REMARKS ON $n$-DIMENSIONAL FEYNMAN DIAGRAMS,
FOR EXAMPLE, WHICH WILL APPEAR
IN M-THEORY AND IN F-THEORY

EIJI OGASA

Abstract. We state some remarks on ‘$n$-dimensional Feynman diagrams’ ($n \in \mathbb{N}$).

‘$n$-dimensional Feynman diagrams’ ($n \in \mathbb{N}$) will be used in physics in the near future. Here, we let 1-dimensional Feynman diagrams mean Feynman diagrams in ‘usual’ QFT (see [3][11]). Furthermore, we let 2-dimensional Feynman diagrams mean world sheets in ‘usual’ superstring theory (see [4][12]). We introduce ‘$n$-dimensional Feynman diagrams’ as a generalization of the 1-, 2-dimensional Feynman diagrams as follows.

F-theory, M-theory (see [6][5][14][17][18][20]) etc. imply that particles are represented by manifolds whose dimensions are greater than one. Here, consider interaction of these particles and use perturbation theory like the 0-,1-dimensional particle case. Then we will be able to use manifolds whose dimensions are greater than two in order to represent the interaction. In this paper we call these manifolds $n$-dimensional Feynman diagrams or $n$-dimensional world membranes if the manifolds are $n$-dimensional ones. We may use not only manifolds but also CW complexes for particles and Feynman diagrams (See [2] for CW complexes).

Suppose we will complete the high-dimensional particle theory ($F$-theory, $M$-theory etc.) without using perturbation theory or Feynman diagrams. However, the limit of the theories is ‘usual’ superstring theory or ‘usual’ QFT. Therefore, we will be able to consider $n$-dimensional Feynman diagrams.

Anyway, mathematically we can discuss $n$-dimensional Feynman diagrams.

In this paper we state some remarks on ‘$n$-dimensional Feynman diagrams.’

We consider the case where we make two 3-vertex functions into a 4-vertex function. In this case there is a different feature between in the case of high dimensional Feynman diagrams and in the 1-, 2-dimensional case.

Let $m$ be any integer greater than two. Let $M$ be a compact oriented connected $m$-manifold with boundary. Let $\partial M = A_1 \amalg A_2 \amalg A_3$, where $\amalg$ denotes a disjoint union and $A_1$ (resp.$A_2$, $A_3$) is a connected closed oriented manifold. Let the orientation of $A_i$ be induced from $M$. Suppose that there is an orientation reversing diffeomorphism $A_i \to A_j$, where we do not assume $i = j$ or $i \neq j$. Let $M'$ be diffeomorphic to $M$. Let $\partial M' =$

Keywords: Feynman diagrams. PACS nos. 11-25w, 11-25Uv.
$A'_1 \amalg A'_2 \amalg A'_3$ and $A_l = A'_l(l = 1, 2, 3)$. Take $M$ and $M'$. Identify $A_l$ and $A'_l$ by an orientation reversing diffeomorphism $f : A_l \to A'_l$ and obtain a compact oriented connected $m$-manifold $W_f$ from $M$ and $M'$. Let $S_M$ be the set whose elements are the diffeomorphism classes of such $W_f$

**Theorem 1.** For $m \geq 3$, there is a compact oriented connected $m$-manifold $M$ such that the above set $S_M$ is an infinite set.

**Note.** Theorem 1 means that, for $m \geq 3$, two 3-vertex functions can make infinitely many kinds of 4-vertex functions under some conditions. If $m = 2$, such $M$ does not exist as string theorists and topologists know. For $m = 1$, they can say such does not.

**Proof.** The $m = 3$ case: Take a solid torus. Remove two open 3-balls from the solid torus, call it $M$. Note $\partial M = S^2 \amalg S^2 \amalg T^2$. Let $f : T^2 \to T^2$ be a diffeomorphism. Let $Z$ be any 3-dimensional Lens space (see [13] for Lens spaces). Take any oriented manifold which is made from $Z$ by removing four open 3-balls, call it $Z'$. Then $Z' \in S$. Hence there are countably infinitely many $Z'$, using the homology groups of $Z'$. Hence Theorem 1 is true in the $m = 3$ case.

The $m > 3$ case: Take $D^2 \times T^{m-2}$, where $T^{m-2}$ is an $(m-2)$-dimensional torus. Remove two open $m$-balls from $D^2 \times T^{m-2}$, call it $M$. Then $S$ includes all manifolds which are made from all $[(\text{Lens spaces}) \times T^{m-3}]$ by removing four open $m$-balls, call it $Z'$. Hence there are countably infinitely many $Z'$, using the homology groups of $Z'$. Note $S$ is an infinite set. Hence Theorem 1 is true in the $m > 3$ case. This completes the proof of Theorem 1.

We consider the case where we make 3-vertex functions into an $l$-vertex function $(l \in \mathbb{N} \cup \{0\})$. Here, let the number of kinds of 3-vertex functions be finite. In this case there is a different feature between in the case of high dimensional Feynman diagrams and in the 1-, 2-dimensional case, too.

Let $m$ be any integer greater than two. Let $M_i(i = 1, ..., \mu)$ be a compact oriented connected $m$-dimensional manifold with boundary, where $\mu \in \mathbb{N}$. Let $\partial M_i = \bigcup_{j=1}^{\nu_i} M_{ij}$, where $\nu_i \in \mathbb{N}$, $\nu_i \geq 3$, $M_{ij}$ is a connected closed oriented manifold, and the orientation of $M_{ij}$ is induced from that of $M_i$. Let $B = \prod B_z(z \in \mathbb{N}, z \geq 3)$ be a closed oriented $(m-1)$-manifold.

We define a set $\mathcal{W}_{\{M_i\}, B}$: An arbitrary element $\in \mathcal{W}_{\{M_i\}, B}$ is a compact connected oriented manifold with boundary $B$ with the following properties: There are embedded closed $(m-1)$ manifolds $Y_1, ..., Y_\alpha \subset \text{Int} W$, where $\text{Int} W$ means the interior of $W$ and $Y_i \cap Y_j = \phi$ for $i \neq j$. Let $N(Y_i) = Y_i \times [-1, 1]$ be the neighborhood of $Y_i$ in $W$. Take $W - \text{Int} N(Y_i) = W_1 \amalg \ldots \amalg W_\alpha$. Then each $W_i$ is diffeomorphic to $M_i$ for an $i$.

Let $\mathcal{X}_B$ be a set of all compact oriented connected $m$-manifolds with boundaries $B$.

**Theorem 2.** Let $m$, $M_i$, $B$, $\mathcal{W}_{\{M_i\}, B}$, and $\mathcal{X}_B$ be as above. Then, for any $B$ and any $M_i$, we have $\mathcal{W}_{\{M_i\}, B} \neq \mathcal{X}_B$.
Theorem 2 means that we may need a new discussion to divide complex Feynman diagrams into fundamental parts.

Proof. Let $W \in \mathcal{W}_{\{M_i\},B}$. Then we can divide $W$ into pieces $W_i, \: N(Y_i′)$ as above and can regard $W = W_1 \cup \ldots \cup W_w$. Consider the Meyer-Vietoris exact sequence (see \[2\] \[9\] \[16\] for the Meyer-Vietoris exact sequence):

\[ H_j(\Pi_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \rightarrow H_j(\Pi^w_{i=1}W_i; \mathbb{Q}) \rightarrow H_j(W; \mathbb{Q}). \]

Here, $\Pi_{i,i'}$ means the disjoint unions of $W_i \cap W_{i'}$ for all $(i, i')$. Consider

\[ H_1(W; \mathbb{Q}) \rightarrow H_0(\Pi_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \rightarrow H_0(\Pi W_i; \mathbb{Q}) \rightarrow H_0(W; \mathbb{Q}) \rightarrow 0. \]

Note. If $m = 2$, for any $B$ there exists a manifold $M$ such that, $\mathcal{W}_{\{M\},B} = \mathcal{X}_B$, as topologists and string theorists know. Here, $\{M\}$ is a set which has only one element $M$. Theorem 2 means that we may need a new discussion to divide complex Feynman diagrams into fundamental parts.

Proof. Let $W \in \mathcal{W}_{\{M_i\},B}$. Then we can divide $W$ into pieces $W_i, \: N(Y_i′)$ as above and can regard $W = W_1 \cup \ldots \cup W_w$. Consider the Meyer-Vietoris exact sequence (see \[2\] \[9\] \[16\] for the Meyer-Vietoris exact sequence):

\[ H_j(\Pi_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \rightarrow H_j(\Pi^w_{i=1}W_i; \mathbb{Q}) \rightarrow H_j(W; \mathbb{Q}). \]

Here, $\Pi_{i,i'}$ means the disjoint unions of $W_i \cap W_{i'}$ for all $(i, i')$. Consider

\[ H_1(W; \mathbb{Q}) \rightarrow H_0(\Pi_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \rightarrow H_0(\Pi W_i; \mathbb{Q}) \rightarrow H_0(W; \mathbb{Q}) \rightarrow 0. \]

Note. If $m = 2$, for any $B$ there exists a manifold $M$ such that, $\mathcal{W}_{\{M\},B} = \mathcal{X}_B$, as topologists and string theorists know. Here, $\{M\}$ is a set which has only one element $M$. Theorem 2 means that we may need a new discussion to divide complex Feynman diagrams into fundamental parts.
Theorem 3. Let $Q_g$ be a compact oriented 3-manifold whose handle decomposition is $(F_g \times [0,1]) \cup (a 1$-handle), where $F_g$ denotes a closed oriented surface with genus $g$. Let $Q_{g,h}$ be a compact connected oriented 3-manifold whose handle decomposition is $(F_g \times [0,1]) \cup (F_h \times [0,1]) \cup (a 1$-handle). Note $\partial Q_g = F_g \sqcup F_g$ and $\partial Q_{g,h} = F_g \sqcup F_h \sqcup F_{g+h}$.

Take a set $Q = \{B^3, Q_{g,h}, Q_g, h \in \mathbb{N} \cup \{0\}\}$, where $B^3$ is a 3-ball. Let $M$ be an arbitrary compact oriented 3-manifold with boundary. Then $M$ is made from elements of a finite subset of $Q$ in the similar manner to make $W$ from $\{M_i\}$ above Theorem 2.

One way to use $n$-dimensional Feynman diagrams ($n \geq 3$) is to restrict what kind of compact oriented $n$-manifolds to represent Feynman diagrams. Indeed, in the $n = 1$ case, we restrict what kind of ‘CW-complexes made from 0-cells and 1-cells’ represent the Feynman diagrams. See [2] for CW complexes.

We might note the following: Suppose that we use only elements of $\mathcal{R} = \{B^3, Q_g, Q_{g,h} | g, h \in A\}$, where $A$ is a finite subset of $\mathbb{N} \cup \{0\}$. Then any element of $\mathcal{R}$ includes a submanifold which is diffeomorphic to $Q_k$ (resp. $Q_{k,l}$) for any $k, l \in \mathbb{N} \cup \{0\}$. It might not be a good idea to restrict what kind of compact oriented 3-manifolds to represent Feynman diagrams.

Although it is one way of saying, Witten’s Chern-Simons theory (see [19]) on 3-manifolds $M$ with the gauge group $G$ are regarded as the theory $M \to \mathcal{G}$, where $\mathcal{G}$ is the Lie ring of $G$. Recall that $\mathcal{G}$ is a vector space. Note that, in this case, we can regard all compact oriented 3-manifolds with boundaries as 3-dimensional world membranes. It might not be a good idea to restrict unnaturally what kind of compact oriented 3-manifolds to represent Feynman diagrams. For example, we might need an idea that such restriction make a sense in only low-energy case.

In the two dimensional case (i.e. ‘usual’ string theory) particles are represented both by closed manifolds (i.e. closed strings) and by compact manifolds with boundaries (i.e. open strings). In the $n$-dimensional case ($n \geq 3$) particles will be represented both by closed manifolds and by compact manifolds with boundaries. In this paper we concentrate on the case of closed manifolds.

It might be good to suppose that $n$-dimensional Feynman diagrams are complex manifolds, symplectic ones, Kähler ones, toric ones, hyperbolic ones, or something. However, in these cases, there exist underlying smooth manifolds (and underlying topological manifolds). Hence our theorems in this paper are fundamental restrictions to such the theories, as Pauli exclusion rule and Coleman-Mandula NO-GO theorem are. Because in our theorems $n$-dimensional Feynman diagrams are just smooth manifolds.

Research on $n$-dimensional Feynman diagrams in a time-space is connected with that on submanifolds in a manifold. Submanifold theory includes $n$-dimensional knot theory as an important field. See [1] [7] [10].
REFERENCES

[1] T. D. Cochran and K. E. Orr: Not all links are concordant to boundary links Ann. of Math., 138, 519–554, 1993.
[2] J. Davis and P. Kirk: Lecture notes in algebraic topology. Graduate Studies in Mathematics, 35. American Mathematical Society, Providence, RI, 2001.
[3] R. P. Feynman: Space-time approach to non-relativistic quantum mechanics. Rev. Modern Physics 20, (1948), 367–387.
[4] M. B. Green, J. H. Schwarz, and E. Witten: Superstring theory Vol. 1, 2. Cambridge, Uk: Univ. Pr. (Cambridge Monographs On Mathematical Physics). 1987.
[5] C. V. Johnson: D-branes. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2003.
[6] C. M. Hull and P. K. Townsend: Unity of superstring dualities. Nuclear Phys. B 438 (1995), no. 1-2, 109–137.
[7] J. Levine and K. Orr: A survey of applications of surgery to knot and link theory. Surveys on surgery theory: surveys presented in honor of C.T.C. Wall Vol. 1, 345–364, Ann. of Math. Stud., 145, Princeton Univ, 2000.
[8] J. Milnor: Lectures on the h-cobordism theorem Princeton University Press 1965
[9] J. W. Milnor and J. D. Stasheff: Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press 1974.
[10] E. Ogasa: (i) [math.GT/0004008 University of Tokyo Preprint series UTMS 97-35]; (ii)Mathematical Research Letters, 5 (1998), 577-582, UTMS 95-50; (iii)Journal of knot theory and its ramifications, 10 (2001), 121–132 UTMS 97-34, [math.GT/0003088]; (iv)Mathematical Proceedings of Cambridge Philosophical Society 126, 1999, 511-519; (v)UTMS 97-63; (vi)[math.GT/0011163 UTMS00-65; (vii)[math.GT/0004007 UTMS 00-22; (viii)hep-th/0311136
[11] M. E. Peskin and D. V. Schroeder: An introduction to quantum field theory. Addison-Wesley Publishing Company, 1995.
[12] J. Polchinski: String theory. Vol. 1, 2. Cambridge, UK: Univ. Pr. 1998.
[13] D. Rolfsen: Knots and links Publish or Perish, Inc. 1976.
[14] J. H. Schwarz: The power of M theory. Phys. Lett. B 367 (1996), no. 1-4, 97–103.
[15] S. Smale: Generalized Poincare's conjecture in dimensions greater than four. Ann. of Math. (2) 74 1961 391–406.
[16] N. Steenrod: The topology of fibre bundles. Reprint of the 1957 edition. Princeton University Press 1999.
[17] P. K. Townsend: M-theory: a new paradigm for quantum gravity. Frontiers in quantum physics (Kuala Lumpur, 1997), 15–23, Springer, Singapore, 1998.
[18] C. Vafa: Evidence for F-theory. Nuclear Phys. B 469 (1996), no. 3, 403–415.
[19] E. Witten: Quantum field theory and the Jones polynomial, Comm. Math. Phys., 121 (1989), 351-399.
[20] E. Witten: String theory dynamics in various dimensions. Nuclear Phys. B 443 (1995), no. 1-2, 85–126.

HIGH ENERGY PHYSICS THEORY GROUP, DEPARTMENT OF PHYSICS, OCHANOMIZU UNIVERSITY, BUNKYO-KU, TOKYO 112-8610, JAPAN,
E-mail address: ogasa@phys.ocha.ac.jp