Note on Quantum Periods and a TBA System

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Abstract

There is an interesting relation between the quantum periods on a certain limit of local \( \mathbb{P}^1 \times \mathbb{P}^1 \) Calabi-Yau space and a TBA (Thermodynamic Bethe Ansatz) system appeared in the studies of ABJM (Aharony-Bergman-Jafferis-Maldacena) theory. We propose a one-parameter generalization of the relation. Furthermore, we derive the differential operators for quantum periods and the TBA system in various limits of the generalized relation.
1 Introduction and Summary

In geometry, we often compute period integrals, which are the integrals of differential forms in certain cohomological classes over cycles of the geometry. For Calabi-Yau three-folds, the classical periods of the holomorphic 3-form over 3-cycles play important roles in mirror symmetry [1]. The classical periods in mirror symmetry satisfy a set of differential equations known as the Picard-Fuchs equations. Basically, because the cohomology space is finite dimensional, we can construct Picard-Fuchs operators by linear combinations of derivatives of complex structure moduli, whose actions on the differential form are exact forms, so that the integrals vanish over cycle.

There have been many works to generalize the notion to quantum geometry and quantum periods, see e.g. [2]. In the context of mirror symmetry, we consider local Calabi-Yau geometries which can be described by complex one-dimensional curves. Quantization of the geometry amounts to promoting the complex coordinates of the curves to canonical conjugate operators $\hat{x}, \hat{p} = i\hbar$. For notational convenience with the $i$ factor we also denote $\epsilon \equiv i\hbar$. The wave function of the quantum system has a standard WKB expansion $\psi(x) = \exp\left[\epsilon \int x w(x')dx'\right]$, where the integrand in the exponent has a power series expansion in $\epsilon$ parameter. The quantum periods can be computed by integrals of $w(x)dx$ over cycles, which in this case are contour over the complex $x$ plane. In the classical limit $\epsilon \to 0$, this reduces the classical periods, which are integrals of the canonical differential one-form $pdx$ over cycles. An important property of the quantum periods is that the higher order contributions can be computed by certain differential operators acting on the classical periods [3, 4, 5].

The classical periods provide a solution for the prepotential of the Calabi-Yau geometry. The free energy of topological string theory also includes higher genus contributions. More generally, motivated by Nekrasov’s calculations [6] of instanton partition function of Seiberg-Witten theory, one can define refined topological string theory which has expansion over two small parameters $\epsilon_{1,2}$ [7]. The conventional genus expansion corresponds to $\epsilon_1 + \epsilon_2 = 0$, while another special limit of setting one parameter to zero e.g. $\epsilon_2 = 0$ is known as the Nekrasov-Shatashvili (NS) limit [8].
One application of the quantum periods is that they can compute the topological free energy in NS limit, in the same way as the classical periods compute the prepotential. This has been studied in Seiberg-Witten theories as well as topological string theory \[3, 4\]. Furthermore, exact quantization conditions for quantum systems of mirror curves including novel non-perturbative contributions are conjectured in \[9, 10, 11\].

In this note, we consider the case of a well studied local $\mathbb{P}^1 \times \mathbb{P}^1$ Calabi-Yau space, which in a special limit is related to the computations of partition functions of ABJM theory on 3-sphere \[12, 13\]. On the other hand, the partition function of ABJM theory can be also formulated in terms of fermion gas, and is related to a TBA system, which are studied in many papers \[14, 15, 16, 17, 18, 21, 22\].

In particular, a novel relation between the quantum periods and the TBA system was conjectured and later derived in \[19, 20\]. The derivation uses ABJM theory as well as its many sophisticated technical ingredients. However, in order to have a deeper understanding as well as exploring possible generalizations to more Calabi-Yau spaces, it is worthwhile to directly study the relation which by itself can be formulated independently without ABJM theory.

The paper is organized as the followings. In section 2 we introduce the notations and propose a one-parameter generalization of the relation between quantum periods and TBA system in \[19\]. In sections 3 and 4 we then take another perspective of the relation by expanding perturbatively for small $\epsilon$ parameter, but keep the coefficients exactly as a function of $z$ in terms of differential operators. We will compute the differential operators to the first few orders for various limits in the generalized setting, and verify that they are the same for the TBA system and the quantum periods.

## 2 A one-parameter generalization

The mirror curve of the local $\mathbb{P}^1 \times \mathbb{P}^1$ Calabi-Yau model is

\[
e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 1, \tag{2.1}
\]

where $x, p$ are the complex coordinates and $z_1, z_2$ are the complex structure moduli parameters. We promote the complex coordinates to canonical operators $\hat{x} = x, \hat{p} = \epsilon \partial_x$. The mirror curve acts on a wave function $\psi(x)$ so that

\[
(-1 + e^x + z_1 e^{-x})\psi(x) + \psi(x + \epsilon) + z_2 \psi(x - \epsilon) = 0. \tag{2.2}
\]

It is convenient to use exponential variables $q = e^\epsilon, X = e^x$. A particular choice of $z_1 \to q^{1/2}z, z_2 \to q^{-1/2}z$ corresponds to the calculations of ABJM partition function. Here we will instead consider a one-parameter generalization of the ABJM setting, by keeping the general $\mathbb{P}^1 \times \mathbb{P}^1$ geometry. We introduce the following parametrization with an additional $m$ parameter

\[
z_1 = e^m z, \quad z_2 = e^{-m} z. \tag{2.3}
\]
The original conjecture in [19] corresponds to a special choice \( m \to \frac{x}{2} \) in our set up.

One also introduces a function \( V(X = e^x) = \frac{\psi(x + \epsilon)}{\psi(x)} \). Instead of the standard WKB expansion of the wave function \( \psi(x) = \exp[\frac{1}{\epsilon} \int_x^x w(x')dx'] \), we consider the logarithmic function \( \log V(X) =\frac{1}{\epsilon} \int_x^x \log w(x')dx' \), which may differ with \( w(x) \) only by some total derivatives. So their residue are actually the same. We have a function of four parameters \( V(X, \epsilon, z, m) \) which satisfies the equation

\[-1 + X + \frac{e^m z}{X} + V(X, \epsilon, z, m) + \frac{e^{-m} z}{V(q^{-1}X, \epsilon, z, m)} = 0 \]  

(2.4)

We can compute the function \( V(X, \epsilon, z, m) \) recursively as a perturbative expansion of small \( z \). The first few terms are

\[ V(X, \epsilon, z, m) = 1 - X - \left(\frac{e^{-m} q}{q - X} + \frac{e^m}{X}\right)z - \left(\frac{e^{-2m} q}{q^2 - X} + \frac{1}{X}\right)\frac{q^3 z^2}{(q - X)^2} + \mathcal{O}(z^3). \]  

(2.5)

The contour integral of quantum \( \Lambda \)-period is given by the residue around \( X = 0 \) plus a log term by the formula

\[ \Pi_A = \log(z) - \text{Res}_{X=0} \frac{2}{X} \log(V(X, \epsilon, z, m)) \]  

(2.6)

\[ = \log(z) + 2(e^m + e^{-m})z + [3(e^{2m} + e^{-2m}) + 2(4 + \frac{1}{q} + q)]z^2 + \mathcal{O}(z^3), \]

where one needs to add the \( \log(z) \) term which is not captured by the residue calculations, but is well known to be present in the classical periods and responsible for their monodromy. Instead of computing residue of \( x \), it is more convenient here to compute equivalently the residue of \( X = e^x \) with an extra factor of \( \frac{1}{X} \).

On the other hand, the TBA system corresponding to the ABJM theory of the choice \( z_1 \to q^{\frac{1}{2}} z, z_2 \to q^{-\frac{1}{2}} z \) is described by a function \( \eta(X, \epsilon, z) \) of the three variables, which also satisfies a difference equation

\[ 1 + z[\eta(qX) + \eta(X)][\eta(q^{-1}X) + \eta(X)](X + X^{-1} + q^{\frac{1}{2}} + q^{-\frac{1}{2}}) = \eta(X)^2. \]  

(2.7)

It is reasonable to expect that there should be a one-parameter deformation which corresponds to the general local \( \mathbb{P}^1 \times \mathbb{P}^1 \) Calabi-Yau model described above. After some guessworks, we find such a deformation, which is to simply replace the \( q^{\frac{1}{2}} + q^{-\frac{1}{2}} \) term with \( e^m + e^{-m} \), where \( m \) is the extra deformation mass parameter. In the generalized model, we now have a four-parameter function \( \eta(X, \epsilon, z, m) \), defined by a deformed equation

\[ 1 + z[\eta(qX) + \eta(X)][\eta(q^{-1}X) + \eta(X)](X + X^{-1} + e^m + e^{-m}) = \eta(X)^2. \]  

(2.8)

One can also solve the function \( \eta(X, \epsilon, z, m) \) recursively as a perturbative series of \( z \). With the choice of plus sign for the leading term, the first few terms are

\[ \eta(X, \epsilon, z, m) = 1 + 2(e^m + e^{-m} + X + X^{-1})z + 2(e^m + e^{-m} + X + X^{-1}) \left[3(e^m + e^{-m}) + (q + 1 + q^{-1})(X + X^{-1})\right]z^2 + \mathcal{O}(z^3). \]  

(2.9)
The relation between quantum \( A \)-period and the TBA system is then
\[
\text{Res}_{X=0} \frac{1}{X} \eta(X, \epsilon, z, m) = \theta_z \Pi_A(\epsilon, z, m), \quad (2.10)
\]
where the differential operator is defined \( \theta_z \equiv z \partial_z \). Although the equations for quantum periods \( (2.4) \) and for TBA system \( (2.8) \) look quite different and the expansions are also different, after taking residue, one can check the relation \( (2.10) \) perturbatively for small \( z \) where the coefficients are rational functions of \( q \) and \( e^m \). Thus we have provided a generalization of the relation in [19].

3 Differential operators for a TBA system

We now consider a different expansion, by solving the equations as a perturbative series of \( \epsilon \). We can treat the extra parameter \( m \) in two ways, either as an independent finite parameter, or it can depend also on \( \epsilon \), e.g. scaling like \( m = \tilde{m} \epsilon \) with \( \tilde{m} \) finite. We will compute in both cases.

In this section, we study the the TBA equation \( (2.8) \). First we consider the case of \( m \) as an independent finite parameter. We denote the perturbative series and the residue as
\[
\eta(X, \epsilon, z, m) = \sum_{n=0}^{\infty} \eta_n(X, z, m) \epsilon^n, \quad p_n(z, m) \equiv \text{Res}_{X=0} \frac{1}{X} \eta_n(X, z, m). \quad (3.1)
\]
Since the equation is invariant under the sign switch \( \epsilon \rightarrow -\epsilon \), the coefficients vanish \( \eta_n(X, z) = 0 \) for odd integers \( n \). So we only need to consider even terms.

The leading term can be solved by a simple quadratic equation. Our convention use the solution with plus sign
\[
\eta_0(X, z, m) = \frac{1}{\sqrt{1 - 4(e^m + e^{-m} + X + X^{-1})z}}. \quad (3.2)
\]
We can compute the leading period perturbatively for small \( z \) expansion
\[
p_0 = 1 + 2(e^m + e^{-m})z + 6(e^{2m} + e^{-2m} + 4)z^2 + 20[e^{3m} + e^{-3m} + 9(e^m + e^{-m})]z^3 + O(z^4). \quad (3.3)
\]
The exact expression is determined by a Picard-Fuchs differential operator \( \mathcal{L} \) so that its action on the leading term \( \mathcal{L} \eta_0(X, z, m) \) is a total derivative of \( x \). After some computations we determine the operator
\[
\mathcal{L} = [16(e^m - e^{-m})^2 z^2 - 8(e^m + e^{-m})z + 1] \theta_z^2 + 8z[4(e^m - e^{-m})^2 z - e^m - e^{-m}] \theta_z + 2z[6(e^m - e^{-m})^2 z - e^m - e^{-m}] \quad (3.4)
\]
So the leading order period satisfies a second order differential equation $Lp_0 = 0$. The coefficient of $\theta_2^0$ term is the discriminant of the curve, and is denoted

$$\Delta = 16(e^m - e^{-m})^2z^2 - 8(e^m + e^{-m})z + 1. \quad (3.5)$$

The higher order terms $\eta_n(X, z, m)$ can then be written as linear combinations of $\eta_0(X, z, m)$ and $\theta z\eta_0(X, z, m)$, plus a total derivative of $x$. So $p_0$ and $\theta z p_0$ provide a linear basis for the higher period $p_n$. After some computations, we determine the differential operators for next two orders

$$p_2 = \left[16(e^m + e^{-m})(e^m - e^{-m})^2z^2 - 8(e^m + e^{-m} - 6)z + e^m + e^{-m}\right]\frac{z\theta z p_0}{6\Delta}$$

$$+ [4(e^m + e^{-m})(e^m - e^{-m})^2z - e^m - e^{-m} + 10]\frac{z^2 p_0}{3\Delta}, \quad (3.6)$$

$$p_4 = \left\{2048(e^m - e^{-m})^4[4^m + e^{-3m} - 85(e^m + e^{-m})]z^6 - 512(e^m - e^{-m})^2[3(e^m + e^{-4m}) + 494(e^m + e^{-2m}) + 3038]z^5 + 2048[107(e^m + e^{-3m}) - 191(e^m + e^{-m})]z^4ight.$$  

$$+ 64[5(e^4m + e^{-4m}) - 658(e^2m + e^{-2m}) + 2474]z^3 - 8[15(e^3m + e^{-3m}) + 109(e^m + e^{-m})]z^2 + 2[9(e^2m + e^{-2m}) + 292]z - (e^m + e^{-m})\right\}\frac{z\theta z p_0}{360\Delta^3}$$

$$+ \left\{[512(e^m - e^{-m})^4[4^m + e^{-3m} - 85(e^m + e^{-m})]z^5 - 128(e^m - e^{-m})^2$$

$$[2(e^4m + e^{-4m}) + 519(e^2m + e^{-2m}) + 2990]z^4 - 64[5^m + e^{-5m} - 733(e^3m + e^{-3m})$$

$$+ 1404(e^m + e^{-m})]z^3 + 8[8(e^4m + e^{-4m}) - 907(e^2m + e^{-2m}) + 3798]z^2$$

$$- 2[7(e^3m + e^{-3m}) + 79(e^m + e^{-m})]z + e^2m + e^{-2m} + 62\right\}\frac{z^2 p_0}{180\Delta^3}, \quad (3.7)$$

Next we study another parametrization $m = \tilde{m}\epsilon$ with $\tilde{m}$ finite, where a choice of $\tilde{m} = \frac{1}{2}$ corresponds to the ABJM case. In this case we denote the perturbative series and the residue as

$$\eta(X, \epsilon, z, \tilde{m}\epsilon) = \sum_{n=0}^{\infty} \eta_n(X, z, \tilde{m})\epsilon^n, \quad p_n(z, \tilde{m}) \equiv \text{Res}_{X=0} \frac{1}{X} \eta_n(X, z, \tilde{m}). \quad (3.8)$$

The leading order calculations and the Picard-Fuchs operator are obtained by simply setting $m \to 0$ in the above equations (3.2, 3.3, 3.4). The higher order periods are different since the $\epsilon$ dependence in $m$ contributes in the expansion. Again after some computations, we find the differential operators that determine the higher order
periods

\[
p_2 = \frac{z}{3(1 - 16z)} \left[ 2(3 \tilde{m}^2 + 4z - 48 \tilde{m}^2 z)p_0 + (1 + 12 \tilde{m}^2 + 16z - 192 \tilde{m}^2 z)\theta_z p_0 \right],
\]

\[
p_4 = \left[ 15 \tilde{m}^4 - 4(-8 + 15 \tilde{m}^2 + 45 \tilde{m}^4)z - 4(43 - 1680 \tilde{m}^2 + 2880 \tilde{m}^4)z^2 \\
+320(25 - 432 \tilde{m}^2 + 816 \tilde{m}^4)_z^3 - 6144(7 - 120 \tilde{m}^2 + 240 \tilde{m}^4)_z^4 \right] \frac{z p_0}{90(1 - 16z)^3} \\
- [1 - 30 \tilde{m}^2 + 60 \tilde{m}^4 + (310 - 1080 \tilde{m}^2)z - 32(31 - 1200 \tilde{m}^2 + 2160 \tilde{m}^4)z^2 \\
+512(73 - 1260 \tilde{m}^2 + 2400 \tilde{m}^4)_z^3 - 24576(7 - 120 \tilde{m}^2 + 240 \tilde{m}^4)_z^4 \right] \frac{z \theta_z p_0}{180(1 - 16z)^3}.
\]

(3.9)

## 4 Differential operators for quantum periods

In this section, we solve the equation for quantum periods \((2.4)\) as a perturbative series of \(\epsilon\) and the results can be compared with those of the previous section \(3\). In practice it turns out the calculations in this section are much more complicated than those in the previous section using the TBA system. In this sense, the relation with the TBA system provides a simpler way to compute the quantum periods and their associated differential operators for Calabi-Yau geometries.

Again we first consider the case of \(m\) as an independent finite parameter. We denote the perturbative series as

\[
\log V(X, \epsilon, z, m) = \sum_{n=0}^\infty w_n(X, z, m) \epsilon^n.
\]

(4.1)

We expand the quantum A-period \((2.6)\) as

\[
\Pi_A(\epsilon, z, m) = \sum_{n=0}^\infty \Pi_A^{(n)}(z, m) \epsilon^n.
\]

(4.2)

Unlike in previous section, the odd \(n\) power terms \(w_n(X, z, m)\) do not simply vanish, but are still total derivative of \(x\), as familiar in calculations of quantum periods \([4]\). So the terms \(\Pi_A^{(n)}\) with odd \(n\) in the expansion of quantum A-period vanish.

In order to later compare with the TBA system, we denote the derivative of the quantum period \(\tilde{p}_n(z, m) = \theta_z \Pi_A^{(n)}(z, m)\), where we use the tilde symbol to distinguish from the notation of TBA system in the previous section, though they are equivalent through our generalized proposal. These coefficients are

\[
\tilde{p}_0(z, m) = \text{Res}_{X=0} \frac{1}{X} \left[ 1 - 2 \theta_z w_0(X, z, m) \right],
\]

\[
\tilde{p}_n(z, m) = -\text{Res}_{X=0} \frac{2}{X} \theta_z w_n(X, z, m), \quad n \geq 1.
\]

(4.3)
The functions $w_n(X, z, m)$ can be solved recursively. The leading term is solved by a simple quadratic equation. Since we will take residue, we assume $X \sim 0$ and take the branch which have the same leading term as (2.5) for $z = 0$

$$w_0(X, z, m) = \log\left[\frac{1}{2}(1 - X - \frac{emz}{X} + \frac{e^{-m}}{X} \sqrt{-4X^2z + em(-X + X^2 + emz)^2}\right].$$

(4.4)

The Picard-Fuchs operator is more complicated to derive than the previous section. Since the leading order period $\tilde{p}_0$ should be the same as $p_0$ in the previous section, we have $\mathcal{L}\theta_z \Pi_A^{(0)} = \mathcal{L}\tilde{p}_0 = 0$ with the operator $\mathcal{L}$ in (3.4). So the Picard-Fuchs operator for the classical period is simply $\mathcal{L}\theta_z$, as one can directly verify that $\mathcal{L}\theta_z(\log z - 2w_0)$ is indeed a total derivative of $x = \log(X)$, with no residue around $X \sim 0$ after dividing by $X$. We note that a constant is also a total derivative but still has non-vanishing residue $\text{Res}_{X=0}\frac{1}{X} = 1$, due to the monodromy of $\log(X)$ function. So here the $\log(z)$ term is needed so that the total contribution has no monodromy in our case.

We consider the higher order periods. The differential operators for the local $\mathbb{P}^2 \times \mathbb{P}^1$ Calabi-Yau model are computed in our previous paper [4]. The results are

$$\Pi_A^{(2)} = -\frac{z_1 + z_2}{6}\theta_z \Pi_A^{(0)} + \frac{1 - 4z_1 - 4z_2}{12}\theta_z^2 \Pi_A^{(0)},$$

$$\Pi_A^{(4)} = \frac{1}{360\Delta^2}\left\{2(z_1^2(1 - 4z_1)^3 + z_2^2(1 - 4z_2)^3 + 4z_1z_2(8 - 37z_1 - 37z_2 - 328z_1^2 + 1528z_1 z_2 - 328z_1^2 + 1392z_2^3 - 1376z_1 z_2^2 + 1392z_2^3))\theta_z \Pi_A^{(0)}
+ [-z_1(1 - 4z_1)^4 - z_2(1 - 4z_2)^4 + 4z_1z_2(69 - 192z_1 - 192z_2 - 1712z_1^2 + 6880z_1 z_2 - 1712z_2^2 + 5568z_1^3 - 5504z_1^2 z_2 - 5504z_1^2 z_2^2 + 5568z_2^3))\theta_z^2 \Pi_A^{(0)}\right\},$$

where the discriminant is (3.5) and the parametrization is $z_1 = e^{-m}z, z_2 = e^{-m}z$. We act the operator $\theta_z$ on both sides of the equations, and use the Picard-Fuchs operator $\mathcal{L}$ in (3.4) to eliminate the second derivatives of $\tilde{p}_0$. In this way, we can derive the differential operators for $\tilde{p}_2, \tilde{p}_4$ as linear combinations of $\tilde{p}_0$ and $\theta_z\tilde{p}_0$. We verify that the results are the same as from the TBA system in (3.6), (3.7).

Finally we consider the parametrization $m = \tilde{m}e$ with $\tilde{m}$ finite. However for general $\tilde{m}$, the computations for higher order periods are quite complicated. Instead of working out the general case, as an illustrative example, we study a special case $\tilde{m} = \frac{1}{2}$, corresponding to the ABJM theory. We determine the differential operators for $\tilde{p}_2, \tilde{p}_4$ in this special case and verify the results are again the same from the TBA system in (3.9).

Some other well studied local Calabi-Yau models, such as the local $\mathbb{P}^2$ model, also have the same feature that the Picard-Fuchs operator for classical periods can be written as $\mathcal{L}\theta_z$, where $\mathcal{L}$ is a second order differential operator. It would be interesting to study whether the correspondence of quantum periods with the TBA like equations...
can be simply generalized to such Calabi-Yau models. A connection of ABJM theory to local \( \mathbb{P}^2 \) model is found by studying the partition functions on ellipsoid \(^{[23]}\). It would be interesting to explore whether such a connection can be found for quantum periods as well.

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