TRIVIALITY AND CLASSIFICATION OF GRADIENT CONFORMAL SOLITONS

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ABSTRACT. A. W. Cunha and E. L. Lima considered gradient $k$-Yamabe solitons and showed that the $\sigma_k$-curvature is constant under some assumption (Ann. Glob. Anal. Geom. 62 (2022), 73-82.). Gradient conformal solitons are generalizations of gradient $k$-Yamabe solitons. In this paper, we give some triviality and classification results of gradient conformal solitons under the same assumption. As a corollary, we can get some triviality and classification results of gradient $k$-Yamabe solitons. Furthermore, we give an affirmative partial answer to the Yamabe soliton version of Perelman’s conjecture.

1. Introduction

An $n$-dimensional Riemannian manifold $(M^n, g)$ is called a gradient Yamabe soliton if there exists a smooth function $F$ on $M$ and a constant $\rho \in \mathbb{R}$, such that

$$\nabla \nabla F = (R - \rho)g,$$

where, $R$ is the scalar curvature on $M$. If $F$ is constant, $M$ is called trivial.

One of the most interesting problem of the Yamabe soliton is the Yamabe soliton version of Perelman’s conjecture, that is, “Any complete gradient Yamabe soliton with positive scalar curvature under some natural assumption is rotationally symmetric”. The problem was first considered by P. Daskalopoulos and N. Sesum [7]. They showed that any locally conformally flat complete gradient Yamabe soliton with positive sectional curvature is rotationally symmetric. Later, G. Catino, C. Mantegazza and L. Mazzieri [5] and H.-D. Cao, X. Sun and Y. Zhang [4] also considered the same problem. In particular, Cao, Sun and Zhang showed that any locally conformally flat complete gradient

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Yamabe soliton with positive scalar curvature is rotationally symmetric.

To understand the gradient Yamabe soliton, many generalizations of it have been introduced.

1. Almost gradient Yamabe solitons [2]:
   For smooth functions $F$ and $\rho$ on $M$,
   \[(R - \rho)g = \nabla \nabla F.\]

2. Gradient $k$-Yamabe solitons [5]:
   For a smooth function $F$ on $M$ and $\rho \in \mathbb{R}$,
   \[2(n - 1)(\sigma_k - \rho)g = \nabla \nabla F,\]
   where, $\sigma_k$ denotes the $\sigma_k$-curvature of $g$, that is,
   \[\sigma_k = \sigma_k(g^{-1}A) = \sum_{i_1 < \cdots < i_k} \mu_{i_1} \cdots \mu_{i_k} \text{ (for } 1 \leq k \leq n),\]
   where, $A = \frac{1}{n-2}(\text{Ric} - \frac{1}{2(n-1)}Rg)$ is the Schouten tensor and $\mu_1, \cdots, \mu_n$ are the eigenvalues of the symmetric endomorphism $g^{-1}A$. Here, Ric is the Ricci tensor of $M$.

3. $h$-almost gradient Yamabe solitons [17]:
   For smooth functions $F$, $\rho$ and $h \ (h > 0 \text{ or } h < 0)$ on $M$,
   \[(R - \rho)g = h\nabla \nabla F.\]

To consider all these solitons, we consider the gradient conformal soliton defined by G. Catino, C. Mantegazza and L. Mazzieri [5]:

**Definition 1.1** ([5]). Let $(M, g)$ be an $n$-dimensional Riemannian manifold. For smooth functions $F$ and $\varphi$ on $M$, $(M, g, F, \varphi)$ is called a gradient conformal soliton if it satisfies

\[(1.1) \quad \varphi g = \nabla \nabla F.\]

If $F$ is constant, $M$ is called trivial. The function $F$ is called the potential function.

**Remark 1.2.** By the definition, all results in this paper can be applied to gradient Yamabe solitons, gradient $k$-Yamabe solitons, almost gradient Yamabe solitons and $h$-almost gradient Yamabe solitons.

We also remark that gradient conformal solitons were studied by Cheeger-Colding ([6], see also [16]).

Recently, A. W. Cunha and E. L. Lima [1] showed the following.
Theorem 1.3 ([1]). Let $(M, g)$ be a complete gradient steady $k$-Yamabe soliton. If $\sigma_k \geq 0$, $F \geq 0$, and
\[
\int_{M \setminus B(x_0, R_0)} \frac{F}{d(x_0, x)^2} < +\infty,
\]
for some $x_0 \in M$ and $R_0 > 0$, then $F$ is harmonic.

In this paper, under the same assumption, we completely classify gradient conformal solitons. In particular, we give an affirmative partial answer to the Yamabe soliton version of Perelman’s conjecture.

Theorem 1.4. Any nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ with $\varphi, F \geq 0$ and
\[
\int_{M \setminus B(x_0, R_0)} \frac{F}{d(x_0, x)^2} < +\infty,
\]
for some $x_0 \in M$ and $R_0 > 0$ is
\[(0, \infty) \times S^{n-1}, ds^2 + a^2 \tilde{g}_S, as + b, 0),\]
where, $\tilde{g}_S$ is the round metric on $S^{n-1}$, and $a > 0$ and $b \geq 0$.

The remaining sections are organized as follows. Section 2 is devoted to the proof of Theorem 1.4. In Section 3, we show some triviality results under the assumption in Cunha and Lima [1]. In Appendix, we give some classifications of complete gradient conformal solitons.

2. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. To show Theorem 1.4, we use the following result shown by the author [13].

Theorem 2.1 ([13]). A nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ is either
\begin{enumerate}
\item compact and rotationally symmetric, or
\item the warped product
\[(\mathbb{R}, ds^2) \times_{|\nabla F|} (N^{n-1}, \tilde{g}) ,\]
where, the scalar curvature $\tilde{R}$ of $N$ satisfies
\[|\nabla F|^2 \tilde{R} = R - (n - 1)(n - 2)\varphi^2 - 2(n - 1)g(\nabla F, \nabla \varphi),\]
or
\item rotationally symmetric and equal to the warped product
\[(0, \infty), ds^2) \times_{|\nabla F|} (S^{n-1}, \tilde{g}_S),\]
where, $\tilde{g}_S$ is the round metric on $S^{n-1}$.
Furthermore, the potential function $F$ depends only on $s$.\end{enumerate}
Therefore, to consider the Yamabe soliton version of Perelman’s conjecture, we only have to consider (2) of Theorem 2.1.

**Proof of Theorem 2.1** We first take a cut off function $\eta$ on $M$ satisfying that

\[
\begin{align*}
0 &\leq \eta(x) \leq 1 \quad (x \in M), \\
\eta(x) &= 1 \quad (x \in B(x_0, R)), \\
\eta(x) &= 0 \quad (x \notin B(x_0, 2R)), \\
|\nabla \eta| &\leq \frac{C}{R} \quad (x \in M), \quad \text{for some constant } C \text{ independent of } R, \\
\Delta \eta &\leq \frac{C}{R^2} \quad (x \in M), \quad \text{for some constant } C \text{ independent of } R,
\end{align*}
\]

where, $B(x_0, R)$ and $B(x_0, 2R)$ are the balls centered at a fixed point $x_0 \in M$ with radius $R$ and $2R$, respectively. By $\Delta F = n\varphi$, one has

\[
\begin{align*}
0 &\leq \int_{B(x_0, 2R)} \eta \varphi \\
&= \frac{1}{n} \int_{B(x_0, 2R)} \eta \Delta F \\
&= \frac{1}{n} \int_{B(x_0, 2R) \setminus B(x_0, R)} \Delta \eta F \\
&\leq \frac{C}{nR^2} \int_{B(x_0, 2R) \setminus B(x_0, R)} F.
\end{align*}
\]

Take $R \to +\infty$. By the assumption, we have that the right hand side of (2.2) goes to 0. Therefore, we have

\[
\int_M \varphi = 0.
\]

Since $\varphi \geq 0$, one has $\varphi = 0$, hence

\[
\begin{align*}
\nabla \nabla F &= 0.
\end{align*}
\]

By Theorem 2.1 we have 3 types of gradient conformal solitons.

**Case 1.** $M$ is compact and rotationally symmetric.

By (2.3), $\Delta F = 0$. Since $M$ is compact, by the standard Maximum principle, we have that $F$ is constant.

**Case 2.** $M$ is the warped product

\[
(\mathbb{R}, ds^2) \times_{|\nabla F|} (N^{n-1}, \bar{g}).
\]
By (2.3), we have

$$\nabla|\nabla F|^2 = 2\nabla_j \nabla_i F \nabla_i F = 0.$$ 

Hence, $\nabla F$ is a constant vector field. Since $F$ depends only on $s \in \mathbb{R}$, one can get that

$$\nabla F = F'(s) \partial_s,$$

where, $F'(s)$ is a constant, say $a$. If $a \neq 0$, one has $F(s) = as + b \geq 0$ on $\mathbb{R}$, which cannot happen. Hence, $F$ is constant. Therefore, $M$ is trivial.

Case 3. $M$ is rotationally symmetric and equal to the warped product $([0, \infty), ds^2) \times_{|\nabla F|} (S^{n-1}, \bar{g}_S)$. By the same argument as in Case 2, we have $F(s) = as + b \geq 0$ on $[0, \infty)$. Therefore, one has $a > 0$ and $b \geq 0$.

$$\Box$$

3. Some triviality results of gradient conformal solitons

Cunha and Lima [1] also showed that the $\sigma_k$-curvature is constant under some assumptions. We first recall parabolicity of $M$: A Riemannian manifold $M$ is parabolic, if every subharmonic function on $M$ which is bounded from above is constant (see [9]).

**Theorem 3.1** ([1]). Let $(M, g)$ be a complete gradient $k$-Yamabe soliton. If one of the following is satisfied, then $\sigma_k = \lambda$.

(A) The potential function satisfies that $F \geq K > 0$ for some $K \in \mathbb{R}$, $\sigma_k \leq \lambda$, and one of the following is satisfied (i) $M$ is parabolic, (ii) $|\nabla F| \in L^1(M)$, (iii) $F^{-1} \in L^p(M)$ for some $p > 1$, or (iv) $M^n$ has linear volume growth.

(B) The soliton $M$ is nontrivial, $\text{Ric}(\nabla F, \nabla F) \leq 0$, and $|\nabla F| \in L^p(M)$ for $p > 1$.

In this section, we give triviality results of gradient conformal solitons under the same assumption. As a corollary, one can also get triviality results of gradient $k$-Yamabe solitons under the assumption of Theorem 3.1

**Proposition 3.2.** There exists no nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ with one of the following is satisfied.

(A) The potential function satisfies that $F \geq K > 0$ for some $K \in \mathbb{R}$, $\varphi \leq 0$, and one of the following is satisfied (i) $M$ is parabolic, (ii) $|\nabla F| \in L^1(M)$, (iii) $F^{-1} \in L^p(M)$ for some $p > 1$, or (iv) $M^n$ has linear volume growth.

(B) The Ricci curvature satisfies that $\text{Ric}(\nabla F, \nabla F) \leq 0$, and $|\nabla F| \in L^p(M)$ for $p > 1$. 

We recall formulas which will be used later. For any gradient conformal soliton, we have
\[
\Delta \nabla_i F = \nabla_i \Delta F + R_{ij} \nabla_j F,
\]
\[
\Delta \nabla_i F = \nabla_k \nabla_k \nabla_i F = \nabla_k (\varphi g_{ki}) = \nabla_i \varphi,
\]
and
\[
\nabla_i \Delta F = \nabla_i (n \varphi) = n \nabla_i \varphi.
\]
Hence, we have
\[
(n - 1) \nabla_i \varphi + R_{ij} \nabla_j F = 0,
\]
where, \( R_{ij} \) is the Ricci tensor of \( M \). Therefore, one has
\[
\langle \nabla \varphi, \nabla F \rangle = -\frac{1}{n-1} \text{Ric}(\nabla F, \nabla F).
\]
By applying \( \nabla_l \) to the both sides of (3.1), we obtain
\[
(n - 1) \nabla_l \nabla_i \varphi + \nabla_l R_{ij} \cdot \nabla_j F + R_{ij} \nabla_l \nabla_j F = 0.
\]
Taking the trace, we obtain
\[
(n - 1) \Delta \varphi + \frac{1}{2} g(\nabla R, \nabla F) + R \varphi = 0.
\]

Proof of Proposition 3.2
(A) If \( M \) is compact, by \( \Delta(-F) = -n \varphi \geq 0 \), the standard Maximum principle shows that \( F \) is constant. Therefore, we assume that \( M \) is noncompact.

(i) Since \(-F \leq -K\), and \( \varphi \leq 0 \), we have
\[
\Delta(-F) = -n \varphi \geq 0.
\]
Since \( M \) is parabolic, \(-F \) is constant. Therefore, \( M \) is trivial.

(ii) By a direct computation,
\[
\text{div}\nabla(-F) = \Delta(-F) = -n \varphi \geq 0.
\]
By Theorem 2.1 we have 3 types of gradient conformal solitons.

Case 1. \( M \) is compact and rotationally symmetric.
This case cannot happen.

To consider Cases 2 and 3, let us recall the following:

**Lemma 3.3** ([3]). Let \( X \) be a smooth vector field on a complete noncompact Riemannian manifold, such that, \( \text{div} X \) does not change the sign on \( M \). If \( |X| \in L^1(M) \), then \( \text{div}_M X = 0 \).
By Lemma 3.3, we have \( \Delta F = 0 \). Hence, we have \( \varphi = 0 \), and \( \nabla \nabla F = 0 \).

Case 2. \( M \) is the warped product
\[ (\mathbb{R}, ds^2) \times_{|\nabla F|} (N^{n-1}, \bar{g}). \]

Since \( \nabla \nabla F = 0 \), we have
\[ \nabla |\nabla F|^2 = 2 \nabla_j \nabla_i F \nabla_i F = 0. \]
Hence, \( \nabla F \) is a constant vector field. Set \( |\nabla F| = a \). If \( a \neq 0 \),
\[ a \text{Vol} (\mathbb{R} \times N^{n-1}) = +\infty. \]
From this and the assumption, we have \( a = 0 \). Therefore, \( F \) is constant, and \( M \) is trivial.

Case 3. \( M \) is rotationally symmetric and equal to the warped product
\[ ([0, \infty), ds^2) \times_{|\nabla F|} (\mathbb{S}^{n-1}, \bar{g}_S). \]
By the same argument as in Case 2, we have that \( F \) is constant.

(iii) Since \( \Delta (-F) \geq 0 \), one has
\[ \Delta F^{-1} = 2F^{-3}|\nabla F|^2 - \Delta FF^{-2} \geq 0. \]
By the Yau’s Maximum principle, one has that \( F^{-1} \) is constant.

(iv) By Lemma 6.3 in [15] and the assumption, we have
\[ \int_{B(x_0, R)} |\nabla F^{-1}|^2 \leq \frac{C}{R^2} \int_{B(x_0, 2R)} F^{-2} \]
\[ \leq \frac{C}{R^2 K^2} \text{Vol}(B(x_0, 2R)) \]
\[ \leq \frac{C}{R K^2}. \]
Take \( R \nearrow +\infty \). The righthand side of (3.6) goes to 0. Therefore, we have that \( F \) is constant.

(B) By \( \Delta F = n \varphi \) and (3.2), we have
\[ \frac{1}{2} \Delta |\nabla F|^2 = |\nabla \nabla F|^2 + \text{Ric}(\nabla F, \nabla F) + \langle \nabla F, \nabla \Delta F \rangle \]
\[ = |\nabla \nabla F|^2 - \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F). \]
From this and $\text{Ric}(\nabla F, \nabla F) \leq 0$, we have
\[ \Delta |\nabla F|^2 \geq 0. \]

If $M$ is compact, by the standard Maximum principle, we have that $|\nabla F|$ is constant.

Assume that $M$ is noncompact. By the Yau’s Maximum principle, $|\nabla F|$ is constant.

By (3.7),
\[ |\nabla \nabla F|^2 - \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F) = 0. \]
Therefore, we have $\nabla \nabla F = 0$.

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. $M$ is compact and rotationally symmetric.

Since $\Delta F = 0$ and $M$ is compact, by the standard Maximum principle, we have that $F$ is constant.

Cases 2 and 3 are considered by the same argument as in (A)-(ii).

\[ \square \]

4. Appendix

In this section, we also give classification results under the similar assumption of Cunha and Lima’s paper [1]. Cunha and Lima also showed the following.

**Theorem 4.1** ([1]). Let $(M, g)$ be a complete gradient $k$-Yamabe soliton. If one of the following is satisfied, then $\sigma_k = \lambda$.

(A) The manifold $M^n$ is noncompact, $|\sigma_k - \lambda| \in L^1(M), \int_M \text{Ric}(\nabla F, \nabla F) \leq 0$, and $F$ has at most quadratic growth on $M$.

(B) The soliton $M^n$ $(n \geq 3)$ is nontrivial, $\sigma_k \geq \lambda$, and $|\nabla F|$ has at most linear growth on $M$, that is, for some $x_0 \in M$, it holds $|\nabla F(x)| \leq Cd(x, x_0)$ near infinity, where $C$ is some uniform constant. Let $u$ be a non-constant solution of

\[ \begin{cases} 
\Delta u + h(u) = 0, \\
\int_M h(u) \langle \nabla u, \nabla F \rangle \geq 0,
\end{cases} \]

for $h \in C^1(\mathbb{R})$, and the function $|\nabla u|$ satisfies
\[ \int_{B(x_0, R)} |\nabla u|^2 = o(\log R), \text{ as } R \to +\infty. \]
(C) The manifold $M^n$ is parabolic and nontrivial with $\text{Ric}(\nabla F, \nabla F) \leq 0$ and $|\nabla F| \in L^\infty(M)$.

**Proposition 4.2.** Let $(M^n, g, F, \varphi)$ be a nontrivial complete gradient conformal soliton. Assume that $M$ satisfies the following (A), (B) or (C).

(A) The function $\varphi$ satisfies that $|\varphi| \in L^1(M)$, $\int_M \text{Ric}(\nabla F, \nabla F) \leq 0$, and $|\nabla F|$ has at most linear growth on $M$.

(B) The function $\varphi$ is nonnegative, and $|\nabla F|$ has at most linear growth on $M$. Let $u$ be a non-constant solution of

\[
\begin{cases}
\Delta u + h(u) = 0, \\
\int_M h(u) \langle \nabla u, \nabla F \rangle \geq 0,
\end{cases}
\]

for $h \in C^1(\mathbb{R})$, and the function $|\nabla u|$ satisfies

\[
\int_{B(x_0, R)} |\nabla u|^2 = o(\log R), \quad \text{as } R \to +\infty.
\]

(C) The manifold $M$ is parabolic and nontrivial with $\text{Ric}(\nabla F, \nabla F) \leq 0$ and $|\nabla F| \in L^\infty(M)$.

Then, $M$ is either

(1) $(\mathbb{R} \times N^{n-1}, ds^2 + a^2 \bar{g}, as + b, 0)$, where $a, b \in \mathbb{R}$, or

(2) $([0, \infty) \times S^{n-1}, ds^2 + a^2 \bar{g}_S, as + b, 0)$, where $\bar{g}_S$ is the round metric on $S^{n-1}$, and $a, b \in \mathbb{R}$.

We observe the following lemma.

**Lemma 4.3.** Any compact gradient conformal soliton with

\[
\int_M \text{Ric}(\nabla F, \nabla F) \leq 0
\]

is trivial.

**Proof.** By $\Delta F = n\varphi$ and (3.2), we have

\[
\int_M \varphi^2 = \frac{1}{n} \int_M \varphi \Delta F = \frac{1}{n} \int_M \langle \nabla \varphi, \nabla F \rangle = \frac{1}{n(n-1)} \int_M \text{Ric}(\nabla F, \nabla F) \leq 0.
\]
Thus, one has \( \phi = 0 \) and \( \nabla \nabla F = 0 \). Hence, we have \( \Delta F = 0 \). By the standard maximum principle, we have that \( M \) is trivial. \( \square \)

**Remark 4.4.** From this lemma, the assumption that \( M \) is noncompact is not needed in (A) of Theorem 4.1.

**Proof of Proposition 4.2**

(A) If \( M \) is compact, by Lemma 4.3 \( M \) is trivial.

Therefore, we assume that \( M \) is noncompact. By \( \Delta F = n \phi \) and (3.2), we have

\[
\int_{B(x_0, R)} \phi^2 = \frac{1}{n} \int_{B(x_0, R)} \phi \Delta F
\]

\[
= -\frac{1}{n} \left\{ \int_{B(x_0, R)} \langle \nabla \phi, \nabla F \rangle + \int_{\partial B(x_0, R)} \phi \langle \nu, \nabla F \rangle \right\}
\]

\[
= -\frac{1}{n} \left\{ \int_{B(x_0, R)} \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F) + \int_{\partial B(x_0, R)} \phi \langle \nu, \nabla F \rangle \right\}
\]

\[
\leq \frac{1}{n(n-1)} \int_{B(x_0, R)} \text{Ric}(\nabla F, \nabla F) + CR \int_{\partial B(x_0, R)} |\phi|,
\]

where, \( \nu \) is the outward unit normal to the boundary \( \partial B(x_0, R) \). Since \( |\phi| \in L^1(M) \), by taking \( R \nearrow +\infty \), one has

\[
CR \int_{\partial B(x_0, R)} |\phi| \rightarrow 0.
\]

Therefore, by the assumption, we have

\[
\int_M \phi^2 = 0,
\]

hence, \( \phi = 0 \).

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. \( M \) is compact. This case cannot happen.

Case 2. \( M \) is the warped product

\[
(\mathbb{R}, ds^2) \times |\nabla F| (N^{n-1}, \bar{g}).
\]

Since \( \nabla \nabla F = 0 \), we have

\[
\nabla |\nabla F|^2 = 2 \nabla_j \nabla_i F \nabla_i F = 0.
\]

Hence, \( \nabla F \) is a constant vector field. Therefore, we have \( F(s) = as + b \).
Case 3. $M$ is rotationally symmetric and equal to the warped product 

$([0, \infty), ds^2) \times |\nabla F| (S^{n-1}, \bar{g}_s)$.

By the same argument as in Case 2, we have that $M$ is 

$([0, \infty) \times S^{n-1}, ds^2 + a^2 \bar{g}_s, as + b, 0)$.

(B) We take a cut off function $\eta$ on $M$ satisfying that

$$\eta(r) = \begin{cases} 
1 & r \leq R, \\
2 - \frac{\log r}{\log R} & r \in [R, R^2], \\
0 & r \geq R^2.
\end{cases}$$

By the soliton equation, we have

$$\int_M \varphi |\nabla u|^2 \eta^2 = \int_{B(x_0, R^2)} \nabla_i \nabla_j F \nabla_i u \nabla_j u \eta^2$$

$$= -\int_{B(x_0, R^2)} \nabla_i F \nabla_j \nabla_i u \nabla_j u \eta^2 + \nabla_i F \Delta u \nabla_j u \eta^2$$

$$- \int_{B(x_0, R^2)} \nabla_i F \nabla_i u \nabla_j u \nabla_j \eta^2.$$

Substituting

$$- \int_{B(x_0, R^2)} \nabla_i F \nabla_j \nabla_i u \nabla_j \eta \eta^2 = \frac{1}{2} \int_{B(x_0, R^2)} n \varphi |\nabla u|^2 \eta^2 + |\nabla u|^2 \langle \nabla F, \nabla \eta \rangle,$$

into (4.2), we have

$$\int_M \varphi |\nabla u|^2 \eta^2 = \frac{1}{2} \int_{B(x_0, R^2)} n \varphi |\nabla u|^2 \eta^2 + 2\eta |\nabla u|^2 \langle \nabla F, \nabla \eta \rangle$$

$$\quad + \int_{B(x_0, R^2)} \langle \nabla F, \nabla u \rangle \eta \eta^2 - 2\eta \langle \nabla F, \nabla u \rangle \langle \nabla u, \nabla \eta \rangle.$$
Therefore, we have
\[
\frac{n-2}{2} \int_M \varphi |\nabla u|^2 \eta^2 = - \int_{B(x_0,R^2)} \eta |\nabla u|^2 \langle \nabla F, \nabla \eta \rangle \\
- \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta |\nabla \eta|^2 + 2 \int_{B(x_0,R^2)} \eta \langle \nabla F, \nabla u \rangle \langle \nabla \eta, \nabla \eta \rangle \\
\leq 3 \int_{B(x_0,R^2)} \eta |\nabla u|^2 |\nabla F| |\nabla \eta| - \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta |\nabla \eta|^2 \\
\leq \frac{C}{\log R} \int_{B(x_0,R^2) \setminus B(x_0,R)} |\nabla u|^2 - \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta |\nabla \eta|^2,
\]
where, the last inequality follows from $|\nabla F| \leq Cr$ near infinity and the definition of the cut off function $\eta$. Take $R \rightarrow +\infty$. From this and the assumption, we have
\[
\frac{n-2}{2} \int_M \varphi |\nabla u|^2 \eta^2 \leq - \int_M \langle \nabla F, \nabla u \rangle \eta |\nabla \eta|^2 \leq 0.
\]
Since $u$ is a non-constant solution, we have $\varphi = 0$. Therefore, we have $\nabla \nabla F = 0$.

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. $M$ is compact and rotationally symmetric.

Since $M$ is compact, by the standard Maximum principle, we have that $F$ is constant. Therefore, $M$ is trivial.

Cases 2 and 3 are considered by the same argument as in $(A)$.

$(C)$ Since (3.2) and $\Delta F = n \varphi$, by a direct computation,
\[
1/2 \Delta |\nabla F|^2 = |\nabla \nabla F|^2 + \text{Ric}(\nabla F, \nabla F) + \langle \nabla F, \nabla \Delta F \rangle \\
= |\nabla \nabla F|^2 - \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F) \\
\geq 0,
\]
where, the last inequality follows from the assumption. Since $M$ is parabolic and $\sup |\nabla F| < +\infty$, we have that $|\nabla F|$ is constant. Therefore, we have $\nabla \nabla F = 0$. By the same argument as in $(B)$, we complete the proof. \qed

**Conflict of interest**

There is no conflict of interest in the manuscript.
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