Curvatures of Embedded Minimal Disks Blow Up on Subsets of $C^1$ Curves

Brian White

Introduction

Let $D_n$ be a sequence of minimal disks that are properly embedded in an open subset $U$ of $\mathbb{R}^3$ or more generally of a 3-dimensional Riemannian manifold. By passing to a subsequence, we may assume that there is a relatively closed subset $K$ of $U$ such that the curvatures of the $D_n$ blow up at each point of $K$ (i.e., such that for each $p \in K$, there are points $p_n \in D_n$ converging to $p$ such that curvature of $D_n$ at $p_n$ tends to infinity as $n \to \infty$) and such that $D_n \setminus K$ converges smoothly on compact subsets of $U \setminus K$ to a minimal lamination $L$ of $U \setminus K$. It is natural to ask what kinds of singular sets $K$ and laminations $L$ can arise in this way. In this paper, we prove:

Theorem 1. Every point of $K$ contains a neighborhood $W$ such that $K \cap W$ is (after a rotation of $\mathbb{R}^3$) contained in the graph of a $C^1$ function from $\mathbb{R}$ to $\mathbb{R}^2$.

This extends previous results of Colding-Minicozzi and of Meeks. In particular, if one replaces “$C^1$” by “Lipschitz” in Theorem 1 then the result is implicit in the work of Colding and Minicozzi. (See [CM04c, Section I.1], and [CM04c, Theorem 0.1] for a very similar result.) Thus if $K$ is a curve, it must be a Lipschitz curve. Meeks later showed that if $K$ is a Lipschitz curve then it must be a $C^{1,1}$ curve [Mee04].

Meeks and Weber [MW07] showed that every $C^{1,1}$ curve arises as such a blow-up set $K$. Hoffman and White [HW11] showed that every closed subset of a line arises as such a blow-up set. (Kleene [Kle09] gave another proof of the Hoffman-White result. Special cases had been proved earlier by Colding-Minicozzi [CM04], Brian Dean [Dea06], and Siddique Kahn [Kah08].)

The following questions remain open:

1. Can $C^1$ in Theorem 1 be replaced by $C^{1,1}$? The Meeks-Weber examples show that one cannot prove more regularity than $C^{1,1}$.

Date: March 28, 2011.

2000 Mathematics Subject Classification. Primary: 53A10; Secondary: 49Q05.

The author would like to thank David Hoffman for helpful suggestions. The research was partially supported by NSF grant DMS-0104049.
(2) If $C^1$ can be replaced by $C^{1,1}$, does every closed subset of a $C^{1,1}$ curve arise as the blow-up set $K$ of some sequence $D_n$? If $C^1$ cannot be replaced by $C^{1,1}$, does every closed subset of a $C^1$ curve arise as such a $K$?

1. Results

We begin with some definitions. For simplicity, we work in $\mathbb{R}^3$, although the results generalize easily to arbitrary smooth Riemannian 3-manifolds; see the remark at the end of the paper. A configuration is a triple $(U, K, L)$ where $U$ is an open ball in $\mathbb{R}^3$, an open halfspace in $\mathbb{R}^3$ or all of $\mathbb{R}^3$, where $K$ is a relatively closed subset of $U$, and where $L$ is a minimal lamination of $U \setminus K$. Here $K$ should be thought of as a singular set: the configurations $(U, K, L)$ we are most interested in arise as limits of smooth, properly embedded minimal surfaces, in which case $K$ will be the set of points where the curvature blows up.

We define the curvature of a configuration $(U, K, L)$ at a point $p \in L$ to be the norm of the second fundamental form at $p$ of the leaf that contains $p$. We define the curvature of the configuration $(U, K, L)$ to be $\infty$ at each point of $K$.

A plane $P$ (i.e., a two-dimensional linear subspace of $\mathbb{R}^3$) is said to be tangent to $(U, K, L)$ at a point $p$ if and only if

1. $p \in L$ and $P$ is the tangent plane at $p$ to the leaf of the lamination that contains $p$, or
2. $p \in K$.

Thus each point in $L$ has a unique tangent plane, whereas each point in $K$ has (by definition) every plane as a tangent plane.

If $(U, K, L)$ is a configuration, the lift of $(U, K, L)$ is

$$\Phi(U, K, L) = \{(x, P) : x \in K \cup L \text{ and } P \text{ is tangent plane to } (U, K, L) \text{ at } x\}.$$ 

Note that the lift is a relatively closed subset of the Grassmann bundle $U \times G$, where $G$ is the set of all 2-dimensional linear subspaces of $\mathbb{R}^3$. Note also that a configuration is determined by its lift: if $\Phi(U, K, L) = \Phi(U', K', L')$ then $K = K'$ and $L = L'$.

**Theorem 2.** Let $(U_n, K_n, L_n)$ be a sequence of configurations such that $U_n$ converges to a nonempty open set $U$. Suppose also that the lifts $\Phi(U_n, K_n, L_n)$ converge in the Gromov Hausdorff sense to a relatively closed subset $V$ of $U \times G$. Then $V$ is the lift $\Phi(U, K, L)$ of a configuration $(U, K, L)$. Furthermore,

1. For each point $q \in K$, the curvatures of the $(U_n, K_n, L_n)$ blow up at $q$, meaning that there is a sequence $q_n \in K_n \cup L_n$ such that $q_n$ converges to $q$ and such that the curvature of $(U_n, K_n, L_n)$ at $q_n$ tends to $\infty$ as $n \to \infty$.
2. For each compact subset $C$ of $U \setminus K$, the curvatures of the $(U_n, K_n, L_n)$ are uniformly bounded on $C$ as $n \to \infty$.
3. The laminations $L_n$ converge to the lamination $L$ on compact subsets of $U$. 

Here (and throughout the paper) convergence of open sets $U_n$ to open set $U$ means convergence of $\mathbb{R}^3 \setminus U_n$ to $\mathbb{R}^3 \setminus U$ in the Gromov-Hausdorff topology. In particular, if $U_n$ and $U$ are balls, convergence of $U_n$ to $U$ means that the centers and radii of the $U_n$ converge to the center and radius of $U$.

**Proof.** Let $K$ be the set of points $q$ in $U$ such that
\[
\{q\} \times G \subset V.
\]
First we prove that (1) holds. For suppose it fails at a point $q \in K$. By passing to a subsequence, we may assume (for some ball $W$ centered at $q$) that the curvatures of the $(U_n, K_n, L_n)$ are uniformly bounded on $W$. In other words, $W$ is disjoint from each $K_n$ and the curvatures of the lamination $L_n \cap W$ are uniformly bounded. By replacing $W$ by a smaller ball, we can then ensure that the tangent planes to $L_n$ at any two points of $L_n \cap W$ make an angle of at most $\pi/20$ (for example) with each other. It follows that if $(x, P)$ and $(x', P')$ are points of $V$ with $x, x' \in W$, then the angle between $P$ and $P'$ is at most $\pi/20$. But this contradicts the fact that $\{q\} \times G \subset V$, thus proving (1).

Next we prove that (2) holds. Suppose that $q \in U \setminus K$. Then
\[
(*) \quad \{P \in G : (q, P) \in V\}
\]
is a closed subset of $G$ but is not equal to $G$. Thus there is a closed set $\Sigma \subset G$ with nonempty interior such that $\Sigma$ is disjoint from the set $(*)$. In other words,
\[
\{(q) \times \Sigma\} \cap V = \emptyset.
\]
By the Gromov-Hausdorff convergence $\Phi(U_n, K_n, L_n) \to V$, it follows that there is an open ball $W$ centered at $q$ and compactly contained in $U$ such that
\[
(W \times \Sigma) \cap \Phi(U_n, K_n, L_n) = \emptyset
\]
for all sufficiently large $n$, say $n \geq N$. It follows immediately that
(i) $K_n \cap W = \emptyset$ for $n \geq N$, and
(ii) the Gauss map of $L_n \cap W$ omits $\Sigma$ for $n \geq N$.

By a theorem of Osserman [Oss60], (i) and (ii) imply that the curvatures of the $L_n$ are uniformly bounded (for $n \geq N$) on compact subsets of $W$. This together with (i) implies that the curvatures of the $(U_n, K_n, L_n)$ are uniformly bounded on compact subsets of $W$. This proves (2).

It remains only to prove (3). Note that the curvature bounds in (2) imply that every subsequence of the $L_n$ has a further subsequence that converges on compact subsets of $U \setminus K$ to a lamination $L$ of $U \setminus K$. But clearly $L$ is determined by $V$.

Thus the limit $L$ is independent of the subsequence, which means that the original sequence $L_n$ converges to $L$ on compact subsets of $U \setminus K$. \hfill $\Box$

\[1\] In fact, $V \cap ((U \setminus K) \times G)$ is the lift of $(U \setminus K, \emptyset, L)$, so the latter may be recovered from the former using the projection map from $U \times G$ to $U$. 
We say that configurations \((U_n, K_n, L_n)\) converge to configuration \((U, K, L)\) provided \(U_n\) converges to \(U\) and \(\Phi(U_n, K_n, L_n)\) converges in the Gromov-Hausdorff topology to \(\Phi(U, K, L)\). From Theorem 2 together with compactness of the space of closed sets under Gromov-Hausdorff convergence, we deduce

**Corollary 3 (Compactness of configurations).** Suppose \((U_n, K_n, L_n)\) is a sequence of configurations such that \(U_n\) converges to a nonempty open set \(U\). Then a subsequence of \((U_n, K_n, L_n)\) converges to a configuration \((U, K, L)\).

A configuration of disks is a configuration \((U, \emptyset, L)\) in which each leaf of \(L\) is a properly embedded minimal disk in \(U\). We let \(\mathcal{D}\) be the set of all configurations of disks. We let \(\overline{\mathcal{D}}\) be the set of all configurations that are limits of configurations of disks. Note that \(\overline{\mathcal{D}}\) is closed under sequential convergence.

**Theorem 4.** Suppose that \((U, K, L)\) \(\in\) \(\overline{\mathcal{D}}\). Then \(U\) is covered by open balls \(B\) with the following properties:

1. For each point \(p \in K \cap B\), there is a leaf \(L_p\) of \(L \cap B\) such that \(L_p \cup \{p\}\) is a minimal graph over a planar region and is properly embedded in \(B\).
2. If \(q_n \in K \cap B\) converges to \(q \in K \cap B\), then \(L_{q_n} \cup \{q_n\}\) converges smoothly to \(L_q \cup \{q\}\).
3. The singular set \(K \cap B\) is contained in a \(C^1\) embedded curve \(\Gamma\) such that at each point \(q\) of \(K \cap B\), the curve \(\Gamma\) is orthogonal to \(L_q \cup \{q\}\) at \(q\).

(See Remark 7 for the generalization to arbitrary Riemannian 3-manifolds.)

**Proof.** Assertion (1) is due to Colding and Minicozzi [CM04b, Theorem 5.8]. Assertion (2) follows immediately from Assertion (1). To prove Assertion (3), we use the following theorem due to Colding-Minicozzi and Meeks:

**Theorem 5.** If \((\mathbb{R}^3, K, L)\) \(\in\) \(\overline{\mathcal{D}}\) and if \(K\) is nonempty, then \(K\) is a line and the lamination \(L\) is the foliation consisting of all planes perpendicular to \(L\).

(According to [CM04c, Theorem 0.1], \(L\) is a foliation of consisting of parallel planes and \(K\) is a Lipschitz curve transverse to those planes. According to [Mee04], the Lipschitz curve must be a straight line perpendicular to those planes.)

We also use the following proposition, which is a restatement of the \(C^1\) case of Whitney’s Extension Theorem [Whi34, Theorem I]:

**Proposition 6.** Let \(K\) be a relatively closed subset of an open subset \(B\) of \(\mathbb{R}^n\). Suppose \(\mathcal{V}\) is a continuous line field on \(K\), i.e., a continuous function that assigns to each \(p \in K\) a line \(\mathcal{V}(p)\) in \(\mathbb{R}^n\). Suppose also that if \(p_i, q_i \in K\) with \(p_i \neq q_i\) converge to \(p \in K\), then \(p_i q_i\) converges to \(\mathcal{V}(p)\).

Then each point \(p \in K\) has a neighborhood \(W\) such that \(K \cap W\) is contained in the graph \(\Gamma\) of a \(C^1\) function \(\mathcal{V}(p)\) to \((\mathcal{V}(p))^+\) such that at each point \(q \in W \cap K\), \(\mathcal{V}(q)\) is tangent to \(\Gamma\) at \(q\).
We will apply Proposition \[\alpha\] with \(\mathcal{V}(p) = (\text{Tan}_p L_p)^\perp\). By assertion (2) of Theorem \[\beta\], \(\mathcal{V}(p)\) depends continuously on \(p \in K\). Let \(p_j, q_j \in K \cap B\) with \(p_j \neq q_j\) converge to \(p \in K \cap B\). It suffices to prove that \(\frac{p_j - q_j}{|p_j - q_j|}\) converges to \((L_p)^\perp\).

Let \(\phi_n : \mathbb{R}^3 \to \mathbb{R}^3\) be translation by \(-q_n\) followed by dilation by \(1/|p_n - q_n|:\)
\[
\phi_n(x) = \frac{x - q_n}{|p_n - q_n|}.
\]
By passing to a subsequence, we may assume that \(\phi_n(p_n)\) converges to a point \(p^*\) with \(|p^*| = 1\). Thus
\[
\frac{p_n q_n}{|p_n q_n|} = \frac{\phi_n(p_n)\phi_n(q_n)}{|\phi_n(p_n)\phi_n(q_n)|} = \phi_n(p_n)O \to p^*O.
\]
Note that \(\phi_n(U_n) \to \mathbb{R}^3\). Now consider the configurations \((\phi_n(U), \phi_n(K), \phi_n(L))\).

By passing to a further subsequence, we may assume that these configurations converge to a configuration \((\mathbb{R}^3, K', L') \in \overline{\mathcal{D}}\). Note that \(K'\) is nonempty since 0 and \(p^*\) are in \(K\). Thus by Theorem \[\gamma\], \(K'\) is a line and \(L'\) consists of all planes perpendicular to \(K'\). Since \(K'\) contains 0 and \(p^*\), in fact \(K'\) is the line through 0 and \(p^*\).

Now by Assertion (2) of the theorem, the leaves \(\phi_n(L_{q_n} \cup \{q_n\})\) converge smoothly to \(\text{Tan}_q L_q\). Thus \(\text{Tan}_q L_q\) is one of the leaves of \(L'\), which means that \(\text{Tan}_q L_q\) is perpendicular to \(K'\). In other words, \(K'\) is the line \(\text{Tan}(q)\). \(\Box\)

**Remark 7.** The definitions and theorems in this paper generalize to arbitrary smooth Riemannian 3-manifolds. In particular, Theorem \[\delta\] remains true if \(U\) is an open geodesic ball of radius \(r\) in a 3-dimensional Riemannian manifold, provided all the geodesic balls of radius \(\leq r\) centered at points in \(U\) are mean convex. (This guarantees that if \(D\) is a minimal disk properly embedded in \(U\), then the intersection of \(D\) with any geodesic ball in \(U\) is a union of disks.) The proof is almost identical to the proof in the Euclidean case.

**References**

[CM04a] Tobias H. Colding and William P. Minicozzi II, *Embedded minimal disks: proper versus nonproper—global versus local*, Trans. Amer. Math. Soc. 356 (2004), no. 1, 283–289 (electronic). MR2020033 [2004k:53005]

[CM04b] __________, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks*, Ann. of Math. (2) 160 (2004), no. 1, 69–92, DOI 10.4007/annals.2004.160.69. MR2119718 [2006a:53005]

[CM04c] __________, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected*, Ann. of Math. (2) 160 (2004), no. 2, 573–615. MR2123933 [2006e:53013]

[Dea06] Brian Dean, *Embedded minimal disks with prescribed curvature blowup*, Proc. Amer. Math. Soc. 134 (2006), no. 4, 1197–1204 (electronic). MR2196057 [2007d:53009]

[HW11] David Hoffman and Brian White, *Sequences of embedded minimal disks whose curvatures blow up on a prescribed subset of a line*, Communications in Analysis and Geometry (2011), to appear, available at [arXiv:0905.0851v3[math.DG]]

[Kah08] Siddique Kahn, *A minimal lamination of the unit ball with singularities along a line segment*, Illinois J. of Math. (2008), to appear, available at [arXiv:0902.3541v2[math.DG]]

[Kle09] Stephen J. Kleene, *A minimal lamination with Cantor set-like singularities* (2009), available at [arXiv:0910.0199[math.DG]]
[Mee04] William H. Meeks III, *Regularity of the singular set in the Colding-Minicozzi lamination theorem*, Duke Math. J. 123 (2004), no. 2, 329–334. MR2066941 (2005d:53014)

[MW07] William H. Meeks III and Matthias Weber, *Bending the helicoid*, Math. Ann. 339 (2007), no. 4, 783–798. MR2341900 (2008k:53020)

[Oss60] Robert Osserman, *On the Gauss curvature of minimal surfaces*, Trans. Amer. Math. Soc. 96 (1960), 115–128. MR0121723 (22 #12457)

[Whi34] Hassler Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89, DOI 10.2307/1989708. MR1501735

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CURVATURES OF EMBEDDED MINIMAL DISKS BLOW UP ON SUBSETS OF $C^1$ CURVES

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Abstract. Assuming results of Colding-Minicozzi and an extension due to Meeks, we prove that a sequence of properly embedded minimal disks in a 3-ball must have a subsequence whose curvature blow-up set lies in a union of disjoint $C^1$ curves.

Introduction

Let $D_n$ be a sequence of minimal disks that are properly embedded in an open subset $U$ of $\mathbb{R}^3$ or more generally of a 3-dimensional Riemannian manifold. By passing to a subsequence, we may assume that there is a relatively closed subset $K$ of $U$ such that the curvatures of the $D_n$ blow up at each point of $K$ (i.e., such that for each $p \in K$, there are points $p_n \in D_n$ converging to $p$ such that curvature of $D_n$ at $p_n$ tends to infinity as $n \to \infty$) and such that $D_n \setminus K$ converges smoothly on compact subsets of $U \setminus K$ to a minimal lamination $L$ of $U \setminus K$. It is natural to ask what kinds of singular sets $K$ and laminations $L$ can arise in this way. In this paper, we prove:

Theorem 1. Every point of $K$ contains a neighborhood $W$ such that $K \cap W$ is (after a rotation of $\mathbb{R}^3$) contained in the graph of a $C^1$ function from $\mathbb{R}$ to $\mathbb{R}^2$.

This extends previous results of Colding-Minicozzi and of Meeks. In particular, if one replaces “$C^1$” by “Lipschitz” in Theorem 1 then the result is implicit in the work of Colding and Minicozzi. (See [CM04c, Section I.1], and [CM04c, Theorem 0.1] for a very similar result.) Thus if $K$ is a curve, it must be a Lipschitz curve. Meeks later showed that if $K$ is a Lipschitz curve then it must be a $C^{1,1}$ curve [Mee04].

Meeks and Weber [MW07] showed that every $C^{1,1}$ curve arises as such a blow-up set $K$. Hoffman and White [HWT11] showed that every closed subset of a line arises as such a blow-up set. (Kleene [Kle09] gave another proof of the Hoffman-White result. Special cases had been proved earlier by Colding-Minicozzi [CM04], Brian Dean [Dea06], and Siddique Kahn [Kah08].)
The following questions remain open:

(1) Can $C^1$ in Theorem 1 be replaced by $C^{1,1}$? The Meeks-Weber examples show that one cannot prove more regularity than $C^{1,1}$.

(2) If $C^1$ can be replaced by $C^{1,1}$, does every closed subset of a $C^{1,1}$ curve arise as the blow-up set $K$ of some sequence $D_n$? If $C^1$ cannot be replaced by $C^{1,1}$, does every closed subset of a $C^1$ curve arise as such a $K$?

1. Results

We begin with some definitions. For simplicity, we work in $\mathbb{R}^3$, although the results generalize easily to arbitrary smooth Riemannian 3-manifolds; see the remark at the end of the paper. A configuration is a triple $(U, K, L)$ where $U$ is an open ball in $\mathbb{R}^3$, an open halfspace in $\mathbb{R}^3$ or all of $\mathbb{R}^3$, where $K$ is a relatively closed subset of $U$, and where $L$ is a minimal lamination of $U \setminus K$. Here $K$ should be thought of as a singular set: the configurations $(U, K, L)$ we are most interested in arise as limits of smooth, properly embedded minimal surfaces, in which case $K$ will be the set of points where the curvature blows up.

We define the curvature of a configuration $(U, K, L)$ at a point $p \in L$ to be the norm of the second fundamental form at $p$ of the leaf that contains $p$. We define the curvature of the configuration $(U, K, L)$ to be $\infty$ at each point of $K$.

A plane $P$ (i.e., a two-dimensional linear subspace of $\mathbb{R}^3$) is said to be tangent to $(U, K, L)$ at a point $p$ if and only if

(1) $p \in L$ and $P$ is the tangent plane at $p$ to the leaf of the lamination that contains $p$, or

(2) $p \in K$.

Thus each point in $L$ has a unique tangent plane, whereas each point in $K$ has (by definition) every plane as a tangent plane.

If $(U, K, L)$ is a configuration, the lift of $(U, K, L)$ is

$$\Phi(U, K, L) = \{(x, P) : x \in K \cup L \text{ and } P \text{ is tangent plane to } (U, K, L) \text{ at } x\}.$$ 

Note that the lift is a relatively closed subset of the Grassmann bundle $U \times G$, where $G$ is the set of all 2-dimensional linear subspaces of $\mathbb{R}^3$. Note also that a configuration is determined by its lift: if $\Phi(U, K, L) = \Phi(U', K', L')$ then $K = K'$ and $L = L'$.

**Theorem 2.** Let $(U_n, K_n, L_n)$ be a sequence of configurations such that $U_n$ converges to a nonempty open set $U$. Suppose also that the lifts $\Phi(U_n, K_n, L_n)$ converge in the Gromov Hausdorff sense to a relatively closed subset $V$ of $U \times G$. Then $V$ is the lift $\Phi(U, K, L)$ of a configuration $(U, K, L)$. Furthermore,

(1) For each point $q \in K$, the curvatures of the $(U_n, K_n, L_n)$ blow up at $q$, meaning that there is a sequence $q_n \in K_n \cup L_n$ such that $q_n$ converges to $q$ and such that the curvature of $(U_n, K_n, L_n)$ at $q_n$ tends to $\infty$ as $n \to \infty$. 

(2) For each compact subset $C$ of $U \setminus K$, the curvatures of the $(U_n, K_n, L_n)$ are uniformly bounded on $C$ as $n \to \infty$.

(3) The laminations $L_n$ converge to the lamination $L$ on compact subsets of $U$.

Here (and throughout the paper) convergence of open sets $U_n$ to open set $U$ means convergence of $\mathbb{R}^3 \setminus U_n$ to $\mathbb{R}^3 \setminus U$ in the Gromov-Hausdorff topology. In particular, if $U_n$ and $U$ are balls, convergence of $U_n$ to $U$ means that the centers and radii of the $U_n$ converge to the center and radius of $U$.

Proof. Let $K$ be the set of points $q$ in $U$ such that

$$\{q\} \times G \subset V.$$

First we prove that (1) holds. For suppose it fails at a point $q \in K$. By passing to a subsequence, we may assume (for some ball $W$ centered at $q$) that the curvatures of the $(U_n, K_n, L_n)$ are uniformly bounded on $W$. In other words, $W$ is disjoint from each $K_n$ and the curvatures of the lamination $L_n \cap W$ are uniformly bounded. By replacing $W$ by a smaller ball, we can then ensure that the tangent planes to $L_n$ at any two points of $L_n \cap W$ make an angle of at most $\pi/20$ (for example) with each other. It follows that if $(x, P)$ and $(x', P')$ are points of $V$ with $x, x' \in W$, then the angle between $P$ and $P'$ is at most $\pi/20$. But this contradicts the fact that $\{q\} \times G \subset V$, thus proving (1).

Next we prove that (2) holds. Suppose that $q \in U \setminus K$. Then

$$(*) \quad \{P \in G : (q, P) \in V\}$$

is a closed subset of $G$ but is not equal to $G$. Thus there is a closed set $\Sigma \subset G$ with nonempty interior such that $\Sigma$ is disjoint from the set $(*)$. In other words,

$$\{q\} \times \Sigma \cap V = \emptyset.$$

By the Gromov-Hausdorff convergence $\Phi(U_n, K_n, L_n) \to V$, it follows that there is an open ball $W$ centered at $q$ and compactly contained in $U$ such that

$$(W \times \Sigma) \cap \Phi(U_n, K_n, L_n) = \emptyset$$

for all sufficiently large $n$, say $n \geq N$. It follows immediately that

(i) $K_n \cap W = \emptyset$ for $n \geq N$, and

(ii) the Gauss map of $L_n \cap W$ omits $\Sigma$ for $n \geq N$.

By a theorem of Osserman [Oss60], (i) and (ii) imply that the curvatures of the $L_n$ are uniformly bounded (for $n \geq N$) on compact subsets of $W$. This together with (i) implies that the curvatures of the $(U_n, K_n, L_n)$ are uniformly bounded on compact subsets of $W$. This proves (2).

It remains only to prove (3). Note that the curvature bounds in (2) imply that every subsequence of the $L_n$ has a further subsequence that converges on compact
subsets of $U \setminus K$ to a lamination $L$ of $U \setminus K$. But clearly $L$ is determined by $V$.

Thus the limit $L$ is independent of the subsequence, which means that the original sequence $L_n$ converges to $L$ on compact subsets of $U \setminus K$. □

We say that configurations $(U_n, K_n, L_n)$ converge to configuration $(U, K, L)$ provided $U_n$ converges to $U$ and $\Phi(U_n, K_n, L_n)$ converges in the Gromov-Hausdorff topology to $\Phi(U, K, L)$. From Theorem 2 together with compactness of the space of closed sets under Gromov-Hausdorff convergence, we deduce

**Corollary 3** (Compactness of configurations). Suppose $(U_n, K_n, L_n)$ is a sequence such that $U_n$ converges to a nonempty open set $U$. Then a subsequence of the $(U_n, K_n, L_n)$ converges to a configuration $(U, K, L)$.

A configuration of disks is a configuration $(U, \emptyset, L)$ in which each leaf of $L$ is a properly embedded minimal disk in $U$. We let $\mathcal{D}$ be the set of all configurations of disks. We let $\overline{\mathcal{D}}$ be the set of all configurations that are limits of configurations of disks. Note that $\overline{\mathcal{D}}$ is closed under sequential convergence.

**Theorem 4.** Suppose that $(U, K, L) \in \overline{\mathcal{D}}$. Then $U$ is covered by open balls $B$ with the following properties:

1. For each point $p \in K \cap B$, there is a leaf $L_p$ of $L \cap B$ such that $L_p \cup \{p\}$ is a minimal graph over a planar region and is properly embedded in $B$.

2. If $q_n \in K \cap B$ converges to $q \in K \cap B$, then $L_{q_n} \cup \{q_n\}$ converges smoothly to $L_q \cup \{q\}$.

3. The singular set $K \cap B$ is contained a $C^1$ embedded curve $\Gamma$ such that at each point $q$ of $K \cap B$, the curve $\Gamma$ is orthogonal to $L_q \cup \{q\}$ at $q$.

(See Remark for the generalization to arbitrary Riemannian 3-manifolds.)

**Proof.** Assertion (1) is due to Colding and Minicozzi [CM04b, Theorem 5.8]. Assertion (2) follows immediately from Assertion (1). To prove Assertion (3), we use the following theorem due to Colding-Minicozzi and Meeks:

**Theorem 5.** If $(\mathbb{R}^3, K, L) \in \overline{\mathcal{D}}$ and if $K$ is nonempty, then $K$ is a line and the lamination $L$ is the foliation consisting of all planes perpendicular to $L$.

(According to [CM04c, Theorem 0.1], $L$ is a foliation of consisting of parallel planes and $K$ is a Lipschitz curve transverse to those planes. According to [Mee04], the Lipschitz curve must be a straight line perpendicular to those planes.)

We also use the following proposition, which is a restatement of the $C^1$ case of Whitney’s Extension Theorem [Whi34, Theorem 1]:

---

1 In fact, $V \cap \{(U \setminus K) \times G\}$ is the lift of $(U \setminus K, \emptyset, L)$, so the latter may be recovered from the former using the projection map from $U \times G$ to $U$. 

...
Proposition 6. Let $K$ be a relatively closed subset of an open subset $B$ of $\mathbb{R}^n$. Suppose $\mathcal{V}$ is a continuous line field on $K$, i.e., a continuous function that assigns to each $p \in K$ a line $\mathcal{V}(p)$ in $\mathbb{R}^n$. Suppose also that if $p_i, q_i \in K$ with $p_i \neq q_i$ converge to $p \in K$, then $\mathcal{V}(p_i)$ converges to $\mathcal{V}(p)$.

Then each point $p \in K$ has a neighborhood $W$ such that $K \cap W$ is contained in the graph $\Gamma$ of a $C^1$ function from $\mathcal{V}(p)$ to $(\mathcal{V}(p))^\perp$ such that at each point $q \in W \cap K$, $\mathcal{V}(q)$ is tangent to $\Gamma$ at $q$.

We will apply Proposition 6 with $\mathcal{V}(p) = (\text{Tan}_p B_p)^\perp$. By assertion (2) of Theorem 4, $\mathcal{V}(p)$ depends continuously on $p \in K$. Let $p_j, q_j \in K \cap B$ with $p_j \neq q_j$ converge to $p \in K \cap B$. It suffices to prove that $p_j q_j$ converges to $(L_p)^\perp$.

Let $\phi_n : \mathbb{R}^3 \to \mathbb{R}^3$ be translation by $-q_n$ followed by dilation by $1/|p_n - q_n|:

$$
\phi_n(x) = \frac{x - q_n}{|p_n - q_n|}.
$$

By passing to a subsequence, we may assume that $\phi_n(p_n)$ converges to a point $p^*$ with $|p^*| = 1$. Thus

$$
p_n q_n = \phi_n(p_n) \phi_n(q_n) = \phi_n(p_n) \overrightarrow{O} \to \overrightarrow{p^* O}.
$$

Note that $\phi_n(U_n) \to \mathbb{R}^3$. Now consider the configurations $(\phi_n(U), \phi_n(K), \phi_n(L))$. By passing to a further subsequence, we may assume that these configurations converge to a configuration $(\mathbb{R}^3, K', L') \in \overline{D}$. Note that $K'$ is nonempty since $0$ and $p^*$ are in $K'$. Thus by Theorem 5, $K'$ is a line and $L'$ consists of all planes perpendicular to $K'$. Since $K'$ contains $0$ and $p^*$, in fact $K'$ is the line through $0$ and $p^*$.

Now by Assertion (2) of the theorem, the leaves $\phi_n(L_{q_n} \cup \{q_n\})$ converge smoothly to $\text{Tan}_q L_q$. Thus $\text{Tan}_q L_q$ is one of the leaves of $L'$, which means that $\text{Tan}_q L_q$ is perpendicular to $K'$. In other words, $K'$ is the line $\mathcal{V}(q)$.

Remark 7. The definitions and theorems in this paper generalize to arbitrary smooth Riemannian 3-manifolds. In particular, Theorem 4 remains true if $U$ is an open geodesic ball of radius $r$ in a 3-dimensional Riemannian manifold, provided all the geodesic balls of radius $\leq r$ centered at points in $U$ are mean convex. (This guarantees that if $D$ is a minimal disk properly embedded in $U$, then the intersection of $D$ with any geodesic ball in $U$ is a union of disks.) The proof is almost identical to the proof in the Euclidean case.

References

[CM04a] Tobias H. Colding and William P. Minicozzi II, Embedded minimal disks: proper versus nonproper—global versus local, Trans. Amer. Math. Soc. 356 (2004), no. 1, 283–289 (electronic). MR2020033 (2004k:53011)

[CM04b] , The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks, Ann. of Math. (2) 160 (2004), no. 1, 69–92, DOI 10.4007/annals.2004.160.69. MR2119718 (2006a:53005)
[CM04c] ______, The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected, Ann. of Math. (2) 160 (2004), no. 2, 573–615. MR 2123933 (2006c:53013)

[Dea06] Brian Dean, Embedded minimal disks with prescribed curvature blowup, Proc. Amer. Math. Soc. 134 (2006), no. 4, 1197–1204 (electronic). MR 2196057 (2007d:53009)

[HW11] David Hoffman and Brian White, Sequences of embedded minimal disks whose curvatures blow up on a prescribed subset of a line, Communications in Analysis and Geometry (2011), to appear, available at [arXiv:0903.0851v3[math.DG]]

[Kah08] Siddique Kahn, A minimal lamination of the unit ball with singularities along a line segment, Illinois J. of Math. (2008), to appear, available at [arXiv:0902.3641v2[math.DG]]

[Kle09] Stephen J. Kleene, A minimal lamination with Cantor set-like singularities (2009), available at [arXiv:0910.0199[math.DG]]

[Mee04] William H. Meeks III, Regularity of the singular set in the Colding-Minicozzi lamination theorem, Duke Math. J. 123 (2004), no. 2, 329–334. MR 2066941 (2005d:53014)

[MW07] William H. Meeks III and Matthias Weber, Bending the helicoid, Math. Ann. 339 (2007), no. 4, 783–798. MR 2341900 (2008k:53020)

[Oss60] Robert Osserman, On the Gauss curvature of minimal surfaces, Trans. Amer. Math. Soc. 96 (1960), 115–128. MR 0121723 (22 #12457)

[Whi34] Hassler Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89, DOI 10.2307/1989708. MR 1501735

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