Multiple scattering by cylinders randomly located in a fluid: effective properties

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Abstract. We consider the interface between a homogeneous fluid and that same fluid with a random distribution of \( n_0 \) cylinders per square meter inside. A harmonic plane wave, frequency \( \nu \) and wavenumber \( k \), is incident upon that boundary under incidence angle \( \alpha \). The reflection coefficient obtained with the Fikioris and Waterman approach is expanded into powers of \( n_0/k^2 \) up to order 2, using Linton and Martin’s expansion of the wavenumber of the coherent wave. This coefficient is then compared to that obtained when a homogeneous viscous fluid replaces the random medium. When the two reflection coefficients are equal, the random fluid is acoustically equivalent to the viscous one, which is called in that case the effective fluid. The coherent wave in the random medium is thus described as the acoustic mode in the effective fluid. Equating the two reflection coefficients provides expressions for the effective properties of the random medium: mass density \( \rho_{\text{eff}} \), and coefficient of dilatation viscosity \( \eta_{\text{eff}} \), as the shear viscosity is set to zero. Both depend on \( \alpha \) and \( \nu \), unless low frequencies only are considered, in which case the dependence on \( \alpha \) vanishes.

1. Introduction

It is well known that a harmonic plane wave that propagates through a dilute random collection of identical elastic cylinders in an ideal fluid undergoes multiple scattering and that the average field may be described as a plane heterogeneous wave propagating with a complex wavenumber \( K \). The most recent analytic expression of the effective wavenumber \( K \) was given in 2005 by Linton and Martin [1],

\[
\frac{K^2}{k^2} = 1 - 4i \nu \left( \frac{n_0}{k^2} \right) f(\theta) f(-\theta) d\theta
\]

with \( k = \omega/c \) the real wavenumber in the absence of scatterers, \( n_0 \) the number of cylinders per unit surface, and \( f(\theta) \) the far-field amplitude of the wave scattered by a single cylinder in direction \( \theta \). Letting \( a \) denote the radius of the cylinders, the concentration of scatterers is \( \varphi = n_0 \pi a^2 \).
Linton and Martin’s equation (Eq.(1)) may be recovered [2] from the Fikioris and Waterman’s equation of dispersion [3], under the assumption that \( n_0/k \) and \( kb \), with \( b \) the radius of the hole correction in Ref.[3], are small. As \( b \) represents the minimum distance between centres of scatterers, \( b > 2a \), and thus Eq.(1) is a low concentration \( (n_0/k^2<<1) \) and low frequency \( (ka<<1) \) approximation. Its range of validity, however, may be larger in frequency than expected, as will be shown in the next section, and the question arises whether it is possible to define an effective medium (acoustic equivalent of the random one) with that same range of validity.

The effective medium in the following is supposed to be a viscous fluid with no shear viscosity, so that the coherent wave of wavenumber \( K \) given by Eq.(1) is supposed to be the only propagating mode, i.e. the acoustic mode [4],

\[
\frac{K^2}{k^2} = \left( 1 - \frac{\omega}{\rho_{\text{eff}} c^2 \eta_{\text{eff}}} \right)^{-1}.
\]  

(2)

In Eq.(2), \( \eta_{\text{eff}} \) is the dilatation viscosity, \( \rho_{\text{eff}} \) the effective mass density, and the adiabatic sound speed of the viscous fluid is supposed to be that of the ideal fluid in the absence of scatterers. Both \( \eta_{\text{eff}} \) and \( \rho_{\text{eff}} \) are sought, from the equality of Eq.(1) and Eq.(2), and from the equality of the reflection coefficient \( R \) at the interface between an ideal fluid and a viscous one [5],

\[
R = \frac{-\rho K \cos \alpha + \rho_{\text{eff}} k \cos \alpha}{\rho K \cos \alpha + \rho_{\text{eff}} k \cos \alpha},
\]  

(3)

with that at the interface between the same ideal fluid and the random medium [6],

\[
R = \frac{1 + \sum_{\alpha \neq 0} (-1)^\alpha D_\alpha T_\alpha e^{-i(n_\alpha a)}}{1 + \sum_{\alpha \neq 0} D_\alpha T_\alpha e^{i(n_\alpha a)}} K \cos \alpha - k \cos \alpha K \cos \alpha + k \cos \alpha.
\]  

(4)

In Eq.(3), \( K \) is given by Eq.(2), while in Eq.(4), it is given by Eq.(1). \( \alpha \) is the incidence angle in the ideal fluid, \( \alpha_r \) the refraction angle, \( K \sin \alpha = k \sin \alpha_r \), \( \rho \) the mass density of the ideal fluid, and the \( T_n \) are the scattering coefficients of the cylinders. The \( D_n \) coefficients are given from Eqs.(5,6),

\[
\forall n \neq 0, \quad X_n = D_n T_0 X_0, \quad \sum_{\alpha \neq 0} \left( \delta_{\alpha \alpha_r} - \frac{2\pi b n_\alpha}{(K^2 - k^2)\alpha} A_{\alpha r} T_\alpha \right) X_\alpha = \frac{2\pi b n_\alpha}{(K^2 - k^2)\alpha} A_{\alpha r} T_0 X_0,
\]  

(5)

\[
A_{\alpha r} = K a J_{\alpha r} (Kb)H_{\alpha r}^{(1)}(kb) - ka J_{\alpha r} (Kb)H_{\alpha r}^{(1)}(kb).
\]  

(6)

In Ref.[6], the Fikioris-Waterman’s approach was used to calculate \( R \), under the assumption that the average acoustic field in the random medium was given by the coherent wave at arbitrary distance from the boundary. According to Refs.[2,3], however, the situation is much more complicated in the vicinity of the interface, namely at distances smaller or the order of \( b \), and neglecting these complications should be relevant only at low frequency \( (ka<<1) \), as long as \( b>2a \) is assumed. Eqs.(5,6) will thus be simplified in section 3 by assuming low concentration \( (n_0/k^2<<1) \) and letting \( b \) tend to zero, and the effective parameters will be derived from here in section 4. The next section, as for it, is devoted to the discussion of the validity of Eq.(1) at intermediate frequencies.

All numerical studies are performed for steel cylinders in water. In water, \( c=1480 \text{m/s} \), \( \rho=1000 \text{kg/m}^3 \). Steel has a mass density \( \rho_s=7816 \text{kg/m}^3 \), a longitudinal velocity \( c_l=6000 \text{m/s} \), and a shear velocity \( c_s=3100 \text{m/s} \). Unless otherwise specified, the concentration, with \( a=1 \text{mm} \), is set to 0.1, so that \( n_0 b^2=0.03 \), \( n_0/k^2 =0.98 \) at \( ka=0.18 \) and \( n_0/k^2 =0.10 \) at \( ka=0.56 \).

2. Validity range of Linton-Martin’s expression of the effective wavenumber

As stated in the introduction section, Eq.(1) is the approximation of the solution \( K_{FW} \) of Fikioris and Waterman’s dispersion equation for \( kb<<1 \) and \( n_0/k^2<<1 \). \( K_{FW} \) is found by setting the determinant of the following homogeneous system to zero:
\[
\sum_{m=0}^{\infty} \left( \delta_{mn} - \frac{2\pi b n_0}{(K^2 - \kappa^2)} a A_{nm} T_m \right) X_m = 0.
\]  

(7)

Fig. 1 shows the comparison between \(K\), as given in Eq. (1), and \(K_{FW}\), for different values of \(b\), over the frequency range \(ka \in [0, 5]\). The \(K_{FW}\) values for \(b=0\) are obtained from Eqs. (7, 8) (see section 3).

Figs. 1 Comparison between \(K_{FW}/k\) (black, thin, and colours) and \(K/k\) (black, thick) for different values of \(b\).

At a concentration \(\varphi=0.1\), the difference between \(K_{FW}/k\) and \(K/k\) is very small (at most 10\% on the imaginary part for \(b=3a\), otherwise less than 6\% on the imaginary part and less than 1\% on the real part), so that Eq. (1) still provides a reasonable approximation to \(K_{FW}\) at intermediate frequencies and \(b\) not too far from its minimum value \(2a\). The situation is completely different at concentration \(\varphi=0.2\), for which \(n_0/k^2\) is less than 0.1 for \(ka\) values larger than 0.77. The difference between the imaginary parts of \(K_{FW}\) and \(K\) exceeds 10\% practically on the whole range of frequency even for \(b=2.05a\). This is the reason why the concentration \(\varphi\) in the following is kept constant at \(\varphi=0.1\). For this concentration, and even at intermediate frequency (1<\(ka<5\)) letting \(b\) tend to zero independently of \(a\) thus provides a reasonable analytic approximation of the “actual” effective wavenumber (supposed here to be \(K_{FW}\), which can be found only numerically from Eq. (7)) via Eq. (1), which, in that case, is a low concentration approximation only.

3. The reflection coefficient of the random medium at low concentration

In order to get an analytic expression of \(\rho_{eff}\) and \(\eta_{eff}\), we need a more explicit expression of the reflection coefficient given in Eq. (4). This is the objective of this section. Following Linton and Martin, we look for an expansion of \(R\) in powers of \(n_0/k^2\) at low concentration. Letting \(b\) tend to zero, Eq. (6) turns into

\[
A_{nm} = \frac{2a}{i\pi b} \left( \frac{K}{\kappa} \right)^{b-a}.
\]

(8)

The following integrals are first introduced, as \(J(0)\) appears in Eq. (1) and both \(I(\alpha)\) and \(J(\alpha)\) will be useful later,

\[
I(\alpha) = \sum_{n}^{\infty} \left[ T_n e^{i\alpha} - \frac{1}{2\pi} \int_{0}^{\pi} \cot \frac{\theta}{2} d\theta \left[ f(\alpha + \theta) + f(\alpha - \theta) \right] d\theta \right],
\]

(9a)
\[ J(\alpha) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \cot \frac{\theta}{2} \frac{d}{d\theta} \left[ f(\theta) f(\alpha - \theta) + f(\theta) f(\alpha + \theta) \right] d\theta. \]  \tag{9b}

Now Eqs.(6,8) are written in a matrix form, \( X = T_{o0} M^{-1} S X_0 \), after letting \( X = (\ldots X_m = \ldots)^{T} \) and \( S = (\ldots S_m = \ldots)^{T} \), with

\[ S_m = \frac{1}{f(0)} \left[ 1 + 2i \frac{n_0}{k^2} f(0) \right] J(0) - m[f(0)] \right] M = I - \left[ \frac{1}{f(0)} B + 2i \frac{n_0}{k^2} f(0) \frac{1}{f(0)} B \right] \right] C, \tag{10} \]

\( I \) the identity matrix, and matrices \( B \) and \( C \) defined from \( B_{nm} = T_{nm}, C_{nm} = |n - m| T_{nm} (n, m) \in \mathbb{Z}^2 \). The inverse of matrix \( M \) is expanded as

\[ M^{-1} = I + \sum_{q=1}^{\infty} \left( \frac{B}{f(0)} \right)^q + 2i \frac{n_0}{k^2} f(0) \sum_{q=1}^{\infty} \left( \frac{B}{f(0)} \right)^q - \sum_{q=1}^{\infty} \left( \frac{B}{f(0)} \right)^q C \right] \tag{11} \]

so that finally the approximate \( D_n \) coefficients are obtained as

\[ D_n = \frac{1}{T_0} - 2i \frac{n_0}{k^2} \left( \frac{f(0)}{f(0)} \right) + \frac{1}{f(0)} \sum_{m \in \mathbb{Z}} |n - m| T_{nm} - \frac{1}{f(0)} f(0) + 2T_0 I(0). \tag{12} \]

Use of the Snell-Descartes laws \( K \sin \alpha = k \sin \alpha \) along with Eq.(1) and Eq.(9b) leads to, for any incidence other than grazing,

\[ \frac{K \cos \alpha}{k \cos \alpha} = 1 - 2i \frac{n_0}{k^2} \frac{f(0)}{\cos^2 \alpha} + 2 \frac{n_0^2}{k^2} \left( \frac{f(0)}{\cos^2 \alpha} \right)^2 - 2 \frac{J(0)}{\cos^2 \alpha}, \tag{13} \]

so that

\[ R = i \frac{n_0}{k^2} \frac{f(\pi - 2\alpha)}{\cos^2 \alpha} + 2 \frac{n_0^2}{k^2} \frac{T_0}{\cos^2 \alpha} \left[ \frac{f(0)}{f(0)} - f'(\pi - 2\alpha) \tan \alpha \right] - \frac{f(0) - f(\pi - 2\alpha) - f(\pi - 2\alpha) - f(0)}{f(0)} I(0) + \frac{f(\pi - 2\alpha)}{f(0)} J(\pi - 2\alpha) \tag{14} \]

Fig.2 shows the comparison at normal incidence between \( R \), as given in Eq.(14), and \( R_{FW} \), obtained from Eq.(4), with \( K = K_{FW} \). For \( b = 0 \), Eq.(8) is used instead of Eq.(6).

![Fig. 2](image_url)

**Fig.2 Normal incidence.** Comparison between \( R_{FW} \) (black, thin and colours) and \( R \) (black, thick) for different values of \( b \).

Black, thin : \( b = 0 \) - Blue : \( b = 2.05a \) - Green : \( b = 2.1a \) - Red : \( b = 3.0a \)

Solid lines : Modulus. Dotted lines : phase.
Eq.(14) is clearly quite a good approximation to $R_{FW}$, even for values of $b$ as high as 3, whatever the frequency. Figs.3a,b show the same comparison, versus the incidence angle, at low ($ka=0.5$) and intermediate ($ka=1.5$) frequency. The effect of the value of $b$ on the curve is more visible than at normal incidence. At both frequencies, however, Eq.(14) gives practically the same result as Fikioris and Waterman with $b=0$. Eq.(14), then, is assumed correct in the following for $b$ tending to zero, for low and intermediate frequencies.

Figs.3 Comparison between $R_{FW}$ (black, thin and colours) and $R$ (black, thick) for different values of $b$.

- Black, thin : $b=0$
- Blue : $b=2.05$
- Green : $b=2.1$
- Red : $b=3.0$

Solid lines : Modulus. Dotted lines : phase.

4. The effective mass density and viscosity

Equating the reflection coefficient $R$ of the viscous fluid (Eq.(3)) to that of the random medium given by Eq.(14) provides the expression of the effective mass density as a function of the incidence angle,

$$\rho_{eff} = \frac{\rho}{1 - 2i \frac{n_0}{k^2} \frac{f(0) - f(\pi - 2\alpha)}{\cos^2 \alpha} + 4 \frac{n_0^2}{k^2} \frac{1}{\cos^2 \alpha} + \frac{f'(\pi - 2\alpha) f(0)}{\cos \alpha} + \frac{f(\pi - 2\alpha) - T_0 f(0)}{f(0)} I(0)}$$

The angle dependence of $\rho_{eff}$ is exhibited in Figs.4a-b at low and intermediate frequencies.
At low frequency ($ka=0.5$), the real part of $\rho_{\text{eff}}$ varies only by about 2% from normal to (quasi-) grazing incidence, and the imaginary part varies by about 5%. Neglecting this variation provides the following approximation to $\rho_{\text{eff}}$, obtained from Eq.(15) at normal incidence,

$$\frac{\rho_{\text{eff}}}{\rho} = 1 - 2i \frac{n_0}{k^2} (f(0) - f(\pi)) \left[ \frac{1}{2} (f(0)^2 - f(\pi)^2) + \frac{f(\pi) - f(0)}{f(0)} \right] + T_0 \frac{f(0) - f(\pi)}{f(0)} \left[ T_0 \frac{J(\pi) - T_0}{f(0)} - T_0 \right],$$

which, at first order in $n_0/k^2$, is identical to that given in Ref.[7].

At $ka=1.5$, however, the dependence of $\rho_{\text{eff}}$ on the incidence angle cannot be neglected any more, as the variation of its real part is about 17% from normal to grazing incidence, and its imaginary part at normal incidence is multiplied roughly by 20 at grazing incidence.

Once $K$ and $\rho_{\text{eff}}$ are known, $\eta_{\text{eff}}$ is given by Eq.(2). Using Eq.(1) for $K$ and Eq.(16) for $\rho_{\text{eff}}$ provides

$$\eta_{\text{eff}} = 4 \frac{n_0 \omega \rho}{k^2} \frac{f(0)}{k^2} \left[ -2f(0)^2 + f(0) + \frac{f(0) - f(\pi - 2\alpha)}{\cos^2 \alpha} \right].$$

The dependence of the nondimensional effective viscosity $\frac{k^2 \omega \rho}{\eta_{\text{eff}}}$ on the incidence angle is shown in Figs.5. While rather negligible at $ka=0.5$, it increases again at $ka=1.5$ (20% variation of the real part).

5. Conclusion

At low frequency, the random medium may be considered as a viscous fluid, the parameters of which being complex, frequency and angle dependent. The dependence on the angle, however, is negligible enough to provide some demonstration of the consistency of the approach followed in the previous sections. At intermediate frequencies, the problem is still open, and the validity of our approach is subject to the possibility of neglecting the boundary layer mentioned in section 1 (see also Refs.[8-10]).
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