CELL MODULES FOR $A_n$ WEBS

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ABSTRACT. We examine the cell modules for the category of type $A_n$ webs and their natural cellular forms. We modify the bases of these modules, due to Elias, to form an orthogonal basis of each cell module. Hence, we calculate the determinant of the Gram matrix in these bases.

These Gram determinants are given in terms of certain traces of clasps — higher order Jones-Wenzl morphisms. Additionally, the modified basis is constructed using these clasps, and each clasp is constructed using traces of smaller clasps.

There is a conjectured value for these traces, or “intersection forms” given by Elias and this paper concludes with a proof of the conjecture in type $A_n$. Thus the results on the Gram matrices are exact.

1. INTRODUCTION

The category of (tensor products of) fundamental representations of $U_q(sl_n)$ is a monoidal, cellular, diagrammatic category. It is cellular because it is an OACC [EMTW20] or, more simply, that each algebra $\text{End}(X)$ is cellular in the sense of Graham and Lehrer [GL96]. We say it is diagrammatic, due to the pleasing presentation by planar diagram generators and relations [CKM14].

Cellular categories are equipped with cell modules. These can be thought of as the cell modules of each of the cellular endomorphism algebras $\text{End}(X)$. If the category is semi-simple, these cell modules are indecomposable. If it is not, they provide us with a complete list of indecomposable objects as their heads.

To be exact, each module inherits a bilinear pairing from the category structure and the quotients of the cell modules by this radical (where the quotient is not zero) are a complete set of indecomposable modules.

These categories also admit an integral lattice, and through this, their modular representation theory can be studied. For example, we may take the indecomposable cell modules over characteristic zero\footnote{In fact, the characteristic should be a “semisimple mixed characteristic”} and study their decompositions in positive characteristic. This decomposition is controlled, to the first order, by the bilinear form on the cell modules. For example, it gives a Jantzen or Scharper filtration.

Since the integral lattice forms a basis of these modules, even under specialisation, we can study certain properties of the bilinear form over semisimple characteristic in order to deduce properties over positive characteristic. One such property is the determinant of the form, which this paper investigates.

In a semisimple cellular category, each object with a weight has an associated idempotent known as a clasp. Indeed, if $X$ is an object of weight $\lambda$, there is a morphism
$JW_X \in \text{End } X$ which is idempotent and which is killed by left- or right-composition by any morphism factoring through an object of weight less than $\lambda$.

In this paper we will frequently refer to “simple” examples as both inspiration and touchstones. The simplest example will be that of the Temperley-Lieb category, $\mathcal{T}L$. In this category (which manifests when studying $U_q(sl_2)$) the clasps are known as Jones-Wenzl elements. They all exist, which is to say the category is semisimple iff $q$ is not a root of unity.

The diagrammatic category equivalent to the category of fundamental representations of $U_q(sl_n)$ is known as the category of webs or spiders. The concepts were introduced by Kuperberg for the rank 2 Cartan types [Kup96]. Type $A_n$ webs were developed by Cautis, Kamnitzer and Morrison using a variation on Howe duality [CKM14]. Thereafter Bodish et. al. extended the construction to type $C_n$ [BERT21].

These presentations are in terms of generators and relations — a quotient of a certain category by a tensor ideal. This doesn’t describe the category completely though, and the problem of determining a basis for the morphism spaces was started by Elias (in type $A_n$) [Eli15], and continued by Bodish for type $C_2$ [Bod20] and Bodish and Wu in type $G_2$ [BW21].

Due to a very similar construction, it turns out that calculating clasps in the web categories can be done nearly concurrently to finding bases. Indeed, we now have constructions for clasps in type $A_n$ [Eli15], $C_2$ [Bod21] and $G_2$ [BW21].

The familiar construction of Jones-Wenzl idempotents forms an important archetype for these higher order clasps. Indeed, the Jones-Wenzl idempotents satisfy

\begin{equation}
\begin{aligned}
\begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
\end{tikzpicture}
\quad = \quad \begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
  \end{tikzpicture}
\quad \begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
  \end{tikzpicture}
\quad - \quad \frac{[n]}{[n+1]} \begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
  \end{tikzpicture}\quad \begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
  \end{tikzpicture}.\end{aligned}
\end{equation}

Two features of this construction generalise. Firstly, clasps are inductively defined in terms of smaller clasps. This is sometimes known as the “triple-clasp formula” (each of the two morphisms in the right hand side of eq. (1.1) is hiding a further $JW_{n-1}$) and comes from plethysm rules.

Secondly, the factor $[n]/[n+1]$ comes from a tracing rule:

\begin{equation}
\begin{aligned}
\begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
\end{tikzpicture}
\quad = \quad \frac{[n+1]}{[n]} \begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,1);
  \draw (0.5,0) -- (0.5,1);
  \draw (0,0.5) -- (1,0.5);
  \draw (1-\pgflinewidth,0.5) -- (1+\pgflinewidth,0.5);
  \draw (0,0) -- (1,1);
  \draw (0,1) -- (1,0);
  \draw (0.5,0.5) arc (90:180:0.25);
\end{tikzpicture}.
\end{aligned}
\end{equation}

Indeed, the is the reciprocal of the coefficient of the identity diagram. In the more general case, this trace is called an intersection form and calculating their numerical values can be difficult. Though these have been calculated for types $A_n$ for $n \leq 4$, $C_2$ and $G_2$, in general we must revert to a conjecture due to Elias for these constants.

In this paper we elucidate the connection between these intersection forms and the Gram determinants of cell modules in the category. In particular we give expressions for all the cell Gram determinants in terms of these intersection forms. We then go on to show that Elias’ conjecture holds in type $A_n$, by finding pre-images of two distinguished bases across the quantum skew Howe duality.
This paper concerns type $A_n$ webs, but should be read with a more general view in mind. Some results, such as Lemma 3.1 hold in more general “object adapted cellular categories” [Spe22]. These include web categories for other Coxeter types, and diagrammatic Soergel bimodule categories. If such a category has a monoidal structure which interfaces with the category structure suitably, then further constructions (such as those of the ladder basis and triple-clasps) should generalise. However, a discussion of monoidal, cellular categories is outside the scope of this paper.

The remainder of this paper is laid out as follows. In section 2 we review the definition the web category and its intersection forms, and introduce Elias’ conjecture for these forms. Then, in section 3 we examine the cell module of this category and recount the ladder basis. In section 4, new bases are found which give recurrant solutions to the determinant of the cell module. We also determine an orthogonal basis. Finally, section 5 shows that a basis and triple-clasps) should generalise. However, a discussion of monoidal, cellular categories is outside the scope of this paper.

## 2. The Category of Type $A_{n-1}$ Webs

Let us recall the definition of type-$A$ webs. This work was initiated by Kuperberg [Kup96] who derived the diagrammatic form for rank two Lie type, and completed by Cautis, Kamnitzer and Morrison who gave diagrammatics for type $A_n$ [CKM14].

The aim is to study the representation theory of $U_q(sl_n)$ over $\mathbb{Q}(q)$. To do this, it will be sufficient to study the category of (tensor products of) fundamental representations of $U_q(sl_n)$, denoted $\text{Fund}_n$. This category encodes the full category of finite dimensional representations through its Karoubian envelope, and the reader is directed to [Eli15, §3] for a full description of the appropriate idempotents, known as clasps.

**Remark 2.1.** Throughout, the reader is encouraged to ignore the quantum deformation on a first reading. The appropriate mental notational substitutions are

$$U_q(sl_n) \mapsto sl_n, \quad q \mapsto 1, \quad [n] \mapsto n.$$ 

Though often results that hold for undeformed $sl_n$ hold for $U_q(sl_n)$ with appropriate notation (see for example the definition of a Jones-Wenzl idempotent using quantum numbers), we will be careful to keep our results correct for the general case.

Recall that $U_q(sl_n)$ has $n-1$ fundamental representations, $\{V_i := \wedge^i \mathbb{C}^n\}_{i=1}^{n-1}$. Here $\mathbb{C}^n$ is given the standard action of $U_q(sl_n)$. In this way we can also consider $V_0 \cong V_n \cong \mathbb{C}$. If $i + j = k$ then there are natural $U_q(sl_n)$-maps $V_i \otimes V_j \to V_k$ and $V_k \to V_i \otimes V_j$. Note that these maps are not inverse.

The objects in the category $\text{Fund}_n$ are of the form $V_i \otimes V_j \otimes \cdots \otimes V_k$ and the morphisms are generated (through composition and tensor product) by the aforementioned maps between $V_i \otimes V_j$ and $V_k$. If $i = i_1 i_2 \cdots i_k$ is a sequence of elements of $\{1, \ldots, n-1\}$ we will write $V_i$ for $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k}$.

**Remark 2.2.** Strictly, we should define the objects to also permit tensor factors of the duals of the $V_i$, but since $V_i^* \cong V_{n-i}$ what we have constructed is sufficient. In [Eli15] our category is denoted by $\text{Fund}^+$.

The composition of the map from $V_k$ to $V_i \otimes V_j$ and the map from $V_i \otimes V_j$ to $V_k$ again is a multiple of the identity map on $V_k$. Indeed, let $[n] = (q^n - q^{(n-1)})/(q - q^{-1})$ be the $n$-th
quantum number. Then if $[n]! = [n][n−1] \cdots [1]$ and $[n]_r! = [n]!/(r![n−r]!)$, the multiple is $[^i_r] = [^i_1]$.

Remark 2.3. Our analysis will actually hold over any pointed ring wherein $q$ is not a root of
unity. In these characteristics, all the quantum numbers are invertible and so the necessary
objects will be well defined. When $q$ is specialised to a root of unity, certain quantum
numbers and quantum binomial coefficients will vanish, and the required clasps will not
be well defined. See [MS21] for the correct idempotents in the $A_1$ case.

To provide an easier-to-work with category, we introduce the category of webs or
spiders. This category denoted $\mathbf{Sp}_n$ is diagrammatic and is equivalent as a category to
$\mathbf{Fund}_n$. Thus we can think of these interchangeably.

The objects are taken from the set of finite words in letters $\{1, \ldots, n−1\}$ and the object
$i$ in $\mathbf{Sp}_n$ corresponds to the object $V_i$ in $\mathbf{Fund}_n$. As such, tensor product on words is simply
concatenation.

Morphisms are linear combinations of diagrams$^2$. A diagram is an oriented, decorated,
planar graph with edges labeled by the letters $\{1, \ldots, n−1\}$. These graphs are taken up to
isotopy. Each nontrivial vertex is trivalent and if, when read left-to-right, one edge splits
into two, the sum of the outgoing labels equal that of the incoming. A similar result holds
for the reverse.

For example, the two classes of generating morphisms are represented as

\begin{equation}
\begin{array}{c}
\includegraphics{diag1.png} \\
\end{array}
\end{equation}

We take the precaution of removing all edges that are labeled with $n$ or 0. Indeed, since
tensoring with $\mathbf{C}$ is a null operation, we lose no information on the object level. On the
morphism level, vertices with an $n$ or 0 are turned into bivalent vertices (simply edges).

Example 2.1. When $n = 2$ so we are examining the representation theory of $U_q(\mathfrak{sl}_2)$, the
well known category to consider is the Temperley-Lieb category. Here, the object set is simply
$\{1\}^* \simeq \{\bullet\}^* \simeq \mathbb{N}$.

Since $n = 2$, there is only a single label for edges and the only allowed vertex is bivalant.
However, the statement that graphs are taken up to isotopy means that we recover the usual
definition of the Temperley-Lieb category as planar matchings of points.

Example 2.2. When considering $U_q(\mathfrak{sl}_3)$ there are two objects which are dual to each other. We
will often consider the set $\{+, -\}^*$ instead of $\{1, 2\}^*$. Instead of labeling the edges with $\{1, 2\}$
we will orient them with the understanding that bivalent vertices have one incident edge of each
orientation (and hence can be taken up to isotopy or ignored) and trivalent vertices are all sources
or sinks.

$^2$Technically we are working with the ladder incarnation of diagrams from [CKM14], however no generality
is lost.
An example diagram from \((++--)\) to \((+++)\) might be

\[
\begin{array}{c}
+ \\
- \\
+ \\
- \\
+ \\
\end{array}
\]

There are relations on the morphisms in this category, as suggested by the comment after remark 2.3 but their exact forms are not important for our study. They can be found in [CKM14].

2.1. Weights. Each fundamental representation \(V_i\) has highest weight a fundamental dominant weight \(\varpi_i\). Viewed as a partition, \(\varpi_i = (x^i, 0^{n-i})\). For any object \(x\) in \(\text{Fund}\), let \(\text{wt}(x) = \sum \varpi_i x_i\). If this is viewed as a \(n+1\)-part partition \((\lambda_1, \cdots, \lambda_n, 0)\) of \(\sum x_i\) then \(x\) contains \(\lambda_i - \lambda_{i+1}\) copies of \(i\). The set of all weights, denoted \(\Lambda\), is equipped with the dominance partial order.

2.2. Compatible Families. Let \(\lambda \in \Lambda\) and pick \(\varphi^\lambda = \{\varphi_{z,y} : x \rightarrow y\}\) be a set of morphisms between objects of weight \(\lambda\). Let \(J_{<\lambda}\) be the ideal spanned by morphisms that factor through objects of weight less than \(\lambda\). We say that \(\varphi^\lambda\) is a compatible family if

- for each pair of objects, \(x\) and \(y\), of weight \(\lambda\), there is a unique \(\varphi_{x,y} : x \rightarrow y\) in \(\varphi^\lambda\),
- \(\varphi_{z,y} \circ \varphi_{x,z} = \varphi_{z,y}\) modulo \(J_{<\lambda}\) for any three composable elements of \(\varphi^\lambda\), and
- \(\varphi_{z,z} = \text{id}_z\) modulo \(J_{<\lambda}\).

The two examples of compatible families we will meet are clasps and neutral ladders.

**Lemma 2.3.** There is a compatible family of morphisms such that \(\varphi_{z,z} = \text{id}_z\) exactly.

These morphisms are known as (families of) neutral ladders.

**Proof.** We present the construction here and signpost the reader to the proof of the claims that it suffices.

Set \(\varphi_{z,z} = \text{id}_z\). Now, suppose \(x\) and \(y\) differ in only two adjacent values, which is to say

\[
x = x^{(0)} \cdots x^{(i-1)} x^{(i)} x^{(i+1)} x^{(i+2)} \cdots x^{(k)}
\]

and

\[
y = x^{(0)} \cdots x^{(i-1)} x^{(i+1)} x^{(i+2)} \cdots x^{(k)}
\]

where without loss \(x^{(i+1)} > x^{(i)}\) (indeed, if the reverse holds, simply apply \(\iota\) to the following construction). Then we construct morphism \(\varphi_{z,y}\), which we term a simple
morphism, as

\[ \varphi_{x,y} = \]

```
+----------------+-----------------+
|                |                 |
|                |                 |
|                |                 |
|                |                 |
| +----------------+-----------------+
```

where the middle rung is labeled \( x^{(i+1)} - x^{(i)} \). Then construct the family by composing these maps. These are known as neutral ladders [Eli16, §2.3]. Though there might be multiple ways to compose simple morphisms to obtain a morphism between two given objects, these are all equal modulo the ideal [Eli16, Corollary 2.24].

Remark 2.4. The objects of weight \( \lambda \vdash k \) are in bijection with the elements of \( \mathfrak{S}_k / \mathfrak{S}_\lambda \) where \( \mathfrak{S}_\lambda \) is the Young subgroup of shape \( \lambda \). In this case, the simple morphisms correspond to the (images of) simple reflections in \( \mathfrak{S}_k / \mathfrak{S}_\lambda \), and by picking reduced words we obtain that the neutral ladders can be given by diagrams.

When used in an equation, we will denote neutral ladders by grey boxes with a 0. They are predominantly used to move between objects of the same weight in our category.

Example 2.4. When \( n = 2 \), the weights and objects can both be identified with \( \mathbb{N} \). The dominance ordering is generated by \( n \succ n - 2 \) and so odd numbers cannot be compared to even numbers. Since there are unique objects of a given weight, all neutral ladders are identities.

Example 2.5. When \( n = 3 \), objects are simply elements of \( \{ +, - \}^* \) and neutral ladders are compositions of transpositions:

```
\[
\ell \\
\vdots \\
\]
```

(2.2)

\[ s_i = \]

```
+----------------+
|                 |
| \( i + 1 \)    |
| \( i \)       |
| \( i + 1 \)    |
|                 |
+----------------+
```

The relations on nets ensure that the braid relations are held by the \( s_i \), up to terms morphism factoring through objects of lower weight.

The second critical family is that of clasps.

Theorem 2.6. If \( x \) and \( y \) are both of weight \( w \), then there is a unique nonzero morphism \( \text{JW}_{x,y} : x \to y \) which is killed by all left (or right) morphisms that factor through objects of weight less than \( w \). Further, the coefficient of the diagram given by \( \varphi_{x,y} \) in \( \text{JW}_{x,y} \) is 1.
When the objects \( x \) and \( y \) are clear or do not matter, we will simply index them by the weight, \( JW_w \). Note that \( JW_{\lambda,\lambda} \) is an idempotent for each \( \lambda \).

**Example 2.7.** When \( n = 2 \) these are the celebrated Jones-Wenzl idempotents.

**Lemma 2.8.** If \( f : x \to y \), then

\[
(2.3) \quad JW_{y',y} \circ f \circ JW_{\lambda',\lambda} = \begin{cases} 
(\text{coefficient of } q_{\lambda,\lambda} \text{ in } f) \times JW_{y',\lambda} & \text{wt}(\lambda) = \text{wt}(y) \\
0 & \text{otherwise}
\end{cases}
\]

for any \( y' \) of the same weight as \( y \) and \( \lambda' \) of the same weight as \( \lambda \).

**Corollary 2.9.** Any morphism factoring through both a clasp of weight \( \mu \) and an object of weight \( \lambda \) vanishes unless \( \mu = \lambda \), in which case it is a multiple of a clasp.

Finally, we will require the elementary light ladders. If \( \omega_i \) is a fundamental weight, then there are \( \binom{n}{i} \) weights appearing in the fundamental representation \( V_i \). To each of these weights \( \mu \), we associate the elementary light ladder \( E_{\mu} \). This is a set of diagram morphisms from an object of weight \( \lambda + \omega_i \) to \( \lambda + \mu \), whenever \( \lambda + \mu \) is dominant. If it is not dominant, the light ladder is not defined for that \( \lambda \). The precise definition appears in [Eli15, §2.5].

**Remark 2.5.** Though we do not need to deal explicitly with the elementary light ladders in this paper, we will note later that they are the images of the monomial basis under the skew quantum Howe duality at play.

**2.3. Computation of intersection forms.** The computation of the clasps for \( A_n \) webs hinge on the calculation of so-called intersection forms.

**Definition 2.10.** If \( \lambda \in \Lambda \) and \( a \) a fundamental dominant weight, let \( \Omega(a) \) be the set of weights appearing in the representation \( V_a \). Then the intersection form \( \kappa_{\lambda,\mu} \) for \( \mu \in \Omega(a) \) is the coefficient of the identity diagram in

\[
(2.4) \quad \lambda
\]

Note this is a morphism from an object of weight \( \lambda + \mu \) to itself and so the intersection form is only defined when \( \lambda + \mu \) is dominant.

In [Eli15] these are denoted \( \kappa_{\lambda,\mu} \).

**Remark 2.6.** Note that the intersection form \( \kappa_{\lambda,\mu} \) requires us to know the clasp on \( \lambda \) to calculate. However, in [Eli15], the clasp on \( \lambda \) is given in terms of intersection forms, much like the form found for generalised Jones-Wenzl elements in [MS21].

This is not circular logic, however, but part of the inductive definition of the clasp elements. Indeed, to define the clasp on \( \lambda \) we need only know all intersection forms \( \kappa_{\lambda,\mu} \) for \( \mu \in \Omega(a) \).
There is a conjectured form of these intersection forms in [Eli15, Conjecture 3.16]. This conjecture is known to hold for \( n \leq 4 \). In section 5 we will prove this conjecture for all \( n \).

**Conjecture 1** (Elias). Let \( \lambda \) be a dominant weight, \( a \) a fundamental weight and \( \mu \in \Omega(a) \). Then if \( \lambda + \mu \) is dominant,

\[
\kappa_{\lambda}^{\lambda+\mu} = \prod_{\alpha \in \Phi(\mu)} \frac{[\langle \lambda + \rho, \alpha \rangle]}{[\langle \lambda + \rho, \alpha \rangle - 1]}
\]

where \( \Phi(\mu) = \{ \alpha \in \Phi^+ : w_{\mu}^{-1}(\alpha) \in \Phi^- \} \) and \( w_{\mu} \) is the minimal element of \( S_n \) taking \( \mu \) to a dominant weight.

To effectively compute these conjectured intersection forms it will help to introduce some notation. Let \( \omega_i \) denote the \( i \)-th fundamental weight so that \( V_i \) has highest weight \( \omega_i \). We identify \( \omega_i \) with the \( n \)-part partition (composition) of \( n \) given by \( (1^0(0^{n-i})) \). For brevity in equations, we may denote this similarly to the below, for example.

\[
\omega_4 = \begin{array}{c} \blacksquare \\
\end{array}
\]

Note that the highest weight of \( V_{i_1} \otimes \cdots \otimes V_{i_k} \) is \( \omega_{i_1} + \cdots + \omega_{i_k} \), which is a partition of \( \sum_i i_s \) into \( n \) parts \(^3\).

In this notation and basis, the roots of type \( A_n \) are of the form \( \alpha_{i,j} = -\omega_{i-1} + \omega_i + \omega_{j-1} - \omega_j \) for distinct \( i \) and \( j \) or, more succinctly,

\[
\alpha_{i,j} = \begin{cases} 
\text{\downarrow} & i < j \\
\text{\uparrow} & i > j 
\end{cases}
\]

where there is a positive box in the \( i \)-th row and a negative in the \( j \)-th. We use red to visually distinguish roots from weights. The inner product is the usual inner product on \( n \)-part tuples.

Now, given a weight (composition in \( n \) parts), \( \mu \), note that it consists of a sequence of 1’s and 0’s. It is not hard to see then that \( \Phi(\mu) \) contains all \( \alpha_{i,j} \) where \( i < j \) and \( \langle \mu_i, \mu_j \rangle = (0, 1) \).

**Example 2.11.** We calculate some of the sets \( \Phi \) for strategically chosen weights to demonstrate this calculation. Here, \( n = 5 \) so all compositions are in 5 parts.

| \( \Phi([\text{\downarrow}] ) \) | \( \Phi([\text{\uparrow}] ) \) | \( \Phi([\text{\blacksquare}] ) \) |
|----------------|----------------|----------------|
| \{ \} | \{ \} | \{ \} |
| \{ \text{\downarrow} \} | \{ \text{\uparrow} \} | \{ \text{\blacksquare} \} |
| \{ \text{\downarrow}, \text{\uparrow} \} | \{ \text{\downarrow}, \text{\uparrow} \} | \{ \text{\downarrow}, \text{\uparrow} \} |
| \{ \text{\downarrow}, \text{\uparrow}, \text{\downarrow} \} | \{ \text{\downarrow}, \text{\uparrow}, \text{\downarrow} \} | \{ \text{\downarrow}, \text{\uparrow}, \text{\downarrow} \} |

It is immediate that if \( \mu \) is dominant (i.e. a partition) the set is empty and so \( \kappa_{\lambda}^{\lambda+\mu} = 1 \). This corresponds to adding a single strand labeled \( \mu \) in the construction of the clasp and so it is expected that the intersection form is trivial.

Now \( \langle \lambda + \rho, \alpha_{i,j} \rangle = \lambda_i - \lambda_j + \rho_i - \rho_j = \lambda_i - \lambda_j - i + j \).

\(^3\)However, the weight of \( i_1 \cdots i_s \) described in section 2.1 is slightly different.
Definition 2.12. The axial distance between $i$ and $j$ in partition $\lambda$ is
$$c_{ij}^\lambda = \lambda_i - \lambda_j + j - i.$$ 

These axial distances occur frequently in the literature of the representation theory of $\text{GL}_n$ and related algebras. They can be interpreted as the differences between contents in the last boxes of row $i$ and $j$, where the contents of a box with coordinates $(x, y)$ is $x - y$. Axial distances are also intimately related to the ratio of (quantum) hook lengths [Kos03]. Note that, for example, $c_{ij}^\lambda + c_{jk}^\lambda = c_{ik}^\lambda$ and that $c_{ij}^\lambda > 0$ if $j > i$.

Example 2.13. We continue our computation of strategically chosen constants and present some conjectured intersection forms.

The easiest to exhibit are those with empty $\Phi$ sets:
$$\kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = 1$$

The rest is simply a matter of working through the calculations.

$$\kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \frac{[c_{34}]}{[c_{34} - 1]} \frac{[c_{24}]}{[c_{24} - 1]} = \frac{[2]}{[1]} \frac{[3]}{[2]} = [3]$$

$$\kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \frac{[c_{34}]}{[c_{34} - 1]} = [2]$$

Note that these intersection forms evaluate to a quantum integer, that is, a polynomial instead of a rational function. However, this is not always the case:

$$\kappa \begin{array}{c} \begin{array}{c} \end{array} \end{array} = \frac{[c_{23}]}{[c_{23} - 1]} \frac{[c_{24}]}{[c_{24} - 1]} = \frac{[2]}{[1]} \frac{[4]}{[3]} = [2][4][3]$$

We finish this example with five further intersection forms that will be useful in later calculations.
\[
\begin{align*}
\kappa_{\lambda^1} &= \frac{[c_{12}]}{[c_{12} - 1]} \frac{[c_{13}]}{[c_{13} - 1]} = \frac{[2][5]}{[4]} \\
\kappa_{\lambda^2} &= \frac{[c_{12}]}{[c_{12} - 1]} \frac{[c_{14}]}{[c_{14} - 1]} \frac{[c_{34}]}{[c_{34} - 1]} = \frac{[3][7]}{[6]} \\
\kappa_{\lambda^3} &= \frac{[c_{12}]}{[c_{12} - 1]} = [2] \\
\kappa_{\lambda^4} &= \frac{[c_{23}]}{[c_{23} - 1]} \frac{[c_{13}]}{[c_{13} - 1]} = \frac{[4]}{[2]} \\
\kappa_{\lambda^5} &= \frac{[c_{34}]}{[c_{34} - 1]} \frac{[c_{24}]}{[c_{24} - 1]} \frac{[c_{14}]}{[c_{14} - 1]} = \frac{[3][6]}{[2][4]}
\end{align*}
\]

We can use this experience to rephrase conjecture 1:

**Conjecture 2** (Elias). Let \( \lambda' \subset \lambda \) be dominant weights, such that \( \lambda \setminus \lambda' \) has no parts larger than 1. Then

\[
\kappa_{\lambda'}^{\lambda} = \prod_{i<j \text{ new box in row } i \text{ new box in row } j} \frac{[c_{ij}^\lambda]}{[c_{ij}^{\lambda'} - 1]}.
\]

3. **CELL MODULES**

For \( x \succ y \), let the cell module \( S(x, y) \) be the set of all morphisms \( x \to y \) modulo those that factor through objects of weight less than \( y \).

**Lemma 3.1.** If \( \text{wt}(y_1) = \text{wt}(y_2) \) then \( S(x, y_1) \simeq S(x, y_2) \) as \( \text{Fund} \)-modules.

**Proof.** There is an isomorphism consisting of composing on the right by neutral ladders (or clasps). Then lemma 2.3 part (ii) gives the result. \( \square \)

For this reason, we may refer to \( S(x, \lambda) \) where \( \lambda \) is a weight. Here, one can pick whichever object of weight \( \lambda \) one prefers as the target for the morphisms in \( S(x, \lambda) \).

We recall that \( \text{Fund} \) is equipped with an anti-involution \( \iota \). In particular, this endows the cell modules with canonical forms \( \langle - , - \rangle \) given by

\[
(\langle m_2 \rangle \circ m_1 = \langle m_1, m_2 \rangle \text{id}_y) \quad m_1, m_2 \in S(x, y)
\]
modulo morphisms factoring through lower weights than \( w(y) \). In the language of clasps, we can write this as

\[
(3.2) \quad JW \circ (m_2) \circ m_1 \circ JW = (m_1, m_2) JW.
\]

3.1. A basis. We recall the construction of a basis for the cell module set out in [Eli15]. This basis is built inductively on the length of the source object.

Suppose we wish to build a basis for cell module \( S(x, y) \) and we can write \( x = \hat{x}x_0 \) where \( x_0 \in \{1, \ldots, n−1\} \). We build this out of basis elements for \( S(\hat{x}, z) \) for various \( z \).

To be exact, let \( \mu \in \Omega(x_0) \) and pick an object \( z_\mu \) of weight \( w(y) − \mu \). Suppose that \( L_t \) is a basis element for \( S(\hat{x}, z_\mu) \). We construct basis element \( L_t \) for \( S(x, y) \) diagrammatically as follows:

\[
(3.3)
\]

Recall that grey boxes denote neutral ladders. In this case they are being used to ensure that the output of \( L_t \), i.e. \( z_\mu \) is post-composable by \( E_\mu \) in the manner shown above, and also to obtain the correct object as the target. Then the set of morphisms \( L_{ta} \), over all the \( \mu \) and \( L_t \) for each \( \mu \), form a basis for \( S(x, y) \).

Example 3.2. If \( n = 2 \) so we are in the Temperley-Lieb case, the only option for \( a \) is \( \bullet \). The two maps \( E_\mu \) corresponding to this weight are

\[
E_+ = \quad \quad E_- = .
\]

There are no neutral ladders. The basis described then is

\[
\left\{ L_t, L_{ta} \right\}
\]

which is exactly the usual diagram basis for the Temperley-Lieb cell modules.

Now for \( \mu \in \Omega(a) \), the morphism \( E_\mu \) increases the weight by \( \mu − a \). Thus in each step we increase the weight by \( a \) (by adding a strand labeled \( a \)) and then again by \( \mu − a \) through an application of \( E_\mu \).

We thus obtain a labeling of the basis by \emph{miniscule Littleman paths}, \( E(x, w(y)) \). This is the set of sequences of weights \( \mu_1\mu_2\cdots\mu_k \) with each \( \mu_i \in \Omega(x^{(i)}) \) and their sum \( w(y) \) such that the partial sum \( \mu_1 + \cdots + \mu_i \) is always dominant.

Lemma 3.3. The set \( E(V_1^a, \lambda) \) is in natural bijection with standard tableaux of shape \( \lambda \).

Proof. The weights of \( V_1^a \) are of the form

\[
\Omega(1) = \{ \emptyset, \bullet, \hat{\bullet}, \hat{\hat{\bullet}} \}
\]
and so a step in a miniscule Littleman path corresponds to adding a box to a partition without making it non-dominant (i.e. no longer a partition).

**Corollary 3.4.** When \( n = 1 \), the dimension of \( S(\mathbf{1}_a, \lambda) \) is

\[
\begin{equation}
    d_{\lambda} = \binom{a}{\lambda_2} - \binom{a}{\lambda_2 - 1}.
\end{equation}
\]

When \( n = 2 \), the dimension of \( S(\mathbf{1}_a, \lambda) \) is

\[
\begin{equation}
    d_{\lambda} = \frac{a!(\lambda_1 - \lambda_2 + 1)(\lambda_1 - \lambda_3 + 2)(\lambda_2 - \lambda_3 + 1)}{\lambda_1!(\lambda_2 + 1)!\lambda_3!}.
\end{equation}
\]

**Proof.** This is just the hook length-formula. \( \Box \)

Note that eq. (3.4) is the dimensions of the cell modules for the Temperley-Lieb algebra on \( a \) strands, as expected.

**Example 3.5.** In the same vein as example 3.2, let us construct the basis for \( S(\mathbf{+}_a, \lambda) \) for \( n = 2 \). The three elements of \( \Omega(\mathbf{+}) \) correspond to

\[
E_p = \begin{array}{c}
\rightarrow\\
\end{array}, \quad E_p = \begin{array}{c}
\nearrow\\
\end{array}, \quad E_b = \begin{array}{c}
\searrow\\
\end{array}.
\]

Note that we would not be able to insert \( E_p \) if \( \mathbb{I}_t \) didn’t have a rightward facing strand at its target, and we wouldn’t be able to use \( E_b \) unless it had a leftward facing one. This is exactly the statement that \( \lambda + \mu \) must be dominant.

Now, since we are looking for a basis of \( S(\mathbf{+}_a, \lambda) \), we are in bijection with standard tableaux of shape \( \lambda \). Given such a standard tableau \( s \in T(\lambda) \), we will denote its basis element by \( \mathbb{I}_s \).

Note that \( L_t \) is a map from \( (\mathbf{+}_n, 0) \) to some object of weight \( w(\lambda_i^-) \). The grey box labeled 0 is a neutral map that simply rearranges the output suitably.

There is a natural extension of lemma 3.3.

**Lemma 3.6.** The set \( E(\overline{x}, \lambda) \) is in natural bijection with row-semistandard tableaux of shape \( \lambda \) and weight \( \overline{x} \).

Here a row-semistandard tableau is one in which the elements increase strictly along rows and weakly along columns. A (column) semistandard tableau is strictly increasing along columns and weakly along rows. We will denote the set of all row-semistandard tableaux of shape \( \lambda \) and weight \( \overline{x} \) as \( \text{r-SSYT}(\lambda, \overline{x}) \), and the set of all semistandard tableaux of that shape and weight by \( \text{SSYT}(\lambda, \overline{x}) \).
A useful construction is the \textit{reduced Young poset} $\mathcal{Y}_n^\lambda$ \cite{CSST10}. This consists of all the partitions
\begin{equation}
\bigcup_{i=0}^{\text{length } \lambda} \{ \mu \text{ a partition of at most } n \text{ parts : r-SSYT}(\mu, x_1 \cdots x_i) \neq \emptyset \}
\end{equation}
arranged in natural layers, with edges between a partition in layer $i$ and one in layer $i+1$ if $x_i$ boxes can be added to obtain it.

It is clear an element of $E(\chi, \lambda)$ or equivalently r-SSYT$(\lambda, \chi)$ is equivalent to a path from the empty partition $\emptyset$ to $\lambda$ in $\mathcal{Y}_n^\lambda$. Further, in the case $\chi = 1^n$, we recover exactly the Young graph.

\textbf{Example 3.7.} We present the graph $\mathcal{Y}_5^{3231}$.

\begin{center}
\includegraphics[width=\textwidth]{example_graph.png}
\end{center}

4. Gram Determinants

Our main result concerns the Gram determinants of the inner form on $S(\chi, \lambda)$.

Now, let
\begin{equation}
\lambda_i^- = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_n + 1)
\end{equation}
where if $\lambda_i^-$ is not a partition (i.e. some part is negative or the parts are not non-decreasing) we omit any symbol involving $\lambda_i^-$ from the equation. For example, we may write $d_\lambda = \sum_i d_{\lambda_i^-}$ without worrying for the edge cases. Notice that
\begin{equation}
w(\lambda_{n+1}^-) \prec w(\lambda_n^-) \prec \cdots \prec w(\lambda_1^-).
\end{equation}

We should think of standard tableaux of shape $\lambda$ as paths from the empty partition, $\emptyset$, to $\lambda$ in the truncated Young graph $\mathcal{Y}_n$. We will write the set of such paths (standard tableaux) as $T(\lambda)$. Here, $\mathcal{Y}_n$ simply consists of the graph of all $n$-part partitions with edges between those which differ by a single box. Each edge in $\mathcal{Y}_n$ has a “type” which is one of $\{1, 2, \ldots, n\}$ giving the row number that the box was added in.

\footnote{Our convention for row-semistandard tableaux results in the graph of transposed partitions to those in the literature.}
Our observation is a change of basis from the construction in eq. (3.3). In this, we simply replace one of the neutral ladders with the appropriate clasp.

\[
\text{(4.3)}
\]

This is an upper unitriangular change of basis and so the Gram matrix for this basis has the same determinant as that for the original basis.

However, if we evaluate an inner product,

\[
\text{(4.4)}
\]

we must compute the coefficient of the identity diagram in

\[
\text{(4.5)}
\]

However, invoking lemma 2.8, we see that this vanishes identically if \( L_s \) and \( L_s \) have different targets. But if they have the same targets and this morphism is between two objects of the same weight, we must have \( \mu = \nu \). If they have the same target, of weight \( \lambda' = \lambda - \mu \) then we are left with

\[
\text{(4.6)}
\]

We have thus shown that in this basis, the Gram matrix is block diagonal, with one block for each weight \( \lambda' \) one level lower down in the Young graph. Each block is the Gram matrix for the corresponding cell module \( S(\hat{x}, \lambda') \) multiplied by the constant \( \kappa_{\lambda'}^{\lambda} \). We have thus shown

\[
\text{(4.7)}
\]

**Example 4.1.** For \( U_q(\mathfrak{sl}_2) \) this change of basis and subsequent calculation of the determinant is one of the standard methods of determining the usual Gram determinant. Here we show the
calculation for $S(6,2)$. The usual basis is

Here we have drawn both the diagram basis and the corresponding tableaux. Note that we have partitioned the basis by the row in which the box labeled 6 occurs.

The Gram matrix in this basis is

$$G(6,2) = \begin{pmatrix}
\delta^2 & \delta & \delta & 1 & \delta & 0 & 0 & 0 & 0 \\
\delta & \delta^2 & 1 & \delta & 1 & 0 & 0 & 0 & 0 \\
\delta & 1 & \delta^2 & \delta & 1 & \delta & 1 & 0 & 0 \\
1 & \delta & \delta & \delta^2 & \delta & 1 & \delta & 1 & \delta \\
\delta & 1 & 1 & \delta & \delta^2 & 0 & 1 & \delta & 1 \\
0 & 0 & \delta & 1 & 0 & \delta^2 & \delta & 0 & 0 \\
0 & 0 & 1 & \delta & 1 & \delta & \delta^2 & \delta & 1 \\
0 & 0 & 0 & 1 & \delta & 0 & \delta & \delta^2 & \delta \\
0 & 1 & 0 & \delta & 1 & 0 & 1 & \delta & \delta^2 \\
\end{pmatrix}.$$ (4.8)

Our construction gives a new basis in terms of the idempotent $JW_3$ and $JW_2$. In this case the $JW_1$ is “absorbed” completely by the condition on elements of cell modules.

Now, in this new basis,

$$G'(6,2) = \begin{pmatrix}
\delta^2 & \delta & \delta & 1 & \delta & 0 & 0 & 0 & 0 \\
\delta & \delta^2 & 1 & \delta & 1 & 0 & 0 & 0 & 0 \\
\delta & 1 & \delta^2 & \delta & 1 & 0 & 0 & 0 & 0 \\
1 & \delta & \delta & \delta^2 & \delta & 0 & 0 & 0 & 0 \\
\delta & 1 & 1 & \delta & \delta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & \frac{\delta^3 - 2\delta}{\delta^2 - 1} & \frac{\delta^3 - 2\delta}{\delta^2 - 1} \\
\end{pmatrix}.$$ (4.9)

$$= \begin{pmatrix}
G(5,1) & 0 \\
0 & \frac{\delta^3 - 2\delta}{\delta^2 - 1} G(5,3) \\
\end{pmatrix}. $$ (4.10)

The factor $(\delta^3 - 2\delta)/(\delta^2 - 1)$ comes from the trace of the Jones-Wenzl idempotent, $JW_3$. 

Remark 4.1. We could in fact bubble this construction down and use clasps instead of the neutral ladders in eq. (3.3) always. This would give us an orthogonal basis of the cell.
modules and would preserve the Gram determinant. We denote this basis by $\mathbb{L}_t$ and it will be critical for the sequel.

Remark 4.2. The importance of preserving the Gram determinant comes from having a view to the modular theory of webs. The relations that webs satisfy are given in terms of polynomials\(^5\) in $[\delta]$. Thus webs form a $\mathbb{Z}[\delta]$-spanning set for the category, and the basis constructed by Elias gives a $\mathbb{Z}[\delta]$ form.

As such, given a suitable $p$-modular system (or rather, an $(\ell, p)$-modular system as in [MS21]), we can consider the modular representation theory of this category. Here, the cell modules are almost never simple, and the Gram determinant identifies this, vanishing when the cell module is not. Moreover there are filtrations of the cell modules that can be derived from the elementary divisors of the Gram matrix.

Of course eq. (4.7) is recursive in nature. It behoves us to write down a “closed form”. To do this, we must simply enumerate all the paths from $\emptyset$ to $\lambda$ in $\mathcal{Y}_n$. Let this be denoted $T(x, \lambda)$. Then

\begin{equation}
\det G(x, \lambda) = \prod_{t \in T(x, \lambda)} \prod_{e : \mu_1 \to \mu_2} (k_{\mu_2}^{\mu_1})^d(\mu_1, \mu_2).
\end{equation}

We can write this in a different way. Instead of counting edges by paths, we can count paths with edges. Let $d(x, \lambda \setminus \mu)$ be the number of paths in $\mathcal{Y}_n^\lambda$ from $\lambda$ to $\mu$. Then

\begin{equation}
\det G(x, \lambda) = \prod_{\mu_1 \to \mu_2} (k_{\mu_2}^{\mu_1})^{d(x, \mu_1)d(x, \lambda \setminus \mu_2)}
\end{equation}

We note that $d(x, \mu_1)d(x, \lambda \setminus \mu_2)$ is exactly the number of paths from $\emptyset$ to $\lambda$ passing through both $\mu_1$ and (subsequently) $\mu_2$.

Example 4.2. If $n = 2$ so we are in the Temperley-Lieb case, then we must have $x = \bullet^n$ and the truncated Young poset $\mathcal{Y}_n$ is simply the branching graph

\begin{itemize}
\item[\footnote{This is true despite them appearing to be given as rational functions because the relations are in the form of $[a/b]$ which are all polynomials in $[2]$ and so defined (though possibly vanishing) in any given field.}]\end{itemize}
The intersection forms can be evaluated by tracing the Jones-Wenzl idempotents and are known to be
\[
\kappa_{(n,m)}^{(n+1,m+1)} = 1 \quad \kappa_{(n,m)}^{(n+1,m-1)} = \frac{m+1}{m}
\]

**Example 4.3.** More generally, if \( x = \chi \), so that paths are standard tableaux, \( d(x, \mu_1) d(x, \lambda \setminus \mu_2) \) counts the number of standard tableaux of shape \( \lambda \) with a \( |\mu_2| \) in the box \( \mu_2 \setminus \mu_1 \).

For a concrete example, if \( \mu_1 = (7, 6, 4, 3, 1) \) and \( \mu_2 = (7, 6, 5, 3, 1) \) for \( \lambda = (8, 7, 6, 6, 3, 2, 1) \), then we are counting all standard tableaux of the form below, where the numbers 1 through 21 = \( |\mu_1| \) are placed in the pink boxes, 22 is as indicated in the green box and the remaining 23 through 33 are in the blue boxes.

**Remark 4.3.** This process should work for other categories. In particular, we only needed

1. a category with a suitable poset of weights,
2. neutral ladders (which are often not even necessary because the object set and the weight set coincide),
3. clasps, which are often inductively definable (see other paper),
(4) a reasonable way of constructing bases for cell modules which looks like the ladder construction (often a byproduct of the construction of clasps being constructed by an iterative ladder)

In particular, we should be able to calculate the determinants for categories such as the planar rook monoid category (where we expect the answer to be “not zero”, since all cell modules are irreducible over arbitrary character) and coloured TL algebras.

As a final remark, we note that in any path through the Young graph, the number of parts of the partition is non-decreasing. Hence we can remove the restriction $e \in \mathcal{Y}_n$ in eq. (4.12) and replace it with $e \in \mathcal{Y}_2$, eliminating the dependence on $n$.

**Example 4.4.** Let us compute the Gram determinant for $S(32312, 4232)$ for $n = 5$. That is, we are considering the space

$$\text{Hom}_{U_q(sl_5)}(V_3 \otimes V_2 \otimes V_3 \otimes V_1 \otimes V_2, V_4 \otimes V_2 \otimes V_3 \otimes V_2)$$

modulo morphisms factoring through representations of weight lower than the target, where $V_i = \Lambda^i C^5$.

The weight of the target is

$$\text{wt}(4232) = \text{wt}(4232)$$

and so should consider the truncated Young poset $\mathcal{Y}_5_{32312}$ of weights at most $\text{wt}(4232)$.

We will draw this poset, where each node has the number of paths from $\emptyset$ to that node annotated in blue and the number of paths to the top in red. Each edge is labeled with the appropriate value of $\kappa$, which we can read from example 2.13.
From this, the algorithm for computing the determinant is simple. Simply multiply together all the coefficients on the edges with multiplicity given by the product of the blue number at their bottom and red number at their top. In our example, only one edge is simultaneously labeled with a number other than 1 and appears in multiple paths from the bottom to the top:

\[
\det G(32312, 4232) = [3] \cdot [2][4] \cdot [2]^2 \cdot [2] \cdot [3][6] \cdot [4] \cdot [2][4] \cdot [3][7] \cdot [2][5].
\]

(4.15)

(4.16)

We note that this is in fact a polynomial despite being expressed as a product of rational functions.

Lemma 4.5. The Gram determinant \( \det G(x, y) \) is a polynomial in \([2]\).

Proof. Consider calculating the Gram matrix in the \( L_t \) basis. The elements of this basis are diagrams and the composition of diagrams lies in the \( \mathbb{Z}[\delta] \)-span of other diagrams. Hence the coefficient of the identity diagram lies in \( \mathbb{Z}[\delta] \) so the determinant of this matrix must too.

Remark 4.4. In example 4.4 the result was expressible as a product of quantum integers. However, this is not always the case.
5. Howe Duality

Consider the space $\mathbb{C}^n \otimes \mathbb{C}^m$. This carries an action of $U_q(\mathfrak{sl}_n)$ by action on the first component, and of $U_q(\mathfrak{gl}_m)$ by action on the second. These can be extended to commuting actions on $\bigwedge^k(\mathbb{C}^n \otimes \mathbb{C}^m)$ for any $k$.

Theorem 5.1. The actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_m)$ on $\bigwedge^k(\mathbb{C}^n \otimes \mathbb{C}^m)$ generate each other’s commutants.

Now,

$$\bigwedge^t (\mathbb{C}^n \otimes \mathbb{C}^m) \simeq \bigwedge^t (\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n)$$

$$\simeq \bigoplus_{\ell(t) = m} \left( \bigwedge^{t_1} \mathbb{C}^n \otimes \bigwedge^{t_2} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{t_m} \mathbb{C}^n \right) = \bigoplus_{\ell(t) = m} V_t$$

as $U_q(\mathfrak{sl}_n)$ representations. Here the direct sum is over all compositions $t$ of $t$ in $m$ parts with no part longer than $n$. Thus,

$$\text{End}_{U_q(\mathfrak{sl}_n)} \left( \bigwedge^t (\mathbb{C}^n \otimes \mathbb{C}^m) \right) \simeq \left( \begin{array}{cccc} \text{End}(V_a) & \text{Hom}(V_a, V_b) & \cdots & \text{Hom}(V_a, V_{\xi}) \\ \text{Hom}(V_b, V_a) & \text{End}(V_b) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \text{Hom}(V_{\xi}, V_a) & \text{Hom}(V_{\xi}, V_b) & \cdots & \text{End}(V_{\xi}) \end{array} \right)$$

Where $\{a, b, \ldots, \xi\}$ are all the appropriate partitions of $t$.

Theorem 5.1 states that there is a surjection $\Psi$ of $U_q(\mathfrak{gl}_m)$ onto $\text{End}_{U_q(\mathfrak{sl}_n)} \left( \bigwedge^k (\mathbb{C}^n \otimes \mathbb{C}^m) \right)$ and its kernel is computed in [CKM14, §4.4]. In fact, this surjection and its kernel are instrumental in deducing the correct relations for the web categories.

If we compose $\Psi$ with projection onto the appropriate component in eq. (5.1) we obtain the families of surjections

$$\Psi_{a,b} : U_q(\mathfrak{gl}_m) \to \text{Hom}_{U_q(\mathfrak{sl}_n)}(V_a, V_b)$$

and

$$\Psi_{a,b}^{\text{wt}} : U_q(\mathfrak{gl}_m) \to \text{Hom}_{U_q(\mathfrak{sl}_n)}^{\text{wt}}(V_a, V_b) = S(a, b).$$

5.1. Cellular forms. Consider now a morphism $V_b \to V_a$ where $b$ is of $(\mathfrak{sl}_n)$ weight $\lambda$. We know that there is a functional $f_\lambda : \text{End}_{U_q(\mathfrak{sl}_n)}(V_b) \to \mathbb{C}$ sending a morphism to the coefficient of its identity diagram. Moreover this is a morphism of algebras as it is in fact the quotient by the maximal ideal in $\text{End}(V_b)$. Recall that this gives rise to the cellular form on $S(a, \lambda)$ by $\langle x, y \rangle = f_\lambda(\rho(x) \circ y)$.

---

6Strictly they should be in not more than $n$ parts, but we take them to be exactly $n$ parts by padding by zeros.

7We could, should we desire, extend this to a functor $f_\lambda$ on each $\text{Hom}(V_a, V_{\xi})$ of the same weight $\lambda$ taking a function to its neutral ladder coefficient. However, we will not need this generality.
On the other hand, consider the image of the torus of $U_q(\mathfrak{gl}_m)$, called $U_q(\mathfrak{h})$, in eq. (5.1). The element $h \in \mathfrak{h}$ gets sent to a diagonal matrix where the component on End($V_h$) is simply $h(h)1$, where $h$ is considered a weight of $\mathfrak{gl}_m$. Under the quotient by the maximal ideal, this is again sent to $h(h)$ in $C$.

Let $\pi : U_q(\mathfrak{gl}_m) \to U_q(\mathfrak{h})$ be the projection in the decomposition $U_q(\mathfrak{gl}_m) = (n_-,U_q(\mathfrak{gl}_m) + U_q(\mathfrak{gl}_m)n_+) \oplus U_q(\mathfrak{h})$.

Suppose $\Psi \in \text{End}(V_h)$ is the image of $g \in U_q(\mathfrak{gl}_m)$. Then the image of $\pi(g)$ in End($V_h$) is $h(\pi(g))1$. Hence we see that $h(\pi(g))1 = f_A(\Psi)1$ and thus $h(\pi(g)) = f_A(\Psi)$.

There is an anti-automorphism, $\sigma$, of $U_q(\mathfrak{gl}_m)$ that fixes $U_q(\mathfrak{h})$ pointwise and sends $e_i$ to $f_i$ and vice-versa. The $h$-weighted bi-linear pairing is then the map

$(-,-)_h : U_q(\mathfrak{gl}_m) \times U_q(\mathfrak{gl}_m) \to C$

$(g_1,g_2) \to h(\pi(\sigma(g_1) \cdot g_2))$.

This pairing is also invariant in that $(xg_1,g_2) = (g_1,\sigma(x)g_2)$. The above discussion shows that

(5.2) $\langle g_1,g_2 \rangle_h = \langle \Psi_{\sigma,h}(g_1), \Psi_{\sigma,h}(g_2) \rangle$.

In particular, if we are able to find pre-images of the clasp basis $\mathbb{I}_\ell$ under $\Psi$, we can evaluate the form $(-,-)_h$ on these pre-images to determine the form $\langle -,- \rangle$.

5.2. Notation for Tableaux. In what follows, we will be manipulating row- and column-semi-standard tableaux, and we set out the required notation here.

Let $T$ be a (row) semi-standard tableaux and let

$N_{ij}^T = \text{number of entries of } j \text{ in row } i \text{ of } T$

Recall that semi-standard tableaux describe paths in the restricted Young graph. Let $T^{(i)}$ be the partition described by all boxes in $T$ of entries at most $i$.

**Example 5.2.** If

$T = \begin{array}{ccc}
1 & 1 & 1 & 3 \\
2 & 2 & 4 \\
3 & 3 \\
5 & 5 \\
\end{array}$

Then $T^{(1)} = (3)$, $T^{(2)} = (3,2)$, $T^{(3)} = (4,2,2)$, $T^{(4)} = (4,3,2)$ and $T^{(5)} = (4,3,2,2)$.

If $T$ has $n$ rows, then its maximal value, $s$, is at most $n$. We will write, for any $i \leq j$,

$M^j_{ik} = (N_{ij+1} + \cdots + N_{ks}) - (N_{ij+1} + \cdots + N_{ik})$

$M^j_{ik} = M^{j-1}_{ik} + N_{ij}$

Note that $M^j_{ik} = T^{(s)}_i - T^{(j)}_i - T^{(s)}_i + T^{(j)}_i$ which can also be expressed as a difference of two axial distances (see definition 2.12).
5.3. **Lowering and Raising Operators.** We now turn to distinguished elements of $U_q(\mathfrak{gl}_m)$ known as raising- and lowering-operators. They are designed as an orthonormal set of operators that move between the weights of Verma modules.

Raising and lowering operators were introduced by Nagel and Moshinsky [NM65] and exposited by Carter at the Acarta conference [Car87]. In these papers they were defined for $U(\mathfrak{gl}_m)$, but in 1990, Tolstoy determined the appropriate elements for $U_q(\mathfrak{gl}_m)$ [Tol90], which we present below.

**Lemma 5.3.** There is a Gelfand-Tsetlin basis of the Verma module of $U_q(\mathfrak{gl}_m)$ with highest weight $\lambda$, written

$$|T\rangle = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1 - N_{1n}^T & \lambda_2 - N_{2n}^T & \cdots & \lambda_{n-1} - N_{n-1n}^T \\ \vdots \\ \lambda_1 - (N_{12}^T + \cdots + N_{1n}^T) \end{pmatrix} = \begin{pmatrix} T^{(n)} \\ T^{(n-1)} \\ \vdots \\ T^{(1)} \end{pmatrix}$$

as $T$ ranges over all semi-standard tableaux of weight $\lambda$. If $|\lambda_n\rangle$ is a (normal) highest weight vector, then

$$|T\rangle = F_- \left( T^{(1)}; T^{(2)} \right) F_- \left( T^{(2)}; T^{(3)} \right) \cdots F_- \left( T^{(n-1)}; T^{(n)} \right) |\lambda_n\rangle$$

where

$$F_- \left( T^{(j-1)}; T^{(j)} \right) = N \left( T^{(j-1)}; T^{(j)} \right)^{-1} P_q(\mathfrak{gl}; j-1) \prod_{i=1}^j (e_{ji})^{\lambda_j - \lambda_{ij-1}}$$

and $P_q(\mathfrak{gl}; j)$ is an extremal projector (clasp) and the constants $N$ are given by

$$N \left( T^{(j-1)}; T^{(j)} \right) = \left\{ \prod_{k=1}^{j-1} \left[ T^{(j)}_k - T^{(j-1)}_k \right]! \prod_{1 \leq i < k \leq j-1} \frac{[T^{(j-1)}_i - i - T^{(j-1)}_k + k]!}{[T^{(j)}_i - i - T^{(j)}_k + k]!} \right\}^{1/2} \prod_{1 \leq i < k \leq j} \frac{[T^{(j)}_i - i - T^{(j)}_k + k - 1]!}{[T^{(j-1)}_i - i - T^{(j-1)}_k + k - 1]!}$$

We’ve given eq. (5.3) in form given in [Tol90] (but with indices $j$ and $k$ swapped), to aide in reconciling the lemmata, but now use our notation to simplify the form of the $N$. Note that $T^{(j)}_i = \lambda_i - (N_{ij+1}^T + \cdots + N_{in}^T)$ and $T^{(j)}_k - T^{(j-1)}_k = N_k$, so that $T^{(j)}_i - i - T^{(j)}_k + k =$
We can then write
\[ \mathcal{N}(T^{(j-1)}; T^{(j)}) = \left\{ \prod_{k=1}^{j-1} [N_{kj}]! \prod_{1 \leq i k \leq r_{k-1}} \left[ c_{ik} + M_{ik}^{j-1} \right]! \prod_{1 \leq i k \leq j} \left[ c_{ik} + M_{ik}^{j-1} - 1 - N_{kj} + N_{ij} \right]! \right\}^{1/2} \]
\[ = \left\{ \left( c_{ik} + M_{ik}^{j-1} - 1 \right) / \prod_{1 \leq i k \leq j} \left( c_{ik} + M_{ik}^{j-1} \right) \right\}^{1/2} \]
\[ = \left\{ \prod_{1 \leq i k \leq j} \left( c_{ik} + M_{ik}^{j} - 1 \right) / \prod_{1 \leq i k \leq j} \left( c_{ik} + M_{ik}^{j} \right) \right\}^{1/2} \]
\[ = \left\{ \prod_{k=1}^{j-1} [N_{kj}]! \prod_{1 \leq i k \leq j} \left[ c_{ik} + M_{ik}^{j} - N_{ij} \right]! \prod_{1 \leq i k \leq j} \left[ c_{ik} + M_{ik}^{j} - 1 \right]! \right\}^{1/2} \]
\[ = \left\{ \prod_{1 \leq i k \leq j} \left( c_{ik} + M_{ik}^{j} - 1 \right) / \prod_{1 \leq i k \leq j} \left( c_{ik} + M_{ik}^{j} \right) \right\}^{1/2} \]

Let \( T' = T^{(s-1)} \) be the semistandard tableau which is the parent of \( T \). That is, it contains the same boxes except those labeled \( s \). These are the “new” boxes in \( T \). Let \( T' \) have shape \( \lambda' \subset \lambda \) and the numbers \( c_{ij}' \) and \( N_{ij}' \) have their definition with respect to the primed variables.

Now, recall Elias’ conjecture concerned the shape of successive column semi-standard tableaux from conjecture 2. To bring this in line with the above calculation we will need to take transposes which we denote \( \lambda' \perp \).

**Lemma 5.4.** With the above notation, \( \mathcal{N}(T'; T)^2 = \kappa_{\lambda' \perp}^{\lambda' \perp} \).

**Proof.** Elias’ conjectural form for \( \kappa_{\lambda' \perp}^{\lambda' \perp} \) is given as
\[
\kappa_{\lambda' \perp}^{\lambda' \perp} = \prod_{1 \leq a} \prod_{a < b} \left[ \lambda_a^{\perp} - \lambda_b^{\perp} + b - a \right] / \left[ \lambda_a^{\perp} - \lambda_b^{\perp} + b - a - 1 \right]
\]

Now, by the fact that column \( a \) has no new boxes, \( \lambda_a^{\perp} = \lambda_a^{\perp} \). If we now group the columns labeled by \( b \) by the row they appear in, which we will index by \( i \), we get
\[
= \prod_{1 \leq i \leq s} \prod_{\text{no new box in column } a} \prod_{a \leq \lambda_i} \prod_{b = \lambda_i - N_{ia} + 1} \left[ \lambda_a^{\perp} - \lambda_b^{\perp} + b - a \right] / \left[ \lambda_a^{\perp} - \lambda_b^{\perp} + b - a - 1 \right]
\]

Note that \( N_{ia} = 0 \) for all \( i > s \), hence the bounds on the first product. We have also subtly swapped the order of product here. But since the box indexed by \( b \) is always in the \( i \)-th row of \( \lambda \), we have \( \lambda_b^{\perp} = i - 1 \), so
\[
= \prod_{1 \leq i \leq s} \prod_{\text{no new box in column } a} \prod_{a \leq \lambda_i} \prod_{b = \lambda_i - N_{ia} + 1} \left[ \lambda_a^{\perp} - i + b - a + 1 \right] / \left[ \lambda_a^{\perp} - i + b - a \right]
\]
and we can collapse the telescoping product. Thus

\[
\prod_{1 \leq i \leq s} \prod_{1 \leq a \leq \lambda_i \text{ no new box in column } a} \frac{[\lambda_a^+ - i + \lambda_i - a + 1]}{[\lambda_a^+ - i + \lambda_i - a + 1 - N_{is}]} \]

Now we repeat the row/column index swapping trick, by grouping the columns \(a\) by the rows they appear in. For notational purposes, we will let \(k\) be the row below the last box in column \(a\) so that \(\lambda_{\perp a} = k - 1\).

\[
\prod_{1 \leq i < k \leq s} \prod_{a=\lambda_k+1}^{\lambda_{k-1} - N_{k-1s}} \frac{[k - i + \lambda_i - a]}{[k - i + \lambda_i - a - N_{is}]} \]

Finally, we adjust our index ranges and clean up the formula:

\[
\prod_{1 \leq i < k \leq s} \prod_{a=\lambda_k+1}^{\lambda_{k-1} - N_{k-1s}} \frac{[k - i + \lambda_i - \lambda_k - \ell]}{[k - i + \lambda_i - \lambda_k - \ell - N_{is}]} \frac{[c_{ik} - \ell]}{[c_{ik} - N_{is} - \ell]}.
\]

(5.4)

On the other hand, Tolstoy gives

\[
\mathcal{N}(T^\prime; T)^2 = \prod_{1 \leq i < k \leq j \leq n} \left( \frac{c_{ik} + M_{ik}^j - 1}{N_{ij}} \right) / \prod_{1 \leq i < k \leq j \leq n} \left( \frac{c_{ik} + M_{ik}^j}{N_{ij}} \right) \]

We will manipulate this until it matches eq. (5.4) exactly. Note that

\[
c_{ik} = c_{ik}' + N_{is} - N_{ks} \]

\[
M_{ik}^j = \begin{cases} M_{ik}^j + N_{ks} - N_{is} & j < s \\ M_{ik}^j = 0 & j \geq s \end{cases} \]

so that

\[
c_{ik} + M_{ik}^j = \begin{cases} c_{ik}' + M_{ik}^j & j < s \\ c_{ik}' + N_{is} - N_{ks} & j \geq s \end{cases} \]

\[
c_{ik} + M_{ik}^j = \begin{cases} c_{ik}' + M_{ik}^j & j < s \\ c_{ik}' + N_{is} & j \geq s \end{cases} \]

As such, in \(\mathcal{N}(T^\prime; T)^2\), all the terms with \(j < s\) cancel and we are left with

\[
\mathcal{N}(T^\prime; T)^2 = \prod_{1 \leq i < k \leq s} \left( \frac{c_{ik} - 1}{N_{is}} \right) / \prod_{1 \leq i < k \leq s} \left( \frac{c_{ik} + N_{ks}}{N_{is}} \right) \]
Putting in a \( k = s \) case to the second product (noting \( \lambda_s = N_{is} \)), and splitting the indices \( i \) and \( k \), we obtain

\[
= \prod_{1 \leq i < s} \left( \frac{(c_is + \lambda_s)}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} - 1)}{N_{is}} \right) \left( \frac{(c_{ik} + N_{ks})}{N_{is}} \right)
\]

Clearly the only terms contributing anything are those where \( N_{is} \neq 0 \)

\[
= \prod_{1 \leq i \leq s} \left( \frac{(c_is + \lambda_s)}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} - 1) \cdots (c_{ik} - N_{is})}{(c_{ik} + N_{ks}) \cdots (c_{ik} + N_{ks} - N_{is})} \right)
\]

We introduce and remove a factor

\[
= \prod_{1 \leq i \leq s} \left( \frac{(c_is + \lambda_s)}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} - 1) \cdots (c_{ik} - N_{is})}{(c_{ik} + N_{ks}) \cdots (c_{ik} + N_{ks} - N_{is})} \right)
\]

to allow us to telescope the product

\[
= \prod_{1 \leq i \leq s} \left( \frac{(c_is + \lambda_s)}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ii+1} - 1) \cdots (c_{ii+1} - N_{is})}{(c_{ii+1} + N_{ks}) \cdots (c_{ii+1} + N_{ks} - N_{is})} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} + N_{ks})}{(c_{ik} + N_{ks} - N_{is})} \right)
\]

but \( c_{is+1} = c_is + \lambda_s + 1 \) and \( c_{ii+1} - 1 = \lambda_i - \lambda_{i+1} \), so we can simplify

\[
= \prod_{1 \leq i \leq s} \left( \frac{(\lambda_i - \lambda_{i+1})}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} + N_{ks})}{(c_{ik} + N_{ks} - N_{is})} \right)
\]

and writing \( c_{ik+1} = c_{ik} + \lambda_k - \lambda_{k+1} + 1 \),

\[
= \prod_{1 \leq i \leq s} \left( \frac{(\lambda_i - \lambda_{i+1})}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} + \lambda_k - \lambda_{k+1}) \cdots (c_{ik} + \lambda_k - \lambda_{k+1} - 1)}{(c_{ik} + N_{ks}) \cdots (c_{ik} + N_{ks} - N_{is})} \right)
\]

But now \( N_{ks} \leq \lambda_k - \lambda_{k-1} \) so that \( c_{ik} + N_{ks} \leq c_{ik} + \lambda_k - \lambda_{k+1} \) and so (introducing some more terms if necessary)

\[
= \prod_{1 \leq i \leq s} \left( \frac{(\lambda_i - \lambda_{i+1})}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik} + \lambda_k - \lambda_{k+1}) \cdots (c_{ik} + \lambda_k - \lambda_{k+1} + N_{is} + 1)}{(c_{ik} + N_{ks}) \cdots (c_{ik} + N_{ks} - N_{is})} \right)
\]

We can re-introduce the \( c_{ik+1} \)

\[
= \prod_{1 \leq i \leq s} \left( \frac{(\lambda_i - \lambda_{i+1})}{N_{is}} \right) \times \prod_{i < k \leq s} \left( \frac{(c_{ik+1} - 1) \cdots (c_{ik+1} - \lambda_k + \lambda_{k+1} + N_{ks})}{(c_{ik+1} - N_{is} - 1) \cdots (c_{ik+1} - \lambda_k + \lambda_{k+1} + N_{ks} - N_{is} - 1)} \right)
\]

and write the product using a dummy index.

\[
= \prod_{1 \leq i \leq s} \left( \frac{(\lambda_i - \lambda_{i+1})}{N_{is}} \right) \times \prod_{i < k \leq s} \prod_{\ell=1}^{\lambda_k - \lambda_{i+1} - N_{is}} \left( \frac{c_{ik+1} - \ell}{c_{ik+1} - N_{is} - \ell} \right)
\]
Now, if we replace all the $k + 1$s with $k$s we pick up two terms on each side of the product range.

$$\prod_{1 \leq i \leq s, N_{is} \neq 0} \left( \lambda_i - \lambda_{i+1} \right) \times \prod_{i < k \leq s} \prod_{\ell=1}^{\lambda_k-1-\lambda_N-k} \frac{c_{ik} - \ell}{c_{ik} - N_{is} - \ell} \times \prod_{\ell=1}^{\lambda_s - \lambda_{i+1} - N_{is}} \frac{c_{is+1} - \ell}{c_{is+1} - N_{is} - \ell} \times \prod_{\ell=1}^{\lambda_i - \lambda_{i+1} - N_{is}} \frac{c_{ii+1} - N_{is} - \ell}{c_{ii+1} - \ell}$$

Note that $\lambda_{s+1} = 0$ and $\lambda_s = N_{ss}$ so the second-to-last product is actually empty. Hence

$$\prod_{1 \leq i \leq s, N_{is} \neq 0} \left( \lambda_i - \lambda_{i+1} \right) \times \prod_{i < k \leq s} \prod_{\ell=1}^{\lambda_k-1-\lambda_N-k} \frac{c_{ik} - \ell}{c_{ik} - N_{is} - \ell} \times \prod_{\ell=1}^{\lambda_i - \lambda_{i+1} - N_{is}} \frac{c_{ii+1} - N_{is} - \ell}{c_{ii+1} - \ell}$$

Finally, recall that $c_{ii+1} = \lambda_i - \lambda_{i+1} + 1$ and so

$$\prod_{1 \leq i \leq s, N_{is} \neq 0} \left( \lambda_i - \lambda_{i+1} \right) \times \prod_{i < k \leq s} \prod_{\ell=1}^{\lambda_k-1-\lambda_N-k} \frac{c_{ik} - \ell}{c_{ik} - N_{is} - \ell} \times \prod_{\ell=1}^{\lambda_i - \lambda_{i+1} - N_{is}} \frac{(N_{is})!(\lambda_i - \lambda_{i+1} - N_{is})!}{(\lambda_i - \lambda_{i+1})!}$$

$$= \prod_{1 \leq i \leq s, N_{is} \neq 0} \lambda_k - \lambda_N - k \times \prod_{i < k \leq s} \prod_{\ell=1}^{\lambda_k-1-\lambda_N-k} \frac{c_{ik} - \ell}{c_{ik} - N_{is} - \ell}$$

This matches exactly with eq. (5.4) and so we deduce that

$$\mathcal{N}(T'; T)^2 = \kappa_{\lambda, \lambda}^{\lambda, \lambda}$$
as desired. \qed

**Theorem 5.5.** Elias’ conjecture (conjecture 1) holds.

**Proof.** Note the construction of $|T|$ as a product of successive $F_-(T^{(i-1)}; T^{(i)})$. Each $F_-(T^{(i-1)}; T^{(i)})$ is in turn (up to a normalising factor) simply a extremal projector composed with the preimage of $E_\mu$.

An extremal projector $P$ in $U_q(\mathfrak{gl}_m)$ an idempotent such that $Pe_{a_i} = e_{a_i}P = 0$ for any $i$. Under the action of $\Phi$, we obtain an element that is idempotent and that is killed by all $e_{a_i}$. This element must be the sum (in the right hand side of eq. (5.1)) of such elements in the diagonal - which is to say the sum of clasps. Moreover, in the construction of the extremal element in [Tol90] we see that $1 \in U_q(\mathfrak{gl}_m)$ appears with coefficient 1. Thus the image under $\Phi$ must be the sum of all the clasps. In particular, its projection onto some $\text{End}(V_g)$ must be the clasp $\text{JW}_g$.

Thus the construction of the Gelfand-Tsetlin basis in lemma 5.3 is exactly (under the action of $\Psi$) the basis we called $\mathcal{L}_T$, up to a normalising factor. In fact, the normalising factor must be the intersection form, in order to ensure the new basis is normal.

However, we have seen in lemma 5.4 that the normalising factor is exactly the conjectured intersection form. \qed
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