PULLBACK ATTRACTORS OF FITZHUGH-NAGUMO SYSTEM ON THE TIME-VARYING DOMAINS

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ABSTRACT. The existence and uniqueness of solutions satisfying energy equality is proved for non-autonomous FitzHugh-Nagumo system on a special time-varying domain which is a (possibly non-smooth) domain expanding with time. By constructing a suitable penalty function for the two cases respectively, we establish the existence of a pullback attractor for non-autonomous FitzHugh-Nagumo system on a special time-varying domain.

1. Introduction. There are many papers devoted to studying the semilinear parabolic equation on a time-varying domain, we refer the readers to [1, 2, 5, 6, 7, 8, 16] and the references therein, most of them studied the nonlinear parabolic equation on the time-varying domains. Recently, Kloeden, Marín-Rubio and Real in [7] followed the penalty function method developed by J.Lions in [9] to establish the existence of the pullback attractor for a semilinear heat equation with a homogeneous Dirichlet boundary condition defined on the domains expanding in time. Moreover, Kloeden, Real and Sun in [6] established the existence of a global pullback attractor for the same equation defined on general time-varying domain by using a $C^2$-diffeomorphism transformation.

In this paper, we will study the dynamics of the non-autonomous FitzHugh-Nagumo system with homogeneous Dirichlet boundary condition defined on the time-varying domains $Q_\tau$ by

\[
\begin{aligned}
\frac{\partial u}{\partial \tau} - \nu \Delta u + \lambda u + h(u) + v &= f(t), \text{ in } Q_\tau, \\
\frac{\partial v}{\partial \tau} - \epsilon(u - \gamma v) &= \epsilon g(t), \text{ in } Q_\tau, \\
u &= 0, \text{ on } \Sigma_\tau, \\
(u(\tau, x), v(\tau, x)) &= (u_\tau(x), v_\tau(x)), x \in \mathcal{O}_\tau.
\end{aligned}
\]

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where the notations are defined later in section 2. We establish the existence of a pullback attractor for non-autonomous FitzHugh-Nagumo system. FitzHugh-Nagumo equations introduced by FitzHugh [4] and Nagumo, Arimoto and Yosimawa [14] are intended to describe the signal transmission across axons. The long time behavior of FitzHugh-Nagumo equations on bounded and unbounded domains was presented by some authors in [10, 11, 12, 13, 17, 18] and the references therein.

The main novelty of the present paper is to construct a new penalty function to show the uniqueness and existence of the variational solution for equations (1) on the time-varying domain expanding in time, which is motivated by the idea in [7] and [9], and establish the existence of a pullback attractor of non-autonomous FitzHugh-Nagumo system on the time-varying domain.

The rest of the paper is arranged as follows. In section 2, some notations and functions setting are introduced. In section 3, the existence of variational solution for FitzHugh-Nagumo system on the time-varying domain which increases with time are obtained by the penalty function method. In section 4, the existence of pullback attractor of FitzHugh-Nagumo system on the time-varying domain are obtained.

2. Preliminaries. Let \( \{O_t\}_{t \in [\tau, T]} \) be a family nonempty bounded open subset of \( \mathbb{R}^N \), and

\[
\begin{align*}
Q_{\tau, T} &:= \bigcup_{t \in (\tau, T)} O_t \times \{t\} \text{ for all } T > \tau, \\
Q_{\tau} &:= \bigcup_{t \in (\tau, \infty)} O_t \times \{t\} \text{ for all } \tau \in \mathbb{R}, \\
\Sigma_{\tau, T} &:= \bigcup_{t \in (\tau, T)} \partial O_t \times \{t\} \text{ for all } T > \tau, \\
\Sigma_{\tau} &:= \bigcup_{t \in (\tau, \infty)} \partial O_t \times \{t\} \text{ for all } \tau \in \mathbb{R}.
\end{align*}
\]

For any \( T > \tau \), the set \( Q_{\tau, T} \) is an open subset of \( \mathbb{R}^{N+1} \) with boundary \( \partial Q_{\tau, T} := \Sigma_{\tau, T} \cup (\partial O_{\tau} \times \{\tau\}) \cup (\partial O_T \times \{T\}) \).

We consider the following auxiliary problem to equations (1)

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + \lambda u + h(u) + v = f(t), & \text{in } Q_{\tau, T}, \\
\frac{\partial v}{\partial t} - \epsilon (u - \gamma v) = \epsilon g(t), & \text{in } Q_{\tau}, \\
u = 0, & \text{on } \Sigma_{\tau}, \\
u(\tau, x) = u_\tau(x), v(\tau, x) = v_\tau(x), & x \in O_\tau,
\end{cases}
\]

where \( \tau \in \mathbb{R}, u_\tau, v_\tau : O_\tau \to \mathbb{R} \) and \( f, g : Q_\tau \to \mathbb{R} \) are given, \( \nu, \lambda, \epsilon, \gamma \) are positive constants, \( h \) is a smooth nonlinear function such that for some positive constant \( \alpha, \beta, l \) and \( p \geq 2 \),

\[
h'(s) \geq -C, \ s \in \mathbb{R},
\]

and

\[
-\beta + \alpha_2 |s|^p \leq h(s)s \leq \beta + \alpha_1 |s|^p, \ s \in \mathbb{R}.
\]

For later observe, from [4], we have for some positive constant \( \tilde{\alpha}, \tilde{\beta} \)

\[
-\tilde{\beta} + \tilde{\alpha}_2 |s|^p \leq H(s) \leq \tilde{\beta} + \tilde{\alpha}_1 |s|^p, \ s \in \mathbb{R}
\]

where \( H(x) := \int_0^x h(s)ds \).
Since $\epsilon$ is small, we can assume that

$$0 < \epsilon \leq \epsilon_0, \text{ where } \epsilon_0 = \min\{1, \frac{\lambda}{\gamma}\}. \quad (7)$$

Denote $\sigma$ be a positive number given by

$$\sigma = \frac{1}{2}\epsilon\gamma. \quad (8)$$

Here, we consider the domains expanding with time, let $\{O_t\}_{t \in [\tau, T]}$ satisfy that

if $s < t$, then $O_s \subset O_t$. \quad (9)

For each $t < T$, consider $H^1_0(O_t)$ as a closed subspace of $H^1_0(O_T)$ with the function belonging to $H^1_0(O_t)$ being trivially extended by zero. It follows from \cite{6} that $\{H^1_0(O_t)\}_{t \in [\tau, T]}$ can be considered as a family of $H^1_0(O_T)$ for each $T > \tau$ with

$$s < t \rightarrow H^1_0(O_s) \subset H^1_0(O_t). \quad (10)$$

Similarly, we have

$$s < t \rightarrow L^2(O_s) \subset L^2(O_t). \quad (11)$$

For the sake of simplicity, we denote

$$\bar{Q}_{\tau, T} := O_T \times (\tau, T)$$

and

$$\bar{Q}_\tau := \bigcup_{T \geq \tau} O_T \times (\tau, T).$$

Denote by $(\cdot, \cdot)_t$, $| \cdot |_t$ the usual inner product and associated norm in $L^2(O_t)$ or $(L^2(O_t))^N$, and by $(\cdot, \cdot)_t$, $| \cdot |_t$ the usual inner product and associated norm in $H^1_0(O_t)$. Notice that $(\cdot, \cdot)_t$ is also used to denote the duality product between $L^p/O_t$ and $L^p(O_t)$. More notations and properties about time-varying domains can be found in \cite{6} and \cite{7}.

Now, we recall the basic concept of pullback attractor for non-autonomous dynamical systems. For each $t \in \mathbb{R}$, and $D_1$, $D_2$ nonempty subsets of $L^2(O_t) \times L^2(O_t)$. Let us denote $dist_t(D_1, D_2)$ the Hausdorff semi-distance defined as

$$dist_t(D_1, D_2) := \sup_{u \in D_1} \inf_{v \in D_2} |u - v|_{L^2(O_t) \times L^2(O_t)}.$$  

Let $R_\sigma$ be the set of all functions $\rho : \mathbb{R} \rightarrow [0, \infty)$ such that

$$e^{\sigma \tau} \rho(\tau) \rightarrow 0 \text{ as } \tau \rightarrow -\infty.$$  

$D_\sigma$ be the class of all families $\hat{D} := \{D(t), t \in \mathbb{R} : D(t) \subset L^2(O_t), D(t) \neq \emptyset\}$ such that $D(t) \subset \{u, v \in L^2(O_t) : \epsilon|u|^2 + |v|^2 \leq \rho_D^2(t)\}$ for some $\rho_D \in R_\sigma$.

**Definition 2.1.** A family $\hat{A} = \{\mathcal{A}(t) : \mathcal{A}(t) \in L^2(O_t) \times L^2(O_t), \mathcal{A}(t) \neq \emptyset, t \in \mathbb{R}\}$ is said to be a pullback $D_\sigma$-attractor for the process $\hat{Y}(t, \tau) := L^2(O_{\tau}) \times L^2(O_{\tau}) \rightarrow L^2(O_t) \times L^2(O_t)$, if

1. $\mathcal{A}(t)$ is compact in $L^2(O_t) \times L^2(O_t)$ for all $t \in \mathbb{R}$;
2. $\hat{A}$ is $L^2$-pullback $D_\sigma$-attracting, i.e.

$$\lim_{\tau \rightarrow -\infty} dist_t(\hat{Y}(t, \tau)D(\tau), \mathcal{A}(t)) = 0 \text{ for all } \hat{D} \in D_\sigma \text{ and all } t \in \mathbb{R};$$
3. $\hat{A}$ is invariant, i.e.

$$\hat{Y}(t, \tau)A(\tau) = \mathcal{A}(t) \text{ for all } -\infty < \tau \leq t < \infty.$$
Lemma 3.2. The process $\Upsilon(t, \tau)$ is said to be pullback $\mathcal{D}_\sigma$-asymptotically compact if the sequence $\{\Upsilon(t, \tau_n)\}_{(u_n, v_n)}$ is relatively compact in $(L^2(\mathcal{O}_t))^2$ for any $t \in \mathbb{R}$, and $\hat{D} = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_\sigma$, any sequences $\{\tau_n\}$ and $(u_n, v_n)$ with $\tau_n \to -\infty$ and $(u_n, v_n) \in D(\tau_n)$.

Theorem 2.3. Suppose that the process $\Upsilon(.\cdot)$ is $L^2$-pullback $\mathcal{D}_\sigma$-asymptotically compact and that $\hat{D}_0$ is a family of $L^2$-Pullback $\mathcal{D}_\sigma$-attracting set for $\Upsilon(.\cdot)$. Then the family $\hat{A} = \{A(t), t \in \mathbb{R}\}$ defined by

$$A(t) = \Lambda(\hat{D}_0, t) \in \mathbb{R},$$

where for any $\hat{D} \in \mathcal{D}_\sigma,$

$$\Lambda(\hat{D}, t) := \bigcap_{s \leq t} \left( \bigcup_{\tau \leq s} \Upsilon(t, \tau) D(\tau) \right), \quad t \in \mathbb{R}, \text{ closure in } L^2(\mathcal{O}_t),$$

is the unique pullback $\mathcal{D}_\sigma$-attractor for process $\Upsilon(t, \tau)$ belonging to $\mathcal{D}_\sigma$. In addition, $\hat{A}$ satisfies

$$\Lambda(t) = \bigcup_{\hat{D} \in \mathcal{D}_{\sigma}} \Lambda(\hat{D}, t) \quad \forall t \in \mathbb{R}.$$

Furthermore, $\hat{A}$ is minimal in the sense the if $\hat{C} = \{C(t), t \in \mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of $(L^2(\mathcal{O}_t))^2$ and

$$\lim_{\tau \to -\infty} \text{dist}_\sigma(\Upsilon(t, \tau), D_0(\tau), C(t)) = 0$$

for all $t \in \mathbb{R}$, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

3. Variational solution on domains expanding in time. In this section, we consider Fitzhugh-Nagumo system on bounded spatial domains (possibly non-smooth) which are expanding in time. For each $T > \tau$, denote

$$U_{\tau, T} := \{ \vartheta \in L^2(\tau, T; H_0^1(\mathcal{O}_T)) \cap \mathcal{L}^p(\tau, T; \mathcal{L}^p(\mathcal{O}_T)) : \vartheta' \in L^2(\tau, T; \mathcal{L}^2(\mathcal{O}_T)), \vartheta(\tau) = \vartheta(T) = 0, \vartheta(t) \in H_0^1(\mathcal{O}_t) \text{ a.e. in } (\tau, T)\}.$$ 

Definition 3.1. A variational solution of (3) on the bounded spatial domain expanding in time is a function $(u, v)$ such that

1. $(u, v) \in \left(L^2([\tau, T]; H_0^1(\mathcal{O}_T)) \cap \mathcal{L}^p(\mathcal{O}_T)) \times L^2(\mathcal{O}_T),
2. \text{ for all } \vartheta \in U_{\tau, T},$

$$\int_{\tau}^{T} - (u, \vartheta')_T + \nu((u, \vartheta)_T + \lambda(u, \vartheta)_T + (v, \vartheta)_T + (h(u), \vartheta)_T dt = \int_{\tau}^{T} (f, \vartheta)_T dt,$$

$$\int_{\tau}^{T} - (v, \vartheta')_T - \epsilon(u, \vartheta)_T + \epsilon(t, \vartheta)_T + \epsilon(t, \vartheta)_T dt = \epsilon \int_{\tau}^{T} (g, \vartheta)_T dt;$$

3. $u(t) \in H_0^1(\mathcal{O}_t)$ and $v(t) \in L^2(\mathcal{O}_t)$ a.e. in $(\tau, T),$
4. $\lim_{t \to T} u \vartheta(t) - \vartheta |_{T} - v(t) - v_T |_{T} dt = 0.$

Lemma 3.2. Let $(u, v)$ be a variational solution of (3) on bounded spatial domains which are expanding in time and suppose that there exists a sequence $\{t_n\} \subset (\tau, T)$ of Lebesgue points of $|u|_T^2$ and $|v|_T^2$ such that $t_n \to T$, then

$$\limsup_{n \to \infty} |u(t_n)|_T^2 \leq |u_T|_T^2 + 2 \int_{\tau}^{T} (f(t, u(t))_T - v_2(u(t)|_T - \lambda|u(t)|_T - (v(t), u(t))_T (h(u), u(t))_T dt$$
and
\[
\limsup_{n \to \infty} |v(t_n)|^2_T \leq |v_T|^2_T + 2 \int_\tau^T \epsilon(g(t), u(t))T - \gamma \epsilon|v(t)|T + \epsilon(v(t), u(t))T dt.
\]
Then, \((u, v)\) satisfies the energy equality a.e. in \((\tau, T)\),
\[
|u(t)|^2_T + 2 \int_\tau^t \nu|u(s)|^2_T + \lambda|u(s)|^2_T + (v(s), u(s))T ds
\]
and
\[
|v(t)|^2_T + 2 \int_\tau^t \epsilon(v(s))T - \epsilon(v(s), u(s))T ds = |v_T|^2_T + 2 \epsilon \int_{\tau}^t (g(s), v(s))T ds.
\]

Proof. The proof is similar to Lemma 10 in [7], and is omitted here. \(\square\)

**Proposition 1.** Let \((u_i, v_i)\), \((i = 1, 2)\) be two variational solutions of (3) corresponding to the initial data \((u_{\tau,i}, v_{\tau,i})\), \((i = 1, 2)\) \(\in \mathbb{L}^2(O_\tau)\) respectively, and satisfy the energy equality a.e. in \((\tau, T)\), then
\[
eq |u_{\tau,1} - u_{\tau,2}|^2 + |v_{\tau,1} - v_{\tau,2}|^2,
\]
a.e. \(t \in (\tau, T)\).

Proof. The proof is similar to Lemma 11 in [7], and is omitted here. \(\square\)

**Proposition 1** guarantees the uniqueness of the variational solutions for (3) on domain expending in time immediately.

3.1. **Penalty method.** In this subsection, we follow the penalty method in [7], which developed by [9] to establish the existence and uniqueness of the variational solution for Fitzhugh-Nagumo system on the domain expending in time.

Fixed \(T > \tau\) and for each \(t \in [\tau, T]\), denote \(H_0^1(\mathbb{O}_t)\) the orthogonal subspace of \(H_0^1(\mathbb{O}_T)\) i.e.
\[
H_0^1(\mathbb{O}_t) := \{ \psi \in H_0^1(\mathbb{O}_T) : ((\psi, \omega))_T = 0 \ \forall \omega \in H_0^1(\mathbb{O}_t) \}.
\]

Let \(P(T) \in \mathcal{L}(H_0^1(\mathbb{O}_T))\) the orthogonal projection operator from \(H_0^1(\mathbb{O}_T)\) onto \(H_0^1(\mathbb{O}_t)\) defined by
\[
P(T) \psi \in H_0^1(\mathbb{O}_t), \quad \psi - P(T) \psi \in H_0^1(\mathbb{O}_t),
\]
for each \(\psi \in H_0^1(\mathbb{O}_T)\). For all \(t > T\), define \(P(t) = P(T)\), where \(P(T)\) is the zero of \(\mathcal{L}(H_0^1(\mathbb{O}_T))\).

Let \(Q(T) \in \mathcal{L}(L^2(\mathbb{O}_T))\) be the orthogonal projection operator from \(L^2(\mathbb{O}_T)\) onto \(L^2(\mathbb{O}_t)\). We now approximate \(P(t)\) and \(Q(t)\) by operators which are more regular in time. Consider the family \(P(t, \cdot, \cdot)\) of symmetric bilinear forms on \(H_0^1(\mathbb{O}_T)\) defined by
\[
P(t, \psi, \omega) := ((P(t) \psi, \omega))_T \ \forall \psi, \omega \in H_0^1(\mathbb{O}_T), \ \forall t \geq \tau,
\]
and family \(Q(t, \cdot, \cdot)\) of symmetric bilinear forms on \(L^2(\mathbb{O}_T)\) defined by
\[
Q(t, \varphi, \phi) := (Q(t) \varphi, \phi)_T \ \forall \varphi, \phi \in L^2(\mathbb{O}_T), \ \forall t \geq \tau.
\]
It can be shown that the mapping \([\tau, \infty) \ni t \to P(t, \psi, \omega) \in \mathbb{R}, (\forall \psi, \omega \in H_0^1(\mathcal{O}_T))\) and the mapping \([\tau, \infty) \ni t \to Q(t, \varphi, \phi), (\forall \varphi, \phi \in L^2(\mathcal{O}_T))\) are measurable. Moreover, \(|P(t, \psi, \omega)| \leq \|\psi\|_T \|\omega\|_T\) and \(|Q(t, \phi, \varphi)| \leq |\phi|_T |\varphi|_T\). For each integer \(k \geq 1\) and \(t \geq \tau\), define
\[
P_k(t; \psi, \omega) := k \int_0^{1/k} P(t + r; \psi, \omega) dr \forall \psi, \omega \in H_0^1(\mathcal{O}_T), \forall t \geq \tau,
\]
and
\[
Q_k(t, \varphi, \phi) := k \int_0^{1/k} Q(t + r; \varphi, \phi) dr \forall \varphi, \phi \in L^2(\mathcal{O}_T), \forall t \geq \tau.
\]
Denote \(P_k(t) \in \mathcal{L}(H_0^1(\mathcal{O}_T))\) the associated operator defined by
\[
(P_k(t) \psi, \omega) := P_k(t; \psi, \omega) \forall \psi, \omega \in H_0^1(\mathcal{O}_T), \forall t \geq \tau,
\]
and \(Q_k(t) \in \mathcal{L}(L^2(\mathcal{O}_T))\) the associated operator defined by
\[
(Q_k(t) \varphi, \phi) := Q_k(t, \varphi, \phi) \forall \varphi, \phi \in L^2(\mathcal{O}_T), \forall t \geq \tau.
\]
By using the proof similar to [7], we can get the following lemmas.

**Lemma 3.3.** For any integers \(1 \leq h \leq k\), any \(t \geq \tau\), \(\psi, \omega \in H_0^1(\mathcal{O}_T)\), we have
\[
P_k(t; \psi, \omega) = P_k(t; \omega, \psi),
\]
\[
0 \leq P_k(t; \psi, \psi) \leq P(t; \psi, \psi) \leq \|P(t)\psi\|_T^2 \leq \|\psi\|_T^2,
\]
\[
P_k(t; \psi, \psi) := \frac{d}{dt} P_k(t; \psi, \psi) = k(P(t + \frac{1}{k}; \psi, \psi) - P(t; \psi, \psi)) \leq 0,
\]
\[
((P_k(t) \psi, \varphi))_T = 0 \forall \varphi \in H_0^1(\mathcal{O}_T).
\]
Moreover, for every sequence \(\{\psi_k\} \subset L^2(\tau, T; H_0^1(\mathcal{O}_T))\) weakly convergent to \(\psi\) in \(L^2(\tau, T; H_0^1(\mathcal{O}_T))\),
\[
\liminf_{k \to +\infty} \int_\tau^T P_k(t; \psi, \omega) dt \geq \int_\tau^T P(t; \psi, \omega) dt.
\]

**Lemma 3.4.** For any integers \(1 \leq h \leq k\), any \(t \geq \tau\), and every \(\phi, \varphi \in L^2(\mathcal{O}_T)\), we have
\[
Q_k(t; \varphi, \phi) = Q_k(t; \phi, \varphi),
\]
\[
0 \leq Q_k(t; \phi, \phi) \leq Q(t; \phi, \phi) \leq |Q(t)\phi|_T^2 \leq |\phi|_T^2,
\]
\[
Q_k(t; \phi, \phi) := \frac{d}{dt} Q_k(t; \phi, \phi) = k(Q(t + \frac{1}{k}; \phi, \phi) - Q(t; \phi, \phi)) \leq 0,
\]
\[
((Q_k(t) \phi, \kappa))_T = 0 \forall \kappa \in L^2(\mathcal{O}_T).
\]
Moreover, for every sequence \(\{\phi_k\} \subset L^2(\tau, T; L^2(\mathcal{O}_T))\) weakly convergent to \(\phi\) in \(L^2(\tau, T; L^2(\mathcal{O}_T))\),
\[
\liminf_{k \to +\infty} \int_\tau^T Q_k(t; \phi, \varphi) dt \geq \int_\tau^T Q(t; \phi, \varphi) dt.
\]

For each integer \(k \geq 1\) and each \(t \in [\tau, T]\), we consider the following symmetric bilinear operators
\[
A_k(t)(\psi, \omega) := \nu((\psi, \omega))_T + k((P_k(t) \psi, \omega))_T \psi, \forall \omega \in H_0^1(\mathcal{O}_T)
\]
and
\[
B_k(t)(\phi, \varphi) := k(Q_k(t) \phi, \varphi)_T \forall \phi, \varphi \in L^2(\mathcal{O}_T).
\]
We have the following estimates
\[
A_k(t)(\psi, \psi) \geq \nu\|\psi\|_T^2 \forall \psi \in H_0^1(\mathcal{O}_T), \forall t \in [\tau, T],
\]
Proof. Now, let \((u_\tau, v_\tau) \in (L^2(\Omega_T))^2\) be given and \(k \geq 1\). We consider the problem

\[
\begin{aligned}
\frac{1}{2 + T - \tau} \int_\tau^T |u_k|^2 \, ds + \frac{1}{2 + T - \tau} \nu \|u_k\|^2_{L^\infty(\tau, T; H^1_0(\Omega_T))} &
+ \frac{1}{2 + T - \tau} \int_\tau^T k((P_k(t)u_k(t), u_k)) \, dt \\
&\leq \nu\|u_\tau\|^2_T + k((P_k(\tau)u_\tau, u_\tau))_T + 2\alpha_2\|u_\tau\|^p_{L^p(\Omega_T)} + 4\beta|\Omega_T| \\
&\quad + 2(T - \tau) \left( \epsilon\|u_\tau\|^2_T + v_\tau^2 \right) \\
&\quad + \int_\tau^T \left( \frac{\epsilon}{\lambda}|f(s)|^2_T ds + \frac{\epsilon}{\gamma}|g(s)|^2_T + 2\epsilon\beta|\Omega_T| \right) ds \\
&\quad + 2 \int_\tau^T (|f(s)|^2_T ds + \lambda|u_k(\tau)|^2_T a.e. t \in (\tau, T),
\end{aligned}
\]

\[
\frac{1}{1 + T - \tau} \int_\tau^T |v_k|^2 \, ds + \frac{k}{1 + T - \tau} \int_\tau^T (Q_k(t)u_k(t), u_k) \, dt \\
\leq k(Q_k(\tau)v_\tau, v_\tau) + \epsilon|v_\tau|^2_T + 2\epsilon^2(\int_\tau^T |g(s)|^2_T ds) \\
+ 2\epsilon[T - \tau] \left( \epsilon\|u_\tau\|^2_T + v_\tau^2 \right) \\
+ \int_\tau^T \left( \frac{\epsilon}{\lambda}|f(s)|^2_T ds + \frac{\epsilon}{\gamma}|g(s)|^2_T + 2\epsilon\beta|\Omega_T| \right) ds.
\]

Theorem 3.5. Assume that (9) hold, \(\partial \Omega\) is \(C^2\) and \(N \leq 2p/(p - 2)\) or \(\partial \Omega\) is \(C^m\) with \(m \geq 2\) integer such that \(m \geq N(p - 2)/2p \) and \(h(s)\) satisfy (2.7)-(2.8). Then for any \((u_\tau, v_\tau) \in (H^1_0(\Omega_T) \cap L^p(\Omega_T)) \times L^2(\Omega_T)(-\infty < \tau < T < +\infty)\) and fixed \(k \geq 1\), there exists a unique variational solution \((u_k, v_k)\) of (28). Moreover, for a.e. \(t \in (\tau, T)\),

\[
\begin{aligned}
\frac{1}{2 + T - \tau} \int_\tau^T |u_k|^2 \, ds + \frac{1}{2 + T - \tau} \nu \|u_k\|^2_{L^\infty(\tau, T; H^1_0(\Omega_T))} &
+ \frac{1}{2 + T - \tau} \int_\tau^T k((P_k(t)u_k(t), u_k)) \, dt \\
&\leq \nu\|u_\tau\|^2_T + k((P_k(\tau)u_\tau, u_\tau))_T + 2\alpha_2\|u_\tau\|^p_{L^p(\Omega_T)} + 4\beta|\Omega_T| \\
&\quad + 2(T - \tau) \left( \epsilon\|u_\tau\|^2_T + v_\tau^2 \right) \\
&\quad + \int_\tau^T \left( \frac{\epsilon}{\lambda}|f(s)|^2_T ds + \frac{\epsilon}{\gamma}|g(s)|^2_T + 2\epsilon\beta|\Omega_T| \right) ds \\
&\quad + 2 \int_\tau^T (|f(s)|^2_T ds + \lambda|u_k(\tau)|^2_T a.e. t \in (\tau, T),
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{1}{1 + T - \tau} \int_\tau^T |v_k|^2 \, ds + \frac{k}{1 + T - \tau} \int_\tau^T (Q_k(t)u_k(t), u_k) \, dt \\
\leq k(Q_k(\tau)v_\tau, v_\tau) + \epsilon|v_\tau|^2_T + 2\epsilon^2(\int_\tau^T |g(s)|^2_T ds) \\
+ 2\epsilon[T - \tau] \left( \epsilon\|u_\tau\|^2_T + v_\tau^2 \right) \\
+ \int_\tau^T \left( \frac{\epsilon}{\lambda}|f(s)|^2_T ds + \frac{\epsilon}{\gamma}|g(s)|^2_T + 2\epsilon\beta|\Omega_T| \right) ds.
\end{aligned}
\]

Proof. The existence of solution of (28) can be obtained by the Galerkin method (3, 9, 15). Let \(e_j = e_j(x) \in H^2(\Omega_T) \cap H^1_0(\Omega_T)(j = 1, 2, \ldots)\). For each fixed positive integer \(m\), let

\[
\begin{aligned}
&u_{k_m}(t) := \sum_{j=1}^m a_{k_m, j}(t)e_j, \quad v_{k_m}(t) := \sum_{j=1}^m b_{k_m, j}(t)e_j
\end{aligned}
\]

to be the unique solution of the equations

\[
\begin{aligned}
&\frac{1}{2 + T - \tau} \int_\tau^T |u_k|^2 \, ds + \frac{1}{2 + T - \tau} \nu \|u_k\|^2_{L^\infty(\tau, T; H^1_0(\Omega_T))} + \frac{1}{2 + T - \tau} \int_\tau^T k((P_k(t)u_k(t), u_k)) \, dt \\
\end{aligned}
\]

\[
\begin{aligned}
&\frac{1}{1 + T - \tau} \int_\tau^T |v_k|^2 \, ds + \frac{k}{1 + T - \tau} \int_\tau^T (Q_k(t)u_k(t), u_k) \, dt \\
\leq k(Q_k(\tau)v_\tau, v_\tau) + \epsilon|v_\tau|^2_T + 2\epsilon^2(\int_\tau^T |g(s)|^2_T ds) \\
+ 2\epsilon[T - \tau] \left( \epsilon\|u_\tau\|^2_T + v_\tau^2 \right) \\
+ \int_\tau^T \left( \frac{\epsilon}{\lambda}|f(s)|^2_T ds + \frac{\epsilon}{\gamma}|g(s)|^2_T + 2\epsilon\beta|\Omega_T| \right) ds.
\end{aligned}
\]
where
\[
\begin{align*}
\tau_m(t) &= \sum_{j=1}^{m} (u_{\tau}, e_j) e_j, \\
\tau_m(t) &= \sum_{j=1}^{m} (v_{\tau}, e_j) e_j,
\end{align*}
\]
and \( j \in \mathbb{N} \). Moreover, the solution \((u_{k_m}, v_{k_m})\) satisfies the energy equality
\[
\begin{align*}
\|u_{k_m}(t)\|^2_T + 2 \int_{\tau}^{T} A_k(s)(u_{k_m}(s), u_{k_m}(s))ds + 2\lambda \int_{\tau}^{T} |u_{k_m}(s)|^2_T ds \\
+ 2 \int_{\tau}^{T} (u_{k_m}(s), u_{k_m}(s))_T ds + 2 \int_{\tau}^{T} (h(u_{k_m}(s)), u_{k_m}(s))_T dr
\end{align*}
\]
\[
\begin{align*}
= |u_{\tau_m}(\tau)|^2_T + 2 \int_{\tau}^{T} (f(s), u_{k_m}(s))_T ds, \\
|v_{k_m}(t)|^2_T + 2 \int_{\tau}^{T} B_k(s)(v_{k_m}(s), v_{k_m}(s))_T ds - 2\epsilon \int_{\tau}^{T} (u_{k_m}, v_{k_m}(s))_T ds \\
+ 2\epsilon \int_{\tau}^{T} |v_{k_m}(s)|^2_T ds = |v_{\tau_m}(\tau)|^2_T + 2 \int_{\tau}^{T} (g(s), v_{k_m}(s))_T ds.
\end{align*}
\]
It follows from (5), (26) and (27) that
\[
\begin{align*}
|u_{k_m}(t)|^2_T + 2 \nu \int_{\tau}^{T} \|u_{k_m}(s)\|^2_T ds + 2\lambda \int_{\tau}^{T} |u_{k_m}(s)|^2_T ds \\
+ 2 \int_{\tau}^{T} (u_{k_m}(s), u_{k_m}(s))_T ds + 2\alpha \int_{\tau}^{T} \|u_{k_m}(s)\|^p_T ds
\end{align*}
\]
\[
\leq |u_{\tau_m}(\tau)|^2_T + 2 \int_{\tau}^{T} (f(s), u_{k_m}(s))_T ds + 2\beta (t - \tau)|\Omega_T|, \\
|v_{k_m}(t)|^2_T - 2 \int_{\tau}^{T} (u_{k_m}, v_{k_m}(s))_T ds + 2\epsilon \int_{\tau}^{T} |v_{k_m}(s)|^2_T ds \\
\leq |v_{\tau_m}(\tau)|^2_T + 2 \int_{\tau}^{T} (g(s), v_{k_m}(s))_T ds.
\]
Due to (8), (34) and (35), we obtain
\[
\epsilon |u_{k_m}(t)|^2_T + |v_{k_m}(t)|^2_T + 2 \epsilon \int_{\tau}^{T} \|u_{k_m}(s)\|^2_T ds + 2\epsilon \int_{\tau}^{T} |u_{k_m}(s)|^2_T ds \\
\leq \epsilon |u_{\tau_m}(\tau)|^2_T + \epsilon |v_{\tau_m}(\tau)|^2_T + \int_{\tau}^{T} \frac{\epsilon}{2} |f(s)|^2_T ds + \frac{\epsilon^2}{2} |g(s)|^2_T + 2\beta |\Omega_T|ds.
\]
\[
(36)
\]
Since \( \{u_{\tau_m}\} \) and \( \{v_{\tau_m}\} \) are bounded in \( L^2(\Omega_T) \), then
\[
\{u_{k_m}\}, \text{is bounded in } L^\infty(\tau, T; L^2(\Omega_T)) \cap L^2(\tau, T; H^1(\mathcal{O}_T)) \cap L^\infty(\tau, T; L^p(\mathcal{O}_T))
\]
and
\[
\{v_{k_m}\} \text{ is bounded in } L^\infty(\tau, T; \mathcal{O}_T).
\]
Multiplying the equation by \( a'_{k_m} \), yields
\[
(u_{k_m}(t), e_j)_T + A_k(t)(u_{k_m}(t), e_j) + (h(u_{k_m}), e_j)_T + \lambda(u_{k_m}, e_j)_T + (v_{k_m}, e_j)_T \\
= (f(t), e_j)_T \text{ a.e. } t \in (\tau, T).
\]
Summing the above equations from \( k = 1 \) to \( k = m \) implies
\[
\begin{align*}
|u_{k_m}(t)|^2_T + \nu \|(u_{k_m}(t), u'_{k_m}(t))\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + (h(u_{k_m}(t)), u'_{k_m}(t)) + \\
+ \lambda(u_{k_m}(t), u_{k_m}(t))_T + (v_{k_m}, u'_{k_m}(t))_T = (f(t), u'_{k_m}(t))_T \text{ a.e. } t \in (\tau, T).
\end{align*}
\]
(39)
It follows from (27) that
\[
((P_k(t)u_{k_m}(t), u'_{k_m}(t)))_T \geq \frac{1}{2} \frac{d}{dt} \|(P_k(t)u_{k_m}(t), u_{k_m}(t))\|_T.
\]
(40)
Thus, we have
\[
\begin{align*}
\int_{\tau}^{T} \|u'_{k_m}\|_T^2 dr + \nu \|u_{k_m}(t)\|^2_T + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\alpha_1 \|u_{k_m}(t)\|^p_T
\leq \nu \|u_{\tau_m}\|^2_T + k((P_k(\tau)u_{\tau_m}, u_{\tau_m}))_T + 2\alpha_2 \|u_{\tau_m}\|^p_T + 4\beta |\Omega_T|.
\end{align*}
\]
and
\[
\{u'_{k_m}\} \text{ is bounded in } L^2(\tau, T; \mathcal{O}_T) \text{ and } \{u_{k_m}\} \text{ is bounded in } L^\infty(\tau, T; H_0^1(\mathcal{O}_T)).
\]

Since \( u_\tau \in H_0^1(\mathcal{O}_T) \cap L^p(\mathcal{O}_T) \), then we derive that there exists a sequence \( \{u_{\tau_m}\} \) converging to \( u_\tau \) in \( H_0^1(\mathcal{O}_T) \cap L^p(\mathcal{O}_T) \). By (42), extracting a subsequence if necessary, we assume that \( \{u_{k_m}\} \) converges weakly-star to \( u_k \) in \( L^\infty(\tau, T; H_0^1(\mathcal{O}_T)) \), and \( \{u'_{k_m}\} \) converges weakly to \( \{u'_k\} \) in \( L^2(\tau, T; \mathcal{O}_T) \). Using the weak and weak-star lower semicontinuity of the norms and (41), we derive

\[
\int_\tau^T |u'_k|^2 dt + \nu \|u_k\|_{L^\infty(\tau, T; H_0^1(\mathcal{O}_T))}^2 
\leq \nu \|u_\tau\|_{L^2(\tau, T; \mathcal{O}_T)}^2 + k((P_k(\tau)u_\tau, u_\tau)) + 2\tilde{\alpha}_2 \|u_\tau\|_{L^p(\mathcal{O}_T)}^p + 4\tilde{\beta}|\mathcal{O}_T|
\]

\[
+ 2(T - \tau) \left( (|u_\tau|_{\mathcal{O}_T}^2 + |v_\tau|_{\mathcal{O}_T}^2) + \int_\tau^T (|f(s)|_{\mathcal{O}_T}^2) ds + \frac{\epsilon}{\tau} |g(s)|_{\mathcal{O}_T}^2 + 2\epsilon|\mathcal{O}_T| \right)
\]

\[
+ 2 \int_\tau^T (|f(s)|_{\mathcal{O}_T}^2) ds + \lambda |u_k(\tau)|_{\mathcal{O}_T}^2 \quad \text{a.e. } t \in (\tau, T),
\]

and

\[
\frac{1}{T - \tau} \int_\tau^T k((P_k(t)u_{k_m}(t), u_{k_m})) dt 
\leq \nu \|u_{\tau_m}\|_{L^2(\tau, T; \mathcal{O}_T)}^2 + k((P_k(\tau)u_{\tau_m}, u_{\tau_m})) + 2\tilde{\alpha}_2 \|u_{\tau_m}\|_{L^p(\mathcal{O}_T)}^p + 4\tilde{\beta}|\mathcal{O}_T|
\]

\[
+ 2(T - \tau) \left( (|u_{\tau_m}|_{\mathcal{O}_T}^2 + |v_{\tau_m}|_{\mathcal{O}_T}^2) + \int_\tau^T (|f(s)|_{\mathcal{O}_T}^2) ds + \frac{\epsilon}{\tau} |g(s)|_{\mathcal{O}_T}^2 + 2\epsilon|\mathcal{O}_T| \right)
\]

\[
+ 2 \int_\tau^T (|f(s)|_{\mathcal{O}_T}^2) ds + \lambda |u_{k_m}(\tau)|_{\mathcal{O}_T}^2 \quad \text{a.e. } t \in (\tau, T).
\]

Define the functional \( \Phi : L^2(\tau, T; H_0^1(\mathcal{O}_T)) \to \mathbb{R} \) by

\[
\Phi(z) = \int_\tau^T ((P_k(t)z(t), z(t))) dt, \quad z \in L^2(\tau, T; H_0^1(\mathcal{O}_T)).
\]

Then \( \Phi \) is continuous and convex and

\[
\frac{k}{T - \tau} \int_\tau^T ((P_k(t)u_k(t), u_k)) dt 
\leq \nu \|u_\tau\|_{L^2(\tau, T; \mathcal{O}_T)}^2 + k((P_k(\tau)u_\tau, u_\tau)) + 2\tilde{\alpha}_2 \|u_\tau\|_{L^p(\mathcal{O}_T)}^p + 4\tilde{\beta}|\mathcal{O}_T|
\]

\[
+ 2(T - \tau) \left( (|u_\tau|_{\mathcal{O}_T}^2 + |v_\tau|_{\mathcal{O}_T}^2) + \int_\tau^T (|f(s)|_{\mathcal{O}_T}^2) ds + \frac{\epsilon}{\tau} |g(s)|_{\mathcal{O}_T}^2 + 2\epsilon|\mathcal{O}_T| \right)
\]

\[
+ 2 \int_\tau^T (|f(s)|_{\mathcal{O}_T}^2) ds + \lambda |u_k(\tau)|_{\mathcal{O}_T}^2 \quad \text{a.e. } t \in (\tau, T).
\]
Similarly, we have the following estimates
\[
|v'_{km}|^2_T + k(Q_k(t)u_{km}(t), u_{km}'(t))_T + \epsilon\gamma(v_{km}(t), v'_{km}(t))_T \\
= \epsilon(g(t), v'_{km})_T + \epsilon(u_{km}(t), v'_{km}(t))_T \text{ a.e. } t \in (\tau, T),
\]
and
\[
(Q_k(t)v_{km}(t), v'_{km}(t))_T \geq \frac{1}{2} \frac{d}{dt}(Q_k(t)v_{km}(t), v_{km}(t))_T.
\]
By using the Hölder inequality and the Cauchy inequality, we have
\[
\int_{\tau}^{T} |v'_{km}(s)|^2_T ds + k(Q_k(t)u_{km}(t), u_{km})_T \\
\leq k(Q_k(\tau)v_{\tau}, v_{\tau})_T + \epsilon\gamma|v_{\tau}|_T^2 + 2\epsilon^2(\int_{\tau}^{T} |g(s)|_T^2 ds),
\]
and
\[
\int_{\tau}^{T} |v'_{km}(s)|^2_T ds + k(Q_k(t)u_{km}(t), u_{km})_T \\
\leq k(Q_k(\tau)v_{\tau}, v_{\tau})_T + \epsilon\gamma|v_{\tau}|_T^2 + 2\epsilon^2(\int_{\tau}^{T} |g(s)|_T^2 ds) \\
+ 2\epsilon[T - \tau]\left((\epsilon|u_{\tau}|_T^2 + |v_{\tau}|_T^2) + \int_{\tau}^{T} (\frac{\epsilon}{\lambda}|f(s)|_T^2 ds + \frac{\epsilon}{\gamma}|g(s)|_T^2 + 2\epsilon\beta|O_T|) ds \right)
\]
for a.e. \( t \in (\tau, T). \)

Hence, we have
\[
\{v'_{km}\} \text{ is bounded in } L^2(\tilde{Q}_{\tau,T}). \tag{46}
\]
Since \( v_{\tau} \in L^2(O_T), \) \( \{v'_{km}\} \) indicates that there exists a sequence \( v_{\tau_m} \) converging to \( v_{\tau} \) in \( L^2(O_T). \) Due to \( (46), \) extracting a subsequence if necessary, we assume that \( \{v_{km}\} \) converges weakly-star to \( \{v_k\} \) in \( L^\infty(\tau, T; L^2(O_T)) \) and \( \{v'_{km}\} \) converges weakly to \( \{v'_k\} \) in \( L^2(\tilde{Q}_{\tau,T}), \) and a.e. \( t \in (\tau, T), \)
\[
\int_{\tau}^{T} |v'_k(s)|_T^2 ds \\
\leq k(Q_k(\tau)v_{\tau}, v_{\tau})_T + \epsilon\gamma|v_{\tau}|_T^2 + 2\epsilon^2(\int_{\tau}^{T} |g(s)|_T^2 ds) \\
+ 2\epsilon[T - \tau]\left((\epsilon|u_{\tau}|_T^2 + |v_{\tau}|_T^2) + \int_{\tau}^{T} (\frac{\epsilon}{\lambda}|f(s)|_T^2 ds + \frac{\epsilon}{\gamma}|g(s)|_T^2 + 2\epsilon\beta|O_T|) ds \right).
\]

Define the functional \( \Psi : L^2(\tau, T; L^2 O_T) \rightarrow \mathbb{R} \) by
\[
\Psi(z) = \int_{\tau}^{T} (Q_k(t)z(t), z(t))_T dt, \quad z \in L^2(\tau, T; L^2 O_T).
\]
Then \( \Psi \) is continuous and convex, and it holds
\[
k \int_{\tau}^{T} (Q_k(t)u_k(t), u_k)_T dt \\
\leq k \liminf_{m \rightarrow +\infty} \int_{\tau}^{T} (Q_k(t)u_{km}(t), u_{km})_T dt \\
\leq [t - \tau]\left[k(Q_k(\tau)v_{\tau}, v_{\tau})_T + \epsilon\gamma|v_{\tau}|_T^2 + 2\epsilon^2(\int_{\tau}^{T} |g(s)|_T^2 ds) \right] \tag{47}
\]
Thus, the proof is completed.

Next, we will establish the existence of variational solutions satisfying the energy equations.

**Theorem 3.6.** Assume \([10]\) holds, \(\partial \Omega\) is \(C^2\) and \(N \leq 2p/(p-2)\) or \(\partial \Omega\) is \(C^m\) with \(m \geq 2\) integer such that \(m \geq N(p-2)/2p\) and \(h(s)\) satisfy (2.7)-(2.8). Then for any \((u_\tau, v_\tau) \in L^2(\Omega_\tau) \times L^2(\Omega_\tau)\) and any \(-\infty < \tau \leq T < +\infty\), there exists a unique variational solution \((u, v)\) of [3].

**Proof.** We prove Theorem 3.6 by the following three steps:

**Step 1.** Assume that \(u_\tau \in H^1_0(\Omega_\tau) \cap L^p(\Omega_\tau), v_\tau \in L^2(\Omega_\tau)\). It follows from \([17]\) and \([22]\) that

\[
(P_k(\tau) u_\tau, u_\tau)_\Omega = 0, \quad (Q_k(\tau) v_\tau, v_\tau)_\Omega = 0 \quad \forall k \geq 1.
\]

By estimates \([29]\) and \([31]\), we have

\[ u_k' \text{ is bounded in } L^2(\bar{Q}_\tau, \tau), \quad u_k \text{ is bounded in } L^\infty(\sigma, T; H^1_0(\Omega_T)), \]

\[ v_k' \text{ is bounded in } L^2(\bar{Q}_\tau, \tau), \quad v_k \text{ is bounded in } L^2(\bar{Q}_\tau, \tau). \]

Extracting a subsequence if necessary, we assume that

\[ \{u_k\} \rightharpoonup u \text{ weakly star in } L^\infty(\tau, T; H^1_0(\Omega_T)), \quad \{u_k'\} \rightharpoonup u' \text{ weakly in } L^2(\bar{Q}_\tau, \tau); \]

\[ \{v_k\} \rightharpoonup v \text{ weakly star in } L^\infty(\tau, T; L^2(\Omega_T)), \quad \{v_k'\} \rightharpoonup v' \text{ weakly in } L^2(\bar{Q}_\tau, \tau). \]

Since \(u_k \rightharpoonup u\) in \(L^2(\tau, T; H^1_0(\Omega_T))\), then we have

\[
\int_\tau^T \|P(t)u(t)\|_\tau dt \leq \liminf_{k \to +\infty} \int_\tau^T P_k(t; u(t), u(t))_\tau dt \leq \liminf_{k \to +\infty} \frac{M_1}{k} = 0,
\]

where

\[ M_1 = (T - \tau) \left[ |\nu| u_\tau \|^2 + 2\alpha_2 |u_\tau| \|^2_{L^p(\Omega_T)} + 4\beta |\Omega_T| + 2 \int_\tau^T |f(s)|^2 ds + 2\epsilon \lambda |u_k(\tau)|^2 \right] \]

\[ + 2(T - \tau) \left[ (\epsilon |u_\tau|^2 + |v_\tau|^2) + \int_\tau^T \left( \frac{\epsilon}{\lambda} |f(s)|^2 ds + \frac{\epsilon}{\gamma} |g(s)|^2 ds + 2\epsilon \beta |\Omega_T| ds \right) \right]. \]

Similarly, we have

\[
\int_\tau^T |Q(t) v(t)|_\tau dt \leq \liminf_{k \to +\infty} \int_\tau^T Q_k(t; v(t), v(t))_\tau dt \leq \liminf_{k \to +\infty} \frac{M_2}{k} = 0,
\]

where

\[ M_2 = [T + 1 - \tau] \left[ |\epsilon |v_\tau|^2 + 2\epsilon^2 \int_\tau^T |g(s)|^2 ds + 2\epsilon (T - \tau) \left| (\epsilon |u_\tau|^2 + |v_\tau|^2) \right| \right] \]

\[ + \int_\tau^T \left( \frac{\epsilon}{\lambda} |f(s)|^2 ds + \frac{\epsilon}{\gamma} |g(s)|^2 ds + 2\epsilon \beta |\Omega_T| ds \right). \]

\([50]\) and \([52]\) implies that \(P(t)u(t) = 0, Q(t) v(t) = 0\) a.e. in \((\tau, T)\), i.e.,

\[
u(t) \in H^1_0(\Omega_t), \quad v(t) \in L^2(\Omega_t), \quad \text{a.e. in } (\tau, T).
\]
We deduce from (29) and the equality $u_k(t) - u_k(s) = \int_s^t u'_k(r)dr (\forall s, t \in [\tau, T], \forall k \geq 1)$ that
\[
|u_k(t) - u_k(s)|_T \leq M_1^{1/2}|t - s|^{1/2} \forall s, t \in [\tau, T], \forall k \geq 1.
\]
Thus
\[
\|u_k(t)\|_T^2 \leq M_1, \quad \text{for all } t \in [\tau, T] \quad \text{and each } k \geq 1.
\]
Since the injection of $H_0^1(\Omega_T)$ into $L^2(\Omega_T)$ is compact, the set \( \{ z \in H_0^1(\Omega_T) : \|z\|_T^2 \leq M_1 \} \) is compact in $L^2(\Omega_T)$. Then by Ascoli-Arzelà Lemma, there exists a subsequence (still denoted as \( \{u_k\} \) such that
\[
u_k \to u \text{ in } C([\tau, T]; L^2(\Omega_T)) \text{ as } k \to +\infty.\tag{55}
\]
Hence, we can extract a subsequence if necessary, $u_k(x, t) \to u(x, t)$ a.e. in $\Omega_T \times (\tau, T)$ such that
\[
h(u_k(x, t)) \to h(u(x, t)) \text{ a.e. in } \Omega_T \times (\tau, T).\tag{56}
\]

**Step 2.** Next, we consider the equality
\[
\frac{\partial(v_1 - v_2)}{\partial t} - \epsilon[(u_1 - u_2) - \gamma(v_1 - v_2)] = 0,
\]
Since $v_{1,\tau} = v_{2,\tau} = v_\tau$, then
\[
|v_1(t) - v_2(t)| = \epsilon \gamma \int_\tau^T \epsilon \gamma(s-t)|u_1(s) - u_2(s)|ds.
\]
We can derive from (55) that
\[
u_k \to v \text{ in } C([\tau, T]; L^2(\Omega_T)) \text{ as } k \to +\infty.\tag{57}
\]
Recall the fact the sequence $h(u_k)$ is bounded in $L^{\tilde{p}}(\tilde{Q}_T)$, and it follows from Lemma 1.2 in [9] that
\[
h(u_k) \to h(u) \text{ weakly in } L^{\tilde{p}}(\tilde{Q}_T).\tag{58}
\]
Combining (28), (50) and (52), we obtain
\[
\begin{cases}
\int_\tau^T -(u_k(x, t), \vartheta'(t))_T + \nu((u_k, \vartheta))_T + \lambda(u_k(x, t), \vartheta(x, t))_T \\
+(v_k(x, t), \vartheta(x, t))_T + h(u_k(x, t), \vartheta(x, t))_T dt \\
= \int_\tau^T f(x, t), \vartheta(x, t))_T dt, \\
- \int_\tau^T (v_k(x, t), \vartheta(x, t))_T - \epsilon(u_k(x, t), \vartheta(x, t))_T + \epsilon \gamma(v_k(x, t), \vartheta(x, t))_T \\
= \epsilon \int_\tau^T (g(x, t), \vartheta(x, t))_T dt;
\end{cases} \tag{59}
\]
for all $\vartheta \in \mathcal{U}_{\tau, T}$. By taking $k \to +\infty$ in (59), we derive that $u$ is a variational solution of (9).

It follows from the energy equality for $u_k, v_k$ that
\[
\|u_k(t)\|_T^2 \leq \|u_\tau\|_T^2 + 2 \int_\tau^T (f(t), u_k(t))_T - \nu\|u_k\|_T - \lambda|u_k(t)|T \\
-(v_k(t), u_k(t))_T - (h(u_k), u_k(t))_T dt, \tag{60}
\]
and
\[
\|v_k(t)\|_T^2 \leq \|v_\tau\|_T^2 + 2 \int_\tau^T \epsilon(g(t), u_k(t))_T - \gamma\epsilon|v_k(t)|T + \epsilon(v_k(t), u_k(t))_T. \tag{61}
\]
Due to (4), then we have
\[
\int_{\tau}^{T} (h(u_k(r)), u_k(r))_T dr \geq -C \int_{\tau}^{T} |u_k(r) - u(r)|^2 dr + \int_{\tau}^{T} (h(u_k(r)), u(r))_T dr \\
+ \int_{\tau}^{T} (h(u(r)), u_k(r))_T dr - \int_{\tau}^{T} (h(u(r)), u(r))_T dr.
\]
Thus, we obtain
\[
|u_k(t)|^2 + 2\nu \int_{\tau}^{T} \|u_k\|_T + 2\lambda |u_k(t)|_T + 2(v_k(t), u_k(t))_T dt \\
\leq |u(t)|^2 + 2\int_{\tau}^{T} (f(t), u_k(t))_T dt + 2C \int_{\tau}^{T} |u_k(r) - u(r)|^2 dr \\
-2\int_{\tau}^{T} (h(u_k(r)), u(r))_T dr - 2\int_{\tau}^{T} (h(u(r)), u_k(r))_T dr \\
+2\int_{\tau}^{T} (h(u(r)), u(r))_T dr. \tag{62}
\]
Thus, we obtain
\[
|u(t)|^2 \leq |u(t)|_T^2 + 2\int_{\tau}^{T} (f(t), u(t))_T - \nu \|u_k\|_T - \lambda |u(t)|_T - (v(t), u(t))_T - (h(u), u)|_T dt.
\]
Since $u_k \rightharpoonup v$ weakly in $L^2(\tau, T; \mathcal{L}_2(\mathcal{O}_T))$, then we have
\[
|v(t)|^2 \leq |v(t)|_T^2 + 2\int_{\tau}^{T} \epsilon(g(t), u(t))_T - \gamma |v(t)|_T + \epsilon(v(t), u(t))_T.
\]
Recall the fact $(u, v) \in C(\tau, T; L^2(\mathcal{O}_T)) \times C(\tau, T; L^2(\mathcal{O}_T))$, we have obtained that $(u, v)$ satisfies the energy equality for all $t \in [\tau, T]$.

**Step 3.** Assume that $u_\tau \in L^2(\mathcal{O}_\tau)$ and $v_\tau \in L^2(\mathcal{O}_\tau)$, then there exists a sequence \{\{u_{\tau,n}\} \subset H^1_0(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)\} such that $u_{\tau,n} \rightharpoonup u_\tau$ in $L^2(\mathcal{O}_\tau)$ as $n \to \infty$. Denote $(u_n, v_n)$ be the unique variational solution of \[3\], which satisfies the energy equality in $[\tau, T]$ with initial value $(u_{\tau,n}, v_\tau)$. Then we have
\[
\{u_n\} \text{ is bounded in } L^2(\tau, T; H^1_0(\mathcal{O}_T)) \cap L^2(\tau, T; \mathcal{L}_2(\mathcal{O}_T)) \cap L^p(\tau, T; \mathcal{L}_p(\mathcal{O}_T)), \tag{63}
\]
\[
\{v_n\} \text{ is bounded in } L^2(\tau, T; \mathcal{L}_2(\mathcal{O}_T)). \tag{64}
\]
By proposition 4.1, we have
\[
e(u_n(t) - u_m(t))_T^2 + |v_n(t) - v_m(t)|_T^2 + 2\nu \int_{\tau}^{T} \|u_n(r) - u_m(r)\|_T^2 dr \leq \epsilon^{2\nu}(t - \tau)(\epsilon |u_{\tau,n} - u_{\tau,m}|),
\]
for all $t \in [\tau, T]$ and any $n, m \geq 1$. Notice that $u_{\tau,n} \rightharpoonup u_\tau$ in $L^2(\mathcal{O}_\tau)$ as $n \to \infty$, then we get \{(u_n, v_n)\}
\[
is a cauchy sequence in \ L^2(\tau, T; H^1_0(\mathcal{O}_T)) \cap C(\tau, T; L^2(\mathcal{O}_T)) \times C(\tau, T; L^2(\mathcal{O}_T)). \tag{65}
\]
It follows from \[63\], \[64\] and \[65\] that $u_n \rightharpoonup u$ in $L^2(\tau, T; H^1_0(\mathcal{O}_T)) \cap C(\tau, T; L^2(\mathcal{O}_T))$ as $n \to \infty$. Repeating the previous arguments, we can show that $h(u_n) \rightharpoonup h(u)$ weakly in $L^{\infty}(\mathbb{Q}_T)$ as $n \to \infty$. By passing to the limit, we obtain that $(u, v)$ is a variational solution of \[3\]. Since $(u_n, v_n)$ satisfies the energy equality in $[\tau, T]$, Lemma 4.1 indicates that $u, v$ also satisfies the energy equality in $[\tau, T]$. Thus, the proof is completed.
4. Existence of the pullback $D_\sigma$-attractor. Let $\mathcal{O}_r := \bigcup_{t \in \mathbb{R}} \mathcal{O}_t$ be bounded, and assume that $C_f := \sup_{t \in \mathbb{R}} \int_{t}^{t+1} |f|^2_T ds < \infty$, $C_g := \sup_{t \in \mathbb{R}} \int_{t}^{t+1} |g|^2_T ds < \infty$, and
\[
\sup_{t \in \mathbb{R}} \int_{t}^{t+1} |\nu|^2_T ds < \infty.
\]
The energy equality of $(u_{km}, v_{km})$ in Theorem 4.1 implies that there exists a $\bar{T}(u_{\tau_m}, v_{\tau_m})$ with $\bar{T} \leq (t - \tau)$ such that
\[
\epsilon |u_{km}|^2_T + |v_{km}|^2_T \leq 2 \int_{-\infty}^{t} e^{\alpha (s-t)} \left( \frac{\epsilon}{\lambda} |f(s)|^2_T + \frac{\epsilon}{\gamma} |g(s)|^2_T + 2\epsilon \beta |\mathcal{O}_T| \right) ds. \tag{66}
\]
Combining weak with weak-star lower semicontinuity of norm, and letting $m, k \to +\infty$, we can obtain a pullback $D_\sigma$-attracting set as in Lemma 3.7.

Next, we will establish pullback $D_\sigma$-asymptotic compactness.

Lemma 4.1. There exists a $\bar{T}(u_\tau, v_\tau)$ with $t - \tau > \bar{T}$ and $M > 0$ such that $|u(t)|^2_T \leq M$, where $M$ is a constant which depends on $l, p, N, \nu, \sigma, \epsilon, \gamma, \lambda, f, g$.

Proof. Assume $u_\tau \in H_0^1(\mathcal{O}_\tau)$, $v_\tau \in L^2(\mathcal{O}_\tau)$, and denote
\[
y_{km} := \nu |u_{km}|^2_T + k((p_k(t)u_{km}(t), u_{km}(t)))T + 2 \int_{\tau}^{T} H(u_{km}) ds + \lambda |u_{km}|^2_T + 2\epsilon k |\mathcal{O}_T|,
\]
where $u_{km}$ are defined in theorem 4.1. Then we deduce from (39) that
\[
y_{km} \geq 0 \text{ and } y_{km} \leq 2(|v_{km}(t)|^2_T + |f(t)|^2_T). \tag{67}
\]
The energy equality of $(u_{km}, v_{km})$ gives
\[
\frac{d}{dt} \left( \epsilon |u_{km}|^2_T + |v_{km}|^2_T + 2k((p_k(u_{km}), u_{km}))T + 2\nu |u_{km}|^2_T \right) + 2\epsilon \alpha_2 |u_{km}|^2_T + \epsilon \lambda |u_{km}|^2_T \leq \frac{\epsilon}{\lambda} |f(s)|^2_T + \frac{\epsilon}{\gamma} |g(s)|^2_T + 2\epsilon \beta |\mathcal{O}_T|.
\]
Integrating the above inequality from $t$ to $t + 1$ yields
\[
\int_{t}^{t+1} 2k((p_k(u_{km}), u_{km}))T + 2\nu |u_{km}|^2_T + 2\nu |u_{km}|^2_T + 2\epsilon \alpha_2 |u_{km}|^2_T + \epsilon \lambda |u_{km}|^2_T \right) ds \leq M_0(\sigma, \epsilon, \gamma, \lambda, \bar{\beta})(C_f + C_g + |\mathcal{O}_T|), \quad \forall t \geq \bar{T}. \tag{68}
\]
Recall that
\[
y_{km} \leq \nu |u_{km}|^2_T + k((p_k(t)u_{km}(t), u_{km}(t)))T + 2\epsilon \alpha_2 |u_{km}|^2_T + \epsilon \lambda |u_{km}|^2_T + 4\epsilon \beta |\mathcal{O}_T|.
\]
Combining above inequality with (68), we have
\[
\int_{t}^{t+1} y_{km}(s) ds \leq M_1(\sigma, \epsilon, \gamma, \lambda, \bar{\beta}, \alpha_2)(C_f + C_g + |\mathcal{O}_T|), \quad \forall t \geq \bar{T}.
\]
We can derive from (66) and (67) that
\[
y_{km}(t+1) \leq y_{km}(s) + 2 \int_{t}^{t+1} (|u_{km}(s)|^2_T + |f(s)|^2_T) ds, \quad \forall t \leq s \leq t + 1,
\]
and
\[
|u_{km}(t)|^2_T \leq M_2(p, N, \nu, \sigma, \epsilon, \gamma, \lambda)(C_f + C_g + |\mathcal{O}_T|), \quad \forall t \geq \bar{T} + 1.
\]
By passing to the limits $m, k \to +\infty$ and combining the weak with weak-star lower semicontinuity of norm, we deduce that $u$ satisfies lemma. Repeating the similar arguments in Theorem 3.6, we can show Lemma 4.1 also hold for the solution of (1) corresponding to any $u_\tau, v_\tau \in L^2(\mathcal{O}_\tau)$. Thus, the proof is completed. \qed
Split \( \varphi_k \) as \( \varphi_k = \varphi_{k,1} + \varphi_{k,2} \), where \( \varphi_{k,1} \) is the solution of initial value problem for \( t \geq 0 \)

\[
\frac{d}{dt} \varphi_{k,1}(t) + kQ_k(\varphi_{k,1}(t)) + \epsilon\gamma(\varphi_{k,1}(t)) = 0, \quad \varphi_{k,1}(0) = \varphi_0,
\]

and \( \varphi_{k,2} \) is the solution of

\[
\frac{d}{dt} \varphi_{k,2}(t) + kQ_k(\varphi_{k,2}(t)) - \epsilon(\varphi_{k,1}(t)) + \epsilon\gamma(\varphi_{k,2}(t)) = 0, \quad \varphi_{k,2}(0) = 0
\]

for any \( \varphi \in V_T \cap L^p(\Omega_T) \). Direct calculation shows that \( \varphi_{k,1} \) satisfies

\[
\varphi_{k,1}(t) \leq e^{-\epsilon\gamma t}|\varphi_{k,1}(0)|, \quad \text{for all } t \in [0, T].
\]

Next, we will give some uniform estimates for \( \varphi_{k,2} \) in \( V_T \).

**Lemma 4.2.** Suppose \( \varphi \in V_t \), then for all \( t \in [0,T] \), we have that

\[
Q_k(t, \varphi, -\Delta \varphi) = P_k(t, \varphi, \varphi),
\]

for all \( k \geq 1 \).

**Proof.** In fact, we can find that the orthogonal projection operator \( Q(t) \) is the closure of orthogonal projection operator \( P(t) \) in the space of \( H_T \). Thus, we can finish the proof.

**Lemma 4.3.** Suppose that (1)-(6) hold, \( f(t), g(t) \in L^2(0,T;V_T) \) given, \( u_k, v_k \) are the solutions of (28) and \( \varphi_{k,2} \) defined as above, then \( \varphi_{k,2} \) satisfies that

\[
\|\varphi_{k,2}\|^2 \leq 2C \int_0^t e^{-\epsilon\gamma t}(\|f\|^2_T + \|g\|^2_T)ds.
\]

**Proof.** Here, we take \( \eta = -\Delta \varphi_{k,2} \) in (70), then we have

\[
\frac{1}{2} \frac{d}{dt} \|\varphi_{k,2}\|^2_T + kQ_k(\varphi_{k,2}, \varphi_{k,2}) + \epsilon\gamma \|\varphi_{k,2}\|^2_T = \epsilon(g, -\Delta \varphi_{k,2}) + \epsilon((\varphi_k(s), u_k)))_T.
\]

Combining it with Young's inequality, we obtain that

\[
\epsilon(g, -\Delta \varphi_{k,2}) \leq \frac{C}{4} \|\varphi_{k,2}\|^2_T + C(\|g\|^2_T)
\]

and

\[
\epsilon((\varphi_k(s), \varphi_{k,2}))_T \leq \frac{C}{4} \|\varphi_{k,2}\|^2_T + \frac{\epsilon}{\gamma} \|\varphi_k\|^2_T,
\]

where \( C \) is a constant depending on \( \epsilon, \gamma \). Thus, it implies that

\[
\frac{d}{dt} \|\varphi_{k,2}\|^2_T + \epsilon \gamma \|\varphi_{k,2}\|^2_T \leq 2C(\|f\|^2_T + \|g\|^2_T) + \frac{2\epsilon}{\gamma} \|\varphi_k(s)\|^2_T.
\]

Using Gronwall's inequality, we can get that

\[
\|\varphi_{k,2}\|^2_T \leq 2C \int_0^t e^{-\epsilon\gamma t}(\|f\|^2_T + \|g\|^2_T)ds + \frac{2\epsilon}{\gamma} \int_0^t e^{-\epsilon\gamma t} \|\varphi_k(s)\|^2_T ds.
\]

Thus, we have

\[
\|\varphi_{k,2}\|^2_T \leq 2C \int_0^t e^{-\epsilon\gamma t}(\|f\|^2_T + \|g\|^2_T)ds.
\]

**Theorem 4.4.** The process generated by the variational solutions of FitzHugh-Nagumo System (3) on variable domain has a unique non-autonomous pullback \( \mathcal{D}_\sigma \) attractor \( \mathcal{A} \).
Proof. we split the process $\Upsilon$ as $\Upsilon = \Upsilon_1 + \Upsilon_2$, processes $\Upsilon_1, \Upsilon_2$ defined by
$$\Upsilon_1(t, \tau; \omega)(u_\tau, v_\tau) = (0, v_1(t, \tau; \omega; u_\tau, v_\tau)) = (0, v_1(t))$$
and
$$\Upsilon_2(t, \tau; \omega)(u_\tau, v_\tau) = (u(t, \tau; \omega; u_\tau, v_\tau), v_2(t, \tau; \omega; u_\tau, v_\tau)) = (u(t), v_2(t)).$$
Then, combining it with Lemma 4.1, Lemma 4.3 and (71), it is easy to see that there exists a compact $D-$attracting $K$ for the non-autonomous process $\Upsilon$ defined above, which attracts bounded subsets of $\{(L^2(\Omega_t))^2\}_{t \in \mathbb{R}}$. Thus, using Theorem 4.1, we can obtain a unique non-autonomous pullback attractor in $\{(L^2(\Omega_t))^2\}_{t \in \mathbb{R}}$. □

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