Swap-invariant and exchangeable random measures

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Abstract

In this work we analyze the concept of swap-invariance, which is a weaker variant of exchangeability. A random vector $\xi$ in $\mathbb{R}^n$ is called swap-invariant if $E|\sum_j u_j \xi_j|$ is invariant under all permutations of $(\xi_1, \ldots, \xi_n)$ for each $u \in \mathbb{R}^n$. We extend this notion to random measures. For a swap-invariant random measure $\xi$ on a measure space $(S, \mathcal{S}, \mu)$ the vector $(\xi(A_1), \ldots, \xi(A_n))$ is swap-invariant for all disjoint $A_j \in \mathcal{S}$ with equal $\mu$-measure. Various characterizations of swap-invariant random measures and connections to exchangeable ones are established. We prove the ergodic theorem for swap-invariant random measures and derive a representation in terms of the ergodic limit and an exchangeable random measure. Moreover we show that diffuse swap-invariant random measures on a Borel space are trivial. As for random sequences two new representations are obtained using different ergodic limits.

Keywords: ergodic theorem; exchangeability; random measure; swap-invariance; zonoid equivalence.

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1 Introduction

Two integrable random vectors $\xi$ and $\eta$ in $\mathbb{R}^n$ are called zonoid equivalent if $E|\langle u, \xi \rangle| = E|\langle u, \eta \rangle|$ for every $u \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. A vector $\xi$ of integrable random variables is called swap-invariant if $\xi \circ \pi$ and $\xi$ are zonoid equivalent for all permutations $\pi$ of $\{1, \ldots, n\}$. Swap-invariance is weaker than exchangeability. Both exchangeability and swap-invariance are extended to random sequences by requiring the respective property for all finite-dimensional distributions. Swap-invariant sequences are introduced and analyzed in [5]. An important property of a swap-invariant sequence $\xi$ is that $n^{-1} \sum_{j=1}^n \xi_j \rightarrow X$ almost surely as $n \rightarrow \infty$ for some random variable $X$ (cf. [5, Theorem 17]). In contrast to the ergodic theorem for integrable exchangeable sequences (see for example [3, Theorem 10.6]), this convergence is not necessarily in $L^1$.

The definitions of exchangeability and swap-invariance can both be extended to random measures on a measure space $(S, \mathcal{S}, \mu)$ where $\mu$ is a deterministic reference measure. Whereas exchangeable random measures are well known (see e.g. [4 and [3 Chapter 10]),
our definition of swap-invariance is new, and it is strictly weaker than exchangeability. If a random measure $\xi$ on a measure space $(S, \mathcal{S}, \mu)$ is $\mu$-exchangeable, then the vector $(\xi(A_1), \ldots, \xi(A_n))$ is exchangeable for all disjoint $A_j \in \mathcal{S}$ with equal $\mu$-measure; if $\xi$ is $\mu$-swap-invariant, then this vector is merely swap-invariant. Exchangeable or swap-invariant sequences arise as special cases where $S = \mathbb{N}$ and $\mu$ is the counting measure.

We show the ergodic theorem for swap-invariant random measures, that is

$$\frac{\xi(A_n)}{\mu(A_n)} \to X \quad \text{a.s. as } n \to \infty$$

for some integrable random variable $X$. Here $(A_n)_{n \geq 1}$ is any increasing sequence of measurable sets such that $\mu(A_n) < \infty$ and $\mu(A_n) \to \infty$ as $n \to \infty$. The ergodic limit always exists if $\mu$ is atomless, and it is independent of the sequence of sets under additional assumptions. As an important consequence we obtain the representation

$$\xi = X \eta \quad \text{(1)}$$

where the random measure $\eta$ is exchangeable under a certain probability measure.

In Section 2 we first give five characterizations of swap-invariant random measures and their counterparts in the exchangeable case. Then a construction method for swap-invariant random measures is provided, which is based on a change of the probability measure. We give an example of a swap-invariant non-exchangeable point process derived from a Poisson process. For finite $\mu$ we demonstrate that a swap-invariant random measure can be represented as $\xi = \xi(S) \eta$ where $\eta$ is exchangeable under a certain probability measure. This formula resembles (1); however in the case of finite $\mu$ no ergodic limit is involved. If the reference measure $\mu$ is $\sigma$-finite and atomless, and if the intensity measure $E\xi$ is $\sigma$-finite, then $E\xi = c\mu$ for some $c \geq 0$. Finally it is shown that a swap-invariant diffuse random measure on a Borel space has the form $\alpha\mu$ where $\alpha \geq 0$ is a random variable. This fact is known for exchangeable random measures.

Section 3 of this paper is devoted to a detailed analysis of the connection between swap-invariant and exchangeable sequences. These results are the basis for the proof of (1) and are interesting by themselves. First it is shown that each swap-invariant sequence of random variables with only two positive values is exchangeable. We then present a simple construction method for swap-invariant sequences, which consists in a multiplication of a given swap-invariant or exchangeable sequence by a random factor and a simultaneous change of the probability measure. In the remainder of Section 3 we prove that two large classes of swap-invariant sequences can always be represented in this way, namely by using an ergodic limit as the random factor; in one case we use the ergodic limit of the sequence itself, in the other case the limit of $p$-norms of the means.

Finally in Section 4 the ergodic theorem for swap-invariant random measures is shown and (1) is derived. We also prove a variant of this theorem for the special case of exchangeable random measures.
2 Swap-invariant random measures

2.1 Preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((S, \mathcal{S})\) a measurable space. A random measure \(\xi\) on \(S\) is a map \(\xi : \Omega \times S \to \mathbb{R}_+\) such that \(\xi(\cdot, M)\) is a \(\mathbb{R}_+\)-valued random element for each \(M \in \mathcal{S}\), and \(\xi(\omega, \cdot)\) is a measure on \(S\) for each \(\omega \in \Omega\). The intensity measure of \(\xi\) is \(\mathbb{E}\xi(M)\) for \(M \in \mathcal{S}\).

If \((A, \mathcal{A})\) and \((B, \mathcal{B})\) are measurable spaces, \(\mu\) is a measure on \(A\), and \(f : A \to B\) is measurable, then the image of \(\mu\) under \(f\) is denoted by \(\mu \circ f^{-1}\), which is a measure on \(B\). If \(\xi\) is a random measure on \(A\), \(\xi \circ f^{-1}\) is defined pointwise for each \(\omega\), and therefore is a random measure on \(B\).

A random measure \(\xi\) on \(S\) is called \(\sigma\)-finite if there exists a fixed measurable partition \((S_j)_{j \geq 1}\) of \(S\) such that \(\xi(S_j) < \infty\) almost surely for every \(j \geq 1\). An atom \(M\) of a measure \(\mu\) is a measurable set such that \(\mu(M) > 0\) and for each measurable \(C \subset M\) either \(\mu(C) = 0\) or \(\mu(C) = \mu(M)\) holds; \(\mu\) is called atomless if it has no atoms. If \((A, \mathcal{A})\) and \((B, \mathcal{B})\) are measurable spaces, then \(f : A \to B\) is called a Borel isomorphism if \(f\) is a bijection and both \(f\) and \(f^{-1}\) are measurable. A measurable space \((S, \mathcal{S})\) is called a Borel space if \(S\) is Borel isomorphic to a Borel subset of \(([0, 1], \mathcal{B}_[0, 1])\). A measure \(\mu\) on a Borel space \(S\) is called diffuse if \(\mu(\{s\}) = 0\) for each \(s \in S\).

For \(n \geq 1\) we write \(\text{Perm}(n)\) for the family of permutations of \(\{1, \ldots, n\}\).

2.2 Exchangeability

The definition of exchangeable random measures relies on the following lemma.

**Lemma 2.1.** Let \(\xi\) be a random measure on a measure space \((S, \mathcal{S}, \mu)\). Consider the following statements:

(i) \((\xi(A_1), \ldots, \xi(A_n)) \overset{d}{=} (\xi(B_1), \ldots, \xi(B_n))\) for any disjoint \(A_1, \ldots, A_n \in \mathcal{S}\), \(n \geq 1\), and any disjoint \(B_1, \ldots, B_n \in \mathcal{S}\) with \(\mu(A_j) = \mu(B_j)\) for \(1 \leq j \leq n\).

(ii) \(\int f \, d\xi \overset{d}{=} \int g \, d\xi\) for each two measurable functions \(f, g : S \to \mathbb{R}_+\) with \(\mu \circ f^{-1} = \mu \circ g^{-1}\).

(iii) \(\xi \circ f^{-1} \overset{d}{=} \xi \circ g^{-1}\) for each two measurable functions \(f, g : S \to \mathbb{R}_+\) with \(\mu \circ f^{-1} = \mu \circ g^{-1}\).

(iv) \(\xi \circ f^{-1} \overset{d}{=} \xi\) for each measurable function \(f : S \to S\) with \(\mu \circ f^{-1} = \mu\).

(v) There exists a sequence \(c_k \downarrow 0\) such that for each \(k \geq 1\), \(n \geq 1\), any disjoint \(A_1, \ldots, A_n \in \mathcal{S}\) with \(\mu(A_j) = c_k\) for \(1 \leq j \leq n\), and each \(\pi \in \text{Perm}(n)\), we have \((\xi(A_1), \ldots, \xi(A_n)) \overset{d}{=} (\xi(A_{\pi(1)}), \ldots, \xi(A_{\pi(n)}))\).

Statements (i), (ii), and (iii) are equivalent, and they imply (iv) and (v). If \(\mu\) is atomless and \(\sigma\)-finite, \(\mu(S) = \infty\), and \(\xi\) is \(\sigma\)-finite, then (v) implies (i). If, in addition, \((S, \mathcal{S})\) is Borel, then also (iv) implies (i).
**Definition 2.2.** Let \( \xi \) be a random measure on a measure space \((S, \mathcal{S}, \mu)\). If \( \xi \) satisfies either of the three conditions (i), (ii), (iii) in Lemma 2.1 then it is called \( \mu \)-exchangeable.

It is easily seen that the \( \mu \)-exchangeable random measures on \( \mathbb{N} \), where \( \mu \) is the counting measure, are exactly the exchangeable sequences in \( \mathbb{R}_+^\infty \).

For the proof of Lemma 2.1 some technical prerequisites are required. We need the following version of the Cramér-Wold device for \( \mathbb{R}_+^n \). To this end the definition of the Euclidean inner product \( \langle \cdot, \cdot \rangle \) is extended to \( \mathbb{R}_+^n \times \mathbb{R}_+^n \) in the canonical way.

**Lemma 2.3.** Two random elements \( \xi \) and \( \eta \) in \( \mathbb{R}_+^n \) are equal in distribution if and only if \( \langle u, \xi \rangle \overset{d}{=} \langle u, \eta \rangle \) for all \( u \in \mathbb{R}_+^n \).

**Corollary 2.4.** Let \( \xi \) and \( \eta \) be two random measures on a measurable space \((S, \mathcal{S})\). Then \( \xi \overset{d}{=} \eta \) under each of the following conditions:

(i) \( \langle \xi(A_1), \ldots, \xi(A_n) \rangle \overset{d}{=} \langle \eta(A_1), \ldots, \eta(A_n) \rangle \) for any disjoint \( A_1, \ldots, A_n \in \mathcal{S}, \, n \geq 1 \).

(ii) \( \int f \, d\xi \overset{d}{=} \int f \, d\eta \) for each measurable function \( f \geq 0 \).

The following result is required in Lemma 2.1 for the implication from \( [\text{iv}] \) to \( [\text{i}] \) and for the corresponding implication in Lemma 2.9.

**Lemma 2.5.** Let \((S, \mathcal{S}, \mu)\) be an atomless measure space, \( J \geq 1, A_j \in \mathcal{S} \) with \( 0 < \mu(A_j) < \infty \) for \( 1 \leq j \leq J \), and \( c_k \downarrow 0 \) as \( k \to \infty \). Then there exist measurable sequences \((A^n_j)_{n \geq 1}\) with \( A_j^n \uparrow A_j \) as \( n \to \infty \) for \( 1 \leq j \leq J \), and a subsequence \((c_{k(n)})_{n \geq 1}\) such that

\[
0 < \mu(A_j \setminus A_j^n) \leq c_{k(n)}, \quad \mu(A_j^n) = m_{jn} c_{k(n)}
\]

for \( 1 \leq j \leq J, \, n \geq 1 \), and some integers \( m_{jn} \geq 0 \).

**Proof.** For \( n = 1 \) let \( k_1 = 1 \) and choose \( A^1_j \subset A_j \) and \( m_{j1} \in \mathbb{Z}_+ \) for \( 1 \leq j \leq J \) such that \( \mu(A^1_j) = m_{j1} c_1 \) and \( 0 < \mu(A_j \setminus A^1_j) \leq c_1 \). Note that we may have \( m_{j1} = 0 \). Now assume that, for some \( N \geq 1 \), we have already found \( A^n_j, m_{jn} \), and \( k_n \) for \( 1 \leq j \leq J, \, 1 \leq n \leq N \). Set

\[
k_{N+1} = \min \left\{ k \geq 1 ; \mu(A_j \setminus A_j^N) > c_k (1 \leq j \leq J) \right\}
\]

and choose \( A_j^{N+1} \) and \( m_{j(N+1)} \) such that

\[
A_j^N \subset A_j^{N+1}, \quad \mu(A_j^{N+1}) = m_{j(N+1)} c_{k(N+1)}, \quad 0 < \mu(A_j \setminus A_j^{N+1}) \leq c_{k(N+1)}.
\]

In order to show that \( [\text{iv}] \) implies \( [\text{i}] \) in Lemma 2.1 and the corresponding implication in Lemma 2.9, we make use of the following result, which can be derived e.g. using Proposition 9.1.11 in [1] and Lemma 3.22 in [3].

**Lemma 2.6.** Let \((C, \mathcal{C}, \mu)\) and \((D, \mathcal{D}, \nu)\) be measure spaces where \( \mu \) is atomless, \( D \) is Borel, and \( \mu(C) = \nu(D) \in (0, \infty) \). Then there is a measurable function \( h : C \to D \) such that \( \mu \circ h^{-1} = \nu \).
Proof of Lemma 2.1. We first show that (i) implies (iii). Let \( f \) and \( g \) be as in (iii). If \( n \geq 1 \) and \( A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}_+) \) are disjoint, then also \( f^{-1}(A_1), \ldots, f^{-1}(A_n) \) are disjoint, and \( g^{-1}(A_1), \ldots, g^{-1}(A_n) \) are disjoint. Moreover \( \mu(f^{-1}(A_j)) = \mu(g^{-1}(A_j)) \) for \( 1 \leq j \leq n \). Hence by (i)

\[
(x(f^{-1}(A_1)), \ldots, x(f^{-1}(A_n))) \overset{d}{=} (x(g^{-1}(A_1)), \ldots, x(g^{-1}(A_n))) .
\]

By Corollary 2.4 (i) the two random measures \( \xi \circ f^{-1} \) and \( \xi \circ g^{-1} \) on \( \mathbb{R}_+ \) have the same distribution.

In order to show that (iii) implies (ii), let \( f \) and \( g \) be as required and note that

\[
\int_S f \, d\xi = \int_{\mathbb{R}_+} x \, d(\xi \circ f^{-1})(x) \overset{d}{=} \int_{\mathbb{R}_+} x \, d(\xi \circ g^{-1})(x) = \int_S g \, d\xi .
\]

To see that (ii) implies (i), let \( n, (A_j)_{1 \leq j \leq n} \), and \( (B_j)_{1 \leq j \leq n} \) be as in (ii), \( u \in \mathbb{R}_+^n \), and define the functions \( f = \sum_{j=1}^n u_j 1_{A(j)} \) and \( g = \sum_{j=1}^n u_j 1_{B(j)} \). We have, for \( B \in \mathcal{B}(\mathbb{R}_+) \)

\[
\mu\{ f \in B \} = \sum_{j=1}^n \mu(A_j) 1_{B}(u_j) = \sum_{j=1}^n \mu(B_j) 1_{B}(u_j) = \mu\{ g \in B \} .
\]

Hence by assumption

\[
\sum_{j=1}^n u_j \xi(A_j) = \int f \, d\xi = \int g \, d\xi = \sum_{j=1}^n u_j \xi(B_j) .
\]

Now (i) follows by Lemma 2.3.

The implications from (i) to (iv) and from (i) to (v) are rather straightforward to prove.

We next prove that (i) follows from (v) under the additional assumptions. Let \( n, (A_j)_{1 \leq j \leq n} \), and \( (B_j)_{1 \leq j \leq n} \) be as in (i), and \( (c_k)_{k \geq 1} \) as in (v). First assume that \( \mu(A_j) \in (0, \infty) \) for \( 1 \leq j \leq n \). We may additionally assume that \( A_i \cap B_j = \emptyset \) for \( 1 \leq i, j \leq n \). For otherwise we could choose a third family of measurable disjoint sets \( (C_j)_{1 \leq j \leq n} \) such that \( \mu(C_j) = \mu(A_j) \) for \( 1 \leq j \leq n \), and \( C_j \cap A_i = \emptyset \) and \( C_j \cap B_i = \emptyset \) for \( 1 \leq i, j \leq n \), and then conclude that

\[
(x(A_1), \ldots, x(A_n)) \overset{d}{=} (x(C_1), \ldots, x(C_n)) \overset{d}{=} (x(B_1), \ldots, x(B_n)) .
\]

By Lemma 2.5 we may choose sequences of measurable sets \( (A_j^N)_{N \geq 1} \) and \( (B_j^N)_{N \geq 1} \) for \( 1 \leq j \leq n \), a subsequence \( (c_k(N))_{N \geq 1} \) of \( (c_k) \), and integers \( m_{j,N} \geq 0 \) for \( 1 \leq j \leq n, N \geq 1 \), such that \( A_j^N \uparrow A_j \) and \( B_j^N \uparrow B_j \) as \( N \to \infty \) and

\[
\mu(A_j^N) = m_{j,N} c_k(N) , \quad 0 < \mu(A_j \setminus A_j^N) \leq c_k(N) ,
\]

\[
\mu(B_j^N) = m_{j,N} c_k(N) , \quad 0 < \mu(B_j \setminus B_j^N) \leq c_k(N) .
\]

For large \( N \) we have \( m_{j,N} \geq 1 \) for all \( j \). For such \( N \) we may partition the sets \( A_j^N \) and \( B_j^N \) each into \( m_{j,N} \) sets of \( \mu \)-measure \( c_k(N) \), and therefore obtain by (v)

\[
(x(A_1^N), \ldots, x(A_n^N)) \overset{d}{=} (x(B_1^N), \ldots, x(B_n^N)) .
\]
Since \( \xi(A_j^N) \uparrow \xi(A_j) \) and \( \xi(B_j^N) \uparrow \xi(B_j) \) almost surely as \( N \to \infty \) for \( 1 \leq j \leq n \), we have
\[
(\xi(A_1), \ldots, \xi(A_n)) \overset{d}{=} (\xi(B_1), \ldots, \xi(B_n)).
\]
So far we have assumed that \( \mu(A_j) \in (0, \infty) \) for \( 1 \leq j \leq n \). We next relax this assumption and assume only that \( \mu(A_j) \in \mathbb{R}_+ \) for \( 1 \leq j \leq n \), i.e. some sets may have \( \mu \)-measure zero. It is sufficient to show that, for each \( A \in S \), \( \mu(A) = 0 \) implies \( \xi(A) = 0 \) almost surely. Let \( A \in S \) with \( \mu(A) = 0 \). By the \( \sigma \)-finiteness of \( \xi \) we may assume that \( \xi(A) < \infty \) almost surely. If \( A = \emptyset \), then clearly \( \xi(A) = 0 \). If \( A \neq \emptyset \), choose measurable sequences \( (A_k)_{k \geq 1} \) and \( (B_k)_{k \geq 1} \) with \( A_k \downarrow A \) and \( B_k \downarrow \emptyset \) as \( k \to \infty \), \( \mu(A_k) = \mu(B_k) \in (0, \infty) \) for \( k \geq 1 \), and \( \xi(A_k) < \infty \), \( \xi(B_k) < \infty \) almost surely. The proof for positive \( \mu \)-measure implies that \( \xi(A_k) \overset{d}{=} \xi(B_k) \) for \( k \geq 1 \). Since \( \xi(A_k) \downarrow \xi(A) \) and \( \xi(B_k) \downarrow 0 \) almost surely as \( k \to \infty \), we obtain \( \xi(A) = 0 \) almost surely. The general case \( \mu(A_j) \in \mathbb{R}_+ \) for \( 1 \leq j \leq n \) is now easily obtained by approximation with sets of finite measure.

Finally we show the implication from \([V] \) to \([I] \) under the stated conditions. So let \( n \), \( (A_j)_{1 \leq j \leq n} \), and \( (B_j)_{1 \leq j \leq n} \) be as in \([I] \). As for the preceding implication we first consider the case \( \mu(A_j) \in (0, \infty) \) for \( 1 \leq j \leq n \). Define \( A = \bigcup_{j=1}^n A_j \) and \( B = \bigcup_{j=1}^n B_j \) and choose a set \( F \in S \) disjoint from \( A \) and \( B \) with \( \mu(F) \in (0, \infty) \). Set \( E = A \cup B \cup F \), \( A_0 = E \setminus A \), and \( B_0 = E \setminus B \). It follows that \( \mu(E) \in (0, \infty) \) and
\[
\mu(A_0) = \mu(E) - \mu(A) = \mu(E) - \mu(B) = \mu(B_0) < \infty.
\]
Moreover, since \( F \subset A_0 \), we have \( \mu(A_0) = \mu(B_0) \in (0, \infty) \). Now note that, by Lemma 2.6, for any \( C, D \in S \) with \( \mu(C) = \mu(D) \in (0, \infty) \), there exists a measurable function \( h : C \to D \) such that \( \mu(h^{-1}(M)) = \mu(M) \) for each \( M \in S \cap D \). Applying this result to \( A_j \) and \( B_j \) for \( j \in \{0, 1, \ldots, n\} \), we can choose respective functions \( h_j : A_j \to B_j \). As \( (A_j)_{0 \leq j \leq n} \) together with \( E \) form a partition of \( S \), the following function is well-defined:
\[
f : S \to S, \quad f = \begin{cases} h_j & \text{on } A_j \ (0 \leq j \leq n) \\ id & \text{on } E \end{cases}.
\]
Since \( (B_j)_{0 \leq j \leq n} \) together with \( E \) also form a partition of \( S \), we have, for \( M \in S \),
\[
f^{-1}(M) = (M \cap E) \cup \bigcup_{j=0}^n h_j^{-1}(M \cap B_j).
\]
Hence \( f \) is measurable. Moreover we compute \( \mu(f^{-1}(M)) = \mu(M) \). By the assumption we therefore find that \( \xi \circ f^{-1} \overset{d}{=} \xi \). Consequently
\[
(\xi(A_1), \ldots, \xi(A_n)) = (\xi(f^{-1}(B_1)), \ldots, \xi(f^{-1}(B_n))) \overset{d}{=} (\xi(B_1), \ldots, \xi(B_n)).
\]
So far we have assumed that \( \mu(A_j) \in (0, \infty) \) for \( 1 \leq j \leq n \). The extension to general \( A_j \in B(\mathbb{R}_+) \) is now the same as in the proof of the implication from \([V] \) to \([I] \).
2.3 Swap-invariance

We now investigate random measures that have a weaker property than $\mu$-exchangeability, namely $\mu$-swap-invariance. As in the case of random sequences our definition is based on that of zonoid equivalence.

**Definition 2.7.** Two random measures $\xi$ and $\eta$ on a measurable space $(S, \mathcal{S})$ are called **zonoid equivalent** if and only if $E\left| \sum_{j=1}^{n} u_j \xi(A_j) \right| = E\left| \sum_{j=1}^{n} u_j \eta(A_j) \right|$ for all $n \geq 1$, $u \in \mathbb{R}^n$, and any disjoint $A_1, \ldots, A_n \in \mathcal{S}$ with $E\xi(A_j) < \infty$ and $E\eta(A_j) < \infty$ for $1 \leq j \leq n$.

Note that the integrands in Definition 2.7 are undefined for those points $\omega \in \Omega$ where two terms in the sum are infinite with opposite signs. However this can happen only with probability zero due to the integrability assumptions. Clearly, for $n \geq 1$, two finite zonoid equivalent random measures on $\{1, \ldots, n\}$ are zonoid equivalent random vectors in $\mathbb{R}^n_+$. Zonoid equivalence of two random measures can be characterized as follows.

**Lemma 2.8.** Two random measures $\xi$ and $\eta$ on a measurable space $(S, \mathcal{S})$ are zonoid equivalent if and only if $E\int f \, d\xi = E\int f \, d\eta$ for each measurable function $f : S \to \mathbb{R}$ with $E\int |f| \, d\xi < \infty$ and $E\int |f| \, d\eta < \infty$.

**Proof.** Assume that $\xi$ and $\eta$ are zonoid equivalent. We first prove the implication for simple functions $f = \sum_{j=1}^{n} u_j \mathbb{1}_{A(j)}$ where $A_j \in \mathcal{S}$ and $u_j \in \mathbb{R}$ for $1 \leq j \leq n$ such that $E\int |f| \, d\xi < \infty$ and $E\int |f| \, d\eta < \infty$. We may assume that the sets $(A_j)_{1 \leq j \leq n}$ are disjoint and $u_j \neq 0$ for $1 \leq j \leq n$. It follows that $E\xi(A_j) < \infty$ and $E\eta(A_j) < \infty$, and therefore

$$E\int f \, d\xi = E\sum_{j=1}^{n} u_j \xi(A_j) = E\sum_{j=1}^{n} u_j \eta(A_j) = E\int f \, d\eta.$$  

Now let $f : S \to \mathbb{R}$ be an arbitrary measurable function that satisfies the integrability conditions. Choose a sequence of simple functions $f_n$ such that $f_n \to f$ and $|f_n| \uparrow |f|$ pointwise as $n \to \infty$. By dominated convergence we obtain $E\int f_n \, d\xi \to E\int f \, d\xi$ as $n \to \infty$, and similarly for $\eta$.

Before defining swap-invariance in Definition 2.10 below, we formulate five statements about first absolute moments of random measures, that mirror their counterparts in the exchangeable case in Lemma 2.1. The first three statements are equivalent and serve as definition.

**Lemma 2.9.** Let $\xi$ be a random measure on a measure space $(S, \mathcal{S}, \mu)$. Consider the following statements:

(i) $E\left| \sum_{j=1}^{n} u_j \xi(A_j) \right| = E\left| \sum_{j=1}^{n} u_j \xi(B_j) \right|$ for all $n \geq 1$, $u \in \mathbb{R}^n$, disjoint $A_1, \ldots, A_n \in \mathcal{S}$, and disjoint $B_1, \ldots, B_n \in \mathcal{S}$ such that $\mu(A_j) = \mu(B_j)$, $E\xi(A_j) < \infty$, and $E\xi(B_j) < \infty$ for $1 \leq j \leq n$.

(ii) $E\int f \, d\xi = E\int g \, d\xi$ for each two measurable functions $f, g : S \to \mathbb{R}$ such that $\mu \circ f^{-1} = \mu \circ g^{-1}$, $E\int |f| \, d\xi < \infty$, and $E\int |g| \, d\xi < \infty$.

(iii) The random measures $\xi \circ f^{-1}$ and $\xi \circ g^{-1}$ on $\mathbb{R}$ are zonoid equivalent for each two measurable functions $f, g : S \to \mathbb{R}$ such that $\mu \circ f^{-1} = \mu \circ g^{-1}$, $E\int |f| \, d\xi < \infty$, and $E\int |g| \, d\xi < \infty$.

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(iv) The random measures $\xi \circ f^{-1}$ and $\xi$ are zonoid equivalent for each measurable function $f : S \to S$ with $\mu \circ f^{-1} = \mu$.

(v) There exists a sequence $c_k \downarrow 0$ such that for each $k \geq 1$, $n \geq 1$, any disjoint $A_1, \ldots, A_n \in S$ with $\mu(A_j) = c_k$ and $E \xi(A_j) < \infty$ for $1 \leq j \leq n$, and each $\pi \in \text{Perm}(n)$, we have $E \left| \sum_{j=1}^n u_j \xi(A_j) \right| = E \left| \sum_{j=1}^n u_j \xi(A_{\pi(j)}) \right|$ for all $u \in \mathbb{R}^n$.

Statements (i), (ii), and (iii) are equivalent, and they imply (iv) and (v). If $\mu$ is atomless and $\sigma$-finite, the intensity measure $E \xi$ is $\sigma$-finite, and $\mu(S) = \infty$, then (v) implies (i). If, in addition, $(S, \mathcal{S})$ is Borel, then also (iv) implies (i).

Proof. We only show that (iii) implies (ii), the proofs of the other implications being similar to Lemma 2.1. Assume that $f$ and $g$ satisfy the conditions of (iii). Then $\xi \circ f^{-1}$ and $\xi \circ g^{-1}$ are zonoid equivalent by (iii). Moreover

$$E \left\| \int_{\mathbb{R}} x \, d(\xi \circ f^{-1})(x) \right\| < \infty,$$

and similarly for $g$. Hence we may apply Lemma 2.8 to the identity map on $\mathbb{R}$ and obtain

$$E \left\| \int_{\mathbb{R}} x \, d(\xi \circ f^{-1})(x) \right\| = E \left\| \int_{\mathbb{R}} x \, d(\xi \circ g^{-1})(x) \right\|,$$

and therefore $E \left\| \int f \, d\xi \right\| = E \left\| \int g \, d\xi \right\|$. \hfill $\Box$

**Definition 2.10.** Let $\xi$ be a random measure on a measure space $(S, \mathcal{S}, \mu)$. If $\xi$ satisfies either of the properties (i), (ii), (iii) in Lemma 2.9, then $\xi$ is called $\mu$-swap-invariant. Clearly, every $\mu$-exchangeable random measure is $\mu$-swap-invariant. Moreover the $\sigma$-finite $\mu$-swap-invariant random measures on $\mathbb{N}$, where $\mu$ is the counting measure, are exactly the non-negative swap-invariant sequences. Note that if $\xi$ is a $\mu$-swap-invariant random measure on $S$, then, for each $A \in \mathcal{S}$ with $E \xi(A) < \infty$, $\mu(A) = 0$ implies $\xi(A) = 0$ almost surely. Note also that statement (iii) in Lemma 2.9 is equivalent to the stronger statement where the restriction to functions satisfying $E \int |f| \, d\xi < \infty$ and $E \int |g| \, d\xi < \infty$ is dropped, as can be seen from the proof. The same is not true for statement (ii).

### 2.4 Swap-invariance vs. exchangeability

The close connection between the concepts of swap-invariance and exchangeability can already be seen from the similarity of the characterizations (i) to (v) in Lemmas 2.1 and 2.9. Two more results are now established. Proposition 2.11 provides a construction of $\mu$-swap-invariant random measures from a given $\mu$-swap-invariant random measure. In particular, one can use a $\mu$-exchangeable random measure to construct a $\mu$-swap-invariant random measure that is not $\mu$-exchangeable. Afterwards we show in Theorem 2.14 how a swap-invariant random measure on a space of finite measure can be expressed through an exchangeable random measure. A similar representation of swap-invariant random measures on a space of infinite measure can only be proven in Section 4 because it is based on the ergodic theorem and on the representation result for sequences derived in Section 3.
**Proposition 2.11.** Let \((S, S, \mu)\) be a measure space, and \(\eta\) a random measure on \(S\) that is \(\mu\)-swap-invariant under a probability measure \(Q\). Further let \(X\) be a random variable with \(X > 0\) \(Q\)-almost surely and \(E_Q[X^{-1}] < \infty\), and \(P\) another probability measure defined by

\[
\frac{dP}{dQ} = \frac{1}{X E_Q[X^{-1}]}.
\]

Then the random measure \(\xi\) defined by \(\xi = X \eta\) is \(\mu\)-swap-invariant under \(P\).

**Example 2.12.** Let \((S, S)\) be a measurable space, \(\mu\) an atomless \(\sigma\)-finite measure on \(S\), and \(\eta\) a Poisson process on \(S\) with intensity measure \(\mu\) under a probability measure \(Q\). It follows by Lemma 2.11 that \(\eta\) is \(\mu\)-exchangeable under \(Q\). Assuming \(\mu(S) \geq 2\) we may choose two disjoint measurable sets \(K\) and \(L\) with \(\mu(K) = \mu(L) = 1\). Define the random variable \(X = 1 + \mathbb{1}\{\eta(K) > 0\}\) and a new probability measure \(P\) by

\[
\frac{dP}{dQ} = \frac{1}{c X}, \quad c = E_Q[X^{-1}] = \frac{1}{2} (1 + e^{-1}).
\]

By Proposition 2.11 the point process \(\xi = X \eta\) is \(\mu\)-swap-invariant under \(P\). We calculate

\[
P(\xi(L) = 0) = \frac{1}{e} \neq \frac{2}{1 + e} = P(\xi(K) = 0).
\]

This shows that \(\xi\) is not \(\mu\)-exchangeable under \(P\).

For the next theorem the following lemma is needed, which is a consequence of [5, Theorem 2] and [3, Lemma 1.35].

**Lemma 2.13.** Let \(\|\cdot\|\) be a norm on \(\mathbb{R}^d\) where \(d \geq 1\), and \(\xi_1, \xi_2\) two random vectors in \(\mathbb{R}^d\) that are integrable under a probability measure \(P\), and either both symmetric or both supported by \(\mathbb{R}^d_+\). For \(i \in \{1, 2\}\) assume \(E_P[\|\xi_i\|] > 0\) and define a probability measure \(Q_i\) by

\[
\frac{dQ_i}{dP} = \frac{\|\xi_i\|}{E_P[\|\xi_i\|]}.
\]

If \(\xi_1\) and \(\xi_2\) are zonoid equivalent under \(P\), then \(E_P[\|\xi_1\|] = E_P[\|\xi_2\|]\) and \(Q_1(\|\xi_1\| / \|\xi_2\| \in A) = Q_2(\|\xi_2\| / \|\xi_2\| \in A)\) for all \(A \in \mathcal{B}(\mathbb{R}^d)\).

**Theorem 2.14.** Let \((S, S, \mu)\) be a space of finite measure and \(\xi\) a random measure on \(S\) that is \(\mu\)-swap-invariant under a probability measure \(P\) with \(E_P[\xi(S)] < (0, \infty)\). Then there exists a random measure \(\eta\) that is \(\mu\)-exchangeable under the probability measure \(Q\) defined by

\[
\frac{dQ}{dP} = \frac{\xi(S)}{E_P[\xi(S)]}
\]

such that \(\xi = \xi(S) \eta\).

**Proof.** Let \(n \geq 1\) and \(A_j, B_j \in S\) with \(\mu(A_j) = \mu(B_j)\) for \(1 \leq j \leq n\) such that \((A_j)_{1 \leq j \leq n}\) are disjoint and \((B_j)_{1 \leq j \leq n}\) are disjoint. Define \(A_0 = S \setminus \bigcup_{j=1}^n A_j\) and \(B_0 = S \setminus \bigcup_{j=1}^n B_j\). Clearly, \(\mu(A_0) = \mu(B_0)\). Now we define random vectors \(\xi_A\) and \(\xi_B\) in \(\mathbb{R}^{n+1}_+\) by

\[
\xi_{A_j} = \xi(A_j) \mathbb{1}\{\xi(A_j) < \infty\}, \quad \xi_{B_j} = \xi(B_j) \mathbb{1}\{\xi(B_j) < \infty\}
\]
for $0 \leq j \leq n$. Here the indicator functions are $P$-almost surely equal to 1. Note that $\xi_A$ and $\xi_B$ are zonoid equivalent under $P$. Moreover $\|\xi_A\|_1 = \xi(S) = \|\xi_B\|_1$ $P$-almost surely where $\|x\|_1 = \sum_{j=0}^n |x_j|$ for $x \in \mathbb{R}^{n+1}$. Define
\[
\eta = \begin{cases} 
\frac{\xi}{\xi(S)} & \text{on } \{\xi(S) > 0\} \\
0 & \text{on } \{\xi(S) = 0\}
\end{cases}
\]
Applying Lemma 2.13, it follows that, under $Q$,
\[
(\eta(A_1), \ldots, \eta(A_n)) \stackrel{d}{=} (\eta(B_1), \ldots, \eta(B_n))
\]
Hence $\eta$ is $\mu$-exchangeable under $Q$. \qed

Note that in the special case if $\xi$ is a random probability measure, i.e. $\xi(S) = 1$, which is $\mu$-swap-invariant under $P$, Theorem 2.14 says that $\xi$ is even $\mu$-exchangeable under $P$. Theorem 2.14 is also needed in the proof of Theorem 2.19 that characterizes diffuse swap-invariant random measures.

### 2.5 Intensity measure

If $\mu$ is atomless, a simple consequence of $\mu$-swap-invariance (and therefore also of $\mu$-exchangeability) is that the intensity measure is proportional to $\mu$.

**Theorem 2.15.** Let $(S, \mathcal{S}, \mu)$ be a measure space where $\mu$ is atomless and $\sigma$-finite, and $\xi$ a $\mu$-swap-invariant random measure on $S$ such that $E\xi$ is $\sigma$-finite. Then $E\xi = c\mu$ for some $c \in \mathbb{R}^+$.

**Proof.** First note that $\mu(S) = 0$ implies $E\xi(S) = 0$ by the $\sigma$-finiteness of $E\xi$ and $\mu$-swap-invariance.

Now assume that $\mu(S) \in (0, \infty)$ and $E\xi(S) < \infty$. There is a function $h : [0, \mu(S)] \to \mathbb{R}^+$ such that $E\xi(A) = h(\mu(A))$ for each $A \in \mathcal{S}$. Since $\mu$ is atomless, $h$ is uniquely defined on $[0, \mu(S)]$. We next show that $h$ is additive. Let $x, y \in \mathbb{R}^+$ such that $x + y \in [0, \mu(S)]$. Choose sets $A, B \in \mathcal{S}$ with $\mu(A) = x$, $\mu(B) = y$, and $A \cap B = \emptyset$. Then
\[
h(x + y) = h((\mu(A) + \mu(B)) = h(\mu(A) \cup B) = E\xi(A \cup B) = E\xi(A) + E\xi(B) = h(\mu(A)) + h(\mu(B)) = h(x) + h(y).
\]
Consequently for $x, y \in \mathbb{R}^+$ with $0 \leq x < y \leq \mu(S)$, we have $h(x) \leq h(y)$. Moreover $h(0) = h(\mu(\emptyset)) = E\xi(\emptyset) = 0$. Now we obtain for $p, q \in \mathbb{N}$ with $1 \leq p \leq q$:
\[
h\left(\frac{p}{q} \mu(S)\right) = p h\left(\frac{1}{q} \mu(S)\right) = \frac{p}{q} \mu(S) = \frac{p}{q} h(\mu(S)),
\]
that is, $h(z\mu(S)) = z h(\mu(S))$ for $z \in \mathbb{Q} \cap [0, 1]$. Since $h$ is non-decreasing, this implies that $h$ is continuous at 0. As $h$ is additive, it follows that $h$ is continuous on $[0, \mu(S)]$, and therefore $h(z\mu(S)) = z h(\mu(S))$ for $z \in [0, 1]$. We obtain, for each $A \in \mathcal{S}$,
\[
E\xi(A) = \frac{E\xi(S)}{\mu(S)} \mu(A).
\]
Now consider the case that $\mu(S)$ or $\mathbb{E} \xi(S)$ or both quantities are infinite. Choose a measurable partition $(S_j)_{j \geq 1}$ of $S$ such that $\mu(S_j) < \infty$ and $\mathbb{E} \xi(S_j) < \infty$ for all $j \geq 1$. Define $J = \{j \in \mathbb{N}; \mu(S_j) > 0\}$. Clearly $\xi(S_j) = 0$ almost surely for $j \in \mathbb{N} \setminus J$. For each $j \in J$ define

$$\mu_j(\cdot) = \mu(S_j \cap \cdot), \quad \xi_j(\cdot) = \xi(S_j \cap \cdot),$$

so that $\mu_j$ is a finite measure on $S$ and $\xi_j$ is a random measure on $S$ with finite intensity measure. Now fix $j \in J$. Note that $\xi_j$ is $\mu_j$-swap-invariant and $\mu_j$ is atomless. Hence it follows by the first part of the proof for finite measure space that

$$\mathbb{E} \xi_j = \frac{\mathbb{E} \xi_j(S)}{\mu_j(S)} \mu_j.$$

If $|J| \geq 2$, let $i, j \in J$ with $i \neq j$ and consider the partition

$$\mathcal{M} = \{S_k; k \geq 1, k \neq i, k \neq j\} \cup \{S_i \cup S_j\}$$

of $S$. We define

$$\mu_{ij}(\cdot) = \mu((S_i \cup S_j) \cap \cdot), \quad \xi_{ij}(\cdot) = \xi((S_i \cup S_j) \cap \cdot).$$

Applying the previous result to the new partition $\mathcal{M}$ we obtain, in particular,

$$\mathbb{E} \xi(S_i) = \mathbb{E} \xi_{ij}(S_i) = \frac{\mathbb{E} \xi_{ij}(S)}{\mu_{ij}(S)} \mu_{ij}(S_i) = \frac{\mathbb{E} \xi_{ij}(S)}{\mu_{ij}(S)} \mu(S_i), \quad \mathbb{E} \xi(S_j) = \frac{\mathbb{E} \xi_{ij}(S)}{\mu_{ij}(S)} \mu(S_j).$$

Thus $c = \mathbb{E} \xi(S_k)/\mu(S_k)$ for some $c \in \mathbb{R}_+$ and all $k \in J$, and therefore

$$\mathbb{E} \xi = \sum_{j \in J} \mathbb{E} \xi_j = c \sum_{j \in J} \mu_j = c \mu. \quad \square$$

In the $\mu$-exchangeable case the assumptions in Theorem 2.15 can be slightly weakened as follows.

**Corollary 2.16.** Let $(S, S, \mu)$ be a measure space where $\mu$ is atomless and $\sigma$-finite, and $\xi$ a $\mu$-exchangeable random measure on $S$ such that there exists $A \in S$ with $\mu(A) \in (0, \infty)$ and $\mathbb{E} \xi(A) < \infty$. Then $\mathbb{E} \xi = c \mu$ for some $c \in \mathbb{R}_+$.\[Proof.\] We may choose a measurable finite cover or countable partition $(S_j)_{j \geq 1}$ of $S$ such that $\mu(S_j) = \mu(A)$ for all $j \geq 1$. Then $\mathbb{E} \xi(S_j) = \mathbb{E} \xi(A)$ for $j \geq 1$, which shows the $\sigma$-finiteness of $\mathbb{E} \xi$. Hence the conditions of Theorem 2.15 are satisfied. \[\square\]

**Example 2.17.** In Example 2.12 we have $\mathbb{E}_{\mathbb{Q}} \eta = \mu$, which implies $\mathbb{E}_{\mathbb{P}} \xi = c^{-1} \mu$.

### 2.6 Borel space

If $S$ is Borel and $\mu$ is atomless, Theorem 2.19 below characterizes diffuse $\mu$-swap-invariant random measures. It is based on the following proposition which is a direct consequence of Proposition 1.22 in [4]. It is formulated here for a general Borel space.
\textbf{Proposition 2.18.} Let \((S, S)\) be a Borel space, \(\hat{\mu}\) an atomless probability measure on \(S\), and \(\eta\) a random measure on \(S\) that is \(\sigma\)-finite, almost surely diffuse, and \(\hat{\mu}\)-exchangeable. Then \(\eta = a\hat{\mu}\) almost surely for some random variable \(a \geq 0\).

\begin{proof}
Since \(S\) is Borel, there is a Borel isomorphism \(h : S \rightarrow [0, 1]\). Let \(\nu = \hat{\mu} \circ h^{-1}\). Clearly \(\nu\) is an atomless probability measure on \([0, 1]\). It is known (see e.g. Theorem 2.1 in [7]) that the probability spaces \(([0, 1], \mathcal{B}[0, 1], \nu)\) and \(([0, 1], \mathcal{B}[0, 1], \lambda)\) where \(\lambda\) denotes the Lebesgue measure are isomorphic in the sense that there are \(L, M \in \mathcal{B}[0, 1]\) with \(\nu(L) = 1\) and \(\lambda(M) = 1\) and a Borel isomorphism \(f : L \rightarrow M\) such that \(\nu \circ f^{-1} = \lambda\). Define \(K = h^{-1}(L)\) and the map \(g : K \rightarrow M, g = f \circ (h|K)\). Then \(g\) is a Borel isomorphism and

\[(\hat{\mu} \circ g^{-1})(M \cap \cdot) = \lambda(\cdot), \quad (\lambda \circ g)(K \cap \cdot) = \hat{\mu}(\cdot).\]

Now define the random measure \(\zeta(\cdot) = (\eta \circ g^{-1})(M \cap \cdot)\) on \([0, 1]\). We show that \(\zeta\) is \(\lambda\)-exchangeable. Let \(n \geq 1\) and \(A_j, B_j \in \mathcal{B}[0, 1]\) with \(\lambda(A_j) = \lambda(B_j)\) for \(1 \leq j \leq n\) such that \((A_j)_{1 \leq j \leq n}\) are disjoint and \((B_j)_{1 \leq j \leq n}\) are disjoint. Then, for \(1 \leq j \leq n\),

\[\hat{\mu}(g^{-1}(A_j \cap M)) = \lambda(A_j) = \lambda(B_j) = \hat{\mu}(g^{-1}(B_j \cap M)).\]

The sets \((g^{-1}(A_j \cap M))_{1 \leq j \leq n}\) are disjoint, and the sets \((g^{-1}(B_j \cap M))_{1 \leq j \leq n}\) are disjoint as well. By the \(\hat{\mu}\)-exchangeability of \(\eta\),

\[\big(\eta(g^{-1}(A_1 \cap M)), \ldots, \eta(g^{-1}(A_n \cap M))\big) \overset{d}{=} \big(\eta(g^{-1}(B_1 \cap M)), \ldots, \eta(g^{-1}(B_n \cap M))\big),\]

and therefore

\[\big(\zeta(A_1), \ldots, \zeta(A_n)\big) \overset{d}{=} \big(\zeta(B_1), \ldots, \zeta(B_n)\big).\]

This shows that \(\zeta\) is \(\lambda\)-exchangeable. Clearly \(\zeta\) is \(\sigma\)-finite. Moreover \(\eta(\cdot) = (\zeta \circ g)(K \cap \cdot)\) almost surely. This implies, in particular, that \(\zeta\) is almost surely diffuse. By Proposition 1.22 in [4], \(\zeta = a\lambda\) almost surely for some random variable \(a \geq 0\). Hence,

\[\eta(\cdot) = (\zeta \circ g)(K \cap \cdot) = a(\lambda \circ g)(K \cap \cdot) = a\hat{\mu}(\cdot)\ a.s. \quad \square\]

\textbf{Theorem 2.19.} Let \((S, S)\) be a Borel space, \(\mu\) an atomless and \(\sigma\)-finite measure on \(S\), and \(\xi\) an almost surely diffuse random measure on \(S\) such that \(E\xi\) is \(\sigma\)-finite. Then \(\xi\) is \(\mu\)-swap-invariant if and only if \(\xi = \alpha\mu\) almost surely for some \(\mathbb{R}_+\)-valued integrable random variable \(\alpha\).

\begin{proof}
It is easy to see that \(\xi\) is \(\mu\)-swap-invariant if it has the stated form, so we prove only the converse implication.

First note that, if \(\xi\) is \(\mu\)-swap-invariant and \(\mu(S) = 0\), then \(\xi(S) = 0\) almost surely by Theorem \ref{2.15}

Next assume that \(\mu(S) \in (0, \infty)\) and define the measure \(\hat{\mu} = \mu/\mu(S)\). We denote the basic probability measure by \(P\). We may assume that \(E_P\xi(S) \in (0, \infty)\). Now define \(Q\) and \(\eta\) as in Theorem \ref{2.14}. By this theorem \(\eta\) is \(\hat{\mu}\)-exchangeable under \(Q\). It follows from the definition of \(\eta\) that it is \(Q\)-almost surely diffuse. Hence we can apply Proposition \ref{2.18} and obtain \(\eta = a\hat{\mu}\) \(Q\)-almost surely for some random variable \(a \geq 0\). Since \(\eta(S) = 1\) \(Q\)-almost surely, we may set \(a = 1\). Therefore \(\xi = \alpha\mu\) \(P\)-almost surely where \(\alpha = \xi(S)/\mu(S)\).

\end{proof}
Now assume that $\mu(S) = \infty$. Choose a measurable partition $(S_j)_{j \geq 1}$ of $S$ such that $\mu(S_j) \in (0, \infty)$. By Theorem 2.15 we may assume that $\mathbb{E}\xi(S_j) \in (0, \infty)$ for all $j \geq 1$. Define, for $j \geq 1$,

$$
\mu_j(\cdot) = \mu(S_j \cap \cdot), \quad \xi_j(\cdot) = \xi(S_j \cap \cdot).
$$

Fix $j \geq 1$. Note that $\xi_j$ is $\mu_j$-swap-invariant. Moreover $\mu_j$ is atomless and $\xi_j$ is almost surely diffuse. Hence it follows by the first part of the proof for finite measure space that

$$
\xi_j = \frac{\xi_j(S)}{\mu_j(S)}\mu_j \quad \text{a.s.}
$$

Now fix $i, j \geq 1$ with $i \neq j$ and consider the measurable partition

$$
\mathcal{M} = \{S_k; k \geq 1, k \neq i, k \neq j\} \cup \{S_i \cup S_j\}
$$

of $S$. Define

$$
\mu_{ij}(\cdot) = \mu((S_i \cup S_j) \cap \cdot), \quad \xi_{ij}(\cdot) = \xi((S_i \cup S_j) \cap \cdot).
$$

Applying the previous result to the new partition $\mathcal{M}$ we obtain, in particular,

$$
\xi(S_i) = \xi_{ij}(S_i) = \frac{\xi_{ij}(S)}{\mu_{ij}(S)}\mu_{ij}(S_i) = \frac{\xi_j(S)}{\mu_j(S)}\mu(S_i), \quad \xi(S_j) = \frac{\xi_{ij}(S)}{\mu_{ij}(S)}\mu_{ij}(S_j) \quad \text{a.s.}
$$

Thus $\alpha = \xi(S_k)/\mu(S_k)$ almost surely for some random variable $\alpha \geq 0$ and all $k \geq 1$, and therefore

$$
\xi = \sum_{j \geq 1} \xi_j = \alpha \sum_{j \geq 1} \mu_j = \alpha \mu \quad \text{a.s.}
$$

Since $\mathbb{E}\xi(S_1) < \infty$, $\alpha$ is integrable.

\[ \Box \]

### 3 Swap-invariant sequences

#### 3.1 Sequences with finitely many values

**Theorem 3.1.** Let $\xi$ be a swap-invariant random vector whose coordinates take only values $a$ and $b$ with $|a| \neq |b|$. Then $\xi$ is exchangeable.

The following lemma is needed to cover the case of negative values.

**Lemma 3.2.** Let $d \geq 1$ and $\xi$ and $\xi^*$ be integrable random vectors in $\mathbb{R}^d$. If $\xi$ and $\xi^*$ are zonoid equivalent, then also the vectors $(|\xi_1|, \ldots, |\xi_d|)$ and $(|\xi^*_1|, \ldots, |\xi^*_d|)$ are zonoid equivalent.

**Proof.** Note that for each $u \in \mathbb{R}^d$ the function $f(x) = \sum_{j=1}^d u_j x_j$ on $\mathbb{R}^d$ is measurable, even, and positively homogeneous. By [5, Theorem 2] we obtain $\mathbb{E}f(\xi) = \mathbb{E}f(\xi^*)$ for such $f$, which proves the assertion. \[ \Box \]
Proof of Theorem 3.1.} First assume that $0 \leq a < b$. Fix $n \geq 2$. For $1 \leq d \leq n$ we denote by $N_d$ the set of vectors $k \in \{1, \ldots, n\}^d$ such that $1 \leq k_1 < k_2 < \ldots < k_d$. Furthermore we define $N = \bigcup_{1 \leq d \leq n} N_d$. For $k \in N$ the dimension of the vector $k$ is denoted by $|k|$. For a random vector $\xi$ in $\{a, b\}^n$ define

$$z_\xi(k) = \mathbf{E} \max \{ \xi_{k_1}, \ldots, \xi_{k_{|k|}} \}, \quad k \in N.$$  

We first prove that the distribution of $\xi$ is uniquely determined by $z_\xi$ and then show that $z_\xi$ is invariant under permutations of $\xi$ if $\xi$ is swap-invariant. The marginal distributions of $\xi$ are denoted as follows:

$$p(k; m) = \mathbf{P}(\xi_{k_1} = m_1, \ldots, \xi_{k_d} = m_d)$$

where $1 \leq d \leq n$, $k \in N_d$, and $m \in \{a, b\}^d$. Then, for $k \in N$,

$$p(k; (a, \ldots, a)) = \frac{b - z_\xi(k)}{b - a}. \quad (2)$$

We show that all marginal probabilities are functions of $z_\xi$. This is obvious for $k \in N_1$ since (2) gives

$$p(k_1; a) = \frac{b - z_\xi(k_1)}{b - a}, \quad \text{and} \quad p(k_1; b) = 1 - p(k_1; a).$$

Now let $1 \leq d \leq n - 1$ and suppose that the probabilities $p(k; m)$ are known for all $k \in N_d$ and $m \in \{a, b\}^d$. Fix $k \in N_{d+1}$. For $1 \leq j \leq d + 1$ and $m \in \{a, b\}^{d+1}$, we obtain by summation over the $j$th coordinate that

$$p\left((k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_{d+1}); (m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_{d+1})\right)$$

$$= p\left(k; (m_1, \ldots, m_{j-1}, a, m_{j+1}, \ldots, m_{d+1})\right) + p\left(k; (m_1, \ldots, m_{j-1}, b, m_{j+1}, \ldots, m_{d+1})\right). \quad (3)$$

By (2) and (3) all probabilities for the chosen $k$ can be calculated iteratively. This shows that all marginal probabilities are determined by $z_\xi$. Now let $\pi \in \text{Perm}(n)$. If $\xi$ is swap-invariant, then $z_\xi(k) = \mathbf{E} \max \{ \xi_1, \ldots, \xi_{|k|} \}$ for $k \in N$ by [3, Theorem 2], and therefore $z_\xi = z_{\xi \circ \pi}$. By the above argument $\xi$ and $\xi \circ \pi$ have the same distribution. Therefore $\xi$ is exchangeable.

To show the claim for general $a$ and $b$ note that the random vector $|\xi|$ is swap-invariant by Lemma 3.2 and hence it is exchangeable by the proof above. Since $|a| \neq |b|$, it follows that $\xi$ is exchangeable. \qed

It is not difficult to construct a random sequence in $\{-1, +1\}$ that is swap-invariant but not exchangeable, e.g. using the method of Proposition 3.3. Moreover Example 3.5 provides a swap-invariant but non-exchangeable sequence whose coordinates take three values.

### 3.2 Construction method

The next proposition provides a method to construct swap-invariant sequences from another swap-invariant sequence. It is a direct consequence of the definition of swap-invariance.

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Proposition 3.3. Let $\eta$ be a sequence that is swap-invariant under a probability measure $Q$. Further let $X$ be a random variable with $X \neq 0$ $Q$-almost surely and $c = E_Q[|X|^{-1}] < \infty$. Define another probability measure $P$ by
\[
dP = \frac{1}{c |X|} \, dQ.
\]
Then the sequence $\xi_j = X\eta_j$ ($j \geq 1$) is swap-invariant under $P$.

Note that in Proposition 3.3 we have $P \sim Q$. By Theorem 5, Theorem 17 there exists a random variable $Y$ such that $E_Q[Y] < \infty$ and
\[
n^{-1} \sum_{j=1}^{n} \eta_j \rightarrow Y \quad \text{a.s.} \quad (4)
\]
Then
\[
n^{-1} \sum_{j=1}^{n} \xi_j \rightarrow XY \quad \text{a.s.} \quad (5)
\]
If the convergence (4) is in $L^1(Q)$ (which is the case, for example, if $\eta$ is exchangeable and integrable under $Q$), then the convergence (5) is in $L^1(P)$.

Example 3.4 (Division by first member of i.i.d. sequence). Let $\eta$ be a random sequence that is i.i.d. under $Q$ with $E_Q[|\eta_1|] < \infty$ and $\eta_1 \neq 0$ $Q$-almost surely. Set $X = \eta_1^{-1}$ in Proposition 3.3. Then the sequence $\xi_1 = 1$, $\xi_j = \eta_j/\eta_1$ ($j \geq 2$) is swap-invariant under $P$. $\xi$ is not exchangeable under $P$. For the exchangeability of $\xi$ under $P$ would imply that $\xi_2 \overset{d}{=} 1$ under $P$ and $Q$, whence $\eta_1 = \eta_2$ $Q$-almost surely.

The following example shows that there exists a swap-invariant but not exchangeable sequence that takes only three values, in contrast to Theorem 3.1.

Example 3.5. Consider the special case of Example 3.4 where $\eta_1$ takes values 1 and 2 with probability 1/2. We obtain the following finite-dimensional distributions of $\xi$:
\[
P(\xi_2 = m_2, \ldots, \xi_n = m_n)
= \frac{2^{1-n}}{3} \left( 1 \{ m_2, \ldots, m_n \in \{1,2\} \} + 2 \cdot 1 \{ m_2, \ldots, m_n \in \{1/2,1\} \} \right)
\]
where $n \geq 2$ and $m_2, \ldots, m_n \in \{1/2, 1\}$. An explicit calculation yields
\[
E_P \left| \sum_{j=1}^{n} u_j \xi_j \right| = \frac{2^{1-n}}{3} \sum_{m_1, \ldots, m_n \in \{1,2\}} \left| \sum_{j=1}^{n} u_j m_j \right|.
\]
So $\xi$ is swap-invariant under $P$. 

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3.3 Ergodic representation

As stated in [5, Theorem 17] for each swap-invariant sequence, the mean converges almost surely to an integrable random variable. We now demonstrate that, if the ergodic limit is different from zero and if the convergence is in $L^1$, the limit can be used to characterize swap-invariant sequences as scaled exchangeable sequences under another probability measure.

**Theorem 3.6.** Let $\xi$ be a random sequence that is swap-invariant under a probability measure $P$ such that $n^{-1} \sum_{j=1}^{n} \xi_j \to X$ $P$-almost surely and in $L^1(P)$ as $n \to \infty$ with $P(X \neq 0) = 1$. Then there exists a random sequence $\eta$ that is exchangeable and integrable under the probability measure $Q$ defined by

$$\frac{dQ}{dP} = \frac{|X|}{E_P |X|}$$

such that $\xi = X \eta$ $P$-almost surely.

**Proof.** Define

$$\eta_j = \begin{cases} \frac{\xi_j}{X} & \text{on } \{X \neq 0\} \\ 0 & \text{on } \{X = 0\} \end{cases}, \quad j \geq 1.$$

For $j \geq 1$, $E_P |\xi_j| < \infty$ implies $E_Q |\eta_j| < \infty$. Let $n \geq 1$, $u \in \mathbb{R}^n$, and $\pi \in \text{Perm}(n)$. For $m \geq n$, the swap-invariance of $\xi$ under $P$ yields

$$E_P \left| m^{-1} \sum_{k=1}^{m} \xi_k + \sum_{j=1}^{n} u_j \xi_j \right| = E_P \left| m^{-1} \sum_{k=n+1}^{m} \xi_k + \sum_{j=1}^{n} (u_j + m^{-1}) \xi_j \right|$$

$$= E_P \left| m^{-1} \sum_{k=n+1}^{m} \xi_k + \sum_{j=1}^{n} (u_{\pi(j)} + m^{-1}) \xi_j \right|$$

$$= E_P \left| m^{-1} \sum_{k=1}^{m} \xi_k + \sum_{j=1}^{n} u_{\pi(j)} \xi_j \right| .$$

Letting $m \to \infty$ we obtain

$$E_P \left| X + \sum_{j=1}^{n} u_j \xi_j \right| = E_P \left| X + \sum_{j=1}^{n} u_{\pi(j)} \xi_j \right| .$$

After change of measure this gives $E_Q \left| 1 + \sum_{j=1}^{n} u_j \eta_j \right| = E_Q \left| 1 + \sum_{j=1}^{n} u_{\pi(j)} \eta_j \right|$. Since this holds for all $u \in \mathbb{R}^n$, it follows by [2, Theorem 1.1] that $\eta$ is exchangeable under $Q$. \qed

**Example 3.7.** Let $(Z_j)_{j \geq 1}$ be i.i.d. standard normal random variables under a probability measure $P$, and let $(b_j)_{j \geq 1}$ be a sequence of real numbers such that $\beta = \sum_{j \geq 1} b_j^2 < \infty$. Define a random sequence $(\xi_j)_{j \geq 1}$ by $\xi_j = \exp \zeta_j$ where

$$\zeta_j = Z_j + \sum_{k=1}^{\infty} b_k Z_k + \mu_j , \quad \mu_j = -\frac{1}{2} (1 + \beta + 2b_j) .$$
Note that $\xi_i \overset{d}{=} \xi_j$ if and only if $b_i = b_j$. In [5, Examples 15 and 25] it is shown that $\xi$ is swap-invariant under $P$ and that the limit

$$X = \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} \xi_j = \exp \left( \sum_{k=1}^{\infty} b_k Z_k - \frac{1}{2} \beta \right)$$

exists $P$-almost surely and in $L^1(P)$. By Theorem 3.6 the random sequence $\eta_j = \frac{\xi_j}{X} = \exp \left( Z_j - b_j - \frac{1}{2} \right)$

is exchangeable under $Q$.

In the context of Theorem 3.6 Birkhoff’s ergodic theorem implies a relation between $\xi_1$ and $X$ that we state in the following corollary. As usual the $\sigma$-algebra generated by the open sets in $\mathbb{R}^\infty$ is denoted by $\mathcal{B}(\mathbb{R}^\infty)$, the tail $\sigma$-algebra of $\mathcal{B}(\mathbb{R}^\infty)$ by $\mathcal{T}$, the shift-invariant $\sigma$-algebra by $\mathcal{I}$, and the $\sigma$-algebra of sets that are invariant under all finite permutations by $\mathcal{E}$. For a random sequence $\eta$ we define the corresponding $\sigma$-algebras on the basic probability space by

$$\mathcal{T}_\eta = \eta^{-1} \mathcal{T}, \quad \mathcal{I}_\eta = \eta^{-1} \mathcal{I}, \quad \mathcal{E}_\eta = \eta^{-1} \mathcal{E}.$$ 

It is well known that, if $\eta$ is exchangeable, then $\mathcal{T}_\eta = \mathcal{I}_\eta = \mathcal{E}_\eta$ almost surely, i.e. their completions are equal (see for example [4, Corollary 1.6]). The sign of $x \in \mathbb{R}$ is denoted by $\text{sign}(x)$.

**Corollary 3.8.** Under the conditions of Theorem 3.6

$$\mathbb{E}_P[\text{sign}(X)\xi_j|\mathcal{T}_\eta] = \mathbb{E}_P[|X||\mathcal{T}_\eta] \quad \text{a.s.,} \quad j \geq 1,$$

where $\eta$ is defined as in Theorem 3.6

**Proof.** First note that $P$ and $Q$ are equivalent, so we may say that an equality or convergence holds ‘almost surely’ without specifying the probability measure. Now on the one hand the definition of $\eta$ implies that $n^{-1} \sum_{k=1}^{n} \eta_k \to 1$ almost surely as $n \to \infty$. On the other hand [3, Theorem 10.6] yields

$$n^{-1} \sum_{k=1}^{n} \eta_k \to \mathbb{E}_Q[\eta_j|\mathcal{T}_\eta] = \frac{\mathbb{E}_P[|X|\eta_j|\mathcal{T}_\eta]}{\mathbb{E}_P[|X||\mathcal{T}_\eta]} = \frac{\mathbb{E}_P[\text{sign}(X)\xi_j|\mathcal{T}_\eta]}{\mathbb{E}_P[|X||\mathcal{T}_\eta]} \quad \text{a.s.,} \quad j \geq 1. \quad \Box$$

### 3.4 $p$-norm representation

The connection between swap-invariant sequences and exchangeable sequences established in the preceding section is restricted to cases where the ergodic limit is attained in $L^1$ and is almost surely different from zero. Now we present another method, using $p$-norms, where the second condition can be dropped. However $L^1$-convergence is still required.
For a sequence \( x \in \mathbb{R}^\infty \) the vector of the first \( n \) components of \( x \) is denoted by \( x^{(n)} = (x_1, \ldots, x_n) \). We define, for \( x \in \mathbb{R}^\infty \), \( n \geq 1 \), and \( 1 \leq p < \infty 
 
 \|x\|^{(n)}_p = \left(n^{-1} \sum_{j=1}^n |x_j|^p\right)^{1/p}, \quad \|x\|_p = \limsup_{n \to \infty} \|x\|^{(n)}_p \in \mathbb{R}^+,
 and
 \n \|x\|^{(n)}_\infty = \max \{|x_j|; 1 \leq j \leq n\}, \quad \|x\|_\infty = \limsup_{n \to \infty} \|x\|^{(n)}_\infty \in \mathbb{R}^+.

For the proof of the main result in this section, Theorem 3.11, the lemma below is required. For its proof we need the following convergence result.

**Proposition 3.9.** Let \( X, Y, X_n, Y_n \ (n \geq 1) \) be non-negative random variables such that \( X_n \to X \) as \( n \to \infty \) almost surely and in \( L^1 \), \( Y_n \) and \( Y \) are bounded by \( K \) for some \( K > 0 \), and \( Y_n \to Y \) almost surely on the event \( \{X > 0\} \). Then \( X_nY_n \to XY \) almost surely and in \( L^1 \).

**Proof.** We have
\n\[ \mathbb{E}|X_nY_n - XY| = \mathbb{E}|(X_n - X)(Y_n - Y)| + \mathbb{E}|X(Y_n - Y)| + \mathbb{E}|(X_n - X)| \leq 2K \mathbb{E}|X_n - X| + \mathbb{E}[|X(Y_n - Y)| 1\{X > 0\}] + K \mathbb{E}|X_n - X|. \]

All three terms on the right-hand side converge to zero as \( n \to \infty \).

**Lemma 3.10.** Fix \( p \in [1, \infty] \). Let \( \xi_1, \xi_2 \) be two symmetric sequences of random variables such that \( \xi_1^{(n)} \) and \( \xi_2^{(n)} \) are zonoid equivalent under \( \mathbb{P} \) for each \( n \geq 1 \), and

\[ \|\xi_1\|^{(n)}_p \to \|\xi_1\|_p \quad \text{\( \mathbb{P} \)-a.s. and in \( L^1(\mathbb{P}) \)} \quad \text{as \( n \to \infty \).} \]

Then \( \mathbb{E}_p\|\xi_1\|_p = \mathbb{E}_p\|\xi_2\|_p \), and if \( \mathbb{E}_p\|\xi_1\|_p > 0 \), the probability measures \( \mathbb{Q}_i \) defined by

\[ \frac{d\mathbb{Q}_i}{d\mathbb{P}} = \frac{\|\xi_1\|_p}{\mathbb{E}_p\|\xi_1\|_p}, \quad i = 1, 2, \]

satisfy \( \mathbb{Q}_1(\xi_1/\|\xi_1\|_p \in B) = \mathbb{Q}_2(\xi_2/\|\xi_2\|_p \in B) \) for all \( B \in \mathcal{B}(\mathbb{R}^\infty) \).

**Proof.** To simplify notation we omit the subscript \( p \) at \( \|x\|^{(n)}_p \) and \( \|x\|_p \) for \( x \in \mathbb{R}^\infty \) and \( n \geq 1 \) throughout the proof.

Let \( n \geq 1 \). Since \( \xi_1^{(n)} \) and \( \xi_2^{(n)} \) are zonoid equivalent, Lemma 2.13 implies that \( \mathbb{E}_p\|\xi_1\|^{(n)} = \mathbb{E}_p\|\xi_2\|^{(n)} \). Letting \( n \to \infty \) yields \( \mathbb{E}_p\|\xi_1\| = \mathbb{E}_p\|\xi_2\| \).

Now assume \( \mathbb{E}_p\|\xi_1\| > 0 \). Choose \( N \geq 1 \) such that \( \mathbb{E}_p\|\xi_1\| > 0 \) for \( n \geq N \). For \( n \geq N \) and \( i \in \{1, 2\} \) define probability measures \( \mathbb{Q}_i^{(n)} \) by

\[ \frac{d\mathbb{Q}_i^{(n)}}{d\mathbb{P}} = \frac{\|\xi_i\|^{(n)}}{\mathbb{E}_p\|\xi_i\|^{(n)}}. \]

By Lemma 2.13

\[ \mathbb{Q}_1^{(n)}(\xi_1^{(n)}/\|\xi_1\|^{(n)} \in A) = \mathbb{Q}_2^{(n)}(\xi_2^{(n)}/\|\xi_2\|^{(n)} \in A) \]

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for $A \in B(\mathbb{R}^n)$. It follows that, for $n \geq N$, $1 \leq k \leq n$, and $A \in B(\mathbb{R}^k)$,

$$Q_1^{(n)}\left(\frac{\xi_i^{(k)}}{\|\xi_i\|^{(n)}} \in A\right) = Q_1^{(n)}\left(\frac{\xi_i^{(n)}}{\|\xi_i\|^{(n)}} \in A \times \mathbb{R}^{n-k}\right)$$

$$= Q_2^{(n)}\left(\frac{\xi_2^{(n)}}{\|\xi_2\|^{(n)}} \in A \times \mathbb{R}^{n-k}\right) = Q_2^{(n)}\left(\frac{\xi_2^{(k)}}{\|\xi_2\|^{(n)}} \in A\right).$$

(6)

Now let $f$ be a bounded continuous function from $\mathbb{R}^k$ to $\mathbb{R}_+$. Then, for $i \in \{1, 2\}$,

$$E_i^{(n)}f\left(\frac{\xi_i^{(k)}}{\|\xi_i\|^{(n)}}\right) = \frac{1}{E_P\|\xi_i\|^{(n)}} \int_{\{\|\xi_i\|^{(n)}>0\}} \|\xi_i\|^{(n)} f\left(\frac{\xi_i^{(k)}}{\|\xi_i\|^{(n)}}\right) dP$$

(7)

where $E_i^{(n)}$ denotes the expectation with respect to $Q_i^{(n)}$. Now we apply Proposition 3.9 to the random variables

$$X_n = \|\xi_i\|^{(n)}, \quad X = \|\xi_i\|,$$

$$Y_n = \begin{cases} f\left(\frac{\xi_i^{(k)}}{\|\xi_i\|^{(n)}}\right) & \text{on } \{\|\xi_i\|^{(n)}>0\}, \\ 0 & \text{on } \{\|\xi_i\|^{(n)}=0\} \end{cases},$$

$$Y = \begin{cases} f\left(\frac{\xi_i^{(k)}}{\|\xi_i\|}\right) & \text{on } \{\|\xi_i\|>0\}, \\ 0 & \text{on } \{\|\xi_i\|=0\} \end{cases}.$$

In particular, note that for a point $\omega$ with $\|\xi_i(\omega)\| > 0$ we have $\|\xi_i(\omega)\|^{(n)}>0$ for all sufficiently large $n$. Hence the continuity of $f$ implies that $Y_i \to Y$ as $n \to \infty$ $P$-almost surely on the event $\{\|\xi_i\|>0\}$. We conclude that the right-hand side of (7) converges to

$$\frac{1}{E_P\|\xi_i\|} \int_{\{\|\xi_i\|>0\}} \|\xi_i\| f\left(\frac{\xi_i^{(k)}}{\|\xi_i\|}\right) dP = E_i f\left(\frac{\xi_i^{(k)}}{\|\xi_i\|}\right)$$

where $E_i$ denotes the expectation with respect to $Q_i$. Using equality of the distributions in (6) yields $E_1 f\left(\xi_1^{(k)}/\|\xi_1\|\right) = E_2 f\left(\xi_2^{(k)}/\|\xi_2\|\right)$. By approximation it follows that $Q_1(\xi_1^{(k)}/\|\xi_1\| \in A) = Q_2(\xi_2^{(k)}/\|\xi_2\| \in A)$ for each $A \in B(\mathbb{R}^k)$. Since this holds for all $k \geq 1$, we obtain $Q_1(\xi_1/\|\xi_1\| \in B) = Q_2(\xi_2/\|\xi_2\| \in B)$ for each $B \in B(\mathbb{R}^\infty)$. \qed

**Theorem 3.11.** Fix $p \in [1, \infty]$. Let $\xi$ be a random sequence that is swap-invariant under a probability measure $P$ such that $\|\xi\|^{(n)}_p \to \|\xi\|_p$ as $n \to \infty$ $P$-almost surely and in $L^1(P)$, and $E_P\|\xi\|_p > 0$. Define another probability measure $Q$ by

$$\frac{dQ}{dP} = \frac{\|\xi\|_p}{E_P\|\xi\|_p}.$$
(i) There exists a random sequence $\zeta$ that is exchangeable under $Q$ such that
\[ \varepsilon \xi_j = \|\xi\|_p \zeta_j \quad P\text{-a.s.}, \; j \geq 1 \]
where $\varepsilon$ is a random variable that takes values $\pm 1$ with probabilities $1/2$ and is independent of $\xi$ under $P$.

(ii) If $\xi$ is either symmetric or non-negative, then there exists a random sequence $\eta$ that is exchangeable under $Q$ such that
\[ \xi_j = \|\xi\|_p \eta_j \quad P\text{-a.s.}, \; j \geq 1. \]

Proof. To simplify notation we again omit the subscript $p$ at $\|x\|_p^{(n)}$ and $\|x\|_p$ for $x \in \mathbb{R}^\infty$ and $n \geq 1$.

First assume that $\xi$ is symmetric under $P$. Let $d \geq 1$ and $\pi \in \text{Perm}(d)$, and denote by $\xi \circ \pi$ the random sequence that results from $\xi$ by applying the permutation $\pi$ to the first $d$ members. Then $\|\xi\| = \|\xi \circ \pi\|$. Moreover $\xi^{(n)}$ and $(\xi \circ \pi)^{(n)}$ are zonoid equivalent under $P$ for all $n \geq 1$. Define
\[
\eta_j = \begin{cases} \frac{\xi_j}{\|\xi\|} & \text{on } \{\|\xi\| > 0\}, \\ 0 & \text{on } \{\|\xi\| = 0\} \end{cases}, \quad j \geq 1.
\]

By Lemma 3.10
\[ Q(\eta \in B) = Q\left(\frac{\xi}{\|\xi\|} \in B\right) = Q\left(\frac{\xi \circ \pi}{\|\xi\|} \in B\right) = Q(\eta \circ \pi \in B). \]

for all $B \in \mathcal{B}(\mathbb{R}^\infty)$. This shows that $\eta$ is exchangeable under $Q$. It remains to show that $\xi_j = 0$ $P$-almost surely on the event $\{\|\xi\| = 0\}$ for all $j \geq 1$. Define random sequences $\xi^>$ and $\xi^=$ by
\[ \xi^>_j = \xi_j 1\{\|\xi\| > 0\}, \quad \xi^=_j = \xi_j 1\{\|\xi\| = 0\} \]
for $j \geq 1$. For $n \geq 1$ and $u \in \mathbb{R}^n$ we have
\[
\mathbb{E}_P\left[\sum_{j=1}^n u_j \xi^>_j\right] = \mathbb{E}_P 1\{\|\xi\| > 0\} \left|\sum_{j=1}^n u_j \xi_j\right| = \mathbb{E}_P \|\xi\| \left|\sum_{j=1}^n u_j \eta_j\right| = \mathbb{E}_P \|\xi\|_Q \left|\sum_{j=1}^n u_j \eta_j\right|.
\]

Hence $\xi^>$ is swap-invariant under $P$. This implies that also $\xi^=$ is swap-invariant under $P$. Since $\|\xi^{(n)}\| \to \|\xi\|_P$ $P$-almost surely and in $L^1(P)$,
\[ \|\xi^=\|^{(n)} \to \|\xi\|_1 1\{\|\xi\| = 0\} = 0 \quad P\text{-a.s.} \text{ and in } L^1(P). \]

Since $\xi^=\|^{(n)}_1 \leq \|\xi^=\|^{(n)}_1$ by the Hölder inequality, we conclude that $\|\xi^=\|^{(n)}_1 \to 0$ $P$-almost surely and in $L^1(P)$. The swap-invariance of $\xi^=$ implies that $\mathbb{E}_P \|\xi^=\|^{(n)}_1 = \mathbb{E}_P \|\xi^=\|_1$ for all $n \geq 1$. Therefore $\mathbb{E}_P|\xi^>_j| = \mathbb{E}_P|\xi^=| = 0$, hence $\xi^>_j = 0$ $P$-almost surely.
If \( \xi \) is not symmetric, we may define a random sequence \( \rho \) by \( \rho_j = \varepsilon \xi_j \) for \( j \geq 1 \) where \( \varepsilon \) has the stated properties. Then \( \rho \) is symmetric and swap-invariant under \( \mathbf{P} \). Applying the preceding proof to \( \rho \) proves \( \square \). In particular, if \( \xi \) is non-negative, then also the sequence \((|\xi_1|, |\xi_2|, \ldots)\) is exchangeable under \( \mathbf{Q} \), which shows the second statement of \( \square \).

In the following example a symmetric random sequence \( \xi \) is defined that is swap-invariant but not exchangeable. Since each component as well as the ergodic limit is zero with positive probability, neither [5, Theorem 21] nor Theorem 3.6 can be used to obtain a representation in terms of an exchangeable sequence. However Theorem 3.11 can be applied.

**Example 3.12.** Let \( \rho \) be a random sequence that is i.i.d. under a probability measure \( \mathbf{R} \) such that \( \rho_1 \) takes values \(-1, 0, +1\) with equal probability \(1/3\). Further let \( X = 1 + |\rho_1| \). Define another probability measure \( \mathbf{P} \) by

\[
\frac{d\mathbf{P}}{d\mathbf{R}} = \frac{3}{2} X^{-1},
\]

and a random sequence \( \xi_j = X \rho_j \) \((j \geq 1)\). Then \( \xi \) is swap-invariant under \( \mathbf{P} \) by Proposition 3.3 and has finite-dimensional distributions

\[
\begin{align*}
\mathbf{P}(\xi_1 = m_1, \ldots, \xi_n = m_n) & = 3^{1-n} \cdot \left( \frac{1}{2} \mathbf{1}\{m_1 = 0, \ m_2, \ldots, m_n \in \{-1, 0, 1\}\} \right. \\
& \quad + \left. \frac{1}{4} \mathbf{1}\{m_1 \in \{-2, 2\}, \ m_2, \ldots, m_n \in \{-2, 0, 2\}\}\right)
\end{align*}
\]

where \( n \geq 1 \) and \( m_1, \ldots, m_n \in \{-2, -1, 0, 1, 2\} \). In particular,

\[
\begin{align*}
\mathbf{P}(\xi_1 = 0) & = \frac{1}{2}, \quad \mathbf{P}(\xi_1 = \pm 2) = \frac{1}{4}, \\
\mathbf{P}(\xi_j = 0) & = \frac{1}{3}, \quad \mathbf{P}(\xi_j = \pm 1) = \mathbf{P}(\xi_j = \pm 2) = \frac{1}{6}.
\end{align*}
\]

Thus \( \xi \) is symmetric and not exchangeable under \( \mathbf{P} \). Since \( \mathbf{P}(\xi_j = 0) > 0 \) for all \( j \), we cannot divide the sequence by one of its members in order to obtain an exchangeable sequence as done in [5, Theorem 21]. Further note that

\[
n^{-1} \sum_{j=1}^{n} \rho_j \to \mathbf{E}_\mathbf{R} \rho_1 = 0 \quad \mathbf{R}\text{-a.s.},
\]

which implies \( n^{-1} \sum_{j=1}^{n} \xi_j \to 0 \) \( \mathbf{P} \)-almost surely. Thus Theorem 3.6 is not applicable here either. In order to apply Theorem 3.11 fix \( p = 1 \). Since the sequence \((|\rho_j|)_{j \geq 1}\) is i.i.d. under \( \mathbf{R} \),

\[
n^{-1} \sum_{j=1}^{n} |\rho_j| \to \mathbf{E}_\mathbf{R} |\rho_1| = \frac{2}{3} \quad \mathbf{R}\text{-a.s. and in } L^1(\mathbf{R}).
\]

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It follows that
\[ \|\xi\|_1^{(n)} = Xn^{-1}\sum_{j=1}^{n} |\rho_j| \to \frac{2}{3}X = \|\xi\|_1 \quad \text{P-a.s. and in } L^1(P) \]
and \( E_P\|\xi\|_1 = 1 \). Thus the conditions of Theorem 3.11 are satisfied. We apply the definitions in Theorem 3.11,
\[ \frac{dQ}{dP} = \frac{2}{3}X, \quad \eta_j = \frac{3}{2}\rho_j, \quad j \geq 1, \]
and find that \( Q = R \). Theorem 3.11 says that \( \eta \) is exchangeable under \( Q \), which can be immediately confirmed here.

An interesting special case of Theorem 3.11 is that of non-negative sequences and \( p = 1 \). In this case the limit in Theorem 3.11 is the ordinary ergodic limit and the probability measure \( Q \) is defined as in Theorem 3.6. However the conditions are weaker than in Theorem 3.6 because the ergodic limit can be zero with positive \( P \)-probability here.

Example 3.13. The lognormal sequence in Example 3.7 satisfies the assumptions of Theorem 3.6, and therefore also those of Theorem 3.11 for \( p = 1 \).

A slightly more general case is \( p = 1 \) and no sign restrictions on \( \xi \). Application of Theorem 3.11 requires the mean of absolute values \( n^{-1}\sum_{j=1}^{n} |\xi_j| \) to converge almost surely and in \( L^1 \). However if \( \xi \) is swap-invariant, it follows from Lemma 3.2 that also the sequence of absolute values, \((|\xi_1|, |\xi_2|, \ldots)\), is swap-invariant, so the almost sure convergence is guaranteed by [5, Theorem 17]. Therefore only the \( L^1 \)-convergence remains to be checked. By Scheffé’s lemma, this reduces to the condition \( E|\xi_1| = E\|\xi\|_1 \). Also note that the inequality \( E|\xi_1| \geq E\|\xi\|_1 \) is always guaranteed by Fatou’s lemma. We summarize the situation for the case \( p = 1 \) as follows:

Proposition 3.14. Let \( \xi \) be a swap-invariant random sequence. Then \( n^{-1}\sum_{j=1}^{n} |\xi_j| \) converges almost surely to an integrable random variable \( \|\xi\|_1 \) as \( n \to \infty \). If \( E|\xi_1| = E\|\xi\|_1 \), then this convergence is in \( L^1 \).

From Theorem 3.11 we finally derive a representation of the ergodic limit of symmetric or non-negative swap-invariant sequences. Again the general case is obtained by noting that, for a swap-invariant sequence \( \xi \), the symmetric sequence \( \varepsilon \xi \) is swap-invariant as well. In the derivation of Theorem 3.16 the formula for the conditional expectation under a change of the probability measure is used in the following form where the Radon-Nikodým derivative may be zero with non-zero probability.

Proposition 3.15. Let \((\Omega, \mathcal{F}, P)\) be a probability space, \( \mathcal{G} \) a sub-\( \sigma \)-algebra of \( \mathcal{F} \), \( Z \) a random variable with \( Z \geq 0 \) and \( E_PZ = 1 \), \( Q \) another probability measure defined by \( dQ/dP = Z \), and \( Y \) a random variable with \( E_Q|Y| < \infty \). Then \( E_P|ZY| < \infty \), and
\[ E_P\mathbbm{1}_{\{Z_0 > 0\}}|Y|\mathcal{G} = \hat{Z}_0 E_P|ZY|\mathcal{G} \quad \text{Q-a.s.} \]
where
\[ Z_0 = E_P[Z|\mathcal{G}], \quad \hat{Z}_0 = \begin{cases} Z_0^{-1} & \text{on } \{Z_0 > 0\} \\ 0 & \text{on } \{Z_0 = 0\} \end{cases} \]
Theorem 3.16. Let $\xi$ be a symmetric or non-negative sequence of random variables that satisfies the conditions of Theorem 3.11 for some $p \in [1, \infty]$, and choose $\eta$ as in Theorem 3.11 (ii). Then

$$n^{-1} \sum_{j=1}^{n} \xi_j \to \|\xi\|_p \mathbb{E}_P[\xi_1|\mathcal{T}_\eta] \quad \text{P-a.s. and in } L^1(P) \text{ as } n \to \infty,$$  

(8)

where

$$Y_0 = \mathbb{E}_P[\|\xi\|_p|\mathcal{T}_\eta], \quad \hat{Y}_0 = \begin{cases} \frac{Y_0^{-1}}{} & \text{on } \{Y_0 > 0\} \\ 0 & \text{on } \{Y_0 = 0\} \end{cases}. $$

Moreover,

$$\{\|\xi\|_p > 0\} \in \mathcal{T}_\eta, \quad \mathbb{P}\left(\{Y_0 > 0\} \Delta \{\|\xi\|_p > 0\}\right) = 0. $$

Theorem 3.16 says that under the stated conditions

$$n^{-1} \sum_{j=1}^{n} \xi_j \to \frac{\|\xi\|_p \mathbb{E}_P[\xi_1|\mathcal{T}_\eta]}{\mathbb{E}_P[\|\xi\|_p|\mathcal{T}_\eta]} \quad \text{P-a.s. on } \{\|\xi\|_p > 0\} \text{ as } n \to \infty. $$

In contrast to the representations of the ergodic limit in Theorem 21 and Proposition 22 in [5], we may allow $\mathbb{P}(\xi_j = 0) > 0$ for all $j$ here. For non-negative $\xi$ and $p = 1$, we summarize the results of Theorems 3.11 and 3.16 in the following corollary; note that this includes variants of Theorem 3.6 and Corollary 3.8.

Corollary 3.17. Let $\xi$ be a sequence of non-negative random variables that is swap-invariant under a probability measure $P$ such that

$$n^{-1} \sum_{j=1}^{n} \xi_j \to X \quad \text{P-a.s. and in } L^1(P) \text{ as } n \to \infty$$

with $\mathbb{E}_P X > 0$. Define the probability measure $Q$ by

$$\frac{dQ}{dP} = \frac{X}{\mathbb{E}_P X}. $$

Then there exists a random sequence $\eta$ that is exchangeable under $Q$ such that $\xi_j = X \eta_j$ $P$-almost surely for $j \geq 1$. Moreover $\mathbb{E}_P[X|\mathcal{T}_\eta] = \mathbb{E}_P[\xi_1|\mathcal{T}_\eta]$ $P$-almost surely.

Proof of Theorem 3.16. Since $\eta$ is $Q$-integrable and exchangeable under $Q$, it follows by [3, Theorem 10.6] that

$$n^{-1} \sum_{j=1}^{n} \eta_j \to \mathbb{E}_Q[\eta_1|\mathcal{T}_\eta] \quad \text{Q-a.s. and in } L^1(Q). $$

By Proposition 3.15

$$\mathbb{1}\{Y_0 > 0\} \mathbb{E}_Q[\eta_1|\mathcal{T}_\eta] = \hat{Y}_0 \mathbb{E}_P[\xi_1|\mathcal{T}_\eta] \quad \text{Q-a.s.} \quad (9)$$

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We write $\|\xi\|$ for $\|\xi\|_p$ in the following. Multiplying both sides of (9) by $\|\xi\|$ shows that

$$1 \{Y_0 > 0\} \, n^{-1} \sum_{j=1}^n \xi_j \to \|\xi\| \, \hat{Y}_0 \, E_P[\xi_1|T_\eta] \quad Q\text{-a.s. as } n \to \infty .$$

In order to see that the convergence (10) holds $P$-almost surely, define $E = \{\|\xi\| > 0\}$. On $E^c$ we have $\xi_j = 0$ $P$-almost surely for all $j \geq 1$. On $E$ the measures $P$ and $Q$ are equivalent, so the convergence holds also $P$-almost surely on $E$.

We now show that $E \in T_\eta$. We may assume that $\eta_j = 0$ on $E^c$ for $j \geq 1$. Define $A_n = \{\eta_j = 0; j \geq n\}$ for $n \geq 1$. Note that

$$A_n \subset \{\xi_j = 0; j \geq n\} \subset E^c,$$

and $E^c \subset A_n$ by assumption. Hence $E^c = A_n$ for all $n \geq 1$, and therefore $E^c \in T_\eta$.

We next show the last statement, which then implies that the convergence (8) holds $P$-almost surely. Now $E_P[Y_0 \mathbb{1}_A] = E_P[\|\xi\| \mathbb{1}_A]$ for each $A \in T_\eta$ by definition of conditional expectation. Define $F = \{Y_0 > 0\}$. Note that $E_P[Y_0 \mathbb{1}_A] > 0$ if and only if $P(F \cap A) > 0$, and $E_P[\|\xi\| \mathbb{1}_A] > 0$ if and only if $P(E \cap A) > 0$. We conclude that $P(\mathbb{1}_A) > 0$ if and only if $P(E \cap A) > 0$. It follows that $P(F^c \setminus E^c) = P(F^c \cap E) = 0$ because $F^c \in T_\eta$. Moreover $P(E^c \setminus F^c) = P(F^c \cap F) = 0$ because $E^c \in T_\eta$. Thus $P(F^c \Delta E^c) = 0$.

To see that the convergence (8) holds in $L^1(P)$ note that $1 \{Y_0 > 0\} = 1\{\|\xi\| > 0\}$ $P$-almost surely and therefore also $Q$-almost surely. It follows that

$$E_P\left| n^{-1} \sum_{j=1}^n \xi_j - \|\xi\| \hat{Y}_0 \, E_P[\xi_1|T_\eta] \right| = E_P\|\xi\| \left| n^{-1} \sum_{j=1}^n \eta_j - \hat{Y}_0 \, E_P[\xi_1|T_\eta] \right|$$

$$= \frac{1}{E_P\|\xi\|} \, E_Q\left| n^{-1} \sum_{j=1}^n \eta_j - \hat{Y}_0 \, E_P[\xi_1|T_\eta] \right|$$

$$= \frac{1}{E_P\|\xi\|} \, E_Q\left| n^{-1} \sum_{j=1}^n \eta_j - 1 \{Y_0 > 0\} \, E_Q[\eta_1|T_\eta] \right|$$

$$= \frac{1}{E_P\|\xi\|} \, E_Q\left| n^{-1} \sum_{j=1}^n \eta_j - 1\{\|\xi\| > 0\} \, E_Q[\eta_1|T_\eta] \right|$$

$$= \frac{1}{E_P\|\xi\|} \, E_Q\left| n^{-1} \sum_{j=1}^n \eta_j - E_Q[\eta_1|T_\eta] \right| .$$

The right-hand side converges to zero as $n \to \infty$.

4 Ergodic theorem

To formulate our ergodic theorem for swap-invariant random measures, we need to introduce some notions.

Definition 4.1. Let $(S, S, \mu)$ be a measure space. An increasing sequence $A_n \in S$ with $\mu(A_n) < \infty$ for $n \geq 1$ and $\mu(A_n) \to \infty$ as $n \to \infty$ is called $\mu$-sequence. For a $\mu$-sequence
\((A_n)_{n \geq 1}\) we write

\[
\Delta A_n = \begin{cases} 
A_1 & \text{if } n = 1 \\
A_n \setminus A_{n-1} & \text{if } n \geq 2
\end{cases}
\]

A \(\mu\)-sequence \((A_n)_{n \geq 1}\) is called \(\mu\)-sequence with constant increments if \(\mu(\Delta A_n) = c\) for all \(n \geq 1\) and some \(c \in (0, \infty)\). Moreover, given a random measure \(\xi\) on \(S\), a \(\mu\)-sequence \((A_n)_{n \geq 1}\) is called \(\xi\)-integrable if \(\mathbb{E}\xi(A_n) < \infty\) for \(n \geq 1\).

Clearly, if the conditions of Theorem 2.15 are satisfied, each \(\mu\)-sequence is \(\xi\)-integrable.

For a fixed \(\mu\)-sequence with constant increments it is straightforward to derive an ergodic theorem by applying the result for swap-invariant sequences as follows.

**Proposition 4.2.** Let \((S, \mathcal{S}, \mu)\) be a measure space, \(\xi\) a \(\mu\)-swap-invariant random measure on \(S\), and \((A_n)_{n \geq 1}\) a \(\xi\)-integrable \(\mu\)-sequence with constant increments. Then there exists an integrable random variable \(X\) such that \(\xi(A_n)/\mu(A_n) \to X\) almost surely as \(n \to \infty\).

**Proof.** We have \(\mathbb{E}\left|\sum_{j=1}^{n} u_j \xi(\Delta A_j)\right| = \mathbb{E}\left|\sum_{j=1}^{n} u_j \xi(\Delta A_{\pi(j)})\right|\) for \(n \geq 1\), \(u \in \mathbb{R}^n\), and \(\pi \in \text{Perm}(n)\). Therefore the random sequence \((\xi(\Delta A_j))_{j \geq 1}\) is almost surely equal to a swap-invariant sequence of integrable \(\mathbb{R}_+\)-valued random variables. By [5] Theorem 17, there exists an integrable random variable \(X\) such that

\[
\frac{\xi(A_n)}{\mu(A_n)} = \frac{1}{\mu(A_1)} \frac{1}{n} \sum_{j=1}^{n} \xi(\Delta A_j) \to X \quad \text{a.s. as } n \to \infty. \quad \square
\]

**Example 4.3.** Let \((S, \mathcal{S}, \mu)\) be a measure space where \(\mu\) is \(\sigma\)-finite and \(\mu(S) = \infty\). Further let \(\eta\) be a Poisson process on \(S\) with intensity measure \(\mu\). Since \(\eta\) is \(\mu\)-exchangeable, it is \(\mu\)-swap-invariant. Now let \((A_n)_{n \geq 1}\) be a \(\mu\)-sequence with constant increments and define \(c = \mu(A_1)\). Then \((\eta(\Delta A_j))_{j \geq 1}\) is an i.i.d. sequence of \(\mathbb{R}_+\)-valued integrable random variables. Thus we obtain almost surely and in \(L^1\) as \(n \to \infty\)

\[
\frac{\eta(A_n)}{\mu(A_n)} = \frac{1}{\mu(A_n)} \frac{1}{c} \frac{1}{n} \sum_{j=1}^{n} \eta(\Delta A_j) \to \frac{1}{c} \mathbb{E}\eta(A_1) = \frac{1}{c} \mu(A_1) = 1.
\]

**Example 4.4.** In Example 2.12 let \(\mu(S) = \infty\). Further let \((A_n)_{n \geq 1}\) be a \(\mu\)-sequence with constant increments and define \(c = \mu(A_1)\). Then as in Example 4.3 we have \(\eta(A_n)/\mu(A_n) \to 1\) as \(n \to \infty\) \(\mathbb{Q}\)-almost surely and in \(L^1(\mathbb{Q})\). It follows that \(\xi(A_n)/\mu(A_n) \to X\) \(\mathbb{Q}\)-almost surely. Since \(\mathbb{P}\) and \(\mathbb{Q}\) are equivalent, this convergence holds also \(\mathbb{P}\)-almost surely. It can be shown by direct computation that the convergence is also in \(L^1(\mathbb{P})\).

We now show that the ergodic limit also exists if the increments are not necessarily constant and that the limit is unique under certain assumptions. This allows us to perform a change of the probability measure and to construct a random measure that is \(\mu\)-exchangeable under the new probability measure in order to obtain (1).

**Theorem 4.5.** Let \((S, \mathcal{S}, \mu)\) be an atomless measure space and \(\xi\) a random measure on \(S\) that is \(\mu\)-swap-invariant under a probability measure \(\mathbb{P}\).

(i) For each \(\xi\)-integrable \(\mu\)-sequence \((A_n)_{n \geq 1}\) there exists an integrable random variable \(X\) such that \(\xi(A_n)/\mu(A_n) \to X\) almost surely as \(n \to \infty\).
Example 4.6. Let \((\omega, \mathcal{E}, \mu)\) and \(\xi\) be a Borel space and \(\mu\) an atomless and \(\sigma\)-finite measure on \(\mathcal{E}\). Further let \(\xi\) be an almost surely diffuse random measure on \(\mathcal{E}\), surely and in \(\mathcal{E}\) sense. We now prove two lemmas on which Theorem 4.5 is based.

\begin{align*}
\text{Lemma 4.9.} \quad \text{Let} \quad (\omega, \mathcal{E}, \mu) \quad \text{be a Borel space and} \quad (\xi, \mathcal{E}) \quad \text{be a random measure on} \quad \mathcal{E}. \quad \text{Further assume that there exists a} \quad \mu\text{-sequence} \quad (\xi_n)_{n \geq 1} \quad \text{with constant increments and limit} \quad A \quad \text{such that} \quad \mu(S \setminus A) = \infty, \quad \xi_n(A_n) / \mu(A_n) \to X \quad \text{in} \quad L^1(\mathcal{E}), \quad \text{and} \quad \mathbb{E}_X X > 0. \quad \text{Then} \quad \xi_n(B_n) / \mu(B_n) \to X \quad \text{in} \quad L^1(\mathcal{E}) \quad \text{for each} \quad \mu\text{-sequence} \quad (B_n)_{n \geq 1}.
\end{align*}

Example 4.7. In Example 2.12 let \(\mu(S) = \infty\). Since \(\xi\) satisfies the conditions of Theorem 4.5 (ii), (see Examples 2.17 and 4.4), it follows that \(\xi_n(B_n) / \mu(B_n) \to X \quad \mathbb{P}\text{-almost surely} \quad \text{and in} \quad L^1(\mathcal{E}) \quad \text{for each} \quad \mu\text{-sequence} \quad (B_n)_{n \geq 1}.

In the proof of Theorem 4.5 we make use of the following notion.

Definition 4.8. Let \((\omega, \mathcal{E}, \mu)\) be a measure space and \((A_n)_{n \geq 1}\) a \(\mu\text{-sequence}. \quad \text{A} \quad \mu\text{-sequence} \quad (C_n)_{n \geq 1} \quad \text{with constant increments is called} \quad \text{compatible sequence with constant increments (CSCI) of} \quad (A_n) \quad \text{if there exists a} \quad \mu\text{-sequence} \quad (B_n)_{n \geq 1} \quad \text{such that} \quad (A_n) \quad \text{and} \quad (C_n) \quad \text{are subsequences of} \quad (B_n).

Obviously in this definition the sequences \((A_n), (B_n), \text{and} \quad (C_n) \quad \text{have the same limit set}. \quad \text{We now prove two lemmas on which Theorem 4.5 is based.}

Lemma 4.9. Let \((\omega, \mathcal{E}, \mu)\) be a measure space where \(\mu\) is atomless and \(\mu(S) = \infty\), \((A_n)_{n \geq 1}\) a \(\mu\text{-sequence}, \quad \xi\) a random measure on \(\mathcal{E}\), and \(X\) a random variable.

\begin{align*}
\text{(i) For each} \quad c \in (0, \infty), \quad \text{there exists a CSCI} \quad (C_n) \quad \text{of} \quad (A_n) \quad \text{with} \quad \mu(C_1) = c.
\end{align*}

\begin{align*}
\text{(ii) If} \quad \xi_n(C_n) / \mu(C_n) \to X \quad \text{almost surely as} \quad n \to \infty \quad \text{for some CSCI} \quad (C_n) \quad \text{of} \quad (A_n), \quad \text{then also} \quad \xi_n(A_n) / \mu(A_n) \to X \quad \text{almost surely.}
\end{align*}

\begin{align*}
\text{(iii) If} \quad \xi_n(C_n) / \mu(C_n) \to X \quad \text{in} \quad L^1 \quad \text{as} \quad n \to \infty \quad \text{for some CSCI} \quad (C_n) \quad \text{of} \quad (A_n), \quad \text{then also} \quad \xi_n(A_n) / \mu(A_n) \to X \quad \text{in} \quad L^1.
\end{align*}

Proof. \(\Box\) is clear because \(\mu\) is atomless. In order to prove (ii) and (iii) let \((C_n)\) be a CSCI of \((A_n)\). Define \(c = \mu(C_1)\) and \(m_k = \min \{m \geq 1; A_k \subset C_m\}\) for \(k \geq 1\). It follows that \(1 \leq m_1 \leq m_2 \leq \ldots\), and \(m_k \to \infty\) as \(k \to \infty\). For large \(k\) we have \(m_k \geq 2\) and \(C_{m(k)-1} \subset A_k \subset C_{m(k)}\) where the first inclusion is strict and the second may not. Hence, for large \(k\),

\begin{align*}
\mu(C_{m(k)-1}) \leq \mu(A_k) \leq \mu(C_{m(k)}), \quad \xi(C_{m(k)-1}) \leq \xi(A_k) \leq \xi(C_{m(k)}),
\end{align*}

\(26\)
and therefore
\[
\left( \frac{m_k - 1}{m_k} \right) \frac{\xi(C_{m(k) - 1})}{\mu(C_{m(k) - 1})} = \frac{\xi(C_{m(k) - 1})}{\mu(C_{m(k) - 1})} \leq \frac{\xi(A_k)}{\mu(A_k)} \leq \frac{\xi(C_{m(k)})}{\mu(C_{m(k) - 1})} = \left( \frac{m_k}{m_k - 1} \right) \frac{\xi(C_{m(k)})}{\mu(C_{m(k)})}.
\]

Thus if \( \xi(C_n)/\mu(C_n) \to X \) almost surely, then \( \xi(A_n)/\mu(A_n) \to X \) almost surely as \( n \to \infty \). This proves (iii). From the same estimate we obtain, for large \( k \):
\[
\mathbb{E} \left| \frac{\xi(A_k)}{\mu(A_k)} - X \right| \leq \mathbb{E} \left| \frac{\xi(A_k)}{\mu(A_k)} - \left( \frac{m_k}{m_k - 1} \right) \frac{\xi(C_{m(k)})}{\mu(C_{m(k)})} \right| + \mathbb{E} \left| \left( \frac{m_k}{m_k - 1} \right) \frac{\xi(C_{m(k)})}{\mu(C_{m(k)})} - X \right|
\]
\[
\leq \mathbb{E} \left| \left( \frac{m_k}{m_k - 1} \right) \frac{\xi(C_{m(k)})}{\mu(C_{m(k)})} - \left( \frac{m_k - 1}{m_k} \right) \frac{\xi(C_{m(k) - 1})}{\mu(C_{m(k) - 1})} \right| + \mathbb{E} \left| \left( \frac{m_k}{m_k - 1} \right) \frac{\xi(C_{m(k)})}{\mu(C_{m(k) - 1})} - X \right|
\]
\[
\leq 2\mathbb{E} \left| \left( \frac{m_k}{m_k - 1} \right) \frac{\xi(C_{m(k)})}{\mu(C_{m(k)})} - X \right| + \mathbb{E} \left| X - \left( \frac{m_k - 1}{m_k} \right) \frac{\xi(C_{m(k) - 1})}{\mu(C_{m(k) - 1})} \right|
\]

The right-hand side converges to zero as \( k \to \infty \) if \( \xi(C_n)/\mu(C_n) \to X \) in \( L^1 \) as \( n \to \infty \). This proves (iii). □

**Lemma 4.10.** Let \((S, \mathcal{S}, \mu)\) be a measure space with \( \mu(S) = \infty \), and \( \xi \) a random measure on \( S \) that is \( \mu \)-swap-invariant under a probability measure \( P \). Further let \((A_n)_{n \geq 1}\) be a \( \xi \)-integrable \( \mu \)-sequence with constant increments and limit \( A \) such that \( \xi(A_n)/\mu(A_n) \to X \) \( P \)-almost surely and in \( L^1(P) \) as \( n \to \infty \) for some random variable \( X \) with \( \mathbb{E}_P X > 0 \). Define the random measure \( \eta \) by
\[
\eta = \begin{cases} \xi/X & \text{on } \{ X > 0 \} \\ 0 & \text{on } \{ X = 0 \} \end{cases}
\]

and \( Q \) by (11).

(i) The sequence \((\eta(\Delta A_n))_{n \geq 1}\) is exchangeable under \( Q \).

(ii) \( \xi(A_n) = 0 \) \( P \)-almost surely on \( \{ X = 0 \} \) for \( n \geq 1 \).

(iii) For each \( m \geq 1 \) and disjoint measurable sets \((B_j)_{1 \leq j \leq m}\) with \( \mu(B_j) = \mu(A_1) \), \( B_j \cap A = \emptyset \), and \( \mathbb{E}_P \xi(B_j) < \infty \), we have, under \( Q \),
\[
(\eta(B_1), \ldots, \eta(B_m)) \overset{d}{=} (\eta(\Delta A_1), \ldots, \eta(\Delta A_m)).
\]

**Proof.** The sequence \((\xi(\Delta A_n))_{n \geq 1}\) is \( P \)-almost surely equal to a swap-invariant sequence of \( \mathbb{R}_+ \)-valued random variables, that satisfies the conditions of Corollary 3.17; this proves (i) and (ii).

Now choose a sequence \( \zeta \) of random variables in \( \mathbb{R}_+ \) that are swap-invariant under \( P \) such that \( P \)-almost surely \( \zeta_j = \xi(B_j) \) for \( 1 \leq j \leq m \) and \( \zeta_{j+m} = \xi(\Delta A_j) \) for \( j \geq 1 \). We have \( n^{-1} \sum_{j=1}^{n} \zeta_j \to \mu(A_1)X \) \( P \)-almost surely and in \( L^1(P) \). Another application of Corollary 3.17 shows (iii). □
Proof of Theorem 4.5. We first show (i). By Lemma 4.9 (i) there is a CSCI \((c_n)\) of \((A_n)\) with \(\mu(C_1) = 1\). In particular \((c_n)\) is a \(\xi\)-integrable \(\mu\)-sequence, so by Proposition 4.2 there is an integrable random variable \(X\) such that \(\xi(c_n)/\mu(C_n) \to X\) \(\mathbb{P}\)-almost surely. By Lemma 4.9 (ii), we know that also \(\xi(A_n)/\mu(A_n) \to X\) \(\mathbb{P}\)-almost surely.

In order to show (ii), assume that \((A_n)\), \(A\), and \(X\) have the stated properties, and let \(c = \mu(A_1)\). Now let \((B_n)\) be another \(\mu\)-sequence, say with limit \(B\). First we assume that \((B_n)\) has constant increments with \(\mu(B_1) = c\), and that \(A \cap B = \emptyset\). Clearly \(\mathbb{E}_P \xi(B_n) < \infty\) for all \(n\). By swap-invariance

\[
\mathbb{E}_P \left| n^{-1} \sum_{j=1}^n \xi(\Delta B_j) - m^{-1} \sum_{k=1}^m \xi(\Delta A_k) \right| = \mathbb{E}_P \left| n^{-1} \sum_{j=1}^n \xi(\Delta A_j) - m^{-1} \left( \sum_{k=1}^n \xi(\Delta A_k) + \sum_{k=n+1}^m \xi(\Delta A_k) \right) \right|
\]

for \(m > n \geq 1\). Letting \(m \to \infty\), it follows that

\[
\mathbb{E}_P \left| n^{-1} \sum_{j=1}^n \xi(\Delta B_j) - c X \right| = \mathbb{E}_P \left| n^{-1} \sum_{j=1}^n \xi(\Delta A_j) - c X \right|.
\]

Letting \(n \to \infty\) shows that \(\xi(B_n)/\mu(B_n) \to X\) in \(L^1(\mathbb{P})\), and (i) implies that this convergence is also \(\mathbb{P}\)-almost surely. Now let \((B_n)\) be an arbitrary \(\mu\)-sequence with limit \(B\), i.e. we may have \(A \cap B \neq \emptyset\). We may choose a \(\mu\)-sequence with constant increments \((E_n)\) with limit \(E\) such that \(\mu(E_1) = c\) and \(A \cap E = \emptyset\). By the first part of the proof it follows that \(\xi(E_n)/\mu(E_n) \to X\) \(\mathbb{P}\)-almost surely and in \(L^1(\mathbb{P})\). Now we distinguish the cases \(\mu(A \cap B) < \infty\) and \(\mu(A \cap B) = \infty\). In the first case we choose another \(\mu\)-sequence with constant increments \((F_n)\) with limit \(F\) such that \(\mu(F_1) = c\) and \(F \subset A \setminus B\). Let \((c_n)\) be a CSCI of \((B_n)\) with \(\mu(C_1) = c\), which exists by Lemma 4.9 (ii). From the convergence of \(\xi(E_n)/\mu(E_n)\) we consecutively conclude that the same convergence holds for \((F_n)\) and \((c_n)\). Finally \(\xi(B_n)/\mu(B_n) \to X\) \(\mathbb{P}\)-almost surely and in \(L^1(\mathbb{P})\) by Lemma 4.9 (ii) and (iii). In the second case, \(\mu(A \cap B) = \infty\), we may choose a CSCI \((c_n)\) of \((B_n)\) with \(\mu(C_1) = 2c\), and two \(\mu\)-sequences with constant increments \((C^{1}_{n})\) \((i = 1, 2)\) such that, for \(n \geq 1\),

\[
C_n = C_1^{1} \cup C_2^{2}, \quad C_1^{1} \cap C_2^{2} = \emptyset, \quad \mu(\Delta C_1^{1}) = \mu(\Delta C_2^{2}) = c.
\]

For \(i \in \{1, 2\}\) let \(C_i^{1}\) be the limit set of \((C^{i}_{n})\). Without loss of generality we may assume that \(\mu(A \cap C^2) = \infty\). Then there is a \(\mu\)-sequence with constant increments \((F_n)\) and limit \(F\) such that \(\mu(F_1) = c\) and \(F \subset A \cap C^2\). By the first part of the proof we consecutively conclude that \(\xi(F_n)/\mu(F_n)\), \(\xi(C^{1}_{n})/\mu(C_1^{1})\), and \(\xi(C^{2}_{n})/\mu(C_2^{2})\) converge to \(X\) \(\mathbb{P}\)-almost surely and in \(L^1(\mathbb{P})\). It follows that

\[
\frac{\xi(C_n)}{\mu(C_n)} = \frac{1}{2} \left( \frac{\xi(C_1^{1})}{c n} + \frac{\xi(C_2^{2})}{c n} \right) = \frac{1}{2} \left( \frac{\xi(C_1^{1})}{\mu(C_1^{1})} + \frac{\xi(C_2^{2})}{\mu(C_2^{2})} \right) \to X \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})
\]

Finally we find that \(\xi(B_n)/\mu(B_n) \to X\) \(\mathbb{P}\)-almost surely and in \(L^1(\mathbb{P})\) by Lemma 4.9 (ii) and (iii). This shows statement (iii).
In order to prove (iii) first note that for any $D \in \mathcal{S}$ with $\mu(D) < \infty$ we may choose a $\mu$-sequence with constant increments $(E_n)$ such that $E_1 = D$. Applying Lemma 4.10 to $(E_n)$ gives $\xi(D) = 0$ $P$-almost surely on $\{X = 0\}$. Now define $\eta$ as in (12). It remains to show that $\eta$ is $\mu$-exchangeable under $Q$. Let $d \in (0, \infty)$, $m \geq 1$, and $(B_j)_{1 \leq j \leq m}$ disjoint measurable sets with $\mu(B_j) = d$. Define $B = \bigcup_{j=1}^m B_j$. Since $\mu(S \setminus B) = \infty$, we may choose a $\mu$-sequence $(C_n)$ with constant increments and limit in $S \setminus B$ such that $\mu(C_1) = d$. Statement (ii) implies that $\xi(C_n)/\mu(C_n) \to X$ $P$-almost surely and in $L^1(P)$. Applying Lemma 4.10 to the sequence $(C_n)$ and sets $(B_j)_{1 \leq j \leq m}$, we obtain that, under $Q$,

$$
(\eta(B_1), \ldots, \eta(B_m)) \overset{d}{=} (\eta(\Delta C_1), \ldots, \eta(\Delta C_m)).
$$

Now let $\pi \in \text{Perm}(m)$. Deriving the same relation for the permuted sets, we get, under $Q$,

$$
(\eta(B_1), \ldots, \eta(B_m)) \overset{d}{=} (\eta(B_{\pi(1)}), \ldots, \eta(B_{\pi(m)})).
$$

By Lemma 2.1, this shows that $\eta$ is $\mu$-exchangeable under $Q$.

Note that if $X$ in Theorem 4.5 (iii) is almost surely constant, then $\xi$ is $\mu$-exchangeable under $P$, similarly to [5, Corollary 24] for random sequences. For easier comparison with existing results we give a variant of Theorem 4.5 for $\mu$-exchangeable random measures.

**Theorem 4.11.** Let $(S, \mathcal{S}, \mu)$ be an atomless measure space and $\xi$ a $\mu$-exchangeable random measure on $S$.

(i) For each $\xi$-integrable $\mu$-sequence $(A_n)_{n \geq 1}$ there exists a random variable $X$ such that $\xi(A_n)/\mu(A_n) \to X$ almost surely and in $L^1$ as $n \to \infty$.

(ii) Assume that $\mu$ is $\sigma$-finite and that there is $C \in \mathcal{S}$ with $\mu(C) \in (0, \infty)$ and $E \xi(C) < \infty$. Then the limit in (i) is unique for all $\mu$-sequences.

**Proof.** Statement (i) is a consequence of Lemma 4.9 and Birkhoff’s ergodic theorem, see e.g. [3, Theorem 10.6]. In order to show (ii) we may assume that $\mu(S) = \infty$. Note that Corollary 2.16 applies, so that $E \xi$ is $\sigma$-finite and each $\mu$-sequence is $\xi$-integrable. Clearly there exists a $\mu$-sequence $(A_n)_{n \geq 1}$ with unit increments and limit $A$ such that $\mu(S \setminus A) = \infty$. By (i) there exists a random variable $X$ such that $\xi(A_n)/\mu(A_n) \to X$ almost surely and in $L^1$. Now let $(B_n)_{n \geq 1}$ be another $\mu$-sequence, say with limit $B$. We may assume that $(B_n)$ has unit increments and that $A \cap B = \emptyset$. The general case is then proven as in Theorem 4.5. Define random sequences $\xi_j = \xi(\Delta A_j)$ and $\xi_j^* = \xi(\Delta B_j)$ for $j \geq 1$, and $g(x) = \limsup_{n \to \infty} n^{-1} \sum_{j=1}^n x_j$ for $x \in \mathbb{R}^\infty_+$. By the $\mu$-exchangeability of $\xi$,

$$
(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots) \overset{d}{=} (\xi_1^*, \ldots, \xi_m^*, \xi_{m+1}, \xi_{m+2}, \ldots),
$$

for $m \geq 1$, and consequently

$$
(\xi_1, \ldots, \xi_m, g(\xi)) \overset{d}{=} (\xi_1^*, \ldots, \xi_m^*, g(\xi_1^*, \ldots, \xi_m^*, \xi_{m+1}, \xi_{m+2}, \ldots)) \overset{a.s.}{=} (\xi_1^*, \ldots, \xi_m^*, g(\xi))
$$

It follows that $(\xi, g(\xi)) \overset{d}{=} (\xi^*, g(\xi))$. Now [3, Corollary 6.11] implies that $g(\xi) = g(\xi^*)$ almost surely.

\qed
Example 4.12. Let $\xi$ be a $\lambda$-exchangeable random measure on $\mathbb{R}^d$ with $E \xi([0,1]^d) < \infty$ where $\lambda$ denotes the Lebesgue measure. It follows that $\xi$ is stationary (cf. [3, p. 189]). Hence the ergodic theorem [3, Corollary 10.19], which is based on [6], implies that $\xi(A_n)/\lambda(A_n) \to X$ almost surely and in $L^1$ as $n \to \infty$ for a certain subclass of $\lambda$-sequences, namely all sequences of increasing bounded convex Borel sets $(A_n)_{n \geq 1}$ such that the inner radius $r(A_n) \to \infty$. Theorem 4.11 generalizes existence and uniqueness to all $\lambda$-sequences.

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