General Solution of the Consistency Equation

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Abstract

We produce the general solution of the Wess-Zumino consistency condition for gauge theories of the Yang-mills type, for any ghost number and form degree.
We resolve the problem of the cohomological independence of these solutions. In other words we fully describe the local version of the cohomology of the BRS operator, modulo the differential on space–time. This in particular includes the presence of external fields and non–trivial topologies of space–time.

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1 Introduction

We investigate the Wess-Zumino consistency equation in non abelian gauge theories on an arbitrary space-time. We produce their general solution, taking due account of the possible non-trivial cohomology of space-time.

Motivated by the problem of quantum anomalies in renormalization theory and more recently by the discovery of their central rôle in topological quantum field theories there has been a number of publications on the subject, many of these being a step towards the solution. We obtain the full resolution by an elaboration of our results of and .

The proper way to look at the Wess-Zumino consistency equation is to view them as a problem of local cohomology. In the general problem, one has to accept the presence of external fields, and the existence of a non trivial cohomology for space–time.

The specificity of the cohomological problem we want to solve is the following: we deal with the algebra of form–valued polynomials in the fields and a finite number of their derivatives. Two differentials are defined on this algebra, the exterior derivative and the B.R.S. operator . These two differentials verify:

\[ \delta d + d\delta = \delta^2 = d^2 = 0 \] (1)

The consistency condition on means there exists such that

\[ \delta Q + dR = 0 \] (2)

and any solution of of the form

\[ Q = \delta A + dB \quad A, B \in \mathcal{A} \] (3)

is to be considered as trivial. Notice that in eq. the objects , , , are globally defined on as differential forms. This is the cohomology of modulo on , denoted .

The very existence of non-trivial solutions originates in the locality condition on . Indeed relaxing this condition may wipe out the cohomology.

We first introduce the necessary objects (certain algebras of polynomials in the fields, equipped with their differentials) and recall what are the so-called descent equations. We then calculate and the cohomology of on . This step is crucial since it allows to show why the Ansatz made in , namely that the solutions are built out of the differential forms and (gauge potential and curvature) rather than their individual components, does not restrict the generality of the solution, up to elements of . We conclude by constructing the independent elements of , i.e. giving the general solution of the Wess-Zumino consistency equation.

We shall assume that the principal fiber bundle where the gauge potential lives is trivial, i.e. identified with . We thus avoid the intricacies of handling a reference connection which are believed to be inessential, but we will return to this problem elsewhere.

Here is any –dimensional space–time and is a compact Lie group. The Lie algebra of is denoted with structure constants in some basis.
2 Preliminaries

We denote by $\mathcal{A}$ the algebra of form–valued functionals $\omega(A, \chi)$ of the fields $A$ (gauge–potential) and $\chi$ (ghost) such that for any $x \in M$, $\omega(A, \chi)(x)$ depends only on the fields and a finite number of their derivatives at $x$. They may depend on external fields such as a metric on $M$.

In a local chart $U$, $\mathcal{A}$ is generated by $\Omega(U)$ the differential forms on $U$, and the fields $A^i_{\mu}$, $\chi^i$ and their derivatives (including in particular the field–strength $F^{i}_{\mu\nu}$). We shall use the subalgebra $\mathcal{B}$ of $\mathcal{A}$ generated by the forms $A^i$, $F^i$, $\chi^i$, $d\chi^i$, $\Omega(M)$, and the subalgebra $\mathcal{H}$ of $\mathcal{B}$ generated by the $G$–invariant polynomials $P(\chi)$, $Q(F)$.

The action of the differentials $d$ and $\delta$ is easily defined on the generators of $\mathcal{A}$:

$$dA^i_{\mu} = A^i_{\mu,\nu} dx^\nu, \quad d\chi^i = \chi^j_{,\nu} dx^\nu$$  \hfill (4)

and so on, where $A^i_{\mu,\nu}$, $\chi^j_{,\nu}$, $dx^\nu$ are independent generators and $d$ is the exterior differential on $\Omega(M)$, while

$$\delta A^i_{\mu} = \chi^i_{,\mu} + f^i_{jk} A^j_{\mu} \chi^k, \quad \delta \chi^i = -\frac{1}{2} f^i_{jk} \chi^j \chi^k, \quad \delta \alpha = 0 \forall \alpha \in \Omega(M).$$  \hfill (5)

Then $d$, $\delta$ are extended as antiderivations of $\mathcal{A}$ in such a way that eq. (1) is verified.

In order to fix the notations we shall call $\mathcal{K}$ the subalgebra of $\mathcal{B}$ generated by $\mathcal{H}$ and $\Omega(M)$ and $\mathcal{K}_c$ (resp. $\mathcal{K}_b$) the subalgebra generated by $\mathcal{H}$ and the closed (resp. $d$–exact) forms on $M$. They will appear in the computation of $H(\mathcal{B}, \delta)$ and $H(H(\delta), d)$ in the following.

One of the main tools used from the early days of the study of the Wess–Zumino equation are the so called descent equations. If $Q$ is some representative of $H(\delta|d)$ then eq. (3) produces for us a local polynomial $R$ which turns out to be a representative for some element of $H(\delta|d)$. Indeed $R$ verifies $\delta(dR) = 0$ and thus there exists a local polynomial $S$ such that $\delta R + dS = 0$. This is a consequence of the triviality of $H(d)$ in form degree strictly smaller than $n = \dim M$ noticing that $Q$ is of form degree $\leq n$ whence $R$ is of degree $\leq n - 1$.

By definition $\partial$ is defined only in cohomology by: $\partial[Q] = [R]$. This definition makes sense since if $Q$ is trivial, i.e. $Q = \delta A + dB$ then $\delta Q = \delta dB$ and $R$ is of the form $\delta B + dC$ that is to say $[R] = 0$. Notice that $\partial$ decreases the form degree by one and consequently $\partial^{n+1} = 0$.

Choose a representative $Q$ for some element of $H(\delta|d)$ and let $k$ be the smallest integer such that $\partial^{k+1}[Q] = 0$. We may write the descent equations (or ladder):

$$\begin{align*}
\delta Q + dQ_1 &= 0 \\
\delta Q_1 + dQ_2 &= 0 \\
\vdots \\
\delta Q_{k-1} + dQ_k &= 0 \\
\delta Q_k &= 0
\end{align*}$$  \hfill (6)
3 Computation of $H(\delta)$

Clearly the problem is of a local nature since $\delta$ acts trivially on $\Omega(M)$. We may thus work in a coordinate patch $U$, where $A$ is the tensor product of $\Omega(U)$ with the algebra generated by $A^i_{\mu}, \chi^i$ and their derivatives. The $\delta$ cohomology becomes obvious if one takes as a system of generators:

\[ A^i_{(\mu, \nu_1, \ldots, \nu_p)}, \quad \delta A^i_{(\mu, \nu_1, \ldots, \nu_p)}, \quad \chi^i, \quad (D_{\nu_1} \ldots D_{\nu_p} F_{\mu})^i, \quad p = 0, 1, \ldots \quad (7) \]

where $(\cdot)$ means symmetrization over indices, and $D_{\rho}$ means covariant derivation. This proper choice makes explicit the splitting of the algebra generated by the fields into a tensor product of differential algebras. The $\delta$–cohomology of the factor generated by $(A, \delta A)$ is trivial. On the factor generated by $(\chi, D p F)$, $H(\delta)$ is the cohomology of $G$ acting on the module of polynomials in the components of $F$ and their covariant derivatives.

It is known [24, 25] to be isomorphic to the tensor product of the invariant forms on $G$ by the $G$–invariant part of the above module. Finally in the chart $U$, and denoting $\{\cdot \cdot \cdot\}^G$ the $G$–invariant part:

\[ H(\delta) \simeq \{\text{polynomials in } \chi\}^G \otimes \{\text{polynomials in } F_{\mu\nu}, D_{\rho} F_{\mu\nu}, \ldots\}^G \otimes \Omega(U) \quad (8) \]

This result was partly stated in [13], but this proof inspired by the one of [12] appears to be much simpler.

The algebra appearing in eq. (8) has an intrinsic meaning, i.e. is invariant under changes of coordinates. Moreover given a global $\delta$–cocycle on $M$, its representatives in this algebra over two coordinates patches $U_\alpha$ and $U_\beta$ match on $U_\alpha \cap U_\beta$. Indeed their difference being a $\delta$–coboundary must vanish. So the global result (on $M$) is immediately obtained by restricting oneself to objects of the above type globally defined on $M$. For instance the lagrangian $\text{Tr}(F \wedge \ast F)$ which explicitely contains the metric on $M$ as an external field belongs to $H(\delta)$.

Proposition. $H(\delta)$ is the skew–tensor product of the algebra of invariant polynomials in $\chi$ by the algebra of globally defined forms on $M$ constructed with invariant polynomials in $F_{\mu\nu}$ and its covariant derivatives.

Similarly the algebra $B$ is generated by the forms $A^i, \delta A^i, \chi^i, F^i, \Omega(M)$ so that $H(B, \delta) \simeq K$ with $K$ defined in the previous section. We see that $H(B, \delta)$ is naturally included in $H(A, \delta)$.

4 Computation of $H(H(\delta), d)$

We shall prove here a beautiful result: in form degree smaller than $n$ only the differential forms $A, F$ and not their individual components nor their derivatives survive the calculation of the cohomology of $d$ on $H(\delta)$. As will be shown later, this calculation validates the older analysis of [13].

The computation proceeds in steps:

- the abelian case ($G = U(1)$) in ghost degree zero
− abelian case with many $U(1)$ factors and a possible global symmetry, ghost degree zero.
− the non abelian case in ghost degree zero.
− non abelian case, unrestricted ghost degree.

**Proposition.** $H(H(\delta), d)$ is generated in form degree smaller than $n$ by $\mathcal{H}$ and $H_{DR}(M)$, where $H_{DR}(M)$ is the de Rham cohomology of space–time.

**Proof.**

**Step 0.** In ghost degree zero, and for just one abelian potential $A_\mu$ and its field–strength $F_{\mu\nu}$ we show that in form degree smaller than $n$, the cohomology of $d$ on the space of polynomials in $F_{\mu\nu}$ and its derivatives (with coefficients forms on $M$) is generated by polynomials in $F = F_{\mu\nu} dx^\mu dx^\nu$ and the de Rham cohomology of $M$.

In other words suppose $Q$ is a polynomial in the components of $F$ and their derivatives, and verifies $dQ = 0$. Then

$$Q = dR(F_{\mu\nu}, \partial_\rho F_{\mu\nu}, \ldots) + U$$

where $U$ is a polynomial in the form $F$, i.e. $U = \sum_k (F)^k \omega_k$, with the $\omega_k$ representatives of the de Rham cohomology of $M$. The proof uses the algebra $\mathcal{A}$ and the descent equation, whose existence comes from the triviality of $d$ on $\mathcal{A}$. The last non trivial term $Q_k$ in the ladder is a representative of $H(\delta)$, i.e a polynomial in $\chi, F_{\mu\nu}$ and its derivatives (no derivatives of $\chi$). Since $\chi^2 = 0$, $k$ is at most one.

The ladder takes the form:

$$Q = dQ_0(A_\mu, A_{\mu,\nu}, \ldots)$$

$$0 = \delta Q_0 + dQ_1$$

$$0 = \delta Q_1$$

$Q_1$ is necessarily of the form $Q_1 = \chi P(F_{\mu\nu}, \partial_\rho F_{\mu\nu}, \ldots)$, from which $\delta Q_0 + d\chi P - \chi dP = 0$. Thus $\delta(Q_0 + AP) = \chi dP$, from which $dP = 0$. As a consequence

$$Q_0 + AP = R(F_{\mu\nu}, \partial_\rho F_{\mu\nu}, \ldots) + \delta S$$

Since $Q_0$ and $P$ are of ghost degree 0, $\delta S = 0$, and $Q = dQ_0 = dR - FP$. One gets the desired result by an induction on the degree of $P$ in $F$. Notice that one gets a polynomial $U$ of the stated form.

During this proof one does not create non–locality in the fields: if for example $dQ = 0$ and $Q$ depends in a local way on some other auxiliary fields, $Q$ is $d$ equivalent to some polynomial $Q'$ in $F$ up to quantities of the form $dR$ with both $Q'$ and $R$ local in these auxiliary fields.

**Step 1.** Let us consider several abelian fields $A_i$.

a) **Cocycle condition.** Let $Q(F_{\mu\nu}, F_{\mu\nu,\rho}, \ldots)$ be a polynomial with form coefficients such that $dQ = 0$. To show that $Q = dR + U(F^i)$ where $R$ is a similar polynomial, while $U$ involves only the forms $F^i$, we first write $d = d_1 + d'$ with $d_1$ acting only on $F^1$ and $d'$ acting on all the other fields. Suppose $Q$ is of maximal order $k$ in $F^1$ (total
number of derivatives of $F^1$). Let $Q = Q^k + Q'$ where $Q^k$ contains the terms of order $k$. Since $d_1Q^k = 0$ we may apply the previous result and get $Q^k = d_1R_1 + U_1$ where $U_1$ depends on $F^1_{\mu\nu}$ only through the form $F^1$, $R_1$ may be chosen of order at most $(k - 1)$ and both are local in $F^2_{\mu\nu}, F^2_{\mu\rho}, \ldots$. In $Q - dR_1$ the field $F_1$ is of order at most $(k - 1)$ in $F^1$. By induction we get $Q = dR + U(F^1, F^2_{\mu\nu}, \ldots)$. The polynomial $U$ may be expanded in exterior powers of $F^1$: $U = \sum (F^1)^k \alpha_k$. Each of the terms $\alpha_k$ is local in $F^2_{\mu\nu}, \ldots$ and separately verifies $d\alpha_k = 0$ since $dU = \sum(F^1)^k d\alpha_k = 0$ identically in $F^1$, from which we can proceed similarly for the other fields. Finally:

$$Q = \sum_i Q_i(F)\alpha_i \quad \text{with } \alpha_i \in \Omega(M), \quad d\alpha_i = 0.$$

b) Coboundary condition. We want to solve $U = dV$ with $V$ a polynomial in $F^i_{\mu\nu}$ and their derivatives with coefficients in $\Omega(M)$ and $U$ some polynomial in the forms $F^i$. We may split $d$ in $d = d_F + d'$ where $d_F$ acts only on the fields $F^i_{\mu\nu}$ and $d'$ acts on the other fields. Let $V^m$ the part of $V$ of maximal order $(m)$ in the derivatives of $F$ so that $d_F V^m = 0$. The previous analysis allows to write: $V^m = d_F W + Q(F)$ with $Q$ a polynomial in the forms $F^i$. Then $V$ may be replaced by $V - dW$ which is of order $(m - 1)$. One reaches $U = dV$ with $V$ of order zero. At this point $d_F V = 0$ hence $V = \sum_i Q_i(F)\beta_i$ where the $\beta_i$ are arbitrary forms on $M$, from which:

$$U = dV = \sum_i Q_i(F) d\beta_i.$$

This is the place where the de Rham cohomology of $M$ appears. Notice that these arguments could be turned into a spectral sequence argument similar to the one used in [21] in the calculation of $H(d)$.

c) Global invariance. Assume now that $G$ acts as a global transformation group of the fields $F^i$, and suppose $P$ is an invariant polynomial in the $F^i$ such that $dP = 0$. Then $P = dR + U$. Taking the mean over the group yields $P = d\overline{R} + \overline{U}$ with $\overline{R}$ and $\overline{U}$ invariant by $G$. We may do the same for the coboundary condition.

**Step 2.** We may now go to the non-abelian case with local invariance. Suppose $P$ is a $G$-invariant polynomial in the $F^i_{\mu\nu}$ and their covariant derivatives, and that its degree as a form is strictly smaller than $n$. The cocycle condition $dP = 0$ implies the existence of $G$–invariant $R_1$ and $U_1$ such that

$$P = dR_1(F^i_{\mu\nu}, \partial_\rho F^i_{\mu\nu}, \ldots) + U_1(F^i)$$

Replacing ordinary derivatives by covariant ones

$$P_1 = P - dR_1(F^i_{\mu\nu}, D_\rho F^i_{\mu\nu}, \ldots)$$

is again a $d$–closed invariant polynomial in the $F^i_{\mu\nu}$ and their covariant derivatives, but of lower order of derivatives of $F$. We will arrive at the conclusion that

$$P = dR(F^i_{\mu\nu}, D_\rho F^i_{\mu\nu}, \ldots) + U(F)$$

with $R$ and $U$ being $G$–invariant. The coboundary condition may be analyzed along the same lines.
To sum up this discussion, we see that cocycles in form degree strictly smaller than \( n \) are cohomologically equivalent to some \( U(F) = \sum_i P_i(F) \alpha_i \). Here \( P_i(F) \) are independent invariant polynomials of the forms \( F^i \) with numerical coefficients and \( \alpha_i \) closed differential forms on \( M \). Such a cocycle is trivial, i.e. \( U = dV + \delta W \) and 

\( V \in H(\delta) \) if and only if: \( \alpha_i = d\beta_i \) (of course \( \delta W \) is irrelevant here since we are living in ghost degree zero).

**Step 3.** To treat the situation with non zero ghost degree, one first notices that the action of \( d \) on invariant polynomials in \( \chi \) vanishes in \( H(\delta) \). Indeed for such a \( P(\chi) \) there exists some \( Q \in B \) such that \( \delta Q + dP = 0 \). Consequently \( H(\delta) \) is a tensor product of a purely ghost part (invariant polynomials in \( \chi \)) on which \( d \) vanishes, by the part which we have just analyzed. By Künneth theorem, the cohomology of \( d \) on \( H(\delta) \) is thus obtained from the one we have just calculated by taking its tensor product with the algebra of invariant polynomials in \( \chi \) (i.e. invariant forms on \( \mathcal{G}^* \)).

In dimension strictly smaller than \( n \) the cycles (resp. boundaries) of the action of \( d \) on \( H(\delta) \) may be identified with elements of \( K_c \) (resp. \( K_b \)) keeping in mind that \( d \) is the induced \( d \) on \( H(\delta) \).

It is straightforward to see that the computation of \( H(H(B, \delta), d) \) leads to the same answer, i.e. the cycles are identified with \( K_c \) and the boundaries to \( K_b \) in dimension smaller than \( n \).

## 5 Computation of \( H(\delta|d) \)

### 5.1 Cocycle condition.

We want to solve eq. (4), by analyzing in some detail the descent homomorphism \( \partial \) of the ladder (7). What we want to show is that we may choose representatives \( Q_k, Q_{k-1}, \ldots, Q_1 \) in the algebra \( B \) rather than \( A \). The idea is to go up from the bottom of the ladder and fix the choice of these representatives. This will permit us to make contact with our results of [13, 14] and produce the desired cohomology.

If \( k = 0 \), \( Q \) is in \( H(\delta) \) and the general solution of the condition \( \delta Q = 0 \) is known (see section 4).

If \( k \geq 1 \) \( Q_k \) verifies:

\[
\delta Q_{k-1} + dQ_k = 0 \quad (9)
\]
\[
\delta Q_k = 0 \quad (10)
\]

The previous equations indicate that \( Q_k \) is a \( d \)-cocycle in \( H(\delta) \). Since by hypothesis on \( k \), \( Q_k \) is non trivial in \( H(\delta|d) \) it cannot be trivial in \( H(H(\delta), d) \). It is equivalent in \( H(H(\delta), d) \) to some element \( B_k \) of \( K_c \). We may choose \( Q_k = B_k \).

We have \( \delta(\delta Q_{k-1} + dB_k) = \delta(dB_k) = 0 \) hence there exists \( B_{k-1} \in B \) and \( X \) a representative of \( H(B, \delta) \) in \( K \) such that

\[
\delta B_{k-1} + dB_k = X \quad (11)
\]

From equations (7,11) we see that

\[
\delta(B_{k-1} - Q_{k-1}) = X \quad (12)
\]
meaning that $X$ is trivial in $H(A, \delta)$. Since we have a natural \textit{inclusion} of $H(B, \delta)$ in $H(A, \delta)$ this forces $X$ to vanish. In conclusion there exists $B_{k-1}$ in $B$ such that:

$$\delta B_{k-1} + dB_k = 0. \quad (13)$$

If $k = 1$, then $\delta(Q - B_0) = 0$ and $Q$ is the sum of an arbitrary $\delta$–cocycle and of some element $B_0$ in $B$ satisfying the consistency condition. These elements have been completely described in [13].

If $k \geq 2$ we are going to show that $Q_{k-1}$ may be taken in $B$. Indeed there exists $Q_{k-2}$ such that

$$\delta Q_{k-2} + dQ_{k-1} = 0 \quad (14)$$

Since from eq.(13), $\delta dB_{k-1} = 0$, there exists $B_{k-2} \in B$ and $Y$ a representative of $H(B, \delta)$ in $K$ such that

$$\delta B_{k-2} + dB_{k-1} = Y. \quad (15)$$

Equations (12,14,15) yield the conditions:

$$\delta(B_{k-2} - Q_{k-2}) + dB_{k-1} - Q_{k-1} = Y \quad (16)$$

$$\delta(B_{k-1} - Q_{k-1}) = 0 \quad (17)$$

The previous equations indicate that $Y$ is trivial in $H(H(\delta), d)$ and is thus of the form $Y = \sum_l \delta \beta_l \wedge P_l$ with $\beta_l \in \Omega(M), \ P_l \in \mathcal{H}$. It is possible to construct $R_l \in B$ such that $dP_l = \delta R_l$, from what we know of $H(H(B, \delta), d)$.

The replacement of $B_{k-1}$ by $B_{k-1} - \sum_l \beta_l \wedge P_l$ and of $B_{k-2}$ by $B_{k-2} - \sum_l \beta_l \wedge R_l$ leaves eq. (13,17) unchanged while eq. (14) becomes

$$\delta(B_{k-2} - Q_{k-2}) + dB_{k-1} - Q_{k-1} = 0. \quad (18)$$

From equations (18,17) we see that $(B_{k-1} - Q_{k-1})$ is of the form $U + \delta R + dS$ with $U$ in $K_c$ and $\delta S = 0$. If we redefine again $B_{k-1}$ as $B_{k-1} - U$ we see that the class $[Q_{k-1}]$ in $H(\delta|d)$ has a representative in $B$ (namely $B_{k-1}$).

\textbf{Remark.} We see here how going up the descent equation is not uniquely defined: there is an ambiguity due to the non triviality of $H(\delta)$. This ambiguity appears in the freedom of choice of $B_{k-1}$. The successive redefinitions of $B_{k-1}$ may change its class in $H(\delta|d)$.

Now we can set $Q_{k-1} = B_{k-1}$ and change $Q_{k-2}$ into $Q_{k-2} + dR$ and no further modification of the ladder. We are brought back to the same situation with $k$ replaced by $(k-1)$. Without further ado we see that $Q$ is equivalent to the sum of an arbitrary $\delta$–cocycle and one of the known solutions in $B$ of the consistency condition.

Notice that, in the course of the proof, we have proved (and used) the fact that if some $Y \in K$ verifies $Y = \delta A + dH$ with $A \in A$ and $H \in H(\delta)$, then there exist $\alpha$ in $B$ and $\beta$ in $K$, such that $Y = \delta \alpha + d\beta$.

\section{5.2 Coboundary condition.}

Since the solutions of the consistency condition are defined up to a $\delta$–cocycle we start from a solution belonging to $B$. In reference [13, 14] we have produced the list of all cohomologically \textit{independent} solutions of the cocycle condition in $B$.
We shall now see that a non trivial solution in \( \mathcal{B} \) remains non trivial in \( \mathcal{A} \). Let us take \( Q \in \mathcal{B} \) and the smallest integer \( k \) such that \( \partial^k + 1 \) \( Q = 0 \). By hypothesis \( k \geq 0 \) and \( P = \partial^k Q \in \mathcal{K} \) is non trivial for the cohomology of \( \delta \) modulo \( d \) computed in \( \mathcal{B} \). We shall prove that \( P \) remains non trivial in \( \mathcal{A} \) hence \([Q]\) is not zero in \( \mathcal{A} \) since \([P] = \partial^k [Q]\) in \( \mathcal{A} \).

**Proposition.** If \( P \in \mathcal{K} \) is of the form \( P = \delta A + dB \), with \( A, B \in \mathcal{A} \), then there exist \( \alpha \) and \( \beta \) in \( \mathcal{B} \) such that \( P = \delta \alpha + d\beta \).

**Proof.** Since \( \delta P = 0 \), we know that \( d(\delta B) = 0 \) and thus there exists \( C \in \mathcal{A} \) such that \( \delta B + dC = 0 \), i.e. \( B \) verifies the consistency condition we have just solved. We know \( B \) is of the form: \( B = G + \delta F + H \) where \( G \in \mathcal{B} \) is a solution of the consistency condition, \( F \in \mathcal{A} \) and \( H \) is a representative of \( H(\delta) \). From this

\[
P - dG = \delta(A - dF) + dH
\]

showing that \( \delta(P - dG) = 0 \). As a consequence, \( (P - dG) \) is a \( \delta \)–cocycle of \( \mathcal{B} \), and may thus be written \( P - dG = X + \delta Y \), with \( X \in \mathcal{K} \), and \( Y \in \mathcal{B} \).

Setting \( A' = A - dF - Y \), we get \( \delta A' + dH = X \). We know from the last remark of the previous section that this implies the existence of \( U \) and \( V \) in \( \mathcal{B} \) such that \( X = \delta U + dV \), concluding the proof if we set \( \alpha = Y + U \) and \( \beta = G + V \).

### 6 Conclusion

We have produced the calculation of various related cohomologies. The \( \delta \)–cohomology has been easily calculated, and we have shown the rôle played by the cohomology of \( d \) on \( H(\delta) \). We have in particular shown where the de Rham cohomology of space–time enters the calculations exactly as it is the case in the evaluation of the de Rham cohomology of the orbit space of connections [29, 27].

By bringing back the problem to the one solved in [13], we have shown that up to the addition of a non \( d \)–trivial \( \delta \)–cocycle in \( \mathcal{A} \), the general solutions are obtained applying a “generalized transgression” to products of elements of \( \mathcal{H} \) with representatives of the de Rham cohomology of space–time.

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