On the history of geometrization of space-time:
From Minkowski to Finsler geometry. (100 years after Minkowski’s Cologne adress.)
Hubert Goenner
Institute for Theoretical Physics
University of Göttingen
Germany

1 Introduction: on the geometrization of physics

This tribute to Hermann Minkowski will consist of three parts: a brief historical introduction concerning geometrization of physics, a middle catering to mathematical themes, and a final chapter dealing with a (speculative) endeavour at applying Finsler geometry to physics loosely connected to Minkowski.

From the history of physics we know that, at first, physical systems were described in a given space and by a given time which both were regarded as independent of matter or any physical influence - not just by the philosopher Kant. At his time, the idea of geometrizing space would have been absurd. Johannes Kepler who for some period in his life had related physical bodies, the planets, to geometric objects, i.e., to the five regular polyhedra, certainly was far from what we now understand by geometrization of physics, i.e. the embedding of physical objects (matter, fields) into a geometrical framework. A weakening of the rigid understanding of space seems to have occurred when the notion of non-euclidean geometry came up, in the 19th century (C. F. Gauss, N. I. Lobachevski, J. Bolyai). The answer to the question of what kind of geometry the space we live in exhibits, now could be delegated to an empirical test [1]. Whether the anecdote about Gauss with his geodesic measurement of the angles of a triangle formed by three hills is true or not, in any case the astronomer K. Schwarzschild investigated the question scientifically with bodies far away in the heavens (1900) [2]. Also in the 19th century, the mechanics of rigid bodies became reformulated within non-euclidean geometry (F. Klein, W. A. Clifford, R. S. Heath) [3]). Yet, with the exception of Clifford, there still was no question about space or time being
influenced by material systems.

As is well known, the joinder of space and time to space-time by Hermann Minkowski whose famous speech about the “union of space and time” was commemorated in September 2008, became a first step in this still ongoing process of geometrization. As holds for many innovations in science, the idea of space-time did not appear like a shooting star. A few mathematicians, fiction writers, and philosophers presented it quite clearly before Minkowski, but not as a mathematical theory. In the 18th century D’Alembert mentioned time as a fourth dimension in 1754 in his encyclopedia (written with Diderot); then again this was done by Lagrange in 1797.

For the 19th century, let me first mention Charles Howard Hinton’s article of 1880: “What is the fourth dimension?” which considered a fourth spatial dimension. In his reply to it in the journal Nature of 1885, an anonymous letter writer signing “S.”, introduced time as the fourth dimension and dealt with a 4-dimensional “time-space”. S. mastered (verbally) what we now call the space-time picture, and even managed to correctly describe the hypercube by looking at the motion of a cube in time-space. In another article, Hinton tried to geometrize electrical charge and currents. Better known is H. G. Wells’ novel “Time Machine” of 1894 in which again a 4-dimensional junction of time and space called “Space” is considered. There, it is made clear that time is not considered as a fourth spacelike dimension. As the German translation of Wells’ book came out in 1904, Minkowski could have read it, in principle. Finally, a philosopher of Hungarian origin, Menyhért (Melchior) Palágyi, who had become a professor in Darmstadt published his “New theory of space and time” in 1901. He joined space and time rather vaguely to a 4-dimensional entity named “flowing space” (fließender Raum), drew a Minkowski-diagram and introduced a “time angle” between the worldline of a moving particle and the time axis. He abstained from giving a mathematical scheme except for pointing out that “the coordinates of a point in flowing space could be represented by \(x + it, y + it, z + it\)” (\[11\], p. 32). After he had become aware of special relativity and, then, of Minkowski’s famous speech, Palágyi claimed priority for the space-time picture but rejected Minkowski’s space-time manifold. From his writings it

\[1^{1}\text{Ainsi on peut regarder la mécanique comme une géométrie à quatre dimensions [...]}.\]

\[2\text{This is a curious combination of time with one absolute and two relative spacelike coordinates. Let } x' = x + it, y' = y + it, z' = z + it; \text{ then } y' - x' = y - x; z' - x' = z - x \text{ are the relative coordinates. } x' - x = it \text{ could at best describe part of the light cone, a concept Palágyi did not have. For the philosophy of Palágyi cf. [15].}\]
is obvious that his thoughts remain within psychology, and that he was incompetent both in mathematics and physics. [12]. In sharp contrast, around 1905, and before Minkowski, Poincaré also had a 4-dimensional (space-time) formalism for the wave equation and electrodynamics [13]. Possibly, due to his epistemological position as a conventionalist, he might not have been interested at all in the issue of geometrization.

2 Minkowskian spaces

Did Minkowski geometrize electrodynamics by formulating it on a space-time manifold? Not in the sense of having found a geometry in which the electromagnetic field corresponded to a geometric object. This would come only later - after the geometrization of the gravitational field - in the framework of unified field theory. Right after Minkowski, M. Planck (1906) [14], G. Herglotz (1910), F. Kottler (1912), G. N. Lewis & R. C. Tolman (1909) among others put mechanics and electrodynamics into a space-time picture: they relativized such topics.

The first geometrization in the narrower meaning was achieved by Einstein and Großmann (1913-15) [16]. At first, Einstein had wanted to keep the constancy of the velocity of light only for “areas of almost constant gravitational potential” ([17], p. 713) Thus, in his attempt toward a relativistic theory of gravitation, he assumed the velocity of light to be a function of the (Newtonian) gravitational potential $\Phi$. He replaced the space-time metric of Minkowski by

$$ds^2 = c(\Phi)^2 dt^2 - \delta_{\alpha\beta} dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3),$$

In an important next step, with Großmann’s help, he introduced a (semi-)Riemannian metric and identified its components with the now more numerous gravitational potentials.

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3In Germany, at the time, Poincaré’s papers seem to have been neglected. Many well known scientists then (e.g., M. Planck, in his paper on relativistic mechanics) and even later historians of science do not refer to Poincaré’s short paper of 1905 - before Einstein’s -, but only to Poincaré’s extended presentation of 1906.

4For the most detailed and expert history of the formation of general relativity cf. the 4 volumes of [18], [19].
3 Minkowski space-time and Minkowski Space

In this part, we first distinguish between the physicist’s and the mathematician’s use of the expression “Minkowski space” and present some of Minkowski’s results concerning the geometry of normed spaces.

3.1 Minkowski space-time

The introduction of an imaginary time-coordinate $T = i \, ct$ by Minkowski into the line element of space-time

$$ds^2 = dT^2 + \delta_{\alpha\beta}dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3)$$

(2)

turned out to be a little misleading for physics and more so for the general public. On the surface, (2) looked as if physics now played in a 4-dimensional euclidean space - with four spatial dimensions corresponding exactly to what Riemann had had in mind. The 19th century had been full of talk and papers about 4-dimensional space with its striking possibility to enter a locked (3-dimensional) room without breaking a seal \[27\]. It was quickly realized, though, that a real representation of the metric suited physics better, i.e., by a Lorentz-metric with signature $\pm 2$ (null cone, Cauchy problem etc.):

$$ds^2 = \eta_{ij}dx^idx^j = c^2dt^2 - \delta_{\alpha\beta}dx^\alpha dx^\beta \quad (i, j = 0, 1, 2, 3).$$

(3)

Space-time, as Minkowski had introduced it, became the framework for all physical theories in which velocities comparable to the velocity of light could occur: it is a natural representation space of the Lorentz (Poincaré-) group. Also in curved space-time it plays a role: as the tangent space at any point of the manifold of events. This is all too well known such that nothing more needs to be said.

3.2 Minkowski Space

The second meaning of the term “Minkowski Space” in the way mathematicians use it, is barely known to physicists. On this occasion of commemorating Minkowski in a tribute to him it is mandatory to include some of his mathematical achievements.

\[5\] For a history of 4-dimensional space and its modern uses cf. \[28\].
Minkowski Space is a real, finite-dimensional ($d \geq 2$) normed (vector) space $V(=R^d)$ ([29], p. 138).

If completeness is added, then it is just a special case of a Banach space. The norm $\|X\|$ of an element $X \in V$ satisfies the following conditions:

i) a) $\|X\| \geq 0$; b) $\|X\| = 0$ if and only if $X = 0$;

ii) $\|\lambda X\| = |\lambda| \|X\|$; $\lambda \in R$, $X \in V$;

iii) $\|X + Y\| \leq \|X\| + \|Y\|$.

If 1) b) does not hold, a semi(pseudo)-norm of the kind needed in space-time obtains.

The unit ball $B \subset M^d$ with $B := \{X \in V| \|X\| \leq 1\}$ is a (compact) convex and symmetric set. With the help of the norm, a (canonical) metric (semi-metric)

$$\delta(X,Y) := \|X - Y\|$$

(4)

can always be introduced. The unit ball may be very different of what we imagine in an euclidean situation. In fact, if and only if the unit ball is an ellipsoid then Minkowski Space turns out to be euclidean space ([20], p. 38). The topology of any $d$-dimensional Minkowski Space is euclidean topology ([21], section 1.2). When Minkowski Space is seen as a metrical space we speak of Minkowski geometry.

Let me give an example for Minkowski geometry: Let $K$ be a compact, convex set in euclidean space, $x \neq y$ two points of $K$ and $\xi \neq \eta$ points on the boundary $\partial K$ of $K$ met by straight lines joining $x$ and $y$ with the zero-point $O \neq x$; $O \neq y$. Then a distance function $F$ on $K$ is defined by

$$F(x - y) := \frac{\|x - y\|}{\|\xi - \eta\|} , \quad F(0) = 0 .$$

(5)

If the euclidean norm is used, then

$$F(x - y) = \sqrt{\frac{\Sigma(x - y)^2}{\Sigma(\xi - \eta)^2}} .$$

(6)

The distance function is a convex function with $F(x) \leq 1$.

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6Minkowski called it “Eichkörper”. A set $W$ is called symmetric (with regard to the zero point $O$) if $-W = W$. This means that all straight line segments passing through $O$ of the set are halved by $O$.

7A convex function satisfies the same conditions like a norm, i.e., the triangle inequality $F(x + y) \leq F(x) + F(y)$, etc.
The study of norms other than the euclidean (and not derived from a metric) is primarily due to Minkowski. Because of a 1-1 correspondence between norms on the linear space $V$ and symmetric, closed convex sets in $V$ with non-empty interior, convexity becomes an essential ingredient for the study of Minkowski Space\(^8\) [21]. In this context, Minkowski introduced a vector sum of two convex sets (bodies) now called “Minkowski sum”. Its combination with the concept of volume led him to important results. We realize that, with only a norm available, “volume” and “orthogonality” are not immediately at hand. As to volume, we assume that $V$ is equipped with an auxiliary euclidean structure and that the volume is the Lebesque measure induced by this structure\(^9\). In connection with his research in number theory, Minkowski used the concepts “volume” and “area” of convex bodies (cf. his “geometry of numbers”, [22] and [23]). To make progress, he introduced fundamental quantities as, e.g., the support function of convex bodies. Another one is mixed volume generalizing the euclidean volume: it comprises and connects the concepts of volume, area, and total mean curvature\(^10\).

In this context, one of the well-known results of Minkowski among mathematicians perhaps is the Brunn-Minkowski inequality concerning the (n-dimensional) volumes $\lambda_n(K_0)$ and $\lambda_n(K_1)$ of 2 compact convex sets $K_0$, $K_1$ in n-dimensional euclidean space $E^n$. Let $0 \leq t \leq 1$, then:

$$[\lambda_n((1-t)K_0 + tK_1)]^\frac{1}{n} \geq (1-t)[\lambda_n(K_0)]^\frac{1}{n} + t[\lambda_n(K_1)]^\frac{1}{n}.$$  \tag{7}

Equality for some $0 < t < 1$ holds if and only if $K_0$ and $K_1$ lie in parallel hyperplanes or are homothetic.

An example is given like follows: We place $K_0$ and $K_1$ in two parallel hyperplanes described by $x = 0$, $x = 1$, respectively, in (n+1)-dimensional euclidean space $E^{n+1}$. Then

$$(1-t)K_0 + tK_1 = \text{conv}(K_0 \cup K_1) \cap \{x|x = t\}.$$  

Let $D_t$ be the n-dimensional ball in $E^{n+1}$ contained in the hyperplane $x = t$ centered on the $x$-axis, and with n-dimensional volume equal to that of $(1-t)K_0 + tK_1$.

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\(^8\)The correspondence reverses order: $B_1 \subseteq B_2$ implies $\| \cdot \|_{B_1} \geq \| \cdot \|_{B_2}$ and $\| \cdot \|_{\alpha \alpha B} = \alpha^{-1} \| \cdot \|_{\alpha B}$.

\(^9\)Closed convex sets being Borel sets, an alternative would be to use a Haar measure ([21], p. 53). - As to orthogonality in normed spaces, see the glossary for a definition.

\(^{10}\)See glossary.
Then the Brunn-Minkowski inequality states that the union of all $D(0 \leq t \leq 1)$ is a convex set. An illustration of this example is to be found on page 416 of [24].

Although I am discussing Minkowski Space mainly because of its connection with Finsler spaces to be introduced soon, I shall dwell on Minkowski’s geometrical insights a bit further. Minkowski introduced his geometry when looking at a lattice formed by integers. An immediate result is Minkowski’s theorem about lattice points:

Let $\Gamma$ be a lattice in $\mathbb{R}^d$, $K \in \mathbb{R}^d$ a bounded, convex set symmetric with regard to the zero-point $O$ as its center. If its volume $\lambda_{n}(K) \geq 2^n \lambda_{n}(\Gamma)$, then $K$ contains at least one further lattice point (different from $O$).

$\lambda_{n}(\Gamma)$ is the volume of the elementary cell; in a cubic lattice with spacing 1 consequently $\lambda_{n}(\Gamma) = 1$. An immediate consequence is that the volume of a convex body the center of which is a lattice point (and which is the only interior such one) cannot be greater than $2^n$. Hilbert calls Minkowski’s theorem “one of the most applicable theorems in arithmetics”[12]. According to him, by using methods of Dirichlet, Minkowski was able to conclude from this result that: “[..] for an algebraic number field exists at least one prime number divisible by the square of a prime ideal, a so-called branching number [..]” ([25], p. 452-53).

4 Finsler Space

Now we progress from Minkowski- to Finsler geometry, a geometry also being used for a geometrization of physics. Physicists who want to learn something about Finsler geometry and look into one or the other textbook known to them, very likely will find themselves in a situation described by the mathematician H. Busemann almost 60 years ago, i.e., in: “an impenetrable forest whose entire vegetation consists of tensors” ([26], p. 5). This is a result of Riemann’s mentioning of one type of such kind of space and the ensuing almost exclusive application of methods used in (pseudo-) Riemannian geometry. General Relativity has set the stage for theoretical physicists; hence up to now most of the research on Finsler space by relativists consisted in

[11] The name “Minkowski geometry” was given only later by S. Mazur ([21], p. 43).
[12] One of the applications is the approximation of real numbers by fractions.
an extension of the space-time metric $g_{ij}(x^l)$ to metrics also dependent on direction $g_{ij}(x^l, dx^m)$.

The extension of Riemannian “point”-space $\{x^i\}$ into a “line-space” $\{x^i, dx^i\}$ by L. Berwald and E. Cartan\textsuperscript{13} did make things clearer but not easier: how do you explain to a physicist a geometry supporting at least 3 curvature tensors and five torsion tensors? Not to speak of its usefulness for physics! Fortunately, the “impenetrable forest” by now has become a real, enjoyable park: through the application of the concepts of fibre bundle and non-linear connection. The different curvatures and torsion tensors result from vertical and horizontal parts of geometric objects in the tangent bundle, or in the Finsler bundle of the underlying manifold. This will be explained in 4.2.3.

4.1 Family lines

Before three different approaches to Finsler geometry will be discussed, the academic ancestry of both Paul Finsler (1894-1970) and Herbert Busemann (1905-1994) is presented. They stand for two fundamental approaches to Finsler geometry. Finsler was a doctoral student of Theodor Carathéodory (1873-1950) who himself had obtained his PhD with Hermann Minkowski, both in Göttingen. On the other hand, David Hilbert (1862-1943) had Richard Courant (1888-1972) as one of his doctoral students. With him Busemann wrote his PhD. This again happened in Göttingen. Finsler started from the calculus of variations; infinitesimal length, and the length of a curve are fundamental concepts. Busemann followed are more axiomatic path by widening the definition of distance.

4.2 Finsler geometry

In essence, Finsler geometry is analogous to Riemannian geometry: there, the tangent space in a point $p$ is euclidean space; here, the tangent space is just a normed space, i.e., Minkowski Space. Put differently: A Finsler metric for a differentiable manifold $M$ is a map that assigns to each point $x \in M$ a norm on the tangent space $T_x M$ ([29], p. 38). When I refered to the almost exclusive use of methods from Riemannian geometry it meant that this norm is demanded to derive from the length of a smooth path $\gamma : [a, b] \to M$ defined

\textsuperscript{13}Cartan called $\{dx^i\}$ a supporting element.
by $\int_a^b \| \frac{dx(t)}{dt} \| dt$. Then Finsler space becomes an example for the class of length spaces \[29\].

### 4.2.1 Following Finsler and Cartan

In this spirit, P. Finsler \[30\] and E. Cartan \[31\] started from the length of the curve

$$d_\gamma(p,q) := \int_p^q L(x(t), \frac{dx(t)}{dt}) dt .$$

The variational principle $\delta d_\gamma(p,q) = 0$ leads to the Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0 ,$$

which may be rewritten into

$$\frac{d^2 x^i}{dt^2} + 2G^i(x^l, \dot{x}^m) = 0 ,$$

with $G^i(x^l, \dot{x}^m) = \frac{1}{4} g^{kl} (-\frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial x^l \partial x^m} \dot{x}^m)$, and $2g_{ik} = \frac{\partial^2 L}{\partial x^i \partial x^k}$, $g^{ik}g_{ij} = \delta^i_j$. The theory then is developed from the “Lagrangian” defined in this way. This includes an important object $N^i_l := \frac{\partial G^i}{\partial y^l}$, the geometric meaning of which as a non-linear connection we shall recognize in section 4.2.3.

In general, a Finsler structure $L(x,y)$ with $y := \frac{dx(t)}{dt} = \dot{x}$ and homogeneous of degree 1 in $y$ is introduced, from which the Finsler metric follows as:

$$f_{ij} = f_{ji} = \frac{\partial (\frac{1}{2} L^2)}{\partial y^i \partial y^j} , \quad f_{ij} y^i y^j = L^2 , \quad y^i \frac{\partial L}{\partial y^i} = L , \quad f_{ij} y^j = L \frac{\partial L}{\partial y^i} .$$

A further totally symmetric tensor $C_{ijk}$ ensues:

$$C_{ijk} := \frac{\partial (\frac{1}{2} L^2)}{\partial y^i \partial y^j \partial y^k} ,$$

which will be interpreted as a torsion tensor. As an example for a Finsler metric related to physics is the Randers metric:

$$L(x,y) = b_i(x) y^i + \sqrt{a_{ij}(x) y^i y^j} .$$
The Finsler metric following from (13) is:

\[ f_{ik} = b_i b_k + a_{ik} + 2 b_{(i} a_{k)} \dot{y}^l a_{lm} \dot{y}^m (b_n \dot{y}^n) \]  

(14)

with \( \dot{y}^k := y^k (a_{lm} (x) y^l y^m)^{-\frac{1}{2}} \). Setting \( a_{ij} = \eta_{ij} \), \( y^k = \dot{x}^k \), and identifying \( b_i \) with the electromagnetic 4-potential \( eA_i \) leads back to the Lagrangian for the motion of a charged particle.

In this context, a Finsler space thus is called a locally Minkowskian space if there exists a coordinate system, in which the Finsler structure is a function of \( y^i \) alone. The use of the “element of support” \( (x^i, dx^k = y^k) \) by Cartan essentially amounts to a step towards working in the tangent bundle \( TM \) of the manifold \( M \).

### 4.2.2 Following Minkowski and Busemann

In order to define a norm in Minkowski Space, H. Busemann did replace the homogeneity condition by the relation:

\[ \frac{\| PQ \|}{\| PQ' \|} = \frac{|PQ|}{|PQ'|}, \]

(15)

where \( P, Q \) and \( Q' \) are points on a line, \( \| PQ \| \) is the Minkowski distance between \( P \) and \( Q \), while \( |PQ| \) measures the euclidean distance. An example in two dimensions with coordinates \( x, y \) is given by:

\[ \| PQ \| = \Phi \left[ \frac{\nu_1 (x_1 - x_2) + \nu_2 (y_1 - y_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right] |PQ| \]

(16)

with an arbitrary function \( \Phi \) and constants \( \nu_1, \nu_2 \).

Thus, more generally, from Busemann’s point of view, “the Minkowskian distance originates from the euclidean distance […] by multiplying it with a factor which depends only on the direction of the segment from \( x \) to \( y \)” This means that the Minkowski distance reads as

\[ F(x - y) = F(u) |y - x|, \]

(17)

where \( u \) is a unit vector in the direction of \( y - x \) ([20], p. 9). By transporting Busemann’s idea to space-time, we arrive at a (pseudo-) “Minkowski”-metric

\[ ds^2 := \Phi \left[ \frac{a_{1l} dx^l}{\sqrt{\eta_{ab} dx^a dx^b}}, \frac{a_{2m} dx^m}{\sqrt{\eta_{ab} dx^a dx^b}}, \ldots \right] \eta_{ij} dx^i dx^j \]

(18)
with vectors $a_1, a_2$ (with constant components in a particular coordinate system) and the Minkowskian flat space-time metric $\eta_{ik}$. We will see in section 5.3 that Bogoslovsky’s Finsler metric is a subcase of this class.

It is here that the routes of researchers applying the methods of 4.2.1 and of Minkowski and Busemann separate. Following Finsler and Cartan - as most of the relativists interested in Finsler geometry have done -, we would use $ds^2 \simeq \frac{1}{2} L^2$ as a Finsler structure and derive the metric from it according to the first equation in (11).

4.2.3 Following Kawaguchi and Matsumoto

Here, a Finsler connection is defined as a pair $(N, \Gamma)$ of a non-linear connection $N$ in $T(M)(TM, \pi_T, V^n)$ and a connection $\Gamma$ in the Finsler bundle $F(M)(TM, \pi_1, GL(n, R))$ linked to the tangent bundle $T(M)$. Here, $\pi_1$, $\pi_T$ are the projection maps from $F(M)$ to $T(M)$, and from $T(M)$ to $M$, respectively. The bundle of linear frames comes in as soon as the directional elements are no longer restricted to the $dx^i$ in $M$ (Cartan’s “supporting elements”) but are considered as arbitrary vectors $y^i$ in some vector space. $V^n = R^n$ is the fibre of $T(M)$ over the manifold $M$. The projection maps from $F(M)$ to the bundle of linear frames $L(M)$ and from $L(M)$ to $M$ are named $\pi_2$, and $\pi_L$. The following relationship is demanded:

$$\pi_T \cdot \pi_1 = \pi_L \cdot \pi_2 .$$

(19)

From the tangent bundle $T(M)$ and the decomposition of the Finsler bundle $F(M)$ into a horizontal and a vertical subspace, the construction of three distinct connections is always guaranteed: $FT = (N^i_k, F_{jk}^i, C_{jk}^i)$. With the three connections, three different curvature tensors $R_{ijk}^l, P_{ijk}^l, S_{ijk}^l$ and 8 torsion tensors, three of which vanish, may be formed.

With $F_{jk}^i$ and $C_{jk}^i$, respectively, the horizontal and vertical covariant derivatives can be defined [32], [33], [34]:

h(orizontal)-covariant derivative:

$$\nabla_{\delta/\delta x^i} (\partial/\partial y^i) = F_{ijk}^l \frac{\partial}{\partial y^k}$$

(20)

v(ertical)-covariant derivative:

$$\nabla_{\partial/\partial y^i} (\partial/\partial y^i) = C_{ij}^k \frac{\partial}{\partial y^k}$$

(21)
Now we can find the link to the Finsler-Cartan approach of 4.2.1. The Cartan connection is defined as $FC = (N^i_k, F^*_{jk}^i, C^*_{jk}^i)$ with

$$F^*_{jk}^i := \frac{1}{2} f^i_l (\delta_j f^k_l + \delta_k f^j_l - \delta_l f^j_k), \quad \delta_i := \frac{\partial}{\partial x_l} - N^i_k \frac{\partial}{\partial y_l},$$

$$C^*_{jk}^i := \frac{1}{2} f^i_l \partial_k f^j_l \partial y_k.$$  \hspace{1cm} (22)

It can be shown that the Cartan-connection is metric compatible:

$$f^i_{ij||k} = 0, \quad f^i_{ijk} = 0.$$ \hspace{1cm} (24)

Here, the first covariant derivative (“$\parallel$”) corresponds to the v-covariant derivative, the second (“$|$”) to the h-covariant derivative.\hspace{1cm} (24)

5 Application of Finsler Geometry to physics

In the last part of this talk, an application to relativistic physics is presented, i.e., a possible break of Lorentz invariance modeled by “Finslerian relativity”

5.1 Generalities

Finsler geometry has been applied to many areas in classical physics and also to biological systems. After looking at many of such papers I get the impression that, up to now, in physics, this geometry was applied to systems with some sort of anisotropy (matter, fields) as an auxiliary device supposed to lead to a better understanding.\hspace{1cm} (15) This is far away from the use of Finsler geometry in a geometrization of physics. For biological systems described by certain sets of ordinary differential equations, these equations have been brought into the form of (10), and then interpreted within Finsler geometry. In my view, in both cases, no new insights into the physics or mathematics of the systems described has been reached which could not have reached without Finsler geometry. Perhaps, recent speculations about a possible break of Lorentz-invariance make a difference. In connection with Finsler geometry, the key idea due to G. Yu. Bogoslovsky is more than 20 years
old\textsuperscript{16} \textsuperscript{38}, \textsuperscript{39}. Its importance is just about to be discovered by mainstream physics \textsuperscript{41}, \textsuperscript{42}.

5.2 A possible break of Lorentz invariance?

At first, the possible break of Lorentz invariance was motivated by the presumed Greisen-Zatzevin-Kuz’m’min cut off in the energy of the observed particle spectrum resulting from inelastic scattering of photons at ultra-high-energy cosmic-ray protons (production of Pions) calculated from special relativity \textsuperscript{43}. Although this has not yet been fully cleared up, the recent measurements from the AUGER collaboration (detector array in Argentina) seem to indicate that the UHE-particles can be linked to Active Galactic Nuclei (AGN) thought to be powered by supermassive black holes. This would weaken the applicability of the GZK-cut off \textsuperscript{44}. Other tests, e.g., using the polarization of cosmic background radiation have been suggested\textsuperscript{17} \textsuperscript{45}.

From the point of view of theory, ad-hoc changes in the dispersion relation for high-energy particles implying the break of Lorentz-invariance have been discussed \textsuperscript{47}, \textsuperscript{48} as well as the inclusion of direction dependent “background fields” into the quantum field theory vacuum \textsuperscript{46}.

5.3 Finslerian special relativity

The basic idea of G. Bogoslovsky was to study a metric having as an isometry group the largest subgroup of the Poincaré-group, an 8-parameter Lie group (4-parameter subgroup of Lorentz group). It is now given the fancy notation ISIM(2). In (1+1)-dimensions, the line element of homogeneity degree 1 turned out to be:

\[ ds = \left( \frac{dx^0 - dx}{dx^0 + dx} \right)^{\frac{1}{2}} \sqrt{(dx^0)^2 - (dx)^2}, \]  

(25)

with \( 0 \leq r < 1 \). The velocity addition law of special relativity remains unaltered. It is easy to extend (25) to (1+3)-dimensions and to write it manifestly covariant:

\[ ds = \left( \frac{a_l dx^l}{\sqrt{\eta_{rs} dx^r dx^s}} \right)^r \sqrt{\eta_{nm} dx^n dx^m}, \]  

(26)

\textsuperscript{16}For a more recent presentation cf. \textsuperscript{40}.

\textsuperscript{17}Such data also are used to test a possible CPT-violation. Up to now, the data are not good enough to resolve the question.
In (26), \( \eta_{lm}a_la^m = 0 \), i.e., \( a^l = (1, a) \), \( a \cdot a = 1 \) is a null direction in Minkowski space-time.

The “generalized Lorentz transformations” now are

\[
x'^i = x^i + t^i_0, \quad x'^i = D(\vec{v}, \vec{a}) \quad R^i_j(\vec{v}, \vec{a}) \quad L^j_k(\vec{v}) \quad x^k = L(\vec{v}, \vec{a})x^k,
\]

with

- \( R^i_j(\vec{v}, \vec{a}) \): rotation of space axes about \( \vec{v} \times \vec{a} \) through an angle determined by \( \vec{v} \) and \( \vec{a} \).
- \( \vec{v} \): velocity of moving frame

From the equation for the mass shell, a highly non-linear “modified” dispersion relation as compared to \( \eta_{ij}p^ip^j = m^2c^2 \) follows:

\[
\left[ \frac{\eta_{ij}p^ip^j}{(\eta_{lm}p^l)^2} \right] \eta_{sk}p^sp^t = m^2c^2(1 + r)^{1+r}(1 - r)^{1-r}.
\]

As the figure shows, for growing \( r \) the mass shell becomes more and more anisotropic. In some of the papers on the breaking of Lorentz symmetry, polynomial additions were suggested: \( \eta_{ij}p^ip^j + a_{ijk}p^ip^jp^k + b_2(\eta_{ij}p^ip^j)^2 + ... = m^2c^2 \).

The kinematics of this Finslerian special relativity disagrees from what we are used to. The expressions for energy and linear momentum of a relativistic particle now are given by

\[
E = \vec{p} \cdot \vec{v} - L, \quad \vec{p} = \frac{\partial L}{\partial \vec{v}}.
\]

\[
E = \frac{mc^2}{\sqrt{1 - \vec{v}^2/c^2}} \left( \frac{1 - \vec{a} \cdot \vec{v}/c}{\sqrt{1 - \vec{v}^2/c^2}} \right)^r \left[ 1 - r + r \frac{1 - \vec{v}^2/c^2}{1 - \vec{a} \cdot \vec{v}/c} \right]
\]

\[
\vec{p} = \frac{mc}{\sqrt{1 - \vec{v}^2/c^2}} \left( \frac{1 - \vec{v} \cdot \vec{a}/c}{\sqrt{1 - \vec{v}^2/c^2}} \right)^r \left[ (1 - r)\vec{v}/c + r\vec{a} \frac{1 - \vec{v}^2/c^2}{1 - \vec{v} \cdot \vec{a}/c} \right].
\]

In the non-relativistic limit, we obtain

\[
E = mc^2 + (1 - r) \frac{m\vec{v}^2}{2} + r(1 - r) \frac{m(\vec{v} \cdot \vec{a})^2}{2} + O \left( \frac{\vec{v}^3}{c} \right)
\]

\[
\vec{p} = rmc\vec{a} + (1 - r)m\vec{v} + r(1 - r)m(\vec{v} \cdot \vec{a})\vec{a} + O \left( \frac{\vec{v}^2}{c} \right)
\]
With a non-vanishing anisotropy-parameter $r$, there exists a “rest-momentum” even for vanishing velocity\(^{18}\).

It might be possible to limit $r$ by measurements of the transverse Doppler effect; from the estimates for the so-called “ether wind” $rc < 5 \times 10^{-10}$ holds \(^{49}\).

6 Conclusion

In this lecture, geometrizations of physics were mentioned some of which are highly successful while others were not. It seems that all of them can be related to exterior / interior symmetry groups (extension of Klein’s “Erlanger Programm”?\(^{19}\)). Why follow such an approach at all? There are some advantages:

- Geometrization helps to obtain new results in physics;
- Geometrization makes possible proofs of exact theorems in mathematical physics;
- Geometrization helps to ease (or even make possible) calculations in physics.

Perhaps, it is possible to distinguish two kinds of geometrization. The first type leads to a geometry forming only a framework for physical systems as does special relativity. Nevertheless, the projective structure of Minkowski’s space-time can be related to the paths of free test particles; its conformal structure to light and electromagnetic test signals. But the second type has more structure. In it physical fields are related with geometric objects (general relativity, Kaluza’s theory). In the mixed geometry of the Einstein-Schrödinger unified field theory there is too much structure to be useful for physics. This may also turn out to be the case with geometrization in the framework of Finsler geometry. This is yet as speculative as are geometrizations involving supersymmetry. \(^{20}\)

One central motive behind the urge for geometrization seems to be a wish for the unification of all fundamental interactions. Nevertheless, we

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\(^{18}\)In fact, the theory is built for a relativistic situation.
\(^{19}\)In this context, general relativity is seen as a gauge theory of some group.
\(^{20}\)We have only mentioned but not dealt with areas in material physics in which Finsler geometry can be adapted to the structure of matter.
must insist that “unification” and “geometrization” are separate concepts not necessarily forming a logical union.

Whether his philosophical conclusions about space and time are accepted, or not, the unification of the temporal and spatial aspects of physical reality in space-time by mathematician Hermann Minkowski was a decisive step for all later geometrizations. In this tribute to him I wanted to point out that, in addition to his highly successful geometrization of space and time, he also is indirectly connected - via Finsler geometry - to another type of geometrization.

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7 Glossary

- **Algebraic numbers**
  A complex number which is a root of a non-zero polynomial with ratio-
  nal (or integer) coefficients (e.g., $\sqrt{3}$). An algebraic integer is a number
  which is a root of a non-zero polynomial with integer coefficients (e.g.,
  $1 \pm \sqrt{5}; a + bi, a, b$ integers). Non-algebraic complex numbers are said
to be transcendental (e.g., $\pi, e$).

- **Algebraic number field**
  A finite extension of the rational numbers $\mathbb{Q}$ is a ring of algebraic
  integers $\mathcal{O}$ in an algebraic number field $K/\mathcal{O}$. The unique factorization
  of integers into prime numbers can fail in $K/\mathcal{O}$ (e.g., $6 = 2 \cdot 3 =
  (1 + i\sqrt{5}) \cdot (1 - i\sqrt{5})$).

- **Borel algebra**
  The Borel algebra (or Borel $\sigma$-algebra) on a topological space $X$ is
  a $\sigma$-algebra of subsets of $X$ associated with the topology of $X$. In the
  mathematical literature, there are at least two nonequivalent definitions
  of this $\sigma$-algebra, either “the minimal $\sigma$-algebra containing the open
  sets”, or “the minimal $\sigma$-algebra containing the compact sets”.

- **Convex set**
  A set $A \in E^n$ is convex if together with any two points $x, y; x \neq y \in A$
it contains the segment $[x, y]$, thus if $(1 - \lambda)x + \lambda y \in A$ for $0 \leq \lambda \leq 1$.

- **Convex hull** $\text{conv}A$:
  The intersection of all closed convex sets containing a given set $A$. Or:
  For $A \in E^n$ the set of all convex combinations of any finitely many
  elements of $A$. $\text{conv}(A + B) = \text{conv}A + \text{conv}B$.

- **Homothetic**
  Sets $A, B$ are called homothetic if $A = \lambda B + d$ with $d \in E^n, \lambda > 0$.

- **Mixed volume**
  Take a Minkowski sum of $r$ convex bodies: $\alpha_1 K_1 + \alpha_2 K_2 + \ldots + \alpha_r K_r$,
  $\alpha_i \geq 0$. The volume $\lambda(K)$ may be written as a polynomial in the $\alpha_i$ with coef-
ficients $V(K_1, K_2, \ldots, K_r) = V_{i_1i_2\ldots i_r}$, such that $\lambda(K) = \Sigma_{i_1, i_2, \ldots, i_r} V_{i_1i_2\ldots i_r} \alpha_1 \alpha_2 \ldots \alpha_r$. \\


The coefficients \( V_{i_1i_2...i_r\alpha_1\alpha_2...\alpha_r} \) may be taken as totally symmetric in their indices; they are termed the \textit{mixed volume}.

- **Length spaces**
  A metric space \( X \) is a length space if for every \( x, y \) in \( X \)
  \[ |x - y| = \inf_{\gamma} L(\gamma) \]
  with \( L \) being the integral over the path \( \gamma : [a, b] \to E^n \):
  \[ L(\gamma) = \int_a^b \| \frac{d\gamma(t)}{dt} \| \, dt \]
  The infimum is taken over the set of paths \( \gamma \) joining \( x \) and \( y \).

- **Measure and volume**
  For a finite dimensional space \( V \) there exists a single Hausdorff linear topology. Thus the concept of Borel set (i.e., an element of a Borel algebra) is intrinsic to \( V \). A translation invariant measure on the Borel \( \sigma \)-algebra is the so-called \textit{Haar}-measure. This can be used for the \textit{volume} \( \lambda(\cdot) \) in \( V \).

  A simpler way to introduce a volume is to assume that \( V \) is equipped with an auxiliary euclidean structure and that the volume \( \lambda(\cdot) \) is the Lebesque measure induced by this structure. The particular choice of Haar-measure is immaterial. (A scalar multiple corresponds to a basis change for the euclidean structure.)

  The volume of the \( d \)-dimensional euclidean unit ball is
  \[ \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \]

- **Orthogonality in normed spaces**
  If \( V \) is a normed linear space and if \( x, y \in V \), then \( x \) is defined to be orthogonal to \( y \) if \( \| x + \alpha y \| \geq \| x \| \) for all \( \alpha \in R \).

- **Prime ideal**
  A prime ideal is a subset of a ring sharing important properties of a prime number in the ring of integers. Any prime ideal of \( \mathbb{Z} \) is of the form \( p\mathbb{Z} \), with \( p \) a prime number. Ideals in \( \mathbb{O} \) formed with a prime number may no longer be a prime ideal, e.g., \( 2\mathbb{Z}[i] \), because \( 2\mathbb{Z}[i] = ((1 + i))\mathbb{Z}[i])^2 \). Fermat’s theorem says that for an odd prime number \( p \)
$p \mathbb{Z}[i]$ is a prime ideal if $p \equiv 3 \pmod{4}$

$p \mathbb{Z}[i]$ is not a prime ideal if $p \equiv 1 \pmod{4}$.

Algebraic number theory generalizes this result to more general rings of integers.

• Support function

Let $K \in E^n$ be a convex closed, non-empty body and $u = (u_1, u_2, ..., u_n) \neq (0, 0, ..., 0)$ a vector, and $x \in K$. Then the equation of the support function can be written as $sup \{ \Sigma x \cdot u | x \in K \} = h(K, u)$. $\Sigma x \cdot u$ is the interior product in $E^n$.

• Support plane

Let $A \in E^n$ be a subset and $H \in E^n$ a hyperplane; let $H^+, H^-$ denote the two closed half spaces bounded by $H$. We say “$H$ supports $A$ at $x$” if $x \in A \cap H$ and either $A \in H^+$ or $A \in H^-$. $H$ is a support plane of $A$ or supports $A$ if $H$ supports $A$ at some point $x$ which is necessarily a boundary point.

Let $A \in E^n$ be convex and closed. Then through each boundary point of $A$ there is a support (hyper)plane of $A$. If $A \neq 0$ is bounded, then to each vector $u \in E^n \{0\}$ there is a support plane to $A$ with exterior normal vector $u$. 