THE STRONG MAXIMAL RANK CONJECTURE AND HIGHER RANK BRILL–NOETHER THEORY

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ABSTRACT. By viewing a rank two vector bundle as an extension of line bundles we may re-interpret cohomological conditions on the vector bundles (e.g., number of sections) as rank conditions on multiplication maps of sections of line bundles. In this paper, we apply this philosophy to relate the Brill–Noether theory of rank two vector bundles with canonical determinant to the Strong Maximal Rank Conjecture for quadrics. By verifying that certain “special maximal-rank loci” are nonempty, we are able to produce candidates for rank two linear series of large dimension. We then show that the underlying vector bundles are stable, in order to conclude the existence portion of certain instances of a well-known conjecture due to Bertram, Feinberg and independently Mukai.

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1. Introduction

Ever since the inception of moduli of (complex) curves as an area of investigation in its own right, linear series have served as crucial tools for probing the intrinsic geometry of the moduli space via the extrinsic properties of a (variable) curve’s embeddings in projective spaces. Classically, a linear series is defined by a vector subspace of holomorphic sections of a line bundle $L$; in that case, the celebrated Brill–Noether theorem of Griffiths and Harris gives a complete description of the space of series on a curve that is general in moduli. There are many interesting variations on the basic Brill–Noether paradigm, however; one of these involves replacing $L$ by a vector bundle of some higher rank $e\geq 2$, and produces a theory of rank-$e$ linear series.

The case $e=2$ is already substantially more complicated than the classical situation. First of all, rank-two bundles are naturally stratified according to their determinants. A celebrated conjecture of Bertram, Feinberg and Mukai (referred to hereafter as the BFM conjecture) predicts that the moduli space of rank-two bundles with canonical determinant and at least $k$ independent holomorphic sections is of dimension $\rho_{g,k} := 3g - 3 - \binom{k+1}{2}$. To date, many instances of the BFM conjecture have been verified, while very little is known about spaces of rank-two linear series with non-canonical determinants.

In his Ph.D thesis, the second author succeeded in verifying new instances of the BFM conjecture. In order to do so, he appealed to a theory of higher-rank limit linear series on reducible curves of compact type. This theory, introduced initially by Teixidor i Bigas, and developed further by Osserman–Teixidor i Bigas, is itself a generalization of theories of (traditional) limit linear series on curves of compact type due to Eisenbud–Harris and Osserman, respectively.

In this paper, we study the BFM conjecture from a somewhat different point of view. In a nutshell, we view the existence of higher-rank (stable) linear series as dependent on two distinct phenomena, namely a) multiplication maps along (not-necessarily complete) linear series being of prescribed rank; and b) moduli spaces of inclusions of linear series being reasonably well-behaved. The first phenomenon produces bundles that are extensions of line bundles, while the second phenomenon certifies that the bundles in question are stable.

Multiplication maps for general linear series on general curves are the focus of the Maximal Rank Conjecture, or MRC, now a theorem thanks to the work of Eric Larson. A strong form of MRC, or SMRC, addresses the dimensionality of spaces of special linear series whose multiplication maps
fail to be of maximal rank. These loci are already interesting from the perspective of the BFM conjecture in the first nontrivial case, that of quadratic multiplication maps. The upshot is that, modulo stability considerations, we may try to verify new instances of the BFM conjecture by certifying the positivity of (classes of) non-maximal rank loci for quadratic multiplication maps of (traditional) linear series on a general curve. The culmination of our efforts is Corollary 7.2, which establishes the non-emptiness of BFM loci in several new cases. In doing so, we leverage Proposition 8.26, which establishes the positivity of SMRC loci, modulo a couple of explicit exceptions, whenever a natural indexing parameter $N$ is small.

Roadmap. The material following this introduction is structured as follows. In section 2, we list some notations and conventions that we will use systematically throughout the entire paper. In section 3.1, we review intersection theory on the Grassmann bundle $\text{Gr}(k, E)$. An important fact that we use later is that the Gysin (pushforward) morphism is induced by the Lagrange–Sylvester symmetrizer on the Chern roots of the bundle $E$; this is recalled in Lemma 3.3.

Our exploration of the strong maximal rank conjecture starts in earnest in section 4, where we explicitly describe quadratic SMRC loci $M^r_d(C)$ as degeneracy loci for maps of vector bundles over the Picard variety of a general curve $C$. Subsection 4.3 introduces a dichotomy between the injective and surjective ranges, depending upon how relatively large the (dimensions of the) source and target of quadratic multiplication are. Insofar as BFM loci are concerned, injective cases correspond to cases where the genus is small, while the surjective range describes the “generic” case. Proposition 4.8 establishes that SMRC loci are always non-empty, and in fact contain excessively large components, in a particular regime of parameters $(g, r, d)$. We refer to these as trivial instances as they arise from the failure of the associated linear series to be very ample.

In section 5, we write down explicit formulae for the Gysin pushforwards to the Picard variety of the intersections of quadratic SMRC loci with complementary powers of the theta divisor. The basic shape of these formulae depends on whether the associated triple $(g, r, d)$ belongs to the injective or surjective range, and they involve ancillary functions $d_I$ and $\psi_I$ introduced by Laksov, Lascoux, and Thorup in their determination of the characteristic classes of the symmetric square of a bundle.

Section 6, which deals with the BFM conjecture and its relationship with the SMRC, is the heart of this paper. In order to realize stable rank-two bundles with canonical determinant and prescribed numbers of sections as extensions $e$ of line bundles $L$ by their Serre duals, the crucial fact is that the map that sends an extension to its coboundary is dual to the quadratic multiplication map on sections of $L$. An extension $e$ is trivial if and only if the quadratic multiplication map is surjective; so stable rank-two bundles arise from (extensions of) line bundles that belong to quadratic SMRC loci.

A nontrivial extension does not necessarily give rise to a stable bundle, however, so accordingly we develop additional tools for understanding when this happens. Proposition 6.2 gives a necessary geometric criterion: certain secant divisors to the image of $C$ under $|L|$ are obstructions to stability. On the other hand, Theorem 6.3, due to Mukai–Sakai, gives (see Corollary 6.4) an upper bound on the degree of the minimal quotient line bundle of a rank-two vector bundle with
canonical determinant. It allows us to identify a critical range of possible degrees for line bundles. Proposition 6.7 establishes that a nontrivial extension of $L$ by its Serre dual gives rise to a stable bundle when its degree is at least $g$ and is minimal among degrees of line bundles $L$ that have sufficiently many sections; its corollary 6.8 establishes new cases of the (existence portion of the) BFM conjecture. In subsection 6.2 we give a solution to the “BFM existence problem” in the injective range via extensions, which gives a simpler alternative to earlier work of Bertram and Feinberg.

In Section 7, we establish a regime of parameters $(g, r, d)$ in which the results of Section 6 yield the existence portion of the BFM conjecture; see Corollary 7.2. We also discuss a significant case at the numerological border of the surjective range, that of $g = 13, k = 8$, in which our arguments are at present inconclusive. Our Claim 1 establishes that the BFM conjecture holds in the $g = 13, k = 8$ case provided that on a general curve i) the multiplication map $\mu_2$ associated with a complete $g^5_{16}$ is always surjective; and ii) there exist very ample complete $g^6_{18}$ for which $\mu_2$ fails to be surjective.

Finally, Section 8 is devoted to the combinatorics of SMRC class formulae. Our point of departure is a formula due to Laksov, Lascoux, and Thorup (see Theorem 8.2) that realizes the Chern polynomial of the symmetric square of a vector bundle $E$ as a linear combination of the Segre classes of $E$, whose coefficients are multiples of certain minors $d_I$ of an infinite matrix. The functions $d_I$, in turn, are multiples of the shifted Schur functions of Okounkov–Olshanski, evaluated along staircase partitions. Proposition 8.8 establishes that these functions vary polynomially (of predictable degree) in the size parameter $r$ of the staircase; in particular, they can be interpolated explicitly. In Subsection 8.4, we relate the large-$r$ asymptotics of our class formulae to the Plancherel distribution on partitions (see Corollary 8.16 and Lemma 8.20) and we realize (the degrees of) SMRC classes as inner products in the ring $\Lambda_\mathbb{Q}$ of symmetric functions with rational coefficients. Proposition 8.25 establishes that for every fixed value of $N$, our SMRC class formulae are positive for all $r$ greater than an explicit cutoff function in $N$. Finally, in Proposition 8.26, we show that SMRC classes are unconditionally positive for all $N \leq 7$, outside of a small number of (explicitly given) exceptions when $N = 1$ or $N = 2$.

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2. Notations and conventions

Notation 2.1. Hereafter, $C$ will denote an irreducible smooth projective curve over an algebraically closed field $K$.

Notation 2.2. $G^r_d(C)$ will denote the moduli scheme of linear series $g^r_d$ over $C$.

Notation 2.3. $\mathcal{L}$ will denote a Poincaré line bundle over $C \times \text{Pic}^d(C)$.

Fix an effective, reduced divisor $D$ of $C$ of degree $\geq 2g - 1 - d$, and let $D' = D \times \text{Pic}^d(C)$. The space $G^r_d(C)$ is naturally a closed subscheme of $\text{Gr}(r + 1, \pi_2^*\mathcal{L}(D'))$. Indeed, it is the zero locus
of the morphism
\[ \mathcal{U} \hookrightarrow p^* \pi_2_* L(D') \to p^* \pi_2_* (L(D')|_{D'}) \]
where \( \mathcal{U} \) is the tautological subbundle, \( p : \text{Gr}(r+1, \pi_2_* L(D')) \to \text{Pic}^d(C) \) is the structure morphism and \( \pi_2 : C \to C \times \text{Pic}^d(C) \) is the second projection.

**Notation 2.4.** Let \( i : G^r_d(C) \to \text{Gr}(r+1, \pi_2_* L(D')) \) denote the corresponding closed immersion.

**Notation 2.5.** Let \( p' : G^r_d(C) \to \text{Pic}^d(C) \) denote the canonical inclusion, and let \( \pi'_2 : C \times G^r_d(C) \to G^r_d(C) \) denote the projection induced by \( \pi_2 \) above.

**Notation 2.6.** Let \( M \) denote the pull-back of \( L \) to \( C \times G^r_d(C) \) along \( \text{Id}_C \times p' \).

**Notation 2.7.** Let \( V \) denote the universal family over \( G^r_d(C) \), i.e. the pull-back of \( U \) to \( G^r_d(C) \) along \( i \).

For the sake of convenience, we summarize the maps and spaces mentioned above in one commutative diagram:

\[
\begin{array}{ccc}
C \times G^r_d(C) & \xrightarrow{\text{Id}_C \times p'} & C \times \text{Pic}^d(C) \\
\downarrow \pi'_2 & & \downarrow \pi_2 \\
G^r_d(C) & \xrightarrow{p'} & \text{Pic}^d(C) \\
\downarrow i & & \downarrow p \\
\text{Gr}(r+1, \pi_2_* L(D')) & & \\
\end{array}
\]

**Notation 2.8.** In numerical examples, \( g \) will always denote the genus of the underlying curve, \( r \) the (projective) dimension of a rank-one linear series, \( d \) the degree of a line bundle, \( \chi \) the Euler characteristic of a line bundle, \( k \) the dimension of a rank 2 linear series, and
\[
\rho = \rho(g,r,d) := g - (r + 1)(r + g - d).
\]

**Notation 2.9.** Given non-negative integers \( g,r,d \), we let
\[
D(g,r,d) := \rho(g,r,d) - 1 - \left( \binom{r+2}{2} - (1 - g + 2d) \right).
\]
When \( \binom{r+2}{2} \geq 1 - g + 2d \), we also set
\[
N(g,r,d) := \binom{r+2}{2} - 2d + g.
\]

**Notation 2.10.** Fix a reference point \( P_0 \) on \( C \). Let \( w_j \) denote the corresponding class of \( W_{g-j} \) for \( j = 1, \ldots, g - 1 \), given by the image of the map \( w_j : \text{Sym}^j C \to J(C) : D \mapsto D - dP_0 \). It is a codimension-\( j \) class.

In order to describe the Chern classes of the Poincaré line bundle, we single out certain cohomology classes of \( C \times \text{Pic}^d(C) \).

**Notation 2.11.** Let \( \theta \) denote the class of the pull-back of the theta divisor, and let \( \eta \) denote the pullback of the class of a point on \( C \).
Notation 2.12 ([OO97],(5.2)). Let $\mu$ be any Young diagram. The $\mu$-th generalized raising factorial of $n$ is defined by
\[
(n \uparrow \mu) := \prod_j [n(n+1) \ldots (n+\mu_j - j)].
\]

Notation 2.13 ([OO97],(11.1)). Similarly, the $\mu$-th generalized falling factorial of $n$ is defined by
\[
(n \downarrow \mu) := \prod_j [n(n-1) \ldots (n-\mu_j + j)].
\]

More generally, given any skew diagram $\mu/\nu$, we set $(n \downarrow \mu/\nu) := (n \downarrow \mu) \cdot (n \downarrow \nu)^{-1}$.

Conventions for Schur functions. Schur functions in $n$ variables are symmetric functions labeled by partitions of length at most $n$. Two equivalent conventions for Schur functions appear in the literature and are convenient for different purposes. We introduce both of them here and comment on their equivalence.

Definition 2.14. A partition of length $n$ is a finite sequence $\mu = (\mu_1, \ldots, \mu_n)$ of non-negative integers arranged in non-increasing order. The conjugate of a partition is a partition whose corresponding Young diagram is obtained from the original diagram by interchanging rows and columns.

Definition 2.15. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of length $n$. The Schur function $s_\lambda(x_1, \ldots, x_n)$ is the symmetric polynomial
\[
s_\lambda(x_1, \ldots, x_n) = \det \frac{\prod_{1 \leq j < k \leq n} (x_j - x_k)}{x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}}.
\]

Definition 2.16. Let $I = (i_1 < \ldots < i_n)$ be a strictly increasing sequence of non-negative integers. Now set
\[
s_I(x_1, \ldots, x_n) := \det(s_{i_{\ell} - k + 1}, k, \ell \in [n])
\]
where $s_j$ is the $j$-th coefficient in the formal expansion
\[
\prod_{i=1}^n \frac{1}{1 - x_i t} = \sum_{j=0}^{\infty} s_j(x_1, \ldots, x_n) t^j.
\]

Definition 2.17. For $I = (i_1 < \ldots < i_n)$, we call $n$ the length of $I$ and denote $\ell(I)$. We also write $|I| = \sum_{k=1}^n i_k$.

Inasmuch as there is a bijection between strictly-increasing sequences of positive integers and partitions, definitions 2.15 and 2.16 need to be reconciled. The Jacobi–Trudi lemma does the trick.

Lemma 2.18. [Lemma A.9.3 in Ful98] Let $\lambda$ be the partition $i_n - n + 1, i_{n-1} - n + 2, \ldots, i_1$. We have
\[
s_\lambda(x_1, \ldots, x_n) = \det(s_{\lambda_k + \ell - k}; 1 \leq k, \ell \leq n) = s_I(x_1, \ldots, x_n).
\]

Remark 2.19. Note $\deg(s_I(x_1, \ldots, x_n)) = \sum_{k=1}^n i_k - \binom{n}{2}$. In particular, $s_{0,1,\ldots,n-1}(x_1, \ldots, x_n) = 1$. 

Notation 2.20. For any given partition $\lambda$ of length $n$, and any given integer $m \geq n$, there is a unique strictly-increasing $m$-term sequence $(i_1, \ldots, i_m)$ defined by
\[ i_k = \lambda_{m-k+1} + k - 1, \quad \text{for } k = 1, \ldots, m \]
in which we set $\lambda_k = 0$ whenever $k > n$. We denote this sequence by $I_m(\lambda)$.

Notation 2.21. Conversely, given a strictly-increasing sequence $I = (i_1, \ldots, i_n)$ of non-negative integers, let
\[ \lambda(I) = (i_n - n + 1, \ldots, i_{n-k+1} - n + k, \ldots, i_1) \]
denote the corresponding partition of length at most $n$.

Notation 2.22. Given a rank-$n$ vector bundle $E$ with Chern roots $a_1, \ldots, a_n$, set $s_I(E) := s_I(a_1, \ldots, a_n)$.

Definition 2.23. More generally, given any finite not-necessarily-increasing sequence of non-negative integers $I = (i_1, \ldots, i_n)$, we set
\[ s_I(x_1, \ldots, x_n) := \det(x_1^{i_k-k+1})_{1 \leq k, \ell \leq n} \prod_{1 \leq i < j \leq n} (x_i - x_j). \]
The fact that $s_I(x_1, \ldots, x_n)$ agrees with Definition 2.16 whenever $I$ is a strictly increasing sequence follows from Lemma 2.18.

Notation 2.24. Fix a positive integer $n$. Let $S = \{s_0, s_1, s_2, \ldots\}$ be a set of elements in some commutative ring and let $\mu$ denote a partition of length at most $n$. We set
\[ \Delta_\mu(S) := \det([s_{\mu_i+j-i}]_{1 \leq i, j \leq n}). \]

3. Review of intersection theory on Grassmann bundles

In this section, we first review some well-known facts about intersection theory on Grassmann bundles, which we will apply freely later.

3.1. The Chow ring of a Grassmann bundle and the Gysin morphism. Let $E$ be a locally-free sheaf of rank $n$ on a smooth projective variety $X$, and let $q : \text{Gr}(k,E) \to X$ denote the natural morphism. The following structure theorem for the Chow ring of a Grassmann bundle is well-known.

Theorem 3.1 ([Ful98], 14.6.6). Let $\mathcal{U}$ and $\mathcal{Q}$ denote the tautological subbundle and tautological quotient bundle of $q^*E$, respectively. The Chow ring $A(\text{Gr}(k,E))$ is an algebra over $A(X)$ generated by the tautological classes
\[ c_1(\mathcal{U}), \ldots, c_k(\mathcal{U}); \quad \text{and } c_1(\mathcal{Q}), \ldots, c_{n-k}(\mathcal{Q}) \]
modulo the relations $\sum_{i=0}^{k} c_i(\mathcal{U}) \cdot c_{k-i}(\mathcal{Q}) = c_k(E)$.

Let $a_1, \ldots, a_k$ and $a_{k+1}, \ldots, a_n$ denote the Chern roots of $\mathcal{U}$ and $\mathcal{Q}$, respectively. The Chern classes $c_i(\mathcal{U})$ and $c_j(\mathcal{Q})$ are elementary symmetric functions $c_i(a_1, \ldots, a_k)$ and $c_j(a_{k+1}, \ldots, a_n)$. Consequently, if we think of $A(X)$ as a subring of $A(\text{Gr}(k,E))$, $c_k(E)$ is a polynomial in $a_1, \ldots, a_n$ which is symmetric in $a_1, \ldots, a_k$ and in $a_{k+1}, \ldots, a_n$ separately. The upshot is that we may express
any intersection product involving \(c_i(\mathcal{Z})\), \(c_j(\mathcal{Q})\) and \(c_k(E)\) as a product of symmetric functions in the Chern roots of \(\mathcal{Z}\) and \(\mathcal{Q}\). We will put this observation to work in writing down the Gysin map \(q_* : A(\text{Gr}(k, E)) \to A(X)\). But first we recall another well-known fact, which we will also use.

**Theorem 3.2** ([Gro58], Theorem 3.1). Let \(\text{Fl}(E)\) denote the complete flag bundle associated to a vector bundle \(E\) of rank \(n\) and \(a_1, \ldots, a_n\) be the Chern roots of \(E\). The Chow ring \(A(\text{Fl}(E))\) is an \(A(X)\)-algebra, generated by the elements \(a_1, \ldots, a_n\) modulo the relations

\[
e_i(a_1, \ldots, a_n) = c_i(E), i = 1, 2, \ldots, n.
\]

**Lemma 3.3** ([Pra88], Lemma 2.5). The Gysin morphism \(q_* : A(\text{Gr}(k, E)) \to A(X)\) is induced by the map

\[
p : \mathbb{Z}[a_1, \ldots, a_n]^{S_k \times S_{n-k}} \to \mathbb{Z}[a_1, \ldots, a_n]^{S_n} : f(a_1, \ldots, a_n) \mapsto \sum_{\sigma \in S_n / S_k \times S_{n-k}} \sigma \left( \frac{f(a_1, \ldots, a_n)}{\prod_{1 \leq i \leq j \leq n} (a_j - a_i)} \right)
\]

where \(\sigma\) acts on a polynomial by permuting the indices of the variables.

**Remark 3.4.** The map \(p\) is known as the Lagrange-Sylvester symmetrizer. See also [Tu17].

**3.2. Combinatorial properties of the Lagrange-Sylvester symmetrizer.** We next review some well-known properties of the Lagrange-Sylvester symmetrizer that we will use. The first is a combinatorial formula that describes the action of \(p\) on Schur functions. For a reference, see [Las88].

**Lemma 3.5.** Given two sequences of non-negative integers \(I = (i_1 < \ldots < i_k)\) and \(J = (j_{k+1} < \ldots < j_n)\), we have

\[
p(s_I(a_1, \ldots, a_k)s_J(a_{k+1}, \ldots, a_n)) = (-1)^{k(n-k)} s_{I+J}(a_1, \ldots, a_n)
\]

where \(J, I\) denotes the concatenation of \(J\) and \(I\).

**Remark 3.6.** By definition, \(p\) is clearly additive. Moreover, because

\[
\mathbb{Z}[x_1, \ldots, x_n]^{S_k \times S_{n-k}} \cong \mathbb{Z}[x_1, \ldots, x_k]^{S_k} \otimes_{\mathbb{Z}} \mathbb{Z}[x_{k+1}, \ldots, x_n]^{S_{n-k}}
\]

and the Schur polynomials form a \(\mathbb{Z}\)-basis for the ring of symmetric functions [Mac98, I.3.2]), the formula in Lemma 3.5 completely determines \(p\).

**Corollary 3.7.** The map \(p\) satisfies the following properties:

1. \(p(fg) = f \cdot p(g)\) for every \(S_n\)-invariant polynomial \(f\).
2. \(p(s_I(a_1, \ldots, a_k)) = s_{0, \ldots, n-k-1,I}(a_1, \ldots, a_n)\) for every \(n\)-tuple \(I\); hence, \(p(s_I(a_1, \ldots, a_k)) = 0\) whenever \(i_1 < n - k\).
3. \(p(s_I(a_1, \ldots, a_k))\) is either a Schur polynomial in \(a_1, \ldots, a_n\) or zero for every \(n\)-tuple \(I\).

**Proof.** Whenever \(f\) is symmetric, we have \(\sigma(fg) = f \sigma(g)\) and thus \(p(fg) = f \cdot p(g)\), which is claim (1). On the other hand, clearly \(s_{0, \ldots, n-k-1}(a_{k+1}, \ldots, a_n) = 1\). The first part of claim (2) follows now from Lemma 3.5. For the second part of claim (2), note that whenever \(i_1 < n - k\), we have \(s_{0, \ldots, n-k-1}(a_1, \ldots, a_n) = \frac{F(a_1, \ldots, a_n)}{F(a_1, \ldots, a_n)}\), where \(F\) is the determinant of a matrix with two identical rows and hence must be zero.
From (2) we know that either \( p(s_1(a_1, \ldots, a_k)) = 0 \) or 0, 1, \ldots, \( n-k-1, I \) is a strictly increasing sequence of non-negative integers. In the latter case, Lemma 2.18 establishes that \( p(s_1(a_1, \ldots, a_k)) \) is a Schur polynomial, which is claim (3).

For our main application, \( X \) will be the Picard variety \( \text{Pic}^d(C) \) of a smooth curve \( C \) of genus \( g \) and \( E \) will be the pushforward \( \pi_2_*(\mathcal{L}(Z')) \) of the twist of a Poincaré line bundle \( \mathcal{L} \) over \( C \times \text{Pic}^d(C) \), by the pullback of an effective divisor \( Z' \) on \( C \) of degree at least \( \max\{2g-1-d,0\} \).

Since ultimately we are interested in whether certain cohomology classes over \( \text{Pic}^d(C) \) are non-zero, we work up to numerical equivalence. We will apply the following well-known result of Mattuck.

**Lemma 3.8** ([Mat65], Example 14.4.5 of [Ful98]). Suppose \( d > 2g-2 \). In this case, the Segre class of the pushforward of a Poincaré line bundle is given by \( s_k(\pi_2_*\mathcal{L}) = w_k \) (see Notation 2.10). Moreover, \( k! w_k \) is numerically equivalent to \( \theta^k \), where \( \theta = w_1 \) is the theta divisor class.

It follows that, up to numerical equivalence, we have \( c_k(\pi_2_*\mathcal{L}(Z')) = \frac{(-1)^k \theta^k}{k!} \), and the relations in \( A(\text{Gr}(r+1, \pi_2_*\mathcal{L}(Z'))) \) are concisely expressed by the equation

\[
\sum_{i=0}^{k} c_i(\mathcal{L})c_{k-i}(\mathcal{Q}) = \frac{(-1)^k \theta^k}{k!}.
\]

Moreover, it does no harm to assume that \( \deg(D) = 2g-1-d \) so that \( E = \pi_2_*\mathcal{L}(D') \) is a rank-\( g \) vector bundle over \( \text{Pic}^d(C) \). Consequently, we have \( \theta = a_1 + \ldots + a_g \) and \( \theta^{g+1} = 0 \). Representing elements in \( A(\text{Gr}(k, E)) \) by elements in \( (\mathbb{Z}[a_1, \ldots, a_k]^{S_k} \otimes \mathbb{Z}[a_{k+1}, \ldots, a_g]^{S_{g-k}})[\theta] \), we may think of the Gysin map \( q \) as a map

\[
\frac{\langle e_j(a_1, \ldots, a_k) \rangle - \langle \frac{-\theta}{j} \rangle_{g+1} }{\langle \theta \rangle_{g+1}} \rightarrow Z[\theta]/\langle \theta^{g+1} \rangle, \text{ via } \left[ \sum_{j=0}^{g} f_j(a_1, \ldots, a_g) \theta^j \right] \mapsto \left[ \sum_{j=0}^{g} p(f_j) \theta^j \right].
\]

Here \( e_j(a_1, \ldots, a_g) \) is the \( j \)-th elementary symmetric function in \( a_1, \ldots, a_g, f_0, \ldots, f_g \) are arbitrary elements in \( \mathbb{Z}[a_1, \ldots, a_k]^{S_k} \otimes \mathbb{Z}[a_{k+1}, \ldots, a_g]^{S_{g-k}} \), and \([\cdot]\) denotes an equivalence class.

**Lemma 3.9.** For any strictly increasing sequence \( I = (i_1 < \ldots < i_k) \) with \( i_1 \geq g-k \), we have

\[
q([s_I(a_1, \ldots, a_k)]) = \frac{(-1)^{|I|-(\frac{k}{2})}}{\prod_{1 \leq j < \ell \leq k} (i_\ell - i_j)} \cdot \theta^{\frac{|I|}{2} - k(g-k)}.
\]

**Proof.** Applying Lemma 3.5, we see that

\[
q([s_I(a_1, \ldots, a_k)]) = [p(s_I(a_1, \ldots, a_k))] = (-1)^{(g-k)}s_{0,\ldots,g-k-1,I}(a_1, \ldots, a_g).
\]
Substituting

\[ s_0, \ldots, s_{g-k-1}, f(a_1, \ldots, a_g) \]

\[
\begin{bmatrix}
  s_0(-E) & s_1(-E) & \cdots & s_{g-k-1}(-E) & s_i(-E) & \cdots & s_k(-E) \\
  0 & s_0(-E) & \cdots & s_{g-k-2}(-E) & s_{i-1}(-E) & \cdots & s_{k-1}(-E) \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & s_0(-E) & s_i(-E) & \cdots & s_k(-E) \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & s_i(-E) & \cdots & s_k(-E) \\
\end{bmatrix} = \det
\begin{bmatrix}
  s_{i-1}(-E) & \cdots & s_{i-k+1}(-E) & s_{g+k}(-E) \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & s_{g+k}(-E) \\
\end{bmatrix}.
\]

Substituting \( s_k(-E) = (\theta^k)_k \), we get

\[
q([s_1(a_1, \ldots, a_k)]) = \det \begin{bmatrix}
\frac{1}{(t_1-g+k)!} & \cdots & \frac{1}{(t_k-g+k)!} \\
\vdots & \ddots & \vdots \\
\frac{1}{(t_1-g+1)!} & \cdots & \frac{1}{(t_k-g+1)!}
\end{bmatrix} (-\theta^1)^{t_1} \cdots (-\theta^k)^{t_k (g-k)}
\]

It is then well-known that the determinant

\[
\det \begin{bmatrix}
\frac{1}{(t_1-g+k)!} & \cdots & \frac{1}{(t_k-g+k)!} \\
\vdots & \ddots & \vdots \\
\frac{1}{(t_1-g+1)!} & \cdots & \frac{1}{(t_k-g+1)!}
\end{bmatrix}
\]

is a multiple of the Vandermonde determinant and can be computed as \( \prod_{1 \leq j < \ell \leq k} (t_j-t_{\ell})/(t_j-t_{\ell}+t_{\ell}) \). (For a similar calculation, see [ACGH85, Ch. 7].) The claim follows. \( \square \)

4. The Strong Maximal Rank Conjecture for Quadrics

4.1. The statement of the conjecture. The focus of this section is the Strong Maximal Rank Conjecture for quadrics, which we state as follows:

**Conjecture 1** (Strong Maximal Rank Conjecture). Fix \( g, r, d \geq 1 \) such that \( g - d + r \geq 0 \) and \( \rho(g, r, d) = g - (r + 1)(g - d + r) \geq 0 \). Let \( C \) denote a (Brill–Noether and Petri) general curve of genus \( g \).

1. Suppose \( D = \rho - 1 - \left( \binom{r+2}{2} - (2d+1-g) \right) \geq 0 \). The determinantal locus

\[
\mathcal{M}_d'(C) := \{(L, V) \in G_d^r(C)| v_2 : \text{Sym}^2 V \rightarrow H^0(L^\otimes 2) \text{ does not have maximal rank}\}
\]

is non-empty and every irreducible component is at least \( D \)-dimensional.

2. When \( D < 0 \), for all \( g^r_d \) in \( G_d^r(C) \), the multiplication map \( v_2 \) has maximal rank.

Note that our formulation of the conjecture differs from the original version stated by Aprodu and Farkas in [AF11], where they impose the restriction \( \rho < r - 2 \). Aprodu and Farkas conjectured
that with this extra assumption $\mathcal{M}_{d}^r(C)$ should be exactly $D$-dimensional. However, we are mainly concerned with the non-emptiness of $\mathcal{M}_{d}^r(C)$, and for our main application to rank two Brill-Noether theory it is necessary to remove the restriction $\rho < r - 2$. However, we do not expect that in this generality the dimension of $\mathcal{M}_{d}^r(C)$ is exactly $D$; see Example 4.10.

Hereafter, we mainly focus on part (1) of the conjecture. First of all, we recall how one may realize $\mathcal{M}_{d}^r(C)$ as the degeneracy locus of a vector bundle morphism $\phi : E \to F$ over $G^r_d(C)$. To this end, fix an effective divisor $Z$ of degree $\max\{2g - 1 - d, 0\}$ on $C$ and let $Z'$ and $Z''$ denote its pullbacks to $C \times \text{Pic}^d(C)$ and $C \times G^r_d(C)$, respectively. Recall also that $G^r_d(C)$ is the closed subscheme of $\text{Gr}(r + 1, \pi_{2*}(\mathcal{L}'(Z'))) = 0$, for every $s \in G^r_d(C)$; see Lemma 4.1 below. Thus $h^0(\mathcal{M}_s^{\otimes 2}) = 2d - g + 1$, and it follows from Grauert’s theorem that $\pi_{2*}(\mathcal{M}_s^{\otimes 2})$ is locally free of rank $2d - g + 1$ over $G^r_d(C)$ and $R^1\pi_{2*}\mathcal{M}^{\otimes 2} = 0$. From the long exact sequence in cohomology, it follows that the sequence

$$0 \to \pi_{2*}(\mathcal{M}^{\otimes 2}) \to \pi_{2*}(\mathcal{M}^{\otimes 2}(2Z'')) \to \pi_{2*}(\mathcal{M}^{\otimes 2}(2Z''))_{|2Z''} \to 0$$

is exact.

We now describe the morphism $\phi : E \to F$.

Let $i$ be the pull-back of $\mathcal{U} \to p^*\pi_{2*}\mathcal{L}(Z')$ to $G^r_d(C)$. Composing $i$ with the natural morphism

$$p'^*\pi_{2*}\mathcal{L}(Z') \to \pi_{2*}(\text{Id}_C \times p')^*\mathcal{L}(Z') = \pi_{2*}'(\mathcal{M}(Z''')),$$

we get a morphism $\mathcal{V} \to \pi_{2*}'(\mathcal{M}(Z'''))$, and hence $\mathcal{V}^{\otimes 2} \to (\pi_{2*}'(\mathcal{M}(Z''')))^{\otimes 2}$. Composing the latter with the natural morphism $(\pi_{2*}'(\mathcal{M}(Z''')))^{\otimes 2} \to \pi_{2*}'(\mathcal{M}^{\otimes 2}(2Z''))$ yields $\mathcal{V}^{\otimes 2} \to \pi_{2*}'(\mathcal{M}^{\otimes 2}(2Z'))$.

By the definition of $G^r_d(C)$, $\mathcal{V} \to p'^*\pi_{2*}(\mathcal{L}(Z')) \to p'^*\pi_{2*}(\mathcal{L}(Z'))_{|Z'}$ is the zero map and thus $\mathcal{V}^{\otimes 2} \to \pi_{2*}'(\mathcal{M}^{\otimes 2}(2Z'')) \to \pi_{2*}'(\mathcal{M}^{\otimes 2}(2Z''))_{|2Z''}$ is zero. Hence, the morphism $\mathcal{V}^{\otimes 2} \to \pi_{2*}'(\mathcal{M}^{\otimes 2}(2Z''))$ factors through $\pi_{2*}'(\mathcal{M}^{\otimes 2}) = F$. The morphism $\phi : E \to F$ is then the descent of the morphism of locally-free sheaves $\mathcal{V}^{\otimes 2} \to F$. As a consequence, $\mathcal{M}_{d}^r(C)$ is scheme-theoretically defined as the degeneracy locus of $\phi$ (cf. [Far09], [FO11]), which is closed in $G^r_d(C)$.

---

1. A relaxed version of the conjecture was also stated in [FO12], removing the assumption that $\rho < r - 2$. 
**Lemma 4.1.** Let $L$ be a line bundle on a Petri-general curve $C$ for which $h^0(L) \geq 2$; then $h^1(L^2) = 0$.

*Proof.* By the Gieseker-Petri theorem, the multiplication map $\mu : H^0(L) \otimes H^0(\omega \otimes L^{-1}) \to H^0(\omega)$ is injective. By Serre duality, we have $H^1(L^\otimes 2) \cong H^0(\omega \otimes L^{-2})^\vee \cong \text{Hom}(L, \omega \otimes L^{-1})^\vee$. If $h^1(L^\otimes 2) \neq 0$, there exists an injection from $L$ to $\omega \otimes L^{-1}$, and the induced linear map $H^0(L) \to H^0(\omega \otimes L^{-1})$ is multiplication by some rational function $f \in K(C)$. Since $h^0(L) \geq 2$, there exist linearly independent sections $s, s'$ of $L$ for which $s \otimes (f \cdot s') - s' \otimes (f \cdot s)$ is a non-zero element in the kernel of $\mu$, a contradiction. \qed

**Definition 4.2.** We call $\mathcal{M}_d(C)$ the SMRC locus for quadrics.

4.2. **The regime** $d \geq g$. Hereafter, we focus on the special case where $d \geq g$. As we shall see later, this will be the case relevant to the Bertram–Feinberg–Mukai conjecture; it is also cohomologically simpler than the general case.

**Lemma 4.3.** When $d \geq g$, we have $\pi^*_2 s(\mathcal{M}^\otimes 2) = p^* \pi_2 s(\mathcal{L}^\otimes 2)$.

*Proof.* This essentially follows from standard properties of cohomology and base change, as applied to the fibered square

\[
\begin{array}{ccc}
C \times G^d_d(C) & \xrightarrow{\text{Id}_C \times p'} & C \times \text{Pic}^d(C) \\
\downarrow \pi'_2 & & \downarrow \pi_2 \\
G^d_d(C) & \xrightarrow{p'} & \text{Pic}^d(C)
\end{array}
\]

and the invertible sheaf $\mathcal{L}^\otimes 2$ on $C \times \text{Pic}^d(C)$. Recall first $\pi^*_2 s(\mathcal{M}^\otimes 2) = \pi'_2 s(\text{Id}_C \times p')^* \mathcal{L}^\otimes 2$. Note that $d \geq g$ implies $h^1(C, \mathcal{L}_s^\otimes 2) = 0$ for all $s \in \text{Pic}^d(C)$. Consequently, cohomology and base change commute for $\mathcal{L}^\otimes 2$ in degree 0, i.e. $\pi'_2 s(\text{Id}_C \times p')^* \mathcal{L}^\otimes 2 = p'^* \pi_2 s(\mathcal{L}^\otimes 2)$.

*Remark 4.4.* The same argument shows that $\pi^*_2 s(\mathcal{M}(Z'')) = p^* \pi_2 s(\mathcal{L}(Z'))$ without making any assumption on $d$ relative to $g$. Without the hypothesis $d \geq g$, we do not have $F = p'^* \pi_2 s(\mathcal{L}^\otimes 2)$ and in general $\pi_2 s(\mathcal{L}^\otimes 2)$ is not locally free.

Hereafter we adopt the following approach to studying the non-emptiness of the SMRC locus.

**Strategy.** Let $\tilde{\mathcal{D}}$ be the class of the degeneracy locus of $\tilde{\phi} : \text{Sym}^2 \mathcal{W} \to p^* \pi_2 s(\mathcal{L}^\otimes 2(2Z'))$ in $A(\text{Gr}(r+1, \pi_2 s(\mathcal{L}^\otimes 2)))$ (in particular $\phi$ is the pullback of $\tilde{\phi}$ along $i$), and let $\mathcal{D}$ be the class of the degeneracy locus of $\phi$ in $A(G^d_d(C))$. If $[G^d_d(C) : \tilde{\mathcal{D}}] \neq 0$, then $\mathcal{D} \neq 0$; in other words, $\mathcal{M}_d(C)$ is non-empty. Furthermore, if $q([G^d_d(C) : \tilde{\mathcal{D}}]) \neq 0$, then $\mathcal{M}_d(C)$ is non-empty.

We briefly outline how to calculate $\tilde{\mathcal{D}}$ and $[G^d_d(C)]$. The former may be calculated using Porteous' formula, for which we need to determine the Chern classes of $\text{Sym}^2 \mathcal{W}$ and $p^* \pi_2 s(\mathcal{L}^\otimes 2(2Z'))$. To handle $\text{Sym}^2 \mathcal{W}$, we appeal to a formula of Laksov, Lascaux and Thorup that expresses the Chern classes of $\text{Sym}^2 \mathcal{W}$ in terms of the Chern classes of $\mathcal{W}$; see Subsection 8.1. To calculate the Chern classes of $\pi_2 s(\mathcal{L}^\otimes 2(2Z'))$, on the other hand, we apply the Grothendieck–Riemann–Roch formula. Following Theorem 3.1, (the pullbacks of) these classes naturally belong to $A(\text{Gr}(r+1, \pi_2 s(\mathcal{L}^\otimes 2)))$. 

Meanwhile, \( G_d^r(C) \) is the zero locus of the bundle map

\[
\Upsilon \hookrightarrow p^* \pi_2^* \mathcal{L}(Z') \to p^* \pi_2^* (\mathcal{L}(Z')|_{Z'})
\]

so \([G_d^r(C)]\) itself may be computed using Porteous’ formula.

4.3. The injective and surjective ranges. Hereafter, we shall refer to cases for which \( (r+2) < 2d - g + 1 \) as cases within the injective range and cases for which \( (r+2) \geq 2d - g + 1 \) as cases within the surjective range. For future reference, we list some simple numerical consequences of our assumptions for cases within the surjective range, which will be the main focus of the remainder of the paper.

**Lemma 4.5.** Let \( N = (r+2) - (2d - g) \geq 1 \) and suppose \( D \geq 0 \). We have

\[
N + (r+1)(r+g-d) \leq g \leq \min\{r+g, 2d-g\} \leq 2d-g = \left( \frac{r+2}{2} \right) - N.
\]

The inequality (1) implies that \( d-g \) is bounded from below by an explicit function of \( N \).

**Corollary 4.6.** \( r \geq d-g \geq \lceil 1 + \sqrt{16N-7} \rceil \).

**Proof.** Let \( a = d-g \). Inequality (1) implies

\[
N + (r+1)(r-a) \leq g \leq g + 2a \leq \left( \frac{r+2}{2} \right) - N
\]

from which we obtain a quadratic inequality in \( r \), namely

\[
\frac{1}{2}r^2 + \left( -\frac{1}{2} - a \right) r + (a+2N-1) \leq 0.
\]

For the latter inequality to hold, the associated \( r \)-discriminant \( (\frac{1}{2} + a)^2 - 2(a+2N-1) = (a - \frac{1}{2})^2 - (4N-2) \) must be non-negative, and in fact at least \( \frac{1}{4} \) (because the \( r \)-discriminant itself is never an integer). Since \( a \) is a non-negative integer, the result follows. \( \square \)

The following easy numerical fact will be useful for our purposes.

**Lemma 4.7.** Let \( a,N \) be integers such that \( a \geq 0 \) and \( N > 0 \). For every \( r > 0 \), there is at most one pair of non-negative integers \( (d,g) \) for which the following conditions hold:

1. \( r + g - d = a \);
2. \( \left( \frac{r+2}{2} \right) - 2d + g = N \); and
3. \( g - a(r+1) - N \geq 0 \).

Moreover, when \( r \) is sufficiently large, there is exactly one such pair of integers.

**Proof.** Items (1) and (2) imply that \( g = \left( \frac{r+2}{2} \right) + 1 - N + 2a \) and \( d = \left( \frac{r+1}{2} \right) + 1 - N + a \). Since \( g \) is quadratic in \( r \) and \( a,N \) are fixed, it follows that when \( r \) is sufficiently large, (3) holds automatically. \( \square \)

4.4. Some known cases. Various cases of part (2) of Conjecture 1 have already been established.

(1) Aprodu and Farkas show in [AF11, Prop. 5.7] that when \( \rho < r-2 \) and \( r + g - d = 0 \), part (2) of Conjecture 1 holds; that is, the special maximal-rank locus \( \mathcal{M}_d^r(C) \) is empty.

In other words, the multiplication map \( v_2 : \text{Sym}^2 H^0(L) \to H^0(L^{\otimes 2}) \) is surjective for every
degree $d$ line bundle $L$. (Note that in this case, $d \geq 2g + 3$, and consequently every $g^r_d$ is a complete linear series.) In fact, the same is true for the $n$-th multiplication map $v_n : \text{Sym}^n H^0(L) \to H^0(L^{\otimes n})$, for all $n \geq 2$.

(2) Farkas and Ortega show in [FO11, Prop. 2.3] that when $d \leq g + 1$ and $r = 3$, part (2) of Conjecture 1 holds; that is, $\mathcal{M}^d_3(C)$ is empty. In other words, $v_2|_V : \text{Sym}^2 V \to H^0(L^\otimes 2)$ is injective, for any 4-dimensional subspace $V$ of sections of a degree-$d$ line bundle $L$.

(3) More recently, two separate groups [JP18, LOTiBZ18] have shown (working independently, and using different methods) that for $r = 6$, $g = 22, 23$ and $d = g + 3$, the map $v_2$ is injective for every line bundle $L$ of degree $d$ on a general curve of genus $g$. This means, in particular, that the respective loci where $v_2$ fails to be injective determine divisors in the space of linear series $G^6_d$ and in $\mathcal{M}_g$. This potentially has important implications for the birational geometry of the moduli space of curves in genus 22 and 23. Indeed, in [Far09, Far18]), Farkas computed the classes of the corresponding virtual divisors, and showed that their (virtual) slopes are strictly less than $6 + \frac{12}{g+1}$. To conclude, it remains to establish that the natural forgetful projections from $G^6_d$ to $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$ are generically finite along SMRC divisors.

4.5. Excess components of the SMRC locus. We now turn to part (1) of Conjecture 1. We describe a family of cases within the surjective range for which the associated SMRC loci are always non-empty. In fact, it is easy to see that whenever they exist, non-very ample linear series contribute components of larger-than-expected dimension to SMRC loci.

**Proposition 4.8.** Let $g, r, d$ be non-negative integers satisfying the following conditions:

1. $r \geq r + g - d \geq 0$;
2. $\rho(g, r + 1, d) < 0 \leq \rho(g, r, d - 1)$; and
3. $\left(\frac{r+1}{2}\right) \geq 1 + 2d - g$.

For every Brill–Noether general curve $C$, the SMRC locus $\mathcal{M}^d_g(C)$ has an excessively large component.

**Proof.** The second condition, coupled with the fact that $C$ is Brill–Noether general, implies that there exist $g^r_{d-1}$’s (and hence $g^r_d$’s) on $C$, and that every $g^r_d$ (and every $g^r_{d-1}$) is a complete linear series. Now let $|L|$ denote a $g^r_{d-1}$ on $C$. Then for every point $P$, $|L(P)|$ is a $g^r_d$ on $C$ which is not base-point free, since the inclusion $H^0(L) \to H^0(L(P))$ is an isomorphism.

On the other hand, condition (1) implies that $\deg(L^2(P)) \geq 2g$, and hence $|L^2(P)|$ must be base-point free. But the image of the multiplication map $v_2 : H^0(L(P))^\otimes 2 \to H^0(L^2(2P))$ is contained in $H^0(L^\otimes 2)$. The upshot is that $v_2$ is not surjective, and $|L(P)|$ belongs to $\mathcal{M}^d_g(C)$.

Finally, condition (3) implies that $(g, r, d)$ falls in the surjective range, and moreover, that

$$\rho(g, r, d - 1) > \rho(g, r, d) - 1 - \left(\left\lfloor\frac{r+2}{2}\right\rfloor - (1 + 2d - g)\right).$$

But by construction, $\mathcal{M}^d_g(C)$ contains an isomorphic image of $G^r_{d-1}(C)$, of dimension $\rho(g, r, d - 1)$. It follows, in particular, that $\mathcal{M}^d_g(C)$ contains an excessively large component. \qed
Remark 4.9. Whenever \((g, r, d)\) satisfies the condition in Proposition 4.8, we say that \((g, r, d)\) is a trivial instance within the surjective range. This naturally raises the question of whether non-empty non-trivial SMRC loci exist.

Example 4.10. It is easy to check that \((g, r, d) = (16, 7, 22)\) is a trivial instance in the surjective range. In this case \(\rho(16, 7, 21) = 0\), so there are finitely many \(g_{21}^7\)'s on \(C\). Each one of these generates a 1-dimensional family of linear series in \(G_{22}^8(C)\) along which \(\nu_2\) fails to be surjective.

More generally, whenever \(G_d^r(C)\) contains a large component consisting of non-very ample linear series, \(M_{d}^r(C)\) may have excessively large components.

Lemma 4.11. Suppose \(|L|\) is a non-very ample \(g_d^r\) on a general genus \(g\) curve such that \(d \geq g + 1\). Then \(\nu_2\) is not surjective.

Proof. In light of Proposition 4.8, it suffices to consider the case where \(|L|\) is base-point free. As \(|L|\) is not very ample, there is some pair of points \(R_1, R_2 \in C\) (not necessarily distinct) for which

\[h^0(L(-R_1 - R_2)) = h^0(L(-R_1)) = h^0(L(-R_2)) = r.\]

Consequently, \(\text{Im}(\nu_2)\) has no section which vanishes to order exactly 1 at \(R_1\) and does not vanish at \(R_2\). But \(d \geq g + 1\), so \(|L^\otimes 2|\) is very ample. It follows that \(\nu_2\) is not surjective. \(\square\)

Example 4.12. Suppose \((g, r, d) = (13, 6, 18)\). Every \(g_{18}^6\) on a general genus 13 curve is a base-point free complete linear series \(|L|\) with \(h^1(L) = 1\). It follows that every \(g_{18}^6\) is of the form \(|\omega_C(-Z)|\), where \(Z\) is an effective divisor of degree 6. Since a general curve of genus 13 has no \(g_{18}^6\), we have \(\omega_C(-Z_1) \cong \omega_C(-Z_2)\) if and only if \(Z_1 = Z_2\).

On the other hand, a \(g_{18}^6\) fails to be very ample if and only if it contains a \(g_{16}^5\). In other words, \(|\omega_C(-Z)|\) fails to be very ample if and only there exist \(R_1, R_2 \in C\) such that \(|\Theta(Z + R_1 + R_2)|\) is a \(g_{16}^5\).

Now let \(I \subset G_b^1(C) \times \text{Sym}^6 C\) denote the incidence variety of degree 6 effective divisors contained in \(g_{16}^5\). It is easy to see that the projection \(I \rightarrow G_b^1(C)\) over the curve \(G_b^1(C)\) has one-dimensional fibers, so \(I\) is irreducible and 2-dimensional. Furthermore, a 6-tuple of points can be contained in at most one \(g_{16}^5\), because there is no \(g_{16}^2\) on a general curve of genus 13. It follows that \(\mathcal{M}_{d}^r(C)\) has a component that is at least 2-dimensional. However, \(\rho(13, 6, 18) - 1 - (|g_{18}^6| - 1 + 36 - 13)| = 1\). So \(\mathcal{M}_{d}^r(C)\) has an excessively large component in this case.

In light of Proposition 4.8 and Lemma 4.11, the following question is fundamental.

Question 4.13. When \(D(g, r, d) \geq 0\), does \(\mathcal{M}_{d}^r(C)\) contain a very ample \(g_d^r\)?

It is worth mentioning that in the original version of SMRC as proposed by Aprodu and Farkas, the condition \(\rho < r - 2\) implies that every \(g_d^r\) on a general curve is very ample. (Indeed, this follows from [Far08, Thm 0.1].) So answers to question 4.13 will naturally extend the work of Aprodu and Farkas. As we will see later, affirmative answers to question 4.13 will also lead to solutions to existence problems in higher-rank Brill-Noether theory.

5. Enumerative calculations

5.1. Chern classes of \(\text{Sym}^2 \omega\) and \(\pi_{2*}(\mathcal{I}^\otimes 2)\), and the degeneracy class of \(\delta\).
Situation 5.1. In this section, we make the running assumption that \( d \geq g \).

Our first goal is to determine the Chern classes of \( \pi_2_*(L^\otimes 2(2Z')) \). Recall that \( Z' \) is the pull-back of an effective divisor on \( C \) to \( C \times \text{Pic}^d(C) \) and \( L(Z') \) is a Poincaré line bundle on \( C \times \text{Pic}^{d+\deg(Z)}(C) \). So it suffices to compute the Chern classes of \( \pi_2_*(L^\otimes 2) \). Notice also under our running assumption that \( h^1(C, L^\otimes 2) = 0 \) for all \( s \in \text{Pic}^d(C) \), and hence \( R^i \pi_2_*(L^\otimes 2) = 0 \) for every \( i > 0 \). Grothendieck–Riemann–Roch now yields

\[
\text{ch}(\pi_2_*(L^\otimes 2)) \cdot \text{td}(\text{Pic}^d(C)) = \pi_2_*(\text{ch}(L^\otimes 2) \cdot \text{td}(C \times \text{Pic}^d(C)))).
\]

The Todd class of an abelian variety is trivial, so \( \text{td}(C \times \text{Pic}^d(C)) \) is the pull-back of \( \text{td}(C) \). Accordingly, (2) reduces to

\[
\text{ch}(\pi_2_*(L^\otimes 2)) = \pi_2_*(\text{ch}(L^\otimes 2) \cdot \text{td}(C)) = \pi_2_*(\text{ch}(L^\otimes 2) \cdot \left(1 + \frac{1}{2}c_1(T_C)\right)).
\]

We still need to compute \( \text{ch}(L^\otimes 2) \), or equivalently \( c_1(L^\otimes 2) = 2c_1(L) \). The latter is, however, given explicitly in [ACGH85, Ch. VIII]. The upshot is that up to numerical equivalence

\[
\text{ch}(\pi_2_*(L^\otimes 2)) = \pi_2_*(\text{ch}(L)^2) \cdot \left(1 + \frac{1}{2}c_1(T_C)\right) = (1 - g + 2d) - 4\theta.
\]

Equivalently, we have \( c_2(\pi_2_*(L^\otimes 2)) = e^{-4\theta} \).

Meanwhile, from the Laksov–Lascoux–Thorup formula, we get

\[
s_k(\text{Sym}^2 \mathcal{U}) = (-1)^k \sum_{I} \psi_I s_I(\mathcal{W}) \quad \text{and} \quad c_k(\text{Sym}^2 \mathcal{U}) = (-1)^{\left(\begin{array}{c} k+1 \\ 2 \end{array}\right)} 2^{-\tau(r+1)} \sum_{I} (-2)^{|I|} d_I s_I(\mathcal{W})
\]

where the summation is over all degree-\( k \) Schur functions in Chern roots of \( \mathcal{U} \) and \( \psi_I \), and the \( d_I \) are particular non-negative combinatorial coefficients defined by determinantal formulae; see Appendix 8.1 for their precise definition.

By Porteous’ formula, the class \( \overline{D} \in A_*(\text{Gr}(r + 1, \pi_2_*(L(Z'))) \) of the locus over which the vector bundle map \( \hat{\phi} : \text{Sym}^2 \mathcal{W} \to p^* \pi_2_*(L^\otimes 2(2Z')) \) fails to be of maximal rank is given by

\[
\Delta_{1+2d-g-c}(S)
\]

in which \( c = \min\{(r+2)/2, 1+2d-g\} - 1 \), \( S \) denotes the set of variables

\[
S = \{c_n(p^* \pi_2_*(L^\otimes 2(2Z')) - \text{Sym}^2 \mathcal{W}) \mid n \geq 0\} = \left\{(-1)^{n} \sum_{I: |I| \leq \left(\begin{array}{c} r+1 \\ 2 \end{array}\right)+n} \frac{(4\theta)^{n-\deg(s_I)}}{(n-\deg(s_I))!} s_I(\mathcal{W}) \mid n \geq 0\right\}
\]

and \( \Delta_{q}(\cdot) = \Delta_{q_1,\ldots,q}(\cdot) \) (see Notation 2.24).

5.2. \( G^s_q(C) \) as a zero locus in \( \text{Gr}(r + 1, \pi_2_*(L(Z'))) \). In order to apply Porteous’ formula, we now turn to the two bundles \( \mathcal{W} \) and \( p^* \pi_2_*(L(Z')) \). Note that the Chern classes of \( \mathcal{W} \) show up among the generators of \( A_*(\text{Gr}(r + 1, \pi_2_*(L(Z')) \)) over \( A_*(\text{Pic}^d(C)) \). On the other hand, \( \pi_2_*(L(Z')) \cong \pi_2_*(L(Z')) \) is a direct sum of line bundles algebraically equivalent to the trivial bundle [ACGH85, Sec. VII.2]. So, modulo algebraic equivalence (and hence up to numerical
equivalence), \( \pi_{2s}(\mathcal{L}(Z')|_{Z'}) \) has trivial Chern classes. It follows that \( c(p^*\pi_{2s}(\mathcal{L}(Z')|_{Z'}) - \mathcal{U}) = s(\mathcal{U}) \).

Assume \( \deg(Z) = 2g - 1 - d \geq 0 \). Porteous’ formula yields

\[
[G'_d(C)] = \det([s_{2g-1-d+j-i}(\mathcal{U})]_{i,j \leq r+1}) = (-1)^{(r+1)(2g-1-d)}s_{2g-1-d,...,2g-1-d+r}(\mathcal{U}).
\]

**Example 5.2.** Applying Lemma 3.9 in tandem with (3), we recover the well-known expression for the class of \( W'_d(C) \) inside \( \text{Pic}^d(C) \):

\[
[W'_d(C)] = q([G'_d(C)]) = \left[ \frac{\prod_{i \leq j \leq r+1}(i^j - i^j)}{\prod_{j=1}^{r+1}(i_j - g + r + 1)!} \cdot \theta |\cdot (\binom{r+1}{i} - (r+1)(g-r-1) \right].
\]

where \( I = (i_1, i_2, \ldots, i_{r+1}) = (2g - 1 - d, 2g - d, \ldots, 2g - 1 - d + r) \). Equivalently,

\[
[W'_d(C)] = \frac{\prod_{\alpha=0}^{r} \alpha!}{\prod_{\alpha=0}^{r+1}(r + g - d + \alpha)!} g^{(r+1)(r+g-d)}.
\]

### 5.3. Classes of SMRC loci.
Combining our formulæ from the previous two subsections, we deduce that up to numerical equivalence the class of \( \mathcal{M}'_d(C) \) is given by

\[
S := [G'_d(C)] : \mathcal{D} = (-1)^{(r+1)(2g-1-d)}s_{2g-1-d,...,2g-1-d+r}(\mathcal{U}) \cdot \Delta^{(1+2d-g-m)}(S)
\]

where \( S = \{(1)^n \sum_{|I| \leq (r+2)} \frac{(4\theta)^n - \deg(s_I)}{(n - \deg(s_I))} \cdot s_{1}(\mathcal{U}) \mid n \geq 0 \} \) and \( m = \min \{\binom{r+2}{2}, 1+2d-g \} - 1 \).

As we mentioned in section 4, there is now a basic dichotomy depending on the sign of \( \binom{r+2}{2} - (1+2d-g) \).

#### 5.3.1. The surjective range: \( \binom{r+2}{2} \geq 1+2d-g \).
This is the case of primary interest to us. In this case, \( 1+2d-g-m = 1 \) and by applying [Ful98, Lemma 14.5.1] we may rewrite (4) as

\[
\Delta_1^{(N)}(S) = (-1)^{N} \Delta_1^{(1)}(S^{-1})
\]

where \( N = \binom{r+2}{2} - m = \binom{r+2}{2} - 2d + g \) and \( S^{-1} \) denotes the set of variables

\[
\{ s_n(p^*\pi_{2s}(\mathcal{L} \otimes 2D')) - \text{Sym}^2 \mathcal{U} \mid n \geq 0 \} = \left\{ \sum_{k=0}^{n} c_{n-k}(\text{Sym}^2 \mathcal{U}) \frac{(4\theta)^k}{k!} \mid n \geq 0 \right\}.
\]

Simplifying, we find that the class of the SMRC locus is given by

\[
S = (-1)^{(r+1)(d+1)+N} s_{J}(\mathcal{U}) \cdot \sum_{|I| \leq \binom{r+1}{2} + N} \frac{(-1)^{\deg(s_I)} 2^{2N-|I|} d_I}{(N - \deg(s_I))!} s_{1}(\mathcal{U}) \cdot \theta^{N - \deg(s_I)}
\]

where \( J = (2g - 1 - d, 2g - d, \ldots, 2g - 1 - d + r) \). To go further, will explicitly rewrite the products \( s_J(\mu) \cdot s_{1}(\mathcal{U}) \) by appealing to the following ancillary result.

**Lemma 5.3.** Let \( \Lambda_n \) be the ring of symmetric polynomials in \( n \) variables. Fix \( \lambda = (a,a,\ldots,a) \) with \( a > 0 \) and suppose \( \mu \) is any partition for which \( |\mu| \leq n \). We then have \( s_\lambda \cdot s_\mu = s_{\lambda+\mu} \) in \( \Lambda_n \).

**Proof.** We begin by writing

\[
s_\lambda \cdot s_\mu = \sum_{\nu \in \mathcal{P}_{\lambda+|\nu|,n}} c_{\lambda,\mu}^{\nu} s_{\nu}
\]

where \( c_{\lambda,\mu}^{\nu} \) are the coefficients of the expansion of \( s_\lambda \cdot s_\mu \) in terms of \( s_{\nu} \).
where \( P_{|\lambda|+|\mu|,n} \) denotes the set of all partitions of length at most \( n \) and size \( |\lambda|+|\mu| \), and where
the coefficients \( c'_{\lambda,\mu} \) are non-negative integers. Now suppose \( c'_{\lambda,\mu} \neq 0 \). Since \( \lambda \) is rectangular with
\( n \) rows, the length of \( \nu \) is then necessarily exactly \( n \) and \( \nu - \lambda \) is still a partition.

According to the Littlewood-Richardson rule, \( c'_{\lambda,\mu} \) counts the number of Littlewood-Richardson
tableaux. These are semi-standard Young tableaux of shape \( \nu - \lambda \) and weight \( \mu \) with the characteristic property that concatenating each of these's reversed rows yields a word which is a lattice permutation; that is, in every initial part of the word, any number \( i < n \) occurs at least as many times as \( i+1 \). The characteristic property immediately forces all the entries in the first row of
such a tableau to be 1, and no further 1's can occur in this tableau, since entries in every column form a strictly increasing sequence. Thus, \( \nu_1 - a = \mu_1 \).

Now remove the first row; the result is a Littlewood-Richardson tableau of shape \( (\nu_2, \ldots, \nu_n) - (a, \ldots, a) \) and weight \( (\mu_2, \ldots, \mu_n) \) on the alphabet \( \{2, \ldots, n\} \). Applying the same argument as
before, we get \( \nu_2 - a = \mu_2 \). By induction, we conclude that \( \nu = \mu \) and in particular \( c'_{\lambda,\mu} = 1 \). \( \square \)

Remark 5.4. An alternative proof of the same result may be derived from Corollary 7.15.2 in [SF99],
which establishes that \( c'_{\lambda,\mu} \) is equal to the coefficient of \( x^{\nu+\delta} \) in \( V(x_1, \ldots, x_n)s_\lambda(x_1, \ldots, x_n)s_\mu(x_1, \ldots, x_n) \),
where \( \delta = (n-1, n-2, \ldots, 1, 0) \).

To wit, note that \( s_\lambda(x_1, \ldots, x_n) = x_1^a \ldots x_n^a \) when \( \lambda = (a, \ldots, a) \), as the only semi-standard
Young tableau on the alphabet \( \{1, 2, \ldots, n\} \) is the one with every column equal to \( (1, 2, \ldots, n)^t \).
It follows that
\[
V(x_1, \ldots, x_n)s_\lambda(x_1, \ldots, x_n)s_\mu(x_1, \ldots, x_n) = (x_1^a x_2^a \ldots x_n^a) \cdot \sum_{\sigma \in S_n} \text{sgn}(\sigma)\sigma(x_1^{\mu_1+n-1} x_2^{\mu_2+n-2} \ldots x_n^{\mu_n})
\]
\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma)x_1^{a+\mu_{\sigma(1)}+n-\sigma(1)} x_2^{a+\mu_{\sigma(2)}+n-\sigma(2)} \ldots x_n^{a+\mu_{\sigma(n)}}.
\]
Note also that \( \nu + \delta \) is a strictly decreasing sequence. But \( (a + \mu_{\sigma(1)} + n - \sigma(1), \ldots, a + \mu_{\sigma(n)}) \)
is strictly decreasing if and only if the permutation \( \sigma \in S_n \) is the identity. So if \( c'_{\lambda,\mu} \neq 0 \) then
necessarily \( \nu = \lambda + \mu \), in which case \( c'_{\lambda,\mu} = 1 \).

Using the alternative convention for Schur polynomials as in Definition 2.16, Lemma 5.3 becomes
the following statement.

Corollary 5.5. Let \( I = (i_1, \ldots, i_n) \) be an arbitrary strictly increasing sequence of non-negative
tegers and \( J = (a, a+1, \ldots, a+n-1) \) (\( a \geq 0 \)). We then have
\[
s_I(x_1, \ldots, x_n)s_J(x_1, \ldots, x_n) = s_K(x_1, \ldots, x_n)
\]
where \( K = I + J - (0, 1, \ldots, n-1) = I + (a, a, \ldots, a) \).

Now assume \( 2g - 1 - d \geq 0 \). Let \( a = 2g - 1 - d \) and \( n = r + 1 \). Applying Corollary 5.5 and
Lemma 3.9 to the intersection product \( S \) given by equation (5), we get
\[
q(S) = (-1)^N \left[ \sum_{|I||J| \leq (r+1)^2 + N} \frac{2^{2N-|I|}d_I}{(N - \deg(s_I))!} \cdot \prod_{1 \leq j < \ell \leq r+1} (i_\ell - i_j) \right] g^{N+g-\rho}.
\]
To prove the corresponding special maximal-rank locus $\mathcal{M}_q'(C)$ is non-empty, for any smooth projective curve $C$ of genus $g \leq d$, it suffices to show that

$$S(g,r,d) := (-1)^N \sum_{I: |I| \leq \left(\frac{r+1}{2}\right) + N} \frac{2^{2N-|I|}d_I}{(N - \deg(s_I))!} \prod_{1 \leq j < \ell \leq r+1} \frac{(i_\ell - i_j)}{\prod_{j=1}^{r+1}(i_j + r + g - d)!} 
eq 0. \tag{7}$$

**Example 5.6.** An important case is that in which $g = 13$, $d = 18$, and $r = 6$. Here

$$S(13,6,18) = - \sum_{I: |I| \leq 26} \frac{2^{10-|I|}d_I}{(26 - |I|)!} \prod_{1 \leq j < \ell \leq 17} \frac{(i_\ell - i_j)}{\prod_{j=1}^{17}(i_j + 1)!},$$

5.3.2. The injective range: $(\frac{r+1}{2}) < 1 + 2d - g$. In this case, $(\frac{r+2}{2}) - m = 1$ and we get

$$S = [G'_q(C)] \cdot \Delta_{(s)}^{(1)}(S) = (-1)^{(r+1)(d+1) + N'} \sum_{I: |I| \leq \left(\frac{r+1}{2}\right) + N'} \frac{(4d)^{N' - \deg(s_I)} \psi_I(s_I)}{(N' - \deg(s_I))!}$$

where $N' = 2 + 2d - g - \left(\frac{r+2}{2}\right)$ and $J = (2g - 1 - d, 2g - d, \ldots, 2g - 1 - d + r)$. Consequently,

$$q(S) = \sum_{I: |I| \leq \left(\frac{r+1}{2}\right) + N'} \frac{(-4)^{N' - \deg(s_I)} \psi_I}{(N' - \deg(s_I))!} \prod_{1 \leq j < \ell \leq r+1} \frac{(i_\ell - i_j)}{\prod_{j=1}^{r+1}(i_j + r + g - d)!}.$$

To show the corresponding SMRC locus is non-empty, it suffices to show that

$$S'(g,r,d) := \sum_{I: |I| \leq \left(\frac{r+1}{2}\right) + N'} \frac{(-4)^{N' - \deg(s_I)} \psi_I}{(N' - \deg(s_I))!} \prod_{1 \leq j < \ell \leq r+1} \frac{(i_\ell - i_j)}{\prod_{j=1}^{r+1}(i_j + r + g - d)!}$$

is nonzero.

**Example 5.7.** Consider the case where $g = 6$, $d = 8$, and $r = 3$. Here

$$S'(6,3,8) = \sum_{I: |I| \leq 8} \frac{(-4)^{2-\deg(s_I)} \psi_I}{(2 - \deg(s_I))!} \prod_{1 \leq j < \ell \leq 4} \frac{(i_\ell - i_j)}{\prod_{j=1}^{4}(i_j + 1)!}$$

and $q(S)$ is numerically equivalent to a zero-cycle of degree 10.

6. **The Bertram-Feinberg-Mukai Conjecture and its connection with the SMRC**

In this section, we explain the connection between the BFM conjecture and the Strong Maximal Rank Conjecture for quadrics. We first recall the statement of the BFM conjecture.

**Conjecture 2.** (Bertram-Feinberg-Mukai [BF98, Muk95]) Set $\rho_{g,k} := 3g - 3 - \binom{k+1}{2}$. On a general curve $C$ of genus $g \geq 2$, the moduli space of stable rank two vector bundles with canonical determinant and $k$ sections is non-empty, and has expected dimension $\rho_{g,k}$ whenever $\rho_{g,k}$ is non-negative. When $\rho_{g,k} < 0$, the moduli space is empty.

The existence portion of the conjecture has been verified for many cases: for small genera, see [BF98] and [Muk93]; while for results asymptotic with respect to $(g, k)$, see [TiB04], [LNP16] and [Zha16]. The non-existence portion is a theorem of Teixidor i Bigas [TiB08]. However, the general case remains very much open.
Our point of departure is the fact that every stable rank two vector bundle $E$ with canonical determinant fits into a short exact sequence of the form

$$0 \to \omega \otimes L^{-1} \to E \to L \to 0.$$  

Every extension $(8)$ naturally determines an element $e \in \text{Ext}^1(L, \omega \otimes L^{-1}) \cong H^0(L^\otimes 2)^\vee$. Given any such extension $e$, we have $h^0(E) = h^1(L) + \dim \ker(u_e)$, where $u_e$ is the linear map $H^0(L) \to H^1(\omega \otimes L^{-1}) \cong H^0(L^\otimes 2)^\vee$ in the cohomological long exact sequence induced by $(8)$.

In order to verify (the existence portion of) Conjecture 2, our aim is to produce extensions $(8)$ of line bundles $[L] \in \text{Pic}^d(C)$ whose associated rank-2 vector bundles $E$ are stable and satisfy $h^0(E) \geq k$, whenever $k(k + 1) \leq 6g + 6$. In doing so, we try to simultaneously ensure that $d$ is as small as possible relative to $k$ and that $u_e$ has small rank. Heuristically speaking, we specify a stable vector bundle by identifying one of its minimal quotient line bundles, to the extent that this is possible given the cohomological condition $h^0(E) \geq k$.

The impulse to consider extensions of (relatively) small degree line bundles comes from two sources. First of all, maximal sub-line bundles (and hence minimal quotient bundles) of a vector bundle are in some sense canonical. For example, it is known that the space of maximal sub-bundles of any vector bundle is at most one-dimensional [LN83, Cor. 4.6], and is further known to be finite or even a singleton [LN83, Cor. 3.2 and Prop. 3.3] for general rank-two vector bundles with certain prescribed degrees and Segre invariants. Second, it is easier to certify that rank-2 bundles determined by extensions of line bundles of relatively small degrees are stable. In fact, we shall see that in many such cases stability is automatic.

On the other hand, since we focus on extensions $e$ of line bundles $e$ in which $u_e$ has relatively large cokernel dimension, quite often the loci of the line bundles being extended are related to the SMRC loci for quadrics we defined in Section 4. To see this, we analyze the two respective conditions:

1. $h^1(L) \geq p$; and
2. $\dim \ker(u_e) \geq k - p$.

The locus of line bundles $L$ in Pic$^d(C)$ with $h^1(L) \geq p$ is easy to describe: it is precisely the Brill-Noether locus $W_d^{p+g-d}(C)$.

We now turn to the dimension of $\ker(u_e)$. The assignment that takes an extension $e$ to the corresponding coboundary map $u_e$ describes a linear map from $\text{Ext}^1(L, \omega_C \otimes L^{-1}) \cong H^0(L^\otimes 2)^\vee$ to $\text{Hom}(H^0(L), H^0(L)^\vee) \cong H^0(L)^\vee \otimes H^0(L)^\vee$, which is dual to the multiplication map $\mu_L : H^0(L)^\otimes 2 \to H^0(L^\otimes 2)$. It is easy to see that the condition $\dim \ker(u_e) \geq k - p$ may be reformulated as the statement that some $(k-p)$-dimensional subspace $V \subset H^0(L)$ is such that $\text{Im}(\mu_L \cdot V) \subset \ker(e)$, where $\mu_L \cdot V$ is the restriction of $\mu_L$ to $H^0(L) \otimes V$, and $e$ is viewed as a linear function on $H^0(L^\otimes 2)$. (See [CF15, Remark 5.7] for a more general statement.)

The upshot is that in order to show existence of rank two linear series with canonical determinant and many sections, it is useful to study loci in $\text{Ext}^1(L, \omega \otimes L^{-1})$ of specified rank.

**Definition 6.1.** Let $W_t$ denote the locus of $e$ in $\text{Ext}^1(L, \omega \otimes L^{-1})$ where $\dim \ker(u_e) \leq h^0(L) - t$. 

It is not hard to see that $\mathcal{W}_t$ is a determinantal scheme of expected codimension $t^2$. Indeed, it is precisely the $((r+1)-t)$-th degeneracy locus of the pull-back of the universal linear map on $\text{Hom}(H^0(L), H^0(L)^\vee)$. Accordingly, we have a natural filtration

$$\text{Ext}^1(L, \omega \otimes L^{-1}) \supset \mathcal{W}_1 \supset \mathcal{W}_2 \supset \ldots \supset \mathcal{W}_{r+1}.$$ 

Note that $\mathcal{W}_{r+1} = 0$ if and only if the multiplication map $\mu_L$ is surjective.

For our application to higher-rank Brill–Noether theory, we will mainly consider non-general extensions of (possibly) non-general linear series by their Serre duals. In some cases, the liftability of invertible subsheaves is related to the existence of secant divisors.

**Proposition 6.2.** Suppose $e \in \text{Ext}^1(L, \omega \otimes L^{-1})$ is nonzero, where $\deg(L) = g + a$ with $a > 0$. Let $E$ denote the rank two vector bundle obtained from $e$, and suppose further that $h^0(E) = h^0(L) + h^1(L)$. Under these circumstances, if a subbundle $L(-D)$ lifts to a subbundle of $E$, then the image of $D$ in (the projectivization of the dual of) $|L|$ is of submaximal dimension, and $\deg(D) \geq 2 \dim(H^0(L)/H^0(L(-D)))$. In particular, $P$ belongs to the base locus $\text{bs}(|L|)$ whenever $L(-P)$ lifts to a subbundle of $E$. Moreover, if there exists an effective divisor $D$ of degree $b \leq a/2$ for which $\mu_{L(-D)}$ is surjective, then $E$ is not stable.

**Proof.** The assumption that $L(-D)$ lifts to a subbundle of $E$ implies that $h^0(L(-D)) + h^1(L(-D)) = h^0(L) + h^1(L)$. Notice that

$$h^0(L(-D)) + h^1(L(-D)) = h^0(L) + h^1(L) + \deg(D) - 2 \dim(H^0(L)/H^0(L(-D)))$$

and the first statement follows.

Now suppose $D$ is an effective divisor of degree $b \leq a/2$ and that $\mu_{L(-D)}$ is surjective. The multiplication map $\nu_2$ factors as

$$H^0(L)^{\otimes 2} \to \text{Im}(\mu_L) \hookrightarrow H^0(L^{\otimes 2}).$$

The fact that $h^0(E) = h^0(L) + h^1(L)$ implies that $e$ is in the kernel of $H^0(L^2)^\vee \to \text{Im}(\mu_L)^\vee$; in particular, $e$ vanishes on $H^0(L^2(-2D))$. It follows that $e$ is in the kernel of

$$\text{Ext}^1(L, \omega \otimes L^{-1}) \cong H^0(L^2)^\vee \to H^0(L^2(-2D))^\vee \cong \text{Ext}^1(L(-2D), \omega \otimes L^{-1})$$

i.e. $L(-2D)$ lifts. Since $b \leq a/2$, we have $\deg(L(-2D)) \geq g$ and consequently $E$ is not stable. $\square$

Finally, a classical result of Mukai and Sakai gives a lower bound on the degree of a maximal subbundle of a vector bundle:

**Theorem 6.3** ([MS85]). Let $C$ be a non-singular projective curve of genus $g$ over an algebraically closed field. Let $E$ be a vector bundle on $C$ and let $F$ be a maximal proper subbundle of $E$. Then

$$\mu(E/F) - \mu(F) \geq g$$

where $\mu(F) = \frac{\deg(F)}{\text{rank}(F)}$ denotes the slope of $F$.

Applying this theorem to the case where $E$ is a stable rank-two vector bundle with canonical determinant yields the following useful statement.

**Corollary 6.4.** The degree of the minimal quotient line bundle of a rank-two vector bundle with canonical determinant is at most $\left\lfloor \frac{3g}{2} \right\rfloor - 1$. 
So by replacing a line bundle $L$ with its Serre dual if necessary, it suffices to consider $L$ for which $g - 1 \leq \text{deg}(L) \leq \lfloor \frac{3g}{2} \rfloor - 1$ when constructing stable rank-two vector bundles with canonical determinant via extensions as in (8).

6.1. The search for minimal quotient line bundles. In order to prove the existence portion of the BFM conjecture it suffices to show that for every $g \geq 2$ and for the maximal integer $k = k(g)$ for which $\rho_{g,k} \geq 0$, there exists a stable rank-two vector bundle $E$ over a general genus $g$ curve for which $\det(E) = \omega$ and $h^0(E) \geq k$.

Recall that we are interested in extensions of the form (8) over a curve $C$ that is (Brill-Noether and Petri-) general. We would also like to minimize the degree of $L$ in such extensions. In light of this, Corollary 6.4 motivates the following definition.

**Definition 6.5.** The minimal BN-compatible degree with respect to $k$ is

$$d_k^{*} := \min \{d : g - 1 \leq d \leq \lfloor \frac{3g}{2} \rfloor - 1 \mid h^0(L) + h^1(L) \geq k\}$$

in which the minimum is taken over all $L \in \text{Pic}^d(C)$.

Now suppose $\text{deg}(L) = d_k^*$. By the Brill-Noether theorem and Serre duality, we then have

$$(9) \begin{cases} (r + 1)(r + g - d) \leq g \\ (r + 1) + (r + g - d) \geq k \end{cases}$$

where $h^0(L) = r + 1$, $\text{deg}(L) = d$ and $h^1(L) = r + g - d$.

**Definition 6.6.** Fix positive integers $(g, k)$ such that $\rho_{g,k} \geq 0$. Suppose $g - 1 \leq d \leq \lfloor \frac{3g}{2} \rfloor - 1$ and $(r, d)$ satisfies the constraints in (9). We shall say $(r, d)$ is BN-compatible with respect to $(g, k)$.

It is not always the case that a stable rank-two vector bundle $E$ for which $\det(E) \cong \omega_C$ and $h^0(E) = k$ has a quotient line bundle of degree $d_k^*$. For example, if for all line bundles $L$ of degree $d_k^*$, the multiplication map $\mu_L$ is surjective and $h^0(L) + h^1(L) = k$, then every extension of the form

$$0 \to \omega \otimes L^{-1} \to E \to L \to 0$$

splits (and hence $E$ is not stable). However, on the positive side we have the following result.

**Proposition 6.7.** Suppose $\epsilon : 0 \to \omega_C \otimes L^{-1} \to E \to L \to 0$ is a non-trivial extension of line bundles such that $h^0(E) \geq k$ and $\text{deg}(L) = d_k^* \geq g$, then $E$ is stable and $L$ is a minimal quotient line bundle of $E$.

**Proof.** Let $0 \to F \to E$ be a sub-line bundle of $E$. The composition of morphisms of line bundles $F \to E \to L$ is either zero or injective.

If it is zero, the morphism $F \to E$ must factor through the kernel of $E \to L$, which is $\omega_C \otimes L^{-1}$. Then $\text{deg}(F) \leq 2g - 2 - \text{deg}(L) \leq g - 2$.

If it is injective, let $d' = \text{deg}(F) \leq \text{deg}(L) = d_k^*$. Since $h^0(F) + h^1(F) \geq k$, by construction, we must have either $\text{deg}(F) < g - 1$ or $\text{deg}(F) = \text{deg}(L)$. However, the second situation is impossible as otherwise we would get $F \cong L$ and thus violate our assumption that $e$ is a non-trivial extension.
Therefore, $E$ does not admit a sub-line bundle of degree $g - 1$ or greater. Hence, $E$ is stable. The fact that $L$ is minimal follows from the definition of $d^*_k$ and the stability of $E$. 

**Corollary 6.8.** The existence portion of the BFM conjecture holds under either of the following two circumstances.

1. There exists a BN-compatible pair $(r, d^*_k)$ such that $2r + 1 - g + d^*_k = k$ and $(r, d^*_k)$ falls within the injective range.

2. There exists a BN-compatible pair $(r, d^*_k)$ such that $2r + 1 - g + d^*_k = k$, $(r, d^*_k)$ falls within the surjective range, and $\mathcal{M}^*_{d^*_k}(C) \neq \emptyset$.

**Proof.** In both cases, the multiplication map $\mu_L$ fails to be surjective and hence its dual $\mu^*_L$ fails to be injective. Case (1) follows from the classical maximal rank conjecture, which is now a theorem of Eric Larson; see [Lar17]. Larson’s theorem implies that there exists a nonzero extension $e$ in $\text{Ext}^1(L, \omega_C \otimes L^{-1})$ with $u_e = 0$, in which case $\dim \ker(u_e) = h^0(L)$ and hence $h^0(E) = k$. It then follows from Proposition 6.7 that $E$ is stable. 

**Example 6.9.** The following are two examples of those cases covered by Corollary 6.8.

1. Set $(g, k) = (14, 8)$. In this case, we have $d^*_8 = 17$, $(5, 17)$ is a BN-compatible pair that falls within the surjective range, and $N(g, r, d) = 1$. It follows from our calculation of the SMRC class in Subsection 8.5 that $\mathcal{M}^*_{17}(C)$ is nonempty. Consequently, the existence portion of the BFM conjecture holds for $(g, k) = (14, 8)$.

2. Set $(g, k) = (18, 9)$. In this case, we have $d^*_9 = 20$, and $(5, 20)$ is a BN-compatible pair that falls within the injective range. Thus, the existence portion of the BFM conjecture also holds for $(g, k) = (18, 9)$.

### 6.2. Existence in small genera.

The existence portion of the BFM conjecture for small genera ($g \leq 12$) was first established by Bertram and Feinberg in [BF98]. Here, we show how these cases of the conjecture can be easily recovered from our MRC-based viewpoint.

The $k$-values listed here are maximal with respect to the given $g$-values such that $\rho_{g,k} = 3g - 3 - \binom{k+1}{2} \geq 0$. As we shall see, the case $g = 2$ is exotic in the sense that $\rho_{2,2} = 0$, but there is no stable bundle of rank two with two sections on a genus 2 curve. We nevertheless describe the situation in this case, since the approach is the same as for other low genera cases.

**$g = 2, k = 2$.** We will show that every semi-stable rank two vector bundle with canonical determinant and two sections is strictly semi-stable. The minimal quotient line bundle of a rank two vector bundle $E$ with canonical determinant on a genus 2 curve $C$ has degree at most $\lfloor \frac{4g}{2} \rfloor - 1 = 2$. For $h^0(E) \geq 2$ to hold, $E$ must fit into an extension $e$ of the form

$$0 \to \mathcal{O}_C \to E \to \omega_C \to 0$$

since $h^0(L) + h^1(L) \geq 2$ and $\deg(L) = 2$ together imply that $L = \omega_C$.

Now consider $e$ as a point in $\mathbb{P}(H^0(\omega_C^{\otimes 2})^\vee)$. By [LN83, Prop. 1.1], $e$ lies in $\text{Sec}_1(X) = X$ whenever $E$ is not stable, where $X$ is the image of the morphism $C \to \mathbb{P}(H^0(\omega_C^{\otimes 2})^\vee)$. In particular, a general extension $e$ will contribute a stable bundle $E$. On the other hand, it is easy to check
directly that $\mu_{\omega_C} : \text{Sym}^2 H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$ is an isomorphism, since $h^0(\omega_C) = 2$. It follows that the dual $\mu_{\omega_C}^\vee$ is also an isomorphism. Consequently, every point in $\mathbb{P}(H^0(\omega_C^{\otimes 2})^\vee)$ corresponds to a symmetric bilinear map on $H^0(\omega_C)$. We claim that the locus where the bilinear map has rank one is precisely $X$. Indeed, we have $\text{rank}(\mu_{\omega_C}^\vee(e)) = 1$ if and only if $\ker(e) = \text{Im}(\mu_{\omega_C,V})$, where $V$ is some 1-dimensional subspace of $H^0(\omega_C)$, in which case we have $V = H^0(\omega_C(-P))$ for some $P$. But then $\ker(e) = H^0(\omega_C^{\otimes 2}(-P))$; that is, $e$ is the image of $P$ on $X$. It follows that $\mathcal{O}_C(P)$ is a sub-bundle of $E$, and that $E$ is semi-stable.

The same argument also yields the existence of stable rank two bundles with canonical determinant and one global section.

$g = 3, k = 3$. In this case $d_3^* = 3$. One can apply Corollary 6.8(1) to a general $\mathfrak{g}_3^1$ on a general genus 3 curve to conclude.

$g = 4, k = 3$. In this case $d_3^* = 3$. However, we consider a general extension of a general $\mathfrak{g}_3^1$ on a genus 4 curve by its Serre dual. The resulting vector bundle will have three sections and since $d_3^* = 3$, it is at least semi-stable. If we further assume the $\mathfrak{g}_3^1$, $(L, H^0(L))$, is base-point free, the vector bundle is in fact stable. Indeed, if $(r, 3)$ is BN-compatible then necessarily $r = 1$. On the other hand, if $L$ is base-point free, then its associated complete series has no sub-$\mathfrak{g}_3^1$, and hence $E$ has no destabilizing subbundle of degree 3, which means that $E$ is stable.

It is easy to see that a base-point free $\mathfrak{g}_3^1$ exists on a general genus 4 curve. Namely, consider the natural (addition) map $W^1_3(C) \times C \to W^1_3(C) \subset \text{Pic}^4(C)$, where $W^1_3(C)$ is the Brill-Noether locus of $\mathfrak{g}_3^1$'s inside $\text{Pic}^3(C)$. By the Brill-Noether theorem, the domain is 1-dimensional, while the target $W^1_3(C)$ is 2-dimensional.

$g = 5, k = 4$. In this case $d_4^* = 4$ and the smallest $d \geq 5$ for which some pair $(r, d)$ is BN-compatible is $d = 6$ with $r = 2$. Now let $(L, H^0(L))$ be a general $\mathfrak{g}_5^2$, and $e$ be a general extension of $L$ by its Serre dual. Similar to the $g = 4$ case one can conclude there is some non-trivial extension $e$ that produces a semi-stable vector bundle with 4 sections. In fact, since $\ker(\mu_L^\vee)$ is 2-dimensional, the locus $S$ in $\mathbb{P}(H^0(L^{\otimes 2})^\vee)$ of such $e$ is 1-dimensional. By [LN83, Prop. 1.1], to conclude that the resulting vector bundle is stable, we need to certify that $e \notin \text{Sec}_2(X)$, where $X$ is the image of $C$ under the morphism $C \to \mathbb{P}(H^0(L^{\otimes 2})^\vee)$.

We now claim that $S$ is not contained in $\text{Sec}_2(X)$. To see this, note first that for $e$ to lie on a secant line of $X$ spanned by $x_1$ and $x_2$ means that $\ker(e) \supset H^0(L^{\otimes 2}(-x_1 - x_2))$. (In this case $L^{\otimes 2}$ is very ample, so we may identify points on $X$ with points on $C$.) Moreover, $h^0(L^{\otimes 2}(-x_1 - x_2)) = \dim(\text{Im}(\mu_L)) = 6$. By the Brill-Noether theorem, a $\mathfrak{g}_5^2$ on a general genus 5 curve is base-point free, so $\text{Im}(\mu_L) \neq H^0(L^{\otimes 2}(-x_1 - x_2))$, for any $x_1, x_2$ (not necessarily distinct). It thus suffices to show that there exists a codimension-1 subspace of $H^0(L^{\otimes 2})$ containing $\text{Im}(\mu_L)$ but not of the form $H^0(L^{\otimes 2}(-P))$. And indeed, the closed immersion $C \to \mathbb{P}(H^0(L^{\otimes 2})^\vee)$ identifies $P \in C$ with $H^0(L^{\otimes 2}(-P))$ (further identified with a line in the dual vector space), while the locus of codimension-1 subspaces of $H^0(L^{\otimes 2})$ containing $\text{Im}(\mu_L)$ is a line in $\mathbb{P}(H^0(L^{\otimes 2})^\vee)$. Since a positive genus curve cannot contain a line, there is some codimension-1 subspace of $H^0(L^{\otimes 2})$ containing $\text{Im}(\mu_L)$ but not of the form $H^0(L^{\otimes 2}(-P))$; the claim follows, and we conclude.
\(g = 6, k = 5\). In this case, \(d^*_5 = 6\) and we can apply Corollary 6.8(1) to a general \(g^2_6\) on a general genus 6 curve to conclude.

\(g = 7, k = 5\). In this case, \(d^*_5 = 7\) and we can apply Corollary 6.8(1) to a general \(g^2_7\) on a general genus 7 curve to conclude.

\(g = 8, k = 6\). In this case, \(d^*_6 = 8\) and we can apply Corollary 6.8(1) to a general \(g^3_8\) on a general genus 8 curve to conclude. This is essentially the example produced by Mukai in [Muk93].

\(g = 9, k = 6\). In this case \(d^*_6 = 8\) and the smallest \(d \geq 9\) for which some pair \((r, d)\) is BN-compatible is \(d = 10\) with \(r = 3\). Similar to the \(g = 5\) case, we consider a general extension of some \(g^3_{10}\) on a general genus 9 curve by its Serre dual. Let \(E\) be the resulting vector bundle. To conclude that \(E\) is stable, it suffices to start with a \(g^3_{10}\) that admits no sub-\(g^3_5\). A well-known theorem of Farkas on inclusion of linear series with base points implies that such \(g^3_{10}\)'s exist; see [Far08, Thm 0.1].

\(g = 10, k = 6\). Much as in the \(g = 9\) case, here we apply [Far08, Thm 0.1] to conclude the existence of a general \(g^3_{11}\) on a genus 10 curve that admits no sub-\(g^3_5\), and then proceed with the same argument as before.

\(g = 11, k = 7\). In this case, \(d^*_7 = 13\) and we can apply Corollary 6.8(1) to a general \(g^3_{13}\) on a genus 11 curve to conclude.

\(g = 12, k = 7\). In this case, \(d^*_7 = 12\) and we can apply Corollary 6.8(1) to a general \(g^3_{12}\) on a genus 12 curve to conclude.

7. The BFM Conjecture from the SMRC

7.1. New nonemptiness certificates for BFM loci. We begin by giving a list of previously unknown cases (of the nonemptiness portion) of the BFM conjecture that follow directly from our non-emptiness result for special maximal rank loci. The running numerical assumptions are that (i) \(g \geq 13, k \geq 8\); and (ii) \((k+2)/2 > 3g - 3 \geq (k+1)/2\). It is easy to see that for any \(k\), there always exists some \(r\) such that \(2r + 1 + g - d^*_k = k\). We deduce the following result.

**Theorem 7.1.** Suppose \(a\) is an integer solution to one of the following systems of inequalities:

\[
\begin{align*}
(x - 1)^2 - k(x - 1) + g &< 0 \\
7 \geq x^2 - 7x + 2(2k - g + 2) &> 0 \\
x^2 + (7 - 2k)x + 4(g - k - 1) &\geq 0 \\
2x &\geq k
\end{align*}
\]

Then, over a general genus \(g\) curve, the moduli space of stable rank two vector bundles with canonical determinant and \(k\) sections is non-empty.

**Proof.** In the system on the left, the first inequality implies \(\rho(g, a - 2, 2a - 3 + g - k) < 0\); the second inequality implies that \((a - 1, 2a - 1 + g - k)\) falls within the surjective range; and the third inequality implies that

\[
\rho(g, a - 1, 2a - 1 + g - k) \geq D(g, a - 1, 2a - 1 + g - k) \geq 0.
\]
In particular, \( d^*_k = 2a - 1 + g - k \). Similarly, in the system on the right, we have \( d^*_k = 2a - 1 + g - k \) and \((a - 1, 2a - 1 + g - k)\) falls within the injective range.

It then follows from Corollary 6.8 and Proposition 8.26 that the corresponding moduli spaces of rank two vector bundles are non-empty.

Using Theorem 7.1, we obtain some sharp existence results for the BFM conjecture, all but one of which were previously unknown.

**Corollary 7.2.** The existence portion of the BFM conjecture holds for \( g = 14, 17, 18, 19, 22, 26, 31 \).

**Remark 7.3.** The \( g = 19 \) case was previously established in [LNP16].

We now turn our attention to a case of particular interest, which lies at the numerological border of the surjective range. While we do not manage to definitively settle this case, we uncover some interesting geometry and in the process discover that BFM and SMRC do not agree in general.

### 7.2. The case of \( g = 13, k = 8 \).

For the BFM conjecture to hold in this case, we must produce an extension \( e \) of the form

\[
0 \to \omega_C \otimes L^{-1} \to E \to L \to 0
\]

such that \( E \) is stable and \( h^0(E) \geq 8 \) over a Brill–Noether–Petri general curve. By the theorem of Mukai–Sakai, it suffices to search within the degree range \([13, 18]\). Imposing \( h^0(L) + h^1(L) \geq 8 \) reduces the possibilities to \( d = 16 \) and \( d = 18 \).

When \( d = 16 \), the Brill–Noether theorem forces \( h^0(L) = 6 \) and \( h^1(L) = 2 \). In this case, any stable \( E \) is associated with a nontrivial extension \( e \) belonging to the kernel of \( H^0(L^2)\) → \( H^0(L^2) \otimes H^0(L) \). In particular, if \( E \) is stable, the multiplication map \( \mu_L \) cannot be surjective. However, in this case

\[
\rho - 1 - \left\lfloor \frac{r + 2}{2} \right\rfloor - (2d + 1 - g) = 1 - 1 - |21 - 20| < 0
\]

so the SMRC predicts that \( \mu_L \) is always surjective. Note that the image of any \( g_{16}^5 \) whose quadratic multiplication map fails to be surjective lies on a Fano threefold \( X \) of type \((2, 2)\). Explicitly, \( \mathbb{P}^3 \) is obtained from the blow-up of \( X \) along a line \( \ell \in X \) via a morphism that contracts the proper transforms of those lines on \( X \) that intersect \( \ell \). Deciding whether the SMRC holds in this case amounts to a statement about how linear series \( g_{16}^5 \) transform under the birational isomorphism \( X \twoheadrightarrow \mathbb{P}^3 \); we intend to pursue this line further in a subsequent paper.

When \( d = 18 \), the Brill-Noether theorem forces \( h^0(L) = 7 \) and \( h^1(L) = 1 \). By the same argument as in the case \( d = 16 \), if a stable bundle \( E \) with sufficiently many sections exists, the corresponding multiplication map \( \mu_L \) cannot be surjective. However, we expect the SMRC locus to have dimension

\[
\rho - 1 - \left( \frac{r + 2}{2} \right) - (2d + 1 - g) = 6 - 1 - (28 - 24) = 1
\]

so the BFM and SMRC conjectures are compatible in this case.

Notice that in this case the Brill–Noether theorem also implies that (the complete linear series determined by) any destabilizing subbundle of \( E \) must be a \( g_{16}^5 \). On the other hand, from Proposition 6.2 it follows that no \( g_{18}^6 \) giving rise to a stable vector bundle \( E \) contains a \( g_{16}^5 \). Moreover,
The image of the multiplication map is of codimension one inside $H^0(L^\otimes 2)$. Putting all of this together, we get the following claim.

**Claim 1.** Assume that Conjecture 1 (SMRC) holds for $(g, r, d) = (13, 5, 16), (13, 6, 18)$. Then the existence portion of the BFM conjecture holds for $(g, k) = (13, 8)$ on a general curve if and only if there exists some $g^6_{18}$ in $\mathcal{M}^6_{18}(C)$ which is very ample.

By [Far08, Thm 0.1], the locus of $g^6_{18}$ containing a sub-$g^5_{16}$ is at most 2-dimensional.\(^2\) So if we can show that the locus in $G^6_{18}(C)$ where the multiplication map $\psi_2$ fails to have maximal rank is non-empty and is not contained inside the locus of $g^6_{18}$ admitting a sub-$g^5_{16}$, we will verify the existence of a rank two linear series with 8 sections and canonical determinant.

## 8. Combinatorics of SMRC classes

In order to certify the nonemptiness of SMRC loci, we use a formula due to Laksov, Lascoux and Thorup (hereafter, the LLT formula) that expresses the Chern classes of the symmetric square of a bundle in terms of its Segre classes.

### 8.1. The Laksov–Lascoux–Thorup formula

The coefficients of the LLT formula are indexed by sequences of strictly-increasing sequences of non-negative integers, and are described by the following definition.

**Definition 8.1.** Let $I = (i_1, \ldots, i_n)$ be a sequence of strictly-increasing non-negative integers. The combinatorial number $\psi_I$ is defined via the expansion $\prod_{1\leq i\leq j\leq n} \frac{1}{1-(x_i + x_j)} = \sum_I \psi_I s_I(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are formal variables, and the sum is taken over all strictly increasing sequences of non-negative integers with $n$ entries.

**Theorem 8.2** (Laksov-Lascoux-Thorup Formula). [Proposition 2.8.4, [LLT89]] \(\text{Let } E \text{ be a vector bundle of rank } n. \) We have

$$c(\text{Sym}^2 E) = (-1)^{\binom{n}{2}} 2^{-n(n-1)} \sum_I (-2)^{|I|} d_I s_I(E)$$

where

$$d_I = \det \begin{bmatrix} (-1)^{i_1} \cdot \binom{2n-1}{i_1} & (-1)^{i_1} \cdot \binom{2n-3}{i_1} & \cdots & (-1)^{i_1} \cdot \frac{1}{i_1} \\ (-1)^{i_2} \cdot \binom{2n-1}{i_2} & (-1)^{i_2} \cdot \binom{2n-3}{i_2} & \cdots & (-1)^{i_2} \cdot \frac{1}{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{i_n} \cdot \binom{2n-1}{i_n} & (-1)^{i_n} \cdot \binom{2n-3}{i_n} & \cdots & (-1)^{i_n} \cdot \frac{1}{i_n} \end{bmatrix}.$$
8.2. Computing \( d_I \). In the preceding subsection, we reduced the problem of computing \( d_I \) to computing certain minors of an infinite matrix \( B \). These minors are in fact related to special values of the shifted-Schur functions of Okounkov–Olshanski, whose definition we recall now.

**Definition 8.3.** Let \( \mu \) be a partition of length at most \( n \). Define the shifted Schur polynomial in \( n \) variables with respect to \( \mu \) as

\[
\ell \mu(x_1, \ldots, x_n) = \frac{\det((x_i + n - i)_{1 \leq i, j \leq n})}{\det((x_i + n - i)_{1 \leq i, j \leq n})},
\]

where \((i_1, i_2, \ldots, i_n) = I_n(\mu) \) (cf. Notation 2.20).\(^3\)

In [OO97], Okounkov and Olshanski introduced and studied the ring \( \Lambda^*(n) \) of shifted polynomials in \( n \) variables to be the algebra consisting of all \( n \)-variable polynomials that become symmetric after taking a shift in variables

\[
x'_i = x_i - i + \text{const}, \quad i = 1, \ldots, n.
\]

**Theorem 8.4.** (Theorem 4.1, 4.2, [OO97]) There exists an involution automorphism \( \omega : \Lambda^* \to \Lambda^* \) satisfying the following properties:

1. \( \omega(f)(\lambda) = f(\lambda') \), for all \( f \in \Lambda^* \).
2. \( \omega(s_\mu) = s_\nu^* \).

Here, \( \lambda' \) denotes the conjugate of an arbitrary partition \( \lambda \).

Notice that for every strictly increasing sequence \( I \), \( |d_I| \) can be written in the form

\[
\det(B_{i_1}^{1,3,\ldots,2n-1}) = \prod_{j=1}^{n} \frac{1}{i_{j-2}} \det((2i-1) \cdot \ldots \cdot (2i - i_j))_{1 \leq i, j \leq n}
\]

in which by convention we interpret the empty product as 1.

**Corollary 8.5.** Let \( I = (i_1, \ldots, i_n) \) be a strictly increasing sequence of integers such that \( i_k \leq 2k - 1 \).

We then have

\[
\det((2i-1) \cdot \ldots \cdot (2i - i_j))_{1 \leq i, j \leq n} = \det((2i-1) \cdot \ldots \cdot (2i - i'_j))_{1 \leq i, j \leq n}
\]

where \((i'_1, \ldots, i'_{n})\) denotes the sequence whose corresponding partition \((i'_1 - n + 1, i'_2 - n + 2, \ldots, i'_n)\) is conjugate to the partition \((i_1 - n + 1, i_2 - n + 2, \ldots, i_n)\) corresponding to \( I \).

Equivalently, we have

\[
\ell \lambda(e) = \ell \lambda'(e)
\]

where \( \lambda = \lambda(i_1, \ldots, i_n) \) and \( e = (n, \ldots, 1) \), for all \( I = (i_1, \ldots, i_n) \) with \( i_k \leq 2k - 1 \).

**Proof.** Notice that the condition \( i_k \leq 2k - 1 \) implies that the partition \((i_1 - n + 1, i_2 - n + 2, \ldots, i_n)\) is of length at most \( n \) and \( \lambda_k \leq k \) for \( k = 1, \ldots, n \). It follows that its conjugate \( \lambda' \) is also a partition of length at most \( n \) such that \( \lambda'_k \leq k \) for \( k = 1, \ldots, n \). So \( I' \) is the sequence \( I_n(\lambda') \) defined in Notation 2.20.

\(^3\)Hereafter, \((a)_n\) will always denote the falling factorial \((a)(a-1)\ldots(a-n+1)\).
By definition, we have
\[
s^*_i \cdot \cdots \cdot s^*_i \cdot \cdots \cdot s^*_i = \frac{\det \left( [2(n-i+1)-1] \cdots [2(n-i+1)-i] \right)}{\det \left( [2(n-i+1)-1] \cdots [2(n-i+1)-j+1] \right)}
\]
We may further re-write Equation (10) as
\[
S(g, r, d) = (-1)^N \sum_{I \in P_{N,r+1}} (-1)^{\deg(s_i)} \cdot 2^{2N-|I|} \cdot D_I \cdot f^{\lambda(I)}
\]
where \(N = (r+2) - (2d - g) \geq 0\), \(D_I = \det \left( [2i-1] \cdots [2i-j] \right)\) and \(f^{\lambda(I)} = \frac{|\lambda(I)|! \prod_{j=1}^{r+1} (i_j + r + g - d)!}{\prod_{j=1}^{|\lambda(I)|} (i_j)!}
\]
is the dimension of the irreducible representation of \(S_{r+1}\) indexed by \(\lambda(I)\), by the hook-length formula; see [FH13, 4.11]. Notice further that
\[
\det \left( [2i-1] \cdots [2i-j+1] \right) = \det((2i-1)^{j-1}) = V(1, 3, ..., 2r+1) = \prod_{1 \leq i < j \leq r+1} 2(j-i) = 2^{r+1} \prod_{j=0}^r j!.
\]
Since \(D_I = s^*_I \cdot \cdots \cdot s^*_I \cdot \cdots \cdot s^*_I\), we may further re-write Equation (10) as
\[
S(g, r, d) = (-4)^N \sum_{I \in P_{N,r+1}} \frac{(-2)^{-|\lambda(I)|} \cdot s^*_I \cdot (r+1, \ldots, 1) \cdot f^{\lambda(I)} \cdot \prod_{j=1}^{r+1} (j-1)!}{(N - \lambda(I))! \cdot |\lambda(I)|! \cdot \prod_{j=1}^{r+1} (i_j + r + g - d)!}
\]
\[
= \frac{(-4)^N}{N!} \sum_{m=0}^{N} (-2)^{-m} \binom{N}{m} \sum_{\lambda \subseteq \alpha} \frac{s^*_\lambda \cdot f^{\lambda} \cdot \prod_{j=1}^{r+1} (j-1)!}{(N - \lambda(I))! \cdot |\lambda(I)|! \cdot \prod_{j=1}^{r+1} (i_j + r + g - d)!}
\]
\[ \frac{(-1)^N}{N!} \prod_{\alpha=0}^{N} \alpha! \sum_{m=0}^{N} (-2)^{-m} \binom{N}{m} \sum_{\lambda \subseteq (\alpha + r + g - d)} \frac{s^\ast_\lambda(\epsilon) \cdot f^\lambda}{\prod_{j=1}^{r} (\lambda_j + r + 1 - j + (r + g - d))!} \]

where \( \epsilon = (r + 1, \ldots, 1) \). In what follows, we shall apply some known results on shifted Schur functions to express \( S(g, r, d) \) in a more explicit form. To this end, note that Okounkov and Olshanski characterize \( s^\ast_\lambda \) as a function determined by its evaluations at partitions (thought of as integer sequences) and give an expression for \( s^\ast_\lambda(\mu) \) in terms of the number of standard Young tableaux (hereafter, SYT) of certain shapes, whenever \( \lambda \subseteq \mu \).

**Theorem 8.6 (Theorem 8.1, [OO97]).** Let \( \lambda \vdash K, \mu \vdash L \) be two partitions such that \( K \leq L \) and \( \lambda \subseteq \mu \). We have

\[ \frac{\dim \mu/\lambda}{\dim \mu} = \frac{s^\ast_\lambda(\mu)}{L(L-1) \cdots (L-K+1)} \]

where \( \dim \mu/\lambda \) denotes the number of SYT of (skew) shape \( \mu/\lambda \).

For us, the most relevant shifted Schur special value is \( s^\ast_\lambda(\epsilon) \), where \( \epsilon = (r + 1, \ldots, 1) \) denotes the \( r \)-staircase. Its dependence on the parameter \( r \) is codified by the following function.

**Definition 8.7.** Given a partition \( \lambda \), let \( F_\lambda : \mathbb{Z}_{\geq -1} \to \mathbb{Q} \) denote the function \( r \mapsto s^\ast_\lambda(r+1, r, \ldots, 1) \).

In [OO97], Okounkov and Olshanski compute a generating function for \( s^\ast_\lambda(k) \), which they use to deduce a Jacobi–Trudi type formula for shifted Schur functions. Their results lead to the following proposition.

**Proposition 8.8.** Given any partition \( \lambda \) and an integer \( r \geq -1 \), we have \( F_\lambda(r) = g_\lambda(r) \) for some \( g_\lambda(x) \in \mathbb{Q}[x] \). Futhermore, the polynomials \( g_\lambda(x) \) are such that

1. \( g(k)(x) = \frac{1}{k!} (x + k + 1) \cdot (x + k) \cdot \ldots \cdot (x - k + 2) \);
2. \( \deg g_\lambda(x) = \deg g_\lambda(x) + 1 \);
3. \( \deg g_\lambda(x) = 2|\lambda| \).

**Proof.** Our point of departure is the generating series for \( s^\ast_\lambda(k) \) given in [OO97, Thm 12.1]:

\[ H^\ast(u) = \sum_{k=0}^{\infty} \frac{s^\ast_\lambda(x_1, x_2, \ldots)}{(u)_k} = \prod_{i=1}^{r+1} \frac{u + i}{u + i - x_i} \]

From (11), we deduce that

\[ \sum_{k=0}^{\infty} \frac{F_\lambda(k)}{(u)_k} = \sum_{k=0}^{\infty} \frac{s^\ast_\lambda(r + 1, r, \ldots, 1)}{(u)_k} = \prod_{i=1}^{r+1} \frac{u + i}{u + 2i - 2 - r} \]

for all \( r \geq -1 \).

\[ \sum_{k=0}^{\infty} \frac{F_\lambda(k)}{(u)_k} = \frac{s^\ast_\lambda(r + 1, r, \ldots, 1)}{(u)_k} = \prod_{i=1}^{r+1} \frac{u + i}{u + 2i - 2 - r} \]

\[ \text{for all } r \geq -1. \]

\[ \text{To clarify, we have } s^\ast_\lambda(r + 1, r, \ldots, 1) = s^\ast_\lambda(0) \text{ when } r = -1. \]
Now let \( t = u^{-1} \). We then get 
\[
\frac{1}{(u)_k} = t^k \prod_{c=0}^{k-1} (\sum_{n=0}^{\infty} (it)^n) \tag{12}
\]
and the left-hand side of (12) can be written as
\[
\sum_{k=0}^{\infty} F_k(r) t^k \prod_{c=1}^{k-1} \left( \sum_{p=0}^{\infty} (it)^p \right) = F_0(r) + F_1(r)t + \sum_{k=2}^{\infty} \left( \sum_{p=2}^{k} (F_p(r)) \cdot \left( \sum_{\sum_{a_i=k-p}}^{p-1} \prod_{a_i=k-p}^{j=a_i} \right) \right) \tag{13}
\]
Notice that
\[
\sum_{\sum_{a_i=k-p}}^{p-1} \prod_{a_i=k-p}^{j=a_i} = (1 + 2 + \ldots + (p - 1))^{k-p} = \left( \frac{p}{2} \right)^{k-p} \tag{14}
\]
Accordingly, we have
\[
F_0(r) + F_1(r)t + \sum_{k=2}^{\infty} \left( \sum_{p=2}^{k} \left( \frac{p}{2} \right)^{k-p} \right) F_p(r) t^k = \prod_{i=1}^{r+1} \frac{u + i}{u + 2i - 2 - r} \tag{15}
\]
Meanwhile, we have
\[
\prod_{i=1}^{r+1} \frac{u + i}{u + 2i - 2 - r} = (1 + t) \cdot \ldots \cdot (1 + (r + 1)t) \frac{1}{\prod_{i=1}^{r+1} (1 - (r + 2 - 2i)t) \tag{16}
\]
Let \( G_r(t) \) denote the latter \( t \)-meromorphic function; note that
\[
\log G_r(t) = \sum \log(1 + it) - \sum \log(1 + (2i - 2 - r)t) \tag{17}
\]
It follows that \( \frac{d^k}{dr^k} \log G_r(t) \mid_{t=0} = (-1)^{k-1}(k - 1)! \left( \sum i^k - \sum (2i - 2 - r)^k \right) \). Applying classical formulae for sum of consecutive powers and alternating sum of consecutive powers as in [Knu93], we deduce that
\[
\frac{d^k}{dr^k} \log G_r(t) \mid_{t=0} = f_k := \begin{cases} 
\phi_k((r + 2)(r + 1)) & \text{if } k \text{ is odd} \\
\frac{1}{2} E_k(r + 2) & \text{if } k \text{ is even}
\end{cases} \tag{18}
\]
where \( \phi_k \) is a polynomials of degree \( \frac{k+1}{2} \) such that \( g_k(0) = 0 \), and \( E_k(x) \) is the \( k \)-th Euler polynomial.\(^5\) At this stage, it is worth remarking that \( f_k(-x - 3) = f_k(x) \) always holds.

Notice that \( G_r(0) = 1 \) and \( \frac{d^k}{dr^k} \log G_r(t) \mid_{t=0} \) can always be written as an integer linear combination of \( G_r^{(\mu)}(0) \), where \( \mu \) is a partition of \( k \) and \( G_r^{(\mu)}(0) := \prod_{i=1}^{\ell(\mu)} G_r^{(i)}(0) \). An induction on \( k \) now shows that there exist polynomials \( g_k(x) \) such that \( G_r^{(k)}(0) = g_k(r) \) for all \( r \geq -1 \) and \( \deg g_k(x) \leq 2k \). And using
\[
F_0(r) + F_1(r)t + \sum_{m=2}^{\infty} \left( \sum_{k=2}^{m} \left( \frac{m}{2} \right) F_k(r) \right) t^m = \sum_{k=0}^{\infty} g_k(r) k^k \tag{19}
\]
we conclude that there exist polynomials \( g_k(x) \) such that \( g_k(r) = F_k(r) \) for \( r \geq -1 \) and \( \deg g_k(x) \leq 2k \). Another induction on \( k \) shows, moreover, that \( g_k(-x - 3) = g_k(x) \).

We now show that \( \deg g_k(x) = 2k \). To this end, note that when \( 0 \leq r + 1 < k \), [OO97, Thm 3.1] implies that \( g_k(r) = 0 \). In particular, \( x = -1, \ldots, k - 2 \) are roots of \( g_k(x) \). Using \( g_k(-x - 3) = g_k(x) \), it follows immediately that \( g_k(x) = 0 \) for \( x = -1, k - 1, \ldots, k - 2 \). On the other hand, since \( \deg g_k(x) \leq 2k \), we further get that \( g_k(x) = c_k \cdot (\prod_{i=k-1}^{k-2} (x - i)) \). To determine the coefficient \( c_k \), we evaluate \( g_k(x) \) at \( x = k - 1 \) and apply Theorem 8.6 to get
\[
c_k \cdot (2k)! = s_{k}^{*}(k, k - 1, \ldots, 1) = \frac{r^{(k-1)}}{r^{(k)}} \cdot \left( \frac{k + 1}{2} \right) = (2k - 1)!!.
\]

\(^5\) Some sources call \( \phi_k \) the \( \left( \frac{m-1}{2} \right) \)-th Faulhaber polynomial.
Thus $c_k = \frac{(2k-1)!!}{(2k-1)} = \frac{1}{2^{k-1}}$, which proves item (1).

To draw a similar conclusion for $F_\lambda(r)$ for an arbitrary partition $\lambda$, we apply the Jacobi–Trudi formula [OO97, Thm 13.1] for shifted Schur functions that relates arbitrary partitions to rectangular ones:

$$F_\mu(r) = \det \left[ \sum_{p=0}^{j-1} \binom{j-1}{p} (\mu_i - i + j - 1)_p F_{\mu_i-i+j-p}(r) \right]_{1 \leq i,j \leq \ell(\mu)}.$$  

Using the polynomiality of the $F_{(k)}$, we conclude immediately from (13) that there exist $g_\lambda(x) \in \mathbb{Q}[x]$ for which $\deg g_\lambda(x) \leq 2|\lambda|$, $g_\lambda(x) = F_\lambda(r)$ for $r \geq -1$ and $g_\lambda(-x-3) = g_\lambda(x)$, which proves item (2).

Finally, note that the $(i,j)$-th entry of the determinant in Equation 13 is a linear combination of $F_{(k)}$'s, among which $F_{\mu_i-i+j}(r)$ is of highest degree in $r$. Thus, when we expand the determinant, we find that the leading coefficient is precisely $\det(c_{\mu_i-i+j})_{1 \leq i,j \leq \ell(\mu)}$, where $c_s = \frac{1}{2^{|s|}}$. But

$$\det(c_{\mu_i-i+j})_{1 \leq i,j \leq \ell(\mu)} = \frac{1}{2^{|\mu|}} \det(\frac{1}{(\mu_i - i + j)!})_{1 \leq i,j \leq \ell(\mu)} = \frac{V(\mu_1,\mu_2-1,...,\mu_{\ell(\mu)}-\ell(\mu)+1)}{2^{|\mu|} \prod_{i=1}^{\ell(\mu)} (\mu_i + \ell(\mu) - i)!} > 0$$

which proves item (3). \hfill \Box

Evaluations of shifted Schur functions along $r$-staircase partitions are closely related to the following statistic on tableaux contained in the staircase.

**Definition 8.9.** Given any partition $\lambda$, let

$$\Pi^s_\lambda(r) := \frac{\dim \lambda \dim \epsilon(r)/\lambda}{\dim \epsilon(r)}$$

where $\epsilon(r) = (r+1,r,...,1)$ denotes the $r$-staircase.

According to Theorem 8.6, we have $\Pi^s_\lambda(r) = \frac{\ell(\lambda) s_\lambda^{r+1,r,...,1}}{\binom{\lambda}{2|^\lambda|}}$. The following result from the representation theory of symmetric groups is also standard.

**Lemma 8.10.** Given any nonnegative integer $m \leq \binom{r+2}{2}$, we have $\sum_{\lambda \vdash m} \Pi^s_\lambda(r) = 1$. Moreover, we have $\Pi^s_\lambda(r) = \Pi^s_{\lambda'}(r)$ whenever $\lambda,\lambda'$ are conjugate to one another.

The coefficients $\Pi^s_\mu(r)$ satisfy a branching rule, as follows.

**Lemma 8.11.** $\Pi^s_\mu(r) = \sum_{\mu^+ \vdash m} \binom{m}{j^\mu} \Pi^s_{\mu^+}(r)$, where $\mu^+$ runs through all partitions of $|\mu| + 1$ such that $\mu^+ = (\mu+0,...,0,\underbrace{1}_{j-th},0,...,0)$, for some $j$.

**Proof.** Given a partition $\mu \subset \epsilon$, let $T(\epsilon/\mu)$ denote the diagram corresponding to $\epsilon/\mu$. By definition, an SYT of shape $\epsilon/\mu$ must have 1 lying at an inner corner of $T(\epsilon/\mu)$, that is, at some $(i,j) \in T(\epsilon/\mu)$ for which $(i-1,j)$ and $(i,j-1)$ lie outside $T(\epsilon/\mu)$. Thus $T(\mu) \cup \{(i,j)\}$ is a diagram corresponding to some $\mu^+ \subset \epsilon$. It is not hard to see that SYT of shape $\epsilon/\mu$ and with 1 at $(i,j)$ are in bijection with SYT of shape $\epsilon/\mu^+$. Running through all inner corners of $T(\epsilon/\mu)$, we obtain a bijection between SYT of shape $\epsilon/\mu$ and SYT of shape $\epsilon/\mu^+$, for some $\mu^+ \subset \epsilon$. In other words, we have

$$f^{\epsilon/\mu} = \sum_{\mu^+, \mu^+ \subset \epsilon} f^{\epsilon/\mu^+}.$$
It follows immediately that
\[ \text{Pl}_\mu^s = \left( \sum_{\mu^+ : \mu^+ \subset \epsilon} f^{\mu^+} / f^{\epsilon} \right) \cdot f^{\mu} = \sum_{\mu^+ : \mu^+ \subset \epsilon} \left( \frac{f^{\mu^+}}{f^{\epsilon}} \right) \text{Pl}_\mu^s \cdot f^{\mu^+} . \]

\[ \square \]

The polynomials \( F_\lambda(r) \) correspondingly obey a branching rule.

**Corollary 8.12.** With the same notation as in Lemma 8.11, we have
\[ F_\mu(r) = \sum_{\mu^+} F_{\mu^+}(r) \left( \binom{r+2}{2} - |\mu| \right) . \]

**Remark 8.13.** This is a special case of [OO97, Thm 9.1]. We state it in the above form for the convenience of the reader.

Meanwhile, Okounkov and Olshanski proved yet another branching rule for shifted Schur polynomials in [OO97] which is also useful in our calculation of the special values of shifted Schur functions \( s^*_\lambda(\epsilon) \) associated with partitions \( \lambda \subset \epsilon \).

**Theorem 8.14** (Thm 11.1, [OO97]). The shifted Schur functions associated to partitions \( \nu < \mu \) (i.e., \( \mu_i \leq \nu_i \leq \mu_{i+1} \) for every \( i \)) satisfy
\[ s^*_\mu(x_1, x_2, ..., x_n) = \sum_{\nu<\mu} (x_1 \downarrow \mu / \nu) s^*_\nu(x_2, x_3, ..., x_n) \]
where \( (x_1 \downarrow \mu / \nu) \) is the generalized falling factorial (see Notation 2.13).

Consequently, we deduce:

**Corollary 8.15.**
\[ F_\mu(r) = \sum_{\nu < \mu} F_\nu(r - 1)(r \downarrow \mu / \nu) . \]

### 8.3. Large-\( r \) asymptotics of \( F(g, r, d) \) and \( S(g, r, d) \).

It follows from Lemma 8.10 that for every non-negative integer \( k \), and for sufficiently large \( r \),
\[ \text{Pl}_k^s(r) := \{ \text{Pl}_\lambda^s(r) \mid \lambda \vdash k \} \]
defines a probability distribution on the finite set of partitions of \( k \). Moreover, part (3) of Proposition 8.8 implies that \( \lim_{r \to \infty} \text{Pl}_k^s(r) > 0 \) for all \( \lambda \vdash k \). In fact, in the large-\( r \) limit, the distribution \( \text{Pl}_k^s(r) \) becomes the Plancherel distribution.

**Corollary 8.16.** The limit distribution of \( \text{Pl}_k^s(r) \) as \( r \) tends to infinity is the Plancherel distribution; that is, we have
\[ \lim_{r \to \infty} \text{Pl}_k^s(r) = \left( \frac{(f^{\lambda})^2}{k!} \right) \]
for all \( \lambda \vdash k \).

**Proof.** By part (3) of Proposition 8.8 and the usual formula for the Vandermonde polynomial, we get
\[ \lim_{r \to \infty} F_\lambda(r) = \frac{1}{2^k} \prod_{1 \leq i < j \leq \ell(\lambda)} (\lambda_i - \lambda_j - i + j) \cdot \frac{\prod_{i=1}^{\ell(\lambda)} (\lambda_i + \ell(\lambda) - i)!}{\prod_{i=1}^{\ell(\lambda)} (\lambda_i + i)!} = \frac{f^\lambda}{2^k k!} \]
in which the second equality follows from the hook-length formula [FH13, 4.11]. Since $\Pi^*_\lambda(r) = \frac{f^\lambda_{\lambda'}(r+1, r+1, \ldots)}{(r+1)!_{\lambda'}}$, we deduce that

$$\lim_{r \to \infty} \Pi^*_\lambda(r) = \lim_{r \to \infty} \frac{f^\lambda_{\lambda'}(r)}{(r+1)!_{\lambda'}} = \lim_{r \to \infty} \frac{f^\lambda_{\lambda'}(r)}{r^{2k}} = \frac{(f^\lambda)^2}{k!}. $$

Remark 8.17. The fact that we have a (probability) distribution on Young tableaux here follows from the basic identity $\sum_{\lambda' \vdash m} f^\lambda_{\lambda'} = k!$ from the representation theory of symmetric groups.

Example 8.18 (Plancherel-type statistics for small partitions). We have

1. $\lim_{r \to \infty} \Pi^*_1(r) = \Pi^*_1(r) = \{1\}$, for $k = 0, 1$ and all $r \geq 1$;
2. $\lim_{r \to \infty} \Pi^*_2(r) = \Pi^*_2(r) = \{\frac{1}{2}, \frac{1}{2}\}$, for all $r \geq 2$; and
3. $\Pi^*_3(3) = \{\Pi^*_1(3) = \frac{7}{48}, \Pi^*_2(3) = \frac{44}{48}, \Pi^*_3(3) = \frac{7}{48}\}; \lim_{r \to \infty} \Pi^*_3(r) = \{\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\}$.

We will now rewrite our class formulae for SMRC loci in a way that makes its asymptotic behavior and positivity (somewhat) more transparent. Namely, for given integers $g, r, d$, let

$$F_{g,r,d}(m) := \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} = \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!}$$

where $A = 2r + 2 + g - d$ and $(A \mid \lambda)$ is the generalized raising factorial (Notation 2.12).

Now let $\langle \bullet, \bullet \rangle_\lambda$ denote the scalar product on the ring of symmetric functions and $p_\lambda$ be the power sum symmetric functions. Applying the standard identity $\langle s_\lambda, p_1^m \rangle = \delta_{\lambda, m} f^\lambda$, we obtain

$$F_{g,r,d}(m) = \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} \left( \sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'} \right) = \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} \left( \sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'} \right) .$$

Notice that $F_{\lambda}(r) = 0$ for all but finitely many partitions, so $\sum_{\lambda} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}$ is an element in the ring $A_Q$ of symmetric functions with rational coefficients.

It follows that

$$S(g, r, d) = \left( \sum_{\lambda} \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} \right) \left( \sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} \right) = \frac{2N}{N} \left( \sum_{\lambda} \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} \right) .$$

Applying the hook content formula, we can further write

$$F_{g,r,d}(m) = \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!^2} \left( \sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} \frac{F_{\lambda}(r)}{A(\lambda)} f^\lambda_{\lambda'}}{(N-m)!m!} \right) .$$

Remark 8.19. It follows from Schur-Weyl duality for $GL_A$ and $S_m$ that $\sum_{\lambda^{\leq t} \vdash \lambda \vdash m} f^\lambda \dim S_{\lambda} C^A = A^m$. So there are in fact three probability distributions implicated in the definition of $F_{g,r,d}(m)$, each indexed by partitions of $m$. These are

1. the Plancherel distribution $\Pi_m$ for $S_m$ given by $\Pi_\lambda = \frac{(f^\lambda)^2}{m!}$, for all $\lambda \vdash m$; the
2. the $r$-th staircase-complement distribution $\Pi^*_m$ for $S_m$ given by $\Pi^*_\lambda(r) = \frac{(f^\lambda)^2}{r^{2k}}$, for all $\lambda \vdash m$; and
(3) the Schur-Weyl distribution $\text{SW}_m(A)$ for $\text{GL}_A$ and $S_m$ given by $\text{SW}_\lambda(A) = \frac{\ell^\lambda \dim S_\lambda C^A}{A^m}$, for all $\lambda \vdash m$.

As a result, our formula for $F_{g,r,d}$ may be rewritten as

$$F_{g,r,d}(m) = \frac{22N-m\binom{r+2}{2}}{(N-m)!m!A^m} \sum_{\lambda \vdash m} \frac{\Pi_{\lambda} \cdot \Pi_{r}(r)}{\text{SW}_\lambda(A)}.$$  

Clearly $S(g, r, d)$ is nonzero if and only if $\sum_{m=0}^{N}(-1)^N - m F_{g, r, d}(m) \neq 0$. Note that the class calculation carried out for $W'_r(C)$ in Example 5.2 implies that the class $q(S)$ of equation (6) is precisely

$$q(S) = \left[ \sum_{m=0}^{N} (-1)^N - m F_{g, r, d}(m) \right] \cdot [W'_r(C)] \cdot A^N.$$  

**Lemma 8.20.** The limit distribution of $\text{SW}_m$ as $A \to \infty$ is the Plancherel distribution $\Pi_m$.

**Proof.** The Weyl dimension formula yields

$$\dim S_\lambda C^A = \prod_{1 \leq i < j \leq A} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i - \lambda_j + j - i}{j - i} \cdot \prod_{k=1}^{\ell(\lambda)} \frac{(A + \lambda_k - k)\lambda_k}{(A + \lambda_k - k)\lambda_k}$$

in which $\lambda_k = 0$ for all $k > \ell(\lambda)$.

It is not hard to see that $\prod_{k=1}^{\ell(\lambda)} (\lambda_k + \ell(\lambda) - k)\lambda_k = c_\lambda(\ell(\lambda))$, where $c_\lambda(x)$ is the content polynomial of $\lambda$. Meanwhile, according to the hook-content formula we have $\prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i - \lambda_j + j - i}{j - i} = s_\lambda(1^{\ell(\lambda)}) = \frac{c_\lambda(\ell(\lambda))\ell^A}{m!}$. It follows that

$$\dim S_\lambda C^A = \frac{f_\lambda}{m!} \left( \prod_{k=1}^{\ell(\lambda)} (A + \lambda_k - k)\lambda_k \right)$$

and therefore

$$\lim_{A \to \infty} \frac{f_\lambda \dim S_\lambda C^A}{A^m} = \frac{(f_\lambda)^2}{m!} = \Pi_\lambda.$$  

Lemma 8.20 leads, in turn, to an asymptotic estimate for $S(g, r, d)$.

**Lemma 8.21.** Let $a \geq 0, N > 0$ be fixed integers. When $r \to \infty$ under the restriction that $r + g - d = a, (r+2)/2 - 2d + g = N$ always hold, we have

$$\lim_{r \to \infty} \frac{F_{g,r,d}(m)}{r^m} = \frac{4^{N-m}}{(N-m)!m!}.$$  

**Proof.** Lemma 4.7 implies that under the given restriction the values of $d, g$ are determined by the value of $r$. Moreover, $A$ tends to infinity with $r$, so by applying Corollary 8.16 and Lemma 8.20 we deduce that $\lim_{r \to \infty} \frac{\Pi_{\lambda} \cdot \Pi_{r}(r)}{\text{SW}_\lambda(A)} = \Pi_\lambda$. Here $A^m = (r + 1 + a)^m$, so by (substituting in) Equation 14 we are done.  

**Remark 8.22.** Lemma 8.21 suggests that $S(g, r, d)$ is asymptotically $\sum \frac{(-4)^{N-m}r^m}{(N-m)!m!} = \frac{1}{a}(r - 4)^N$.  

□
8.4. $F_{g,r,d}(m)$ for small values of $m$. We now apply the technical results of the final part of Subsection 8.2 to compute $F_{g,r,d}(m)$ whenever $m \leq 7$.

Lemma 8.23. Let $A = 2r + 1 + g - d$, $B = \binom{r+2}{2}$. We have

\begin{enumerate}
\item $F_{g,r,d}(0) = \frac{2^{2N}}{N!}$;
\item $F_{g,r,d}(1) = \frac{2^{2N-1}B}{N(N-1)!}$;
\item $F_{g,r,d}(2) = \frac{2^{2N-3}B_A}{(A+1)(N-2)!2!}$;
\item $F_{g,r,d}(3) = \frac{2^{2N-4}B_A}{(A+2)(N-3)!3!}[B_3(A^2 - 2) - 2B_2]$;
\item $F_{g,r,d}(4) = \frac{2^{2N-5}B_A}{(A+3)(N-4)!4!}[B_4(A^2 - 1)(A^2 - 9) + B_2(B - 3)(B - 6)(2A^2 - 3)]$;
\item $F_{g,r,d}(5) = \frac{2^{2N-6}B_A}{(A+4)(N-5)!5!}[B(B - 1)(B - 3)(72 + A^4(B - 4)(B - 2) - 20A^2(B - 4)(B - 1) + 6B(13B - 48))]$;
\item $F_{g,r,d}(6) = \frac{2^{2N-7}B_A}{(A+5)(N-6)!6!}[B(B - 1)(B - 3)((B^3 - 11B^2 + 38B - 40)A^8 - (41B^3 - 411B^2 + 1198B - 840)A^6 + 2(229B^3 - 2009B^2 + 4636B - 1680)A^4 - 2(629B^3 - 4629B^2 + 8256B - 1280)A^2 + 240B^3 - 1440B^2 + 4800B)]$; and
\item $F_{g,r,d}(7) = \frac{2^{2N-8}B_A}{(A+6)(N-7)!7!}[B(B - 1)(B - 3)(B - 6)((B^3 - 11B^2 + 38B - 40)A^8 - (71B^3 - 711B^2 + 2068B - 1440)A^6 + 14(112B^3 - 977B^2 + 2233B - 840)A^4 - 2(5699B^3 - 40149B^2 + 67266B - 9680)A^2 + 15780B^3 - 73200B^2 + 105300B - 9000)]$
\end{enumerate}

Proof. When $m \leq 2$, the results follow directly from the definition of $F_{g,r,d}(m)$. Now say $m = 3$. Lemma 8.10 implies that $2F_{3,2,1}(r) = \binom{r+2}{2}3 - 2F_{3,0,0}(r) = B_3 - 2F_{3,0,0}(r)$, and it follows that

\[
F_{g,r,d}(3) = \frac{2^{2N-3}}{(N-3)!3!} \left( \frac{F_{3,0,0}(r)}{(A + 2)_3} + \frac{B_3 - 2F_{3,0,0}(r)}{(A + 1)_3} + \frac{F_{3,1,1}(r)}{(A)_3} \right)
\]

\[
= \frac{2^{2N-3}}{(N-3)!3!} \left( \frac{12F_{3,0,0}(r)}{(A + 2)_3} + \frac{B_3}{(A + 1)_3} \right)
\]

\[
= \frac{2^{2N-3}}{(N-3)!3!} \left( \frac{(r + 4)_6}{4(A + 2)_5} + \frac{B_3}{(A + 1)_3} \right)
\]

\[
= \frac{2^{2N-3}}{(A + 2)_5(N-3)!3!}(2B_3 - 2B_2 + B_3(A^2 - 4))
\]

\[
= \frac{2^{2N-3}}{(A + 2)_5(N-3)!3!}(B_3(A^2 - 2) - 2B_2).
\]

The determination of $F_{g,r,d}(m)$ for $4 \leq m \leq 7$ is similar, but more involved. Essentially, we apply Proposition 8.8, Corollary 8.12 and Corollary 8.15 to compute all the functions $F_{3,0,0}(r)$ with $|\lambda| \leq 7$. In this process, we used MATLAB to solve some of the linear systems of equations thus arise. \hfill \Box

Now, given any partition $\lambda$ of length at most $r + 1$, let

\[
h_{\lambda}(g, r, d) := \prod_{j=1}^{r+1} ((2r + 1 + g - d) + \lambda_j - j)^{-1}.
\]

Notice that the rational function $h_{\lambda}(g, r, d)$ is precisely the reciprocal of the product of all the shifted contents $A + j - i$, for $(i, j) \in D(\lambda)$. The following result is straightforward to verify.

Lemma 8.24. We have

\[
h_{\lambda}(g, r, d) = [\lambda + 1 + j + (2r + 1 + g - d)]h_{\lambda^+}(g, r, d).
\]
for all partitions $\lambda$ and $\lambda^+$ related by $\lambda^+ - \lambda = (0, \ldots, 0, \frac{1}{j-\text{th}}, 0, \ldots, 0)$.

Calculating the class $q(S)$ explicitly becomes difficult as soon as $N \geq 8$. However, applying the branching rule of Lemma 8.11 for $\Pi^*_\mu$, we may conclude that $q(S)$ is a positive class when $r$ is large.

**Proposition 8.25.** Fix a choice of $N = \binom{r+2}{2} - 2d + g \geq 1$. The SMRC degree $S(g, r, d)$ is a positive rational number whenever $r \geq 12N - 2$.

**Proof.** Let $a_{g,r,d}(m) := \frac{2^{N-m}(\binom{r+2}{2})}{(N-m)!}$, where $N = \binom{r+2}{2} - 2d + g \geq 1$, whenever $m \leq N$. Whenever $m < N$, we further set

$$c_{m+1}(r) := \frac{a_{g,r,d}(m + 1)}{a_{g,r,d}(m)} = \frac{N-m}{2} \cdot \frac{\binom{r+2}{2} - m}{m + 1}.$$ 

For any fixed value of $N$, $c_{m+1}(r)$ is a quadratic polynomial in $r$ with highest-degree coefficient equal to $\frac{N-m}{4(m+1)} > 0$.

Applying Lemma 8.11, we now write

$$\sum_{\lambda^+ \in \epsilon} r_{\lambda} c_{m}(\mu) = \sum_{\lambda^+ \in \epsilon} \left( \sum_{\lambda^+ \in \epsilon} [\lambda_j + 1 - j + (2r + 1 + g - d)] \cdot \left( \frac{f_{\lambda}}{f_{\lambda^+}} \right) \cdot r_{\lambda} h_{\lambda^+} \right)$$

where $\lambda^+ - \lambda = (0, \ldots, 0, \frac{1}{j-\text{th}}, 0, \ldots, 0)$. Equivalently, we have

$$\sum_{\lambda^+ \in \epsilon} r_{\lambda} h_{\lambda} = \sum_{\mu^+ \in \epsilon} \left( \sum_{\mu^- \in \epsilon} [\mu_j - j + (2r + 1 + g - d)] \cdot \left( \frac{f_{\mu^+}}{f_{\mu^-}} \right) \cdot r_{\mu} h_{\mu} \right)$$

where $\mu - \mu^- = (0, \ldots, 0, \frac{1}{j-\text{th}}, 0, \ldots, 0)$.

Now say $N = 2p - 1$ is odd. Define $A_i := -F_{g,r,d}(2i - 2) + F_{g,r,d}(2i - 1)$, for $i = 1, \ldots, p$. $S(g, r, d)$ then decomposes as $S(g, r, d) = A_1 + \ldots + A_p$, and equation (16) implies that

$$A_i = a_{g,r,d}(2i - 2) \cdot \sum_{\lambda^+ \in \epsilon} r_{\lambda} \left( c_{2i-1}(r) - \sum_{\lambda^-} [\lambda_j - j + (2r + 1 + g - d)] \cdot \left( \frac{f_{\lambda^-}}{f_{\lambda}} \right) h_{\lambda}(g, r, d). \right)$$

Here $r_{\lambda} \in \mathbb{Q} \cap (0, 1]$, while the quotients $\frac{f_{\lambda^-}}{f_{\lambda}}$ are positive rational numbers such that $\sum_{\lambda^-} \frac{f_{\lambda^-}}{f_{\lambda}} = 1$, by the usual branching rule. More importantly, we have

$$\lambda_j - j + (2r + 1 + g - d) \leq 3r + 1$$

since $\lambda \in \epsilon$ and $d \geq g$. Therefore, provided $c_{2i-1}(r) > 3r + 1$ holds for all $i$, we get $S(g, r, d) > 0$. Since $c_{2i-1}(r) \geq c_N(r)$, it further suffices to have $c_N(r) > 3r + 1$. Now $c_N(r)$ is a quadratic polynomial in $r$, with positive highest-degree coefficient, the desired positivity follows in this case.

For $N = 2p$ even, decompose $S(g, r, d)$ as $B_0 + B_1 + \ldots + B_p$, where $B_0 = F_{g,r,d}(0)$ and $B_i = -F_{g,r,d}(2i - 1) + F_{g,r,d}(2i)$ for $i \geq 1$. An argument analogous to that used in the case of odd $N$ yields the desired conclusion.
Finally, the inequality $c_N(r) > 3r + 1$, which can be written as $r^2 + r(3 - 12N) + 4 - 6N > 0$, give us the desired result. The largest root of this quadratic polynomial in $r$ is less than $12N - 2$. □

8.5. **Positivity of SMRC class formulae for small values of $N$.** In this section, we obtain explicitly positive universal formulae for $q(S)$ whenever $N \leq 7$.

**Proposition 8.26.** The class $q(S)$ of equation (6) is strictly positive when $N \leq 2$ except when either

(i) $N = 1$, and $(g, r, d) \in \{(1, 2, 5), (5, 3, 7)\}$; or

(ii) $N = 2$, and $(g, r, d) \in \{(2, 3, 5), (7, 4, 10)\}$

and $q(S)$ is unconditionally strictly positive whenever $3 \leq N \leq 7$.

**Proof.** In Lemma 8.23 we computed the functions $F_{g, r, d}(m)$ explicitly for $m \leq 7$. Since $S(g, r, d)$ is an alternating sum of all $F_{g, r, d}(m)$ with $m \leq N$, we thus obtain explicit formulas for $S(g, r, d)$ whenever $N \leq 7$.

Applying Proposition 8.25, we conclude that $S(g, r, d)$ is positive whenever $r \geq 12N - 2$.

Finally, we check the positivity for the finitely many remaining feasible cases of $(g, r, d)$ using MATLAB (see the ancillary file at https://drive.google.com/file/d/1Az5WOZyoa_UzQvktT7Kui1TpbddL4ANn/view?usp=sharing). For each of these cases, we compute

$$S^*(g, r, d) = S(g, r, d) \cdot \frac{\prod_{\alpha=0}^{r} (\alpha + r + g - d)!}{\prod_{\alpha=0}^{r} \alpha!}$$

and conclude. □

**References**

[ACGH85] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, *Geometry of algebraic curves, volume 1*, Springer, 1985.

[AF11] M. Aprodu and G. Farkas, *Koszul cohomology and applications to moduli*, Grassmannians, Moduli Spaces and Vector Bundles 14 (2011), 25–50.

[BF98] A. Bertram and B. Feinberg, *On stable rank two bundles with canonical determinant and many sections*, Algebraic Geometry, Papers for Europroj Conferences in Catania and Barcelona (1998), 259–269.

[CF15] C. Ciliberto and F. Flamini, *Extensions of line bundles and Brill-Noether loci of rank-two vector bundles on a general curve*, Rev. Roumaine Math. Pures Appl. 60 (2015), 201–255.

[Far08] G. Farkas, *Higher ramification and varieties of secant divisors on the generic curve*, J. Lond. Math. Soc. 78(2) (2008), 418–440.

[Far09] ———, *Koszul divisors on moduli spaces of curves*, Amer. J. Math. 131, No. 3 (2009), 819–867.

[Far18] ———, *Effective divisors on hurwitz spaces and moduli of curves*, arXiv:1804.01898 (2018).

[FH13] W. Fulton and J. Harris, *Representation theory: a first course*, vol. 129, Springer Science & Business Media, 2013.

[FO11] G. Farkas and A. Ortega, *The maximal rank conjecture and rank two Brill-Noether theory*, Pure Appl. Math. Q. 7 (2011), 1265–1296.

[FO12] ———, *Higher rank Brill-Noether theory on sections of K3 surfaces*, Int. J. Math. 23 (2012).

[Ful98] W. Fulton, *Intersection Theory*, 2nd ed., Springer, 1998.

[Gro58] A. Grothendieck, *Sur quelques propriétés fondamentales en théorie des intersections*, Séminaire Claude Chevalley 3 (1958), no. 4, 1–36.
[JP18] D. Jensen and S. Payne, *On the strong maximal rank conjecture in genus 22 and 23*, arXiv:1808.01285v2 (2018).

[Knu93] D. E. Knuth, *Johann Faulhaber and sums of powers*, Math. Comp. 61 (1993), no. 203, 277–294.

[Lar17] E. Larson, *The maximal rank conjecture*, arXiv:1711.04906 (2017).

[Las88] A. Lascoux, *Interpolation de Lagrange*, On Orthogonal Polynomials and their Applications 1 (1988), 95–101.

[LLT89] D. Laksov, A. Lascoux, and A. Thorup, *On Giambelli’s theorem on complete correlations*, Acta Math. 162 (1989), no. 1, 143–199.

[LN83] H. Lange and M. S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves*, Math. Ann. 266 (1983), 55–72.

[LNP16] H. Lange, P. Newstead, and S. Park, *Non-emptiness of Brill-Noether loci in $M(2,k)$*, Comm. Algebra 44(2) (2016), 746–767.

[LOTiBZ18] F. Liu, B. Osserman, M. Teixidor i Bigas, and N. Zhang, *The strong maximal rank conjecture and moduli spaces of curves*, arXiv:1808.01290v3 (2018).

[Mac98] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford university press, 1998.

[Mat65] A. P. Mattuck, *Secant bundles on symmetric products*, Amer. J. Math. 87 (1965), 779–797.

[MS85] S. Mukai and F. Sakai, *Maximal subbundles of vector bundles on a curve*, Manuscripta Math. 52 (1985), 251–256.

[Muk93] S. Mukai, *Curves and Grassmannians*, Algebraic Geometry and Related Topics (1993), 19–40.

[Muk95] ______, *Vector bundles and Brill-Noether theory*, Current Topics in Complex Algebraic Geometry MSRI Publications, vol. 28 (1995), 145–158.

[OO97] A. Okounkov and G. Olshanski, *Shifted Schur functions*, Algebra i Analiz 9 (1997), no. 2, 73–146.

[Pra88] P. Pragacz, *Enumerative geometry of degeneracy loci*, Ann. Sci. Éc. Norm. Supér., vol. 21, Elsevier, 1988, pp. 413–454.

[SF99] R. P. Stanley and S. Fomin, *Enumerative combinatorics, volume 2*, Cambridge Studies in Advanced Mathematics, Cambridge university press, 1999.

[TiB04] M. Teixidor i Bigas, *Rank two vector bundles with canonical determinant*, Math. Nachr. 265 (2004), 100–106.

[TiB08] ______, *Petri map for rank two vector bundles with canonical determinant*, Compos. Math. 144, no.3 (2008), 705–720.

[Tu17] L. W. Tu, *Computing the Gysin map using fixed points*, Algebra Number Theory, Springer, 2017, pp. 135–160.

[Zha16] N. Zhang, *Towards the Bertram-Feinberg-Mukai conjecture*, J. Pure Appl. Algebra 220 (2016), 1588–1654.