LOGICO-ALGEBRAIC APPROACH TO SPACETIME QUANTIZATION

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Abstract
In the last decades the logico-algebraic approach to quantum mechanics turned to be a successful tool to render the quantum mechanical formalism on a steady operationalistic background [7]. The algebraic approach to general relativity first proposed by Geroch [1] is used to build a smearing procedure for events in the quantized theory of spacetime. It stems from the notion of basic algebra which possesses both the differential structure (as functional algebras in [1]) and non-commutativity (as algebras of observables in quantum theories). The main essence of this formalism is that it deprives the notion of spacetime as event support of its fundamental status making it relative to the measurement performed. Before the observation is executed, it is pointless to speak of events as points of spacetime: there is no underlying manifold. Only when an eigen-subalgebra of the basic algebra is outlined as eigenstate of the observable, it gives rise to its functional representation on the appropriate set thought of as spacetime.

1 INTRODUCTION

In general relativity an event in spacetime is idealized to a point of a four-dimensional manifold. Such idealization is adequate within clas-
sical physics, but is unsatisfactory from the operationalistic point of view. In quantum theory the influence of a measuring apparatus on the object being observed can not in principle be removed. We could expect the metric of a quantized theory to be subject to fluctuations, whereas the primary tool to separate individual events is just the metric [1]. Thus a sort of smearing procedure for events is to be imposed into the theory.

An essential step in this direction was the idea to build the differential geometry in terms of abstract algebras. Geroch [1] proposed to generalize the notion of algebra of smooth functions on a manifold to that of Einstein algebra whose elements are not yet functions. This generalization was successful since the entire content of general relativity can be reformulated in such a way that the underlying spacetime manifold is used only once: to define the collection of smooth functions.

Consider the ideas of Geroch in more detail. The basis of the differential geometry is the notion of vector field. It is known any vector field $v$ can be associated with the differential operator in the algebra $\mathcal{A}$ of smooth functions on the manifold acting as the derivation along this vector field. This operator $v$ is linear, and its main feature is the Leibniz rule:

$$v(ab) = v(a)b + av(b)$$

It is known that linear operators in $\mathcal{A}$ satisfying (1) are exhausted by that induced by actions of vector fields. That is why the difference is not drawn between such operators and vector fields: this is the essence of the coordinate-free account of differential geometry. As a matter of fact, coordinates appear only once: to specify the algebra $\mathcal{A}$ of smooth functions, since the notion of smoothness is referred to local maps. The forthcoming notions such as connection, torsion, curvature and others need no local coordinates in their definition.

I emphasize that at the mere level of definitions the principle notions of differential geometry require no coordinates, nor even points: the fact that $\mathcal{A}$ is the algebra of functions on a set is never used. Thus the global geometry per se does not confine us by set-theoretical concept of space.

Although, since the commutative case is considered, the absence
of points is, roughly speaking, an illusion. As a matter of fact, a
commutative algebra can always be represented by functions on an
underlying space. Such representation is, for instance, the Gel’fand
construction (for normed algebra) which is the special case of repre-
sentation of commutative algebra on its spectrum. So, in the case of
the commutative algebra points implicitly exist.

The standpoint of the suggested approach is to essentially remove
points from the theory. This happens automatically when we pass
to non-commutative Einstein algebras. Whereas the reproduction of
geometrical constructions causes a number of purely mathematical
obstacles [2,3].

2 NONCOMMUTATIVITY AND THE
SPATIALIZATION PROCEDURE

In this section the obstacles arising in the non-commutative gener-
alization of the algebraic construction of differential geometry are
overviewed giving rise to some new concepts such as the spatializa-
tion procedure.

Basic algebra. The first question is why non-commutativity is to
be fetched to geometry? The rough answer is that we have to follow
the tradition of quantization. An amount of non-commutativity in
the geometry itself is needed to quantize it. This produces the follow-
ing problem: the lack of points in this quantum geometry requires a
"spatialization" procedure to be imposed into the general scheme to
describe the observable entities.

So, by basic algebra of the model I shall mean an associative
and, in generally, non-commutative algebra \( \mathcal{A} \) over real (or complex)
numbers which will play the role analogous to that of the algebra of
smooth functions.

Spatialization procedure. Let us try to extract the geometry from
the basic algebra \( \mathcal{A} \) on its coarsest level, that is, set-theoretical one.
As it is usually done, we must consider the elements of \( \mathcal{A} \) as functions
defined on a certain set \( M \), and perhaps, taking the values in a non-
commutative domain \( R \). That is, the representation of \( \mathcal{A} \) by means of
homomorphism \( \hat{\text{hom}} \) is introduced:
where \( \hat{a} \) is a function \( M \to R \). Thus each point \( m \in M \) is associated with the two-sided ideal \( I(m) \subseteq \mathcal{A} \):

\[
I(m) = \{ a \in \mathcal{A} | \hat{a}(m) = 0 \}
\]

Now we see that the resources of spatialization are bounded by the number of two-sided ideals in \( \mathcal{A} \). Whereas, if \( \mathcal{A} \) contains two-sided ideals, it can be, as a rule, decomposed into mutually commuting components. So, each point can be associated with at least a simple component of the decomposition of \( \mathcal{A} \). The conclusion is that spatialization and non-commutativity are in some sense complementary: commutation relations cannot be described in terms of points.

When the basic algebra \( \mathcal{A} \) is commutative and satisfies some additional requirements (is Banach algebra), the proposed construction is just the Gel’fand representation endowing the set \( M \) by a natural topology. So, the commutative case makes it possible to store the topological space \( M \) so that \( \mathcal{A} \) is represented by continuous functions on \( M \). However, the Gel’fand construction does not yield the differential structure for \( M \).

Although, the lack of points is not an obstacle to introduce differential structure with all its attributes. As in the commutative case, it is introduced in terms of the collection \( \text{Der} \mathcal{A} \) of derivations of the basic algebra \( \mathcal{A} \).

On the coarse-grained level spacetime is replaced by its finitary substitute such as Regge space [4] to restore the metric or pattern space [5] to restore only the topology. In the latter case the points of the space can be restored by a purely algebraic construction [6].

**Scalars.** In the commutative case we can multiply a vector by any element of the basic algebra \( \mathcal{A} \). In general, an element \( v \in \text{Der} \mathcal{A} \) multiplied by an element \( a \in \mathcal{A} \) does not enjoy the Leibniz rule. However, to define such object as, say, connection, multiplicators are necessary: they play the role of scalars. So, in the case of non-commutative basic algebra the notion of scalar is to be carefully redefined [3,7].

\[\text{recall that a derivation of } \mathcal{A} \text{ is the linear mapping } v : \mathcal{A} \to \mathcal{A} \text{ enjoying the Leibniz rule (1)}\]
**Einstein equation.** Now consider the point-free counterpart of the Einstein equation. In conventional theory it postulates the equality between the Einstein tensor depending on geometry only and the momentum-energy tensor.

To form the left side, the analog of the scalar curvature $R$ is introduced. In classical geometry $R$ is the contraction of the contravariant metric tensor with the Ricci tensor. In differential algebras we have neither contraction nor tensors, but only operators. Although we have the trace of operators in our disposal. In conventional relativity the operator form of the Einstein equation is:

\[
R^i_k - \frac{1}{2} R\delta^i_k = \kappa T^i_k
\]

In differential algebras the Einstein equation takes the operator form which requires the introduction of the momentum operator $\mathcal{T}$, which acts as follows. Recall that $V$ is interpreted as the set of virtual shifts, So, if $v \in V$ is associated with a shift of the observer, $\mathcal{T}v$ yields the energy flow he observes.

CONCLUDING REMARKS

The models of point-free differential geometries are proposed as pairs $(A, V)$, called *differential algebras* which are the non-commutative generalization of Einstein algebras [1]. The substantial feature of non-commutativity is the discrepancy between the elements of the basic algebra (analog of the smooth functions) and scalars.

Luckily, the geometry of affine connection survives in non-commutative differential algebras, including the notions of torsion and curvature (as it was shown in [2]). The scalar curvature can also be defined under certain circumstances. If it becomes possible, the operator analog of Einstein equation is introduced.

The idea to consider vectors as differential operators applied to functional algebras, but defined on a broader class of spaces than manifolds, called *differential spaces*, were used to implement the spaces with singularities to general relativity. Heller et al. [8] have shown that the reasonable definition of differential structure can be formulated in terms of certain algebra $A$ of functions so that even the analog of Lorentz structure can be introduced. In particular, when $A$ is an
algebra of smooth functions on a manifold, the standard differential geometry is restored.

The approach we suggest can be considered as a reasonable way to quantize the gravity. The main problem arising here is to find the appropriate representations of the basic algebras. What about the source of basic algebras, this is the Wheeler’s suggestion to consider logic as pregeometry which could work here. The first step along these lines was made by Isham [9]: the lattice of all topologies over a set was considered, and the analog of creation and annihilation operators was suggested. The appropriate algebra could be taken as basic one. Moreover, starting from an arbitrary property lattice as background object, one can always build the semigroup (called generating [10]) whose annihilator lattice restore this property lattice. Then the algebra spanned on this semigroup could play the role of the basic algebra $\mathcal{A}$.

The next step is the spatialization procedure. When a differential structure $V$ and a metric on it are set up, the problem arises to extract usual (i.e. point) geometry from the triple $(\mathcal{A}, V, g)$. To return to points, we must consider a subalgebra $\mathcal{C} \subseteq \mathcal{A}$ such that $\mathcal{C}$ would be commutative (to enable functional representation) and in some sense concerted with $V$ and $g$. We did not yet tackle this problem of eigen-subalgebras in detail, whereas it looks as direct way to reveal events within our scheme. It is noteworthy that whenever the triple $(\mathcal{A}, V, g)$ is set up, there still may exist several functionally representable eigen-subalgebras associated with possibly non-isomorphic geometries. That means that the observed geometry depends on observation which is in complete accordance with quantum mechanical point of view.

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