We obtain new recurrence relations, an explicit formula, and convolution identities for higher-order geometric polynomials. These relations generalize known results for geometric polynomials and lead to congruences for higher-order geometric polynomials and, in particular, for $p$-Bernoulli numbers.

1. Introduction

For a complex variable $y$, the geometric polynomials $w_n(y)$ of degree $n$ are defined by [31]

$$w_n(y) = \sum_{k=0}^{n} \binom{n}{k} k! y^k,$$

where $\binom{n}{k}$ is the Stirling number of the second kind [15]. These polynomials have been studied from analytic, combinatoric, and number-theoretic points of view. Analytically, they are used in evaluating the geometric series of the form [4]

$$\sum_{k=0}^{\infty} k^n y^k$$

with

$$\left(y \frac{d}{dy}\right)^n \frac{1}{1-y} = \sum_{k=0}^{\infty} k^n y^k = \frac{1}{1-y} w_n \left(\frac{y}{1-y}\right)$$

for every $|y| < 1$ and every $n \in \mathbb{Z}$, $n \geq 0$. Combinatorially, they are related to the total number of preferential arrangements of $n$ objects

$$w_n(1) := w_n = \sum_{k=0}^{n} \binom{n}{k} k!,$$

i.e., the number of partitions of an $n$-element set into $k$ nonempty distinguishable subsets (cf. [10]). The number-theoretic studies of the geometric polynomials are mostly originated from their exponential generating function

$$\sum_{n=0}^{\infty} w_n(y) \frac{t^n}{n!} = \frac{1}{1-y(e^t-1)}.$$

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For example, setting $y = -\frac{1}{2}$ gives

$$w_n \left( -\frac{1}{2} \right) = \frac{2}{n+1} \left( 1 - 2^{n+1} \right) B_{n+1} = -\frac{T_n}{2^n},$$

where $B_n$ are Bernoulli numbers and $T_n$ are tangent numbers. Bernoulli numbers also occur in integrals involving geometric polynomials, namely, we have \[24\]

$$\int_0^1 w_n(-y)dy = B_n, \quad n > 0.$$ Moreover, we note that \[21\]

$$\int_0^1 (1 - y)^p w_n(-y)dy = \frac{1}{p+1} B_{n,p},$$

where $B_{n,p}$ are $p$-Bernoulli numbers \[30\] (see Section 2 for definitions). The analysis of congruence identities of geometric numbers is also one of the subjects studied. Gross \[16\] showed that

$$w_{n+4} = w_n \pmod{10},$$

which was later generalized by Kauffman \[19\]. Mezö \[27\] also gave an elementary proof for the Gross identity. Moreover, Diagana and Maïga \[11\] used $p$-adic Laplace transform and $p$-adic integration to give some congruences for geometric numbers. We refer the reader to the papers \[5–7, 12, 20, 29\] and the references therein for more information about geometric numbers and polynomials.

In the literature, there are numerous studies aimed at the generalization of geometric polynomials (see, e.g., \[13, 14, 22, 23\]). As one of the natural extension of geometric polynomials, we can mention the higher-order geometric polynomials \[4\]

$$w_n^{(r)}(y) = \sum_{k=0}^n \binom{n}{k} (r)_k y^k, \quad r > 0,$$ (1.2)

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = x(x+1) \cdots (x+n-1) \quad \text{with} \quad (x)_0 = 1.$$ It is evident that

$$w_n^{(1)}(y) = w_n(y).$$ The polynomials $w_n^{(r)}(y)$ have the property \[4\]

$$\left( \frac{d}{dy} \right)^n \frac{1}{(1-y)^{r+1}} = \sum_{k=0}^\infty \binom{k+r}{k} k^n y^k = \frac{1}{(1-y)^{r+1}} w_n^{(r+1)} \left( \frac{y}{1-y} \right).$$ (1.3)
for any \( n, r = 0, 1, 2, \ldots \), and can be defined by means of the exponential generating function [4]

\[
\sum_{n=0}^{\infty} w_n^{(r)}(y) \frac{t^n}{n!} = \left( \frac{1}{1 - y(e^t - 1)} \right)^r.
\]

On the other hand, the higher-order geometric polynomials and exponential (or single-variable Bell) polynomials

\[
\varphi_n(y) = \sum_{k=0}^{n} \binom{n}{k} y^k
\]

are connected by

\[
w_n^{(r)}(y) = \frac{1}{\Gamma(r)} \int_{0}^{\infty} \varphi_n(y) e^{-\lambda} d\lambda
\]

(cf. [4, 8]). According to this integral representation, several generating functions and recurrence relations for higher-order geometric polynomials were obtained in [8]. Namely, \( w_{n+m}^{(r)}(y) \) admits a recurrence relation according to the family \( \{ y^j w^{(r+j)}_n(y) \} \) as follows:

\[
w_n^{(r)}(y) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} \varphi_j y^{n-k} w_k^{(r+j)}(y).
\]

Setting \( y = 1 \) in (1.2), we get higher-order geometric numbers \( w_n^{(r)} \). The higher-order geometric numbers and geometric numbers are connected by \( w_1^{(1)} = w_n \) and the formula

\[
w_n^{(r)} = \frac{1}{r! 2^r} \sum_{k=0}^{r+1} \binom{r+1}{k+1} w_{n+k},
\]

which was proved by a combinatorial method in [1] (Theorem 2). Here, \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) is the Stirling number of the first kind [15]. Moreover, some congruence identities for the higher-order geometric numbers can be also found in the recent work [11].

In the present paper, dealing with two-variable geometric polynomials defined in [25] by

\[
\sum_{n=0}^{\infty} w_n^{(r)}(x; y) \frac{t^n}{n!} = \left( \frac{1}{1 - y(e^t - 1)} \right)^r e^{xt},
\]

we obtain new recurrence relations, an explicit formula, and a result generalizing (cf. [8])

\[
\sum_{k=0}^{n} \binom{n}{k} w_k^{(r)}(y) w_{n-k}(y) = \frac{w_{n+1}^{(r)}(y) + r w_n^{(r)}(y)}{r(1 + y)}
\]
for higher-order geometric polynomials. In particular, we use the explicit formula to obtain an integral representation similar to (1.4) involving the \( r \)-Bell polynomials defined in [26] as follows:

\[
\varphi_{n,r}(y) = \sum_{k=0}^{n} \binom{n + r}{k + r} y^k,
\]

where \( \binom{n + r}{k + r} \) are \( r \)-Stirling numbers of the second kind [3]. The resulting integral representation enables us to utilize some properties of \( r \)-Bell polynomials for higher-order geometric polynomials. In particular, we evaluate the infinite sum

\[
\sum_{k=0}^{\infty} (k + r)^n \binom{k + r - 1}{k}
\]

in terms of higher-order geometric polynomials, obtain an ordinary generating function for higher-order geometric polynomials, introduce a new recurrence for \( w_{n+m}^{(r)}(y) \), and generalize (1.6). We also give an integral representation relating the higher-order geometric polynomials and \( p \)-Bernoulli numbers, and express the properties of \( p \)-Bernoulli numbers originating from the properties of the higher-order geometric polynomials. In addition, by using some of theses results, we prove congruences for higher-order geometric polynomials and \( p \)-Bernoulli numbers. In particular, we state a von-Staudt–Clausen-type congruence for \( p \)-Bernoulli numbers.

The paper is organized as follows: In Section 2, we summarize known results that we need throughout the paper. We state and prove the aforementioned results for higher-order geometric polynomials and \( p \)-Bernoulli numbers in Section 3. In Section 4, we deal with some congruences for higher order geometric polynomials and \( p \)-Bernoulli numbers.

2. Preliminaries

The Stirling numbers of the first kind \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) can be defined as follows:

\[
x(x + 1) \ldots (x + n - 1) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k
\]
or by the generating function

\[
(- \log(1 - x))^k = k! \sum_{n=k}^{\infty} \left[ \begin{array}{c} n \\ k \end{array} \right] x^n \frac{x^k}{k!}
\]

(cf. [9, 15]). It follows from either of these definitions that

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = (n - 1) \left[ \begin{array}{c} n - 1 \\ k \end{array} \right] + \left[ \begin{array}{c} n - 1 \\ k - 1 \end{array} \right]
\]

with

\[
\left[ \begin{array}{c} n \\ 0 \end{array} \right] = 0, \quad \text{if} \quad n > 0,
\]
and
\[
\binom{n}{k} = 0, \quad \text{if} \quad k > n \quad \text{or} \quad k < 0.
\]

We mention the following special values, which will be used in the sequel:
\[
\binom{n}{1} = 1, \quad \binom{n}{1} = (n - 1)! \quad \text{if} \quad n > 0,
\]
\[
\binom{n}{n-1} = \binom{n}{2}, \quad \binom{n}{n-2} = \frac{3n-1}{4} \binom{n}{3}, \quad \binom{n}{n-3} = \binom{n}{2} \binom{n}{4}.
\]

Many properties of \(\binom{n}{k}\) can be found in [9, p. 214–219]. In particular, we have
\[
k \binom{n}{k} = \sum_{i=k-1}^{n-1} \binom{n}{i} (n-i-1)! \binom{i}{k-1}.
\]

This equality can be used to obtain some congruences for \(\binom{n}{k}\). Thus, if we take \(n = q\), where \(q\) is a prime number, then
\[
\binom{q}{k} \equiv 0 \pmod{q}, \quad k = 2, 3, \ldots, q - 1,
\]
(2.2)
since
\[
\binom{q}{i} \equiv 0 \pmod{q}, \quad i = 1, 2, \ldots, q - 1.
\]

The Stirling numbers of the second kind \(\{\binom{n}{k}\}\) can be defined by means of
\[
x^n = \sum_{k=0}^{n} \binom{n}{k} x(x-1)\ldots(x-k+1),
\]
or by the generating function
\[
(e^x - 1)^k = k! \sum_{n=k}^{\infty} \binom{n}{k} \frac{x^n}{n!}
\]
(cf. [9, 15]). It follows from the generating function that
\[
\binom{n}{k} = \binom{n-1}{k-1} + k \binom{n-1}{k}.
\]
with

\[
\begin{align*}
\binom{n}{0} &= 0, \quad \text{if } n > 0, \\
\binom{n}{k} &= 0, \quad \text{if } k > n \text{ or } k < 0, \\
\binom{n}{n} &= 1, \\
\binom{n}{1} &= 1, \quad \text{if } n > 0.
\end{align*}
\]

We mention the following known identity for \(\binom{n}{k}\) for future reference:

\[
\binom{n}{k} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n.
\]

By finding the product of two generating functions for \(\binom{n}{k}\), we obtain the convolution formula [18]

\[
\binom{k_1 + k_2}{k_1} \binom{n}{k_1 + k_2} = \sum_{m=0}^{n} \binom{n}{m} \binom{m}{k_1} \binom{n-m}{k_2}.
\]

Let \(k = k_1 + k_2\) and let \(n = q\) be a prime number. This yields

\[
\binom{q}{k} \equiv 0 \pmod{q}, \quad k = 2, 3, \ldots, q - 1,
\]

since again

\[
\binom{q}{i} \equiv 0 \pmod{q}, \quad i = 1, 2, \ldots, q - 1, \quad \text{and} \quad 1 < k < q.
\]

The Stirling numbers have been generalized in numerous ways. One of them is called \(r\)-Stirling numbers (or weighted Stirling numbers). The \(r\)-Stirling numbers of the second kind \(\binom{n}{k}_r\) can be defined by means of the generating function (see [3])

\[
(e^x - 1)^k e^{rx} = k! \sum_{n=k}^{\infty} \binom{n}{k}_r \frac{x^n}{n!}.
\]

The Bernoulli numbers \(B_n\) are defined by the generating function

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
\]
or by the equivalent recursion

\[ B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{B_k}{k!(n-k)!} = 0 \quad \text{for} \quad n \geq 2. \]

The first values are

\[ B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \]

and \( B_{2k+1} = 0 \) for \( k \geq 1 \). The denominators of the Bernoulli numbers can be completely determined according to the von-Staudt–Clausen theorem: For any integer \( n \geq 1 \), \( B_{2n} \) can be represented as

\[ B_{2n} = A_{2n} - \sum_{q \equiv (q-1)/2n} \frac{1}{q}, \]

where \( A_{2n} \) is an integer and the sum runs over all prime numbers such that \( (q-1)/2n \). This can be equivalently formulated as follows:

\[ qB_{2n} \equiv \begin{cases} 0 \pmod{q} & \text{if} \quad (q-1) \nmid 2n, \\ -1 \pmod{q} & \text{if} \quad (q-1) \mid 2n. \end{cases} \tag{2.6} \]

We note that this classification is also valid for \( B_1 \).

Numerous generalizations of Bernoulli numbers appear in the literature. As one of these generalizations, we can mention, the \( p \)-Bernoulli numbers \( B_{n,p} \), which are due to Rahmani [30]. These numbers are defined by means of the generating function

\[ \sum_{n=0}^{\infty} B_{n,p} \frac{x^n}{n!} = 2F1(1, 1; p + 2, 1 - e^t), \]

where \( 2F1(a, b; c; z) \) is the Gaussian hypergeometric function

\[ 2F1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}. \]

The \( p \)-Bernoulli numbers are related to Bernoulli numbers as follows: \( B_{n,0} = B_n \) and

\[ \sum_{k=0}^{p} \binom{p}{k} (-1)^kB_{n+k} = \frac{p!}{p+1}B_{n,p} \quad \text{for} \quad n, p \geq 0, \tag{2.7} \]

and satisfy an explicit formula of the form

\[ B_{n,p} = \frac{p+1}{p!} \sum_{k=0}^{n} \binom{n+p}{k+p} \frac{(-1)^k(k+p)!}{k+p+1}. \tag{2.8} \]
3. Recurrence Relations

From the generating function for higher-order two-variable geometric polynomials (1.7), we have

\[ w_n^{(r)}(x; y) = \sum_{k=0}^{n} \binom{n}{k} w_k^{(r)}(y) x^{n-k}. \] (3.1)

Thus, it is obvious that

\[ w_n^{(r)}(0; y) = w_n^{(r)}(y), \quad w_n^{(1)}(x; y) = w_n(x; y), \]

\[ w_n^{(r)}(0; 1) = w_n^{(r)} \quad \text{and} \quad w_n^{(1)}(0; 1) = w_n. \]

Setting \( x + r \) instead of \( x \) in (1.7), we get

\[ w_n^{(r)}(x + r; y) = (-1)^n w_n^{(r)}(-x; -y - 1) \quad \text{for} \quad n \geq 0. \]

Hence, for \( x = 0 \), we conclude that

\[ w_n^{(r)}(r; y) = (-1)^n w_n^{(r)}(-y - 1). \] (3.2)

This is a relationship between two- and single-variable higher-order geometric polynomials.

**Proposition 3.1.** For \( n \geq 0 \) and \( r > 0 \), the following recurrence relations are true:

\[ \sum_{k=0}^{n} \binom{n}{k} w_k^{(r)}(y) r^{n-k} = (-1)^n w_n^{(r)}(-y - 1) \] (3.3)

and

\[ \sum_{k=0}^{n} \binom{n}{k} w_k^{(r+1)}(y) = \frac{1}{ry} w_{n+1}^{(r)}(y). \] (3.4)

**Proof.** Combining (3.1) and (3.2), we get (3.3). Furthermore, taking \( x = 1 \) in (3.1) and using the recurrence relation presented in [25] (Theorem 3.4)

\[ u_n^{(r)}(x; y) = w_n^{(r)}(x; y) + ry w_n^{(r+1)}(x + 1; y) \]

for \( x = 1 \), we obtain (3.4).

**Theorem 3.1.** For every \( n \geq 0 \) and every \( r_1, r_2 > 0 \), the convolution identity

\[ \sum_{k=0}^{n} \binom{n}{k} w_k^{(r_1)}(y) w_k^{(r_2)}(y) = \frac{w_n^{(r_1+r_2-1)}(y) + (r_1 + r_2 - 1) w_n^{(r_1+r_2-1)}(y)}{(r_1 + r_2 - 1)(1 + y)} \] (3.5)

is true.
**Proof.** We first note that

$$
y^r \frac{e^{(x+1)t}}{(1 - y(e^t - 1))^{r+1}} = \frac{d}{dt} \left( \left( \frac{1}{1 - y(e^t - 1)} \right)^r e^{xt} \right) - \frac{x e^{xt}}{(1 - y(e^t - 1))^r}. $$

Let $x = x_1 + x_2 - 1$ and $r = r_1 + r_2 - 1$. Thus, by using (1.7), the product of two infinite series, and the operation of formal differentiation under summation, we obtain

$$
y^r \frac{e^{(x+1)t}}{(1 - y(e^t - 1))^{r+1}} = y(r_1 + r_2 - 1) \frac{e^{(x_1 + x_2)t}}{(1 - y(e^t - 1))^{r_1+r_2}}
$$

$$
= y(r_1 + r_2 - 1) \sum_{n=0}^{\infty} w_n^{(r_1)}(x_1; y) \frac{t^n}{n!} \sum_{n=0}^{\infty} w_n^{(r_2)}(x_2; y) \frac{t^n}{n!}
$$

$$
= y(r_1 + r_2 - 1) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} w_k^{(r_1)}(x_1; y) w_{n-k}^{(r_2)}(x_2; y) \frac{t^n}{n!},
$$

and

$$
\frac{x e^{xt}}{(1 - y(e^t - 1))^r} = (x_1 + x_2 - 1) \sum_{n=0}^{\infty} w_n^{(r_1+r_2-1)}(x_1 + x_2 - 1; y) \frac{t^n}{n!},
$$

and

$$
\frac{d}{dt} \left( \left( \frac{1}{1 - y(e^t - 1)} \right)^r e^{xt} \right) = \sum_{n=0}^{\infty} w_n^{(r_1+r_2-1)}(x_1 + x_2 - 1; y) \frac{t^n}{n!}.
$$

Equating coefficients of $\frac{t^n}{n!}$ on both sides, we derive

$$
\sum_{k=0}^{n} \binom{n}{k} w_k^{(r_1)}(x_1; y) w_{n-k}^{(r_2)}(x_2; y)
$$

$$
= \frac{1}{y(r_1 + r_2 - 1)} \left[ w_n^{(r_1+r_2-1)}(x_1 + x_2 - 1; y) - (x_1 + x_2 - 1) w_n^{(r_1+r_2-1)}(x_1 + x_2 - 1; y) \right].
$$

Setting $x_1 = r_1$, $x_2 = r_2$ and using (3.2), we arrive at the convolution formula (3.5).

In the next theorem, we give a new explicit expression for the higher-order geometric polynomials and numbers.

**Theorem 3.2.** For $n \geq 0$,

$$
w_n^{(r)}(y) = \sum_{k=0}^{n} \binom{n+1}{k+r} (r)_k (-1)^{n+k}(y+1)^k. \quad (3.6)
$$

In particular,

$$
w_n^{(r)} = \sum_{k=0}^{n} \binom{n+1}{k+r} (r)_k (-1)^{n+k}2^k.
$$
Proof. Writing $x = r$ in (1.7), employing the generalized binomial formula, and using the generating function of $r$-Stirling numbers (2.5), we find

$$
\sum_{n=0}^{\infty} w_n^{(r)}(r; y) \frac{t^n}{n!} = \left( \frac{1}{1-y(e^t-1)} \right)^r e^{rt}
$$

$$
= \sum_{k=0}^{\infty} (r)_k y^k \frac{(e^t-1)^k}{k!} e^{rt}
$$

$$
= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n+r}{k+r} (r)_k y^k \frac{t^n}{n!}
$$

$$
= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \binom{n+r}{k+r} (r)_k y^k \right] \frac{t^n}{n!}
$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain

$$
w_n^{(r)}(r; y) = \sum_{k=0}^{n} \binom{n+r}{k+r} (r)_k y^k.
$$

Using (3.2) and replacing $y$ with $-(y + 1)$, we arrive at the desired equation.

Further, with the help of Theorem 3.2, we connect higher-order geometric polynomials and $r$-Bell polynomials in the following lemma, which proves to be useful in getting our subsequent results.

Lemma 3.1. For every $n \geq 0$ and every $r > 0$, the following integral representation is true:

$$
(-1)^n w_n^{(r)}(-y - 1) = \frac{1}{\Gamma(r)} \int_0^{\infty} \lambda^{r-1} \varphi_{n,r}(y\lambda)e^{-\lambda} d\lambda.
$$

Proof. By (1.8) we have

$$
\int_0^{\infty} \lambda^{r-1} \varphi_{n,r}(y\lambda)e^{-\lambda} d\lambda = \sum_{k=0}^{n} \binom{n+r}{k+r} y^k \int_0^{\infty} \lambda^{r+k-1} e^{-\lambda} d\lambda
$$

$$
= \sum_{k=0}^{n} \binom{n+r}{k+r} \Gamma(r + k) y^k.
$$

By using (3.6) in this relation, we get the desired equation.
Higher-order geometric polynomials are encountered in the evaluation of the infinite series (1.3). If we apply Lemma 3.1 to the Dobinski formula for \( r \)-Bell polynomials

\[
\varphi_{n,r}(y) = \frac{1}{e^y} \sum_{n=0}^{\infty} \frac{(k + r)^n}{k!} x^k,
\]

then we can evaluate a new infinite series in terms of higher-order geometric polynomials.

**Theorem 3.3.** For every \( n \geq 0 \) and every \( r > 0 \), \( |y| < 1 \),

\[
\sum_{k=0}^{\infty} (k + r)^n \binom{k + r - 1}{k} y^k = \frac{(-1)^n}{(1-y)^r} w^{(r)}_n \left( \frac{1}{1-y} \right).
\]

Further, we introduce an ordinary generating function for higher order geometric polynomials.

**Theorem 3.4.** For real \( y < -\frac{1}{2} \), the higher-order geometric polynomials have the generating function

\[
\sum_{n=0}^{\infty} w_{n}^{(r)}(y)t^n = \frac{(-1)^r}{(1+rt)y^r} {}_2F_1 \left( \frac{rt+1}{t}, \frac{rt+t+1}{t}; \frac{y+1}{y} \right).
\]

**Proof.** We start by observing the ordinary generating function for \( r \)-Bell polynomials [26] (Theorem 3.2):

\[
\sum_{n=0}^{\infty} \varphi_{n,r}(y)t^n = \frac{-1}{rt-1} e^{y} {}_1F_1 \left( \frac{rt-1}{t}; \frac{rt+t-1}{t}; y \right).
\]

In view of equation (3.7), this equation can be rewritten as

\[
\sum_{n=0}^{\infty} (-1)^n w^{(r)}_n (-y-1)t^n = \frac{1}{(1 - rt) \Gamma(r)} \int_0^\infty \lambda^{r-1} e^{-(y+1) \lambda} {}_1F_1 \left( \frac{rt-1}{t}; \frac{rt+t-1}{t}; y \lambda \right) d\lambda
\]

\[
= \frac{1}{(1 - rt) \Gamma(r)} \sum_{k=0}^{\infty} \frac{(rt-1)^k}{t^k} \frac{y^k}{k!} \int_0^\infty \lambda^{r+k-1} e^{-(y+1) \lambda} d\lambda
\]

\[
= \frac{1}{(1 - rt)(1+y)^r} \sum_{k=0}^{\infty} \frac{(rt-1)^k}{t^k} \frac{(r^k)_k}{k!} \left( \frac{y}{1+y} \right)^k
\]

\[
= \frac{1}{(1 - rt)(1+y)^r} {}_2F_1 \left( \frac{rt-1}{t}, \frac{rt+t-1}{t}; \frac{y}{1+y} \right).
\]

Then we replace \( -(y+1) \) with \( y \) and \( -t \) with \( t \) in order to obtain the desired equation.
We now present an alternative representation for $w_n^{(r)}(y)$, which also generalizes (1.6), in the following theorem:

**Theorem 3.5.** For all nonnegative integers $n$, $m$, $r$, and $p$,

\[
 w_n^{(r)}(y) = \sum_{k=0}^{m} \binom{m + r}{k + r} (r)_k (-1)^{m+k} (y + 1)^k w_n^{(r+k)}(y) \tag{3.8}
\]

and

\[
 w_n^{(r+p)}(y) = \frac{1}{(r)p(1 + y)^p} \sum_{k=0}^{p} \binom{p + r}{k + r} w_n^{(r)}(y). \tag{3.9}
\]

**Proof.** We first prove (3.8). By using the following property of $r$-Bell polynomials [presented in [28], Eq. (8)]:

\[
 \varphi_{n+m,r}(y) = \sum_{k=0}^{m} \binom{m + r}{k + r} y^k \varphi_{n,r+k}(y)
\]

in relation (3.7), we obtain

\[
 (-1)^{n+m} w_n^{(r)}(-y - 1) = \sum_{k=0}^{m} \binom{m + r}{k + r} y^k \frac{\Gamma(k + r)}{\Gamma(r)} \frac{1}{\Gamma(k + r)} \int_0^\infty \lambda^{r+k-1} \varphi_{n,r+k}(y\lambda)e^{-\lambda}d\lambda
\]

\[
 = \sum_{k=0}^{m} \binom{m + r}{k + r} (r)_k y^k (-1)^n w_n^{(r)}(-y - 1),
\]

which is equal to (3.8).

To prove (3.9), we use the formula

\[
 y^p \varphi_{n,r+p}(y) = \sum_{k=0}^{p} \binom{p + r}{k + r} (-1)^{p-k} \varphi_{n+k,r}(y)
\]

{see [28], Eq. (11) in (3.7).}

By using (3.3) in (3.8), we obtain the following similar result, which is slightly different from (1.5).

**Corollary 3.1.** The following relation is true:

\[
 w_n^{(r)}(y) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m + r}{j + r} \binom{n}{k} (j + r)^{n-k} (-1)^{n+m+j} (r)_j (y + 1)^j w_k^{(r+j)}(-y - 1)
\]

Note that it is also possible to obtain this result by applying (1.4) and (3.7) in

\[
 \varphi_{n+m,r}(y) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m + r}{j + r} (j + r)^{n-k} y^j \varphi_k(y);
\]
this is a formula given in [28], Eq. (9). Moreover, for \( r = 1 \), (3.9) can be written as

\[
(1 + y)^p w_n^{(p+1)}(y) = \frac{1}{p!} \sum_{k=0}^{p} \binom{p+1}{k+1} w_n^{k}(y),
\]

which is also a polynomial extension of (1.6). Replacing \( y \) by \( -y \) and integrating both sides with respect to \( y \) from 0 to 1, we get

\[
\int_0^1 (1 - y)^p w_n^{(p+1)}(-y) \, dy = \frac{1}{p!} \sum_{k=0}^{p} \binom{p+1}{k+1} B_n^{k}.
\]

Thus, using (2.7), we obtain the following integral representation for \( p \)-Bernoulli numbers:

**Theorem 3.6.** For \( n \geq 1 \) and \( p \geq 0 \),

\[
\int_0^1 (1 - y)^p w_n^{(p+1)}(-y) \, dy = (-1)^{n-1} \frac{p+1}{p+2} B_{n-1,p+1}.
\]

The explicit formula (2.8) for \( p \)-Bernoulli numbers can be also deduced by applying this integral representation to (3.6).

The following theorem generalizes identities (2.8) and (2.7):

**Theorem 3.7.** For \( n, p, m \geq 0 \),

\[
B_{n+m,p} = (p+1) \sum_{k=0}^{m} \binom{m+p}{k+p} \frac{(-1)^k (p+1)k}{k+p+1} B_{n,p+k}.
\]

For \( n, r \geq 1 \) and \( p \geq 0 \),

\[
B_{n,p+r} = \frac{r(p+r+1)}{(r+1)(p+r)(r)p} \sum_{k=0}^{p} \binom{p+r}{k+r} (-1)^k B_{n+k,r}.
\]

**Proof.** First, we replace \( y \) with \( -y \) in (3.9), multiply both sides by \( (1 - y)^{r-1} \), and integrate with respect to \( y \) from 0 to 1. The result is as follows:

\[
\int_0^1 (1 - y)^{p+r-1} w_n^{(r+p)}(-y) \, dy = \frac{1}{(p)r} \sum_{k=0}^{p} \binom{p+r}{k+r} \int_0^1 (1 - y)^{r-1} w_n^{(r)}(-y) \, dy.
\]

From (3.11), this equation turns into

\[
B_{n-1,p+r} = \frac{r(p+r+1)}{(r+1)(p+r)(r)p} \sum_{k=0}^{p} \binom{p+r}{k+r} (-1)^k B_{n+k-1,r}.
\]

Replacing \( n \) with \( n+1 \) in this equation, we complete the proof of (3.13).

Applying the same method to identity (3.8), we get (3.12).
4. Congruences

In this section, we first consider the congruences modulo a prime number \( q \) for higher-order geometric polynomials. We start with two auxiliary results.

**Lemma 4.1.** Let \( q \) be an odd prime and let \( y \) be an integer. Then

\[
w_q(y) \equiv y \pmod{q}.
\]

**Proof.** From (1.1), we obtain

\[
w_q(y) = \binom{q}{0} + \binom{q}{1} y + \binom{q}{q} q! y^q + \sum_{k=2}^{q-1} \binom{q}{k} k! y^k.
\]

Since

\[
\binom{q}{0} = 0, \quad \binom{q}{1} = 1
\]

and, in view of (2.4),

\[
k! \binom{q}{k} \equiv 0 \pmod{q}, \quad k = 2, 3, \ldots, q - 1,
\]

we get the desired result

\[
w_q(y) \equiv y + q! y^q \equiv y \pmod{q}.
\]

**Lemma 4.2.** Let \( q \) be a prime and let \( y \) be an integer. Then, for all \( n \geq 1 \),

\[
w_{q+n-1}(y) \equiv w_n(y) \pmod{q}.
\]

**Proof.** If \( q = 2 \), then, by (1.1), we obtain

\[
w_{n+1}(y) - w_n(y) = \sum_{k=0}^{n+1} \binom{n+1}{k} k! y^k - \sum_{k=0}^{n} \binom{n}{k} k! y^k
\]

\[
= (n + 1)! y^{n+1} + \sum_{k=2}^{n} \left( \binom{n+1}{k} - \binom{n}{k} \right) k! y^k \equiv 0 \pmod{2},
\]

since \( \binom{n}{k} = 0 \) and \( \binom{n}{1} = 1 \) for \( n > 0 \).

Further, suppose that \( q \) is an odd prime and \( n \geq q - 1 \). Then, again by (1.1), we can write

\[
w_{q+n-1}(y) - w_n(y) = \sum_{k=0}^{q+n-1} \binom{q+n-1}{k} k! y^k - \sum_{k=0}^{n} \binom{n}{k} k! y^k
\]
\[
\sum_{k=0}^{q-1} \binom{q+n-1}{k} - \binom{n}{k} k!y^k
\]

\[
+ \sum_{k=q}^{q+n-1} \binom{q+n-1}{k} k!y^k - \sum_{k=q}^{q-1} \binom{n}{k} k!y^k
\]

\[
\equiv \sum_{k=0}^{q-1} \binom{q+n-1}{k} - \binom{n}{k} k!y^k \pmod{q}.
\]

By using (2.3), we obtain

\[
w_{q+n-1}(y) - w_n(y) \equiv \sum_{k=0}^{q-1} k!y^k \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n (j^{q-1} - 1) \equiv 0 \pmod{q},
\]

since

\[
(j, q) = 1 \quad \text{and} \quad j^{q-1} - 1 \equiv 0 \pmod{q}.
\]

If \(1 \leq n < q - 1\), then we can write

\[
w_{q+n-1}(y) - w_n(y) = \sum_{k=0}^{q+n-1} \binom{q+n-1}{k} k!y^k - \sum_{k=0}^{n} \binom{n}{k} k!y^k
\]

\[
= \sum_{k=0}^{q-1} \binom{q+n-1}{k} - \binom{n}{k} k!y^k
\]

\[
+ \sum_{k=q}^{q+n-1} \binom{q+n-1}{k} k!y^k - \sum_{k=q}^{q-1} \binom{n}{k} k!y^k + \sum_{k=n+1}^{q-1} \binom{n}{k} k!y^k
\]

\[
\equiv \sum_{k=0}^{q-1} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n (j^{q-1} - 1) \equiv 0 \pmod{q},
\]

since \(\binom{n}{k} = 0\) for \(k > n\).

Hence, for \(n \geq 1\), we get

\[
w_{q+n-1}(y) \equiv w_n(y) \pmod{q}.
\]

Note that a more general result can be found in [2] for Fubini numbers.

**Theorem 4.1.** Let \(q\) be an odd prime. If \(1 + y\) is not a multiple of \(q\), then \(w_q^{(q)}(y) \equiv 0 \pmod{q}\).
Proof. We set $p = q - 1$ and $n = q$ in (3.10) to obtain

$$(1 + y)^{q-1}(q - 1)!w^q(y) = \sum_{k=1}^{q} \begin{bmatrix} q \\ k \end{bmatrix} w_{q+k-1}(y)$$

$$= \begin{bmatrix} q \\ 1 \end{bmatrix} w_q(y) + \begin{bmatrix} q \\ q \end{bmatrix} w_1(y) + \sum_{k=2}^{q-1} \begin{bmatrix} q \\ k \end{bmatrix} w_{q+k-1}(y)$$

$$= (q - 1)!w_q(y) + y + \sum_{k=2}^{q-1} \begin{bmatrix} q \\ k \end{bmatrix} w_{q+k-1}(y).$$

By Lemmas 4.1 and 4.2, we find

$$(1 + y)^{q-1}(q - 1)!w^q(y) \equiv (-1)y + y + \sum_{k=2}^{q-1} \begin{bmatrix} q \\ k \end{bmatrix} w_k(y) \equiv 0 \pmod{q},$$

since, by (2.2),

$$\begin{bmatrix} q \\ k \end{bmatrix} \equiv 0 \pmod{q} \quad \text{for} \quad 2 \leq k \leq q - 1$$

and $w_k(y)$ is an integer if $y$ is an integer. The result now follows from Fermat’s and Wilson’s theorems.

It is clear from (1.2) that if $y$ is an integer which is a multiple of $q$, then

$$w_n^{(r)}(y) \equiv 0 \pmod{q},$$

since $\begin{bmatrix} n \\ k \end{bmatrix} (r)_k$ is an integer. Note that Theorem 4.1 is a special case that can be obtained from the following result:

**Theorem 4.2.** If $y$ is an integer that is not a multiple of $q$, then

$$w_n^{(r)}(y) \equiv 0 \pmod{q} \quad \text{for} \quad n \geq 1 \quad \text{and} \quad r \equiv 0 \pmod{q}.$$ 

Proof. Let $r = tq$ for some integer $t$. By virtue of (1.2), we get

$$w_n^{(r)}(y) = \sum_{k=0}^{n} k! \begin{bmatrix} t+1 \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} y^k.$$ 

Since

$$k! \begin{bmatrix} t+1 \\ k \end{bmatrix} = (t+1)(t+2)\cdots(t+1)(t) \equiv 0 \pmod{q},$$

we get the required result.
Theorem 4.3. If \( y \) is an integer such that \( y \) and \( 1 + y \) are not multiples of an odd prime \( q \), then

\[
w_{q-1}^{(r)}(y) \equiv 0 \pmod{q} \quad \text{for} \quad r \equiv 1 \pmod{q}.
\]

Proof. Let \( r = 1 + tq \) for some integer \( t \). By (1.2), we have

\[
w_{q-1}^{(r)}(y) = \sum_{k=0}^{q-1} k! \left( \begin{array}{c} tq + k \\ k \end{array} \right) \left( \begin{array}{c} q - 1 \\ k \end{array} \right) y^k.
\]

Since

\[
\left( \begin{array}{c} tq + k \\ k \end{array} \right) = \frac{(tq + k)(tq + k - 1) \ldots (tq + 1)}{k!} = \frac{k(k - 1) \ldots 1}{k!} = 1 \pmod{q},
\]

we conclude that

\[
w_{q-1}^{(r)}(y) \equiv \sum_{k=0}^{q-1} k! \left( \begin{array}{c} q - 1 \\ k \end{array} \right) y^k \pmod{q}.
\]

It follows from (2.3) that

\[
k! \left( \begin{array}{c} q - 1 \\ k \end{array} \right) \equiv (-1)^{k-1} \pmod{q}
\]

for \( 1 \leq k \leq q - 1 \). Since \( \left\{ \begin{array}{c} q - 1 \\ 0 \end{array} \right\} = 0 \), we get

\[
w_{q-1}^{(r)}(y) \equiv \sum_{k=0}^{q-1} (-1)^{k-1} y^k = 1 - \sum_{k=0}^{q-1} (-1)^k y^k = 1 - \frac{1 + y^q}{1 + y} \pmod{q},
\]

which implies that

\[
(1 + y)w_{q-1}^{(r)}(y) \equiv 1 + y - 1 + y^q \equiv 0 \pmod{q}
\]

and the required result.

These results and their proofs are direct generalizations of the corresponding congruences for higher-order geometric numbers given in [11] (Corollary 4.2).

We conclude the study of congruences for higher-order geometric polynomials by a similar result.

Theorem 4.4. If \( y \) is an integer that is not a multiple of an odd prime \( q \), then

\[
w_{q+1}^{(r)}(y) \equiv 0 \pmod{q} \quad \text{for} \quad r \equiv 0 \pmod{q}
\]

and

\[
w_{q+1}^{(r)}(y) \equiv -y \pmod{q} \quad \text{for} \quad r \equiv -1 \pmod{q}.
\]
Proof. For a prime $q$ and a nonnegative integer $m$, we have

$$\binom{q+m}{k} \equiv \binom{m+1}{k} + \binom{m}{k-q} \pmod{q}.$$ 

This result was presented by Howard in [17] and can be easily verified by induction on $m$. This implies that

$$\binom{q+1}{k} \equiv 0 \pmod{q} \quad \text{for} \quad k = 3, 4, \ldots, q \quad \text{and} \quad \binom{q+1}{2} \equiv 1 \pmod{q}.$$ 

We can now represent (1.2) as follows:

$$w_{q+1}^{(r)}(y) = \sum_{k=0}^{q+1} k! \binom{r + k - 1}{k} \binom{q + 1}{k} y^k,$$

$$= ry + r(r + 1) \binom{q + 1}{2} y^2 + (q + 1)! \binom{r + q}{q} y^{q+1}$$

$$+ \sum_{k=3}^{q} k! \binom{r + k - 1}{k} \binom{q + 1}{k} y^k,$$

$$\equiv ry + r(r + 1)y^2 \pmod{q},$$

which yields the required results.

In the remaining part of this section we consider congruences for the $p$-Bernoulli numbers. In particular, the following theorem establishes a von-Staudt–Clausen-type result for $p$-Bernoulli numbers.

**Theorem 4.5.** Let $n$ be a positive integer. Then

$$4B_{2n,2} \equiv -1 \pmod{2}$$

and if $(q - 1) | 2n$ for an odd prime $q$, then

$$qB_{2n,q} \equiv -\frac{1}{2} \pmod{q}.$$ 

Proof. First, we take $p = 2$ in (2.6). This gives

$$\frac{2}{3} B_{2n,2} = B_{2n+2},$$

or, equivalently,

$$4B_{2n,2} = 3 \cdot 2B_{2n+2}.$$ 

The result then follows from the von-Staudt–Clausen theorem.
Further, let $q$ be an odd prime. Then we replace $n$ by $2n$ and $p$ by $q$ in (2.6) and obtain

$$\frac{q!}{q + 1}B_{2n,q} = \sum_{k=0}^{q} \binom{q}{k} (-1)^k B_{2n+k}$$

$$= \binom{q}{0} B_{2n} - \binom{q}{1} B_{2n+1} + \binom{q}{q-1} B_{2n+q-1} - \binom{q}{q} B_{2n+q} + \sum_{k=2}^{q-2} \binom{q}{k} (-1)^k B_{2n+k}. $$

Since

$$\binom{q}{0} = 0, \quad \binom{q}{1} = (q - 1)!, \quad \binom{q}{q-1} = \frac{q(q-1)}{2}, \quad \binom{q}{q} = 1$$

and

$$B_{2n+1} = 0, \quad n \geq 1,$$

the above equality turns into

$$\frac{q!}{q + 1}B_{2n,q} = \frac{q - 1}{2} q B_{2n+q-1} + \sum_{k=2}^{q-2} \frac{1}{q} \binom{q}{k} (-1)^k q B_{2n+k}.$$ 

If $(q - 1) | 2n$, then $(q - 1) | 2n + q - 1$. Hence, $q B_{2n+q-1} \equiv -1 \pmod{q}$ by (2.6). We also have $(q - 1) \nmid 2n + k$ for $k = 2, 3, \ldots, q - 2$ and, thus, $q B_{2n+k} \equiv 0 \pmod{q}$ again by (2.6). Noting that

$$\binom{q}{k} \equiv 0 \pmod{q} \quad \text{for} \quad 2 \leq k \leq q - 2,$$

we observe that the sum vanishes modulo $q$. Therefore,

$$\frac{q!}{q + 1}B_{2n,q} \equiv -\frac{q - 1}{2} \pmod{q},$$

or, equivalently,

$$q B_{2n,q} \equiv -\frac{1}{2} \pmod{q}$$

by Wilson’s theorem.

In the following theorem, we establish a congruence for $B_{q,q}$, where $q > 3$ is a prime.

**Theorem 4.6.** For a prime $q > 3$, $q B_{q,q} \equiv \frac{1}{12} - B_{q+1} \pmod{q}$.

**Proof.** Let $q > 3$ be a prime. Writing $n = p = q$ in (2.7), we get

$$\frac{q!}{q + 1}B_{q,q} = \binom{q}{0} B_q - \binom{q}{1} B_{q+1} + \binom{q}{q-1} B_{2q-1} - \binom{q}{q} B_{2q}$$
\[-\left[ \begin{array}{c} q \\ q - 2 \end{array} \right] B_{2q-2} + \sum_{k=2}^{q-3} \left[ \begin{array}{c} q \\ k \end{array} \right] (-1)^k B_{q+k} \]

\[= -(q-1)! B_{q+1} - B_{2q} - \left[ \begin{array}{c} q \\ q - 2 \end{array} \right] B_{2q-2} + \sum_{k=2}^{q-3} \left[ \begin{array}{c} q \\ k \end{array} \right] (-1)^k B_{q+k}, \]

or, equivalently,

\[q! B_{q,q} = -(q-1)! (q+1) B_{q+1} - (q+1) B_{2q} \]

\[\quad - \frac{(q+1)(3q-1)(q-1)(q-2)}{24} q B_{2q-2} + (q+1) \sum_{k=2}^{q-3} \frac{1}{q} \left[ \begin{array}{c} q \\ k \end{array} \right] (-1)^k q B_{q+k}, \]

since

\[\left[ \begin{array}{c} q \\ q - 2 \end{array} \right] = \frac{(3q-1)q(q-1)(q-2)}{24}. \]

Now \((q-1) \nmid (q+k)\) for \(k = 2, 3, \ldots, q-3\). Hence, by the von-Staudt–Clausen theorem,

\[q B_{q+k} \equiv 0 \quad (\text{mod } q). \]

Moreover,

\[\left[ \begin{array}{c} q \\ k \end{array} \right] \equiv 0 \quad (\text{mod } q) \quad \text{for} \quad k = 2, 3, \ldots, q-3. \]

Hence, the sum in the relation presented above vanishes modulo \(q\). The von-Staudt–Clausen theorem also implies that \(q B_{q+1} \equiv 0 \ (\text{mod } q)\), \(q B_{2q} \equiv 0 \ (\text{mod } q)\) and \(q B_{2q-2} \equiv -1 \ (\text{mod } q)\). All these facts and Wilson’s theorem imply that

\[q B_{q,q} \equiv \frac{1}{12} - B_{q+1} + B_{2q} \ (\text{mod } q). \]

The result readily follows by virtue of Adam’s theorem, which states that \(q \mid n\) implies that \(B_n \equiv 0 \ (\text{mod } q)\) for the primes \((q-1) \nmid n\).

Finally, we present a congruence for \(B_{q,q+1}\), where \(q > 3\) is a prime.

**Theorem 4.7.** For a prime \(q > 3\), \(q B_{q,q+1} \equiv 0 \ (\text{mod } q)\).

**Proof.** Letting \(n = p = q\) and \(r = 1\) in (3.13) and using \(B_{n,1} = -2B_{n+1}\), we get

\[(q+1)! B_{q,q+1} = (q+2) \sum_{k=0}^{q} \left[ \begin{array}{c} q + 1 \\ k + 1 \end{array} \right] (-1)^{k-1} B_{q+k+1} \]

\[= q(q+2) \sum_{k=0}^{q} \left[ \begin{array}{c} q \\ k \end{array} \right] (-1)^k B_{q+k} + (q+2) \sum_{k=0}^{q} \left[ \begin{array}{c} q \\ k \end{array} \right] (-1)^{k-1} B_{q+k+1}. \]
Equation (2.7) enable us to rewrite the first sum as follows:

\[ q(q + 2) \sum_{k=0}^{q} \sum_{k=0}^{q} \binom{q}{k} (-1)^kB_{q+k} = \frac{(q + 2)q^1}{(q + 1)} qB_{q,q}. \]

By Theorem 4.6, we conclude that

\[ q(q + 2) \sum_{k=0}^{q} \binom{q}{k} (-1)^kB_{q+k} = \frac{(q + 2)q^1}{(q + 1)} \left( \frac{1}{12} - B_{q+1} \right) \equiv 0 \pmod{q}, \]

by virtue of the von-Staudt–Clausen theorem.

Further, we separate the terms of the second sum as follows:

\[ (q + 2) \sum_{k=0}^{q} \binom{q}{k} (-1)^kB_{q+k} = - (q + 2) \sum_{k=0}^{q} \binom{q}{k} B_{2q} - (q + 2) \sum_{k=0}^{q} \binom{q}{k} B_{2q,2} \]

\[ + \frac{q + 2}{q} \sum_{k=2}^{q-4} \binom{q}{k} (-1)^kB_{q+k+1} \]

\[ = -(q + 2) \frac{q-1}{2} qB_{2q} - q + 2 \left( \frac{q}{4} \right) \frac{q-1}{2} qB_{2q,2} \]

\[ + \frac{q + 2}{q} \sum_{k=2}^{q-4} \binom{q}{k} (-1)^kB_{q+k+1}, \]

since

\[ \binom{q}{q-1} = \binom{q}{2} \quad \text{and} \quad \binom{q}{q-3} = \binom{q}{2} \binom{q}{4}. \]

For \( k = 2, 3, \ldots, q - 4 \), we have \((q - 1) \nmid (q + k + 1)\). Hence, by the von-Staudt–Clausen theorem,

\[ qB_{q+k+1} \equiv 0 \pmod{q}. \]

Moreover,

\[ \binom{q}{k} \equiv 0 \pmod{q} \]

in the same range. Hence, we conclude that the sum presented above vanishes modulo \( q \). The result now follows if we note that

\[ \binom{q}{4} \equiv 0 \pmod{q}, \quad qB_{2q,2} \equiv -1 \pmod{q}, \quad \text{and} \quad qB_{2q} \equiv 0 \pmod{q}. \]
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