EFFECTIVE EQUIDISTRIBUTION OF EXPANDING
TRANSLATES IN THE SPACE OF AFFINE LATTICES

WOOYEON KIM

Abstract. We prove a polynomially effective equidistribution result
for expanding translates in the space of d-dimensional affine lattices for
any $d \geq 2$.

1. Introduction

1.1. Backgrounds. In the theory of unipotent dynamics on homogeneous
spaces, a fundamental result is Ratner’s theorems [Rat91a, Rat91b] on mea-
sure rigidity, topological rigidity, and orbit equidistribution. They have nu-
umerous and diverse applications ranging from number theory to mathemat-
ical physics, see e.g. [EMS98, EM04, EO06, Sha09, Sha10, Mrk10, MS10].
We refer the reader to [Mor05] for expositions and references.

A limitation of Ratner’s results is that they do not tell us an explicit
rate of density or equidistribution for unipotent orbits. As pointed out in
[Mrg00, Problem 7], establishing effective versions of Ratner’s results has
been a central topic in homogeneous dynamics.

We say that a subgroup $H \subset G$ is (unstable) horospherical if there exists
an element $a \in G$ such that

$$H = \left\{ g \in G : a^k g a^{-k} \to e \quad \text{as} \quad k \to -\infty \right\}.$$  

For horospherical subgroups, effective equidistribution results have been
established with a polynomially strong error term [KM96, KM12, Ven10,
DKL16, KSW17, Shi19] using the mixing property of $a$ on $G/\Gamma$ and the so
called “thickening” method, originates in Margulis’ thesis [Mrg04]. We also
refer the reader to the effective equidistribution results for horocycle orbits
in $\mathrm{SL}_2(\mathbb{R})/\Gamma$ [Sar81, Bur90, FF03, Str04, Str13] via direct representation-
theoretic approaches.

Here we state an effective equidistribution result of the horospherical or-
bits in a concrete setting as Theorem 1.1 below; it is closely related to the
main results of the present paper. Let $G = \mathrm{SL}_d(\mathbb{R})$, $\Gamma = \mathrm{SL}_d(\mathbb{Z})$, and
$X = G/\Gamma$ for fixed $d \geq 2$. Let $\{a_t\} < G$ be a 1-parameter diagonal subgroup
$\text{diag}(e^{mt} \text{Id}_m, e^{-mt} \text{Id}_n)$, where $\text{Id}_m$ and $\text{Id}_n$ are $m \times m$ and $n \times n$ identity
matrices, respectively, and $m + n = d$. Then the horospherical subgroup for
a := a_1 in G is indeed expressed as
\[ H = \left\{ \begin{pmatrix} \text{Id}_m & A \\ 0 & \text{Id}_n \end{pmatrix} \in G : A \in \text{Mat}_{m,n}(\mathbb{R}) \right\}. \]

We denote by \( m_H \) the Haar measure of \( H \) and \( m_X \) the \( G \)-invariant probability measure on \( X \). In [KM96], an effective equidistribution theorem of expanding horospherical translates in \( X \) was established as follows.

**Theorem 1.1.** [KM96, Proposition 2.4.8] Let \( V \subset H \) be a fixed neighborhood of the identity in \( H \) with smooth boundary and compact closure. Then there exists a constant \( \delta_0 > 0 \) only depending on \( m \) and \( n \) so that the following holds. For any compact set \( K \subset X \), there exists a constant \( T(K) \geq 0 \) such that
\[ \frac{1}{m_H(V)} \int_V f(a_tux)dm_H(u) = \int_X fdm_X + O(S(f)e^{-\delta_0 t}) \]
for any \( t \geq T(K) \), \( f \in C^\infty_c(X) \), and \( x \in K \). Here, \( S \) is a suitable Sobolev norm of \( C^\infty_c(X) \) and the implied constant depends only on \( m,n \), and \( V \).

For non-horospherical subgroups, several effective equidistribution results have been proved in the last decade. When the ambient group \( G \) is semisimple, an effective equidistribution theorem was established for closed orbits of semisimple group in general homogeneous spaces [EMV09] (see also [AELM20]), and was recently extended to the adelic setting [EMMV20]. Also, effective results for non-horospherical unipotent subgroups are known in some concrete settings [Mo12, LM14, SU15, Ubi16, CY19]. When \( G \) is unipotent, [GT12] settled effective equidistribution of polynomial orbits in nilmanifolds.

In general, let \( \hat{G} = G \times W \) be the ambient group where \( G \) is a semisimple group and \( W \) is the unipotent radical of \( \hat{G} \). If \( \hat{G} \) is neither semisimple nor unipotent, the methods of the aforementioned results do not seem directly applicable for general homogeneous spaces. However, in special settings effective equidistribution results have been obtained e.g. [Str15, BV16] for \( \hat{G} = \text{SL}_2(\mathbb{R}) \rtimes \mathbb{R}^2 \), [SV20] for \( \hat{G} = \text{SL}_2(\mathbb{R}) \rtimes (\mathbb{R}^2)^k \), and [Pri18] for \( \hat{G} = \text{SL}_d(\mathbb{R}) \rtimes \mathbb{R}^d \). The purpose of the present paper is to establish an effective equidistribution result for \( \hat{G} = \text{SL}_d(\mathbb{R}) \rtimes \mathbb{R}^d \).

**1.2. Main results.** For fixed \( d \geq 2 \), denote
\[ G = \text{SL}_d(\mathbb{R}), \quad \hat{G} = \text{SL}_d(\mathbb{R}) \rtimes \mathbb{R}^d, \]
\[ \Gamma = \text{SL}_d(\mathbb{Z}), \quad \hat{\Gamma} = \text{SL}_d(\mathbb{Z}) \rtimes \mathbb{Z}^d = \text{Stab}_G(\mathbb{Z}^d). \]

We define \( X = G/\Gamma \) and \( Y = \hat{G}/\hat{\Gamma} \). It is sometimes convenient to view \( \hat{G} \) as a subgroup of \( \text{SL}_{d+1}(\mathbb{R}) \) by \( \hat{G} = \left\{ \begin{pmatrix} g & b \\ 0 & 1 \end{pmatrix} : g \in \text{SL}_d(\mathbb{R}), b \in \mathbb{R}^d \right\} \). Then there exists a natural projection \( \pi : Y \to X \) sending \( \begin{pmatrix} g & b \\ 0 & 1 \end{pmatrix} \hat{\Gamma} \in Y \) to
\[ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Gamma \in X. \] We denote by \( W \simeq \mathbb{R}^d \) the unipotent radical of \( \hat{G} \), which consists of all translations on \( \mathbb{R}^d \). Since \( W = \left\{ w(b) := \begin{pmatrix} \text{Id}_d & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}^d \right\} \) acts simply transitively on \( \mathbb{R}^d \), we can identify \( \hat{G}/W \simeq G \). We take a lift of the element \( g \in G \) to \( \hat{G} \subset SL_{d+1}(\mathbb{R}) \) given by \( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \) and by abuse of notation we denote it again by \( g \). Following this notation, we have a relation \( gw(b) = w(gb)g \) for \( g \in G \) and \( b \in \mathbb{R}^d \). For \( g \Gamma \in X \) and \( b \in \mathbb{R}^d \), \( gw(b)\hat{\Gamma} \) is invariant under the integer translations of \( b \). Thus, the fiber \( \pi^{-1}(g\Gamma) \) can be seen as \( \left\{ gw(b)\hat{\Gamma} : b \in \mathbb{T}^d \right\} \).

Let \( \{a_t\} \) be a 1-parameter diagonal subgroup of \( G \) defined by

\[
    a_t = \text{diag} \left( e^{nt} \text{Id}_n, e^{-mt} \text{Id}_n \right),
\]

\( H \) be the unstable horospherical subgroup in \( G \) for \( a = a_1 \), and \( m_H \) be the Haar measure of \( H \) as before. We remark that \( H \) is a full horospherical subgroup in \( G \), but not in \( \hat{G} \), which renders the methods in the proof of Theorem 1.1 irrelevant. We denote by \( m_X \) the \( G \)-invariant probability measure on \( X \) and \( m_Y \) the \( \hat{G} \)-invariant probability measure on \( Y \). Let \( V \subset H \) be a fixed neighborhood of the identity in \( H \) with smooth boundary and compact closure. For \( y \in Y \) and \( t \geq 0 \), \( a_tVy \subset Y \) is a lift of a piece of a full horospherical orbit \( a_tVx \subset X \), where \( x = \pi(y) \in X \). In this paper, we study the equidistribution of the expanding translates \( \{a_tVy\}_{t \geq 0} \) in \( Y \) as \( t \to \infty \).

For any positive integer \( q \), set

\[
    X_q := \left\{ gw(b)\hat{\Gamma} \in Y : g \in G, b \in \mathbb{Q}^d, q(b) = q \right\},
\]

where for any \( b \in \mathbb{Q}^d \), \( q(b) \) denote its common denominator, i.e. the smallest \( q \in \mathbb{N} \) such that \( b \in q^{-1}\mathbb{Z}^d \). Clearly \( X_q \) is a closed \( G \)-invariant subset of \( Y \) for any \( q \). We denote by \( m_X \) the \( G \)-invariant probability measure on \( X_q \).

We start by discussing the ineffective equidistribution of \( \{a_tVy\}_{t \geq 0} \) in \( Y \). For \( y \in Y \) and \( t \geq 0 \), we denote by \( \mu_{y,t} \in \mathcal{P}(Y) \) the normalized probability measure on the orbit \( a_tVy \), i.e. \( \mu_{y,t}(f) := \frac{1}{m_{a_tVy}(f)} \int_Y f(a_tuy)dm_H(u) \) for any \( f \in C_b(Y) \). The following equidistribution result is a special case of \cite[Thm. 1.4]{Sha96}:

\[
\mu_{y,t} \xrightarrow{w} m_{Y} \quad \text{for all } y \notin \bigcup_{q \in \mathbb{N}} X_q,
\]

\[
\mu_{y,t} \xrightarrow{w} m_{X_q} \quad \text{for all } y \in X_q,
\]

as \( t \to \infty \). We refer to \cite[proof of Thm. 5.2]{MS10} and \cite[§1.1]{Str15} for the proof and more detailed discussions. Here we just make a remark that the proof of \cite[Thm. 1.4]{Sha96} crucially relies on Ratner’s measure classification theorem.
The main result of the present paper is Theorem 1.3 below, which is an effective refinement of (1.4). Observe that the weak* limit of \( \{ \mu_{y,t} \}_{t \geq 0} \) is determined by the Diophantine properties of the initial point \( y \). Also, the condition \( y \notin \bigcup_{q \in \mathbb{N}} X_q \) in (1.4) is equivalent to \( b \in \mathbb{T}^d \setminus \mathbb{Q}^d \), where \( y = gw(b)\hat{\Gamma} \). Based on these observations, we can expect that the rate of convergence in (1.4) might also depend on the Diophantine properties of \( y \), or equivalently that of \( b \).

We say that a vector \( b \in \mathbb{T}^d \) is of (Diophantine) type \( M \geq 1 \) if there exists \( c > 0 \) such that \( |b - \frac{p}{q}| > cq^{-M} \) for all \( q \in \mathbb{N} \) and \( p \in \mathbb{Z}^d \). We first state a simplified version of the main result.

**Theorem 1.2.** Let \( V \subset H \) be a fixed neighborhood of the identity in \( H \) with smooth boundary and compact closure. Then there exists a constant \( \delta > 0 \) only depending on \( m \) and \( n \) such that

\[
\frac{1}{m_H(V)} \int_V f(a_1 uy) dm_H(u) = \int_Y f dm_Y + O \left( S(f) e^{-\frac{\delta t}{M+1}} \right)
\]

for any \( t \geq 0, f \in C^\infty_c(Y) \), and \( y = gw(b)\hat{\Gamma} \), where \( b \in \mathbb{T}^d \) is of Diophantine type \( M \). Here, \( S \) is a suitable Sobolev norm of \( C^\infty_c(Y) \) and the implied constant depends only on \( m, n, V, \) and \( y \).

More generally, in order to express the Diophantine properties of \( b \) in a quantitative sense, let us define a function \( \zeta : (\mathbb{T}^d \setminus \mathbb{Q}^d) \times \mathbb{R}^+ \to \mathbb{N} \) by

\[
\zeta(b, T) := \min \left\{ N \in \mathbb{N} : \min_{1 \leq q \leq N} \|qb\| \leq \frac{N^2}{T} \right\},
\]

where \( \| \cdot \|_{\mathbb{Z}} \) denotes the supremum distance from \( 0 \in \mathbb{T}^d \). We note that \( \zeta \) is non-decreasing, unbounded, and has the following properties:

\[
\zeta(b, cT) \leq [c^\frac{1}{c} \zeta(b, T)],
\]

(1.8)

\[
\zeta(b, \| \gamma^{-1} \|_{op}^{-1} T) \leq \zeta(\gamma b, T) \leq \zeta(b, \| \gamma \|_{op} T),
\]

(1.9)

\[
\zeta(b, T) \leq [T^{\frac{d}{3M+1}}].
\]

for any \( \gamma \in \Gamma \) and \( c > 1 \). Here \( \| \cdot \|_{op} \) denotes the operator norm of \( G \) w.r.t. the supremum norm on \( \mathbb{R}^d \). The inequality (1.9) is equivalent to Dirichlet’s approximation theorem.

**Theorem 1.3.** Let \( V \) and \( S \) be as in Theorem 1.2. Then there exists a constant \( \delta' > 0 \) only depending on \( m \) and \( n \) such that

\[
\frac{1}{m_H(V)} \int_V f(a_1 uy) dm_H(u) = \int_Y f dm_Y + O \left( S(f) \zeta(b, e^{\frac{\delta t}{M}})^{-\delta'} \right)
\]

for any \( t \geq 0, f \in C^\infty_c(Y) \), and \( y = gw(b)\hat{\Gamma} \) with \( \| g \| \leq \zeta(b, e^{\frac{\delta t}{M}})^{\delta'} \) and \( b \in \mathbb{T}^d \). Here, the implied constant depends only on \( m, n \) and \( V \).
By definition of $\zeta$, we have
\begin{equation}
\zeta(b, T) \gg T^{1+\epsilon}
\end{equation}
for any $T > 0$ and $b$ of Diophantine type $M$. Hence, Theorem 1.3 directly implies Theorem 1.2 with $\delta = \frac{\omega_d}{2}$. Moreover, Theorem 1.3 implies (1.4) directly as $\zeta(b, T) \to \infty$ as $T \to \infty$ for any irrational $b \in \mathbb{T}^d$.

We give more general versions of Theorem 1.2 and Theorem 1.3 for other diagonal subgroups in Section 5.

Strömbergsson [Str15] proved Theorem 1.3 for $\text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 / \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ using Fourier analysis and methods of from analytic number theory, and later Prinyasart [Pri18] obtained an effective equidistribution result in $\text{SL}_3(\mathbb{R}) \ltimes \mathbb{R}^3 / \text{SL}_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ building on similar methods. In particular, their proofs rely on delicate analysis on the explicit expressions of unitary representations of $\text{SL}_2(\mathbb{R})$ or $\text{SL}_3(\mathbb{R})$ and cancellations from Kloosterman sums. In order to prove Theorem 1.3 for arbitrary dimension $d \geq 2$, we will use a different approach from the previous works. The method in the present paper is geometric in a sense that we do not rely on representation theoretic properties of the semisimple component $G = \text{SL}_d(\mathbb{R})$. It proceeds via a geometric decomposition of $X = G/T$ and an analysis on the Fourier coefficients of probability measures on $\mathbb{T}^d$. We will use the effective equidistribution result in $X$ (Theorem 1.1 above), and a strategy which is similar in spirit to the idea given in [BFLM11].

1.3. Sketch of the proof of Theorem 1.3. We fix an initial point $y_0 = g_0 w(b_0) \Gamma$ and let $\mu_t \in \mathcal{P}(Y)$ be the normalized probability measure on the orbit $a_t Y y_0$, for any $t \geq 0$. Then Theorem 1.3 is equivalent to obtaining an upper bound of $|\mu_t(f) - m_Y(f)|$ for $f \in C_c^\infty(Y)$.

For any $x \in X$, $\pi^{-1}(x) \subset Y$ can be identified with $\mathbb{T}^d$, so $Y$ is a torus bundle of $X$. Moreover, if we fix a fundamental domain $F \subset G$ of $X$, any $y \in Y$ can be uniquely parametrized as $y = gw(b) \Gamma$, where $g \in F$ and $b \in \mathbb{T}^d$. Hence, we have a bijective measurable map $y \mapsto (g \Gamma, b)$ between $Y$ and $X \times \mathbb{T}^d$. The projection to $X$ of this map is the canonical projection $\pi : Y \to X$. We denote by $\sigma : Y \to \mathbb{T}^d$ the projection to $\mathbb{T}^d$ of this map, which depends on the choice of $F$.

We first decompose the orbit $a_t Y y_0 \subset Y$ to small pieces so that each piece of orbit is approximately on the same fiber torus. To simplify notations assume that $b_0 \in \mathbb{T}^d$ is of Diophantine type $M$. Let $\{\omega_i\}_{i \in I}$ be a partition of unity of $X$ such that each $\omega_i$ is supported on a ball $B^X(z_i, r)$, where $z_i \in X$, $r \asymp e^{-\kappa M}$, and $\kappa > 0$ will be an appropriately chosen small constant. Then the measure $\mu_t$ is decomposed by $\sum_i \pi^* \mu_t(\omega_i) \mu_{t, \omega_i}$, where $\mu_{t, \omega_i}(f) = \pi_* \mu_t(\omega_i)^{-1} \int f \omega_i \circ \pi d\mu_t$. Considering $\mu_{t, \omega_i}$ as a measure on $B^X(z_i, r) \times \mathbb{T}^d$, we may also look at $\nu_{t, \omega_i} := \sigma_* \mu_{t, \omega_i}$ which is a probability measure on $\mathbb{T}^d$ (approximating $\mu_{t, \omega_i}$ up to error $\ll r$).

Hence to prove the effective equidistribution in $Y$, it is enough to show that for each $\omega$, the measure $\nu_{t, \omega}$ effectively equidistribute toward the
Lebesgue measure on the torus. In other words, it suffices to prove that any nontrivial Fourier coefficient \( \hat{\nu}(\mathbf{m}) \) is small for \( \mathbf{m} \in \mathbb{Z}^d \setminus \{0\} \). Most of this paper will be devoted to prove the decay of nontrivial Fourier coefficients, Proposition 4.5.

Our basic approach to prove Proposition 4.5 is viewing the horospherical action in \( Y \) as the combination of the horospherical action in \( X \) and a discrete \( \Gamma = \text{SL}_d(\mathbb{Z}) \)-action on \( \mathbb{T}^d \). For example, for \( s \geq 0 \), \( u \in V \), and \( x = g\Gamma \in X \) such that \( g \in F \), there uniquely exist \( \xi_s(x, u) \in F \) and \( \gamma_s(x, u) \in \Gamma \) such that \( a_s u g = \xi_s(x, u) \gamma_s(x, u) \). Then for \( y = gw(b) \Gamma \), a point on the horospherical orbit \( a_s V y \) is described by \( a_s u y = \xi_s(x, u) w(\gamma_s(x, u) b) \Gamma \). Observe that the dynamics on the torus part is given by \( \text{SL}_d(\mathbb{Z}) \)-action.

We already have an effective equidistribution result in \( X \) thanks to [KM96]: expanding horospherical translates in \( X \) are polynomially effectively equidistributed by Theorem 1.1. Our main argument to deal with the dynamics on the torus is inspired by [BFLM11], which proved a quantitative equidistribution result for \( \text{SL}_d(\mathbb{Z}) \)-random walk on the torus. Following the strategy of [BFLM11] (see also [HdS19, HLL20]), we will prove:

1. (Proposition 3.5) If \( |\hat{\nu}(\mathbf{m})| \) is large for \( \mathbf{m} \in \mathbb{Z}^d \setminus \{0\} \), then for appropriately chosen \( s < t \), we can find a measure \( \nu \in \mathcal{P}(\mathbb{T}^d) \) such that \( \nu \ll \nu_{t-s} \) and \( \nu \) has a rich set of Fourier coefficients which are larger than a polynomial in \( |\hat{\nu}(\mathbf{m})| \).

2. (Proposition 4.4) If the torus-coordinate \( b_0 \in \mathbb{T}^d \) of the initial point \( y_0 \in Y \) is not well-approximated by rationals of low denominator, then \( \nu_{t-s} \) cannot be highly concentrated on a \( \rho \)-radius ball, i.e., \( \nu_{t-s}(B(\mathbb{T}^d(p, \rho))) \ll \rho^{2\kappa'} \) for any \( \rho \in \mathbb{T}^d \), where \( \rho \approx e^{-\frac{\kappa'(t-s)}{M+1}} \) for an appropriately chosen constant \( \kappa', \kappa'' > 0 \).

Then Proposition 4.5 is obtained from (1) and (2) as follows. Making use of [BFLM11, Proposition 7.5] which is a quantitative analogue of Wiener’s lemma, we deduce from Proposition 3.5 that if \( |\hat{\nu}(\mathbf{m})| \) is large, then \( \nu_{t-s} \) is highly concentrated, namely \( \nu_{t-s}(B(\mathbb{T}^d(p, \rho))) \geq \rho^{\kappa'} \) for some \( p \in \mathbb{T}^d \) (Proposition 4.2). Combining this with Proposition 4.4, we conclude that \( |\hat{\nu}(\mathbf{m})| \) must be small if \( b_0 \) is not well-approximated by rationals of low denominator, i.e. Proposition 4.5.

We now briefly explain how to prove (1) and (2) above. In the setting of random walks as [BFLM11], the dynamics on \( \mathbb{T}^d \) is determined by the law of the random walk. In the case of horospherical translates, the function \( \gamma_s(x, u) \) defined above plays a similar role as the law of the random walk. Proposition 3.3 describes the dynamics on \( \mathbb{Z}^d \) induced by \( \gamma_s(x, u) \), and is a crucial ingredient to prove both (1) and (2). It asserts that for any \( \mathbf{m} \in \mathbb{Z}^d \setminus \{0\} \) and \( x \in X \), the orbit \( \gamma_s(x, u)^{\dagger} \mathbf{m} \) with respect to \( u \in V \) is well-separated on \( \mathbb{Z}^d \).
For (1), the Fourier coefficient \( \widehat{\nu_t}(m_0) \) can be expressed as the average of Fourier coefficients at this \( \Gamma^{tr} \)-orbit, so well-separateness of such \( \Gamma^{tr} \)-orbit implies a rich set of large Fourier coefficients. Unlike \([BFLM11]\) which studied the random walk on a fixed torus, a technical difficulty in our setting is that the function \( \gamma_s(x,u) \) and the fiber tori \( \pi^{-1}(x) \) varies with the base point \( x \in X \). For this reason we decompose the measure \( \mu_{t,s} \) again with \( r' \ll \text{re}^{-3ds} \)-radius balls in \( X \) so that \( \gamma_s(x,u) \) is stable with respect to \( x \) in each balls.

For (2), we have the well-separateness of the orbit \( \gamma_{t-s}(x,u)^{tr}e_1 \) from Proposition 3.3, and it deduces the well-separateness of the first coordinate of the orbit \( \gamma_{t-s}(x,u)b_0 \) under some Diophantine condition of \( b_0 \). It implies that the measure \( \nu_{t,s} \) cannot be highly concentrated, so we obtain (2). An effective version of well-known Weyl’s equidistribution criterion for an irrational rotation on the one-dimensional torus will be needed, and here is the place where we use the Diophantine properties of \( b_0 \).

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2. Preliminaries

Throughout this paper, we fix the dimension \( d = m + n \), the diagonal group \( a_t = \text{diag} \left( e^{nt}Id_m, e^{-mt}Id_n \right) \), and the neighborhood \( V \subset H \). We use the notation \( \kappa_1, \kappa_2, \cdots \) to denote positive exponents that depend only on the dimensions \( m, n \). We shall use the standard notation \( A \ll B \) or \( A = O(B) \) to mean that \( A \leq CB \) for some constant \( C > 0 \) that depends only on \( m, n \) and \( V \subset H \). We shall also write \( A \asymp B \) to mean both \( A \ll B \) and \( B \ll A \). The notation \( C_1, C_2, \cdots \) will denote positive constants that depend only on \( m, n \) and \( V \).

2.1. Metrics and norms. Let \( d^X(\cdot,\cdot) \) be a right invariant Riemannian metric on \( \hat{G} \). Then this metric induces metrics on \( G, Y, \) and \( X \). We denote by \( B^G(\hat{g}, r), B^G(g,r), B^Y(y,r), \) and \( B^X(x, r) \) the ball with radius \( r \) and the center \( \hat{g} \in \hat{G}, g \in G, y \in Y, \) and \( x \in X \) with respect to the metric on \( G, G, Y, X \), respectively. Also, we set \( \|g\| := \max_{1 \leq i,j \leq d} \left( |g_{ij}|, |(g^{-1})_{ij}| \right) \) for \( g \in G \).

Then for some constants \( C_1, C_2 > 0 \), we have (See [EMV09, §3.3]):

\[
\|g^{-1}\| = \|g\| = \|g^{tr}\|, \quad \|g_1g_2\| \leq C_1 \|g_1\| \|g_2\|, \quad (2.1)
\]

\[
\|g(x_1, x_2)\| \leq C_2 \|g\|^2 d^X(x_1, x_2), \quad \|g(x_1, x_2)\| \leq C_2 \|g\|^2 d^Y(y_1, y_2) \quad (2.2)
\]

for any \( g \in G, x_1, x_2 \in X, y_1, y_2 \in Y \). For \( v = (v_1, \cdots, v_d) \in \mathbb{R}^d \), we use the supremum norm \( \|v\| = \max_{1 \leq i \leq d} |v_i| \). We denote by \( \|g\|_{\text{op}} \) the operator norm of
$g \in G$ with respect to the supremum norm of $\mathbb{R}^d$. Then for some $C_3 > 1$,
\begin{equation}
\|g v\| \leq \|g\|_\infty \|v\|, \quad \|g\|_\infty \leq C_3 \max_{1 \leq i, j \leq d} |(g_{ij})| \leq C_3 \|g\|
\end{equation}
for any $g \in G$ and $v \in \mathbb{R}^d$.

2.2. Compactness, height, and injectivity. For $x \in X$ we set:
\begin{equation}
ht(x) \overset{\text{def}}{=} \sup \left\{ \| g v \|^{-1} : x = g \Gamma, v \in \mathbb{Z}^d \setminus \{0\} \right\}
\end{equation}
\begin{equation}
K(\epsilon) \overset{\text{def}}{=} \left\{ x \in X : \ht(x) \leq \epsilon^{-1} \right\}.
\end{equation}
Note that $\ht(x) \geq 1$ for all $x \in X$. By Mahler’s compact criterion, $K(\epsilon)$ is a compact set of $X$ for all $\epsilon > 0$. Moreover, one can show that
\begin{equation}
m_X(X \setminus K(\epsilon)) \asymp \epsilon^d
\end{equation}
for $\epsilon > 0$ by using the following Siegel’s integral formula [Sie45]:
\begin{equation}
\int_X \tilde{f} dm_X = \int_{\mathbb{R}^d} f dm_{\mathbb{R}^d}
\end{equation}
for any bounded any compactly supported function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, where $\tilde{f} : X \to \mathbb{R}_{\geq 0}$ is the Siegel transform of $f$ defined by $\tilde{f}(g \Gamma) = \sum_{v \in g \mathbb{Z}^d \setminus \{0\}} f(g v)$.

Applying the Siegel’s integral formula with the characteristic function of $\epsilon$-radius ball centered at $0$ in $\mathbb{R}^d$, (2.4) follows.

We also have the following estimate of the injectivity radius of $K(\epsilon)$ from [KM12, Proposition 3.5]. There exists a constant $C_4 > 0$ such that for any $0 < r < \frac{1}{2}$ and $x \in K(C_4 r^{\frac{1}{2}})$, the map $g \mapsto gx$ is injective on the ball $B^G(id, r)$. In other words, there exist a bijective isometry between $B^G(id, r) x$ and $B^X(x, r)$.

2.3. Sobolev norms. Fix a basis $\mathcal{B}$ for the Lie algebra $\mathfrak{g}$ of $G$, and a basis $\hat{\mathcal{B}}$ for the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ extended from $\mathcal{B}$. The basis $\hat{\mathcal{B}}$ defines differentiation action of $\hat{\mathfrak{g}}$ on $C_c^\infty(X)$ by $Z f(x) = \frac{d}{dt}f(\exp(tZ)x)|_{t=0}$ for $f \in C_c^\infty(X)$ and $Z \in \hat{\mathcal{B}}$. The differentiation action of $\mathfrak{g}$ on $C_c^\infty(Y)$ is defined similarly. We denote by $\nabla f$ the gradient vector field on $G$ with respect to the basis $\mathcal{B}$. Following [EMV09, §3.7], we define $L^2$-Sobolev norms on $C_c^\infty(X)$ and $C_c^\infty(Y)$ for $k \in \mathbb{N}$ as
\begin{equation}
S_k^X(f)^2 := \sum_{D} \ht(x)^k \|Df\|_{L^2}^2,
\end{equation}
\begin{equation}
S_k^Y(f)^2 := \sum_{D} \ht(y)^k \|Df\|_{L^2}^2,
\end{equation}
where $D$ ranges over all monomials in $\mathcal{B}$, $\hat{\mathcal{B}}$ of degree $\leq k$, respectively. Let $l_0$ be an integer such that Theorem 1.1 holds with the Sobolev norm $S^X = S_{l_0}^X$. Throughout this paper, we fix a sufficiently large integer $l \in \mathbb{N}$.
and the corresponding $l$-th degree Sobolev norm $S = S^l_f$ on $C^\infty_c(Y)$ with the following properties: for $f \in C^\infty_c(Y)$,

\[(2.5) \quad S^X(\mathcal{F}) \leq S(f), \quad \|Df\|_{L^\infty(Y)} \leq S(f)\]

for $\mathcal{D}$ of degree $\leq d + 2$. The function $\mathcal{F} \in C^\infty_c(X)$ is the average function over a fiber defined by $\mathcal{F}(x) = \int_{\pi^{-1}(x)} f(y) dm_{\pi^{-1}(x)}(y)$ for $x \in X$, where $m_{\pi^{-1}(x)}$ is the normalized Haar measure of $\pi^{-1}(x)$. Note that one can view $\pi^{-1}(x)$ and $m_{\pi^{-1}(x)}$ as the torus $\mathbb{T}^d$ and the Lebesgue measure on the torus.

2.4. Effective equidistribution in $X$ and measure estimates. Considering the dependence on the initial point $x_0 \in X$, Theorem 1.1 indeed can be formulated as follows: there exists a constant $\kappa_1 > 0$ such that

\[(2.6) \quad \left| \frac{1}{m_H(V)} \int_V f(a_t u x_0) dm_H(u) - \int_X f dm_X \right| \ll ht(x_0)^{\kappa_1} S^X(f) e^{-\delta_0 t}\]

for any $t > 0$, $x_0 \in X$ and $f \in C^\infty_c(X)$. Keeping track of the dependence on $x_0 \in X$ in the proof of [KM96], one can check that the error bound polynomially depends on the injectivity radius of $x_0$, which also polynomially depends on $ht(x_0)$.

Recall that $\mu_{y,t}$ denote the normalized orbit measure on $a_t V y \subset Y$, and $\pi_* \mu_{y,t}$ is the normalized horospherical orbit measure on $a_t V x \subset X$, where $x = \pi(y)$. Thus, (2.6) enables us to estimate the measure of a small ball in $X$ with respect to $\pi_* \mu_{y,t}$: there exist $0 < \kappa_2 \leq \frac{1}{2}$ such that for any $e^{-\kappa_2 t} < r < \frac{1}{2}$ and $x \in X(C_4 r^{\frac{1}{4}})$,

\[(2.7) \quad \pi_* \mu_{y,t}(B^X(x, r)) \asymp r^{d-1}\]

if $ht(\pi(y)) < e^{\kappa_2 t}$. This estimate is from the fact $m_X(B^X(x, r)) \asymp r^{d-1}$ for $x \in X(C_4 r^{\frac{1}{4}})$ and the use of (2.6) with an approximating function of $B^X(x, r)$.

Also, we can estimate how much measure appears near the cusp with respect to $\pi_* \mu_{y,t}$. We may assume that $\kappa_2 > 0$ was taken sufficiently small so that for any $e^{-\kappa_2 t} < \epsilon < \frac{1}{2}$,

\[(2.8) \quad \pi_* \mu_{y,t}(X \backslash K(\epsilon)) \asymp m_X(X \backslash K(\epsilon)) \asymp \epsilon^d\]

if $ht(\pi(y)) < e^{\kappa_2 t}$. For this estimate, we used (2.4) and (2.6) with an approximating function of $K(\epsilon)$.

2.5. The Horospherical subgroup. For the unstable horospherical subgroup $H < G$ and the fixed neighborhood of identity $V \subset H$, $a_t V a_{-t}$ can be considered as the $e^{(m+n)t}$-dilated set of $V$. Thus, we have $m_H(a_t V a_{-t}) = e^{(m+n)mt} m_H(V)$ for any $t \in \mathbb{R}$, since $\dim H = mn$. Also, this expanding property provides us an inductive structure of the measures of expanding translates $\{\mu_{y,t}\}_{t \geq 0}$ as follows, where $y$ is fixed.
Proposition 2.1. For any $0 \leq s < t$, $u_0 \in V$, $y \in Y$ and $f \in L^\infty(Y)$, we have

$$\mu_{y,t}(f) = (a_su_0)_*\mu_{y,t-s}(f) + O(\|f\|_{L^\infty} e^{-(m+n)(t-s)}).$$

Proof. First, we have

$$(a_su_0)_*\mu_{y,t-s}(f) = \frac{1}{m_H(V)} \int_V f(a_su_0a_{t-s}uy)dm_H(u)$$
$$= \frac{1}{m_H(V)} \int_V f(a_{-(t-s)}u_0a_{(t-s)}uy)dm_H(u).$$

Denote by $\triangle$ the symmetric difference $A\triangle B = (A\setminus B) \cup (B\setminus A)$. Let $u' = a_{-(t-s)}u_0a_{(t-s)}$. Note that $a_{-(t-s)}Va_{(t-s)}$ is the $e^{(m+n)(t-s)}$-contracted set of $V$ and $u' \in a_{-(t-s)}Va_{(t-s)}$. Since $V$ has a smooth boundary, it follows that $m_H(u'\triangle V) \ll e^{-(m+n)(t-s)}$. By a change of variable, we get

$$(a_su_0)_*\mu_{y,t-s}(f) = \frac{1}{m_H(V)} \int_{u'V} f(a_{t}uy)dm_H(u)$$
$$= \frac{1}{m_H(V)} \left( \int_{V} f(a_{t}uy)dm_H(u) + O(\|f\|_{L^\infty} m_H(u'\triangle V)) \right)$$
$$= \mu_{y,t}(f) + O(\|f\|_{L^\infty} e^{-(m+n)(t-s)}).$$

2.6. A fundamental domain of $X = \text{SL}_d(\mathbb{R})/\text{SL}_d(\mathbb{Z})$. We will fix a fundamental domain of $X$ in order to parametrize elements in $Y$. The standard Siegel domain has been commonly used as a fundamental domain of $\text{SL}_d(\mathbb{R})/\text{SL}_d(\mathbb{Z})$. For $d \geq 3$, the standard Siegel domains are a weak fundamental domain which allows finite numbers of $g$ satisfying $x = g\Gamma$ for fixed $x \in X$, but it is not sufficient for our purpose. To avoid technical complexity, we construct a strong fundamental domain $\mathcal{F}$ as follows in order to allow unique $g \in \mathcal{F}$ satisfying $x = g\Gamma$. We define a continuous function $F : G \to \mathbb{R}_{>0}$ by

$$F(g)^2 = \frac{\sum_{i,j} |g_{ij}|^2(\sum_{i,j} |(g^{-1})_{ij}|^2)}{\sum_{i,j} |g_{ij}|^2 + \sum_{i,j} |(g^{-1})_{ij}|^2},$$

where $g_{ij}$’s are matrix components of $g \in \text{SL}_d(\mathbb{R})$. Hereafter, we fix a connected fundamental domain $\mathcal{F} \subset G$ of $X$ which consists of every elements in \{ $g \in G : F(g) < F(g\gamma)$ for all $\gamma \in \Gamma$ \} and properly chosen elements from its boundary. Denote by $\phi : G \to X$ the canonical projection. Then $\mathcal{F}$ satisfies the following properties. See Appendix for the concrete construction and the proofs.

1. For any $x \in X$, there uniquely exists $g \in \mathcal{F}$ such that $x = g\Gamma$. Moreover, we can define a measure-preserving map $\iota : X \to \mathcal{F}$ such that $\phi \circ \iota = \text{id}_{X \to X}$ and $\iota|_{\phi(\mathcal{F}^\circ)}$ is continuous, where $\mathcal{F}^\circ$ denotes the interior of $\mathcal{F}$. We also have $m_G(\mathcal{F}) = m_G(\mathcal{F}^\circ) = m_X(\phi(\mathcal{F}^\circ)) = m_X(X) = 1.$
(2) There exists $C_5 > 0$ such that for any $x \in X$,
\begin{equation}
\|\iota(x)\| \leq C_5 \text{ht}(x)^{d-1}.
\end{equation}
Hence, for any $\epsilon > 0$,
\begin{equation}
m_G\left(\{g \in \mathcal{F} : \|g\| > \epsilon^{-1}\}\right) \leq m_X\left(\left\{x \in X : \text{ht}(x) > C_5^{-1} \epsilon^{\frac{1}{d-1}}\right\}\right) \leq C_6 \epsilon^{d-1}
\end{equation}
for some constant $C_6 > 0$ by (2.4).
(3) For $r > 0$ and $\epsilon > 0$, let $\mathcal{F}(r, \epsilon) := \{g \in \mathcal{F} : \text{ht}(g \Gamma) \leq \epsilon^{-1}, \text{dist}(g, \partial \mathcal{F}) \geq r\}$, where $\partial \mathcal{F}$ denotes the boundary of $\mathcal{F}$. Then there exist constants $0 < \kappa_3 \leq 1$ and $C_7 > 0$ satisfying
\begin{equation}
m_G(\mathcal{F} \setminus \mathcal{F}(r, \epsilon)) \leq C_7 \max(r^{\kappa_3}, \epsilon^d)
\end{equation}
for any $r > 0$ and $\epsilon > 0$.

2.7. Parametrization of $Y$. For a point $y \in Y$, it is often convenient to view $y$ as a point on the fiber $\pi^{-1}(x)$, where $x = \pi(y)$. The fiber $\pi^{-1}(x)$ can be identified with the torus $\mathbb{T}^d$ by a map $b \mapsto gw(b)\hat{\Gamma}$, where $b \in \mathbb{T}^d$ and $g \in G$ with $x = g\Gamma$, but this mapping is depending on the choice of $g \in G$. For example, $gw(b)\hat{\Gamma} = (g\gamma^{-1})w(\gamma b)\hat{\Gamma}$ for any $\gamma \in \hat{\Gamma}$.
However, if we fix a fundamental domain, then we have a well-defined parametrization of $Y$. Throughout this paper, we fix $\mathcal{F} \subset G$ and $\iota$ as in §2.6. For any $y \in Y$, there exists unique $b \in \mathbb{T}^d$ such that $y = \iota(\pi(y))w(b)\hat{\Gamma}$. Denote this map $y \mapsto b$ by $\sigma : Y \rightarrow \mathbb{T}^d$. Then we have a parametrization $y = \iota(\pi(y))w(\sigma(y))\hat{\Gamma}$. We remark that the map $\sigma$ is not defined canonically unlike $\pi : Y \rightarrow X$ and depends on the choice of the fundamental domain $\mathcal{F}$.

2.8. A partition of unity of $X$. In this subsection, we will partition $X$ to small boxes, and construct a partition of unity of $X$ for given parameter $0 < r < \frac{1}{2}$.

**Proposition 2.2.** Given $0 < r < \frac{1}{2}$, there exists a set $\{x_1, \ldots, x_{N_r}\} \subset K(C_8 r^{\frac{3}{2}})$ with $N_r \asymp r^{-d(d-1)}$ and a partition of unity $\{\psi_{r,i}\}_{i \in \mathcal{I}_r}$ of $X$ with $\mathcal{I}_r := \{1, \ldots, N_r\} \cup \{\infty\}$ satisfying the following properties:
- $0 \leq \psi_{r,i} \leq 1$ for all $i \in \mathcal{I}_r$,
- $1_{B^X(x_i, r)} \leq \psi_{r,i} \leq 1_{B^X(x_i, 5r)}$ for all $i \in \mathcal{I}_r \setminus \{\infty\}$,
- $\text{Supp} \psi_{r,x} \subseteq X \setminus K(C_8 r^{\frac{3}{2}})$,
- $\sum_{i \in \mathcal{I}_r} \psi_{r,i} = 1_X$,
- $\|\nabla \psi_{r,i}\|_{L^\infty(X)} \leq C_9 r^{-d}$ for all $i \in \mathcal{I}_r \setminus \{\infty\}$
for some constants $C_8, C_9 > 0$ which are not depending on $r$.

**Proof.** Take $\{x_1, \ldots, x_{N_r}\}$ to be a maximal $4r$-separated subset of $K(C_8 r^{\frac{3}{2}})$, where $C_8 := 5^4 C_4$. Then for any $1 \leq i \leq N_r$, the map $g \mapsto gx_i$ is injective
on the ball $B^G(id, 5r)$. We get a partition $\mathcal{P}_r = \{P_i\}_{i \in I_r}$ of $X$ by letting

$$P_i = B^X(x_i, 4r) \setminus \left( \bigcup_{j=1}^{i-1} P_j \cup \bigcup_{j=i+1}^{N_r} B^X(x_j, 2r) \right)$$

for $1 \leq i \leq N_r$ and $P_\infty = X \setminus \bigcup_{j=1}^{N_r} P_j$ inductively. Then for $1 \leq i \leq N_r$, $B^X(x_i, 2r) \subseteq P_i \subseteq B^X(x_i, 4r)$ by the construction. Since $m_X(P_i) \approx r^{d^2-1}$ for each $1 \leq i \leq N_r$, we have $N_r \approx r^{-d^2}$. To construct a partition of unity, choose a nonnegative smooth approximating function $\psi \in C_c^\infty(G)$ supported on $B^G(id, r)$ and satisfying $\int_G \psi_r(g)m_G(g) = 1$. Moreover, it is also possible to take such $\psi_r$ to satisfy $\|\nabla \psi_r\|_{L^\infty(X)} \leq C_9 r^{-d^2}$ for some constant $C_9 > 0$ which is not depending on $r$. Then $\psi_{r,i} := \psi_r * 1_{P_i}$'s satisfy the desired properties. \hfill \Box

### 3. Inductive Structures of the Measures of Expanding Translates

#### 3.1. Well-separateness of $\Gamma^r$-orbit in $\mathbb{Z}^d$.

For fixed $s > 0$, we define two maps $\xi = \xi_s : X \times V \to \mathcal{F}$ and $\gamma = \gamma_s : X \times V \to \Gamma$ as follows: For any $x \in X$ and $u \in V$, there uniquely exist $\xi(x, u) \in \mathcal{F}$ and $\gamma(x, u) \in \Gamma$ such that

$$a_s u(x) = \xi(x, u) \gamma(x, u)$$

by the definition of the fundamental domain $\mathcal{F}$. The map $\xi(x, u)$ encodes the projected orbit of $a_s u$-action onto $X$ as $a_s u x = \xi(x, u) \Gamma$. Also, the map $\gamma(x, u)$ describes the orbit of $a_s u$-action on the fiber tori as in the following lemma.

**Lemma 3.1.** For any $y \in Y$ and $u \in V$, we have $\sigma(a_s u y) = \gamma(x, u) \sigma(y)$, where $x = \pi(y)$.

**Proof.** It is a direct consequence from the definitions of $\sigma$ and $\gamma$:

$$\sigma(a_s u y) = \sigma(a_s u w(x) w(\sigma(y)) \tilde{\Gamma}) = \sigma(\xi(x, u) \gamma(x, u) w(\sigma(y)) \tilde{\Gamma}) = \sigma(\xi(x, u) w(\gamma(x, u) \sigma(y)) \tilde{\Gamma}) = \gamma(x, u) \sigma(y).$$

\hfill \Box

For fixed $m_0 \in \mathbb{Z}^d \setminus \{0\}$, we define two maps $x : X \times V \to \mathbb{R}^m$ and $y : X \times V \to \mathbb{R}^n$ satisfying the following:

$$\left(\xi(x, u)^{tr}\right)^{-1} m_0 = \begin{pmatrix} x(x, u) \\ y(x, u) \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n.$$

**Lemma 3.2.** For $0 < \epsilon < \frac{1}{2}$ and $x \in X$ denote by $V_{x, \epsilon}$ the set of elements $u \in V$ satisfying:

1. $\|\xi(x, u)\| < \epsilon^{-1}$,
(2) \( \|x(u, v)\| > \varepsilon^2 |m_0| \).

If \( \varepsilon > e^{-\kappa_2 s} \) and \( \text{ht}(x) < e^{\kappa_2 s} \), then \( m_H(V \setminus V_{x, \varepsilon}) \leq C_{11} \varepsilon^{\frac{s}{2}} \) for some constants \( C_{11} > 0 \).

**Proof.** We first estimate the measure of the subset violating (1). If \( \|\xi(x, u)\| \geq \varepsilon^{-1} \), then \( \text{ht}(a_u x) = \text{ht}(\xi(x, u) \Gamma) \geq C_{3}^{-\frac{1}{1+s}} \varepsilon^{-\frac{1}{1+s}} \) by (2.9), hence \( a_u x \notin K(C_{5}^{-\frac{1}{1+s}} \varepsilon^{-\frac{1}{1+s}}) \). As we obtained the estimate (2.8), we have

\[
(3.3) \quad m_H \left( \left\{ u \in V : a_u x \notin K(C_{5}^{-\frac{1}{1+s}} \varepsilon^{-\frac{1}{1+s}}) \right\} \right) \approx \varepsilon^{\frac{d}{s+1}}
\]

if \( \varepsilon > e^{-\kappa_2 s} \), \( \text{ht}(x) < e^{\kappa_2 s} \). Thus, \( m_H \left( \left\{ u \in V : \|\xi(x, u)\| \geq \varepsilon^{-1} \right\} \right) \ll \varepsilon^{\frac{d}{s+1}} \).

Now we estimate the measure of the subset satisfying (1) but violating (2). If \( u \in V \) satisfies (1), then \( \|\xi(x, u)\| \leq C_{3}^{-1} \|m_0\| \). Assume from now on \( \|x(x, u)\| \ll C_{3}^{-1} \). Therefore \( \|y(x, u)\| \ll \|m_0\| \). Moreover, we can find some \( h \in B^{c}(\text{id}, C_{10}^{-1}) \) so that \( h(\xi(x, u)) = h \left( \begin{pmatrix} x(x, u) \\ y(x, u) \end{pmatrix} \right) \in \{0_m\} \times \mathbb{R}^n \), for some constant \( C_{10} > 1 \).

Let \( \mathcal{M} := \{ g \in G : (g^{\frac{1}{s}})^{-1}m_0 \in \{0_m\} \times \mathbb{R}^n \} \), then \( \xi(x, u) \) is in the \( C_{10}^{-1} \)-neighborhood of \( \mathcal{M} \). Denote by \( \mathcal{C}(\varepsilon) \subset \mathcal{F} \) the intersection of this neighborhood and \( \mathcal{F} \). It follows that \( \xi(x, u) \Gamma \in \mathcal{C}(\varepsilon) \Gamma \).

We now need to get an upper bound of \( m_X(\mathcal{C}(\varepsilon) \Gamma) \). Note that \( \epsilon^{\frac{1}{2}} \)-neighborhood of \( \mathcal{F}(\epsilon^{\frac{1}{2}}, C_{4} \epsilon^{\frac{3}{2s}}) \) is still contained in \( \mathcal{F} \). Since \( \mathcal{M} \) is a \( (d^2 - m - 1) \)-dimensional analytic closed submanifold of \( G \), we have

\[
m_G(\mathcal{C}(\varepsilon) \cap \mathcal{F}(\epsilon^{\frac{1}{2}}, C_{4} \epsilon^{\frac{3}{2s}})) \ll (\epsilon^{\frac{1}{2}})^m m_G(\mathcal{C}(\varepsilon) \cap \mathcal{F}(\epsilon^{\frac{1}{2}}, C_{4} \epsilon^{\frac{3}{2s}})) \ll \epsilon^{\frac{d}{2}}.
\]

On the other hand, by (2.11) we have \( m_G(\mathcal{F}(\epsilon^{\frac{1}{2}}, C_{4} \epsilon^{\frac{3}{2s}})) \ll \epsilon^{\frac{d}{2}} \). Therefore, we obtain \( m_X(\mathcal{C}(\varepsilon) \Gamma) = m_G(\mathcal{C}(\varepsilon)) \ll \epsilon^{\frac{d}{2}} + \epsilon^{\frac{d}{2s}} \ll \epsilon^{\frac{d}{2}} \). We remark that the implied constant is uniform on the choice of \( m_0 \in \mathbb{Z}^d \setminus \{0\} \) since \( \mathcal{M} \) is determined by \( \frac{m_0}{|m_0|} \in \mathbb{S}^{d-1} \) and \( \mathbb{S}^{d-1} \) is compact.

On the other hand, if \( u \in V \) satisfies (1) and violates (2), then \( a_u x = \xi(x, u) \Gamma \in \mathcal{C}(\varepsilon) \Gamma \cap K(C_{5}^{-\frac{1}{1+s}} \epsilon^{-\frac{1}{1+s}}) \). Let \( f \in L^\infty(X) \) be the characteristic function on the set \( \mathcal{C}(\varepsilon) \Gamma \cap K(C_{5}^{-\frac{1}{1+s}} \epsilon^{-\frac{1}{1+s}}) \) and \( \psi_{\frac{10}{10}} \in C^\infty_c(X) \) be a nonnegative smooth approximating function on \( B^{c}(\text{id}, \frac{1}{10}) \) as defined in the proof of 2.2. Applying (2.6) with \( f \ast \psi_{\frac{10}{10}} \) as we obtained (2.7), we have

\[
m_H \left( \left\{ u \in V : a_u x \in \mathcal{C}(\varepsilon) \Gamma \cap K(C_{5}^{-\frac{1}{1+s}} \epsilon^{-\frac{1}{1+s}}) \right\} \right) \ll m_X \left( \mathcal{C}(\varepsilon) \Gamma \cap K(C_{5}^{-\frac{1}{1+s}} \epsilon^{-\frac{1}{1+s}}) \right) \ll \epsilon^{\frac{s}{2}}
\]

if \( \varepsilon > e^{-\kappa_2 s} \) and \( \text{ht}(x) < e^{\kappa_2 s} \). \( \square \)
The following density property of $\gamma(x,u)^{tr}$-orbit of $m_0 \in \mathbb{Z}^d$, where $x$ and $m_0$ are fixed, is a key ingredient to obtain an effective equidistribution on fiber tori.

**Proposition 3.3.** Let $s > 0$ and $m_0 \in \mathbb{Z}^d \setminus \{0\}$. For any $e^{-\kappa^2 s} \leq \epsilon \leq \frac{1}{2}$ and $x \in X$ with $\text{ht}(x) < e^{-\kappa^2 s}$, let $V_{x,\epsilon}$ be the set as in Lemma 3.2. Then

1. $\{\gamma(x,u)^{tr}m_0 \in \mathbb{Z}^d : u \in V_{x,\epsilon}\} \subseteq B^{2d}(0,R)$, where $R = C_{13}\epsilon^{-1}\text{ht}(x)^{d-1}\|m_0\|e^ns$,

2. For any $m \in \mathbb{Z}^d$,

$$m_H(\{u \in V_{x,\epsilon} : \gamma(x,u)^{tr}m_0 = m\}) \leq C_{14}\epsilon^{-3n}e^{-dns}$$

for some constants $C_{13}, C_{14} > 1$.

We remark that the estimate in this proposition is tight in the following sense: If we consider $\epsilon$ and $\text{ht}(x)$ as constants, as the points $\gamma(x,u)^{tr}m_0$ belongs to $B^{2d}(0,R)$ and the latter contains $\approx R^d$ points, the estimate in (2) is compatible with each $m \in B^{2d}(0,R)$ obtaining the same weight $R^{-d} \approx e^{-dns}$. In other words, roughly speaking, Proposition 3.3 asserts that the orbit $\gamma(x,u)^{tr}m_0$ is well-distributed on $B^{2d}(0,R)$ if $e^ns$ is much bigger than $\epsilon^{-1}$ and $\text{ht}(x)$.

**Proof.** Recall that there is the canonical measure-preserving bijection $\varphi$ between $M_{m,n}(\mathbb{R})$ and $H$ defined by $\varphi(A) = \begin{pmatrix} \text{Id}_m & A \\ 0 & \text{Id}_n \end{pmatrix}$. Since $V$ is a neighborhood of identity with compact closure, there exists a constant $C_{12} > 1$ such that $\varphi(B^{R^{mn}}(0,C_{12}^{-1})) \subseteq V \subseteq \varphi(B^{R^{mn}}(0,C_{12}))$. Note also that $\|u\| \leq C_{12}$ for any $u \in V$.

1. Recall that we have $\|\iota(x)\| \leq C_{15}\text{ht}(x)^{d-1}$, $\|\xi(x,u)\| \leq \epsilon^{-1}$, $\|u\| \leq C_{12}$, and $\|a_s\|_{\text{op}} = e^ns$. By the definition of $\xi$ and $\gamma$ we have a relation $\xi(x,u)^{-1}a_s u(x) = \gamma(x,u)$. Hence,

$$\|\gamma(x,u)^{tr}m_0\| = \|\iota(x)^{tr}a^{tr}_s (\xi(x,u)^{tr})^{-1}m_0\|
= \|\iota(x)^{tr}\|_{\text{op}}\| u^{tr}\|_{\text{op}}\| a^{tr}_s\|_{\text{op}}\| (\xi(x,u)^{tr})^{-1}\|_{\text{op}}\| m_0\|
\leq C_{15}^3\|\iota(x)\|\| u\|\|a_s\|_{\text{op}}\|\xi(x,u)\|\| m_0\|
\leq C_{15}^3 C_{12}e^{-1}\text{ht}(x)^{d-1}\| m_0\|e^ns.$$

2. The equation $\gamma(x,u)^{tr}m_0 = m$ can be written

$$a^{tr}_s (\xi(x,u)^{tr})^{-1}m_0 = (u^{tr})^{-1}(\iota(x)^{tr})^{-1}m.$$

Let $v_+ \in \mathbb{R}^m$ and $v_- \in \mathbb{R}^n$ be the vectors such that $\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = (\iota(x)^{tr})^{-1}m$.

Here, $v_+$ and $v_-$ are only depending on $m$, not on $u$. For any $A \in \varphi^{-1}(V)$, by a straightforward computation we have

$$u^{tr})^{-1}(\iota(x)^{tr})^{-1}m = \begin{pmatrix} v_+ \\ -Av_+ + v_- \end{pmatrix}.$$
where $u = \varphi(A)$. On the other hand, the left hand side of (3.4) is equal to 
\[ \left( e^{ims}x(x,u) - e^{-ms}y(x,u) \right). \]
Combining this with (3.5), if $u = \varphi(A)$ is a solution of (3.4), then
\[
\| A^tr v_+ - v_- \| \leq e^{-m}\|y(x,u)\|. \tag{3.6}
\]
If $u \in V_{x,\varepsilon}$, we have $\|x(x,u)\| > e^2\|m_0\|$ and
\[
\|y(x,u)\| \leq \|(\xi(x,u)^{tr})^{-1}m_0\| \leq C_3\|\xi(x,u)\|\|m_0\| \leq C_3e^{-1}\|m_0\|
\]
by (5.5) and the definition of $V_{x,\varepsilon}$ in Lemma 3.2. Moreover, $\|v_+\| > e^2\|m_0\|e^{ns}$ since $v_+ = e^{ns}x(x,u)$. To sum up, a solution $u = \varphi(A)$ must be in a thin tube $\{ A \in M_{m,n}(\mathbb{R}) : \| A^tr v_+ - v_- \| \leq C_3e^{-1}\|m_0\|e^{-ms} \},$ where $\|v_+\| > e^2\|m_0\|e^{ns}$. We can consider $A$ as in the Euclidean space $\mathbb{R}^{mn}$, then the volume of the intersection of this tube and the ball $B^{\mathbb{R}^{mn}}(0,C_{12})$ is
\[
\ll (C_{12}^{-1}\|v_+\|^{-1}(C_3e^{-1}\|m_0\|e^{-ms}))^n \ll e^{-3n}e^{-dns}. \]
Therefore, for any $m \in \mathbb{Z}^d$, the $m_H$-measure of the set of $u \in V_{x,\varepsilon}$ satisfying $\gamma_x(u)^{tr}m_0 = m$ is less than $C_{14}e^{-3n}e^{-dns}$ for some constant $C_{14} > 1$. \hfill \Box

### 3.2. A relation between Fourier coefficients of measures

In this subsection, we explore a relation between Fourier coefficients of measures. We start with the following lemma.

**Lemma 3.4.** Let $s > C_{15}$, $0 < r \leq \frac{1}{2}$ and $r' := re^{-3ds}$ for some constant $C_{15} > 0$. For $x \in X$ and $u \in V$, suppose that $\iota(x) \in F(6r', 6r')$ and $\xi(x,u) \in F(r, r^{\frac{1}{2}r})$. If $d^X(x, x') < 5r'$, then $d^X(a_sux, a_sux') \leq C_{16}re^{-ds}$ and $\gamma(x,u) = \gamma(x',u)$ for some constant $C_{16} > 0$.

**Proof.** Recall that
\[
a_sux(x) = \xi(x,u)\gamma(x,u), \quad a_sux(x') = \xi(x',u)\gamma(x',u).
\]
Since $\iota(x)$ is in $F(6r', 6r')$, $\iota$ is an isometry on $B^X(x, 5r')$, so $d^G(\iota(x), \iota(x')) = d^X(x, x') < 5r'$. Using $\|a_s\| = \max(e^{ms}, e^{ns}) \leq e^{ds}$ and $\|u\| \leq C_{12}$, we have
\[
d^G(a_sux(x), a_sux'(x')) \leq C_{16}C_1C_2\|a_s\|^2d^G(\iota(x), \iota(x')) \leq C_{16}re^{-ds},
\]
where $C_{16} := 5C_1C_2C_{12}$. Since $a_sux(x) = \xi(x,u)$ is in $F(r, r^{\frac{1}{2}r})$,
\[
d^G(a_sux(x), a_sux'(x')) \geq d^X(\xi(x,u)\Gamma, \xi(x',u)\Gamma) = d^G(\xi(x,u), \xi(x',u)).
\]
Combining (3.7) and (3.8), we get
\[
d^G(\xi(x,u), \xi(x',u)) \leq C_{16}re^{-ds}.
\]
Moreover, \( d^X(a_sux, a_sux') \leq C_16re^{-ds} \) from (3.7), and
(3.10)
\[
d^G(\gamma(x, u), \gamma(x', u)) = d^G(\xi(x, u)^{-1}a_sux(x), \xi(x', u)^{-1}a_sux(x'))
\]
\[
\leq d^G(\xi(x, u)^{-1}a_sux(x), \xi(x, u)^{-1}a_sux(x'))
\]
\[
+ d^G(\xi(x, u)^{-1}a_sux(x'), \xi(x', u)^{-1}a_sux(x'))
\]
\[
\leq C_2\|\xi(x, u)\|d^G(a_sux(x), a_sux(x')) + d^G(\xi(x, u)^{-1}, \xi(x', u)^{-1}),
\]
using the right-invariance of \( d^G \) in the last inequality. From (2.9) and (3.11) we have \( \|\xi(x, u)\| \leq C_5r^{-1} \). By (3.7) and (3.9), the last line of (3.10) is bounded by \( C_2C_5C_16e^{-ds} + C_16r^{e^{-ds}} < \frac{1}{100} \) if \( s \) is bigger than some constant \( C_{15} > 0 \). We deduce \( \gamma(x, u) = \gamma(x', u) \) from this since \( \Gamma \) is a discret subgroup of \( G \).

For \( t > 0 \), let \( 0 < s < \frac{\kappa_2}{100}t \) and \( e^{-\kappa_2s} < r < \frac{1}{2} \) so that \( r' = re^{-3ds} > e^{-\frac{\kappa_2}{2}t} \). In the rest of this paper, we fix \( y_0 \in Y \) with \( \text{ht}(\pi(y_0)) < e^{\frac{\kappa_2}{2}t} \) and denote \( \mu_t := \mu_{y_0,t} \) and \( \mu_{t-s} := \mu_{y_0,t-s} \) for simplicity. In this section, we also fix a partition of unity \( \{\psi_j\}_{j \in J} \) using Proposition 2.2 with a radius \( r' \), where \( J = \{1, \ldots, N_r\} \) and \( N_r = (r')^{-(d-1)} \). Let \( \{x_1, \ldots, x_{N_r}\} \) be the corresponding \( 4r' \)-separated set as in Proposition 2.2.

For \( z \in X \) with \( \text{ht}(z) < e^{\frac{\kappa_2}{2}t} \) and \( \omega \in C_c^\infty(X) \) such that \( 1_{B^X(z, r)} \leq \omega \leq 1_{B^X(z, 5r)} \) and \( \|\nabla \omega\|_{L^\infty(X)} \leq C_g r^{-d} \), let \( \mu_{t,\omega} \) be the probability measure on \( Y \) defined by
(3.11)
\[
\mu_{t,\omega}(f) := \pi_*\mu_t(\omega)^{-1} \int_Y \omega \circ \pi(y)f(y)d\mu_t(y).
\]
Note that \( \pi_*\mu_t(B^X(z, r)) \leq \pi_*\mu_t(\omega) \leq \pi_*\mu_t(B^X(z, 5r)) \) and \( \kappa_2 > 0 \) was taken to be sufficiently small so that we can apply (2.7), hence we have
(3.12)
\[
C_{17}^{-1}e^{-d-1} \leq \mu_*\mu_t(\omega) \leq C_{17}r^{d-1}
\]
for some \( C_{17} > 1 \). For \( 1 \leq j \leq N_r \), let \( \mu_{t-s,j} \) be the probability measure on \( Y \) defined by
(3.13)
\[
\mu_{t-s,j}(f) := \alpha_j^{-1} \int_Y \psi_j \circ \pi(y)f(y)d\mu_{t-s}(y),
\]
where \( \alpha_j := \mu_{t-s}(\psi_j \circ \pi) \). We can apply (2.7) again since \( r' > e^{-\frac{\kappa_2}{2}t} \). Hence, we have
(3.14)
\[
C_{18}^{-1}(r')^{d-1} \leq \alpha_j \leq C_{18}(r')^{d-1}, \quad \forall 1 \leq j \leq N_r,
\]
for some \( C_{18} > 1 \).

Taking push-forward of these measures under \( \sigma : Y \to \mathbb{T}^d \), we define probability measures on the torus as follows:
(3.15)
\[
\nu_t := \sigma_*\mu_t, \quad \nu_{t,\omega} := \sigma_*\mu_{t,\omega}
\]
\[
\nu_{t-s} := \sigma_*\mu_{t-s}, \quad \nu_{t-s,j} := \sigma_*\mu_{t-s,j}, \quad 1 \leq j \leq N_r.
\]
We now observe the relation between $\tilde{\nu}_{t,\omega}$ and $\tilde{\nu}_{t-s,j}$'s. The following proposition is the main purpose of this section.

**Proposition 3.5.** Let $t > 0$, $C_{15} < s \leq \kappa_4 t$, $e^{-\kappa_4 s} \leq r \leq \frac{1}{2}$, $r' = re^{-3ds}$, $e^{-\kappa_4 s} \leq \varepsilon \leq \frac{1}{2}$ and $m_0 \in \mathbb{Z}^d\backslash\{0\}$, where $\kappa_4 := \min\left(\frac{\kappa_2}{10d}, 2\kappa_3\right)$. For $\iota(z) \in \mathcal{F}(10r, C_{15}\frac{1}{d})$, let $\omega \in C_\mathcal{E}(X)$ be a function satisfying $\|\partial\omega\|_{L^\infty(X)} \leq C_{0r'^{-d^2}}$ as above. For some constants $C_{19}, C_{22} > 0$, if $|\tilde{\nu}_{t,\omega}(m_0)| \geq C_{19}r^{-(d^2-1)}e^{\frac{\kappa_3}{2}}$, then there exists $J' \subseteq J\backslash\{\infty\}$ with $|J'| \geq \frac{|J|}{3}$ and

$$\left|\left\{m \in B^{2d}(R) : |\tilde{\nu}(m)| \geq \eta\right\}\right| > \eta^3|B^{2d}(0, R)|,$$

where $\nu := \frac{1}{|J'|} \sum_{j \in J'} \nu_{t-s,j} \in \mathcal{P}(T^d)$, $\eta := C_{22}e^{\frac{\kappa_3}{2}+\kappa_3}\|m_0\|^{-d}$, and $R = C_{13}e^{-d}\|m_0\|e^{\kappa_3}$. 

**Remark 3.6.** Note that $\nu \leq C_{23}\nu_{t-s}$ for some $C_{23} > 0$. Since $|J'| \approx |J| \approx (r')^{-(d^2-1)}$ and $\alpha_j \approx (r')^{d^2-1}$ for all $1 \leq j \leq N_{r'}$, we have

$$\nu = \frac{1}{|J'|} \sum_{j \in J'} \nu_{t-s,j} \leq \sum_{j \in J'} \alpha_j \nu_{t-s,j} \leq \sum_{j \in J} \alpha_j \nu_{t-s} = \nu_{t-s}.$$ 

**Proof.** From the definitions (3.11) and (3.15),

$$\tilde{\nu}_{t,\omega}(m_0) = \int_{\mathbb{T}^d} e^{-2\pi im_0 \cdot b} d\nu_{t,\omega}(b)$$

(3.16)

$$= \int_Y e^{-2\pi im_0 \cdot \sigma(y)} d\mu_{t,\omega}(y)$$

$$= \pi_s \mu_t(\omega)^{-1} \int_Y \omega \circ \pi(y) e^{-2\pi im_0 \cdot \sigma(y)} d\mu_t(y).$$

Recall that for any $f \in L^\infty(Y)$,

$$\mu_t(f) = \frac{1}{m_H(V)} \int_V (a_u \pi f) dm_H(u) + O(\|f\|_{L^\infty(Y)}e^{-(m+n)(t-s)})$$

from Proposition 2.1. Applying this to (3.16) and integrating over $V$,

(3.17)

$$\tilde{\nu}_{t,\omega}(m_0) = \frac{\pi_s \mu_t(\omega)^{-1}}{m_H(V)} \int_V \int_Y \omega(a_u \pi y) e^{-2\pi im_0 \cdot \sigma(a_u y)} d\mu_{t-s}(y) dm_H(u)$$

$$+ O(\pi_s \mu_t(\omega)^{-1} e^{-(m+n)(t-s)})$$

$$= \frac{\pi_s \mu_t(\omega)^{-1}}{m_H(V)} \sum_{j \in J} \int_V \int_Y \omega(a_u \pi y) \psi_j(\pi(y)) e^{-2\pi im_0 \cdot \sigma(a_u y)} d\mu_{t-s}(y) dm_H(u)$$

$$+ O(\pi_s \mu_t(\omega)^{-1} e^{-(m+n)(t-s)}).$$

Let $J_{int} \subset J$ be the set of $j \in J$ such that $\iota(x_j) \in \mathcal{F}(6r', \epsilon)$. Here, $\epsilon^d \gg (r')^{\kappa_3}$ since we are assuming $\epsilon \gg e^{-\kappa_4 s} \gg e^{-2\kappa_4 s}$. Then we have $m_G(\mathcal{F} \backslash \mathcal{F}(7r', \frac{\epsilon}{2})) \ll$
\( \epsilon^d \) from (2.11). For any \( j \in J \setminus J_{\text{int}} \), each \( \nu(B^X(x_j, r')) \) is in \( \mathcal{F} \setminus \mathcal{F}(\tau', \frac{1}{2}) \) and has \( m_{\mathbb{Z}} \)-measure \( \approx (r')^{d-1} \). Thus, \( |J \setminus J_{\text{int}}| \ll \epsilon^d (r')^{-(d-1)} \) since \( \nu(B^X(x_j, r')) \)'s are disjoint. It follows that
\[
\frac{\pi_* \mu_t(\omega)^{-1}}{m_H(V)} \sum_{j \in J \setminus J_{\text{int}}} \int_V \int_Y \omega(a_s u \pi(y)) \psi_j(\pi(y)) e^{-2\pi i m_0 \sigma(a_s uy)} d\mu_{t-s}(y) dm_H(u)
\leq \frac{\pi_* \mu_t(\omega)^{-1}}{m_H(V)} \sum_{j \in J \setminus J_{\text{int}}} \int_V \int_Y \omega(a_s u \pi(y)) \psi_j(\pi(y)) d\mu_{t-s}(y) dm_H(u)
\leq \pi_* \mu_t(\omega)^{-1} \sum_{j \in J \setminus J_{\text{int}}} \pi_* \mu_{t-s}(\psi_j)
= \pi_* \mu_t(\omega)^{-1} |J \setminus J_{\text{int}}| (r')^{d-1} = O(\pi_* \mu_t(\omega)^{-1} \epsilon^d).
\]

We now estimate the summation of (3.17) over \( j \in J_{\text{int}} \). For each \( j \in J_{\text{int}} \) and \( y \) as in the integral with respect to \( \mu_{t-s} \), we may assume \( a_s u \pi(y) \in \text{Supp} \omega \) and \( \pi(y) \in \text{Supp} \psi_j \). It implies that \( \xi(\pi(y), u) \in \text{Supp} \omega \subseteq \mathcal{F}(r, r \frac{1}{2}) \) and \( d^X(x_j, \pi(y)) < 5r' \), so the assumptions in Lemma 3.4 are satisfied for \( x_j \) and \( \pi(y) \in \text{Supp} \psi_j \). By Lemma 3.4, we have
\[
d^X(a_s u x_j, a_s u \pi(y)) \leq C_{16} r e^{-ds},
\]
\[
\gamma(x_j, u) = \gamma(\pi(y), u)
\]
for any \( u \in V, j \in J_{\text{int}} \), and \( y \in \text{Supp}(\psi_j \circ \pi) \). Since we are assuming \( \| \nabla \omega \|_{L^\infty} \leq C_9 r^{-d^2} \), by (3.19) we have
\[
|\omega(a_s u x_j) - \omega(a_s u \pi(y))| \leq C_9 C_{16} r^{-(d^2-1)} e^{-ds}.
\]
By Lemma 3.1 and (3.20), we also have
\[
\sigma(a_s uy) = \gamma(\pi(y), u) \sigma(y) = \gamma(x_j, u) \sigma(y).
\]
It follows from (3.21) and (3.22) that
\[
\frac{\pi_* \mu_t(\omega)^{-1}}{m_H(V)} \sum_{j \in J_{\text{int}}} \int_V \int_Y \omega(a_s u \pi(y)) \psi_j(\pi(y)) e^{-2\pi i m_0 \sigma(a_s uy)} d\mu_{t-s}(y) dm_H(u)
= \frac{\pi_* \mu_t(\omega)^{-1}}{m_H(V)} \sum_{j \in J_{\text{int}}} \int_V \int_Y \omega(a_s u x_j) \psi_j(\pi(y)) e^{-2\pi i \gamma(x_j, u) \tau r \sigma(y)} d\mu_{t-s}(y) dm_H(u)
+ O(\pi_* \mu_t(\omega)^{-1} r^{-(d^2-1)} e^{-ds}).
\]
For \( j \in J_{\text{int}} \) and \( m \in \mathbb{Z}^d \), let \( V_{x_j, \epsilon} \) be the set as in Lemma 3.2 and \( E_{j, m} \subseteq V \) be the set \{ \( u \in V_{x_j, \epsilon} : \gamma(x_j, u) \tau r \sigma(m_0 \pi) = m \} \) which was the set of (2) in Proposition 3.3. For each \( j \in J_{\text{int}} \), \( \text{ht}(x_j) \leq \epsilon^{-1} \) by the definition of \( J_{\text{int}} \), so we have a disjoint partition \( V_{x_j, \epsilon} = \bigcup_{m \in B_{\mathbb{Z}^d}^{\epsilon}(0, R)} E_{j, m} \) by Proposition 3.3, where
\( R = C_{13} \epsilon^{-d} \| \mathbf{m}_0 \| e^{ns} \). By Lemma 3.2 and Proposition 3.3, we have the following measure estimates:

\[
(3.24) \quad m_H(V \setminus V_{x_j, \epsilon}) \leq C_{11} \epsilon^2,
\]

\[
(3.25) \quad m_H(E_{j, \mathbf{m}}) \leq C_{14} \epsilon^{-3n} e^{-dns}
\]

for any \( j \in J_{\text{int}} \) and \( \mathbf{m} \in \mathbb{Z}^d \). Denoting

\[
(3.26) \quad \beta_{j, \mathbf{m}} := \frac{1}{m_H(V)} \int_{E_{j, \mathbf{m}}} \omega(a_s u x_j) dm_H(u),
\]

each summand of the last line in (3.23) can be expressed by

\[
(3.27) \quad \int_V \int_Y \omega(a_s u x_j) \psi_j(\pi(y)) e^{-2\pi i \gamma(x_j, u)^\top \mathbf{m}_0 \cdot \sigma(y)} d\mu_{t-s}(y) dm_H(u)
\]

\[
= \sum_{\mathbf{m} \in B_{2^d}(0, R)} \left( \int_{E_{j, \mathbf{m}}} \omega(a_s u x_j) dm_H(u) \right) \left( \int_Y \psi_j(\pi(y)) e^{-2\pi i \mathbf{m} \cdot \sigma(y)} d\mu_{t-s}(y) \right)
\]

\[
+ O(\epsilon^2 m_X(\psi_j))
\]

\[
= m_H(V) \left( \sum_{\mathbf{m} \in B_{2^d}(0, R)} \alpha_j \beta_{j, \mathbf{m}} \tilde{\nu}_{t-s, j}(\mathbf{m}) + O(\epsilon^2 m_X(\psi_j)) \right)
\]

for any \( j \in J_{\text{int}} \). Combining (3.17), (3.18), (3.23) and (3.27) all together,

\[
(3.28) \quad \pi_{s, \mu_t}(\omega) \tilde{\nu}_{t, \omega}(\mathbf{m}_0) = \sum_{j \in J_{\text{int}}} \sum_{\mathbf{m} \in B_{2^d}(0, R)} \alpha_j \beta_{j, \mathbf{m}} \tilde{\nu}_{t-s, j}(\mathbf{m})
\]

\[
+ O(\max(\epsilon^2, r^{-(d^2-1)} e^{-ds}, e^{-(m+n)(t-s)}))
\]

\[
= \sum_{j \in J_{\text{int}}} \sum_{\mathbf{m} \in B_{2^d}(0, R)} \alpha_j \beta_{j, \mathbf{m}} \tilde{\nu}_{t-s, j}(\mathbf{m}) + O(\epsilon^2)
\]

if \( \epsilon \geq e^{-\kappa_4 s}, r \geq e^{-\kappa_4 s}, \) and \( s \leq \kappa_4 t \). Recall that \( \pi_{s, \mu_t}(\omega) \asymp r^{d^2-1} \) from (3.12), hence we can find constants \( C_{19}, C_{20} > 0 \) so that if \( |\tilde{\nu}_{t, \omega}(\mathbf{m}_0)| \geq C_{19} r^{-(d^2-1)} \epsilon^{\frac{m}{2}} \), then

\[
(3.29) \quad \left| \sum_{j \in J_{\text{int}}} \sum_{\mathbf{m} \in B_{2^d}(0, R)} \alpha_j \beta_{j, \mathbf{m}} \tilde{\nu}_{t-s, j}(\mathbf{m}) \right| > C_{20} r^{(d^2-1)} |\tilde{\nu}_{t, \omega}(\mathbf{m}_0)| \geq C_{19} C_{20} \epsilon^\frac{m}{2}.
\]
It means that a weighted average of $\tilde{\nu}_{t-s,j}(m)$’s is large. To deduce the desired conclusion from this, we need to “flatten” the weights $\alpha_j \beta_{j,m}$’s using the following technical lemma.

**Lemma 3.7.** Let $I$ and $J$ be finite sets. For each $(i, j) \in I \times J$, $a_{ij} \geq 0$ and $b_{ij} \in \mathbb{C}$ with $|b_{ij}| \leq 1$ are given. If $a_{ij}$ and $b_{ij}$ satisfy

- $0 \leq a_{ij} \leq \frac{\lambda}{|I \times J|}$,
- $\sum_{(i,j) \in I \times J} a_{ij}b_{ij} \geq \tau$

for some $\lambda, \tau > 0$, then there exists $J' \subseteq J$ such that $|J'| \geq \frac{|J|}{4}$ and

$$\left\{ i \in I : \left| \frac{1}{|J|} \sum_{j \in J} b_{ij} \right| \geq \frac{\tau}{2^6 \lambda} \right\} \geq \frac{\tau^3}{2^{17} \lambda^3} |I|.$$

Apply Lemma 3.7 for $I = B^{Z^d}(0, R)$ and $J = J_{\text{int}}$ so that $\alpha_j \beta_{j,m} \geq 0$ and $\tilde{\nu}_{t-s,j}(m) \in \mathbb{C}$ be the corresponding $a_{ij}$ and $b_{ij}$ in the lemma. By (3.14), (3.25) and (3.26), we have

$$\alpha_j \beta_{j,m} \leq \alpha_j (C_{14} e^{-3n} e^{-dn\nu}) \leq \frac{C_{21} e^{-(d^2+3n)} \|m_0\|^d}{|B^{Z^d}(0, R)| |J_{\text{int}}|}$$

for some constant $C_{21} > 0$ since $|B^{Z^d}(0, R)| \asymp R^d$ and $|J_{\text{int}}| \asymp (r')^{-(d^2-1)}$. In (3.29) and (3.30), we showed that $\alpha_j \beta_{j,m}$ and $\tilde{\nu}_{t-s,j}(m)$ satisfy the conditions in Lemma 3.7 for $\lambda = C_{21} e^{-(d^2+3n)} \|m_0\|^d$ and $\tau = C_{19} C_{20} e^{\frac{5a}{2}}$. Therefore, there exists $J' \subseteq J$ such that $|J'| \geq \frac{|J|}{2} \geq \frac{|J|}{4}$ and

$$\left\{ m \in B^{Z^d}(R) : \left| \frac{1}{|J_{\text{int}}|} \sum_{j \in J'} \tilde{\nu}_{t-s,j}(m) \right| \geq \frac{\tau}{2^6 \lambda} \right\} \geq \frac{\tau^3}{2^{17} \lambda^3} |B^{Z^d}(R)|.$$

Let $C_{22} := \frac{C_{20} \sqrt{2} \alpha}{2 C_{21}}$ and $\eta = \frac{\tau}{2^6 \lambda} = C_{22} e^{d^2+3n+\frac{5a}{2}} \|m_0\|^{-d}$. For $\nu = \frac{1}{|J'|} \sum_{j \in J'} \nu_{t-s,j}$ we have

$$\left| \left\{ m \in B^{Z^d}(R) : |\hat{\nu}(m)| \geq \eta \right\} \right| > \eta^3 |B^{Z^d}(R)|$$

as claimed. \hfill \Box

We now give the proof of Lemma 3.7 we postponed.

**Proof of Lemma 3.7.** Let $E \subset I \times J$ be the set of $(i, j)$ such that $a_{ij} \geq \frac{\tau}{2|I \times J|}$ and $|b_{ij}| \geq \frac{\tau}{25}$. Then for any $(i, j) \notin E$, $|a_{ij}b_{ij}| \leq \frac{\tau}{2|I \times J|}$. It follows that

$$\left| \sum_{(i,j) \notin E} a_{ij}b_{ij} \right| \leq |I \times J| \frac{\tau}{2|I \times J|} = \frac{\tau}{2}. $$
hence \( \left| \sum_{(i,j) \in \mathcal{E}} a_{ij} b_{ij} \right| \geq \frac{\tau}{2} \). For \( 0 \leq \theta < 2\pi \), denote by \( \mathcal{E}_\theta \subseteq \mathcal{E} \) the set of \((i,j)\) such that \( \theta - \frac{\pi}{4} \leq \arg b_{ij} \leq \theta + \frac{\pi}{4} \) modulo \( 2\pi \). By the pigeonhole principle, there exists \( 0 \leq \theta < 2\pi \) satisfying \( \left| \sum_{(i,j) \in \mathcal{E}_\theta} a_{ij} b_{ij} \right| \geq \frac{\tau}{8} \). Note that the argument of \( \sum_{(i,j) \in \mathcal{E}_\theta} a_{ij} b_{ij} \) is still between \( \theta - \frac{\pi}{4} \) and \( \theta + \frac{\pi}{4} \). Taking the real part, we get \( \sum_{(i,j) \in \mathcal{E}_\theta} a_{ij} \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{16} \). It follows from the assumptions \( 0 \leq a_{ij} \leq \frac{1}{\lambda |I \cap J|} \) and \( |b_{ij}| \leq 1 \) that

\[
|\mathcal{E}_\theta| \geq \sum_{(i,j) \in \mathcal{E}_\theta} \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{16\lambda} |I \times J|.
\]

For \( i \in I \), let \( \mathcal{E}_{\theta,i} := \{ j : (i,j) \in \mathcal{E}_\theta \} \) and \( I' := \{ i \in I : |\mathcal{E}_{\theta,i}| \geq \frac{\tau}{32\lambda} |J| \} \), then \( |\mathcal{E}_\theta| = \sum_{i \in I} \left| \mathcal{E}_{\theta,i} \right| \leq |I'| |J| + |I \setminus I'| \left( \frac{\tau}{32\lambda} |J| \right) \), hence we have

\[
|I'| \geq \frac{\tau}{32\lambda} |I|
\]

from (3.31). Also, for any \( i \in I' \),

\[
\sum_{(i,j) \in \mathcal{E}_{\theta,i}} \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{4\lambda} |\mathcal{E}_{\theta,i}| \geq \frac{\tau^2}{2\lambda^2} |J|
\]

since \( \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{4\lambda} \) for any \((i,j) \in \mathcal{E}_\theta \). Passing over subsets \( J_0 \subset J \) such that \( \frac{|J|}{4} \leq |J_0| \leq \frac{|J|}{2} \), we may choose a subset \( J_0 \subset J \) so that \( |J_0| \leq \frac{|J|}{4} \) and

\[
\sum_{i \in I'} |\mathcal{E}_{\theta,i} \cap J_0| \geq \frac{1}{4} \sum_{i \in I'} |\mathcal{E}_{\theta,i}|.
\]

Define independent random variable \( \{Z_j\}_{j \in J} \) so that \( Z_j \) is identically distributed to the Bernoulli random variables which takes values 1 and 0 with \( p = \frac{1}{2} \) if \( j \in J_0 \), and the constant random variable with \( P(Z_j = 1) = 1 \) if
For the inequality we are using the independence of \( \{Z_j\}_{j \in J} \) and the fact that \( \mathbb{E}(|Z - c|^2) \geq \mathbb{E}(|Z - \mathbb{E}(Z)|^2) \) holds for any random variable \( Z \) and \( c \in \mathbb{R} \). Note that the last line is the variance of \( \frac{1}{|J|} \sum_{j \in J_0 \cap \mathcal{E}_{\theta,i}} \text{Re}(b_{ij}e^{-i\theta})Z_j \). It follows that

\[
\mathbb{E} \left( \left\{ \frac{1}{|J|} \sum_{j \in J} \text{Re}(b_{ij}e^{-i\theta})Z_j \right\}^2 \right) \geq \mathbb{V} \left( \frac{1}{|J|} \sum_{j \in J_0 \cap \mathcal{E}_{\theta,i}} \text{Re}(b_{ij}e^{-i\theta})Z_j \right) \\
= \frac{1}{|J|} \sum_{j \in J_0 \cap \mathcal{E}_{\theta,i}} \text{Re}(b_{ij}e^{-i\theta}) \mathbb{V}(Z_j) \\
\geq \frac{\tau |\mathcal{E}_{\theta,i} \cap J_0|}{16\lambda |J|} \\
\]  

since \( \text{Re}(b_{ij}e^{-i\theta}) \geq \frac{\tau}{4\lambda} \) for any \( j \in \mathcal{E}_{\theta,i} \). Summing over \( i \in I' \), by (3.33) and the definition of \( I' \), we obtain

\[
\mathbb{E} \left( \left\{ \frac{1}{|I'|} \sum_{i \in I'} \left\{ \frac{1}{|J|} \sum_{j \in J} \text{Re}(b_{ij}e^{-i\theta})Z_j \right\}^2 \right\} \right) \geq \frac{\tau}{16\lambda |I'||J|} \sum_{i \in I'} |\mathcal{E}_{\theta,i} \cap J_0| \\
\geq \frac{\tau}{64\lambda |I'||J|} \sum_{i \in I'} |\mathcal{E}_{\theta,i}| \\
\geq \frac{\tau}{64\lambda |I'||J|} \frac{|I'|}{32\lambda |J|} = \frac{\tau^2}{211\lambda^2}. \\
\]  

Hence, the event \( \frac{1}{|I'|} \sum_{i \in I'} \left\{ \frac{1}{|J|} \sum_{j \in J} \text{Re}(b_{ij}e^{-i\theta})Z_j \right\}^2 \leq \frac{\tau^2}{211\lambda^2} \) happens, i.e. there exists \( J_0 \subseteq J' \subseteq J \) such that

\[
\frac{1}{|I'|} \sum_{i \in I'} \left\{ \frac{1}{|J|} \sum_{j \in J} \text{Re}(b_{ij}e^{-i\theta}) \right\}^2 \geq \frac{\tau^2}{211\lambda^2}. \\
\]
Let $I''$ be the set of $i \in I'$ such that $\frac{1}{|I'|} \sum_{j \in I'} \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{2^6 \lambda}$. Then

$$\frac{\tau^2}{2^{11} \lambda^2} \leq \frac{1}{|I'|} \sum_{i \in I''} 1 + \frac{1}{|I'|} \sum_{i \in I\setminus I''} \frac{\tau^2}{2^{12} \lambda^2} \leq \frac{|I''|}{|I'|} + \frac{\tau^2}{2^{12} \lambda^2},$$

hence $|I''| \geq \frac{\tau^2}{2^{13} \lambda^2} |I'|$. Combining with (3.32), we have $|I''| \geq \frac{\tau^3}{2^{17} \lambda^3} |I|$. On the other hand, the condition $\frac{1}{|I'|} \sum_{j \in J'} \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{2^6 \lambda}$ implies

$$\frac{1}{|I|} \sum_{j \in J'} b_{ij} = \max_{0 \leq \theta < 2\pi} \frac{1}{|J|} \sum_{j \in J'} \Re(b_{ij} e^{-i\theta}) \geq \frac{\tau}{2^6 \lambda}.$$

This concludes the proof. \qed

4. Proof of Theorem 1.3

4.1. High concentration of $\nu_{t-s}$ from a large Fourier coefficient. The aim of this subsection is to show that a large nontrivial Fourier coefficient $\hat{\nu}_{t,\omega}(m_0)$ implies the existence of a highly concentrated ball in $\mathbb{T}^d$ with respect to $\nu_{t-s}$. We make use of the following purely harmonic analytic lemma proved in [BFLM11], which is a quantitative analogue of Wiener’s lemma.

**Proposition 4.1.** [BFLM11, Proposition 7.5.] Let $N(E; S)$ denotes the covering number of $E \subset \mathbb{Z}^d$ by $S$-balls. There exists $c > 0$ so that if a probability measure $\nu$ on $\mathbb{T}^d$ satisfies

\begin{equation}
N\left(\left\{ m \in B^{Z^d}(0, R) : |\hat{\nu}(m)| > \tau \right\} ; S \right) > \lambda \left( \frac{R}{S} \right)^d
\end{equation}

with $S < \text{const}_d \cdot R$, then there exists an $S^{-1}$-separated set $A \subset \mathbb{T}^d$ with

$$\nu\left( \bigcup_{p \in A} B^{\mathbb{T}^d}(p, R^{-1}) \right) > c(\tau \lambda)^3.$$

In particular, there exists $p_0 \in \mathbb{T}^d$ with $\nu(B^{\mathbb{T}^d}(p_0, R^{-1})) > c(\tau \lambda)^3 S^{-d}$.

Combining Proposition 3.5 and 4.1, we deduce the following proposition about the concentration of $\nu_{t-s}$.

**Proposition 4.2.** Let $t > 0$, $C_{15} < s \leq \kappa_5 t$, and $r = e^{-\kappa_5 s}$, where $\kappa_5 := \min\{\frac{\kappa_4}{2d^2}, \frac{\kappa_3^2}{900d^2(d^2 + 3n + 1)}\}$. Let $z$ and $\omega$ be as in Proposition 3.5. If $|\hat{\nu}_{t,\omega}(m_0)| \geq C_{19} e^{-\kappa_5 s}$ for some $0 < \|m_0\| < e^{-\frac{\kappa_4}{4800d^2} n s}$, then there exists $p \in \mathbb{T}^d$ with

$$\nu_{t-s}(B^{\mathbb{T}^d}(p, e^{-ns})) > C_{24} e^{-\frac{\kappa_3}{39d} ns}$$

for some $C_{24} > 0$. 

Proof. Let $\epsilon = \frac{2d}{r^2}$. The constant $\kappa_5$ was taken to satisfy $r \geq e^{-\kappa_4 s}$, $\epsilon \geq e^{-\kappa_4 s}$, and
\[
|\hat{\nu}_{\omega}(m_0)| \geq C_1 r e^{-\kappa_5 s} = C_1 r e^{-\kappa_4 s},
\]
so the conditions of Proposition 3.5 is satisfied. Applying Proposition 3.5, we have
\[
|\left\{ m \in B^{\mathbb{Z}^d}(0, R) : |\hat{\nu}(m)| \geq \eta \right\}| > \eta^3 |B^{\mathbb{Z}^d}(R)|,
\]
where $\nu$, $\eta$, $R$ are as in Proposition 3.5. We apply Proposition 4.1 for the same $\nu$ and $R$. We take $S = 1$, $\tau = \eta$, and $\lambda \approx \eta^3$ so that the condition (4.1) is satisfied by a trivial bound
\[
N(\left\{ m \in B^{\mathbb{Z}^d}(0, R) : |\hat{\nu}(m)| > \tau \right\} ; S) \geq | \left\{ m \in B^{\mathbb{Z}^d}(0, R) : |\hat{\nu}(m)| > \tau \right\} |.
\]
Thus, there exists $p \in \mathbb{T}^d$ with $\nu(B^{\mathbb{T}^d}(p, R^{-1})) > c\eta^2$. Recall from Remark 3.6 that $\nu \leq C_{23} \nu_{t-s}$. By the assumptions we also have
\[
\eta = C_{22} e^{d^2 + 3a + 2} \|m_0\|^d > C_{22} e^{d^2 + 3a + 1} \|m_0\|^d > C_{22} e^{-\kappa_4 n s}
\]
and $R^{-1} = C_{13} e^d \|m_0\|^{-1} e^{-n s} \leq e^{-n s}$. Therefore, we get
\[
\nu_{t-s}(B^{\mathbb{T}^d}(p, e^{-n s})) \geq \nu_{t-s}(B^{\mathbb{T}^d}(p, R^{-1})) \geq C_{23}^{-1} \nu(B^{\mathbb{T}^d}(p, R^{-1})) > cC_{23}^{-1} C_{22} e^{-\kappa_4 n s}.
\]
4.2. Low concentration of $\nu_{t-s}$ from a Diophantine condition. The aim of this subsection is to show that if $b_0 \in \mathbb{T}^d$ is not approximated by a rational number with a small denominator, then the concentration of any ball in $\mathbb{T}^d$ is low with respect to $\nu_{t-s} = \nu_{y_0, t-s}$, where $y_0 = g_0 w(b_0)$. Recall the definition
\[
\zeta(b, T) := \min \left\{ N \in \mathbb{N} : \min_{1 \leq q \leq N} \| qb \|_Z \leq \frac{N^2}{T} \right\}
\]
for $b \in \mathbb{R}^d$ and $T > 0$.

Lemma 4.3. Let $\Xi$ be an interval in $\mathbb{T}$ of length $2\rho$. For $b \in \mathbb{T}^d$, if $\zeta(b, T) \geq \rho^{-2d}$, then
\[
| \left\{ m \in B^{\mathbb{Z}^d}(0, R) : b \cdot m \in \Xi \right\}| \leq C_{25} \rho |B^{\mathbb{Z}^d}(0, R)|
\]
for some $C_{25} > 1$.

Proof. Let $b = (b_1, \ldots, b_d)$ and write
\[
\zeta(b, R) := \min \left\{ N \in \mathbb{N} : \min_{1 \leq q \leq N} \| qb \|_Z \leq \frac{N^2}{R} \right\}
\]
for each $1 \leq i \leq d$. We first claim $\zeta(b, R) \leq \prod_{j=1}^{d} \zeta(b_j, R)$. By definition, for each $i$ there exists $1 \leq q_i \leq \zeta(b_i, R)$ such that $\|q_i b_i\| \leq \frac{\zeta(b_i, R)^2}{R}$. It follows that for any $1 \leq i \leq d$, we have

$$
\|q_1 \cdots q_d b\| \leq \|q_1 b_1\| \prod_{j \neq i} q_j \leq \frac{\zeta(b_i, R)^2}{R} \prod_{j \neq i} \zeta(b_j, R) \leq \frac{1}{R} \prod_{j=1}^{d} \zeta(b_j, R)^2,
$$

hence $\|q_1 \cdots q_d b\| \leq \frac{1}{R} \prod_{j=1}^{d} \zeta(b_j, R)^2$. Since $1 \leq q_1 \cdots q_d \leq \prod_{j=1}^{d} \zeta(b_j, R)$, the claim follows.

By the claim and the assumption $\zeta(b, R) \geq \rho^{-2d}$, there exists $1 \leq i \leq d$ such that $\zeta(b_i, R) \geq \rho^{-2}$. For the convenience of notations, assume that such $i$ is 1 without loss of generality. Note that $b_1$ is an irrational number. Write $b = (b_1, b') \in T \times T^{d-1}$ and $m = (k, m') \in \mathbb{Z} \times \mathbb{Z}^{d-1}$. The condition $b \cdot m \in \Xi$ is equivalent to $kb_1 \in \Xi'$, where $\Xi' = \Xi - b' \cdot m'$. Thus, it suffices to prove the following statement: for any interval $\Xi' \subset T$ of length $2\rho$ and $\zeta(b_1, R) \geq \rho^{-2}$,

\begin{equation}
(4.2) \quad |k \in B^{\Xi}(0, R) : kb_1 \in \Xi'| \ll \rho R.
\end{equation}

One can modify famous Weyl’s equidistribution criterion to be an effective version, and obtain the following upper bound of the density of the irrational rotational orbit on $T$ (See Lemma B.1 for a detailed computation):

\begin{equation}
(4.3) \quad \frac{1}{2R} \left| \left\{ k \in B^{\Xi}(0, R) : kb_1 \in \Xi' \right\} \right| \ll |\Xi'| + |\Xi'|^{-1} \zeta(b_1, R)^{-1}.
\end{equation}

Under the assumptions $|\Xi'| = |\Xi| = 2\rho$ and $\zeta(b_1, R) \geq \rho^{-2}$ it yields (4.2), so concludes the proof. \hfill \Box

In the rest of this subsection, the maps $\xi = \xi_{t-s} : X \times V \to \mathcal{F}$ and $\gamma = \gamma_{t-s} : X \times V \to \Gamma$ denote the same maps as in the Section 3, but for the parameter $t-s$ instead of $s$, i.e.

$$a_{t-s}(x) = \xi(x, u)\gamma(x, u).$$

The following proposition relates the concentration of the measure $\nu_{t-s} = \nu_{y_0, t-s}$ and the Diophantine condition of $y_0$.

**Proposition 4.4.** Let $0 < s < \frac{t}{2}$ and $\rho \geq e^{-n(t-s)}$. For $g_0 \in G$ and $b_0 \in T^d$, let $y_0 = g_0 w(b_0) \hat{\Gamma} \in Y$. Let $x_0 = \pi(y_0) \in X$ and assume $ht(x_0) \leq \rho^{-\frac{1}{108(d-1)}}$. For some constants $C_{26}, C_{27} > 0$, if $\zeta(b_0, \|g_0\|^{-1} e^{n(t-s)}) \geq C_{26} \rho^{-2d}$, then

$$\nu_{t-s}(B^{T^d}(p, \rho)) \leq C_{27} \rho^{\frac{1}{108}}$$

for any $p \in T^d$. 
Proof. Let $\gamma_0$ be the element of $\Gamma$ such that $g_0 = \iota(x_0)\gamma_0$. Then $y_0$ can be expressed by $y_0 = g_0w(b_0)\tilde{\Gamma} = \iota(x_0)w(\gamma_0b_0)\tilde{\Gamma}$. Let $b = \gamma_0b_0$, then $\sigma(y_0) = \gamma_0b_0 = b$. Denote by $\Theta : \mathbb{T}^d \to \mathbb{T}$ the projection onto the first coordinate. Let $\Xi = \Theta(B^\mathbb{T}((p, \rho)))$, then $\Xi \subseteq \mathbb{T}$ is an interval of length $2\rho$. By the definitions of $\nu_{t-s}$ and $\mu_{t-s}$, we have

$$\nu_{t-s}(\Theta^{-1}(\Xi)) = \frac{1}{m_H(V)}m_H(\{u \in V : \sigma(a_{t-s}uy_0) \in \Theta^{-1}(\Xi)\}).$$

By Lemma 3.1 with $\xi = \xi_{t-s}$ and $\gamma = \gamma_{t-s}$,

$$\sigma(a_{t-s}uy_0) = \gamma(x_0, u)\sigma(y_0) = \gamma(x_0, u)b$$

holds. Hence, the set in (4.4) can be expressed as

$$\{u \in V : \sigma(a_{t-s}uy_0) \in \Theta^{-1}(\Xi)\} = \{u \in V : e_1 \cdot \sigma(a_{t-s}uy_0) \in \Xi\}$$

$$= \{u \in V : \gamma(x_0, u)^{tr}e_1 \cdot b \in \Xi\}.$$ 

Let $\epsilon = \rho^{\frac{1}{m_0}}$ and $V_{x_0, \epsilon}$ be the set as in Lemma 3.2. Note that the conditions

$$\epsilon = \rho^{\frac{1}{m_0}} > e^{-\kappa_2(t-s)}$$

and $\text{ht}(x_0) < e^{\kappa_2(t-s)}$ of Lemma 3.2 and Proposition 4.3 for $\xi_{t-s}$ and $\gamma_{t-s}$ are satisfied since $\kappa_5 \leq \frac{e^{\kappa_2}}{10d^2}$. Then we have

$$m_H(V \setminus V_{x_0, \epsilon}) \leq C_{11}\epsilon m_H(V)$$

by Lemma 3.2. We now apply Proposition 3.3 for $t-s$ and $m_0 = e_1$. Let $R = C_{13}\epsilon^{-1}\text{ht}(x_0)^{d-1}e^{n(t-s)}$ as in Proposition 3.3 and $Q = B^{\mathbb{Z}^d}(0, R)$. Denoting $\Omega := \{m \in Q : b \cdot m \in \Xi\}$ and applying Proposition 3.3, we have

$$\{u \in V : \gamma(x_0, u)^{tr}e_1 \cdot b \in \Xi\} \subseteq (V \setminus V_{x, \epsilon}) \cup \bigcup_{m \in \Omega} \{u \in V_{x_0, \epsilon} : \gamma(x_0, u)e_1 = m\},$$

(4.7) 

$$m_H(\{u \in V_{x_0, \epsilon} : \gamma(x_0, u)e_1 = m\}) \leq C_{14}\epsilon^{-3n}e^{-dn(t-s)}$$

for any $m \in \mathbb{Z}^d$. We also have

$$|\Omega| \leq C_{25}\rho |Q| \leq \rho^{-d}\text{ht}(x_0)^{d(d-1)}e^{dn(t-s)}$$

by Lemma 4.3. Let $C_{26} := \max(C_1^2C_3^2C_{13}^{-\frac{2}{5}}, 1)$. Using (1.7), (1.8), (2.1), and (2.3), the assumption $\zeta(b_0, \|g_0\|^{-1}e^{n(t-s)}) \geq C_{26}\rho^{-2d}$ implies

$$\zeta(b, R) \geq \zeta(b_0, \|g_0\|^{-1}R) \geq \zeta(b_0, C_1^{-1}\|g_0\|^{-1}\|\iota(x_0)\|^{-1}R)$$

$$\geq \zeta(b_0, C_1^{-1}C_5^{-1}\|g_0\|^{-1}\text{ht}(x_0)^{-(d-1)}R)$$

$$\geq \min(C_1^{-\frac{1}{2}}C_3^{-\frac{1}{2}}C_{13}^{-\frac{2}{5}}, 1)\zeta(b_0, \|g_0\|^{-1}e^{n(t-s)}) \geq \rho^{-2d},$$

(4.10)
so the condition of Lemma 4.3 is satisfied. Combining (4.4)-(4.9),
\begin{equation}
\nu_{t,s}(\Theta^{-1}(\Xi)) = \frac{1}{m_H(V)} \min_{(V) \varepsilon H(V)} \{ \{ u \in V : \gamma(x_0, u)^r e_1 \cdot b \in \Xi \} \}
\leq \frac{m_H(V \varepsilon H(V \varepsilon x, \varepsilon)}{m_H(V)} + \sum_{m \in \Omega} \frac{m_H(\{ u \in V_{x,\varepsilon} : \gamma(x_0, u)e_1 = m \})}{m_H(V)}
\leq C_1 \epsilon + C_2 |\Omega| \epsilon^{-3n} e^{-dn(t-s)}
\ll \epsilon + \rho e^{-(d+3n)ht(x_0)^{d(d-1)}} \ll \rho^{\frac{\kappa_3}{10d}}.
\end{equation}
It completes the proof since $B^{t,d}(p, \rho) \subseteq \Theta^{-1}(\Xi)$.

4.3. Proof of Theorem 1.3. Combining Proposition 4.2 and 4.4, we deduce the following Fourier decay estimate.

**Proposition 4.5.** Let $t > 0$, $\|g_0\| \leq e^{\frac{\mu}{T}}$, $b \in \mathbb{T}^d$, and $y_0 = g_0 w(b_0)\hat{\Gamma}$. Let $\rho = \max \left( e^{-\kappa_5 t}, C_2^{\frac{1}{26}} \zeta(b_0, e^{\frac{\mu}{T}})^{-\frac{1}{3}} \right)$ and $r = \rho^{\frac{\kappa_3}{4}}$. Assume $\rho > C_2$ and
$\min_{\Omega} \{ \nu_{t,\omega}(m_0) \} < C_19 \rho^{\frac{\kappa_5}{4}}$
for any $0 < \|m_0\| < \rho^{-\frac{\kappa_3}{10d \nu}}$.

**Proof.** This is a direct consequence of Proposition 4.2 and 4.4, so it suffices to check the conditions in the propositions. Set $s = -\frac{t}{n} \log \rho$ so that $\rho = e^{-ns}$, then $\rho \geq e^{-\kappa_5 t}$ and $s \leq \kappa_5 t$. Since $\|g_0\| \leq e^{\frac{\mu}{T}}$, $\|g_0\|^{-1} e^{n(t-s)} \geq e^{\frac{\mu}{2}}$, hence the condition $\zeta(b_0, \|g_0\|^{-1} e^{n(t-s)}) \geq C_2 n^{-d(1-\epsilon)}$ is satisfied. It follows from Proposition 4.4 that
\begin{equation}
\nu_{t,s}(B^{t,d}(p, \rho)) \leq C_{27} \rho^{\frac{\kappa_3}{10d \nu}}
\end{equation}
for any $p \in \mathbb{T}^d$. Suppose that there exists some $0 < \|m_0\| < \rho^{-\frac{\kappa_3}{10d \nu}} = e^{\frac{\kappa_3}{480d^2}n}$ with $|\nu_{t,\omega}(m_0)| \geq C_19 \rho^{\frac{\kappa_3}{4}} = C_19 e^{-\kappa_5 s}$ for contradiction. For sufficiently large $\rho$, it follows from Proposition 4.2 that
\begin{equation}
\nu_{t,s}(B^{t,d}(p, \rho)) > C_{24} \rho^{\frac{\kappa_3}{10d \nu}} \geq C_{27} \rho^{\frac{\kappa_3}{10d \nu}},
\end{equation}
which contradicts to (4.12).

We now prove Theorem 1.3.

**Proof of Theorem 1.3.** We fix $y_0 = g_0 w(b_0)\hat{\Gamma} \in Y$ with $b_0 \in \mathbb{T}^d \backslash \mathbb{Q}^d$ and denote $\mu_t = \mu_{y_0,t}$ the same as we did. To prove Theorem 1.3, it suffices to find a constant $\delta' > 0$ such that if $\|g\| \leq \zeta(b_0, e^{\frac{\mu}{T}})^{-\delta}$, then
$\mu_t(f) = m_Y(f) + O(S(f)\zeta(b_0, e^{\frac{\mu}{T}})^{-\delta})$
for any $t \geq 0$, $f \in C^\infty_c(Y)$. Recall the notations that $l$ is the degree of the Sobolev norm $S$, $\delta_0$ is the constant of Theorem 1.1, and $\kappa_1, \cdots, \kappa_5$ are the constants we have chosen throughout the present paper. Given $t > 0$, let

$$\rho = \min \left( e^{-\kappa_6 t}, C_{2\delta}^{-\frac{1}{\kappa_6}} \zeta(b_0, e^{\frac{\delta_0}{2}}) - \frac{1}{\delta_0} \right)$$

as in Proposition 4.5, $\kappa_6 := \min\left( \frac{1}{4s0^2}, \frac{\kappa_4 \alpha}{3dn} \right)$ and $\epsilon = \rho^{\frac{\kappa_4}{2(d-1)}}$. We can also take $\delta_1 > 0$ to satisfy $\zeta(b_0, e^{\frac{\delta}{2}}) \delta_1 \leq \min\left( \rho^{-\frac{1}{10d^2(d-1)}}, e^{\frac{\delta}{2\kappa_1}} \right)$. The desired constant $\delta'$ will be taken to be $0 < \delta' < \delta_1$ so that $\|g_0\| \leq \zeta(b_0, e^{\frac{\delta}{2}}) \delta'$ implies $\text{ht}(x_0) \leq \rho^{-\frac{1}{10d^2(d-1)}}$ and $\|g_0\| \leq e^{\frac{\delta}{2}}$, which are the conditions in Proposition 4.5.

We fix a partition of unity $\{\omega_i\}_{i \in I}$ using Proposition 2.2 with a radius $r = \rho^{-\frac{\kappa_3}{2}}$, where $I = \{1, \cdots, N_f\} \cup \{\infty\}$. Let $\{z_1, \cdots, z_{N_f}\} \subset K(C^1 r^{-\frac{1}{2}})$ be the corresponding set as in Proposition 2.2. Let $I_{\text{int}} \subset I$ be the set of $i \in I$ such that $\iota(z_i) \in F(10r, \epsilon)$. For any $i \in I \setminus I_{\text{int}}$, each $\iota(B^X(z_i, r))$ is in $F \setminus F(11r, \frac{\epsilon}{2})$ and has $m_G$-measure $\ll r^{-d(d-1)}$. Thus, $|I \setminus I_{\text{int}}| \ll e^d r^{-d(d-1)}$ since $\iota(B^X(z_i, r))$’s are disjoint.

For any $i \in I_{\text{int}}$, $z_i$ and $\omega_i$ satisfy the conditions $\iota(z_i) \in F(10r, C_6 r^\frac{1}{2})$, $1_{B^X(z_i, r)} \ll \omega_i \ll 1_{B^X(z_i, 5r)}$, and $\|\nabla \omega_i\|_{L^\infty(X)} \ll C_9 r^{-d}$, which we have assumed in the previous propositions. Write $\mu_{t,i} := \mu_{t, \omega_i}$ and $\nu_{t,i} := \nu_{t, \omega_i}$ for simplicity, where $\mu_{t, \omega_i}$ and $\nu_{t, \omega_i}$ is defined as (3.11) and (3.15). Since $r^{\kappa_3} \ll \rho^{-\frac{\kappa_3 \kappa_4}{2}} \ll e^d$, $m_G(F \setminus F(11r, \frac{\epsilon}{2})) \ll e^d$ by (2.11).

Recall that we defined $\overline{f} \in C^\infty_c(X)$ by $\overline{f}(x) = \int_{\pi^{-1}(x)} f(y) dm_{\pi^{-1}(x)}(y)$ for $x \in X$, $f \in C^\infty_c(Y)$. Let us define $h \in C^\infty_c(Y)$ by $h(y) = f(y) - \overline{f}(\pi(y))$ for $y \in Y$. Note that

$$S(h) \ll S(f), \quad \|h\|_{L^\infty(Y)} \ll \|f\|_{L^\infty(Y)}, \quad |h(yy) - h(y)| \ll r S(f)$$

by definition. We also have

$$\int_{\pi^{-1}(x)} h(y) dm_{\pi^{-1}(x)}(y) = 0 \tag{4.14}$$

for any $x \in X$. We first decompose $\mu_t(f)$ as

$$\mu_t(f) = \mu_t(f \circ \pi) + \mu_t(h) = \pi_* \mu_t(\overline{f}) + \mu_t(h). \tag{4.15}$$

By (2.6) and $\text{ht}(x_0) \ll \|g_0\| \leq \zeta(b_0, e^{\frac{\delta}{2}}) \delta' \ll e^{\frac{\delta}{2\kappa_1}}$, we have

$$\pi_* \mu_t(\overline{f}) = m_X(\overline{f}) + O(\text{ht}(x_0)^{\kappa_1} S^X(\overline{f}) e^{-\delta_0 t}) = m_Y(f) + O(\text{ht}(x_0)^{\kappa_1} S(f) e^{-\delta_0 t})$$

$$= m_Y(f) + O(S(f) e^{-\frac{\delta_0 t}{2}}), \tag{4.16}$$
Split the second term of (4.15) by
\[
\mu_{t,i}(h) = \sum_{i \in I} \pi_i \mu_t(\omega_i) \mu_{t,i}(h)
\]
(4.17)
\[
= \sum_{i \in I \setminus I_{\text{int}}} \pi_i \mu_t(\omega_i) \mu_{t,i}(h) + \sum_{i \in I_{\text{int}}} \pi_i \mu_t(\omega_i) \mu_{t,i}(h).
\]
By (3.12) and a trivial bound $\mu_{t,i}(h) \leq \|h\|_{L^\infty(Y)} \ll S(f)$,
(4.18)
\[
\sum_{i \in I \setminus I_{\text{int}}} \pi_i \mu_t(\omega_i) \mu_{t,i}(h) \ll \|\mathcal{I}\mathcal{I}_{\text{int}}|S(f)|_{r(d-1)} \ll S(f) \epsilon^d = S(f) \rho^{d\kappa_6}.
\]
For each $i \in I_{\text{int}}$, denote by $\vartheta_i : \mathbb{T}^d \to \pi^\perp(z_i)$ the bijective Lipschitz function defined by $\vartheta_i(b) := \iota(z_i)w(b)\hat{\Gamma}$. Note that $\vartheta_i \circ \sigma = \text{id}_{\pi^\perp(z_i)}$ and the Lipschitz constant of $\vartheta_i$ is bounded above by $\|\iota(z_i)\| \leq C_5 \text{ht}(z_i)^{d-1}$ and below by $\|\iota(z_i)\|^{-1} \geq C_5^{-1} \text{ht}(z_i)^{-(d-1)}$. Let $h_i := h \circ \vartheta_i \in C^\infty_c(\mathbb{T}^d)$.

For any $i \in I_{\text{int}}$, Supp $\mu_{t,i} \subseteq \pi^\perp(B^Y(z_i, 5r))$ and $\iota(B^X(z_i, 5r)) \subseteq \mathcal{F}$. It follows that $d^\varphi(g, \iota(z_i)) \leq 5r$ for any $g \in \text{Supp} \omega_i \circ \phi$, so
(4.19)
\[
|h(gw(b)\hat{\Gamma}) - h(\iota(z_i)w(b)\hat{\Gamma})| \ll rS(f),
\]
i.e. $|h(y) - h_i(\sigma(y))| \ll rS(f)$ for any $y \in \text{Supp} \mu_{t,i}$. Hence,
(4.20)
\[
\mu_{t,i}(h) = \mu_{t,i}(h_i \circ \sigma) + O(rS(f)) = \mu_{t,i}(h_i) + O(rS(f)).
\]

For each $h_i \in C^\infty_c(\mathbb{T}^d)$ and $b \in \mathbb{T}^d$, we consider the Fourier expansion
(4.21)
\[
h_i(b) = \sum_{m \in \mathbb{Z}^d} \hat{h}_i(m)e^{-2\pi im \cdot b}.
\]
By (4.14), the Fourier coefficient at zero vanishes as
(4.22)
\[
\hat{h}_i(0) = \int_{\mathbb{T}^d} h_i(\iota(z_i)w(b)\hat{\Gamma})dm_{\mathbb{T}^d}(b) = 0.
\]

The smoothness of $h$ provides us a decay of nonzero Fourier coefficients. Let $S^d$ be a $l$-th degree Sobolev norm on $\mathbb{T}^d$ as $S^Y$. Then $\mathcal{S}^{\mathbb{T}^d}(h_i)$ is bounded above by $S^d(h \circ \vartheta_i) \ll \text{ht}(z_i)^{(d-1)}S(h)$. Hence, for any $m \in \mathbb{Z}^d \setminus \{0\}$,
(4.23)
\[
|\hat{h}_i(m)| \leq S^d(h_i)\|m\|^{-(d+2)} \ll \text{ht}(z_i)^{(d-1)}S(f)\|m\|^{-(d+2)}
\]
since $S$ is a $l$-th degree Sobolev norm and it controls the $L^\infty$ norm of $(d+2)$-th derivatives.

On the other hand, we obtained a upper bound of $|\hat{\nu}_{t,i}(m)|$ in Proposition 4.5. Recall $\kappa_5 = \min\left(\frac{\kappa_3}{\kappa_6}, \frac{\kappa_5}{\kappa_4}\right)$. By Proposition 4.5, we have
(4.24)
\[
|\hat{\nu}_{t,i}(m)| < C_1^9 \rho^{3\kappa_6}
\]
for any $0 < \|m\| < \rho^{-2\kappa_6}$. Expand $\nu_{t,i}(h)$ as

$$|\nu_{t,i}(h)| = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} |\hat{h}_i(m)\hat{\nu}_{t,i}(m)|$$

$$\leq \sum_{0 < |m| < \rho^{-2\kappa_6}} |\hat{h}_i(m)||\hat{\nu}_{t,i}(m)| + \sum_{|m| \geq \rho^{-2\kappa_6}} |\hat{h}_i(m)||\hat{\nu}_{t,i}(m)|.$$  

(4.25)

Then the first term in the last line is $< (\rho^{-2\kappa_6})^t S(f) \rho^{3\kappa_6} = \rho^{\kappa_6} S(f)$, using (5.9) and a trivial bound $|\hat{h}_i(m)| \leq \|h\|_{L^\infty(Y)} \ll S(f)$. Using (4.23) and a trivial bound $|\hat{\nu}_{t,i}(m)| \leq 1$, the second term in the last line is bounded by $ht(z_i)^{l(d-1)} S(f) \sum_{|m| \geq \rho^{-2\kappa_6}} |m|^{-(d+2)} \ll ht(z_i)^{l(d-1)} S(f) \rho^{2\kappa_6}$. Hence, we have

$$|\nu_{t,i}(h)| \ll |ht(z_i)^{l(d-1)} S(f) \rho^{2\kappa_6} \ll S(f) \rho^{\kappa_6}.$$  

(4.26)

since $ht(z_i) \leq \epsilon^{-1} = \rho^{-\frac{\kappa_6}{2(d-1)}}$. Taking summation over $i \in \mathcal{I}_{int}$ and using (4.20) and (4.26),

$$\sum_{i \in \mathcal{I}_{int}} \pi_i \mu_i(\omega) \mu_{t,i}(h) = \sum_{i \in \mathcal{I}_{int}} \pi_i \mu_i(\omega) \nu_{t,i}(h) + O(r S(f))$$

$$\leq \sum_{i \in \mathcal{I}_{int}} \pi_i \mu_i(\omega) S(f) \rho^{\kappa_6} + O(r S(f))$$

$$= O((\rho^{\kappa_6} + \rho^{\frac{\kappa_6}{n}} S(f))).$$

Combining (4.15)-(4.18) and (4.27), we obtain the estimate

$$\mu_t(f) = m_Y(f) + O(S(f)(\epsilon^{-\frac{\delta_0}{2}} + \rho^{\frac{2\kappa_6}{2d}})).$$

We note that $\zeta(b_0, e^{\frac{\delta_0}{2}}) \leq (e^{\frac{\delta_0}{2}})^{\frac{2d}{d+1}}$ by (1.9) and $\zeta(b_0, e^{\frac{\delta_0}{2}}) \ll \rho^2$. Hence we can find $\delta' > 0$ such that $\epsilon^{-\frac{\delta_0}{2}} + \rho^{\frac{2\kappa_6}{2d}} \ll \zeta(b_0, e^{\frac{\delta_0}{2}})^{-\delta'}$ and $\delta' < \delta_1$. This completes the proof of the main theorem.

\[\square\]

5. GENERAL DIAGONAL SUBGROUPS

In this section, we extend Theorem 1.3 for general 1-dimensional diagonal subgroups and give a sketch of the modification. Following the notations of [KW08] and [KMI12], let us denote by $a^+$ the set of $d$-tuples $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ such that

$$t_1 \geq \cdots \geq t_m > 0, \quad t_d \geq \cdots \geq t_{m+1} > 0, \quad \text{and} \quad \sum_{i=1}^{m} t_i = \sum_{j=1}^{n} t_{m+j},$$

and for $t \in a^+$ define

$$a_t := \text{diag}(e^{t_1}, \ldots, e^{t_m}, e^{-t_{m+1}}, \ldots, e^{-t_d}) \in G$$

and

$$|t| := \min_{1 \leq i \leq d} t_i.$$
In general the horospherical subgroup for $a_t$ may be bigger than

$$H = \left\{ \begin{pmatrix} I_{m} & A \\ 0 & I_{n} \end{pmatrix} \in G : A \in \text{Mat}_{m,n}(\mathbb{R}) \right\},$$

but we still have an effective equidistribution result for $a_t$-translates of $H$-orbits. The following theorem is a reformulation of [KM12, Theorem 1.3] which is a generalization of Theorem 1.1.

**Theorem 5.1.** Let $V \subset H$ be a fixed neighborhood of the identity in $H$ with smooth boundary and compact closure. Then there exist constants $\delta_0, \kappa_1 > 0$ only depending on $m$ and $n$ so that

$$\frac{1}{m_H(V)} \int_V f(a_tux)dm_H(u) = \int_X dm_X + O\left(ht(x)^{\kappa_1}S^X(f)e^{-\delta_0|t|}\right)$$

for any $t \in \mathfrak{a}^+$, $x \in X$ and $f \in C^\infty_c(X)$. Here, the implied constant depends only on $m, n$ and $V$.

The following is a generalization of the main result of this paper, Theorem 1.3, for general diagonal elements.

**Theorem 5.2.** Let $V$ be as in Theorem 5.1. Then there exists a constant $\delta' > 0$ only depending on $m$ and $n$ such that

$$\frac{1}{m_H(V)} \int_V f(a_tuy)dm_H(u) = \int_X dm_Y + O\left(S(f)\zeta(b,e^{\frac{|t|}{\delta'}})^{\delta'}\right)$$

for any $t \geq 0$, $f \in C^\infty_c(Y)$, and $y = gw(b)\hat{\Gamma}$ with $\|g\| \leq \zeta(b,e^{\frac{|t|}{\delta'}})^{\delta'}$ and $b \in \mathbb{T}^d$. Here, the implied constant depends only on $m, n$ and $V$.

We now give a sketch of the proof of Theorem 5.2. The proof is almost same with the proof of Theorem 1.3. We denote by $\mu_t = \mu_{y,t} \in \mathcal{P}(Y)$ the normalized probability measure on the orbit $a_tVy$ and decompose $\mu_t$ by $\sum_i \pi_i \mu_{t,\omega_i}(\omega_i)\mu_{t,\omega_i}$ with respect to a partition of unity \{\omega_i\}_{i \in I} of $X$, where $\mu_{t,\omega_i}(f) := \pi_i \mu_t(\omega_i)^{-1}\int f \omega_i \circ \pi d\mu_t$. We also write $\nu_t := \sigma_t \mu_t \in \mathcal{P}(\mathbb{T}^d)$ and $\nu_{t,\omega_i} := \sigma_t \mu_{t,\omega_i} \in \mathcal{P}(\mathbb{T}^d)$. As in the proof of Theorem 1.3, it suffices to prove that for each $\omega_i$, any nontrivial Fourier coefficient $\hat{\nu}_{t,\omega_i}(m_0)$ is small for $m_0 \in \mathbb{Z}^d \setminus \{0\}$ (Proposition 4.5).

Recall that we combined Proposition 4.2 and Proposition 4.4 to prove Proposition 4.5. Proposition 4.2 claimed that if $\hat{\nu}_{t,\omega_i}(m_0)$ is large for some small $m_0$, then $\nu_{t-s} = \sigma_s \mu_{t-s} = \sigma_s(a_{-s} \mu_t)$ is highly concentrated for any $C_{15} < s \leq \kappa_5 t$. For general diagonal elements, we consider $\mu_t$ and $\mu_{t'} = a_{-t} \mu_t$ with $0 < s \leq \frac{|t|}{\sigma}$, instead of $\mu_t$ and $\mu_{t-s} = a_{-s} \mu_t$. Here we denote by $t' \in \mathfrak{a}^+$ the vector $(t_1 - ns, \cdots, t_m - ns, t_{m+1} - ms, \cdots, t_d - ms) \in \mathbb{R}^d$.

We remark that the preliminaries in Section 2 still hold for general diagonal subgroup. Here we only state the following analogue of Proposition 2.1 which holds without any change of the proof. The rest of Section 2 clearly holds by replacing the use of Theorem 1.1 by Theorem 5.1.
Proposition 5.3. For any $0 \leq s < \frac{t_1}{d}$, $u_0 \in V$, $y \in Y$, and $f \in L^\infty(Y)$, we have

$$
\mu_{y,t}(f) = (a_s u_0) \ast \mu_{y,t}(f) + O(\|f\|_{L^\infty} e^{t_1 \frac{t_1}{d} ds}).
$$

Throughout Section 3 and Subsection 4.1, we investigated a relation between $\mu_{y,t}$ and $\mu_{y,t-s}$. We did not use any particular property of the measure $\mu_{y,t-s}$ and only used the fact that $\mu_{y,t-s}$ is approximately $a_s$-translate of $\mu_{y,t}$. Thus, the arguments in Section 3 and Subsection 4.1 still work for the measures $\mu_{y,t}$ and $\mu_{y,t} \ast$ without any modification. Hence, we get the following proposition as we obtained Proposition 4.2.

Proposition 5.4. Let $t \in a^+$, $C_{15} < s \leq \kappa_5 \frac{|t|}{d}$, and $r = e^{-\kappa_5 s}$, where $\kappa_5$ is as in Proposition 4.2. Let $z$ and $\omega$ be as in Proposition 3.5. If $|\nu_{t,\omega}(m_0)| \geq C_{19} e^{-\kappa_5 s}$ for some $0 < \|m_0\| < e^{\kappa_5 ns}$, then there exists $p \in T^d$ with

$$
\nu_{t'}(B^{zd}(p, e^{-ns})) > C_{24} e^{-\kappa_5 ns},
$$

where $\nu_{t'} = \sigma_* \mu_{t'} = \sigma_*(a_s \mu_t)$.

Now it remains to prove an analogue of Proposition 4.4. We made use of the results of Subsection 3.1 for the proof of Proposition 4.4, so we first need some modifications of Subsection 3.1. For

$$
t' = (t_1 - ns, \cdots, t_m - ns, t_{m+1} - ms, \cdots, t_d - ms)
$$

we define two maps $\xi = \xi_t : X \times V \to \mathcal{F}$ and $\gamma = \gamma_t : X \times V \to \Gamma$ as follows:

For any $x \in X$ and $u \in V$, there uniquely exist $\xi(x,u) \in \mathcal{F}$ and $\gamma(x,u) \in \Gamma$ such that

$$
(5.4) \quad a_t' u(x) = \xi(x,u) \gamma(x,u)
$$

by the definition of the fundamental domain $\mathcal{F}$.

For fixed $m_0 \in Z^d \setminus \{0\}$, we define two maps $x : X \times V \to \mathbb{R}^m$ and $y : X \times V \to \mathbb{R}^n$ satisfying the following:

$$
(5.5) \quad (\xi(x,u)^{tr})^{-1} m_0 = \begin{pmatrix} x(x,u) \\ y(x,u) \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n.
$$

Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_n)$.

Lemma 3.1 and Lemma 3.2 still hold in this setting as follows.

Lemma 5.5. For any $y \in Y$ and $u \in V$, we have $\sigma(a_{t'} y) = \gamma(x,u) \sigma(y)$, where $x = \pi(y)$.

Lemma 5.6. For $0 < \epsilon \leq \frac{1}{2}$ and $x \in X$ denote by $V_{x,\epsilon}$ the set of elements $u \in V$ satisfying:

1. $\|\xi(x,u)\| < \epsilon^{-1}$,
2. $|x_1(x,u)| > \epsilon^2 \|m_0\|.$

If $\epsilon > e^{-\kappa_2 |t'| |d|}$ and $ht(x) < e^{\kappa_2 |t'| |d|}$, then $m_H(V \setminus V_{x,\epsilon}) \leq C_{11} \epsilon^{\kappa_2}$ for some constants $C_{11} > 0.$
In (2) of Lemma 5.6, we replaced \( \|x(x, u)\| \) of Lemma 3.2 by \( |x_1(x, u)| \). However, the proof of Lemma 3.2 still works if one replace \( \{0_m\} \times \mathbb{R}^n \) in the definition of \( \mathcal{M} \) by \( \{0\} \times \mathbb{R}^{d-1} \).

Denote by \( \mathfrak{B}(t', D) \) the box 
\([-De'^{t_1}, De'^{t_1}] \times \cdots \times [-De'^{t_m}, De'^{t_m}] \times [-De'^{t_1}, De'^{t_1}] \times \cdots [-De'^{t_1}, De'^{t_1}] \subset \mathbb{R}^d \)
for \( C > 0 \). The following is a generalization of Proposition 3.3 and here is the place where a nontrivial modification is needed.

**Proposition 5.7.** Let \( s > 0 \) and \( m_0 \in \mathbb{Z}^d \setminus \{0\} \). For any \( e^{-\kappa_2 s} \leq \varepsilon \leq \frac{1}{2} \) and \( x \in X \) with \( \text{ht}(x) < e^{-\kappa_2 s} \), let \( V_{x, \varepsilon} \) be the set as in Lemma 3.2. Then
\[
(1) \quad \{ \gamma(x, u)^{\text{tr}} m_0 \in \mathbb{Z}^d : u \in V_{x, \varepsilon} \} \subseteq \iota(x)^{\text{tr}} \mathfrak{B}(t', C_{12} e^{-1} \|m_0\|),
\]
\[
(2) \quad \text{for any } m \in \mathbb{Z}^d,
\]
\[
m_H(\{ u \in V_{x, \varepsilon} : \gamma(x, u)^{\text{tr}} m_0 = m \}) \leq C_{14} e^{-3\alpha - nt'_1 - (t'_{m+1} + \cdots + t'_d)}.
\]

We remark that the estimate in this proposition is still tight as Proposition 3.3: If we consider \( \varepsilon \) and \( \text{ht}(x) \) as constants, as the points \( \gamma(x, u)^{\text{tr}} m_0 \) belongs to \( \iota(x)^{\text{tr}} \mathfrak{B}(t', C_{12} e^{-1} \|m_0\|) \) and the latter contains \( e^{nt'_1 + (t'_{m+1} + \cdots + t'_d)} \) points, the estimate in (2) is compatible with each \( m \in \iota(x)^{\text{tr}} \mathfrak{B}(t', C_{12} e^{-1} \|m_0\|) \) obtaining the same weight \( e^{-nt'_1 - (t'_{m+1} + \cdots + t'_d)} \).

**Proof.** Recall that there is the canonical measure-preserving bijection \( \varphi \) between \( M_{m,n}(\mathbb{R}) \) and \( H \) defined by \( \varphi(A) = \begin{pmatrix} \text{Id}_m & A \\ 0 & \text{Id}_n \end{pmatrix} \), there exists a constant \( C_{12} > 1 \) such that \( \varphi(B_{\mathbb{R}^{mn}}(0, C_{12}^{-1})) \subseteq V \subseteq \varphi(B_{\mathbb{R}^{mn}}(0, C_{12})) \), and \( \|u\| \leq C_{12} \) for any \( u \in V \).

(1) Recall that we have \( \|\xi(x, u)\| \leq \varepsilon^{-1} \), \( \|u\| \leq C_{12} \). Hence, \( \|x(x, u)\| \) and \( \|y(x, u)\| \) are bounded by \( e^{-1\|m_0\|} \). By explicit matrix computation, it follows that
\[
u^{\text{tr}} a_t^{(t)} (\xi(x, u)^{\text{tr}})^{-1} m_0 \in \mathfrak{B}(t', C_{12} e^{-1} \|m_0\|).
\]
By the definition of \( \xi \) and \( \gamma \) we have \( \xi(x, u)^{-1} a_t^t w u(x) = \gamma(x, u) \). Therefore,
\[
\gamma(x, u)^{\text{tr}} m_0 = \iota(x)^{\text{tr}} u^{\text{tr}} a_t^{(t)} (\xi(x, u)^{\text{tr}})^{-1} m_0 \in \iota(x)^{\text{tr}} \mathfrak{B}(t', C_{12} e^{-1} \|m_0\|).
\]
(2) The equation \( \gamma(x, u)^{\text{tr}} m_0 = m \) can be written
\[
(5.6) \quad a_t^{(t)} (\xi(x, u)^{\text{tr}})^{-1} m_0 = (u^{\text{tr}})^{-1} (\iota(x)^{\text{tr}})^{-1} m.
\]
Let \( v_+ \in \mathbb{R}^m \) and \( v_- \in \mathbb{R}^n \) be the vectors such that \( \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = (\iota(x)^{\text{tr}})^{-1} m \).
Here, \( v_+ \) and \( v_- \) only depend on \( m \), not on \( u \). For any \( A \in \varphi^{-1}(V) \), by a straightforward computation we have
\[
(5.7) \quad (u^{\text{tr}})^{-1} (\iota(x)^{\text{tr}})^{-1} m = \begin{pmatrix} v_+ \\ -A^{\text{tr}} v_+ + v_- \end{pmatrix},
\]
where \( u = \varphi(A) \). On the other hand, the left hand side of (5.6) is equal to \( (e^{t_1} x_1(x, u), \ldots, e^{t_m} x_m(x, u), e^{-t'_m} y_1(x, u), \ldots, e^{-t'_d} y_n(x, u)) \). Combining
this with (5.7), if \( u = \varphi(A) \) is a solution of (5.6), then

\[
(5.8) \quad A^tr v_+ - v_- \in \mathcal{I}(t', \|y(x, u)\|),
\]

denoting by \( \mathcal{I}(t', D) \) the small box

\[
[-De^{-t_{m+1}}, De^{-t_{m+1}}] \times \cdots \times [-De^{-t_1}, De^{-t_1}] \subset \mathbb{R}^n.
\]

If \( u \in V_{x, \epsilon} \), we have \( \|x_1(x, u)\| > \epsilon^2 \|m_0\| \) and

\[
\|y(x, u)\| \leq \|\xi(x, u)^tr\|^{-1} \|m_0\| \leq C_3 \|\xi(x, u)^tr\| \leq C_3 \epsilon^{-1} \|m_0\|
\]

by (5.5) and the definition of \( V_{x, \epsilon} \) in Lemma 5.6. Moreover, \( \|v_+\| > \epsilon^2 \|m_0\| \) since \( \|v_+\| \geq \epsilon^2 |x_1(x, u)| \). To sum up, a solution \( u = \varphi(A) \) must be in a thin tube \( \{A \in M_{m,n}(\mathbb{R}) : A^tr v_+ - v_- \in \mathcal{I}(t', C_3 \epsilon^{-1} \|m_0\|)\} \), where \( \|v_+\| > \epsilon^2 \|m_0\| \epsilon^{t_1} \). We can consider \( A \) as in the Euclidean space \( \mathbb{R}^m \), then the volume of the intersection of this tube and the ball \( B^{R^m}(0, C_{12}) \) is

\[
\ll \prod_{j=1}^n \left(C_{12}^{-1} \|v_+\|^{-1} (C_3 \epsilon^{-1} \|m_0\| \epsilon^{-t_{m+j}}) \right) \ll \epsilon^{-3n \epsilon^{-nt_1 - (t_{m+1} + \cdots + t_d)}. \]

Therefore, for any \( m \in \mathbb{Z}^d \), the \( m_H \)-measure of the set of \( u \in V_{x, \epsilon} \) satisfying \( \gamma_x(A)^tr \|m_0\| = m \) is less than \( C_{14} \epsilon^{-3n \epsilon^{-nt_1 - (t_{m+1} + \cdots + t_d)} \) for some constant \( C_{14} > 1 \).

We are now ready to prove an analogue of Proposition 4.4. The proof is almost same with Proposition 4.4, but here we count the number of certain points in \( \iota(x)^tr \mathcal{B}(t', C_{12} \epsilon^{-1}) \), instead of \( B^\mathbb{Z}(0, R) \).

**Proposition 5.8.** Let \( 0 < s < \frac{h \|u\|}{2} \) and \( \rho \geq e^{-\kappa \frac{|u|}{s}} \). For \( g_0 \in G \) and \( b_0 \in \mathbb{T}^d \), let \( y_0 = g_0 w(b_0) \hat{\Gamma} \in Y \). Let \( x_0 = \pi(y_0) \in X \) and assume \( h(x_0) \leq \rho^{1/(10d^2+1)} \).

If \( \zeta(b_0, \|g_0\|^{-1} e^{n(h(u, s))} \geq C_{26} \rho^{-2d} \), then

\[
\nu_t(B^\mathbb{T}(p, \rho)) \leq C_{27} \rho^{\frac{\kappa a}{10d^2}}
\]

for any \( p \in \mathbb{T}^d \).

**Proof.** We define \( \Theta : \mathbb{T}^d \to \mathbb{T} \) and \( \Xi = \Theta(B(\mathbb{T}^d, p, \rho)) \) as in Proposition 5.8, then we also have

\[
\nu_t(\Theta^{-1}(\Xi)) = \{u \in V : \gamma(x_0, u)^tr e_1 \cdot b \in \Xi \}.
\]

Let \( \epsilon = \rho^{\frac{s}{2}} \) and \( V_{x_0, \epsilon} \) be the set as in Lemma 5.6. Note that the conditions \( \epsilon = \rho^{\frac{s}{2}} > e^{-\kappa_2 \frac{|u|}{s}} \) and \( h(x_0) < e^{\kappa_2 \frac{|u|}{s}} \) of Lemma 5.6 and Proposition 4.3 are satisfied since \( \kappa_5 \leq \frac{\kappa_7}{10d^2} \) and \( |t'| > |t| - ds \). Then we have

\[
(5.9) \quad m_H(V \setminus V_{x_0, \epsilon}) \leq C_{11} \epsilon m_H(V)
\]

by Lemma 5.6. We now apply Proposition 5.7 for \( m_0 = e_1 \). Denoting \( \Omega := \{m \in \iota(x)^tr \mathcal{B}(t', C_{12} \epsilon^{-1}) : b \cdot m \in \Xi \} \) and applying Proposition 5.7, we
have
\begin{equation}
\{ u \in V : \gamma(x_0, u) e_1 \cdot b \in \mathbb{Z} \} \subseteq (V \setminus V_{x_0}) \cup \bigcup_{m \in \Omega} \{ u \in V_{x_0, \epsilon} : \gamma(x_0, u) e_1 = m \},
\end{equation}
for any \( m \in \mathbb{Z}^d \).

Write \( \mathcal{B} := \mathcal{B}(t', C_{12} \epsilon^{-1}) \) for simplicity. Since \( \mathcal{B} \) can be covered with \( \ll e^{n t'_1 + (t'_1 + \ldots + t'_m) - d |t'|} \) number of cubes with length \( \epsilon^{-1} e^{|t'|} \), we can also cover \( \nu(x) t' \mathcal{B} \) with \( \nu(x) t' Q'_1, \ldots, \nu(x) t' Q'_N \), where \( Q'_i \)'s are cubes with length \( \epsilon^{-1} e^{|t'|} \) and \( N \ll e^{n t'_1 + (t'_1 + \ldots + t'_m) - d |t'|} \). Since \( \|\nu(x)\| \leq C_{13} \text{ht}(x_0)^{d-1} \), the region \( \nu(x) t' \mathcal{B} \) is covered with \( Q_1, \ldots, Q_N \), where \( Q_i \)'s are cubes with length \( R := C_{13} \epsilon^{-1} \text{ht}(x_0)^{d-1} e^{|t'|} \).

Applying Lemma 4.3 for each \( Q_i \), we obtain
\begin{equation}
|\Omega| \leq C_{25} \rho \sum_{i=1}^N |Q_i| \ll \rho e^{-d \text{ht}(x_0)^{d(d-1)} e^{|t'|} N}
= \rho e^{-d \text{ht}(x_0)^{d(d-1)} e^{n t'_1 + (t'_1 + \ldots + t'_m)}}.
\end{equation}
Then the rest of the proof is the same as in Proposition 4.4.

Combining Proposition 5.4 and Proposition 5.8, we deduce the following Fourier decay estimate.

**Proposition 5.9.** Let \( t \in \mathfrak{a}^+, \|g_0\| \leq e^{\frac{1}{10}}, b \in \mathbb{T}^d \), and \( y_0 = g_0 w(b_0) \hat{\Gamma} \). Let \( \rho = \max \left( e^{-\kappa t}, C_{26} \frac{1}{\zeta(b_0, e^{\frac{1}{10}})^{\frac{1}{2}} - \frac{1}{2}} \right) \) and \( r = \rho^{\frac{\kappa s}{n}} \). Assume \( \rho > C_{28} \) and \( \text{ht}(\pi(y_0)) \leq \rho^{\frac{1}{10d^2(d-1)}} \). Let \( z \) and \( \omega \) be as in Proposition 3.5. Then
\[ |\widehat{u_{t\omega}}(m_0)| < C_{19} \rho^{\frac{\kappa s}{n}} \]
for any \( 0 < \|m_0\| < \rho^{\frac{\kappa s}{100d^2}} \).

Theorem 5.2 follows from Proposition 5.9 by the same procedure as in the proof of Theorem 1.3.

**Appendix A. Proofs of properties in §2.6**

In this subsection, we verify the properties we stated in §2.6. Recall that a continuous function \( F : G \to \mathbb{R}_{>0} \) is given by
\[ F(g) = \left( \sum_{i,j} |g_{ij}|^2 \right) \left( \sum_{i,j} |(g^{-1})_{ij}|^2 \right)^{-\frac{1}{2}} \sum_{i,j} |g_{ij}|^2 + \sum_{i,j} |(g^{-1})_{ij}|^2, \]
and satisfies \( F(g) \leq \min \left( \sum_{i,j} |g_{ij}|^2 \right)^{\frac{1}{2}}, \left( \sum_{i,j} |(g^{-1})_{ij}|^2 \right)^{\frac{1}{2}} \leq 2F(g) \) for any \( g \in G \). We will construct a fundamental domain \( \mathcal{F} \) of \( X = G/\Gamma \) so that it consists of every elements in \( \mathcal{F}^\circ := \{ g \in G : F(g) < F(g_{\gamma}) \}, \forall \gamma \in \Gamma \) and
properly chosen elements from its boundary. We can observe that the closure of $\mathcal{F}^\circ$ is $\{g \in G : F(g) \leq F(gG), \ \forall g \in \Gamma\}$ and the boundary $\overline{\mathcal{F}^\circ} \setminus \mathcal{F}^\circ$ has measure zero with respect to $m_G$.

(1) Let us first show that for any $x \in X$, there exists $g \in \overline{\mathcal{F}^\circ}$ such that $x = g\Gamma$. For any $x \in X$, let $\Lambda_x$ be the corresponding lattice in $\mathbb{R}^d$, and $\Lambda_x^*$ be the dual lattice of $\Lambda_x$, i.e.

$$\Lambda_x^* := \left\{ v^* \in \mathbb{R}^d : v^* \cdot v \in \mathbb{Z}^d, \ \forall v \in \Lambda_x \right\}.$$  

We refer [Cas71] for basic properties of the dual lattice. For any $g \in \phi^{-1}(x)$, the column vectors of $g$ are vectors in $\Lambda_x$ and the row vectors of $g^{-1}$ are vectors in $\Lambda_x^*$, where $\phi : G \to X$ is the canonical projection. Observe that for any $R > 0$, there are only finite vectors in either $\Lambda_x$ or $\Lambda_x^*$ with Euclidean length $\leq R$. It follows that the function $F|_{\phi^{-1}(x)} : \phi^{-1}(x) \to \mathbb{R}_{>0}$ that is restricted to the subset $\phi^{-1}(x)$ attains its minimum. Hence, for any $x \in X$, there exists $g \in \overline{\mathcal{F}^\circ}$ such that $x = g\Gamma$.

For each $\gamma \in \Gamma$, let $\mathcal{F}_\gamma := \{g \in G : F(g) \leq F(g\gamma)\}$. Note that $\mathcal{F} = \bigcap_{\gamma \in \Gamma} \mathcal{F}_\gamma$, $\partial \mathcal{F}_\gamma = \{g \in G : F(g) = F(g\gamma)\}$, and $\partial \mathcal{F} \subseteq \bigcup_{\gamma \in \Gamma} \partial \mathcal{F}_\gamma$. If $g$ is in the interior $\mathcal{F}^\circ$, then $F(g) < F(g\gamma)$ for any $\gamma \in \Gamma$. Hence, for any $x \in \phi(\mathcal{F}^\circ)$, $g \in G$ such that $x = g\Gamma$ is unique, where $\phi : G \to X$ is the canonical projection map. Let $\iota(x)$ be the unique element in $G$, then $\iota : \phi(\mathcal{F}^\circ) \to \mathcal{F}^\circ$ is a continuous measure-preserving map satisfying $\phi \circ \iota = \text{id}_{\phi(\mathcal{F}^\circ)}$.

We can choose elements from the boundary so that for any $x \in X$ there uniquely exists $g \in \mathcal{F}$ such that $x = g\Gamma$. Since the $m_G$-measure of the boundary $\overline{\mathcal{F}^\circ} \setminus \mathcal{F}^\circ$ is zero, the map $\iota$ is extended to be a measure preserving map from $X$ to $\mathcal{F}$. It also follows that $m_G(\mathcal{F}) = m_G(\mathcal{F}^\circ) = m_X(\phi(\mathcal{F}^\circ)) = m_X(X) = 1$.

(2) We next prove the estimate (2.9). Let $\lambda_j(\Lambda)$ denote the $j$-th successive minimum of a lattice $\Lambda \subset \mathbb{R}^d$ i.e. the infimum of $\lambda$ such that the ball $B^{\mathbb{R}^d}(0, \lambda)$ contains $j$ independent vectors. By Minkowski’s second theorem [Cas71, Theorem I in Chapter VIII], $\lambda_1(\Lambda_x) \lambda_2(\Lambda_x) \cdots \lambda_d(\Lambda_x) \asymp 1$ for any $x \in X$.

Since $\lambda_j(\Lambda_x) \geq \lambda_1(\Lambda_x) \geq \text{ht}(x)^{-1}$ for any $1 \leq j \leq d-1$, we have $\lambda_d(\Lambda_x) \ll \text{ht}(x)^{d-1}$. For $x \in \phi(\mathcal{F}^\circ)$ let $g = \iota(x)$, i.e. $g \in \mathcal{F}$ is the element with $x = g\Gamma$. Then $\lambda_d(\Lambda_x) \ll \text{ht}(x)^{d-1}$ implies that there exists $\gamma \in \Gamma$ such that the Euclidean length of each column vectors of $g\gamma$ is $\ll \text{ht}(x)^{d-1}$. Hence, We have $|g_{ij}| \leq F(g) \leq F(g\gamma) \ll \text{ht}(x)^{d-1}$ for any $i, j$. In order to estimate the components of $g^{-1}$, we may consider the dual lattice $\Lambda_x^*$. By Mahler’s inequality [Cas71, Theorem VI in Chapter VIII], $\lambda_1(\Lambda_x) \lambda_d(\Lambda_x^*) \ll 1$, so we have $\lambda_d(\Lambda_x^*) \ll \lambda_1(\Lambda_x) \ll \text{ht}(x)$. In other words, there exists $\gamma \in \Gamma$ such that the Euclidean length of each row vectors of $(g\gamma)^{-1}$ is $\ll \text{ht}(x)$. It follows that $|(g^{-1})_{ij}| \leq F(g) \leq F(g\gamma) \ll \text{ht}(x)$ for any $i, j$. Thus, $\|\iota(x)\| \ll \text{ht}(x)^{d-1}$ for any $x \in \phi(\mathcal{F}^\circ)$. 

(3) We will prove the estimate (2.10). Since we have
\[ m_G(\{g \in F : \text{ht}(g \Gamma) > \epsilon^{-1}\}) = m_X(X \setminus K(\epsilon)) \ll \epsilon \]
from (2.4), it is enough to show that \( m_G(\{g \in F : \text{dist}(g, \partial F) < r\}) \ll r^{\kappa_3} \)
for some \( \kappa_3 > 0 \). For each \( R > 0 \), denote by \( \Upsilon(R) \) the set of elements \( g \in G \)
such that \( (\sum_{i,j} |g_{ij}|^2)^{\frac{1}{2}} \ll R \). We make use of the following volume estimate
of \( \Upsilon(R) \) and counting estimate of \( \Upsilon(R) \cap \Gamma \) (See [DRS93, Example 1.6 and Appendix 1] or [EM93]):
\[ m_G(\Upsilon(R)) \asymp R^{\frac{d(d-1)}{2}}, \quad |\Upsilon(R) \cap \Gamma| \asymp R^{\frac{d(d-1)}{2}}. \]

In (2) we showed that if \( \text{ht}(x) \ll \epsilon^{-1} \), then the column vectors of \( g = \iota(x) \)
has Euclidean length \( \ll \epsilon^{-(d-1)} \), so \( (\sum_{i,j} |g_{ij}|^2)^{\frac{1}{2}} \ll \epsilon^{-(d-1)} \). It means that
the set \( F \setminus \Upsilon(R) \) is contained in \( \{ g \in F : \text{ht}(g \Gamma) \gg R^{\frac{d(d-1)}{2}} \} \), hence

\[ m_G(F \setminus \Upsilon(R)) \ll R^{-\frac{d}{d-1}}. \]

Observe that if \( g \in \Upsilon(R) \) and \( \gamma \notin \Upsilon(R^3) \), then \( F(g) < F(g\gamma) \), so \( g \) is not
on the boundary \( \partial F \). It follows that
\[ \partial F \cap \Upsilon(R) \subseteq \bigcup_{\gamma \in \Upsilon(R^3) \cap \Gamma} (\partial F \gamma \cap \Upsilon(R)). \]
Denote by \( F(r) \) and \( F_\gamma(r) \) the set of \( g \) such that \( \text{dist}(g, \partial F) < r \) and
\( \text{dist}(g, \partial F_\gamma) < r \), respectively. Note that each \( \partial F_\gamma \) is the set of zeroes of the equation \( F(g) = F(g\gamma) \). It is a \((\dim G - 1)\)-dimensional hypersurface,
so \( m_G(F_\gamma(r) \cap \Upsilon(R)) \ll m_G(\Upsilon(R)) r \ll R^{\frac{d(d-1)}{2}} r \) by (A.1). Therefore, using
(A.1) and (A.2), we have
\[
m_G(F(r)) = m_G(F(r) \cap \Upsilon(R)) + m_G(F(r) \setminus \Upsilon(R))
\leq m_G(\bigcup_{\gamma \in \Upsilon(R^3) \cap \Gamma} (F_\gamma(r) \cap \Upsilon(R))) + m_G(F \setminus \Upsilon(R))
\leq \sum_{\gamma \in \Upsilon(R^3) \cap \Gamma} m_G(F_\gamma(r) \cap \Upsilon(R)) + R^{-\frac{d}{d-1}}
\ll |\Upsilon(R^3) \cap \Gamma| R^{\frac{d(d-1)}{2}} r + R^{-\frac{d}{d-1}}
\ll R^{2d(d-1)} r + R^{-\frac{d}{d-1}}
\]
Choosing \( R \) to satisfy \( r = R^{-2(d-1)\frac{d}{d-1}} \), we obtain \( m_G(F(r)) \ll r^{\kappa_3} \) for \( \kappa_3 = \frac{1}{2(d-1)^2+1} \).

Appendix B. Effective Weyl’s criterion on \( \mathbb{T} \)

This subsection is devoted to prove the estimate (4.3) using an effective version of Weyl’s criterion. Neither the result nor the proof is new, but we prove it here to express the estimate in a suitable form using the function \( \zeta \). To prove the estimate (4.3) it is enough to show the following statement.
Lemma B.1. Let \( \Xi = [x_0 - \rho, x_0 + \rho] \) be an interval in \( T \), where \( x \in T \) and \( 0 < \rho < \frac{1}{2} \). For any \( \alpha \in T \setminus \mathbb{Q} \) and \( T \in \mathbb{N} \),

\[
\frac{1}{T} \left| \left\{ 0 \leq k \leq T - 1 : k\alpha \in \Xi \right\} \right| \ll \rho + \rho^{-1} \zeta(\alpha, T)^{-1},
\]

where \( \zeta(\alpha, T) = \min \left\{ N \in \mathbb{N} : \min_{1 \leq q \leq N} \| q\alpha \|_2 \leq \frac{N^2}{T} \right\} \).

Proof. For any \( 0 < \epsilon < \rho \), there exists an nonnegative approximation function \( \psi_\rho \in C^\infty_c(T) \) such that \( \text{Supp} \ \psi_\rho \subseteq [-\rho, \rho] \), \( \int_T \psi_\rho(x) dx = 1 \), and \( \| \psi_\rho \|_T \ll \rho^{-2} \). Then for any \( m \in \mathbb{Z} \setminus \{0\} \), we have \( |\hat{\psi}_\rho(m)| \ll \min(1, |m|^{-2} \rho^{-2}) \). Let \( \phi = \psi_\rho \cdot 1_{[x_0 - 2\rho, x_0 + 2\rho]} \), then \( \phi(x) \geq 1_{\Xi}(x) \) for any \( x \in T \). We also have

\[
|\hat{\phi}(m)| \ll \rho \min(1, |m|^{-2} \rho^{-2}) = \min(\rho, |m|^{-2} \rho^{-1})
\]
for \( m \in \mathbb{Z} \setminus \{0\} \). In order to get an upper bound of the left hand side of (B.1), it suffices to estimate \( \frac{1}{T} \sum_{0 \leq k \leq T-1} \phi(k\alpha) \). We can expand this summation as follows using the Fourier expansion of \( \phi \):

\[
\frac{1}{T} \sum_{0 \leq k \leq T-1} \phi(k\alpha) = \frac{1}{T} \sum_{m \in \mathbb{Z}} \hat{\phi}(m) \sum_{0 \leq k \leq T-1} e^{2\pi imk}\alpha
\]

\[
= \hat{\phi}(0) + \frac{1}{T} \sum_{0 < |m| < \zeta(\alpha, T)} \hat{\phi}(m) \frac{1 - e^{2\pi im\alpha}}{1 - e^{2\pi im\alpha}}
\]

\[
+ \sum_{|m| \gg \zeta(\alpha, T)} \hat{\phi}(m) \left( \frac{1}{T} \sum_{0 \leq k \leq T-1} e^{2\pi imk} \right).
\]

We now estimate each terms as follows:

\[
\hat{\phi}(0) = \int_T 1_{[x_0 - 2\rho, x_0 + 2\rho]}(x) dx = 4\rho,
\]

\[
\frac{1}{T} \sum_{0 < |m| < \zeta(\alpha, T)} \hat{\phi}(m) \frac{1 - e^{2\pi im\alpha}}{1 - e^{2\pi im\alpha}} \ll \rho \zeta(\alpha, T)^{-1},
\]

\[
\sum_{|m| \gg \zeta(\alpha, T)} \hat{\phi}(m) \left( \frac{1}{T} \sum_{0 \leq k \leq T-1} e^{2\pi imk} \right) \ll \rho^{-1} \sum_{|m| \gg \zeta(\alpha, T)} |m|^{-2} \ll \rho^{-1} \zeta(\alpha, T)^{-1},
\]

using \( |\hat{\phi}(m)| \ll \rho \) and \( \left| \frac{1 - e^{2\pi im\alpha}}{1 - e^{2\pi im\alpha}} \right| \ll \| m\alpha \|_T^{-1} \leq T(\zeta(\alpha, T) - 1)^{-2} \) for the second estimate, and \( |\hat{\phi}(m)| \ll \rho^{-1} |m|^{-2} \) and \( \left| \frac{1}{T} \sum_{0 \leq k \leq T-1} e^{2\pi imk} \right| \leq 1 \) for the third estimate. Hence, we obtain

\[
\frac{1}{T} \left| \left\{ 0 \leq k \leq T - 1 : k\alpha \in \Xi \right\} \right| \ll \frac{1}{T} \sum_{0 \leq k \leq T-1} \phi(k\alpha) \ll \rho + \rho^{-1} \zeta(\alpha, T)^{-1}.
\]

\[ \square \]
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**Wooyeon Kim. Department of Mathematics, ETH Zürich, wooyeon.kim@math.ethz.ch**