Fermionic $R$-Operator and Algebraic Structure of 1D Hubbard Model: Its application to quantum transfer matrix

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Abstract

The algebraic structure of the 1D Hubbard model is studied by means of the fermionic $R$-operator approach. This approach treats the fermion models directly in the framework of the quantum inverse scattering method. Compared with the graded approach, this approach has several advantages. First, the global properties of the Hamiltonian are naturally reflected in the algebraic properties of the fermionic $R$-operator. We want to note that this operator is a local operator acting on fermion Fock spaces. In particular, $SO(4)$ symmetry and the invariance under the partial particle hole transformation are discussed. Second, we can construct a genuinely fermionic quantum transfer transfer matrix (QTM) in terms of the fermionic $R$-operator. Using the algebraic Bethe Ansatz for the Hubbard model, we diagonalize the fermionic QTM and discuss its properties.
1 Introduction

Since the discovery of high $T_c$ superconductivity, strongly correlated electron models in low dimensionals have attracted much more interest. Among them the 1D Hubbard model is one of the most interesting solvable models. The Hamiltonian consists of the hopping term and the on-site Coulomb interaction term,

$$\mathcal{H} = -\sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} (c_{j+1\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{j+1\sigma}) + U \sum_{j=1}^{L} (n_{j\uparrow} - \frac{1}{2})(n_{j\downarrow} - \frac{1}{2}).$$ (1)

Lieb and Wu diagonalized the Hamiltonian [1] by using coordinate Bethe ansatz method under the periodic boundary condition (PBC). They found that the ground state at the finite correlation strength ($U$) shows property of the insulator in the half-filled case, and that of conductor in the other case. Some excited states are studied in [2, 3]. Takahashi introduced the string hypothesis to classify the states [4] and obtained the thermodynamic formulation. The bulk properties, such as the specific heat and the magnetic susceptibility, were calculated using this formulation [5, 6].

In the framework of the Inverse Scattering method [7, 8, 9, 10, 11], the Yang-Baxter equation plays an important role. Integrability can be discussed in the sense of an infinite family of conserved currents which are created from the transfer matrix. Then the transfer matrix is diagonalized by using algebraic Bethe ansatz. At the same time, this procedure can be applied to diagonalize the quantum transfer matrix which shows thermodynamic properties. Shastry introduced the Jordan-Wigner transformation [15] for the Hubbard model to show the integrability of this model [12, 13, 14]. This corresponding spin model is called the coupled spin model (the Shastry model). Mapping to the spin model is a usual method since the 1D quantum spin model can be easily reduced to the 2D classical spin model. Furthermore the quantum transfer matrix method was also applied to the coupled spin model [16]. The bulk properties, such as the specific heat and the magnetic susceptibility, were successfully calculated. They found that the spin-charge separation still remains even at the finite temperature.

We have to note, however, that there exist several differences between the fermion system and the spin system. Because of the non-locality of this transformation, the PBC for the fermion system is not equivalent to the PBC for the corresponding spin system. Even when the system size is infinite, the correlation function, such as the one particle green function $<c_{j\sigma}^\dagger c_{k\sigma}> (\sigma = \uparrow, \downarrow)$, can not be expressed locally in the spin system and it is difficult to evaluate the thermodynamic properties by using quantum transfer matrix method. Thus another approach is required to investigate local properties.

We introduced recently the fermionic $R$-operator approach, where the genuinely fermionic quantum transfer matrix can be constructed [21]. The fermionic $R$-operator is constituted from fermion operators. The integrability is proved for the fermion models under the PBC [17, 18, 19] and the open boundary condition [20]. Here the super trace (Str) and the super transpose (st) for the fermion operators are introduced. For the spinless fermion model, the correlation length and the $k_F$ oscillation are obtained at a finite temperature [21] through
the quantum transfer matrix method. This oscillation is a intrinsically fermionic property and the conformal field theory has already expected it. The correlation length also agrees with the finite temperature correction \((T \to 0)\) [22]. For the Hubbard model, we showed the integrability using the fermionic \(R\)-operator in the previous study [19]. In this paper, we proceed this study and diagonalize the quantum transfer matrix algebraically to get the thermodynamic properties. The algebraic properties are also examined.

We want to mention another attempt to treat fermion models directly, without mapping to the spin model. It is based on the graded Yang Baxter equation, where the quantum space is a fermion fock space and the auxiliary space is a graded vector space. The integrability [23, 24, 25] was first proved by Wadati et al. exactly for the Hubbard model. The \(SO(4)\) symmetry [26, 27, 28, 29, 30] of the transfer matrix is shown in [31, 32]. Ramos and Martins diagonalize the transfer matrix algebraically [33, 34], which agrees with the coordinate Bethe ansatz method [1]. Note that several differences exist between this approach and the fermionic \(R\)-operator approach. First, the initial condition of the transfer matrix is not a shift operator in the graded approach and it is difficult to examine the energy momentum. Second, the fermionic \(R\)-operator approach is more applicable when we apply the quantum transfer matrix method and diagonalize it algebraically. In this approach, both the quantum space and the auxiliary space are the fermion fock spaces. On the other hand, the quantum transfer matrix acts on the auxiliary space while the transfer matrix acts on the quantum space. Then both transfer matrices have a similar algebraic property and can be diagonalized almost in the same way.

This paper is constituted as follows. In the next section, we review the fermionic \(R\)-operator and the integrability of this model. In §3, the properties of the fermionic \(R\)-operator and the relation with global symmetry is clarified. In §4, we diagonalize the transfer matrix by means of algebraic BA approach. Then we generalize the eigenvalue of the vacuum state which is useful in diagonalizing the quantum transfer matrix in §5. The twisted boundary condition is also considered. In §5, we show that the monodromy matrix for the quantum transfer matrix is intertwined by the fermionic \(R\)-operator just in the same as that for the transfer matrix. Then we diagonalize it algebraically and compare with [16]. The last section is devoted to concluding remarks.

## 2 Fermionic \(R\)-operator and Yang Baxter equation

In this section we review the fermionic \(R\)-operator approach for the 1D Hubbard model (1). The integrability can be discussed by using the Yang-Baxter equation [19],

\[
R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2).
\]  

The monodromy operator \(T_a(u)\) can be intertwined by the fermionic \(R\)-operator,

\[
R_{ab}(u_a, u_b)T_a(u_a)T_b(u_b) = T_b(u_b)R_a(u_a)R_{ab}(u_a, u_b),
\]

\[
T_a(u) = R_{aL}(u, 0) \cdots R_{a1}(u, 0)
\]
which leads to the commutativity of the transfer matrix $\tau(u)$,
\[
[\tau(u), \tau(v)] = 0, \quad \tau(u) = \text{Str}_a \mathcal{T}_a(u).
\] (2.4)

Here Str is defined as (B.8)/(B.9). And the commuting family can be produced from the transfer matrix,
\[
[I^{(m)}, I^{(n)}] = 0, \quad I^{(n)} = \frac{d^n}{du^n} \{\tau(0)^{-1}\tau(u)\}|_{u=0},
\] (2.5)

where $I^{(1)}$ is the Hamiltonian. The fermionic $R$-operator which satisfies (2.1) \cite{35, 36, 37, 19} is given by
\[
\mathcal{R}^{b}_{12}(u, v) = \mathcal{R}^{(\uparrow)}_{12}(u-v)\mathcal{R}^{(\downarrow)}_{12}(u-v) + \frac{\cos(u-v)}{\cos(u+v)} \tanh(h(u) - h(v))
\times \mathcal{R}^{(\uparrow)}_{12}(u+v)\mathcal{R}^{(\downarrow)}_{12}(u+v)(2n_1^\uparrow - 1)(2n_1^\downarrow - 1),
\] (2.6)

where $\mathcal{R}^{(\sigma)}_{12}(u) = a(u)(-n_{1\sigma}n_{2\sigma} + (1 - n_{1\sigma})(1 - n_{2\sigma})) - b(u)(n_{1\sigma}(1 - n_{2\sigma}) + (1 - n_{1\sigma})n_{2\sigma})$
\[
+ c(u)(c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma}), \quad (\sigma = \uparrow, \downarrow)
\] (2.7)

which is slightly different from the one in \cite{19} because of the different JW trans. Note that the initial condition for this $R$-operator ($u=v=0$) gives the permutation operator ($\mathcal{P}$), which means that the initial condition for the transfer matrix $\tau(0)$ is a left shift operator,
\[
\mathcal{P}_{jk}c_{j\sigma} = c_{k\sigma}\mathcal{P}_{jk}, \quad \mathcal{P}_{jk}c_{k\sigma} = c_{j\sigma}\mathcal{P}_{jk} \quad \text{and} \quad \mathcal{P}^2_{jk} = 1,
\] (2.9)

\[
\tau(0)c_{j\sigma} = c_{j+1\sigma}\tau(0), \quad \text{with PBC} \quad (c_{L+j} = c_j),
\] (2.10)

The Hamiltonian (1) is expressed as
\[
H = \tau(0)^{-1} \frac{d}{du} \tau(u)|_{u=0} = \sum_{j=1}^{L} \mathcal{P}_{j,j+1}(u, 0)|_{u=0}.
\] (2.11)

This is the fermionic $R$-operator approach for the integrability. We see that (2.2)(2.3) can be generalized \cite{19} to $\mathcal{T}_a(u) = \mathcal{R}_{aL}(u, u_0) \cdots \mathcal{R}_{a1}(u, u_0)$ where $\mathcal{R}_{jk}(u_0, u_0) = \mathcal{P}_{jk}$. Then the generalized Hamiltonian is obtained. It was generally believed that we demand the deference property $\mathcal{R}(u, v) = \mathcal{R}(u-v)$ to get the boost operator. The Hubbard model does not have this property. Recently J. Links et al. found the boost operator by using this generalized formula \cite{38}.

3. $SO(4)$ Symmetry and Partial Particle-Hole Transformation

In this section we discuss the two important symmetries of the Hubbard model by means of the fermionic $R$-operator. The first one is the $SO(4)$ symmetry which consists of the
spin-$SU(2)$ and the charge-$SU(2)$ ($\eta$-pairing $SU(2)$). The generators of the spin-$SU(2)$ and the charge-$SU(2)$ are given by

$$S^x = \frac{1}{2} \sum_{j=1}^{L} (c_{j\uparrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow}), \quad S^y = \frac{1}{2i} \sum_{j=1}^{L} (c_{j\uparrow}^\dagger c_{j\downarrow} - c_{j\downarrow}^\dagger c_{j\uparrow}), \quad S^z = \frac{1}{2} \sum_{j=1}^{L} (n_{j\uparrow} - n_{j\downarrow}),$$

(3.1)

$$\eta^x = \frac{1}{2} \sum_{j=1}^{L} ((-1)^j (c_{j\uparrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow})), \quad \eta^y = \frac{1}{2} \sum_{j=1}^{L} ((-1)^j (c_{j\uparrow}^\dagger c_{j\downarrow} - c_{j\downarrow}^\dagger c_{j\uparrow})), \quad \eta^z = \frac{1}{2} \sum_{j=1}^{L} (n_{j\uparrow} + n_{j\downarrow} - 1).$$

(3.2)

These generators constitute the Lie algebra of $SO(4)$, i.e., $so(4) \equiv su(2) \oplus su(2)$. It is well known that the six generators (3.1) and (3.2) commute with the Hubbard Hamiltonian (1),

$$[H, S^\alpha] = [H, \eta^\alpha] = 0, \quad (\alpha = x, y, z)$$

(3.3)

if we assume the even number of sites and the PBC.

The second one is the so called partial particle hole transformation,

$$c_{j\uparrow} \rightarrow c_{j\uparrow}, \quad c_{j\downarrow} \rightarrow (-1)^j c_{j\downarrow}^\dagger, \quad U \rightarrow -U.$$  

(3.4)

It is also well known that the Hubbard Hamiltonian is invariant under the partial particle-hole transformation, while the generators $S^\alpha$ and $\eta^\alpha$ are exchanged to each other.

We discuss the above two symmetries in terms of the fermionic $R$-operator. Here we omit some calculations, which we will show in Appendix A. The Lie algebra $so(4)$ symmetry of the fermionic $R$-operator was already discussed in [19], which is represented by the relations,

$$[\mathcal{R}_{jk}(u, v), S_j^\alpha + S_k^\alpha] = 0, \quad (\alpha = x, y, z)$$

(3.5)

$$[\mathcal{R}_{jk}(u, v), \eta_j^\alpha + \eta_k^\alpha] = 0,$$

(3.6)

$$\{\mathcal{R}_{jk}(u, v), \eta_j^\alpha - \eta_k^\alpha\} = 0. \quad (\alpha = x, y)$$

(3.7)

Here we have introduced the local generators as

$$S_j^x = \frac{1}{2} (c_{j\uparrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow}), \quad S_j^y = \frac{1}{2i} (c_{j\uparrow}^\dagger c_{j\downarrow} - c_{j\downarrow}^\dagger c_{j\uparrow}), \quad S_j^z = \frac{1}{2} (n_{j\uparrow} - n_{j\downarrow}),$$

$$\eta_j^x = \frac{1}{2} (c_{j\uparrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow}), \quad \eta_j^y = \frac{1}{2i} (c_{j\uparrow}^\dagger c_{j\downarrow} - c_{j\downarrow}^\dagger c_{j\uparrow}), \quad \eta_j^z = \frac{1}{2} (n_{j\uparrow} + n_{j\downarrow} - 1).$$

The relations (3.5)–(3.7) can be extended to the those for the transfer matrix, from which it is possible to prove the $so(4)$ symmetry of the local conserved currents.
We shall generalize the symmetry relations (3.5)–(3.7) into exponentiated forms, which corresponds to the generalization of the Lie algebra \(so(4)\) symmetry into Lie group \(SO(4)\) symmetry. By direct calculation, we have found the following identities,

\[
\begin{align*}
[\mathcal{R}_{jk}(u, v), \exp(i\theta S^\alpha_u) \exp(i\theta S^\alpha_v)] &= 0, \quad (\alpha = x, y, z) \tag{3.8} \\
[\mathcal{R}_{jk}(u, v), \exp(i\theta \eta^\alpha_u) \exp(i\theta \eta^\alpha_v)] &= 0, \tag{3.9} \\
\mathcal{R}_{jk}(u, v) \exp(i\theta \eta^\alpha_u) \exp(-i\theta \eta^\alpha_v) &= \exp(-i\theta \eta^\alpha_v) \exp(i\theta \eta^\alpha_u) \mathcal{R}_{jk}(u, v), \quad (\alpha = x, y) \tag{3.10}
\end{align*}
\]

Here \(\theta\) is an arbitrary (real) parameter. From the symmetry relations (3.8)–(3.10) for the fermionic \(R\)-operator, we can derive the following identity for the transfer matrix, (see Appendix A)

\[
\begin{align*}
&\exp(-i\theta S^\alpha) \tau(u, u_0) \exp(i\theta S^\alpha) = \tau(u, u_0), \quad (\alpha = x, y, z) \tag{3.11} \\
&\exp(-i\theta \eta^\alpha) \tau(u, u_0) \exp(i\theta \eta^\alpha) = \tau(u, u_0), \tag{3.12} \\
&\exp(i\theta \eta^\alpha) \tau(u, u_0) \exp(i\theta \eta^\alpha) = \tau(u, u_0). \quad (\alpha = x, y) \tag{3.13}
\end{align*}
\]

When we think about the local conserved currents \((I^{(m)})\) as was discussed in (2.5), we consider the product \(\tau^{-1}(u_0, u_0) \tau(u, u_0)\) and it satisfies the relation

\[
e^{-i\theta X} \tau^{-1}(u_0, u_0) \tau(u, u_0) e^{i\theta X} = \tau^{-1}(u_0, u_0) \tau(u, u_0), \tag{3.14}
\]

where \(X\) is any generator of \(SO(4)\), i.e., \(S^\alpha\) or \(\eta^\alpha\) \((\alpha = x, y, z)\). Namely the local conserved currents which contain the Hamiltonian are therefore invariant under the \(SO(4)\) rotation.

Next we consider the partial particle-hole transformation (3.4). The relevant relations for the fermionic \(R\)-operator are found to be

\[
\begin{align*}
\mathcal{R}_{jk}(u, v; U) V_j V_k &= V_k V_j \mathcal{R}_{jk}(u, v; -U), \tag{3.15} \\
\mathcal{R}_{jk}(u, v; U) \bar{V}_j \bar{V}_k &= \bar{V}_k \bar{V}_j \mathcal{R}_{jk}(u, v; -U), \tag{3.16}
\end{align*}
\]

where \(V_j = (1 - 2n_{ji})(c_{ji}^\dagger + c_{ji})\), \(\bar{V}_j = i(1 - 2n_{ji})(c_{ji}^\dagger - c_{ji}) = i(2n_{ji} - 1)V_j\) \((3.17)\). Here we write the \(U\)-dependence of the \(R\)-operator explicitly. Note that \(V_j\) and \(\bar{V}_j\) are Grassmann odd operators and \(V_j^2 = \bar{V}_j^2 = 1\). The operators \(V_j\) and \(\bar{V}_j\) induce the following transformation for the fermion operators,

\[
\begin{align*}
V_j c_{ji} V_j &= c_{ji}, \quad V_j c_{ji} V_j = c_{ji}, \quad V_j c_{ji} V_j = c_{ji}, \quad V_j c_{ji} V_j = c_{ji}, \\
\bar{V}_j c_{ji} \bar{V}_j &= c_{ji}, \quad \bar{V}_j c_{ji} \bar{V}_j = c_{ji}, \quad \bar{V}_j c_{ji} \bar{V}_j = -c_{ji}, \quad \bar{V}_j c_{ji} \bar{V}_j = -c_{ji}. \tag{3.18}
\end{align*}
\]

Now let us define the operator \(V\) by

\[
V = \prod_{k=1}^{L/2} V_{2k} \bar{V}_{2k-1} = V_L \bar{V}_{L-1} \cdots V_2 \bar{V}_1, \tag{3.19}
\]
From the property (3.18), one can see that the transformation for an arbitrary operator \( X(\{c_{j\sigma}\}, U) \)

\[
X(\{c_{j\sigma}\}, U) \rightarrow V^{-1}X(\{c_{j\sigma}\}, -U)V
\]  

(3.20)
is nothing but the partial particle-hole transformation (3.4). From the properties of the fermionic \( R \)-operator (3.15)(3.16) the product \( \tau^{-1}(u_0, u_0)\tau(u, u_0) \) is invariant under the partial particle-hole transformation, (see Appendix A)

\[
V^{-1}\tau^{-1}(u_0, u_0)\tau(u, u_0; -U)V = \tau^{-1}(u_0, u_0)\tau(u, u_0; U).
\]  

(3.21)

Then we can conclude that all the local conserved currents are invariant under the partial particle-hole transformation (3.4).

4 Algebraic Bethe Ansatz

In this section, we propose the brief note of the algebraic bethe ansatz. As Ramos and Martins did in the graded case [33], we construct the eigenstates algebraically although a little difference exists. The eigenvalue of the transfer matrix and the nested bethe ansatz equation are obtained. Then we apply the generalized eigenvalue of the vacuum state, which we will use in §5. Twisted boundary condition is also discussed.

We will diagonalize the transfer matrix (B.8)(B.9),

\[
\tau(u) = \text{Str}_a \mathcal{T}_a(u) = D_{11}(u) + D_{22}(u) - A_{11}(u) - A_{22}(u).
\]  

(4.1)
The eigenstate is constructed algebraically. We have five creation operators \( (C_{21}, C_{22}, D_{21}, B_{11}, B_{21}) \) defined in (B.7). By using three creation operators \( (C_{21}, C_{22}, D_{21}) \) and taking linear combination of these products, we define the \( n \)-particle eigenstate by

\[
|\Phi_n(\lambda_1, \cdots, \lambda_n)\rangle = \tilde{\Phi}_n(\lambda_1, \cdots, \lambda_n)\tilde{F}|0\rangle,
\]  

(4.2)

\[
\tilde{\Phi}_n(\lambda_1, \cdots, \lambda_n) = \tilde{C}(\lambda_1) \otimes \tilde{\Phi}_{n-1}(\lambda_2, \cdots, \lambda_n)
\]  

\[+ \sum_{j=2}^n \tilde{\xi} \otimes D_{21}(\lambda_1)\tilde{\Phi}_{n-2}(\lambda_2, \cdots, \lambda_{j-1}, \lambda_j, \cdots, \lambda_n)D_{22}(\lambda_j)]\tilde{g}^{(n)}_{j-1}(\lambda_1, \cdots, \lambda_n)
\]  

(4.3)

where \( \tilde{C} = (C_{21}, C_{22}) \) and \( \tilde{\xi} = (0, 1, -1, 0) \). \( C_{21} \) and \( C_{22} \) are 1-particle creation operators and \( D_{21} \) is 2-particle creation operator. \( \hat{g}_j \) is an operator defined recursively,

\[
\hat{g}^{(n)}_{j-1}(\lambda_1, \cdots, \lambda_n) = \frac{a^+(\lambda_{j-1}, \lambda_j)}{a(\lambda_{j-1}, \lambda_j)} \hat{g}^{(n)}_{j-2}(\lambda_1, \cdots, \lambda_j, \cdots, \lambda_n)\hat{r}_{j-1, j}(\lambda_{j-1}, \lambda_j),
\]  

(4.4)

\[
\hat{g}^{(n)}_{1}(\lambda_1, \cdots, \lambda_n) = -\frac{f(\lambda_1, \lambda_2)}{c^+(\lambda_1, \lambda_2)} \prod_{k=3}^n \frac{a^+(\lambda_k, \lambda_2)}{b^-(\lambda_k, \lambda_2)}.
\]  

(4.5)
The vacuum state \( (|0\rangle) \) is a fermion Fock space defined as,

\[
|0\rangle = \prod_{j=1}^L |00\rangle_j, \text{ where } c_{j\sigma}|00\rangle_j = 0 \quad (\sigma = \uparrow, \downarrow),
\]  

(4.6)
The main point of this algebraic construction is its exchange property,

\[ \Phi_n(\lambda_1, \cdots, \lambda_n) = \frac{a^-(\lambda_{j-1}, \lambda_j)}{a^+(\lambda_{j-1}, \lambda_j)} \Phi_n(\lambda_1, \cdots, \lambda_j, \lambda_{j-1}, \cdots, \lambda_n). \]

which simplifies this construction. After a long patient calculations (see Appendix B) using the algebraic relations (B.10)–(B.19), we get the eigenvalue of transfer matrix

\[ \Lambda(u, \{\lambda_l\}) = (a^+(u))^L \Pi_j^n \left( \frac{a^-(u, \lambda_j)}{b^-(u, \lambda_j)} \right) + (c^+(u))^L \Pi_j^n \left( \frac{b^+(u, \lambda_j)}{c^+(u, \lambda_j)} \right) \]

where \( a^\pm(u) = a^\pm(u, 0), \) \( b^\pm(u) = b^\pm(u, 0) \) and \( c^\pm(u) = c^\pm(u, 0) \). And the Bethe Ansatz equations are,

\[ \left( \frac{a^+(\lambda_j)}{b^-(\lambda_j)} \right)^L = \prod_{i=1}^m \frac{1}{b(\mu_i, \lambda_j)}, \quad \prod_{j=1}^n \bar{b}(\mu_i, \lambda_j) = \prod_{k \neq i}^m \bar{b}(\mu_k, \mu). \]

To see the agreement with the coordinate one [1], we examine the energy spectrum and the momentum, after the reparametrization \( e^{2n(\lambda_j)} \tan \lambda_j = e^{ik_j}, \)

\[ E = \frac{d}{du} \log \Lambda(u, \{\lambda_l\}) |_{u=0} = -2\sum_{j=1}^m \cos k_j + \frac{U}{4}(L - 2n) \]  
\[ P = i \log(\Lambda(u = 0, \{\lambda_l\})) = -\sum_{j=1}^m k_j. \]

Now we compare with the Ramos and Martins’ results. As we mentioned in §2, the initial condition of the transfer matrix doesn’t give the shift operator in the graded case. Then the energy momentum is not given as (4.11). Beside the agreement of (4.9)(4.10), (4.8) differs from their results by the overall factor. As is discussed in [33], the completeness can be discussed in the similar way [39, 40]. Here the regular bethe ansats are the highest-weight states of the \( SO(4) \) symmetry. The other states are obtained by the lowering operators.

Next we generalize the eigenvalue of the vacuum state in the following way,

\[ D_{11}(u)|0> = d_{11}(u)|0>, \quad D_{22}(u)|0> = d_{22}(u)|0>, \]
\[ A_{11}(u)|0> = a_{11} a_{xx}(u)|0>, \quad A_{22}(u)|0> = a_{22} a_{xx}(u)|0>, \]

where \( a_{11} \) and \( a_{22} \) are constant. The eigenvalue of transfer matrix \( \Lambda(u, \{\lambda_l\}) \) and the Bethe Ansatz equations are,

\[ \Lambda(u, \{\lambda_l\}) = d_{22}(u) \Pi_j^n \left( \frac{a^+(u, \lambda_j)}{b^-(u, \lambda_j)} \right) + d_{11}(u) \Pi_j^n \left( \frac{b^+(u, \lambda_j)}{c^+(u, \lambda_j)} \right) - a_{xx}(u) \left( a_{11} \Pi_j^n \left( \frac{a^-(u, \lambda_j)}{b^-(u, \lambda_j)} \right) \Pi_{i=1}^m \frac{1}{b(\mu_i, u)} + a_{22} \Pi_j^n \left( \frac{b^+(u, \lambda_j)}{c^+(u, \lambda_j)} \right) \Pi_{i=1}^m \frac{1}{b(\mu_i, u)} \right), \]

\[ \left( \frac{d_{22}(\lambda_j)}{a_{xx}(\lambda_j)} \right)^L = a_{11} \Pi_{i=1}^m \frac{1}{b(\mu_i, \lambda_j)}, \quad \Pi_j^n \bar{b}(\mu_i, \lambda_j) = \frac{a_{11} \Pi_{k \neq i}^m \bar{b}(\mu_k, \mu)}{a_{22} \Pi_{k \neq i}^m \bar{b}(\mu_k, \mu)}. \]

\[ \text{(4.14)} \]
\[ \text{(4.15)} \]
Then we think about the twisted boundary condition,
\[ c_{L+1\uparrow} = e^{i\theta_1}c_{1\uparrow}, \quad c_{L+1\downarrow} = e^{i\theta_2}c_{1\downarrow}, \]  
(4.16)
where the transfer matrix is defined as
\[ \tau^{\text{twist}}(u) = \text{Str}_a \mathcal{J}^{\text{twist}}(u), \]  
(4.17)
\[ \mathcal{J}^{\text{twist}}(u) = \mathcal{V} \mathcal{T}^{\text{twist}}(u) \]  
where \( \mathcal{V} = e^{-\frac{i}{2}(\theta_1 + \theta_2)(n_{a\uparrow} + n_{a\downarrow}) + i\frac{\theta_2 - \theta_1}{2}(n_{a\uparrow} - n_{a\downarrow})} \).  
(4.18)
Because of the properties of the fermionic \( R \)-operator (3.8), the global Yang Baxter relation (2.2) is satisfied with this monodromy matrix \( \mathcal{T}^{\text{twist}} \) although the eigenvalues of its elements for the vacuum state change to
\[ d_{22}(u) = a^+(u)^L, \quad d_{11}(u) = e^{i\theta_1 + \theta_2}c^+(u)^L, \]  
\[ d_{11}(u) = e^{-i\theta_1}, \quad d_{22} = e^{-i\theta_2}, \quad d_{xx}(u) = b^-(u)^L. \]  
(4.19)
Here \( \theta_1 = \theta_2 = 0 \) gives the PBC \( (c_{j+L\sigma} = c_{j\sigma}) \) and \( \theta_1 = \theta_2 = \pi \) gives the anti periodic boundary condition \( (c_{j+L\sigma} = -c_{j\sigma}) \).

5 Algebraic Bethe Ansatz for the Quantum Transfer Matrix

In this section, we will diagonalize the quantum transfer matrix in the same way as §4. First we compare the quantum transfer matrix with the usual transfer matrix. Then we make use of the similar algebraic structure to diagonalize it.

In the row to row case where the transfer matrix,
\[ \tau(u) = \text{Str}_a \mathcal{R}_{aL}(u, 0) \cdots \mathcal{R}_{a1}(u, 0) = \tau(0)(1 + uH + O(u^2)), \]  
(5.1)
is diagonalized, we use the Yang Baxter equation to intertwine with respect to auxiliary space, where \( 1 \to a, 2 \to b \) and \( 3 \to j \) in (2.1),
\[ \mathcal{R}_{ab}(u_a, u_b)\mathcal{R}_{aj}(u_a, u_j)\mathcal{R}_{bj}(u_b, u_j) = \mathcal{R}_{bj}(u_b, u_j)\mathcal{R}_{aj}(u_a, u_j)\mathcal{R}_{ab}(u_a, u_b). \]  
(5.2)
But in the case of the quantum transfer matrix, it operates on the auxiliary space and the monodromy matrix should be intertwined with respect to the quantum space. We use the two equations,
\[ \mathcal{R}_{jk}(u_j, u_k)\mathcal{R}_{ak}(u_a, u_k)\mathcal{R}_{aj}(u_a, u_j) = \mathcal{R}_{aj}(u_a, u_j)\mathcal{R}_{ak}(u_a, u_k)\mathcal{R}_{jk}(u_j, u_k), \]  
(5.3)
\[ \mathcal{R}_{jk}(u_j, u_k)\mathcal{R}_{ak}(u_k, u_a)\mathcal{R}_{aj}(u_j, u_a) = \mathcal{R}_{aj}(u_j, u_a)\mathcal{R}_{ak}(u_k, u_a)\mathcal{R}_{jk}(u_j, u_k), \]  
(5.4)
which are equivalent to (2.1). In (5.3), l.h.s. and r.h.s are reversed and 1 → a, 2 → j and 3 → k. In (5.4), we take the super transpose (st) with respect to 3 and 1 → j, 2 → k and 3 → a, where the super transpose (st) is defined as st_a = st_a a and

$$ (X_1u\sigma + X_2\sigma_j^+ + X_3\sigma_j + X_4(1 - n_\sigma))^{st_a} $$

$$ = X_1u\sigma - X_2\sigma_j + X_3\sigma_j^+ + X_4(1 - n_\sigma), \quad (\sigma = \uparrow, \downarrow) $$

(5.5)

where \( \sigma \) operators in space a is expressed explicitly. \( \mathcal{R}_{aj}(u_j, u_a) \) can be expressed in the following way,

$$ \mathcal{R}_{aj}(u_j, u_a) = \mathcal{R}^{st_a}_{ja}(u_j, u_a) $$

$$ = \mathcal{R}^{(\uparrow)}_{aj}(u_j - u_a)\mathcal{R}^{(\downarrow)}_{aj}(u_j - u_a) + \frac{\cos(u_j - u_a)}{\cos(u_j + u_a)} \tanh(h(u_j) - h(u_a)) $$

$$ \times \mathcal{R}^{(\uparrow)}_{aj}(u_j + u_a)\mathcal{R}^{(\downarrow)}_{aj}(2n_j) - 1)(2n_j - 1), $$

(5.7)

(5.8)

Because of the properties for \( \mathcal{R}^{\sigma}_{aj}(0) \),

$$ \frac{d}{du} \mathcal{R}^{\sigma}_{aj+1}(u)|_{u=0} \mathcal{R}^{\sigma}_{aj} = \mathcal{H}^{\sigma}_{jj+1} \mathcal{R}^{\sigma}_{aj+1} \mathcal{R}^{\sigma}_{aj}, $$

(5.9)

$$ \mathcal{R}^{\sigma}_{aj+1} \mathcal{R}^{\sigma}_{aj} = c_j \mathcal{R}^{\sigma}_{aj+1} \mathcal{R}^{\sigma}_{aj}, $$

(5.10)

where \( \mathcal{R}^{\sigma}_{aj} = \mathcal{R}^{\sigma}_{aj}(0), \mathcal{H}^{\sigma}_{jj+1} = -(c_j c_j^+ c_j + c_j^+ c_j), \quad (\sigma = \uparrow, \downarrow) $$

(5.11)

we have the properties,

$$ \tau(u) = \text{Str}_a \mathcal{R}_{aL}(0, u) \cdots \mathcal{R}_{a1}(0, u) = (1 - u\mathcal{H} + O(u^2))\tau(0), $$

(5.12)

$$ \tau(0)c_j = c_j \tau(0). $$

(5.13)

Here \( \tau(0) \) is a right shift operator. Then

$$ \tau(0)\tau(0) = 1, $$

(5.14)

$$ \tau(-u)\tau(u) = 1 - 2u\mathcal{H} + O(u^2). $$

(5.15)

Next we evaluate the partition function by using \( \mathcal{R} \) and \( \mathcal{R} \),

$$ \text{Tr} e^{-\beta \mathcal{H}} = \lim_{N \to \infty} \text{Tr}(\tau(-u)\tau(u))^{N/2} $$

(5.16)

$$ = \lim_{N \to \infty} \text{Tr} \prod_{m=1}^{N/2} \text{Str}_a \mathcal{R}_{a2m,a2m-1}[\mathcal{R}_{a2mL}(-u, 0) \cdots \mathcal{R}_{a2m1}(-u, 0) $$

$$ \times \mathcal{R}_{a2m-1L}(u, 0) \cdots \mathcal{R}_{a2m-1}(u, 0)] $$

(5.17)

$$ = \lim_{N \to \infty} \text{Tf}_{j}^{QT\mathcal{M}}(u, 0), \quad (u = \beta / N) $$

(5.18)
where $N$ is called Trotter number which we to be even and

$$
\tau_{QT M}(u, v) = Tr_j \tau_{QT M}^j(u, v).
\tag{5.20}
$$

Here $\tau_{QT M}(u, v)$ is the QTM. From (5.3)(5.4), this monodromy operator is intertwined as

$$
\mathcal{R}_{jk}(u_j, u_k)\tau_{QT M}^j(u, u_j)\tau_{QT M}^k(u, u_j) = \tau_{QT M}^j(u, u_j)\tau_{QT M}^k(u, u_k)\mathcal{R}_{jk}(u_j, u_k).
\tag{5.21}
$$

After gauge transformation (B.3), we have the property for the fermionic $R$-operator $\mathcal{R}_{jk}(u_j, u_k) = \mathcal{R}_{k,j}(-u_k, -u_j)$. Then we have

$$
\mathcal{R}_{jk}(u_j, u_k)\tau_{QT M}^j(u, -u_j)\tau_{QT M}^k(u, -u_k) = \tau_{QT M}^j(u, -u_j)\tau_{QT M}^k(u, -u_k)\mathcal{R}_{jk}(u_j, u_k).
\tag{5.22}
$$

We see the similar relation for the usual transfer matrix (2.2)(2.4). And the QTM is diagonalized in the same way except the generalization of the vacuum eigenvalue. Next we introduce the magnetic field ($H = 2\hbar$) and the chemical potential ($\mu$),

$$
H_{\text{ext}} = -\sum_{j=1}^{L} \{ h(n_j^\uparrow - n_j^\downarrow) - \mu(n_j^\uparrow + n_j^\downarrow) \},
\tag{5.23}
$$

and evaluate the QTM algebraically. $Tr_j$ in (5.20) is nothing but the anti periodic boundary condition ($\theta_1 = \theta_2 = \pi$ in (4.16) and $V_{\text{apbc}}$ in (4.17)) and the introduction of the magnetic field and the chemical potential is also applied to (4.17) as $V_{\text{m,c}}$,

$$
V = V_{\text{apbc}}V_{\text{m,c}},
\tag{5.24}
$$

$$
V_{\text{apbc}} = e^{-\pi i (n_u + n_d)}, \quad V_{\text{m,c}} = e^{-\beta \mu (n_u + n_d) + \beta h (n_u - n_d)}.
\tag{5.25}
$$

And the vacuum state ($|\Omega\rangle_{\text{aux}}$) in the auxiliary Fock space is defined as

$$
|\Omega\rangle_{\text{aux}} = \prod_{m=1}^{N/2} |0\rangle_{a_{2m}}|1\rangle_{a_{2m-1}}, \quad \text{where} \quad |1\rangle_a = c_{a1}^{\dagger}c_{a1}^{\dagger}|00\rangle_a
\tag{5.26}
$$

and $|00\rangle_a$ is given in (4.6). Then $C_{21}, C_{22}$ and $D_{21}$ remain to be creation operators. Then the generalized eigenvalue for the vacuum state is given by

$$
d_{22}(v) = (a^+(-u, -v)c^+(-v, u))^N/2, \quad d_{11}(v) = e^{-2\beta \mu} (c^+(-u, -v)a^+(-v, u))^N/2,\label{eqn:5.27}
$$

$$
a_{11} = e^{\beta(-\mu + h)}, \quad a_{22} = e^{\beta(-\mu - h)}, \quad a_{xx}(v) = (-b^+(-u, -v)b^-(v, u))^N/2.
\tag{5.28}
$$

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And the eigenvalue and the bethe ansatz equations (4.14)(4.15) are

\[ \Lambda(u, \{\lambda_i\}) = (a^+(-u, -v)c^+(-v, u))^{N/2}\Pi_{j=1}^n\frac{a^+(u, \lambda_j)}{b^-(u, \lambda_j)} \]

\[ + e^{-2\beta \mu}(c^+(-u, -v)a^+(-v, u))^{N/2}\Pi_{j=1}^n\left(-\frac{b^+(u, \lambda_j)}{c^+(u, \lambda_j)}\right) \]

\[ - (-b^+(-u, -v)b^-(v, u))^{N/2}\left(e^{\beta(-\mu + h)}\Pi_{j=1}^n\left(-\frac{a^-(u, \lambda_j)}{b^-(u, \lambda_j)}\right)\Pi_{l=1}^m\frac{1}{b(\mu_l, u)}\right) \]

\[ + e^{\beta(-\mu - h)}\Pi_{j=1}^n\left(-\frac{b^+(u, \lambda_j)}{c^+(u, \lambda_j)}\right)\Pi_{l=1}^m\frac{1}{b(u, \mu_l)} \],

\[ (\frac{-a^+(u, -\lambda_j)c^+(-\lambda_j, u)}{b^+(-u, -\lambda_j)b^-(\lambda_j, u)})^{N/2} = -e^{\beta(-\mu + h)}\Pi_{l=1}^m\frac{1}{b(\mu_l, \lambda_j)}, \]

\[ \Pi_{j=1}^n\bar{b}(\mu_l, \lambda_j) = e^{-2\beta h}\Pi_{k\neq l}^m\frac{\bar{b}(\mu_l, \mu_k)}{\bar{b}(\mu_k, \mu_l)}. \] (5.31)

When the system size is infinite \((L \to \infty)\), we see that the largest eigenvalue of the QTM with \((v = 0)\) gives the partition function because (5.18) can be expressed as a summation over the eigenstates (Str),

\[ \text{Tr} e^{-\beta H} = \Lambda_1^L + (-1)^{p_2} \Lambda_2^L + (-1)^{p_3} \Lambda_3^L + \cdots, \] (5.32)

where \(p_l\) is a parity of \(n\). The largest eigenvalue \((\Lambda_1)\) is given by the sector \((n = N\) and \(m = N/2)\) and this agrees with [16] after the partial particle hole transformation. Furthermore we can deal with the intrinsically fermionic properties which can not be expected in [16]. The asymptotics of the correlation functions are given by another eigenvalue. For example, the eigenvalue of \((n = N - 1\) and \(m = N/2 - 1)\) sector gives the one particle Green function \(<c_j^\dagger \sigma c_j^\sigma>\). (see [21])

6 Concluding Remarks

We studied the algebraic structure for the 1D Hubbard model in terms of the fermionic \(R\)-operator. This approach is a powerful method when we treat fermion system directly. The quantum transfer matrix (QTM) is naturally constructed. Then it can be diagonalized just in the same way as that for the usual transfer matrix. It has been a difficult problem and mapping to the spin system was a usual method, where intrinsically fermion properties can not be observed. Our result can be applied to evaluate any correlation function in addition to the bulk properties.

In this paper, we review the fermionic \(R\)-operator approach for the 1D Hubbard model and show the supertrace and supertranspose for the fermion operator. Then the algebraic aspect of this approach are discussed. The properties of the \(R\)-operator itself reflect the global properties of the model. Then we construct the eigenstate algebraically and diagonalize the transfer matrix. There appears naturally the \(r\)-structure which leads to the
nested bethe ansatz equations. The QTM is made from the fermionic $R$-operator and the supertrace of it. By using the generalized algebraic bethe ansatz equations, we diagonalize the quantum transfer matrix (QTM).

For the future work, we will investigate more details about the solutions of this Bethe ansatz equations for the QTM to obtain the thermodynamic properties of this model. Here the auxiliary space is finite and the Trotter limit ($N \to \infty$) is desirable to obtain the exact result. For the bulk properties, Juttner et al. [16] obtained the nonlinear integral equations (NLIEs) to evaluate the eigenvalue of the QTM, where the infinite space can be easily considered. We expect that the NLIEs for the another eigenvalue can be the small deformation of the original one [21]. The one particle Green function $c_j^\dagger c_k^\sigma$ and the spin correlation function $S_j^+ S_k^-$ are our first problems, where we expect no singularity with respect to the temperature. The spin correlation function $S_j^z S_k^z$ is our next problem. We expect it to be more complicated. When the Coulomb interaction is strong in the half-filed case, it behave as the XXX Heisenberg chain, where the crossover phenomena in $S_j^z S_k^z$ is observed [41].

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Appendix A; $SO(4)$ symmetry

Here we will show some calculations which are omitted in §3. The global properties are obtained from the local properties of the fermionic $R$-operator. As for the $SO(4)$ symmetry, it is from (3.8)–(3.10) to (3.11)–(3.13). (3.11) is proved from (3.8),

$$\exp(-i\theta S^\alpha) \tau(u, u_0) \exp(i\theta S^\alpha) = \text{Str}_c \prod_{j=1}^L e^{-i\theta S_j^\alpha} R_{aj}(u, u_0) e^{i\theta S_j^\alpha}$$

$$= \text{Str}_c \prod_{j=1}^L e^{i\theta S_j^\alpha} R_{aj}(u, u_0) e^{-i\theta S_j^\alpha}$$

$$= \tau(u, u_0). \quad \text{(A.1)}$$
In the same way, (3.12) is obtained from (3.9). And (3.13) is from (3.10),
\[
\exp(i\theta^a)\tau(u, u_0)\exp(i\theta^a)
\]
\[
= \text{Str}_a \prod_{k=1}^{L/2} e^{i\eta^a_{2k}} R_{a,2k}(u, u_0)e^{i\eta^a_{2k-1}} R_{a,2k-1}(u, u_0)e^{-i\eta^a_{2k-1}}
\]
\[
= \text{Str}_a \prod_{k=1}^{L/2} e^{i\eta^a_a} R_{a,0}(u, u_0)e^{i\eta^a_a} R_{a,0}(u, u_0)e^{-i\eta^a_a}
\]
\[
= \tau(u, u_0).
\] (A.2)

When we consider \(\tau^{-1}(u_0, u_0)\tau(u, u_0)\), it has \(SO(4)\) symmetry (3.14).

As for the invariance under the partial particle hole transformation, we introduce \(\bar{V}\) defined by
\[
\bar{V} = \prod_{k=1}^{L/2} \bar{V}_{2k} \bar{V}_{2k-1} = \bar{V}_L \bar{V}_{L-1} \cdots \bar{V}_2 \bar{V}_1.
\] (A.3)

Now using (3.15)(3.16), we obtain
\[
\bar{V}^{-1} \tau(u, u_0; -U)V = \text{Str}_a \prod_{k=1}^{L/2} \bar{V}_{2k} R^f_{a,2k}(u, u_0; -U) \bar{V}_{2k} V_{2k-1} R^f_{a,2k-1}(u, u_0) \bar{V}_{2k-1}
\]
\[
= \text{Str}_a \prod_{k=1}^{L/2} (-V_a R^f_{a,2k}(u, u_0; U)V_a)(-\bar{V}_a R^f_{a,2k-1}(u, u_0)V_a)
\]
\[
= -\tau(u, u_0; U).
\] (A.4)

Here in the last equality we have used a property of the supertrace
\[
\text{Str}_a \{V_a XV_a\} = -\text{Str}_a \{X\}.
\] (A.5)

for an arbitrary operator \(X\). The transformation law (A.4) means that the transfer matrix itself is not invariant under the partial particle hole transformation (3.4). When we consider the product \(\tau^{-1}(u_0, u_0)\tau(u, u_0)\), as before, it is invariant (3.21) under this transformation.

**Appendix B; Fermionic \(R\)-operator and Algebraic relations among the elements of the Monodromy matrix**

Here we show the simplified fermionic \(R\)-operator and the algebraic relations among the elements of the monodromy matrix \(\mathcal{T}\) which are necessary when diagonalizing the transfer matrix. The relations are obtained from the global Yang Baxter relation,
\[
R_{ab}(u_a, u_b)\mathcal{T}_a(u_a)\mathcal{T}_b(u_b) = \mathcal{T}_b(u_b)\mathcal{T}_a(u_a)R_{ab}(u_a, u_b)
\] (B.1)
\[
\mathcal{T}_a(u_a) = R_{aN}(u_a, 0) \cdots R_{a1}(u_a, 0).
\] (B.2)
Then some calculations that are omitted in § 4 are supplemented. We gauge transformed the fermionic $R$-operator (2.6),

$$I_a(u_a)I_b(u_b)\mathcal{R}_{ab}(u_a, u_b)I_a(u_a)^{-1}I_b(u_b)^{-1} \to \mathcal{R}_{ab}(u_a, u_b),$$

(B.3)\]

where

$$I_a(u_a) = e^{i(h(u_a)(2n_{\uparrow} - 1)(2n_{\downarrow} - 1)},$$

(B.4)

to the simple form,

$$\mathcal{R}_{12}(u_1, u_2) = a^+(u_1, u_2)(n_{1\uparrow}n_{1\downarrow}n_{2\uparrow}n_{2\downarrow} + (1 - n_{1\uparrow})(1 - n_{1\downarrow})(1 - n_{2\uparrow})(1 - n_{2\downarrow}))$$

$$- a^-(u_1, u_2)(n_{1\uparrow}(1 - n_{1\downarrow})n_{2\uparrow}(1 - n_{2\downarrow}) + (1 - n_{1\uparrow})n_{1\downarrow}(1 - n_{2\uparrow})n_{2\downarrow})$$

$$+ c^+(u_1, u_2)(n_{1\uparrow}n_{1\downarrow}(1 - n_{2\uparrow})(1 - n_{2\downarrow}) + (1 - n_{1\uparrow})(1 - n_{1\downarrow})n_{2\uparrow}n_{2\downarrow})$$

$$+ c^-(u_1, u_2)((1 - n_{1\uparrow})n_{1\downarrow}n_{2\uparrow}(1 - n_{2\downarrow}) + n_{1\uparrow}(1 - n_{1\downarrow})(1 - n_{2\uparrow})n_{2\downarrow})$$

$$+ b^+(u_1, u_2)((1 - n_{1\uparrow})(1 - n_{1\downarrow}) - n_{1\uparrow}n_{1\downarrow})$$

$$\times (n_{2\uparrow}(1 - n_{2\downarrow}) + (1 - n_{2\uparrow})n_{2\downarrow})$$

$$+ b^-(u_1, u_2)(n_{1\uparrow}(1 - n_{1\downarrow}) + (1 - n_{1\uparrow})n_{1\downarrow})$$

$$\times ((1 - n_{2\uparrow})(1 - n_{2\downarrow}) - n_{2\uparrow}n_{2\downarrow})$$

$$- d^+(u_1, u_2)(c_{1\uparrow}^t c_{1\downarrow}^t c_{2\uparrow} c_{2\downarrow} + c_{1\uparrow} c_{1\downarrow} c_{2\uparrow}^t c_{2\downarrow}^t)$$

$$+ d^-(u_1, u_2)(c_{1\uparrow}^t c_{1\downarrow} c_{2\uparrow} c_{2\downarrow}^t + c_{1\uparrow} c_{1\downarrow}^t c_{2\uparrow}^t c_{2\downarrow}^t)$$

$$+ e(u_1, u_2)((c_{1\uparrow}^t c_{2\uparrow} + c_{2\uparrow}^t c_{1\uparrow})(1 - n_{1\downarrow})(1 - n_{2\downarrow}) - n_{1\downarrow}n_{2\downarrow})$$

$$+ (c_{1\downarrow}^t c_{2\downarrow} + c_{2\downarrow}^t c_{1\downarrow})(1 - n_{1\uparrow})(1 - n_{2\uparrow}) - n_{1\uparrow}n_{2\uparrow})$$

$$+ f(u_1, u_2)((c_{1\uparrow}^t c_{2\uparrow} + c_{2\uparrow}^t c_{1\uparrow})(1 - n_{1\downarrow})(1 - n_{2\downarrow}) + (1 - n_{1\downarrow})n_{2\downarrow})$$

$$+ (c_{1\downarrow}^t c_{2\downarrow} + c_{2\downarrow}^t c_{1\downarrow})(1 - n_{1\uparrow})(1 - n_{2\uparrow}) + (1 - n_{1\uparrow})n_{2\downarrow}),$$

(B.5)

where

$$a^\pm(u_j, u_k) = \cos^2(u_j - u_k) \left[ 1 \pm \tanh \left\{ h(u_j) - h(u_k) \right\} \frac{\cos(u_j + u_k)}{\cos(u_j - u_k)} \right],$$

$$b^\pm(u_j, u_k) = -\sin(u_j - u_k) \cos(u_j - u_k) \left[ 1 \pm \frac{\sin(u_j + u_k)}{\sin(u_j - u_k)} \frac{\sin(u_j + u_k)}{\sin(u_j - u_k)} \right],$$

$$c^\pm(u_j, u_k) = \sin^2(u_j - u_k) \left[ 1 \pm \frac{\sin(u_j + u_k)}{\sin(u_j - u_k)} \right],$$

$$d^\pm(u_j, u_k) = 1 \pm \tanh \left\{ h(u_j) - h(u_k) \right\} \frac{\cos(u_j - u_k)}{\cos(u_j + u_k)},$$

$$= 1 \pm \tanh \left\{ h(u_j) + h(u_k) \right\} \frac{\sin(u_j - u_k)}{\sin(u_j + u_k)},$$

$$e(u_j, u_k) = \frac{\cos(u_j - u_k)}{\cosh \left\{ h(u_j) - h(u_k) \right\}},$$

$$f(u_j, u_k) = \frac{\sin(u_j - u_k)}{\cosh \left\{ h(u_j) + h(u_k) \right\}}.$$  

(B.6)
We define the Monodromy operator as follows,

\[ \mathcal{T}(u) = D_{11} n_\uparrow n_\downarrow + D_{22}(1 - n_\uparrow)(1 - n_\downarrow) + A_{11} n_\uparrow (1 - n_\downarrow) + A_{22}(1 - n_\uparrow)n_\downarrow + D_{12} c_\uparrow c_\downarrow + D_{21} c_\uparrow c_\downarrow - A_{12} c_\uparrow c_\downarrow + A_{21} c_\uparrow c_\downarrow + C_{11} n_\uparrow c_\downarrow + C_{12} c_\uparrow n_\downarrow - C_{21} c_\uparrow (1 - n_\downarrow) + C_{22}(1 - n_\uparrow)c_\downarrow + B_{11} n_\uparrow c_\downarrow - B_{12} c_\uparrow (1 - n_\downarrow) + B_{21} c_\uparrow n_\downarrow + B_{22}(1 - n_\uparrow)c_\downarrow. \]  

(B.7)

Note that the sign is a little tricky as to get the simple r-structure which is the most interesting point of this story. As the definition of the supertrace \( \text{Str} \), the transfer matrix is expressed as,

\[ \tau(u) = \text{Str}_a \mathcal{T}_a(u) = D_{11}(u) + D_{22}(u) - A_{11}(u) - A_{22}(u) \]  

(B.8)

(B.9)

Relations among \( A_{11} \cdots D_{22} \) are

\[ C_{2a}(u)C_{2b}(u') = \frac{a^-(u, u')}{a^+(u, u')} \hat{\pi}_{ab}(u, u') C_{2c}(u')C_{2d}(u) \]

\[ + \frac{f(u, u')}{c^+(u, u')} \xi_{ab}(D_{21}(u)D_{22}(u') - D_{21}(u')D_{22}(u)), \]  

(B.10)

\[ A_{ab}(u)C_{2c}(u') = -\frac{a^-(u, u')}{b^-(u, u')} \hat{\pi}_{bc}(u, u') C_{2e}(u')A_{ad}(u) + \frac{e(u, u')}{b^-(u, u')} C_{2b}(u)A_{ac}(u') \]

\[ + \frac{f(u, u')}{c^+(u, u')} \{B_{a1}(u)D_{22}(u') - \frac{e(u, u')}{b^-(u, u')} D_{21}(u)B_{a2}(u') \]

\[ + \frac{a^+(u, u')}{b^-(u, u')} D_{21}(u')B_{a2}(u), \]  

(B.11)

\[ D_{11}(u)C_{2a}(u') = -\frac{b^+(u, u')}{c^+(u, u')} C_{2a}(u')D_{11}(u) - \frac{e(u, u')}{c^+(u, u')} D_{21}(u')C_{1a}(u) \]

\[ + \frac{d^+(u, u')}{c^+(u, u')} D_{21}(u)C_{1a}(u') \]

\[ + \frac{f(u, u')}{c^+(u, u')} \{B_{11}(u)A_{2a}(u') - B_{21}(u)A_{1a}(u') \}, \]  

(B.12)

\[ D_{22}(u')C_{2a}(u) = \frac{a^+(u, u')}{b^-(u, u')} C_{2a}(u)D_{22}(u') - \frac{e(u, u')}{b^-(u, u')} C_{2a}(u')D_{22}(u), \]  

(B.13)
\[ A_{ab}(u) D_{21}(u') = -D_{21}(u') A_{ab}(u) \]
\[ - \frac{e(u,u')}{b^{-}(u,u')} (C_{2b}(u') B_{a1}(u) - C_{2b}(u) B_{a1}(u')) , \]  
\[ (B.14) \]

\[ D_{11}(u) D_{21}(u') = - \frac{d^{+}(u,u')}{c^{+}(u,u')} D_{21}(u) D_{11}(u') + \frac{a^{+}(u,u')}{c^{+}(u,u')} D_{21}(u') D_{11}(u) \]
\[ + \frac{f(u,u')}{c^{+}(u,u')} \xi_{ab} B_{a1}(u) B_{b1}(u'), \]  
\[ (B.15) \]

\[ D_{22}(u') D_{21}(u) = - \frac{d^{+}(u,u')}{c^{+}(u,u')} D_{21}(u') D_{22}(u) + \frac{a^{+}(u,u')}{c^{+}(u,u')} D_{21}(u) D_{22}(u') \]
\[ - \frac{f(u,u')}{c^{+}(u,u')} \xi_{ab} C_{2a}(u') C_{2b}(u), \]  
\[ (B.16) \]

\[ B_{a1}(u) C_{2b}(u') = \frac{b^{+}(u,u')}{b^{-}(u,u')} C_{2b}(u') B_{a1}(u) \]
\[ + \frac{e(u,u')}{b^{-}(u,u')} (D_{21}(u') A_{ab}(u) - D_{21}(u) A_{ab}(u')) , \]  
\[ (B.17) \]

\[ B_{a2}(u) C_{2b}(u') = \frac{-b^{+}(u,u')}{b^{-}(u,u')} C_{2b}(u') B_{a2}(u) \]
\[ - \frac{e(u,u')}{b^{-}(u,u')} (D_{22}(u') A_{ab}(u) - D_{22}(u) A_{ab}(u')) , \]  
\[ (B.18) \]

\[ C_{1a}(u) C_{2b}(u') = \frac{-d^{+}(u,u')}{c^{+}(u,u')} C_{2a}(u) C_{1b}(u') \]
\[ + \frac{f(u,u')}{c^{+}(u,u')} (A_{1a}(u) A_{2b}(u') - A_{2a}(u) A_{1b}(u')) \]
\[ + \frac{a^{-}(u,u')}{c^{+}(u,u')} x_{ab}(u,u') C_{2c}(u') C_{1d}(u) \]
\[ - \frac{f(u,u')}{c^{+}(u,u')} \xi_{ab} (D_{22}(u') D_{11}(u) - D_{21}(u') D_{12}(u)), \]  
\[ (B.19) \]

where

\[ \hat{r} = \begin{pmatrix} 1 \\ \bar{a} \\ \bar{b} \\ \bar{a} \end{pmatrix} \]
\[ \hat{x} = \begin{pmatrix} 1 \\ \frac{d^{+}}{a} \\ -\frac{c^{-}}{a} \\ \frac{-c^{-}}{a} \end{pmatrix}, \]  
\[ (B.20) \]

\[ \bar{b}(u,v) = \frac{b^{+}(u,v)b^{-}(u,v)}{a^{-}(u,v)c^{+}(u,v)}, \quad \bar{a}(u,v) = 1 - \bar{b}(u,v) \]  
\[ (B.21) \]

By using these relations, we operate the transfer matrix (B.9) to the eigenstate defined
by (4.2). Here show the elements separately

\[ D_{22}(u)|\Phi_n(\lambda_1, \ldots, \lambda_n) \]
\[ = (a^+(u))^L \Pi_{j=1}^n a^+(\lambda_j, u) |\Phi_n(\lambda_1, \ldots, \lambda_n) \]
\[ + \sum_{j=1}^n (a^+(\lambda_j))^L |\Psi_{n-1}^{(1)}(u, \lambda_j; \{\lambda_l\}) \]
\[ + \sum_{j=2}^n \sum_{l=1}^{j-1} H_1(u, \lambda_l, \lambda_j) (a^+(\lambda_l)a^+(\lambda_j))^L |\Psi_{n-1}^{(3)}(u, \lambda_j, \lambda_l; \{\lambda_k\}) \}, \]
\[ (B.22) \]

\[ D_{11}(u)|\Phi_n(\lambda_1, \ldots, \lambda_n) \]
\[ = (c^+(u))^L \Pi_{j=1}^n (\frac{-b^+(u, \lambda_j)}{c^+(u, \lambda_j)}) |\Phi_n(\lambda_1, \ldots, \lambda_n) \]
\[ + \sum_{j=1}^n (b^-(\lambda_j))^L \Lambda^{(1)}(u = \lambda_j, \{\lambda_l\}) |\Psi_{n-1}^{(2)}(u, \lambda_j; \{\lambda_l\}) \]
\[ + \sum_{j=2}^n \sum_{l=1}^{j-1} H_2(u, \lambda_l, \lambda_j) (b^- (\lambda_l)b^- (\lambda_j))^L \]
\[ \times \Lambda^{(1)}(u = \lambda_j, \{\lambda_k\}) \Lambda^{(1)}(u = \lambda_l, \{\lambda_k\}) |\Psi_{n-1}^{(3)}(u, \lambda_j, \lambda_l; \{\lambda_k\}) \}, \]
\[ (B.23) \]

\[ \Sigma_a A_{aa}(u)|\Phi_n(\lambda_1, \ldots, \lambda_n) \]
\[ = (b^-(u))^L \Pi_{j=1}^n (\frac{-a^-(u, \lambda_j)}{b^-(u, \lambda_j)}) |\Lambda^{(1)}(u, \{\lambda_l\}) |\Phi_n(\lambda_1, \ldots, \lambda_n) \]
\[ + \sum_{j=1}^n (b^-(\lambda_j))^L \Lambda^{(1)}(u = \lambda_j, \{\lambda_l\}) |\Psi_{n-1}^{(1)}(u, \lambda_j; \{\lambda_l\}) \]
\[ + \sum_{j=1}^n (a^+(\lambda_j))^L |\Psi_{n-1}^{(2)}(u, \lambda_j; \{\lambda_l\}) \]
\[ + \sum_{j=2}^n \sum_{l=1}^{j-1} H_3(u, \lambda_l, \lambda_j) (\bar{a}(\lambda_l, \lambda_j) - \bar{a}(\lambda_j, \lambda_l))(a^+(\lambda_l)b^- (\lambda_j))^L \]
\[ \times \Lambda^{(1)}(u = \lambda_j, \{\lambda_k\}) |\Psi_{n-1}^{(3)}(u, \lambda_j, \lambda_l; \{\lambda_k\}) \]
\[ + \sum_{j=2}^n \sum_{l=1}^{j-1} H_4(u, \lambda_l, \lambda_j) (\bar{a}(\lambda_j, \lambda_l) - \bar{a}(\lambda_j, \lambda_l))(a^+(\lambda_j)b^- (\lambda_l))^L \]
\[ \times \Lambda^{(1)}(u = \lambda_l, \{\lambda_k\}) |\Psi_{n-1}^{(3)}(u, \lambda_j, \lambda_l; \{\lambda_k\}) \}, \]
\[ (B.24) \]
where

$$|\Psi^{(1)}_{n-1}(u, \lambda_j; \{\lambda_k\})⟩ = -\frac{e(\lambda_j, u)}{b^- (\lambda_j, u)} \prod_{k \neq j} a^+(\lambda_k, \lambda_j) \overline{c}(u)$$

$$\otimes \Phi_{n-1}^{-1}(\lambda_1, \ldots, \hat{\lambda}_j, \ldots, \lambda_n), \hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}) \overline{F}|0⟩,$$

$$|\Psi^{(2)}_{n-1}(u, \lambda_j; \{\lambda_k\})⟩ = \frac{f(u, \lambda_j)}{c^+(u, \lambda_j)} \prod_{k \neq j} (-\frac{a^- (\lambda_k, \lambda_j)}{b^-(\lambda_k, \lambda_j)}) [\overline{\xi}(\overline{B}(u) \otimes 1)]$$

$$\otimes \Phi_{n-1}^{-1}(\lambda_1, \ldots, \hat{\lambda}_j, \ldots, \lambda_n), \hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}) \overline{F}|0⟩,$$

$$|\Psi^{(3)}_{n-1}(u, \lambda_j, \lambda_l; \{\lambda_k\})⟩ = \prod_{k \neq j,l} a^+(\lambda_k, \lambda_j) a^+(\lambda_k, \lambda_l) D_{21}(u) \overline{ξ}$$

$$\otimes \Phi_{n-2}^{-1}(\lambda_1, \ldots, \hat{\lambda}_j, \ldots, \hat{\lambda}_l, \ldots, \lambda_n), \hat{O}_{jl}^{(2)}(\lambda_l, \lambda_j; \{\lambda_k\}) \overline{F}|0⟩,$$

$$\hat{O}_{jl}^{(1)}(\lambda_j; \{\lambda_k\}) = \prod_{k=1}^{l-1} \frac{a^- (\lambda_k, \lambda_j)}{a^+(\lambda_k, \lambda_j)} \hat{r}_{k,k+1}(\lambda_k, \lambda_j),$$

$$\hat{O}_{jl}^{(2)}(\lambda_l, \lambda_j; \{\lambda_k\}) = \prod_{k=1}^{l-1} \frac{a^- (\lambda_k, \lambda_j)}{a^+(\lambda_k, \lambda_j)} \hat{r}_{k+1,k+2}(\lambda_k, \lambda_j)$$

$$\times \prod_{k=1}^{l-1} \frac{a^- (\lambda_k, \lambda_j)}{a^+(\lambda_k, \lambda_j)} \hat{r}_{k,k+1}(\lambda_k, \lambda_j) \prod_{k=1}^{l-1} \frac{a^- (\lambda_k, \lambda_j)}{a^+(\lambda_k, \lambda_j)} \hat{r}_{k,k+1}(\lambda_k, \lambda_j),$$

$$H1(x, y, z) = \frac{a^+(y, x) e(z, x) f(y, x)}{b^-(y, x) b^- (z, x) c^+(y, x)} + \frac{d^+(y, x) f(y, z)}{c^+(y, x) c^+(y, z)},$$

$$H2(x, y, z) = \frac{d^+(x, y) f(y, z)}{c^+(x, y) c^+(y, z)} - \frac{e(x, y) b^+(x, z)}{c^+(x, y) c^+(x, z)},$$

$$H3(x, y, z) = \frac{f(x, y) e(x, y) e(y, z)}{c^+(x, y) b^- (x, y) b^- (y, z)} - \frac{a^+(x, y) e(x, z) f(x, y)}{b^-(y, x) b^- (x, z) c^+(x, y)},$$

$$H4(x, y, z) = -\frac{f(x, y) e(x, y) e(y, z)}{c^+(x, y) b^- (x, y) b^- (y, z)} - 2 \frac{e(x, y) e(x, y) f(y, z)}{b^+(y, x) b^- (x, y) c^+(y, z)}$$

$$- \frac{a^- (x, y) f(x, z) e(x, y)}{b^- (x, y) c^+(x, z) b^+(x, y)} (1 + \bar{a}(x, y)),$$

using the identities,

$$\Phi_n(\lambda_1, \ldots, \lambda_n) = \Phi_n(\lambda_1, \lambda_j, \ldots, \hat{\lambda}_j, \ldots, \lambda_n), \hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}),$$

$$\Phi_n(\lambda_1, \ldots, \lambda_n) = \Phi_n(\lambda_l, \lambda_j, \lambda_1, \ldots, \hat{\lambda}_l, \ldots, \hat{\lambda}_j, \ldots, \lambda_n), \hat{O}_{jl}^{(2)}(\lambda_l, \lambda_j; \{\lambda_k\}),$$

$$H1(x, y, z) + H2(x, y, z)$$

$$= (\bar{b}(y, z) - \bar{a}(y, z)) H3(x, y, z) + (\bar{b}(z, y) - \bar{a}(z, y)) H4(x, y, z).$$

The first term of r.h.s in (B.24) is obtained by using r-structure of the (B.11) and the
following calculation. \( F \) in (4.2) is fixed as follows,

\[
\text{tr}_a \, \mathcal{F}_a^{(1)}(u, \{ \lambda_l \}) \cdot \mathcal{F} = \Lambda^{(1)}(u, \{ \lambda_l \}) \mathcal{F},
\]

\[
\mathcal{F}_a^{(1)}(u, \{ \lambda_l \}) = \mathcal{L}_a^{(1)}(u, \lambda_a) \cdots \mathcal{L}_a^{(1)}(u, \lambda_1),
\]

\[
\mathcal{L}^{(1)}(u, \lambda_j) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \tilde{b}(u, \lambda_j) & \tilde{a}(u, \lambda_j) & 0 \\
0 & \tilde{a}(u, \lambda_j) & \tilde{b}(u, \lambda_j) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (B.35)

This \( L \) matrix \( \mathcal{L}^{(1)} \) is a permutation of the \( r \)-matrix. After the reparametrization,

\[
\tilde{a}(\lambda, \mu) = -\frac{U/2}{\hat{\mu} - \lambda - U/2}, \quad \tilde{b}(\lambda, \mu) = \frac{\hat{\mu} - \hat{\lambda}}{\hat{\mu} - \hat{\lambda} - U/2},
\]

\[
\hat{\mu} = \frac{z^{-}(\mu) - 1/z^{-}(\mu)}{2} + \frac{U}{4}, \quad z^{-}(\mu) = -\exp(2h(\mu))/\tan \mu,
\] (B.36)

we see that this is equivalent to the problem of diagonalizing the transfer matrix for the XXX model and we know the solution,

\[
\Lambda^{(1)}(u, \{ \lambda_j \}, \{ \mu_l \}) = \Pi^{m}_{j=1} \frac{1}{b(\mu_i, u)} + \Pi^{n}_{j=1} \tilde{b}(u, \lambda_j) \Pi^{m}_{i=1} \frac{1}{b(u, \mu_i)},
\] (B.37)

where \( \Pi^{n}_{j=1} \tilde{b}(\mu_j, \lambda_j) = \Pi^{m}_{k \neq i} \frac{\tilde{b}(\mu_i, \mu_k)}{b(\mu_i, \mu_k)}. \) (B.38)

Now the eigenvalue of the transfer matrix is obtained,

\[
\Lambda(u, \{ \lambda_l \}) = (a^+(u))^L \Pi^{m}_{j=1} \frac{a^+(\lambda_j, u)}{b^-(\lambda_j, u)} + (c^+(u))^L \Pi^{n}_{j=1} (-\frac{b^+(u, \lambda_j)}{c^+(u, \lambda_j)}) - (b^-(u))^L \Pi^{n}_{j=1} (-\frac{a^-(u, \lambda_j)}{b^-(u, \lambda_j)}) \Lambda^{(1)}(u, \{ \lambda_l \}).
\] (B.39)

And the unwanted terms in (B.22), (B.23) and (B.24) give the another constraint,

\[
\left(\frac{a^+(\lambda_j)}{b^-(\lambda_j)}\right)^L = \Lambda^{(1)}(u = \lambda_j, \{ \lambda_l \}).
\] (B.40)

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