Uniqueness in MHD in divergence form: right nullvectors and well-posedness

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ABSTRACT

Magnetohydrodynamics in divergence form describes a hyperbolic system of covariant and constraint-free equations. It comprises a linear combination of an algebraic constraint and Faraday’s equations. Here, we study the problem of well-posedness, and identify a preferred linear combination in this divergence formulation. The limit of weak magnetic fields shows the slow magnetosonic and Alfvén waves to bifurcate from the contact discontinuity (entropy waves), while the fast magnetosonic wave is a regular perturbation of the hydrodynamical sound speed. These results are further reported as a starting point for characteristic based shock capturing schemes for simulations with ultra-relativistic shocks in magnetized relativistic fluids.

Subject headings: MHD, divergence form, characteristics

1. Introduction

Highly relativistic astrophysical fluids have been observed as highly energetic outflows, e.g.: jets in active galactic nuclei, including a few optical radio-jets such as 3C273 (Pearson et al. 1981, Thomson et al. 1993, Bhacall et al. 1995), 3C346 (Dey & van Breugel 1994), M87 (Biretta et al. 1995) and PKS 1229-21 (Le Brun et al. 1996), microquasars in our galaxy (Hjellming & Rupen 1995; Mirabel & Rodriguez 1995; Levinson & Blandford 1996), pulsar winds (Kennel & Coroniti 1984), and fireballs in recent models of γ-ray bursts (Rees & Meszaros 1995). These flows are generally time-dependent, or have been produced in a strongly time-variable episode, and hence are relativistically shocked fluid flows. In most cases, shocks are responsible for brightest emission features at the highest energies.
The evolution of strongly magnetized flows can be markedly different from unmagnetized flows. This is already apparent from small amplitude wave-motion in ideal magnetohydrodynamics compared with hydrodynamics, and their distinct shock structures. The nonlinear development of large scale morphology of strongly magnetized jets can result in features such as the formation of a nose cone (Clarke et al. 1986), which is absent in hydrodynamical evolution. Of particular interest is the role of magnetic fields in the large scale, three-dimensional stability of jets and their knotted structures.

Time-dependent simulations may provide the link between the observed emission features and the internal structure such as magnetized field distribution, and boundary conditions at the source. It is hoped that simulations ultimately provide constraints on the flow parameters, perhaps also derived from stability criteria. Higher dimensional simulations of jets are performed by a number of groups in the approximation of relativistic hydrodynamics (van Putten 1993b; Duncan & Hughes 1994; Martí et al. 1995, Gómez et al. 1997) and relativistic magnetohydrodynamics (van Putten 1994ab, 1996; Nishikawa et al. 1997; Koide et al. 1996, 1998).

The earliest approach for time-dependent simulations on shocked relativistic magnetohydrodynamic flows with dynamically significant magnetic fields uses the equations of magnetohydrodynamics (MHD) in divergence form (van Putten 1991, 1993a). The divergence technique obtains hyperbolic systems from partial differential-algebraic systems of equations, and applies more generally to the case of Yang-Mills magnetohydrodynamics in SU(N) (van Putten 1994cd, Choquet-Bruhat 1994ab), and general relativity (van Putten & Eardley 1996). A linear smoothing method has been used as a shock capturing scheme for this formulation (van Putten 1993a, 1994ab, 1995). Both one- and two-dimensional simulations on astrophysical jets are performed (van Putten 1993b, 1996, Levinson & van Putten 1997). This method is accurate and stable, and generally performs well for
relativistic shocked fluid flow with up to moderately strong shock strengths (van Putten 1993), and preserves divergence free magnetic fields to within machine round-off error (van Putten 1995). A smoothing method, therefore, is appropriate for simulations on the large scale morphology of astrophysical jets.

Advanced shock capturing schemes are commonly based on characteristics, however, such as Roe’s method (1981) and its extensions. These methods are generally more stable than smoothing methods for flows with ultra-relativistic shocks, such as in calculations of fire-balls for $\gamma$-ray bursts (Rees & Meszaros 1994; Wen et al. 1997). It is therefore of interest to explore applications of these shock capturing schemes to relativistic MHD. Here, we describe a first step in this direction is given by studying the computational stability of normalized right nullvectors (i.e., the right eigenvectors) of the characteristic matrix.

The divergence technique incorporates a constraint $c = 0$ into a divergence equation of a two-form, $\nabla^a \omega_{ab} = 0$, as in Faraday’s equation, through the linear combination

$$\nabla^a (\omega_{ab} + \lambda g_{ab} c) = 0, \quad \lambda \neq 0.$$  (1)

In the context of an Cauchy problem, (1) conserves $c = 0$ in the future domain of dependence of the initial hypersurface with physical Cauchy data (van Putten 1991).

In this paper, we identify a preferred linear combination in (1), i.e.: a choice of $\lambda$ and overall sign of (1)) in its application to the equations of ideal MHD. This follows from two separate analysis: a derivation of the right nullvectors of the characteristic matrix and well-posedness. Somewhat remarkably, both analysis agree in their preferred linear combinations. This suggests to consider this preferred linear combination in future applications in characteristic based methods to MHD in divergence form.

The problem of linearized perturbations in relativistic MHD has been considered previously by Anile (1989) and that of Alfvén waves by Komissarov (1997). The
well-posedness proof uses an extension to the Friedrichs-Lax symmetrization procedure from earlier work on Yang-Mills magnetohydrodynamics (van Putten, 1994cd).

Section 2 describes non-uniqueness in the original version of the divergence technique. In Section 3, a new derivation of the right nullvectors is given. Section 4 briefly summarizes well-posedness obtained by embedding of physical solutions in a symmetric hyperbolic system of equations.

2. MHD in divergence form

Ideal MHD describes an inviscid, perfectly conductive plasma in a single fluid description with velocity four-vector, \( u^b \ (u^c u_c = -1) \). It is given by energy-momentum conservation, \( \nabla_a T^{ab} = 0 \), where \( T^{ab} \) is the stress-energy tensor of both the fluid and the electromagnetic field, Faraday’s equations, \( \nabla_a (u^{[a} h^{b]} ) = 0 \) subject to the algebraic constraint \( u^c h_c = 0 \), and conservation, \( \nabla_a (ru^a) = 0 \), of baryon number, \( r \). For a polytropic equation of state with polytropic index \( \gamma \), we have

\[
T^{ab} = \left( r + \frac{\gamma - 1}{\gamma - 1} \frac{\gamma}{\rho} + h^2 \right) u^a u^b + \left( P + \frac{h^2}{2} \right) g^{ab} - h^a h^b,
\]

\( P \) is the hydrostatic pressure and \( g^{ab} \) is the metric tensor. The theory of relativistic magnetohydrodynamics is contained in the conservation laws of energy-momentum, \( \nabla_a T^{ab} = 0 \), and baryon number, \( \nabla_a (ru^a) = 0 \), together with Faraday’s equations and a constraint,

\[
\nabla_a (h^a u^b) = 0, \quad u^c h_c = 0. \tag{2}
\]

The divergence technique considers a constraint-free formulation by taking a linear combination

\[
\nabla_a (h^a u^b + \lambda u^c h_c) = 0. \tag{3}
\]

Provided \( \lambda \neq 0 \), (3) preserves \( u^c h_c = 0 \) during dynamical evolution in response to physical initial data (van Putten 1991), and no constraint violating wave-motion occurs.
Algebraically, the linear combination (3) establishes a rank-one update to its Jacobian, and hence of that of the full equations of MHD. Clearly, symmetry conditions of the Jacobian may enter a particular choice of $\lambda$. Below, we consider the choice

$$\lambda = 1,$$

so that

$$\begin{align*}
\nabla_a T^{ab} &= 0, \\
-\nabla_a (h^{[a} u^{b]} + g^{ab} u^c h_c) &= 0, \\
\nabla_a (r u^a) &= 0, \\
\nabla_a (\xi^a (u^2 + 1)) &= 0,
\end{align*}$$

where $\xi$ is any time-like vector field and $U = (u^b, h^b, r, P)$. The minus sign in front of the present linear combination is chosen also in regards to the structure of the Jacobian of (5). This will be made explicit below.

Upon expansion, (5) obtains the system

$$A^a \partial_a U + \cdots = 0,$$

where the matrices $A^a_B = A^a_B(U) = \frac{\partial F^a}{\partial U^B}$ are 10 by 10, and the dots refer coupling terms to the Christoffel symbols. The infinitesimal wave-structure is given by characteristic wave-fronts at given $U$ (since the $A^a$ are coordinate independent). The simple wave ansatz $U = U(\phi)$ obtains

$$A^a \partial_a \phi U' + \cdots = 0.$$ 

The wave-fronts are characteristic surfaces, whenever the matrix $A^a \partial_a \phi$ is singular. The directions $\nu_a = \partial_a \phi$ then are the normals to these surfaces. The small amplitude perturbations in these simple waves are given by the right nullvectors of $A^a \nu_a$. Stated
differently, the small amplitude perturbations are right eigenvectors, $R$, of $(A^t)^{-1}A^x \nu_x$, when the wave moves along the $x$–direction, following

$$((A^t)^{-1}A^x - v) R = 0,$$

where $v$ is the velocity of propagation.

The divergence technique provides an embedding of the theory of ideal MHD in a system of ten equations. Physical initial data are properly propagated by it, without exiting non-physical wave-modes. The physical waves (entropy waves, Alfvén and magnetohydrodynamic waves) are all contained within the light cone. Here, adding $g^{ab} u^c h_c$ (or a multiple thereof) to Faraday’s equations provides a rank-one update to the characteristic matrix $A^c \nu_c$. On the light cone, however, $\nu^2 = 0$, and this linear combination no longer regularizes of the characteristic determinant. (This results from insisting on covariance in the divergence formulation.) Komissarov (1997) attempts to discusses MHD in divergence form outside the context of the initial value problem with physical initial data, and hence erroneously concludes the presence of non-physical wave-modes.

3. The characteristic matrix

We have

$$A^{aA}(U) = \frac{\partial F^{aA}}{\partial U^B} \nu_a = \frac{\partial F^{aA} \nu_0}{\partial U^B},$$

with $\rho = r + \frac{\gamma}{\gamma-1} P + h^2 = r f + h^2$, we have

$$F^{cA} \nu_c = \begin{cases} 
\rho(u^c \nu_c) u^a + \left(P + h^2/2\right) \nu^a - (h^c \nu_c) h^a, \\
\{h^c \nu_c\} u^a - (u^c \nu_c) h^a + \nu^a u^c h_c, \\
r(u^c \nu_c), \\
(\xi^c \nu_c)(u^2 + 1). 
\end{cases}$$

(10)
The system of 10×10 equations for \( U^B = (u^b, h^b, r, P) \) can be reduced to 8×8 in the variables \( V^B = (v^s, h^b, r) \) by expressing \( u^b \) in terms of the spatial three-velocity \( u^b = \Gamma(1, v^s) \),

\[
\Gamma = \frac{1}{\sqrt{1 - v^2(1, v^s)}}, \quad s = 1, 2, 3,
\]

noting that linearized wave-motion conserves entropy, so that \( dP = \gamma \frac{\rho}{\rho} \frac{P}{r} dr \). In \( V^B \), the equation of energy conservation, \( \nabla_a T^{at} = 0 \) and the last equation of (5) are automatically satisfied, whence they can be ignored. In what follows, \( A^a \) shall denote the resulting 8×8 matrix, obtained from the original 10×10 matrix by deletion of the first and last row, addition of the last column (multiplied by \( \gamma \frac{P}{r} \)) to the one-but last column (associated with \( r \)), followed by deletion of the first and last columns.

The linearized wave-structure is given by the characteristic problem

\[
A^c \nu_c z = 0 \tag{11}
\]

for the right null-vectors \( z = U' \). Without loss of generality, (11) can be studied in a co-moving frame, in which \( u^b = (1, 0, 0, 0) \). In this event, \( \Gamma = 1 \) and \( \Gamma \frac{P}{r} = 0 \). Furthermore, the \( x \)-axis of the local coordinate system can be aligned with the magnetic field, so that \( h^b = (0, H, 0, 0) \). Given the two orientations \( u^s \) and \( h^b \), the wave-structure is rotationally symmetric about the \( x \)-axis, and hence \( \nu_y \) and \( \nu_z \) act symmetrically as \( \sqrt{\nu_y^2 + \nu_z^2} \); we will put \( \nu_z = 0 \). For \( A^c \nu_c \), we have

\[
\begin{bmatrix}
\rho \nu_1 & 0 & 0 & -\nu_1 H & -H \nu_2 & -H \nu_3 & 0 & \frac{\gamma P \nu_2}{\rho} \\
0 & \rho \nu_1 & 0 & 0 & H \nu_3 & -H \nu_2 & 0 & \frac{\gamma P \nu_3}{\rho} \\
0 & 0 & \rho \nu_1 & 0 & 0 & 0 & -H \nu_2 & 0 \\
\nu_1 H & 0 & 0 & -\nu_1 & -\nu_2 & -\nu_3 & 0 & 0 \\
-H \nu_2 & H \nu_3 & 0 & \nu_2 & \nu_1 & 0 & 0 & 0 \\
-H \nu_3 & -H \nu_2 & 0 & \nu_3 & 0 & \nu_1 & 0 & 0 \\
0 & 0 & -H \nu_2 & 0 & 0 & 0 & \nu_1 & 0 \\
\nu_2 & \nu_3 & 0 & 0 & 0 & 0 & \nu_1 & 1
\end{bmatrix}. \tag{12}
\]

Note that the lower diagonal block is \( \nu_1 \) times the 4×4 identity matrix. This results from the sign choice in the given combination of Faraday’s equations and the constraint in (5) and (10). Furthermore, notice that the third and seventh rows and columns act
independently to give rise to the Alfvén waves. The remaining waves are described by the reduced problem

\[(A^c \nu_c)' z' = 0,\]  

where \((A^c \nu_c)'\) is obtained from \(A^c \nu_c\) by deleting the third and seventh rows and columns, thereby obtaining a problem in the 6-dimensional variable \(z'\). Introducing

\[z' = \begin{pmatrix} x \\ y \end{pmatrix},\]  

(14)

(11) takes the form of a coupled system of \(3 \times 3\) equations

\[\nu_1 Z x + X y = 0, \quad Y x + \nu_1 y = 0,\]  

(15)
in which

\[Z = \begin{bmatrix} \rho & 0 & -H \\ 0 & \rho & 0 \\ H & 0 & -1 \end{bmatrix},\]  

\[X = \begin{bmatrix} -H \nu_2 & -H \nu_3 & \frac{\gamma p \nu_2}{r} \\ H \nu_3 & -H \nu_2 & \frac{\gamma p \nu_3}{r} \\ -\nu_2 & -\nu_3 & 0 \end{bmatrix},\]  

(16)

\[Y = \begin{bmatrix} -H \nu_2 & H \nu_3 & \nu_2 \\ -H \nu_3 & -H \nu_2 & \nu_3 \\ r \nu_2 & r \nu_3 & 0 \end{bmatrix}.\]

This obtains a single \(3 \times 3\) eigenvalue problem in \(x\), given by

\[XY x = \nu_1^2 Z x \iff Z^{-1} XY x = \nu_1^2 x.\]  

(17)
Here, $Z^{-1}XY - \nu^2_1$ is given by the matrix

$$
\begin{bmatrix}
  w_{1,1} & w_{1,2} & 0 \\
  w_{2,1} & w_{2,2} & 0 \\
  \rho (\gamma \rho \nu_2^2 - \gamma \nu_2^2 - \gamma \nu_3^2) & \rho \nu_2 \nu_3 & \nu_2^2 + \nu_3^2 - \nu_1^2 \\
\end{bmatrix}
$$

(18)

where the upper diagonal 2×2 matrix $W$ is given by

$$
W = W_{ij}.
$$

(19)

The two zeros in the third column of (18) result from $\lambda = 1$. Upon substitution $\nu_3^2 = \nu^2_1 + \nu^2_2$, the determinant assumes the covariant expression

$$
\rho \det W = (rf - \gamma P)(u^c \nu_c)^4
$$

$$
- (h^2 + \gamma P)\nu^2(u^c \nu_c)^2 + \frac{2P}{f}(h^c \nu_c)^2 \nu^2.
$$

(20)

**Alfvén waves.** The eigenvalues for the Alfvén waves are given by

$$
\nu_1 = \pm |h^c \nu_c| \sqrt{\rho}
$$

(21)

with null-vector

$$
z = (0, 0, H \nu_2, 0, 0, \rho \nu_1, 0)^T,
$$

(22)

associated with Alfvén waves; covariantly,

$$
U^A = (v^a, \pm \sqrt{\rho}v^a, 0, 0)^T,
$$

(23)

where $v_a$ may be taken to be

$$
H(0, 0, \nu_4, -\nu_3) = \epsilon_{abcd}u^b h^c \nu^d \equiv v_a.
$$

(24)

Thus, the Alfvén wave is transversal in which $h^2$ is conserved ($\delta h^b$ is orthogonal to $h^b$).

**Magnetohydrodynamic waves.** The eigenvalues for the magnetohydrodynamic waves are given by the roots of the characteristic determinant (20). Writing

$$
n^b = \nu^b + (u^c \nu_c)u^c,
$$

(25)
we have \( \nu^2 = -t^2 + n^2, \ t = u^n_c, \ n^2 = n^n_c. \) Let \( \alpha = \frac{r f}{\gamma P} \) and \( \beta = \frac{h^2}{\gamma P}. \) Then

\[
\frac{(h^n_c)^2}{rfn^2} = \frac{\beta (h^n_c)^2}{\alpha h^n_c n^2} = \frac{\beta}{\alpha} \cos^2 \phi.
\] (26)

Consequently, (20) becomes

\[
(\alpha - 1)v^4 - (1 + \beta)v^2(1 - v^2) + \beta \alpha^{-1} \cos^2(1 - v^2) = 0,
\] (27)

where \( v^2 = \frac{t^2}{n^2}. \) (27) has real solutions \( v \) for any given \( n^b, \) whenever

\[
(\alpha + \beta)v^4 - (1 + \beta \alpha^{-1})v^2 + \beta \alpha^{-1} = 0
\] (28)

has real solutions \( v. \) But (28) has discriminant

\[
D = (\alpha + \beta - \alpha \beta)^2 \geq 0.
\] (29)

Weak magnetic fields are described by small \( \beta \) expansions as follows.

**Proposition 3.1.** Fast magnetosonic waves are a regular perturbation of sound waves in pure hydrodynamics, while the Alfvén and slow magnetosonic waves bifurcate from entropy waves (contact discontinuities), whose propagation velocities satisfy

\[
v_f^2/v_h^2 \sim 1 + \beta 2^{-1} \alpha \sin^2 \phi + O(\beta^2),
\]

\[
v_A^2/v_h^2 \sim \beta \cos^2 \phi[1 - \beta \alpha^{-1} + O(\beta^2)],
\] (30)

\[
v_s^2/v_h^2 \sim \beta \cos^2 \phi[1 - \beta (1 - \alpha^{-1} \cos^2 \phi) + O(\beta^2)],
\]

where \( v^2_h = \alpha^{-1} \) is the square of the hydrodynamical velocity, and which obey the inequalities

\[
v_s^2 \leq v_A^2 \leq v_f^2.
\] (31)

Inequalities (31) remain valid for general \( \beta \) (e.g. Bazer & Ericson 1959; Lichnerowicz 1967; Anile 1989).
4. Right nullvectors

Inspection of (19), together with (15), shows the null-vector

\[ z = \begin{pmatrix}
\nu_1 \nu_2 \nu_3 \\
-\nu_1 \nu_3 (\nu_2^2 - \alpha \nu_1^2) \\
0 \\
H \nu_1 \nu_2 \nu_3^2 \\
H \nu_3^2 (\nu_2^2 - \alpha \nu_1^2) \\
-H \nu_2 \nu_3 (\nu_2^2 - \alpha \nu_1^2) \\
0 \\
-\alpha \nu_3^2 \nu_1^2
\end{pmatrix}. \tag{32} \]

Of course, (32) can be stated covariantly by noting that

\[ H^2 = h^2, \quad H \nu_2 = h^c \nu_c, \quad \nu_1 = u^c \nu_c, \]

and introducing

\[ H(0, \nu_4^2, \nu_2^2, -\nu_2 \nu_4) = \epsilon_{abcd} u^b u^c \nu^d \equiv w_a. \tag{34} \]

Since \(-\alpha \nu_3^2 \nu_1^2\) is a scalar, \(\nu^3\) is to be treated as

\[ H^2 (\nu_3^2 + \nu_1^2) = h^2 n^2 - (h^c \nu_c)^2 \equiv h^2 k_1, \tag{35} \]

were \(n_a = \nu_a + (u^c \nu_c) u_a\). Note that

\[ k_1 = n^2 (\cos^2 \phi - \alpha v^2), \quad k_2 = n^2 \sin^2 \phi, \tag{36} \]

where \(v = v_s, v_f\). Clearly, \(z\) is formed from

\[ \delta u^b = -t (k_1 n^b - (k_2 + k_1) (\hat{h}^c n_c) \hat{h}^b) \]
\[ \delta h^b = k_1 w^b + k_2 (h^c n_c) u^b, \]
\[ \delta r = -\alpha r k_2 t^2, \]
\[ \delta P = -rf k_2 t^2, \tag{37} \]
where $\hat{h}^b = h^b/|h|$, and

$$v_a = \epsilon_{abcd} u^b h^c v^d, \quad w_a = \epsilon_{abcd} u^b v^c v^d. \tag{38}$$

We thus have the following.

**Proposition 3.2.** Given a unit vector $n^b$ orthogonal to $u^b$, and a root $\nu^b = n^b + v u^b$, $v = u^c \nu_c$ of (28), the right nullvectors for the hydrodynamical waves of (11), $U^A = (\delta u^b, \delta h^b, \delta r, \delta P)$, are

$$\delta u^b = v \left[ \sin^2 \phi \ n^b - (1 - \alpha v^2)(n^b - \cos \phi \ \hat{h}^b) \right],$$

$$\delta h^b = |h| \left[ (\cos^2 \phi - \alpha v^2)\hat{w}^b + v \sin^2 \phi \cos \phi \ u^b \right],$$

$$\delta r = -v^2 \alpha r \sin^2 \phi,$$

$$\delta P = -v^2 r f \sin^2 \phi.$$

where $\hat{w}^b = w^b/|h|$.

Anile (1989) gives a different form of these right nullvectors. By Proposition 3.1, our weak magnetic field limits show that

$$\cos^2 \phi - \alpha v_j^2 < 0 \tag{40}$$

for fast magnetosonic waves, while

$$\cos^2 \phi - \alpha v_s^2 > 0 \tag{41}$$

for slow magnetosonic waves. Inspection of (34) shows that therefore the tangential component of the magnetic field is strengthened in fast magnetosonic waves, while it
is weakened in slow magnetosonic waves. This distinguishing aspect of fast and slow magnetosonic waves was first noted by Bazer & Ericson (1959) in their analysis of shocks in nonrelativistic MHD.

The limit of small $\beta$ is of particular interest to computation. For example, in various settings a magnetized fluid streams into a nearly unmagnetized environment. A characteristics based scheme must therefore reliably treat a large dynamic range in $\beta$. Clearly, a full set of nullvectors (including those of contact discontinuities) obtains for nonzero $\beta$. However, the behavior of these nullvectors is somewhat nontrivial as $\beta$ becomes small. In what follows, we consider the small $\beta$ limit, in the sense of small $|h|/\sqrt{\gamma P}$, while keeping the direction $\hat{h}^b$ constant. In this limit,

\begin{align}
1 - \alpha v^2 &\sim -\beta \frac{\alpha - 1}{\alpha} \sin^2 \phi + O(\beta^2), \\
1 - \alpha v^2 &\sim 1 + O(\beta)
\end{align}

for the fast and slow magnetosonic speeds, respectively.

**Corollary 4.1.** In the limit of low magnetic field strength, the fast magnetosonic waves are described by the right nullvectors

\begin{align}
\delta u^b &= v_f n^b + \beta \frac{\alpha - 1}{\alpha} (n^b - \cos \phi \hat{h}^b)v_f + O(\beta^2), \\
\delta h^b &= |h|(-\hat{w}^b + v_f \cos \phi \hat{u}^b) + \beta \frac{\alpha - 1}{\alpha} w^b + O(\beta^2), \\
\delta r &= -v_f^2 \alpha r, \\
\delta P &= -v_f^2 r f,
\end{align}

and the slow magnetosonic waves by

\begin{align}
\delta u^b &= \cos \phi (\hat{h}^b - \cos \phi n^b) + O(\beta), \\
\delta h^b &= \sqrt{\gamma P}(\cos \phi \hat{w}^b + v_s \sin^2 \phi \hat{u}^b) + O(\beta), \\
\delta r &= -v_s \alpha r \sin^2 \phi, \\
\delta P &= -v_s \alpha r f \sin^2 \phi.
\end{align}

The small $\beta$ limit of the nullvectors can now be normalized.
4.1. Bifurcations from entropy waves

The behavior of the nullvectors in the limit of weak magnetic fields can be derived from (23) and Corollary 4.1. To this end, note that

\[ v^a = |h|\tilde{v}^a = \sin \phi |h|\tilde{v}^a, \]

where \( \tilde{v}^c \tilde{v}_c = 1 \), and \( \phi \) denotes the angle between \( n^c \) and \( h^c \),

\[ n^b = \cos \phi \hat{h}^b + \sin \phi y^b, \]

\( y^c u_c = h^c y_c = 0, y^c y_c = 1 \) (\( n^b \) is normalized to be unit, as in the assumptions of Proposition 3.2). It follows that the Alfvén nullvectors may be normalized to

\[ \delta \hat{U}^A = (\dot{v}^a, \pm \sqrt{\rho} \dot{v}^a, 0, 0). \]

In the limit of vanishingly small \( \beta \), the pair of slow magnetosonic waves collapse to the single normalized nullvector

\[ \delta \hat{U}^A = (y^b, \sqrt{\gamma P} y^b, 0, 0). \]

Note that \( y^c \dot{v}_c = 0 \), so that (47) and (48) are independent. Division by \( \sin \phi \) thus provides a normalization of the original expressions (23) and (44).

The nullvector associated with entropy waves \( (u^c \nu_c = 0) \) is

\[ \delta U^A = (0, 0, \delta r, 0) \]

if \( h^c \nu_c \neq 0 \), and

\[ (0, \delta h^c, \delta r, \delta P), \; (\delta u^c, 0, 0, 0), \]

if \( h^c \nu_c = 0 \), subject to

\[ \delta P + h_c \delta h^c = 0, \; \nu_c \delta h^c = 0, \; \nu_c \delta u^c = 0. \]
The second case refers to transverse MHD for which continuity must hold of total pressure, zero orthogonal magnetic field and transverse velocity. Note that transverse MHD has two nullvectors, and corresponds to the case of pure hydrodynamics. With the exception of transverse MHD, therefore, the contact discontinuity provides one nullvector.

Transverse MHD or pure hydrodynamics allows for shear along contact discontinuities, which is responsible for the two independent nullvectors. Whenever magnetic field lines cross a contact discontinuity, however, the persistent coupling to the magnetic field lines in ideal MHD prohibits shear. In ideal MHD, the response to the original two-dimensional degree of freedom in shear is two new wave-modes. These wave modes are the Alfvén wave and the slow magnetosonic wave. These two wave-modes are indeed different, as (47) and (48) show. The Alfvén and slow magnetosonic wave may be regarded as one pair, bifurcating from the contact discontinuity. This has been illustrated in Fig. 6 of van Putten (1993a). Indeed, the limit of vanishing $\beta$ recovers the two shear modes from the independent Alfvén and slow magnetosonic waves. Of course, the Alfvén wave is purely rotational, while the slow magnetosonic wave is slightly helical, including a longitudinal variation of $\pm v_s \sin^2 \phi = \pm \beta \sin^2 \phi \cos \phi$. The fast magnetosonic wave remains a regular perturbation of the ordinary sound wave.

The weak magnetic field limit thus obtains two nullvectors from the fast magnetosonic waves, two from the Alfvén waves, one from the slow magnetosonic waves and generally one from the contact discontinuity, a total of six. This leaves an apparent degeneracy of one.

The degeneracy stems from the neighboring to order $v_s$ of the two nullvectors of the slow magnetosonic waves. This would suggest ill-posedness to this order in projections. However, characteristic based methods consider the product of the projections on the nullvectors and the associated eigenvectors. In the present case, therefore, the order of the degeneracy is precisely cancelled by multiplication with the eigenvalue $v_s$, which is
computationally stable. The limit of arbitrarily small $\beta$ in the application of characteristic based methods is computationally well-posed.

5. Well-posedness

The theory of ideal relativistic MHD was first shown to be well-posed by Friedrichs (1974), using the Friedrichs-Lax symmetrization procedure (1971). The problem of constraints was circumvented by a reduction of variables. The symmetrization procedure of Friedrichs and Friedrichs and Lax (1971) applies to hyperbolic systems of equations of the form

$$\nabla_a F^{aB} = f^B$$

which satisfy a certain convexity condition. The presence of conserved constraints, however, can be treated also by an extension of the Friedrichs-Lax symmetrization procedure, with no need for an additional reduction of variables, developed in earlier work on Yang-Mills magnetohydrodynamics in SU(N) (van Putten 1994cd). Once in symmetric hyperbolic form, well-posedness results from standard energy arguments (e.g. Fisher & Marsden 1972). The main arguments of symmetrization in the presence of constraints are briefly recalled here, to highlight the same linear combination of (5), now from the point of view of well-posedness.

5.1. Symmetrization with constraints

Variations $\delta V^A$ of $(u^b, h^b, r, P)$ can be unconstraint (with respect to all ten degrees of freedom), and constraint, i.e., those obeying the constraints. For example, $\delta c \neq 0$ results from a total variation, while $\delta c = 0$ is a constraint variation. Symmetrization in the presence of constraints follows if there exists a vector field $W_A$ which produces a total
derivative in the modified main dependency relation

\[
YI : \ W_A \delta F^a A \equiv \delta z^a ,
\]

and which obtains constraint positive definiteness in

\[
YII : \ \delta W_A \delta F^{a A} \xi_a > 0
\]

for some time-like vector \( \xi^a \). Of course, the source terms \( f^B \) must satisfy the consistency condition

\[
W_A f^A = 0
\]

whenever the constraints are satisfied. Allowing a possible nonzero total derivative in YI defines an extension (van Putten 1994cd) to the Friedrichs-Lax (1971) symmetrization procedure.

Differentiation by \( V^C \) of the unconstraint identity YI obtains

\[
\frac{\partial W_A \partial F^{a A}}{\partial V^C \partial V^D} \nabla_a V^D + \frac{W_A \partial^2 F^{a A}}{\partial V^C \partial V^D} \nabla_a V^D = \frac{\partial^2 z}{\partial V^C \partial V^D} \nabla_a V^D.
\]

This establishes symmetry of the matrices

\[
A^a_{CD} = \frac{\partial W_A \partial F^{a A}}{\partial V^C \partial V^B}
\]

Also,

\[
\delta V^C A^a_{CD} \xi_a \delta V^D = (\delta V^C \frac{\partial W_A}{\partial V^A}) (\frac{\partial F^{a A}}{\partial V^D} \delta V^D) = \delta W_A \delta F^{a A} \xi_a > 0
\]

for all constraint variations \( \delta V^A \). Of course, given \( V^A \), the constraint variations \( \delta V^A \) define a linear subspace \( \mathcal{V} \) of dimension \( N - m \), where \( m \) is the number of constraints \( c = 0 \), each giving rise to

\[
0 = \delta c = \frac{\partial c}{\partial V^A} \delta V^A.
\]
We have the following construction (van Putten, 1994cd).

**Lemma 5.1** Given a real-symmetric $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ which is positive definite on a linear subspace $V \subset \mathbb{R}^n$, there exists a real-symmetric, positive definite $A^* \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$A^* y = Ay \quad (y \in V).$$

(60)

This may be seen as follows. Consider $A^* = A + \mu x^T x$, where $x$ is a unit element from $V^\perp$. Then $A^*$ is symmetric positive definite on $V' = \{z = y + \lambda x \mid y \in V, \lambda \in \mathbb{R} \}$:

$$z^T A^T z \geq c' \|z\|^2 = c' (\|y\|^2 + \lambda^2 \|x\|^2)$$

with $c' > 0$ upon choosing $\mu > M$, where $M = \|A\|$ denotes the norm of $A$. This construction may be repeated until $V^\perp$ is exhausted, leaving $A^*$ symmetric positive definite on $\mathbb{R}^n$ as an embedding of $A$ on $V$.

The real-symmetric matrix $A_{\xi D}^\alpha \xi_a$ is positive definite on the subspace of constraint variations $V$; let $(A_{\xi D}^\alpha \xi_a)^*$ be the positive definite, symmetric matrix obtained from the Lemma. It follows that solutions to (52) (and its constraints) satisfy the *symmetric positive definite* system of equations

$$-(A_{\xi D}^\alpha \xi_a)(\xi^c \nabla_c) V_A + A_{\xi D}^\alpha (\nabla_\Sigma)_a V_A = f^B,$$

(61)

where

$$\nabla_a = -\xi_a (\xi^c \nabla_c) + (\nabla_\Sigma)_a.$$

(62)

It remains to show that ideal MHD satisfies properties YI and YII.
5.2. Symmetrization of hydrodynamics

Relativistic hydrodynamics has been shown to be symmetrizable by Friedrichs (1974), Ruggeri & Strumia (1981), and Anile (1989). This uses the equations in the form

\[ \nabla_a F_f^{aA} = \begin{cases} 
\nabla_a (r f u^a u^b + P g^{ab}) = 0, \\
\nabla_a (ru^a) = 0, \\
\nabla_a (r S u^a) = 0
\end{cases} \tag{63} \]

away from entropy generating shocks. Then \( W_f^A = (u_a, f - TS, T) \) and \( V_C^f = (v_a, T, f) \) with a reduction of variables on the velocity four-vector by \( u^b = \Gamma (1, v^a) \), where \( \Gamma \) is the Lorentz factor. With \( F_f^{aA} \) denoting the fluid dynamical equations \( \nabla_a T_f^{ab} = 0, T_f^{ab} = r f u^a u^b + P g^{ab} \) with \( f \) the specific enthalpy, and \( \nabla_a (ru^a) = 0 \), it has been shown that (Ruggeri & Strumia 1981; Anile 1989)

\[ W_f^A \delta F_f^{aA} \equiv 0, \quad Q_f = \delta W_A \delta F_f^{aA} \xi_a > 0 \tag{64} \]

provided that the free enthalpy \( G(T, P) = f - TS - 1 \) is concave, and the sound velocity is less than the speed of light. Under this conditions, the hydrodynamical equations alone, therefore, satisfy YI and YII, and in fact the original Friedrichs-Lax conditions CI and CII of Friedrichs & Lax (1971), so that they satisfy a symmetric hyperbolic system of equations.

5.3. Symmetrization of ideal MHD

In what follows, we set

\[ \omega_{ab} = h^a u^b - u^a h^b + g^{ab} u^c h_c, \]
\[ T_{m}^{ab} = h^2 u^a u^b + \frac{1}{2} h^2 g^{ab} - h^a h^b. \tag{65} \]
We then have the expansions
\[
\begin{align*}
    u_b \delta T_{ab}^m &= u_b(h^2 u^a \delta u^b + h^2 u^b \delta u^a + 2u^a u^b h_c \delta h^c \\
    &\quad + g^{ab} h_c \delta h^c - h^a \delta h^b - h^b \delta h^a) \\
    &= -h^2 \delta u^a - u^a(h_c \delta h^c) - h^a(u_c \delta h^c) \\
    &\quad - c \delta h^a, \\
    h_b \delta \omega^{ab} &= h_b(h^a \delta u^b + u^b \delta h^a - h^ab \delta u^a - u^a \delta h^b \\
    &\quad + g^{ab} \delta c) \\
    &= h^a(h_c \delta u^c) + c \delta h^a - h^2 \delta u^a - u^a(h_c \delta h^c) \\
    &\quad + h^a \delta c.
\end{align*}
\]

(66)

We hereby arrive at the identity
\[
    u_b \delta T_{ab}^m - h_b \delta \omega^{ab} \equiv \delta z^a,
\]

(67)

where \( z^a = -2h^a c \). The total derivative in (67) follows by the unique linear combination \( \omega^{ab} = h^a u^b - h^b u^a + g^{ab} c \), as in (5). With \( W_A = (u_a, h_a, f - TS, S) \) and \( F^{aA} \) given by (5) [rewritten according to (63)], it follows that
\[
    W_A \delta (F^{aA} + F_m^{aA}) \equiv \delta z^a.
\]

(68)

A similar calculation (van Putten 1994cd) shows that quadratic of constraint variations \( Q_m \) given by
\[
\begin{align*}
    \delta u_b \delta T_{m}^{ab} \xi_a &- \delta h_b \delta \omega^{ab} \xi_a = (u^c \xi_c)[h^2(\delta u)^2 + (\delta h)^2] \\
    &+ 2[(\xi_c \delta u^c)(h_c \delta h^c) - (h^c \xi_c)(\delta u_c \delta h^c)]
\end{align*}
\]

(69)

is positive definite (for \( \delta h^a \neq 0 \)). Therefore, the sum
\[
    Q = \delta W_A \delta F^{aA} \xi_a = Q_f + Q_m
\]

(70)

is constraint positive definite, whenever \( Q_f \) is such (with respect to the fluid dynamical variables). It follows that both YI and YII are satisfied (with \( W_A = (u_a, h_a, f - TS, S) \) and
\( V_A = (v_\alpha, h_a, T, f) \), and hence physical solutions to (5) satisfy the symmetric hyperbolic system (61) with \( f^B = 0 \).

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