Bihamiltonian Structure of the Two-component Kadomtsev-Petviashvili Hierarchy of type B

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Abstract

We employ a Lax pair representation of the two-component BKP hierarchy and construct its bihamiltonian structure with \( R \)-matrix techniques.

Key words: BKP hierarchy, Hamiltonian structure, \( R \)-matrix

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy of type B (BKP for short) was introduced in \([6, 7]\), and generalized to multi-component cases by Date, Jimbo, Kashiwara, Miwa \([4]\) in the form of bilinear equations. Among these multi-component integrable systems, the two-component BKP hierarchy is of special interest.

In the original definition of the two-component BKP hierarchy, the solution space of tau functions can be regarded as the vacuum orbit in the two-component neutral free fermionic Fock representation of the infinite dimensional Lie algebra \( D_\infty \) \([5, 14]\), which corresponds to the infinite Dynkin diagram of type D \([15]\). The Lie algebra \( D_\infty \) can be reduced to the affine Lie algebra \( D_n^{(1)} \) under the so-called \((2n-2,2)\)-reduction in \([5]\), see also \([14, 17]\). This reduction reduces the two-component BKP hierarchy to a hierarchy that is equivalent with the Kac-Wakimoto hierarchy corresponding to the principal vertex operator realization of the basic representation of \( D_n^{(1)} \), the Drinfeld-Sokolov hierarchy associated to the Lie algebra \( D_n^{(1)} \) and the zeroth vertex \( c_0 \) of its Dynkin diagram, as well as the Givental-Milanov hierarchy satisfied by the total descendant for the \( D_n \) singularity,
see [9] [12] [13] [16] [19] [26] and references therein. Such a reduction is analogous to the one that reduces the KP hierarchy to the $n$th Gelfand-Dickey hierarchy (see e.g. [8]) that corresponds to the reduction of Lie algebras: $A_{\infty} \mapsto A_n^{(1)}$. So in this sense to compare the two-component BKP hierarchy with the KP hierarchy would deepen our understanding of integrable hierarchies and relevant theories, such as Jacobi/Prym varieties in algebraic geometry and Landau-Ginzburg Models of topological strings, see e.g. [22] [23] [24].

In this article our aim is to study the two-component BKP hierarchy from the viewpoint of Hamiltonian structures. To our best knowledge, this topic has not been considered in the literature, possibly for the reason that the KP-analogue Lax pair representation of the two-component BKP hierarchy was unknown. Recall that the two-component BKP hierarchy was defined to be the bilinear equation of a single tau function:

$$\text{res}_z z^{-1} X(t; z) \tau(t, \hat{t}) X(t', -z) \tau(t', \hat{t}') = \text{res}_z z^{-1} X(t; z) \tau(t, \hat{t}) X(t'; -z) \tau(t', \hat{t}'), \quad (1.1)$$

where $t = (t_1, t_3, t_5, \cdots)$, $\hat{t} = (\hat{t}_1, \hat{t}_3, \hat{t}_5, \cdots)$, and $X$ is a vertex operator given by

$$X(t; z) = \exp \left( \sum_{k \in \mathbb{Z}^{\text{odd}}} t_k z^k \right) \exp \left( - \sum_{k \in \mathbb{Z}^{\text{odd}}} \frac{2}{k} z^k \frac{\partial}{\partial t_k} \right).$$

Here the residue of a Laurent series is taken as $\text{res}_z (\sum_{i \in \mathbb{Z}} f_i z^i) = f_{-1}$. In [22] Shiota proposed a scalar Lax representation of the hierarchy (1.1), though this did not attract much attention as it contains pseudo-differential operators with derivations of two spatial variables. Recently, a Lax pair representation of the two-component BKP hierarchy was found by Liu, Zhang and one of the authors [19]. It was shown that the hierarchy (1.1) can be redefined by certain extension of the following Lax equations (see Section 3 below):

$$\frac{\partial P}{\partial t_k} = [(P^k)_+, P], \quad \frac{\partial \hat{P}}{\partial t_k} = [(P^k)_+, \hat{P}], \quad (1.2)$$

$$\frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, P], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{P}] \quad (1.3)$$

with $k \in \mathbb{Z}^{\text{odd}}$, where

$$P = D + \sum_{i \geq 1} u_i D^{-i}, \quad \hat{P} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 1} \hat{u}_i D^i \quad \text{with} \quad D = \frac{d}{dx}.$$
are pseudo-differential operators such that \( P^* = -DPD^{-1}, \hat{P}^* = -D\hat{P}D^{-1} \). Note that the first equation in (1.2) is just the Lax formulation of the BKP hierarchy appearing in [6]. Our arguments will be based on the Lax pair representation \((1.2), (1.3)\) of the two-component BKP hierarchy.

Observe that the expression \((1.2), (1.3)\) is similar to the Lax pair representation of the two-dimensional Toda hierarchy [25], which carries a tri-Hamiltonian structure [1]. Following the idea of [1], we want to use the \(R\)-matrix theory to construct Hamiltonian structures of the two-component BKP hierarchy \((1.2), (1.3)\).

We are also motivated by the recent work [2], in which Carlet, Dubrovin and Mertens constructed an infinite-dimensional Frobenius manifold underlying the two-dimensional Toda hierarchy. Due to the similarity of the Lax representations mentioned above, we expect that there also exists an infinite dimensional Frobenius manifold that underlies the two-component BKP hierarchy. A hint is that the potential \( F \) (in the notion of [23], namely the dispersionless limit of the logarithm of the tau function, see Section 3 below) of the dispersionless two-component BKP hierarchy was discovered to satisfy certain infinite-dimensional WDVV-type associativity equation [3]. While in the finite-dimensional case, the concept of Frobenius manifolds [10] is known as a geometric description of the WDVV equations, and associated to certain nondegenerate Frobenius manifold there lies a Poisson pencil so that a bihamiltonian hierarchy can be constructed [11]. We hope that this article and follow-up work might help to understand the theory of infinite-dimensional manifolds.

This article is arranged as follows. In next section we recall the definition and some properties of pseudo-differential operators introduced in [19], and in Section 3 we recall the Lax pair representation of the two-component BKP hierarchy. In Sections 4 and 5, an \(R\)-matrix will be used to construct Poisson brackets on an algebra of pseudo-differential operators, and then after appropriate reductions of the Poisson brackets we obtain a bihamiltonian structure of the two-component BKP hierarchy. In Section 6 we compute the dispersionless limit of this bihamiltonian structure. Finally some remarks are given in Section 7.

## 2 Pseudo-differential operators

For preparation we recall the notion of pseudo-differential operators over a ring with certain gradation as introduced in [19].

Let \( \mathcal{A} \) be a ring, and \( D : \mathcal{A} \to \mathcal{A} \) be a derivation. The algebra of usual
pseudo-differential operators is
\[ D^- = \left\{ \sum_{i<\infty} f_i D^i \mid f_i \in \mathcal{A} \right\}. \tag{2.1} \]
This algebra is topologically complete with a topological basis given by the following filtration:
\[ \cdots \subset D^-_{(d-1)} \subset D^-_{(d)} \subset D^-_{(d+1)} \subset \cdots, \quad D^-_{(d)} = \left\{ \sum_{i\leq d} f_i D^i \mid f_i \in \mathcal{A} \right\}, \]
and in this algebra two elements are multiplied as series of the following product of monomials:
\[ f D^i \cdot g D^j = \sum_{r \geq 0} \binom{i}{r} f D^r(g) D^{i+j-r}, \quad f, g \in \mathcal{A}. \tag{2.2} \]
Assume there is a gradation on \( \mathcal{A} \) such that
\[ \mathcal{A} = \prod_{i \geq 0} \mathcal{A}_i, \quad \mathcal{D} : \mathcal{A}_i \rightarrow \mathcal{A}_{i+1}, \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}, \]
and consider the linear space
\[ \mathcal{D} = \left\{ \sum_{i \in \mathbb{Z}} f_i D^i \mid f_i \in \mathcal{A} \right\}. \]
Obviously \( \mathcal{D}^- \subset \mathcal{D} \).
For any \( k \in \mathbb{Z} \), denote by \( \mathcal{D}_k \) the set of homogeneous operators with degree \( k \) in \( \mathcal{D}^- \), i.e.,
\[ \mathcal{D}_k = \left\{ \sum_{i \leq k} f_i D^i \mid f_i \in \mathcal{A}_{k-i} \right\}. \]
Let \( \mathcal{D}^+ \) be a subspace of \( \mathcal{D} \) that reads
\[ \mathcal{D}^+ = \bigcup_{d \in \mathbb{Z}} \mathcal{D}^+_{(d)}, \quad \mathcal{D}^+_{(d)} = \prod_{k \geq d} \mathcal{D}_k, \tag{2.3} \]
and \( \mathcal{D}^+ \) have a topological basis given by the filtration
\[ \cdots \supset \mathcal{D}^+_{(d-1)} \supset \mathcal{D}^+_{(d)} \supset \mathcal{D}^+_{(d+1)} \supset \cdots. \]
In fact, every element \( A \in \mathcal{D}^+ \) has the following normal expansion \[19\]
\[ A = \sum_{i \in \mathbb{Z}} \left( \sum_{j \geq \max\{0, m-i\}} a_{i,j} \right) D^i, \quad a_{i,j} \in \mathcal{A}_j \]
with some integer $m$. Note that $D_k \cdot D_l \subset D_{k+l}$ according to the multiplication defined by (2.2), then this multiplication can be naturally extended to $D^+$ such that $D^+$ becomes an associative algebra.

**Definition 2.1 (19)** Elements of $D^-$ (resp. $D^+$) are called pseudo-differential operators of the first type (resp. the second type) over $A$. The intersection of $D^-$ and $D^+$ in $D$ is denoted by

$$D^b = D^- \cap D^+,$$

and its elements are called bounded pseudo-differential operators.

Sometimes to indicate the ring $A$ and the derivation $D$, we will use the notations $D^\pm(A, D)$ instead of $D^\pm$.

Pseudo-differential operators of the second type have similar properties to those of the operators in $D^-$. For any operator

$$A = \sum_{i \in \mathbb{Z}} f_i D^i \in D^\pm,$$

its positive part, negative part, residue and adjoint operator are defined to be respectively

$$A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i,$$

$$\text{res} A = f_{-1}, \quad A^* = \sum_{i \in \mathbb{Z}} (-D)^i \cdot f_i.$$

Note that the formulae (2.5) give two projections of $D$, and they induce the following decompositions of spaces

$$D^\pm = (D^\pm)_+ \oplus (D^\pm)_-.$$

Particularly one sees that

$$(D^-)_+ \subset D^b, \quad (D^+)_- \subset D^b.$$

An element $A$ of $(D^\pm)_+$ is called a **differential operator**. Let $A(f)$ denote the action of a differential operator $A$ on $f \in A$.

Elements of the quotient space $\mathcal{F} = A / (D(A) \oplus \mathbb{C})$ are called **local functionals**, which are denoted as

$$\int f \, dx = f + D(A), \quad f \in A.$$
Then the pairing
\[ \langle A, B \rangle = \langle AB \rangle \] (2.10)
defines an inner product on each of \( \mathcal{D}^\pm \).

Given any subspace \( S \subset \mathcal{D}^\pm \), we denote by \( S^\ast \) the dual space of \( S \) (c.f. the notation of adjoint operators). Via the above inner product, we have the following identification of dual spaces
\[ (\mathcal{D}^\pm)^* = \mathcal{D}^\pm. \] (2.11)

Consider the decompositions (2.7), it is easy to see that
\[ ((\mathcal{D}^\pm)_\pm)^* = (\mathcal{D}^\pm)_\mp. \]

We also decompose \( \mathcal{D}^\pm \) as
\[ \mathcal{D}^\pm = \mathcal{D}^\pm_0 \oplus \mathcal{D}^\pm_1, \] (2.12)
where
\[ \mathcal{D}^\pm_\nu = \{ A \in \mathcal{D}^\pm \mid A^* = (-1)^\nu A \}, \quad \nu = 0, 1. \]

Since \( \langle A \rangle = -\langle A^* \rangle \) for any \( A \in \mathcal{D}^\pm \), then the dual subspaces of \( \mathcal{D}^\pm_\nu \) read
\[ (\mathcal{D}^\pm_\nu)^* = \mathcal{D}^\pm_{1-\nu}, \quad \nu = 0, 1. \] (2.13)

For more details on properties of pseudo-differential operators one can refer to [8, 19].

3 The two-component BKP hierarchy

The two types of pseudo-differential operators serve in [19] to give a scalar Lax pair representation of the two-component BKP hierarchy, which is reviewed as follows.

Let \( \tilde{M} \) be an infinite-dimensional manifold with local coordinates
\[ (a_1, a_3, a_5, \ldots, b_1, b_3, b_5, \ldots), \]
and \( \tilde{A} \) be the algebra of differential polynomials on \( \tilde{M} \):
\[ \tilde{A} = C^\infty(\tilde{M})[[a_k^{(s)}, b_k^{(s)} \mid k \in \mathbb{Z}^\text{odd}, s \geq 1]]. \]

We assign a gradation on \( \tilde{A} \) by
\[ \deg f = 0 \text{ for } f \in C^\infty(\tilde{M}), \quad \deg a_k^{(s)} = \deg b_k^{(s)} = s \]
which make $\mathcal{A}$ a topologically complete algebra:

$$\mathcal{A} = \prod_{i \geq 0} \mathcal{A}_i, \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}.$$ 

Note that this gradation is induced from the derivation $D : \mathcal{A} \rightarrow \mathcal{A}$, 

$$D = \sum_{s \geq 0} \sum_{k \in \mathbb{Z}^+} \left( a_k^{(s+1)} \frac{\partial}{\partial a_k^{(s)}} + b_k^{(s+1)} \frac{\partial}{\partial b_k^{(s)}} \right)$$

with $a_k^{(0)} = a_k$, $b_k^{(0)} = b_k$. So one can define the algebras $\mathcal{D}^\pm = D^\pm(\mathcal{A}, D)$ of pseudo-differential operators as was done in last section.

Introduce two operators

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i} \in \mathcal{D}^-, \quad \Psi = 1 + \sum_{i \geq 1} b_i D^i \in \mathcal{D}^+,$$ 

(3.1)

where $a_2, a_4, a_6, \ldots, b_2, b_4, b_6, \ldots \in \mathcal{A}$ are determined by the following conditions

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \Psi^* = D\Psi^{-1}D^{-1}. \quad (3.2)$$

Then the two-component BKP hierarchy (1.1) can be redefined to be

$$\frac{\partial \Phi}{\partial t_k} = -(P_k)^{-1} \Phi, \quad \frac{\partial \Psi}{\partial t_k} = (P_k + \delta_{k1} \hat{P}^{-1}) \Psi, \quad (3.3)$$

$$\frac{\partial \Phi}{\partial \hat{t}_k} = -(\hat{P}_k)^{-1} \Phi, \quad \frac{\partial \Psi}{\partial \hat{t}_k} = (\hat{P}_k)^{-1} \Psi, \quad (3.4)$$

where $k \in \mathbb{Z}^{odd}_+$, and the operators $P, \hat{P}$ read

$$P = \Phi D \Phi^{-1} \in \mathcal{D}^-, \quad \hat{P} = \Psi D^{-1} \Psi^{-1} \in \mathcal{D}^+.$$ 

(3.5)

The operators $P, \hat{P}$ have the following expressions:

$$P = D + \sum_{i \geq 1} u_i D^{-i}, \quad \hat{P} = D^{-1} u_{-1} + \sum_{i \geq 1} \hat{u}_i D^i,$$ 

(3.6)

with $u_{-1} = (\Psi^{-1})^*(1)$, and they satisfy

$$P^* = -DPD^{-1}, \quad \hat{P}^* = -D\hat{P}D^{-1}, \quad (3.7)$$

which implies

$$(P_k)^-(1) = 0, \quad (\hat{P}_k)^-(1) = 0, \quad k \in \mathbb{Z}^{odd}_+. \quad (3.8)$$
Observe that the coefficients of $P$ and $\hat{P}$ are elements of the algebra $\bar{A}$, and among these coefficients the ones with odd subscript are independent, while the others are determined by the conditions (3.7). Assume that

$$u = (u_1, u_3, \ldots, \hat{u}_{-1}, \hat{u}_1, \hat{u}_3, \ldots) \quad (3.9)$$

serves as a coordinate of some infinite-dimensional manifold $M$, then the algebra $\mathcal{A}$ of differential polynomials on $M$ reads

$$\mathcal{A} = C^\infty(M)[[u^{(s)} | s \geq 1]],$$

which is a subalgebra of $\bar{A}$. Similarly as above, one can assign a gradation to $\mathcal{A}$ that is induced from the derivation

$$D : \mathcal{A} \to \mathcal{A}, \quad D = \sum_{s \geq 0} u^{(s+1)} \frac{\partial}{\partial u^{(s)}}$$

with $u^{(0)} = u$, and then define the algebras $\mathcal{D}^\pm = \mathcal{D}^\pm(\mathcal{A}, D)$ of pseudo-differential operators over $\mathcal{A}$.

Clearly $P \in \mathcal{D}^-$, $\hat{P} \in \mathcal{D}^+$. When the two-component BKP hierarchy (3.3), (3.4) is restricted from $\bar{A}$ to $\mathcal{A}$, it becomes

\[
\begin{align*}
\frac{\partial P}{\partial t_k} &= [(P^k)_+, P], \quad \frac{\partial \hat{P}}{\partial t_k} = [(P^k)_+, \hat{P}], \\
\frac{\partial P}{\partial \hat{t}_k} &= [-(\hat{P}^k)_-, P], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{P}]
\end{align*} \quad (3.10)
\]

\[
\begin{align*}
\frac{\partial \hat{P}}{\partial t_k} &= [-(\hat{P}^k)_-, \hat{P}], \quad \frac{\partial P}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, P]
\end{align*} \quad (3.11)
\]

with $k \in \mathbb{Z}_{\text{odd}}$. In the present article we regard the two-component BKP hierarchy as the evolutionary equations (3.10), (3.11) defined on the algebra $\mathcal{A}$.

In fact, the hierarchy (3.10), (3.11) possesses a tau function $\tau = \tau(t, \hat{t})$ defined by

$$\omega = d(2 \partial_x \log \tau) \quad \text{with} \quad x = t_1, \quad (3.12)$$

where $\omega$ is the following closed 1-form:

$$\omega = \sum_{k \in \mathbb{Z}_{\text{odd}}} (\text{res } P^k \, dt_k + \text{res } \hat{P}^k \, d\hat{t}_k).$$

This tau function solves the bilinear equation (1.1), which is the original definition of the two-component BKP hierarchy.

**Remark 3.1** The dispersionless limit of the flows (3.10), (3.11) first exists in [23], where Takasiki also considered the dispersionless limit of the logarithm of the tau function as given in (3.12). Inspired by [23], Chen and Tu [3] discovered that the leading term of $\log \tau$ solves an infinite-dimensional associativity equation of WDVV type.
4  \( R \)-matrix and pseudo-differential operators

To show that the two-component BKP hierarchy (3.10), (3.11) possesses a bihamiltonian structure, we need to construct a Poisson pencil for it. The method is to use the standard \( R \)-matrix theory and introduce Poisson brackets on a Lie algebra (see [21], [18], [20] and references therein), then restrict the Poisson brackets to certain submanifold of the Lie algebra. Our approach is similar with that used by Carlet [1] for the two-dimensional Toda hierarchy.

We first recall the \( R \)-matrix formalism. Let \( g \) be a Lie algebra, and \( R : g \to g \) be a linear transformation. Then \( R \) is called an \( R \)-matrix [21] on \( g \) if it defines a Lie bracket by

\[
[X, Y]_R = [R(X), Y] + [X, R(Y)], \quad X, Y \in g.
\] (4.1)

A sufficient condition for a transformation \( R \) being an \( R \)-matrix is that \( R \) solves the modified Yang-Baxter equation [21]

\[
[R(X), R(Y)] - R([X, Y]_R) = -[X, Y]
\] (4.2)

for all \( X, Y \in g \).

Assume that \( g \) is an associative algebra, with the Lie bracket defined naturally by commutators, and there is a map \( \langle \rangle : g \to \mathbb{C} \) that defines a non-degenerate symmetric invariant bilinear form (inner product) \( \langle \rangle \) by

\[
\langle X, Y \rangle = \langle XY \rangle = \langle YX \rangle, \quad X, Y \in g.
\]

Via this inner product one can identify \( g \) with its dual space \( g^* \). The tangent and the cotangent bundles of \( g \) are denoted by \( Tg \) and \( T^*g \) respectively, with fibers \( T_Ag = g \) and \( T^*_Ag = g^* \) at every point \( A \in g \).

Let \( R^* \) be the adjoint transformation of \( R \) with respect to the above inner product. We introduce the notations of the symmetric and the anti-symmetric parts of \( R \) respectively as

\[
R_s = \frac{1}{2}(R + R^*), \quad R_a = \frac{1}{2}(R - R^*).
\]

The \( R \)-matrix formalism is briefly stated as follows. Given an \( R \)-matrix \( R : g \to g \) that satisfies certain conditions, there define three compatible Poisson brackets on \( g \), say, the linear, the quadratic and the cubic brackets in the notion of [18], [20].

In particular, let us recall the quadratic bracket, which will be used to construct a Poisson pencil for the two-component BKP hierarchy.

**Lemma 4.1** ([18], [20]) Let \( f, g \) be two arbitrary smooth functions on \( g \), and \( \nabla f, \nabla g \in T^*_Ag \) be their gradients at any point \( A \in g \). Given a linear
transformation $R : \mathfrak{g} \to \mathfrak{g}$, if both $R$ and its anti-symmetric part $R_a$ satisfy the modified Yang-Baxter equation (4.2), then the quadratic bracket

$$\{ f, g \}(A) = \frac{1}{4} \left( \langle [A, \nabla f], R(A \nabla g + \nabla g \cdot A) \rangle - \langle [A, \nabla g], R(A \nabla f + \nabla f \cdot A) \rangle \right)$$

(4.3)
defines a Poisson bracket on $\mathfrak{g}$.

Note that the bracket (4.3) can be rewritten as

$$\{ f, g \}(A) = \langle \nabla f, \mathcal{P}_A(\nabla g) \rangle,$$

where $\mathcal{P} : T^* \mathfrak{g} \to T \mathfrak{g}$ is a Poisson tensor given by

$$\mathcal{P}_A(\nabla g) = -\frac{1}{2} A \{ R_a(A \nabla g) + R_a(\nabla g \cdot A) \} + \frac{1}{2} \{ R_a(A \nabla g) + R_a(\nabla g \cdot A) \} A.$$

(4.4)

Henceforth we take $\mathfrak{g}$ to be the algebra

$$\mathfrak{D} = \mathfrak{D}^- \times \mathfrak{D}^+,$$

where $\mathfrak{D}^-$ and $\mathfrak{D}^+$ are the sets of pseudo-differential operators of the first type and the second type over some differential algebra $\mathfrak{A}$ as defined in Section 2. In $\mathfrak{D}$ the elements read $X = (X, \hat{X})$, and the operations are defined diagonally as

$$(X, \hat{X}) + (Y, \hat{Y}) = (X + Y, \hat{X} + \hat{Y}), \quad (X, \hat{X})(Y, \hat{Y}) = (XY, \hat{X}\hat{Y}).$$

So $\mathfrak{D}$ is indeed an associative algebra. Moreover, the algebra $\mathfrak{D}$ is equipped with an inner product defined by

$$\langle (X, \hat{X}), (Y, \hat{Y}) \rangle = \langle (X, \hat{X})(Y, \hat{Y}) \rangle = \langle X, Y \rangle + \langle \hat{X}, \hat{Y} \rangle,$$

see (2.9), (2.10). Via this inner product we have the identification of dual spaces as above:

$$\mathfrak{D}^* = (\mathfrak{D}^-)^* \times (\mathfrak{D}^+)^* = \mathfrak{D}^- \times \mathfrak{D}^+ = \mathfrak{D}.$$

Inspired by [1], we introduce a linear transformation of $\mathfrak{D}$ as follows

$$R : \mathfrak{D} \to \mathfrak{D}, \quad (X, \hat{X}) \mapsto (X_+ - X_-, 2\hat{X}_- - \hat{X}_+ + 2X_+).$$

(4.5)

Since $R = \Pi - \hat{\Pi}$, where

$$\Pi(X, \hat{X}) = (X_+ + \hat{X}_-, \hat{X}_- + X_+), \quad \hat{\Pi}(X, \hat{X}) = (X_- - \hat{X}_-, \hat{X}_+ - X_+).$$
are two projections of $\mathcal{D}$ onto its subalgebras, more exactly,

$$\Pi\mathcal{D} = \{(X, X) \mid X \in \mathcal{D}^b\}, \quad \tilde{\Pi}\mathcal{D} = (\mathcal{D}^-)_- \times (\mathcal{D}^+)_+,$$

$$\Pi^2 = \Pi, \quad \tilde{\Pi}^2 = \tilde{\Pi}, \quad \tilde{\Pi}\Pi = 0 = \Pi\tilde{\Pi}, \quad \Pi + \tilde{\Pi} = \text{id},$$

then transformation $R$ satisfies the modified Yang-Baxter equation (4.2). Hence $R$ is an $R$-matrix on $\mathcal{D}$.

On the other hand, with respect to the inner product on $\mathcal{D}$ the adjoint transformation of $R$ reads

$$R^*: \mathcal{D} \rightarrow \mathcal{D}, \quad (X, \hat{X}) \mapsto (X_--X_+ + 2\hat{X}_-, \hat{X}_+ - \hat{X}_- + 2X_+).$$

Then the symmetric and anti-symmetric parts of $R$ are given by

$$R_s(X, \hat{X}) = 2(\hat{X}_- - \hat{X}_+), \quad R_a(X, \hat{X}) = (X_+ - X_-, \hat{X}_- - \hat{X}_+). \quad (4.6)$$

Observe that $R_a$ can be expressed as the difference of two projections onto subalgebras of $\mathcal{D}$, hence $R_a$ also solves the Yang-Baxter equation (4.2). Thus the $R$-matrix given in (4.5) fulfills the condition of Lemma 4.1.

We regard $\mathcal{D}$ as an infinite-dimensional manifold, whose coordinate is given by the coefficients of the general expression of its elements

$$\mathbf{A} = \left( \sum_{i \in \mathbb{Z}} w_i D^i, \sum_{i \in \mathbb{Z}} \hat{w}_i D^i \right) \in \mathcal{D}. \quad (4.7)$$

The set $\mathcal{F}$ of local functionals over the differential algebra $\mathcal{A}$ (see Section 2) plays the role of $C^\infty(\mathfrak{g})$. For any $F = \int f \, dx \in \mathcal{F}$, the variational gradient of $F$ at $\mathbf{A}$ given in (4.7) is defined to be

$$\frac{\delta F}{\delta \mathbf{A}} = \left( \sum_{i \in \mathbb{Z}} D^{-i-1} \frac{\delta F}{\delta w_i(x)} \sum_{i \in \mathbb{Z}} D^{-i-1} \frac{\delta F}{\delta \hat{w}_i(x)} \right),$$

where $\delta F/\delta w(x) = \sum_{j \geq 0} (-D)^j \left( \frac{\partial f}{\partial w^{(j)}} \right)$. Note that $\delta F/\delta \mathbf{A}$ is not contained in $\mathcal{D}^* = \mathcal{D}$ in general, so to go forward we need to do some restriction.

It shall be indicated that, in this paper we only consider functionals with variational gradients lying in $\mathcal{D}$. Let $\mathcal{F}_0$ denote the set of such functionals.

Now we can use Lemma 4.1 and the formulae (4.4), (4.6) to obtain the following result.

**Lemma 4.2** Let $F$ and $G$ be two arbitrary functionals in $\mathcal{F}_0$. On the algebra $\mathcal{D}$ there is a quadratic Poisson bracket

$$\{F, G\}(\mathbf{A}) = \left\langle \frac{\delta F}{\delta \mathbf{A}}, \mathcal{P}_\mathbf{A} \left( \frac{\delta G}{\delta \mathbf{A}} \right) \right\rangle, \quad \mathbf{A} = (A, \hat{A}) \in \mathcal{D}, \quad (4.8)$$
where the Poisson tensor $\mathcal{P} : T \mathfrak{D}^* \to T \mathfrak{D}$ is defined by

$$\mathcal{P}(A, \hat{A})(X, \hat{X}) = A(XA) - (AX)A - A(\hat{A}\hat{X}) + (\hat{X}\hat{A})A, \quad \hat{A}(X\hat{A}) - (\hat{A}X)\hat{A} - \hat{A}(AX) + (XA)\hat{A}.$$  \hspace{1cm} (4.9)

Aiming at Hamiltonian structures of the two-component BKP hierarchy, we need to reduce the Poisson structure (4.9) to an appropriate submanifold of $\mathfrak{D}$. Recall the decompositions (2.12), let us decompose the space $\mathfrak{D}$ as

$$\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}_1,$$ \hspace{1cm} (4.10)

where $\mathfrak{D}_\nu = \mathfrak{D}^- \times \mathfrak{D}^+$ for $\nu = 0, 1$. Since the subspaces $\mathfrak{D}_0$ and $\mathfrak{D}_1$ are dual to each other with respect to the inner product on $\mathfrak{D}$, then for any $A \in \mathfrak{D}_\nu$ we have $T^*_\mathfrak{A} \mathfrak{D}_\nu = (\mathfrak{D}_\nu)^* = \mathfrak{D}_{1-\nu}$ for $\nu = 0, 1$. It is straightforward to verify the following lemma.

**Lemma 4.3** The Poisson structure (4.9) on $\mathfrak{D}$ can be properly restricted to each of its subspaces $\mathfrak{D}_0$ and $\mathfrak{D}_1$.

### 5 Bihamiltonian representation of the two-component BKP hierarchy

In this section, we are to find a submanifold of $\mathfrak{D}$ where the Poisson pencil for the two-component BKP hierarchy lies, then after a further reduction of the Poisson structure constructed in last section we will express the hierarchy (3.10), (3.11) to the form of Hamiltonian equations.

Recall the operators $P \in \mathfrak{D}^-$, $\hat{P} \in \mathfrak{D}^+$ given in (3.5), we let

$$A = (P^2D^{-1}, D\hat{P}^2).$$ \hspace{1cm} (5.1)

It is easy to see that $A \in \mathfrak{D}_1$ (see (4.10)), and $A = (A, \hat{A})$ has the following expression:

$$A = P^2D^{-1} = D + \sum_{i \geq 0}(v_{-i}D^{-2i-1} + f_{-i}D^{-2i-2}),$$ \hspace{1cm} (5.2)

$$\hat{A} = D\hat{P}^2 = \rho D^{-1}\rho + \sum_{i \geq 1}(\hat{v}_iD^{2i-1} + \hat{f}_iD^{2i-2}), \quad \rho = \hat{u}_{-1}. \hspace{1cm} (5.3)$$

Denote $v = (v_0, v_{-1}, \ldots, \hat{v}_0, \hat{v}_1, \ldots)$ with $\hat{v}_0 = \rho^2$. Then the coordinate $v$ is related to $u$ given in (4.10) by a Miura-type transformation, while $f_{-i}$ and $\hat{f}_i$ are linear functions of derivatives of $v$ determined by the symmetry property $(A^*, \hat{A}^*) = -(A, \hat{A})$. Hence the flows of the hierarchy (3.10), (3.11) can be described in the coordinate $v$.
Given any local functional $F \in \mathcal{F}_0$ (remind the notation $\mathcal{F}_0$ in last section), its variational gradient with respect to $A$, say $\delta F / \delta A$, is defined to be $X = (X, \hat{X}) \in \mathcal{D}$ with

$$X = \frac{1}{2} \sum_{i \geq 0} \left( \frac{\delta F}{\delta v_{-i}(x)} D^{2i} + D^{2i} \frac{\delta F}{\delta v_{-i}(x)} \right), \quad (5.4)$$

$$\hat{X} = \frac{1}{2} \sum_{i \geq 0} \left( \frac{\delta F}{\delta \hat{v}_{i}(x)} D^{-2i} + D^{-2i} \frac{\delta F}{\delta \hat{v}_{i}(x)} \right). \quad (5.5)$$

In a coordinate-free way, $\delta F / \delta A = X$ can be defined by

$$\delta F = \langle X, \delta A \rangle, \quad X \in \mathcal{D}_0. \quad (5.6)$$

Note that in the latter definition, the variational gradient is determined up to a kernel part $Z = (Z, \hat{Z}) \in \mathcal{D}_0$ such that $Z_+ = 0$, $\hat{Z}_- = 0$, $\hat{Z}_+(\rho) = 0$. \quad (5.7)

Let us consider the coset $(D, 0) + \mathcal{U}$ consisting of operators of the form \((5.1)\), namely,

$$\mathcal{U} = (D^-_1)_- \times ((D^+_1)_+ \times \mathcal{M}), \quad \mathcal{M} = \{ \rho D^{-1} \rho \mid \rho \in \mathcal{A} \}. \quad (5.8)$$

Here $\mathcal{M}$ is regarded as a 1-dimensional manifold with coordinate $\rho$, and this manifold has tangent spaces of the form

$$T_{\rho} \mathcal{M} = \{ \rho D^{-1} f + f D^{-1} \rho \mid f \in \mathcal{A} \}.$$

So the tangent bundle, denoted by $T \mathcal{U}$, of the coset $(D, 0) + \mathcal{U}$ has fibers

$$T_A \mathcal{U} = (D^-_1)_- \times ((D^+_1)_+ \times T_{\rho} \mathcal{M}), \quad A \in (D, 0) + \mathcal{U}, \quad (5.9)$$

while the cotangent bundle $T^* \mathcal{U}$ of $(D, 0) + \mathcal{U}$ is composed of

$$T^*_A \mathcal{U} = (D^+_0)_+ \times ((D^-_0)_- \times T^*_{\rho} \mathcal{M}), \quad T^*_A \mathcal{M} = \mathcal{A}. \quad (5.10)$$

From \((5.4)\), \((5.5)\) one sees that $\delta F / \delta A \in T^*_A \mathcal{U}$ for any $F \in \mathcal{F}_0$.

Now we are ready to do the desired reduction of the Poisson structure.

**Lemma 5.1** The map

$$\mathcal{P} : T^* \mathcal{U} \to T \mathcal{U} \quad (5.11)$$

defined by the formula \((4.9)\) is a Poisson tensor on the coset $(D, 0) + \mathcal{U}$ that consists of operators of the form \((5.1)\).
Proof. We only need to show that the map defined by (4.9) admits the restriction to the coset $(D, 0) + \mathcal{U}$, i.e., the following map is well defined:

$$\mathcal{P}_A : T^*_A \mathcal{U} \to T_A \mathcal{U}, \quad A \in (D, 0) + \mathcal{U}. \quad (5.12)$$

Assume $X = (X, \hat{X}) \in T^*_A \mathcal{U} \subset \mathcal{D}_0$. It follows from Lemma 4.3 that $\mathcal{P}_A(X) \in \mathcal{D}_1$. More precisely, the first component of $\mathcal{P}_A(X)$ belongs to $(D^-_1)^-$. On the other hand, for any $\hat{Y} \in (D^+_1)$ we have

$$(\hat{A} \hat{Y} + \hat{Y}^* \hat{A})_- = (\rho D^{-1} \rho \hat{Y} + \hat{Y}^* \rho D^{-1} \rho)_-$$

$$= - (\hat{Y}^* \rho D^{-1} \rho)_-^* + \hat{Y}^*(\rho) D^{-1} \rho$$

$$= \rho D^{-1} \hat{Y}^*(\rho) + \hat{Y}^* \rho D^{-1} \rho \in T_p \mathcal{M},$$

then by taking $\hat{Y} = (\hat{X} \hat{A})_+$, $(AX)_+$ it follows that the second component of $\mathcal{P}_A(X)$ lies in $(D^+_1)^+ \times T_p \mathcal{M}$. Thus $\mathcal{P}_A(X) \in T_A \mathcal{U}$, i.e., the map (5.12) is well defined. The lemma is proved. \[\square\]

**Remark 5.2** The proof of this lemma is the simplest case of the Dirac reduction procedure for Poisson tensors, see e.g. [20]. In fact, one can express the manifolds $\mathcal{D}_1$ and $\mathcal{D}_1^*$ as

$$\mathcal{D}_1 = \mathcal{U} \times \mathcal{V} = T_A \mathcal{U} \times V_A, \quad \mathcal{D}_1^* = \mathcal{D}_0 = T^*_A \mathcal{U} \times V^*_A, \quad (5.13)$$

where

$$\mathcal{V} = V_A = (D^-_1)^+ \times \mathcal{N}, \quad \mathcal{N} = \{X \in (D_1^+)^- \mid \text{res} \mathcal{X} = 0\},$$

$$V^*_A = (D^-_0)^- \times (T^*_0)^\perp \mathcal{M}, \quad (T^*_0)^\perp \mathcal{M} = \{\hat{Y} \in (D^+_0)^+ \mid \hat{Y}(\rho) = 0\}.$$

Similar as the proof of Lemma 5.1 one can show that the map

$$\mathcal{P}_A = \left( \begin{array}{cc} p^{\mathcal{U} \mathcal{U}}_A & p^{\mathcal{U} \mathcal{V}}_A \\ p^{\mathcal{V} \mathcal{U}}_A & p^{\mathcal{V} \mathcal{V}}_A \end{array} \right) : T^*_A \mathcal{U} \times V^*_A \to T_A \mathcal{U} \times V_A$$

defined by (4.9) is diagonal. Hence from Lemma 4.3 it follows that the map (4.9) gives a Poisson tensor on the coset $(D, 0) + \mathcal{U} \subset \mathcal{D}_1$.

**Lemma 5.3** On the coset $(D, 0) + \mathcal{U}$ there are two compatible Poisson tensors defined by the following formulae:

$$\mathcal{P}_1(X, \hat{X}) = (AXD^{-1})_- + D^{-1}(XA)_- - (D^{-1}X)_- - (AX)_-\hat{D}^{-1} - \hat{A}(\hat{X} \hat{D})_+ + D(\hat{X} \hat{A})_+ - (\hat{X} \hat{A})_+ \hat{D}$$

$$- \hat{A}(D^{-1}X)_+ - D(AX)_+ + (XD^{-1})_+ \hat{A} + (AX)_+ \hat{D}, \quad (5.14)$$

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\[ \mathcal{P}_2(X, \hat{X}) = (A(XA)_- - (AX)_- A - A(\hat{A}X)_- + (\hat{X}A)_- A, \]  
\[ \hat{A}(\hat{X}A)_+ - (\hat{A}X)_+ \hat{A} - \hat{A}(AX)_+ + (XA)_+ \hat{A}) \]  
\[ \text{with } (X, \hat{X}) \in T^*_A U \text{ at any point } A = (A, \hat{A}) \in (D, 0) + U. \]

**Proof.** Lemma 5.1 shows that \( \mathcal{P}_2 \) is a Poisson tensor on the coset \((D, 0) + U\). Introduce a shift transformation on \((D, 0) + U\) as
\[ T : (A, \hat{A}) \mapsto (A + sD^{-1} \hat{A}, \hat{A} + sD) \]
with \( s \) being a parameter. Then the push-forward of the Poisson tensor \( \mathcal{P}_2 \) reads
\[ (T_\ast \mathcal{P}_2)(X, \hat{X}) = \mathcal{P}_2(X, \hat{X}) + s \mathcal{P}_1(X, \hat{X}) + s^2 \mathcal{P}_0(X, \hat{X}), \]
where
\[ \mathcal{P}_0(X, \hat{X}) = (D^{-1}(XD^{-1})_- - (D^{-1}X)_- D^{-1} - D^{-1}(D\hat{X})_- + (\hat{X}D)_- D^{-1}, \]
\[ D(\hat{X}D)_+ - (D\hat{X})_+ D - D(D^{-1}X)_+ + (XD^{-1})_+ D). \]

By virtue of the symmetry property \((X^+, \hat{X}^+) = (X, \hat{X})\) that yields the formulae
\[ (XD^{-1})_\pm = X_\pm D^{-1} = X_\pm (1) D^{-1}, \]
\[ (D^{-1}X)_\pm = D^{-1}X_\pm = D^{-1} \cdot X_\pm (1), \]
\[ (D\hat{X})_\pm = D\hat{X}_\pm, \quad (\hat{X}D)_\pm = \hat{X}_\pm D, \]
one can check \( \mathcal{P}_0(X, \hat{X}) = 0 \). Hence the expansion (5.16) implies that \( \mathcal{P}_1 \) is a Poisson tensor that is compatible with \( \mathcal{P}_2 \). The lemma is proved. \( \square \)

Let \( \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2 \) denote the Poisson brackets given in (1.8) with Poisson tensors being \( \mathcal{P}_{1,2} \) respectively. We arrive at the main result of this article.

**Theorem 5.4** The two-component BKP hierarchy (3.10), (3.11) can be expressed in the following bihamiltonian recursion form
\[ \frac{\partial F}{\partial t_k} = \{F, H_{k+2}\}_1(A) = \{F, H_k\}_2(A), \]
\[ \frac{\partial F}{\partial \hat{t}_k} = \{\hat{F}, \hat{H}_{k+2}\}_1(A) = \{\hat{F}, \hat{H}_k\}_2(A) \]
with \( k \in \mathbb{Z}_{\text{odd}}^+ \), where \( F \in \mathcal{F}_0, \ A = (P^2D^{-1}, D\hat{P}^2) \) as given in (5.1), and the Hamiltonians are
\[ H_k = \frac{2}{k} \langle P^k \rangle, \quad \hat{H}_k = -\frac{2}{k} \langle \hat{P}^k \rangle, \quad k \in \mathbb{Z}_{\text{odd}}^+. \]
Proof. First let us compute the variational gradients of the Hamiltonian functionals. Since
\[ \delta H_k = (P^{k-2}, \delta P^2) = (DP^{k-2}, \delta (P^2 D^{-1})) = ((DP^{k-2}, 0), \delta A) \]
and similarly
\[ \delta \hat{H}_k = (0, -\hat{P}^{k-2} D^{-1}), \delta A \]
then up to kernel parts given in (5.7) we have the variational gradients of the Hamiltonians:
\[ \frac{\delta H_k}{\delta A} = (DP^{k-2}, 0), \quad \frac{\delta \hat{H}_k}{\delta A} = (0, -\hat{P}^{k-2} D^{-1}) \] (5.20)
One can easily see that different choices of the kernel parts do not change the definition of the Poisson tensors \( P_{1,2} \).

According to the flows (3.10), (3.11) one has
\[ \frac{\partial A}{\partial t} = \left( [(P^k)_+, P^2] D^{-1}, D[(P^k)_+, \hat{P}^2] \right). \]
Note that
\[ \frac{\partial F}{\partial t} = \left\langle \frac{\delta F}{\delta A}, \frac{\partial A}{\partial t} \right\rangle, \]
then to show (5.17) we only need to verify the equations
\[ \frac{\partial A}{\partial t} = \mathcal{P}_1 \left( \frac{\delta H_{k+2}}{\delta A} \right) = \mathcal{P}_2 \left( \frac{\delta H_k}{\delta A} \right). \] (5.21)
The verification is straightforward by substituting (5.20) into (5.14), (5.15) with the help of the following formulae induced from (3.8):
\[ (DP^k D^{-1})_\pm = D(P^k)_\pm D^{-1}, \quad (D\hat{P}^k D^{-1})_\pm = D(\hat{P}^k)_\pm D^{-1}, \quad k \in \mathbb{Z}^{\text{odd}}_. \]
The equations (5.18) can be checked similarly. The theorem is proved. \( \square \)

Remark 5.5 One can also construct Hamiltonian structures of the two-component BKP hierarchy by reducing the linear and the cubic Poisson brackets induced from the \( R \)-matrix mentioned in last section. However, from these brackets we have not found bihamiltonian recursion relations like (5.17), (5.18).
6 Dispersionless limit of the bihamiltonian structure

Let us compute the leading term of the bihamiltonian structure in (5.17), (5.18) of the two-component BKP hierarchy, which would make sense in studying the corresponding Frobenius manifold if there be.

First we replace the pseudo-differential operators by Laurent series of symbols. In the dispersionless case, the operator \( A = (P^2 D^{-1}, D \hat{P}^2) \) becomes

\[
(a(z), \hat{a}(z)) = \left( z + \sum_{i \geq 0} v_{-i} z^{-2i-1}, \sum_{i \geq 0} \hat{v}_i z^{2i-1} \right), \quad \text{(6.1)}
\]

and the coordinate-type local functionals \( v_{-i}(y), \hat{v}_j(y) \) have variational gradients \( (z^{2i} \delta(x-y), 0), (0, z^{-2j} \delta(x-y)) \) respectively. Substituting these Laurent series into the Poisson brackets defined by the formulae (4.8), (5.14), (5.15), we obtain the following result.

For the convenience of expression we set \( v_1 = 1, v_i = 0 \) when \( i \geq 2 \), and \( \hat{v}_j = 0 \) when \( j \leq -1 \).

i) The first bracket: for \( i, j \geq 0 \),

\[
\{ v_{-i}(x), v_{-j}(y) \}^{[0]}_1 = (1 - \delta_{i0} - \delta_{j0}) (2(i + j - 1) v_{-i-j+1}(x) \delta'(x-y) \nonumber \\
\nonumber + (2j - 1) v'_{i-j+1}(x) \delta(x-y)), \quad \text{(6.2)}
\]

\[
\{ \hat{v}_i(x), \hat{v}_j(y) \}^{[0]}_1 = -(1 - \delta_{i0} - \delta_{j0}) (2(i + j - 1) \hat{v}_{i+j}(x) \delta'(x-y) \nonumber \\
\nonumber + (2j - 1) \hat{v}'_{i+j}(x) \delta(x-y)), \quad \text{(6.3)}
\]

\[
\{ v_{-i}(x), \hat{v}_j(y) \}^{[0]}_1 = 2(i - j) (1 - \delta_{i0}) v_{j-i}(x) (1 - \delta_{j0}) \hat{v}_{j-i+1}(x) \delta'(x-y) \nonumber \\
\nonumber - (2j - 1) (1 - \delta_{j0}) \hat{v}'_{j-i}(x) (1 - \delta_{i0}) v_{j-i+1}(x) \delta(x-y). \quad \text{(6.4)}
\]

ii) The second bracket: for \( i, j \geq 0 \),

\[
\{ v_{-i}(x), v_{-j}(y) \}^{[0]}_2 = \sum_{r=1}^{-1} \left( 2(i + j - 2r - 1) v_{-r}(x) v_{-i-j+r+1}(x) \delta'(x-y) \nonumber \\
\nonumber + (2j - 2r - 1) v_{-r}(x) v'_{-i-j+r+1}(x) \delta(x-y) \right) \nonumber \\
\nonumber + (2i - 2r - 1) v'_{-r}(x) v_{-i-j+r+1}(x) \delta(x-y)), \quad \text{(6.5)}
\]

\[
\{ \hat{v}_i(x), \hat{v}_j(y) \}^{[0]}_2 = - \sum_{r=0}^{-i} \left( 2(i + j - 2r + 1) \hat{v}_r(x) \hat{v}_{i+j-r+1}(x) \delta'(x-y) \nonumber \\
\nonumber \right.
\]

\[
\left. + (2j - 2r + 1) \hat{v}'_r(x) \hat{v}_{i+j-r+1}(x) \delta(x-y)) \right) \nonumber 
\]

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\begin{align*}
+ (2j - 2r + 1) \hat{v}_r(x) \hat{v}'_{i+j-r+1}(x) \delta(x - y) \\
+ (2i - 2r + 1) \hat{v}'_r(x) \hat{v}_{i+j-r+1}(x) \delta(x - y)
\end{align*}
\right),
\tag{6.6}
\end{equation}

\begin{equation}
\{v_{-i}(x), \hat{v}_j(y)\}_{2}^{[0]}
= \sum_{r = \max\{-1, i-j-1\}}^{i-1} \left( (2-i) v_{-r}(x) \hat{v}_{-i+j+r+1}(x) \delta'(x - y) \\
+ (2r - 2j + 1) v_{-r}(x) \hat{v}'_{-i+j+r+1}(x) \delta(x - y) \\
+ (2r - 2i + 1) v'_r(x) \hat{v}_{-i+j+r+1}(x) \delta(x - y) \right).
\tag{6.7}
\end{equation}

7 Concluding remarks

Based on the Lax pair representation (3.10), (3.11) of the two-component BKP hierarchy, we obtain a bihamiltonian structure of this hierarchy. Our method in the construction of the Poisson brackets is to employ the standard R-matrix formalism, which is analogous to that for the two-dimensional Toda hierarchy [1]. In comparison with the two-dimensional Toda hierarchy, we expect that there would be an infinite-dimensional Frobenius manifold underlying the two-component BKP hierarchy.

As shown in [19], the two-component BKP hierarchy (3.10), (3.11) is reduced to the Drinfeld-Sokolov hierarchy of type \((D^{(1)}_n, c_0)\) under the constraint \(P_{2n-2} = P^2\). Whether such a constraint induces a reduction of the bihamiltonian structure is unclear yet. We hope that considering this example would help to understand the relations between Frobenius manifolds of infinite and finite dimensions.

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