Generalised Witt algebras and idealizers

Citation for published version:
Sierra, SJ & Špenko, Š 2017, 'Generalised Witt algebras and idealizers', Journal of Algebra, vol. 483, pp. 415-428. https://doi.org/10.1016/j.jalgebra.2017.03.042

Digital Object Identifier (DOI):
10.1016/j.jalgebra.2017.03.042

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Journal of Algebra

General rights
Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.
GENERALISED WITT ALGEBRAS AND IDEALIZERS

S. J. SIERRA AND Š. ŠPENKO

Abstract. Let \( k \) be an algebraically closed field of characteristic zero, and let \( \Gamma \) be an additive subgroup of \( k \). Results of Kaplansky-Santharoubane and Su classify intermediate series representations of the generalised Witt algebra \( W_\Gamma \) in terms of three families, one parameterised by \( k^2 \) and two by \( \mathbb{P}^1 \). In this note, we use the first family to construct a homomorphism \( \Phi \) from the enveloping algebra \( U(W_\Gamma) \) to a skew extension \( k[\mathcal{A}^2] \times \Gamma \) of the coordinate ring of \( \mathcal{A}^2 \). We show that the image of \( \Phi \) is contained in a (double) idealizer subring of this skew extension and that the representation theory of idealizers explains the three families. We further show that the image of \( U(W_\Gamma) \) under \( \Phi \) is not left or right noetherian, giving a new proof that \( U(W_\Gamma) \) is not noetherian.

We construct \( \Phi \) as an application of a general technique to create ring homomorphisms from shift-invariant families of modules. Let \( G \) be an arbitrary group and let \( A \) be a \( G \)-graded ring. A graded \( A \)-module \( M \) is an intermediate series module if \( M_g \) is one-dimensional for all \( g \in G \). Given a shift-invariant family of intermediate series \( A \)-modules parameterised by a scheme \( X \), we construct a homomorphism \( \Phi \) from \( A \) to a skew extension of \( k[X] \). The kernel of \( \Phi \) consists of those elements which annihilate all modules in \( X \).

1. Introduction

Fix an algebraically closed ground field \( k \) of characteristic zero, and let \( \Gamma \) be a finitely generated additive subgroup of \( k \). The generalised Witt algebra \( W_\Gamma \) is the Lie algebra generated by elements \( e_\gamma : \gamma \in \Gamma \), with \( [e_\gamma, e_\delta] = (\delta - \gamma)e_{\delta + \gamma} \). Recall that an intermediate series representation of \( W_\Gamma \) is an indecomposable representation all of whose \( c_0 \)-eigenspaces are 1-dimensional. It is a theorem of Kaplansky and Santharoubane [KS85] (if \( \Gamma = \mathbb{Z} \)) and of Su [Su94] (for general \( \Gamma \)) that intermediate series representations of \( W_\Gamma \) come in three families (with two modules represented twice): one family parameterised by \( k^2 \) and two parameterised by \( \mathbb{P}^1 \). In this note we use the first family to construct a homomorphism \( \Phi \) from \( U(W_\Gamma) \) to \( T = k[\mathcal{A}^2] \times \Gamma \), and show that the existence of the other two families is a consequence of the fact that the image of \( U(W_\Gamma) \) is a sub-idealizer in \( T \). We further use the homomorphism \( \Phi \) to give a new proof that the enveloping algebra of \( U(W_\Gamma) \) is not noetherian, a fact originally proved in [SW14].

Since our main method is to construct and then analyze a homomorphism from \( U(W_\Gamma) \) to an idealizer in \( T \), we recall some facts about idealizers. We first define \( T \): as a vector space we write \( T = \bigoplus \gamma \in \Gamma k[a, b]t^\gamma \), with \( t^\gamma t^\delta = t^{\gamma + \delta} \) and \( t^\gamma f(a, b) = f(a + \gamma, b)t^\gamma \). Note that \( T \) is a bimodule over \( k[a, b] \).

An intermediate series module \( M \) over a \( \Gamma \)-graded ring is an indecomposable \( \Gamma \)-graded module with each \( M_\gamma \), a one-dimensional vector space. It is a generalisation of a point module over an \( \mathbb{N} \)-graded ring, which is a cyclic graded module with Hilbert series \( 1/(1-t) \).

For \( p = (\alpha, \beta) \in k^2 \), let \( I(p) \) be the ideal \( (a - \alpha, b - \beta) \) of \( k[a, b] \). Let \( V(p) = T/I(p)T \). It is easy to see that the \( V(p) \) are all of the intermediate series right \( T \)-modules; more precisely, the right ideals \( J \) of \( T \) such that \( T/J \) is an intermediate series module are precisely the \( I(p)T \). Likewise, the intermediate series left \( T \)-modules are the \( T/IT(p) \). These families are preserved under degree shifting.

We now consider a subring of \( T \). Fix \( p_0 \in k^2 \), and let \( S = S/p_0 = k \oplus I(p_0)T \). The ring \( S \) is an idealizer in \( T \): the largest subalgebra of \( T \) such that the right ideal \( I(p_0)T \) becomes a two-sided ideal in \( S \). It is known [Rog(14)] that the representation theory of idealizers involves blowing up. Here for \( p = p_0 \), we have that \( V(p) \cong S/(S \cap I(p)T) \) is an intermediate series right \( S \)-module. On the other hand, to define an intermediate series right \( S \)-module at \( p_0 \), we need to consider a point \( q \) infinitely near to \( p_0 \): that is, an ideal \( I(q) \) with \( I(p_0)^2 \subseteq I(q) \subseteq I(p_0) \) of \( k[a, b] \) such that \( I(p_0)/I(q) \) is one-dimensional. Such ideals are parameterised by the exceptional \( \mathbb{P}^1 \) in the blowup \( Bl_{p_0}(\mathcal{A}^2) \); more specifically, we can write

\(* Date: March 10, 2017.
\* 2010 Mathematics Subject Classification. 16W50,17B35.
\* Key words and phrases. generalised Witt algebra, intermediate series representation, idealizer.
\[ I(q) = (y(a - \alpha) - x(b - \beta), (a - \alpha)^2, (a - \alpha)(b - \beta), (b - \beta)^2) \text{ for some } [x : y] \in \mathbb{P}^1. \] For such \( I(q) \) we have that \( I(p_0) + I(q)T \) is a right ideal of \( S \). Let
\[ P(q) = S/(I(p_0) + I(q)T). \]
Then \( P(q) \) is a intermediate series right \( S \)-module. In fact, we have constructed all right ideals \( J \) of \( S \) such that \( S/J \) is an intermediate series \( S \)-module; they are parameterised by \( \text{Bl}_{p_0}(\mathbb{A}^2) \) but it is sometimes more convenient to consider them as parameterised by \( \mathbb{A}^2 \) \( \setminus \{p_0\} \) together with \( \mathbb{P}^1 \).

Left intermediate series \( S \)-modules are also parameterised by \( \text{Bl}_{p_0}(\mathbb{A}^2) \). For \( p \in \mathbb{A}^2 \setminus \{p_0\} \), the left intermediate series module \( T/\text{TI}(p) \) is isomorphic to \( \left( I(p_0) \oplus \bigoplus_{0 \neq \nu \in \Gamma} \mathbb{k}[a,b]t^\nu \right) / \left( (I(p_0) \cap I(p)) \oplus \bigoplus_{0 \neq \nu \in \Gamma} t^\nu I(p) \right) \).

We can extend this construction to a family of modules parameterised by \( \text{Bl}_{p_0}(\mathbb{A}^2) \) by adding the \( \mathbb{P}^1 \) of points \( q \) infinitely near to \( p_0 \):
\[ Q(q) = \frac{I(p_0) \oplus \bigoplus_{0 \neq \nu \in \Gamma} \mathbb{k}[a,b]t^\nu}{I(q) \oplus \bigoplus_{0 \neq \nu \in \Gamma} t^\nu I(p_0)}. \]

Consider now right intermediate series modules over the double idealiser
\[ R = \mathbb{k}[a,b] + (I(p_0)T \cap \text{TI}(p_1)) \]
and assume for simplicity that \( p_0, p_1 \in \mathbb{A}^2 \) have distinct \( \Gamma \)-orbits. These correspond to points of the double blowup \( \text{Bl}_{p_0,p_1}(\mathbb{A}^2) \). More precisely, the \( V(p) \) are intermediate series modules for \( p \in \mathbb{A}^2 \setminus \{p_0, p_1\} \). From the inclusion \( R \subseteq \mathbb{k} \oplus I(p_0)T \) we obtain a family \( P(q) \) parameterised by the \( \mathbb{P}^1 \) of points infinitely near to \( p_0 \). Finally, from the inclusion \( R \subseteq \mathbb{k} \oplus \text{TI}(p_1) \) we obtain a family \( Q(q) \) of right modules parameterised by the \( \mathbb{P}^1 \) of points infinitely near to \( p_1 \) and constructed similarly to the construction of the left modules \( Q(q) \) over \( S \).

Let \( \Gamma \) now be an arbitrary group (more generally, a monoid) and let \( A \) be a \( \Gamma \)-graded ring. We give a general result in Theorem 2.2 (respectively, Theorem 2.5) which constructs a ring homomorphism (respectively, an anti-homomorphism) \( \Phi : A \rightarrow \mathbb{k}[X] \times \Gamma \), where \( X \) is a shift-invariant family of right (respectively, left) intermediate series \( A \)-modules; this generalises constructions in \([\text{ATV91}, \text{RZ08}, \text{V96}]\).

When we apply this technique to \( U(W_T) \), we show that the image of \( \Phi \) is contained in a double idealizer \( R \) inside the ring \( T \) defined in the second paragraph, and we show in Propositions \([3.5], [3.6]\) that the right intermediate series \( R \)-modules constructed above restrict to precisely the intermediate series representations of \( W_T \). This gives a unified geometric description of what have until now been seen as three distinct families of representations.

We further show in Proposition 4.3 that the image of \( U(W_T) \) under \( \Phi \) is neither right nor left noetherian. For \( \Gamma = \mathbb{Z} \) this was proved in \([\text{SW15}]\) as the main step in proving the non-noetherianity of \( U(W) \). It follows that \( U(W_T) \) is neither right nor left noetherian; other proofs are given in \([\text{SW15}, \text{SW14}]\).

The general behaviour of idealizers leads one to expect that at idealizers in \( T \) at ideals of points in \( \mathbb{k}[a,b] \) will not be noetherian since no points have dense \( \Gamma \)-orbits; see \([\text{Se11}]\) for a precise statement of a related result for \( N \)-graded rings. However, infinite orbits are dense in \( \mathbb{A}^1 \). Thus one expects that the factors \( \Phi(U(W_T))|_{b=\beta} \), which live on the \( \Gamma \)-invariant line \( \{b = \beta\} \) in \( \mathbb{A}^2 \), are noetherian for all \( \beta \in \mathbb{k} \) and we also show in Proposition 4.6 that this is indeed the case.

Acknowledgements: We thank Jacques Alev and Lance Small for useful discussions. The second author is supported, and the first author is partially supported, by EPSRC grant EP/M008460/1.

2. Intermediate series modules and ring homomorphisms

It is well-known that ring homomorphisms can be constructed from shift-invariant families of modules. Let \( A \) be a (connected \( N \)-) graded ring, generated in degree 1. A point module over \( A \) is a cyclic graded \( A \)-module with Hilbert series \( 1/(1-t) \). Suppose that (right) \( A \)-point modules are parameterised by a projective scheme \( X \). Let the point module corresponding to \( x \in X \) be \( M_x \). Then the shift functor \( \Psi : M \mapsto M[1]_{\geq 0} \) induces an automorphism \( \sigma \) of \( X \) so that \( \Psi(M^\sigma_\tau) \cong M^{\sigma(x)} \).

The following result goes back to \([\text{ATV90}]\) (see also \([\text{V96}]\)), although in this form it is due to Rogalski and Zhang.
Theorem 2.1. ([RZ05] Theorem 4.4) There is an invertible sheaf $L$ on $X$ so that there is a homomorphism $\phi : A \to B(X, L, \sigma)$ of graded rings, where $B(X, L, \sigma)$ is the twisted homogeneous coordinate ring defined in [AV90]. If $A$ is noetherian then $\phi$ is surjective in large degree.

The kernel of $\phi$ is equal in large degree to

$$J = \bigcap \{ \text{Ann}_A(M) \mid M \text{ is a } C\text{-point module for some commutative } k\text{-algebra } C \}.$$

The purpose of this section is to give a version of this theorem for a (not necessarily connected graded) algebra graded by an arbitrary monoid $\Gamma$.

We first need some notation. Let $\Gamma$ be a monoid and let $A$ be a $\Gamma$-graded ring. If $M$ is a $\Gamma$-graded right $A$-module and $\gamma \in \Gamma$, we define the shift $M(\gamma)$ of $M$ by $\gamma$ as:

$$M(\gamma) = \bigoplus_{\delta \in \Gamma} M(\gamma)_{\delta},$$

where $M(\gamma)_{\delta} = M_{\gamma \delta}$. We note that

$$(2.1) \quad M(\gamma) A_\epsilon = M_{\gamma \delta} A_\epsilon \subseteq M_{\gamma \delta \epsilon} = M(\gamma)_{\epsilon},$$

so $M(\gamma)$ is again a $\Gamma$-graded right $A$-module. Note that

$$(M(\gamma))(\delta)_\epsilon = M(\gamma)_{\delta \epsilon} = M_{\gamma \delta \epsilon} = M(\gamma)_{\epsilon},$$

and so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\gamma \delta)$.

If $M$ is a left module we define $M(\gamma)_{\delta} = M_{\delta \gamma}$. Then (2.1) becomes:

$$A_\epsilon M(\gamma)_{\delta} = A_\epsilon M_{\delta \gamma} \subseteq M_{\delta \gamma} = M(\gamma)_{\epsilon \delta},$$

as needed. We have

$$(M(\gamma))(\delta)_\epsilon = M(\gamma)_{\epsilon \delta} = M_{\epsilon \delta \gamma} = M(\delta \gamma)_{\epsilon},$$

so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\delta \gamma)$.

If $A$ is a $\Gamma$-graded ring, an intermediate series module over $A$ is a $\Gamma$-graded left or right $A$-module $M$ so that $\dim M_{\gamma} = 1$ for all $\gamma \in \Gamma$. We will use a shift-invariant family of intermediate series modules to construct a ring homomorphism from $A$ to a $\Gamma$-graded ring, giving a version of Theorem 2.1 in this setting.

Our notation for smash products is that if $\Gamma$ acts on $X$, then $A \times_{\gamma} = \bigoplus_{\gamma \in \Gamma} A t^\gamma$, where $t^\gamma t^\delta = t^{\gamma + \delta}$ and $t^0 r = r t^0$ for all $r \in A$, $\gamma \in \Gamma$.

Theorem 2.2. Let $\Gamma$ be a monoid with identity $e$ and let $A$ be a $\Gamma$-graded ring. Let $X$ be a reduced affine scheme that parameterises a set of intermediate series right $A$-modules, in the sense that for $x \in X$ there is a module $M^x \cong \bigoplus_{\gamma} A_{\gamma}$ with basis $\{ v_{\gamma}^x \mid \gamma \in \Gamma \}$, and that there is a $k$-linear function $\phi : A \to k[X]$ so that

$$v_{\gamma}^x r = \phi(r)(x) v_{\gamma}^x$$

for all $\gamma \in \Gamma, r \in A_{\gamma}$. Further suppose that shifting defines a group antihomomorphism $\sigma : \Gamma \to \text{Aut}_k(X), \gamma \mapsto \sigma^\gamma$ so that $M^{\sigma^\gamma}(\gamma) \cong M^{\sigma^\gamma(x)}$. Here we require that the isomorphism maps $v_{\gamma}^{\sigma^\gamma(x)} \mapsto v_{\gamma}^{\sigma^\gamma(x)}$.

In this setting the map

$$\Phi : A \to k[X] \times_{\Gamma} \gamma \mapsto \phi(r) t^\gamma, \quad r \in A_{\gamma} \mapsto \phi(r) t^\gamma$$

is a graded homomorphism of algebras. Further,

$$\ker \Phi = \bigcap_{x \in X} \text{Ann}_A M^x.$$

Proof. Let $\Gamma$ act on $k[X]$ by $f^\gamma = (\sigma^\gamma)^*(f)$, so $\sigma$ defines a homomorphism from $\Gamma \to \text{Aut}_k(k[X])$.

Let $r \in A_{\gamma}, s \in A_{\delta}$, and let $\alpha : V^\gamma(\gamma) \to V^{\sigma^\gamma(x)}$ be the given isomorphism. Then:

$$\alpha(v_{\gamma}^{\sigma^\gamma(x)} s) = v_{\gamma}^{\sigma^\gamma(x)} s = \phi(s)(\sigma^\gamma(x)) v_{\gamma}^{\sigma^\gamma(x)} = \alpha(\phi(s)(\sigma^\gamma(x)) v_{\gamma}^{\sigma^\gamma(x)}).$$

So

$$(2.2) \quad v_{\gamma}^{\sigma^\gamma(x)} s = \phi(s)^{\gamma}(x) v_{\gamma}^{\sigma^\gamma(x)}.$$

Now, using (2.2), we obtain:

$$\phi(rs)(x) v_{\gamma}^{\sigma^\gamma(x)} = v_{\gamma}^{\sigma^\gamma(x)} rs = \phi(r)(x) v_{\gamma}^{\sigma^\gamma(x)} = \phi(r)(x) \phi(s)(\sigma^\gamma(x)) v_{\gamma}^{\sigma^\gamma(x)}.$$
and so
\[
(2.3) \quad \phi(rs) = \phi(r)\phi(s)^\gamma.
\]
Then by \((2.3)\) we have
\[
\Phi(rs) = \phi(rs)t^\gamma t^\delta = \phi(r)\phi(s)^\gamma t^\delta = \phi(r)\phi(s)^\gamma = \Phi(r)\Phi(s).
\]
Since \(\Phi\) is graded, \(\ker \Phi\) is a graded ideal of \(A\). If \(r \in A\) is homogeneous then
\[
\Phi(r) = 0 \iff \phi(r) = 0 \iff v^x_r = 0 \text{ for all } x \in X.
\]
Let \(\gamma \in \Gamma\). Then
\[
v^x_\gamma r = 0 \text{ for all } x \in X \iff v^\sigma^\gamma(x)_r = 0 \text{ for all } x \in X \iff v^x_\gamma r = 0 \text{ for all } x \in X,
\]
using the isomorphism between \(M^\delta(\gamma)\) and \(M^{\sigma^\gamma(x)}\). So
\[
\Phi(r) = 0 \iff v^x_\gamma r = 0 \text{ for all } x \in X, \gamma \in \Gamma \iff r \in \bigcap_{x \in X} \text{Ann}_A M^x.
\]
\(\square\)

(The reason we require \(X\) in the theorem statement to be reduced is that we are constructing \(\Phi\) from the closed points of \(X\), and so effectively from the reduced induced structure on \(X\).)

**Remark 2.3.** We need the map \(\sigma\) in Theorem 2.2 to be an antihomomorphism because of the equations:
\[
M^{\sigma^\gamma(x)}(\delta) \cong M^{\delta}(\gamma)(\delta) \cong M^{\sigma^\gamma(x)}(\delta) = M^{\sigma^\delta(\sigma^\gamma(x))}.
\]

**Remark 4.** There is a universal module \(M\) for the family \(\{M^x \mid x \in X\}\), which is isomorphic as a \(k[X]\)-module to \(\bigoplus_{\gamma \in \Gamma} k[X]v^x_\gamma\). The module structure is given by
\[
\gamma s = \phi(s)^\gamma v^x_\gamma
\]
for \(s \in A_\gamma\). If we consider the natural right action of \(A\) on \(M = k[X] \rtimes \Gamma\) then we have \(t^\gamma \cdot s = t^\gamma \phi(s) = t^\gamma \phi(s)^\gamma t^\delta = \phi(s)^\gamma t^\delta\) for \(s \in A_\delta\). This agrees with \((2.4)\) if we set \(v^x_\gamma = t^\gamma\), and so \(M \cong k[X] \rtimes \Gamma\).

The theorem for left modules is:

**Theorem 2.5.** Let \(\Gamma\) be a monoid with identity \(e\) and let \(A\) be a \(\Gamma\)-graded ring. Let \(X\) be a reduced affine scheme that parameterises a set of intermediate series left \(A\)-modules, in the sense that the left module \(N^x\) has a basis \(\{v^x_\gamma \mid \gamma \in \Gamma\}\) and that there is a \(k\)-linear function \(\phi : A \to k[X]\) so that
\[
rv^x_e = \phi(r)(x)v^x_\gamma
\]
for all \(\gamma \in \Gamma, r \in A_\gamma\). Further suppose that shifting defines a group homomorphism \(\sigma : \Gamma \to \text{Aut}_k(X), \gamma \mapsto \sigma^\gamma\) so that \(N^x(\gamma) \cong N^{\sigma^\gamma(x)}\). Here we require that the isomorphism maps \(v^x_\gamma \mapsto v^{\sigma^\gamma(x)}_\delta\).

In this setting the map
\[
\Phi : A \to k[X] \rtimes \Gamma^{\text{op}} \quad r \in A_\gamma \mapsto \phi(r)t^\gamma
\]
is a graded antihomomorphism of algebras. Further,
\[
\ker \Phi = \bigcap_{x \in X} \text{Ann}_A N^x.
\]

**Proof.** We repeat the proof above to ensure that the change of notation from right to left is handled correctly. Again, let \(f^\gamma = (\sigma^\gamma)^* f\), so \(\sigma\) defines a homomorphism from \(\Gamma^{\text{op}} \to \text{Aut}_k k[X]\). Let \(r \in A_\gamma, s \in A_\delta\), and let \(\alpha : V^x(\delta) \to V^{\sigma^\delta(x)}\) be the given isomorphism. Then:
\[
\alpha(rv^x_\gamma) = rv^{\sigma^\delta(x)}_e = \phi(r)(\sigma^\delta(x))v^{\sigma^\delta(x)}_\gamma = \alpha(\phi(r)(\sigma^\delta(x))v^{\sigma^\delta(x)}_\gamma).
\]
So
\[
(2.5) \quad rv^x_\delta = \phi(r)(\sigma^\delta(x))v^{\sigma^\delta(x)}_\gamma.
\]
Now, using \((2.5)\), we obtain:
\[
\phi(rs)(x)v^{\sigma^\delta(x)}_\gamma = rsv^x_e = \phi(s)(x)rv^x_\delta = \phi(s)(x)\phi(r)(\sigma^\delta(x))v^{\sigma^\delta(x)}_\gamma.
\]

and so

\[ \phi(rs) = \phi(s)\phi(r)\delta. \]

Then by \((2.6)\) we have

\[ \Phi(rs) = \phi(s)\phi(r)\delta t^\delta = \phi(s)\phi(r)\delta t^{\delta_{\alpha \nu \gamma}} = \phi(s)t^\delta \phi(r)t^\gamma = \Phi(s)\Phi(r). \]

The proof of the last statement is identical to the proof in Theorem 2.2. \(\square\)

**Remark 2.6.** We need the map \(\sigma\) in Theorem 2.5 to be a homomorphism because:

\[ N^{\sigma^\delta(x)} = N^x(\gamma \delta) = (N^x(\delta))(\gamma) = N^{\sigma^\delta(x)}(\gamma) = N^{\sigma^\gamma}(\sigma^\delta(x)). \]

Note also that a graded anti-homomorphism from a \(\Gamma\)-graded algebra should map to a \(\Gamma^{op}\)-graded algebra, as we indeed have.

**Remark 2.7.** We likewise obtain the universal left module for the \(N^x\) from \(\Phi\). Set \(N = k[X] \rtimes \Gamma^{op}\). The left action induced by \(\Phi\) is \(r \cdot \delta = \delta \Phi(r)\) because \(\Phi\) is an anti-homomorphism, so we get

\[ r \cdot t^\delta = t^\delta \Phi(r) = t^\delta \phi(r)t^\gamma = \phi(r)t^\delta t^{\delta_{\alpha \nu \gamma}} = \phi(r)t^\delta t^{\gamma \delta} \]

for \(r \in A_\gamma\), which is the structure we expect.

**Remark 2.8.** Let \(\text{Bir}(X)\) be the group of birational self-maps of \(X\). In the settings above, suppose that shifting defines elements of \(\text{Bir}(X)\), in the sense that \(\sigma\) maps \(\Gamma\) to \(\text{Bir}(X)\). We get a generalization of Theorems 2.2 and 2.3 by replacing \(k[X]\) and \(\text{Aut}(k[X])\) with \(\mathfrak{a}(X)\) and \(\text{Bir}(X)\), respectively.

### 3. Intermediate series modules over higher rank Witt algebras

Let \(\Gamma\) be a rank \(n\) \(\mathbb{Z}\)-submodule of \(\mathbb{A}\). The *rank \(n\) Witt algebra\( W_{\Gamma} \) (or **higher rank Witt algebra** if \(n \geq 2\), sometimes called the centerless higher rank Virasoro algebra) is the Lie algebra with \(k\)-basis \(\{e_\nu \mid \nu \in \Gamma\}\) and bracket

\[ [e_\mu, e_\nu] = (\nu - \mu)e_{\nu + \mu} \]

for \(\nu, \mu \in \Gamma\). The rank one Witt algebra is the “usual” Witt algebra, which we denote by \(W\).

As \(U(W_{\Gamma})\) is \(\Gamma\)-graded one can consider intermediate series modules as in Section 2. They are the standard intermediate series modules of Lie algebras, called also Harish-Chandra modules over \((W_{\Gamma}, k[e_0])\); i.e., modules of the form \(\bigoplus_{\gamma \in \Gamma} V_\gamma\), where \(V_\gamma\) is the \(\gamma\)-eigenspace for \(e_0\) and has dimension 1.

The intermediate series \(W_{\Gamma}\)-modules have been classified in [KSS94] Theorem 2.1, generalizing the classification [KSS55] for the Witt algebra. There are three families of indecomposable intermediate series \(W_{\Gamma}\)-modules:

\[
V_{(\alpha, \beta)} = \bigoplus_{\nu \in \Gamma} k e_{\nu}, \quad e_{\mu}v_{\nu} = (\alpha + \beta \mu + \nu)v_{\mu + \nu},
\]

\[
A_{(\alpha, \beta)} = \bigoplus_{\nu \in \Gamma} k e_{\nu}, \quad e_{\mu}v_{\nu} = \begin{cases} 
u v_{\mu + \nu} & \nu \neq 0, \mu + \nu \neq 0, \\ (\alpha + \beta \mu)v_\mu & \nu = 0, \\ 0 & \mu + \nu = 0, \end{cases}
\]

\[
B_{(\alpha, \beta)} = \bigoplus_{\nu \in \Gamma} k e_{\nu}, \quad e_{\mu}v_{\nu} = \begin{cases} (\mu + \nu)v_{\mu + \nu} & \nu \neq 0, \mu + \nu \neq 0, \\ 0 & \nu = 0, \\ (\alpha + \beta \mu)v_0 & \mu + \nu = 0, \end{cases}
\]

where \((\alpha, \beta) \in \mathbb{A}^2\). Note that \(A_{(\alpha, \beta)}, B_{(\alpha, \beta)}\) are only defined where \((\alpha, \beta) \neq (0,0)\) and depend up to isomorphism (rescaling of \(v_0\)) only on \([\alpha : \beta] \in \mathbb{P}^1\). We will therefore denote them by \(A_{[\alpha : \beta]}, B_{[\alpha : \beta]}\). Note also that we have \(A_{[1:0]} \cong V_{(0,1)}\) (by \(v_0 \mapsto v_0\) and \(v_\nu \mapsto -\nu v_0\) when \(\nu \neq 0\)) and \(B_{[1:0]} \cong V_{(0,0)}\) (by \(v_0 \mapsto -v_0\) and \(v_\nu \mapsto v_\nu\) when \(\nu \neq 0\)).

**Remark 3.1.** Note that \(A_{[\alpha : \beta]}\) contains a simple submodule \(\bigoplus_{\mu \neq \nu \in \Gamma} k e_{\nu}\) with a 1-dimensional trivial quotient. On the other hand, \(B_{[\alpha : \beta]}\) has the 1-dimensional trivial submodule \(k e_0\), and the quotient is a simple module. This is explained by the isomorphism \(B_{[\alpha : \beta]} \cong A_{[\alpha : \beta]}\), where \(^t\) denotes the adjoint. (If \(M = \bigoplus_{\gamma \in \Gamma} k e_\gamma\) is a left \(\Gamma\)-graded \(W_{\Gamma}\)-module, the **adjoint** (or **restricted dual**) of \(M\) is the left \(\Gamma\)-graded \(W_{\Gamma}\)-module \(M'\) with \(M'_\gamma = \text{Hom}_k(M_{-\gamma}, k), v'_\gamma = v_{-\gamma}^*, e_{\mu}v'_\nu = -v'^*_{-\gamma}e_{\mu}\).)
Remark 3.2. We use a slightly different presentation of the families \(A_{[\alpha; \beta]}\) and \(B_{[\alpha; \beta]}\) than in [Su04]. In loc.cit the last two families are replaced by \(\tilde{A}(a')\) defined by
\[
e_{\mu} v_{\nu}' = (\nu + \mu)_{\mu + \nu}, \quad \nu \neq 0, \quad e_{\mu} v_0 = \mu (1 + (\mu + 1) a')_{\mu},
\]
and by \(\tilde{B}(a')\) defined by
\[
e_{\mu} v_{\nu}' = \nu_{\mu + \nu}, \quad \nu \neq -\mu, \quad e_{\mu} v_{-\mu} = -\mu (1 + (1 + a')_{\mu}) v_0,
\]
for \(a' \in \mathbb{k} \cup \{\infty\}\). If \(a' = \infty\) then \(1 + (1 + a')_{\mu}\) in the above definition is regarded as \(\mu + 1\). Note that \(\tilde{A}(a')\) (resp. \(\tilde{B}(a')\)) is isomorphic to \(A_{[1 + a', a']}(\text{resp. } B_{[1 + a', a']})\) if \(a' \neq \infty\) and to \(A_{[1; 1]}\) (resp. \(B_{[1; 1]}\)) if \(a' = \infty\), for \(v_{\nu} = \nu v_{\nu}'\) (resp. \(v_{\nu} = \frac{1}{\nu} v_{\nu}'\)) if \(\nu \neq 0\), and \(v_0 = v_0'\).

For the Witt algebra the choice of the basis is the same in [KSS5], however there \(a' \in \mathbb{k}\) and modules are classified up to inversion: replacing \(v_0\) by \(-v_{-\nu}\).

Let us show how to obtain the intermediate series modules using results of Section 2

Proposition 3.3. Let \(\Gamma = \mathbb{Z}\) act on \(\mathbb{k}[a, b]\) as \(t^\nu p(a, b) = p(a + \nu, b)t^\nu\), and let \(T := \mathbb{k}[a, b] \rtimes \Gamma\). The map \(\phi : W_T \to T, \phi(e_{\mu}) = (a + b\mu)t^\mu,\) induces an anti-homomorphism \(\Phi : U(W_T) \to T\). Consequently, \(T\) is a left \(U(W)\)-module via \(e_{\mu} p(a, b)t^\nu = (a + \nu + b\mu)p(a, b)t^{\mu + \nu}\).

Proof. Note that \(\mathbb{k}^2\) parametrises a set of intermediate series modules \(N(\alpha, \beta) := V(\alpha, \beta)\) and \(e_{\mu} v_0^{(\alpha, \beta)} = (a + b\mu)(\nu, \beta)v_{(\alpha, \beta)}\). Further, \(N(\alpha, \beta)(\nu) \cong N(\alpha + \nu, \beta)\) and hence \(\sigma'(\nu, \beta) = (\alpha + \nu, \beta)\) (using the notation of Section 2). The proposition therefore follows by Theorem 2.5 and Remark 2.7. \(\square\)

Remark 3.4. Let \(\Gamma = \mathbb{Z}\) and \(T = \mathbb{k}[a, b] \rtimes \mathbb{Z}\). We may compose the map \(\Phi\) of Proposition 3.3 with the canonical anti-automorphism \(e_n \mapsto -e_n\) of \(U(W)\) to obtain a homomorphism \(\Phi' : U(W) \to T, e_n \mapsto (-a - bn)t^n\).

Recall that in [SW15] a homomorphism \(\hat{\phi}\) was constructed from \(U(W)\) to \(T' := \mathbb{k}[u, v, v^{-1}, w] / (uv - vu - v^2, uw - wu - vw, vw - wv)\), defined by \(\hat{\phi}(e_n) = (u - (n - 1)w)v^{n-1}\). The reader may verify that \(\alpha : T' \to T\) defined by \(u \mapsto (b - a)t, v \mapsto t, w \mapsto bt\)
is an isomorphism of graded rings and that \(\alpha \hat{\phi} = \Phi'\). Thus Proposition 3.3 generalises the construction of \(\hat{\phi}\).

We now discuss applications of \(\Phi\) to the representation theory of \(W_T\). For \(p = (\alpha, \beta) \in \mathbb{A}^2\) we denote by \(I(p)\) the ideal \((a - \alpha, b - \beta)\) in \(\mathbb{k}[a, b]\). For \(q\) infinitely near to \(p\), corresponding to \([x : y] \in \mathbb{P}^1\), we denote by \(I(q)\) the ideal \((y(a - \alpha) - x(b - \beta), (a - \alpha)^2, (a - \alpha)(b - \beta), (b - \beta)^2)\).

Let \(B = \Phi(U(W_T))\), and note that \(B\) is contained in the double idealizer \(R = \mathbb{k}[a, b] + (I(0, 0)T \cap TI(0, 1))\). From the discussion in the introduction, then, we expect three families of intermediate series \(U(W_T)\)-modules, one parameterised by \(\mathbb{A}^2 \setminus \{(0, 0), (0, 1)\}\) and two parameterised by \(\mathbb{P}^1\). Note that because \(\Phi\) is an anti-homomorphism, right \(B\)-modules will correspond to left \(U(W_T)\)-modules.

By construction of \(\Phi\) we have \(V(\alpha, \beta) := T/I(p)T\), considered as a \(B\)-module. Removing \(V(0, 0)\) and \(V(0, 1)\) we obtain the two-dimensional family we expect. We next show that we also obtain the two \(\mathbb{P}^1\)-families \(A_{[\alpha; \beta]}\) and \(B_{[\alpha; \beta]}\).

Proposition 3.5. Let \([x : y] \in \mathbb{P}^1\) and let \(I(q) = (ya - xb, a^2, ab, b^2)\) define a point infinitely near to \((0, 0)\). Let
\[
P(q) = \frac{\mathbb{k}[a, b] + I(0, 0)T}{I(0, 0) + I(q)T}.
\]
Then \(A_{[x:y]} \cong P(q)\).

Proof. If \(w \in \mathbb{k}[a, b] + I(0, 0)T\) let \(w\) be the image of \(w\) in \(P(q)\). If \(x \neq 0\) we choose a basis
\[
v_{\nu} = \begin{cases} \alpha T \nu \neq 0, \\ 1 \nu = 0 \end{cases}
\]
for \(P(q)\).
Using the anti-homomorphism, we compute for $\nu \neq 0$

$$e_\mu^\nu v_\nu = \overline{at^\nu(a + b\mu)t^\mu} = a(a + b\mu + \nu)t^{\mu + \nu} = \nu a t^{\mu + \nu} = \begin{cases} \nu v_{\nu + \mu} & \nu + \mu \neq 0, \\ 0 & \nu + \mu = 0. \end{cases}$$

and

$$e_\mu^\nu v_0 = (a + b\mu)t^\mu = (a + \frac{y}{x}a\mu)t^\mu = \left(1 + \frac{y}{x}\mu\right)v_\mu,$$

so $P(q) \cong A_{[x:y]}$ as claimed.

If $y \neq 0$ we pick a basis

$$v_\nu = \begin{cases} b \nu & \nu \neq 0, \\ 1 & \nu = 0, \end{cases}$$

and obtain $e_\mu^\nu v_\nu = \nu v_{\nu + \mu}$, $e_\mu^\nu v_0 = (\frac{y}{x} + \mu)v_\mu$, $e_\mu^\nu v_{-\mu} = 0$. Thus $P(q) \cong A_{[x:y]}$ again. \hfill $\square$

In the next result, note the change of sides from the left modules $Q(q)$ defined in the introduction.

**Proposition 3.6.** Let $[x : y] \in \mathbb{P}^1$ and let $I(q) = (ya - x(b - 1), a^2, a(b - 1), (b - 1)^2)$ define a point infinitely near to $(0, 1)$. Let

$$Q(q) = \frac{I(0, 1)}{T(q)} \oplus \bigoplus_{0 \neq \nu \in \Gamma} \kappa[a, b]t^\nu.$$

Then $B_{[x:y]} \cong Q(q)$.

**Proof.** If $x \neq 0$ we choose a basis

$$v_\nu = \begin{cases} t^\nu & \nu \neq 0, \\ 0 & \nu = 0, \end{cases}$$

for $Q(q)$. We compute for $\nu + \mu \neq 0, \nu \neq 0$

$$e_\mu^\nu v_\nu = (a + b\mu + \nu)t^{\mu + \nu} = (\mu + \nu)t^{\mu + \nu} = (\mu + \nu)v_{\nu + \mu},$$

and

$$e_\mu^\nu v_0 = a(a + b\mu)t^\mu = 0,$$

$e_\mu^\nu v_{-\mu} = a + b\mu - \mu = \left(1 + \frac{y}{x}\mu\right)v_0$.

If $y \neq 0$ we pick a basis

$$v_\nu = \begin{cases} \nu t^\nu & \nu \neq 0, \\ 0 & \nu = 0. \end{cases}$$

We get $e_\mu^\nu v_\nu = \nu v_{\nu + \mu}$, $e_\mu^\nu v_0 = 0$, $e_\mu^\nu v_{-\mu} = \left(\frac{y}{x} + \mu\right)v_0$. \hfill $\square$

4. **Factors of** $U(W_T)$

In this section we generalise techniques from [SW15] to show that $B = \Phi(U(W_T))$ is not left or right noetherian. This in particular implies that $U(W_T)$ is not left or right noetherian, which was proved earlier in [SW13 SW13].

For $0 \neq \mu \in \Gamma$, let

$$p_\mu = e_\mu e_{3\mu} - e_{2\mu} - \mu e_{4\mu}.$$

**Lemma 4.1.** We have $\Phi(p_\mu) = \mu^2b(1 - b)t^{4\mu}$.

**Proof.** Let us compute

$$\Phi(e_\mu e_{3\mu} - e_{2\mu} - \mu e_{4\mu}) = ((a + 3\mu b)(a + \mu b + 3\mu) - (a + 2\mu b)(a + 2\mu b + 2\mu) - \mu(a + 4\mu b)) t^{4\mu} = \mu^2b(1 - b)t^{4\mu}.$$

$\square$

Fix $0 \neq \mu \in \Gamma$ and let $I = B\Phi(p_\mu)B$.

**Lemma 4.2.** For all $\nu \in \Gamma$ we have $b(1 - b)t^\nu \in I$. In particular, $I$ does not depend on the choice of $\mu$. Consequently, $I = b(1 - b)\kappa[a, b] \times \Gamma$. 7
Proof. We have

$$\Phi(e_{v-4\mu})b(1-b)t^{4\mu} - b(1-b)t^{4\mu}\Phi(e_{v-4\mu}) = (\Phi(e_{v-4\mu}) - \Phi(e_{v-4\mu}) - 4\mu)b(1-b)t^\nu = -4\mu b(1-b)t^\nu.$$ 

Thus the first claim follows by Lemma 4.1. Note that

$$\text{Note that } I \subseteq b(1-b)[a, b] \times \Gamma, \text{ and as } b(1-b) \in I \text{ and } a \in B, \text{ we have } b(1-b)k[a] \times \Gamma \subseteq I \text{. Since also } (a + b\mu)t^\nu \in B, \text{ we easily obtain by induction on } n \text{ that } b(1-b)b^n k[a] \times \Gamma \subseteq I \text{ for all } n \geq 0, \text{ and thus the last claim.} \blacksquare$$

**Proposition 4.3.** The ideal $I$ is not finitely generated as a left or right ideal of $B$.

Proof. We first compute

$$(1) \quad (a + b\nu_1)t^{\nu_1} \cdots (a + b\nu_t)t^{\nu_t} p(a, b)b(1-b)t^\lambda =$$

$$(a + b\nu_1) \cdots (a + b\nu_t + \nu_1 + \cdots + \nu_{t-1}) p(a + \nu_1 + \cdots + \nu_{t-1} + \nu_t, b)b(1-b)t^{\nu_1 + \cdots + \nu_t + \lambda},$$

$$(2) \quad p(a, b)b(1-b)t^\lambda (a + b\nu_1) \cdots (a + b\nu_t)t^{\nu_t} =$$

$$p(a, b)b(1-b)(a + b\nu_1 + \lambda) \cdots (a + b\nu_t + \lambda + \nu_1 + \cdots + \nu_{t-1})t^{\lambda + \nu_1 + \cdots + \nu_t}.$$

Let us assume that $I$ is finitely generated as a left ideal of $B$. Then there exist $\mu_1, \ldots, \mu_k \in \Gamma$ such that $I = B(I_{\mu_1} + \cdots + I_{\mu_k})$. Let us take $\mu \neq \mu_i, 1 \leq i \leq k$. It follows from (1) that $(B(I_{\mu_1} + \cdots + I_{\mu_k}))_{\mu}$ is contained in $(a, b)b(1-b)t^\mu$, a contradiction to Lemma 4.2.

Let us assume now that $I$ is finitely generated as a right ideal in $B$. Then there exist $\mu_1, \ldots, \mu_k \in \Gamma$ such that $I = (I_{\mu_1} + \cdots + I_{\mu_k})B$. For $\mu \neq \mu_i, 1 \leq i \leq k$, we obtain from (2) that $((I_{\mu_1} + \cdots + I_{\mu_k})B)_{\mu}$ is contained in $(a + \mu, b - 1)b(1-b)t^\mu$, which again contradicts Lemma 4.2. \blacksquare

**Remark 4.4.** Note that the same proof works if $\Gamma$ is a submonoid of $k$. Lemma 4.2 yields in this case $b(1-b)t^{\mu n} \in I$, for $n \geq 4$. The proof of Proposition 4.3 can then be adapted in an obvious way to apply to this a slightly more general situation. In particular, $\Phi(U(W_+))$ is not noetherian, where $W_+$ is the subalgebra of $W$ generated by $\{e_n : n \geq 1\}$. (This last statement is proved in [SW15].)

We now show that the image $B_{\beta}$ of the map $\phi_{\beta} : U(W) \rightarrow B/(b - \beta)$ induced from $\Phi$ is noetherian for every $\beta \in k$. This is an analogue of [SW15 Proposition 2.1].

**Lemma 4.5.** We have $B_0 \cong k + a(k[a] \times \Gamma), B_1 \cong k + (k[a] \times \Gamma)a, B_\beta \cong k[a] \times \Gamma$ for $\beta \neq 0, 1$.

Proof. The lemma is obvious for $\beta = 0, 1$. Assume therefore that $\beta \neq 0, 1$. Let us compute

$$(a + \beta \mu)t^\mu(a + \beta \nu)t^\nu - a(a + \beta(\mu + \nu))t^{\mu + \nu} = (\mu a + \beta \mu(\beta \nu + \mu))t^{\mu + \nu} \in B_{\beta}.$$ 

Subtracting $\mu(a + b(\mu + \nu)t^{\mu + \nu}$, we thus have $\beta \mu \nu(\beta - 1)t^{\mu + \nu} \in B_{\beta}$, and hence our claim. \blacksquare

**Proposition 4.6.** $B_{\beta}$ is noetherian for every $\beta \in k$.

Proof. For $\beta \neq 0, 1$ this follows by [MR01 Theorem 4.5] using Lemma 4.5. Let us note that $B_0 \cong B_1$ by conjugation with $a$. It thus suffices to prove that $B_0$ is right noetherian and $B_1$ is left noetherian. We show that $B_0$ is right noetherian, and following the same argument one can show that $B_1$ is left noetherian.

We first note that $I = a(k[a] \times \Gamma)$ is a maximal right ideal in $C = k[a] \times \Gamma$. To see this, let $J \neq I$ be a right ideal which contains $I$. Take an element $c = \sum \alpha_i t^i \neq 0$ in $J$ with the minimal number of nonzero coefficients. Since $ca = \sum \alpha_i (a + \mu)t^i \in J$ and hence $\sum \alpha_i \mu t^i \in J$, the minimality assumption implies that $J = k[a] \times \Gamma$.

The proposition now follows by [Rob72 Theorem 2.2] using Lemma 4.5. \blacksquare

**Remark 4.7.** We remark that for any $\beta$ the modules $V(\alpha, \beta)$ are all faithful over $B_{\beta}$, and it follows easily that the $B_{\beta}$ are primitive. In general, the primitive factors of $U(W_\Gamma)$ are unknown, even for $\Gamma = \mathbb{Z}$.
References

[ATV90] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 33–85.

[ATV91] ———, Modules over regular algebras of dimension 3, Invent. Math. 106 (1991), no. 2, 335–388.

[AV90] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, J. Algebra 133 (1990), no. 2, 249–271.

[KS85] I. Kaplansky and L. J. Santharoubane, Harish-Chandra modules over the Virasoro algebra, Infinite-dimensional groups with applications (Berkeley, Calif., 1984), Math. Sci. Res. Inst. Publ., vol. 4, Springer, New York, 1985, pp. 217–231.

[MR01] J. C. McConnell and J. C. Robson, Noncommutative Noetherian rings, revised ed., Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001, With the cooperation of L. W. Small.

[Rob72] J. C. Robson, Idealizers and hereditary Noetherian prime rings, J. Algebra 22 (1972), 45–81.

[Rog04] D. Rogalski, Idealizer rings and noncommutative projective geometry, J. Algebra 279 (2004), no. 2, 791–809.

[RZ08] D. Rogalski and J. J. Zhang, Canonical maps to twisted rings, Math. Z. 259 (2008), no. 2, 433–455.

[Sie11] S. J. Sierra, Geometric idealizer rings, Trans. Amer. Math. Soc. 363 (2011), no. 1, 457–500.

[SW14] S. J. Sierra and C. Walton, The universal enveloping algebra of the Witt algebra is not noetherian, Adv. Math. 262 (2014), 239–260.

[SW15] ———, Maps from the enveloping algebra of the positive Witt algebra to regular algebras, Pacific J. Math., 284 (2016), no. 2, 475–509.

[Su94] Y. C. Su, Harish-Chandra modules of the intermediate series over the high rank Virasoro algebras and high rank super-Virasoro algebras, J. Math. Phys. 35 (1994), no. 4, 2013–2023.

[V96] M. Van den Bergh, A translation principle for the four-dimensional Sklyanin algebras, J. Algebra 184 (1996), no. 2, 435–490.