On the Degrees of Freedom of Asymmetric MIMO Interference Broadcast Channels

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Abstract—In this paper, we study the degrees of freedom (DoF) of the asymmetric multi-input-multi-output interference broadcast channel (MIMO-IBC). By introducing a notion of connection pattern chain, we generalize the genie chain proposed in [11] to derive and prove the necessary condition of IA feasibility for asymmetric MIMO-IBC, which is denoted as irreducible condition. It is necessary for both linear interference alignment (IA) and asymptotic IA feasibility in MIMO-IBC with arbitrary configurations. In a special class of asymmetric two-cell MIMO-IBC, the irreducible condition is proved to be the sufficient and necessary condition for asymptotic IA feasibility, while the combination of proper condition and irreducible condition is proved to the sufficient and necessary condition for linear IA feasibility. From these conditions, we derive the information theoretic maximal DoF per user and the maximal DoF per user achieved by linear IA, and these DoFs are also the DoF per user upper-bounds of asymmetric G-cell MIMO-IBC with asymmetric IA and linear IA, respectively.

I. INTRODUCTION

The degrees of freedom (DoF) can reflect the potential of interference networks, which is the first-order approximation of sum capacity in high signal-to-noise ratio regime [1][2]. Recently, significant research efforts have been devoted to find the information theoretic DoF for multi-input-multi-output (MIMO) interference channel (MIMO-IC) [1][8] and MIMO interference broadcast channel (MIMO-IBC) [9][11].

For a symmetric G-cell MIMO-IC where each base station (BS) and each mobile station (MS) have M antennas, the study in [3] showed that the information theoretic maximal DoF per user is M/2, which can be achieved by asymptotic interference alignment (IA) (i.e., with infinite time/frequency extension). It implies that the sum DoF can linearly increase as G, and the interference networks are not interference-limited [5]. Encouraged by such a promising result, many recent works strive to analyze the DoF for MIMO-IC and MIMO-IBC with various settings and devise interference management techniques to achieve the maximal DoF.

So far, the existing studies focus on the symmetric system where each BS and each MS have M and N antennas, respectively. For a three-cell symmetric MIMO-IC, the information theoretic maximal DoF was obtained in [1], which can be achieved by linear IA (i.e., without any symbol extension or only with finite spatial extension) [3]. For a G-cell symmetric MIMO-IC, the information theoretic maximal DoF was only obtained for some configurations [5][6]. For symmetric MIMO-IBC, the information theoretic maximal DoF was obtained in [11] for arbitrary configurations. The result indicates that the information theoretic maximal DoF can be divided into two regions according to the ratio of M/N. In the first region, the sum DoF of the system linearly increases with the number of cells, which can be achieved by asymptotic IA. In the second region, the DoF is a piecewise linear function of M and N alternately, which can be achieved by linear IA. Considering that asymmetric MIMO-IBC is more complex than symmetric MIMO-IBC and expecting that the results for asymmetric cases may be extended from symmetric ones, there are only a few results on asymmetric systems. So far, only the proper condition is obtained and proved to be necessary for linear IA feasibility in asymmetric MIMO-IC [8] and asymmetric MIMO-IBC [10]. The following questions are still open for asymmetric MIMO-IC and MIMO-IBC: what is the information theoretic maximal DoF? when linear IA can achieve the information theoretic maximal DoF?

To understand the potential of practical networks, including both homogeneous and heterogeneous networks, we need to investigate the information theoretic maximal DoF for asymmetric systems. Moreover, in symmetric systems, all BSs or users have the same spatial resources. Consequently, it is hard to know which resources of BSs or users participate in managing interference, how the BSs or users remove the interference jointly, and what is the impact on the DoF if some nodes increase or reduce some resources (e.g., turning on or turning off antennas for energy saving)?

In this paper, we investigate the DoF of the asymmetric MIMO-IBC. By finding the difference of deriving the necessary conditions between symmetric and asymmetric systems and introducing a notion of connection pattern chain, we generalize the genie chain proposed in [11] into asymmetric MIMO-IBC and derive a necessary condition for linear IA and asymptotic IA feasibility, denoted as irreducible condition. From the irreducible condition and the combination of the proper condition and irreducible condition, we obtain the information theoretic DoF outer-bound and the DoF outer-bound achieved by linear IA in G-cell asymmetric MIMO-IBC, respectively. In addition, we also prove that these DoF bounds of a special class of two-cell MIMO-IBC can be achieved by asymptotic IA and linear IA, respectively.

II. NECESSARY CONDITIONS FOR ASYMMETRIC G-CELL MIMO-IBC

Consider a G-cell MIMO system, where BS$_i$ equipped with $M_i$ antennas transmits to $K_i$ users each with $N_{i1}, \cdots, N_{iK_i}$ antennas in the $i$th cell, $i = 1, \cdots, G$. BS$_i$ respectively transmits $d_{i1}, \cdots, d_{iK_i}$ data streams to its $K_i$ users, then the total number of data streams in the $i$th cell is $d_i = \sum_{k=1}^{K_i} d_{ik}$. Assume that there are no data sharing among the BSs and
every BS has perfect CSIs of all links. This is a scenario of asymmetric MIMO-IBC, and the configuration is denoted as \( \prod_{i=1}^{G} (M_i \times \prod_{k=1}^{K_i} (N_k, d_{ik})) \). When \( M_i = M \), \( K_i = K \), \( N_k = N \) and \( d_{ik} = d \), \( \forall i, k \), the system becomes a symmetric MIMO-IBC denoted as \( (M \times (N, d))^G \).

Because both the IA with and without symbol extension will be addressed, we define two terminologies to be used throughout the paper.

**Definition 1:** Linear IA is the IA without any symbol extension or only with finite spatial extension [1].

**Definition 2:** Asymptotic IA is the IA with infinite time or frequency extension [1].

In this section, we study two necessary conditions of IA feasibility.

### A. Proper Condition

When the channels are generic (i.e., drawn from a continuous probability distribution), the proper condition is one necessary condition for linear IA feasibility, which has been obtained for asymmetric MIMO-IBC in [10]. To find the difference of deriving the necessary conditions between symmetric and asymmetric systems, we first review the proper condition in brief, which is

\[
\sum_{j:(i_k,j)\in I} (M_j - d_j)d_j + \sum_{i_k:(i_k,j)\in I} (N_{i_k} - d_{i_k})d_{i_k} \geq \sum_{(i_k,j)\in I} d_{i_k}d_{j}, \forall \mathcal{I} \subseteq \mathcal{J}
\]  

(1)

where

\[
\mathcal{J} \triangleq \{(i_k,j) | 1 \leq i \neq j \leq G, 1 \leq k \leq K_i \}
\]

(2)

denotes the set of all MS-BS pairs that mutually interfering each other and \( \mathcal{I} \) is an arbitrary subset of \( \mathcal{J} \).

From (2), it is not hard to obtain that the set \( \mathcal{I} \) has

\[
L_{\mathcal{J}} = 2^{L_{\mathcal{J}}} - 1 = 2^{G \sum_{i=1}^{G} \sum_{j=1}^{G} K_i} - 1
\]

(3)

nonempty subsets, where \( |\mathcal{J}| \) is the cardinality of \( \mathcal{J} \). As a result, (1) includes \( L_{\mathcal{J}} \) inequalities. By contrast, the proper condition for symmetric MIMO-IBC includes only one inequality, which is [10]

\[
M + N \geq (GK + 1)d
\]

(4)

and obtained by only considering \( \mathcal{I} = \mathcal{J} \) in [1].

Then, we introduce another terminology.

**Definition 3:** Connection pattern is a graph that represents which BSs and users that mutually interfering each other in an arbitrarily connected MIMO-IBC.

Each subset \( \mathcal{I} \) corresponds to a connection pattern. Take a two-cell MIMO-IBC where \( K_1 = 2 \), \( K_2 = 1 \) as an example, denoted as Ex. 1. Since \( L_{\mathcal{J}} = 2^{K_1 + K_2} - 1 = 7 \), there are seven connection patterns in Ex. 1 as shown in Fig. 1. According to (2), we know that the subset \( \mathcal{I} = \mathcal{J} \) corresponds to a fully connected pattern, i.e., Pattern I, and the subset \( \mathcal{I} \subseteq \mathcal{J} \) corresponds to a partially connected pattern, e.g., each one in Patterns II~VII.

Comparing (1) and (4), we find that for the symmetric MIMO-IBC, it is enough to only consider the fully connected pattern. However, for the asymmetric MIMO-IBC, since all BSs or users have different number of antennas and suffer from different interference, it is necessary to consider all possible connection patterns rather than only consider the fully connected pattern.

### B. Irreducible Condition

In [11], another necessary condition for both linear IA and asymptotic IA feasibility has been found for symmetric MIMO-IBC, which leads to an information theoretic DoF upper-bound. It was called irreducible condition in [10] since it can ensure to eliminate a kind of irreducible inter-cell interference (ICI).

A usual way to derive the information theoretic DoF upper-bound is introducing a genie [1]. It is not easy to find a useful and smart genie to provide the tightest possible upper-bound. The analysis in [11] indicates that when dividing the interference subspace at each BS or user into two linearly independent subspaces, i.e., resolvable subspace and irresolvable subspace, we can construct a wise genie from the irresolvable subspace. For some antenna configurations, there may exist multiple irresolvable subspaces that constitute an irresolvable subspace chain, called subspace chain. Correspondingly, the genies in the subspace chain constitute a genie chain. To derive the DoF upper-bound (equivalently the irreducible condition) for asymmetric MIMO-IBC, we need to first investigate the subspace chain and genie chain. Although the principle of constructing the genie chain for asymmetric MIMO-IBC is similar with that for symmetric MIMO-IBC, the results in [11] cannot be extended in a straightforward manner, which is shown in the forthcoming analysis.

1) **Subspace Chain:** The subspace chain depends on how BSs and users eliminate the interference cooperatively. When BSs first eliminate one part of ICIs and then return the remaining part to users, the exists a subspace chain that starts from the BS side, denoted as Chain A. Meanwhile, when users first eliminate one part of ICIs and then return the remaining to BSs, there also exists another subspace chain that starts from the user side, denoted as Chain B. To derive the whole necessary condition, we need to consider Chains A and B simultaneously.

For easy understanding, we illustrate the subspace chain for
asymmetric MIMO-IBC by a subspace chain of Ex. 1 in Fig. 2

![Diagram](image)

Fig. 2. A subspace chain of Ex. 1.

As shown in the figure, the connection pattern in each step is Pattern IV in Fig. 1, i.e., BS$_2$ and MS$_{1_1}$, MS$_{1_2}$ mutually interfere with each other. For BS$_2$, there are two interfering users and the ICIIs generated by the two users $(N_{1_1} + N_{1_2})$-dimensional interference subspace at BS$_2$. Since BS$_2$ has $M_2$ antennas, only the minimum $(N_{1_1} + N_{1_2}, M_2)$-dimensional subspace of the interference subspace at BS$_2$ is resolvable at BS$_2$ and the remaining $(N_{1_1} + N_{1_2} - M_2)^+$-dimensional subspace is irreversible. If $N_{1_1} + N_{1_2} - M_2 \leq 0$, there does not exist an irreversible subspace so that the subspace chain stops. When $N_{1_1} + N_{1_2} - M_2 > 0$, the irreversible subspace at BS$_2$ is nonempty and the subspace chain continues.

For MS$_{1_1}$ and MS$_{1_2}$, there is only one interfering BS and the irreversible ICIIs occupy $(N_{1_1} + N_{1_2} - M_2)$-dimensional subspace. Since MS$_{1_1}$ and MS$_{1_2}$ have $N_{1_1}$ and $N_{1_2}$ antennas to resolve $N_{1_1}$ and $N_{1_2}$-dimensional subspace, the remaining $(N_{1_1} - M_2)^+$ and $(N_{1_2} - M_2)^+$-dimensional subspaces are irreversible at MS$_{1_1}$ and MS$_{1_2}$, respectively.

We use $S_{\alpha,i}^{(n)}$ or $S_{\alpha,j}^{(n)}$ to denote $S_{\alpha,i}$ or $S_{\alpha,j}$’s irreversible subspace in the $n$th step of Chain $\alpha$, $\alpha = A, B$. For the subspace chain in Fig. 2, the dimension of the irreversible subspace at BS$_2$ in Step 1 is

$$|S_{A,2}^{(1)}| = (N_{1_1} + N_{1_2} - M_2)^+$$  \hspace{1cm} (5)

and the dimensions of the irreversible subspaces at MS$_{1_1}$ and MS$_{1_2}$ in Step 2 are

$$|S_{A,1_2}^{(2)}| = (S_{A,2}^{(1)} - N_{1_1})^+ = (N_{1_2} - M_2)^+$$  \hspace{1cm} (6a)

$$|S_{A,1_2}^{(2)}| = (S_{A,2}^{(1)} - N_{1_2})^+ = (N_{1_1} - M_2)^+$$  \hspace{1cm} (6b)

If the considered connection pattern of the subspace chain in Fig. 2 is Pattern III but not Pattern IV, the dimension of the irreversible subspace at BS$_2$ in Step 1 becomes $S_{A,2}^{(1)} = (N_{1_2} - M_2)^+$. It indicates that the dimension of irreversible subspace also depends on the connection pattern.

To describe all connection patterns in the subspace chain, we introduce the fourth terminology.

**Definition 4:** Connection pattern chain is a chain constituted by the connection patterns in different steps of a subspace chain, denoted as $\mathcal{I}^{(1)} \leftrightarrow \mathcal{I}^{(2)} \leftrightarrow \cdots$, where $\mathcal{I}^{(n)}$ is the $n$th pattern in the chain. If a connection pattern chain satisfies $\mathcal{I}^{(n)} = \mathcal{I}$, $\forall n$, it is equally connected pattern chain (ECPC), otherwise it is unequally connected pattern chain (UCPC). If an ECPC satisfies $\mathcal{I}^{(n)} = \emptyset$, $\forall n$, it is a full ECPC, otherwise it is a partial ECPC.

As shown in Fig. 2, the connection pattern chain is Pattern IV$\leftrightarrow$Pattern IV$\leftrightarrow \cdots$, which is an ECPC.

Since all BSs or users have different number of antennas, the different BS or user nodes have different dimensions of irreversible subspace. We know that in one step, the irrecoverable subspace at some nodes may become empty but that at other nodes is nonempty. Consequently, only the nonempty subspaces appear in the next step of the subspace chain. As a result, the connection pattern for an arbitrary asymmetric MIMO-IBC in the next step is always a subset of that in the current step, i.e.,

$$\mathcal{I}^{(n)} \subseteq \mathcal{I}^{(n+1)}, \forall n \geq 1$$  \hspace{1cm} (7)

From the above analysis, we know that to derive the necessary condition for asymmetric systems, all possible connection pattern chains satisfying (7) need to be taken into account to construct the genie chain. In [11], only the fully ECPU is investigated in deriving the genie chain so that the obtained genie chain is also a special case of the genie chain considered for asymmetric systems. In other word, by introducing a notion of connection pattern chain, we generalize the genie chain in [11] into asymmetric MIMO-IBC.

Following the similar way in [11] and considering the difference mentioned above, for an arbitrary connection pattern chain satisfying (7), we can obtain the dimension of irrecoverable subspace in each step for asymmetric MIMO-IBC. To express the dimension of irreversible subspace in a unified way, we define $|S_{A,ik}^{(n)}| = |S_{B,ik}^{(n-1)}| \leq N_{ik}$, $|S_{A,j}^{(n-1)}| = |S_{B,j}^{(n)}| \leq M_j$. Then, the dimension of irreversible subspace in each step of Chains A and B can be expressed as

$$|S_{A,ik}^{(2n+1)}| = \left|\sum_{i \in I_{2n+1}^{(n+1)}} |S_{A,ik}^{(2n)}| - |S_{A,j}^{(2n-1)}|\right|$$  \hspace{1cm} (8a)

$$|S_{A,ik}^{(2n+2)}| = \left|\sum_{i \in I_{2n+2}^{(n+1)}} |S_{A,ik}^{(2n+1)}| - |S_{A,j}^{(2n)}|\right|$$  \hspace{1cm} (8b)

$$\forall 0 \leq n \leq n_{\text{max}}, \text{ where } I_{2n}^{(n)} = \{i \mid (i,k,j) \in \mathcal{I}^{(n)}\} \text{ is the set of MSs’ index who are connected with } BS_j \text{ in the connection pattern } \mathcal{I}^{(n)} \text{, and } I_{2n+1}^{(n)} = \{j \mid (i_k,j) \in \mathcal{I}^{(n)}\} \text{ is the set of BSs’ index who are connected with } MS_{ik} \text{ in } \mathcal{I}^{(n)}, n_{\text{max}} \text{ reflects the maximal length of subspace chain and satisfies } |S_{A,ik}^{(2n+1)}| = 0, \forall (i_k,j) \in I_{2n+1}^{(n+1)} \text{ or } |S_{A,ik}^{(2n+2)}| = 0, \forall (i_k,j) \in I_{2n+2}^{(n+2)}$$

2) Genie Chain: From the dimension of the irreversible subspace in (8), we can determine the corresponding dimension of the genie and then obtain the irreducible condition.

We use $G_{\alpha,i}^{(n)}$ or $G_{\alpha,j}^{(n)}$ to denote the introduced genie at BS$_j$ or MS$_{ik}$ in the $n$th step of Chain $\alpha$, $\alpha = A, B$. In Step 1, if the irreversible subspace at BS$_2$ $S_{A,2}^{(1)}$ in (5) is nonempty, we can introduce a genie in $S_{A,2}^{(1)}$ (denoted as $G_{A,2}^{(1)}$) to help BS$_2$ resolve all ICIIs in Step 1. Then, we have

$$d_1 + d_2 \leq (M_2 - d_2) + |G_{A,2}^{(1)}|$$  \hspace{1cm} (9)
If $S_{A,1}^{(2)}$ and $S_{A,12}^{(2)}$ are nonempty, we can introduce two genies in $S_{A,1}^{(2)}$ and $S_{A,12}^{(2)}$ (denoted as $G_{A,1}^{(2)}$ and $G_{A,12}^{(2)}$) to help $MS_{11}$ and $MS_{12}$ to resolve the remaining ICIs in Step 2. Then, we have

$$|G_{A,2}^{(2)}| \leq d_{11} + |G_{A,1}^{(2)}|, \quad |G_{A,2}^{(1)}| \leq d_{11} + |G_{A,12}^{(2)}|$$

(10)

Following the same way, we can obtain the dimension of the genie for the general asymmetric MIMO-IBC. To describe the genie’s dimension in a unified way, we define $|G_{A,ik}^{(j)}| \triangleq d_{ik} + |G_{A,ik}^{(j-1)}| = M_j - d_j$, $|G_{B,ik}^{(j)}| \triangleq d_j$, $|G_{A,ik}^{(j-1)}| \triangleq N_{ik} - d_{ik}$. Then, the dimension of the genie in each step satisfies

$$\begin{align*}
\sum_{i \in T_1^{2(n+1)}} |G_{A,ik}^{(2n)}| &\leq |G_{A,j}^{(2n)}| + |G_{A,j}^{(2n+1)}| \\
\sum_{j \in T_2^{2(n+2)}} |G_{A,ik}^{(2n+2)}| &\leq |G_{A,ik}^{(2n)}| + |G_{A,ik}^{(2n+2)}| \\
\sum_{j \in T_2^{2(n+1)}} |G_{B,ik}^{(2n+1)}| &\leq |G_{A,ik}^{(2n+1)}| + |G_{A,ik}^{(2n+1)}| \\
\sum_{i \in T_1^{2(n+2)}} |G_{A,ik}^{(2n+2)}| &\leq |G_{A,j}^{(2n+2)}| + |G_{A,j}^{(2n+2)}| 
\end{align*}$$

(11a)

(11b)

$\forall 0 \leq n \leq n_{\max}$.

Moreover, since the genie in each step lie in its corresponding irresolvable subspace, the dimension of the genie does not exceed that of the irresolvable subspace, i.e.,

$$\begin{align*}
|G_{A,ik}^{(2n)}| &\leq |S_{A,ik}^{(2n)}|, \quad |G_{A,ik}^{(2n)}| \leq |S_{A,ik}^{(2n)}| \\
|G_{B,ik}^{(2n)}| &\leq |S_{B,ik}^{(2n)}|, \quad |G_{B,ik}^{(2n)}| \leq |S_{B,ik}^{(2n)}| 
\end{align*}$$

(12a)

(12b)

$\forall 0 \leq n \leq n_{\max}$.

Combining the inequalities in (11) and (12), we obtain the whole irreducible condition for asymmetric MIMO-IBC.

**Remark 1:** Since $|S_{A,ik}^{(0)}| = N_{ik}$ and $|G_{A,ik}^{(0)}| = d_{ik}$, $|S_{B,ik}^{(0)}| = N_j$ and $|G_{B,ik}^{(0)}| = d_j$, from (12) we have $d_{ik} \leq N_{ik}$ and $d_j \leq M_j$. As a result, the irreducible condition contains the condition that ensuring each BS or MS with enough antennas to convey the desired signals.

**C. Comparison of Two Necessary Conditions**

The irreducible condition is one necessary condition for both linear IA and asymmetric IA feasibility, while the proper condition is necessary for linear IA feasibility but no necessary for asymmetric IA feasibility. Therefore, the information theoretical outer-bound of DoF region can be derived from the irreducible condition, while the outer-bound of DoF region achieved by linear IA needs to be derived from both the irreducible condition and the proper condition.

**Remark 2:** If the configuration of a MIMO-IBC satisfies the proper condition but not the reducible condition, the system is poor but infeasible for linear IA.

Since it is very difficult to obtain a general result for arbitrary configurations of asymmetric MIMO-IBC, in the following we use an example to show how to obtain the information theoretical outer-bound from the irreducible condition. The outer-bound achieved by linear IA can be obtained in a similar way.

Consider Ex.1 again, if the configuration satisfies $|S_{A,2}^{(1)}| > 0$, $|S_{A,1}^{(2)}| = 0$ and $|S_{A,12}^{(2)}| = 0$, i.e., $N_{11} + N_{12} > M_2 \geq max\{N_{11}, N_{12}\}$, from (5) we can introduce a genie to help BS$_2$, but cannot introduce any genie to help MS$_{11}$ and MS$_{12}$. Substituting into (11) and (12), the reducible condition is

$$\begin{align*}
d_1 &\leq N_{11}, d_{11} \leq N_{12}, d_2 \leq N_2 \\
d_{11} + d_{12} + d_2 &\leq M_2 + |G_{A,2}^{(1)}| \\
|G_{A,2}^{(1)}| &\leq d_{11} + |G_{A,2}^{(1)}| \leq d_{12} \\
|G_{A,2}^{(1)}| &\leq |S_{A,2}^{(1)}| = N_1 + N_{12} - M_2
\end{align*}$$

(13)

By solving (13), the outer-bound of DoF region is obtained as

$$\begin{align*}
d_1 &\leq N_{11}, d_{11} \leq N_{12}, d_2 \leq N_2 \\
d_{11} + d_{12} + d_2 &\leq M_2, d_{11} + d_2 \leq M_2 \\
1 &\leq d_{11} + d_{12} + d_2 \leq N_{11} + N_{12}
\end{align*}$$

(14)

In the chain shown in Fig. 2 we only consider the connection pattern chain where $T^{(n)}$ is Pattern IV, $\forall n$. If we consider other connection pattern chain, we can obtain the other outer-bound of DoF region in a similar way.

**III. DoF per User for Two-cell Asymmetric MIMO-IBC**

Because the proper condition and the irreducible condition for general asymmetric MIMO-IBC are too complicated for analysis, we consider a class of special configurations to obtain the closed-form expression of the DoF. We consider that all users have the same number of data streams, i.e., $d_{ik} = d$, $\forall k, i$, then the outer-bound of DoF region can be characterized by the upper-bound of DoF per user. Besides, we consider that all users in one cell have the same number of receive antennas, i.e., $N_{ik} = N_i$, $\forall k$, then the DoF of these users can be analyzed in a unified way.

For such a special configuration, we can derive the closed-form DoF upper-bound per user for two-cell asymmetric MIMO-IBC and prove that it is the upper-bound of DoF per user for G-cell asymmetric MIMO-IBC. Note that such a two-cell asymmetric MIMO-IBC represents a typical heterogeneous network where a macro-cell and a micro-cell interfere each other, so that its DoF results can provide useful insights into interference management in heterogeneous networks.

In the following, we investigate the two-cell MIMO-IBC, denoted as $\prod_{i=1}^2(M_i \times (N_i, d)^K_i)$.

**A. Information Theoretic Maximal DoF**

To understand the potential of the two-cell asymmetric MIMO-IBC, we first investigate the information theoretic maximal DoF per user.

**Theorem 1 (Information Theoretic Maximal DoF):** For a two-cell MIMO-IBC $\prod_{i=1}^2(M_i \times (N_i, d)^K_i)$, the information theoretic maximal DoF per user is

$$\min_{i,j} \left\{ d_{\text{info}}(M_j, N_i, K_j, K_i) \right\}$$

(15)

where

$d_{\text{info}}(M_j, N_i, K_j, K_i) \triangleq \begin{cases} d_{\text{Decom}}(M_j, N_i, K_j), \text{ Region I} \\ d_{\text{Quan}}(M_j, N_i, K_j, K_i), \text{ Region II} \end{cases}$
\[
d^{\text{Decom}}(M_j, N_i, K_j) \triangleq \frac{M_j N_i}{M_j + K_j N_i}
\]  \hspace{1cm} (16)

\[
d^{\text{Quan}}(M_j, N_i, K_j, k) \triangleq \min \left\{ \frac{M_j}{K_j + C^{-1}_n(k)}, \frac{N_i}{1 + C^{-1}_n(k)} \right\}, \forall C_n^A (k) \leq \frac{M_j}{N_i} < C_n^{A-1}(k) \hspace{1cm} (17)
\]

Region I: \( C_n^B(K_j) < \frac{M_j}{N_i} < C_n^A(K_i), \forall K_i \geq 4 \)

Region II: \( \begin{cases} \text{Arbitrary } M_j, N_i, \forall K_i \leq 3 & \text{or } \frac{M_j}{N_i} \leq C_n^B(K_i), \forall K_i \geq 4 \\ \frac{M_j}{N_i} \geq C_n^A(K_i) & \end{cases} \)

For the considered two-cell MIMO-IBC, the DoF bounds in (16) and (17) reduce to the decomposition DoF bound and quantity DoF bound in (11), respectively. Therefore, we also call (16) and (17) as decomposition DoF bound and quantity DoF bound.

Due to the lack of space, we only present the proof for all theorems.

**Proof Skeleton:** We first prove that (15) is the information theoretic DoF upper-bound and then prove that it is achievable.

For the conclusion that the quantity DoF bound is the information theoretic upper-bound in Region II, a rigorous proof was provided in (11) for symmetric MIMO-IBC. Using similar derivations, the quantity DoF bound can be derived from the irreducible condition where the full ECP is considered, which is the information theoretic DoF upper-bound.

For the conclusion that the decomposition DoF bound is the information theoretic upper-bound in Region I, there is no rigorous proof in existing studies. From the analysis in last section, we know that the subspace chain depends on the connection pattern chain. This means that for different connection pattern chains, we can obtain different DoF upper-bounds from the irreducible condition. Moreover, we show that when considering the full ECP, the DoF upper-bound derived from the irreducible condition is only applicable for the antenna configurations in Region II. However, if considering the partial ECPs or UCPs, the derived DoF upper-bound is applicable for the antenna configurations in Region I. With more derivations, we find that for some antenna configurations in Region I, the derived DoF upper-bound is equal to the decomposition DoF bound. It means that the decomposition DoF bound is the DoF upper-bound for these antenna configurations.

To help understand the proof, we illustrate the result by an example in Fig. 3. From the boundary of feasible region, the information theoretic DoF upper-bound is shown. From the boundaries of infeasible regions, the DoF upper-bounds derived from the irreducible condition are shown. When considering the full ECP, the DoF upper-bound is obtained only for the antenna configurations in Region II. However, when considering a given partial ECP, the DoF upper-bound is obtained for antenna configurations in Region I, which is equal to the decomposition DoF bound for some configurations.

![Fig. 3. Feasible and infeasible regions for a two-cell MIMO-IBC](image)

According to the properties of the generalized continued fraction and generalized Fibonacci sequence-pairs in [12], we prove that for an arbitrary antenna configuration in Region I, there always exists a connected pattern chain ensuring that the DoF upper-bound obtained from the reducible condition is equal to the decomposition DoF bound. As a result, this proves that the decomposition DoF bound is the information theoretic DoF upper-bound in Region I.

In the following, we prove the DoF upper-bound is achievable. Following similar derivations in (11), we can show that for the antennas configuration when both \( M_1/N_2 \) and \( M_2/N_1 \) fall in Region II, the DoF upper-bound can be achieved by linear IA and the closed-form solution of linear IA exists. Obviously, the DoF can also be achieved by asymmetric IA but the infinite extension is not necessary. For other configurations, the DoF upper-bound can only be achieved by asymptotic IA but not by linear IA.

**Remark 3:** For the considered two-cell MIMO-IBC, the information theoretic DoF upper-bound for arbitrary configurations can be obtained from the irreducible condition and can always be achieved by asymmetric IA. Therefore, the irreducible condition is the sufficient and necessary condition for asymptotic IA feasibility.

In (11), the irreducible condition is obtained as the sufficient and necessary condition of asymptotic IA feasibility for the antenna configurations in Region II and what is the condition for the antenna configurations in Region I is still unknown. In this study, we prove that the irreducible condition obtained from the generalized genie chain is the sufficient and necessary condition of asymptotic IA feasibility for arbitrary antenna configurations.

**B. Maximal Achievable DoF of Linear IA**

Considering that asymptotic IA requires infinite time/frequency extension, which is not feasible for practical
systems, this motivates us to find the maximal DoF per user achieved by linear IA.

**Theorem 2 (Maximal DoF achieved by Linear IA):** For a two-cell MIMO-IBC \( \prod_{i=1}^{2} (M_i \times (N_i, d) K_i) \), the maximal DoF per user achieved by linear IA is

\[
\min_{i \neq j} \left\{ d^{\text{prop}}(M_j, N_j, K_j, K_i), d^{\text{Info}}(M_j, N_j, K_j, K_i) \right\} \quad (18)
\]

where

\[
d^{\text{prop}}(M_j, N_j, K_j, K_i) = \frac{K_j M_j + K_i N_i}{K_j^2 + K_j K_i + K_i} \quad (19)
\]

Proof Skeleton: We first prove that \( (18) \) is the DoF upper-bound and then prove that it is achievable.

After some tedious but regular manipulations, we can show that in the considered two-cell MIMO-IBC, the proper condition in \( (1) \) becomes

\[(M_j - K_j d) K_j d + (N_i - d) K_i d \geq K_j K_i d^2 \quad \forall i \neq j \quad (20)\]

From (20), we can obtain \( d \leq d^{\text{Info}}(M_j, N_j, K_j, K_i) \) in (19) directly.

Since \( (19) \) is one DoF upper-bound obtained from the proper condition, it is called *proper DoF bound*. Moreover, \( d^{\text{Info}}(M_j, N_j, K_j, K_i) \) is another DoF upper-bound obtained from the irreducible condition, which is proved in Theorem (11). Consequently, \( (18) \) is the upper-bound of the DoF per user achieved by linear IA.

Following the similar analysis in (11), we can show that if the proper DoF bound is not higher than the information theoretic maximal DoF, the DoF can be achieved by a closed-form linear IA, otherwise, there exist at least one feasible solution for linear IA following a similar proof as in (10).

**Remark 4:** In the considered two-cell MIMO-IBC, the maximal DoF per user achieved by linear IA is obtained from the proper condition and the irreducible condition simultaneously. Therefore, the combination of the proper condition and the irreducible condition is the sufficient and necessary condition for linear IA feasibility.

**Remark 5:** When the proper DoF bound is achievable, (19) implies that to achieve the desired number of data streams, if one BS increases or reduces the transmit antennas, the MSs in another cell can reduce or should increase the receive antennas. Specifically, let \( L = K_j / K_i \) where \( K_j > K_i \). If the number of transmit antennas at BSs is \( M_j \pm \Delta \), the number of receive antenna at MSs in cell i should be \( N_i \pm L \Delta \). As a result, when a cell that supports more users (or equivalently that needs to transmit more data streams) has redundant antennas to help eliminate ICI, the overall antenna resource in the network can be reduced. This suggests that a heterogeneous network with different downlink traffic and antenna resources among multiple cells is more spectrally efficient than a homogeneous network, if we allow the more powerful cell (such as a macro-cell whose BS is equipped with more antennas) to help remove the ICI for the resource-limited cell (such as a micro-cell).

**Remark 6:** Since a two-cell MIMO-IBC can be treated as a partially connected case of a \( G \)-cell MIMO-IBC, the necessary condition for the two-cell MIMO-IBC must be the necessary condition for the \( G \)-cell MIMO-IBC. Consequently, the DoF per user upper-bounds in Theorems (1) and (2) are the DoF per user upper-bounds for \( G \)-cell MIMO-IBC \( \prod_{i=1}^{G} (M_i \times (N_i, d) K_i) \) with asymptotic and linear IA, respectively.

**IV. Conclusion**

In this paper, we analyzed the DoF of the asymmetric MIMO-IBC. By generalizing the genie chain considered in (11), we found that irreducible condition is necessary for both linear IA and asymptotic IA feasibility. From the irreducible condition, we derived the information theoretic DoF outer-bound for arbitrary \( G \)-cell MIMO-IBC and the information theoretic maximal DoF per user for a special class of two-cell MIMO-IBC with the antenna configurations in both Regions I and II. By contrast, the reducible condition derived in (11) only leads to the information theoretic maximal DoF for symmetric MIMO-IBC with the antenna configurations in Region II. By combining the proper condition and irreducible condition, we obtained the DoF outer-bound for arbitrary MIMO-IBC achieved by linear IA and the maximal achievable DoF per user for the special two-cell MIMO-IBC. By comparing the information theoretic maximal DoF and the maximal DoF achieved by linear IA, we showed when the linear IA can achieve the information theoretic maximal DoF.

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