CONSTRUCTION OF TYPE II$_1$ FACTORS WITH PRESCRIBED COUNTABLE FUNDAMENTAL GROUP

CYRIL HOUDAYER

Abstract. In the context of Free Probability Theory, we study two different constructions that provide new examples of factors of type II$_1$ with prescribed countable fundamental group. First we investigate state-preserving group actions on the almost periodic free Araki-Woods factors satisfying both a condition of mixing and a condition of free malleability in the sense of Popa. Typical examples are given by the free Bogoliubov shifts. Take an ICC $w$-rigid group $G$ such that $\mathcal{F}(L(G)) = \{1\}$ (e.g., $G = \mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z})$). For any countable subgroup $S \subset \mathbb{R}^*_+$, we construct an action of $G$ on $L(F_\infty)$ such that the associated crossed product $L(F_\infty) \rtimes G$ is a type II$_1$ factor and its fundamental group is $S$. The second construction is based on a free product. Take $(B(H), \psi)$ any factor of type I endowed with a faithful normal state and denote by $S \subset \mathbb{R}^*_+$ the subgroup generated by the point spectrum of $\psi$. We show that the centralizer $(L(G) \ast B(H))^{\tau \ast \psi}$ is a type II$_1$ factor and its fundamental group is $S$. Our proofs rely on Popa’s deformation/rigidity strategy using his intertwining-by-bimodules technique.

1. Introduction

Popa introduced in [21, 22, 24] the following remarkable concept: a state-preserving action $\sigma$ of a group $G$ on a von Neumann algebra $(\mathcal{N}, \varphi)$ is said to be malleable if there exists a continuous action $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{N} \otimes \mathcal{N}, \varphi \otimes \varphi)$ which commutes with the diagonal action $(\sigma_g \otimes \sigma_g)$ and such that $\alpha_1(a \otimes 1) = 1 \otimes a$, for any $a \in \mathcal{N}$. It is said to be $s$-malleable if there moreover exists an automorphism $\beta$ of $(\mathcal{N} \otimes \mathcal{N}, \varphi \otimes \varphi)$ commuting with $(\sigma_g \otimes \sigma_g)$ such that $\beta \alpha_t = \alpha_{-t} \beta$ for all $t \in \mathbb{R}$ and $\beta(a \otimes 1) = a \otimes 1$ for all $a \in \mathcal{N}$ and such that $\beta$ has period 2: $\beta^2 = \text{id}$. Typical examples of such actions are given by the commutative Bernoulli shifts. The remarkable idea of Popa was to combine this deformation property of the action with a rigidity property, namely the relative property (T) of Kazhdan and Margulis ([18, 19]) of the group. Using this “tension” between deformation and rigidity, Popa solved longstanding problems in the theory of von Neumann algebras. We refer to the papers of Popa and his coauthors [16, 20, 21, 22, 23, 24, 25], and to Vaes’ Bourbaki Seminar [34] for the stunning applications of these deformation/rigidity phenomena.

In the context of Free Probability Theory, Popa introduced another notion of malleability where the free product naturally replaces the tensor product.

Definition 1.1 (Popa, [24]). Let $G$ be a countable discrete group. Let $\sigma$ be a state-preserving action of $G$ on the von Neumann algebra $(\mathcal{N}, \varphi)$.

- The action is said to be malleable if there exists a continuous action $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{N} \ast \mathcal{N}, \varphi \ast \varphi)$ which commutes with the diagonal action $(\sigma_g \ast \sigma_g)$ and such that $\alpha_1(a \ast 1) = 1 \ast a$, $\forall a \in \mathcal{N}$.

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It is said to be \textit{s-malleable} if there moreover exists a period 2 automorphism \( \beta \) of \((\mathcal{N} \rtimes \mathcal{N}, \varphi \circ \varphi)\) commuting with \((\sigma_g \ast \sigma_g)\) such that \( \beta \alpha_t = \alpha_{-t} \beta \) for all \( t \in \mathbb{R} \) and \( \mathcal{N} \rtimes \mathcal{C} \subset (\mathcal{N} \rtimes \mathcal{N})^\beta \).

**Important Convention.** In the rest of this paper, the notion of \textit{malleability} or \textit{s-malleability} will always be taken in the sense of Definition [1,1] i.e. in the “free” sense.

We introduce the following important notation:

**Notation 1.2.** Let \((\mathcal{N}, \varphi)\) be a von Neumann algebra endowed with a faithful normal state, \( \mathcal{N}^\varphi \) denotes the centralizer of the state \( \varphi \). We set \( \mathcal{N} \ominus \mathcal{C} = \mathcal{N} \cap \ker(\varphi) \). More generally, let \( \mathcal{B} \subset \mathcal{N} \) be a von Neumann subalgebra globally invariant under the modular group \((\sigma_t^\varphi)\); if \( E_B : \mathcal{N} \to \mathcal{B} \) denotes the unique state-preserving conditional expectation, we set \( \mathcal{N} \ominus \mathcal{B} = \mathcal{N} \cap \ker E_B \).

In the context of free probability, we present, for a state-preserving group action, a stronger mixing property than the usual one.

**Definition 1.3.** Let \( G \) be a countable discrete group. Let \( \sigma \) be a state-preserving action on the von Neumann algebra \((\mathcal{N}, \varphi)\). The action is said to be \textit{freely mixing} if for all \( n \in \mathbb{N}^*, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n \in \mathcal{N} \ominus \mathcal{C} \), except possibly \( y_0 \) and/or \( y_n \) are equal to 1, we have

\[
\lim_{g_1, \ldots, g_n \to -\infty} \varphi(y_0\sigma_{g_1}(x_1)y_1 \cdots \sigma_{g_n}(x_n)y_n) = 0.
\]

Obviously, a \textit{freely mixing} action is \textit{strongly mixing}. We shall show (see Section 3) that free Bernoulli actions and free Bogoliubov shifts on the almost periodic free Araki-Woods factors are typical examples of (s-)malleable, freely mixing actions.

**Terminology 1.4.** A \textit{w-rigid} group \( G \) is a group that admits an infinite normal subgroup \( H \), such that the pair \((G, H)\) has the relative property \((T)\) of Kazhdan and Margulis [15, 19]. The example par excellence of such a pair is \((G, H) = (\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)\). Other examples include \((\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)\) where \( \Gamma \) is any nonamenable subgroup of \( \text{SL}(2, \mathbb{Z}) \) acting on \( \mathbb{Z}^2 \) by its given embedding in \( \text{SL}(2, \mathbb{Z}) \) (see [2,28]). Of course, any group \( G \) with the property \((T)\) is \( w \)-rigid.

In this paper, we present two different constructions that produce new examples of type \( \text{II}_1 \) factors with a prescribed countable fundamental group. Remind that for a type \( \text{II}_1 \) factor \( M \), the \textit{fundamental group} of \( M \) is defined as follows:

\[
\mathcal{F}(M) := \{ \tau(p)/\tau(q) : pMp \simeq qMq \},
\]

where \( p, q \) are projections in \( M \). The first construction is based on a \textit{crossed product}. In [21], we remind that Popa proved several results of intertwining of rigid subalgebras in crossed products. Using the so-called Connes-Størmer Bernoulli shifts, he constructed actions of ICC \( w \)-rigid groups on the hyperfinite type \( \text{II}_1 \) factor \( \mathcal{R} \) such that the associated crossed products have prescribed countable fundamental group. We prove an analogue of this result with \( L(\mathcal{F}_\infty) \) instead of \( \mathcal{R} \). The type \( \text{II}_1 \) factor \( L(\mathcal{F}_\infty) \) appears naturally as the centralizer of the free quasi-free state for the almost periodic free Araki-Woods factors. We obtain the following result:

**Theorem 1.5.** Let \( S \subset \mathbb{R}_+^* \) be a countable subgroup. Let \( G \) be an ICC \( w \)-rigid group. Assume that \( \mathcal{F}(L(G)) = \{1\} \). Then, there exists an action of \( G \) on the type \( \text{II}_1 \) factor \( L(\mathcal{F}_\infty) \) such that the crossed product \( L(\mathcal{F}_\infty) \rtimes G \) is a type \( \text{II}_1 \) factor with fundamental group equal to \( S \).
The typical example of a group $G$ satisfying the conditions of Theorem 1.5 is $G = \mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z})$ [23]. Once again, the actions considered in Theorem 1.5 are concrete: they are the free Bogoliubov shifts.

The second construction is based on a free product. In [16], among other remarkable results, Ioana, Peterson & Popa gave several examples of type II$_1$ factors with a prescribed fundamental group using the free product construction in the tracial case. We shall generalize some of their techniques to the almost periodic case. For $(\mathcal{A}, \psi)$, a von Neumann algebra endowed with an almost periodic faithful normal state, denote by $\text{Sp}(\mathcal{A}, \psi)$ the point spectrum of $\psi$ and by $\Gamma_{\text{Sp}(\mathcal{A}, \psi)} \subset \mathbb{R}^*_+$ the subgroup generated by $\text{Sp}(\mathcal{A}, \psi)$. We obtain the following (see Theorem 5.10 for a more general version):

**Theorem 1.6.** Let $G$ be an ICC w-rigid group such that $\mathcal{F}(L(G)) = \{1\}$. Let $(\mathcal{A}, \psi)$ be a von Neumann algebra endowed with an almost periodic state such that the centralizer $\mathcal{A}^\psi$ has the Haagerup property. Write $M = (L(G) * \mathcal{A})^{\tau \psi}$. Then $M$ is a type II$_1$ factor and its fundamental group is $\Gamma_{\text{Sp}(\mathcal{A}, \psi)}$.

Many examples of such von Neumann algebra $\mathcal{A}$ do exist: amenable von Neumann algebras endowed with a faithful normal almost periodic state, almost periodic free Araki-Woods factors with their free quasi-free state, but also all the free products studied by Dykema in [8].

This paper is organized as follows. Section 2 is devoted to a few preliminaries. In Section 3 we give the main examples of s-malleable freely mixing actions on these factors. In Section 4 we show some technical results about the intertwining of subalgebras. At last, in Section 5 we use the deformation/rigidity strategy “à la” Popa to prove Theorems 1.5 and 1.6. The Appendix is devoted to prove some well known facts about the polar decomposition of a vector.

**Notation 1.7.** Throughout this paper, we shall use the following notation: $G$ is any countable discrete group and $(\mathcal{N}, \varphi)$ is any von Neumann algebra endowed with a faithful normal almost periodic state [9]. Any group action is always assumed to be state-preserving. Any von Neumann algebra $\mathcal{N}$ is always assumed to have separable predual. Any state, conditional expectation is assumed to be normal and faithful. If $(\mathcal{M}, \varphi)$ are von Neumann algebras endowed with states, whenever we write $(\mathcal{M}, \varphi) \cong (\mathcal{N}, \psi)$, we mean that there exists a $*$-isomorphism $\theta : \mathcal{M} \rightarrow \mathcal{N}$ such that $\psi \circ \theta = \varphi$. For $n \in \mathbb{N}^*$ and a von Neumann algebra $\mathcal{M}$, we set $\mathcal{M}^n = M_n(C) \otimes \mathcal{M}$. The canonical normalized trace on $M_n(C)$ is usually denoted by $tr_n$.

We mention that recently, Popa & Vaes [26] proved the existence of free ergodic measure-preserving actions of $\mathbf{F}_\infty$ on the standard non-atomic probability space $(X, \mu)$ whose type II$_1$ factors and orbit equivalence relations have prescribed fundamental group in a large class $S$ of subgroups of $\mathbb{R}_+^*$ that contains all countable subgroups and many uncountable subgroups. In particular, they obtained the first examples of separable type II$_1$ factors and orbit equivalence relations with uncountable fundamental group different from $\mathbb{R}_+^*$.

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2. Preliminaries

2.1. Von Neumann Algebras Endowed with Almost Periodic States. Most of the time, the von Neumann algebra $(\mathcal{M}, \varphi)$ will be assumed to have a faithful normal almost periodic state $\varphi$. We regard $\mathcal{M} \subset B(L^2(\mathcal{M}, \varphi))$ through the GNS construction. We denote
by \( \hat{\cdot} : \mathcal{M} \to L^2(\mathcal{M}, \varphi) \) the canonical embedding. Let \( S^0_\varphi \) be the antilinear operator defined by

\[
S^0_\varphi : \hat{\mathcal{M}} \to L^2(\mathcal{M}, \varphi) \quad \xrightarrow{\hat{x}} \quad \hat{x}^\ast.
\]

Thanks to Tomita-Takesaki theory, \( S^0_\varphi \) is closable and we denote by \( S_\varphi \) its closure. Write \( S_\varphi = J_\varphi \Delta_\varphi^{1/2} \) for its polar decomposition. The modular automorphism group \( (\sigma^t_\varphi) \) of \( \mathcal{M} \) w.r.t. the state \( \varphi \) is defined by \( \sigma^t_\varphi = \Delta_\varphi^{it} \cdot \Delta_\varphi^{-it} \). Moreover, note that \( L^2(\mathcal{M}, \varphi) \) comes naturally equipped with a structure of \( \mathcal{M} \)-\( \mathcal{M} \) bimodule:

\[
x \cdot \xi := x \xi, \quad \xi \cdot x := J_\varphi x^* J_\varphi \xi, \forall \xi \in L^2(\mathcal{M}, \varphi), \forall x \in \mathcal{M}.
\]

In the sequel, we shall simply write \( x \xi \) and \( \xi x \) instead of \( x \cdot \xi \) and \( \xi \cdot x \).

Denote by \( \text{Sp}(\mathcal{M}, \varphi) \) the point spectrum of the modular operator \( \Delta_\varphi \). For \( \gamma \in \text{Sp}(\mathcal{M}, \varphi) \), denote by \( \mathcal{M}^\gamma \) the vector subspace of \( \mathcal{M} \) of all \( \gamma \)-eigenvectors for \( \varphi \), i.e.

\[
\mathcal{M}^\gamma = \{ x \in \mathcal{M} : \sigma^t_\varphi(x) = \gamma^t x \} = \{ x \in \mathcal{M} : \varphi(xy) = \gamma \varphi(yx), \forall y \in \mathcal{M} \}.
\]

Denote by \( \mathcal{M}_{\text{alg}} = \text{span}\{ \mathcal{M}^\gamma : \gamma \in \text{Sp}(\mathcal{M}, \varphi) \} \), the linear span of the \( \mathcal{M}^\gamma \)'s. It is clear that \( \mathcal{M}_{\text{alg}} \) is a unital \( * \)-subalgebra of \( \mathcal{M} \). Since the state \( \varphi \) is assumed to be almost periodic, \( \mathcal{M}_{\text{alg}} \) is \( \sigma \)-weakly dense in \( \mathcal{M} \). We can also write:

\[
L^2(\mathcal{M}, \varphi) = \bigoplus_{\gamma \in \text{Sp}(\mathcal{M}, \varphi)} L^2(\mathcal{M}^\gamma).
\]

It is straightforward to check that if \( \gamma \in \text{Sp}(\mathcal{M}, \varphi) \) and \( \xi \in L^2(\mathcal{M}^\gamma) \), then for any \( x \in \mathcal{M}^\lambda \), \( x \xi \) and \( \xi x \in L^2(\mathcal{M}^{\lambda \gamma}) \).

The \( L^2 \)-norm w.r.t. the state \( \varphi \), simply denoted by \( \| \cdot \|_2 \), is defined as follows: \( \| x \|_2 = \varphi(x^* x)^{1/2} \), for any \( x \in \mathcal{M} \). Remind that the topology given by the norm \( \| \cdot \|_2 \) coincides with the strong topology on bounded sets of \( \mathcal{M} \).

For \( a \in \mathcal{M} \), set

\[
L_a : L^2(\mathcal{M}, \varphi) \to L^2(\mathcal{M}, \varphi) \quad \xrightarrow{b} \quad \hat{a} b.
\]

The operator \( L_a \) is always bounded and \( \| L_a \| = \| a \| \). For \( b \in \mathcal{M} \), set

\[
R_b : L^2(\mathcal{M}, \varphi) \to L^2(\mathcal{M}, \varphi) \quad \xrightarrow{a} \quad \hat{a} \hat{b}.
\]

One must pay attention to the fact that the operator \( R_b \) is unbounded in general. However, if \( b \in \mathcal{M}^\gamma \) for some \( \gamma \in \text{Sp}(\mathcal{M}, \varphi) \), the Tomita-Takesaki theory claims that

\[
R_b = J_{\varphi} L_{a_{\gamma,2}(b^*)} J_{\varphi} = \gamma^{-1/2} J_{\varphi} L_{b^*} J_{\varphi}.
\]

We refer for example to Lemma VIII.3.18 in [33] for further details. Consequently, in this case, \( R_b \) is bounded and \( \| R_b \| = \gamma^{-1/2} \| b \| \). Thus, for any \( x, z \in \mathcal{M} \), and for any \( y \in \mathcal{M}^\gamma \), we have

\[
\| xzy \|_2 = \| L_x R_y (z) \|_2 = \gamma^{-1/2} \| L_x J_{\varphi} L_y J_{\varphi}(z) \|_2 \leq \gamma^{-1/2} \| x \| \| y \| \| z \|_2.
\]

We shall repeatedly use this inequality in the sequel.
2.2. Hilbert (Bi)modules and the Basic Construction. We reproduce here Appendix A of Vaes’ Bourbaki seminar. Let \((\mathcal{N}, \varphi)\) be a von Neumann algebra endowed with an almost periodic state, and let \(\mathcal{B} \subset \mathcal{N}\) be a von Neumann subalgebra globally invariant under the modular group \((\sigma_t^\mathcal{N})\). We regard \(\mathcal{N} \subset B(L^2(\mathcal{N}, \varphi))\) through the GNS construction. We shall simply denote \(L^2(\mathcal{N}, \varphi)\) by \(L^2(\mathcal{N})\). We denote by \(E_B : \mathcal{N} \to \mathcal{B}\) the unique state-preserving conditional expectation (see [33] for further details). The basic construction \((\mathcal{N}, \varepsilon_B)\) is defined as the von Neumann subalgebra of \(B(L^2(\mathcal{N}))\) generated by \(\mathcal{N}\) and the orthogonal projection \(\varepsilon_B\) of \(L^2(\mathcal{N})\) onto \(L^2(\mathcal{B}) \subset L^2(\mathcal{N})\). The relationship between \(\varepsilon_B\) and \(E_B\) is as follows:

\[
\varepsilon_B x \varepsilon_B = E_B(x) \varepsilon_B, \forall x \in \mathcal{N}.
\]

It can be checked that \((\mathcal{N}, \varepsilon_B)\) consists of those operators \(T \in B(L^2(\mathcal{N}))\) that commute with the right module action of \(\mathcal{B}\): \(T(\xi b) = T(\xi) b, \forall \xi \in L^2(\mathcal{N}), \forall b \in \mathcal{B}\). In other words,

\[
\langle \mathcal{N}, \varepsilon_B \rangle = (J_{\varphi} B J_{\varphi})' \cap B(L^2(\mathcal{N})).
\]

The basic construction comes equipped with a canonical normal semifinite faithful weight \(\widehat{\varphi}_B\) which satisfies

\[
\widehat{\varphi}_B(x \varepsilon_B y) = \varphi(xy), \forall x, y \in \mathcal{N}.
\]

It can be checked that \(\widehat{\varphi}_B\) is also almost periodic and \(\text{Sp}(\langle \mathcal{N}, \varepsilon_B \rangle, \widehat{\varphi}_B) = \text{Sp}(\mathcal{N}, \varphi)\).

Let \((\mathcal{B}, \tau)\) be any finite von Neumann algebra endowed with a faithful normal trace \(\tau\). Let \(\mathcal{K}\) be a right \(\mathcal{B}\)-module. Denote by \(\mathcal{B}'\) the commutant of \(\mathcal{B}\) on \(\mathcal{K}\), i.e. \(\mathcal{B}'\) consists of the operators \(T \in B(\mathcal{K})\) that commute with the right module action of \(\mathcal{B}\). One can construct a faithful semifinite normal positive linear map

\[
E' : (\mathcal{B}')_+ \to \{\text{positive self-adjoint operators affiliated with } Z(\mathcal{B})\}
\]

satisfying \(E'(x^* x) = E'(xx^*), \forall x \in \mathcal{B}'\). Moreover, whenever \(T : L^2(\mathcal{B}) \to \mathcal{K}\) is bounded and right \(\mathcal{B}\)-linear, \(T^* T \in \mathcal{B}, TT^* \in \mathcal{B}'\), and we have

\[
E'(TT^*) = E_{Z(\mathcal{B})}(T^* T).
\]

The positive self-adjoint operator \(E'(1_{\mathcal{B}'})\) affiliated with the center \(Z(\mathcal{B})\) provides a complete invariant for right \(\mathcal{B}\)-modules. It should be noted here that the right \(\mathcal{B}\)-module \(\mathcal{K}\) is finitely generated, i.e. of the form \(pL^2(\mathcal{B})^{\oplus n}\) for some projection \(p \in M_n(\mathcal{C}) \otimes \mathcal{B}\), iff \(E'(1_{\mathcal{B}'})\) is bounded. In that case, \(E'(1_{\mathcal{B}'}) = (\text{tr}_n \otimes E_{Z(\mathcal{B})})(p)\). Write \(\text{Tr} = \tau \circ E'\). It follows that \(\text{Tr}\) is a faithful semifinite normal trace on \(\mathcal{B}'\). If \(\text{Tr}(1_{\mathcal{B}'}) < \infty\), we shall say that the right \(\mathcal{B}\)-module \(\mathcal{K}\) is of finite trace over \(\mathcal{B}\). Thus, \(E'(1_{\mathcal{B}'})\) is not bounded a priori but is \(\tau\)-integrable. This implies that \(E'(1_{\mathcal{B}'})z\) is bounded for projections \(z \in Z(\mathcal{B})\) with trace arbitrary close to 1. So, we have the following lemma:

**Lemma 2.1.** Let \(\mathcal{K}\) be a right \(\mathcal{B}\)-module of finite trace over \(\mathcal{B}\). Then, for any \(\varepsilon > 0\), there exists a projection \(z \in Z(\mathcal{B})\) with \(\tau(1 - z) \leq \varepsilon\), and such that the right \(\mathcal{B}\)-module \(\mathcal{K}z\) is finitely generated over \(\mathcal{B}\).

Let’s come back to the basic construction for the inclusion \(\mathcal{B} \subset \mathcal{N}\), and assume now that \(\mathcal{B} \subset \mathcal{N}^\varphi\). We observe that the restriction of \(\varphi\) to \(\mathcal{B}\) defines a tracial state and \(\langle \mathcal{N}, \varphi_B \rangle\) is precisely the commutant of \(\mathcal{B}\) on the right \(\mathcal{B}\)-module \(L^2(\mathcal{N})\). Using the previous paragraph, \(\langle \mathcal{N}, \varphi_B \rangle\) comes equipped with a canonical faithful semifinite normal trace \(\text{Tr}\). For \(\gamma \in \text{Sp}(\mathcal{N}, \varphi)\), denote by \(p_\gamma\) the orthogonal projection of \(L^2(\mathcal{N})\) onto \(L^2(\mathcal{N}^\gamma)\). Let \(\lambda \in \text{Sp}(\mathcal{N}, \varphi)\) and \(\xi \in L^2(\mathcal{N}^\lambda)\). Since \(\mathcal{B} \subset \mathcal{N}^\varphi\), one has \(\xi b \in L^2(\mathcal{N}^\lambda)\), and so \(p_\gamma(\xi b) = \xi b\).
\[ \delta_{\gamma,x} b = p(x)b. \] Consequently, \( p_\gamma \in \langle N, e_B \rangle \). One can check that
\[
\hat{\varphi}_B(x) = \sum_{\gamma \in \text{Sp}(N, \varphi)} \hat{\varphi}_B(p_\gamma xp_\gamma)
\]
\[
\text{Tr}(x) = \sum_{\gamma \in \text{Sp}(N, \varphi)} \gamma^{-1} \hat{\varphi}_B(p_\gamma xp_\gamma), \forall x \in \langle N, e_B \rangle_+.
\]

In particular, on \( p_\gamma \langle N, e_B \rangle p_\gamma \), \( \hat{\varphi}_B \) is tracial and a multiple of \( \text{Tr} \), \( \forall \gamma \in \text{Sp}(N, \varphi) \). In Section 4, we shall encounter the following situation. Let \( p \in p_\gamma \langle N, e_B \rangle p_\gamma \) be a projection such that \( \hat{\varphi}_B(p) < \infty \). Let \( K \) be the right \( B \)-module defined by \( K = pL^2(N) \). We can check that the commutant of \( B \) on \( K \) is exactly \( p\langle N, e_B \rangle p \). Since \( \text{Tr}(p) = \gamma^{-1} \hat{\varphi}_B(p) < \infty \), it follows that as a right \( B \)-module, \( K \) is of finite trace over \( B \). Thus, one can apply Lemma 2.1.

### 2.3. Free Araki-Woods Factors of Shlyakhtenko

Let \( H_{\mathbb{R}} \) be a real Hilbert space and let \( (U_t) \) be an orthogonal representation of \( \mathbb{R} \) on \( H_{\mathbb{R}} \). Let \( H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C} \) be the complexified Hilbert space. Let \( A \) be the infinitesimal generator of \( (U_t) \) on \( H_{\mathbb{C}} \). Define another inner product on \( H_{\mathbb{C}} \) by
\[
\langle \xi,\eta \rangle_U = \left\langle \frac{2}{1 + A^{-1}}, \xi,\eta \right\rangle.
\]

Note that for any \( \xi \in H_{\mathbb{R}}, ||\xi||_U = ||\xi|| \); also, for any \( \xi,\eta \in H_{\mathbb{R}}, \Re(\langle \xi,\eta \rangle_U) = \langle \xi,\eta \rangle \). We refer to [32] for the main properties of this inner product. Denote by \( H \) the completion of \( H_{\mathbb{C}} \) for this new inner product. Introduce now the full Fock space of \( H \):
\[
\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^\otimes n.
\]

The unit vector \( \Omega \) is called the vacuum vector. For any \( \xi \in H \), we have the left creation operator
\[
l(\xi) : \mathcal{F}(H) \to \mathcal{F}(H) : \left\{ \begin{array}{ll}
l(\xi)\Omega = \xi, \\
l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.
\end{array} \right.
\]

For any \( \xi \in H \), we denote by \( s(\xi) \) the real part of \( l(\xi) \) given by
\[
s(\xi) = \frac{l(\xi) + l(\xi)^*}{2}.
\]

The crucial result of Voiculescu [37] claims that the distribution of the operator \( s(\xi) \) w.r.t. the vacuum vector state \( \varphi_U(x) = \langle x\Omega, \Omega \rangle_U \) is the semicircular law of Wigner supported on the interval \([\|\xi\|, \|\xi\|]\).

**Definition 2.2** (Shlyakhtenko, [32]). Let \( (U_t) \) be an orthogonal representation of \( \mathbb{R} \) on the real Hilbert space \( H_{\mathbb{R}} \) (\( \dim H_{\mathbb{R}} \geq 2 \)). The free Araki-Woods factor associated with \( H_{\mathbb{R}} \) and \( (U_t) \), denoted by \( \Gamma(H_{\mathbb{R}}, U_t)^\prime\prime \), is defined by
\[
\Gamma(H_{\mathbb{R}}, U_t)^\prime\prime = \{ s(\xi) : \xi \in H_{\mathbb{R}} \}^\prime\prime.
\]

The vector state \( \varphi_U(x) = \langle x\Omega, \Omega \rangle_U \) is called the free quasi-free state.

The free Araki-Woods factors provided many new examples of full factors of type III [1, 7, 29]. We can summarize some of their general properties in the following theorem (see also Vaes’ Bourbaki seminar [35]):

**Theorem 2.3** (Shlyakhtenko, [29, 30, 31, 32]). Let \( (U_t) \) be an orthogonal representation of \( \mathbb{R} \) on the real Hilbert space \( H_{\mathbb{R}} \) with \( \dim H_{\mathbb{R}} \geq 2 \). Denote by \( \mathcal{N} = \Gamma(H_{\mathbb{R}}, U_t)^\prime\prime \).
(1) \( \mathcal{N} \) is of type II\(_1\) if and only if \( U_t = id \) for every \( t \in \mathbb{R} \).

(2) \( \mathcal{N} \) is of type III\(_\lambda\) (\( 0 < \lambda < 1 \)) if and only if \( (U_t) \) is periodic of period \( \frac{2\pi}{\log \lambda} \).

(3) \( \mathcal{N} \) is of type III\(_1\) in the other cases.

(4) If \( (U_t) \) is almost periodic, then \( \varphi_U \) is an almost periodic state.

Remark 2.4 \((\cite{32})\). Explicitly the value of \( \varphi_U \) on a word in \( s(h_i) \) is given by

\[
\varphi_U(s(h_1) \cdots s(h_n)) = 2^{-n} \sum_{(\{\beta_i, \gamma_i\}) \in NC(n), \beta_i \leq \gamma_i} \prod_{k=1}^{n/2} (h_{\beta_k}, h_{\gamma_k})_U.
\]

for \( n \) even and is zero otherwise. Here \( NC(2p) \) stands for all the noncrossing pairings of \( \{1, \ldots, 2p\} \), i.e. pairings for which whenever \( a < b < c < d \), and \( a, c \) are in the same class, then \( b, d \) are not in the same class. The total number of such pairings is given by the \( p \)-th Catalan number

\[
C_p = \frac{1}{p+1} \binom{2p}{p}.
\]

In the almost periodic case, using a powerful tool called the matricial model, Shlyakhtenko obtained the following remarkable result:

Theorem 2.5 \((\text{Shlyakhtenko, } \cite{29, 32})\). Let \( (U_t) \) be a nontrivial almost periodic orthogonal representation of \( \mathbb{R} \) on the real Hilbert space \( H_R \) with \( \dim H_R \geq 2 \). Let \( A \) be the infinitesimal generator of \( (U_t) \) on \( H_C \), the complexified Hilbert space of \( H_R \). Denote by \( \mathcal{N} = \Gamma(H_R, U_t)^0 \). Let \( \Gamma \subset \mathbb{R}^*_+ \) be the subgroup generated by the point spectrum of \( A \). Then, \( \mathcal{N} \) only depends on \( \Gamma \) up to state-preserving isomorphisms.

Conversely, the group \( \Gamma \) coincides with the \( Sd \) invariant of the factor \( \mathcal{N} \). Consequently, \( Sd \) completely classifies the almost periodic free Araki-Woods factors. Moreover, the centralizer of the free quasi-free state \( \varphi_U \) is isomorphic to the type II\(_1\) factor \( L(F_\infty) \).

Let \( K_R \) be an infinite dimensional separable real Hilbert space, and let \( 0 < \lambda < 1 \). We define on \( K_R \oplus K_R \) the following one-parameter family of orthogonal transformations:

\[
U_t^\lambda = \begin{pmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{pmatrix}.
\]

Notation 2.6. Write \( (T_\lambda, \varphi_\lambda) := (H_R, U_t^\lambda)^\nu \) where \( (U_t^\lambda) \) is given by Equation (3). It is (up to state-preserving isomorphism) the only free Araki-Woods factor of type III\(_\lambda\).

Notation 2.7. More generally, for any nontrivial countable subgroup \( \Gamma \subset \mathbb{R}^*_+ \), we shall denote by \( (T_\Gamma, \varphi_\Gamma) \) the unique (up to state-preserving isomorphism) almost periodic free Araki-Woods factor whose \( Sd \) invariant is exactly \( \Gamma \). Of course, \( \varphi_\Gamma \) is its free quasi-free state. If \( \Gamma = \lambda \mathbb{Z} \) for \( \lambda \in ]0, 1[ \), then \( (T_\Gamma, \varphi_\Gamma) \) is of type III\(_\lambda\); in this case, it will be simply denoted by \( (T_\lambda, \varphi_\lambda) \), as in Notation 2.6. Theorem 6.4 in \( \cite{32} \) gives the following formula:

\[
(T_\Gamma, \varphi_\Gamma) \cong \bigoplus_{\gamma \in \Gamma} (T_{\gamma}, \varphi_{\gamma}).
\]

2.4. Haagerup Property for Groups and Finite von Neumann Algebras. Remind that a countable group \( G \) is said to have the Haagerup property if there exists a sequence \( (\varphi_n) \) of normalized (i.e. \( \varphi_n(1) = 1 \), \( \forall n \in \mathbb{N} \)) positive definite functions on \( G \) such that each \( \varphi_n \) vanishes at infinity and \( \lim_{n \to \infty} \varphi_n(g) = 1, \forall g \in G \). This property was proven by Haagerup in \( \cite{12} \) for the free groups \( F_n, 2 \leq n \leq \infty \). Other examples include \( \text{SL}(2, \mathbb{Z}) \), and more generally \( \text{SL}(2, F) \) for any number field \( F \) (see \( \cite{3} \) for a more comprehensive list of groups with the Haagerup property).
This notion can be extended to finite von Neumann algebras. Let \((N, \tau)\) be a finite von Neumann algebra. Let \(\phi : N \to N\) be a completely positive map, and assume that there exists \(c > 0\) such that \(\tau \circ \phi \leq c\tau\). Let \(T_\phi\) be the linear operator on \(L^2(N, \tau)\) defined by

\[
T_\phi x = \widehat{\phi(x)}, \forall x \in N.
\]

We can check that \(T_\phi\) is bounded and precisely \(\|T_\phi\| \leq c\|\phi(1)\|\). We have the following definition:

**Definition 2.8** (Choda, [3]). Let \(N\) be a finite von Neumann algebra. The von Neumann algebra \(N\) is said to have the **Haagerup property** if there exist a faithful normal trace \(\tau\) on \(N\) and a sequence of normal completely positive maps \(\phi_n : N \to N\) such that

1. \(\tau \circ \phi_n \leq \tau, \phi_n(1) \leq 1, \forall n \in \mathbb{N};\)
2. the corresponding operator \(T_{\phi_n}\) on \(L^2(N, \tau)\) is compact, \(\forall n \in \mathbb{N};\)
3. \(\|\phi_n(x) - x\|_2 \to 0, \forall x \in N.\)

It was shown by Jolissaint in [17] that this property does not depend on the faithful normal trace on \(N\), i.e. if \(N\) has the Haagerup property, then for any faithful normal trace \(\tau\) on \(N\), there exists a sequence \((\phi_n)\) of completely positive maps \(\phi_n : N \to N\) such that conditions \((1 - 3)\) are satisfied. It was proven in [4] that a group \(G\) has the Haagerup property iff \(L(G)\) has. If \(N\) has the Haagerup property, and \(p\) is a nonzero projection in \(N\), then \(pNp\) has the Haagerup property.

Any amenable finite von Neumann algebra has the Haagerup property. Moreover, any interpolated free group factor \(L(F_t)\) \((1 < t \leq \infty)\) has the Haagerup property [3][27].

### 2.5. Relative Property \((T)\) for Groups and Finite von Neumann Algebras

Let \(G\) be a countable discrete group and let \(H \subset G\) be a subgroup. The pair \((G, H)\) is said to have the **relative property** \((T)\) if one of the following equivalent conditions is satisfied [13]:

- Any unitary representation \(\pi\) of \(G\) which has almost invariant vectors, has a nonzero \(H\)-invariant vector.
- Whenever a sequence \((\varphi_n)\) of normalized, positive definite functions on \(G\) converges to 1, then it converges to 1 uniformly on \(H\).

A group \(G\) has **property** \((T)\) if the pair \((G, G)\) has the relative property \((T)\). As we mentioned in the introduction, the pair \((\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z}), \mathbb{Z}^2)\) has the relative property \((T)\), and more generally the pair \((\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)\) has the relative property \((T)\), for any non-amenable subgroup \(\Gamma \subset \text{SL}(2, \mathbb{Z})\) (see [2][28]). Other examples were given by Valette [36] and Fernós [10]. Remind that \(\text{SL}(n, \mathbb{Z})\) has property \((T)\), for every \(n \geq 3\). Clearly, relative property \((T)\) is an obstruction to the Haagerup property; more precisely if the group \(G\) contains an infinite subgroup \(H\) with the relative property \((T)\), then \(G\) cannot have the Haagerup property.

Property \((T)\) was defined by Connes and Jones in [5] for type II\(_1\) factors. More generally, in [28], Popa defined the relative property \((T)\) for inclusions of finite von Neumann algebras. It naturally involves the language of correspondences and completely positive maps.

**Terminology 2.9.** Let \((P, \tau)\) be a finite von Neumann algebra endowed with a faithful normal trace. Let \(\mathcal{H}\) be a \(P\)-\(P\) bimodule. Let \((\xi_n)\) be a sequence of unit vectors in \(\mathcal{H}\). We say that

- \((\xi_n)\) is almost central if \(\|x\xi_n - \xi_n x\| \to 0, \forall x \in P.\)
- \((\xi_n)\) is almost \(\tau\)-tracial if \(\|\langle \xi_n, \xi_n \rangle - \tau\| \to 0\) and \(\|\langle \xi_n, \xi_n \rangle - \tau\| \to 0.\)

Let \(Q \subset P\) be a von Neumann subalgebra. A unit vector \(\xi \in \mathcal{H}\) is said to be \(Q\)-central if \(x\xi = \xi x, \forall x \in Q.\)
Definition 2.10 (Popa, [23]). Let $P$ be a finite von Neumann algebra. Let $Q \subset P$ be a von Neumann subalgebra. Denote by $(Q)_1$ the unit ball of $Q$ (w.r.t. the operator norm). The inclusion $Q \subset P$ is said to be rigid or to have the relative property $(T)$ if one of the following equivalent conditions holds:

1. There exists a faithful normal trace $\tau$ on $P$, such that for any $P$-$P$ bimodule $\mathcal{H}$, if $\mathcal{H}$ contains a sequence of almost central, almost $\tau$-tracial vectors, then it contains a sequence of $Q$-central, almost $\tau$-tracial vectors.
2. There exists a faithful normal trace $\tau$ on $P$, such that for any sequence $(\phi_n)$ of normal completely positive maps $\phi_n : P \to \mathcal{L}(\mathcal{K})$, such that for any $n \in \mathbb{N}$, $\phi_n(1) \leq 1$, $\tau \circ \phi_n \leq \tau$, the following holds true:

$$\text{if } \forall x \in P, \|\phi_n(x) - x\|_2 \to 0, \text{ then } \sup_{x \in (Q)_1} \|\phi_n(x) - x\|_2 \to 0.$$  

3. Condition (1) above is satisfied for any faithful normal trace $\tau$ on $P$.
4. Condition (2) above is satisfied for any faithful normal trace $\tau$ on $P$.

The von Neumann algebra $P$ is said to have property $(T)$ if the inclusion $P \subset P$ is rigid.

It was proven in [23] that the pair $(G,H)$ has the relative property $(T)$ iff the inclusion $L(H) \subset L(G)$ is rigid. This notion of rigid inclusion behaves well w.r.t. compressions. Namely, take $q \in Q$ a nonzero projection. If the inclusion $Q \subset P$ is rigid, then the inclusion $qQq \subset qPq$ is rigid [23]. Finally, we remind the following theorem which will be needed in Section 3 for finite von Neumann algebras, relative property $(T)$ is an obstruction to Haagerup property.

Theorem 2.11 (Popa, [24]). Let $P,Q$ be finite von Neumann algebras. Assume that $Q$ is diffuse and $Q \subset P$ is a rigid inclusion. Then $P$ cannot have the Haagerup property.

3. Main Examples of (S-)Malleable Freely Mixing Actions

3.1. Bogoliubov Shifts on the Almost Periodic Free Araki-Woods Factors. Let $G$ be any countable discrete group, let $H_R$ be a separable real Hilbert space of infinite dimension. Let $\pi : G \to O(H_R)$ be an orthogonal representation. Let $(U_t)$ be an almost periodic orthogonal representation of $R$ on $H_R$ and denote by $(N,\varphi) = (\Gamma(H_R,U_t)^\gamma,\varphi_U)$ the associated free Araki-Woods factor. We shall always assume that $\pi$ and $(U_t)$ commute. Remind that the representation $\pi$ is said to be $C_0$, if for any $\xi,\eta \in H_R$, $\langle \pi(g)\xi,\eta \rangle \to 0$, as $g \to \infty$. The following construction gives an example where $\pi$ and $(U_t)$ commute.

Example 3.1. Let $\rho : G \to O(K_R)$ be any orthogonal representation of $G$ on an infinite dimensional separable real Hilbert space $K_R$. Let $\Gamma \subset R_+^\times$ be a countable subgroup. Let $(U_t^\Gamma)$ be the orthogonal representation on $H_R^\Gamma = \bigoplus_{\gamma \in \Gamma} (K_R \oplus K_R)$ defined by $U_t^\Gamma = \bigoplus_{\gamma \in \Gamma} U_t^\gamma$ (see Notation [2,4]). We know that $(T_\Gamma,\varphi_T^\Gamma) \cong (\Gamma(H_R,U_t^\Gamma)^\gamma,\varphi_{U_t^\Gamma})$. Consider now $\pi = \bigoplus_{\gamma \in \Gamma} (\rho \oplus \rho)$ on $H_R$. It is clear that the representations $\pi$ and $(U_t^\Gamma)$ commute. Note that if the representation $\rho$ is $C_0$, then $\pi$ is $C_0$.

Denote by $H_C$ the complexified Hilbert space of $H_R$. The complexified representations are still denoted by $\pi$ and $(U_t)$. Denote by $A$ the infinitesimal generator of $(U_t)$ on $H_C$:

$$U_t = A t, \forall t \in R.$$
Proposition 3.3. From [32], we know that \( \Gamma(H) \). Thus the (bounded) operator \[ \frac{2}{1+A} \] belongs to \( \pi(G) \). Consequently, for any \( g \in G \) and any \( \xi, \eta \in H_C \), we have
\[
\langle \pi(g)\xi, \pi(g)\eta \rangle_U = \langle \frac{2}{1+A} \pi(g)\xi, \pi(g)\eta \rangle_U \\
= \langle \frac{2}{1+A} \xi, \eta \rangle_U \\
= \langle \xi, \eta \rangle_U.
\]
Thus, the representation \( \pi \) is unitary w.r.t. the inner product \( \langle \cdot, \cdot \rangle_U \). Note that if \( \pi \) is \( C_0 \) w.r.t. the inner product \( \langle \cdot, \cdot \rangle \), then \( \pi \) is still \( C_0 \) w.r.t. the inner product \( \langle \cdot, \cdot \rangle_U \). Denote as before, by \( H \) the completion of \( H_C \) w.r.t. \( \langle \cdot, \cdot \rangle_U \). For any \( g \in G \), set
\[
w_g = 1 \oplus \bigoplus_{n \geq 1} \pi(g)^{\otimes n}.
\]
It is clear that for every \( g \in G \), \( w_g \) is a unitary of \( \mathcal{F}(H) \), the full Fock space of \( H \). For every \( g \in G \), write \( \sigma^g = \text{Ad}(w_g) \). If no confusion is possible, \( \sigma^g \) will be simply denoted by \( \sigma \). It is straightforward to check that for any \( \xi \in H \), \( \sigma_g(l(\xi)) = l(\pi(g)\xi) \). Therefore, \( (\sigma_g) \) defines an action on \( (\mathcal{N}, \varphi) := (\Gamma(H_R \oplus U_t), \varphi_U) \), called the free Bogoliubov shift. Moreover, since \( w_g \Omega = \Omega, \forall g \in G \), the action \( (\sigma_g) \) is \( \varphi \)-preserving.

**Proposition 3.2.** The action \( (\sigma_g) \) is \( s \)-malleable for \( \varphi \).

**Proof.** From [32], we know that \( \Gamma(H_R \oplus H_R \oplus U_t) \) \( = (\mathcal{N}, \varphi) * (\mathcal{N}, \varphi) \). Consider on the real Hilbert space \( H_R \oplus H_R \), the following family of orthogonal elements:
\[
V_t = \begin{pmatrix}
\cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\
\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)
\end{pmatrix}, \forall t \in \mathbb{R}.
\]
It is clearly an orthogonal representation of \( \mathbb{R} \) on \( H_R \oplus H_R \). Consider now the canonical action \( (\alpha_t) \) on \( (\mathcal{N}, \varphi) * (\mathcal{N}, \varphi) \) associated with \( (V_t) \):
\[
\alpha_t(s(\xi \eta)) = s(V_t \left( \begin{array}{c}
\xi \\
\eta
\end{array} \right)), \forall t \in \mathbb{R}, \forall \xi, \eta \in H_R.
\]
We can easily see that \( (V_t) \) commutes with \( (U_s \oplus U_s) \) and with \( \pi \oplus \pi \); consequently, the action \( (\alpha_t) \) is \( \varphi \)-\( \varphi \)-preserving and commute with the diagonal action \( (\sigma_x * \sigma_y) \). Moreover, \( \alpha_1(a \ast 1) = 1 \ast a \), for every \( a \in \mathcal{N} \). At last, consider the automorphism \( \beta \) defined on \( (\mathcal{N}, \varphi) * (\mathcal{N}, \varphi) \) by:
\[
\beta(s(\xi \eta)) = s(\left( \begin{array}{c}
\xi \\
-\eta
\end{array} \right)), \forall \xi, \eta \in H_R.
\]
It is straightforward to check that \( \beta \) commutes with the diagonal action \( (\sigma_x * \sigma_y) \), \( \beta^2 = \text{Id}, \) \( (\mathcal{N} \ast \mathbb{C}) \subset (\mathcal{N} \ast \mathcal{N})^\beta \), and \( \beta \alpha_t = \alpha_{-t} \beta, \forall t \in \mathbb{R} \). We are done.

**Proposition 3.3.** The action \( (\sigma_g) \) is freely mixing for the state \( \varphi \) if and only if the representation \( \pi \) is \( C_0 \).

**Proof.** Assume first that \( (\sigma_g) \) is freely mixing. Define \( x = s(\xi), y = s(\eta) \) for \( \xi, \eta \in H_R \). We know that \( \varphi(x) = \varphi(y) = 0 \). Moreover, we have
\[
\langle \pi(g)\xi, \eta \rangle = 4 \varphi(s(\pi(g)\xi)s(\eta)) \\
= 4 \varphi(s(\pi(g)\xi)s(\eta)) \\
= 4 \varphi(s(\pi(g)\xi)s(\eta)).
\]
Consequently, \( \langle \pi(g)\xi, \eta \rangle \to 0 \), as \( g \to \infty \) and the representation \( \pi \) is \( C_0 \).

Conversely, assume now that the representation \( \pi \) is \( C_0 \). We have to prove that for any \( n \in \mathbb{N}^* \), and for any \( x_1, \ldots, x_n, y_0, \ldots, y_n \in \mathcal{N} \oplus \mathbb{C} \), except possibly \( y_0 \) and/or \( y_n \) are equal to 1,

\[
\lim_{g_1, \ldots, g_n \to \infty} \varphi(y_0\sigma_{g_1}(x_1)y_1 \cdots \sigma_{g_n}(x_n)y_n) = 0.
\]

It suffices to show this property for \( x_i \) and \( y_j \) words in \( s(h_i) \). For \( 1 \leq i \leq n \) and \( 0 \leq j \leq n \), take

\[
x_i = s(\xi_1^i) \cdots s(\xi_{r_i}^i),
\]

\[
y_j = s(\eta_1^j) \cdots s(\eta_{s_j}^j).
\]

Then for any \( 1 \leq i \leq n \) and any \( g_1, \ldots, g_n \in G \),

\[
\sigma_{g_i}(x_i) = s(\pi(g_i)\xi_1^i) \cdots s(\pi(g_i)\xi_{r_i}^i).
\]

For \( 1 \leq i \leq n \) and \( 0 \leq j \leq n \), define

\[
K_{2i-1} = \{1, \ldots, r_i\}, \quad K_{2j} = \{1, \ldots, s_j\}.
\]

Let \( m = \sum_i r_i + \sum_j s_j \). We will use the following identification

\[
\{1, \ldots, m\} = K_0 \sqcup K_1 \sqcup K_2 \sqcup \cdots \sqcup K_{2n-1} \sqcup K_{2n}.
\]

If \( m \) is odd, there is nothing to prove. If \( m \) is even, then from Equation (2), we know that

\[
(4) \quad \varphi(y_0\sigma_{g_1}(x_1)y_1 \cdots \sigma_{g_n}(x_n)y_n) = 2^{-m} \sum_{\{\beta_i, \gamma_i\} \in NC(m), \beta_i < \gamma_i} \prod_{k=1}^{m/2} \langle h_{\beta_k}, h_{\gamma_k} \rangle_U,
\]

where the letter \( h \) stands for \( \eta \) or \( \pi(g)\xi \). Let \( \nu = \{\{\beta_i, \gamma_i\}\} \in NC(m) \). Write \( z_{2i-1} = x_i, z_{2j} = y_j \), for \( i \in \{1, \ldots, n\}, j \in \{0, \ldots, n\} \).

(a) Either there exist for the noncrossing pairing \( \nu \), some \( i \in \{1, \ldots, n\}, j \in \{0, \ldots, n\} \), \( p \in K_{2i-1} \), and \( q \in K_{2j} \) such that \( p, q \) are in the same class. Thus, the inner product \( \langle \pi(g_i)\xi_1^i, \eta_1^j \rangle_U \) or \( \langle \pi(g_i)\xi_p^i, \eta_q^j \rangle_U \) necessarily appears in the product \( \prod_{k=1}^{m/2} \langle h_{\beta_k}, h_{\gamma_k} \rangle_U \) of Equation (4). Since the representation \( \pi \) is \( C_0 \), \( \langle \pi(g_i)\xi_1^i, \eta_1^j \rangle_U \to 0 \), as \( g_i \to \infty \).

(b) Or for any \( p, q \in \{1, \ldots, m\} \), if \( p, q \) are in the same class, then it means that necessarily \( p \in K_{2i-1}, q \in K_{2q-1} \) or \( p \in K_{2j}, q \in K_{2q-1} \) for some \( i, i', j, j' \). Then there must exist a maximal \( r \geq 1 \) and integers \( 0 \leq k_1 < \cdots < k_r \leq 2n \), such that when one restricts the noncrossing pairing \( \nu \) to the subsets \( K_{k_1}, \ldots, K_{k_r} \), one still has a noncrossing pairing on each of those subsets. Now if we sum up over all the noncrossing pairings \( \nu' \) of \( \{1, \ldots, m\} \) such that \( \nu' \) and \( \nu \) agree on \( \{1, \ldots, m\}\backslash (K_{k_1} \sqcup \cdots \sqcup K_{k_r}) \), and \( \nu' \) can be any noncrossing pairing on each of the subsets \( K_{k_1}, \ldots, K_{k_r} \), we obtain

\[
(5) \quad 2^{-m} \sum_{\nu'} \prod_{k=1}^{m/2} \langle h_{\beta_k}, h_{\gamma_k} \rangle_U = C \varphi(z_{k_1}) \cdots \varphi(z_{k_r}),
\]

with the constant \( C \) given by

\[
C = 2^{-m'} \prod_{\{(\beta_k, \gamma_k)\} \in \nu'} \langle h_{\beta_k}, h_{\gamma_k} \rangle_U,
\]
where $\nu^c$ denotes the restriction of $\nu$ on $\{1, \ldots, m\}\setminus(K_{k_1} \sqcup \cdots \sqcup K_{k_r})$ and $m'$ is the cardinality of $\{1, \ldots, m\}\setminus(K_{k_1} \sqcup \cdots \sqcup K_{k_r})$. By choice of $x_i, y_j$, we have $\varphi(z_{k_i}) = \cdots = \varphi(z_{k_r}) = 0$, so that the sum in Equation (3) is 0. Finally if we sum up over all the noncrossing pairings, according to (a) and (b), we are done. \hfill \Box

For further applications (see Section 5), we shall always take $\pi = \lambda_G$ the left regular representation of $G$ which is $C_0$ as soon as the group $G$ is infinite.

3.2. Free Bernoulli Shifts. Let $G$ be any infinite countable discrete group and let $(N, \varphi)$ be any von Neumann algebra. Write

$$(\mathcal{M}, \Phi) = \ast_{g \in G} (N, \varphi)_g.$$  

For any $g, h \in G$ and any $x_h \in \mathcal{N}_h$, let $\sigma_g(x_h) = x_g^{-1}h \in \mathcal{N}_g^{-1}h$. The action $\sigma$ extends to the whole von Neumann algebra $\mathcal{M}$ and is called the free Bernoulli shift with base $(N, \varphi)$. The action $\sigma$ is obviously $\Phi$-preserving.

**Proposition 3.4.** The action $(\sigma_g)$ is freely mixing for $\Phi$.

**Proof.** For $h \in G$, denote as usual $\mathcal{N}_h^C = \mathcal{N}_h \cap \ker(\Phi)$. We have to prove that for any $n \in \mathbb{N}^*, x_1, \ldots, x_n, y_0, \ldots, y_n \in \mathcal{M} \cap \mathcal{C}$, except possibly $y_0$ and/or $y_n$ are equal to 1,

$$\lim_{g_1, \ldots, g_n \to \infty} \Phi(y_0\sigma_{g_1}(x_1)y_1 \cdots \sigma_{g_n}(x_n)y_n) = 0.$$

Actually, it suffices to show this property for $x_i$ and $y_j$ of the form

$$x_i = x_{h_1}^{i_1} \cdots x_{m_i}^{i_m}, \quad y_j = y_{k_1}^{j_1} \cdots y_{k_j}^{j_l},$$

with $1 \leq i \leq n$, $0 \leq j \leq n$ and

1. $h_1 \neq \cdots \neq h_{m_i}$ and for $1 \leq p \leq m_i$, $x_{h_p} \in \mathcal{N}_{h_p} \cap \mathcal{C}$;
2. $k_1 \neq \cdots \neq k_{j_l}$ and for $1 \leq q \leq j_l$, $y_{k_q} \in \mathcal{N}_{k_q} \cap \mathcal{C}$.

For any $g_1, \ldots, g_n \in G$, we have

$$\sigma_{g_i}(x_i) = x_{(g_i)^{-1}h_1}^{i_1} \cdots x_{(g_i)^{-1}h_{m_i}}^{i_m}.$$  

With all these notations, it is clear now that for any $g_1, \ldots, g_n \in G$ large enough, the $(g_i)^{-1}h_p$’s are pairwise distinct from the $k_q$’s. In particular, using the freeness property of the state $\Phi$, for any $g_1, \ldots, g_n \in G$ large enough, we get

$$\Phi(y_0\sigma_{g_1}(x_1)y_1 \cdots \sigma_{g_n}(x_n)y_n) = 0.$$

\hfill \Box

**Proposition 3.5.** Take $(N, \varphi) = (\Gamma(H_R, U_t)^n, \varphi_U)$ an almost periodic free Araki-Woods factor. Then, the associated free Bernoulli shift is $s$-malleable.

**Proof.** To check that the free Bernoulli action is $s$-malleable, it suffices to produce an action $(\alpha_t)$ of $R$ on $(N, \varphi)^* (N, \varphi)$ and a period 2 automorphism $\beta$ of $(N, \varphi)^* (N, \varphi)$ such that all the conditions of Definition 1.1 are satisfied. One can then take the infinite free product of these $(\alpha_t)$ and $\beta$. But, we have already proven this property in Proposition 3.2 \hfill \Box

**Proposition 3.6.** Take $(N, \varphi) = (M_n(C), \omega)$, where $\omega$ is any faithful state on $M_n(C)$. Then, the associated free Connes-Størmer Bernoulli shift is malleable.
**Definition 4.2**

Non-unital. Let \( \| \cdot \| \) be almost periodic state. Denote by \( Q \) and \( M, \tau \) be von Neumann subalgebras of a finite von Neumann algebra \((M_\infty, \omega)\). Dykema proved in [8], among other results, that the centralizer \((M_n(C) \ast M_n(C))^{\omega \omega} \) is the type II_1 factor \( L(F_\infty) \). We should mention that we obtained in [14] classification results for some of these free products using free Araki-Woods factors of Shlyakhtenko.

Denote by \((e_{ij})\) for \( i, j \in \{0, \ldots, n - 1\} \) a system of matrix unit in \( M_n(C) \). Prove the following lemma:

**Lemma 3.7.** Let \((\mathcal{P}, \psi)\) be a von Neumann algebra endowed with a faithful normal state, such that the centralizer \( \mathcal{P}^\psi \) is a factor. For \( i = 1, 2 \), let \( \rho_i : M_n(C) \rightarrow (\mathcal{P}, \psi) \) be a modular embedding, i.e. \( \rho_i \) is state-preserving and \( \rho_i(M_n(C)) \) is globally invariant under the modular group \((\sigma^\psi_t)\). Then, there exists a unitary \( u \in \mathcal{P}^\psi \) such that \( u\rho_1(e_{ij})u^* = \rho_2(e_{ij}), \forall i, j \in \{0, \ldots, n - 1\} \).

**Proof of Lemma 3.7** Let \( i \in \{1, 2\} \). Denote by \( p_i = \rho_i(e_{00}) \). Since \( \rho_i \) is modular, we have \( p_{1,2} \in \mathcal{P}^\psi \) and \( \psi(p_1) = \psi(p_2) = \omega(e_{00}) \). Since \( \mathcal{P}^\psi \) is a factor, there exists a partial isometry \( v \in \mathcal{P}^\psi \) such that \( p_1 = v^* v \) and \( p_2 = vv^* \). Denote by \( u = \sum_{i=0}^{n-1} \rho_2(e_{00}) \rho_1(e_{00}) \). An easy computation shows that \( u \) is a unitary and \( u \in \mathcal{P}^\psi \), since \( p_{1,2} \) are modular. Moreover, for any \( 0 \leq k, l \leq n - 1 \), \( u\rho_1(e_{kl})u^* = \rho_2(e_{kl}) \).

Write \((\mathcal{P}, \psi) = (M_n(C), \omega) \ast (M_n(C), \omega) \). Define \( \rho_{1,2} : M_n(C) \rightarrow \mathcal{P} \), by \( \rho_{1,2}(x) = x \ast 1 \), \( \rho_{2}(x) = 1 \ast x \), \( \forall x \in M_n(C) \). The embeddings \( \rho_{1,2} \) are modular (see [8]). Applying Lemma 3.7 there exists \( u \in \mathcal{U}(\mathcal{P}^\psi) \) such that \( u\rho_1(e_{ij})u^* = \rho_2(e_{ij}) \), for any \( 0 \leq i, j \leq n - 1 \). Write \( u = \exp(ith) \) with \( h \) a selfadjoint element in \( \mathcal{P}^\omega \). Denote by \( u_t = \exp(ith), \forall t \in \mathbb{R} \). We can then define on \( \mathcal{P} \), \( \alpha_t = \text{Ad}(u_t) \). Since \( u_t \in \mathcal{P}^\omega \), the action \((\alpha_t)\) is \( \omega \)-preserving. By definition, \( \alpha_1(x \ast 1) = 1 \ast x \), \( \forall x \in M_n(C) \). We are done.

4. Popa’s Intertwining Techniques

4.1. Intertwining Techniques for von Neumann Algebras Endowed with Almost Periodic States. We remind Popa’s intertwining-by-bimodules technique: it is a very strong method to prove that two von Neumann subalgebras of a von Neumann algebra are unitarily conjugate. Roughly, Definition 4.2 below says the following. Let \( A, B \subset M \) be von Neumann subalgebras of a finite von Neumann algebra \((M, \tau)\). The following conditions are equivalent:

- A corner of \( A \) can be conjugated into a corner of \( B \).
- The \( A-B \) bimodule \( L^2(M, \tau) \) contains a nonzero \( A-B \) subbimodule which is finitely generated as a right \( B \)-module.
- The basic construction \( \langle M, e_B \rangle \) contains a positive element \( a \) commuting with \( A \) and satisfying \( 0 < \hat{\tau}(a) < \infty \), where \( \hat{\tau} \) denotes the canonical semifinite trace on the basic construction \( \langle M, e_B \rangle \).

**Terminology 4.1.** Let \( M \) be a von Neumann algebra. For a possibly non-unital subalgebra \( Q \subset M \), we shall denote by \( 1_Q \) the unit of \( Q \). Obviously, \( 1_Q \) is a projection in \( M \) and \( Q \subset 1_QM1_Q \). We shall always mention when a von Neumann subalgebra is possibly non-unital.

**Definition 4.2** (Popa, [24, 23]). Let \((M, \varphi)\) be a von Neumann algebra endowed with an almost periodic state. Denote by \( \| \cdot \|_2 \) the \( L^2 \)-norm w.r.t. the state \( \varphi \). Assume that

- \( A \subset M^\varphi \) is a possibly non-unital von Neumann subalgebra, and denote by \( 1_A \) its unit;
\* \( B \subset \mathcal{M}^{\varphi} \) is a unital von Neumann subalgebra.

We say that \( A \) embeds into \( B \) inside \( \mathcal{M} \) and write \( A \preceq B \), if one of the following equivalent conditions is satisfied:

1. There exist \( n \geq 1 \), \( \gamma > 0 \), \( v \in M_{1,n}(C) \otimes 1_A \mathcal{M} \), a projection \( p \in B^n \) and a (unital) \(*\)-homomorphism \( \theta : A \to pB^n p \) such that \( v \) is a nonzero partial isometry which is a \( \gamma \)-eigenvector for \( \varphi \), \( v^*v \leq p \) and
   \[ xv = v\theta(x), \forall x \in A. \]

2. There exists a nonzero element \( w \in 1_A \mathcal{M} \) such that \( Aw \subset \sum_{k=1}^n w_k B \) for finitely many \( w_k \in 1_A \mathcal{M} \).

3. There exists a nonzero element \( a \in 1_A \langle M, e_B \rangle^+ \cap A' \) with \( \widehat{\varphi}_B(a) < \infty \). Here \( \langle M, e_B \rangle \) denotes the basic construction for the inclusion \( B \subset \mathcal{M} \), with its canonical almost periodic semifinite weight \( \widehat{\varphi}_B \).

4. There is no sequence of unitaries \( (u_k) \) in \( P \) such that \( \| E_B(a^*u_kb) \|_2 \to 0 \) for all \( a, b \in 1_A \mathcal{M} \).

We refer to Theorem 2.1 in [21] for the proof of these properties (see also Proposition C.1 in [34]). Note that if \( B = C \), then \( A \not\preceq C \) if and only if \( A \) is diffuse. Indeed, if \( B = C \), we simply have \( E_B = \varphi 1 \). Since the von Neumann algebra \( A \) is finite, \( A \) is diffuse if and only if there exists a sequence of unitaries \( (u_k) \) in \( A \) that weakly tends to 0. But,

\[ u_k \to 0 \text{ weakly} \iff \langle u_k \xi, \eta \rangle \to 0, \forall \xi, \eta \in 1_A L^2(\mathcal{M}, \varphi) \]
\[ \iff \langle u_k b, \overline{a} \rangle \to 0, \forall a, b \in 1_A \mathcal{M} \text{ (since \( (u_k) \) is bounded)} \]
\[ \iff \varphi(a^*u_kb) \to 0, \forall a, b \in 1_A \mathcal{M}. \]

For our purpose, we need a generalization of this technique. Indeed, we want to allow the subalgebra \( B \) to be globally invariant under the modular group \( (\sigma_t^\varphi) \), and not just included in the centralizer \( \mathcal{M}^{\varphi} \). We prove the following theorem:

**Theorem 4.3.** Let \( (\mathcal{M}, \varphi) \) be a von Neumann algebra endowed with an almost periodic state. Denote by \( \| \cdot \|_2 \) the \( L^2 \)-norm w.r.t. the state \( \varphi \). Assume that

- \( P \subset \mathcal{M}^{\varphi} \) is a possibly non-unital von Neumann subalgebra, and denote by \( 1_P \) its unit;
- \( B \subset \mathcal{M} \) is a unital von Neumann subalgebra globally invariant under the modular group \( (\sigma_t^\varphi) \).

Denote by \( B = B^{\varphi} = B \cap \mathcal{M}^{\varphi} \). The following two conditions are equivalent:

1. There exist \( n \geq 1 \), \( \gamma > 0 \), \( v \in M_{1,n}(C) \otimes 1_P \mathcal{M} \), a projection \( p \in B^n \) and a (unital) \(*\)-homomorphism \( \theta : P \to pB^n p \) such that \( v \) is a nonzero partial isometry which is a \( \gamma \)-eigenvector for \( \varphi \), \( v^*v \leq p \) and
   \[ xv = v\theta(x), \forall x \in P. \]

2. There is no sequence of unitaries \( (u_k) \) in \( P \) such that \( \| E_B(a^*u_kb) \|_2 \to 0 \) for all \( a, b \in 1_P \mathcal{M} \).

If one of the conditions holds, we shall still write \( P \preceq \mathcal{B}_\mathcal{M} \).

**Proof.** We must pay attention to the following fact: there are two different basic constructions here. The one with \( \mathcal{B} \) and the other one with \( B = B^{\varphi} \). Of course, we have the inclusion \( \langle \mathcal{M}, e_B \rangle \subset \langle \mathcal{M}, e_B \rangle \), but the associated weights are not equal on \( \langle \mathcal{M}, e_B \rangle \). For this reason, we shall denote by \( \widehat{\varphi}_B \) the weight for the basic construction \( \langle \mathcal{M}, e_B \rangle \) and by \( \widehat{\varphi}_B \) the weight for \( \langle \mathcal{M}, e_B \rangle \).
(1) \implies (2). Suppose that we have all the data of (1). Let \((u_k)\) be a sequence of unitaries in \(P\) such that \(\|E_B(a^*u_k b)\|_2 \to 0\) for all \(a, b \in \mathcal{M}\). Then \(\|E_B(v^*u_k v)\|_{\text{tr}_n \otimes \varphi} \to 0\). But for every \(k \in \mathbb{N}\), \(v^*u_k v = \theta(u_k)v^*v\). Moreover, \(\theta(u_k) \in U(pB^np)\) and \(v^*v \leq p\). Thus,

\[
\|E_B(v^*v)\|_{\text{tr}_n \otimes \varphi} = \|E_B(v^*v)\|_{\text{tr}_n \otimes \varphi} = \|E_B(v^*v)\|_{\text{tr}_n \otimes \varphi} = 0.
\]

We conclude that \((id \otimes E_B)(v^*v) = 0\) and so \(v = 0\). Contradiction.

(2) \implies (1). We prove this implication in three steps. For any \(\gamma, \lambda \in \text{Sp}(\mathcal{M}, \varphi)\), denote by \(\mathcal{M}^\gamma\) the vector space of all \(\gamma\)-eigenvectors for \(\varphi\) in \(\mathcal{M}\). Since \(\varphi\) is almost periodic,

\[
L^2(\mathcal{M}, \varphi) = \bigoplus_{\gamma \in \text{Sp}(\mathcal{M}, \varphi)} L^2(\mathcal{M}^\gamma).
\]

Denote by \(p_\gamma\) the orthogonal projection from \(L^2(\mathcal{M})\) onto \(L^2(\mathcal{M}^\gamma)\).

**Step (1): Proving that for any \(\gamma, \lambda \in \text{Sp}(\mathcal{M}, \varphi)\), and for any \(a \in \mathcal{M}^\lambda\), we have**

\[
\widehat{\varphi}_B(p_\gamma a e_B a^* p_\gamma) \leq \varphi(aa^*).
\]

First of all, note that for any \(\gamma \in \text{Sp}(\mathcal{M}, \varphi)\), \(\mathcal{B}^\gamma = \mathcal{M}^\gamma \cap \mathcal{B}\). Since \(\mathcal{B}\) is globally invariant under \(\sigma^\varphi\), for every \(t \in \mathbb{R}\), we have \(\sigma^\varphi_t \circ E_B = E_B \circ \sigma^\varphi_t\). It follows immediately that \(E_B(\mathcal{M}^\gamma) = \mathcal{B}^\gamma\). It is straightforward to check that \(e_B p_\gamma = p_\gamma e_B\); we shall denote this projection by \(e_B^\gamma\). In fact, it is nothing but the orthogonal projection of \(L^2(\mathcal{M})\) onto \(L^2(\mathcal{B}^\gamma)\). Take now \(\gamma \in \text{Sp}(\mathcal{B}, \varphi)\). Since \(\mathcal{B}^\gamma \neq 0\), take \((v_i)_{i \in I}\) a maximal family of nonzero partial isometries in \(\mathcal{B}^\gamma\) such that the final projections \(p_i = v_i^*v_i\) are pairwise orthogonal. We assume that \(I = \{1, \ldots, n\}\) with \(1 \leq n \leq \infty\). Denote by \(v = [v_1 \cdots v_n] \in M_{1,n}(\mathbb{C}) \otimes \mathcal{B}^\gamma\) and by \(p = \sum_i v_i^*v_i\). It is easy to see that

\[
eq e_B^\gamma \eta = v(1 \otimes e_B) v^* \eta = 0, \forall \eta \in (L^2(\mathcal{M}) \otimes L^2(\mathcal{B})) \oplus \bigoplus_{\lambda \neq \gamma} L^2(\mathcal{B}^\lambda).
\]

Assume now that there exists \(x \in \mathcal{B}^\gamma\) such that \(\hat{x} \neq v(1 \otimes e_B) v^* \hat{x}\). Thus, \((1 - p)x \neq 0\). Write \((1 - p)x = w b\) its polar decomposition. Since \((1 - p)x \in \mathcal{B}^\gamma\), it follows that \(w \in \mathcal{B}^\gamma\). But \(w \neq 0\), and \(ww^* \leq 1 - p\). Since the family \((v_i)_{i \in I}\) is assumed to be maximal, we have a contradiction. Consequently, we have just proven that

\[
eq e_B^\gamma = v(1 \otimes e_B) v^*.
\]

Since \(B \subseteq \mathcal{M}^\varphi\), it follows that \(p_\gamma \in \langle M, e_B \rangle\), for all \(\gamma \in \text{Sp}(\mathcal{M}, \varphi)\). Take \(\gamma, \lambda \in \text{Sp}(\mathcal{M}, \varphi)\). We want to prove now that \(\widehat{\varphi}_B(p_\gamma a e_B a^* p_\gamma) \leq \varphi(aa^*)\). It is easy to see that \(a^* p_\gamma = p_\gamma a^* a^*\). Consequently, we have

\[
\gamma p_\gamma a e_B^\lambda a^* = a e_B^\lambda a^*.
\]

If \(\gamma^{-1} \notin \text{Sp}(\mathcal{B}, \varphi)\), then \(e_B^\lambda = 0\) and so \(\widehat{\varphi}_B(p_\gamma a e_B a^* p_\gamma) = 0\). If \(\gamma^{-1} \in \text{Sp}(\mathcal{B}, \varphi)\), take as before \(v = [v_1 \cdots v_n] \in M_{1,n}(\mathbb{C}) \otimes \mathcal{B}^\gamma \) such that \(e_B^\lambda = v(1 \otimes e_B) v^*\) and \((v_i)\) is a family of nonzero partial isometries such that the projections \(v_i v_i^*\) are pairwise orthogonal. Thus,
\[ \hat{\varphi}_B(p, ae_B a^* p) = \hat{\varphi}_B(ae_B^{\lambda^{-1}} a^*) \]
\[ = \hat{\varphi}_B(\alpha B(1 \otimes e_B) v^* a^*) \]
\[ = \hat{\varphi}_B(a \left( \sum_i v_i e_B v_i^* \right) a^*) \]
\[ = \sum_i \hat{\varphi}_B(\alpha v_i e_B v_i^* a^*) \] (\( \hat{\varphi}_B \) is normal)
\[ = \sum_i \varphi(\alpha v_i^* a^*) \] (\( \hat{\varphi}_B \) is normal)
\[ = \varphi(a \left( \sum_i v_i^* \right) a^*) \] (\( \hat{\varphi}_B \) is normal)
\[ \leq \varphi(aa^*). \]

**Step (2): Finding a nonzero element** \( d \in 1_P \langle M, e_B \rangle^+ 1_P \cap P' \) such that for any \( \gamma \in \text{Sp}(M, \varphi) \), we have
\[ \hat{\varphi}_B(p, dp_\gamma) < \infty. \]

By (2), we can take \( \varepsilon > 0 \) and \( K \subset 1_P M \) finite subset such that for all unitaries \( u \in P \), \( \max_{a, b \in K} \| E_B(a^* ub) \|_2 \geq \varepsilon \). Note that
\[ \| E_B(a^* ub) \|^2 = \varphi(E_B(a^* ub)^* E_B(a^* ub)) \]
\[ = \hat{\varphi}_B(E_B(a^* ub)^* e_B E_B(a^* ub)) \]
\[ = \hat{\varphi}_B(e_B(a^* ub)^* e_B(a^* ub) e_B). \]

Since the functional \( \hat{\varphi}_B(e_B \cdot e_B) \) is a normal state on the basic construction \( \langle M, e_B \rangle \) and since \( \varphi \) is almost periodic, we can assume that all the elements of \( K \) are eigenvectors for \( \varphi \). Define now the element \( c = \sum_{a \in K} a e_B a^* \) in \( 1_P \langle M, e_B \rangle^+ 1_P \). Note that \( \hat{\varphi}_B(c) = \sum_{a \in K} \varphi(aa^*) \), and so \( \hat{\varphi}_B(c) < \infty \). Moreover, since the elements of \( K \) are eigenvectors for \( \varphi \), we get \( c \in \langle M, e_B \rangle^{\hat{\varphi}_B} \). Denote by \( C \) the convex hull of \( \{ u^* cu : u \in U(P) \} \). Denote now by \( \overline{C} \) the closure of \( C \) for the weak topology. It should be noted that since \( C \) is bounded, \( \overline{C} \) is also closed for the \( \sigma \)-weak topology. Let \( d \in 1_P \langle M, e_B \rangle^+ 1_P \) be the element of minimal \( L^2 \)-norm \( \| . \|_2, \hat{\varphi}_B \) (w.r.t. the weight \( \hat{\varphi}_B \)) in \( \overline{C} \). By uniqueness of the element of minimal \( L^2 \)-norm, it follows that \( u^* du = d \), \( \forall u \in U(P) \), and so \( d \in 1_P \langle M, e_B \rangle^+ 1_P \cap P' \). Obviously, we have \( \sigma(t)^{\hat{\varphi}_B}(d) = d, \forall t \in \mathbb{R} \). We show now that \( d \neq 0 \). For all \( u \in U(P) \), we have
\[ \sum_{b \in K} \hat{\varphi}_B(e_B b^* (u^* cu) b e_B) = \sum_{a, b \in K} \hat{\varphi}_B(e_B (a^* ub)^* e_B (a^* ub) e_B) \]
\[ = \sum_{a, b \in K} \hat{\varphi}_B(E_B(a^* ub)^* e_B E_B(a^* ub)) \]
\[ = \sum_{a, b \in K} \varphi(E_B(a^* ub)^* E_B(a^* ub)) \]
\[ = \sum_{a, b \in K} \| E_B(a^* ub) \|^2 \geq \varepsilon^2. \]

Consequently, we have
\[ \sum_{b \in K} \hat{\varphi}_B(e_B b^* y b e_B) \geq \varepsilon^2, \forall y \in C. \]
Since the functional $\hat{\varphi}_B(e_B \cdot e_B)$ is a normal state on the basic construction $\langle \mathcal{M}, e_B \rangle$, we get

$$\sum_{b \in K} \hat{\varphi}_B(e_B b^* d e_B) \geq \varepsilon^2.$$ 

It follows that $d \neq 0$. At last, using the result of Step (1) and since $P \subset \mathcal{M}^\varphi$, $\forall \gamma, \lambda \in \text{Sp}(\mathcal{M}, \varphi), \forall u \in U(P), \forall a \in 1_p \mathcal{M}^\lambda$, we have

$$\hat{\varphi}_B(p_\gamma u^* a e_B a^* u p_\gamma) \leq \varphi(u^* a a^* u) = \varphi(aa^*).$$

Consequently, summing over $a \in K$ and using the convexity of $C$, we obtain

$$\hat{\varphi}_B(p_\gamma y p_\gamma) \leq \sum_{a \in K} \varphi(aa^*), \forall \gamma \in \text{Sp}(\mathcal{M}, \varphi), \forall y \in C.$$ 

Using the $\sigma$-weak lower semi continuity of the weight $\hat{\varphi}_B$ (see for example Theorem VII.1.11 in [33]), for every $\gamma \in \text{Sp}(\mathcal{M}, \varphi)$, we have

$$\hat{\varphi}_B(p_\gamma dp_\gamma) \leq \sum_{a \in K} \varphi(aa^*) < \infty.$$ 

**Step (3): Constructing a nonzero $P$-$B$-subbimodule $\mathcal{H} \subset 1_p L^2(\mathcal{M}^\gamma)$ finitely generated over $B$ to conclude.** We remind that for any $x \in \langle \mathcal{M}, e_B \rangle^+$,

$$\hat{\varphi}_B(x) = \sum_{\gamma \in \text{Sp}(\mathcal{M}, \varphi)} \hat{\varphi}_B(p_\gamma x p_\gamma).$$

Since $d \neq 0$ and thanks to Step (2), there exists $\gamma \in \text{Sp}(\mathcal{M}, \varphi)$ such that

$$0 < \hat{\varphi}_B(p_\gamma dp_\gamma) < \infty.$$ 

Since $p_\gamma \in (\mathcal{M}^\varphi)'$, we have $p_\gamma \in P'$. Thus $p_\gamma dp_\gamma \in 1_p \langle \mathcal{M}, e_B \rangle^+ 1_p \cap P'$. Take now $q$ a nonzero spectral projection of the element $p_\gamma dp_\gamma$. We get that $\mathcal{K} = q L^2(\mathcal{M})$ is a nonzero $P$-$B$-subbimodule of $1_p L^2(\mathcal{M}^\gamma)$ with finite trace over $B$ (see the discussion in Section 2). Thus, cutting down by a central projection of $B$ (see Lemma 2.1), we get a nonzero $P$-$B$-subbimodule $\mathcal{H} \subset 1_p L^2(\mathcal{M}^\gamma)$ which is finitely generated over $B$. Now, the rest of the proof is exactly the same as the proof of Theorem 2.1 in [21]. For the sake of completeness, we proceed in order to obtain condition (1). Hence, we can take $n \geq 1$, a projection $p \in B^n$ and a right $B$-module isomorphism

$$\psi : p L^2(B)^{\otimes n} \to \mathcal{H}.$$ 

Since $\mathcal{H}$ is a $P$-module, we get a (unital) $*$-homomorphism $\theta : P \to p B^n p$ satisfying $x \psi(\eta) = \psi(\theta(x) \eta)$ for all $x \in P$, and $\eta \in p L^2(B)^{\otimes n}$. Define now $e_j \in L^2(B)^{\otimes n}$ as $e_j = (0, \ldots, 1, \ldots, 0)$ and $\xi = (\xi_1, \ldots, \xi_n) \in M_{1,n}(C) \otimes \mathcal{H}$, with $\xi_j = \psi(pe_j)$. Let $j \in \{1, \ldots, n\}$. 
For any \( x \in P \), write \( \theta(x) = (\theta_{kl}(x))_{kl} \in pB^np \). We have
\[
  x \xi_j = x \psi(pe_j) \\
  = \psi(\theta(x) pe_j) \\
  = \psi(p \theta(x) e_j) \\
  = \psi(p \sum_{i=1}^{n} \theta_{ij}(x) e_i) \\
  = \sum_{i=1}^{n} \psi(p(0, \ldots, \theta_{ij}(x), \ldots, 0)) \\
  = \sum_{i=1}^{n} \psi(pe_i) \theta_{ij}(x) \\
  = \sum_{i=1}^{n} \psi(pe_i) \theta_{ij}(x) (\psi \text{ is a right } B\text{-module isomorphism}) \\
  = \sum_{i=1}^{n} \xi_i \theta_{ij}(x).
\]

Consequently, for every \( x \in P \), \( x \xi = \xi \theta(x) \). In the von Neumann algebra \( \mathcal{M}^{n+1} \subset B(L^2(\mathcal{M}) \oplus L^2(\mathcal{M})^{\otimes n}) \), define
\[
  X_x = \begin{pmatrix} x & 0 \\ 0 & \theta(x) \end{pmatrix}, \forall x \in P.
\]

In the space \( L^2(\mathcal{M}^{n+1}) \), define
\[
  \Xi = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}.
\]

We still denote by \( 1_P \) the unit of \( P^{n+1} \subset \mathcal{M}^{n+1} \). Note that \( X_x \in 1_P \mathcal{M}^{n+1} 1_P, \forall x \in P \), and \( \Xi \in 1_P L^2(\mathcal{M}^{n+1}) \). We obtain \( X_x \Xi = \Xi X_x \), for every \( x \in P \). Since \( \Xi \) is a \( \gamma \)-eigenvector in \( \mathcal{M}^{n+1} \) for the state \( tr_{n+1} \otimes \varphi \), we can define (as in the Appendix Proposition [4.1]) \( T_\Xi \) and write \( T_\Xi = V |T_\Xi| \) the polar decomposition of \( T_\Xi \). We get \( X_x V = V X_x \), for every \( x \in P \), and \( VV^* \leq 1_P \). Write
\[
  V = \begin{pmatrix} u & v \\ v' & w \end{pmatrix}.
\]

It is straightforward to check that \( v \in M_1, n(\mathbb{C}) \otimes 1_P \mathcal{M} \) is a partial isometry from \( \ker w \) onto \( \ker u^* \) such that \( xv = v \theta(x) \), for every \( x \in P \). Moreover, \( v \) is a \( \gamma \)-eigenvector for \( \varphi \) and \( v^* v \leq p \). \( \square \)

As a consequence of Definition [4.2] and Theorem [4.3] we prove the following proposition which will be needed in the next section.

**Proposition 4.4.** Let \( (\mathcal{M}, \varphi) \) be a von Neumann algebra endowed with a faithful normal almost periodic state. Let \( \mathcal{M}_i \subset \mathcal{M}, (i = 1, 2) \), be two von Neumann subalgebras globally invariant under the modular group \( (\sigma_i^\varphi) \). Let \( Q \subset \mathcal{M}^\varphi \) be a possibly non-unital von Neumann subalgebra, and denote by \( 1_Q \) its unit. Assume that \( Q \neq \mathcal{M}_1 \) and \( Q \neq \mathcal{M}_2 \).

Then, there exists a sequence of unitaries \( (u_k) \) in \( Q \) such that \( ||E_{\mathcal{M}_i}(x^* u_k y)||_2 \to 0 \) for all \( x, y \in 1_Q \mathcal{M} \) and all \( i \in \{1, 2\} \).
Proof. Denote by

\[ B = \left( \begin{array}{cc} M_1 & 0 \\ 0 & M_2 \end{array} \right) \subset M_2(\mathbb{C}) \otimes M, \]

and \( B = B^{tr_2 \otimes \varphi} \). Define \( \rho : Q \to M_2(\mathbb{C}) \otimes M \) in the following way

\[ \rho(x) = \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \forall x \in Q. \]

We still denote by \( 1_Q \) the unit of \( M_2(\mathbb{C}) \otimes Q \subset M_2(\mathbb{C}) \otimes M \). Assume that there is no sequence of unitaries \( (u_k) \) in \( Q \) such that for all \( x, y \in 1_QM \) and all \( i \in \{1, 2\} \), \( \|E_{M_i}(x^*u_ky)\|_2 \to 0 \). It is equivalent to saying that there is no sequence of unitaries \( (V_k) \) in \( \rho(Q) \) such that \( \|E_B(X^*V_kY)\|_{tr_2 \otimes \varphi} \to 0 \) for all \( X, Y \in 1_Q(M_2(\mathbb{C}) \otimes M) \). Using our notation, we get

\[ \rho(Q) \lessdot M_2(\mathbb{C}) \otimes M. \]

Combining Theorem 4.3 and Definition 4.2 (second point), we know that there exists \( n \geq 1 \), there exist a nonzero element \( W \) in \( 1_Q(M_2(\mathbb{C}) \otimes M) \) and finitely many \( W_1, \ldots, W_n \in 1_Q(M_2(\mathbb{C}) \otimes M) \), such that \( \rho(Q)W \subset \sum_{k=1}^{n} W_kB \). Write

\[ W = \begin{pmatrix} w^a & w^b \\ w^c & w^d \end{pmatrix}, W_k = \begin{pmatrix} w_k^a & w_k^b \\ w_k^c & w_k^d \end{pmatrix}. \]

Thus, we obtain

\[ \begin{pmatrix} Qw^a & Qw^b \\ Qw^c & Qw^d \end{pmatrix} \subset \begin{pmatrix} \sum_{k=1}^{n} w_k^aM_1^{\varphi_1} & \sum_{k=1}^{n} w_k^bM_2^{\varphi_2} \\ \sum_{k=1}^{n} w_k^cM_1^{\varphi_1} & \sum_{k=1}^{n} w_k^dM_2^{\varphi_2} \end{pmatrix}. \]

Since \( W \neq 0 \), there exists a letter \( z \in \{a, b, c, d\} \) such that \( w^z \neq 0 \). So, there exists \( i \in \{1, 2\} \), such that \( Qw^z \subset \sum_{k=1}^{n} w_k^zM_1^{\varphi_1} \), and \( w^z, w_1^z, \ldots, w_n^z \in 1_QM \). Thus, combining once again Definition 4.2 (second point) and Theorem 4.3, we have proven that there exists \( i \in \{1, 2\} \) such that \( Q \lessdot M_i \).

\[ \square \]

4.2. Controlling Quasi-Normalizers of Subalgebras of Free Products with Amalgamation. Let \( Q \subset M \) be a von Neumann subalgebra of \( M \). An element \( x \in M \) is said to quasi-normalize \( Q \) inside \( M \) if there exist \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_r \) in \( M \) such that

\[ xQ \subset \sum_{i=1}^{k} Qx_i \quad \text{and} \quad Qx \subset \sum_{j=1}^{r} y_jQ. \]

The elements quasi-normalizing \( Q \) inside \( M \) form a unital *-subalgebra of \( M \) and their weak closure is called the quasi-normalizer of \( Q \) inside \( M \). The inclusion \( Q \subset M \) is said to be quasi-regular if \( M \) is the quasi-normalizer of \( Q \) inside \( M \).

A typical example arises as follows: let \( G \) be a countable group and let \( H \) be an almost normal subgroup, which means that \( ghg^{-1} \cap H \) is a finite index subgroup of \( H \) for every \( g \in G \). It is straightforward to check that the inclusion \( L(H) \subset L(G) \) is quasi-regular.

The next result is already known for finite von Neumann algebras: it is a result of Ioana, Peterson & Popa (see Theorem 1.2.1 in [16]). For our purpose, we need to extend it to von Neumann algebras endowed with almost periodic states.

**Theorem 4.5.** Let \((M_1, \varphi_1)\) and \((M_2, \varphi_2)\) be von Neumann algebras with faithful normal almost periodic states and let \( N \subset M_i^{\varphi_i} \) be a von Neumann subalgebra for \( i = 1, 2 \). Set \( M = M_1 \ast_{N} M_2 \). Let \( Q \subset M_i^{\varphi_i} \) be a possibly non-unital von Neumann subalgebra, and denote by \( 1_Q \) its unit. Assume that \( Q \not\preceq N \). Then, every \( Q \cdot M_i^{\varphi_i} \) subbimodule \( H \) of
than \(1_Q L^2(M)\) with finite trace over \(M_1^{φ_1}\), as a right \(M_1^{φ_1}\)-module, is contained in \(1_Q L^2(M_1)\).

In particular, the quasi-normalizer of \(Q\) inside \(1_Q M_1 Q\) is included in \(1_Q M_1 Q\) and \(1_Q M_1 Q \cap Q' \subset 1_Q M_1 Q\).

**Proof.** The free product state will be denoted by \(φ\). Let \(A\) be the linear subspace of \(M \otimes M_1\) defined by

\[
(6) \quad A = \text{span}\{M_2 \ominus N, (M_{i_1} \ominus N) \cdots (M_{i_n} \ominus N) : n \geq 2, i_1 \neq \cdots \neq i_n \in \{1, 2\}\}
\]

It is a well-known fact that \(A\) is \(σ\)-weakly dense in \(M \otimes M_1\). Moreover, since \((M_i, φ_i)\) is almost periodic (for \(i = 1, 2\) and \(N \subset M_1^{φ_i}\)), we have

\[
(7) \quad \mathcal{M}_i \ominus N = \overline{\text{span}}^w \{M_1^{φ_i} \ominus N, M_1^{λ} : λ \in \text{Sp}(M_i, φ_i) \setminus \{1\}\}.
\]

Since \(Q \not\subseteq N\), we know from Definition 4.2 that there exists a sequence of unitaries \((u_k)\) in \(Q,\) such that for any \(a, b \in 1_Q M_1, \|E_N(a^* u_k b)\|_2 \to 0.\)

**Claim 4.6.** \(∀x, y \in M \otimes M_1, \|E_{M_1}(x u_k y)\|_2 \to 0.\)

**Proof of Claim 4.6.** Let \(x\) and \(y\) be reduced words in \(M\) with letters alternatingly from \(M_1 \ominus N\) and \(M_2 \ominus N\). We assume that \(x\) and \(y\) contain at least a letter from \(M_2 \ominus N\).

We moreover assume that all the letters of \(y\) are eigenvectors for \(φ\). We set \(x = x' a\) with \(a = 1\) if \(x\) ends with a letter from \(M_2 \ominus N\) and \(a\) equal to the last letter of \(x\) otherwise. Note that either \(x'\) equals 1 or is a reduced word ending with a letter from \(M_2 \ominus N\).

In the same way, we set \(y = b y'\) with \(b = 1\) if \(y\) begins with a letter from \(M_2 \ominus N\) and \(b\) equal to the first letter of \(y\) otherwise. Note that either \(y'\) equals 1 or is a reduced word beginning with a letter from \(M_2 \ominus N\).

Moreover, note that we cannot have at the same time \(x' = y' = 1\) and \(a z b - E_N(a z b) \in M_1 \ominus N\). Then for \(z \in Q\), we have

\[
E_{M_1}(x z y) = E_{M_1}(x' E_N(a z b) y').
\]

Since all the letters of \(y\) are eigenvectors for \(φ\), there exists \(λ > 0\) such that \(y' \in M_1^{λ}\). Then

\[
\|E_{M_1}(x z y)\|_2 \leq \|x' E_N(a z b) y'\|_2 \leq λ^{-1/2} \|x'\| \|y'\| \|E_N(a z b)\|_2.
\]

It follows that \(\|E_{M_1}(x u_k y)\|_2 \to 0.\) More generally, with the same \(y\), for any \(x \in A\), we have \(\|E_{M_1}(x u_k y)\|_2 \to 0.\)

We keep the same \(y\), but now we take \(x \in M \otimes M_1\). We can find a sequence \((x_i)\) in \(A\) such that \(\|x - x_i\|_2 \to 0.\) Since \(u_k \in Q \subset M_1^{φ_1}\), it follows that \(u_k y \in M_1^{λ}\), \(∀n \in \mathbb{N}\). We get

\[
\|(x - x_i) u_n y\|_2 = \|R_{u_n y}(x - x_i)\|_2 \leq λ^{-1/2} \|u_n y\| \|x - x_i\|_2 \leq λ^{-1/2} \|y\| \|x - x_i\|_2
\]

Take now \(ε > 0.\) Choose \(i\) such that \(\|x - x_i\|_2 \leq ε/(2λ^{-1/2} \|y\|)\). Choose now \(k_0 \in \mathbb{N}\), such that for any \(k \geq k_0, \|E_{M_1}(x_i u_k y)\|_2 \leq ε/2.\) Write \(E_{M_1}(x u_k y) = E_{M_1}((x - x_i) u_k y) + E_{M_1}(x_i u_k y).\) For any \(n \geq n_0\), we get

\[
\|E_{M_1}(x u_k y)\|_2 \leq \|E_{M_1}((x - x_i) u_k y)\|_2 + \|E_{M_1}(x_i u_k y)\|_2 \leq \|(x - x_i) u_k y)\|_2 + \|E_{M_1}(x_i u_k y)\|_2 \leq ε.
\]
Denote by $E$ the linear span of the $y$'s which are reduced words in $M$ containing at least a letter from $M_2 \oplus N$ and such that all the letters of $y$ are eigenvectors for $\varphi$. We finally get that for any $x \in M \ominus M_1$ and any $y \in E$, $\|E_{M_1}(xu_ky)\|_2 \to 0$. Note that from (6) and (7), it is straightforward to check that $E$ is ultraweakly dense in $M \ominus M_1$.

At last, take $x,y \in M \ominus M_1$. As before, take $\varepsilon > 0$ and choose $z \in E$, such that $\|xu_k(y - z)\|_2 \leq \varepsilon/2$, uniformly in $k \in \mathbb{N}$. Choose now $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, $\|E_{M_1}(xu_kz)\|_2 \leq \varepsilon/2$. Write $E_{M_1}(xu_ky) = E_{M_1}(xu_k(y - z)) + E_{M_1}(xu_kz)$. For any $k \geq k_0$, we get

$$\|E_{M_1}(xu_ky)\|_2 \leq \|E_{M_1}(xu_k(y - z))\|_2 + \|E_{M_1}(xu_kz)\|_2 \leq \|xu_k(y - z)\|_2 + \|E_{M_1}(xu_kz)\|_2 \leq \varepsilon.$$ 

Consequently, for any $x,y \in M \ominus M_1$, $\|E_{M_1}(xu_ky)\|_2 \to 0$. The claim is proven. $\square$

Let $H$ be a $Q \cdot M_1^{\varphi_1}$-submodule of $1_QL^2(M)$ with finite trace over $M_1^{\varphi_1}$. Since $\varphi$ is almost periodic, we can write

$$H = \bigoplus_{\gamma \in \text{Sp}(M, \varphi)} H^\gamma$$

where all the elements of $H^\gamma$ are $\gamma$-eigenvectors for $\varphi$. Note that $H^\gamma$ is nothing but $p_\gamma H$. Since $Q, M_1^{\varphi_1} \subset M^\varphi$, it follows that $p_\gamma \in \langle M, e_{M_1^{\varphi_1}} \rangle \cap Q'$ and thus, each of the $H^\gamma$'s is a $Q \cdot M_1^{\varphi_1}$-submodule of $1_QL^2(M)$ with finite trace over $M_1^{\varphi_1}$, as a right $M_1^{\varphi_1}$-module. So, we can assume that $H = H^\gamma$ for some $\gamma \in \text{Sp}(M, \varphi)$. From Lemma 2.1, we know that there exists a sequence $(z_k)$ of central projections in $M_1^{\varphi_1}$ such that $Hz_k$ is finitely generated as a right $M_1^{\varphi_1}$-module and $\varphi(z_k) \to 1$. If we prove that $Hz_k \subset 1_QL^2(M_1)$, $\forall k \in \mathbb{N}$, we are done. Indeed, assume that $Hz_k \subset 1_QL^2(M_1)$, $\forall k \in \mathbb{N}$. Since $\varphi(z_k) \to 1$, it follows that $z_k \to 1$ strongly. Thus, $\forall \xi \in H$, $\xi = \lim_{k \to \infty} \xi z_k \in 1_QL^2(M_1)$. Consequently, $H \subset 1_QL^2(M_1)$.

From now on, we assume that $H \subset 1_QL^2(M_1)$, and $H$ is finitely generated as a right $M_1^{\varphi_1}$-module. Then, there exist $n \geq 1$, a projection $p \in (M_1^{\varphi_1})^n$ and a right $M_1^{\varphi_1}$-module isomorphism

$$\psi : pL^2(M_1^{\varphi_1})^\oplus n \to H.$$ 

Since $H$ is a left $Q$-module, there exists a (unital) $*$-homomorphism $\theta : Q \to p(M_1^{\varphi_1})^n p$ such that for every $\eta \in pL^2(M_1^{\varphi_1})^\oplus n$, and every $x \in Q$, $x\psi(\eta) = \psi(\theta(x)\eta)$. For $i \in \{1, \ldots, n\}$, let $e_i = (0, \ldots, 1, \ldots, 0) \in L^2(M_1^{\varphi_1})^\oplus n$ and $\xi_i = \psi(pe_i)$. Let $\xi = (\xi_1, \ldots, \xi_n) \in M_{1,n}(C) \otimes H$. As in Theorem 4.3, we can prove that $x\xi = \xi \theta(x)$, for every $x \in Q$. In the von Neumann algebra $M^{n+1} \subset B(L^2(M) \oplus L^2(M)^\oplus n)$, define as before

$$X_x = \begin{pmatrix} x & 0 \\ 0 & \theta(x) \end{pmatrix}, \forall x \in Q.$$ 

In the space $L^2(M^{n+1})$, define

$$\Xi = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}.$$ 

Thus, we obtain $X_x \Xi = \Xi X_x$, for every $x \in Q$. We still denote by $1_Q$ the unit of $Q^{n+1} \subset M^{n+1}$. Since $\Xi$ is a $\gamma$-eigenvector in $M^{n+1}$ for the state $\text{tr}_{n+1} \otimes \varphi$, we can define as before $T_\Xi$ and write $T_\Xi = V|T_\Xi|$ the polar decomposition of $T_\Xi$. We get $X_x V = VX_x$, for every $x \in Q$. Let $f : R_+ \to C$ be a bounded Borel function with compact support. By
functional calculus, \(T_{\Xi}f(|\Xi|) \in 1_QM^{n+1}\). Write
\[
T_{\Xi}f(|\Xi|) = \begin{pmatrix}
y & a' \\
a & z
\end{pmatrix},
\]
with \(a \in M_{1,n}(C) \otimes 1_QM\). It is straightforward to check that \(xa = a\theta(x)\), for every \(x \in Q\). Since \(Q,M_{1}^{n+1} \subset M_{1,1}\), \((1 \otimes E_{M_{1}})(a) = (1 \otimes E_{M_{1}})(a)\theta(x)\), for every \(x \in Q\). Write \(b = a - (1 \otimes E_{M_{1}})(a)\). Note that \(b \in M_{1,n}(C) \otimes 1_QM\) and \((1 \otimes E_{M_{1}})(b) = 0\). We have \(xb = b\theta(x)\), for every \(x \in Q\) (note that range\((b^*b) \subset \text{range}\,p\)). Since \((1 \otimes E_{M_{1}})(b) = 0\), we have \(||(1 \otimes E_{M_{1}})(b^*u_k b)||_{tr_n \otimes \varphi} \to 0\), thanks to Claim 4.6. Since \(b^*u_k b = \theta(u_k) b^*b\) and \(\theta(u_k) \in U(p(M_{1}^{n+1})_p)\), we get
\[
||\theta(u_k)(1 \otimes E_{M_{1}})(b^*b)||_{tr_n \otimes \varphi} = ||\theta(u_k)(1 \otimes E_{M_{1}})(b^*b)||_{tr_n \otimes \varphi} = ||(1 \otimes E_{M_{1}})(b^*u_k b)||_{tr_n \otimes \varphi} \to 0.
\]
Consequently, \((1 \otimes E_{M_{1}})(b^*b) = 0\) and so \(b = 0\). Thus, \(a = (1 \otimes E_{M_{1}})(a)\) and so \(a \in M_{1,n}(C) \otimes 1_QM_{1}\).

Take now \(f_k = \chi[0,k]\), the characteristic function of the interval \([0,k]\), for each \(k \in N^*\) and write
\[
T_{\Xi}f_k(|\Xi|) = \begin{pmatrix} y_k & a_k \\ a_k' & z_k \end{pmatrix}.
\]
Applying what we have done, we get \(a_k \in M_{1,n}(C) \otimes 1_QM_{1}\), for every \(k \geq 1\). Denote by \(P_{1}\) the orthogonal projection from \(1_QL^2(M)\) onto \(1_QL^2(M_{1})\). Then, for any \(k \geq 1\), we have \((1 \otimes P_{1})a_k = a_k\). Since \(a_k \to \xi\), as \(k \to \infty\), we get \((1 \otimes P_{1})\xi = \xi\), and so \(\xi \in M_{1,n}(C) \otimes 1_QL^2(M_{1})\). But the \(\xi_i\)'s generate \(H\) as a right \(M_{1}^{n+1}\)-module. Thus, \(H \subset 1_QL^2(M_{1})\).

Take now \(x \in 1_QM_{1}L_{1}\) that quasi-normalizes \(Q\) inside \(1_QM_{1}L_{1}\). In particular, there exist \(r \geq 1\), \(y_1,\ldots,y_r \in 1_QM_{1}L_{1}\) such that \(Qx \subset \sum_{k=1}^r y_k Q\). Define \(H\) the \(Q\-M_{1}^{n+1}\) subbimodule of \(1_QL^2(M)\) by \(H = QxM_{1}^{n+1}\). Since \(H \subset \sum_{k=1}^r y_k M_{1}^{n+1}\), it follows that \(H\) is of finite trace over \(M_{1}^{n+1}\) as a right \(M_{1}^{n+1}\)-module. Thus, \(H \subset 1_QL^2(M_{1})\). So, \(x \in M_{1}\). Consequently, the quasi-normalizer of \(Q\) inside \(M\) is included in \(1_QM_{1}L_{1}\). Obviously, we get also \(1_QM_{1}L_{1} \cap Q' \subset 1_QM_{1}L_{1}\).

5. Type II\(_1\) Factors with Prescribed Countable Fundamental Group

5.1. Intertwining Rigid Subalgebras of Crossed Products.

**Notation 5.1.** Let \(\sigma : G \to \text{Aut}(N,\varphi)\) be a state-preserving action, with \(\varphi\) an almost periodic state. We adopt the following notation:

1. \(M = N \rtimes G\), with the action \((\sigma_g)\).
2. \(M_1 = (N \rtimes C) \rtimes G\), with the action \((\sigma_g \circ \text{id})\).
3. \(M_2 = (C \rtimes N) \rtimes G\), with the action \((\text{id} \circ \sigma_g)\).
4. \(M = (N \rtimes N) \rtimes G\), with the diagonal action \((\sigma_g \circ \sigma_g)\).
5. \(N = N^{\varphi}\) and \(\tilde{N} = (N \rtimes N)^{\varphi \varphi}\).
6. \(M = N \rtimes G\) and \(\tilde{M} = \tilde{N} \rtimes G\).

It is clear that \(M = M^{\varphi}\) and \(\tilde{M} = \tilde{M}^{\varphi \varphi}\). We shall identify \(M\) with \(M_1\). We regard \(M_{1,2}\) as subalgebras of \(\tilde{M}\) by considering \(N \rtimes C\) and \(C \rtimes N \subset N \rtimes N\). Moreover, canonically we have the following isomorphism:
\[
\tilde{M} \cong \frac{M_1 \rtimes L(G)}{M_2}.
\]
The next theorem is an analogue of a result by Popa (see Theorem 4.4 in [21]). In the context of free malleable actions, a gauged extension for the action $\sigma$ (see Section 1 in [21]) no longer makes sense. However, regarding a crossed product as a free product with amalgamation (Notation 5.1), and using free etymology techniques as in the proof of Theorem 4.5 we are able to prove the following result.

**Theorem 5.2.** Let $\sigma : G \to \text{Aut}(N, \varphi)$ be a state-preserving s-malleable (freely) mixing action. We shall freely use Notation 5.1. Let $Q \subset M$ be a diffuse subalgebra with the relative property $(T)$. Denote by $P$ the quasi-normalizer of $Q$ inside $M$.

Then, there exist $\gamma > 0$, $n \geq 1$ and a nonzero partial isometry $v \in M_{1,n}(C) \otimes M$ which is a $\gamma$-eigenvector for $\varphi$ and satisfies

$$vv^* \in P \cap Q', \quad v^*v \in L(G)^n, \quad v^*Qv \subset v^*Pv \subset v^*v(M_n(C) \otimes L(G))v^*. $$

**Proof.** We take $(\alpha_t)$ and $\beta$ as in Definition 1.1. We extend $(\alpha_t)$ and $\beta$ to $\tilde{M}$.

**Step (1): Using the relative property (T).** For every $t \in R$, we have the following $Q$-$Q$-bimodule $\mathcal{H}_t = L^2(\tilde{M})$, with

$$x \cdot \xi = x\xi, \quad \xi \cdot x = \xi \alpha_t(x),$$

for all $x \in Q$, $\xi \in L^2(\tilde{M})$. Since the action $(\alpha_t)$ is continuous, we have $\mathcal{H}_t \to \mathcal{H}_0$ as $t \to 0$, in the sense of correspondences. The relative property $(T)$ yields $t = 2^{-s}$, $s \in N^*$ and $\xi \in \mathcal{H}_t$, $\xi \neq 0$, such that

$$x\xi = \xi \alpha_t(x), \forall x \in Q. $$

Taking the polar decomposition of the vector $\xi$ (see Proposition A.1), we find a nonzero partial isometry $v \in \tilde{M}$ satisfying

$$xv = v\alpha_t(x), \forall x \in Q. $$

**Step (2): Proving $Q \not\prec L(G)$ using the amalgamation over $L(G)$.** Assume that $Q \not\prec L(G)$. We shall obtain a contradiction. Definition 4.2 yields a sequence of unitaries $(u_k)$ in $Q$ such that for any $a, b \in M_1$, $\|E_{L(G)}(au_k b)\|_2 \to 0$.

First of all, we shall find a nonzero partial isometry in $\tilde{M}$ satisfying Equation (8) for $t = 1$. In order to do so, it suffices to prove the existence of a nonzero partial isometry $w \in \tilde{M}$ satisfying $xw = w\alpha_{2t}(x)$ for all $x \in Q$. Indeed, iterating the procedure then allows to continue till $t = 1$. Thanks to Theorem 4.5 with $N = L(G)$, we get $\tilde{M} \cap Q' \subset M_1$ and so $\tilde{M} \cap Q' \subset M$. In particular, $vv^* \in M$. We write $vv^* = p$, with $p \in M$. Using the properties of $\beta$ (in particular, $\beta(x) = x$ for all $x \in M$) and Equation (8), one checks that $w := \alpha_t(\beta(v^*)v)$ is an element of $M$ satisfying $xw = w\alpha_{2t}(x)$ for all $x \in Q$. Indeed, for any $x \in Q$,

$$w\alpha_{2t}(x) = \alpha_t(\beta(v^*)v\alpha_t(x))$$

$$= \alpha_t(\beta(v^*)xv)$$

$$= \alpha_t(\beta(v^*)v)$$

$$= \alpha_t(\beta(\alpha_t(x)v^*)v)$$

$$= \alpha_t\beta\alpha_t(x)\alpha_t(\beta(v^*)v)$$

$$= \beta(x)v$$

$$= xv.$$
Moreover, 
\[ \text{ww}^* = \alpha_1(\beta(v^*)p \beta(v)) = \alpha_1(\beta(v^*p)\beta(v)) = \alpha_1(\beta(v^*)v) \].

The last term is a nonzero projection. So, w is the required nonzero partial isometry. Thus, we have found a nonzero partial isometry \( v \in \mathcal{M} \) satisfying 
\[ xv = v\alpha_1(x), \forall x \in Q, \]

Observe now that using the second point of Definition 4.2 we get \( \alpha_1(Q) \neq L(G) \), since \( \alpha_1(L(G)) = L(G) \) and \( \alpha_1(M_1) = M_2 \). Thus, by Theorem 4.5 we get \( \mathcal{M} \cap \alpha_1(Q)^\prime \subset \mathcal{M}_2 \) and so \( \mathcal{M} \cap \alpha_1(Q)^\prime \subset \alpha_1(M) \). In particular, \( v^*v \in \alpha_1(M) \).

**Claim 5.3.** \( \forall x, y \in \mathcal{M}, \|E_{M_2}(xuy)\|_2 \to 0. \)

**Proof of Claim 5.3.** Regarding \( \mathcal{M} = M_1 \ast M_2 \), let \( x, y \in \mathcal{M} \) be either in \( L(G) \) or reduced words in \( \mathcal{M} \) with letters alternatingly from \( M_1 \ominus L(G) \) and \( M_2 \ominus L(G) \). We assume as in the proof of Claim 4.6 that all the letters of \( y \) are eigenvectors for \( \varphi \). We set \( a = x \) with \( a = x \) if \( x \in L(G) \), \( a = 1 \) if \( x \) ends with a letter from \( M_2 \ominus L(G) \) and \( a \) equal to the last letter of \( x \) otherwise. Note that \( x' \) is either equal to \( 1 \) or a reduced word ending with a letter from \( M_2 \ominus L(G) \). In the same way, we set \( y = by' \) with \( b = y \) if \( y \in L(G), b = 1 \) if \( y \) begins with a letter from \( M_2 \ominus L(G) \) and \( b \) equal to the first letter of \( y \) otherwise. Note that \( y' \) is either equal to \( 1 \) or a reduced word beginning with a letter from \( M_2 \ominus L(G) \).

Then for \( z \in Q \), we have \( azb - E_{L(G)}(azb) \in M_1 \ominus L(G), \) and thus 
\[ E_{M_2}(xzy) = E_{M_2}(x'E_{L(G)}(azb)y'). \]

Since all the letters of \( y \) are eigenvectors for \( \varphi \), there exists \( \lambda > 0 \) such that \( y' \in \mathcal{M}^\lambda \). Therefore, 
\[ \|E_{M_2}(xzy)\|_2 \leq \|x'E_{L(G)}(azb)y'\|_2 \leq \lambda^{-1/2}\|x\|\|y'\|\|E_{L(G)}(azb)\|_2. \]

It follows that \( \|E_{M_2}(xuy)\|_2 \to 0. \) We can proceed exactly the way we did in the proof of Claim 4.6, in order to obtain that \( \|E_{M_2}(xuy)\|_2 \to 0, \) for every \( x, y \in \mathcal{M}. \)

We remind that for any \( x \in Q \), \( v^*xv = \alpha_1(x)v^*v. \) Moreover, \( v^*v \in \alpha_1(M) \subset \mathcal{M}_2. \) So, for any \( x \in Q \), \( v^*xv \in \mathcal{M}_2. \) Since \( \alpha_1(u_k) \in \mathcal{U}(\mathcal{M}_2) \), we get 
\[ \|v^*v\|_2 = \|\alpha_1(u_k)v^*v\|_2 = \|E_{M_2}(\alpha_1(u_k)v^*v)\|_2 = \|E_{M_2}(v^*u_kv)\|_2 \to 0. \]

Thus \( v = 0, \) which is a contradiction.

**Step (3): Using the mixing property of the action to conclude.** From Definition 4.2 we get \( \gamma > 0 \), \( n \geq 1 \), \( p \) a projection in \( L(G)^n \), a unital \(*\)-homomorphism \( \theta : Q \to pL(G)^n p \) and a nonzero partial isometry \( w \in M_1 \cap (C) \cap \mathcal{M} \) such that \( w \) is a \( \gamma \)-eigenvector for \( \varphi \) and \( xw = w\theta(x) \) for all \( x \in Q. \) It follows that \( w^*w \in pM^n p \cap \theta(Q)' \). Since \( \theta(Q) \) is diffuse and since the action is mixing, the quasi-normalizer of \( \theta(Q) \) inside \( pM^n p \) is included in \( pL(G)^n p \) by Theorem 3.1 of [21] (see also Theorem D.4 of [33]). Take now \( x \) that quasi-normalizes \( Q \) inside \( M. \) Thus, there exist \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_r \) in \( M \) such that 
\[ xQ = \sum_{i=1}^{k} Qx_i \text{ and } Qx = \sum_{j=1}^{r} y_jQ. \]
Observe moreover that \( ww^* \in M \cap Q' \). We get
\[
\theta(Q)w^*xw \subset w^*Qxw \\
\subset \sum w^*y_jQw \\
\subset \sum w^*y_jw\theta(Q).
\]

Exactly in the same way, we prove that
\[
w^*xw\theta(Q) \subset \sum \theta(Q)w^*x_iw.
\]

Thus \( w^*Pw \) is included in the quasi-normalizer of \( \theta(Q) \) inside \( pM^np \), and so \( w^*Pw \subset pL(G)^np \). Since obviously, \( M \cap Q' \subset P \), we can take \( v = w \) to conclude. \( \square \)

**Remark 5.4.** Note that we used the period 2 automorphism \( \beta \) in a very crucial way. We do not know if the result still holds true for a malleable action, even if \( P \) (the quasi-normalizer of \( Q \) in \( M = N \rtimes G \)) is assumed to be a factor. Remark 4.5 in [15] may shed light on this problem.

5.2. **Intertwining Rigid Subalgebras of Free Products with Amalgamation.**

**Notation 5.5.** Let \( (M_1, \varphi_1) \) and \( (M_2, \varphi_2) \) be von Neumann algebras endowed with almost periodic states. Assume that \( N \subset M_i^{\sim} \) for \( i = 1, 2 \). We write

1. \( M = M_1 \ast M_2 \), with \( \varphi \) the free product state.
2. \( M \) denotes the centralizer of \( M \).
3. \( \tilde{M}_i = M_i \ast (N \otimes L(Z)) \), denoting by \( u_i \in L(Z) \) the canonical generating unitary sitting in \( \tilde{M}_i \).
4. \( \tilde{M} = M \ast (N \otimes L(F_2)) = \tilde{M}_1 \ast \tilde{M}_2 \).
5. \( \tilde{M} \) denotes the centralizer of \( M \).

Note that canonically, we have the following isomorphism:

\[
M^n \cong M_1^n \ast M_2^n, \forall n \in \mathbb{N}^*.
\]

The next theorem can be viewed as a generalization to the almost periodic case of a result by Ioana, Peterson & Popa. They proved (see Theorem 0.1 in [16]) that any relatively rigid von Neumann subalgebra \( Q \subset (M_1, \tau_1) \ast (M_2, \tau_2) \) can be intertwined into one of the \( M_i \)'s. We prove a similar result replacing the faithful normal trace \( \tau_1, \tau_2 \) by any almost periodic faithful normal state \( \varphi_1, \varphi_2 \). The beautiful idea of the proof of Theorem 5.6 was given to us by Stefaan Vaes. We gratefully thank him for allowing us to present it here.

**Theorem 5.6.** Let \( (M_1, \varphi_1) \) and \( (M_2, \varphi_2) \) be von Neumann algebras with faithful normal almost periodic states and \( N \subset M_i^{\sim} \) for \( i = 1, 2 \). Set \( M = M_1 \ast M_2 \) and \( \varphi \) the free product state. We shall freely use Notation 5.5. If \( Q \subset M^\varphi \) is rigid, then there exists \( i \in \{1, 2\} \) such that \( Q \not\prec M_i \).

**Proof.** We assume that \( Q \not\prec M_1 \) and \( Q \not\prec M_2 \); we shall obtain a contradiction.

**Step (0): Defining the deformation property.** Consider \( L(F_2) = L(Z) \ast L(Z) \) with its canonical unitaries \( u_1 \) and \( u_2 \). Let \( f : S^1 \to ]-\pi, \pi] \) be the Borel function satisfying \( \exp(if(z)) = z \) for all \( z \in S^1 \). Define the self-adjoint elements \( h_i = f(u_i) \) for \( i = 1, 2 \).
Regarding $\tilde{M} = \tilde{M}_1 \ast N \tilde{M}_2$, define the one-parameter group of automorphisms $(\alpha_t)$ on $\tilde{M}$ by:

$$\alpha_t = (\text{Ad} \exp(i th_1)) \ast (\text{Ad} \exp(i th_2)).$$

Note that $\alpha_1 = (\text{Ad} u_1) \ast (\text{Ad} u_2)$. Define now the period 2 automorphism $\beta$ on $L(F_2)$ by:

$$\beta(u_1) = u_1^*, \beta(u_2) = u_2^*.$$

Regarding $\tilde{M} = M \ast (N \otimes L(F_2))$, extend $\beta$ to $\tilde{M}$ using the identity automorphism on $M$. We know from Lemma 2.2.2 in [16] that $\beta \alpha_t = \alpha_{-t} \beta$ for every $t \in R$. Thus this deformation if of malleable type as in Definition [1.1].

**Step (1): Using the relative property (T).** Recall that $\tilde{M}$ denotes the centralizer of $\tilde{M}$ and $M$ the centralizer of $\tilde{M}$. For every $t \in R$, define $H_t = L^2(\tilde{M})$ the following $Q \otimes Q$ bimodule:

$$x : \xi = x \xi, \quad \xi : x = \xi \alpha_t(x), \forall x \in Q, \forall \xi \in L^2(\tilde{M}).$$

Since the action $(\alpha_t)$ is continuous, $H_t \to H_0$ as $t \to 0$, in the sense of correspondences. Thus the relative property (T) (and Proposition A.1) yields $t = 2^{-s}, s \in N$ and a nonzero partial isometry $v \in M$ satisfying

$$x v = v \alpha_t(x), \forall x \in Q. \tag{10}$$

**Step (2): Going till $t = 1$ using the deformation property.** We shall find a nonzero partial isometry in $\tilde{M}$ satisfying Equation (10) for $t = 1$. As in the proof of Theorem 5.2 in order to do so, it suffices to prove the existence of a nonzero partial isometry $w \in \tilde{M}$ satisfying $x w = w \alpha_2(x)$ for all $x \in Q$. Indeed, iterating the procedure then allows to continue till $t = 1$.

Since $Q \not\subset M_1$, certainly $Q \not\subset N$. Regarding $\tilde{M} = M \ast (N \otimes L(F_2))$, Theorem 4.5 implies that $\tilde{M} \cap Q' \subset M$ and therefore $\tilde{M} \cap Q' \subset M$. From Equation (10), we get $v v^* \in \tilde{M} \cap Q'$, thus $v v^* \in M$. We write $v v^* = p$ with $p \in M$. Using the properties of $\beta$ (in particular, $\beta(x) = x$ for all $x \in M$), one checks, as in the proof of Theorem 5.2, that $w := \alpha_2(\beta(v^*) v)$ is a nonzero partial isometry satisfying $x w = w \alpha_2(x)$ for all $x \in Q$.

**Step (3): Using the amalgamation over $N$ to obtain the contradiction.** We write $\theta = \alpha_1$. We have found a nonzero partial isometry $v \in \tilde{M}$ satisfying $x v = v \theta(x)$ for all $x \in Q$. Since $Q \not\subset M_1$, certainly $\theta(Q) \not\subset \theta(M_1)$, and thus $\theta(Q) \not\subset N$. Regarding $\tilde{M} = M \ast (N \otimes L(F_2))$, Theorem 4.5 implies that $\tilde{M} \cap \theta(Q)' \subset \theta(M)$ and therefore $\tilde{M} \cap \theta(Q)' \subset \theta(M)$. Since $v^* v \in \tilde{M} \cap \theta(Q)'$, we get $v^* v \in \theta(M)$.

Set $A = L(F_2)$ and define the subspace $H_{alt} \subset L^2(\tilde{M})$ as the closed linear span of $N$ and the words in $M_1 \ast M_2 \ast (N \otimes A)$ with letters alternatingly from $M_1 \otimes N, M_2 \otimes N, N \otimes (A \otimes C1)$ and such that two consecutive letters never come from $M_1 \otimes N, M_2 \otimes N$. This means that letters from $M_1 \otimes N$ and $M_2 \otimes N$ are always separated by a letter from $N \otimes (A \otimes C1)$.

By the definition of $\theta$, it follows that $\theta(M) \subset H_{alt}$. Denote by $P_{alt}$ the orthogonal projection of $L^2(\tilde{M})$ onto $H_{alt}$. Since $Q \not\subset M_1$ and $Q \not\subset M_2$, thanks to Proposition 4.4 we know that there exists a sequence of unitaries $(u_k)$ in $Q$ such that $\|E_{M_k} (x u_k y)\|_2 \to 0$ for all $x, y \in M$ and all $i \in \{1, 2\}$. Moreover, we have the following:
Claim 5.7. \( \forall c, d \in \tilde{\mathcal{M}}, \|P_{alt}(cu_kd)\|_2 \to 0. \)

Proof of Claim 5.7. Let \( c, d \in \tilde{\mathcal{M}} = \mathcal{M} \ast (N \otimes A) \) be either in \( N \) or reduced words with letters alternatingly from \( \mathcal{M} \otimes N \) and \( N \otimes (A \oplus C_1). \) As we did in the proof of Claim 4.6, we assume that all the letters of \( L \) are assumed to be eigenvectors for \( \varphi. \) Set \( c = c'a, \) with \( a = c \) if \( c \in N, a = 1 \) if \( c \) ends with a letter from \( N \otimes (A \oplus C_1) \) and \( a \) equal to the last letter of \( c \) otherwise. Note that either \( c' \) is equal to 1 or is a reduced word ending with a letter from \( N \otimes (A \oplus C_1). \) Exactly in the same way, set \( d = bd', \) with \( b = d \) if \( d \in N, b = 1 \) if \( d \) begins with a letter from \( N \otimes (A \oplus C_1) \) and \( b \) equal to the first letter of \( d \) otherwise. For \( x \in \mathcal{M}, \) write \( cxd = c'(axb)d' \), and note that \( axb \in \mathcal{M}. \) Note that either \( d' \) is equal to 1 or is a reduced word beginning with a letter from \( N \otimes (A \oplus C_1). \) Since \( \mathcal{M} = \mathcal{M}_1 \ast \mathcal{M}_2, \) recall that

\[ \mathcal{M} = \text{span}^\oplus \{ N, \mathcal{M}_1 \otimes N, \mathcal{M}_2 \otimes N, (\mathcal{M}_{i_1} \otimes N) \cdots (\mathcal{M}_{i_n} \otimes N); n \geq 2, i_1 \neq \cdots \neq i_n \in \{1, 2\} \}. \]

By definition of the projection \( P_{alt}, \) it is clear that

\[ P_{alt}(c'zd') = 0, \forall z \in \text{span}^\oplus \{ (\mathcal{M}_{i_1} \otimes N) \cdots (\mathcal{M}_{i_n} \otimes N); n \geq 2, i_1 \neq \cdots \neq i_n \in \{1, 2\} \}. \]

Denote by \( P_1 \) the orthogonal projection of \( L^2(\mathcal{M}) \) onto the space \( L^2(\mathcal{M}_1 \otimes N) \oplus L^2(\mathcal{M}_2 \otimes N). \) By definition of the conditional expectations \( E_{\mathcal{M}_i} (i = 1, 2), \) it is easy to see that

\[ P_1(z) = E_{\mathcal{M}_1}(z) + E_{\mathcal{M}_2}(z) - E_N(z), \]

\[ \|P_1(z)\|_2^2 \leq \|E_{\mathcal{M}_1}(z)\|_2^2 + \|E_{\mathcal{M}_2}(z)\|_2^2, \forall z \in \mathcal{M}. \]

We recall that all the letters of \( d \) are assumed to be eigenvectors for \( \varphi. \) Thus, there exists \( \lambda > 0 \) such that \( d' \) is a \( \lambda \)-eigenvector. Thanks to (11) and (12), we get for any \( z \in Q \)

\[ \|P_{alt}(cxd)\|_2^2 = \|P_{alt}(c'P_1(azb)d')\|_2^2 \]

\[ \leq \|c'P_1(azb)d'\|_2^2 \]

\[ \leq \lambda^{-1}\|c'\|^2\|d'\|^2\|P_1(azb)\|_2^2 \]

\[ \leq \lambda^{-1}\|c'\|^2\|d'\|^2 (\|E_{\mathcal{M}_1}(azb)\|_2^2 + \|E_{\mathcal{M}_2}(azb)\|_2^2). \]

Since \( \|E_{\mathcal{M}_i}(xu_ky)\|_2 \to 0 \) for all \( x, y \in \mathcal{M} \) and all \( i \in \{1, 2\}, \) the inequality (13) implies that \( \|P_{alt}(cu_kd)\|_2 \to 0 \) for \( c, d \) chosen as before. We can now proceed exactly the same way we did in the proof of Claim 4.6 in order to obtain that \( \|P_{alt}(cu_kd)\|_2 \to 0, \) for every \( c, d \in \tilde{\mathcal{M}}. \)

At last, for any \( x \in Q, \) \( v^*xv = \theta(x)v^*v \in \theta(M) \subset H_{alt}. \) So, since \( \theta(u_k) \in U(M), \) we get

\[ \|v^*v\|_2 = \|\theta(u_k)v^*v\|_2 = \|P_{alt}(\theta(u_k)v^*v)\|_2 = \|P_{alt}(v^*uv)\|_2 \to 0. \]

It follows that \( v = 0, \) which is a contradiction.

5.3. Fundamental Groups of Type II\(_1\) Factors. We denote by \( \mathcal{F}(M) \subset \mathbb{R}^*_+ \) the fundamental group of a type \( II_1 \) factor \( M, \) and by \( \text{Sp}(N, \varphi) \subset \mathbb{R}^*_+ \) the point spectrum of the modular operator \( \Delta_\varphi \) of an almost periodic state \( \varphi \) on \( N. \) We shall denote by \( \Gamma_{\text{Sp}(N, \varphi)} \subset \mathbb{R}^*_+ \) the subgroup generated by \( \text{Sp}(N, \varphi). \) Note that if the centralizer \( N^{\varphi} \) is a factor, then \( \text{Sp}(N, \varphi) \) is a multiplicative subgroup and then \( \Gamma_{\text{Sp}(N, \varphi)} = \text{Sp}(N, \varphi) \) (see [8]). As a consequence of Theorem 5.2 we obtain the following result.

Theorem 5.8. Let \( G \) be an ICC \( w \)-rigid group. Let \( \sigma : G \to \text{Aut}(N, \varphi) \) be a state-preserving \( s \)-malleable (freely) mixing action with \( \varphi \) an almost periodic state. Assume that
the centralizer $N^\circ$ is a factor. Denote by $M$ the crossed product $N^\circ \rtimes G$ as in Notation 5.4. Then, $M$ is a type $\Pi_1$ factor and one has

$$\text{Sp}(N, \varphi) \subset \mathcal{F}(M) \subset \text{Sp}(N, \varphi)\mathcal{F}(L(G)).$$

In particular, if $\mathcal{F}(L(G)) = \{1\}$, then $\mathcal{F}(M) = \text{Sp}(N, \varphi)$.

Proof. We refer to the proof of Theorem 5.2 and Corollary 5.4 in [21] (see also Theorem 7.1 in [34]). The arguments are exactly the same. However, we shall give the proof for the sake of completeness. We should mention here that controlling quasi-normalizers will rely on a result of Popa (Theorem 3.1 in [21]). We remind that Theorem 3.1 in [21] uses in a crucial way the mixing property of the action. Denote as in Notation 5.4, $N \rtimes G$ by $M$ and $N^\circ \rtimes G$ by $M$. Since the group $G$ is ICC and $N^\circ$ is a factor, $M = N^\circ \rtimes G$ is necessarily a type $\Pi_1$ factor. Note that $\text{Sp}(M, \varphi) = \text{Sp}(N, \varphi) \subset \mathbb{R}_+$ is a multiplicative subgroup.

It was shown in [11] that the inclusion $\text{Sp}(N, \varphi) \subset \mathcal{F}(M)$ holds. Indeed, take $\gamma \in \text{Sp}(N, \varphi)$ and $v$ a nonzero partial isometry in $N^\gamma$. Write $p = v^*v$ and $q = vv^*$. Then, $p, q \in N^\circ \subset M$, $\varphi(q) = \gamma\varphi(p)$, and $\text{Ad}(v)$ yields a $\ast$-isomorphism between $pMp$ and $qMq$. Therefore, $\gamma \in \mathcal{F}(M)$.

Conversely, assume that $t \in \mathcal{F}(M)$ and let $\theta : M \to M^t$ be a $\ast$-isomorphism. We assume that $t \geq 1$. Realize $M^t := p(M_n(C) \otimes M)p$. Let $H \subset G$ be an infinite normal subgroup with the relative property (T). Since $H \subset G$ is normal, it is clear that $L(G)$ is contained in the quasi-normalizer of $L(H)$ inside $M$. Moreover since $L(H)$ is diffuse, Theorem 3.1 of [21] implies that the quasi-normalizer of $L(H)$ inside $M$ is exactly $L(G)$. Write $Q = \theta(L(H))$ and $P = \theta(L(G))$. The inclusion $Q \subset P$ still has the relative property (T) and $P$ is the quasi-normalizer of $Q$ inside $M^t$. Since $s = 1/t \leq 1$, choose a projection $q \in Q$ with trace $s$. Write $Q^s := qQq$ and $P^s := qPq$. We regard $Q^s \subset P^s \subset M$. The inclusion $Q^s \subset P^s$ has the relative property (T), $Q^s$ is diffuse and $P^s$ is the quasi-normalizer of $Q^s$ inside $M$.

We can apply Theorem 5.2 in order to obtain $\gamma > 0$ and a nonzero partial isometry $w \in M_{1,r}(C) \otimes M$ which is a $\gamma$-eigenvector for $\varphi$ and such that $w^*w \in M_{r}(C) \otimes L(G)$, $ww^* \in M \cap (Q^s)^c \subset P^s$.

$$w^*Q^sw \subset w^*P^sw \subset w^*w(M_{r}(C) \otimes L(G))w^*w.$$

Since $P$ is a factor, we can find partial isometries $x_1, \ldots, x_m \in P$ such that $x_i^*x_i \leq ww^*$, for every $i \in \{1, \ldots, m\}$ and $\sum_i x_i^*x_i = 1$. Let $k = mr$, and write $v = [x_1w \cdots x_mw]$. Since $P \subset M^t = p(M_n(C) \otimes M)p$, we can regard $v \in M_{n,k}(C) \otimes M$, and $v$ is a $\gamma$-eigenvector for $\varphi$. Moreover, we have

$$v^*Qv \subset v^*Pv \subset L(G)^{1/\gamma},$$

with $vv^* = p$, $v^*v := q \in M_{k}(C) \otimes L(G)$, and $L(G)^{1/\gamma} := q(M_{k}(C) \otimes L(G))q$. Note that increasing $n$ or $k$ if necessary, we may assume $k = n$. We want to prove that in fact, $v^*Pv = q(M_{n}(C) \otimes L(G))q = L(G)^{1/\gamma}$. If we do so, we are done. Indeed, we have $L(G) \simeq L(G)^{1/\gamma}$, and so $t/\gamma \in \mathcal{F}(L(G))$. Consequently, $t \in \text{Sp}(N, \varphi)\mathcal{F}(L(G))$.

Changing $q$ to an equivalent projection in $M_{n}(C) \otimes L(H)$, we may assume that $q \in M_{n}(C) \otimes L(H)$. Define

$$Q_1 := \theta^{-1}(v(M_{n}(C) \otimes L(H))v^*) \quad \text{and} \quad P_1 := \theta^{-1}(v(M_{n}(C) \otimes L(H))v^*).$$

The inclusion $Q_1 \subset P_1$ has the relative property (T), $P_1$ is the quasi-normalizer of $Q_1$ inside $M$, and Equation (14) yields $L(G) \subset P_1$. We want to prove that $P_1 \subset L(G)$. Once again using Theorem 5.2, we get that there exist $k \geq 1$, $\lambda > 0$ and a nonzero partial isometry $w \in M_{1,k}(C) \otimes \mathcal{M}^{\lambda}$, such that $ww^* = 1$, $w^*w \in M_{k}(C) \otimes L(G)$ and
$w^*P_1w \subset L(G)^{1/\lambda}$, where we have realized $L(G)^{1/\lambda} := w^*w(M_k(C) \otimes L(G))w^*w$. Since $w w^* = 1$, we have $P_1w \subset wL(G)^{1/\lambda}$. Since $L(G) \subset P_1$, it follows that $L(G)w \subset wL(G)^{1/\lambda}$.

From Theorem 3.1 in [21], since $L(G)$ is diffuse, we know that any $L(G)$-L($G$) subbimodule $\mathcal{H}$ of $L^2(M)$ such that $\dim(\mathcal{H}_L(G)) < \infty$ (as a right $L(G)$-module) is contained in $L^2(L(G))$.

In particular, this implies that $w \in M_{1,k}(C) \otimes L(G)$. Since $P_1 \subset wL(G)^{1/\lambda}w^*$, we get $P_1 \subset L(G)$. We are done. □

As a corollary, we obtain the result we mentioned in the introduction.

**Corollary 5.9.** Let $G$ be an ICC $w$-rigid group such that $\mathcal{F}(L(G)) = \{1\}$. Let $\Gamma \subset \mathbb{R}_+^d$ be a countable subgroup. Set $(N, \varphi) = (T_\Gamma, \varphi_\Gamma)$ the unique almost periodic free Araki-Woods factor whose $Sd$ invariant equals $\Gamma$. Assume that $G$ acts on $(N, \varphi)$ by free Bogoliubov shifts w.r.t. to the left regular representation $\lambda_G$ (see Section 3). Write $M = N^\varphi \rtimes G$. Then $M$ is a type $\Pi_1$ factor and $\mathcal{F}(M) = \Gamma$.

We prove at last the second result we mentioned in the introduction. A type $\Pi_1$ factor $N$ is said to be $w$-rigid if it contains a diffuse von Neumann subalgebra $B$ such that the inclusion $B \subset N$ is quasi-regular and has the relative property (T). Of course, if $G$ is an ICC $w$-rigid group, $L(G)$ is a $w$-rigid type $\Pi_1$ factor.

**Theorem 5.10.** Let $N$ be a $w$-rigid type $\Pi_1$ factor such that $\mathcal{F}(N) = \{1\}$. Let $(A, \psi)$ be a von Neumann algebra endowed with an almost periodic state. Assume that the centralizer $A^\psi$ has the Haagerup property. Write $M = (N \ast A)^{\tau \psi}$. Then $M$ is a type $\Pi_1$ factor and $\mathcal{F}(M) = \Gamma_{Sp(A, \psi)}$.

**Proof.** The proof will go as the one of Theorem 5.8 but on Theorem 4.5 of the present paper. However, we shall sketch the proof for completeness. Denote by $(M, \varphi) = (N, \tau) \ast (A, \psi)$, $M = M^\varphi$ and $A = A^\psi$. First of all, we prove that $M$ is a factor of type $\Pi_1$. Note that $N \subset M$. Take now $x \in Z(M) = M \cap M'$. Since $N$ is diffuse, $N \not\subset C$. Theorem 4.5 implies $M \cap M' \subset N$. Consequently, $x \in N \cap N' = Z(N) = C1$.

It follows that $Z(M) = C1$. Note that in this case, one has $Sp(M, \varphi) = \Gamma_{Sp(A, \psi)}$.

We already know that the inclusion $\Gamma_{Sp(A, \psi)} \subset \mathcal{F}(M)$ holds. Conversely, let $t \in \mathcal{F}(M)$ and let $\theta : M \rightarrow M^t$ be a $*$-isomorphism. We assume that $t \geq 1$. Realize $M^t := p(M_n(C) \otimes M)p$. Let $B \subset G$ be a diffuse von Neumann subalgebra such that the inclusion $B \subset N$ is quasi-regular and has the relative property (T). Since $B \subset N$ is quasi-regular, $N$ is contained in the quasi-normalizer of $B$ inside $M$. Moreover, since $B$ is diffuse, Theorem 4.5 implies that the quasi-normalizer of $B$ inside $M$ is exactly $N$. Write $Q = \theta(B)$ and $P = \theta(N)$. The inclusion $Q \subset P$ still has the relative property (T) and $P$ is the quasi-normalizer of $Q$ inside $M'$. Since $s = 1/t \leq 1$, as we did before, choose a projection $q \in Q$ with trace $s$. Write $Q^s := qQq$ and $P^s := qPq$. We regard $Q^s \subset P^s \subset M$. The inclusion $Q^s \subset P^s$ has the relative property (T), $Q^s$ is diffuse and $P^s$ is the quasi-normalizer of $Q^s$ inside $M$. We know from Theorem 5.6 that either $Q^s \preceq M$ or $Q^s \preceq M$.

**Claim 5.11.** $Q^s \not\preceq M$.

**Proof of Claim 5.11.** Assume that $Q^s \not\preceq M$. We shall obtain a contradiction. By Theorem 4.3, we know that there exist $m \geq 1$, $\gamma > 0$, a projection $e \in M_m(C) \otimes A$, a nonzero partial isometry $v \in M_{1,m}(C) \otimes M^\psi$ and a (unital) $*$-homomorphism $\rho : Q^s \rightarrow e(M_m(C) \otimes A)e$ such that $v^*v \leq e$ and $xv = \rho(x), \forall x \in Q^s$. 

Note that \( vv^* \in M \cap (Q^*)' \subset P^\ast \). In the same way, \( v^*v \in e(M_n(C) \otimes M)e \cap \rho(Q^*)' \). Since \( \rho(Q^*) \) is diffuse, Theorem 4.3 and Equation 13) tell us that the quasi-normalizer of \( \rho(Q^*) \) inside \( e(M_n(C) \otimes M)e \) is contained in \( e(M_n(C) \otimes A)e \). Consequently \( v^*v \in e(M_n(C) \otimes A)e \) so that we may assume \( e = v^*v \). In the same way, since \( e\rho(Q^*) \) is still diffuse, Theorem 4.3 tells us that the quasi-normalizer of \( e\rho(Q^*) \) inside \( eMe \) is contained in \( e(M_n(C) \otimes A)e \). In particular, \( v^*P^2v \subset e(M_n(C) \otimes A)e \). But the inclusion \( v^*Q^2v \subset V^*P^2v \) has the relative property (T) and the von Neumann algebra \( v^*Q^2v \) is diffuse. Since A is assumed to have the Haagerup property, this cannot happen thanks to Theorem 5.4 in [23] (see also Theorem 2.11 in Section 2). We have a contradiction. \( \square \)

Consequently, we obtain that \( Q^2 \preceq N \). We proceed as in the proof of Claim 5.11 and the proof of Theorem 5.8. Increasing \( n \) if necessary, we obtain \( \gamma > 0 \) and a nonzero partial isometry \( v \in M_n(C) \otimes M \), which is a \( \gamma \)-eigenvector for \( \varphi \) such that

\[
(15) \quad v^*Qv \subset v^*Pv \subset N^{1/\gamma},
\]

with \( vv^* = p, v^*v := q \in M_n(C) \otimes N \), and \( N^{1/\gamma} := q(M_n(C) \otimes N)q \). We want to prove that in fact, \( v^*Pv = q(M_n(C) \otimes N)q = N^{1/\gamma} \). If we do so, we are done. Indeed, we have \( N \approx N^{1/\gamma} \), and so \( t > \gamma \in \mathcal{F}(N) = \{1\} \). Consequently, \( t = \gamma \in \Gamma_{Sp(A,\psi)} \).

Changing \( q \) to an equivalent projection in \( M_n(C) \otimes B \), we can always assume that \( q \in M_n(C) \otimes B \). Define

\[
Q_1 := \theta^{-1}(v(M_n(C) \otimes B)v^*) \quad \text{and} \quad P_1 := \theta^{-1}(v(M_n(C) \otimes N)v^*).
\]

The inclusion \( Q_1 \subset P_1 \) has the relative property (T), and \( P_1 \) is the quasi-normalizer of \( Q_1 \) inside \( M \), and Equation 15) yields \( N \subset P_1 \). We want to prove that \( P_1 \subset N \). Once again using Theorem 5.6 and Claim 5.11 we get \( Q_1 \subset N \). Thus, there exists \( k \geq 1 \), \( \lambda > 0 \) and a nonzero partial isometry \( w \in M_{1,k}(C) \otimes M^\lambda \), such that \( ww^* = 1 \), \( w^*w \in M_k(C) \otimes N \) and \( w^*P_1w \subset N^{1/\lambda} \), where we have realized \( N^{1/\lambda} := w^*w(M_k(C) \otimes N)w^*w \). Since \( ww^* = 1 \), we have \( P_1w \subset wN^{1/\lambda} \). Since \( N \subset P_1 \), it follows that \( Nw \subset wN^{1/\lambda} \). From Theorem 4.5 since \( N \) is diffuse, we know that any \( N \)-\( N \) subbimodule \( \mathcal{H} \) of \( L^2(N) \) such that \( \text{dim}(\mathcal{H}) < \infty \) (as a right \( N \)-module) is contained in \( L^2(N) \). In particular, this implies that \( w \in M_{1,k}(C) \otimes N \). Since \( P_1 \subset w(N^{1/\lambda})w^* \), we get \( P_1 \subset N \). We are done.

**Appendix A. On the Polar Decomposition of a Vector**

The von Neumann algebra \( M \) is assumed to be endowed with a faithful normal almost periodic state \( \varphi \). We regard \( M \subset B(L^2(M,\varphi)) \). For \( \gamma \in \text{Sp}(M,\varphi) \), denote as usual by \( M^\gamma \subset M \) the subspace of all \( \gamma \)-eigenvectors of the state \( \varphi \). Denote by \( M_{alg} := \text{span}\{M^\gamma : \gamma \in \text{Sp}(M,\varphi)\} \). Let \( \gamma \in \text{Sp}(M,\varphi) \). Let \( \xi \in L^2(M^\gamma) \) such that \( \xi \neq 0 \). Let \( T^0_\xi : \widetilde{M}_{alg} \to L^2(M,\varphi) \) be the linear operator defined by

\[
T^0_\xi(\tilde{x}) = \lambda^{1/2}\xi x, \forall x \in M^\lambda.
\]

The aim of this Appendix is to prove the following proposition: it is well known from specialists, but we give a proof for the sake of completeness.

**Proposition A.1.** The densely defined operator \( T^0_\xi \) is closable. Denote by \( T^1_\xi \) its closure. The operator \( T^1_\xi \) is affiliated with \( M \). Write \( T^1_\xi = \nu[T^1_\xi] \) for its polar decomposition. Then, \( \nu \in M^\gamma \) and \( [T^1_\xi] \) is affiliated with the centralizer \( M^{\nu} \). Moreover, if \( B \subset M^\gamma \) is a von Neumann subalgebra such that for every \( x \in B \), \( x\xi = \xi x \), then for every \( x \in B \), we have \( xv = vx \).
Proof. First, we prove that the operator \( T^0_\xi \) is closable. It suffices to show that \((T^0_\xi)^*\) is densely defined. Let \( \alpha \in \text{Sp}(\mathcal{M}, \varphi) \). Set \( \beta = \gamma \alpha \). Let \( y \in \mathcal{M}^\beta \) and \( z \in \mathcal{M}^\gamma \). Then,
\[
(T^0_\xi(y), z) = \alpha^{-1/2} \langle \xi y, z \rangle \\
= \alpha^{-1/2} \langle J_\varphi y J_\varphi \xi, z \rangle \\
= \alpha^{-1/2} \langle z^* J_\varphi y^* J_\varphi \xi, \hat{1} \rangle \\
= \alpha^{-1/2} \langle J_\varphi y^* J_\varphi z^* \xi, \hat{1} \rangle \\
= \alpha^{-1/2} \langle \hat{y}, J_\varphi z^* \xi \rangle \\
= \alpha^{-1/2} \langle \hat{y}, (J_\varphi \xi)z \rangle \\
= (\gamma \alpha)^{-1/2} \langle \hat{y}, (S_\varphi \xi)z \rangle \\
= \langle \hat{y}, T^0_\xi(\xi z) \rangle.
\]
If \( \beta \neq \gamma \alpha \), then \( \langle T^0_\xi(y), z \rangle = \langle \hat{y}, T^0_\xi(\xi z) \rangle = 0 \). Consequently, for every \( y, z \in \mathcal{M}_{\text{alg}}, \)
\[
\langle T^0_\xi(y), z \rangle = \langle \hat{y}, T^0_\xi(\xi z) \rangle.
\]
Then \( T^0_\xi \) is closed and we denote by \( T_\xi \) its closure. We prove now that \( T_\xi \) is affiliated with \( \mathcal{M} \). Let \( \alpha, \lambda \in \text{Sp}(\mathcal{M}, \varphi) \). Let \( a \in \mathcal{M}^\alpha \) and \( x \in \mathcal{M}^\lambda \). On the one hand,
\[
T^0_\xi J_\varphi a^* J_\varphi (\hat{x}) = \alpha^{1/2} T^0_\xi(\hat{x}a) = \lambda^{-1/2} \xi xa.
\]
On the other hand,
\[
J_\varphi a^* J_\varphi T^0_\xi(\hat{x}) = \lambda^{-1/2} \xi xa.
\]
Consequently, we have \( J_\varphi a^* J_\varphi T^0_\xi \subset T^0_\xi J_\varphi a^* J_\varphi \), for every \( a \in \mathcal{M}^\lambda \). Since \( J_\varphi \mathcal{M}_{\text{alg}} J_\varphi \) is \( \sigma \)-weakly dense in \( \mathcal{M}' \), it follows that \( T_\xi \) is affiliated with \( \mathcal{M} \).

Write \( T_\xi = v|T_\xi| \) for the polar decomposition of \( T_\xi \). We know that \( v \in \mathcal{M} \). Let \( \lambda \in \text{Sp}(\mathcal{M}, \varphi) \) and \( x \in \mathcal{M}^\lambda \). Then, for every \( t \in \mathbb{R}, \)
\[
\Delta^it T^0_\xi \Delta^{-it}_\varphi (\hat{x}) = \lambda^{-it} \Delta^it T^0_\xi(\hat{x}) = \lambda^{-1/2} \lambda^{-it} \Delta^it_\varphi (\xi x) = \gamma^it \lambda^{-1/2} \xi x = \gamma^it T^0_\xi(\hat{x}).
\]
Thus, it follows that \( \Delta^it T_\xi \Delta^{-it}_\varphi = \gamma^it T_\xi \), for any \( t \in \mathbb{R} \). But, we also have
\[
\Delta^it T_\xi \Delta^{-it}_\varphi = (\Delta^it v \Delta^{-it}_\varphi)(\Delta^it|T_\xi| \Delta^{-it}_\varphi).
\]
By uniqueness of the polar decomposition, we get for every \( t \in \mathbb{R}, \)
\[
\Delta^it v \Delta^{-it}_\varphi = \gamma^it v.
\]
\[
\Delta^it|T_\xi| \Delta^{-it}_\varphi = |T_\xi|.
\]
Consequently, \( \sigma^t_\varphi(v) = \gamma^it v \), for every \( t \in \mathbb{R} \), and so \( v \in \mathcal{M}^\gamma \). Since \( \mathcal{M}^\varphi = \mathcal{M} \cap \{ \Delta^it v : t \in \mathbb{R} \} \), it follows that \( |T_\xi| \) is affiliated with \( \mathcal{M}^\varphi \).

At last, let \( B \subset \mathcal{M}^\varphi \) be a von Neumann subalgebra such that for any \( x \in B \), \( x\xi = \xi x \). Fix \( x \in B \). It is straightforward to check that \( x T_\xi \subset T_\xi x \). We also have \( x(T_\xi)^* \subset (T_\xi)^*x \), and so \( x(T_\xi)^*T_\xi \subset (T_\xi)^*T_\xi x \). By functional calculus, it follows that \( x|T_\xi| \subset |T_\xi|x \). Moreover, since \( \mathcal{M}^\varphi \) is a finite von Neumman algebra, since \( x \in \mathcal{M}^\varphi \) and \( |T_\xi| \) is affiliated
with \(M^\circ\), it follows that \(x|T_\xi|\) and \(|T_\xi|x\) are closed, affiliated with \(M^\circ\) and consequently the equality \(x|T_\xi| = |T_\xi|x\) holds. Thus,

\[
xv|T_\xi| = xT_\xi \\
\quad \subset T_\xi x \\
\quad \subset v|T_\xi|x \\
\quad \subset vx|T_\xi|.
\]

It follows that \(xv\) and \(vx\) coincide on the range of \(|T_\xi|\), and so \(xv = vx\). Thus, \(xv = vx\), for every \(x \in B\).

\[\square\]

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UCLA, Department of Mathematics, Los Angeles, CA 90095, USA
E-mail address: cyril@math.ucla.edu