Polynomial invariants which can distinguish the orientations of knots

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Abstract

This paper contains the first knot polynomials which can distinguish the orientations of classical knots and which make no explicit use of the knot group. But they make extensive use of the meridian and of the longitude in a geometric way.

Let $M$ be the topological moduli space of long knots up to regular isotopy, and for any natural number $n > 1$ let $M_n$ be the moduli space of all $n$-cables $nK$ of framed long knots $K$ which are twisted by a given string link $T$ to close to a knot in the solid torus, with a marked point on the knot at infinity. First we construct integer valued combinatorial 1-cocycles for $M_n$ by using Gauss diagram formulas for finite type invariants. We observe then that our 1-cocycles allow to fix certain crossings of $nK$ as local parameters of the 1-cocycles. Finally, we transform the local parameter into an unordered set of global parameters by following the crossings in the isotopy. We evaluate now the 1-cocycles on a canonical loop in $M_n$. The outcome are polynomial valued invariants of $K$, where the variables are indexed by finite type invariants and by regular isotopy types of string links $T$.

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1 Introduction

1.1 Some history

The first mathematical paper which has mentioned knots is “Remarques sur les problèmes de situation” by A.T. Vandermonde in 1771. Since then, mathematicians try to solve the main problem in knot theory: for two given smooth knots in 3-space (often called classical knots), decide in an effective way whether or not they are smoothly isotopic, i.e. can they be deformed one into the other in a differentiable way? The main tool to achieve this are knot invariants. Most knot invariants are defined by using diagrams of knots.

A more recent approach is to use moduli spaces of knots or of diagrams of knots. The topology of moduli spaces of classical knots and of long knots was much studied in [17], [3], [4], [3], [6]. In particular, there is a complete description of the homotopy type and of the homology groups (even with an additional structure of certain Gerstenhaber-Poisson algebras) for the space of long knots in $\mathbb{R}^3$ (the same space, which was studied by Vassiliev using singularity theory) in the subsequent papers [17], [4], [4], [6]. Vassiliev [29] had constructed certain universally defined 0-dimensional cohomology classes. They give the finite type invariants of long knots (or...
equivalently of compact knots in the 3-sphere) when they are evaluated on
the 0-dimensional homology classes of the disconnected space. It is known
that all finite type knot invariants, also called Vassiliev-Goussarov invariants,
have diagrammatic formulas, which allow the computation of the invariant
from an arbitrary generic knot diagram [15].

Combinatorial integer 0-cocycles are usually called Gauss diagram for-
mulas, compare [25], [7], [8], [2], and also [9]. They correspond to finite type
invariants and are solutions of the 4T- and 1T-relations. Dror Bar-Natan has
shown in [1] that such solutions can be constructed systematically by using
the representation theory of Lie algebras.

It is known that all quantum knot invariants can be decomposed into
series of finite type invariants, compare [7] as a good example. Thirty years
after the discovery of finite type invariants there are still not known any finite
type invariants which could distinguish the orientations of classical knots.
An observation of Greg Kuperberg [21] says, that if finite type invariants can
not distinguish the orientations of knots then they fail even to distinguish
non-oriented knots as well.

Victor Vassiliev has started the combinatorial study of the space of long
knots in [30] by studying the simplicial resolution of the discriminant of long
singular knots in \( \mathbb{R}^3 \). Our approach is very different, because it is based on
the study of another discriminant, namely the discriminant of non-generic
diagrams of knots in the solid torus [10]. There is also a big difference for \( H_1 \)
of the moduli spaces. Let \( K_1 \# K_2 \) be the long knot which corresponds to the
connected sum of two different knots. In the moduli space of long knots we
have a loop which consists of pushing the knot \( K_1 \) from the left to the right
through the knot \( K_2 \) and then pushing \( K_2 \) through \( K_1 \) from the left to the
right too [13]. In \( M_n \) we can just push a knot \( T \cup nK_1 \) through \( nK_2 \) from the
left to the right and then sliding without any Reidemeister moves \( T \cup nK_1 \)
again in its initial position along the remaining part of the solid torus. This
is already a loop.

Arnaud Mortier has constructed for 1-cocycles of finite type for long knots
the analogue of the Kontsevich integral in [24]. The 4T-relations are now re-
placed by three 16T- and three 28T-relations and 4x4T-relations! There is
actually no representation theory which could help to construct such solu-
tions. The reason for this is simple. The well known representation theory
related to the tetrahedron equation, see e.g. [19], is as usual of a local nature.
But $H_1$ of the moduli space of (closed) non-satellite knots in $\mathbb{R}^3$ is only torsion (in contrast to $H_0$), see [17], [5] and [6], and hence all integer 1-cocycles from local solutions of the tetrahedron equation are trivial (in contrast to the local solutions from the Yang-Baxter equation, see e.g. [20]), no matter we consider closed knots in $S^3$ or long knots in $\mathbb{R}^3$! We have to construct therefore in a combinatorial way solutions of the global tetrahedron equation, i.e. the contribution of a R III move depends on the whole knot in the solid torus and not only on the local picture of the move.

It seems that before [10] only the moduli space of long knots was studied. The Teiblum-Turchin 1-cocycle $v_3^1$ was the only known 1-cocycle for long knots which represents a non-trivial cohomology class and which has an explicit diagrammatic formula for its computation. The Teiblum-Turchin 1-cocycle is an integer valued 1-cocycle of degree 3 in the sense of Vassiliev’s theory [29]. Its reduction mod 2 had the first diagrammatic description of a 1-cocycle, see [30] and [28]. Sakai has defined a $\mathbb{R}$ valued version of the Teiblum-Turchin 1-cocycle via configuration space integrals [26] and [18]. The most beautiful diagrammatic formula for an integer valued 1-cocycle for long knots which extends $v_3^1$ mod 2 and which probably coincides with $v_3^1$ was found by Mortier [22], [23]. Budney has defined another integer valued 1-cocycle (but not in a combinatorial way) on the space of long knots in [5].

In the monograph [10] and in its sequel [12] (compare also [11], which contains a 1-cocycle without a marked point on the knot) we have laid the foundation of the theory of combinatorial 1-cocycles which depend on integer parameters for knots in the solid torus. In particular, these 1-cocycles give invariants for knots in $\mathbb{R}^3$, when they are evaluated on certain canonical loops in the topological moduli spaces of knots in the solid torus, i.e. loops which are universally defined in connected components of the moduli space. (We will often use [10] and [12] as a reference for definitions, notations, conventions, figures and even for proofs in order to keep the present paper relatively short.)

### 1.2 New results

It turns out to be relatively simple to distinguish the orientations of a knot with our invariant, which is probably a big step towards distinguishing all classical knots. But we have to change the paradigm: knot theory in $\mathbb{R}^3$
is not mainly part of algebraic topology and of representation theory but of global analysis. The global analysis in this paper takes the form of combinatorics.

More precisely, in the present paper we refine further the 1-cocycle \( R_{i,j,k}^{(2)} \)
from Chapter 2.5 in [12] to a 1-cocycle \( qR_{i,j,k}^{(2)} \) and apply it to the loop
\( \text{push}(3_1^+, K) \) which consists of pushing the 2-cable of the standard diagram
of the long positive trefoil (i.e. writhe \( w = 3 \) and Whitney index \( n = -1 \),
i.e. after smoothing all double points of the underlying planar curve we stay
with a clock-wise oriented circle) through the 2-cable of a long knot \( K \). 
We consider knots always up to regular isotopy. In particular, we have calculated
that for the standard diagram of the knot \( K = 8_{17} \) and its inverse
\(-K = -8_{17} \) we have
\[
qR^{(2)}_{2,1,2}(\text{push}(3_1^+, 8_{17})) = x^{-5}
\]
\[
qR^{(2)}_{2,1,2}(\text{push}(3_1^+, -8_{17})) = -x^{-3} + 2x^{-4}.
\]

Consequently, the invariant distinguishes the knot \( 8_{17} \) from its inverse
(i.e. with opposite orientation).

So far, the orientation of knots could only be distinguished by using covering invariants of certain irregular metabelian covering spaces, see [27] and [16].

We can see \( qR^{(2)}_{i,j,k}(\text{push}) \) as a naive quantization of finite type invariants
through 1-cocycles, because its derivative evaluated at \( x = 1 \) is very likely to
be a finite type invariant for framed knots (which of course does not distinguish \( 8_{17} \) from its inverse). The invariant \( qR^{(2)}_{i,j,k}(\text{push}) \) is extremely effective,
because it can be calculated with quadratic complexity with respect to the
number of crossings of \( K \). Moreover, each time when we replace the knot
which we push through (as e.g. \( 3_1^+ \) in the example which we have calculated)
by another knot then it adds a new variable to each of the monomials in the
(Laurent) polynomial \( qR^{(2)}_{i,j,k}(\text{push}) \) for the knot \( K \). (Hence, the variable \( x \)
in the above formulas should be denoted by \( x_{v_2,3_1^+,w=3,n=-1} \), where \( v_2 \) is the
Vassiliev invariant of order 2.)

In Section 3 we show how to add simultaneously all the 1-cocycles \( R_{a,m,l,n,m}^{2k} \)
(which are lifts of the coefficients of the Conway polynomial of \( K \)) from Chapters 2.2 and 2.4 in [12] in order to obtain a polynomial invariant of the knot
\( K \). In particular, we have calculated
Here $z$ is the new variable corresponding to $R^{(4)}_{ij,hm}$. Notice that for $x = 1$ or $z = 1$ there is a cancellation in the polynomial for $8_{17}$, but no longer for the 2-variable polynomial.

In Section 4 we give some details for the calculation of the examples. This could be helpful for the reader to become familiar with the new invariants.

In Section 5 we show how to generalize $qR^{(2)}_{i,j,k}(push)$ to an invariant which can be calculated with cubical complexity, in order to be sure that we can distinguish mirror images of knots as well.

We call all the invariants, which are introduced in this paper, the polynomial 1-cocycle invariants. Each of them is calculable with polynomial complexity with respect to the number of crossings of $K$ and of $T$ if we fix the number $n$ of strands of the cable.

**Question 1** The present paper contains already so many polynomial 1-cocycle invariants, which are polynomials with arbitrary many variables and which can at least sometimes distinguish the orientations of the knot as well as the orientations of its complement in 3-space, that it is natural to ask if they can perhaps distinguish all classical knots?

Let us mention that our simplest polynomial 1-cocycle invariant gives a new result even for the trefoil. In a regular isotopy we can follow the crossings with what is called the trace graph, see Section 2.3 and [10]:

Let $K$ be the standard diagram of the positive (or negative) trefoil as a long knot. Then each regular isotopy which brings $K$ to itself acts as the identity on the set of crossings of $K$.

It also gives a new result for crossing numbers of knots up to regular isotopy:

Let $\text{aug}_+$ (respectively $\text{aug}_-$) be the sum of all positive (respectively negative) coefficients in the polynomial $qR^{(2)}_{i,j,k}(push(K', K))$. Then each diagram which represents the knot type $K$ up to regular isotopy has at least $\text{aug}_+$ positive crossings and at least $-\text{aug}_-$ negative crossings.
There is also a finer estimation for diagrams of long knots, because there is a finer distinction of crossings (see Section 2.4), and the above example for $-8_{17}$ shows that it can be sharp at least sometimes.

2 The polynomial invariant $qR_{i,j,k}^{(2)}(\text{push})$

All the ingredients in order to define the invariant are already in [12], Chapter 2.5 and [10], Chapters 2.2 and 4.1.3., we have only to combine them. For the convenience of the reader we repeat the main constructions here.

2.1 The 1-cocycle $R_{i,j,k}^{(2)}$

Definition 1 We fix an orthogonal projection $pr : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ together with standard coordinates $(x, y, z)$ of $\mathbb{C} \times \mathbb{R}$. A long knot $K$ is an oriented smoothly embedded copy of $\mathbb{R}$ in $\mathbb{C} \times \mathbb{R}$ which coincides with the real $x$-axis in $\mathbb{C} \times 0$ outside a compact set. A parallel $n$-cable of a framed long knot $nK$ is a $n$-component link with fixed endpoints where each component is parallel to the framed long knot with respect to the blackboard framing given by $pr$ (the $z$-coordinate) and with the same orientation on each component. A $n$-string link $T$ is a $n$-component link with fixed endpoints where each component is parallel to a long knot in $\mathbb{C} \times 0$ outside some compact set.

We put a base point $\infty$ on the lowest component of $T$ with respect to the $y$-coordinate at infinity.

We cut now $nK$ with a very big 3-ball $B^3$. The endpoints of the $n$-cable are respectively in two big discs (called the disc at infinity) which we glue together in order to obtain a solid torus $V^3$. We chose a string link $T$ which we glue to $nK$ near to the disc at infinity at $x = -\infty$ and such that $T \cup nK$ is an oriented knot in the solid torus and we numerate the components of $T \cup nK$ in the complement of the disc at infinity in the solid torus by 1, 2, ..., $n$ starting from the point $\infty$. We can see this solid torus as the complement of the meridian of the knot $K$ in the 3-sphere.

First of all, we have to replace isotopy by regular isotopy. It is well known that two knot diagrams in 3-space with the same writhe and the same Whitney index (the regular homotopy class of the underlying immersed planar curve) represent the same knot type if and only if there is a regular isotopy which connects them, i.e. a sequence of only Reidemeister moves of
type II and III (compare e.g. [9]). This can always be achieved by adding appropriate small curls to the diagrams and it is still true for long knots.

To be more precise, let $M_n$ be the space of all knots in the solid torus which we obtain for a fixed string link $T$ with the above construction, and such that the projection into the annulus is always an immersion. Because we have fixed a projection this space is a space of knot diagrams. We have to add one more condition, called the separation condition: a crossing can not move over the point $\infty$ at the same moment as a Reidemeister III move happens somewhere in the diagram (i.e. Reidemeister III moves and sliding crossings over the point $\infty$ are separated, in particular they are not allowed to commute, compare [12]). We call the resulting space still simply the moduli space $M_n$.

If a framed long knot $K'$ is regularly isotopic to $K$ (i.e. an isotopy without Reidemeister I moves) then the corresponding knots $K_1 = T \cup nK'$ and $K_2 = T \cup nK$ are in the same component of $M_n$ (compare e.g. [9]). Consequently, each 1-cocycle of $M_n$ evaluated on an universally defined loop in $M_n$ gives an invariant of the classical knot $K$. (It is often convenient for calculations to represent a long knot $K$ as a closed braid with just one strand opened to go to infinity.)

To each Reidemeister move of type III corresponds a diagram with a triple crossing $p$: three branches of the knot (the highest, middle and lowest with respect to the projection $pr : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$) have a common point in the projection into the plane. A small perturbation of the triple crossing leads to an ordinary diagram with three crossings near $pr(p)$.

**Definition 2** We call the crossing between the highest and the lowest branch of the triple crossing $p$ the distinguished crossing of $p$ and we denote it by $d$ ("d" stands for distinguished). The crossing between the highest branch and the middle branch is denoted by $hm$ and that of the middle branch with the lowest is denoted by $ml$, compare Fig. 1. For better visualization we draw the crossing $d$ always with a thicker arrow.

**Definition 3** For an ordinary crossing $q$ we call $D_q^+$ the knot which is obtained by smoothing the crossing $q$ from the under-cross to the over-cross (and the remaining knot is called $D_q^-$), see Fig. 2. The homological marking of $q$, denoted by $[q]$, is the homology class in $\mathbb{Z} \cong H_1(V^3)$ represented by $D_q^+$.
Figure 1: The names of the crossings in a R III-move

\[ \begin{array}{c}
\text{ml} \\
\text{hm}
\end{array} \]

\[ d \]

Figure 2: The two ordered knot diagrams associated to a crossing \( q \)

\[ q \rightarrow \begin{array}{c}
D_q^-\\
D_q^+
\end{array} \]

Figure 3: The Polyak-Viro formula \( C_2 \) for \( \nu_2(K) \) of long knots

\[ C_2 = \begin{array}{c}
\text{0} \\
\text{1}
\end{array} \]

Figure 4: The coorientation for Reidemeister III-moves
, i.e. by the knot which corresponds to the arc of the circle, which goes from
the over-cross to the under-cross.

A crossing \( q \) in \( T \cup nK \) is called a 1-crossing, if the point at infinity is in
\( D^+_q \), otherwise \( q \) is called a 0-crossing.

The sign (or writhe) of a crossing \( q \) is denoted as usual by \( w(q) \).

An invariant of long knots which is given by a Gauss diagram formula
(for the definition of Gauss diagram formulas see [25], [7], [8]) which uses at
most \( k \)-tuples of arrows, is an invariant of order \( k \). For example, for \( k = 2 \)
it can be defined by the beautiful formulas of Michael Polyak and Oleg Viro
for \( \nu_2(K) \) of long knots [25], see Fig. 3. The point on the circle corresponds
to the point at infinity on the knot.

**Definition 4** The coorientation for a Reidemeister III move is the direction
from two intersection points of the corresponding three arrows to one intersec-
tion point and of no intersection point of the three arrows to three intersection
points, compare Fig. 4. (We see in the cube equations for \( \Sigma^{(2)}_{\text{trans-self}} \) that the
two co-orientations for triple crossings fit together for the strata of \( \Sigma^{(1)}_{\text{tri}} \) which
come together in \( \Sigma^{(2)}_{\text{trans-self}} \), see [10].) Evidently, our co-orientation is com-
pletely determined by the corresponding planar curves and therefore we can
draw just chords instead of arrows in Fig. 4. We call the side of the comple-
ment of \( \Sigma^{(1)}_{\text{tri}} \) in \( M_n \) into which points the co-orientation, the positive side of
\( \Sigma^{(1)}_{\text{tri}} \).

The co-orientation for Reidemeister II and Reidemeister I moves is the
direction from no crossings to the diagram with two respectively one new
crossing.

Each transverse intersection point \( p \) of an oriented generic arc in \( M \) with
\( \Sigma^{(1)}_{\text{tri}} \) has now an intersection index +1 or −1, called \( \text{sign}(p) \), by comparing
the orientation of the arc with the co-orientation of the stratum \( \Sigma^{(1)}_{\text{tri}} \).

The following 1-cocycle makes use of the Polyak-Viro formula.

**Definition 5** Let \( i, j, k \in \{1, 2, ..., n\} \) be fixed integers, not necessarily all
distinct. We define the 1-cochain \( R^{(2)}_{i,j,k} \) in Fig. 5.

We use here our usual notation conventions (compare [10] or [11] or [12]):
we sum up over all signed with \( \text{sign}(p) \) \( R \) III moves of the given types in the
loop. To each of the moves we associate the sum of the signs of the crossings which correspond to the arrow, which is not in the triangle. These crossings cut always the crossing $h_m$ in the move in a configuration $C_2$. The sign of the crossing $h_m$ does not enter. The letter $i, j, k$ at the foot of an arrow means that the corresponding crossing has its foot (the under-cross) on the corresponding component of the string link $T \cup nK$.

If $i = k$, then we deform the triple crossing first into a regular diagram, no matter on the positive or negative side of the move, and we count also the contributions of $d$ or $ml$ if they form a configuration $C_2$ with the crossing $h_m$ (which happens only for the first two configurations in Fig. 5).

Let us call the arrow, which is not in the triangle, a $k$-crossing. We can now formulate Theorem 2.7 from [12].

**Proposition 1** $R^{(2)}_{i,j,k}$ is a 1-cocycle in $M_n$ for each natural numbers $n > 0$, $1 \leq i, j, k \leq n$. It represents a non-trivial integer cohomology class.

In order to prove that a 1-cochain is a 1-cocycle we have to prove three sorts of equations, which say that the 1-cochain is 0 on the meridian of the corresponding strata of codimension 2, compare [12] Chapter 1:

- the commutation relations
the positive global tetrahedron equation
the cube equations.

For the commutation relations the crossing $ml$ of the contributing $R$ moves is evidently the same in the meridian.

In the positive global tetrahedron equation we have to study the meridian of quadruple crossings. The six involved ordinary crossings, which correspond to the edges of the tetrahedron, are determined by the numbers of the local branches in the quadruple crossing. It follows that the crossing $ml = (ij)$ in the configurations which enter $R_{i,j,k}^{(2)}$ in the meridian of the quadruple crossing is always the same crossing (in the trace graph). Consequently, we can fix this crossing as a local parameter for $R_{i,j,k}^{(2)}$. This rigidity is of crucial importance.

In the cube equations the crossings $ml$ are either the same, or they are two crossings which appear or disappear together in an auto-tangency. Hence, they belong to the same component of the trace graph or to two identical components of the trace graph but with different orientations (see Section 2.3).

Notice, that to each crossing $q$ in $nK$ in $T \cup nK$ we can associate five elementary invariants.

**Definition 6** The following quantities are invariants of crossings $q$ of $nK$ under all isotopies, which preserve $q$ and such that the crossing $q$ stays disjoint from the disc at infinity:

(a) the number of the branch of the under-cross of $q$ and the number of the branch of its over-cross,

(b) 0 or 1 in correspondence to $q$ is a 0-crossing or a 1-crossing in $T \cup nK$,

(c) the homology class $[q] = [D_q^+]$ in the solid torus $V^3$,

(d) the sign $w(q)$,

(e) 0 or $n$ in correspondence with $q$ belongs to a bunch of $n^2$ crossings in $nK$ and which contain crossings of homological marking 0 or $n$.

Only (e) needs some explanation. Indeed, a 0-crossing of $K$ gives rise to a bunch of $n^2$ crossings in $T \cup nK$, where on each of the $n$ branches of over-crossings and on each of the $n$ branches of under-crossings we find all homological markings from 0 to $n - 1$ like in a sudoku. The same is true for each 1-crossing of $K$, but the homological markings are now from 1 to $n$ (see e.g. the example in Section 4).

Of course, (a) determines (b) and (c), but not (d) and (e) (see again the example in Section 4).
2.2 The loop $\text{push}(K', K)$

We define a very effective loop, which makes calculations rather short.

Let $K'$ be a fixed long knot diagram (up to regular isotopy) and let $\beta$ in $B_n$ be a fixed cyclic permutation braid, e.g. $\beta = \sigma_1 \sigma_2 \ldots \sigma_{n-1}$. We consider $T = \beta \cup nK'$, where $nK'$ is the parallel $n$-cable of $K'$.

Let $K$ be the framed long knot in which we are interested. We define the loop $\text{push}(K', K)$ in Fig. 6. First we slide the tangle $nK'$ along the tangle $nK$. This is in fact the only part of the loop which will contribute to the invariant! Then we slide $nK'$ along the solid torus $V^3$ to be in front of the point $\infty$. We slide now the point $\infty$ along $nK'$. After this we have just to
slide $\beta$ along $nK'$ in order to obtain finally a loop in $M_n$.

It is evident, that if we replace $K$ by a knot $\tilde{K}$, which is regularly isotopic to $K$, then the loops $\text{push}(K', K)$ and $\text{push}(K', \tilde{K})$ are even homotopic in $M_n$.

The existence of the loop $\text{push}$ makes of course use of the fact that we replace $K$ by very specific satellites.

If we would consider more general string links $T$ than we would have to slide them $n$ times along $nK$ and along the solid torus in order to obtain a loop in $M_n$ and calculations would be much longer (compare [12]).

Notice also, that $\text{push}$ is very different from the generalization of Gra- main’s loop (compare e.g. [12]) which would consist of sliding $\beta \cup nK'$ along the parallel $n$-cable of a curl (instead of $nK$) and then along the solid torus into its initial position. Our invariant $qR^{(2)}_{i,j,k}$ would be in this case only an invariant of the trivial knot!

2.3 The knot invariant $qR^{(2)}_{i,j,k}(\text{push})$ from the trace graph

We will not repeat the definition and the properties of the trace graph for an isotopy in its full generality (compare Chapter 4 in [10] for a detailed study), because we restrict ourself to regular isotopies from the very beginning.

Let $\gamma : t \in [0,1]$ or $S^1 \to K_t \in M_n$ in the case of a loop, be a generic regular isotopy of knots in the solid torus $V^3$. For each fixed generic $t$ we consider the set of the double points $\text{pr}(c_i(t)) \in \mathbb{C}$ which corresponds to the crossings $c_i(t)$ of $K_t$. The union of all these double points for all $t \in I$ forms a singular tangle $TL(\gamma) \subset \mathbb{C} \times I$ or respectively in $\mathbb{C}^* \times S^1$ (i.e. we forget the coordinate $z(c_i(t))$, but we remember the time-coordinate $t \in I$ or $S^1$). $TL(\gamma)$ is non-singular besides ordinary triple points which correspond exactly to the triple points in the family $\text{pr}(K_t)$, i.e. to the Reidemeister III moves. A generic point of $TL(\gamma)$ corresponds just to an ordinary crossing $c_i(t)$ of some knot $K_t$. Let $t : TL(\gamma) \to I$ be the natural projection. We orient the set of all generic points in $TL(\gamma)$ (which is a disjoint union of embedded arcs) in such a way that the local mapping degree of $t$ at $c_i(t)$ is $+1$ if $c_i(t)$ is a positive crossing and it is $-1$ if $c_i(t)$ is a negative crossing. The trace graph $TL(\gamma)$ is now the oriented singular tangle, which we obtain by gluing the arcs together in the ordinary tangent points which correspond to the Reidemeister II moves and in the triple points which correspond to the Reidemeister III moves in the family $K_t$. 

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As already mentioned, the arcs of generic points come together in the triple points and in points corresponding to an ordinary self-tangencies. But one easily sees that the above defined orientations fit together to define an orientation on the natural resolution $T\bar{L}(\tilde{\gamma})$ of $T\bar{L}(\gamma)$, i.e. we separate the three branches in a triple point, compare [10]. $T\bar{L}(\tilde{\gamma})$ is a union of oriented arcs (or circles), called trace arcs or trace circles. Evidently, we can decorate now the trace arcs by the invariants (a), (b), (c) and (e) of the previous section (the sign in (d) is already determined by the orientation of the arc and it can change on the arc). The boundaries of the trace arcs are mapped under the projection $t$ to the boundaries of the interval $I = \{t\}$. Consequently, each trace arc (and each trace circle) has a mapping degree for the projection $t$.

However, in general one has to be careful about the changes of the trace graph in generic homotopies of generic isotopies $\gamma_s, s \in I$ in $M_n$. If for a fixed $s_0$ the arc $\gamma_{s_0}(t), t \in I$, is tangential from the positive side (i.e. where the generic diagrams have two more crossings) in $M_n$ to a stratum $\Sigma^{(1)}_{tan}$ of auto-tangencies, then the trace graph $TG(\gamma_{s_0})$ changes uncontrollable by a Morse modification of index 1, when $s$ passes through $s_0$, compare Fig. 4.8 and Fig. 4.11 in [10]. (If it is tangential from the negative side, then the trace graph changes by a Morse modification of index 0 or 2, i.e. just a small circle appears or disappears.)

We will apply now trace graphs to our 1-cocycle $R_{i,j,k}^{(2)}$, but only for the very special loop $push(K', K)$. The following definition is very important.

**Definition 7** Let us fix $1 \leq i, j, k \leq n$ in such a way that all crossings of $nK$ with the under-cross on $i$ and the over-cross on $j$ are 0-crossings with respect to (b) from Definition 6. Let $\{a_1, \ldots, a_m\}$ be the (unordered) set of all crossings of $nK$ with the under-cross on $i$ and the over-cross on $j$ which are also 0-crossings with respect to (e) (i.e. belong to a bunch of $n^2$ crossings where the homological marking 0 appears). We consider the corresponding trace circles, which we denote by the same names $a_l$. Analogue, let $\{b_1, \ldots, b_r\}$ be the (unordered) set of all crossings of $nK$ with the under-cross on $i$ and the over-cross on $j$ and which are also $n$-crossings with respect to (e). We denote the trace circles by the same names $b_l$.

For each fixed $l \in \{1, \ldots, m\}$ let $R_{a_l,k}^{(2)}(push(K', K))$ be the value of the 1-cocycle $R_{i,j,k}^{(2)}(push(K', K))$, but where we sum up only over those $R$ III moves, where the crossing $ml = (i, j)$ belongs to the trace circle $a_l$. Analogue, for each fixed $l \in \{1, \ldots, r\}$ let $R_{b_l,k}^{(2)}(push(K', K))$ be the value of the 1-cocycle
We define now \( q_{i,j,k}^{(2)}(\text{push}(K',K)) \) by

\[
q_{i,j,k}^{(2)}(\text{push}(K',K)) = \sum_{l=1}^{t=m} w(a_l)x^{w(a_l)}R_{a_l,k}^{(2)}(\text{push}(K',K)) + \sum_{l=1}^{t=r} w(b_l)y^{w(b_l)}R_{b_l,k}^{(2)}(\text{push}(K',K))
\]

The following theorem is our main result.

**Theorem 1** The polynomial \( q_{i,j,k}^{(2)}(\text{push}(K',K)) \) in \( \mathbb{Z}[x,x^{-1},y,y^{-1}] \) is an invariant of the knot \( K \) up to regular isotopy.

**Proof.**

Let \( K_s, s \in I \), be a generic regular isotopy which connects \( K = K_0 \) with \( K_1 \). It induces a generic regular isotopy which connects \( T \cup nK_0 \) with \( T \cup nK_1 \). For each generic \( s \) we consider the loop \( \text{push}(K',K_s) \). Evidently, this loop is never tangential to \( \Sigma^{(1)}_{\text{tan}} \), if we slide \( nK' \) with a constant speed through \( nK_s \), but we do not even need this, because we are only interested in the crossings of \( nK_s \). Moreover, the crossings \( c_i(t) \) of \( nK_s \) do not depend on \( t \) at all, i.e. they do not move in the loop. Consequently, each crossing \( c_i(t) \) of \( nK_s \) defines a trace circle, for which the mapping degree is just the sign \( w(c_i(t)) \) of the crossing \( c_i(t) \). We are only interested in the circles \( a_l \) and \( b_l \), because the other circles do not contribute to \( R_{i,j,k}^{(2)}(\text{push}(K',K)) \) in any case. If in the family \( K_s \) appears a R III move, then the loop \( \text{push}(K',K_s) \) passes through a quadruple crossing and hence \( R_{i,j,k}^{(2)}(\text{push}(K',K)) \) changes by the value on the meridian of the quadruple crossing. But it follows from the rigidity in Proposition 1 that \( R_{a_l,k}^{(2)}(\text{push}(K',K)) \) and \( R_{b_l,k}^{(2)}(\text{push}(K',K)) \) stay both invariant for each \( l \).

A regular isotopy of \( K \) brings 0-crossings to 0-crossings and 1-crossings to 1-crossings. Consequently, it brings 0-crossings of \( T \cup nK \) with respect to (e) from Definition 6 to 0-crossings with respect to (e) and n-crossings with respect to (e) to n-crossings with respect to (e). Therefore we can split \( q_{i,j,k}^{(2)}(\text{push}(K',K)) \) into two polynomials, each of which is already an invariant.

If in the family \( K_s \) appears a R II move, then two new crossings appear or disappear together. Evidently, the two crossings, say \( c_1 \) and \( c_2 \), have all the same local invariants from Definition 6, besides different signs. They can contribute only simultaneously to \( q_{i,j,k}^{(2)}(\text{push}(K',K)) \). It follows from the cube equations, compare [10] and [12], that
\( R_{c_1,k}^{(2)}(\text{push}(K', K)) = -R_{c_2,k}^{(2)}(\text{push}(K', K)) \), otherwise \( R_{i,j,k}^{(2)} \) wouldn’t be a 1-cocycle. It follows that the exponents

\[ w(c_1)R_{c_1,k}^{(2)}(\text{push}(K', K)) \text{ and } w(c_2)R_{c_2,k}^{(2)}(\text{push}(K', K)) \]

are the same, but the monomials enter with different signs into \( qR_{i,j,k}^{(2)}(\text{push}(K', K)) \) and cancel out together.

This finishes the proof.

**Remark 1** In fact, all crossings of \( T \cup nK \) give rise to trace circles which are well defined too. However, because the point \( \infty \) moves over these crossings the decorations (a) \((i,j)\), and (b) 0- or 1-crossings, are not well defined on the trace circles and we can not use them to define an invariant, because of the separation condition.

**Remark 2** If we replace \( K' \) by another knot \( K'' \), then the loop \( \text{push}(K'', K) \) is completely different, it is even in another component of \( M_n \). However, the trace circles \( a_l \) and \( b_l \) are still the same! Hence, we can add a new variable \( x_{K''}^{w(a_l)R_{a_l,k}^{(2)}(\text{push}(K'', K))} \) as a factor to the monomial which corresponds to \( a_l \) (respectively \( b_l \)). The same is true, if we replace \( R_{i,j,k}^{(2)} \) by another 1-cocycle , but with the same rigidity, i.e. \( a_l \) and \( b_l \) are still the trace circles such that the 1-cocycle is trivial for each trace circle on the meridian of each quadruple crossing.

We will give examples of such refinements in Section 3.

### 2.4 First applications

The standard diagram of 8\(_{17}\) as a long knot has one negative 0-crossing and two positive 0-crossings, exactly as \(-8_{17}\). Already the \( x \)-part of the invariant \( qR_{2,1,2}^{(2)}(\text{push}(3^+_1, K)) \) distinguishes the orientations, see Section 4.

The standard diagram of 3\(_{1}^+\) as a long knot has two positive 0-crossings and one positive 1-crossing. We have calculated \( qR_{2,1,2}^{(2)}(\text{push}(3^+_1, K)) \). We indicate the crossings \( a_1, a_2 \) and \( b_1 \), which give rise to the contributing trace circles, in Fig. 7. A calculation yields

\[ qR_{2,1,2}^{(2)}(\text{push}(3^+_1, 3^+_1)) = x^{-6} + x^{-5} + y^{-4} \]

We consider the trace graph associated to a regular isotopy of \( 3^+_1 \). Any regular isotopy of a knot has to respect the 0-crossings and the 1-crossings.
But a priori, there could be a regular isotopy of the knot $3_1^+$ to itself which exchanges the two 0-crossings. However, than it has to exchange the crossings $a_1$ and $a_2$ in $\sigma_1 \cup 2.3_1^+ \cup 2.3_1^+$ as well. But this is not possible, because the trace circle $a_1$ contributes $x^{-6}$ and the trace circle $a_2$ contributes $x^{-5}$ to $qR_{2,1,2}^{(2)}(\text{push}(3_1^+, 3_1^+))$!

One easily sees, that we need $n > 1$. Indeed, without cabling the two 0-crossings give the same contribution. Hence, the information which comes from using the longitude is essential.

It is also amazing that $qR_{2,1,2}^{(2)}(\text{push}(3_1^+, 3_1^-))$ gives exactly the same result. (Attention, we have to add six curls to $3_1^-$ in order to get the same writhe and the same Whitney index as for $3_1^+$.) This means that $qR_{2,1,2}^{(2)}(\text{push}(3_1^+, K))$ does not distinguish the two trefoils. It’s calculation is of quadratic complexity with respect to the number of crossings of $K$. It is well known that the Vassiliev invariant $v_3(K)$ and the knot signatures can distinguish the trefoils. However, their calculations are at least of cubical complexity with respect to the number of crossings. This suggests, that the complexity of our invariant $qR_{i,j,k}^{(2)}(\text{push})$ is just to low for doing so. Therefore we increase the complexity in Section 5.

Each trace circle contributes a monomial to $qR_{i,j,k}^{(2)}(\text{push}(K', K))$ with the sign of the crossing. Consequently, the number of positive 0-crossings is not less than $\text{aug}_+$ of the x-part of $qR_{i,j,k}^{(2)}(\text{push}(K', K))$ and the number of negative 0-crossings is not less than $-\text{aug}_-$ of the x-part for each diagram of a knot which is regularly isotopic to $K$. The analogue statement holds also for the
Figure 8: The Chmutov-Khoury-Rossi formulas for $k = 1$ and $k = 2$

1-crossings and the y-part. The example of $-8_{17}$ shows that this estimation can be sharp. It would not be sharp for $8_{17}$. However, the corresponding estimate from the 2-variable polynomial $qR^{(2)}_{2,1,2}R^{(4)}_{a=1,ml-.,hm}(push(3^+, 8_{17}))$ shows that it is now sharp as well.

For knots $K$ in $S^3$ there are no longer 0-crossings and 1-crossing. However, we can always transform the knot into a long knot without creating new crossings. Consequently, $\arg_+$ and $-\arg_-$ of $qR^{(2)}_{i,j,k}(push(K^+,K))$ without distinguishing the x-part from the y-part is still an estimate for the minimal number of positive, respectively negative, crossings of $K$ up to regular isotopy.
The monograph [12] contains several lifts of the coefficients of the Conway polynomial to 1-cocycles. For the convenience of the reader we repeat here the definition of the simplest one.

First of all, we need to put one more condition on \( M_n \), but which is satisfied automatically for the loops \( \text{push}(K',K) \).

Given a generic knot diagram \( K \subset V^3 \) we consider the oriented curve \( \text{pr}(K) \) in the annulus. A loop in \( \text{pr}(K) \) is a piecewise smoothly oriented immersed circle in \( \text{pr}(K) \) which respects the orientation of \( \text{pr}(K) \). In other words, we go along \( \text{pr}(K) \) following its orientation and at a double point we are allowed to switch perhaps to the other branch, but still following the orientation of \( \text{pr}(K) \). Naturally, a loop in \( \text{pr}(K) \) is called negative (respectively positive) if it represents a negative (respectively positive) homology class in \( H_1(V^3) \) (by definition the closure of a long knot represents the positive generator of \( H_1(V^3) \)). One easily sees that knots which arise as cables of long knots by our above construction, contain never negative loops. We put this as a new condition on \( M_n \), (compare [12]): the projection of the knots does never contain negative loops.

Let \( T \cup nK \) be in \( M_n \). We consider the cyclic \( n \)-fold covering of \( V^3 \) and we lift \( T \cup nK \) to a knot \( \tilde{K} \) in the covering. \( \tilde{K} \) can be naturally identified with a long knot \( (\infty \text{ is on } \tilde{K}) \) and we call it the underlying long knot for \( T \cup nK \).

The crossings of \( \tilde{K} \) are called the persistent crossings. These are exactly the crossings of \( T \cup nK \) with the homological markings 0 and \( n \).

We will make use of the generalization of the Polyak-Viro formula by Chmutov-Khoury-Rossi [8] and Brandenbursky [2] to Gauss diagram formulas for all coefficients of the Conway polynomial.

For the convenience of the reader we repeat these formulas here, compare [8]. Let \( k \in \mathbb{N} \) be fixed and let \( A_{2k} \) be an arrow diagram (i.e. an abstract Gauss diagram without signs on the arrows, we call it often a configuration) with one (oriented) circle, \( 2k \) arrows and a base point. Following Chmutov-Khoury-Rossi, \( A_{2k} \) is called ascending one-component if by going along the oriented circle starting from the base point and each time jumping along the arrow, we meet each arrow first at its foot and the travelling meets the whole
circle.

The Gauss diagram formula $C_{2k}$ is simply the sum of all ascending one-component arrow diagrams $A_{2k}$. Each arrow diagram enters of course with the sign which is the product of all signs of crossings in it. We show as examples $C_2$ and $C_4$ in Fig. 8. By definition $C_0 = 1$.

We apply these formulas to the persistent crossings. Notice, that if we travel on the oriented circle starting from the point at infinity then we meet in each configuration $A_{2k}$ always first the foot of an $n$-crossing, called first $n$-crossing, and we meet last the foot of an 0-crossing (this is an immediate consequence of their definition).

Let us fix again branches $i$ and $j$ with $i \neq j$, such that the crossings with the foot (under-cross) on $i$ and the head (over-cross) on $j$ are 0-crossings. Let the homological marking of these crossings be $[ij] = a$. Because $i \neq j$ the homological marking $a$ is fixed between 1 and $n - 1$. We will consider now only those R III moves which are of the global type as shown in Fig. 9, i.e. the distinguished crossing $d$ has the homological marking $a$, the crossing $hm$ has the homological marking $n$ and the crossing $ml$ is a $ij$-crossing. (The crossing $ml$ has then automatically the homological marking $a$ too.)

**Definition 8** The weight $W_{2k-1}(hm)$ for a diagram with a triple crossing $p$ of type $r(a,n,ij)$ is defined by the sum of all $A_{2k}$ where the crossing $hm$ in $p$ is the first $n$-crossing in $A_{2k}$.

The sign of a R III move was already defined in the previous section.

**Definition 9** Let $\gamma$ be a generic oriented loop in $M_n$. Let $k > 0$.

The integer-valued 1-cochain $R_{ij,hm}^{(2k)}$ is defined by
\[
R^{(2k)}_{ij,hm}(\gamma) = \sum_{p=r(a,n,ij)\in\gamma} \text{sign}(p)W_{2k-1}(hm)w(hm).
\]

This means that we consider only those contributions to \( A_{2k} \), where \( hm \) is the first \( n \)-crossing.

**Proposition 2** \( R^{(2k)}_{ij,hm} \) is an integer 1-cocycle in \( M_n \) which represents a non-trivial integer cohomology class.

Compare [12], Definition 2.9, Theorem 2.4 and Theorem 2.6.

The definition of this 1-cocycle is the closest to the definition of the Conway polynomial as a 0-cocycle by Chmutov-Khoury-Rossi.

**Theorem 2** The polynomial

\[
qR^{(2k)}_{ij,hm}(\text{push}(K', K)) = \sum_{l=m} w(a_l)z^{w(a_l)R^{(2k)}_{al,hm}(\text{push}(K', K))}_{2k} + \sum_{l=r} w(b_l)\bar{z}^{w(b_l)R^{(2k)}_{al,hm}(\text{push}(K', K))}_{2k}
\]

in \( \mathbb{Z}[z_{2k}, z_{-2k}^{-1}, \bar{z}_{2k}, \bar{z}_{-2k}^{-1}] \) is an invariant of the knot \( K \) up to regular isotopy for each \( k > 0 \).

The proof is exactly the same as the proof of Theorem 1, because the 1-cocycles \( R^{(2k)}_{ij,hm} \) have exactly the same rigidity in the quadruple crossings. Moreover, as already explained in Remark 2, we can consider them all simultaneously and add the corresponding contributions of each of them as factors to the monomials which correspond to the trace circles in \( qR^{(2k)}_{ij,hk}(\text{push}(K', K)) \), compare the next section for an example.

There are more sophisticated lifts of the coefficients of the Conway polynomial to 1-cocycles in [12]. They use combinations with complicated weights of different types of strata of \( \Sigma^{(1)}_{nti} \). But they can all be adapted (by a slight change of the definition of certain linking numbers) to give polynomial 1-cocycle invariants exactly as in Theorem 1 and 2. But we left this to the interested reader.
Figure 10: The 0-crossings in $8_{17}$ and in $-8_{17}$

Figure 11: The crossings corresponding to the trace circles $a_1$, $a_2$, $a_3$ in $\text{push}(3^+_1, 8_{17})$ for $n = 2$ and the homological markings of the crossings
Figure 12: The two contributing R III moves for $a_1$

Figure 13: A couple of crossings which does not contribute to $R_{ij=a_1, hm}^{(4)}(\text{push}(K', K))$
4 The example

We show the diagrams of \(8_{17}\) and \(-8_{17}\) together with the 0-crossings in Fig. 10. The 2-cable for \(8_{17}\), which we use for the loop \textit{push}, together with the homological markings (we do not write the markings 1) and the names of the trace circles, is shown in Fig. 11. (The names of the corresponding trace circles of \(-8_{17}\) are induced by the rotation of the long knot \(8_{17}\) by \(\pi\) around the \(y\)-axes.) A trace circle can only contribute to the invariant, when the trefoil moves \textit{over} the crossing corresponding to the trace circle, because it has to be the crossing \(ml\) in a R III move. One easily sees, that each trace circle contributes with exactly two R III moves. We show the relevant fragment for \(a_1\) in Fig. 12 as an example. In the first of the two moves the crossing \(hm\) has the homological marking \(n = 2\). Hence the crossings in \(R_{i,j,k}^{(2)}(\text{push}(K',K))\) which cut \(hm\) (i.e. the corresponding arrows in the Gauss diagram intersect) come all from the trefoil. In the second of the two moves the crossing \(hm\) has the marking 1 and the crossings in \(R_{i,j,k}^{(2)}(\text{push}(K',K))\) which cut \(hm\) come all from the longitude of \(8_{17}\). On the other hand, the first move is the only move for \(a_1\) which contributes also to \(R_{ij,hm}^{(2k)}(\text{push}(K',K))\). One easily sees, that \(R_{ij,hm}^{(2)}(\text{push}(K',K))\) is not interesting, because there is always only one crossing which cuts \(hm\) and which contributes and it comes from the trefoil. However, \(R_{ij,hm}^{(4)}(\text{push}(K',K))\) is interesting, because the non-connected configurations in the last line of Fig. 8 can contribute non-trivially. One easily sees that all the connected configurations in Fig. 8 can not contribute, because there are no such configurations in the trefoil. On the other hand, we indicate in Fig. 13 a couple of crossings, which do \textit{not} contribute to \(R_{ij=3,hm}^{(4)}(\text{push}(K',K))\), because \(hm\) would be no longer the first 2-crossing in the configuration. \((hm,0_1,2_1,0_2\) corresponds to the third configuration in the last line in Fig. 8 but it does not contribute to the 1-cocycle, because the foot of \(2_1\) is on the circle before the foot of \(hm\).) The crossing \(2_1\) belongs to \(8_{17}\) as other couples of crossings which do contribute. This shows, that in \(R_{ij,hm}^{(4)}(\text{push}(K',K))\) the longitude of \(K\) contributes in a very different way in comparison with \(R_{ij,k}^{(2)}(\text{push}(K',K))\)!

We give now the contributions of all the relevant R III moves. The reader can check this easily by drawing the corresponding pictures.

\(8_{17}\):

move 1: \(R_{a_1,2}^{(2)} = 1, R_{a_1,hm}^{(4)} = -1\)
move 2: $R_{a_1,2}^{(2)} = 3$
move 3: $R_{a_2,2}^{(2)} = -1$, $R_{a_2,hm}^{(4)} = 1$
move 4: $R_{a_2,2}^{(2)} = -4$
move 5: $R_{a_2,2}^{(2)} = -1$, $R_{a_3,hm}^{(4)} = 0$
move 6: $R_{a_3,2}^{(2)} = -3$

Remember that $w(a_1) = -1$ and $w(a_2) = w(a_3) = +1$. It follows that

$$qR_{2,1,2}^{(2)}R_{21,hm}^{(4)}(push(3^+, 8_{17})) = -zx^{-4} + zx^{-5} + x^{-4}.$$

The first monomial is the contribution of the trace circle $a_1$, the second of the trace circle $a_2$ and the third of the trace circle $a_3$.

$-8_{17}$:

move 1: $R_{a_1,2}^{(2)} = 1$, $R_{a_1,hm}^{(4)} = 0$
mov 2: $R_{a_2,2}^{(2)} = 2$
mov 3: $R_{a_2,2}^{(2)} = -1$, $R_{a_2,hm}^{(4)} = 0$
mov 4: $R_{a_2,2}^{(2)} = -3$
mov 5: $R_{a_3,2}^{(2)} = -1$, $R_{a_3,hm}^{(4)} = 0$
mov 6: $R_{a_3,2}^{(2)} = -3$

We have ordered here the moves in analogy to the knot $8_{17}$. Remember that $w(a_1) = -1$ and $w(a_2) = w(a_3) = +1$ too. It follows that

$$qR_{2,1,2}^{(2)}R_{21,hm}^{(4)}(push(3^+_{1}, -8_{17})) = -x^{-3} + 2x^{-4}.$$

There are five trace circles in the $y$-part for each of the two knots and we haven’t calculated the corresponding invariants.

5 The polynomial invariant $qR_{i,j}^{(3)}(push)$ of cubical complexity

In [7] Sergei Chmutov and Michael Polyak have extracted from the HOMFLYPT polynomial a very nice Gauss diagram formula $C_3$ for the Vassiliev
invariant $v_3(K)$. We repeat here the version of $C_3$ adapted to our approach, see Fig. 14 with the usual conventions. It has again the property, that starting from the point $\infty$ we come in each configuration of $C_3$ first to the foot of a crossing, called the \textit{first crossing}. We show in Fig. 14 also the identity which comes from different expressions of the same linking number. Our version of $C_3$ differs by this identity from the original formula of Chmutov and Polyak. This does not make any difference as a 0-cocycle, i.e. an expression for the Vassiliev invariant $v_3$. But surprisingly, it is of crucial importance for the definition of the 1-cocycle!

It was shown in [14] that this formula gives rise to a 1-cocycle exactly along the same lines as in Section 3, if we apply it only to the persistent crossings.

But we want to make a stronger use of the longitude, compare the example in the previous section, and therefore we want to generalize also the invariant from Section 2. Let us fix again branches $i$ and $j$ (but we allow now $i = j$ as well), such that the crossings with the foot (under-cross) on $i$ and the head (over-cross) on $j$ are 0-crossings (with respect to (b) of Definition 6 of
\[ r(i, j) = \begin{array}{c}
    \infty \\
    i \\
    \quad j
\end{array} \quad \quad \quad l(i, j) = \begin{array}{c}
    \infty \\
    j \\
    \quad i
\end{array} \]

Figure 15: The two global types of triple crossings which contribute to \( R_{i,j}^{(3)} \) course. As for \( R_{i,j,k}^{(2)} \), we will use only two global types of triple crossings. We give names to them in Fig. 15.

**Definition 10** Let \( p \) be a triple crossing of global type \( r(ij) \) or \( l(ij) \). First we deform it into a regular diagram, no matter on the positive or the negative side of the \( R III \) move. The weight \( W_2(hm) \) is then defined by the sum of all those configurations in \( C_3 \) where the (now regular) crossing \( hm \) is the first crossing in the configuration.

**Definition 11** Let \( \gamma \) be a generic oriented loop in \( M_n \).

The integer-valued 1-cochain \( R_{i,j}^{(3)} \) is defined by

\[
R_{i,j}^{(3)}(\gamma) = \sum_{p=r(ij) \in \gamma} \text{sign}(p)W_2(hm)w(hm) - \sum_{p=l(ij) \in \gamma} \text{sign}(p)W_2(hm)w(hm).
\]

**Proposition 3** \( R_{i,j}^{(3)} \) is an integer 1-cocycle in \( M_n \) which represents a non-trivial integer cohomology class.

**Proof.**

The proof follows the same lines as in \([12]\) and \([14]\) and uses a lot of figures from \([12]\) and \([10]\). Therefore we consider here only some cases. The remaining cases are quite similar and we left the verification to the reader.

First of all we observe that the cube equations force us to transform the triple crossing into a regular diagram before considering the contributing configurations. Let us consider e.g. the edge \( r : 1 \to 6 \) from Fig. 6.60 in \([12]\). In Fig. 16 we show the two triple crossings together with their deformations.
We consider just two examples of the twenty four positive global tetrahedron equations. (The crossings from the quadruple crossing are positive and are denoted by the number of the two branches. The under-cross is always on the branch with the lower number, compare [12].) All other cases are quit similar.

Let us consider the case $I$ with $\infty = 4$ shown in Fig. 3.5 and 3.6 in [12].

Only the strata $P_1$, $P_3$ and $P_4$ have the right global type to contribute. One easily sees, that non of the three crossings from the tetrahedron and which are not in the triple crossing can contribute for $P_1$ neither for $\bar{P}_1$, because they have always a foot or a head between $\infty = 4$ and the foot of the crossing $hm$. The strata $P_3$ and $P_4$ share the same crossing $ml = 12$. 

on the positive side for both of them, compare Fig. 4. We see that without deforming the triple crossings, the local type 1 would contribute 0 and the local type 6 would contribute $+1$ and they would not cancel out together. After the deformation, the local type 1 contributes now $+1$ and the local type 5 contributes $+1 + 1 - 1 = +1$ as well and they cancel now out together.
Figure 17: A tetrahedron equation for the type $I_4$

Figure 18: A tetrahedron equation for the type $II_2$
We deform each triple crossing in a regular diagram on the positive side and we show the corresponding diagrams with their signs in the meridian in Fig. 17 where we draw the crossing $hm$ with a thicker arrow. We write the contributions in the following form: (crossing with first foot after $\infty$, crossing in the middle if any, crossing with last foot before $\infty$). The contributions to $R^{(3)}_{1,2}$ are the following:

$P_3$: $(hm, 12), (hm, 12, 14)$

$-\bar{P}_3$: $(hm, 12)$

$P_4$: $(hm, 12), (hm, 13), (hm, 13, 14), (hm, 13, 12)$

$-\bar{P}_4$: $(hm, 12), (hm, 13), (hm, 12, 13), (hm, 23, 13), (hm, 34, 13)$.

It follows that $R^{(3)}_{1,2}$ vanishes on the meridian for this stratum of quadruple crossings. (Of course, we have still to check that this is still the case if besides $hm$ exactly one of the crossings in the configurations is from the tetrahedron and one other crossing comes from the rest of the diagram. But this is easier to check.)

Let us consider the case $II$ with $\infty = 2$ shown in Fig. 3.7 and 3.8 in [12]. Only the strata $P_3$ and $P_4$ have the right global type to contribute, and which share the same crossing $ml = 12$. Notice that $P_3$ and $P_4$ have now different global types.

We deform again each triple crossing in a regular diagram on the positive side and we show the corresponding diagrams with their signs in the meridian in Fig. 18 where we draw the crossing $hm$ with a thicker arrow. The contributions to $R^{(3)}_{1,2}$ are now the following:

$-P_3$: $(hm, 34)$

$\bar{P}_3$: 0

$P_4$: $(hm, 12), (hm, 12, 34)$

$-\bar{P}_4$: $(hm, 12), (hm, 34), (hm, 12, 34)$.

Remember, that the global type of $P_3$ (and hence of $\bar{P}_3$) enters with a minus sign into the formula. It follows that $R^{(3)}_{1,2}$ vanishes on the meridian.
Figure 19: Gramain’s loop in the solid torus for the right trefoil

for this stratum of quadruple crossings.

Notice, that in our formula we use the right side of the identity in Fig. 14. If we would use instead the left side, than \((hm, 12, 34)\) would no longer contribute to \(P_4\) and \(R_{1,2}^{(3)}\) would not vanish on the meridian.

To see that \(R_{i,j}^{(3)}\) is non-trivial, we consider just Gramain’s loop for the trefoil with \(n = 1\) in the solid torus. We show it in Fig. 19. There is only one R III move which has the right global type to contribute. We show its Gauss diagram in Fig. 19 too, together with its deformation on the positive side of the discriminant (we ignore two crossings which cancel out in a R II move). The sign of the R III move is \(-1\) and \(W_2(hm) = +1\) as shown in the figure too. This finishes the proof.

Remark 3 In fact, we could fix a third branch \(k\) exactly as in \(R_{i,j,k}^{(2)}\) and consider only those configurations in \(C_3\) where in addition the foot of the last crossing (before \(\infty\)) is on the branch \(k\). This would give a splitting of \(R_{i,j}^{(3)}\) into finer 1-cocycles. However, it seems that the case \(k = i\) is always the
most interesting one and it is already included in $R_{i,j}^{(3)}$.

It follows immediately from the structure of the 1-cocycle (we can fix again the crossing $ml$ in the trace graph for the quadruple crossings) that it has the same rigidity in the quadruple crossings as all the other 1-cocycles in this paper. Consequently, it can be used to define a polynomial 1-cocycle invariant in exactly the same way as previously and we obtain the following theorem.

**Theorem 3** The polynomial

$$qR_{i,j}^{(3)}(\text{push}(K', K)) = \sum_{l=m} w(a_l)x_3^{w(a_l)R_{i,j}^{(3)}(\text{push}(K', K))} + \sum_{l=r} w(b_l)y_3^{w(b_l)R_{i,j}^{(3)}(\text{push}(K', K))}$$

in $\mathbb{Z}[x_3, x_3^{-1}, y_3, y_3^{-1}]$ is an invariant of the knot $K$ up to regular isotopy.

This invariant is evidently of cubical complexity with respect to the number of crossings. Calculations by hand of examples are too complicated for $n > 1$. A computer program would be very helpful.

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