |x|) dx) \), the continuity of the reflection and transmission coefficients at \( k = 0 \) (which would follow if these coefficients were to belong to \( \mathcal{W}_{n \times n}^{n \times n} \)) is known for \( n = 1 \) \(^{45}\) and for self-adjoint potentials.\(^ {46}\)

### 4 INVERSE SCATTERING PROBLEM

In this section, we introduce the Marchenko integral equations for the matrix Schrödinger equation (6) with Faddeev class potential \( Q \) satisfying (2). We make use of the hypothesis that the reflection coefficients \( R_{r,l} \in \mathcal{W}_{0 \times 0}^{n \times n} \), something that we have not been able to prove in the most general exceptional case. For the sake of simplicity we assume that the poles of \( A_{r,l}(k)^{-1} \) in \( \mathbb{C}^+ \) are simple. The extension to multiple pole situations is rather technical but straightforward.\(^ {56}\) Inverse scattering theory is well documented in the scalar case,\(^ {14,17,37–39}\) as it is in the matrix half-line case.\(^ {40–42}\) Without any details on the small energy asymptotics and the integrability properties of the Marchenko integral kernels, the matrix full-line case is discussed in some detail in Ref. 43.

Let us first write the transmission coefficients in the form
\[
A_r(k)^{-1} = A_{r0}(k) + \sum_{s=1}^{N} \frac{\tau_{r,s}}{k - k_s}, \quad A_l(k)^{-1} = A_{l0}(k) + \sum_{s=1}^{N} \frac{\tau_{l,s}}{k - k_s},
\] (56)
where $k_1, \ldots, k_N$ are the distinct simple poles of $A_{r,l}(k)^{-1}$ in $\mathbb{C}^+$, $\tau_{r,s}$ and $\tau_{l,s}$ are the residues of $A_r(k)^{-1}$ and $A_l(k)^{-1}$ at $k = k_s$ ($s = 1, \ldots, N$), and $A_{r0}(k)$ and $A_{l0}(k)$ are continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $k \in \mathbb{C}^+$, and tend to $I_n$ as $k \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$.

Letting $\tau_{r,s}$ and $\tau_{r,s}^*$ (or $\tau_{l,s}$ and $\tau_{l,s}^*$) stand for the residues of $A_r(k)^{-1}$ (or $A_l(k)^{-1}$) at $k_s$ and $-k_s^*$, we get with the help of (32)

$$
\tau_{r,s}^* = \lim_{k \to -k_s^*} (k + k_s^*)A_r(k)^{-1} = \sigma_3 \lim_{k \to -k_s^*} (k + k_s^*)A_l(-k^*)^{-1} - \sigma_3
$$

$$
= -\sigma_3 \left( \lim_{\zeta \to k_s} (\zeta - k_s)A_l(\zeta)^{-1} \right)^\dagger \sigma_3 = -\sigma_3 \tau_{l,s}^* \sigma_3.
$$

(57)

Similarly, we get $\tau_{l,s} = -\sigma_3 \tau_{r,s} \sigma_3$.

Next, let us consider the Riemann–Hilbert problem

$$
\begin{pmatrix}
F_l(x, -k) & F_r(x, -k)
\end{pmatrix}
= \begin{pmatrix}
F_r(x, k) & F_l(x, k)
\end{pmatrix}
\begin{pmatrix}
A_r(k)^{-1} & -R_l(k)
-A_r(k) & A_l(k)^{-1}
\end{pmatrix},
$$

(58)

where $\det A_{r,l}(k) \neq 0$ for $0 \neq k \in \mathbb{R}$. In the generic and superexceptional cases while assuming the absence of spectral singularities, we write

$$
R_r(k) = \int_{-\infty}^{\infty} d\alpha e^{-ik\alpha} \hat{R}_r(\alpha), \quad R_l(k) = \int_{-\infty}^{\infty} d\alpha e^{ik\alpha} \hat{R}_l(\alpha),
$$

(59)

where the entries of the $n \times n$ matrix functions $\hat{R}_r,l(\alpha)$ belong to $L^1(\mathbb{R})$. In fact, we only need to assume the absence of spectral singularities as well as $R_{r,l} \in W_{0,n,n}$ for the derivations below to be valid. Rewriting the left half of (58) we obtain

$$
e^{ikx} F_l(x, -k) = e^{ikx} F_r(x, k)A_r(k)^{-1} - e^{-ikx} F_l(x, k)e^{2ikx} R_r(k)
$$

$$
= e^{ikx} F_r(x, k)A_{r0}(k) + \sum_{s=1}^{N} \frac{e^{ikx} F_l(x, k_s) - e^{ikx} F_r(x, k_s)}{k - k_s} \tau_{r,s}
$$

$$
+ i \sum_{s=1}^{N} \frac{e^{2ikx} F_l(x, k_s)}{k - k_s} N_{r,s} - e^{-ikx} F_l(x, k)e^{2ikx} R_r(k),
$$

(60)

where the norming constants $N_{r,s}$ ($s = 1, \ldots, N$) are defined by

$$
F_r(x, k_s) \tau_{r,s} = i F_l(x, k_s) N_{r,s}, \quad s = 1, \ldots, N.
$$

(61)

By the same token, rewriting the right half of (58) we get

$$
e^{-ikx} F_r(x, -k) = e^{-ikx} F_l(x, k)A_l(k)^{-1} - e^{ikx} F_r(x, k)e^{-2ikx} R_l(k)
$$

$$
= e^{-ikx} F_l(x, k)A_{l0}(k) + \sum_{s=1}^{N} \frac{e^{-ikx} F_l(x, k) - e^{-ikx} F_r(x, k_s)}{k - k_s} \tau_{l,s}
$$

$$
+ i \sum_{s=1}^{N} \frac{e^{-2ikx} F_r(x, k_s)}{k - k_s} N_{l,s} - e^{ikx} F_r(x, k)e^{-2ikx} R_l(k),
$$

(62)
where the norming constants \( N_{r,s} \) (\( s = 1, \ldots, N \)) are defined by

\[
F_l(x, k_s) r_{l,s} = i F_r(x, k_s) N_{l,s}, \quad s = 1, \ldots, N. \tag{63}
\]

For every \( x \in \mathbb{R} \) the identities (60) and (62) are equations in \( \mathcal{W}_{n \times n} \). Using \( \Pi_- \) to project these two equations onto \( \mathcal{W}_{n \times n}^{-} \) along \( \mathcal{W}_{n \times n}^{+} \), we obtain

\[
e^{ikx} F_l(x, -k) = I_n + i \sum_{s=1}^{N} e^{2ik_s x} F_l(x, k_s) N_{r,s} k - k_s r_{s}
- \Pi_- \left[ e^{-ikx} F_l(x, k) e^{2ikx} R_r(k) \right],
\]

\[
e^{-ikx} F_r(x, -k) = I_n + i \sum_{s=1}^{N} e^{-2ik_s x} F_r(x, k_s) N_{l,s} k - k_s l_s
- \Pi_- \left[ e^{ikx} F_r(x, k) e^{-2ikx} R_l(k) \right]. \tag{64}
\]

respectively. Using (39), (59), and the identity

\[
\int_0^\infty d\omega e^{-iw} e^{ik_s \omega} = -i k - k_s, \quad s = 1, 2, \ldots, N, \tag{65}
\]

we strip off the Fourier transforms and arrive at the Marchenko integral equations

\[
K(x, y) + \Omega_r(x + y) + \int_x^\infty dz K(x, z) \Omega_r(z + y) = 0_{n \times n}, \tag{66a}
\]

\[
J(x, y) + \Omega_l(x + y) + \int_{-\infty}^x dz J(x, z) \Omega_l(z + y) = 0_{n \times n}, \tag{66b}
\]

where the Marchenko integral kernels are given by

\[
\Omega_r(w) = \hat{R}_r(w) + \sum_{s=1}^{N} e^{ik_s w} N_{r,s}, \tag{67a}
\]

\[
\Omega_l(w) = \hat{R}_l(w) + \sum_{s=1}^{N} e^{-ik_s w} N_{l,s}. \tag{67b}
\]

Indeed, to prove (66a) and (67a), we compute

\[
\int_0^\infty d\alpha e^{-ik_\alpha} K(x, x + \alpha)
= - \int_0^\infty d\alpha e^{-ik_\alpha} \sum_{s=1}^{N} \left[ e^{ik_s(\alpha + 2x)} N_{r,s} + \int_0^\infty d\beta e^{ik_s(\alpha + \beta + 2x)} K(x, x + \beta) N_{r,s} \right]
- \int_0^\infty d\alpha e^{-ik_\alpha} \left[ \hat{R}_r(\alpha + 2x) + \int_0^\infty d\beta K(x, x + \beta) \hat{R}_r(\alpha + \beta + 2x) \right]. \tag{68}
\]
Stripping off the Fourier transform and putting $y = x + \alpha$ and $z = x + \beta$, we get (66a) and (67a).

Analogously, to prove (66b) and (67b), we compute

$$\int_0^\infty d\alpha e^{-ik\alpha} J(x, x - \alpha)$$

$$= - \int_0^\infty d\alpha e^{-ik\alpha} \sum_{s=1}^N e^{-ik_s(2x-\alpha)} N_{l,s} + \int_0^\infty d\beta e^{-ik_s(2x-\alpha-\beta)} J(x, x - \beta) N_{l,s}$$

$$- \int_0^\infty d\alpha e^{-ik\alpha} \left[ \hat{R}_l(2x - \alpha) + \int_0^\infty d\beta J(x, x - \beta) \hat{R}_l(2x - \alpha - \beta) \right].$$

(69)

Stripping off the Fourier transform and putting $y = x - \alpha$ and $z = x - \beta$, we get (66b).

As in Ref. 56, we can prove the adjoint symmetry relations

$$\Omega_{r,l}(w) = \sigma_3 \Omega_{r,l}(w)^\dagger \sigma_3,$$

(70)

thus implying the following symmetry relations for the norming constants:

$$N_{r,s} = \sigma_3 N_{r,s}^\dagger \sigma_3, \quad N_{l,s} = \sigma_3 N_{l,s}^\dagger \sigma_3.$$

(71)

For the rather tedious details, we refer to Appendix B.

5  |  **AKNS VERSUS MATRIX SCHRÖDINGER DATA**

Let us define the Jost solutions of the AKNS system (3) as those solutions of (3) that for $k \in \mathbb{R}$ satisfy the asymptotic relations

$$\mathbb{F}_l(x, k) = e^{-ikx\sigma_3} [I_n + o(1)], \quad x \to +\infty,$$

(72a)

$$\mathbb{F}_r(x, k) = e^{-ikx\sigma_3} [I_n + o(1)], \quad x \to -\infty.$$

(72b)

Because we are dealing with a first-order system, there exist nonsingular matrices $\mathbb{A}_l(k)$ and $\mathbb{A}_r(k)$ such that

$$\mathbb{F}_r(x, k) = \mathbb{F}_l(x, k) \mathbb{A}_r(k), \quad \mathbb{F}_l(x, k) = \mathbb{F}_r(x, k) \mathbb{A}_l(k),$$

(73)

and hence $\mathbb{A}_l(k)$ and $\mathbb{A}_r(k)$ are each other's inverses. Equations (72) and (73) imply the asymptotic relations

$$\mathbb{F}_l(x, k) = e^{-ikx\sigma_3} \mathbb{A}_l(k) [I_n + o(1)], \quad x \to -\infty,$$

(74a)

$$\mathbb{F}_r(x, k) = e^{-ikx\sigma_3} \mathbb{A}_r(k) [I_n + o(1)], \quad x \to +\infty.$$

(74b)

Because the matrix $-ik\sigma_3 + Q$ in (3) has trace $-ik(m_1 - m_2)$, the determinants of $\mathbb{F}_l(x, k)$ and $\mathbb{F}_r(x, k)$ equal $e^{-ik(m_1 - m_2)x}$. As a result, $\det \mathbb{A}_{r,l}(k) = 1$.

Let us relate the AKNS and matrix Schrödinger Jost solutions. Putting

$$e_1 = I_{m_1} \oplus 0_{m_2 \times m_2}, \quad e_2 = 0_{m_1 \times m_1} \oplus I_{m_2},$$

(75)
we obtain from the principal asymptotic relations
\[ F_l(x, k) = F_l(x, -k) e_1 + F_l(x, k) e_2, \]
\[ F_r(x, k) = F_r(x, k) e_1 + F_r(x, -k) e_2, \]
as well as
\[ F_l(x, k) = F_l(x, -k) e_1 + F_l(x, k) e_2, \]
\[ F_r(x, k) = F_r(x, k) e_1 + F_r(x, -k) e_2. \]

Using \( e_1 e_2 = e_2 e_1 = 0_{n\times n} \) and \( e_1^2 + e_2^2 = I_n \), it is easy to derive (77) from (76) and vice versa. Using (76) we obtain from (39) the triangular representations for the AKNS system
\[ F_l(x, k) = e^{-ikx\sigma_3} + \int_x^\infty dy K(x, y) e^{-iky\sigma_3}, \]
\[ F_r(x, k) = e^{-ikx\sigma_3} + \int_{-\infty}^x dy J(x, y) e^{-iky\sigma_3}, \]
where the kernel functions \( K \) and \( J \) coincide with those for the matrix Schrödinger system.

Let us now relate the AKNS and matrix Schrödinger scattering coefficients. Let us introduce the standard partitioning
\[ H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \]
of an \( n \times n \) matrix \( H \) into the \( m_1 \times m_1 \) matrix \( H_1 \), the \( m_1 \times m_2 \) matrix \( H_2 \), the \( m_2 \times m_1 \) matrix \( H_3 \), and the \( m_2 \times m_2 \) matrix \( H_4 \). Using the asymptotic relations (18) and (74) we obtain for \( k \in \mathbb{R} \)
\[ e^{ikx} A_l(k) + e^{-ikx} B_l(k) = \begin{pmatrix} e^{ikx} a_{l1}(-k) & e^{-ikx} a_{l2}(k) \\ e^{-ikx} a_{l3}(-k) & e^{ikx} a_{l4}(k) \end{pmatrix}, \]
\[ e^{-ikx} A_r(k) + e^{ikx} B_r(k) = \begin{pmatrix} e^{-ikx} a_{r1}(k) & e^{ikx} a_{r2}(-k) \\ e^{ikx} a_{r3}(k) & e^{-ikx} a_{r4}(-k) \end{pmatrix}. \]

Consequently, for \( 0 \neq k \in \mathbb{R} \), we have
\[ A_l(k) = \begin{pmatrix} a_{l1}(-k) & 0_{m_1 \times m_2} \\ 0_{m_2 \times m_1} & a_{l4}(k) \end{pmatrix}, \quad A_r(k) = \begin{pmatrix} a_{r1}(k) & 0_{m_1 \times m_2} \\ 0_{m_2 \times m_1} & a_{r4}(-k) \end{pmatrix}, \]
\[ B_l(k) = \begin{pmatrix} 0_{m_1 \times m_1} & a_{l2}(k) \\ a_{l3}(-k) & 0_{m_2 \times m_2} \end{pmatrix}, \quad B_r(k) = \begin{pmatrix} 0_{m_1 \times m_1} & a_{r2}(-k) \\ a_{r3}(k) & 0_{m_2 \times m_2} \end{pmatrix}. \]

Conversely, for \( 0 \neq k \in \mathbb{R} \) we get in terms of blocks of \( A_{r,l}(k) \) and \( B_{r,l}(k) \)
\[ a_{l}(k) = \begin{pmatrix} a_{l1}(k) & B_{l2}(k) \\ B_{l3}(-k) & a_{l4}(k) \end{pmatrix}, \]
\[ a_{r}(k) = \begin{pmatrix} A_{r1}(k) & B_{r2}(-k) \\ B_{r3}(k) & A_{r4}(-k) \end{pmatrix}. \]
Because \( A_l(k) \) and \( A_r(k) \) are continuous in \( k \in \mathbb{R} \) and tend to \( I_n \) as \( k \to \pm \infty \), we see that \( A_l(k) \), \( B_l(k) \), \( A_r(k) \), and \( B_r(k) \) are continuous in \( k \in \mathbb{R} \) and have finite limits as \( k \to \pm \infty \). Of course, the block entries of \( A_{li,r}(k) \) have all the usual continuity and analyticity properties, also at \( k = 0 \). Thus, we are always in the exceptional case. We are in the superexceptional case iff \( k = 0 \) is not a spectral singularity of the AKNS system (3). Further, the matrix Schrödinger reflection coefficients are given by

\[
R_l(k) = \begin{pmatrix}
0_{m_1 \times m_1} & R_{l1}(k) \\
R_{l4}(k) & 0_{m_2 \times m_2}
\end{pmatrix} = \begin{pmatrix}
0_{m_1 \times m_1} & A_{l2}(k)A_{l4}(k)^{-1} \\
A_{l2}(-k)A_{l1}(-k)^{-1} & 0_{m_2 \times m_2}
\end{pmatrix}, \quad (83a)
\]

\[
R_r(k) = \begin{pmatrix}
0_{m_1 \times m_1} & R_{r4}(k) \\
R_{r1}(k) & 0_{m_2 \times m_2}
\end{pmatrix} = \begin{pmatrix}
0_{m_1 \times m_1} & A_{r2}(-k)A_{r4}(-k)^{-1} \\
A_{r2}(k)A_{r1}(k)^{-1} & 0_{m_2 \times m_2}
\end{pmatrix}. \quad (83b)
\]

Finally, the AKNS system is reflectionless iff the matrix Schrödinger system is reflectionless.

Let \( \tau_{r,s} \) and \( \tau_{l,s} \) be the residues of \( A_r(k)^{-1} \) and \( A_l(k)^{-1} \) at the simple pole \( k_s \in \mathbb{C}^+ \). Now let \( \tau_{l4,s} \) and \( \tau_{r1,s} \) be the residues of \( A_{l4}(k)^{-1} \) and \( A_{r1}(k)^{-1} \) at \( k = k_s \) and \( \tau_{l1,s} \) and \( \tau_{r4,s} \) the residues of \( A_{l1}(k)^{-1} \) and \( A_{r4}(k)^{-1} \) at the simple pole \( k = -k_s \). Then, (81a) implies that

\[
\tau_{r,s} = \tau_{r1,s} \oplus (-\tau_{r4,s}), \quad \tau_{l,s} = (-\tau_{l1,s}) \oplus \tau_{l4,s}. \quad (84)
\]

Let us define the AKNS norming constants as follows:

\[
F_r(x, k_s) e_1 \tau_{r1,s} = i F_l(x, k_s) e_2 N_{r1,s}, \quad (85a)
\]

\[
F_r(x, -k_s) e_2 \tau_{r4,s} = -i F_l(x, -k_s) e_1 N_{r4,s}, \quad (85b)
\]

\[
F_l(x, k_s) e_2 \tau_{l4,s} = i F_r(x, k_s) e_1 N_{l4,s}, \quad (85c)
\]

\[
F_l(x, -k_s) e_1 \tau_{l1,s} = -i F_r(x, -k_s) e_2 N_{l1,s}. \quad (85d)
\]

We then obtain

\[
N_{r,s} = \begin{pmatrix}
0_{m_1 \times m_1} & N_{r4,s} \\
N_{r1,s} & 0_{m_2 \times m_2}
\end{pmatrix}, \quad N_{l,s} = \begin{pmatrix}
0_{m_1 \times m_1} & N_{l1,s} \\
N_{l4,s} & 0_{m_2 \times m_2}
\end{pmatrix}, \quad (86)
\]

6 | TIME EVOLUTION

In this section, we establish the time evolution of the scattering data of the matrix Schrödinger equation and the accompanying AKNS system. We then go on to derive, in either case, the Marchenko integral kernels as a function of time. These results allow us, in the next section, to derive the reflectionless solutions of the nonlocal integrable equation (14) and hence of the focusing matrix NLS equation.
6.1 Matrix Schrödinger time evolution

The Lax pair \( \{L, A\} \) for the matrix Schrödinger equation (14) is given by (10), where \( B_0 = Q \). Equation (14) is compatible with the linear system

\[
L v = k^2 v, \quad v_t = A v,
\]  

(87)

where \( L = -\partial_x^2 + Q \). We may therefore write

\[
v_t = A v = 2i \sigma_3 v_{xx} - 2i \sigma_3 Q v_x - i \sigma_3 Q v
\]

\[
= 2i \sigma_3 (Q - k^2 1) v - 2i \sigma_3 Q v_x - i \sigma_3 Q v
\]

\[
= i \sigma_3 \left\{ (Q - 2k^2 1) v - 2Q v_x \right\},
\]  

(88)

where \( 1 \) stands for the identity operator on a suitable function space. Next, we compute

\[
(v_x)_t = (Av)_x = i \sigma_3 \left\{ (Q - 2k^2 1) v_x + Q_x v - 2Q_x v_x - 2Q (Q - k^2 1)v \right\}
\]

\[
= i \sigma_3 (Q_x - 2QQ + 2k^2 Q) v + i \sigma_3 (Q - 2k^2 1 - 2Q_x) v_x.
\]  

(89)

Hence, putting \( V = \begin{pmatrix} v \\ v_x \end{pmatrix} \), we get the linear system

\[
V_x = X(x, k; t) V, \quad V_t = T(x, k; t) V,
\]

(90)

where

\[
X(x, k; t) = \begin{pmatrix} 0_{n \times n} & I_n \\ Q(x; t) - k^2 I_n & 0_{n \times n} \end{pmatrix},
\]

(91a)

\[
T(x, k; t) = \begin{pmatrix} i \sigma_3 (Q - 2k^2 I_n) & -2i \sigma_3 Q \\ i \sigma_3 (Q_x - 2QQ + 2k^2 Q) & i \sigma_3 (Q - 2k^2 I_n - 2Q_x) \end{pmatrix}.
\]

(91b)

Then, we easily compute

\[
i(\sigma_3 \oplus \sigma_3)(X_t - T_x + XT - TX) = \begin{pmatrix} 0_{n \times n} & 0_{n \times n} \\ E_{21} & 0_{n \times n} \end{pmatrix},
\]

(92)

where

\[
E_{21} = i \sigma_3 Q_t + Q_{xx} - 2QQ_x + 2k^2 Q
\]

\[
- \sigma_3 (Q - k^2 I_n) \sigma_3 (Q - k^2 I_n) + (Q - 2k^2 I_n - 2Q_x)(Q - k^2 I_n)
\]

\[
= i \sigma_3 Q_t + Q_{xx} + Q^2 - \sigma_3 Q \sigma_3 Q - 2QQ_x - 4Q_x Q
\]

\[
+ k^2 (2Q_x + 2\sigma_3 Q \sigma_3 + Q - Q - 2Q + 2Q_x)
\]

\[
= i \sigma_3 Q_t + Q_{xx} + Q^2 - \sigma_3 Q \sigma_3 Q - 2QQ_x - 2(Q^2 - \sigma_3 Q \sigma_3 Q)
\]

\[
= i \sigma_3 Q_t + Q_{xx} - Q^2 + \sigma_3 Q \sigma_3 Q - 2QQ_x,
\]  

(93)
as claimed. Thus, the zero curvature condition for the AKNS pair \( \{X, T\} \) is equivalent to the nonlinear evolution equation (14).

Because \( F_l(x, k; t), F_r(x, k; t), \) and \( V(x, k; t) \) all satisfy the same linear homogeneous first-order system in \( x \in \mathbb{R} \), there exist nonsingular matrices \( C_{F_l}(k; t) \) and \( C_{F_r}(k; t) \) not depending on \( k \in \mathbb{R} \) such that
\[
F_l(x, k; t) = V(x, k; t)C_{F_l}(k; t)^{-1}, \quad F_r(x, k; t) = V(x, k; t)C_{F_r}(k; t)^{-1}. \tag{94}
\]

Then, a simple differentiation yields
\[
\left[ C_{F_l}(k; t) \right]_t C_{F_l}(k; t)^{-1} = F_l^{-1} TF_l - F_l^{-1} [F_l]_t, \tag{95}
\]
\[
\left[ C_{F_r}(k; t) \right]_t C_{F_r}(k; t)^{-1} = F_r^{-1} TF_r - F_r^{-1} [F_r]_t, \tag{95}
\]
where the two left-hand sides do not depend on \( x \in \mathbb{R} \). We may therefore compute and equate the \( x \to \pm \infty \) limits of the two right-hand sides and obtain
\[
\left[ C_{F_l}(k; t) \right]_t C_{F_l}(k; t)^{-1} = \left[ C_{F_r}(k; t) \right]_t C_{F_r}(k; t)^{-1} = -2ik^2(\sigma_3 \oplus \sigma_3). \tag{96}
\]

Consequently,
\[
C_{F_l}(k; t) = \left(e^{-2ik^2t}\sigma_3 \oplus e^{-2ik^2t}\sigma_3\right)C_{F_l}(k; 0), \tag{97a}
\]
\[
C_{F_r}(k; t) = \left(e^{-2ik^2t}\sigma_3 \oplus e^{-2ik^2t}\sigma_3\right)C_{F_r}(k; 0). \tag{97b}
\]

Relating \( F_{r,l}(x, k; t) \) by means of the equalities
\[
F_r(x, k; t) = F_l(x, k; t)A_r(k; t), \quad F_l(x, k; t) = F_r(x, k; t)A_l(k; t), \tag{98}
\]
where the factors \( A_{r,l}(k; t) \) are given by the matrices [cf. (21)]
\[
A_r(k; t) = \begin{pmatrix} A_r(k; t) & B_r(-k; t) \\ B_r(k; t) & A_r(-k; t) \end{pmatrix}, \quad A_l(k; t) = \begin{pmatrix} A_l(-k; t) & B_l(k; t) \\ B_l(-k; t) & A_l(k; t) \end{pmatrix}, \tag{99}
\]
we compute
\[
[A_r]_t = -F_l^{-1}[F_l]_t F_l^{-1} F_r + F_l^{-1}[F_r]_t
\]
\[
= -F_l^{-1} \left( TF_l - F_l \left[ C_{F_l}(k; t) \right]_t \right) C_{F_l}(k; t)^{-1} A_r
\]
\[
+ F_l^{-1} \left( TF_r - F_r \left[ C_{F_r}(k; t) \right]_t \right) C_{F_r}(k; t)^{-1}
\]
\[
= \left[ C_{F_l}(k; t) \right]_t C_{F_l}(k; t)^{-1} A_r - A_r \left[ C_{F_l}(k; t) \right]_t C_{F_l}(k; t)^{-1}
\]
\[
= 2ik^2 (A_r(\sigma_3 \oplus \sigma_3) - (\sigma_3 \oplus \sigma_3) A_r). \tag{100}
\]
Using that \( A_{r,l}(k; t) \) are each other's inverses, we get
\[
A_{r,l}(k; t) = \left[e^{-2ik^2t}\sigma_3 \oplus e^{-2ik^2t}\sigma_3\right]A_{r,l}(k; 0) \left[e^{2ik^2t}\sigma_3 \oplus e^{2ik^2t}\sigma_3\right]. \tag{101}
\]
Therefore,
\[
A_{r,l}(k; t) = e^{-2ik^2\sigma_3} A_{r,l}(k; 0) e^{2ik^2\sigma_3},
\] (102a)
\[
B_{r,l}(k; t) = e^{-2ik^2\sigma_3} B_{r,l}(k; 0) e^{2ik^2\sigma_3}.
\] (102b)

For the reflection coefficients, we get
\[
R_{r,l}(k; t) = e^{-2ik^2\sigma_3} R_{r,l}(k; 0) e^{2ik^2\sigma_3}.
\] (103)

Moreover, if \(k_s\) is a simple pole of \(A_{r,l}(k; t)^{-1}\) in \(\mathbb{C}^+\), (102a) implies that
\[
\tau_{r,s}(t) = e^{-2ik^2_s\sigma_3} \tau_{r,s}(0) e^{2ik^2_s\sigma_3},
\]
\[
\tau_{l,s}(t) = e^{-2ik^2_s\sigma_3} \tau_{l,s}(0) e^{2ik^2_s\sigma_3}.
\] (104)

As a result,
\[
[\tau_{r,s}]_t = 2ik^2_s (\tau_{r,s} \sigma_3 - \sigma_3 \tau_{r,s}),
\]
\[
[\tau_{l,s}]_t = 2ik^2_s (\tau_{l,s} \sigma_3 - \sigma_3 \tau_{l,s}).
\] (105)

Let us now study the time evolution of the norming constants corresponding to a simple pole \(k_s\) of \(A_r(k; t)^{-1}\) in \(\mathbb{C}^+\). Equations (97) imply
\[
[F_{r,l}]_t = T F_{r,l} + 2ik^2 F_{r,l}(\sigma_3 \oplus \sigma_3).
\] (106)

Now recall (61) and (63) in the form
\[
F_r(x, k_s; t) \tau_{r,s}(t) = iF_l(x, k_s; t) N_{r,s}(t),
\] (107a)
\[
F_l(x, k_s; t) \tau_{l,s}(t) = iF_r(x, k_s; t) N_{l,s}(t).
\] (107b)

Using the right/left upper block of (106) for \(k = k_s\) to differentiate (107) with respect to \(t\) for \(k = k_s\) and then substituting (105) and (107), we get
\[
F_l(x, k_s; t) \{ [N_{r,s}]_t - 2ik^2_s (N_{r,s} \sigma_3 - \sigma_3 N_{r,s}) \} = 0_{n \times n},
\]
\[
F_r(x, k_s; t) \{ [N_{l,s}]_t - 2ik^2_s (N_{l,s} \sigma_3 - \sigma_3 N_{l,s}) \} = 0_{n \times n}.
\] (108)

Using (16) we obtain
\[
[N_{r,s}]_t = 2ik^2_s (N_{r,s}(t) \sigma_3 - \sigma_3 N_{r,s}(t)),
\]
\[
[N_{l,s}]_t = 2ik^2_s (N_{l,s}(t) \sigma_3 - \sigma_3 N_{l,s}(t)).
\] (109)

Consequently, we have arrived at the time evolution identities
\[
N_{r,s}(t) = e^{-2ik^2_s\sigma_3} N_{r,s}(0) e^{2ik^2_s\sigma_3},
\] (110a)
\[
N_{l,s}(t) = e^{-2ik^2_s\sigma_3} N_{l,s}(0) e^{2ik^2_s\sigma_3}.
\] (110b)
6.2 | AKNS time evolution

Equations (102a) and (81a), applied for \( k \in \mathbb{R} \) and then extended by analytic continuation, imply that \( \mathbb{A}_{l4}(k; t) \) and \( \mathbb{A}_{r1}(k; t) \) for \( k \in \mathbb{C}^+ \cup \mathbb{R} \) and \( \mathbb{A}_{l1}(k; t) \) and \( \mathbb{A}_{r4}(k; t) \) for \( k \in \mathbb{C}^- \cup \mathbb{R} \) are time independent. Equations (102b) and (81b) imply

\[
\mathbb{A}_{l2}(k; t) = e^{-4ik^2t} \mathbb{A}_{l2}(k; 0), \quad \mathbb{A}_{l3}(k; t) = e^{4ik^2t} \mathbb{A}_{l3}(k; 0),
\]

(111a)

\[
\mathbb{A}_{r2}(k; t) = e^{-4ik^2t} \mathbb{A}_{r2}(k; 0), \quad \mathbb{A}_{r3}(k; t) = e^{4ik^2t} \mathbb{A}_{r3}(k; 0),
\]

(111b)

where \( k \in \mathbb{R} \). Equations (103) and (83) imply

\[
\mathbb{R}_{l1}(k; t) = e^{-4ik^2t} \mathbb{R}_{l1}(k; 0), \quad \mathbb{R}_{l4}(k; t) = e^{4ik^2t} \mathbb{R}_{l4}(k; 0),
\]

(112a)

\[
\mathbb{R}_{r1}(k; t) = e^{4ik^2t} \mathbb{R}_{r1}(k; 0), \quad \mathbb{R}_{r4}(k; t) = e^{-4ik^2t} \mathbb{R}_{r4}(k; 0),
\]

(112b)

where \( k \in \mathbb{R} \). Finally, (110) and (86) imply

\[
\mathbb{N}_{l1; s}(t) = e^{-ik^2st} \mathbb{N}_{l1; s}(0), \quad \mathbb{N}_{l4; s}(t) = e^{ik^2st} \mathbb{N}_{l4; s}(0),
\]

(113a)

\[
\mathbb{N}_{r1; s}(t) = e^{ik^2st} \mathbb{N}_{r1; s}(0), \quad \mathbb{N}_{r4; s}(t) = e^{-ik^2st} \mathbb{N}_{r4; s}(0),
\]

(113b)

where \( k_s \) is a simple pole of \( A_{r,l}(k; t)^{-1} \) in \( \mathbb{C}^+ \).

6.3 | Marchenko kernels and time evolution

The Marchenko integral kernels of the matrix Schrödinger system can be expressed in terms of those of the AKNS system as follows:

\[
\Omega_r(w; t) = \begin{pmatrix} 0_{m_1 \times m_1} & \Omega_{r4}(w; t) \\ \Omega_{r1}(w; t) & 0_{m_2 \times m_2} \end{pmatrix}, \quad \Omega_l(w; t) = \begin{pmatrix} 0_{m_1 \times m_1} & \Omega_{l1}(w; t) \\ \Omega_{l4}(w; t) & 0_{m_2 \times m_2} \end{pmatrix},
\]

(114)

where in the case of simple poles

\[
\Omega_{r1}(w; t) = \hat{R}_{r1}(w; t) + \sum_{s=1}^{N} e^{ik^2sw} N_{r1; s}(0),
\]

(115a)

\[
\Omega_{r4}(w; t) = \hat{R}_{r4}(w; t) + \sum_{s=1}^{N} e^{ik^2sw} e^{-4ik^2st} N_{r4; s}(0),
\]

(115b)

\[
\Omega_{l1}(w; t) = \hat{R}_{l1}(w; t) + \sum_{s=1}^{N} e^{ik^2sw} e^{-4ik^2st} N_{l1; s}(0),
\]

(115c)

\[
\Omega_{l4}(w; t) = \hat{R}_{l4}(w; t) + \sum_{s=1}^{N} e^{-ik^2sw} e^{4ik^2st} N_{l4; s}(0).
\]

(115d)

Consequently, for simple poles we obtain
\[ \Omega_r(w; t) = \hat{R}_r(w; t) + \sum_{s=1}^{N} e^{ik_s w} e^{-4i k_s^2 t} \sigma_3 N_{r,s}(0), \]  
\[ \Omega_l(w; t) = \hat{R}_l(w; t) + \sum_{s=1}^{N} e^{-ik_s w} e^{-4i k_s^2 t} \sigma_3 N_{l,s}(0), \]

which we could also have obtained directly from (67) and (114). As a result, we have arrived at the linear partial differential equations (PDEs)

\[ [\Omega_r]_t = 4i \sigma_3 [\Omega_r]_{\alpha\alpha}, \quad [\Omega_l]_t = 4i \sigma_3 [\Omega_l]_{\alpha\alpha}, \]

in accordance with (105).

Let us discuss the adjoint symmetry relations of the Marchenko integral kernels. According to Corollary B.1, those of the matrix Schrödinger equation read

\[ \Omega_r(w; t)^\dagger = \sigma_3 \Omega_r(w; t) \sigma_3, \quad \Omega_l(w; t)^\dagger = \sigma_3 \Omega_l(w; t) \sigma_3. \]

Using (114) this yields the adjoint symmetry relations for the AKNS system

\[ \Omega_{r1}(w; t)^\dagger = -\Omega_{r4}(w; t), \quad \Omega_{r4}(w; t)^\dagger = -\Omega_{r1}(w; t), \]
\[ \Omega_{l1}(w; t)^\dagger = -\Omega_{l4}(w; t), \quad \Omega_{l4}(w; t)^\dagger = -\Omega_{l1}(w; t). \]

### 7 | MULTISOLITON SOLUTIONS

In this section, we apply the matrix triplet method to write the Marchenko integral kernels in separated form and solve them by separation of variables. This method has been successfully applied to the KdV equation,\(^7\) the NLS equation,\(^9\) the sine-Gordon equation,\(^6\) the mKdV equation,\(^3\) and the Toda lattice equation.\(^4\) An introduction to this method can be found in Ref. 50.

In the reflectionless case, we can write the Marchenko integral kernels of the AKNS system as follows:

\[ \Omega_{r1}(w; t) = C_r e^{-w A_r} e^{-4i t A_r^2} B_r, \quad \Omega_{r4}(w; t) = -B_r^\dagger e^{-w A_r^\dagger} e^{4i t A_r^\dagger} C_r^\dagger, \]
\[ \Omega_{l1}(w; t) = C_l e^{w A_l} e^{4i t A_l^2} B_l, \quad \Omega_{l4}(w; t) = -B_l^\dagger e^{w A_l^\dagger} e^{-4i t A_l^\dagger} C_l^\dagger, \]

where we have taken into account the adjoint symmetry relations (119). Here, \((A_r, B_r, C_r)\) and \((A_l, B_l, C_l)\) are matrix triplets, where all of the eigenvalues of \(A_r\) and \(A_l\) have a positive real part. Let us now define

\[ A_r = \begin{pmatrix} A_r^\dagger & 0_{p \times p} \\ 0_{p \times p} & A_r \end{pmatrix}, \quad B_r = \begin{pmatrix} 0_{p \times m_1} & -C_r^\dagger \\ B_r & 0_{p \times m_2} \end{pmatrix}, \quad C_r = \begin{pmatrix} B_r^\dagger & 0_{m_1 \times p} \\ 0_{m_2 \times p} & C_r \end{pmatrix}, \]
\[ A_l = \begin{pmatrix} A_l^\dagger & 0_{p \times p} \\ 0_{p \times p} & A_l \end{pmatrix}, \quad B_l = \begin{pmatrix} 0_{p \times m_1} & B_l \\ -C_l^\dagger & 0_{p \times m_2} \end{pmatrix}, \quad C_l = \begin{pmatrix} C_l & 0_{m_1 \times p} \\ 0_{m_2 \times p} & B_l^\dagger \end{pmatrix}, \]

(121)
where \( \mathcal{A}_r \) and \( \mathcal{A}_l \) are both matrices of order \( p \) whose eigenvalues have a positive real part. Putting 
\[
\delta_3 = \sigma_3 \otimes I_p = I_p \oplus (-I_p),
\]
we obtain
\[
\Omega_r(w; t) = C_r e^{-w \mathcal{A}_r} e^{4i \delta_3 \mathcal{A}_r^2} B_r,
\]
\[
\Omega_l(w; t) = C_l e^{w \mathcal{A}_l} e^{4i \delta_3 \mathcal{A}_l^2} B_l,
\]
where
\[
\delta_3 \mathcal{A}_{r,l} = \mathcal{A}_{r,l} \delta_3, \quad B_{r,l} = -\delta_3 B_{r,l}, \quad \sigma_3 \mathcal{C}_{r,l} = \mathcal{C}_{r,l} \delta_3.
\]

Solving the Marchenko integral equations (66) we get
\[
K(x, y, t) = -W_r(x; t) e^{-y \mathcal{A}_r} e^{4i \delta_3 \mathcal{A}_r^2} B_r,
\]
\[
J(x, y, t) = -W_l(x; t) e^{y \mathcal{A}_l} e^{4i \delta_3 \mathcal{A}_l^2} B_l,
\]
where
\[
W_r(x; t) = C_r e^{-x \mathcal{A}_r} \left[ I_{2p} + e^{-x \mathcal{A}_r} e^{4i \delta_3 \mathcal{A}_r^2} P_r e^{-x \mathcal{A}_r} \right]^{-1},
\]
\[
W_l(x; t) = C_l e^{x \mathcal{A}_l} \left[ I_{2p} + e^{x \mathcal{A}_l} e^{4i \delta_3 \mathcal{A}_l^2} P_l e^{x \mathcal{A}_l} \right]^{-1},
\]
provided the inverse matrices exist. Here,
\[
P_r = \int_0^\infty dz e^{-z \mathcal{A}_r} B_r C_r e^{-z \mathcal{A}_r} = \begin{pmatrix} 0_{p \times p} & Q_{r, \Sigma} \\ N_{r, \Xi} & 0_{p \times p} \end{pmatrix},
\]
\[
P_l = \int_0^\infty dz e^{-z \mathcal{A}_l} B_l C_l e^{-z \mathcal{A}_l} = \begin{pmatrix} 0_{p \times p} & N_{l, \Xi} \\ Q_{l, \Sigma} & 0_{p \times p} \end{pmatrix},
\]
are the unique solutions of the Sylvester equations
\[
\mathcal{A}_r P_r + P_r \mathcal{A}_r = B_r C_r, \quad \mathcal{A}_l P_l + P_l \mathcal{A}_l = B_l C_l.
\]

They are easily seen to anticommute with \( \delta_3 \). More precisely, given \( (x, t) \in \mathbb{R}^2 \), the Marchenko integral equations (66) are uniquely solvable (in an \( L^1 \)-setting) iff the algebraic equations for \( W_{r,l}(x; t) \) are uniquely solvable. Consequently,
\[
K(x, y, t) = -C_r e^{-x A_r} \left[ I_{2p} + e^{-x A_r} e^{4i \sigma_3 A_r^2} P_r e^{-x A_r} \right]^{-1} e^{-y A_r} e^{4i \sigma_3 A_r^2} B_r
\]
\[
= -C_r \left[ I_{2p} + e^{-2x A_r} e^{4i \sigma_3 A_r^2} P_r \right]^{-1} e^{-(x+y) A_r} e^{4i \sigma_3 A_r^2} B_r
\]
\[
= -C_r \Pi_r(x; t)^{-1} e^{-(y-x) A_r} B_r, \quad (129a)
\]

\[
J(x, y, t) = -C_l e^{x A_l} \left[ I_{2p} + e^{x A_l} e^{4i \sigma_3 A_l^2} P_l e^{x A_l} \right]^{-1} e^{y A_l} e^{4i \sigma_3 A_l^2} B_l
\]
\[
= -C_l \left[ I_{2p} + e^{2x A_l} e^{4i \sigma_3 A_l^2} P_l \right]^{-1} e^{x A_l} e^{4i \sigma_3 A_l^2} B_l
\]
\[
= -C_l \Pi_l(x; t)^{-1} e^{-(x-y) A_l} B_l, \quad (129b)
\]

where

\[
\Pi_r(x; t) = e^{2x A_r} e^{-4i \sigma_3 A_r^2} + P_r, \quad \Pi_l(x; t) = e^{-2x A_l} e^{-4i \sigma_3 A_l^2} + P_l. \quad (130)
\]

Observe that, for \((x, t) \in \mathbb{R}^2\), the matrices \(\Pi_r, \Pi_l(x; t)\) are invertible iff the Marchenko integral equations (66) are uniquely solvable. Using (21) [with \(Q = Q^2 + Q_x\) instead of \(Q\)] and (42), we get

\[
-Q(x; t) + \int_x^\infty dy \, Q(y; t)^2 = 2K(x, x; t) = -2C_r \Pi_r(x; t)^{-1} B_r, \quad (131a)
\]
\[
Q(x; t) + \int_{-\infty}^x dy \, Q(y; t)^2 = 2J(x, x; t) = -2C_l \Pi_l(x; t)^{-1} B_l, \quad (131b)
\]

where \(\Pi_r(x; t)\) and \(\Pi_l(x; t)\) are defined by (130). The matrix NLS solution then equals the \(\mp\) block off-diagonal parts of (131), where we recover the explicit (matrix) NLS solutions obtained before by the matrix triplet method.\(^{50,59,60}\) Differentiating (131) with respect to \(x\) and using (5), we obtain the following solutions of the nonlocal integrable equation (12):

\[
Q(x; t) = -\frac{d}{dx} \left( -2C_r \Pi_r(x; t)^{-1} B_r \right)
\]
\[
= -4C_r \Pi_r(x; t)^{-1} A_r e^{2x A_r} e^{-4i \sigma_3 A_r^2} \Pi_r(x; t)^{-1} B_r, \quad (132a)
\]
\[
Q(x; t) = +\frac{d}{dx} \left( -2C_l \Pi_l(x; t)^{-1} B_l \right)
\]
\[
= -4C_l \Pi_l(x; t)^{-1} A_l e^{-2x A_l} e^{-4i \sigma_3 A_l^2} \Pi_l(x; t)^{-1} B_l. \quad (132b)
\]

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Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

ORCID
Francesco Demontis https://orcid.org/0000-0001-5479-342X

REFERENCES
1. Zakharov VE, Shabat AB. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Sov Phys JETP. 1972;34:62-69.
2. Ablowitz MJ, Segur H. Solitons and the Inverse Scattering Transform. SIAM; 1981.
3. Ablowitz MJ. Nonlinear Dispersive Waves. Asymptotic Analysis and Solitons, Cambridge Texts in Applied Mathematics 47. Cambridge University Press; 2011.
4. Hasegawa A, Tappert F. Trans:1 mission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Appl Phys Lett. 1973;23(3):142-144; II. Normal dispersion. Appl Phys Lett. 1973;23(4):171-172.
5. Hasegawa A. Optical Solitons in Fibers, Springer Series in Photonics 9. Springer; 2002.
6. Shaw JK. Mathematical Principles of Optical Fiber Communications: CBMS-NSF Regional Conference Series in Applied Mathematics 76, SIAM, Philadelphia, 2004.
7. Pethick CJ, Smith H. Bose-Einstein Condensation in Dilute Gases. 2nd ed.. Cambridge University Press; 2008.
8. Pitaevskii LP, Stringari S. Bose-Einstein Condensation and Superconductivity. Oxford University Press; 2016.
9. Kevrekidis PG, Frantzeskakis DJ, Carretero-González R. Emergent Non-linear Phenomena in Bose-Einstein Condensates. Springer; 2008.
10. Zakharov VE. Hamilton formalism for hydrodynamic plasma models. Sov Phys JETP. 1971;33:927-932.
11. Zakharov VE, Popkov AF. Contribution to the nonlinear theory of magnetostatic spin waves. Sov Phys JETP. 1983;57:350-355.
12. Chen M, Tsankov MA, Nash JM, Patton CE. Backward-volume-water microwave-envelope solitons in yttrium iron garnet films. Phys Rev B. 1994;49:12773-12790.
13. Ablowitz MJ, Kaup DJ, Newell AC, Segur H. The inverse scattering transform—Fourier analysis for nonlinear problems. Stud Appl Math. 1974;53:249-315.
14. Calogero F, Degasperis A. Spectral Transform and Solitons, I, Studies in Mathematics and Its Applications 13. Elsevier; 1982.
15. Faddeev LD, Takhtajan LA. Hamiltonian Methods in the Theory of Solitons, Classics in Mathematics. Springer; 1987.
16. Ablowitz MJ, Clarkson PA. Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press; 2003.
17. Ablowitz MJ, Clarkson PA. Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press; 1991.
18. Berkolaiko G. An elementary introduction to quantum graphs. In: Geometric and Computational Spectral Theory, Contemporary Mathematics 700. American Math Society; 2017:41-72.
19. Berkolaiko G, Carlson R, Fulling SA, Kuchment P. (eds.) Quantum Graphs and Their Applications, Contemporary Mathematics. Vol 415. American Math Society; 2006.
20. Berkolaiko G, Kuchment P. Introduction to Quantum Graphs, Mathematical Surveys and Monographs 186. Amer Math Society; 2013.
21. Berkolaiko G, Liu W. Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph. J Math Anal Appl. 2017;445:803-818.
22. Boman J, Kurasov P. Symmetries of quantum graphs and the inverse scattering problem. Adv Appl Math. 2005;35:58-70.
23. Exner P, Keating JP, Kuchment P, Sunada T, Teplyaev A, eds. Analysis on graphs and its applications. Proceedings of Symposia in Pure Mathematics 77. American Math Society; 2008.
24. Gerasimenko NI. The inverse scattering problem on a noncompact graph. Theor Math Phys. 1988;75:460-470.
25. Gerasimenko I, Pavlov BS. A scattering problem on a noncompact graphs. *Theor Math Phys.* 1988;74:230-240.
26. Gutkin B, Smilansky U. Can one hear the shape of a graph? *J Phys A: Math Gen.* 2001;34:6061-6068.
27. Harmer MS. Inverse scattering for the matrix Schrödinger operator and Schrödinger operator on graphs with general self-adjoint boundary conditions. *ANZIAM J.* 2002;44:161-168.
28. Harmer MS. *The Matrix Schrödinger Operator and Schrödinger Operator on Graphs.* Ph.D. thesis. University of Auckland; 2004.
29. Harmer MS. Inverse scattering on matrices with boundary conditions. *J Phys A: Math Gen.* 2005;38:4875-4885.
30. Kostrykin V, Schrader R. Kirchhoff’s rule for quantum wires. *J Phys A: Math Gen.* 1999;32:595-630.
31. Kostrykin V, Schrader R. Kirchhoff’s rule for quantum wires. II. The inverse problem with possible applications to quantum computers. *Fortsch Phys.* 2000;48:703-716.
32. Kuchment P. Quantum graphs. I. Some basic structures. *Waves Random Media.* 2004;14:S107-S128.
33. Kuchment P. Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. *J Phys A: Math Gen.* 2005;38:4877-4900.
34. Kurasov P, Nowaczyk M. Inverse spectral problem for quantum graphs. *J Phys A: Math Gen.* 2005;38:4901-4915.
35. Kurasov P, Nowaczyk M. Geometric properties of quantum graphs and vertex scattering matrices. *Opusc Math.* 2010;30:295-309.
36. Kurasov P, Stenberg F. On the inverse scattering problem on branching graphs. *J Phys A: Math Gen.* 2002;35:101-121.
37. Faddeev LD. Properties of the $S$-matrix of the one-dimensional Schrödinger equation. *Amer Math Soc Transl Series.* 1964;2(65):139-166.
38. Deift P, Trubowitz E. Inverse scattering on the line. *Commun Pure Appl Math.* 1979;32:121-251.
39. Chada K, Sabatier P. *Inverse Problems in Quantum Scattering Theory.* 2nd ed. Springer; 1989.
40. Agranovich ZS, Marchenko VA. *The Inverse Problem of Scattering Theory.* Gordon and Breach; 1963.
41. Aktosun T, Werner R. Inverse scattering on the half line for the matrix Schrödinger equation. *J Math Anal Appl.* 2018;457:237-269.
42. Aktosun T, Werner R. *Direct and inverse scattering for the matrix Schrödinger equation.* Appl Math Sci. Vol 203. Springer; 2020.
43. Wadati M, Kamijo T. On the extension of inverse scattering method. *Progr Theor Phys.* 1974;52:397-414.
44. Martinez AL, Olmedilla E. Trace identities in the inverse scattering transform method associated with matrix Schrödinger operators. *J Math Phys.* 1982;23(11):2116-2121.
45. Klaus M. Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line. *Inv Prob.* 1988;4:505-512.
46. Aktosun T, Klaus M, van der Mee C. Small-energy asymptotics of the scattering matrix for the self-adjoint matrix Schrödinger operator on the line. *J Math Phys.* 2001;42:4627-4652.
47. Demontis F, van der Mee C. A matrix Schrödinger approach to focusing nonlinear Schrödinger equations with nonvanishing boundary conditions. *J Nonlin Sci.* 2022;32(4):57.
48. Demontis F, Prinari B, van der Mee C, Vitale F. The inverse scattering transform for focusing nonlinear Schrödinger equation with asymmetric boundary conditions. *J Math Phys.* 2014;55:101505, 40 pp.
49. Ortiz AK, Prinari B. Inverse scattering transform and solitons for square matrix nonlinear nonlinear Schrödinger equations with mixed sign reductions and nonzero boundary conditions. *J Nonlin Math Phys.* 2020;27(1):130-161.
50. van der Mee C. Nonlinear evolution models of integrable type. In: *SIMAI e-Lecture Notes* 11, Vol 13 SIMAI, Torino; 2013.
51. Lax P. Integrals of nonlinear equations of evolution and solitary waves. *Commun Pure Appl Math.* 1968;21:467-490.
52. Horn RA, Johnson CJ. *Topics in Matrix Analysis.* Cambridge University Press; 1994.
53. Klaus M, van der Mee C. Wave operators for the matrix Zakharov-Shabat system. *J Math Phys.* 2010;51:053503, pp. 26.
54. Bilman D, Miller PD. A robust inverse scattering transform for the focusing nonlinear Schrödinger equation. *Comm Pure Appl Math.* 2019;72(8):1722-1805.
55. Gelfand IM, Raikov DA, Shilov GE. *Commutative Normed Rings.* Chelsea Publications; 1964.
56. Demontis F, van der Mee C. Marchenko equations and norming constants of the matrix Zakharov-Shabat system. *Oper Matrices.* 2008;2:79-113.
APPENDIX A: DERIVING THE NONLOCAL INTEGRABLE EQUATION

In this appendix, we present an alternative proof of (14). Indeed, let us start from the matrix NLS equation

\[ i\sigma_3 Q_t + Q_{xx} - 2Q^3 = 0_{n\times n}, \]  

(A.1)

where we seek solutions \( Q \) that anticommute with \( \sigma_3 \). Putting

\[ Q = Q^2 + Q_x, \]  

(A.2)

so that

\[ Q_x = Q_x Q + QQ_x + Q_{xx}, \]  

(A.3)

\[ Q_{xx} = Q_{xx} Q + 2Q^2 + QQ_{xx} + Q_{xxx}, \]  

(A.4)

we obtain

\[-i\sigma_3 Q_t = -(i\sigma_3 Q_t)Q + Q(i\sigma_3 Q_t) - (i\sigma_3 Q_t)_x \]

\[ = (Q_{xx} - 2Q^3)Q - Q(Q_{xx} - 2Q^3) + Q_{xxx} - 2(Q^3)_x \]

\[ = Q_{xx} Q - QQ_{xx} + Q_{xxx} - 2Q_x Q^2 - 2QQ_x Q - 2Q^2 Q_x \]

\[ = (Q_x - Q_x Q - QQ_x)Q - Q(Q_x - Q_x Q - QQ_x) \]

\[ + (Q_{xx} - Q_{xx} Q - 2Q^2_x - QQ_{xx}) - 2Q_x Q^2 - 2QQ_x Q - 2Q^2 Q_x \]

\[ = Q_x Q - QQ_x + Q_{xx} \]
\[-3Q_xQ^2 - 2QQ_xQ - Q^2Q_x - Q_{xx}Q - 2Q_x^2 - QQ_{xx}\]

\[= Q_xQ - QQ_x + Q_{xx}Q - QQ_x - 2Q_xQ^2 - 2Q_x^2\]

\[= Q_{xx} - 2QQ_x - 2Q_xQ^2 - 2Q_x^2\]

\[= Q_{xx} - 2QQ_x - 2Q_xQ.\]  \hspace{1cm} (A.5)

where \(Q_x = \frac{1}{2}(Q - \sigma_3Q\sigma_3)\) and hence

\[Q(x; t) = - \int_x^{\infty} dy \frac{1}{2}(Q - \sigma_3Q\sigma_3) = \int_{-\infty}^x dy \frac{1}{2}(Q - \sigma_3Q\sigma_3).\]  \hspace{1cm} (A.6)

Here, the integrand depends on \((y, t) \in \mathbb{R}^2\). We have thus reproduced (14) by a more direct method.

**APPENDIX B: ADJOINT SYMMETRY OF MARCHENKO KERNELS**

In this appendix, we prove the adjoint symmetry relations (70) for the Marchenko integral kernels and (71) for the norming constants. Even though these relations are easily established for scalar Schrödinger equations \(^{14,37–39}\) or for matrix Schrödinger equations without discrete eigenvalues (where they are immediate from (35)), they require considerably more effort in the general matrix Schrödinger case.

Let \(G_l(x, k)\) and \(G_r(x, k)\) be the \(n \times n\) matrix solutions of the dual matrix Schrödinger equation

\[-\phi''(x, k) + \phi(x, k)Q(x) = k^2\phi(x, k)\]  \hspace{1cm} (B.1)

under the asymptotic conditions

\[G_l(x, k) = e^{ikx}[I_n + o(1)], \quad x \to +\infty,\]  \hspace{1cm} (B.2a)

\[G_r(x, k) = e^{-ikx}[I_n + o(1)], \quad x \to -\infty.\]  \hspace{1cm} (B.2b)

Then, the dual Faddeev functions satisfy the Volterra integral equations

\[e^{-ikx}G_l(x, k) = I_n + \int_x^{\infty} dy \frac{e^{2ik(y-x)} - 1}{2ik} e^{-iky}G_l(y, k)Q(y),\]  \hspace{1cm} (B.3a)

\[e^{ikx}G_r(x, k) = I_n + \int_{-\infty}^x dy \frac{e^{2ik(x-y)} - 1}{2ik} e^{iky}G_r(y, k)Q(y).\]  \hspace{1cm} (B.3b)

In analogy with Proposition 1, we prove the following:

**Proposition B.1.** The dual Faddeev functions \(e^{-ikx}G_l(x, k)\) and \(e^{ikx}G_r(x, k)\) are continuous in \(k \in \mathbb{C}^+ \cup \mathbb{R}\), are analytic in \(k \in \mathbb{C}^+\), and tend to \(I_n\) as \(k \to \infty\) from within \(\mathbb{C}^+ \cup \mathbb{R}\), irrespective of the choice of \(x \in \mathbb{R}\).

Using the symmetry relation (2) in the matrix Schrödinger equation (6) with \(k\) replaced by \(-k^*\), we see that \(\sigma_3\Psi(x, -k^*)^\dagger \sigma_3\) satisfies the dual matrix Schrödinger equation (B.1) whenever \(\Psi(x, k)\) satisfies the original matrix Schrödinger equation (6). Equations (16) and (B.2) then imply
the following adjoint symmetry relations:

\[ G_{r,l}(x, k) = \sigma_3 F_{r,l}(x, -k^*)^\dagger \sigma_3. \]  

(B.4)

These relations can also be derived directly from the Volterra integral equations (17) and (B.3). With the help of (18), (31), and (B.4), we obtain for \( 0 \neq k \in \mathbb{R} \) the asymptotic relations

\[ G_l(x, k) = e^{ikx} A_r(k) - e^{-ikx} B_r(-k) + o(1), \quad x \to -\infty, \]  

(B.5)

\[ G_r(x, k) = e^{-ikx} A_l(k) - e^{ikx} B_l(-k) + o(1), \quad x \to +\infty. \]  

(B.6)

Putting

\[ G_l(x, k) = \begin{pmatrix} G'_r(x, k) & -G_r(x, k) \\ -G'_r(x, k) & G_r(x, k) \end{pmatrix} = (\sigma_2 \otimes \sigma_3) F_r(x, -k)^\dagger (\sigma_2 \otimes \sigma_3), \]  

(B.7)

\[ G_r(x, k) = \begin{pmatrix} G'_l(x, k) & -G_l(x, k) \\ -G'_l(x, k) & G_l(x, k) \end{pmatrix} = (\sigma_2 \otimes \sigma_3) F_l(x, -k)^\dagger (\sigma_2 \otimes \sigma_3), \]  

(B.8)

where \( \sigma_2 \otimes \sigma_3 = \left( \begin{smallmatrix} 0_{n \times n} & -i\sigma_3 \\ i\sigma_3 & 0_{n \times n} \end{smallmatrix} \right) \), we get for \( 0 \neq k \in \mathbb{R} \)

\[ G_l(x, k) = \begin{pmatrix} A_l(-k) & B_l(k) \\ B_l(-k) & A_l(k) \end{pmatrix} G_r(x, k), \]  

(B.9a)

\[ G_r(x, k) = \begin{pmatrix} A_l(k) & B_l(-k) \\ B_l(k) & A_l(-k) \end{pmatrix} G_l(x, k), \]  

(B.9b)

where the matrices containing the \( A \) and \( B \) quantities are each other’s inverses. By analytic continuation, we obtain for \( k \in \mathbb{C}^+ \cup \mathbb{R} \)

\[ G_{r,l}(x, k) = (\sigma_2 \otimes \sigma_3) F_{r,l}(x, -k^*)^\dagger (\sigma_2 \otimes \sigma_3). \]  

(B.10)

Using (B.9) and (33) we easily derive the Riemann–Hilbert problem

\[ \begin{pmatrix} G_r(x, -k) \\ G_l(x, -k) \end{pmatrix} = \begin{pmatrix} A_r(k)^{-1} & -R_l(k) \\ -R_r(k) & A_l(k)^{-1} \end{pmatrix} \begin{pmatrix} G_l(x, k) \\ G_r(x, k) \end{pmatrix}, \]  

(B.11)

where we are either in the generic case or in the superexceptional case and there are no spectral singularities.

Assume that the poles \( k_s \) of \( A_r(k)^{-1} \) are simple and hence (56) is true. Rewriting the bottom half of (B.11) we get
\[ e^{ikx}G_l(x, -k) = A_{l0}(k)^{-1}e^{ikx}G_r(x, k) - e^{2ikx}R_r(k)e^{-ikx}G_l(x, k) \]

\[ = A_{l0}(k)e^{ikx}G_r(x, k) + \sum_{s=1}^{N} \tau_{l,s} \frac{e^{ikx}G_r(x, k) - e^{ikx}G_r(x, k_s)}{k - k_s} \]

\[ + i \sum_{s=1}^{N} \frac{e^{2ikx}N_{l,s}}{k - k_s}e^{-ikx}G_l(x, k) - e^{2ikx}R_r(k)e^{-ikx}G_l(x, k), \quad \text{(B.12)} \]

where

\[ \tau_{l,s}G_r(x, k_s) = iN_{r,l}G_l(x, k_s) \quad \text{(B.13)} \]

for the “dual” norming constants \( N_{r,s} \). By the same token,

\[ e^{-ikx}G_r(x, -k) = A_r(k)^{-1}e^{-ikx}G_l(x, k) - e^{-2ikx}R_l(k)e^{ikx}G_r(x, k) \]

\[ = A_{r0}(k)e^{-ikx}G_l(x, k) + \sum_{s=1}^{N} \tau_{r,s} \frac{e^{-ikx}G_l(x, k) - e^{-ikx}G_l(x, k_s)}{k - k_s} \]

\[ + i \sum_{s=1}^{N} \frac{e^{-2ikx}N_{r,s}}{k - k_s}e^{ikx}G_r(x, k) - e^{-2ikx}R_l(k)e^{ikx}G_r(x, k), \quad \text{(B.14)} \]

where

\[ \tau_{r,s}G_l(x, k_s) = iN_{l,r}G_r(x, k_s) \quad \text{(B.15)} \]

for another set of “dual” norming constants \( N_{l,s} \). For every \( x \in \mathbb{R} \) the identities (B.12) and (B.14) are equations in \( \mathcal{W}^{n \times n} \). Using \( \Pi_- \) to project these two equations onto \( \mathcal{W}^{n \times n}_{-,0} \) along \( \mathcal{W}^{n \times n}_{+,0} \), we obtain

\[ e^{ikx}G_l(x, -k) = I_n + i \sum_{s=1}^{N} \frac{e^{2ikx}N_{l,s}}{k - k_s}e^{-ikx}G_l(x, k) \]

\[ - \Pi_- [e^{2ikx}R_r(k)e^{-ikx}G_l(x, k)], \quad \text{(B.16a)} \]

\[ e^{-ikx}G_r(x, -k) = I_n + i \sum_{s=1}^{N} \frac{e^{-2ikx}N_{l,s}}{k - k_s}e^{ikx}G_r(x, k) \]

\[ - \Pi_- [e^{-2ikx}R_l(k)e^{ikx}G_r(x, k)]. \quad \text{(B.16b)} \]

Using the triangular representations

\[ e^{-ikx}G_l(x, k) = I_n + \int_{-\infty}^{\infty} dy e^{ik(y-x)}\mathcal{J}(y, x), \quad \text{(B.17a)} \]

\[ e^{ikx}G_r(x, k) = I_n + \int_{-\infty}^{\infty} dy e^{ik(x-y)}\mathcal{J}(y, x), \quad \text{(B.17b)} \]

in (B.16) and stripping off the Fourier transforms, we get the dual Marchenko integral equations...
\[ \mathbb{K}(y, x) + \omega_r(y + x) + \int_x^\infty dz \omega_r(y + z)\mathbb{K}(z, x) = 0_{n \times n}, \]  
(B.18a)

\[ \mathbb{J}(y, x) + \omega_l(y + x) + \int_{-\infty}^x dz \omega_l(y + z)\mathbb{J}(z, x) = 0_{n \times n}, \]  
(B.18b)

where the dual Marchenko integral kernels are given by

\[ \omega_r(w) = \hat{R}_r(w) + \sum_{s=1}^N e^{ik_s w} \mathbb{N}_{r,s}, \]  
(B.19)

\[ \omega_l(w) = \hat{R}_l(w) + \sum_{s=1}^N e^{-ik_s w} \mathbb{N}_{l,s}. \]  
(B.20)

**Theorem B.1.** Let the generic case or the superexceptional case be satisfied and assume there do not exist any spectral singularities. Then, the dual Marchenko kernels are related to the Marchenko kernels as follows:

\[ \omega_r(w) = \sigma_3 \Omega_r(w) \dagger \sigma_3, \quad \omega_l(w) = \sigma_3 \Omega_l(w) \dagger \sigma_3. \]  
(B.21)

**Proof.** Taking the adjoint in (66a), we see that the matrix functions \( \sigma_3 \Omega_{r,l}(w) \dagger \sigma_3 \) satisfy the dual Marchenko equations (B.18), where

\[ \mathbb{K}(y, x) = \sigma_3 K(x, y) \dagger \sigma_3, \quad \mathbb{J}(y, x) = \sigma_3 J(x, y) \dagger \sigma_3. \]  
(B.22)

Because the Marchenko equation (B.18a) is uniquely solvable in \( L^1(x, +\infty)^{n \times n} \) for large enough \( x \) and (B.18b) is uniquely solvable in \( L^1(-\infty, x)^{n \times n} \) for large enough \( -x \), we obtain the symmetry relations

\[ \omega_r(w) = \sigma_3 \Omega_r(w) \dagger \sigma_3, \quad \omega_l(w) = \sigma_3 \Omega_l(w) \dagger \sigma_3, \]  
(B.23)

where \( w \geq x_0 > -\infty \) in the first identity and \( w \leq x_0 < +\infty \) in the second identity. Because the Fourier transformed reflection coefficients satisfy

\[ \hat{R}_r(\alpha) \dagger = \sigma_3 \hat{R}_r(\alpha) \sigma_3, \quad \hat{R}_l(\alpha) \dagger = \sigma_3 \hat{R}_l(\alpha) \sigma_3, \]  
(B.24)

we see that \( \Omega_r(w) - \hat{R}_r(w) \) is an entire analytic matrix function of \( w \) that satisfies (B.23) for \( w \geq x_0 > -\infty \). Similarly, \( \Omega_l(w) - \hat{R}_l(w) \) is an entire analytic matrix function that satisfies (B.23) for \( w \leq x_0 < +\infty \). By analytic continuation, (B.21) holds true for every \( w \in \mathbb{R} \). \[ \square \]

From (B.21), we immediately have the symmetry relations for the norming constants

\[ N_{r,s} = \sigma_3 N_{r,s} \dagger \sigma_3, \quad N_{l,s} = \sigma_3 N_{l,s} \dagger \sigma_3, \]  
(B.25)

provided \( k_s \) is a simple pole of \( A_r(k) \dagger \).
Theorem B.2. Let the generic case or the superexceptional case be satisfied and assume there do not exist any spectral singularities. Then, the dual Marchenko kernels satisfy

$$\omega_r(w) = \Omega_r(w), \quad \omega_l(w) = \Omega_l(w). \quad (B.26)$$

Proof. Using (39) and (B.17) we easily verify that for $$0 \ne k \in \mathbb{R}$$

$$F_r(x, k) G_r(x, -k) - F_r(x, -k) G_r(x, k)$$

$$= F_r(x, k) \left[ A_r(k)^{-1} G_l(x, k) - R_l(k) G_r(x, k) \right]$$

$$- \left[ -F_r(x, k) R_l(k) + F_l(x, k) A_l(k)^{-1} \right] G_r(x, k)$$

$$= F_r(x, k) A_r(k)^{-1} G_l(x, k) - F_l(x, k) A_l(k)^{-1} G_r(x, k), \quad (B.27)$$

which can be meromorphically extended to the upper half-plane. In the same way, we get

$$F_l(x, k) G_l(x, -k) - F_l(x, -k) G_l(x, k)$$

$$= F_l(x, k) \left[ -R_r(k) G_l(x, k) + A_l(k)^{-1} G_r(x, k) \right]$$

$$- \left[ F_r(x, k) A_r(k)^{-1} - F_l(x, k) R_r(k) \right] G_l(x, k)$$

$$= F_l(x, k) A_l(k)^{-1} G_r(x, k) - F_r(x, k) A_r(k)^{-1} G_l(x, k), \quad (B.28)$$

which can be meromorphically extended to the upper half-plane. If $$k_s$$ is a simple pole of $$A_r(k)^{-1}$$ and hence of $$A_l(k)^{-1}$$, we get by taking the residues

$$F_r(x, k_s) \tau_{r,s} G_l(x, k_s) - F_l(x, k_s) \tau_{l,s} G_r(x, k_s) \quad (B.29)$$

as well as the negative of this expression. Note that either term in the expression (B.29) is exponentially decaying as $$x \to \pm \infty$$. Using (61), (63), (B.13), and (B.15), we can write the latter expression in the two equivalent forms

$$iF_r(x, k_s) \left[ N_{r,s} - N_{r,s} \right] G_l(x, k_s) = iF_r(x, k_s) \left[ N_{l,s} - N_{l,s} \right] G_r(x, k_s). \quad (B.30)$$

Utilizing the asymptotic behavior of the Jost and dual Jost matrices as $$x \to \pm \infty$$, we obtain

$$e^{2ik_s x} \left[ N_{r,s} - N_{r,s} \right]$$

as $$x \to +\infty$$ and

$$e^{-2ik_s x} \left[ N_{l,s} - N_{l,s} \right]$$

as $$x \to -\infty$$. Therefore,

$$N_{r,s} = N_{r,s}, \quad N_{l,s} = N_{l,s}. \quad (B.31)$$

Consequently, Theorem B.2 is true if the poles of $$A_{r,l}(k)^{-1}$$ are simple. \[\blacksquare\]

Corollary B.1. Let the generic case or the superexceptional case be satisfied and assume there do not exist any spectral singularities. Then, the Marchenko kernels satisfy

$$\Omega_r(w) = \sigma_3 \Omega_r(w)^\dagger \sigma_3, \quad \Omega_l(w) = \sigma_3 \Omega_l(w)^\dagger \sigma_3. \quad (B.32)$$

In other words, the Marchenko kernels are $$\sigma_3$$-hermitian.
APPENDIX C: DIRECT SUBSTITUTION

In this appendix, we derive the multisoliton solution \( Q(x; t) \) of the nonlocal nonlinear evolution equation (14) and the multisoliton solution \( \varphi(x; t) \) of the matrix NLS equation (7) in the reflectionless case by substituting the derivatives of the expressions

\[
\int_{x}^{\infty} d y Q(y; t) = 2K(x, x; t) = -2C_r \Pi_r(x; t)^{-1} B_r, \tag{C.1a}
\]

\[
\int_{-\infty}^{x} d y Q(y; t) = 2J(x, x; t) = -2C_l \Pi_l(x; t)^{-1} B_l, \tag{C.1b}
\]

with respect to \( x \in \mathbb{R} \) directly into (14), where \( \Pi_{r,l}(x; t) \) are defined by (130).

Recall that \( \sigma_3 = I_{m_1} \oplus (-I_{m_2}) \) and \( \tilde{\sigma}_3 = I_p \oplus (-I_p) \), where \( p \) is the order of the matrices \( A_{r,l} \) in (C.1) and (123) is satisfied. Then, (123) implies the intertwining relations

\[
\tilde{\sigma}_3 P_{r,l} = -P_{r,l} \tilde{\sigma}_3, \tag{C.2a}
\]

\[
\tilde{\sigma}_3 \Pi_{r,l}(x; t) = \Pi_{r,l}(x; t) \tilde{\sigma}_3, \tag{C.2b}
\]

where

\[
\Pi_r(x; t) = e^{2x A_r} e^{-4i \tilde{\sigma}_3 A_r^2} - P_r, \quad \Pi_l(x; t) = e^{-2x A_l} e^{-4i \tilde{\sigma}_3 A_l^2} - P_l. \tag{C.3}
\]

Theorem C.1. Let \((A_r, B_r, C_r)\) be a matrix triplet satisfying (123) for which the Sylvester equation (128) has a unique solution \( P_r \). Then, the matrix function \( Q(x; t) \) defined by

\[
Q(x; t) = -4C_r \Pi_r(x; t)^{-1} A_r e^{2x A_r} e^{-4i \tilde{\sigma}_3 A_r^2} \Pi_r(x; t)^{-1} B_r \tag{C.4}
\]

satisfies the nonlinear evolution equation (14) in those \((x, t) \in \mathbb{R}^2\) for which the matrix \( \Pi_r(x; t) \) is nonsingular.

Proof. Let us introduce the abbreviations

\[
E = e^{2x A_r} e^{-4i \tilde{\sigma}_3 A_r^2}, \quad \ldots = \Pi(x; t) = E + P_r = e^{2x A_r} e^{-4i \tilde{\sigma}_3 A_r^2} \Pi_r(x; t)^{-1} B_r \tag{C.5}
\]

so that

\[
[\cdot] = \Pi(x; t) = \tilde{\sigma}_3 \ldots \tilde{\sigma}_3 = E - P_r = e^{2x A_r} e^{-4i \tilde{\sigma}_3 A_r^2} - P_r. \tag{C.6}
\]

Then, \( E_x = 2A_r E = 2EA_r \). In terms of these abbreviations we now define the product

\[
\Gamma_{j_1,j_2,...,j_r} = \ldots^{-1} A_{j_1}^{j_1} E \ldots^{-1} A_{j_2}^{j_2} E \ldots^{-1} \ldots^{-1} A_{j_r}^{j_r} E \ldots^{-1} \tag{C.7}
\]

consisting of \( r + 1 \) factors \( \ldots^{-1} \) interrupted by the consecutive factors \( A_{j_s}^{j_s} E \), where \( j_s \) is a nonnegative integer \((s = 1, 2, \ldots, r)\). Then, (132a) implies that

\[
Q(x; t) = -4C_r \ldots^{-1} A_{r} E \ldots^{-1} B_r = -4C_r \Gamma_{1,1} B_r. \tag{C.8}
\]
In the subscript string \( j_1, \ldots, j_r \), we replace a comma by a semicolon if the factor \([\cdot]^{-1}\) takes the place of \([\ldots]^{-1}\). Using (123), we then get from (C.8)

\[
\sigma_3 Q(x; t) \sigma_3 = 4 C_r [\cdot]^{-1} A_r E[\cdot]^{-1} B_r = 4 C_r \Gamma_{1;1} B_r. \tag{C.9}
\]

Using (C.8), we easily derive the identity

\[
\begin{align*}
\left[ \Gamma_{j_1, j_2, \ldots, j_r} \right]_{x} &= -2 \Gamma_{j_1, j_2, \ldots, j_r} - 2 \Gamma_{j_1, j_2, \ldots, j_r} - \cdots - 2 \Gamma_{j_1, j_2, \ldots, j_r} + 2 \Gamma_{j_1+1, j_2, \ldots, j_r} + 2 \Gamma_{j_1, j_2+1, j_3, \ldots, j_r} + \cdots + 2 \Gamma_{j_1, \ldots, j_{r-1}, j_r+1},
\end{align*}
\]

where we have first differentiated the \( r + 1 \) factors \([\ldots]^{-1}\) and then the factors \( A_s^j E \) \((s = 1, 2, \ldots, r)\). A similar differentiation rule holds if some (or all) of the subscripted commas are replaced by semicolons.

Using the above \( x \)-differentiation rule, we get

\[
Q_x = 8 C_r (2 \Gamma_{1,1,1} - \Gamma_{1,2}) B_r. \tag{C.11}
\]

Using the above \( x \)-differentiation rule again, we get

\[
Q_{xx} = 16 C_r (-6 \Gamma_{1,1,1} + 3 \Gamma_{1,2} + 3 \Gamma_{2,1} - \Gamma_{3,3}) B_r. \tag{C.12}
\]

Next, using (C.8) and (C.9), we compute

\[
Q_x = \frac{1}{2} (Q - \sigma_3 Q \sigma_3) = -2 C_r \Gamma_{1,1} B_r - 2 C_r \Gamma_{1;1} B_r
\]

\[
= \frac{\partial}{\partial x} C_r \left[ \left( e^{2x A_r} e^{-4 i t \sigma_3 A_r^2} + P_r \right)^{-1} + \left[ e^{2x A_r} e^{-4 i t \sigma_3 A_r^2} - P_r \right]^{-1} \right] B_r, \tag{C.13}
\]

implying that

\[
Q(x; t) = 2 C_r \left[ e^{2x A_r} e^{-4 i t \sigma_3 A_r^2} \pm P_r \right]^{-1}
\]

\[
\times e^{2x A_r} e^{-4 i t \sigma_3 A_r^2} \left[ e^{2x A_r} e^{-4 i t \sigma_3 A_r^2} \mp P_r \right]^{-1} B_r
\]

\[
= 2 C_r \Gamma_{0;1} B_r = 2 C_r \Gamma_{0;1} B_r, \tag{C.14}
\]

where we have used the identity

\[
[\ldots]^{-1} E[\cdot]^{-1} = [\cdot]^{-1} E[\ldots]^{-1} = \frac{1}{2} \left( [\ldots]^{-1} + [\cdot]^{-1} \right). \tag{C.15}
\]

Using (128), we get

\[
B_r C_r = A_r [\ldots] + [\ldots] A_r - 2 A_r E, \tag{C.16a}
\]

\[
B_r C_r = A_r [\ldots] - [\cdot] A_r. \tag{C.16b}
\]

Equations (C.8) and (C.16a) imply the identity

\[
Q(x; t)^2 = 16 C_r \left( \Gamma_{1,2} + \Gamma_{2,1} - 2 \Gamma_{1,1,1} \right) B_r. \tag{C.17}
\]
Equations (C.16b) and (123) imply the identity

$$\sigma_3 Q \sigma_3 Q = -16C_\Gamma,1;1, B_r C_\Gamma,1, B_r$$

$$= -16C_\Gamma,1;1, (A_\Gamma, \ldots - [\bullet] A_\Gamma) \Gamma,1, B_r$$

$$= -16C_\Gamma, [\bullet]^{-1} A_\Gamma E[\bullet]^{-1} (A_\Gamma, \ldots - [\bullet] A_\Gamma)[\ldots]^{-1} A_\Gamma E[\ldots]^{-1} B_r$$

$$= -16C_\Gamma, [\bullet]^{-1} (A_\Gamma E[\bullet]^{-1} A_\Gamma^2 E[\ldots]^{-1} A_\Gamma E)[\ldots]^{-1} B_r$$

$$= -16C_\Gamma, \Gamma,1;2, B_r + 16C_\Gamma, \Gamma,2,1, B_r. \quad (C.18)$$

Equations (C.11), (C.15), and (C.16b) imply the identity

$$Q Q_x = 16C_\Gamma, [\cdot]^{-1} E[\cdot]^{-1} B_r C_\Gamma (2\Gamma,1,1, - \Gamma,2,1, B_r)$$

$$= 32C_\Gamma, [\cdot]^{-1} E[\cdot]^{-1} (A_\Gamma, \ldots - [\bullet] A_\Gamma)[\ldots]^{-1} A_\Gamma E[\ldots]^{-1} B_r$$

$$- 16C_\Gamma, [\cdot]^{-1} E[\cdot]^{-1} (A_\Gamma, \ldots - [\bullet] A_\Gamma)[\ldots]^{-1} A_\Gamma^2 E[\ldots]^{-1} B_r$$

$$= 32C_\Gamma, \Gamma,0;2,1, B_r - 32C_\Gamma, \Gamma,1,1,1, B_r - 16C_\Gamma, \Gamma,0;3, B_r + 16C_\Gamma, \Gamma,1,2, B_r. \quad (C.19)$$

Finally, using $\bar{\sigma}_3[\cdot] \bar{\sigma}_3 = [\bullet]$ as well as

$$E_t = -4i \bar{\sigma}_3 A_\Gamma^2 E,$$

$$([\cdot]^{-1})_t = -[\cdot]^{-1} (-4i \bar{\sigma}_3 A_\Gamma^2 E)[\cdot]^{-1} = 4i \bar{\sigma}_3 [\cdot]^{-1} A_\Gamma^2 E[\cdot]^{-1} = 4i \bar{\sigma}_3 \Gamma,2;$$

$$([\cdot]^{-1})_t = -[\cdot]^{-1} (-4i \bar{\sigma}_3 A_\Gamma^2 E)[\cdot]^{-1} = 4i \bar{\sigma}_3 [\cdot]^{-1} A_\Gamma^2 E[\cdot]^{-1} = 4i \bar{\sigma}_3 \Gamma,2; \quad (C.20)$$

we compute with the help of (123) and (C.8)

$$i \sigma_3 Q_t = -4i \sigma_3 C_\Gamma, (\Gamma,1,1,1, B_r) = -4i C_\Gamma, \bar{\sigma}_3 (\Gamma,1,1,1, B_r)$$

$$= -4i C_\Gamma, \bar{\sigma}_3 (\Gamma,1,1,1, B_r)$$

$$= 16C_\Gamma, \bar{\sigma}_3 (\sigma_3 \Gamma,2; A_\Gamma E[\ldots]^{-1} - [\cdot]^{-1} \sigma_3 A_\Gamma^2 E[\ldots]^{-1})$$

$$+ [\cdot]^{-1} A_\Gamma E[\cdot]^{-1} \bar{\sigma}_3 \Gamma,2; B_r$$

$$= 16C_\Gamma, \Gamma,2,1, B_r - 16C_\Gamma, \Gamma,3, B_r + 16C_\Gamma, \Gamma,1,2, B_r. \quad (C.21)$$

To prove (14), we now employ (C.22), (C.12), (C.18), (C.19), and (C.20) to compute

$$i \sigma_3 Q_t + Q_{xx} - Q^2 + \sigma_3 Q \sigma_3 Q - 2Q Q_x$$

$$= 16C_\Gamma, \Gamma,2,1, B_r - 16C_\Gamma, \Gamma,3, B_r + 16C_\Gamma, \Gamma,1,2, B_r$$

$$- 96C_\Gamma, \Gamma,1,1,1, B_r + 48C_\Gamma, \Gamma,1,2, B_r + 48C_\Gamma, \Gamma,2,1, B_r - 16C_\Gamma, \Gamma,3, B_r$$

$$- 16C_\Gamma, \Gamma,1,2, B_r - 16C_\Gamma, \Gamma,2,1, B_r + 32C_\Gamma, \Gamma,1,1,1, B_r$$

$$- 16C_\Gamma, \Gamma,1,2, B_r + 16C_\Gamma, \Gamma,2,1, B_r$$

$$- 64C_\Gamma, \Gamma,0,2,1, B_r + 64C_\Gamma, \Gamma,1,1,1, B_r + 32C_\Gamma, \Gamma,0,3, B_r - 32C_\Gamma, \Gamma,1,2, B_r.$$
\[
= 32C_r\Gamma;2,1,B_r - 16C_r\Gamma;3,B_r + 32C_r\Gamma;2,1,B_r - 16C_r\Gamma;3,B_r
- 64C_r\Gamma;0,2,1,B_r + 32C_r\Gamma;0,3,B_r.
\]

We now regroup the terms in the last member of (C.23) as follows:

\[
i\sigma_3Q_t + Q_{xx} - Q^2 + \sigma_3Q\sigma_3Q - 2QQ_x
= 32C_r\Gamma;2,1,B_r + 32C_r\Gamma;2,1,B_r - 64C_r\Gamma;0,2,1,B_r
- 16C_r\Gamma;3,B_r - 16C_r\Gamma;3,B_r + 32C_r\Gamma;0,3,B_r.
\]

Using (C.16) to derive the identities

\[
\Gamma;2,1, + \Gamma,2,1, - 2\Gamma;0,2,1, = 0_{2p\times2p}, \quad \Gamma;3, + \Gamma,3, - 2\Gamma;0,3, = 0_{2p\times2p},
\]

it appears that the right-hand side of (C.24) equals the zero matrix. Thus, the expression for \(Q(x; t)\) in the statement of Theorem C.1 satisfies (14), as claimed.

By applying Theorem C.1 to the matrix triplet \((-A_l, B_l, -C_l)\), we prove the following:

**Theorem C.2.** Let \((A_l, B_l, C_l)\) be a matrix triplet satisfying (123) for which the Sylvester equation (128) has a unique solution \(P_l\). Then, the matrix function \(Q(x; t)\) defined by

\[
Q(x; t) = -4C_l\Pi_l(x; t)^{-1}A_l e^{-2xA_l} e^{-4it\partial_2A_l^2} \Pi_l(x; t)^{-1}B_l
\]

satisfies the nonlinear evolution equation (14) in those \((x, t) \in \mathbb{R}^2\) for which the matrix \(\Pi_l(x; t)\) is nonsingular.

Let us now partition the matrices \(A_{r,l}, B_{r,l},\) and \(C_{r,l}\) as in Section 7 and prove that the off-diagonal blocks of \(Q(x; t)\) are the \(x\)-derivatives of the matrix NLS solutions \(\Theta(x; t)\). Writing

\[
\int_x^\infty dy \, Q(y; t) = \begin{pmatrix}
\int_x^\infty dy \, Q_{11}(y; t) & -Q^{-}(x; t) \\
-Q^{+}(x; t) & \int_x^\infty dy \, Q_{22}(y; t)
\end{pmatrix},
\]

we employ (C.1a) to arrive at the identities

\[
\int_x^\infty dy \, Q_{11}(y; t) = 2B_r^\dagger \left[I_p - e^{-2xA_l^\dagger} e^{4itA_l} Q_{r,\Sigma} e^{-2xA_l} e^{-4itA_l^2} N_{r,\Xi}ight]^{-1}
\]

\[
\times e^{-2xA_l^\dagger} e^{4itA_l} Q_{r,\Sigma} e^{-2xA_l} e^{-4itA_l^2} B_r,
\]

\[
Q^{-}(x; t) = -2B_r^\dagger \left[e^{2xA_r^\dagger} e^{-4itA_r^\dagger} - Q_{r,\Sigma} e^{-2xA_r} e^{-4itA_r^2} N_{r,\Xi}\right]^{-1} C_r^\dagger,
\]

\[
Q^{+}(x; t) = 2C_r \left[e^{2xA_r} e^{4itA_r^\dagger} - N_{r,\Sigma} e^{2xA_r^\dagger} e^{4itA_r} Q_{r,\Xi}\right]^{-1} B_r,
\]
∫_{x}^{\infty} dy \, Q_{22}(y; t) = -2C_r \left[ I_p - e^{-2x A_r} e^{-4it\delta^3 A_r^2} N_r,\Xi e^{-2x A_r} e^{4it\delta^3 A_r^2} Q_{r,\Sigma} \right]^{-1} \\
\times e^{-2x A_r} e^{-4it\delta^3 A_r^2} N_r,\Xi e^{-2x A_r} e^{4it\delta^3 A_r^2} C_r^\dagger, \tag{C.27d}

where $Q_{r,\Sigma}$ and $N_r,\Xi$ are the off-diagonal blocks of $P_r$, [cf. (127a)].

**Theorem C.3.** Let $Q^-(x; t)$ and $Q^-(x; t)$ be given by (C.27b) and (C.27c). Then, the $n \times n$ matrix function

$$Q(x, t) = \begin{pmatrix} Q^{-}(x; t) \\ Q^{--}(x; t) \end{pmatrix} \tag{C.28}$$

satisfies the matrix NLS equation

$$i Q_t + Q_{xx} - 2Q^3 = 0_{n \times n}. \tag{C.29}$$

**Proof.** Let us write $Q(x; t)$ in the form

$$Q(x; t) = -4C_r e^{-x A_r} \left[ I_{2p} + e^{-x A_r} e^{4it\delta^3 A_r^2} P_r e^{-x A_r} \right]^{-1} \\
\times A_r \left[ I_{2p} + e^{-x A_r} e^{4it\delta^3 A_r^2} P_r e^{-x A_r} \right]^{-1} e^{-x A_r} e^{4it\delta^3 A_r^2} B_r. \tag{C.30}$$

Now put

$$\rightarrow = e^{-x A_r^\dagger} e^{4it\delta^3 A_r^2} Q_{r,\Sigma} e^{-x A_r}, \quad \leftarrow = e^{-x A_r} e^{-4it\delta^3 A_r^2} N_r,\Xi e^{-x A_r^\dagger}, \tag{C.31}$$

where $e^{-x A_r} e^{4it\delta^3 A_r^2} P_r e^{-x A_r} = (0_{p \times p} \rightarrow 0_{p \times p})$. Using that

$$\begin{pmatrix} I_p & \rightarrow \\ \leftarrow & I_p \end{pmatrix}^{-1} = \begin{pmatrix} \left( I_p - \rightarrow \leftarrow \right)^{-1} & \left( I_p - \leftarrow \rightarrow \right)^{-1} \\ -(I_p - \leftarrow \rightarrow)^{-1} \leftarrow & (I_p - \rightarrow \leftarrow)^{-1} \rightarrow \end{pmatrix},$$

where, according to the Sherman–Morrison–Woodbury formula,\(^65\)

$$\det \begin{pmatrix} I_p & \rightarrow \\ \leftarrow & I_p \end{pmatrix} = \det(I_p - \rightarrow \leftarrow) = \det(I_p - \leftarrow \rightarrow), \tag{C.33}$$

we obtain the identities

$$Q_{11}(x; t) = 4B_r e^{-x A_r^\dagger} (I_p - \rightarrow \leftarrow)^{-1} \left( A_r^\dagger \rightarrow + \leftarrow A_r \right) \\
\times (I_p - \leftarrow \rightarrow)^{-1} e^{-x A_r} e^{4it\delta^3 A_r^2} B_r, \tag{C.34a}$$
\(Q_{22}(x; t) = -4C_r e^{-x A_r} (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} \left( \begin{array}{c} \longrightarrow A_r^\dagger + A_r \longrightarrow \end{array} \right) \times (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} e^{-x A_r^\dagger} e^{4it A_r^2} C_r^\dagger. \quad \text{(C.34b)}\)

Let us now write (C.27b) and (C.27c) in the form
\(Q^-(x; t) = -2B_r^\dagger e^{-x A_r^\dagger} (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} e^{-x A_r^\dagger} e^{4it A_r^2} C_r^\dagger, \quad \text{(C.35a)}\)
\(Q^-(x; t) = 2C_r e^{-x A_r} (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} e^{-x A_r} e^{-4it A_r^2} B_r. \quad \text{(C.35b)}\)

Using the block off-diagonal representations of \(P_r\) in the first Sylvester equation (128) and exploiting the block representations of the matrices \(A_r, B_r, \) and \(C_r,\) we arrive at the Sylvester equations
\[A_r^\dagger \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} + \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} A_r = -e^{-x A_r^\dagger} e^{4it A_r^2} C_r^\dagger C_r e^{-x A_r}, \quad \text{(C.36a)}\]
\[A_r \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} + \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} A_r^\dagger = e^{-x A_r} e^{-4it A_r^2} B_r B_r e^{-x A_r^\dagger}. \quad \text{(C.36b)}\]

Multiplying the expressions (C.38) in either order and using the Sylvester equations (C.39) to simplify the resulting product matrices, we obtain
\[Q^-(x; t)Q^-(x; t) = 4B_r^\dagger e^{-x A_r^\dagger} (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} \left( A_r^\dagger \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} + \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} A_r \right) \times (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} e^{-x A_r^\dagger} e^{4it A_r^2} B_r = Q_{11}(x; t), \quad \text{(C.37)}\]
\[Q^-(x; t)Q^-(x; t) = -4C_r e^{-x A_r} (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} \left( A_r \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} + \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} A_r^\dagger \right) \times (I_p - \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array})^{-1} e^{-x A_r^\dagger} e^{4it A_r^2} C_r^\dagger = Q_{22}(x; t), \quad \text{(C.38)}\]

where we have applied (C.37) at the last equality sign to prove that \(Q(x; t)^2\) is the diagonal component of \(Q(x; t).\)

Finally, let us write \(Q = Q^2 + Q_x\) and substitute it into the nonlocal evolution equation (14). We get
\[i \sigma_3 (Q^2)_t + (Q_x Q - QQ_x)_x + (i \sigma_3 Q_t + Q_{xx} - 2Q^3)_x = 0_{n \times n}. \quad \text{(C.39)}\]

Limiting ourselves to the off-diagonal part, we obtain
\[(i \sigma_3 Q_t + Q_{xx} - 2Q^3)_x = 0_{n \times n}, \quad \text{(C.40)}\]

where the expression between brackets vanishes as \(x \to \pm \infty.\) Consequently, \(Q\) satisfies the matrix NLS equation (C.29).

Using (C.1b) instead of (C.1a), we can compute the x-antiderivative
\[\int_{-\infty}^x dy \, Q(y; t) = \left( \begin{array}{c} \int_{-\infty}^x dy \, Q_{11}(y; t) \\ \int_{-\infty}^x dy \, Q_{22}(y; t) \end{array} \right) \right), \quad \text{(C.41)}\]
where

\[
\int_{-\infty}^{x} dy \, Q_{11}(y; t) = -2C_I \left[ I_p - e^{2x \cdot A_I} e^{4it \cdot A_I^2} N_{I,\Xi} e^{2x \cdot A_I^\dagger} e^{-4it \cdot A_I^\dagger^2} Q_{I,\Sigma} \right]^{-1} \\
\times e^{2x \cdot A_I} e^{4it \cdot A_I^2} N_{I,\Xi} e^{2x \cdot A_I^\dagger} e^{-4it \cdot A_I^\dagger^2} C_I^\dagger, \tag{C.42a}
\]

\[
Q_{\to}(x; t) = -2C_I \left[ e^{-2x \cdot A_I} e^{-4it \cdot A_I^\dagger^2} - N_{I,\Xi} e^{2x \cdot A_I^\dagger} e^{-4it \cdot A_I^\dagger^2} Q_{I,\Sigma} \right]^{-1} B_I, \tag{C.42b}
\]

\[
Q_{\leftarrow}(x; t) = 2B_I^\dagger \left[ e^{-2x \cdot A_I^\dagger} e^{4it \cdot A_I^2} - Q_{I,\Sigma} e^{2x \cdot A_I} e^{4it \cdot A_I^2} N_{I,\Xi} \right]^{-1} C_I^\dagger, \tag{C.42c}
\]

\[
\int_{-\infty}^{x} dy \, Q_{22}(y; t) = 2B_I^\dagger \left[ I_p - e^{2x \cdot A_I^\dagger} e^{-4it \cdot A_I^\dagger^2} Q_{I,\Sigma} e^{2x \cdot A_I} e^{4it \cdot A_I^2} N_{I,\Xi} \right]^{-1} \\
\times e^{2x \cdot A_I^\dagger} e^{-4it \cdot A_I^\dagger^2} Q_{I,\Sigma} e^{2x \cdot A_I} e^{4it \cdot A_I^2} C_I^\dagger. \tag{C.42d}
\]

Then, following the proof of Theorem C.3, we derive the following:

**Theorem C.4.** Let \( Q_{\to}(x; t) \) and \( Q_{\leftarrow}(x; t) \) be given by (C.42b) and (C.42c). Then, the \( n \times n \) matrix function

\[
Q(x, t) = \begin{pmatrix} Q_{\to}(x; t) & 0_{m_1 \times m_2} \\
0_{m_2 \times m_1} & Q_{\leftarrow}(x; t) \end{pmatrix} \tag{C.43}
\]

satisfies the matrix NLS equation (C.29).