Effective Lagrangian of QED

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From the Euler-Heisenberg formula we calculate the exact real part of the one-loop effective Lagrangian of Quantum Electrodynamics in a constant electromagnetic field, and determine its strong-field limit.

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1. Introduction

The possibility of electron-positron pair production from the vacuum of Quantum Electrodynamics (QED) vacuum was first pointed out by Sauter, Euler, Heisenberg and Schwinger [1,2,3] who studied the behavior of the Dirac vacuum in a strong external electric field. If the field is stronger than the critical value $E_c = m_e^2 c^3 / e \hbar$, the energy of the vacuum can be lowered by spontaneously creating an electron-positron pair. This is the Sauter-Euler-Heisenberg-Schwinger process. For many decades, both theorists and experimentalists have been interested in the aspects of electron-positron pair production from the QED vacuum by an external electromagnetic field. There are many reasons for the interest in the phenomenon of pair-production in a strong electric field. In addition to understand the behavior of QED in the strong coupling regime, the most compelling one is that both laboratory conditions and astrophysical events may now exist for observing this process.

The above process is governed by the imaginary part of the one-loop effective action in QED vacuum by an external electromagnetic field. In this article, we supplement the existing results by exact expressions for electrons.

2. QED in external electromagnetic fields

The QED Lagrangian describing the interacting system of photons, electrons and positrons reads

$$\mathcal{L} = \mathcal{L}_0^\infty + \mathcal{L}_0^{\text{ee}} + \mathcal{L}_\text{int},$$  

where the free Lagrangians $\mathcal{L}_0^\infty$ and $\mathcal{L}_0^{\text{ee}}$ for electrons and photons are expressed in terms of Dirac field $\psi(x)$ and an electromagnetic field $A_\mu(x)$ as follows:

$$\mathcal{L}_0^\infty = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m_e)\psi(x),$$  

$$\mathcal{L}_0^{\text{ee}} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \text{gauge-fixing term}.$$

Here $\gamma^\mu$ are the $4 \times 4$ Dirac matrices, $\bar{\psi}(x) \equiv \psi(x)^\dagger \gamma^0$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the electromagnetic field tensor. Minimal coupling gives rise to the interaction Lagrangian

$$\mathcal{L}_\text{int} = -e\bar{\psi}(x)\gamma^\mu \psi(x) A_\mu(x).$$

We are using natural units $\hbar = c = 1$.

An external electromagnetic field is incorporated by adding to the quantum field $A_\mu$ in (4) an unquantized external vector potential $A_\mu^e$, so that the total interaction becomes

$$\mathcal{L}_\text{int} + \mathcal{L}_\text{int}^e = -e\bar{\psi}(x)\gamma^\mu \psi(x) \left[ A_\mu(x) + A_\mu^e(x) \right].$$

Quantum field theory is defined by a functional integral formulation for the quantum mechanical partition function

$$Z[A^e] = \int \mathcal{D}A^e \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ i \int dt \mathcal{L} + \mathcal{L}_\text{int}^e \right],$$

which may now exist for observing this process.

The above process is governed by the imaginary part of the one-loop effective action in QED vacuum by an external electromagnetic field. In this article, we supplement the existing results by exact expressions for electrons.
external field \( A^e(x) \) varies smoothly over a finite region spacetime, we may define an approximately local effective Lagrangian \( \Delta \mathcal{L}_{\text{eff}}[A^e(x)] \):

\[
\Delta \mathcal{A}_{\text{eff}}[A^e] \approx \int d^4x \Delta \mathcal{L}_{\text{eff}}[A^e(x)] \approx V \Delta t \Delta \mathcal{L}_{\text{eff}}[A^e],
\]

where \( V \) is the spatial volume and time interval \( \Delta t = t_+ - t_- \).

For a large time interval \( \Delta t = t_+ - t_- \to \infty \), the amplitude of the vacuum to vacuum transition \( \langle \text{out}, 0 | 0, \text{in} \rangle \) has the form,

\[
\langle \text{out}, 0 | 0, \text{in} \rangle = e^{-i(\Delta \mathcal{E}_0 - \Gamma \Delta t)/2},
\]

where \( \Delta \mathcal{E}_0 = \mathcal{E}_0(A^e) - \mathcal{E}_0(0) \) is the difference between the vacuum energies in the presence and the absence of the external field, \( \Gamma \) is the vacuum decay rate, and \( \Delta t \) the time over which field is non zero. The probability that the vacuum remains as it is in the presence of the external classical electromagnetic field is

\[
\langle \text{out}, 0 | 0, \text{in} \rangle^2 = e^{-2i \Delta \mathcal{A}_{\text{eff}}[A^e]}. \tag{11}
\]

This determines the decay rate of the vacuum in an external electromagnetic field:

\[
\frac{\Gamma}{V} = \frac{2 \text{Im} \Delta \mathcal{A}_{\text{eff}}[A^e]}{\Delta t V} \approx \text{Im} \Delta \mathcal{L}_{\text{eff}}[A^e]. \tag{12}
\]

The finite lifetime is caused by the production of electron and positron pairs. The external field changes the energy density by

\[
\frac{\Delta \mathcal{E}_0}{V} = -\frac{\text{Re} \Delta \mathcal{A}_{\text{eff}}[A^e]}{\Delta t V} \approx -\text{Re} \Delta \mathcal{L}_{\text{eff}}[A^e]. \tag{13}
\]

3. Schwinger formula for pair production

The Dirac field appears quadratically in the partition functional \( \mathcal{Z} \) and can be integrated out, leading to

\[
\mathcal{Z}[A^e] = \int DA \text{Det}(i\bar{\sigma} - e\mathcal{A}(x) - m_e + i\eta)\; \; \nu \equiv \gamma^\mu \partial_\mu,
\]

where \( \nu \) denotes the functional determinant of the Dirac operator. Ignoring the fluctuations of the electromagnetic field, the result is a functional of the external vector potential \( A^e(x) \):

\[
\mathcal{Z}[A^e] \approx \text{const.} \times \text{Det}(i\bar{\sigma} - e\mathcal{A}(x) - m_e + i\eta). \tag{15}
\]

which is the one-loop approximation of \( \mathcal{O}(\hbar) \). The infinitesimal constant \( i\eta \) with \( \eta > 0 \) specifies the treatment of singularities in energy integrals. From Eqs. \( 7 - 15 \), the effective action \( \Delta \mathcal{A}_{\text{eff}}[A^e] \) is given by

\[
\Delta \mathcal{A}_{\text{eff}}[A^e] = -\text{Tr} \ln \left\{ [i\bar{\sigma} - eA^e(x) - m_e + i\eta] \right\}.
\]

where \( \text{Tr} \) denotes the functional and Dirac trace. In physical unit, this is of order \( \hbar \). The result may be expressed as a one-loop Feynman diagram, so that one speaks of a one-loop approximation. More convenient will be the equivalent expression

\[
\Delta \mathcal{A}_{\text{eff}}[A^e] = -\frac{i}{2} \text{Tr} \ln \left\{ ([i\bar{\nu} - eA^e(x)]^2 - m_e^2 + i\eta) \right\} \cdot \frac{1}{-\partial^2 - m_e^2 + i\eta}, \tag{17}
\]

where

\[
[i\bar{\nu} - eA^e(x)]^2 = [i\partial_\mu - eA^e_\mu(x)]^2 + \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}^e.
\]

Using the identity

\[
\ln \frac{a_2}{a_1} = \int_0^\infty \frac{ds}{s} \left( e^{is(a_1 + i\eta)} - e^{is(a_2 + i\eta)} \right), \tag{19}
\]

Eq. \( 17 \) becomes the Schwinger form

\[
\Delta \mathcal{A}_{\text{eff}}[A^e] = \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{-is(m_e^2 - \eta)} \cdot \text{Tr} \left\{ [i\partial_\mu - eA^e_\mu(x)]^2 + \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}^e \right\} - e^{-is\partial^2} |x|, \tag{20}
\]

where \( \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \), \( F_{\mu\nu}^e = \partial_\mu A^e_\nu - \partial_\nu A^e_\mu \), and \( \langle \cdots |x\rangle \) are the diagonal matrix elements in the local basis \( |x\rangle \). This is defined \( \bar{\mathcal{L}} \) Dirac by the matrix elements with the momentum eigenstates \( |k\rangle \) being plane waves: \( \langle x|k\rangle = e^{-ikx} \). The symbol \( \text{Tr} \) denotes integral \( \int d^4x \) in spacetime and the trace in spinor space. For constant electromagnetic fields, the integrand in \( 20 \) does not depend on \( x \), and \( \sigma^{\mu\nu} F_{\mu\nu}^e \) commutes with all other operators. This will allow us to calculate the exponential in Eq. \( 20 \) explicitly. The presence of \( -i\eta \) in the mass term ensures the convergence of integral for \( s \to \infty \).

If only a constant electric field \( E \) is present, it may be assumed to point along the \( z \)-axis, and one can choose a gauge such that \( A^e_z = -Et \) is the only nonzero component of \( A^e_\mu \). Then one finds

\[
\text{tr} \exp is \left( \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}^e \right) = 4 \cosh(seE), \tag{21}
\]

where the symbol \( \text{tr} \) denotes the trace in spinor space. Using commutation relation \( [\partial_\mu, x^0] = 1 \), one computes the exponential term in the effective action \( 20 \) (c.e.g. \( 3 \))

\[
\langle x| \exp is \left[ (i\partial_\mu - eA^e_\mu(x))^2 + \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}^e \right] |x\rangle = \frac{eE}{(2\pi)^2is} \coth(eEs).
\]

The second term in Eq. \( 20 \) is obtained by setting \( E = 0 \) in Eq. \( 22 \), so that then the effective action \( 20 \) yields,

\[
\Delta \mathcal{A}_{\text{eff}} = \frac{1}{2(2\pi)^2} \int d^4x \int_0^\infty \frac{ds}{s^3} [eEs \coth(eEs) - 1] \cdot e^{-is(m_e^2 - \eta)}. \tag{23}
\]
Since the field is constant, the integral over $x$ gives a volume factor, and so the effective action (20) can be attributed to the spacetime integral over an effective Lagrangian (21)
\[
\Delta \mathcal{L}_{\text{eff}} = \frac{1}{2(2\pi)^2} \int_0^\infty \frac{ds}{s^3} \left[ eEs \coth(eEs) - 1 \right] e^{-is(m^2 - \eta^2)}.
\]

By expanding the integrand in Eq. (21) in powers of $e$, one obtains,
\[
\frac{1}{s^3} \left[ eEs \coth(eEs) - 1 \right] e^{-is(m^2 - \eta^2)} = \left[ \frac{e^2}{3} E^2 - \frac{e^4}{45} E^4 + \mathcal{O}(e^6) \right] e^{-is(m^2 - \eta^2)}.
\]

The small-$s$ divergence in the integrand,
\[
\frac{e^2}{3} E^2 \frac{1}{2(2\pi)^2} \int_0^\infty \frac{ds}{s^3} e^{-is(m^2 - \eta^2)},
\]

is proportional to the electric field in the original Maxwell Lagrangian. The divergent term (20) can therefore be removed by a renormalization of the field $E$. Thus, we add subtract counterterm in Eq. (24) and form,
\[
\Delta \mathcal{L}_{\text{eff}} = \frac{1}{2(2\pi)^2} \int_0^\infty \frac{ds}{s^3} \left[ eEs \coth(eEs) - 1 - \frac{e^2}{3} E^2 s^2 \right] e^{-is(m^2 - \eta^2)}.
\]

Remembering Eq. (22), we find from (20) the decay rate of the vacuum per unit volume
\[
\Gamma = \frac{1}{(2\pi)^2} \text{Im} \int_0^\infty \frac{ds}{s^3} \left[ eEs \coth(eEs) - 1 - \frac{e^2}{3} E^2 s^2 \right] e^{-is(m^2 - \eta^2)}.
\]

The integral (25) can be evaluated analytically by proceeds by the method of residues. Since the integrand is even, the integral can be extended to the entire $s$-axis. After this, the integration contour is deformed to enclose the negative imaginary axis and to pick up the contributions of the poles of the coth function at $s = n\pi/eE$. The result is
\[
\Gamma = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \exp \left( -\frac{n\pi E_e}{E} \right),
\]

where $E_e = m^2/e$ is the value of critical field. This result due to Schwinger (3) is valid to lowest order in $\hbar$ for arbitrary constant electric field strength.

An analogous calculation for a charged scalar field yields
\[
\Gamma = \frac{\alpha E^2}{2\pi^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} \exp \left( -\frac{n\pi E_e}{E} \right).
\]

These Schwinger results complete the development. The leading $n = 1$ -terms agree with the WKB results obtained by Sauter [1] and Heisenberg-Euler [2].

4. Pair production in constant electromagnetic fields

If the constant external field has both $E$ and $B$ nonzero, the effective action will be a function of the two Lorentz invariants. We can now go to an arbitrary Lorentz frame by expressing the result in terms of the scalar $S$ and the pseudoscalar $P$, that can be formed from arbitrary electromagnetic fields $B$ and $E$ fields:
\[
S \equiv \frac{1}{4} F_{\mu\nu} F_{\mu\nu} = \frac{1}{2} (E^2 - B^2); \quad P \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = E \cdot B,
\]

where $F_{\mu\nu} = \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa}$ is the dual field tensor. It is useful to define the related invariants $\varepsilon$ and $\beta$ as the solutions of the invariant equations
\[
\varepsilon^2 - \beta^2 \equiv E^2 - B^2 \equiv 2S, \quad \varepsilon \equiv E \cdot B \equiv P,
\]

which read explicitly
\[
\left\{ \begin{array}{c}
\varepsilon \\
\beta
\end{array} \right\} = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \pm (E^2 - B^2)} = \sqrt{(S^2 + P^2)^{1/2} \pm S}.
\]

Then exists a special Lorentz frame to be called the center-of-fields frame, where the fields $B_{\text{CF}}$ and $E_{\text{CF}}$ are parallel. In this frame, $\beta = |B_{\text{CF}}|$ and $\varepsilon = |E_{\text{CF}}|$.

Relations (32) are invariant under the discrete duality transformation:
\[
|B| \to -i|E|, \quad |E| \to i|B|,
\]

i.e., under
\[
\beta \to -i\varepsilon, \quad \varepsilon \to i\beta.
\]

This implies that effective action for a pure magnetic field can be simply obtained by replacing $E \to iB$ in Eqs. (22,27), so that
\[
\langle x \mid \exp \{ i \partial_\mu - e A_\mu^e(x) \}^2 + \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} \mid x \rangle = \frac{eB}{(2\pi)^2 i\varepsilon} \cot(eBs),
\]

and
\[
\Delta \mathcal{L}_{\text{eff}} = \frac{1}{2(2\pi)^2} \int_0^\infty \frac{ds}{s^3} \left[ eBs \cot(eBs) - 1 + \frac{e^2}{3} B^2 s^2 \right] e^{-is(m^2 - \eta^2)}.
\]

If both electric and magnetic fields $E$ and $B$ are present, we assume the parallel $E_{\text{CF}}$ and $B_{\text{CF}}$ fields point along the $z$-axis in center-of-fields frame. We further choose a gauge such that only $A_x^e = -E_{\text{CF}}$, $A_y^e = B_{\text{CF}}$, $A_z^e = x^1$ are nonzero. The exponential in the effective action Eq. (20) can then be factorized into a product of the magnetic part and the electric part. Based on the duality, we obtain directly
\[
\text{tr} \exp \{ i \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} \} = 4 \cosh(seE_{\text{CF}}) \cos(seB_{\text{CF}}),
\]
Performing the same substraction as before, we obtain the effective Lagrangian

\[
\Delta L_{\text{eff}} = \frac{1}{2(2\pi)^2} \int_0^\infty \frac{ds}{s^3} \left[ e^2 E_C B_C s^2 \coth(eE_C s \cot(eB_C s)) - 1 - \frac{e^2}{3} (E_C^2 - B_C^2 s^2) \right] \cdot e^{-i(s^2 - i\eta)}. \tag{40}
\]

In an arbitrary Lorentz frame, \(E_C\) and \(B_C\) are replaced by the invariants \(\varepsilon\) and \(\beta\):

\[
\Delta L_{\text{eff}} = \frac{1}{2(2\pi)^2} \int_0^\infty \frac{ds}{s^3} \left[ e^2 \varepsilon \beta s^2 \coth(e\varepsilon s \cot(e\beta s)) - 1 - \frac{e^2}{3} (e^2 - \beta^2) s^2 \right] e^{-i(s^2 - i\eta)}. \tag{41}
\]

The decay rate becomes

\[
\Gamma = \frac{\alpha e^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 \tanh n\pi \beta / \varepsilon} \exp \left( -\frac{n\pi E_c}{\varepsilon} \right), \tag{43}
\]

which reduces for \(\beta \to 0\) \((\mathbf{B} = 0)\) correctly to eq. \(\mathbf{[1]}\).

The analogous result for bosonic fields is

\[
\Gamma = \frac{\alpha e^2}{2\pi} \sum_{n=1}^{\infty} (-1)^n \frac{n\pi \beta / \varepsilon}{\sinh n\pi \beta / \varepsilon} \exp \left( -\frac{n\pi E_c}{\varepsilon} \right), \tag{44}
\]

Note that the magnetic field produces in the fermionic case a extra factor \((n\pi \beta / \varepsilon) / \tanh(n\pi \beta / \varepsilon) > 1\) in each term which enhances the decay rate. The bosonic series \(\mathbf{(1)}\), on the other hand, carries in each term a suppression factor \((n\pi \beta / \varepsilon) / \sinh n\pi \beta / \varepsilon < 1\).

The decay rate \(\Gamma / V\) gives the number of electron-positron pairs produced per unit volume. The prefactor can be estimated on dimensional grounds. It has the dimension of \(E_c^2 / h\), i.e., \(m^4 e^5 / \hbar^4\). This arises from the energy of a pair \(2m_e c^2\) divided by the volume whose diameter is the Compton wavelength \(\hbar / m_e c\) produced within a Compton time \(\hbar / m_e c^2\). The exponential factor suppresses pair production as long as the electric field is much smaller than the critical electric field \(E_c\). The general results \(\mathbf{(3)}\) was first obtained by Schwinger for scalar and spinor electrodynamics.

5. Effective nonlinear Lagrangian

In this section, we evaluate further the effective Lagrangian \(\mathbf{[1]}\) for arbitrary constant electromagnetic fields \(\mathbf{E}\) and \(\mathbf{B}\). Making the expansions,

\[
e\varepsilon s \coth(e\varepsilon s) = \sum_{n=-\infty}^{\infty} \frac{s^2}{(s^2 + \tau_n^2)} \cdot \tau_n = n\pi / e\varepsilon, \tag{45}
\]

\[
e\beta s \cot(e\beta s) = \sum_{m=-\infty}^{\infty} \frac{s^2}{(s^2 - \tau_m^2)} \cdot \tau_m = m\pi / e\beta, \tag{46}
\]

we obtain the finite effective Lagrangian,

\[
\Delta L_{\text{eff}} = \frac{1}{2(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_0^\infty ds \frac{s}{s^2 + \tau_n^2} \left[ \delta_{mn} \right] e^{-i(s^2 - i\eta)} \tag{47}
\]

where divergent terms \(n \neq m = 0\) and \(n = m = 0\) are excluded from the sum, as indicated by a prime. The symbol \(\delta_{ij} \equiv 1 - \delta_{ij}\) denotes the complementary Kronecker-\(\delta\) which vanishes for \(i = j\). The divergent term with \(n = m = 0\) is eliminated by the zero-field subtraction in Eq. \(\mathbf{[1]}\), while the divergent terms \(n \neq m = 0\) and \(n = m \neq 0\) are eliminated by the second subtraction in Eq. \(\mathbf{[1]}\).

This can be seen by performing the sums

\[
\sum_{n=1}^{\infty} \frac{1}{\tau_n^2} = \left( \frac{e\varepsilon}{\pi} \right)^k \zeta(k); \quad \sum_{n=1}^{\infty} \frac{1}{\tau_n^2} = \left( \frac{e\beta}{\pi} \right)^k \zeta(k), \tag{49}
\]

where \(\zeta(k) = \sum_n 1/n^k\) is the Riemann function.

The infinitesimal \(i\eta\) accompanying the mass term in the \(s\)-integral \(\mathbf{[1]}\) is equivalent to replacing \(e^{-i(s^2 - i\eta)}\) by \(e^{-i(1-i\eta)m_e^2}\). This implies that \(s\) is to be integrated slightly below (above) the real axis for \(s > 0\) \((s < 0)\). Equivalently one may shift the \(\tau_m\) \((-\tau_m)\) variables slightly upwards (downwards) to \(\tau_m + i\eta\) \((-\tau_m - i\eta)\) in the complex plane. In order to calculate the finite effective Lagrangian \(\mathbf{[1]}\), the factor \(e^{-i(1-i\eta)m_e^2}\) is divided into its sin and cos parts:

\[
\Delta L_{\text{sin}}^\text{eff} = \frac{-i}{4(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_{-\infty}^{\infty} ds ds \frac{\delta_{mn}}{s^2 + \tau_n^2} \left( \frac{\delta_{mn}}{s^2 - \tau_m^2} \right) \sin[s(1-i\eta)m_e^2]; \tag{50}
\]

\[
\Delta L_{\text{cos}}^\text{eff} = \frac{1}{4(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_0^\infty ds ds \frac{\delta_{mn}}{\tau_n^2 + \tau_m^2} \left( \frac{\delta_{mn}}{s^2 - \tau_m^2} \right) \cos[s(1-i\eta)m_e^2]. \tag{51}
\]
The sin part has an even integrand allowing for an extension of the s-integral over the entire s-axis. The contours of integration can then be closed by infinite semicircles in the half-plane, the integration receives contributions from poles $\pm \tau_m, \pm i\eta$, so that the residue theorem leads to

$$\Delta L_{\text{eff}}^{\sin} = \frac{\varepsilon \varepsilon \beta}{2(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{m} \coth \left( \frac{m \pi \varepsilon \beta}{\varepsilon} \right) \exp(-n\pi E_c/\varepsilon)(52)$$

$$-i \frac{\varepsilon \varepsilon \beta}{2(2\pi)^2} \sum_{m=1}^{\infty} \frac{1}{m} \coth \left( \frac{m \pi \varepsilon \beta}{\beta} \right) \exp(i m \pi E_c/\beta)(53)$$

The first part produces the exact non-perturbative Schwinger rate for pair production.

Shifting $s \to s - i\eta$, we rewrite the cos part of effective Lagrangian as

$$\Delta L_{\text{eff}}^{\cos} = \frac{1}{2(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_{0}^{\infty} ds \cos(sm^2) \frac{\tau_n^2 + \tau_m^2}{\tau_n^2 + \tau_m^2} \cdot \left( \frac{s \delta_{n0}}{s^2 - \tau_n^2 - i\eta} - \frac{s \delta_{n0}}{s^2 + \tau_n^2 - i\eta} \right). (54)$$

In the first term of magnetic part, singularities $s = \tau_m, (m > 0)$ and $s = -\tau_m, (m < 0)$ appear in integrating s-axis. We decompose,

$$\frac{s}{s^2 - \tau_m^2 - i\eta} = \frac{\pi}{2} \delta(s - \tau_m) + i \frac{\pi}{2} \delta(s + \tau_m) + \mathcal{P} \frac{s}{s^2 - \tau_m^2}, (55)$$

where $\mathcal{P}$ indicates the principle value under the integral. The integrals over the $\delta$-functions give

$$\Delta L_{\text{eff}}^{\cos} = \frac{\varepsilon \varepsilon \beta}{2(2\pi)^2} \sum_{m=1}^{\infty} \frac{1}{m} \coth \left( \frac{m \pi \varepsilon \beta}{\beta} \right) \exp(i m \pi E_c/\beta), (56)$$

which exactly cancel the second part of the sin part $\Delta L_{\text{eff}}^{\sin}$. It remains to find the principal-value integrals in Eq. (54),

$$\Delta L_{\text{eff}}^{\cos} = \frac{1}{2(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_{0}^{\infty} ds \cos(sm^2) \cdot \left( \frac{s \delta_{n0}}{s^2 - \tau_n^2} - \frac{s \delta_{n0}}{s^2 + \tau_n^2} \right). (57)$$

We rewrite the cos function as $\cos(sm^2) = \cos(1m^2) + e^{-im \eta}/2$ and make the rotations of integration contours by $\pm \pi/2$ respectively,

$$\Delta L_{\text{eff}}^{\cos} = \frac{1}{2(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_{0}^{\infty} d\tau \cos(\tau^2 - (i\tau m_e^2)/2) \cdot \left( \frac{\delta_{n0} \tau e^{-\tau}}{\tau^2 - (i\tau m_e^2)^2} - \frac{\delta_{n0} \tau e^{-\tau}}{\tau^2 - (\tau m_e^2)^2} \right)(58)$$

Using the formulas (see 3.354, 8.211.1 and 8.211.2 in Ref. [6])

$$J(z) = \mathcal{P} \int_{0}^{\infty} ds \frac{se^{-s}}{s^2 - z^2} = -\frac{1}{2} \left[ e^{-z} \text{Ei}(z) + e^z \text{Ei}(-z) \right], (59)$$

where $\text{Ei}(z)$ is the exponential-integral function,

$$\text{Ei}(z) \equiv \mathcal{P} \int_{-\infty}^{z} dt \frac{e^t}{t} = \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k k!}, (60)$$

we obtain the principal-value integrals

$$\left( \Delta L_{\text{eff}}^{\cos} \right)_{\mathcal{P}} = \frac{1}{2(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_{0}^{\infty} d\tau \cos(\tau^2 - (i\tau m_e^2)/2) \cdot \left[ \delta_{n0} \tau \left( \delta_{n0} \tau \right) - \delta_{n0} \delta_{n0} \right]. (61)$$

With the help of the series and asymptotic representation (see formula 8.215 in Ref. [6]) of the exponential-integral function $\text{Ei}(z)$ for large $z$, corresponding to weak electromagnetic fields ($\varepsilon \ll 1, \beta \ll 1$),

$$J(z) = \frac{1}{z^2} - \frac{6}{z^4} - \frac{120}{z^6} - \frac{5040}{z^8} - \frac{362880}{z^{10}} + \cdots, (62)$$

and Eq. (51), we find,

$$\left( \Delta L_{\text{eff}}^{\cos} \right)_{\mathcal{P}} = \frac{1}{2(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \int_{0}^{\infty} d\tau \cos(\tau^2 - (i\tau m_e^2)/2) \cdot \left[ \delta_{n0} \left( \frac{1}{\tau_n^2 m_e^2} + \frac{6}{\tau_n^4 m_e^4} + \frac{120}{\tau_n^6 m_e^6} + \cdots \right) \right] (63)$$

Applying the summation formulas, the weak-field expansion is seen to agree with the Heisenberg and Euler effective Lagrangian.

$$\left( \Delta L_{\text{eff}} \right)_{\mathcal{P}} = \frac{2\alpha^2}{45m_e^2} \left\{ (E^2 - B^2)^2 + 7(E \cdot B)^2 \right\}$$

$$+ \frac{64\pi^2}{315m_e^2} \left\{ 2(E^2 - B^2)^3 + 13(E^2 - B^2)(E \cdot B)^2 \right\}$$

$$+ \cdots, (64)$$

which is expressed in terms of a powers series of weak electromagnetic fields up to $O(\alpha^3)$. The expansion coefficients of the terms of order $n$ have the general form $m_n^4/(E_c)^n$. As long as the fields are much smaller than $E_c$, the expansion converges.

On the other hand, using the series and asymptotic representation (formulas 8.214.1 and 8.214.2 in Ref. [6]) of the exponential-integral function $\text{Ei}(z)$ for small $z \ll 1$, corresponding to strong electromagnetic fields ($\varepsilon \gg 1, \beta \gg 1$),

$$J(z) = -\frac{1}{2} \left[ e^{-z} \ln(z) + e^z \ln(-z) \right] + \gamma + O(z), (65)$$
with $\gamma = 0.577216$ being the Euler-Mascheroni constant, we obtain the leading terms in the strong-field expansion of Eq. (66),

$$\left(\Delta L_{\text{eff}}^{\text{cos}}\right)^{p} = \frac{1}{2(2\pi)^{2}} \sum_{n,m=-\infty}^{\infty} \frac{1}{\tau_{m}^{2} + \tau_{n}^{2}} \left[ \delta_{n0} \ln(\tau_{n} m_{e}^{2}) - \delta_{m0} \ln(\tau_{m} m_{e}^{2}) \right] + \cdots. \quad (66)$$

In the case of vanishing magnetic field $B = 0$ and $m = 0$ in Eq. (66), we have

$$\left(\Delta L_{\text{eff}}^{\text{cos}}\right)^{p} = \frac{1}{2(2\pi)^{2}} \sum_{n=1}^{\infty} \frac{1}{\tau_{n}^{2}} \ln(\tau_{n} m_{e}^{2}) + \cdots = \frac{e^{2}E^{2}}{8\pi^{4}} \sum_{n=1}^{\infty} \frac{1}{m_{e}^{2}} \ln \left( \frac{n\pi m_{e}}{E} \right) + \cdots. \quad (67)$$

for a strong electric field $E$. In the case of vanishing electric field $E = 0$ and $n = 0$ in Eq. (66), we obtain for a strong magnetic field $B$,

$$\left(\Delta L_{\text{eff}}^{\text{cos}}\right)^{p} = -\frac{1}{2(2\pi)^{2}} \sum_{m=1}^{\infty} \frac{1}{\tau_{m}^{2}} \ln(\tau_{m} m_{e}^{2}) + \cdots = -\frac{e^{2}B^{2}}{8\pi^{4}} \sum_{m=1}^{\infty} \frac{1}{m_{e}^{2}} \ln \left( \frac{m\pi E_{c}}{B} \right) + \cdots. \quad (68)$$

a result first considered by Weisskopf [7].

6. Concluding remarks

We have presented in Eqs. (52, 53, 56, 61) closed form result for the one-loop effective Lagrangian $\Delta L_{\text{eff}}$ [11] for arbitrary strength of constant electromagnetic fields. The results will receive fluctuation corrections from higher loop diagrams. These carry one or more factors $\alpha, \alpha^{2}, \ldots$ and are thus suppressed by factors $1/137$. Thus results are valid for all field strenghts with an error no larger than roughly 1%. If we include, for example, the two-loop correction, the first term in the Heisenberg and Euler effective Lagrangian becomes [8]

$$\left(\Delta L_{\text{eff}}^{\text{cos}}\right)^{p} = \frac{2e^{2}}{45m_{e}^{2}} \left\{ \left( 1 + \frac{40\alpha}{9\pi} \right) (E^{2} - B^{2})^{2} + 7 \left( 1 + \frac{1315\alpha}{252\pi} \right) (E \cdot B)^{2} \right\}. \quad (69)$$

Readers can consult the recent review article [2], where one finds discussions and computations of the effective Lagrangian at tow-loop levels, in homogeneous and inhomogeneous fields.

The interaction of an external electromagnetic field with the QED vacuum leads to the appearance of nonlinear real and imaginary parts in the effective Lagrangian $\Delta L_{\text{eff}}$ of the electromagnetic field. Electron and positron pair production is only one aspect of the phenomena connecting to these nonlinearities. As we remarked before the rate $\Gamma/V$ of pair production [20] is exponentially small when $E \lesssim E_{c}$.

Unfortunately, it seems inconceivable to produce a macroscopic static field with electric field strengths of the order of the critical value in any ground laboratory to directly observe the Sauter-Euler-Heisenberg-Schwinger process of electron-positron pair production in vacuum. In Ref. [14], we discuss some ideas: (i) heavy-ion collisions [11], (ii) the focus of coherent laser beams [12] and (iii) electron beam-laser collisions [13], to experimentally create a transient electric field $E \lesssim E_{c}$ in earth-bound laboratories, whose lifetime is expected to be long enough (larger than $\hbar/m_{e}c^{2}$) for the pair production process to take place. In Ref. [11], we also discuss (i) the idea that the supercritical conditions can be reached on the time scale of gravitational collapse ($GM/c^{2}$, $10^{-2}$ sec. for $M = 10M_{\odot}$) around an astrophysical object [17] and (ii) the application of Sauter-Euler-Heisenberg-Schwinger process to the black hole physics in connection with the observed astrophysical phenomenon of gamma ray bursts [13].

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