The sAKNS Hierarchy

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ABSTRACT

We study, systematically, the properties of the supersymmetric AKNS (sAKNS) hierarchy. In particular, we discuss the Lax representation in terms of a bosonic Lax operator and some special features of the equations and construct the bosonic local charges as well as the fermionic nonlocal charges associated with the system starting from the Lax operator. We obtain the Hamiltonian structures of the system and check the Jacobi identity through the method of prolongation. We also show that this hierarchy of equations can equivalently be described in terms of a fermionic Lax operator. We obtain the zero curvature formulation as well as the conserved charges of the system starting from this fermionic Lax operator which suggests a connection between the two. Finally, starting from the fermionic description of the system, we construct the soliton solutions for this system of equations through Darboux-Bäcklund transformations and describe some open problems.

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1. Introduction:

The AKNS hierarchy[1-3] is an important bosonic, integrable hierarchy which has played a fundamental role in the development of many interesting ideas. This hierarchy, among other things, includes all the well known integrable models such as the KdV equation, the mKdV equation, the nonlinear Schrödinger equation, sine-Gordon equation. In its original formulation, the AKNS hierarchy was described in terms of a matrix Lax operator

\[ L = \frac{\partial}{\partial x} - q\sigma_+ - r\sigma_- + i\zeta\sigma_3 \]  

where \( q \) and \( r \) represent the dynamical variables while \( \zeta \) represents the spectral parameter. Furthermore, the \( \sigma \)'s represent the \( 2 \times 2 \) Pauli matrices and, in particular, \( \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \). Different integrable hierarchies would result from this Lax operator with different identifications of the dynamical variables. Thus, for example, \( r \sim \psi \) and \( q \sim \psi^* \) would lead to the hierarchy associated with the nonlinear Schrödinger equation.

The AKNS hierarchy can also be described in terms of the more conventional scalar Lax operator of the form[4-5]

\[ L = \partial + q\partial^{-1}r \]  

which can also be seen to follow from the linear equation associated with the matrix Lax operator in eq. (1). This way of describing the AKNS hierarchy, however, has led to further interesting properties associated with the system. Among other things, it has led to a new understanding of the gradings associated with the zero curvature formulation of the integrable models.

In recent years, there has been a lot of interest in understanding the properties of supersymmetric integrable models[6]. Several such models have already been constructed[7-9] and more recently, a supersymmetric formulation of the AKNS (sAKNS) hierarchy has also been given[10-11]. It is expected to be at least as important in the study of supersymmetric integrable models as the AKNS hierarchy is within the context of the bosonic integrable systems. Therefore, it is worth studying the properties of the sAKNS hierarchy systematically which we carry out in this paper. In section 2, we describe the sAKNS hierarchy in terms of the conventional bosonic Lax operator in superspace giving the explicit form of the dynamical equations upto the first few orders. We construct the local, bosonic conserved charges as well as the nonlocal, fermionic conserved charges associated with the system starting from the Lax operator. We also review very briefly the zero curvature formulation of the integrable models.
curvature formulation of this system[12-13] in this section. In section 3, we construct the Hamiltonian structures associated with this system pointing out some of the subtleties. We check Jacobi identity for these structures using the method of prolongation[14-15]. We also show that the Hamiltonian structures are compatible making it a genuinely bi-Hamiltonian system. We also construct, in this section, the recursion operator, with a vanishing Nijenhuis torsion tensor, which relates the different Hamiltonians of the theory. In section 4, we show that the sAKNS hierarchy can also be described in terms of a fermionic Lax operator. This is, in fact, the only integrable system we know of which allows a description in terms of a bosonic as well as a fermionic Lax operator. We construct the local conserved charges and the zero curvature formulation starting with this fermionic Lax operator. However, the construction of the nonlocal, fermionic conserved charges, from the fermionic Lax operator, as well as a transformation relating the two Lax operators directly remains an open question. In section 5, we construct soliton solutions associated with this system starting from the fermionic Lax operator through a Darboux-Bäcklund transformation and compare its properties with those of such solutions obtained from an algebraic dressing method[13]. Finally, we close with some conclusions in section 6.

2. The sAKNS System:

The supersymmetric AKNS hierarchy can be described by the bosonic Lax operator

\[ L = D^2 + \phi D^{-1} \psi \]  

where the supercovariant derivative is defined to be

\[ D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \]  

Here \( \theta \) is the Grassmann coordinate of the superspace and \( \phi \) and \( \psi \) represent a bosonic and a fermionic superfield respectively. It is also straightforward to see that \( D^2 = \partial_x \). The hierarchy of equations (the sAKNS hierarchy) can be obtained from the scalar Lax equation

\[ \frac{\partial L}{\partial t_n} = [(L^n)_+, L] \]  

where \( n = 0, 1, 2, \cdots \) represents the flow of the hierarchy. We note here the first few flows of the hierarchy, namely,
\[ \frac{\partial \phi}{\partial t_0} = \phi \]
\[ \frac{\partial \psi}{\partial t_0} = -\psi \]
\[ \frac{\partial \phi}{\partial t_1} = (D^2 \phi) \]
\[ \frac{\partial \psi}{\partial t_1} = (D^2 \psi) \]
\[ \frac{\partial \phi}{\partial t_2} = (D^4 \phi) + 2\phi(D\phi\psi) \]
\[ \frac{\partial \psi}{\partial t_2} = -(D^4 \psi) - 2\psi(D\phi\psi) \]
\[ \frac{\partial \phi}{\partial t_3} = (D^6 \phi) + 3\phi(D(D^2 \phi)\psi) + 3(D^2 \phi)(D\phi\psi) \]
\[ \frac{\partial \psi}{\partial t_3} = (D^6 \psi) + 3\phi(D^2 \psi(D\psi)) \]

and so on. It is worth noting here a peculiarity of this hierarchy of equations, namely, the lowest order equation, which arises from \((L^0)_+ = 1\) is nontrivial mainly because the Lax operator in eq. (3) is composite in the dynamical variables as opposed to the usual cases where the coefficients of the pseudodifferential operators are linear in the dynamical variables leading to the fact that nontrivial flows exist only for \(n = 1, 2, \ldots\).

Given the Lax operator in eq. (3), we can immediately construct the local conserved quantities of the system. They are simply given by

\[ H_n = -\frac{1}{n} sTrL^n = -\frac{1}{n} \int dz \, sResL^n \]

where \(z\) represents the coordinates of the superspace and the superTrace is defined as the integral over the superspace of the superResidue which corresponds to the coefficient of the \(D^{-1}\) term of the pseudodifferential operator. Because \(L\) is bosonic, it is clear from eq. (10), that these conserved quantities are bosonic as well. The first few of these conserved quantities have the explicit form

\[ H_1 = \int dz \, \phi\psi \]
\[ H_2 = \int dz \, (D^2 \phi)\psi \]
\[ H_3 = \int dz \, [(D^4 \phi)\psi + \phi^2\psi(D\psi)] \]
\[ H_4 = \int dz \, [(D^6 \phi)\psi + 3\phi\psi(D\psi)(D^2 \phi)] \] (11)
These quantities can be easily seen to be conserved from the form of the Lax equation in eq. (4) or by explicit computation.

In addition to the local conserved charges, the theory does contain conserved, nonlocal fermionic charges which can also be obtained from the Lax operator as

$$Q_n = (-1)^{n+1} sTrL^{(n+\frac{1}{2})}$$

(12)

It is clear from the definition of these charges that they are fermionic since the Lax operator is bosonic and the explicit form of the first two is given by

$$Q_0 = \int dz (D^{-1}\phi\psi)$$

$$Q_1 = \int dz \left[ \phi (D\psi) + \frac{1}{2} (D^{-1}\phi\psi)^2 - (D^{-1}(D^2\phi)\psi) + 2\phi\psi(D^{-2}\phi\psi) \right]$$

(13)

It is clear that they are manifestly fermionic and nonlocal. The fact that these charges are conserved is not obvious from the Lax equation. However, a little bit of algebra (tedious) shows that they are, indeed, conserved under the flows of the hierarchy in eqs. (5)–(9). The theory is supersymmetric and so, we expect one of these fermionic charges to generate the supersymmetry transformations and this depends on the particular Hamiltonian structure used.

For the sake of completeness as well as for later use, we discuss here, briefly, the zero curvature formulation of this hierarchy[12-13]. The Lax operator of eq. (3) leads to the linear equation

$$(D^2 + \phi D^{-1}\psi)\chi = \lambda\chi$$

(14)

Here $\lambda$ is the spectral parameter of the theory. Starting with this, as discussed in [12], we can obtain an $OSp(2|2) (SL(2|1))$ valued potential, $A_1$,

$$A_1 = \left(\begin{array}{ccc}
\lambda & -\phi & 0 \\
(D\psi) & 0 & -\psi \\
-(\phi\psi) & -(D\phi) & \lambda
\end{array}\right)$$

(15)

such that we can write the linear equation of (14) also as a matrix equation

$$\partial_x\left(\begin{array}{c}
\chi \\
(D^{-1}\psi\chi) \\
(D\chi)
\end{array}\right) = A_1 \left(\begin{array}{c}
\chi \\
(D^{-1}\psi\chi) \\
(D\chi)
\end{array}\right)$$

(16)

It is, then, easy to show that if we define another $OSp(2|2) (SL(2|1))$ valued potential, $A_0$ as

$$A_0 = \left(\begin{array}{ccc}
A & B & C \\
E + C\psi & F & G \\
H + \lambda C & J - \phi C & A + F
\end{array}\right)$$

(17)
then, the zero curvature condition associated with these potentials, namely,

\[ \partial_t A_1 - \partial_x A_0 - [A_0, A_1] = 0 \]

leads to the sAKNS hierarchy of equations provided

\[
\begin{align*}
E &= -(DG) \\
F &= (DC) \\
H &= (DA) + B\psi \\
J &= (DB) \\
C_x &= -G\phi + B\psi \\
A_x &= \phi(DG) - B(D\psi) + 2C\phi\psi \\
\end{align*}
\]

(18)

with the dynamical equations given by

\[
\begin{align*}
\phi_t &= -[B_x - \phi A + (D\phi C) - \lambda B] \\
\psi_t &= -[G_x + \psi A - (D\psi)C + \lambda G] \\
\end{align*}
\]

(19)

3. Hamiltonian Structures:

In this section, we describe the Hamiltonian structures of the theory. Our goal is to find the Hamiltonian structure \( \mathcal{D} \) such that the dynamical equations of the hierarchy can be written as

\[
\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mathcal{D} \begin{pmatrix} \delta H \\ \delta \phi \delta \psi \end{pmatrix}
\]

(20)

where \( H \) represents the appropriate Hamiltonian for the flow. From the form of the dynamical equations in (7)–(9) as well as the forms of the conserved quantities in eq. (12), it is straightforward to write down the simplest of the Hamiltonian structures, namely, we see that with

\[
\mathcal{D}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(21)

we can write all the equations of the hierarchy as

\[
\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \delta H_1 \\ \delta \phi \delta \psi \end{pmatrix}
\]

(22)

The structure \( \mathcal{D}_1 \) is a simple structure with constant elements and, therefore, trivially satisfies the Jacobi identity. Thus, we can identify this with the first Hamiltonian structure of the hierarchy.
However, as we know, integrable hierarchies have, in general, several distinct Hamiltonian structures. Finding the higher ones, though, can be tricky as was already noticed in [9, 16]. Let us point out how one can run into problems here also. With some algebra, one can show that the structure

\[
\tilde{D}_2 = \begin{pmatrix}
-2\phi D^{-1}\phi & D^2 + 2\phi D^{-1}\psi \\
D^2 + 2\psi D^{-1}\phi & -2\psi D^{-1}\psi 
\end{pmatrix}
\]

(23)
gives equations (8)–(9) through the relation (for \(n = 0, 1\))

\[
\partial_{t_{n+1}} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \tilde{D}_2 \begin{pmatrix} \frac{\delta H_n}{\delta \phi} \\ \frac{\delta H_n}{\delta \psi} \end{pmatrix}
\]

(24)
This is, however, a much more complicated structure than \(D_1\) and Jacobi identity is not obvious from the form of this structure. A careful calculation shows that \(\tilde{D}_2\) does not satisfy the Jacobi identity and, therefore, cannot represent a true Hamiltonian structure of the theory. In fact, if we try to derive eq. (10) with this structure, it fails showing that the structure is not quite complete. It misses some terms which give a vanishing contribution for the low order equations and, therefore, do not make their presence manifest.

The true second Hamiltonian structure of the hierarchy can be determined with some work to be

\[
D_2 = \begin{pmatrix}
-\phi D^{-2}\phi D - D\phi D^{-2}\phi & D^2 + D\phi D^{-2}\psi + \phi D^{-2}(D\psi) \\
-2\phi D^{-2}\phi \psi D^{-2}\phi & +2\phi D^{-2}\phi \psi D^{-2}\psi \\
D^2 + \psi D^{-2}\phi D + (D\psi)D^{-2}\phi & -\psi D^{-2}(D\psi) - (D\psi)D^{-2}\psi \\
+2\psi D^{-2}\phi \psi D^{-2}\phi & -2\psi D^{-2}\phi \psi D^{-2}\psi
\end{pmatrix}
\]

(25)
It is now easy to check that the equations of the hierarchy (except for the lowest one) can now be written as \((n = 0, 1, 2, \cdots)\)

\[
\partial_{t_{n+1}} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = D_2 \begin{pmatrix} \frac{\delta H_n}{\delta \phi} \\ \frac{\delta H_n}{\delta \psi} \end{pmatrix}
\]

(26)
This is, of course, a much more complicated structure. However, the Jacobi identity can be verified through the method of prolongation which we describe briefly. Let us consider a graded matrix one form \(\Omega\) defined as

\[
\Omega = \begin{pmatrix} \Omega_f \\ \Omega_b \end{pmatrix}
\]

(27)
where \(\Omega_b\) and \(\Omega_f\) represent respectively bosonic and fermionic components of the matrix one form. Given this and the structure \(D_2\), we can now construct the bivector associated
with the structure $\mathcal{D}_2$ as
\[
\Theta_{\mathcal{D}_2} = \frac{1}{2} \int dz \left( D_2 \Omega \right)_\alpha \wedge \Omega_\alpha \\
= \int dz \left[ (D^2 \Omega_b) \wedge \Omega_f + (\phi \Omega_f - \psi \Omega_b) \wedge \\
\left\{ D^{-2}(\phi (D \Omega_f) - (D \psi) \Omega_b + \phi \psi (D^{-2}(\phi \Omega_f - \psi \Omega_b))) \right\} \right]
\]
(28)

Here $\alpha = 1, 2$ runs over the two components of the matrix and we have used integration by parts. The structure $\mathcal{D}_2$ can be shown to satisfy Jacobi identity provided the prolongation of this bivector vanishes.

The prolongation can be calculated by noting that the prolongations for the basic variables are defined to be (see [14-15] for details)
\[
\text{pr} \mathcal{D}_2 \Omega \left( \Theta_{\mathcal{D}_2} \right) = \left( \begin{array}{c} (\mathcal{D}_2 \Omega)_1 \\ (\mathcal{D}_2 \Omega)_2 \end{array} \right)
\]
(29)

With some tedious algebra and using eq. (29), we can show that (upto surface terms)
\[
\text{pr} \mathcal{D}_2 \Omega (\Theta_{\mathcal{D}_2}) = 0
\]
(30)

This shows that $\mathcal{D}_2$ satisfies the Jacobi identity and, indeed, represents the true second Hamiltonian of the theory. Furthermore, it is also easy to show that the structures $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible. Namely, let
\[
\mathcal{D} = \mathcal{D}_2 + \alpha \mathcal{D}_1
\]
(31)

where $\alpha$ is an arbitrary constant parameter. Then, from the simple structure of $\mathcal{D}_1$ in eq. (21), it is straightforward to check that
\[
\text{pr} \mathcal{D}_1 \Omega (\Theta_{\mathcal{D}_2}) = 0
\]
(32)

showing that any linear combination of $\mathcal{D}_1$ and $\mathcal{D}_2$ is also a Hamiltonian structure.

The compatibility of $\mathcal{D}_1$ and $\mathcal{D}_2$, suggests that we can define the recursion operator for the theory which corresponds to
\[
\mathcal{R} = D_1^{-1} \mathcal{D}_2
\]
\[
= \left( \begin{array}{cc} -D^2 - \psi D^{-2} \phi D - (D \psi) D^{-2} \phi & \psi D^{-2}(D \psi) + (D \psi) D^{-2} \phi \\
-2\psi D^{-2} \phi D^{-2} \phi & +2\psi D^{-2} \phi D^{-2} \phi \\
-\phi D^{-2} \phi D - D \phi D^{-2} \phi & \phi D^2 + D \phi D^{-2} \phi + \phi D^{2}(D \psi) \\
-2\phi D^{-2} \phi D^{-2} \phi & +2\phi D^{-2} \phi D^{-2} \phi \end{array} \right)
\]
(33)
which will give the recursion relation between the Hamiltonians as

\[
\begin{pmatrix}
\frac{\delta H_{n+1}}{\delta \phi} \\
\frac{\delta H_{n+1}}{\delta \psi}
\end{pmatrix}
= \mathcal{R}
\begin{pmatrix}
\frac{\delta H_n}{\delta \phi} \\
\frac{\delta H_n}{\delta \psi}
\end{pmatrix}
\] (34)

as well as the higher order Hamiltonian structures, for example,

\[D_3 = D_2 \mathcal{R}\]

and so on. Furthermore, the Nijenhuis torsion tensor associated with this recursion operator would vanish implying again the integrability of this hierarchy of equations.

It is also worth pointing out here that we could try to obtain an alternate supersymmetry hierarchy associated with the AKNS hierarchy as follows. If we choose as the Lax operator

\[L = D^2 + \phi D^{-2}(D \psi)\] (35)

which differs only slightly from the Lax operator in eq. (3), the equations

\[\frac{\partial L}{\partial t_n} = [(L^n)_+, L]\] (36)

would correspond to the sAKNS-B hierarchy which is an alternate supersymmetrization\[17\] of the AKNS hierarchy. Thus, for example, the second order equation following from eq. (36) would be

\[\begin{align*}
\frac{\partial \phi}{\partial t} &= (D^4 \phi) + 2 \phi^2 (D \psi) \\
\frac{\partial \psi}{\partial t} &= -(D^4 \psi) - 2(D^{-1} \phi (D \psi)^2)
\end{align*}\] (37)

which can be compared with eq. (8). The second equation of (37), however, is nonlocal, as was also noted in \[18\] and so, the alternate supersymmetrization for the case of AKNS hierarchy does not appear to be particularly useful.

We also note from the form of the Lax operator in (3) that the terms are not linear in the dynamical variables. Therefore, it is not clear how one can define a dual which will give rise to a linear functional of the dynamical variables. Consequently, the question of obtaining the Hamiltonian structures of the theory from the Gelfand-Dikii brackets or through the R-matrix approach remains open.
4. A Fermionic Lax Description:

The sAKNS hierarchy discussed in the above sections is quite peculiar in that, in addition to its description in terms of a bosonic Lax operator in eq. (3), it can also be described equivalently by a fermionic Lax operator. Let us consider the Lax operator

\[ \Lambda = D + \phi D^{-2} \psi \]  

where \( \phi \) and \( \psi \) denote the superfields discussed in the earlier sections. It is clear that this Lax operator, unlike the one in eq. (3), is fermionic. Furthermore, it is straightforward to check that the Lax equation (\( n = 0, 1, 2, \cdots \))

\[ \frac{\partial \Lambda}{\partial t_n} = \left[ (\Lambda^{2n})_+, \Lambda \right] \]  

(39)

gives the sAKNS hierarchy of equations in eq. (4) or more explicitly in eqs. (6)–(9). This is, therefore, the only hierarchy of equations that we know of which can be described in terms of a bosonic as well as a fermionic Lax operator. Furthermore, there is no obvious relation between the two Lax operators in eqs. (3) and (38) although we will describe some equivalences later in this section. We note here that the odd powers of \( \Lambda \), namely, \((\Lambda^{2n+1})_+\) in eq. (39) do not lead to any consistent equation (with anti-commutation relation, of course).

Since this is a new Lax operator, it is worth studying its properties systematically. First, let us look at the conserved quantities following from this Lax operator. It is straightforward to check that

\[ sTr(\Lambda)^n = 0 \quad \text{for any} \ n \]  

(40)

On the other hand, we have checked explicitly up to the fourth term and conjecture that

\[ Res\Lambda^{2n} = -(DsResL^n) \]  

(41)

so that the local conserved quantities in eq. (10) can equivalently be written as

\[ H_n = \frac{1}{n} \int dz \left( D^{-1}Res\Lambda^{2n} \right) \]  

(42)

Since \( \Lambda \) is a fermionic operator, it would appear that the fermionic conserved quantities should follow in a simple manner. However, we have not succeeded in obtaining the

fermionic, nonlocal conserved quantities of eq. (12)–(13) from Λ. That remains an open question.

It would be interesting if we can relate the two Lax operators in some way so that the calculations with either of them would simplify. However, we have not, so far, found a transformation that would directly relate the two except for the observation that, under

\[ \phi \to D^{-1} \phi D \] (43)

we have

\[ \Lambda \to D + D^{-1} \phi D^{-1} \psi \]
\[ = D^{-1}(D^2 + \phi D^{-1} \psi) \]
\[ = D^{-1} L \] (44)

This is, however, not very useful in practical calculations. We note here that we can write

\[ \Lambda^2 = D^2 + D \phi D^{-2} \psi + \phi D^{-2} \psi D + \phi D^{-2} \phi \psi D^{-2} \psi \]
\[ = D^2 + \tilde{\phi} D^{-2} \psi - \phi D^{-2} \tilde{\psi} \] (45)

where

\[ \tilde{\phi} = (D\phi) + (D^{-2} \phi \psi)\phi \]
\[ \tilde{\psi} = -(D\psi) + (D^{-2} \phi \psi)\psi \] (46)

are the variables already defined in [12] in connection with a Hamiltonian description of this system. This way of defining \( \Lambda^2 \) simplifies the calculation of the local conserved quantities of the theory greatly.

Let us next note that if we start with the Lax operator in eq. (38), we can write a linear equation of the form

\[ \Lambda \chi = (D + \phi D^{-2} \psi)\chi = 0 \] (47)

From eq. (38), we can write the matrix equation

\[ D \begin{pmatrix} \phi D^{-2} \psi & \chi \\ \chi & (D^{-2} \phi D^{-2} \psi) \end{pmatrix} = \begin{pmatrix} 0 & \psi & 0 \\ 0 & 0 & -\phi \\ 1 & 0 & 0 \end{pmatrix} \] (48)

It now follows from this, as well as the properties of the supercovariant derivative, that

\[ \partial_x \begin{pmatrix} \phi D^{-2} \psi & \chi \\ \chi & (D^{-2} \phi D^{-2} \psi) \end{pmatrix} = A_1 \begin{pmatrix} (D^{-1} \phi \psi) \\ \chi \\ (D^{-2} \phi D^{-2} \psi) \end{pmatrix} \] (49)
where

\[ A_1 = \begin{pmatrix} 0 & (D\psi) & \phi\psi \\ -\phi & 0 & -(D\phi) \\ 0 & \psi & 0 \end{pmatrix} \]  \tag{50}

Furthermore, if we define a graded matrix

\[ A_0 = \begin{pmatrix} A & E - C\psi & H \\ B & F & J + \phi C \\ C & G & A + F \end{pmatrix} \]  \tag{51}

then, it is straightforward to see that the zero curvature condition

\[ \partial_t A_1 - \partial_x A_0 - [A_0, A_1] = 0 \]

leads to the sAKNS hierarchy of equations provided

\[
\begin{align*}
F &= -(DC) \\
E &= (DG) \\
J &= (DB) \\
H &= (DA) - B\psi \\
A_x &= (DG)\phi + B(D\psi) - 2C\phi\psi \\
C_x &= G\phi + B\psi
\end{align*}
\]  \tag{52}

It is easy now to check from eqs. (13)–(18) that these are the same as before with proper identification (and \( \lambda = 0 \)). This is an alternate way to see that the fermionic Lax operator gives rise to the entire hierarchy of sAKNS equations.

An alternate way of establishing this equivalence with the spectral parameter is to note that if we start with the linear matrix equation

\[ \tilde{\Lambda} \begin{pmatrix} \chi_b \\ \chi_f \end{pmatrix} = \sqrt{\lambda} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_b \\ \chi_f \end{pmatrix} \]  \tag{53}

where \( \tilde{\Lambda} \) is obtained from \( \Lambda \) by conjugation

\[ \tilde{\Lambda} = -\Lambda^\dagger = D + \psi D^{-2}\phi \]  \tag{54}

and \( \chi_b \) and \( \chi_f \) are two bosonic and fermionic superfield wave functions respectively, then, this set of two coupled equations is equivalent to the two uncoupled equations of the form

\[ \tilde{\Lambda}^2\chi = (D^2 - \bar{\psi} D^{-2}\phi - \psi D^{-2}\bar{\phi})\chi = \lambda\chi \]  \tag{55}
It is now easy to obtain from this that
\[
\partial_x \begin{pmatrix}
  e^{(-\lambda x)}(D^2\phi\chi)
  \\
  e^{(-\lambda x)}\chi
  \\
  e^{(-\lambda x)}(D^2\bar{\psi}\chi)
\end{pmatrix}
= \begin{pmatrix}
  -\lambda & \phi & 0 \\
  \bar{\psi} & 0 & \psi \\
  0 & \bar{\phi} & -\lambda
\end{pmatrix}
\begin{pmatrix}
  e^{(-\lambda x)}(D^2\phi\chi)
  \\
  e^{(-\lambda x)}\chi
  \\
  e^{(-\lambda x)}(D^2\bar{\psi}\chi)
\end{pmatrix}
\]
(56)

This is the same as the matrix obtained earlier in [12] in connection with the conventional bosonic Lax operator. This establishes a connection between the matrix eigenvalue problems of two descriptions although a more direct connection remains an open question.

5. Soliton Solutions:

The soliton solutions for the sAKNS hierarchy have already been constructed earlier [13] from the conventional, bosonic Lax operator through the algebraic dressing method. Here, we would like to discuss the construction of soliton solutions for the fermionic Lax operator \( \tilde{\Lambda} \) as elements on the Darboux-Bäcklund orbit. Let us consider a general Darboux-Bäcklund transformation generated by

\[
T = \xi \partial \xi^{-1} = \xi D^2\xi^{-1}
\]
(57)

where \( \xi \) is an arbitrary bosonic superfield. From this, we obtain

\[
TDT^{-1} = D + \xi \left(D^3 \ln \xi\right) D^{-2}\xi^{-1}
\]
(58)

Similarly, we can also show that

\[
T\psi D^{-2}\phi T^{-1} = \xi \left(\xi^{-1}\psi \int (\phi \xi)\right)' D^{-2}\xi^{-1} - \xi \left(\xi^{-1}\psi\right)' D^{-2} \left(\int (\phi \xi)\right) \xi^{-1}
\]
(59)

We want the sum of the terms on right hand sides of (58) and (59) representing \( T\tilde{\Lambda}T^{-1} \) to have the same form as \( \tilde{\Lambda} = D + \psi D^{-2}\phi \) in (54).

It is clear that there are only two possible solutions. First, if \( \xi \) satisfies \( (\xi^{-1}\psi)' = 0 \) or \( \xi^{-1}\psi = f_0 \) where \( f_0 \) is an odd constant, then, we have

\[
\tilde{\Lambda} = D + \left[\xi \left(D^3 \ln \xi\right) + \xi^2 f_0\right] D^{-2}\xi^{-1}
\]
(60)

which is of the same form as \( \tilde{\Lambda} \).

The second possibility, on the other hand, is more interesting. Let us choose \( \xi \) such that

\[
\xi \left(D^3 \ln \xi\right) + \xi \left(\xi^{-1}\psi \int (\phi \xi)\right)' = 0
\]
(61)
With this choice it follows from (58) and (59) that
\[ T \tilde{\Lambda} T^{-1} = D - \xi \left( \xi^{-1} \psi \right)' D^{-2} \left( \int (\phi \xi) \right) \xi^{-1} \] (62)
has the desired form.

Condition (61) can be rewritten as (by ignoring one odd integration constant)
\[ D \xi + \psi \int (\phi \xi) = 0 \] (63)
Remarkably, this is nothing but a condition \( \tilde{\Lambda}(\xi) = 0 \). We will use this fact later. Define now
\[ \tilde{\Lambda}_2 = T_1 \tilde{\Lambda}_1 T_1^{-1} = T_1 T_0 DT_0^{-1} T_1^{-1} ; \quad T_0 = \xi_0 \partial \xi_0^{-1} ; \quad T_1 = \xi_1 \partial \xi_1^{-1} \] (64)
One sees that \( \tilde{\Lambda}_1 \) is given by the right hand side of (58). The condition (63) \( \tilde{\Lambda}_1(\xi_1) = 0 \) for \( \xi_1 \) has a solution of the form \( \xi_1 = \text{const} \times \partial \ln \xi_0 \). For this solution we find (with \( \text{const} = 1 \)):
\[ \tilde{\Lambda}_2 = T_1 \tilde{\Lambda}_1 T_1^{-1} = D + (\partial \xi_0) \left( D^3 \ln(\partial \xi_0) \right) D^{-2}(\partial \xi_0)^{-1} \] (65)
These observations inspire us to define a chain of the Darboux-Bäcklund transformations
\[ \tilde{\Lambda}_n = T_{n-1} \tilde{\Lambda}_{n-1} T_{n-1}^{-1} ; \quad T_n = \xi_n \partial \xi_n^{-1} \] (66)
such that the constraint \( \tilde{\Lambda}_n(\xi_n) = 0 \) is satisfied at each level. Correspondingly,
\[ \tilde{\Lambda}_n = D + \psi_n D^{-2} \phi_n \] (67)
\[ \phi_n = (T_{n-1})^\dagger \cdots (T_1)^\dagger (\xi_0^{-1}) \] (68)
\[ \psi_n = T_{n-1} \cdots T_1 (\xi_0 D^3 \ln \xi_0) = T_{n-1} \cdots T_0 (D \xi_0) \] (69)
The closed expressions for \( \xi_n, \phi_n, \psi_n \)
\[ \xi_n = \partial \ln \left( \partial^n \xi_0 \right) \] (70)
\[ \phi_n = \left( \partial^n \xi_0 \right)^{-1} \] (71)
\[ \psi_n = \left( \partial^n \xi_0 \right) D^3 \ln \left( \partial^n \xi_0 \right) \] (72)
can be verified by inspection. One also finds that \( \tilde{\phi}_n = \tilde{\Lambda}_n^\dagger(\phi_n) = 0 \) for all \( n \) and therefore the bosonic Lax operator
\[ \tilde{\Lambda}_n^2 = \partial + \left( \partial^{n-1} \xi_0 \right) \left( \ln \left( \partial^{n-1} \xi_0 \right) \right)^{\prime \prime} \partial^{-1} \left( \partial^{n-1} \xi_0 \right)^{-1} \] (73)
obtained by squaring $\tilde{\Lambda}_n$ like in (55) will only have bosonic coefficients, while fermionic coefficients do not appear. Similarly, in the s-AKNS equations of motion for these solutions the quadratic fermionic terms will be absent. This feature is reminiscent of the soliton solutions found for the sAKNS model in [13] using the algebraic dressing method.

6. Conclusion:

As we have argued, the sAKNS hierarchy is clearly an important supersymmetric integrable hierarchy. In this paper, we have worked out various properties associated with this hierarchy systematically. We have constructed the conserved charges (both bosonic and fermionic), the Hamiltonian structures verifying the Jacobi identity and the recursion operator. We have also shown that this is a unique hierarchy which allows a description in terms of a bosonic as well as a fermionic Lax operator. Starting from the fermionic Lax operator, we have shown how the bosonic conserved charges as well as the zero curvature formulation can be obtained. The Darboux-Bäcklund transformation associated with the fermionic Lax operator is also worked out leading to soliton solutions. There remain, however, some open questions. Namely, we do not yet know how to obtain the fermionic conserved quantities starting from the fermionic Lax operator nor is a direct connection between the bosonic and the fermionic Lax operator clear at this point. However, we would like to emphasize that it is quite worth understanding in detail the description of this hierarchy in terms of the fermionic Lax operator since it might give some insight into the Manin-Radul description of the supersymmetric KP hierarchy[7].

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