NEW GENERAL INTEGRAL INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRALS

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Abstract. In this paper, the author obtains new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for Lipschitzian functions via Hadamard fractional integrals. Some applications to special means of positive reals numbers are also given.

1. Introduction

Let real function \( f \) be defined on some nonempty interval \( I \) of real line \( \mathbb{R} \). The function \( f \) is said to be convex on \( I \) if inequality

\[
 f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \).

Following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality respectively:

**Theorem 1.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following double inequality holds

\[
 f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.2}
\]

**Theorem 2.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping differentiable in \( I^0 \), the interior of \( I \), and let \( a, b \in I^0 \) with \( a < b \). If \( |f'(x)| \leq M, x \in [a,b] \), then we the following inequality holds

\[
 \left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{M}{b - a} \left[ \frac{(x - a)^2 + (b - x)^2}{2} \right]
\]

for all \( x \in [a,b] \).

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} \|f^{(4)}(x)\| < \infty \). Then the following inequality holds:

\[
 \left| \frac{1}{3} \left[ f(a) + f(b) \right] + 2f\left(\frac{a + b}{2}\right) \right| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b - a)^4.
\]

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In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see \cite{1, 2, 5, 6, 7, 12, 13} and references therein.

The following definitions are well known in the literature.

**Definition 1.** A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is called an \( M \)-Lipschitzian function on the interval \( I \) of real numbers with \( M \geq 0 \), if
\[
|f(x) - f(y)| \leq M |x - y|
\]
for all \( x, y \in I \).

For some recent results connected with Hermite-Hadamard type integral inequalities for Lipschitzian functions, see \cite{3, 4, 14, 15}.

**Definition 2** (\cite{10, 11}). A function \( f : I \subseteq (0, \infty) \to \mathbb{R} \) is said to be GA-convex (geometric-arithmetically convex) if
\[
f(x^t y^{1-t}) \leq t f(x) + (1 - t) f(y)
\]
for all \( x, y \in I \) and \( t \in [0, 1] \).

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

**Definition 3.** Let \( f \in L[a, b] \). The right-sided and left-sided Hadamard fractional integrals \( J^\alpha_{a^+} f \) and \( J^\alpha_{b^+} f \) of order \( \alpha > 0 \) with \( b > a \geq 0 \) are defined by
\[
J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b
\]
and
\[
J^\alpha_{b^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b
\]
respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \) (see \cite{9}).

In \cite{8}, Iscan established Hermite-Hadamard’s inequalities for GA-convex functions in Hadamard fractional integral forms as follows.

**Theorem 4.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a function such that \( f \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is a GA-convex function on \([a, b]\), then the following inequalities for fractional integrals hold:
\[
f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2 \left( \ln \frac{b}{a} \right)^\alpha} \left\{ J^\alpha_{a^+} f(b) + J^\alpha_{b^+} f(a) \right\} \leq \frac{f(a) + f(b)}{2}
\]
with \( \alpha > 0 \).

In the inequality \(1.3\), if we take \( \alpha = 1 \), then we have the following inequality
\[
f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq \frac{f(a) + f(b)}{2}.
\]
Moreover in [8], Iscan obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for quasi-geometrically convex functions via Hadamard fractional integrals as related the inequality (1.3).

In this paper, the author obtains new general inequalities for Lipschitzian functions via Hadamard fractional integrals as related the inequality (1.3).

2. Main Results

Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \), the interior of \( I \), throughout this section we will take

\[
I_f (x, \lambda, \alpha, a, b) = (1 - \lambda) \left[ \ln^\alpha \frac{x}{a} + \ln^\alpha \frac{b}{x} \right] f(x)
\]

\[
+ \lambda \left[ f(a) \ln^\alpha \frac{x}{a} + f(b) \ln^\alpha \frac{b}{x} \right] - \Gamma (\alpha + 1) \left[ J^\alpha_{x}f(a) + J^\alpha_{x}f(b) \right]
\]

and

\[
S_f (x, y, \lambda, \alpha, a, b) = \lambda^\alpha f(x) + (1 - \lambda)^\alpha f(y) - \frac{\Gamma (\alpha + 1)}{\ln^\alpha \frac{b}{a}} \left[ J^\alpha_{C}f(a) + J^\alpha_{C}f(b) \right]
\]

where \( a, b \in I \) with \( a < b \), \( x, y \in [a, b] \), \( \lambda \in [0, 1] \), \( C = a^{1 - \lambda} b^\lambda \), \( \alpha > 0 \) and \( \Gamma \) is Euler Gamma function.

**Theorem 5.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a \( M \)-Lipschitzian function on \( I \) and \( a, b \in I \) with \( a < b \). then for all \( x \in [a, b] \), \( \lambda \in [0, 1] \) and \( \alpha > 0 \) we have the following inequality for Hadamard fractional integrals

\[
|I_f (x, \lambda, \alpha, a, b)| 
\leq M \left\{ \left[ (1 - \lambda) x - \lambda a \right] \left( \ln^\alpha \frac{x}{a} \right)^\alpha + \alpha (2\lambda - 1) \int_a^x \left( \ln^\alpha \frac{t}{a} \right)^{\alpha - 1} dt
\right.
\]

\[
+ \left[ \lambda b - (1 - \lambda) x \right] \left( \ln^\alpha \frac{b}{x} \right)^\alpha + \alpha (1 - 2\lambda) \int_x^b \left( \ln^\alpha \frac{t}{x} \right)^{\alpha - 1} dt \right\}.
\]

**Proof.** Using the hypothesis of \( f \), we have the following inequality

\[
|I_f (x, \lambda, \alpha, a, b)| = (1 - \lambda) \left[ \ln^\alpha \frac{x}{a} + \ln^\alpha \frac{b}{x} \right] f(x)
\]

\[
+ \lambda \left[ f(a) \ln^\alpha \frac{x}{a} + f(b) \ln^\alpha \frac{b}{x} \right] - \alpha \left[ \int_a^x \left( \ln^\alpha \frac{t}{a} \right)^{\alpha - 1} f(t) \frac{dt}{t} + \int_x^b \left( \ln^\alpha \frac{t}{x} \right)^{\alpha - 1} f(t) \frac{dt}{t} \right]
\]

\[
\leq (1 - \lambda) \left| f(x) \ln^\alpha \frac{x}{a} - \alpha \int_a^x \left( \ln^\alpha \frac{t}{a} \right)^{\alpha - 1} f(t) \frac{dt}{t} + f(x) \ln^\alpha \frac{b}{x} - \alpha \int_x^b \left( \ln^\alpha \frac{t}{x} \right)^{\alpha - 1} f(t) \frac{dt}{t} \right|
\]

\[
+ \lambda \left[ f(a) \ln^\alpha \frac{x}{a} - \alpha \int_a^x \left( \ln^\alpha \frac{t}{a} \right)^{\alpha - 1} f(t) \frac{dt}{t} + f(b) \ln^\alpha \frac{b}{x} - \alpha \int_x^b \left( \ln^\alpha \frac{t}{x} \right)^{\alpha - 1} f(t) \frac{dt}{t} \right]
\]
In Theorem 5, if we take Corollary 1.

In this inequality,

\[ \leq \alpha (1 - \lambda) \left[ \int_a^x \left( \ln \frac{t}{a} \right)^{\alpha - 1} |f(x) - f(t)| \frac{dt}{t} + \int_x^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} |f(x) - f(t)| \frac{dt}{t} \right] \]

\[ + \alpha \lambda \left[ \int_a^x \left( \ln \frac{t}{a} \right)^{\alpha - 1} |f(a) - f(t)| \frac{dt}{t} + \int_x^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} |f(b) - f(t)| \frac{dt}{t} \right] \]

\[ \leq \alpha (1 - \lambda) M \left[ \int_a^x \left( \ln \frac{t}{a} \right)^{\alpha - 1} (x - t) \frac{dt}{t} + \int_x^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} (t - x) \frac{dt}{t} \right] \]

\[ + \alpha \lambda M \left[ \int_a^x \left( \ln \frac{t}{a} \right)^{\alpha - 1} (t - a) \frac{dt}{t} + \int_x^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} (b - t) \frac{dt}{t} \right] \]

\[ \leq M \left\{ [(1 - \lambda) x - \lambda a] \left( \ln \frac{x}{a} \right)^{\alpha} + \alpha (2\lambda - 1) \int_a^x \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt \right. \]

\[ + |\lambda b - (1 - \lambda) x| \left( \ln \frac{b}{x} \right)^{\alpha} + \alpha (1 - 2\lambda) \int_x^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} dt \right\}. \]

\[ \square \]

Corollary 1. In Theorem 5, if we take \( \lambda = 0 \), then we get

\[ (2, \ln \frac{a}{b})^\alpha \left[ \frac{f(x)}{\ln b} - \frac{f(a)}{\ln a} \right] \]

\[ \leq \frac{M}{(\ln \frac{b}{a})^\alpha} \left\{ x \left[ \left( \ln \frac{x}{a} \right)^{\alpha} - \left( \ln \frac{b}{x} \right)^{\alpha} \right] + \alpha \left[ \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} dt - \int_a^x \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt \right] \right\}. \]

In this inequality,

(i) If we take \( \alpha = 1 \), then

\[ \left| f(x) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \right| \]

\[ \leq \frac{M}{\ln b - a} \left\{ x \ln \frac{x^2}{ab} + (a + b - 2x) \right\}. \]

(ii) If we take \( x = \sqrt{ab} \), then

\[ \left| f(\sqrt{ab}) - \frac{2^{\alpha - 1} (\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J_{\sqrt{ab}}^\alpha f(a) + J_{\sqrt{ab}}^\alpha f(b) \right] \right| \]

\[ \leq \frac{2^{\alpha - 1} M \alpha}{(\ln \frac{b}{a})^\alpha} \left\{ \int_{\sqrt{ab}}^b \left( \ln \frac{b}{t} \right)^{\alpha - 1} dt - \int_a^{\sqrt{ab}} \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt \right\}. \]
(iii) If we take \( x = \sqrt{ab} \) and \( \alpha = 1 \) then

\[
(2.2) \quad \left| f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} \, dt \right| 
\leq \frac{M}{\ln b - \ln a} (a + b - 2\sqrt{ab}).
\]

Corollary 2. In Theorem 3, if we take \( \lambda = 1 \), then we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{2^{\alpha - 1} \Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J^\alpha_{\sqrt{ab}-f(a)} + J^\alpha_{\sqrt{ab}+f(b)} \right] \right|
\leq \frac{2^{\alpha - 1} M}{(\ln \frac{b}{a})^\alpha} \left\{ \frac{b - a}{2} \left( \ln \frac{b}{a} \right)^\alpha - \alpha \left[ \int_{\sqrt{ab}}^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \, dt - \int_a^\sqrt{ab} \left( \ln \frac{t}{a} \right)^{\alpha-1} \, dt \right] \right\}.
\]

Specially if we take \( \alpha = 1 \) in this inequality, then we have

\[
(2.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} \, dt \right| 
\leq \frac{M}{\ln b - \ln a} \left\{ \frac{b - a}{2} \left( \ln \frac{b}{a} \right) - (a + b - 2\sqrt{ab}) \right\}.
\]

Corollary 3. In Theorem 3

(1) If we take \( x = \sqrt{ab} \) and \( \lambda = 1/3 \), then

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f(\sqrt{ab}) \right] - \frac{2^{\alpha - 1} \Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J^\alpha_{\sqrt{ab}-f(a)} + J^\alpha_{\sqrt{ab}+f(b)} \right] \right|
\leq \frac{2^{\alpha - 1} M}{3 (\ln \frac{b}{a})^\alpha} \left\{ \frac{b - a}{2^\alpha} \left( \ln \frac{b}{a} \right)^\alpha - \alpha \left[ \int_{\sqrt{ab}}^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \, dt - \int_a^\sqrt{ab} \left( \ln \frac{t}{a} \right)^{\alpha-1} \, dt \right] \right\}.
\]

Specially if we take \( \alpha = 1 \) in this inequality, then we have

\[
(2.5) \quad \left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f(\sqrt{ab}) \right] - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} \, dt \right| 
\leq \frac{M}{3 (\ln b - \ln a)} \left\{ \frac{b - a}{2} \left( \ln \frac{b}{a} \right) - (a + b - 2\sqrt{ab}) \right\}.
\]
(2) If we take \( x = \sqrt{ab} \) and \( \lambda = 1/2 \), then
\[
\left| \frac{1}{2} \left[ f(a) + f(b) \right] + 2 f\left( \sqrt{ab} \right) \right| - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{\left( \ln \frac{b}{a} \right)^{\alpha}} \left[ J_{\sqrt{ab}}^{a} f(a) + J_{\sqrt{ab}}^{b} f(b) \right] \leq \frac{M (b - a)}{4}.
\]

Specially if we take \( \alpha = 1 \) in this inequality, then we have
\[
(2.6) \quad \left| \frac{1}{2} \left[ f(a) + f(b) \right] + 2 f\left( \sqrt{ab} \right) \right| - \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(t)}{t} \, dt \leq \frac{M (b - a)}{4}.
\]

**Corollary 4.** In Theorem 4 If we take \( \alpha = 1 \), then
\[
\left| (1 - \lambda) f(x) + \lambda \left( \frac{f(a) \ln \frac{a}{x} + f(b) \ln \frac{b}{x}}{\ln b - \ln a} \right) - \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(t)}{t} \, dt \right| \leq \frac{M (b - a)}{\ln b - \ln a} \left[ (1 - \lambda) x - \lambda a \right] \left( \ln \frac{x}{a} \right) + \left| \lambda b - (1 - \lambda) x \right| \left( \ln \frac{b}{x} \right). + (2) If we take \( x = \sqrt{ab} \) in this inequality, then we have
\[
(2.7) \quad \left| (1 - \lambda) f\left( \sqrt{ab} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(t)}{t} \, dt \right| \leq \frac{M (b - a)}{\ln b - \ln a} \left[ \frac{\lambda (b - a)}{2} \ln \frac{b}{a} + (1 - 2\lambda) \left( a + b - 2 \sqrt{ab} \right) \right].
\]

We note that if we take \( \lambda = 0, \lambda = 1, \lambda = 1/3 \) and \( \lambda = 1/2 \) in the inequality (2.7) we obtain the inequalities (2.2a), (2.4a), (2.5a) and (2.6a) respectively.

Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be an \( M \)-Lipschitzian function. In the next theorem, let \( \lambda \in [0, 1], C = a^{1-\lambda} b^{\lambda}, x, y \in [a, b] \) and define \( C_{\alpha, \lambda}, \alpha > 0 \), as follows:

1. If \( a \leq C \leq x \leq y \leq b \), then
\[
C_{\alpha, \lambda}(x, y) = \frac{x}{\alpha} \lambda^{\alpha} \left( \ln \frac{b}{a} \right)^{\alpha} + \frac{y}{\alpha} \left( 1 - \lambda \right)^{\alpha} \left( \ln \frac{b}{a} \right)^{\alpha} - 2 \left( \ln \frac{b}{y} \right)^{\alpha}
\]
\[
+ \int_{y}^{b} \left( \ln \frac{b}{t} \right)^{\alpha-1} \, dt - \int_{C}^{y} \left( \ln \frac{b}{t} \right)^{\alpha-1} \, dt - \int_{a}^{C} \left( \ln \frac{t}{a} \right)^{\alpha-1} \, dt.
\]

2. If \( a \leq x \leq C \leq y \leq b \), then
\[
C_{\alpha, \lambda}(x, y) = \frac{x}{\alpha} \left( 2 \left( \ln \frac{x}{a} \right)^{\alpha} \lambda^{\alpha} \left( \ln \frac{b}{a} \right)^{\alpha} \right) + \frac{y}{\alpha} \left( 1 - \lambda \right)^{\alpha} \left( \ln \frac{b}{a} \right)^{\alpha} - 2 \left( \ln \frac{b}{y} \right)^{\alpha}
\]
\[
+ \int_{x}^{C} \left( \ln \frac{t}{a} \right)^{\alpha-1} \, dt - \int_{a}^{C} \left( \ln \frac{t}{a} \right)^{\alpha-1} \, dt + \int_{y}^{b} \left( \ln \frac{b}{t} \right)^{\alpha-1} \, dt - \int_{C}^{y} \left( \ln \frac{b}{t} \right)^{\alpha-1} \, dt.
\]
(3) If \( a \leq x \leq y \leq C \leq b \), then

\[
C_{\alpha, \lambda}(x, y) = \frac{x}{\alpha} \left\{ 2 \left( \frac{\ln x}{a} \right)^\alpha - \lambda^\alpha \left( \frac{\ln b}{a} \right)^\alpha \right\} - \frac{y}{\alpha} (1 - \lambda)^\alpha \left( \frac{\ln b}{a} \right)^\alpha \\
+ \int_a^C \left( \frac{\ln b}{t} \right)^{\alpha - 1} dt + \frac{1}{\alpha} \int_x^y \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt - \frac{1}{\alpha} \int_a^y \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt.
\]

Now we shall give another result for Lipschitzian functions as follows.

**Theorem 6.** Let \( x, y, \alpha, \lambda, C, C_{\alpha, \lambda} \) and the function \( f \) be defined as above. Then we have the following inequality for Hadamard fractional integrals

\[
|S_f(x, y, \lambda, \alpha, a, b)| \leq \frac{\alpha M C_{\alpha, \lambda}(x, y)}{(\ln \frac{b}{a})^\alpha}
\]

**Proof.** Using the hypothesis of \( f \), we have the following inequality

\[
|S_f(x, y, \lambda, \alpha, a, b)| = \frac{\alpha}{(\ln \frac{b}{a})^\alpha} \left| \int_a^C \frac{|f(x) - f(t)|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt + \frac{1}{\alpha} \int_x^y \frac{|f(y) - f(t)|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt \right|
\]

\[
\leq \frac{\alpha}{(\ln \frac{b}{a})^\alpha} \left[ \int_a^C \frac{|f(x) - f(t)|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt + \frac{1}{\alpha} \int_x^y \frac{|f(y) - f(t)|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt \right]
\]

\[
= \frac{\alpha M}{(\ln \frac{b}{a})^\alpha} \left[ \int_a^C \frac{|x - t|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt + \frac{1}{\alpha} \int_x^y \frac{|y - t|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt \right]
\]

Now using simple calculations, we obtain the following identities \( \int_a^C \frac{|x - t|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt \) and \( \int_x^y \frac{|y - t|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt \).

(1) If \( a \leq C \leq x \leq y \leq b \), then

\[
\int_a^C \frac{|x - t|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt = \frac{x}{\alpha} \lambda^\alpha \left( \frac{\ln b}{a} \right)^\alpha - \frac{1}{\alpha} \int_a^C \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt
\]

and

\[
\int_x^y \frac{|y - t|}{t} \left( \frac{\ln t}{a} \right)^{\alpha - 1} dt = \frac{y}{\alpha} \left\{ (1 - \lambda)^\alpha \left( \ln \frac{b}{a} \right)^\alpha - 2 \left( \ln \frac{b}{y} \right)^\alpha \right\} + \frac{1}{\alpha} \int_y^b \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt - \frac{1}{\alpha} \int_x^C \left( \ln \frac{t}{a} \right)^{\alpha - 1} dt.
\]
(2) If \(a \leq x \leq C \leq y \leq b\), then
\[
\int_a^C \frac{|x-t|}{t} \left(\ln \frac{t}{a}\right)^{\alpha-1} dt = \frac{x}{\alpha} \left\{2 \left(\ln \frac{x}{a}\right)^\alpha - \lambda^\alpha \left(\ln \frac{b}{a}\right)^\alpha\right\} + \int_x^C \left(\ln \frac{t}{a}\right)^{\alpha-1} dt - \int_a^x \left(\ln \frac{t}{a}\right)^{\alpha-1} dt
\]
and
\[
\int_C^b \frac{|y-t|}{t} \left(\ln \frac{t}{b}\right)^{\alpha-1} dt = \frac{y}{\alpha} \left\{(1-\lambda)^\alpha \left(\ln \frac{b}{a}\right)^\alpha - 2 \left(\ln \frac{b}{y}\right)^\alpha\right\} + \int_y^b \left(\ln \frac{t}{b}\right)^{\alpha-1} dt - \int_C^y \left(\ln \frac{t}{b}\right)^{\alpha-1} dt.
\]

(3) If \(a \leq x \leq y \leq C \leq b\), then
\[
\int_a^C \frac{|x-t|}{t} \left(\ln \frac{t}{a}\right)^{\alpha-1} dt = \frac{x}{\alpha} \left\{2 \left(\ln \frac{x}{a}\right)^\alpha - \lambda^\alpha \left(\ln \frac{b}{a}\right)^\alpha\right\} + \int_x^C \left(\ln \frac{t}{a}\right)^{\alpha-1} dt - \int_a^x \left(\ln \frac{t}{a}\right)^{\alpha-1} dt
\]
and
\[
\int_C^b \frac{|y-t|}{t} \left(\ln \frac{t}{b}\right)^{\alpha-1} dt = \int_C^y \left(\ln \frac{t}{b}\right)^{\alpha-1} dt - \frac{y}{\alpha} (1-\lambda)^\alpha \left(\ln \frac{b}{a}\right)^\alpha.
\]

Using the inequality (2.9) and the above identities \(\int_a^C \frac{|x-t|}{t} \left(\ln \frac{t}{a}\right)^{\alpha-1} dt\) and \(\int_C^b \frac{|y-t|}{t} \left(\ln \frac{t}{b}\right)^{\alpha-1} dt\), we derive the inequality (2.8). This completes the proof.

Under the assumptions of Theorem 6, we have the following corollaries and remarks:

**Corollary 5.** In Theorem 6, if we take \(\alpha = 1\), then the inequality (2.8) reduces to the following inequality:
\[
\left| (1-\lambda) f(x) + \lambda f(y) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \right| \leq \frac{MC_{1,\lambda}(x, y)}{\ln \frac{a}{b}}
\]

**Corollary 6.** In Theorem 6 let \(\delta \in \left[\frac{1}{2}, 1\right]\), \(x = a^\delta b^{1-\delta}\) and \(y = a^{1-\delta} b^\delta\). Then, we have the inequality
\[
\left| \lambda^\alpha f(a^\delta b^{1-\delta}) + (1-\lambda)^\alpha f(a^{1-\delta} b^\delta) - \frac{\Gamma(\alpha + 1)}{\ln \frac{a}{b}} J_{\alpha}^{-\delta} f(a) + J_{\alpha}^{\delta} f(b) \right| \leq ML(\alpha, \lambda, \delta)
\]
where
(i) If \( \lambda \leq 1 - \delta \), then
\[
L(\alpha, \lambda, \delta) = a^\delta b^{1-\delta} \lambda^\alpha + a^{1-\delta} b^\delta [(1-\lambda)^\alpha - 2(1-\delta)^\alpha] + \frac{\alpha}{(\ln \frac{b}{a})^\alpha} \left\{ \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{b^{1-\delta}}{t^{a-1}} dt - \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{a^{1-\delta}}{t^{a-1}} dt - C \left( \ln \frac{b}{t} \right)^{\alpha-1} dt \right\}
\]

(ii) If \( 1 - \delta \leq \lambda \leq \delta \), then
\[
L(\alpha, \lambda, \delta) = a^\delta b^{1-\delta} [2(1-\delta)^\alpha - \lambda^\alpha] + a^{1-\delta} b^\delta [(1-\lambda)^\alpha - 2(1-\delta)^\alpha] + \frac{\alpha}{(\ln \frac{b}{a})^\alpha}
\times \left\{ \int_a^C \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{b^{1-\delta}}{t^{a-1}} dt - \int_a^C \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{a^{1-\delta}}{t^{a-1}} dt + \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{a^{1-\delta}}{t^{a-1}} dt - \int_a^C \left( \ln \frac{b}{t} \right)^{\alpha-1} dt \right\}
\]

(iii) If \( \delta \leq \lambda \), then
\[
L(\alpha, \lambda, \delta) = a^\delta b^{1-\delta} [2(1-\delta)^\alpha - \lambda^\alpha] - a^{1-\delta} b^\delta (1-\lambda)^\alpha + \frac{\alpha}{(\ln \frac{b}{a})^\alpha} \left\{ \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{b^{1-\delta}}{t^{a-1}} dt + \int_a^C \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{a^{1-\delta}}{t^{a-1}} dt - \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha-1} \frac{a^{1-\delta}}{t^{a-1}} dt \right\}
\]

Corollary 7. In Theorem 7, if we take \( x = y = C \), then we have the inequality
\[
(2.11) \quad \left| \lambda^\alpha (1 - \lambda)^\alpha f(x) - \frac{\Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J_{x-}^\alpha f(a) + J_{x+}^\alpha f(b) \right] \right|
\]
\[\leq M \left\{ [\lambda^\alpha - (1 - \lambda)^\alpha] x + \frac{\alpha}{(\ln \frac{b}{a})^\alpha} \left[ \int_x^b \left( \ln \frac{b}{t} \right)^{\alpha-1} dt - \int_a^x \left( \ln \frac{b}{t} \right)^{\alpha-1} dt \right] \right\}.
\]

Remark 1. In the inequality (2.11), if we choose \( \lambda = \frac{1}{2} \), then we get the inequality (2.1).

Corollary 8. In the inequality (2.10), if we take \( \delta = 1 \), then we have the following weighted Hadamard-type inequalities for Lipschitzian functions via Hadamard fractional integrals
\[
(2.12) \quad \left| \lambda^\alpha f(a) + (1 - \lambda)^\alpha f(b) - \frac{\Gamma(\alpha + 1)}{(\ln \frac{b}{a})^\alpha} \left[ J_{x-}^\alpha f(a) + J_{x+}^\alpha f(b) \right] \right|
\]
\[\leq M \left\{ [(b - 1) - \lambda a^\alpha + a^{1-\delta} b^\delta \left( \ln \frac{b}{a} \right)^{\alpha-1} \left[ \int_a^b \left( \ln \frac{b}{t} \right)^{\alpha-1} dt - \int_a^C \left( \ln \frac{b}{t} \right)^{\alpha-1} dt \right] \right\}.
\]

Remark 2. In the inequality (2.12), if we choose \( \lambda = \frac{1}{2} \), then we get the inequality (2.3).

3. Application to Special Means

Let us recall the following special means of two positive number \( a, b \) with \( b > a \):

\[
\begin{align*}
(i) & \quad \frac{a + b}{2}, \\
(ii) & \quad \frac{b - a}{\ln b - \ln a} + a, \\
(iii) & \quad \frac{a^\alpha - b^\alpha}{\ln b - \ln a} + a.
\end{align*}
\]
(1) The arithmetic mean
\[ A = A(a, b) := \frac{a + b}{2}. \]

(2) The geometric mean
\[ G = G(a, b) := \sqrt{ab}. \]

(3) The harmonic mean
\[ H = H(a, b) := \frac{2ab}{a + b}. \]

(4) The logarithmic mean
\[ L = L(a, b) := \frac{b - a}{\ln b - \ln a}. \]

(5) The identric mean
\[ I = I(a, b) = \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}. \]

To prove the results of this section, we need the following lemma:

**Lemma 1** ([14]). Let \( f : [a, b] \to \mathbb{R} \) be differentiable with \( \| f' \|_\infty < \infty \). Then \( f \) is an \( M \)-Lipschitzian function on \( [a, b] \) where \( M = \| f' \|_\infty \).

**Proposition 1.** For \( b > a > 0 \), \( \lambda \in [0, 1] \), and \( n \geq 1 \), we have.
\[
| (1 - \lambda) G^n(a, b) + \lambda A^n(a^n, b^n) - L(a^n, b^n) | \leq \frac{2nb^{n-1}}{\ln b - \ln a} \left[ \frac{b - a}{2} \ln \frac{b}{a} + 2 (1 - 2\lambda) (A(a, b) - G(a, b)) \right].
\]

**Proof.** The proof follows by the inequality (2.7) applied for the Lipschitzian function \( f(x) = x^n \) on \([a, b]\). \( \square \)

**Remark 3.** Let \( \lambda = 0 \) and \( \lambda = 1 \) in the inequality (3.1). Then, using the inequality \([1,4]\), we have the following inequalities respectively
\[
0 \leq L(a^n, b^n) - G^n(a, b) \leq \frac{2nb^{n-1}}{\ln b - \ln a} (A(a, b) - G(a, b)),
\]
\[
0 \leq A(a^n, b^n) - L(a^n, b^n) \leq \frac{nb^{n-1}}{\ln b - \ln a} \left[ \frac{b - a}{2} \ln \frac{b}{a} - 2 (A(a, b) - G(a, b)) \right].
\]

**Proposition 2.** For \( b > a > 0 \) and \( \lambda \in [0, 1] \), we have
\[
| (1 - \lambda) G(ae^a, be^b) + \lambda A(ae^a, be^b) - L(e^a, e^b) L(a, b) | \leq \frac{e^b (b + 1)}{\ln b - \ln a} \left[ \frac{\lambda (b - a)}{2} \ln \frac{b}{a} + 2 (1 - 2\lambda) (A(a, b) - G(a, b)) \right].
\]

**Proof.** The proof follows by the inequality (2.7) applied for the Lipschitzian function \( f(x) = xe^x \) on \([a, b]\). \( \square \)
Remark 4. Let \( \lambda = 0 \) and \( \lambda = 1 \) in the inequality \([1,4]\). Then, using the inequality \([1,4]\), we have the following inequalities respectively
\[
0 \leq L (e^a, e^b) L (a, b) - G (ae^a, be^b) \leq \frac{2e^b (b + 1)}{\ln b - \ln a} (A (a, b) - G (a, b)) ,
\]
\[
0 \leq A (ae^a, be^b) - L (e^a, e^b) L (a, b)
\leq \frac{e^b (b + 1)}{\ln b - \ln a} \left[ \frac{b - a}{2} \ln \frac{b}{a} - 2 (A (a, b) - G (a, b)) \right] .
\]

Proposition 3. For \( b > a > 0 \), \( \lambda \in [0,1] \) and \( n \geq 1 \), we have.
\[
(3.3) \quad \left| (1 - \lambda) G^{-1} (a, b) + \lambda H^{-1} (a, b) - L (a, b) G^{-2} (a, b) \right|
\leq \frac{1}{a^2 (\ln b - \ln a)} \left[ \frac{\lambda (b - a)}{2} \ln \frac{b}{a} + 2 (1 - 2\lambda) (A (a, b) - G (a, b)) \right] .
\]

Proof. The proof follows by the inequality \([4]\) applied for the Lipschitzian function \( f(x) = 1/x \) on \([a,b] \).

Remark 5. Let \( \lambda = 0 \) and \( \lambda = 1 \) in the inequality \([1,4]\). Then, using the inequality \([1,4]\), we have the following inequalities respectively
\[
0 \leq L (a, b) - G (a, b) \leq \frac{G^2 (a, b)}{a^2 (\ln b - \ln a)} (A (a, b) - G (a, b)) ,
\]
\[
0 \leq G^2 (a, b) - L (a, b) H (a, b)
\leq \frac{G^2 (a, b) H (a, b)}{a^2 (\ln b - \ln a)} \left[ \frac{b - a}{2} \ln \frac{b}{a} - 2 (A (a, b) - G (a, b)) \right] .
\]

Proposition 4. For \( b > a > 0 \) and \( \lambda \in [0,1] \), we have
\[
\ln G (a, b) \leq \frac{1}{a (\ln b - \ln a)} \left[ \frac{\lambda (b - a)}{2} \ln \frac{b}{a} + 2 (1 - 2\lambda) (A (a, b) - G (a, b)) \right] .
\]

Proof. The proof follows by the inequality \([4]\) applied for the Lipschitzian function \( f(x) = \ln x \) on \([a,b] \).

Proposition 5. For \( b > a > e^{-1} \) and \( \lambda \in [0,1] \), we have
\[
\left| (1 - \lambda) G (a, b) \ln G (a, b) + \lambda \ln G (ae^a, be^b) - L (a, b) \ln I (a, b) \right|
\leq \frac{1 + \ln b}{\ln b - \ln a} \left[ \frac{\lambda (b - a)}{2} \ln \frac{b}{a} + 2 (1 - 2\lambda) (A (a, b) - G (a, b)) \right] .
\]

Proof. The proof follows by the inequality \([4]\) applied for the Lipschitzian function \( f(x) = x \ln x \) on \([a,b] \).

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