Integrable Isotropic Profiles for Polarized Light

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Abstract—We consider the propagation of polarized light in the medium with isotropic refraction index profile and show that polarization violates the additional symmetries of the medium. Then we suggest a scheme for the construction of polarization-dependent refraction index which restores all symmetries of the initial profile. We illustrate the proposed scheme on the examples of Luneburg and Maxwell’s fisheye profiles.

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1. INTRODUCTION

It is well-known that the minimal action principle came in physics from geometric optics. Initially, it was invented for the description of the propagation of light and is presently known as the Fermat principle

$$S_{\text{Fermat}} = \int n(r) \sqrt{\frac{dr}{d\tau}} d\tau,$$  \hspace{1cm} (1)

where $n(r)$ is the refraction index, and $\lambda_0$ is the wavelength in vacuum. This action could be interpreted as the action of the system on the three-dimensional curved space equipped with the “optical metrics” of Euclidean signature (see [1])

$$dt^2 = n^2(r)dr \cdot dr.$$  \hspace{1cm} (2)

Thus, the symmetries of the system which describe the propagation of light in a particular medium are coming from the symmetries of the respective optical metrics. On the other hand, in accordance with Mau-pertuis principle, one can relate any non-relativistic systems describing with the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} g(r) \dot{r}^2 - V(r),$$  \hspace{1cm} (3)

can be related with the action (1), with the refraction index ($E$ is the value of the system’s energy)

$$n(r) = \lambda_0 \sqrt{2g(r)(E - V(r))}.$$  \hspace{1cm} (4)

Clearly such an optical system inherits all symmetries of the initial system.

In the superintegrable systems, i.e. in the systems with a maximal number of functionally independent integrals ($2N - 1$ integrals for $N$-dimensional system), all the trajectories become closed. The closeness of the trajectories makes respective optical profiles highly relevant in the study of cloaking and perfect imaging phenomena.

The most well-known profile of this sort is the so-called “Maxwell’s fisheye” profile which is defined by the metrics of (three-dimensional) sphere or pseudosphere (under pseudosphere we mean the upper (or lower) sheet of the two-sheet hyperboloid).

$$n_{\text{MFE}}(r) = \frac{n_0}{\sqrt{1 + \kappa r^2}}, \hspace{1cm} \kappa = \pm \frac{1}{4r_0^2},$$  \hspace{1cm} (5)

where the plus/minus sign in the expression for $\kappa$ corresponds to the sphere/pseudosphere with the radius $r_0$, and $n_0 > 0$. Apart from applications in cloaking and perfect imaging phenomena [2], Maxwell’s fish eye has numerous different applications, see, e.g. [3].

Another well-known example of a superintegrable profile is the Luneburg lens, which is related with three-dimensional isotropic oscillator,

$$n_{\text{Lun}}(r) = n_0 \sqrt{1 - \left(\frac{r}{r_0}\right)^2}.$$  \hspace{1cm} (6)

However, the optical systems describing by the action (1) do not take into account polarization of light.

Introduction of spin (polarization) results to the rotation of this plane by a constant angle proportional to spin, moreover, it breaks the non-rotational symmetries of the optical systems superintegrable integrable profile, so that photon trajectories no longer remain closed [4]. In these systems, the ray trajectories
belong to the plane which is orthogonal to the angular momentum. Thus, the key property of superintegrable isotropic profiles which makes them relevant in cloaking and perfect imaging phenomena becomes violated.

In this paper, following [5] we propose a general scheme of the deformation of isotropic refraction index profiles. It allows us to restore the initial symmetries of the system after one takes the light polarization into consideration. Namely, to preserve the qualitative properties of scalar wave trajectories for the propagating polarized light, we suggest replace the refraction index \( n(r) \) with the modified index \( n'(r) \) which is the solution (with respect to \( p \)) of the following equation:

\[
p = \frac{1}{\lambda_0} n \left( \sqrt{r^2 + s^2} - \frac{s^2}{p^2} \right), \quad \Rightarrow p = \frac{1}{\lambda_0} n'(r),
\]

where \( s \) is polarization of light.

For the particular cases of Maxwell’s fisheye and Luneburg profiles it yields the following expressions

\[
n_{\text{Mfe}}'(r) = \frac{n_{\text{Mfe}}(r)}{2} \left( 1 + \sqrt{1 - \frac{(2s\lambda_0)^2}{\kappa n_0 n_{\text{Mfe}}(r)}} \right), \tag{8}
\]

\[
n_{\text{Lun}}'(r) = \frac{n_{\text{Lun}}(r)}{\sqrt{2}} \left( 1 + \sqrt{1 + \frac{2s\lambda_0 n_0}{r_0 n_{\text{Lun}}(r)}} \right), \tag{9}
\]

where \( n_{\text{Mfe}}(r) \) and \( n_{\text{Lun}}(r) \) are the original Maxwell’s fisheye and Luneburg profiles given, respectively, by (5) and (6), while \( s \) is the light polarization. The proposed deformations restores all the symmetries of the initial systems with Maxwell’s fisheye and Luneburg profiles, which were broken after the inclusion of polarization.

Let us notice, that for the inclusion of polarization one should extend the initial physical space by additional, isospin degrees of freedom [6] (see also [7] and refs therein), which seems intuitively artificial. Otherwise, one can extend the initial optical Lagrangian by the additional terms depending on higher derivatives [8]. However, the latter, aesthetically attractive, approach describes very particular class of optical systems. By this reason in our study we will mostly use the Hamiltonian framework, where taking into account of polarization is very natural.

The paper is organized as follows. In Section 2, we present the Hamiltonian formulation of the optical system given by the action (1). In Section 3, we present the Hamiltonian formalism for the polarized light propagating in an optical medium and propose the general scheme of the deformation of isotropic refraction index which allows us to restore the initial symmetries after the inclusion of polarization.

Through the text we will use the notation \( r := |\mathbf{r}| \), \( \mathbf{r} := (x_1, x_2, x_3) \), \( \mathbf{p} := (p_1, p_2, p_3) \), \( p := |\mathbf{p}| \), and so on.
struct the physically non-equivalent optical Hamiltonians (and refraction indices) with the same symmetry algebra.

3. POLARIZED LIGHT

When taking into account light polarization we should add to the scalar Lagrangian $L_0 = p \dot{r} - p + \lambda_0^{-1} n$ the additional term $L_i = -s A(p) \dot{p}_i$, where $s$ is spin of photon, and $A$ is the the vector-potential of “Berry monopole” (i.e., the potential of the magnetic (Dirac) monopole located at the origin of momentum space) [9]

$$F := \frac{\partial}{\partial p} \times A(p) = \frac{p}{\rho}. \tag{16}$$

From the Hamiltonian viewpoint this means to preserve the form of the Hamiltonian (12) and replace the canonical Poisson brackets (11) by the twisted ones

$$\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = se_{ijk} F_k(p), \quad \{x_i, x_j\} = 0, \tag{17}$$

where $i, j, k = 1, 2, 3$, and $F_k$ are the components of Berry monopole (16). On this phase space the rotation generators take the form

$$J = r \times p + s \frac{p}{p}. \tag{18}$$

while the equations of motion read

$$\frac{dp}{dt} = \lambda_0^{-1} \nabla n(r), \quad \frac{dr}{dt} = p - \frac{s}{\lambda_0} F \times \nabla n(r). \tag{19}$$

However, the above procedure, i.e., twisting the Poisson bracket with preservation of the Hamiltonian, violates the non-kinematical (hidden) symmetry of the system. To get the profiles admitting the symmetries in the presence of polarization, we use the following observation [11]. Assume we have the three-dimensional rotationally-invariant system

$$\mathcal{H}_0 = \frac{p^2}{2g(r)} + V(r), \quad \{p_i, x_j\} = \delta_{ij}, \tag{20}$$

$$\{p_i, p_j\} = \{x_i, x_j\} = 0. \tag{21}$$

For the inclusion of interaction with magnetic monopole, we should switch from the canonical Poisson brackets to the twisted ones:

$$\{p_i, x_j\} = \delta_{ij}, \quad \{p_i, p_j\} = se_{ijk} \frac{x_k}{r^3}, \quad \{x_i, x_j\} = 0. \tag{21}$$

The rotation generators then read

$$J = r \times p + s \frac{p}{r}. \tag{22}$$

By modifying the initial Hamiltonian to

$$\mathcal{H}_s = \frac{p^2}{2g(r)} + \frac{s^2}{2g(r)r^2} + V(r), \tag{23}$$

we find that trajectories of the system preserve their form, but the plane which they belong to, fails to be orthogonal to the axis $J$. Instead, it turns to the constant angle

$$\cos \theta_s = \frac{s}{|J|}. \tag{24}$$

For the systems with hidden symmetries one can find the appropriate modifications of the hidden symmetry generators respecting the inclusion of the monopole field.

For applying this observation on the systems with polarized light, we should choose the appropriate integrable system with magnetic monopole, and then perform simple canonical transformation which yields the Poisson brackets for polarized light (21):

$$(p, r) \rightarrow (-r, p). \tag{25}$$

Afterwards we need to solve the following equation

$$r^2 + \frac{s^2}{p^2} - 2g(r)(E - V(r)) = 0, \quad \Rightarrow p = \frac{n'(r)}{\lambda_0}. \tag{26}$$

So, to preserve the qualitative properties of scalar wave trajectories for the propagating polarized light, we should replace it with the modified index $n'(r)$ which is the solution (with respect to $p$) of the following equation:

$$p = \frac{1}{\lambda_0} n \left( r^2 + \frac{s^2}{p^2} \right), \quad \Rightarrow p = \frac{n'(r)}{\lambda_0}, \tag{27}$$

where $s$ is polarization of light.

For example, to get the “polarized Coulomb profile” we have to start from the free-particle Hamiltonian on three-dimensional sphere/hiperboloid. Then, after fixing the energy surface $H_s = E$ and performing canonical transformation (25) we arrive to the third-order algebraic equation which has either one real and two complex solutions or three real solutions, which describe the “polarized Coulomb profiles”. Conversely, when we start from the Coulomb problem we will arrive to the “polarized Maxwell’s fish eye” (8), i.e. the deformation of the “Maxwell fish eye” which preserves, in the presence of polarized light, all symmetries of initial scalar system. While to get “polarized Luneburg profile” (9) we should choose, to the role of initial Hamiltonian, the isotropic oscillator.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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