THE SPECTRAL PROBLEM, SUBSTITUTIONS
AND ITERATED MONODROMY

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Abstract. We provide a self-similar measure for the self-similar group $G$ acting faithfully on the binary rooted tree, defined as the iterated monodromy group of the quadratic polynomial $z^2 + i$. We also provide an $L$-presentation for $G$ and calculations related to the spectrum of the Markov operator on the Schreier graph of the action of $G$ on the orbit of a point on the boundary of the binary rooted tree.

INTRODUCTION

It was observed recently that the class of self-similar groups naturally appears in mathematics. The most recent examples come from combinatorics and are related to one of the most famous combinatorial problems known as Hanoi Towers Game [13]. Slightly older examples, due first of all to Nekrashevych [18] (see also [2]), are related to holomorphic dynamics and random walks.

Self-similar groups can be defined as groups generated by all the states of a (not necessarily finite) Mealey automaton [11]. Of particular interest and importance is the case when the automaton is finite, since in that case the obtained group is finitely generated.

Self-similar groups have many interesting and important properties. The class of self-similar groups contains exotic examples, such as groups of Burnside type or groups of intermediate growth, as well as familiar examples, such as free groups or free products of finite groups, that are well known and are regular objects of study in combinatorial group theory.

One of the most remarkable discoveries in the recent years is the observation, due to Nekrashevych, that the so-called iterated monodromy groups (IMG), which can be related to any self-covering map, belong to the class of self-similar groups and that, in the most natural situations, there is an explicit procedure representing them by finite automata.

Even in the case of quadratic maps over $\mathbb{C}$ one gets a rich theory with wonderful applications both to holomorphic dynamics and to group theory [1, 4, 8].

Already the simplest examples of quadratic polynomials, such as $z^2 - 1$ or $z^2 + i$, show that the corresponding groups can be quite complicated and can have extraordinary properties.

The group IMG($z^2 - 1$) is called the Basilica group after the Julia set of $z^2 - 1$ which (mildly) resembles the roof of the San Marco Basilica in Venice (the top part of the Julia set is the roof and the bottom part is its reflection in the water).
Basilica group is torsion free, of exponential growth, amenable but not elementary (and not even subexponentially) amenable \[5, 15\], has trivial Poisson boundary, is weakly branch, and has many other interesting properties.

The main object of this article is the group \(IMG(z^2 + i)\) (denoted by \(G\) in the rest of the text), introduced in \[2\] and later studied by Bux and Perez \[6\], who proved that \(G\) has intermediate growth. This is not the first example of a self-similar group of intermediate growth (the first examples were constructed in \[7, 8\]), but it is the first example of a group of intermediate growth that naturally appears in the area of applications of group theory.

We start with a quick introduction to the theory of self-similar groups and, in particular, to iterated monodromy groups. We are aiming for a self-contained treatment, which would make it possible for the reader to understand the context of the paper completely without reading other sources. In particular we give very detailed calculation of the action of \(G = IMG(z^2 + i)\) on the binary rooted tree. Then we show that the group \(G\) is a regular branch group, thus presenting an example of a branch group which naturally appears in holomorphic dynamics. The main body of the article is devoted to the calculation of an L-presentation for \(G\) (i.e., a presentation of a group by generators and relations which involves a finite set of relations and their iterations by substitution). Although it is known that L-presentations are quite common for groups of branch type the number of examples in which explicit computation is possible is rather small.

The presence of L-presentations is important from different points of view. Such presentations are at the first level of complexity after the finite presentations and quite often provide the simplest way to describe a group that is not finitely presented \((G\) is not finitely presented \[19\]). Further, such presentations can be used to embed a group into a finitely presented group in a way that preserves many properties of the original group. We use the obtained L-presentation of \(G\) to embed \(G\) into a finitely presented group with 4 generators and 10 relators, which is amenable but not elementary amenable (this approach has been used for the first time in \[9\]).

The rest of the article deals with finding a self-similar measure on \(G\). The notion of a self-similar measure was introduced by Kaimanovich in \[17\], who extends some ideas (in particular the idea of self-similarity of a random walk) that appeared in the work of Bartholdi and Virág \[5\].

The self-similar measure is closely related to the problem of computation of the spectrum of a Hecke type operator that can be related to any group acting on a rooted tree and to the problem of computation of the spectrum of the discrete Laplace operator (or, what is almost the same, the Markov operator) on the boundary Schreier graph of a group (i.e., the graph of the action of the group on the orbit of a point of the boundary). A general approach to the spectral problem (which extends the ideas outlined in \[11, 14\]) based on a renormalization principle and leading to questions on amenability, multidimensional dynamics and multiparametric self-similarity of operators is described in \[12\]. Unfortunately, the spectral problem is not solved yet in our situation. What we are able to construct is a rational map on \(\mathbb{R}^3\) whose proper invariant set (shaped as a “strange attractor”) gives the spectrum of the Markov operator after intersection by a corresponding line. Here we have a situation analogous to the case of Basilica group \[16\]. Further efforts in the description of the shape of the attractor (and hence of the spectrum) are needed.
The Schreier graph in this case, viewed through a macroscope, has a form of a dendrite and this is a reflection of a general fact relating the geometry of Schreier graphs and Julia sets proved by Nekrashevych [18].

In any case, our computations allow us to find a self-similar nondegenerate measure on $G$, which gives a self-similar random walk on the group. The study of asymptotic properties of such random walks is a promising direction and will be one of our subsequent subjects of investigation.

1. Iterated Monodromy Groups

The theory of iterated monodromy groups was developed mostly by Nekrashevych. A very detailed exposition can be found in his monograph [18]. Here we give a definition and some basic properties of these groups.

Consider a path connected and locally path connected topological space $M$. Let $M_1$ be an open and path connected subset of $M$ and $f: M_1 \to M$ be a $d$-fold covering map. Fix a base point $t \in M$ and let $\pi_1(M, t)$ be the corresponding fundamental group. The set of iterated preimages of $t$ under $f$ has a natural structure of a $d$-ary rooted tree $T$. Namely, each point $s$ from this set has exactly $d$ preimages $s_1, \ldots, s_d$ and these preimages are declared to be adjacent to $s$ in $T$. The $n$th level of the tree $T$ consists of the $d^n$ points in the set $f^{-n}(t)$. Note that although the intersection of $f^{-n}(t)$ and $f^{-m}(t)$ may be nonempty for $m \neq n$, we formally consider the set of vertices of $T$ to be a disjoint union of the sets $f^{-n}(t)$, $n \geq 0$.

There is a natural action of $\pi_1(M, t)$ on the tree $T$. Let $\gamma \in \pi_1(M, t)$ be a loop based at $t$. For any point $s$ of $f^{-n}(t)$, there is a unique preimage $\gamma[s]$ of $\gamma$ under $f^n$ which starts at $s$ and ends at a point $s'$, which also belongs to $f^{-n}(t)$. We define an action of $\gamma$ on $T$ by setting $\gamma(s) = s'$. This action induces a permutation of $f^{-n}(t)$ because the preimages of $\gamma^{-1}$ starting at the points of $f^{-n}(t)$ are defined uniquely as well. The group of all permutations of $f^{-n}(t)$ induced by all elements of $\pi_1(M, t)$ is called the $n$th monodromy group of $f$. If $\gamma(s) = s'$ then $\gamma(f(s)) = f(s')$ since $f(\gamma[s]) = \gamma[f(s)]$, so $\gamma$ acts on $T$ by a tree automorphism.

The action of $\pi_1(M, t)$ on $T$ is not necessary faithful. Let $N$ be the kernel of this action.

**Definition 1.1.** The group $\text{IMG}(f) = \pi_1(M, t)/N$ is called the **iterated monodromy group of $f$**.

It can be shown (see [18] for details) that, up to isomorphism, $\text{IMG}(f)$ does not depend on the choice of the base point $t$.

In order to describe the automorphisms induced on $T$ by the loops from $\pi_1(M, t)$ we need to come up with a “coordinate system” on $T$. Let $X = \{0, 1, \ldots, d - 1\}$ be a standard alphabet of cardinality $d$. Then the set $X^*$ of all finite words over $X$ also has the structure of a $d$-ary rooted tree, where $v$ is adjacent to $vx$, for any $v \in X^*$ and $x \in X$.

The group $\text{Aut} X^*$ of all automorphisms of $X^*$ has the structure of an infinite iterated permutational wreath product $\otimes_{i \geq 1} \text{Sym}(d)$ (because $\text{Aut} X^* \cong \text{Aut} X^* \sqcup_X \text{Sym}(d)$, where $\text{Sym}(d)$ acts naturally on $X$ by permutations). This gives a convenient way to express automorphisms from $\text{Aut} X^*$ in the form

$$g = (g|_0, g|_1, \ldots, g|_{d-1})\sigma_g,$$

(1.1)
where \( g|_0, g|_1, \ldots, g|_{d-1} \) are automorphisms of the subtrees of \( X^* \) with roots at the vertices 0, 1, \ldots, \( d-1 \) (these subtrees are canonically identified with \( X^* \)) induced by \( g \), and \( \sigma_g \) is the permutation of \( X \) induced by \( g \) (i.e., \( \sigma_g(x) = g(x) \) — the action of \( g \) on \( x \in X \)). More generally, for every \( u \in X^* \) we define \( g|_u \) to be the automorphism of the subtree of \( X^* \) rooted at \( u \) (identified with \( X^* \)) induced by \( g \).

The automorphism \( g|_u \) is called the section of \( g \) at \( u \) and is uniquely determined by \( g(uw) = g(u)g|_u(w) \), for all \( w \in X^* \).

The product of automorphisms written in form \( \sigma_1 \circ \sigma_2 \) is performed in the following way. If \( h = (h|_0, h|_1, \ldots, h|_{d-1}) \sigma_h \) then

\[
gh = (g|_0h|_{\sigma(0)}, \ldots, g|_{d-1}h|_{\sigma(d-1)}) \sigma_g \sigma_h.
\]

By definition \( gh(w) = h(g(w)) \).

**Definition 1.2.** A group \( G \leq \text{Aut} \ X^* \) is called self-similar if \( g|_u \in G \) for all \( g \in G \) and \( u \in X^* \).

A convenient way to describe a particular finitely generated self-similar group \( G \) generated by automorphisms \( g_1, g_2, \ldots, g_n \) is through a, so-called, wreath recursion. In this presentation we simply write the action of each \( g_i \) in the form

\[
g_i = (w_1(g_1, \ldots, g_n), \ldots, w_d(g_1, \ldots, g_n)) \sigma_{g_i},
\]

where \( w_i, i = 1, \ldots, n \), are words in the free group of rank \( n \).

Another language which describes self-similar groups is the language of automaton groups (see the survey paper [11] for details).

**Definition 1.3.** A Mealy automaton is a tuple \( (Q, X, \pi, \lambda) \), where \( Q \) is a set (a set of states), \( X \) is a finite alphabet, \( \pi: Q \times X \to Q \) is a transition function and \( \lambda: Q \times X \to X \) is an output function. If the set of states \( Q \) is finite the automaton is called finite.

One can think of an automaton as a sequential machine which, at each moment of time, is in one of its states. Given a word \( w \in X^* \) the automaton acts on it as follows. It “eats” the first letter \( x \) in \( w \) and depending on this letter and on the current state \( q \) it “spits out” a new letter \( \lambda(q, x) \in X \) and changes its state to \( \pi(q, x) \). The new state then handles the rest of word \( w \) in the same fashion. Thus the map \( \lambda \) can be extended to \( \lambda: Q \times X^* \to X^* \) — we just feed the automaton with letters of \( u \in X^* \) one by one. Each state \( q \) of the automaton defines a map, also denoted by \( q \), from \( X^* \) to itself defined by \( q(w) = \lambda(q, w) \). In the special case when, for all \( q \in Q \), the map \( \lambda(q, \cdot) \) is a permutation of \( X \) the map \( q: X^* \to X^* \) is invertible and hence, an automorphism of the tree \( X^* \). In this case the automaton is called invertible.

**Definition 1.4.** A group of automorphisms of \( X^* \) generated by all the states of an invertible automaton \( \mathcal{A} \) is called the automaton group generated by \( \mathcal{A} \).

The class of automaton groups coincides with the class of self-similar groups. Indeed, the action on \( X^* \) of every element \( g \) of a self-similar group can be encoded by an automaton whose states are the sections of \( g \) on the words from \( X^* \), transition and output functions are derived from the representation \( \lambda(g, x) \). Namely, for each \( u \in X^* \), set \( \pi(g|_u(x), x) = g|_ux \) and \( \lambda(g|_u, x) = g|_u(x) \).

Important subclass of automaton groups consists of groups generated by finite automata. For example, we know that for groups in this class the word problem is solvable.
A standard way to visualize automata is by so-called Moore diagrams. Such a diagram is an oriented graph where the set of vertices is \( Q \) and for every \( q \in Q, x \in X, \) there is an edge from \( q \) to \( \pi(q, x) \) labeled by \( (x, \lambda(q, x)) \). In case of invertible automata it is common to label states by the corresponding permutations of \( X \) and leave only the first coordinate on the edge labels. An example of a Moore diagram is presented in Figure 5.

We go back now to iterated monodromy groups and construct an isomorphism \( \Lambda: X^* \to T \) such that the induced action of \( \pi_1(M, t) \) on \( X^* \) becomes particularly nice (self-similar).

We construct \( \Lambda \) level by level. Set \( \Lambda(\emptyset) = t \). For each vertex \( v \) in \( X^n \) we will construct a path \( l_v \) in \( M \) joining \( t \) to one of its preimages \( s_v \) from \( f^{-n}(t) \) and define \( \Lambda(v) = s_v \). Choose arbitrarily paths \( l_0, \ldots, l_{d-1} \) in \( M \) connecting \( t \) to its \( d \) preimages in \( f^{-1}(t) \) and, for \( x \in X \), define \( \Lambda(x) \) to be the end of the path \( l_x \). Now assume we have already defined \( \Lambda(v) \) and corresponding paths \( l_v \) for all \( v \in X^m \), \( m \leq n \) and \( \Lambda \) is an isomorphism between the first \( n \) levels of \( X^* \) and \( T \) such that, for all vertices \( v \) on the first \( n \) levels, \( \Lambda(v) \) is the endpoint of \( \ell_v \). For any word \( xu \in X^{n+1} \) with \( x \in X \) and \( u \in X^n \) define

\[
l_{xu} = l_u f_{\Lambda(u)}^{-n}(l_x),
\]

where \( f_{\Lambda(u)}^{-n}(l_x) \) is the unique preimage of the path \( l_x \) under \( f^n \) starting at the vertex \( \Lambda(u) \) (composition of paths is performed from left to right, i.e., the path on the left is traversed first). Define \( \Lambda(xu) \) to be the end of the path \( l_{xu} \).

In order to prove that \( \Lambda \) is an isomorphism of trees we need to show that \( f(\Lambda(xy)) = \Lambda(xy) \), for all \( x, y \in X \) and \( v \in X^* \). Indeed,

\[
f(l_{xy}) = f(l_v f_{\Lambda(v)}^{-n}(l_x)) = f(l_v) f_{\Lambda(v)}^{-\ell_1}(l_x).
\]

By definition, \( f_{\Lambda(v)}^{-\ell_1}(l_x) \) is a path going from \( \Lambda(v) \) to \( \Lambda(x) \), so the end \( \Lambda(xy) \) of the path \( l_{xy} \) is mapped to \( \Lambda(xy) \) under \( f \). Abusing the notation, we often identify the trees \( T \) and \( X^* \) and write \( v \) for \( \Lambda(v) \) (see Figure 1 where solid lines represent edges in the tree \( T \) and dashed lines represent paths in \( M \)).

**Definition 1.5.** The action of IMG(\( f \)) on \( X^* \) induced by the isomorphism \( \Lambda \) is called the standard action of IMG(\( f \)).

The tree isomorphism \( \Lambda \) allows us to compute iterated monodromy groups using the language of self-similar groups [13]. We provide the details here in order to

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**Figure 1.** Isomorphism \( \Lambda \) between \( T \) and \( X^* \).
keep the paper relatively self-contained and to help the understanding of the computations that follow. Recall, that for any loop $\gamma$ based at $t$ and any $u \in f^{-n}(t)$ we denote by $\gamma_{[u]}$ the unique preimage of $\gamma$ under $f^n$ that starts at the point $u$. Similarly, $f^{-n}(l_x)$ denotes the unique preimage of the path $l_x$ starting at $u$.

**Theorem 1.6.** The standard action of $\text{IMG}(f)$ is self-similar. More precisely, the section $\gamma|_x$ of $\gamma \in \text{IMG}(f)$ at $x \in X$ is given by

$$\gamma|_x = l_x \gamma_{[x]}(l_{\gamma(x)})^{-1}. \quad (1.2)$$

**Proof.** Let $v \in X^n$ be an arbitrary word and suppose $\gamma(xv) = yu$, for $y \in X$ and $u \in X^n$. Then vertices $v$ and $u$ are connected by the path

$$p = f^{-n}_{[u]}(l_x) \cdot \gamma_{[xv]} \cdot (f^{-n}_{[u]}(l_y))^{-1},$$

which goes through the vertices $v \to xv \to yu \to u$ (see Figure 2, where solid curves represent paths in $M$ and dashed lines represent paths in the tree $X^*$). We have

$$f^n(p) = l_x \gamma_{[x]} l_y^{-1}.$$ 

Thus the loop $\ell = l_x \gamma_{[x]} l_y^{-1}$ based at $t$ represents the element of $\text{IMG}(f)$ which moves $v$ to $u$. The loop $\ell$ is independent of $v$ (and $u$). Thus we have $\gamma|_x = l_x \gamma_{[x]} l_y^{-1}$. \hfill \Box

We are now ready to compute the standard action of $\text{IMG}(z^2 + i)$ on $\{0,1\}^*$. The only critical point of this map is $z = 0$, which is preperiodic:

$$0 \xrightarrow{f} i \xrightarrow{f} (i - 1) \xrightarrow{f} -i.$$

and, hence, the postcritical set of $f$ is $\{i, i-1, -i\}$. Therefore the restriction of $f$ on $M_1 = \mathbb{C} \setminus \{i, i-1, -i, 0\}$ is a 2-fold covering of $M = \mathbb{C} \setminus \{i, i-1, -i\}$.

Set $t = 0 \in \mathbb{C}$ as the base point. It has two preimages $e^{i3\pi/4}$ and $e^{i7\pi/4}$ which are identified with the letters 0 and 1, respectively (more precisely, we set $\Lambda(0) = e^{i3\pi/4}$ and $\Lambda(1) = e^{i7\pi/4}$). For the paths $l_0$ and $l_1$ connecting $t$ to its preimages we choose the straight segments shown in Figure 3(a).

The fundamental group $\pi_1(M,t)$ is generated by the 3 loops $a, b, c$ shown in Figure 3(b) going around $i$, $-i$ and $i-1$ respectively. Each of these loops has two preimages $a_{[i]}$, $b_{[i]}$ and $c_{[i]}$, $i = 0, 1$, shown in Figure 4.
According to formula (1.2) and Figures 3 and 4 the sections of the generators $a$, $b$, $c$ at 0 and 1 satisfy:

$$a|_0 = l_0 a|_0 l_0^{-1} = 1, \quad a|_1 = l_1 a|_1 l_0^{-1} = 1, \quad b|_0 = l_0 b|_0 l_0^{-1} = a, \quad b|_1 = l_1 b|_1 l_1^{-1} = c, \quad c|_0 = l_0 c|_0 l_0^{-1} = b, \quad c|_1 = l_1 c|_1 l_1^{-1} = 1,$$

where 1 denotes the trivial loop at $t$, which represents the identity element of $\text{IMG}(z^2 + i)$.

Since $a$ permutes the elements of $f^{-1}(t)$, while $b$ and $c$ do not, we obtain the following wreath recursion for the generators of $\text{IMG}(z^2 + i)$

(1.3) \hspace{1cm} a = (1, 1)\sigma, \quad b = (a, c), \quad c = (b, 1),

where $\sigma$ is the nontrivial transposition in $\text{Sym}(2)$.

These relations show that the set of all sections of the generators $a, b, c$ of $\mathcal{G} = \text{IMG}(z^2 + i)$ is $\{1, a, b, c\}$ and that the group $\mathcal{G}$ is generated by the states of the finite automaton shown in Figure 5.

We now say a few words about the relation between the dynamics of the map $z \mapsto z^2 + i$ and the combinatorial properties of the action of $\mathcal{G}$ on the tree $T$. 
Recall that if a group $G$ acts on a set $Y$ then the Schreier graph of this action is an oriented graph, whose set of vertices is $Y$ and there is an edge from $y \in Y$ to $z \in Y$ labeled by $g \in G$ if and only if $g(y) = z$. It is convenient sometimes to forget about the labels and/or the orientation of the edges.

Every group acting on a rooted tree acts on each level of the tree. The Schreier graphs of such actions are of particular interest, since in many situations (such as the one we are in) they can be used to find the spectrum of the Markov operator on the boundary of the tree.

Recent results of Nekrashevych [18] show that the Schreier graphs of $\text{IMG}(f)$ on the levels of the tree converge to the Julia set of the map $f$. Therefore the structure of the Julia set of $f$ provides understanding of the structure of the Schreier graphs of $\text{IMG}(f)$ (and vice versa). In our case the Julia set of $z^2 + i$ is the dendrite shown in the left half of Figure 6. The right half of this figure displays the Schreier graph of $\mathcal{G}$ on level 8. The set of vertices of this graph is just $f^{-8}(0)$ and the vertices are connected according to the action of $\mathcal{G}$ (no loops are drawn through).

2. Branch groups

Another important class of subgroups of $\text{Aut } X^*$ is the class of branch groups [3, 10]. Here we give basic definitions and prove that $\text{IMG}(z^2 + i)$ is a regular branch group. This shows that branch groups arise naturally in mathematics (not just as a way to construct groups with unusual properties).

Let $G$ be a subgroup of $\text{Aut } X^*$. Then for any vertex $v \in X^*$ one can define the subgroup of $G$ consisting of all the elements in $G$ fixing all words in $X^*$ that do
not have $v$ as a prefix. This subgroup of $G$ is called the \textit{rigid stabilizer} of $v$ and it is denoted by $\text{rist}_G(v)$. Furthermore, the subgroup

$$\text{rist}_G(n) = \left\langle \bigcup_{v \in X^n} \text{rist}_G(v) \right\rangle$$

generated by the union of the rigid stabilizers of vertices at level $n$, is called the \textit{rigid stabilizer of the $n$th level}. Since elements of rigid stabilizers of different vertices on the same level commute we have

$$\text{rist}_G(n) = \prod_{v \in X^n} \text{rist}_G(v).$$

Note that, if $G$ acts transitively on the levels of the tree, then all rigid stabilizers of the vertices on a fixed level are conjugate and, hence, isomorphic.

**Definition 2.1.** A group $G$ of tree automorphisms of $X^*$ that acts transitively on the levels of $X^*$ is called a \textit{branch group} if all rigid level stabilizers $\text{rist}_G(n), n \geq 0$, have finite index in $G$. If all rigid stabilizers are nontrivial then $G$ is called a \textit{weakly branch group}.

It is often easier to prove that a given group belongs to a more narrow class of regular (weakly) branch groups. Consider a self-similar group $G$ and its normal subgroup $\text{St}_G(1)$ consisting of all elements in $G$ that stabilize the first level of $X^*$. There is a natural embedding

$$\Psi: \text{St}_G(1) \hookrightarrow G \times G \times \cdots \times G$$

given by

$$g \mapsto (g|_0, g|_1, \ldots, g|_{d-1}).$$

**Definition 2.2.** Let $K, K_0, \ldots, K_{d-1}$ be subgroups of a self-similar group $G$ acting on $X^*$. We say that $K$ \textit{geometrically contains} $K_0 \times \cdots \times K_{d-1}$ and write $K_0 \times \cdots \times K_{d-1} \leq K$ if $K_0 \times \cdots \times K_{d-1} \leq \Psi(\text{St}_G(1) \cap K)$.

**Definition 2.3.** A group $G$ of tree automorphisms of $X^*$ that acts transitively on the levels of the tree $X^*$ is called a \textit{regular weakly branch group} over its normal subgroup $K$ if

$$K \times \cdots \times K \leq K.$$ 

If, in addition, the index of $K$ in $G$ is finite then $G$ is called a \textit{regular branch group} over $K$.

It can be shown that if $G$ is a regular (weakly) branch group than it is a (weakly) branch group.

**Definition 2.4.** A self-similar group $G$ is called \textit{self-replicating} if, for every vertex $u$ in $X^*$, the map $\varphi_u: G_u \rightarrow G$ given by $\varphi_u(g) = g_u$ is onto (where $G_u$ is the stabilizer of the vertex $u$ in $G$).

Note that $G = \text{IMG}(z^2 + i)$ is self-replicating. This is clear from the equalities

$$b = (a, c), \quad c = (b, 1), \quad aba = (c, a), \quad aca = (1, b).$$

Consider the normal subgroup $N$ of $G$ defined by

$$N = \langle [a, b], [b, c] \rangle^G.$$
By definition, \([g, h] = g^{-1}h^{-1}gh\) and \(\langle \cdot \rangle^G\) denotes normal closure in \(G\).

**Theorem 2.5.** The group \(G\) is a regular branch group over \(N\).

**Proof.** First we observe that \(N\) has finite index in \(G\). Direct computation shows that
\[
\begin{align*}
a^2 &= b^2 = c^2 = (ac)^4 = (ab)^8 = (bc)^8 = 1,
\end{align*}
\]
so \(G/N\) is a homomorphic image of
\[
\langle a, b, c | a^2 = b^2 = c^2 = (ac)^4 = [a, b] = [b, c] = 1 \rangle \cong C_2 \times D_4,
\]
where \(C_2\) is the cyclic group of order 2 and \(D_4\) is the dihedral group of order 8.

Further, we have
\[
[b, c] = ([a, b], 1) \quad [c, b a] = ([b, c], 1).
\]
Since \([c, b a] = cb[b, a] = [b, a][b][b, a] \in N\) we have that \(([a, b], 1)\) and \(([b, c], 1)\) are elements in \(N\). The fractalness of \(G\) enables us to conjugate the sections in \(([a, b], 1)\) and \(([b, c], 1)\) by arbitrary elements in \(G\) without leaving \(N\).

Thus we get the inclusion \(N \times 1 \leq N\). The transitivity of \(G\) on the first level then implies
\[
N \times N \leq N.
\]

The level transitivity of \(G\) can be obtained almost for free. The fact that \(N\) is nontrivial along with the fact that \(N \times N \leq N\) implies that \(N\) is infinite, and hence so is \(G\). But one can prove that a self-similar group acting on a binary rooted tree is infinite if and only if it acts transitively on all levels. \(\square\)

3. **L-PRESENTATION**

The goal of this section is to prove the following result.

**Theorem 3.1.** The group \(G\) has the following L-presentation
\[
(3.1) \quad G \cong \langle a, b, c \mid \phi^n(a^2), \phi^n((ac)^4), \phi^n([c, bab]^2), \phi^n([c, babab]^2), n \geq 0 \rangle,
\]
where \(\phi\) is the substitution defined on words in the free monoid over the alphabet \(\{a, b, c\}\) by
\[
\phi: \begin{cases} 
a \rightarrow b, 
\quad b \rightarrow c, 
\quad c \rightarrow aba.
\end{cases}
\]

In order to prove Theorem 3.1 we introduce some notation and prove a few intermediate results.

The group
\[
\Gamma = \langle a, b, c \mid a^2, b^2, c^2, (ac)^4 \rangle
\]
covers \(G\) (the relators of \(\Gamma\) are relators of \(G\)). The action of \(G\) on the binary tree induces an action of the covering group \(\Gamma\) on the same tree, which is not faithful.

Let \(\Omega\) be the kernel of this action. Then, obviously, a set of generators of \(\Omega\) as a normal subgroup in \(\Gamma\), together with the relators in \(\Gamma\) constitutes a presentation for \(G\).

The embedding \(G \rightarrow G \wr \text{Sym}(2)\) induces a homomorphism
\[
\Psi: \Gamma \rightarrow \Gamma \wr \text{Sym}(2)
\]
defined by

\[
\Psi : \begin{cases}
    a \mapsto (1, 1)\sigma, \\
    b \mapsto (a, c), \\
    c \mapsto (b, 1).
\end{cases}
\]

Indeed, the relations of \( \Gamma \) are mapped to the trivial element \((1, 1)\) of \( \Gamma \wr \text{Sym}(2) \):

\[
\begin{align*}
\Psi(a^2) &= (1, 1)\sigma(1, 1)\sigma = (1, 1), \\
\Psi(b^2) &= (a, c)^2 = (a^2, c^2) = (1, 1), \\
\Psi(c^2) &= (b, 1)^2 = (b^2, 1) = (1, 1), \\
\Psi((ac)^4) &= ((1, b)\sigma)^4 = (b^2, b^2) = (1, 1).
\end{align*}
\]

The homomorphism \( \Psi \) induces homomorphisms \( \Psi_n : \Gamma \rightarrow \Gamma \wr (\text{Sym}(2))^l \) (here \( (\text{Sym}(2))^l \) denotes the iterated permutational wreath product) defined recursively by \( \Psi_1 = \Psi \) and

\[
\Psi_n : \Gamma \xrightarrow{\Psi^n-1} \Gamma \wr (\text{Sym}(2))^l \xrightarrow{\Psi} (\Gamma \wr \text{Sym}(2))^l \xrightarrow{\Psi} \Gamma \wr \left(\bigcup_{i=1}^{n-1} \text{Sym}(2)\right) \xrightarrow{\Psi} \Gamma \wr \left(\bigcup_{i=1}^{n-1} \text{Sym}(2)\right).
\]

If, for \( g \in \Gamma \), we have \( \Psi_n(g) = (g|_u, u \in X^n)\sigma_n \), with \( g|_u \in \Gamma \) and \( \sigma_n \in \Gamma \wr \left(\bigcup_{i=1}^{n-1} \text{Sym}(2)\right) \) we call \( g|_u \) the section of \( g \) at \( u \).

For every \( g \in \Gamma \) denote by \( l(g) \) the length of the shortest word in \( a, b, c \) representing \( g \) in \( \Gamma \). The following lemma shows that \( \Gamma \) possesses the so called contraction property.

**Lemma 3.2.** For every \( g \in \Gamma \) and \( u \in X^2 \)

\[
l(g|_u) \leq \frac{l(g) + 1}{2}.
\]

**Proof.** Observe that, because of the self-similarity, all generators satisfy inequality (3.2). Indeed,

\[
\begin{align*}
\Psi_2(a) &= (1, 1, 1, 1)(02)(13), & \Psi_2(b) &= (1, 1, b, 1)(01), & \Psi_2(c) &= (a, c, 1, 1),
\end{align*}
\]

where, for ease of notation, the vertices on the second level are renamed by using the identifications 00 ↔ 0, 01 ↔ 1, 10 ↔ 2, and 11 ↔ 3.

All pairwise products of generators also satisfy inequality (3.2):

\[
\begin{align*}
\Psi_2(ab) &= (\Psi_1(c), \Psi_1(a))\sigma = (b, 1, 1, 1)(0213), \\
\Psi_2(ba) &= (\Psi_1(a), \Psi_1(c))\sigma = (1, 1, b, 1)(0312), \\
\Psi_2(ac) &= (\Psi_1(1), \Psi_1(b))\sigma = (1, 1, a, c)(02)(13), \\
\Psi_2(ca) &= (\Psi_1(b), \Psi_1(1))\sigma = (a, c, 1, 1)(02)(13), \\
\Psi_2(bc) &= (\Psi_1(ab), \Psi_1(c)) = (c, a, b, 1)(01), \\
\Psi_2(cb) &= (\Psi_1(ba), \Psi_1(c)) = (a, c, b, 1)(01).
\end{align*}
\]

Any word \( w \) in \( a, b, c \) of length \( n \) can be split into a product of at most \((n + 1)/2\) products of pairs of generators (if the length of \( w \) is odd one can pair the last letter in \( w \) with 1). Therefore the sections of \( w \) on the vertices of the second level are products of at most \((n + 1)/2\) letters. Thus the inequality (3.2) holds for \( w \) as well. \( \square \)
Define an increasing sequence of subgroups of $\Gamma$ by
$$\Omega_n = \ker \Psi_n.$$  

**Lemma 3.3.** The kernel $\Omega$ of the canonical epimorphism $\Gamma \to \mathcal{G}$ satisfies
$$\Omega = \bigcup_{n \geq 1} \Omega_n.$$  

**Proof.** Let $h$ be a word in $a, b, c$ of length at most $2^n + 1$ representing the trivial element in $\mathcal{G}$. Then, since for any words $u, v \in X^*$
\begin{equation}
(3.4) \quad h_{|uv} = h_{|u} h_{|v},
\end{equation}
by Lemma 3.2 we obtain that all sections of $h$ have length at most 1 on the $2(n+1)$th level. Therefore they must be trivial, because $h$ acts trivially on the tree. Hence, $h \in \Omega_{2(n+1)}$. \hfill \Box

The last lemma reduces the problem of finding generators for $\Omega$ to finding generators for $\Omega_n$. We start from $\Omega_1 = \ker \Psi$ and, based on it, derive generators for $\Omega_n$.

Let $H = \text{St}_1(1)$ be the stabilizer of the first level of the tree in $\Gamma$.

**Lemma 3.4.** The group $H$ has the following presentation
$$H = \langle \beta, \delta, \gamma, \rho \mid \beta^2 = \delta^2 = \gamma^2 = \rho^2 = (\rho \delta)^2 = 1 \rangle,$$
where $\beta = b$, $\delta = c$, $\gamma = aba$, $\rho = aca$.

**Proof.** The index of $H$ in $\Gamma$ is 2 and the coset representatives are $\{1, a\}$. The Reidemeister–Schreier procedure gives the above presentation. \hfill \Box

Obviously, each $\Omega_n$ is a subgroup of $H$. Therefore one can restrict $\Psi$ to $H$. Since $H$ stabilizes the first level one can think of $\Psi$ as a homomorphism $H \to \Gamma \times \Gamma$. This map is given by
$$\Psi: \begin{cases} 
\beta = b \to (a, c), \\
\gamma = aba \to (c, a), \\
\delta = c \to (b, 1), \\
\rho = aca \to (1, b), 
\end{cases}$$
which mimics the corresponding embedding of the generators $b, aba, c, aca$ of $\text{St}_G(1)$ into $G \times G$.

Define the following words in $\Gamma$:
$$U_1 = (ba)^8, \quad U_2 = [c, ab]^2, \quad U_3 = [c, bab]^2, \quad U_4 = [c, ababa]^2, \quad U_5 = [c, ababab]^2, \quad U_6 = [c, bababab]^2.$$  

**Lemma 3.5.** $\Omega_1 = \langle U_1, U_2, U_3, U_4, U_5, U_6 \rangle^\Gamma$.

**Proof.** In order to find a generating set for $\Omega_1 = \ker \Psi$ we first describe $\Psi(H)$. Since $\Psi(\delta) = (b, 1)$ and $\Psi(\rho) = (1, b)$, we get
$$B \times B \leq \Psi(H),$$
where $B = (b)^\Gamma$. Furthermore, $\Psi(H)/(B \times B) \cong \langle (a, c), (c, a) \rangle \cong D_4$. Therefore
$$\Psi(H) \cong (B \times B) \rtimes D_4.$$  

Now we provide a presentation for $B$. 

Define
\[ \xi_1 = b, \quad \xi_2 = b^a, \quad \xi_3 = b^c, \quad \xi_4 = b^{ac}, \]
\[ \xi_5 = b^{ac}, \quad \xi_6 = b^{aca}, \quad \xi_7 = b^{cac}, \quad \xi_8 = b^{acac}. \]

Since \( \Gamma = C_2 \ast D_4 \) where the cyclic group \( C_2 \) of order 2 is generated by \( b \) and the dihedral group \( D_4 \) of order 8 is generated by \( a \) and \( c \), it is clear that \( B \) is generated by all conjugates of \( b \) by the elements in \( D_4 = \langle a, c \rangle \). Thus \( \{ \xi_i \mid i = 1, \ldots, 8 \} \) is a generating set for \( B \). Moreover, it is clear that
\[
B = \langle \xi_i, i = 1, \ldots, 8 \mid \xi_i^2 = 1, i = 1, \ldots, 8 \rangle,
\]
i.e., \( B \) is a free product of 8 copies of the cyclic group of order 2 (indeed, none of the \( b \)'s in an expression of the form \( \xi_i, \xi_i, \ldots, \xi_i \) can be canceled in \( \Gamma \) when \( i_j \neq i_{j+1} \) for \( j = 1, \ldots, m-1 \).

Therefore \( B \times B \) is generated by 16 elements, namely, \( \tilde{\xi}_i = (\xi_i, 1) \) and \( \hat{\xi}_i = (1, \xi_i) \) and is presented by
\[
B \times B = \langle \tilde{\xi}_i, \hat{\xi}_i, i = 1, \ldots, 8 \mid \tilde{\xi}_i^2 = 1, \hat{\xi}_i^2 = 1, i = 1, \ldots, 8 \rangle.
\]

Now we compute the action of \( D_4 \) generated by \( x = (a, c) \) and \( y = (c, a) \) on \( B \times B \).

\[
\begin{align*}
\tilde{\xi}_1^x &= (b, 1)^{(a,c)} = (aba, 1) = \tilde{\xi}_2, \\
\tilde{\xi}_2^x &= (aba, 1)^{(a,c)} = (b, 1) = \tilde{\xi}_1, \\
\tilde{\xi}_3^x &= (abc, 1)^{(a,c)} = (acbc, 1) = \tilde{\xi}_4, \\
\tilde{\xi}_4^x &= (acbc, 1)^{(a,c)} = (cba, 1) = \tilde{\xi}_3, \\
\tilde{\xi}_5^x &= (cabac, 1)^{(a,c)} = (acabac, 1) = \tilde{\xi}_7, \\
\tilde{\xi}_6^x &= (acabac, 1)^{(a,c)} = (acabac, 1) = \tilde{\xi}_8, \\
\tilde{\xi}_7^x &= (acabacac, 1)^{(a,c)} = (acabacac, 1) = \tilde{\xi}_5, \\
\tilde{\xi}_8^x &= (acabacac, 1)^{(a,c)} = (acabacac, 1) = \tilde{\xi}_6, \\
\hat{\xi}_1^y &= (b, 1)^{(c,a)} = (cvc, 1) = \hat{\xi}_3, \\
\hat{\xi}_2^y &= (cba, 1)^{(c,a)} = (cabac, 1) = \hat{\xi}_5, \\
\hat{\xi}_3^y &= (cabac, 1)^{(c,a)} = (b, 1) = \hat{\xi}_1, \\
\hat{\xi}_4^y &= (acbc, 1)^{(c,a)} = (cacbcac, 1) = \hat{\xi}_6, \\
\hat{\xi}_5^y &= (acbc, 1)^{(c,a)} = (aba, 1) = \hat{\xi}_2, \\
\hat{\xi}_6^y &= (acbcac, 1)^{(c,a)} = (acbcac, 1) = \hat{\xi}_4, \\
\hat{\xi}_7^y &= (acbcac, 1)^{(c,a)} = (cacbcacac, 1) = \hat{\xi}_8, \\
\hat{\xi}_8^y &= (acbcacac, 1)^{(c,a)} = (acbcacac, 1) = \hat{\xi}_7.
\end{align*}
\]  

The action on \( \tilde{\xi}_i \)'s can be determined from the action on \( \xi_i \). Namely, if \( \tilde{\xi}_p = \hat{\xi}_q \) and \( \tilde{\xi}_i = \hat{\xi}_j \), then
\[
\tilde{\xi}_i^x = \tilde{\xi}_q \quad \text{and} \quad \tilde{\xi}_i^y = \hat{\xi}_p.
\]
We use the fact that $w$ where $w^2 = 1$, we immediately get $\tilde{\xi}_i \to c^z$, where $z$ runs over all elements of $\langle b \rangle_{D_8}$. In the same way the lifts of $\tilde{\xi}_i$’s look like $\tilde{\xi}_i \to (aca)^z = c^{az} = c^w$, where $w$ runs over the complement of $\langle b \rangle_{D_8}$ in $D_8$, which is just the coset $a\langle b \rangle_{D_8}$.

Hence we get the remaining lifts of relators

- $\tilde{\xi}_i^2 \to (c^z)^2 = z^{-1}czz^{-1}c = 1$,
- $\tilde{\xi}_i^2 \to (c^w)^2 = w^{-1}cww^{-1}c = 1$,
- $[\tilde{\xi}_i, \tilde{\xi}_j], i, j = 1, \ldots, 8 \to [c^z, c^w], z \in \langle b \rangle_{D_8}, w \in a\langle b \rangle_{D_8}$.

Now we would like to simplify the generators we obtained. First of all, since $c^2 = 1$ we immediately get

$$[c^z, c^w] = [c, c^{aw^{-1}}]z$$

so we can get rid of $z$ (because $wz^{-1} \in a\langle b \rangle_{D_8}$). Furthermore,

$$[c, c^w] = cw^{-1}cw \cdot cw^{-1}cw = [c, w]^2$$

We can discard 3 more generators since

$$[c, ab]^2 = ([c, ab]^{-2})c, \quad [c, bababa]^2 = ([c, ababab]^{-2})c, \quad [c, a]^2 = (ca)^4 = 1.$$ 

Thus we get 5 more generators for $\ker \Psi$:

$$[c, ab]^2 = U_2, \quad [c, bab]^2 = U_3, \quad [c, abab]^2 = U_4,$$
$$[c, ababab]^2 = U_5, \quad [c, bababab]^2 = U_6.$$ 

These generators, together with $U_1 = (ba)^8$ generate $\ker \Psi$ as a normal subgroup in $\Gamma$.

\[\boxed{\text{Lemma 3.6.} \ \Omega_n = \langle \phi^i(U_j), i = 0, \ldots, n - 1, j = 1, \ldots, 6 \rangle_\Gamma}\]
Proof. We will use induction on $n$. For $n = 1$ the statement holds by Lemma 3.5. Assume it is true for some fixed $n$.

By the definition of $\Omega_{n+1}$ we have $\Psi(\Omega_{n+1}) \leq \Omega_n \times \Omega_n$. We will show that $\Psi(\Omega_{n+1}) \geq \Omega_n \times \Omega_n$. Observe that

$$\tag{3.7} \varphi_1\left(\Psi(\phi^i(U_j))\right) = 1$$

in $\Gamma$ (recall that, for an element $h = (h_0, h_1)$ in $H$, $\varphi_1(h) = h_1$. Indeed, since $\varphi_1\left(\Psi(\phi(\Gamma))\right) \leq \langle a, c \rangle = D_4$ it’s sufficient to check only that all $U_i$’s are trivial in $D_4$. But this is true since all these words are squares of commutators and $[D_4, D_4] \cong \mathbb{Z}/2\mathbb{Z}$.

Equation (3.7) for $i = n + 1$, together with the inductive assumption yields $\Omega_n \times 1 \leq \Psi(\Omega_{n+1})$. Since $\Omega_{n+1}$ is normal in $\Gamma$ conjugation by $a$ yields $1 \times \Omega_n \leq \Psi(\Omega_{n+1})$. Therefore

$$\Psi(\Omega_{n+1}) = \Omega_n \times \Omega_n.$$  

Equation (3.7) also implies that

$$\Psi(\phi^{n+1}(U_j)) = (\phi^n(U_j), 1), \quad \Psi(\phi^{n+1}(U_j)^n) = (1, \phi^n(U_j)), $$

i.e.,

$$\Psi(\langle \phi^i(U_j), i = 1, \ldots, n, j = 1, \ldots, 6 \rangle^\Gamma) = \Omega_n \times \Omega_n.$$  

Therefore

$$\Omega_{n+1} = \ker \Psi \cdot \langle \phi^i(U_j), i = 1, \ldots, n, j = 1, \ldots, 6 \rangle^\Gamma$$

$$= \langle \phi^i(U_j), i = 0, \ldots, n, j = 1, \ldots, 6 \rangle^\Gamma. \qed$$

Lemmas 3.6 and 3.3 prove Theorem 3.1. Since $\phi((ac)^4) = (ba)^8 = U_1$, $\phi(a^2) = b^2$, $\phi(b^2) = c^2$ and $\phi(c^2) = (aba)^2 = 1$ the presentation in (3.1) is slightly simplified.

**Corollary 3.7.** The group $\mathcal{G}$ embeds into an amenable finitely presented group of exponential growth

$$\mathcal{G} = \langle a, b, c, s \mid a^2, (ac)^4, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, bababab]^2, a^8 = b^8 = c^8 = aba \rangle,$$

which is an ascending HNN-extension of $\mathcal{G}$.

**Proof.** By Theorem 3.1 and the fact that $\varphi_0(\Psi(\phi(u))) = u$ the substitution $\phi$ induces an injective endomorphism of $\mathcal{G}$. Thus the HNN-extension construction can be applied. Since $\mathcal{G}$ is an extension of the amenable group $\mathcal{G}$ (see Section 3) by the amenable group $\mathbb{Z}$ generated by $s$, $\mathcal{G}$ is amenable. The growth of $\mathcal{G}$ is exponential because it contains a free semigroup of rank 2 (it follows from the HNN-extension construction that, for example, $s$ and $sa$ generate such a semigroup). \qed

### 4. Self-affine measures and amenability

Although the amenability of $\mathcal{G}$ follows from the intermediate growth of this group, which was established in [6], we present here a different approach based on the tools developed in [5][7]. More precisely, we construct a particular self-affine measure on $\mathcal{G}$, which proves the vanishing of the asymptotic entropy and, hence, amenability.

Let $G$ be a self-similar group acting spherically transitively on a $d$-ary tree. Consider a nondegenerate probability measure $\mu$ on $G$ (the support of $\mu$ generates...
Then for any \(x \in X\) one can define a new probability measure \(\mu\lvert_x\) on \(G\), which is called the **restriction** of \(\mu\) on \(x\). The details of the definition and proofs of relevant statements are given in [17] and here we only give the basic idea.

We consider a right random walk \(g_n = h_1h_2 \ldots h_n\) on \(G\) determined by \(\mu\), i.e., \(\{h_n\}\) is a sequence of independent variables identically distributed according to the measure \(\mu\). We consider \(G\) as embedded in \(G \rtimes \text{Sym}(d)\) and keep track of the \(x\)th coordinate of the image of \(g_n\) in \(G \rtimes \text{Sym}(d)\). Recall that \(h_n\) is an automorphism of \(X^*\). For \(x \in X\), \(h_n(x)\) denotes the action of \(h_n\) on \(x\). Since

\[
g_{n+1}\lvert_x = g_n\lvert_x \cdot h_{n+1}\lvert_{g_n(x)}
\]

the probability distribution of \(g_{n+1}\lvert_x\) is completely determined by \(g_n\lvert_x\) and \(g_n(x)\). Therefore the induced random walk

\[
(g_n\lvert_x, g_n(x))
\]

on \(G \times X\) is again a Markov chain. The last random walk is called a **random walk with internal degrees of freedom**. Since \(X\) is finite and \(G\) acts transitively on \(X\), the subset \(G \times \{x\} \subset G \times X\) is recurrent with respect to (4.1). Therefore one can consider the trace of the random walk (4.1) on \(G \times \{x\}\), which is also a random walk. Finally, we define the measure \(\mu\lvert_x\) as the transition law for the last random walk on \(G \times \{x\}\) considered as a copy of \(G\).

There is a convenient way to compute \(\mu\lvert_x\) using the properties of the random walk (4.1). The random walk with internal degrees of freedom is governed by the matrix

\[
M = (\mu_{xy})_{x,y \in X}
\]

whose entries \(\mu_{xy}\) are subprobability measures on \(G\) such that \(\mu_{xy}(h)\) is a transition probability of getting to the state \((gh, y)\) from the state \((g, x)\).

With slight abuse of notation, we denote by \(g\) the \(\delta\)-measure concentrated at \(g\). Then the matrix \(M\) can be expressed as

\[
M = \sum_{g \in \text{supp} \mu} \mu(g)M^g,
\]

where

\[
M^g_{xy} = \begin{cases} 
  g\lvert_x, & y = g(x), \\
  0, & y \neq g(x),
\end{cases}
\]

The following theorem is proved in [17].

**Theorem 4.1.** The measure \(\mu\lvert_x, x \in X\) can be expressed in terms of the matrix \(M\) as

\[
\mu\lvert_x = \mu_{xx} + M_{xx}(1 - M_{xx})^{-1}M_{x\cdot},
\]

where \(M_{x\cdot}\) (resp., \(M_{\cdot x}\)) denotes the \(x\)th row (column) of \(M\) from which the entry \(\mu_{xx}\) is removed, and \(M_{x\cdot}\) is the matrix obtained from \(M\) by removing the \(x\)th row and the \(x\)th column.

One can define \(\mu\lvert_w\) for any \(w = x_1x_2 \ldots x_n \in X^*\) by

\[
\mu_w = (\cdots (\mu\lvert_{x_1})\lvert_{x_2} \cdots)\lvert_{x_n}.
\]
Definition 4.2. The nondegenerate probability measure $\mu$ on a self-similar group $G$ is called \textit{self-affine} \cite{self-affine} if there is a word $w \in X^*$ such that
$$
\mu|_w = \alpha e + (1 - \alpha)\mu,
$$
where $0 < \alpha < 1$ and $e$ is the identity element in $G$.

For simplicity, we write $\alpha$ instead of $\alpha e$.

Theorem 4.3 \cite{self-affine}. If a self-similar group $G$ carries a self-affine nondegenerate measure $\mu$ with finite entropy, then it is amenable.

In this section we construct such a measure on $G$. Since this measure should be nondegenerate and have finite entropy the most natural place to look for it is the space $Q$ of positive convex linear combinations of $\delta$-measures concentrated on the generators $a, b, c$, i.e., measure $\mu$ of the form
$$
\mu = xa + yb + zc, \quad x + y + z = 1, \quad x, y, z > 0.
$$
Suppose we want this measure to be self-affine with respect to $x \in X$. By definition this means $\mu|_x = \alpha + (1 - \alpha)\mu$ or, equivalently,
$$
\mu = \frac{\mu|_x - \alpha}{1 - \alpha}.
$$

Since $\mu(e) = 0$ we get $\alpha = \mu|_x(e)$. Thus the measure $\mu$ is a fixed point of the transformation
$$
(4.3) \quad \Phi: \mu \mapsto \frac{\mu|_x - \mu|_x(e)}{1 - \mu|_x(e)},
$$
which is defined in \cite{self-affine} and used to prove amenability of a family of groups generalizing Basilica group $\text{IMG}(z^2 - 1)$.

Let’s compute $\mu|_0$ and the corresponding transformation $\Phi$ in the case of $G$. The support of $\mu$ is $\{a, b, c\}$ and the corresponding matrices $M^0$ are given by
$$
(4.4) \quad M^a = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \quad M^b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \quad M^c = \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}.
$$

Hence,
$$
M = xM^a + yM^b + zM^c = \begin{pmatrix} ya + zb & x \\ x & yc + z \end{pmatrix}.
$$

By Theorem 4.1
$$
\mu|_0 = (ya + zb) + x^2(1 - yc - z)^{-1}.
$$

Since $c$ has order 2 in $G$ it is easy see that in the group algebra $\mathbb{R}G$
$$
(1 - yc - z)^{-1} = \frac{y}{z^2 - 2z + 1 - y^2} \cdot c - \frac{z - 1}{z^2 - 2z + 1 - y^2}.
$$

Therefore
$$
\mu|_0 = y \cdot a + z \cdot b + \frac{yx^2}{z^2 - 2z + 1 - y^2} \cdot c - \frac{(z - 1)x^2}{z^2 - 2z + 1 - y^2}.
$$
and the transformation $\Phi$ takes the form

$$
\Phi(xa + yb + zc) = \frac{y \cdot a + z \cdot b + yx^2/(z^2 - 2z + 1 - y^2) \cdot c}{1 + (z - 1)x^2/(z^2 - 2z + 1 - y^2)}
$$

$$
= \frac{y(z^2 - 2z + 1 - y^2)}{z^2 - 2z + 1 - y^2 + x^2} \cdot a
$$

$$
+ \frac{z(z^2 - 2z + 1 - y^2)}{z^2 - 2z + 1 - y^2 + x^2} \cdot b
$$

$$
+ \frac{yx^2}{z^2 - 2z + 1 - y^2 + x^2} \cdot c.
$$

Now we are interested in a fixed point of the rational map $F : \mathbb{R}^3 \to \mathbb{R}^3$ induced by $\Phi$, which maps $(x, y, z)$ to the coefficients of $\Phi(xa + yb + zc)$. Moreover, we are searching for such a fixed point only in the invariant simplex $x + y + z = 1$, $x, y, z > 0$. Fortunately, there is such a fixed point. If $\zeta \approx 0.4786202932$ is the unique real root of the polynomial $Z^3 - 6Z^2 + 11Z - 4$, then the point

$$(\zeta, \zeta^2 - 4\zeta + 2, -1 + 3\zeta - \zeta^2)$$

is fixed under $F$, which produces a self-affine nondegenerate probabilistic measure on $G$ with finite support, proving amenability of $G$. This point is unique in the simplex of nondegenerate measures. Indeed, from the equation $F_1(x, y, 1 - x - y) = x$, where $F_1$ is the first coordinate of $F$, we get

$$(4.5) \quad y = \frac{1}{4}(-x^2 + x \pm x \sqrt{x^2 - 10x + 9}).$$

Substitution in $F_2(x, y, 1 - x - y) = y$ yields

$$(4.6) \quad (-x^4 + 12x^3 - 39x^2 + 40x - 12) \pm \sqrt{x^2 - 10x + 9(x^3 - 7x^2 + 12x - 4)} = 0.$$

Moving the second summand to the right-hand side and squaring both sides produces the equation $x(x - 1)(x^3 - 6x^2 + 11x - 4) = 0$, whose unique real root on the interval $(0, 1)$ is $\zeta$. The graphs of the two functions in (4.6) are shown in Figure 7: (a) for “plus” and (b) for “minus.” The solution comes from (a), so in (4.5) “plus” should be used. It is a routine to check that indeed $y = \zeta^2 - 4\zeta + 2$. 

![Figure 7](image-url)
5. Spectral properties and Schur complement

Let $H$ be a Hilbert space and $M$ be an operator on $H$. Let $H = H_0 \oplus H_1$ and there are operators $A \in B(H_0)$, $D \in B(H_1)$, $B : H_1 \to H_0$ and $C : H_0 \to H_1$, such that the matrix of $M$ in the basis consisting of the bases of $H_0$ and $H_1$ takes the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

The following fact is of folklore type.

**Proposition 5.1.** Let $D$ be invertible. The operator $M$ is invertible if and only if $S_1(M) = A - BD^{-1}C$ is invertible.

The matrix $S_1(M)$ is called the first Schur complement of $M$.

**Proof.** Indeed, the matrix

$$L = \begin{pmatrix} I_0 & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix}$$

is invertible. Therefore $M$ is invertible if and only if

$$ML = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I_1 \end{pmatrix}$$

is invertible, which is equivalent to the nonsingularity of $S_1(M)$. \qed

In our case the action of $G$ on the boundary $X^\omega$ (the set of infinite sequences over $X$) of the tree $X^*$ induces a unitary representation $\pi_g(f)(x) = f(g^{-1}x)$ of $G$ in $\mathcal{H} = B(L_2(X^\omega))$. The Markov operator $M = \frac{1}{3}(\pi_a + \pi_b + \pi_c)$ corresponding to this unitary representation plays an important role (we do not include inverse elements because all generators are of order 2). The usual method to find the spectrum of $M$ for a self-similar group $G$ is to approximate $M$ with finite dimensional operators arising from the action of $G$ on the levels of the tree $X^*$. For more on this see [1]. For simplicity we write $g$ for $\pi_g$.

Let $H_n$ be the subspace of $\mathcal{H}$ spanned by the $2^n$ characteristic functions $f_v$, $v \in X^n$, of the cylindrical sets, corresponding to the $2^n$ vertices of the nth level. Then $H_n$ is invariant under the action of $G$ and $H_n \subset H_{n+1}$. Also $H_n$ can be naturally identified with $L_2(X^n)$. By $\pi_g^{(n)}$ (or, with a slight abuse of notation, by $g^n$) we denote the restriction of $\pi_g$ on $H_n$. Then, for $n \geq 0$,

$$M_n = \frac{1}{3}(a_n + b_n + c_n)$$

are finite-dimensional operators whose spectra converge to the spectrum of $M$ in the sense

$$\text{sp}(M) = \bigcup_{n \geq 0} \text{sp}(M_n).$$

Moreover, if $P$ is the stabilizer of an infinite word from $X^\omega$, then one can consider the Markov operator $M_{G/P}$ on the Schreier graph of $G$ with respect to $P$. The following fact is observed in [1] and can be applied in the case of $G$.

**Theorem 5.2.** If $G$ is amenable then

$$\text{sp}(M_{G/P}) = \text{sp}(M).$$
Common practice for finding the spectrum of $M$, initiated in [1], is to consider a pencil of operators
\[ \widetilde{M}(y, z, \lambda) = a + yb + zc - \lambda \]
and find the set $\text{sp}(y, z, \lambda)$ of points $(y, z, \lambda)$ such that $\widetilde{M}(y, z, \lambda)$ is not invertible. Then the spectrum of $M$ is just the intersection of $\text{sp}(y, z, \lambda)$ with the line $y = z = 1$, shrunk by a factor of $\frac{1}{3}$. We take 1 as the coefficient at $a$ to simplify the computation. Otherwise one can divide it out (we restrict our attention to the case when $x, y, z$ are nonzero).

Let us consider the corresponding pencil $\widetilde{M}_n(y, z, \lambda) = a_n + yb_n + zc_n - \lambda$ and find its matrix in the basis $\{f_v : v \in X^n\}$. The orthogonal subspaces $H_n^{(i)} = \text{span}(f_v, v \in X^{n-1}), i = 0, 1$ span $H_n$ and are naturally isomorphic to $H_{n-1}$. The self-similar structure of $G$ gives the following operator recursion (which coincides with the recursion (4.4))

\[
\begin{align*}
\frac{1}{1} a_n &= \begin{pmatrix} 0 & I_{n-1} \\ I_{n-1} & 0 \end{pmatrix}, & \frac{1}{1} b_n &= \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix}, & \frac{1}{1} c_n &= \begin{pmatrix} b_{n-1} & 0 \\ 0 & I_{n-1} \end{pmatrix},
\end{align*}
\]

for $n > 0$, where $I_{n-1}$ denotes the identity matrix of size $2^{n-1}$. The matrices $a_0$, $b_0$ and $c_0$ are equal to the $1 \times 1$ matrix [1]. For any constant $r$, we write $r$ instead of $rI_n$. Thus we have,

\[ \widetilde{M}_n(y, z, \lambda) = a_n + yb_n + zc_n - \lambda = \begin{pmatrix} ya_{n-1} + zb_{n-1} - \lambda & 1 \\ 1 & yc_{n-1} + z - \lambda \end{pmatrix}. \]

By Proposition 5.1 in case $yc_{n-1} + z - \lambda$ is invertible the operator $\widetilde{M}_n(y, z, \lambda)$ is invertible if and only if $S_1(\widetilde{M}_n(y, z, \lambda))$ is invertible. The inverse of $yc_{n-1} + z - \lambda$ in $\mathbb{R}G$ is

\[ \frac{y}{-y^2 + z^2 - 2z\lambda + \lambda^2} \cdot c_{n-1} + \frac{-z + \lambda}{-y^2 + z^2 - 2z\lambda + \lambda^2}. \]
Hence, $yc_{n-1} + z - \lambda$ is not invertible if and only if $-y^2 + z^2 - 2z\lambda + \lambda^2 = (z - \lambda - y) \times (z - \lambda + y) = 0$. Denote the union of these 2 planes by $Z_1$. Note, that $\tilde{M}_n(y, z, \lambda)$ is not necessary singular at each point of $Z_1$.

The first Schur complement of $\tilde{M}$ is

$$S_1(\tilde{M}_n(y, z, \lambda)) = y a_{n-1} + z b_{n-1} - \lambda - (yc_{n-1} + z - \lambda)^{-1}$$

$$= y \cdot a_{n-1} + z \cdot b_{n-1} + \frac{y}{-y^2 + z^2 - 2z\lambda + \lambda^2} \cdot c_{n-1} + \frac{-z + \lambda}{-y^2 + z^2 - 2z\lambda + \lambda^2} - \lambda.$$

If $y = 0$ we get $S_1(\tilde{M}_n(0, z, \lambda)) = zb_{n-1} + \frac{1}{z} - \lambda$ is not invertible if and only if

$$\det \left( \begin{array}{cc} 1/(\lambda - z) - \lambda & 1/(\lambda - z) - \lambda \\ z & 1/(\lambda - z) - \lambda \end{array} \right) = \frac{1}{(\lambda - z)^2} (1 - (\lambda - z)(\lambda + z))(1 - \lambda + z)(1 + \lambda - z) = 0.$$

Denote corresponding union of a hyperbola and two lines in $\mathbb{R}^3$ by $Z_2$. Note that $Z_2 \cap Z_1 = \emptyset$.

If $y \neq 0$ then

$$\frac{1}{y} S_1(\tilde{M}_n(y, z, \lambda)) = a_{n-1} + \frac{z}{y} \cdot b_{n-1} + \frac{1}{-y^2 + z^2 - 2z\lambda + \lambda^2} \cdot c_{n-1}$$

$$- \frac{-\lambda y^2 + \lambda z^2 - 2z\lambda^2 + \lambda^3 + z - \lambda}{y(-y^2 + z^2 - 2z\lambda + \lambda^2)}$$

$$= \tilde{M}_n-1(F(y, z, \lambda)),$$

where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the rational map defined by

$$F : (y, z, \lambda) \to \left( \frac{z}{y}, \frac{1}{-y^2 + z^2 - 2z\lambda + \lambda^2}, \frac{-\lambda y^2 + \lambda z^2 - 2z\lambda^2 + \lambda^3 + z - \lambda}{y(-y^2 + z^2 - 2z\lambda + \lambda^2)} \right).$$

Therefore the set $\text{sp}_n(y, z, \lambda)$ of points $(y, z, \lambda)$ where $\tilde{M}_n(y, z, \lambda)$ is not invertible in this case ($y \neq 0$) is a preimage under $F$ of the corresponding set $\text{sp}_{n-1}(y, z, \lambda)$. To summarize,

$$Z_2 \cup F^{-1}(\text{sp}_{n-1}(y, z, \lambda)) \subset \text{sp}_n(y, z, \lambda) \subset Z_1 \cup Z_2 \cup F^{-1}(\text{sp}_{n-1}(y, z, \lambda)).$$

Since $\tilde{M}_0(y, z, \lambda) = (1 + y + z - \lambda)$ we have

$$\text{sp}_0(y, z, \lambda) = \{(y, z, \lambda) : 1 + y + z - \lambda = 0\}.$$
One can easily check that $P \subset F^{-1}(P)$ and, hence, $F^{-n}(P) = \bigcup_{i=0}^{n} F^{-i}(P)$. Therefore equation (5.4) transforms to

$$Z_3 \cup \bigcup_{i=0}^{n} F^{-i}(P) \subset \text{sp}_n(y, z, \lambda) \subset F^{-n}(P) \cup Z_3 \cup \bigcup_{i=0}^{n-1} F^{-i}(P \cup Z_1).$$

Thus, the spectrum of the operator $\tilde{M}$ on $L^2(X^{\omega})$ satisfies

$$Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P) \subset \text{sp}(\tilde{M}(y, z, \lambda)) = \bigcup_{i=0}^{\infty} \text{sp}_n(y, z, \lambda) \subset Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P \cup Z_1).$$

Note, that the sets $A = Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P)$ and $B = Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P \cup Z_1)$ are almost invariant with respect to $F$, in the sense that

$$F^{-1}(A) \cup Z_2 = A, \quad F^{-1}(B) \cup Z_1 \cup Z_2 = B,$$

which is an analog of [16, Theorem 4.1] for the Basilica group.

The preimages of the plane $P$ under $F^4$ and $F^5$ are shown in Figure 9.

Note that there are points in the spectrum of $\tilde{M}(y, z, \lambda)$ which do not belong to any preimage of the plane $P$. In particular, the point $(-\frac{1}{2}, 0, \frac{1}{2})$ belongs to $Z_1$ so it is not in the domain of $F$, but

$$\det \tilde{M}_n(-\frac{1}{2}, 0, \frac{1}{2}) = \det((a_{n-1} + 1)(c_{n-1} + 1) - 4) = 0$$

since 4 is an eigenvalue of $(a_{n-1} + 1)(c_{n-1} + 1)$. However, this point could be in the closure of the union of all preimages of $P$.

On the other hand we can formulate a conjecture that the spectrum of $M = \frac{1}{3}(a + b + c)$ is the intersection of the line $y = z = 1$ with $A = Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P)$, shrunk by a factor of $\frac{1}{3}$. This conjecture survives at least up to the 6th level.

Note also that the map $F$ is conjugate to a simpler map

$$G: (y, z, \lambda) \to \left(\frac{z}{y}, \frac{\lambda}{y}(-2 + y\lambda), \frac{1}{\lambda}(-y + y\lambda^2 - \lambda)\right)$$
by the conjugator map

\[(y, z, \lambda) \mapsto \left(\frac{1}{y}, \frac{1}{z}, y + z - \lambda\right).\]

The histogram for the spectral density of the operator \(M_n\) acting on 9th level is shown in Figure 8.

Further steps are required to identify the spectrum of the pencil \(\tilde{M}(y, z, \lambda)\) and of \(M\) more precisely. This is related to the problem of finding invariant subsets of the rational map \(F\). Perhaps the spectrum of \(M\) is just the intersection of the “strange attractor” of \(F\) with the line \(y = z = 1\), shrunk by factor of \(\frac{1}{3}\). In any case here we have one more example when the spectral problem is related to the dynamics of a multidimensional rational map. There is a hope that the methods developed for this type of transformations (see, for instance [20]) could help to handle this case.

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