Comment on “Extension of neoclassical rotation theory for tokamaks to realistically account for the geometry of magnetic flux surfaces”

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Abstract. The derivation presented in the paper (C. Bae et al 2013 Nucl. Fusion 53 043011) relies heavily on the approximate solution of the electron momentum balance equation for the poloidal component of the electric field. One can show that, within the assumptions specified by the model, the exact solution of the resulting equation leads to an unphysical expression for the electrostatic potential relative to the magnetic axis. Remarks on its treatment of the radial and toroidal components of the electric field also appear.
Comment on “Extension of neoclassical rotation theory for tokamaks to realistically account for the geometry

In their recent paper [1], Bae, Stacey, and Solomon present a model for plasma rotation within a tokamak. Their work focuses on the adaptation of the concentric circular flux surface model [2] to a geometry which better represents what is believed to exist within a D shaped tokamak. While considerable attention is paid to the ion fluid equations, the expressions derived rely heavily on the approximate solution of the poloidal electron equation of motion in circular geometry. With respect to the assumptions specified by the model, one can show that the exact solution of the resulting equation leads to an unphysical expression for the electrostatic potential relative to the magnetic axis.

The authors assume in their Eqn. (41) a form for the electrostatic potential of
\[ \Phi_{\text{model}}(r, \theta) \equiv \bar{\Phi}(r)[1 + \Phi^c(r) \cos(\theta) + \Phi^s(r) \sin(\theta)] , \tag{1} \]
where the approximation is between the physical (unknowable) potential and that of the model \( \Phi_{\text{physical}} \approx \Phi_{\text{model}} \), not between the symbol for the model’s potential and its definition in terms of the Fourier degrees of freedom considered by the model. The same form is assumed for the electron density in Eqn. (22a) while the electron temperature is assumed to be constant over the flux surface, thus the model for the electron pressure is defined as
\[ p_e(r, \theta) \equiv T_e(r)\bar{n}_e(r)[1 + n_e^c(r) \cos(\theta) + n_e^s(r) \sin(\theta)] . \tag{2} \]
These expressions are then put into the electron poloidal equation of motion, retaining only “the pressure and electric field terms” in their Eqn. (19),
\[ 0 = h_\theta^{-1} \partial_\theta p_e(r, \theta) + e n_e(r, \theta) E_\theta(r, \theta) , \tag{3} \]
which reduces for \( E_\theta = -h_\theta^{-1} \partial_\theta \Phi \) in the static case to
\[ 0 = T_e[n_e^s \cos(\theta) - n_e^c \sin(\theta)] - e\bar{\Phi}[1 + n_e^c \cos(\theta) + n_e^s \sin(\theta)][\Phi^s \cos(\theta) - \Phi^c \sin(\theta)] , \tag{4} \]
where the poloidal measure factor \( h_\theta \) and the mean electron density \( \bar{n}_e \) cancel out of the equation and the dependence on \( r \) is implicit. The authors then claim in their Eqn. (42) that the approximate solution
\[ \begin{bmatrix} \Phi^c \\ \Phi^s \end{bmatrix} \approx \begin{bmatrix} n_e^c \\ n_e^s \end{bmatrix} \frac{T_e}{e\bar{\Phi}} \tag{5} \]
is valid over the entire region of consideration. Let us now examine under what conditions the RHS expression constitutes a valid solution.

Consider first the concentric circular flux surface approximation [2] upon which the model [1] is based. In this case, the unity, cosine, and sine moments of the flux surface average are given by the integral expressions
\[ \langle X \rangle_{\{U,C,S\}} = (2\pi)^{-1} \int_{-\pi}^{\pi} d\theta \{1, \cos(\theta), \sin(\theta)\}[1 + \varepsilon \cos(\theta)]X(r, \theta) , \tag{6} \]
where $\varepsilon \equiv r/R_0$ is the ratio of the minor radial location to the central major radius.

Taking those moments of Eqn. (4) yields the system

$$
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\propto
\begin{bmatrix}
4\varepsilon \bar{\Phi} (n^c_e \Phi^c - n^s_e \Phi^s) + 4\varepsilon (T_e n^s_e - e\bar{\Phi} \Phi^s) \\
4(T_e n^s_e - e\bar{\Phi} \Phi^s) + \varepsilon e\bar{\Phi} (n^c_e \Phi^c - 3n^c_s \Phi^s) \\
4(e\bar{\Phi} \Phi^c - T_e n^c_s) + \varepsilon e\bar{\Phi} (n^c_e \Phi^c - n^c_s \Phi^s)
\end{bmatrix}
\equiv
\begin{bmatrix}
U \\
C \\
S
\end{bmatrix},
$$

(7)

where the constant of proportionality is equal to 8. The terms with a factor of $\varepsilon$ arise explicitly from consideration of the toroidal measure factor $R/R_0 = [1 + \varepsilon \cos(\theta)]$ and would not be present for a cylindrical containment vessel $R_0 \to \infty$ with vanishing aspect ratio. In the limit $\varepsilon \to 0$ one can indeed say that Eqn. (5) gives a solution of equations $C$ and $S$ with $U$ satisfied identically; however, in that limit $n^c_s \to 0$ by continuity. For finite $R_0$, Eqn. (5) is valid only on the magnetic axis $r = 0$, yet the authors apply this model throughout the confinement region where, at its greatest, the flux surface aspect ratio $\varepsilon$ approaches 1/3. Passing the full system of equations $[U, C, S]$ to one’s favorite computer algebra software, one finds that no explicit solution can be found for $[\bar{\Phi}, \Phi^c, \Phi^s]$. The expression for $U$ is conspicuous by its absence from the model [1, 2], when the unity moment of every other equation is considered.

One can linearize the form of Eqn. (7) by rewriting $[U, C, S]$ in terms of $T_e/e\bar{\Phi}$, yielding the equivalent system

$$
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\propto
\begin{bmatrix}
4\varepsilon n^c_e & 4n^s_e & -4(n^c_e + \varepsilon) \\
4n^c_e & \varepsilon n^s_e & -(4 + 3\varepsilon n^c_e) \\
-4n^c_e & (4 + \varepsilon n^c_e) & -\varepsilon n^s_e
\end{bmatrix}
\begin{bmatrix}
T_e/e\bar{\Phi} \\
\Phi^c \\
\Phi^s
\end{bmatrix}.
$$

(8)

In matrix form, one should instantly recognize a set of linear, homogeneous algebraic equations whose only (unique) solution is trivial, $[T_e/e\bar{\Phi}, \Phi^c, \Phi^s] = [0, 0, 0]$. If one asserts that $T_e$ is not zero, then $\bar{\Phi}$, defined as the mean potential difference between locations on the flux surface and the magnetic axis, must be infinite. One can achieve the same result without recourse to the flux surface average simply by evaluating Eqn. (4) at $\theta_i$ and $\bar{\Phi}$, defined as the mean potential difference between locations on the magnetic axis, must be infinite. One can achieve the same result without recourse to the flux surface average simply by evaluating Eqn. (4) at

$$
0 \propto E_i \equiv 2[n^c_e \sin(\theta_i) - n^s_e \cos(\theta_i)]T_e/e\bar{\Phi}
$$

\[ - [n^c_e \sin(2\theta_i) - n^s_e \cos(2\theta_i) + 2 \sin(\theta_i) + n^c_s \Phi^c

+ [n^c_e \sin(2\theta_i) + n^c_s \cos(2\theta_i) + 2 \cos(\theta_i) + n^c_s \Phi^c],
\]

(9)

will always be linearly independent on account of the trigonometric functions. Since the secondary authors (Stacey and Solomon) were informed of this issue as early as 2007 [3], their continued support for the use of a model with such obvious difficulties is hard to understand. A detailed investigation of these equations appears in Ref. [4].

Some further, ancillary remarks now follow. Since the integrals over the Miller flux surfaces have to be done numerically, one wonders why the authors do not work directly with the 2-D equilibrium data provided by the experimentalists rather than its 1-D summary. The evaluation of $\nu^1_{ij}$, their Eqn. (34), for the Stacey-Sigmar model in the concentric circular flux surface approximation has been available in the literature since
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2011 [5] (and earlier on the arXiv); a term arising from the change in the gyroviscous coefficient dependent upon the gyrofrequency still is missing from their $\nu^2_{ij}$. In their evaluation of the unity moment of the radial momentum balance, Eqn. (B4), they neglect the contributions of the inertial term \(-\langle \hat{r} \cdot \mathbf{nm} (V \cdot \nabla)V \rangle\), and of the radial shear viscous force \(-\langle \hat{r} \cdot \nabla \cdot \Pi_S \rangle\) on account of their assumption in Eqn. (A7) that $|B_\theta/B| \approx 0$, which together are sufficient to balance the force from the pressure gradient and $V \times B$ terms [5] without invoking the presence of a radial electric field $E_r$.

Finally, the authors in several places mention comparison of their model [1] with results obtained in the concentric circular flux surface approximation [2]. Readers should be made aware that the calculations presented in that paper did not include the effect of the toroidal electric field despite its Eqn. (27) implying the contrary. The relevant lines of the code used for [2] read:

202: $Y_i = \text{extmomhat}_\text{tor}_i + \beta_\text{av} \cdot \mathbf{v}_\text{th}_i \cdot \mathbf{r}_\text{hat}_i/nustar_iz$
203: $Y_z = \text{extmomhat}_\text{tor}_z + \beta_\text{av} \cdot \mathbf{v}_\text{th}_z \cdot \mathbf{r}_\text{hat}_z/nustar_izi$

which account for the NBI momentum input and the radial flux term but not the toroidal electric field. Whether or not the authors are accounting for $E_\phi$ in [1] is unclear, since the only experimental profiles displayed are those of the poloidal and toroidal components of the carbon velocity. The presence of the loop voltage in its Table 1 suggests that it is; however, correspondence with the primary author [6] indicates that such may not be the case. In order to assess the reproducibility of their results, the authors should clearly state how the toroidal electric field is evaluated in their numerical calculation. Unpublished investigations (available on the arXiv) indicate that its effect on the toroidal velocities is not insignificant, and a method for its evaluation given the time rate of change of the current through the central solenoid and poloidal field shaping coils is available in the literature [7].

In closing, a few remarks on the source of the difficulties faced by the neoclassical model for plasma physics are in order. No theory of electromagnetism is complete without the inclusion of Gauss’s law, which is conspicuous by its absence from the model equations [1] [2]. Much confusion abounds over what many call the quasi-neutral approximation, which might be better nominated as the neutral fluid limit. For comparison, the quasi-static approximation states that $\nabla \cdot \mathbf{J}_\text{model} = 0$ when $\nabla \cdot \mathbf{J}_\text{physical} = -\partial_t \mathbf{j}_\text{physical}$ is vanishingly small; its electrostatic analogue is $\nabla \cdot \epsilon_0 \mathbf{E}_\text{model} = 0$ when $\nabla \cdot \epsilon_0 \mathbf{E}_\text{physical} = \mathbf{j}_\text{physical}$ is vanishingly small, such that $\mathbf{E}_\text{physical} \approx \mathbf{E}_\text{model} = 0$. The classical equations of Maxwell can be expressed most succinctly [8, 9, 10] as $d \star dA = J$ for $dA \equiv 0$, in terms of the exterior derivative $d$, the Hodge dual $\star$, the connection 1-form $A$, and the current 3-form $J$. Those equations in coordinate-free notation, while impractical for any particular calculation, demonstrate the physical equivalence of the inhomogeneous Maxwell equations; the Maxwell-Ampere relation is nothing but Gauss’s law in a different frame of reference. One can easily show that even the most basic of derivations, such as that for the cold plasma dispersion relation [11], when done in the field formulation without Gauss’s law, are not consistent with the potential formulation.
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Simply put, no one can do better than Maxwell by doing less.

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