THE ACDM MODEL IN THE LEAD—A BAYESIAN COSMOLOGICAL MODEL COMPARISON

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ABSTRACT

Recent astronomical observations indicate that our universe is undergoing a period of accelerated expansion. While there are many cosmological models that explain this phenomenon, the main question remains which is the best one in light of available data. We consider 10 cosmological models of the accelerating universe and select the best one using the Bayesian model comparison method. We demonstrate that the ACDM model is most favored by the Bayesian statistical analysis of the SNIa, CMB, baryon acoustic oscillation, and \( H(z) \) data.

Subject headings: cosmological parameters — cosmology: miscellaneous — methods: statistical

1. INTRODUCTION

Recent observations of Type Ia supernovae (SNIa) provide the main evidence that the current universe is in an accelerating phase of expansion (Riess et al. 1998; Perlmutter et al. 1999). There are many different cosmological models used to explain the accelerating phase of evolution of the current universe. They can be divided into two groups of models according to “philosophical” assumptions about the cause of the accelerated expansion. In the first type of explanation, the concept of mysterious dark energy of an unknown form is used, while in the second one some modification of the Friedmann equation is postulated. Here we choose five models which belong to the former group as well as five ones which belong to the latter one. All the chosen models are assumed to be spatially flat.

If we assume the Friedmann-Robertson-Walker (FRW) model in which effects of nonhomogeneities are neglected, then acceleration can be driven by a dark energy component \( \Lambda \) (matter fluid violating the strong energy condition). This kind of energy represents roughly 70% of the matter content of the present universe. The model with the cosmological constant (the ACDM model) has the equation of state for dark energy \( p_X = -\rho_X \) (Weinberg 1989). The model with phantom dark energy has \( p_X = w_X \rho_X \), where \( w_X < -1 \) is a negative constant (Caldwell 2002; Dabrowski et al. 2003).

The next one is the model with a dynamical coefficient of the equation of state, parameterized by the scale factor \( a \), \( w(a) = w_0 + w_1(1 - a) \) (Chevallier & Polarski 2001; Linder 2003). The other simple approach is to represent dark energy in the form of a minimally coupled scalar field \( \phi \) with the potential \( V(\phi) \). In cosmology, the quintessence idea is important in understanding the role of the scalar field in the current universe. We consider the power-law parameterized quintessence model (Peebles & Ratra 1988; Ratra & Peebles 1988). In this case, the density of dark energy changes with the scale factor as \( \rho_X = \rho_0 a^{-3(1+w_X)} \), where \( w_X(a) \) is the mean of a coefficient of the equation of state in the logarithmic scale factor

\[
\bar{w}_X(a) = \int \frac{w_X(a) d \ln a}{\int d \ln a}
\]

and has the form \( \bar{w}_X = w_0 a^\alpha \) (Rahvar & Movahed 2007). The first group is completed with the model with the generalized Chaplygin gas, where \( p_X = -A/\rho_X^\gamma \) (here \( A > 0 \) and \( \gamma = \text{const} \)). We gathered the above models together with their Hubble functions (with the assumption that the universe is spatially flat) in Table 1.

As we have written above we also consider five models offering an explanation of the current acceleration of the universe in an alternative way to dark energy. The brane models have postulated that the observer is embedded on the brane in a larger space in which gravity can propagate; these include the Dvali-Gabadadze-Porrati model (DGP; Dvali et al. 2000) and the Sahni-Shtanov brane 1 model (Shtanov 2000). The Cardassian model, in which the universe is flat, is matter dominated and accelerating as a consequence of the modification of the Friedmann first integral as \( 3H^2 = \rho + Ba^\gamma \), where \( B \) is a constant and the energy density contains only dust matter and radiation (Freese & Lewis 2002). We also include in the analysis the bouncing model arising in the context of loop quantum gravity (the BACDM model; Singh & Vandersloot 2005; Szydłowski et al. 2005) and the model with energy transfer between the dark matter and dark energy sectors (the A decaying vacuum model; Szydłowski et al. 2006b). We gathered the above models together with their Hubble functions (with the assumption that the universe is spatially flat) in Table 2.

The main goal of this paper is to compare all these models in light of SNIa, CMB, baryon acoustic oscillation (BAO), and \( H(z) \) data. We use the Bayesian model comparison method, which we describe in §2. This method is commonly used in the context of cosmological model selection (see, e.g., Liddle 2004, 2007; John & Narlikar 2002; Saini et al. 2004; Parkinson et al. 2005; Mukherjee et al. 2006a, 2006b; Beltran et al. 2005; Szydłowski & Godlowski 2006; Godlowski & Szydłowski 2005; Szydłowski et al. 2006a; Liddele et al. 2006; Sahlen et al. 2007; Serra et al. 2007; Kunz et al. 2006; Trotta 2007a, 2007b). Recently, the Bayesian information criteria were applied in the context of choosing an adequate model of acceleration of the universe (Davis et al. 2007). The authors showed a preference for models beyond the standard FRW cosmology (so-called exotic cosmological models) whose best-fit parameters reduce them to the cosmological constant model.

2. MODEL COMPARISON IN BAYES THEORY

Let us consider the set of \( K \) models \( \{ M_1, \ldots, M_K \} \). In the Bayes theory, the best model from the set under consideration is
the one which has the largest value of the probability in light of the data \((D)\), the so-called posterior probability (Jeffreys 1961),

\[
P(M_i|D) = \frac{P(D|M_i)P(M_i)}{P(D)}.
\]

(1)

where \(P(M_i)\) is the prior probability for the model indexed by \(i\), the value of which depends on our previous knowledge about the model under consideration, that is to say without information coming from data \(D\), and \(P(D)\) is the normalization constant. If we have no foundation to favor one model over another one from the set, we usually assume the same values of this quantity for all of them, i.e., \(P(M_i) = 1/K, i = 1,\ldots,K\).

To obtain the form of \(P(D)\), it is required that a sum of the posterior probabilities for all models from the set is equal to one,

\[
\sum_{i=1}^{K} P(M_i|D) = 1 \rightarrow P(D) = \sum_{i=1}^{K} P(D|M_i)P(M_i).
\]

Therefore, conclusions based on the values of posterior probabilities strongly depend on the set of models and can change when the set of models is different.

The quantity \(P(D|M_i)\) is the marginal likelihood (also called the evidence) and has the form

\[
P(D|M_i) = \int L(\theta|D, M_i)P(\theta|M_i)d\theta = E_i,
\]

(2)

where \(L(\theta|D, M_i)\) is the likelihood of the model under consideration, \(\theta\) is the vector of the model parameters, and \(P(\theta|M_i)\) is the prior probability for the model parameters.

In the case under consideration we cannot obtain the value of the evidence by analytical computation. We need a numerical method or an approximation to this quantity.

Schwarz (1978) showed that for independent and identically distributed observations \((D = \{x_i\}, i = 1,\ldots,N)\) coming from a linear exponential family distribution, defined as

\[
f(x_i|\theta) = \exp \left[ \sum_{k=1}^{S} w_k (\theta) t_k(x_i) + b(\theta) \right], \quad S = d,
\]

where \(w_1,\ldots,w_S, b\) are functions of only \(\theta \in \mathbb{R}^d\) and \(t_1,\ldots,t_S\) are functions of only \(x_i\), the asymptotic approximation \((N \rightarrow \infty)\) to the logarithm of the evidence is given by

\[
\ln E = \ln \mathcal{L} - \frac{d}{2} \ln N + O(1),
\]

(3)

where \(\mathcal{L}\) is the maximum likelihood and \(O(1)\) is the term of order unity in \(N\). In this case, the likelihood function has the form

\[
\mathcal{L}(\bar{\theta}|D, M) = \prod_{i=1}^{N} f(x_i|\bar{\theta}) = \exp \left\{ N \left[ \sum_{k=1}^{S} w_k (\bar{\theta}) t_k(D) + b(\bar{\theta}) \right] \right\},
\]

where \(t_k(D) = (1/N) \sum_{i=1}^{N} t_k(x_i)\). The integral from equation (2) can be written as

\[
\int \exp \left[ N g(\bar{\theta}) \right] P(\bar{\theta}|M) d\bar{\theta},
\]

(4)

where \(g(\bar{\theta}) = \sum_{k=1}^{S} w_k(\bar{\theta}) t_k(D) + b(\bar{\theta})\). This integral has the form of the so-called Laplace integral. Assume that \(g(\theta)\) has a maximum at \(\theta_0\) and \(\mathcal{L}(\theta_0|M) \neq 0\). When \(N \rightarrow \infty, \exp[N g(\theta)]\) will be a sharp function peaked at \(\theta_0\). Then the main contribution to the integral from equation (4) comes from the small neighborhood of \(\theta_0\). In this region \(\mathcal{L}(\theta|M) \approx \mathcal{L}(\theta_0|M)\), we can also replace the \(g(\theta)\) function by its Taylor expansion around \(\theta_0\), \(g(\theta) = g(\theta_0) - \frac{1}{2} (\theta - \theta_0)^T C^{-1}(\theta - \theta_0)\), where \([C^{-1}]_{ij} = [\partial^2 g(\theta)/\partial \theta_i \partial \theta_j]_{\theta = \theta_0}\), and

\[
\Omega \approx 10^{-9}
\]

\[
\Omega_0 \approx 1 - \Omega_{\Lambda 0} + 2 \sqrt{\Omega_{\Lambda 0}/\Omega_{\Lambda 0} (1+z^3)}^3 + \Omega_{\Lambda 0} + \Omega_{0 0} + \Omega_{3 0}
\]

\[
\Omega_{\Lambda 0} = 10^{-9}
\]

\[
\Omega_{\Lambda 0} = 1 - \Omega_{\Lambda 0} + 2 \sqrt{\Omega_{\Lambda 0}/\Omega_{\Lambda 0} (1+z^3)}^3 + \Omega_{\Lambda 0} + \Omega_{0 0} + \Omega_{3 0}
\]
extend the integration region to all of \( R^d \). One can get the asymptotic of the integral from equation (4), \( E = \exp \left( N g(\theta_0) \right) / \left( 2\pi / N \right)^{d / 2} \text{det}(C) \) and \( \ln E = N g(\theta_0) - (d / 2) \ln N + R \), where \( R \) is the term which does not depend on \( N \). One can see that \( N g(\theta_0) = \ln \left( L(\theta_0 | D, M) \right) \), where \( \theta_0 \) is the point which maximizes \( g(\theta) = (1 / N) \ln L(\theta | D, M) \), so it is equivalent to \( \theta_{\text{MLE}} \) (the maximum likelihood estimator of \( \theta \)). Finally one can obtain the result from equation (3). According to this result, Schwarz (1978) introduced a criterion for the model selection: the best model is that which minimizes the Bayes information criterion (BIC) quantity, defined as

\[
\text{BIC} = -2 \ln L + d \ln N. \tag{5}
\]

This criterion can be derived in such a way that it is not required to assume any specific form for the likelihood function but it is only necessary that the likelihood function satisfies some nonrestrictive regularity conditions. Moreover, data do not need to be independent and identically distributed. This derivation requires one to assume that a prior for model parameters is not equal to zero in the neighborhood of the point where the likelihood function under a given model reaches a maximum and that it is bound in the whole parameter space under consideration (Cavanaugh & Neath 1999). It should be pointed out that an asymptotic assumption is satisfied when the sample size used in the analysis is large with respect to the number of unknown model parameters.

It is useful to choose one model from our model set (here indexed by \( s \)) and compare the rest of the models with this one. We can define the \( \Delta \text{BIC}_{ls} \) quantity, which is the difference of the BIC quantity for the models indexed by \( i \) and \( s \), as \( \Delta \text{BIC}_{ls} = \text{BIC}_l - \text{BIC}_s \) and present the posterior probability in the form

\[
P(M_i | D) = \frac{\exp \left( -1/2 \Delta \text{BIC}_{ls} \right) P(M_i)}{\sum_{k=1}^{K} \exp \left( -1/2 \Delta \text{BIC}_{ks} \right) P(M_k)}. \tag{6}
\]

Let us assume that we have computed the probabilities in light of data \( D \) for models from the set under consideration. Then we gather new data \( D_l \) and want to update the probabilities which we already have. We can compute probabilities in light of the new data using information coming from the previous analysis, which allows us to favor one model over another; we can use posterior probabilities for models obtained in earlier computations as the prior probabilities for models in the next analysis. We apply this method in evaluating the posterior probabilities for models described in \( \S \) 1 using the information coming from SNIa, CMB, BAO, and \( H(z) \) data.

### 3. APPLICATION TO COSMOLOGICAL MODEL COMPARISON

We start with the \( N = 192 \) sample of SNIa (Riess et al. 2007; Wood-Vasey et al. 2007; Davis et al. 2007). In this case, the likelihood function has the form

\[
L \propto \exp \left( -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\mu_i - \mu_i^{\text{theor}}}{\sigma_i^2} \right)^2 \right),
\]

where \( \sigma_i \) is known, \( \mu_i^{\text{obs}} = m_i - M (m_i \) is the apparent magnitude, and \( M \) is the absolute magnitude of SNIa, \( \mu_i^{\text{theor}} = 5 \log_{10} D_Li + M \), \( D_Li = H_0 d_Li \), where \( d_Li \) is the luminosity distance, which, with the assumption that \( k = 0 \), is given by

\[
d_Li = (1 + z_i)c \int_0^{z_i} \frac{dz'}{H(z')}.
\]

In this case we used the BIC quantity as an approximation to \(-2 \ln E\) and assumed that all models have equal values of prior probabilities. Based on our previous experiences we have assumed that \( H_0 \in (60, 80) \) for all models and for each model additionally assumed:

| Model | Prior SN | Posterior SN | Posterior + CMB | Posterior + BAO | Posterior + \( H(z) \) | Posterior SN + CMB + BAO + \( H(z) \) |
|-------|--------|-------------|-----------------|----------------|-----------------|----------------------------------|
| 1.    | 0.20   | 0.91        | 0.92 (0.95)     | 0.91 (0.97)    | 0.94 (0.99)     | 0.84                             |
| 2.    | 0.20   | 0.01        | 0.01 (0.01)     | 0.01 (0.01)    | 0.00 (0.00)     | 0.02                             |
| 3.    | 0.20   | 0.07        | 0.06 (0.04)     | 0.07 (0.03)    | 0.05 (0.01)     | 0.06                             |
| 4.    | 0.20   | 0.01        | 0.01 (0.00)     | 0.01 (0.00)    | 0.01 (0.00)     | 0.04                             |
| 5.    | 0.20   | 0.00        | 0.00 (0.00)     | 0.00 (0.00)    | 0.00 (0.00)     | 0.04                             |

The \( \text{BIC} \) is applied to data from a set of SNIa, CMB, BAO, and \( H(z) \) data.

### Table 3

**Posterior Probabilities for Models from Table 1 with \( \Omega_{m,0} \in (0, 1) \)**

| Model | Prior SN | Posterior SN | Posterior + CMB | Posterior + BAO | Posterior + \( H(z) \) | Posterior SN + CMB + BAO + \( H(z) \) |
|-------|--------|-------------|-----------------|----------------|-----------------|----------------------------------|
| 6.    | 0.20   | 0.89        | 0.92 (0.97)     | 0.92 (0.98)    | 0.93 (0.98)     | 0.07                             |
| 7.    | 0.20   | 0.01        | 0.00 (0.00)     | 0.00 (0.00)    | 0.00 (0.00)     | 0.03                             |
| 8.    | 0.20   | 0.01        | 0.01 (0.01)     | 0.01 (0.01)    | 0.01 (0.01)     | 0.13                             |
| 9.    | 0.20   | 0.08        | 0.07 (0.01)     | 0.07 (0.00)    | 0.06 (0.00)     | 0.74                             |
| 10.   | 0.20   | 0.01        | 0.00 (0.01)     | 0.00 (0.01)    | 0.00 (0.01)     | 0.03                             |

### Table 4

**Posterior Probabilities for Models from Table 2 with \( \Omega_{m,0} \in (0, 1) \)**

| Model | Prior SN | Posterior SN | Posterior + CMB | Posterior + BAO | Posterior + \( H(z) \) | Posterior SN + CMB + BAO + \( H(z) \) |
|-------|--------|-------------|-----------------|----------------|-----------------|----------------------------------|
| 6.    | 0.20   | 0.89        | 0.92 (0.97)     | 0.92 (0.98)    | 0.93 (0.98)     | 0.07                             |
| 7.    | 0.20   | 0.01        | 0.00 (0.00)     | 0.00 (0.00)    | 0.00 (0.00)     | 0.03                             |
| 8.    | 0.20   | 0.01        | 0.01 (0.01)     | 0.01 (0.01)    | 0.01 (0.01)     | 0.13                             |
| 9.    | 0.20   | 0.08        | 0.07 (0.01)     | 0.07 (0.00)    | 0.06 (0.00)     | 0.74                             |
| 10.   | 0.20   | 0.01        | 0.00 (0.01)     | 0.00 (0.01)    | 0.00 (0.01)     | 0.03                             |
Tables 6, 7, and 8 for set 1, set 2, and set 3, respectively. The best one from the group of models with modified gravity; the DGP model is the best one from all the models under consideration; the DGP model is the best one from the set of models with dark energy as well as the best one conclude that in light of SNIa data the

Tables 3, 4, and 5 for set 1, set 2, and set 3, respectively. One can

R

where

H

(Lexp)

posterior probabilities are obtained using equation (6). We analyze three sets of models: (1) the set of models with dark energy (Table 1), (2) the set of models with a modified theory of gravity (Table 2), and (3) the set of all models (Tables 1 and 2 together).

The results for the case with \( \Omega_m,0 \in (0, 1) \) are presented in Tables 3, 4, and 5 for set 1, set 2, and set 3, respectively. One can conclude that in light of SNIa data the \( \Lambda \)CDM is the best model from the set of models with dark energy as well as the best one from all the models under consideration; the DGP model is the best one from the group of models with modified gravity.

The results for the case with \( \Omega_{m,0} \in (0.25, 0.31) \) are gathered in Tables 6, 7, and 8 for set 1, set 2, and set 3, respectively. The conclusion changed for the set of models with modified gravity; here the best one is the Cardassian model.

In the next step we included information coming from CMB data. Here the likelihood function has the form

\[
L \propto \exp \left( -\frac{(R^{\text{theor}} - R^{\text{obs}})^2}{2\sigma_R^2} \right),
\]

where \( R \) is the so-called shift parameter, \( R^{\text{theor}} = \int_0^{\infty} f_{\text{th}}(z) H_0/\exp[H(z)dz] \), and \( R^{\text{obs}} = 1.70 \pm 0.03 \) for \( z_{\text{dec}} = 1089 \) (Spergel et al. 2007; Wang & Mukherjee 2006). It should be pointed out that the parameter \( R \) is independent of \( H_0 \).

The values of the evidence were obtained by numerical integration. We assumed a flat prior for all model parameters. It is known that evidence depends on the prior probabilities for model parameters. Assumptions for the model parameter intervals which we made in a previous analysis could be inappropriate here. Because of this we made a stricter analysis for models with parameters for which the interval width exceeds one. This width was used for convenience. We computed the evidence for different parameter intervals, which do not exceed the range assumed above, and with a minimal width equal to one. There are of course extremely many possibilities. We limited our analysis to intervals \((a, b)\), where \( a \) and \( b \) are integers. Finally, we chose the case with the greatest evidence.

We consider the situation with \( \Omega_{m,0} \in (0, 1) \) and \( \Omega_{m,0} \in (0.25, 0.31) \). The range for parameters which change after stricter analysis in the first case are:

\[
\text{Model 3.} - w_X \in (-2, -1),
\]

\[
\text{Model 4.} - w_0 \in (-1, 0), \; w_1 \in (-2, 0),
\]

\[
\text{Model 5.} - w_0 \in (-3, -2), \; \alpha \in (1, 2),
\]

\[
\text{Model 7.} - \Omega_{m,0} \in (0, 1), \; n \in (3, 4),
\]

\[
\text{Model 8.} - \Omega_{int,0} \in (-1, 0), \; n \in (-10, -9),
\]

\[
\text{Model 9.} - n \in (0, 1),
\]

\[
\text{Model 10.} - \Omega_{0,0} \in (0, 1), \; \Omega_{ab,0} \in (0, 1);
\]

and in the second case are:

\[
\text{Model 3.} - w_X \in (-2, -1),
\]

\[
\text{Model 4.} - w_0 \in (-1, 0), \; w_1 \in (-2, -1),
\]

\[
\text{Model 5.} - w_0 \in (-2, -1), \; \alpha \in (0, 1),
\]

\[
\text{Model 7.} - \Omega_{m,0} \in (0, 1), \; n \in (3, 4),
\]

\[
\text{Model 8.} - \Omega_{int,0} \in (-1, 0), \; n \in (-2, -1),
\]

\[
\text{Model 9.} - n \in (0, 1),
\]

\[
\text{Model 10.} - \Omega_{0,0} \in (0, 1), \; \Omega_{ab,0} \in (3, 4).
\]

Posterior probabilities were obtained using equation (1). Here we treated posterior probabilities evaluated in the analysis with SNIa data as prior probabilities. Results are gathered in tables like in previous analyses. We also show the values of the posterior probabilities obtained for the intervals assumed at the beginning (numbers in the parentheses). As we can see, the \( \Lambda \)CDM model is still the best one from the models with dark energy (for both ranges of \( \Omega_{m,0} \)). The conclusion is the same for the set of models with modified gravity; the DGP model is the best one in the first case and the Cardassian model in the second. When we assume that \( \Omega_{m,0} \in (0, 1) \), there is no evidence to favor the \( \Lambda \)CDM model over the DGP model (they have the same values for the

### TABLE 5

**Posterior Probabilities for Models from Tables 1 and 2 with \( \Omega_{m,0} \in (0, 1) \)**

| Model | Prior | Posterior SNIa | Posterior + CMB | Posterior + BAO | Posterior + \( H(z) \) | Posterior SNIa+CMB+BAO+\( H(z) \) |
|-------|-------|----------------|----------------|----------------|-----------------|----------------------------------|
| 1      | 0.10  | 0.51           | 0.46 (0.48)    | 0.44 (0.46)    | 0.43 (0.45)     | 0.74                             |
| 2      | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.02                             |
| 3      | 0.10  | 0.04           | 0.03 (0.02)    | 0.03 (0.01)    | 0.02 (0.00)     | 0.05                             |
| 4      | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.04                             |
| 5      | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.03                             |
| 6      | 0.10  | 0.39           | 0.46 (0.48)    | 0.47 (0.52)    | 0.50 (0.54)     | 0.01                             |
| 7      | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.005                            |
| 8      | 0.10  | 0.01           | 0.01 (0.01)    | 0.02 (0.01)    | 0.02 (0.01)     | 0.01                             |
| 9      | 0.10  | 0.04           | 0.04 (0.01)    | 0.04 (0.00)    | 0.03 (0.00)     | 0.09                             |
| 10     | 0.10  | 0.01           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.005                            |

### TABLE 6

**Posterior Probabilities for Models from Table 1 with \( \Omega_{m,0} \in (0.25, 0.31) \)**

| Model | Prior | Posterior SNIa | Posterior + CMB | Posterior + BAO | Posterior + \( H(z) \) | Posterior SNIa+CMB+BAO+\( H(z) \) |
|-------|-------|----------------|----------------|----------------|-----------------|----------------------------------|
| 1      | 0.20  | 0.91           | 0.98 (0.88)    | 0.99 (0.99)    | 0.99 (1.00)     | 0.84                             |
| 2      | 0.20  | 0.01           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.02                             |
| 3      | 0.20  | 0.06           | 0.01 (0.11)    | 0.00 (0.01)    | 0.00 (0.00)     | 0.06                             |
| 4      | 0.20  | 0.01           | 0.01 (0.00)    | 0.01 (0.00)    | 0.01 (0.00)     | 0.04                             |
| 5      | 0.20  | 0.01           | 0.00 (0.00)    | 0.00 (0.00)    | 0.00 (0.00)     | 0.04                             |
posterior probabilities), but when we restrict the $\Omega_{m,0}$ range to $(0.25, 0.31)$, the $\Lambda$CDM model still stays as the best one, with even greater probability.

As the third set of observational data we used the measurement of the BAOs from the SDSS luminous red galaxies (Eisenstein et al. 2005). In this case, the likelihood function has the form

$$L \propto \exp \left[ -\frac{(\frac{\mathcal{L}}{\mathcal{L}_{\text{theory}}})^2}{2\sigma^2} \right],$$

where $\frac{\mathcal{L}}{\mathcal{L}_{\text{theory}}}$ is the likelihood, $\mathcal{L}_{\text{theory}} = \frac{\mathcal{L}_{\text{theory}}}{\mathcal{L}_{\text{obs}}}$, and $\sigma$ is the standard deviation.

Here values of the evidence are obtained by numerical integration. We made the analogous analysis with the parameter intervals as above with the additional requirement that obtained intervals must at least cover the intervals obtained in the previous analysis. For most of the models, the conclusions are the same as in the previous analysis. The case where the intervals have changed for $\Omega_{m,0}(0, 1)$ is only model 5, $\Omega_{m,0}(-3, 1)$, $\alpha \in (1, 2)$, while the case where the intervals have changed for $\Omega_{m,0}(0.25, 0.31)$ is only model 8, $\Omega_{m,0}(-1, 0)$, $n \in (-2, 1)$.

Here we used posterior probabilities obtained in the analysis with the CMB data as prior probabilities. Results were again presented in the tables described above. The conclusion is different for the set of all models in the case with $\Omega_{m,0}(0, 1)$; the DGP model becomes the best one from them.

Finally, we used the observational $H(z)$ data ($N = 9$) from Simon et al. (2005; see also Samushia & Ratra 2006; Wei & Zhang 2007 and references therein). These data are based on the differential ages $(dt/dz)$ of the passively evolving galaxies which allow one to estimate the relation $H(z) = \ddot{a}/a = -[1/(1+z)](dz/dt)$. Here the likelihood function has the form

$$L \propto \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} \frac{(H(z)_i - H(z))_j^2}{\sigma^2} \right],$$

where $H(z)$ is the Hubble function, and $H_i$ and $z_i$ are observational data.

The values of the evidence were obtained by numerical integration. As above we assumed flat prior probabilities for model parameters. The ranges for them which changed after analogous to previous analysis are as follows for the case with $\Omega_{m,0}(0, 1)$ and $H_0 \in (60, 80)$:

- Model 5: $\alpha \in (-3, 1)$, $\alpha \in (0, 2)$,
- Model 8: $\Omega_{m,0}(-1, 0)$, $n \in (-10, 0)$;

and for the case with $\Omega_{m,0}(0.25, 0.31)$ and $H_0 \in (60, 80)$:

- Model 4: $\Omega_{m,0}(-1, 0)$, $w_1 \in (-2, 0)$.

Values of posterior probabilities obtained in analysis with the BAO data were treated as prior probabilities in this analysis. The results are presented in the tables.

As one can see in the case with $\Omega_{m,0}(0.25, 0.31)$, the $\Lambda$CDM model is the best one from the set of models with dark energy as well as the best one from all models considered in this paper. The conclusions are different in the set of models with modified gravity; after the analysis with observational $H(z)$ data, the DGP model becomes the best one. In the case with $\Omega_{m,0}(0, 1)$, the $\Lambda$CDM model is still the best model from the set of models with dark energy, while the DGP model is the best one from the set of models with the modified theory of gravity as well as the best one from all models considered.

As one can conclude, final results coming from computation including the shrinking parameter interval procedure are the same as final results coming from computation with parameter intervals assumed at the beginning. The best model (in each set) does not change, but the probability of being the best one are greater for the second case.

As we have written above, we used the BIC quantity as an approximation to $-2 \ln E$ for the SNIa data. This approximation gives good results if, in the set under consideration, there is one favored model. The problem appears when we have two favored models with nearly the same probabilities. Such a situation is in

**TABLE 7**

| Model | Prior | Posterior SNIa | Posterior + CMB | Posterior + BAO | Posterior + $H(z)$ | Posterior SNIa+CMB+BAO+$H(z)$ |
|-------|-------|----------------|----------------|---------------|-------------------|-----------------------------|
| 6     | 0.20  | 0.19           | 0.27 (0.26)    | 0.35 (0.50)   | 0.45 (0.91)       | 0.07                        |
| 7     | 0.20  | 0.04           | 0.00 (0.00)    | 0.00 (0.00)   | 0.00 (0.00)       | 0.03                        |
| 8     | 0.20  | 0.05           | 0.13 (0.04)    | 0.24 (0.06)   | 0.26 (0.04)       | 0.13                        |
| 9     | 0.20  | 0.68           | 0.60 (0.70)    | 0.41 (0.44)   | 0.29 (0.05)       | 0.74                        |
| 10    | 0.20  | 0.04           | 0.00 (0.00)    | 0.00 (0.00)   | 0.00 (0.00)       | 0.03                        |

**TABLE 8**

| Model | Prior | Posterior SNIa | Posterior + CMB | Posterior + BAO | Posterior + $H(z)$ | Posterior SNIa+CMB+BAO+$H(z)$ |
|-------|-------|----------------|----------------|---------------|-------------------|-----------------------------|
| 1     | 0.10  | 0.81           | 0.91 (0.82)    | 0.96 (0.96)   | 0.96 (0.97)       | 0.74                        |
| 2     | 0.10  | 0.01           | 0.00 (0.00)    | 0.00 (0.00)   | 0.00 (0.00)       | 0.02                        |
| 3     | 0.10  | 0.07           | 0.01 (0.12)    | 0.01 (0.01)   | 0.01 (0.00)       | 0.05                        |
| 4     | 0.10  | 0.01           | 0.01 (0.00)    | 0.01 (0.00)   | 0.01 (0.00)       | 0.04                        |
| 5     | 0.10  | 0.01           | 0.01 (0.00)    | 0.01 (0.00)   | 0.01 (0.00)       | 0.03                        |
| 6     | 0.10  | 0.02           | 0.02 (0.02)    | 0.01 (0.02)   | 0.01 (0.02)       | 0.01                        |
| 7     | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)   | 0.00 (0.00)       | 0.005                       |
| 8     | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)   | 0.00 (0.00)       | 0.01                        |
| 9     | 0.10  | 0.07           | 0.04 (0.04)    | 0.01 (0.01)   | 0.01 (0.01)       | 0.09                        |
| 10    | 0.10  | 0.00           | 0.00 (0.00)    | 0.00 (0.00)   | 0.00 (0.00)       | 0.05                        |
the set of all models with $\Omega_m,0 \in (0, 1)$. The $\Lambda$CDM and DGP models have nearly the same values of model probabilities which are the greatest ones in the considered set. This forces us to compute the evidence by numerical integration for these two models (for SNIa data) to check if our previous conclusion is true. In Table 9 we gathered the values of probabilities obtained after computation of the full Bayesian evidence (case 1) as well as such values obtained when the BIC approximation was used (case 2).

As one can see the conclusion changed. The $\Lambda$CDM model is better than the DGP model in light of all data sets used in this paper. The BIC approximation is not enough in these cases, since it gives us the wrong answer.

Finally we compare the considered sets of models, treating all described-above data sets as $N = 192 + 1 + 1 + 9$ independent data. In this case the likelihood function has the form

$$L \propto \exp \left( -\frac{1}{2} \left\{ \sum_{i=1}^{192} \frac{(\mu_i^{\text{theor}} - \mu_i^{\text{obs}})^2}{\sigma_i^2} + \frac{(R^{\text{theor}} - R^{\text{obs}})^2}{\sigma_R^2} + \frac{(A^{\text{theor}} - A^{\text{obs}})^2}{\sigma_A^2} + \sum_{i=1}^9 \frac{[H(z_i) - H(z_i)]^2}{\sigma_i^2} \right\} \right).$$

Here we assumed a flat prior for model parameters in the range described at the beginning of this section. We used the BIC as an approximation to $-2 \ln E$. The results for the cases with $\Omega_m,0 \in (0, 1)$ and $\Omega_m,0 \in (0.25, 0.31)$ for all described-above sets of models are gathered in Tables 3, 4, 5, 6, 7, and 8.

The results in this analysis confirm that the $\Lambda$CDM model is the best one in the set of models with dark energy as well as the best one in the set of all models for both ranges in $\Omega_m,0$. The conclusion that the Cardassian model is the best one in the set of models with modified gravity (for both ranges in $\Omega_m,0$) is contrary to the previous inference in which the DGP model was the best one. The reason for this disagreement is related to the different constraints on the $\Omega_m,0$ parameter for various data sets for the considered model. In Table 10 we presented the results of parameter estimation performed for all data sets independently as well as for all data sets applied simultaneously for the $\Lambda$CDM, DGP, and Cardassian models.

### 4. CONCLUSIONS

In this paper we gathered 10 models of the accelerating universe. Five of them explain the accelerated phase of the universe in terms of dark energy, while the other five explain this phenomenon by the modification of the theory of gravity. We used the Bayesian model comparison method to select the best one in the set of models with dark energy, in the set of models with a modified theory of gravity, as well as the best one of all of them. The selection was based on the SNIa, CMB, BAO, and observational $H(z)$ data; we treat posterior probabilities obtained in one analysis as prior probabilities in the next one. Information coming from the previous analysis allows us to favor one model over another. We used the approximation proposed by Schwarz to $-2 \ln E$ in the case with SNIa data and numerical integration of the likelihood function within an allowed parameter space (we assumed flat prior probabilities for model parameters) in the other cases. We consider separately cases with $\Omega_m,0 \in (0, 1)$ and with $\Omega_m,0 \in (0.25, 0.31)$. We made a stricter analysis for models with parameters for which the interval widths exceed one; we evaluated the evidence for these models for different parameter intervals with the minimal width equal to one, which does not exceed intervals assumed in the analysis with SNIa data, and finally chose the best one from them (with the greatest evidence) for the next analysis. We compare such results with the results obtained in the calculation where we treat all data sets as $N = 192 + 1 + 1 + 9$ independent data and use BIC as an approximation to $-2 \ln E$.

We can conclude that for the case with $\Omega_m,0 \in (0.25, 0.31)$ as well as for case with $\Omega_m,0 \in (0, 1)$:

1. The $\Lambda$CDM model is the best one from the set of models with dark energy as well as the best one from the set of all models considered in this paper.
2. The Cardassian model is the best one from the set with models with a modified theory of gravity.

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