Convex model predictive control for collision avoidance

Saša V. Raković | Sixing Zhang | Li Dai | Yanye Hao | Yuanqing Xia

School of Automation, Beijing Institute of Technology, Beijing 100081, China

Correspondence
Li Dai, School of Automation, Beijing Institute of Technology, Beijing 100081, China.
Email: li.dai@bit.edu.cn

Yuanqing Xia was supported by the National Natural Science Foundation, China Projects of International Cooperation and Exchanges under Grant 61720106010.

Abstract
This manuscript proposes a model predictive control for collision avoidance for the regulation problem of deterministic linear systems, which provides a priori guarantees of strong system theoretic properties, such as positive invariance and asymptotic stability, and high computational efficiency. Notion of safe distance sets is introduced, and also utilized as a novel approach to ensure collision avoidance via suitably defined convex constraints. The proposed convex model predictive control for collision avoidance is obtained by employing interactive strategic-tactical structure for overall decision-making. The strategic stage of the overall algorithm employs direct algebraic manipulations in order to construct safe distance sets that ensure collision avoidance. The tactical stage of the overall algorithm employs strictly convex quadratic programs for the optimization of local finite horizon predicted control processes. The dynamically compatible interaction of strategic and tactical stages of the overall algorithm is ensured by construction, which guarantees structural and computational benefits. These novel and unique features effectively enable both real time implementation and real life utilization of model predictive control for collision avoidance.

1 INTRODUCTION

Collision avoidance is a classical control problem whose theoretical, computational and practical aspects have received a considerable amount of attention [1–19]. This strong interest in collision avoidance control problem should not come as a surprise given its fundamental relevance for classical vehicles within automotive, aerospace and space applications as well as contemporary intelligent robotic systems and unmanned vehicles. The most systematic approach to control synthesis for collision avoidance is to deploy model predictive control (MPC). Indeed, MPC [20, 21] offers a systematic approach to handle static and dynamic constraints while optimizing performance of the considered system. Naturally, MPC is highly effective when its implementation can be executed via convex optimization. Unfortunately, MPC under collision-avoidance constraints is an intrinsically nonconvex problem [1–4, 9, 12]. This intrinsic nonconvexity of collision avoidance control problem diminishes considerably realistic utility of MPC due to prohibitive computational efforts required for the actual implementation of this advanced control methodology.

A shift of paradigm in MPC for obstacle avoidance has been recently reported in [22]. In this predecessor article, separation theorem was utilized in order to obtain suitable separating hyperplanes and construct related “separating closed polyhedra”, with the help of which nonconvex obstacle avoidance constraints were converted to computationally more convenient (convex) polyhedral constraints. The convex MPC for obstacle avoidance proposed in [22] was reduced to solving two strictly convex quadratic programming (QP) problems per iteration of the actual control process (except for the initialization of the actual control process at its very beginning). The solution of the first of these two strictly convex QP problems enabled, via direct algebraic manipulations, construction of sequences of, separating-hyperplanes-based, “separating closed polyhedra”. The use of the related “separating closed polyhedra” enabled optimization of an admissible finite horizon predicted control process via the second of the two above mentioned strictly convex QP problems. The strictly convex QP problems for the construction of “separating closed polyhedra” and optimization of admissible finite horizon predicted control processes were dynamically compatible and consistent, which resulted in convex MPC for obstacle avoidance with a priori guarantees of strong system theoretic properties and considerable computational efficiency. The underlying principles of this recent approach [22] to convex MPC for obstacle avoidance provide silhouettes, and motivate development, of a conceptually new approach to MPC for collision avoidance via convex...
optimization. The further development reported in this article, however, requires a considerable degree of generalizations from conceptual and implementational points of view due to the inevitable differences of obstacle and collision avoidance problems.

To gain clearer insights into the intrinsic features of collision avoidance problem as well as to provide concrete basis for associated interactive strategic-tactical decision-making, the real time air-traffic control is perhaps the most illustrative real life application. At an airport, a number of airplanes arrive and aim to land safely. The airplanes might have different dynamics as well as different actual constraints, but they will surely have different locations allocated for their landing at a given and fixed time period. This can be seen as a collection of dynamically decoupled systems with associated stage constraints, which are typically different (but can possibly be identical as the airplanes share common airspace) and with different terminal constraint sets and different terminal states. Obviously for safety reasons, the airplanes should not collide during the approach and landing stages of their flights. This naturally induces collision-avoidance constraints on the collection of systems, requiring that each system operates at a safe distance relative to all other systems. The air-traffic control center provides detailed and regularly updated paths to each of the airplanes in order to land safely, while each of the airplanes optimizes its flight path according to the directions received from the air-traffic control center. This reveals an interactive strategic-tactical decision-making, in which the air-traffic control center, given the knowledge of all relevant details for each of the airplanes, provides strategic instructions to each of the airplanes, which are tactically implemented by each of the airplanes through local, simultaneous and independent optimization of their flight paths resulting in safe landing. Naturally, this modeling related discussion applies transparently, with possibly minor modifications, to a variety of real-life applications including, inter alia, smart traffic control of general autonomous vehicles as well as motion and mission planning of intelligent robotic systems and smart autonomous vehicles.

In this article, we consider a collection of linear discrete time systems subject to system-wise independent stage and terminal polyhedral constraints and system-wise dependent, but collection-wise independent, collision-avoidance constraints. More precisely, we are concerned with MPC within such a setting with system-wise prescribed strictly convex quadratic stage and terminal cost functions. As already pointed out, such optimal control and, consequently, MPC problems are inherently nonconvex even in the theoretically most flexible centralized setting. To address this challenge, the objectives of this paper are

(i) to propose an interactive strategic-tactical decision-making architecture,
(ii) to introduce the notion of safe distance sets in order to ensure collision-avoidance constraints through (convex) polyhedral constraints, and
(iii) to design a dynamically compatible and consistent MPC.

The resulting convex MPC for collision avoidance has the benefits of a priori guarantees of strong system theoretic properties and computational efficiency.

The article structure is as follows. Section 2 details setting, discusses plausible decision-making structures, and outlines traditional, nonconvex optimization based, approach to centralized MPC for collision avoidance. Section 3 introduces a novel notion of safe distance sets motivated by Voronoi diagrams, which enables utilization of polyhedral constraints to ensure satisfaction of nonconvex collision-avoidance constraints. Section 3 also provides algebraic details enabling the construction of safe distance sets. Section 4 provides the formulation of interactive strategic-tactical decision-making resulting in local convex finite horizon optimal control for collision avoidance, and it outlines a prototype algorithm for convex MPC for collision avoidance. Section 5 establishes main system theoretic properties of the proposed algorithm. Section 6 discusses implementational aspects; this includes construction of terminal constraint sets, modification of the main algorithm to allow for optimality improving subiterations, and outline of plausible alternatives for initialization step. Section 6 also provides a detailed illustration of the proposed convex MPC for collision avoidance. Section 7 delivers concluding remarks and comments on extensions.

Basic Nomenclature: The sets of reals and non-negative integers are denoted by \( \mathbb{R} \) and \( \mathbb{N} \). Given \( a, b \in \mathbb{N} \) such that \( a < b \), we denote \( \mathbb{N}_{[a:b]} := \{a, a + 1, ..., b-1, b\} \); we write \( \mathbb{N}_b \) for \( \mathbb{N}_{[0:b]} \). A polyhedron is the intersection of a finite number of open and/or closed half-spaces and a polytope is a closed and bounded polyhedron. We distinguish row vectors from column vectors only when necessary, and we employ the same symbol for a variable \( x \) and its vectorized form. The scalars appearing in algebraic expressions represent a matrix/vector of compatible dimensions. Proofs of some of the technical results are given in Appendices.

2 | MPC FOR COLLISION AVOIDANCE

2.1 | Setting

We consider a collection of \( r \in \mathbb{N} \) linear discrete time systems given, for each \( i \in \mathbb{N}_{[1:r]} \), by

\[
x_{ij} = A_{ij} x_i + B_{ij} u_i,
\]

(2.1)

where \( x_i \in \mathbb{R}^{n_i} \) and \( u_i \in \mathbb{R}^{m_i} \) are the current state and control of the \( i^{th} \) system, \( x_{ij} \) is the successor state of the \( i^{th} \) system, while the matrix pairs \( (A_{ij}, B_{ij}) \in \mathbb{R}^{n_i \times n_i} \times \mathbb{R}^{n_i \times m_i}, \ i \in \mathbb{N}_{[1:r]} \) are of compatible dimensions. The considered collection of linear discrete time systems is accompanied with a collection of fixed point pairs \( (\bar{x}_i, \bar{u}_i) \), \( i \in \mathbb{N}_{[1:r]} \), which satisfy, for all \( i \in \mathbb{N}_{[1:r]} \),

\[
\bar{x}_i = A_i \bar{x}_i + B_i \bar{u}_i.
\]

(2.2)

The collection of linear discrete time systems is also accompanied with a collection of auxiliary variables \( z_i \in \mathbb{R}^s, \ i \in \mathbb{N}_{[1:r]} \).
specified, for all \(i \in \mathbb{N}_{[1:r]}\), by
\[
z_i = C_i x_i + D_i u_i,
\]
for given matrix pairs \((C_i, D_i) \in \mathbb{R}^{n_i \times n} \times \mathbb{R}^{n_i \times m_i}\).

**Assumption 1.** For each \(i \in \mathbb{N}_{[1:r]}\),

(i) Matrix pair \((A_i, B_i) \in \mathbb{R}^{n_i \times n} \times \mathbb{R}^{n_i \times m_i}\) is strictly stabilizable.

(ii) Fixed point pair \((\overline{x}_i, \overline{u}_i) \in \mathbb{R}^n \times \mathbb{R}^m\) is known.

(iii) Matrix pair \((C_i, D_i) \in \mathbb{R}^{n_i \times n} \times \mathbb{R}^{n_i \times m_i}\) is known.

The states and controls \(x_i\) and \(u_i\) of the \(i\)th system are subject to stage constraints
\[
(x_i, u_i) \in \mathcal{Y}_i, \tag{2.4}
\]
The terminal state of the \(i\)th system is subject to terminal constraints taking the form
\[
x_i \in \mathcal{X}_i, \tag{2.5}
\]
The variables \(z_i\) associated with the \(i\)th systems are subject to collision-avoidance constraints
\[
\forall j \in \mathbb{N}_{[1:r]} \setminus \{i\}, \quad \|z_j - z_i\| \geq \varepsilon_{(i,j)}. \tag{2.6}
\]

**Assumption 2.** For each \(i \in \mathbb{N}_{[1:r]}\),

(i) Stage constraint set \(\mathcal{Y}_i\) is a closed polyhedral subset of \(\mathbb{R}^{n_i \times n_i}\) containing the corresponding fixed point pair \((\overline{x}_i, \overline{u}_i)\) in its interior, and its irreducible representation is given by
\[
\mathcal{Y}_i = \{(x_i, u_i) : Y_{x_i}(x_i - \overline{x}_i) + Y_{u_i}(u_i - \overline{u}_i) \leq 1\}. \tag{2.7}
\]
The matrix pair \((Y_{x_i}, Y_{u_i}) \in \mathbb{R}^{n_i \times n_i} \times \mathbb{R}^{n_i \times m_i}\) is known.

(ii) Terminal constraint set \(\mathcal{X}_i\) is a closed polyhedral subset of \(\mathbb{R}^n\) containing the controlled fixed point \(\overline{x}_i\) in its interior, and its irreducible representation is given by
\[
\mathcal{X}_i = \{x_i : X_i(x_i - \overline{x}_i) \leq 1\}. \tag{2.8}
\]
The matrix \(X_i \in \mathbb{R}^{n \times n_i}\) is known.

(iii) Scalars \(\varepsilon_{(i,j)} \in (0, \infty), j \in \mathbb{N}_{[1:r]} \setminus \{i\}\) are known and, for each \(j \in \mathbb{N}_{[1:r]} \setminus \{i\}\), it holds that \(\varepsilon_{(i,j)} = \varepsilon_{(j,i)}\).

(iv) For each \(j \in \mathbb{N}_{[1:r]} \setminus \{i\}\), it holds that
\[
\|z_j - \overline{z}_j\| > \varepsilon_{(i,j)}, \tag{2.9}
\]
where \(\overline{z}_j := C_j \overline{x}_i + D_j \overline{u}_i\) (and \(\overline{z}_i := C_i \overline{x}_i + D_i \overline{u}_i\)).

The stage and terminal cost functions, \(\ell_i(\cdot, \cdot)\) and \(V_{f_i}(\cdot)\), are given, for each \(i \in \mathbb{N}_{[1:r]}\) and all \(x_i \in \mathbb{R}^n\) and all \(u_i \in \mathbb{R}^m\), by
\[
\ell_i(x_i, u_i) = (x_i - \overline{x}_i)^T Q_i (x_i - \overline{x}_i) + (u_i - \overline{u}_i)^T R_i (u_i - \overline{u}_i)
\]
and \(V_{f_i}(x_i) = (x_i - \overline{x}_i)^T P_i (x_i - \overline{x}_i)\). \tag{2.10}

The terminal control laws \(\kappa_{f_i}(\cdot)\), associated with the \(i\)th systems, are specified, for each \(i \in \mathbb{N}_{[1:r]}\), by
\[
\kappa_{f_i}(x_i) = \overline{u}_i + K_i(x_i - \overline{x}_i), \tag{2.11}
\]
and they induce the related \(i\)th terminal dynamics
\[
x_i^+ = A_i x_i + B_i \kappa_{f_i}(x_i) = \overline{x}_i + (A_i + B_i K_i)(x_i - \overline{x}_i)
\]
where \(o_j := (I - (A_j + B_j K_j))\overline{x}_j\). \tag{2.12}

**Assumption 3.** For each \(i \in \mathbb{N}_{[1:r]}\),

(i) The cost weighting matrices \(P_i \in \mathbb{R}^{n \times n}, Q_i \in \mathbb{R}^{n \times n}\) and \(R_i \in \mathbb{R}^{m_i \times m_i}\) are symmetric and positive definite, that is, \(P_i = P_i^T > 0, Q_i = Q_i^T > 0\) and \(R_i = R_i^T > 0\).

(ii) The control matrix gain \(K_i\) ensures that corresponding matrix \(A_j + B_j K_j\) is strictly stable.

(iii) Terminal constraint set \(\mathcal{X}_i\) is a positively invariant set for related terminal dynamics \(x_i^+ = A_i x_i + B_i \kappa_{f_i}(x_i)\) and constraints \((x_i, \kappa_{f_i}(x_i)) \in \mathcal{X}_i\) is true,
\[
\forall x_i \in \mathcal{X}_i, (x_i, \kappa_{f_i}(x_i)) \in \mathcal{X}_i \text{ and } A_i x_i + B_i \kappa_{f_i}(x_i) \in \mathcal{X}_i. \tag{2.13}
\]

(iv) The terminal cost function \(V_{f_i}(\cdot)\) is a local Lyapunov function for terminal dynamics \(x_i^+ = A_i x_i + B_i \kappa_{f_i}(x_i)\) over the terminal constraint set \(\mathcal{X}_i\) relative to the controlled fixed point \(\overline{x}_i\), and it verifies the decrease condition \(V_{f_i}(A_i x_i + B_i \kappa_{f_i}(x_i)) + \ell_i(x_i, \kappa_{f_i}(x_i)) \leq V_{f_i}(x_i)\) for all \(x_i \in \mathcal{X}_i\), as guaranteed by
\[
(A_i + B_j K_j)^T P_i (A_i + B_j K_j) + (Q_i + K_j^T R_j K_j) \leq P_i. \tag{2.14}
\]

(v) For each \(j \in \mathbb{N}_{[1:r]} \setminus \{i\}\), it holds that
\[
\forall z_j \in \mathcal{Z}_i, \forall z_j \in \mathcal{Z}_i, \quad \|z_j - \overline{z}_j\| \geq \varepsilon_{(i,j)}, \tag{2.15}
\]
where
\[
\mathcal{Z}_i := \{C_i x_i + D_i \kappa_{f_i}(x_i) : x_i \in \mathcal{X}_i\}
\]
and \(\mathcal{Z}_j := \{C_j x_j + D_j \kappa_{f_j}(x_j) : x_j \in \mathcal{X}_j\}\).
All of these assumptions are introduced for stability, and these assumptions are positive invariance and fairly standard in the MPC for the considered setting [20, 21].

2.2 Decision-making structures

While dynamical constraints (2.1), stage and terminal constraints (2.4) and (2.5) as well as stage and terminal cost functions (2.10) of the considered collection of linear systems are effectively independent for each of the systems, the collision-avoidance constraints (2.6) introduce, in general case, dependence between all systems. This dependence calls for characterization of the structure of the decision-making for control synthesis as well as compatibility of informational and dynamical flows so as to enable plausibility of utilized control functions for underlying decision-making. It is possible to consider a wide spectrum of structural architectures for the related decision-making lying between two extreme settings, namely centralized and decentralized decision-making.

The centralized decision-making is performed with a single entity and globally. The global decision maker1 has perfect knowledge of systems, constraints and costs as well as states of all systems when computing control actions for each of the systems. (The global decision maker computes control actions for each of the systems simultaneously.) The global decision maker is allowed to deploy control functions \( u_i(x_{1i}, x_{2i}, x_{3i}, \ldots, x_{ni}) \) in order to generate control actions \( u_i = u_i(x_{1i}, x_{2i}, x_{3i}, \ldots, x_{ni}) \) for the \( i^\text{th} \) system. Equally importantly, the global decision maker is able to address the collision-avoidance constraints (2.6) by taking the states, and related auxiliary variables, of all systems directly into account. The centralized decision-making is a theoretically exact structure, and it yields best results. Since all systems, constraints and cost functions are considered globally and simultaneously, the centralized decision-making is effectively a dimension-wise enlarged and computationally more complex decision-making process, which somewhat offsets its structural exactness and flexibility.

The decentralized decision-making is performed with multiple entities and locally. In particular, a local decision-maker is assigned to each of the systems. Any of the local decision makers has perfect knowledge only of his system and its constraints and cost as well as states of all systems when computing control action for the related system. (The local decision makers compute control actions for their systems simultaneously.) The \( i^\text{th} \) local decision maker is allowed to deploy control functions \( u_i(x_i) \) in order to generate control actions \( u_i = u_i(x_i) \) for the \( i^\text{th} \) system. Since each of the local decision makers considers only his system and its constraints and cost, the decentralized decision-making is effectively decomposed into a number of computationally more convenient decision-making processes, which are distributed among local decision makers. However, collision-avoidance constraints (2.6) are traditionally addressed in a somewhat conservative way, as these constraints require a nontrivial mechanism to ensure their satisfaction.

Evidently, the main challenge is to devise a structural architecture for decision-making which incorporates best of both extreme cases. In this sense, it is highly desirable to allow for systematic handling of the collision-avoidance constraints (2.6) as available in centralized decision-making as well as to achieve computational convenience of decentralized decision-making. This chore, in fact, leads to a number of research questions including design of communication networks, protocols, and their dynamics, compatibility of related informational and dynamical flows as well as customized distributed computations for local decision-making. The decision-making structure in this article adapts this philosophy, and it is illustrated in Figure 1. The utilized architecture for decision-making is composed of a strategist and a number of tacticians, each of which is assigned to a particular system. At the conceptual level, the strategist is in charge of initializing the actual control process as well as providing guidance to tacticians, which are in charge of locally optimizing behavior of the controlled system. Technical analysis of the utilized decision-making structure, which is for obvious reasons referred to as the interactive strategic-tactical decision-making, is elaborated on in detail in what follows.

2.3 Centralized finite horizon optimal control

Centralized MPC employs a centralized finite horizon \( N \) optimal control (FHOC) so as to simultaneously optimize finite horizon \( N \) predicted control processes \( d_{i(N)} \) for each of the considered systems. A finite horizon \( N \) predicted control process \( d_{i(N)} \) of the \( i^\text{th} \) system is simply a pair of state and control sequences \( x_{i(N)} := \{x_{i(k)}\}_{k=0}^{N} \) and \( u_{i(N)} := \{u_{i(k)}\}_{k=0}^{N-1} \), that is, \( d_{i(N)} = (x_{i(N)}, u_{i(N-1)}) \). The overall decision variable \( d_N \) in centralized decision-making is the collection of finite horizon \( N \) predicted control processes \( d_{i(N)} \), that is, \( d_N := (d_{1(N)}, d_{2(N)}, \ldots, d_{i(N)}) \).

The collection of finite horizon \( N \) predicted control process \( d_N \) is, for any given composed state \( x := (x_1, x_2, \ldots, x_{N}, x_N) \in \mathbb{R}^{nC} \) with \( n_C = \sum_{i=1}^{N} n_i \) subject to system-wise independent: dynamical consistency constraints

\[
\forall i \in [1,N], \quad \forall k \in [N-1],
\]

\[
x_{i(k+1)} = A_i x_{i(k)} + B_i u_{i(k)} \quad \text{with} \quad x_{i(0)} = x_i, \quad (2.16)
\]

\[\text{FIGURE 1} \quad \text{The interactive strategic-tactical decision-making}\]

---

1 The term “decision makers” is utilized compatibly with the common use in mathematics and the decision maker is an entity referred to verbally as “him”, but this by no means implies the decision maker is a human.
stage constraints
\[ \forall i \in \mathbb{N}_{[1,r]}, \forall k \in \mathbb{N}_{N-1}, \quad (z_i(k), u_i(k)) \in \mathcal{Y}, \quad (2.17) \]

terminal constraints
\[ \forall i \in \mathbb{N}_{[1,r]}, \quad z_i(N) \in \mathcal{X}, \quad (2.18) \]

and to system-wise dependent, but collection-wise independent, collision-avoidance constraints
\[ \forall i \in \mathbb{N}_{[1,r]}, \quad \forall j \in \mathbb{N}_{[1,r]} \setminus \{i\}, \quad \forall k \in \mathbb{N}_N, \quad \|z_i(k) - z_j(k)\| \geq \varepsilon_{i,j}, \quad (2.19) \]

where \( \varepsilon_{i,j} := C_{i} z_i(k) + D_i u_i(k) \) and, for each \( i \in \mathbb{N}_{[1,r]} \),
\[ \tilde{z}_{i,N} := C_{i} z_i(N) + D_i x_i(N), \]

The set \( D_N(x) \) is defined as the admissible collections of finite horizon \( N \) control processes \( d_N \) for any \( x \in \mathbb{R}^\infty \), by
\[ D_N(x) = \{ d_N : \text{relations (2.16)–(2.19) hold} \}. \]

The cost function \( V_{i,N}(\cdot) \) is associated with the \( i \)-th system, and it is given as the sum of the stage costs \( \ell_i(x_{i,k}, u_{i,k}) \), \( k \in \mathbb{N}_{N-1} \) and terminal cost \( V_{f,i}(z_{i,N}) \), that is
\[ V_{i,N}(d_{i,N}) = \sum_{k=0}^{N-1} \ell_i(x_{i,k}, u_{i,k}) + V_{f,i}(z_{i,N}), \quad (2.21) \]

For any given composed state \( x \in \mathbb{R}^\infty \), the global decision maker aims at simultaneous optimization of cost functions \( V_{i,N}(\cdot) \), \( i \in \mathbb{N}_{[1,r]} \) via selection of an admissible collection of finite horizon \( N \) control processes \( d_N \in D_N(x) \). Clearly, the optimality of centralized decision-making depends on the preference employed by the global decision maker. In this sense, the global decision maker can consider several optimality criteria including, but not exclusively limited to, multiobjective, pareto, and equilibria optimality. A computationally simplifying, and practically reasonable, solution that the global decision maker can consider is to consider a suitable scalar-valued function of cost functions \( V_{i,N}(\cdot) \), \( i \in \mathbb{N}_{[1,r]} \), for example, the sum of the cost functions \( V_{i,N}(\cdot) \), \( i \in \mathbb{N}_{[1,r]} \). Within the intended scope of this manuscript, the latter approach is discussed briefly so as to set the stage for what follows. Thus, the cost function \( V_N(\cdot) \) associated with a collection of finite horizon \( N \) control processes \( d_N \) is simply given by
\[ V_N(d_N) = \sum_{i=1}^{r} V_{i,N}(d_{i,N}). \]

With this concession, the centralized FHNOC problem \( \Phi_N(x) \) takes form, for any given composed state \( x \in \mathbb{R}^\infty \),
\[ V_N^0(x) = \min_{d_N} \{ V_N(d_N) : d_N \in D_N(x) \} \]
and
\[ d_N^0(x) = \arg \min_{d_N} \{ V_N(d_N) : d_N \in D_N(x) \}. \]

The domain of the value function \( V_N^0(x) \) and its optimizer map \( d_N^0(x) \) is the \( N \)-step controllable set \( \tilde{X}_N \) to a terminal constraint set \( \tilde{X} := X_1 \times X_2 \times \ldots \times X_r \), given by
\[ \tilde{X}_N := \{ x : D_N(x) \neq \emptyset \}. \]

In general, under Assumptions 1–3, the sets \( D_N(x), x \in \tilde{X}_N \) are nonempty and closed but nonconvex, while the centralized FHNOC problem \( \Phi_N(x) \) is a well-posed optimization problem, which is, in fact, a problem of minimization of a strictly convex quadratic function over a nonempty, closed and non-convex set.

2.4 Centralized model predictive control

Centralized MPC utilizes solution of the centralized FHNOC problem \( \Phi_N(x) \) so as to implement control laws \( \kappa_{N,i}(\cdot) \) satisfying, for all \( x \in \tilde{X}_N \),
\[ \kappa_{N,i}(x) \in u_{i(0)}^0(x), \quad (2.25) \]

where \( u_{i(0)}^0(x) \) denotes the set of optimal control actions \( u_{i(0)} \) at \( x \), which is not necessarily single valued due to nonconvexity of collision-avoidance constraints (2.19). The control laws \( \kappa_{N,i}(\cdot) \) induce model predictive controlled dynamics taking the form, for each \( i \in \mathbb{N}_{[1,r]} \),
\[ x_i^+ \in F_{N,i}(x), \quad (2.26) \]

Under Assumptions 1–3, the composed controlled fixed point state \( \bar{x} := (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r) \) is asymptotically stable in the strong sense for the collection of model predictive controlled dynamics \( (x_1^+, x_2^+, \ldots, x_r^+) \in (F_{N,1}(x), F_{N,2}(x), \ldots, F_{N,r}(x)) \) with the domain of attraction being the \( N \)-step controllable set \( \tilde{X}_N \) (in fact, exponentially stable if \( \tilde{X}_N \) is bounded), which is also a positively invariant set in the strong sense for the collection of model predictive controlled dynamics \( (x_1^+, x_2^+, \ldots, x_r^+) \in (F_{N,1}(x), F_{N,2}(x), \ldots, F_{N,r}(x)) \) and stage constraints \( (x_i, \kappa_{N,i}(x_i)) \in \mathcal{Y} \) for each \( i \in \mathbb{N}_{[1,r]} \) and collision-avoidance constraints \( \|C_{i} x_i + D_i x_i(x) - (C_{i} x_i + D_i x_i(x))\| \geq \varepsilon_{i,j} \) for each \( i \in \mathbb{N}_{[1,r]} \) and each \( j \in \mathbb{N}_{[1,r]} \setminus \{i\} \).

Unfortunately, even in this simplified version of the centralized decision-making, due to the size and, more prohibitively, nonconvexity of the centralized FHNOC problem \( \Phi_N(x) \), the computational effort of traditional approach to centralized MPC for collision avoidance is overwhelming for its practical utilization.

2.5 Main objective

It is of importance to alleviate prohibitive computational burden to enable realistic implementation of MPC for collision avoidance. This aspect is addressed entirely in this article by
documenting a computationally efficient, convex optimization based, reformulation of traditional approach to MPC for collision avoidance.

3 | VORONOI DIAGRAMS AND SAFE DISTANCE SETS

3.1 | Safe distance sets

The ideas underpinning Voronoi diagrams are utilized in order to derive the notion, and enable computationally efficient construction, of safe distance sets, as discussed next. A basic overview of Voronoi diagrams is recalled in Appendix A, and a detailed study of it can be found in [23, 24].

Consider points \( s_i \in \mathbb{R}^n \), \( i \in \mathbb{N}_{[1,T]} \) and scalars \( \varepsilon_{(i,j)} \in (0, \infty) \), \( i \in \mathbb{N}_{[1,T]} \), \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) satisfying \( \|s_i - s_j\| \geq \varepsilon_{(i,j)} \) and \( \varepsilon_{(i,j)} = \varepsilon_{(j,i)} \) for all \( i \in \mathbb{N}_{[1,T]} \) and all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \). The sets \( S_i \) defined, for all \( i \in \mathbb{N}_{[1,T]} \), by

\[
S_i := \left\{ \xi : \forall j \in \mathbb{N}_{[1,T]} \setminus \{i\}, \alpha_{(i,j)}^T \xi \leq \frac{\varepsilon_{(i,j)}}{2} \right\},
\]

where \( \alpha_{(i,j)} \) and \( \beta_{(i,j)} \) are given in (3.2), (3.3), are referred to as the safe distance sets. The justification of the term safe distance sets and relevant properties of the safe distance sets \( S_i \) are provided by the following:

Proposition 1. Take any collection of scalars \( \varepsilon_{(i,j)} \in (0, \infty) \), \( i \in \mathbb{N}_{[1,T]} \), \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) with \( \varepsilon_{(i,j)} = \varepsilon_{(j,i)} \). Take also any collection of points \( s_i \in \mathbb{R}^n : i \in \mathbb{N}_{[1,T]} \) such that \( \|s_i - s_j\| \geq \varepsilon_{(i,j)} \) for all \( i \in \mathbb{N}_{[1,T]} \) and all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \). Consider the collection of safe distance sets \( \{S_i : i \in \mathbb{N}_{[1,T]}\} \) where each set \( S_i \) is given by (3.1).

(i) Sets \( S_i \), \( i \in \mathbb{N}_{[1,T]} \), are closed polyhedral sets.
(ii) For all \( i \in \mathbb{N}_{[1,T]} \), \( s_i \in S_i \).
(iii) For all \( i \in \mathbb{N}_{[1,T]} \) and all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \),

\[
\forall \xi \in S_i, \quad \|\xi - s_i\| < \|\xi - s_j\|. \tag{3.2}
\]

(iv) For all \( i \in \mathbb{N}_{[1,T]} \) and all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \),

\[
\forall \xi \in S_i, \quad \forall \xi \in \mathcal{S}_j, \quad \|\xi - \xi_j\| \geq \varepsilon_{(i,j)}. \tag{3.3}
\]

3.2 | Sequence of safe distance sets

Consider sequences of points \( \{s_{(i,k)} \in \mathbb{R}^n \}_{k=0}^N \), \( i \in \mathbb{N}_{[1,T]} \) and scalars \( \varepsilon_{(i,k)} \in (0, \infty) \), \( i \in \mathbb{N}_{[1,T]} \), \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) satisfying \( \|s_{(i,k)} - s_{(j,k)}\| \geq \varepsilon_{(i,j)} \) and \( \varepsilon_{(j,i)} = \varepsilon_{(i,j)} \) for all \( i \in \mathbb{N}_{[1,T]} \), \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) and all \( k \in \mathbb{N}_N \). The sets \( S_{i,k} \) defined, for all \( i \in \mathbb{N}_{[1,T]} \) and all \( k \in \mathbb{N}_N \), by

\[
S_{i,k} := \left\{ \xi : \forall j \in \mathbb{N}_{[1,T]} \setminus \{i\}, \alpha_{(i,j)}^T \xi \leq \frac{\varepsilon_{(i,j)}}{2} \right\},
\]

where, in light of (A.3), for all \( i \in \mathbb{N}_{[1,T]} \), all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) and all \( k \in \mathbb{N}_N \),

\[
\alpha_{(i,j)} := \frac{s_{(j,k)} - s_{(i,k)}}{\|s_{(j,k)} - s_{(i,k)}\|}, \quad \mu_{(i,j)} := \frac{\|s_{(i,k)} + s_{(j,k)}\|}{2}, \quad \beta_{(i,j)} := \alpha_{(i,j)}^T \mu_{(i,j)},
\]

are also referred to as the safe distance sets. A direct extension of Proposition 1, provides a formal justification of the term safe distance sets and summarizes relevant properties of \( S_{i,k} \).

Proposition 2. Take any collection of scalars \( \varepsilon_{(i,k)} \in (0, \infty), i \in \mathbb{N}_{[1,T]} \), \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) with \( \varepsilon_{(i,j)} = \varepsilon_{(j,i)} \). Take also any collection of sequences of points \( \{s_{(i,k)} \in \mathbb{R}^n \}_{k=0}^N : i \in \mathbb{N}_{[1,T]} \} \) such that \( \|s_{(i,k)} - s_{(j,k)}\| \geq \varepsilon_{(i,j)} \) for all \( i \in \mathbb{N}_{[1,T]} \), all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) and all \( k \in \mathbb{N}_N \).

(i) Sets \( S_{i,k} \), \( i \in \mathbb{N}_{[1,T]} \), \( k \in \mathbb{N}_N \) are closed polyhedral sets.
(ii) For all \( i \in \mathbb{N}_{[1,T]} \), all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) and all \( k \in \mathbb{N}_N \),

\[
\forall \xi \in S_{i,k}, \quad \|\xi - s_{(i,k)}\| < \|\xi - s_{(j,k)}\| \tag{3.6}
\]

(iii) For all \( i \in \mathbb{N}_{[1,T]} \), all \( j \in \mathbb{N}_{[1,T]} \setminus \{i\} \) and all \( k \in \mathbb{N}_N \),

\[
\forall \xi \in S_{i,k}, \quad \forall \xi \in S_{j,k}, \quad \|\xi - \xi_j\| \geq \varepsilon_{(i,j)}. \tag{3.7}
\]

3.3 | Safe distance sets for collision avoidance

For any \( i \in \mathbb{N}_{[1,T]} \), the sequence of sets \( \{S_{i,k}\}_{k=0}^N \) is referred to as the sequence of safe distance sets for the \( i \)th system, while each of the sets \( S_{i,k} \) is called safe distance set for the \( i \)th system at time \( k \). In light of Proposition 2(i), the collection of the sequences of the safe distance sets for the \( i \)th system, that is, the collection \( \{\{S_{i,k}\}_{k=0}^N : i \in \mathbb{N}_{[1,T]}\} \) can be utilized to ensure satisfaction of collision-avoidance constraints (2.6). In particular, under postulates of Proposition 2, collision-avoidance constraints (2.19) can be replaced by

\[
\forall i \in \mathbb{N}_{[1,T]}, \quad \forall k \in \mathbb{N}_N, \quad \xi_{(i,k)} \in S_{i,k}. \tag{3.8}
\]

Unlike direct form of the collision-avoidance constraints (2.19), constraints based on safe distance sets (3.8) are affine and system-wise independent constraints.

Evidently, knowledge of a sequence of the safe distance sets \( \{S_{i,k}\}_{k=0}^N \) enables the \( i \)th tactician to optimize locally and independently finite horizon control process \( d_{(i,N)} \) for the \( i \)th system. As formally shown in what follows, this can be achieved by solving a strictly convex QP problem. When the strategist provides sequences of the safe distance sets \( \{S_{i,k}\}_{k=0}^N \), the tacticians can optimize locally, independently and
simultaneously, finite horizon $N$ control processes $d_{i,N}$, $i \in \mathbb{N}_{[1,L]}$, for their systems. Within MPC paradigm, once the tacticians optimize finite horizon $N$ control processes $d_{i,N}$, $i \in \mathbb{N}_{[1,L]}$, they can provide the strategist with the collection of sequences of points $\{t_{(i,k)} \in \mathbb{R} \}_N^{k=0}$ satisfying postulates of Proposition 2. As formally shown in what follows, this requires direct and simple algebraic calculations. The strategist can then update sequences of the safe distance sets $\{S_{(i,k)}(x)\}_N^{k=0}$, $i \in \mathbb{N}_{[1,L]}$, and the whole process can be repeated. This brief summary provides insights into the philosophy of the interactive strategic-tactical decision-making, which is the architectural structure employed in this article and which is formally elaborated on next.

4 | CONVEX MPC FOR COLLISION AVOIDANCE

4.1 | Strategic decision-making

At the very beginning of the actual control process, the strategist is tasked with its initialization. Since the initialization stage is performed only once, it is discussed in more detail in Section 6.3. The main effective chore of the strategist is the construction of the sequences of the safe distance sets $\{S_{(i,k)}(x)\}_N^{k=0}$, $i \in \mathbb{N}_{[1,L]}$, which are communicated to tacticians so as to make them available for their decision-making. At the current composed state $x = (x_1, x_2, \ldots, x_r)$, for the construction of the sequences of the safe distance sets $\{S_{(i,k)}(x)\}_N^{k=0}$, $i \in \mathbb{N}_{[1,L]}$, the strategist receives sequences $\{s_{(i,k)}(x)\}_N^{k=0}$ from each $i$th tactician, whose collection $\{s_{(i,k)}(x)\}_N^{k=0}$ satisfies postulates of Proposition 2. The strategist utilizes this collection to construct safe distance sets $S_{(i,k)}(x)$ by using direct algebraic operations specified in (4.4) and (4.5). In particular, the strategist constructs, for all $i \in \mathbb{N}_{[1,L]}$ and all $k \in \mathbb{N}_N$, safe distance sets $S_{(i,k)}(x)$ represented as

$$S_{(i,k)}(x) := \{ z_{(i,k)} : E_{(i,k)}(x)z_{(i,k)} \leq \epsilon_{(i,k)} \},$$  

where the $j$th row $E_{(i,k)}(x)$ of the matrices $E_{(i,k)}(x)$ and $j$th entries $e_{(i,k)}(x)$ of vectors $e_{(i,k)}(x)$ are $\alpha_{(i,k)}(x)$ and $\beta_{(i,k)}(x) - \frac{\epsilon_{(i,k)}}{2}$, respectively.

The values of $\alpha_{(i,k)}(x)$ and $\beta_{(i,k)}(x)$ are obtained by using relations (3.5), in which each $s_{(i,k)}(x)$ is replaced by $s_{(i,k)}(x)$. Thus, each of the safe distance sets $S_{(i,k)}(x)$ is entirely characterized by the related matrix-vector pair $(F_{(i,k)}(x), e_{(i,k)}(x))$, which is constructed by simple and direct algebraic operations. At the current composed state $x = (x_1, x_2, \ldots, x_r)$, the strategist then communicates the sequences of the safe distance sets $\{S_{(i,k)}(x)\}_N^{k=0}$, $i \in \mathbb{N}_{[1,L]}$ to the $j$th tacticians, that is, it provides the related sequences of the matrix-vector pairs $\{F_{(i,k)}(x), e_{(i,k)}(x)\}_N^{k=0}$, $i \in \mathbb{N}_{[1,L]}$ to the tacticians.

4.2 | Tactical decision-making

Each of the $j$th tacticians, optimizes locally, independently and simultaneously finite horizon $N$ control process $d_{j,N}$, which is subject to dynamical consistency, stage and terminal as well as term-wise inclusion in safe distance sets constraints. Thus, for each $i \in \mathbb{N}_{[1,L]}$ and each state $x_i$ and related sequence of safe distance sets $\{S_{(i,k)}(x_i)\}_N^{k=0}$, these constraints reduce to a set of affine equalities and inequalities, as detailed next system-wise, that is, for each $i \in \mathbb{N}_{[1,L]}$. The dynamical consistency is expressed explicitly as

$$\forall k \in \mathbb{N}_{N-1}, \quad x_{(i,k+1)} = A_i x_i + B_i u_i \quad \text{with} \quad x_{(i,0)} = x_i.$$  

(4.3)

The stage constraints are given explicitly as

$$\forall k \in \mathbb{N}_{N-1}, \quad x_{(i,k)} C_i x_{(i,k)} + E_i(x_{(i,k)}) D_i u_i \leq \epsilon_{(i,k)}.$$  

(4.4)

Likewise, the terminal constraints are given explicitly as

$$x_{(i,N)} C_i x_{(i,N)} + E_i(x_{(i,N)}) D_i u_i \leq \epsilon_{(i,N)},$$  

(4.5)

The term-wise inclusion into safe distance sets (3.8), as already elaborated on in Section 3.3, ensures collection-wise collision-avoidance constraints (2.19). These inclusion constraints take the explicit form given by

$$\forall k \in \mathbb{N}_N, \quad E_{(i,k)}(x) C_i x_{(i,k)} + E_{(i,k)}(x) D_i u_i \leq \epsilon_{(i,k)}(x),$$  

(4.6)

where $u_{(i,N)} := x_{(i,N)} C_i x_{(i,N)} + E_i(x_{(i,N)}) D_i u_i$. The set $D_{(i,N)}(x)$ of admissible finite horizon $N$ control processes $d_{(i,N)}(x) = (x_{(i,N)}, u_{(i,N)})$ for the $j$th system is given, for each $i \in \mathbb{N}_{[1,L]}$ and any $x_i \in \mathbb{R}^n$ and related sequence of safe distance sets $\{S_{(i,k)}(x)\}_N^{k=0}$, by

$$D_{(i,N)}(x) := \{ d_{(i,N)} : \text{relations (4.3)-(4.6) hold} \},$$  

(4.7)

and, by definition, it satisfies

$$D_{(i,N)}(x) = C_{(i,N)}(x) \cap S_{(i,N)}(x),$$  

(4.8)

where the sets $C_{(i,N)}(x) := \{ d_{(i,N)} : \text{relations (4.3)-(4.5) hold} \}$ and $S_{(i,N)}(x) := \{ d_{(i,N)} : \text{relation (4.6) holds} \}$.
represent constraints that can be constructed by the \(i\)th tactician without guidance of the strategist (the set \(\mathcal{C}_{\{i,N\}}(x_t)\)) and with guidance of the strategist (the set \(\mathcal{S}_{\{i,N\}}(x_t)\)). In view of relations (4.3)–(4.6), the set \(\mathcal{D}_{\{i,N\}}(x)\) of admissible finite horizon \(N\) control processes for the \(i\)th system is a closed polyhedral set. Furthermore, for each \(i \in \mathbb{N}_{|1:r|}\) and any \(x_t \in \mathbb{R}^{n_i}\) and related sequence of safe distance sets \(\{\mathcal{S}_{\{i,N\}}(x_t)\}_{k=0}^{N}\), the local decision-making process that is solved by the \(i\)th tactician is a FHNOC problem \(\mathbf{P}_{\{i,N\}}(x)\), which takes the form of a computationally efficient strictly convex QP problem

\[
V^0_{\{i,N\}}(x) = \min_{d_{\{i,N\}}} \{V^0_{\{i,N\}}(d_{\{i,N\}}) : \text{d}_{\{i,N\}} \in \mathcal{D}_{\{i,N\}}(x)\}
\]

\[
\text{d}^0_{\{i,N\}}(x) = \arg\min_{\text{d}_{\{i,N\}}} \{V^0_{\{i,N\}}(d_{\{i,N\}}) : \text{d}_{\{i,N\}} \in \mathcal{D}_{\{i,N\}}(x)\}. \tag{4.10}
\]

Each \(i\)th tactician employs MPC \(\mathbf{x}_{\{i,N\}}(\cdot)\), which is simply given by

\[
\mathbf{x}_{\{i,N\}}(x) = \mathbf{x}_{\{i,0\}}(x), \tag{4.11}
\]

where \(\mathbf{x}_{\{i,0\}}(x)\) is, in this case, unique. The related \(i\)th model predictive controlled dynamics are given by

\[
x_{i}^+ = A_{i} x_{i} + B_{i} \mathbf{x}_{\{i,N\}}(x). \tag{4.12}
\]

As shown in what follows, the composed domain of the proposed convex MPC for collision avoidance, is, in fact, the \(N\)-step controllable set \(\mathcal{X}_N\) to a terminal constraint set \(\mathcal{X}\) specified in (2.24) provided that the initialization step is performed as discussed in Section 6.3.

The dependence of the sets \(\mathcal{D}_{\{i,N\}}(\cdot)\) of admissible horizontal \(N\) control processes \(d_{\{i,N\}}\), the value function \(V^0_{\{i,N\}}(\cdot)\) and its optimizer function \(d^0_{\{i,N\}}(\cdot)\) and the MPC laws \(\mathbf{x}_{\{i,N\}}(\cdot)\) on the composed state \(x = (x_1, x_2, \ldots, x_r)\) is indirect, and it is induced by the dependence of the sequences of the safe distance sets \(\{\mathcal{S}_{\{i,N\}}(x_t)\}_{k=0}^{N}\) on the composed state \(x = (x_1, x_2, \ldots, x_r)\). Evidently, the local decision-making processes can be performed without direct knowledge of the composed state \(x = (x_1, x_2, \ldots, x_r)\). More precisely, the strictly convex quadratic programs \(\mathbf{P}_{\{i,N\}}(x)\) can be solved by the \(i\)th tacticians as long as they have the knowledge of the current state \(x_t \in \mathbb{R}^{n_i}\) of the \(i\)th system and related sequence of safe distance sets \(\{\mathcal{S}_{\{i,N\}}(x_t)\}_{k=0}^{N}\). Thus, it is a subtle point that tacticians have an indirect access to, and make locally an indirect use of, global information (i.e. the composed state \(x = (x_1, x_2, \ldots, x_r)\), both of which are enabled due to their interaction with, and guidance of, the strategist; namely, the strategist provides related sequence of safe distance sets \(\{\mathcal{S}_{\{i,N\}}(x_t)\}_{k=0}^{N}\) to each of the \(i\)th tacticians.

Summa summarum, each \(i\)th tactician performs local and independent optimization of an admissible finite horizon \(N\) control process \(d_{\{i,N\}} = (s_{\{i,N\}}(u_{\{i,N-1\}}))\), implements control action \(u^0_{\{i,0\}}(x)\) to the \(i\)th system, constructs the sequence \(\{s_{\{i,k\}}(x^+)^N\}_{k=0}\) by setting

\[
\forall k \in \mathbb{N}_{N-2},
\]

\[
s_{\{i,k\}}(x^+) = C_{i} x^+_{\{i,k+1\}}(x) + D_{i} u^0_{\{i,k+1\}}(x),
\]

\[
s_{\{i,N-1\}}(x^+) = C_{i} x^+_{\{i,N\}}(x) + D_{i} x_{\{i,N\}}(x), \quad \text{and}
\]

\[
s_{\{i,N\}}(x^+) = C_{i} x^+_{\{i,N+1\}}(x) + D_{i} x_{\{i,N+1\}}(x), \tag{4.13}
\]

where \(x_{\{i,N+1\}}(x) := A_{i} x_{\{i,N\}}(x) + B_{i} \mathbf{x}_{\{i,N\}}(x)\), and communicates the sequence \(\{s_{\{i,k\}}(x^+)^N\}_{k=0}\) to the strategist in order to enable him to perform his subsequent decision-making at \(x^+ = (x^+_1, x^+_2, \ldots, x^+_r)\).

### 4.3 Interactive strategic-tactical decision-making

The interactive strategic-tactical decision-making represents effectively a dynamically consistent composition of the strategic and tactical decision-making, and it results in a computationally highly efficient convex MPC for collision avoidance. The overall algorithm constructs, at any time instant, the collection \(\{\mathcal{S}_{\{i,k\}}(x_t)\}_{k=0}^{N}\) of the sequences of the safe distance sets for the \(i\)th system by direct algebraic manipulations, and then optimizes independently and simultaneously the predicted finite horizon \(N\) control processes by solving resulting strictly convex QP problems \(\mathbf{P}_{\{i,N\}}(x)\).

#### Initialization

1. Initialize algorithm with a collection of admissible finite horizon \(N\) control process \(\{x_{\{i,k\}}(x)\}_{k=0}^{N}\) and \(\{u_{\{i,k\}}(x)\}_{k=0}^{N}\) satisfying relations (2.16)–(2.19).

2. Construct sequences of safe distance sets \(\{\mathcal{S}_{\{i,k\}}(x)\}_{k=0}^{N}\) for each \(i \in \mathbb{N}_{|1:r|}\) by utilizing (4.1) and (4.2). (In practical terms, construct sequences of matrix–vector pairs \(\{E_{\{i,k\}}(x), \mathbf{e}_{\{i,k\}}(x)\}\}_{k=0}^{N}\). \(\mathcal{S}_{\{i,k\}}(x)\) for each \(i \in \mathbb{N}_{|1:r|}\), communicate the sequence \(\{\mathcal{S}_{\{i,k\}}(x)\}_{k=0}^{N}\) to the \(i\)th tactician. (In practical terms, communicate the corresponding sequence of matrix-vector pairs \(\{E_{\{i,k\}}(x), \mathbf{e}_{\{i,k\}}(x)\}\}_{k=0}^{N}\).

3. Update the set \(\mathcal{D}_{\{i,N\}}(x)\) in (4.7) by utilizing \(E_{\{i,k\}}(x)\)'s and \(\mathbf{e}_{\{i,k\}}(x)\)'s relation in (4.6).
5. Optimize predicted finite horizon $N$ control process $d_{i}\{N\}$ by solving strictly convex QP problem $\mathcal{P}_{i}\{N\}(x)$ specified in (4.10).

6. Implement value of the control law $x_{i}\{N\}(x) := u_{i}\{0\}(x)$, and obtain $x_{i}^{+} = A_{i}x_{i} + B_{i}x_{i}\{N\}(x)$.

7. Generate related sequence of points $\{s_{i}^{(k)}(x^{+})\}_{k=0}^{N}$ by utilizing (4.13).

8. Communicate the sequence of points $\{s_{i}^{(k)}(x^{+})\}_{k=0}^{N}$ to the strategist.

9. Set $x_{i} = x_{i}^{+}$ and go to step 2.

Except for initialization, which is discussed in more detail in Section 6.3, the algorithm requires, at any time instant of the actual control process, simple algebraic operations for the strategic decision-making and solutions to a collection of strictly convex QP problems for the tactical decision-making. This is in a stark contrast to the centralized MPC formulation outlined in Section 2.4, which requires a solution to single nonconvex optimization problem (taking the form of the minimization of a quadratic function over a nonconvex closed set). This effectively enhances computational aspects considerably and, in fact, enables a real time implementation of MPC for collision avoidance. The algorithm is marginally more complex compared to the conventional decentralized (and centralized) MPC without collision-avoidance constraints, but the analysis of related system theoretic properties does require a more careful consideration as discussed next.

### 5 SYSTEM THEORETIC PROPERTIES

To discuss feasibility, positive invariance, stability, attractivity and consistent improvement properties, we need to slightly generalize standard arguments. More precisely, we need to account adequately for the interaction of the strategic and tactical decision-making processes.

#### 5.1 Feasibility and positive invariance

First thing to observe is that, for any composed state $x = (x_{1}, x_{2}, \ldots, x_{N})$ lying in the $N$-step controllable set $X_{N}$ given by (2.24), the actual control process can be initiated. Namely, for any composed state $x \in X_{N}$, there exists a collection of admissible finite horizon $N$ control process $\{\{x_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$ constructed by setting $s_{i}(x) := C_{i}x_{i}(x) + D_{i}u_{i}(x)$ for each $i \in \mathbb{N}_{1:1}$ and each $k \in \mathbb{N}_{0:N-1}$ and $s_{i}(x_{i}) := C_{i}x_{i}(x) + D_{i}x_{i}\{0\}(x_{i})$ for each $i \in \mathbb{N}_{1:1}$ satisfies postulates of Proposition 2. In other words, the strategic decision-making, that is, construction of the collection of sequences of safe distance sets $\{S_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$, is feasible for any composed state $x \in X_{N}$.

**Proposition 3.** Suppose Assumptions 1, 2 and 3 hold. For any composed state $x \in X_{N}$, and any collection of admissible finite horizon $N$ control process $\{\{x_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$, the strategic decision-making, that is, construction of the collection of sequences of safe distance sets $\{S_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$ via relations (4.1) and (4.2) with properties established in Proposition 2, is feasible.

For each $i \in \mathbb{N}_{1:1}$ any admissible finite horizon $N$ control process $\{\{x_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$ is, by construction, such that for all $k \in \mathbb{N}_{0:N-1}$, $x_{i}\{k+1\}(x) = A_{i}x_{i}(x) + B_{i}u_{i}(x)$ with $u_{i}(x) = \varepsilon$ and, for all $k \in \mathbb{N}_{0:N-1}$, $x_{i}\{k\}(x), u_{i}(x) \in \mathcal{Y}_{i}$ and $x_{i}\{N\}(x) \in \mathcal{X}_{i}$. Furthermore, by the virtue of Proposition 3 in mind, $\gamma_{i}\{0\}(x) = \gamma_{i}\{0\}(x) = C_{i}x_{i}(x) + D_{i}u_{i}(x) \in S_{i}\{0\}(x)$ for each $k \in \mathbb{N}_{0:N-1}$ as well as $\gamma_{i}\{N\}(x) = \gamma_{i}(x_{i}\{N\}(x)) + D_{i}x_{i}\{f_{i}\{f_{i}\{N\}\}(x_{i}\{N\}(x))) \in S_{i}\{N\}(x)$. In plain words, each of the tactical decision-making problems, that is, each of the strictly convex QP problems $\mathcal{P}_{i}\{N\}(x)$ specified in (4.10), is also feasible for any composed state $x \in X_{N}$.

**Proposition 4.** Suppose Assumptions 1, 2 and 3 hold. For any composed state $x \in X_{N}$, and any associated collection of sequences of safe distance sets $\{S_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$ constructed via relations (4.1) and (4.2) satisfying properties established in Proposition 2, the strictly convex QP problems $\mathcal{P}_{i}\{N\}(x)$, $i \in \mathbb{N}_{1:1}$ specified in (4.10) are all feasible.

The unique solutions of the strictly convex QP problems $\mathcal{P}_{i}\{N\}(x)$, $i \in \mathbb{N}_{1:1}$ yield directly optimized finite horizon $N$ control processes $\{\{x_{i}\{N\}(x)\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$.

Since the control applied to each of the $i$th systems in the actual control process is $x_{i}\{N\}(x) = u_{i}\{0\}(x)$, that is, $x_{i}^{+} = A_{i}x_{i} + B_{i}x_{i}\{N\}(x)$ for each $i \in \mathbb{N}_{1:1}$, the feasibility arguments above can be repeated recursively throughout the iterations of the proposed algorithm. To see this point clearly, consider finite horizon $N$ control processes $\{\{s_{i}\{N\}(x^{+})\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$ with, for each $i \in \mathbb{N}_{1:1}$ and each relevant $k$, $x_{i}\{k\}(x^{+}) = \gamma_{i}\{k\}(x^{+})$ and $x_{i}\{N\}(x^{+}) = A_{i}x_{i}\{k\}(x) + B_{i}x_{i}\{f_{i}\{f_{i}\{N\}\}(x_{i}\{N\}(x)))$ as well as $u_{i}(x^{+}) = u_{i}\{f_{i}\{f_{i}\{0\}\}(x^{+}) = \gamma_{i}\{f_{i}\{f_{i}\{0\}\}(x^{+})$. Under Assumptions 1–3, for all $k \in \mathbb{N}_{0:N-1}$, $x_{i}(k+1)(x^{+}) = A_{i}x_{i}(x^{+}) + B_{i}u_{i}(x^{+})$ with $u_{i}(x^{+}) = \varepsilon$. Likewise, for all $k \in \mathbb{N}_{0:N-1}$, $x_{i}(k)(x^{+}), x_{i}(x^{+}) \in \mathcal{Y}^{i}$ and $h_{i}\{N\}(x^{+}) \in \mathcal{X}^{i}$ since $x_{i}\{N\}(x^{+}) = x_{i}(x^{+}) \in \mathcal{X}^{i}$ and $h_{i}\{N\}(x^{+}) = \gamma_{i}\{f_{i}\{f_{i}\{N\}\}(x^{+})$. Furthermore, by construction, $\gamma_{i}(x^{+}) = C_{i}x_{i}(x^{+}) + D_{i}u_{i}(x^{+}) \in S_{i}\{0\}(x^{+}) := S_{i}\{k\}(x^{+})$ for each $k \in \mathbb{N}_{0:N-1}$ and since $x_{i}(x^{+}) \in \mathcal{X}^{i}$, there exists a safe distance set $S_{i}\{N\}(x^{+})$ such that $\gamma_{i}(x^{+}) = C_{i}x_{i}(x^{+}) + D_{i}x_{i}\{f_{i}\{f_{i}\{N\}\}(x_{i}\{N\}(x))) \in S_{i}\{N\}(x^{+})$ due to Assumption 3(e). By definition, the collection of sequences $\{s_{i}\{N\}(x^{+})\}_{k=0}^{N} : i \in \mathbb{N}_{1:1}\}$ satisfies postulates of Proposition 2, that is, for each $i \in \mathbb{N}_{1:1}$, each $j \in \mathbb{N}_{1:1} \backslash \{i\}$ and each $k \in \mathbb{N}_{0:N}$, it holds that $\|s_{i}(x^{+}) - s_{j}(x^{+})\| \geq \varepsilon$; this also ensures that $\|s_{i}(x^{+}) - \gamma_{i}(x^{+})\| \geq \varepsilon$ with $z_{i}(x^{+}) := s_{i}(x^{+})$ (and $z_{i}(x^{+}) := s_{i}(x^{+})$) so that collection-wise collision-avoidance constraints (2.19) are
consistent improvement satisfied. All in all, the arguments preceding Propositions 3 and 4 can be repeated at $x^+$, which effectively verifies the highly desired positive invariance property of the interactive strategic-tactical decision-making process, that is, convex MPC for collision avoidance.

**Proposition 5.** Suppose Assumptions 1, 2 and 3 hold. The $N$-step controllable set $\mathbb{X}_N$ given by (2.24) is a positively invariant set for the composed model predictive controlled dynamics, components of which are given, for each $i \in \mathbb{N}_{[1],[N]}$, by $x_i^+ = A_i x_i + B_i x_{(i+1)}(x_i)$, and stage constraints $(x_i, x_{(i+1)}(x_i)) \in \mathcal{Y}$. Moreover, each of the considered finite horizon $V$ that satisfies $V^0(\mathbb{X}_N) \leq V_{f,j}(x_i) \leq c_\iota(x_i) \|x_i - \mathcal{X}_i\|^2$, (5.4) for some strictly positive and finite scalars $c_\iota(x_i) \in \mathbb{R}$, $i \in \mathbb{N}_{[1],[N]}$. With relation (5.1) of Proposition 5 in mind, relations (5.2), (5.3) and (5.4) suffice to conclude asymptotic stability of the composed controlled fixed point $x = (x_1, x_2, \ldots, x_N)$ for the composed model predictive controlled dynamics, components of which are given, for each $i \in \mathbb{N}_{[1],[N]}$, by $x_i^+ = A_i x_i + B_i x_{(i+1)}(x_i)$, with the domain of attraction being the $N$-step controllable set $\mathbb{X}_N$. We also observe that, under our assumptions, when $\mathbb{X}_N$ is bounded the upper bounds of (5.4) can be extended to $\mathbb{X}_N$ (possibly relative to different scalars $c_\iota(x_i) \in \mathbb{R}$ such that $c_\iota(x_i) \leq c_\iota(x_i) < \infty$ for each $i \in \mathbb{N}_{[1],[N]}$, in which case the composed controlled fixed point $x = (x_1, x_2, \ldots, x_N)$ is exponentially stable.

**Theorem 1.** Suppose Assumptions 1, 2 and 3 hold. The composed controlled fixed point $x = (x_1, x_2, \ldots, x_N)$ is asymptotically stable for the composed model predictive controlled dynamics, components of which are given, for each $i \in \mathbb{N}_{[1],[N]}$, by $x_i^+ = A_i x_i + B_i x_{(i+1)}(x_i)$, with the domain of attraction being the $N$-step controllable set $\mathbb{X}_N$ given by (2.24). (The composed controlled fixed point $x = (x_1, x_2, \ldots, x_N)$ is actually exponentially stable when $\mathbb{X}_N$ is bounded.)

5.2 Stability and attractiveness

The optimized costs at a given composed state $x \in \mathbb{X}_N$ are $V^0(\mathbb{X}_N)(x)$, $i \in \mathbb{N}_{[1],[N]}$, and the above utilized collection of admissible finite horizon $N$ control processes $\{(x_i, x_{(i+1)}(x_i))^N\}_{i=0}^{N-1}$ yields the desired cost decrease. As discussed in the previous subsection, the strategic and tactical decision-making processes are feasible. Furthermore, each of the considered finite horizon $N$ control processes $\{(x_i, x_{(i+1)}(x_i))^N\}_{i=0}^{N-1}$ ensures that $V^0(\mathbb{X}_N)(x_i, x_{(i+1)}(x_i))^N$ is bounded, and it is guaranteed to be an admissible finite horizon $N$ control process for the optimal control problem $\mathcal{P}(\mathbb{X}_N)^N$. Thus, each of the optimized costs $V^0(\mathbb{X}_N)(x_i)$, $i \in \mathbb{N}_{[1],[N]}$, at the composed successor state $x_i^+ = (x_1^+, x_2^+, \ldots, x_N^+)$ with $x_i^+ = A_i x_i + B_i x_{(i+1)}(x_i)$ for each $i \in \mathbb{N}_{[1],[N]}$, satisfies the cost decrease condition.

**Proposition 6.** Suppose Assumptions 1, 2 and 3 hold.

\[
\forall x = (x_1, x_2, \ldots, x_N) \in \mathbb{X}_N, \quad \forall i \in \mathbb{N}_{[1],[N]},
\]

\[
V^0(\mathbb{X}_N)(x_i^+) \leq V^0(\mathbb{X}_N)(x_i) - \ell_i(x_i, x_{(i+1)}(x_i)), \quad (5.2)
\]

where $x_i^+ = (x_1^+, x_2^+, \ldots, x_N^+)$ with $x_i^+ = A_i x_i + B_i x_{(i+1)}(x_i)$ for each $i \in \mathbb{N}_{[1],[N]}$.

Furthermore, under postulated assumptions,

\[
\forall x = (x_1, x_2, \ldots, x_N) \in \mathbb{X}_N, \quad \forall i \in \mathbb{N}_{[1],[N]},
\]

\[
c_\iota(x_i) \|x_i - x_i^+\|^2 \leq \ell_i(x_i, x_{(i+1)}(x_i)) \leq V^0(\mathbb{X}_N)(x_i), \quad (5.3)
\]

for some strictly positive and finite scalars $c_\iota(x_i) \in \mathbb{R}$, $i \in \mathbb{N}_{[1],[N]}$, and $\forall x = (x_1, x_2, \ldots, x_N) \in \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N$. Furthermore, each of the considered finite horizon $N$ control processes $\{(x_i, x_{(i+1)}(x_i))^N\}_{i=0}^{N-1}$ ensures that $V^0(\mathbb{X}_N)(x_i, x_{(i+1)}(x_i))^N$ is bounded, and it is guaranteed to be an admissible finite horizon $N$ control process for the optimal control problem $\mathcal{P}(\mathbb{X}_N)^N$. Thus, each of the optimized costs $V^0(\mathbb{X}_N)(x_i)$, $i \in \mathbb{N}_{[1],[N]}$, at the composed successor state $x_i^+ = (x_1^+, x_2^+, \ldots, x_N^+)$ with $x_i^+ = A_i x_i + B_i x_{(i+1)}(x_i)$ for each $i \in \mathbb{N}_{[1],[N]}$, satisfies the cost decrease condition.
and all $x \in X_0 = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_r$. In layman’s terms, increasing the prediction horizon length $N$ comes with the potential for the enlargement of the domain of attraction and improvement of the optimized cost.

### 6 IMPLEMENTATIONAL ASPECTS

#### 6.1 Construction of terminal constraint sets

For each $i \in \mathbb{N}_{[1:N]}$, the terminal affine state feedbacks $u_i = \bar{n}_i + K_i(x_i - \bar{x}_i)$ and terminal cost functions $V_j(x_i) = (x_i - \bar{x}_i)^T P_j(x_i - \bar{x}_i)$ can be constructed directly based on relation (2.14) by using linear algebra. An optimal choice is to select matrices $K_i$ and $P_j$ as the solution of the infinite horizon linear quadratic regulator problem for the system $(x_i - \bar{x}_i)^T Q(x_i - \bar{x}_i) + (u_i - \bar{u}_i)^T R_i (u_i - \bar{u}_i)$, that is, to get these as the solution to the infinite horizon linear quadratic regulator for the quadruple $(A_i, B_i, Q_i, R_i)$. For each $i \in \mathbb{N}_{[1:N]}$, given a terminal affine state feedback $u_i = \bar{n}_i + K_i(x_i - \bar{x}_i)$, the construction of the terminal constraint set $\mathcal{X}_i$ reduces to the construction of a positively invariant set for the terminal affine dynamics

$$x_i^+ = (A_i + B_i K_i)x_i + o_i$$

with

$$o_i = (I - (A_i + B_i K_i))\bar{x}_i,$$

subject to overall state constraints

$$x_i \in \mathcal{X}_i \cap \mathcal{X}_{i+1} \cap \mathcal{X}_{i+2} \cap \ldots \cap \mathcal{X}_r \cap \mathcal{X}_{r+1} \cap \ldots \cap \mathcal{X}_N$$

In (6.2), $C_{i,j}$ represents stage constraints under terminal affine state feedback $u_i = \bar{n}_i + K_i(x_i - \bar{x}_i)$, that is, $C_{i,j} := \{x_i : (x_i - \bar{x}_i) \in \mathcal{Y}_i\}$ so that

$$C_{i,j} = \{x_i : (Y_{i,j} + Y_{i,k})x_i \leq 1 + (Y_{i,j} + Y_{i,k})\bar{x}_i\}.$$  

(6.3)

L icewise, in (6.2), $R_{i,j}$ represents terminal inclusion in the safe distance sets constraints that, by Proposition 1, ensure collection-wise collision-avoidance constraints, and which are constructed relative to the collection of points $\{\bar{x}_i : i \in \mathbb{N}_{[1:N]}\}$ with $\bar{x}_i = C_{i+1} + D_{i+1}$, that is, $R_{i,j} := \{x_i : C_i x_i + D_i \bar{x}_i \in S_{i,j}\}$ and $S_{i,j}$ is the related terminal safe distance set given by

$$S_{i,j} = \{\bar{x}_i \in \mathcal{X}_i : E_{i,j} x_i \leq \varepsilon_{i,j}\},$$

(6.4)

where the $j$th row $E_{i,j}$ of the matrices $E_{i,j}$ and $j$th entries $\varepsilon_{i,j}$ of vectors $\varepsilon_{i,j}$ are $\alpha_{i,j}^T$ and $\beta_{i,j}$, that is, $\varepsilon_{i,j} := \alpha_{i,j}^T$ and $\varepsilon_{i,j} := \beta_{i,j}$. The values of $\alpha_{i,j}$ and $\beta_{i,j}$ are obtained by using relations (A.2), in which each $s_i$ is replaced by $\bar{x}_i = C_i x_i + D_i \bar{x}_i$. Hence,

$$R_{i,j} = \{x_i : E_{i,j}(C_i + D_i K_i)x_i \leq \varepsilon_{i,j}\}.$$  

(6.5)

Under Assumption 2, the overall state constraint $\mathcal{X}_{i,j}$ is a closed polyhedral subset of $\mathbb{R}^n$ containing the controlled fixed point $\bar{x}_i$ in its interior. Thus, the construction of the terminal constraint set reduces to a well-understood problem of positive invariance for strictly stable affine dynamics subject to proper polyhedral state constraints, for which efficient computational procedures exist and can be employed directly [25]. An optimal choice for the terminal constraint set is the maximal positively invariant set for $x_i^+ = (A_i + B_i K_i)x_i + o_i$ subject to constraints $x_i \in \mathcal{X}_{i,j}$, which is obtained as the limit of the standard set iteration

$$X_{i,j+1} = F^{-1}(X_{i,j}) \cap X_{i,j}$$

(6.6)

and which is, under our assumptions, guaranteed to be finitely determined as well as a bounded closed polyhedral set in $\mathbb{R}^n$ containing the controlled fixed point $\bar{x}_i$ in its interior, provided that either terminal dynamics matrices $A_i + B_i K_i$ and matrix composed by concatenation of $Y_{i,j} + Y_{i,k}$ and $E_{i,j}(C_i + D_i K_i)$ form an observable matrix pair or $\mathcal{X}_{i,j}$ is bounded. When $K_i$ and $P_j$ solve the infinite horizon linear quadratic regulator problem for the system $(x_i - \bar{x}_i)^T Q(x_i - \bar{x}_i) + (u_i - \bar{u}_i)^T R_i (u_i - \bar{u}_i)$, and the terminal constraint set is the maximal positively invariant set for $x_i^+ = (A_i + B_i K_i)x_i + o_i$ subject to constraints $x_i \in \mathcal{X}_{i,j}$, the convex MPC for collision avoidance algorithm can be safely terminated when $x = (x_1, \ldots, x_r) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_r$, as, in this case, $\mathcal{X}_{i,j}(x) = \bar{x}_i + K_i(x_i - \bar{x}_i)$ for each $i \in \mathbb{N}_{[1:N]}$ and all $x \in \mathcal{X}_i \times \mathcal{X}_i \times \ldots \times \mathcal{X}_i$. Furthermore, under such construction, each of tactical decision-making process, that is, each optimization problem $\mathcal{P}_{i,j}(\mathcal{X}(x))$, needs not be solved when $x_i \in \mathcal{X}_i$, since then $\mathcal{X}_{i,j}(x) = \bar{x}_i + K_i(x_i - \bar{x}_i)$.

#### 6.2 Optimality improving subiteration

The strategic decision-making is effectively reduced to the construction of the sequences of safe distance sets $\{S_{i,j}(x)\}_{i=0}^N$, which requires merely simple algebraic operations. Furthermore, the tactical decision-making is effectively reduced to solving the optimization problems $\mathcal{P}_{i,j}(\mathcal{X}(x))$, which are highly structured strictly convex QP problems and, thus, can be solved very efficiently [26]. These two facts allow for a subiteration with the main aim to improve optimality in global sense. This can be attained by modifying steps 2 and 5 of the proposed prototype algorithm for convex MPC for collision avoidance. Step 2 could be replaced by
2(a). Set \( j = 0 \) and, for each \( i \in \mathbb{N}_{[1:N]} \), \( V^0_{i,N}\) \((x) = \sum_{k=0}^{N-1} f_i'(x_{i,k}(x), u_i(k)(x)) + V'_f(x_{i,N}(x)). \)

2(b). Construct sequences of safe distance sets \( S_{i,k}(x) \) for each \( i \in \mathbb{N}_{[1:N]} \) by utilizing (4.1) and (4.2). (In practical terms, construct sequences of matrix-vector pairs \( \{(E_{i,k}(x), e_{i,k}(x))\}_{k=0}^{N-1} \), \( i \in \mathbb{N}_{[1:N]} \).)

Likewise, the step 5. could be replaced by

5(a). Optimize predicted finite horizon \( N \) control process \( d_{i,N}(x) \) by solving strictly convex QP problem \( \mathfrak{P}_{i,N}(x) \) specified in (4.10).

5(b). Set \( j = j + 1 \), and \( V^0_{i,N}(x) = V^0_{i,N}(x) \).

5(c). If termination condition holds, set \( V^0_{i,N}(x) = V^0_{i,N}(x) \) and go to step 6. of the main algorithm. Otherwise, update sequence of points \( \{x_{i,k}(x)\}_{k=0}^{N-1} \) and \( \{u_i(k)(x)\}_{k=0}^{N-1} \) by setting, for all \( k \in \mathbb{N}_{[0:N-1]} \), \( x_{i,k}(x) = x^0_{i,k}(x) \) and, for all \( k \in \mathbb{N}_{[N-1]} \), \( u_i(k)(x) = u_i(k)(x) \), to the strategy and go to step 2(b).

Sensible termination condition for the outlined subiteration could be \( j \geq j_{\text{max}} \) for a prescribed maximal number of iterations \( j_{\text{max}} \), or a desired improvement of the optimized costs \( V^0_{i,N}(x) \), that is, for all \( i \in \mathbb{N}_{[1:N]} \), \( V^0_{i,N}(x) - V^0_{i,N}(x) \) \( \leq \varepsilon_i \), for given scalars \( \varepsilon_i \), \( i \in (0, \infty) \), \( i \in \mathbb{N}_{[1:N]} \), or combination of the two, for example

\[
j \geq j_{\text{max}} \text{ or } \forall i \in \mathbb{N}_{[1:N]}, \quad |V^0_{i,N}(x) - V^0_{i,N-1}(x)| \leq \varepsilon_i.
\]

Note that, within our setting, such a subiteration produces monotonically nonincreasing sequences of optimized costs, that is, \( \forall i \in \mathbb{N}_{[1:N]}, \forall j \in \mathbb{N}, V^0_{i,N-1}(x) \leq V^0_{i,N}(x) \). Since the optimized costs \( V^0_{i,N}(x) \), \( i \in \mathbb{N}_{[1:N]} \) are nonnegative and, thus, lower bounded, the sequences of the optimized costs \( V^0_{i,N}(x) \) are guaranteed to be convergent sequences.

### 6.3 Initialization step

The most direct, but computationally most intensive, way to initialize the algorithm is to solve a feasibility stage of the optimization problem \( \mathfrak{P}_N(x) \) specified in (2.23). This approach is acceptable since feasibility stage of the optimization problem (2.23) is considerably simpler than the related optimality stage. More importantly, this step is performed only once at the very beginning of the actual control process. Thus, its computational burden is, in fact, negligible relative to accumulated computational effort of the proposed algorithm. Namely, as already pointed out, all of the remaining steps require throughout actual control process direct algebraic operations (for the strategic decision-making processes) and solutions to computationally highly efficient, standard and strictly convex quadratic programs (for the tactical decision-making processes).

An alternative is to devise a dedicated algorithm for the initialization step based on successive relaxation of the collision-avoidance constraints; The conceptual prototype formulation can be obtained along the following lines.

0(a). Start with a collection of finite horizon \( N \) control processes \( d_{i,N}(x) = \{x_{i,k}(x)\}_{k=0}^{N-1}, \{u_i(k)(x)\}_{k=0}^{N-1} \), each of which satisfies stage and terminal constraints.

0(b). Obtain a collection of scalars \( \bar{\varepsilon}_{(i,j)} \in [0, \varepsilon_{(i,j)}] \), \( i \in \mathbb{N}_{[1:N]} \), \( j \in \mathbb{N}_{[1:N]} \) \( \{i\} \) for which possibly relaxed collision-avoidance constraints \( \|z_{(i,k)} - z_{(i,k)}\| \geq \bar{\varepsilon}_{(i,j)} \) for each relevant \( i, j \) and \( k \).

0(c). If collision-avoidance constraints (2.19) hold (i.e. \( \bar{\varepsilon}_{(i,j)} \geq \varepsilon_{(i,j)} \) for each relevant \( i, j \)) go to the step 1. of the main algorithm. Otherwise, construct sequences of safe distance sets \( \overline{S}_{(i,k)}(x)\}_{k=0}^{N-1} \) with respect to the relaxed collision-avoidance constraints from the step 0(d). (If degeneracy is observed, perturb utilized sequences so as to handle it.)

0(d). Update a collection of finite horizon \( N \) control processes \( d_{i,N}(x) = \{x_{i,k}(x)\}_{k=0}^{N-1}, \{u_i(k)(x)\}_{k=0}^{N-1} \) by constructing each of \( d_{i,N}(x) \) to satisfy stage and terminal constraints as well as inclusion into relaxed safe distance sets \( \overline{S}_{(i,k)}(x) \) constraints and for which the related sequence \( \{z_{(i,k)}\}_{k=0}^{N-1} \) lies term-wise as deep as possible inside of the corresponding sequence of relaxed safe distance sets \( \overline{S}_{(i,k)}(x) \) (i.e. \( \overline{z}_{(i,k)} \) is as deep as possible inside of \( \overline{S}_{(i,k)}(x) \) for each \( k \)). This can be done by solving a linear programming problem.

0(e). Go to step 0(b).

The outlined conceptual prototype algorithm for initialization step is, by construction, guaranteed to be consistently improving. Its concrete formulation is relatively direct. Naturally, its more detailed analysis, including various modifications and enhancements as well as convergence properties and handling of singular geometries, deserves a study in its own right that, however, lies beyond the intended scope and page limitations of this article.

### 6.4 Academic illustration

An academic illustration of the proposed convex MPC for collision avoidance is provided by a multi-agent system example taken from [14]. The system is a collection of six agents given by

\[
x^+ = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad n, \quad i = 1, 2, \ldots, 6,
\]

where \( x_i = (x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}) \) consists of position \( (x_{i,1}, x_{i,2}) \) and velocity \( (x_{i,3}, x_{i,4}) \) of the \( i \)-th agent. The related fixed
All of the scalars \( \varepsilon \) defining collision-avoidance constraints are identically equal to \( \varepsilon \) with concrete value \( \varepsilon = 0.25 \). The stage and terminal cost function are defined by (2.10) while, for \( i = 1, 2, \ldots, 6 \)

\[
C_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } D_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

Each agent is subjected to the input constraint \( U_i = [-0.5, 0.5] \times [-0.5, 0.5] \). In addition, we limit the position of agents by adding constraints on the first two dimensions of states

\[
X_i = [-10, 10] \times [-10, 10], \quad X_2 = [-9, 11] \times [-10, 10], \quad X_3 = [-8, 12] \times [-9, 11], \quad X_4 = [-9, 11] \times [-8, 12], \\
X_5 = [-10, 10] \times [-8, 12], \quad X_6 = [-11, 9] \times [-9, 11].
\]

All of the scalars \( \varepsilon_{i,j} \) defining collision-avoidance constraints are identically equal to \( \varepsilon \) with concrete value \( \varepsilon = 0.25 \). The stage and terminal cost function are defined by (2.10) with \( Q_i = 100I \) and \( R_i = 100I \) for \( i = 1, 2, \ldots, 6 \). Each of the terminal cost weighting matrices \( P_i \) and terminal control feedback matrix gains \( K_i \) are obtained as the solution of the infinite horizon unconstrained optimal control for the system \( (x_i - \overline{x}_i)^T A_i (x_i - \overline{x}_i) + B_i (a_i - \overline{a}_i) \) and the stage cost \( \ell (x_i, a_i) = (x_i - \overline{x}_i)^T Q_i (x_i - \overline{x}_i) + (a_i - \overline{a}_i)^T R_i (a_i - \overline{a}_i) \). The related matrices \( P_i \) and \( K_i \) are given by:

\[
P_i = \begin{pmatrix} 236,710 & 0 & 111,8034 & 0 \\ 0 & 236,710 & 0 & 111,8034 \\ 111,8034 & 0 & 258,7483 & 0 \\ 0 & 111,8034 & 0 & 258,7483 \end{pmatrix},
\]

\[
K_i = \begin{pmatrix} -0.4345 & 0 & -1.0285 & 0 \\ 0 & -0.4345 & 0 & -1.0285 \end{pmatrix}.
\]

Obstacle avoidance is illustrated in Figure 2. The figure depicts the sequences of restricted safe distance sets \( \{\mathcal{S}_{(i,j)}(x_0) \cap \mathcal{X}_i\}_{j=0}^{\infty}, i = 1, 2, \ldots, 6 \) for feasible state sequences at composed current state \( x = (x_1, x_2, \ldots, x_6) \). The termination condition param-
FIGURE 3  Sequences of restricted safe distance sets \( \{ S_{(1,k)}(x) \cap \mathbb{X}_1 \}_{k=0}^7 \), \( \{ S_{(2,k)}(x) \cap \mathbb{X}_2 \}_{k=0}^7 \), \( \ldots \), \( \{ S_{(6,k)}(x) \cap \mathbb{X}_6 \}_{k=0}^7 \) with feasible and optimized position sequences at composed current state \( x = (x_1, x_2, \ldots, x_6) \).

FIGURE 4  Model predictive controlled position sequences: Time plots.

Considered current composed state, model predictive controlled position sequences generated by these two MPC methods are shown in Figure 6. From a local enlargement of the plot, for this example, it can be seen that position sequences do not coincide. By considering the summation of the costs of all agents over 20 time steps as the control performance index, the performance index of improved convex MPC is reduced by 241.0463, which demonstrates that the subiterations in the improved algorithm play an important role. The computational times are recorded in seconds for online optimization. For each time instant of simulated control processes, the worst-case, average and best-case computational times are 0.0364, 0.0248, 0.0217 s, respectively, for basic convex MPC, and 0.9820, 0.1425, 0.0837 s, respectively, for improved convex MPC, which demonstrates clearly a very high degree of computational efficiency of these two algorithms.

FIGURE 5  Model predictive controlled input sequences: Time plots.

FIGURE 6  Model predictive controlled position sequences: Space plots.
CONCLUSIONS AND EXTENSIONS

This article has reported convex MPC for collision avoidance. The unique feature of the developed algorithm is the utilization of safe distance sets in order to handle nonconvexity induced by collision-avoidance constraints. This feature enables design of convex MPC for collision avoidance, which is computationally efficient, since, except for the initialization step, its implementation reduces to performing simple algebraic operations for the construction of safe distance sets and solving standard and strictly convex QP problems for the optimization of local predicted control processes.

In terms of decision-making architectures, the notion of safe distance sets can be utilized directly as a technique for the design of convex centralized MPC for collision avoidance. The interactive strategic-tactical decision-making structure can be also entirely decentralized. More precisely, the strategist can be removed from the proposed structure at the cost of ensuring information exchange of collection of systems in all-to-all manner. With this modification, tacticians can, in an entirely decentralized fashion, construct sequences of safe distance sets for their systems and optimize their local finite horizon $N$ control processes. The potential drawback of such a modification would be reflected in an increased demand for information exchange and utilization of appropriate communication protocols as well as in an increased local computational effort. Likewise, such modifications can be relatively directly customized for the setting of collection of systems with a priori designed static or dynamic communication networks. In terms of robustness, a highly relevant extension is the development of convex robust MPC for collision avoidance. This extension can be obtained by combining algorithm proposed herein with computationally efficient tube MPC methods [27–32]. In terms of class of systems, an equally relevant extension is MPC of nonlinear systems for collision avoidance, which would require a combination of direct algebraic operations and sequential convex QP. In terms of control problems, the considered convex MPC for stabilization can be extended to tracking of a class of admissible references. This is a relatively standard extension in MPC [20, 21].

PERMISSION STATEMENT TO REPRODUCE THE MATERIALS FROM THE OTHER SOURCES

None.

ORCID
Saša V. Raković https://orcid.org/0000-0002-5402-0483
Sixing Zhang https://orcid.org/0000-0001-6696-2438
Li Dai https://orcid.org/0000-0002-7268-7548
Yanye Hao https://orcid.org/0000-0002-9361-0160
Yuanqing Xia https://orcid.org/0000-0002-5977-4911

REFERENCES
1. Schouwenaars, T., et al.: Mixed integer programming for multi-vehicle path planning. In: 2001 European Control Conference ( ECC), pp. 2603–2608. IEEE, Piscataway (2001)
2. Richards, A., How, J.P.: Aircraft trajectory planning with collision avoidance using mixed integer linear programming. In: Proceedings of American Control Conference, pp. 1936–1941. IEEE, Piscataway (2002)
3. Eze, A., Richards, A.: Multi vehicle avoidance using nonlinear branch and bound optimisation. In: AIAA Guidance, Navigation, and Control Conference, p. 5780. American Institute of Aeronautics and Astronautics, Reston (2009)
4. Kurzhanski, A.B., Varaiya, P.: On synthesizing team target controls under obstacles and collision avoidance. J. Franklin Inst. 347, 130–145 (2010)
5. Zhou, C., et al.: Collision-free UAV formation flight control based on nonlinear MPC. In: 2011 International Conference on Electronics, Communications and Control (ICECC), pp. 1951–1956. IEEE, Piscataway (2011)
6. Werfing, M., Laccardo, D.: Automatic collision avoidance using model-predictive online optimization. In: 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), pp. 6309–6314. IEEE, Piscataway (2012)
7. Shim, T., et al.: Autonomous vehicle collision avoidance system using path planning and model-predictive-control-based active front steering and wheel torque control. Proc. Inst. Mech Eng. D, J. Automob. Eng. 226, 767–778 (2012)
8. Zhou, C., et al.: UAV formation flight based on nonlinear model predictive control. Math. Prob. Eng. 2012, 261367 (2012)
9. Alrifae, B., et al.: Centralized non-convex model predictive control for cooperative collision avoidance of networked vehicles. In: 2014 IEEE International Symposium on Intelligent Control (ISIC), pp. 1583–1588. IEEE, Piscataway (2014)
10. Wang, P., Ding, B.: A synthesis approach of distributed model predictive control for homogeneous multi-agent system with collision avoidance. Int. J. Control 87, 52–63 (2014)
11. Kuriki, Y., Namienkawa, T.: Formation control with collision avoidance for a multi-UAV system using decentralized MPC and consensus-based control. SICE J. Control Meas. System Integr. 8, 285–294 (2015)
12. Rosolita, U., et al.: Autonomous vehicle control: a nonconvex approach for obstacle avoidance. IEEE Trans. Control Syst. Technol. 25, 469–484 (2016)
13. Lopez, B.T., How, J.P.: Aggressive collision avoidance with limited field-of-view sensing. In: 2017 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), pp. 1358–1365. IEEE, Piscataway (2017)
14. Dai, L., et al.: Distributed mpc for formation of multi-agent systems with collision avoidance and obstacle avoidance. J. Franklin Inst. 354, 2068–2085 (2017)
15. Sun, X., et al.: Collision avoidance using finite control set model predictive control for unmanned surface vehicle. Appl. Sci. 8, 926 (2018)
16. Guo, H., et al.: Simultaneous trajectory planning and tracking using an mpc method for cyber-physical systems: a case study of obstacle avoidance for an intelligent vehicle. IEEE Trans. Ind. Inf. 14, 4273–4283 (2018)
17. Denler, J., et al.: Collision avoidance effects on the mobility of a U/A Swarm using chaotic ant colony with model predictive control. J. Intell. Rob. Syst. 93, 227–243 (2019)
18. Hu, Q., et al.: Dynamic path planning and trajectory tracking using mpc for satellite with collision avoidance. ISA Trans. 84, 128–141 (2019)
19. Morgan, D., et al.: Model predictive control of swarms of spacecraft using sequential convex programming. J. Guid. Control Dyn. 37(6), 1725–1740 (2014)
20. Rawlings, J.B., Mayne, D.Q.: Model Predictive Control: Theory and Design. Nob Hill Publishing, Madison (2009)
21. Raković, S.V., Levine, W.S.: Handbook of Model Predictive Control. Springer, Cham (2018)
22. Raković, S.V., et al.: Convex MPC for exclusion constraints. Automatica 127, 109502 (2021).
23. Aurenhammer, F.: Voronoi diagrams—A survey of a fundamental geometric data structure. ACM Comput. Surv. (CSUR) 23(3), 345–405 (1991)
24. Ziegler, G.M.: Lectures on Polytopes, vol. 152. Springer Science & Business Media, New York (2012)
25. Blanchini, F., Miani, S.: Set–Theoretic Methods in Control. Birkhauser, Boston (2008)
26. Rao, C.V., et al.: Application of interior point methods to model predictive control. J. Optim. Theory Appl. 99(3), 723–757 (1998)
27. Mayne, D.Q., et al.: Robust model predictive control of constrained linear systems with bounded disturbances. Automatica 41, 219–224 (2005)
28. Raković, S.V.: Robust Control of Discrete Time Systems: Characterization and Implementation. PhD Thesis, Imperial College London (2005)
29. Raković, S.V., et al.: Homothetic tube model predictive control. Automatica 48, 1631–1638 (2012)
30. Raković, S.V., et al.: Parameterized tube model predictive control. IEEE Trans. Autom. Control 57, 2746–2761 (2012)
31. Raković, S.V., et al.: Equi-normalization and exact scaling dynamics in homothetic tube model predictive control. Syst. Control Lett. 62(2), 209–217 (2013)
32. Raković, S.V.: Robust model predictive control. In: Encyclopedia of Systems and Control, 2nd ed. Springer, Switzerland (2019)

**APPENDIX A: VORONOI DIAGRAMS**

The given distinct points \( s_i \in \mathbb{R}^n, \ i \in \mathbb{N}_{[1:l]} \) represent Voronoi sites, which induce Voronoi cells \( C_i, \ i \in \mathbb{N}_{[1:l]} \) whose collection \( \{C_i : \ i \in \mathbb{N}_{[1:l]}\} \) forms a Voronoi diagram. The Voronoi cells \( C_i \) are given, for all \( i \in \mathbb{N}_{[1:l]} \), by

\[
C_i = \{ z_i : \forall j \in \mathbb{N}_{[1:l]} \setminus \{i\}, \ \alpha_{(i,j)}^T z_i \leq \beta_{(i,j)} \}, \tag{A.1}
\]

where, for all \( i \in \mathbb{N}_{[1:l]} \) and all \( j \in \mathbb{N}_{[1:l]} \setminus \{i\} \),

\[
\alpha_{(i,j)} := \frac{s_i - s_j}{\| s_i - s_j \|}, \quad \mu_{(i,j)} := \frac{s_i + s_j}{2}, \quad \text{and} \quad \beta_{(i,j)} := \alpha_{(i,j)}^T \mu_{(i,j)}. \tag{A.2}
\]

The Voronoi cells \( C_i, \ i \in \mathbb{N}_{[1:l]} \) are closed polyhedral sets such that \( s_i \in C_i \). The Voronoi diagram \( \{C_i : \ i \in \mathbb{N}_{[1:l]}\} \) forms a partition of \( \mathbb{R}^n \). Furthermore, for any \( i \in \mathbb{N}_{[1:l]} \), the Voronoi cell \( C_i \) is the set of all points, distance of which from the Voronoi site \( s_i \) is smaller or equal to distance from any other Voronoi site \( s_j, \ j \in \mathbb{N}_{[1:l]} \setminus \{i\} \).

**APPENDIX B: PROOF OF PROPOSITION 1**

(i): The sets \( S_i \) are closed polyhedral sets by their definition in (3.1).

(ii): For each \( i \in \mathbb{N}_{[1:l]} \) and each \( j \in \mathbb{N}_{[1:l]} \setminus \{i\} \),

\[
\frac{(s_j - s_i)^T}{\| s_j - s_i \|} \mu_{(i,j)} := \frac{s_j + s_i}{2} - \frac{s_j - s_i}{2} \leq \frac{\| s_j - s_i \|}{2},
\]

and \( \| s_i - s_j \| \geq \varepsilon_{(i,j)} \). Thus, in light of definitions in (A.2), it follows that, for all \( i \in \mathbb{N}_{[1:l]} \) and all \( j \in \mathbb{N}_{[1:l]} \setminus \{i\} \),

\[
\alpha_{(i,j)}^T s_i \leq \beta_{(i,j)} - \frac{\varepsilon_{(i,j)}}{2} \quad \text{so that, in view of (3.1),} \ s_i \in S_j.
\]

(iii): For each \( i \in \mathbb{N}_{[1:l]} \), each \( j \in \mathbb{N}_{[1:l]} \setminus \{i\} \), the definition of safe distance sets \( S_i \) ensures that, for all \( z_i \in S_i \),

\[
\| z_i - s_i \| < \| z_i - s_j \|.
\]

(iv): For each \( i \in \mathbb{N}_{[1:l]} \), each \( j \in \mathbb{N}_{[1:l]} \setminus \{i\} \), in light of definition of safe distance sets \( S_i \),

\[
\min\{\| y_i - y_j \| : y_i \in S_i, y_j \in S_j\} \geq \varepsilon_{(i,j)}
\]

and, hence, for all \( z_i \in S_i \) and all \( z_j \in S_j \),

\[
\| z_i - z_j \| \geq \min\{\| y_i - y_j \| : y_i \in S_i, y_j \in S_j\} \geq \varepsilon_{(i,j)}.
\]

**APPENDIX C: PROOF OF PROPOSITION 2**

This result follows a direct application of Proposition 1, and, hence, its proof is omitted.