Identifying metric spinor phase with axion field from axion-fermion coupling and Weyl-Peccei Quinn spin transformations

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Metric spinor phase of the Infeld-van der Waerden $\gamma$-formalism and axion field were identified in Ref. [1], by using Maxwell’s theory. Since axion couples with fermions, we will investigate Dirac’s theory to extend the work given in [1], showing that is possible again to identify this phase with the axion. By searching an exact identification, we will work yet the spin transformations to adapt chiral rotations and so to show that the metric spinor phase behaves exactly as required for the axion field. We will study also the Dirac-Maxwell system which provides a minimal Lagrangian approach for the axion classical sector and, finally, we will obtain an explicit 2-spinor description of magnetic monopole and its charge.

I. INTRODUCTION

Axion theories have provided important results in several areas, such as QCD, CP symmetry problem, condensed matter and string theory [2–5]. The axion is also a good candidate to explain cold dark matter [6–9], even though it is not directly observed in experiments. The axion/Maxwell coupling is given by the following Lagrangian term

$$\alpha F_{\mu\nu} F_{\mu\nu}^\ast.$$ (1)

Since $A_\mu$ is the electromagnetic gauge potential and $\epsilon_{\mu\nu\sigma\rho}$ are the Levi-Civita tensor components, the objects $\alpha$, $F_{\mu\nu}$ and $F_{\mu\nu}^\ast$ represent, respectively, the axion pseudo-scalar, Maxwell’s tensor and its Hodge dual defined by $F_{\mu\nu}^\ast \equiv (1/2)\epsilon_{\mu\nu\sigma\rho} F_{\sigma\rho}$. In the usual cases the term $F_{\mu\nu} F_{\mu\nu}^\ast$ is taken to zero. In [10–13], have been formulated a local dual invariant electrodynamics (LDIE) formalism to add axionic fields in electromagnetic equations.

On another hand, by using the Infeld-van der Waerden formalisms [14–21], geometric sources for Infeld-van der Waerden electromagnetic fields have been defined in [22], such that in [1], magnetic monopoles are defined in a similar way. In this case, the spinor formulation of Maxwell equations in $\gamma$-formalism yields an axion electrodynamics identical with the found by Tiwari [10–13]. Based on Weyl’s representation, the Infeld-van der Waerden $\gamma\varepsilon$-formalisms have been useful for spinor

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formulation of General Relativity \[16\]. Each $\gamma\varepsilon$-formalism is based on its spinor metric: constant in the $\varepsilon$-formalism and depending locally on the coordinates in the $\gamma$-formalism. In these formulations, the spinor decomposition of $g_{\mu\nu}$ admits a local scale/phase freedom given by $\varepsilon_{AB} \mapsto |\gamma| e^{i\Theta} \varepsilon_{AB}$ and $\Sigma^{AA'}_{\mu} \mapsto |\gamma|^{-1} \Sigma^{AA'}_{\mu}$. Here, $\varepsilon_{AB}$ is the metric spinor component and $\Sigma^{AA'}_{\mu}$ a connecting object component (or Infeld-van der Waerden symbol). Thus, the $\gamma$-formalism defines the metric spinor and the connecting objects as follow

$$\gamma_{AB} \equiv |\gamma| e^{i\Theta} \varepsilon_{AB} \quad \text{and} \quad \Upsilon^{AA'}_{\mu} \equiv |\gamma|^{-1} \Sigma^{AA'}_{\mu}. \quad (2)$$

In \[1\], by using electromagnetic theory, the identification $\alpha \sim \Theta$ has been established. It is known that classical “world” theories can be rewritten in a 2-spinor version, since the linear group of unimodular complex $2 \times 2$ matrices has a homomorphism two to one with the orthochronous proper Lorentz group.

The Infeld-van Waerden formalisms was based on the homomorphism between the Weyl and Lorentz groups \[1, 23\]. Originally, this theory provided a geometrical origin of the electromagnetic potential, since it would lead to an imaginary part of the spinor connection trace in which would satisfy the Weyl’s principle of gauge invariance \[24\]. Weyl studied the relationship between the tetrad formalism for curved spacetime and the parameter of the Dirac 4-spinor phase transformation: if the tetrad varies so the parameter varies too \[23, 24\]. Infeld and van der Waerden considered Weyl’s work to implement Dirac’s theory in General Relativity. However, this idea wasn’t consolidated it should not be understood on its original form \[25\]. On this interpretation, the formalism would imply a relation between electric charge and spin, since the scale/phase couples with each type of fermion. Unfortunately, the neutron disabled this idea: as it has spin but no electric charge. Furthermore, the interpretation of the imaginary part of the spinor connection trace impaired some investigations of Maxwell’s theory in the $\gamma$-formalism. In \[1\], has been considered the electromagnetic potential as an external physical entity. Thus, the physical significance of the phase and scale are freedom to be reinterpreted.

Since axion was identificate with Infeld-van der Waerden phase in \[1\], by using electromagnetic fields, we will want to study other fields that interacts with $\alpha$. Fermion-axion coupling is given by the following Lagrangian term \[26\]

$$\bar{\Psi} \gamma^\mu \Psi \partial_\mu \alpha, \quad (3)$$

with $\Psi$ being the Dirac 4-spinor, $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$ its spinor adjoint and $\gamma^\mu$ the Dirac matrices. By wanting to repeat a similar result with the derived in \[1\], we will investigate Dirac’s theory in the $\gamma$-formalism.
We will use $\hbar = c = 1$, as well as the signature $(+ − − −)$. The index symmetry/antisymmetry will be indicated by round/square brackets. The paper will be organized as follows. In section 2 we will review the axion electrodynamics provided by the $\gamma$-formalism. In the section 3, we must obtain the axion-fermion coupling from $\gamma$-formalism and establish the Weyl-Peccei Quinn transformations to identify the phase with the axion, as well as, we will write Maxwell-Dirac theory to obtain an explicit magnetic monopole 2-spinor form and derive its effective magnetic charge.

II. IDENTIFYING AXION WITH METRIC SPINOR PHASE: MAXWELL CASE

In this section, we will review the work given in [1]. The axion electrodynamics was originally postulated by Wilczek [27]. The Lagrangian in which provides the axion electrodynamics can be represented by

$$2F_{\mu \nu} F_{\mu \nu} + 4\alpha F_{\mu \nu} F^*_{\mu \nu} + A^\mu j_\mu, \quad (4)$$

where $\mathcal{L}_M = 2F_{\mu \nu} F_{\mu \nu}$ is the usual Maxwell Lagrangian, $\mathcal{L}_{CS} = 4\alpha F_{\mu \nu} F_{\mu \nu}$ the Chern-Simons term and $\mathcal{L}_1 = A^\mu j_\mu$ the interaction term with the electric current density $j_\mu$. In the 3-vector notation, we have $F_{\mu \nu} F_{\mu \nu} = B^2 - E^2$ and $F_{\mu \nu} F^*_{\mu \nu} = E \cdot B$. E and B are the electric and magnetic fields respectively. The usual field equations for axion electrodynamics are given by the expressions

$$\partial_\mu F_{\mu \nu} = 4\pi j_\nu + (\partial_\mu \alpha) F_{\mu \nu}^* \quad \text{and} \quad \partial_\mu F_{\mu \nu}^* = 0. \quad (5)$$

In [10], by studying a generic local dual electrodynamics, Tiwari found the following axion electrodynamics equations adapt for us as

$$\partial_\mu F_{\mu \nu} = 4\pi j_\nu + (\partial_\mu \alpha) F_{\mu \nu}^* \quad \text{and} \quad \partial_\mu F_{\mu \nu}^* = 4\pi m_\nu - (\partial_\mu \alpha) F_{\mu \nu}. \quad (6)$$

where $m_\mu$ is a magnetic current density. The first expression resumes the effect caused from an axion field. The second can be understood as a solution for the non observation of magnetic monopole in nature, since the axion term can cancel its effects. In general, expressions (6) are invariant when duality rotations

$$\mathcal{F} \mapsto U \mathcal{F} \quad \text{and} \quad \mathcal{S} \mapsto U \mathcal{S}, \quad \mathcal{F} = \begin{pmatrix} F_{\mu \nu} \\ F^*_{\mu \nu} \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} j_\mu \\ m_\nu \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}. \quad (7)$$

are taken into account, with $\phi = \phi(x^\alpha)$ and simultaneously if the axion changes as $\partial_\mu \alpha \mapsto \partial_\mu \alpha + \partial_\mu \phi$.

By considering valid the Maxwell’s equations, we find from (6) the equations

$$(\partial^\mu \alpha) F_{\mu \nu}^* = 0 \quad \text{and} \quad (\partial^\mu \alpha) F_{\mu \nu} = 4\pi m_\nu. \quad (8)$$
On another hand, the spacetime algebra in $\gamma$-formalism is represented as follows

$$g_{\mu\nu} = \Upsilon_{\mu}^{AA'}\Upsilon_{\nu}^{BB'}\gamma_{AB}\gamma_{A'B'}.$$  

The Einstein convention is adopted and each spinor index runs of 0 to 1 (0' to 1'). The object $\gamma_{AB}$ is the metric spinor and $\Upsilon_{\mu}^{AA'}$ the Infeld-van der Waerden symbols $[16, 19, 28]$. Explicity, we have $g_{\mu\nu}$ is the metric tensor component of a generic background. Complex conjugation is denoted by $(\Upsilon_{\mu}^{AA'})^* = \Upsilon_{\mu}^{A'B'}$. Spinors and tensors are related by using a hermitian matrix set $\Upsilon$, such as $v_{\mu} = \Upsilon_{\mu}^{AA'}v_{AA'}$ and $v_{AA'} = \Upsilon_{\mu}^{A_A'B_B'}v_{\mu}$. We use the metric spinor to lower (or raise) the spinor indexes: $\xi^B = \gamma_{BA}\xi^A$, $\xi_A = \gamma_{AB}\xi^B$. The object $\gamma_{AB}$ is a skew-symmetric spinor component. In the matrix form, we have

$$\begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \text{ with } \gamma \doteq |\gamma|e^{\Theta i}. \quad (10)$$

in which $|\gamma|$ and $\Theta$ are real-valued functions of $x^\mu$.

For generic spinors spinors $\xi^A$ and $\zeta_A$, the covariant derivative is given respectively as

$$\nabla_{\mu}\xi^A = \partial_{\mu}\xi^A + \Xi_{\mu B}^A\xi^B$$ and $$\nabla_{\mu}\zeta_A = \partial_{\mu}\zeta_A - \Xi_{\mu A}^B\zeta_B,$$  

with $\Xi_{\mu B}^A$ a spinor connection. The complex component $\Xi_{\mu A}^A$ can be written as

$$\Xi_{\mu A}^A = \partial_{\mu}\ln|\gamma| - 2i|\Xi|, \quad (12)$$

with $\Xi \doteq -(1/2)\text{Im}\,\Xi_{\mu A}^A$. In $\gamma$-formalism, the compatibility metric $\nabla_{\alpha}g_{\mu\nu} = 0$ yields the eigenvalue equations $[19]

$$\nabla_{\mu}\gamma_{AB} = i\beta_{\mu}\gamma_{AB}$$ and $$\nabla_{\mu}\gamma^{AB} = -i\beta_{\mu}\gamma^{AB}.$$  

with $\beta_{\mu}$ defined by

$$\beta_{\mu} \doteq \partial_{\mu}\Theta + 2\Xi_{\mu}.$$  

As it is usual, the spinors $\xi^A$ and $\zeta_A$ transform under the action of the generalized Weyl gauge group, which can be expressed in the component form as

$$\Delta_{A}^B = \sqrt{\rho}e^{i\lambda}\delta_{A}^B,$$  

where $\rho > 0$ is a real function and $\lambda$ the gauge parameter of the group. In flat spacetime, the choice $\Xi_{\mu A}^B = 0$ can be taken in account. Thus, $\Xi_{\mu} = 0$ and $|\gamma| = \text{const} > 0$, so that $\beta_{\mu} = \partial_{\mu}\Theta$. Here, we will assume $|\gamma| = 1$. 

In this formalism, the spinor version of the Maxwell’s tensor and its Hodge are \[ \gamma \] \[19, 28, 29\]

\[ 2F_{A'B'B'} = \gamma_{AB} f_{A'B'} + \gamma_{A'B'} f_{AB}, \quad \text{and} \quad 2F_{{A'A'B'B'}}^* = i (\gamma_{AB} f_{A'B'} - \gamma_{A'B'} f_{AB}), \]

(16)
in which \( f_{AB} \) is called of Maxwell spinor. By defining the complex tensor \( F_{\mu\nu}^{(\pm)} = F_{\mu\nu} \pm i F_{\mu\nu}^* \)
and by taking the spinor forms (16), we find

\[ F_{A'B'B'}^{(\pm)} = \gamma_{A'B'} f_{AB} \quad \text{and} \quad F_{A'A'B'B'}^{(\pm)} = \gamma_{AB} f_{A'B'}. \]

(17)
The complex version of Maxwell equations \( \nabla_{\mu} F_{\mu\nu}^{(\pm)} = 4\pi j_{\nu} \) yield

\[ \nabla_{A'} f_{AB} = 2\pi j_{AA'} + i \beta_{A'} f_{AB}, \]

(18)

Since in the usual \( \varepsilon \)-formalism, the Maxwell equations with magnetic monopoles \( \nabla_{\mu} F_{\mu\nu}^{(\pm)} = 4\pi (j_{\nu} \pm im_{\nu}) \) provides

\[ \nabla_{A'} f_{AB} = 2\pi (j_{AA'} + im_{AA'}), \]

(19)

follows the definition elaborate in \[1\], i. e.,

\[ \beta_{A'} f_{AB} \equiv 2\pi m_{AA'}. \]

(20)

With this definition, the expression (20) implies \( \beta_{A'} F_{\mu\nu}^{(\pm)} = 4\pi m_{\nu} \). Since \( \beta_{\mu} = \partial_{\mu} \Theta \) in flat spacetime,
this equation becomes \( (\partial_{\mu} \Theta) F_{\mu\nu}^{(\pm)} = 4\pi m_{\nu} \). By expliciting \( F_{\mu\nu}^{(\pm)} \), we find

\[ (\partial_{\mu} \Theta) F_{\mu\nu} = 4\pi m_{\nu} \quad \text{and} \quad (\partial_{\mu} \Theta) F_{\mu\nu}^* = 0. \]

(21)

By comparing (21) with (15), the identification \( \alpha \sim \Theta \) is done.

On the Lagrangian viewpoint in Minkowski spacetime, the Maxwell Lagrangian in \( \gamma \)-formalism
is given by

\[ \mathcal{L}_M = 2F_{\mu\nu} F^{\mu\nu} = \Re \left[ \gamma^{AC} \gamma^{BD} f_{AB} f_{CD} \right], \]

(22)

which the right side can be rewritten as follows

\[ \Re \left[ \gamma^{AC} \gamma^{BD} f_{AB} f_{CD} \right] = \Re \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] \cos (2\Theta) + \Im \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] \sin (2\Theta). \]

(23)

By taking the approximation \( \gamma_{AB} \simeq \varepsilon_{AB} (\Theta \simeq 0) \) and thus by considering

\[ \Re \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] \simeq 2F_{\mu\nu}^* F_{\mu\nu}^* \quad \text{and} \quad \Im \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] \simeq 2F_{\mu\nu}^* F_{\mu\nu}, \]

(24)

we find from (22), the theory

\[ \mathcal{L}_M \simeq 2F_{\mu\nu}^* F_{\mu\nu} + 4\Theta F_{\mu\nu}^* F_{\mu\nu}. \]

(25)

Thus, when \( \Theta \simeq 0 \), the invariant form \( \Re \left[ f_{AB} f_{AB} \right] \) in \( \gamma \)-formalism generates the usual Maxwell Lagrangian plus the axionic Chern-Simons term \( \mathcal{L}_{CS} \equiv \Theta F_{\mu\nu}^* F_{\mu\nu}^* \) in which relates the axion like Infeld-van der Waerden phase-electromagnetic coupling.
III. PHASE-FERMION COUPLING AND WEYL-PQ TRANSFORMATIONS

We will follow Ref. [21] to present Dirac’s theory. Other works about Dirac’s theory in the
Infeld-van der Waerden formalisms are found in [16, 18, 30]. In 2-component spinor formalism,
Dirac equations in relativistic spacetimes can be stated as follow

\[ i \nabla_{AA'} \psi^A = \mu \chi_{A'} \quad \text{and} \quad i \nabla^{AA'} \chi_{A'} = \mu \psi^A. \]  

(26)

\( \psi^A \) and \( \chi_{A'} \) are, respectively, right handed and left handed 2-spinors. \( \mu \equiv -m/\sqrt{2} \), where minus
sign is placed according with our purpose. Thanks to the fact that in the \( \gamma \)-formalism we have
eigenvalue equations for \( \gamma_{AB} \), (26) is equivalent to

\[ \nabla_{AA'} \psi^A - i \beta_{AA'} \psi^A = i \mu \chi_{A'} \quad \text{and} \quad \nabla_{AA'} \chi_{A'} - i \beta_{AA'} \chi_{A'} = i \mu \psi^A. \]  

(27)

The \( \varepsilon \)-formalism version of (27) is obtained by taking \( \beta_{AA'} \) to zero. Dirac’s fields which satisfy
respectively (26) and (27) are given by the systems

\[ \mathcal{D} = \left\{ (\psi^A, \chi_{A'}), (\chi_A, \psi_{A'}) \right\} \quad \text{and} \quad \mathcal{D}_0 = \left\{ (\psi^A, \chi_{A'}), (\chi_A, \psi_{A'}) \right\}. \]  

(28)

By using metric spinors depending on the formalism considered, \( \mathcal{D}_0 \) is obtained from \( \mathcal{D} \). We stress
that only \( \mathcal{D}_0 \) couples with \( \beta \)-terms.

If we want to obtain usual covariant Dirac’s equation in the \( \varepsilon \)-formalism, we must define the
4-component Dirac’s field \( \Psi \) as follows

\[ \Psi \equiv \begin{pmatrix} \psi^A \\ \chi_{A'} \end{pmatrix}. \]  

(29)

This choice is valid, since we have \( \Psi \mapsto e^{i\lambda} \Psi \) under the original Weyl’s group action: \( \psi^A \mapsto e^{i\lambda} \psi^A \)
and \( \chi_{A'} \mapsto e^{i\lambda} \chi_{A'} \). Original Weyl’s group is recovered by taking \( \rho = 1 \) in (15). If our interest
concerns only on axion/fermion coupling, we can work in flat spacetime. In this background and
in the \( \varepsilon \)-formalism, equation (27) becomes

\[ \partial_{AA'} \psi^A = i \mu \chi_{A'} \quad \text{and} \quad \partial_{AA'} \chi_{A'} = i \mu \psi^A. \]  

(30)

On the \( \varepsilon \)-formalism, covariant derivative \( \nabla_{AA'} \) is taken by using Infeld-van der Waerden symbols:
\( \nabla_{AA'} = \Sigma^\mu_{AA'} \nabla_\mu \) and \( \nabla^{AA'} = \varepsilon^{AB} \varepsilon^{A'B'} \Sigma^\mu_{BB'} \nabla_\mu \). In Minkowski universe \( \sqrt{2} \Sigma^\mu_{AA'} = \sigma^\mu_{AA'} \) as well as
\( \nabla_\mu = \partial_\mu \cdot \left( \sigma^\mu_{AA'} \right) = (I, \sigma^i_{AA'}) \) where \( \sigma^i_{AA'} \) are the Pauli matrices and \( I \) the 2 \times 2 unity matrix. If
taken in account the Weyl’s representation, the Dirac’s matrices become

\[ \gamma^0 = \begin{pmatrix} \mathbb{1} & \sigma^0 \\ \sigma^0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} \mathbb{1} & \sigma^i \\ -\sigma^i & \mathbb{1} \end{pmatrix}. \]  

(31)
If we use the definition \((\ref{29})\) and Dirac’s matrices in the Weyl’s representation, Dirac’s equation is obtained from \((\ref{30})\). In fact, we have

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} \sigma_{AA'} \partial \mu
\end{pmatrix}
\begin{pmatrix}
\Omega
\frac{1}{\sqrt{2}} \sigma_{AA'} \partial \mu
\end{pmatrix}
\psi_A = \frac{1}{\sqrt{2}} \gamma^\mu \partial \mu \Psi \quad \text{and} \quad \mu \left( \psi_A \chi^{A'} \right) = -\frac{m}{\sqrt{2}} \Psi,
\]

(32)

so that we obtain \((i \gamma^\mu \partial \mu - m) \Psi = 0\), which is the familiar covariant Dirac’s equation.

By considering the Euler-Lagrange equations, \((\ref{30})\) are derived from the Lagrangian

\[
\mathcal{L}_D \left[ \bar{D}, \partial \bar{D} \right] = i \psi^A \partial_{AA'} \psi^A + i \chi_A \partial AA' \chi_{A'} - \mu \left( \psi^A \chi_A + \psi^A \chi_{A'} \right),
\]

(33)

since the relationship

\[
\psi^A \chi_A + \psi^A \chi_{A'} = -\psi_A \chi^A - \psi_A \chi^{A'} = \bar{\Psi} \Psi,
\]

(34)

is satisfied in both formalisms and

\[
i \psi^A \partial_{AA'} \psi^A + i \chi_A \partial AA' \chi_{A'} = i \psi^A \partial AA' \psi^A + i \chi^A \partial AA' \chi^{A'} = \frac{1}{\sqrt{2}} i \bar{\Psi} \gamma^\mu \partial \mu \Psi,
\]

(35)

only in the \(\varepsilon\)-formalism. Notation given in \((\ref{33})\) denotes that \(\mathcal{L}_D\) depends of \(\bar{D}\) and derivatives. In 2-spinor notation, we have yet \(\bar{\Psi} \equiv \psi^{\dagger} \gamma^0 = \left( \chi^A, \psi_A \right)\). If \(\rho = 1\) in \((\ref{15})\), we note that \(\bar{\Psi}\) transforms as \(\bar{\Psi} \mapsto e^{-i \lambda} \bar{\Psi}\), which must be in this specific case. Thus, if we use \((\ref{34})\) and \((\ref{35})\), the Lagrangian \((\ref{33})\) assumes the form \(\mathcal{L}_D \left[ \bar{D}, \partial \bar{D} \right] = \mathcal{L}_D \left[ \bar{D}, \partial \bar{D} \right] = i \bar{\Psi} \gamma^\mu \partial \mu \Psi - m \bar{\Psi} \Psi\), which is the usual Dirac’s Lagrangian. The factor \(\sqrt{2}\) is absorbed by the redefinition \(\mathcal{L}_D \mapsto \sqrt{2} \mathcal{L}_D\).

Since that in the \(\varepsilon\)-formalism, Dirac’s theory is obtained from \((\ref{33})\), we will consider its index configuration as starting form in the \(\gamma\)-formalism. In this formalism, equations \((\ref{27})\) in flat spacetime are rewritten by putting \(\beta_{AA'} = \partial_{AA'} \Theta\). The equations \((\ref{29})\) have the same form that those given in \((\ref{30})\), in both formalisms. Let us study \((\ref{33})\). Equation \((\ref{34})\) is valid also in the \(\gamma\)-formalism. However, a change in the index configuration of \((\ref{35})\) yields, in the \(\gamma\)-formalism, the expression

\[
i \psi^A \partial_{AA'} \psi^A + i \chi_A \partial AA' \chi_{A'} = i \psi^A \partial AA' \psi^A + i \chi^A \partial AA' \chi^{A'} + \psi_A \psi_A \partial AA' \Theta + \chi^A \chi^{A'} \partial AA' \Theta,
\]

(36)

due to eigenvalue equations. Since that

\[
\psi_A \psi_A \partial AA' \Theta + \chi^A \chi^{A'} \partial AA' \Theta = \frac{1}{\sqrt{2}} \bar{\Psi} \gamma^\mu \partial \mu \Theta,
\]

(37)

we obtain \((\ref{33})\) from \((\ref{37})\), by simply indentifying \(\alpha \sim \Theta\). Thus, in the \(\gamma\)-formalism, we have the functional relation

\[
\mathcal{L}_D \left[ \bar{D}, \partial \bar{D} \right] = \mathcal{L}_D \left[ \bar{D}, \partial \bar{D}, \partial \Theta \right] = i \bar{\Psi} \gamma^\mu \partial \mu \Psi - m \bar{\Psi} \Psi + \bar{\Psi} \gamma^\mu \partial \mu \Theta,
\]

(38)

which represents an axion-like coupling between Dirac fields and \(\Theta\). Again, \(\mathcal{L}_D\) absorbed \(\sqrt{2}\).

Therefore, we have identified the Infeld-van der Waerden phase with axion field.
A. Weyl-PQ (Peccei-Quinn) Spin Transformations

We have identified the metric spinor phase with the axion field from Maxwell and Dirac theories. Moreover, such identifications are “poors” since the behavior of Θ on chiral rotation has not been established. By considering (15), we have seen that the Dirac’s spinor is defined in terms of Weyl’s 2-spinors according with the action of the Weyl group. On another hand, we have also the chiral rotations: \( \Psi \mapsto \tilde{\Psi} = e^{i\zeta \gamma_5} \Psi \). The original Peccei-Quinn procedure [2] states that, on a chiral (PQ) transformation, an axion-like pseudo-scalar \( \theta \) transforms as

\[
θ \mapsto \tilde{θ} = θ - 2ζ.
\]

The transformation (39) is an elegant solution for solve the CP-problem. Here, our strategy is identify \( Θ \) with \( α \) by using a similar way with the developed by Infeld and van der Waerden. The key idea of the Infeld-van der Waerden unification is to consider the gauge behavior of the electromagnetic potential \( A_μ \), i.e.,

\[
A_μ \mapsto \hat{A}_μ = A_μ - \partial_μ λ.
\]

Since the object \( Ξ_μ \) transforms as (see for example [19])

\[
\hat{Ξ}_μ = Ξ_μ - \frac{1}{2} \text{Im}(\partial_μ \ln Δ),
\]

by putting \( Δ ≡ \det(Δ_A^B) = e^{2iλ} \) in (41), we find

\[
Ξ_μ \mapsto \hat{Ξ}_μ = Ξ_μ - \partial_μ λ.
\]

By looking (40) and (42), it is suggested the identification \( Ξ_μ \sim A_μ \).

Our strategy is verify if \( Θ \) behaves as (39) when a PQ transformation is taken in account. A similar problem was solved in [31], where the author worked \( Ξ_μ \) as being a mixture of polar and axial vectors. Our first step is to find spin transformations which represent simultaneously Weyl and PQ rotations. In 2-spinor language, \( \Psi \mapsto e^{iζ γ_5} \Psi \) is translated as \( \tilde{ψ}_A = e^{iζ} ψ_A \) and \( \tilde{χ}^A' = e^{-iζ} χ^A' \). Thus, we need to obtain the spin transformations where, in general, \( \bar{ψ}_A = e^{i(λ+ζ)} \) and \( \bar{χ}^A' = e^{i(λ-ζ)} χ^A' \) are verified. The notation \( (n) \) denotes a composite Weyl \( (\hat{n}) \)-PQ \( (\tilde{n}) \) transformation.

Let us consider the general spin transformation of the metric spinor:

\[
\gamma_{AB} = Δ_A^C Δ_B^D γ_{CD}.
\]

Now, we will consider the composition

\[
Δ_A^B = Δ_\circ_A^C Δ_\bullet_C^B = \sqrt{Δ_\circ Δ_\bullet} δ_A^B,
\]

\[
\Delta^\circ_A^B = \Delta_\circ_A^C \Delta_\bullet_C^B = \sqrt{\Delta_\circ \Delta_\bullet} δ_A^B,
\]

(44)
where $\Delta^\circ$ and $\Delta^\bullet$ are the determinants of the Weyl ($\Delta^\circ_A^B$) and PQ ($\Delta^\bullet_A^B$) rotations. For a Weyl transformation we have $\tilde{\gamma}_{AB} = \Delta^\circ \gamma_{AB}$ while for PQ transformation we will suppose $\tilde{\gamma}_{AB} = \Delta^\bullet (\tilde{\gamma} \varepsilon_{AB})$. With such considerations, (43) yields

$$\nabla_{AB} = \Delta^\circ \Delta^\bullet (\tilde{\gamma} \varepsilon_{AB}).$$

(45)

As we will verify later, a PQ transformation can be implemented by requiring

$$\nabla_{AB} = \Delta^\circ \gamma_{AB} \iff \tilde{\gamma} = (\Delta^\bullet)^{-1} \gamma.$$

(46)

From (46), we find $\hat{\gamma}_{AB} = \Delta^\circ \gamma_{AB}$ and $\tilde{\gamma}_{AB} = \Delta^\bullet \gamma_{AB}$.

The transformation of generic spinors $v_A$ and $u^A = \gamma^{AB} u_B$ are then given by

$$\nabla_A = \sqrt{\Delta^\circ \Delta^\bullet} v_A \quad \text{and} \quad \nabla'^A = \sqrt{(\Delta^\circ)^{-1} \Delta^\bullet} u^A.$$  

(48)

Each separated rotation acts as

$$\hat{v}_A = \sqrt{\Delta^\circ} v_A, \quad \hat{u}^A = \sqrt{(\Delta^\circ)^{-1}} u^A, \quad \tilde{v}_A = \sqrt{\Delta^\bullet} v_A \quad \text{and} \quad \tilde{u}^A = \sqrt{\Delta^\bullet} u^A.$$  

(49)

Since $\Delta^\circ = e^{2\lambda i}$ and $\Delta^\bullet = e^{2\zeta i}$, the 2-component fermions transform as

$$\overline{\psi}_A = e^{i(\lambda + \zeta)} \psi_A \quad \text{and} \quad \overline{\chi'}^A = e^{i(\lambda - \zeta)} \chi'^A,$$

(50)

or, particularly, $\hat{\psi}_A = e^{i\lambda} \psi_A$ ($\hat{\chi'}^A = e^{i\lambda} \chi'^A$) (Weyl) and $\tilde{\psi}_A = e^{i\zeta} \psi_A$ ($\tilde{\chi'}^A = e^{-i\zeta} \chi'^A$) (PQ).

As we have seen, the Weyl-PQ transformation is possible if $\tilde{\gamma} = (1/\Delta^\bullet) \gamma$ is satisfied. In Minkowski spacetime we have $\gamma = e^{i\Theta}$, such that from a PQ rotation, we obtain $e^{i\tilde{\Theta}} = (e^{i\Theta}/\Delta^\bullet)$. Therefore $\tilde{\Theta} = \Theta + i \ln \Delta^\bullet + 2n\pi$, which implies

$$\Theta \mapsto \tilde{\Theta} = \Theta - 2\zeta + 2n\pi, \quad n \in \mathbb{Z},$$

(51)

PQ-behavior

since $\Delta^\bullet = e^{2\zeta i}$. Thus, we have demonstrated that $\Theta$ satisfies the PQ requirement (W4).

We must note that the invariant form $\nabla^A \nabla_A$ is broken in our formulation: $\nabla^A \nabla_A = e^{2\zeta i} u^A v_A$. This property is a formal declaration of no chiral symmetry for massive fermion terms. The invariant indexes configurations are now given by

$$AA', \quad AA', \quad AA'^{AA'}, \quad AA'BB', AA'^{BB'}, \ldots.$$
B. Maxwell-Dirac system and magnetic monopoles

When coupled with Maxwell fields, the Dirac equations in curved spacetime are taken by putting \( \nabla A \rightarrow \nabla - ieA \) in (27), i.e.,

\[
\left( \nabla^{A'} - ieA^{A'} - i\beta^{A'} \right) \psi_A = i\mu\chi^{A'} \quad \text{and} \quad \left( \nabla^{A'} - ieA^{A'} - i\beta^{A'} \right) \chi_A = i\mu\psi_A,
\]

with \( A_\mu \) being an electromagnetic potential component. Since \( f_{AB} = \nabla^A (A^B A') \) in the spinor language, equations (18) can be rewritten as follows

\[
\left( \nabla^{B'} - i\beta^{B'} \right) \nabla^{B'} (A^B A') = e (\psi_A \psi_A' + \chi_A \chi_A').
\]

Here, we have used for the electric current density the expression

\[
j_{A'} = e (\psi_A \psi_A' + \chi_A \chi_A'),
\]

where \( e \) is the electric charge.

Let us consider the equations (21) in curved spacetime:

\[
\nabla^{\mu}F^{(\pm)}_{\mu\nu} = 4\pi j_\nu \quad \text{and} \quad \beta^{\mu}F^{(\pm)}_{\mu\nu} = m_\nu.
\]

Since \( \beta^{[\mu}\beta^{\nu]} = 0 \), the expression \( \beta^{\mu}F^{(\pm)}_{\mu\nu} \) provides \( \beta^{\mu}m_\mu = 0 \). By applying the covariant derivative in \( \beta^{\mu}F^{(\pm)}_{\mu\nu} = 4\pi m_\nu \), we obtain

\[
W^{\mu\nu} F_{\mu\nu} + 8\pi (\beta^{\mu}j_\mu + \nabla^{\mu}m_\mu) = 0 \quad \text{and} \quad W^{\mu\nu} F^*_{\mu\nu} = 0,
\]

where \( W^{\mu\nu} = 2\partial_{[\mu} \Xi_{\nu]} \) is the Infeld-van der Waerden curvature bivector. By remembering that \( \Xi_\mu = -(1/2) \text{Im} \Xi_\mu A^A \), if the spacetime is flat, we obtain \( \partial^{\mu}m_\mu = -j_\mu \partial^{\mu}\Theta \) such that

\[
m_\mu = - (\Theta - C) j_\mu,
\]

where \( C \) is a constant and we have used yet the null divergence for electric sources: \( \partial^{\mu}j_\mu = 0 \). Since in this case \( \beta^{\mu}m_\mu = -\Theta j_\mu \partial^{\mu}\Theta = 0 \), we will obtain \( 2\Theta \partial^{\mu} (\Theta j_\mu) = \Theta \partial^{\mu} (\Theta j_\mu) \) implying \( \partial^{\mu}m_\mu = 0 \) and \( \beta^{\mu}j_\mu = 0 \). Thus, in flat background, we have the divergence/orthogonality relationships

\[
\partial^{\mu}j_\mu = 0 = \partial^{\mu}m_\mu \quad \text{and} \quad j_\mu \partial^{\mu}\Theta = 0 = m_\mu \partial^{\mu}\Theta.
\]

By taking (54), the expression (57) in the 2-spinor form is so rewritten as

\[
m_{AA'} = g_{eff} (\psi_A \psi_A' + \chi_A \chi_A'), \quad \text{with} \quad g_{eff} \doteq -e (\Theta - C).
\]

The term \( g_{eff} \) in (59) can be understood as an effective magnetic charge. From a PQ transformation, the effective magnetic charge behaves

\[
g_{eff} \mapsto \tilde{g}_{eff} = g_{eff} + 2e (\zeta - n\pi),
\]

such that \( \tilde{g}_{eff} = g_{eff} \) when \( \zeta = n\pi \), with \( n \in \mathbb{Z} \).
IV. CONCLUSION

We have obtained an axion-fermion coupling from a minimal Lagrangian formulation, since the axion-Dirac Lagrangian density $L_D[D, \partial D, \partial \alpha]$ can be derived from $L_D[D, \partial D]$ if $\alpha \sim \Theta$. As also we recapitulate, the Maxwell Lagrangian density $L_M$ provides an electromagnetic-axion coupling if $\alpha \sim \Theta \simeq 0$: $L_M \simeq 2F^{\mu \nu}F_{\mu \nu} + 4L_{CS}$. Thus, we can derive of the $\gamma$-formalism an axionic classical sector from the compact form:

$$L_D[D, \partial D] + L_M.$$  

Thus, the scientific use of our work can be justified by reductionist arguments. However, the behavior of $\Theta$ by PQ rotations was not clear. Thus, we have elaborate a Weyl-PQ approach in which the PQ rotations are implemented in the usual spin transformations, providing a Weyl-PQ spin unified formalism. By using this new formulation, we have shown that the spacetime phase behaves exactly as the axion is supposed to transform when the PQ group acts. Thus, we can conclude that the metric spinor phase $\Theta$ behaves geometrically as an axion field, as well as, it generates axion-like couplings with Maxwell and Dirac fields. Finally, we obtained an explicit 2-spinor structure for magnetic monopole source such as its effective charge. A rigorous formulation of the Weyl-PQ transformations in the Infeld-van der Waerden formalisms and a more accurate study of the Dirac-Maxwell system must be worked in a future paper.

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