THE INVARIANT RING OF $m$ MATRICES UNDER THE ADJOINT ACTION BY A SUBGROUP OF A PRODUCT OF GENERAL LINEAR GROUPS

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Abstract. Let $V = V_1 \otimes \cdots \otimes V_n$ be a vector space over an algebraically closed field $K$ of characteristic zero. We study the ring of polynomial invariants $K[\text{End}(V)^{\otimes m}]^G$ of $m$ endomorphisms of $V$ under the adjoint action (by conjugation) of a subgroup of $\text{GL}(V_1) \times \cdots \times \text{GL}(V_n)$. We find that the ring is generated by certain generalized trace monomials $\text{Tr}_M^\sigma$ where $M$ is a multiset with entries in $[m] = \{1, \ldots, m\}$ and $\sigma \in S_n^m$ is a choice of $n$ permutations of $[m]$, and give bounds on the degree in which the invariant ring is generated. For rank-one endomorphisms, if $\dim V_i = m = 2$, the $\text{Tr}_M^\sigma$ generate the invariant ring in degree 11, and if $m \geq 3$, in degree $6{m \choose 3} + 16$. For general endomorphisms, the degree bound for $\dim V_i = 2$ is $6{m \choose 3} + 16$ where $r$ is the generic rank of $\text{End}(V)$ as a $4 \times \cdots \times 4$ ($n$ times) tensor.

Keywords: Invariant ring; degree bounds; partial trace

1. Introduction

Let $V = V_1 \otimes \cdots \otimes V_n$ be a vector space over an algebraically closed field $K$ of characteristic zero. We study the ring of polynomial invariants $K[\text{End}(V)^{\otimes m}]^G$ of $m$ endomorphisms of $V$ under the adjoint action (by conjugation) of a local group $G$, that is a subgroup of $\text{GL}(V_1) \times \cdots \times \text{GL}(V_n)$. We find that for local groups including local $\text{GL}$ and $\text{SL}$, this ring is generated by certain generalized trace monomials $\text{Tr}_M^\sigma$ where $M$ is a multiset with entries in $[m] = \{1, \ldots, m\}$ and $\sigma \in S_n^m$ is a choice of $n$ permutations of $[m]$, and give bounds on the degree in which the invariant ring is generated. The case $n = 1$ was solved by Procesi [21]. The local ($n > 1$) version of this problem and its relatives has recently become more important due to applications to understanding quantum states [2, 7, 8, 9, 12, 15, 18], as well as statistical models and computational complexity theory [16].

Given a representation of a group $G$ on a vector space $V$ over a field $K$, various general upper bounds have been given (3, 8, 20, 10) on the generating degree

$$\beta_G(V) := \min\{d \mid K[V]^G \text{ is generated by polynomials of degree } \leq d\}.$$ 

We obtain improved bounds by specializing to the case $\beta_G(\text{End}(V)^{\otimes m})$.

This paper is organized as follows. In Section 2, we discuss preliminaries, the relevant double commutator theorem, and the multilinear invariants. In Section 3 we give the invariant ring $K[\text{End}(V)^{\otimes m}]^G$, and in Section 4 bound the degree of its generators.
2. Preliminaries, the double commutator, and multilinear invariants

Consider a $K$ vector space $V = V_1 \otimes \cdots \otimes V_n$ where $V_i$ is a $t_i$-dimensional complex vector space, so $V^\otimes m \cong V_1^\otimes m \otimes \cdots \otimes V_n^\otimes m$. Any subgroup of the local general linear group $GL_m = \times_{i=1}^n GL(t_i, K)$ acts on $V$ and also on its $m$th tensor power $V^\otimes m$, and this action commutes with the action of $S_m^n$ (the $n$th power of the symmetric group on $m$ letters) on $V^\otimes m$. We are particularly interested in subgroups of $G_t \subseteq GL_t$ whose centralizer is precisely $S_m^n$.

Given a representation $\varphi : G \to \text{End}(V^\otimes m)$, denote by $\langle G \rangle$ the linear span of the image of $G$ under the map $\varphi$. Let us consider $G_t = \times_{i=1}^n G_{t_i} \subseteq GL_t$. Standard representation theory arguments (See e.g. [6, 16]) lead to the following.

**Theorem 2.1.** Suppose $\text{End}_{S_m^n}(V_i^\otimes m) = \langle G_{t_i} \rangle$ for every $i$. Then

(a) $\text{End}_{S_m^n}(V^\otimes m) = \langle G_t \rangle$.
(b) $\text{End}_{G_t}(V^\otimes m) = \langle S_m^n \rangle$.

Throughout the rest of this paper, we assume that $G_t$ has the above property. In the complex case, groups such as $SU_t = \times_{i=1}^n SU(t_i, \mathbb{C})$ and $U_t = \times_{i=1}^n U(t_i, \mathbb{C})$ satisfy this property since each $U(t_i, \mathbb{C})$ generates the same group algebra as $GL(t_i, \mathbb{C})$.

The local special linear group $SL_t = \times_{i=1}^n SL(t_i, K)$ satisfies it as well. As Theorem 2.1 suggests, we describe a surjection $\otimes_{i=1}^n \mathbb{C}[S_m^n] \to K[\text{End}(V)^\otimes m]^{G_t}$, generalizing to local conjugation by $G_t$; see also Leron [17]. We get the following generalization of trace functions:

**Definition 2.2.** For $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_m^n$, let $\sigma_i = (r_1 \cdots r_k)(s_1 \cdots s_l) \cdots$ be a disjoint cycle decomposition. For such a $\sigma \in S_m^n$, define the trace monomials by \(Tr_{\sigma} = T_{\sigma_1} \cdots T_{\sigma_n} \in \text{End}(V)^\otimes m\), where

\[ T_{\sigma_i}(\otimes A_j1, \ldots, \otimes A_jm) = \text{Tr}(A_{ir_1} \cdots A_{ir_k})\text{Tr}(A_{is_1} \cdots A_{is_l}) \cdots \]

and extend multilinearly.

**Theorem 2.3.** The multilinear invariants of $\text{End}(V)^\otimes m$ under the adjoint action of $G_t$ are generated by the $Tr_{\sigma}$.

**Proof.** Let $M$ be the multilinear functions from $\text{End}(V)^\otimes m \cong (V \otimes V^*)^\otimes m \to K$. Then we can identify $M$ with $(V \otimes V^*)^\otimes m$ by the universal property of tensor product. But we have an isomorphism

\[ \alpha : \text{End}(V)^\otimes m \to (V \otimes V^*)^\otimes m \]

given by $\alpha(A)(\varphi \otimes \phi) = \phi(A\varphi)$ which is $GL(V)$-equivariant. So we get a $G_t$-equivariant isomorphism $\text{End}(V)^\otimes m \cong M$. This induces an isomorphism

\[ \text{End}_{G_t}(V^\otimes m) \cong M^{G_t} \]

where $M^{G_t}$ are the $G_t$-invariant multilinear functions.

Since $V^\otimes m \cong V_1^\otimes m \otimes \cdots \otimes V_n^\otimes m$, we can write $\alpha = \otimes \alpha_i$ where $\alpha_i$ are the induced isomorphisms $\text{End}(V_i)^\otimes m \to (V_i \otimes V_i^*)^\otimes m$. Note that the following holds for the isomorphism $\alpha_i$:

(a) $\text{Tr}(\alpha_i(v \otimes \varphi)) = \varphi(v)$
(b) $\alpha_i(v_1 \otimes \varphi_1) \circ \alpha_i(v_2 \otimes \varphi_2) = \alpha_i(v_1 \otimes \varphi_1(v_2)\varphi_2)$

Since $\text{End}_{G_t}(V^\otimes m) \cong M^{G_t}$, by Theorem 2.1 we just need the image of $\sigma \in S_m^n$ under the isomorphism $\alpha$ to find the generators of $M^{G_t}$. So consider $\sigma = \ldots$. 

Figure 1. Illustration of the isomorphism on invariants induced by the $G_t$-equivariant isomorphism $\alpha$ in the proof of Theorem 2.3. This sends a trace monomial $\text{Tr}_\sigma$, $\sigma \in S^n_m$, acting on $(\text{End}V)^{\otimes m}$ (left) to a pairing invariant of $(V \otimes V^*)^{\otimes m}$ (middle), and the simplified cycle decomposition form (right) in Sections 3 and 4. Here $m = n = 2$ and $\sigma = ()$, $(12)$, with $(12)$ acting on the solid wire.

(\sigma_1, \ldots, \sigma_n); we have

$$
\alpha(\sigma)(\bigotimes_{i,j} \varphi_{ij} \otimes \bigotimes_{i,j} \phi_{ij}) = (\bigotimes_{i,j} \phi_{ij})(\bigotimes_{i,j} \varphi_{\sigma^{-1}_i(j)})
$$

$$
= \prod_{i=1}^{n} \phi_{im}(\varphi_{\sigma^{-1}_i(m)}) = T_{\sigma_1^{-1}} \cdots T_{\sigma_n^{-1}} = \text{Tr}_{\sigma}^{-1}
$$

see Figure 1 for an illustration. □

3. Tensor Network Representations of Invariants

Consider an ordered multiset $M = \{m_i\}$ with elements from $[m]$, and denote the group of permutations on $|M|$ letters by $S_{|M|}$. Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in S^n_{|M|}$. Let $(m_1, \ldots, m_r)(m_s, \ldots, m_l)\cdots$ be a disjoint cycle decomposition for $\sigma_i$.

**Definition 3.1.** Given a multiset $M$ and $\sigma \in S^n_{|M|}$, define the trace monomials on $\text{End}(V)^{\otimes m}$ by their action on simple tensors in $\bigotimes_{i=1}^{n} \text{End}(V_i)$,

$$
T^M_{\sigma_j} (\bigotimes_{j=1}^{n} A_{j1}, \ldots, \bigotimes_{j=1}^{n} A_{jm}) = \text{Tr}(A_{im_{r_1}} \cdots A_{im_{r_k}})\text{Tr}(A_{im_{s_1}} \cdots A_{im_{s_l}})\cdots
$$

$$
\text{Tr}^M_{\sigma} = \prod_{i=1}^{d} T^M_{\sigma_i}
$$

and extending multilinearly to $\text{End}(V)^{\otimes m}$.

Note that Definition 3.1 differs from Definition 2.2 in that it allows for repetition (akin to the difference between standard and semi-standard Young tableaux).

For our combinatorial approach to degree bounds, it is convenient to consider invariants as represented by tensor networks. A tensor network is a graph possibly with dangling edges, a vector space associated to each edge, and a complex tensor associated to each vertex. After contractions along the internal (non-dangling) edges, the network represents an element in the tensor product of the vector spaces associated to the dangling edges; this element is called the value of the network.

Alternatively a tensor network is a diagram in the monoidal category of finite-dimensional complex vector spaces and linear transformations. That is, a tensor network is a graphical way to represent formulas in a monoidal category built up
from objects, morphisms, composition, and tensor product \[^{13, 22}\]. In this formalism, objects are represented by arrows and morphisms are represented by boxes with some number of arrows entering and exiting. Tensor product is denoted by placing arrows and boxes in parallel. Composition is achieved simply by matching output wires of one box with the input wires of another.

**Observation 3.2.** Each trace monomial \(\text{Tr}_M^\sigma\) corresponds to a tensor network.

Each \(\text{Tr}(A_1 \ldots A_k)\), where the \(A_i\) are \(2 \times 2\) matrices drawn from a set of \(m\) such matrices has a representation as a tensor network by

\[
\begin{array}{c}
\text{A}_1 \\
\text{A}_2 \\
\text{A}_3 \\
\end{array}
\]

A trace monomial \(\text{Tr}_M^\sigma\) acting on a simple tensor is a product of such loops. For example, let \(V = V_1 \otimes V_2\) and take \(M = \{1, 2, 1\} = \{m_1, m_2, m_3\}\) and \(\sigma = ((m_1m_2)(m_3), (m_1)(m_2m_3)) \in S_3^2\). Then the degree-three trace monomial \(\text{Tr}_M^\sigma(A_1 \otimes B_1, A_2 \otimes B_2)\) is equal to the tensor network:

\[
\begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
B_1 \\
B_2 \\
B_3 \\
\end{array}
\]

Thus \(M\) tells us which elements of \(\text{End}(V)\) are selected and \(\sigma\) how to connect the wires. The trace monomials \(\text{Tr}_M^\sigma\) defined on simple tensors in \(\text{End}(V_1 \otimes \ldots \otimes V_n)\) extend to all of \(\text{End}(V)^{\otimes m}\). In particular, a matrix \(M\) in a tensor network whose wires correspond to copies of the vector space \(\mathbb{C}^2\) decomposes as a sum of simple tensors of \(2 \times 2\) matrices: \(M = \sum_j (\otimes_{i=1}^n A_{ij})\), and we are considering \(m\) such \(M\).

For example, arbitrary \(M_1, M_2 \in \text{End}(V_1 \otimes V_2)^{\otimes m}\) are of rank at most four. So \(M_1 = \sum_{i=1}^4 A_{i1} \otimes B_{i1}\) and \(M_2 = \sum_{i=1}^4 A_{i2} \otimes B_{i2}\), corresponding to a sum of tensor network diagrams as follows:

\[
\begin{array}{c}
M_1 \\
M_2 \\
M_3 \\
\end{array}
\]

Multiset reordering corresponds to the action of an element of \(S_{\{M\}}^n\) as an automorphism of the invariant ring. In the following we assume that \(M\) is in weakly increasing order, so the \(M\) in our example would become \(M = \{1, 1, 2\}\).

3.1. **Description of the Invariant Ring.** It is clear the trace monomials are invariant under the action of \(G_k\). To see that they generate all invariant functions we consider multihomogeneous invariants as restitutions.

**Definition 3.3.** A function \(f \in K[V_1 \oplus \ldots \oplus V_r]\) is multihomogeneous of degree \(d = (d_1, \ldots, d_r)\) if \(f(\lambda_1 v_1, \ldots, \lambda_r v_r) = \lambda_1^{d_1} \cdots \lambda_r^{d_r} f(v_1, \ldots, v_r)\).

The coordinate ring \(K[V_1 \oplus \ldots \oplus V_r]\) can be graded with respect to the multidegree, and the multihomogeneous components of an invariant polynomial are
themselves invariant. So the multihomogeneous components of $K[V_1 \oplus \cdots \oplus V_r]^G$ generate this ring. Details can be found in Kraft and Procesi [14], from which we also need a result on restitution. Suppose $f \in K[V_1^{\otimes d_1} \oplus \cdots \oplus V_r^{\otimes d_r}]$ is a multilinear polynomial. Then the restitution of $f$, $\mathcal{R}f \in K[V_1 \oplus \cdots \oplus V_r]$ is defined by

$$\mathcal{R}f(v_1, \ldots, v_r) = f(v_1, \ldots, v_1, \ldots, v_r, \ldots, v_r).$$

**Proposition 3.4** ([14]). Assume char $K=0$ and $V_1, \ldots, V_m$ are representations of a group $G$. Then every multihomogeneous invariant $f \in K[V_1 \oplus \cdots \oplus V_m]^G$ of degree $d = (d_1, \ldots, d_m)$ is the restitution of a multilinear invariant $F \in K[V_1^{\otimes d_1} \oplus \cdots \oplus V_m^{\otimes d_m}]^G$.

With this we can obtain the generators for the $G_t$-invariants.

**Theorem 3.5.** The ring of $G_t$-invariants of $\text{End}(V)^{\otimes m}$ is generated by the $\text{Tr}_\sigma^M$.

**Proof.** After Definition [13] we observed that the multihomogeneous invariants generate all the invariants. Let $W = \text{End}(V)$. Consider a multihomogeneous invariant function of degree $d = (d_1, \ldots, d_m)$ in $K[W^{\otimes m}]$. It is the restitution of a multilinear invariant in $K[W^{\otimes d_1} \oplus \cdots \oplus W^{\otimes d_m}]$. Let $D = \sum d_i$.

By Proposition 3.4 we need only look at the restitutions of $\text{Tr}_\sigma$, for $\sigma \in S_D$. It suffices to compute the restitution of $\text{Tr}_\sigma = \prod T_{\sigma_i}$ on simple tensors. So consider $T_{\sigma_i}(A_1^{\otimes d_1}, \ldots, A_m^{\otimes d_m})$ where $A_i = A_{i_1} \otimes \cdots \otimes A_{i_{d_i}}$. Let $\sigma_i$ have a disjoint cycle decomposition $(r_1 \cdots r_k)(s_1 \cdots s_l)\cdots$.

However, since the the input matrices are not all distinct, we can think of $\sigma_i$ as an element of $S_M^n$ where $M = \{m_1, \ldots, m_D\} = \{1, \ldots, 1, \ldots, m, \ldots, m\}$ and the number of occurrences of $i$ in $M$ is $d_i$. The disjoint cycle decomposition becomes $(m_{r_1} \cdots m_{r_k})(m_{s_1} \cdots m_{s_l})\cdots$ and we get that $T_{\sigma_i}(A_1^{\otimes d_1}, \ldots, A_m^{\otimes d_m}) = T_{\sigma_i}^{M}$.

This gives us that the restitution of $\text{Tr}_\sigma$ is $\prod T_{\sigma_i}^{M} = \text{Tr}_\sigma^M$. Thus these functions agree on simple tensors and are extended multilinearly, so they agree everywhere and generate the ring of invariants.

Note that $\text{Tr}_\sigma^M \text{Tr}_{\sigma'}^M = \text{Tr}_{\sigma \oplus \sigma'}^M$, where $\sigma \oplus \sigma'$ is the induced permutation on $M \sqcup M'$. So every invariant is a linear combination of the generators above; we have a surjection from the tensor product of group algebras $\bigotimes_{i=1}^n \mathbb{C}[S_m] \rightarrow K[\text{End}(V)^{\otimes m}]^G$. This gives an explicit description of all invariants under $G_t$, namely as linear combinations of tensor networks.

4. Bounding the Degree of the Generators

Of particular interest to us is the case when $V = (K^2)^{\otimes n}$. To draw conclusions about particular networks, it is useful to enumerate or at least bound the degree of the generators of $K[\text{End}(V)^{\otimes m}]^G$. Much is known about ring of invariants of $\text{End}(V)^{\otimes m}$ under the adjoint representation of $\text{GL}(V)$ including that it is Cohen-Macaulay and Gorenstein [11]; see Formanek [5] for an exposition. We will be able to use several classical theorems that will allow us to bound the degrees of generators for our related ring of invariants, beginning with the following:

**Theorem 4.1** ([14]). If $\text{End}_G(V^{\otimes m}) = \langle S_m \rangle$, then $K[\text{End}(V)^{\otimes m}]^G$ under the adjoint action is generated by $\text{Tr}(A_{i_1} \cdots A_{i_k}) \quad 1 \leq i_1, \ldots, i_k \leq m$
where \( k \leq \dim(V)^2 \). If \( n = \dim(V) \leq 3 \), \( k \leq \binom{n+1}{2} \) suffices.

This, however, does not provide a bound on the degree. Note that the degree of \( \text{Tr}_\sigma^M \) as a polynomial in the matrix entries equals \( |M| \). To see the issue, consider the tensor networks depicted in Figure 2. Some trace monomials such as \( \text{Tr}_{(12),23}^{1,2,1} \) factor into trace monomials of smaller degree, \( \text{Tr}_{(12),23}^{1,2,1} = \text{Tr}_{(12),23}^{1,2,1} \text{Tr}_{1,1}^{1,1} \), while others such as \( \text{Tr}_{(12),23}^{2,1,1} \) do not.

We will need to bound the maximal degree of a trace monomial which does not factor. This will require a somewhat detailed combinatorial argument which will occupy the rest of this section. We begin with the following definitions.

**Definition 4.2.** The **size** of \( T^M_{\sigma_i} \) is defined to be the size of the largest cycle in the disjoint cycle decomposition of \( \sigma_i \).

**Definition 4.3.** Given a minimal set of generators, the **girth** of \( K[\text{End}(V)^{\otimes m}]^{G_t} \) is a tuple \((w_1, \ldots, w_n)\) where \( w_i \) is the maximum size of any \( T^M_{\sigma_i} \) appearing in a generator. The girth of a function \( \text{Tr}_\sigma^M \) is a tuple \((s_1, \ldots, s_n)\), where \( s_i \) is the size of \( T^M_{\sigma_i} \).

Note that the girth of the simple case \( K[\text{End}(V)]^{G_t} \) is simply the minimum \( k \) such that the functions \( \{ \text{Tr}(A_{i_1} \cdots A_{i_k}) : 1 \leq i_1, \ldots, i_k \leq m \} \) generate it. We put a partial ordering on girth as follows: \((w_1, \ldots, w_n) < (w'_1, \ldots, w'_n)\) if \( \exists i \) such that \( w_i < w'_i \) and for no \( j \) do we have \( w'_j < w_j \). The girth is bounded locally by the square of the dimension.

**Proposition 4.4.** If \((w_1, \ldots, w_n)\) is the girth of \( K[\text{End}(V)^{\otimes m}]^{G_t} \), then \( w_i \leq y_i \), where \( y_i \) is the girth of \( K[\text{End}(V_i)^{\otimes m}]^{G_{t_i}} \). In particular for \( V = V_1 \otimes \cdots \otimes V_n \), the girth of \( K[\text{End}(V)^{\otimes m}]^{G_t} \) is bounded by \((y_1^2, \ldots, y_n^2)\). If \( t_i \leq 3 \), then the girth is bounded by \((y_1^{t_1+1}, \ldots, y_n^{t_n+1})\).

**Proof.** First note that \( T^M_{\sigma_i} \) lies in the invariant ring \( R_t = K[\text{End}(V_i)^{\otimes m}]^{G_{t_i}}. \) Thus it has size at most \( y_i \), where \( y_i \) is the girth of \( R_t \). Now apply Theorem 4.1. 

As we mentioned above, we are specifically interested in the case where \( V = (K^2)^{\otimes m} \). The case where \( n = 1 \) and the generators of \( K[\text{End}(K^2)^{\otimes m}]^{G_t} \) are well understood. We make use of the following theorem for the two-dimensional case.

**Theorem 4.5 (1).** The ring \( K[\text{End}(K^2)^{\otimes m}]^{G_t} \) is minimally generated by

\[
\begin{align*}
\text{Tr}(A_i) & \quad 1 \leq i \leq m \\
\text{Tr}(A_{i_1} A_{i_2}) & \quad 1 \leq i_1, i_2 \leq m, \text{ and} \\
\text{Tr}(A_{i_1} A_{i_2} A_{i_3}) & \quad 1 \leq i_1 < i_2 < i_3 \leq m.
\end{align*}
\]
So we may assume that $\text{Tr}_\sigma^M$ is written in terms of the trace monomials in Theorem 4.5. The degree bound for generators of $K[\text{End}(V)^{\otimes m}]^G$, we give depends on the generic tensor rank of $\text{End}(V)$ as an element of $\bigotimes_{i=1}^m \text{End}(V_i)$.

We begin the analysis by restricting to the tensors in $\text{End}(V)$ which are rank one in $\bigotimes_{i=1}^m \text{End}(V_i)$ in Sections 4.1 and 4.2 and then will consider linear combinations of these to obtain the general case in Section 4.3.

### 4.1. Girth at most $(2, \ldots, 2)$ invariants operating on rank-one tensors.

Let $S$ be the subvariety of $\text{End}(K^2)^{\otimes n}$ of rank one matrices, i.e. matrices of the form $\bigotimes_{i=1}^n M_i, M_i \in \text{End}(K^2)$. First we bound the degree on a simpler ring, which we call $R_{\text{trans}}$, which is the subring of $R = K[S^{\otimes m}]^G$ generated by functions with girth at most $(2, \ldots, 2)$. Note that in the case $m = 2$, $R_{\text{trans}} = R$.

We want to show that for $|M|$ sufficiently large, for any $\sigma \in S_M^n$, $\text{Tr}_\sigma^M$ factors as $\text{Tr}_\sigma^M = \text{Tr}_{\sigma_a}^M \text{Tr}_{\sigma_b}^M$ for two disjoint multisets $M_a, M_b$ with $M = M_a \cup M_b$.

**Definition 4.6.** Let $M' \subseteq M$ be a sub-multiset. Let $x_1, \ldots, x_k \in M$. We say that $M'$ does not separate the points $x_1, \ldots, x_k$ if either $\{x_i\} \subseteq M'$ or $\{x_i\} \subseteq M \setminus M'$. Otherwise, we say $M'$ separates $x_1, \ldots, x_n$.

**Definition 4.7.** Given a trace polynomial $\text{Tr}_\sigma^M$, we say that $M' \subseteq M$ separates a monomial of $\text{Tr}_\sigma^M$ if there is a trace monomial $\text{Tr}(A_{i_1} \cdots A_{i_{m_k}})$, $1 \leq k \leq 3$, in $\text{Tr}_\sigma^M$ such that $M'$ separates $m_1, \ldots, m_k$. Otherwise we say that $M'$ does not separate monomials of $\text{Tr}_\sigma^M$.

We can now rephrase what it means for a trace polynomial to factor in a more convenient way.

**Definition 4.8.** A trace polynomial $\text{Tr}_\sigma^M$ factors if there exists $M' \subsetneq M$ such that $M'$ does not separate monomials of $\text{Tr}_\sigma^M$.

We describe an tableau-shape for a multiset $M$ that we will use to encode which elements of $M$ we want to be inseparable. For the purposes of this subsection, at most two elements of $M$ will be inseparable.

The table-shape of $M, M,$ will be a collection of pairs, $\lfloor \ast \ast \rfloor$, and singles $\lfloor \ast \rfloor$, which are unfilled, arranged in a particular pattern, see Figure 3 (a). Let $M$ be drawn from $[m]$ and let $s = |M|$. There will be $m + 1$ rows. The first $m$ rows each have $\lfloor \ast \ast \rfloor$ pairs and will be labeled from top to bottom by the elements of $[m]$. The last row, which we call the augmented row, contains $s$ singles followed by $\lfloor \ast \ast \rfloor$ pairs.

In the table-shape, pairs represent inseparable elements in $M$. Singles represent elements that are separable from all other elements.

So given a trace polynomial $T_{\sigma_i}^M$, $1 \leq i \leq n$, we fill $M$ in the following way: Out of the first $m$ rows, we interpret the $k$th row as those inseparable pairs $m_1, m_2 \in M$ where $m_1 = m_2 = k$. We call such pairs *duplicate pairs*. There are at most $\lfloor \frac{m}{2} \rfloor$ of them. In the augmented row, the singles represent separable elements. The pairs in the augmented row represent separable pairs $m_1, m_2 \in M$ where $m_1 \neq m_2$. These are called *non-duplicate pairs*.

So we fill up $M$ by declaring the pairs $m_1, m_2$, inseparable if $\text{Tr}(A_{m_1} A_{m_2})$ appears in $T_{\sigma_i}^M$. We place the inseparable pairs in $M$ in the way described above. Note that there is some non-uniqueness in how to fill the augmented row. We call a filled tableau-shape simply a tableau. This gives a recipe for taking a trace polynomial $T_{\sigma_i}^M$ and filling $M$ to give a tableau which we denote $T_{\sigma_i}$. An example is given in Figure 3 (b).
Let $T$ be a tableau. Then choosing an $1 \leq i \leq n$, we can associate a function $T_{\sigma_i}^M$, for some $\sigma_i$, to $T$. First let us disregard pairs or singles containing no elements. Secondly, if a pair has a single element, move that element to a single in the augmented column. Then for every pair $[m_1, m_2]$ in the tableau, $T_{\sigma_i}^M$ appears in $T_{\sigma_i}^M$. For every single, $[m_1]$, $T_{\sigma_i}$ appears. Let the $f_{i,T}$ denote the function that $T$ represents for choice of $i$. Note that $f_{i,T_{\sigma_i}} = f_{i,T}^M$.

We define an equivalence relation $\sim$ on tableaux in the following way: $T_1 \sim T_2$ if $f_{i,T_1} = f_{i,T_2}$, which will be independent of choice of $i$. Now let $T_1$ and $T_2$ be two different fillings of $M$, $T_1 \sim T_2$. Then if we allow horizontal permutations of the elements in a row and vertical permutations of elements in a column, we can permute $T_1$ into $T_2$.

Now suppose we are considering a tableau $T_{\sigma_i}$ filled from $M$. We make two types of adjustments. First, take any two elements $m_1, m_2$ appearing in singles in the augmented row. We declare them inseparable. We do this until there is at most one element appearing in a single left.

Secondly, look at the non-duplicate pairs. Suppose there are two elements of $[m]$ repeated in this row. So we have the pairs $[m_1, m_2], [m_3, m_4]$ where $m_1 = m_3$. We replace these two pairs with $[m_1, m_3], [m_2, m_4]$ and then move the pair $[m_1, m_3]$. 

Figure 3. For $M = \{1,1,1,2,2,3\}$, $m = 3$, and $\sigma_i = (m_1)(m_2)(m_3m_4)(m_5m_6)$. 

(a) $M$

(b) $T_{\sigma_i}$

(c) $\tilde{T}_{\sigma_i}$
to the appropriate row of duplicate pairs. We repeat until all elements appearing in non-duplicate pairs are distinct. Lastly, we flush all elements are far right as possible. Note that the augmented row has at most $2\left\lceil \frac{m}{2} \right\rceil + 1$ elements. Let us call this adjusted tableau $\tilde{T}_{\sigma_i}$. An example is given in Figure 3 (c).

Now we consider a restricted set of permutations on our adjusted tableaux. Let $P_{\text{aug}}$ be permutations of the elements of the augmented row and $P_{\text{vert}}$ be permutations of elements within a column, for any column. Then our restricted permutations are $P_{\text{aug}} \times P_{\text{vert}}$. In fact, we can insist that the permutation in $P_{\text{aug}}$ is always applied first, followed by the permutation from $P_{\text{vert}}$.

**Observation 4.9.** Consider two functions $T_{\sigma_i}^M$ and $T_{\sigma_j}^M$. Since many types of permutations on $T_{\sigma_i}^M$ are trivial with respect to $\sim$, it is not hard to see that there is a permutation in $P_{\text{aug}} \times P_{\text{vert}}$ that takes $\tilde{T}_{\sigma_i}$ to a tableau $T' \sim T_{\sigma_j}$, although they won’t be equal in general.

**Theorem 4.10.** The $\text{Tr}_{\sigma_i}^M$ with degree at most $2(m+1)\left\lceil \frac{m}{2} \right\rceil + 2m + 1$ generate $R_{\text{trans}}$.

**Proof.** Let us first consider $\tilde{T}_{\sigma_i}$. $\tilde{T}_{\sigma_i}$ differs from a tableau $T'$ such that $T' \sim T_{\sigma_i}$ by first applying a horizontal permutation in the augmented row and then vertical permutations in the columns, for any $i$.

Not suppose that $|M| > 2(m+1)\left\lceil \frac{m}{2} \right\rceil + 2m + 1$. Then there are at least $2\left\lceil \frac{m}{2} \right\rceil + 3$ filled columns in $\tilde{T}_{\sigma_i}$. Indeed, suppose $\tilde{T}_{\sigma_i}$ had only $2\left\lceil \frac{m}{2} \right\rceil + 2$ filled columns, what is the maximum $|M|$? This is the case where there are $\frac{m}{2} + 1$ duplicate pairs for every element of $[m]$ as well as $\frac{m}{2}$ non-duplicate pairs and one single. Thus the duplicate pairs contribute $2m\left\lceil \frac{m}{2} \right\rceil + 2m$ to the size of $|M|$ and the augmented row contributes $2\left\lceil \frac{m}{2} \right\rceil + 1$.

Now let $M'$ be the elements filling the rightmost $2\left\lceil \frac{m}{2} \right\rceil + 2$ columns, so $M' \subsetneq M$. Note that the restricted set of permutations we described above preserve $M'$ as the subset of $M$ filling the rightmost $2\left\lceil \frac{m}{2} \right\rceil + 2$ columns of $T'$ and $f_{\sigma_i, T'} = T_{\sigma_i}^M$. Furthermore, $M'$ does not separate monomials for all $T_{\sigma_i}^M$. So $T_{\sigma_i}^M$ factors. □

**Corollary 4.11.** For $m = 2$, the $\beta_{G_i}(K[S^{\otimes m}]) \leq 11$.

**Proof.** This follows from Theorem 4.10 by substituting in 2 for $m$ and noticing that $R_{\text{trans}} = R$ when $m = 2$. □

### 4.2. General girth, rank-one tensors

For general $m$, $R$ has girth $(3,\ldots,3)$. We adapt the ideas from the previous section to achieve our degree bound.

For a multiset $M$, we define a tableau-shape $M$ the same as before but with extra rows added above. We add $m\choose 3$ rows each with $m\choose 4$ triplets $[i_1, i_2, i_3, i_4]$, $s = |M|$. We will think of each of these rows corresponding to a trace monomial $\text{Tr}(A_{i_1}A_{i_2}A_{i_3})$, $1 \leq i_1 < i_2 < i_3 \leq m$.

We call a filled tableau-shape a tableau and, as before, we can associate to a tableau $T$ a trace polynomial $T_{\sigma_i}^M$ for some $i$, which we denote $f_{\sigma_i, T}$, and we define the same equivalence relation $\sim$ as before. Given a trace polynomial $T_{\sigma_i}^M$, we fill $M$ as before, but now placing the trace monomials $\text{Tr}(A_{i_1}A_{i_2}A_{i_3})$ is the corresponding row. We call this tableau $T_{\sigma_i}$. We have $f_{\sigma_i, T} = T_{\sigma_i}^M$.

Given a tableau $T_{\sigma_i}$, we will form another tableau $\tilde{T}_{\sigma_i}$ by first performing the two adjustments we did before. In addition, suppose there are three non-duplicate pairs in the augmented row: $[m_1, m_2], [m_3, m_4], [m_5, m_6]$, and we can assume...
that $m_1 < m_2 < \cdots < m_6$. Then replace these three pairs with the triplets
$[m_1, m_2, m_3], [m_4, m_5, m_6]$, which are then placed in their appropriate rows.
If there are two non-duplicate pairs and one single left afterwards, then one of the
non-duplicate pairs contains two elements distinct from the element in the single.
We declare this pair and single inseparable and place it in the appropriate right row.
Otherwise, there may be two non-duplicate pairs left and no singles.
We also make adjustments on the duplicate pairs. For $[m_1, m_1], [m_2, m_2]$, and
$[m_3, 3]$, replace them with $[m_1, m_2, m_3], [m_1, m_2, m_3]$
One may object that we separated $m_3$ and $m_4$, for example, while they we clearly inseparable originally. This does not matter however since we do not require
$f_{i, f_{\sigma_i}} = f_{i, \sigma_i}$, and it will still be true that $\tilde{T}_{\sigma_i}$ differs from $T_{\sigma_i}$ by some combination
of horizontal and vertical permutations. Note that the augmented row of $\tilde{T}_{\sigma_i}$ has
at most four elements. There are at most two non-empty rows of duplicate pairs.

**Observation 4.12.** Suppose we have two function $T_{\sigma_i}$ and $T_{\sigma_j}$. Once again, many
of these permutations on $\tilde{T}_{\sigma_i}$ are trivial with respect to $\sim$. Once again, there is a
permutation $P_{\text{aug}} \times P_{\text{vert}}$ that transforms $\tilde{T}_{\sigma_i}$ into a tableau $T'$, $T' \sim T_{\sigma_j}$. We can insist,
as before, that the element of $P_{\text{aug}}$ is applied first.

**Theorem 4.13.** For $m \geq 3$, $\beta_C(K[S^{d \times m}]) \leq 6(m_3) + 16$.

**Proof.** Let us first consider $\tilde{T}_{\sigma_i}$. $\tilde{T}_{\sigma_i}$ differs from a tableau $T'$ such that $T' \sim T_{\sigma_i}$
by first applying a horizontal permutation in the augmented row and then vertical
permutations in the columns, for any $i$.
Not suppose that $|M| > 6(m_3) + 15$. Then there are at least seven filled columns
in $\tilde{T}_{\sigma_i}$. Indeed, suppose $\tilde{T}_{\sigma_i}$ had only six filled columns, what is the maximum
$|M|?$. The augmented row accounts for four elements. Every column (excluding elements
in the augmented row) accounts for $(m_3) + 2$ elements. Thus there are at
most $6(m_3) + 16$ elements in the tableau.
Now let $M'$ be the elements filling the rightmost six columns, so $M' \subsetneq M$. Note
that the restricted set of permutations we described above preserve $M'$ as the subset
of $M$ filling the rightmost six columns of $T'$ and $f_{\sigma_i, T'} = T'_{M'}$. Furthermore, $M'$
does not separate monomials for all $T'_{M'}$. So $T'_{M'}$ factors.

**Example 4.14.** Let $V$ be the subvariety of $\text{End}(C^2)$ of rank one matrices. Let $V^{\otimes 2}$
be acted on by $G = GL(2, C) \times GL(2, C)$. The ring $C[V^{\otimes 2}]^G$ has 23 generators.
For a pair of $4 \times 4$ matrices $(A, B) = (A_1 \otimes A_2, B_1 \otimes B_2)$, the generators are:
$\text{Tr}(A_1B_1)\text{Tr}(A_2B_2), \quad \text{Tr}(A_1)\text{Tr}(A_2), \quad \text{Tr}(A_1)\text{Tr}(A_1B_1)\text{Tr}(B_2)\text{Tr}(B_2)^2$,
$\text{Tr}(A_1)\text{Tr}(B_1)\text{Tr}(A_2B_2), \quad \text{Tr}(B_1)\text{Tr}(B_2), \quad (A_1^2)\text{Tr}(B_1)\text{Tr}(A_2B_2)\text{Tr}(A_2),$  
$\text{Tr}(A_1B_1)^2\text{Tr}(A_2)^2\text{Tr}(A_2B_2), \quad (A_1^2)^2\text{Tr}(A_2)^2\text{Tr}(A_2B_2), \quad \text{Tr}(A_1B_1)^2\text{Tr}(A_2^2)^2\text{Tr}(B_2)^2,$
$\text{Tr}(A_1B_1)^2\text{Tr}(B_2)^2\text{Tr}(B_2)^2, \quad (A_1^2)^2\text{Tr}(A_2)^2\text{Tr}(A_2B_2)^2,$
$\text{Tr}(A_1B_1)^2\text{Tr}(A_2^2)^2\text{Tr}(B_2)^2, \quad \text{Tr}(A_1B_1)^2\text{Tr}(B_2)^2\text{Tr}(B_2)^2,$
$\text{Tr}(A_1B_1)^2\text{Tr}(A_2^2)^2\text{Tr}(B_2)^2, \quad \text{Tr}(A_1B_1)^2\text{Tr}(B_2)^2\text{Tr}(B_2)^2,$
$\text{Tr}(A_1B_1)^2\text{Tr}(A_2B_2)\text{Tr}(A_2)\text{Tr}(B_2)$

We then extend these functions multilinearly to give 23 $T'_{M'}$. By Corollary 4.11
those $T'_{M'}$ of degree at most 11 generate $C[V^{\otimes 2}]^G$. We simply enumerated all $T'_{M'}$
up to degree 11 and removed those that were a product of functions of smaller degree. Notice that the highest degree in this example is 4. We do not know if the bounds given by Theorems 4.10 and 4.13 are sharp.

4.3. General case, arbitrary tensors.

Theorem 4.15. Let \( r \) be the generic rank of \( \text{End}(V) \) as a \( 4 \times \cdots \times 4 = 4^n \) tensor. Then \( \beta_{Gt}(K[\text{End}(V)^{\otimes m}]) \leq 6\binom{rm}{3} + 16 \).

Proof. This proof follows precisely the same logic as Theorem 4.13 with the difference that the tableaux can contain up to \( rm \) different matrices. \( \square \)

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