Monge solutions and uniqueness in multi-marginal optimal transport via graph theory*

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Abstract

We study a multi-marginal optimal transport problem with surplus
\[ b(x_1, \ldots, x_m) = \sum_{\{i,j\} \in P} x_i \cdot x_j, \]
where \( P \subseteq Q := \{\{i, j\} : i, j \in \{1, 2, \ldots, m\}, i \neq j\} \). We reformulate this problem by associating each surplus of this type with a graph with \( m \) vertices whose set of edges is indexed by \( P \). We then establish uniqueness and Monge solution results for two general classes of surplus functions. Among many other examples, these classes encapsulate the Gangbo and Świȩch surplus \([15]\) and the surplus \( \sum_{i=1}^{m-1} x_i \cdot x_{i+1} + x_m \cdot x_1 \) studied in an earlier work by the present authors \([26]\).

1 Introduction

Multi-Marginal optimal transport is the problem of correlating probability measures to maximize a given surplus function. There are two formulations

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of this problem: the Kantorovich formulation and the Monge formulation. In the Kantorovich formulation, given Borel probability measures \( \mu_i \) on open bounded sets \( X_i \subseteq \mathbb{R}^n \), with \( i = 1, \ldots, m \), and \( b \) a real-valued surplus function on the product space \( X_1 \times \ldots \times X_m \), the goal is to maximize

\[
\int_{X_1 \times \ldots \times X_m} b(x_1, \ldots, x_m) \, d\mu, \tag{KP}
\]

among all Borel probability measures \( \mu \) on the product space \( X_1 \times \ldots \times X_m \) whose marginals are the \( \mu_i \); that is, for each fixed \( i \in \{1, \ldots, m\} \), \( \mu(X_1 \times \ldots \times X_{i-1} \times A \times X_{i+1} \times \ldots \times X_m) = \mu_i(A) \) for any Borel set \( A \subseteq X_i \).

In the Monge formulation, one seeks to maximize

\[
\int_{X_1} b(x_1, T_2 x_1, \ldots, T_m x_1) \, d\mu_1, \tag{MP}
\]

among all \((m - 1)\)-tuples of maps \((T_2, \ldots, T_m)\) such that \((T_i)_\sharp \mu_1 = \mu_i\) for all \( i = 2, \ldots, m \), where \((T_i)_\sharp \mu_1\) denotes the image measure of \( \mu_1 \) through \( T_i \), defined by \((T_i)_\sharp \mu_1(A) = \mu_1(T_i^{-1}(A))\), for any Borel set \( A \subseteq X_i \). It is well known that problem (KP) is a relaxation of problem (MP), as for any \((m - 1)\)-tuple of maps \((T_2, \ldots, T_m)\) satisfying the image measure constraint in (MP), we can define \( \mu = (Id, T_2, \ldots, T_m)_\sharp \mu_1 \), which satisfies the constraint in (KP) and

\[
\int_{X_1 \times \ldots \times X_m} b(x_1, \ldots, x_m) \, d\mu = \int_{X_1} b(x_1, T_2 x_1, \ldots, T_m x_1) \, d\mu_1.
\]

Under mild conditions, a maximizer to (KP) exists.

When \( m = 2 \), the classical optimal transport problems of Monge and Kantorovich arise in (KP) and (MP). This case has been widely studied and it is reasonably well understood; in particular, if \( \mu_1 \) is absolutely continuous with respect to Lebesgue measure \( \mathcal{L}^n \) and the twist condition holds (for each fixed \( x_1 \), the map \( x_2 \mapsto D_{x_1}b(x_1, x_2) \) is injective), the Kantorovich solution \( \mu \) induces a Monge solution and it is unique [4][14][16][17]. The classical optimal transportation problem has deep connections with many different areas of mathematics, including analysis, probability, PDE and geometry, and a great wealth of applications in, for example, economics, fluid mechanics, physics, among many other areas [30][31][32].

A wide variety of applications has also recently emerged for the \( m \geq 3 \) case, including, for example, matching in economics [11][13][28], density functional theory in computation [5][6][8][9][12], interpolating among distributions in machine learning and statistics [3][33] (see also [24] for an overview...
and additional references). However, determining whether solutions to the multi-marginal Kantorovich problem (KP) are unique and of Monge form has proven much more challenging, as the answer depends on the form of the surplus $b$ in subtle ways which are still not understood. A fairly general condition, called twist on splitting sets, together with absolute continuity of $\mu_1$, was shown in [22] to guarantee unique Monge solutions, unifying results in [10, 15, 20, 27, 23, 25, 28]. However, this condition is cumbersome and difficult to verify for many surpluses, and, as our recent work reveals, there exist surplus functions outside of this class for which Monge solution and uniqueness results hold (albeit stronger conditions on the marginals are required than for surplus functions which are twisted on splitting sets) [26]. One of our goals here is to shed some light on the general problem of classifying which surplus functions yield unique Monge solutions, by investigating a certain class of surpluses.

One of the best known surplus functions in the multi-marginal setting is the Gangbo and Świącek surplus function [15]

$$\sum_{1 \leq i < j \leq m} x_i \cdot x_j,$$

(1)

In their seminal work, they prove that every solution to (KP) is concentrated on a graph of a measurable map, thus obtaining a unique solution to the Monge-Kantorovich problem (in the subsequently developed terminology of [22], (1) is twisted on splitting sets). In [1], Agueh and Carlier proved that solving the multi-marginal Kantorovich problem with a weighted version of (1) is equivalent to finding the barycenter of the marginals $\mu_1, \ldots, \mu_m$. On the other hand, our recent paper [26] focused on cyclic costs of the form

$$\sum_{i=1}^{m-1} x_i \cdot x_{i+1} + x_m \cdot x_1,$$

(2)

whose origin lies in the time discretization of Arnold’s variational interpretation of the incompressible Euler equation [2, 7]; we showed that unique Monge solutions are obtained for $m \leq 4$ (although when $m = 4$, $\mu_2$, in addition to $\mu_1$, is required to be absolutely continuous – in this case (2) violates the twist on splitting sets condition from [22]), whereas higher dimensional solutions exist when $m \geq 5$. In the present work, we encapsulate the surplus functions (1) and (2) by studying a more general form in which arbitrary
interaction structures between the variables are permitted. More precisely, we set
\[ b(x_1, \ldots, x_m) = \sum_{\{i,j\} \in P} x_i \cdot x_j, \quad (3) \]
where \( P \subseteq Q := \{\{i, j\} : i, j \in \{1, 2, \ldots, m\}, i \neq j\} \) (note that \( (3) \) takes shape \( (1) \) when \( P = Q \), and shape \( (2) \) when \( P = \{\{i, i + 1\} : i = 1, \ldots, m - 1\} \cup \{\{1, m\}\} \); our main goal is then to identify conditions on \( P \) which lead to Monge solutions. For this, we exploit a natural connection to graph theory; in particular, we associate the surplus function \( (3) \) with a graph whose vertices we label \( \{v_1, \ldots, v_m\} \) and whose set of edges is indexed by \( P \). For instance, it is evident that surplus \( (1) \) is associated to a complete graph with vertices \( \{v_1, \ldots, v_m\} \), denoted by \( C_m \). See figure below for the case \( m = 7 \).

![Figure 1: C_m with m = 7.](image)

In this setting every subgraph with \( m \) vertices \( G \) of \( C_m \) is associated to a surplus \( \sum_{\{v_i, v_j\} \in E(G)} x_i \cdot x_j \), where
\[ E(G) = \{\{v_i, v_j\} : G \text{ has an edge between } v_i \text{ and } v_j, v_i \neq v_j\}. \]

For instance, the "border" of \( C_m \), that is, the cycle graph with vertex sequence \( (v_1, \ldots, v_m, v_1) \) (see definition in Subsection 2.2 and Figure 2-b for the case \( m = 7 \)) is associated to surplus \( (2) \). Hence, according to our work in \[26\], we can conclude that if \( m \geq 4 \), the only cycle graph providing unique Monge solution is the graph cycle with \( m = 4 \) (under regularity conditions of the first and fourth marginal). See Figure 2-a below.
The connection between multi-marginal surplus functions and graphs described above recently appeared in a computational setting in [18], where a regularized (through an entropy term) multi-marginal optimal transport problem with surplus associated to a tree was studied. Although the scope of that work is restricted to a more basic graph structure (only trees were considered), the edges \( \{v_i, v_j\} \) are associated to more general symmetric surplus \( c_{ij}(x_i, x_j) \). Also, [19] established an equivalence of the regularized multi-marginal optimal transport and the inference problem for a probabilistic graphical model when both problems are associated to a common graph structure. On the other hand, the same relationship was noted in [20], where connectedness of the graph played an important role in solving a one dimensional multi-marginal martingale optimal transport problem under various assumptions; see Theorem 5.3 in [20].

As we shall see in sections 3 and 4, our main results (Theorem 3.1 and Theorem 4.1, as well as the related Propositions 4.1 and 4.2), provide a broad class of graphs providing unique Monge solutions; some of these are classical, well known graphs, whereas others are less standard and more exotic. In particular, we highlight in Corollary 3.1 a special subclass of graphs encompassed by our theory, offering a generalization of the Gangbo and Święch result which we find conceptually appealing: the class in which each vertex is connected to all, except at most one, of the other vertices. Generally speaking, the graphs for which we establish Monge solution results come in two complementary classes; one (see Section 3) result from the extraction from the complete graph of subgraphs with a particular structure, while the other...
(see Section 4) is obtained by joining complete graphs in a special way.

We would like to emphasize that, in addition to the regularity assumption on $\mu_1$, which is standard in optimal transport, many of our results require extra regularity conditions on certain other marginals; these assumptions are not typical in optimal transport theory, but are necessary in our setting, since many surplus functions counterexamples to Monge solutions and uniqueness exist in their absence (see the second assertion of Proposition 2.1). Note that these examples confirm that the framework developed here reaches well beyond the twist on splitting sets theory, the most general currently known condition implying Monge solution and uniqueness results for multi-marginal problems; indeed, Proposition 2.1 verifies that the twist on splitting sets condition is violated by a wide variety of surplus functions, many of which fall within the scope of either Theorem 3.1 or the results in section 4 (Theorem 4.1 and Propositions 4.1 and 4.2).

In the next section, we recall and formulate some definitions and concepts from the theory of optimal transportation and graph theory, as well as establish some preliminary results connecting these two areas, one of which in particular will be used later on the paper. In section 3, we establish and prove our first main result and provide some examples. In section 4 we state and prove our second main result and provide examples. The short fifth section contains a (standard) proof of the fact that when every solution is of Monge type, as we prove for the graphs considered in sections 3 and 4, the solution must in fact be unique. The final section is reserved for discussion, including examples of graphs which fall outside the scope of the results proved in this paper, and for which the Monge solution and uniqueness questions therefore remain open.

2 Definitions and preliminaries

In this section, we formulate the main concepts used in this work.

2.1 The dual problem

For the surplus function (3), set

$\mathcal{U} = \left\{ (u_1, u_2, \ldots, u_m) \in \prod_{i=1}^m L^1(\mu_i) : b(x_1, \ldots, x_m) \leq \sum_{i=1}^m u_i(x_i), \forall (x_1, \ldots, x_m) \in X_1 \times \ldots \times X_m \right\}.$
The dual of (KP) asks to minimize on $\mathcal{U}$ the map:

$$(u_1, u_2, \ldots, u_m) \mapsto \sum_{i=1}^{m} \int_{X_i} u_i(x_i) d\mu_i(x_i).$$

(DP)

The following subclass of $\mathcal{U}$ plays a key role in multi-marginal optimal transport theory:

**Definition 2.1.** An $m$-tuple of functions $(u_1, u_2, \ldots, u_m)$ is $b$-conjugate if for all $i$,

$$u_i(x_i) = \sup_{x_j \in X_j, j \neq i} \left( b(x_1, \ldots, x_m) - \sum_{j \neq i} u_j(x_j) \right).$$

Since in our setting the surplus $b$ is locally Lipschitz and semi-convex, it follows that for any $b$-conjugate $m$-tuple $(u_1, u_2, \ldots, u_m)$, $u_i$ is locally Lipschitz and semi-convex for each $i$ [14].

We now introduce a duality theorem that shows the connection between (DP) and (KP). We refer to [15][25] for a proof that solutions can be taken to be $b$-conjugate; the remaining assertions can be found in [21].

**Theorem 2.1.** Assume $X_i$ is compact for every $i$ and let $\text{spt}(\mu)$ be the support of $\mu$. Hence, there exists a solution $\mu$ to the Kantorovich problem and a $b$-conjugate solution $(u_1, u_2, \ldots, u_m)$ to its dual. The minimum and maximum values in (DP) and (KP) respectively agree, and $\sum_{i=1}^{m} u_i(x_i) = b(x_1, \ldots, x_m)$ for each $(x_1, \ldots, x_m) \in \text{spt}(\mu)$.

### 2.2 Some graph theory

First, let us recall some definitions from Graph Theory. An **undirected simple graph** $G$ is an ordered pair $(V(G), E(G))$, consisting of a finite set of vertices $V(G)$ and a set of edges $E(G) \subseteq \{\{v, w\} : v, w \in V(G) \text{ and } v \neq w\}$. Throughout this work, every graph $G$ is an undirected simple graph. A **trail** is a finite sequence $\{\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_l}, v_{i_{l+1}}\}\}$ of pairwise distinct edges which joins a sequence of vertices. A **path** is a trail in which all vertices are distinct: $v_{i_j} \neq v_{i_k}$ for all $j \neq k$. A **cycle graph** is a trail in which the first and last vertex are the only one repeated: $\{\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_l}, v_{i_{l+1}}\}\}$, where $\{\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_{l-1}}, v_{i_l}\}\}$ is a path and $v_{i_{l+1}} = v_{i_1}$. A **tree** is a graph where any two distinct vertices are connected by a unique path. A graph $G$ is **connected** if for every $v, w \in V(G)$ there exists a path in the
graph joining them. We will denote by $I(V(G))$ the set of indices of $V(G)$ (that is, for $V(G) = \{v_1, ..., v_m\}$, $I(V(G)) = \{1, 2, ..., m\}$) and $|V(G)|$ the cardinality of $V(G)$.

A subgraph $S$ of a graph $G$ is a graph whose sets of vertices and edges are subsets of $V(G)$ and $E(G)$ respectively. In this case, we call the graph $G\backslash S := (V(G), E(G) \setminus E(S))$ the extraction of $S$ from $G$. Note that if $G$ is complete and $V(G) = V(S)$, $G\backslash S$ coincides with the complement of $S$; that is, $G\backslash S = (V(S), E(S^c))$, where $E(S^c) := \{\{v, w\} : v, w \in V(S) \text{ and } \{v, w\} \notin E(S)\}$.

Given $v, w \in V(G)$, $v$ and $w$ are called adjacent if $\{v, w\} \in E(G)$. The open neighborhood of a vertex $v$, denoted $N_G(v)$ (or simply $N(v)$ if there is not danger of confusion), is the set of vertices that are adjacent to $v$; that is,

\[ N(v) = \{w \in V(G) : \{v, w\} \in E(G)\}. \]

The closed neighborhood of a vertex $v$, denoted $\overline{N}_G(v)$ (or simply $\overline{N}(v)$), is the set $N(v) \cup \{v\}$. A graph $G$ is complete if $N(v) = V(G) \setminus \{v\}$ for every $v \in V(G)$. A clique $\bar{G} = (V(\bar{G}), E(\bar{G}))$ of a graph $G = (V(G), E(G))$ is a complete subgraph of $G$; that is, $\bar{G}$ is a subgraph of $G$ and satisfies $N_{\bar{G}}(v) = V(G) \setminus \{v\}$ for every $v \in V(\bar{G})$. A clique is maximal if it is not a proper subgraph of any other clique of $G$.

The union of two given graphs $G_1$ and $G_2$ (denoted by $G_1 \cup G_2$) is the graph with set of vertices $V(G_1) \cup V(G_2)$ and edges $E(G_1) \cup E(G_2)$. A complete $k$-partite graph $G$ is a graph whose set of vertices $V(G)$, can be partitioned into $k$ subsets $V_1, V_2, \ldots, V_k$ such that for every $v \in V_j$, $N(v) = \bigcup_{\alpha \neq j}^{k} V_{\alpha}$ for any fixed $j \in \{1, \ldots, k\}$. A complete $k$-partite graph $G$ is denoted as $K_{m_1, \ldots, m_k}$, where $|V_j| = m_j$ for every $j \in \{1, \ldots, k\}$.

**Remark 2.1.** Note that the definition of a graph implies $v \notin N(v)$ for every $v \in V(G)$. Furthermore, from now on, if $S$ is a subgraph of $G$ and $v \in V(S)$, we will write $N(v)$ for the open neighborhood of $v$ in $G$; that is, $N(v) = N_G(v)$. Similarly, $\overline{N}(v) = \overline{N}_G(v)$.

The next two concepts are used to facilitate the description of our main results, established in Theorem 3.1 and 4.1.

**Definition 2.2.** We say that a subset $A$ of $V(G)$ is the inner hub of $G$, if $A = V(S_1) \cap V(S_2)$ for any two maximal cliques $S_1$ and $S_2$ of $G$. 
Example 2.1. The picture below shows the graph $G := S_1 \cup S_2 \cup S_3$, where $S_1, S_2$ and $S_3$ are complete graphs with \( V(S_1) = \{v_6, v_7, v_8, v_9\} \), \( V(S_2) = \{v_6, v_7, v_8, v_4, v_5\} \) and \( V(S_3) = \{v_6, v_7, v_8, v_1, v_2, v_3, v_{10}\} \). Clearly, \( \{S_1, S_2, S_3\} \) is the collection of maximal cliques of $G$, and the inner hub of $G$ is the set formed by the vertices of the triangle colored blue; that is, $A = \{v_6, v_7, v_8\}$.

![Figure 3: Graph $G = S_1 \cup S_2 \cup S_3$.](image)

Not all graphs have an inner hub. Letting \( \{S_j\}_{j=1}^l \) be the maximal cliques of a graph $G$, it is clear that $G = \bigcup_{j=1}^l S_j$; $A$ is the inner hub of $G$ if \( V(S_j) \cap V(S_k) = A \) for all $j \neq k$, $j, k \in \{1, \ldots, l\}$. Note that we allow the inner hub $A$ to be the empty set, which is the case when $G$ is disconnected and each connected component is complete. At the other extreme, we could have $A = G$, which is the case when $G$ is complete.

Definition 2.3. Let \( \{S_{1j}\}_{j=1}^{l_1} \) and \( \{S_{2j}\}_{j=1}^{l_2} \) be the collection of maximal cliques of given graphs $G_1$ and $G_2$, respectively. Assume that $G_1$ and $G_2$ have inner hubs $A_1$ and $A_2$, respectively. We say the graphs $G_1$ and $G_2$ are glued on a clique if:

1. There are $j \in \{1, \ldots, l_1\}$ and $k \in \{1, \ldots, l_2\}$ such that $S_{1j} = S_{2k}$.

2. $V(G_1) \cap V(G_2) = V(S_{1j})$

Example 2.2. The picture below shows the graph $S_1 \cup S_2 \cup S_3$, where \( \{S_1, S_2, S_3\} \) is the collection of maximal cliques in Example 2.1, glued with the graph $S'_1 \cup$
\( S_2' \cup S_3' \cup S_4' \) on \( S_2 \), where \( S_1', S_2', S_3' \) and \( S_4' \) are complete graphs with \( V(S_1') = \{ v_4, v_5, v_{14} \} \), \( V(S_2') = \{ v_6, v_7, v_8, v_4, v_5 \} = V(S_2) \), \( V(S_3') = \{ v_4, v_5, v_{15}, v_{16} \} \) and \( V(S_4') = \{ v_4, v_5, v_{11}, v_{12}, v_{13} \} \). Note that the inner hub of \( S_1' \cup S_2' \cup S_3' \cup S_4' \), whose collection of maximal cliques is \( \{ S_j' \}_{j=1}^4 \), is formed by the vertices of the edges colored red; that is, \( \mathcal{A}' = \{ v_4, v_5 \} \).

Figure 4: Graph \( \bigcup_{j=1}^3 S_j \bigcup \bigcup_{j=1}^4 S_j' \)

2.3 Preliminary results connecting graph theory and multi-marginal optimal transport

In this subsection, we establish some initial results connecting solutions of the multi-marginal optimal transport problem \([KP]\) with cost \([3]\) and the structure of the corresponding graph. These include a couple of very basic observations (Proposition 2.1), as well as a technical lemma which will be used throughout the paper (Lemma 2.1).

Proposition 2.1. Let \( G \) be the graph corresponding to some \( P \subseteq Q \) and \( b \) the surplus \([3]\).

1. Assume \( G \) is not connected and let \( x_i \) be any vertex such that there is no path between \( x_1 \) and \( x_i \), and assume that \( \mu_i \) is not a dirac mass.
Then there exist non Monge solutions to (KP), and, if $\mu_1$ is not a dirac mass, the solution to (KP) is not unique.

2. Assume \( \{v_1, v_i\} \) is not an edge of $G$ for some $i$, and all the marginals are dirac measures except $\mu_1$ and $\mu_i$, with $\mu_1$ absolutely continuous with respect to Lebesgue measure. Then, there exist solutions of non-Monge form to (KP) and the solution to (KP) is not unique.

**Proof.** Consider the first assertion. Let $G_1$ be the connected component of $G$ satisfying $v_1 \in Z := V(G_1)$, and $G_2$ the graph union of the other components of $G$, with $W := V(G_2)$. Then the surplus (3) takes the separable form:

\[
b(x_1, \ldots, x_m) = b_Z(x_Z) + b_W(x_W),
\]

where we decompose $x = (x_Z, x_W)$ into components $x_Z$ and $x_W$ whose indices of their coordinates lie in $I(Z)$ and $I(W)$, respectively, and $b_Z(x_Z) = \sum_{\{v_s, v_t\} \in E(G_1)} x_s \cdot x_t$, $b_W(x_W) = \sum_{\{v_s, v_t\} \in E(G_2)} x_s \cdot x_t$. Solutions to (KP) are then exactly measures $\gamma$ whose projections $\gamma_Z$ and $\gamma_W$ onto the appropriate subspaces are optimal for the multi-marginal optimal transport problem with costs $b_Z$ and $b_W$, respectively, and the appropriate marginals. In particular, the dependence structure between $\gamma_Z$ and $\gamma_W$ is completely arbitrary, and so, if $\mu_i$ is not a dirac mass for some $v_i \in W$, we immediately get the existence of non-Monge solutions (for instance, the product measure $\gamma_Z \otimes \gamma_W$), and if in addition $\mu_1$ is not a dirac mass, solutions are non-unique.

Turning to assertion 2, without loss of generality, assume $\{v_1, v_2\}$ is not an edge of $G$. Take $\mu_i$ be absolutely continuous with respect to Lebesgue measure, $\mu_2$ be any measure other than a dirac mass (so that $\mu_2$ charges at least two points) and let all other marginals be dirac masses, $\mu_i = \delta_{\bar{x}_i}$. In this case, measures $\gamma$ whose marginals are the $\mu_i$ all take the form $\gamma = \sigma(x_1, x_2) \otimes \delta_{\bar{x}_3} \otimes \ldots \otimes \delta_{\bar{x}_m}$, where $\sigma \in P(X_1 \times X_2)$ has marginals $\mu_1$ and $\mu_2$. For any such $\gamma$, we have

\[
\int_{X_1 \times X_2 \times \ldots \times X_m} b(x_1, \ldots, x_m) d\gamma(x_1, \ldots, x_m) = \int_{X_1 \times X_2} b(x_1, x_2, \bar{x}_3, \ldots, \bar{x}_m) d\sigma(x_1, x_2)
\]

\[
= \int_{X_1} b_1(x_1, \bar{x}_3, \ldots, \bar{x}_m) d\mu_1(x_1)
\]

\[
+ \int_{X_2} b_2(x_2, \bar{x}_3, \ldots, \bar{x}_m) d\mu_2(x_2)
\]
where \( b_1(x_1, x_3, \ldots, x_m) = \sum_{\{v_s, v_t\} \in E(G) \, s \neq t} x_s \cdot x_t \) and \( b_2(x_2, x_3, \ldots, x_m) = \sum_{s \in I(N(v_2))} x_2 \cdot x_s \). Thus, the Kantorovich functional is independent of \( \sigma \), and so any \( \sigma \) with marginals \( \mu_1 \) and \( \mu_2 \) is optimal. We conclude that solutions are non-unique, and can be of non-Monge form (as is the case when, for example, \( \sigma = \mu_1 \times \mu_2 \) is the product measure).

Let us now establish an immediate consequence of the above proposition. For this, let us invoke the main definitions in [22].

**Definition 2.4.** A set \( S \subseteq \prod_{i=1}^m X_i \) is called a \( b \)-splitting set if there are Borel functions \( u_i : X_i \mapsto \mathbb{R} \) such that

\[
\sum_{i=1}^m u_i(x_i) \geq b(x_1, \ldots, x_m)
\]

for every \((x_1, \ldots, x_m) \in \prod_{i=1}^m X_i\), and whenever \((x_1, \ldots, x_m) \in S\) equality holds.

**Definition 2.5.** Let \( b \) be a continuous semi-convex surplus function. It is called twisted on \( b \)-splitting sets, whenever for each fixed \( x_1 \in X_1 \) and \( b \)-splitting set \( S \subseteq \{x_1\} \times X_2 \times \ldots \times X_m \), the map

\[
(x_2, \ldots, x_m) \mapsto D_{x_1} b(x_1^0, x_2, \ldots, x_m)
\]

is injective on the subset of \( S \) where \( D_{x_1} b(x_1, x_2, \ldots, x_m) \) exists.

**Remark 2.2.** The main result in [22] establishes that if \( b \) is twisted on \( b \)-splitting sets, then every solution to \((KP)\) is induced by a map, whenever \( \mu_1 \) is absolutely continuous with respect to local coordinates.

**Corollary 2.1.** Under the hypothesis in any of assertion 1 or assertion 2 of Proposition 2.1, the surplus \( b \) is not twisted on \( b \)-splitting sets.

**Proof.** Suppose \( b \) is twisted on \( b \)-splitting sets. From the above remark every solution to \((KP)\) is induced by map; that is, every solution to \((KP)\) is of Monge type. This clearly contradicts Proposition 2.1, completing the proof of the corollary.

Clearly, in light of the first assertion there is no hope of obtaining Monge solution results for disconnected graphs (except in the trivial case when each
$\mu_i$ with $x_i$ not connected to $\mu_1$ is a dirac mass, in which case the problem reduces to a problem on the connected component containing $x_1$.) We therefore will focus on connected graphs throughout this paper. On the other hand, our work in [26] suggests that at least for some surplus functions where $\{v_i, v_i\}$ is not an edge of $G$ for some $i$, unique Monge solutions may exist when extra regularity conditions on the marginals are imposed (even though the twist on splitting sets condition fails). Our results in the following sections confirm that this is indeed the case.

The proofs of our main results will require the following technical lemma.

**Lemma 2.1.** Let $G$ be a graph, with $V(G) = \{v_1, \ldots, v_m\}$, and $b(x_1, \ldots, x_m) = \sum_{(v_i, v_i) \in E(G)} x_i \cdot x_i$ be the surplus associated to $G$. Let $(u_1, \ldots, u_m)$ a $b$-conjugate $m$-tuple. Set

$$W := \left\{(x_1, \ldots, x_m) \in X_1 \times \ldots \times X_m : \sum_{i=1}^m u_i(x_i) = b(x_1, \ldots, x_m)\right\}.$$  

Fix $x_0^1 \in X_1$ and for convenience of notation set $x_1^1 = x_1^2 = x_0^1$. Let $(x_1^1, x_2^1, \ldots, x_m^1), (x_1^2, x_2^2, \ldots, x_m^2) \in W$.

1. Assume there are sets $V_1, V_2 \subseteq V(G)$ such that $N(v_s) = V_2$ for every $s \in I(V_1)$, and set

$$y_s = \begin{cases} 
  x_1^s & \text{if } s \in \{1, \ldots, m\} \setminus I(V_1) \\
  x_2^s & \text{if } s \in I(V_1).
\end{cases}$$

If

$$\sum_{s \in I(V_2)} x_1^s = \sum_{s \in I(V_2)} x_2^s,$$  

then $y := (y_1, y_2, \ldots, y_m) \in W$.

2. For all $t \in \{1, \ldots, m\}$ we have

$$\left(x_t^2 - x_t^1\right) \cdot \sum_{s \in I(N(v_t))} (x_s^1 - x_s^2) \leq 0.$$  

3. If there exists $t \in \{1, \ldots, m\}$ such that

$$\sum_{s \in I(N(v_t))} x_1^s = \sum_{s \in I(N(v_t))} x_2^s,$$  

then $x_t^1 = x_t^2$. 

13
4. Assume \( x^1_p = x^2_p \) and \( Du_p(x^1_p) \) exists for some \( p \in \{1, \ldots, m\} \).

(a) For every \( t \in \{2, \ldots, m\} \setminus \{p\} \) satisfying

\[
\mathcal{N}(v_p) = \mathcal{N}(v_t),
\]

we have \( x^1_t = x^2_t \).

(b) Assume there are sets \( F_1, F_2, F_3 \) such that \( F_1, F_2 \subseteq N(v_p) \) and \( N(v_s) = F_2 \cup F_3 \) for every \( s \in I(F_1) \). If \( x^1_s = x^2_s \) for every \( s \in I(N(v_p) \setminus F_1 \cup F_2) \cup I(F_3) \), then \( x^1_s = x^2_s \) for every \( s \in I(F_1) \).

**Proof.** Since for every \( s \in I(V_1) \) we have \( N(v_s) = V_2 \), and \( v \notin N(v) \) for all \( v \in V(G) \), we get \( V_1 \cap V_2 = \emptyset \). Hence, we can write

\[
b(x_1, \ldots, x_m) = g(x_1, \ldots, x_m) + \left( \sum_{s \in I(V_2)} x_s \right) \cdot \left( \sum_{s \in I(V_1)} x_s \right),
\]

where \( g(x_1, \ldots, x_m) \) does not depend on \( \{x_s\}_{s \in I(V_1)} \). Hence

\[
\{x^2_s\}_{s \in I(V_1)} \in \text{Argmax} \left\{ \{x_s\}_{s \in I(V_1)} \mapsto \left( \sum_{s \in I(V_2)} x^2_s \right) \cdot \left( \sum_{s \in I(V_1)} x_s \right) - \sum_{s \in I(V_1)} u_s(x_s) \right\},
\]

as \( (x^2_1, x^2_2, \ldots, x^2_m) \in W \). Then, If (4) holds we get

\[
\{x^2_s\}_{s \in I(V_1)} \in \text{Argmax} \left\{ \{x_s\}_{s \in I(V_1)} \mapsto \left( \sum_{s \in I(V_2)} x^1_s \right) \cdot \left( \sum_{s \in I(V_1)} x_s \right) - \sum_{s \in I(V_1)} u_s(x_s) \right\},
\]

which implies \( y \in W \), as \( (x^1_1, x^1_2, \ldots, x^1_m) \in W \). This completes the proof of part 4. Using the arguments of the previous proof, and taking \( V_1 = \{v_t\} \) and \( V_2 = N(v_t) \) for any fixed \( t \in \{2, \ldots, m\} \), we deduce

\[
x^2_t \in \text{Argmax}\left\{ x_t \mapsto \left( \sum_{s \in I(N(v_t))} x^2_s \right) \cdot x_t - u_t(x_t) \right\}.
\]

Similarly,

\[
x^1_t \in \text{Argmax}\left\{ x_t \mapsto \left( \sum_{s \in I(N(v_t))} x^1_s \right) \cdot x_t - u_t(x_t) \right\}.
\]

Then,

\[
\left( \sum_{s \in I(N(v_t))} x^2_s \right) \cdot x^1_t - u_t(x^1_t) \leq \left( \sum_{s \in I(N(v_t))} x^2_s \right) \cdot x^2_t - u_t(x^2_t)
\]

(8)
and
\[
\left( \sum_{s \in I(N(v_1))} x_s^1 \right) \cdot x_t^2 - u_t(x_t^2) \leq \left( \sum_{s \in I(N(v_1))} x_s^1 \right) \cdot x_t^1 - u_t(x_t^1).
\]

Adding (8) and (9) (and eliminating terms) we obtain inequality (5), completing the proof of the second part. The proof of part 3 follows immediately from part 2 as if there exists \( t \in \{2, \ldots, m\} \) satisfying (6) we get
\[
x_t^1 + \sum_{s \in I(N(v_1))} x_s^1 = x_t^2 + \sum_{s \in I(N(v_1))} x_s^2,
\]
hence, \( \sum_{s \in I(N(v_1))} (x_s^1 - x_s^2) = x_t^2 - x_t^1 \). Substituting it into inequality (3) we get \( \|x_t^2 - x_t^1\|^2 \leq 0 \); that is, \( x_t^2 = x_t^1 \). To prove part 4 first note that
\[
\sum_{s \in I(N(v_1))} x_s^1 = D_x b(x_1, \ldots, x_m) = D u_p(x_p) = D u_p(x_p^2) = D_x b(x_1, \ldots, x_m) = \sum_{s \in I(N(v_1))} x_s^2.
\]

Hence, for any \( t \in \{2, \ldots, m\} \setminus \{p\} \) satisfying (7) we obtain
\[
\sum_{s \in I(N(v_1))} x_s^1 = \sum_{s \in I(N(v_p))} x_s^1 = \sum_{s \in I(N(v_p))} x_s^2 \quad \text{by (10) and the equality } x_p^1 = x_p^2.
\]

Then, by part 3 we conclude \( x_t^1 = x_t^2 \), completing the proof of part 4a. To prove part 4b observe that \( F_1 \cap F_2 = \emptyset \), as \( N(v_s) = F_2 \cup F_3 \) and \( v_s \notin N(v_s) \) for every \( s \in I(F_1) \). Then, from (10) we get
\[
\sum_{s \in I(F_1)} x_s^1 + \sum_{s \in I(F_2)} x_s^1 + \sum_{s \in I(N(v_p) \setminus F_1 \cup F_2)} x_s^1 = \sum_{s \in I(F_1)} x_s^2 + \sum_{s \in I(F_2)} x_s^2 + \sum_{s \in I(N(v_p) \setminus F_1 \cup F_2)} x_s^2,
\]
as \( F_1, F_2 \subseteq N(v_p) \). Since \( x_s^1 = x_s^2 \) for every \( s \in I(N(v_p) \setminus F_1 \cup F_2) \), the above equality reduces to
\[
\sum_{s \in I(F_1)} x_s^1 + \sum_{s \in I(F_2)} x_s^1 = \sum_{s \in I(F_1)} x_s^2 + \sum_{s \in I(F_2)} x_s^2.
\]
and applying part 2 we get
\[(x_t^2 - x_t^1) \cdot \sum_{s \in I(F_2 \cup F_3)} (x_s^1 - x_s^2) \leq 0,\]
for any \( t \in I(F_1) \). Summing over \( t \in I(F_1) \) and using the equalities \( x_s^1 = x_s^2 \) on \( I(F_3) \), we obtain
\[\sum_{t \in I(F_1)} \sum_{s \in I(F_2)} (x_t^2 - x_t^1) \cdot (x_s^1 - x_s^2) \leq 0.\]
Furthermore, by (12) we get
\[\sum_{t \in I(F_1)} (x_t^2 - x_t^1) = \sum_{s \in I(F_2)} (x_s^1 - x_s^2).\]
Substituting it into the above inequality we get
\[\parallel \sum_{s \in I(F_2)} (x_s^1 - x_s^2) \parallel^2 \leq 0; \quad \text{that is,} \]
\[\sum_{s \in I(F_2)} x_s^1 = \sum_{s \in I(F_2)} x_s^2. \quad \text{(13)}\]
Now, fix \( t \in I(F_1) \) and set \( V_1 = F_1 \setminus \{v_t\}, \ V_2 = F_2 \cup F_3 \) and \( y = (y_1, y_2, \ldots, y_m) \) such that
\[y_s = \begin{cases} x_s^1 & \text{if } s \in \{1, \ldots, m\} \setminus I(V_1) \\ x_s^2 & \text{if } s \in I(V_1). \end{cases}\]
Since \( x_s^1 = x_s^2 \) on \( I(F_3) \), (13) can be written as \( \sum_{s \in I(F_2 \cup F_3)} x_s^1 = \sum_{s \in I(F_2 \cup F_3)} x_s^2 \).
Therefore, by part II we get \( y \in W, \ N(v_s) = V_2 \) for every \( s \in I(V_1) \). Hence,
\[\sum_{s \in I(N(v_p))} y_s = Du_p(y_p) = Du_p(x_p^1) = Du_p(x_p^2) = \sum_{s \in I(N(v_p))} x_s^2,\]
or equivalently,
\[y_t + \sum_{s \in I(F_1) \setminus \{t\}} y_s + \sum_{s \in I(F_2)} y_s + \sum_{s \in I(N(v_p)) \setminus F_1 \cup F_2} y_s = x_t^2 + \sum_{s \in I(F_1) \setminus \{t\}} x_s^2 + \sum_{s \in I(F_2)} x_s^2 + \sum_{s \in I(N(v_p)) \setminus F_1 \cup F_2} x_s^2.\]
From (13), construction of \( y \) and the equalities \( x_s^1 = x_s^2 \) on \( I(N(v_p)) \setminus F_1 \cup F_2 \), we get \( x_t^1 = x_t^2 \), completing the proof of part 4b. \( \square \)
3 Monge solutions under extraction of graphs

The main theorem of this section establishes that, roughly speaking, the extraction from $C_m$ of a subgraph with an inner hub provides a unique Monge solution, possibly under an additional regularity condition on one of the marginals.

We will present several examples of graphs obtained in this way later on, but for now we mention that the graph in Figure 2-a is obtained by extracting the edges $\{x_1, x_3\}$ and $\{x_2, x_4\}$ from $C_4$, which can be interpreted as maximal cliques of the graph with edges $\{x_1, x_3\}$ and $\{x_2, x_4\}$, and inner hub $A = \emptyset$.

### 3.1 Monge solutions

We now state and prove our first main result.

**Theorem 3.1.** Let $\{S_j\}_{j=1}^l$ be the collection of maximal cliques of a given subgraph $S$ of $C_m$ with inner hub $A$, for some $m \in \mathbb{N}$. Let $G := C_m \setminus S$ be connected, $b$ the surplus function associated to $G$ and $\mu_i$ be probability measures over $X_i$, $i = 1, \ldots, m$, with $\mu_1$ absolutely continuous with respect to $\mathcal{L}^n$. Assume that one of the following conditions is met:

(i) $v_1 \in V(G) \setminus V(S)$,

(ii) There exists $p \in I(N_G(v_1))$ such that $A \subseteq N_G(v_p)$, with $\mu_p$ is absolutely continuous with respect to $\mathcal{L}^n$, and, if $S$ is not complete, $v_1 \notin A$.

Then every solution to the Kantorovich problem (KP) is induced by a map.

**Proof.** Let $\gamma$ be a solution to the Kantorovich problem with surplus $b$ and $(u_1, \ldots, u_m)$ a $b$-conjugate solution to its dual. Consider:

$$\tilde{W} = \left\{ (x_1, \ldots, x_m) : Du_1(x_1) \text{ exists, } \sum_{i=1}^m u_i(x_i) = b(x_1, \ldots, x_m) \right\}. $$

The function $u_1$ is differentiable $\mathcal{L}^n$-a.e, as it is Lipschitz continuous. Hence, it is differentiable $\mu_1$ a.e, as $\mu_1$ is absolutely continuous. It follows that $\gamma(\tilde{W}) = 1$. Similarly, if in addition there exists $p \in \{2, \ldots, m\}$ such that $\mu_p$ is absolutely continuous with respect to $\mathcal{L}^n$, we get $u_p$ is differentiable $\mu_p$ a.e and $\gamma(\tilde{W}_p) = 1$, where

$$\tilde{W}_p = \left\{ (x_1, \ldots, x_m) : Du_1(x_1) \text{ and } Du_p(x_p) \text{ exist, } \sum_{i=1}^m u_i(x_i) = b(x_1, \ldots, x_m) \right\}. $$
Fix \( x_1^0 \in spt(\mu_1) \), where \( u_1(x_1) \) is differentiable, and \((x_0^0, \ldots, x_m^0)\) such that \((x_1^0, \ldots, x_m^0) \in \tilde{W}\). Note that \( b \) is differentiable with respect to \( x_1 \) at \((x_1^0, \ldots, x_m^0)\) and it satisfies
\[
Du_1(x_1^0) = D_{x_1}b(x_1^0, \ldots, x_m^0). \tag{14}
\]

We will show that the map
\[
(x_2, \ldots, x_m) \mapsto D_{x_1}b(x_1^0, x_2, \ldots, x_m)
\]
is injective on \( \tilde{W}_{x_1^0} := \{(x_2, \ldots, x_m) : (x_0^0, x_2, \ldots, x_m) \in \tilde{W} \} \), if \( v_1 \in V(G) \setminus V(S) \), or on \( \tilde{W}_{x_1^0 \mu_p} := \{(x_2, \ldots, x_m) : (x_0^0, x_2, \ldots, x_m) \in \tilde{W}_{\mu_p} \} \), if there exists \( p \in I(N(v_1)) \) such that \( \mu_p \) is absolutely continuous with respect to \( L^n \) and \( A \subseteq N(v_p) \); this will imply that the equation \( (14) \) defines \((x_0^0, \ldots, x_m^0)\) uniquely from \( x_1^0 \), which will complete the proof. Let \((x_1^1, x_2^1, \ldots, x_m^1), (x_1^2, x_2^2, \ldots, x_m^2) \in \tilde{W}\) and assume
\[
D_{x_1}b(x_1^0, x_1^1, \ldots, x_m^1) = \sum_{s \in I(N(v_1))} x_1^s = \sum_{s \in I(N(v_1))} x_2^s = D_{x_1}b(x_1^0, x_2^2, \ldots, x_m^2). \tag{15}
\]

We want to prove \( x_1^s = x_2^s \) for every \( s \in \{2, \ldots, m\} \).

Set \( x_1^0 := x_1^1 := x_1^2 \) and \( B_j = V(S_j) \setminus A \), with \( j \in \{1, \ldots, l\} \). First, note that \( S = \bigcup_{j=1}^l S_j \) and
\[
N(v_s) = V(G) \setminus \{v_s\} \quad \text{for any } s \in I(V(G) \setminus V(S)), \tag{16}
\]
\[
N(v_s) = V(G) \setminus V(S_j) \quad \text{for any } s \in I(B_j), \quad j \in \{1, \ldots, l\}, \tag{17}
\]
\[
N(v_s) = V(G) \setminus V(S), \quad \text{for any } s \in I(A). \tag{18}
\]

Let us consider two cases:

**Case 1.** Assume \( v_1 \in V(G) \setminus V(S) = \{v_1, \ldots, v_m\} \setminus V(S) \), then by \( (16) \) we get \( N(v_1) = V(G) = N(v_s) \) for any \( s \in I(V(G) \setminus V(S)) \setminus \{1\} \). It follows from part \( 4a \) of Lemma \( 2.1 \) that
\[
 x_1^s = x_2^s \quad \text{for all} \quad s \in I(V(G) \setminus V(S)) \setminus \{1\}. \tag{19}
\]

Fix \( j \in \{1, \ldots, l\} \), and let us consider two sub-cases:
Case 2. Assume $A = \emptyset$. By defining $F_1 = B_j$ and $F_2 = (V(G) \setminus B_j) \setminus \{v_1\}$, we get $F_1 \cup F_2 = V(G) \setminus \{v_1\}$, where $F_3 = \{v_1\}$. Then, we can apply part 4b of Lemma 2.1 to get $x_s^1 = x_s^2$ for every $s \in I(B_j)$; that is, $x_s^1 = x_s^2$ on $\bigcup_{j=1}^l I(B_j) = \bigcup_{j=1}^l I(V(S_j)) = I(V(S))$. Combining this result with (19) we get $x_s^1 = x_s^2$ on $I(V(G)) \setminus \{1\} = \{2, \ldots, m\}$. This completes the proof of sub-case (a).

(b) Assume $A \neq \emptyset$. By setting $V_2 = V(G) \setminus V(S)$ and $V_1 = A$ we can use (19) to get equality (4), and then, by (18) and part 4b of Lemma 2.1 we get $y := (y_1, y_2, \ldots, y_m) \in \tilde{W}$, where

$$
y_s = \begin{cases} 
x_s^1 & \text{if } s \in \{1, \ldots, m\} \setminus I(A) \\
x_s^2 & \text{if } s \in I(A).
\end{cases}
$$

Now, set $F_1 = B_j$, $F_2 = (V(G) \setminus V(S)) \setminus \{v_1\}$ and $F_3 = \{v_1\}$. Clearly, $F_1, F_2 \subseteq N(v_1) = V(G) \setminus \{v_1\}$ and $F_1 \cup F_2 = V(G) \setminus (A \cup \{v_1\})$, then $N(v_1) \setminus (F_1 \cup F_2) = A$. Furthermore, $y$ and $(x_s^1, x_s^2, \ldots, x_m^2)$ trivially satisfies $y_s = x_s^2$ on $I(A)$, and by (17), $N(v_s) = F_2 \cup F_3$ for every $s \in I(F_1)$. Hence, by part 4b of Lemma 2.1 we get $x_s^1 = y_s = x_s^2$ on $I(B_j)$, which proves that, using (19) and the equality $\bigcup_{j=1}^l B_j = V(S) \setminus A$, $x_s^1 = x_s^2$ on $I(V(G) \setminus A) \setminus \{1\}$. We combine this result with (18) and part 4b of Lemma 2.1 to get $x_s^1 = x_s^2$ on $I(A)$; all the conditions needed to apply this part of the lemma are trivially satisfied by setting $F_1 = A$, $F_2 = (V(G) \setminus V(S)) \setminus \{v_1\}$, $F_3 = \{v_1\}$ and $p = 1$. We conclude that $x_s^1 = x_s^2$ on $I(V(G) \setminus \{v_1\}) = \{2, \ldots, m\}$, completing the proof of sub-case (b).

This completes the proof of case 1.

Case 2. Assume $v_1 \in V(S)$ and let $p \in I(N(v_1))$ be such that $\mu_p$ is absolutely continuous with respect to $\mathcal{L}^n$ and $A \subseteq N(v_p)$. Assume $Du_p(x_p)$ and $Du_p(x_p^2)$ exist. If $S$ is complete, $S_j = S$ for all $j = 1, \ldots, l$, and so, $A = V(S)$ . Using (15) and (18) we obtain

$$
\sum_{s \in I(V(G) \setminus A)} x_s^1 = \sum_{s \in I(V(G) \setminus A)} x_s^2.
$$

(20)
Also, using (18) and part 1 of Lemma 2.1 we get \( z := (z_1, \ldots, z_m) \in \tilde{W}_p \), where
\[
z_s = \begin{cases} 
x_s^1 & \text{if } s \in \{1, \ldots, m\} \setminus I(A) \\
x_s^2 & \text{if } s \in I(A).
\end{cases}
\] (21)

Fix \( t \in I(V(G) \setminus A) \). Then
\[
\sum_{s \in I(N(v_t))} z_s = z_t + \sum_{s \in I(N(v_t))} z_s
\]
\[
= z_t + \sum_{s \in I(V(G) \setminus \{v_t\})} z_s
\]
\[
= \sum_{s \in I(V(G))} z_s
\]
\[
= \sum_{s \in I(A)} z_s + \sum_{s \in I(V(G) \setminus A)} z_s
\]
\[
= \sum_{s \in I(A)} x_s^2 + \sum_{s \in I(V(G) \setminus A)} x_s^1
\]
\[
= \sum_{s \in I(A)} x_s^2 + \sum_{s \in I(V(G) \setminus A)} x_s^2
\]
\[
= \sum_{s \in I(V(G))} x_s^2
\]
\[
= x_t^2 + \sum_{s \in I(V(G) \setminus \{v_t\})} x_s^2
\]
\[
= x_t^2 + \sum_{s \in I(N(v_t))} x_s^2
\]
\[
= \sum_{s \in I(N(v_t))} x_s^2.
\]

It follows that \( z_s = x_s^2 \) on \( I(V(G) \setminus A) \), by part 3 of Lemma 2.1, that is,
\[
x_s^1 = x_s^2 \text{ on } I(V(G) \setminus A),
\] (22)
by construction of \( z \). Now, to prove that \( x_s^1 = x_s^2 \) on \( I(A) \) we use part 4b of Lemma 2.1. For this, set \( F_1 = A, F_2 = V(G) \setminus (A \cup \{v_p\}) \) and \( F_3 = \{v_p\} \). Since \( v_p \in N(v_1) = V(G) \setminus V(S) \), hence \( N(v_p) = V(G) \setminus \{v_p\} \)
Next, as in Case 1, let us consider the following sub-cases:

of Lemma 2.1 to get $x^1 = x^2$ and part 1 of Lemma 2.1 to get $y$.

It follows that, by setting $x^1 = x^2$ on $I(A)$. Hence, $x^1 = x^2$ on $I(V(G)) = \{1, \ldots, m\}$.

Let us know assume that $S$ is not complete, then $v_1 \notin A$ by assumption, which implies that $v_1 \in B_k$ for some $k \in \{1, \ldots, l\}$. We first claim that $x^1 = x^2$ on $\bigcup_{j=1}^{l} I(B_j) = I(V(S) \setminus V(S_0))$. Indeed, from (17) and (16) we get

$$
\sum_{s \in I(V(G) \setminus V(S_0))} x^1_s = \sum_{s \in I(V(G) \setminus V(S_0))} x^2_s.
$$

It follows that, by setting $V_1 = B_k$ and $V_2 = V(G) \setminus V(S_k)$, we can use (17) and part 1 of Lemma 2.1 to get $y := (y_1, \ldots, y_m) \in \overline{W}_p$, where

$$
y_s = \begin{cases} 
    x^1_s & \text{if } s \in \{1, \ldots, m\} \setminus I(B_k) \\
    x^2_s & \text{if } s \in I(B_k).
\end{cases}
$$

Fix $j \in \{1, \ldots, l\}$, with $j \neq k$. Set $F_1 = B_j$, $F_2 = V(G) \setminus (V(S_k) \cup B_j)$ and $F_3 = B_k$. Note that from (17) we get $F_1 \cup F_2 = V(G) \setminus V(S_k) = N(v_1)$ and $F_2 \cup F_3 = V(G) \setminus V(S_j) = N(v_s)$, for any $s \in I(F_1)$. Since $y$ and $(x^1_1, x^2_2, \ldots, x^2_m)$ satisfies $y_s = x^2_s$ on $I(F_3)$, we can apply part 1 of Lemma 2.1 to get $x^1_s = y_s = x^2_s$ on $I(B_j)$; that is,

$$
x^1_s = x^2_s \quad \text{on } \bigcup_{j=1, j \neq k}^l I(B_j) = I(V(S) \setminus V(S_k)).
$$

Next, as in Case 1, let us consider the following sub-cases:

(a) If $A = \emptyset$, then $V(S_k) = B_k$. Also, for any $t \in I(V(G) \setminus V(S))$ we get

$$
\sum_{s \in I(N(v_1))} y_s = y_t + \sum_{s \in I(N(v_1))} y_s = y_t + \sum_{s \in I(V(G) \setminus \{v_1\})} y_s \quad \text{by (16)}
$$

$$
= \sum_{s \in I(V(G))} y_s
$$

21
\[
\begin{align*}
\sum_{s \in I(B_k)} y_s + \sum_{s \in I(V(G) \setminus B_k)} y_s &= \sum_{s \in I(B_k)} x_s^2 + \sum_{s \in I(V(G) \setminus B_k)} x_s^1 \quad \text{by construction of } y \\
&= \sum_{s \in I(B_k)} x_s^2 + \sum_{s \in I(V(G) \setminus B_k)} x_s^2 \quad \text{by (23)} \\
&= \sum_{s \in I(V(G))} x_s^2 \\
&= x_t^2 + \sum_{s \in I(V(G) \setminus \{v_t\})} x_s^2 \\
&= x_t^2 + \sum_{s \in I(N(v_t))} x_s^2 \\
&= \sum_{s \in I(N(v_t))} x_s^2
\end{align*}
\]

Thus, by part 3 of Lemma 2.1 we obtain \( x_s^1 = x_s^2 \) on \( I(V(G) \setminus V(S)) \), as \( y_s = x_s^1 \) on \( I(V(G) \setminus V(S)) \). Combining this with (23) we deduce

\[
x_s^1 = x_s^2 \quad \text{on } I(V(G) \setminus V(S_k)) = I(V(G) \setminus B_k) = I(N(v_1)). \tag{26}
\]

To prove that \( x_s^1 = x_s^2 \) on \( I(B_k) \) we use part 4b of Lemma 2.1. Let us first recall that \( p \in I(N(v_1)) \), and then, the above result tell us that \( x_p^1 = x_p^2 \). Furthermore, \( p \in I(V(G) \setminus V(S)) \), or \( p \in I(B_j) \) for some \( j \in \{1, \ldots, l\} \), \( k \neq j \). Thus, from (16), (17) and the disjointness of \( B_k \) and \( B_j \), we deduce \( B_k \subseteq N(v_p) \). Now, set \( F_1 = B_k, F_2 = N(v_p) \setminus B_k \) and \( F_3 = V(G) \setminus N(v_p) \). Then, \( F_1, F_2 \subseteq N(v_p), F_1 \cup F_2 = N(v_p) \) and \( F_2 \cup F_3 = V(G) \setminus B_k = N(v_s) \), for every \( s \in I(F_1) \). Also, from (26) we get \( x_s^1 = x_s^2 \) on \( I(F_3) \), as it is evident that \( B_k \cap F_3 = \emptyset \). We can then apply part 4b of Lemma 2.1 to obtain \( x_s^1 = x_s^2 \) on \( I(B_k) \), which combined with (26) allow us to have \( x_s^1 = x_s^2 \) on \( \{2, \ldots, m\} \). This completes the proof of sub-case (a).

(b) Assume \( A \neq \emptyset \). Let us first prove that \( x_s^1 = x_s^2 \) on \( I(V(G) \setminus V(S)) \); this will be achieved via part 3 of Lemma 2.1.
Using (23) and (25), we can write
\[
\sum_{s \in I(V(G) \setminus V(S))} x^1_s = \sum_{s \in I(V(G) \setminus V(S))} x^2_s,
\]
and defining \( y \) as in (24) we can equivalently write
\[
\sum_{s \in I(V(G) \setminus V(S))} y_s = \sum_{s \in I(V(G) \setminus V(S))} x^2_s.
\]
Hence, from (18) and part 1 of Lemma 2.1 we get
\[y' = (y'_1, \ldots, y'_m) \in \tilde{W}_p,\]
where
\[y'_s = \begin{cases} 
    y_s & \text{if } s \in I(V(G) \setminus A) \\
    x^2_s & \text{if } s \in I(A)
\end{cases} = \begin{cases} 
    x^1_s & \text{if } s \in I(V(G) \setminus V(S_k)) \\
    x^2_s & \text{if } s \in I(V(S_k)).
\end{cases}\]
Then,
\[\sum_{s \in I(V(G) \setminus V(S_k))} y'_s = \sum_{s \in I(N(v_1))} y'_s = Du_1(x^0_1) = \sum_{s \in I(N(v_1))} x^2_s = \sum_{s \in I(V(G) \setminus V(S_k))} x^2_s.
\]
By construction of \( y' \), one has,
\[\sum_{s \in I(V(G))} y'_s = \sum_{s \in I(V(G))} x^2_s.
\]
Hence, using (16) we clearly might express it as
\[\sum_{s \in I(N(v_1))} y'_s = \sum_{s \in I(N(v_1))} x^2_s,
\]
for every fixed \( t \in I(V(G) \setminus V(S)) \). We can now apply part 3 of Lemma 2.1 and get \( y'_t = x^2_t \) on \( I(V(G) \setminus V(S)) \), which implies \( x^1_t = x^2_t \) on \( I(V(G) \setminus V(S)) \), since \( I(V(G) \setminus V(S)) \subseteq I(V(G) \setminus V(S_k)) \) and \( y'_t = x^1_t \) on \( I(V(G) \setminus V(S_k)) \). Thus, from (25),
\[x^1_s = x^2_s \text{ on } I(V(G) \setminus V(S_k)) = I(N(v_1)). \quad (27)\]
It only remains to prove that \( x^1_s = x^2_s \) on \( I(V(S_k)) \). Since \( p \in I(N(v_1)) \), from the above equalities \( p \in I(V(G) \setminus V(S)) \) or \( p \in I(B_j) \) for some \( j \neq k \), and \( x^1_p = x^2_p \). Then \( N(v_p) = V(G) \setminus \{v_p\} \) or \( N(v_p) = V(G) \setminus V(S_j) \). It follows that \( N(v_p) = V(G) \setminus \{v_p\} \), as \( A \subseteq N(v_p) \) and \( A \cap (V(G) \setminus V(S_j)) = \emptyset \). Now, set \( F_1 = A \), \( F_2 = V(G) \setminus (V(S) \cup \{v_p\}) \) and \( F_3 = \{v_p\} \). It is clear that \( F_1, F_2 \subseteq N(v_p) \) and \( F_2 \cup F_3 = V(G) \setminus V(S) = N(v_s) \) for every \( s \in I(A) \). Furthermore, for \( y \) defined as in (24) and \( (x_1^0, x_2^0, \ldots, x_m^0) \), we have \( y_s = x^2_s \) on \( I(B_k) \). It follows from (27) and construction of \( y \) that \( y_s = x^2_s \) on \( I(V(G) \setminus A) \); in particular, \( y_s = x^2_s \) on \( I(N(v_p) \setminus F_1 \cup F_2) \subseteq I(V(G) \setminus A) \). Hence, \( x^1_s = y_s = x^2_s \) on \( I(A) \) by part 4b of Lemma 2.1, and then, we can easily obtain \( x^1_s = x^2_s \) on \( I(B_k) \), by applying again part 4b of Lemma 2.1 this time we set \( F_1 = B_k \), \( F_2 = (V(G) \setminus V(S_k)) \setminus \{v_p\} \) and \( F_3 = \{v_p\} \). This completes the proof of sub-case (b).

This completes the proof of the theorem.

Before presenting some examples, we note the following consequence of the preceding theorem.

**Corollary 3.1.** Let \( G \) be a subgraph of \( C_m \) with \( |G| = m \), and satisfying \( |N(v)| \in \{m - 1, m - 2\} \) for every \( v \in V(G) \). Assume that \( \mu_1 \) is absolutely continuous with respect to Lebesgue measure, and that either \( |N(v_1)| = m - 1 \) or that \( \mu_i \) is absolutely continuous with respect to Lebesgue measure for some \( i \in I(N(v_1)) \). Then every solution to the Kantorovich problem (KP) is induced by a map.

**Proof.** Note that if \( G \neq C_m \), then \( G = C_m \setminus \bigcup S^i_{j=1} \), for some disjoint collection of complete graphs \( \{S_j\}_{j=1} \), where \( |V(S_j)| = 2 \) for every \( j \) (that is, every \( S_j \) consists on a single edge). Clearly, the graph \( \bigcup S_j \) has inner hub \( A = \emptyset \) and maximal cliques \( S_1, \ldots, S_l \). The result then follows from Theorem 3.1.

Note that the Gangbo-Swiech surplus corresponds to a complete graph, or, equivalently, to the graph \( G \) satisfying \( |N(v)| = m - 1 \) for each \( v \in V(G) \); the Corollary is then a generalization to the case where each vertex can be missing at most one edge connecting it to the other vertices.
3.2 Examples

Here, we illustrate the result obtained in Theorem 3.1 throughout several examples.

(i) Let $G$ be a complete $k$-partite graph with set partition $\{V_1, \ldots, V_k\}$ and $m := |V(G)| = |\bigcup_{j=1}^{k} V_j|$. Write $\bigcup_{j=1}^{k} V_j = \{v_1, \ldots, v_m\}$, and let $S_1, \ldots, S_k$ be $k$ complete graphs with sets of vertices $V_1, \ldots, V_k$, respectively. Note that $G := C_m \setminus \bigcup_{j=1}^{k} S_j$ and $N(v_1) = \bigcup_{j \neq \alpha}^{k} V_j$, for some $\alpha \in \{1, \ldots, k\}$. Hence, by assuming $\mu_1$ and $\mu_p$ absolutely continuous, for some $p \in N(v_1)$, we can conclude by Theorem 3.1 that the graph $G$ gives a unique Monge solution, as we can interpret $\{S_j\}_{j=1}^{k}$ as the collection of maximal cliques of the graph $\bigcup_{j=1}^{k} S_j$. Here, $A = \emptyset$ is clearly the inner hub of $\bigcup_{j=1}^{k} S_j$.

A special case is the complete graph $C_k$; several other examples of $k$-partite graphs are below.

- Complete bipartite graphs $K_{m,n}$:

(a) Graph $K_{3,3}$. Known as the *Utility graph*.  

(b) Graph $K_{4,4}$. Known as the *Cayley graph*.

Figure 5
Figure 6: Bipartite graph with set partition \( \{ V_1, V_2 \} \), where \( V_1 = \{ v_1, v_2, v_3, v_4, v_5, v_{10} \} \) and \( V_2 = \{ v_6, v_7, v_8, v_9 \} \).

- Complete Tripartite graphs \( K_{m,n,p} \):

(a) Graph \( K_{1,2,2} \). Known as the 5-wheel graph.

(b) Graph \( K_{1,1,2} \). Known as the Diamond graph.
(ii) A notable special case of Corollary 3.1 occurs when \( m \) is even and 
\(|N(v)| = m - 2\) for all \( v \in V(G) \), in which case \( G = K_{2,2,2}^{m-2} \). 
This graph is known as the \textit{Cocktail Party Graph}. See example below.

(iii) Theorem 3.1 can be used easily to construct many other, more obscure,
graphs leading to Monge solutions. We construct one of such examples here; set
\[ V_1 = \{v_1, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}\}. \]
\[ V_2 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}; \]
\[ V_3 = \{v_6, v_{14}, v_{15}, v_{16}\}; \]
\[ V_4 = \{v_4, v_{14}, v_{15}, v_{16}\}. \]

Consider \( S_1, S_2, S_3, S_4 \) complete graphs with \( V(S_j) = V_j, j = 1, 2, 3, 4 \). Then, the graph \( S = S_1 \cup S_2 \cup S_3 \cup S_4 \) has inner hub \( A = \{v_{14}, v_{15}, v_{16}\} \), with maximal cliques \( S_1, S_2, S_3, S_4 \). See Figure below.

Figure 10: Graph \( S = S_1 \cup S_2 \cup S_3 \cup S_4 \).

Then, \( G = C_{20} \setminus S \) provides a solution of Monge type.
4 Monge solutions for graphs with inner hubs and gluing of them

The main result of this section (Theorem 4.1) ensures that under regularity conditions on two of the marginals, the surplus associated to a graph with inner hub provides a unique solution for the Monge-Kantorovich problem.

Before stating the main result of this section, we present the following simple example, which illustrates part of the motivation for Theorem 4.1 and Propositions 4.1 and 4.2.

Example 4.1. Let $b$ be the surplus associated to the graph $G$ below.
The second assertion of Proposition 2.1 implies that \( b \) is not twisted on splitting sets, and there are in fact choices \( \mu_1, \mu_2, \mu_3 \) and \( \mu_4 \) of marginals such that \( \mu_1 \) is absolutely continuous with respect to Lebesgue measure and the solution to (KP) is of non-Monge form and non-unique (explicitly, take \( \mu_3 \) to be a Dirac mass and the other marginals to be uniform on bounded domains). However, it is clear that the problem does admit a unique, Monge type solution as soon as both \( \mu_1 \) and \( \mu_3 \) are absolutely continuous. The reason for this is one may solve the three marginal problem with \( \mu_1, \mu_2, \mu_3 \) and the surplus \( x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 \) via the Gangbo-Święch theorem [15], obtaining unique optimal maps \( T_2, T_3 \), and then solve independently the two marginal problems between \( \mu_3, \mu_4 \) with surplus \( x_3 \cdot x_4 \), yielding a unique optimal map \( T_4 \), and between \( \mu_1, \mu_5 \) with surplus \( x_1 \cdot x_5 \), yielding a unique optimal map \( T_5 \). Since \( x_4 \) only interacts with \( x_3 \), and \( x_5 \) only interacts with \( x_1 \), \( (T_2, T_3, T_4, T_5) := (T_2, T_3, T_4 \circ T_3, T_5) \) is then the unique Monge solution for the overall problem.

This sort of result is not captured by Theorem 3.1, as the graph extracted from the complete graph \( C_5 \) to yield \( G \), depicted below:
does not have an inner hub; we develop in this section a framework that encapsulates simple examples like this one, as well as more complicated ones which cannot be treated with adhoc arguments like the one sketched above.

4.1 Monge solutions for graphs with inner hubs

We now proceed to state and prove our second main result.

**Theorem 4.1.** Let $G$ be a graph with inner hub $A$ and maximal cliques $S_1, \ldots, S_l$, with $m = |V(G)|$, and $b$ its associated surplus. Let $\mu_i$ be probability measures over $X_i$, $i = 1, \ldots, m$, with $\mu_1$ absolutely continuous with respect to $\mathcal{L}^n$. If there exists $p \in I(A)$ such that $\mu_p$ is absolutely continuous with respect to $\mathcal{L}^n$, then every solution to the Kantorovich problem $(\text{KP})$ with surplus $b$ is induced by a map.

**Proof.** Let $\gamma$ be a solution to the Kantorovich problem with surplus $b$ and $(u_1, \ldots, u_m)$ a $b$-conjugate solution to its dual. Set:

$$\tilde{W}_p = \{(x_1, \ldots, x_m) : Du_1(x_1) \text{ and } Du_p(x_p) \text{ exist, } \sum_{i=1}^{m} u_i(x_i) = b(x_1, \ldots, x_m)\}.$$

As in Theorem 3.1 we obtain $\gamma(\tilde{W}_p) = 1$. Moreover, by fixing $x_1^0$ where $u_1(x_1)$ is differentiable, we get for any $(m-1)$-tuple $(x_2^0, \ldots, x_m^0)$ satisfying $(x_1^0, \ldots, x_m^0) \in \tilde{W}_p$,

$$Du_1(x_1^0) = D_{x_1} b(x_1^0, \ldots, x_m^0).$$
Let us show that the map
\[(x_2, \ldots, x_m) \mapsto D_{x_1}b(x_1, x_2, \ldots, x_m)\]
is injective on \(\tilde{W}_{x_0}^{x_1} := \{(x_2, \ldots, x_m) : (x_1^0, x_2, \ldots, x_m) \in \tilde{W}_p\}\). Indeed, assume
\[D_{x_1}b(x_1^0, x_2, \ldots, x_m) = \sum_{s \in I(N(v_1))} x_s^1 = \sum_{s \in I(N(v_1))} x_s^2 = D_{x_1}b(x_1^0, x_2^0, \ldots, x_m^0),\]
where \((x_1^0, x_2, \ldots, x_m), (x_1^0, x_2^0, \ldots, x_m^0) \in \tilde{W}_p\), and \(x_1^0 := x_1^1 := x_1^2\). Recall that \(G = \bigcup_{j=1}^l S_j\), and without loss of generality assume \(v_1 \in V(S_1)\). Then \(v_1 \in B_1\) or \(v_1 \in A\), where \(B_j = V(S_j) \setminus A\), \(j \in \{1, \ldots, l\}\). For the case \(v_1 \in B_1\), we split the proof into several steps.

**Step 1.** Since \(S_1\) is complete, for every \(s \in I(B_1)\), \(N(v_s) = V(S_1) \setminus \{v_s\}\), which implies \(N(v_1) = N(v_s)\). Then, by part 4a of Lemma 2.1 we get
\[x_s^1 = x_s^2\text{ for all } s \in I(B_1).\]
Hence, the equalities \(N(v_1) = V(S_1) \setminus \{v_1\} = (B_1 \cup A) \setminus \{v_1\}\), and (28), show that
\[\sum_{s \in I(A)} x_s^1 = \sum_{s \in I(A)} x_s^2.\]

**Step 2.** From part 2 of Lemma 2.1 for every \(t \in I(A)\) we get
\[(x_t^2 - x_t^1) \cdot \sum_{s \in I(N(v_t))} (x_s^1 - x_s^2) \leq 0,\]
and by the definition of \(A\),
\[N(v_t) = V(G) \setminus \{v_t\}\]
\[= \left(\bigcup_{j=1}^l V(S_j)\right) \setminus \{v_t\}\]
\[= (A \setminus \{v_t\}) \bigcup \left(\bigcup_{j=1}^l B_j\right).\]
Thus, we can write (31) as
\[
(x_t^2 - x_t^1) \cdot \sum_{s \in I(A) \setminus \{t\}} (x_s^1 - x_s^2) + (x_t^2 - x_t^1) \cdot \sum_{s \in \bigcup_{j=1}^l I(B_j)} (x_s^1 - x_s^2) \leq 0.
\]

It follows from (30) that
\[
\|x_t^2 - x_t^1\|^2 + (x_t^2 - x_t^1) \cdot \sum_{s \in \bigcup_{j=1}^l I(B_j)} (x_s^1 - x_s^2) \leq 0,
\]

hence, one easily deduces
\[
(x_t^2 - x_t^1) \cdot \sum_{s \in \bigcup_{j=1}^l I(B_j)} (x_s^1 - x_s^2) \leq 0.
\]

Summing over \( t \in I(A) \) we get
\[
\sum_{t \in I(A)} (x_t^2 - x_t^1) \cdot \sum_{s \in \bigcup_{j=1}^l I(B_j)} (x_s^1 - x_s^2) \leq 0,
\]

and by (30), we must have equality in (31) for every \( t \in I(A) \). Therefore, from (33) we get
\[
x_t^1 = x_t^2 \quad \text{for all} \quad t \in I(A).
\]

In particular, \( x_p^1 = x_p^2 \) and so, \( x_p^2 \) belongs to
\[
\text{Argmax}\left\{ x_p \mapsto \left( \sum_{s \in I(N(v_p))} x_s^1 \cdot x_p - u_p(x_p) \right) \right\} \cap \text{Argmax}\left\{ x_p \mapsto \left( \sum_{s \in I(N(v_p))} x_s^2 \cdot x_p - u_p(x_p) \right) \right\}.
\]

It follows that
\[
\sum_{s \in I(N(v_p))} x_s^1 = Du_p(x_p^2) = \sum_{s \in I(N(v_p))} x_s^2,
\]
or equivalently, invoking (32),
\[
\sum_{s \in I(A) \setminus \{p\}} x_s^1 + \sum_{s \in \bigcup_{j=1}^l I(B_j)} x_s = \sum_{s \in I(A) \setminus \{p\}} x_s^2 + \sum_{s \in \bigcup_{j=1}^l I(B_j)} x_s^2.
\]

It immediately implies by (35) that
\[
\sum_{s \in \bigcup_{j=1}^l I(B_j)} x_s^1 = \sum_{s \in \bigcup_{j=1}^l I(B_j)} x_s^2.
\]
Step 3. Fix \( k \in \{2, \ldots, l\} \). From definition 22. \( \{B_j\}_{j=1}^{l} \) is a disjoint collection of sets and every \( j \in \{1, \ldots, l\} \) satisfies \( N(v_s) = V(S_j) \setminus \{v_s\} = (B_j \cup A) \setminus \{v_s\} \), for every \( s \in I(B_j) \). Since \((x_1^0, x_2^1, \ldots, x_m^1) \in \widetilde{W}_p\), we get

\[
\{ x_s^1 \}_{s \in \bigcup_{j \neq k}^l I(B_j)} \in \text{Argmax} \left\{ \{ x_s^1 \}_{s \in \bigcup_{j \neq k}^l I(B_j)} \mapsto \left( \sum_{s \in I(A)} x_s^1 \right) \cdot \sum_{s \in \bigcup_{j \neq k}^l I(B_j)} x_s + \sum_{j=1}^l \sum_{s \in I(B_j), s \neq k} x_s \cdot x_t - \sum_{s \in \bigcup_{j \neq k}^l I(B_j)} u_s(x_s) \right\},
\]

and by \((35)\),

\[
\{ x_s^1 \}_{s \in \bigcup_{j \neq k}^l I(B_j)} \in \text{Argmax} \left\{ \{ x_s^1 \}_{s \in \bigcup_{j \neq k}^l I(B_j)} \mapsto \left( \sum_{s \in I(A)} x_s^2 \right) \cdot \sum_{s \in \bigcup_{j \neq k}^l I(B_j)} x_s + \sum_{j=1}^l \sum_{s \in I(B_j), s \neq k} x_s \cdot x_t - \sum_{s \in \bigcup_{j \neq k}^l I(B_j)} u_s(x_s) \right\}.
\]

Hence, setting \( y := (y_1, y_2, \ldots, y_m) \) with

\[
y_s = \begin{cases} 
  x_s^2 & \text{if } s \in \{1, 2, \ldots, m\} \setminus \bigcup_{j=1}^l I(B_j) = I(V(S_k)) \\
  x_s^1 & \text{if } s \in \bigcup_{j=1}^l I(B_j),
\end{cases}
\]

we get \( y \in \widetilde{W}_p \), as \((x_1^0, x_2^1, \ldots, x_m^1) \in \widetilde{W}_p\). Since \((36)\) holds true for every \((x_1^0, x_2^1, \ldots, x_m^1), (x_1^0, x_2^2, \ldots, x_m^2) \in \widetilde{W}_p\); in particular, it is true for \((x_1^0, x_2^1, \ldots, x_m^1)\) and \( y \), which implies that

\[
\sum_{s \in I(B_k)} x_s^1 = \sum_{s \in I(B_k)} x_s^2.
\]

Using \((35)\) we can write the above equality as

\[
\sum_{s \in I(V(S_k))} x_s^1 = \sum_{s \in I(V(S_k))} x_s^2.
\]
Hence, all the elements of $I(B_k)$ satisfy (6), as each $s \in I(B_k)$ satisfies $N(v_s) = V(S_k) \setminus \{v_s\}$. Then, by part 3 of Lemma 2.1, $x_1^s = x_2^s$ for all $s \in I(B_k)$. We thus conclude by (29) and (35) that $x_1^s = x_2^s$ for all $s \in \bigcup_{j=1}^I I(B_j) \cup I(A) = I(G) = \{1, 2, \ldots, m\}$. This completes the proof for the case $v_1 \in B_1$.

Finally, for the case $v_1 \in A$, note that every $s \in I(A)$ satisfies $N(v_s) = V(G) \setminus \{v_s\}$, hence for any $s \in I(A)$ we get

$$\overline{N}(v_s) = \{v_s\} \cup N(v_s) = V(G) = (V(G) \setminus \{v_1\}) \cup \{v_1\} = N(v_1) \cup \{v_1\} = \overline{N}(v_1).$$

Therefore, by part 4a of Lemma 2.1 we get (35), and then, (28) reduces to (36). The rest of the proof runs exactly as the proof in Step 3, but instead of fixing $k$ in $\{2, \ldots, l\}$, we fix it in $\{1, \ldots, l\}$, completing the proof of the theorem. 

4.2 Monge solutions for graphs glued on cliques

We now turn to a natural extension of Theorem 4.1. The next proposition states, roughly speaking, that gluing together several graphs with inner hubs via the procedure formulated in Definition 2.3 leads to a solution of Monge type.

**Proposition 4.1.** Let $S_1$ be a graph with inner hub $A_1$ and $\{S_{1j}\}_{j=1}^l$ its collection of maximal cliques. Let $E \subset \{2, \ldots, l\}$ such that for every $\alpha \in E$, $S_\alpha$ is a graph with inner hub $A_\alpha \neq \emptyset$, and with collection of maximal cliques $\{S_{\alpha j}\}_{j=1}^{k_\alpha}$. Assume $A_\alpha \cap A_1 = \emptyset$ for every $\alpha \in E$, and set $G = \bigcup_{\alpha \in E \cup \{1\}} S_\alpha$ and $m = |V(G)|$. Let $\mu_i$ be probability measures over $X_i$, $i = 1, \ldots, m$ and assume:

1. $S_\alpha$ and $S_1$ are glued on a clique for all $\alpha \in E$.
2. $V(S_\alpha) \cap V(S_\beta) = A_1$ for all $\alpha \neq \beta, \alpha, \beta \in E$.
3. For each $\alpha \in E \cup \{1\}$, there exists $p_\alpha \in I(A_\alpha)$ such that $\mu_{p_\alpha}$ is absolutely continuous with respect to $\mathcal{L}^n$.
4. $\mu_1$ is absolutely continuous with respect to $\mathcal{L}^n$ and $v_1 \in V(S_{11})$.

Then every solution to the Kantorovich problem (KP) with surplus associated to $G$ is concentrated on a graph of a measurable map.
Step 1. We proceed to make a straightforward adaptation of the arguments used in Step 3 of the proof of Theorem 4.1. First, note that $N(v_s) = V(S_{1s}) \setminus \{v_s\}$, for every $s \in I(B_1)$, then, using the differentiability of $u_{p_1}(x_{p_1})$ at $x_{p_1}^1$ and $x_{p_1}^2$, and the equalities (37) and (38), we can mirror steps 1 and 2 in the proof of Theorem 4.1 to get:

$$x_s^1 = x_s^2 \quad \text{for all} \quad s \in I(B_1),$$

Proof. The strategy of the proof is similar to the strategy used in Theorem 4.1. Let $\gamma$ be a solution to the Kantorovich problem with surplus $b(x_1, \ldots, x_m)$, where $b$ is the surplus associated to $G$. Let $(u_1, \ldots, u_m)$ be a $b$-conjugate solution to its dual and set

$$\tilde{W} = \left\{ (x_1, \ldots, x_m) : Du_1(x_1) \quad \text{and} \quad Du_{pa}(x_{pa}) \quad \text{exist} \quad \text{for all} \quad \alpha \in E \cup \{1\}, \right. \left. \sum_{i=1}^{m} u_i(x_i) = b(x_1, \ldots, x_m) \right\}. $$

Fix $x_0^i \in spt(\mu_1)$, where $u_1(x_1)$ is differentiable. Then $Du_1(x_0^i) = D_{x_1} b(x_1^0, \ldots, x_m^0)$, for every $(x_1^0, \ldots, x_m^0) \in \tilde{W}$. We want to prove that the map $(x_2, \ldots, x_m) \mapsto D_{x_1} b(x_1^0, x_2, \ldots, x_m)$ is injective on

$$\tilde{W}_{x_1^0} := \left\{ (x_2, \ldots, x_m) : (x_1^0, x_2, \ldots, x_m) \in \tilde{W} \right\}. $$

Assume

$$D_{x_1} b(x_1^0, x_2, \ldots, x_m) = \sum_{s \in I(N(v_1))} x_s^1 = \sum_{s \in I(N(v_1))} x_s^2 = D_{x_1} b(x_1^0, x_2^0, \ldots, x_m^0), $$

with $(x_1^0, x_2^0, \ldots, x_m^0), (x_1^0, x_2^1, \ldots, x_m^1) \in \tilde{W}$ and $x_1^0 := x_1^1 = x_1^2$. Note that if $E = \emptyset$, we get $G = S_1$, and then, by Theorem 4.1 we get a solution of Monge type. Assume $E \neq \emptyset$ and set $B_j = V(S_{1j}) \setminus A_1$, where $j \in \{1, \ldots, l\}$. Since $A_j \cap A_1 = \emptyset$ for every $j \in E$, $B_j \neq \emptyset$ for every $j \in E$ and

$$N(v_s) = V(S_1) \setminus \{v_s\} = \bigcup_{j=1}^{l} B_j \cup (A_1 \setminus \{v_s\}), \quad \text{for every} \quad s \in I(A_1). \quad (38)$$

Furthermore, by assumption 1 we can assume without lost of generality that $S_{1\alpha} = S_{\alpha 1}$ for every $\alpha \in E$. As in Theorem 4.1 we consider two cases, $v_1 \in B_1$ or $v_1 \in A_1$. Let us divide the proof of case $v_1 \in B_1$ into several steps:
\[ x^1_s = x^2_s \quad \text{for all} \quad s \in I(A_1), \quad (40) \]

and

\[ \sum_{s \in \bigcup_{j=1}^l I(B_j)} x^1_s = \sum_{s \in \bigcup_{j=1}^l I(B_j)} x^2_s. \quad (41) \]

**Step 2.** Fix \( \alpha \in \{2, \ldots, l\} \) and set \( S_\beta = S_{1\beta} \) for any \( \beta \in \{2, \ldots, l\} \setminus E \). Define

\[ \mathcal{I}_1 = \bigcup_{\substack{\beta=2 \\
 \beta \neq \alpha}}^l I(V(S_\beta) \setminus A_1) \cup I(B_1) = \{t_1, \ldots, t_d\} \quad \text{and} \quad \mathcal{I}_2 = I(V(S_\alpha) \setminus A_1) = \{r_1, \ldots, r_e\}. \]

We claim that \( x^1_s = x^2_s \) for all \( s \in \mathcal{I}_2 \), this will complete the proof. Indeed, note that

\[ \{1, \ldots, m\} = \bigcup_{\beta=2}^l I(V(S_\beta)) \cup I(B_1) \]

\[ = \left( \bigcup_{\substack{\beta=2 \\
 \beta \neq \alpha}}^l I(V(S_\beta)) \cup I(B_1) \right) \cup I(V(S_\alpha)) \]

\[ = \left( \bigcup_{\substack{\beta=2 \\
 \beta \neq \alpha}}^l I(V(S_\beta) \setminus A_1) \cup I(B_1) \right) \cup I(V(S_\alpha) \setminus A_1) \cup A_1 \]

\[ = \mathcal{I}_1 \cup \mathcal{I}_2 \cup A_1 \quad (42) \]

Furthermore, the last union is disjoint by assumptions 1 and 2. Now, let \( g_1(x_{t_1}, \ldots, x_{t_d}) \) and \( g_2(x_{r_1}, \ldots, x_{r_e}) \) be the functions formed by all the terms of \( b \) that depend only on the variables with index in \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) respectively. From Definition 2.3 and assumptions 1 and 2, it is not hard to deduces that

\[ \bigcup_{s \in \mathcal{I}_k} N(v_s) = \{v_s\}_{s \in \mathcal{I}_k} \cup A_1, \ k = 1, 2. \]

Combining the above equalities, (42) and (38) we get

\[ b(x_1, \ldots, x_m) = g_1(x_{t_1}, \ldots, x_{t_d}) + g_2(x_{r_1}, \ldots, x_{r_e}) + \left( \sum_{s \in I(A_1)} x_s \right) \cdot \sum_{s \in \bigcup_{j=1}^l I(B_j)} x_s + \sum_{s,t \in I(A_1)} x_s \cdot x_t. \]
\[ g_1(x_{t_1}, \ldots, x_{t_d}) + g_2(x_{r_1}, \ldots, x_{r_e}) + \left( \sum_{s \in I(A_1)} x_s \right) \cdot \sum_{s \in \bigcup_{j=1}^{l} I(B_j)} x_s + \left( \sum_{s \in I(A_1)} x_s \right) \cdot \sum_{s \in I(B_a)} x_s + \sum_{s \leq t} x_s \cdot x_t. \]

Note that
\[ \bigcup_{j=1}^{l} I(B_j) \subset I_1, \quad (43) \]
and the only terms of \( b \) that depend on the variables with index in \( I_1 \) are \( g_1(x_{t_1}, \ldots, x_{t_d}) \) and \( \left( \sum_{s \in I(A_1)} x_s \right) \cdot \sum_{s \in \bigcup_{j=1}^{l} I(B_j)} x_s \). Hence,
\[
\{ x_1^s \}_{s \in I_1} \in \text{Argmax} \left\{ \{ x_s \}_{s \in I_1} \mapsto \left( \sum_{s \in I(A_1)} x_s \right) \cdot \sum_{s \in \bigcup_{j=1}^{l} I(B_j)} x_s + g_1(x_{t_1}, \ldots, x_{t_d}) - \sum_{s \in I_1} u_s(x_s) \right\},
\]
and by (40),
\[
\{ x_1^s \}_{s \in I_1} \in \text{Argmax} \left\{ \{ x_s \}_{s \in I_1} \mapsto \left( \sum_{s \in I(A_1)} x_s \right) \cdot \sum_{s \in \bigcup_{j=1}^{l} I(B_j)} x_s + g_1(x_{t_1}, \ldots, x_{t_d}) - \sum_{s \in I_1} u_s(x_s) \right\}.
\]
Since \((x^0_1, x^2_2, \ldots, x^2_m) \in \widehat{W}\), we obtain \(y := (y_1, y_2, \ldots, y_m) \in \widehat{W}\), where
\[
y_s = \begin{cases} 
  x^2_s & \text{if } s \in \{1, 2, \ldots, m\} \setminus I_1 \\
  x^1_s & \text{if } s \in I_1
\end{cases}
\]

Therefore, (41) holds true for \(y\) and \((x^0_1, x^1_2, \ldots, x^1_m) \in \widehat{W}\); that is,
\[
\sum_{s \in \bigcup_{j=1}^l I(B_j)} y_s = \sum_{s \in \bigcup_{j=1}^l I(B_j)} x^1_s,
\]
or equivalently,
\[
\sum_{s \in I(B_\alpha)} y_s + \sum_{s \in \bigcup_{j=1}^l I(B_j) \setminus I(B_\alpha)} y_s = \sum_{s \in I(B_\alpha)} x^1_s + \sum_{s \in \bigcup_{j=1}^l I(B_j) \setminus I(B_\alpha)} x^1_s.
\]

By the above equality, (43) and construction of \(y\) we get
\[
\sum_{s \in I(B_\alpha)} x^2_s = \sum_{s \in I(B_\alpha)} x^1_s. \quad (44)
\]

**Step 3.** Since \(B_\alpha = V(S_{1\alpha}) \setminus A_1\), by (40) and the above equality we can write,
\[
\sum_{s \in I(V(S_{1\alpha}))} x^1_s = \sum_{s \in I(V(S_{1\alpha}))} x^2_s. \quad (45)
\]

Now, if \(\alpha \in \{2, \ldots, m\} \setminus E\), then \(S_\alpha = S_{1\alpha}\) and \(N(v_s) = V(S_{1\alpha}) \setminus \{v_s\}\) for any \(s \in I(B_\alpha)\). Hence, from (45) we get (6) on \(I(B_\alpha)\), implying \(x^1_s = x^2_s\) on \(I(V(S_\alpha))\), by part 3 of Lemma 2.1 and (40).

On the other hand, if \(\alpha \in E\), the equality \(A_\alpha \cap A_1 = \emptyset\) implies \(A_\alpha \subseteq V(S_{1\alpha}) \setminus A_1 = V(S_{1\alpha}) \setminus A_1 = B_\alpha\). It follows by (45) that equality (6) holds for the elements of \(I(B_\alpha \setminus A_\alpha)\), as every \(s \in I(B_\alpha \setminus A_\alpha)\) satisfies \(N(v_s) = V(S_{1\alpha}) \setminus \{v_s\}\). Then, by part 3 of Lemma 2.1
\[
x^1_s = x^2_s \quad \text{for all } s \in I(B_\alpha \setminus A_\alpha), \quad (46)
\]
and by (44),
\[ \sum_{s \in I(A_\alpha)} x^1_s = \sum_{s \in I(A_\alpha)} x^2_s. \]

Note that by the differentiability of \( u_{p_\alpha}(x_{p_\alpha}) \) at \( x^1_{p_\alpha} \) and \( x^2_{p_\alpha} \), we can apply to the graph \( S_\alpha = \bigcup_{j=1}^{k_\alpha} S_{\alpha_j} \), the same arguments discussed in Step 2 of the proof of Theorem 4.1, getting
\[ x^1_s = x^2_s \quad \text{for all} \quad s \in I(A_\alpha), \quad \text{and} \quad \sum_{s \in \bigcup_{j=1}^{k_\alpha} I(B_{\alpha_j})} x^1_s = \sum_{s \in \bigcup_{j=1}^{k_\alpha} I(B_{\alpha_j})} x^2_s, \]

where \( B_{\alpha_j} = V(S_{\alpha_j}) \setminus A_\alpha \) for all \( j \in \{1, \ldots, k_\alpha\} \). By the left-hand equality of the above results, (40) and (46), we get
\[ x^1_s = x^2_s \quad \text{on} \quad I(V(S_{1\alpha})) = I(V(S_{01})). \]

Next, we fix \( r \in \{2, \ldots, k_\alpha\} \) and proceed to apply the same strategy used in step 2: we set \( I'_1 = \{1, \ldots, m\} \setminus I(V(S_{0r})) = \{e_1, \ldots, e_q\} \) and \( I'_2 = I(B_{0r}) = \{d_1, \ldots, d_j\} \), and consider \( g_1'(x_{e_1}, \ldots, x_{e_q}) \) and \( g_2'(x_{d_1}, \ldots, x_{d_j}) \), the functions formed by all the terms of \( b \) that depend only on the vertices with index in \( I'_1 \) and \( I'_2 \) respectively. Noting that \( \bigcup_{s \in I'_j} N(v_s) = \{v_s\}_{s \in I'_j} \cup A_\alpha, j = 1, 2 \), and using the left-hand equality in (47), we follow the arguments of Step 2 to get
\[ \left\{ x^1_s \right\}_{s \in I'_1} \in \text{Argmax} \left\{ \left\{ x_s \right\}_{s \in I'_1} \mapsto \left( \sum_{s \in I(A_\alpha)} x^1_s \right) \cdot \sum_{s \in I(A_\alpha)} x_s + g_1'(x_{e_1}, \ldots, x_{e_q}) - \sum_{s \in I'_1} u_s(x_s) + g_2'(x^1_{d_1}, \ldots, x^1_{d_j}) + \left( \sum_{s \in I(A_\alpha)} x^1_s \right) \cdot \sum_{s \in I(B_{0r})} x^1_s + \sum_{s \in I(A_\alpha)} x^1_s \cdot x^1_s - \sum_{s \in I'_2 \cup I(A_\alpha)} u_s(x^1_s) \right\} \]
\[ = \text{Argmax} \left\{ \left\{ x_s \right\}_{s \in I'_1} \mapsto \left( \sum_{s \in I(A_\alpha)} x^1_s \right) \cdot \sum_{s \in I(A_\alpha)} x_s + g_1'(x_{e_1}, \ldots, x_{e_q}) - \sum_{s \in I'_1} u_s(x_s) \right\} \]
40
\[
\text{argmax} \left\{ \{x_s\}_{s \in I_1'} \mapsto \left( \sum_{s \in I(A_\alpha)} x_s \right) \cdot \sum_{s \in \bigcup_{j \neq r}^k I(B_{\alpha j})} x_s + g'(x_{e_1}, \ldots, x_{e_q}) \right. \\
\left. - \sum_{s \in I_1'} u_s(x_s) \right\},
\]

and then, using the right-hand equality in (47), we get the equality (44) on \(I(B_{\alpha r})\); that is,
\[
\sum_{s \in I(B_{\alpha r})} x_s = \sum_{s \in I(B_{\alpha r})} x_s^1. \tag{49}
\]

Finally, we combine the above equality with the left-hand equality in (47) to get (6) for all \(s \in I(B_{\alpha r})\), since \(N(v_s) = V(S_{\alpha r}) \setminus \{v_s\}\) for every \(s \in I(B_{\alpha r})\) and \(B_{\alpha j} = V(S_{\alpha j}) \setminus A_\alpha\). Hence, by part 3 of Lemma 2.1, \(x_s^1 = x_s^2\) for all \(s \in I(B_{\alpha r})\). Thus, \(x_s^1 = x_s^2\) for all \(s \in \bigcup_{j=2}^k I(B_{\alpha j}) = I(V(S_\alpha) \setminus V(S_{\alpha 1}))\). Hence, from (48) we conclude \(x_s^1 = x_s^2\) on \(I(V(S_\alpha))\), and so, \(x_s^1 = x_s^2\) for all \(s \in \bigcup_{\alpha=1}^l I(V(S_\alpha)) = \{1, \ldots, m\}\), completing the proof of the case \(v_1 \in B_1\).

For the case \(v_1 \in A_1\), every \(s \in I(A_1)\) satisfies \(N(v_s) = V(S_1) \setminus \{v_s\}\), then any \(s \in I(A_1)\) satisfies \(N(v_s) = N(v_1)\). Therefore, by part 4a of Lemma 2.1 we get (40), and (37) reduces to (41). For the rest of the proof we fix \(\alpha \in \{1, \ldots, l\}\) and mimic the proof of the case \(v_1 \in B_1\), completing the proof of the theorem.

**Remark 4.1.** Th results developed in this section, for graphs with inner hubs glued on their cliques, are neither more or less general than Theorem 3.1, which applies to graphs obtained by extracting subgraphs with inner hubs from complete graphs. To see this, note that in Theorem 3.1, if \(m = 4\) and \(l = 2\), with \(S_1 = \{x_1, x_3\}\) and \(S_2 = \{x_2, x_4\}\), we get the surplus associated to the graph in Figure 2a, which clearly cannot be obtained from the results of Section 4. On the other hand, we can find examples of surplus functions covered by the framework presented in Section 4, but not covered by Theorem 3.1. For instance, Figure 14a and 14b are graphs whose respective surplus are not covered by Theorem 3.1 as we need more than two absolutely continuous measures and clearly, these conditions are necessary.
We next turn to a slight generalization of Proposition 4.1, where, roughly speaking, any two graphs (with inner hubs) can be glued together (unlike in the preceding proposition, where each $S_\alpha, \alpha \in E$ was glued to $S_1$). The proof is a straightforward modification of the proof of Proposition 4.1 and is therefore omitted.

In order to facilitate the description of the next Proposition we will introduce a natural higher level notion of graph. For this, we interpret any collection of graphs with inner hubs $\{G_\alpha\}_{\alpha=1}^l$, as the vertices of a graph $G$, whose edges are glueings on cliques between the $G_\alpha$ and $G_\beta$; that is,

\[ V(G) = \{G_\alpha\}_{\alpha=1}^l \]

and

\[ E(G) = \{\{G_\alpha, G_\beta\} : G_\alpha \text{ is glued on a clique to } G_\beta\}. \]

**Proposition 4.2.** Let $\{G_\alpha\}_{\alpha=1}^l$ be a collection of graphs with inner hubs $A_\alpha$, and $G$ its associated higher order graph (described above). Let $m = |\bigcup_{\alpha=1}^l V(G_\alpha)|$ and $\mu_i$ be probability measures over $X_i$, $i = 1, \ldots, m$, where without loss of generality $v_1 \in V(G_1)$. Assume:

1. For each distinct $\alpha \neq \beta$, $A_\alpha \cap A_\beta = \emptyset$ and $V(G_\alpha) \cap V(G_\beta)$ is either:
   - empty,
   - the vertex set $V(S)$, where $S$ is a maximal clique $S$ of both $G_\alpha$ and $G_\beta$ (in this case $G_\alpha$ and $G_\beta$ are glued on a clique $S$), or
   - $A_\lambda$ for some other $G_\lambda$ (as when $G_\alpha$ and $G_\beta$ are both glued to $G_\lambda$).
2. $\mu_1$ is absolutely continuous with respect to $\mathcal{L}^n$, and, for each $\alpha \in \{1, \ldots, l\}$, there exists $p_\alpha \in I(A_\alpha)$ such that $\mu_{p_\alpha}$ is absolutely continuous with respect to $\mathcal{L}^n$.
3. For at least one maximal clique $S$ having $v_1$ as one of its vertices, $G_1$ is not glued to any other $G_\alpha$ on $S$.
4. $G$ is a tree.

Then every solution to the Kantorovich problem with surplus $\bigcup_{\alpha=1}^l G_\alpha$ is induced by a map.
Remark 4.2. Using the terminology developed above, the assumptions in Proposition 4.1 are equivalent to the assumptions in Proposition 4.2, except that the hypothesis that $G$ is a tree is replaced with the hypothesis that $G$ is a star with internal node $S_1$. Therefore, Proposition 4.2 is a direct generalization of Proposition 4.1.

4.3 Examples

Let us illustrate the results obtained in Section 4 throughout some examples. In what follows, $\mu_1$ is absolutely continuous.

Examples 4.3.1. (i) In Theorem 4.1, if $S_j = S_k$ for every $j, k \in \{1, \ldots, l\}$, then $\bigcup_{j=1}^{l} S_j$ reduces to the Gangbo and Święch surplus.

(ii) By Theorem 4.1, the graph $S_1 \cup S_2 \cup S_3$ in Example 2.1 provides a Monge solution, with $\mu_p$ absolutely continuous for some $p \in \{6, 7, 8\}$.

(iii) In Example 2.2, if there are $p_1 \in I(A)$ and $p_2 \in I(A')$ such that $\mu_{p_1}$ and $\mu_{p_2}$ are absolutely continuous, then by Proposition 4.1 the graph $\left(\bigcup_{j=1}^{3} S_j\right) \cup \left(\bigcup_{j=1}^{l} S'_j\right)$ provides a solution of Monge type.

(iv) By Theorem 4.1 any graph of the form $K_{1,k}$ (known as a star graph) provides a solution of Monge type, under at most two regularity conditions (see pictures below). Note that $|V(K_{1,k})| = k + 1$ and there exists $v \in V(K_{1,k})$ such that $N(v) = \{v_1, \ldots, v_k\}$. Additionally, $N(v_s) = \{v\}$ for all $s \in \{1, \ldots, k\}$. This is one of the most simple graphs providing Monge solutions that we could obtain, since a graph with inner hub have in fact a "star shape". Note that, in the general setting, the single set $\{v\}$ is replaced by the inner hub $A$ and $\{v_j\}$ is replaced by $B_j := V(S_j) \setminus A$, $j = 1, \ldots, k$, where $\{S_j\}_{j=1}^{l}$ is the collection of maximal cliques. See for instance Figure 13.
(a) $K_{1,6}$, with $V_1 = \{v_7\}$ and $V_2 = \{v_i\}_{i=1}^{6}$. Here, we need regularity conditions on $\mu_1$ and $\mu_7$.

(b) $K_{1,6}$, with $V_1 = \{v_1\}$ and $V_2 = \{v_i\}_{i=2}^{7}$. Here, we only need a regularity condition on $\mu_1$.

Figure 12

Figure 13: Graph $G = \bigcup_{j=1}^{5} S_j$ generated by the collection of its maximal cliques $\{S_j\}_{j=1}^{5}$, where $V(S_1) = \{v_2, v_3, v_4, v_1, v_7\}$, $V(S_2) = \{v_2, v_3, v_4, v_6, v_8\}$, $V(S_3) = \{v_2, v_3, v_4, v_{11}, v_{12}\}$, $V(S_4) = \{v_2, v_3, v_4, v_{10}, v_{13}\}$ and $V(S_5) = \{v_2, v_3, v_4, v_5, v_9\}$. Clearly, $A = \{v_2, v_3, v_4\}$ is the inner hub of $G$.

(v) Let $G$ be a graph tree with $V(G) = \{v_1, \ldots, v_m\}$ and $D = \{s \in \{1, \ldots, m\} : |N(v_s)| = 1\}$. Assume $\mu_s$ is absolutely continuous for every $s \in \{2, \ldots, m\} \setminus D$. Monge solutions for these graphs could be
easily deduced by adapting the reasoning presented in Example 4.1; the solution will be the composition of optimal maps for two marginal problems along any path. Alternatively, these can be seen as special cases of Proposition 4.2.

(a) Path with vertex sequence \((x_3, x_7, x_1, x_6, x_5, x_4)\). Here, we need regularity conditions on \(\mu_1, \mu_5, \mu_6, \mu_7\).

(b) Here, we need regularity conditions on \(\mu_k\), for every \(k \in \{5, 6, 7, 8, 9\}\).

Figure 14

5 Uniqueness

Here, we include a standard argument, showing that in situations where all solutions are of Monge type, the solution to (KP) must be unique.

**Corollary 5.1.** Under the hypotheses in any of Theorem 3.1, Theorem 4.1, Proposition 4.1 or Proposition 4.2, the solution to the Kantorovich problem (KP) is unique.

**Proof.** If there are two such solutions, \(\gamma_0\) and \(\gamma_1\), linearity of the Kantorovich functional implies that their interpolant \(\gamma_{1/2} = \frac{1}{2} \gamma_0 + \frac{1}{2} \gamma_1\) is also a solution; under any of the collections of hypotheses listed in the statement of the corollary, the corresponding result then asserts that each of \(\gamma_0\), \(\gamma_1\) and \(\gamma_{1/2}\) must concentrate on the graph of a function. This is clearly not possible, as if \(\gamma_0, \gamma_1\) concentrate on the graphs of \(T_0\) and \(T_1\), respectively, \(\gamma_{1/2}\) concentrates on the union of these two graphs, which is itself a single graph only if \(T_0 = T_1\) \(\mu_1\) almost everywhere, in which case \(\gamma_0 = \gamma_1\).  

45
6 Discussion and negative examples

This paper has identified a wide class of graphs leading to Monge solution and uniqueness results in the multi-marginal optimal transport problem \((\text{MP})\) with corresponding surplus \((3)\), under appropriate conditions on the marginals; see Theorems 3.1 and 4.1 as well as Propositions 4.1 and 4.2. To the best of our knowledge, such results are not known for any graph which is not covered here. Furthermore, Part 2 of Proposition 2.1 verifies that the extra regularity conditions on the marginals imposed here are necessary in order to obtain Monge solution and uniqueness results.

There are many graphs to which none of Theorem 3.1, Theorem 4.1, Proposition 4.1 or 4.2 apply, and for most of these we do not know whether or not Monge solution and uniqueness results might hold, assuming for simplicity that all the marginals are absolutely continuous. A notable exception to this is the cycle graph for \(m \geq 5\) (see Figure 2-(b) for the case \(m = 7\)); in a recent work [26], we showed the existence of absolutely continuous marginals generating non-Monge solutions for the corresponding surplus \((2)\). For illustrative purposes, we close by mentioning a class of graphs falling outside the scope of this paper, for which Monge solution and uniqueness remain completely open. For this, recall that for graphs \(G_1\) and \(G_2\) with disjoint vertex sets \(V_1\) and \(V_2\), the graph join \(G_1 + G_2\) is defined as the graph union \(G_1 \cup G_2\) together with all edges joining vertices in \(V_1\) with vertices in \(V_2\). Also, for any graph \(G\), the graph complement (denoted \(\overline{G}\)) is the graph with vertices \(V(G)\) and set of edges \(E(G) = \{\{v, w\} : v, w \in V(G)\}\) and \(\{v, w\} \notin E(G)\).

**Definition 6.1.** Let \(P_n\) be a path with \(n\) vertices and \(\overline{C_k}\) the complement of the complete graph with \(k\) vertices \(C_k\) (so \(N_{\overline{C_k}}(v) = \emptyset\) for every \(v \in V(\overline{C_k})\)). The fan graph \(F_{k,n}\) is defined as the graph \(\overline{C_k} + P_n\).

**Example 6.1.** Let us illustrate the above definition with some basics examples.

- The graphs \(F_{1,1}\) and \(F_{1,2}\) reduce to complete graphs with two and three vertices respectively.

- The graph \(F_{1,3}\) reduces to the extraction of the graph consisting of only one edge from the complete graph \(C_4\).

- The graphs \(F_{1,6}\) and \(F_{2,5}\), where for \(F_{1,6}\) we denote the only vertex of \(C_1\) as \(v_1\), and for \(F_{2,5}\) we denote the vertices of \(C_2\) as \(v_1\) and \(v_7\). See figures below
Proposition 6.1. Let $F_{k,n}$ be a fan graph.

1. If $n \geq 4$, then $F_{k,n}$ does not belong to the class of graphs in Theorem 3.1, Theorem 4.1, Proposition 4.1 or Proposition 4.2.

2. If $n < 4$, then $F_{k,n}$ belongs to the class of graphs considered in Theorem 3.1.

The proof of Part 1 of the above proposition will be divided into two cases. In both cases the next lemma will be used during the proofs.

Lemma 6.1. Assume $n \geq 4$. Then $F_{1,n}$ does not have an inner hub.

Proof. Assume $F_{1,n}$ has an inner hub. Since $F_{1,n}$ is connected, every vertex in the nonempty hub is adjacent to all the other vertices. Now, the only vertex of $F_{1,n}$ satisfying this property is the vertex of $C_1 = C_1$ (so $V(C_1)$ is the hub of $F_{1,n}$). This implies by definition of inner hub that $P_n$ is complete or it is the disjoint union of complete graphs. This is a contradiction as $n > 2$ (so $P_n$ cannot be complete) and it is connected, completing the proof of the lemma.

Proof of Proposition 6.1. Since Proposition 4.2 generalizes Theorem 4.1 and Proposition 4.1, it suffices to prove Part 1 for Theorem 3.1 and Proposition 4.2. For this, we set $m = k + n$ and consider two cases.

Case 1. Assume $k = 1$. If $F_{1,n} = C_m \setminus S$ for some subgraph $S$ of $C_m$, then $S = P_n$ or $S = C_1 \cup P_n$. Since $n \geq 4$, $P_n$ is connected and there is not a vertex

47
in \( V(P_n) \) adjacent to all the other vertices; that is, \( P_n \) can not have an inner hub. Also, the only way that the disconnected graph \( C_1 \cup P_n \) has an inner hub is when \( P_n \) is complete (as it is connected), which is clearly not the case. Hence, the structure of \( F_{1,n} \) does not correspond to the graphs considered in Theorem 3.1. On the other hand, note that the vertex of \( C_1 = C_1 \) is connected to all the other vertices of \( F_{1,n} \), and the only case in Proposition 4.2 where a vertex of a graph \( \bigcup_{\alpha=1}^{l} G_{\alpha} \) (where \( \{G_{\alpha}\}_{\alpha=1}^{l} \) is a collection of graphs with inner hubs \( A_{\alpha} \) satisfying the conditions in Proposition 4.2) satisfies this condition is when \( l = 1 \); that is, if there exists one of these collections satisfying \( \bigcup_{\alpha=1}^{l} G_{\alpha} = F_{1,n} \), then \( F_{1,n} \) would be a graph with an inner hub, contradicting Lemma 6.1. This proves that \( F_{1,n} \) does not belong to the class of graphs in Proposition 4.2, completing the proof of Case 1.

Case 2. Assume \( k \geq 2 \). If \( F_{k,n} = C_m \setminus S \) for some subgraph \( S \) of \( C_m \), then \( S = C_k \cup P_n \). Note that \( S \) is disconnected with connected components \( C_k \) and \( P_n \) (as \( n \geq 4 \)), so if \( S \) has inner hub then it must be empty, which implies \( P_n \) is complete. This clearly is not possible as \( n > 1 \). Hence, \( F_{k,n} \) does not belong to the class of graphs in Theorem 3.1. For the other part of the assertion, consider \( \{G_{\alpha}\}_{\alpha=1}^{l} \) a collection of graphs with inner hubs \( A_{\alpha} \) satisfying the conditions imposed in Proposition 4.2 and assume \( F_{k,n} = \bigcup_{\alpha=1}^{l} G_{\alpha} \). Fix any vertex \( v \) in \( V(C_k) \subseteq V(F_{k,n}) \), then there exists \( \beta \) such that \( v \in A_{\beta} \) or \( v \in V(S_{\beta}) \setminus A_{\beta} \) for some maximal clique \( S_{\beta} \) of \( G_{\beta} \). If \( v \in A_{\beta} \), then

\[
V(G_{\beta}) = (V(G_{\beta}) \setminus \{v\}) \cup \{v\} \\
= N_{\bigcup_{\alpha=1}^{l} G_{\alpha}}(v) \cup \{v\} \\
= N_{F_{k,n}}(v) \cup \{v\} \\
= V(P_n) \cup \{v\}
\]

as \( v \in V(C_k) \).

This implies that \( G_{\beta} = P_n \cup K_{v,V(P_n)} \) where \( K_{v,V(P_n)} \) is a bi-partite graph with set partition \( \{\{v\}, V(P_n)\} \) (alternatively we can interpret it as a star graph with "center" \( v \)); that is, \( G_{\beta} \) is a graph of the form \( F_{1,n} \) having an inner hub. This is a contradiction by Lemma 6.1. This proves that \( F_{k,n} \) does not satisfy the graph structure condition in Proposition 4.2. Now, assume \( v \in V(S_{\beta}) \setminus A_{\beta} \) and without lost of generality assume \( v \not\in A_{\alpha} \) for any \( \alpha \neq \beta \) (otherwise we apply the same arguments as in the
case \( v \in A_\beta \) above), then \( V(S_\beta) = N_{F_{k,n}}(v) \cup \{v\} = V(P_n) \cup \{v\} \); that is, \( P_n \cup K_{v,V(P_n)} \) is a complete graph (so it has inner hub \( V(P_n \cup K_{v,V(P_n)}) \)), contradicting Lemma 6.1. Hence, \( F_{k,n} \) does not belong to the class of graphs in Proposition 4.2, completing the proof of Part 1.

To prove Part 2, note that if \( n \in \{1, 2, 3\} \) the graph \( \overline{P}_n \) can be trivially expressed as a union of disjoint complete graphs, so \( C_k \cup \overline{P}_n \) is a disjoint union of complete graphs and can be interpreted as a graph with empty inner hub. Since \( F_{k,n} = C_m \setminus (C_k \cup \overline{P}_n) \), we immediately conclude that \( F_{k,n} \) belong to the class of graphs in Theorem 3.1. This completes the Proof of part 2.

We note that the essential ideas in the proposition above can in fact be adapted to a more abstract class of graphs. The next lemma describes such a class, which therefore also falls outside the scope of the results in this paper and for which the Monge solution and uniqueness questions remain open.

**Lemma 6.2.** Let \( G \) be a connected graph satisfying \( N_G(v) \cup \{v\} \neq V(G) \), for all \( v \in V(G) \) and consider the graph \( F_{k,G} := C_k + G \). Then \( F_{k,G} \) does not belong to the classes of graph considered in Theorem 3.1 and Proposition 4.2.

**Proof.** Note that the condition \( N_G(v) \cup \{v\} \neq V(G) \), for all \( v \in V(G) \) implies that \( G \) does not have an inner hub (there is not a vertex in \( V(G) \) adjacent to all the other vertices). Also, since \( G \) is connected, \( N_{\overline{G}}(v) \cup \{v\} \neq V(\overline{G}) \) for all \( v \in V(\overline{G}) = V(G) \), so \( \overline{G} \) also has no inner hub. In particular, \( G \) and \( \overline{G} \) are not complete and can not be expressed as a disjoint union of complete graphs. Knowing this, it is not hard to follow the arguments of Lemma 6.1 to prove that \( F_{1,G} \) does not have inner hub, and then, by mimicking the proof of the above proposition the proof is completed.

Lemma 6.2 allows one to construct many graphs for which Monge solution and uniqueness results are not known, with more adhoc structure than the fan graphs considered above. One such possibility is illustrated in the figure below.
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