Deforming an $\epsilon$-Close to Hyperbolic Metric to a Warp Metric

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Abstract

We show how to deform a metric of the form $g = g_r + dr^2$ to a warp metric $\mathcal{W}g = sinh^2(r)g' + dr^2$ ($g'$ does not depend on $r$), for $r$ less than some fixed $r_0$. Our main result establishes to what extent the warp forced metric $\mathcal{W}g$ is close to being hyperbolic, if we assume $g$ to be close to hyperbolic.

Section 0. Introduction.

First we introduce some notation. Let $(M^n, g)$ be a complete Riemannian manifold with center $o \in M$, that is, the exponential map $exp_o : T_o M \to M$ is a diffeomorphism. Using the exponential map $exp_o$ we shall sometimes identify $M$ with $\mathbb{R}^n$, thus can write the metric $g$ on $M - \{o\} = S^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where $r$ is the distance to $o$. The closed ball of radius $r$ in $M$, centered at $o$, will be denoted by $B_r = B_r(M)$. Let $\rho : \mathbb{R} \to [0, 1]$ be smooth and such that: (i) $\rho|_{(-\infty, 0+\delta]} \equiv 1$, and (ii) $\rho|_{[1/2-\delta, \infty)} \equiv 0$, where $\delta > 0$ is small.

Let $M$ have center $o$ and metric $g = g_r + dr^2$. Fix $r_0 > 0$. We define the metric $\bar{g}_{r_0}$ by:

$$\bar{g}_{r_0} = sinh^2(t)\left(\frac{1}{sinh^2(r_0)}\right)g_{r_0} + dr^2$$

Note that this metric is a warp metric (warped by $sinh$). We now force the metric $g$ to be equal to $\bar{g}_{r_0}$ on $B_{r_0} = B_{r_0}(M)$ and stay equal to $g$ outside $B_{r_0 + \frac{1}{2}}$. For this we define the warp forced (on $B_{r_0}$) metric as:

$$\mathcal{W}_{r_0} g = \rho_{r_0} \bar{g}_{r_0} + (1 - \rho_{r_0})g$$

where $\rho_{r_0}(t) = \rho(t - r_0)$. Hence we have

$$\mathcal{W}_{r_0} g = \begin{cases} 
\bar{g}_{r_0} & \text{on } B_{r_0} \\
g & \text{outside } B_{r_0 + \frac{1}{2}}
\end{cases}$$

We call the process $g \mapsto \mathcal{W}g$ warp forcing. The purpose of this paper is to prove that if $g$ is $\epsilon$-close to a hyperbolic metric then the warp forced metric $\mathcal{W}_{r_0} g$ is also close to hyperbolic. In the next paragraph we explain what we mean by a metric being close to hyperbolic (for more details see Section 1).

Let $B$ be the unit open $(n-1)$-ball, with the flat metric $\sigma_{S^{n-1}}$. Write $I_\xi = (-(1 + \xi), 1 + \xi)$, $\xi > 0$. Our basic models are $T_\xi = B \times I_\xi$, with hyperbolic metric $\sigma = e^{2t}g_{S^{n-1}} + dt^2$. The number $\xi$ is called the excess of $T_\xi$. (The reason for introducing $\xi$ will become clear in the Main Theorem

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below; see also the remark after the Theorem). Let \((M, g)\) be a Riemannian manifold and \(S \subset M\).

We say that \(g\) is \(\epsilon\)-close to hyperbolic on \(S\) if there is \(\xi > 0\) such that for every \(p \in S\) there is an \(\epsilon\)-close to hyperbolic chart with center \(p\), that is, there is a chart \(\phi : T_{\xi} \to M, \phi(0, 0) = p\), such that \(|\phi^*g - \sigma|_{C^2} < \epsilon\). The number \(\xi\) is called the excess of the charts (which is fixed).

If \((M, g)\) has center \(o\) and \(S \subset M\). We say that \(g\) is radially \(\epsilon\)-close to hyperbolic on \(S\) (with respect to \(o\)) if, in addition, the charts \(\phi\) respect the product structure of \(T\) and \(M-o = \mathbb{S}^{n-1} \times \mathbb{R}^+\), that is \(\phi(.; t) = (\phi_1(.), t + a)\), for some \(a\) depending on the \(\phi\) (see Section 2 for details). Here the “radial” directions are \((- (1 + \xi), 1 + \xi)\) and \(\mathbb{R}^+\) in \(T\) and \(M-o\), respectively.

Our main result below shows that if \(g\) is radially \(\epsilon\)-close to hyperbolic then the warp forced metric \(W_0 g\) is radially \(\eta\)-hyperbolic, where \(\eta\) depends on \(\epsilon\) and \(r_0\). In the next Theorem we assume \(\xi > 1\).

**Theorem.** Let \((M, g)\) have center \(o\), and \(S \subset M\). If \(g\) is radially \(\epsilon\)-close to hyperbolic on \(S\), with charts of excess \(\xi\), then \(W_0 g\) is radially \(\eta\)-close to hyperbolic on \(S-B_{r_0-(1+\xi)}\) with charts of excess \(\xi - 1\), provided \(\eta \geq e^{16+6\xi}(e^{-2r_0} + \epsilon)\).

**Remark.** Note that warp forcing reduces the excess of the charts by 1. This was one of the motivations to introduce the excess \(\xi\).

The results in this paper are used to smooth Charney-Davis strict hyperbolizations [1], [2]. In Section 1 we define (with more detail) \(\epsilon\)-close to hyperbolic metrics. In Section 2 we give some estimates on changing warping functions. In Section 3 we do warp forcing locally. In Section 4 we introduce warp forcing (with more detail) and prove the Theorem.

**Section 1. \(\epsilon\)-close to hyperbolic metrics.**

Let \(B = \mathbb{B}^{n-1} \subset \mathbb{R}^{n-1}\) be the unit ball, with the flat metric \(\sigma_{\mathbb{R}^{n-1}}\). Write \(I_\xi = (-(1 + \xi), 1 + \xi)\), \(\xi > 0\). Our basic models are \(T_{\xi} = T_\xi \times I_\xi\), with hyperbolic metric \(\sigma = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2\). In what follows we may sometimes suppress the subindex \(\xi\), if the context is clear. The number \(\xi\) is called the excess of \(T_\xi\).

**Remark.** In the proof of the Main Theorem we actually need warped metrics with warping functions that are multiples of hyperbolic functions. All these functions are close to the exponential \(e^t\) (for \(t\) large), so instead of introducing one model for each hyperbolic function we introduced only the exponential model. In the next section we show what is the effect of changing warping functions.

Let \(|.|\) denote the uniform \(C^2\)-norm of \(\mathbb{R}^l\)-valued functions on \(T_\xi = \mathbb{B} \times I_\xi \subset \mathbb{R}^n\). Given a metric \(g\) on \(T\), \(|g|\) is computed considering \(g\) as the \(\mathbb{R}^{n^2}\)-valued function \((x, t) \mapsto (g_{ij}(x, t))\) where, as usual, \(g_{ij} = g(e_i, e_j)\), and the \(e_i\)'s are the canonical vectors in \(\mathbb{R}^n\). We will say that a metric \(g\) on \(T\) is \(\epsilon\)-close hyperbolic if \(|g-\sigma| < \epsilon\).

A Riemannian manifold \((M, g)\) is \(\epsilon\)-close hyperbolic if there is \(\xi > 0\) such that for every \(p \in M\) there is an \(\epsilon\)-close to hyperbolic chart with center \(p\), that is, there is a chart \(\phi : T_{\xi} \to M, \phi(0, 0) = p\), such that \(\phi^*g\) is \(\epsilon\)-close to hyperbolic. Note that all charts are defined on the same model space \(T_\xi\). The number \(\xi\) is called the excess of the charts (which is fixed). More generally, a subset \(S \subset M\) is \(\epsilon\)-close to hyperbolic if every \(p \in S\) is the center of an \(\epsilon\)-close to hyperbolic chart in \(M\) with fixed excess \(\xi\).

If \(M\) has center \(o\) we say that \(S \subset M\) is radially \(\epsilon\)-close to hyperbolic (with respect to \(o\)) if,
in addition, the charts $\phi$ respect the product structure of $T$ and $M - o = S^{n-1} \times \mathbb{R}^+$, that is $\phi(.t) = (\phi_1(.), t + a)$, for some $a$ depending on the $\phi$. Note also that the term “radially” in the definition above refers to the decomposition of the manifold $M - o$ as a product $S^{n-1} \times \mathbb{R}^+$.

Of course a radially $\epsilon$-close to hyperbolic manifold is $\epsilon$-close to hyperbolic.

Remark. The definition of radially $\epsilon$-close to hyperbolic metrics is well-suited to studying metrics of the form $g_t + dt^2$ for $t$ large, but for small $t$ this definition has some drawbacks because:

(1) we need some space to fit the charts, and (2) the form of our specific fixed model $T$.

An undesired consequence is that punctured hyperbolic space $\mathbb{H}^n - \{o\} = S^{n-1} \times \mathbb{R}^+$ (with warp metric $\sinh^2(t)\sigma_{g^{-1}} + dt^2$) is not radially $\epsilon$-close to hyperbolic for $t$ small.

Lemma 1.1. Let $g_t$ be $\epsilon_i$-close to hyperbolic on $T_{\xi}$, for $i = 1, 2$. Let $\lambda : T_{\xi} \to [0, 1]$ be smooth with $|\lambda|$ finite. Then the metric $\lambda g_1 + (1 - \lambda) g_2$ is $(4(1 + |\lambda|)(\epsilon_1 + \epsilon_2))$-hyperbolic.

Proof. The proof follows from the triangular inequality, Leibniz rules and the equality $(\lambda g_1 + (1 - \lambda) g_2 - \sigma = \lambda (g_1 - \sigma) + (1 - \lambda)(g_2 - \sigma)$. This proves the lemma.

Section 2. Warping with $\sinh t$.

The metric of our basic hyperbolic model $T$ is an exponentially warped metric. Here we show how we can easily change the exponential by multiples of $\sinh(t)$ for $t$ large.

Lemma 2.1. For $t_0 > 2$ we have $|e^{-2t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right)^2 - 1|_{C^2(\mathbb{R}^+)} < (5.2) e^{-2t_0}$.

Proof. Write $e^{-t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right) - 1 = \left(\frac{e^{-t_0}}{1 - e^{-t_0}}\right)(1 - e^{-2t})$. Differentiating twice we get the following two estimates: (1) $|e^{-t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right) - 1|_{C^2(\mathbb{R}^+)} < 1.02 e^{-2t_0}$, (2) $|e^{-t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right)|_{C^2(\mathbb{R}^+)} < 1 + (1.02)e^{-2t_0} < 1.02$. To prove the lemma write

$$e^{-2t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right)^2 - 1 = \left(e^{-t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right) - 1\right)\left(e^{-t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right) + 1\right)$$

and use the previous estimates together with the Leibniz rule. This proves the lemma.

Recall $I_{\xi} = (-1 + \xi, 1 + \xi)$. Let $\nu : I_{\xi} \to \mathbb{R}^+$ be smooth. For a metric $g = g_t + dt^2$ on $T_{\xi} = \mathbb{B}^k \times I_{\xi}$, we write $g_{\nu} = \nu g_{t} + dt^2$.

Lemma 2.2. We have $|g - g_{\nu}|_{C^2} < 4 |1 - \nu|_{C^2} |g|_{C^2}$.

Proof. Just note that $g - g_{\nu} = (1 - \nu)g_{t}$ and differentiate twice. This proves the lemma.

Recall that the metric on our model $T_{\xi}$ is $\sigma = e^{2t} \sigma_{\mathbb{R}^k} + dt^2$. Consider now the metric $\sigma_{t_0} = \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right)^2 \sigma_{\mathbb{R}^k} + dt^2$, $t_0 > 2$.

Lemma 2.3. Let $g = g_{t} + dt^2$ be $\epsilon$-close to hyperbolic on $T_{\xi}$, $\xi > 0$. Let $h = e^{-2t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right)^2 g_{t} + dt^2$. Assume $t_0 > 2$. Then

(1) $|h - g| < (21)(\epsilon + e^{2(1+\xi)}) e^{-2t_0}$.

(2) $h$ is $\eta$-hyperbolic, with $\eta \geq 21 e^{2(1+\xi)} \left(e^{-2t_0} + \epsilon\right)$.

Proof. Note that $h = g_{\nu}$, with $\nu = e^{-2t} \left(\frac{\sinh(t + t_0)}{\sinh(t_0)}\right)^2$. Therefore (1) follows from 2.1, 2.2 and the fact that $|g|_{C^2} \leq |g - \sigma|_{C^2} + |\sigma|_{C^2} < \epsilon + e^{2(1+\xi)}$. Item (2) follows from item (1), the triangular inequality and the fact that $g$ is $\epsilon$-close to hyperbolic. This proves the lemma.
2.4. Remark.
It is straightforward to verify that Lemma 2.1 holds if we replace the variable \( t \) by \( t - s \), for some fixed \( s \). Hence Lemma 2.3 also holds if we replace \( h = g_\nu \) (with \( \nu \) as in the proof of 2.3), by \( g_\nu \), where \( \nu_\nu (t) = \nu(t - s) \), for fixed \( s \).

Section 3. Local warp forcing.
Here we give a kind of a local version the the method of warp forcing.

Let \( a \) be a metric on \( \mathbb{R}^k \). For fixed \( s \in I_\xi = (-1 - \xi, 1 + \xi) \), \( \xi > 0 \), we denote by \( a = a_\xi \), the warped metric \( e^{2(t-s)}a + dt^2 \) on \( \mathbb{R}^k \times I_\xi \).

**Lemma 3.1.** Fix \( s \). Let \( a, b \) be metrics on \( \mathbb{R}^k \) with \( |a - b|_{C^2(\mathbb{R}^k)} < \epsilon \). Then \( |a - b|_{C^2} < 4e^{4(1+\xi)} \epsilon \).

**Proof.** Just compute the derivatives of \( a - b = e^{2(t-s)}(a - b) \). This proves the lemma.

**Lemma 3.2.** Let \( g = g_r + dt^2 \) be an \( \epsilon \)-close to hyperbolic metric on \( T_\xi = \mathbb{R}^k \times I_\xi \). Fix \( s \in I_\xi \) and consider \( g_s = e^{2(t-s)}g_s + dt^2 \). Then we have that the metric \( g_s \) is \((4e^{4(1+\xi)} \epsilon)\)-hyperbolic.

**Proof.** By hypothesis we have \( |(g_r + dt^2) - (e^{2t}g_s + dt^2)|_{C^2} < \epsilon \). Therefore, taking \( t = s \) we get \( |g_s - e^{2s}g_s|_{C^2(\mathbb{R}^k)} < \epsilon \). Lemma 3.1 implies then that \( |g_s - \sigma| < 4e^{4(1+\xi)} \epsilon \). This completes the proof of Lemma 3.2.

Section 4. Spherical Cuts, Warp Forcing and Proof of the Theorem.
As in the Introduction, let \((M^n, g)\) be a complete Riemannian manifold with center \( o \in M \). Recall that we can write the metric on \( M - \{o\} = S^{n-1} \times \mathbb{R}^+ \) as \( g = g_r + dr^2 \). We denote by \( S_r = S_r(M) = S^{n-1} \times \{r\} \) the sphere of radius \( r \).

The metric \( g_r \) on \( S_r \) is called the warped by sinh spherical cut of \( g \) at \( r \), and the metric

\[
\tilde{g}_r = \left( \frac{1}{\sinh^4(r)} \right) g_r
\]

is called the (unwarped by sinh) spherical cut of \( g \) at \( r \).

Fix \( r_0 > 0 \). We define the warped by sinh metric \( \tilde{g}_{r_0} \) by:

\[
\tilde{g}_{r_0} = \sinh^2(t)\tilde{g}_r_0 + dr^2 = \sinh^2(t)\left( \frac{1}{\sinh^4(r_0)} \right) g_\nu + dr^2
\]

We now force the metric \( g \) to be equal to \( \tilde{g}_r \) on \( B_{r_0} = \mathbb{B}_{r_0} \) and stay equal to \( g \) outside \( B_{r_0 + \frac{1}{2}} \). For this we define the warped forced (on \( B_{r_0} \)) metric as:

\[
W_{r_0} g = \rho_{r_0} \tilde{g}_{r_0} + (1 - \rho_{r_0}) g
\]

where \( \rho_{r_0} \) is as in the Introduction. Hence we have

\[
W_{r_0} g = \begin{cases} 
\tilde{g}_{r_0} & \text{on } B_{r_0} \\
g & \text{outside } B_{r_0 + \frac{1}{2}}
\end{cases}
\]

We now prove our Theorem.

**Proof of Theorem.** At some points in the proof we will use the notation in Section 2 and 3. Let \( p = (x, t_0) \in S \) and outside \( B_{r_0 - (1+\xi)} \). We have three cases.
First case. \( p \notin B_{r_0 + \frac{1}{2}}(1 + \xi) \)
In this case we can completely fit a radially \( \epsilon \)-close to hyperbolic chart of \( g \) of excess \( \xi \) outside \( B_{r_0 + \frac{1}{2}} \). But, by (4.1), this chart is also a radially \( \epsilon \)-close to hyperbolic chart for \( W_{r_0}g \). This shows the metric \( W_{r_0}g \) is \( \epsilon \)-close to hyperbolic outside \( B_{r_0 + \frac{1}{2}}(1 + \xi) \), with charts of excess \( \xi \).

Second case. \( p \in B_{r_0 + \frac{1}{2}}(1 + \xi) - B_{r_0 + \frac{1}{2}} + \xi \)
Let \( \phi : T_{\xi} \to M \) be an \( \epsilon \)-close to hyperbolic chart of \( g \) centered at \( p = (x, t_0) \). Then the image of the restriction \( \phi|_{\xi-1} \) of \( \phi \) to \( T_{\xi-1} \) does not intersect \( B_{r_0 + \frac{1}{2}} \), hence as in the first case, by 4.1, the chart \( T_{\xi-1} \) is an \( \epsilon \)-close to hyperbolic chart for \( W_{r_0}g \), but with excess \( \xi - 1 \).

Third case. \( p \in B_{r_0 + \frac{1}{2}} + \xi \)
In this case the interval \( I_{\xi} = (t_0 - (1 + \xi), t_0 + (1 + \xi)) \) contains \( r_0 \) (recall we are assuming \( p \notin B_{r_0 -(1 + \xi)} \)). Let \( \phi : T_{\xi} \to M \) be an \( \epsilon \)-close to hyperbolic chart of \( g \) centered at \( p = (x, t_0) \).
Write \( g' = g' + dt^2 = \phi^*g \). Then \( g' = \phi^*g + t_0 \). Write \( s = r_0 - t_0 \), thus \(-\frac{1}{2} - \xi \leq s < 1 + \xi \). In particular we have \( s \in I_{\xi} \) and \( g'_s = \phi^*g_{r_0} \).

Since \( \sinh^2(t + t_0) = \frac{\sinh^2((t-s) + r_0)}{\sinh^2(r_0)} \) we have that \( e^{2(t-s)}|_s(t) = \frac{\sinh^2((t-s) + r_0)}{\sinh^2(r_0)} \), where \( \nu(t) = e^{-2t\sinh^2(2(t + r_0))} \) and \( \nu_s(t) = \nu(t-s) \). Hence (see notation in 2)
\[ \phi^*(\bar{g}_{r_0}) = (g'_s)_{\nu_s} \] (1)
where we used the notation \( \bar{g} = g_s \), introduced in Section 3. And since \( g' \) is \( \epsilon \)-close to hyperbolic, 3.2 implies that \( g'_s \) is \((4e^{4(1+\xi)}\epsilon)\)-hyperbolic. This together with 2.3 (2) (see also Remark 2.4) imply
\[ (g'_s)_{\nu_s} \text{ is } (21e^{2(1+\xi)}(e^{-2t_0} + 4e^{4(1+\xi)}\epsilon) - \text{hyperbolic}) \] (2)
Note that
\[ \phi^*(W_{r_0}g) = \rho_{r_0}\phi^*(\bar{g}_{r_0}) + (1 - \rho_{r_0})\phi^*g = \rho_{r_0}(g_s)_{\nu_s} + (1 - \rho_{r_0})g' \] (3)
From (1), (2), (3) and Lemma 1.1 we get that \( \phi^*(W_{r_0}g) \) is \( \epsilon' \)-hyperbolic with
\[ \epsilon' = 4 \left( 1 + |\rho_{r_0}| \right) \left( 21e^{2(1+\xi)}(e^{-2t_0} + 4e^{4(1+\xi)}\epsilon) + \epsilon \right) \]
A calculation shows that we can take \(|\rho_{r_0}| < 48 \). This together with \( t_0 > r_0 - (1 + \xi) \) imply that we can take \( \epsilon' < (4)(49)(1 + (21)(4)e^{6(1+\xi)})(e^{-2t_0} + \epsilon) < 196(1 + 84e^{6(1+\xi)})(e^{-2t_0} + \epsilon) < (196)(85)e^{6(1+\xi)}(e^{-2t_0} + \epsilon) < e^{16+6\xi}(e^{-2t_0} + \epsilon) \). Note that the excess of the charts in this third case is also \( \xi \). This proves the theorem.

References

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