On Superstring Disk Amplitudes in a Rolling Tachyon Background

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Abstract

We study the tree level scattering or emission of $n$ closed superstrings from a decaying non-BPS brane in Type II superstring theory. We attempt to calculate generic $n$-point superstring disk amplitudes in the rolling tachyon background. We show that these can be written as infinite power series of Toeplitz determinants, related to expectation values of a periodic function in Circular Unitary Ensembles. Further analytical progress is possible in the special case of bulk-boundary disk amplitudes. These are interpreted as probability amplitudes for emission of a closed string with initial conditions perturbed by the addition of an open string vertex operator. This calculation has been performed previously in bosonic string theory, here we extend the analysis for superstrings. We obtain a result for the average energy of closed superstrings produced in the perturbed background.

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1 Introduction

One of the basic open questions in string theory is understanding the decay of unstable branes. Sen has proposed a CFT description for spatially homogenous decay by deforming the open string worldsheet theory by exactly marginal rolling tachyon backgrounds [1–3]. This process can be interpreted as a spacelike brane localized in time (full S-brane) [4]. An alternative, rescaled rolling tachyon background [5] corresponds to decay starting from past infinity (half S-brane). One can also consider brane decay on a space-time orbifold with a semi-infinite time direction, to obtain a model where the unstable brane is prepared at origin of time and then decays [6]. Basic questions such as computing amplitudes for scattering or emission of strings from decaying branes have turned out to lead into quite complicated calculations rendering it difficult to draw out lessons of physics interest. Several different approaches to this problem have been explored, such as timelike boundary Liouville theory [7] and matrix integrals [8]. In particular, for full S-branes, a prescription based upon analytic continuation to imaginary time where the full S-brane corresponds to an array of smeared D-branes, was proposed in [33]. Further references include [9–30], and the recent reviews [31, 32]. Recently, for half S-branes, this problem was elaborated and mapped into the study of random matrices [34]. In this paper we extend this approach to a study of superstring scattering from unstable branes in superstring theory. It would be interesting to compare the random matrix approach with that of [33].

The general setup is also interesting from the point of view of cosmology. Recently, there has been progress in constructing string theoretic models of inflation. Of particular motivational interest here are models based in Type IIB superstring theory, where inflation arises from interactions of branes in (single or multiple) warped throats [35],[36–39]. In these models, it has been proposed [36–39] that reheating after inflation is associated with KK modes of gravitons that are produced copiously as end decay products of massive closed strings emitted from decaying $D\bar{D}$-systems at the throats. However, the emission of massive closed strings is at present under calculational control only for production of single strings, see [13, 17, 18, 26]. In this paper we aim for progress in calculating closed string $n$-point disk amplitudes in the rolling tachyon background in superstring theory, that could be interpreted as probability amplitudes for multi-string emission. This is a very complicated problem, and we are able to make only partial progress.

One technique to organize these calculations is to map them to a computation in the language of random matrices: the amplitudes turn out to involve power series of expectation values of periodic functions in Circular Unitary Ensembles (CUEs) of $U(N)$ matrices of increasing rank. This was found in [34] in the context of two-point disk amplitudes in bosonic string theory; in this paper we generalize the observation for generic $n$-point disk amplitudes in superstring theory. Such expectation value calculations are a basic question in the theory of random matrices. However, for the particular periodic functions that arise in the calculations, the expectation values are only known\(^1\) as Toeplitz determinants of Fourier coefficients of the function. Further progress, needed for extracting physics lessons from the amplitudes, is then associated with new progress in the field of random matrix theory and mathematical analysis.

In the special case of bulk-boundary disk amplitudes, two-point functions of one bulk

\(^1\)As far as we are aware of.
and one boundary vertex operator, it is known that the calculations can be carried out to the point of actually finding corrections to the one-point amplitude in an analytic form. These results have been derived in bosonic string theory [34], and also in [16, 23] using Liouville theory methods. In this paper we will extend the calculations and results to the case of superstrings.

This paper is organized as follows. In section 2, we consider generic $n$-point superstring disk amplitudes and show how they are related to infinite power series of expectation values in CUEs, or Toeplitz determinants of increasing rank. In section 3, we calculate the bulk-boundary disk amplitudes in superstring theory. In section 4, we interpret the open string vertex operator as an additional initial perturbation on the decaying brane, and calculate how it corrects the average energy of the emitted closed strings in the decay. Finally, section 5 is a brief summary.

## 2 Generic Closed String Disk Amplitudes and Random Matrices

We begin by attempting to compute NS-NS and R-R disk amplitudes in the background of a decaying brane. Depending on the external momentum assignments, these could be interpreted as scattering or emission probability amplitudes. As in [34, 26], we focus on the $1/2$S-brane or rolling tachyon background, which for the non-BPS brane of Type II superstring corresponds to the exactly marginal deformation

$$\delta S_B = -\sqrt{2} \pi \int \frac{dt}{2\pi} \psi^0 e^{X^0/\sqrt{2}} \otimes \sigma_1,$$

where $\psi^0$ is the time component of the worldsheet fermion field and $\sigma_1$ is a Chan-Paton factor associated with the boundary tachyon, which can be related to the one-dimensional boundary fermion $\eta$ [5, 40, 41, 26]; see Appendix for an elaboration on this point. For the bulk closed string vertex operators $V_s$, one can adopt convenient gauge choices [42, 13] (see also [43, 26]), where the dependence on the time component $X^0$ of the bosonic field takes a simple form:

$$V_s = e^{i\omega_c X^0} V^\perp_s (X^i, \psi^i, \bar{\psi}^i, \ldots)$$ (2)

in the NS-NS sector, and

$$V_s = e^{i\omega_c X^0} \Theta_{s_0} \tilde{\Theta}_{s_0} V^\perp_s (X^i, \psi^i, \bar{\psi}^i, \ldots)$$ (3)

in the R-R sector, with the spin fields $\Theta_{s_0} = e^{i\sigma_0 H^0}$ in the bosonized form. (The ellipsis refers to ghosts and superconformal ghosts.) Thus, for a generic $n$-point closed string amplitude, the non-trivial part of the computation due to the presence of the rolling tachyon amounts to the expectation value

$$A_n(\omega_1, \ldots, \omega_n) \equiv \left\langle \prod_{a=1}^n e^{i\omega_a X^0(z_a, \bar{z}_a)} \right\rangle_{\text{deformed}} = e^{\sqrt{2} \pi \lambda \sigma_1 \int_{-\pi}^\pi \frac{dt}{2\pi} \psi^0 e^{X^0/\sqrt{2}}} \prod_{a=1}^n e^{i\omega_a X^0(z_a, \bar{z}_a)}$$ (4)

for vertex operators in the NS-NS sector, and

$$A_n(\omega_1, \ldots, \omega_n) \equiv \left\langle \prod_{a=1}^n e^{i\omega_a X^0(z_a, \bar{z}_a)} \Theta_{s_0}^{(a)} \tilde{\Theta}_{s_0}^{(a)} \right\rangle_{\text{deformed}}$$ (5)
in the R-R sector. Consider for example the NS-NS sector disk amplitude in more detail. Bosonizing the fermionic superpartner $\psi^0$ and expanding, we obtain (odd terms vanish, since $\text{Tr}[\sigma_i^0] = 0$, for $n = \text{odd}$)

\[
A_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{\infty} dx \, e^{ix0} \sum_{a=1}^{n} \omega_a \sum_{N=0}^{\infty} \frac{(\pi \lambda e^{\frac{\sqrt{\alpha}}{2}})^{2N}}{(2N)!} \mathcal{O}(\omega_1, \ldots, \omega_n),
\]

where

\[
\mathcal{O}(\omega_1, \ldots, \omega_n) = \int_{-\pi}^{\pi} \prod_{i=1}^{2N} \frac{dt_i}{2\pi} \left[ e^{iH(t_i)} - e^{-iH(t_i)} \right] e^{\frac{X'(t_i)}{\sqrt{\alpha}}} \prod_{a=1}^{n} e^{i\omega_a X'(z_a, \bar{z}_a)}
\]

and we have separated out the zero mode from the fluctuating part, $X^0 = x^0 + X'^0$, and further dropped the superscript 0. The Wick contractions in (7) are easily calculated and we obtain

\[
\mathcal{O}(\omega_1, \ldots, \omega_n) = \sum_{\{\epsilon_i\} = \pm} \int_{-\infty}^{\infty} \prod_{i=1}^{n} (\epsilon_i e^{i\lambda h}) \int_{-\pi}^{\pi} \prod_{i=1}^{2N} \frac{dt_i}{2\pi} \prod_{1 \leq i < j \leq 2N} |e^{it_i} - e^{it_j}|^{1 + \epsilon_i \epsilon_j} \cdot \prod_{i=1}^{2N} \prod_{a=1}^{n} \prod_{1 \leq a < b \leq n} |z_a - z_b|^{-\omega_a \omega_b} \prod_{a,b=1}^{n} |1 - z_a \bar{z}_b|^{-\frac{\omega_a \omega_b}{2}}
\]

where we have separated out the zero mode $h$ from $H$. The integral over it enforces a constraint

\[
\sum_{i=1}^{2N} \epsilon_i = 0.
\]

In the sum over $\epsilon_i = \pm$, all the combinations subject to the constraint contribute equally to (8), as can be seen by an appropriate relabeling of the $t_i$'s. Thus, we can choose

\[
\epsilon_1, \ldots, \epsilon_N = +1; \quad \epsilon_{N+1}, \ldots, \epsilon_{2N} = -1
\]

and count the number of all equivalent terms. This is a random walk problem, there are $(2N)!/(N!)^2$ such terms. The remaining integrals then factorize and we can write

\[
\mathcal{O}(\omega_1, \ldots, \omega_n) = (-1)^N (2N)! \prod_{1 \leq a < b \leq n} |z_a - z_b|^{-\omega_a \omega_b} \prod_{a,b=1}^{n} |1 - z_a \bar{z}_b|^{-\frac{1}{2} \omega_a \omega_b} I_N(\omega_1, \ldots, \omega_n),
\]

where $I_N$ is the integral

\[
I_N(\omega_1, \ldots, \omega_n) = \frac{1}{N!} \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{1 \leq i < j \leq N} |e^{it_i} - e^{it_j}|^2 \prod_{i=1}^{N} \prod_{a=1}^{n} |1 - z_a e^{-it_i} | |\sqrt{\omega_a} |.
\]

The study of this type of integrals is a central question in the theory of random matrices [44]. We can recognize it as the expectation value of a periodic function with respect to the Circular Unitary Ensemble of $U(N)$ matrices,

\[
I_N \equiv \mathbb{E}_{U(N)} \left\{ \prod_{i=1}^{N} f(t_i) \right\},
\]

\[
(13)
\]
where the periodic function is
\[ f(t) = \prod_{a=1}^{n} |1 - z_a e^{-it}|^{i \sqrt{2} \omega_a}. \]  

(14)

It contains the information about the locations (modular parameters) \( z_a \) of the closed string vertex operators and the on-shell energies \( \omega_a \). Alternatively, because of the factorization, we could have written the result as a \( U(N) \times U(N) \) integral as in [26],
\[ I^2_N = E_{U(N)} \left\{ \prod_{i=1}^{N} f(t_i) \right\} \cdot E_{U(N)} \left\{ \prod_{i=1}^{N} f(t_i) \right\} = E_{U(N) \times U(N)} \left\{ \prod_{i=1}^{2N} f(t_i) \right\}. \]  

(15)

The integrals (13) can then be evaluated by Heine’s identity [34, 45] and rewritten as Toeplitz determinants of the Fourier coefficients of \( f \),
\[ I_N = \text{det}(\hat{f}(k-l))_{1 \leq k,l \leq N} \equiv D_N[\hat{f}], \]  

(16)

where
\[ \hat{f}(k-l) = \int \frac{dt}{2\pi} f(t) e^{i(k-l)t}. \]  

(17)

Thus the amplitude becomes a Fourier transform of an infinite series of Toeplitz determinants,
\[ A_n(\omega_1, \ldots, \omega_n) = \prod_{1 \leq a < b \leq n} |z_a - z_b|^{-\omega_a \omega_b} \prod_{a,b=1}^{n} |1 - z_a \bar{z}_b|^{-\frac{1}{2} \omega_a \omega_b} \cdot \int_{-\infty}^{\infty} dx^0 e^{ix^0 \sum_{a=1}^{n} \omega_a} F(x^0; \omega_1, \ldots, \omega_n), \]  

(18)

where
\[ F(x^0; \omega_1, \ldots, \omega_n) = \sum_{N=0}^{\infty} (-\pi^2 \lambda^2 e^{\sqrt{2} x^0})^N (D_N[\hat{f}])^2. \]  

(19)

Unfortunately, the Toeplitz determinants are in general quite complicated so a more detailed analysis of the infinite series is extremely difficult. For example, the radius of convergence of (19) is difficult to determine. By physics reasons, we expect the infinite series to converge at least for sufficiently early times (as then the amplitude approaches that for scattering from a stable D-brane). It might also be possible to gain some further insight into the behavior of the series from numerical methods. However, in order to perform the Fourier transform in (18), one would need an analytic expression for (19) and then analytically continue beyond its expected convergence radius, a much harder task.

For R-R sector the story is a bit modified. The amplitude (5) becomes
\[ A_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{\infty} dx^0 e^{ix^0 \sum_{a=1}^{n} \omega_a} \sum_{N=0}^{\infty} \left( \frac{\pi \lambda e^{x^0}}{\sqrt{2}} \right)^N \frac{N!}{N!} O(\omega_1, \ldots, \omega_n) \text{Tr}(\sigma_1)^{N+n}, \]  

(20)

\[ \text{For example, in the case of a 2-point function the Fourier coefficients turn out to be related to Hypergeometric functions, see [34].} \]
where

\[ O(\omega_1, \ldots, \omega_n) = \int_{-\pi}^{\pi} \prod_{i=1}^{n} \frac{dt_i}{2\pi} \left( e^{iH(t_i)} - e^{-iH(t_i)} \right) e^{\frac{X'(t_i)}{\sqrt{2}}} \prod_{a=1}^{n} e^{i\omega_a X'(z_a, \bar{z}_a)} e^{i\bar{s}_a H(z_a)} e^{i\bar{s}_a H(\bar{z}_a)} \right) \]  

(21)

In the R-R sector one has to explain why the amplitude with a single R-R vertex operator in the bulk is non-vanishing for an odd number of insertion of boundary tachyon vertex operators \([46–48]\). From (1), it is clear that the vertex operator of the tachyon contains the Chan-Paton matrix \(\sigma_1\). The Chan-Paton Hilbert space is two-dimensional, even for a single non-BPS D-brane, since a non-BPS \(D_p\)-brane of Type IIA(B) theory can be thought of as a bound state of a \(D_p-D_{\bar{p}}\)-pair of Type IIB(A) theory. For an odd number of tachyon vertex operator insertions on the boundary, naively we expect that the amplitude vanishes because of the presence of the factor \(\text{Tr}[\sigma_1^{2n+1}] = 0\), \(n = +\text{ve integer}\). However, this is not the full story, at least for bulk-boundary amplitudes involving R-R sector (which will be considered in section 3).

For concreteness, let us suppose we are considering a non-BPS \(D_p\)-brane in Type IIA theory (so \(p\) is odd). It is obtained by taking a \(D_p-D_{\bar{p}}\)-brane pair in Type IIB and modding it out by \((-1)^{F_L}\), where \(F_L\) is the left-moving spacetime fermion number. The R-R and R-NS sectors of Type IIA can be thought of as ‘twisted sector’ states under \((-1)^{F_L}\) orbifold in Type IIB theory. For diagrams involving R-R operators it is easier if we stick to Type IIB orbifold rather than Type IIA language. The operator \((-1)^{F_L}\) does not act on the matter or ghost part of any open string vertex operator, but it has an action on the \(2 \times 2\) CP Hilbert space. Since under its action a BPS \(D_p\)-brane gets exchanged with a \(\bar{D}_{\bar{p}}\)-brane, the representation of \((-1)^{F_L}\) in the CP Hilbert space is \(\sigma_1\).

So a Type IIA disk diagram with some \(N\) number of boundary tachyon vertex operators and a R-R vertex operator inserted in the bulk, from Type IIB orbifold perspective, is equivalent to a disk diagram with a cut, associated with the \((-1)^{F_L}\) operator, ending on the boundary. Due to above representation of \((-1)^{F_L}\) in the CP Hilbert space, the trace part in the full amplitude gets another factor of \(\sigma_1\), where the cut hits the boundary, \(i.e.,\) now the trace from the CP sector is \(\text{Tr}[\sigma_1^{N+1}]\). This is non-vanishing only when \(N = \text{odd}\). It is straightforward to extend this procedure for \(n\) insertions of R-R vertex operators in the bulk. The amplitude will then be non-vanishing iff \((N + n) = \text{even}\). Thus, if \(N\) and \(n\) are \(\text{even}\) integers separately, the amplitude is still non-vanishing.\(^\text{3}\)

For a correlation function in (21) with \(n\) number of bulk R-R and \(N\) of boundary tachyon operator insertions, the zero mode integral from the temporal part imposes a constraint

\[ \sum_{i=1}^{N} \epsilon_i = -\sum_{a=1}^{n} \left( s_0^{(a)} + \bar{s}_0^{(a)} \right) \equiv k \in \mathbb{Z} \]. \hspace{1cm} (22)

By inspection one can see that \(N\), \(n\) and \(k\) all have the same (even or odd) parity.

Similar considerations as for the NS-NS amplitudes show that the constraint can be satisfied in \(\binom{N + \text{odd}}{\text{odd}}\) equivalent ways. Omitting contractions which are not relevant for our discussion, the source-dependent part of the amplitude again leads to a series of

\(^3\)The whole analysis can be done in terms of the GSO operator \((-1)^F\) instead of \((-1)^{F_L}\), where \(F\) is the left-moving worldsheet fermion number. This is a bit involved; interested readers may consult ref. [47].
expectation values of periodic functions in CUE ensembles,
\[
\mathcal{O}(\omega_1, \ldots, \omega_n) \sim \mathbf{E}_{U(N-k)} \left\{ \prod_{i=1}^{(N-k)/2} f_-(t_i) \right\} \cdot \mathbf{E}_{U(N+k)} \left\{ \prod_{i=1}^{(N+k)/2} f_+(t_i) \right\},
\]
(23)
where the periodic functions \( f_\pm(t) \) resemble (14) but differ in their exponents. The underlying \( U(N-k) \times U(N+k) \) structure was found in [26] in the case of generic 1-point amplitudes. By Heine’s identity, the source-dependent part of the amplitude can again be rewritten as an infinite series of (products of) Toeplitz determinants of increasing rank.

3 Bulk-Boundary Disk Amplitudes

Let us consider the case \( n = 2 \), and place\(^4\) the other vertex operator into the boundary of the disk, thus renaming \( \omega_1 \equiv \omega_c, \omega_2 \equiv \omega_o \). We consider the operator in the boundary to represent an additional open string. In other words, we will consider the amplitudes
\[
A_{\text{NSNS,NS}}(\omega_c, \omega_o) \equiv \left\langle e^{i\omega_c X^0(z, \bar{z})} e^{i\omega_o X^0(t)} \right\rangle_{\text{deformed}}
\]
(24)
and
\[
A_{\text{RR,NS}}(\omega_c, \omega_o) \equiv \left\langle e^{i\omega_c X^0(z, \bar{z})}\Theta_{s_0} \hat{\Theta}_{\bar{s}_0} e^{i\omega_o X^0(t)} \right\rangle_{\text{deformed}}.
\]
(25)
We can choose the bulk vertex operator to be inserted at the origin of the disk, \( z = \bar{z} = 0 \), while the location \( t \) of the boundary vertex operator remains a free modular parameter to be integrated over in the end.

3.1 NS-NS Bulk Vertex Operator

Consider first the case with a NS-NS bulk vertex operator. The amplitude becomes
\[
A_2(\omega_c, \omega_o) = \int_{-\infty}^{\infty} dx^0 e^{ix^0(\omega_o + \omega_c)} \sum_{N=0}^{\infty} \left( -\pi^2 \lambda^2 e^{\sqrt{2}x^0} \right)^N \left[ I_N(\omega_o) \right]^2,
\]
(26)
where \( I_N(\omega_o) \) is the integral
\[
I_N(\omega_o) = \frac{1}{N!} \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{1 \leq i < j \leq N} |e^{it_i} - e^{it_j}|^2 \prod_{i=1}^{N} |1 - e^{it_i} e^{-it_j}|^{i\sqrt{2}\omega_o}.
\]
(27)
\(^4\)The vertex operator cannot be mapped into the boundary by a conformal transformation.
We have removed an apparent divergence resulting from the self-contractions on the boundary, by an appropriate normal ordering [5]. The multiple integrals over \( t_i \) do not depend on \( t_i \), hence we can set \( t_i = 0 \). As noted in [34], the integral \( I_N \) can be evaluated using Selberg’s integral formula. After some algebra, we can then evaluate the amplitude in a closed form in terms of known functions. Defining a “chemical potential” \( \mu = -\log(-\pi^2 \lambda^2 e^{\sqrt{2} t_0}) \), carefully following the calculational strategy in [34], and carrying out the \( x^0 \) integral using the real contour of [13] we obtain

\[
A_2(\omega_c, \omega_o) = \int_{-\infty}^{\infty} dx^0 e^{ix^0(\omega_c+\omega_o)} \sum_{N=0}^{\infty} e^{-N\mu} \left[ \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+i\sqrt{2}\omega_o)}{\Gamma(j+i\omega_o/\sqrt{2})^2} \right]^{2} \\
= \int_{-\infty}^{\infty} dx^0 e^{ix^0(\omega_c+\omega_o)} \sum_{N=0}^{\infty} e^{-N\mu} e^{2\int_0^\infty dt H(t,\omega_o/\sqrt{2})(e^{-Nt}-1)} \\
= \frac{-i\pi}{\sqrt{2}} \frac{(\pi\lambda)^{-i\sqrt{2}(\omega_c+\omega_o)}}{\sinh(\pi(\omega_o+\omega_c)/\sqrt{2})} \exp \left\{ 2G(\frac{\omega_c}{\sqrt{2}}, \frac{\omega_o}{\sqrt{2}}) \right\},
\]

where

\[
G\left(\frac{\omega_c}{\sqrt{2}}, \frac{\omega_o}{\sqrt{2}}\right) \equiv \int_0^\infty dt H\left(t, \frac{\omega_o}{\sqrt{2}}\right) (e^{i(\omega_o+\omega_c)t/\sqrt{2}} - 1),
\]

with

\[
H(t, \omega_o) \equiv \frac{(1 - e^{-i\omega_o t})^2}{2t(1 - \cosh t)},
\]

similar to the result in [34]. As a simple consistency check we can verify that the result reduces\(^5\) to the answer in [26] in the absence of the initial open string perturbation, \( \omega_o = 0 \). This follows easily since \( H(t, \omega_o) \) vanishes in the limit.

### 3.2 R-R Bulk Vertex Operator

Consider then the R-R closed string vertex operator [26]

\[
\Theta_{s_0} \tilde{\Theta}_{\bar{s}_0} e^{i\omega_c X^0 (z_c, \bar{z}_c)},
\]

and bosonize the spin fields

\[
\Theta_{s_0} = e^{is_0 H^0}; \quad \tilde{\Theta}_{\bar{s}_0} = e^{i\bar{s}_0 \bar{H}^0}.
\]

Note that in the series expansion of the amplitude the terms with \( N = \text{even} \) vanish. The relevant Wick contractions now give

\[
\left\langle \prod_i (e^{iH(t_i)} - e^{-iH(t_i)}) e^{is_0 H(0)} e^{i\bar{s}_0 \bar{H}(0)} e^{i\omega_c X'(0,t_0)} e^{i\omega_c X'(t_i)/\sqrt{2}} \right\rangle \\
= -2s_0\delta_{s_0,\bar{s}_0} \frac{(1)^N(2N+1)!}{N!(N+1)!} \left[ \prod_{1\leq i<j\leq N} |e^{it_i} - e^{it_j}|^2 \right] \left[ \prod_{N+1\leq i<j\leq 2N+1} |e^{it_i} - e^{it_j}|^2 \right] \\
\cdot \left[ \prod_i \left| 1 - e^{it_i} e^{-it_i} i|\sqrt{2}\omega_o| \right| \right] \times (\text{irrelevant terms}) . \tag{33}
\]

\(^5\)Apart from an irrelevant overall phase factor \(-i\).
Note that we have already integrated out the zero modes of $H$ and $\tilde{H}$. We have suppressed the details of terms that will ultimately not contribute because the bulk vertex operator has been placed at the origin. The amplitude becomes

$$A_2(\omega_c, \omega_o) = -2s_0 \delta_{s_0, \tilde{s}_0} \int_{-\infty}^{\infty} dx^0 e^{ix^0(\omega_o + \omega_c)} \sum_{N=0}^{\infty} (-1)^N (\pi \lambda e^{x^0/\sqrt{2}})^{2N+1} I_N \cdot I_{N+1}, \quad (34)$$

where $I_N, I_{N+1}$ are the same Selberg integrals as before, giving

$$I_N = \prod_{j=1}^{N} \Gamma(j)\Gamma(j + i\sqrt{2}\omega_o) / (\Gamma(j + i\omega_o/\sqrt{2}))^2. \quad (35)$$

Proceeding as before, we get

$$A_2(\omega_c, \omega_o) = -2s_0 \delta_{s_0, \tilde{s}_0} \int_{-\infty}^{\infty} dx^0 e^{ix^0(\omega_o + \omega_c)} \sum_{N=0}^{\infty} (-1)^N (\pi \lambda e^{x^0/\sqrt{2}})^{2N+1} \cdot \exp \left\{ \int_{0}^{\infty} dt \ H(t, \omega_o/\sqrt{2}) [e^{-Nt}(1 + e^{-t}) - 2] \right\}, \quad (36)$$

and, after some algebra, finally

$$A_2(\omega_c, \omega_o) = -2s_0 \pi \delta_{s_0, \tilde{s}_0} \frac{(\pi \lambda)^{-i\sqrt{2}(\omega_o + \omega_c)}}{\sqrt{2} \cosh(\pi(\omega_o + \omega_c)/\sqrt{2})} \cdot \exp \left\{ G \left( \frac{\omega_c}{\sqrt{2}} - \frac{i}{2} \frac{\omega_o}{\sqrt{2}} \right) + G \left( \frac{\omega_c}{\sqrt{2}} + \frac{i}{2} \frac{\omega_o}{\sqrt{2}} \right) \right\}. \quad (37)$$

Note that this again meets the result in [26] as $\omega_o \to 0$.

We would like to add a few comments on the delta function present in the equation (37). It implies that the left- and right-movers in the R-R field are such that $s_0 = \tilde{s}_0$. It results from the correlation function of the spin field along the temporal direction of the R-R vertex operator, $V_s$, given in equation (3). Apparently, it does not contain any information about the nature of the theory, i.e., whether this result holds in either Type IIA or Type IIB or both. Certainly, such information can not come from the temporal part of the correlation function. These informations are contained in other parts of the vertex operators, which we have suppressed since they do not take part in the physics of the rolling tachyon. First, the full R-R vertex operator has a spatial part, $V_{s}^{\perp}$, as defined in (3), which depends on the spin fields along spatial directions. Second, the R-R vertex operator also has a piece with R-R field strength given by

$$F_{\alpha\beta} \sim F_{\mu_1 \cdots \mu_k} (\Gamma^{\mu_1} \cdots \Gamma^{\mu_k})_{\alpha\beta}, \quad (38)$$

where $\alpha, \beta$ are spinor indices ($\alpha, \beta = 1, \ldots, 32$) and we suppressed the normalization constants which are not so important for our purpose. Finally, there is another piece which also contributes Gamma matrices. This comes from the standard doubling trick procedure for such bulk-boundary correlation function computations, where we extend
the definition of the holomorphic field from upper half-plane (UHP) to lower half-plane (LHP) by equating it to its anti-holomorphic partner:

\[ X(z) = \begin{cases} X(z), & \text{for } z \in \text{UHP}, \\ \pm \bar{X}(z), & \text{for } z \in \text{LHP} \end{cases} \] (39)

where the ± sign is for Neumann (Dirichlet) directions. For a spin field, it gives

\[ \bar{S}^\alpha(z) = (\Gamma^0 \cdots \Gamma^p)^\alpha_\beta S^\beta(\bar{z}), \] (40)

where \( p \) is the dimensionality of the \( D_p \)-brane, and for a non-BPS brane \( p = \text{odd} (\text{even}) \) for Type IIA (IIB).

Once we take all this into consideration, the restriction on the sets \( \{s_0, s_i\} \) and \( \{\bar{s}_0, \bar{s}_i\} \) turns out to be

\[ \begin{align*}
\text{For Type IIA:} & \quad s_0 + \bar{s}_0 = 0 \Rightarrow s_0 = -\bar{s}_0 \\
\sum_{i=1}^{4} (s_i + \bar{s}_i) &= \pm 1 \quad (41)
\end{align*} \]

\[ \begin{align*}
\text{For Type IIB:} & \quad s_0 + \bar{s}_0 = \pm 1 \Rightarrow s_0 = \bar{s}_0 = \pm \frac{1}{2} \\
\sum_{i=1}^{4} (s_i + \bar{s}_i) &= 0, \quad (42)
\end{align*} \]

so that the Type IIA (IIB) spinor chiralities can be satisfied correctly.

### 4 Energy Emission

We are ultimately interested in computing the expectation value of total emitted energy from the decaying brane. For the unperturbed initial state of the D-brane (spatially homogeneous decay), it was found in [26] that the total energy of closed strings emitted was divergent for a \( D_p \)-brane with \( p \leq 2 \). We now shortly examine how this is modified when the initial state is perturbed by addition of the boundary tachyon vertex operator, thus extending the discussion in [34] to superstring.

So the relevant question is, how does the inclusion of an open string perturbation change the asymptotics of brane decay into closed strings? For this we need the asymptotics of \( G(\omega_c, \omega_o) \) for \( \omega_c \gg \omega_o \). Using a method described in [49] we find for large \( n \) that

\[ e^{G(in+is/2,-is/2)} = \prod_{j=1}^{n} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2} \sim n^{(s/2)^2} e^{(s/2)^2(\gamma+1)+\sum_{j=3}^{\infty}(-s)^{j-1}2j-1\zeta(j-1)}. \] (43)

So upon analytic continuation\(^7\) we get the asymptotics with large \( \omega_c \gg \omega_o \),

\[ 2G\left(\frac{\omega_c}{\sqrt{2}}, \frac{\omega_o}{\sqrt{2}}\right) \sim -2\omega_o^2 \log\left(\frac{\omega_c}{\omega_o}\right) \quad (44) \]

\(^6\)For our purpose we choose the chirality of left- and right-moving R sector spinors in such a way that for Type IIA: \( \sum_i s_i = \text{even} \) and \( \sum_i \bar{s}_i = \text{odd} \), whereas for Type IIB it is: \( \sum_i s_i = \sum_i \bar{s}_i = \text{even} \).

\(^7\)Assuming that the ratio \( e^{G(in+is/2,-is/2)/n(s/2)^2} \) is analytic around \( n = \infty \).
for NS-NS bulk amplitude, and
\[
G\left(\frac{\omega_c - i/\sqrt{2}}{\sqrt{2}}\right) + G\left(\frac{\omega_c + i/\sqrt{2}}{\sqrt{2}}\right) \sim -2\omega_o^2 \log \left(\frac{\omega_c + i/\sqrt{2}}{\omega_o}\right) \sim -2\omega_o^2 \log \left(\frac{\omega_c}{\omega_o}\right)
\]

for R-R bulk amplitude. The total emitted energy is calculated by summing over all emitted closed string energies [13, 34]
\[
\frac{E}{V_p} = \sum_s \frac{1}{2} |A_2(\omega_c, \omega_o)|^2 \sim \frac{1}{(2\pi)^p} \int d\omega_c \omega_c^{-p/2-2\omega_o^2},
\]
showing that the result is in close analogy to bosonic case.

The lesson is that, without the perturbation $\omega_o = 0$, emitted energy diverges for $p < 3$ and is finite for $p \geq 3$. But morally speaking, we expect divergencies to indicate that the unstable brane decays completely into closed strings, whereas for finite emitted energy into a lower dimensional brane. The extra factor $\omega_c^{-2\omega_o^2}$ is a suppression (enhancement) if $\omega_o$ is real (imaginary), depending on the dimension of the D$p$-brane. However, for perturbations with imaginary $\omega_o$, decay into closed strings is enhanced, so we expect a complete decay to closed strings for all $p$. See [34] for additional discussion.

5 Summary

We have investigated superstring disk amplitudes in the rolling tachyon background corresponding to an eternally decaying non-BPS brane. Such computations address very basic questions about how these branes decay. We have shown here that the general structure of the amplitudes is a Fourier transform of a power series in the target space time coordinate, where the coefficients are Toeplitz determinants arising from expectation values of a periodic function in Circular Unitary Ensembles of increasing rank. The periodic function encodes the essential information about the amplitude. The determinants of increasing rank $N$ compute disk amplitudes with $N$ open string tachyon vertex operators from the rolling tachyon background. Further progress is related to advance in solving mathematical problems in the context of random matrices, in particular there is a need to investigate grand canonical ensembles, where the rank of the ensemble (corresponding to the number of open string tachyon insertions) can vary. So far, the calculations can be carried out fully only in the special case of bulk-boundary amplitudes. Apart from the more difficult mathematical problems, one tractable direction to pursue could be to study the field theory limit of the power series of the individual terms, and compare it with results computed from the effective Dirac-Born-Infeld field theory, in the spirit of earlier such investigations (see, e.g., [50–53]).

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Appendix

A Relations between boundary fermions and Pauli matrices $\sigma_i$

Recall that $\eta, \bar{\eta}$ are in fact Grassmann variables:

\[
\{\eta, \bar{\eta}\} = 1 \quad (A.1)
\]
\[
\eta^2 = \bar{\eta}^2 = 0 . \quad (A.2)
\]

Their spinorial representation on the two-dimensional Hilbert space can be easily worked out, and is given by

\[
\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\eta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
\[
\eta\bar{\eta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\eta}\eta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (A.3)
\]

Defining $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, we find

\[
\sigma_{\pm} = \eta, \quad \sigma_{-} = \bar{\eta}, \quad \sigma_{3} = [\eta, \bar{\eta}]
\]
\[
\sigma_{1} = \eta + \bar{\eta}, \quad \sigma_{2} = -i(\eta - \bar{\eta}) . \quad (A.4)
\]

The relevant part of the supersymmetric boundary action in terms of $\eta, \bar{\eta}$ on a brane-anti-brane pair is

\[
\delta S_B \sim i\sqrt{\frac{2}{\pi}} \int_{\partial\Sigma} dt \left( \bar{\eta}\psi^\mu D_\mu T - \psi^\mu \eta D_\mu T(t) \right) . \quad (A.5)
\]

On a brane-anti-brane pair, the tachyon $T$ is a complex field. Substituting $T = U + iV$ in the above, we get

\[
\delta S_B \sim i\sqrt{\frac{2}{\pi}} \int_{\partial\Sigma} dt \left[ (\bar{\eta} - \eta)\psi^\mu D_\mu U + i(\bar{\eta} + \eta)\psi^\mu D_\mu V \right] . \quad (A.6)
\]

Next, to obtain a non-BPS D-brane from a brane-anti-brane pair, we choose the $(-1)^{F_L}$ projection in such a way that only the 2nd term in the above equation gets projected in. Choosing $V \sim \sqrt{\frac{1}{2\pi}}\lambda e^{x^0/\sqrt{2}}$, we arrive at (1). The boundary fermion $\eta$ used in [26] is actually $(\eta + \bar{\eta}) = \sigma_1$ in our notation.

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