Classical Nature of the Inflaton Field with Self-Interaction

Takahiro Tanaka and Masa-aki Sakagami

1Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka 560, Japan
2Cosmology Group, Faculty of Integrated Human Studies, Kyoto University, Kyoto 606-01, Japan

Taking into account the effect of self-interaction, the dynamics of the quantum fluctuations of the inflaton field with $\phi^4$ potential is studied in detail. We find that the self interaction efficiently drives the initial pure state into a mixed one, which can be understood as a statistical ensemble. Further, the expectation value of the squared field operator is found to be converted into the variance of this statistical ensemble without giving any significant change in its amplitude. These results verify the ansatz of the quantum-to-classical transition that has been assumed in the standard evaluation of the amplitude of the primordial fluctuations of the universe.

PACS number(s): 98.80.Cq, 04.62.+v, 05.40.+j

I. INTRODUCTION

The inflationary universe scenarios explain satisfactorily various aspects of the universe, such as the homogeneity, isotropy and the amplitude of the primordial fluctuations observed in the microwave background radiation and in the large scale structure [1]. However, the incompleteness in the discussion of the evaluation of the primordial fluctuations has been pointed out in several papers as is explained below.

In most of inflationary universe scenarios, the seed of the inhomogeneity of the universe is traced back to the quantum fluctuations of the inflaton field generated by the accelerated expansion of the cosmic length scale. In this context, the amplification of the quantum fluctuations is characterized by a large amount of squeezing of the state vector. A difficulty in interpreting this state vector exists in the fact that the expectation value of the linear field operator does not show inhomogeneities but vanishes while that of the squared field operator becomes very large. Usually, one interprets this quantum state as if it were equivalent to a statistical ensemble which has the same amount of variance that the corresponding quantum operator has in the sense of an expectation value once the scale of the fluctuations of our interest exceeds the Hubble horizon scale. Here we refer to the calculation based on this ad-hoc classicalization ansatz as “the standard calculation”.

In order to justify the standard calculation, the stochastic approach to the inflationary universe scenario was proposed by Starobinsky [2] and further investigated by many authors [3]. The evolution of the order parameter that is defined as the expectation value of the spatially averaged field operator was studied. If the physical size of the spatial averaging is kept constant, the modes with a large comoving wave number become to contribute to the order parameter as the universe expands. The contribution from the newly added small scale modes affects on the dynamics of the order parameter as a random noise. As a consequence, the evolution of the order parameter mimics the Brownian motion and thus the distribution of the order parameter can be well approximated by a statistical ensemble of a classical system influenced by the stochastic noise.

In these early studies, a free scalar field was investigated and little attention was paid to the effect of interaction. However, the fluctuations of a free scalar field in a Friedmann-Robertson-Walker spacetime can be decomposed into a set of harmonic oscillators which have time dependent spring constants. Then all of these oscillators are decoupled with each other. Hence, no classicalization will be expected in such a decomposed degree of freedom. It follows that the each modes, which are added to the order parameter during the inflation, never lose their quantum nature. This fact implies that the discussion of the classicalization in the context of the stochastic inflation is incomplete [4]. We think that, in the true theory of the quantum-to-classical transition, the classical nature should be observed even if each decomposed mode is considered. In other words, we believe that the change of the number of modes that
contribute to the order parameter cannot play an essential role in the appearance of the classical nature of the inflaton field.

In this direction, recently, Lesgourgues, Polarski and Starobinsky proposed a new idea to explain the quantum-to-classical transition. They claimed that the equivalence between the large squeezing and the classicality of the state can be explained if one discards the tiny contribution due to the decaying mode of the perturbation. However, a free scalar field is considered in their approach, and the effect of interaction was not taken into account manifestly. Thus we think that their elimination of the decaying mode is still artificial and will not be fully justified although their result seems to contain an important suggestion.

Only recently, the importance of interaction has become emphasized. As a useful tool to deal with the effect of interaction with the environmental degrees of freedom, the closed time path formalism has been developed. The total system is divided into “the system” and “the environment” so that the system contains the quantities of our interest while the environment does not. By using the closed time path formalism, one can integrate out the environmental degrees of freedom to obtain the effective action for the system. Many cosmological issues, such as the back reaction to the expansion rate of the universe due to the particle creation, have been investigated.

Especially, in the references, the evolution of the fluctuations of the inflaton field was examined by the aid of the closed time path formalism. It was repeatedly stressed that the correct treatment of the effect of the environmental degrees of freedom may relax the problem of the fine tuning of parameters in the inflaton potential. Let us consider the $\lambda\phi^4$ model of the chaotic inflation. In the standard calculation, $\lambda \sim 10^{-12}$ is required in order to explain the observed value of the primordial density fluctuations. To the contrary, the authors in the above references suggested the possibility that the tuning of the coupling constant can be relaxed to $\lambda \sim 10^{-6}$. One of the main issues of the present paper is to cast a doubt on this statement.

Here we take a conservative picture of the quantum-to-classical transition based on the paper by Joos and Zeh. The basic idea is explained in Sec. 2. There we present two sufficient conditions for the system to possess the classical nature. Based on this picture, we investigate the $\lambda\phi^4$ model, which is one of the simplest models of the chaotic inflation. With the choice of parameter $\lambda \sim 10^{-12}$, the two conditions for the quantum-to-classical transition are proved to be satisfied and the fluctuations of the scalar field, $\delta \phi$ become of $O(1)$ as is predicted in the standard calculation. Hence, we obtain the conclusion that “the standard calculation” is justified in this simple model without any significant modification in the amplitude of the fluctuations as opposed to the prediction given in literature.

This paper is organized as follows. In Sec. 2 we explain our picture of the quantum-to-classical transition and clarify in what situation the system behaves as a classical one. In Sec. 3 we explain our simple model and the basic assumptions. Integrating out the environmental degrees of freedom, we derive the effective action for the system. In Sec. 4, by using the closed time path formalism, the dynamics of the system is analyzed in a less rigorous but a rather intuitive manner. More rigorous treatment based on the quantum master equation for the reduced density matrix is provided in Sec. 5. Section 6 is devoted for summary and discussion.

II. DECOHERENCE MEASURE BETWEEN DIFFERENT WORLDS

In this section, we explain our picture of the quantum-to-classical transition. At first, let us briefly summarize the standard discussion of decoherence based on the analysis of the reduced density matrix. Suppose that the whole system can be divided into two parts: one is the “system”, which contains the observables of our interest, and the other is the “environment”, which is to be integrated out to obtain the reduced density matrix. We set an initial condition for the whole system so that the density matrix is given by that of a pure state and the correlation between the system and the environment is absent. Hence the state can be represented as

$$\rho(t_1 = |t_i\rangle_{sys} \otimes |t_i\rangle_{env} \otimes \langle t_i|_{sys} \otimes \langle t_i|_{env}}.$$ (2.1)

The reduced density matrix is obtained by taking the partial trace over the environment,

$$\tilde{\rho}(x, x'; t) := \text{Tr}_{env} \rho(t|x)x')_{sys},$$ (2.2)

where $x$ and $x'$ are the labels of a complete set of state vectors of the system, say, the eigen state of the coordinate. If the interaction between the system and the environment is absent, the loss of coherence does not take place, i.e., $\rho(x, x'; t)$ keeps the form of a pure state, i.e., $\rho(x, x'; t) = \rho(x, x'; t) = 1$. However, in the presence of interaction, $\rho(x, x'; t)$ no longer keeps the form of a pure state. If $\rho(x, x'; t) = \rho(x, x'; t)$ becomes quite small for $x \neq x'$, we may say that the coherence between different states labeled by $x$ and $x'$ disappears. If $x$ is a continuous parameterization of state vectors, we can determine the typical scale $\Delta x$ such that the coherence between the states labeled by $x$ and $x'$ is lost when $|x - x'| > \Delta x$. As long as we observe the system with a resolution coarser than $\Delta x$, we may think that the different states have no interference between them and so they can be recognized as independent different worlds. However, in the above discussion, the stability of the state through the time evolution was not taken into account. The dynamics of the system itself and the effects from the environment cause the broadening of the wave function of the system in general. If there is a large amount of broadening, it would be difficult to interpret that the state evolves into a statistical ensemble of many different “classical worlds”.

2
Here we propose to take more conservative picture of the quantum-to-classical transition in this paper. As mentioned in Introduction, the basic idea is taken from the paper by Joos and Zeh [12]. We restrict our attention to the case in which the interaction term between the system and the environment does not contain the momentum variable of the system. We suppose that the sufficient conditions in order for the system to possess the classical nature are the following two.

The first condition is that the total system has a set of wave packets which have a sufficiently peaky probability distribution in comparison with the accuracy of our measurement through the whole duration we consider. Say, we label the wave packets by their initial peak position, \( s \), as \(|s; t\rangle\). Since the interaction term does not contain the momentum variable, in an approximate sense each wave packet will be written by the direct product as

\[
|s; t\rangle = |s; t\rangle_{\text{sys}} \otimes |s; t\rangle_{\text{env}}.
\]  

(2.3)

Here we should note that the state of the environment \(|s; t\rangle_{\text{env}}\) is also labeled by \( s \), since the interaction causes the correlation between the system and the environment in the course of their time evolution [13]. Strictly speaking, what we require here is that not \(|s; t\rangle\) but \(|s; t\rangle_{\text{sys}}\) has a sharp peak and that the peak is stable against the evolution. Furthermore, we require that this set of wave packets is complete enough that the initial state, \(|\Psi\rangle\), can be decomposed into a quantum mechanical superposition of these wave packets, i.e.,

\[
|\Psi\rangle = \sum_{s} c_{s} |s; t\rangle.
\]  

(2.4)

This means that each wave packet is sufficiently peaky and stable to be recognized as distinguishable “world”. We refer to this condition as “the classicality of the dynamics of the system”.

The second condition is that the coherence between the different wave packets becomes lost swiftly. We can say that this condition is the one that was roughly discussed at the beginning of this section. We refer to this condition as “the decoherence between different worlds”. For the total system that satisfies the above mentioned first condition, the density matrix at time \( t \) will be given by

\[
\rho(t) = \sum_{s,s'} c_{s} c_{s'}^{*} \langle s; t\rangle \langle s'; t|,
\]  

(2.5)

The partial trace over the environment gives the reduced density matrix,

\[
\tilde{\rho}(t) = \sum_{s,s'} c_{s} c_{s'}^{*} C(s, s'; t) |s; t\rangle_{\text{sys}} \langle s'; t| =: \sum_{s,s'} \tilde{\rho}_{s,s'}(t),
\]  

(2.6)

where we defined the partial reduced density matrix \( \tilde{\rho}_{s,s'}(t) \) and \( C(s, s'; t) \) is given by

\[
C(s, s'; t) := \langle s; t|s'; t\rangle_{\text{env}}.
\]  

(2.7)

If we assume that the initial state is given by a direct product of the state of the system and that of the environment, we find \( C(s, s'; t_{i}) = 1 \). If the interaction between the system and the environment is absent, the loss of coherence does not take place, i.e. \( C(s, s'; t) \) stays time independent constant for all \( s \) and \( s' \). However, in the presence of interaction, \(|s; t\rangle_{\text{env}}\) and \(|s'; t\rangle_{\text{env}}\) evolve differently and hence \( C(s, s'; t) \) no longer stay constant for \( s \neq s' \), while \( C(s, s; t) \equiv 1 \). If \( |C(s, s'; t)| \) becomes quite small for \( s \neq s' \), we can say that the diagonalization of \( \tilde{\rho} \) has been occurred. Thus the quantity \( |C(s, s'; t)| \) characterizes the degree of decoherence between different worlds, if the evolution of the wave packets is not seriously affected by the environment.

Here we note that

\[
|c_{s} c_{s'}^{*} C(s, s'; t)|^{2} = \text{Tr} \left( \tilde{\rho}_{s,s'}(t) \tilde{\rho}_{s,s'}^{T}(t) \right),
\]  

(2.8)

In a strict sense, \(|s; t\rangle\) can not be written by a direct product as \(|s; t\rangle_{\text{sys}} \otimes |s; t\rangle_{\text{env}}\). So the definition of \( C(s, s'; t) \) given in (2.3) is not well defined. However, the expression in the right hand side of Eq. (2.8) makes sense at any time and is expected to give the measure of the decoherence between different worlds.

As a more convenient alternative of the measure of decoherence, we propose to use

\[
R = \frac{\max |\tilde{\rho}_{s,s'}(t)|}{\max |\tilde{\rho}_{s,s'}(t)|},
\]  

(2.9)

If \( \tilde{\rho}_{s,s'}(t) \) takes the Gaussian form, \( R \) equals to \( |C(s, s'; t)| \) besides the small correction due to the determinant factor that arises from the Gaussian integral. If \( R \) becomes very small for \( s \neq s' \), we recognize that the decoherence between different worlds is achieved.

In some sense, our picture is that of the third quantization of the universe [14] or of the decoherence history [15]. Our two requirements for the quantum-to-classical transition may be too strong [16]. But, for the present purpose, we do not have to relax these conditions.
III. SIMPLE MODEL AND THE EFFECTIVE ACTION

We consider the following simple model of inflation consisted of a single real scalar field using an approximation similar to the one that introduced by Matacz [9], although the details are significantly modified. We assume that the space time can be approximated by a spatially flat de Sitter space,

\[ ds^2 = dt^2 - a^2(t)d^2x, \]  

with

\[ a(t) = \frac{1}{H} e^{Ht}, \]  

and that the Lagrangian of the inflaton field is given by

\[ S = \int_{t_i}^t ds \ a^3(s) \int_\Omega d^3x \ \left[ \frac{1}{2} \left( \frac{d\Phi(x,s)}{ds} \right)^2 - \frac{(\nabla \Phi(x,s))^2}{2a^2(s)} - V(\Phi(x,s)) \right], \]  

where \( \Omega \) is a finite comoving volume corresponding to the scale of the fluctuations of our interest and we assumed that the effect from outside of this volume can be neglected. For simplicity, we choose \( \Omega \) as a cube with \( 0 \leq x, y, z \leq L \). The lower boundary of the time integration, \( t_i \), is the time at which some appropriate initial condition is set.

We focus on the dynamics of a spatially averaged field in this volume,

\[ \phi(s) := \frac{1}{\Omega} \int_\Omega d^3x \, \Phi(x,s). \]  

We stress that this averaging is performed only on a finite comoving volume, i.e., a part of the time-constant spatial surface. Hence \( \phi(s) \) does not represent the homogeneous part of the field \( \Phi(x,s) \) but it represents the fluctuation of scale \( L \). Although it is not essential but, for definiteness, we set the periodic boundary condition on \( \Phi(x,s) \). Then \( \Phi(x,s) \) is decomposed as

\[ \Phi(x,s) = \phi(s) + \psi(x,s) = \phi(s) + \sqrt{\frac{2}{\Omega}} \sum_k \left[ q^+ k \cos k \cdot x + q^- k \sin k \cdot x \right], \]  

where \( k = \frac{2\pi}{L}(i,j,k) \) is a non-vanishing vector with integer, \( i,j \), and non-negative integer, \( k \). Assuming that the potential can be approximated by

\[ V(\Phi) \sim V(\phi) + \psi V'(\phi) + \frac{\psi^2}{2} V''(\phi), \]  

the action reduces to

\[ S[\phi, \psi] = \Omega \int_{t_i}^t ds \ a^3(s) \left[ \frac{1}{2} \dot{\phi}^2(s) - V(\phi(s)) \right] - \frac{1}{2} \sum_\sigma \sum_k \int_{t_i}^t ds \ a^3(s) V''(\phi(s)) (q^+_k)^2 \]  

\[ =: S_{\text{sys}}[\phi] + S_{\text{int}}[\phi, \psi] + S_{\text{env}}[\psi], \]  

where dot \( \cdot \) means a derivative with respect to \( t \). Expecting that the interaction term is small, i.e., \( V''(\phi(s)) \) is so, we consider the perturbative expansion with respect to it. As seen from the notation introduced in Eq. \((3.3)\), the spatially averaged field is considered as the system and the other short wave length modes are the environment.

As was performed in Ref. \[9\], we calculate the reduced density matrix for \( \phi \) integrating over the environmental degrees of freedom, \( q^+_k \). Here we assume that the density matrix of the total system is initially represented by the direct product as

\[ \rho_i = \tilde{\rho}(\phi_i, \phi'_i; t_i) \otimes \prod_{\sigma} \prod_k \rho_{q^+_k} (q^+_k, q^+_k'; t_i). \]
In order to define the initial quantum state of \( q_k^s \), we suppose that the interaction is switched off before \( t = t_i \) and we set the quantum state of \( q_k^s \) by

\[
a_k^s |0\rangle = 0, \tag{3.9}
\]

where the annihilation operator \( a_k^s \) is defined by the decomposition of the field operator,

\[
q_k^s = a_k^s u_k^s + a_k^{s*} u_k^{s*}, \tag{3.10}
\]

and the positive frequency function, \( u_k^s \), is taken as

\[
u_k = u_0 := \frac{1}{\sqrt{2k}} e^{-ik\eta} \left( 1 - \frac{i}{k\eta} \right), \tag{3.11}
\]

before \( t = t_i \). Here we introduced the conformal time coordinate by \( \eta := -e^{-Ht} \). Then the reduced density matrix is calculated to the second order in \( S_{\text{int}}[\phi, q] \) as

\[
\tilde{\rho}(\phi, \phi'; t) := \int dq \rho(\phi, \phi', q, q'; t)
\]

\[
= \int d\phi_i \int d\phi' \int_0^{\phi_i} D\phi \int_{\phi_i}^{\phi'} D\phi' \exp \{ i \{ S_{\text{sys}}[\phi] - S_{\text{sys}}[\phi'] \} + iS_{\text{IF}}[\phi, \phi'] \} \tilde{\rho}(\phi_i, \phi_i'; t), \tag{3.12}
\]

and

\[
is_{\text{IF}}[\phi, \phi'] = -i \int_{t_i}^{t} ds \Delta(s) f(s) + i \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Delta(s) \Sigma(s') \mu(s, s')
\]

\[
- \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Delta(s) \Delta(s') \nu(s, s'), \tag{3.13}
\]

where

\[
\Delta(s) = V''(\phi) - V''(\phi'), \quad \Sigma(s) = \frac{1}{2} (V''(\phi) + V''(\phi')), \tag{3.14}
\]

and

\[
f(s) = \frac{a(s)^3}{2} \sum_{\delta} \sum_{k} u_0^s(s) u_0(s),
\]

\[
\mu(s, s') = -i \frac{a(s)^3 a(s')^3}{2} \sum_{\delta} \sum_{k} \left( [u_0^s(s)]^2 [u_0(s')]^2 - [u_0(s)]^2 [u_0^s(s')]^2 \right),
\]

\[
\nu(s, s') = \frac{a(s)^3 a(s')^3}{4} \sum_{\delta} \sum_{k} \left( [u_0^s(s)]^2 [u_0(s')]^2 + [u_0(s)]^2 [u_0^s(s')]^2 \right). \tag{3.15}
\]

In order to evaluate the summation over the discrete modes, we replace it with the integral as

\[
\sum_{\delta} \sum_{k} \rightarrow \frac{\Omega}{2\pi^2} \int_{k_{\text{min}}}^{\infty} k^2 \, dk, \tag{3.16}
\]

where we take \( k_{\text{min}} \) as a constant of \( O(2\pi/L) \). Since the proper length scale, \( a(t)L \), of the fluctuations which contribute to the formation of the large scale structure or the anisotropies of the cosmic microwave background radiation becomes much larger than the Hubble scale, \( H^{-1} \) at the end of the inflation era. Therefore for simplicity we concentrate on the case in which

\[
a(t)H = -\eta^{-1} \gg k_{\text{min}}, \tag{3.17}
\]

is satisfied. Thus we set the initial condition for the reduced density matrix at a time after the length scale of the fluctuations of our interest exceeds the Hubble horizon scale. Or equivalently, we assume that the evolution is free from the effect of the environmental degrees of freedom until \( t = t_i \). Under this condition, we evaluate the functions \( f(s), \mu(s, s') \) and \( \nu(s, s') \) approximately. The evaluation of \( f(s), \mu(s, s') \) and \( \nu(s, s') \) is rather complicated. The
details of computation are given in appendix A. As shown there, \( f \) and \( \mu \)-terms contain the ultraviolet divergences that require renormalization. After subtraction of these divergences, keeping only the leading terms, we obtain

\[
\begin{align*}
f(s) & \sim \frac{a^3(s) \Omega H^2}{8\pi^2} \log \left( \frac{H}{\mu_{\text{min}}(s)} \right), \\
\nu(s, s') & \sim \frac{a^3(s) \Omega H^4}{48\pi^2 \mu_{\text{min}}(s)} e^{-3H(s-s')},
\end{align*}
\]

(3.18)

where we introduced \( \mu_{\text{min}}(s) := a^{-1}(s)k_{\text{min}} \) and \( \mu_{\text{max}}(s) := a^{-1}(s)k_{\text{max}} \). As shown in appendix A, in order to subtract the divergent portion in the \( \mu \)-term of \( S_{\text{IF}} \), a integration by part with respect to \( s' \) is necessary. Then \( \mu \)-term is evaluated as

\[
S^{(\mu)}_{\text{IF}} = \int_{t_i}^t ds \Delta(s) \int_{t_i}^s ds' \left[ 2(\mu_i(s, s') + \mu_a(s, s')) \Sigma(s') + \mu_b(s, s') \Sigma(s') \right],
\]

(3.19)

where

\[
\begin{align*}
\mu_i(s, s') & \sim -\frac{a^3(s) \Omega}{16\pi^2} \log \left( \frac{\mu_{\text{min}}(s)}{H} \right) \delta(s-s'), \\
\mu_a(s, s') & \sim -\frac{a^3(s) \Omega H}{16\pi^2} \log (k_{\text{min}}(\eta-\eta')), \\
\mu_b(s, s') & \sim \frac{a(s) a^2(s') \Omega}{16\pi^2} \log (k_{\text{min}}(\eta-\eta')).
\end{align*}
\]

(3.20)

Precisely speaking, the approximated values of \( \mu \) are different from the true value by a factor of order unity. Thus \( 16\pi^2 \) can be replaced by, say, \( 24\pi^2 \). However, these errors do not change the discussion given below in this paper because we just show that the effect of these terms is small and can be neglected.

**IV. AN INTUITIVE INTERPRETATION OF THE EFFECTIVE ACTION**

Here we give a less rigorous but a rather intuitive analysis of the evolution of the averaged field, \( \phi \), in the model with \( V = \frac{\lambda \Phi^4}{4!} \). We defer a more rigorous treatment to the next section.

The effective action is rewritten in terms of

\[
\phi_+ = \frac{\phi + \phi'}{2}, \quad \varphi_\Delta = \phi - \phi',
\]

(4.1)

as

\[
S[\phi, \phi'] = \Omega \int_{t_i}^t ds \, a^3(s) \left\{ \dot{\phi}_+(s) \varphi_\Delta(s) - V'(\phi(s)) \varphi_\Delta \right\} \\
- \lambda \int_{t_i}^t ds \, \phi_+(s) \varphi_\Delta(s) f(s) + i\lambda^2 \int_{t_i}^t ds \phi_+(s) \varphi_\Delta(s) \int_{t_i}^s ds' \nu(s, s') \phi_+(s') \varphi_\Delta(s') + S^{(\mu)}[\phi, \phi'],
\]

(4.2)

and

\[
S^{(\mu)}[\phi, \phi'] = \lambda^2 \int_{t_i}^t ds \phi_+(s) \varphi_\Delta(s) \int_{t_i}^s ds' \left[ (\mu_i(s, s') + \mu_a(s, s')) \phi_+^2 (s') + \mu_b(s, s') \phi_+(s') \phi_+(s') \right],
\]

(4.3)

where the cubic or higher terms with respect to \( \varphi_\Delta \) are neglected.

Since \( \nu(s, s') \) decays fast as \( s - s' \) becomes large, here we approximate it as

\[
\nu(s, s') \sim \delta(s-s') \frac{\alpha^6(s) \Omega^2 \Xi^2(s) / \phi_+^4(s)}{2\nu(s, s')},
\]

(4.4)

where

\[
\Xi(s) := \frac{\phi_+(s)}{a^3(s) \Omega} \left[ t^s_{t_i} ds' \, 2\nu(s, s') \right]^{1/2} \sim \sqrt{\frac{H^3}{72\alpha^2 \pi^2}} \phi_+(s),
\]

(4.5)
with $\alpha' := \Omega k_{\text{min}}^3$. Then the $\nu$-term in the action reduces to
\begin{equation}
S_\nu[\phi, \phi'] = i\lambda^2 \Omega^2 \int_{t_i}^t ds \ a^6(s) \Xi^2(s) \frac{\varphi^2_{\Delta}(s)}{2}.
\end{equation}

To manage the effect of the $\nu$-term, we introduce an auxiliary field, $\xi$, which represents the Gaussian white noise
\begin{equation}
\langle \xi(s) \xi(s') \rangle_{\text{EA}} = \delta(s - s'),
\end{equation}

where the subscript EA stands for the ensemble average. Further we introduce the action with noise by
\begin{align*}
S_\xi[\phi, \phi'] &= \Omega \int_{t_i}^t ds \ a^3(s) \left\{ \dot{\phi}_+(s) \dot{\varphi}_{\Delta}(s) - V'(\phi(s)) \varphi_{\Delta}(s) \right\} \\
&\quad - \lambda \int_{t_i}^t ds \ (\phi_+(s)f(s) - a^3(s)\Omega \Xi(s) \xi(s)) \varphi_{\Delta}(s) + S^{(b)}[\phi, \phi'],
\end{align*}

Then from the fact that
\begin{equation}
\exp(iS[\phi, \phi']) = \langle \exp(iS_\xi[\phi, \phi']) \rangle_{\text{EA}},
\end{equation}

we can expect that the action with noise, $S_\xi$, gives the evolution of the field including the effect of the fluctuations induced by the environment through the $\nu$-term. Taking the variation of $S_\xi$ with respect to $\varphi_{\Delta}$, we obtain the Heisenberg equation for the operator $\hat{\phi}_+(t)$. Sandwiching thus obtained Heisenberg equation in between the “bra” and “ket” vectors of the initial state, the equation for the expectation value becomes
\begin{align*}
\frac{d^2}{dt^2} \langle \hat{\phi}_+(t) \rangle + 3H \frac{d}{dt} \langle \hat{\phi}_+(t) \rangle + \frac{\lambda f(t)}{3a^3(t)\Omega} \langle \hat{\phi}_+(t) \rangle \\
- \frac{\lambda f(t)}{3a^3(t)\Omega} \int_{t_i}^t ds \left[ (\mu_i(t, s) + \mu_{a}(t, s)) \langle \hat{\phi}_+(t) \hat{\varphi}_{\Delta}^2(s) \rangle + \mu_b(t, s) \langle \hat{\phi}_+(t) \hat{\varphi}_{\Delta}(s) \rangle \right] &= \Xi(t)\xi(t).
\end{align*}

If we set
\begin{equation}
\langle \hat{\phi}_+ \rangle = \langle \hat{\phi}_+ \rangle + \hat{\varphi},
\end{equation}

\langle \hat{\varphi} \rangle = 0 follows by construction. Thus we find
\begin{equation}
\langle \hat{\varphi}_+^2(t) \rangle = \langle \hat{\phi}_+(t) \rangle^2 + 3 \langle \hat{\phi}_+(t) \rangle \langle \hat{\varphi}(t) \rangle^2 + \langle \hat{\varphi}(t) \rangle^3.
\end{equation}

Assuming that the effect of the quantum fluctuations is negligible, i.e., $\langle \hat{\varphi}_+^2(t) \rangle \ll \langle \hat{\phi}_+(t) \rangle^2$, we obtain
\begin{align*}
\frac{d^2}{dt^2} \langle \hat{\phi}_+(t) \rangle + 3H \frac{d}{dt} \langle \hat{\phi}_+(t) \rangle + \frac{\lambda f(t)}{3a^3(t)\Omega} \langle \hat{\phi}_+(t) \rangle \\
- \frac{\lambda f(t)}{3a^3(t)\Omega} \int_{t_i}^t ds \left[ (\mu_i(t, s) + \mu_{a}(t, s)) \langle \hat{\phi}_+(t) \hat{\phi}_+(s) \rangle + \mu_b(t, s) \frac{d \langle \hat{\phi}_+(s) \rangle}{ds} \right] = \Xi(t)\xi(t).
\end{align*}

In the rest of this section, we simply use $\phi(t)$ instead of $\langle \hat{\phi}_+(t) \rangle$. The $f$- and $\mu$-terms give the correction to the evolution of $\phi$ in a deterministic manner while the effect of $\nu$-term gives a stochastic force.

Now we show that the effect of the $f$- and $\mu$-terms are negligible under the present condition. The effect of $\nu$-term is just to modify the potential. Thus we compare the force due to the $f$-term:
\begin{equation}
\frac{\lambda f(t)\phi(t)}{a^3(t)\Omega} \sim \frac{1}{8\pi^2} \lambda \phi(t) H^2 \log \left( \frac{H}{p_{\text{min}}(t)} \right),
\end{equation}

with that due to the bare potential (the third term in Eq. (1.13)):
\begin{equation}
V'(\phi(t)) = \frac{\lambda \phi^3(t)}{6}.
\end{equation}

Hence, the contribution from $f$-term can be neglected if $\phi^2 \gg H^2 \log(H/p_{\text{min}}(t))$. Since in the inflationary universe scenario the typical value of $\phi$ at the time when the comoving scale of the fluctuations of our interest crosses the
Hubble horizon scale during inflation is known to become the order of the Planck scale, $m_{pl}$. Therefore the above inequality holds. Thus we conclude that the $f$-term can be neglected.

The effect of $\mu$-term is not a simple change of the potential but has a hereditary one. Again the force coming from the $\mu_i$ and $\mu_a$-terms is roughly evaluated as

$$
-\frac{\lambda^2}{a^3(t)\Omega} \phi(t) \int_{t_i}^t ds \left( \mu(t,s) + \mu_a(t,s) \right) \phi^2(s)
$$

\[ \sim \left| \frac{\lambda^2}{16\pi^2} \phi^3(t) \int_{t_i}^t ds' \left[ \log \left( \frac{p_{\min}(s')}{H} \right) \delta(t-s') + H \log (k_{\min}(\eta_i - \eta')) \right] \right| 
\]

\[ \lesssim \frac{\lambda^2}{32\pi^2} \phi^3(t) (H\Delta t)^2. \]  

(4.16)

where $\eta_i$ is the conformal time corresponding to the cosmological time $t$ and $\Delta t$ is the maximum value that $t - t_i$ takes. Here we neglected the time-dependence of $\phi(s)$ because the slow rolling condition is expected to be satisfied. The details of evaluation of the integral is shown in appendix B. Equation (4.16) is to be compared with Eq. (4.15).

Then we find that if the $\mu_i$ and $\mu_a$-terms can be neglected as long as the condition $1 \gg \frac{\lambda}{16\pi^2} (H\Delta t)^2$ holds. This condition is satisfied for typical values of the model parameters such as $\lambda \sim 10^{-12}$ and $H\Delta t \sim 60$.

We turn to the contribution from $\mu_\nu$. In the same way, it is evaluated as

$$
-\frac{\lambda^2}{a^3(t)\Omega} \phi(t) \int_{t_i}^t ds \mu_\nu(t,s) \phi(s) \phi(s)
$$

\[ \sim \left| \frac{\lambda^2}{16\pi^2} \phi^2(t) \phi(t) \int_{t_i}^t ds' \frac{\eta^2}{\eta_i^2} \log (k_{\min}(\eta_i - \eta')) \right| 
\]

\[ \lesssim \frac{\lambda^2}{32\pi^2} \phi^2(t) \phi(t) (H\Delta t). \]  

(4.17)

Also, the details of calculation are provided in appendix B. This term should be compared with the friction term due to the cosmic expansion (the second term in Eq. (4.13)):

$$
3H\phi(t).
$$  

(4.18)

Then the ratio of these two terms is evaluated as $\frac{\lambda}{48\pi^2} \frac{(v/H^2)}{H\Delta t}$, and is found to be small. Here we introduced a constant, $v$, as a typical value of the inflaton mass squared: $v \sim \lambda \phi(t)^2/2$. For $\lambda \phi^4$ model, typical value for $v$ is given by $v/H^2 \sim 1/100$. Thus we conclude that $\mu_\nu$-term can be also neglected. Thus we concentrate on the effect of the $\nu$-term neglecting $f$- and $\mu$-terms in the rest of this section.

Under the condition that the fluctuation, $\delta \phi(t) := \phi(t) - \phi(t)|_{\xi=0}$ caused by $\xi(t)$ is small, the above equation reduces

$$
\delta \phi(t) + 3H \delta \phi(t) + V''(\phi(t)) \delta \phi(t) = \lambda \Xi(t) \xi(t).
$$  

(4.19)

Approximating $V''(\phi(t))$ and $\Xi(t)$ by constants $\nu$ and $\Xi$, respectively, the equation can be solved as

$$
\delta \phi(t) = -\frac{\lambda \Xi}{D} \left[ \int_{t_i}^t ds e^{-\lambda_1(t-s)} \xi(s) - \int_{t_i}^t ds e^{-\lambda_2(t-s)} \xi(s) \right],
$$  

(4.20)

where

$$
\lambda_1 = \frac{3H + D}{2}, \quad \lambda_2 = \frac{3H - D}{2},
$$  

$$
D = \sqrt{9H^2 - 4\nu}.
$$  

(4.21)

Since $\nu$ is much smaller than $H^2$ for $\lambda \phi^4$ model, we can approximate as $\lambda_1 \sim 3H$ and $\lambda_2 \sim v/3H$. Then the fluctuation caused by this stochastic field $\xi(t)$ is evaluated as

$$
\langle (\delta \phi)^2 \rangle_{EA} \sim \frac{\lambda^2 \Xi^2}{2D^2} \left( \frac{D^2}{3H\nu} - \frac{1}{\lambda_2} e^{-2\lambda_2(t-t_i)} \right)
$$

\[ \sim \frac{\lambda}{432\alpha'\pi^2} \left( 1 - e^{-2\lambda_2(t-t_i)} \right) H^2,
$$  

(4.22)

which means that the $\nu$-term broadens the peak width of each wave packet as much as $\langle (\delta \phi)^2 \rangle_{EA}$. Hence, the effect is negligible small as long as the width of the packet, $\langle (\delta \phi)^2 \rangle_{WP}$, is much larger than $\langle (\delta \phi)^2 \rangle_{EA}$. In the references
The factor \( \lambda v/H \) where we thought of \( \Xi(t) \) as the real fluctuation which is expected to become classical. Thus our interpretation is totally different from theirs. Here we do not further discuss this issue. A rigorous justification of our interpretation is provided in the next section.

The main effect of the \( \nu \)-term is to reduce the off-diagonal elements of the density matrix and brings the quantum state into a decohered one. In the above, we have shown that the effect of the environment is ineffective on the evolution of the trajectory of the peak, \( \phi(t) \), of a wave packet, \( \Psi_\psi(\phi, t) \), where \( \psi \) is a label to distinguish different wave packets. Here we use the initial peak position as the label of wave packets, i.e., \( \psi = \phi_\psi(t_i) \).

We set the initial condition at a time, \( t_i \), sufficiently after the scale of the fluctuations of our interest, \( L \), crossed the Hubble horizon, namely \( a(t_i)L > H^{-1} \). Assuming that the quantum state is not affected much by the environment before \( t = t_i \), we take the initial quantum state of the system as a pure state represented by a squeezed vacuum state which has a large variance

\[
(\delta \phi)^2_{QF} \sim H^2, \tag{4.23}
\]

which corresponds to a natural vacuum state in the de Sitter space such as the Euclidean vacuum state. We decompose the initial wave function into the superposition of the wave packets, \( \Psi_\psi(\phi, t) \). These wave packets are supposed to have a sharp peak at \( \phi_\psi(t) \) with the width, \( (\delta \phi)_{WP} \), that is much smaller than \( (\delta \phi)_{QF} \sim H \). We write the reduced density matrix at initial time as

\[
\bar{\rho}[\phi, \phi'; t_i] = \int d\psi \int d\psi' C(\psi) \Psi_\psi(\phi, t_i) C^*(\psi') \Psi^*_\psi(\phi', t_i). \tag{4.24}
\]

If the condition, \( (\delta \phi)^2_{WP} \gg (\delta \phi)^2_{QF} \), is not satisfied, this decomposition is of no use because we cannot construct the wave packets that do not lose their shape as time passes. (Later we find that there is another restriction related with the uncertainty relation.) Looking at Eq. (4.6), the evolution of this state under the influence of the environment will be approximately given by

\[
\bar{\rho}[\phi, \phi'; t] \sim \int d\psi \int d\psi' C(\psi) \Psi_\psi(\phi, t) C^*(\psi') \Psi^*_\psi(\phi', t) \exp \left( -\lambda^2 \Omega^2 a^6(t) \frac{(\psi - \psi')^2}{12H} \right), \tag{4.25}
\]

where we thought of \( \Xi(t) \) and \( \phi_\psi(t) - \phi_\psi'(t) \) as constants \( \Xi \) and \( \psi - \psi' \), respectively. The latter replacement will be justified because \( \dot{\phi}_\psi \) is nearly constant when the slow rolling condition is satisfied. The equation (4.25) indicates that the off-diagonal elements are exponentially suppressed when

\[
(\psi - \psi')^2 \gg (\delta \phi)^2_{dec} = \frac{6H}{\lambda^2 \Omega^2 a^6(t) \Xi^2} = \frac{216 \pi^2}{\alpha'} \left( \frac{p_{min}(t)}{H} \right)^6 \left( \frac{\lambda v}{H^2} \right)^{-1} H^2. \tag{4.26}
\]

The factor \( (\lambda v/H^2)^{-1} \) is a large number typically of \( O(10^{14}) \), but \( (p_{min}(t)/H)^6 \) becomes extremely small as \( e^{-6H(t-t_i)} \sim e^{-360} \). Hence, \( (\delta \phi)^2_{QF} \gg (\delta \phi)^2_{dec} \). Therefore even if we require that the wave packets have a peak that is sharp enough to satisfy \( (\delta \phi)^2_{WP} \ll (\delta \phi)^2_{QF} \), it is still possible to choose \( (\delta \phi)^2_{WP} \) so as to satisfy \( (\delta \phi)^2_{WP} \gg (\delta \phi)^2_{dec} \). So, if the width of the wave packets, i.e., the coarse graining scale of our view, is appropriately chosen, they lose the quantum coherence with each other during the inflation. Thus we conclude that the \( \nu \)-term leads the quantum state of the system effectively into a decohered one, which can be recognized as a statistical ensemble of the states represented by wave packets with a sufficiently sharp peak, without any significant distortion of the shape or the peak position of each wave packet.

Before closing this section we must mention the effect related with the uncertainty relation. Here we decomposed the initial quantum state of the system into a superposition of wave packets with a small variance with respect to the variables in configuration space, \( \phi \). We write the width of the wave packet, \( (\delta \phi)_{WP} \), as \( 1/\sqrt{\Gamma} \) for the later convenience. According to the uncertainty principle, the small variance in \( \phi \) necessarily indicates the existence of a large variance in its conjugate variable, \( \Omega a^3(t) \dot{\phi}(t) \). This variance may induce a large effect on the succeeding evolution. The possible presence of the effect of this kind was first pointed out by Matacz [17]. The induced variance in \( (\delta \phi)^2 \) will become of \( O \left( \int_{t_i}^{t} ds \left( \dot{\phi}(s) \right)^2 \right) \). This will be evaluated by using the uncertainty relation, \( \Omega a^3(t)(\dot{\phi}(t))_{UR}(\dot{\phi}(t))_{WP} \sim 1 \), as

\[
(\delta \phi)^2_{UR} := \left( \int_{t_i}^{t} ds \left( \dot{\phi}(s) \right)_{UR} \right)^2 \sim \frac{H^2}{9} \left( \Gamma H^2 \right) \left( \frac{p_{min}(t_i)}{\alpha' H^6} \right). \tag{4.27}
\]

Thus we find that this effect is also small compared with \( (\delta \phi)^2_{QF} \sim H^2 \) if \( t_i \) is set at a time well after the scale of our interest exceeds the Hubble horizon scale and unless \( \Gamma \) is extremely large. Further we note that
Here we introduce new variables, \( \varphi \) and \( \phi' \), as notational simplicity, we abbreviated them.

The restriction to the initial time obtained here is consistent with the general belief that the quantum fluctuations of the inflaton field become classical only after the horizon crossing.

V. MASTER EQUATION

In this section we directly study the evolution of the density matrix. We choose one solution of an approximate equation of motion of \( \phi(t) \) obtained by neglecting the effect of the environment. We denote this classical trajectory as \( \bar{\phi}(t) \). Namely, \( \bar{\phi}(t) \) satisfies

\[
\ddot{\bar{\phi}}(t) + 3H\dot{\bar{\phi}}(t) + V'(\bar{\phi}(t)) = 0.
\]

(5.1)

Here we introduce new variables, \( \varphi \) and \( \varphi' \), which represent the deviations of the \( \phi \) and \( \phi' \) from the classical trajectory \( \bar{\phi} \) by

\[
\phi = \bar{\phi} + \varphi, \quad \phi' = \bar{\phi} + \varphi',
\]

(5.2)

and assume that \( \varphi \) and \( \varphi' \) are small. Then the effective action \( S[\phi, \phi'] = iS_{\text{sys}}[\phi] - iS_{\text{sys}}[\phi'] + iS_{\text{IF}}[\phi, \phi'] \) is reduced to

\[
S[\phi, \phi'] = \Omega \int_{t_i}^{t_f} ds \ a^3(s) \left\{ \left( \frac{1}{2} \dot{\varphi}^2(s) - \frac{1}{2} V''(\bar{\phi}(s))\varphi^2(s) + O(\varphi^3) \right) - \left( \frac{1}{2} \dot{\varphi'}^2(s) - \frac{1}{2} V''(\bar{\phi}(s))\varphi'^2(s) + O(\varphi'^3) \right) \right\}
\]

\[
- \int_{t_i}^{t_f} ds \ \Delta(s) f(s) + i \int_{t_i}^{t_f} ds \ \Delta(s) \int_{t_i}^{s} ds' \nu(s, s') \Delta(s') + S^{(a)}[\phi, \phi']
\]

(5.3)

The time evolution operator for the density matrix can be obtained by constructing the Hamiltonian recognizing \( \bar{\phi} \) and \( \bar{\phi}' \) as two different interacting fields. Neglecting the cubic or higher order terms in \( \varphi \) and \( \varphi' \), the Hamiltonian corresponding to this action is obtained as

\[
H(t) = \frac{1}{\Omega a^3(t)} P_+(t) P_\Delta(t) + \Omega a^3(t) V''(\bar{\phi}(t)) \varphi_+(t) \varphi_\Delta(t)
\]

\[
+ f(t) \Delta(t) - \Delta(t) \int_{t_i}^{t_f} ds \left[ 2 (\mu_i(t, s) + \mu_i(t, s)) \Sigma(s) + \mu_b(t, s) \Sigma(s) + i\nu(t, s) \Delta(s) \right],
\]

(5.4)

where we defined

\[
\varphi_+ := \frac{\varphi + \varphi'}{2}, \quad \varphi_\Delta := \varphi - \varphi',
\]

(5.5)

and

\[
P_+ = P + P', \quad P_\Delta = \frac{P - P'}{2},
\]

(5.6)

are the conjugate momenta of \( \varphi_+ \) and \( \varphi_\Delta \), respectively. We note that \( P \) and \( P' \) are the conjugate momenta of \( \varphi \) and \( \varphi' \). Since \( \varphi \) and \( P \) are quantum mechanical operators, they should be associated with hat, \( \hat{\cdot} \), but in order to keep the notational simplicity, we abbreviated them.

In the above Hamiltonian, there appear Heisenberg operators at a past time. The existence of such operators is problematic in solving the evolution of the density matrix. To overcome this difficulty we replace the Heisenberg operators at a past time with those at present time by using the solution of lowest order Heisenberg equations [ES], which are given by

\[
\dot{P} = -\Omega a^3 V''(\bar{\phi}) \varphi, \quad \dot{\varphi} = \frac{1}{\Omega a^3} P,
\]

\[
\dot{P}' = \Omega a^3 V''(\bar{\phi}) \varphi', \quad \dot{\varphi}' = -\frac{1}{\Omega a^3} P'.
\]

(5.7)
Approximating $V''(\tilde{\phi})$ by a constant $v$, we can solve these equations as
\[ \varphi_+(s) = T_1(t,s)\varphi_+(t) + T_2(t,s)P_\Delta(t), \quad \varphi_\Delta(s) = T_1(t,s)\varphi_\Delta(t) + T_2(t,s)P_+(t), \] (5.8)
and $T_1(s,t)$ and $T_2(s,t)$, are calculated as
\[
T_1(s,t) = \frac{\lambda_1}{D} e^{\lambda_2(t-s)} - \frac{\lambda_2}{D} e^{\lambda_1(t-s)}, \\
T_2(s,t) = \frac{1}{a^3(t)\Omega D} \left( e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)} \right),
\] (5.9)
where $\lambda_1, \lambda_2$ were constants which were defined in Eq. (1.22). For the later purpose, we express $\varphi_+(t)$ in an alternative way as
\[ \varphi_+(t) = O_2(t)e^{\lambda_2(t-s)} - O_1(t)e^{\lambda_1(t-s)}, \] (5.10)
where the operators $O_1(t)$ and $O_2(t)$ are defined by
\[
O_1(t) := \frac{1}{D} \left( \lambda_2 \varphi_+(t) + \frac{1}{a^3(t)\Omega} P_\Delta(t) \right), \\
O_2(t) := \frac{1}{D} \left( \lambda_1 \varphi_+(t) + \frac{1}{a^3(t)\Omega} P_+(t) \right).
\] (5.11)
Then $f$-term, that is the term in $H$ which contains $f$, becomes
\[ H^{(f)} = \lambda(\bar{\phi} + \varphi_+)\varphi_\Delta f(t). \] (5.12)
The $\mu$-term is explicitly written as
\[
H^{(\mu)} = \frac{\lambda^2}{32\pi^2} \frac{a^3(t)\Omega}{H} \log \left( \frac{p_{\text{min}}(t)}{H} \right) \varphi_\Delta(t) \left( \bar{\phi}(t)^3 + 3\bar{\phi}(t)^2\varphi_+(t) \right) \\
- \lambda^2 \varphi_\Delta(t) \left[ \bar{\phi}(t) + \varphi_+(t) \right] \int_{t_1}^t ds \left( \mu_a(t,s)\bar{\phi}^2(s) + \mu_b(t,s)\bar{\phi}(s)\dot{\phi}(s) \right) \\
- \lambda^2 \varphi_\Delta(t) \int_{t_1}^t ds \left( 2\mu_a(t,s)\bar{\phi}(s)\varphi_+(s) + \mu_b(t,s)\bar{\phi}(s)\dot{\varphi}_+(s) + \mu_b(t,s)\dot{\bar{\phi}}(s)\varphi_+(s) \right). 
\] (5.13)
The first line corresponds to the instantaneous part, $\mu_i$-term. In the second line, the ratio between the first and second term in the round bracket is given by
\[
\frac{|\mu_a(t,s)\bar{\phi}(t)|}{|\mu_b(t,s)\bar{\phi}(s)|} \sim \frac{a^2(t)H\bar{\phi}(t)}{\bar{\phi}(t)} \sim \frac{a^2(t)}{\bar{\phi}^2(t)} \left( \frac{H^2}{v} \right), 
\] (5.14)
and is found to be much greater than unity. Thus the second term can be neglected. By the same reason, the last term in the round bracket in the last line can be also neglected. Then the last line is rewritten by using the relation [5.10] as
\[
\rightarrow \lambda^2 \varphi_\Delta(t)\bar{\phi}(t)O_1(t) \int_{t_1}^t ds \left\{ 2\mu_a(t,s)\bar{\phi}(s) - \lambda_1\mu_b(t,s)\bar{\phi}(s) \right\} e^{\lambda_1(t-s)} \\
- \lambda^2 \varphi_\Delta(t)\bar{\phi}(t)O_2(t) \int_{t_1}^t ds \left\{ 2\mu_a(t,s)\bar{\phi}(s) - \lambda_2\mu_b(t,s)\bar{\phi}(s) \right\} e^{\lambda_2(t-s)}. 
\] (5.15)
Substituting the explicit form of $\mu_a(t,s)$ and $\mu_b(t,s)$, and approximating $\bar{\phi}(s)$ by a constant $\bar{\phi}$, we will find that all the dominant contribution arises from the terms that contain $\mu_a(t,s)$. Using the formulas given in appendix B, we obtain
\[
H^{(\mu)} = \mu_1(t)\varphi_\Delta \bar{\phi} + \mu_2(t)\varphi_\Delta \varphi_+(t) + \mu_3(t)\varphi_\Delta O_1(t) + \mu_4(t)\varphi_\Delta O_2(t), 
\] (5.16)
with
\[
|\mu_1(t)|, |\mu_2(t)|, |\mu_4(t)| \lesssim \frac{\lambda^2}{16\pi^2} \bar{\phi}^2 (H\Delta t)^2,
\]
\[ |\mu_3(t)| \lesssim \frac{\lambda^2 a^6(t) \Omega}{16\pi^2 a^3(t)} \tilde{\phi}^2 (H \Delta t), \] (5.17)

where again we approximated \( \tilde{\phi}(t) \) by a constant \( \tilde{\phi} \).

The \( \nu \)-term can be also rewritten by using the relation [5.8] as

\[ H_\nu = -i\nu_1(t)\varphi_\Delta^2(t) - i\nu_2(t)\varphi_\Delta(t)P_+(t) + O(\varphi^3). \] (5.18)

The coefficients are evaluated as

\[ \nu_1(t) \sim \frac{\lambda^2 \tilde{\phi}(t)}{144 \pi^2 \rho_{\text{min}}(t)} a^3(t) \Omega H^3 e^{-\lambda_2(t-t_i)} = e^{\dot{H}t} e^{-\lambda_2(t-t_i)} \tilde{\rho} \]
\[ \nu_2(t) \sim \frac{\lambda^2 \tilde{\phi}(t)}{a^3(t) \Omega} \left( 1 - e^{-\lambda_2(t-t_i)} \right), \] (5.19)

where we used \( H(t-t_i) \gg 1 \) and \( \lambda_2(t-t_i) = O(1) \). Strictly speaking, the former is not the case for \( t \sim t_i \). However, the absolute value of the correct expression does not become much larger than that of this approximate expression. Since in the later calculation we will find that the contribution from \( t \sim t_i \) does not become significant, this approximation is not so bad.

In the above calculations, we used many crude approximations. But small errors caused by these approximations will not significantly affect on the results obtained by the following discussion.

Putting all the results together, we get the Hamiltonian that does not contain any hereditary terms as

\[ H = \frac{1}{\Omega a^3(t)} P_\Delta + \Omega a^3(t) v(t) \varphi_\Delta \varphi_+ + u(t) \varphi_\Delta \]
\[ + \mu_3(t) \varphi_\Delta(t) O_1(t) + \mu_4(t) \varphi_\Delta(t) O_2(t) - i\nu_1(t) \varphi_\Delta^2 - i\nu_2(t) \varphi_\Delta P_+, \] (5.20)

where we defined

\[ v(t) := V''(\tilde{\phi}(t)) + \frac{1}{\Omega a^3(t)} (\lambda f(t) + \mu_2(t)), \]
\[ u(t) := \lambda f(t) \tilde{\phi}(t) + \mu_1(t) \tilde{\phi}(t). \] (5.21)

The master equation can be derived from the above Hamiltonian. In the coordinate representation, it becomes \( i\partial \tilde{\rho}/\partial t = H \tilde{\rho} \), where \( P_+ \) and \( P_\Delta \) in \( H \) are to be replaced by \( -i\frac{\partial}{\partial \varphi_+} \) and \( -i\frac{\partial}{\partial \varphi_\Delta} \), respectively. We note that the same master equation can be derived by means of different methods \[18\]. \( v(t) \) is dominated by the first term due to the same reason explained around Eqs. (4.14), (4.15) and (4.16) in the preceding section. So we approximate \( v(t) \) by a constant \( \nu \) as before. Later \( u(t) \) is found to be compared with \( a^3(t) \Omega V' \). Since the former is much smaller than the latter because of the same reason, the effect of \( u(t) \) is negligible small. However, we keep this term for a while until this fact turns out to be manifest.

In order to solve the evolution of the reduced density matrix \( \tilde{\rho}(\varphi, \varphi'; t) \), it is better to consider in the \( k - \varphi_\Delta \) representation, which is defined by

\[ \zeta(k, \varphi_\Delta; t) = \int d\varphi_+ \exp \left[ -ik\varphi_+ a^3(t) \Omega \right] \tilde{\rho}(\varphi_+, \varphi_\Delta; t). \] (5.22)

Then the evolution equation for \( \zeta \) becomes

\[ \frac{\partial}{\partial t} \zeta = \left[ 3Hk \frac{\partial}{\partial k} - k \frac{\partial}{\partial \varphi_\Delta} + \nu_2 \frac{\partial}{\partial \varphi_\Delta} - iuv_\Delta + \mu_3(t) \varphi_\Delta \tilde{O}_1 + \mu_4(t) \varphi_\Delta \tilde{O}_2 - v_1 \varphi_\Delta^2 - a^3 \Omega v_2 \varphi_\Delta \right] \zeta, \] (5.23)

where we introduced

\[ \tilde{O}_1 := \frac{1}{a^3(t) \Omega D} \left[ \lambda_2 \frac{\partial}{\partial k} - \frac{\partial}{\partial \varphi_\Delta} \right], \]
\[
\hat{O}_2 := \frac{1}{a^3(t)\Omega D} \left[ \lambda_1 \frac{\partial}{\partial k} - \frac{\partial}{\partial \varphi_{\Delta}} \right].
\] (5.24)

Let us assume the Gaussian form of the density matrix as

\[
\zeta = \mathcal{N}_\zeta \exp \left[ -\frac{1}{2} \sum_{i,j} M_{ij} x_i x_j - \sum_i N_i x_i \right],
\] (5.25)

where \( x_i = (k, \varphi_\Delta) \) and \( \mathcal{N}_\zeta \) is a time independent normalization constant. Then the evolution equation for the density matrix reduces to the following set of equations:

\[
\frac{d}{dt} M = \begin{pmatrix}
6H & -2 & 0 \\
v & 3H & -1 \\
0 & 2v & 0
\end{pmatrix} M + \begin{pmatrix}
0 \\
a^3\Omega_2 \\
0
\end{pmatrix} \\
-\frac{1}{a^3(t)\Omega D} \begin{pmatrix}
\mu_1 (M_{k\Delta} - \lambda_2 M_{kk}) + \mu_4 (M_{k\Delta} - \lambda_1 M_{\Delta\Delta}) \\
2\mu_3 (M_{k\Delta} - \lambda_2 M_{k\Delta}) + 2\mu_4 (M_{\Delta\Delta} - \lambda_1 M_{\Delta\Delta})
\end{pmatrix},
\] (5.26)

where

\[
M = \begin{pmatrix}
M_{kk} \\
M_{k\Delta} \\
M_{\Delta\Delta}
\end{pmatrix}, \quad
N = \begin{pmatrix}
N_k \\
N_{k\Delta} \\
N_{\Delta\Delta}
\end{pmatrix}.
\] (5.27)

We first consider the evolution of \( M_{ij} \). For this purpose, we define

\[
\sigma_1 := 3H, \quad \sigma_{2,3} := 3H \pm D, \\
\mathbf{e}_1 := \begin{pmatrix}
2 \\
\sigma_1 \\
2v
\end{pmatrix}, \quad
\mathbf{e}_2 := \begin{pmatrix}
2 \\
\sigma_3 \\
\sigma_3^2/2
\end{pmatrix}, \quad
\mathbf{e}_3 := \begin{pmatrix}
2 \\
\sigma_2 \\
\sigma_2^2/2
\end{pmatrix}.
\] (5.28)

Then

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} = \frac{1}{2D^2} (\mathbf{e}_2 + \mathbf{e}_3 - 2\mathbf{e}_1), \quad
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} = \frac{1}{2D^2} (2\sigma_1 \mathbf{e}_1 - \sigma_2 \mathbf{e}_2 - \sigma_3 \mathbf{e}_3),
\] (5.29)

follows. Introducing new parameterization of \( M_{ij} \) by

\[
M = M_1 \mathbf{e}_1 + M_2 \mathbf{e}_2 + M_3 \mathbf{e}_3,
\] (5.30)

the equations for \( M_j \), where \( j = 1, 2, \) or \( 3 \), are decoupled like

\[
\frac{dM_j}{dt} = \sigma_j M_j + S_{M_j},
\] (5.31)

where

\[
\begin{pmatrix}
S_{M1} \\
S_{M2} \\
S_{M3}
\end{pmatrix} = \frac{\mu_1}{D^2} \begin{pmatrix}
-2 \\
1 \\
1
\end{pmatrix} + \frac{a^3\Omega_2}{D^2} \begin{pmatrix}
\sigma_3 \\
-\sigma_3^2/2 \\
-\sigma_3/2
\end{pmatrix} - \frac{\mu_3}{a^3\Omega D} \begin{pmatrix}
M_1 - 2M_3 \\
-M_1 \\
2M_3
\end{pmatrix} - \frac{\mu_4}{a^3\Omega D} \begin{pmatrix}
M_1 - 2M_2 \\
2M_2 \\
-M_1
\end{pmatrix}.
\] (5.32)

Here \( \mu \)-terms contain \( M_i \) but we can solve the above equation as if \( S_{Mj} \) is a given source term. We solve the above equation perturbatively taking \( \lambda \) as a small parameter. In this sense, \( M_j \) is also expanded in powers of \( \lambda \). At the lowest order, we solve the homogeneous equation without source term in Eq. (5.31). Then to find the next order solution of Eq. (5.31), we need to solve the equation with the source term, \( S_{Mj} \). At this stage, we can substitute the lowest order solution, \( M_j^{(0)}(t) \), into \( S_{Mj} \). Then \( S_{Mj} \) can be considered as a given source term. Here we should keep in mind the limitation of the present analysis. In deriving Eq. (5.31), only the one loop order correction was taken.
into account, and the lowest order Heisenberg equation was used to remove hereditary terms in the Hamiltonian. Thus only the correction up to $O(\lambda^2)$ is valid. If we boldly solve Eq. (5.31) without regard to this limitation, many unphysical pathological features will give arise.

Formally, the solution is given by

$$M_j(t) = e^{\sigma_j(t-t_i)} M_j(t_i) + \delta M_j(t), \quad (5.33)$$

where

$$\delta M_j(t) = \int_{t_i}^t ds e^{\sigma_j(t-s)} S_{M_j}(s). \quad (5.34)$$

The contribution from $\nu$-terms can be evaluated without specifying the lowest order solution, $M_j^{(0)}(t)$. The corresponding inhomogeneous solution is approximately evaluated as

$$\begin{pmatrix} \delta M_1^{(\nu)}(t) \\ \delta M_2^{(\nu)}(t) \\ \delta M_3^{(\nu)}(t) \end{pmatrix} \sim \nu_1(t) \begin{pmatrix} (1 - e^{-\lambda_2(t-t_i)})/\lambda_2 \\ (3H/2\lambda_2^2)(1-e^{-\lambda_2(t-t_i)})^2/1/6H \end{pmatrix}, \quad (5.35)$$

Next we consider $N$-part. In the similar manner, we can get the solution as

$$\begin{pmatrix} N_k(t) \\ N_\Delta(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix}, \quad (5.36)$$

with

$$N_j(t) = e^{\lambda_j(t-t_i)} N_j(t_i) + \delta N_j(t), \quad (j = 1, 2). \quad (5.37)$$

Here $\delta N_j(t)$ is an inhomogeneous solution related to the $u$-term. In the later discussion, we conclude that the effect of these terms can be neglected. Thus for the present purpose, we have only to know their order of magnitude. Hence, we roughly approximate the time dependence of $u(t)$ by $a^3(t)$. Then we obtain

$$\begin{pmatrix} \delta N_1(t) \\ \delta N_2(t) \end{pmatrix} \sim iu(t) \begin{pmatrix} -3H(1-e^{-\lambda_2(t-t_i)})/\lambda_2 \\ 1 \end{pmatrix}, \quad (5.38)$$

Now we discuss the initial condition for the reduced density matrix. We set the initial condition at a time when the size of the fluctuations of our interest, $k^{-1}a(t) \equiv (\alpha^{-1}1/3a(t))$, becomes larger than the horizon scale, $H^{-1}$, where $\alpha = (2\pi)^{-3}$. In our present approximation, the earlier epoch is inaccessible, for we used the evaluation of the effective action, i.e., the coefficients $f, \mu$ and $\nu$, under the assumption that $a(t)H \gg k_{min} \sim k_c$. Here we assume that the evolution of the fluctuations for $t < t_i$ can be approximated by the evolution of a non interacting field. Then the wave function will be given by

$$\Psi(\varphi) \propto \exp \left[ -\frac{A}{2} \varphi^2 \right], \quad (5.39)$$

where

$$A = \frac{\alpha}{H^2} \left( 1 + i \frac{H}{p_c(t_i)} \right), \quad (5.40)$$

and $p_c(t) = k_c/a(t)$.

As mentioned in Sec. 2, we decompose the wave function into a superposition of Gaussian wave packets

$$\Psi_\psi(\varphi) \propto \exp \left[ -\frac{\Gamma}{2}(\varphi - \psi)^2 \right], \quad (5.41)$$

as

$$\Psi(\varphi) \propto \int d\psi e^{-\frac{\Gamma}{2}\psi^2}\Psi_\psi(\varphi), \quad (5.42)$$

where
\[ F = \frac{A \Gamma}{\Gamma - A}. \]  
(5.43)

As before, \( \psi \) represents the initial position of the peak of wave packets and it is real. For simplicity, we set \( \Gamma \) is also real. The peak width of the wave packet, \( \Gamma^{-1/2} \) should be sufficiently small compared with the extension of the wave function, \( \sqrt{\langle \varphi^2 \rangle_{QF}} \sim H \). Here we choose, \( \Gamma \) to satisfy
\[ H^2 \Gamma \gg \frac{\alpha H}{\rho_c(t_i)} > 1. \]  
(5.44)

The latter inequality comes from the fact that the initial condition is set after the scale of the fluctuations of our interest becomes larger than the Hubble horizon scale. With this choice of \( \Gamma \), we find that \( F \sim A \).

Further, we assume that the initial condition is set at a time when
\[ \lambda \ll \frac{p_{\text{min}}(t_i)}{H^3} \ll 1, \]  
(5.45)

is satisfied. This choice of the initial time much simplifies the following analysis. Further we assume
\[ \frac{1}{\lambda} \gg H^2 \Gamma. \]  
(5.47)

We refer to this case specified by Eqs. (5.44) and (5.47) as case A. In case A, as the restriction to \( \Gamma \) is mild, we can examine the dependence of the evolution of the density matrix on the choice of \( \Gamma \).

Alternatively, instead of the limitation on the initial time Eq. (5.45), we can set a rather strong limitation on the initial width of the wave packets as
\[ \frac{1}{\lambda} \gg \frac{H^3}{\Gamma} \]  
(5.50)

where we introduced the notation
\[ \rho_{\psi\psi'}(t) = \sum_{i,j} m_{ij} y_i y_j - \sum_i n_i y_i. \]  
(5.51)

We refer to this case as case B.

It will be possible to examine more general cases but the analysis becomes much more complicated. So here we restrict our attention to these limited two cases. Now the reduced density matrix is also decomposed as Eq. (2.6)
\[ \rho(t) \propto \int d\psi \int d\psi' \exp \left[ \frac{F}{2} \psi^2 + \frac{F^*}{2} \psi'^2 \right] \rho_{\psi\psi'}(t), \]  
(5.48)

where we introduced the partial reduced density matrix, \( \rho_{\psi\psi'}(t) \), which satisfies the same evolution equation as \( \rho(t) \). We refer to this equation as Eq. (5.48).

If we introduce the notation
\[ \rho_{\psi\psi'}(t) = \sum_{i,j} m_{ij} y_i y_j - \sum_i n_i y_i, \]  
(5.50)

where \( y_i = (\varphi_+, \varphi_-) \). The coefficients \( m_{ij} \) and \( n_i \) are related to those in the corresponding \( k - \Delta \) representation, \( M_{ij} \) and \( N_i \), as
\[ m_{++} = \frac{a^6 \Omega^2}{M_{kk}}, \quad m_{+\Delta} = ia^3 \Omega \frac{M_{k\Delta}}{M_{kk}}, \quad m_{\Delta\Delta} = M_{\Delta\Delta} - \frac{M_{k\Delta}^2}{M_{kk}}, \]  
(5.51)

\[ n_+ = ia^3 \Omega \frac{N_k}{M_{kk}}, \quad n_\Delta = N_\Delta - \frac{M_{k\Delta}}{M_{kk}} N_k, \quad M_{\Delta\Delta} = \frac{\varphi_+^2}{m_{++}}. \]

Thus we obtain
\[ M_{kk}(t_i) = \frac{a^6 (t_i) \Omega^2}{2\Gamma}, \quad M_{k\Delta}(t_i) = 0, \quad M_{\Delta\Delta}(t_i) = \frac{\Gamma}{2}, \]  
(5.52)
\[ N_k(t_i) = i a^3(t_i) \Omega \psi_+, \quad N_{\Delta}(t_i) = \frac{\Gamma}{2} \psi_\Delta, \]

where
\[ \psi_+ := \frac{\psi + \psi'}{2}, \quad \psi_\Delta := \psi - \psi', \]

Noting that
\[ \left( \begin{array}{ccc} M_1 \\ M_2 \\ M_3 \end{array} \right) = \frac{1}{D^2} \left( \begin{array}{ccc} -\sigma & \sigma_1 & -1 \\ \sigma_2/8 & -\sigma_2/2 & 1/2 \\ \sigma_3/8 & -\sigma_3/2 & 1/2 \end{array} \right) \left( \begin{array}{ccc} M_{kk} \\ M_{k\Delta} \\ M_{\Delta\Delta} \end{array} \right), \quad \left( \begin{array}{ccc} N_1 \\ N_2 \end{array} \right) = \frac{1}{D} \left( \begin{array}{ccc} \lambda_1 & -1 \\ -\lambda_2 & 1 \end{array} \right) \left( \begin{array}{ccc} N_k \\ N_{\Delta} \end{array} \right), \]

we can calculate \( M_j(t_i) \) as
\[ \left( \begin{array}{ccc} M_1(t_i) \\ M_2(t_i) \\ M_3(t_i) \end{array} \right) \sim \frac{a^6(t_i) \Omega^2}{2 D^2} \left( \begin{array}{ccc} -v & \sigma_1/8 & 1/2 \\ \sigma_2/8 & -\sigma_2/2 & 1/2 \\ \sigma_3/8 & -\sigma_3/2 & 1/2 \end{array} \right) + \frac{\Gamma}{2 D^2} \left( \begin{array}{ccc} -1 \\ 1/2 \\ 1/2 \end{array} \right). \]

Then the lowest order solution in \( \lambda \) is given by \( M_j^{(0)}(t) = e^{\sigma_j(t-t_i)} M_j(t_i) \). Roughly speaking, when \( \Gamma \) is not so large and satisfies
\[ H^3 a^3(t_i) \Omega = \frac{\alpha H^3}{p^3_{\min}(t_i)} \gg H^2 \Gamma, \]

the first term in the right hand side of Eq. (5.55) dominates. In case B, this is always the case if \( \lambda \gg p^3_{\min}(t_i)/H^3 \). For the later convenience, we introduce
\[ \tilde{\Gamma} := \left( \frac{1}{\Gamma} + \frac{\Gamma}{D^2 a^6(t_i) \Omega^2} \right)^{-1}. \]

Then it follows that \( M_2(t_i) \sim a^6(t_i) \Omega^2/2 D^2 \tilde{\Gamma} \). We note that the order of magnitude of the initial value of the other components is the same or smaller than that of \( M_2(t_i) \).

To obtain the next order correction, first we need to evaluate the source term \( S_{M_j} \). The order of magnitude of \( \nu \)-term is estimated as
\[ S_{M_j}^{(\nu)} = O\left( \frac{\lambda^2 \tilde{\omega}^2 a^6(t) \Omega^2 H}{144 \pi^2 a^6(t_i)} \right), \]

while that of \( \mu \)-term is estimated as
\[ \left| \frac{S_{M_1}^{(\mu)}}{S_{M_2}^{(\mu)}} \right| < \lambda^2 \tilde{\omega}^2 a^6(t) \Omega^2 H (H \Delta t)^2 \left( \frac{1}{16 \pi^2 H^2} \right) \left( \frac{1}{a^3(t_i)/a^3(t)} \right). \]

Now it is clear that the contribution from the \( \mu \)-term is 4\( \alpha' (H \Delta t)^2 / (\tilde{\Gamma} H^2) \) times smaller than that from the \( \nu \)-term. This factor can be set small in both cases \( A \) and \( B \) by choosing \( \Gamma \) appropriately. This suppression of the contribution from the \( \mu \)-term is not so trivial. The time dependence of \( \mu_3(t) \) is approximately proportional to \( a^6(t) \). Hence, if there appears the combination, \( a^3(t) \mu_3(t) M_2(t) \), in \( S_{M_j}^{(\mu)} \), it behaves as \( a^3(t) \) and dominates the source term when \( a^3(t)/a^3(t_i) \) becomes exponentially large. The disappearance of this kind of dangerous terms is not manifest in Eqs. (5.31) and (5.32). Here we should note that the contribution of \( \mu \)-term to \( \delta M_3 \), is much more suppressed by the existence of the factor \( a^3(t_i)/a^3(t) \) compared with that of the \( \nu \)-term. Thus, \( \delta M_1 \) and \( \delta M_2 \) might be dominated by the \( \mu \)-term but \( \delta M_3 \) is not.

Under the conditions for case \( A \) or case \( B \), \( M_1(t) \) and \( M_3(t) \) are dominated by the inhomogeneous solution, \( \delta M_j(t) \), while the contribution to \( M_2(t) \) from \( \delta M_2(t) \) gives only a small collection to \( M_2(t) \). Then from Eq. (5.33), we find
\[ M_1(t) \sim \frac{\lambda^2 \tilde{\omega}^2 a^3(t) \Omega}{1296 \pi^2 p_{\min}^3} \frac{H (1 - e^{-\lambda_2(t-t_i)})}{\lambda_2}, \]
\[ M_2(t) \sim e^{\sigma_3(t-t_i)} M_2(t_i) + \delta M_2(t) \sim \frac{a^6(t)}{4\Gamma} e^{-\sigma_3(t-t_i)} + \delta M_2(t), \]

\[ M_3(t) \sim \frac{\lambda^2 e^2 a^3(t) \Omega}{7776 \pi^2 \mu^3(t)} . \]  

(5.60)

Note that they are all real.

If we again substitute this solution into \( S_{M_j}^{(\mu)} \), its time dependence becomes proportional to \( a^6(t) \) while that of \( S_{M_j}^{(\nu)} \) is given by \( a^6(t) \). Thus it seems that \( S_{M_j}^{(\mu)} \) becomes dominant when \( a^3(t)/a^3(t_i) \) becomes large. However, this does not mean the breakdown of the present perturbation scheme. In the present calculation, we used the lowest order Heisenberg equation in deriving Eq. (5.33). Thus as mentioned before, such a substitution of \( \delta M_j \) into \( S_{M_j}^{(\mu)} \) is not allowed.

Using Eqs. (5.33), (5.54), and (5.60), we finally obtain

\[ m_{++}(t) \sim \frac{a^6(t) \Omega^2}{2M_2(t)} \sim e^{\sigma_3(t-t_i)} \frac{a^6(t_i) \Omega^2}{2M_2(t_i)} \sim 2\tilde{\Gamma} e^{\sigma_3(t-t_i)}, \]

\[ m_{+\Delta}(t) \sim \frac{i\nu}{3H} a^3(t) \Omega, \]

\[ m_{\Delta\Delta}(t) = \frac{4D^2 M_2(t) M_3(t) - D^2 M_2^2(t)}{2M_2(t)} \sim 18H^2 M_3(t). \]  

(5.61)

It should be mentioned that \( m_{+\Delta} \) stays purely imaginary. Thus it does not contribute to the absolute magnitude of the density matrix.

For \( n_j \), with the aid of Eqs. (5.37), (5.52) and (5.54), we obtain

\[ n_+(t) \sim \frac{ia^3(t) \Omega N_1(t)}{2M_2(t)} \]

\[ \sim 2\tilde{\Gamma} \left[ -e^{\lambda_2(t-t_i)} \psi_+ \right] + e^{\sigma_3(t-t_i)} \left( \frac{u(t)}{3H^2 a^3(t) \Omega} \right) \left[ \frac{H(1-e^{-\lambda_2(t-t_i)})}{\lambda_2} - i e^{\lambda_2(t-t_i)} \frac{\Gamma \psi_\Delta}{6H a^3(t_i) \Omega} \right] , \]

\[ n_\Delta(t) = \frac{2M_2(t)}{2M_2^2(t)} (2M_2(t) N_2(t) - 2M_2(t) N_1(t) + M_1(t) N_2(t) - N_1(t)) \]

\[ \sim -i \left[ \frac{u(t)}{3H} + e^{\lambda_2(t-t_i)} \frac{12H \tilde{\Gamma} M_1(t)}{a^3(t) \Omega} \psi_+ \right] + e^{\lambda_3(t-t_i)} \left[ \frac{2\tilde{\Gamma} \tilde{\Gamma} M_1(t)}{a^3(t) \Omega} \psi_+ \right] . \]  

(5.62)

Here we note that in the above evaluations of \( m_{ij}(t) \) and \( n_j(t) \) there is no relevant contribution from \( \delta M_1 \) and \( \delta M_2 \). Among \( \delta M_j \) a relevant contribution is provided only by \( \delta M_3 \), which is almost insensitive to the effect of \( \mu \)-term.

To see the degree of decoherence, we consider the following ratio

\[ R := \frac{\max|\rho_{\psi \psi'}(t)|}{\max|\rho_{\psi \psi'}(t_i)|}, \]  

(5.63)

where max means the maximum value when \( \phi_+ \) and \( \phi_\Delta \) are varied. As was discussed in Sec. 2, \( R \) is the quantity that represents how efficiently the coherence between the two worlds (two wave packets) labeled by, \( \psi \) and \( \psi' \), gets lost. With this definition, \( R \) is evaluated by

\[ R = \exp \left( K(t) - K(t_i) \right), \]  

(5.64)

with

\[ K := \frac{1}{2} \left( \frac{\Re(n_\Delta)^2}{m_{\Delta\Delta}} + \frac{\Im(n_+)^2}{m_{++}} \right) . \]  

(5.65)

In deriving this formula, we used the fact that \( \mathcal{N} \) is a constant and we neglected the small logarithmic correction that comes from the Gaussian-integral in Eq. (5.23). Then, after a straightforward calculation, we obtain

\[ K(t_i) = \frac{\Gamma^2 \psi_\Delta^2}{4}, \]  

(5.66)

and

\[ K(t) \sim \left( \frac{M_7^2(t)}{M_2(t) M_3(t)} + 1 \right) \frac{\Gamma^2 \tilde{\Gamma}^2}{32H^2 a^6(t_i) \Omega^2} \psi_\Delta^2, \]  

(5.67)
for $H(t - t_i) \gg 1$. In Eq. (5.67), the first and the second terms in the round bracket represent the contribution from the corresponding terms in Eq. (5.65), respectively. It is clear that the second term dominates $K(t)$.

When the first term in the defining equation of $\tilde{\Gamma}$ dominates, i.e., when Eq. (5.56) is satisfied,

$$\frac{\Gamma \tilde{\Gamma}}{9H^2 a^6(t_i) \Omega^2} \sim \frac{\Gamma^2}{9H^2 a^6(t_i) \Omega^2} \ll 1. \tag{5.68}$$

Hence, $K(t)$ is much smaller than $K(t_i)$, and $R$ is predominantly determined by $K(t_i)$ as

$$R \sim \exp (-K(t_i)). \tag{5.69}$$

This means that the coherence between two wave packets labeled by $\psi$ and $\psi'$ is exponentially suppressed for large $\psi_\Delta$ and the typical scale of decoherence is determined by the width of the wave packets. For two wave packets with $\psi_\Delta^2 < \Gamma^{-1}$, their overlap is large. Hence, it is natural that their coherence is maintained. So the scale of the decoherence depends totally on the width of wave packets that we choose.

In case A, we can set the width of wave packet much smaller so that the second term in the defining equation of $\tilde{\Gamma}$ dominates. Then

$$\frac{\Gamma \tilde{\Gamma}}{9H^2 a^6(t_i) \Omega^2} \sim 1 - \frac{9H^2 a^6(t_i) \Omega^2}{\Gamma^2}, \tag{5.70}$$

and $R$ becomes

$$R \sim \exp \left(- \frac{9H^2 a^6(t_i) \Omega^2}{4\Gamma} \psi_\Delta^2 \right). \tag{5.71}$$

In this case the typical scale of the decoherence between two different wave packets, $(\delta\psi)^2_{\text{dec}}$, will be given by

$$(\delta\psi)^2_{\text{dec}} \sim \frac{2\Gamma}{9H^2 a^6(t_i) \Omega^2}. \tag{5.72}$$

Thus the decoherence does not occur if we set the initial wave packets too narrow, i.e., if $\Gamma$ is too large. The best choice of $\Gamma$ that minimizes $R$ is given by

$$\Gamma H^2 = \frac{3H^3 a^3(t_i) \Omega}{p_c^3(t_i)} = 3\alpha H^3 \tag{5.73}$$

With this choice of $\Gamma$, the typical scale of the decoherence becomes

$$(\delta\psi)^2_{\text{dec}} \sim \frac{4H^2}{3} \left( \frac{p_c^3(t_i)}{\alpha H^3} \right). \tag{5.74}$$

This bound mainly comes from the broadening of the wave packet due to the uncertainty relation. This fact can be understood by seeing that the minimum value of $(\delta\psi)^2_{\text{dec}}$ correspond to the scale given in Eq. (4.27) in the preceding section. To see this fact in the present context, we consider the expectation value of $(\varphi_{++})^2$, which is expected to represent the degree of the broadening of the wave packet. If $H^2 \Gamma \ll \alpha H^3 / p_c^3(t_i)$, we have

$$\langle (\varphi_{++})^2 \rangle \sim 2/m_{++} \sim 1/\tilde{\Gamma}, \tag{5.75}$$

and is found to stay almost constant. Instead, if we assume a very large value of $\Gamma$ such that violates this condition, the broadening of the wave packet at a late time is evaluated as

$$\langle (\varphi_{++})^2 \rangle \sim \frac{1}{\tilde{\Gamma}} \left[ \frac{\Gamma H^2}{3\alpha H^3 / p_c^3(t_i)} \right]^2, \tag{5.76}$$

and becomes much larger than the initial width of the wave packet $\Gamma^{-1}$. It is easy to see that the minimum broadening of the wave packet

$$\langle (\varphi_{++})^2 \rangle \sim \frac{2H^2}{3} \left( \frac{p_c^3(t_i)}{\alpha H^3} \right), \tag{5.77}$$

with this choice of $\Gamma$. This bound mainly comes from the broadening of the wave packet due to the uncertainty relation. This fact can be understood by seeing that the minimum value of $(\delta\psi)^2_{\text{dec}}$ correspond to the scale given in Eq. (4.27) in the preceding section. To see this fact in the present context, we consider the expectation value of $(\varphi_{++})^2$, which is expected to represent the degree of the broadening of the wave packet. If $H^2 \Gamma \ll \alpha H^3 / p_c^3(t_i)$, we have

$$\langle (\varphi_{++})^2 \rangle \sim 2/m_{++} \sim 1/\tilde{\Gamma}, \tag{5.75}$$

and is found to stay almost constant. Instead, if we assume a very large value of $\Gamma$ such that violates this condition, the broadening of the wave packet at a late time is evaluated as

$$\langle (\varphi_{++})^2 \rangle \sim \frac{1}{\tilde{\Gamma}} \left[ \frac{\Gamma H^2}{3\alpha H^3 / p_c^3(t_i)} \right]^2, \tag{5.76}$$

and becomes much larger than the initial width of the wave packet $\Gamma^{-1}$. It is easy to see that the minimum broadening of the wave packet

$$\langle (\varphi_{++})^2 \rangle \sim \frac{2H^2}{3} \left( \frac{p_c^3(t_i)}{\alpha H^3} \right), \tag{5.77}$$
is also achieved when Eq. (5.73) holds. Comparing this result with Eq. (5.74), we find that the scale of decoherence is determined by the broadening of the wave packets.

However, there is another broadening mechanism due to the $\nu$-term. The fluctuations of the environment behave as a stochastic noise, and cause the broadening of the wave packet. In the last section we estimated the fluctuation in $\delta \phi$ responsible for this effect in Eq. (4.22), and we left the task to justify our interpretation of $\langle (\delta \phi)^2 \rangle_{\text{EA}}$. Now we return to this issue. In evaluating Eq. (5.77), we have completely neglected the contribution from the $\nu$-term. As mentioned before $M_2(t)$ has a small correction, $\delta M_2(t)$, due to the effect of $\nu$-term. This gives an additional broadening of the wave packet;

$$
\langle (\delta \varphi)^2 \rangle \sim \frac{2 \delta M_2(t_i)}{\alpha^2(t_i) \Omega^2} \sim \frac{\lambda}{216 \alpha^2 \nu^2} \left( \frac{H(1 - e^{-\lambda_2(t-t_i)})}{\lambda_2} \right)^2 \left( \frac{v}{H^2} \right) H^2,
$$

(5.78)

which corresponds to the expression previously derived in Eq. (4.22) with the aid of the Angusili field and naive approximations. At this point, the meaning of the quantity $\langle (\delta \phi)^2 \rangle_{\text{EA}}$ became transparent. $\langle (\delta \phi)^2 \rangle_{\text{EA}}$ represents the fluctuation caused by the environment. Both in case A and in case B, this broadening effect does not change the width of wave packets much compared with the case in which this effect is neglected. This is simply because we restricted our attention to the case when $M_2^{(0)}(t) \gg \delta M_2(t)$ holds for simplicity.

Then we find that the typical scale of the decoherence is not directly related with $\langle (\delta \phi)^2 \rangle_{\text{dec}}$, which was evaluated in Eq. (1.20). We find this scale in $M_2^{(1)}$, which gives exactly the same expression as $\langle (\delta \phi)^2 \rangle_{\text{dec}}$. To understand the meaning of this result, we focus on one diagonal component of the partial reduced density matrix, $\rho_{\varphi \varphi}(\varphi, \varphi'; t)$. $\rho_{\varphi \varphi}(\varphi, \varphi'; t)$ also has two continuous arguments, $\varphi$ and $\varphi'$. The suppression of the off-diagonal elements of $\rho_{\varphi \varphi}(\varphi, \varphi'; t)$ is determined by $m_{\varphi \varphi}$, and $\rho_{\varphi \varphi}(\varphi, \varphi'; t)$ becomes exponentially small for a large $\varphi\varphi$ which satisfies $\varphi\varphi \gg (\delta \phi)_{\text{dec}}$. Thus we can say that the decoherence occurs within a wave packet on the scale of (\delta \phi)_{\text{dec}}. However, for narrower wave packets, the classicality of the evolution of the system cannot be maintained through the whole duration of our concern. In the present case, this condition of the classicality of the evolution of the system totally determines the minimum width of the wave packet.

Finally we observe the peak location. It is calculated as

$$
\langle \varphi_+(t) \rangle = \frac{\Re(n_+)}{m_{++}} \sim e^{-\lambda_2(t-t_i)} \psi_{+} + \left( \frac{u(t)}{3 H^2 a^2(t) \Omega} \right) \frac{H(1 - e^{-\lambda_2(t-t_i)})}{\lambda_2}.
$$

(5.79)

The first term represents the change of the separation of the different trajectories labeled by the initial position of the peak, $\psi_+$. This is just the term to be attributed to the nature of the model potential. In the present model, congruence of the classical trajectories converges gradually. The second term is independent of $\psi_+$. This term arises because the interaction with the environment was not taken into account when we determine $\delta \phi(t)$. Thus it can be interpreted as the correction to $\delta \phi(t)$ due to the effect of the environment. This correction does not change the motion of $\delta \phi(t)$ so much by the same reason that discussed around Eqs. (4.14), (4.15) and (4.16).

VI. SUMMARY AND DISCUSSION

We investigated the evolution of the perturbation of the inflaton field with $\lambda \phi^4$ potential after the scale of the perturbation exceeds the Hubble horizon scale during the inflation. The effect of the coupling to the smaller scale modes through the self interaction was taken into consideration by using the closed time path formalism. That is, the smaller scale modes are considered as the environment. The initial condition for the quantum state of the fluctuation of the inflaton field was set well after the time of the horizon crossing of the considered mode. The initial state was supposed to be given by a pure state density matrix composed of a direct product of the usual Euclidean vacuum state, which has the variance of $O(H^2)$. This initial quantum state can be recognized as a quantum mechanical superposition of wave packets with a sharp peak. In the present model, we found that the influence of the environment does not distort these wave packets much but it extinguishes the coherence between the wave packets with different peak positions. The efficiency of the decoherence is so high that the state described by the different wave packets can be recognized as completely different worlds. Hence, we can conclude that the initial pure state evolves into a mixed state which can be interpreted as a statistical ensemble.

In the context of the inflationary universe scenario, the primordial fluctuations of the universe are evaluated by using an ad-hoc classicalization ansatz such that the expectation value of the squared field operator can be interpreted as the amplitude of the variance of the statistical ensemble. In this paper we have shown that this ansatz can be verified in a simple model. Thus the result obtained here gives a partial justification of the standard calculation of the primordial fluctuations.

However, the important issue might be whether the inflaton field behaves as classical during the reheating process that successively occurs after inflation. This is because the previous study on reheating is mostly based on the assumption that the fluctuations have already become classical before the reheating occurs. Thus, it will not be necessary that the fluctuations of the inflaton field are kept to be classical during the inflation. In order to prove
that the standard calculation works, we have only to show that the fluctuations of the inflaton become classical not throughout the whole duration of the inflation but at the end of it. In this sense, the condition for the standard calculation to be justified might be weaker than the conditions required in the present paper.

In this paper, we restricted our consideration to a specific model and the effect of the metric perturbation and the process of reheating were not taken into account at all. So further study is still required as future work.

ACKNOWLEDGMENTS

We thank Misao Sasaki for fruitful conversations. This work was supported in part by Monbusho Grant-in-Aid for Scientific Research No. 07304033.

APPENDIX A: EVALUATION OF $F(S)$, $\mu(S, S')$ AND $\nu(S, S')$

In this appendix, we show the details of the calculation to obtain the approximate expression for $f(s)$, $\mu(s, s')$ and $\nu(s, s')$ given in Eqs. (3.18) and (3.20).

1. $f(s)$

We begin with the evaluation of $f(s)$. The explicit expression for $f(s)$ is written down as

$$f(s) = \frac{a(s) \Omega}{4\pi^2} \int_{k_{min}}^{\infty} dk \frac{k}{2} \left[ 1 + \frac{1}{k^2 \eta^2} \right]. \quad (A1)$$

This expression is divergent, and needs renormalization. For this purpose, we use the point splitting technique. The function, $f(s)$, is basically given by using the Weightman function as $f(s) \propto \int d^3x G^{(+)}(x, s, x)$. Now we regularize the expression as $\int d^3x G^{(+)}(x, s, x', s)$, where $x'$ is chosen to satisfy $|x - x'|^2 = s^2$. After this regularization, $f(s)$ is calculated straightforwardly as

$$f(s) = \frac{a^3(s) \Omega}{16\pi^2} \int_{p_{min}(s)}^{\infty} dp \int_{-1}^{1} d(\cos \theta) p \left( 1 + \frac{H^2}{p^2} \right) e^{ip \cos \theta z}$$

$$= \frac{a^3(s) \Omega}{16\pi^2} \left\{ \frac{2}{s^2} - p_{min}^2(s) + 2H^2 - H^2 \left[ Ei(-ip_{min}(s)z) + Ei(ip_{min}(s)z) \right] + O(z) \right\}$$

$$\sim \frac{a^3(s) \Omega}{8\pi^2} \left\{ \frac{1}{s^2} - H^2 \log z - H^2(\gamma - 1) - \frac{p_{min}^2(s)}{2} - H^2 \log(p_{min}(s)) + O(z) \right\}, \quad (A2)$$

where $Ei$ is the exponential integral function. The divergent first term in the last line exists even in the limiting case where the background curvature can be neglected. Hence, it should be subtracted by the renormalization procedure. The second term is to be attributed to the renormalization of the curvature coupling term, $\xi R \phi^2$, where $R$ is the scalar curvature of the spacetime. Setting an appropriate renormalization condition, we obtain the renormalized $f(s)$ as

$$f(s) = \frac{a^3(s) \Omega}{8\pi^2} \left[ -\frac{1}{2} p_{min}^2(s) + H^2 \log \left( \frac{H}{p_{min}(s)} \right) \right]. \quad (A3)$$

2. $\mu(s, s')$

The function $\mu(s, s')$ is explicitly written down as

$$\mu(s, s') = -\frac{a^3(s)a^3(s') \Omega}{4\pi^2} \int_{k_{min}}^{\infty} dk k^2 \left( u^*(s)^2 u(s')^2 - u(s)^2 u^*(s')^2 \right)$$

$$= -\frac{a^3(s)a(s') \Omega}{16\pi^2} \int_{k_{min}}^{\infty} dk \left( e^{2ik(n - n')} \left( 1 + \frac{i}{k \eta} \right)^2 \left( 1 - \frac{i}{k \eta} \right)^2 - \text{(c.c.)} \right). \quad (A4)$$

where $\eta$ and $\eta'$ is the conformal time corresponding to $s$ and $s'$. We divide it into two pieces, $\mu_r(s, s')$ and $\mu_s(s, s')$. 


\[ \mu_s(s, s') := -\frac{a(s)a(s')}{16\pi^2} \int_{k_{\text{min}}}^{\infty} \left( e^{2ik(\eta - \eta')} - (c.c.) \right), \] (A5)

is the portion that contains the ultraviolet divergence corresponding to coupling constant renormalization and \( \mu_r(s, s') \) is the remaining regular terms defined by \( \mu_r(s, s') := \mu(s, s') - \mu_s(s, s') \).

The expression for \( \mu_r(s, s') \) is slightly complicated but there is no technical difficulty in its evaluation. After a straightforward calculation, we obtain

\[ \mu_r(s, s') \sim \frac{a(s)a(s')}{\pi^2} \left\{ \frac{1}{12} \left( \frac{1}{\eta} - \frac{1}{\eta'} \right) + \frac{1}{12} \left[ \frac{\eta'}{\eta^2} - \frac{\eta}{\eta'^2} \right] \left( -\frac{7}{3} + \gamma + \log \left[ 2k_{\text{min}}(\eta - \eta') \right] \right) \right\}. \] (A6)

Here we take into account the fact (3.17) and pick up only the dominant terms in the \( k_{\text{min}} \rightarrow 0 \) limit. However, this expression is still complicated. In this paper, we just aim to show that the effect of \( \mu - \text{term} \) is negligible small. For this purpose, we can simplify the expression by taking the \( |\eta| \ll |\eta'| \) limit as

\[ \mu_r(s, s') \sim -\frac{a(s)\Omega H}{12\pi^2} \log (k_{\text{min}}(\eta - \eta')). \] (A7)

When \( s \sim s' \), this simplified expression is not correct. However, since \( \mu(s, s') \) vanishes in the coincidence limit, this simplification does not underestimate the effect of \( \mu - \text{term} \). Hence, this simplification will be justified.

The singular part \( \mu_s(s, s') \) needs regularization as before. Recalling that \( \mu(s, s') \) is essentially given by the product of the Weightman function as

\[ \mu(s, s') \propto \int d^3x \left[ G^{(+)}(x, t; 0, t') \right]^2 + (c.c.), \] (A8)

we can introduce the point splitting regularization by replacing \( \left[ G^{(+)}(x, t; 0, t') \right]^2 \) by \( G^{(+)}(x, t; 0, t')G^{(+)}(x + \epsilon, t; 0, t') \), where \( |\epsilon|^2 = z^2 \). Then the regularized expression for \( \mu_s(s, s') \) becomes

\[ \mu_s(s, s') = -i\frac{a(s)a(s')}{32\pi^2} \int_{k_{\text{min}}}^{\infty} dk \int_{-1}^{1} d(cos \theta) \left( e^{2ik(\eta - \eta')} - e^{-2ik(\eta - \eta')} \right) e^{-ik\cos \theta/a(s)}. \] (A9)

In order to subtract the singular term which needs renormalization it is necessary to consider the expression including the \( s' \) integral

\[ I := \int_{t_i}^{s} ds' \Sigma(s')\mu_s(s, s'). \] (A10)

Integrating by part, this integral is rewritten as

\[ I = \left[ a^2(s')U(s, s')\Sigma(s') \right]_{t_i}^{s} - \int_{t_i}^{s} ds' \frac{d}{ds'} \left( U(s, s') \frac{d}{ds'} \Sigma(s')a^2(s') \right), \] (A11)

where

\[ U(s, s') := \frac{a(s)\Omega}{64\pi^2} \int_{k_{\text{min}}}^{\infty} \frac{dk}{k} \int_{-1}^{1} d(cos \theta) \left( e^{2ik(\eta - \eta')} + e^{-2ik(\eta - \eta')} \right) e^{-ik\cos \theta z/a(s)} = \int_{s'}^{s} \frac{ds''}{a^2(s'')} \mu_s(s, s''). \] (A12)

In evaluating \( U(s, s') \), when \( s \neq s' \), we can take the \( z \rightarrow 0 \) limit without any trouble, and easily evaluated as

\[ U(s, s') \sim -\frac{a(s)\Omega}{16\pi^2} \log \left( 2k_{\text{min}}(\eta - \eta') \right) + \gamma. \] (A13)

On the other hand, when \( s = s' \), \( U(s, s) \) becomes

\[ U(s, s) \sim \frac{a(s)\Omega}{16\pi^2} [(1 - \gamma - \log z) - \log(p_{\text{min}}(s))], \] (A14)

which contains the logarithmic divergence corresponding to the coupling constant renormalization. After the renormalization, we obtain

\[ U(s, s) \sim -\frac{a(s)\Omega}{16\pi^2} \log \left( \frac{p_{\text{min}}(s)}{H} \right). \] (A15)

Combining all the results, finally we get the expression given in Eq. (3.19).
3. $\nu(s, s')$

In the same way, the dominant terms in $\nu(s, s')$ are evaluated as

$$
\frac{a(s)^3a(s')^3}{8\pi^2} \int_{k_{min}}^{k_{max}} k^2 dk \left( u^*(s)^2 u(t')^2 + u(s)^2 u^*(t')^2 \right) \\
\sim \frac{a(s)a(s')}{32\pi^2} \left( \frac{1}{2} \frac{1}{3\eta'^2 s'^2 k_{min}^3} + \sin \left[ \frac{2k_{max}(\eta - \eta')}{\eta - \eta'} \right] \right).
$$

(A16)

The second term becomes proportional to $\delta(\eta - \eta')$ in the $k_{max} \to \infty$ limit. Since the time dependence of the second term is different from that of the first one, we cannot simply discard the second term. However, the important quantity for the discussion in the present paper is the integral over $s'$ of the product of $\nu(s, s')$ and some function $F(s, s')$ which is always a smooth function with respect to $s - s'$. Thus we can conclude that the first term gives the dominant contribution, and the second term can be neglected.

**APPENDIX B: APPROXIMATE FORMULAS FOR INTEGRALS**

Here we explain the details of the approximation used in evaluating several integrals. In this appendix, we set $\eta = -e^{-Hs}$, $\eta' = -e^{-Hs'}$ and $\eta_i = -e^{-Ht_i}$.

First we evaluate the following integral

$$
I_1 := \int_{t_i}^{s} ds' \log(k_{min}(\eta - \eta')) \\
= -\int_{\eta_i}^{\eta} \frac{d\eta'}{\eta'} \log(k_{min}(\eta - \eta')).
$$

(B1)

Changing the variable into $x := (\eta' - \eta)/\eta$, the integral becomes

$$
I_1 = \int_0^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} \log x + \int_{1}^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} \log(-k_{min}\eta).
$$

(B2)

The second term is explicitly calculated to become $H(s - t_i)\log(-k_{min}\eta)$. When $(\eta_i - \eta)/\eta < 1$ the first term is bounded by

$$
\left| \int_0^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} \log x \right| < \left| \int_0^{1} \frac{dx}{1+x} \log x \right| = \frac{\pi^2}{12}.
$$

(B3)

To the contrary, when $(\eta_i - \eta)/\eta > 1$ the first term is evaluated by

$$
\int_0^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} \log x = -\frac{\pi^2}{12} + \int_1^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} \log x
$$

and

$$
\left| \int_1^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} \log x \right| < \log \frac{\eta_i - \eta}{\eta} \int_1^{(\eta_i - \eta)/\eta} \frac{dx}{1+x} = \log \frac{\eta_i - \eta}{\eta} \log \frac{\eta_i}{2\eta}.
$$

(B5)

Thus we find

$$
|I_1| \lesssim (H\Delta t)^2.
$$

(B6)

Next we consider

$$
I_2 := \int_{t_i}^{s} ds' \frac{\alpha^2(s')}{\alpha^2(s)} \log(k_{min}(\eta - \eta')) \\
= -\eta^2 \int_{\eta_i}^{\eta} \frac{d\eta'}{\eta'} \log(k_{min}(\eta - \eta')).
$$

(B7)
After a straightforward calculation by using the same change of variable as before, we obtain
\[ I_2 = \frac{1}{2} \left[ \frac{\eta^2}{\eta_i} \log \left( \frac{\eta_i}{\eta} - 1 \right) + \log \left( \frac{\eta_i}{\eta - \eta_i} \right) + 2 \frac{\eta - \eta_i}{\eta_i} \right] + \frac{1}{2} \left( 1 - \frac{\eta^2}{\eta_i} \right) \log (-k_{min} \eta). \] (B8)

Thus we find
\[ |I_2| \lesssim H \Delta t. \] (B9)

Finally we consider
\[ I_3 := \int_{t_i}^s ds' H a^3(t_i) \log (k_{min}(\eta - \eta')) \]
\[ = \eta_i^{-3} \int_{\eta}^{\eta_i} d\eta' \eta'^3 \log (k_{min}(\eta - \eta')). \] (B10)

In the same way, after a straightforward calculation, we obtain
\[ I_3 = \frac{\eta^3}{\eta_i} \left[ \frac{1}{3} \left( \frac{\eta^3}{\eta_i^3} - 1 \right) \log \left( \frac{\eta_i}{\eta} - 1 \right) - \frac{1}{9} \left( \frac{\eta_i}{\eta} - 1 \right)^3 - \frac{1}{2} \left( \frac{\eta_i}{\eta} - 1 \right)^2 - \left( \frac{\eta_i}{\eta} - 1 \right) + \frac{1}{3} \left( \eta^3 - 1 \right) \log (-k_{min} \eta) \right], \] (B11)
and it implies
\[ |I_3| \lesssim H \Delta t. \] (B12)

[1] say, A. Linde, Particle Physics and Inflationary Cosmology (Harwood Academic Publishers 1990).
[2] A. Starobinsky, in Field Theory, Quantum Gravity and Strings, ed. H. J. de Vega and N. Sanchez, Lecture notes in Physics Vol. 226 (Springer, Berlin 1986).
[3] A.S. Goncharov and A.D. Linde, Sov J. Part. Nucl. 17, 369 (1986); A.D. Linde and A. Mezhlinian, Phys. Lett. B 307, 25 (1993); A. Linde, D. Linde and A. Mezhlinian, Phys. Rev. D 49, 1783 (1994); J. Garcia-Bellido, A. Linde and D. Linde, Phys. Rev. D 50, 730 (1994); A. Linde and D. Linde, Phys. Rev. D 50, 2456 (1994); I. Yi and E.T. Vishniac, Phys. Rev. D 47, 5280 (1993); Y. Nambu and M. Sasaki, Phys. Lett. B205, 441 (1988); M. Sasaki, Y. Nambu and K. Nakao, Nucl. Phys. B308, 868 (1988); Y. Nambu and M. Sasaki, Phys. Lett. B219, 240 (1989); Y. Nambu, Phys. Lett. B276, 11 (1992); M. Mijic, Phys. Rev. D 49, 6434 (1994);
[4] S. Habib, Phys. Rev. D 46, 2408 (1992).
[5] D. Polarski and A.A. Starobinsky, Class. Quantum Grav. 13, 377 (1996); J. Lesgourgues, D. Polarski and A.A. Starobinsky, preprint gr-qc/9611019.
[6] E. Calzetta and B. L. Hu, Phys. Rev. D 35, 495 (1987); Phys. Rev. D 37, 2878 (1988); Phys. Rev. D 40, 380 (1989); B. L. Hu and Y. Zhang, Phys. Rev. D 37, 2151 (1988);
[7] E. Calzetta and B. L. Hu ?
[8] M. Morikawa, Phys. Rev. D 42, 2929 (1990); Prog. Theor. Phys. 93, 685 (1995).
[9] A. Matacz, Phys. Rev. D 55, 1860 (1997).
[10] R. Feynman and F.L.Vernon, Ann. Phys. 24, 118 (1963); G.W. Ford, M.Kac, and P. Mazur, J. Math. Phys. 6, 504 (1963); A.O. Caldeira and A.J. Leggett, Physica (Utrecht) 121A, 587 (1983); Ann. Phys. (N.Y.) 149, 374 (1983).
[11] M. Morikawa, Phys. Rev. D 42, 1027 (1990);
[12] E. Joos and H. D. Zeh, Z. Phys. B -Condensed Matter 59, 223 (1985).
[13] M. Sakagami, H. Kubotani and T. Okamura, Prog. Theor. Phys. 95, 703 (1996).
[14] A. Hosoya and M. Morikawa, Phys. Rev. D 39, 1123 (1989).
[15] J. B. Hartle, in Quantum Cosmology and Baby Universe, Proceedings of the 7th Jerusalem Winter School, Jerusalem, Israel, 1990, edited by S. Coleman, J.Hartle, T.Piran and S. Weinberg (World Scientific, Singapore, 1991); H. F. Dowker and J.J. Halliwell, Phys. Rev. D 46, 1580 (1992).
[16] M. Morikawa, Phys. Rev. D 42, 2929 (1990); J.P. Paz and W. H. Zurek, Phys. Rev. D 48, 2728 (1993).
[17] A.L. Matacz, Phys. Rev. D 49, 788 (1994).
[18] B.L. Hu, J.P. Paz and Y. Zhang, Phys. Rev. D45, 2843 (1992); B.L. Hu, J.P. Paz and Y. Zhang, Phys. Rev. D47, 1576 (1993); T. Okamura, Prog. Theor. Phys. 91, 219 (1994).