Two loop expansion of the Schrödinger functional coupling $\alpha_{SF}$ in $SU(3)$ lattice gauge theory

A. Bode

a Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstr. 110, D-10115 Berlin, Germany

The two loop coefficient of the expansion of the Schrödinger functional coupling in terms of the lattice coupling is calculated for the $SU(3)$ Yang-Mills theory. This coefficient is required to relate lattice data to the $\overline{MS}$-coupling. As a byproduct of the calculation, the Schrödinger functional is improved to two loop order and the three loop coefficient of the beta function in the SF-scheme is derived.

1. Introduction

In the framework of the Schrödinger functional the renormalized coupling $\alpha_{SF}$ can be traced from low to high energy numerically on the lattice with finite size techniques. At high energy a conversion to $\alpha_{\overline{MS}}$ with perturbation theory is possible. The relation can be calculated by expanding both $\alpha_{SF}$ and $\alpha_{\overline{MS}}$ in $g_0$, the lattice bare coupling.

This program is completed for $SU(2)$ [2,4,5] where the two loop relation between $\alpha_{\overline{MS}}$ and $\alpha_{SF}$ was required in order to avoid a significant source of error in the conversion to $\alpha_{\overline{MS}}$. In the $SU(3)$ case [6] the remaining step is to compute the two loop coefficient between $\alpha_{SF}$ and $g_0$.

This coefficient also enters into an extension of the program to include quenched quarks, which is discussed in Martin Lüscher’s contribution to these proceedings.

The calculation is analogous to the $SU(2)$ case [2]. We have to calculate the perturbative coefficients depending on the box size $I = L/a$, where $\alpha_{SF}$ is defined. From this we are able to extract the continuum results and the $O(a)$ improvement terms. Combining the numerical data and the two loop relations results in the expected error reduction comparable to the $SU(2)$ case.

2. Definition of $\alpha_{SF}$

The coupling $\alpha_{SF}(q) = g_{SF}^2(L)/(4\pi)$, at $q = L^{-1}$ is defined via the effective action $\Gamma$:

$$\exp(-\Gamma) = \int D[U] \exp(-S[U]) \tag{1}$$

where the Wilson action

$$S[U] = \frac{1}{g^2_0} \sum_p w(p) \text{tr}\{1 - U(p)\} \tag{2}$$

is modified by the weight $w$ to achieve the $O(a)$ improvement: $w$ differs from one only for the plaquettes attached to the boundary fields, where $w = c_t(g_0)$ is expanded in a series of $g_0$. The coupling

$$g_{SF}^2(L) = \frac{\Gamma'(L,\eta)}{\Gamma(L,\eta)} \bigg|_{\eta=0} \tag{3}$$

is normalised via the classical action minimum $\Gamma_0$. The derivative is with respect to $\eta$, which parametrises the boundary fields applied at $t = 0, L$. The remaining three space dimensions have periodic boundary conditions. Details about the chosen diagonal and constant boundary fields can be found in [6].

3. Perturbative expansion and calculation

The perturbative expansion has to be taken around the induced nonzero background field. Since the propagators are not known analytically, we calculate them numerically for each $I$ and sum.
them up within each of the 13 diagrams\[10]\). The use of symmetries (for instance translation invariance and the cubic group in space) in this context reduces the numerical effort. The propagators are diagonal in the root basis of SU(3) and in spatial momentum space, and have been calculated there by a recurrence relation in time. Numerical efficiency and preserving maximal precision was taken into account in the program. In particular the \( R \) derivatives were taken analytically.

The expected independence of the gauge parameter \( \lambda_0 \) of \( \Gamma'_2 \), the validity of the symmetries, a comparison of \( \Gamma'_2 \) with a numerical derivative of \( \Gamma_2 \) and the recalculation of the SU(2) results are applied as tests as well as an independent calculation of Peter Weisz for small \( I \).

In 2 months CPU-time on a HP735 with 128MB main memory we obtained \( m_2^X(I) \) in the range \( I = 4 \ldots 32 \), where:

\[
g_s^2(L) = g_0^2 + m_1(I)g_0^4 + m_2(I)g_0^6 \ldots \quad (4)
\]

\[
m_1(I) = m_1^I(I) + c_1^{(1)} m_1^I(I) \quad (5)
\]

\[
m_2(I) = m_2^I(I) + c_1^{(1)} m_2^I(I) + \left[c_1^{(1)} \right]^2 m_2^I(I) + c_2^{(2)} m_2^I(I) + m_1(I)^2 \quad (6)
\]

From \[3\] \( m_1^I \) is known and \( m_1^I, m_2^I, m_2^\beta \) are derived analytically.

Symanzik’s analysis suggests as asymptotic expansion of \( m_2^X \)

\[
m_2^X(I) = \sum_{n=0}^{\infty} \frac{r_n^X + s_n^X \ln(I) + t_n^X \ln^2(I)}{I^n} \quad (7)
\]

where we expect for instance \( t_1^I = 0 \) from tree level improvement.

Using the Lüscher-Weisz \[8\] blocking method, which is based on

\[
R_n[f](L) = f(L) + \frac{L}{n} \frac{\partial}{\partial L} f(L) \quad n \geq 1 \quad (8)
\]

With \( f(L) = a + b \ln^{m}(L)/L^n \) we get:

\[
R_n[f](L) = \begin{cases} 
  a + c/L^{n+1} & m = 0 \\
  a + c \ln^{m-1}(L)/L^n & m > 0
\end{cases} \quad (9)
\]

This enables us to cancel some powers in the residual terms. We used a symmetric lattice derivative and traced the numerical errors during the blocking. The error of the constant term was obtained by a fit to the residual terms.

The universal \( \beta \)-function coefficient \( b_1 \) was subtracted after confirming it within 1.3\%. At present all errors are very conservative. A more detailed error analysis will be presented in \[10\]. The one loop improvement coefficient \( c_1^{(1)} \) was confirmed within errors. This should be considered as an additional test of the calculation.

Up to \( O(I^{-2}) \) times logarithms we get:

\[
m_1(I) = 2b_0 \ln(I) + .368282(11) \quad (10)
\]

\[
m_2(I) = 2b_1 \ln(I) + .048085(63) + m_1(I)^2 \quad (11)
\]

The continuum results are obtained by formally neglecting all negative powers of \( I \), so eqn. \[10\] are the continuum results.

4. Applying the expansion

Together with the expansion of \( \alpha_{\overline{MS}} \) in \( g_0 \) we get:

\[
\alpha_{\overline{MS}}(sq) = \alpha_{SF} + c_1(s) \alpha_{SF}(q)^2 + c_2(s) \alpha_{SF}(q)^3 \quad (12)
\]

\[
c_1(s) = -8\pi b_0 \ln(s) + 1.25562(14) \quad (13)
\]

\[
c_2(s) = c_1(s)^2 - 32\pi^2 b_1 \ln(s) + 1.197(10) \quad (14)
\]

With \( b_{SF}^{\overline{MS}} \) we are able to quote \( b_{SF}^{\overline{MS}} = 0.4827(88)/(4\pi)^3 \), where a fit to the data of numerical simulations gives \( b_{SF}^{\overline{MS}} = 1.58(8)/(4\pi)^3 \).

There is no reason that the coefficient of the fit coincides with \( b_{SF}^{\overline{MS}} \), since \( \alpha_{SF} \) is traced nonperturbatively on the lattice. Nevertheless the order is the same and we expect as quoted in \[11\] the same error reduction as in the SU(2).

The perturbative relation \[12\] between \( \alpha_{SF} \) and \( \alpha_{\overline{MS}} \) involves the scale factor \( s \). Fixing them by demanding \( c_1(s) = 0 \) or \( c_2(s) \) to be minimal gives comparable results:

\[
\alpha_{\overline{MS}}(2.048q) = \alpha_{SF}(q) + 0.271(11) \alpha_{SF}(q)^3 \quad (15)
\]

\[
\alpha_{\overline{MS}}(2.529q) = \alpha_{SF}(q) - 0.36895 \alpha_{SF}(q)^2 + 0.135(11) \alpha_{SF}(q)^3 \quad (16)
\]

where the quantities quoted without error are in all digits significant. Note beside the scale factor the small coefficients. This perturbative relation between the two physical couplings should be safe
for small enough couplings. Using the smallest numerically determined coupling $\alpha_{SF}$ we get for the SU(3) gauge theory:

$$\alpha_{MS}(14.5 \text{ GeV}) = 0.1146(22)(2)$$  \hspace{1cm} (17)

$$\alpha_{MS}(78.2 \text{ GeV}) = 0.08407(121)(5)$$  \hspace{1cm} (18)

where the first error arises from the numerical uncertainties and the scale determination at low energies. The second error results from the residual evolution and the conversion to the $\overline{MS}$ scheme. This should be compared with the results where $c_2$ is unknown and $b_{eff}^2$ is used:

$$\alpha_{MS}(14.06 / a) = 0.1145(23)(15)$$  \hspace{1cm} (19)

$$\alpha_{MS}(78.2 / a) = 0.08380(121)(59)$$  \hspace{1cm} (20)

The error reduction of the conversion by a factor of 10 was the main goal of this calculation.

Using the tadpole improved coupling $\tilde{\alpha}_0 = \alpha_0 / P$, we get a better behaved scale factor, but the coefficients are still relative large.

$$\alpha_{SF}(1.285 / a) = \tilde{\alpha}_0 + 1.914(11)\tilde{\alpha}_0^3$$  \hspace{1cm} (23)

$$\alpha_{SF}(1.586 / a) = \tilde{\alpha}_0 - 0.3695\tilde{\alpha}_0^2 + 1.778(11)\tilde{\alpha}_0^3$$  \hspace{1cm} (24)

As an example we can insert $\tilde{\alpha}_0 = 0.12 (\beta = 6.5 = 6/g_0^2, P = 0.6384)$ and get in eqn. (24) a 1-loop effect of 4.5% and a 2-loop effect of 2.6%. This casts doubt on the accuracy of tadpole improvement in this region.

6. Conclusion

The present two loop lattice calculation completes the connection between $\alpha_{SF}$ and $\alpha_{MS}$ in the SU(3) gauge theory. It has lead to a reduction of systematic errors comparable to the case of SU(2).

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5. Relation to the bare coupling

The relation of $\alpha_{SF}$ to the bare coupling $\alpha_0 = g_0^2/(4\pi)$ involves a large scale factor and large coefficients:

$$\alpha_{SF}(14.06 / a) = \alpha_0 + 4.178(11)\alpha_0^3$$  \hspace{1cm} (21)

$$\alpha_{SF}(17.36 / a) = \alpha_0 - 0.3695\alpha_0^2 + 4.042(11)\alpha_0^3$$  \hspace{1cm} (22)