Tensor surgery and tensor rank

Matthias Christandl and Jeroen Zuiddam

Abstract

We introduce tensor surgery and apply it to tensors defined by graphs and hypergraphs. By splitting vertices, creating and absorbing virtual edges and by inserting new edges, we obtain nontrivial upper bounds on tensor rank, border rank and asymptotic rank of the resulting tensors. We illustrate our method with a number of examples. Tensor surgery on the triangle graph, which corresponds to the matrix multiplication tensor, leads to nontrivial rank bounds for all odd cycle graphs, which correspond to the tensors of iterated matrix multiplication. In the asymptotic setting we obtain upper bounds in terms of the matrix multiplication exponent $\omega$ and the rectangular matrix multiplication parameter $\alpha$ that are optimal if $\omega$ equals two. We also give examples that illustrate that tensor surgery on general graphs might involve the absorption of virtual hyperedges and provide an example of tensor surgery on a hypergraph. In the context of quantum information theory, our results may be interpreted as upper bounds on the rate of Greenberger-Horne-Zeilinger (GHZ) states needed in order to create a graph of Einstein-Podolsky-Rosen states, or, more generally, GHZ states shared among subsets of the vertices.

1. Introduction

This paper introduces a method for proving upper bounds on tensor rank, border rank and asymptotic tensor rank. The method gives particularly clean results when applied to tensors that are defined combinatorially. Let us first illustrate the combinatorial description that we are using, and then explain the method in a basic form. The most famous example of a tensor that fits into our combinatorial framework is the two-by-two matrix multiplication tensor $\langle 2, 2, 2 \rangle$, which is described by the triangle graph $C_3$

$$T\left(\begin{array}{c}
\vdots \\
\end{array}\right) = \sum_{i, j, k \in \{0, 1\}} b_{ij} \otimes b_{jk} \otimes b_{ki} \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4.$$
Here the $b_{ij}$’s denote the standard basis elements of $\mathbb{C}^4$, each vertex of the graph corresponds to a tensor leg and each edge in the graph corresponds to an index to sum over, shared between tensor legs. Another important example is the so-called rank-two unit tensor $\langle 2 \rangle$ of order three, which corresponds to the hypergraph on three vertices with a single hyperedge $\{1, 2, 3\}$

$$T(\bullet) = \sum_{i,j,k \in \{0,1\}} b_i \otimes b_i \otimes b_i \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$$ 

As a final illustrative example consider the complete graph on 4 vertices $K_4$ and the corresponding tensor

$$T(\bullet) = \sum_{i \in \{0,1\}^6} b_{i1i2i3} \otimes b_{i3i4i5} \otimes b_{i2i4i6} \otimes b_{i1i5i6} \in (\mathbb{C}^8)^\otimes 4.$$ 

Our aim is to prove nontrivial upper bounds on tensor rank and asymptotic tensor rank. Let us for now focus on tensor rank. The tensor rank of a tensor in $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ is the smallest number $r$ such that the tensor can be written as a sum of $r$ simple tensors $v_1 \otimes \cdots \otimes v_k$ with $v_i \in \mathbb{C}^{n_i}$. The tensor rank of a tensor $t$ is denoted by $R(t)$. Since tensor rank is invariant under the action of the group $\text{GL}_{n_1} \otimes \cdots \otimes \text{GL}_{n_k}$ we will identify tensors that are in the same orbit under this group action. Going back to our examples, a nontrivial upper bound of seven on the tensor rank of $\langle 2, 2, 2 \rangle$ was obtained by Strassen by constructing an efficient bilinear algorithm for multiplying two-by-two matrices [Str69]. This upper bound was later proven to be tight by Winograd [Win71] and independently by Hopcroft and Kerr [HK71]. The triangle tensor $\langle 2, 2, 2 \rangle$ will play an important role in this paper. The second tensor $\langle 2 \rangle$ is the canonical example of a tensor of rank two. For the third tensor, observe that the graph contains a triangle, and hence Strassen’s decomposition of $\langle 2, 2, 2 \rangle$ can directly be upgraded to a nontrivial decomposition of this tensor.

In its most basic form the tensor surgery method consists of the following steps. Take a tensor $t$ of which we know (an upper bound on) the tensor rank. Then, linearly split up a tensor leg of $t$ into multiple tensor legs, keeping track of the increase in rank that this causes. Denote the new tensor by $t’$. Finally, take the tensor product of $t’$ with some other tensor $s$ to obtain our goal tensor.

We illustrate this idea with the tensor of the five-cycle $C_5$,

$$T(\bigtriangleup) = \sum_{i \in \{0,1\}^5} b_{i1i2} \otimes b_{i2i3} \otimes b_{i3i4} \otimes b_{i4i5} \otimes b_{i5i1} \in (\mathbb{C}^4)^\otimes 5.$$ 

The trivial decomposition of $T(C_5)$ has size 32. Let $t = T(C_3) = \langle 2, 2, 2 \rangle$ be the matrix multiplication tensor. Let $b_0, b_1$ be a basis of $\mathbb{C}^2$ and define
$b_+ = b_0 + b_1$ and $b_- = b_0 - b_1$. For any pair of symbols $x, y \in \{0, 1, +, -\}$ define $b_{xy} = b_x \otimes b_y \in \mathbb{C}^4$. We take Strassen’s decomposition

$$T(C_3) = - b_- \otimes b_{0+} \otimes b_{11} - b_{0+} \otimes b_{11} \otimes b_- - b_{11} \otimes b_- \otimes b_{0+} + b_- \otimes b_{1+} \otimes b_{00} + b_{1+} \otimes b_{00} \otimes b_- + b_{00} \otimes b_- \otimes b_{1+} + (b_{00} + b_{11}) \otimes (b_{00} + b_{11}) \otimes (b_{00} + b_{11}).$$

Viewing the seven summands as 6-tensors in $(\mathbb{C}^2)^\otimes 6$, the first six summands are simple tensors while the last summand is not simple. Let $\phi$ be the linear map $\mathbb{C}^2 \otimes \mathbb{C}^2 \to (\mathbb{C}^2 \otimes \mathbb{C}^2)^\otimes 3$ defined on simple tensors by $v_1 \otimes v_2 \mapsto \sum_{j,k \in \{0,1\}} (v_1 \otimes b_j) \otimes (b_k \otimes v_2)$ which inserts a vertex with two edges.

Let $\psi$ be the map which applies $\phi$ at the first tensor leg of a tensor in $(\mathbb{C}^4)^\otimes 3$. Then $\psi(T(C_3)) = T(C_5)$. In pictures, the effect of applying $\psi$ is

\[
\begin{array}{c}
\text{△} \quad \sim \\
\text{□} \quad \sim
\end{array}
\]

Note that splitting a tensor leg possibly increases tensor rank. To remind us of this, we like to decorate the picture with a ‘virtual edge’ connecting the cut vertices,

\[
\begin{array}{c}
\text{△} \quad \sim \\
\text{□} \quad \sim
\end{array}
\]

The crux is the triangle appearing on the right. Observe that $\phi(b_{xy})$ has rank at most 4 and that $\phi(b_{00} + b_{11})$ equals $T(C_3)$ which has rank 7. Applying $\psi$ to each summand in Strassen’s decomposition thus gives six summands of rank at most 4 and one summand of rank 7. This proves $R(T(C_5)) \leq 31$ and concludes our example.

Before going to more applications of tensor surgery, we will discuss some definitions, preliminaries, lower bound methods and relationships with other work.

**Definitions and preliminaries.** All our vector spaces will be complex vector spaces. However, the ideas in this paper will work over any field. Let $G = (V,E)$ be a graph and let $d$ be a natural number. Let $b_1, \ldots, b_d$ be the standard basis of $\mathbb{C}^d$. We define the tensor of order $|V|$ corresponding to the graph $G$ as

$$T_d(G) := \sum_{i \in [d]^E} \bigotimes_{v \in V} \left( \bigotimes_{e \in E} b_{i_e} \right),$$

where the sum is over all tuples $i$ indexed by $E$ with entries in $[d] := \{1,2,\ldots,d\}$. We write $T$ for $T_2$. We can safely ignore the fact that this tensor depends on the choice of order of the edges and vertices, since the
This value reduces to \(\omega\). See [VC15] and [VC16] for general results on the quantity \(\omega\).

The asymptotic log-rank or **exponent** of a tensor \(t\) is defined as the limit

\[
\omega(t) := \lim_{n \to \infty} \frac{1}{n} \inf \{m \in \mathbb{N} \mid R(t^{\otimes n}) \leq 2^m\},
\]

and satisfies the inequality \(\omega(t) \leq \log_2 R(t) \leq \log_2 R(t)\). For two \(k\)-tensors \(\phi \in U_1 \otimes \cdots \otimes U_k\) and \(\psi \in V_1 \otimes \cdots \otimes V_k\) we say \(\phi \geq \psi\) if there exist linear maps \(A_i : U_i \rightarrow V_i\) such that \(\psi = (A_1 \otimes \cdots \otimes A_k)\phi\). Define the asymptotic conversion rate from \(\phi\) to \(\psi\) as

\[
\omega(\phi, \psi) := \lim_{n \to \infty} \frac{1}{n} \inf \{m \in \mathbb{N} \mid \psi^{\otimes n} \leq \phi^{\otimes m}\}.
\]

This value reduces to \(\omega(\psi)\) by setting \(\phi\) to the rank-2 unit tensor \(T([k])\), namely \(\omega(\psi) = \omega(T([k]), \psi)\). The value \(\omega(\psi)\) thus measures how many copies of \(T([k])\) are asymptotically needed to create a copy of \(\phi\) by local linear maps. On the other hand, setting \(\psi\) to \(T([k])\), the value \(\omega(T([k]), \psi)^{-1}\) measures how many copies of \(T([k])\) can be distilled from one copy of \(\psi\). A well-known result is that \(\omega(T([3]), T(C_3))^{-1} = 2\) [Str87]. It was recently shown that this distillation rate holds for all cycles, that is, \(\omega(T([3]), T(C_k))^{-1} = 2\) for any \(k\) [VC16]. See [VC15] and [VC16] for general results on the quantity \(\omega(\phi, \psi)\).

The matrix multiplication tensor \(\langle a, b, c \rangle\) is defined as

\[
\langle a, b, c \rangle = \sum_{i \in [a], j \in [b], k \in [c]} b_{ij} \otimes b_{jk} \otimes b_{ki} \in \mathbb{C}^{ca} \otimes \mathbb{C}^{ab} \otimes \mathbb{C}^{bc},
\]

and can thus be thought of as the tensor of the triangle graph where the edges are given multiplicities according to \(a, b, c\). It is an open problem in algebraic complexity theory to compute the exponent of matrix multiplication \(\omega = \omega((2, 2, 2))\). The currently best bounds on this number are \(2 \leq \omega < \)
2.3728639 \([\text{LG14}]\). We will not improve the bounds on \(\omega\) in this paper. For any real number \(a\), let \(\langle n, n, n^a \rangle\) denote the tensor

\[
\langle n, n, n^a \rangle := \sum_{i \in \langle n \rangle, j \in \langle n \rangle, k \in \langle n^a \rangle} b_{ij} \otimes b_{jk} \otimes b_{ki},
\]

which one can think of as the tensor of the triangle whose edges have multiplicity \(1, 1, a\). Let \(\alpha\) be the supremum of all real \(a > 0\) for which the exponent of \(\langle n, n, n^a \rangle\) equals 2. We call this number \(\alpha\) the dual exponent of matrix multiplication. The currently best bounds on this number are \(0.3029805 < \alpha \leq 1\) \([\text{LG12}]\). The number \(\omega\) equals 2 if and only if \(\alpha\) equals 1. The dual exponent \(\alpha\) will turn out to be very useful in combination with tensor surgery.

**Lower bound methods.** A flattening is a grouping of the tensor legs into two groups as to obtain a matrix. The rank of the flattening matrix is a lower bound for the rank of the tensor. For graph tensors \(T(G)\), the flattening lower bound on the exponent coincides with the size of a maximum cut of \(G\), which we denote by \(f(G)\). We thus have the following inequality of graph parameters,

\[
f(G) \leq \omega(T(G)) \leq \log_2 R(T(G)) \leq |E(G)|.
\]

This means that for bipartite graphs \(G\), one has \(\omega(T(G)) = \log_2(R(T(G))) = |E(G)|\). For odd cycles we get the flattening lower bounds \(d^{k-1} \leq R(T_d(C_k))\) and \(k - 1 \leq \omega(T_d(C_k))\). There exist more sophisticated flattenings called Young flattenings \([\text{LO11}]\), which in our language correspond to the controlled splitting of a vertex before flattening. Young flattenings were used in \([\text{BCZ16}]\) to show that the flattening lower bound on the rank of \(T_d(C_k)\) is not tight.

**Connections to other work.** Tensor rank has been studied in many fields: in algebraic geometry in the context of rth secant varieties of Segre varieties \([\text{Lan12}]\), in algebraic complexity theory as a complexity measure for matrix multiplication \([\text{BCS97}, \text{Bis13}]\), in quantum information theory as a monotone for stochastic local operations and classical communications (SLOCC) \([\text{CCD}^+10, \text{VC15}]\) and in quantum communication complexity \([\text{DKW09}, \text{BCZ16}]\) to characterize the complexity of communication problems.

In quantum information language, for any graph \(G\) the tensor \(T_d(G)\) is the (unnormalized) quantum state obtained by identifying the vertices of \(G\) with quantum systems and letting each edge of the graph correspond to a dimension-\(d\) Einstein-Podolsky-Rosen (EPR) pair shared among the vertices contained in the edge. For example, if \(G\) contains just a single edge, then \(T_d(G)\) is the EPR pair \(\sum_{i=1}^d |ii\rangle\). For any hypergraph \(H\) consisting of a single edge \(\{1, 2, \ldots, k\}\), the tensor \(T_d(H)\) is the (unnormalized) Greenberger-Horne-Zeilinger (GHZ) state \(\sum_{i=1}^d |i\rangle^\otimes k\). The tensor corresponding to a
general hypergraph is the tensor product of GHZ-states corresponding to the hyperedges.

In the context of entanglement theory, our tensor surgery can be interpreted as a method to obtain upper bounds on the rate of GHZ states needed to create an entangled quantum state under stochastic local operations and classical communication (SLOCC). The study of tensor rank of the cycle graph was initiated in [BCZ16] in the context of quantum communication complexity. There, an explicit decomposition of the ring of 5 entangled pairs was given and nontrivial asymptotic upper bounds were given for all \( k \). However, they approach the trivial upper bound for large \( k \). Some of the results presented in this paper directly lead to better upper bounds on the non-deterministic quantum communication complexity of the cyclic equality problem.

Outline of the paper. In the rest of the paper we are going to show some applications of tensor surgery. In Section 2 we show that for any odd \( k \), the tensor of the \( k \)-cycle has a nontrivial decomposition,

\[
R(T(C_k)) \leq 2^k - 1.
\]

This was previously only known for \( k \leq 5 \). Let \( \omega_k = \omega(T(C_k)) \). We prove a relationship between the exponents of odd cycles,

\[
\omega_{k+\ell-1} \leq \omega_k + \omega_{\ell},
\]

and an upper bound on the exponent of odd cycles in terms of the dual exponent of matrix multiplication \( \alpha \),

\[
\omega_k \leq k - \alpha \left( 1 + \frac{1-\alpha}{k-1+\alpha} \right) \leq k - \alpha.
\]

This means that \( \omega_k \) is bounded away from \( k \) by a constant. In Section 3 we turn to the more general version of tensor surgery, where one splits up a tensor leg into multiple tensor legs and instead of a graph inserts a hypergraph. We build two asymptotic examples that make use of the tensor corresponding to the hypergraph

which we call the dome tensor.

2. Tensor surgery on cycles

**Theorem 1.** For any odd number \( k \), the tensor rank of the tensor corresponding to the cycle graph \( C_k \) is upper bounded by \( R(T(C_k)) \leq 2^k - 1 \).
Theorem 3. The tensor square of Strassen’s decomposition is a size-49 decomposition of whose first tensor leg has local rank 1 and one summand whose first tensor vertices and insert $R(ℓ)$. Next we consider an optimal decomposition of $T(C_k)$; and again this decompositions consists of $2^{k-2} - 2$ summands whose first tensor leg has local rank 1 and one summand whose first tensor leg has local rank 2.

For the smallest interesting case $k = 5$ we can prove the lower bounds $24 \leq R(T(C_5))$ and $25 \leq R(T(C_5))$, not matching the upper bound $R(T(C_5)) \leq 31$ of Theorem 1. We mention that the decomposition of $T(C_5)$ given by the proof of Theorem 1 is different from the decomposition in [BCZ16] in the sense of De Groote [dG78], that is, the decompositions can not be transformed into each other by sandwiching and cyclic permutation. This is because the local ranks of the summands are incompatible.

In general, we can transform decompositions for $⟨n_1, n_2, n_3⟩$ to decompositions of $T_n(C_k)$. We give two nontrivial examples, based on $R(⟨4, 4, 2⟩) \leq 26$, $R(⟨4, 4, 4⟩) \leq 24$, $R(⟨4, 4, 2⟩) ≤ 24$, $R(⟨4, 4, 4⟩) ≤ 46$, see [HK71, Smi13].

Proposition 2. $R(T_4(C_5)) \leq 937 < 31^2$ and $R(T_4(C_5)) \leq 910$.

Proof. Let $φ$ be the map $v_1 ⊗ v_2 \mapsto \sum_{j,k \in [4]} (v_1 ⊗ b_j) ⊗ b_j k ⊕ (b_k ⊗ v_2)$, and let $ψ$ be the map which applies $φ$ to the first tensor leg. Then $ψ(⟨4, 4, 4⟩) = T_4(C_5)$. The tensor square of Strassen’s decomposition is a size-49 decomposition of $⟨4, 4, 4⟩$. Applying $ψ$ to its summands and using $R(⟨4, 4, 2⟩) \leq 26$ we obtain $R(T_4(C_5)) \leq 6^2 \cdot 4^2 + 12 \cdot 26 + 1 \cdot 49 = 937$.

The border rank result follows from the same reasoning combined with $R(⟨4, 4, 2⟩) \leq 24$ and $R(⟨4, 4, 4⟩) \leq 46$.

Theorem 3. Let $ω_k = ω(T(C_k))$. For $k, ℓ$ odd, $ω_k + ℓ - 1 \leq ω_k + ω_ℓ$.

Proof. Let us for a moment identify $T_n(G)$ with its defining graph $G$. The idea of the proof is to take the $k$-cycle $C_k$, split one vertex in $C_k$ into two vertices and insert $ℓ - 2$ new vertices in the graph together with the appropriate $ℓ - 1$ edges in order to create the $(k + ℓ - 1)$-cycle. In pictures, for $k = 5$ and $ℓ = 3$,

Next we consider an optimal decomposition of $T_n(C_k)$. Not only inserting $ℓ - 1$ edges comes with a cost, but also splitting the vertex. The crucial
Theorem 4. \( \omega_k = \omega_{k-2} + \omega_3 \) and thus by induction \( \omega_k \leq \frac{k-1}{2} \omega_3. \)

**Theorem 4.** For any odd \( k \geq 3 \), \( \omega(T(C_k)) \leq k - \alpha \left( 1 + \frac{1-\alpha}{k-1+\alpha} \right) \)

*Proof.* We do edge insertion on the unbalanced triangle \( (n, n, n^\alpha) \), where \( \alpha \) is the dual exponent of matrix multiplication. The crucial observation is that the total cost of splitting a vertex and inserting one vertex with \( \alpha \) respectively, at cost \( k-1 \). To get an evenly weighted \( T_n(C_k) \) we symmetrise cyclically, which gives

\[
\omega_k \leq \frac{k}{k-1+\alpha} (k-1) = k - \alpha \left( 1 + \frac{1-\alpha}{k-1+\alpha} \right). \]

Both Theorem 3 and Theorem 4 imply that if \( \omega = 2 \), then \( \omega_k = k-1 \) for all odd \( k \).

**Other upper bound methods.** In [BCZ16], Strassen’s laser method combined with a distillation result of [VC16] was used to get the nontrivial upper bound \( \omega(T(C_k)) \leq \min_q k \log_q((q+1)^k/4). \) For small \( k \), this bound gives the best results that we know. When \( k \) grows, however, this bound converges to the trivial upper bound.

Another method for obtaining upper bounds on \( R(T(G)) \) or \( \omega(T(G)) \) is, intuitively, to cover the graph \( G \) by triangles, which cost \( \omega \) each and use the distillation result of [VC16] to remove unwanted edges. For example, for \( k = 5 \), the distillation result of [VC16] says that asymptotically one \( T(C_5) \) can locally be converted into the tensor product of two copies of \( T(5) = \sum_{i \in [0,1]} b_i \otimes 5. \) Now cover the complete graph with 10 triangles, then distil 3 unwanted 5-cycles to be left with 3 correct 5-cycles. We thus obtain \( \omega(T(C_5)) \leq (10\omega - 2 \cdot 3)/3. \) One can also cover the cycle \( C_k \) by unbalanced triangles weighted as \( (1, 1, \alpha) \), which cost 2 each, and then distil a \( k \)-cycle weighted by \( \alpha \). This yields the upper bound \( \omega(T(C_k)) \leq k - \alpha \) matching Theorem 4 for large \( k \).
The following table contains the best upper bounds that we know on \( \omega_k = \omega(T(C_k)) \) for some small odd \( k \). From \( k = 11 \) onwards, Theorem 4 gives the best upper bound.

| \( k \) | \( \omega_k \) | reference |
|--------|----------------|-----------|
| 3      | 2.3728639      | [LG14]    |
| 5      | 4.6030870      | [BCZ16]   |
| 7      | 6.6511243      | [BCZ16]   |
| 9      | 8.6786899      | [BCZ16]   |
| 11     | 10.676522      | Theorem 4 |
| 13     | 12.679854      | Theorem 4 |

3. Tensor surgery on general graphs and hypergraphs

In this final section we want to illustrate tensor surgery on general graphs. The first example shows that tensor surgery on a graph might involve absorbing a virtual hyperedge. The second example is an example of general hypergraph surgery. In both examples we use the following hypergraph.

We define \( D_{k,\ell} \) to be the hypergraph on 4 vertices with multi-edges

\[ (\{1, 2, 3\}, k), (\{1, 4\}, \ell), (\{2, 4\}, \ell), (\{3, 4\}, \ell), \]

where \((e, m)\) means that the edge \( e \) has multiplicity \( m \); that is, \( D_{k,\ell} \) is the following hypergraph

![Dome Hypergraph Diagram]

where \( k \) and \( \ell \) denote multiplicities. We call this graph the dome and \( T(D_{k,\ell}) \) the dome tensor with multiplicities \( k, \ell \).

**Lemma 5.** We have \( 3 \leq \omega(T(D_{1,1})) \leq 3\omega/2 \) and \( \omega(T(D_{1,4})) = 12 \).

**Proof.** For the upper bound in the first statement, cover each triangle \((1, 2, 4), (2, 3, 4), (1, 3, 4)\) with a triangle \( T_n(C_3) \) at cost \( n^\omega \). Then use distillation to asymptotically distil two copies of \( T_n([3]) \) on the triangle \((1, 2, 3)\). We end up with \( T_{n^2}(D_{1,1}) \). For the lower bound in the first statement take the cut through the 2-edges.

We now prove the second statement. We will prove the equivalent statement \( \omega(T(H) \otimes 2) = 24 \). Recall that the asymptotic rank \( \omega(\langle n, n^4, n^4 \rangle) \) equals 8, since the dual exponent of matrix multiplication \( a \) is at least 0.3029805 > 1/4 (see the introduction or the proof of Theorem 4). We cover each of the triangles \((1, 2, 4), (2, 3, 4), (1, 3, 4)\) by a rectangular matrix multiplication tensor
\((n, n^4, n^4)\) at cost 8. This makes sure we have the edges of multiplicity 8 in place, and introduces a triangle of multiplicity 1 on the vertices 1, 2, 3. We now asymptotically distil the tensor \(T_{n^4}(\{1, 2, 3\})\) from the triangle at the vertices 1, 2, 3. The total cost is at most \(3 \cdot 8\). On the other hand, the cut of the graph that cuts the three edges of multiplicity 8 yields the lower bound \(\omega(T(H)) \geq 3 \cdot 8\).

For the first example, let \(G\) be the multigraph with edges \((\{1, 2\}, 4), (\{2, 3\}, 3), (\{2, 4\}, 1), (\{1, 5\}, 1), (\{5, 6\}, 8), (\{4, 6\}, 8), (\{3, 6\}, 8)\), that is, \(G\) is the following hypergraph where the numbers denote edge-multiplicity:

Grouping the vertices 1, 3, 4, 5 together and grouping the vertices 2, 6 together shows that the size of a max-cut is at least 32. Therefore, \(\omega(T(G)) \geq 32\). On the other hand, one can cover the 5-cycle 1,2,4,6,5 at cost \(\omega_5\) and the remaining edges at cost 1 each, which implies that \(\omega(T(G)) \leq \omega_5 + 28\). Therefore, by Theorem 3 if \(\omega = 2\) then \(\omega(T(G)) = 32\). We need the following lemma.

**Proposition 6.** \(\omega(T(G)) = 32\).

**Proof.** It remains to show the upper bound. We start off with the rectangular matrix multiplication tensor \(\langle n, n^4, n^4 \rangle\) at cost 8, and, viewing it as a triangle graph

split up one of the low-dimension vertices into three vertices such that the resulting tensor corresponds to the following graph:

We then insert vertex 6 and the incident edges:
Since the rank of a tensor in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is at most $n^2$, the linear map which splits up the vertex and inserts the new vertex together with the appropriate edges with multiplicity 8 has cost at most the cost of creating the tensor corresponding to the hypergraph $H$ of Proposition 6. We thus obtain the upper bound $\omega(T(G)) \leq 8 + 24 = 32$.

For the second example, define $H$ as the hypergraph with edges

\{1, 2, 3\}, \{3, 4, 5\}, \{1, 6\}, \{2, 6\}, \{6, 7\}, \{4, 7\}, \{5, 7\},

which looks like

\[ \text{Proposition 7. } \text{We have } 6 \leq \omega(T(G)) \leq 6\omega/2. \]

\textbf{Proof.} The lower bound follows from grouping the vertices 3, 6, 7 together and grouping the remaining vertices together. For the upper bound, we start off with the dome $D_{1,1}$

We split one of the vertices in the hyperedge, as follows

and insert the remaining vertices and edges as to obtain the goal tensor.

We see that the combined cost of splitting the vertex and inserting the vertices and edges is at most $\omega(T(D_{1,1}))$ which is at most $3\omega/2$ (Lemma 5). We conclude that $\omega(T(G)) \leq 2\omega(T(D_{1,1})) \leq 6\omega/2$.

We believe that one can not obtain the above result by simply covering the graph with cycles and distilling. Of course, by replacing $D_{1,1}$ by $D_{1,4}$ one can obtain an exact result like in Proposition 6.
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