Vector fields on $\Pi$-symmetric flag supermanifolds

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Abstract

The main result of this paper is the computation of the Lie superalgebras of holomorphic vector fields on the complex $\Pi$-symmetric flag supermanifolds, introduced by Yu.I. Manin. We prove that with one exception any vector field is fundamental with respect to the natural action of the Lie superalgebra $q_n(\mathbb{C})$.

1 Introduction

A $\Pi$-symmetric flag supermanifold is a subsupermanifold in a flag supermanifold in $\mathbb{C}^n|n$ that is invariant with respect to an odd involution in $\mathbb{C}^n|n$. This supermanifold possesses a transitive action of the linear classical Lie superalgebra $q_n(\mathbb{C})$, which belongs to one of two “strange series” in the Kac classification [Kac]. It turns out that with one exceptional case all global holomorphic vector fields are fundamental for this action of the Lie superalgebra $q_n(\mathbb{C})$. In the simplest case of super-Grassmannians the similar result was obtained in [Oni].

The main result of this paper was announced in [V4] and the idea of the proof was given in [V2]. The goal of this notes is to give a detailed proof. We also describe the connected component of the automorphism supergroup of this supermanifolds.

2 Flag supermanifolds

We will use the word “supermanifold” in the sense of Berezin and Leites [BL], see also [Oni] for details. Throughout, we will restrict our attention to the complex-analytic version of the theory of supermanifolds. Recall that a complex-analytic superdomain of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space of the form $(U_0, F_{U_0} \otimes \mathbb{C} \wedge(m))$, where $F_{U_0}$ is the sheaf of holomorphic functions on an open set $U_0 \subset \mathbb{C}^n$ and $\wedge(m)$ is the exterior (or Grassmann) algebra with $m$ generators. A complex-analytic supermanifold of dimension $n|m$ is a $\mathbb{Z}_2$-graded locally ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$. Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ be a supermanifold and $\mathcal{J}$ be the subsheaf of ideals generated by odd elements in $\mathcal{O}_\mathcal{M}$. We set $\mathcal{F}_{\mathcal{M}_0} := \mathcal{O}_\mathcal{M}/\mathcal{J}$. Then $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}_0})$ is a usual complex-analytic manifold, it is called the underlying space of $\mathcal{M}$. Usually we will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}_0})$.

In this paper we denote by $F_{k|l}^{m|n}$ a flag supermanifold of type $k|l$ in the vector superspace $\mathbb{C}^{m|n}$. Here we set $k = (k_1, \ldots, k_r)$ and $l = (l_1, \ldots, l_r)$ such that

$$0 \leq k_r \leq \ldots \leq k_1 \leq m, \quad 0 \leq l_r \ldots \leq l_1 \leq n \quad \text{and} \quad 0 < k_r + l_r < \ldots < k_1 + l_1 < m + n.\tag{1}$$

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A $\Pi$-symmetric flag supermanifold $\Pi F^{m,n}_{k|l}$ of type $k = (k_1, \ldots, k_r)$ in $\mathbb{C}^{n|n}$ is a certain subsupermanifold in $F^{m,n}_{k|l}$. Let us give an explicite description of these supermanifolds in terms of charts and local coordinates (see also [Man, V1, V3]).

Let us take two non-negative integers $m, n \in \mathbb{Z}$ and two sets of non-negative integers

$$k = (k_1, \ldots, k_r), \quad \text{and} \quad l = (l_1, \ldots, l_r)$$

such that (1) holds. The underlying space of the supermanifold $F^{m,n}_{k|l}$ is the product $F^{m}_k \times F^n_l$ of two manifolds of flags of type $k = (k_1, \ldots, k_r)$ and $l = (l_1, \ldots, l_r)$ in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. For any $s = 1, \ldots, r$ let us fix two subsets

$$I_{s0} \subset \{1, \ldots, k_{s-1}\} \quad \text{and} \quad I_{s1} \subset \{1, \ldots, l_{s-1}\},$$

where $k_0 = m$ and $l_0 = n$, such that $|I_{s0}| = k_s$, and $|I_{s1}| = l_s$. We set $I_s = (I_{s0}, I_{s1})$ and $I = (I_1, \ldots, I_r)$. Let us assign the following $(k_{s-1} + l_{s-1}) \times (k_s + l_s)$-matrix

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ H_s & Y_s \end{pmatrix}, \quad s = 1, \ldots, r,$$  \hspace{1cm} (2)

where $X_s = (x_{ij}^s) \in \text{Mat}_{k_{s-1} \times k_s}(\mathbb{C})$, $Y_s = (y_{ij}^s) \in \text{Mat}_{l_{s-1} \times l_s}(\mathbb{C})$, and elements of the matrices $\Xi_s = (\xi_{ij}^s), H_s = (\eta_{ij}^s)$ are odd. We also assume that $Z_{I_s}$ contains the identity submatrix $E_{k_s+l_s}$ of size $(k_s + l_s) \times (k_s + l_s)$ in the lines with numbers $i \in I_{s0}$ and $k_{s-1} + i$, $i \in I_{s1}$. For example in case

$$I_{s0} = \{k_{s-1} - k_s + 1, \ldots, k_{s-1}\} \quad \text{and} \quad I_{s1} = \{l_{s-1} - l_s + 1, \ldots, l_{s-1}\}$$

the matrix $Z_{I_s}$ has the following form:

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ E_{k_s} & 0 \\ H_s & Y_s \\ 0 & E_{k_s} \end{pmatrix}.$$

(For simplicity of notation we use here the same letters $X_s, Y_s, \Xi_s$ and $H_s$ as in (2).)

We see that the sets $I_0 = (I_{10}, \ldots, I_{r0})$ and $I_1 = (I_{11}, \ldots, I_{r1})$ determine the charts $U_{I_0}$ and $V_{I_1}$ in the flag manifolds $F^{m}_k$ and $F^n_l$, respectively. We can take the non-trivial elements (i.e., those not contained in the identity submatrix) from $X_s$ and $Y_s$ as local coordinates in $U_{I_0}$ and $U_{I_1}$, respectively. Summing up, we defined an atlas

$$\{U_I = U_{I_0} \times U_{I_1}\} \quad \text{on} \quad F^{m}_k \times F^n_l$$

with charts parametrized by $I = (I_s)$. In addition the sets $I_0$ and $I_1$ determine the super-domain $U_I$ with underlying space $\tilde{U}_I$ and with even and odd coordinates $x_{ij}^s, y_{ij}^s$ and $\xi_{ij}^s, \eta_{ij}^s$, respectively. (As above we assume that $x_{ij}^s, y_{ij}^s, \xi_{ij}^s, \eta_{ij}^s$ are non-trivial. That is they are not contained in the identity submatrix.) Let us define the transition functions between two superdomains corresponding to $I = (I_s)$ and $J = (J_s)$ by the following formulas:

$$Z_{J_1} = Z_1 C^1_{I_1 J_1}, \quad Z_{J_s} = C_{I_{s-1}, J_{s-1}} Z_{I_s} C^1_{I_s J_s}, \quad s \geq 2.$$  \hspace{1cm} (3)
Here $C_{I,s}$ is an invertible submatrix in $Z_{I,s}$ that consists of lines with numbers $i \in J_{s0}$ and $k_{s-1} + i$, where $i \in J_{s1}$. In other words, we choose the matrix $C_{I,s}$ in such a way that $Z_{I,s}$ contains the identity submatrix $E_{k_{s-1}+i}$ in lines with numbers $i \in J_{s0}$ and $k_{s-1} + i$, where $i \in J_{s1}$. These charts and transition functions define a supermanifold that we denote by $F_{kl}$. This supermanifold we will call the supermanifold of flags of type $k|l$. In case $r = 1$ this supermanifold is called the super-Grassmannian and is denoted by $\text{Gr}_{m|n,k|l}$ (see also [Oni, Man]).

Let us take $n \in \mathbb{N}$ and $k = (k_1, \ldots, k_r)$, such that

$$0 < k_1 < \ldots < k_r < n.$$  

We will define the supermanifold of $\Pi$-symmetric flags $\text{IF}_{n|n}^{n|n}$ of type $k$ in $\mathbb{C}^{n|n}$ as a certain subsupermanifold in $F_{kl}$. The underlying space of $\text{IF}_{n|n}^{n|n}$ is the diagonal in $F_{k}^{n} \times F_{n}^{k}$, that is clearly isomorphic to $F_{k}^{n}$. For any $s = 1, \ldots, r$ we fix a set $I_{s0} = I_{s1} \subset \{1, \ldots, k_{s-1}\}$, where $|I_{s0}| = k_s$ and $k_0 = n$. Consider the chart on $F_{k|k}$ corresponding to $I = (I_s)$, where $I_s = (I_{s0}, I_{s1})$. Such charts cover the diagonal in $F_{k}^{n} \times F_{n}^{k}$. Let us define the subsupermanifold of $\Pi$-symmetric flags in these charts by the equations $X_s = Y_s; \, \Xi_s = H_s$. It is easy to see that these equations are well-defined with respect to the transition functions (3). The coordinate matrices in this case have the following form

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ \Xi_s & X_s \end{pmatrix}, \quad s = 1, \ldots, r. \quad (4)$$

(Compare with (2).) As above even and odd local coordinates on $\text{IF}_{n|n}^{n|n}$ are non-trivial elements from $X_s$ and $\Xi_s$, respectively. The transition functions between two charts are defined again by formulas (3). We can consider the supermanifold $\text{IF}_{n|n}^{n|n}$ as the “set of fixed-point” of a certain odd involution $\Pi$ in $\mathbb{C}^{n|n}$ (see [Man]). In case $r = 1$, the supermanifold of $\Pi$-symmetric flags is called also the $\Pi$-symmetric super-Grassmannian. We will denote it by $\Pi\text{Gr}_{n|n,k|k}$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ be a complex-analytic supermanifold. Denote by $\mathcal{T} = \text{Der} (\mathcal{O}_{\mathcal{M}})$ the tangent sheaf or the sheaf of vector fields on $\mathcal{M}$. It is a sheaf of Lie superalgebras with respect to the multiplication $[X,Y] = YX - (-1)^{p(X)p(Y)}XY$. The global sections of $\mathcal{T}$ are called holomorphic vector fields on $\mathcal{M}$. They form a complex Lie superalgebra that we will denote by $\mathfrak{v} (\mathcal{M})$. This Lie superalgebra is finite dimensional if $\mathcal{M}_0$ is compact. The goal of this paper is to compute the Lie superalgebra $\mathfrak{v} (\mathcal{M})$ in the case when $\mathcal{M}$ is a supermanifold of $\Pi$-symmetric flags of type $k$ in $\mathbb{C}^{n|n}$.

We denote by $\mathfrak{q}_n (\mathbb{C})$ the Lie subsuperalgebra in $\mathfrak{gl}_{n|n} (\mathbb{C})$ that consists of the following matrices:

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad \text{where} \quad A, B \in \mathfrak{gl}_n (\mathbb{C}).$$

Denote by $\mathfrak{Q}_n (\mathbb{C})$ the Lie supergroup of $\mathfrak{q}_n (\mathbb{C})$. In [Man] an action of $\mathfrak{Q}_n (\mathbb{C})$ on the supermanifold $\text{IF}_{n|n}^{n|n}$ was defined. In our coordinates this action is given by the following formulas:

$$(L, (Z_{I_1}, \ldots, Z_{I_r})) \mapsto (\tilde{Z}_{I_1}, \ldots, \tilde{Z}_{I_r}), \quad \text{where}$$

$$L \in \mathfrak{Q}_n (\mathbb{C}), \quad \tilde{Z}_{I_1} = LZ_{I_1}C_1^{-1}, \quad \tilde{Z}_{I_s} = C_{s-1}Z_{I_s}C_s^{-1}. \quad (5)$$

3
Here $C_1$ is an invertible submatrix in $LZ_i$, that consists of lines with numbers $i$ and $n+i$, where $i \in J_1$; and $C_s$, $s \geq 2$, is an invertible submatrix in $C_{s-1}Z_i$, that consists of lines with numbers $i$ and $k_{s-1}+i$, where $i \in J_s$. This Lie supergroup action induces a Lie superalgebra homomorphism

$$
\mu : q_n(\mathbb{C}) \to \mathfrak{v}(\Pi F_{k|k}^{n|n}).
$$

In case $r = 1$ in [Oni, Proposition 5.5] it was proven that $\text{Ker } \mu = \langle E_{2n} \rangle$, where $E_{2n}$ is the identity matrix of size $2n$. In general case $r > 1$ the proof is similar. Hence, $\mu$ induces an injective homomorphism of Lie superalgebras $q_n(\mathbb{C})/\langle E_{2n} \rangle \to \mathfrak{v}(\Pi F_{k|k}^{n|n})$. We will show that with one exception this homomorphism is an isomorphism.

## 3 About superbundles

Recall that a morphism of a complex-analytic supermanifold $\mathcal{M}$ to a complex-analytic supermanifold $\mathcal{N}$ is a pair $f = (f_0, f^*)$, where $f_0 : \mathcal{M}_0 \to \mathcal{N}_0$ is a holomorphic map and $f^* : \mathcal{O}_\mathcal{N} \to (f_0)_*(\mathcal{O}_\mathcal{M})$ is a homomorphism of sheaves of superalgebras.

**Definition.** We say that a superbundle with fiber $\mathcal{S}$, base $\mathcal{B}$, total space $\mathcal{M}$ and projection $p = (p_0, p^*) : \mathcal{M} \to \mathcal{B}$ is given if there exists an open covering $\{U_i\}$ on $\mathcal{B}$ and isomorphisms $\psi_i : (p_0^{-1}(U_i), \mathcal{O}_\mathcal{M}) \to (U_i, \mathcal{O}_\mathcal{B}) \times \mathcal{S}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
(p_0^{-1}(U_i), \mathcal{O}_\mathcal{M}) & \xrightarrow{\psi_i} & (U_i, \mathcal{O}_\mathcal{B}) \times \mathcal{S} \\
\downarrow p & & \downarrow pr \\
(U_i, \mathcal{O}_\mathcal{B}) & \xrightarrow{id} & (U_i, \mathcal{O}_\mathcal{B})
\end{array}
$$

where $pr$ is the natural projection.

**Remark.** From the form of transition functions (3) it follows that for $r > 1$ the supermanifold $\Pi F_{k|k}^{n|n}$ is a superbundle with base $\Pi Gr_{n,k|k_1}$ and fiber $\Pi F_{k|k|k}$, where $k' = (k_2, \ldots, k_r)$. In local coordinates the projection $p$ is given by

$$(Z_1, Z_2, \ldots, Z_n) \mapsto (Z_1).$$

Moreover, the formulas (5) tell us that the projection $p$ is equivariant with respect to the action of the supergroup $Q_n(\mathbb{C})$ on $\Pi F_{k|k}^{n|n}$ and $\Pi Gr_{n,k|k_1|k'}$.

Let $p = (p_0, p^*) : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds. A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called projectable with respect to $p$, if there exists a vector field $v_1 \in \mathfrak{v}(\mathcal{N})$ such that

$$p^*(v_1(f)) = v(p^*(f)) \quad \text{for all } f \in \mathcal{O}_\mathcal{N}.$$  

In this case we say that $v$ is projected into $v_1$. Projectable vector fields form a Lie subsuperalgebra $\overline{\mathfrak{v}}(\mathcal{M})$ in $\mathfrak{v}(\mathcal{M})$. In case if $p$ is a projection of a superbundle, the homomorphism $p^* : \mathcal{O}_\mathcal{N} \to p_*(\mathcal{O}_\mathcal{M})$ is injective. Hence, any projectable vector field $v$ is projected into unique vector field $v_1 = P(v)$. The map

$$P : \overline{\mathfrak{v}}(\mathcal{M}) \to \mathfrak{v}(\mathcal{N}), \quad v \mapsto v_1$$
is a homomorphism of Lie superalgebras. A vector field \( v \in \mathfrak{v}(\mathcal{M}) \) is called \( \textit{vertical} \), if \( \mathcal{P}(v) = 0 \). Vertical vector fields form an ideal \( \text{Ker} \mathcal{P} \) in \( \mathfrak{v}(\mathcal{M}) \).

We will need the following proposition proved in [B].

**Proposition 1.** Let \( p : \mathcal{M} \to \mathcal{B} \) be the projection of a superbundle with fiber \( \mathcal{S} \). If \( \mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C} \), then any holomorphic vector field from \( \mathfrak{v}(\mathcal{M}) \) is projectable with respect to \( p \).

For any superbundle \( p : \mathcal{M} \to \mathcal{B} \) with fiber \( \mathcal{S} \) we define the sheaf \( \mathcal{W} \) on \( \mathcal{B}_0 \) in the following way. We assign to any open set \( U \subset \mathcal{B}_0 \) the set of all vertical vector fields on the supermanifold \( (p_0^{-1}(U), \mathcal{O}_\mathcal{M}) \). In [V1] the following statement was proven.

**Proposition 2.** Assume that \( \mathcal{S}_0 \) is compact. Then \( \mathcal{W} \) is a locally free sheaf of \( \mathcal{O}_\mathcal{B} \)-modules and \( \dim \mathcal{W} = \dim \mathfrak{v}(\mathcal{S}) \). The Lie algebra \( \mathcal{W}(\mathcal{B}_0) \) coincides with the ideal of all vertical vector fields in \( \mathcal{v}(\mathcal{M}) \).

Let us describe the corresponding to \( \mathcal{W} \) graded sheaf as in [V1]. Consider the following filtration in \( \mathcal{O}_\mathcal{B} \)

\[
\mathcal{O}_\mathcal{B} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \ldots
\]

where \( \mathcal{J} \) is the sheaf of ideals in \( \mathcal{O}_\mathcal{B} \) generated by odd elements. We have the corresponding graded sheaf of superalgebras

\[
\hat{\mathcal{O}}_\mathcal{B} = \bigoplus_{p \geq 0} (\hat{\mathcal{O}}_\mathcal{B})_p,
\]

where \( (\hat{\mathcal{O}}_\mathcal{B})_p = \mathcal{J}^p / \mathcal{J}^{p+1} \). Putting \( \mathcal{W}_p = \mathcal{J}^p \mathcal{W} \) we get the following filtration in \( \mathcal{W} \):

\[
\mathcal{W} = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \ldots \quad (6)
\]

We define the \( \mathbb{Z} \)-graded sheaf of \( \mathcal{F}_{\mathcal{B}_0} \)-modules by

\[
\hat{\mathcal{W}} = \bigoplus_{p \geq 0} \hat{\mathcal{W}}_p, \quad \text{where} \quad \hat{\mathcal{W}}_p = \mathcal{W}(p) / \mathcal{W}(p+1), \quad (7)
\]

where \( \mathcal{F}_{\mathcal{B}_0} \) is the structure sheaf of the underlying space \( \mathcal{B}_0 \). The \( \mathbb{Z}_2 \)-grading in \( \mathcal{W}_p \) induces the \( \mathbb{Z}_2 \)-grading in \( \hat{\mathcal{W}}_p \). Note that the natural map \( \mathcal{W}(p) \to \hat{\mathcal{W}}_p \) is even.

### 4 Functions on \( \Pi \)-symmetric flag supermanifolds

In this section we show that the superbundle described in Section 2, that is the \( \Pi \)-symmetric flag supermanifold, satisfies conditions of Proposition 1. Holomorphic functions on other flag supermanifolds was considered in [V3].

**Lemma 1.** Let \( \mathcal{M} \) be a superbundle with base \( \mathcal{B} \) and fiber \( \mathcal{S} \). Assume that \( \mathcal{O}_\mathcal{B}(\mathcal{B}_0) = \mathbb{C} \) and \( \mathcal{O}_\mathcal{S}(\mathcal{S}_0) = \mathbb{C} \). Then \( \mathcal{O}_\mathcal{M}(\mathcal{M}_0) = \mathbb{C} \).

In the Lie superalgebra \( \mathfrak{q}_n(\mathbb{C})_0 \simeq \mathfrak{gl}_n(\mathbb{C}) \) we fix the following Cartan subalgebra:

\[
t = \{ \text{diag}(\mu_1, \ldots, \mu_n) \},
\]

the following system of positive roots:

\[
\Delta^+ = \{ \mu_i - \mu_j, \ i < j \}
\]


and the following system of simple roots:

$$\Phi = \{\alpha_1, \ldots, \alpha_{n-1}\}, \quad \alpha_i = \mu_i - \mu_{i+1}.$$  

Denote by $t^* (\mathbb{R})$ a real subspace in $t^*$ spaned by $\mu_j$. Consider the scalar product $(\cdot, \cdot)$ in $t^* (\mathbb{R})$ such that the vectors $\mu_j$ form an orthonormal basis. An element $\gamma \in t^* (\mathbb{R})$ is called dominant if $(\gamma, \alpha) \geq 0$ for all $\alpha \in \Delta^+$. 

We need the Borel-Weyl-Bott Theorem (see for example [A] for details). Let $G \cong \text{GL}_n (\mathbb{C})$ be the underlying space of $Q_n (\mathbb{C})$, $P$ be a parabolic subgroup in $G$ and $R$ be the reductive part of $P$. Assume that $E_\varphi \rightarrow G/P$ is the homogeneous vector bundle corresponding to a representation $\varphi$ of $P$ in $E = (E_\varphi)_p$. Denote by $E_\varphi$ the sheaf of holomorphic section of this vector bundle.

**Theorem 1.** [Borel-Weyl-Bott] Assume that the representation $\varphi : P \rightarrow \text{GL}(E)$ is completely reducible and $\lambda_1, \ldots, \lambda_s$ are highest weights of $\varphi | R$. Then the $G$-module $H^0 (G/P, E_\varphi)$ is isomorphic to the sum of irreducible $G$-modules with highest weights $\lambda_{i_1}, \ldots, \lambda_{i_s}$, where $\lambda_{i_a}$ are dominant highest weights.

The main result of this section is the following theorem.

**Theorem 2.** Let $\mathcal{M} = \Pi \mathfrak{Gr}_{n|k}^{|n}$, then $\mathcal{O}_\mathcal{M} (\mathcal{M}_0) = \mathbb{C}$.

**Proof.** First consider the case $r = 1$. This is $\mathcal{M} = \Pi \mathfrak{Gr}_{n|k}$. Let us prove that $\tilde{\mathcal{O}}_\mathcal{M} (\mathcal{M}_0) = \mathbb{C}$, where $\tilde{\mathcal{O}}_\mathcal{M}$ is defined as in the previous section. We use the Borel-Weyl-Bott Theorem. The manifold $\mathcal{M}_0 = \mathfrak{Gr}_{n,k}$ is isomorphic to $G/H$, where $G = \text{GL}_n (\mathbb{C})$ and

$$H = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \bigg| A \in \text{GL}_{n-k} (\mathbb{C}), \ B \in \text{GL}_k (\mathbb{C}) \right\}. \quad (8)$$

The reductive part $R$ of $H$ is given by the equation $C = 0$. It is isomorphic to $\text{GL}_{n-k} (\mathbb{C}) \times \text{GL}_k (\mathbb{C})$. Let $\rho_1$ and $\rho_2$ be the standard representations of $\text{GL}_{n-k} (\mathbb{C})$ and $\text{GL}_k (\mathbb{C})$, respectively. Denote by $E \rightarrow \mathcal{M}_0$ the holomorphic vector bundle that is determined by the locally free sheaf $E = \mathcal{J} / \mathcal{J}^2$, where $\mathcal{J}$ is the sheaf of ideals generated by odd elements in $\mathcal{O}_\mathcal{M}$. In [Oni, Proposition 5.2] it was proven that $E$ is a homogeneous vector bundle corresponding to the representation $\varphi$ of $H$ such that

$$\varphi | R = \rho_1^* \otimes \rho_2.$$  

Since $(\tilde{\mathcal{O}}_\mathcal{M})_p \cong \Lambda^p E$, we have to find the vector space of global sections of $\Lambda^p E$. This vector bundle corresponds to the representation $\Lambda^p \varphi = \Lambda^p (\rho_1^* \otimes \rho_2)$. We need to find dominant weights of $\Lambda^p \varphi$.  

Let us choose the following Cartan subalgebra

$$t = \{ \text{diag}(\mu_1, \ldots, \mu_n) \}$$

in $\mathfrak{g} = \mathfrak{gl}_n (\mathbb{C}) = \text{Lie}(G)$. For $p > 0$ any weight of this representation has the following form:

$$\Lambda = -\mu_{i_1} - \cdots - \mu_{i_p} + \mu_{j_1} + \cdots + \mu_{j_p},$$

where

$$1 \leq i_1, \ldots, i_p \leq n-k \quad \text{and} \quad n-k+1 \leq j_1, \ldots, j_p \leq n.$$
For \( p = 0 \) the highest weight is equal to 0. The weight \( \Lambda = 0 \) is clearly dominant. For \( p > 0 \) assume that \( \mu^1 = \mu_{i_a} \), where \( i_a = \max\{i_1, \ldots, i_p\} \), and \( \mu^2 = \mu_{j_b} \), where \( j_b = \min\{j_1, \ldots, j_p\} \).

Then, \( (\Lambda, \mu^1 - \mu^2) < 0 \). Therefore, the weight \( \Lambda \) is not dominant and we have

\[
(\hat{O}_M)_0(M_0) = \mathbb{C} \quad \text{and} \quad (\hat{O}_M)_p(M_0) = \{0\} \quad \text{for} \quad p > 0.
\]

Now let us compute the space of global holomorphic functions \( O_M(M_0) \). Clearly, \( J^p(M_0) = \{0\} \) for large \( p \). For \( p \geq 0 \) we have an exact sequence

\[
0 \rightarrow J^{p+1}(M_0) \rightarrow J^p(M_0) \rightarrow (\hat{O}_M)_p(M_0).
\]

By induction we see that \( J^p(M_0) = \{0\} \) for \( p > 0 \). Hence, for \( p = 0 \) our exact sequence has the form:

\[
0 \rightarrow J^0(M_0) = O_M(M_0) \rightarrow \mathbb{C}.
\]

Note that on any supermanifold we have constant functions. Hence, \( O_M(M_0) = J^0(M_0) = \mathbb{C} \). Using Lemma 1 and induction, we get the result. \( \square \)

5 Vector fields on \( \Pi \)-symmetric flag supermanifolds

The Lie superalgebra of holomorphic vector fields on \( \Pi \)-symmetric super-Grassmannian \( \PiGr_{n|n,k|k} \) was computed in [Oni].

**Theorem 3.** Let \( M = \PiGr_{n|n,k|k} \) and \((n, k) \neq (2, 1)\). Then

\[
v(M) \simeq q_n(\mathbb{C})/\langle E_{2n} \rangle.
\]

Assume that \( M = \PiGr_{2|2,1|1} \). Then

\[
v(M) \simeq g \oplus \langle z \rangle.
\]

Here \( g = g_{-1} \oplus g_0 \oplus g_1 \) is a \( \mathbb{Z} \)-graded Lie superalgebra defined in the following way:

\[
g_{-1} = V, \quad g_0 = sl_2(\mathbb{C}), \quad g_1 = \langle d \rangle,
\]

where \( V = sl_2(\mathbb{C}) \) is the adjoint \( sl_2(\mathbb{C}) \)-module, \([g_0, g_1] = \{0\}\) and \([d, -] \) maps identically \( g_{-1} \) to \( g_0 \). Here \( z \) is the grading operator of the \( \mathbb{Z} \)-graded Lie superalgebra \( g \).

**Proof.** The first part of the proof was given in [Oni, Theorem 5.2]. The second part follows from [Oni, Theorem 4.2]. The proof is complete. \( \square \)

**Remark.** For completeness we describe the more general Onishchik result in our particular case \( M = \PiGr_{2|2,1|1} \). In local chart on \( \PiGr_{2|2,1|1} \)

\[
\begin{pmatrix}
x & \xi \\
1 & 0 \\
\xi & x \\
0 & 1
\end{pmatrix}
\]
the Lie superalgebra of vector fields $\mathfrak{v}(\Pi \text{Gr}_{2|2,1|1})$ has the following form:

$$\mathfrak{g}_1 = \langle \partial \partial_x, x^2 \partial \partial_x, x \partial \partial_x + \xi \partial \partial_x \rangle,$$

$$\mathfrak{g}_0 = \langle \partial \partial_x, x^2 \partial \partial_x + 2x \partial \partial_x \partial \partial_x + \xi \partial \partial_x \rangle,$$

$$d = \xi \partial \partial_x, \quad z = \xi \partial \partial_x.$$

We will need another description of the Lie superalgebra of vector fields on $\Pi \text{Gr}_{2|2,1|1}$. We have

$$\mathfrak{v}(\Pi \text{Gr}_{2|2,1|1}) \simeq q_2(\mathbb{C}) \langle E_4 \rangle \oplus \langle z \rangle$$

as $\mathfrak{g}_0$-modules. More precisely, in the local chart (9) the isomorphism is given by

$$(q_2(\mathbb{C}) \langle E_4 \rangle)_0 \simeq \mathfrak{g}_0 = \langle \partial \partial_x, x^2 \partial \partial_x + 2x \partial \partial_x \partial \partial_x + \xi \partial \partial_x \rangle,$$

$$(q_2(\mathbb{C}) \langle E_4 \rangle)_1 \simeq \langle \partial \partial_x, x \partial \partial_x + \xi \partial \partial_x, x^2 \partial \partial_x \partial \partial_x + \xi \partial \partial_x \rangle,$$

$$z = \xi \partial \partial_x.$$

From now on we use the following notations:

$$\mathcal{M} = \Pi \mathcal{F}_{k_1|k_1}, \quad \mathcal{B} = \Pi \text{Gr}_{n|k_1}^{k_1|k_1}, \quad \mathcal{S} = \Pi \mathcal{F}_{k_1|k_1}^{k_1|k_1},$$

where $k' = (k_2, \ldots, k_r)$. We also assume that $r > 1$. By Proposition 1 and Theorem 2 the projection of the superbundle $\mathcal{M} \to \mathcal{B}$ determines the homomorphism of Lie superalgebras

$$\mathcal{P} : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\mathcal{B}).$$

This projection is equivariant. Hence for the natural Lie superalgebra homomorphisms $\mu : q_n(\mathbb{C}) \to \mathfrak{v}(\mathcal{M})$ and $\mu_{\mathcal{B}} : q_n(\mathbb{C}) \to \mathfrak{v}(\mathcal{B})$ we have

$$\mu_{\mathcal{B}} = \mathcal{P} \circ \mu.$$

Note that for $r > 1$ the base $\mathcal{B}$ cannot be isomorphic to $\Pi \text{Gr}_{2|2,1|1}$. Therefore, by Theorem 3, the homomorphisms $\mu_{\mathcal{B}}$ and hence the homomorphism $\mathcal{P}$ is surjective. We will prove that $\mathcal{P}$ is injective. Hence,

$$\mu = \mathcal{P}^{-1} \circ \mu_{\mathcal{B}}$$

is surjective and

$$\mathfrak{v}(\mathcal{M}) \simeq q_n(\mathbb{C}) \langle E_{2n} \rangle.$$

Let us study $\text{Ker} \mathcal{P} \subset \mathfrak{v}(\mathcal{M})$. In previous sections we constructed a locally free sheaf $\mathcal{W}$ on $\mathcal{B}_0$. We have a natural action of $G = \text{GL}_n(\mathbb{C})$ on the sheaf $\mathcal{W}$ that preserves the filtration (6) and induces the action on the sheaf $\mathcal{W}$. Hence, the vector bundle $\mathcal{W}_0 \to \mathcal{B}_0$ corresponding to $\mathcal{W}_0$ is homogeneous. We use notations from the proof of Theorem 2. Let us compute the representation of $H \subset G$ in the fiber of $\mathcal{W}_0$ over the point $o = H \in \mathcal{B}_0$. We will identify $(\mathcal{W}_0)_o$ with the Lie superalgebra of vector fields $\mathfrak{v}(\mathcal{S})$ on $\mathcal{S}$. 

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Consider a local chart that contains \( o \) on the \( \Pi \)-symmetric super-Grassmannian \( \mathcal{B} \). For example we can take the chart corresponding to \( I_{i0} = \{ n - k_1 + 1, \ldots, n \} \). The coordinate matrix (4) in this case has the following form

\[
Z_{I_1} = \begin{pmatrix}
X_1 & \Xi_1 \\
E_{k_1} & 0 \\
\Xi_1 & X_1 \\
0 & E_{k_1}
\end{pmatrix}.
\]

(10)

Let us choose an atlas of \( \mathcal{M} \) in a neighborhood of \( o \) defined by certain \( I_{i0}, \ s = 2, \ldots, r, \) see (4). In notations (8) and (10) the group \( H \) acts on \( Z_{I_1} \) in the following way:

\[
\begin{pmatrix}
A & 0 & 0 & 0 \\
C & B & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & C & B
\end{pmatrix} Z_{I_1} = \begin{pmatrix}
AX_1 & A\Xi_1 \\
CX_1 + B & C\Xi_1 \\
A\Xi_1 & AX_1 \\
C\Xi_1 & CX_1 + B
\end{pmatrix}.
\]

Hence, for \( Z_{I_2} \) we have

\[
\begin{pmatrix}
CX_1 + B & C\Xi_1 \\
C\Xi_1 & CX_1 + B
\end{pmatrix} \begin{pmatrix}
X_2 & \Xi_2 \\
\Xi_2 & X_2
\end{pmatrix} = \begin{pmatrix}
BX_2 + CX_1X_2 + C\Xi_1\Xi_2 & B\Xi_2 + CX_1\Xi_2 + C\Xi_1X_2 \\
B\Xi_2 + CX_1\Xi_2 + C\Xi_1X_2 & BX_2 + CX_1X_2 + C\Xi_1\Xi_2
\end{pmatrix}.
\]

(11)

Note that the local coordinates \( Z_{I_2}, \ s \geq 2, \) determine the local coordinate on \( S \). To obtain the action of \( H \) in the fiber \( (\mathbf{W}_0)_o \) in these coordinates we put \( X_1 = 0, \ \Xi_1 = 0 \) in (11) and modify \( Z_{I_1}, \ s \geq 3, \) accordingly. We see that the nilradical of \( H \) and the subgroup \( \text{GL}_{n-k_1}(\mathbb{C}) \) of \( R \) act trivially on \( S \) and that the subgroup \( \text{GL}_{k_1}(\mathbb{C}) \subset R \) acts in the natural way. This means that \( H \) acts as the even part of the Lie supergroup \( Q_{k_1}(\mathbb{C}) \) on \( \Pi \)-symmetric flag supermanifold \( S \), see (5).

Furthermore, by induction we assume that

\[
\mathbf{v}(S) \simeq q_{k_1}(\mathbb{C})/\langle E_{2n} \rangle \quad \text{or} \quad \mathbf{v}(S) \simeq q_2(\mathbb{C})/\langle E_4 \rangle \oplus \langle z \rangle.
\]

Then the induced action of \( \text{GL}_{k_1}(\mathbb{C}) \) on \( \mathbf{v}(S) \) coincides with the adjoint action of the even part of \( Q_{k_1}(\mathbb{C}) \). Standard computations lead to the following lemma, where we denote by \( \text{Ad}_{k_1} \) the adjoint representation of \( \text{GL}_{k_1}(\mathbb{C}) \) on \( \mathfrak{s}_n(\mathbb{C}) \) and 1 is the one dimensional trivial representation of \( \text{GL}_{k_1}(\mathbb{C}) \).

**Lemma 2.** The representation \( \psi \) of \( H \) in the fiber \( (\mathbf{W}_0)_o = \mathbf{v}(S) \) is completely reducible. If \( \mathbf{v}(S) \simeq q_{k_1}(\mathbb{C})/\langle E_{2n} \rangle \), then

\[
\psi|_{\mathbf{v}(S)} = \text{Ad}_{k_1}, \quad \psi|_{\mathbf{v}(S)_1} = \text{Ad}_{k_1} + 1.
\]

(12)

If \( \mathbf{v}(S) \simeq q_2(\mathbb{C})/\langle E_4 \rangle \oplus \langle z \rangle, \) then

\[
\psi|_{\mathbf{v}(S)} = \text{Ad}_{k_1} + 1, \quad \psi|_{\mathbf{v}(S)_1} = \text{Ad}_{k_1} + 1.
\]

(13)

Further we will use the chart on \( \Pi \Pi \Pi \Pi_{k|k}^{n|n} \) defined by \( I_{i0} \), where \( I_{i0} \) is as above, and

\[
I_{i0} = \{ k_{s-1} - k_s + 1, \ldots, k_{s-1} \}
\]
for \( s \geq 2 \). The coordinate matrix of this chart have the following form

\[
Z_{I_s} = \begin{pmatrix}
X_s & \Xi_s \\
E_{k_s} & 0 \\
\Xi_s & X_s \\
0 & E_{k_s}
\end{pmatrix},
\]

where again the local coordinate are \( X_s = (x^s_{ij}) \) and \( \Xi_s = (\xi^s_{ij}) \). We denote this chart by \( U \).

**Lemma 3.** The following vector fields in \( U \)

\[
\frac{\partial}{\partial x^1_{ij}}, \frac{\partial}{\partial \xi^1_{ij}}, u_{ij} + \frac{\partial}{\partial x^2_{ij}}, v_{ij} + \frac{\partial}{\partial \xi^2_{ij}}
\]

are fundamental. This is they are induced by the natural action of \( Q_n(\mathbb{C}) \) on \( M \). Here \( u_{ij} \) and \( v_{ij} \) are vector field that depend only on coordinates from \( Z_{I_1} \).

**Proof.** Let us prove this statement for example for the vector field \( \frac{\partial}{\partial x^1_{11}} \). This vector field corresponds to the one-parameter subgroup \( \exp(tE_{1,n-k_1+1}) \). Indeed, the action of this subgroup is given by

\[
\begin{pmatrix}
X_1 & \Xi_1 \\
E_{k_1} & 0 \\
\Xi_1 & X_1 \\
0 & E_{k_1}
\end{pmatrix} \mapsto \begin{pmatrix}
\tilde{X}_1 & \Xi_1 \\
E_{k_1} & 0 \\
\Xi_1 & \tilde{X}_1 \\
0 & E_{k_1}
\end{pmatrix}
\]

and \( Z_{I_s} \mapsto Z_{I_s}, s \geq 2 \),

where

\[
\tilde{X}_1 = \begin{pmatrix}
t + x^1_{11} & \cdots & x^1_{1k_1} \\
\vdots & \ddots & \vdots \\
x^1_{n-k_1+1} & \cdots & x^1_{n-k_1,k_1}
\end{pmatrix}
\]

Let us choose a basis \((v_q)\) of \( \mathfrak{v}(S) \). In [VI] it was proven that any holomorphic vector field on \( M \) can be written uniquely in the form

\[
w = \sum_q f_q v_q,
\]

where \( f_q \) are holomorphic functions on \( U \) depending only on coordinates from \( Z_{I_1} \). Further, we will need the following lemma:

**Lemma 4.** Assume that \( \ker P \neq \{0\} \). Then there exists a vector field \( w \in \ker P \setminus \{0\} \), such that \( w = \sum_q f_q v_q \), where \( f_q \) are holomorphic functions depending only on even coordinates from \( Z_{I_1} \).

**Proof.** Assume that in (14) there is a non-trivial vector field \( w \) such that a function \( f_q \) depends for example on \( \xi^1_{ij} \). Then \( w = \xi^1_{ij} w' + w'' \), where \( w' \) and \( w'' \) are vertical vector fields and their coefficients (14) do not depend on \( \xi^1_{ij} \), and \( w' \neq 0 \). Using Lemma 3 and the fact that \( \ker P \) is an ideal in \( \mathfrak{v}(M) \), we see that

\[
w' = [w, \frac{\partial}{\partial \xi^1_{ij}}] \in \ker P.
\]
Hence, we can exclude all odd coordinates $\xi^{1}_{ij}$. □

**Corollary.** We have

$$(\ker \mathcal{P} \neq \{0\}) \implies (\mathcal{W}_{0}(\mathcal{B}_{0}) \neq \{0\}).$$

We will need the following well-known construction for holomorphic homogeneous vector bundles. Let us take any homogeneous vector bundle $\mathbf{E}$ over $G/H$ and $x_{0} = H$. Assume that $v_{x_{0}} \in E_{x_{0}}$ is an $H$-invariant. We can construct the $G$-invariant section of $\mathbf{E}$ corresponding to $v_{x_{0}}$ in the following way. We set

$$v_{x_{1}} := g \cdot v_{x_{0}} \in E_{x_{1}}, \quad \text{where} \quad x_{1} = gx_{0}.$$  

Clearly this definition does not depend on $g \in G$ such that $x_{1} = gx_{0}$. Indeed, assume that $x_{1} = g_{1}x_{0}$ and $x_{1} = g_{2}x_{0}$. Then $g_{2} = g_{1}h$, where $h \in H$. Hence,

$$g_{2}(v_{x_{0}}) = (g_{1}h)(v_{x_{0}}) = g_{1}(v_{x_{0}}).$$

We use this construction to express locally the $G$-invariant section of $\mathbf{W}_{0}$ corresponding to an $H$-invariant in $(\mathbf{W}_{0})_{x_{0}}$. Let us take the following element $g \in G$:

$$g = \begin{pmatrix} E_{n-k_{1}} & A \\ 0 & E_{k_{1}} \end{pmatrix} \times \begin{pmatrix} E_{n-k_{1}} & A \\ 0 & E_{k_{1}} \end{pmatrix},$$

where $A$ is any complex matrix. Then $g$ acts on $\Pi^{n|n}_{k_{1}k}$ in the following way:

$$\begin{pmatrix} E_{n-k_{1}} & A & 0 & 0 \\ 0 & E_{k_{1}} & 0 & 0 \\ 0 & 0 & E_{n-k_{1}} & A \\ 0 & 0 & 0 & E_{k_{1}} \end{pmatrix} \begin{pmatrix} X^{1} & \Xi^{1} \\ E_{k_{1}} & 0 \\ \Xi^{1} & X^{1} \\ 0 & E_{k_{1}} \end{pmatrix} = \begin{pmatrix} X^{1} + A & \Xi^{1} \\ E_{k_{1}} & 0 \\ \Xi^{1} & X^{1} + A \\ 0 & E_{k_{1}} \end{pmatrix},$$

$$Z_{l_{s}} = Z_{l_{s}}, \quad s > 1.$$  

(15)

We see that $x_{0} = H$ has coordinates $X_{1} = \Xi_{1} = 0$. Clearly, $\{x_{1} \mid x_{1} = gx_{0}\}$ is an open set in $\mathcal{B}_{0}$. Moreover, element $g$ does not modify fiber coordinates. Therefore, the corresponding to an $H$-invariant $v_{x_{0}} \in (\mathbf{W}_{0})_{x_{0}}$ section $v$ is the constant section $x_{1} \mapsto v_{x_{0}}$ over the open set $\{x_{1} \mid x_{1} = gx_{0}\}$.

**Theorem 4.** Assume that $r > 1$ and $v(S) \simeq q_{k_{1}}(\mathbb{C})/(E_{2k_{1}})$. Then $\ker \mathcal{P} = \{0\}$ and

$$v(\Pi^{n|n}_{k_{1}k}) \simeq q_{n}(\mathbb{C})/(E_{2n}).$$

**Proof.** First let us compute the vector space of global sections of $\mathbf{W}_{0}$. As in Theorem 2, we use the Borel-Weyl-Bott Theorem. The representation $\psi$ of $H$ in $(\mathbf{W}_{0})_{o}$ is described in Lemma 2. From (12) it follows that the highest weights of $\psi$ have the form:

$$\mu_{n-k_{1}+1} - \mu_{n} \quad \times 2 \quad \text{and} \quad 0.$$

The first highest weight is not dominant because by definition of $\Pi$-symmetric flag supermanifolds $k_{1} < n$. The second highest weight is clearly dominant. Therefore, the vector
space of global sections of $W_0$ is the irreducible $G$-module with highest weight 0. Therefore, $W_0(B_0) \simeq \mathbb{C}$.

Let $v_0$ be the $H$-invariant element form $v(S)$. It is defined by the following one-parameter subsupergroup

$$\beta(\tau) = \left( \begin{array}{cc} E_{k_1} & \tau E_{k_1} \\ \tau E_{k_1} & E_{k_1} \end{array} \right)$$

in the Lie supergroup $Q_{k_1}$. In our chart we have

$$v_0 = 2 \sum_{ij} \xi_{ij}^2 \frac{\partial}{\partial x_{ij}} + u,$$

where $u$ is a vector field depending only on coordinates from $Z_{l_s}$, $s \geq 3$. Above we have seen that the corresponding to $v_0$ global section is a constant section. Therefore, the unique global section $v$ of $W_0$ in our chart has also the form (16).

Assume that $\text{Ker} \mathcal{P} \neq \{0\}$ and $w \in \text{Ker} \mathcal{P}/\{0\}$ is as in Lemma 4. Clearly the vector fields $w$ and $\alpha(w)$, where $\alpha : W \to W_0$, have the same form in our chart. Therefore, $w = av$ for some $a \in \mathbb{C}$. Furthermore,

$$\alpha([w, v_{ij} + \frac{\partial}{\partial \xi_{ij}^2}]) = 2a \frac{\partial}{\partial x_{ij}}.$$

The commutator of these vector fields is an even vector field and it belongs to $\text{Ker} \mathcal{P}$. Hence,

$$\alpha([w, v_{ij} + \frac{\partial}{\partial \xi_{ij}^2}]) = 0,$$

because $W_0$ has no global even sections. Therefore, $a = 0$ and the proof is complete. □

Now consider the case when the fiber of superbundle $\mathcal{M}$ is isomorphic to $\Pi \text{Gr}_{2|2,1\frac{1}{1}}$. 

**Theorem 5.** Assume that $r = 2$ and $S = \Pi \text{Gr}_{2|2,1\frac{1}{1}}$. Then

$$v(\Pi F_n^{3|n}) \simeq q_n(\mathbb{C})/\langle E_{2n} \rangle.$$

**Proof.** As in Theorem 4, let us compute the space of global sections of $W_0$. From (13) it follows that the highest weights of $\psi$ have the form:

$$\mu_{n-k_1+1} - \mu_n \quad (\times 2) \quad \text{and} \quad 0 \quad (\times 2).$$

By the Borel-Weyl-Bott Theorem we get

$$W_0(B_0) \simeq W_0(B_0) \simeq \mathbb{C}.$$

A basic section of $W_0(B_0)$ was obtained in Theorem 4. It has the form $v = 2\xi_{11}^2 \frac{\partial}{\partial \xi_{11}}$ in our case. Furthermore, we can take $z = \xi_{11}^2 \frac{\partial}{\partial \xi_{11}}$ as a basic element of 1-dimensional vector subspace in $v(S)_0$ corresponding to the trivial representation 1. Again in our local chart the unique even section of $W_0(B_0)$ has locally the form $s = \xi_{11}^2 \frac{\partial}{\partial \xi_{11}}$. 

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Let us take the vector field $w$ as in Lemma 4. The vector fields $\alpha(w)$ and $w$ have the same form in our chart. Hence, $w = av + bs$, where $a, b \in \mathbb{C}$. Furthermore,

$$\alpha([w, v_{11} + \frac{\partial}{\partial \xi_{11}^2}]) = \alpha(2a \frac{\partial}{\partial x_{11}^2} + b \frac{\partial}{\partial \xi_{11}^2}) = 2a \frac{\partial}{\partial x_{11}^2} + b \frac{\partial}{\partial \xi_{11}^2} \in \mathcal{W}_0(\mathcal{B}_0) = \langle v, z \rangle.$$

In other words, $2a \frac{\partial}{\partial x_{11}^2} + b \frac{\partial}{\partial \xi_{11}^2}$ must be a linear combination of $v$ and $z$. Hence, $a = b = 0$, and the proof is complete. □

By induction we get our main result:

**Theorem 6.** Assume that $r > 1$. Then $\mathfrak{v}(\Pi F_{k|k}^{n|n}) \simeq \mathfrak{q}_n(\mathbb{C})/\langle E_{2n} \rangle$.

## 6 The automorphism supergroup of $\Pi F_{k|k}^{n|n}$

Our main result has infinitesimal nature. However we can determine the connected component of the automorphism supergroup of $\Pi F_{k|k}^{n|n}$. Let us discuss this statement in details.

Let us take a complex-analytic supermanifold $\mathcal{M}$ with a compact underlying space $\mathcal{M}_0$. Then the Lie superalgebra of holomorphic vector fields $\mathfrak{v}(\mathcal{M})$ is finite dimensional. Denote by $\text{Aut}(\mathcal{M})_0$ the Lie group of even global automorphisms of $\mathcal{M}$. (The fact that $\text{Aut}(\mathcal{M})_0$ is a complex-analytic Lie group with the Lie algebra $\mathfrak{v}(\mathcal{M})_0$ was proven in [BK].) Moreover, we have a natural holomorphic action of $\text{Aut}(\mathcal{M})_0$ on $\mathcal{M}$ (see [BK]) and hence on $\mathfrak{v}(\mathcal{M})$. Therefore, the pair $(\text{Aut}(\mathcal{M})_0, \mathfrak{v}(\mathcal{M}))$ is a super Harish-Chandra pair. (See citeViLieSuper-group for the definition of a super Harish-Chandra pair.) Using the equivalence of complex super Harish-Chandra pairs and complex Lie supergroups obtained in [V5] we determine the complex Lie supergroup $\text{Aut}(\mathcal{M})$. We call this Lie supergroup the *automorphism supergroup* of $\mathcal{M}$.

Consider the case $\mathcal{M} = \Pi F_{k|k}^{n|n}$. Above we described a holomorphic action of $\text{GL}_n(\mathbb{C}) = Q_n(\mathbb{C})_0$ on $\mathcal{M}$. In other words we have a homomorphism of Lie groups

$$Q_n(\mathbb{C})_0 \rightarrow \text{Aut}(\mathcal{M})_0.$$  \hspace{1cm} (17)

This homomorphism induces (almost always, see Theorems 3 and 6) the isomorphism of Lie algebras $\mathfrak{q}_n(\mathbb{C})_0/\langle E_{2n} \rangle$ and $\mathfrak{v}(\mathcal{M})_0$, see Theorems 3 and 6. In Section 2 we have seen that the kernel of the homomorphism (17) is equal to $\{\alpha E_{2n}\}$, where $\alpha \neq 0$, or to the center $\mathcal{Z}(Q_n(\mathbb{C})_0)$ of $Q_n(\mathbb{C})_0$. Therefore, the connected component of the automorphism supergroup $\text{Aut}^0(\Pi F_{k|k}^{n|n})$ is determined by the super Harish-Chandra pair

$$(Q_n(\mathbb{C})_0/\mathcal{Z}(Q_n(\mathbb{C})_0), \mathfrak{q}_n(\mathbb{C})/\langle E_{2n} \rangle).$$

In other words,

$$\text{Aut}^0(\Pi F_{k|k}^{n|n}) \simeq Q_n(\mathbb{C})/\mathcal{Z}(Q_n(\mathbb{C})).$$

In case $\mathcal{M} = \Pi \text{Gr}_{2|2,1|1}$, the connected component of the automorphism supergroup $\text{Aut}^0(\Pi \text{Gr}_{2|2,1|1})$ is given by the following super Harish-Chandra pair:

$$(Q_2(\mathbb{C})_0/\mathcal{Z}(Q_2(\mathbb{C})_0) \times \mathbb{C}^*, \mathfrak{v}(\Pi \text{Gr}_{2|2,1|1}));$$

see Theorem 3 for a description of $\mathfrak{v}(\Pi \text{Gr}_{2|2,1|1})$. 

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