We study the bosonic super Liouville system which is a statistical transmutation of super Liouville system. Lax pair for the bosonic super Liouville system is constructed using prolongation method, ensuring the Lax integrability, and the solution to the equations of motion is also considered via Leznov-Saveliev analysis.

**Key words:** Bosonic super Liouville equation, prolongation method, Leznov-Saveliev analysis
1 INTRODUCTION

Liouville and super Liouville equations are found to be important in quite a vast range of physical problems. For examples, Liouville equation is closely connected to string theory, two-dimensional gravity in the conformal gauge and is a very popular example of two-dimensional integrable field theory with conformal invariance, and the same roles in super analogs of the above problems are played by super Liouville equation.

From the point of view of Toda lattice field theory, the Liouville equation is nothing but the simplest Toda field theory with the Toda lattice denoted by a single node—the Dynkin diagram of the Lie algebra $sl(2)$ (and the super Liouville equation, which gauges the basic Lie superalgebra $osp(1|2)$ (Toppan 1991), is the simplest one from the family of super Toda field theories). It is remarkable that for each underlying Lie algebra $\mathcal{G}$ one can construct a Toda field theory. Analogously, for each basic Lie superalgebra one can construct a super Toda field theory. A more interesting fact is that for each Lie algebra $\mathcal{G}$ of rank $r > 1$ there exists a so-called bosonic super Toda theory (Chao (1993), Chao & Hou (1993, 1994), Hou & Chao (1993)), a kind of lattice field theory which can be viewed as the usual Toda field theory coupled to some bosonic “matter” fields, whose equations of motion looks very similar to the super Toda equations written in component form except the following two points: i) the Cartan matrices entering the equations of motion are different for bosonic super and true super Toda theories since the underlying algebras are different; ii) the bosonic super Toda theory contains only bosonic fields and hence does not yield a true supersymmetry. However, despite these differences, the bosonic super Toda theory does yield very nice mathematical properties both as integrable and conformal field theoretic models, in particular, such a model is intimately related to the $W_n^{(2)}$ algebra if the underlying gauge group is chosen to be $SL(n, \mathbb{R})$ (Chao & Hou (1994)). Moreover, it was recently argued in Ferreria et al (1995) and Gervais & Saveliev (1995) that though classically the extended Toda theories such as the bosonic super Liouville theory contains only bosonic fields, their quantum versions might give rise some fermionic degrees of freedom and may have relevant applications in photo-electronic problems. Both due to the mathematical beauty and the potential physical significance, much efforts have been paid to the study of bosonic super Toda theories (Chao (1993, 1994, 1995), Chao & Hou (1993, 1994), Hou & Chao (1993)).

Two puzzling points are worth of further efforts in the above picture. First, since the equations of motion for bosonic super Toda and true super Toda theories are so much alike, one naturally expects to establish some relationship between these two kinds of theories; Second, though the bosonic super Toda theory exists for almost all underlying Lie algebras, the simplest rank one Lie algebra $sl(2)$ is excluded from this picture and therefore no bosonic super Liouville model exists along the above line.

A naive answer to the first problem might be such that the bosonic super Toda theories are just the statistically transmuted super Toda theories, i.e. by replacing all the fermionic

\footnote{Compare the equations of motion for bosonic super Toda theories in Chao (1993), Chao & Hou (1993, 1994), Hou & Chao (1993) and that of supersymmetric Toda theories in, e.g., Au & Spence (1995).}
fields in super Toda theories by bosonic ones one get a bosonic super Toda theory. But this cannot be true as is mentioned, the Cartan matrices entering the equations of motion are quite different for both kinds of theories. However, this naive idea might be a useful clue to construct a bosonic super Liouville model and in this paper we do adopt such a technique to define a bosonic super Liouville model.

We start from the supersymmetric Liouville equation

\[ D_+ D_- \Phi = \exp(\Phi), \]  

where we have chosen \( D_\pm = \frac{1}{3} \partial_x \pm \theta_\mp \frac{\partial}{\partial x_\pm} \) and

\[ \Phi = \phi + 3\sqrt{2} (\theta_+ \psi_- + \theta_- \psi_+) + 6\theta_+ \theta_- F \]

so that, in component form, equation (1) can be rewritten as follows

\[ \partial_+ \partial_- \phi = 18 \psi_+ \psi_- e^{\phi} + 4e^{2\phi}, \]
\[ \partial_+ \psi_- = 3\psi_+ e^{\phi}, \]
\[ \partial_- \psi_+ = 3\psi_- e^{\phi}. \]

We see that these equations have exactly the same form as some extended Liouville equation obtained by one of the author earlier in Chao (1993, 1994) except that the fields \( \psi_\pm \) in (2) are fermionic. We call equation (2) with \( \psi_\pm \) changed into bosonic fields a statistically transmuted super Liouville equation or bosonic super Liouville equation (BSLE), and the present paper is just devoted to study the integrability of that equation. We stress that, in BSLE, no signature change occur in front of the \( \psi_+ \psi_- \) term while the order of \( \psi_+ \) and \( \psi_- \) is reversed.

Before going into detailed studies, let us mention that, eq. (2), viewed as BSLE, represent the usual Liouville system coupled to a pair of external fields \( \psi_\pm \). Moreover these external fields do not possess mass, because the whole system of equations of motion is conformally invariant, i.e. if the coordinate system \( (x_+, x_-) \) undergo the following conformal transformation

\[ x_\pm \to f_\pm(x_\pm), \]

the equations of motion will be left invariant provided the fields \( \phi, \psi_\pm \) transform as

\[ \begin{align*}
\phi &\to +\ln(f_+')^{1/2}(f_-')^{1/2}, \\
\psi_\pm &\to (f_+'')^{1/2}\psi_\pm.
\end{align*} \]
It is interesting to see that the statistical transmutation from the super Liouville equation to BSLE also changes the fields \( \psi^+ \) and \( \psi^- \) from the \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) of Lorentz group to that of the classical conformal group.

2 LAX-PAIR AND SYMMETRY ALGEBRAS FOR BSLE

In this section we shall address the problem of integrability of the BSLE \((\text{2})\). A system of nonlinear partial differential equations is said to be integrable if it is a Hamiltonian system and possesses an infinite number of Poisson commuting integrals of motion. This is the classical Liouville sense of integrability. Another slightly weaker definition of integrability is that if the system can be identified to the compatibility condition of a system of linear auxiliary problems, i.e. the Lax pair. The Lax integrability will be identical to Liouville integrability if the Lax system admits a fundamental Poisson structure and this Poisson structure can be recast into the form of a classical Yang-Baxter formalism. Therefore the first step to consider the integrability of BSLE either in the Liouville sense or in the Lax sense is to find its Lax formalism, and to this end the famous prolongation approach (Walquist & Estabrook (1975), Lu & Li (1989a, 1989b)) is preferred.

To begin with, we introduce a transformation of independent variables

\[
x_+ \to \frac{x_0 + x_1}{2}, \quad x_- \to \frac{x_0 - x_1}{2},
\]

which leads to the changes \( \partial_\pm \to \partial_0 \pm \partial_1 \) of the derivatives.

Setting \( \pi_0 = \partial_0 \phi, \pi_1 = -\partial_1 \phi \), the system \((\text{2})\) can be expressed by the following set of rank two differential forms on the space of variables \((x_0, x_1, \phi, \psi^+, \psi^-, \pi_0, \pi_1)\),

\[
\begin{align*}
\alpha_1 &= d\psi^+ \wedge dx_1 - dx_0 \wedge d\psi^+ - 3\psi^- e^\phi dx_0 \wedge dx_1, \\
\alpha_2 &= d\psi^- \wedge dx_1 + dx_0 \wedge d\psi^- - 3\psi^+ e^\phi dx_0 \wedge dx_1, \\
\alpha_3 &= d\phi \wedge dx_1 - \pi_0 dx_0 \wedge dx_1, \\
\alpha_4 &= d\phi \wedge dx_0 - \pi_1 dx_0 \wedge dx_1, \\
\alpha_5 &= d\pi_0 \wedge dx_1 - d\pi_1 \wedge dx_0 - (18\psi^+ \psi^- e^\phi + 4e^{2\phi})dx_0 \wedge dx_1.
\end{align*}
\]

(3)

On the intersection with the space of independent variables \((x_0, x_1)\) the system \((\text{2})\) will be reproduced. It is easy to check that the system \((\text{3})\) of two forms generate a closed ideal in the sense that

\[
d\alpha_i = \eta_{ij} \alpha_j
\]

for some one forms \( \eta_{ij} \). Given the system \((\text{3})\), we now assume that the (enlarged) prolongation (Lu & Li (1989a, 1989b)) form takes the form
\[ \omega = -dT + F(\phi, \psi_+, \psi_-, \pi_0, \pi_1)Tdx_0 + G(\phi, \psi_+, \psi_-, \pi_0, \pi_1)Tdx_1, \] (4)

where \( F \) and \( G \) are functions of indicated variables taking values in some undetermined Lie algebra, and the newly introduced “pseudopotential” \( T \) lies in the group generated by that Lie algebra.

From the integrability condition

\[ d\omega \in I(\omega, \alpha), \]

where \( I(\omega, \alpha) \) is an ideal generated by the set \( \{ \alpha_i \} \) and \( \{ \omega \} \), we have the following equations for \( F \) and \( G \)

\[
\begin{align*}
F_{\psi_+} - G_{\psi_+} &= 0, \\
F_{\psi_-} + G_{\psi_-} &= 0, \\
F_{\pi_0} - G_{\pi_1} &= 0, \\
F_{\pi_1} + G_{\pi_0} &= 0, \\
[F, G] + \pi_1 F_{\phi} + \pi_0 G_{\phi} + 3e^\phi (\psi_- F_{\psi_+} + \psi_+ G_{\psi_-}) \\
&- (18\psi_+ \psi_- e^\phi + 4e^{2\phi}) F_{\pi_1} = 0. \\
\end{align*}
\] (5)

where \([F, G] = FG - GF\). Solving the system of equations (5), we get

\[
\begin{align*}
F &= -\frac{1}{2}[\pi_1 L_0 + 3 \psi_+ e^{\phi} L_1 - 3 \psi_- e^{\phi} L_0 - e^{\phi} L_2 - e^{2\phi} L_2], \\
G &= \frac{1}{2}[\pi_0 L_0 + 3 \psi_+ e^{\phi} L_1 + 3 \psi_- e^{\phi} L_0 + e^{\phi} L_2 + e^{2\phi} L_2],
\end{align*}
\]

where \( L_i, i = 0, \pm 1, \pm 2 \), are operators satisfying the following commutation relations:

\[
\begin{align*}
[L_0, L_1] &= -L_1, & [L_0, L_{-1}] &= L_{-1}, \\
[L_0, L_2] &= -2L_2, & [L_0, L_{-2}] &= 2L_{-2}, \\
[L_1, L_{-1}] &= 2L_0, & [L_1, L_{-2}] &= 3L_{-1}, \\
[L_{-1}, L_2] &= -3L_1, & [L_{-2}, L_{-2}] &= 4L_0.
\end{align*}
\] (6)

Notice that the system (5) does not yet generate a closed algebra. However, one can easily see that all relations in (4) can be rewritten in a unified form.
\[ [L_n, L_m] = (n - m)L_{n+m}, \]  
\[ \text{for } n, m = 0, \pm 1, \pm 2. \]

Defining new generators iteratively by

\[ L_{m+2} = \frac{1}{m}[L_{m+1}, L_1], \quad L_{-m-2} = \frac{1}{m}[L_{-1}, L_{-m-1}], \quad m \geq 1, \]

then equation (7) will close over the generators \( L_j, j = 0, \pm 1, \pm 2, \ldots \). This is the well-known Witt algebra or "centerless Virasoro algebra".

Now intersecting the prolongation form (4) on the solution manifold \((x_+, x_-)\), we obtain the Lax pair for BSLE (2)

\[ \partial_+ T = (F+G)T, \]
\[ \partial_- T = (F-G)T. \]

The existence of Lax pair (8) ensures that the BSLE (3) is integrable in the Lax sense. However since no Hamiltonian structure is currently known for BSLE, the Liouville integrability cannot be established at this point.

Notice that the Lax pair (8) involves the generator of Witt algebra with degrees ranging from \(-2\) to 2. It is a well known fact that the Witt algebra does not contain any finite dimensional subalgebra of dimension greater than 3. Therefore the Witt algebra is the only possible gauge algebra of the Lax system (8). Moreover, as there is no nondegenerate symmetric bilinear form on Witt algebra, it is hard to obtain a Lagrangian formulation for BSLE as in the conventional Toda case by taking the trace of \( A_+ A_- \) with \( A_\pm \) being the Lax potentials. Actually if the Lagrangian is indeed in the form of a trace over \( A_+ A_- \) in the case of BSLE, then it would lead to the conclusion that BSLE is a topological theory because the Lagrangian is identically zero. Whether this is true or not still deserves further study.

3 SOLUTION OF BSLE

Given the Lax pair (8), we can now consider the possible solutions of the BSLE (3) using the Leznov-Saveliev analysis.

For convenience we choose the following specific gauges for the Lax pair of BSLE,

\[ \partial_+ T_L = (\partial_+ \phi L_0 + 3\psi_+ L_1 + L_2)T_L, \]
\[ \partial_- T_L = -(3\psi_- e^\phi L_{-1} + e^{2\phi} L_{-2})T_L. \]  

\[ (9) \]
\[
\begin{align*}
\partial_+ T_R &= (3\psi_+ e^\phi L_1 + e^{2\phi} L_2) T_R,
\partial_- T_R &= -(\partial_- \phi L_0 + 3\psi_- L_{-1} + L_{-2}) T_R,
\end{align*}
\]  
(10)

where

\[
T_L = e^{\frac{2}{3} L_0} T, \quad T_R = e^{-\frac{1}{2} \phi L_0} T.
\]  
(11)

Now let us choose some highest weight representation of the Witt algebra with highest weight \( h \) and denote the highest weight vector by \( |h\rangle \). The dual of \( |h\rangle \) is denoted \( \langle h| \). The highest weight conditions read

\[
\begin{align*}
L_0 |h\rangle &= h |h\rangle, \\
\langle h| L_0 &= \langle h|h, \\
L_n |h\rangle &= 0, \quad \langle h| L_{-n} &= 0, \quad (n > 0), \\
\langle h|h\rangle &= 1.
\end{align*}
\]  
(12)

From (9), (10) and (12), it follows that

\[
\langle h| \partial_- T_L = 0, \quad \partial_+ T_R^{-1} |h\rangle = 0,
\]  
(13)

and hence the vectors

\[
\xi(x_+) = \langle h| T_L, \quad \bar{\xi}(x_-) = T_R^{-1} |h\rangle,
\]  
(14)

are chiral, namely

\[
\begin{align*}
\partial_- \xi(x_+) &= 0, \\
\partial_+ \bar{\xi}(x_-) &= 0.
\end{align*}
\]  
(15)

Moreover, defining

\[
\begin{align*}
\tilde{T}_L &= e^{\psi_+ L_{-1}} T_L, \\
\tilde{T}_R &= e^{-\psi_- L_1} T_R,
\end{align*}
\]
an easy calculation leads to

\[ \langle h | L_1 \partial_- \vec{T}_L = 0, \quad \partial_+ \vec{T}_R^{-1} L_{-1} | h \rangle = 0, \tag{16} \]

showing that the vectors

\[ \zeta(x_+) = \langle h | L_1 \vec{T}_L, \quad \bar{\zeta}(x_-) = \vec{T}_R^{-1} L_{-1} | h \rangle, \tag{17} \]

also are chiral

\[ \partial_- \zeta(x_+) = 0, \quad \partial_+ \bar{\zeta}(x_-) = 0. \tag{18} \]

From equations (13-18), a straightforward calculation gives

\[ \xi(x_+) \bar{\xi}(x_-) = e^{h \phi}, \]
\[ \zeta(x_+) \bar{\xi}(x_-) = 2h \psi_+ e^{h \phi}, \]
\[ \xi(x_+) \bar{\zeta}(x_-) = 2h \psi_- e^{h \phi}, \]

which in turn, gives a formal solution to BSLE

\[ \phi = \frac{1}{h} \ln(\xi(x_+) \bar{\xi}(x_-)), \tag{19} \]
\[ \psi_+ = \frac{1}{2h} \frac{\zeta(x_+) \bar{\xi}(x_-)}{\xi(x_+) \bar{\xi}(x_-)}, \tag{20} \]
\[ \psi_- = \frac{1}{2h} \frac{\xi(x_+) \bar{\zeta}(x_-)}{\xi(x_+) \bar{\xi}(x_-)}. \tag{21} \]

Some remarks are in due course. First, one could be quite dubious on the correctness of the assumption of the highest weight conditions (12). Indeed, it is known from the study of conformal field theory that no nontrivial \textit{unitary} highest weight representations exists for Virasoro algebra at center \( c = 0 \). However, as we are using the Witt algebra as a gauge algebra of our Lax system, we do not concern the unitarity of the representation and so are free to choose the non-unitary representations in (12). Actually, the choice of non-unitary representations in (12) is not unavoidable if we introduce an extra auxiliary field, say \( \rho \), and modify the Lax system \( [8] \) to the form
\[ \partial_+ T = (\partial_+ \rho c + F + G)T, \]
\[ \partial_- T = (-\partial_- \rho c + F - G)T, \]
and, in the mean time, change the gauge algebra (\[7\]) into the full Virasoro algebra
\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m, 0}. \]

One can show that such modifications do not change the equations of motion for \( \phi, \psi_\pm \) and only give rise to a new equation for the auxiliary field \( \rho, \)
\[ \partial_+ \partial_- \rho + 2\exp(2\phi) = 0. \]

The modified Lax system (\[22\]) can then be treated in exactly the same way as above and one can choose unitary highest weight representations of the Virasoro algebra in place of the non-unitary representations in (\[12\]).

Another remark is as follows. Though the solution of BSLE (\[2\]) can be expressed in the form of (\[21\]), the chiral vectors cannot be regarded as arbitrary because they are defined from the non-chiral objects \( T_L, T_R \) and \( \bar{T}_L, \bar{T}_R \) subjecting nontrivial constraints (the Lax pair). The explicit solution of BSLE therefore cannot be obtained in this way. In conventional Liouville and Toda cases, one can, however, make a similar construction starting not from the specific gauges (\[9\]) and (\[10\]) of the Lax pair but from the set of so-called Drinfeld-Sokolov systems. In the present case such systems would look like
\[ \partial_+ Q = (\partial_+ k(x_+)L_0 + 3p(x_+)L_1 + L_2)Q, \quad \partial_- Q = 0, \]
\[ \partial_- \bar{Q} = \bar{Q}(\partial_- k(x_-)L_0 + 3p(x_-)L_{-1} + L_{-2}), \quad \partial_+ \bar{Q} = 0 \]
with some arbitrary chiral functions \( k(x_\pm) \) and \( p(x_\pm) \). Unfortunately we have been unable to obtain exact solutions to (\[2\]) using the above Drinfeld-Sokolov systems.

4 DISCUSSION

In this paper, we have identified the Lax integrability for BSLE (\[2\]) using the enlarged prolongation approach. The same set of Lax pair will be obtained if we used the original scalar form of prolongation forms as did by Estabrook and Walquist in their classical paper Walquist & Estabrook (1975).

On the other hand we expressed the solution of BSLE in terms of some chiral vectors obtained from the action of solution of the Lax system in some specific gauges which is
in complete analogy to the conventional Toda and bosonic Toda cases. However, as is mentioned in the end of the last section, the Drinfeld-Sokolov construction of solutions for BSLE is not established and this may be one of the subtle points where the BSLE behaves different from the bosonic super Toda theories.

To conclude this paper, we briefly point out some other related open problems:

(1) It is easy to see that, the quantity $\frac{1}{2} \partial_+ \phi \partial_- \phi + 18 \psi_+ \psi_- e^\phi + 2 e^{2\phi}$ is a Lagrangian for the “Liouville part” $\phi$ in BSLE (4). However no Lagrangian expression for $\psi_\pm$ is currently known. A principal reason is that $\psi_\pm$ are chiral fields of first order, and it seems interesting to see whether one can construct a Hamiltonian or Lagrangian formalism for the whole BSLE via the Dirac method.

(2) It was shown in the introduction that the BSLE is conformal invariant and thus admit a $Witt_L \otimes Witt_R$ symmetry algebra. Is there any relationship between the conformal symmetry algebra and the gauged Witt algebra?
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