Off-mass-shell Sudakov-like suppression factor for the fermionic four-point function in QCD

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Abstract

We consider a four-point process, associated with a wide-angle elastic scattering of two off-mass-shell spin-1/2 matter particles, in a non-abelian gauge theory. On the basis of a worldline approach, which reverts the functional to a path-integral description of the system, we factorize an eikonal (“soft”) subsector of the full theory and calculate the Sudakov-like suppression factor for the four-point function as a whole, once we have extracted the associated anomalous dimensions and taken into account the renormalization-group controlled evolution.

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1. Next to confinement, factorization defines the single most important issue associated with the theoretical confrontation of QCD. In particular, a clean passage from partonic to hadronic modes of description entails the formation of infrared-safe quantities, wherein long-distance (nonperturbative) contributions appear in a factorized form. A prime example of isolating long-distance behavior in the context of a purely field theoretical calculation, involving quarks and gluons, is provided by the Sudakov form factor (see, for example, [1,2,3] for reviews). Recently, two of us have established [4] an analogous mode of behavior which refers to a four-point process, pertaining to on-mass-shell elastic quark-quark scattering at a fixed angle, viewing this process not in terms of Feynman graph lines or as part of an operator product expansion (OPE) based treatment but rather as a whole.

Our approach to factorization relies on the so-called worldline casting of a non-abelian gauge field theory with spin-1/2 matter fields [3], which results from the reformulation

\[ \int D\bar{\psi}(x) D\psi(x) e^{S[\bar{\psi}(x),\psi(x),A_{\mu}(x)]} \ldots \rightarrow \int Dx(\tau) Dp(\tau) e^{S[x(\tau),p(\tau),A_{\mu}(x(\tau))]} \ldots , \]

.. taking us from a functional to a path-integral description of the system. In particular, working explicitly in the Feynman gauge, we have argued [3] that contributions to the path integral, which take into consideration only those paths that are straight lines almost everywhere (allowing, therefore, for the presence of cusps) and with the Dirac determinant set to unity, factorize in a most natural manner. In physical terms, the above specifications enable the isolation of a subsector of the full theory wherein the “live” gauge field exchanges can neither derail matter particles from their propagation paths nor create virtual pairs from the vacuum. Any derailment occurs only on a sudden-impulse basis and corresponds to the presence of cusps on the propagation contour.

One may now ask whether such a procedure is sufficient to accomplish the full factorization of the soft physics entering a given process, and to what extent the resulting construction depends on particular gauge choices. The answer to the first part of the question is implicit in the methodology by which Sudakov-like suppression factors (Sudakov logarithms) were successfully derived in Refs. [7,4]. As it turns out, the relevant, nonperturbative expression that factorizes out corresponds to a LLA result which is swept by an appropriate renormalization-group (RG) controlled running. Confronting the second part of the question, on the other hand, has not been directly addressed to before and will become one of the focal points in this investigation, as we intend to elucidate the precise role played by the Feynman gauge in our computations.

On the physical front, our present effort will direct itself towards the derivation of the Sudakov-like suppression factor for the four-point process associated with the wide angle “elastic scattering” involving two off-mass-shell spin-1/2 matter particles in a non-abelian gauge theory. Looking, at the same time, beyond this specific objective, we view the present investigation as constituting an attempt to connect parton-model based factorization schemes [1,8,9,10,11,12,13,14,15,16,17,18] with results founded in basic field theoretical calculations. Our discussion will therefore naturally connect to previous works on the understanding of Sudakov effects in elastic scattering [19,20], though the present approach is generically different from those, as here open, i.e., Wilson lines of finite length are not used in correspondence with the OPE but as fundamental ingredients of the formalism.

2. We commence our exposition by displaying the basic formulas, in worldline form, entering the analysis to follow. Working in Euclidean space we introduce the four-point function
\[
\frac{\delta^4}{\delta \eta_i^f(x_1) \delta \eta_j^f(x_2) \delta \eta_i^{f'}(y_1) \delta \eta_j^{f'}(y_2)} \ln Z(\eta, \eta)|_{\eta=\eta=0} = \mathcal{M} (x_1, x_2; y_1, y_2)^{ii'}_{jj'}, \\
-\mathcal{M} (x_1, x_2; y_2, y_1)^{ij'}_{ji'},
\]

where \(Z(\eta, \eta)\) is the partition function for the non-abelian gauge field theory with spin-1/2 matter fields, whose sources are \(\eta, \eta\), whereas \(f\) denotes flavor and \(i, j \ldots\) are group representation indices.

The expression for the fermion-fermion amplitude in the worldline formalism is given by

\[
\mathcal{M}^{ii'}_{jj'} = \sum_{C^I_{x_1,x_2}} \sum_{C^{II}_{x_2,y_2}} I^{[\dot{x}^I]} I^{[\dot{x}^{II}]} \left< \text{P exp} \left[ ig \int_0^{T_1} d\tau \dot{x}^I(\tau) \cdot A (x^I(\tau)) \right]_{ii'} \right. \\
\times \left. \text{P exp} \left[ ig \int_0^{T_2} d\tau \dot{x}^{II}(\tau) \cdot A (x^{II}(\tau)) \right]_{jj'} \right>_A ,
\]

where \(< \ldots >_A\) denotes functional averaging in the gauge field sector with whatever this entails (ghosts, gauge choice prescription, Dirac determinant, etc.) and, where

\[
\sum_{C^I_{x,y}} I^{[\dot{x}]} \equiv \int_0^\infty dT \int_{x(0)=x}^{x(T)=y} Dx(\tau) \int Dp(\tau) \text{P exp} \left[ -\int_0^T d\tau (ip(\tau) \cdot \gamma + M) \right] \\
\times \exp \left[ i \int_0^T d\tau p(\tau) \cdot \dot{x}(\tau) \right] .
\]

The configuration to be studied in this work pertains to two incoming worldlines of finite extent with four-velocity vectors \(u_1\) and \(u_2\), respectively, which are derailed via a mutual interaction operating on a sudden-impulse basis. The latter induces a cusp on each line so that the colliding fermions exit with four-velocities \(u'_1\) and \(u'_2\). This situation is depicted in Fig. 1a and corresponds, when the matter particles are on-mass-shell, to an elastic-scattering process at very high energies and large momentum transfers, i.e., to a process that probes the limit \(s \to \infty, |t| \to \infty\) at fixed ratio \(s/t\). The on-mass-shell version has already been considered in the context of the worldline formalism \[4\] and has led to the derivation of the on-mass-shell expression, giving rise to a Sudakov-like suppression factor for the elastic amplitude.

In the present investigation, the spin-1/2 entities entering the four-point process are taken to be off-mass-shell. In our formalism the off-mass-shellness is of the order of \(1/\sigma\), where \(\sigma\) stands for the length of each (broken) straight-line path. Indeed, the finite length of the matter particle’s propagation contour serves to cut-off all gauge field modes with momentum less than \(1/\sigma\), participating in its full, on-mass-shell description.

A useful kinematical parametrization for the on-mass-shell problem is

\[
p_1 = \left( \sqrt{Q^2 + M^2}, 0, 0, Q \right), \quad p_2 = \left( \sqrt{Q^2 + M^2}, 0, 0, -Q \right) \\
p'_1 = \left( \sqrt{Q^2 + M^2}, 0, Q \sin \theta, Q \cos \theta \right), \quad p'_2 = \left( \sqrt{Q^2 + M^2}, 0, -Q \sin \theta, -Q \cos \theta \right).
\]

In turn, the variables \(s\) and \(t\) are given, respectively, by
Fig. 1. Worldline configurations for forward high-energy scattering of spin-1/2 matter particles with associated four-velocities. (a) shows the configuration which corresponds to particles being derailed via their mutual interaction on a sudden-impulse basis. (b) shows the double-cusped worldline configuration which mixes with the previous case under renormalization-group evolution.

\[ s = (p_1 + p_2)^2 = 4 \left(Q^2 + M^2\right) \]  

and

\[ t = (p_1 - p'_1)^2 = -2Q^2 \left(1 - \cos \theta\right) . \]  

The limit \( s, |t| \to \infty \) with \( s/t \) fixed is now taken in the sense that \( |Q| \to \infty \), and \( \theta \) is held fixed. Though the above relations have only an implicit meaning for the off-mass-shell situation in consideration here, we shall continue to employ them in order to characterize the kinematical region under study by a single large-momentum scale, namely \( Q \). A precise characterization of the (effective) mass parameter \( M \), entering the off-mass-shell analysis, will be made in the end. It should, in any case, be anticipated that the off-mass-shellness will somehow be related to an IR cutoff.

With the above definitions/clarifications in place, let us now introduce the various scales that will enter our analysis. At the very top, we place the c.m.-energy/momentum-transfer \( |Q| \) at and above which the truly hard, perturbative, physics acts. At the very bottom lies an IR scale \( \mu_{\text{IR}} \) which marks the point from which our RG-running commences. Note that, to the extent that we shall rely on perturbative estimates of the various quantities entering the RG equation, we must demand that \( \mu_{\text{IR}} \gg \Lambda_{\text{QCD}} \). Finally, an in-between scale \( \Lambda \) serves to isolate that subsector of the full theory which is associated with the (almost everywhere) straight-line propagation of matter particles. We shall refer to the latter as the eikonal (sub)sector. Clearly, all the potentially factorizable soft physics reside in energy scales below \( |Q| \) and in this sense \( \Lambda \) can “run” in the interval \( [\mu_{\text{IR}}, |Q|] \). It is worth noting that \( |Q| \) also provides a measure for angles between a pair of four-velocity vectors that enter and exit a given cusped formation.

3. Generically speaking, the factorization of a quantity \( W \) of physical interest, associated with an (energy) separation scale \( \Lambda \), is based on a casting of the form

\[ W = W_{\text{SOFT}}W_{\text{HARD}} + \mathcal{O}(1/\Lambda) . \]  

In the framework of QCD and the language of Feynman diagrams, such an assertion calls for intricate considerations which take into account contributions from homogeneously soft gluons, on the one hand, and collinear gluons (in jets), on the other.
To the extent that a problem, such as the one in hand, involves a large momentum scale $Q$ (representing, e.g., a large-momentum transfer), it becomes convenient in order to extract the asymptotic behavior of $W$ as $|Q| \rightarrow \infty$ to work with the quantity $\frac{d}{d \ln Q^2} W$, whose independence from $\Lambda$, at $\mathcal{O}(1/\Lambda)$, leads to the RG equation

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) \frac{d}{d \ln Q^2} \ln W_{\text{HARD}} = - \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) \frac{d}{d \ln Q^2} \ln W_{\text{SOFT}} = \Gamma_S ,
\]  

where $\Gamma_S$ stands for anomalous dimensions pertaining to $\frac{d}{d \ln Q^2} W_{\text{SOFT}}$, and where we have replaced $\Lambda$ by $\mu$ adhering to the conventional choice for the running scale.

Our obvious intention is to calculate anomalous dimensions associated with the configuration of Fig. 1a, wherein what is emitted or absorbed by the straight-line segments are uniformly soft gluons, with respect to $\Lambda$, i.e., employing a no-impulse approximation. The existence of these anomalous dimensions can be understood on the basis that, from the perspective of the IR cut-off $\mu_{\text{IR}}$, the upper momentum scale $\Lambda$ appears as being infinitely remote – after all, it is the ratio between UV and IR cutoffs that really matters, as far as RG-running is concerned. To what extent, on the other hand, the physics operating within this eikonal subsector can fully account for the long-distance behavior of $W$ constitutes an open question which will not be addressed here.

In what follows, we shall focus our efforts on the disconnected four-point function which we denote by $(W_1)_{jj'}^{ii'}$. Non-abelian group considerations lead us to define the following invariant quantities, see, e.g., \[22,23\],

\[ W_1^{(a)} \equiv \langle \text{tr} P_I \text{tr} P_{II} \rangle_A = \delta_{ii'} \delta_{jj'} (W_1)_{jj'}^{ii'} \]  

and

\[ W_1^{(b)} \equiv \langle \text{tr} (P_I P_{II}) \rangle_A = \delta_{ij'} \delta_{ji'} (W_1)_{jj'}^{ii'} , \]

where $P_I$ denotes a line configuration parametrized by $\hat{x}^I(s)$ and $P_{II}$ another one parametrized by $\hat{x}^{II}(s)$. In the situation shown in Fig. 1a, of course, $\hat{x}^I(s) = u_1$ in $[0, s_1]$, $\hat{x}^{I'}(s) = u_1'$ in $[s_1, \sigma]$, $\hat{x}^{II}(s) = u_2$ in $[0, s_2]$ and $\hat{x}^{II'}(s) = u_2'$ in $[s_2, \sigma]$, where $s_1$ and $s_2$ mark the derailment points on the respective contours.

It follows that

\[ (W_1)_{jj'}^{ii'} = \frac{NW_1^{(a)} - W_1^{(b)}}{N(N^2 - 1)} \delta_{ii'} \delta_{jj'} + \frac{NW_1^{(b)} - W_1^{(a)}}{N(N^2 - 1)} \delta_{ij'} \delta_{ji'} . \]  

Referring to the invariant quantities $W_1^{(a,b)}$, the RG equation (8) reads, up to possible mixing with other operators,

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) \frac{d}{d \ln Q^2} \ln \left( W_1^{(a,b)} \right)_{\text{HARD}} = - \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) \frac{d}{d \ln Q^2} \ln \left( W_1^{(a,b)} \right)_{\text{SOFT}} = \Gamma_S^{(a,b)} .
\]

But, provided we find the appropriate anomalous dimensions associated with the soft factor, then $\frac{d}{d \ln Q^2} W_1^{(a,b)}$ can be determined via a two-step procedure which addresses itself first to its soft and second to its hard component.
The crucial point now is that the worldline scheme allows us to compute unambiguously anomalous dimensions associated with the homogeneously soft gluon region. In our case, the relevant configuration is that represented graphically in Fig. 1. As it will turn out, these will be the anomalous dimensions denoted $\Gamma$ in (13), or stated equivalently: the soft physics that we shall be in the position to factorize pertains to the eikonal subsector. Our first task, however, is to establish that it is enough to perform the relevant calculation in the Feynman gauge.

4. Consider the perturbative expansion of the expectation value, with respect to the gauge field sector, entering the expression for the four-point function under study, cf. Eq. (2), albeit now for the disconnected part. A typical term, which involves an interaction point, is of the form

$$[V]_{ij}^{\mu
u} = (ig)^2 \int_0^{T_1} d\tau \int_0^{T_2} d\tau' x^I_\mu(\tau) x^{HH}_\nu(\tau') \left\langle A_\mu \left( x^I(\tau) \right)_{ii}, A_\nu \left( x^{HH}(\tau) \right)_{jj} \right\rangle_A .$$  \hspace{1cm} (13)

Let us first treat the case of covariant gauge choices. Anticipating the fact that the eikonal subsector of the full theory has its own UV domain, we work with a (dimensionally) regularized form of the propagator, which, generically, reads

$$D_{\mu\nu}(|x|) = \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot x} \frac{1}{k^2} \left[ \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right] = D^{(F)}_{\mu\nu} - (1 - \xi) D^{(S)}_{\mu\nu} ,$$  \hspace{1cm} (14)

where $D^{(F)}_{\mu\nu}$ stands for the Feynman gauge propagator.

We readily determine

$$D^{(F)}_{\mu\nu}(|x|) = \delta_{\mu\nu} \frac{\mu^{4-D}}{4\pi^{D/2}} \Gamma \left( \frac{D}{2} - 1 \right) \frac{1}{|x|^{D-2}}$$  \hspace{1cm} (15)

and

$$D^{(S)}_{\mu\nu}(|x|) = \mu^{4-D} \frac{4\pi^2}{4\pi^{D/2}} \left[ \frac{\delta_{\mu\nu}}{2} \frac{1}{|x|^{D-2}} - (D - 2) \frac{x_\mu x_\nu}{|x|^D} \right] .$$  \hspace{1cm} (16)

Consequently, we have

$$D_{\mu\nu}(x - x') = \frac{\mu^{4-D}}{4\pi^{D/2}} \Gamma \left( \frac{D}{2} - 1 \right) \left[ \frac{\delta_{\mu\nu}}{|x - x'|^{D-2}} - \frac{1}{2} (1 - \xi) \frac{1}{D - 4} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x'_\nu} \frac{1}{|x - x'|^{D-4}} \right] .$$  \hspace{1cm} (17)

Inserting in Eq. (13) the second term in the square brackets of this expression, leads to integrals having the typical form

$$\frac{\mu^{4-D}}{D - 4} \int_{x_1}^{y_1} dx_\mu \int_{x_2}^{y_2} dx'_\nu \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x'_\nu} |x - x'|^{4-D}$$

$$= \frac{\mu^{4-D}}{D - 4} \left[ |y_1 - y_2|^{4-D} - |x_1 - y_2|^{4-D} - |y_1 - x_2|^{4-D} - |x_1 - x_2|^{4-D} \right] ,$$  \hspace{1cm} (18)

where $x_1, x_2$ denote the initial and $y_1, y_2$ the final points of the two fermionic paths. The latter are obliged to pass through the respective deflection (equivalently, interaction) points $z_1, z_2$ under the condition that they respect the kinematics of the process.
Asymptotically, the length of each of the four branches (two per cusped contour) is very large, the order of which will be denoted by $|L|$. Moreover, the projection of a given branch on another in the vicinity of the deflection point is of order $Q^2/m^2$. As a consequence, the rhs of the above relation becomes

$$\frac{1}{D-4} \left[ \mu |L| \frac{|Q|}{m} \right]^{4-D} \rightarrow \ln(\mu |L|) + \ln \frac{|Q|}{m} \quad (19)$$

and hence does not yield an anomalous-dimension contribution to $\frac{d}{d \ln Q^2} [V]_{ijj}^{ii'}$. One surmises that, for an arbitrary covariant gauge choice, the anomalous dimensions in the eikonal subsector are exclusively associated with the expression furnished by the Feynman propagator.

Turning our attention now to an axial gauge, we write

$$D_{\mu\nu}(|x|) = D^{(F)}_{\mu\nu}(|x|) - D^{(\eta)}_{\mu\nu}(|x|), \quad (20)$$

where

$$D^{(\eta)}_{\mu\nu}(|x|) = \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} e^{-ik \cdot x} \left[ \frac{\eta_{\mu k_\nu} + \eta_{\nu k_\mu}}{\eta \cdot k} - \eta^2 \frac{k_\mu k_\nu}{(\eta \cdot k)^2} \right]. \quad (21)$$

The last term in the square brackets amounts to a double derivative action and leads to the same result as the $\xi$-dependent part of the covariant gauge propagator. The remainder, when inserted into Eq. (15), leads to the following expression

$$\int_0^\infty ds \int_{x_1}^{y_1} dx_\mu \int_{x_2}^{y_2} dx'_\mu \left[ \frac{\eta_{\mu} \partial}{\partial x'_\mu} + \frac{\eta_{\nu} \partial}{\partial x'_\nu} \right] |x - x' + s\eta|^{D-2}$$

$$= 2 \int_{x_1}^{y_1} dx_\mu \left[ \int_{-\infty}^{y_2} dw_\mu \frac{1}{|x - w|^{D-2}} - \int_{-\infty}^{x_2} d\bar{w}_\mu \frac{1}{|x - \bar{w}|^{D-2}} \right], \quad (22)$$

where $w_\mu \equiv y_2 - \eta_\mu$, $\bar{w}_\mu \equiv x_2 - \eta_\mu$.

Now, the main contribution in the above two integrals comes, respectively, from the points $x_\mu \simeq w_\mu$, $x'_\mu \simeq \bar{w}_\mu$ and is of order $\ln(\Lambda) \ln(Q)$. However, the corresponding terms register in the overall expression with opposite signs and consequently cancel each other.

Generalizing to the full disconnected four-point function $[W_1]^{ii'}_{jj'}$, one concludes that, for an arbitrary choice of gauge condition, the leading behavior giving rise to anomalous dimensions for $\frac{d}{d \ln Q^2} [W_1]^{ii'}_{jj'}$ comes from the insertion of the Feynman propagator in the full expression for the four-point function. It is, moreover, of crucial importance to observe that the relevant singularity structure associated with interaction points will be invariably picked up by the (broken) straight-line configurations. Indeed, as becomes evident from Eq. (15), the main contribution to the terms involving two different branches exiting a point of interaction comes from the immediate vicinity of the latter$^2$. In addition, such singularities will be picked

1Clearly, $\ln(\mu |L|)$ furnishes the divergent logarithmic factor in the eikonal subsector.

2It is perhaps helpful to rewrite the relevant factor in Eq. (15) as $\frac{1}{|x(\tau) - x(\tau')|^{D-2}}$ and let $\tau$, $\tau'$ run, respectively, along branches with different four-velocities.
up by an arbitrary path, entering the full expression for \([W_{1}]^{\alpha}_{ij}\) on account of the constraint imposed by the momentum transfer injected at the cusp. This implies that the eikonal sector, we have isolated via the worldline casting of the system, automatically factorizes carrying with it the relevant, overall renormalization factor. For four-point processes, in general, such factors, induced by singularities associated with interaction points, have been identified as being of the cusp [23] or the cross [23] type. For the present case, in which a large momentum transfer is involved, our concern will be with cusp-type singularities.

5. The preceding analysis has led us to the specific task of determining anomalous dimensions for the quantity \(\frac{d}{dt} [W_{1}]^{\alpha}_{ij} \). The explicit expression for \([W_{1}]^{\alpha}_{ij} \) to \(\mathcal{O}(g^2)\) reads

\[
([W_{1}]_{\text{EIK}}^{\alpha}_{ij}) = \delta_{ii'}\delta_{jj'} + (ig)^2 \int_{0}^{\sigma} d\tau \int_{0}^{\sigma} d\tau' \left\{ C_F \delta_{ii'}\delta_{jj'} \left[ u_{1\mu}u_{1\nu}D_{\mu\nu}(|\tau u_1 - \tau' u_1|) + (1 \leftrightarrow 2) \right. \right.
\]

\[+ \ldots \] \(\left. + t_{ii'}^{\alpha}t_{jj'}^{\alpha} \left[ u_{1\mu}u_{2\nu}D_{\mu\nu}(|\tau u_1 - \tau' u_2|) + \ldots \right] \right\} \]

where the eclipse denotes terms which comprise combinations of each worldline branch carrying four-velocity \(u_{1}^{(i)} = p_{1}^{(i)}/M\) either with itself (second bracket) or with the other three branches, while involving, at the same time, a gluon exchange across the point of interaction. Note that for those terms which contain \(u_{i}\), the argument of \(D_{\mu\nu}\) has the reverse sign. Note also that, as we have already established, the gluon propagator can be taken in the Feynman gauge without loss of generality.

Our basic calculational task amounts to dealing with integrals of the form

\[
I^{(\pm)} = u_i \cdot u_j \int_{0}^{\sigma} d\tau \int_{0}^{\sigma} d\tau' D(\tau u_i \pm \tau' u_j) \]

(24)

where \(u_{i,j}\) stands for \(u_{1}^{(i)}\) and \(u_{2}^{(i)}\).

One finds, as \(D \to 4\),

\[
\int_{-\sigma}^{+\sigma} d\tau \int_{-\sigma}^{+\sigma} d\tau' u_i \cdot u_j D(|u_i \tau \pm u_j \tau'|) = \frac{1}{4\pi^{2}} \left( \frac{\mu^{2}}{\lambda^{2}} \right) \frac{\epsilon}{2} f_{4-2\epsilon}(w_{ij}),
\]

(25)

where we have used \(w_{ij} \equiv u_i \cdot u_j, \bar{\lambda} \equiv 1/\sigma\) and \(\epsilon = 4 - D(> 0)\). Here

\[
f_{4-2\epsilon}^{(+)} = 2w^{2}F\left(1, 1; \frac{3}{2}; 1 - w^{2}\right) - [2(1 + w)]^{\epsilon} wF\left(1, 1; \frac{3}{2}; \frac{1-w}{2}\right)
\]

(26)

and

\[
f_{4-2\epsilon}^{(-)} = 2w^{2}F\left(1, 1; \frac{3}{2}; w^{2}\right) + [2(1 - w)]^{\epsilon} wF\left(1, 1; \frac{3}{2}; \frac{1-w}{2}\right),
\]

(27)

with \(w = u_1 \cdot u_2/|u_1||u_2|\).

Taking now into account that, in the fundamental representation,

\[
t_{ii'}^{\alpha}t_{jj'}^{\alpha} = -\frac{1}{2N} \delta_{ii'}\delta_{jj'} + \frac{1}{2} \delta_{ij'}\delta_{jj'},
\]

(28)
we determine

\[(W_1^{(a)})_{\text{EIK}} = N^2 C_{11} + NC_{12}\]  

and

\[(W_1^{(b)})_{\text{EIK}} = NC_{11} + N^2 C_{12}\]  

with \(C_{11}\) and \(C_{12}\) remaining to be calculated. In fact, we are actually interested in \(\frac{\partial}{\partial \ln Q^2} C_{1i}\) and, as far as the leading-order estimate is concerned, in the singular terms entering the respective (asymptotic) expressions from which the anomalous dimensions will be extracted. For that reason, we shall restrict ourselves to a minimal exposition of relevant mathematical manipulations by explicitly treating only the case of \(C_{11}\).

Setting \(w_{ij} = \cos \phi_{ij}\) and going over to Minkowski space \((\phi_{ij} \to -i\gamma_{ij})\), we obtain (by using Eqs. (25)-(27), removing pole terms via the \(\overline{\text{MS}}\) subtraction scheme, and going to the limit \(\epsilon \to 0\))

\[C_{11} = 1 - \frac{\alpha_s}{\pi} \left\{ \ln(\mu \bar{\sigma} |u|) \left[ 2C_F (\gamma_{11}' \coth \gamma_{11} - 2) - \frac{1}{N} ((i\pi - \gamma_{12}) \coth \gamma_{12} + \gamma_{12}' \coth \gamma_{12}') \right] 
+ C_F h^+(\gamma_{11}') - \frac{1}{2N} \left[ h^+(\gamma_{12}') + h^-(\gamma_{12}) - 4C_F \right] \right\} + \mathcal{O}(g^4), \]  

\[\text{(31)}\]

where

\[h^\pm(w) \simeq \pm \frac{1}{4} \ln^2(\pm 2w) + \mathcal{O}\left(\frac{1}{w}\right). \]  

\[\text{(32)}\]

These functions enter, typically, off-mass-shell expressions in the worldline scheme \([4]\).

In the asymptotic regime that is of interest to us, we get from Eq. (4) and subsequent transcription to Minkowski space,

\[\gamma_{12} \simeq \ln \left(\frac{2Q^2}{M^2}\right), \quad \gamma_{11}' \simeq \ln \left(\frac{2Q^2}{M^2}\right) + \ln \left(\frac{\sin^2 \theta}{2}\right), \quad \gamma_{12}' \simeq \ln \left(\frac{2Q^2}{M^2}\right) + \ln \left(\frac{\cos^2 \theta}{2}\right). \]  

\[\text{(33)}\]

We, thereby, obtain

\[\frac{\partial}{\partial \ln Q^2} C_{11} \simeq - \frac{\alpha_s}{\pi} \left[ 2C_F \ln(\mu \bar{\sigma}) + C_F \ln \left( \frac{2 \sin^2 \theta}{2} \right) - \frac{1}{N} \ln \left( \frac{2 \cos^2 \theta}{2} \right) - i\pi \right]. \]  

\[\text{(34)}\]

where \(\bar{\sigma} \equiv |u| \sigma \left(\frac{2 \sigma^2}{M^2}\right)^{1/2}\) properly defines the lowest scale of the RG running, thereby facilitating the identification \(\bar{\sigma} \equiv \mu_{\text{IR}}^{-1}\).

From the above result, we read off the relevant contribution to the anomalous dimensions, namely the coefficient in front of \(\ln(\mu \bar{\sigma})\): \(-2C_F \frac{\alpha_s}{\pi}\). On the other hand, it turns out that there is no contribution to the anomalous dimensions from \(\frac{\partial}{\partial \ln Q^2} C_{12}\).

6. The reason for carrying all along the index 1 on \(W\) is that under the RG running the four-point function associated with the scattering process mixes with \([W_2]_{\text{EIK}}^{uv}\), which corresponds to the Wilson-line arrangement shown in Fig. 1b [23]. One may represent this quantity by
\[(W_2)_{\text{EIK}}^{i'j'} = \left\langle \text{P} \exp \left[ ig \int_{-\sigma}^{0} d\tau u_1 \cdot A(\tau u_1) + ig \int_{0}^{\sigma} d\tau u'_2 \cdot A(\tau u'_2) \right] \right\rangle_{ij'}, \]
\[
\times \text{P} \exp \left[ ig \int_{-\sigma}^{0} d\tau u_2 \cdot A(\tau u_2) + ig \int_{0}^{\sigma} d\tau u'_1 \cdot A(\tau u'_1) \right] \right\rangle_A, \tag{35}\]

with the agreement that it acquires literal meaning in the Feynman gauge.

Corresponding invariant functions \((W_2^{(a,b)})_{\text{EIK}}\) can now be introduced in a similar fashion as in Eqs. (9), (10). Moreover, we may write
\[
(W_2^{(a)})_{\text{EIK}} = N^2 C_{21} + N C_{22}, \quad (W_2^{(b)})_{\text{EIK}} = N C_{21} + N^2 C_{22} \tag{36}\]

and face, for the determination of the \(C_{2i}\), similar tasks to those involved for the determination of \(C_{1i}\). The encountered integrals are the same as before. The end result is that we have a contribution to the anomalous dimensions from \(\partial \frac{\partial}{\partial \ln Q^2} C_{22}\) being equal to \(-2C_F\frac{\alpha_s}{\pi}\), i.e., it is the same as for \(\partial \frac{\partial}{\partial \ln Q^2} C_{11}\), whereas \(\partial \frac{\partial}{\partial \ln Q^2} C_{21}\) provides a null contribution.

Before carrying through the RG analysis, we need to adjust our formalism in such a way as that it applies to the invariant quantities \(W^{(i)}\), \((i = a, b)\). To this end, let us set
\[
\tilde{W}^{(i)}_{\text{EIK}} \equiv \left( \begin{array}{c} W_1^{(i)} \\ W_2^{(i)} \end{array} \right)_{\text{EIK}}. \tag{37}\]

We define
\[
\tilde{F}^{(i)}_{\text{EIK}} \left( \frac{\mu}{\mu_{\text{IR}}} , \frac{Q^2}{M^2} \right) \equiv \frac{d}{d \ln Q^2} \ln \tilde{W}^{(i)}_{\text{EIK}} \left( \frac{\mu}{\mu_{\text{IR}}} , \frac{Q^2}{M^2} \right) = -\ln \left( \frac{\mu}{\mu_{\text{IR}}} \right) \tilde{\Gamma}^{(i)}_{\text{EIK}} - \tilde{D}^{(i)}_{\text{EIK}}, \quad i = a, b, \tag{38}\]

where the quantities \(\tilde{\Gamma}^{(i)}_{\text{EIK}}\) and \(\tilde{D}^{(i)}_{\text{EIK}}\) can be surmised from the expressions for the \(C_{ij}\).

The RG equation, then, reads
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \tilde{F}^{(i)}_{\text{EIK}} = -2C_F\frac{\alpha_s}{\pi} \tilde{\Gamma}^{(i)}_{\text{EIK}}. \tag{39}\]

We readily determine that the associated initial conditions are given by
\[
\tilde{F}^{(i)}_{\text{EIK}} \left( \frac{\mu}{\mu_{\text{IR}}} , \frac{Q^2}{M^2} \right)_{\mu = \mu_{\text{IR}}} = -\tilde{D}^{(i)}_{\text{EIK}}(\alpha_s(\mu_{\text{IR}})). \tag{40}\]

As we are primarily interested in isolating the leading-order result, we write
\[
\tilde{F}^{(i)}_{\text{EIK}} \left( \frac{\mu}{\mu_{\text{IR}}} , \frac{Q^2}{m^2} \right) = -2C_F\frac{\alpha_s}{\pi} \int_{\mu_{\text{IR}}}^{\mu} d\tau \alpha_s(\tau) + \text{nonleading terms} \tag{41}\]
in which the nonleading terms correspond to \(\tilde{D}^{(a)}_{\text{EIK}}(\alpha_s(\mu_{\text{IR}})) + \mathcal{O}(\alpha_s^2)\).

According to our procedure, what we have factorized with respect to the energy scale \(\mu \in [\mu_{\text{IR}}, |Q|]\), is eikonal vs. non-eikonal physics operative in this range. The latter contains configurations that go beyond the no-impulse approximation. Referring to Eq. (12), after adjusting to the present situation, we have
\[ \frac{d}{d \ln Q^2} \ln(W_1^{(i)})_{\text{NON-EIK}} = -\frac{2C_F}{\pi} \int_{\mu}^{Q^2} \frac{d\tau}{\tau} \alpha_s(\tau) + \text{nonleading terms}, \] (42)

where now the nonleading terms are associated with initial conditions pertaining to the running of the non-eikonal contributions with respect to \( \mu \).

Putting everything together, we finally obtain

\[ \frac{d}{d \ln Q^2} \ln(W_1^{(i)}) = -\frac{2C_F}{\pi} \int_{\mu}^{Q^2} \frac{d\tau}{\tau} \alpha_s(\tau) + \text{nonleading terms}. \] (43)

Next, introducing the mass scale \( \bar{M} \equiv (\mu^2 Q^2)^{1/4} = (\frac{M^2}{2\pi^2 |W_1^2|})^{1/4} \), which should be greater than \( \Lambda_{\text{QCD}} \) for the perturbative analysis to be valid, we may express the leading order RG result as follows

\[ \ln W_1^{(i)} = -\frac{4C_F}{\beta_0} \left[ \ln \frac{Q^2}{\Lambda_{\text{QCD}}^2} \ln \ln \left( \frac{Q^2}{\Lambda_{\text{QCD}}^2} \right) + \ln \frac{\bar{M}^4}{\Lambda_{\text{QCD}}^2 Q^2} \ln \ln \left( \frac{\bar{M}^4}{\Lambda_{\text{QCD}}^2 Q^2} \right) \right], \] (44)

which, after a number of routine algebraic manipulations, can be cast into a form that helps isolating its dominant term. The relevant expression then reads

\[ \ln W_1^{(i)} = -\frac{8C_F}{\beta_0} \ln \frac{Q^2}{\Lambda_{\text{QCD}}^2} \ln \ln \left( \frac{Q^2}{\Lambda_{\text{QCD}}^2} \right) + \ln \frac{\bar{M}^4}{\Lambda_{\text{QCD}}^2 Q^2} \ln \ln \left[ 1 - 2 \frac{\ln \frac{\bar{M}^4}{\Lambda_{\text{QCD}}^2 Q^2}}{\ln \frac{\bar{M}^4}{\Lambda_{\text{QCD}}^2 Q^2}} \right]. \] (45)

Given the hierarchy of scales \( Q^2 > M^2 > \frac{M^4}{Q^2} > \Lambda_{\text{QCD}}^2 \), we identify the first term on the rhs of this equation as the dominant one. Further, it follows from the relation \( \sigma \sim \frac{m}{p^2 - m^2} \), which can be deduced from the original path integral, that the effective mass scale entering our final results is related to the off-mass-shellness: \( \bar{M}^2 \sim (p^2 - m^2) \).

Just as in the case of the three-point function, we have calculated earlier \[7\], one witnesses the fact that the off-mass-shell result for the Sudakov suppression has an exponent twice as large as the one for the on-mass-shell case \[26\]. Moreover, it becomes obvious that the asymptotic, long-range behavior of \( W_1^{(a)} \) and \( W_1^{(b)} \) is the same, so that in fact it characterizes the long-range behavior of the total expression \( [W_1]_{\text{ij}}'' \). We have, in other words, extracted the Sudakov-like suppression factor for the four-point function as a whole. This extends the hitherto known result which refers to three-point vertices, i.e., the one pertaining to form factors. If, now, we wish to go to the connected four-point function, then we must subtract a term comprising the product of two Sudakov form factors. But, then, we shall obtain precisely the same leading factor as in Eq. (45), see, e.g., \[4,7\].

In conclusion, we have established the Sudakov behavior for four-point processes through a computational process that has taken place within the framework of a pure, field theoretical treatment with a relatively painless effort.

\[ ^3 \text{Actually, the term “Sudakov form factor” is a misnomer since it does not account for structure but merely for long-distance behavior of the vertex function in the limit of large-momentum transfer.} \]
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