ON THE CONSISTENCY PROBLEM FOR MODULAR LATTICES AND RELATED STRUCTURES

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Abstract. The consistency problem for a class of algebraic structures asks for an algorithm to decide for any given conjunction of equations whether it admits a non-trivial satisfying assignment within some member of the class. By Adyan (1955) and Rabin (1958) it is known unsolvable for (the class of) groups and, recently, by Bridson and Wilton (2015) for finite groups. We derive unsolvability for (finite) modular lattices and various subclasses; in particular, the class of all subspace lattices of finite dimensional vector spaces over a fixed or arbitrary field of characteristic 0. The lattice results are used to prove unsolvability of the consistency problem for (finite) rings and (finite) representable relation algebras. These results in turn apply to equations between simple expressions in Grassmann-Cayley algebra and to functional and embedded multivalued dependencies in databases.

1. Introduction

A solution of the consistency problem for a class $C$ of structures and a set $\Sigma$ of constraints consists in an algorithm which, given any $\varphi \in \Sigma$, decides whether there is a structure $A \in C$ and a non-trivial assignment in $A$ satisfying $\varphi$. Here, in the context of a fixed set $\Sigma$, an assignment is “trivial” if it satisfies all constraints $\psi \in \Sigma$. For classes of algebraic structures, the familiar constraints are conjunctions of equations. In the case $\Sigma$ consists of all of them, the complement of the consistency problem is known as the triviality problem: to decide for a given conjunction of equations whether every satisfying assignment within the class generates a singleton subalgebra (that is, whether the associated finitely presented algebra is trivial) — a problem reducing to the word problem. A famous instance of unsolvability is given by

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the class of all groups, Adyan [2, 3] and Rabin [25]; the case of all finite groups is due to Bridson and Wilton [5].

We use these to show in Section 4 the consistency problem unsolvable for classes of modular lattices and subclasses of the quasivariety generated by finite modular lattices; these classes are supposed to satisfy certain richness conditions concerning the presence of ‘sufficiently many’ subspace lattices \( L(V) \) of vector spaces \( V \).

This applies, in particular, to the class of all \( L(V) \) where \( V \) is finite dimensional over a fixed field of characteristic 0. In case the \( V \) are real or complex Hilbert spaces, unsolvability extends to the associated ortholattices (Section 8), thus giving a negative answer to the question raised in [16, §III.C], the decision problem for “quantum satisfiability in indefinite (yet finite) dimensions” (see also [17]).

Section 3 recalls our central tool: the interpretation via (von Neumann) frames, translating group presentations into such for modular lattices. This provides a reduction of word problems if the encoding is also applied to the equations to be decided, cf. Lipshitz [22] and Freese [6]. However, such encoding turns trivial assignments in groups into non-trivial assignments within frame generated sublattices. Thus, we devise in Lemma 7 of Section 4 a bit more sophisticated encoding based on lattice relations specific for fixed-point free actions of linear groups — after asserting in Lemma 5 of Section 2 faithful such representation to indeed exist.

The methods and results of [14, 15] allow to transfer unsolvability of the consistency problem to (finite) representable relation algebras (Section 5); further to (finite) relational databases with conjunctions of functional and embedded multivalued dependencies as constraints (Section 6). In particular, there is no algorithm to decide for every finite conjunction of functional dependencies and embedded multivalued dependencies whether it implies that all attributes are keys. Consistency for conditional inclusion and functional dependencies has been studied in [23]; undecidability has been shown for the combination of both.

Using the description of joins and meets of principal right ideals in regular rings, the consistency problem for classes of (finite) modular lattice can be reduced to that for classes of (finite) rings in Section 7. The special case of endomorphism rings gives a further reduction to satisfiability of conjunctions of equations between simple expressions in Grassmann-Cayley algebras (Section 9). Thus, these problems turn out algorithmically unsolvable, too.
2. Algebraic structures

We consider classes $C$ of structures and sets $\Sigma$ of constraints, that is formulas $\pi$ in a language associated with $C$ — we write $\pi(\bar{x})$ where $\bar{x}$ denotes the list of free variables in $\pi$. An assignment within $A \in C$ is a map $\bar{x} \mapsto \bar{a}$ into $A$; it satisfies $\pi(\bar{x})$ if $A \models \pi(\bar{a})$; it is trivial if it satisfies all constraints $\pi(\bar{x}) \in \Sigma$. A solution of the consistency problem for $C$ and $\Sigma$ consists in an algorithm which decides for any constraint whether there is a non-trivial satisfying assignment within $C$, that is, within some $A \in C$.

Primarily, we consider classes $C$ of algebraic structures of finite signature; here, the usual constraints are conjunctions of equations. If $\Sigma$ consists of exactly these, we speak just of the consistency problem for $C$; trivial assignments are those within singleton subalgebras of some $A \in C$; if $C$ is a class of rings with unit or bounded lattices, $A$ must be trivial (requiring $0 = 1$).

Of course, unsolvability with respect to $\Sigma$ for $C$ is inherited by any expansion $C'$ of $C$ (that is, the language of $C'$ has some operation symbols in addition to that of $C$ and the members of $C'$ arise from those of $C$ by adding operations denoted by these additional symbols). But, trivial assignments may generate non-trivial subalgebras in the expansion. Though, if within $C$ trivial assignments require trivial algebras, unsolvability of the consistency problem is inherited by any expansion.

A quasi-variety is a class of algebraic structures definable by sets of quasi-identities: sentences of the form $\forall \bar{x}. \pi(\bar{x}) \Rightarrow \psi(\bar{x})$ where $\pi$ and $\psi$ are conjunctions of equations, $\pi$ being possibly empty. Given a class $C$, the smallest quasi-variety $Q_C$ containing $C$ (i.e. generated by $C$) is the model class of the set of quasi-identities valid in $C$. The consistency problems for $C$ and $Q_C$ are equivalent due to the following.

Fact 1. A conjunction $\pi(\bar{x})$ of equations admits a non-trivial satisfying assignment in $C$ if and only if it does so in $Q_C$.

Proof. Consider the quasi-identity $\forall \bar{x}. \pi(\bar{x}) \Rightarrow \psi(\bar{x})$ where $\psi$ is the conjunction of all equations $x_i = x_1$ and $f(x_1, \ldots, x_1) = x_1$, $f$ an operation symbol. This is valid in $A$ if and only if $A$ admits only trivial satisfying assignments for $\pi(\bar{x})$.

Recall that a positive primitive formula is of the form $\exists \bar{x} \alpha(\bar{x})$ where $\alpha$ is a conjunction of atomic formulas. By a basic equation we mean an equation of the form $y = x$ or $y = f(\bar{x})$ where $f$ is an operation symbol. An unnested pp-formula is of the form $\exists \bar{y}. \varphi(\bar{x}, \bar{y})$ where $\varphi(\bar{x}, \bar{y})$ is a conjunction of basic equations. For the following compare [18, Theorem 2.6.1].
Fact 2. Every conjunction $\pi(\bar{x})$ of equations is logically equivalent to an unnested pp-formula $\exists \bar{y}. \varphi(\bar{x}, \bar{y})$. Moreover, in the case of a (bounded) lattice $L$, if $L \models \pi(\bar{a})$ only for single valued $\bar{a}$ then $L \models \varphi(\bar{a}, \bar{b})$ only for single valued $\bar{a}, \bar{b}$.

Unsolvability for the classes of structures to be considered, here, is shown by reducing from the following deep results of Adyan [2, 3], Rabin [25], and Bridson and Wilton [5].

Theorem 3. The consistency problems for the class of all groups and the class of all finite groups are unsolvable.

Corollary 4. Let $C$ be a class of (finite) semigroups or monoids such that any (finite) group embeds into some member of $C$. Then the consistency problem for $C$ is unsolvable.

In particular, we may consider groups just with multiplication.

Proof. In the case of monoids with unit $e$, given a conjunction $\pi(\bar{x})$ of group equations, form the conjunction $\hat{\pi}(\bar{x}, \bar{y})$ of $\pi(\bar{x})$ and the $x_i y_i = e = y_i x_i$ with new variables $y_i$. Thus, $\pi(\bar{x})$ admits a non-trivial assignment within some (finite) group if and only $\hat{\pi}$ does so within some (finite) member of $C$, namely within the group of units. In the absence of constant $e$, mimic it by a new variable $u$ adding the equations $x_i u = x_i = u x_i$, $y_i u = y_i = u y_i$.

The following is the intermediate step when deriving a lattice from a group. Supposedly, it is well known. To some extent it could be replaced by use of Maschke’s Theorem. For a vector space $V$ over a division ring $F$ of characteristic $c$ we write $\chi(F) = \chi(V) = c$. Let $\mathcal{V}_F$ denote the class of all $F$-vector spaces. A representation $\rho$ of $G$ in $V$ is fixed point free if $\rho(g)(v) = v$ for all $g \in G$ only if $v = 0$.

Lemma 5. Let $G$ be a group and $V$ a vector space where either $\dim V \geq |G|$ and $G$ is infinite or $\dim V = |G| - 1 > 0$ is finite and $\chi(V)$ does not divide $|G|$. Then there exists a fixed point free faithful representation of $G$ in $V$.

Proof. For infinite $G$, and $\dim V = |G|$, we use the regular representation: We may assume $G$ a basis of $V$ and define $\rho(g)$ given by the basis permutation $h \mapsto gh$. The claim follows from the fact that this action of $G$ on $G$ is transitive: $v \neq 0$ in $V$ has the form $v = \sum_{h \in H} r_h h$ with some finite $H \subseteq G$ and $r_h \neq 0$; choosing $k \in G \setminus H$, there is $g \in G$ with $gh = k$ whence $gv = \sum_{h \in H} r_h gh \neq v$. For $\dim V > |G|$ we use a suitable direct multiple of this representation.
For $G$ the 2-element group $\{e, g\}$, define the action on $V$ by $gv = -v$. For finite $G$ of order $> 2$, again assuming $G$ a basis of $V$, we have the 1-dimensional invariant subspace $U$ spanned by $\sum_{g \in G} g$ and the induced action of $G$ on $V/U$. The $g \in G^+ := G \setminus \{e\}$ form a basis of $V/U$; that is, any $v + U$ in $V/U$ has a unique representation

$$v + U = \sum_{h \in G^+} r_h h + U = (\sum_{h \in G^+, g \neq e} r_{gh} h) + r_g g + U$$

for any $g \in G^+$, in particular,

$$e + U = \sum_{h \in G^+} -h + U = (\sum_{h \in G^+, g \neq e} -gh) - g + U.$$

Thus, for $v + U = \sum_{h \in G^+} r_h h + U$ and $g \in G^+$ one has $g(v + U) = (\sum_{h \in G^+, g \neq e} r_{gh} h) - r_{gh} e + U = (\sum_{h \in G^+, g \neq e} (r_h - r_{gh}) h) - r_{gh} e + U$ and the last expression returns $g(v + U)$ as a linear combination of basis vectors of $V/U$. Assume $v + U = g(v + U)$ for all $g \in G$; that is, for all $g, h \in G^+$ one has $r_g = -r_{gh}$ and

$$r_{gh} = r_h - r_{gh} - r_g \text{ if } gh \neq e.$$

For each $h \in G^+$, it follows $r_{hh} = kr_h$, by induction, for all $1 \leq k < \ell$ where $\ell$ is the order of $h$; in particular, $-r_h = r_{h^{-1}} = r_{h^k} = (\ell - 1)r_h$, whence $\ell r_h = 0$ and $r_h = 0$, due to the assumption on the characteristic. Thus, $v + U = 0 + U$. \hfill \Box

3. Coordinates in modular lattices

We consider lattices as algebraic structures with operations join $a + b$ and meet $a \cap b$; in particular, with respect to a suitable partial order $\leq$, one has $a + b = \sup\{a, b\}$ and $a \cap b = \inf\{a, b\}$. A lattice is modular if $a \geq b$ implies $a \cap (b + c) = b + a \cap c$. Let $\mathcal{M}_f$ denote the class of all finite modular lattices. The lattice of all equivalence relations on the set $S$ is denoted by Eq($S$). A sublattice $L$ of the latter is a lattice of permuting equivalences if, for the relational product $\alpha \circ \beta = \{(x, z) : \exists y, (x, y) \in \alpha, (y, z) \in \beta\}$, of any $\alpha, \beta \in L$, one has $\alpha \circ \beta = \beta \circ \alpha$; that is, $\alpha \circ \beta$ is the join $\alpha + \beta$ in Eq($S$) and $L$ and, in particular, transitive. In that case, $L$ is a modular lattice $[19]$. The lattices $L(V)$ of all linear subspaces of the vector space $V$ are isomorphic to such: associate with a subspace $U$ the equivalence relation on $V$ defined by $x - y \in U$. Bounds of a lattice, if considered as constants, will be denoted by $0$ (bottom) and $1$ (top); in case of $L(V)$ these are $\{0\}$ and $V$. We write $a \oplus b = c$ if $a + b = c$ and $a \cap b = 0$. 
An \( n \)-frame in a lattice \( L \) is a system \( \bar{a} \) of elements \( a_1, \ldots, a_n, a_{ij} = a_{ji} \) (\( 1 \leq i < j \leq n \)), and \( a_\perp, a_T \) of \( L \) such that, where \( \sum_{i \in I} a_i := a_\perp \),

\[
\left( \sum_{i \in I} a_i \right) \cap \sum_{j \in J} a_j = \sum_{k \in I \cap J} a_k \quad \text{for } I, J \subseteq \{1, \ldots, n\},
\]

\( a_T = \sum_{t} a_t \), and, for pairwise distinct \( i, j, k \),

\[
a_i + a_j = a_i + a_{ij}, \quad a_i \cap a_{ij} = a_\perp, \quad a_{ik} = (a_i + a_k) \cap (a_{ij} + a_{jk}).
\]

Define

\[
G(L, \bar{a}) = \{ g \in L \mid g + a_1 = g + a_2 = a_1 + a_2, \ g \cap a_1 = g \cap a_2 = a_\perp \}.
\]

If \( L \) has bounds 0, 1 and if \( a_\perp = 0 \) and \( a_T = 1 \) then we speak of an \( n \)-frame of \( L \). We use \( \bar{z} \) to denote a system of variables to be interpreted by 4-frames. Items (i)–(iii)(b) of the following are well known in a broader context \[24, 4, 22, 6\]; our modest amendment of Item (iv) will turn out as crucial to establish Lemma 7 below. All can be generalized to any fixed \( n \geq 4 \). We state and prove what is relevant, here. We say that a subgroup \( G \) of \( \text{GL}(V) \) acts fixed point free if \( gv = v \) for all \( g \in G \) only if \( v = 0 \).

**Lemma 6.**  
(i) For any vector space \( V \) with \( \dim V = nd \), \( d \) any cardinal, and subspace \( V_1 \) of \( V \) of \( \dim V_1 = d \) there is a \( n \)-frame \( \bar{a} \) of \( L(V) \) such that \( a_1 = V_1 \).

(ii) For any \( n \)-frame \( \bar{a} \) in a modular lattice, \( a_\perp = a_{12} \) implies \( a_\perp = a_T \).

(iii) There is a lattice term \( t(x, y, \bar{z}) \) such that for any modular lattice \( L \) and 4-frame \( \bar{a} \) in \( L \) the following hold:

(a) \( G(L, \bar{a}) \) is a group under the multiplication \( (g, h) \mapsto t(g, h, \bar{a}) \) and with neutral element \( a_{12} \).

(b) If \( V \) is a vector space and \( L = \text{L}(V) \), then there is a unique isomorphism \( \varepsilon_{\bar{a}} : a_1 \to a_2 \) such that \( \Gamma_{\bar{a}}(f) := \{ v - \varepsilon_{\bar{a}}(f(v)) \mid v \in a_1 \} \) defines an isomorphism of \( \text{GL}(a_1) \) onto \( G(L, \bar{a}) \).

(c) In (b), the subgroup \( G \) generated by \( f_1, \ldots, f_k \) in \( \text{GL}(a_1) \) acts fixed point free on \( V_1 = a_1 \) if and only if \( a_{12} \cap \bigcap_{i=1}^{k} g_i = a_\perp \) where \( g_i = \Gamma_{\bar{a}}(f_i) \).

(iv) With any conjunction \( \pi(\bar{x}) \) of group equations one can effectively associate a conjunction \( \pi^#(\bar{x}, \bar{z}) \) of lattice equations such that for any modular lattice \( L \) and \( \bar{g} = (g_1, \ldots, g_k) \) and \( \bar{a} \) in \( L \) one has \( L \models \pi^#(\bar{g}, \bar{a}) \) if and only if \( \bar{a} \) is a 4-frame in \( L \), \( \bar{g} \) in \( G(L, \bar{a}) \), \( G(L, \bar{a}) \models \pi(\bar{g}) \), and \( a_{12} \cap \bigcap_{i=1}^{k} g_i = a_\perp \).
Proof. (i) We may assume $V = \bigoplus_{i=1}^{n} V_i$ with isomorphisms $\varepsilon_j : V_1 \to V_j$ for $j > 1$. Put $a_i = V_i$, $a_{ij} = \{v - \varepsilon_j(v) \mid v \in V_i\}$ and $a_{kj} = (a_k + a_j) \cap (a_{kj} + a_{ij})$ for $j \neq k$ in $\{2, \ldots, n\}$.

(ii) Given a $n$-frame $a$ in $L$ we may assume $a_\bot = 0$ and $a_\top = 1$. For readability, we write meets as $a \cap b = ab$. Now, if $a_{12} = 0$, then $a_1 = a_1(a_{12} + a_2) = a_1a_2 = 0$ and then $a_2 = a_j(a_1 + a_{1j}) = a_ja_{1j} = 0$ for all $j > 1$. Thus $a_\top = 0$.

(iii) We deal with an arbitrary 4-frame $a$, uniformly, so that it is obvious which terms govern the construction. Again, we may assume $a_\bot = 0$ and $a_\top = 1$. Let $a'_i = \sum_{j \neq i} a_{ij}$ and observe that, for $i \neq k$, $a_{ik} \oplus a'_i = 1$. By modularity, $x \mapsto x + a_{ik}$ and $y \mapsto ya'_i$ are isomorphisms $[0, a'_k] \to [a_{ik}, 1]$ and $[a_{ik}, 1] \to [0, a'_i]$ between intervals and compose to the isomorphism $\pi_k^i : [0, a_k'] \to [0, a'_i]$, that is $\pi_k^i(x) = (x + a_{ik})a'_i$, with inverse $\pi_k^i$. Observe that $\pi_k^i(a_i) = a_k$ and $\pi_k^i(a_{ij}) = a_{kj}$ for $j \neq i, k$. Moreover, $\pi_k^k$ is identity on $[0; \sum_{j \neq i, k} a_{ij}]$; indeed, by modularity, $(x + a_{ik})a'_i = x + a_{ik}a'_i = x$ if $x \leq a'_i$.

Now, let $G_{ij} = \{x \in L \mid a_i \oplus x = a_i \oplus x = a_i + a_j\}$ for $i \neq j$ and observe that $\pi_k^i$ restricts to a bijection $\pi_k^{ij} : G_{ij} \to G_{kj}$ for $k \neq i, j$. Observe that for $x \in G_{ij}$ one has $xa'_i = x(a_i + a_j)a'_i = xa_j = 0$ and, similarly, $xa'_j = 0$.

For $r \in G_{12}$ define $r_{12} = r$, $r_{1j} = \pi_j^2(r) \in G_{1j}$, and $r_{ij} = \pi_i^1(r_{1j}) \in G_{ij}$ where $1 < i < j$. Observe that, for $r = a_{12}$, this is consistent with the notation for the $a_{ij}$. Given $r, s \in G_{12}$ one has $x = (r_{12} + s_{23})(a_1 + a_3) \in G_{13}$. Namely, by modularity, $xa_1 = [r_{12} + s_{23}(a_1 + a_2)]a_1 = r_{12}a_1 = 0$, $x + a_1 = (r_{12} + a_{23})(a_1 + a_3) = (a_1 + a_{23})(a_1 + a_3) = a_1 + a_3$, and, similarly, $xa_3 = 0$ and $x + a_3 = a_1 + a_3$. Thus, one obtains a well defined multiplication on $G_{12}$ such that
\[ r \otimes s = t \iff (r_{12} + s_{23})(a_1 + a_3) = t_{13}. \]

$a_{12}$ is a right unit since $(r_{12} + a_{23})(a_1 + a_3) = \pi_3^2(r) = r_{13}$. Given $r \in G_{12}$ has image $s = \pi_3^2\pi_1^2\pi_3^1(r)$ in $G_{12}$ and, applying the inverse maps, it follows $s_{23} = \pi_2^1\pi_3^2(s) = \pi_3^1(r)$. Thus, by modularity, $(r_{12} + s_{23})(a_1 + a_3) = (r_{12} + (r_{12} + a_{13})(a_2 + a_3))(a_1 + a_3) = (r_{12} + (r_{12} + a_{13})(a_2 + a_3))a_1 + a_3 = (r_{12} + a_{13})(a_1 + a_3) = a_{13}$ which shows that $s$ is a right inverse of $r$.

In order to establish $G_{12}$ as a group it remains to prove associativity. Preparing for this, we show for pairwise distinct $i, j, k, h$
\[ \pi_{hj}^k \pi_{kj}^{ij} = \pi_{kj}^{ij}, \quad \pi_{hk}^i \pi_{kj}^{ij} = \pi_{kh}^i \pi_{hi}^j. \]

Indeed, for $x \in G_{ij}$, modularity yields $[(x + a_{ik})(a_j + a_k) + a_{kh})(a_j + a_h)](a_j + a_h) = [(x + a_{ik})(a_j + a_k) + a_{kh})(a_j + a_h)](a_j + a_h) = (x + a_{ik} + a_{kh})(a_j + a_h) = (x + a_{ih})(a_i + a_h)$ and
relations are equivalent via the isomorphisms \( \pi_4^3, \pi_3^2, \text{ and } \pi_1^2 \), respectively, which match the elements associated with \( r \) and, similarly, for \( s \) and \( t \). Indeed, \( \pi_4^3(s_{23}) = \pi_4^3 \pi_2^1 \pi_3^2(s_{12}) = \pi_2^1 \pi_4^3 \pi_3^2(s_{12}) = s_{24}, \pi_4^3(t_{13}) = \pi_4^3 \pi_2^1(s_{12}) = t_{14}, \pi_4^3(s_{24}) = \pi_3^2 \pi_4^3(s_{12}) = \pi_3^2 \pi_4^3(s_{12}) = s_{34}, \) while the remaining equalities are obvious.

Now, for \( r, s, t \in G_{12} \), it follows by modularity \([ (r \otimes s) \otimes t ]_{14} = [(r \otimes s)_{13} + t_{34}] (a_1 + a_4) = [(r_{12} + s_{23})(a_1 + a_3) + t_{34}] (a_1 + a_4) = (r_{12} + s_{23})(a_1 + a_3 + a_4) + t_{34}] (a_1 + a_4) \).

Observe that \( G_{12} = G(L, \hat{a}) \) as sets. We turn \( G(L, \hat{a}) \) into the group opposite to \( G_{12} \) defining multiplication as \( (g, h) \mapsto t(g, h, \hat{a}) \) where \( t(x, y, z) \) is the term \( (y + ((x + z_{23}) (z_1 + z_3) + z_{12}) (z_2 + z_3)) (z_1 + z_3) + z_{23} \).

In (b) observe that for any \( i \neq j \) there is a 1-1-correspondence between elements \( r \in G_{ij} \) and isomorphisms \( f : a_i \rightarrow a_j \) given by \( f(v) = w \) if and only if \( v - w \in r \). We write \( f = \hat{r} \). For \( s \in G_{jk} \) and \( t = (r + s) \cap (a_i + a_k) \) it follows \( t = s \circ \hat{r} \). In particular we have the \( \hat{a}_{ij} : a_i \rightarrow a_j \) with \( \hat{a}_{ji} = \hat{a}_{ji}^{-1} \) and \( \hat{a}_{ij} \circ \hat{a}_{ij} = \hat{a}_{ik} \). We have to put \( \varepsilon_{a} = \hat{a}_{12}. \) Now, given \( g, h \in GL(a_1) \) and \( r = \Gamma_{a}(g), s = \Gamma_{a}(h) \) one has \( \hat{r}_{12} = \hat{a}_{12} \circ g \) and \( \hat{s}_{23} = \hat{a}_{23} \circ h \).\( \hat{a}_{21} = \hat{a}_{13} \circ h \circ \hat{a}_{21} \) whence , for \( t = r \otimes s, \hat{t}_{13} = \hat{s}_{23} \circ \hat{r}_{12} = \hat{a}_{13} \circ h \circ g \) and \( \hat{t} = \hat{a}_{31} \circ \hat{t}_{13} = h \circ g \).

(c) Define

\[
U = a_{12} \cap \bigcap_{i=1}^{k} g_i = \{ v - \varepsilon_{a}(v) \mid v \in V_1 \} \cap \bigcap_{i=1}^{k} \{ v - \varepsilon_{a}(f_i v) \mid v \in V_1 \}.
\]

Consider \( v \) such that \( v - \varepsilon_{a} v \in U. \) For every generator \( f_i \) of \( G \) there is some \( w \in V_1 \) such that \( v - \varepsilon_{a}(v) = w - \varepsilon_{a}(f_i w) \) whence \( w = v \) and \( v = f_i v \) since the sum \( a_1 \oplus a_2 \) is direct. Thus, \( v \) is fixed under the
action of $G$ on $V_1$. Conversely, if $v = f_i v$ for all $i$ then $v - \varepsilon_a v \in U$.
Now, observe that $U = 0$ if and only if \{ $v \in V_1 \mid v - \varepsilon_a v \in U$ \} = 0.

(iv) is obvious by (iii). □

4. Consistency in modular lattices

Lemma 7. There is a recursive set $\Sigma$ of conjunctions of lattice equations such that the following hold for any $\varphi \in \Sigma$.

(i) Given a division ring $F$ and a cardinal $\kappa \geq \aleph_0$. If $\varphi$ admits a non-trivial satisfying assignment in some modular lattice, then it does so within $L(V)$ for some $V \in \mathcal{V}_F$ with $\dim V = \kappa$.

(ii) If $\varphi$ admits a non-trivial satisfying assignment in the finite modular lattice $L$ and if $F$ is a division ring of $\chi(F) = 0$ or $\chi(F) > |L|$ (it suffices to require $\chi(F) > |G(L, \bar{a})|$ for all 4-frames $\bar{a}$ in $L$) then $\varphi$ admits a non-trivial satisfying assignment within $L(V)$ for some $V \in \mathcal{V}_F$ with $\dim V = 4d < \aleph_0$ for some $d$.

(iii) $\varphi$ is of the form $\varphi(\bar{x}, \bar{z})$, with $\bar{z}$ referring to 4-frames. $\varphi^3$ given by $\exists \bar{x} \exists \bar{z} \varphi(\bar{x}, \bar{z}) \land z_\perp \neq z_\top$ is valid within the modular lattice $L$ if and only if $\varphi$ admits a non-trivial satisfying assignment within $L$.

(iv) The sets of all $\varphi \in \Sigma$ with $\varphi^3$ valid in some, respectively, some finite modular lattice are not recursive.

The statements remain valid with constants $0 = z_\perp$ and $1 = z_\top$.

Proof. Let $\Sigma$ consist of all $\pi^#(\bar{x}, \bar{z})$, according to Lemma 6(iv), with $\pi(\bar{x})$ a conjunction of group equations. We claim:

(*) $\pi(\bar{x})$ admits a non-trivial satisfying assignment within some (finite) group $G$ if and only if $\pi^#(\bar{x}, \bar{z})$ does so within some (finite) modular lattice $L$; moreover, given a non-trivial satisfying assignment within $G$, $L$ can be chosen as $L(V)$ as in (i) respectively (ii).

Clearly, for a modular lattice $L$, $(\bar{g}, \bar{a})$ is a satisfying assignment within $L$ if and only if $\bar{g}$ is a satisfying assignment for $\pi(\bar{x})$ in the group $G = G(L, \bar{a})$ — which is finite if $L$ is finite.

If the assignment $\bar{g}$ in $G$ is trivial, then $g_i = e_G = a_{12}$ for all $i$. On the other hand, $a_{12} \cap \bigcap_i g_i = a_\perp$ whence $a_{12} = a_\perp$ and the assignment $(\bar{g}, \bar{a})$ is trivial in view of Lemma 6(ii). In other words, if $\pi(\bar{x})$ admits only trivial satisfying assignments within (finite) groups, then $\pi^#(\bar{x}, \bar{z})$ does so within (finite) modular lattices.

Now, assume that $\pi(\bar{x})$ has a non-trivial assignment $\bar{h}$ in some (finite) group $G$: in particular $G$ is not the trivial group. We have to
find assignments in suitable $L(V)$. We may assume that $G$ is at most countable. If $G$ is finite, in (i) we may consider some countably infinite extension, instead. Let $\kappa \geq \aleph_0$ resp. $\kappa = 4(|G| - 1)$ if $G$ is finite and choose $V$ of dim $V = \kappa$ as required in (i) and (ii), respectively. By Lemma 6(i), there is a 4-frame $\bar{a}$ of $L = L(V)$ such that dim $V_1 = \kappa$ for $V_1 = a_1$. Fact 3 provides a fixed point free faithful representation $\rho$ of $G$ in $V_1$; that is, the $f_i = \rho(h_i)$ generate a subgroup of $GL(V_1)$ acting fixed point free on $V_1$. Recall Lemma 6(iii)–(iv) and let $g_i = \Gamma_a(f_i)$ to obtain a non-trivial assignment $(\bar{g}, \bar{a})$ in $L$ such that $L \models \pi^#(\bar{h}, \bar{a})$.

This proves (*), (i), and (ii). (iii) is obvious and (iv) follows from (*), Theorem 3 and the reduction $\pi \mapsto \pi^#$. □

In view of these Facts, we consider the following richness conditions on a class $\mathcal{C}$ of lattices respectively $\mathcal{V}$ of vector spaces.

(I) $L(V) \in Q\mathcal{C}$ for some vector space $V$ of dim $V \geq \aleph_0$.
(II) For every $0 < d < \aleph_0$ there are a division ring $F$ of characteristic not dividing $d + 1$ and an $F$-vector space $V$ of dim $V = nd$ such that $L(V) \in Q\mathcal{C}$.
(III) $\mathcal{V}$ consists of finite dimensional vector spaces over division rings which are finite dimensional over the center; moreover, for any $0 < d < \aleph_0$ there is $V \in \mathcal{V}$ of characteristic not dividing $d + 1$ such that dim $V = nd$.

Clearly, (III) implies (II) for $L(\mathcal{V}) := \{L(V) \mid V \in \mathcal{V}\}$. We refer of (II) and (III) just as (II) and (III); except for Section 9, these are the ones to be used.

**Theorem 8.** The consistency problem is unsolvable for any class $\mathcal{C}$ of modular lattices (with or without bounds) satisfying (I) or satisfying $\mathcal{C} \subseteq Q\mathcal{M}_f$ and (II). In the bounded case, unsolvability also persists in any expansion of $\mathcal{C}$.

In particular, in Corollary 24 below, we will show that case (II) applies to $\mathcal{C} = L(\mathcal{V})$ where $\mathcal{V}$ satisfies (III).

**Proof.** In view of Lemma 7 and the richness condition a conjunction of lattice equations admits a non-trivial satisfying assignment within $\mathcal{C}$ if and only if it does so within some (finite) modular lattice — in case (II) use Fact 1 □

We conclude the section discussing restricted variants of the consistency problem for modular lattices. These are not needed for the applications to other structures.

**Corollary 9.** The decision problems of Theorem 8 remain unsolvable if restricted to conjunctions $\pi(x_1, \ldots, x_5)$ of equations.
Proof. Recall from [13] that the modular lattice freely generated by a 
\((k + 1)\)-frame is finitely presented as a modular lattice with four 
generators, the frame given by a system \(\bar{b}(\bar{y})\) of terms, 
\(\bar{y} = (y_1, y_2, y_3, y_4)\), and finitely many relations. Dealing with a conjunction of group equations 
in \(k\) variables \(\bar{x}\), encode these adding to \(\bar{y}\) a single lattice variable 
\(y_5\) and finitely many relations. Namely, considering a \((k + 1)\)-frame \(\bar{b}\) in a modular lattice 
\(L\), let the \(4\)-frame \(\bar{a}\) given by the \(b_i, b_{1j}, i, j \leq 4\) 
and \(L' = [0, \sum_i a_i]\). Then the \(x_i\) correspond to \(g_1, \ldots, g_k\) in \(G(L', \bar{a})\). Let 
\(g'_1 = g_1\) and \(g'_i = (b_1 + b_{i+1}) \cap (b_{2,i+1} + g_i)\) for \(i > 1\). Then 
\(g_i = (b_1 + b_{i+1}) \cap (b_{2,i+1} + g'_i)\) for \(i > 1\) and \(g'_i = (b_1 + b_{i+1}) \cap c\) 
where \(c = \sum_{i=1}^k g'_i\). Introducing the variable \(y_5\) for \(c\) and the associated 
equations, this yields the conjunction \(\psi\) of \(5\)-variable lattice equations 
replacing \(\pi^#\) from Lemma 6(iv). In view of Lemma 6(ii), \(a_\perp = a_\top\) 
implies \(b_\perp = b_\top\). □

For a field \(F\) and \(\mathcal{V} = \{F^d \mid d < \aleph_0\}\), if satisfiability of conjunctions 
of ring equations is decidable for \(F\), then the reasoning of [17, Theorem 4.10] shows that the consistency problem for \(L(\mathcal{V})\) is solvable if and 
only if there is a recursive function \(\delta\) that for every conjunction \(\psi\) of lattice equations one has the following: If \(\psi\) is of binary length \(n\) and satisfiable in \(L(F^d)\) for some \(d\) then \(\psi\) is also satisfiable in \(L(F^{d'})\) for 
some \(d' \leq \delta(n)\). By Theorem 8, no such \(\delta\) exists if \(F\) is the field of real 
or complex numbers.

On the other hand, in the presence of an orthocomplementation, Example 4.2(b) in [17] gives a recursively defined sequence \(t_k(\bar{x})\) of terms 
of length \(O(k)\) in \(2k + 1\) variables such that \(t_k(\bar{x}) = 1\) is satisfiable in 
\(L(F^d)\) if \(d = 2^k\) but not for \(d < 2^k\). We provide an analogous recursive 
sequence without orthocomplementation and with fixed number of variables.

In [9] the bit length of a group presentation is defined as the total 
number of bits required to write the presentation; in particular, 
words are considered as strings of powers of generators and inverses 
of generators, the exponents encoded in binary. Transferring this to 
lattice presentations, we allow the use of recursively defined subterms, 
encoding the number of iteration steps in binary.

**Corollary 10.** There is a recursively defined sequence of conjunctions 
\(\psi_n(\bar{y})\), \(n > 7\), of bounded lattice equations in 5 variables \(\bar{y}\) such that 
\(\psi_n\) is of bit length \(O(\log n)\) and such that, for any field of \(F\) of characteristic 0, \(\psi_n(\bar{y})\) is satisfiable in some \(L(V_F)\), with \(\dim V_F = d > 0\) for 
\(d = 4(n - 1)\) but not for \(d < 4(n - 1)\).
Proof. By [9, Theorem C] the alternating groups $A_n$, $n > 7$, have presentations of bit length $O(\log n)$ in 3 generators $\bar{x} = (x_1, x_2, x_3)$; and any non-trivial irreducible representation of $A_n$ has degree $\geq n - 1$ [28]. Put $z_\perp = 0$, $z_\top = 1$ and define $\psi_n$ as $\pi_n^\#(\bar{x}, \bar{z})$ associated with such presentation of $A_n$ according to Lemma 6(iv). By Lemma 6(iii)(a) there is a constant $K$ such that for every group word $w(\bar{x})$ one has a lattice term $w_\perp(\bar{x})$ (in the extended sense) such that $|w_\perp(\bar{x})| \leq K|w(\bar{x})|$ and $w_\perp(\bar{x})$ evaluates as $w(\bar{x})$ in any $G(L, \bar{a})$. Use the proof of Corollary 9 to replace the 13 variables $\bar{x}, \bar{z}$ by 5 new ones.

Now recall Lemma 6(iii) and observe that for any 4-frame $\bar{a}$ of $L(V)$ and subgroup $G$ of $\text{GL}(a_1)$ one has a $G$-invariant subspace $U_1 = \{v \in a_1 \mid v - \varepsilon_a(v) \in U\}$ of $V_1 = a_1$ where $U$ is as in the proof of Lemma 7. Now, $U_1 = 0$ if and only if $U = 0$. Thus, any non-trivial irreducible representation of $A_n$ in some $V_1$ gives rise to a non-trivial satisfying assignment for $\psi_n$ in $L(V)$, $V = V_1^4$. Conversely, any non-trivial satisfying assignment $\bar{g}, \bar{a}$ for $\psi_n$ in some $L(V)$, $V$ a finite dimensional $F$-vector space, we may assume $a_\perp = 0$ and $a_\top = V$ and $\bar{g}$ defines a non-trivial representation of $A_n$ in $V_1 = a_1$ which, by Maschke’s Theorem, has a non-trivial direct summand, whence $\dim V \geq 4(n - 1)$. □

5. Relation Algebras

A pre-relation algebra is an algebraic structure $A$ with two binary operations written as $\cap$ and $\circ$, a unary operation $^{-1}$, and constant $\Delta$. We write $\alpha \in \text{Eq}(A)$ if $\Delta \cap \alpha = \Delta$, $\alpha^{-1} = \alpha$, and $\alpha \circ \alpha = \alpha$. We also consider the partial algebra $A^\#$ where $\circ$ is replaced by the partial operation given by $\alpha + \beta = \gamma$ if and only if $\alpha, \beta \in \text{Eq}(A)$ and $\alpha \circ \beta = \beta \circ \alpha = \gamma$. We write $\alpha \oplus \beta = \gamma$ if $\alpha + \beta = \gamma$ and $\alpha \cap \beta = \Delta$. A system $\bar{a}$ in $\text{Eq}(A)$ is a permuted 4-frame of $A$ if the equations defining a 4-frame in a lattice are satisfied by $\bar{a}$, being evaluated within $A^\#$, and if $a_\perp = \Delta$. Define

$$G(A, \alpha) = \{\beta \in \text{Eq}(A) \mid \beta \oplus a_1 = \beta \oplus a_2 = a_1 \oplus a_2\}$$

Given a set $S$ we consider the pre-relation algebras on sets of binary relations on $S$ with the following operations: intersection $\cap$, relational product $\circ$, inversion $^{-1}$, and $\Delta = \text{id}_S$. We say that $A$ is represented on $S$ - Jónsson [20] calls $A$ an algebra of relations. Let $\mathcal{R}$ denote the class of all algebras isomorphic to such — the class of representable pre-relation algebras; $\mathcal{R}$ is quasi-variety by [20, Theorem 1]. By $\mathcal{R}_f$ we denote the class of finite members of $\mathcal{R}$. The following are immediate by [14, Corollary 2] and Lemma 6(i).
Fact 11. If $\alpha$ is a permuting 4-frame of $A$ which is represented on $S$, then the subalgebra $L = L(A, \alpha)$ generated by $\alpha$ together with $G(A, \alpha)$ consists of pairwise permuting equivalence relations on $S$; with operations $\cap$ and $+$ it forms a modular sublattice $L(A, \alpha)$ of $\text{Eq}(S)$; in particular, $L \subseteq \text{Eq}(A)$ and $L^\# = L$. Moreover, $\alpha$ is a 4-frame of $L$ and $G(A, \alpha) = G(L, \alpha)$.

Fact 12. For any vector space $V$ of $\dim V = 4d$ there is a pre-relation algebra $A = A(V)$ represented on $V$ and a permuting 4-frame of $A$ such that $L(V) = L(A, \alpha)$.

We consider particular formulas in the language of pre-relation algebras: $\text{Eq}(\bar{y})$ is the conjunction of equations such that $A \models \text{Eq}(\bar{\xi})$ implies $\beta \in \text{Eq}(A)$ for any $\beta$ in the list $\bar{\xi}$. A type-1-formula $\psi(\bar{y})$ is a conjunction of basic equations in the $\cap$-$\circ$-fragment. Let $\tau(u, \bar{y}, \bar{v})$ denote the obvious type-1-formula such that $A \models \tau(\gamma, \bar{\delta}, \bar{\varepsilon})$ if and only if $\gamma = \delta_1 \circ \ldots \circ \delta_n$ and that, in this case, $\gamma = \varepsilon_j = \delta$ for all $j$ if $\delta_i = \delta$ for all $i$. The richness conditions (I) and (II) are modified replacing $L(V)$ by $A(V)$.

Theorem 13. The consistency problem is unsolvable for any class $A$ of representable pre-relation algebras satisfying (I) or contained in $Q^{\mathcal{R}_f}$ and satisfying (II). More precisely, there is a recursive set $\Sigma'$ of type-1-formulas such that there is no algorithm which on input $\psi(\bar{y}) \in \Sigma'$ would decide whether $\psi$ given as

$\exists \bar{y} \exists u \exists \bar{v}. \text{Eq}(\bar{y}) \land \bigwedge_i y_i = \Delta \land \psi(\bar{y}) \land \tau(u, \bar{y}, \bar{v}) \land u \neq \Delta$

is satisfied in some member of $A$.

Corollary 14. Unsolvability persists if the total relation is given as constant $\nabla$; in this case, $u = \nabla$ has to be added as a conjunct in forming $\psi$. Also unsolvability persists in any expansion of $A$.

Proof of Theorem 13. Recall $\Sigma$ from Lemma 7 to obtain $\Sigma'$ replace $\varphi(\bar{x}, \bar{z})$ by an equivalent unnested pp-formula in lattice language (Fact 2). In the latter, replace any equation $u = v + w$ by $u = v \circ w \land u = w \circ v$. This associates with $\varphi(\bar{x}, \bar{z})$ a type-1-formula $\varphi'(\bar{x}, \bar{z}, \bar{y})$ such that in any lattice $L$ of permuting equivalences one has $(L, \cap, +) \models \varphi(\bar{\xi}, \bar{\alpha})$ if and only if $(L, \cap, \circ) \models \exists \bar{y}, \varphi'(\bar{\xi}, \bar{\alpha}, \bar{y})$. Observe that, if $\varphi^3$ is valid in $A \in \mathcal{R}$, then $\varphi^3$ is valid in $L(A, \alpha)$ for some witnessing permuting 4-frame $\alpha$ in $A$. Conversely, if $\varphi^3$ is valid in $L = L(A, \alpha)$, then $\varphi^3$ is valid in $A \in \mathcal{R}$ witnessed by the values for $\bar{z}$ and $\bar{x}$ in $L \subseteq A$. Thus, put $\Sigma' = \{ \varphi' \mid \varphi \in \Sigma \}$. 

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Now, if \( \varphi^3 \) is valid within some (finite) modular lattice, then it is so within suitable \( L(V) \) (Lemma 7(i),(ii)) whence \( \varphi^3 \) is valid in \( A(V) \in \mathcal{A} \) (cf. Fact 12). Conversely, if \( \varphi^3 \) is valid in \( A \in \mathcal{A} \) (by Fact 11 we may assume \( A \) finite in case (II)), then \( \varphi^3 \) is valid in the (finite) modular lattice \( L(A, \alpha) \) where \( \alpha \) is witness for \( \bar{z} \). Thus, the claim follows from Lemma 7(iii),(iv) and the reduction \( \varphi \mapsto \varphi' \).

\[ \square \]

6. Databases

We follow [21] for database concepts, though adapting notation to common use in mathematics. Fix a countably infinite set \( X_\infty \) of variables and use \( x, y, \ldots \) to denote elements of \( X_\infty \). Under the pure universal relation assumption, a database \( D \) is given by a finite non-empty \( U \subseteq X_\infty \) of attributes, for each \( x \in U \) a domain \( \Delta[x] \) of values of the attribute \( x \), and a non-empty subset (relation) \( R \) of the direct product \( \prod_{x \in U} \Delta[x] \). For a tuple \( t \) in \( R \) and \( X \subseteq U \) let \( t[X] \) be the restriction of \( t \) to \( X \).

The atomic sentences to be considered are the functional dependencies (fd’s) \( X \to Y \) and the embedded multivalued dependencies (emvd’s) \( [X,Y] \) with non-empty finite \( X,Y \subseteq U_\infty \). Validity is defined as follows: \( D \models X \to Y \) if and only if for all \( s,t \in R \), if \( s[X] = t[X] \) then \( s[Y] = t[Y] \) - provided \( X \) and \( Y \) are subsets of \( U \), at all. \( D \models [X,Y] \) if and only if for every \( t_1, t_2 \in R \) with \( t_1[X \cap Y] = t_2[X \cap Y] \) there exists \( t \in R \) with \( t[X] = t_1[X] \) and \( t[Y] = t_2[Y] \); in other words, the restriction of \( R \) to \( XY = X \cup Y \) is the natural join of the restrictions of \( X \) and \( Y \).

A database \( D \) with attribute set \( U \) is trivial if it satisfies all fd’s and emvd’s (with attributes from \( U \)). \( D \) is almost trivial if each of its attributes is a key, that is if \( D \) satisfies all fd’s.

**Fact 15.** A database is almost trivial if \( s[x] \neq t[x] \) for all \( s \neq t \) in \( R \) and \( x \in U \). A database is trivial if and only if its attribute set \( U \) or its relation \( R \) is a singleton set.

**Proof.** The first claim is obvious and so is the inverse implication in the second. Now, assume \( D \) trivial and \( U = X \cup Y \) the disjoint union of non-empty \( X,Y \). Consider \( t_1, t_2 \) in \( R \). In view of the emvd \( [X,Y] \) there must be \( t \in R \) such that \( t_1[X] = t[X] \) and \( t_2[Y] = t[Y] \). The fd’s \( x \to y \) and \( y \to x \) \( (x \in X, y \in Y) \) imply \( t_1[Y] = t[Y] \) and \( t_2[X] = t[X] \) whence \( t_1 = t = t_2 \).

For a vector space \( V \) and injective map \( f : U \to L(V), U \subseteq X_\infty \) finite, consider the database \( D(V,f) \) where \( \Delta[x] \) is given as the set of cosets of \( f(x) \) and \( R = \prod_{x \in U} \Delta[x] \). Let \( D(V) \) denote the set of all
The consistency problem (I) and (II) are modified replacing \( L(V) \) resp. \( A(V) \) by \( D(V) \).

**Theorem 16.** Consider a class \( \mathcal{D} \) of databases which either satisfies (I) or which consists of finite databases and satisfies (II). Then there is no algorithm which on input of a conjunction of fd’s and emv d’s decides whether there is a member of \( \mathcal{D} \) which is not almost trivial (resp. non-trivial).

While consistency (as a special case of implication) is decidable for fd’s alone, the question for emvd’s alone remains open. By [10, Section 2.2] the analogue of the Theorem follows for models in independence logic w.r.t. inclusion and conditional independence atoms.

**Proof of Theorem 16.** Given a database \( D \), one has projection maps \( \pi_x : R \to \Delta[x] \) yielding for each \( t \in R \) its \( x \)-component \( \pi_x(t) = t[x] \) in \( \Delta[x] \). With each of these maps one has its kernel equivalence relation \( \theta_x \). For a set \( X \) of attributes write \( \theta_X = \bigcap_{x \in X} \theta_x \). Thus, \( (s, t) \in \theta_X \) if and only if \( s[X] = t[X] \).

Let \( S \) denote the factor set \( R/\theta_U \) of equivalence classes modulo \( \theta_U \) and, for \( X \subseteq U \), \( \eta_X^D \) the equivalence relation on \( S \) corresponding to \( \theta_X \), that is for any \( \theta_U \)-classes one has

\[
(t/\theta_U, s/\theta_U) \in \eta_X^D \text{ iff } (t, s) \in \theta_X.
\]

Clearly, \( \eta_X^D = \bigcap_{x \in X} \eta_x^D \). Let \( A(D) \) the pre-relation algebra represented on \( S \) which is generated by the \( \eta_x^D \), \( x \in U \). It follows from [15, Lemma 11]

\[
\eta_X^D \subseteq \eta_Y^D \text{ iff } D \models X \to Y
\]

\[
\eta_Z^D = \eta_X^D \cap \eta_Y^D \text{ iff } D \models XY \to Z \land Z \to XY
\]

and, if \( X, Y, Z \) are pairwise disjoint, then

\[
\eta_Z^D = \eta_X^D \circ \eta_Y^D \text{ iff } D \models [XZ, YZ] \land X \to Z \land Y \to Z.
\]

For a tuple \( \bar{y} \) from \( U \) write \( \eta^D(\bar{y}) = (\eta_{y_1}^D, \ldots, \eta_{y_n}^D) \).

Conversely, any pre-relation algebra \( A \), represented on a set \( S \), together with an assignment \( \eta \) within \( \text{Eq}(A) \) for variables from a finite \( U \subseteq X_\infty \) such that \( \bigcap_{x \in U} \eta(x) = \Delta \), gives rise to a database \( D = D(A, \eta) \) with \( \Delta[x] \) the set of classes of \( \eta(x) \) and \( R \) the image of \( S \) under the map \( a \mapsto \{ a/\eta(x) \mid x \in U \} \). For such, \( \theta_U = \Delta \) and \( (A, \eta) \equiv (A(D), \eta^D) \). Moreover, \( D \) is almost trivial if and only if \( \text{im } \eta \) is a singleton set, namely \( \{ s \} \).

Given a type-1-formula \( \psi(\bar{y}) \) let \( U \) consist of all variables occurring in \( \psi^3 \) and translate \( \psi(\bar{y}) \land \tau(\bar{u}, \bar{y}, \bar{v}) \) into a conjunction \( \psi' \) of fd’s and
emvd's in attributes $\bar{y}, u, \bar{v}$ such that $D \models \psi'$ if and only if

$$A(D) \models \bigcap_i \eta^D(y_i) = \Delta \wedge \psi(\eta^D(\bar{y})) \wedge \tau(\eta^D(u), \eta^D(\bar{y}), \eta^D(\bar{v})).$$

Observe that for a database $D = D(V, f)$ one has $(A(D), \eta^D) \cong (A(V), \hat{f})$ where $\hat{f}(x)$ is the equivalence relation associated with $f(x)$. Thus, Theorem \[13\] applies to $A = \{A(D) \mid D \in D\}$. We rephrase its statement: There is no algorithm which, on input of a type-1-formula $\psi(\bar{y})$ decides whether there is $A$ in $A$ and an assignment $\eta$ for $\bar{y}, u, \bar{v}$ in Eq($A$) such that $A \models \psi(\eta(\bar{y})) \wedge \tau(u, \bar{y}, \bar{v})$ and $\bigcap_i \eta(y_i) = \Delta$ but $\eta(u) \neq \Delta$, the latter being equivalent to $\text{im} \eta$ not to be singleton. Deciding the latter reduces to deciding whether there is a non almost trivial database $D$ with attributes $\bar{y}, u, \bar{v}$ satisfying the conjunction $\psi'$ of fd's and emvd's. This shows unsolvability of the first decision problem in the Theorem. For the second problem, we have to refer to Corollary \[14\] and translate $u = \nabla$ by $[u, U \setminus u] \wedge U \setminus u \rightarrow u$. Indeed, if the latter holds in $D$, then, by the emvd, for any $t_1, t_2 \in R$ there is $t \in R$ such that $t[u] = t_1[u]$ and $t[U \setminus u] = t_2[U \setminus u]$ whence, by the fd, $t_2[u] = t[u] = t_1[u]$ and so $(t_1, t_2) \in \theta_u$. \[\square\]

7. Rings

We consider rings $R$ with constants 0, 1. Let $L(R)$ denote the (modular) lattice of all right ideals. A ring $R$ is (von Neumann) regular if for any $a \in R$ there is $x \in R$ such that $axa = a$; equivalently, any of its principal right ideals is generated by an idempotent. The principal right ideals of a regular ring form a sublattice $L(R)$ of $L(R)$. The following is well known and easy to prove.

**Fact 17.** The endomorphism ring $R = \text{End}(V)$ of an vector space $V$ is regular and one has $\overline{L}(R) \cong L(V)$ via $\varphi R \mapsto \text{im} \varphi$.

**Fact 18.** There are positive primitive $\sigma(x, y, z)$ and $\mu(x, y, z)$ in the language of rings such that the following hold for any idempotents $e, f, g$ in a ring $R$:

(a) $R \models \sigma(e, f, g)$ if and only if $gR = eR + fR$.

(b) $R \models \mu(e, f, g)$ implies $gR = eR \cap fR$.

If $R$ is regular, then in (b) holds the converse, too.

**Proof.** Concerning (a) observe that $gR = eR + fR$ if and only

$$R \models ge = e \wedge gf = f \wedge \exists r \exists s. g = er + fs.$$
Concerning (b) observe that, according to [27, Lemma 8-3.12 (ii)], if
\[ (f - ef)r(f - ef) = f - ef \land g(f - fr(e - ef)) = f - fr(e - ef) \land g = (f - fr(e - ef))s. \]
holds in \( R \) then \( gR = eR \cap fR \) — the first equation encodes that \( r \) is a quasi-inverse of \( f - ef \) while the last two state that
\[ gR = (f - fr(e - ef))R. \]
\[ \square \]

In the richness conditions replace \( L(V) \) by \( \text{End}(V) \). Let \( \mathcal{R}_f \) denote the class of all finite rings.

**Theorem 19.** The consistency problem is unsolvable for any class \( \mathcal{C} \) of rings satisfying (I) or contained in \( \mathbb{Q} \mathcal{R}_f \) and satisfying (II).

**Proof.** Consider the language of bounded lattices and recall \( \Sigma \) from Lemma 7. Given \( \varphi(\bar{x}) \in \Sigma \), replace it by the equivalent unnested pp-formula \( \exists \bar{y}. \varphi'(\bar{x}, \bar{y}) \) (Fact 2); associate with each variable \( x, y \) in the latter a ring variable \( \hat{x}, \hat{y} \) and let \( \chi \) be the conjunction of all equations \( \hat{x}^2 = \hat{x} \) and \( \hat{y}^2 = \hat{y} \). Use Fact 18 to replace the basic lattice equations by existentially quantified conjunctions of ring equations, each with new rings variables. The conjunction of these and of \( \chi \) is equivalent to a positive primitive ring formula \( \hat{\varphi}(\bar{x}, \bar{y}) \) such that \( L(R) \models \varphi \) if \( R \models \varphi^R \) where \( \varphi^R \) is given as \( \exists \bar{x} \exists \bar{y}. \varphi(\bar{x}, \bar{y}) \land 0 \neq 1 \). Conversely, if \( \varphi^3 \) holds in \( L(V) \) then, in view of Fact 17, \( \varphi^R \) holds in \( \text{End}(V) \).

In view of the richness conditions and Fact 1, the claim follows from Lemma 7(iii),(iv) as in the proof of Theorem 13. \[ \square \]

Let \( \mathcal{N}_f \) denote the class of all finite regular rings. By the Artin-Wedderburn Theorem \( \mathcal{N}_f \) consists, up to isomorphism, just of the direct products of matrix rings \( F^{d \times d}, d < \aleph_0, F \) a finite field.

**Fact 20.** \( \text{End}(V) \in \mathbb{Q} \mathcal{N}_f \) if \( V \) is an \( F \)-vector space of \( \dim V < \aleph_0 \) and if \( F \) is finite dimensional over its center.

**Proof.** This can be seen as a variant of Lemma 3.5 in Lipshitz [22]. Since \( \text{End}V_F \) embeds into \( \text{End}V_C, C \) the center of \( F \), we may assume that \( F \) is a field and consider \( F^{d \times d} \cong \text{End}(V_F) \). By tensoring with \( \bar{F} \), the algebraic closure of \( F \), we have \( F^{d \times d} \) embedded into \( \bar{F}^{d \times d} \). The algebraic closure \( P \) of the prime subfield \( P \) of \( F \) is elementarily equivalent to \( \bar{F} \), and it follows that \( \bar{F}^{d \times d} \) and \( F^{d \times d} \) are elementarily equivalent, too. Now, \( \bar{F}^{d \times d} \) is the directed union (whence in the quasi-variety) of the \( K^{d \times d} \) where \( K \) is a subfield of \( P \) of finite degree, — and finite if \( P \) is finite. Finally, observe that \( \bar{F} \) embeds into a suitable ultraproduct of
the $P$, $P$ finite, since that is algebraically closed and of characteristic $0$. □

In particular, if $\mathcal{F}$ is a class of division rings which are finite dimensional over the center and if $\mathcal{F}$ contains members of characteristic $0$ or infinitely many finite characteristics, then there is no algorithm to decide, for a given finite family of multi-variate polynomials $p_i$ (in non-commuting variables) with integer coefficients, whether there is a common zero in the matrix ring $F^{d \times d}$ for some $F \in \mathcal{F}$ and $0 < d < \aleph_0$. In view of Fact 2 one may restrict to families of quadratic polynomials.

If the matrix rings $F^{d \times d}$ are endowed with an involution $A \mapsto A^*$ such that $\sum_i A_i A_i^* = 0$ implies $A_i = 0$ for all $i$ then a family $(p_i)$ can be replaced by the single $\sum_i p_i p_i^*$, which can be considered a polynomial in variables $x_i, x_i^*$, to be interpreted such that $x_j^* \mapsto B_j^*$ if $x_j \mapsto B_j$. Again, it suffices to consider a single quartic such polynomial. In particular this applies if $\mathcal{F}$ consists of subfields of the complex numbers, closed under conjugation, and if $A^*$ is the conjugate transpose of $A$.

In the context of the categorical approach to Quantum Theory (cf. [1, 11]), Theorem 19 yields the following: Let $F$ be a division ring of characteristic $0$ and $C$ the additive category, possibly enriched with additional structure, of finite dimensional $F$-vector spaces. Consider $C$ as a partial algebraic structure the underlying “set” of which is the class of all morphisms. Then there is no algorithm to decide, for any given conjunction $\pi(\bar{x})$ of equations, whether $\pi(\bar{x})$ admits an assignment in $C$ which is satisfying (in a particular, having all terms in $\pi(\bar{x})$ evaluated) and non-trivial (that is, not having a 0-morphism as single value). Indeed, the problem of Theorem 19 can be encoded so that satisfying assignments must have values which are endomorphism of a single object.

8. Complemented modular lattices

A modular lattice $L$ with bounds $0, 1$ as constants is complemented if for any $a$ there is $b$ such that $a \oplus b = 1$ (in the sequel, we consider $0, 1$ as constants). Here, consistency problems can be given a more special form.

**Fact 21.** Within the class of complemented modular lattices, any conjunction of equations is equivalent to a formula $\exists \bar{y}. \ s(\bar{x}, \bar{y}) = 0 \land t(\bar{x}, \bar{y}) = 1$ with terms $s, t$.

**Proof.** Given a conjunction of equations $s_j = t_j$, observe each $s_j = t_j$ equivalent to $\exists v : \bar{s}_j = 0 \land \bar{t}_j = 1$ for $\bar{s}_j := (s_j + t_j) \cap v$ and $\bar{t}_j := (s_j \cap t_j) + v$ (due to modularity and existence of complements);
and \( \tilde{s}_j = 0 \land \tilde{t}_j = 1 \land \tilde{s}_i = 0 \land \tilde{t}_i = 1 \) equivalent to \( \tilde{s}_j + \tilde{s}_i = 0 \land \tilde{t}_j \cap \tilde{t}_i = 1 \). □

In particular, the lattices \( \mathbb{L}(V) \) of all linear subspaces of vector spaces are complemented modular and so are the lattices \( \mathbb{L}(R) \) of principal right ideals of regular rings. For the latter, the following is useful in case (II).

**Fact 22.** For any class \( \mathcal{R} \) of regular rings one has \( \mathbb{Q}\{\mathbb{L}(R) \mid R \in \mathcal{R}\} \subseteq \{\mathbb{L}(R) \mid R \in \mathbb{Q}\mathcal{R}\} \)

**Proof.** Since any \( R \in \mathbb{Q}\mathcal{R} \) embeds into some direct product \( P \) of ultraproductions \( S_i \) of members \( R_{ij} \) of \( \mathcal{R} \) (cf. [8, Corollary 2.3.4]) it suffices to observe that \( \mathbb{L}(R) \) embeds into \( \mathbb{L}(P) \) via \( eR \mapsto eP \) and \( \mathbb{L}(P) \) into the direct product of the \( \mathbb{L}(S_i) \) via \( (e_i S_i \mid i \in I) \mapsto (e_i \mid i \in I)P \) (cf. [27] Corollaries 8-3.14-15] and that the \( \mathbb{L}(S_i) \) satisfy all quasi-identities valid in the \( \mathbb{L}(R_{ij}) \) (which, by Fact 18 translate into sentences in the language of rings). □

The following is a lattice theoretic variant of Fact 20 and can be proved in the same fashion. It follows, immediately, if one combines Facts 17, 20, and 22. Together with Theorem 8 it implies Corollary 24.

**Fact 23.** \( \mathbb{L}(V) \in \mathbb{Q}\mathcal{M}_f \) if \( V \) is an \( \mathbb{F} \)-vector space of \( \dim V < \aleph_0 \) and if \( \mathbb{F} \) is finite dimensional over its center.

**Corollary 24.** Theorem 8 and Corollary 9 apply to \( \mathcal{C} = \{\mathbb{L}(V) \mid V \in \mathcal{V}\} \) where \( \mathcal{V} \) satisfies the richness condition (III).

Of course, if for a class \( \mathcal{C} \) of complemented modular lattices choice of some complement is added as fundamental operation, Theorem 8 applies. If \( d = \dim V < \aleph_0 \), if \( F \) is a division ring with involution, and if \( V \) is endowed with an anisotropic form \( \Phi \) hermitean with respect to this involution, then \( U \mapsto U^\perp = \{v \in V \mid \forall u \in U.\Phi(v, u) = 0\} \) turns \( \mathbb{L}(V) \) into the ortholattice \( \mathbb{L}^+(V) \). Here, an ortholattice is a bounded lattice endowed with a dual automorphism \( x \mapsto x^\perp \) of order 2 such that \( x \oplus x^\perp = 1 \).

In order to have Corollary 24 available, we consider a class \( \mathcal{V} \) of such spaces where the class of underlying vector spaces satisfies condition (III). Then, by Corollary 9 and Fact 26 below, we obtain the following.

**Corollary 25.** There is no algorithm which, given a 5-variable term \( t(\bar{x}) \) in the language of ortholattices, decides whether \( \exists \bar{x}. t(\bar{x}) = 1 \) is valid in the ortholattice \( \mathbb{L}^+(V) \) for some \( V \in \mathcal{V} \) with \( \dim V > 0 \).
Natural examples for $\mathcal{V}$ are the classes of all finite dimensional real, complex, and quaternionian, respectively, Hilbert spaces. Here, in contrast, deciding whether $\exists x \ t(\bar{x}) > 0$ holds in some $L^+(\mathcal{V})$ (“weak satisfiability”) is decidable (cf. [12]) and an upper complexity bound has been derived in [17].

**Fact 26.** Within the class of modular ortholattices, any conjunction $\pi(\bar{x})$ of equations is equivalent to an equation $t(\bar{x}) = 1$.

**Proof.** Observe that the following are equivalent for any given $x, y$: $x + x^\perp y^\perp = 1$; $x^\perp = x^\perp(x + x^\perp y^\perp)$; $x^\perp = xx^\perp + x^\perp y^\perp$ (by modularity); $x^\perp \leq y^\perp$; $y \leq x$. Thus, $s_j \leq t_j$ is equivalent to some $u_j = 1$ and $t_j \leq s_j$ to some $v_j = 1$; and $\bigwedge_j s_j = t_j$ to $\bigcap_j u_j \cap v_j = 1$. \hfill $\square$

9. **Grassmann-Cayley algebra**

Recall, that for a finite dimensional vector space $V$ the Grassmann-Cayley algebra $\text{GC}(V)$ (cf [20]) has, in particular, operations $\wedge$ and $\vee$ and terms built from them and 0, 1: the simple expressions. These operations are related to the lattice $L(V)$ as follows: 0, 1 are the bounds of $L(V)$, $A \wedge B = A \cap B$ if $A + B = V$ and $A \vee B = A + B$ if $A \cap B = 0$.

**Theorem 27.** Let $\mathcal{V}$ be a class of vector spaces which satisfies (III$_{16}$). There is no algorithm to decide for any given conjunction of equations $t_i(\bar{x}) = s_i(\bar{x})$, with simple expressions $t_i, s_i$, whether it admits a satisfying assignment within $\text{GC}(V)$ for some $V \in \mathcal{V}$, $V \neq 0$.

The proof needs some preparation. We consider lattices with bound 0, 1. For a term $t(\bar{x})$, call the assignment $\bar{x} \mapsto \bar{a}$ in $L$ admissible if, for any occurrence of subterms $s(\bar{x}), s_1(\bar{x}),$ and $s_2(\bar{x})$ in $t(\bar{x})$, the following hold in $L$:

- If $s(\bar{x}) = s_1(\bar{x}) + s_2(\bar{x})$ then $s_1(\bar{a}) \cap s_2(\bar{a}) = 0$;
- If $s(\bar{x}) = s_1(\bar{x}) \cap s_2(\bar{x})$ then $s_1(\bar{a}) + s_2(\bar{a}) = 1$.

We say that $\bar{a}$ is admissible for a conjunction $\pi(\bar{x})$ of equations if it is so for any subterm occurrence in $\pi(\bar{x})$. If, in addition, $L \models \pi(\bar{a})$ then we write $L \models_a \pi(\bar{a})$. For an $n$-frame $\bar{a}$ of a modular lattice $L$ and $i \neq j$ put

$$R_{ij}(L, \bar{a}) = \{x \in L \mid x \oplus a_j = a_i + a_j\}.$$ 

**Fact 28.** Fix $n \geq 3$.

(i) There is a conjunction $\varphi(\bar{z})$ of lattice equations such that $\bar{a}$ is a 4-frame of the modular lattice $L$ if and only if $L \models_a \varphi_n(\bar{a})$.

(ii) There are lattice terms $\otimes(x, y, \bar{z})$ and $\ominus(x, y, \bar{z})$ such that for any vector space $V$ and 4-frame $\bar{a}$ of $L(V)$ there is a (unique) isomorphism $\varepsilon_{\bar{a}} : a_1 \to a_2$ such that $\Gamma_{\bar{a}}(f) := \{v - \varepsilon_{\bar{a}}(f(v)) \mid v \in$
Proof of Theorem 27. Observe that the class C of \( \text{End}(V_1) \) where \( V_1 \cong V \in \mathcal{V} \) satisfies condition (II) for rings while \( C \subseteq \mathcal{Q} R_f \) follows from Fact 20. Thus, the claim follows from the reduction in Fact 28(iii) and Theorem 19 where \( C = \{ \text{End}(V_1) \mid V_1 \in L(V), \dim V = 4 \dim V_1 \} \). \( \square \)

Proof of Fact 28. (i) Consider a 4-frame of the modular lattice \( L \). Due to the first condition defining a frame, the assignment \( \bar{z} \mapsto \bar{a} \) is admissible for any term arising from \( \sum_{k \in K} z_k \) by insertion of brackets. It only remains to deal with the last condition. Observe that \( a_{ij} \cap a_{jk} \leq a_{ij} \cap (a_i + a_j) \cap (a_j + a_k) = a_{ij} \cap a_j = 0 \), \( b + a_{ij} + a_{jk} = 1 \) where \( b = \sum_{\ell \not= j} a_{\ell} \), and that the condition is equivalent to \( b \cap (a_{ij} + a_{jk}) = a_{ik} \).

(ii) Recall the approach in the proof of Lemma 6(iii)(a), in particular the terms \( \pi_i \). We write \( R_{ij} = R_{ij}(L, \bar{a}) \). The assignment \( x, \bar{z} \mapsto r, \bar{a} \) where \( r \in R_{ij} \) or \( r \in R_{ji} \) is admissible for \( \pi_i \) since \( r \cap a_{ik} \leq (a_i + a_j) \cap a_{ik} = 0 \) and \( \pi_i^2(r) = (r + a_{ik}) \cap b \) where \( b = \sum_{\ell \not= i} a_\ell \) and \( b + r + a_{ik} = 1 \). Also, observe that the map \( r \mapsto \pi_i^2(r) \) restricts to a bijection \( \pi_{ij}^{kl} \) of \( R_{ij} \) onto \( R_{kl} \) and to a bijection \( \pi_{ij}^{kl} \) of \( R_{ji} \) onto \( R_{jk} \). For \( r \in R_{12} \) and \( s \in R_{23} \) one has \( r \cap s \leq r \cap 2 = 0 \) and \( (r + s) \cap b = (r + s) \cap (a_1 + a_3), r + s + b = 1 \) where \( b = \sum_{i \not= 1, 3} a_i \). For \( r \in R_{12} \) define \( r_{ij} \) as in the proof of Lemma 6. Then the definition \( \otimes(s, r, \bar{a}) = (r_{12} + s_{23}) \cap b \) of multiplication is given by a term for which the \( x, y, \bar{z} \mapsto s, r, \bar{a} \) with \( r, s \in R_{12} \) are admissible. To obtain the same kind of term for difference, put

\[
r \ominus s = \left( [s_{13} + a_2 + a_3] \cap (r + a_{23}) \right) + a_3 + a_4 \cap (a_1 + a_2).
\]

Thus, the lattice terms and equations used in the proof of Lemma 6 and Lemma 7 can be modified to become admissible. That \( \Gamma_\bar{a} \) is an isomorphism (of rings), is shown by easy Linear Algebra calculations (cf. 21). (iii) follows, immediately. \( \square \)
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