Solving Problems with Inconsistent Constraints with a Modified Augmented Lagrangian Method

Martin P. Neuenhofen\textsuperscript{1} and Eric C. Kerrigan\textsuperscript{2}

Abstract—We present a numerical method for the minimization of constrained optimization problems where the objective is augmented with large quadratic penalties of inconsistent equality constraints. Such objectives arise from quadratic integral penalty methods for the direct transcription of optimal control problems. The Augmented Lagrangian Method (ALM) has a number of advantages over the Quadratic Penalty Method (QPM). However, if the equality constraints are inconsistent, then ALM might not converge to a point that minimizes the bias of the objective and penalty term. Therefore, we present a modification of ALM that fits our purpose. We prove convergence of the modified method and bound its local convergence rate by that of the unmodified method. Numerical experiments demonstrate that the modified ALM can minimize certain quadratic penalty-augmented functions faster than QPM, whereas the unmodified ALM converges to a minimizer of a significantly different problem.

I. INTRODUCTION

A. Problem Statement

This paper describes and analyzes a modified augmented Lagrangian method (MALM) for the numerical solution of a quadratic penalty program:

\[
\min_{x \in \mathcal{B}} \Phi_\omega(x) := f(x) + \frac{1}{2\omega}\|c(x)\|_2^2, \quad \text{(QPP)}
\]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}, c : \mathbb{R}^n \rightarrow \mathbb{R}^m, g : \mathbb{R}^n \rightarrow \mathbb{R}^p \) are possibly non-convex and nonlinear functions; \( \omega \) is meant for each vector component; \( \mathcal{B} := \{ x \in \mathbb{R}^n | g(x) \geq 0 \} \) is the feasible set; \( m, n, p \in \mathbb{N} \) are dimensions; \( \omega \in \mathbb{R}_{>0} \) is part of the problem data.

We define the associated Lagrangian function \( \mathcal{L}(x, \lambda, \eta) := f(x) - \lambda^T c(x) - \eta^T g(x) \), with Lagrange multipliers \( \lambda \in \mathbb{R}^m, \eta \in \mathbb{R}^p_{\geq 0} \). We explain the meaning of a Lagrange multiplier \( \lambda \) below.

1) Relation to Constrained Programs (CP): When \( \omega > 0 \) is close to zero then the penalty forces \( c(x) \approx 0 \), provided such a point exists. Hence, the problem may be considered to be related to:

\[
\min_{x \in \mathbb{R}} f(x) \quad \text{s.t.} \quad c(x) = 0 \quad \text{(CP)}
\]

In (CP), \( c, g \) are equality and inequality constraint functions with Lagrange multipliers \( \lambda \in \mathbb{R}^m, \eta \in \mathbb{R}^p_{\geq 0} \).

2) Inconsistency: (CP) only makes sense when \( c(x) = 0 \) is consistent. However, for the scope of this work we are particularly interested in the case when \( c \) is inconsistent. Experiments show that for inconsistent \( c \) the solution of (QPP) depends significantly on the value of \( \omega \); cf. Section II-B.2 and Figure I.

3) Optimality Conditions: From [14, Thm 12.1]:

\[
\nabla f(x) - \nabla c(x) \cdot \frac{1}{\omega} \cdot c(x) - \nabla g(x) \cdot \eta = 0 \quad \text{(KKT1)}
\]

\[
\begin{align*}
& g_i(x) = 0 \quad \text{and} \quad \eta_i \geq 0 & & \forall i \in \mathcal{A} \quad \text{(KKT2a)} \\
& g_i(x) > 0 \quad \text{and} \quad \eta_i = 0 & & \forall i \notin \mathcal{A} \quad \text{(KKT2b)}
\end{align*}
\]

where \( \mathcal{A} \subseteq \{1, \ldots, p\} \) is the active set, and \( g_i \) is the \( i \)-th component of the vector \( g(x) \).

Substituting \( \lambda = \frac{1}{\omega} c(x) \), we can re-express (KKT1):

\[
\nabla x \mathcal{L}(x, \lambda, \eta) = 0, \quad c(x) + \omega \cdot \lambda = 0 \quad \text{(KKT1)}
\]

(KKT) (i.e., (KKT1) and (KKT2)) determines \( x, \lambda, \eta \). (KKT) are the optimality conditions of (QPP) when \( \omega > 0 \) and the optimality conditions of (CP) when \( \omega = 0 \).

B. Motivation

1) Necessity of Tailored Solvers for (QPP): Minimizing (QPP) directly appears natural but, unless \( c \) is affine, will result in many iterations. This is caused by the bad scaling of the penalties.

As a demonstration, consider the instance

\[
\begin{align*}
& f(x) := -x_1, \quad g(x) := \begin{bmatrix} x_1 \\ x_2 - x_1 \end{bmatrix} \in \mathbb{R}^2 \quad \text{(1a)} \\
& c(x) := \begin{bmatrix} (x_1 + \varepsilon)^2 + x_2^2 - 2 \\ (x_1 - \varepsilon)^2 + x_2^2 - 2 \end{bmatrix} \in \mathbb{R}^2 \quad \text{(1b)}
\end{align*}
\]

with primal and dual initial guesses \( x_0 := [2 \quad 1]^T \) and \( \lambda_0 := 0 \), for \( \varepsilon = 0 \). We discuss later with Table I that minimization of (QPP) of I with a direct minimization method takes 134 iterations when \( \omega = 10^{-6} \). This is inefficient when compared to our later proposed MALM, which solves the same instance in only 39 iterations.

2) Relevant Instances of (QPP): Integral penalty methods [1], [8], [13] are an alternative to collocation methods for solving dynamic optimization problems. Integral penalty methods can solve dynamic optimization
problems with singular arcs and high-index differential-algebraic path-constraints as a problem of form \((\text{QPP})\). Consider the bang-singular example

\[
\begin{align*}
\min_{y,u} & \quad J := \int_0^5 (y(t)^2 + tu(t)) \, dt, \\
\text{s.t.} & \quad y(0) = 0.5, \quad \dot{y}(t) = \frac{1}{2}y(t)^2 + u(t), \quad (\text{OCP}) \\
& \quad y(t), u(t) \in [-1,1] \quad \forall t \in [0,5].
\end{align*}
\]

In integral-penalty-methods, the idea is to force \(y(t)^2/2 + u - y = 0\) not only at collocation points, but instead add an integral penalty \(r = \int_0^5 \|y^2/2 + u - y\|^2_2 \, dt\) to the objective.

Consider using continuous piecewise linear finite elements \(y_h\) for \(y\) and discontinuous ones \(u_h\) for \(u\) on a uniform mesh of \(N \in \mathbb{N}\) intervals (mesh size \(h = 5/N\)), represented with \(x := [y_h(0), \ldots, y_h(Nh), u_h^+(0), u_h^-(0), u_h^+(h), \ldots, u_h^-(Nh)]^T \in \mathbb{R}^n\), \(n := 3N\). \(y_h(0) = 0.5\) is fixed and removed from \(x\). We can minimize a quadrature approximation of \(J + \frac{1}{2^2}r\) by solving an instance of \((\text{QPP})\), where

\[
\begin{align*}
& f(x) := \sum_{j=1}^{N_q} \alpha_j \left(y_h(\tau_j)^2 + \tau_j u_h(\tau_j)\right) \\
& c(x) := \left[\sqrt{\alpha_j(y_h(\tau_j)^2/2 + u_h(\tau_j) - y_h(\tau_j))} \right] \in \mathbb{R}^m \\
& g(x) := \begin{bmatrix} 1 - x \\ 1 + x \end{bmatrix} \in \mathbb{R}^p
\end{align*}
\]

with \(q\) quadrature points \(\tau_j\) and weights \(\alpha_j > 0\) per mesh-interval. \(\omega \in \mathbb{R}_{>0}\) is ideally chosen in \(O(1/N)\) [8], [11]. Figure 1 plots numerical solution \(y_h, u_h\) for different values of \(\omega\) against the analytic solution. The numerical solutions are vastly different for different \(\omega\). Problem \((\text{CP})\) is infeasible for (2) because \(c(x) \neq 0\) \(\forall x \in \mathcal{B}\).

In conclusion: Problems \((\text{QPP})\) and \((\text{CP})\) have different solutions. Solutions of \((\text{QPP})\) depend on \(\omega\). Due to page limitations, for a discussion on integral penalty methods, implementation, and choice of \(\omega\), we refer to [1], [8], [13]. The experiments in [11], [13] present singular-arc and differential-algebraic optimal control problems where collocation methods fail to converge, but integral penalty methods converge.

C. Literature Review

We saw in Section I-B.1 that straightforward numerical minimization of \((\text{QPP})\) is inefficient due to bad scaling when \(\omega\) is close to zero, hence necessitating tailored algorithms.

1) Quadratic Penalty Method (QPM): QPMs compensate for the bad scaling by iteratively minimizing a sequence of problems \((\text{QPP})\). Therein, \(\omega\) is replaced by a sequence of values \(\{\rho_k\}_{k \in \mathbb{N}_0}\) that converges to \(\omega\) from above. Due to page limitations, we refer to [5], [6] for details. Actually, these methods have been proposed for problem \((\text{CP})\), i.e. when \(\omega = 0\); but they can also be used for \((\text{QPP})\). This is so because QPMs solve penalty problems of form \((\text{QPP})\). QPMs can converge slowly due to bad scaling [10].

2) Augmented Lagrangian Method (ALM): ALMs have been developed as a replacement for QPMs when solving \((\text{CP})\). They work like QPMs, but augment \(\Phi_{\rho_k}(x)\) with the term \(-\lambda_k^T \cdot c(x)\). This term with \(\lambda_k \in \mathbb{R}^m\) creates the right amount of slope such that the inequality constrained minimizer of \(\Psi_k\) eventually matches with the minimizer of \((\text{CP})\). Due to page limitations, we refer to [14, Alg. 17.3] and the references therein for all details on how \(\lambda_k \in \mathbb{R}^m\) is iteratively refined to achieve this. Convergence of \(\lambda_k\) is asserted under suitable conditions.
proposes a modified scheme (MALM) for (QPP) when treated only (CP) as opposed to (QPP). The work [17] augmentations [4]. Augmentations can suffer from non-smooth, non-differentiable, or low-order smooth penalties/barriers, and can converge slower or less reliably.

4) Extensions of ALM to (QPP): Originally, ALM treated only equality constraints [9], [15] by means of quadratic penalties of c and update schemes for \( \lambda \). In this case, inequalities can be subjected [3], i.e. minimize the sequence \( \Psi_k \) subject to \( x \in B \). Alternatively, penalty or barrier terms of \( g \) can be augmented [16] with according update strategies for \( \eta \). Subjections are considered more efficient in practice than augmentations [4]. Augmentations can suffer from non-smooth, non-differentiable, or low-order smooth penalties/barriers, and can converge slower or less reliably.

D. Challenges

Our goal is in devising a method that solves (QPP) by solving a sequence of penalty problems with moderate penalty parameter \( \rho \gg \omega \), and prove its convergence. In the limit \( \omega \to 0 \), MALM should match ALM due to the relation of the problems (QPP) and (CP) as described in Section I-A.1. Proving convergence for non-convex \( \Phi_\omega \) is challenging because solutions of sub-problems may be non-unique and hence alternating. Convergence of \( \eta_k \) may be challenging to prove because the solution \( \eta \) of (KKT) may be non-unique. We will assert uniqueness of \( \eta \) from a strict complementarity assumption. Striking the right balance between mild assumptions and strong convergence assertions appears non-trivial in this context.

E. Contributions

We present MALM for general functions \( f, c, g \) (Algorithm 1). We prove convergence for the case when \( f, c \) are twice continuously differentiable and \( g \) is linear (Theorem 1). Furthermore, we give a local rate-of-convergence result for the case when \( f, c, g \) are twice local Lipschitz-differentiable (Theorem 2).

Theorem 1 is not easily extendable to nonlinear \( g \) because it uses a result for ALM on (CP) for linear \( g \). Theorem 2 works for general \( g \) but assumes convergence and Lipschitz-continuous second derivatives of \( f, c, g \). In the iteration limit, convergence of ALM can only be guaranteed to be at least at a linear rate [14, Thm 17.6]. Likewise, our rate-of-convergence result for MALM asserts only a linear rate. However, this linear rate is slightly better than the linear rate of ALM. Hence, our work draws connections between the rate of convergence between MALM and ALM.

F. Structure of the Paper

Section II derives the proposed algorithm. Section III presents the convergence analysis. Section IV gives numerical experiments.

II. Derivation of the Algorithm of MALM

MALM is a solution method for (QPP). MALM has been presented in [17] for the special case when \( f \) is quadratic, \( c \) linear, and \( \Phi_\omega \) convex. The method has been presented for the case where \( f, c \) are general in [12] but without inequality constraints and without a convergence analysis. Here, we derive MALM for general nonlinear non-convex \( f, c, g \), and in a stronger relation to its origins in ALM [9], [15]. For the method presented here, we give global and local convergence proofs.

The derivation poses an auxiliary problem, applies ALM to it, and then eliminates variables.

A. Auxiliary Problem

The following problem is equivalent to (QPP) but of the form (CP):

\[
\begin{align*}
\min_{\hat{x}, (\xi, \eta) \in \mathbb{R}^{(n+m)}} & \quad \hat{f}(\hat{x}) := f(x) + \frac{\omega}{2} \|\xi\|_2^2 \\
\text{s.t.} & \quad \hat{c}(\hat{x}) := c(x) + \omega \xi = 0,
\end{align*}
\]

The optimality conditions of (3) are (KKT2) and

\[
\begin{align*}
\nabla f(x) - \nabla c(x) \lambda - \nabla g(x) \eta &= 0 \\
\omega \xi &= 0.
\end{align*}
\]

B. Augmented Optimality System

Since (3) is of form (CP), we can apply ALM with augmented inequality constraints as in [3]. To this end, we introduce an auxiliary vector \( z \in \mathbb{R}^m \) and a moderate penalty parameter \( \rho > 0 \). These are added to (3b) and in the gradient of the Lagrangian function:

\[
\begin{align*}
\nabla f(x) - \nabla c(x) (\lambda + z) - \nabla g(x) \eta &= 0 \\
\omega \xi + \rho z &= 0.
\end{align*}
\]

We could use (5) directly in order to form an ALM iteration. That iteration would consist of two alternating steps: 1) solving the optimality system (5) together with (KKT2) for \( (x, \xi, z, \eta, \lambda) \) where \( \lambda \) is fixed; 2) updating \( \lambda \leftarrow \lambda + z \), being equivalent to \( \lambda \leftarrow \lambda - \frac{1}{\rho} (c(x) + \omega \xi) \).

C. Elimination of the Auxiliary Vector

Instead, we propose to eliminate \( \xi = \lambda + z \) to obtain

\[
\begin{align*}
\nabla f(x) - \nabla c(x) (\lambda + z) - \nabla g(x) \eta &= 0 \\
(c(x) + \omega \lambda + (\omega + \rho) z) &= 0.
\end{align*}
\]

As in ALM, we solve (6a) and (KKT2) with an iteration of two alternating steps:
Algorithm 1 Modified Augmented Lagrangian Method

1: procedure MALM$(f, g, c, w, x_0, \lambda_0, \text{tol})$
2: \hspace{1em} $\rho \leftarrow \rho_0$
3: for $k = 1, 2, 3, \ldots, k_{\text{max}}$ do
4: \hspace{1em} Compute $x_k$, and optionally $\eta_k$, by solving
5: \hspace{2em} $\min_{x \in \mathbb{R}^n} \Psi_k(x)$ s.t. $g(x) \geq 0$. \hspace{1em} (8)
6: \hspace{1em} Update $\lambda_k \leftarrow \lambda_{k-1} - \frac{1}{\omega + \rho} \langle c(x_k) + \omega \lambda_{k-1} \rangle$
7: \hspace{1em} if $\|c(x_k) + \omega \lambda_k\|_\infty \leq \text{tol}$ then
8: \hspace{2em} return $x_k, \lambda_k$ and optionally $\eta_k$
9: \hspace{1em} else
10: \hspace{2em} Decrease $\rho \leftarrow c_\rho \rho$ to promote convergence.
11: \hspace{2em} end if
12: end for
13: end procedure

1) Keep the value of $\lambda$ fixed, and solve (3) and (KKT2) for $(x, z, \eta, A)$.
2) Update $\lambda \leftarrow \lambda + z$.

Analogous to ALM, the first step can be realized by minimizing an augmented Lagrangian function for $x$ at fixed $\lambda$ subject to $x \in B$, whereas in the second step $z$ can be expressed in terms of $x$ from (6b). Using this, the method can be expressed in Algorithm 1 where

$$
\Psi_{k+1}(x) := L(x, \lambda_k, 0) + \frac{0.5}{\omega + \rho} \|c(x) + \omega \lambda_k\|_2^2 \quad (7)
$$

is the augmented Lagrangian function, with $L(x, \lambda, 0) \equiv f(x) - \lambda^T \cdot c(x)$.

D. Practical Aspects

Values that we have found work well in practice are $\text{tol} = 10^{-8}$, $c_\rho = 0.1$, $\rho_0 = 0.1$. Care must be taken that $\Psi_k$ in (3) is bounded below. To this end, practical methods impose box constraints $x_L \leq x \leq x_U$ [4, eq. 3.2.2], expressible via $g(x) \geq 0$, or a trust-region constraint $g(x) = \Delta^2 - \|x - x_k\|_2^2 \geq 0$ [4, eq. 3.2.4] with trust-region radius $\Delta > 0$.

In order to minimize (3), one can use any numerical method for inequality constrained nonlinear minimization; e.g. an interior-point method like IPOPT [18] or an active set method like SNOPT [7].

We refer to [14, eq. 17.21] for details on how the quasi-Newton direction for the quadratic penalty function can be computed in a more numerically stable fashion from a saddle-point linear equation system.

E. Discussion

1) True Generalization of ALM: MALM is a true generalization of ALM, because they differ only by the parameter $\omega$. In particular, if $\omega = 0$ then MALM in Algorithm 1 is identical to ALM in [3, Algorithm 3.1]. In contrast, when selecting $\omega > 0$, we show below that MALM converges to critical points of (QPP) with the given $\omega$.

2) Benefit: MALM solves the penalty function $\Phi_{\omega}$ in (QPP) by minimizing a sequence of penalty functions $\Psi_k$. When does this make sense? If we select $\rho \gg \omega$. Thereby, the penalty functions $\Psi_k$ have better scaling and hence can often be minimized more efficiently in comparison to a single minimization of $\Phi_{\omega}$. The computational performance results in Section IV verify this claim.

III. Convergence Analysis

The below analyses assume that all sub-problems are solved exactly, and that computations are performed in exact arithmetic. Throughout this subsection, MALM means the callback-function in Algorithm 1 wherein any black-box method can be used to solve (8).

A. Global Convergence

For the analyses, we consider a call of Algorithm 1 with instance $I := (f, c, g, w, x_0, \lambda_0, \text{tol})$. MALM will create a sequence of iterates $x_k, \lambda_k$.

Lemma 1 (Equivalence): MALM on the instance $J := (\hat{f}, \hat{c}, \hat{g}, 0, \hat{x}_0, \hat{\lambda}_0, \text{tol})$ from [3] will generate the same iterates $(x_k, \xi_k), \lambda_k$ as MALM on the instance $I$ in terms of $x_k, \lambda_k$.

Proof: By induction over $k$. Base: For $k = 0$ the proposition holds by construction of the initial guesses. Step: Let the proposition hold for $k - 1$. We now show that the proposition holds for $k$. The iterate $\hat{x}_k$ from $J$ in line 4 necessarily satisfies $\nabla \hat{x} \psi_k(\hat{x}_k) - \nabla \hat{g}(\hat{x})^T \cdot \eta_k = 0$, which is equivalent to (5a) after elimination of $z$ by means of (5b). From the second component of (5a) it follows that

$$
\xi_k = \xi(x_k, \lambda_{k-1}) := \frac{1}{\omega + \rho} \left( \rho \lambda_{k-1} - c(x_k) \right). \quad (9)
$$

I.e. $\xi_k$ is uniquely determined, hence $\hat{x}_k$ takes on the form $x_k = (x_k, \xi(x_k, \lambda_{k-1}))$ for some $x_k$.

Substituting (9) into the first component of (5a) yields $\nabla \hat{x} \psi_k(\hat{x}_k) = 0$, which is indeed identical to what $x_k$ in line 4 of $I$ satisfies. Thus, $\hat{x}_k = (x_k, \xi(x_k, \lambda_{k-1}))$ with $x_k$ from $I$ is a valid $k$th iterate of $J$. Finally, notice that $\lambda_k$ in $I$, $J$ are identical because

$$
\frac{1}{\rho} \left( c(x_k) + \omega \xi(x_k, \lambda_{k-1}) \right) = \frac{1}{\rho + \omega} (c(x_k) + \omega \lambda_{k-1})
$$

In turn, MALM with $\omega = 0$ is identical to ALM in [3, Algorithm 3.1]. We can hence use the convergence result from [3, Thm 4.6]:

Theorem 1 (Global Convergence): Choose a bounded domain $\Omega \subset \mathbb{R}^n$. Let $\omega > 0$, $c$ be bounded on $\Omega$, and let $f, c$ be twice continuously differentiable in $\Omega$, and $g$ affine. Suppose all iterates $\{x_k\}_{k \in \mathbb{N}_0}$ of MALM live in $\Omega$. If $\rho_0$ is sufficiently small then $\{x_k\}_{k \in \mathbb{N}_0}$ converges to a critical point of (QPP).

Proof: [3, Thm 4.6] shows convergence of $J$ under four assumptions (AS1)-(AS4). It suffices to show that $f, \hat{c}, \hat{g}$ satisfy these assumptions.
Feasibility [3, AS1] of \( \{3\} \) holds naturally by \( \xi = \frac{1}{\rho}c(x) \). Twice continuous differentiability [3, AS2] of \( f, \hat{\epsilon} \) holds per requirement. Boundedness [3, AS3] of all \( \hat{x}_k \in \Omega \times c(\Omega) \) follows from boundedness of \( \Omega \) and \( c \) on \( \Omega \).

The last assumption [3, AS4] is more technical. Since \( g \) is affine, we can express \( g(x) = A \cdot x - b \), and likewise \( \hat{g}(\hat{x}) = \hat{A} \cdot \hat{x} - \hat{b} \), where \( \hat{A} = [A \ 0] \). We define the matrix \( Z \) of orthonormal columns that span the null-space of \( \hat{A} \), i.e. the matrix of sub-rows of \( \hat{A} \) of the active constraints at \( \hat{x} \). (AS4) requires \( \nabla \hat{c}(\hat{x}) \cdot Z \) to be of column rank \( \geq m \). Due to the special structure of \( \hat{A} \), we see that \( Z \) has a structure like

\[
Z = \begin{bmatrix}
0 \\
1 \\
\vdots \\
\end{bmatrix}.
\]

Since \( \nabla \hat{c}(\hat{x})^T = [\nabla c(x)^T \ \omega I] \) has full row rank, the rank of \( \nabla \hat{c}(\hat{x})^T \cdot Z \) is bounded below by the number of columns of \( Z \), i.e. bounded below by \( m \).

Some of the requirements in Theorem 1 may be forcible: Section [II-D] explains how \( B \) can be bounded. In this case, choosing \( \Omega = B \) yields \( \{x_k\} \subset \Omega \). Also, \( c \) may be bounded over \( \Omega \) by approximating \( c(x) \) with arctan \( c(x) \). If \( ||c(x)||_2 \) is very small at the minimizer of \( f \), then the approximation error of arctan is negligible. To make \( g \) affine, several practical ALM implementations (Lancelot, MINOS) convert inequalities to equalities via the addition of slack variables \( s \geq 0 \) [14, Sec. 17.4]. The constraints \( g(x) - s = 0 \) (as in [14, eqn 17.47]) can be merged into \( c \) and scaled such that they hold tightly. Also, interior-point methods like IPOPT [18] use slacks to ensure iterates are strictly interior.

### B. Local Convergence

[14, Thm 17.6] asserts linear convergence of ALM when \( \nabla^2 \psi_c \), \( \nabla c \), \( \nabla^2 x \psi_c \) are local Lipschitz-continuous and \( \rho = const \). Section [V-A.2.2] and Figure 3 show this. Likewise, MALM attains a linear rate in the limit when \( \rho = const \). Upper bounds for these rates can be computed. In this section we prove that the rate of MALM is strictly smaller than that of ALM.

For the following result, we compare the iteration of ALM and MALM from the same initial guess \( x_0, \lambda_0 \) and the same problem-defining functions \( f, c, g \). We assume that \( \{x_k\} \subset U \), where \( U \subset \mathbb{R}^n \) is an open neighborhood which contains unique local minimizers of both \( (CP) \) and \( (QPP) \).

**Theorem 2 (Local Convergence):** Let \( \nabla f, \nabla c, \nabla^2 x \psi_c \) be Lipschitz-continuous \( \forall x \in U \) and let all iterates of ALM and MALM remain in \( U \). Let the local minimizers satisfy strict complementarity. Apply ALM and MALM with fixed penalty parameter \( \rho \) to solve either problem, each starting from \( x_k \). If both methods converge and if \( x_k \) is sufficiently close to the local minimizer of \( (QPP) \), then the linear rates of convergence of MALM and ALM satisfy the relation \( C_{MALM} = \frac{\rho}{\rho + \omega} C_{ALM} < C_{ALM} \).

**Proof:** We use the Taylor series

\[
\nabla \psi_k(x_k, \lambda_{k-1}) = Hx_k + g - \frac{1}{\omega + \rho} \mathbf{J}^T(\rho \lambda_{k-1} + c) + R_L(x_k, \lambda_{k-1})
\]

with \( \mathbf{J} := \nabla c(x_\omega) \), \( \mathbf{H} := \nabla^2 c(x_\omega, \lambda_\omega, 0) + \frac{1}{\omega + \rho} \mathbf{J}^T \mathbf{J} \), \( c := \nabla c(x_\omega) \) and \( g := \nabla f(x_\omega) \) has the Lagrange remainder \( ||R_L(x_k, \lambda_k)||_2 \leq \frac{\rho}{\rho + \omega} ||(x_k - x_\omega)||_2 + ||\lambda_k - \lambda_\omega||_2^2 \), where \( L \) is the Lipschitz constant.

We now first consider the case where \( p = 0 \), i.e. when there are no inequality constraints. Since \( x_k \) is convergent by requirement, \( \mathbf{H} \) must be positive semi-definite and, if \( x_k \) is locally unique, \( \mathbf{H} \) must be positive definite. Clearly, local convergence to a unique point depends quantitatively on uniqueness, hence we imply \( \lambda_{\text{min}}(\mathbf{H}) \geq \mu > 0 \). For the induced 2-norm it follows that \( ||\mathbf{H}^{-1}||_2 \leq \mu \), hence

\[
\left\| x_k - \mathbf{H}^{-1} \left( g - \frac{1}{\omega + \rho} \mathbf{J}^T(\rho \lambda_{k-1} + c) \right) \right\|_2 \leq \frac{L}{\mu(\rho + \omega)} ||\lambda_k - \lambda_\omega||^2_2.
\]

Inserting the estimate for \( x_k \) into line 5 in Algorithm 1 gives a formula for \( \lambda_k \) that only depends on \( \lambda_{k-1} \):

\[
\lambda_k = M \cdot \lambda_{k-1} + f + R(\lambda_k)
\]

with \( M \in \mathbb{R}^{m \times m} \) below, some \( f \in \mathbb{R}^m \), and \( ||R(\lambda_k)||_2 \leq \frac{1}{\rho} \left( \frac{\rho}{\rho + \omega} \right) ||\lambda_k - \lambda_\omega||^2_2 \).

Rearranging reveals

\[
M = \frac{\rho}{\omega + \rho} \left( I - \frac{1}{\omega + \rho} \mathbf{J}^T \mathbf{H}^{-1} \mathbf{J} \right).
\]

Since Theorem 1 asserts convergence of \( \lambda_k \), the second order terms become negligible compared to the first-order terms and can hence be ignored in the limit. Then, [10] is a Banach iteration. Thus, in the limit, the rate of convergence for \( \lambda_k \) is linear with contraction \( ||M||_2 < 1 \). The analysis holds regardless of whether \( \omega = 0 \) or \( \omega > 0 \).

We see that in the limit MALM converges faster than ALM because \( \frac{\rho}{\rho + \omega} < 1 \) when \( \omega > 0 \), whereas \( \frac{\rho}{\rho + \omega} = 1 \) when \( \omega = 0 \). Hence, in the limit \( k \to \infty \) MALM yields a stronger contraction for the errors per iteration than ALM. This is in particular an advantage in cases where ALM would converge slowly. For instance, choosing \( \rho = 10\omega \) guarantees convergence in the limit with at least a rate of contraction of \( \frac{\rho}{\rho + \omega} < 0.91 \).

From the above, when dropping the Lagrange remainder terms, we can identify the local rate of convergence by that of the following quadratic model iteration: 1) Solve

\[
\min \ \frac{1}{2} x^T H x + \left( g + \frac{1}{\omega + \rho} \mathbf{J}^T (Jx_{k-1} - c - \omega \lambda_{k-1}) \right)^T x.
\]

2) Update \( \lambda_k := \lambda_{k-1} - \frac{1}{\omega + \rho} (Jx_k - c - \omega \lambda_{k-1}) \).

We discuss the case when \( p > 0 \), i.e. when inequality constraints are present. We use our assumption on strict complementarity, i.e. \( i \in A \iff \eta_i > \beta \) for some real
Once subsequent iterations the active set $A$ converge in order to yield $\beta > 0$. Since $x_k$ converges by requirement, $\lambda_{k-1}$ converges and thus also $\nabla \Psi_k(x_k)$ converges. Hence, $\eta_k$ must converge in order to yield $\nabla_{x_k} \Psi_k(x_k) - \nabla g(x) \eta_k = 0$.

Once $\eta_k$ changes less than $\beta$ at some finite $k_0 \in \mathbb{N}$, the active set $A_k$ will remain unchanged $A_{k0}$ for all subsequent iterations $k \geq k_0$. We use $g_{\infty}$ for only the active constraints of $g$ and define $A_{\infty} := \nabla g_{\infty}(x_{\infty})^T$, $b_{\infty} := \nabla g_{\infty}(x_{\infty})^T \cdot x_{\infty}$; hence $g_{\infty}(x) = A_{\infty} \cdot x - b + \mathcal{O}(\|x - x_{\infty}\|^2)$.

Given the above intermezzo, the appropriate model iteration in the limit becomes obvious: 1) Solve

$$\min_{x} \frac{1}{2} x^T H x + (g + \frac{1}{\omega + \rho} J^T (Jx_{k-1} - c - \omega \lambda_{k-1})^T \cdot x$$

s.t. $A_{\infty} \cdot x = b_{\infty}$.

2) Update $\lambda_k := \lambda_{k-1} - \frac{1}{\omega + \rho} (Jx_k - c - \omega \lambda_{k-1})$.

This is just a projection of the iteration above. Thus, we can project the iteration for $x_k$ onto the nullspace of $A_{\infty}$, identifying $x_k = x_r + N \tilde{x}_k \forall k \geq k_0$, where $x_r \in \mathbb{R}^n$ has active set $A_r$, $\tilde{x}_k \in \mathbb{R}^{n - \dim(A_r)}$ and $N$ is a matrix of orthogonal columns that span the nullspace of $\nabla g_{\infty}(x_{\infty})$.

Defining $H := N^T H N$, $J := JN$, and $g, c$ appropriately, we arrive at the former unconstrained quadratic model iteration form, but with $H, g, J, c, x_k$ replaced by the tilded quantities. Accordingly, the Banach iteration matrix $M$ is replaced with the matrix

$$M = \frac{\rho}{\omega + \rho} \left( I - \frac{1}{\omega + \rho} J H^{-1} J^T \right).$$

The resulting contraction matrix $M$ for the Banach iteration of the inequality constrained case has a factor $\frac{\rho}{\omega + \rho}$ in front, just like for the case when $\rho = 0$. Thus, for $\rho > 0$ the method converges locally faster in the limit $k \to \infty$.

IV. NUMERICAL EXPERIMENTS

For our tests we use two instances: $(1)$ and $(2)$. Each instance will be considered once as $(\text{CP})$ and once as $(\text{QPP})$. Both instances are parametric: The inconsistency of $(1)$ grows in the order of $\varepsilon$ and inconsistency of $(2)$ grows in the order of the mesh size $h$. The subproblems in $(3)$ are solved with IPOPT version 12.0.3.

For tests on examples with equality constraints only, we refer to [12].

A. Circle Problem

1) Setting:

a) Initial Guess and Numerical Methods: We use the initial guess $x_0 = [2 \ 1]^T$, $\lambda_0 = 0$. Fig. 2 shows the instance’s geometry. The figure also shows two points $x_A := [0 \ \sqrt{2}]^T$, $x_B := [1 \ 1]^T$.

b) Expected Minimizers: When considering the instance as $(\text{CP})$ then we expect that $x_A$ would be the solution. To see this, notice that $c(x) = 0$ is only satisfied at $x = x_A$. When $\varepsilon \to 0$, (KKT) becomes ill-conditioned for $x_A$. Once $\varepsilon = 0$, the minimizer is suddenly $x_B$.

When considering the instance as $(\text{QPP})$ then a point close to $x_B$ should be the solution unless $\varepsilon$ becomes large relative in comparison to $\omega$. To see this, note that $x_B$ minimizes $f$ among all points in $B$ that yield $\|c(x)\|_2$ small relative to $\omega$.

c) Scope: Both ways $(\text{CP})$ and $(\text{QPP})$ of interpreting the instance $(1)$ and both solutions $x_A, x_B$ make sense in their own right. We want to find out which solver works best for solving a respective combination $\omega, \varepsilon$.

2) Computational Results: We observe that all iterates of all methods remain in $\Omega = B \cap \{x \in \mathbb{R}^2 | x_2 \leq 2\}$. Hence, Theorem 1 asserts that MALM and ALM converge because $f, c$ are twice continuously differentiable on $\Omega$ and $g$ is affine.

We solve the instance with MALM and QPM, for various values of $\varepsilon, \omega$, including 0. We implement QPM by solving $(\text{QPP})$ directly in IPOPT with objective $\Phi_\omega$.

Recall that MALM = ALM for $\omega = 0$ and that QPM is not applicable (n.a.) when $\omega = 0$, since $\Phi_\omega$ is undefined.

a) Confirmation of Expected Minimizers: We first analyze the limit points $x_{\infty}$ (which are identical for both tested methods throughout all tests) for each $\varepsilon, \omega$, by measuring the quantities

$$e_A := \|x_{\infty} - x_A\|_2, \quad e_B := \|x_{\infty} - x_B\|_2.$$
Fig. 3. Comparison of convergence for ALM and MALM for the circle problem. Solid lines are measured convergence rates. Dotted lines indicate the theoretical rates of ALM and MALM in the limit $k \to \infty$.

TABLE I

| $h, \omega$ | $1.0 \times 10^{-6}$ | $1.0 \times 10^{-4}$ | $1.0 \times 10^{-2}$ | $1.0 \times 10^{-1}$ | $\varepsilon$ |
|------------|--|--|--|--|----------|
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |

TABLE II

| Total number of IPOPT iterations for MALM and QPM for the Circle Problem with respect to $\varepsilon, \omega$. Fewer iterations mean better computational efficiency; highlighting best in slanted (QPM) or bold (MALM). |
|---|---|---|---|---|---|
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\omega$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $\varepsilon$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |

then ALM should converge to $x_A$ but its iteration count blows up for small $\varepsilon > 0$. In two instances ALM did not converge (n.c.) within 1000 iterations. In conclusion, ALM is inefficient when $c$ has small inconsistencies.
outperforms QPM when minimizing quadratic penalty in \( \omega > 0 \) method when the inequalities are affine. A local rate of convergence is very small.

TABLE III
Solution of the Optimal Control Problem with respect to \( N, \omega \). For a given mesh size \( N \), the value for \( \omega \) is suitable when \( \delta J \) (optimality gap) and \( r \) (feasibility residual) have similar magnitude.

| \( \delta J \) | \( r \) | \( h \) |
|---|---|---|
| \( \omega \) | 1.0e-2 | 2.0e-2 | 1.0e-2 | 5.0e-3 | 2.5e-3 |
| 1.0e-2 | -1.7e-1 | -1.7e-1 | -1.7e-1 | -1.7e-1 | -1.7e-1 |
| 1.0e-3 | 4.3e-2 | 4.3e-2 | 4.3e-2 | 4.3e-2 | 4.3e-2 |
| 1.0e-4 | 7.6e-2 | 7.6e-2 | 7.6e-2 | 7.6e-2 | 7.6e-2 |
| 1.0e-5 | 2.5e-2 | 2.5e-2 | 2.5e-2 | 2.5e-2 | 2.5e-2 |
| 1.0e-6 | 8.0e-2 | 8.0e-2 | 8.0e-2 | 8.0e-2 | 8.0e-2 |
| 0.0 | 7.2e+0 | 7.2e+0 | 7.2e+0 | 7.2e+0 | 7.2e+0 |

TABLE IV
Total number of IPOPT iterations for MALM and QPM for the Optimal Control Problem with respect to \( N, \omega \). Fewer iterations mean better computational efficiency; highlighting best in slanted (QPM) or bold (MALM).

| \#MALM | \#QPM |
|---|---|
| \( \omega \) | 1.0e-2 | 2.0e-2 | 1.0e-2 | 5.0e-3 | 2.5e-3 |
| 1.0e-2 | 39 | 63 | 99 | 69 | 69 |
| 1.0e-3 | 17 | 24 | 44 | 26 | 44 |
| 1.0e-4 | 47 | 65 | 90 | 44 | 44 |
| 1.0e-5 | 61 | 66 | 80 | 93 | 121 |
| 1.0e-6 | 102 | 110 | 124 | 124 |
| 0.0 | n. n. | n. n. | n. n. | n. n. | n. n. |

optimal control solutions require \( h, \omega \) very small; thus MALM seems very attractive for solving these classes of problems.

The last row shows that ALM does not converge (n.c.) within 500 iterations for any mesh size.

V. Conclusions

We presented a modified augmented Lagrangian method (MALM), generalized to non-convex optimization problems with additional inequality constraints. We proved global convergence for our generalized method when the inequalities are affine. A local rate of convergence result shows that MALM inherits all the local convergence results of ALM while the regularization in \( \omega > 0 \) also yields a slight benefit to its rate of local convergence in the iteration limit.

Our numerical experiments demonstrate that MALM outperforms QPM when minimizing quadratic penalty programs \( \text{QPP} \) in those situations where \( \omega \) is very small, in a similar manner as ALM outperforms QPM when solving equality constrained programs \( \text{CP} \). The experiments further show that ALM cannot solve \( \text{QPP} \), but solves \( \text{CP} \) instead. Hence, MALM is the best candidate for solving \( \text{QPP} \) when \( \omega \) is very small.

In this paper we have assumed that the subproblems \( \text{CP} \) are solved to high accuracy. Future work could extend the approach to inexact iterations and sub-iterations to mild tolerances. This could reduce computations at sub-iterations where the dual is far from converged. Another open subject is the extension of global convergence analysis to the cases when \( g \) is convex nonlinear or non-convex nonlinear.

References

[1] A. V. Balakrishnan. On a new computing technique in optimal control. SIAM J. Control, 6:149–173, 1968.
[2] D. P. Bertsekas. Constrained optimization and Lagrange multiplier methods. Computer Science and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.
[3] A. R. Conn, N. Gould, A. Sartenaer, and Ph. L. Toint. Convergence properties of an augmented Lagrangian algorithm for optimization with a combination of general equality and linear constraints. SIAM J. Optim., 6(3):674–703, 1996.
[4] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. LANCELOT, volume 17 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 1992. A Fortran package for large-scale nonlinear optimization (release A).
[5] R. Courant. Variational methods for the solution of problems of equilibrium and vibrations. Bull. Amer. Math. Soc., 49:1–23, 1943.
[6] A.V. Fiacco and G.P. McCormick. Nonlinear Programming: Sequential unconstrained minimization techniques. John Wiley & Sons, 1968.
[7] Philip E. Gill, Walter Murray, and Michael A. Saunders. SNOPT: An SQP algorithm for large-scale constrained optimization. SIAM Rev., 47:99–131, 2005.
[8] W. W. Hager. Multiplier methods for nonlinear optimal control. SIAM J. Numer. Anal., 27(4):1061–1080, 1990.
[9] M. R. Hestenes. Multiplier and gradient methods. J. Optim. Theory Appl., 4:303–320, 1969.
[10] W. Murray. Analytical expressions for the eigenvalues and eigenvectors of the Hessian matrices of barrier and penalty functions. J. Optim. Theory Appl., 7:189–196, 1971.
[11] M. P. Neuenhofen and E. C. Kerrigan. Dynamic optimization with convergence guarantees. arXiv:1810.04059, 2018.
[12] M. P. Neuenhofen and E. C. Kerrigan. A direct method for solving integral penalty transcriptions of optimal control problems. Proceedings of the IEEE Conference on Decision and Control 2020, 2020.
[13] M. P. Neuenhofen and E. C. Kerrigan. An integral penalty-barrier direct transcription method for optimal control. Proceedings of the IEEE Conference on Decision and Control 2020, 2020.
[14] J. Nocedal and S. J. Wright. Numerical optimization. Springer Series in Operations Research and Financial Engineering, Springer, New York, second edition, 2006.
[15] M. J. D. Powell. A method for nonlinear constraints in minimization problems. In Optimization (Sympos., Univ. Keele, Keele, 1968), pages 283–298. Academic Press, London, 1969.
[16] R. T. Rockafellar. The multiplier method of Hestenes and Powell applied to convex programming. J. Optim. Theory Appl., 2020.
[17] M.H.B.M. Shariff and J.R. Dormand. A modified augmented Lagrangian method for a class of constrained problems. Journal of Computational and Applied Mathematics, 151(2):257–270, 2003.
[18] Andreas Wächter and Lorenz T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Math. Program., 106(1, Ser. A):25–57, 2006.