ALEXANDER POLYNOMIAL OBSTRUCTION OF BI-ORDERABILITY FOR RATIONALLY HOMOLOGICALLY FIBERED KNOT GROUPS

TETSUYA ITO

Abstract. We show that if the fundamental group of the complement of a rationally homologically fibered knot in a rational homology 3-sphere is bi-orderable, then its Alexander polynomial has at least one positive real root. Our argument can be applied for a finitely generated group which is an HNN extension with certain properties.

1. Introduction

A total ordering $\leq_G$ on a group $G$ is a bi-ordering if $a \leq_G b$ implies both $ga \leq_G gb$ and $ag \leq_G bg$ for all $a, b, g \in G$. A group is called bi-orderable if it admits a bi-ordering.

The Alexander polynomial provides a useful criterion for the (non) bi-orderability. In [CR], Clay-Rolfsen proved that if the knot $K$ is fibered (actually their argument can be applied for a finitely generated group whose commutator subgroup is finitely generated), then its Alexander polynomial $\Delta_K(t)$ has at least one positive real root. In [CGW] Chiswell-Glass-Wilson showed the same result under the assumption that the group admits a certain two generators, one relator presentation.

In this note we prove the following (non)-bi-orderability criterion for a rationally homologically fibered knot.

Definition 1. [GS] A knot $K$ in a rational homology 3-sphere $M$ is rationally homologically fibered if $\deg \Delta_K(t) = 2g(K)$, where $g(K)$ denotes the genus of the knot $K$.

Theorem 2. Let $K$ be a rationally homologically fibered knot in a rational homology 3-sphere $M$. If the Alexander polynomial $\Delta_K(t)$ has no positive real root, then the knot group $\pi_1(M \setminus K)$ is not bi-orderable.

Although not all knots are rationally homologically fibered, compared with fibered knots the class of rationally homologically fibered knots are much larger. For example, the alternating knots (in $S^3$) are rationally homologically fibered [Cro, Mur], and all knots with less than or equal to 11 crossings are rationally homologically fibered, except $11_{n34}, 11_{n42}, 11_{n45}, 11_{n67}, 11_{n73}, 11_{n07}, 11_{n152}$ (in the table Knotinfo [CL]).

Example 3. An alternating knot $K = 11a_1$ has the Alexander polynomial $\Delta_K(t) = 2 - 12t + 30t^2 - 39t^3 + 30t^4 - 12t^5 + 2t^6$ which has no positive real root. Thus the fundamental group of its complement is not bi-orderable. ($K$ is not fibered and [CDN] fails to find a presentation that satisfies the assumption of Chiswell-Glass-Wilson’s criterion so they could not detect the non-bi-orderability)

Our argument relies on the rationally homologically fibered condition which in particular forces the Alexander polynomial to be non-trivial. Thus it is interesting to ask whether $\pi_1(M \setminus K)$ bi-orderable or not when $\Delta_K(t) = 1$.

2. Proof of Theorem

Let $X = M \setminus K$ be the knot complement and $G = \pi_1(M \setminus K)$ be the knot group. Let $\pi : \tilde{X} \to X$ be the infinite cyclic covering of $X$ which corresponds to the kernel of the abelianization map $\phi : G \to \mathbb{Z} = \langle t \rangle$.

2010 Mathematics Subject Classification. Primary 57M05, Secondary 20F60, 06F15.

Key words and phrases. bi-orderable group, Alexander polynomial.
Then the homology group of infinite cyclic covering $H_1(\widetilde{X};\mathbb{Q})$ has a structure of $\mathbb{Q}[t, t^{-1}]$ module, where $t$ acts on $\widetilde{X}$ as a deck translation. There exists $p_1(t), \ldots, p_n(t) \in \mathbb{Q}[t, t^{-1}]$ and $f \in \mathbb{Z}_{\geq 0}$ such that

$$H_1(\widetilde{X};\mathbb{Q}) \cong \mathbb{Q}[t, t^{-1}]^f \oplus \bigoplus_{i=1}^{n} \mathbb{Q}[t, t^{-1}]/(p_i(t)).$$

The Alexander polynomial $\Delta_K(t)$ is defined by

$$\Delta_K(t) = \begin{cases} p_1(t)p_2(t) \cdots p_n(t) & (f = 0) \\ 0 & (f > 0). \end{cases}$$

Thus $\Delta_K(t) \cdot h = 0$ for every $h \in H_1(\widetilde{X};\mathbb{Q})$.

Let $\Sigma$ be a minimum genus Seifert surface of $K$, and let $Y = M \setminus \Sigma$, where $N(\Sigma) \cong \Sigma \times (-1, 1)$ denotes a regular neighborhood of $\Sigma$.

Let $\iota^n : \Sigma \hookrightarrow \Sigma \times \{\pm 1\} \subset Y$ denote the inclusion maps. As is well-known, the infinite cyclic covering $\widetilde{X}$ is obtained by gluing infinitely many copies $\{Y_i\}_{i \in \mathbb{Z}}$ of $Y$, where the $i$-th copy $Y_i$ and the $(i+1)$-st copy $Y_{i+1}$ are glued by identifying $\iota^-(\Sigma) \subset Y_i$ and $\iota^+(\Sigma) \subset Y_{i+1}$. In the rest of the argument, we will always take a base point of $\widetilde{X}$ so that it lies in $Y_0$.

For $N \geq 0$, let $Y_{[-N,N]} = \bigcup_{i=-N}^{N} Y_i \subset \widetilde{X}$, and let $i_N : Y_0 \hookrightarrow Y_{[-N,N]}$ and $j_N : Y_{[-N,N]} \hookrightarrow \widetilde{X}$ be the inclusion maps. We denote the fundamental group $\pi_1(Y_{[-N,N]})$ and $\pi_1(\widetilde{X}) = \ker \phi$ by $K_N$ and $K$, respectively. Since $Y_{[-N,N]}$ is compact, $K_N$ is finitely generated.

Since $\iota^+ : \pi_1(\Sigma) \to \pi_1(\widetilde{X})$ are injective, by van-Kampen theorem it follows that both $(i_N)_* : K_0 \to K_N$ and $(j_N)_* : K_N \to K$ are injective. By these inclusion maps we will always regard $K_0$ as a subgroup of $K_N$, and $K_N$ as a subgroup of $K$. For $x \in K_0$, we will often write $(i_N)_*(x) \in K_N$ simply by the same symbol $x$, by abuse of notation.

**Proof of Theorem 2** Assume that $K$ is rationally homologically fibered, and the Alexander polynomial $\Delta_K(t)$ has no positive real root.

A Theorem of Dubickas [Dub] says that a one-variable polynomial $f(t) \in \mathbb{Q}[t, t^{-1}]$ has no positive real root, if and only if there is a non-zero polynomial $g(t) \in \mathbb{Q}[t, t^{-1}]$ such that all the non-zero coefficients of $g(t)f(t)$ are positive. Thus there is a non-zero polynomial $\Delta'(t)$ such that all the non-zero coefficient of $\Delta'(t)\Delta_K(t)$ are positive. We take such $\Delta'(t)$ so that $\Delta'(t)\Delta_K(t) = \sum_{i \geq 0} a_i t^i$ with $a_0 > 0$ and $a_i \geq 0$ ($i > 0$).

For $x \in K = \pi_1(\widetilde{X})$, we denote by $[x] \in H_1(\widetilde{X};\mathbb{Q})$ the homology class represented by $x$. Then $[x] = 0$ if and only if $x^r \in [K,K]$ for some $r > 0$.

Let $s \in \pi_1(X)$ be an element represented by a meridian of the knot $K$. Then $t^i[x] = [s^{-i}xs^i]$. By definition of the Alexander polynomial, for each $x \in K$

$$\Delta'(t)\Delta_K(t)[x] = \sum_{i \geq 0} a_it^i[x] = \sum_{i \geq 0} [s^{-i}xs^i] = \left[\prod_{i \geq 0} (s^{-i}xs^i)^{r(x)}\right] = 0 \in H_1(\widetilde{X};\mathbb{Q}).$$

This implies that there is $r(x) > 0$ such that

$$\left[\prod_{i \geq 0} (s^{-i}xs^i)^{r(x)}\right] \in [K,K].$$

Moreover, since $K = \bigcup_{n \geq 0} K_n$, there is $N(x) \in \mathbb{Z}$ such that

$$\left[\prod_{i \geq 0} (s^{-i}xs^i)^{r(x)}\right] \in [K_{N(x)}K_{N(x)}].$$

Take a finite symmetric generating set $\mathcal{X}$ of $K_0$. Here symmetric we mean that $x \in \mathcal{X}$ implies $x^{-1} \in \mathcal{X}$. Let $N = \max\{N(x) \mid x \in \mathcal{X}\}$, and let $r$ be the least common multiple of $r(x)$ for $x \in \mathcal{X}$.
Then for every $x \in \mathcal{X}$ we have

\[
(2.1) \quad \left( \prod_{i \geq 0} (s^{-i} x^a s^i) \right)^r \in [K_N, K_N].
\]

Now assume to the contrary that, $G$ is bi-orderable. Let $<_{K_N}$ be a bi-ordering on $K_N$ which is the restriction of a bi-ordering of $G$. Since $K_N$ is finitely generated, there is a $<_{K_N}$ convex normal subgroup $C$ of $K_N$ such that the quotient group $A_N := K_N/C$ is a non-trivial, torsion-free abelian group. Then $A_N$ has the bi-ordering $<_{A_N}$ coming from $<_{K_N}$: $a <_{A_N} a'$ if and only if $a = P(k)$, $a' = P(k')$ ($k, k' \in K_N$) with $k <_{K_N} k'$, where $P : K_N \to A_N$ denotes the quotient map (see [CR] [10] for details on abelian, bi-ordered quotients).

**Lemma 1.** Let $q = P \circ (i_N)_* : K_0 \xrightarrow{(i_N)_*} K_N \xrightarrow{P} A_N$. If both $(i^\pm)_* : H_1(\Sigma; \mathbb{Q}) \to H_1(Y; \mathbb{Q})$ are surjections, then $q$ is a surjection.

**Proof.** By Meyer-Vietoris sequence, the surjectivity of $(i^\pm)_*$ shows the surjectivity of $(i_N)_* : H_1(Y_0; \mathbb{Q}) \to H_1(Y_{-N,N}; \mathbb{Q})$. Thus $(i_N)_* : H_1(Y_0; \mathbb{Z}) \to H_1(Y_{-N,N}; \mathbb{Z})$ is a surjection modulo torsion elements.

On the other hand, $A_N$ is an abelian group so the map $q$ is written as compositions

\[
K_0 = \pi_1(Y_0) \to H_1(Y_0; \mathbb{Z}) \xrightarrow{(i_N)_*} H_1(Y_{-N,N}; \mathbb{Z}) = K_N/[K_N, K_N] \to K_N/C = A_N.
\]

All maps are surjection modulo torsion elements and $A_N$ is torsion-free so $q$ is a surjection. □

The following lemma clarifies a role of the rationally homologically fibered assumption (cf. [GS], Proposition 2).

**Lemma 2.** Both $(i^\pm)_* : H_1(\Sigma; \mathbb{Q}) \to H_1(Y; \mathbb{Q})$ are surjection if and only if $K$ is rationally homologically fibered.

**Proof.** Let $g$ be the genus of $K$. By Alexander duality, $\dim H_1(\Sigma; \mathbb{Q}) = \dim H_1(Y; \mathbb{Q}) = 2g$. This shows that $(i^\pm)_*$ are surjection if and only if $(i^\pm)_*$ are isomorphism, that is, they are invertible.

By Meyer-Vietoris sequence, $H_1(\mathcal{X}; \mathbb{Q})$ is written as

\[
H_1(\mathcal{X}; \mathbb{Q}) = \mathbb{Q}[t, t^{-1}]/\{t(i^+)_, (h) = (i^-)_*(h) \ \forall h \in H_1(\Sigma)\}
\]

Thus $\Delta_K(t)$ is equal to the determinant of $t(i^+)_* - (i^-)_* : \mathbb{Q}^{2g} = H_1(\Sigma; \mathbb{Q}) \to H_1(Y; \mathbb{Q}) \cong \mathbb{Q}^{2g}$.

If $(i^\pm)_*$ are surjective, then they are invertible so $\Delta_K(t) = \det(t - (i^+)_*^{-1}(i^-)_*)\det(i^+)$. Since $(i^+)_*^{-1}(i^-)_*$ is invertible, $\deg \Delta_K(t) = 2g$. Conversely, if $\deg \Delta_K(t) = 2g$ then $\Delta_K(0) = \det((i^-)_*) = \det((i^+)_*) \neq 0$ so both $(i^\pm)_*$ are invertible. □

Now we are ready to complete the proof of Theorem.

By Lemma 1 and Lemma 2 if $K$ is rationally homologically fibered, then $q$ is surjective. Since $\mathcal{X}$ is a symmetric generating set, the surjectivity of $q$ implies that there exists $x \in \mathcal{X}$ such that $1 <_{A_N} q(x)$. By definition of the quotient ordering $<_{A_N}$, $1 <_{K_N} x$. The ordering $<_{K_N}$ is the restriction of a bi-ordering of $G$ and $0 < x_i$ so $1 <_{K_N} s^{-1} x^a s^i$. Therefore $1 <_{A_N} P(s^{-1} x^a s^i)$ for all $i \geq 0$. Since $a_0 > 0$, as a consequence we get

\[
1 <_{A_N} q(x) \leq_{A_N} P \left( \prod_{i \geq 0} (s^{-i} x^a s^i) \right)^r.
\]

On the other hand, $[K_N, K_N] \subset C$ so (2.1) implies

\[
P \left( \prod_{i \geq 0} (s^{-i} x^a s^i) \right)^r = 1 \in K_N/C = A_N.
\]

This is a contradiction. □
We state and prove our main theorem for the case that the group is the fundamental group of a knot complement. However, our proof can be applied for finitely generated group represented by a certain HNN extension.

For a finitely generated group $G$ and a surjection $\phi : G \to \mathbb{Z} = \langle t \rangle$, $H_1(\text{Ker } \phi; \mathbb{Q})$ has a structure of finitely generated $\mathbb{Q}[t, t^{-1}]$-module and the Alexander polynomial $\Delta^\phi_G(t)$ (with respect to $\phi$) is defined similarly, and have the same property that $\Delta^\phi_G(t) \cdot h = 0$ for all $h \in H_1(\text{Ker } \phi; \mathbb{Q})$.

In the proof of Theorem 2 besides the assumption that the Alexander polynomial has no positive real roots, what we really needed and used can be stated in terms of the groups $\text{Ker } \phi$, $\pi_1(\Sigma)$ and $\pi_1(Y)$: we used the amalgamated product decomposition

$$\text{(2.2) } \text{Ker } \phi = \pi_1(\tilde{X}) = \cdots *_{\pi_1(\Sigma)} \pi_1(Y) *_{\pi_1(\Sigma)} \pi_1(Y) *_{\pi_1(\Sigma)} \cdots$$

having the properties

$$\pi_1(Y) \text{ is finitely generated.}$$

$$\text{(2.3) } \pi_1(Y) \text{ is finitely generated.}$$

$$\text{(2.4) The inclusion } \iota^\pm : \pi_1(\Sigma) \to \pi_1(Y) \text{ induce surjections } \iota^\pm : H_1(\pi_1(\Sigma); \mathbb{Q}) \to H_1(\pi_1(Y); \mathbb{Q}).$$

Note that we used the topological assumption that $K$ is a rationally homologically fibered knot in a rational homology sphere $M$ only at Lemma 2, which is used to show the property (2.4).

In a language of group theory, the amalgamated product decomposition (2.2) comes from an expression of $\pi_1(M \setminus K)$ as an HNN extension

$$\pi_1(M \setminus K) = *_{\pi_1(\Sigma)} \pi_1(Y) = \langle t, \pi_1(Y) \mid t^{-1}\iota^+(s)t = \iota^-(s) \ (\forall s \in \pi_1(\Sigma)) \rangle.$$ 

In summary, our proof of Theorem 2 actually shows the following non-bi-orderability criterion.

**Theorem 4.** Let $H$ be a finitely generated group and $A$ be a group (not necessarily a finitely generated). Let $\iota^\pm : A \to H$ be homomorphisms such that

$$(\iota^\pm)_* : H_1(A; \mathbb{Q}) \to H_1(H; \mathbb{Q})$$

are surjective. Let $G$ be a finitely generated group given by an HNN extension

$$G = *_{A}H = \langle t, H \mid t^{-1}\iota^+(a)t = \iota^-(a) \ (\forall a \in A) \rangle.$$ 

Let $\phi : G \to \mathbb{Z}$ is a surjection given by $\phi(t) = 1$, $\phi(h) = 0$ for all $h \in H$. If the Alexander polynomial $\Delta^\phi_G(t)$ has no positive real root, then $G$ is not bi-orderable.

**Acknowledgements**

The author thanks for Eiko Kin for valuable comments on earlier draft of the paper. Also, the author thank for Stefan Friedl for pointing out that the proof of Theorem 2 requires the hypothesis that $K$ is rationally homologically fibered. This work was supported by JSPS KAKENHI Grant Number 15K17540.

**References**

[CL] J. C. Cha and C. Livingston, Knotinfo: Table of knot invariants. http://www.indiana.edu/knotinfo

[CGW] I. Chiawell, A. Glass and J. Wilson, Residual nilpotence and ordering in one-relator groups and knot groups, Math. Proc. Camb. Phil. Soc. 158 (2015), 275–288.

[CDN] A. Clay and C. Desmarais and P. Naylor, Testing bi-orderability of knot groups, arXiv:1410.5774

[CR] A. Clay and D. Rolfsen, Ordered groups, eigenvalues, knots, surgery and L-spaces, Math. Proc. Camb. Phil. Soc. 152 (2012), 115–129.

[Cro] R. Crowell, Genus of alternating link types, Ann. of Math. (2) 69 (1959), 258–275.

[Dub] A. Dubickas, On roots of polynomials with positive coefficients, Manuscripta Math. 123(3) (2007) 353–356.

[GS] H. Goda and T. Sakasai, Homology cylinders and sutured manifolds for homologically fibered knots, Tokyo J. Math. 36 (2013) 85–111.

[Ito] T. Ito, A remark on the Alexander polynomial criterion for the bi-orderability of fibered 3-manifold groups, Int. Math. Res. Not. IMRN, (1) (2013) 156–169.

[Mur] K. Murasugi, On the genus of the alternating knot $H$, J. of Math. Soc. Japan. 10,(1958) 235–248.
Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama Toyonaka, Osaka 560-0043, JAPAN

E-mail address: tetito@math.sci.osaka-u.ac.jp

URL: http://www.math.sci.osaka-u.ac.jp/~tetito/