TOPOLOGY OPTIMIZATION FOR INCREMENTAL ELASTOPLASTICITY

STEFANO ALMI AND ULISSE STEFANELLI

ABSTRACT. We discuss a topology optimization problem for an elastoplastic medium. The distribution of material in a region is optimized with respect to a given target functional taking into account compliance. The incremental elastoplastic problem serves as state constraint. We prove that the topology optimization problem admits a solution. First-order optimality conditions are obtained by considering a regularized problem and passing to the limit.

1. INTRODUCTION

Topology Optimization is concerned with determining the optimal shape of a mechanical piece against a number of criteria. Besides mechanical performance, these criteria may include weight, manufacturing costs, topological, and geometrical features. The distribution of material within an a-priori given region $\Omega \subset \mathbb{R}^n$ is the control parameter of the process. Given the portion $E \subset \Omega$ to be filled with material, one determines the mechanical response of the body and minimizes a target functional depending on $E$ and such response. This very general setting applies to a number of different shape design problems, from mechanical engineering, to aerospace and automotive, to architectural engineering, to biomechanics [7].

The applicative interest in topology optimization has triggered an intense research activity, which in turn generated a wealth of results at the engineering and computational level. Starting from the pioneering paper [6], see also [2], the literature has developed to cover a wide range of different mechanical settings [49], including strain-gradient theories [27], finite strains [17], thermoelectricity [48], material interfaces [43], surface effects [35], graded materials [15], stochastic effects [14], and fluid-structure interactions [30]. The reader is referred to the monographs [18, 50] for additional material. Correspondingly, extensive numerical experimental campaigns have been developed. Among the different computational approaches in use one can mention finite elements [12, 37], NURBS [21], smoothed-particle hydrodynamics [28], shape-derivatives [40], and level-set methods [3].

On the more theoretical side, the linear elastic setting has been considered in a number of contributions. In [10] the existence of optimal shapes is tackled by introducing a penalization of interfaces between solid $E$ and void $\Omega \setminus E$ and considering an additional phase-field regularization. The actual position of the body is hence modeled as a level set of a scalar order parameter $z \in [0, 1]$ and gradients of this parameter are penalized. By removing such penalization, a solution of the original sharp-interface limit is recovered by means of a $\Gamma$-convergence argument [16]. Existence under additional stress constraints is discussed in [13] and the extension to hyperelasticity is presented in [30]. More recently, first-order optimality conditions have been obtained in the linear elastic setting [3, 9, 15] for both the sharp-interface model and its phase-field approximation, also for graded materials.

Date: April 15, 2020.
2010 Mathematics Subject Classification. 74C05, 74P10, 49Q10, 49J20, 49K20.
Key words and phrases. Topology optimization, elastoplasticity, first-order conditions.
Inelastic effects such as large or cyclical stresses, permanent deformations, damage, and fracture appear ubiquitously in applications. Topology optimization in the inelastic setting is for instance paramount to the bending, punching, and machining of steel sheets, which are tasks of the utmost applicative relevance. Correspondingly, topology optimization in inelastic settings has already attracted strong attention in the engineering community, see the pioneering [42] and [3, 26, 29, 31, 34, 47] among others. On the contrary, the mathematical theory seems to be less developed, with no rigorous existence and optimality result available to date.

We intend to fill this gap in this paper, by investigating a topology optimization problem in the setting of incremental, linearized elastoplasticity with hardening. Referring to Section 2 for all necessary assumptions and details, let us anticipate here that we aim at investigating the sharp-interface problem

$$\min_z \{ J(z, u) : (u, p) \in \text{argmin} \mathcal{E}(\cdot, z) \}$$

where the compliance-type target functional is defined as

$$J(z, u) := \int_\Omega \ell(z) f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, d\mathcal{H}^{n-1} + \frac{1}{6} \text{Per}(\{z = 1\}; \Omega),$$

and the incremental elastoplastic state functional reads

$$\mathcal{E}(z, u, p) := \frac{1}{2} \int_\Omega C(z) \varepsilon(u) \cdot \varepsilon(u) \, dx + \frac{1}{2} \int_\Omega \mathbb{H}(z)p \cdot p \, dx + \int_\Omega d(z)|p| \, dx$$

$$- \int_\Omega \ell(z) f \cdot u \, dx - \int_{\Gamma_N} g \cdot u \, d\mathcal{H}^{n-1}.$$  

The order parameter $z: \Omega \to \{0, 1\}$ determines the actual position of the solid, to be identified with the set $\{z = 1\}$. The smooth nonnegative function $\ell(\cdot)$ represents the density of the body, in dependence of $z$. In particular, $\ell(0) = 0$. We assume the whole container $\Omega$ to deform under the action of the body force $\ell(z)f$ and the boundary traction $g$. The elasticity tensor $C$, the linear hardening tensor $\mathbb{H}$, and the yield stress $d$ hence depend on $z$. The fields $u: \Omega \to \mathbb{R}^n$ and $p: \Omega \to \mathbb{R}^{n \times n}$ are the displacement and the plastic strain, respectively, and $\varepsilon(u)$ denotes the symmetrized gradient.

Given the solid $\{z = 1\}$, the minimization of the incremental elastoplastic state functional $\mathcal{E}$ gives a unique elastoplastic state $(u, p)$. Note that $\mathcal{E}$ is nothing but the complementary elastic energy of the body, augmented by the linear dissipation term $d(z)|p|$, modeling the plastic work from a prior nonplasticized state. Then, the displacement component $u$ enters into the definition of $J$, which ultimately depends just on $z$. Note that the functional $J$ features the total-variation norm of $z$, which indeed corresponds to the perimeter of the solid $\{z = 1\}$ in $\Omega$. In particular, the boundedness of $J$ implies that $z \in BV(\Omega; \{0, 1\})$, the space of functions of bounded variation [4].

In addition to the sharp-interface problem (1.1), we study its phase-field approximation

$$\min_z \{ J_\delta(z, u) : (u, p) \in \text{argmin} \mathcal{E}(\cdot, z) \}$$

where, for $\delta > 0$, the phase-field target functional $J_\delta$ is defined as

$$J_\delta(z, u) := \int_\Omega \ell(z) f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, d\mathcal{H}^{n-1} + \int_\Omega \frac{\delta}{2} |\nabla z|^2 + \frac{z^2(1-z)^2}{2\delta} \, dx,$$

with order parameter $z$ now assumed to belong to $H^1(\Omega; [0, 1])$ and referred to as phase-field.

We prove that problems (1.1) and (1.2) admit solutions, see Propositions 2.1 and 2.3 respectively. In addition, we prove that optimal phase fields $z_\delta$ solving problem (1.2) converge, up to
subsequences, to a solution to the sharp-interface problem (1.1), see Corollary 2.7. Indeed, this relies on a general \( \Gamma \)-convergence result as \( \delta \to 0 \), see Theorem 2.6.

The phase-field problem (1.2) is then further regularized by replacing the dissipation term \( d(z)|p| \) by \( d(z) h_{\gamma}(p) \) in \( \mathcal{E} \), where \( h_{\gamma} \) is a smooth approximation of the norm, namely \( h_{\gamma}(p) \to |p| \) as \( \gamma \to +\infty \). The ensuing regularized phase-field problem admits solutions \( z_{\gamma} \). For fixed \( \delta > 0 \), as \( \gamma \to +\infty \) the sequence \( z_{\gamma} \) converges, up to subsequences, to a solution of the phase-field problem (1.2), see Proposition 2.8.

In addition, owing to the smoothness of the corresponding control-to-state map for all \( \gamma \), one can derive first-order optimality conditions for the regularized phase-field problem, see Theorem 3.1 and Corollary 3.4. Eventually, by letting \( \gamma \to +\infty \) we derive first-order optimality conditions for the phase-field problem (1.2), see Theorem 4.1.

Before closing this introduction, let us mention that the literature on existence and optimality conditions for elastoplastic problem is rather scant and, to our knowledge, always focusing on force and traction control. The incremental elastoplastic problem is discussed in \cite{17, 25, 23} whereas the quasistatic setting is addressed in the series \cite{11, 44, 45, 46}, see also \cite{41}. Some related results on elastoplasticity are in \cite{11}, whereas \cite{1} and \cite{20, 19} deal with quasistatic adhesive contact and shape memory alloys, respectively. The reader is referred to \cite{38, 39} for some abstract optimal control theory for rate-independent systems.

As concerns our quest for first-order optimality conditions, our analysis follows some argument from \cite{17}. Here, the optimal control problem in incremental elastoplasticity is discussed, with elastoplastic state \((u, p)\) controlled via \( f \) and traction \( g \) and the coefficients \( C, H, \) and \( d \) are given constants. In particular, the incremental elastoplastic state functional \( \mathcal{E} \) in \cite{17} is linear with respect to the controls. Here instead the order parameter \( z \) appears nonlinearly in \( \mathcal{E} \). This generates some additional difficulty which we overcome in a series of technical lemmas in Section 3 below.

This is the plan of the paper: we introduce our setting and state and prove existence of both sharp-interface problems and phase-field approximations in Section 2, where we also discuss \( \Gamma \)-convergence. The \( \gamma \)-regularized problem is then discussed in Section 3 where the corresponding first-order optimality conditions are derived. Eventually, in Section 4 we pass to the limit as \( \gamma \to +\infty \) and obtain first-order optimality conditions for the phase-field approximation.

2. Setting of the problem

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) with Lipschitz boundary \( \partial \Omega \) containing the elastoplastic body under kinematic hardening. The body is identified by means of the order parameter \( z \in [0, 1] \) at each point. The region \( \{ z > 0 \} \) hence represents the body, whereas \( \{ z = 0 \} \) on the portion of \( \Omega \) which is not occupied by the body. As already mentioned, we assume also the portion \( \{ z = 0 \} \) to deform elastoplastically, being extremely soft however. This would correspond to the situation where the elastoplastic body is immersed in a soft polymeric matrix filling the remainder of the domain \( \Omega \).

The elastic and plastic properties at a point \( x \in \Omega \) hence depend on the value \( z \) at that point. We describe such properties via prescribing the elastic and hardening tensors \( C(z): \mathbb{M}_S^n \to \mathbb{M}_S^n \) and \( H(z): \mathbb{M}_P^d \to \mathbb{M}_P^d \), where \( \mathbb{M}_S^n \) and \( \mathbb{M}_P^d \) denote the spaces of symmetric and deviatoric and symmetric square matrices of order \( n \), respectively. In the sequel, we will also use the symbol \( \mathbb{M}^n \) for the space of square matrices of order \( n \). As usual, we assume \( C \) and \( H \) to be positive definite, uniformly w.r.t. \( z \in [0, 1] \), namely, there exists \( 0 \leq \alpha_C \leq \beta_C < +\infty \) and \( 0 \leq \alpha_H \leq \beta_H < +\infty \) such that

\[
(2.1) \quad \alpha_C |E|^2 \leq C(z)E \cdot E \leq \beta_C |E|^2 \quad \text{for every } z \in [0, 1] \text{ and every } E \in \mathbb{M}_S^n,
\]
\( \alpha_H |Q|^2 \leq H(z) Q \cdot Q \leq \beta_H |Q|^2 \) for every \( z \in [0, 1] \) and every \( Q \in M_D^n \),

where the dot indicates the scalar product between matrices. For technical reasons that will be clear in the proofs of Theorems 3.1 and 4.1, we assume that \( C \) and \( H \) are elements of \( C^1([0, 1]; L(M^n; M^n)) \), where \( L(M^n; M^n) \) denotes the space of linear and continuous functions from \( M^n \) to \( M^n \). We further suppose that \( C(\cdot) \) and \( H(\cdot) \) can be extended constantly outside the interval \([0, 1]\), that is, \( C(z) = C(0) \) (resp. \( H(z) = H(0) \)) for \( z < 0 \) and \( C(z) = C(1) \) (resp. \( H(z) = H(1) \)) for \( z > 1 \), keeping the \( C^1 \)-regularity on \( R \). This last requirement allows us to drop the usual constraint \( z \in [0, 1] \) on the order parameter, as it can be renormalized a posteriori without changing the solution of the problem (see (2.12)–(2.13) for further details).

Finally, we assume that the elasticity tensor \( C(\cdot) \) can be split into a volumetric and a deviatoric part, i.e., there exists \( C_D : \mathbb{R} \rightarrow L(M_D^n; M_D^n) \) and \( \kappa : \mathbb{R} \rightarrow \mathbb{R} \) joining the same properties of \( C(\cdot) \) and such that

\[
C(z) = C_D(z)E_D + \kappa(z)\text{tr}(E)I \quad \text{for every } z \in \mathbb{R} \text{ and every } E \in M_S^n ,
\]

where \( \text{tr}(E) \) is the trace of \( E \), \( I \) stands for the identity matrix, and \( E_D := E - \frac{\text{tr}(E)}{n}I \in M_D^n \) denotes the deviatoric part of \( E \). In particular, (2.3) means that \( C_D(z) \) maps \( M_D^n \) in \( M_D^n \).

A typical example of \( C(\cdot) \) and \( H(\cdot) \) is the convex combination

\[
C(z) = \theta(z)C_{\text{mat}} + (1 - \theta(z))C_{\text{void}} \quad \text{and} \quad H(z) = \theta(z)H_{\text{mat}} + (1 - \theta(z))H_{\text{void}}
\]

with \( \theta \in C^1([0, 1]; [0, 1]) \) such that \( \theta(0) = 0 \), \( \theta(1) = 1 \), \( \theta'(0) = \theta'(1) = 0 \), and \( C_{\text{mat}}, C_{\text{void}} \) (resp. \( H_{\text{mat}}, H_{\text{void}} \)) satisfying (2.1) (resp. (2.2)). In this case, \( C_{\text{void}} \) is the elasticity tensor of the material under consideration, while \( C_{\text{void}} \) is interpreted as a residual elasticity tensor when no material is present \( (z = 0) \). As it is natural to expect, the tensor \( C(z) \) degrades when the parameter \( z \) decreases. In order to maintain the coerciveness of the problem, the residual tensor \( C_{\text{void}} \) has to be considered. The very same interpretation holds for \( H_{\text{mat}} \) and \( H_{\text{void}} \).

Assume to be given \( w \in H^1(\Omega; \mathbb{R}^n) \), whose trace on the subset \( \Gamma_D \subset \partial \Omega \) represents a prescribed boundary displacement. Here, \( \Gamma_D \) is assumed to be open in the relative topology of \( \partial \Omega \) and \( H^{n-1}(\Gamma_D) > 0 \), where the latter is the \((n-1)\)-Hausdorff surface measure in \( \mathbb{R}^n \). Let us define the set of admissible states \( A(w) \) as

\[
A(w) := \{(u, \varepsilon, p) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_S^n) \times L^2(\Omega; M_D^n) : Eu = \varepsilon + p, u = w \text{ on } \Gamma_D \},
\]

where \( Eu \) is the symmetric part of \( \nabla u \), and \( \varepsilon \) and \( p \) are usually referred to as the elastic and the plastic strain, respectively.

As for the plastic dissipation, for simplicity of exposition we consider here a von Mises model. Hence, given an order parameter \( z \in L^\infty(\Omega; [0, 1]) \) and \( p \in L^2(\Omega; M_D^n) \), the plastic dissipation associated to \( p \) reads

\[
\int_\Omega d(z)|p|\, dx ,
\]

where \( d \in C^1([0, 1]; [0, +\infty)) \) is such that

\[
d(z) \geq \lambda > 0 \quad \text{for every } z \in [0, 1] .
\]

As we did for the tensors \( C \) and \( H \), we assume that \( d \) can be extended to a \( C^1 \)-function on the whole \( \mathbb{R} \) in such a way that \( d(z) = d(0) \) for \( z < 0 \) and \( d(z) = d(1) \) for \( z > 1 \). We notice that in (2.5) the presence of material, expressed through the order parameter \( z \), directly affects the plastic dissipation. In a typical situation, \( d \) is also assumed to be increasing in the interval \([0, 1]\), which implies that the plastic yield stress is lower for \( z \sim 0 \) and higher for \( z \sim 1 \). Assumption (2.6), in agreement with (2.1)–(2.2), says plastic dissipation occurs also in \( \{z = 0\} \), although to possibly very small extent.
Besides the prescribed boundary displacement at $\Gamma_D$, the system is subjected to an applied body force $f \in L^2(\Omega; \mathbb{R}^n)$ and an applied boundary traction $g \in L^2(\Gamma_N; \mathbb{R}^n)$. Here, $\Gamma_N \subset \partial \Omega$ is open in the topology of $\partial \Omega$ and $\Gamma_N \cap \Gamma_D = \emptyset$, where $\Gamma_N$ and $\Gamma_D$ are closures in $\partial \Omega$. Furthermore, we assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger [22, Definition 2], that is, for every $x \in \partial \Omega$ there exists an open neighborhood $U_x \subseteq \mathbb{R}^n$ of $x$ and a bi-Lipschitz map $\Psi_x: U_x \rightarrow \Psi(U_x)$ such that $\Psi_x$ coincides with one of the following sets:

$$V_1 := \{y \in \mathbb{R}^n : |y| \leq 1, y_n < 0\},$$
$$V_2 := \{y \in \mathbb{R}^n : |y| \leq 1, y_n \leq 0\},$$
$$V_3 := \{y \in V_2; y_n < 0 \text{ or } y_1 > 0\},$$

where $y_i$ is the $i$-th component of $y \in \mathbb{R}^n$. This last assumption turns to be useful in the proof of Theorem 3.1.

Given a state $(u, \varepsilon, p) \in \mathcal{A}(w)$ and an order parameter $z \in L^\infty(\Omega)$, we define the energy functional

$$E(z, u, \varepsilon, p) := \frac{1}{2} \int_{\Omega} C(z) \varepsilon : \varepsilon \, dx + \frac{1}{2} \int_{\Omega} H(z) p : p \, dx + \int_{\Omega} d(z) |p| \, dx - \int_{\Omega} \ell(z) f : u \, dx - \int_{\Gamma_N} g \cdot u \, d\mathcal{H}^{n-1}.$$  

As we did for $C$, $H$, and $d$, in (2.7) we assume that $\ell \in C^1(\mathbb{R})$ is such that $\ell$ is constant out of the interval $[0, 1]$ with $\ell(0) = 0$, $\ell(1) = 1$. At a static (or time discrete) level, the functional $E$ represents the total energy of a linearly elastoplastic material occupying $\Omega$ with density $\ell(z)$ and subjected to a displacement $u$, an elastic strain $\varepsilon$, and a plastic strain $p$. In particular, we notice that because of our assumptions on the tensors $C$ and $H$ and on the functions $d$ and $\ell$, we have $E(z, u, \varepsilon, p) = E(0 \lor z \land 1, u, \varepsilon, p)$ for every $z \in H^1(\Omega) \cap L^\infty(\Omega)$. Thus, the natural constraint $z \in [0, 1]$ needs not to be explicitly imposed at this level, for it will follow directly from the minimization of a cost functional, as shown in (2.12)–(2.13) below. This avoids some technicalities in Sections 3 and 4.

We now turn to the topology optimization problem. The primal aim of topology optimization is to detect the optimal distribution of a material inside a known box, which is the open set $\Omega$ in our context. From a mathematical standpoint, such a problem can be recast as an optimal control problem, in which the set $E \subseteq \Omega$ to be filled with the material serves as a control parameter, while the physical properties of the material act as a constraint through the equilibrium and constitutive equations. While such a mathematical approach has been already presented in many papers in the setting of linear or nonlinear elasticity (see, e.g., [9] [8] [13] [15]), topology optimization in presence of inelastic structures has been considered only at an engineering and computational level [20, 54]. The aim of our work is to present and discuss a model for topology optimization which accounts for possible plastic behaviors. The target functional we want to minimize here reads

$$J(E, u) := \int_E f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, d\mathcal{H}^{n-1} + \frac{1}{6} \text{Per}(E; \Omega),$$

where $u \in H^1(\Omega; \mathbb{R}^n)$, $E \subseteq \Omega$ is a set of finite perimeter in $\Omega$, i.e., such that the characteristic function $1_E$ of $E$ belongs to the space $BV(\Omega)$ of functions of bounded variations [4], and the symbol $\text{Per}(E; \Omega)$ denotes the perimeter of $E$ in $\Omega$. The constant $1/6$ in front of the perimeter term in $J$ is just intended to ease notations with respect to the phase-field approximation argument, see Theorem 2.6. In particular, the analysis is independent from the specific value of this constant, which can be changed with no difficulty. With an abuse of notation, we write
\[ E \in BV(\Omega) \text{ to indicate that } E \text{ is a set of finite perimeter in } \Omega. \text{ Moreover, for a sequence } E_k \in BV(\Omega) \text{ we say that } E_k \text{ converges to } E \in BV(\Omega) \text{ in } L^1(\Omega) \text{ and write } E_k \to E \text{ if } |E_k \Delta E| \to 0, \text{ where } \Delta \text{ stands for the symmetric difference of sets.} \]

Let us briefly comment on the cost functional \( J \). The first two integrals appearing in \( (2.8) \) represent the compliance and measure the ability of the body to resist to the applied loads \( f \) and \( g \). The perimeter functional, instead, penalizes sets \( E \) with large boundary, limiting the onset of a very fine microstructure. We notice that the functional \( J \) could be modified to take into account, for instance, accumulation of plastic strain \( p \) or a cost of production depending on the size of \( E \). We stay with the specific form above for the sake of definiteness.

The sharp-interface optimization problem \( (1.1) \) is therefore specified as
\[
\begin{align*}
\min_{E \in BV(\Omega)} & \quad J(E,u), \\
\text{subjected to } & \quad \min_{(u,\varepsilon,p) \in A(w)} \mathcal{E}(1_E,u,\varepsilon,p) .
\end{align*}
\]

The forward problem \( (2.10) \) represents the elastoplastic equilibrium condition, and is formulated at a static, incremental level. Under this constraint, we look for the optimal shape \( E \subset \Omega \) by minimizing \( J \) in \( (2.9) \).

We now briefly discuss the existence of solutions to \( (2.9) - (2.10) \).

**Proposition 2.1** (Existence of optimal sets). Let \( f \in L^2(\Omega;\mathbb{R}^n) \), \( g \in L^2(\Gamma_N;\mathbb{R}^n) \), and \( w \in H^1(\Omega;\mathbb{R}^n) \). Then, there exists a solution \( E \in BV(\Omega) \) to the optimization problem \( (2.9) - (2.10) \).

**Proof.** The existence of a solution follows from an application of the Direct Method. Indeed, denoted with \( (u_E,\varepsilon_E,p_E) \in A(w) \) the unique solution of \( (2.13) \) for \( E \in BV(\Omega) \), we clearly have
\[
\mathcal{E}(1_E,u_E,\varepsilon_E,p_E) \leq \mathcal{E}(1_E,w,0,0).
\]

From \( (2.1) - (2.2) \) and the previous inequality we deduce that \( (u_E,\varepsilon_E,p_E) \) is bounded in \( H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}_D^0) \times L^2(\Omega;\mathbb{M}_S^0) \) uniformly w.r.t. \( E \).

Let \( E \in BV(\Omega) \) be a minimizing sequence. By definition of \( J \) in \( (2.3) \), we immediately infer that \( E_k \) is bounded in \( BV(\Omega) \), so that, up to a subsequence, \( E_k \to E \) in \( L^1(\Omega) \). Furthermore, also \( (u_{E_k},\varepsilon_{E_k},p_{E_k}) \) admits a weak limit \( (u,\varepsilon,p) \) in \( H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}_D^0) \times L^2(\Omega;\mathbb{M}_S^0) \) with \( (u,\varepsilon,p) \in A(w) \).

By the lower semicontinuity of \( \mathcal{E} \) we have that the triple \( (u,\varepsilon,p) \) minimizes \( \mathcal{E}(\cdot,\cdot,\cdot,\cdot) \) in \( A(w) \).

Finally, the lower semicontinuity of \( J \) implies that \( E \) is a solution of \( (2.9) - (2.10) \). \( \square \)

**Remark 2.2.** We remark that the forward problem \( (2.10) \) admits a unique solution for every control \( E \in BV(\Omega) \), since \( \mathcal{E}(1_E,\cdot,\cdot,\cdot) \) is strictly convex in the state variable. The same holds for the two phase-field optimization problem we consider below, namely \( (2.13) \) and \( (2.21) \).

In what follows we focus on a phase-field approximation of the optimization problem \( (2.9) - (2.10) \). Precisely, for \( \delta > 0 \) we consider the target functional
\[
J_\delta(z,u) := \int_{\Omega} \ell(z)f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, d\mathcal{H}^{n-1} + \int_{\Omega} \frac{\delta}{2} |\nabla z|^2 + \frac{\delta^2 (1-z)^2}{2\delta} \, dx
\]
defined for \( z \in H^1(\Omega) \cap L^\infty(\Omega) \) and \( u \in H^1(\Omega;\mathbb{R}^n) \). The first two integrals in \( (2.11) \) are an approximation of the compliance appearing in \( (2.8) \). The last integral, instead, is a typical Modica-Mortola term \([32, 33]\), which, according to the value of \( \delta \), forces \( z \) to take the values 0 or 1 and penalizes the transition between material \( \{z = 1\} \) and void \( \{z = 0\} \) regions. In particular, such a term \( \Gamma \)-converges to the perimeter functional \( \text{Per}(\cdot;\Omega)/6 \).
In a similar way we may define \( G \) \( \mathcal{H}_p \). We start by proving the Remark (2.13) steps of the proof of Proposition 2.1, we get that Theorem 2.6 \( \Gamma \)-convergence result, as it can be imposed a posteriori on a solution. However, such a constraint has \( [0, 1] \) is described by characteristic functions taking values in \( \mathcal{H}_p \). We remark that the usual constraint \( \mathcal{H}_p \) imposed in (2.12). However, if \( z \in H^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \) is a solution of (2.12)–(2.13), then also \( \mathcal{H}_p \) satisfies \( \mathcal{H}_p \) and \( \mathcal{H}_p \) imposed in (2.12). However, if \( z \in H^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \) is a solution of (2.12)–(2.13), then also \( \mathcal{H}_p \).

Remark 2.3. We remark that the usual constraint \( z \in H^1(\Omega; [0, 1]) \) has not been explicitly imposed in (2.12). However, if \( z \in H^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \) is a solution of (2.12)–(2.13), then also \( \mathcal{H}_p \) satisfies \( \mathcal{H}_p \).

In the next proposition we prove the existence of solutions for (2.12)–(2.13) and show the \( \Gamma \)-convergence of the phase-field problem to the sharp-interface one as \( \delta \to 0 \).

**Proposition 2.4** (Existence of optimal phase fields). Let \( f \in L^2(\Omega; \mathbb{R}^n) \), \( g \in L^2(\mathcal{G}_N; \mathbb{R}^n) \), and \( w \in H^1(\Omega; \mathbb{R}^n) \). Then, there exists a solution \( z \in H^1(\Omega; [0, 1]) \) to the optimization problem (2.12)–(2.13).

**Proof.** The existence of solutions follows from the same argument of Proposition 2.1. We only have to notice that, by Remark 2.3, we may assume, without loss of generality, that a minimizing sequence \( z_k \in H^1(\Omega) \cap \mathcal{L}^\infty(\Omega) \) satisfies \( 0 \leq z_k \leq 1 \) in \( \Omega \). \( \square \)

In order to give a compact statement of the \( \Gamma \)-convergence result, we introduce \( G_\delta : H^1(\Omega; [0, 1]) \times H^1(\Omega; \mathbb{R}^n) \to \mathbb{R} \cup \{ +\infty \} \) defined as

\[
G_\delta(z, u) = \begin{cases} J_\delta(z, u) & \text{if } (u, \varepsilon, p) \in \mathcal{A}(w) \text{ is a solution of } (2.13), \\ +\infty & \text{elsewhere}. \end{cases}
\]

In a similar way we may define \( G : BV(\Omega; [0, 1]) \times H^1(\Omega; \mathbb{R}^n) \to \mathbb{R} \cup \{ +\infty \} \) by replacing \( J_\delta \) with \( \mathcal{J} \) and (2.13) with (2.10).

**Remark 2.5.** Notice that in the definition (2.14) of \( G_\delta \) we have imposed \( z \in H^1(\Omega; [0, 1]) \). As pointed out in Remark 2.3 such a requirement is not necessary when we look for minimizers of (2.12)–(2.13), as it can be imposed a posteriori on a solution. However, such a constraint has to be considered to state a precise \( \Gamma \)-convergence result, as the space of sets of finite perimeter is described by characteristic functions taking values in \( [0, 1] \) only.

**Theorem 2.6** (Phase-field approximation of the sharp-interface model). Let \( f \in L^2(\Omega; \mathbb{R}^n) \), \( g \in L^2(\mathcal{G}_N; \mathbb{R}^n) \), and \( w \in H^1(\Omega; \mathbb{R}^n) \). Then, \( G_\delta \) \( \Gamma \)-converges to \( G \) as \( \delta \to 0 \) in the \( L^1 \)-topology.

**Proof.** We start by proving the \( \Gamma \)-liminf inequality. Let \( \delta_k \to 0 \), \( E \in BV(\Omega) \), and \( z_k \in H^1(\Omega; [0, 1]) \) be such that \( z_k \to 1_E \) in \( L^1(\Omega) \). Let us denote by \((u_E, \varepsilon_E, p_E), (u_{z_k}, \varepsilon_{z_k}, p_{z_k}) \in \mathcal{A}(w)\) the corresponding state variables, solutions of (2.10) and (2.13), respectively. Following the steps of the proof of Proposition 2.1 we get that \((u_{z_k}, \varepsilon_{z_k}, p_{z_k}) \rightharpoonup (u_E, \varepsilon_E, p_E)\) weakly in \( H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_D^n) \times L^2(\Omega; \mathbb{M}_D^n) \). Furthermore, from the \( \Gamma \)-convergence of the Modica-Mortola term to the perimeter functional \[ G(E, u_E) \leq \liminf_{k \to \infty} G_\delta(z_k, u_{z_k}). \]

As for the \( \Gamma \)-limsup inequality, for every \( E \in BV(\Omega) \) we consider the recovery sequence \( z_k \in H^1(\Omega; [0, 1]) \) constructed in \[ G_\delta(z_k, u_{z_k}) \] and such that \( z_k \to 1_E \) in \( L^1(\Omega) \). Again, let
For later use, we set
\[ h_\gamma(Q) := \sqrt{|Q|^2 + \frac{1}{\gamma^2} - \frac{1}{\gamma}} \quad \text{for every } Q \in M^n_D, \]
and notice that \( h_\gamma \in C^\infty(M^n_D) \) is convex and such that
\[ h_\gamma(0) = 0, \quad |Q| - \frac{1}{\gamma} \leq h_\gamma(Q) \leq |Q|, \]
\[ |h_\gamma(Q_1) - h_\gamma(Q_2)| \leq \gamma |Q_1 - Q_2|, \]
\[ |\nabla_Q h_\gamma(Q_1) - \nabla_Q h_\gamma(Q)| \leq 2\gamma |Q_1 - Q_2| \]
for every \( \gamma \in (0, +\infty) \) and every \( Q, Q_1, Q_2 \in M^n_D \).

We now consider the regularized optimization problem
\[ \min_{z \in H^1(\Omega) \cap L^\infty(\Omega)} J_{\delta}(z, u), \]
\[ \text{subjected to } \min_{(u, \varepsilon, P) \in A(w)} E_\gamma(z, u, \varepsilon, P). \]

We notice that the minimization problem (2.20)–(2.21) is expected to be regular w.r.t. \( z \) (see Theorem 3.1) as the regularization of the plastic dissipation
\[ \int_{\Omega} d(z) h_\gamma(P) \, dx \]
is differentiable w.r.t. \( P \in L^2(\Omega; M^n_D) \).

In the following proposition we state the existence of solutions of (2.20)–(2.21) and the convergence as \( \gamma \to +\infty \) to solutions of (2.12)–(2.13).
Proposition 2.8 (Existence and convergence of regularized optimal controls). Let \( f \in L^2(\Omega; \mathbb{R}^n) \), \( g \in L^2(\Gamma_N; \mathbb{R}^n) \), and \( w \in H^1(\Omega; \mathbb{R}^n) \). Then, for every \( \gamma \in (0, +\infty) \) there exists a solution \( z_\gamma \in H^1(\Omega; [0, 1]) \) to \((2.20)–(2.21)\).

Moreover, every sequence \( z_\gamma \) of solutions of \((2.20)–(2.21)\) admits a subsequence which is weakly convergent to some \( z \in H^1(\Omega; [0, 1]) \) solving \((2.12)–(2.13)\), and the corresponding state configuration \((u_\gamma, \varepsilon_\gamma, p_\gamma)\) converges to \((u, \varepsilon, p)\) solving \((2.13)\) in \( H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_\delta) \times L^2(\Omega; M_\delta^2) \).

Proof. The existence of solutions of \((2.20)–(2.21)\) can be shown as in Propositions 2.1 and 2.3.

Let us prove the second part of the statement. Let \( z_\gamma \in H^1(\Omega; [0, 1]) \) be a sequence of solutions of \((2.20)–(2.21)\) for \( \gamma \in (0, +\infty) \), and let us denote with \((u_\gamma, \varepsilon_\gamma, p_\gamma)\) the corresponding solution of the forward problem \((2.21)\). Then, we have that

\[
\mathcal{E}_\gamma(z_\gamma, u_\gamma, \varepsilon_\gamma, p_\gamma) \leq \mathcal{E}_\gamma(z, w, 0, 0) \leq \beta c \|w\|_{H^1}^2 + (\|f\|_2 + \|g\|_2)\|w\|_{H^1}.
\]

Hence, \((u_\gamma, \varepsilon_\gamma, p_\gamma)\) is bounded in \( H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_\delta) \times L^2(\Omega; M_\delta^2) \) independently of \( \gamma \) and admits, up to a subsequence, a weak limit \((u, \varepsilon, p)\) in \( A(w) \). By optimality of \( z_\gamma \) we also infer that \( z_\gamma \) is bounded in \( H^1(\Omega; [0, 1]) \) and, up to a further subsequence, \( z_\gamma \rightarrow z \) weakly in \( H^1(\Omega; [0, 1]) \).

Let us show that \((u, \varepsilon, p)\) is a minimizer of \( \mathcal{E}(z, \cdot, \cdot, \cdot) \) in \( A(w) \). Thanks to \((2.17)\) we have that for every \((v, \eta, q)\) in \( A(w) \) it holds

\[
\mathcal{E}(z_\gamma, u_\gamma, \varepsilon_\gamma, p_\gamma) - \frac{M_d}{\gamma} |\Omega| \leq \mathcal{E}(z_\gamma, u_\gamma, \varepsilon_\gamma, p_\gamma) \leq \mathcal{E}(z_\gamma, v, \eta, q) \leq \mathcal{E}(z_\gamma, v, \eta, q),
\]

where we have set \( M_d = \max\{|d(z) : z \in [0, 1]| \} \). Passing to the liminf as \( \gamma \rightarrow +\infty \) on both sides of \((2.22)\) we get

\[
\mathcal{E}(z, u, \varepsilon, p) \leq \mathcal{E}(z, v, \eta, q)
\]

for every \((v, \eta, q)\) in \( A(w) \).

Hence, \((u, \varepsilon, p)\) is a minimizer of \( \mathcal{E}(z, \cdot, \cdot, \cdot) \) in \( A(w) \). Moreover, from the argument above we deduce that \( \mathcal{E}(z_\gamma, u_\gamma, \varepsilon_\gamma, p_\gamma) \) converges to \( \mathcal{E}(z, u, \varepsilon, p) \), which implies the strong convergence of \((u_\gamma, \varepsilon_\gamma, p_\gamma)\) to \((u, \varepsilon, p)\) in \( H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_\delta) \times L^2(\Omega; M_\delta^2) \).

Remark 2.9. We remark that the solutions of \((2.12)–(2.13)\) and \((2.20)–(2.21)\) are not unique, since the functional \( J_\delta \) is not convex. In particular, we can not ensure that all the solutions of \((2.12)–(2.13)\) can be approximated by solutions of \((2.20)–(2.21)\). Since in the next sections we are going to deduce the optimality conditions for \((2.12)–(2.13)\) as limit of those for \((2.20)–(2.21)\), these will be valid only for a subclass of solutions of \((2.12)–(2.13)\). Clearly, if \((2.12)–(2.13)\) admits a unique solution, in view of Proposition 2.8 it can be approximated by solutions of \((2.20)–(2.21)\), and its first order optimality conditions follows from Theorems 3.1 and 4.1.

3. Optimality of the regularized problem

This section is devoted to the computation of the first-order optimality conditions for the regularized problem \((2.20)–(2.21)\). In particular, we follow here the main lines of [17].

In order to state the main result of this section, we need some additional notation. For \( \gamma \in (0, +\infty) \), \( z, \varphi \in L^\infty(\Omega) \), and \((u, \varepsilon, p), (v, \eta, q) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_\delta) \times L^2(\Omega; M_\delta^2) \), we set

\[
\mathcal{F}_\gamma(u, \varepsilon, p)(v, \eta, q) = \frac{1}{2} \int_\Omega C(z) \eta \cdot \eta \, dx + \frac{1}{2} \int_\Omega \mathbb{H}(z) \cdot q \cdot q \, dx + \int_\Omega (C'(z) \varphi) \varepsilon \cdot \eta \, dx
+ \int_\Omega (\mathbb{H}(z) \varphi) p \cdot q \, dx + \int_\Omega \varphi d'(z) \nabla h_\gamma(p) \cdot q \, dx
\]
Lemma 3.5. For every $F_{\gamma}(z) := \int_{\Omega} \varphi \ell'(z) f \cdot v \, dx - \int_{\Omega} \varphi' \ell'(z) \cdot q \, dx$.

Theorem 3.1 (Differentiability of the control-to-state map). Let $p \in (2, +\infty)$, $f \in L^p(\Omega; \mathbb{R}^n)$, $g \in L^p(\Gamma_N; \mathbb{R}^n)$, $w \in W^{1,p}(\Omega; \mathbb{R}^n)$, and $\gamma \in (0, +\infty)$. Then, the control-to-state operator $S_{\gamma} : L^\infty(\Omega) \rightarrow A(w)$ is defined as

$$S_{\gamma}(z) := \arg\min \left\{ \mathcal{E}_{\gamma}(z, \varepsilon, p) : (u, \varepsilon, p) \in A(w) \right\}$$

is Fréchet differentiable. Its derivative in the direction $\varphi \in L^\infty(\Omega)$ is given by

$$S'_{\gamma}(z)[\varphi] = \arg\min \left\{ \mathcal{F}_{\gamma}(u, \varepsilon, p) : (v, \eta, q) \in A(0) \right\}.$$

Remark 3.2. We notice that the minimization problem (3.2) admits a unique solution, since the functional $\mathcal{F}_{\gamma}(u, \varepsilon, p)$ is strictly convex as a consequence of assumptions (2.1)–(2.2), of the positivity of $\mathcal{F}$, and of the convexity of $h_{\gamma}$.

Remark 3.3. We remark that the additional $p$-integrability of the applied force $f$, of the applied traction $g$, and of the boundary datum $w$ is necessary to prove the differentiability of the control-to-state operator $S_{\gamma}$, while they are not needed to show existence of solutions of the optimality problems (2.1)–(2.2), of the positivity of $d$, and of the convexity of $h_{\gamma}$.

Corollary 3.4 (First-order conditions for the regularized problem). In the framework of Theorem [3.2], assume that $z_{\gamma} \in H^1(\Omega; [0, 1])$ is a solution of (2.20)–(2.21) with corresponding displacement $(u_{\gamma}, \varepsilon_{\gamma}, p_{\gamma}) \in A(w)$. Then, for every $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ there exists $(\overline{w}_{\gamma}, \overline{\varepsilon}_{\gamma}, \overline{p}_{\gamma}) \in A(0)$ such that for every $(v, \eta, q) \in A(0)$

$$\int_{\Omega} \mathcal{C}(z_{\gamma}) \overline{\varepsilon}_{\gamma} \cdot \eta \, dx + \int_{\Omega} \mathbb{H}(z_{\gamma}) \overline{p}_{\gamma} \cdot q \, dx + \int_{\Omega} (\mathcal{C}'(z_{\gamma}) \varphi) \varepsilon_{\gamma} \cdot \eta \, dx + \int_{\Omega} (\mathbb{H}'(z_{\gamma}) \varphi) p_{\gamma} \cdot q \, dx$$

$$+ \int_{\Omega} \varphi \ell'(z_{\gamma}) \mathbb{Q} h_{\gamma}(p_{\gamma}) \cdot q \, dx + \int_{\Omega} d(z_{\gamma}) (\nabla^2 h_{\gamma}(p_{\gamma}) \mathbb{P}_{\gamma}) \cdot q \, dx - \int_{\Omega} \varphi \ell'(z_{\gamma}) f \cdot v \, dx = 0,$$

$$\int_{\Omega} \varphi \ell'(z_{\gamma}) f \cdot u_{\gamma} \, dx + \int_{\Omega} \ell(z_{\gamma}) f \cdot \overline{w}_{\gamma} \, dx + \int_{\Gamma_N} g \cdot \overline{w}_{\gamma} \, d\mathcal{H}^{n-1}$$

$$+ \int_{\Omega} \delta \nabla z_{\gamma} \cdot \nabla \varphi + \frac{1}{\delta} \varphi(z_{\gamma}(1-z_{\gamma})^2 - z_{\gamma}^2(1-z_{\gamma})) \, dx = 0.$$

In order to prove the Fréchet differentiability of the control-to-state operator $S_{\gamma}$ stated in Theorem 3.1, we first need to investigate the integrability and continuity properties of $S_{\gamma}(z)$ for $z \in L^\infty(\Omega)$. This is the subject of the following three lemmata.

Lemma 3.5. For every $z \in \mathbb{R}$ and every $\gamma \in (0, +\infty)$ let $F_{z,\gamma} : M^n_D \rightarrow M^n_D$ be the map defined as

$$F_{z,\gamma}(Q) := \mathcal{C}(z) Q + \mathbb{H}(z) Q + d(z) \nabla Q h_{\gamma}(Q)$$

for every $Q \in M^n_D$.

Then, there exist three constants $C_1, C_2, C_3 > 0$ independent of $\gamma$ and $z$ and a constant $C_{\gamma} > 0$ (depending only on $\gamma$) such that the following inequalities hold:

$$|F_{z,\gamma}(Q_1) - F_{z,\gamma}(Q_2)| \leq C_{\gamma}|Q_1 - Q_2|,$$

$$(F_{z,\gamma}(Q_1) - F_{z,\gamma}(Q_2)) \cdot (Q_1 - Q_2) \geq C_1|Q_1 - Q_2|^2,$$

$$(F_{z,\gamma}(Q_1) - F_{z,\gamma}(Q_2)) \cdot (Q_1 - Q_2)$$
Then, for every $z, z_1, z_2 \in \mathbb{R}$ and every $Q_1, Q_2 \in M^n_D$:

Moreover, $F_{z, \gamma}$ is invertible for every $z \in \mathbb{R}$ and every $\gamma \in (0, +\infty)$, and its inverse satisfies

(3.9) \[ |F_{z, \gamma}^{-1}(Q_1) - F_{z, \gamma}^{-1}(Q_2)| \leq \tilde{C}|Q_1 - Q_2| \]

for every $Q_1, Q_2 \in M^n_D$, for a positive constant $\tilde{C}$ independent of $\gamma$ and $z$.

Proof. Inequality (3.6) follows from assumptions (2.1)–(2.2), from the Lipschitz continuity of $d$, and from inequality (2.19). In particular, $C_\gamma$ depends on $\gamma$ because the Lipschitz constant in (2.19) degenerates with $\gamma$. Inequality (3.7) is a consequence of (2.11–2.2), of the convexity of $h_\gamma$, and of the sign of $d$. The constant $C_1$ is independent of $\gamma \in (0, +\infty)$ as (2.11–2.2) are.

Let us now show (3.8). By definition of $F_{z, \gamma}$, for every $z_1, z_2 \in \mathbb{R}$ and every $Q_1, Q_2 \in M^n_D$ we have

(3.10) \[
(F_{z_1, \gamma}(Q_1) - F_{z_2, \gamma}(Q_2)) \cdot (Q_1 - Q_2) \\
= \left( (C(z_1)Q_1 + H(z_1)Q_1 + d(z_1)\nabla h_\gamma(Q_1) - C(z_2)Q_2) \cdot (Q_1 - Q_2) + (\mathbb{H}(z_2)Q_2 - d(z_2)\nabla h_\gamma(Q_2)) \cdot (Q_1 - Q_2) \\
(F_{z_1, \gamma}(Q_1) - F_{z_2, \gamma}(Q_2)) \cdot (Q_1 - Q_2) + (d(z_1) - d(z_2))\nabla h_\gamma(Q_2)) \cdot (Q_1 - Q_2) .
\]

Inequality (3.8) can be deduced from (3.10) by taking into account the assumptions (2.1)–(2.2), the regularity of $C$, $\mathbb{H}$, and $d$ w.r.t. $z$, inequality (3.7), and the bound $|\nabla h_\gamma(Q_2)| \leq 1$. \hfill \Box

Lemma 3.6. For every $\gamma \in (0, +\infty)$ and every $z \in \mathbb{R}$ let the map $b_{z, \gamma} : M^n_S \to M^n_S$ be defined as

(3.11) \[
b_{z, \gamma}(E) := C(z)(E - F_{z, \gamma}^{-1}(\Pi_{M^n_D}(C(z)E))) \quad \text{for every } E \in M^n_S,
\]

where $\Pi_{M^n_D} : M^n_S \to M^n_D$ denotes the projection operator on $M^n_D$. Then, there exist two positive constants $c_1, c_2$ such that for every $\gamma \in (0, +\infty)$, every $z \in \mathbb{R}$, and every $E_1, E_2 \in M^n_S$

(3.12) \[
|b_{z, \gamma}(E_1) - b_{z, \gamma}(E_2)| \leq c_1|E_1 - E_2| ,
\]

(3.13) \[
(b_{z, \gamma}(E_1) - b_{z, \gamma}(E_2)) \cdot (E_1 - E_2) \geq c_2|E_1 - E_2|^2 .
\]

Proof. Property (3.12) is a direct consequence of (3.9). Let us prove (3.13), instead. Following the ideas of [17, Section 5], let us define for $z \in \mathbb{R}$, $E \in M^n_S$, and $Q \in M^n_D$ the auxiliary function

\[
G_{z, \gamma}(E, Q) := \left( C(z)(E - Q) \right) .
\]

Then, for every $z \in \mathbb{R}$, every $E_1, E_2 \in M^n_S$, and every $Q_1, Q_2 \in M^n_D$ we have

(3.14) \[
(G_{z, \gamma}(E_1, Q_1) - G_{z, \gamma}(E_2, Q_2)) \cdot \left( E_1 - E_2 \right) \left( Q_1 - Q_2 \right) \\
= C(z)((E_1 - Q_1) - (E_2 - Q_2)) \cdot (E_1 - E_2) \\
+ \Pi_{M^n_D}(C(z)(Q_1 - Q_2) + H(z)(Q_1 - Q_2)) \cdot (Q_1 - Q_2) \\
+ (d(z)\nabla h_\gamma(Q_1) - d(z)\nabla h_\gamma(Q_2)) \cdot (Q_1 - Q_2) .
\]
By the convexity of $h_\gamma$ and the assumptions (2.1)–(2.2) we deduce from (3.14) that there exists $C > 0$ independent of $\gamma$, $z$, $\mathbf{E}_1, \mathbf{E}_2$, and $Q_1, Q_2$ such that

\begin{equation}
(G_{z,\gamma}(\mathbf{E}_1, Q_1) - G_{z,\gamma}(\mathbf{E}_2, Q_2)) \cdot \left( \begin{array}{c}
\mathbf{E}_1 - \mathbf{E}_2 \\
Q_1 - Q_2
\end{array} \right) \geq C(|\mathbf{E}_1 - \mathbf{E}_2|^2 + |Q_1 - Q_2|^2).
\end{equation}

By rewriting (3.15) with the particular choice $Q_i = F_{z,\gamma}^{-1}(\Pi_{M_D^n}(C(z)\mathbf{E}_i))$ for $i = 1, 2$, we get (3.13), as $C(z)(\mathbf{E}_i - Q_i) = b_{z,\gamma}(\mathbf{E}_i)$ and $F_{z,\gamma}(Q_i) - \Pi_{M_D^n}(C(z)\mathbf{E}_i) = 0$.

\[ \square \]

**Lemma 3.7** (Bounds on the regularized control-to-state map). Let $p \in (2, +\infty)$, $f \in L^p(\Omega; \mathbb{R}^n)$, $g \in L^p(\Gamma_N; \mathbb{R}^n)$, and $w \in W^{1,p}(\Omega; \mathbb{R}^n)$. Then, there exist $\tilde{p} \in (2, p)$ and a positive constant $C > 0$ such that for every $q \in (2, \tilde{p})$, every $z, z_1, z_2 \in L^\infty(\Omega)$, and every $\gamma \in (0, +\infty)$, the following holds:

\begin{align}
\|S_\gamma(z)\|_{W^{1,p} \times L^p} &\leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p}}), \\
\|S_\gamma(z_1) - S_\gamma(z_2)\|_{W^{1,q} \times L^q} &\leq C\|z_1 - z_2\|_r(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p} + 1}),
\end{align}

where $1/r + 1/\tilde{p} = 1/q$.

**Proof.** The proof of (3.16) and (3.17) follows from an application of [24, Theorem 1.1]. To apply such result, we first have to recast the Euler-Lagrange equations associated to the minimization problem (2.21) in terms of the sole displacement variable $u$.

Let us fix $\gamma > 0$ and $z \in L^\infty(\Omega)$. For simplicity of notation, let $(u, \mathbf{E}_1, \mathbf{p}) = S_\gamma(z)$ and $(u_i, \mathbf{E}_1, \mathbf{p}_i) = S_\gamma(z_i)$, $i = 1, 2$. From the minimization problem (2.21) we deduce that the following Euler-Lagrange equation holds: for every $(v, \eta, \mathbf{q}) \in \mathcal{A}(0)$

\begin{equation}
\int_\Omega C(z)(\mathbf{E}u - \mathbf{p}) \cdot \eta \, dx + \int_\Omega \mathbb{H}(\mathbf{p}) \cdot \mathbf{q} \, dx + \int_\Omega d(z)\nabla \mathbf{h}_\gamma(\mathbf{p}) \cdot \mathbf{q} \, dx \\
- \int_\Omega \ell(z)f \cdot v \, dx - \int_{\Gamma_N} g \cdot v \mathcal{H}^{n-1} = 0,
\end{equation}

where $\mathbf{E}u$ denotes the symmetric part of the gradient of $u$. By testing (3.18) with $(0, \eta, -\eta) \in \mathcal{A}(0)$ for $\eta \in L^2(\Omega; \mathbb{M}^n_D)$ we get that

\begin{equation}
C(z)\mathbf{p} + \mathbb{H}(\mathbf{p}) + d(z)\nabla \mathbf{h}_\gamma(\mathbf{p}) = \Pi_{M_D^n}(C(z)\mathbf{E}u) \quad \text{a.e. in } \Omega.
\end{equation}

In view of the definition (5.5) of $F_{z,\gamma}$, we have $F_{z(x),\gamma}(\mathbf{p}(x)) = \Pi_{M_D^n}(\{C(z(x))\mathbf{E}u(x)\})$ and $\mathbf{p}(x) = F_{z(x),\gamma}^{-1}(\Pi_{M_D^n}(\{C(z(x))\mathbf{E}u(x)\}))$ for a.e. $x \in \Omega$.

Recalling definition (3.11), we define for $x \in \Omega$

\[ b_{z,\gamma}(x, \mathbf{E}) := b_{z,\gamma}(x, \mathbf{E}) = C(z(x))(\mathbf{E} - F_{z(x),\gamma}^{-1}(\Pi_{M_D^n}(\{C(z(x))\mathbf{E}\}))) \quad \text{for } x \in \Omega \text{ and } \mathbf{E} \in \mathbb{M}^n_S. \]

From now on, when not explicitly needed, we drop the dependence on the spatial variable $x \in \Omega$ in the definition of $F_{z,\gamma}^{-1}$, since all the arguments discussed below are valid uniformly in $\Omega$. We rewrite the Euler-Lagrange equation (3.18) in terms of the sole displacement $u$ and for test functions of the form $(\psi, \mathbf{E}\psi, 0) \in \mathcal{A}(0)$ for $\psi \in H^1(\Omega; \mathbb{R}^n)$ with $\psi = 0$ on $\Gamma_D$:

\begin{equation}
\int_\Omega b_{z,\gamma}(x, \mathbf{E}u) \cdot \mathbf{E} \psi \, dx = \int_\Omega \ell(z)f \cdot \psi \, dx + \int_{\Gamma_N} g \cdot \psi \mathcal{H}^{n-1}.
\end{equation}

In view of (3.12)–(3.13), the nonlinear operator $B_{z,\gamma}: W^{1,p}(\Omega; \mathbb{R}^n) \to W^{-1,p}(\Omega; \mathbb{R}^n)$ defined as $B_{z,\gamma}(u) = b_{z,\gamma}(x, \mathbf{E}u)$ satisfies the hypotheses of [24, Theorem 1.1]. Since $\Omega \cup \Gamma_N$ is Gröger regular, $p \in (2, +\infty)$, $f \in L^p(\Omega; \mathbb{R}^n)$, $g \in L^p(\Gamma_N; \mathbb{R}^n)$, and $w \in W^{1,p}(\Omega; \mathbb{R}^n)$, we infer from [24]
Theorem 1.1] applied to equation (3.20) that there exist \( \tilde{p} \in (2, p) \) and a constant \( C > 0 \) such that

\[
\|u\|_{W^{1,q}} \leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p}})
\]

for every \( q \in (2, \tilde{p}) \). In particular, \( C \) is independent of \( z \in L^\infty(\Omega) \), of \( \gamma \in (0, +\infty) \), and of \( q \in (2, \tilde{p}) \). Inequality (3.16) can be deduced by combining (3.7) and (3.21). Indeed, we have that

\[
\|F_{z,\gamma}^{-1}(\Pi_{M_D}^n(C(z)Eu))\|_q \leq C\|\Pi_{M_D}^n(C(z)Eu)\|_q \leq C\|u\|_{W^{1,q}}.
\]

The last inequality implies (3.16).

In order to prove (3.17), we first rewrite the Euler-Lagrange equation (3.20) satisfied by \( u_2 \). Namely, for every \( \psi \in W^{-1,\tilde{p}}(\Omega; \mathbb{R}^n) \) with \( \psi = 0 \) on \( \Gamma_D \) we have, after a simple algebraic manipulation,

\[
\int_{\Omega} B_{z_1,\gamma}(u_2) \cdot \psi \, dx = \int_{\Omega} C(z_1)(F_{z_1,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)) - F_{z_1,\gamma}^{-1}(\Pi_{M_D}^n(C(z_1)Eu_2))) \cdot \psi \, dx \\
+ \int_{\Omega} C(z_1)(F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)) - F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2))) \cdot \psi \, dx \\
+ \int_{\Omega} C(z_1) - C(z_2) \cdot (Eu_2 - F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2))) \cdot \psi \, dx \\
+ \int_{\Omega} \ell(z_2)f \cdot \psi \, dx + \int_{\Gamma_N} g \cdot \psi \, d\mathcal{H}^{n-1}.
\]

Comparing (3.22) with (3.20) written for \((z_1, u_1)\), we deduce that \( u_1 \) and \( u_2 \) solve the same kind of equation, with a different right-hand side, always belonging to \( W^{-1,\tilde{p}}(\Omega; \mathbb{R}^n) \). Thus, applying once more [24, Theorem 1.1], we infer that there exists \( C > 0 \) independent of \( z_1, z_2 \) and of \( \gamma \) such that for every \( q \in (2, \tilde{p}) \)

\[
\|u_1 - u_2\|_{W^{1,q}} \leq C\left( \|C(z_1)(F_{z_1,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)) - F_{z_1,\gamma}^{-1}(\Pi_{M_D}^n(C(z_1)Eu_2)))\|_{W^{-1,q}} \\
+ \|\Pi_{M_D}^n(C(z_1)(F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)) - F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)))\|_{W^{-1,q}} \\
+ \|C(z_1) - C(z_2)\|_{\Pi_{M_D}^n(C(z_2)Eu_2))} \|_{W^{-1,q}} \\
+ \|\ell(z_1) - \ell(z_2)\|_1 \|u\|_{W^{1,q}} \right) \\
=: C(I_1 + I_2 + I_3 + I_4).
\]

By the Lipschitz continuity (3.9) of \( F_{z,\gamma}^{-1} \), by the Hölder inequality, and by (3.16) we deduce that

\[
I_1 \leq C\|\Pi_{M_D}^n(C(z_1) - C(z_2))Eu_2\|_q \leq C\|z_1 - z_2\|_r \|u\|_{W^{1,\tilde{p}}}
\]

\[
\leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,\tilde{p}}})\|z_1 - z_2\|_r,
\]

where \( 1/r + 1/\tilde{p} = 1/q \).

Rewriting (3.8) for \( Q_i = F_{z_i,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)) \) we get that for a.e. \( x \in \Omega \)

\[
C_1|F_{z_1,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2)) - F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2))| \\
\leq C_2|F_{z_2,\gamma}^{-1}(\Pi_{M_D}^n(C(z_2)Eu_2))|\|z_1 - z_2\|_r + C_3|z_1 - z_2|.
\]
The identification \( p_\gamma = S_{\gamma,3}(z_2) = F_{z,\gamma}^{-1}(\Pi_{\mathcal{M}_D}(\mathbb{C}\{z_2\}E_\mu_2)) \), inequalities (3.16) and (3.20), and an application of the Hölder inequality imply that

\[
I_2 \leq C(\|p_\gamma\|_p + 1)\|z_1 - z_2\|_r \leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p}} + 1)\|z_1 - z_2\|_r.
\]

As for \( I_3 \), we simply use the Lipschitz continuity of \( C(\cdot) \) and inequality (3.13), to show that

\[
I_3 \leq C\|z_1 - z_2\|_r(\|u_2\|_{W^{1,p}} + \|p_2\|_p) \leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p}})\|z_1 - z_2\|_r.
\]

In a similar way, since \( \ell \) is Lipschitz continuous we obtain

\[
I_4 \leq C\|z_1 - z_2\|_r\|f\|_p.
\]

Finally, inserting (3.24)–(3.28) into (3.23) we infer

\[
\|u_1 - u_2\|_{W^{1,q}} \leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p}})\|z_1 - z_2\|_r.
\]

In order to conclude for (3.17), we notice that inequality (3.7) tested with \( Q_i = F_{z_\gamma(\cdot),\gamma}^{-1}(\Pi_{\mathcal{M}_D}(\mathbb{C}\{z(x)\}E_\mu(x))) = p_i(x) \) for a.e. \( x \in \Omega \) and integrated over \( \Omega \) implies

\[
\|p_1 - p_2\|_q \leq C\|u_1 - u_2\|_{W^{1,q}} \leq C(\|f\|_p + \|g\|_p + \|w\|_{W^{1,p}})\|z_1 - z_2\|_r.
\]

By the triangle inequality, we also estimate \( \|\varepsilon_1 - \varepsilon_2\|_q \), and the proof of (3.17) is complete. \( \Box \)

We are now ready to prove Theorem 3.1.

**Proof of Theorem (3.1)** Let us fix \( \gamma \in (0, +\infty) \) and \( z, \varphi \in L^\infty(\Omega) \). For \( t \in \mathbb{R} \), let \( z_\gamma = x + t\varphi \), \((u_\gamma, \varepsilon_\gamma, p_\gamma) := S_{\gamma}(z_\gamma) \). The solution for \( t = 0 \) will be simply denoted with \((u, \varepsilon, p)\). Moreover, we denote with \((\vec{u}_\gamma, \vec{\varepsilon}_\gamma, \vec{p}_\gamma) \) the solution of (3.2) and we set

\[
\vec{u}_\gamma := u_\gamma - u - t\vec{\varepsilon}_\gamma, \quad \vec{\varepsilon}_\gamma := \varepsilon - \varepsilon - t\vec{\varepsilon}_\gamma, \quad \vec{p}_\gamma := p - p - t\vec{\varepsilon}_\gamma.
\]

In what follows, we show that

\[
\|u_\gamma, \varepsilon_\gamma, p_\gamma\|_{H^1 \times L^2 \times L^2} = o(t),
\]

which implies the statement of the Theorem.

Writing the Euler-Lagrange equations satisfied by \((u_\gamma, \varepsilon_\gamma, p_\gamma)\), \((u, \varepsilon, p)\), and \((\vec{u}_\gamma, \vec{\varepsilon}_\gamma, \vec{p}_\gamma)\) and subtracting the second and the third from the first one, we obtain, for every \((\varphi, \eta, \gamma) \in \mathcal{A}(0)\),

\[
\int_\Omega \mathbb{C}(z_\gamma)e \cdot \eta \ dx - \int_\Omega \mathbb{C}(z)\varepsilon \cdot \eta \ dx - t \int_\Omega \mathbb{C}(z)\vec{\varepsilon}_\gamma \cdot \eta \ dx - t \int_\Omega (\mathbb{C}'(z)\varphi)\varepsilon \cdot \eta \ dx + \int_\Omega \mathbb{H}(z_\gamma)p_\gamma \cdot q \ dx
\]

\[
- \int_\Omega \mathbb{H}(z)p \cdot q \ dx - t \int_\Omega \mathbb{H}(z)\vec{p}_\gamma \cdot q \ dx - t \int_\Omega (\mathbb{H}'(z)\varphi)p \cdot q \ dx + \int_\Omega d(z_\gamma)\nabla \mathcal{Q}_{h_\gamma}(p_\gamma) \cdot q \ dx
\]

\[
- \int_\Omega d(z)\nabla \mathcal{Q}_{h_\gamma}(p) \cdot q \ dx + t \int_\Omega \varphi \nabla \mathcal{Q}_{h_\gamma}(p) \cdot q \ dx - t \int_\Omega d(z)\nabla \mathcal{Q}_{h_\gamma}(\vec{p}_\gamma) \cdot q \ dx - t \int_\Omega \ell'_{\gamma}\varphi \cdot q \ dx + t \int_\Omega \ell'(z)\varphi \cdot q \ dx = 0.
\]

By a simple algebraic manipulation, we rewrite the previous equality as

\[
0 = \left( \int_\Omega \mathbb{C}(z_\gamma)e \cdot \eta \ dx - \int_\Omega \mathbb{C}(z)\varepsilon \cdot \eta \ dx - t \int_\Omega \mathbb{C}(z)\vec{\varepsilon}_\gamma \cdot \eta \ dx - t \int_\Omega (\mathbb{C}'(z)\varphi)\varepsilon \cdot \eta \ dx \right)
\]

\[
+ \left( \int_\Omega \mathbb{H}(z_\gamma)p_\gamma \cdot q \ dx - \int_\Omega \mathbb{H}(z)p \cdot q \ dx - t \int_\Omega \mathbb{H}(z)\vec{p}_\gamma \cdot q \ dx - t \int_\Omega (\mathbb{H}'(z)\varphi)p \cdot q \ dx \right)
\]

\[
+ \left( \int_\Omega (d(z_\gamma) - d(z) - t\varphi \cdot d'_{\gamma}(z))\nabla \mathcal{Q}_{h_\gamma}(p) \cdot q \ dx \right)
\]
In a similar way, we have that
\begin{equation}
\ell (3.32)
\end{equation}
for some positive constant \( C \) of \((3.31)\). Let us now estimate \( I_{t,j} \), \( j = 1, \ldots, 5 \). We write \( I_{t,1} \) as
\begin{align*}
I_{t,1} = & \int_{\Omega} C(z) \eta_t \cdot \eta \, dx + \int_{\Omega} (C(z_t) - C(z))(\epsilon_t - \epsilon) \cdot \eta \, dx + \int_{\Omega} (C(z_t) - C(z) - t(C'(z) \varphi)) \epsilon \cdot \eta \, dx.
\end{align*}
In a similar way, we have that
\begin{align*}
I_{t,2} = & \int_{\Omega} \mathbb{H}(z) \eta_t \cdot q \, dx + \int_{\Omega} (\mathbb{H}(z_t) - \mathbb{H}(z))(p_t - p) \cdot q \, dx + \int_{\Omega} (\mathbb{H}(z_t) - \mathbb{H}(z) - t(\mathbb{H}'(z) \varphi)) p \cdot q \, dx.
\end{align*}
As for \( I_{t,4} \), since \( h_\gamma \in C^\infty(M^0_D) \), for every \( t > 0 \) there exists \( \xi_t \) laying on the segment \([p, p_t]\) such that
\begin{align*}
I_{t,4} = & \int_{\Omega} d(z)(\nabla^2_Q h_\gamma(\xi_t)(p_t - p) - t\nabla^2_Q h_\gamma(p) \bar{\xi}_t) \cdot q \, dx
\end{align*}

Inserting the previous equalities in \((3.30)\), choosing the test function \((v, \eta, q) = (\tau_t, \eta_t, \xi_t) \in A(0)\), using \((2.1)-(2.2)\), the Lipschitz continuity of \( C(\cdot), \mathbb{H}(\cdot), d(\cdot), \) and \( \nabla Q h_\gamma \), the convexity of \( h_\gamma \), and Lemma \((3.7)\), we obtain the estimate
\begin{equation}
(3.31) \quad \| (\tau_t, \eta_t, \xi_t) \|^2 \leq C_t t^2 \| \varphi \|^2 H^\infty(\| (\eta_t, \xi_t, \xi_t) \|_2)
\end{equation}
for some positive constant \( C_t \) dependent on \( \gamma \in (0, +\infty) \). In view of the regularity of \( C(\cdot), \mathbb{H}(\cdot), d(\cdot), \) and \( \ell(\cdot) \), we can continue in \((3.31)\) with
\begin{equation}
(3.32) \quad \| (\tau_t, \eta_t, \xi_t) \|^2 \leq \tilde{C}_t t^2 \| \varphi \|^2 H^\infty(\| (u, \epsilon, p) \|_2 + 1) \| (\tau_t, \eta_t, \xi_t) \|_2
\end{equation}
for some $\tilde{C}_\gamma > 0$ and some $\nu \in (1, +\infty)$. In order to conclude for (3.29) we are led to show that the last term on the right-hand side of (3.32) tends to 0 as $t \to 0$. To do this, we explicitly write $\nabla^2_Q h_{\gamma}(Q)$ for $Q \in M^n$:  
\begin{equation}
\nabla^2_Q h_{\gamma}(Q) = \frac{1}{\sqrt{|Q|^2 + \frac{1}{\gamma^2}}} \left(id - \frac{Q \otimes Q}{|Q|^2 + \frac{1}{\gamma^2}}\right).
\end{equation}

Since $p_t \to p$ in $L^2(\Omega; M^n)$ as $t \to 0$, up to a subsequence we can assume that $p_t \to p$ a.e. in $\Omega$. Hence, $\xi_t \to p$ and $\nabla^2_Q h_{\gamma}(\xi_t) \to \nabla^2_Q h_{\gamma}(p)$ a.e. in $\Omega$. In view of formula (3.33) of $\nabla^2_Q h_{\gamma}$, we have that $|\nabla Q h_{\gamma}(\xi_t)| \leq 2\gamma$ in $\Omega$. Thus, the dominated convergence theorem implies that $\|\nabla^2_Q h_{\gamma}(\xi_t) - \nabla^2_Q h_{\gamma}(p)\|_\nu \to 0$ as $t \to 0$. This, together with (3.32), concludes the proof of (3.29). \hfill \Box

We conclude this section with the proof of Corollary 3.4.

\textbf{Proof of Corollary 3.4.} Given $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, we set $(\overline{u}_\gamma, \overline{\varepsilon}_\gamma, \overline{p}_\gamma) := S'_\gamma(z_\gamma)[\varphi]$. By Theorem 3.1 $(\overline{u}_\gamma, \overline{\varepsilon}_\gamma, \overline{p}_\gamma)$ is a solution of (3.2), which in turn is equivalent to the Euler-Lagrange equation (3.3).

The equality (3.4) also follows from Theorem 3.1. Indeed, for $t \in \mathbb{R}$ we consider $z^t_\gamma := z_\gamma + t\varphi$ and denote with $(u^t_\gamma, \varepsilon^t_\gamma, p^t_\gamma) = S'_\gamma(z^t_\gamma)$ the corresponding solution of the forward problem (2.12). By optimality of $z_\gamma$ we have that $J_\delta(z_\gamma, u_\gamma) \leq J_\delta(z^t_\gamma, u^t_\gamma)$. Equality (3.4) is therefore obtained by dividing the previous expression by $t$ and passing to the limit as $t \to 0$, by taking into account the differentiability of the control-to-state operator $S_\gamma$ from Theorem 3.1. \hfill \Box

4. Optimality conditions for $\gamma \to +\infty$

This section is devoted to the computation of the optimality conditions for (2.12)–(2.13). Such conditions are obtained by passing to the limit in (3.3) and (3.4). As the control-to-state operator for (2.12)–(2.13) is not differentiable anymore and quantities such as $\nabla_Q h_{\gamma}$ and $\nabla^2_Q h_{\gamma}$ appearing in formulas (3.1)–(3.2) degenerate as $\gamma \to +\infty$, the limit passage is not completely trivial and deserves some further analysis. This is the content of the following theorem.

\textbf{Theorem 4.1 (First-order conditions).} Let $p \in (2, +\infty)$, $f \in L^p(\Omega; \mathbb{R}^n)$, $g \in L^2(\Gamma_N; \mathbb{R}^n)$, and $w \in W^{1,p}(\Omega; \mathbb{R}^n)$. For every $\gamma \in (0, +\infty)$ let $z_\gamma \in H^1(\Omega; [0, 1])$ be a solution of (2.20)–(2.21), with corresponding state variable $(u_\gamma, \varepsilon_\gamma, p_\gamma) \in \mathcal{A}(w)$. Then, there exists $z \in H^1(\Omega; [0, 1])$ solution of (2.12)–(2.13) with corresponding state variable $(u, \varepsilon, p) \in \mathcal{A}(w)$ such that, up to a subsequence, $z_\gamma \to z$ weakly in $H^1(\Omega)$ and $(u_\gamma, \varepsilon_\gamma, p_\gamma) \to (u, \varepsilon, p)$ in $H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M^n) \times L^2(\Omega; M^n)$ as $\gamma \to +\infty$.

Moreover, there exists $p \in L^\infty(\Omega)$ such that for every $(v, \eta, q) \in \mathcal{A}(0)$
\begin{equation}
\int_\Omega C(z)v \cdot \varepsilon \, dx + \int_\Omega H(z)p \cdot q \, dx + \int_\Omega \rho \cdot q \, dx - \int_\Omega f \cdot v \, dx - \int_{\Gamma_N} g \cdot v \, dH^{n-1} = 0.
\end{equation}

\begin{equation}
\rho \cdot p = d(z)\|p\| \quad \text{in } \Omega, \quad p = 0 \quad \text{in } \{|p| < d(z)\}.
\end{equation}

Finally, for every $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ there exist a state variable $(\overline{u}_\varphi, \overline{\varepsilon}_\varphi, \overline{p}_\varphi) \in \mathcal{A}(0)$ and an adjoint variable $\pi_\varphi \in L^2(\Omega)$ such that the following hold:
\begin{equation}
\int_\Omega C(z)\overline{\varepsilon}_\varphi \cdot \eta \, dx + \int_\Omega H(z)\overline{p}_\varphi \cdot q \, dx + \int_\Omega (C'(z)\varphi)\varepsilon \cdot \eta \, dx + \int_\Omega (H'(z)\varphi)p \cdot q \, dx
\end{equation}
of (4.8) is positive. Recalling that 
\( C \)

In view of the convergences shown above, from (4.6) we infer that

\[ \therefore \text{passing to the limit in (3.3) we obtain (4.3).} \]

\( \psi \)

\[ \text{We now prove the second inequality in (4.5). For every } \gamma \in (0, +\infty), \text{in view of Corollary 3.3 we know that for every } \varphi \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_{1\ast}^N) \times L^2(\Omega; M_{1\ast}^N) \text{ and } z \to z \text{ weakly in } H^1(\Omega; [0, 1]), \text{we deduce from the previous equalities that } d(z_\gamma) \nabla Q h_{\gamma}(p_{\gamma}) \to 0 \text{ in } L^2(\Omega; M_{1\ast}^N). \]

\( \psi \)

Let us prove (4.3)–(4.4). For \( \gamma \to +\infty \) we infer that (4.3)–(4.4) hold. From (3.3) (or, equivalently, (3.2)) we immediately deduce that \((\pi_{\gamma}, \xi_{\gamma}, \ell_{\gamma})\) is bounded in \( H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; M_{1\ast}^N) \times L^2(\Omega; M_{1\ast}^N) \). Hence, up to a subsequence, \((\pi_{\gamma}, \xi_{\gamma}, \ell_{\gamma})\) converges weakly to some \((\pi_\gamma, \xi_\gamma, \ell_\gamma)\) \( \in A(0) \). In particular, passing to the limit in (4.4) as \( \gamma \to +\infty \) we infer (4.4).

By testing (3.3) with \((0, \eta, -\eta)\) for \( \eta \in L^2(\Omega; M_{1\ast}^N) \), we get that

\[ \pi_{\gamma} = \Pi_{M_{1\ast}^N} \left( C(z_\gamma) \xi_\gamma - \mathbb{H}(z_\gamma) p_{\gamma} \right). \]

In view of the convergences shown above, from (4.6) we infer that \( \pi_{\gamma} \) converges weakly in \( L^2(\Omega) \) to

\[ \pi_{\gamma} = \Pi_{M_{1\ast}^N} \left( C(z) \xi_{\gamma} - \mathbb{H}(z_{\gamma}) p_{\gamma} + (C'(z_{\gamma}) \varphi) \xi_{\gamma} - (H'(z_{\gamma}) \varphi) p_{\gamma} - \varphi d'(z_{\gamma}) \nabla Q h_{\gamma}(p_{\gamma}) \right). \]

Therefore, passing to the limit in (4.3), we obtain (4.3).

We now prove the second inequality in (4.4). For every \( \gamma \in (0, +\infty) \) and every \( \psi \in C_c^\infty(\Omega) \) with \( \psi \geq 0 \), we test (3.3) with the triple \((\psi \ell_{\gamma}, \psi \xi_{\gamma} + \nabla \psi, \psi \ell_{\gamma}) \in A(0)\), obtaining

\[ 0 = \int_\Omega C(z_\gamma) \xi_{\gamma} \cdot (\psi \xi_{\gamma} + \nabla \psi) \; dx + \int_\Omega H(z_\gamma) \ell_{\gamma} \cdot (\psi \ell_{\gamma}) \; dx \]

\[ + \int_\Omega (C'(z_\gamma) \varphi) \xi_{\gamma} \cdot (\psi \xi_{\gamma} + \nabla \psi) \; dx + \int_\Omega (H'(z_\gamma) \varphi) p_{\gamma} \cdot (\psi \ell_{\gamma}) \; dx \]

\[ + \int_\Omega \varphi \cdot (z_\gamma \nabla Q h_{\gamma}(p_{\gamma}) \cdot (\psi \ell_{\gamma})) \; dx - \int_\Omega \varphi \cdot (z_\gamma \ell_{\gamma} \cdot (\psi \ell_{\gamma})) \; dx + \int_\Omega \pi_{\gamma} \cdot (\psi \ell_{\gamma}) \; dx. \]

Since \( \psi \geq 0 \), \( h_{\gamma} \) is convex, and (4.4) holds, we notice that the last term on the right-hand side of (4.8) is positive. Recalling that \( C, \mathbb{H}, \) and \( d \) are of class \( C^1 \), we can pass to the liminf in (4.8) obtaining

\[ 0 \geq \int_\Omega C(z) \xi_{\gamma} \cdot (\psi \xi_{\gamma} + \nabla \psi) \; dx + \int_\Omega H(z) \ell_{\gamma} \cdot (\psi \ell_{\gamma}) \; dx \]
+ \int_{\Omega} (C'(z) \varphi) \varepsilon \cdot (\psi \tau + \tau \varphi \circ \nabla \psi) dx + \int_{\Omega} (H' \varphi) \psi (\psi \tau) \tau dx

+ \int_{\Omega} \nu' (z) \rho (\psi \tau \tau) dx - \int_{\Omega} \nu' (z) (\psi \tau \varphi) dx = - \int_{\Omega} \tau \varphi (\psi \tau) \tau dx \),

where, in the last equality, we have used (4.3) with the test \((\psi \tau, \psi \tau \tau + \tau \varphi \circ \nabla \psi, \psi \tau) \in A(0)\).

The arbitrariness of \(\psi \geq 0\) in (4.9) implies that \(\pi \varphi \cdot \tau \varphi \geq 0\) a.e. in \(\Omega\).

Let us show the first equality in (4.5). To do this, we set \(J_1 = \{ |\pi \tau| \leq \frac{1}{\gamma} \} \) and estimate \(\| \pi \tau \cdot \tau \|_1^2 \) as follows:

\[
(4.10) \quad \| \pi \tau \cdot \tau \|_1^2 = \left( \int_{\Omega} \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left( |p \gamma \cdot \tau \gamma| - \frac{|p \gamma|^2 (\tau \gamma \cdot \tau \gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \right) dx \right)^2
\]

\[
\leq 2 \left( \int_{J_1} \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left( |p \gamma \cdot \tau \gamma| - \frac{|p \gamma|^2 (\tau \gamma \cdot \tau \gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \right) dx \right)^2
\]

\[
+ 2 \left( \int_{\Omega \setminus J_1} \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left( |p \gamma \cdot \tau \gamma| - \frac{|p \gamma|^2 (\tau \gamma \cdot \tau \gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \right) dx \right)^2 =: I_{1,1} + I_{1,2}.
\]

We now show that \(I_{1,1}\) and \(I_{1,2}\) tend to 0 separately. For \(I_{1,1}\), by Hölder and Cauchy inequalities and by the continuity of \(d\) we have that there exists \(C > 0\) independent of \(\gamma\) such that

\[
(4.11) \quad I_{1,1} \leq 2 |\Omega| \int_{J_1} \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left| 1 - \frac{|p \gamma|^2}{|p \gamma|^2 + \frac{1}{\gamma^2}} \right|^2 dx
\]

\[
\leq C |\Omega| \int_{J_1} \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left| 1 - \frac{|p \gamma|^2}{|p \gamma|^2 + \frac{1}{\gamma^2}} \right| dx
\]

\[
\leq C \gamma^{-1} \int_{J_1} \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left| \tau \gamma \right|^2 - \frac{(p \gamma \cdot \tau \gamma)^2}{|p \gamma|^2 + \frac{1}{\gamma^2}} dx.
\]

In order to conclude that \(I_{1,1} \to 0\), we write explicitly \(\| \pi \tau \cdot \tau \|_1^2\):

\[
\| \pi \tau \cdot \tau \|_1^2 = \frac{d(z_\gamma)}{|p \gamma|^2 + \frac{1}{\gamma^2}} \left| \tau \gamma \right|^2 - \frac{(p \gamma \cdot \tau \gamma)^2}{|p \gamma|^2 + \frac{1}{\gamma^2}}.
\]

Therefore, we can continue in (4.11) with

\[
I_{1,1} \leq C \frac{|\Omega|}{\sqrt{\gamma}} \int_{\Omega} \| \pi \tau \cdot \tau \|_{1} dx
\]

which implies that \(I_{1,1} \to 0\) as \(\gamma \to +\infty\), as \(\pi \tau\) and \(\tau \gamma\) are bounded in \(L^2(\Omega; \mathbb{M}_{D})\).

As for \(I_{1,2}\), instead, by the Cauchy inequality we have

\[
(4.12) \quad I_{1,2} \leq C \left( \int_{\Omega \setminus J_1} \left| \tau \gamma \right|^2 \left| \tau \gamma \right|^2 + \frac{1}{\gamma^2} dx \right)^2 \leq C \left( \int_{\Omega \setminus J_1} \frac{\left| \tau \gamma \right|^2}{\gamma^2 |p \gamma|^2 + 1} dx \right)^2 \leq C \frac{\gamma^2}{\gamma^2} \left| \tau \gamma \right|^2.\]

Thus, also \(I_{1,2}\) tends to 0 as \(\gamma \to +\infty\).
All in all, we deduce from (4.10) that \( \pi^\gamma_\epsilon \cdot p_\gamma \to 0 \) in \( L^1(\Omega) \). Furthermore, we know that \( \pi^\gamma_\epsilon \to \pi_\varphi \) weakly in \( L^2(\Omega; M^n_D) \) and \( p_\gamma \to p \) in \( L^2(\Omega; M^n_D) \), so that \( \pi^\gamma_\epsilon \cdot p_\gamma \to \pi_\varphi \cdot p \) weakly in \( L^1(\Omega) \), which implies \( \pi_\varphi \cdot p = 0 \) in \( \Omega \).

We now check with the third inequality in (4.5). Since \( |\rho| \leq d(z) \), the function

\[
q \mapsto \int_\Omega |q|(d(z) - |\rho|) \, dx
\]

is lower semicontinuous w.r.t. the weak topology of \( L^2(\Omega; M^n_D) \). Hence,

\[
0 \leq \int_\Omega |\mathcal{P}_\epsilon|(d(z) - |\rho|) \, dx \leq \liminf_{\gamma \to +\infty} \int_\Omega |\mathcal{P}_\gamma^\epsilon|(d(z) - |\rho|) \, dx
\]

\[
\leq \limsup_{\gamma \to +\infty} \int_\Omega |\mathcal{P}^\epsilon_\gamma| (d(z) - d(z_\gamma)) \, dx + \limsup_{\gamma \to +\infty} \int_\Omega |\mathcal{P}^\epsilon_\gamma| (d(z_\gamma) - d(z_\gamma)|\nabla Q h_\gamma(p_\gamma)|) \, dx
\]

\[
+ \limsup_{\gamma \to +\infty} \int_\Omega |\mathcal{P}^\epsilon_\gamma| (d(z_\gamma)|\nabla Q h_\gamma(p_\gamma)| - |\rho|) \, dx.
\]

Since \( \mathcal{P}^\epsilon_\gamma \) is bounded in \( L^2(\Omega; M^n_D) \), \( d(z_\gamma)\nabla Q h_\gamma(p_\gamma) \to \rho \) in \( L^2(\Omega; M^n_D) \), and \( z_\gamma \to z \) weakly in \( H^1(\Omega; [0,1]) \), the previous inequality reduces to

\[
0 \leq \int_\Omega |\mathcal{P}_\gamma| (d(z) - |\rho|) \, dx \leq \liminf_{\gamma \to +\infty} \int_\Omega |\mathcal{P}^\epsilon_\gamma| (d(z_\gamma) - d(z_\gamma)|\nabla Q h_\gamma(p_\gamma)|) \, dx
\]

\[
= \limsup_{\gamma \to +\infty} \int_{J_\gamma} d(z_\gamma)|\mathcal{P}^\epsilon_\gamma| \left( 1 - \frac{|p_\gamma|}{\sqrt{|p_\gamma|^2 + \frac{1}{\gamma}} \right) \, dx
\]

\[
\leq \limsup_{\gamma \to +\infty} \int_{J_\gamma} d(z_\gamma)|\mathcal{P}^\epsilon_\gamma| \left( 1 - \frac{|p_\gamma|^2}{\sqrt{|p_\gamma|^2 + \frac{1}{\gamma}} \right) \, dx
\]

\[
+ \limsup_{\gamma \to +\infty} \int_{\Omega \setminus J_\gamma} d(z_\gamma)|\mathcal{P}^\epsilon_\gamma| \left( 1 - \frac{|p_\gamma|}{\sqrt{|p_\gamma|^2 + \frac{1}{\gamma}} \right) \, dx,
\]

where in the second line we have used the explicit expression of \( \nabla Q h_\gamma(p_\gamma) \). Arguing as in (4.12) we deduce that the second integral on the right-hand side of (4.13) tends to 0 as \( \gamma \to +\infty \).

As for the integral on \( J_\gamma \), instead, by the Cauchy inequality and the continuity of \( d \) there exists a positive constant \( C \) independent of \( \gamma \) such that

\[
\int_{J_\gamma} d(z_\gamma)|\mathcal{P}^\epsilon_\gamma| \left( 1 - \frac{|p_\gamma|}{\sqrt{|p_\gamma|^2 + \frac{1}{\gamma}} \right) \, dx \leq \int_{J_\gamma} d(z_\gamma) \left( |\mathcal{P}^\epsilon_\gamma| - \frac{|p_\gamma|}{\sqrt{|p_\gamma|^2 + \frac{1}{\gamma}} \right) \, dx
\]

\[
\leq C \int_{J_\gamma} \left( |\mathcal{P}^\epsilon_\gamma|^2 - \frac{|p_\gamma|^2}{|p_\gamma|^2 + \frac{1}{\gamma}} \right) \, dx.
\]
In view of (2.6) and of the boundedness of $\pi_\gamma^c$ and $p_\gamma^c$ are bounded in $L^2(\Omega; M)_{H}$ we get
\[
\int J_\gamma |\pi_\gamma^c \cdot p_\gamma^c| \, dx = \int J_\gamma \frac{d(z_\gamma)}{\sqrt{\|p_\gamma^c\|^2 + \frac{1}{\gamma^2}}} \|p_\gamma^c\|^2 - \frac{(p_\gamma^c \cdot \bar{p}_\gamma^c)^2}{\|p_\gamma^c\|^2 + \frac{1}{\gamma^2}} \, dx 
\geq \frac{\gamma \lambda}{\gamma + 1} \int J_\gamma \|p_\gamma^c\|^2 - \frac{(p_\gamma^c \cdot \bar{p}_\gamma^c)^2}{\|p_\gamma^c\|^2 + \frac{1}{\gamma^2}} \, dx ,
\]
we deduce that
\[
\lim_{\gamma \to +\infty} \int J_\gamma \|\bar{p}_\gamma^c\|^2 - \frac{(p_\gamma^c \cdot \bar{p}_\gamma^c)^2}{\|p_\gamma^c\|^2 + \frac{1}{\gamma^2}} \, dx = 0.
\]
Combining (4.13)–(4.15) we get that
\[
\int \Omega |\bar{p}_\phi|(d(z) - |\rho|) \, dx = 0
\]
which implies $|\bar{p}_\phi| = 0$ a.e. in $\{|\rho| \leq d(z)\}$. This concludes the proof of the theorem. \qed

**ACKNOWLEDGMENT**

US is partially supported by the FWF projects F 65, I 2375, and P 27052 and by the Vienna Science and Technology Fund (WWTF) through Project MA14-009.

**References**

[1] L. Adam, J. Outrata, and T. Roubiček, Identification of some nonsmooth evolution systems with illustration on adhesive contacts at small strains, Optimization, 66 (2017), pp. 2025–2049.

[2] G. Allaire, *The homogenization method for topology and shape optimization*, in Topology optimization in structural mechanics, vol. 374 of CISM Courses and Lect., Springer, Vienna, 1997, pp. 101–133.

[3] ———, *Topology optimization with the homogenization and the level-set methods*, in Nonlinear homogenization and its applications to composites, polycrystals and smart materials, vol. 170 of NATO Sci. Ser. II Math. Phys. Chem., Kluwer Acad. Publ., Dordrecht, 2004, pp. 1–13.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[5] O. Amir, Stress-constrained continuum topology optimization: a new approach based on elasto-plasticity, Struct. Multidiscip. Optim., 55 (2017), pp. 1797–1818.

[6] M. P. Bendsoe and N. Kikuchi, Generating optimal topologies in structural design using a homogenization method, Comput. Methods Appl. Mech. Engrg., 71 (1988), pp. 197–224.

[7] M. P. Bendsoe and O. Sigmund, *Topology optimization*, Springer-Verlag, Berlin, 2003. Theory, methods and applications.

[8] L. Blank, H. Garcke, M. H. Farshbaf-Shaker, and V. Styles, Relating phase field and sharp interface approaches to structural topology optimization, ESAIM Control Optim. Calc. Var., 20 (2014), pp. 1025–1058.

[9] L. Blank, H. Garcke, C. Hecht, and C. Rupprecht, Sharp interface limit for a phase field model in structural optimization, SIAM J. Control Optim., 54 (2016), pp. 1558–1584.

[10] B. Bourdin and A. Chambolle, Design-dependent loads in topology optimization, ESAIM Control Optim. Calc. Var., 9 (2003), pp. 19–48.

[11] M. Brokate and P. Krejčí, Optimal control of ODE systems involving a rate independent variational inequality, Discr. Contin. Dyn. Syst. Ser. B, 18 (2013), pp. 331–348.

[12] M. Brugger and P. Venini, A mixed FEM approach to stress-constrained topology optimization, Internat. J. Numer. Methods Engrg., 73 (2008), pp. 1693–1724.

[13] M. Burger and R. Stainko, Phase-field relaxation of topology optimization with local stress constraints, SIAM J. Control Optim., 45 (2006), pp. 1447–1466.

[14] M. Carrasso, B. Ivorra, and A. M. Ramos, Stochastic topology design optimization for continuous elastic materials, Comput. Methods Appl. Mech. Engrg., 289 (2015), pp. 131–154.
TOPOLOGY OPTIMIZATION FOR INCREMENTAL ELASTOPLASTICITY 21

[15] M. Carraturo, E. Rocca, E. Bonetti, D. Hömberg, A. Reali, and F. Auricchio, Graded-material design based on phase-field and topology optimization, Comput. Mech., 64 (2019), pp. 1589–1600.

[16] G. Dal Maso, An introduction to Γ-convergence, vol. 8 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, Inc., Boston, MA, 1993.

[17] J. C. de los Reyes, R. Herzog, and C. Meyer, Optimal control of static elastoplasticity in primal formulation, SIAM J. Control Optim., 54 (2016), pp. 3016–3039.

[18] J. D. Deaton and R. V. Grandhi, A survey of structural and multidisciplinary continuum topology optimization: post 2000, Struct. Multidiscip. Optim., 49 (2014), pp. 1–38.

[19] M. Eleuteri and L. Lussardi, Thermal control of a rate-independent model for permanent inelastic effects in shape memory materials, Evol. Equ. Control Theory, 3 (2014), pp. 411–427.

[20] M. Eleuteri, L. Lussardi, and U. Stefanelli, Thermal control of the Souza-Auricchio model for shape memory alloys, Discrete Contin. Dyn. Syst. Ser. S, 6 (2013), pp. 369–386.

[21] J. Gao, Z. Luo, M. Xiao, L. Gao, and P. Li, A NURBS-based multi-material interpolation (N-MMI) for isogeometric topology optimization of structures, Appl. Math. Model., 81 (2020), pp. 818–843.

[22] K. Gröger, A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations, Math. Ann., 283 (1989), pp. 679–687.

[23] R. Herzog and C. Meyer, Optimal control of static plasticity with linear kinematic hardening, ZAMM Z. Angew. Math. Mech., 91 (2011), pp. 777–794.

[24] R. Herzog, C. Meyer, and G. Wachsmuth, Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions, J. Math. Anal. Appl., 382 (2011), pp. 802–813.

[25] C-stationarity for optimal control of static plasticity with linear kinematic hardening, SIAM J. Control Optim., 50 (2012), pp. 3052–3082.

[26] K. A. James and H. Waisman, Topology optimization of viscoelastic structures using a time-dependent adjoint method, Comput. Methods Appl. Mech. Engrg., 285 (2015), pp. 166–187.

[27] L. Li, G. Zhang, and K. Khandelwal, Topology optimization of structures with gradient elastic material, Struct. Multidiscip. Optim., 56 (2017), pp. 371–390.

[28] J. Lin, Y. Guan, G. Zhao, H. Naceur, and P. Lu, Topology optimization of plane structures using smoothed particle hydrodynamics method, Internat. J. Numer. Methods Engrg., 110 (2014), pp. 726–744.

[29] J. Liu, K. Duke, and Y. Ma, Multi-material plastic part design via the level set shape and topology optimization method, Eng. Optim., 48 (2016), pp. 1910–1931.

[30] C. Lundgaard, J. Alexandersen, M. Zhou, C. S. Andreasen, and O. Sigmund, Revisiting density-based topology optimization for fluid-structure-interaction problems, Struct. Multidiscip. Optim., 58 (2018), pp. 969–995.

[31] K. Maute, S. Schwarz, and E. Ramm, Structural optimization—the interaction between form and mechanics, ZAMM Z. Angew. Math. Mech., 79 (1999), pp. 651–673.

[32] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, Arch. Rational Mech. Anal., 98 (1987), pp. 123–142.

[33] L. Modica and S. Mortola, Un esempio di Γ-convergenza, Boll. Un. Mat. Ital. B (5), 14 (1977), pp. 285–299.

[34] P. B. Nakhstratrala and D. A. Tortorelli, Nonlinear structural design using multiscale topology optimization. Part II: transient formulation, Comput. Methods Appl. Mech. Engrg., 304 (2016), pp. 605–618.

[35] S. S. Nanthakumar, N. Valizadeh, H. S. Park, and T. Rabczuk, Surface effects on shape and topology optimization of nanostructures, Comput. Mech., 56 (2015), pp. 97–112.

[36] P. Penzler, M. Rumpf, and B. Wirth, A phase-field model for compliance shape optimization in nonlinear elasticity, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 229–258.

[37] J. Petersson, Finite element analyses of topology optimization of elastic continua, in ENUMATH 97 (Heidelberg), World Sci. Publ., River Edge, NJ, 1998, pp. 503–510.

[38] F. Rindler, Optimal control for nonconvex rate-independent evolution processes, SIAM J. Control Optim., 47 (2008), pp. 2773–2794.

[39] F. Rindler, Approximation of rate-independent optimal control problems, SIAM J. Numer. Anal., 47 (2009), pp. 3884–3909.

[40] J. Sokolowski and A. Żochowski, Optimality conditions for simultaneous topology and shape optimization, SIAM J. Control Optim., 42 (2003), pp. 1198–1221.

[41] U. Stefanelli, D. Wachsmuth, and G. Wachsmuth, Optimal control of a rate-independent evolution equation via viscous regularization, Discrete Contin. Dyn. Syst. Ser. S, 10 (2017), pp. 1467–1485.
[42] C. C. Swan and I. Kosaka, Voigt-Reuss topology optimization for structures with nonlinear material behaviors, Internat. J. Numer. Methods Engrg., 40 (1997), pp. 3785–3814.

[43] N. Vermaak, G. Michailidis, G. Parry, R. Estevez, G. Allaire, and Y. Bréchet, Material interface effects on the topology optimization of multi-phase structures using a level set method, Struct. Multidiscip. Optim., 50 (2014), pp. 623–644.

[44] G. Wachsmuth, Optimal control of quasi-static plasticity with linear kinematic hardening, Part I: Existence and discretization in time, SIAM J. Control Optim., 50 (2012), pp. 2836–2861 + loose erratum.

[45] Optimal control of quasistatic plasticity with linear kinematic hardening II: Regularization and differentiability, Z. Anal. Anwend., 34 (2015), pp. 391–418.

[46] Optimal control of quasistatic plasticity with linear kinematic hardening III: Optimality conditions, Z. Anal. Anwend., 35 (2016), pp. 81–118.

[47] M. Wallin and M. Ristimaa, Finite strain topology optimization based on phase-field regularization, Struct. Multidiscip. Optim., 51 (2015), pp. 305–317.

[48] W. Zhang, J. Yang, Y. Xu, and T. Gao, Topology optimization of thermoelastic structures: mean compliance minimization or elastic strain energy minimization, Struct. Multidiscip. Optim., 49 (2014), pp. 417–429.

[49] X. Zhang and B. Zhu, Topology optimization of compliant mechanisms, Springer, Singapore, 2018.

[50] J.-H. Zhu, W.-H. Zhang, and L. Xia, Topology optimization in aircraft and aerospace structures design, Arch. Comput. Methods Eng., 23 (2016), pp. 595–622.

(Stefano Almi) Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria.

E-mail address: stefano.almi@univie.ac.at
URL: http://www.mat.univie.ac.at/~almi

(Ulisse Stefanelli) Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria, Vienna Research Platform on Accelerating Photoreaction Discovery, University of Vienna, Währingerstrasse 17, 1090 Wien, Austria, & Istituto di Matematica Applicata e Tecnologie Informatiche E. Magenes, via Ferrata 1, I-27100 Pavia, Italy

E-mail address: ulisse.stefanelli@univie.ac.at
URL: http://www.mat.univie.ac.at/~stefanelli