Conformal field theory approach to gapless 1D fermion systems and application to the edge excitations of 
\( \nu = 1/(2p + 1) \) quantum Hall sequences

P. Degiovanni
Laboratoire de Physique Théorique ENSLAPP
ENS Lyon, 46 allée d’Italie 69364 Lyon cedex 07, France
Email: Pascal.Degiovanni@ens-lyon.fr

C. Chaubet
GES, Université des Sciences et Techniques du Languedoc
Place Eugène Bataillon, 34095 Montpellier cedex 05, France
Email: cris@pollux.ges.fr

R. Mélin
Centre de Recherche sur les Très Basses Températures
BP166, 25 avenue des martyrs, 38042 Grenoble Cedex 09, France
Email: melin@crtbt.polycnrs-gre.fr

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Abstract

We present a comprehensive study of the effective Conformal Field Theory (CFT) describing the low energy excitations of a gas of spinless interacting fermions on a circle in the gapless regime (Luttinger liquid). Functional techniques and modular transformation properties are used to compute all correlation functions in a finite size and at finite temperature. Forward scattering disorder is treated exactly. Laughlin experiments on charge transport in a Quantum Hall Fluid on a cylinder are reviewed within this CFT framework. Edge excitations above a given bulk excitation are described by a twisted version of the Luttinger effective theory. Luttinger CFTs corresponding to the \( \nu = 1/(2p + 1) \) filling fractions appear to be rational CFTs (RCFT). Generators of the extended symmetry algebra are identified as edge fermions creators and annihilators, thus giving a physical meaning to the RCFT point of view on edge excitations of these sequences.

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1 Introduction

The many-body problem in condensed matter physics has been a subject of intensive research for several decades. The first of these theories dates back to 1956 and is the so-called Landau Fermi liquid theory. It provides a powerful phenomenological description of three dimensional normal systems of interacting fermions \cite{58, 59}. One of the central concepts of this theory is the notion of quasiparticle excitations that are in a one-to-one correspondence with excitations of the non-interacting gas, via the adiabatic continuation principle \cite{7}. The Landau-Fermi liquid theory consists in a description of the many-body state in terms of a few phenomenological parameters (the effective mass and the so-called Landau parameters) that can be extracted from experimental measurements such as specific heat, compressibility, susceptibility and sound velocity. For a detailed description of this theory, we refer the reader to the many books on this subject (see for instance \cite{7, 1}). The phenomenological theory of Landau was put on a microscopic basis by means of quantum field theory methods \cite{1, 73, 74}. The adiabatic continuation of \cite{7} requires the inverse particle life time \(1/\tau(k)\) to be much smaller than the quasiparticle energy \(\epsilon(k)\) so that the switching rate of interactions \(R\) can be safely chosen such that

\[
\frac{1}{\tau(k)} \ll R \ll \epsilon(k).
\] (1)

In three dimensions, the quasiparticle life time can be evaluated from the “golden rule” \cite{7}: \(\tau(k) \sim \epsilon^{-2}(k)\). Henceforth \cite{7} is valid at arbitrarily low energies. It means that the quasiparticle should not disappear before the end of the interaction switching and that the interaction switching rate should be compatible with Heisenberg uncertainty relations. This condition breaks down in one dimension (1D), indicating that the 1D interacting fermionic system is not of the Fermi liquid type.

The failure of the Fermi liquid description for 1D interacting fermions dates back to the pioneer work by Dzialoshinskii and Larkin \cite{25}. These authors showed that this 1D system has no Fermi surface in the sense that the quasiparticle residue vanishes and the fermionic occupation numbers have a power law behavior without discontinuity. Other striking features of 1D interacting fermionic systems were derived latter. For instance, with local interactions, the Green function vanishes \cite{41}, in an orthogonality catastrophe-like scenario \cite{3}. Besides this point, the correlations decay algebraically and are governed by anomalous exponents \cite{41, 65, 64, 39}. Also, the spin \(1/2\) liquid shows spin and charge separation \cite{92}: an additional electron in this system dynamically breaks into a spin packet and a charge packet that propagate at different speeds. See \cite{81} and references therein for a review on the Fermi liquid/non-Fermi liquid problematics.

The first steps in deriving exactly soluble 1D fermionic systems were performed by Tomonaga \cite{90}, Luttinger \cite{66} and Mattis and Lieb \cite{69}. Technically, a linearization of the free electrons dispersion relation around the two Fermi points provides a mapping to a two dimensional Dirac theory with interactions (the massless Thirring model) which can then be bosonized. It was pointed out by Haldane \cite{11} that these low energy properties are the ones of a wide class of 1D systems, for instance fermionic 1D liquids or some massless regimes of spin-1/2 chains. In fact, it is possible to identify precisely the low energy physics of some models with the Luttinger Liquid (LL). For instance, this was done in \cite{40} for the spin 1/2 Heisenberg chain, with a mapping to a fermionic liquid via a Jordan-Wigner transformation; and in \cite{82} for the 1D Hubbard model. In this sense, the LL is analogous to the Landau theory in 3D systems since it provides a solvable universal effective low energy theory with a few parameters.

Two dimensional Conformal Field Theories (CFT) provide another unifying framework for one dimensional quantum systems. Originally studied in depth as a generic description of effective field theories for two dimensional critical points \cite{4}, they proved to be useful for studying one dimensional quantum systems. This is nothing but the equivalence between a \(D\)-dimensional quantum system at finite temperature and the appropriate classical system in dimension \(D + 1\) \cite{26}. Of course, since CFTs essentially describe “massless” field theories, they are appropriate for describing one-dimensional gapless systems. Then, one could expect, on these general grounds, that the LL admits such an effective low-energy description.

Many references now exist on CFTs. The interested reader is of course referred to original papers (\cite{4} for CFT on a plane, and \cite{16} for CFT on the torus) but also to various reviews on this vast subject: for example, Les Houches lectures \cite{17} and \cite{33}, Itzykson and Drouffe book \cite{46, Chapter 9}, or more recently
Ketov book \cite{52}. Of special interest for the present paper, let us quote \cite{23} which treats in great details $c = 1$ CFT. Obviously, this small conformal bibliography is far from being exhaustive but should help the non specialist of this subject!

The aim of this article is to present a comprehensive study of the relation between LLs and CFTs. More precisely, we identify the spinless LL in a finite size and at finite temperature but without Umklapp terms with the field theory of a compact boson on a two-dimensional torus with an appropriate topological weight. Let us recall that, as far as 1D quantum liquids are concerned, the most general model consists of the full $g$-ology. It is shown in \cite{86} that carrying out the poor man scaling analysis (originally applied by Anderson to the Kondo problem \cite{4}), there exists a region of interactions such that only the $(g_2, g_4)$ interactions survive at the fixed point. In this paper, we shall limit ourselves to $g_2$ and $g_4$ interactions. These couplings are marginal in the sense of the renormalization group, just like the Landau parameters for the three dimensional interacting Fermi system \cite{85}. We shall see precisely how the effective CFT of the LL changes when these $g_{2,4}$ parameters are varied.

Identification between the LLs and the compactified boson was already proposed in \cite{2} for quantum spin chains, and in the late 80s, string theorists also studied the correspondence between Dirac theory and the compactified boson but in modular invariant sectors \cite{4}. By contrast, we shall deal with a specific sector of the fermionic LL theory which is obviously not modular invariant. Therefore, bosonization formulae need a careful treatment of boundary conditions and this is why a topological term is needed. For the sake of pedagogy, the appropriate formulae will be derived here in a elementary way. Nevertheless, they have already appeared sporadically in the literature (see \cite{3, 21}) but have not been yet exploited within the context of one-dimensional condensed matter systems. Nevertheless, we have found a few papers, unfortunately not very well-known, that contain explicit references to these subtle questions: in \cite{96}, Wu and Yu characterize the Luttinger CFT starting from the study of an ideal exclusion gas, following Haldane’s ideas on generalized statistics \cite{41}. In \cite{54}, the Luttinger CFT is extracted from locality considerations on the operator product algebra and its specific modular properties and duality properties are noticed. This paper also contains a deep discussion of the relation between the Thirring model and the Sine-Gordon theory and provides a complementary view to the present article.

Focusing back on more formal matters, a deep and quite complete reference on bosonization is \cite{31}. See also the work by Jolicoeur and Le Guillou \cite{48} who also use functional methods in the thermodynamic and zero temperature limits, that is to say on the plane. And of course, classical works by Coleman \cite{19} and Mandelstam \cite{68} provide operator based approaches to the bosonization of the massive Thirring model.

To conclude this small review on bosonization approaches to the Luttinger liquid, let us quote \cite{77} which contains a numerical study of an XXZ spin $1/2$ quantum chain away from half-filling on a circle with periodic or open spin boundary conditions. According to the authors of this paper, CFT predictions for the spectrum agree with numerical computations based on a density matrix renormalization group method.

In the present article we shall explicitly show that all physical quantities of the LL that can be calculated within the framework of the bosonization used by Haldane in \cite{41} can also be explicitly calculated by functional integral techniques. The only restriction is that the interactions should be local so that conformal invariance holds. As shown in \cite{71}, non-local interactions induce a breakdown of the Fermi liquid by loss of coherence for the quasiparticle but they do not preserve conformal invariance because of their length scale. However, at sufficiently large distance all non-singular interactions can be viewed as local.

One of the motivations for developing finite size functional bosonization of the LL, besides clarifying the literature on the subject, is to deal with an external magnetic field and with forward-scattering disorder. By forward-scattering disorder, we mean a random potential coupled to long-wavelength density fluctuations. The problem of a disordered LL was already considered in \cite{32}, within the framework of renormalization. Compared to the present article, the model of \cite{32} involves backscattering and presents a localized phase. But then, it is not possible to perform a direct functional integral computation of correlation functions. Forward scattering does not lead to a localized phase but, in this case, averages over disorder of all products of correlation functions can be explicitly computed.

The CFT formulation of the LL surprisingly provides a simpler point of view on the effective theory for edge excitations of a Fractional Quantum Hall (FQH) fluid. Wen originally pointed out that edge excitations
of a FQH fluid were described by a LL \[93\]. The relation has been studied in depth for edge states of a disk-shaped FQH fluid (in this case, a chiral LL arises at the boundary). In \[43\], Haldane and Rezayi studied numerically cylindrical geometries that may lead to a FQH fluid. Henceforth, it is interesting to look for a precise identification of the field theory of edge excitations on the cylinder. Here, we argue that the non-chiral LL studied here corresponds to edge states of a FQH fluid on a cylinder (see Laughlin’s paper \[61\]). More precisely, Luttinger CFTs corresponding to the \(\nu = 1/(2p + 1)\) filling fraction are Rational Conformal Field Theories (RCFTs) \[29\]. Their partition functions appear to coincide with the ones recently obtained by Cappelli and Zemba \[15\]. While they were obtained from their modular invariant properties by these authors, here they arise from a one dimensional effective theory of interacting fermions. Of course, in FQH fluids, these interactions are induced by the strong correlations of electrons in the Hall fluid. We have pursued this analysis by introducing an edge charge on the boundary in order to describe the tower of edge excitations above a given gaped bulk excitation. The partition function of this “twisted” Luttinger CFT should reflect how states above a given bulk excitation are organized. Following Fisher and Stone \[87\], it is also interesting to identify edge excitations that correspond to introducing a physical electron on one of the two edges of the cylinder. We show that the associated conformal fields generate an extended maximal symmetry algebra for Luttinger CFTs, thus providing a simple view on edge excitations of the FQH fluid.

The present article is organized as follows: section 2 recalls the basics of Haldane work and connect them to the massless Thirring model on the torus. Section 3 studies the coupling to a gauge field, and gives the proof of our bosonization formulae by functional techniques. In section 4, we explain how forward scattering can explicitly be treated. Section 5 and 6 are devoted to the computation of correlation functions: first, charge and current densities are computed. This provides us with the charge and current response to an external electric potential and to a magnetic flux. Next, correlation functions of all vertex operators are computed. Among these are the Luttinger fermion operators.

Having completed the study of the generic Luttinger CFT, we shall show how some of these CFTs provide a description for edge excitations of a FQH fluid on a cylinder. Section 7 explains how Laughlin gedanken experiments on charge transport in a Hall sample can be understood within the framework of Luttinger CFTs. This enables relating the interaction parameter of the Luttinger CFT with the filling fraction of the Hall fluid. In section 8, Luttinger CFTs corresponding to the Laughlin series are considered as RCFTs. This analysis is extended in section 9 where we introduce a fractional charge on the edge in order to study edge excitations above bulk excitations of the Hall fluid. The resulting theory is again a RCFT and is a twisted version of the Luttinger CFT. The physical meaning of the maximal symmetry algebra of these RCFTs is discussed in the light of Stone and Fisher’s identification of edge fermions in the FQH effect.

Last, most of the technical details and computations are in the appendices. In particular, appendix E contains all the necessary definitions and technicalities on elliptic and modular functions needed in this paper. Appendix F deals with the simple case of non-relativistic free fermions, and shows that these results are compatible with the CFTs ones. Appendix G contains operator computations à la Haldane in the interacting theory.
2 CFT of the LL

In this section, we recall how the long range physics of the LL is described by an effective conformal field theory, namely the theory of a free bosonic field compactified on a circle.

2.1 The Luttinger model

The LL is a theory of one dimensional interacting fermions on a circle of perimeter \( L \). Haldane has given a detailed discussion of the low energy physics of the LLs \([11]\). For non relativistic fermions, the quadratic dispersion relation \( \epsilon(k) = k^2/(2m) \) of the one dimensional Fermi gas is linearized in the vicinity of the Fermi surface: \( \epsilon_{\text{lin}}(k) = v_F(\alpha k - k_F) \), where \( \alpha = +1, -1 = R,L \) denote the right and left Fermi points, and \( k_F \) is the Fermi wave vector. The Fermi velocity is nothing but \( v_F = \frac{\partial \epsilon}{\partial k}(k_F) \). This approximation is valid provided the interactions are not too strong compared to the Fermi velocity. We shall come back to this point in section \( \ref{sec:interaction-energy} \). The linear dispersion relation is then extrapolated to arbitrary energies: an infinite number of fermions (Dirac sea) is then present in the ground state. One ends up with two linear branches. The system obtained in this way should capture all the long range physics of the initial model. Results obtained through the Bethe Ansatz technique are compatible with this effective theory \([2]\).

2.1.1 Free theory

In the infrared limit, the Fermi sea is replaced by a Dirac sea, and we choose \( k_F = 0 \) in what follows (\( k_F \neq 0 \) would lead to minor corrections). The free Hamiltonian is then:

\[
H^{(0)} = \frac{2\pi v_F}{L} \sum_{n,\alpha} \alpha n : c_{n,\alpha}^\dagger c_{n,\alpha} : ,
\]

where the \( c \)s are fermionic creation and destruction operator at momenta \( 2\pi \alpha n/L \). These operators satisfy canonical anti-commutation relations:

\[
\{ c_{n,\alpha}, c_{m,\alpha'} \} = \{ c_{n,\alpha}^\dagger, c_{m,\alpha'}^\dagger \} = 0 \]

\[
\{ c_{n,\alpha}^\dagger, c_{m,\alpha'} \} = \delta_{n,m} \delta_{\alpha,\alpha'} \mathbf{1} .
\]

Fermionic normal ordering is defined by:

\[
\begin{align*}
\psi_R^\dagger(\sigma) &= \frac{1}{\sqrt{L}} \sum_n c_{n,R}^\dagger e^{-2\pi i n \sigma / L} \\
\psi_L^\dagger(\sigma) &= \frac{1}{\sqrt{L}} \sum_n c_{n,L}^\dagger e^{2\pi i n \sigma / L} .
\end{align*}
\]

As recalled in appendix \[A]\, these fields are the spatially slow-varying components of the initial Fermi fields, the fast varying part originating from oscillations at the Fermi wave-vector. The charge and current density operators are defined by:

\[
\begin{align*}
\rho(\sigma) &= \psi_R^\dagger(\sigma)\psi_R(\sigma) + \psi_L^\dagger(\sigma)\psi_L(\sigma) : \\
j(\sigma) &= v_F : \psi_R^\dagger(\sigma)\psi_R(\sigma) - \psi_L^\dagger(\sigma)\psi_L(\sigma) :
\end{align*}
\]

The interaction energy between the electrons and an external electrostatic or vector potential is linear:

\[
H_{\text{ext}} = \int_0^L d\sigma \left( V(\sigma) \rho(\sigma) - A(\sigma) j(\sigma) \right) .
\]
The central tool in the operator formulation of the model is the current algebra. Let us define the following operators \((n \in \mathbb{Z})\):

\[
J_n = \int_0^L J(\sigma) e^{-2\pi in\sigma/L} d\sigma, \quad (10)
\]

\[
\mathcal{J}_n = \int_0^L \mathcal{J}(\sigma) e^{2\pi in\sigma/L} d\sigma, \quad (11)
\]

with \(J(\sigma) = :\psi_R^\dagger(\sigma)\psi_R(\sigma) :\) and \(\mathcal{J}(\sigma) = :\psi_L^\dagger(\sigma)\psi_L(\sigma) :\). One can show that the modes (10) and (11) satisfy the following commutation relations

\[
[J_n, J_m] = n \delta_{n,-m} 1, \quad (12)
\]

this anomalous commutator being central to mode bosonization. It can be derived either from a point splitting regularization in real space (see for instance [28]) or directly in Fourier space (see for instance [41]). In both cases, the anomaly originates from normal ordering of operators with respect to the Dirac sea. Moreover, the \(J_n\)s commute with \(J_m\)s which also satisfy commutation relations (12). These relations define the infinite dimensional Heisenberg algebra [49], also called the affine \(U(1)\) algebra and denoted by \(\hat{U}(1)\).

The symmetry of the model is therefore given by two commuting copies of the Heisenberg algebra. The vacuum state of Dirac’s theory \(|0\rangle\) satisfies the highest weight conditions which implement Pauli’s exclusion principle:

\[
\forall n > 0, \ J_n |0\rangle = \mathcal{J}_n |0\rangle = 0. \quad (13)
\]

Finally, as shown by Haldane [41], the free Hamiltonian (3) can be expressed in terms of the currents:

\[
H^{(0)} = \frac{\pi v_F}{L} \sum_{n \in \mathbb{Z}} (J_n J_{-n} + \mathcal{J}_n \mathcal{J}_{-n}) = \pi v_F \int_0^L \left( J^2(\sigma) + \mathcal{J}^2(\sigma) \right) d\sigma. \quad (14)
\]

This bosonized form of the free Hamiltonian is the key point of mode bosonization.

### 2.1.2 Interacting theory

In the following, local interactions between fermions are assumed, although Haldane has given a solution for non local interactions satisfying certain conditions. Short range interactions fall into this class, and the local case should be considered as a limiting case of his analysis. The local interaction term of the spinless LL is

\[
H_{\text{int}} = \frac{\pi}{L} \sum_{\alpha, \alpha', k, k'} (g_4 \delta_{\alpha, \alpha'} + g_2 \delta_{\alpha, -\alpha'}) c^\dagger_{k+q, \alpha} c_{k, \alpha} c^\dagger_{k', -q, \alpha'} c_{k', \alpha'}. \quad (15)
\]

The total Hamiltonian is diagonalized via a bosonization procedure [41], using the current modes introduced in (10) and (11). The interactions are first re-written in terms of the currents:

\[
H_{\text{int}} = \frac{\pi}{L} \sum_{n \in \mathbb{Z}} (2g_2 J_n \mathcal{J}_n + g_4 (J_n \mathcal{J}_{-n} + \mathcal{J}_n \mathcal{J}_{-n})) \quad (16)
\]

\[
= 2\pi g_2 \int_0^L J(\sigma) \mathcal{J}(\sigma) + \pi g_4 \int_0^L \left( J^2(\sigma) + \mathcal{J}^2(\sigma) \right) d\sigma. \quad (17)
\]

The Bogoliubov transformation then consists in a redefinition of current algebra’s generators. At the appropriate angle, it diagonalizes the Hamiltonian. More precisely, let us define

\[
J_n = \cosh(\varphi) J_n - \sinh(\varphi) \mathcal{J}_{-n} \quad (18)
\]

\[
\mathcal{J}_n = \cosh(\varphi) \mathcal{J}_n - \sinh(\varphi) J_{-n}, \quad (19)
\]

\(^1\)This factorization between a left and a right symmetry algebra is quite common in CFT.
where
\[ \tanh (2\varphi) = -\frac{g_2}{v_F + g_4}. \tag{20} \]

In the interacting theory, these are the new symmetry generators. They also satisfy the commutation relations \[^{12}\]. The \( J \)s and \( \overline{J} \)s should be understood as bare symmetry generators, whereas the \( J \)s and \( \overline{J} \)s are renormalized operators that implement the infinite dimensional affine symmetry in the interacting theory. It is useful to Fourrier transform Laurent’s modes \((J_n)_n\) and \((\overline{J}_n)_n\) to \( J(\sigma) \) and \( \overline{J}(\sigma) \). Then we have
\[ \left( \begin{array}{c} J(\sigma) \\ \overline{J}(\sigma) \end{array} \right) = \left( \begin{array}{cc} \cosh (\varphi) & -\sinh (\varphi) \\ -\sinh (\varphi) & \cosh (\varphi) \end{array} \right) \times \left( \begin{array}{c} J(\sigma) \\ \overline{J}(\sigma) \end{array} \right). \tag{21} \]

Using these notations, the free Hamiltonian plus the interaction term is nothing but the one of a free bosonic field theory:
\[ H_{\text{tot}} = \frac{\pi v_S}{L} \sum_{n \in \mathbb{Z}} (J_n J_{-n} + \overline{J}_n \overline{J}_{-n}) = \pi v_S \int_{0}^{L} (J^2(\sigma) + \overline{J}^2(\sigma)). \tag{22} \]

The density operator is not modified by switching on interactions since it consists in counting particles:
\[ \rho(\sigma) =: \psi_R^\dagger(\sigma) \psi_R(\sigma) : + : \psi_L^\dagger(\sigma) \psi_L(\sigma) : = \alpha^{-1/2} (J(\sigma) + \overline{J}(\sigma)), \]

where we introduce the interaction parameter \( \alpha = e^{-2\varphi} \). To obtain the current, one should, as Haldane suggested in \[^{11}\], use charge conservation. Using the total interacting Hamiltonian, and the commutation relations of the \( J(\sigma) \) and \( \overline{J}(\sigma) \) operators, one easily finds the time evolution of these operators in the Heisenberg representation\[^{3}\]
\[ J(\sigma, t) = \frac{1}{L} \sum_{n \in \mathbb{Z}} J_n e^{2\pi i n (\sigma - v_S t)/L} \tag{23} \]
\[ \overline{J}(\sigma, t) = \frac{1}{L} \sum_{n \in \mathbb{Z}} \overline{J}_n e^{-2\pi i n (\sigma + v_S t)/L}. \tag{24} \]

Interactions are taken into account through the modification of the Fermi velocity \( v_F \rightarrow v_S \), with
\[ v_S = \sqrt{(v_F + g_4)^2 - g_2^2} \tag{25} \]
\[ v_N = v_F + g_2 + g_4 \tag{26} \]
\[ v_J = v_F - g_2 + g_4. \tag{27} \]

Then \( \alpha = \sqrt{v_N/v_J} \). To summarize, we have:
\[ \rho(\sigma) = \alpha^{-1/2} (J(\sigma) + \overline{J}(\sigma)) \tag{28} \]
\[ j(\sigma) = v_S \alpha^{-1/2} (J(\sigma) - \overline{J}(\sigma)). \tag{29} \]

Of course, these computations assume that the interactions are local, i.e., \( \varphi \) does not depend on \( n \). Since we are interested in the low energy properties of the model, we may assume that this hypothesis holds, at least for all modes that are likely to be excited within the appropriate temperature range.

The vacuum of the interacting theory \(|O_{\text{Latt}}\rangle\) also satisfies the highest weight conditions \[^{13}\] but for the \( J \) and \( \overline{J} \) operators (see \[^{15}\] for more information on representation theory of infinite dimensional Lie algebras). As is well known, this state is orthogonal to the original vacuum state. For non local interactions, the \( \varphi \) angle depends on \( n \). For the sake of regularization, let us reestablish the dependence over \( n \) of \( \varphi^n \)2

\[ |O_{\text{Latt}}\rangle = Z^{-1/2} \exp \left( -\sum_{n=1}^{+\infty} \frac{\tanh (\varphi_n)}{n} J_{-n} \overline{J}_{-n} \right) |0\rangle, \tag{30} \]

\(^{2}\)The propagation of a wave packet created at time zero by \( \psi_R^\dagger(\varphi(\sigma))(t = 0) = \int \varphi(\sigma) \psi_R^\dagger(\sigma, 0) d\sigma \) is given by \( e^{-iH/\hbar} \psi_R^\dagger(\varphi(\sigma)) e^{iH/\hbar} = \psi_R^\dagger[\phi(\sigma - v_S t)] \). As should be, a wave-packet of right movers propagate to the right.

\(^{3}\)One should remember that there is always some ultra violet cut-off that truncates the sum over \( n \).
the prefactor \( Z \) being

\[
Z = \prod_{n=1}^{+\infty} \cosh^2(\varphi_n). \tag{31}
\]

The \( Z \) prefactor diverges in the thermodynamic limit where more and more modes accumulate below some ultra violet cut-off. It means that the vacuum of the interacting theory is orthogonal to the free theory’s one. However, let us stress that the Hilbert space is still a representation of a \( \hat{U}(1)_R \times \hat{U}(1)_L \) algebra generated by the “renormalized” currents \( J_n \) and \( \tilde{J}_n \). Turning on the interactions preserves the symmetry of the Luttinger effective theory.

The diagonalized Hamiltonian is then

\[
H_{\text{Lutt}} = E_0 + \frac{2\pi v_S}{L} \sum_{n>0} n(N_{n,R} + N_{n,L}) + \frac{\pi v_S}{2L}(\alpha N^2 + \frac{1}{\alpha} J^2), \tag{32}
\]

where \( E_0 \) denotes the energy of the vacuum. In (32), \( N_{n,R} \) and \( N_{n,L} \) denote occupation numbers of bosonic modes:

\[
\forall n > 0, \quad N_{R,n} = \frac{1}{n} J_{-n} J_n, \quad N_{n,L} = \frac{1}{n} \tilde{J}_{-n} \tilde{J}_n. \tag{33}
\]

The operators \( N \) and \( J \) are the charge and current numbers in the \( q = 0 \) mode: \( N = J_0 + \tilde{J}_0 \), and \( J = J_0 - \tilde{J}_0 \).

Let us notice that the physics of the LL can be characterized through two parameters: a renormalized Fermi velocity \( v_S \) and a dimensionless interaction parameter \( \alpha \). At the Fermi liquid point, \( v_S = v_F \) and \( \alpha = 1 \).

### 2.2 Field theoretical description

The aim of this section is to introduce the field theoretical approach to the LL. As explained before, all physical quantities of this quantum system at finite temperature can be derived from a Euclidian quantum field theory in two dimensions, which will be determined in the following sections.

#### 2.2.1 CFT in two dimensions

The basic idea consists in representing a one dimensional quantum system at finite temperature by a two dimensional statistical field theory. This old idea proves to be useful in the present context since the LL has gapless excitations. Therefore, one would naively expect the corresponding two dimensional statistical theory to be scale invariant, and even more, to be conformally invariant\(^4\). Before studying in detail the effective CFT that describes the low energy physics of LLs, let us recall a few basic facts about CFT.

As usual in CFT, the objects we shall be interested in are partition functions with a special twist: besides \( \exp(-\beta H) \), a translation operator in the spatial direction \( \exp(\imath \theta P) \) is introduced. More precisely, we shall compute partition functions of the type

\[
Z = \text{Tr} \left( e^{-\beta H} e^{\imath \theta P} \right). \tag{34}
\]

As explained by Cardy in [14], this corresponds to computing functional integrals on a torus. In our context, the relevant modular parameter \( \tau \) of this torus is

\[
\tau = \frac{\theta}{L} + \imath \frac{\beta v_S}{L}. \tag{35}
\]

\(^4\)It is important not to forget that these arguments are quite heuristic. Their relevance in the present context relies on the fact that we can recover Haldane’s spectrum and other known low energy properties of LLs using CFT.
As is usually done in CFT \cite{16}, the Hamiltonian and momentum operators can be expressed in terms of Virasoro generators \( L_0 \) and \( \overline{L}_0 \) by

\[
H = \frac{2\pi v_s}{L} \left( L_0 + \overline{L}_0 - \frac{c}{12} \right) \tag{36}
\]

\[
P = \frac{2\pi v_s}{L} \left( L_0 - \overline{L}_0 \right). \tag{37}
\]

The number \( c \) (called the central charge) corresponds to the ground state energy. Results are usually expressed using the complex parameters \( q = \exp(2i\pi\tau) \) and \( \overline{q} = \exp(-2i\pi\overline{\tau}) \). For example, the partition function

\[
Z = \text{Tr} \left( q^{L_0 + \overline{L}_0} \right) \tag{38}
\]

can be expressed as a double Puiseux expansion in \( q \) and \( \overline{q} \). Its coefficients are nothing but the degeneracy of states at a given energy and momentum.

### 2.2.2 Free fermions on a torus

The Hamiltonian (2) is nothing but the one arising from Dirac theory of free fermions. Let us introduce the Minkowskian gamma matrices: \( \gamma_0 = \sigma_x \) and \( \gamma_1 = -i\sigma_y \). The Dirac Lagrangian density is nothing but

\[
\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi, \tag{39}
\]

where \( \bar{\psi} = \psi^\dagger \gamma^0 \). We shall be interested in the thermodynamics of the LL. Besides the operator approach in which the density operator \( \exp(-\beta H) \) plays a central role, we shall use mainly the functional approach which involves functional Berezin integrals of the form:

\[
\int \mathcal{D}[\bar{\psi}, \psi] \exp \left( -\int \bar{\psi} \gamma^\mu_E \partial_\mu \psi \right),
\]

where \( (\gamma^\mu_E)_\mu \) denote Euclidian gamma matrices \( (\gamma^0_E = \sigma_x \) and \( \gamma^1_E = \sigma_y ) \).

In the fermionic integral representing the partition function of our system, the Euclidean action is an integral over a torus \( S_1 \times S_1 \). The spatial direction corresponds to the first circle and the other one is associated with the imaginary time. Obviously, boundary conditions for fermionic fields on the torus play an important role which we shall discuss now.

#### Spatial boundary conditions

In the space direction, fermions may be periodic \( (n \in \mathbb{Z}) \) or anti-periodic \( (n \in \mathbb{Z} + 1/2) \) \cite{80}. In the latter case, the ground state is unique. In the former case, it has a four-fold degeneracy since the occupation of the fermionic levels located on the Fermi surface does not change the total energy. In \cite{41}, the Fermi wave vector is an half-integer. Henceforth, we choose fermions in the anti-periodic sector.

Nevertheless, it is interesting to consider both cases. These two sectors, called "periodic" and "anti-periodic" have their own Hilbert space denoted respectively by \( \mathcal{H}_{P,A} \). As we shall see later, other boundary conditions may be obtained by tuning a magnetic flux through the circle.

#### Imaginary time boundary conditions

In the functional integral formalism, it is well known (see \cite{47}, chapter 9) that computing the trace of an operator acting on a single fermionic mode requires the evaluation of a Berezin integral for its representing kernel with anti-periodic boundary conditions. Remember also that periodic boundary conditions amounts to introducing a \((-1)^F\) factor, with \( F \) the fermion number:

\[
\text{Tr} (A) = \int d\xi d\xi e^{\xi \bar{\xi}} A(-\xi, \xi) \tag{40}
\]

\[
\text{Tr}((-1)^F A) = \int d\xi d\xi e^{\xi \bar{\xi}} A(\xi, \xi). \tag{41}
\]
Therefore, in our setting, we end up with four different boundary conditions for fermions on the torus: “PP”, “PA”, “AP” and “AA” where the first letter gives the periodicity in the spatial direction and the second one in the imaginary time direction. They are usually referred to as “spin sectors”. Finally we have:

\[ Z_{AP} = \text{Tr}_A \left( (-1)^F e^{-\beta H_A} \right) = \int_{AP} D[\bar{\psi}, \psi] \exp \left( -\int \bar{\psi} \gamma^\mu \partial_\mu \psi \right) \] (42)

\[ Z_{PP} = \text{Tr}_P \left( (-1)^F e^{-\beta H_P} \right) = \int_{PP} D[\bar{\psi}, \psi] \exp \left( -\int \bar{\psi} \gamma^\mu \partial_\mu \psi \right) \] (43)

\[ Z_{PA} = \text{Tr}_P \left( e^{-\beta H_P} \right) = \int_{PA} D[\bar{\psi}, \psi] \exp \left( -\int \bar{\psi} \gamma^\mu \partial_\mu \psi \right) \] (44)

\[ Z_{AA} = \text{Tr}_A \left( e^{-\beta H_A} \right) = \int_{AA} D[\bar{\psi}, \psi] \exp \left( -\int \bar{\psi} \gamma^\mu \partial_\mu \psi \right) \] (45)

We are indeed studying the two dimensional massless Dirac theory, which is a CFT. But, contrarily to the usual stringy point of view, we consider here only one specific spin sector instead of summing over all of them.

2.2.3 The massless Thirring model

The infrared behavior is described by an interacting theory as expressed by the sum of Hamiltonians (2) and (16). In Dirac’s theory, the conserved current is given by \( J^\mu = \bar{\psi} \gamma^\mu \psi \). The interaction term of Haldane’s Hamiltonian can be expressed in terms of this current and therefore corresponds, in a Lagrangian formulation, to a quadratic current interaction of generic Thirring form:

\[ S_{\text{int}}[\bar{\psi}, \psi] = -\frac{\kappa}{2} \int (\bar{\psi} \gamma^\mu \psi)^2. \] (46)

This suggests that the Euclidian quantum field theory to be considered in this paper therefore describes the massless Thirring model \(^5\) in a specific spin sector, which we call the Luttinger CFT. Its Lagrangian is given by:

\[ \mathcal{L} = i\bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{\kappa}{2} (\bar{\psi} \gamma^\mu \psi)^2. \] (47)

Most of the analysis carried in sections \(^3\) and \(^4\) will consist in performing explicit computations of correlation functions using a functional bosonization procedure. Let us point out that here, the relevant parameters are the Fermi velocity and the coupling constant \( \kappa \). These are to be related with \( v_S \) and \( \alpha \) in Haldane’s analysis. In order to obtain a perfect dictionary between the Haldane’s approach, the bosonization computation, and Thirring’s model, one should of course be able to perform direct computations of correlation functions in the massless Thirring model\(^5\). Reference papers on this particular point are \(^43\) and \(^44\). The main point of these computations rely on the proper definition for the current composite operators \( \bar{\psi} \gamma^\mu \psi \) and \( \bar{\psi} \gamma^\mu \gamma^5 \psi \). However, since we know the spectra and the asymptotics of fermion correlators in Haldane and – as we shall see later – in the bosonic approach, all is needed here are relations \(^27\) to \(^27\) and \( \alpha \)’s definition. It would however be interesting to clarify the relation between these three approaches.

2.2.4 Explicit computations in the operator formalism

It is quite interesting to compute explicitly the partition functions \(^12\) to \(^13\) of the Dirac theory, using the operator formalism. Here, computations are given at zero magnetic field and chemical potential. But, anticipating our needs, let us notice the interested reader that a more complete discussion can be found in appendix \(^4\).

\(^5\)This was pointed out to us by R. Stora.
**Definition of the operators**  In order to symmetrize both branches, we note $b_{n,R}^\dagger = c_{n,R}^\dagger$ and $b_{n,L}^\dagger = c_{-n,L}^\dagger$, and $N_{n,\alpha} = b_{n,\alpha}^\dagger b_{n,\alpha}$ if $n > 0$, $N_{n,\alpha} = b_{n,\alpha} b_{-n,\alpha}$ if $n < 0$. Then, using a zeta function renormalization to get rid of infinities, we obtain:

$$H = \frac{2\pi v_F}{L} \sum_n |n| (N_{n,R} + N_{n,L}) - 2 \zeta_a(-1),$$

where $a = 0$ corresponds to the periodic sector and $a = -1/2$ to the anti-periodic one. Here, $\zeta_a$ denotes the analytic continuation of

$$\zeta_a(s) = \sum_{n=1}^{+\infty} \frac{1}{(n+a)^s}.$$  

(49)

The right hand side of equation (49) is defined for $\Re(s)$ sufficiently large. We recall that (18):

$$\zeta_0(-1) = -\frac{1}{12} \text{ and } \zeta_{1/2}(-1) = \frac{1}{24}. \quad (50)$$

The Hamiltonian is now completely determined in each sector:

$$H_P = \frac{2\pi v_F}{L} \sum_{n \in \mathbb{Z}} |n| (N_{n,R} + N_{n,L}) + \frac{1}{6}, \quad (51)$$

$$H_A = \frac{2\pi v_F}{L} \sum_{n \in \mathbb{Z}+1/2} |n| (N_{n,R} + N_{n,L}) - \frac{1}{12}. \quad (52)$$

The momentum operator does not depend on the regularization and is given by:

$$P = \frac{2\pi v_F}{L} \sum_n |n| (N_{n,R} - N_{n,L}). \quad (53)$$

**Explicit expressions of the partition functions**  In the free case, all traces can be explicitly computed to obtain

$$Z_{AA} = (q^2)^{-1/24} \left( \prod_{n=0}^{+\infty} (1 + q^{n+1/2})^2 \right)^2 \quad (54)$$

$$Z_{PA} = 4 (q^2)^{-1/24} \left( q^{1/8} \prod_{n=1}^{+\infty} (1 + q^n)^2 \right)^2 \quad (55)$$

$$Z_{AP} = (q^2)^{-1/24} \left( \prod_{n=0}^{+\infty} (1 - q^{n+1/2})^2 \right)^2 \quad (56)$$

$$Z_{PP} = 0. \quad (57)$$

The vanishing of $Z_{PP}$ is due to the fact that, in the doubly periodic sector, the occupation number of a state lying at the Fermi surface does not change the energy nor the momentum (since $k_F = 0$). In the functional integral language, it corresponds to the existence of fermionic zero modes in the “PP” spin sector.

Let us conclude that these partition functions can also be obtained by a zeta regularization for the determinant of the Dirac operator on the torus in each spin sector.

**Partition functions of the LL**  Let us now infer from Haldane’s spectrum the partition function of the interacting Luttinger theory. Using the $\zeta$ renormalization prescription, we obtain

$$E_0 = -\frac{\pi v_s}{6L}. \quad (58)$$
Let us introduce the Dedekind function
\[
\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{+\infty} (1 - q^n).
\] (59)

Then, the partition function of the LL is given by:
\[
Z_{AA} = \frac{1}{|\eta(q)|^2} \left( \sum_{(n,m) \in \mathbb{Z}^2} q^{\frac{1}{2} \left( n \sqrt{\alpha} + m \mu \right)^2} + \sum_{(n,m) \in (\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z}} q^{\frac{1}{2} \left( n \sqrt{\alpha} + m \mu \right)^2} \right) q^{\frac{1}{2} \left( n \sqrt{\alpha} + m \mu \right)^2},
\] (60)

where we have simply replaced the sum over \(J\) and \(N\) of same parity by a sum over \((n,m)\) where \(N = 2n + m\) and \(J = 2n - m\). Our problem is now to find a convenient way to describe the physics of the LL. More precisely, we look for an effective field theory that would reproduce the partition function \(Z_{AA}\) by functional techniques.

### 2.3 Bosonic description of the Luttinger effective theory

We explain here how to recover the partition function of the interacting Luttinger theory from a free bosonic theory. All subtleties lie in the boundary conditions of the bosonic field and we shall focus on this point. First of all, we recall how to compute some partition functions in the compactified bosonic theory, and then, the identification of the “AA” fermionic partition function will be performed. A standard argument of modular covariance enables to find explicit expressions in two of the remaining sectors.

#### 2.3.1 The compactified boson

In this section, we give explicit expressions of partition functions of the theory of a free boson, compactified on a circle of radius \(R\). By this, we mean that \(\varphi\) and \(\varphi + 2\pi R\) are identified. We work on the torus \(T_\Gamma = \mathbb{C}/\Gamma\), where \(\Gamma\) denotes the lattice \(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}\). The modular parameter \(\tau = \omega_2/\omega_1\) is assumed to have a positive imaginary part. The action reads
\[
S[\varphi] = \frac{g}{2\pi} \int_{T_\Gamma} |\nabla \varphi|^2.
\] (61)

We consider the possible boundary conditions \([\epsilon, \epsilon']\), where \(\epsilon\) and \(\epsilon'\) are given modulo 1, defined by the monodromy conditions
\[
\begin{align*}
\varphi(z + \omega_1) &\equiv \varphi(z) + 2\pi R \epsilon \quad (\text{mod } 2\pi R) \\
\varphi(z + \omega_2) &\equiv \varphi(z) + 2\pi R \epsilon' \quad (\text{mod } 2\pi R).
\end{align*}
\] (62)

The partition function
\[
Z_{\epsilon, \epsilon'} = \int_{[\epsilon, \epsilon']} D[\varphi] \ e^{-S[\varphi]}
\] (63)

is obviously Gaussian. It can be explicitly computed using a quadratic expansion around classical solutions (also called instantons) compatible with the imposed boundary conditions. Such a computation is straightforward and well known. Details are briefly recalled in appendices C and D.

**Expression of the bosonic partition functions**  As shown in appendix E, the partition function of the compact boson is
\[
Z_{[\epsilon, \epsilon']}(gR^2) = \frac{1}{|\eta(q)|^2} \sum_{m \in \mathbb{Z}} e^{-2\pi m \epsilon'} q^{\frac{1}{2} \mu^2 \epsilon' m^2 + \frac{1}{2} \mu^2 \mu^2 m^2}.
\] (64)
where we have introduced the momenta:

\[ p_{n,m} = n\sqrt{\alpha} + \frac{m}{2\sqrt{\alpha}} \]  \hspace{1cm} (65)  
\[ \overline{p}_{n,m} = n\sqrt{\alpha} - \frac{m}{2\sqrt{\alpha}} \]  \hspace{1cm} (66)  

Expression (64) is the building block of all the computations done in this paper. As we shall see later, it will be used in the computation of correlation functions of the so-called vertex operators, among which renormalized fermions.

### 2.3.2 Identification of the LL to the compact boson

String theorists are usually interested in the modular invariant partition function \[ Z_{\text{Dirac}} = \frac{1}{2} (Z_{AA} + Z_{AP} + Z_{PA} + Z_{PP}) \]  \hspace{1cm} (67)  

This sum can be readily evaluated by the usual techniques \[ \text{[33]} \] and is equal to

\[ Z_{\text{Dirac}} = \frac{1}{|\eta(q)|^2} \sum_{(m,n) \in \mathbb{Z}^2} q^{\frac{1}{2}(n+2m)^2} q^{\frac{1}{2}(n-2m)^2}. \]  \hspace{1cm} (68)  

Comparing to (64) gives \[ Z_{\text{Dirac}} = Z_{[0,0]}(1). \]  \hspace{1cm} (69)  

Dirac theory thus coincides with the theory of a free compactified boson for \( gR^2 = 1. \) We now wish to express each of the partition functions (54), (55) and (56) in terms of bosonic functional integrals with \( gR^2 = 1. \) These formulae already exist in the literature, at least in [21] and [3]. A very detailed study of bosonization in a general charge and background metric may be found in [31]. Nevertheless these results do not seem to be widely known and it is interesting to derive them in an elementary way.

**Bosonic expressions of the free partition functions**  

Jacobi triple product identity reads \[ \text{[95]}:\]

\[ \sum_{n=-\infty}^{+\infty} y^n q^{n^2/2} = \prod_{n=1}^{+\infty} (1 - q^n) \prod_{n=0}^{+\infty} (1 + yq^{n+1/2})(1 + y^{-1}q^{n+1/2}). \]  \hspace{1cm} (70)  

Specializing to \( y = 1, y = q^{1/2} \) and \( y = -1 \) and combining with (54), (53) and (56) leads to expressions that, as we shall explain, can be recovered from a bosonic theory:

\[ Z_{AA} = \frac{1}{|\eta(q)|^2} \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2} \]  \hspace{1cm} (71)  
\[ Z_{PA} = \frac{1}{|\eta(q)|^2} \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}(n+1/2)^2} \]  \hspace{1cm} (72)  
\[ Z_{AP} = \frac{1}{|\eta(q)|^2} \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{1}{2}n^2} \]  \hspace{1cm} (73)
The LL with charge velocity $v_N$ and current velocity $v_J$ may be identified to the compact boson with $gR^2 = \sqrt{\alpha}$, where $\alpha = \sqrt{v_N/v_J}$. The reader should keep in mind the very special boundary conditions that are imposed to the bosonic field for this identification to hold.

Conformal spins appearing in (81) have the form

$$\Delta_{n,m} = \frac{1}{2} \left( \frac{n\sqrt{\alpha} + m}{2\sqrt{\alpha}} \right)^2 - \frac{1}{2} \left( \frac{n\sqrt{\alpha} - m}{2\sqrt{\alpha}} \right)^2 = nm.$$  

(82)

Therefore, no exotic statistics are involved in the interacting problem in the following sense: all fields have integer of half-integer conformal spin.

Equation (81) describes the identification of spectra between the LL studied by Haldane and the spectrum of a certain bosonic CFT. It is based on an explicit comparison of partition functions. We will also need to couple the Luttinger system to an electromagnetic field in order to understand the effects of a magnetic field and of an electric potential. Fermionic computations can be found in appendix \[E.\] As we show there, the $AA$ sector does not lead to any surprise. But, as explained in this appendix, the $PP$ sector deserves some care. In order to go further in this analysis, the next section will be devoted to the study of the bosonic Luttinger field theory coupled to a gauge field.

Functional integral expressions  Now, these bosonic partition functions will be identified with bosonic functional integrals of the type (64). For this purpose, let us write explicitly (64) in the four sectors:

$$Z_{[0,0]}(1) = \frac{1}{|\eta(q)|^2} \sum_{(n,m) \in \mathbb{Z}^2} q^{\frac{1}{2}(n+\frac{1}{2})^2} \bar{q}^{\frac{1}{2}(n-\frac{1}{2})^2}$$  

(74)

$$Z_{[0,\pm]}(1) = \frac{1}{|\eta(q)|^2} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^m q^{\frac{1}{2}(n+\frac{1}{2})^2} \bar{q}^{\frac{1}{2}(n-\frac{1}{2})^2}$$  

(75)

$$Z_{[\pm,0]}(1) = \frac{1}{|\eta(q)|^2} \sum_{(n,m) \in \mathbb{Z}^2} q^{\frac{1}{2}(n+\frac{1}{2}+\frac{\eta}{\bar{\eta}})^2} \bar{q}^{\frac{1}{2}(n+\frac{1}{2}-\frac{\eta}{\bar{\eta}})^2}$$  

(76)

$$Z_{[\pm,\pm]}(1) = \frac{1}{|\eta(q)|^2} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^m q^{\frac{1}{2}(n+\frac{1}{2}+\frac{\eta}{\bar{\eta}})^2} \bar{q}^{\frac{1}{2}(n+\frac{1}{2}-\frac{\eta}{\bar{\eta}})^2}.$$  

(77)

After some algebra, the "AA" partition function of the free Dirac theory can be expressed as:

$$Z_{AA} = \frac{1}{2} \left( Z_{[0,0]}(1) + Z_{[0,\pm]}(1) + Z_{[\pm,0]}(1) - Z_{[\pm,\pm]}(1) \right).$$  

(78)

Using the modular transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$, equation (78) leads to

$$Z_{AP} = \frac{1}{2} \left( Z_{[0,0]}(1) + Z_{[0,\pm]}(1) - Z_{[\pm,0]}(1) + Z_{[\pm,\pm]}(1) \right)$$  

(79)

$$Z_{PA} = \frac{1}{2} \left( Z_{[0,0]}(1) - Z_{[0,\pm]}(1) + Z_{[\pm,0]}(1) + Z_{[\pm,\pm]}(1) \right).$$  

(80)

The key point is that any sector of the fermionic theory can be obtained from the "AA" sector using modular transformations. On the other hand, the modular properties of the $Z_{\left[\epsilon,\epsilon^\prime\right]}$ bosonic partition functions are obvious.

Interacting case  As far as the interacting case in concerned, we use the form (64) of the partition function of the LL to obtain

$$Z_{AA}^{(\text{Lutt})} = \frac{1}{2} \left( Z_{[0,0]}(\sqrt{\alpha}) + Z_{[0,\pm]}(\sqrt{\alpha}) + Z_{[\pm,0]}(\sqrt{\alpha}) - Z_{[\pm,\pm]}(\sqrt{\alpha}) \right).$$  

(81)

The LL with charge velocity $v_N$ and current velocity $v_J$ may be identified to the compact boson with $gR^2 = \sqrt{\alpha}$, where $\alpha = \sqrt{v_N/v_J}$. The reader should keep in mind the very special boundary conditions that are imposed to the bosonic field for this identification to hold.
3 Coupling to a gauge field

The aim of this section is to give a bosonization prescription of the LL coupled to a gauge field. We first examine the case of the compact boson coupled to a gauge field, and we end up with an identification of the charge and current in terms of the bosonic field. As an application, we can compute the partition function of the LL in the presence of a magnetic flux through the Luttinger ring. In section 5, explicit expressions the generating functionals of charge and current density correlators will be obtained.

3.1 Compact boson coupled to a constant gauge field

In the bosonic theory, the gauge field is coupled via an exterior product, namely the action is

\[ S_{\wedge}[\phi, A] = \frac{g}{2\pi} \int (\nabla \phi)^2 - \frac{i}{\pi R} \int A \wedge d\phi. \]  

(83)

We first consider the case of a constant gauge potential with holonomies

\[ \int_{(a)} A = 2\pi a \text{ and } \int_{(b)} A = 2\pi b. \]  

(84)

In a second step, non constant terms in the gauge potential will be introduced.

We are interested in a partition function with boundary conditions given by (62). A computation detailed in appendix D gives the following result:

\[ Z[\epsilon, \epsilon'] = \frac{1}{|\eta(q)|^2} \sum_{(m,n) \in \mathbb{Z}^2} e^{2i\pi m\epsilon'} e^{4i\pi (\epsilon + n)\beta} q^{\frac{1}{2} p_{n+m,2}^2} \vartheta[s-a, m+2a, q^2 \vartheta[s-a, m+2a]. \]  

(85)

where \( p_{n,m} \) and \( \vartheta[s-a, m+2a] \) are defined by (65) and (66).

As a warm up exercise, let us focus on the non interacting case, that is \( \alpha = gR^2 = 1 \) and let us compute the partition function

\[ Z[A] = \frac{1}{|\eta(q)|^2} \sum_{(\epsilon, \epsilon') \in \{0, 1/2\}} (-1)^{4\epsilon \epsilon'} Z[\epsilon, \epsilon'][A]. \]  

(86)

The signs appearing here correspond to the ones in identity (81). We notice that

\[ \sum_{\epsilon' \in \{0, 1/2\}} (-1)^{4\epsilon \epsilon'} e^{i\pi m \epsilon'} = \frac{1}{2} (1 + (-1)^{2\epsilon + m}). \]  

(87)

Therefore, \( m \equiv 2\epsilon \pmod{2} \). Separating the two cases \( \epsilon = 0 \) and \( \epsilon = 1/2 \), and performing a change of indices leads to:

\[ Z[A] = \frac{1}{|\eta(q)|^2} \sum_{(\epsilon, \epsilon') \in \{0, 1/2\}} (-1)^{4\epsilon \epsilon'} \vartheta[s-a, \beta] \vartheta[s-a, \beta]. \]  

(88)

Finally, expression (88) can be rewritten in terms of Riemann’s theta function with characteristics (see appendix E), thus recovering the known results [23, 4, 78]:

\[ Z[A_{a,b}] = \frac{1}{|\eta(q)|^2} \theta \left[ \begin{array}{c} a \\ -b \end{array} \right] \vartheta \left[ \begin{array}{c} a \\ -b \end{array} \right] (0, \tau) \]  

(89)

Here, we have assumed that \( b \) is complex and \( a \) real, since the chemical potential corresponds to a purely imaginary \( b \). This point is important in the comparison of bosonic and fermionic partition functions performed in appendix E. As we shall see later, the interacting case is a straightforward generalization of the non interacting case, provided the theta function is replaced by its suitable generalization.

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3.2 Gauge transformations

Strictly speaking, the previous computation has been done with a constant gauge field. Knowing the transformation properties of $Z_{[\epsilon,\epsilon']}[A]$ in normal ($A \mapsto A + d\chi$) and chiral ($A \mapsto A + d^*\lambda$) gauge transformations provide us with $Z[A]$ for any gauge field $A$, since by the Hodge decomposition theorem, any 1-form $A$ can be decomposed in a unique way as:

$$A = h + d\chi + d^*\lambda.$$  \hfill (90)

Here, $h$ denotes an harmonic 1-form, or equivalently on the torus, a constant connection. We recall that gauge parameters $\chi$ and $\lambda$ are well defined functions on the torus (no monodromy).

We first consider normal gauge transformations. Using relation (282), and the exactness of $d\chi$, we obtain:

$$S_\lambda[\varphi, A + d\chi] - S_\lambda[\varphi, A] = \frac{i}{\pi R} \int d\chi \wedge d\varphi = 0.$$  \hfill (91)

The partition function is exactly gauge invariant under normal gauge transformations.

We now turn to chiral gauge transformations:

$$A \rightarrow A + d^*\lambda,$$  \hfill (92)

where $d^*\lambda = \epsilon^{\mu\nu}(\partial_\nu \lambda)dx^\mu$. Elementary manipulations lead to:

$$S_\lambda[\varphi, A + d^*\lambda] = S_\lambda[\varphi - \frac{i}{gR} \lambda, A] + \frac{1}{2\pi} \int (d\lambda)^2 + \frac{1}{\pi\alpha} \int A \wedge d\lambda.$$  \hfill (93)

To summarize, the partition function behaves as follows under normal and chiral gauge transformations:

$$Z[A + d\chi] = Z[A]$$  \hfill (94)

$$Z[A + d^*\lambda] = Z[A] \exp \left( -\frac{1}{2\pi\alpha} \int (d\lambda)^2 - \frac{1}{\pi\alpha} \int A \wedge d\lambda \right).$$  \hfill (95)

It is important to notice that the transformation properties of bosonic functional integrals are completely independent of boundary conditions in the bosonic functional integrals. Identities (94) and (95) are the Ward identities of the Luttinger CFT.

3.3 Non chiral bosonization prescription

The aim of this section is to establish, by means of functional integrals manipulations, the identification between fermionic functional integrals of the massless Thirring model in the “AA” spin sector and an appropriate bosonic functional integral. This formula encodes the bosonization of the massless Thirring model in a functional way. This bosonization approach is different from the usual one which relies on chiral bosonization of left and right movers in the operator formalism (see [49, Chapter 14]). Here, both chiralities are treated at the same time. This is why we call this procedure a non-chiral bosonization.

3.3.1 Statement of the result

Let us introduce the following notation:

$$\int_C \mathcal{D}[\varphi] = \frac{1}{2} \sum_{(\epsilon,\epsilon') \in \{0,1/2\}} (-1)^{\epsilon\epsilon'} \int_{[\epsilon,\epsilon']} \mathcal{D}_R[\varphi],$$  \hfill (96)

where the boundary conditions in the right hand side path integral are

$$\begin{cases}
\varphi(z+1) \equiv \varphi(z) + 2\pi \epsilon R \pmod{2\pi R} \\
\varphi(z+\tau) \equiv \varphi(z) + 2\pi \epsilon' R \pmod{2\pi R}
\end{cases}$$  \hfill (97)
and where $D_R[\varphi]$ denotes the integration measure for a bosonic field compactified on a circle of radius $R$ (see appendix C). We will show that the following equality between partition function holds:

$$
\int_{\AA} D[\bar{\psi}, \psi] e^{-S_L[\bar{\psi}, \psi]} + \int A_\mu \bar{\psi} \gamma_\mu E \psi = \int_{\mathcal{C}} D[\varphi] e^{-S_\wedge[\varphi, A]},
$$

(98)

where $S_L$ denotes the action of the massless Thirring model, defined by the Lagrangian (47), and $S_\wedge$ is the action (83) of the compact bosonic field coupled to a gauge field via an exterior product. This formula expresses a non-chiral bosonization of the Luttinger CFT in the presence of a gauge background. It also shows that the fermionic current can be expressed in terms of the bosonic field:

$$
\bar{\psi} \gamma_\mu E \psi = \frac{\epsilon_{\mu\nu} \partial_\nu \varphi}{\pi R}.
$$

(99)

This identity should be understood as an equality within correlation functions.

3.3.2 The interacting case

We shall now prove (98) in full generality. The method goes as follows: noticing that this equation is true for a uniform gauge background, we first prove it for the Dirac theory in a generic gauge background by using Ward identities. Then, the interacting case will be analyzed with the help of a suitable Hubbard-Stratanovich transformation. Let us now prove equation (98) for an arbitrary gauge field. The proof goes as follows: we know the transformation law of fermionic partition functions both under chiral gauge transformations and normal gauge transformations. They are invariant under the latter whereas the so-called chiral anomaly term appears under chiral gauge transformations (see equations (94) and (95)). Thus, the fermionic partition functions and the bosonic functional integral appearing in (98) have the same transformation properties in normal and chiral gauge transformations and coincide for constant gauge fields. Therefore, using the Hodge decomposition theorem for 1-forms on the torus, these functionals do indeed coincide.

We now introduce the interaction term (46) of the Thirring type. The four fermions interaction can be usefully decoupled by introducing an auxiliary field $b_\mu$:

$$
\exp \left(-\frac{\kappa}{2} (\bar{\psi} \gamma_\mu_E \psi)^2\right) = \int D[b_\mu(x)] \exp \left(\frac{1}{2\kappa} \int b(x)^2 + i \int b_\mu \bar{\psi} \gamma_\mu_E \psi\right).
$$

(100)

Therefore, we are back to the previous case in presence of a fluctuating vector potential:

$$
Z_{\text{Lutt}}[A] = \int_{\AA} D[\bar{\psi}, \psi] e^{-S_L[\bar{\psi}, \psi]} + \int A_\mu \bar{\psi} \gamma_\mu E \psi
$$

(98)

$$
= \int_{\AA} D[\bar{\psi}, \psi] \int D[b_\mu(x)] e^{-S_L[\bar{\psi}, \psi]} - \frac{\kappa}{2} \int (b(x)^2 + i \int (b_\mu + A_\mu) \bar{\psi} \gamma_\mu E \psi).
$$

Let us now apply identity (98), which, up to now, has been proven to be valid just in the free theory and in the presence of a fluctuating gauge potential. We then obtain

$$
Z_{\text{Lutt}}[A] = \int_{\mathcal{C}} D[\varphi] \int D[b_\mu(x)] e^{-\frac{\kappa}{2\pi} \int (\partial \varphi)^2 + \frac{1}{\pi R} \int (A + b)^\wedge d\varphi - \frac{1}{\pi} \int (\partial (b_\mu + A_\mu) \varphi_E)^2}.
$$

(101)

We now integrate again over the auxiliary field $b_\mu$ and obtain:

$$
Z_{\text{Lutt}}[A] = \int_{\mathcal{C}} D[\varphi] e^{-\frac{\kappa}{2\pi} \int (\partial \varphi)^2 + \frac{1}{\pi R} \int (A + b)^\wedge d\varphi - \frac{1}{\pi} \int (A - b)^\wedge d\varphi}.
$$

(102)

The effect of interactions as a change of the coupling constant (or equivalently of the compactification radius) is summarized by:

$$
g' = g + \frac{\kappa}{\pi R}.
$$

(103)
Discussion of instability  Let us notice that the interacting theory is only defined for $\kappa > -\pi$. When $\kappa \leq -\pi$, then the bosonic action is no longer positive. Such a limitation already arises in the operator formalism. Formula (20) shows that for $|g_2| \geq |v_F + g_4|$, the Bogoliubov transformation does not exist anymore. More precisely, one easily sees that the spectrum of the bosonic Hamiltonian (16) is no longer bounded from below. This is the sign of an instability, which we recover in the functional integral language since $\alpha^2 = (v_F + g_4 - g_2)/(v_F + g_4 + g_2)$.

It would be interesting to understand the origin of this instability directly in the fermionic framework. Let us recall that in the massive Thirring model, Korepin [57] has shown that for the so-called repulsive regime ($-\pi < g < -\pi/2$), the spectrum includes bound states. Coming back to the massless case, when $\kappa \to -\pi$, the fermions tend to form a bound state condensate and the model becomes unstable at $\kappa = -\pi$. However, the precise relation between this instability (in the massive regime) and the one we find in the bosonic theory is still an open question.

Finally, such instabilities are not surprising from the solid state physics point of view. Indeed, they appear when the interactions, measured in suitable units, are of the order of the Fermi velocity $v_F$. In this case, the electronic fluid is expected to develop instabilities (see for instance Anderson’s book [7]).

3.4 Evaluation of the effective action

In this section, integration over matter fields of the Luttinger model will be explicitly performed, thus providing an effective action for the vector potential. From the field theoretical point of view, this boils down to solving the Schwinger model [83] on the torus in a specific spin sector. This effective action can be understood as a generating functional for charge and current densities correlation functions as we shall see in the next section.

All computations will be done within the bosonic theory. We shall first of all isolate the contribution of the zero modes of the vector potential. After this separation, the computation will be straightforward.

3.4.1 Separation of zero modes

Let $A = A_\mu dx^\mu$ be a vector potential. Hodge theory tells us that it can be decomposed in a unique way as $A = h_A + d\chi + d^*\lambda$, with $h_A$ a uniform vector potential. We then have:

$$Z[A] = Z[h_A] \exp\left(-\frac{1}{2\pi\alpha} \int (d\lambda)^2\right),$$

where we have used the Ward identity (105).

3.4.2 Explicit evaluation

We are now going to compute $\lambda$ in terms of the vector potential $A$. Let us introduce $B = A - h_A$, such that each component of $B$ has zero average on the torus. The one-form $d^*\lambda - B$ is exact and therefore $d d^*\lambda = dB$. However, since $d d^*\lambda = -(\Delta\lambda)^*$, the scalar curvature $F_B$ of $B$ is nothing but

$$F_B = \Delta\lambda.$$

The Laplacian is invertible on the kernel of the normalized integral over the torus:

$$\int_N : f \mapsto \frac{1}{A} \int f,$$

with $A$ the total area of the torus. Finding an inverse of the Laplacian involves solving a Green’s equation:

$$\Delta_x G(x, y) = \delta(x - y) - \frac{1}{A}.$$

Different solutions to this equation differ by a constant which is irrelevant since the Green’s function is always applied to a zero average function. Let $\Delta^{-1}$ denote the Laplacian’s inverse on this vector space. Its
explicit expression is obtained in appendix E. Inverting equation (105) and introducing complex coordinates leads to

$$Z[A] = Z[h_A] \exp \left( K[B_z, B_{\bar{z}}] \right), \quad (107)$$

where

$$K[B_z, B_{\bar{z}}] = \frac{2}{\pi \alpha} \int d^2 z d^2 \xi \left( B_z B_\xi (\partial^2 \Delta^{-1})(z - \xi) + B_{\bar{z}} B_\xi (\partial^2 \Delta^{-1})(z - \xi) \right) - \frac{1}{\pi \alpha} \int B_z B_{\bar{z}} d^2 z. \quad (108)$$

As recalled in appendix E, both second derivatives $\partial^2 \Delta^{-1}$ and $\partial^2 \Delta^{-1}$ should be understood as derivatives of distributions (see equation (304) of appendix E). These distributions are related to Weierstrass $\wp$ function by formula (310) and its complex conjugate. This subtlety is important when computing density-density correlation functions in presence of an external potential (see section 5.1).

Other authors work on the plane, namely in the zero temperature and thermodynamics limits. Under this circumstance, there are two differences with the results we have obtained: first of all the contribution of $A$’s zero modes is not present. Next, the Green’s function has a simpler expression. Nevertheless, the idea is the same and explicit integration over matter fields has long been a useful trick in these two-dimensional field theories [83].

4 LL with a forward scattering disorder

The formalism developed in the previous sections shall now be applied to the study the LL in presence of a magnetic flux and of a very special kind of disorder: a Gaussian external potential coupled to the fermionic density $\rho(\sigma)$, as defined in equation (9). Let us stress that this disorder does not include backscattering effects and therefore localization is not present in this toy-model. This is of course a severe limitation of our study but taking into account backscattering requires a detailed study of Sine-Gordon like interactions, which goes beyond the scope of the present work. When these interactions are relevant, one is driven away from the conformal regime and other methods should be used. A disorder more suited for systems such as quasi-one-dimensional conductors would involve impurities along the chains and therefore the existence of backscattering. Several approaches, such as Berezinskii diagram techniques (see for instance [10, 24, 51, 76, 82, 11, 38]) were developed in the seventies by the Russian school to treat such a kind of disorder in the absence of electron-electron interactions. The conjugated effects of disorder and interactions were analyzed in [12] by means of renormalization group techniques. In these different works, the presence of disorder easily gives way to massive phases, or to massless phases with a singular low energy density of states. In both cases, the resulting disordered system is obviously not conformal and its treatment using methods similar to the ones of the present article is an open question.

Nevertheless, in this framework, correlation functions and their averages over the aforementioned toy-disorder can be computed without using the replica trick. The method used here relies on transformation properties of the Lagrangian under chiral gauge transformations and has been used by D. Bernard [13] to solve the random vector potential model. As we shall see in section 3, transport properties are not affected by the toy-disorder, as expected since no localization is involved. We shall also compute the specific heat of the LL in the regime where the temperature is much larger than the inter-level spacing $2\pi v_S/L$, showing that it is not altered by the simple toy-disorder considered here.

4.1 General setting and notations

In this section, a random classical potential $V(\sigma)$ will be introduced. We also introduce a constant magnetic field with a magnetic flux $\Phi$ through the ring, and $\chi = \Phi/\Phi_0$ denotes the number of flux quanta, with $\Phi_0 = 2\pi/e$ the flux quantum. The reason why we treat simultaneously a coupling to an external potential and a coupling to a magnetic field is that, as we shall see, both can be naturally analyzed in the same framework. In fact, as we will see later, they transform into one another by duality.
The random potential distribution is taken to be Gaussian:

$$P[V(\sigma)] = \exp\left(-\frac{1}{2\gamma} \int_0^L d\sigma V(\sigma)^2\right).$$  \hspace{1cm} (109)

Thermal and quantum averages with respect to the Luttinger system are denoted by $\langle\ldots\rangle$:

$$\langle O[\overline{\psi},\psi]\rangle[\chi,V(\sigma)] = \int D[\overline{\psi},\psi] e^{-S[\overline{\psi},\psi,V,\chi]} O[\overline{\psi},\psi] \int D[\overline{\psi},\psi] e^{-S[\overline{\psi},\psi,V,\chi]},$$  \hspace{1cm} (110)

where fermionic fields live in the “AA” sector. We denote by $\overline{X}$ the average of $X$ over the toy-disorder, and we are interested in correlation functions of the type

$$\prod_{k=1}^N \langle O_k[\overline{\psi},\psi]\rangle = \int D[V] e^{-\frac{1}{2\gamma} \int_0^L d\sigma V(\sigma)^2} \prod_{k=1}^N \langle O_k[\overline{\psi},\psi]\rangle[\chi,V(\sigma)].$$  \hspace{1cm} (111)

As we shall see in the following, these correlation functions can be explicitly computed without using the replica trick. In order to achieve this goal, we shall first of all bosonize the system as explained in the previous sections and then use the quadraticity of the action both in the bosonic field and in the random potential. At this point, the choice of a Gaussian distributed random potential is crucial.

**Feynman weight contribution of disorder and magnetic field** The statistical weight associated with the disorder and magnetic field is given by:

$$W[\overline{\psi},\psi] = \exp \left(i \int_0^\beta \int_0^L d\sigma \left[ \frac{eV}{L} \overline{\psi} \gamma^1 \psi + i eV(\sigma) \overline{\psi} \gamma^0 \psi \right] \right).$$  \hspace{1cm} (112)

It may be usefully rewritten as $\exp \left(i \int j.A \right)$ where $j^\mu = \overline{\psi} \gamma^\mu \psi$, and $A_0 = ieV(\sigma)$ and $A_1 = evf/\Phi/L$.

### 4.2 Correlation functions averages over disorder

Let us now derive the general formula for computing averages over disorder of any product of correlation functions. We first split the potential $V(\sigma)$ into its average part and a fluctuating part:

$$V(\sigma) = \mathcal{V}(\sigma) + \frac{Q}{L},$$  \hspace{1cm} (113)

and we introduce $\eta(\sigma)$ defined by

$$\eta(\sigma) = \int_0^\sigma \mathcal{V}(x) dx.$$  \hspace{1cm} (114)

The observable $O$ considered here is assumed to be expressed as a functional of the bosonic field. In this case, we introduce the following notation:

$$Z_O[\chi,V(\sigma)] = \int_{\mathcal{C}} D[\varphi] e^{-S[\varphi,A]} O[\varphi],$$  \hspace{1cm} (115)

The idea is now to incorporate the fluctuating part $\eta(\sigma)$ of the potential in a redefinition of $\varphi$. Readers familiar with CFTs can recognize here the abelian version of Polyakov-Wiegmann’s identity \[^{55}\]. After simple algebraic manipulations, we get

$$Z_O[\chi,V(\sigma)] = e^{\frac{2\gamma}{2\pi gR} \int_0^L d\sigma \eta'(\sigma)^2} Z_{O[\varphi + \eta]}[\chi,Q/L].$$  \hspace{1cm} (116)
The field $\varphi$ has been shifted by $en/gR$. This shift does not modify the boundary conditions for our bosonic field since $\eta(0) = \eta(L) = 0$. Since the exponential prefactor appearing in the right hand side of (114) is independent of the observable $O$, it cancels when averages over $O$ are taken:

$$\langle O[\varphi]|_{X,V(\sigma)} \rangle = \langle O[\varphi + \frac{e}{gR}\eta]|_{X,Q/L} \rangle.$$  \hspace{1cm} (117)

Using these equations, averages over the toy-disorder can be computed:

$$\prod_{k=0}^{N} \langle O_k[\phi] \rangle = \int_{-\infty}^{+\infty} dQ \frac{1}{\sqrt{2\pi} L} e^{-\frac{m^2}{2} \int D[\eta(\sigma)] \delta(\eta(0))} e^{-\frac{1}{2} \int_0^L d\sigma \eta'(\sigma)^2} \prod_{k=0}^{N} \langle O_k[\phi + \frac{e}{gR}\eta]|_{X,Q/L} \rangle.$$  \hspace{1cm} (118)

We have therefore achieved our goal: averages over the toy-disorder can be computed without the replica trick. In the forthcoming sections, we shall give explicit examples of correlation functions, averaged over the toy-disorder. But before doing that, we turn now to the problem of computing the average free energy over the toy-disorder.

### 4.3 Average free energy, thermal capacity

We first rewrite the partition function of the Luttinger CFT (60) as

$$Z_{\text{Lutt}} = \frac{1}{\eta(q)^2} \sum_{(m,m') \in \mathbb{Z}^2} q^{\frac{1}{2} \left( \frac{\pi^2 + \pi^2 - 1}{m + \pi^2} \right)^2} \left( \frac{\pi^2 + \pi^2 - 1}{m + \pi^2} \right)^2.$$  \hspace{1cm} (119)

We are interested in the behavior of the partition function in the limit $\tau = \beta vs/L \to 0^+$. Physically, it means that the temperature is much larger than the inter-level spacing. Henceforth, a high number of energy levels are excited. We recall that a real system may not be described at all energies by the effective Luttinger theory considered here. If $E^*$ denotes the energy scale where the effective description by an interacting massless Thirring model breaks down, we assume that $E^* >> 2\pi vs/L$. The energy scale $E^*$ may originate from band curvature, non local interactions,. . . Our analysis will be valid in a temperature regime such that

$$\frac{2\pi vs}{L} \ll k_B T \ll E^*.$$  \hspace{1cm} (120)

In terms of $x = 2\pi vs/\beta L$, the relevant limit is $x \to 0^+$. Without any twist in the spatial direction (in other words, $\theta = 0$ in (13)), we have $q = \theta = e^{-x}$. In terms of $x$, the partition function is

$$Z_{\text{Lutt}}(\alpha, [A]) = \frac{1}{\eta(x/2\pi)^2} \sum_{(n,m) \in \mathbb{Z}^2} \exp \left( -\frac{x}{4} (\alpha(n + m)^2 + \alpha^{-1}(n - m - 2a)^2) \right) \exp(2ib(n + m)).$$  \hspace{1cm} (121)

Its limiting behavior for $x \to 0^+$ is easily obtained with the help of Dedekind’s function asymptotics in the limit $\tau \to i0^+$. Substituting $b = i\beta vsQ/2\pi L$, we finally obtain:

$$\ln (Z_{\text{Lutt}}(\alpha, [A])) \sim \frac{\pi L}{6\beta vs} + \frac{\beta vsQ^2}{2\pi \alpha L}.$$  \hspace{1cm} (122)

Before averaging over the quenched toy-disorder, let us notice that the effect of the magnetic field and of the electric potential boils down to a free energy decrease of $\nu_1 Q^2/2\pi \alpha L$, independent on the temperature and on the magnetic field. In particular, the specific heat is not affected neither by the electric potential nor by the magnetic field. Of course, the same conclusions are valid after averaging over the disorder.

### 5 Charge-charge and current-current correlations

As an application of previous formalism, correlations of charge and current densities in Luttinger CFT will be computed. Our strategy will rely on the evaluation of the generating functional for charge density correlators
and current density correlators. We are interested in correlation functions in a regime of a fixed total charge in the system. This total charge is denoted here by \( q \). Averaging over \( q \) amounts to considering a set of many isolated Luttinger rings, some of which contain an odd number of charge carriers, and others carry an even number of carriers. Again, our aim is not to describe in a “realistic” way the physics of permanent currents in mesoscopic rings, would involve multichannel effects, impurities, spin effects, etc., and goes much beyond the scope of the present formalism. We rather want to demonstrate that the formalism developed in the previous sections is an operational tool from which physical quantities such as permanent currents in a Luttinger ring can be calculated. In order to be more general, we will first calculate the generating functional of the \( n \) points density and current correlations.

5.1 Charge and current density correlators

**Statement of the results**

The generating functional for charge density correlators is defined by

\[
W^{(0)}_{V(\sigma), \chi, q}[b(z)] = \langle \exp \left( \int d\sigma \, dt \, b(\sigma, t) \rho(\sigma, t) \right) \rangle. \tag{123}
\]

An explicit expression can be derived from the effective action (107). It has the form

\[
W^{(0)}[V(\sigma), \chi, q] = \exp \left( L_0[V(\sigma), b(z)] + F_0[b(z)] \right), \tag{124}
\]

involving a linear contribution:

\[
L_0[V(\sigma), b(z)] = \int d\sigma \, du \left( \frac{q}{L} - \frac{1}{\pi \alpha} V(\sigma) \right) b(\sigma, u), \tag{125}
\]

and a quadratic contribution

\[
F_0[b(z)] = \exp \left( \frac{1}{4 \pi \alpha} \int d^2 \! z \, d^2 \! \xi \, b(z) b(\xi) G_0(z - \xi) \right), \tag{126}
\]

the kernel \( G_0 \) being

\[
G_0(z - \xi) = -\Re \left( \frac{\psi(z - \xi)}{\pi} \right) + \delta(z - \xi). \tag{127}
\]

Of course, the Weierstrass function contribution should be understood as a regularized distribution, just like in formula (110) in appendix B. Details of the computation will be given in the next subsection.

The generating functional for current correlators is defined by

\[
W^{(1)}_{V(\sigma), \chi, q}[c(z)] = \langle \exp \left( \int d\sigma \, dt \, c(\sigma, t) j(\sigma, t) \right) \rangle. \tag{128}
\]

It can be explicitly computed and we obtain

\[
W^{(1)}_{V(\sigma), \chi, q}[c(z)] = \frac{Z_q[0, \chi, q]}{Z_q[0, \chi]} \times \exp \left( F_1[c(z)] \right), \tag{129}
\]

where \( \tilde{c}(z) = c(z) - \frac{1}{L} \int c(\xi) d^2 \! \xi \). The quadratic contribution \( F_1 \) is

\[
F_1[c(z)] = -\frac{1}{4 \pi \alpha} \int d^2 \! z \, d^2 \! \xi \, \tilde{c}(z) \tilde{c}(\xi) \left( \Re \left( \frac{\psi(z - \xi)}{\pi} \right) + \delta(z - \xi) \right). \tag{130}
\]
Details of computation Let us now show how to obtain formulae (1.29) to (1.27). We shall start from the following potential: $A_\sigma = 0$ and $A_u = i(\mathcal{V}(\sigma) + b(\sigma, u))$. This implies $B_z = -B_{\bar{z}} = (\mathcal{V} - b)/2$, with $b(z) = b(\bar{z}) = \frac{1}{\pi} \int b(\xi) d^2\xi$. Plugging this in formula (108) leads to three distinct contributions:

$$\begin{align*}
\frac{1}{2\pi \alpha} \int \left( (\partial^2 + \bar{\partial}^2) \Delta^{-1} + \delta/2 \right) (z - \xi) \frac{\partial}{\partial\xi} b(\xi) d^2 z d^2 \xi \\
\frac{1}{2\pi \alpha} \int \left( (\partial^2 + \bar{\partial}^2) \Delta^{-1} + \delta/2 \right) (z - \xi) \mathcal{V}(\sigma_\xi) d^2 z d^2 \xi \\
\frac{1}{\pi \alpha} \int \left( (\partial^2 + \bar{\partial}^2) \Delta^{-1} + \delta/2 \right) (z - \xi) \frac{\partial}{\partial\xi} b(\xi) \mathcal{V}(\sigma_\xi) d^2 z d^2 \xi.
\end{align*}$$

(131)

The first one directly gives $F_0[b(z)]$ and the second one compensates while dividing by the partition function. Only the third one deserves attention. The main point is to remember that derivatives of $\Delta^{-1}$ should be understood as derivatives of a distribution as explained in appendix E.2. For simplicity, computations use the $\mathbb{Z} \oplus \tau \mathbb{Z}$ lattice. Let $f$ be a periodic function, of period 1, of a real variable. We define a $\mathbb{Z} \oplus \tau \mathbb{Z}$ periodic function $f_1$ by $f_1(z) = f(\mathbb{R}(z))$. The problem is now to evaluate the distribution $T$ defined by

$$T . f = \partial_z^2 \Delta^{-1} . f_1.$$  

(132)

Noticing that $f_1$ only depends on $\mathbb{R}(z)$, we thus get $\partial_z f_1 = (\partial_z + \partial_{\bar{z}}) f_1$ and therefore $\partial_z^2 \Delta^{-1} . f_1 = \partial_z \partial_z \Delta^{-1} . f_1$. It is now possible to apply formula (31) to get

$$T . f = \partial_z^2 \Delta^{-1} . f_1 = \frac{1}{4} (\delta - 1) . f,$$

(133)

where the distributions are now acting on $f$, which depends on a real variable. In other words, for a zero-average test function $f$:

$$\int f(\sigma_\xi) g(z - \xi) \frac{\phi(z - \xi)}{\pi} d^2 z d^2 \xi = \int f(\sigma) g(\sigma, u) d\sigma du.$$  

This shows that the $\mathcal{V} \times b$ contribution is given by the $\mathcal{V}$-dependent part of $L_0[\mathcal{V}, b]$ and that the second contribution is nothing but

$$\exp \left( \frac{1}{2\pi \alpha} \int d^2 z \mathcal{V}(\sigma_z)^2 \right),$$

a result that can easily be deduced from the Ward identity (35).

This computation deserves some comments: it shows that properly considering correlation functions of a CFT as distributions enables to extract some ultra violet properties of the model from its infrared effective description. More explicitly, the coupling between $b$ and the potential, which indeed contains the charge response to an external potential, is local in the CFT approximation. Nevertheless, it can be extracted from a long distance effective theory since it relies on the symmetry properties of the system, that is to say, its transformation properties under chiral gauge transformations, which we expect to be independent of the description of the system.

The proof of equations (129) and (130) goes along the same lines. We have to compute $K[C + D, C - D]$ where $C = \hat{c}(z)/2$ and $D = \mathcal{V}(\sigma)/2$. This gives two terms, the first one being

$$F_1[\hat{c}(z)] + \frac{1}{2\pi \alpha} \int d^2 z \mathcal{V}(\sigma_z)^2,$$

and the second one contains a crossed term of the type $\hat{c} \times \mathcal{V}$:

$$\frac{i}{2\pi \alpha} \lim_{\epsilon \to 0^+} \int_{|z - \xi| > \epsilon} d^2 z d^2 \xi \hat{c}(z) \mathcal{V}(\sigma_\xi) \Im \left( \frac{\phi(z - \xi)}{\pi} \right).$$

The point is then to notice that, taking the imaginary part of $\phi$ in equation (133) gives zero. Therefore, the crossed term vanishes.
5.2 Density response to an external potential

Density modifications induced by the external potential  Differentiating once with respect to $b(\sigma, t)$ provides us with the average density in the presence of the potential $V(\sigma)$. We obtain:

$$\rho_{av}(\sigma) = \frac{q}{L} - \frac{1}{\pi v_N} \left( V(\sigma) - \frac{1}{L} \int_0^L V(\sigma) \, d\sigma \right).$$  \hfill (134)

The first term in (134) ensures charge conservation, with a total charge of $q$. The response function to an external potential is thus purely local. This result is expected from the density response of non interacting, non relativistic fermions in one dimension (see appendix F where the calculations in this limit are recalled). Turning on local interactions in the LL only amounts to replacing the Fermi velocity $v_F$ by the charge velocity $v_N$ defined in equation (26) in terms of the interaction parameters. This result could maybe have been anticipated from the beginning on physical grounds.

Correlation functions  The two point connected correlator of the densities has a singularity at equal times due to the delta function in time and space. It is interesting to understand the origin of this short distance singularity. For this reason, we have recalled the corresponding computation for non relativistic free fermions on a circle in appendix F. This computation has been performed at zero temperature for simplicity but may easily be extended to a finite temperature. In the non relativistic model, the Fermi sea has a finite depth $k_F$ and the density-density correlations show a peak spreading over a distance $1/k_F$. The two points correlations also show some short distance oscillations of the form $\sin^2(k_F \sigma)$ which are not present in our CFT computation. These two apparent discrepancies with conformal results have the following interpretation: first of all, observed at distances small compared to the size $L$ of the circle but large compared to the “microscopic” length scale $1/k_F$, the short distance peak becomes a delta function, the coefficient of which is proportional to $k_F$ (and therefore diverges in the infrared limit). This corresponds to the double delta function in the conformal result. Next, the average of $\sin^2(k_F \sigma)$ over distance much greater than $k_F^{-1}$ is $1/2$ and we recover the conformal result.

Finally, no orthogonality-like scenario is operational in the density-density correlations, unlike the case of fermion correlations, as we shall see later. The reason for this is that, even though density operators are defined in terms of chiral fermions, density operators are composite operators. It is also a-priori obvious that no orthogonality-like behavior is expected since integrating the $n$-point correlations with respect to all the coordinates leads to a non vanishing term even in the thermodynamic limit, thus excluding the possibility of exponentially small prefactors.

In a sector of fixed charge, the generating functional of density correlations is Gaussian (see equation (124)). The same is true for the current density fluctuations. Apart from the term associated with the permanent current, the functional (129) is Gaussian. The reason for that lies in the fact that the fermionic density can be viewed as a “coordinate” for this quantum field theory. The current density is then conjugated to it. In term of these coordinates, the Luttinger CFT is essentially free, therefore explaining the Gaussian nature of these functionals.

5.3 Permanent currents in the Luttinger ring

As emphasized by many authors, and as observed experimentally, mesoscopic rings exhibit permanent currents of quantum origin in the presence of a magnetic flux (see for instance [63, 67]). In the LL, these currents can be exactly computed at a finite temperature. They have a “universal” expression in the sense that all the interaction dependence is encoded in the $\alpha$ and $v_S$ Luttinger parameters, as expected on general grounds exposed in the introduction.

The permanent current in an isolated system of charge $q$ is proportional to the derivative of the free energy with respect to the magnetic flux:

$$I[V(\sigma), \chi, q] = \left( \frac{\partial F}{\partial \Phi} \right) = \frac{e}{\hbar} \left( \frac{\partial F}{\partial \chi} \right).$$  \hfill (135)
A straightforward computation leads to

\[ I[V(\sigma), \chi, q] = -\frac{e}{\beta h} \frac{\partial}{\partial \chi} \left( \log \left( \sum_{n \in \mathbb{Z}} \exp \left( -\frac{x}{\alpha} \left( \frac{q}{2} + \chi + n \right)^2 \right) \right) \right). \]  

Equation (136)

In the zero temperature limit, this obviously reduces to the famous “saw-tooth” curve. The maximal value is given by \( \pi e v_f J / L \). The current variations as a function of \( \chi \) are shown on figure [1] as a function for different values of the prefactor \( x/\alpha \). As this ratio decreases the “saw-tooth” curve is more and more smoothened. The currents are periodic in the magnetic flux, with a unit flux quantum periodicity, as can also be seen from figure [1]. Even though we stress that the LL is far from a proper modeling of mesoscopic rings, this unit quantum flux periodicity was observed experimentally in a single ring in [67]. The currents also show a periodic behavior in the total charge \( q \) but with a period of two unit charges, as expected. Of course, if an average over the parity of the charges is taken, the current shows a half flux quantum periodicity. This situation is the analogous of the experimental realization of a large ensemble of mesoscopic ring, where half-flux-quantum periodic currents were observed [63]. Finally, the current decays exponentially as soon as the temperature exceeds the level spacing \( 2\pi v_f S / L \), as expected on physical grounds.

### 6 Correlation functions of vertex operators

The aim of this section is to compute correlation functions of vertex operators. Let us recall that in a free bosonic CFT, \( \hat{U}(1) \) primary fields are in a one to one correspondence with chiral operators. They constitute the building blocks of the Luttinger CFT. As we shall see in section 6.4, the renormalized fermion in the interacting theory correspond to one of these vertex operators.

After having identified the effective theory for edge excitations of a FQH fluid on a cylinder as some of the Luttinger CFTs (see section [6]), we shall see that physical fermions localized on one edge of the Hall
sample are created and destroyed by specific vertex operators. Therefore computations performed in this
section give access to physically important correlation functions.

We start by recalling a definition of vertex operators well-suited to functional integral computations. Then, we derive correlation functions on the sphere for warming up and on the torus. Finally, the identification of Luttinger fermion and the effect of the toy-disordered are discussed.

### 6.1 Vertex operators

Instead of studying vertex operators themselves, we shall be interested in specific products of them. More precisely, we shall compute objects such as averages of

\[
\prod_{k=1}^{n} \exp \left( \frac{ie_k}{R} \int_{C_k} d\varphi + \frac{e_k^*}{R} \int_{C_k} d^* \varphi \right). \tag{137}
\]

The parameters \((e_k, e_k^*)\) are fixed real numbers and \(C_k\) are some open oriented curves starting at \(a_k\) and ending at \(b_k\). Of course, the object (137) should be renormalized precisely, and this will be explained in the forthcoming subsections. Nevertheless, it is first interesting to understand its meaning.

The first integral in (137) defines a product of usual vertex operators since one can easily recognize

\[
\exp (ie_k (\varphi(b_k) - \varphi(a_k)))/R).
\]

To understand the physical meaning of the second contribution in (137), let us choose locally some coordinates near a point \(x\) on the curve \(C_k\). The spatial coordinate \(l_x\) will be chosen locally along the curve \(C_k\) and the imaginary time coordinate \(u_x\) is chosen normally to \(C_k\) at point \(x\). Then, the local contribution to the second integral is

\[-(\partial_{u_x} \varphi)(l_x, u_x) dl_x.\]

But the imaginary time derivative is proportional to the moment associated with coordinate \(\varphi\). More precisely, we know that the conjugate moment of \(\varphi\) is nothing but

\[\Pi_{\varphi} = \frac{g}{\pi} \partial_t \varphi.\]

Therefore, if we denote by \(\Pi_{C_k}\) the moment on the curve \(C_k\), we have

\[
\exp \left( \frac{e_k^*}{R} \int_{C_k} d^* \varphi \right) = \exp \left( -i \frac{\pi e_k^*}{gR} \Pi_{C_k} \right). \tag{138}
\]

The second integral thus corresponds to introducing a defect along curve \(C_k\). It is nothing but a disorder operator.

With this interpretation, we immediately see that both \(e_k\) and \(e_k^*\) are quantized. Taking into account \(\varphi\)'s compactification at radius \(R\) immediately gives \(e_k \in \mathbb{Z}\). The disorder part introduces a cut of \(\pi e_k^*/gR\) for the field \(\varphi\) along curve \(C_k\). Then, imposing that this shift be a multiple of \(2\pi R\) gives the second quantization condition: \(e_k^* \in 2\alpha \mathbb{Z}\). In fact these are the quantization conditions of the usual bosonic theory, in the modular invariant sector. In order to understand this, and as a warm up exercise, let us first compute correlation functions on the sphere.

### 6.2 Correlation functions on the sphere

Let us compute the two points function on the Riemann sphere:

\[
\frac{1}{Z_{P_1(C)}} \int D[\varphi] \exp \left( \frac{i e}{R} \int_C d\varphi + \frac{e^*}{R} \int_C d^* \varphi - \frac{g}{2\pi} \int (d\varphi)^2 \right), \tag{139}
\]

where \(C\) starts at point \(a\) and ends at point \(b\). This is a Gaussian integral to be computed using a saddle point method. In the following, \(\delta_C\) will denote the delta distribution located on the oriented curve \(C\) and
\( \partial_n \delta C \) will denote its derivative with respect to the normal coordinate on the curve. The saddle point equation is then:

\[
\Delta \varphi = \frac{i \pi e}{gR} (\delta_a - \delta_b) - \frac{\pi e^*}{gR} \partial_n \delta C.
\] (140)

As expected, when crossing the curve \( C \) in the direct sense with respect to \( C \)'s orientation, \( \varphi \) drops by \( \pi e^*/gR \). Up to an additive constant, the solution to this equation can be expressed in terms of a determination of the complex logarithm. Let us define \( h_{a,b} \) by:

\[
h_{a,b}(z) = z - \frac{a}{z - b}.
\] (141)

If \( \log_C \) denotes the determination of the logarithm such that \( \log_C(h_{a,b}(z)) \) has a cut along \( C \), we have:

\[
\varphi_0(z) = \frac{i}{4gR} \left( (e - e^*) \log_C(h_{a,b}(z)) + (e + e^*) \log_C(h_{a,b}(z)) \right).
\] (142)

The Gaussian integral to be computed here is of the form

\[
\int dX \exp \left( -\frac{1}{2} t X.A.X + t X.J. \right).
\]

The saddle point evaluation of the exponential gives

\[
\exp \left( \frac{1}{2} t J.X_0 \right) \quad \text{with} \quad A.X_0 = J.
\] (143)

As we shall see, using this Ansatz requires some care because of divergences.

Let us compute the first contribution to the saddle point evaluation: we have to renormalize

\[
\exp \left( \frac{i e}{2R} (\varphi_0(b) - \varphi_0(a)) \right).
\] (144)

The prescription chosen here is to replace \( \varphi_0(a) \) and \( \varphi_0(b) \) by their averages of circles of radius \( \varepsilon \), centered on \( a \) and \( b \) respectively. This will provide us with regularized quantities depending on \( \varepsilon \). Then, we shall use a minimal renormalization scheme, that is we will forget about divergences. Let us denote by \( \Gamma^{(1)}_{\varepsilon}(a, b, e, e^*) \) the regularized quantity defined by

\[
\langle e^{\frac{R}{i\varepsilon} (\varphi_0(b) - \varphi_0(a))} \rangle_{\varepsilon} = e^{-\Gamma^{(1)}_{\varepsilon}(a, b, e, e^*)}.
\] (145)

We have, at dominant orders:

\[
\Gamma^{(1)}_{\varepsilon}(a, b, e, e^*) \simeq -\frac{e^2}{2\alpha} \log (\varepsilon) + \frac{e^2}{2\alpha} \log ( |h_{a,b}(z)|) - \frac{e e^*}{2\alpha} \log \left( \frac{h_{a,b}(z)}{h_{a,b}(z)} \right).
\] (146)

In order to obtain the other contribution, we simply notice that

\[
d\varphi = \partial_z \varphi \, dz + \partial_{\bar{z}} \varphi \, d\bar{z}
\]

\[
d^* \varphi = i (\partial_{\bar{z}} \varphi \, d\bar{z} - \partial_z \varphi \, dz),
\] (147)

and therefore, the second contribution is obtained from the first by exchanging \( e \) and \( e^* \). In the end:

\[
\Gamma^{(2)}_{\varepsilon}(a, b, e, e^*) \simeq -\frac{(e^*)^2}{2\alpha} \log (\varepsilon) + \frac{(e^*)^2}{2\alpha} \log ( |h_{a,b}(z)|) - \frac{e e^*}{2\alpha} \log \left( \frac{h_{a,b}(z)}{h_{a,b}(z)} \right).
\] (149)
Taking into account the two contributions and eliminating the overall divergence $\varepsilon^{-(e^2+(e^*)^2)/2\alpha}$ leads to the required correlation function since the partition function on the sphere cancels between the numerator and the denominator. We finally get:

$$\langle e^{i\int_C d\varphi + \int_C d^* \varphi} \rangle_{\mathbb{P}(C)} = (b-a)^{-(e-e^*)^2/4\alpha} \left(\frac{1}{b-a}\right)^{-(e+e^*)^2/4\alpha}.$$  \hfill (150)

Remembering the quantization conditions obtained in section 6.1 and the usual form of two point correlation functions in 2D CFT, we recover the spectrum of the free compactified boson in the modular invariant sector, as expected: $e^* = 2n\pi$ and $e = -m\pi$. In the following, $\phi_{n,m}(z,\bar{z})$ will denote the $(p_{n,m},\overline{p}_{n,m}) U(1)$ primary field. Equation (150) simply gives $\langle \phi_{n,m}^\dagger(b) \phi_{n,m}(a) \rangle$.

### 6.3 Correlation functions on the torus

In order to perform this computation on the torus, one has to proceed in three steps. The idea is to split the bosonic field into several parts:

- The usual instanton contribution, depending on winding numbers along the two homology cycles of the torus.
- A saddle point contribution with vanishing monodromy along homology cycles of the torus but which takes into account vertex operator insertions.
- The fluctuation contribution.

Renormalization issues only appear in the computation of the fluctuation contribution. As for the computation of partition functions, a Poisson resummation has to be performed. Here, only the final results will be given and commented. Some details may be found in appendix D.

Let us introduce

$$Z(a,b) = \omega_1 \frac{\partial_1}{\partial_1^2(0,\tau)};$$  \hfill (151)

and

$$F_{[(e_k)a]}(z_1,\ldots, z_n) = \prod_{k\neq l} Z(z_k, z_l)^{e_k e_l / 8\alpha}. $$  \hfill (152)

The following divisors on the torus $(e_k^\pm = e_k \pm \overline{e_k})$:

$$D_- = \sum_{k=1}^n e_k^- \frac{z_k}{\omega_1} \quad \text{and} \quad \overline{D_+} = \sum_{k=1}^n e_k^+ \frac{\overline{z_k}}{\omega_1}$$  \hfill (153)

are needed in order to define a Theta function contribution:

$$\Theta_{\Gamma_L}(\tau, \overline{\tau}, D_-, \overline{D_+}) = \sum_{(p, \overline{p}) \in \Gamma_L} q^{p \cdot \overline{p}} \tau^p \overline{\tau}^p \exp \left( i \frac{\pi}{\sqrt{\alpha}} (p D_- + \overline{p} D_+) \right).$$  \hfill (154)

Then the correlation function is given by the rather complicated expression:

$$\langle \prod_{k=1}^n V_{(e_k^-, e_k^+)}(z_k, \overline{z_k}) \rangle_{\mathbb{T}^2} = \frac{\Theta_{\Gamma_L}(\tau, \overline{\tau}, D_-, \overline{D_+})}{\Theta_{\Gamma_L}(\tau, \overline{\tau}, 0, 0)} \times F_{[(e_k^-)k]}(z_1,\ldots, z_n) F_{[(e_k^+)k]}(\overline{z_1},\ldots, \overline{z_n}).$$  \hfill (155)

It captures all finite size effects in correlation functions. Let us determine the asymptotics of this correlation function in the limit of coinciding points, that is when

$$\forall k \neq l, \quad |z_k - z_l| < \min(|\omega_1|, |\omega_2|).$$

30
In this limit, $D_-$ and $\overline{D}_+$ vanish, and $Z(a, b) \simeq b - a$. Therefore, we recover the expression obtained before for correlation functions on the Riemann sphere.

Let us notice the familiar CFT structure \[29\] of these correlation functions which appear as sesquilinear combinations of conformal blocks:

$$\sum_{l, I} \mathcal{F}_l(x) \overline{\mathcal{F}}_l(x), \quad (156)$$

where $x$ denotes some complex coordinates over the relevant Teichmüller space. Here this space is nothing but the one associated with complex tori with $n + 1$ marked points\[4\] and the appropriate spin structure on the elliptic curve. The expression just obtained assumes that this correlation has been computed using some special region in Teichmüller space: namely all points $z_{k}$ and $z_0$ belong to the fundamental cell of the period lattice of the torus. Locally, $\hat{U}(1)$ conformal blocks are given by:

$$\mathcal{F}_{(c_k)_L, p} = \frac{q^2 r^2}{\eta(\tau)} e^{\frac{i}{\alpha} p D_-} \mathcal{F}_{(c_k)_{\overline{c}}} (z_1; \ldots, z_n), \quad (157)$$

where $p$ is of the form $p_{(2, m)}$ with $l \equiv m \pmod{2}$. During the eighties, very similar expressions appeared in the context of string theory and in CFT \[23\]. However, exactly as for partition functions, people were interested in the modular invariant sector. More precisely, expression obtained for correlators were exactly the same, except for the sum over all the moments $(p, \overline{p})$. In the modular invariant sector, one has to use the lattice $\{(p_{(n,m), \bar{p}_{(n,m)}}) / (n, m) \in \mathbb{Z}^2\}$ which corresponds to the moments appearing $Z_{0,0}$. In the Luttinger case, these moments belong to the lattice

$$\Gamma_{\text{Lutt}} = \{(p_{(2,m), \bar{p}_{(2,m)}}) / (l, m) \in \mathbb{Z}^2 \quad l \equiv m \pmod{2}\}.$$

### 6.4 Free fermions

It is interesting to specialize these expressions to the case of free fermions ($\alpha = 1$), in order to compute the correlation functions of fermionic operators. Right fermions have conformal spin $1/2$. They are $\hat{U}(1)$ primary fields, \emph{id est} chiral vertex operators with some specific choice of $(e, e^*)$. The following choice has conformal spin $1/2$: $e = 1$ and $e^* = -\alpha$ ($n = -1/2$ and $m = 1$). Let us now compute the two point function on the torus:

$$G_{\alpha=1}(a, b) = \langle \exp \left( \frac{i}{R} \int_{C_{ab}} d\varphi - \frac{\alpha}{R} \int_{C_{ab}} d^* \varphi \right) \rangle_{\omega_1, \mathbb{Z} \times \mathbb{Z}}. \quad (158)$$

We specialize to $\alpha = 1$. In this case $e_- = 2$ and $e_+ = 0$. The fluctuation contribution only contains a holomorphic contribution

$$\omega_1^{-1} \frac{\vartheta_4'(0, \tau)}{\vartheta_4'(z, \tau)} \vartheta_3^3(z, \tau),$$

where $z = (b - a)/\omega_1$. The theta function contribution is a quotient of two theta functions. For $\alpha = 1$, expression \[154\] specializes to:

$$\sum_{(p, \overline{p}) \in \Gamma_L} \frac{q^2 r^2}{\overline{q}^2 \overline{r}^2} e^{2\pi p z}. \quad (159)$$

In the free case, the lattice $\Gamma_{\text{Lutt}}$ coincides with $\mathbb{Z}^2$ and therefore this theta function factorizes between holomorphic and anti-holomorphic parts.

Putting all parts together we get:

$$G_{\alpha=1}(a, b) = \omega_1^{-1} \frac{\vartheta_4'(0, \tau)}{\vartheta_4'(z, \tau)} \vartheta_3^3(z, \tau), \quad (160)$$

The $n + 1$-th point is the reference point $P_0$. It does not appear in the expression.
which may be rewritten as

\[ G_{\alpha=1}(a,b) = \frac{\pi}{\omega_1} N(\tau) H(z,\tau), \]  

(161)

where \( y = e^{2\pi iz} \)

\[ N(\tau) = \prod_{n=1}^{+\infty} \left( 1-q^n \right)^2 \]

(162)

\[ H(z,\tau) = \frac{1}{\sin(\pi z)} \prod_{n=1}^{+\infty} \frac{\left( 1+yq^{n-1/2} \right)\left( 1+y^{-1}q^{n-1/2} \right)}{\left( 1-yq^n \right)\left( 1-y^{-1}q^n \right)}. \]

(163)

This is nothing but the two point correlator of the free Dirac fermions on the torus. This may be understood by showing that expression (160) satisfies the Dyson-Schwinger equations of Dirac theory. A direct fermionic computation also gives the same result.

Of course, when \( \alpha \neq 1 \), these correlation functions mix holomorphic and anti-holomorphic parts. Moreover, vertex operators considered here \((n = \pm 1/2, m = \pm 1)\) are renormalized fermions in the interacting theory. More precisely, if \( \psi_{0,R}^\dagger \) creates a bare right fermion, we have \( \phi_{1/2,1} = Z^{-1/2} \psi_{0,R}^\dagger \) where \( Z \) is a multiplicative renormalization constant. \( Z \) vanishes for local interactions, in an orthogonality catastrophe-like scenario \[5\]. Its physical meaning is quite clear: \( \phi_{1/2,1} \) creates a normed state whereas \( Z \) is the square norm of the state obtained by adding a bare electron to the system. \( Z = 0 \) simply means that adding a bare electron to the system drives it in a state orthogonal to all eigenstates of the interacting Hamiltonian. Saying this another way round: the eigenstates of the interacting Hamiltonian have nothing to do with the original fermions. The renormalisation constant \( Z \) is also related to the discontinuity of the momentum distribution function at \( k_F \) through Migdal’s theorem (see \[1\] Page 42 and 63). Therefore, the vanishing of \( Z \) means that there is no discontinuity in the electron’s momentum distribution at the Fermi surface.

### 6.5 Inclusion of the toy-disorder

Thanks to the results obtained in section \[4\], the effect of the toy-disorder on vertex operators correlators can be explicitly studied. Let us consider the product of vertex operators (139). We have

\[ O[\varphi + \frac{\eta}{g_R}] = O[\varphi] \times \exp \left( \sum_{k=1}^{n} \frac{i e_k}{\alpha} (\eta(\sigma_k) - \eta(\sigma_0)) + \frac{e_k^2}{\alpha} \int_{C_{ok}} d^* \eta \right). \]

(164)

Of course \( d^* \eta = \eta'(\sigma) d\sigma \) and \( d^* \eta = \eta'(\sigma) dt \). The functional integral

\[ F_1(\sigma_1, \ldots, \sigma_n) = \int D[\eta(\sigma)] \delta(\eta(0)) e^{-\frac{\alpha}{4}\int_0^L (\eta'(\sigma))^2 d\sigma} e^\sum_k \frac{\alpha}{2} (\eta(\sigma_k) - \eta(\sigma_0)) \]

(165)

is easy to compute by the saddle point method. The result is expressed in terms of the \( \sigma_{ki} \)s which are the representatives of \( \sigma_k - \sigma_l \in \mathbb{R}/L\mathbb{Z} \) that belong to \([0,L[\).

\[ F_1(\sigma_1, \ldots, \sigma_n) = \exp \left( -\frac{\gamma}{4\alpha^2} \sum_{k \neq l} \frac{e_k e_l}{\sigma_{kl}} (L - \sigma_{kl}) \right). \]

(166)

The other contribution is

\[ F_2(z_1, \ldots, z_n) = \int D[\eta(\sigma)] \delta(\eta(0)) e^{-\frac{\alpha}{4}\int_0^L (\eta'(\sigma))^2 d\sigma} e^\sum_k \frac{\alpha}{2} \int_{C_{ok}} \eta'(\sigma) dt. \]

(167)

For the sake of simplicity, let us give the result for a two point function. The curve \( C_{ab} \) we have used in our computations is nothing but the line connecting \( z_1 \) to \( z_2 \). We have

\[ \int_{C_{z_1}} d^* \eta = \frac{3(z_{12})}{\Re (z_{12})} (\eta(\Re(z_2)) - \eta(\Re(z_2))). \]
and using this expression we obtain:

$$F_2(z_1, z_2) = \exp \left( \gamma (e^*)^2 u_{12}^2 (L - \sigma_{12}) \right).$$  \hspace{1cm} (168)

These results deserve some comments. First of all, an exponential decrease of correlations in the spatial direction is observed. This is the net effect of the toy-disorder. Let us stress that the exponents of algebraic decays are not modified. Restoring a finite wave-vector $k_F$ would show that this exponential decrease is in fact a $2k_F$ contribution and is not a localization effect.

7 Laughlin’s experiment and LLs

The aim of this section is to discuss the relation between some incompressible Hall fluids and the LL. The basic idea is that, in a certain limit, the Hall fluid in the annular geometry is equivalent to a LL with suitable parameters. In particular, we discuss the behavior of FQH fluids within the framework of the Luttinger CFT.

We begin by recalling how to compute the physical charge and current densities in the LL from the conserved $U(1)$ currents of the underlying CFT. Then, revisiting Laughlin gedanken experiment [21], we relate the filling fraction $\nu = 1/(2p + 1)$ of the QHE to the interaction parameter of the LL. We are then able to express the correspondence between quantum Hall fluids and LLs. In the forthcoming sections, we discuss rational LLs. Some of these RCFTs describe Laughlin Hall states (at filling fraction $\nu = 1/(2p + 1)$). After identifying these RCFTs, we introduce a fractional charge on the edge in order to describe the tower of excited states above a given bulk excitation, and discuss in a precise manner the resulting twisted theories.

7.1 Physical charge and current densities of the LL

As we have already explained in previous sections, the LL admits an infrared description by an effective CFT. This effective theory is nothing but the theory of a free massless compactified boson, with special boundary conditions that take into account the boundary conditions of the underlying bare fermionic fields. As a free bosonic CFT, it has a $\hat{U}(1) \times \hat{U}(1)$ symmetry generated by a spin one current.

As explained in section 2.1.2, the physical electric charge and current densities are directly related to the CFT $U(1)$ currents of the Luttinger CFT. A careful restitution of the speed and charge factors into (28) and (29) yields

$$\rho = \frac{e}{\sqrt{\alpha}} (J(\sigma, t) + \overline{J}(\sigma, t))$$  \hspace{1cm} (169)

$$j = \frac{v S e}{\sqrt{\alpha}} (J(\sigma, t) - \overline{J}(\sigma, t)),$$  \hspace{1cm} (170)

$e$ being the unit charge. These densities can be expressed in terms of the Laurent modes $(J_n)_{n \in \mathbb{Z}}$ and $(\overline{J}_n)_{n \in \mathbb{Z}}$ of currents on the Riemann sphere $\mathbb{P}^1$:

$$\rho(\sigma, t) = \frac{e}{L \sqrt{\alpha}} \sum_{n \in \mathbb{Z}} \left( J_n e^{2\pi i \frac{\sigma - v_S t}{L}} + \overline{J}_n e^{-2\pi i \frac{\sigma + v_S t}{L}} \right)$$  \hspace{1cm} (171)

$$j(\sigma, t) = \frac{e v S}{L \sqrt{\alpha}} \sum_{n \in \mathbb{Z}} \left( J_n e^{2\pi i \frac{\sigma - v_S t}{L}} - \overline{J}_n e^{-2\pi i \frac{\sigma + v_S t}{L}} \right).$$  \hspace{1cm} (172)

Therefore, the total charge and average electric current around the circle are given in terms of Fourier modes of the currents in the underlying CFT by

$$q = \frac{e}{\sqrt{\alpha}} (J_0 + \overline{J}_0)$$  \hspace{1cm} (173)

$$I = \frac{e v S}{L \sqrt{\alpha}} (J_0 - \overline{J}_0).$$  \hspace{1cm} (174)
It is now convenient to introduce some charge densities on each branch $\rho_R$ and $\rho_L$ such that
\[
\rho = \rho_L + \rho_R, \quad j = v_S (\rho_R - \rho_L) + 2e \frac{v_S}{L\alpha} \chi.
\] (175)
The second relation takes into account the effect of the potential vector on the fermion momenta and therefore on the electric current. Let us recall the usual non relativistic expression of the current:
\[
J_\mu = e \frac{\hbar}{2m} (\psi \partial_\mu \psi^* - \psi^* \partial_\mu \psi) - e^2 \frac{2}{2m} |\psi|^2 A_\mu.
\] (176)
The second term, proportional to the vector potential, is imposed by gauge invariance of the current. Formula (173) may be guessed from the study of a free fermion gas in two dimensions, confined on a cylinder. Chiral charges are expressed in terms of CFT operators:
\[
q_R = e \sqrt{2\alpha} \ J_0 - e \chi \alpha \quad q_L = e \sqrt{2\alpha} \ J_0 + e \chi \alpha.
\] (177)
These expressions will now be used for understanding the relation between the LL physics and the QHE.

### 7.2 Charge transport between the chiralities in a LL

Laughlin [61] and Halperin [44] considered a quantum Hall fluid localized on a ring or on a cylinder threaded by a magnetic flux $\Phi$. In fact, even though the cylinder geometry cannot be realized experimentally, it is more convenient for our purposes to consider the quantum Hall effect on the cylinder. These authors study charge transport between the two edges when this flux is adiabatically switched on. In the integer QHE, when the flux is increased by one flux quantum $\Phi_0$, the system returns in the same state except that some electrons have moved from one edge to the other.

In the FQHE, a fractional charge is transferred [88], and we have to shift the magnetic flux by several flux quanta before the system gets back on a state with an integer number of transferred electrons. More precisely, for a filling fraction $\nu$, the charge transferred during an increase $\Phi \rightarrow \Phi + \Omega_0$ is equal to $\nu e$. For $\nu = q/p$, after $\Phi \rightarrow \Phi + \Phi_0$, $q$ electrons are transferred. This may be understood as the effect of the electric field induced by the flux variation, in presence of a Hall conductivity $\sigma_H = \nu e^2/h$.

Let us now consider the LL with an interaction parameter $\alpha$. The partition function with a magnetic flux $\chi\Phi_0$ is equal to:
\[
Z_{\text{Lutt}}(\chi) = \frac{1}{|\eta(\tau)|^2} \sum_{\substack{(n,m) \in (\mathbb{Z}/2) \times \mathbb{Z} \\ 2n \equiv m \pmod{2}}} q^{\frac{1}{2} p^2_{n,m+2\chi}} \tilde{p}^{\chi}_{n,m+2\chi},
\] (178)
where
\[
p_{n,m} = n\sqrt{\alpha} + \frac{m}{2\sqrt{\alpha}} \quad \tilde{p}_{n,m} = n\sqrt{\alpha} - \frac{m}{2\sqrt{\alpha}}.
\] (179)
States of interest will be $\hat{U}(1) \times \hat{U}(1)$ highest weight states. Let us introduce the following notation:
\[
|n, m\rangle_\chi = |p_{n,m+2\chi}, \tilde{p}_{n,m+2\chi}\rangle.
\] (180)
The $|n, m\rangle_\chi$ can be viewed as the $|p_{n,m}, \tilde{p}_{n,m}\rangle$ state continuously deformed by the external magnetic flux. The partition function (178) gives the spectrum of charges on both edges. For the $|n, m\rangle_\chi$ state, we have:
\[
\begin{cases}
q_R = e \left(n + \frac{m}{2\alpha}\right) \\
q_L = e \left(n - \frac{m}{2\alpha}\right).
\end{cases}
\] (181)
The total charge is therefore $2ne$ which is always an integer. But charges on each edge can be non integer. Note that charge density fluctuations, which corresponds to $\hat{U}(1)$ descendants, are globally neutral: they don’t change left and right total charges.

We now study in more details charge transport between chiralities in the Luttinger CFT. This will enable us to guess the correspondence between edges excitations of the FQH fluid at $\nu = 1/q$ and an appropriate Luttinger CFT.
7.3 Laughlin experiment revisited

Let us now start from a Luttinger CFT with an interaction parameter $\alpha$. At zero temperature and zero magnetic flux, the system is in the $|p_0, 0, 0, \bar{p}_0, 0, 0\rangle = |0, 0\rangle_{\chi=0}$ state. The chiral charges $q_R$ and $q_L$ both vanish in this state. If we increase adiabatically the magnetic flux from $\chi = 0$ to a finite value $\chi \in [0, 1/2]$, the system goes adiabatically to the state $|0, 0\rangle_{\chi}$. In this intermediate state, we still have $q_L = q_R = 0$ but there is a non-zero current: $I = \frac{2 e \nu}{h \alpha} \chi$. When $\chi$ reaches $1/2$, the lowest energy state is no longer $|0, 0\rangle_{\chi}$ but for $1/2 < \chi < 3/2$, it is $|0, -2\rangle_{\chi}$. But in this new state, a charge has been transferred from one chirality to the other:

$$ \begin{cases} 
q_R = e/\alpha \\
q_L = -e/\alpha.
\end{cases} $$

We immediately notice that this behavior is very similar to the one of a FQH fluid as described by Laughlin \cite{Laughlin1983} and by Tao and Wu \cite{Tao1988}. If we now identify the left and right branches of the LL with the two different edges of the cylindrical sample, the charge transferred from one edge to the other is equal to $e/\alpha$ as Laughlin considered. Therefore, the filling fraction $\nu$ should be related to the interaction parameter by $\alpha \nu = 1$.

Assuming this relation, it is straightforward to compute the Hall conductivity from Luttinger CFTs: let us assume that the two edges are not connected to any charge reservoir and assume that we increase the magnetic flux $\chi$. While increasing the flux, the system goes through various states:

$$ |0, 0\rangle_{\chi} \rightarrow |0, -2\rangle_{\chi} \rightarrow |0, -4\rangle_{\chi} \rightarrow \ldots $$

The charge on the edge is therefore a step function of the magnetic flux. More precisely, for $k + 1/2 < \chi < k + 3/2$, we have $q_R = -q_L = e \nu k$. When $k$ reaches $p - 1/2$, we have $q_R = -q_L = e \nu p = q e$, as expected from Tao and Wu. As we shall see in section 9.3, the state $|0, -2q\rangle_{\chi}$ is obtained by destroying electrons on the left edge and re-creating them on the right edge. Let the magnetic flux be linearly increased with time, then by induction, an orthonodal electric field appears. As we have just seen, charges are then transferred from one edge to the other. The voltage between the two edges is given by

$$ U = \frac{h}{e} \frac{d\chi(t)}{dt}, $$

and charges are transferred at the following rate:

$$ I_\perp = e \nu \frac{d\chi(t)}{dt}.$$

Therefore, we find:

$$ I_\perp = \frac{e^2}{h} \nu U, \quad (183) $$

which is the correct form of Hall conductivity. As we have already seen, the longitudinal intensity is a periodic function of the flux, the average of which is zero. Therefore, the longitudinal conductivity is zero.

7.4 Another transport experiment

Another transport experiment consists in bringing the two edges of the sample at different potentials. In this case, a current should appear along the Hall ring. We shall see how to recover this result from the effective Luttinger CFT, thus supporting our identification of the effective theory of edge excitations by an appropriate Luttinger CFT.

Let us couple the two edges of our cylinder to different electric potentials:

$$ H_{elec} = - \int_0^L \left( V_R(\sigma) \rho_R(\sigma) + V_L(\sigma) \rho_L(\sigma) \right) d\sigma. $$. 

(184)
Assuming $\chi = 0$, this can be rewritten as

$$H_{\text{elec}} = -\int_0^L V(\sigma) \rho(\sigma) d\sigma - \int_0^L \frac{V_R(\sigma) - V_L(\sigma)}{2v_S} j(\sigma) d\sigma,$$

where $V(\sigma) = (V_R(\sigma) + V_L(\sigma))/2$ and $U_\perp = V_R - V_L$. Therefore, the electric potential between the two edges can be viewed as a vector potential $A_{\text{eff}} = U_\perp/2v_S$ in the effective 1D LL. The effective magnetic flux is given by

$$\chi_{\text{eff}} = \frac{eL}{2\hbar v_S} U_\perp.$$  \hspace{1cm} (185)

We may apply our previous results about the electric current induced by a magnetic flux in a LL. However, as we computed in section 5.3, the current in our mesoscopic cylinder should be periodic in the magnetic flux. But that is not the expected behavior in a Hall experiment. The solution to this apparent paradox lies in the fact that, in a Hall experiment, edges are connected to an electric generator which, besides imposing $U_\perp$, can bring charges into the system. Let us discuss this point more precisely, assuming $\alpha = q \in 2N + 3$.

When the flux reaches $1/2$, a charge $\pm e/q$ appears at the edge. The external generator does not bring any charge in to neutralize it, since only integer charges can be brought by the external generator. Hence, nothing special happens compared to the discussion of an isolated Luttinger ring. Current periodicity breaks down when the total charge becomes $\pm e$ on each edge. Then, the generator may neutralize the charge on each edge. For $\chi$ slightly less than $q - 1/2$, the system is in the state

$$|0, 2(1-q)\rangle_\chi$$

Then when $\chi$ becomes larger than $q - 1/2$, instead of going back in the state

$$|p_{0,0}, \bar{p}_{0,0}\rangle = |0, -2q\rangle_\chi,$$

with left and right charges $q_L = -q_R = 1$, the system jumps to the state $|0, 0\rangle_\chi$. In this state, we again have $q_{L,R} = 0$ but the current is non zero. Notice that the energy varies due to edge effects. The energy change is brought in by the external electric generator. The electric Hall conductivity can easily be computed. In terms of the effective magnetic flux, the “classical” Hall current is given by

$$I = \frac{2v_S e}{L\alpha} \chi_{\text{eff}}.$$  \hspace{1cm} (186)

Therefore, reintroducing the potential difference between the two edges of the system, we obtain

$$I = \frac{e^2}{\hbar} \alpha^{-1} U_\perp = \frac{e^2}{\hbar} \nu U_\perp.$$  \hspace{1cm} (187)

Of course, the result obtained above crucially relies on the charge transfer mechanism between the electric generator and the Hall sample. A correct treatment of this situation should use an explicit description of the connection of the two edges to charge reservoirs at different chemical potentials and of course a quantum description of these two reservoirs. For completeness, let us recall that tunneling between chiralities of an Hall bar has been widely studied these last years: see [51] and also section V.A of reference [27]. Recent experimental work [79] has been performed on tunneling experiments in the $\nu = 1/3$ case. This experiment, based on a measurement of the current noise due to charge tunneling between two edges, seem to establish the existence of fractionally charged edge excitations.

To summarize, we have shown that transport properties of the quantum Hall fluid can be understood within the framework of the Luttinger CFT. It is important to notice that not all LLs can be obtained from quantum Hall fluids: the interaction parameter $\alpha$ must be rational. Of course, our approach does not describe the transition between Hall plateaux. We only provide an effective theory for edge excitations within a given plateau.

It is interesting to understand more precisely why these rational values are special. This is our main motivation for studying Luttinger CFTs from the point of view of RCFTs.
8 Rational Luttinger CFTs

RCFTs arose originally from Friedan and Shenker work in the late 80s [29]. Roughly speaking, these are CFTs with a very large symmetry algebra; so large that the Hilbert space of states is a direct sum of a finite number of irreducible representations of this symmetry algebra (see [72] for a review). By analogy, a LL is called rational if and only if the effective theory is a RCFT.

We begin by deriving a rationality condition for the LL. Generalized characters with respect to the extended symmetry algebra will be computed. This will enable us to re-express the partition function in terms of the extended characters. As we shall see, partition functions recently discovered by Cappelli and collaborators [15] belong to our list.

8.1 Rationality criterion for the Luttinger CFT

The partition function for the Luttinger CFT is given by equations (178) and (179). If the effective theory is a RCFT, then there should exist some conformal primary fields of dimensions \((h,0)\) which extend the \(\hat{U}(1)\) algebra. This implies a condition similar to the one found for the usual free modular invariant bosonic theory: \(\alpha/2 \in \mathbb{Q}\). In the present situation, we find that \(\alpha\) is rational. Let us write \(\alpha = q/p\) where \(q\) and \(p\) are coprime positive integers.

The conformal dimensions \((h,\bar{h})\) of \(\phi_{l/2,m}\) are given by:

\[
\left(\frac{(lq + mp)^2}{8pq}, \frac{(lq - mp)^2}{8pq}\right).
\]  

(188)

Here \(l\) and \(m\) are integers of same parity (congruence condition). From Gauss lemma [84], we deduce that if \(\phi_{l/2,m}\) has a vanishing \(\bar{h}\) dimension, then there exists \(k \in \mathbb{Z}\) such that \(l = kp\) and \(m = kq\). In this case, we have

\[
h_{kp/2,kq} = \frac{pq}{2}k^2.
\]

Nevertheless, in the Luttinger model, one has the congruence condition \(l \equiv m \pmod{2}\) and therefore, two cases will be discussed separately:

Case \(p \equiv q \pmod{2}\) Since \(p\) and \(q\) are coprime, both of them are odd. The congruence condition is satisfied for all \(k \in \mathbb{Z}\). Holomorphic \(\hat{U}(1)\) primaries are \(\phi_{kp/2,kq}\) for any \(k \in \mathbb{Z}\). For \(k\) even, the \(\bar{h}\) dimension is an integer but for \(k\) odd, it is half integer. Let us denote by \(\mathcal{A}\) the chiral algebra generated by the \(\hat{U}(1)\) current \(J(z)\) and all fields \(\phi_{kp/2,ku}\) for \(k \in \mathbb{Z}\). This operator algebra \(\mathcal{A}\) is clearly \(\mathbb{Z}_2\) graduated by \(2\bar{h}\) \((\text{mod } 2)\) and we write \(\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-\). Fields \(\phi_{kp/2,ku}\) for \(k \in 2\mathbb{Z} + 1\) belong to \(\mathcal{A}_-\) and fields \(\phi_{kp,2ku}\) belong to \(\mathcal{A}_+\).

Case \(p \equiv q + 1 \pmod{2}\) One of these two integers is even. Therefore, the congruence condition is satisfied only when \(k\) is even. The holomorphic \(\hat{U}(1)\) primaries are \(\phi_{kp,2kq}\) for all \(k \in \mathbb{Z}\). The chiral algebra only consists of integer spin fields.

8.2 Extended characters

Extended characters are defined in CFT as follows: let us consider \(\mathcal{V}\) an irreducible representation of a chiral algebra\(^7\), then

\[
\chi_{\mathcal{V}}(\tau) = \text{Tr}_{\mathcal{V}}(q^{L_0 - c/24}),
\]

(189)

where \(q = \exp(2\pi i \tau)\). As shown by Cardy [16], these characters are building blocks for the partition function of CFTs on the torus.

\(^7\)Which of course contains the Virasoro algebra.
Here, the extended algebra is formed by the integer spin holomorphic fields of the Luttinger CFT. It is therefore quite obvious to obtain extended characters. Fields $\phi_{kp,2kq}$ and their $\hat{U}(1)$ (right) descendants generate the chiral algebra $A_c$. In terms of $J_0$'s eigenvalue $p$, they correspond to $p \in 2\sqrt{pq}\mathbb{Z}$. An irreducible representation of $A_c$ is a direct sum of $\hat{U}(1)$ irreducible representations, and may be characterized by a set of $J_0$ eigenvalues of the form $p_0 + 2\sqrt{pq}\mathbb{Z}$. Let us now determine what are the possible values of $p_0$. For this, we impose that the Laurent modes of all chiral vertex operators generating $A_c$ and of the chiral vertex operator corresponding to $p_0$ satisfy commutation relations. The operator product expansion of chiral vertex operators

$$V_{p_1}(z)V_{p_2}(\xi) = (z - \xi)^{p_1p_2}V_{p_1+p_2}(\xi)$$

provides us with such a condition. The Laurent modes of $V_{p_1}$ and $V_{p_2}$ satisfy commutation relations if and only if $p_1p_2 \in \mathbb{Z}$. Therefore, we have $p_0 \in (2\sqrt{pq})^{-1}\mathbb{Z}$. The extended characters are therefore given by

$$\chi_\lambda(\tau) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(\lambda + 4npq)^2/8pq},$$

where $\lambda \in \mathbb{Z}/4pq\mathbb{Z}$. In terms of theta functions with characteristic, we have:

$$\chi_\lambda(\tau) = \frac{\theta \left[ \begin{array}{c} \lambda/4pq \\ 0 \end{array} \right](0,4pq\tau)}{\eta(\tau)}.$$

When $p \equiv q \pmod{2}$, it is convenient to introduce extended characters $\phi_\lambda^{(\pm)}$ defined by:

$$\phi_\lambda^{(\pm)} = \chi_\lambda \pm \chi_{\lambda+2pq},$$

where $\lambda \in \mathbb{Z}/2pq\mathbb{Z}$. Their modular transformation properties can easily be computed. A subset of them, namely the $\phi_\lambda^{(+)}$ where $\lambda \in \mathbb{Z}/pq\mathbb{Z}$, forms a unitary representation of the modular group $\Gamma(S,T^2)$ [56]. The corresponding $S$ matrix [44] is given by:

$$\phi_\lambda^{(+)}(-1/\tau) = \frac{1}{\sqrt{pq}} \sum_{\lambda' \in \mathbb{Z}/pq\mathbb{Z}} e^{-2\pi i \lambda\lambda'/pq} \phi_\lambda^{(+)}(\tau).$$

We also have:

$$\phi_\lambda^{(+)}(\tau + 2) = e^{2\pi i (\lambda^2 \tau + \lambda)} \phi_\lambda^{(+)}(\tau).$$

In conclusion, we notice that the underlying CFT is nothing but a $\mathbb{Z}/N\mathbb{Z}$ CFT in the sense of [20] where $N = 4pq$. However, since the boundary conditions to be used in the Luttinger model are not modular invariant with respect to the full modular group $SL(2,\mathbb{Z})$, the partition function of a Luttinger CFT does not belong to the list obtained in [20]. Instead of giving a complete classification of possible partition functions (see [61] for classification results), we shall express the partition function of the rational Luttinger model in terms of these extended characters.

### 8.3 Partition functions of rational Luttinger CFTs

We now express the partition function of the rational Luttinger models in terms of the extended characters. These formulae will give us more insight into the structure of these theories. Of course, two cases will be analyzed separately according to $p$ and $q$'s relative parity.

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8 Analogous to the super-characters of super-conformal field theories.
8.3.1 Case $p$ and $q$ odd

We can decompose the Luttinger partition function into two terms:

$$Z_1 = \frac{1}{|\eta(\tau)|^2} \sum_{(r,s) \in \mathbb{Z}^2} q^{\frac{(r+p)^2}{2}} q^{\frac{(r-p)^2}{2}},$$

and

$$Z_2 = \frac{1}{|\eta(\tau)|^2} \sum_{(r,s) \in \mathbb{Z}^2} q^{\frac{(r+p+q)^2}{2}} q^{\frac{(r-p)^2}{2}}.$$ 

Since $p$ and $q$ are coprime, there exists $(u,v) \in \mathbb{Z}^2$ such that $qu - pv = 1$. These integers are not unique and indeed $qu + pv$ is only defined modulo $2pq$. Let us denote its class modulo $2pq$ by $\omega \in \mathbb{Z}/2pq\mathbb{Z}$. Let us now introduce the following subsets of $\mathbb{Z}^2$:

$$C_\lambda = \{(\lambda + 2pq\mathbb{Z}) \times (\overline{\lambda} + 2pq\mathbb{Z})\} \quad (196)$$

$$R_1 = \{(rq + sp, rq - sp) : (r,s) \in \mathbb{Z}^2\} \quad (197)$$

$$R_2 = \{(rq + sp, rq - sp + pq) : (r,s) \in \mathbb{Z}^2\} \quad (198)$$

where $(\lambda, \overline{\lambda}) \in (\mathbb{Z}/2pq\mathbb{Z})^2$. It is obvious to prove that $(C_\lambda, \omega)_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}}$ is a partition of $R_1$ and that $(C_\lambda, \omega + pq)_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}}$ is a partition of $R_2$. Therefore, we get:

$$Z_1 = \sum_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}} \chi_{2\lambda}(\tau) \overline{\chi_{2\omega\lambda}(\tau)} \quad (199)$$

$$Z_2 = \sum_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}} \chi_{2\lambda + 2pq}(\tau) \overline{\chi_{2\omega\lambda}(\tau)} \quad (200)$$

Summing these two contributions gives

$$Z_{\text{Lutt}}(\tau, \overline{\tau}) = \sum_{\lambda \in \mathbb{Z}/pq\mathbb{Z}} \varphi^{(+)}_{2\lambda}(\tau) \overline{\varphi^{(+)}_{2\omega\lambda}(\tau)}, \quad (201)$$

which is exactly the partition function found by Cappelli and his collaborators in [13].

8.3.2 Case $p$ or $q$ even

The same kind of computation can be carried out in this case. Again, let us decompose the partition function into two terms:

$$Z_1 = \frac{1}{|\eta(\tau)|^2} \sum_{(r,s) \in \mathbb{Z}^2} q^{\frac{(r+p)^2}{2}} q^{\frac{(r-p)^2}{2}},$$

and

$$Z_2 = \frac{1}{|\eta(\tau)|^2} \sum_{(r,s) \in \mathbb{Z}^2} q^{\frac{(2r+q+p+q)^2}{2}} q^{\frac{(2r-q+p)^2}{2}}.$$ 

Exactly as in the previous case, let us apply Bezout theorem to find $(u,v) \in \mathbb{Z}^2$ such that $qu - pv = 1$. These two numbers are not unique: $(u + kp, v + kq)$ where $k \in \mathbb{Z}$ satisfies the same property. If we define $\omega = pu + qv$, we have $\omega (p + q) = q - p + 2pq(u + v)$. Since $p$ and $q$ do not have the same parity, it is possible to choose $(u,v)$ such that $u + v$ is even. With such a choice, $\omega (p + q) \equiv q - p \pmod{4pq}$. But now, the freedom on $u$ and $v$ is restricted and therefore $\omega \in \mathbb{Z}/4pq\mathbb{Z}$. We also have $\omega^2 = 1$ in $\mathbb{Z}/4pq\mathbb{Z}$. Finally we get:

$$Z_1 = \sum_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}} \chi_{2\lambda}(\tau) \overline{\chi_{2\omega\lambda}(\tau)} \quad (202)$$

$$Z_2 = \sum_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}} \chi_{2\lambda+1}(\tau) \overline{\chi_{\omega(2\lambda+1)}(\tau)}, \quad (203)$$

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and therefore
\[
Z_{\text{Lutt}}(\tau, \bar{\tau}) = \sum_{\lambda \in \mathbb{Z}/4pq\mathbb{Z}} \chi_{\lambda}(\tau) \overline{\chi_{\omega\lambda}(\bar{\tau})}.
\] (204)

We have thus obtained the field contents of all rational Luttinger CFTs.

9 Introduction of an edge charge on the boundary

The purpose of this section is to present a CFT description of edge excitations above a given bulk excitation in a quantum Hall fluid. We begin by explaining the motivations of our computations. They are based on the introduction of a fractional edge charge in the Luttinger CFT. Then, we shall compute the partition functions of these “twisted” Luttinger CFTs with edge charges and express them in terms of generalized characters in the rational cases. Then, it will be appropriate to provide a physical interpretation of the chiral algebra of the theory. Following Fisher and Stone [87], fermions localized on one edge of the sample will enter the game. Duality properties of the Luttinger CFT will also be discussed. Finally, examples such as the Fermi liquid and Laughlin Hall fluids will be considered.

9.1 Why introducing an edge charge?

It has been known since Laughlin that a quantum Hall fluid at filling fraction $1/(2p + 1)$ is incompressible [62]. This means that all bulk excitations have a gap in energy: it is therefore difficult to create them contrarily to the case of the Fermi liquid for example, which has a finite compressibility [1] since low lying quasi-particle quasi-hole excitations can be easily excited. As explained by Laughlin, bulk excitations can be described and correspond to objects with fractional charge and anyonic statistics [94].

However, some low energy excitations do exist in a quantum Hall fluid: these are the edge excitations. They are present above any bulk excitation. We have argued in the previous sections that edge excitations above the bulk ground state were described by the Luttinger CFT at $\alpha = \nu^{-1}$. Here, we would like to give a description of edge excitations above the excited bulk states.

Starting from a quantum Hall fluid on a cylinder and creating a bulk excitation implies creating a fractional charge excess or defect in the bulk [62]. But by charge conservation, the opposite charge should appear on the edge. Therefore a fractional charge appears on the edge. But then, creating edge excitations should not change the charge on the edge (except by integers). This leads us to considering some twisted sectors of the Luttinger CFT. These sectors are defined by a congruence condition on the total charge:
\[
\frac{q_R + q_L}{e} \equiv r \pmod{1}.
\] (205)

The aim of this section is to study these twisted sectors.

9.2 Partition functions with an edge charge

We now compute the partition function of the Luttinger model in presence of an edge charge satisfying the condition (205). We will first of all compute it for generic $\alpha$. Then we specialize $\alpha$ to some rational value and express the partition function in terms of generalized characters.

9.2.1 Computation of the partition function

Because of the identification of the physical charge in the Luttinger model, we modify our bosonic field boundary conditions to impose a given charge modulo 1 on the boundary. More precisely, we impose
\[
\int_{(\alpha)} d\varphi \equiv \pi R r \pmod{\pi R}.
\]
In the previously considered cases, $r = 0$. The modified functional integral will then be

$$
\int_{\Omega(r)} D[\varphi] W[\varphi] = \frac{1}{2} \sum_{(\epsilon, \epsilon') \in \{0, 1/2\}^2} (-1)^{4\epsilon\epsilon'} \int_{\Delta_{\varphi=0}} W[\varphi] = 1
$$

All computations can be carried out exactly as in the usual case and we obtain, in the presence of a magnetic flux $\chi \Phi_0$ through the ring:

$$
Z_{Lutt}^{(r)}(\tau, \bar{\tau}, \chi) = \frac{1}{|\eta(\tau)|^2} \sum_{(n, m) \in (\mathbb{Z}/2 \mathbb{Z}) \times \mathbb{Z}} q^{\frac{1}{2} p^2 n^2 + m^2} \chi a \omega n + \frac{1}{2} pq \chi a \omega m + 1.
$$

The $(n, m)$ term in this sum corresponds to a total charge of $(2n + r)e$ as expected.

### 9.2.2 Expression in terms of generalized characters

Exactly as before, it is quite interesting to express these partition functions for $\alpha$ rational and in a zero magnetic field in terms of generalized characters. Of course, although the vacuum structure depends on the sector, fields of the chiral algebra can still be used for going from one representation of $\hat{U}(1)$ to the other.

We choose $\alpha = q/p$. Again, two cases must be treated separately according to $p + q$’s parity.

**p and q odd** Conformal dimensions of $\hat{U}(1)$ primaries are given by

$$
((r + l)q \pm (m + 2\chi))^2 = \frac{8pq}{8pq},
$$

where $(l, m) \in \mathbb{Z}^2$ have the same parity. The interesting case arises when $r = a/q$ with $a \in \mathbb{Z}$. Then for $\chi = 0$, the spectrum of conformal dimensions becomes

$$
\frac{(a + lq \pm m)^2}{8pq}.
$$

When $a$ is even, we note that $lq \pm mp + q \equiv 0 \pmod{2}$ and therefore, exactly as in section 8.3, we expect the partition function to be a sesquilinear combination of the generalized characters $\varphi^{(+)}_{2\lambda}$. We can easily prove that in general:

$$
Z_{Lutt}^{(a)}(\tau, \bar{\tau}) = \sum_{\lambda \in \mathbb{Z}/2pq\mathbb{Z}} (\chi a + 2\lambda(\tau) + \chi a + 2\lambda + 2pq(\tau)) \chi a + 2\omega \lambda(\tau).
$$

This partition function can be rewritten in terms of generalized characters

$$
Z_{Lutt}^{(a)}(\tau, \bar{\tau}) = \sum_{\lambda \in \mathbb{Z}/pq\mathbb{Z}} \varphi_{a+2\lambda}^{(+)}(\tau) \varphi_{a+2\omega \lambda}^{(+)}(\tau).
$$

**p or q even** The same kind of analysis yields the following result:

$$
Z_{Lutt}^{(a)}(\tau, \bar{\tau}) = \sum_{\lambda \in \mathbb{Z}/4pq\mathbb{Z}} \chi a + \lambda(\tau) \chi a + \omega \lambda(\tau),
$$

which reduces to (204) when $a = 0$. We have therefore obtained the operator content of the twisted sectors of Luttinger rational models in terms of the characters relative to the maximal chiral algebra. The study of this partition function tells us how gapless are excitations are organized above bulk excitations.
9.3 Fusion rules in the rational Luttinger models

Let us now discuss the fusion rules [72] of these models. The partition functions are of the following form:

\[ \sum_i \chi_i(\tau) \overline{\chi_{\sigma(i)}(\tau)}, \]

where \( i \) runs over a finite set of indices and \( \sigma \) is a bijection of this set. Of course, this decomposition is relative to the maximal symmetry algebra of the model \( \mathcal{A} \otimes \mathcal{A} \) [71]. In this case, the generalized primary field can be indexed by a chiral index \( i \). We would like to find the selection rules for the operator product algebra [71] of the Luttinger CFT.

Invariance under \( \tau \mapsto -1/\tau \) shows, thanks to Verlinde formula [91], that \( i \mapsto \sigma(i) \) defines an automorphism of the fusion algebra [24]. Therefore, the selection rules of the operator algebra of our Luttinger CFT are given by the fusion rules for the chiral fields, exactly as in usual CFTs.

**p and q odd** Here we may consider the theory with respect to \( \mathcal{A}_+ \) or to the extended algebra \( \mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \). For the sake of simplicity, we consider here the maximal algebra \( \mathcal{A} \).

Let us introduce \( \Phi_\lambda(a) \) corresponding to the extended character \( \phi_{\alpha+2\lambda}^{(+)} \). It is then clear that the following fusion rules hold:

\[ \Phi_\lambda(a) \Phi_{\lambda'}(a') = \Phi_{\lambda+\lambda'}(a+a'). \tag{211} \]

There are \( \mathbb{Z}/pq\mathbb{Z} \) fusion rules in the \( a = 0 \) sector as noticed by Cappelli and his collaborators [15]. However let us point out that the chiral algebra \( \mathcal{A} \) contains fields with half integer spins. Taking a chiral algebra containing only integer spin fields gives us \( \mathbb{Z}/4pq\mathbb{Z} \) fusion rules.

**p or q even** Let us denote by \( \Phi_\lambda(a) \) the primary field of indices \( (a + \lambda, a + \omega \lambda) \). It lives in the sector of charge \( a/q \). Using the fusion rules for the \( \tilde{U}(1) \) chiral algebra, we obtain:

\[ \Phi_\lambda(a) \Phi_{\lambda'}(a') = \Phi_{\lambda+\lambda'}(a+a'), \tag{212} \]

which expresses both the conservation of the boundary charge \( a/q \) and the fact that we have a \( \mathbb{Z}/4pq\mathbb{Z} \) CFT. Starting from \( \Phi_\lambda(a) \), other fields \( \Phi_{\lambda'}(a) \) are obtained by fusion with \( \Phi_{\mu}(0) \).

**Physical interpretation** The field \( \Phi_\lambda(a) \) not only creates a boundary excitation but also changes the sector of our Luttinger CFT. In some sense these fields create bulk excitations. All descendants of the state created by \( \Phi_\lambda(a) \) satisfy \( q_L + q_R \equiv a/q \) (mod 1). We can shift \( \lambda \) at a fixed \( a \) with an operator of the form \( \Phi_{\mu}(0) \) that does not change the sector.

Finally, let us stress that even in the twisted sectors, the Hilbert space of states is still a representation of the same extended chiral algebra \( \mathcal{A} \) as in the bulk vacuum sector. In this sense, this chiral algebra may be viewed as the symmetry algebra for single branch FQH fluid edge excitations. This automatically raises the question of the physical meaning of the fields that generate the extended symmetry algebra. It turns out that this is related to the question of fermions localized on one edge.

In the cylindrical geometry, it should be physically possible to introduce electrons on one of the edges of the system (through a tunnel junction connected to one of the edges). Fields corresponding to the creation or destruction of these edge fermions should therefore be present in all twisted Luttinger CFTs corresponding to the \( \nu = 1/(2p + 1) \) sequence. Moreover, they should carry an electric charge localized only on one edge. Because of the relation between the conformal dimensions and the edge charge of \( \tilde{U}(1) \) primary fields, fields that carry a zero charge on the left (respectively right) edge correspond to holomorphic (respectively anti-holomorphic) primary fields, and therefore to the chiral algebra of the Luttinger CFT.

Let us recall that, for \( a = q \in 2n + 3 \) (corresponding to Laughlin fluids), the chiral algebra is generated by the two fields \( \phi_{1/2,q} \) and \( \phi_{1/2,-q} \). The following table gives the \( (n, m) \) parameters, the left and right charges and the conformal dimensions of these fields and their hermitian conjugates:\[7]

\[ \text{The } \epsilon \text{ subscript recalls that we are dealing with edge fermions, not to be confused with the Luttinger fermions.} \]
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Field} & n & m & q_R & q_L & h & h' \\
\hline
\psi_{R,c}^1 & 1/2 & q & 1 & 0 & q/2 & 0 \\
\psi_{R,c}^{-1/2} & q & 0 & 1 & 0 & q/2 & 0 \\
\psi_{L,c}^1 & -1/2 & q & 0 & 1 & 0 & q/2 \\
\psi_{L,c}^{-1/2} & q & 0 & -1 & 0 & q/2 \\
\hline
\end{array}
\]

As expected, they carry a unit charge localized on one of the two edges, which is a strong support for their identification with edge fermions of the FQH fluid. The first column of the table details the action of these fields. They were identified with edge fermions of the FQH fluid by Stone and Fisher. Their argumentation was based on the problem of establishing a relation between correlation functions for the Luttinger CFT and Laughlin wave function.

Finally, we recall that the \( U(1) \) currents, which belong to the chiral algebra of the Luttinger CFT for all values of \( \alpha \), generate neutral excitations which are incompressible deformations of the FQH fluid. Therefore, the structure of edge excitations appears to be remarkably simple: they are all obtained from a finite set of elementary excitations corresponding to primary fields \( \Phi_\lambda \) with respect to the extended algebra \( A \) by adding or subtracting fermions localized on the edges, and deforming the edges boundary through the use of \( U(1) \) currents. Luttinger fermions generate all the excitations in the untwisted (\( a = 0 \)) Luttinger CFT. Only for \( \alpha = 1 \) do they coincide with edge fermions. The same table as above reads for Luttinger fermions:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Field} & n & m & q_R & q_L & h & h' \\
\hline
\psi_{R,Lutt}^1 & 1/2 & 1 & (1 + q^{-1})/2 & (1 - q^{-1})/2 & (q^{1/2} + q^{-1/2})^2/8 & (q^{1/2} - q^{-1/2})^2/8 \\
\psi_{R,Lutt}^{-1/2} & -1 & -1 & -(1 + q^{-1})/2 & -(1 - q^{-1})/2 & (q^{1/2} + q^{-1/2})^2/8 & (q^{1/2} - q^{-1/2})^2/8 \\
\psi_{L,Lutt}^1 & 1/2 & -1 & (1 + q^{-1})/2 & (1 - q^{-1})/2 & (q^{1/2} - q^{-1/2})^2/8 & (q^{1/2} + q^{-1/2})^2/8 \\
\psi_{L,Lutt}^{-1/2} & -1 & 1 & -(1 - q^{-1})/2 & -(1 + q^{-1})/2 & (q^{1/2} + q^{-1/2})^2/8 & (q^{1/2} - q^{-1/2})^2/8 \\
\hline
\end{array}
\]

9.4 Duality for the Luttinger CFT

The free bosonic modular invariant theory possesses an electro-magnetic duality which plays an important role in the discussion of \( c = 1 \) CFTs. It is therefore interesting to determine if such a duality is present in the Luttinger CFT. Not surprisingly, the answer is yes but there are a few differences with the usual modular invariant theory.

In order to understand duality properties of the usual free boson and Luttinger CFTs coupled to a gauge field, we derive them using functional methods. This involves performing a Hubbard-Stratanovich transformation which introduces a 1-form \( b = b_\mu dx^\mu \). Integration over the original bosonic field constrains the harmonic, exact and co-exact parts of \( b \) which are used to build the dual field. For the sake of clarity, all necessary technical details are given in appendix E.

The main point is that duality acts differently on coupling constants of the usual bosonic theory and of the Luttinger theory. In the usual, modular invariant bosonic theory, \( \alpha \) is sent on \( 1/4\alpha \) whereas in the Luttinger CFT \( \alpha \) is sent on \( 1/\alpha \). Dirac theory now appears as the fixed points for duality transformations acting on Luttinger CFTs instead of \( \alpha = 1/2 \) in the modular invariant sector. Of course, in both cases, duality exchanges the roles of electric and magnetic potentials. For the Luttinger CFT, we have:

\[
Z_{\text{Lutt}}^r[\alpha, A] = Z_{\text{Lutt}}^{(0)}[\alpha^{-1}, \frac{iA^*}{\alpha} + b_r] \times \exp \left( \frac{1}{2\pi \alpha} \int A^2 \right),
\]

(213)

where \( b_r \) corresponds to a magnetic flux of \( r \Phi_0/2 \). This shows that a given edge charge in the original theory becomes a magnetic flux in the dual theory.

In an operator approach to the Luttinger CFT, the original field derivative with respect to the spatial coordinate appears to be the charge density. Therefore, the bosonic field may be interpreted as a spatial deformation. A strong coupling regime is a regime of low fluctuations of the charge density. On the other

\[^{10}\text{Which is known to be the SU(2) Wess-Zumino-Witten model at level one.}\]
Finally, we may ask for the structure of the moduli space of Luttinger CFTs. For modular invariant $c = 1$ CFTs, this moduli space has been described in the late eighties (see [33] for a review). Here, the moduli space is simpler.

In a zero magnetic field, we have a line of effective theories indexed by $\alpha > 0$. As usual, we may look for marginal integrable operators. Such an operator gives rise to the line of orbifolds in the modular invariant case [22]. But it is not present in the Luttinger case: looking for operators of conformal dimensions $(1,1)$ gives $nm = 0$ and $n^2\alpha + \frac{m^2}{\alpha} = 2$. Besides that, the integrability condition, which only relies on fusion rules of the bosonic model, forces us to limit to the $n = 0$, $m = 1$ operator which is not present in the spectrum of a Luttinger CFT. Taking into account the magnetic field would also eliminate this possibility since there is no operator that has conformal dimensions $(1,1)$ for all values of $\chi$.

To conclude, the only deformation parameters of Luttinger CFT are the interaction strength $\alpha$, the magnetic flux $\chi$, and the external potential. Of course, there exists relevant perturbations which drive us away from CFT. As an example, $\psi_R^\dagger \psi_L^\dagger + \psi_L^\dagger \psi_R$ corresponds to $\cos(2R^{-1}\varphi)$ and becomes relevant for $\alpha > 1/2$ in zero magnetic field. This operator corresponds to a mass perturbation of the Thirring model [89]. One could also get interested in other operators, corresponding to the Umklapp processes, such as $(\psi_L^\dagger \psi_R)^2$ which correspond to a momentum transfer of $-4k_F$. This operator appears in a fermion lattice model as explained in appendix A. It is responsible for the phase transition of the XXZ antiferromagnetic spin 1/2 chain from the massless phase to the Ising massive phase as the anisotropy increases. It becomes relevant for $\alpha > 2$.

Analogous criteria are used in the study of the resonant tunneling of a LL through an impurity. In this problem, the total density (including 2$p_k F$ Fourier components) is coupled to a local potential $U(\sigma)$ localized near a particular value of $\sigma$: $H_{\text{imp}} = \int U(\sigma) \rho_{\text{total}}(\sigma) d\sigma$ (see [50] and section IV.E of [27]).

### 9.5 Examples

#### 9.5.1 The Fermi liquid

This basic example corresponds to $\alpha = 1$, that is to say $p = q = 1$. The underlying CFT is a $\mathbb{Z}/4\mathbb{Z}$ theory which has four basic primary fields. The $\lambda = 2$ field has dimension $1/2$ and corresponds to the Dirac fermion. In this example, they coincide with Luttinger fermions and with edge fermions discussed in the previous section. The $\lambda = \pm 1$ fields have dimensions $1/8$ and are twists fields. The $\lambda = 0$ field is the identity field.

The partition function for the Dirac theory in the sector of charge $a$ is given by:

$$Z^{(a)}_{\text{Fermi}}(\tau, \bar{\tau}) = |\chi_a(\tau) + \chi_{a+2}(\tau)|^2 = |\varphi_a^{(+)}(\tau)|^2.$$  \hfill (214)

Here $\omega = 1$ and the partition functions are factorized as expected since these are squared modulus of fermionic determinants [33]. Explicit computations give the following formulae:

$$\varphi_0^{(+)} = \chi_0 + \chi_2 = \vartheta_3/\eta$$ \hfill (215)
$$\varphi_1^{(+)} = \chi_1 + \chi_3 = \vartheta_2/\eta$$ \hfill (216)
$$\varphi_0^{(-)} = \chi_0 - \chi_2 = \vartheta_4/\eta$$ \hfill (217)
$$\varphi_1^{(-)} = \chi_1 - \chi_3 = i \vartheta_1/\eta,$$  \hfill (218)

which gives the correspondence between the $\mathbb{Z}/4\mathbb{Z}$ formulation of this model and classical theta functions. The $a = 1$ partition function is nothing but $|\vartheta_2/\eta|^2$ which is nothing but $Z_{PA}(\tau)$. This result is physically expected since adding an extra electron to the free Dirac theory in the $(A,A)$ sector is equivalent to considering it in the $(P,A)$ sector. Clearly, the charged excitations considered here are nothing but free electrons.
9.5.2 Laughlin general Hall fluid

Here, we assume $\nu = 1/q$ and therefore $\alpha = q$ is an odd integer. We can consider that $p = 1$ and therefore easily get $\omega \equiv -1 \pmod{2q}$. The underlying CFT is a $\mathbb{Z}/4q\mathbb{Z}$ field theory. Its partition function can be expressed quite easily in terms of generalized characters:

$$ Z_{\text{Lutt}}^{(a/q)}(\tau, \bar{\tau}) = \sum_{\lambda \in \mathbb{Z}/q\mathbb{Z}} \varphi_{2\lambda+a}^{(+)}(\tau) \varphi_{-2\lambda+a}^{(+)}(\bar{\tau}) $$

All states corresponding to $\varphi_{2\lambda+a}^{(+)} \varphi_{-2\lambda+a}^{(+)}$ have left and right charges given, up to integers, by:

$$ \left( \frac{a}{2q} + \frac{\lambda}{q}, \frac{a}{2q} - \frac{\lambda}{q} \right) $$

We see that the charge $a/q$ of Laughlin excitation is equally shared by the two edges. The partition function is periodic in $a$ with a period of $2q$. This factor of two can be understood quite easily: only in this case, the contribution to the charge on each edge is given by an integer.

Let us illustrate these considerations in the $\nu = 1/3$ case: the underlying CFT is a $\mathbb{Z}/12\mathbb{Z}$ CFT. It has twelve primary fields with respect to $A_+$. In $a$ even sectors, the renormalized Luttinger fermion corresponds to $\chi_4^{1/2}$ and $\chi_{10}^{1/2}$ which has conformal dimensions $(h, \bar{h}) = (2/3, 1/6)$ (and therefore spin $1/2$). The spin $3/2$ conserved current corresponds to edge fermions. The two point function exchange properties are governed by $(z - \xi)^3$, exactly as for Laughlin wave function.

10 Conclusion

In this paper, we have described in an accurate and comprehensive way the effective conformal field theory of the spinless Luttinger liquid in a gapless regime. Our description is valid in a finite size and at finite temperature. The partition function and correlation functions of charge and current densities have been computed for any value of the coupling constant, within the framework of a non-chiral bosonization scheme. The conserved currents generate the symmetry algebra of all Luttinger CFTs. Their correlation functions give access to the response function of the system and to permanent currents induced by an external magnetic flux. Primary fields with respect to this $\widehat{U}(1)_R \times \widehat{U}(1)_L$ symmetry are vertex operators whose correlation functions are exactly computed for the Luttinger CFT.

Using this description of Luttinger CFTs, we have shown how the physics of edge excitations of a fractional quantum Hall fluid on an annulus can be recovered. The filling fraction is related in a simple way to the Luttinger interaction parameter which plays the role of a moduli for the Luttinger CFT. The solution of the Luttinger CFT has been extended to twisted sectors in order to describe edge excitations above a given bulk excitation. With the $c = 1$ Luttinger CFT, only Laughlin’s fluids ($\nu^{-1} \in 2\mathbb{N} + 1$) can be recovered but an extension of our analysis to several boson fields will give access to other filling fractions. It turns out that the Luttinger CFTs that describe edge excitations of a FQH fluid are rational: they possess extra conserved currents which generate an extended symmetry algebra so huge that the Hilbert space of states is decomposed in a finite direct sum of irreducible representations. For Laughlin’s FQH fluids, this algebra is generated by creators and annihilators of edge fermions. This provides a nice and simple physical picture of edge excitations in Laughlin’s fluids: they are all obtained from a finite number of them through the addition or substraction of fermions located on the left or right edge, and through neutral deformations induced by $U(1)$ currents.

The formalism developed in the present article can now be used for studying other aspects of 1D strongly interacting systems such as impurity problems (for which a large amount of work has already been done), coupling of a Luttinger circle or of a Hall cylinder to various external systems such as phonons, quantized electromagnetic fields, tunnel junctions, other Luttinger systems etc. Performing computations in finite size and at finite temperature gives a direct access to the interplay between various effects such as discreteness.

\[11\text{Work in progress}\]
of charges, interference effects and how the physics changes with temperature. Other strongly interacting systems may be described by the same techniques, although the structure of the underlying Hilbert space may be different. The most natural example of such systems is the XXZ spin 1/2 chain in its massless regime. Introduction of backscattering disorder should lead to a different physics such as localization. However, this is a difficult problem for which conformal methods may not be trivially applied. Maybe the study of the quantum group symmetries of the massive phase or the use of the quantum inverse scattering method could help to understand the transition from a gapless regime to an insulating one and its interplay with finite size effects.

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A Infrared limit Hamiltonians

The aim of this appendix is to recall the derivation of the infrared limit of the Hamiltonian of a one dimensional fermionic gas. We begin with the free Hamiltonian and then treat the interacting case.

A.1 Free Hamiltonian

We have explained at the beginning of section 2 how the dispersion relation can be linearized around the two Fermi points. Let us derive the expression (2) of Dirac’s Hamiltonian $H^{(0)}$ in terms of right and left moving fields. This is a well-known procedure which we recall here for the sake of pedagogy.

A.1.1 Lattice fermions

Let us start with the second quantized lattice fermions Hamiltonian

$$H^{(0)} = \frac{t}{2} \sum_{n=1}^{N} (c_n^+ c_{n+1} + c_{n+1}^+ c_n)$$

with cyclic boundary conditions. This is also the Hamiltonian of Jordan-Wigner fermions of the antiferromagnetic XX0 spin chain, with $t = J$, the exchange constant. The dispersion relation is $\epsilon(k) = t \cos(k)$. The continuum limit is obtained by factoring out the $k_F$ dependence of the fermion field as follows:

$$c_n = \sqrt{\frac{a}{2}} \left( e^{ik_F \sigma} \psi_R(\sigma) + e^{-ik_F \sigma} \psi_L(\sigma) \right),$$

where $\psi_R, L$ are slowly varying fields of the variable $\sigma = na$. Notice that $c_n$ being dimensionless, $\psi_R$ and $\psi_L$ have a scaling dimension 1/2. Moreover, once $\sigma$ becomes a continuous variable, the canonical anticommutation relations $\{\psi_{R,L}(\sigma), \psi_{R,L}(\sigma')\} = \delta(\sigma - \sigma')$ are compatible with the ones of the lattice fermions $\{c_n^+, c_m\} = \delta_{n,m}$ provided the following convention for the continuous and lattice delta functions:

$$\delta(\sigma - \sigma') = \delta_{n,n'}/a.$$

\[12\] Work in progress by P. Degiovanni and S. Dusuel.
Expanding (220) in terms of the right and left moving fields in (221) leads to
\[
2H^{(0)} = v_F \int_0^L d\sigma \left( \psi_R^+(\sigma) i \partial_\sigma \psi_R(\sigma) - \psi_L^+(\sigma) i \partial_\sigma \psi_L(\sigma) \right),
\]
where \( v_F = \partial \epsilon(k_F)/\partial k = a t \sin(k_F a) \). The right hand side of (222) is nothing but the Hamiltonian of Dirac’s theory. In deriving (222), we have (i) neglected the fast varying fields with prefactors \( \exp(\pm 2ik_F \sigma) \) since these processes have a negligible probability, \( 2k_F \) not being a vector of the reciprocal lattice. Notice that in the presence of a dimerization (i.e. \( t_{2i} = t(1 - \delta) \) and \( t_{2i+1} = t(1 + \delta) \)), these processes are no longer negligible since \( 2k_F \) is then a reciprocal lattice vector. In this dimerized case, the \( 2k_F \) processes are responsible for the opening of a gap. (ii) considered \( \sigma \) as a continuous variable (iii) expanded \( \psi_R^+(\sigma + a) = \psi_R^+(\sigma) + a \partial_\sigma \psi_R^+(\sigma) \), and neglected the higher order derivatives which do not contribute to the infrared limit (iv) dropped out a constant ground state energy.

### A.1.2 Non relativistic fermions

We notice that the same analysis can be carried out starting from the microscopic Hamiltonian
\[
H^{(0)} = \int_0^L \Psi^+(\sigma) \frac{\Delta^2}{2m} \Psi(\sigma) d\sigma.
\]

The fermion field \( \Psi(\sigma) \) is related to the chiral fields by
\[
\Psi(\sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i k_F \sigma} \psi_R(\sigma) + e^{-i k_F \sigma} \psi_L(\sigma) \\ e^{i k_F \sigma} \psi_R(\sigma) - e^{-i k_F \sigma} \psi_L(\sigma) \end{pmatrix}
\]
and the low energy effective Hamiltonian is nothing but (222).

### A.2 A lattice model with only \( g_2 \) and \( g_4 \) interactions

The aim of this section is to point out that, most of the time, the Hamiltonian (14) containing only \( g_2 \) and \( g_4 \) interactions is not the most general low energy Hamiltonian of one dimensional spinless fermions. Additional interactions may arise due to the possibility of Umklapp processes when an underlying lattice is present. These Umklapp processes correspond to scattering events where the transferred momentum is a multiple of the reciprocal lattice vector. These terms give in general birth to Gross-Neveu type interactions that may drastically change the physics. For instance, in the XXZ chain, these Umklapp processes are responsible for the flow to the massive Ising fixed point when \( J_x > J_y = J_z \). These terms can be neglected away from half-filling since, under this condition, scattering processes across the Fermi sea cannot transfer a momentum belonging to the reciprocal lattice. However, under some very special circumstances, these terms may vanish even at half-filling, and the long range physics is then exactly the one described in the present article.

To be more concrete, we consider a spinless lattice one-dimensional fermion system with a hopping term (220) plus an interaction term involving nearest-neighbor and next-nearest-neighbor interactions
\[
H^{(1)}(U, V) = U \sum_{n=1}^N c_n^+ c_{n+1}^+ c_{n+1} + V \sum_{n=1}^N c_n^+ c_{n+2}^+ c_{n+2}.
\]

Using the decomposition (221) into fast and slowly varying modes, we get
\[
c_n^+ c_n = \frac{a}{2} \begin{pmatrix} \psi_R^+(\sigma) \psi_R(\sigma) + \psi_L^+(\sigma) \psi_L(\sigma) + e^{-2ikF \sigma} \psi_R^+(\sigma) \psi_L(\sigma) + e^{2ikF \sigma} \psi_L^+(\sigma) \psi_R(\sigma) \end{pmatrix}.
\]

Using this last equation, we derive the low energy interaction Hamiltonian:
\[
2H^{(1)}(U, V) = \frac{(U + V) v_F}{2t \sin(k_F a)} \int_0^L \left( J^2(\sigma) + \overline{J}^2(\sigma) \right) d\sigma + \frac{(U + V) v_F}{t \sin(k_F a)} \int_0^L J(\sigma) \overline{J}(\sigma) d\sigma + \left\{ \frac{v_F e^{-2ikF a} (U + V) e^{-2ikF a}}{2t \sin(k_F a)} \int e^{-4ikF \sigma} \psi_R^+(\sigma) \psi_L(\sigma)^2 + h.c. \right\},
\]

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where the currents $J$ and $\overline{J}$ are defined in section 2.1.1. The last term of this Hamiltonian correspond to the Umklapp processes with momentum transfer $4k_F$. At half filling ($k_F = \pi/2a$), these terms vanish when $U = V$. Under these conditions, the remaining interaction Hamiltonian only involves $g_2$ and $g_4$ interactions:

$$2H^{(1)}(U,U) = \frac{Uv_F}{L} \int_0^L \left( \overline{J}^2(\sigma) + \overline{J}^2(\sigma) \right) \, d\sigma + \frac{2Uv_F}{L} \int_0^L J(\sigma)\overline{J}(\sigma) \, d\sigma. \quad (227)$$

## B Explicit computations for the Dirac theory

This appendix details the computation of free fermionic partition functions, in the presence of a magnetic flux and a chemical potential. The identification of fermionic determinants with bosonic partition functions is discussed in great details. Emphasis is put on modular properties of all partition functions and on their behavior under various shifts of the magnetic flux and of chemical potential.

### B.1 Anti-periodic fermions coupled to magnetic field and chemical potential

**Operator computations with fermions** The computation of the partition function for Dirac’s theory in the presence of a magnetic flux and a chemical potential can be performed using a zeta function renormalisation prescription. The contribution of left and right modes to the vacuum energy are given by:

$$\beta_R = -\frac{\pi v_F}{L} \left( \frac{1}{12} - (\lambda + a)^2 \right) \quad \text{and} \quad \beta_L = -\frac{\pi v_F}{L} \left( \frac{1}{12} - (\lambda - a)^2 \right) \quad (228)$$

where $\lambda = \frac{L\mu}{2\pi v_F}$. The contribution of excitations is easily computed using a particle-hole picture and, introducing the fugacity $y = e^{\beta\mu}$, the final result is:

$$Z^{(0)}_{AA}(a,\mu) = (q\overline{q})^{a^2/2} \prod_{n=0}^{+\infty} (1 + y q^{n+a+1/2})(1 + y^{-1} q^{n-a+1/2}) \times \prod_{n=0}^{+\infty} (1 + y q^{n-a+1/2})(1 + y^{-1} q^{n+a+1/2}) \quad (229)$$

Let us remark immediately that $(q\overline{q})^{a^2/2}$ correspond to an extensive contribution to the free energy. All other contributions come from energy levels that scale as $L^{-1}$. Therefore, we shall forget about the extensive contribution which is understood as a zero energy. The $(q/\overline{q})^{\lambda a}$ contribution, which scales as $L^0$ is also discarded. Therefore, the final expression for the partition function is:

$$Z_{AA}(a,\mu) = (q\overline{q})^{a^2/2} \prod_{n=0}^{+\infty} (1 + y q^{n+a+1/2})(1 + y^{-1} q^{n-a+1/2}) \prod_{n=0}^{+\infty} (1 + y q^{n-a+1/2})(1 + y^{-1} q^{n+a+1/2}) \quad (230)$$

Using Jacobi’s triple product identity, we may express it as follows:

$$Z_{AA}(a,\mu) = \frac{1}{|q(\tau)|^2} \sum_{(m,\overline{m}) \in \mathbb{Z}^2} q^{\frac{1}{2}(m+a)^2} \overline{q}^{\frac{1}{2}(-a)^2} y^{m+\overline{m}} \quad (231)$$

It is quite instructive to compare these expressions with the ones obtained by Sachs and Wipf in [13]. Their computation is done using a fermionic functional integral, and therefore involves the study of the zeta function for the spectrum of the Dirac operator. They compare it to an operator computation performed in an appendix of their paper and their results agree with ours.

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13 They assume that $\tau$ is real and the extensive contribution to the energy is also discarded.
Fermion/boson comparison in the \((A, A)\) sector

We also need to compare \(Z_{AA}(a, \mu)\) and \(K_{AA}(a, \mu)\) defined as follows:

\[
K_{AA}(a, \mu) = \frac{1}{2} \left( Z_{[0,0]}(a, -\frac{i\mu\beta}{2\pi}) + Z_{[+,0]}(a, -\frac{i\mu\beta}{2\pi}) + Z_{[0,\pm]}(a, -\frac{i\mu\beta}{2\pi}) - Z_{[+\pm]}(a, -\frac{i\mu\beta}{2\pi}) \right)
\]

This expression is equal to (see section 3.1):

\[
K_{AA}(a, \mu) = \frac{1}{|\eta(\tau)|^2} \sum_{l, k \in \mathbb{Z}} y^l q^{\frac{(l+k+2a)^2}{4\beta^2}} q^{\frac{(l-k-2a)^2}{4\beta^2}}
\]

Therefore, using \(m = (l + k)/2\) and \(m' = (l - k)/2\), the right hand side of equation (231) is easily recovered since \(m\) and \(m'\) now belong to \(\mathbb{Z}^2\) because of the parity condition \(l \equiv k \text{ (mod } 2)\). Therefore, the final relation between the fermionic zeta-renormalized partition function and the bosonic one has the form \(\mathfrak{z}^\lambda\):

\[
Z^{(0)}_{AA}(a, \mu) = (q\bar{q})^{\frac{\mathfrak{z}^\lambda}{\beta}} \left( \frac{q}{\bar{q}} \right)^\lambda K_{AA}(a, \mu)
\]

This proves our bosonization formula for the \((A, A)\) sector, in the presence of a magnetic flux and a uniform chemical potential. Let us now turn to the \(AP\) sector.

The \(AP\) sector

This sector is obtained from the \(AA\) sector by changing \(y\) into \(-y\) in the particle-hole contribution to \(Z_{AA}\). The vacuum contribution is not modified. Therefore, we have:

\[
Z^{(0)}_{AP}(a, \mu) = (q\bar{q})^{\frac{\mathfrak{z}^\lambda}{\beta}} \left( \frac{q}{\bar{q}} \right)^\lambda K_{AA} \left( a, \mu + \frac{1}{\beta} \right)
\]

B.2 Shifting and modular invariance properties

It is also useful to shift \(a\) and the chemical potential (or equivalently \(b\)) for the fermionic determinant. We shall first of all details the effects of such shifts on bosonic partition functions, and then on fermionic partition functions.

Effects of shifts on bosonic partition functions

First of all, let us perform the shifts on the bosonic expressions. We introduce three new bosonic expressions:

\[
K_{AP}(a, \mu) = K_{AA} \left( a, \mu + \frac{i\pi}{\beta} \right)
\]

\[
= \frac{1}{|\eta(\tau)|^2} \sum_{(m, m') \in \mathbb{Z}^2} (-y)^{m+\bar{m}+1} q^{\frac{1}{4}(m+a)^2} q^{\frac{1}{4}(m-a)^2}
\]

\[
K_{PA}(a, \mu) = K_{AA} \left( a + \frac{1}{2}, \mu \right)
\]

\[
= \frac{1}{|\eta(\tau)|^2} \sum_{(m, m') \in \mathbb{Z}^2} y^{m+\bar{m}+1} q^{\frac{1}{4}(m+a+1/2)^2} q^{\frac{1}{4}(m-a+1/2)^2}
\]

\[
K_{PP}(a, \mu) = K_{PP} \left( a + \frac{1}{2}, \mu + \frac{i\pi}{\beta} \right)
\]

\[
= \frac{1}{|\eta(\tau)|^2} \sum_{(m, m') \in \mathbb{Z}^2} (-y)^{m+\bar{m}+1} q^{\frac{1}{4}(m+a+1/2)^2} q^{\frac{1}{4}(m-a+1/2)^2}
\]
Of course, these partition functions can also be expressed as:

\[ K_{AP}(a, \mu) = \frac{1}{2} \left( Z_{[0,0]} + Z_{[0,1/2]} - Z_{[1/2,0]} + Z_{[1/2,1/2]} \right) \]  
\[ K_{PA}(a, \mu) = \frac{1}{2} \left( Z_{[0,0]} - Z_{[0,1/2]} + Z_{[1/2,0]} + Z_{[1/2,1/2]} \right) \]  
\[ K_{PP}(a, \mu) = \frac{1}{2} \left( Z_{[0,0]} - Z_{[0,1/2]} - Z_{[1/2,0]} - Z_{[1/2,1/2]} \right) \]

as easily follows from the obvious properties of \( Z_{[e,e']}(a, b) \):

\[ Z_{[e,e']} \left( a + \frac{1}{2}, b \right) = e^{2\pi i e'} Z_{[e,e']}(a, b) \]  
\[ Z_{[e,e']} \left( a, b + \frac{1}{2} \right) = e^{2\pi i e} Z_{[e,e']}(a, b) \]

**Compatibility properties** It is interesting to study the interplay between shifts on \( a \) and \( b \), modular transformations properties of partition functions and bosonization formulae.

To be more precise, let us introduce a vector notation, which gathers simultaneously the four fermionic sectors and the four bosonic sectors. With this notation, the non-chiral bosonization has the following expression (valid for \( (a, b) \) **real**):

\[
    \begin{pmatrix}
        A & F \\
        P & A
    \end{pmatrix}
    = \frac{1}{2} \left( \begin{array}{cccc}
        1 & 1 & 1 & -1 \\
        1 & -1 & -1 & 1 \\
        1 & -1 & -1 & 1 \\
        1 & -1 & -1 & -1
    \end{array} \right) \Lambda
    \begin{pmatrix}
        Z_{[0,0]} \\
        Z_{[0,1/2]} \\
        Z_{[1/2,0]} \\
        Z_{[1/2,1/2]}
    \end{pmatrix}
\]

In this framework, shifts by \( 1/2 \) on \( a \) and \( b \) are represented by matrices, which we denote by \( B_{(a,b)} \) for bosons\(^{15}\) and \( F_{[a,b]} \) for fermions. Then, if \( \Lambda \) denotes the matrix relating bosonic partition functions to the fermionic ones, we have:

\[ F_{(a)} \times \Lambda = \Lambda \times B_{(a)} \quad \text{and} \quad F_{(b)} \times \Lambda = \Lambda \times B_{(b)} \]  

It is an easy exercise to check that these relations are compatible with the modular properties of bosonic partition functions \( Z_{[e,e']} \). The \( S \) and \( T \) modular transformations can be represented by \( 4 \times 4 \) matrices on bosonic and fermionic partition functions. More explicitly, these modular transformation matrices are:

\[ S^{(B)} = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad S^{(F)} = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}, \]  
\[ T^{(B)} = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad T^{(F)} = \begin{pmatrix}
    0 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}. \]

Of course, we have

\[ T^{(B)} \times \Lambda = \Lambda \times T^{(F)} \quad \text{and} \quad S^{(B)} \times \Lambda = \Lambda \times S^{(F)} \]  
\(^{15}\)Diagonal with eigenvalues given by equation (243) and (244).
which expresses the compatibility of our bosonization matrix $\Lambda$ natural action of $SL(2,\mathbb{Z})$ on partition functions. Then, we also check that the shifting matrices are compatible with modular transformation matrices:

$$\begin{align*}
S(B(a)) &= B(b)S(B(b)) \\
S(B(b)) &= B(a)S(B(a))
\end{align*}$$

and similar equations for $F_{(a,b)}$.

Imposing compatibility between modular transformations and the bosonization matrix only partially fixes it. To be precise, imposing (251) fixes $\Lambda$'s form as follows:

$$\Lambda = \begin{pmatrix} a & b & b & c \\ a & b & c & b \\ a & c & b & b \\ d & e & e & e \end{pmatrix} \quad (253)$$

The bosonization formula for the $PP$ sector thus appears to be independent from the one of the three other fermionic sectors. Only imposing equations (248) provides extra conditions on $\Lambda$ as was already noticed in [3]. With notations of (253), we obtain $b = -c$, $e = c$ and $d = a$. Therefore $\Lambda$ now depends only on two coefficients, as could easily be expected since $Z_{[0,0]}$ is modular invariant:

$$\Lambda = \begin{pmatrix} a & b & b & -b \\ a & b & -b & b \\ a & -b & b & b \\ a & -b & -b & -b \end{pmatrix} \quad (254)$$

Henceforth, the bosonization formula for the $AA$ fermionic sector fixes bosonization formulae for all other sectors.

However, this nice picture is disturbed when coupling fermions to a chemical potential or electric potential. The main point lies in the fact that such a potential corresponds to an imaginary twist of boundary conditions $b$. But, as noticed by Sachs and Wipf [78, Section 3.1], evaluating the zeta renormalized partition function is not direct. The zeta function of Dirac's operator has a well-defined analytic continuation through a Poisson resummation only for $a$ and $b$ real (relatively to the $AA$ sector). Therefore, introducing a chemical potential forces us to use the analytic continuation of the result for $b$ complex. But in this case, one should take care of contributions arising from the part sensitive to these analytic continuations, that is to say to vacuum energy contributions.

It is now time to turn to the $PA$ and $PP$ sector which need special care, as we shall see in the next section.

### B.3 The $PA$ and $PP$ sectors

**Shifting fermionic partition functions** Fermionic partition functions can also be directly computed. The computation in the anti-periodic fermionic sector have already been performed and in this case, there Zeta renormalisation of the vacuum energy gives:

$$E_R = -\frac{\pi v_F}{L} \left( (a+\lambda)^2 + a + \lambda + \frac{1}{b} \right) \quad \text{and} \quad E_L = -\frac{\pi v_F}{L} \left( (a-\lambda)^2 + \lambda - a + \frac{1}{b} \right) \quad (255)$$

Using these vacuum energies, and taking the trace over the fermionic Fock’s space gives $Z_{PA}$ as a product of left and right contributions:

$$Z^{(R)}_{PA}(a,\mu) = q^{\frac{1}{2}(\lambda-a)^2-a-\lambda} \prod_{n=0}^{+\infty} (1 + y^q n + a) (1 + y^{-1} q^{n+1-a}) \quad (256)$$

$$Z^{(L)}_{PA}(a,\mu) = q^{\frac{1}{2}(\lambda-a)^2+a-\lambda} \prod_{n=0}^{+\infty} (1 + y^q n - a) (1 + y^{-1} q^{n+1+a}) \quad (257)$$
Jacobi’s identity transforms the infinite products in power series:

\[ Z_{PA}^{(R)}(a, \mu) Z_{PA}^{(L)}(a, \mu) = (q\bar{q})^{\lambda^2/2} \left( \frac{q}{\bar{q}} \right)^{\lambda a} (q\bar{q})^{-\lambda/2} \sum_{(m, \bar{m}) \in \mathbb{Z}^2} y^{m+\bar{m}} q^{\frac{1}{2} \beta m} q^{-\frac{1}{2} \beta \bar{m}} (m-a-1/2)^2 (m-a-1/2)^2 \]  

(258)

Then, let us shift \( m \mapsto m + 1 \) and \( \bar{m} \mapsto \bar{m} + 1 \), and remember that \( (q\bar{q})^{-\lambda/2} = y^{-1} \). We finally obtain:

\[ Z_{PA}^{(0)}(a, \mu) = (q\bar{q})^{\lambda^2/2} (q\bar{q})^{\lambda a} \times \sum_{(m, \bar{m}) \in \mathbb{Z}^2} y^{m+\bar{m}+1} q^{\frac{1}{2} \beta (m+a+1/2)^2} \bar{q}^{\frac{1}{2} \beta (m-a+1/2)^2} \]  

(259)

Therefore, comparison with bosonic expressions is described by

\[ Z_{PA}(a, \mu) = (q\bar{q})^{\lambda^2/2} (q\bar{q})^{\lambda a} \times K_{PA}(a, \mu) \]  

(260)

It is now instructing to compute the fermionic partition function in the PP sector. In this case, the vacuum contribution remains the same than for the PA partition function. The only change is a shift from \( y \) to \(-y\) in the particle-hole contributions to the partition functions. Henceforth, we have:

\[ Z_{PP}^{(R)}(a, \mu) = q^{\frac{1}{2} \beta (\lambda+a^2-a-\lambda+1)} \prod_{n=0}^{+\infty} (1 - yq^{a+n})(1 - y^{-1}q^{n+1-a}) \]  

(261)

\[ Z_{PP}^{(L)}(a, \mu) = \bar{q}^{\frac{1}{2} \beta (\lambda-a^2+a-\lambda+1)} \prod_{n=0}^{+\infty} (1 - \bar{q}^{a+n})(1 - \bar{y}^{-1}q^{n+1+a}) \]  

(262)

Using Jacobi’s triple product identity again, we obtain:

\[ Z_{PP}^{(R)}(a, \mu) Z_{PP}^{(L)}(a, \mu) = (q\bar{q})^{\frac{1}{2} (\lambda^2-\lambda^2)} \left( \frac{q}{\bar{q}} \right)^{\lambda a} \sum_{(m, \bar{m}) \in \mathbb{Z}^2} (-y)^{m+\bar{m}+1} q^{\frac{1}{2} \beta (m+a-1/2)^2} \bar{q}^{\frac{1}{2} \beta (m-a+1/2)^2} \]  

(263)

And the same shifts as before give us:

\[ Z_{PP}^{(0)}(a, \mu) = (q\bar{q})^{\lambda^2/2} \left( \frac{q}{\bar{q}} \right)^{\lambda a} \sum_{(m, \bar{m}) \in \mathbb{Z}^2} (-1)^{m+\bar{m}} y^{m+\bar{m}+1} q^{\frac{1}{2} \beta (m+a+1/2)^2} \bar{q}^{\frac{1}{2} \beta (m-a+1/2)^2} \]  

(264)

This can be compared with the corresponding bosonic partition function:

\[ Z_{PP}^{(0)}(a, \mu) = - (q\bar{q})^{\lambda^2/2} \left( \frac{q}{\bar{q}} \right)^{\lambda a} \times K_{PP}(a, \mu) \]  

(265)

This sign deserves some comments although we shall not use the PP sector in this paper, which is mainly concerned with the AA sector of the massless Thirring model. First of all, let us stress that the signs we have obtained ensure positivity of the partition functions \( Z_{PP} \) and \( Z_{PA} \). In zero magnetic field, the dominant terms at vanishing temperature are

\[ \left\{ \begin{array}{ll}
Z_{PA} \simeq (q\bar{q})^{\frac{1}{2} \beta} (y + y + 2) = 4(q\bar{q})^{\frac{1}{2} \beta} \cosh^2 \left( \frac{\beta a}{2} \right) \\
Z_{PP} \simeq (q\bar{q})^{\frac{1}{2} \beta} (y - y - 2) = 4(q\bar{q})^{\frac{1}{2} \beta} \sinh^2 \left( \frac{\beta a}{2} \right)
\end{array} \right. \]

whereas \( K_{PP} \) behaves like \( 2 - y - y^{-1} = -\sinh^2 (\beta \mu/2) \). The sign in (264) technically originates from the extraction of a \( |\mu| \) factor coming from vacuum contributions to build the \( K_{PP} \) functions. A more formal way to express this invokes, as explained before, the analyticity properties of zeta regularized fermionic determinants and their behavior near vanishing points.

Last but not least, let us point out that taking into account properly this sign is important for finding the correct genus one effective CFT describing the finite size XXZ model at finite temperature\(^6\).

\(^6\) S. Dusuel and P. Degiovanni, work in progress.
C Normalization condition for bosonic functional integrals

C.1 Normalizations

For a non compactified bosonic field $\phi$, the integration measure $D_g[\phi]$ is usually (see (33)) normalized by:

$$\int D_g[\phi] \exp \left( -\frac{g}{2\pi} \int \phi^2 \right) = 1. \quad (266)$$

Let us decompose $\phi$ into a constant part $\phi_0$ and a zero-average part $\tilde{\phi}$: $\phi = \phi_0 + \tilde{\phi}$. We introduce $D_{\perp,g}$ the restriction of our measure to the space of zero-average functions. Then, equation (266) implies

$$\int D_{\perp,g}[\tilde{\phi}] \exp \left( -\frac{g}{2\pi} \int \tilde{\phi}^2 \right) = \sqrt{g} A \frac{2\pi}{A^2}. \quad (267)$$

On the space of zero-average functions, the measures $D_{\perp,g}$ are related to each other by

$$D_{\perp,g} = \sqrt{g} g' D_{\perp,g'}, \quad (268)$$

as can be easily inferred from equation (267).

For compactified fields over a circle of radius $R$, we have to specify the integration over the zero mode $\phi_0 = A^{-1} \int \phi$ of the field. We choose the usual volume $d\phi_0$ of $\mathbb{R}/2\pi R \mathbb{Z}$. There is an instanton sum and a zero-average fluctuating part $\xi$ remains. The integration measure is thus decomposed as:

$$D_{R,g}[\phi] = \sum_{\text{instantons}} d\phi_0 D_{\perp,g}[\xi]. \quad (269)$$

C.2 Fluctuation contribution

We briefly recall how to compute the fluctuation contribution

$$Z_{\text{fluct}} = \int D_{\perp,g}[\phi] \exp \left( -\frac{g}{2\pi} \int (d\tilde{\phi})^2 \right). \quad (270)$$

Using equation (267), this partition function can be expressed in terms of a determinant of the Laplacian on the space of zero-average functions as:

$$Z_{\text{fluct}} = \sqrt{\frac{g A}{2\pi^2}} \times \text{Det}_{\perp}^{-1/2} (-\Delta). \quad (271)$$

The simplest way to compute the determinant is to use the so-called zeta function renormalization procedure. From the spectrum of the Laplacian on the torus we get

$$Z_{\text{fluct}} = \frac{1}{\sqrt{2\pi^2 A}} \frac{1}{|\eta(\tau)|^2}, \quad (272)$$

the Dedekind $\eta$ function being

$$\eta(\tau) = e^{i\pi \tau/12} \prod_{n=1}^{+\infty} (1 - q^n). \quad (273)$$

This final result is independent on $g$ as can be expected from the Jacobian (268).

D Explicit computations in the bosonic theory

In this section, we recall how to compute some partition functions of the theory of a free boson, compactified on a circle of radius $R$. The functional integral we want to compute is defined by equation (63). In order to “define” it, a precise normalization prescription for functional integrals is needed. All details and definitions relative to these matters have been gathered in appendix C. The reader is referred to this section for details. The present appendix focuses on the “instanton” aspects of the computation.
D.1 Instantons

We look for instantons with the choice of boundary conditions \([\epsilon, \epsilon']\). A given complex number \(z\) can be uniquely written as \(z = x\omega_1 + t\omega_2\), where \((x, t) \in \mathbb{R}^2\) are given by:

\[
\begin{align*}
x &= \Re \left( \frac{z}{\omega_1} \right) - \frac{\Re(\tau)}{\Im(\tau)} \Im \left( \frac{z}{\omega_1} \right), \\
t &= \frac{\Im(z/\omega_1)}{\Im(\tau)}. 
\end{align*}
\]

(274)

(275)

Instantons are classical solutions, that is harmonic functions of the torus with \([\epsilon, \epsilon']\) boundary conditions, and therefore have the form

\[
\varphi^{(I)}_{n+\epsilon, m+\epsilon'}(x, t) = 2\pi R ((n+\epsilon)x + (m+\epsilon)t),
\]

(276)

with \((m, n) \in \mathbb{Z}^2\). The solution (276) is the only one with monodromy \((2\pi(n+\epsilon)R, 2\pi(m+\epsilon')R)\). For the sake of simplicity, let us introduce \(v = 2\pi(n+\epsilon)R\) and \(w = 2\pi(m+\epsilon')R\), so that \(\varphi_I(x, t) = xv + tw\). The action of a the configuration (276) is thus

\[
S[\varphi^{(I)}_{n+\epsilon, m+\epsilon'}] = \frac{9}{2\pi} \int_{D_T} (\nabla \varphi^{(I)}_{n+\epsilon, m+\epsilon'})^2 = \frac{2g}{\pi} \int_{D_T} (\partial_z \varphi^{(I)}_{n+\epsilon, m+\epsilon'}) (\partial_z \varphi^{(I)}_{n+\epsilon, m+\epsilon'}) \frac{d\tau \wedge dz}{2i},
\]

(277)

where \(D_T\) is an elementary cell of the lattice \(\Gamma\). We obtain easily

\[
\begin{align*}
\partial_z \varphi^{(I)}_{n+\epsilon, m+\epsilon'} &= \frac{1}{2\omega_1} \left( iv + \frac{w}{\Im(\tau)} - \frac{R\tau}{\Im(\tau)} vu \right), \\
\partial_z \varphi^{(I)} &= \frac{1}{2\omega_1} \left( iv - \frac{w}{\Im(\tau)} + \frac{R\tau}{\Im(\tau)} vu \right).
\end{align*}
\]

(278)

Finally, the action of a \((n+\epsilon, m+\epsilon')\) instanton is

\[
S[\varphi^{(I)}_{\epsilon+n, \epsilon'+m}] = 2\pi R^2 g\Im(\tau) \left( (n+\epsilon)^2 + \left( m+\epsilon' - \frac{R(\tau)(n+\epsilon)}{\Im(\tau)} \right)^2 \right).
\]

(279)

D.2 Calculation of the partition function

If \(\varphi\) and \(\varphi^{(I)}_{\epsilon+n, \epsilon'+m}\) both have the same monodromies, then \(\phi = \varphi - \varphi^{(I)}_{\epsilon+n, \epsilon'+m}\) has a vanishing monodromy. Moreover, since the action is quadratic and since \(\varphi^{(I)}\) extremalizes the action, we have

\[
S[\varphi^{(I)}_{\epsilon+n, \epsilon'+m} + \phi] = S[\phi] + S[\varphi^{(I)}_{\epsilon+n, \epsilon'+m}].
\]

(280)

Henceforth, the partition function factorizes:

\[
Z[\epsilon, \epsilon'](g,R) = Z_f \sum_{(m, n) \in \mathbb{Z}^2} e^{-S[\varphi^{(I)}_{\epsilon+n, \epsilon'+m}]},
\]

(281)

where \(Z_f\) contains the contribution of the fluctuations without monodromy, and we now compute the instanton contribution, the fluctuation contribution having been already computed in appendix \(\mathbb{3}\).

D.3 Introduction of a gauge field and instanton contribution

The following property will be useful: let \(A\) and \(B\) be closed 1-forms on the torus, then:

\[
\int_T A \wedge B = \int_{(a)} A \int_{(b)} B - \int_{(b)} A \int_{(a)} B,
\]

(282)
where \( (a) \) and \( (b) \) are the cycles

\[
(a) : t \in [0,1] \rightarrow t \quad \text{and} \quad (b) : t \in [0,1] \rightarrow \tau t.
\]  

(283)

Using (282), taking the holonomies (84)

\[
\int (a) A = 2\pi a \quad \text{and} \quad \int (b) A = 2\pi b,
\]  

(284)

and the monodromies (97) of \( \varphi \) along the \( (a) \) and \( (b) \) cycles, we obtain

\[
S[\varphi, A] = \frac{g}{2\pi} \int (\nabla \varphi)^2 + 4\pi i((\epsilon + n)b - (\epsilon' + m)a).
\]  

(285)

Clearly, the coupling to the gauge field adds a topological term to the action.

**Contribution of the instantons:** If \( f \) is a square integrable function on \( \mathbb{R} \), let the Fourier transform be defined as

\[
(\mathcal{F}.f)(y) = \int_{-\infty}^{+\infty} dx \, e^{2\pi i xy} f(x).
\]  

(286)

With this convention, the Poisson resummation formula reads:

\[
\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} (\mathcal{F}.f)(n).
\]  

(287)

Using (287), we deduce an expression for the summation over \( m \) in (281), and finally the contribution of the instantons reads

\[
\left( \frac{\Im(\tau)}{2R^2g} \right)^{1/2} \sum_{(m,n)\in\mathbb{Z}^2} e^{-2\pi i m\epsilon'} \frac{1}{\eta^{(a)}} \left( (n+\epsilon)R\sqrt{\Im(\tau)} + \frac{\pi i}{\eta^{(b)}} \right)^2 \frac{1}{\eta^{(b)}} \left( (n+\epsilon)R\sqrt{\Im(\tau)} - \frac{\pi i}{\eta^{(a)}} \right)^2.
\]  

(288)

**E Elliptic and theta functions**

The aim of this appendix is to recall a few definitions and basic formulae about elliptic and theta functions. Numerous references are available on this vast subject, but [84], [60] and [95] cover the material needed in this paper.

**E.1 Basic definitions**

Let us denote by \( \tau \) an element of the upper-half plane, and \( q = e^{2\pi i \tau} \). Let \( \Gamma = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z} \) (\( \Im(\omega_2/\omega_1) > 0 \)) be a lattice in \( \mathbb{C} \), and we define by \( \tau = \omega_2/\omega_1 \) the modular parameter. An elliptic curve is associated with each lattice by \( T_\Gamma = \mathbb{C}/(\omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}) \). Its complex structure is parametrized by the modular parameter \( \tau \) modulo the action of \( PSL(2,\mathbb{Z}) \) through homographies.

The Weierstrass function associated with the lattice \( \Gamma \) is defined by

\[
\wp_{\Gamma}(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).
\]  

(289)

It is doubly periodic, thus defining a meromorphic function on the torus \( T_\Gamma \). The limit \( \tau \rightarrow +i\infty \) is interesting:

\[
\lim_{\tau \rightarrow +i\infty} (\wp_{\mathbb{Z}\oplus\mathbb{Z}}(z)) = \frac{1}{z^2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin(\pi z)^2}.
\]  

(290)
Riemann theta functions with characteristics are defined by:

\[ \vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (z, \tau) = \sum_{n \in \mathbb{Z}+a} q^{n^2/2} e^{2\pi i n (z+b)}. \]  

(291)

Specializing to \((a, b) \in \{0, 1/2\}^2\) gives the famous Jacobi functions. Using the conventions of [46, Appendix 9.A]:

\vartheta_1(z, \tau) = \vartheta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z, \tau) \quad (292)

\vartheta_2(z, \tau) = \vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (z, \tau) \quad (293)

\vartheta_3(z, \tau) = \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (z, \tau) \quad (294)

\vartheta_4(z, \tau) = \vartheta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (z, \tau). \quad (295)

Technically, these are holomorphic sections of spin bundles over the torus \(T_{(\mathbb{Z} \oplus \tau \mathbb{Z})}\) and they appear in the theory of chiral fermions on a torus [3].

E.2 Relations between elliptic and theta functions and the inverse of the Laplacian on the torus

The inverse of the Laplacian on the torus \(T_{\Gamma}\) can be expressed in terms of \(\vartheta_1\) [46]:

\[ \Delta^{-1}(z) = \frac{1}{2\pi} \log \left( \left| \omega_1 e^{-iz} + \frac{\vartheta_1(z/\omega_1, \tau)}{\vartheta_1(0, \tau)} \right| \right) - \frac{1}{2} \left( \Im \left( \frac{z}{\omega_1} \right) + \frac{\Im \left( \frac{\tau}{\omega_1} \right)^2}{\Im(\tau)} \right). \]  

(296)

It satisfies the following Green equation:

\[ \Delta(\Delta^{-1}) = \delta - \frac{1}{A}, \]  

(297)

where \(A = \Im(\omega_1 \omega_2)\) is the area of the torus. Connection with Weierstrass’ \(\wp\) function arises through

\( (\partial^2 \Delta^{-1})(z) = \frac{1}{4\omega_1^2 \Im(\tau)} - \frac{1}{4\pi} \wp(z) \) \quad (298)

\( (\partial^2 \Delta^{-1})(z) = \frac{1}{4\omega_1^2 \Im(\tau)} - \frac{1}{4\pi} \wp(z). \) \quad (299)

Computing correlators in the operator formalism requires to dissymmetrize the role of \(\Re(z)\) and \(\Im(z)\), which are space and imaginary time components on the torus. Therefore, it proves useful to have a Fourier expansion for Weierstrass’ function. If we note \(x = \exp(2\pi iz)\), we have for \(|q| < |x| < |q|^{-1}\) [60]:

\[ -\frac{\varphi(z)}{4\pi^2} = \frac{1}{12} - 2 \sum_{n=1}^{+\infty} \frac{q^n}{(q^n - 1)^2} + \frac{x}{(1-x)^2} + \sum_{n=1}^{+\infty} \frac{nq^n}{1-q^n} (x^n + x^{-n}). \]  

(300)

The function

\[ \frac{x}{(1-x)^2} = \frac{-1}{4 \sin^2(\pi z)} \]

may be expanded in powers series in \(x\) on each domain \(|x| > 1\) and \(|x| < 1\).

The Weierstrass function admits a complex primitive, known as the \(\xi\) function: \(\xi’ = -\varphi\). Its expansion is given by

\[ \xi(z) = \frac{1}{z} + \sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{z - \omega} + \frac{z + \omega}{\omega^2} \right). \]  

(301)
This function is multivalued on the torus. If \( \eta_{1,2} = \xi(z + \omega_{1,2}) - \xi(z) \) denotes its monodromies,
\[
\omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i.
\]
Using the Fourier expansion of \( \varphi \), we easily get \( \eta_1 \) as a function of \( \omega_1 \) and \( \omega_2 \):
\[
\eta_1 = \frac{4\pi^2}{\omega_1} \left( \frac{1}{12} - 2 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \right).
\]
In the functional integral approach of the Luttinger CFT, one needs to remember that derivatives of \( \Delta^{-1} \) are to be understood as derivatives of distributions. Let us recall that if \( T \) be a distribution and \( f \) a test function, \( \partial_\mu T \) is defined as:
\[
(\partial_\mu T). f = -T. (\partial_\mu f).
\]
For the sake of precision, we shall denote by \([\varphi] \) the regular distribution defined by the integrable function \( \varphi \):
\[
[\varphi]. f = \int \varphi f.
\]
On the plane, the Laplacian Green’s function is
\[
\Delta^{-1}_C(z) = \frac{1}{2\pi} \log (|z|).
\]
Its first derivative \( \partial_z [\Delta^{-1}_C] \), as a distribution, is \( \frac{1}{4\pi^2} \). Even if \( 1/z^2 \), not being integrable, does not define a regular distribution, the second derivative of \( [\Delta^{-1}_C] \) may however be related to \( z \mapsto z^{-2} \) through an integration by parts formula. Let \( K \) be a bounded region of \( \mathbb{C} \) with boundary \( \partial K \), then:
\[
\int_K (\partial_z f)g = -\int_K f (\partial_z g) + \frac{i}{2} \int_{\partial K} (\bar{f}g)(z, \bar{z}) \, d\bar{z}.
\]
Now, splitting the integral \( \int (\partial_z f)/z \) into two contributions (\( |z| > \varepsilon \) and \( |z| < \varepsilon \)) and applying the above formula, one easily gets, for any test function \( f \),
\[
\partial_z \left[ \frac{1}{z} \right]. f = \lim_{\varepsilon \to 0^+} \left( \int_{|z| > \varepsilon} \frac{f(z, \bar{z})}{z^2} \, d^2z \right).
\]
This “principal value” formula connects \( \partial^2_z [\Delta^{-1}_C] \) to \( z \mapsto z^{-2} \). In the same way, formula (306) immediately leads to the well-known formula
\[
\partial_z \left[ \frac{1}{z} \right] = \pi \delta,
\]
where the right hand side arises from \( \partial \{z, |z| > \varepsilon\} \). It is quite easy to generalize these formulæ on the torus \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \). Taking into account periodicity relative to the lattice, the same kind of analysis as before around singularity gives
\[
\partial_z [\Delta^{-1}] = \frac{1}{4\pi} \left[ \partial'_1(z, \tau) + 2\pi i \frac{\Theta(z)}{\Theta(\tau)} \right] = \frac{1}{4\pi} \left[ \xi(z) - \eta_1 z + 2\pi i \frac{\Theta(z)}{\Theta(\tau)} \right].
\]
Differentiating once again with respect to \( z \) gives, for any test function \( f \) on the torus,
\[
\partial^2_z [\Delta^{-1}] \cdot f = \lim_{\varepsilon \to 0^+} \left( \int_{|z| > \varepsilon} \partial^2_z \left( \frac{1}{4\pi \Theta(\tau)} - \frac{1}{4\pi} \varphi(z) \right) f(z, \bar{z}) \right),
\]
where the integral is performed over a unit cell of the lattice \( \mathbb{Z} + \tau \mathbb{Z} \) centered on zero, with the restriction \( |z| > \varepsilon \). In the same way, we recover that
\[
\partial_z \partial_{\bar{z}} [\Delta^{-1}] = \frac{1}{4} \left( \delta - \frac{1}{\lambda} \right).
\]
\(^{17}\)Of course, the integral is restricted to a fundamental cell of the \( \mathbb{Z} + \tau \mathbb{Z} \) lattice, which for convenience, is supposed to be centered on 0.
### E.3 Modular properties of theta functions

Modular properties of Riemann theta functions are easily found. Let us recall that the modular group $SL(2, \mathbb{Z})$ is generated by two matrices [Chapter 7]:

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

We then have

$$\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + \tau + 1) = e^{i \pi a (1 - \frac{1}{2})} \vartheta \left[ \begin{array}{c} b + a + 1/2 \\ b + a + 1 \end{array} \right] (z, \tau),$$  

(312)

$$\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] \left( \frac{z}{\tau}, \frac{-1}{\tau} \right) = (-i \tau)^{1/2} e^{-2 \pi i z^2/\tau} \vartheta \left[ \begin{array}{c} b \\ -a \end{array} \right] (z, \tau).$$  

(313)

The Dedekind $\eta$ function, defined in (273) transforms as

$$\eta(\tau + 1) = e^{i \pi \tau/12} \eta(\tau)$$  

(314)

$$\eta(-1/\tau) = (-i \tau)^{1/2} \eta(\tau).$$  

(315)

Combining Dedekind’s function with Riemann’s theta functions leads to finite dimensional unitary representations of the modular group. For $N \geq 1$ and $n \in \mathbb{Z}/N\mathbb{Z}$, let us define

$$\chi_n(\tau) = \vartheta \left[ \begin{array}{c} n/N \\ 0 \end{array} \right] (0, N\tau) \eta(\tau).$$  

(316)

Then, one has

$$\chi_n(\tau + 1) = e^{2 \pi i (\frac{n^2}{4N} - \frac{n}{2})} \chi_n(\tau)$$  

(317)

$$\chi_n(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{m \in \mathbb{Z}/N\mathbb{Z}} e^{-2 \pi i m n/N} \chi_m(\tau).$$  

(318)

This unitary linear representation of $SL(2, \mathbb{Z})$ arises in many conformal field theories, such as the rational Gaussian model [22] which is heavily used in the present paper, but also in the $SU(N)$ Wess-Zumino-Witten models at level one [12]. The kernel of this representation contains one of the congruence subgroups $\Gamma(12N)$ of $SL(2, \mathbb{Z})$. Remember that

$$\Gamma(N) = \left\{ M \in SL(2, \mathbb{Z}), \quad M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$  

and therefore, it appears as a representation of a quotient of $SL(2, \mathbb{Z})/\Gamma(12N) \simeq SL(2, \mathbb{Z}/12\mathbb{Z}).$

### F Non relativistic fermions

In this section, we compute density-density correlations of a gas of non relativistic fermions. For the sake of simplicity, we work at zero temperature, thus avoiding to use the detailed dynamics of fermions. Our goal is to shed light on the Luttinger CFT results obtained in section 5.1.

#### F.1 Response to an external potential

Let us compute the linear response of the charge density to an external potential for non-relativistic free fermions. The external potential may conveniently be treated as a stationary perturbation. Standard stationary perturbation theory gives us the first order correction to the eigenstate $|\psi_n\rangle$:

$$|\psi_n^{(1)}\rangle = |\psi_n\rangle + \sum_{m \neq n} \frac{V_{nm}}{E_n - E_m} |\psi_m\rangle,$$  

(320)
with $V$ the perturbation. This gives the first order correction to the matrix element $A_{mn} = \langle \psi_n^{(1)} | A | \psi_m^{(1)} \rangle$:

$$\sum_{l \neq m} V_{lm} A_{nl} \frac{E_m - E_l}{E_n - E_l} + \sum_{l \neq n} V_{nl} A_{lm},$$

(321)

The Fourier transform of the density response $\rho(q)$ to an external potential $V(q)$ modulated at a wave vector $q$ can therefore be expressed at the linear order in terms of the density of states of particle-hole excitations by $\rho(q) = \nu(q)V(q)$, with

$$\nu(q) = \int_0^{+\infty} \frac{d\omega}{\omega} \nu(q, \omega).$$

(322)

The density $\nu(q, \omega)$ of particle-hole excitations is defined by:

$$\nu(q, \omega) = \sum q n(q)(1 - n(q))\delta(\omega - \epsilon(k) - \epsilon(q + k)).$$

(323)

Evaluating this expression leads to the explicit expression of the density response to an external potential:

$$\nu(q) = -\frac{L}{\pi v_F} \frac{1}{2x} \log \left( \frac{1 + x}{1 - x} \right), \quad x = \frac{q}{2k_F}.$$  

(324)

This expression deserves two comments: first, there is a logarithmic singularity (Kohn singularity) at $q = 2k_F$, which is the sign of the so-called Peierls instability [76]. This instability is responsible for the opening of a gap and a transition to a charge density wave if, for instance, the system is coupled to lattice distortions. Electronic interactions by themselves can also be responsible for such a transition. Next, at small momenta, we have $\nu(q) \approx -L/\pi v_F$, which is, as expected, nothing but the CFT result [83].

**F.2 Density-density correlations**

We consider here fermionic modes of momenta $2\pi n/L$. The charge density operator is given in terms of creation and annihilation operators by the operator

$$\hat{\rho}(\sigma) = \frac{1}{L} \sum_{k,q} e^{iq\sigma} \hat{c}^+_k \hat{c}_{k+q}.$$ 

(325)

At zero temperature, the system is in the Fermi vacuum $|F\rangle$ where all fermionic states of momenta $k$ such that $|k| < k_F$ are occupied, and all other states are empty. We have $N = 2n_F + 1$ particles in the system and the Fermi wave vector is equal to $k_F = 2\pi(n_F + 1/2)/L$. A ultra violet cut-off is introduced in the system: momenta are bounded in absolute value by $\pi/a$, where $a$ is the lattice spacing. We want to compute the equal time two point function:

$$\langle F | \hat{\rho}(\sigma) \hat{\rho}(0) | F \rangle = \frac{1}{L^2} \sum_{k,k',q,q'} e^{iq\sigma} \langle F | \hat{c}^+_k \hat{c}^+_{k+q} c_{k+q} c_k | F \rangle,$$ 

(326)

and we separate between two cases: (i) $q = 0$ and (ii) $q \neq 0$.

**Case** $q = 0$ Non vanishing contributions arise from $|k'| < k_F$ and $q' = 0$. The contribution to (326) is $(N/L)^2 = (k_F/\pi)^2$, which is nothing but the the square of the average of $\rho(\sigma)$. Taking normal ordered products cancels this contribution, which is therefore not relevant in our discussion.

**Case** $q \neq 0$ Then $|k + q| > k_F$, and we should have $k' = k + q$ and $q' = -q$ otherwise the matrix element in (326) vanishes. After a straightforward calculation of the different sums, we obtain an explicit expression for the connected density-density correlation:

$$\langle F | \hat{\rho}(\sigma) \hat{\rho}(0) | F \rangle_c = \frac{2k_F}{\pi L} \cos \left( \frac{2\sigma}{L} \right) \sin \left( \frac{\sigma}{L} \right) \sin \left( \frac{\sigma}{L} \right) + \frac{k_F}{\pi L} \sin \left( \frac{2\sigma}{L} \right) \sin \left( \frac{\sigma}{L} \right) - \frac{1}{L^2} \sin^2 \left( \frac{\sigma}{L} \right).$$ 

(327)
The first term arises from the contribution $|q| \geq 2k_F$ (all of the $k$-matrix elements are non-zero, with $k$ inside the Fermi sea), and the last two terms arise from the contribution $|q| < 2k_F$ (only some of these matrix elements are non zero).

In order to recover the CFT results, one should take the thermodynamic limit and consider only long distance correlations: $1/k_F \ll \sigma \ll L$ and $1/a \ll \sigma \ll L$, $k_F$ and $a$ being kept fixed. The connected density-density correlation in this limit is then

$$
\langle F | \hat{\rho}(\sigma) \hat{\rho}(0) | F \rangle_c \sim \frac{2k_F}{\pi} \delta(\sigma) - \frac{1}{2L^2} \sin^{-2} \left( \frac{\pi \sigma}{L} \right).
$$

(328)

In order to derive the $\delta(\sigma)$ contribution, the following identity is needed:

$$
\frac{1}{L} \sin \left( \frac{\pi \lambda \sigma}{L} \right) \sin \left( \frac{\pi \sigma}{L} \right) \to \delta(\sigma)
$$

(329)

in the limit $\lambda \to +\infty$, $\lambda$ being an odd integer. The other contribution in (328) has been obtained by performing an average over length scales $l$ such that $1/k_F \ll l \ll \sigma$: $\sin^2 \left( k_F \sigma \right)$ has been replaced by $1/2$.

The CFT prediction for this correlation function are given by equations (126) and (127). Without any external potential, the CFT result reads

$$
\langle \rho(\sigma,t)\rho(0,0) \rangle_c = -\frac{1}{2\pi^2 \alpha} \Re \phi(\sigma + ivst) + \frac{1}{2\pi \alpha} \delta(\sigma)\delta(t).
$$

(330)

Since we are interested here in the zero temperature limit, we make use of the limiting behavior (290) of the Weierstrass $\wp$ function in the limit $\tau \to +i\infty$, from which the equal time and zero temperature correlator can be deduced:

$$
\frac{1}{2\alpha L^2} \sin^{-2} \left( \frac{\pi \sigma}{L} \right) + \frac{1}{2\pi \alpha} \delta(\sigma)\delta(0).
$$

(331)

The second term is rather ill-defined and deserves some comments: the $\delta(0)$ contribution arises from the equal time condition $t = 0$. In fact, as we have just seen, this delta distribution in the complex plane must be regularized. The detailed computation we have just performed from the non relativistic fermions problem shows that the divergence is smeared on a length scale of order $k_F^{-1}$. The first term in (331) is nothing but the average of the non relativistic correlator over length scales large compared to the microscopic scale $k_F^{-1}$, as we have already seen. This completes the identification of the CFT result in the non interacting limit with the non relativistic free fermions result.

G  Operator computations and comparison with the CFT results

This Appendix is devoted to recover some of the CFT results in terms of chiral mode bosonization. Contrary to appendix I, we work here from the beginning with relativistic fermions, but computations in the interacting theory can be carried out by making use of the Bogoliubov transformation (18) and (19). As an example, we reconsider the density-density correlations within this chiral framework. More precisely, we compute the generating functional for equal time density correlators. The functional result will of course be recovered.

Introducing an external potential coupled linearly to the density, or equivalently a source field for density correlations amounts to solve a bosonic Hamiltonian of the generic form $h = \omega a^+ a + \lambda (a^+ a)$, the second term arising from the coupling to the potential. This problem is then diagonalized by a unitary transformation. The unitary operator $U[V]$ to be used is

$$
U[V(\sigma)] = \exp \left( \frac{i}{2eS\sqrt{\alpha}} \int_0^L d\eta(\sigma) \left( J(\sigma) + \bar{J}(\sigma) \right) \right),
$$

(332)
where \( \eta'(\sigma) = \mathcal{V}(\sigma) \). Its action on the \( U(1) \) currents is given by:

\[
J'(\sigma) = U[\mathcal{V}(\sigma)] J(\sigma) U[\mathcal{V}(\sigma)]^{-1} = J(\sigma) + \frac{\mathcal{V}(\sigma)}{2\pi v_S \sqrt{\alpha}}
\] (333)

\[
\mathcal{J}'(\sigma) = U[\mathcal{V}(\sigma)] \mathcal{J}(\sigma) U[\mathcal{V}(\sigma)]^{-1} = \mathcal{J}(\sigma) + \frac{\mathcal{V}(\sigma)}{2\pi v_S \sqrt{\alpha}}
\] (334)

These transformed operators satisfy \( \hat{U}(1) \) commutation relations. The total Hamiltonian (involving the kinetic term plus the fermion-fermion interactions plus the coupling of the density to the external potential) transforms as follows under the unitary transformation (332):

\[
H[V(\sigma), J, \mathcal{J}] = H[0, J', \mathcal{J}'] + \frac{1}{2\pi \alpha v_S} \int_0^L \mathcal{V}(\sigma)^2 \, d\sigma,
\] (335)

where we have used the form (22) of the free plus fermion-fermion interaction terms. We also have:

\[
\int_0^L b(\sigma) \rho_{[J, J]}(\sigma) = \int_0^L b(\sigma) \rho_{[J', \mathcal{J}']}(\sigma) - \frac{1}{\pi \alpha v_S} \int_0^L \mathcal{V}(\sigma) b(\sigma) \, d\sigma.
\] (336)

Therefore, using equations (333) to (336) and the cyclicity of the trace, the generating functional (123) reads

\[
W_{[V(\sigma), J', \mathcal{J}']}^{(0)}[b(\sigma)] = W_{[0, X, q]}^{(0)}[b(\sigma)] e^{-\frac{1}{\pi \alpha} \int_0^L b(\sigma) \mathcal{V}(\sigma) \, d\sigma}.
\] (337)

The average \( b_0 \) of \( b \) couples to the total charge and contributes to the linear term (125) through

\[
\int_0^L \rho(\sigma) b_0(\sigma) \, d\sigma = \frac{q}{L} \int_0^L b(\sigma) \, d\sigma.
\]

Therefore, we have recovered the linear part corresponding to (125). Notice that the \( \frac{1}{\pi \alpha v_S} \int \mathcal{V}(\sigma)^2 \, d\sigma \) term that appears in the right hand side of equation (335) corresponds to the chiral gauge transformation anomaly, up to a factor of \( v_s \) that has been put to unity in (22).

The quadratic contribution to the generating functional may easily be computed by noticing that \( e^{\int b(\sigma) \rho(\sigma) \, d\sigma} \) is of the form \( e^{A^+ A} \) where \( [A, A^+] \) is proportional to the identity. More precisely\(^18\)

\[
A = \frac{1}{\sqrt{\alpha}} \sum_{n=1}^{+\infty} (b_{-n} J_n + b_n \mathcal{J}_n).
\] (338)

and the commutator of interest reads

\[
[A, A^+] = 2 \sum_{n=1}^{+\infty} n |b_n|^2
\]

Glauber formula leads to computing the finite temperature average of \( < e^{A^+ A} > \). But for a single harmonic oscillator, the following identity between finite temperature averages holds:

\[
\langle e^{\lambda a^+} e^{\mu a} \rangle = e^{\lambda \mu} \langle a^+ a \rangle.
\] (339)

Applied to the present context, this leads to

\[
W_{[0, X, q]}^{(0)}[b(\sigma)] = \exp \left( \frac{1}{\alpha} \sum_{n=1}^{+\infty} n b_n b_{-n} \frac{1 + q^n}{1 - q^n} \right),
\]

where \( q = \exp(-2\pi v_S / L) \). Then, one recognizes the Fourier expansion of the Weierstrass function. More precisely, one formally has

\[
\lim_{\varepsilon \to 0^+} \left( \frac{\psi(\sigma + i \varepsilon) + \psi(\sigma - i \varepsilon)}{2} \right) = -\eta_1 + 2\pi^2 \sum_{n=1}^{+\infty} n \frac{1 + q^n}{1 - q^n} (x^n + x^{-n}).
\] (341)

where \( \eta_1 \) denotes \( \xi \)'s monodromy as in appendix E. However, since \( b \)'s constant term is not taken in account in equation (340), equations (126) and (127) are recovered.

\(^18\)In this paragraph, \( b \) has a vanishing average over the circle.
H  Duality for the free boson revisited

The explicit expression of the partition function of the free boson on the torus shows that theories at \( \alpha = gR^2 \) and \( \alpha^{-1/4} \) have the same partition functions. This symmetry is called duality by conformal field theorists \cite{23}. We analyze here the interplay between this duality and the boundary conditions that are imposed on the bosonic field. Of course we focus on the generating functional corresponding to the massless Thirring model coupled to a gauge field as described in section \( \text{C} \). Our method will rely on direct functional integrals manipulation. The normalization prescription is important for these computations.

H.1  Introduction of auxiliary fields

The starting point has been known for a long time and consists in introducing a vector field so that:

\[
\exp \left( -\frac{g}{2\pi} \int (\partial_{\mu} \varphi)^2 \right) = \int D[b_{\mu}] \exp \left( -\frac{\pi}{2g} \int b_{\mu} b^\mu + i \int b_{\mu} \epsilon^{\mu\nu} \partial_{\nu} \varphi \right).
\]  

(342)

The next step consists in integrating over \( \varphi \). An effective action for the field \( b_{\mu} \) comes out. More precisely, let us consider

\[
Z_{C}[A] = \int_{C} D_{R,g}[\varphi] \int D[b_{\mu}] \exp \left( -\frac{\pi}{2g} \int (b - \frac{A}{R})^2 + i \int b \wedge d\varphi \right).
\]  

(343)

Let us now decompose \( \varphi \) in a classical solution \( \varphi_c \) having the required monodromy, the constant part (zero mode), and a fluctuating part \( \xi : \varphi = \varphi_c + \varphi_0 + \xi \). Integrating by parts shows that

\[
\int b \wedge d\varphi = \int b \wedge d\varphi_c + \int \xi db.
\]

Roughly speaking, the integration over \( \xi \) shows that \( b \) is a flat connection. We can therefore decompose it as \( b = h + da \) where \( h \) is a constant 1-form on the torus and \( \alpha \) a zero form. Part of the dual bosonic field will arise from this zero form. In the next section of this appendix, this idea will be expressed more precisely.

H.2  Compactification of the dual field

Formula (282) implies the vanishing of the integral of \( da \wedge d\varphi_c \) since \( da \) is exact. Therefore, if \( b \) is closed,

\[
\int b \wedge d\varphi_c
\]

can be expressed in terms of the \( \varphi_c \)'s monodromies and the \( b \)'s holonomies, or equivalently the \( h \)'s ones. Let us denote by \( h_{(a,b)} \) these holonomies along the \( (a) \) and \( (b) \) cycles respectively. Let us also introduce \( n_{(a,b)} \) such that the \( \varphi_c \)'s monodromies are \( 2\pi R n_{(a,b)} \). We then have:

\[
\int b \wedge d\varphi_c = 2\pi R (h_{(a)} n_{(b)} - h_{(b)} n_{(a)}).
\]

The summation over the discrete set of values for \( n_{(a,b)} \) discretizes the \( h_{(k,a)} \):

\[
\sum_{n \in \mathbb{Z}} \exp (2\pi i R n) = \frac{1}{R} \sum_{m \in \mathbb{Z}} \delta(h - m/R).
\]  

(344)

This is the reason why a compactified discretized field also appears in the dual theory.

At this stage, a careful treatment of the integration measures is necessary. Definitions are given in appendix \( \text{C} \). We assume that \( A \) is a constant gauge field. The case of a general gauge field will be treated in section \( \text{H.4} \).
Let us now perform all manipulations on the modular invariant partition function

\[ Z[A] = 2\pi R \sum_{(m,n)\in\mathbb{Z}^2} \int \mathcal{D}[\phi] e^{-\frac{1}{\pi R} \int \mathcal{D}_g \phi \int \mathcal{D}_{A,h} \phi \exp \left( i \int (b + \frac{A}{\pi R}) \wedge d(\phi_{n,m} + \chi) \right) \}
\]

\[ = 2\pi R \int \mathcal{D}[\phi] e^{-\frac{1}{\pi R} \int (b + \frac{A}{\pi R}) e^i \int b \wedge d\xi} \cdot (345) \]

The functional \( W_{\text{inst}}[b] \) arises from the sum over instantons. Let us decompose \( b = h + \tilde{b} \), where the constant 1-form \( h \) is \( \tilde{b} \)'s average over the torus. The integration measure for the auxiliary fields factorizes as \( \mathcal{D}[\phi] = \mathcal{D}_{A,h} \mathcal{D}_g \). We then have

\[ W_{\text{inst}}[h] = \frac{1}{R^2} \sum_{(m_a,m_b)\in\mathbb{Z}^2} \delta \left( \int_a h - \frac{m_a}{R} \right) \delta \left( \int_b h - \frac{m_b}{R} \right). \]

Therefore, denoting by \( h_{(a,b)} \) the constant 1-form of holonomies \( (l_a,l_b) \), we obtain:

\[ \int d^2h W_{\text{inst}}[h] \Phi[h] = \frac{1}{A R^2} \sum_{(m_a,m_b)\in\mathbb{Z}^2} \Phi[h_{R^{-1}(m_a,m_b)}]. \]

We shall now compute the integrals over \( \tilde{b} \) and \( \xi \). Let

\[ F[A,h] = \int \mathcal{D}_{A,h} [\phi] e^i \int b \wedge d\xi \exp \left( -\frac{\pi}{2g} \int (h - \frac{A}{\pi R} + \tilde{b})^2 \right). \]

Integrating over \( \tilde{b} \), taking into account the normalization condition over \( b \) and using that \( A \) is a constant gives:

\[ F[A,h] = \frac{A}{2g} \int \mathcal{D}_g [\phi] \exp \left( -\frac{\pi}{2g} \int (h - \frac{A}{\pi R})^2 \right). \]

Changing variables for \( \tilde{\chi} = g\xi \) and using the Jacobian \( \frac{1}{2\pi g^2} \) leads to:

\[ F[A,h] = \frac{A}{2} \int \mathcal{D} \tilde{\chi} \exp \left( -\frac{1}{2\pi g} \int (d\tilde{\chi} + \tau h - \frac{A}{R})^2 \right). \]

Putting all parts together, we finally get

\[ Z[A] = \int \mathcal{D}_{A,h} [\phi] \exp \left( -\frac{1}{2\pi g} \int (d\tilde{\chi} - \frac{A}{R})^2 \right), \]

and \( \tilde{\chi} \) is compactified over a circle of radius \( R' = 1/2R \). The resulting parameter is \( \alpha' = R'^2 g^{-1} = 1/4\alpha \) as expected.

### H.3 Duality for the Luttinger Conformal Field Theory

We shall now go along the computation for the Luttinger case. Here, the functional integral boundary conditions are defined by equations \( (200) \). The summation over monodromies at fixed \( (\varepsilon,\varepsilon') \) are performed first. The \( h_{(a,b)} \) holonomies should therefore be integer multiples of \( 1/R \). We shall then perform the summation over \( (\varepsilon,\varepsilon') \) and obtain the following result:

\[ Z^{(r)}_{\text{Lutt}}[\alpha,A] = \sum_{(u_a,u_b)\in\{0,1\}^2} (-1)^{u_a u_b} e^{i\pi r u_b} \int_{(\pi R^{-1} u_a,\pi R^{-1} u_b)} \mathcal{D}_{A,h} [\phi] \exp \left( -\frac{1}{2\pi} \int (\partial_a \chi - \frac{A}{R})^2 \right). \]

Then, by shifting to a field compactified on a circle of radius \( 1/R \), we restore a 1/2 factor due to the zero mode measure. Developing the exponential, we obtain:

\[ Z^{(r)}_{\text{Lutt}}[\alpha,A] = Z^{(0)}_{\text{Lutt}}[\alpha^{-1},\frac{iA^*}{\alpha} + b_r] \times \exp \left( \frac{1}{2\pi\alpha} \int A^2 \right), \]

\[ (352) \]
where \( b_r \) corresponds to a magnetic flux of \( r \Phi_0/2 \). This arises from

\[
e^{i \pi r u_b} = e^{i \pi r} \int b_r \wedge d\chi \quad \text{where} \quad b_r = \frac{\pi r}{L} d\sigma.
\]

It is interesting to notice that duality still holds but \( \alpha' = \alpha^{-1} \) instead of \( 4/\alpha \). Contrarily to the modular invariant case, the free Dirac theory is the self dual point\(^\text{19}\). We also notice that the roles of electric and magnetic fields are exchanged. The presence of a given charge in the system corresponds to a magnetic flux in the dual theory. Therefore, although the partition function in the \( r = 0 \) sector, without any external field, is invariant under \( \alpha \mapsto 1/\alpha \), the physics is not the same.

### H.4 Ward identities in the dual theory

Finally, we have obtained duality formulae for the compactified boson and the Luttinger theory in its bosonic form coupled to a constant gauge field. Let us now extend these formulae to any gauge field. Let us first recall that the original theory is gauge invariant under the transformation

\[
\begin{cases}
A \mapsto A + d\beta \\
\varphi \mapsto \varphi.
\end{cases}
\]

But the dual theory is also gauge invariant under

\[
\begin{cases}
A \mapsto A + d\beta \\
\chi \mapsto \chi + \pi \beta.
\end{cases}
\]

In the dual theory, chiral gauge transformations correspond to:

\[
\begin{cases}
A \mapsto A + d^\ast \beta \\
\chi \mapsto \chi,
\end{cases}
\]

whereas, in the dual theory \( \varphi \) is changed into \( \varphi + i\beta/gR \). It is obvious that, under chiral gauge transformations, both the original and the dual theory have the same transformation properties (95). They are also invariant under normal gauge transformations. Therefore, using Hodge’s theorem, we can extend formulae (350) and (352) to any vector potential \( A \).

### References

\[1\] A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, *Methods of quantum field theory in statistical physics*, Dover, 1963.

\[2\] I. Affleck, *Field theory methods and quantum critical phenomena*, Fields, strings and critical phenomena (XLIX Les Houches session) (E. Brézin and J. Zinn-Justin, eds.), Elsevier, 1988, pp. 565 – 640.

\[3\] L. Alvarez-Gaume, J.B. Bost, G. Moore, and C. Vafa, *Bosonization on higher Riemann surfaces*, Comm. Math. Phys. 112 (1987), 503.

\[4\] L. Alvarez-Gaume, G. Moore, and C. Vafa, *Theta functions, modular invariance and strings*, Comm. Math. Phys. 106 (1986), 1–40.

\[5\] P.W. Anderson, *Infrared catastrophe in Fermi gases with local scattering potentials*, Phys. Rev. Lett. 18 (1967), 1049 – 1051.

\[6\] ———, *A poor man’s derivation of scaling laws for the Kondo problem*, J. Phys. C 3 (1970), 2436 – 2441.

\(^{19}\) This little difference with the usual modular invariant case had indeed been noticed by Klassen et al. some time ago.
Basic notions of condensed matter physics, Frontiers in Physics, Addison-Wesley, 1984.

T.N. Antsygina, L.A. Pastur, and V.A. Slyusarev, Localization of states and kinetic properties of one-dimensional disordered systems, Sov. J. Low. Temp. Phys. 7 (1981), 1 – 21.

A.A. Belavin, A.B. Polyakov, and A.B. Zamolodchikov, Infinite conformal symmetry in 2d field theory, Nucl. Phys. B. 241 (1984), 333–380.

V.L. Berezinskii, Kinetics of a quantum particle in a one-dimensional random potential, Sov. Phys. JETP 38 (1973), 620 – 627.

V.L. Berezinskii and L.P. Gorkov, On the theory of electrons localized in the field of defects, Sov. Phys. JETP 50 (1979), 1209 – 1218.

D. Bernard, On the random vector potential model in two dimensions, Nucl. Phys. B 441 (1995), 471–482.

Perturbed conformal field theory applied to 2D disordered models: an introduction, Low dimensional applications of quantum field theory (Cargese), 1995, [hep-th/9509137].

J.-B. Bost, Introduction to compact Riemann surfaces, jacobians and abelian varieties, Number theory and physics (Les Houches, 1989), Springer Verlag, 1992.

A. Cappelli and G.R. Zemba, Modular invariant partition functions in the quantum Hall effect, Nucl. Phys. B 490 (1997), 595 – 632.

J. Cardy, Operator content of 2d conformal field theories, Nucl. Phys. B. 270 (1986), 186–204.

Conformal invariance and statistical mechanics, Fields, Strings and Critical phenomena, (XLIX Les Houches session) (E. Brézin and J. Zinn-Justin, eds.), Elsevier, 1989, pp. 169–246.

P. Cartier, An introduction to zeta functions, From number theory to physics (Les Houches, 1989), Springer-Verlag, 1992.

S. Coleman, Quantum sine-Gordon equation as the massive Thirring model, Phys. Rev. D 11 (1975), 2088 – 2097.

P. Degiovanni, Z/NZ Conformal Field Theories, Comm. Math. Phys. 127 (1990), 71–99.

Ph. Di Francesco, H. Saleur, and J.B. Zuber, Critical Ising correlation functions in the plane and on the torus, Nucl. Phys. B 290 (1987), 527.

R. Dijkgraaf, A geometric approach to 2D conformal field theory, Ph.D. thesis, Utrecht University., 1989.

R. Dijkgraaf, E. Verlinde, and H. Verlinde, c = 1 conformal field theories on Riemann surfaces, Comm. Math. Phys. 115 (1987), 649.

Modular invariance and the fusion algebra, Conformal Field Theories and Related Topics (P. Binétruy, P. Sorba, and R. Stora, eds.), Nucl. Phys. B. (Proc. Suppl 5), North Holland, 1988, pp. 87–97.

I.E. Dzyaloshinskii, Correlation functions for a one-dimensional Fermi system with a long range interaction (Tomonaga model), Sov. Phys. JETP 38 (1974), 202 – 208.

R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals, McGraw-Hill, 1965.

M.P.A. Fisher and L.I. Glazman, Transport in a one-dimensional Luttinger liquid, Mesoscopic electron transport (L. Kowenhoven, G. Schoen, and L. Solin, eds.), NATO ASI Series, 1997.
[28] E. Fradkin, *Field theories of condensed matter systems*, Frontiers in Physics, vol. 82, Addison Wesley, 1991.

[29] D. Friedan and S. Shenker, *The analytic formulation of 2D conformal field theories*, Nucl. Phys. B 281 (1987), 509–545.

[30] T. Gannon, *U(1)^m modular invariants, N = 2 minimal models and the quantum Hall effect*, Nucl. Phys. B 491 (1997), 659 – 688.

[31] K. Gawedzki and E. Charpentier, *Bosonization in background of metric and charge*, J. Math. Phys. 34 (1993), 381–436.

[32] T. Giamarchi and H.J. Schulz, *Anderson localization and interactions in one-dimensional metals*, Phys. Rev. B 37 (1988), 325–340.

[33] P. Ginsparg, *Applied conformal field theory*, Fields, Strings and Critical phenomena, (XLIX Les Houches session) (E. Brézin and J. Zinn-Justin, eds.), Elsevier, 1988, pp. 1–168.

[34] V. Glaser, *An explicit solution of the Thirring model*, Nuovo Cimento 9 (1958), 2812 – 2827.

[35] A.A. Gogolin and V.I. Mel’nikov, *Conductivity of disordered one-dimensional metal with half-filled band*, Sov. Phys. JETP 46 (1977), 369 – 376.

[36] A.A. Gogolin, V.I. Mel’nikov, and E.I. Rashba, *Conductivity in a disordered one-dimensional system induced by electron-phonon interaction*, Sov. Phys. JETP 42 (1975), 168 – 178.

[37] L.P. Gorkov and O.N. Dorokhov, *Singularities in the density of electronic states of one-dimensional conductors with disorder*, Solid State Comm. 20 (1976), 789 – 792.

[38] L.P. Gorkov, O.N. Dorokhov, and F.V. Prigara, *Structure of wave functions and ac conductivity in disordered one-dimensional conductors*, Sov. Phys. JETP 58 (1983), 852 – 862.

[39] F.D.M Haldane, *Coupling between charge and spin degrees of freedom in the one-dimensional Fermi gas with backscattering*, J. Phys. C 12 (1979).

[40] F.D.M. Haldane, *General relation of correlation exponents and spectral properties of one-dimensional Fermi systems: application to the anisotropic S = 1/2 Heisenberg chain*, Phys. Rev. Lett. 45 (1980).

[41] ______, *Luttinger liquid theory of one dimensional quantum fluids (I). properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas*, J. Phys. C: Solid State Phys. 14 (1981), 2585–2609.

[42] ______, *Fractional statistics in arbitrary dimensions: a generalization of the Pauli principle*, Phys. Rev. Lett. 67 (1991), 937 – 940.

[43] F.D.M. Haldane and E.H. Rezayi, *Laughlin state on stretched and squeezed cylinders and edge excitations in the quantum Hall effect*, Phys. Rev. B 50 (1994), 17199 – 17207.

[44] B.I. Halperin, *Quantized Hall conductance, current carrying edge states, and the existence of extended states in a two dimensional disordered potential*, Phys. Rev. B 25 (1982), 2185 – 2190.

[45] C. Itzykson, *Level one Kac-Moody characters and modular invariance*, Conformal Field Theories and related topics (P. Binetruy, P. Sorba, and R. Stora, eds.), Nucl. Phys. B (Proc Suppl. 5), North Holland, 1988, pp. 150–165.

[46] C. Itzykson and J.M. Drouffe, *Théorie statistique des champs*, Interéditions-CNRS, 1990.

[47] C. Itzykson and J.B. Zuber, *Quantum field theory*, Mac Graw Hill, 1980.

[48] T. Jolicoeur and J.C. Le Guillou, *Abelian bosonization in a path integral framework*, Int. Jour. of Modern Physics A 8 (1993), 1923.
10.1016/0022-247X(84)90110-9.
[73] P. Nozières and J.M. Luttinger, *Derivation of the Landau theory of Fermi liquids I: Formal preliminaries*, Phys. Rev. **127** (1962), 1423 – 1431.

[74] ______, *Derivation of the Landau theory of Fermi liquids II: Equilibrium properties and transport equation*, Phys. Rev. **127** (1962), 1431–1440.

[75] A.A. Ovchinnikov and N.S. Erikhman, *Density of states in a one dimensional random potential*, Sov. Phys. JETP **46** (1977), 340 – 346.

[76] R.E. Peierls, *Quantum theory of solids*, Oxford University Press, 1995.

[77] S. Qin, M. Fabrizio, L. Yu, M. Oshikawa, and I. Affleck, *Impurity in a Luttinger liquid away from half-filling: a numerical study*, Preprint [cond-mat/9705269](cond-mat/9705269), 1997.

[78] I. Sachs and A. Wipf, *Generalized Thirring models*, Annals of Physics **249** (1996), 380–429.

[79] L. Saminadayar, D.C. Glattli, Y. Jin, and B. Etienne, *Observation of the e/3 fractionally charged Laughlin quasi-particle*, Phys. Rev. Lett. **79** (1997), 2526 – 2529.

[80] J. Scherk, *An introduction to the theory of dual models and strings*, Review of Modern Physics **47** (1975), 429–444.

[81] H.J. Schulz, *Correlated fermions in one dimension*, Int. J. Mod. Phys. **B 5** (1991), 57 – 74.

[82] ______, *Interacting fermions in one dimension: from weak to strong correlation*, Lecture notes at the Jerusalem winter school on theoretical physics, Dec. 1991 - Jan. 1992, [cond-mat/9302006](cond-mat/9302006), 1993.

[83] J. Schwinger, *Gauge invariance and mass (II)*, Phys. Rev. **128** (1962), 2425–2429.

[84] J.P. Serre, *Cours d’arithmétique*, P.U.F., 1970.

[85] R. Shankar, *Renormalization group approach to interacting fermions*, Rev. Mod. Phys. **66** (1994), 129.

[86] J. Solyom, *The Fermi gas model of one-dimensional conductors*, Adv. in Physics **28** (1979), 201 – 303.

[87] M. Stone and M.P.A. Fisher, *Laughlin states at the edge*, Int. J. Mod. Phys. **B 8** (1994), 2539 – 2553.

[88] R. Tao and Y.-S. Wu, *Gauge invariance and fractional quantum Hall effect*, Phys. Rev. **B 30** (1984), 1097–1098.

[89] W. Thirring, *On interacting spinor fields in one dimension*, Nuovo Cimento **9** (1958), 2829 – 2837.

[90] S. Tomonaga, *Remarks on Bloch’s method of sound waves applied to many fermions problems*, Progr. Theor. Phys. **5** (1950), 544 – 569.

[91] E. Verlinde, *Fusion rules and modular transformations in 2D CFT’s*, Nucl. Phys. **B 300 (FS 22)** (1988), 360–376.

[92] J. Voit, *Charge-spin separation and the spectral properties of Luttinger liquids*, Phys. Rev. **B 47** (1993), 6740–6743.

[93] X.G. Wen, *Chiral Luttinger liquid and the edge excitations in the fractional quantum Hall states*, Phys. Rev. **B 41** (1990), 12838 – 12844.

[94] F. Wilczek, *Fractional statistics and anyon superconductivity*, World Scientific, 1990.

[95] E.T. Wittaker and G.N. Watson, *A course of modern analysis*, fourth ed., Cambridge University Press, 1980.

[96] Y.-S. Wu and Y. Yu, *Bosonization of one-dimensional exclusions and characterization of Luttinger liquids*, Phys. Rev. Lett. **75** (1995), 890 – 893.