Deformations of Gabor Frames

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ABSTRACT

The quantum mechanical harmonic oscillator Hamiltonian $H = (t^2 - \partial^2_t)/2$ generates a one–parameter unitary group $W(\theta) = e^{i\theta H}$ in $L^2(\mathbf{R})$ which rotates the time–frequency plane. In particular, $W(\pi/2)$ is the Fourier transform. When $W(\theta)$ is applied to any frame of Gabor wavelets, the result is another such frame with identical frame bounds. Thus each Gabor frame gives rise to a one–parameter family of frames, which we call a deformation of the original. For example, beginning with the usual tight frame $F$ of Gabor wavelets generated by a compactly supported window $g(t)$ and parameterized by a regular lattice in the time–frequency plane, one obtains a family $\{F_\theta : 0 \leq \theta < 2\pi\}$ of frames generated by the non–compactly supported windows $g^\theta = W(\theta)g$, parameterized by rotated versions of the original lattice. This gives a method for constructing tight frames of Gabor wavelets for which neither the window nor its Fourier transform have compact support. When $\theta = \pi/2$, $F_\theta$ is the well–known Gabor frame generated by a window with compactly supported Fourier transform. The family $\{F_\theta\}$ therefore interpolates these two familiar examples.

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1. Introduction

If $f(t)$ is a complex–valued, differentiable function, let

$$ (Qf)(t) = tf(t), \quad (Pf)(t) = -if'(t). \quad (1) $$

$Q$ and $P$ extend to unbounded, self–adjoint operators on $L^2(\mathbb{R})$ which satisfy the (Heisenberg) commutation relation $[Q, P] \equiv QP - PQ = iI$, where $I$ is the identity operator. Being self–adjoint, $Q$ and $P$ generate one–parameter unitary groups operating on $L^2(\mathbb{R})$,

$$ U(\omega) = e^{i\omega Q}, \quad V(s) = e^{-isP}. \quad (2) $$

$U(\omega)$ and $V(s)$ act by modulation and translation, respectively:

$$ (U(\omega)f)(t) = e^{i\omega t} f(t), \quad (V(s)f)(t) = f(t - s). \quad (3) $$

The local (Lie–algebraic) commutation relation $[Q, P] = iI$ has the global (Lie–group) equivalent

$$ U(\omega) V(s) = e^{i\omega s} V(s) U(\omega). \quad (4) $$

Hence the set of operators $G(\phi, \omega, s) = e^{i\phi} U(\omega) V(s)$ satisfies the relation

$$ G(\phi, \omega, s) G(\phi', \omega', s') = G(\phi + \phi' - \omega' s, \omega + \omega', s + s'), \quad (5) $$

so it forms a group of unitary operators on $L^2(\mathbb{R})$, known as (a representation of) the Weyl–Heisenberg group. The Gabor transform (or windowed Fourier transform) may be viewed entirely in terms of this group. (See Daubechies [1] or Kaiser [2] for general background on the windowed Fourier transform.) Namely, given a window function $g(t)$ in $L^2(\mathbb{R})$ with $\|g\| = 1$, define

$$ g_{\omega, s} \equiv U(\omega) V(s) g. \quad (6) $$
Then (3) shows that \( g_{\omega,s}(t) = e^{i\omega t} g(t - s) \), which gives the translated and modulated windows that form the basis of the continuous Gabor transform and its inverse:

\[
\tilde{f}(\omega, s) \equiv \langle g_{\omega,s}, f \rangle = \int_{-\infty}^{\infty} dt \ e^{-i\omega t} \overline{g}(t - s) f(t), \tag{7}
\]

\[
f(t) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} d\omega ds \ g_{\omega,s}(t) \tilde{f}(\omega, s).
\]

Under certain conditions, a discrete subset of such “Gabor wavelets” \( g_{\omega,s} \) is sufficient to reconstruct \( f(t) \) from \( \tilde{f}(\omega, s) \). For example, suppose that \( g \) has compact support in \([0, \tau]\) and choose an interval \( 0 < T \leq \tau \). Suppose that the periodic function

\[
H(t) \equiv \tau \sum_{n \in \mathbb{Z}} |g(t - nT)|^2 \tag{8}
\]

is bounded above and below by positive constants, i.e.,

\[
0 < A \leq H(t) \leq B. \tag{9}
\]

Then it can be shown (cf. Daubechies et al. [3], Kaiser [4]) that

\[
f(t) = H(t)^{-1} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} g_{\omega_m,s_n}(t) \tilde{f}(\omega_m, s_n), \tag{10}
\]

where \( \omega_m = 2\pi m/\tau \) and \( s_n = nT \). In fact, (9) is equivalent to

\[
A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\langle g_{\omega_m,s_n}, f \rangle|^2 \leq B\|f\|^2, \tag{11}
\]

which is just the condition that the set of vectors \( \{g_{\omega_m,s_n} : m, n \in \mathbb{Z}\} \) form a frame in \( L^2(\mathbb{R}) \) with frame bounds \( A, B \). This frame can be made tight if we replace \( g \) by the new window

\[
h(t) \equiv H(t)^{-1/2} g(t). \tag{12}
\]
Since \( h \) is also compactly supported, and \( \tau \sum_{n \in \mathbb{Z}} |h(t-nT)|^2 \equiv 1 \), the above shows that \( \{h_{\omega_m, s_n}\} \) form a tight frame with bounds \( A' = B' = 1 \), i.e., a resolution of unity. Note, however, that \( \|h\|^2 = T/\tau \leq 1 \), so this frame is an orthonormal basis if only if \( T = \tau \). But if \( T = \tau \), then \( g \) (and therefore also \( h \)) must be discontinuous in order to satisfy (9); this means that \( \hat{g}(\omega) \) and \( \hat{h}(\omega) \) have slow decay, giving \( \tilde{f}(\omega, s) \) poor frequency resolution. (This is a special case of the Balian-Low theorem; see Daubechies [1].)

The above construction depends crucially on the fact that \( g \) has compact support, since it expands \( \hat{g}(t-s)f(t) \) in a Fourier series. A similar construction can be made starting with a window which is compactly supported in frequency (expand in a Fourier series in the frequency domain). This leads again to a resolution of unity in terms of Gabor wavelets. For windows that are neither compactly supported in time nor in frequency, both constructions fail. For such windows, it is generally difficult to give explicit constructions of resolutions of unity. The theorem proved in the next section provides a method for generating a one-parameter family of Gabor frames starting from an arbitrary Gabor frame. If the original frame is tight or orthonormal, then so is each member of the family it generates. A general member of this family will have a window which is neither compactly supported in time nor in frequency, even when the original window has compact support.

2. Deformations of Frames

Consider the harmonic oscillator Hamiltonian

\[
H \equiv \frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(t^2 - \partial_t^2),
\]

which is an unbounded and self-adjoint operator on \( L^2(\mathbb{R}) \). Note that

\[
[H, Q] = -iP, \quad [H, P] = iQ.
\]
Let \( W(\theta) \equiv e^{i\theta H} \) be the one-parameter group of unitary operators generated by \( H \), and define the one-parameter families of operators

\[
Q(\theta) \equiv W(\theta) Q W(-\theta), \quad P(\theta) \equiv W(\theta) P W(-\theta).
\]  

Then

\[
\frac{d}{d\theta} Q(\theta) = i [H, Q(\theta)] = i W(\theta) [H, Q] W(-\theta) = P(\theta), \quad \frac{d}{d\theta} P(\theta) = i [H, P(\theta)] = i W(\theta) [H, P] W(-\theta) = -Q(\theta).
\]

These operator equations (with the initial conditions \( Q(0) = Q, P(0) = P \)) can be integrated to give

\[
Q(\theta) = Q \cos \theta + P \sin \theta, \quad P(\theta) = P \cos \theta - Q \sin \theta.
\]

Hence, \( W(\theta) \) “rotates” the operators \( Q \) and \( P \). \( W(\theta) \) acts on the “global” operators \( U(\omega) \) and \( V(s) \) by

\[
W(\theta) U(\omega) = W(\theta) e^{i\omega Q} = e^{i\omega Q(\theta)} W(\theta),
\]

\[
W(\theta) V(s) = W(\theta) e^{-isP} = e^{-isP(\theta)} W(\theta).
\]

From (17) and (18), we can compute the action of \( W(\theta) \) on the Gabor wavelets \( g_{\omega,s} \):

\[
W(\theta) g_{\omega,s} = W(\theta) e^{i\omega Q} e^{-isP} g = e^{i\omega Q(\theta)} e^{-isP(\theta)} W(\theta) g.
\]

To obtain an explicit expression, let \( g^\theta \equiv W(\theta) g \). This is a new window function which generate a new family of Gabor wavelets \( g_{\omega,s}^\theta \equiv U(\omega) V(s) g^\theta \).

**Theorem 1.** \( W(\theta) \) acts on the Gabor wavelets \( g_{\omega,s} \) by replacing the window \( g \) with \( g^\theta \), rotating the labels \( (\omega, s) \), and multiplying by a phase factor \( \gamma \):

\[
W(\theta) g_{\omega,s} = \gamma(\omega, s, \theta) g_{\omega(\theta),s(\theta)}^\theta,
\]
where
\[ \gamma(\omega, s, \theta) = \exp \left[ \frac{i}{4} (\omega^2 - s^2) \sin(2\theta) + i\omega s \sin^2 \theta \right] \] (21)
and
\[ s(\theta) = s \cos \theta + \omega \sin \theta, \quad \omega(\theta) = \omega \cos \theta - s \sin \theta. \] (22)

The new window \( g^\theta \) is given by
\[ g^\theta(u) = (-2\pi i \sin \theta)^{1/2} \int_{-\infty}^{\infty} dt \exp \left[ iut \csc \theta - i(u^2 + t^2) \cot \theta \right] g(t) \] (23)
for \( \theta \neq n\pi \). When \( \theta = \pi/2 \), (23) reduces to \( g^{\pi/2}(u) = e^{i\pi/4} \hat{g}(u) \), where \( \hat{g} \) is the Fourier transform of \( g \). The singular cases are given by \( g^{n\pi}(t) = i^n g((-1)^n t), n \in \mathbb{Z} \).

**Proof:** We need the identity
\[ e^{i(aQ + bP)} = e^{i(ab/2)} e^{iQ} e^{ibP}, \quad a, b \in \mathbb{R}, \] (24)
which is closely related to (4). Setting \( \alpha = \cos \theta \) and \( \beta = \sin \theta \), we have
\[ e^{i\omega Q(\theta)} = e^{i\omega \alpha Q + i\omega \beta P} = \exp \left[ i\omega^2 \alpha \beta / 2 \right] e^{i\omega \alpha Q} e^{i\omega \beta P}, \]
\[ e^{-is P(\theta)} = e^{is \beta Q - is \alpha P} = \exp \left[ -is^2 \alpha \beta / 2 \right] e^{is \beta Q} e^{-is \alpha P}. \] (25)
Substituting this into (19) and using
\[ e^{i\omega \beta P} e^{is \beta Q} = V(-\omega \beta) U(s \beta) = \exp \left[ i\omega s \beta^2 \right] e^{is \beta Q} e^{i\omega \beta P}, \] (26)
which follows from (4), we obtain

\[ W(\theta) g_{\omega, s} = \exp \left[ i(\omega^2 - s^2) \alpha \beta / 2 + i\omega s \beta^2 \right] e^{i(\alpha \omega + \beta s)Q} e^{-i(\alpha s - \omega \beta)P} g^\theta, \] (27)
which is (20). Eq. (23) follows from Mehler’s formula for the kernel of \( W(\theta) \) (cf. Merzbacher [5], p.159, where \( \theta = -t \)). That \( g^{\pi/2} = e^{i\pi/4} \hat{g} \) follows from (23) by taking the limit \( \theta \to \pi/2 \). The expression for \( g^{n\pi} \) then follows from \( W(n\pi) = (W(\pi/2))^{2n} \) and \( \hat{g}(t) = g(-t) \).
Corollary 2. Let \( \mathcal{F} = \{ g_{\omega,s} : (\omega, s) \in \Gamma \} \) be any frame of Gabor wavelets, labeled by a discrete set \( \Gamma \subset \mathbb{R}^2 \), with window \( g \) and frame bounds \( 0 < A \leq B < \infty \). For every real \( \theta \), let \( \Gamma_\theta = \{ (\omega(\theta), s(\theta)) : (\omega, s) \in \Gamma \} \) be the rotated version of \( \Gamma \) and let \( g^\theta = W(\theta)g \). Then the set of vectors \( \mathcal{F}_\theta \equiv \{ g^\theta_{\omega,s} : (\omega, s) \in \Gamma_\theta \} \) forms a Gabor frame with window \( g^\theta \) and identical frame bounds \( A, B \).

Proof: For any \( (\omega, s) \in \mathbb{R}^2 \), Theorem 1 gives

\[
g^\theta_{\omega(\theta), s(\theta)} = \gamma(\omega, s, \theta)^{-1}W(\theta)g_{\omega,s}.
\]

Hence the Gabor transform with respect to the vectors in \( \mathcal{F}_\theta \) is

\[
\langle g^\theta_{\omega(\theta), s(\theta)} , f \rangle = \gamma(\omega, s, \theta) \langle W(\theta)g_{\omega,s} , f \rangle = \gamma(\omega, s, \theta) \langle g_{\omega,s} , W(-\theta) f \rangle = \gamma(\omega, s, \theta) \langle g_{\omega,s} , f^\theta \rangle
\]

by the unitarity of \( W(\theta) \), where \( f^\theta \equiv W(-\theta)f \in L^2(\mathbb{R}) \). Thus

\[
\sum_{(\omega,s)\in\Gamma_\theta} |\langle g^\theta_{\omega,s} , f \rangle|^2 = \sum_{(\omega,s)\in\Gamma} |\langle g^\theta_{\omega(\theta), s(\theta)} , f \rangle|^2 = \sum_{(\omega,s)\in\Gamma} |\langle g_{\omega,s} , f^\theta \rangle|^2.
\]

Since \( \mathcal{F} \) is a frame,

\[
A\|f^\theta\|^2 \leq \sum_{(\omega,s)\in\Gamma} |\langle g_{\omega,s} , f^\theta \rangle|^2 \leq B\|f^\theta\|^2.
\]

By the unitarity of \( W(-\theta) \), \( \|f^\theta\|^2 = \|f\|^2 \). Hence (30) shows that \( \mathcal{F}_\theta \) is a frame as well, with the same frame bounds \( A \) and \( B \).

For example, if \( \mathcal{F} \) is the frame generated by a compactly supported window \( g \) as in (10), then \( \mathcal{F}_\theta \) is generated by the window \( g^\theta \). This gives a one–parameter family of frames \( \mathcal{F}_\theta \), where only \( \mathcal{F}_n\pi \) have compactly supported windows \( (n \in \mathbb{Z}) \). However, the frames \( \mathcal{F}_\theta \) with \( \theta = (n + \frac{1}{2})\pi \) have
windows which are compactly supported in the frequency domain. The family \( \{ \mathcal{F}_\theta \} \) therefore interpolates the two familiar types of easily constructed Gabor frames having windows with compact supports in time and in frequency, respectively. We call the family \( \mathcal{F}_\theta \) a deformation of the frame \( \mathcal{F} \). Note that since \( g^{2\pi}(t) = -g(t) \), the frame \( \mathcal{F}_{2\pi} \) is essentially equivalent to \( \mathcal{F}_0 = \mathcal{F} \), and we only get distinct frames for \( 0 \leq \theta < 2\pi \).

After this note was completed, I learned that work along similar (but inequivalent) lines has been done by R. G. Baraniuk and D. L. Jones [6]. They consider deformations of Gabor frames by operators that shear or chirp the time–frequency plane. I thank Bruno Torresani for pointing this work out to me.

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