Constant-Time Testing and Learning of Image Properties
(full version)

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Abstract

We initiate a systematic study of sublinear-time algorithms for image analysis that have access only to labeled random samples from the input. Most previous sublinear-time algorithms for image analysis were query-based, that is, they could query pixels of their choice. We consider algorithms with two types of input access: sample-based algorithms that draw independent uniformly random pixels, and block-sample-based algorithms that draw pixels from independently random square blocks of the image. We investigate three basic properties: being a half-plane, convexity, and connectedness. For the first two, our algorithms are sample-based; for connectedness, they are block-sample-based. All our algorithms have low sample complexity that depends polynomially on the inverse of the error parameter and is independent of the input size.

We design algorithms that approximate the distance to the three properties within a small additive error or, equivalently, tolerant testers for being a half-plane, convexity and connectedness. Tolerant testers for these properties, even with query access to the image, were not investigated previously. Tolerance is important in image processing applications because it allows algorithms to be robust to noise in the image. We also give (non-tolerant) testers for convexity and connectedness with better complexity than our distance approximation algorithms and previously known query-based testers.

To obtain our algorithms for convexity, we design two fast proper PAC learners of convex sets in two dimensions that work under the uniform distribution: non-agnostic and agnostic.

1 Introduction

Image processing is a particularly compelling area of applications for sublinear-time algorithms and, specifically, property testing. Images are huge objects, and our visual system manages to process them very quickly without examining every part of the image. Moreover, many applications in image analysis have to process a large number of images online, looking for an image that satisfies a certain property among images that are generally very far from satisfying it. Or, alternatively, they look for a subimage satisfying a certain property in a large image (e.g., a face in an image where

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most regions are part of the background.) There is a growing number of proposed rejection-based algorithms that employ a quick test that is likely to reject a large number of unsuitable images (see, e.g., citations in [15]).

Property testing [20, 11] is a formal study of fast algorithms that accept objects with a given property and reject objects that are far. Testing of properties of images in this framework was first considered in [19]. Ron and Tsur [21] initiated property testing of images with a different input representation, suitable for testing properties of sparse images. Since these models were proposed, there have been several sublinear-time algorithms for visual properties that were implemented and used: namely, those by Kleiner et al. and Korman et al. [15, 16, 17].

However, for sublinear-time algorithms to reach their full potential in image processing, they have to be robust to noise: images are often noisy, and it is undesirable to reject images that differ only on a small fraction of pixels from an image satisfying the desired property. Tolerant testing was introduced by Parnas, Ron and Rubinfield [18] exactly with this goal in mind—to deal with noisy objects. It builds on the property testing model and calls for algorithms that accept objects that are close to having a desired property and reject objects that are far. Another related task is approximating distance of a given object to a nearest object with the property within additive error $\epsilon$. (Distance approximation algorithms imply tolerant testers in a straightforward way: see the remark after Definition 2.2). The only image problem for which tolerant testers were studied is the image partitioning problem investigated by Kleiner et al. [15].

Another feature that may make sublinear algorithms more applicable is operability with simple access to input. Most property testing algorithms, and, in particular, nearly all known algorithms for properties of images, need oracle access to input. However, in some applications, it may simply be impossible to provide such access. Goldreich and Ron [10] advocate investigating algorithms that rely only on a random sample. We embark on such an investigation for properties of images.

**Our results.** We consider algorithms for properties of images with two types of input access: sample-based algorithms that draw independent uniformly random pixels, and block-sample-based algorithms that draw pixels from independently random square blocks of the image. (Formal definitions are provided in Section 2.) Most previous sublinear-time algorithms for image analysis were query-based, that is, they could query pixels of their choice. Moreover, they were adaptive, i.e., their queries depended on answers to previous queries.

We investigate three basic properties: being a half-plane, convexity, and connectedness. For the first two, our algorithms are sample-based, and for connectedness, they are block-sample-based.

We design algorithms that approximate the distance to the three properties within a small additive error $\epsilon$, or, equivalently, tolerant testers for being a half-plane, convexity and connectedness. Tolerant testers for these properties, even with query access to the image, were not investigated previously. For all three properties, we give $\epsilon$-additive distance approximation algorithms that run in constant time (i.e., dependent only on $\epsilon$, but not the size of the image). We remark that even though it was known that these properties can be tested in constant time [19], this fact does not necessarily imply constant-query tolerant testers for these properties. E.g., Fischer and Fortnow [8] exhibit a property (of objects representable with strings of length $n$) which is property testable with a constant number of queries, but for which every tolerant tester requires $n^{\Omega(1)}$ queries.

Moreover, for connectedness and convexity, we improve the previously known property testing algorithms from [19, 4]. (The previously known algorithm for being a half-plane is optimal, albeit
Table 1: Complexity of known algorithms for half-plane, convexity, and connectedness (a./s.b./b.s.b. = adaptive/sample-based/block-sample-based, 1-s./2-s. = 1-sided error/2-sided error). To get complexity of $(\epsilon_1, \epsilon_2)$-tolerant testing, substitute $\epsilon = (\epsilon_2 - \epsilon_1)/2$ in the first two columns.

| Property          | Distance Approximation | Property Testing |
|-------------------|-----------------------|------------------|
|                   | Our Results           | Query and Time Complexity |
|                   | Sample Complexity     | Run Time         | Previous Work | Our Results |
| Half-plane        | s.b. $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ | $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ | a. 1-s. $O\left(\frac{1}{\epsilon}\right)$ [19] | — |
| Convexity         | s.b. $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ | $O\left(\frac{1}{\epsilon^2}\right)$ | a. 2-s. $O\left(\frac{1}{\epsilon}\right)$ [19] | s.b. 1-s. $O\left(\frac{1}{\epsilon^2}\right)$ |
| Connectedness     | b.s.b. $O\left(\frac{1}{\epsilon^2}\right)$ | $\exp\left(O\left(\frac{1}{\epsilon}\right)\right)$ | a. 1-s. $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ [4] | b.s.b. 1-s. $O\left(\frac{1}{\epsilon^2}\right)$ |

adaptive\(^1\)) First testers for these properties were given in [19]. Berman et al. [4] improved the technique that [19] used to obtain a connectedness tester, thereby decreasing its running time by a log $1/\epsilon$ factor. Our results on distance approximation and property testing are summarized and compared to previous work in Table 1. Our algorithms for convexity are the most technically difficult of our results, requiring a large number of new ideas.


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Our algorithms for convexity and half-plane work by first implicitly learning the object\(^2\). PAC learning was defined by Valiant [22], and agnostic learning, by Kearns et al. [14] and Haussler [12]. As a corollary of our analysis, we obtain fast proper PAC learners of convex sets in two dimensions and of half-planes that work under the uniform distribution. For convex sets, we get two PAC learners: non-agnostic and agnostic. For half-planes, we get an agnostic PAC learner. The sample and time complexity\(^3\) of agnostic PAC learners is as indicated in Table 1 for distance approximation algorithms for corresponding properties; the complexity of non-agnostic PAC learner for convex sets in two dimensions corresponds to that of the property tester for convexity.

While the sample complexity of our agnostic half-plane learner (and hence our distance approximation algorithm for half-planes) follows from the VC dimension bounds, its running time does not. Aagnostically learning half-spaces under the uniform distribution has been studied by [13], but only for the hypercube $\{-1, 1\}^d$ domains, not the plane. Our PAC learners of convex sets, in contrast to our half-plane learner, beat the VC dimension lower bounds on sample complexity.

\(^1\)One can also get a sample-based 2-sided error tester with sample complexity $O(1/\epsilon \log(1/\epsilon))$ from a proper PAC learner for half-planes, as described in Footnote 2. This sample complexity is achievable because Vapnik-Chervonenkis (VC) dimension of half-planes is 3. It is open whether the extra log $1/\epsilon$ is necessary for sample-based testers.

\(^2\)There is a known implication from learning to testing. As proved in [11], a proper PAC learning algorithm for property $P$ with sampling complexity $q(\epsilon)$ implies a 2-sided error (sample-based) property tester for $P$ that takes $q(\epsilon/2) + O(1/\epsilon)$ samples. There is an analogous implication from proper agnostic PAC learning to distance approximation with an overhead of $O(1/\epsilon^2)$ instead of $O(1/\epsilon)$. We choose to present our testers first and get learners as corollary because our focus is on testing and because we want additional features for our testers, such as 1-sided error, that do not automatically follow from the generic relationship.

\(^3\)All our results are stated for error probability $\delta = 1/3$. To get results for general $\delta$, by standard arguments, it is enough to multiply the complexity of an algorithm by $\log 1/\delta$. 

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(The sample complexity of a PAC learner for a class is at least proportional to the VC dimension of that class [7].) Since VC dimension of convexity of $n \times n$ images is $\Theta(n^{2/3})$ (this is the maximum number of vertices of a convex lattice polygon in an $n \times n$ lattice [2]), proper PAC learners of convex sets in two dimensions (that work under arbitrary distributions) must have sample complexity $\Omega(n^{2/3})$. However, one can do much better with respect to the uniform distribution. Baum [3] gave an algorithm for that task that takes $O(\epsilon^{-2})$ samples and polynomial in $1/\epsilon$ running time. We improve the sample complexity and the running time to $O(\epsilon^{-3/2})$. Surprisingly, it appears that this question has not been studied at all for agnostic learners.

Finally, we note that for connectedness, we take a different approach. Our algorithms do not try to learn the object first; instead they rely on a combinatorial characterization of distance to connectedness. We show that distance to connectedness can be represented as an average of distances of sub-images to a related property.

Comparison to other related work. The only previously known tolerant tester for image properties was given by Kleiner et al. [15]. They consider the following class of image partitioning problems, each specified by a $k \times k$ binary template matrix $T$ for a small constant $k$. The image satisfies the property corresponding to $T$ if it can be partitioned by $k - 1$ horizontal and $k - 1$ vertical lines into blocks, where each block has the same color as the corresponding entry of $T$. Kleiner et al. prove that $O(1/\epsilon^2)$ samples suffice for tolerant testing of image partitioning properties. Note that VC dimension of such a property is $O(1)$, so by Footnote 2, we can get a $O(1/\epsilon^2 \log 1/\epsilon)$ bound. Our algorithms required numerous new ideas to significantly beat VC dimension bounds (for convexity and connectedness) and to get low running time.

Another line of work potentially relevant for understanding connectedness of images is on connectedness of bounded degree-graphs. Goldreich and Ron [9] gave a tester for this property, subsequently improved by Berman et al. [4]. Campagna et al. [6] gave a tolerant tester for this problem. Even though we view our image as a graph in order to define connectedness of images, there is a significant difference in how distances between instances are measured (see [19] for details). We also note, that unlike in [6], our tolerant tester for connectedness is fully tolerant, i.e., it has no restrictions on the parameters for which it works.

Open questions. In this paper we give sample-based and block-sample-based testers for several important testing and tolerant testing problems for images. It is open whether these testers are optimal. No nontrivial lower bounds are known for these problems (for any non-trivial property, an easy lower bound on the query complexity of a tester is $\Omega(1/\epsilon)$; for a distance approximation algorithm, it is $\Omega(1/\epsilon^2)$. It is also open whether ability to ask arbitrary queries and adaptivity help in testing these properties. Our (block-)sample-based algorithms achieve the best known query complexity, even for adaptive algorithms. The exception is the half-plane property. The best known algorithm for it [19] is adaptive and has optimal query complexity of $O(1/\epsilon^2)$, while the best known sample-based tester for this property has complexity $O(1/\epsilon \log 1/\epsilon)$ (see Footnote 1).

Organization. We give formal definitions and notation in Section 2. Algorithms for being a half-plane, convexity, and connectedness are given in Sections 3, 4, and 5. Each of the sections presenting algorithms for being a half-plane and convexity starts by giving a distance approximation or testing algorithm and concludes with the corollary about the corresponding PAC learner.
2 Definitions and Notation

We use \([0..n]\) to denote the set of integers \(\{0, 1, \ldots, n-1\}\) and \([n]\) to denote \(\{1, 2, \ldots, n\}\). By \(\log\) we mean the logarithm base 2, and by \(\ln\), the logarithm base \(e\).

**Image representation.** We focus on black and white images. For simplicity, we only consider square images, but everything in this paper can be easily generalized to rectangular images. We represent an image by an \(n \times n\) binary matrix \(M\) of pixel values, where 0 denotes white and 1 denotes black. To keep the correspondence with the plane, we index the matrix by \([0..n]^2\) so that left/right refers to low/high values of the first coordinate and bottom/top to low/high values of the second coordinate. The object is a subset of \([0..n]^2\) corresponding to black pixels; namely, \(\{(i, j) \mid M[i, j] = 1\}\). The left border of the image is the set \(\{(0, j) \mid j \in [0..n]\}\). The right, top and bottom borders are defined analogously. The image border is the set of pixels on all four borders.

For any region \(R\) we use \(A(R)\) to denote its area. For a set \(S \subset [0..n]^2\) and \((i, j) \in [0..n]^2\) we define \(S + (i, j) = \{(x + i, y + j) : (x, y) \in S\}\).

**Distance to a property.** The absolute distance, \(\text{Dist}(M_1, M_2)\), between matrices \(M_1\) and \(M_2\) is the number of the entries on which they differ. The relative distance between them is \(\text{dist}(M_1, M_2) = \text{Dist}(M_1, M_2)/n^2\). A property \(\mathcal{P}\) is a subset of binary matrices. The distance of an image represented by matrix \(M\) to a property \(\mathcal{P}\) is \(\text{dist}(M, \mathcal{P}) = \min_{M' \in \mathcal{P}} \text{dist}(M, M')\). An image is \(\epsilon\)-far from the property if its distance to the property at least \(\epsilon\); otherwise, it is \(\epsilon\)-close to it.

**Computational Tasks.** We consider several computational tasks: property testing \([20, 11]\), tolerant testing \([18]\), additive approximation of the distance to the property, and proper (agnostic) PAC learning \([22, 14, 12]\). Here we define them specifically for properties of images.

**Definition 2.1** (Property tester). An \(\epsilon\)-tester for a property \(\mathcal{P}\) is a randomized algorithm that, given a proximity parameter \(\epsilon \in (0, 1/2)\) and access to an \(n \times n\) binary matrix \(M\),

1. accepts with probability at least \(2/3\) if \(M \in \mathcal{P}\);
2. rejects with probability at least \(2/3\) if \(\text{dist}(M, \mathcal{P}) \geq \epsilon\).

The tester has 1-sided error if it always accepts \(M \in \mathcal{P}\). Otherwise, it has 2-sided error.

A \((\epsilon_1, \epsilon_2)\)-tolerant tester is defined similarly to the property tester, except that it gets two proximity parameters \(\epsilon_1, \epsilon_2 \in (0, 1/2)\), and it must accept if \(\text{dist}(M, \mathcal{P}) \geq \epsilon_1\) and reject if \(\text{dist}(M, \mathcal{P}) \geq \epsilon\) (both with probability at least \(2/3\), as before).

**Definition 2.2** (Distance approximation algorithm). An \(\epsilon\)-additive distance approximation algorithm for a property \(\mathcal{P}\) is a randomized algorithm that, given an error parameter \(\epsilon \in (0, 1/4)\) and access to an \(n \times n\) binary matrix \(M\), outputs a value \(\hat{\epsilon} \in [0, 1/2]\) that with probability at least \(2/3\) satisfies \(|\hat{\epsilon} - \text{dist}(M, \mathcal{P})| \leq \epsilon\).

As observed in \([18]\), we can obtain an \((\epsilon_1, \epsilon_2)\)-tolerant tester for any property \(\mathcal{P}\) by running a distance approximation algorithm for \(\mathcal{P}\) with \(\epsilon = (\epsilon_2 - \epsilon_1)/2\). Thus, all our distance approximation algorithms directly imply tolerant testers.
Theorem 3.1. Distance Approximation To Being a Half-Plane

Definition 2.3 (Proper PAC learner). A proper PAC learning algorithm for class \( \mathcal{P} \) that works under the uniform distribution is given a parameter \( \epsilon \in (0, 1/2) \) and access to an image \( M \in \mathcal{P} \). It can draw independent uniformly random samples \((i, j)\) and obtain \((i, j)\) and \(M[i, j]\). With probability at least \(2/3\), it must output an image \(M' \in \mathcal{P}\) such that \(\text{dist}(M, M') \leq \epsilon\).

Definition 2.4 (Proper agnostic PAC learner). A proper agnostic PAC learning algorithm for class \( \mathcal{P} \) that works under the uniform distribution is defined as proper PAC learner, except that \( M \) is not required to be in \( \mathcal{P} \) and the output \( M' \) must satisfy \(\text{dist}(M, M') \leq \text{dist}(M, \mathcal{P}) + \epsilon\) with probability at least \(2/3\).

Access to the input. Given access to an image represented by an \( n \times n \) matrix \( M \), a query-based algorithm is allowed to query a pixel \((i, j)\) of its choice and obtain \(M[i, j]\). The query complexity of the algorithm is the number of pixels it queries. A query-based algorithm is adaptive if its queries depend on answers to previous queries and nonadaptive otherwise.

A sample-based algorithm can draw independent samples \((i, j)\) from the uniform distribution over the domain (i.e., \([0..n]^2\)) and obtain \((i, j)\) and \(M[i, j]\). A block-sample-based algorithm can specify a block length \( r \in [n] \). For \(x, y \in [0..n]\), where \(x\) and \(y\) are divisible by \(r\), an \(r\)-block \(B_r(x, y)\) is the set \([0..r]^2 + (x, y)\). The \(r\)-block partition\(^4\) of \([0..n]^2\) is the set of all valid \(r\)-blocks. For a block length \(r\) of its choice, the algorithm can draw a uniformly random \(B_r(x, y)\) from the \(r\)-block partition and obtain \(B_r(x, y)\) and the submatrix of \(M\) that specifies \(M[i, j]\) for all \((i, j) \in B_r(x, y)\). The sample complexity of a sample-based or a block-sample-based algorithm is the number of pixels of the image it examines.

Remark 2.1. Sample-based algorithms have access to independent (labeled) samples from the uniform distribution over the domain. Sometimes it is more convenient to design Bernoulli algorithms that only have access to (labeled) Bernoulli samples from the image: namely, each pixel appears in the sample with probability \(s/n^2\), where \(s\) is the sample parameter that controls the expected sample complexity. By standard arguments, a Bernoulli algorithm with the sample parameter \(s\) can be used to obtain a sample-based algorithm that takes \(O(s)\) samples and has the same guarantees as the original algorithm (and vice versa).

3 Algorithms for the Half-plane Property

An image is a half-plane if there exist an angle \(\varphi \in [0, 2\pi]\) and a real number \(c\) such that \(M[x, y] = 1\) (i.e., pixel \((x, y)\) is black) iff \(x \cos \varphi + y \sin \varphi \geq c\). In other words, an image is a half-plane if there is a line, called a separating line, that separates black and white pixels of the image. For all \(\varphi\) and \(c\), let \(M_{c, \varphi}^0\) denote the half-plane that satisfies the above inequality with parameters \(\varphi\) and \(c\), and let \(L_{c, \varphi}^0\) be the segment of the separating line that belongs to the image. We call \(\varphi\) the direction of \(M_{c, \varphi}^0\) (and \(L_{c, \varphi}^0\)). Note that \(\varphi\) is the oriented angle between the \(x\)-axis and a line perpendicular to \(L_{c, \varphi}^0\).

3.1 Distance Approximation To Being a Half-Plane

Theorem 3.1. There is a sample-based \(\epsilon\)-additive distance approximation algorithm for the half-plane property with sample complexity \(O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)\) and time complexity \(O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)\).

\(^4\)If \(n\) is not divisible by \(r\), we pad \(M\) with 0s on the bottom and right to obtain an \(n' \times n'\) matrix \(M'\), where \(n' = \lceil n/r \rceil r\), and let the algorithm sample \(r\)-blocks from \(M'\) instead of \(M\).
Proof. At a high level, our algorithm for approximating the distance to being a half-plane (Algorithm 1) constructs a small set \( H_\varepsilon \) of reference half-planes. It samples pixels uniformly at random, and outputs the empirical distance to the closest reference half-plane. The core property of \( H_\varepsilon \) is that the smallest empirical distance to a half-plane in \( H_\varepsilon \) can be computed quickly.

**Definition 3.1** (Reference directions and half-planes). Given \( \varepsilon \in (0, \frac{1}{4}) \), let \( a = \varepsilon n/\sqrt{2} \). Let \( D_\varepsilon \) be the set of directions of the form \( i \epsilon \) for \( i \in [0, 2\pi/\varepsilon] \), called reference directions. The set of reference half-planes, denoted \( H_\varepsilon \), consists of half-planes of the form \( M_\varphi \), where \( \varphi \in D_\varepsilon \), the reference line intersects \([0,n-1]^2\), and \( c \) is an integer multiple of \( a \).

In other words, for every reference direction, we space reference half-planes distance \( a \) apart. By definition, there are at most \( \sqrt{2} n/a = 2/\varepsilon \) reference half-planes for each direction in \( D_\varepsilon \) and, consequently, \( |H_\varepsilon| \leq 2\pi/\varepsilon \cdot (2/\varepsilon) < 13/\varepsilon^2 \).

**Algorithm 1:** Distance approximation to being a half-plane.

```plaintext
input : parameters \( n \in \mathbb{N} \) and \( \varepsilon \in (0, 1/4) \); sample access to an \( n \times n \) binary matrix \( M \).

1. Sample a set \( S \) of \( s = \frac{n^2}{\varepsilon^2} \ln \frac{9}{\varepsilon} \) pixels uniformly at random with replacement.
2. Let \( D_\varepsilon, H_\varepsilon \) be the sets of reference directions and half-planes (see Def. 3.1) and \( a = \varepsilon n/\sqrt{2} \).

// Compute \( \hat{\varepsilon} = \min \{ \hat{\varepsilon}(M') \} \), where \( \hat{\varepsilon}(M') = \frac{1}{s} \cdot |\{p \in S : M[p] \neq M'[p]\}| \) as follows:
3. foreach \( \varphi \in D_\varepsilon \) do
   // Lines with direction \( \varphi \) partition the image. Bucket sort samples by position in the partition:
   4. Assign each sample \((x, y)\) to bucket \( j = \lfloor (x \cos \varphi + y \sin \varphi)/a \rfloor \).
   5. For each bucket \( j \), compute \( w_j \) and \( b_j \), the number of white and black pixels it contains.
   6. For each \( j \), where \( M_{ja}^\varphi \in H_\varepsilon \), compute \( \hat{\varepsilon}(M_{ja}^\varphi) = \frac{1}{s} \sum_{k<j} b_k + \frac{1}{s} \sum_{k\geq j} w_k \).
   7. Output \( \hat{\varepsilon} \), the minimum of the values computed in Step 6.
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**Lemma 3.2.** For every half-plane matrix \( M \), there exists \( M' \in H_\varepsilon \) such that \( \text{dist}(M, M') \leq \varepsilon/2 \).

**Proof.** Consider a half-plane \( M_{\phi,c} \). Let \( \phi \) be a reference direction closest to \( \varphi \). Then \( |\varphi - \phi| \leq \varepsilon/2 \). We consider two cases. See Figures 3.1 and 3.2.

Case 1: suppose that there is a reference half-plane \( M_{\phi,d} \) such that the separating line segments \( L_{\phi,c} \) and \( L_{\phi,d} \) intersect. Then the symmetric difference of \( M_{\phi,c} \) and \( M_{\phi,d} \) is contained in two regions formed by line segments \( L_{\varphi,c} \) and \( L_{\varphi,d} \). Each of these regions is either a triangle or (if it contains a corner of the image) a quadrilateral. First, suppose both regions are triangles. The sum of lengths of their bases is at most \( \sqrt{2} n \), whereas the sum of their heights is at most \( \sin(\varepsilon/2) \times \sqrt{2} n \leq \varepsilon n/\sqrt{2} \). Hence, the sum of their areas is at most \( \varepsilon n^2/2 \).

If exactly one of the regions is a quadrilateral, we add a line through the corner of the image contained in the quadrilateral and the intersection point of \( L_{\varphi,c} \) and \( L_{\phi,d} \). It partitions the symmetric difference of \( M_{\varphi,c} \) and \( M_{\phi,d} \) into two pairs of triangular regions. Let \( \phi_1 \) (respectively, \( \phi_2 \)) be the angle between the new line and \( L_{\varphi,c} \) (respectively, \( L_{\phi,d} \)). Then \( \phi_1 + \phi_2 \leq \varepsilon/2 \). Applying the same reasoning as before to each pair of regions, we get that the sum of their areas is at most \( \phi_1 n^2 + \phi_2 n^2 \leq \varepsilon n^2/2 \). If both regions are quadrilaterals, we add a line as before for each of them
and apply the same reasoning as before to the three resulting pairs of regions. Again, the area of the symmetric difference of $M_{\phi,c}$ and $M_{\phi,d}$ is at most $\epsilon n^2/2$. Thus, $M_{\phi,d}$ is the required $M'$.

**Case 2:** there exist reference half-planes with separating line segments $L = L_{\phi,d}$ and $L' = L_{\phi,d} + a$ such that the line segment $L_{\phi,c}$ is between $L$ and $L'$. The region between $L$ and $L'$ has length at most $\sqrt{2}n$ and width $a$. Thus, its area is at most $\epsilon n^2$. Partition it into two regions: between $L$ and $L_{\phi,c}$ and between $L'$ and $L_{\phi,c}$. One of the two regions has area at most $\epsilon n^2/2$. Thus, $M_{\phi,d}$ or $M_{\phi,d} + a$ is the required $M'$.

**Analysis of Algorithm 1.** Let $\epsilon_M$ be the distance of $M$ to being a half-plane. Then there exists a half-plane matrix $M^*$ such that $\text{dist}(M, M^*) = \epsilon_M$. By a uniform convergence bound (see, e.g., [5]), since $s \geq (2/\epsilon^2)(\ln |H_\epsilon| + \ln 6)$ for all $\epsilon \in (0,1/4)$, we get that with probability at least $2/3$, $|\text{dist}(M, M') - \hat{\epsilon}(M')| \leq \epsilon/2$ for all $M' \in H_\epsilon$. Suppose this event happened. Then $\hat{\epsilon} \geq \epsilon_M - \epsilon/2$ because $\text{dist}(M, M') \geq \epsilon_M$ for all half-planes $M'$. Moreover, by Lemma 3.2, there is a matrix $\hat{M} \in H_\epsilon$ such that $\text{dist}(M, \hat{M}) \leq \text{dist}(M, M^*) + \text{dist}(M^*, \hat{M}) \leq \epsilon_M + \epsilon/2$. For this matrix, $\hat{\epsilon}(\hat{M}) \leq \text{dist}(M, \hat{M}) + \epsilon/2 \leq \epsilon_M + \epsilon$. Thus, $\epsilon_M - \epsilon/2 \leq \hat{\epsilon} \leq \epsilon_M + \epsilon$. That is, $|\epsilon_M - \hat{\epsilon}| \leq \epsilon$ with probability $2/3$, as required.

**Sample and time complexity.** The number of samples, $s$, is $O(1/\epsilon^2 \log 1/\epsilon)$. To analyze the running time, recall that $|D_\epsilon| = O(1/\epsilon)$. For each direction in $D_\epsilon$, we perform a bucket sort of all samples in $O(s)$ time. The remaining steps in the foreach loop of Step 3 can also be implemented

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\[5\] For simplicity of presentation, we equate the area of a convex region and a number of pixels in it, thus ignoring additional small-order terms. By Pick’s theorem, the number of pixels could be at most the area plus $2n$. It does not affect the asymptotic analysis of our algorithms.
to run in \(O(s)\) time. The overall running time of Algorithm 1 is thus \(O(1/\varepsilon \cdot s) = O(1/\varepsilon^3 \log 1/\varepsilon)\), as claimed in Theorem 3.1.

**Corollary 3.3.** The class of half-plane images is properly agnostically PAC-learnable with sample complexity \(O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})\) and time complexity \(O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})\) under the uniform distribution.

**Proof.** We can modify Algorithm 1 to output, along with \(\hat{\varepsilon} = \min_{M' \in H_{\varepsilon}} \hat{\varepsilon}(M')\), a reference half-plane \(\hat{M}\) that minimizes it. By the analysis of Algorithm 1, with probability at least 2/3, the output \(\hat{M}\) satisfies \(\text{dist}(M, \hat{M}) \leq \varepsilon M + \varepsilon\). □

### 4 Algorithms for Convexity

An image is **convex** if the convex hull of all black pixels contains only black pixels.

#### 4.1 Tester for Convexity

**Theorem 4.1.** There is a sample-based (1-sided error) \(\varepsilon\)-tester for convexity with sample and time complexity \(O(\varepsilon^{-3/2})\).

**Proof.** By Remark 2.1, to prove Theorem 4.1, it suffices to design a Bernoulli tester with expected \(O(\varepsilon^{-3/2})\) sample and time complexity. Our Bernoulli tester is Algorithm 2.

**Algorithm 2:** Bernoulli \(\varepsilon\)-tester for convexity.

```plaintext
input : parameters \(n \in \mathbb{N}\) and \(\varepsilon \in (0, 1/2)\); Bernoulli access to a \(n \times n\) binary matrix \(M\).
1. Set \(s = 20 \cdot \varepsilon^{-3/2}\). Include each pixel from the image in the sample with probability \(s/n^2\).
2. Bucket sort sampled black pixels by the \(x\)-coordinate into \(s\) bins to obtain list \(S_B\).
   Similarly, compute \(S_W\) for the sampled white pixels.
   // Check if the convex hull of \(S_B\) contains a pixel from \(S_W\).
3. Use Andrew’s monotone chain convex hull algorithm [1] to compute \(\text{UH}(S_B)\) and \(\text{LH}(S_B)\), the upper and the lower hulls of \(S_B\), respectively, sorted by the \(x\)-coordinate.
4. Merge sorted lists \(S_W, \text{UH}(S_B)\) and \(\text{LH}(S_B)\) to determine for each pixel \(w\) in \(S_W\) its left and right neighbors in \(\text{UH}(S_B)\) and \(\text{LH}(S_B)\). If \(w\) lies between the corresponding line segments of the upper and lower hulls, reject.
5. Accept.
```

Let \(\text{Hull}(V)\) denote the convex hull of a point set \(V\). If image \(M\) is convex, \(\text{Hull}(S_B)\) contains only black pixels, and Algorithm 2 always accepts.

Consider an image \(M\) that is \(\varepsilon\)-far from convexity. We show that Algorithm 2 rejects \(M\) with probability at least 2/3. To do that, we use black samples to construct a region \(W\) and a set \(B \subseteq S_B\) such that (1) \(\text{Hull}(B) \cup W\) covers almost all area of the image and (2) with high probability \(W\) contains almost no black points. Since the image is \(\varepsilon\)-far from convexity, this implies that \(\text{Hull}(B)\) contains enough white pixels so that at least one of them is likely to be sampled. (Notice that with the Bernoulli algorithm, we can view sampling black pixels and sampling white pixels as independent processes.) If it happens, Algorithm 2 rejects since \(\text{Hull}(B) \subset \text{Hull}(S_B)\).

We start by giving a high-level description of the construction of \(W\), which is at the core of the analysis of the algorithm. Initially, \(B = W = \emptyset\), and the whole image is the “gray zone”, the
region between Hull($B$) and $W$. To classify a new portion of the gray zone we conduct the following mental experiment: we “sweep” a line through that zone until it hits a sampled black pixel $b$. Let $\ell$ be the final position of the line and $W_\ell$ be the region of the gray zone that $\ell$ passed through. We add $b$ to $B$ and $W_\ell$ to $W$. We will argue that the number of black pixels in $W_\ell$ is upper-bounded by a geometrically distributed random variable. To guarantee that final $W$ is likely to contain a small number of black pixels, we ensure that the number of regions $W_\ell$ is small, namely, $O(1/\sqrt{\epsilon})$.

We do that by carefully choosing the directions of the sweeping lines. The first four sweeping lines are always the same. The main idea for choosing the subsequent lines is to make them parallel to the sides of the current Hull($B$).

Now we construct the set $B \subseteq S_B$ and the set of lines $L$ together with the associated regions $W_\ell$ for each line $\ell \in L$. Let $R$ be the minimum axis-parallel rectangle that contains all pixels in $S_B$. Let $\ell_0$ (respectively, $\ell_1, \ell_2, \ell_3$) be the lines that pass through its top (respectively, left, bottom, right) side. Let $b_0$ (respectively, $b_2$) be the leftmost sampled black pixel on $\ell_0$ (respectively, $\ell_2$). Let $b_1$ (respectively, $b_3$) be a sampled black pixel on $\ell_1$ (respectively, $\ell_3$). Initially, $B = \{b_0, b_1, b_2, b_3\}$ and $L = \{\ell_0, \ell_1, \ell_2, \ell_3\}$. Let $W_{\ell_0}$ (resp., $W_{\ell_2}$) be the set of pixels of the image $M$ that lie either above $\ell_0$ or to the left of $b_0$ on $\ell_0$ (resp., either below $\ell_2$ or to the left of $b_2$ on $\ell_2$). Let $W_{\ell_1}$ (resp., $W_{\ell_3}$) be the set of pixels of $M - W_{\ell_0} - W_{\ell_2}$ to the left of $\ell_1$ (resp., to the right of $\ell_3$), as shown in Figure 4.1. Let $T_0$ be the set of triangles of the region $R - \text{Hull}(B)$ (see Figure 4.2).

Next we explain how to update sets $B$ and $L$. Let $m = \log \sqrt{4/\epsilon}$. W.l.o.g. assume that $1/\epsilon$ is an even power of 2. We inductively construct sets $T_i$ for $i \in [m]$ and add new points from $S_B$ to $B$ and new lines to $L$ as follows. Initially, $T_i = \emptyset$. For every triangle $\triangle vb'b'' \in T_{i-1}$ such that $b', b'' \in B$ do the following: Let $\ell$ be the line parallel to the side $b'b''$, at the largest possible distance from $b'b''$, which passes through a point in $S_B \cap \triangle vb'b''$. Call this point $b$. Intuitively, we sweep a line parallel to the side $b'b''$ through the triangle $\triangle vb'b''$, starting at point $v$ until it hits a sampled black pixel $b$, and line $\ell$ is the final position of the sweeping line. Let $v'$ and $v''$ be the points where $\ell$ intersects the sides $vb'$ and $vb''$, as shown in Figure 4.3. Let $W_\ell$ be the pixels of $M$ in $\triangle vv'v''$ which are not on the line $\ell$. Add pixel $b$ to $B$, line $\ell$ to $L$, and triangles $\triangle bb'v'$ and $\triangle bb''v''$ to $T_i$. Finally, after we obtain $T_m$, define $W = \bigcup_{\ell \in L} W_\ell$. The final gray zone is the union of all triangles
Claim 4.3. For the triangles $T$ inside it. Let

$$\sum_{T \in T_i} \Delta^2 = \frac{a(h_a + h_c) - h_c}{2} = \frac{(a-c)h_c}{2}.$$  

Since triangles $T$ and $\Delta vv'v''$ are similar, $\frac{h_c}{h_a} = \frac{c}{a}$. Thus, $\frac{A(T') + A(T''')}{A(T)} = \frac{(a-c)h_c}{2} \cdot \frac{2}{ah_a} = (1 - \frac{c}{a})^2 \leq \frac{1}{4}$, as claimed. The last inequality holds since $(1 - x)\frac{1}{4}$ is maximized when $x = 1/2$. \qed

By Claim 4.3, $\sum_{T \in T_i} A(T) \leq \frac{1}{4} \sum_{T \in T_i} A(T')$ for all $i \in [m]$. The sum of the areas of all triangles in $T_0$ is at most $n^2$. Thus, the sum of the areas of all triangles in $T_m$ is at most $\frac{1}{4} \cdot n^2 = \frac{a}{4}$, completing the proof of Lemma 4.2. \qed

Lemma 4.4. With probability at least 3/4, the number of black pixels in $W$ is less than $\frac{3n^2}{4}$.

Proof. First, we give an upper bound on $|L|$, which is equal to the number of regions in $W$. Recall that $L$ consists of the lines that define the sides of $R$ and one line for each triangle in $T_i$ for all $i \in [0..m)$. Therefore,

$$|L| = 4 + \sum_{i=0}^{m-1} |T_i| \leq 4 + \sum_{i=0}^{m-1} 4 \cdot 2^i = 4 \cdot 2^m = \frac{8}{\sqrt{\epsilon}}.$$  

For every line $\ell \in L$, let random variable $X_\ell$ denote the number of black pixels in $W_\ell$. By definition of $W_\ell$, we did not sample any black pixels from this set, i.e., $W_\ell \cap S_B = \emptyset$. Recall that to define $W_\ell$, we were sweeping a line through this region until it hit a black pixel from the sample. Equivalently, we can consider the black pixels in $W_\ell$ in the order the line $\ell$ sweeps through them, ordering the pixels that are swept at the same time from left to right. (Ordering from left to right
is important only for $\ell_0$ and $\ell_1$; for the remaining lines it can be arbitrary.) Each black pixel we are considering, gets caught in the sample with probability $p = s/n^2$. As soon as we succeed in sampling a black pixel, we stop forming the region $W_\ell$. In other words, for every black pixel in $W_\ell$, we failed to sample it. Let $Y_\ell$ be a geometric random variable that counts the number of failed Bernoulli trials before the first success, where each trial is successful with probability $p = s/n^2$. Then $X_\ell \leq Y_\ell$ for all $\ell \in L$. (Note that $X_\ell$ could be less than $Y_\ell$ because, with the exception of the first two lines, we do not add pixels from line $\ell$ to $W_\ell$.) Since the regions $W_\ell$ are disjoint, random variables $X_\ell$ are independent. (A subtle point here is that each region $W_\ell$ depends on previously constructed regions. However, the number of black points in $W_\ell$ does not.) The expectation and variance of each $Y_\ell$ are $\mathbb{E}[Y_\ell] = \frac{1-p}{p} \leq \frac{1}{p}$ and $\text{Var}[Y_\ell] = \frac{1-p}{p^2} \leq \frac{1}{p^2}$.

Let $Y = \sum_{\ell \in L} Y_\ell$. Then the number of black pixels in $W$ is at most $Y$. Recall that sets $W_\ell$ do not intersect and, consequently, all variables $Y_\ell$ are independent. Thus, $\mathbb{E}[Y] \leq \frac{1}{p} \cdot \frac{8}{\sqrt{\epsilon}} = \frac{2\alpha^2}{\sqrt{\epsilon}}$ and $\sigma_Y = \sqrt{\text{Var}[Y]} \leq \sqrt{\frac{1-p}{p^2} \cdot \frac{8}{\sqrt{\epsilon}}} \leq \frac{\alpha^2}{\sqrt{10}}$. This implies that $\mathbb{E}[Y] + 2\sigma_Y \leq \frac{3\alpha^2}{\sqrt{\epsilon}}$. By Chebyshev’s inequality, $\Pr[Y \geq \frac{3\alpha^2}{\sqrt{\epsilon}}] \leq \frac{\Pr[Y \geq \mathbb{E}[Y] + 2\sigma_Y]}{4} \leq \frac{1}{4}$. \hfill \qed

Let $F$ denote the region formed by all triangles in $T_m$. Let $A$ be the event that the number of black pixels in $W$ is less than $\frac{3\alpha^2}{\sqrt{\epsilon}}$. By Lemma 4.4, it occurs with probability at least $3/4$. If $A$ occurs then by Lemma 4.2, the number of black pixels in $W \cup F$ is at most $\frac{17\alpha^2}{20}$. Therefore, the number of white pixels in Hull($B$) is at least $\frac{3\alpha^2}{20}$, since we can obtain a convex image by making all white pixels in Hull($B$) black and all black pixels in $W \cup F$ white. Algorithm 2 fails if it samples no white pixel from Hull($S_B$). But Hull($B$) $\subseteq$ Hull($S_B$). Let $E$ be the event that no white pixels from Hull($B$) were sampled. Recall that our construction of $W$ depends only on black samples. We can view sampling as two independent processes: first we decide for each black pixel whether to take it into the sample, then we do the same thing for white pixels. Then $\Pr[\bar{E} \cap A] = (1 - \frac{36}{20})^{-3/2} \leq e^{-3} < 1/12$. Thus, the success probability of the algorithm is at least $\Pr[\bar{E} \cap A] = \Pr[\bar{E}] \cdot \Pr[A] \geq 11/12 \cdot 3/4 > 2/3$, as desired.

**Sample and time complexity of Algorithm 2.** The expected size of the sample is $s = O\left(\frac{1}{\epsilon^{1/2}}\right)$. Because the samples are uniformly distributed, the expected running time of the bucket sort is $O(s)$. Conditioned on drawing $s'$ samples, Andrew’s algorithm runs in time $O(s')$ on a lexicographically sorted set of 2-dimensional points. Finally, merging also takes $O(s')$ time on list of length at most $s'$. Thus, the expected sample and time complexity are $O(s) = O\left(\frac{1}{\epsilon^{1/2}}\right)$. \hfill \qed

**Corollary 4.5.** The class of convex images is properly agnostically PAC-learnable with sample and time complexity $O\left(\frac{1}{\epsilon^{1/2}}\right)$ under the uniform distribution.

**Proof.** We can modify Algorithm 2 the image $M'$ whose black part is the convex hull of black samples. Suppose the algorithm is given a convex image $M$. By the analysis of Algorithm 2, with probability at least $3/4$, the constructed region $W$ contains less than $\frac{3\alpha^2}{\sqrt{\epsilon}}$ pixels. By construction, the gray region contains at most $en^2/4$ pixels. Thus, with probability at least $3/4$, $\text{dist}(M, M') \leq (3/5 + 1/4)en^2 \leq en^2$, as required. \hfill \qed

### 4.2 Distance Approximation to Convexity

**Theorem 4.6.** There is a sample-based $\epsilon$-additive distance approximation algorithm for convexity with sample complexity $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ and running time $O\left(\frac{1}{\epsilon^2}\right)$. 

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Proof. The starting point for our algorithm for approximating the distance to convexity (Algorithm 3) is similar to that of Algorithm 1 that approximates the distance to a nearest half-plane. We define a small set of reference polygons. Algorithm 3 outputs the empirical distance to the closest reference polygon. The key features of $P_\epsilon$ is that (1) every convex image has a nearby polygon in $P_\epsilon$ and (2) one can use dynamic programming (DP) to quickly compute the smallest empirical distance to a polygon in $P_\epsilon$.

We start by defining reference directions, lines, points, and line-point pairs that are later used to specify our DP instances. Reference directions are almost the same as in Definition 3.1.

**Definition 4.1** (Reference lines, line-point pairs). Fix $\epsilon_0 = \epsilon/144$. The set of reference directions is $D_\epsilon = \{\pi/2\} \cup \{i\epsilon_0 : \ i \in [0, \lfloor 2\pi/\epsilon_0 \rfloor] \}$. For every $\varphi \in D_\epsilon$, the set $L_\varphi$ of reference lines contains lines satisfying the equation $x \cos \varphi + y \sin \varphi = c$, where $c$ is an integer multiple of $\epsilon_0 n$. For each reference line, the set of reference points on $\ell$ contains points on $\ell$, which are inside $[0, n - 1]^2$, spaced exactly $\epsilon_0 n$ apart (it does not matter how the initial point is picked). A line-point pair is a pair $(\ell, b)$, where $\ell$ is a reference line and $b$ is a reference point on $\ell$.

Roughly speaking, a reference polygon is a polygon whose vertices are defined by line-point pairs. Reference polygons are built starting from reference boxes, which are defined next.

**Definition 4.2** (Reference box). A reference box is a set of four line-point pairs $(\ell_i, b_i)$ for $i = 0, 1, 2, 3$, where $\ell_0, \ell_2$ are distinct horizontal lines, such that $\ell_0$ is above $\ell_2$, and $(\ell_1, \ell_3)$ are distinct vertical lines, such that $\ell_1$ is to the left of $\ell_3$. The reference box defines a set of vertices $B_0 = \{b_0, b_1, b_2, b_3\}$ and a set of triangles $T_0$, formed by removing the quadrilateral $b_0 b_1 b_2 b_3$ from the rectangle delineated by the reference box, as shown on Figure 4.2.

Note that unlike in the analysis of Algorithm 2, line-point pairs here do not depend on the input. Intuitively, by picking a reference box, we decide to keep the area inside the quadrilateral $b_0 b_1 b_2 b_3$ black, the area outside the rectangle formed by $\ell_0, \ell_1, \ell_2, \ell_3$ white, and the triangles in $T_0$ gray, i.e., undecided for now.

Reference polygons are defined next. Intuitively, to obtain a reference polygon, we keep subdividing “gray” triangles in $T_0$ into smaller triangles and deciding to color the smaller triangles black or white or keep them gray (i.e., undecided for now).

**Definition 4.3** (Reference polygon). A reference polygon is an image of a polygon Hull$(B)$, where the set $B$ can be obtained from a reference box by the following recursive process. Initially, $T_{end} = \emptyset$ and $B = B_0$. While $T_0 \neq \emptyset$, move a triangle $T$ from $T_0$ to $T_{end}$ and perform the following steps:

1. (Base Change.) Let $T = \Delta b'b''$, where $b', b'' \in B$. Select reference points $b'_0, b''_0$ on the sides $b'v, b''v$, respectively, where $b'_0 = b', b''_0 = b''$ is a valid option. (This corresponds to coloring the quadrilateral $b'b''_0b''_0b'$ black.) Let $h$ be the height of $T$ w.r.t. $b'_0b''_0$.

2. (Recursive Step.) If $h > 6\epsilon_0 n$, choose whether to proceed with this step or go to Step 3 (both choices correspond to a legal reference polygon). Let $\varphi$ be the angle that $b'_0b''_0$ forms with the $x$-axis, and $\varphi \in D_\epsilon$ be such that $|\varphi - \varphi| \leq \epsilon_0/2$. Select a reference line-point pair $(\ell, b)$, where the line $\ell \in L_\varphi$ crosses the line segments $v_0b'_0$ and $v_0b''_0$ at $v'$ and $v''$, respectively, whereas $b$ is in the triangle $\Delta v_0b'_0b''_0$. Let $T' = \Delta b'_0v'b$, $T'' = \Delta b''_0v''b$, as shown on Figure 4.4. Add $b$ to $B$ and triangles $T', T''$ to $T_0$. (This represents coloring the top triangle white, the bottom triangle black, and keeping $T'$ and $T''$ gray.)
3. (Base Case.) Finish processing $T$ (this represents coloring $\triangle b_0 b'' v$ white).

By Remark 2.1, to prove Theorem 4.6, it suffices to design a Bernoulli tester that takes $s = O\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ samples in expectation and runs in time $O\left(\frac{1}{\epsilon^8}\right)$. Our Bernoulli tester is Algorithm 3. In Algorithm 3, we use the following notation for the (relative) empirical error with respect to an input image $M$, a set of sampled pixels $S$, and the size parameter $s$. Let $\hat{\epsilon}(M') = \frac{1}{s} \cdot |\{u \in S : M[u] \neq M'[u]\}|$. For every region $R \subseteq [0..n]^2$, we let $\hat{\epsilon}_+(R) = \frac{1}{s} \cdot |\{u \in S \cap R : M[u] = 0\}|$, and $\hat{\epsilon}_-(R) = \frac{1}{s} \cdot |\{u \in S \cap R : M[u] = 1\}|$, i.e., the empirical error if we make $R$ black/white, respectively.

**Algorithm 3:** Bernoulli approximation algorithm for distance to convexity.

**input**: parameters $n \in \mathbb{N}$ and $\epsilon \in (0, 1/4)$; Bernoulli access to an $n \times n$ binary matrix $M$.

1. Set $s = \Theta\left(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}\right)$. Include each image pixel in the sample $S$ with probability $p = s/n^2$.
2. Use dynamic programming to find $\hat{\epsilon}$, the smallest fraction of samples misclassified by a reference polygon in $P_{\epsilon}$.
3. foreach line-point pair $\langle \ell_1, b_1 \rangle$, where $\ell_1$ is a vertical line do
4. Set $\hat{\epsilon}_{\text{left}} = 1$.
5. Set $\hat{\epsilon}_{\text{right}} = 1$.
6. Let $v_1$ (respectively, $v_2$) be the point where $\ell_1$ intersects $b_0$ (respectively, $b_2$).
7. Compute $\hat{\epsilon}_{\text{left}} = \min(\hat{\epsilon}_{\text{left}}, \hat{\epsilon}_-(W_{\ell_1}) + \hat{\epsilon}_+(\triangle b_0 b_1 b_2) + \text{Best}(\triangle v_1 b_0 b_1) + \text{Best}(\triangle v_2 b_1 b_2))$
8. Compute $\hat{\epsilon}_{\text{right}} = \min(\hat{\epsilon}_{\text{right}}, \hat{\epsilon}_+(W_{\ell_2}) + \hat{\epsilon}_{\text{left}} + \hat{\epsilon}_{\text{right}})$.
9. return $\hat{\epsilon}$.

Subroutine **Best**, presented next, uses dynamic programming to choose the option with the smallest empirical relative error among those given in Definition 4.3, items 1-3.

Our set of reference polygons has two critical features. First, for each convex image there is a nearby reference polygon. This is proved in Section 4.2.1. It turns out that the empirical error for a region is proportional to the square root of its area. The second key feature of our reference polygons is that, for each of them, the set of considered triangles, $T_{\text{end}}$, has small $\sum_{T \in T_{\text{end}}} \sqrt{A(T)}$. 

![Figure 4.4: Triangle $\triangle b'b''v$.](image-url)
Proof. Consider a convex image $M$. We will show how to construct a nearby reference polygon $M'$ using the recursive process in Definition 4.3. First, we obtain a reference box (see Definition 4.2) for $M'$ as follows. Let $(\ell_0, b_0)$ be a line-point pair, where $b_0$ is black in $M$ and $\ell_0$ is the topmost horizontal line that contains such a reference point. Similarly, define $(\ell_1, b_2)$, replacing “topmost” with “bottommost”. Define the two line-point pairs $(\ell_2, b_1)$, $(\ell_3, b_3)$ with vertical lines analogously. The four line-point pairs $(\ell_i, b_i)$ for $i \in \{0, 4\}$ define the reference box for $M'$, as shown in Figure 4.5.

Next we construct the set $B$ from the reference box, as in Definition 4.3. In the description below, we specify the choices we make at each step of the recursive process. We also maintain two sets of line segments, $F_1$ and $F_2$, that are used in the analysis. Initially, $F_1 = F_2 = \emptyset$. The colors of the points in the description are with respect to the convex image $M$:

1. (Base Change.) Choose $b_0', b_0''$ to be the furthest from $b'b''$ black reference points on the sides $b'v, b''v$, respectively.

2. (Recursive Step.) If $h > 6\epsilon_0 n$, let $B_M$ denote the black component of $M$ and points $b_1', b_1''$ be the intersection points of $B_M$ with $b_0'v$ and $b_0''v$, respectively. Choose $(\ell, b)$ such that $\ell$ is the furthest line from $b_1', b_1''$ in $L_{\varphi'}$ that contains a black reference point, and let $b$ be this point. Let $\ell$ intersect $B_M$ at $y'$ and $y''$ as in Figure 4.6. Put the line segment $y'y''$ in $F_1$. If no line in $L_{\varphi'}$ (“above” $b_1' b_1''$) contains a black reference point or if $h \leq 6\epsilon_0 n$ go to Step 3.

The proof of this fact, as well as the analysis of the empirical error appears in Section 4.2.2. Finally, Section 4.2.3 completes the analysis of the algorithm, gives details of its implementation and presents the corollary about agnostic PAC learning of convex objects.

### 4.2.1 Existence of a nearby reference polygon

**Lemma 4.7.** For every convex image $M$, there exists $M' \in P_{\ell}$ such that $\text{dist}(M, M') \leq \epsilon/6$.

**Proof.** Consider a convex image $M$. We will show how to construct a nearby reference polygon $M'$ using the recursive process in Definition 4.3. First, we obtain a reference box (see Definition 4.2) for $M'$ as follows. Let $(\ell_0, b_0)$ be a line-point pair, where $b_0$ is black in $M$ and $\ell_0$ is the topmost horizontal line that contains such a reference point. Similarly, define $(\ell_2, b_2)$, replacing “topmost” with “bottommost”. Define the two line-point pairs $(\ell_1, b_1)$, $(\ell_3, b_3)$ with vertical lines analogously. The four line-point pairs $(\ell_i, b_i)$ for $i \in \{0, 4\}$ define the reference box for $M'$, as shown in Figure 4.5.

Next we construct the set $B$ from the reference box, as in Definition 4.3. In the description below, we specify the choices we make at each step of the recursive process. We also maintain two sets of line segments, $F_1$ and $F_2$, that are used in the analysis. Initially, $F_1 = F_2 = \emptyset$. The colors of the points in the description are with respect to the convex image $M$:

1. (Base Change.) Choose $b_0', b_0''$ to be the furthest from $b'b''$ black reference points on the sides $b'v, b''v$, respectively.

2. (Recursive Step.) If $h > 6\epsilon_0 n$, let $B_M$ denote the black component of $M$ and points $b_1', b_1''$ be the intersection points of $B_M$ with $b_0'v$ and $b_0''v$, respectively. Choose $(\ell, b)$ such that $\ell$ is the furthest line from $b_1', b_1''$ in $L_{\varphi'}$ that contains a black reference point, and let $b$ be this point. Let $\ell$ intersect $B_M$ at $y'$ and $y''$ as in Figure 4.6. Put the line segment $y'y''$ in $F_1$. If no line in $L_{\varphi'}$ (“above” $b_1' b_1''$) contains a black reference point or if $h \leq 6\epsilon_0 n$ go to Step 3.

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3. (Base Case.) Put the line segment $b_1' b_1''$ in $F_2$. Triangle $\triangle b_0' b_0'' v$ is not subdivided and is called a final triangle.

Observe that $M$ and $M'$ differ only on three types of regions: outside of the reference box, inside the final triangles, and “above” the reference lines used for subdivision in the construction of $M'$. To show that $\text{Dist}(M, M') \leq \epsilon_6$, we prove in Claims 4.8, 4.9, and 4.12 that the number of pixels in each of the three regions is small. For any region $R \subseteq [0..n)^2$, let $\text{Err}(R) = |\{u \in R : M[u] \neq M'[u]\}|$.

Next claim follows from the analysis of the convexity tester in [19].

Claim 4.8. The number of black pixels in $M$ outside the reference box is at most $12 \cdot \epsilon_0 n^2$.

Definition 4.4. For points $x$ and $y$, let $\ell(x, y)$ denote the line that passes through $x, y$, and $|xy|$ denote the distance between $x$ and $y$. Let $d(z, \ell)$ be the distance from a point $z$ to a line $\ell$.

Claim 4.9. Let $\triangle b_0' b_0'' v$ be a final triangle and points $b_1', b_1''$ be the points of intersection of $B_M$ with $b_0' v$ and $b_0'' v$, respectively. Then $\text{Err}(\triangle b_0' b_0'' v) \leq 3 \cdot |b_1' b_1''| \epsilon_0 n$.

Proof. If $h \leq 6 \epsilon_0 n$ then $\text{Err}(\triangle b_0' b_0'' v) \leq A(\triangle b_1' b_1'' v) \leq 3 \cdot |b_1' b_1''| \epsilon_0 n$. Now assume that $h > 6 \epsilon_0 n$ and no line in $L_{\phi'}$ with a black reference point intersects the line segments $b_1' v$ and $b_1'' v$ in $\triangle b_0' b_0'' v$.

Proposition 4.10. Let $T$ be a triangle with sides $a, b$ and $c$. Let $\alpha$ be the angle opposite to side $a$, and $h_a$ be the height w.r.t. base $a$ in $T$. If $\alpha \geq \pi/2$ then $h_a \leq a/2$.

Proof. By the cosine theorem, $a^2 = b^2 + c^2 - 2bc \cdot \cos \alpha \geq b^2 + c^2 \geq 4 \cdot A(T) = 2 \cdot a \cdot h_a$. Thus, $h_a \leq a/2$, as claimed.  

Proposition 4.11. Let $\triangle b_0' b_0'' v$ be a triangle in which $\angle b_0' v b_0''$ is obtuse and $B_M$ intersects the sides $b_0' v$ and $b_0'' v$. Let $\ell \in L_{\phi'}$ be a line that intersects $B_M$ at $y'$ and $y''$ as in Figure 4.6. Then the area of the portion of $B_M$ above $\ell$ is at most $\frac{|y'y''|^2}{4}$.
The area of $y$ shown in Figure 4.6. Let $\angle \text{be the line segment } \ell \text{ and that passes through point } b_0' \text{ (if } \angle v''v \geq \angle b_0'b_0''v). \text{ As shown in Figure 4.6. Let } v_1' \text{ be the intersection point of } \ell \text{ and the line segment } b_0'. \text{ Denote the angle between } \ell \text{ and } \ell(b_0', b_0'') \text{ by } \gamma. \text{ The distance between } \ell \text{ and } \ell \text{ is at most } 2\epsilon_0n. \text{ Otherwise there are two distinct lines from } L_{y'} \text{ that pass through the line segment } b_0'b_0'' \text{. Since } |b_0', b_0''| \leq \epsilon_0n \text{ the distance between the two lines is less than } \epsilon_0n, \text{ contradiction.}

Now we find an upper bound on the area of $B_M$ inside $\triangle b_0'b_0''v$. Let $B_M$ intersect $\ell$ at $y'$ and $y''$. Then $|y'y''| \leq \epsilon_0n$. By Propositions 4.11 and 4.10, the area of $B_M$ above $\ell$ is at most $(\epsilon_0n)^2/4$. The area of $B_M$ below $\ell$ is at most $A(b_0'b_0''v'v'')$. The area

$$A(b_0'b_0''v'v') = A(v_1'b_0''v'v') + A(\triangle b_0'b_0''v_1') \leq |v_1'b_0''| \cdot 2\epsilon_0n + A(\triangle b_0'b_0''v_1') \leq 2|b_0'b_0''|\epsilon_0n + A(\triangle b_0'b_0''v_1').$$

The last inequality holds since $\angle v_1'b_0''v_1'$ is obtuse. We find an upper bound on $A(\triangle b_0'b_0''v_1')$:

$$A(\triangle b_0'b_0''v_1') = \frac{|b_0'b_0''|d(v_1', \ell(b_1', b_1''))}{2} = \frac{|b_0'b_0''| \cdot |b_0''v_1'| \cdot \sin \gamma}{2} \leq \frac{|b_0'b_0''| \cdot \sqrt{2n(\epsilon_0/2)}}{2} < 0.4|b_0'b_0''| \cdot \epsilon_0n.$$

Thus, $A(b_0'b_0''v'v') \leq 2.4 \cdot |b_0'b_0''|\epsilon_0n$. The height $h = d(v, \ell(b_1', b_1'')) + \epsilon_0n$. If $|b_1'b_1''| \leq 10\epsilon_0n$ then, by Proposition 4.10, $d(v, \ell(b_1', b_1'')) \leq 5\epsilon_0n$. It implies that $h \leq 6\epsilon_0n$, contradiction. Therefore, $|b_1'b_1''| > 10\epsilon_0n$. By the triangle inequality $|b_0'b_0''| \leq |b_1'b_1''| + 2\epsilon_0n$. Thus,

$$A(b_0'b_0''v'v') \leq 2.4 \cdot (|b_1'b_1''| + 2\epsilon_0n)\epsilon_0n$$

and

$$\text{Err}(\triangle b_0'b_0''v) \leq 2.4 \cdot (|b_1'b_1''| + 2\epsilon_0n)\epsilon_0n + (\epsilon_0n)^2/4 \leq 3 \cdot |b_1'b_1''|\epsilon_0n.$$

The last inequality holds since $|b_1'b_1''| > 10\epsilon_0n$. This completes the proof of Claim 4.9. \qed
Claim 4.12. Let triangle $\triangle b_0b''v$ and line $\ell$ be as defined in Step 2 of the recursive construction of $M'$. Then the area of $B_M$ above $\ell$ is at most $3 \cdot |y'y''|\epsilon_0 n$.

Proof. If $|y'y''| \leq \epsilon_0 n$ then, by Propositions 4.11 and 4.10, the area of $B_M$ above $\ell$ is at most $|y'y''|^2/4 \leq 3 \cdot |y'y''|\epsilon_0 n$. Now assume that $|y'y''| > \epsilon_0 n$. Let $\ell' \in L_{\phi'}$ be the line above $\ell$ at distance $\epsilon_0 n$ from it, as in Figure 4.7. The portion of $B_M$ between $\ell$ and $\ell'$ is at most $|y'y''|\epsilon_0 n$. The distance between the points of intersection of $B_M$ with $\ell'$ is at most $\epsilon_0 n$. Thus, by Propositions 4.11 and 4.10, the portion of $B_M$ above $\ell'$ is at most

$$|y'y''|\epsilon_0 n + \frac{(\epsilon_0 n)^2}{4} \leq 3 \cdot |y'y''|\epsilon_0 n.$$ 

The last inequality holds since $|y'y''| > \epsilon_0 n$. This completes the proof of Claim 4.12. \qed

Observe that all points in $B$ lie on the boundary of a convex polygon. Images $M$ and $M'$ differ only on pixels outside of the reference box and “above” the line segments in $F_1 \cup F_2$. All the line segments in $F_1 \cup F_2$ are the sides of a convex polygon that is inside an $n \times n$ square. Thus, the sum of the lengths of the line segments in $F_1 \cup F_2$ is at most $4n$. By Claims 4.8, 4.9 and 4.12,

$$\text{Dist}(M, M') \leq \left( \sum_{v'_1v'_2 \in F_2} |b'_1b'_2| + \sum_{y'y'' \in F_1} |y'y''| \right) \cdot 3\epsilon_0 n + 12\epsilon_0 n^2 \leq 24\epsilon_0 n^2.$$ 

This completes the proof of Lemma 4.7. \qed

4.2.2 Error analysis

Lemma 4.13. For each set $T_{\text{end}}$ obtained in the construction of a reference polygon in Definition 4.3,

$$\sum_{T \in T_{\text{end}}} \sqrt{A(T)} < 11n.$$ 

Proof. All triangles in $T_{\text{end}}$ are obtained by partitioning the four initial triangles in $T_0$. The following claim analyzes how the area is affected by one step of partitioning.
Figure 4.8: Triangle $\triangle b_0' b_0'' v$.

**Claim 4.14.** Let $T'$ and $T''$ be two gray triangles obtained from a triangle $T$ during one step of the recursive process in Definition 4.3. Then $\sqrt{A(T')} + \sqrt{A(T'')} \leq \sqrt{A(T)}$.

**Proof.** Observe that $\sqrt{A(T')} + \sqrt{A(T'')} = 6$ is maximized when $b_0' = b'$ and $b_0'' = b''$. W.l.o.g. we prove the lemma for this case. We use notation from Figure 4.8. Recall that a triangle is partitioned only if its height $h \geq 6\varepsilon_0 n$. Since the sides of $T$ are of length at most $\sqrt{2}n$, the height is large only if both angles adjacent to the base $b_0'b_0''$ are greater than $4\varepsilon_0$. (To see this, consider an angle $\alpha$ between the base and a side of length $a$. We get $6\varepsilon_0 n \leq h = a \cdot \sin \alpha \leq \sqrt{2}n \cdot \alpha$. Thus, $\alpha \geq 6\varepsilon_0/\sqrt{2} > 4\varepsilon_0$.)

First, we find the maximum value of $\sqrt{A(T')} + \sqrt{A(T'')} = 6$ at a fixed line $\ell$ on which position of point $b$ varies. Let $\alpha = \angle b_0'b_0''v$, $\beta = \angle b_0'b_0''v$ and $\gamma$ be the angle between lines $\ell$ and $\ell(b_0', b_0'')$. W.l.o.g. assume that $\angle v'v''v \leq \beta$. Then $\angle v'v''v = \alpha + \gamma$ and $\angle v'v''v = \beta - \gamma$. By the construction of triangles in $T_{\text{end}}$, $\alpha + \beta \leq \frac{\pi}{2}$ and $\gamma \leq \frac{\alpha}{2}$. Let $q = |b_0'v|$, $r = |b_0''v|$, $t = |v'v''|$ and $q x = |v'v|$, $r y = |v''v|$, $tz = |b''v|$ $(x, y, z \in [0, 1])$. Let

$$f(z) = \sqrt{A(T')} + \sqrt{A(T'')} = \sqrt{\frac{q(1 - x) + t(1 - z) \sin(\alpha + \gamma)}{2}} + \sqrt{\frac{r(1 - y) + tz \sin(\beta - \gamma)}{2}}.$$  

Thus, $f(z) = \sqrt{C_1 \cdot (1 - z)} + \sqrt{C_2 \cdot z}$, where $C_1 = A(\triangle b_0'v'v'')$, $C_2 = A(\triangle b_0''v'v')$ are constants. By the Cauchy-Schwarz inequality, $f(z) = \sqrt{C_1 \cdot (1 - z)} + \sqrt{C_2 \cdot z} \leq \sqrt{(C_1 + C_2)(1 - z + z)} = \sqrt{C_1 + C_2}$.

Next, we find the maximum value of $C_1 + C_2$ varying position of $\ell$ inside $T$. We use the fact that

$$C_1 = A(\triangle b_0'v''v) - A(\triangle v'v''v) = \frac{(q - qx)ry \cdot \sin(\alpha + \beta)}{2},$$

$$C_2 = A(\triangle b_0''v'v) - A(\triangle v'v''v) = \frac{(r - ry)qx \cdot \sin(\alpha + \beta)}{2}$$

to obtain

$$C_1 + C_2 = \frac{(x + y - 2xy)qr \cdot \sin(\alpha + \beta)}{2} = (x + y - 2xy)A(T).$$

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We need to show that $x + y - 2xy \leq 2/3$. Let $\hat{c} = \frac{y}{x} = \frac{\sin \beta \sin (\alpha + \gamma)}{\sin \alpha \sin (\beta - \gamma)}$. Since $\hat{c}$ is constant and the geometric mean of two numbers is at most their arithmetic mean

$$\sqrt{x + y - 2xy} = \sqrt{2\hat{c}} \cdot \sqrt{x(1 + \frac{\hat{c}}{2\hat{c}} - x)} \leq \sqrt{2\hat{c}} \cdot \frac{1}{2} \cdot (x + \frac{1 + \hat{c}}{2\hat{c}} - x) = \frac{1 + \hat{c}}{\sqrt{8\hat{c}}}.$$

We prove that $\frac{(1 + \hat{c})^2}{8\hat{c}} \leq \frac{2}{3}$ which is equivalent to $(3\hat{c} - 1)(\hat{c} - 3) \leq 0$. The latter inequality holds if $1 \leq \hat{c} \leq 3$. Function $\sin \theta$ is increasing on $[0, \pi/2]$ thus, $1 \leq \hat{c}$. Now we show that $\hat{c} \leq 3$. If $\gamma = 0$ the inequality holds. Let us assume that $\gamma > 0$. We need to prove that

$$\frac{\sin \beta \sin (\alpha + \gamma)}{\sin \alpha \sin (\beta - \gamma)} = \frac{\cot \gamma + \cot \alpha}{\cot \gamma - \cot \beta} \leq 3.$$

Function $\cot \theta$ is decreasing on $(0, \pi/2]$ thus,

$$\frac{\cot \gamma + \cot \alpha}{\cot \gamma - \cot \beta} \leq \frac{\cot \gamma + \cot 4\epsilon_0}{\cot \gamma - \cot 4\epsilon_0} \leq \frac{\cot \gamma + \cot \epsilon_0}{\cot \gamma - \cot \epsilon_0} \leq 3.$$

The last inequality is equivalent to $2\cot \epsilon_0 \leq \cot \gamma$ which is true since $2\cot \epsilon_0 \leq \cot \frac{\pi}{2} \leq \cot \gamma$. This completes the proof of Claim 4.14.

Let $A_1, \ldots, A_4$ be the areas of the first four triangles in $T_0$. Then $A_1 + \cdots + A_4 \leq n^2$. By construction of triangles in $T_{\text{end}}$, Claim 4.14, and concavity of the square root function,

$$\sum_{T \in T_{\text{end}}} \sqrt{A(T)} \leq K \cdot \sum_{j=1}^{4} \sqrt{A_j} \leq 2K \sqrt{A_1 + A_2 + A_3 + A_4} \leq 2K \cdot n,$$

where $K = \sum_{m=0}^{\infty} (\sqrt{2/3})^m = (1 - \sqrt{2/3})^{-1} < 5.5$. This completes the proof of Lemma 4.13. \hfill \Box

Let $M$ be an input image, $S$ be the set of samples obtained by the algorithm, and $s$ be the parameter in the algorithm. For any image $M'$, let $\epsilon(M') = \text{dist}(M, M')$ and $\hat{\epsilon}(M') = \frac{1}{s} \cdot \{|u \in S : M[u] \neq M'[u]\}$. Also, for any region $R \subseteq [0..n]^2$, let $\epsilon(M'|_R) = \frac{1}{n^2} \cdot \{|u \in R : M[u] \neq M'[u]\}$ and $\hat{\epsilon}(M'|_R) = \frac{1}{s} \cdot \{|u \in S \cup R : M[u] \neq M'[u]\}$.

**Lemma 4.15.** With probability at least $2/3$ over the choice of the samples taken by Algorithm 3, $|\hat{\epsilon}(M') - \text{dist}(M, M')| \leq 5\epsilon/6$ for all reference polygons $M'$.

**Proof.** Consider a region $R = (R_+, R_-)$, partitioned into two regions $R_+$ and $R_-$, such that in some step of the algorithm we are checking the assumption that $R_+$ is black and $R_-$ is white, i.e., evaluating $\hat{\epsilon}_+(R_+) + \hat{\epsilon}_-(R_-)$. Let $R$ be the set of all such regions $R$. We will show that with probability at least $2/3$, the estimates $\hat{\epsilon}_+(R_+) + \hat{\epsilon}_-(R_-)$ are accurate on all regions in $R$.

Fix $R = (R_+, R_-) \in R$. Let $\Gamma$ be the set of misclassified pixels in $R$, i.e., pixels in $R_+$ which are white in $M$ and pixels in $R_-$ which are black in $M$. Define $\gamma = |\Gamma|/n^2$. Algorithm 3 approximates $\gamma$ by $\hat{\epsilon}_+(R_+) + \hat{\epsilon}_-(R_-) = \frac{1}{s} |\Gamma \cap S|$. Equivalently, it uses the estimate $\frac{1}{p} |\Gamma \cap S|$ for $|\Gamma|$. The error of the estimate is $\text{err}_S(R) = \frac{1}{p} |\Gamma \cap S| - |\Gamma|$.

**Claim 4.16.** $\Pr[|\text{err}_S(R)| > \sqrt{\gamma} \cdot c n^2] \leq 2 \exp(-\frac{3}{8} c^2 \epsilon^2 s)$, where $c = 1/21$. 

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Proof. For each pixel \( u \), we define random variables \( \chi_u \) and \( X_u \), where \( \chi_u \) is the indicator random variable for the event \( u \in S \) (i.e., a Bernoulli variable with the probability parameter \( p \)), whereas \( X_u = \frac{\lambda_u}{p} - 1 \). Then our estimate of \( |\Gamma| \) is \( \frac{1}{p} |\Gamma \cap S| = \frac{1}{p} \sum_{u \in \Gamma} \chi_u \), whereas \( \text{err}_S(R) = \sum_{u \in \Gamma} X_u \). We use Bernstein inequality (Theorem 4.18) with parameters \( m = \gamma n^2 \) and \( z = \sqrt{\gamma} \cdot \text{cen}^2 \) to bound \( \Pr[\sum_{u \in \Gamma} X_u > \sqrt{\gamma} \cdot \text{cen}^2] \). The variables \( X_u \) are identically distributed. The maximum value of \( |X_u| \) is \( a = \frac{1-p}{p} \). Note that \( \mathbb{E}[X_u^2] = \frac{1}{p^2} \mathbb{E}[(\chi_u - p)^2] = \frac{1}{p^2} \text{Var}[\chi_u] = \frac{1-p}{p} = a \). We assume w.l.o.g. that \( z < |\Gamma| \). (If \( z \geq |\Gamma| \) then \( \sum_{u \in \Gamma} X_u \) can never exceed \( z \), and the probability we are bounding is 0.) By Bernstein inequality,

\[
\Pr \left[ \sum_{u \in \Gamma} X_u > z \right] \leq \exp \left( - \frac{z^2/2}{a|\Gamma| + a \cdot z/3} \right) < \exp \left( - \frac{3}{8} \cdot \frac{z^2}{|\Gamma|} \right) = \exp \left( - \frac{3}{8} \cdot \frac{\gamma \cdot c^2 \epsilon n^4}{\gamma n^2} \cdot \frac{s}{n^2} \right)
\]

The second inequality holds because \( a < 1/p \) and \( z < |\Gamma| \). The equalities are obtained by substituting the expressions for \( z, |\Gamma|, \) and \( p \), and simplifying. By symmetry, \( \Pr[|\text{err}_S(R)| \geq z] \leq 2 \exp(-\frac{3}{8} c^2 \epsilon^2 s) \).

Claim 4.17. The number of regions in \( \mathbb{R} \) is at most \( 50/\epsilon_0^8 \).

Proof. Let \( k = 1/\epsilon_0 \). There are four types of regions in \( \mathbb{R} \), each corresponding to a different call of the form \( \epsilon_+(R_+) \) or \( \epsilon_-(R_-) \) in the algorithm. The first type is a horizontal double strip of the form \( R_+ = \emptyset \) and \( R_- = W_{b_0} \cup W_{b_2} \). There are \( \binom{k+1}{2} \) such strips. The second type is where \( R_+ \) is a black triangle \( \triangle b_0 b_1 b_2 \) (or \( \triangle b_0 b_3 b_2 \)) and \( R_- \) is a vertical strip \( W_{b_1} \) (respectively, \( W_{b_3} \)). For each horizontal double strip, there are \( 2k - 1 \) vertical strips. For each of them, there are \( k \) ways to choose a reference point on the vertical line that delineates the strip. So, overall, there are \((k + 1)k(k - 1)/2 \) regions of type 2. Type 1 and 2 together have at most \( 5k^8 \) regions. Regions of type 3 are black quadrilaterals of the form \( R_+ = b'b''b'_0 b''_0 \). Each quadrilateral is defined by two reference lines, \( b'b'_0 \) and \( b''b''_0 \), with two reference points on each. There are \( \binom{k^8}{2} \) ways to choose reference directions for the two lines. For each of them, there are at most \( \sqrt{2k} \cdot \left( \sqrt{\frac{z}{k^8}} \right) \) ways to choose a reference line and two reference points. Overall, the number of quadrilaterals in \( \mathbb{R} \) is at most \( \pi^2 k^8 \). Finally, regions of type 4 are contained in triangles of the form \( \triangle b_0 b'_0 b''_0 \); they are of the form either \( R_+ = \triangle v'v''b''_0, R_- = \emptyset \) or \( R_+ = \triangle b_0 b'_0 b''_0, R_- = \triangle v'v'' \). In the former case, there are very few of them. In the latter case, they are defined by three reference line-point pairs: \( (v'b'_0, b'_0), (v'b''_0, b''_0), \) and \( (v''b''_0, b''_0) \), but the direction of the line through \( v''b''_0 \) is determined by \( b'_0, b''_0 \). There are \( \left( \frac{k^8}{2} \right) \) pairs of reference directions. For each of them, there are at most \( \sqrt{2k} \) choices for each reference line and \( \sqrt{2} \cdot k \) choices for each reference point. Overall, the number of regions of type 4 is upper-bounded by \( 4\pi^2 k^8 \). Overall, \( |\mathbb{R}| \leq (5\pi^2 + 0.5)k^8 < 50k^8 = 50/\epsilon_0^8 \), as claimed.

By taking a union bound over all regions in \( \mathbb{R} \) and applying Claims 4.16–4.17, we get that the probability that for one or more of them the error is larger that stated in Claim 4.16 is at most \( |\mathbb{R}| \cdot 2 \exp(-\frac{3}{8} c^2 \epsilon^2 s) \leq \frac{100}{c_0^8} \cdot \exp(-\frac{3}{8} c^2 \epsilon^2 s) \leq 1/3 \), where the last inequality holds provided that \( s \geq C \cdot \ln \frac{1}{\varepsilon} \) for some sufficiently large constant \( C \). We get that

\[
\Pr[|\text{err}_S(R)| \leq \sqrt{\gamma} \cdot \text{cen}^2 \text{ for all } R \in \mathbb{R}] \geq 2/3.
\]
Now suppose that event in (1) holds, that is, the error is low for all regions. Fix a reference polygon $M'$. Consider the partition of $M'$ into regions from $R = (R_+, R_-) \in \mathbf{R}$ on which Algorithm 3 evaluates $\hat{\epsilon}_+(R_+) + \hat{\epsilon}_-(R_-)$ while implicitly computing $\hat{\epsilon}(M')$. Let $\mathbf{R}_{M'} \subset \mathbf{R}$ be the set of regions in the partition. Recall the four types of regions from the proof of Claim 4.17. Then $\mathbf{R}_{M'}$ contains one region of type 1 and two regions of type 2, defined by the reference box of $M'$. Denote their areas by $A_1, A_2, A_3$. For each triangle $T \in \mathbf{T}_{\text{end}}$ created during the construction of $M'$ in Definition 4.3, the set $\mathbf{R}_{M'}$ contains at most one region of type 3 and at most one region of type 4. They were implicitly colored, respectively, in Item 1 and Items 2-3 of Definition 4.3, when triangle $T$ was processed. Let $A_T$ and $A_T'$ denote their respective areas.

Recall that $A(R)$ denotes the area of $R$, and observe that the number of misclassified pixels in $R$ is at most $A(R)$. Since the event in (1) holds,

$$\text{err}_S(M') \leq \sum_{R \in \mathbf{R}_{M'}} \text{err}_S(R) \leq c\text{en} \sum_{R \in \mathbf{R}_{M'}} \sqrt{A(R)} \leq c\text{en} \left( \sum_{t=1}^{3} \sqrt{A_t} + \sum_{T \in \mathbf{T}_{\text{end}}} (\sqrt{A_T} + \sqrt{A_T'}) \right).$$

Since $\sum_{t=1}^{3} A_t \leq n^2$ and $A_T + A_T' \leq A(T)$ for all $T \in \mathbf{T}_{\text{end}}$, by concavity of the square root function,

$$\sum_{t=1}^{3} \sqrt{A_t} \leq \sqrt{3} \sum_{t=1}^{3} A_t \leq 3n \text{ and } \sqrt{A_T} + \sqrt{A_T'} \leq \sqrt{2(A_T + A_T')} \leq \sqrt{2A(T)}.$$

We substitute these expressions in the previous inequality, use Lemma 4.13 and recall that $c = 1/21$:

$$\text{err}_S(M') \leq c\text{en} (\sqrt{3n} + \sqrt{2} \sum_{T \in \mathbf{T}_{\text{end}}} \sqrt{A(T)}) \leq c\text{en}^2 (\sqrt{3} + 11\sqrt{2}) \leq \frac{5}{6} c\text{en}^2.$$

This holds for all reference polygons $M'$ as long as the event in (1) happens, i.e., with probability at least $2/3$. This completes the proof of Lemma 4.15.

For completeness, we state Bernstein’s inequality, which was used in the proof of Lemma 4.15.

**Theorem 4.18** (Bernstein’s inequality). Let $X_1, \ldots, X_m$ be $m$ independent zero-mean random variables, where $|X_i| \leq a$ for all $i \in [m]$. Then for all positive $z$,

$$\Pr \left[ \sum_{i=1}^{m} X_i > z \right] \leq \exp \left( -\frac{z^2/2}{\sum_{i=1}^{m} \mathbb{E}[X_i^2] + a \cdot z^2/3} \right).$$

**4.2.3 Wrapping it up: proof of Theorem 4.6 and corollary on agnostic learning**

**Analysis of Algorithm 3.** Let $\epsilon_M$ be the distance of $M$ to convexity. Then there exists a convex image $M^*$ such that $\text{dist}(M, M^*) = \epsilon_M$. By Lemma 4.7, there is a reference polygon $\hat{M}$ such that $\text{dist}(M^*, \hat{M}) \leq \epsilon/6$, and consequently, $\epsilon_M \leq \text{dist}(M, \hat{M}) \leq \epsilon_M + \epsilon/6$. By Lemma 4.15, with probability at least $2/3$ over the choice of the samples taken by Algorithm 3, $|\hat{\epsilon}(M') - \text{dist}(M, M')| \leq 5\epsilon/6$ for all reference polygons $M'$. Suppose this event happened. Then $\hat{\epsilon} \geq \epsilon_M - 5\epsilon/6$ because $\text{dist}(M, M') \geq \epsilon_M$ for all convex images $M'$. Moreover, $\hat{\epsilon}(M) \leq \text{dist}(M, \hat{M}) + 5\epsilon/6 \leq \epsilon_M + \epsilon$. Thus, $\epsilon_M - 5\epsilon/6 \leq \hat{\epsilon} \leq \epsilon_M + \epsilon$. That is, $|\epsilon_M - \hat{\epsilon}| \leq \epsilon$ with probability $2/3$, as required.

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Sample and time complexity of Algorithm 3. The number of samples taken by the algorithm is \( s = O(\epsilon^{-2} \log \epsilon^{-1}) \).

Next we explain how to implement it to run in time \( O(\epsilon^{-8}) \). Refer to Figure 4.4. Each instance triangle \( \triangle b'b''v \) of the dynamic programming in subroutine \textbf{Best} is specified by two line-point pairs: \((vb', b'), (vb'', b'')\). The number of line-point pairs is \( O(\epsilon^{-3}) \) because for each we select the reference direction, the shift of the line, and the reference point, each in \( O(\epsilon^{-1}) \) ways. Hence, we have \( O(\epsilon^{-6}) \) entries in the dynamic programming table for \textbf{Best}.

In the process of solving an instance of \textbf{Best}, we consider \( O(\epsilon^{-2}) \) possibilities for points \( b'_0, b''_0 \), that is, \( O(\epsilon^{-8}) \) possibilities over all instances. We show how to evaluate each of the possibilities in amortized time \( O(1) \). For that, we count white and black sample pixels in each sub-area in Figure 4.4 in amortized time \( O(1) \).

First, we show how to do it for the entire triangle \( \triangle b'b''v \). We have \( O(\epsilon^{-6}) \) triangles that can be partitioned into \( O(\epsilon^{-5}) \) groups by specifying the first line-point pair \((vb', b')\) and the second line (through \( vb'' \)). That is, within each group, we vary only point \( b''_0 \) on the second line. We sort all sample points \( p \in S \) according to the angle of the segment \( \overrightarrow{p, b'} \). Similarly, we sort the reference points \( b''_0 \) on the second line according to the angle of the segment \( \overrightarrow{b'', b'} \). After sorting, a single scan can establish the counts of white and black pixels in the triangles. Clearly, we can sort in time \( o(\epsilon^{-3}) \). Thus, we compute white/black counts for all instance triangles of \textbf{Best} in time \( o(\epsilon^{-8}) \).

When we consider a possibility in \textbf{Best}, the area above \( b'_0 b''_0 \) is also an instance triangle, so we can find the white/black counts for the quadrilateral \( b'_0 b''_0 b' b'' \) by computing the difference between the counts for entire triangle \( \triangle b'b''v \) and triangle \( \triangle b'_0 b''_0 v \), that is, in time \( O(1) \).

When we consider a possibility in subroutine \textbf{Best For Fixed Base}, we need the counts for the four parts of \( \triangle b'_0 b''_0 v \). Since we already calculated the counts for \( \triangle b'_0 b''_0 v \) and because we can perform subtractions, it is enough to do it for three parts. Two of them, \( \triangle b'_0 b' v \) and \( \triangle b''_0 b'' v \), are instance triangles for \textbf{Best}, so we already calculated their counts. The third we choose is the triangle above \( \ell \). Note that this triangle is specified by three reference lines, so there are \( O(\epsilon^{-6}) \) such triangles. We make a table for all of them. To fill the table, we consider \( O(\epsilon^{-5}) \) groups: we group together triangles for which line \( \ell \) has a common direction. By sorting samples in \( S \), we can compute the counts for each group in time \( o(\epsilon^{-3}) \). Thus, the time for filling the second table is \( o(\epsilon^{-8}) \).

To summarize, Algorithm 3 runs in time \( O(\epsilon^{-8}) \).

This completes the proof of Theorem 4.6. \( \square \)

**Corollary 4.19.** The class of convex images is properly agnostically PAC-learnable with sample complexity \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \) and time complexity \( O(\frac{1}{\epsilon^2}) \) under the uniform distribution.

**Proof.** We can modify Algorithm 3 to output, along with \( \hat{\epsilon} \), a reference polygon \( \hat{M} \) with \( \hat{\epsilon}(\hat{M}) = \hat{\epsilon} \). With an additional DP table, we can compute which points became its vertices. By the analysis of Algorithm 3, with probability at least \( 2/3 \), the output \( \hat{M} \) satisfies \( \text{dist}(\hat{M}, \hat{M}) \leq \epsilon_M + \epsilon \). \( \square \)

## 5 Algorithms for Connectedness

To define \textbf{connectedness}, we consider the \textit{image graph} \( G_M \) of an image \( M \). The vertices of \( G_M \) are \( \{(i, j) \mid M[i, j] = 1\} \), and two vertices \((i, j)\) and \((i', j')\) are connected by an edge if \(|i-i'|+|j-j'| = 1\). In other words, the image graph consists of black pixels connected by the grid lines. The image is \textit{connected} if its image graph is connected.
The first idea in our algorithms for connectedness is that we can modify an image in a relatively few places by superimposing a grid on it (as shown in Figures 5.1 and 5.2), and as a result obtain an image whose distance to connectedness is determined by the properties of individual squares into which the grid lines partition the image. The squares and the relevant property of the squares are defined next.

**Definition 5.1** (Squares and grid pixels). Fix a side length \( n \equiv 1 \pmod{r} \). For all integers \( i, j \in [0..n) \), where \( i \) and \( j \) are divisible by \( r \), the \((r-1) \times (r-1)\) image that consists of all pixels in \([r-1]^2 + (i, j)\) is called an \( r \)-square of \( M \). The set of all \( r \)-squares of \( M \) is denoted \( S_r \).

The pixels that do not lie in any squares of \( S_r \), i.e., pixels \((i, j)\) where \( i \) or \( j \) is divisible by \( r \), are called grid pixels. The set of all grid pixels is denoted by \( GP_r \).

**Claim 5.1.** \( |GP_r| \leq 2n^2/r \).

**Proof.** \( |GP_r| = 2\left(\frac{n-1}{r} + 1\right)n - \left(\frac{n-1}{r} + 1\right)^2 \leq 2n^2/r \). \(\square\)

Note that a square consists of pixels of an \( r \)-block, with the pixels of the first row and column removed. Therefore, a block-sample-based algorithm can obtain a uniformly random \( r \)-square.

Recall the definition of the border of an image from Section 2.

**Definition 5.2** (Border connectedness). A (sub)image \( S \) is border-connected if for every black pixel \((i, j)\) of \( S \), the image graph \( G_S \) contains a path from \((i, j)\) to a pixel on the border. The property border connectedness, denoted \( C' \), is the set of all border-connected images.

### 5.1 Distance Approximation to Connectedness

**Theorem 5.2.** There is a block-sample-based \( \epsilon \)-additive distance approximation algorithm for connectedness with sample complexity \( O(1/\epsilon^4) \).

**Proof.** The main idea behind Algorithm 6, used to prove Theorem 5.2, is to relate the distance to connectedness to the distance to another property, which we call grid connectedness. The latter
Lemma 5.3. Let \( |\varepsilon| \in C \) without changing whether it is connected and adjust the accuracy parameter.

W.l.o.g. assume that \( n \equiv 1 \pmod{4/\varepsilon} \). (Otherwise, we can pad the image with white pixels without changing whether it is connected and adjust the accuracy parameter.)

Algorithm 6: Distance approximation to connectedness.

```plaintext
input : \( n \in \mathbb{N} \) and \( \varepsilon \in (0, 1/4) \); block-sample access to an \( n \times n \) binary matrix \( M \).
1 Sample \( s = 4/\varepsilon^2 \) squares uniformly and independently from \( S_{4/\varepsilon} \) (see Definition 5.1).
   // This can be done by drawing random blocks from the \( 4/\varepsilon \)-partition of \( [0..n] \).
2 For each such square \( S \), compute \( \text{dist}(S, C') \), where \( C' \) is border connectedness (see Definition 5.2). Let \( \bar{d} \) be the average of \( \text{dist}(S, C') \).
3 return \( \hat{\varepsilon} = ((1 - \frac{\varepsilon}{4})(1 - \frac{1}{n}))^2 \cdot \bar{d} \).
```

Definition 5.3. Fix \( \varepsilon \in (0, 1/4) \). An image \( M_C \) is a grided image obtained from image \( M \) if

\[
M_C[i, j] = \begin{cases} 
1 & \text{if } (i, j) \text{ is a grid pixel from GP}_{4/\varepsilon}; \\
M[i, j] & \text{otherwise}.
\end{cases}
\]

Let \( C \) be the set of all connected images. For \( \varepsilon \in (0, 1/4) \), define grid connectedness \( C_{\varepsilon} = \{ M | M \in C, \text{ and } M[i, j] = 1 \text{ for all } (i, j) \in \text{GP}_{4/\varepsilon} \} \).

Lemma 5.3. Let \( \varepsilon = \text{dist}(M, C) \) and \( \varepsilon_{\varepsilon} = \text{dist}(M_{\varepsilon}, C_{\varepsilon}) \). Then \( \varepsilon - \frac{\varepsilon}{2} \leq \varepsilon_{\varepsilon} \leq \varepsilon \). Moreover,

\[
\varepsilon_{\varepsilon} = \left((1 - \frac{\varepsilon}{4})(1 - \frac{1}{n})\right)^2 \cdot \frac{1}{|S_{4/\varepsilon}|} \sum_{S \in S_{4/\varepsilon}} \text{dist}(S, C').
\]

Proof. First, we prove \( \varepsilon_{\varepsilon} \leq \varepsilon \). Let \( M' \) be a connected image such that \( \text{dist}(M, M') = \varepsilon \). Note that \( M' \), the grided image obtained from \( M' \), satisfies \( C_{\varepsilon} \). Since \( \text{dist}(M_{\varepsilon}, M''_{\varepsilon}) \leq \varepsilon \), it follows that \( \varepsilon_{\varepsilon} \leq \varepsilon \).

Now we show that \( \varepsilon - \frac{\varepsilon}{2} \leq \varepsilon_{\varepsilon} \). Let \( M''_{\varepsilon} \subset C_{\varepsilon} \) be such that \( \text{dist}(M_{\varepsilon}, M''_{\varepsilon}) = \varepsilon_{\varepsilon} \). Then \( M''_{\varepsilon} \in C_{\varepsilon} \) and, by Claim 5.1, \( \text{dist}(M, M''_{\varepsilon}) \leq |\text{GP}_{4/\varepsilon}|/n^2 + \varepsilon_{\varepsilon} \leq \varepsilon/2 + \varepsilon_{\varepsilon} \), implying \( \varepsilon \leq \varepsilon/2 + \varepsilon_{\varepsilon} \), as required.

Finally, observe that to make \( M_{\varepsilon} \) satisfy \( C_{\varepsilon} \), it is necessary and sufficient to ensure that each square satisfies \( C' \). In other words, \( \varepsilon_{\varepsilon} \cdot n^2 = \sum_{S \in S_{4/\varepsilon}} \text{Dist}(S, C') = (4/\varepsilon - 1)^2 \sum_{S \in S_{4/\varepsilon}} \text{dist}(S, C') \). Since \( |S_{4/\varepsilon}| = (n-1)^2 \), the desired expression for \( \varepsilon_{\varepsilon} \) follows.

Analysis of Algorithm 6. Let \( \bar{d} = \frac{1}{|S_{4/\varepsilon}|} \sum_{S \in S_{4/\varepsilon}} \text{dist}(S, C') \). Recall that \( \bar{d} \) is the empirical average computed by the algorithm. By Chernoff-Hoeffding bound, \( \Pr[|d - \bar{d}| > \varepsilon/2] \leq 2 \exp(-2\varepsilon^2 s) \leq 1/3 \). So, with probability at least 2/3, we have \( |d - \bar{d}| \leq \varepsilon/2 \). If this event happens then \( |\hat{\varepsilon} - \varepsilon_{\varepsilon}| \leq \varepsilon/2 \) because by Algorithm 6 and Lemma 5.3, respectively, \( \hat{\varepsilon} = A \cdot \bar{d} \) and \( \varepsilon_{\varepsilon} = A \cdot \hat{\varepsilon} \), where \( A = (\left((1 - \frac{\varepsilon}{4})(1 - \frac{1}{n})\right)^2 \leq 1 \). By Lemma 5.3, \( |\varepsilon_{\varepsilon} - \varepsilon| \leq \varepsilon/2 \). Thus, \( |\hat{\varepsilon} - \varepsilon| \leq |\hat{\varepsilon} - \varepsilon_{\varepsilon}| + |\varepsilon_{\varepsilon} - \varepsilon| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \) holds with probability at least 2/3, as required.

Query complexity. Algorithm 6 samples \( O(1/\varepsilon^2) \) squares containing \( O(1/\varepsilon^2) \) pixels each. Thus, the sample complexity is \( O(1/\varepsilon^4) \).
5.2 Tester for Connectedness

**Theorem 5.4.** There is a block-sample-based (1-sided error) \( \epsilon \)-tester for connectedness with sample and time complexity \( O(\frac{1}{\epsilon^2}) \).

**Proof.** Observe that a tester can safely reject if it finds a small connected component and a black pixel outside it. Our tester (Algorithm 7) looks for squares that contain small connected components, that is, are not border-connected, which we call witnesses. To find a witness, it samples \( r \)-squares for \( \log \frac{1}{\epsilon} + 1 \) values of \( r \), which we call levels. In each subsequent level, the number of samples is doubled, but the side length of the squares is halved, i.e., the number of pixels in them is divided by 4. If it finds a witness, it samples pixels to look for black pixels outside the witness.

For simplicity of the analysis of the algorithm we assume\(^6\) that \( n - 1 \) and \( 1/\epsilon \) are powers of 2.

**Definition 5.4** (Levels, witnesses). For \( t \in [0..\log \frac{1}{\epsilon} + 1) \), let \( r_t = \frac{4}{\epsilon} \cdot 2^{-t} \) denote the length of level \( t \). Pixels of the set \( GP_{r_t} \) are called grid pixels of level \( t \), and squares in the set \( S_{r_t} \) are called squares of level \( t \). A square of level \( t \) which is not border-connected (see Definition 5.2) is called a witness.

---

**Algorithm 7:** \( \epsilon \)-tester for connectedness.

```plaintext
input : n ∈ \( \mathbb{N} \) and \( \epsilon \in (0,1/2) \); block-sample access to an \( n \times n \) binary matrix \( M \).

1. For \( t = 0 \) to \( \log \frac{1}{\epsilon} \)
   (a) Sample \( 2^{t+1} \) squares of level \( t \) (see Definition 5.4) uniformly at random with replacement.
   (b) For every sampled square \( S \) from Step 1a, use a breadth-first search (BFS) to find all connected components of the image graph \( G_S \). If there is a connected component that does not have a pixel on the border of \( S \), mark \( S \) is a witness, and proceed to Step 2.

2. Sample \( \frac{4}{\epsilon} \) pixels uniformly at random. If a witness is found in Step 1 and the sample contains a black pixel outside the witness, reject; otherwise, accept.
```

---

We start the analysis of Algorithm 7 by relating the distance of an image to connectedness to the fraction of witnesses it contains at different levels.

**Lemma 5.5.** Consider an \( n \times n \) image \( M \), and let \( w_t \) denote the fraction of witnesses among all squares in \( S_{r_t} \) for all \( t \in [0..\log \frac{1}{\epsilon} + 1) \). Then \( \text{dist}(M,C) \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} \sum_{t=0}^{\log \frac{1}{\epsilon}} 2^{t+1} w_t \).

**Proof.** We will show how to make \( M \) connected by changing the required fraction of pixels. First, we make all pixels in \( GP_{r_0} \) black. By Claim 5.1, there are at most \( 2n^2/r_0 = \epsilon n^2/2 \) of them. Then we recursively “repair” the witnesses, starting from the witnesses in \( S_{r_0} \), as follows. Inside each witness \( S \in S_{r_t} \), we make the pixels in \( GP_{r_{t+1}} \) black, and call the set of modified pixels the cross of \( S \). Then we call the repair procedure on all witnesses in \( S_{r_{t+1}} \) which are inside \( S \). At each level, all pixels that belong to the central cross of a witness are connected to the grid pixels of level 0. After processing witnesses of level \( \log \frac{1}{\epsilon} \), which are \( 3 \times 3 \) squares, the image is connected. There are \( (\frac{n-1}{r_t})^2 \) squares in \( S_{r_t} \). In each witness of level \( t \) we modify at most \( 2r_t \) pixels. All together, at level

\(^6\)This assumption can be made w.l.o.g. because if \( n \in (2^{i-1} + 1, 2^i + 1) \) for some \( i \), instead of the original image \( M \) we can consider a \( (2^i + 1) \times (2^i + 1) \) image \( M' \), which is equal to \( M \) on the corresponding coordinates and has white pixels everywhere else. Let \( \epsilon' = \epsilon n^2/(2^i + 1)^2 \). To \( \epsilon \)-test \( M \) for connectedness, it suffices to \( \epsilon' \)-test \( M' \) for connectedness. The resulting tester for \( M \) has the desired query complexity because \( \epsilon' = \Theta(\epsilon) \). If \( \epsilon \in (1/2^j, 1/2^{j-1}) \) for some \( j \), to \( \epsilon \)-test a property \( \mathcal{P} \), it is sufficient to run an \( \epsilon'' \)-test for \( \mathcal{P} \) with \( \epsilon'' = 1/2^j < \epsilon \).
we modify at most \(2rt \cdot wt \cdot \left(\frac{n-1}{rt}\right)^2 \leq \frac{2wtn^2}{rt} = \frac{\epsilon n^2 2^{t+1} w_t}{4} \) pixels. Overall, at most \(\frac{\epsilon}{2} + \frac{\epsilon}{4} \sum_{i=0}^{\log \frac{1}{\epsilon}} 2^{t+1} w_t \) fraction of pixels is modified in \(M\) to obtain a connected image, as claimed.

**Analysis of Algorithm 7.** If the input image \(M\) is connected, the algorithm always accepts because there are no witnesses.

Consider an image \(M\) that is \(\epsilon\)-far from connected, i.e., \(\text{dist}(M,C) \geq \epsilon\). Then by Lemma 5.5, \(\sum_{i=0}^{\log \frac{1}{\epsilon}} 2^{t+1} w_t \geq 2\). For fixed \(t \in [0..\log \frac{1}{\epsilon} + 1)\), the probability that Algorithm 7 fails to detect a witness of level \(t\) is \((1 - wt)^{2^{t+1}} \leq e^{-wt2^{t+1}}\). Thus, the overall probability that it fails to detect a witness is at most \(\prod_{i=0}^{\log \frac{1}{\epsilon}} e^{-\frac{1}{4} \epsilon^2} \leq \frac{1}{6}\).

Assume that the algorithm detects a witness \(S\). There are at most \(16 \cdot \epsilon^2 < \epsilon n^2\) pixels in \(S\). Since \(\text{dist}(M,C) \geq \epsilon\), there are at least \(\epsilon n^2\) black pixels in \(M\), and at least \(\epsilon n^2\) of them are outside \(S\). The probability that Algorithm 7 fails to detect at least one of these black pixels in Step 2 is at most \((1 - \frac{\epsilon}{2})^2 < e^{-2} < 1/6\). Thus, the algorithm detects a witness and a black pixel outside it with probability at least \(1 - (1/6 + 1/6) = 2/3\), as required.

**Query and time complexity.** The algorithm samples \(2^{t+1}\) squares of \(S_{rt}\), for \(t \in [0..\log \frac{1}{\epsilon}]\), and inside each square queries \((\frac{1}{\epsilon} \cdot 2^{-t} - 1)(\frac{1}{\epsilon} \cdot 2^{-t} - 1) < \frac{16}{\epsilon^2 2^t}\) pixels. Thus, the query complexity of the algorithm is

\[
\sum_{t=0}^{\log \frac{1}{\epsilon}} 2^{t+1} \cdot \frac{16}{\epsilon^2 2^t} < \sum_{t=0}^{\log \frac{1}{\epsilon}} \frac{32}{\epsilon^2 2^t} = O\left(\frac{1}{\epsilon^2}\right).
\]

The time complexity is linear in the number of samples, since BFS takes time linear in the number of vertices on bounded-degree graphs, and the remaining steps can be easily implemented to run in the time proportional to the number of sampled pixels.

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