REGULARITY AND CAPACITY FOR THE FRACTIONAL DISSIPATIVE OPERATOR

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ABSTRACT. This note is devoted to exploring some analytic-geometric properties of the regularity and capacity associated to the so-called fractional dissipative operator $\partial_t + (-\Delta)^\alpha$, naturally establishing a diagonally sharp Hausdorff dimension estimate for the blow-up set of a weak solution to the fractional dissipative equation $(\partial_t + (-\Delta)^\alpha)u(t, x) = F(t, x)$ subject to $u(0, x) = 0$.

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References

1. INTRODUCTION AND THE MAIN RESULTS

This beginning part is designed to describe the principal results of this article.

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1.1. The fractional dissipative equation. For \( n = 1, 2, 3, \ldots, \alpha \in (0, 1) \) and \( \mathbb{R}_{+} := (0, \infty) \), let \( \mathbb{R}_{+}^{1+n} := \mathbb{R}_{+} \times \mathbb{R}^{n} \) be the upper half space of the \( 1+n \) dimensional Euclidean space \( \mathbb{R}_{+}^{1+n} \) and \( (-\Delta)^{\alpha} \) be the fractional Laplace operator which is determined by

\[
(-\Delta)^{\alpha} u(\cdot, x) := \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F} u(\cdot, \xi))(x), \quad \forall x \in \mathbb{R}^{n},
\]

where \( \mathcal{F} \) denotes the Fourier transform and \( \mathcal{F}^{-1} \) its inverse:

\[
\begin{align*}
\mathcal{F}(g)(x) &:= (2\pi)^{-n/2} \int_{\mathbb{R}^{n}} e^{-ix\cdot y} g(y) \, dy; \\
\mathcal{F}^{-1}(g)(x) &:= (2\pi)^{-n/2} \int_{\mathbb{R}^{n}} e^{ix\cdot y} g(y) \, dy.
\end{align*}
\]

From the celebrated Duhamel’s principle it follows that a weak solution \( u(t, x) \) to the fractional dissipative equation living in fluid dynamics via the so-called fractional dissipative operator \( L^{(\alpha)} := \partial_{t} + (-\Delta)^{\alpha} \):

\[
\begin{align*}
&\left\{ \begin{array}{l}
L^{(\alpha)} u(t, x) = F(t, x), \quad \forall (t, x) \in \mathbb{R}_{+}^{1+n}, \\
u(0, x) = f(x), \quad \forall x \in \mathbb{R}^{n},
\end{array} \right.
\end{align*}
\]

namely (cf. [9, 7]),

\[
\begin{align*}
\mathbb{E} \int_{\mathbb{R}_{+}^{1+n}} u \mathcal{L}^{(\alpha)} \phi \, dx dt &= - \int_{\mathbb{R}_{+}^{1+n}} F \phi \, dx dt - \int_{\mathbb{R}^{n}} f(x) \phi(0, x) \, dx, \quad \forall \phi \in \mathcal{C}_{0}^{\infty}(\mathbb{R}_{+}^{1+n}) \\
\text{with}
\end{align*}
\]

\[
\mathcal{L}^{(\alpha)} \phi(t, x) = -\partial_{t} \phi(t, x) + \left( \frac{(1-\alpha)2^{2\alpha}\Gamma(\frac{n+2\alpha}{2})}{\pi^{n/2}\Gamma(1-\alpha)} \right) \lim_{\epsilon \to 0} \int_{|y|<\epsilon} \frac{\phi(t,x+y) - \phi(t,x)}{|y|^{n+2\alpha}} \, dy,
\]

can be written as

\[
u(t, x) = R_{\alpha} f(t, x) + S_{\alpha} F(t, x),
\]

where

\[
\begin{align*}
R_{\alpha} f(t, x) &:= e^{-t(-\Delta)^{\alpha}} f(x) ; \\
S_{\alpha} F(t, x) &:= \int_{0}^{t} e^{-t(s)(-\Delta)^{\alpha}} F(s, x) \, ds,
\end{align*}
\]

for which

\[
\begin{align*}
e^{-t(-\Delta)^{\alpha}} v(\cdot, x) &:= K_{t}^{(\alpha)}(x) \ast v(\cdot, x) ; \\
K_{t}^{(\alpha)}(x) &:= (2\pi)^{-n/2} \int_{\mathbb{R}^{n}} e^{ix\cdot y - |y|^{2\alpha}} \, dy,
\end{align*}
\]

and * represents the convolution operating on the space variable. Here it is perhaps appropriate to mention that

\[
\begin{align*}
K_{t}^{(1)}(x) &:= (4\pi)^{-n/2} e^{-|x|^{2}/(4t)} \\
\text{and}
K_{t}^{(1,2)}(x) &:= \pi^{-(1+n)/2} \Gamma((n+1)/2) t^{n/2} + |x|^{2} - (1+n)/2
\end{align*}
\]

are the heat and Poisson kernels, respectively. Of course, \( \Gamma(\cdot) \) is the classical gamma function. Although an explicit formula of \( K_{t}^{(\alpha)}(x) \) for \( \alpha \in (0, 1) \setminus \{1/2, 1\} \) is unknown (cf. [8, 6, 15, 14, 4] and [19, 9, 10, 11, 12] for some
related information), one has the following basic estimate (cf. [17, 5]): under $\alpha \in (0, 1)$

$$K^{(\alpha)}_t(x) \approx t^{\frac{1}{2\alpha}} + |x|^{-(\alpha + 2\alpha)}, \quad \forall (t, x) \in \mathbb{R}_{++}^n.$$  

In the above and below, $X \approx Y$ means $Y \leq X \leq Y$ where the second estimate
means that there is a positive constant $c$, independent of main parameters, such that $X \leq cY$. From now on, $\alpha$ will be always assumed to be in the interval $(0, 1)$.

1.2. **Regularity for the fractional dissipative operator.** The following function space regularity results of Strichartz type, plus [11], actually induce the research objective of this current paper.

**Theorem 1.1.**

(i) [8] Lemma 3.2] If

$$\begin{cases}
1 \leq p \leq \tilde{p} < \frac{np}{n - \min(n, 2\alpha)}; \\
\frac{1}{q} = \left(\frac{\alpha}{2\alpha}\right)(\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}}),
\end{cases}$$

then

$$\|R_{\alpha}f\|_{L^q_tL^{\tilde{p}}_x(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$  

(ii) [18, Theorem 1.4] If

$$\begin{cases}
1 \leq p < \tilde{p} \leq \infty; \\
1 < q < \tilde{q} < \infty; \\
\left(\frac{1}{q} - \frac{1}{\tilde{q}}\right) + \left(\frac{\alpha}{2\alpha}\right)(\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}}) = 1,
\end{cases}$$

then

$$\|S_{\alpha}F\|_{L^q_tL^{\tilde{p}}_x(\mathbb{R}^{1+n})} \lesssim \|F\|_{L^p_tL^{\tilde{p}}_x(\mathbb{R}^{1+n})}.$$  

Here and henceforth: $L^p_x(\mathbb{R}^n)$ denotes the usual Lebesgue $1 \leq p \leq \infty$-space with respect to the space variable $x$; $L^p_tL^q_x(\mathbb{R}^{1+n})$ is the mixed $(1 \leq p_1, p_2 < \infty)$-Lebesgue space of all functions $F$ on $\mathbb{R}^{1+n}$ with

$$\|F\|_{L^{p_1}_tL^{p_2}_x(\mathbb{R}^{1+n})} := \left(\int_{\mathbb{R}_+} \left[\int_{\mathbb{R}^n} |F(t, x)|^{p_1}dx\right]^{\frac{p_2}{p_1}}dt\right)^{\frac{1}{p_2}} < \infty,$$

where a suitable modification is needed whenever $p_1$ or $p_2$ is $\infty$; for $\mathbb{X} = \mathbb{R}^n$ or $\mathbb{R}^{1+n}$ the symbols $C^\infty(\mathbb{X})$, $C_0^\infty(\mathbb{X})$ and $C(\mathbb{X})$ stand for all infinitely smooth functions in $\mathbb{X}$, all infinitely smooth functions with compact support in $\mathbb{X}$ and all continuous functions in $\mathbb{X}$, respectively.

Throughout the paper, for each $(t_0, x_0) \in \mathbb{R}^{1+n}$ and $r > 0$, the parabolic ball is defined as

$$B^{(\alpha)}_r(t_0, x_0) := \{(t, x) \in \mathbb{R}^{1+n} : |t - t_0| < r^{2\alpha} \& |x - x_0| < r\}$$
and its volume is denoted by $|B_{r_0}^{(\alpha)}(t_0, x_0)| \approx r_0^{n+2\alpha}$.

The first main result of this paper appears as an essential extension or complement of Theorem 1.1.

**Theorem 1.2.**

(i) If $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^n)$, then $R_\alpha f$ is continuous on $\mathbb{R}^{1+n}_+$. 

(ii) If

$$\begin{align*}
p \in [1, \infty); \\
1 < q < \infty; \\
\frac{n}{p} + \frac{2\alpha}{q} = 2\alpha; \\
(t_0, x_0) \in \mathbb{R}^{1+n}_+; \\
r_0 = \frac{2\alpha}{q}; \\
0 < \|F\|_{L_q^\alpha L_p^q(\mathbb{R}^{1+n}_+)} < \infty,
\end{align*}$$

then there exists $C > 0$ such that

$$\frac{1}{|B_{r_0}^{(\alpha)}(t_0, x_0)|} \int_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp \left( \frac{S_\alpha F(t, x)}{C\|F\|_{L_q^\alpha L_p^q(\mathbb{R}^{1+n}_+)}} \right)^{\frac{q}{2q - n}} \, dx \, dt \leq 1.$$ 

(iii) If

$$\begin{align*}
p \in [1, \infty); \\
1 < q < \infty; \\
\frac{n}{p} + \frac{2\alpha}{q} < 2\alpha; \\
(t, x) \in \mathbb{R}^{1+n}_+; \\
\|F\|_{L_q^\alpha L_p^q(\mathbb{R}^{1+n}_+)} < \infty,
\end{align*}$$

then $S_\alpha F$ is Hölder continuous in the sense that

$$|S_\alpha F(t, x) - S_\alpha F(t_0, x_0)| \leq (|t - t_0|^{\frac{2\alpha}{q} - \frac{\alpha}{n} + \frac{n}{p}} + |x - x_0|^{\frac{2\alpha}{q} - \frac{\alpha}{n} - \frac{2\alpha}{q}})\|F\|_{L_q^\alpha L_p^q(\mathbb{R}^{1+n}_+)}$$

holds for any two sufficient close points $(t_0, x_0), (t, x) \in \mathbb{R}^{1+n}_+$.

1.3. **Capacity for the fractional dissipative operator.** From Theorems 1.1, 1.2, we know that it is necessary to estimate the size of the blow-up set of the so-called fractional dissipative potential $S_\alpha F$ below:

$$\mathcal{B}[S_\alpha F; p, q] := \{(t, x) \in \mathbb{R}^{1+n}_+ : S_\alpha F(t, x) = \infty\} \quad \text{for} \quad 0 \leq F \in L_q^\alpha L_p^q(\mathbb{R}^{1+n}_+).$$

To handle this issue, let us introduce a new type of capacity. For a compact subset $K$ of $\mathbb{R}^{1+n}_+$, let

$$C^{(\alpha)}_{p, q}(K) := \inf \left\{ \|F\|_{L_q^\alpha L_p^q(\mathbb{R}^{1+n}_+)} : F \geq 0 \& S_\alpha F \geq 1_K \right\}$$

be the $(\alpha, p, q)$-capacity of $K$ for the fractional dissipative operator $L^{(\alpha)}$, where $1_K$ is the characteristic function of $K$, $p \land q := \min(p, q)$, and $1 \le
$p, q < \infty$. Moreover, the definition of $C_{p,q}^{(\alpha)}$ extends to any subset of $\mathbb{R}^{1+n}$ in a similar way as \cite{[2]} Definitions 2.2.2 & 2.2.4.

Next, for
\[
\begin{align*}
0 < \varepsilon \leq \infty; \\
0 < d < \infty; \\
K \subset \mathbb{R}^{1+n}; \\
B_{r_j}^{(\alpha)}(t_j, x_j) := \{(s, y) \in \mathbb{R}^{1+n} : |s - t_j| < r_j^{2\alpha} \& |y - x_j| < r_j\}; \\
(t_j, x_j, r_j) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+; \\
\phi : [0, \infty) \mapsto [0, \infty] \text{ an increasing function with } \phi(0) = 0,
\end{align*}
\]

let
\[
H_{\varepsilon}^{\phi, \alpha}(K) := \inf \left\{ \sum_{j=1}^{\infty} \phi(r_j) : K \subseteq \bigcup_{j=1}^{\infty} B_{r_j}^{(\alpha)}(t_j, x_j); \quad \text{with } r_j \in (0, \varepsilon) \right\}
\]

be the $L^\alpha$-based $(\phi, \varepsilon)$-Hausdorff capacity of $K$. Then the $L^\alpha$-based $\phi$-Hausdorff measure of $K$ is defined by
\[
H^{\phi, \alpha}(K) := \lim_{\varepsilon \to 0} H_{\varepsilon}^{\phi, \alpha}(K).
\]

If $\phi(r) := r^d$ for all $r \in (0, \infty)$, then
\[
\begin{align*}
H_{\varepsilon}^{\phi, \alpha}(K) &\equiv H_{\varepsilon}^{d, \alpha}(K); \\
H^{\phi, \alpha}(K) &\equiv H^{d, \alpha}(K); \\
\dim_{H}^{(\alpha)}(K) &:= \inf\{d : H^{d, \alpha}(K) = 0\},
\end{align*}
\]

where the last quantity is called the $L^{(\alpha)}$-based Hausdorff dimension of $K$.

Below is our second theorem.

**Theorem 1.3.**

(i) If
\[
\begin{align*}
1 \leq p < \infty; \\
1 < q < \infty; \\
\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha > 0,
\end{align*}
\]

then
\[
C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0)) \approx r_0^{(p,q)\left(\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha\right)} \quad \text{as } r_0 \to 0 \quad \text{& } (r_0, x_0) \in \mathbb{R}^{1+n}.
\]

(ii) If
\[
\begin{align*}
1 \leq p < \infty; \\
1 < q < \infty; \\
\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha = 0,
\end{align*}
\]
then
\[ C_{p,q}^{(\alpha)}(B_r^{(\alpha)}(f_0, x_0)) \approx \left( \ln \frac{1}{r_0} \right)^{(p\wedge q)(\frac{n}{p} - 1)} \text{ as } r_0 \to 0 \text{ and } (r_0, x_0) \in \mathbb{R}^{1+n}. \]

As an immediate consequence of Theorems 1.1-1.2-1.3, we get not only three geometric inequalities linking two types of capacity, but also some Haussdorff dimension estimates for the blow-up sets which are sharp in the diagonal case \( p = q \).

**Corollary 1.4.**

(i) Let \( \mathcal{L}^1(A) \) and \( \mathcal{L}^n(B) \) stand for the 1-dimensional and \( n \)-dimensional Lebesgue measures of bounded Borel sets \( A \subset \mathbb{R}_+ \) and \( B \subset \mathbb{R}^n \), respectively. If

\[
\begin{align*}
1 \leq p < \tilde{p} < \infty; \\
1 < q < \tilde{q} < \infty; \\
\beta := (p \wedge q)(\frac{p + 2\alpha}{p} - 2\alpha) > 0; \\
\left( \frac{1}{q} - \frac{1}{q} \right) + \left( \frac{1}{p} - \frac{1}{p} \right) = 1,
\end{align*}
\]

then there is a \( \delta \in (0, 1) \) such that

\[
(\mathcal{L}^1(A))^{\frac{p\wedge q}{\tilde{p}\wedge q}} (\mathcal{L}^n(B))^{\frac{p\wedge q}{\tilde{p}\wedge q}} \leq C_{p,q}^{(\alpha)}(A \times B) \leq H^\beta_{\delta}(A \times B).
\]

(ii) Let \( K \) be a compact subset of \( \mathbb{R}_{1+n}^+ \). If

\[
\begin{align*}
1 \leq p < \infty; \\
1 < q < \infty; \\
\beta := (p \wedge q)(\frac{p + 2\alpha}{p} - 2\alpha) > 0;
\end{align*}
\]

then there is a \( \delta \in (0, 1) \) such that

\[ C_{p,q}^{(\alpha)}(K) \leq H^\beta_{\delta}(K), \]

and hence

\[ \dim_H^\alpha(\mathcal{B}[S_\alpha F; p, q]) \leq n - 2\alpha(p \wedge q - 1) \text{ provided } n - 2\alpha(p \wedge q - 1) > 0. \]

(iii) Let \( K \) be a compact subset of \( \mathbb{R}_{1+n}^+ \). If

\[
\begin{align*}
1 \leq p < \infty; \\
1 < q < \infty; \\
\frac{p}{p} + \frac{2\alpha}{q} - 2\alpha = 0; \\
\phi(r) := (\ln r)^{-1}(p\wedge q)(\frac{1}{p} - \frac{1}{q}), \forall r \in \mathbb{R}_+; \\
\ln t := \max\{0, \ln t\}, \forall t \in \mathbb{R}_+,
\end{align*}
\]

then there is a \( \delta \in (0, 1) \) such that

\[ C_{p,q}^{(\alpha)}(K) \leq H^\phi_{\delta}(K), \]

where \( \phi(r) \) is the capacity of the diagonal case \( p = q \).
and hence

\[ H^{\phi_k}\left(\mathcal{B}[S_{\alpha}F; p, q]\right) = 0 \text{ provided } \begin{cases} n - 2\alpha(p \land q - 1) = 0; \\ \phi_k(r) := \left(\log\frac{1}{r}\right)^{-p/q - \epsilon}, \forall r \in \mathbb{R}_+; \\ \epsilon > 0. \end{cases} \]

2. Basics of the \((\alpha, p, q)\)-capacity

In order to demonstrate Theorems 1.2, 1.3 and Corollary 1.4 we need to know some basic facts on the \((\alpha, p, q)\)-capacity.

2.1. Duality of the \((\alpha, p, q)\)-capacity. To establish the adjoint formulation of \(C_{p,q}^{(\alpha)}\), we need to find out adjoint operator \(S_{\alpha}^*\) of \(S_{\alpha}\). Note that for any \(F, G \in C_0^{\infty}(\mathbb{R}_+^{1+n})\) one has

\[
\int_{\mathbb{R}_+^{1+n}} S_{\alpha}^* G(t,x) dx dt = \int_{\mathbb{R}_+^{1+n}} F(t,x) \left( \int_0^\infty e^{-(s-t)(-\Delta)^\alpha} G(s,x) ds \right) dx dt.
\]

Thus, \(S_{\alpha}^* G\) is given by setting, for all \((t,x) \in \mathbb{R}_+^{1+n}\),

\[
(S_{\alpha}^* G)(t,x) := \int_0^\infty e^{-(s-t)(-\Delta)^\alpha} G(s,x) ds, \forall G \in C_0^{\infty}(\mathbb{R}_+^{1+n}).
\]

The definition of \(S_{\alpha}^*\) can be extended to the family of Borel measures \(\mu\) with compact support in \(\mathbb{R}_+^{1+n}\). In fact, note that if \(F\) is continuous and has a compact support in \(\mathbb{R}_+^{1+n}\) and \(\|\mu\|_1\) stands for the total variation of \(\mu\) then a simple calculation with the equivalent estimate

\[
K_{\alpha}^{(\theta)}(x) \approx \theta^{1/(2\alpha)} |x|^{-n-2\alpha}, \forall (t,x) \in \mathbb{R}_+^{1+n},
\]

gives

\[
\left| \int_{\mathbb{R}_+^{1+n}} S_{\alpha} F d\mu \right| \leq \|\mu\|_1 \sup_{(t,x) \in \mathbb{R}_+^{1+n}} |F(t,x)|.
\]

Hence an application of the Riesz representation theorem yields a Borel measure \(\nu\) on \(\mathbb{R}_+^{1+n}\) such that

\[
\int_{\mathbb{R}_+^{1+n}} S_{\alpha} F d\mu = \int_{\mathbb{R}_+^{1+n}} F d\nu.
\]

This indicates that \(S_{\alpha}^* \mu\) may be defined by \(\nu\).

The above analysis leads to a dual description of the \((\alpha, p, q)\)-capacity.

**Proposition 2.1.** For a compact subset \(K\) of \(\mathbb{R}_+^{1+n}\) let \(\mathcal{M}_0(K)\) be the class of all positive measures on \(\mathbb{R}_+^{1+n}\) supported by \(K\). If

\[
\begin{cases} 1 < p, q < \infty; \\ p' = p/(p - 1); \\ q' = q/(q - 1), \end{cases}
\]

then \(S_{\alpha}^* \mu k = \mathcal{C}^\infty_{p,q}\).
then
\[ C_{p,q}^{(\alpha)}(K) = \sup \{ \| \mu \|_{1}^{p/q} : \mu \in \mathcal{M}_{+}(K) \text{ and } \| S_{\alpha}^{*} \mu \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})} \leq 1 \} =: \tilde{C}_{p,q}^{(\alpha)}(K). \]

**Proof.** Since
\[
\| \mu \|_{1} = \mu(K) \\
\leq \int_{\mathbb{R}_{+}^{1+n}} S_{\alpha} F \, d\mu \\
= \int_{\mathbb{R}_{+}^{1+n}} F S_{\alpha} \mu \, dx dt \\
\leq \| F \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})} \| S_{\alpha} \mu \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})},
\]
on one has
\[
\tilde{C}_{p,q}^{(\alpha)}(K) \leq C_{p,q}^{(\alpha)}(K)
\]
for any compact set \( K \subset \mathbb{R}_{+}^{1+n} \). Moreover, this last inequality is actually an equality - in fact, if
\[
X = \{ \mu : \mu \in \mathcal{M}_{+}(K) \text{ and } \mu(K) = 1 \}; \\
Y = \{ F : 0 \leq F \in L_{p}^{q}(\mathbb{R}_{+}^{1+n}) \text{ and } \| F \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})} \leq 1 \}; \\
Z = \{ F : 0 \leq F \in L_{p}^{q}(\mathbb{R}_{+}^{1+n}) \text{ and } S_{\alpha} F \geq 1_{K} \}; \\
E(\mu, F) = \int_{\mathbb{R}_{+}^{1+n}} (S_{\alpha}^{*} \mu) F \, dx dt = \int_{\mathbb{R}_{+}^{1+n}} S_{\alpha} F \, d\mu,
\]
then an easy computation, along with an application of [2, Theorem 2.4.1], gives
\[
\min_{\mu \in \mathcal{M}_{+}(K)} \frac{\| S_{\alpha}^{*} \mu \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})}}{\mu(K)} = \min_{\mu \in X} \sup_{F \in Y} E(\mu, F) \\
= \sup_{F \in Y} \min_{\mu \in X} E(\mu, F) \\
= \sup_{F \in Z} \min_{F \in L_{p}^{q}(\mathbb{R}_{+}^{1+n})} \| F \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})}^{-1} \\
= \sup_{F \in Z} \min_{F \in L_{p}^{q}(\mathbb{R}_{+}^{1+n})} \| F \|_{L_{p}^{q}(\mathbb{R}_{+}^{1+n})}^{-1} \\
= (C_{p,q}^{(\alpha)}(K))^{-1/q},
\]
and hence
\[
\tilde{C}_{p,q}^{(\alpha)}(K) \geq C_{p,q}^{(\alpha)}(K),
\]
thereby the desired equality follows. \( \square \)
2.2. **Essentialness of the** $(\alpha, p, q)$-**capacity.** Some fundamental properties of the $(\alpha, p, q)$-capacity are stated in the following proposition.

**Proposition 2.2.**  
(i) $C^{(\alpha)}_{p,q}(\emptyset) = 0$. Moreover, under $\emptyset \neq K \subset \mathbb{R}^{1+n}$, $C^{(\alpha)}_{p,q}(K) = 0$ if and only if there exists $0 \leq F \in L^q_\alpha L^p(\mathbb{R}^{1+n})$ such that 

$$K \subseteq \{(t, x) \in \mathbb{R}^{1+n} : S_aF(t, x) = \infty\}.$$  

(ii) $K_1 \subseteq K_2 \subset \mathbb{R}^{1+n} \implies C^{(\alpha)}_{p,q}(K_1) \leq C^{(\alpha)}_{p,q}(K_2)$.  

(iii) 

$$C^{(\alpha)}_{p,q}\left(\bigcup_{j=1}^{\infty} K_j\right) \leq \sum_{j=1}^{\infty} C^{(\alpha)}_{p,q}(K_j)$$  

for any sequence $(K_j)_{j=1}^{\infty}$ of subsets of $\mathbb{R}^{1+n}$.  

(iv) $C^{(\alpha)}_{p,q}(K + (0, x_0)) = C^{(\alpha)}_{p,q}(K)$ for any $K \subset \mathbb{R}^{n+1}$ and any $x_0 \in \mathbb{R}^n$.

**Proof.** (i) Only the ‘iff’ part needs an argument. To do so, note that for $\lambda > 0$ the inequality

$$C^{(\alpha)}_{p,q}\{(t, x) \in \mathbb{R}^{1+n} : F \geq 0 \& S_aF(t, x) \geq \lambda\} \leq \lambda^{-p/q}||F||^{p/q}_{L^q_\alpha L^p(\mathbb{R}^{1+n})}$$

follows from the definition of $C^{(\alpha)}_{p,q}$. Clearly, this implies

$$C^{(\alpha)}_{p,q}(\mathcal{B}[S_aF; p, q]) = 0.$$  

Therefore, if $0 \leq F \in L^q_\alpha L^p(\mathbb{R}^{1+n})$ enjoys $K \subset \mathcal{B}[S_aF; p, q]$, then $C^{(\alpha)}_{p,q}(K) = 0$ follows from (ii) - the monotonicity of capacity.

Conversely, if $C^{(\alpha)}_{p,q}(K) = 0$ then taking nonnegative functions $F_j$ such that

$$\begin{cases} 
S_aF_j(t, x) \geq 1, \ \forall(t, x) \in K \\
||F_j||_{L^q_\alpha L^p(\mathbb{R}^{1+n})} < 2^{-j} 
\end{cases}$$

derives that $F = \sum_{j=1}^{\infty} F_j$ enjoys the required properties.

(ii) This follows from the definition of $(\alpha, p, q)$-capacity.

(iii) The forthcoming argument is standard; see also \[1, 16, 3\].

**Case 1:** $p \geq q$. If we choose $F_j$ with $S_aF_j \geq 1$ on $K_j$, then $F = \sup_{j=1,2,3,...} F_j$ satisfies $S_aF \geq 1$ on $\bigcup_{j=1}^{\infty} K_j$ and

$$||F||^{p}_{L^q_\alpha L^p(\mathbb{R}^{1+n})} \leq \int_{0}^{\infty} \left(\sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |F_j|^p dx\right)^{\frac{2}{p}} dt \leq \sum_{j=1}^{\infty} \int_{0}^{\infty} \left(\int_{\mathbb{R}^n} |F_j|^p dx\right)^{\frac{2}{p}} dt.$$  

So, the desired inequality follows.
Case 2: $p < q$. Now, the Minkowski inequality implies that
\[
\|F\|_{L^q_tL^p_x(\mathbb{R}^{1+n})}^p \leq \left[ \int_0^\infty \left( \sum_{j=1}^\infty \int_{\mathbb{R}^n} |F_j|^p dx \right)^{\frac{q}{p}} dt \right]^{\frac{1}{q}}.
\]
whence deducing the desired inequality.

(iv) This is a consequence of the following implication:
\[
F_{x_0}(t, x) = F(t, x + x_0) \implies \|F_{x_0}\|_{L^q_tL^p_x(\mathbb{R}^{1+n})} = \|F\|_{L^q_tL^p_x(\mathbb{R}^{1+n})},
\]
which completes the proof of Proposition 2.2.

3. Proofs of Theorems 1.2, 1.3 and Corollary 1.4

Now, we are ready to carry out the task as just mentioned in the title of Section 3.

3.1. Proof of Theorem 1.2  
(i) Let
\[
\begin{cases}
(t, x) \in \mathbb{R}^{1+n}, \\
(t_0, x_0) \in \mathbb{R}^{1+n}, \\
f \in L^p(\mathbb{R}^n), \\
p \in [1, \infty), \\
0 \leq t_1 < t_2 < \infty.
\end{cases}
\]

Since $K^{(\alpha)}_0(\cdot)$ is of $C^\infty(\mathbb{R}^n)$, one has that $R_{\alpha}f(t_0, x) = e^{-t_0(-\Delta)^\alpha}f(x)$ is of $C^\infty(\mathbb{R}^n)$ too. Meanwhile, for $x \in \mathbb{R}^n$ one gets
\[
R_{\alpha}f(t_1, x) - R_{\alpha}f(t_2, x) = \int_{t_1}^{t_2} (-\Delta)^\alpha e^{-t(-\Delta)^\alpha}f(x) \, dt.
\]
Note that the kernel $\hat{K}^{(\alpha)}_t(\cdot)$ of $(-\Delta)^\alpha e^{-t(-\Delta)^\alpha}$ obeys $|\hat{K}^{(\alpha)}_t(x)| \leq (t + |x|)^{-(n-2\alpha)}$; see also [8 Lemma 2.2 & (2.5)]. So, an application of [8 Lemma 3.1] gives
\[
\left\| (-\Delta)^\alpha e^{-t(-\Delta)^\alpha}f \right\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases}
t^{-1-\frac{n}{2p}} \|f\|_{L^p(\mathbb{R}^n)} & \text{for } p \in [1, \infty), \\
\|f\|_{L^\infty(\mathbb{R}^n)} & \text{for } p = \infty,
\end{cases}
\]
and hence
\[
|R_{\alpha}f(t_1, x) - R_{\alpha}f(t_2, x)| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \begin{cases}
t_1^{-\frac{n}{2p}} - t_2^{-\frac{n}{2p}} & \text{for } p \in [1, \infty), \\
\ln t_1 - \ln t_2 & \text{for } p = \infty.
\end{cases}
\]
Putting the above facts together yields
\[
|R_{\alpha}f(t, x) - R_{\alpha}f(t_0, x_0)| \leq |R_{\alpha}f(t_0, x) - R_{\alpha}f(t_0, x_0)| + |R_{\alpha}f(t, x) - R_{\alpha}f(t_0, x)|
\]

Therefore $R_\alpha f$ is of $C(\mathbb{R}^{1+n}_+)$.

(ii) Let $(t, x) \in \mathbb{R}^{1+n}_+$ be fixed. Then we have

$$|S_\alpha F(t, x)| \leq \int_0^\infty \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y)|F(s, y)| dy \, ds = I + II,$$

where

$$I := \int_0^\infty \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y)|F(s, y)| dy \, ds;$$

$$II := \int_0^\infty \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y)|F(s, y)| dy \, ds.$$

From the Hölder inequality and the assumption $\frac{2}{p} + \frac{2\alpha}{q} = 2\alpha$ it follows that

$$I \lesssim \int_0^\infty \left( \int_{\mathbb{R}^n} \frac{|t-s|}{|t-s|^{\frac{n}{p}} + |x-y|^{n+2\alpha}} |F(s, y)| dy \right)^{\frac{p}{p-1}} ds$$

$$\lesssim \int_0^\infty \left( \int_{\mathbb{R}^n} \frac{ds}{|t-s|^{\frac{n}{p}} + |x-y|^{n+2\alpha}} \right)^{\frac{p}{p-1}} ds$$

$$\lesssim \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n}_+)} \left( \int_0^\infty \frac{ds}{|t-s|^{\frac{n}{p}} + |x-y|^{n+2\alpha}} \right)^{\frac{p-1}{p}}$$

$$\lesssim \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n}_+)} \left( \ln \frac{t-r}{t} \right)^{\frac{n-1}{n}}.$$

Similarly, by using $M_\mathbb{R}$ - the Hardy-Littlewood maximal function on $\mathbb{R}$, we obtain

$$II \lesssim \int_r^\infty \int_{\mathbb{R}^n} \frac{|t-s|}{|t-s|^{\frac{n}{p}} + |x-y|^{n+2\alpha}} |F(s, y)| dy \, ds$$

$$\lesssim \int_r^\infty \left( \int_{\mathbb{R}^n} \frac{ds}{|t-s|^{\frac{n}{p}} + |x-y|^{n+2\alpha}} \right)^{\frac{p}{p-1}} ds$$

$$\lesssim \sum_{k=0}^{\infty} \left( \int_{t-2^k|r|}^{t-2^{k+1}|r|} \frac{1}{|t-s|^{\frac{n}{p}}} ds \right)^{\frac{p}{p-1}}$$

$$\lesssim \sum_{k=0}^{\infty} \left( 2^k |t-r|^{1-\frac{n}{p}} M_\mathbb{R}(\|F(\cdot, \cdot)\|_{L^p_x(\mathbb{R}^n)})(t) \right)$$

$$\lesssim |t-r|^{1/q} M_\mathbb{R}(\|F(\cdot, \cdot)\|_{L^p_x(\mathbb{R}^n)})(t).$$
Via choosing $r \in (0, t)$ such that

$$|t - t|^{1/q} = \min \left\{ t^{1/q}, \frac{\|F\|_{L^q_tL^\infty_x(\mathbb{R}^n)}}{M_\mathbb{R}(\|F(\cdot, \cdot)\|_{L^\infty_x(\mathbb{R}^n)})(t)} \right\},$$

we see that

$$|S_\alpha F(t, x)| \leq \|F\|_{L^q_tL^\infty_x(\mathbb{R}^n)} \max \left\{ 1, \left[ \ln \frac{et^{1/q}M_\mathbb{R}(\|F\|_{L^\infty_x(\mathbb{R}^n)})(t)}{\|F\|_{L^q_tL^\infty_x(\mathbb{R}^n)}} \right]^{\frac{q-1}{q}} \right\}.$$

Letting $r_0 = t_0^{\frac{1}{2q}}$ yields a constant $C > 0$ such that

$$\int_{B^{(a)}(t_0, r_0)} \exp \left( \frac{S_\alpha F(t, x)}{C\|F\|_{L^q_tL^\infty_x(\mathbb{R}^n)}} \right)^{\frac{q^2}{q+1}} dx dt$$

$$\leq \int_{B^{(a)}(t_0, r_0)} \frac{et^{1/q}M_\mathbb{R}(\|F\|_{L^\infty_x(\mathbb{R}^n)})(t)}{\|F\|_{L^q_tL^\infty_x(\mathbb{R}^n)}} dx dt$$

$$\leq r_0^{1/4} \int_0^{2t_0} M_\mathbb{R}(\|F\|_{L^\infty_x(\mathbb{R}^n)})(t) dt$$

$$\leq t_0^{1/4} r_0^{2\alpha - 2q/2}$$

$$\approx |B^{(a)}(t_0, x_0)|,$$

which completes the proof of (ii).

(iii) Given a point $(t_0, x_0) \in \mathbb{R}^{1+n}$, let $x \in \mathbb{R}^n$ be sufficient close to $x_0$ and $\delta = |x - x_0|$. Then

$$|S_\alpha F(t_0, x_0) - S_\alpha F(t_0, x)|$$

$$\leq \int_0^{t_0} \int_{\mathbb{R}^n} |K^{(a)}_{t_0-s}(x_0 - y) - K^{(a)}_{t_0-s}(x - y)| |F(y, s)| dy ds$$

$$\leq \int_0^{t_0} \int_{B(x_0, 3\delta)} \cdots dy ds + \int_0^{t_0} \int_{\mathbb{R}^n \setminus B(x_0, 3\delta)} \cdots dy ds$$

$$=: I + II.$$

Note that

$$\int_0^{t_0} \int_{B(x_0, 3\delta)} K^{(a)}_{t_0-s}(x_0 - y)|F(y, s)| dy ds$$

$$\leq \int_0^{t_0-(2\delta)^{2\alpha}} \int_{B(x_0, 3\delta)} \left( \frac{|t - s|}{|t - s|^{1+\frac{1}{2\alpha}}} \right) |F(y, s)| dy ds$$

$$+ \int_{t_0-(2\delta)^{2\alpha}}^{t} \int_{B(x_0, 3\delta)} \left( \frac{|t - s|}{(|t - s|^{1/2\alpha} + |x_0 - y|)^{n+\alpha}} \right) |F(y, s)| dy ds$$
To estimate the second term II, notice that

$$\int_0^{t} \frac{\|F(\cdot, s)\|_{L_t^p(\mathbb{R}^n)} ds}{|t-s|^{\frac{n}{2p}}} \lesssim \int_0^{t} \frac{ds}{|t-s|^{\frac{n}{2p}}} \lesssim \int_0^{t} \frac{ds}{|t-s|^{\frac{n}{2p+1}}} \lesssim \frac{1}{|t-s|^{\frac{n}{2p+1}}} \cdot$$

Thus the first term I is bounded from above as

$$I \leq \int_0^{\delta_0-(2\delta)^{2a}} \int_{B(x_0, \delta_0)} K^{(a)}_{K_0} (x_0 - y) \|F(y, s)\| dy ds$$

$$+ \int_0^{\delta_0-(2\delta)^{2a}} \int_{B(x, \delta_0)} K^{(a)}_{K_0} (x - y) \|F(y, s)\| dy ds$$

$$\leq \|F\|_{L_t^q L_r^r(\mathbb{R}^{1+n})} \frac{2a(q-1)}{q} \delta \frac{n}{q} + \frac{2a(q-1)}{q} \frac{n}{p} \cdot$$

To estimate the second term II, notice that

$$|\nabla K_1^{(a)}(x)| \lesssim (1 + |x|)^{-n-1};$$

see also [S, Remark 2.1]. Using this and the Hölder inequality, we have

$$II \leq \int_0^{\delta_0-(2\delta)^{2a}} \int_{\mathbb{R}^n \setminus B(x_0, \delta_0)} \|K^{(a)}_{K_0} (x_0 - y) - K^{(a)}_{K_0} (x - y)\| \|F(y, s)\| dy ds$$

$$\leq \int_0^{\delta_0-(2\delta)^{2a}} \int_{\mathbb{R}^n \setminus B(x_0, \delta_0)} \left( \frac{\delta}{|t-s|^{\frac{n}{2p}}} \left( \frac{|t-s|^{\frac{1}{2}}}{|t-s|^{\frac{n}{2p}} + |x_0 - y|^{n+1}} \right) \|F(y, s)\| dy ds$$

$$+ \int_0^{\delta_0-(2\delta)^{2a}} \int_{\mathbb{R}^n \setminus B(x_0, \delta_0)} \left( \frac{|x_0 - y|^{n+1}}{\delta} \|F(y, s)\| dy ds$$

$$\leq \int_0^{\delta_0-(2\delta)^{2a}} \frac{\delta}{|t-s|^{\frac{n}{2p}}} \left( \frac{|t-s|^{\frac{1}{2}}}{|t-s|^{\frac{n}{2p}} + |x_0 - y|^{n+1}} \right) \|F(y, s)\| dy ds$$

$$\leq \int_0^{\delta_0-(2\delta)^{2a}} \frac{\delta}{|t-s|^{\frac{n}{2p}} + \frac{1}{2} |t-s|^{\frac{n}{2p}}} ds$$

$$+ \int_0^{\delta_0-(2\delta)^{2a}} \|F(\cdot, s)\|_{L_t^p(\mathbb{R}^n)} ds$$
\[ \|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} \|x - x_0\|^{\frac{2(q-1)}{q} - \frac{n}{p}}. \]

Thus, we conclude that
\[ |S_a F(t_0, x_0) - S_a F(t_0, x)| \leq \|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} \|x - x_0\|^{\frac{2(q-1)}{q} - \frac{n}{p}}. \]

Let \((x, t_1), (x, t_2) \in \mathbb{R}^{1+n}_+.\) Without loss of generality we may assume \(t_1 > t_2,\) and then write
\[ |S_a F(t_1, x) - S_a F(t_2, x)| \leq \int_{t_2}^{t_1} \left| (e^{-(t_1-s)(-\Delta)^{\mu}} - e^{-(t_2-s)(-\Delta)^{\mu}}) F(x, s) \right| \, ds \]
\[ + \int_{t_2}^{t_1} \left| e^{-(t_1-s)(-\Delta)^{\mu}} F(x, s) \right| \, ds \]
\[ =: \text{III} + \text{IV}. \]

By using the mapping property of the semigroup, we obtain
\[ \text{III} \leq \int_0^1 \int_{t_2}^{t_1} |(-\Delta)^{\mu} e^{-r(-\Delta)^{\mu}} F(x, s)| \, dr \, ds \]
\[ \leq \int_0^1 \int_{t_2}^{t_1} r^{-1 - \frac{m}{2p}} \|F(\cdot, s)\|_{L^p_x(\mathbb{R}^n)} \, dr \, ds \]
\[ \leq \int_0^1 \int_{t_2}^{t_1} (t_2 - s + r)^{-1 - \frac{m}{2p}} \|F(\cdot, s)\|_{L^p_x(\mathbb{R}^n)} \, ds \]
\[ \leq \int_0^1 \int_{t_2}^{t_1} (t_2 - s + r)^{-1 - \frac{m}{2p}} \|F(\cdot, s)\|_{L^p_x(\mathbb{R}^n)} \, ds \]
\[ \leq \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n})} \int_0^1 (t_2 - t_1)^{1 - \frac{1}{q} - \frac{m}{2p}} \, dr \]
\[ \leq |t_2 - t_1|^{1 - \frac{1}{q} - \frac{m}{2p}} \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n})}, \]

and
\[ \text{IV} \leq \int_{t_2}^{t_1} (t_1 - s)^{-\frac{m}{2p}} \|F(s, \cdot)\|_{L^p_x(\mathbb{R}^n)} \, ds \]
\[ \leq |t_2 - t_1|^{1 - \frac{1}{q} - \frac{m}{2p}} \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n})}. \]

Hence
\[ |S_a F(t_1, x) - S_a F(t_2, x)| \leq |t_2 - t_1|^{1 - \frac{1}{q} - \frac{m}{2p}} \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n})}. \]

The difference estimates on \(S_a\) give us that if \((t, x)\) is close to \((t_0, x_0)\) then
\[ |S_a F(t, x) - S_a F(t_0, x_0)| \]
\[ \leq |S_a F(t, x) - S_a F(t_0, x)| + |S_a F(t_0, x) - S_a F(t_0, x_0)| \]
\[ \leq \left( |t - t_0|^{1 - \frac{1}{q} - \frac{m}{2p}} + |x - x_0|^{\frac{2(q-1)}{q} - \frac{n}{p}} \right) \|F\|_{L^p_t L^p_x(\mathbb{R}^{1+n})}, \]
which completes the proof of (iii).

3.2. **Proof of Theorem 1.3** (i) In the sequel, let \( \beta = (p \wedge q)(\frac{2}{p} + \frac{2\alpha}{q} - 2\alpha) \).
Also, assume that \( F \geq 0 \) satisfies \( S_\alpha F \geq 1_{B^\alpha_{r_0}(0,0)} \). Then, according to the definition of operator \( S_\alpha \), the following transform
\[
\begin{cases}
s = \frac{r_0}{r}; \\
y = \frac{s}{r_0}; \\
F_{r_0}(s, y) = F(r_0^\alpha s, r_0y); \\
G(s, y) = r_0^{2\alpha} F_{r_0}(s, y),
\end{cases}
\]
extends the property \( S_\alpha G \geq 1_{B^\alpha_{r_0}(0,0)} \). Thus,
\[
C^{(\alpha)}_{p,q}(B_1^{(\alpha)}(0,0)) \leq \|r_0^{2\alpha} F_{r_0}\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} = r_0^{-\beta} \|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}.
\]
This implies that
\[
C^{(\alpha)}_{p,q}(B_1^{(\alpha)}(0,0)) \leq r_0^{-\beta} C^{(\alpha)}_{p,q}(B_0^{(\alpha)}(0,0)).
\]
In fact, the last inequality is an equality since changing the order of \( B_1^{(\alpha)}(0,0) \) and \( B_0^{(\alpha)}(0,0) \) derives
\[
C^{(\alpha)}_{p,q}(B_0^{(\alpha)}(0,0)) \leq r_0^{\beta} C^{(\alpha)}_{p,q}(B_1^{(\alpha)}(0,0)).
\]
Next, we consider the desired equivalence estimate. If \( F \geq 0 \) and \( S_\alpha F \geq 1_{B^\alpha_{r_0}(0,0)} \), then, for \( 1 \leq p < \infty \) and \( 1 < q < \infty \), there exist \( \bar{p} \) and \( \bar{q} \) such that
\[
\begin{cases}
1 \leq p < \bar{p} < \infty; \\
1 < q < \bar{q} < \infty; \\
\left(\frac{1}{\bar{q}} - \frac{1}{q}\right) + \frac{2\alpha}{\bar{q}}\left(\frac{1}{p} - \frac{1}{\bar{p}}\right) = 1.
\end{cases}
\]
Consequently, according to Theorem 1.1(ii) we have
\[
\|S_\alpha F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} \leq \|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}.
\]
This, along with the definition of \( C^{(\alpha)}_{p,q}(\cdot) \), implies that
\[
r_0^{\beta} \leq C^{(\alpha)}_{p,q}(B_0^{(\alpha)}(t_0, x_0))
\]
thanks to
\[
\frac{n}{\bar{p}} + \frac{2\alpha}{\bar{q}} = \frac{n}{p} + \frac{2\alpha}{q} - 2\alpha.
\]
To get the corresponding upper bound of \( C^{(\alpha)}_{p,q}(B_0^{(\alpha)}(t_0, x_0)) \), we consider
\[
B_{r_0,q}^{(\alpha)}(t_0, x_0) := \{(t, x) \in \mathbb{R}^{1+n} : |t - t_0| < \eta r_0 \quad \& \quad |x - x_0| < r_0\}.
\]
for some sufficiently large \( \eta > 0 \) which will be determined later. Note that 
\((t, x) \in B^{(\alpha)}(t_0, x_0)\) ensures 
\[
S_\alpha 1_{B^{(\alpha)}_{(t_0, x_0)}}(t, x) \\
= \int_0^t \int_{\mathbb{R}^n} K^{(\alpha)}(x-y) 1_{B^{(\alpha)}_{(t_0, x_0)}}(s, y) \, dy \, ds
\]
\[
= \int_{(0,t] \cap |x| < (2r_0)^{2\alpha}} \int_{|y| < r_0} K^{(\alpha)}(x-y) \, dy \, ds
\]
\[
\geq \int_{(0,t] \cap |x| < (2r_0)^{2\alpha}} \int_{|x-y| < \frac{2r_0 \sqrt{2} \alpha}{2} \cap |y| < r_0} K^{(\alpha)}(x-y) \, dy \, ds
\]
for sufficiently small \( r_0 > 0 \). According to [13, Proposition 1], there are positive constants \( \sigma \) and \( \kappa \), depending only \( n \) and \( \alpha \), such that 
\[
\inf \{ K^{(\alpha)}_t(x) : |x| \leq \sigma t^{\frac{1}{2\alpha}} \} \geq \kappa t^{-\frac{\alpha}{2\alpha}}. 
\]
Under 
\[
(t, x) \in B^{(\alpha)}_{(t_0, x_0)}; \\
|y - x_0| < r_0; \\
t - s > (\frac{2\alpha}{2} - 1) r_0^{2\alpha},
\]
one has 
\[
|x-y| \leq |x-x_0| + |y-x_0| < 2r_0 < 2 \left( \frac{2}{\eta^{2\alpha} - 1} \right) \frac{1}{\sigma} |t-s|^{\frac{1}{2\alpha}} < \sigma |t-s|^{\frac{1}{2\alpha}}
\]
for some large enough \( \eta \) with 
\[
2 \left( \frac{2}{\eta^{2\alpha} - 1} \right)^{\frac{1}{2\alpha}} < \sigma.
\]
Thus, one gets that if \( |t-t_0| < r_0^{2\alpha} \) then 
\[
S_\alpha 1_{B^{(\alpha)}_{(t_0, x_0)}}(t, x) \\
\geq \int_{(0,t] \cap |x| < (2r_0)^{2\alpha}} \int_{|x-y| < \frac{2r_0 \sqrt{2} \alpha}{2} \cap |y| < r_0} |t-s|^{\frac{1}{2\alpha}} \, dy \, ds
\]
\[
\geq c r_0^{2\alpha}
\]
holds for some constant \( c > 0 \) independent of \( r_0 \). Consequently, 
\[
S_\alpha \left( \frac{1_{B^{(\alpha)}_{(t_0, x_0)}}}{cr_0^{2\alpha}} \right)(t, x) \geq 1, \quad \forall (t, x) \in B^{(\alpha)}_{(t_0, x_0)}.
\]
This gives 
\[
C_{p,q}^{(\alpha)}(B^{(\alpha)}_{(t_0, x_0)}) \leq \left\| \frac{1_{B^{(\alpha)}_{(t_0, x_0)}}}{cr_0^{2\alpha}} \right\|_{L^{p,q}_0(\mathbb{R}^{1+\alpha})} \lesssim r_0^\beta.
\]
(ii) For an arbitrarily fixed point \((t_0, x_0) \in \mathbb{R}_+^{1+n}\). Let \(r_0 << \min\{t_0, 1\} \). Suppose that \(S\alpha F(t, x) \geq 1\) on \(B_{r_0}^{(\alpha)}(t_0, x_0)\). Then by Theorem 1.2(ii), we have a constant \(C > 0\) such that

\[
\int \int_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp \left( \frac{S\alpha F(t, x) \cdot \frac{x}{C\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}}} {\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}} \right) dx dt \\
\leq \int \int_{B_{r_0}^{(\alpha)}(t_0, x_0)} \frac{et^{1/q}M_{\mathbb{R}}(\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})})}{\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}} dx dt \\
\leq r_0^{n+2\alpha} + r_0^{\frac{n+1}{q}} \int_{t_0-r_0}^{t_0+t_0} \frac{M_{\mathbb{R}}(\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})})}{\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}} dt \\
\leq r_0^{\frac{n+2\alpha}{q}} r_0^{t_0^{n+2\alpha-2\alpha/q}}.
\]

On the other hand, as \(S\alpha F(\cdot, \cdot) \geq 1\) on \(B_{r_0}^{(\alpha)}(t_0, x_0)\), it follows that for a constant \(c > 0\),

\[
\int \int_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp \left( \frac{S\alpha F(t, x) \cdot \frac{x}{c(\ln \frac{1}{r_0})^{\frac{1}{a}}}} {\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})}} \right) dx dt \\
\geq r_0^{n+2\alpha} \exp \left( c^{-\frac{q}{p\alpha}} \ln \frac{1}{r_0} \right) \geq r_0^{n+2\alpha-c^{-\frac{q}{p\alpha}}},
\]

which implies that

\[
\|F\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} \geq \left( \ln \frac{1}{r_0} \right)^{\frac{1}{a}}
\]

and hence

\[
C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0)) \geq \left( \ln \frac{1}{r_0} \right)^{\frac{1}{q}(\rho/q)} \quad \text{as} \quad r_0 \to 0.
\]

Next, we prove the converse form of the last inequality. Let

\[E := \{(t, x) \in \mathbb{R}_+^{1+n} : (2r_0)^{2\alpha} < t_0 - t < (2r_0)^{\alpha} \& |t - t_0|^{\frac{1}{2\alpha}} < |x_0 - x| < 2\}.
\]

Define

\[F(x, t) := \begin{cases} 
\frac{1}{(t_0-x)^{2\alpha} + |x-x_0|^{2\alpha}} & \forall (t, x) \in E; \\
0 & \text{otherwise}.
\end{cases}
\]

By using the known estimate below

\[K^{(\alpha)}_t(x - y) \approx \frac{t}{(t + |x-y|^{2\alpha})},
\]

we see that for each \((t, x) \in B_{r_0}^{(\alpha)}(t_0, x_0)\),

\[
S\alpha F(t, x) = \int_0^t \int_{\mathbb{R}^n} K^{(\alpha)}_{t-s}(x - y)F(y, s) dy ds \\
\approx \int \int_E \frac{|t - s|}{(|t - s|^{2\alpha} + |x - y|^{n+2\alpha})} F(y, s) dy ds
\]
The above two estimates give

\[ \| F \|_{L^q_x L^r_y(\mathbb{R}^{1+n})}^{\frac{q}{p}} \leq \int_{t_0-(2r_0)^n}^{t_0-(2r_0)^n} \left( \int_{B(x_0, 2r_0)} \left| \log \left( \frac{1}{|t_0 - s|^{\frac{1}{4}} + |x_0 - y|^{2r_0}} \right) \right|^{q/p} dy \right)^{1/p} ds \]

Moreover, noticing that \( 2p\alpha > n \), we have

\[ \| F \|_{L^q_x L^r_y(\mathbb{R}^{1+n})}^{\frac{q}{p}} \leq \int_{t_0-(2r_0)^n}^{t_0-(2r_0)^n} \left( \int_{B(x_0, 2r_0)} \left| \log \left( \frac{1}{|t_0 - s|^{\frac{1}{4}} + |x_0 - y|^{2r_0}} \right) \right|^{q/p} dy \right)^{1/p} ds \]

The above two estimates give

\[ C_{\rho,q}(B^\alpha_{t_0}(t_0, x_0)) \leq \left\| \frac{F}{\ln \frac{1}{r_0}} \right\|_{L^q_x L^r_y(\mathbb{R}^{1+n})}^{\frac{q}{p}} \leq \left( \frac{1}{\ln \frac{1}{r_0}} \right)^{\frac{q}{p}} \]

as \( r_0 \to 0 \).

3.3. **Proof of Corollary 1.4** (i) This follows from Theorem 1.1(ii), Theorem 1.3(i) and Proposition 2.2 (iii).

(ii)-(iii) The comparison inequalities for the two capacities follow from Proposition 2.2(iii) and (i)-(ii) of Theorem 1.3. To get the dimension inequality, we firstly keep in mind the fact

\[ C_{\rho,q}(B^\alpha_{t_0}(t_0, x_0)) = 0 \quad \text{for} \quad 0 \leq F \in L^q_x L^r_y(\mathbb{R}^{1+n}), \]

and secondly recall Proposition 2.1 and the following Frostman type theorem (cf. [2, Theorem 5.1.12]): if \( \phi : [0, \infty) \to [0, \infty] \) increases with
\( \phi(0) = 0 \) then for a given compact \( K \subset \mathbb{R}^{1+n} \) there is a measure \( \mu \in \mathcal{M}^+ (K) \) obeying \( \mu(B_r^{(a)} (t, x)) \leq \phi(r) \) such that \( \mu(K) \approx H^{\phi}_{\infty} (K) \).

Now, let \( K \) be any compact subset of the blow-up set \( \mathcal{B}[S_aF; p, q] \) and be contained in a ball \( B_R^{(a)} (t_0, x_0) \). Taking \( 0 \leq G \in L_2^q L_2^q (\mathbb{R}^{1+n}) \) such that \( S_aG \geq 1 \) on \( K \), we use the dyadic decomposition of a set and the Hölder inequality to get that if \( 0 < R_0 < 1 \wedge R \) then

\[
\mu(K) \leq \int_K S_a G(t, x) \, d\mu(t, x)
\]

\[
\leq \int_{\mathbb{R}^{1+n}} G(s, y) \int_{K \cap ((-\infty, x) \times \mathbb{R}^n)} K_{t-s}^{(a)} (x - y) \, d\mu(t, x) \, dyds
\]

\[
\leq \int_{\mathbb{R}^{1+n}} G(s, y) \int_{K \cap ((-\infty, x) \times \mathbb{R}^n)} \frac{|t - s|}{|t - s|^{\frac{1}{p} + |x - y|^{n+2\alpha}}} \, d\mu(t, x) \, dyds
\]

\[
\leq \mu(K) \int_{\mathbb{R}^{1+n}} G(s, y) \left( \sum_{j=0}^{\infty} \frac{\mu(B_{R_0}^{(a)} (s, y))}{(2^{-j} R_0)^n} \right) \, dyds
\]

\[
\leq \mu(K) R_0^{2\alpha - \frac{2}{p} - \frac{2\alpha}{q}} \|G\|_{L_2^q L_2^q (\mathbb{R}^{1+n})}
\]

\[
+ \int_{B_{2R_0} (t_0, x_0)} G(s, y) \int_0^{R_0} \frac{\mu(B_{r}^{(a)} (s, y))}{r^{1+n}} \, dr \, dyds
\]

\[
\leq \mu(K) R_0^{2\alpha - \frac{2}{p} - \frac{2\alpha}{q}} \|G\|_{L_2^q L_2^q (\mathbb{R}^{1+n})}
\]

\[
+ \int_0^{R_0} \int_{B_{2R_0} (t_0, x_0)} G(s, y) \mu(B_{r}^{(a)} (s, y)) \, dyds \, \frac{dr}{r^{1+n}}
\]

For \( p \leq q \), we have

\[
\int_0^{R_0} \int_{B_{2R_0} (t_0, x_0)} G(s, y) \mu(B_{r}^{(a)} (s, y)) \, dyds \, \frac{dr}{r^{1+n}}
\]

\[
\leq \|G\|_{L_2^q L_2^q (\mathbb{R}^{1+n})} \int_0^{R_0} \left( \int_{B_{2R_0} (t_0, x_0)} \mu(B_{r}^{(a)} (s, y)) \frac{1}{r^{1/n}} \, dyds \right)^{\frac{q}{p}} \, \frac{dr}{r^{1+n}}
\]

\[
\leq R_0^{2\alpha - \frac{1}{p} - \frac{1}{q}} \|G\|_{L_2^q L_2^q (\mathbb{R}^{1+n})} \int_0^{R_0} \left( \int_{B_{2R_0} (t_0, x_0)} \mu(B_{r}^{(a)} (s, y)) \, dyds \right)^{\frac{q}{p}} \, \frac{\phi(r)^{\frac{1}{p}} \, dr}{r^{1+n}}
\]

\[
\leq R_0^{2\alpha - \frac{1}{p} - \frac{1}{q}} \|G\|_{L_2^q L_2^q (\mathbb{R}^{1+n})} \mu(K) \int_0^{R_0} \phi(r)^{\frac{1}{p}} \, r^{-1+2\alpha - \frac{2\alpha}{p}} \, dr.
\]
Meanwhile, for \( p > q \), it holds that

\[
\int_0^{R_0} \iint_{B_{2R_0}^c(0, x_0)} G(s, y) \mu(B_r^\alpha(s, y)) \, dy \, ds \, \frac{dr}{r^{1+n}}
\]

\[
\lesssim \|G\|_{L_p^bL_q^r((2R_0)_c(0, x_0))} \int_0^{R_0} \left( \iint_{B_{2R_0}^c(0, x_0)} \mu(B_r^\alpha(s, y)) \frac{dy}{s^{1+\alpha}} \right)^{\frac{1}{p-1}} \, dr \frac{dr}{r^{1+n}}
\]

\[
\lesssim R_0^{n\left(\frac{1}{q} - \frac{1}{p}\right)} \|G\|_{L_p^bL_q^r((R_0)_c)} \mu(K)^{\frac{1}{p-1}} \int_0^{R_0} \phi(r)^{\frac{1}{q}} r^{-1 + 2\alpha - \frac{n\alpha}{p+\sigma}} \, dr.
\]

The above estimates induce a constant \( c_0 := C(R_0, p, q, \alpha) > 0 \), depending on \( R_0 \) and \( p, q, \alpha \), such that

\[
\mu(K) \lesssim c_0 \|G\|_{L_p^bL_q^r((R_0)_c)} \left( \mu(K) + \mu(K)^{1-\frac{1}{p-\sigma}} \int_0^{R_0} \phi(r)^{\frac{1}{q}} r^{-1 + 2\alpha - \frac{n\alpha}{p+\sigma}} \, dr \right).
\]

Therefore, if

\[
\text{III} := \int_0^{R_0} \phi(r)^{\frac{1}{q}} r^{-1 + 2\alpha - \frac{n\alpha}{p+\sigma}} \, dr < \infty,
\]

then by the fact \( c_0^\alpha(K) = 0 \) it follows that \( \mu(K) = 0 \), and hence \( H_\alpha^{b, \alpha}(K) \approx \mu(K) = 0 \). This in turn implies \( H_\alpha^{b, \alpha}(K) = 0 \) thanks to

\[
H_\alpha^{b, \alpha}(\cdot) = 0 \iff H_\alpha^{b, \alpha}(\cdot) = 0.
\]

Consequently, \( H_\alpha^{b, \alpha}(\mathcal{B}[S_\alpha F; p, q]) = 0 \).

The remaining is to consider two situations as follows.

\textbf{Case 1:} \( n - 2\alpha(p \wedge q - 1) > 0 \). Under this condition, we choose

\[
\phi(r) := r^\eta, \ \forall r \in (0, \infty) \ & \ \eta > n - 2\alpha(p \wedge q - 1)
\]

to obtain \( \text{III} < \infty \), thereby reaching

\[
\dim_H^{(\alpha)}(\mathcal{B}[S_\alpha F; p, q]) \leq n - 2\alpha(p \wedge q - 1).
\]

\textbf{Case 2:} \( n - 2\alpha(p \wedge q - 1) = 0 \). Under this condition, we select

\[
\phi_\epsilon(r) := \left( \ln_+ \frac{1}{r} \right)^{-\eta_\epsilon}, \ \forall r \in (0, \infty) \ & \ \eta_\epsilon = p \wedge q + \epsilon > p \wedge q
\]

to ensure \( \text{III} < \infty \) and thus \( H^{b, \alpha}(\mathcal{B}[S_\alpha F; p, q]) = 0 \).
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