The Quantitative Characterization of Finite Simple Groups *

In celebration of the 80 birthday of Professor John Thompson

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Abstract
In this report we summarize this work, all finite simple groups \( G \) can determined uniformly using their orders \(|G|\) and the set \( \pi_e(G) \) of their element orders.

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1 Introduction

Group theory is an important branch of mathematics. It has a wide range of applications in other mathematics, physics, chemistry, and other fields. Completed (announced) in 1980, the classification theorem of finite simple groups, is one of the most important mathematical achievements of the 20th century. It is long time to prove this theorem (D. Gorenstein called it the "Thirty Years War"[1]), participants in many countries in hundreds of group theory scientist, many articles (close to 500), length (more than 15,000 pages), is unprecedented in the history of mathematics, with landmark significance.

For a finite group, the order of group and the element order are two of the most important basic concepts. Let \( G \) be a finite group and \( \pi_e(G) \) be the set of element orders in \( G \). In 1987, the author of this paper posed the following conjecture[2]:

\[
\text{Conjecture.} \quad \text{Let} \ G \ \text{be a group and} \ M \ \text{a finite simple group. Then} \ G \cong M \ \text{if and only if} \ (a) \ \pi_e(G) = \pi_e(M), \ \text{and} \ (b) \ |G| = |M|.
\]

That is, for all finite simple groups we may characterize them using only their orders and the sets of their element orders (briefly, "two orders").

After I wrote some letters to Prof. John G. Thompson and reported the above conjecture. Thompson pointed that, "Good luck with your conjecture about simple groups. I

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hope you continue to work on it”, ”This would certainly be a nice theorem”, in his reply letters. The warmly encouragement prompt us to finish this work.

Moreover, Thompson posed the following problem and conjecture:

For each finite group $G$ and each integer $d \geq 1$, let $G(d) = \{x \in G; x^d = 1\}$. $G_1$ and $G_2$ are of the same order type if and only if $|G_1(d)| = |G_2(d)|$, $d = 1, 2, \cdots$.

**Thompson’s Problem (1987).** Suppose $G_1$ and $G_2$ are groups of the same order type. Suppose also $G_1$ is solvable. Is it true that $G_2$ is also necessarily solvable?

If $G$ is a finite group, set $N(G) = \{n \in N; G$ has a conjugacy class $C$ with $|C| = n\}$.

**Thompson’s Conjecture (1988).** If $G$ and $M$ are of finite groups and $N(G) = N(M)$, and if in addition, $M$ is a non-Abelian simple group while the center of $G$ is 1, then $G$ and $M$ are isomorphic.

From 1987 to 2003, the authors of [2-8] proved that this conjecture is correct for all finite simple groups except $B_n$, $C_n$ and $D_n$ ($n$ even). In the end of 2009, the authors of [9] proved that this conjecture is correct for $B_n$, $C_n$ and $D_n$ ($n$ even). Thus, this conjecture is proved and become a theorem, that is, **all finite simple groups can determined by their ”two orders”**.

**Question 1.** Find the application for all finite simple groups can determined by their ”two orders”.

Let $G$ be a finite group and $B(G)$ be Burnside ring of $G$. We have the following application:

**Corollary 1.** Let $G$ be a finite simple group. Then $B(G)$ determines $G$ up to isomorphism. **Proof.** See [10, Theorem 5.3.].

**Question 2.** Proving this conjecture, can or not independent on the classification theorem of finite simple groups? For a small number of nonabelian simple group, for example, $A_5$, we may do it (see [11]).

**Question 3.** Weaken the condition of ”two orders”, characterize all finite simple groups.

**References**

[1] R. Solomon, On finite simple groups and their classification, Notices of the AMS, 42:2(1995), 231-239.

[2] W.J. Shi, A new characterization of the sporadic simple groups, Group Theory - Proc. Singapore Group Theory Conf. 1987, Walter de Gruyter Berlin-New York, 1989 531-540.

[3] W.J. Shi and J.X. Bi, A characteristic property for each finite projective special linear group (with J.X. Bi), Lecture Notes in Math., Springer-Verlag, 1456(1990), 171-180.
[4] W.J. Shi and J.X. Bi, A characterization of Suzuki-Ree groups, Sci. in China, Ser. A, 34 (1991), 14-19.

[5] W.J. Shi and J.X. Bi, A new characterization of the alternating groups, Southeast Asian Bull. Math., 16 (1992), 81-90.

[6] W.J. Shi, The pure quantitative characterization of finite simple groups (I), Prog. Nat. Sci., 4 (1994), 316-326.

[7] H.P. Cao and W.J. Shi, Pure quantitative characterization of finite projective special unitary groups, Sci. China, Ser. A, 45 (2002), 761-772.

[8] M.C. Xu and W.J. Shi, Pure quantitative characterization of finite simple groups $^2D_n(q)$ and $D_l(q)$ (l odd), Alg. Coll., 10 (2003), 427-443.

[9] A.V. Vasilev, M.A. Grechkoseeva, and V.D. Mazurov, Characterization of the finite simple groups by spectrum and order, Algebra and Logic, 48 (2009), 385-409.

[10] W. Kimmerle, F. Luca and A.G. Raggi-Cárdenas, Irreducible components and isomorphisms of the Burnside ring, J. Group Theory, 11(2008), 831-844.

[11] W.J. Shi, A characteristic property of $A_5$ (in Chinese), J. Southwest-China Normal University, 11:3(1986), 11-14.