Dirac Analysis and Integrability of Geodesic Equations for Cylindrically Symmetric Spacetimes

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Abstract. Dirac’s constraint analysis and the symplectic structure of geodesic equations are obtained for the general cylindrically symmetric stationary spacetime. For this metric, using the obtained first order Lagrangian, the geodesic equations of motion are integrated, and found some solutions for Lewis, Levi-Civita, and Van Stockum spacetimes.

PACS numbers: 04.20

12 November 2018

1. Introduction

Dirac first worked out the theory of quantizing constrained systems in general [1], and general relativity in particular [2], and his pioneering work continues to serve as the foundation of current efforts to canonically quantize gravity. Besides the hamiltonian formalism utilizing the Schrödinger representation, there is an alternative Hamiltonian approach which is based on Dirac’s analysis [3] of constrained systems. Many more extensive studies of the subject can be found in the literature; see, for example, Refs. 4 and 5. In this context, we discuss Dirac analysis of geodesic equations for the general cylindrically symmetric stationary spacetimes, and later integrate the obtained first order equations of motion for these spacetimes.

Because of both the mathematical simplicity and the physical relevance to our realistic world [6, 7], space-times with cylindrical symmetry have been extensively studied, and their relativistic applications have been further discussed recently [8–12]. The general form of this metric in vacuum case was given by Lewis [13]. Lewis stationary vacuum metric is usually presented with four parameters [14] which admits a specific physical interpretation when matched to a particular source. These four parameters which are related to topological defects [9, 15] not entering into the expression of the physical components of curvature tensor may be real (Weyl class) or complex (Lewis class). In recent years, the physical meaning of these parameters have been discussed for both classes [9, 10]. The corresponding static limit of the the Lewis class was obtained...
by Levi-Civita (LC)\cite{16}. Even in the simplest case of the LC solution, its physical interpretation is not completely understood, yet. In general, it contains two independent parameters, in which there is only one mass parameter. One of them is associated with the topological defects while the second parameter is connected with the mass per unit length. Another special case of Lewis metric is the van Stockum solution\cite{17} which represents the gravitational field produced by a rigidly rotating dust cylinder with a finite thickness. The matching of this space-time to the vacuum Lewis space-time was also completed in Ref.12, and studied in detail by Bonnor\cite{18}.

The paper is organized as follows. In the next section we present the stationary cylindrical spacetime in general and give the familiar spacetimes at the exterior of the boundary of the source. In Sec. 3, we shall present the symplectic structure of geodesic equations for the general cylindrically symmetric stationary metric. For the first order form of the Lagrangian that yields the geodesic equations for this metric, we shall apply Dirac’s theory of constraints\cite{3} to the degenerate the Lagrangian. We find that the constraints are second class as in the case of all integrable systems. The constraint analysis yields the Dirac brackets, or the Hamiltonian operators in the language of integrable systems. The symplectic 2-form is obtained by the Poison bracket of Dirac’s constraints which is also the inverse of the Hamiltonian operator. In Sec. 4, using the first order Lagrangian for the general cylindrically symmetric metric, the geodesic equations are integrated, and the obtained results are worked in cases of the Lewis spacetime for the Weyl class, the exterior van Stockum and LC spacetimes.

2. Spacetime

The general line element for a cylindrically symmetric stationary spacetime is given by

\[ ds^2 = -f dt^2 + 2kdtd\phi + e^\mu (dr^2 + dz^2) + \ell d\phi^2, \]  

(1)

where \( f, k, \mu \) and \( \ell \) are functions only of \( r \), and \( x^i = (t, r, z, \phi, t), i = 0, 1, 2, 3 \) are the usual cylindrical coordinates with

\[ -\infty \leq t, z \leq \infty, \quad r \geq 0, \quad 0 \leq \phi \leq 2\pi \]  

(2)

and the hypersurfaces \( \phi = 0, 2\pi \) being identified. Einstein’s field equations for vacuum are

\[ R_{ij} = 0. \]  

(3)

The general solution of (3) for (1) is the stationary Lewis metric\cite{14}, which can be written as

\[ f = ar^{-n+1} - \frac{c^2}{n^2a} r^{n+1}, \]  

(4)

\[ k = -Af, \]  

(5)

\[ \ell = \frac{r^2}{f} - A^2f, \]  

(6)

\[ e^\mu = r^{\frac{1}{2}}(n^2-1) \]  

(7)
with

\[ A = \frac{c r^{n+1}}{naf} + b. \]  

(8)

The constants \( n, a, b \) and \( c \) can be either real or complex, the corresponding solutions belong to the Weyl or Lewis classes, respectively. For the Weyl class, the above parameters have the following physical interpretations. The parameter \( n \) is associated with the Newtonian mass per unit length of an uniform line mass \( \sigma \) when it produces the low density regime. The parameter \( a \) is related to the constant arbitrary potential that exist in the corresponding Newtonian solution, while the parameters \( b \) and \( c \) are responsible for the non-staticity of the spacetime, since when we take \( b = 0 \) and \( c = 0 \) the Weyl class reduces to the static LC metric. The parameter \( b \) is related, in locally flat limit, with the angular momentum of a spinning string. The parameter \( c \) measures the vorticity of the source when it represented by a stationary completely anisotropic fluid. For further details see Ref. 9. For the Lewis class, the physical and geometrical meaning of the four parameters of the Lewis metric was given in Ref. 10.

For the line element of the LC static vacuum spacetime in the Weyl form, the functions \( f, k, \mu, \) and \( \ell \) have the following expressions\[6, 9, 19\]

\[ f = r^{4\sigma} \quad k = 0, \quad \ell = C^{-2}r^{2-4\sigma}, \quad e^\mu = r^{4\sigma(2\sigma-1)} \]  

(9)

where \( \sigma \) and \( C \) are two arbitrary constants and both of them are fixed by the internal composition of the physical source. The constant \( C \) refers to the angular defect\[9\], and cannot be removed by scale transformation. This constant is related, in the locally flat limit, with the parameter \( a \) in the Lewis solution for the Weyl class, given by

\[ a = C^2. \]  

(10)

The physical importance of the other parameter \( \sigma \) is mostly understood in accordance with the Newtonian analogy of the LC solution, i.e. the parameter \( \sigma \) represents the mass per unit length\[9, 19\]. The parameter \( \sigma \) is connected to the parameter \( n \) in the Lewis spacetime for the Weyl class as

\[ n = 1 - 4\sigma. \]  

(11)

The functions \( f, k, \mu, \) and \( \ell \) for the van Stockum solution\[17\] are given by

\[ f = 1, \quad k = \alpha r^2, \quad \ell = r^2(1 - \alpha^2 r^2), \quad \mu = -\alpha^2 r^2 \]  

(12)

with \( \alpha \) being an arbitrary positive constant. The energy density and the four velocity of the dust are

\[ \rho = \frac{\alpha^2}{2\pi G} e^{\alpha^2 r^2}, \quad u^\mu = \delta^\mu_4 \]

where \( G \) is the gravitational constant. The angular velocity to the fluid with respect to a locally nonrotating frame is \( \omega = \alpha(1 - \alpha^2 r^2)^{-1} \). Since near the axis, \( \omega \rightarrow \alpha \), it can be interpreted that \( \alpha \) is the angular velocity of the fluid on the axis\[18\].

The van Stockum exterior solution \[12\], which is a particular case of the Lewis metric, contains the globally Minkowski spacetime as a special case. Therefore the
van Stockum solution [12] must be a particular case of the Weyl class [9]. Since the van Stockum spacetime cannot be reduced to the globally static LC metric [9], it is a particular case of the Weyl class with \( b \neq 0 \) and \( c \neq 0 \), since for \( b = 0 \) and \( c = 0 \) the Weyl class can be globally reduced to the static LC metric [9].

3. Dirac Analysis for Geodesic Equations

The equations governing the geodesics can be derived from the lagrangian

\[
2\mathcal{L} = g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}
\]

where \( \tau \) is an affine parameter along the geodesics. From the external problem it emerges the Euler-Lagrange equations

\[
\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0
\]

and from them follow the geodesics given by

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0
\]

where the overdot denotes differentiation with respect to \( \tau \). For spacetime [11] the Lagrangian [13] is

\[
\mathcal{L}_L = \frac{1}{2f} \dot{t}^2 - k \dot{\phi}^2 - \frac{1}{2} e^\mu (\dot{r}^2 + \dot{z}^2) - \frac{1}{2} \ell \dot{\phi}^2.
\]

This Lagrangian is second order and therefore not suitable to a discussion of symplectic structure. For purposes of Hamiltonian analysis we need to start with first order Lagrangian and it can be verified that

\[
\mathcal{L} = -\frac{1}{2f^2} P Q + \frac{1}{2f} \left[ k(P + Q) - D(P - Q) \right] \dot{t} + \frac{1}{2} (P + Q) \dot{\phi}
\]

\[
- \frac{1}{2} e^{-\mu} (M^2 + N^2) + M \dot{r} + N \dot{z}
\]

gives rise to the equations of motion

\[
P = \ell \dot{\phi} + (k + D) \dot{t}, \quad Q = \ell \dot{\phi} + (k - D) \dot{t},
\]

\[
M = e^\mu \dot{r}, \quad N = e^\mu \dot{z},
\]

\[
\dot{M} - \frac{1}{2} e^{-\mu} \mu'(M^2 + N^2) - \frac{\ell}{2f^2} PQ - \frac{(P - Q)}{4D} F = 0,
\]

\[
\left[ \frac{k(P + Q) - D(P - Q)}{\ell} \right] \dot{t} = 0,
\]

\[
\dot{P} + \dot{Q} = 0,
\]

\[
\dot{N} = 0,
\]

which together result equations (A.1-A.3) in Appendix A, where we have defined

\[
D^2 \equiv k^2 + \ell f
\]
and

\[ F \equiv (P + Q) \left( \frac{k}{\ell} \right)' - (P - Q) \left( \frac{D}{\ell} \right)' , \]

and the prime represents derivative with respect to \( r \). Dirac quantization is canonically quantize the original phase space which is usually even dimensional symplectic manifold, and then imposed the gauge constraints as operator conditions on the physical quantum states. In this first order formulation we have introduced \( x^a \equiv (P, Q, M, N) \), \( a = 4, 5, 6, 7 \), as new variables which is double the number required. So we consider a symplectic manifold spanned by variables \( X^A = (x^i, x^a) \), \( A = 0, ..., 7 \), where \( x^i \)'s and \( x^a \)'s are, respectively, spacetime and configuration space variables. Then, first order field equations become

\[ \dot{X}^A = X(X^A) , \quad (24) \]

with the vector field defining the flow

\[
X = \frac{1}{2D}(P - Q) \frac{\delta}{\delta t} + e^{-\mu} \left( M \frac{\delta}{\delta r} + N \frac{\delta}{\delta z} \right) \\
+ \frac{1}{2\ell} \left[ P + Q - \frac{k}{D}(P - Q) \right] \frac{\delta}{\delta \phi} + \frac{M \ell e^{-\mu}}{2D} F \left( \frac{\delta}{\delta P} - \frac{\delta}{\delta Q} \right) \\
+ \left[ \frac{1}{2} e^{-\mu} \mu' (M^2 + N^2) + \ell' P Q + \frac{(P - Q)}{4D} F \right] \frac{\delta}{\delta M} , \quad (25)
\]

for the geodesic equation (15).

The Lagrangian (17) is degenerate because its Hessian

\[
\det \left| \frac{\partial^2 \mathcal{L}}{\partial X^A \partial X^B} \right| = 0 \quad (26)
\]

vanishes identically. Hence it is a system subject to constraints and the passage to its Hamiltonian structure requires the use of Dirac’s theory of constraints[3]. We introduce the canonical momenta of the test particle defined by

\[ \Pi_A \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^A} \quad (27) \]

which cannot be inverted due to equation (26). The definition of the momenta therefore gives rise to the constraints

\[
\Phi_0 = \Pi_t - \left[ \frac{k}{2\ell} (P + Q) - \frac{D}{2\ell} (P - Q) \right] , \\
\Phi_1 = \Pi_r - M , \\
\Phi_2 = \Pi_z - N , \\
\Phi_3 = \Pi_\phi - \frac{1}{2} (P + Q) , \\
\Phi_4 = \Pi_P , \\
\Phi_5 = \Pi_Q , \\
\Phi_6 = \Pi_M , \\
\Phi_7 = \Pi_N , \quad (28)
\]
which must vanish weakly, i.e on shell. In order to determine the class of these constraints we need to obtain the Poisson bracket of the constraints

\[ C_{AB}(\tau, \tilde{\tau}) = \{\Phi_A(\tau), \Phi_B(\tilde{\tau})\} \]  

using the canonical Poisson brackets

\[ \{X^A(\tau), \Pi_B(\tilde{\tau})\} = \delta^A_B \delta(\tau - \tilde{\tau}) \]  

between the dynamical variables and their conjugate momenta. The result

\[ C_{AB}(\tau, \tilde{\tau}) = \frac{1}{2} \begin{pmatrix} 0 & -F & 0 & 0 & -\frac{k+D}{\ell} & -\frac{(k+D)}{\ell} & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \frac{k-D}{\ell} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{k+D}{\ell} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta(\tau - \tilde{\tau}) \]  

shows that the constraints (28) are second class as in the case of all integrable systems\textsuperscript{[22]}.

In order to obtain the Hamiltonian for the degenerate Lagrangian (17) we first construct the free Hamiltonian obtained by Legendre transformation

\[ H_0 = \Pi_A \dot{X}^A - \mathcal{L} = \frac{1}{2\ell} PQ + \frac{1}{2} e^{-\mu} (M^2 + N^2) \]  

and the total Hamiltonian density of Dirac is given by

\[ H_T = H_0 + \lambda^A \Phi_A \]  

where \( \lambda^A \) are Lagrange multipliers. Since we have second class constraints the Lagrange multipliers will be determined from the solution of

\[ \{H_T, \Phi_A\} = 0 \]  

which ensure that the constraints hold for all values of \( \tau \). Since the constraints are linear in the momenta the Lagrange multipliers are given by

\[ \lambda^0 = \frac{1}{2D} (P - Q), \]
\[ \lambda^1 = M e^{-\mu}, \]
\[ \lambda^2 = N e^{-\mu}, \]
\[ \lambda^3 = \frac{1}{2\ell} (P + Q) - \frac{k}{2\ell D} (P - Q), \]
\[ \lambda^4 = \frac{M \ell e^{-\mu}}{2D} F, \]
\[ \lambda^5 = -\lambda^4, \]
\[ \lambda^6 = \frac{\ell}{2\ell^2} P Q + \frac{1}{2} e^{-\mu} \mu' (M^2 + N^2) + \frac{F}{4D} (P - Q), \]
\[ \lambda^7 = 0 \]
which follows directly from the flow \( \text{Eq. 25} \).

The Dirac bracket is a modification of the Poisson bracket designed to vanish on the surface defined by the constraints. For two smooth functionals \( A, B \) of the canonical variables we have

\[
\{ A, B \}_D = \{ A, B \} - \{ A, \Phi_A \} J^{AB} \{ \Phi_B, B \} \tag{36}
\]

where \( J \) is obtained by inverting the matrix of the Poisson bracket of the constraints \( C \)

\[
\int C_{AB}(\tau, \tilde{\tau})J^{BC}(\tilde{\tau}, \tilde{\tau})d\tilde{\tau} = \delta_A^C \delta(\tau - \tilde{\tau}) \tag{37}
\]

The inverse of the Poisson bracket of the constraints is known as the Hamiltonian operator in the literature of integrable systems \[22\]. From Eq. (37) we obtain

\[
J^{AB}(\tau, \tilde{\tau}) = \frac{1}{2}
\begin{pmatrix}
0 & 0 & 0 & 0 & -\ell D & \ell D & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & k + D & -k + D & 0 & 0 \\
0 & 0 & 0 & -\ell D & 0 & 0 & \ell F & 0 \\
0 & 0 & 0 & k - D & 0 & 0 & -\ell F & 0 \\
0 & -1 & 0 & 0 & -\ell F & \ell F & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\delta(\tau - \tilde{\tau}) \tag{38}
\]

and Eqs. (18)-(23) can be written in Hamiltonian form

\[
\dot{X}^A = J^{AB} \frac{\delta H_0}{\delta X_B} \tag{39}
\]

where integration over dotted variables is implied.

The symplectic 2-form is given by \[22\]

\[
\omega_D = \frac{1}{2} \delta X^A \wedge C_{AB} \delta X^B \tag{40}
\]

and using Eq. (31) we find that

\[
\omega_D = \frac{1}{2} \delta P \wedge \delta \phi + \frac{1}{2} \delta Q \wedge \delta \phi + \delta M \wedge \delta r + \delta N \wedge \delta z + \frac{(k - D)}{2\ell} \delta P \wedge \delta \phi + \frac{(k + D)}{2\ell} \delta Q \wedge \delta t + \frac{1}{2} \left[ (P + Q)\left(\frac{k}{\ell}\right)' - (P - Q)\left(\frac{D}{\ell}\right)' \right] \delta r \wedge \delta t \tag{41}
\]

Finally, Hamilton’s equations can be written in the form

\[
i_X \omega_D = -\delta H_0 \tag{42}
\]

where \( i_X \) denotes contraction with respect to the vector field \( \text{Eq. 25} \) of the symplectic 2-form \[11\].

We shall now turn to the Witten-Zuckerman\[20, 21\] formulation of symplectic 2-form vector density \( \omega \) which is closed

\[
\delta \omega = 0 \tag{43}
\]

and conserved

\[
\dot{\omega} = 0. \tag{44}
\]
Starting with the Lagrangian (17) we find that the Witten-Zuckerman symplectic 2-form is the same expression for the symplectic 2-form (41) obtained from Dirac’s theory of constraints, i.e.

\[ \omega = \omega_D. \]

4. Integration

The geodesic equations (15) are four equations for the four unknowns \( \dot{t}, \dot{r}, \dot{z} \) and \( \dot{\phi} \) (see Appendix A). Firstly, from the Eqs. (18) and (19), we obtain

\[
\dot{t} = \frac{1}{2D} (P - Q),
\]
\[
\dot{r} = Me^{-\mu},
\]
\[
\dot{z} = Ne^{-\mu},
\]
\[
\dot{\phi} = \frac{1}{2F} \left[ (P + Q) - \frac{k}{\ell} (P - Q) \right].
\]

Later, integrating Eqs. (21)-(23) we get

\[
N = K,
\]
\[
P + Q = 2L,
\]
\[
\frac{D}{\ell} (P - Q) - \frac{k}{\ell} (P + Q) = 2E,
\]

where \( K, L, \) and \( E \) are integration constants. Using these results in Eq. (20) and rearranging, yields the following Riccati type differential equation

\[
\dot{M} = p(\tau) M^2 + q(\tau)
\]

where \( p \) and \( q \) are given by

\[
p = \frac{\mu'}{2} e^{-\mu},
\]
\[
q = \frac{1}{2} K^2 p + \frac{\ell'}{2 \ell F} \left[ (L f - E k)^2 - E^2 D^2 \right] + \frac{(L k + E \ell)}{2D^2} F,
\]

where \( F \equiv 2 \left[ L (\frac{k}{\ell})' - \frac{(L k + E \ell)}{D} (\frac{p}{q})' \right] \). Then, the solution of Eq. (52) is obtained as

\[
M = \left( \left| \frac{q}{p} \right| \right)^{1/2} U
\]

where the following integral equation must still be satisfied for \( U \),

\[
\int \frac{dU}{U^2 + \beta U + 1} = \int q \left( \frac{p}{q} \right)^{1/2} d\tau
\]

where \( c_1 \) is an integration constant, and \( \beta \) is given by

\[
\beta = \frac{1}{2p} \left( \left| \frac{q}{p} \right| \right)^{1/2} \left( \left| \frac{p}{q} \right| \right).
\]

Using (46) in (55), we find

\[
\dot{r} = e^{-\mu} M.
\]
Thus, we have obtained the generic expression for the radial speed of the test particle. A detailed examination of the solutions of the Eq. (52) is beyond the scope of this paper; therefore, we shall only consider some special orbits here. Let us first assume that \( p = -q = -1 \). In this case, we find the following solutions:

\[
e^{-\mu} = 2r + m,
\]

\[
r(\tau) = \begin{cases} \\
\frac{1}{2} \left[ n^2 e^{2(\tau - \tau_0)} - m \right], & \text{for } M = \pm 1, \\
\frac{1}{2} \left[ n^2 \cosh^2(\tau - \tau_0) - m \right], & \text{for } M = \tanh(\tau - \tau_0), \\
\frac{1}{2} \left[ n^2 \sinh^2(\tau - \tau_0) - m \right], & \text{for } M = \coth(\tau - \tau_0)
\end{cases}
\]

where \( m, n \), and \( \tau_0 \) are integration constants. Second, we assume that \( p = q = b \tau^n \), where \( b \) and \( n \) are nonzero constants. In this case, it follows from Eqs. (52) and (53) that for \( n \neq -1 \)

\[
e^{-\mu} = a \cos^2 \left( \frac{b}{n+1} \tau^{n+1} - \tau_0 \right), \quad M = \tan \left( \frac{b}{n+1} \tau^{n+1} - \tau_0 \right),
\]

\[
r(\tau) = \frac{ab \text{sign}(b) \cos(2\tau_0) \tau^{2+n}}{(1+n)(2+n)\text{sign}(1+n)} \\
\times \ 1F_2 \left[ \{ \frac{1}{2} + \frac{1}{2(1+n)} \}, \{ \frac{3}{2}, \frac{3}{2} + \frac{1}{2(1+n)} \} ; -\frac{b^2 \tau^{2(1+n)}}{(1+n)^2} \right] \\
- \frac{a \sin(2\tau_0) \tau}{2} \ 1F_2 \left[ \{ \frac{1}{2(1+n)} \}, \{ \frac{1}{2}, 1 + \frac{1}{2(1+n)} \} ; -\frac{b^2 \tau^{2(1+n)}}{(1+n)^2} \right]
\]

and for \( n = -1 \)

\[
e^{-\mu} = a \cos^2 \left( b \ln(\tau_0 \tau) \right), \quad M = \tan[b \ln(\tau_0 \tau)],
\]

\[
r(\tau) = \frac{a \tau}{2(1 + 4b^2)} \cos[2b \ln(\tau_0 \tau)] \left[ \tan[2b \ln(\tau_0 \tau)] - 2b \right],
\]

where \( a \) and \( \tau_0 \) are constants of integration, and \( 1F_2 \) is the generalized hypergeometric function.

Considering the Eq. (27), the momenta of the test particle for metric (1) using the Lagrangian (17) is given by

\[
\Pi_r \equiv M, \\
\Pi_z \equiv K, \\
\Pi_\phi \equiv \frac{1}{2}(P + Q) = L, \\
\Pi_\tau \equiv \frac{1}{2\ell} [k(P + Q) - D(P - Q)] = -E
\]

Hence \( E \) can be interpreted as the total energy of the particle, and will be always taken nonnegative. \( K \) can be interpreted as its momentum along \( z \) and \( L \) its angular momentum. In terms of these conserved quantities, \( \dot{i}, \dot{r}, \dot{z} \) and \( \dot{\phi} \) become

\[
\dot{i} = \frac{-(E\ell + Lk)}{D^2},
\]

\[
\dot{r} = \left( \frac{q}{p} \right)^{1/2} e^{-\mu} U,
\]
\[ \dot{z} = Ke^{-\mu}, \]
\[ \dot{\phi} = \frac{Ek - Lf}{D^2}. \]

As is well known, the Eq. (15) has a first integral that is equivalent to \( g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\epsilon \), where \( \epsilon = 0, 1, \) or \(-1\) if the geodesics are respectively null, timelike or spacelike. This implies that
\[ -\epsilon = -fi^2 + 2kt\dot{\phi} + e^\mu (i^2 + \dot{z}^2) + \ell \dot{\phi}^2 \]
so that \( L_L \) in (16) is equal to \( \frac{1}{2} \) along the path. For the Lagrangian (17), this equation becomes
\[ \frac{1}{\ell} PQ + e^{-\mu} (M^2 + N^2) = -\epsilon. \]

Now, substituting (45), (47) and (48) into (70), we have an expression for the radial speed \( \dot{r}^2 \) of the test particle:
\[ \dot{r}^2 = e^{-\mu} [W_0(r) - W(r)], \]
where \( W_0(r) \) and \( W(r) \) are defined as
\[ W_0(r) = E^2 \frac{\ell}{D^2} + 2ELkD^2 - \epsilon, \quad W(r) = L^2 \frac{f}{D^2} + K^2 e^{-\mu}. \]

From the Eqs. (67) and (72), we find
\[ U = \pm e^{\mu/2} \sqrt{\frac{p}{q}} [W_0(r) - W(r)] \]
which enables us to relate Eqs. (67) and (72). On the other hand, setting \( W_0(r) = G(r) [V_0 - V_1(r)] \) and \( W = G(r)V_2(r) \), then Eq. (72) becomes
\[ \dot{r}^2 = e^{-\mu} G(r) [V_0 - V(r)], \]
where \( V = V_1 + V_2 \). Then, one can obtain the functions \( G(r) \) and \( V(r) \), and the constant \( V_0 \) for spacetimes (1) with (4)-(7), (9) and (12) given in Sec. 2.

In the Lewis spacetime for the Weyl class, which means that the parameters \( n, a, b \) and \( c \) appearing in (4)-(7) are real, we obtain \( V_0, V(r) \) and \( G(r) \) as
\[ G(r) = r^{n-1}, \]
\[ V_0 = \frac{1}{an^2} [(Eb + L)c - En]^2, \]
\[ V(r) = cr^{1-n} + a(Eb + L)^2 r^{-2n} + K^2 r^{(1-n)(3+n)/2}. \]

In the general case, i.e. \( b \neq 0 \) and \( c \neq 0 \), we see that the parameter \( c \) only effects \( V_0 \) by modifying the energy of the test particle, and leaving otherwise the geodesics indistinguishable from the static LC spacetime. In two different subcases, when \( b = 0, c \neq 0 \) and \( b \neq 0, c = 0 \), the physical interpretations are given by Herrera and Santos [15]. Furthermore, for the Lewis solution (4)-(7), the differential equation (75) can be written in the integral form as
\[ \int \frac{r^{\frac{n-1}{2}} dr}{\sqrt{\frac{1}{n} [\Gamma^2 - E]^2 - a\Gamma^2 r^{-2n} - \epsilon r^{1-n} - K^2 r^{(1-n)(3+n)}}} = \pm (\tau - \tau_0), \]
where \( \Gamma \equiv Eb + L \), and \( \tau_0 \) is a constant of integration. The \( \pm \) signs in Eq. (79) correspond to outgoing and ingoing geodesics, respectively.

In the case of the LC solution (9), for \( V_0, V(r) \) and \( G(r) \), we have

\[
G(r) = r^{-4\sigma},
\]

\[
V_0 = E^2,
\]

\[
V(r) = cr^{4\sigma} + C^2 L^2 r^{2(4\sigma-1)} + K^2 r^{8\sigma(1-\sigma)}.
\]

Then, it is seen that the parameter \( C \), or \( a \) due to (10), does appear in (77), but does not in (81). This is a different result obtained from the Lewis spacetime for the Weyl class taking \( b = 0 \) and \( c = 0 \), in which the spacetime reduces to the static LC spacetime.

For the van Stockum solution (12), we find that \( V_0, V(r) \) and \( G(r) \) are as follows [24]

\[
G(r) = 1,
\]

\[
V_0 = E^2 + 2\alpha EL - \epsilon,
\]

\[
V(r) = \alpha^2 E^2 r^2 + L^2 r^{-2} + K^2 e^{\alpha^2 r^2}.
\]

In this case, \( V(r) \) is non-negative and, in order to have Eq. (75) meaningful for real \( r \), we must have \( V_0 > 0 \), which is equivalent to

\[
E > (\alpha^2 L^2 + \epsilon)^{1/2} - \alpha L.
\]

Finally, we note that Eqs. (66), (68), (69), and (75) describe the motion of test particles in the background of the general cylindrically symmetric stationary spacetimes. Using these equations, it can easily be obtained the orbits for the particles, and the effective radial potential.

5. Conclusion

In this paper, we have presented the Dirac analysis of geodesic equations of the general cylindrically symmetric stationary spacetimes in explicit form. Using Dirac’s theory of constraints and the covariant Witten-Zuckerman approach we have obtained the Hamiltonian operators. The results for the symplectic 2-form coincide in both of these theories. We note that the original Lagrangian (16) which is second-order is non-degenerate, and gives second-order geodesic equations which are given in Appendix A. However, the first-order Lagrangian (17) is degenerate and produces first-order geodesic equations. Therefore, if we consider this first-order Lagrangian given by (17), then we can easily find the solution of the obtained first-order geodesic equations of motion for the considered spacetimes. In the previous section, Sec. 4 using this degenerate Lagrangian approach, we have integrated the geodesic equations of motion for the general cylindrically symmetric stationary spacetimes, and found some solutions for Lewis, LC, and Van Stockum spacetimes. Also, for the radial speed of the test particle, we have found a generic expression given in (58) which is depend on the solution of Eq. (52). In some spacial cases, we have solved the Eq. (52) and found some solutions for the radial speed of the test particle.
Acknowledgements

This work was done in the Geometry and Integrability Research Semester at Feza Gürsey Institute. I wish to thank Prof. Dr. Y. Nutku for helpful discussions and I would like to express profound gratitude to the Feza Gürsey Institute for the hospitality.

Appendix A. Second-Order Geodesic Equations of Motion

For the spacetime (11), it follows from the Lagrangian (16) that the geodesic equations of motion (15) are

\[ D\ddot{t} + \frac{\ell f' + kk'}{D} \dot{r} + \frac{k\ell' - \ell k'}{D} \dot{r} \dot{\phi} = 0, \quad (A.1) \]

\[ 2\dot{r} + e^{-\mu}(f'\dot{t}^2 - 2k'\dot{t}\dot{\phi} - l'\dot{\phi}^2) = 0, \quad (A.2) \]

\[ \ddot{z} + \mu' \dot{r} \dot{z} = 0, \quad D\ddot{\phi} + \frac{fk' - k\ell'}{D} \dot{r} + \frac{f\ell' + kk'}{D} \dot{r} \dot{\phi} = 0. \quad (A.3) \]

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