THE TOPOLOGICAL ASPECT OF THE HOLONOMY DISPLACEMENT ON THE PRINCIPAL $U(n)$ BUNDLES OVER GRASSMANNIAN MANIFOLDS

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Abstract. Consider the principal $U(n)$ bundles over Grassmann manifolds $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$. Given $X \in U_{m,n}(\mathbb{C})$ and a 2-dimensional subspace $m' \subset m \subset u(m + n)$, assume either $m'$ is induced by $X, Y \in U_{m,n}(\mathbb{C})$ with $X^*Y = \mu I_n$ for some $\mu \in \mathbb{R}$ or by $X, iX \in U_{m,n}(\mathbb{C})$. Then $m'$ gives rise to a complete totally geodesic surface $S$ in the base space. Furthermore, let $\gamma$ be a piecewise smooth, simple closed curve on $S$ parametrized by $0 \leq t \leq 1$, and $\tilde{\gamma}$ its horizontal lift on the bundle $U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S$, which is immersed in $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$. Then

$$\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot (e^{i\theta I_n}) \quad \text{or} \quad \tilde{\gamma}(1) = \tilde{\gamma}(0),$$

depending on whether the immersed bundle is flat or not, where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$ and $\theta = 2 \cdot \frac{\pi + m}{2n} A(\gamma)$.

1. Introduction

For two natural numbers $n, m \in \mathbb{N}$, let

$$U_{m,n}(\mathbb{C}) := \{X \in M_{m,n}(\mathbb{C}) \mid X^*X = \lambda I_n \text{ for some } \lambda \in \mathbb{C} - \{0\}\},$$

which may be regarded as a generalization of a unitary group. It plays an important role in studying the principal $U(n)$ bundles $U(n) \to U(n + m)/U(m) \to G_{n,m}$ over Grassmannian manifolds, where, for $k \in \mathbb{N}$, $U(k)$ has a metric, related to the Killing-Cartan form, given by

$$\langle A, B \rangle = \frac{1}{k} \text{Re}(\text{Tr}(A^*B)), \quad A, B \in u(k),$$

and each quotient space has the induced metric which makes the projection a riemannian submersion.

Consider the Hopf fibration $S^1 \to S^3 \to S^2$. Let $\gamma$ be a simple closed curve on $S^2$. Pick a point in $S^3$ over $\gamma(0)$, and take the unique horizontal lift $\tilde{\gamma}$ of $\gamma$. Since $\gamma(1) = \gamma(0)$, $\tilde{\gamma}(1)$ lies in the same fiber as $\tilde{\gamma}(0)$ does. We are interested in understanding the difference between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$. The following equality was already known $[3]$: 

$$V(\gamma) = e^{-\frac{1}{2}A(\gamma)i},$$

where $V(\gamma)$ is the holonomy displacement along $\gamma$, and $A(\gamma)$ is the area of the region surrounded by $\gamma$. 

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In this paper, we generalize this fact to the following higher dimensional Stiefel bundle over the Grassmannian manifold through $U_{m,n}(\mathbb{C})$

$$U(n) \rightarrow U(n + m)/U(m) \xrightarrow{\gamma} G_{n,m},$$

where $G_{n,m} = U(n + m)/(U(n) \times U(m))$. The main results are stated as follows: For $\hat{X} \in \mathfrak{u}(n + m)$ which is induced by $X \in U_{m,n}(\mathbb{C})$, consider a 2-dimensional subspace $\mathfrak{m}' \subset \mathfrak{m} \subset \mathfrak{u}(m + n)$ with $\hat{X} \in \mathfrak{m}'$. Assume either

$$\mathfrak{m}' = \text{Span}_{\mathbb{R}}\{\hat{X}, \hat{Y}\}$$

for some $Y \in U_{m,n}$ with $X^*Y = \mu I_n$ for some $\mu \in \mathbb{R}$ or

$$\mathfrak{m}' = \text{Span}_{\mathbb{R}}\{\hat{X}, i\hat{X}\}.$$  

Then $\mathfrak{m}'$ gives rise to a complete totally geodesic surface $S$ in the base space. Furthermore, let $\gamma$ be a piecewise smooth, simple closed curve on $S$ parametrized by $0 \leq t \leq 1$, and $\tilde{\gamma}$ its horizontal lift on the bundle $U(n) \rightarrow \pi^{-1}(S) \xrightarrow{\tilde{\gamma}} S$, which is immersed in $U(n) \rightarrow U(n + m)/U(m) \xrightarrow{\tilde{\gamma}} G_{n,m}$. Then

$$\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot (e^{i\theta} I_n) \quad \text{or} \quad \tilde{\gamma}(1) = \tilde{\gamma}(0),$$

depending on whether the immersed bundle is flat or not, where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$ and $\theta = 2 \cdot \frac{n+m}{2m} A(\gamma)$. See Theorem 3.11.

2. The bundle $S^1 \rightarrow SU(2) \rightarrow \mathbb{C}P^1$

It will be studied not only the holonomy displacement of the bundle $S^1 \rightarrow SU(2) \rightarrow \mathbb{C}P^1$ but also its isomorphic equivalence to the one

$$S(U(1) \times U(1)) \rightarrow SU(1 + 1) \rightarrow SU(1 + 1)/S(U(1) \times U(1)),$$

not the isometric equivalence. In fact, a conformal map $h : SU(1 + 1)/S(U(1) \times U(1)) \rightarrow \mathbb{C}P^1$ will be constructed such that the identity map on $SU(2)$ is the bundle map covering it. The latter bundle will play an important role for the case $\mathfrak{m}' = \text{Span}_{\mathbb{R}}\{\hat{X}, i\hat{X}\}$.

Of course,

$$S^3 \cong SU(2) = \{A \in \text{GL}(2, \mathbb{C}) : A^*A = I \text{ and } \det(A) = 1\}$$

for $S^3 = \{(z_1, z_2)| |z_1|^2 + |z_2|^2 = 1\}$ under the map

$$(z_1, z_2) \mapsto \begin{bmatrix} \bar{z}_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} : S^3 \rightarrow SU(2).$$

From now on, we use the convention of $\mathfrak{gl}(k, \mathbb{C}) \subset \mathfrak{gl}(2k, \mathbb{R})$ by

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \rightarrow \begin{bmatrix} x_{11} + iy_{11} & x_{12} + iy_{12} \\ x_{21} + iy_{21} & x_{22} + iy_{22} \end{bmatrix} \rightarrow \begin{bmatrix} x_{11} & -y_{11} & x_{12} & -y_{12} \\ y_{11} & x_{11} & y_{12} & x_{12} \\ x_{21} & -y_{21} & x_{22} & -y_{22} \\ y_{21} & x_{21} & y_{22} & x_{22} \end{bmatrix},$$
which is an isometric monomorphism with respect to the metric on $GL(k, \mathbb{C})$ and on $GL(2k, \mathbb{R})$, given by

$$\langle A, B \rangle = \frac{1}{k} \text{Re}(\text{Tr}(A^* B)), \quad A, B \in \mathfrak{gl}(k, \mathbb{C})$$

and

$$\langle C, D \rangle = \frac{1}{2k} \text{Tr}(C^t D), \quad C, D \in \mathfrak{gl}(2k, \mathbb{R}),$$

respectively.

The group $SU(2)$ has the following natural representation into $GL(4, \mathbb{R})$:

$$w = \begin{bmatrix} w_1 & w_2 & -w_3 & -w_4 \\ -w_2 & w_1 & w_4 & -w_3 \\ w_3 & -w_4 & w_1 & w_2 \\ w_4 & w_3 & -w_2 & w_1 \end{bmatrix}$$

with the condition $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1$. In fact, the map

$$w_1 + w_2i + w_3j + w_4k \mapsto w$$

is an isometric monomorphism from the unit quaternions into $GL(4, \mathbb{R})$. The circle group

$$S(U(1) \times U(1)) = \left\{ \begin{bmatrix} e^{-iz} & 0 \\ 0 & e^{iz} \end{bmatrix} : 0 \leq z \leq 2\pi \right\}$$

is a subgroup of $SU(2)$, and acts on $SU(2)$ as right translations, freely with quotient $SU(1 + 1)/S(U(1) \times U(1))$, which is an affine symmetric space and produces a principal circle bundle

$$S(U(1) \times U(1)) \to SU(1 + 1) \to G_{1,1} = SU(1 + 1)/S(U(1) \times U(1)).$$

Let $\bar{w}$ be the “i-conjugate” of $w$ (replace $w_2$ by $-w_2$). That is,

$$\bar{w} = \begin{bmatrix} w_1 & -w_2 & -w_3 & -w_4 \\ w_2 & w_1 & w_4 & -w_3 \\ w_3 & -w_4 & w_1 & w_2 \\ w_4 & w_3 & -w_2 & w_1 \end{bmatrix}.$$

Then,

$$\bar{w}w = \begin{bmatrix} w_1^2 + w_2^2 - w_3^2 - w_4^2 & 0 \\ 2(w_1w_3 + w_2w_4) & 2w_1^2 - 2w_2^2 - w_3^2 - w_4^2 \\ -2w_1w_3 + 2w_2w_4 & 2(w_1w_3 + w_2w_4) \end{bmatrix} \begin{bmatrix} 2w_1w_3 - 2w_2w_4 \\ 2w_1w_3 + 2w_2w_4 \\ 2w_1w_3 - w_2^2 - w_4^2 \\ 2w_1w_3 + w_2w_4 \end{bmatrix}$$

and

$$(w_1^2 + w_2^2 - w_3^2 - w_4^2)^2 + (2w_1w_3 + 2w_2w_4)^2 + (-2w_1w_3 + 2w_1w_4)^2 = 1.$$
which is a subset of $SU(2)$ such that $i$-conjugate on $\mathbb{CP}^1$ is the identity map of $\mathbb{CP}^1$. And the map \[ p : SU(2) \longrightarrow \mathbb{CP}^1 \] defined by \[ p(w) = w\bar{w} \]
has the following properties:
\[ p(wv) = wp(v)\bar{w} \quad \text{for all } w, v \in SU(2) \]
\[ p(wv) = p(w) \quad \text{if and only if } v \in S(U(1) \times U(1)) \cong S^1 \]
under the convention of $S(U(1) \times U(1)) \hookrightarrow GL(4, \mathbb{R})$. This shows that the map $p$ is, indeed, the orbit map of the principal bundle \[ S^1 \longrightarrow SU(2) \xrightarrow{p} \mathbb{CP}^1. \]
But we have to be careful that the inclusion map $\mathbb{CP}^1 \hookrightarrow SU(2)$ is not a cross-section in this bundle. In fact, \[ p(v) = v^2 \in \mathbb{CP}^1 \] for any $v \in \mathbb{CP}^1$.

Define a map \[ h : SU(2)/S(U(1) \times U(1)) \longrightarrow \mathbb{CP}^1 \] by \[ h(vH) = v^2 = p(v) \quad v \in \mathbb{CP}^1, \]
where $H = S(U(1) \times U(1))$. Then, the identity map of $SU(2)$ is a trivially isomorphic bundle map which covers the map $h$. Under the identification \[ (x, y, z) = \begin{bmatrix} x & 0 & -y & -z \\ 0 & x & z & -y \\ y & -z & x & 0 \\ z & y & 0 & x \end{bmatrix} : S^2 \cong \mathbb{CP}^1, \]
give the metric $\langle \cdot, \cdot \rangle_{S^2}$ of $S^2$ to $\mathbb{CP}^1$ and consider a metric space $\left(\mathbb{CP}^1, \langle \cdot, \cdot \rangle_{S^2}\right)$. Will $h$ be an isometry?

The Lie group $SU(2)$ will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra $\mathfrak{su}(2)$
\[ E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \]
which correspond to
\[ e_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]
in $\mathfrak{gl}(2k, \mathbb{R})$, respectively. Notice that $[e_1, e_2] = 2e_3$. 

In order to understand the map $h$ between base spaces and the projection map $p$ better, consider the subset of $SU(2)$:

$$T = \left\{ \begin{bmatrix} \cos x & -(\sin x)e^{-iy} \\ (\sin x)e^{iy} & \cos x \end{bmatrix} : 0 \leq x \leq \pi, \ 0 \leq y \leq 2\pi \right\}$$

which is the exponential image of

$$m = \left\{ \begin{bmatrix} 0 & -\xi^t \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{C} \right\}.$$ 

Furthermore, it is exactly same as $\mathbb{CP}^1$, so the map $p$ restricted to $T$ is just the squaring map; that is,

$$p(w) = w^2, \quad w \in T.$$ 

To check $h$ is a conformal map: given

$$w = (\cos x, (\sin x)(\cos y), (\sin x)(\sin y)) \in T = \mathbb{CP}^1,$$

$$|D_1(wH)| = |(D_1w)^h| = \left|((\cos y)L_{w^*}e_1 + (\sin y)L_{w^*}e_2)^h\right| = 1$$

and

$$|D_2(wH)| = |(D_2w)^h| = \left|(-\frac{1}{2}(\sin 2x)(\sin y)L_{w^*}e_1 + \frac{1}{2}(\sin 2x)(\cos y)L_{w^*}e_2 - (\sin^2 x)L_{w^*}e_3)^h\right| = \frac{1}{2} |\sin 2x|,$$

while, under the expression $\langle a, b, c \rangle$ of vectors in $\mathbb{R}^3$,

$$|D_1 \ h(wH)| = |D_1 \ w^2| = \left|(-2\sin 2x, 2(\cos 2x)(\cos y), 2(\cos 2x)(\sin y))\right| = 2$$

and

$$|D_2 \ h(wH)| = |D_2 \ w^2| = \left|(0, -(\sin 2x)(\sin y), (\sin 2x)(\cos y))\right| = |\sin 2x|.$$

Thus $h$ is a conformal map.
Theorem 2.1 (2). Let \( S^1 \to SU(2) \to \left( \mathbb{CP}^1, \langle \cdot, \cdot \rangle_{S^2} \right) \) be the natural fibration. Let \( \gamma \) be a piecewise smooth, simple closed curve on \( \mathbb{CP}^1 \). Then the holonomy displacement along \( \gamma \) is given by

\[
V(\gamma) = e^{\frac{1}{2}A(\gamma)i} = e^{2A(h^{-1}o\gamma)} = \Phi \in S^1 \cong S(U(1) \times U(1))
\]

where \( A(\gamma) \) is the area of the region on \( \mathbb{CP}^1 \) enclosed by \( \gamma \) and

\[
\Phi = \begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix}.
\]

Proof. Let \( \gamma(t) \) be a closed loop on \( \mathbb{CP}^1 \) with \( \gamma(0) = p(I_4) \). Therefore,

\[
\gamma(t) = \begin{bmatrix}
\cos 2x(t) & 0 & -\sin 2x(t) \cos y(t) & -\sin 2x(t) \sin y(t) \\
\sin 2x(t) \cos y(t) & \cos 2x(t) & -\sin 2x(t) \sin y(t) & -\sin 2x(t) \cos y(t) \\
\sin 2x(t) \sin y(t) & -\sin 2x(t) \cos y(t) & \cos 2x(t) & 0 \\
\sin 2x(t) \cos y(t) & -\sin 2x(t) \sin y(t) & \sin 2x(t) \cos y(t) & \cos 2x(t)
\end{bmatrix}
\]

Let

\[
\tilde{\gamma}(t) = \begin{bmatrix}
\cos x(t) & 0 & -\sin x(t) \cos y(t) & -\sin x(t) \sin y(t) \\
\sin x(t) \cos y(t) & \cos x(t) & -\sin x(t) \sin y(t) & -\sin x(t) \cos y(t) \\
\sin x(t) \sin y(t) & -\sin x(t) \cos y(t) & \cos x(t) & 0 \\
\sin x(t) \cos y(t) & -\sin x(t) \sin y(t) & \sin x(t) \cos y(t) & \cos x(t)
\end{bmatrix}
\]

with \( 0 \leq x(t) \leq \pi/2 \) so that \( p(\tilde{\gamma}(t)) = \gamma(t) \) (\( \tilde{\gamma} \) is a lift of \( \gamma \)), and let

\[
\omega(t) = \begin{bmatrix}
\cos z(t) & -\sin z(t) & 0 & 0 \\
\sin z(t) & \cos z(t) & 0 & 0 \\
0 & 0 & \cos z(t) & \sin z(t) \\
0 & 0 & -\sin z(t) & \cos z(t)
\end{bmatrix}.
\]

Put

\[
\eta(t) = \tilde{\gamma}(t) \cdot \omega(t).
\]

Then still \( p(\eta(t)) = \gamma(t) \), and \( \eta \) is another lift of \( \gamma \). We wish \( \eta \) to be the horizontal lift of \( \gamma \). That is, we want \( \eta'(t) \) to be orthogonal to the fiber at \( \eta(t) \).

The condition is that \( \langle \eta'(t), (\ell_{\eta(t)})_*(e_3) \rangle = 0 \), or equivalently, \( \langle (\ell_{\eta(t)})^{-1}_*, \eta'(t), e_3 \rangle = 0 \). That is,

\[
\eta(t)^{-1} \cdot \eta'(t) = \alpha_1 e_1 + \alpha_2 e_2
\]

for some \( \alpha_1, \alpha_2 \in \mathbb{R} \). From this, we get the following equation:

(2–1)

\[
z'(t) = \sin^2 x(t)y'(t).
\]

Since any piecewise smooth curve can be approximated by a sequence of piecewise linear curves which are sums of boundaries of rectangular regions, it will be enough to prove the statement for a particular type of curves as follows [2]: Suppose we are given a rectangular region in the \( xy \)-plane

\[
p \leq x \leq p + a, \quad q \leq y \leq q + b.
\]

Consider the image \( R \) of this rectangle in \( \mathbb{CP}^1 \) by the map

\[
(x, y) \mapsto r(x, y) = (\cos 2x, (\sin 2x)(\cos y), (\sin 2x)(\sin y)).
\]
Then $||\mathbf{r}_x \times \mathbf{r}_y|| = 2 \sin 2x$, (because $0 \leq x \leq \pi/2$). Thus, the area of $R$ is
\[
\int_{q}^{q+b} \int_{p}^{p+a} 2 \sin 2x \, dx \, dy = 2b(\sin^2(p + a) - \sin^2(p)).
\]

On the other hand, the change of $z(t)$ along the boundary $\gamma(t)$ of this region can be calculated using condition (2-1). Let $\gamma(t)$ be represented by
\[
(p + 4at, q) \text{ for } t \in [0, \frac{1}{4}], (p + a, q + b(4t - 1)) \text{ for } t \in [\frac{1}{4}, \frac{1}{2}], (p + a(3 - 4t), q + b) \text{ for } t \in [\frac{1}{2}, \frac{3}{4}], (p, q + b(4 - 4t)) \text{ for } t \in [\frac{3}{4}, 1].
\]
Then
\[
z(1) - z(0) = \int_{0}^{1} z'(t) \, dt = b \cdot \sin^2(p + a) - b \cdot \sin^2(p).
\]

Thus the total vertical change of $z$-values, $z(1) - z(0)$, along the perimeter of this rectangle is
\[
b \cdot (\sin^2(p + a) - \sin^2(p))
\]
which is $\frac{1}{2}$ times the area. Hence we get the conclusion. \qed

3. The bundle $U(n) \to U(n + m)/U(m) \to G_{n,m}$

To deal with the bundle
\[
U(n) \to U(n + m)/U(m) \to G_{n,m},
\]
we investigate the bundle
\[
U(n) \times U(m) \to U(n + m) \to G_{n,m}.
\]

The Lie algebra of $U(n + m)$ is $\mathfrak{u}(n + m)$, the skew-Hermitian matrices, and has the following canonical decomposition:
\[
\mathfrak{g} = \mathfrak{h} + \mathfrak{m},
\]
where
\[
\mathfrak{h} = \mathfrak{u}(n) + \mathfrak{u}(m) = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A \in \mathfrak{u}(n), B \in \mathfrak{u}(m) \right\}
\]
and
\[
\mathfrak{m} = \left\{ \hat{X} := \begin{bmatrix} 0 & -X^* \\ X & 0 \end{bmatrix} : X \in M_{m,n}(\mathbb{C}) \right\}.
\]

Define an Hermitian inner product $h : \mathbb{C}^m \to \mathbb{C}$ by
\[
h(v, w) = v^* w,
\]
where $v$ and $w$ are regarded as column vectors.

**Lemma 3.1.** If a matrix $X \in M_{m,n}$ satisfies $X^*X = \lambda I_n$ for some $\lambda \in \mathbb{C}$, then $\lambda$ will be a nonnegative real number and $\lambda = 0$ only if $X$ is trivial.

**Proof.** Given any column vector $v$ of $X$, $\lambda = v^* v = h(v, v) \geq 0$ and the equality holds only if $v = 0$, which shows the claim. \qed
From the lemma 3.1 we obtain that
\[ U_{m,n}(\mathbb{C}) = \{ X \in M_{m,n}(\mathbb{C}) \mid X^*X = \lambda I_n \text{ for some } \lambda \in \mathbb{C} - \{0\} \} \]
\[ = \{ X \in M_{m,n}(\mathbb{C}) \mid X^*X = \lambda I_n \text{ for some } \lambda > 0 \}. \]

**Lemma 3.2.** Let
\[ X = \left( a_k^r + ib_k^r \right), Y = \left( c_k^r + id_k^r \right) \in M_{m,n}(\mathbb{C}) \]
for \( r = 1, \cdots, m, \) and \( k = 1, \cdots, n. \) Suppose that for their induced \( \hat{X}, \hat{Y} \in \mathfrak{m}, \]
\[ [[\hat{X}, \hat{Y}], \hat{X}] = \hat{Z} \in \mathfrak{m} \]
for some \( Z = \left( \alpha_k^r \right) \in M_{m,n}(\mathbb{C}) \) for \( r = 1, \cdots, m, \) and \( k = 1, \cdots, n. \) Then we have
\[ \alpha_k^r = \sum_{j=1}^{n} \left( a_j^r + ib_j^r \right) \left( -2h(Y_j, X_k) + h(X_j, Y_k) \right) + \sum_{j=1}^{n} \left( c_j^r + id_j^r \right) h(X_j, X_k), \]
where \( X_k \) and \( Y_k \) are \( k \)-column vectors of \( X \) and \( Y \) for \( k = 1, \cdots, n. \)

**Proof.** It is easily obtained from
\[ [[\hat{X}, \hat{Y}], \hat{X}] = \hat{X}(2\hat{Y}X - \hat{X}\hat{Y}) - \hat{Y}\hat{X}\hat{X}. \]

Recall the following proposition, which gives the clue for the holonomy displacement in the principal \( U(n) \) bundles over Grassmaniann manifolds \( U(n) \rightarrow U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}. \)

**Proposition 3.3.** [1] Let \((G, H, \sigma)\) be a symmetric space and \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \) the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces \( \mathfrak{m}' \) of \( \mathfrak{m} \) such that \( [[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}' \) and the set of complete totally geodesic submanifolds \( M' \) through the origin 0 of the affine symmetric space \( M = G/H, \) the correspondence being given by \( \mathfrak{m}' = T_0(M'). \)

Note that \( \mathfrak{m}' \) in the Proposition 3.3 will make a bunch of complete totally geodesic submanifolds, each of which is obtained from another one by a translation, in the affine symmetric space \( G/H. \)

The role of \( U_{m,n}(\mathbb{C}) \) in this paper will be seen from now on.

**Theorem 3.4.** Given \( X \in U_{m,n}(\mathbb{C}) \) and the natural fibration \( U(n) \times U(m) \rightarrow U(n+m) \rightarrow G_{n,m}(\mathbb{C}), \) assume a 2-dimensional subspace \( \mathfrak{m}' = \text{Span}_\mathbb{R}\{\hat{X}, \hat{Y}\} \) of \( \mathfrak{m} \subset \mathfrak{u}(n+m) \) satisfies
\[ X^*X = \lambda I_n, \quad X^*Y = \mu I_n, \quad \mu \in \mathbb{C} \]
for \( Y \in M_{m,n}(\mathbb{C}). \) Then \( \mathfrak{m}' \) gives rise to a complete totally geodesic surface \( S \) in \( G_{n,m}(\mathbb{C}) \) if and only if \( \text{Im}\mu = 0 \) and \( Y \in U_{m,n}(\mathbb{C}) \)
Proof. To begin with, note that $\lambda > 0$. Assume that $m'$ gives rise to a complete totally geodesic surface $S$ in $G_{n,m}(\mathbb{C})$. By a translation, without loss of generality, we can assume that $S$ passes through the origin of the affine symmetric space $G_{n,m}(\mathbb{C}) = U(n + m)/ (U(n) \times U(m))$.

To show $\text{Im}\mu = 0$ by contradiction, suppose that $\text{Im}\mu \neq 0$. Let $e_k \in \mathbb{C}^m$, $k = 1, \ldots, m$, be an elementary vector which has all components 0 except for the $k$-component with 1. Then
\[ h(X_k, Y_j) = h(X e_k, Y e_j) = e_k^* (X^* Y) e_j, \]
so the condition (3.1) is equivalent to
\[ h(X_k, Y_k) = \mu, \quad h(X_k, X_k) = \lambda, \quad h(X_k, X_j) = 0, \quad h(X_k, Y_j) = 0 \]
for $k \neq j$ in $\{1, \ldots, n\}$. From $h(X_k, Y_k) = \mu$, we obtain
\[ -2h(Y_k, X_k) + h(X_k, Y_k) = -\text{Re}\mu + 3i\text{Im}\mu. \]
Thus Lemma 3.2, Proposition 3.3 and the hypothesis of totally geodesic say that
\[ a \hat{X} + b \hat{Y} = \hat{([X, Y], \hat{X}]} = (-\text{Re}\mu + 3i\text{Im}\mu) \hat{X} + \lambda \hat{Y} = 3i\text{Im}(i\hat{X}) + (-\text{Re}\mu \hat{X} + \lambda \hat{Y}). \]
for some $a, b \in \mathbb{R}$. Since $\text{Im}\mu \neq 0$, $i\hat{X}$ will lie in $\text{Span}_\mathbb{R}\{\hat{X}, \hat{Y}\} = m' \subset u(n + m)$, and then
\[ -i \hat{X} = -(i\hat{X})^* = -i\hat{X}^* = i\hat{X}, \]
which implies $\hat{X} = O_{n+m}$, a contradiction.

From $\text{Im}\mu = 0$,
\[ -X^* Y + Y^* X = -X^* Y + (X^* Y)^* = -2i\text{Im}\mu I_n = O_n, \]
so
\[ [\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & -XY^* + YX^* \end{bmatrix} \in u(m) \subset u(n + m). \]
Let $M = -XY^* + YX^*$. Then
\[ [\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & M \end{bmatrix} \]
and $[[\hat{Y}, \hat{X}], \hat{Y}] = -M \hat{Y} \in m'$ from the hypothesis of the condition of totally geodesic and from Proposition 3.3. Note that
\[ -MY = XY^* Y - Y X^* Y = XY^* Y - Y \mu I_n = XY^* Y - (\text{Re}Y). \]
Thus $XY^* Y = aX + bY$ for some $a, b \in \mathbb{R}$. Then $\lambda Y^* Y = X^*(XY^* Y) = X^*(aX + bY) = (a \lambda + b \text{Re}\mu) I_n$ and so
\[ Y^* Y = \frac{a \lambda + b \text{Re}\mu}{\lambda} I_n, \quad \frac{a \lambda + b \text{Re}\mu}{\lambda} \in \mathbb{R}. \]
Since $m' = \text{Span}_\mathbb{R}\{\hat{X}, \hat{Y}\}$ is 2-dimensional, $Y$ is not a zero matrix and so from Lemma 3.4, $Y \in U_{m,n}(\mathbb{C})$. 

Conversely, assume the necessary part holds and let $Y^*Y = \eta I_n$, where $\eta > 0$. Then, the condition $\text{Im} \mu = 0$ says that

$$[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & M \end{bmatrix}, \quad [[\hat{X}, \hat{Y}], \hat{X}] = M\hat{X} \quad \text{and} \quad [[\hat{Y}, \hat{X}], \hat{Y}] = -MY,$$

where $M = -XY^* + YX^*$. It suffices to show that $[[\hat{X}, \hat{Y}], \hat{X}] \in \mathfrak{m}'$ and $[[\hat{Y}, \hat{X}], \hat{Y}] \in \mathfrak{m}'$. Since

$$MX = -XY^*X + YX^*X = -X\mu I_n + Y\lambda I_n = -\text{Re} \mu X + \lambda Y,$$

we get $[[\hat{X}, \hat{Y}], \hat{X}] \in \mathfrak{m}'$. We also get $[[\hat{Y}, \hat{X}], \hat{Y}] \in \mathfrak{m}'$ since

$$-MY = XY^*Y - YX^*Y = X\eta I_n - Y\mu I_n = \eta X - \mu Y.$$

Hence we get the conclusion.

**Corollary 3.5.** Given $X, Y \in U_{m,n}(\mathbb{C})$ and given the natural fibration $U(n) \times U(m) \to U(n+m) \to G_{n,m}(\mathbb{C})$, assume $\mathfrak{m}' = \text{Span}_\mathbb{R}\{\hat{X}, \hat{Y}\}$ produce a 2-dimensional subspace of $\mathfrak{m} \subset \mathfrak{u}(n+m)$. If $X^*Y = \mu I_n$ for some $\mu \in \mathbb{R}$, then $\mathfrak{m}'$ will give rise to a complete totally geodesic surface $S$ in $G_{n,m}(\mathbb{C})$.

**Remark 3.6.** Given $X \in U_{m,n}(\mathbb{C})$, if $n \leq m$, then $X: \mathbb{C}^n \to \mathbb{C}^m$ is a conformal one-one linear map. In view of $\hat{X} \in \mathfrak{u}(n+m) \subset \text{End}(\mathbb{C}^{n+m})$, $\hat{X}$ sends the subspace $\mathbb{C}^n$ to its orthogonal subspace $\mathbb{C}^m$ conformally. And the condition of the relation between $X$ and $Y$ in Theorem 3.4 says that

$$h_{\mathbb{C}^m}(Xv, Yw) = \mu h_{\mathbb{C}^n}(v, w) \quad \text{for } v, w \in \mathbb{C}^n,$$

where $h_{\mathbb{C}^k}$ is an Hermitian on $\mathbb{C}^k$, $k = 1, 2, \ldots$, given by

$$h_{\mathbb{C}^k}(u_1, u_2) = u_1^*u_2 \quad \text{for } u_1, u_2 \in \mathbb{C}^k.$$

When $n = 1$, the condition (3.1) is satisfied automatically for any two vectors in $\mathbb{C}^m$ by identifying $M_{m,1}(\mathbb{C})$ with $\mathbb{C}^m$. So we get

**Corollary 3.7.** A 2-dimensional subspace $\mathfrak{m}'$ of $\mathfrak{m} \subset \mathfrak{u}(m+1)$ gives rise to a complete totally geodesic submanifold in the affine symmetric space $\mathbb{C}P^m = U(1+m)/(U(1) \times U(m))$ if $\mathfrak{m}'$ has two linearly independent tangent vectors $\hat{v}$ and $\hat{w}$ such that $\text{Im} h_{\mathbb{C}^m}(\hat{v}, \hat{w}) = 0$.

We return to the bundle $U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$. Any submanifold $A \subset G_{n,m}$ induces a bundle $U(n) \to \pi^{-1}(A) \to A$, which is immersed in the original bundle and diffeomorphic to the pullback bundle with respect to the inclusion of $A$ into $G_{n,m}$. In fact, in the bundle $U(n) \times U(m) \to U(n+m) \xrightarrow{\tilde{\pi}} G_{n,m}$, the induced distribution in $\tilde{\pi}^{-1}(A)$ from $\mathfrak{u}(m)$ in $U(n+m)$ is integrable and preserved by the right multiplication of $U(n)$, so this induces the bundle $U(n) \to \pi^{-1}(A) \to A$. 
Theorem 3.8. Given a complete totally geodesic surface $S$ in $G_{m,n}$ which is induced by a 2-dimensional subspace $m' \subset m$ with the necessary condition in Theorem 3.4 satisfied, the bundle $U(n) \to \pi^{-1}(S) \to S$, which is immersed in the original bundle $U(n) \to U(n + m)/U(m) \to G_{n,m}$, is flat.

Proof. By a left translation, without loss of generality, assume that $S$ passes through the origin of the affine symmetric space $G_{n,m}$.

Consider the bundle $U(n) \times U(m) \to U(n + m) \xrightarrow{\tilde{\pi}} G_{n,m}$. Then $S$ induces a bundle $U(n) \times U(m) \to \tilde{\pi}^{-1}(S) \to S$. Totally geodesic condition says that the distribution induced from $\text{Span}_\mathbb{R}\{\hat{X}, \hat{Y}, [\hat{X}, \hat{Y}]\}$ is integrable. Since $[\hat{X}, \hat{Y}]$ is contained in the Lie algebra $u(m)$ of $U(m)$ from the proof of Theorem 3.4, the conclusion is obtained. □

Theorem 3.9. Given $X \in U_{m,n}(\mathbb{C})$ and the natural fibration $U(n) \times U(m) \to U(n + m) \xrightarrow{\tilde{\pi}} G_{n,m}(\mathbb{C})$, consider the 2-dimensional subspace $m' = \text{Span}_\mathbb{R}\{\hat{X}, i\hat{X}\}$. Then,

1. $m'$ gives rise to a complete totally geodesic surface $S$ in $G_{n,m}(\mathbb{C})$,
2. $m'$ induces a $U(1)$-subbundle of a bundle

$$U(n) \times U(m) \to \tilde{\pi}^{-1}(S) \to S,$$

which is an immersion of the bundle $S(U(1) \times U(1)) \to SU(1 + 1) \to SU(1 + 1)/SU(1) \times U(1)$

into

$$U(n) \times U(m) \to U(n + m) \xrightarrow{\tilde{\pi}} G_{n,m}.$$

such that it is isomorphic to the Hopf bundle $S^1 \to S^3 \to S^2$,

3. the immersion is conformal, and isometric in case of $n = m$. In fact,

$$|\tilde{f}_*v| = \sqrt{\frac{2n}{n+m}}|v|$$

under the expression $\tilde{f} : SU(2) \to U(n + m)$ for the immersion.

Proof. From Lemma 3.1 let $X^* X = \lambda I_n$ for some $\lambda > 0$.

By a left translation, without loss of generality, assume that $S$ passes through the origin of the affine symmetric space $G_{n,m}$.

Note that, for $K = \begin{bmatrix} -i\lambda I_n & 0 \\ 0 & iXX^* \end{bmatrix} \in u(n) \times u(m)$,

$$[\hat{X}, i\hat{X}] = 2K, \quad [K, \hat{X}] = 2\lambda \hat{X}, \quad [K, i\hat{X}] = -2\lambda \hat{X},$$

which implies $[m', m', m'] \subset m'$ and the conclusion (1).

Consider an orthonormal basis of $\mathfrak{su}(1 + 1)$:

$$E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$
and a Lie algebra monomorphism \( f : su(1 + 1) \to u(n + m) \), given by

\[
f(aE_1 + bE_2 + cE_3) = \frac{a}{\sqrt{\lambda}} \hat{X} + \frac{b}{\sqrt{\lambda}} \hat{iX} + \frac{c}{\lambda} K
\]

for \( a, b, c \in \mathbb{R} \), from

\[
[E_1, E_2] = 2E_3, \quad [E_3, E_1] = 2E_2, \quad [E_3, E_2] = -2E_1.
\]

For any \( \theta \in \mathbb{R} \),

\[
e^{\theta E_3} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in S(U(1) \times U(1)).
\]

Thus \( f \) will induce a Lie group monomorphism \( \tilde{f} : SU(1 + 1) \to U(n + m) \) with \( \tilde{f}(SU(1) \times U(1)) \subset U(n) \times U(m) \) since \( SU(2) \) is simply connected and \( SU(1) \times U(1) \) is connected. Furthermore, it is the bundle map from 

\[
S(U(1) \times U(1)) \to SU(1 + 1) \to G_{1,1} = SU(1 + 1)/SU(1) \times U(1)
\]

to

\[
U(n) \times U(m) \to U(n + m) \to G_{n,m},
\]

so the connected component of the integral manifold of the distribution induced by \( \text{Span}_\mathbb{R}\{K, \hat{X}, \hat{iX}\} \), which is the image of \( \tilde{f} \), shows (2).

Note that \( \{\frac{1}{\sqrt{\lambda}} \hat{X}, \frac{1}{\sqrt{\lambda}} \hat{iX}, \frac{1}{\lambda} K\} \) is an orthogonal basis of the image of \( \tilde{f} \) such that

\[
\sqrt{\frac{n}{n+m}} = \left|\frac{1}{\sqrt{\lambda}} \hat{X}\right| = \left|\frac{1}{\sqrt{\lambda}} \hat{iX}\right| = \left|\frac{1}{\lambda} K\right|
\]

which shows (3).

\[\square\]

**Remark 3.10.** Let \( \dot{\theta} = \frac{\theta}{\lambda} \). Then, for \( \Phi = -E_3 \),

\[
\tilde{f}(e^{\theta \Phi}) = \tilde{f}(e^{-\theta E_3}) = e^{-\dot{\theta}K} = \begin{bmatrix} e^{i\theta}I_n & 0 \\ 0 & I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda} \end{bmatrix}
\]

from

\[
(-i\dot{\theta}XX^*)^j = \left(\frac{-i\theta}{\lambda}\right)^j X(X^*X)^j-1X^* = \frac{(-i\theta)^j}{\lambda} XX^*
\]

for \( j = 1, 2, \ldots \). Furthermore,

\[
\left(I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda}ight)\left(I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda}\right)
\]

\[
= I_m + \frac{e^{-i\theta + i\theta - 2}XX^* + e^{-i(\theta + \phi)} - e^{-i\theta - e^{-i\theta + 1}}X(X^*X)X^*}{\lambda}
\]

\[
= I_m + \frac{e^{-i(\theta + \phi - 1)}XX^*}{\lambda},
\]

from which it is also obtained that

\[
I_m = \left(I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda}\right)\left(I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda}\right)^*.
\]
We return to the bundle $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$. In fact, Remark 3.10 implies that the immersed $U(1)$-subbundle, which is the image of $\tilde{\gamma}$, gives two $U(1)$-bundles, one of which is an immersed $U(1)$-subbundle in the bundle $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$ and the other one is an immersed $U(1)$-subbundle in the bundle $U(m) \to U(n + m)/U(n) \xrightarrow{\pi} G_{n,m}$.

**Theorem 3.11.** Assume the same condition for a complete totally geodesic surface $S$ of either Theorem 3.9 or Theorem 3.4, and consider the immersed bundle $U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S$ in the bundle $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$. Then the holonomy displacement along $\gamma$, 

$$\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot V(\gamma),$$

is given by the right action of 

$$V(\gamma) = e^{i\theta} I_n \quad \text{or} \quad e^{i\bar{\theta}} I_n \in U(n),$$

depending on whether the immersed bundle is flat or not, where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$ and $\theta = 2 \cdot \frac{n + m}{2n} A(\gamma)$. Especially, $\theta = 2 \cdot A(\gamma)$ in case of $n = m$.

**Proof.** If the immersed bundle is flat, then it is obvious that the holonomy displacement is trivial. Assume the condition of Theorem 3.9 for the immersed $U(1)$-subbundle, which is the image of $\tilde{\gamma}$. Consider the induced map $\hat{f} : B \to S \subset G_{m,n}$ between base spaces from the bundle map $\hat{f} : SU(2) \to \text{Im}(\hat{f}) \subset U(n + m)$, which is a monomorphism, where $B = SU(2)/S(U(1) \times U(1))$. Let $\alpha = \sqrt{\frac{2n}{n+m}}$, $\theta = 2 \cdot \frac{n + m}{2n} A(\gamma) = \frac{2\alpha^2 - 2}{8} A(\gamma)$ and $\hat{\theta} = \frac{\theta}{\alpha}$. The Theorem 3.9, Theorem 2.11 and Remark 3.10 say that the holonomy displacement of $\gamma$ in the bundle $U(n) \times U(m) \to \pi^{-1}(S) \xrightarrow{\pi} S$, which is immersed in the bundle $U(n) \times U(m) \to U(n + m) \xrightarrow{\pi} G_{n,m}$, is given by the right action of 

$$V(\gamma) = \hat{f}(V(\hat{f}^{-1} \circ \gamma)) = \hat{f}(e^{2A(\hat{f}^{-1} \circ \gamma)} \Phi) = \hat{f}(e^{\hat{\theta} \Phi}) = \begin{bmatrix} e^{i\theta} I_n & 0 \\ 0 & I_m + e^{-i\theta - 1} XX^* \end{bmatrix}. $$

Thus in the bundle $U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S$, which is immersed in the bundle $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$, the holonomy displacement is given by the right action of 

$$V(\gamma) = e^{i\theta} I_n.$$

$\Box$
Remark 3.12. For \( n = 1 \), we have the following Hopf bundle \( S^1 \to S^{2m+1} \to \mathbb{C}P^m \), where \( \mathbb{C}P^m \) is given by the quotient metric, so the projection is a Riemannian submersion. Let \( S \) be a complete totally geodesic surface in \( \mathbb{C}P^m \) and \( \gamma \) be a piecewise smooth, simple closed curve on \( S \). Identify \( \mathbb{C}^m \cong M_m(\mathbb{C}) \). If \( S \) is induced by \( \text{Span}\{v, w\} \subset \mathbb{C}^m \) with \( \text{Im}_{\mathbb{C}^m}(v, w) = 0 \), then the holonomy displacement along \( \gamma \) is trivial. See Corollary 3.7 and Theorem 3.8. If \( S \) is induced by a two dimensional subspace with complex structure in \( \mathbb{C}^m \), then the holonomy displacement depends not only on the area of the region surrounded by \( \gamma \) but also on \( m \) unless \( m = 1 \). In case of \( m = 1 \), here, \( \mathbb{C}P^m \) is isometric to \( S^2\left(\frac{1}{2}\right) \). Refer to the map \( h \) defined in Section 2.

Remark 3.13. Let \( U(m) \to U(n+m)/U(n) \xrightarrow{\hat{\pi}} G_{n,m} \) be the natural fibration. Assume the same condition for a complete totally geodesic surface \( S \) of Theorem 3.9, and consider the bundle \( U(m) \to \hat{\pi}^{-1}(S) \xrightarrow{\hat{\pi}} S \). Let \( \gamma \) be a piecewise smooth, simple closed curve on \( S \). Then the holonomy displacement along \( \gamma \) is given by the right action of

\[
V(\gamma) = I_m + e^{-id_{\lambda}X}XX^* \in U(m),
\]

which depends on \( X \), not only on \( n \) and \( m \), where \( \theta = 2 \cdot \frac{m+n}{2n} A(\gamma) \).

References

1. S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. II, Reprint of the 1969 original. Wiley Classics Library, A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, 1996.
2. Y. Choi, K.B. Lee, Holonomy displacements in the Hopf bundles over \( \mathbb{C}H^n \) and the complex Heisenberg groups, J. Korean Math. Soc. 49, 733–743 (2012)
3. U. Pinkall, Hopf tori in \( S^3 \), Invent. Math. 81, 379–386 (1985)

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