Rate-Equivocation Optimal Spatially Coupled LDPC Codes for the BEC Wiretap Channel

Vishwambhar Rathi‡, Rüdiger Urbanke†, Mattias Andersson*, and Mikael Skoglund*
*School of Electrical Engineering and the ACCESS Linnaeus Center,
Royal Institute of Technology (KTH),
Stockholm, Sweden
Email: {vish, amattias, skoglund}@ee.kth.se
†School of Computer and Communication Sciences
EPFL, Lausanne, Switzerland
Email: ruediger.urbanke@epfl.ch

Abstract—We consider transmission over a wiretap channel where both the main channel and the wiretapper’s channel are Binary Erasure Channels (BEC). We use convolutional LDPC ensembles based on the coset encoding scheme. More precisely, we consider regular two edge type convolutional LDPC ensembles. We show that such a construction achieves the whole rate-equivocation region of the BEC wiretap channel. Convolutional LDPC ensemble were introduced by Felström and Zigangirov and are known to have excellent thresholds. Recently, Kudekar, Richardson, and Urbanke proved that the phenomenon of “Spatial Coupling” converts MAP threshold into BP threshold for transmission over the BEC.

The phenomenon of spatial coupling has been observed to hold for general binary memoryless symmetric channels. Hence, we conjecture that our construction is a universal rate-equivocation achieving construction when the main channel and wiretapper’s channel are binary memoryless symmetric channels, and the wiretapper’s channel is degraded with respect to the main channel.

I. INTRODUCTION

The wiretap channel was introduced by Wyner in [1]. The basic diagram is depicted in Figure 1. We consider the setting when both channels are Binary Erasure Channels (BEC). We note a BEC with erasure probability $\epsilon$ by $\text{BEC}(\epsilon)$. In a wiretap channel, Alice is communicating a message $W$ to Bob. The message is uniformly chosen from the message set $\mathcal{W}_n$ and it is sent through the main channel, which is a $\text{BEC}(\epsilon_m)$. Alice encodes $W$ as an $n$ bit vector $\underline{X}$ and transmits it. Bob receives a partially erased version of $\underline{X}$, denote it by $\underline{Y}$. Eve is observing $\underline{X}$ via the wiretapper’s channel, which is a $\text{BEC}(\epsilon_w)$. Let $\underline{Z}$ denote the observation of Eve. We denote this wiretap channel by $\text{BEC-WT}(\epsilon_m, \epsilon_w)$. In order to fulfill the requirement of degradation of the wiretapper’s channel w.r.t. the main channel, we assume that $\epsilon_w \geq \epsilon_m$. We denote the capacity of the main channel and wiretapper’s channel by $C_m = 1 - \epsilon_m$ and $C_w = 1 - \epsilon_w$, respectively. The encoding of the message $W$ by Alice should be such that Bob is able to decode $\underline{W}$ reliably and that $\underline{Z}$ provides as little information to Eve as possible about $W$.

Assume that transmission takes place using the code $G_n$ and let $\hat{W}$ be the message decoded by Bob. We define the performance metric for reliability to be the average error probability $P_e(G_n)$.

$$P_e(G_n) = \frac{1}{|\mathcal{W}_n|} \sum_{w \in \mathcal{W}_n} P(\hat{W} \neq w \mid W = w).$$

We use the normalized equivocation $R_e$ as the performance metric for secrecy.

$$R_e(G_n) = \frac{1}{n} H(W \mid \underline{Z}).$$

The rate $R$ of the coding scheme for the intended receiver Bob is given by

$$R(G_n) = \frac{\log_2(|\mathcal{W}_n|)}{n}.$$  

We say that a rate-equivocation pair $(R, R_e)$ is achievable using a sequence of codes $G_n$ if

$$\lim_{n \to \infty} R(G_n) = R, \lim_{n \to \infty} P_e(G_n) = 0, R_e \leq \liminf_{n \to \infty} R_e(G_n).$$

The achievable rate-equivocation pair $(R, R_e)$ for the BEC-WT$(\epsilon_m, \epsilon_w)$ is given by

$$R_e \leq R \leq C_m, \quad 0 \leq R_e \leq C_m - C_w.$$
Note that we consider weak notion of secrecy as opposed to the strong notion [3], [4].

From Figure 2, we see that the boundary of the achievable rate-equivocation region is composed of two branches, namely AB and BC. The branch AB corresponds to achieving perfect secrecy, i.e., $R_e = R \leq C_m - C_w$. The point B corresponds to the secrecy capacity, the highest rate at which perfect secrecy is possible. The branch BC corresponds to achieving information rates higher than secrecy capacity. However, in this case some information “leaks” to Eve (the equivocation in this case is strictly smaller than the rate).

Recently, it has been shown that, using Arikan’s polar codes [5], it is possible to achieve the whole rate-equivocation region [6–9]. In this paper, we show that convolutional LDPC codes achieve the whole rate-equivocation region for the BEC wiretap channel. Why might this be of interest? Compared to polar codes, convolutional LDPC ensembles have two potential advantages. First, these codes are not only asymptotically very good but they are know to be competitive with the best known codes already for modest lengths. Second, convolutional LDPC ensembles have the potential of being universal, i.e., one and the same code is optimal for a large class of channels. Before discussing this point in more detail, let us first quickly review the literature on convolutional LDPC codes.

Convolutional LDPC codes were introduced by Felström and Zigangirov and were shown to have excellent thresholds [10]. There has been a significant amount of work done on convolutional-like LDPC ensembles [11–16], and see in particular the literature review in [17]. The explanation for the excellent performance of convolutional-like or “spatially coupled” codes over the BEC was given by Kudekar, Richardson, and Urbanke in [17]. (In the following, we also use the term spatially coupled codes when we refer to convolutional like codes.) More precisely, it was shown in [17] that the phenomenon of spatial coupling has the effect of converting MAP threshold of underlying ensemble to BP threshold for BEC and regular LDPC codes. This phenomenon has been observed to hold in general over Binary Memoryless Symmetric (BMS) channels, see [18], [19].

Thus, when point-to-point transmission is considered over BMS channels, regular convolutional-like LDPC ensembles are conjectured to be universally capacity achieving. This is because the MAP threshold of regular LDPC ensembles converges to the Shannon threshold for BMS channels as their left and right degrees are increased by keeping the rate fixed. To date there is only empirical evidence for this conjecture. But should in the future a proof be found that spatially coupled codes are indeed universal for point-to-point channels, then this would immediately imply that our construction for the wiretap channel is also universal.

Let us summarize. Our two main motivations for considering code constructions for the wire-tap channel based on spatially coupled codes is that these codes perform very well already for modest code lengths and that they have the potential to be universal.

In [20] and [21] coset encoding scheme based sparse graph codes were given. It was shown in [22] that a two edge type LDPC code is a natural candidate for the coset encoding scheme and optimized degree distributions were presented. In the next section we describe our code design method using spatially coupled codes.

II. CODE CONSTRUCTION

We first describe the coset encoding scheme. Let $H$ be an $(1 - \tau)n \times n$ LDPC matrix and let $H_1$ and $H_2$ be the submatrices of $H$ such that

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

where $H_1$ is an $(1 - \tau_1)n \times n$ and $H_2$ is an $\tau n \times n$ matrix. Let $G_n^{(1)}$ be the code with parity-check matrix $H_1$, and let $G_n^{(1,2)}$ be the code whose parity-check matrix is $H$. Assume that Alice wants to transmit an $nR$-bit message $\mathbf{S}$. To do this she transmits $X$, which is a randomly chosen solution of

$$H_1 X = [0 \cdots 0 \mathbf{S}]^T.$$ 

As shown in [20], if $H$ is capacity achieving over the wire-tapper’s channel then $\mathbf{S}$ is perfectly secure from Eve. Also, if the threshold of the code $G_n^{(1)}$ is higher than the main channel erasure probability $\epsilon_m$ then Bob can recover $\mathbf{S}$ reliably. We call this wiretap code $G_n$.

The code described by the LDPC matrix $H$ given in (6) is a two edge type LDPC code. The two types of edges are the edges connected to check nodes in $H_1$ and those connected to check nodes in $H_2$. An example of a two edge type LDPC code is shown in Figure 5.

For our purpose it is sufficient to focus on regular two edge type LDPC ensembles.

**Definition II.1** \{(1, 2, r_1, r_2)\} Two Edge Type LDPC Ensemble. A \{(1, 2, r_1, r_2)\} two edge type LDPC ensemble of blocklength $n$ contains all the bipartite graphs (allowing multiple edges between a variable node and a check node) where all the $n$ variable nodes are connected to $1_1$, check nodes of type $i$ and all the type $i$ check nodes have degree $r_i$, $i \in \{1, 2\}$.

A protograph of a regular two edge type LDPC code is shown in Figure 4.

Based on the definition of an \{1, r, L, w\} ensemble from [17], we define the regular spatially coupled two edge type
LDPC ensemble. Before giving this definition, we define $T(1)$ to be the set of $w$-tuple of non-negative integers which sum to $1$. More precisely, $T(1) = \{ (t_0, \cdots , t_{w-1}) : \sum_{j=0}^{w-1} t_j = 1 \}$.

Remark: Note that the $w$-tuple $(t_0, \cdots , t_{w-1})$ is called a type in [17]. We avoid this terminology as we refer to different edges in two edge type LDPC ensemble by their type.

Definition II.2 $(\{l_1, l_2, r_1, r_2, L, w\}$ Spatially Coupled Two Edge Type LDPC Ensemble). Assume that there are $M$ variable nodes at positions $[-L, L]$, $L \in \mathbb{N}$. The blocklength of a code in the ensemble is $n = M(2L + 1)$. Every variable node has degree $l_1$ with respect to type $1$ edges and $l_2$ with respect to type $2$ edges. At each position there are $M$ variable nodes, $\frac{1}{l_1} M$ check nodes of type $1$ which has degree $r_1$, and $\frac{1}{l_2} M$ check nodes of type $2$ which has degree $r_2$.

Assume that for each variable node we order its edges in an arbitrary but fixed order. A constellation $c$ of type $j$ is an $1_j$-tuple, $c = (c_1, \cdots , c_{c_j})$ with elements in $\{0, 1, \cdots , w-1\}$, $j \in \{1, 2\}$. Its operational significance is that if a variable node at position $i$ has type $j$ constellation as $c_j$ then its $k$-th edge of type $j$ is connected to a check node at position $i + c_k$, $j \in \{1, 2\}$. We denote the set of all the type $j$ constellations by $C_j$. Let $\tau(c)$ be the $w$-tuple which counts the occurrence of $0, 1, \cdots , w-1$ in $c$. Clearly, if $c$ is a type $j$ constellation then $\tau(c) \in T(1_j)$. We impose uniform distribution over both the type of constellations. This imposes the following distribution over $t \in T(1_j)$

$$p^{(j)}(t) = \left\{ \begin{array}{ll}
\frac{|\{c \in C_j : \tau(c) = t\}|}{w^j}, & j \in \{1, 2\}.
\end{array} \right.$$ 

Now we pick $M$ so that $M p^{(1)}(t_1) p^{(2)}(t_2)$ is a natural number for $\forall t_1 \in T(1_1), \forall t_2 \in T(1_2)$. For each position $i$ we pick $M p^{(1)}(t_1) p^{(2)}(t_2)$ which have their type $j$ edges assigned according to $t_j$, $j \in \{1, 2\}$. We use a random permutation for each variable and type $j$ edge over $l_j$ letters to map $t_j$ to a constellation, $j \in \{1, 2\}$. Ignoring boundary effects, for each check position $i$, the number of type $j$ edges that come from variables at position $i - k$, $k \in \{0, \cdots , w-1\}$, is $M \frac{l_j}{w}$, $j \in \{1, 2\}$. This implies, it is exactly a fraction $\frac{1}{w}$ of the total number $M l_j$ of sockets at position $i$. At the check nodes, we distribute this edges by randomly choosing a permutation over $M l_j$ letters, to the $M \frac{l_j}{w}$ check nodes of type $j$, $j \in \{1, 2\}$.

Remark: Each of the $l_1$ (resp. $l_2$) type $1$ (resp. $2$) connections of a variable node at position $i$ is uniformly and independently chosen from the range $[i, \cdots , i + w - 1]$, where $w$ is a “smoothing” parameter. Similarly, as was remarked in [17], for each check node each edge is roughly independently chosen to be connected to one of its nearest $w$ “left” neighbors. More precisely, the corresponding probability deviates at most by a term of order $1/M$ from the uniform distribution.

To summarize, a $\{l_1, l_2, r_1, r_2, L, w\}$ spatially coupled two edge type LDPC ensemble is obtained by replacing the standard regular LDPC ensemble in the $(1, r, L, w)$ ensemble (defined in [17]) by a $\{l_1, l_2, r_1, r_2\}$ two edge type LDPC ensemble. The spatial coupling is done such that only the edges of the same type are coupled together. An example of a protograph of a two edge type LDPC code is shown in Figures 4 and its spatially coupled version is shown in Figure 5.

In the next lemma we show that if the degrees of the two types of check nodes are the same, i.e. if $r_1 = r_2 = r$, then the $\{l_1, l_2, r, r, L, w\}$ spatially coupled two edge type LDPC ensemble has the same asymptotic performance as that of the spatially coupled ensemble $(l_1 + l_2, r, L, w)$.

Lemma II.3. The $\{l_1, l_2, r, r, L, w\}$ spatially coupled two edge type LDPC ensemble has the same BP threshold as the spatially coupled ensemble $(l_1 + l_2, r, L, w)$.

Proof: Let $x_i^{(l_1)}$ be the average erasure probability which is emitted by a variable node at position $i$ in the $i^{th}$ iteration along an edge of type $j$, $j \in \{1, 2\}$. For $i \in \left[-L, L\right]$, we set $x_i^{(l_1)} = 0$. For $i \in \left[-L, L\right]$, $j \in \{1, 2\}$, and $l = 0$, we set $x_i^{(l_0)} = \epsilon$.

As in [17], the density evolution recursion for the $\{l_1, l_2, r, r, L, w\}$ two edge type spatially coupled LDPC ensemble is given by

$$x_i^{(l_1)} = \epsilon \left( 1 - \frac{1}{w} \sum_{p=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x^{((l-1,1))}_{i+p-k} \right)^{r_1-1} \right)^{l_1-1} \left( 1 - \frac{1}{w} \sum_{p=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x^{((l-1,2))}_{i+p-k} \right)^{r_2-1} \right)^{l_2}, \quad (7)$$
ensemble grows linearly in $E\{ \text{term} \}$, see [23, Appendix D], we obtain Lemma II.4. This proves the lemma.

This recursion is same as that of the standard regular edge type LDPC ensembles $\{1, l_2, r, L, w\}$. Indeed, for $l = 1$ and $i \in [-L, L]$, $x_i^{(1)} = x_i^{(1,2)} = \epsilon$ and for $i \notin [-L, L]$, $x_i^{(1)} = x_i^{(1,2)} = 0$. Thus, by induction on number of iterations $l$, $x_i^{(l,1)} = x_i^{(l,2)}$. Hence we drop the superscript corresponding to the type of edge and write the density evolution recursion as

$$
x_i^{(l)} = \epsilon \left( 1 - \frac{1}{w} \sum_{p=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+p-k}^{(l-1)} \right)^{r-1} \right) \ln (1 - \frac{1}{w} \sum_{p=0}^{w-1} x_{i+p-k}^{(l-1)}). \quad (8)
$$

This recursion is same as that of $\{(1, l_2, r, L, w)\}$ spatially coupled ensemble given in [17]. This proves the lemma. ■

Before proving the main result, we show that regular two edge type LDPC ensembles $\{1, l_2, r, x\}$ have the same growth rate of the average stopping set distribution as that of the standard regular $\{1, l_2, x\}$ LDPC ensemble.

**Lemma II.4.** Consider the $\{1, l_2, r, x\}$ regular two edge type LDPC ensemble with blocklength $n$, $l_1 \geq 3$, and positive design rate. Let $N(n, w\omega)$ be the stopping set distribution of a randomly chosen code from this ensemble and let $E(N(n, \omega n))$ be its average. Then the growth rate of $E(N(n, \omega n))$ is the same as that of the standard regular $\{1, l_2, r\}$ ensemble. In particular, the minimum stopping set distance of the $\{1, l_2, r, x\}$ regular two edge type LDPC ensemble grows linearly in $n$.

**Proof:** Using standard counting arguments we obtain

$$
E(N(n, \omega n)) = \binom{n}{n\omega} \frac{\text{coef} \left( p(x) \frac{1}{1-x}, x^{\omega n} \right) \text{coef} \left( p(x) \frac{1}{1-x}, x^{\omega 2n} \right)}{(\frac{1}{\omega_1 n}, \frac{1}{\omega_2 n}}),
$$

where $p(x) = (1+x)^r - \omega x$. Using Stirling’s approximation for binomial terms and the Hayman expansion for the coef term, see [23] Appendix D, we obtain

$$
\lim_{n \to \infty} \frac{\ln \left( E(N(n, \omega n)) \right)}{n} = (1 - l_1 - l_2) h(\omega) + \frac{1}{x} \ln \left( p(x) \right) - \omega_1 \ln(t) + \frac{1}{x_2} \ln \left( p(x) \right) - \omega_2 \ln(t), \quad (11)
$$

where $h(x) \triangleq -x \ln(x) - (1-x) \ln(1-x)$ is the binary entropy function, all the logarithms are natural logarithms, and $t$ is a positive solution of

$$
x_2 \left( (1+x)^{r-1} - \frac{1}{(1+x)^r - \omega x} \right) = \omega. \quad (12)
$$

From (11), we see that the growth rate is the same as that of the average stopping set distribution of the standard $\{1, l_2, x\}$ regular LDPC ensemble [24 Thm. 2]. Now, the linearity of minimum stopping set distance immediately follows from [24 Cor. 7]. ■

**Remark:** We could have come to this conclusion by specializing the general result contained in [25 Thm. 5]. But for the convenience of the reader, and since the above proof is so short, we decided to include a complete proof.

Lemma II.4 and [17 Lemma 1] imply that $\{(1, l_2, r, x, L, w)\}$ spatially coupled two edge type LDPC ensembles with variable node degree at least three have a linear minimum stopping set distance. This gives us the following lemma on the block error probability of the $\{1, l_2, r, x, L, w\}$ ensemble under iterative decoding.

**Lemma II.5.** Consider transmission over the BEC(ε) using the $\{1, l_2, r, x, L, w\}$ spatially coupled two edge type LDPC ensembles with BP threshold $e^*$ and blocklength $n$. Let $l_1 \geq 3$. Assume that $\epsilon < e^*$. Denote by $P_e(B)$ the block error probability under iterative decoding. Then

$$
\lim_{n \to \infty} n P_e(B) = 0.
$$

**Proof:** In fact, a much stronger result is true – the block error probability converges to 0 exponentially fast. But for our purpose we only need that it converges to zero faster than linearly.

To see why this is correct, fix $\epsilon < e^*$. Then, for any $\delta > 0$, there exists an $l$ so that after $l$ iterations of DE, the bit error probability is below $\delta/3$. Further, for $n = n(l)$, sufficiently large, the expected behavior over all instances of the code and the channel deviates from the density evolution predictions by at most $\delta/3$. Finally, by standard concentration results (see [23 Thm. 3.30]) it follows that the probability that a particular instance deviates more than $\delta/3$ from its average decays exponentially fast in the blocklength.

We summarize, with a probability which converges exponentially fast (in the blocklength) to 1, an individual instance will have reached a bit error probability of at most $\delta$ after a fixed number of iterations.

If $\delta$ is chosen sufficiently small, in particular smaller than the relative minimum stopping set distance, then we know that the decoder can correct the remaining erasures with probability 1. ■

In the following lemma we calculate the design rate of the spatially coupled two edge type ensemble.

**Lemma II.6 (Design Rate).** The design rate of the spatially coupled two edge type ensemble $\{(1, l_2, r_1, r_2, L, w)\}$ with
$w \leq 2L$ is given by

$$R(1, 1, 2, r_1, r_2, L, w) = \left(1 - \frac{1}{r_1} - \frac{1}{r_2}\right) - \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \frac{w + 1 - 2\sum_{i=0}^{w} (\frac{w}{r})r}{2L + 1}. \quad (13)$$

The design rate of the coset encoding scheme for the wiretap channel is given by

$$R_{\text{des}} = \frac{1}{r_2} - \frac{1}{r_2} \frac{w + 1 - 2\sum_{i=0}^{w} (\frac{w}{r})r}{2L + 1}. \quad (14)$$

Proof: Let $C_1(C_2)$ be the number of type one (two) check nodes connected to variable nodes and let $V$ be the number of variable nodes. Then $R(1, 1, 2, r_1, r_2, L, w) = 1 - C_1/V - C_2/V$ and $R_{\text{des}} = C_2/V$. The calculations then follow from the proof of [17] Lemma 3.

The number of possible messages $\mathcal{S}$ of the coset encoding scheme is given by the number of cosets of $G_n(1,2)$ in $G_n(1)$. For a standard LDPC ensemble the design rate is a lower bound on the average (over the channel and ensemble) equivocation of variable nodes. Then

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr(R(G_n) > C_w + R) = 0. \quad (17)$$

This implies

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \Pr(R(G_n) < R) = 0. \quad (19)$$

Proof: Let $C_1(C_2)$ be the number of type one (two) check nodes connected to variable nodes and let $V$ be the number of variable nodes. Then $R(1, 1, 2, r_1, r_2, L, w) = 1 - C_1/V - C_2/V$ and $R_{\text{des}} = C_2/V$. The calculations then follow from the proof of [17] Lemma 3.

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Theorem II.7. Consider transmission over the BEC-WT($\epsilon_m$, $\epsilon_w$) using spatially coupled regular $\{1, 1, 2, r, r, L, w\}$ two edge type LDPC ensemble. Assume that the desired rate of information transmission from Alice to Bob is $R, \; R \leq C_m - C_w$. Let $1_1 = [(1 - C_m - R)\mathbf{r}]$ and $1_2 = [(1 - C_w)\mathbf{r}] - [(1 - C_m - R)\mathbf{r}]$. Let $R_e$ be the average (over the channel and ensemble) equivocation achieved for the wiretapper. Then,

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(P_e(G_n)) = 0,$$

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} R_e = R. \quad (18)$$

Let $R(G_n)$ be the rate from Alice to Bob of a randomly chosen code in the ensemble. Then

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr(R(G_n) < R) = 0. \quad (19)$$

Proof: We first show that the rate from Alice to Bob is $R$ almost surely. Let $G_n(1,2)$ be a two edge type spatially coupled code, and let $G_n(1)$ be the code induced by its type 1 edges only. Then

$$R(G_n) = R(G_n(1)) - R(G_n(1,2)). \quad (16)$$

Since both the two edge type spatially coupled ensemble and the ensemble induced by its type 1 edges are capacity achieving we must have

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr(R(G_n(1)) > C_w + R) = 0, \quad (17)$$

and

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr(R(G_n(1,2)) > C_w) = 0. \quad (18)$$

This implies

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr(R(G_n) < R) = 0. \quad (19)$$

The reliability part easily follows from the capacity achieving property of the spatially coupled ensemble. This is because the rate of the ensemble corresponding to type 1 edges approaches $C_w + R$. As this ensemble is capacity achieving, its threshold is $1 - C_w - R$. As $R < C_m - C_w$, we see that the threshold is greater than $\epsilon_m$. This proves reliability.

To bound the equivocation of Eve, using the chain rule we expand the mutual information $I(X; S; Z)$ in two different ways

$$I(X; S; Z) = I(X; Z) + I(S; Z | X) \quad (20)$$

and

$$I(S; Z) = I(S; Z | X) + I(S; Z). \quad (21)$$

As $S \to X \to Z$ is a Markov chain, $I(S; Z | X) = 0$. Using $I(S; Z) = H(S) - H(S | Z)$, we obtain,

$$\frac{1}{n} H(S | Z) = \frac{1}{n} (H(S) + I(X; Z | S) - I(X; Z)) \quad (22)$$

and

$$\frac{1}{n} (H(S) + I(X; Z | S) - I(X; Z))$$

where we have used that $H(S) + I(X; Z | S) = H(S, X) = H(X)$ and that $I(X; Z)/n \leq C_w$. Thus

Since the ensemble induced by type 1 edges is capacity achieving its rate must equal its design rate asymptotically, so

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} H(X)/n = R + C_w. \quad (25)$$

Denote the block error probability of decoding $X$ from $Z$ and $S$ by $P_e(X; S, Z)$. From Fano’s inequality we obtain,

$$\frac{H(X|S,Z)}{n} \leq \frac{1}{n} P_e(X; S, Z) + P_e(X|S, Z)(1 - \epsilon_w). \quad (26)$$

Note that, as the two edge type spatially coupled construction is capacity achieving over the wiretapper’s channel, $\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr_e(X; S, Z) = 0$.

We now obtain the desired bound on the equivocation by substituting (26) and (25) in (24), and taking the limit $r, w, L, M \to \infty$. \qedsymbol

Note that in the previous theorem our requirement was to have perfect secrecy. Hence we constructed spatially coupled two edge type matrix such that it was capacity achieving over the wiretapper’s channel. In the next theorem we prove that using spatially coupled two edge LDPC codes, it is possible
to achieve an information rate equal to $C_m$, the capacity of the main channel, and equivocation equal to $C_m - \epsilon_w$.

**Theorem II.8.** Consider transmission over the BEC-WT($\epsilon_m, \epsilon_w$) using spatially coupled regular $\{1, 1_2, r, x, L, w\}$ two edge type LDPC ensemble. Assume that the desired rate of information transmission from Alice to Bob is $R$, $R > C_m - C_w$ and $R \leq C_m$. Let $l_1 = \lceil (1 - C_m) r \rceil$ and $l_2 = \lceil R x \rceil$. Let $R_e$ be the average (over the channel and ensemble) equivocation achieved for the wiretapper. Then,

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E} \{ P_e(G_n) \} = 0,$$

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} R_e = C_m - C_w.$$

Let $R(G_n)$ be the rate from Alice to Bob of a randomly chosen code in the ensemble. Then

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \Pr( R(G_n) < R ) = 0.$$

**Proof:** The proof that the rate is $R$ asymptotically is the same as in the proof of Theorem [II.7].

The reliability part easily follows from the capacity achieving property of the spatially coupled ensemble corresponding to type 1 edges. This is because the rate of the ensemble corresponding to type 1 edges approaches $C_m$. As this ensemble is capacity achieving, its threshold is $\epsilon_m$. This proves reliability.

The proof for equivocation is very similar to that of Theorem [II.7]. From (24), we know

$$\frac{1}{n} \sum_{Z} H(S | Z) \geq \frac{1}{n} \left( H(X) - \sum_{Z} H(X | Z, S) \right) - C_w. \quad (27)$$

Since the code induced by type 1 edges is capacity achieving we have

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \sum_{Z} H(S | Z) / n = C_m. \quad (28)$$

Note that as the two edge type code has rate $C_m - R$ and is capacity achieving, its threshold for the BEC is $1 - C_m + R$. As $R > C_m - C_w$, the threshold is higher than $\epsilon_w$. As in Theorem [II.7] given $S$ the error probability of decoding $X$ from $Z$, denoted by $P_e(X | S, Z)$ goes to zero. Thus (26) holds and we obtain

$$\lim_{r \to \infty} \lim_{w \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} \sum_{Z} H(X | S, Z) / n = 0. \quad (29)$$

We obtain the desired bound on the equivocation by substituting (28) and (29) in (27), and taking the limit $r, w, L, M \to \infty$.

**III. Numerical Results**

We have rigorously shown the optimality of the $\{1_1, 1_2, r, x, L, w\}$ ensemble. In this section, we briefly discuss the performance of the $\{1, 1_2, r, x, L, w\}$ ensemble, which is the two edge type extension of the $\{1, r, L\}$ ensemble discussed in [II.7]. Based on the method in [22], we numerically evaluate the equivocation of the $\{3, 3, 6, 12, L\}$ ensemble for the BEC-WT(0.5, 0.75). The results are given in Table [I]. We observe that as $L$ increases, the equivocation $R_e$ converges to $R$, the rate from Alice to Bob. Thus, the optimality of secrecy performance of the $\{1_1, 1_2, r, x, L, w\}$ ensemble seems to hold for the wiretap channel. The optimality of reliability performance has been conjectured to hold in [17].

| $L$  | 20   | 30   | 40   | 50   | 60   | 70   |
|------|------|------|------|------|------|------|
| $R$  | 0.2642 | 0.2582 | 0.2562 | 0.2545 | 0.2535 | 0.2535 |
| $R_e$ | 0.2276 | 0.235 | 0.2357 | 0.241 | 0.2425 | 0.2436 |

**TABLE I**

Rate from Alice to Bob ($R$) and equivocation of Eve ($R_e$) for different values of $L$, $M = 1000$ for $\{3, 3, 6, 12, L\}$ ensemble.

**IV. Conclusion**

We showed how to achieve the whole rate-equivocation region using spatially coupled regular two edge type LDPC codes over the binary erasure wiretap channel. As the spatially coupled two edge type LDPC codes are conjectured to achieve capacity over general erasure wiretap channel, we conjecture that our code construction is also universally optimal for the class of wiretap channel where the main channel and wiretapper’s channel are BMS channels and wiretapper’s channel is physically degraded with respect to the main channel.

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