Describing hereditary properties by forbidden circular orderings*

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Abstract

Each hereditary property can be characterized by its set of minimal obstructions; these sets are often unknown, or known but infinite. By allowing extra structure it is sometimes possible to describe such properties by a finite set of forbidden objects. This has been studied most intensely when the extra structure is a linear ordering of the vertex set. For instance, it is known that a graph $G$ is $k$-colourable if and only if $V(G)$ admits a linear ordering $\leq$ with no vertices $v_1 \leq \cdots \leq v_{k+1}$ such that $v_i v_{i+1} \in E(G)$ for every $i \in \{1, \ldots, k\}$. In this paper, we study such characterizations when the extra structure is a circular ordering of the vertex set. We show that the classes that can be described by finitely many forbidden circularly ordered graphs include forests, circular-arc graphs, and graphs with circular chromatic number less than $k$. In fact, every description by finitely many forbidden circularly ordered graphs can be translated to a description by finitely many forbidden linearly ordered graphs. Nevertheless, our observations underscore the fact that in many cases the circular order descriptions are nicer and more natural.

1 Introduction

We follow [1] for terminology and notation not defined here, and we consider simple finite graphs; when needed, we will work with loopless oriented graphs.

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as well. A hereditary property is a class of graphs $\mathcal{P}$ such that whenever $G \in \mathcal{P}$ and $H$ is an induced subgraph of $G$, then $H \in \mathcal{P}$. A minimal obstruction to a hereditary property $\mathcal{P}$ is a graph $G$ that does not belong to $\mathcal{P}$ but every proper induced subgraph does. A natural way to characterize or define a hereditary property is by exhibiting its set of minimal obstructions. For instance, bipartite graphs are characterized as those graphs with no induced odd cycles, while the class of evenhole-free graphs is defined as the class of graphs that contain no even cycle as an induced subgraph. Unfortunately exhibiting the set of minimal obstructions might be a highly complex task; as of today, the sets of minimal obstructions to the class of $k$-colourable graphs is unknown for every positive integer $k$ greater than 2.

A linearly ordered graph $(G, \leq)$ is a graph $G$ together with a linear ordering $\leq$ of its vertex set. Given two linearly ordered graphs, $(G, \leq_G)$ and $(H, \leq_H)$, we say that $(H, \leq_H)$ is a linearly ordered subgraph of $(G, \leq_G)$ if $H$ is a subgraph of $G$, and $\leq_H$ is the restriction of $\leq_G$ to $V(H)$; if additionally $H$ is an induced subgraph of $G$, we say that $(H, \leq_H)$ is an induced linearly ordered subgraph of $(G, \leq_G)$. Consider a set of linearly ordered graphs $F$. An $F$-free linear ordering of a graph $G$ is a linear ordering $\leq$ of $V(G)$ such that none of the linearly ordered graphs in $F$ is an induced linearly ordered subgraph of $(G, \leq)$. Given a linear order $\leq$ of some set $X$, we define the dual $\leq^*$ of $\leq$, by letting $x \leq^* y$ if and only if $y \leq x$.

In 1990 Damaschke [3] proposed to study characterizations of hereditary properties $\mathcal{P}$ by exhibiting a finite set of linearly ordered graphs $F$ such that $\mathcal{P}$ is the class of graphs that admit an $F$-free linear ordering. He observed that, for instance, chordal graphs, bipartite graphs and interval graphs are characterized by a forbidden set of linearly ordered graphs on three vertices; also in [3] he asked if the class of circular-arc graphs can be described by finitely many forbidden linearly ordered graphs. We will see that we can reinterpret a (known) characterization of circular-arc graphs in our context to obtain a positive answer to Damaschke’s question.

Around 2014, Hell, Mohar and Rafiey [7] showed that for every set $F$ of linearly ordered graphs on three vertices, the class of graphs that admit an $F$-free linear ordering can be recognized in polynomial time. Recently, Habib and Feuilloley published a thorough survey [4] on the subject, where they characterized all hereditary properties defined by forbidden linear ordering on three vertices. Moreover, they showed that all of these classes (except for two of them) can be recognized in linear time. In their work, Habib and Feuilloley, stated that an obvious next step is to study graph properties described by forbidden linear orderings on more vertices. All of our results can be translated to this context.

In this work we start the study of circularly ordered graphs, in an attempt to obtain a development parallel to the one described in the above paragraphs for linearly ordered graphs. We also present an interesting result relating strict upper bounds on the circular chromatic number of graphs to certain forbidden circular orderings.

This work is structured as follows. For the rest of this section we mention
some definitions and results on circular chromatic number of graphs that we will use in Section 4. In Section 2 we introduce basic definitions and notation to work with circularly ordered graphs, and we observe that circular-arc graphs and outerplanar graphs can be described by finitely many forbidden circularly ordered graphs. In Section 3 we study how forbidden circular orderings and forbidden linear orderings are related; moreover, we exhibit some properties expressible by finitely many forbidden circularly ordered graphs. In Section 4 we explore a nice relation between forbidden circularly ordered graphs and integer circular chromatic numbers. Finally, in Section 5 we discuss some computational aspects of finding admissible circular orderings of a given input graph.

Recall that a homomorphism between a pair of relational structures, $G$ and $H$, is a function $\varphi: V(G) \to V(H)$ that preserves all relations. If such a function exists we write $G \to H$. In particular, homomorphisms between graphs are functions that preserve adjacencies, so a graph $G$ is $k$-colourable if and only if $G \to K_k$. (We will later work with homomorphisms of linearly and circularly ordered graphs as well.) A bijective homomorphism such that its inverse is also a homomorphism is an isomorphism.

Given a pair of positive integers $p$ and $q$, $q \leq p$, the rational complete graph $K_{p/q}$ has vertices $\{0, 1, \ldots, p-1\}$ and there is an edge $ij$ if and only if the circular distance between $i$ and $j$ is at least $q$. In particular, if $p < 2q$ then $K_{p/q}$ is the empty graph on $p$ vertices, and $K_{p/1}$ is the complete graph on $p$ vertices. These graphs have a nice monotonic property with respect to the natural ordering of rational numbers and graph homomorphisms.

**Theorem 1.** [8] Consider a pair of positive integers $p$ and $q$ such that $p/q \geq 2$. Then $K_{p/q} \to K_{p'/q'}$ if and only if $p/q \leq p'/q'$.

A graph $G$ is $(p/q)$-colourable if $G \to K_{p/q}$. The circular chromatic number of a graph $G$, denoted by $\chi_c(G)$, is defined as

$$\chi_c(G) = \inf \{p/q: p \leq n, G \to K_{p/q} \}$$

where $n = |V(G)|$. It turns out that this infimum is always reached.

**Proposition 2.** [8] For a graph $G$ on $n$ vertices, we have

$$\chi_c(G) = \min \{p/q: p \leq n, G \to K_{p/q} \}.$$

As a nice consequence of these results, for every graph $G$ the inequalities $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ hold.

There are several interpretations of the circular chromatic number of graph, the following one will be useful for our work. Before stating it, recall that given a (possibly closed) walk $W = v_1v_2\cdots v_n$ in an oriented graph $G'$, an arc $(v_i, v_{i+1})$ is a forward arc of $W$ while an arc $(v_{i+1}, v_i)$ is a backward arc of $W$. We denote by $W^+$ ($W^-$) the set of forward (backward) arcs of $W$.

**Theorem 3.** [8] If $G$ is a forest, then $\chi_c(G) = 2$. Otherwise, $\chi_c(G)$ is the minimum over all acyclic orientations $G'$ of $G$, of the maximum, over all cycles $C$ of $G$, of

$$1 + \frac{|C^+|}{|C^-|}.$$
2 Circularly ordered graphs

A circular ordering of a set $X$ is a ternary relation $C \subseteq X^3$ such that for any four elements $x, y, z, w \in X$ the following statements hold:

- if $(x, y, z) \in C$ then $(y, z, x) \in C$,
- if $(x, y, z) \in C$ then $(x, z, y) \notin C$,
- if $(x, y, z) \in C$ and $(x, z, w) \in C$, then $(x, y, w) \in C$, and
- either $(x, y, z) \in C$ or $(x, z, y) \in C$.

A circularly ordered graph $G$ is an ordered pair $G = (U_G, C)$, where $U_G$ is a graph (the underlying graph of $G$) and $C$ is a circular ordering of $V(U_G)$. We will often abuse nomenclature and say that a circularly ordered graph $(G, C)$ is a circular ordering of the graph $G$. Notice that each graph on two or three vertices defines a unique circularly ordered graph; in Figure 1 we depict all circularly ordered graphs on four vertices.

Since circularly ordered graphs are relational structures, we use the isomorphism definition for relational structures to compare circularly ordered graphs. In particular, the underlying graphs of two isomorphic circularly ordered graphs are isomorphic (via graph isomorphism). We say that $H$ is an (induced) circularly ordered subgraph of $G$ if $U_H$ is an (induced) subgraph of $U_G$ and $C_H$ is the restriction of $C_G$ to $V(U_H)$. We also say that $H$ is a spanning circularly ordered supergraph of $G$ if $G$ is a circularly ordered subgraph of $H$, and $V(U_H) = V(U_G)$. If a circularly ordered graph isomorphic to $H$ is an induced circularly ordered subgraph of $G$, we will say that $G$ contains (an induced copy of) $H$. For a set $F$ of circularly ordered subgraphs, we say that a circularly ordered graph $G$ avoids $F$, or is $F$-free, if $G$ does not contain any of the circularly ordered subgraphs in $F$; if $F = \{F\}$, we will abuse notation and say that $G$ avoids $F$ or is $F$-free. A graph $G$ admits an $F$-free circular ordering if there exists an $F$-free circularly ordered graph $(G, C)$.

Rather than representing a circular ordering by the ternary relation itself, we will choose two simpler representations. Given a linear order $\leq$ of a set $X$, the circular closure of $\leq$ is a circular ordering $c(\leq)$ defined as follows. For every $x, y$ and $z$ in $X$ such that $x \leq y \leq z$, we have $(x, y, z) \in c(\leq)$, and then, take the cyclic closure of these triples, i.e., $(x, y, z) \in c(\leq)$ if either $(y, z, x) \in c(\leq)$ or $(z, x, y) \in c(\leq)$. Conversely, consider a circular ordering $C$ and any element $x \in X$. Define $\leq_x$ as $x \leq_x y$ for any $y \in X$, and $y \leq_x z$ if $(x, y, z) \in C$. It is not hard to observe that $c(\leq_x) = C$ for any $x \in X$. So we can always describe a circular ordering of $X$ as the circular closure of a linear ordering on $X$.

**Remark 4.** Consider a pair of linearly ordered graphs $(G, \leq_G)$ and $(H, \leq_H)$, and let $(G, C_G)$ and $(H, C_H)$ be a pair of circularly ordered graphs. Then, the following statements hold:

- if $(H, \leq_H)$ is a linearly ordered subgraph of $(G, \leq_G)$, then $(H, c(\leq_H))$ is a circularly ordered subgraph of $(G, c(\leq_G))$, and
• if \((H, C_H)\) is a circularly ordered subgraph of \((G, C_G)\) then, for any \(u \in V_G\) there is a vertex \(v \in H\) such that \((H, \leq_u)\) is a linearly ordered subgraph of \((G, \leq)\).

Moreover, analogous statements when induced linearly ordered subgraphs and induced circularly subgraphs are considered also hold.

Consider the unit circle \(S^1 \subseteq \mathbb{R}^2\) and a finite set \(X\). Let \(f : X \to S^1\) be an injective function. Consider the ternary relation \(C(f)\) on \(X\) defined by the ordered triples \((x, y, z)\) such that when traversing \(S^1\) in a clockwise direction starting in \(f(x)\) we see \(f(y)\) before \(f(z)\). It is not hard to convince ourselves that \(C(f)\) is a circular ordering of \(X\). Conversely, let \(C_X\) be any circular ordering on \(X\), choose \(x_1\) in \(X\) so we have \(C_X = c(\leq_{x_1})\), where \(\leq_{x_1}\) is the linear order \(x_1 \leq x_2 \leq x_3 \cdots \leq x_n\). Define the function \(f : X \to S^1\) by 
\[
f(x_k) = (\cos \frac{2\pi k}{n}, -\sin \frac{2\pi k}{n}).
\]
Clearly then, \(f : X \to S^1\) is an injective function of \(X\) into the unit circle and \(C(f) = C_X\). This representation is specially useful when picturing a circular ordering. Due to the arguments in these paragraphs we will refer to circular orderings as circular arrangements as well. Similarly, we will use the verb arrange to mean that we are constructing or defining a circular ordering for a set (usually the vertex set of a graph).

Note that there are two natural operations on circularly ordered graphs. Let \((G, c(\leq))\) be a circularly ordered graph. The complement \((\overline{G}, c(\leq))\) of \((G, c(\leq))\) is obtained by taking the complement of \(G\) and respecting the circular order of the vertices, i.e., \((\overline{G}, c(\leq)) = (\overline{G}, c(\leq))\). The reflection or dual \((G, c(\leq))^*\) of \((G, c(\leq))\) is obtained by considering the dual \(\leq^*\) of the linear order \(\leq\) and taking its circular closure, i.e., \((G, c(\leq))^* = (\overline{G}, c(\leq^*))\). The latter operation can be interpreted geometrically as follows. Consider the reflection \(r : S^1 \to S^1\) over the \(y\)-axis and let \((G, c(\leq))\) be a circularly ordered graph. If \(f : V(G) \to S^1\) is an embedding such that \(c(\leq)\) is recovered by traversing \(S^1\) in a clockwise motion, then the circular ordering of \(V(G)\) in \((G, c(\leq))^*\) is recovered by the embedding \(r \circ f : V(G) \to S^1\) and traversing \(S^1\) in a clockwise motion; equivalently, it is recovered by the embedding \(f : V(G) \to S^1\) and traversing the circle in an anti-clockwise motion.

For a positive integer \(k\), a simple \(k\)-path, \(SP_k\), is the \(k\)-path \(v_1 \cdots v_k\) together with the circular ordering obtained from the circular closure \(v_1 \leq \cdots \leq v_k\). Analogously, if \(k \geq 3\) a simple \(k\)-cycle, \(SC_k\), is the \(k\)-cycle \(C_k = v_1 \cdots v_k v_1\) together with the circular ordering obtained from the circular closure of \(v_1 \leq \cdots \leq v_k\). The simple path \(SP_3\) and the simple cycle \(SC_4\) are depicted in Figure 1 labelled \((m)\) and \((g)\), respectively. Consider now the five cycle \(C_5 = v_1 \cdots v_5 v_1\) and define \(C_5\)-star as the circularly ordered graph \((C_5, c(v_1 \leq v_4 \leq v_2 \leq v_5 \leq v_3))\). Note that the complement of \(C_5\)-star is \(SC_5\), and the dual of a simple path or a simple cycle is a simple path or a simple cycle, respectively.

To use a technique analogous to the one used in [4] for depicting families of linearly ordered graphs, we introduce circularly ordered patterns. A pattern consists of a set \(V\) together with a set of edges \(E\) and a set of non-edges \(NE\) with the restriction that \(NE \cap E = \emptyset\). A pattern \((V, E, NE)\) represents all graphs \((V(G), E(G))\) such that \(V(G) = V\) and \(E \subseteq E(G)\) but \(E(G) \cap NE = \emptyset\).
Figure 1: All circularly ordered graphs on 4 vertices. In all cases, the circular order is the circular closure of \( v_1 \preceq v_2 \preceq v_3 \preceq v_4 \).

So a circularly ordered pattern \((G, c(\leq))\) consists of a pattern \(G\) together with a circular ordering of its vertices, and it represents all circularly ordered graphs obtained by a graph represented by \(G\) and ordering its vertices by \(c(\leq)\). Given a set \(\mathcal{P}\) of patterns, we say that \(\mathcal{P}\) generates all the graphs represented by some pattern in \(\mathcal{P}\). When depicting a pattern we will use straight lines for edges and dashed lines for no edges. For instance, in Figure 2 we depict a single
circularly ordered pattern and the family of circularly ordered graphs that it represents. Finally, we say that a circularly ordered graph \((G, c(\leq))\) avoids a circularly ordered pattern \((H, c(\leq'))\) if \((G, c(\leq))\) avoids every circularly ordered graph represented by \((H, c(\leq'))\).

As a consequence of a result due to Tucker [10], we describe a circularly ordered pattern \(CA\) (top of Figure 2), such that the family of circular-arc graphs is the class of graphs that admit a \(CA\)-free circular ordering.

**Theorem 5.** [10] If \(G\) is a graph, then \(G\) is a circular-arc graph if and only if the vertices of \(G\) can be arranged in a circular ordering \(v_1, \ldots, v_n\) such that, for \(i < j\), if \(v_iv_j \in E(G)\) then either \(v_{i+1}, \ldots, v_j \in N(v_i)\) or \(v_{j+1}, \ldots, v_i \in N(v_j)\).

Denote by \(CP\) the property of circularly ordered graphs described in Theorem 5.
Proposition 6. A graph $G$ admits a CA-free circular ordering if and only if it is a circular-arc graph.

Proof. Note that a circular ordering $c(\leq)$ of $V(G)$ fails to satisfy $CP$ if and only if there are four vertices $v_i \leq v_k \leq v_j \leq v_l$ such that $v_iv_j \in E(G)$, and $v_iv_k, v_jv_l \notin E(G)$. Thus, $(G,c(\leq))$ satisfies $CP$ if and only if every of its induced circularly ordered subgraphs on four vertices satisfy $CP$. The statement of this proposition follows since the family represented by $CA$ corresponds to those circularly ordered graphs on four vertices that do not satisfy $CP$.

We denote by $cr$ the circularly ordered graph labelled $(f)$ in Figure 2 and by $CR$ the set of circularly ordered spanning supergraphs of $cr$.

Proposition 7. A graph $G$ is an outerplanar graph if and only if it admits a $CR$-free circular ordering.

Proof. Suppose that a graph $G$ admits a circular ordering $C_G$ of $V(G)$ that avoids $CR$. Represent the circular ordering $C_G$ by an injective function $f : V(G) \rightarrow S^1$. Consider the embedding of $G$ into $\mathbb{R}^2$ obtained from $f$ and representing every edge $xy$ by the segment joining $f(x)$ and $f(y)$. Since $(G,C_G)$ is $CR$-free, then the previously mentioned embedding has no crossing edges and thus is a planar embedding of $G$. Moreover, as all edges are represented by a line segment in the interior of $S^1$ and all vertices are represented by a point on $S^1$, then the embedding is an outerplanar embedding of $G$. Thus, $G$ is an outerplanar graph.

On the other hand, let $G$ be an outerplanar graph and $G'$ be an outerplanar embedding of the graph resulting of adding edges to $G$ until it is a biconnected outerplanar graph. If $C$ is a hamiltonian cycle of $G'$, then a circular ordering $C_G$ of $V(G)$ is obtained by traversing $C$ in a clockwise motion. The fact that $(G,C_G)$ is a $CR$-free circular ordering of $G$ follows from the definition of $CR$ and the fact that $G'$ is an outerplanar embedding of a supergraph of $G$.

3 Circular arrangements and linearly ordered patterns

As noted by Habib and Feuilloley [4], an obvious line of research in the context of forbidden linearly ordered graphs, is to study hereditary properties characterized by forbidden sets of linearly ordered graphs on four vertices or more. To this end, we notice that for any hereditary property described by a finite set of forbidden circularly ordered graphs, there is a set of linearly ordered graphs (with the same size of vertex sets) that describes the same property. Let $c$ be the function that maps a linearly ordered graph $(G, \leq)$ to the circularly ordered graph $(G, c(\leq))$, i.e., $c(G, \leq) = (G, c(\leq))$. The function $c$ can be naturally extended to take linearly ordered patterns as an argument if we think a linearly ordered pattern as the set of linearly ordered graphs that it represents. As the following observation shows, the inverse image of a set of circularly ordered graphs $F$ under $c$, directly relates the families of graphs admitting an $F$-free circular ordering and those
admitting an \((c^{-1}[F])\)-circular ordering. For this reason, it is convenient to define the “linearizing operator” \(L\) for a set of circularly ordered graphs \(F\) as \(L(F) = c^{-1}[F]\). Again, \(L\) can take a circularly ordered pattern as an argument if we think it as the set of circularly ordered graphs it represents.

**Observation 8.** Let \(F\) be a set of circularly ordered graphs and let \(P\) be the class of graphs that admit an \(F\)-free circular ordering. Then, \(P\) is the class of graphs that admit a \(L(F)\)-free linear ordering.

*Proof.* Recall that every circular ordering can be described as the circular closure of some linear ordering. So let \((G, c(\leq))\) be an \(F\)-free circular ordering of a graph \(G\). Then, \((G, \leq)\) is a \(L(F)\)-free linear ordering of \(G\). Conversely, if \((G, \leq)\) is a \(L(F)\)-free linear ordering of \(G\) then \((G, c(\leq))\) is an \(F\)-free circular ordering of \(G\).

In particular, since we already showed that outerplanar graphs can be naturally described by forbidden circularly ordered graphs (Proposition 7) by Observation 8 we recover an observation mentioned in [4] that states that there is a finite set of linearly ordered patterns that characterizes outerplanar graphs. The class of circular-arc graphs is also described by finitely many forbidden circularly ordered graphs (Proposition 6) so there is a set of linearly ordered patterns on four vertices \(F_C\) such that the class of graphs that admit an \(F_C\)-free linear ordering is the class of circular-arc graphs. This remark positively answers a question posed by Damaschke: is there a finite set of linearly ordered graphs that describes the class of circular-arc graphs? [3]. To precisely determine \(F_C\), let \(L(CA)\) be the set of ordered graphs \((G, \leq)\) such that \((G, c(\leq))\) is represented by the circularly ordered pattern \(CA\). We depict a pair of linearly ordered patterns that generate \(L(CA)\) in Figure 3.

![Figure 3: A pair of linearly ordered patterns that generate \(L(CA)\).](image)

**Proposition 9.** A graph \(G\) admits a \(L(CA)\)-free linear ordering if and only if \(G\) is a circular-arc graph.

Note that in this case, describing the class of circular-arc graphs by forbidden circular arrangements yields a simpler set of forbidden structures than describing them by forbidden linearly ordered patterns.

Observation 8 gives us the opportunity to propose what we think is a really interesting question: Is its “converse” true? This is, given a finite set of linearly ordered patterns, \(F\), is there a finite set of circular ordered graphs \(F'\) such that the class of graphs that admit an \(F\)-free linear ordering is precisely the class
of graphs that admit an $F'$-free circular ordering? We do not have an example where it does not hold, but the existence of one seems to be likely, so it would be interesting to see one. In contrast, it is not hard to find examples of some nice classes where the converse holds, we now present a handful.

Linear forests, caterpillar forests and forests are examples of graph classes characterized by a set of forbidden linearly ordered patterns on three vertices [4].

Let $LF$ be the set of circularly ordered graphs that consists of the simple triangle, both circular orderings of $C_4$, the simple $P_4$, the crossed $P_4$, and the unique circular ordering of the claw. We illustrate these graphs in Figure 4.

Figure 4: All circularly ordered graphs in $LF$. If we do not use the last graph (the claw), the resulting family is $CF$.

It is convenient to define the following class of circularly ordered paths. First note that there are four possible non-isomorphic circular orderings of $P_4$, namely $SP_4$ (Figure 1.m), the crossed $P_4$ (Figure 1.n) and two more which we will denote by $Z$ (Figure 1.o) and $Z^*$ (Figure 1.p). Note that the dual of $Z$ is $Z^*$ (which justifies our choice of notation). Given a positive integer $k$ greater than 3, a $k$-zigzag is a circular ordering of the $k$-path $P$ such that every induced copy of $P_4$ in $P$ is ordered as $Z$ or $Z^*$. In particular, $Z$ and $Z^*$, are the unique 4-zigzags. Finally, we say that a circularly ordered graph $G' = (G, C)$ has a pair of crossing edges if there is a pair of edges $v_1v_3$ and $v_2v_4$ of $G$, such that $(v_1, v_2, v_3) \in C$ and $(v_3, v_4, v_1) \in C$; otherwise, we say that $G'$ has no crossing edges. For instance, the crossed $P_4$ is a circular ordering of $P_4$ with crossing edges, while the other three circular orderings of $P_4$ have no crossing edges.

**Observation 10.** Let $k$ be a positive integer, $k \geq 4$. A circular arrangement $P_k'$ of $P_k$ is $LF$-free if and only if it is a $k$-zigzag.
Proof. Since there are exactly four circular orderings of $P_4$, and two of them, namely the simple $P_4$ and the crossed $P_4$ are members of $LF$, then the desired result follows directly from the definition of a $k$-zigzag.

This simple observation yields the following statement.

**Proposition 11.** A graph $G$ is a linear forest if and only if it admits an $LF$-free circular ordering.

Proof. One implication follows from the fact that for every positive integer $k$, a $k$-zigzag is an $LF$-free circular ordering of $P_k$. To prove the converse implication first note that if a graph $G$ has a vertex of degree at least 3, then $G$ contains either a claw or a triangle. Since the unique circular ordering of both of these graphs belongs to $LF$, if a graph $G$ admits an $LF$-free circular ordering then $G$ is a disjoint union of paths and cycles with no triangles. Again, as both circular orderings of $C_4$ belong to $LF$, any graph that admits an $LF$-free ordering is $C_4$-free. Now we show that for every positive integer $k$, $k \geq 5$, the $k$-cycle, $C_k = v_1 \cdots v_k v_1$, does not admit an $LF$-free circular ordering. By Observation 10 if $C_k$ admits an $LF$-free circular ordering, $C'_k$, then the induced path $v_1 \cdots v_{k-1}$ must be arranged as a $k$-zigzag. But then, wherever $v_k$ is placed in the circular ordering it forces $C'_k$ to have an induced copy of $P_4$ with crossing edges, contradicting the fact that $C'_k$ is an $LF$-free ordering.

Let $CF$ be the set obtained from $LF$ by removing the claw (see Figure 4) and let $T_2$ be the graph obtained from the claw by subdividing every edge. Recall that a graph $G$ is a caterpillar if and only if it is a $T_2$-free tree.

**Proposition 12.** A graph $G$ is a caterpillar forest if and only if it admits a $CF$-free circular ordering.

Proof. First note that every caterpillar forest is an induced subgraph of a caterpillar, thus it suffices to observe that every caterpillar admits a $CF$-free circular ordering. We order the largest dominating path, $P = v_1 \cdots v_k$, as a $k$-zigzag. Note that for every $j \in \{2, \cdots, k-1\}$ one of the circular segments delimited by $v_{j-1}$ and $v_{j+1}$ contains no vertices of $P$. We place the leaves adjacent to $v_j$ in this circular segment. It is not hard to observe that this circular ordering of a caterpillar if $CF$-free. On the contrary if $G$ is not a caterpillar forest then it must contain a cycle or a $T_2$. With the same arguments as in the proof of Proposition 11 one can notice that no cycle admits a $CF$-free circular ordering. It is also not hard to observe that $T_2$ does not admit a $CF$-free circular ordering, which concludes the proof.

Now we show that forests can be characterized by a finite set of forbidden circularly ordered graphs. Let $F$ be the set of all circular patterns depicted in Figure 5. In particular, every $F$-free circular ordered graph must avoid crossing edges. Thus, if we were to obtain an $F$-free circular ordering of a cycle, we should order its vertices cyclicly, but then we would obtain either one of the cycles in $F$ or the simple $P_5$. Hence, no cycle admits an $F$-free circular ordering. We will show that every forest does admit an $F$-free circular ordering.
Figure 5: All circularly ordered graphs in $F$.

**Theorem 13.** A graph $G$ admits an $F$-free circular ordering if and only if $G$ is a forest.

**Proof.** We have already shown that if $G$ admits an $F$-free circular ordering, then $G$ is an acyclic graph. To observe that every forest admits such a circular ordering it suffices to note that every tree does. Indeed, every forest is an induced subgraph of some tree and the class of graphs that admit an $F$-free circular ordering is a hereditary class of graphs.

Given a pair of vertices, $x$ and $y$, whenever we say we place $y$ "ahead of" ("behind") $x$, we think we are traversing the circle in a clockwise motion starting from $x$ and we place $y$ before seeing any other vertex (after seeing all other vertices).

Let $T$ be a tree. We will describe the circular ordering of $V(T)$ by arranging the vertices of $T$ around the circle and we will construct this arrangement recursively. Let $\{v_0, \ldots, v_{n-1}\}$ be an enumeration of the vertices of $T$ in such a way that if $i \leq j$ then $d(v_i, v_0) \leq d(v_j, v_0)$. In particular, the graph $T_k$ induced by $\{v_0, \ldots, v_k\}$ is a tree for every $k \in \{0, \ldots, n-1\}$ where $v_k$ is a leaf of $T_k$.

We first place the root $v_0$ anywhere in the circle. Suppose we have arranged $V(T_{k-1})$, now we arrange $V(T_k)$ by respecting the ordering of $V(T_{k-1})$ and simply including $v_k$ as follows. Let $a$ be the ancestor of $v_k$. If $a = v_0$ then incorporate $v_k$ behind $v_0$. On the other hand, let $b$ be the ancestor of $a$. If $b = v_0$ then include $v_k$ behind $a$. Finally, if $b \neq v_0$ let $c$ be the ancestor of $b$. There are two cases:

- when traversing the circle in a clockwise motion we see $(a, b, c)$, in this case we include $v_k$ ahead of $a$, or
- when traversing the circle in a clockwise motion we see $(a, c, b)$, in this case we include $v_k$ behind $a$. 

12
We illustrate this construction in Figure 6.

Let $T_c$ be the tree $T$ together with the previously constructed circular ordering of $V(T)$. Now we prove that $T_c$ is $\mathcal{F}$-free. Since $T$ is a tree, $T_c$ avoids every cycle in $\mathcal{F}$. Also, at every step of the recursive construction, we place the new vertex either ahead or behind its parent, so there are no crossing edges in $T_c$. Thus, it only remains to verify that $T_c$ contains no simple $P_5$.

First assume that $T_c$ contains a simple $P_4$, $P_4 = v_1v_2v_3v_4$. Let $i \in \{1, 2, 3, 4\}$ be the vertex of $P_4$ closest to $v_0$ in $T$ (this index is unique because otherwise there would be a cycle in $T$). Due to the recursive rule, it is not hard to notice that $i \in \{2, 3\}$. Thus, if $T_c$ contains a simple $P_5$, say $P_5 = w_1w_2w_3w_4w_5$, then $w_3$ must be the closest vertex in $P_5$ to $v_0$. So when we added $w_1$ to $T_c$, we included it behind its parent, $w_2$. Which means that $w_3 = v_0$ or when traversing the circle in a clockwise motion we see $(w_2, c, w_3)$ where $c$ is the parent of $w_3$. In both cases, when $w_5$ was included in the arrangement, it was added behind $w_4$. Then, it means that when we included $w_5$ in the recursion process $w_3$ was not included yet, but this contradicts the fact the $w_3$ is closer to $v_0$ and the choice of the order in which we process the vertices of $T$. Therefore, $T_c$ is $\mathcal{F}$-free.

Note that the recursive construction of the circular ordering exposed in the proof of Theorem 13 yields an algorithm to construct an $\mathcal{F}$-free circular ordering.
of a tree. This algorithm runs in polynomial time as we process every vertex only once, and every time we process a vertex we make a constant amount of operations.

The descriptions by forbidden circular arrangements of outerplanar graphs and circular-arc graphs proposed in this section are simpler (and somewhat more intuitive) than their descriptions by forbidden linearly ordered graphs. On the contrary, describing forests, linear forests and caterpillar forest by linearly ordered graphs yield simpler expressions (and proofs) than describing these classes by forbidden circularly ordered graphs. But this should be expected since these classes are characterized by forbidden linear patterns on three vertices. Every graph on three vertices has a unique circular ordering, thus, forbidding induced circularly ordered graphs on three vertices is equivalent to forbidding induced graphs on three vertices (without orderings), but none of these families can be characterized by forbidding induced subgraphs on three vertices. Nonetheless the statements of this section show that circularly ordered graphs can describe several natural graph classes. Moreover, these observations raise the question of whether for any finite set of linearly ordered patterns there is a finite set of (possibly larger) circular arrangements such that and describe the same classes by forbidden linearly ordered patterns and forbidden circular arrangements, respectively.

4 Circular chromatic number and circular orderings

In this section we study how certain forbidden circular orderings relate to the circular chromatic number of graphs. These forbidden orderings stem from the following characterization of -colourable graphs in terms of forbidden linear orderings.

Proposition 14. ([4], [7]) Let be a positive integer. A graph is -colourable if and only if there is a linear ordering of such that there are no vertices such that for every .

This result can be restated in terms of homomorphisms. Recall that for graphs and , we denote the existence of a homomorphism from to by ; we also denote by the fact that there is no homomorphism from to .

For a positive integer denote by the straight path on vertices, i.e., has vertex set with the natural ordering of their indices and with edge set . Now Proposition 14 can be restated as follows: a graph is -colourable if and only if there is a linear ordering of such that there is no homomorphism (of linearly order graphs) from to .

We are interested in proving an analogous version of this result for circular orderings. Instead of the straight path we consider the simple path .
(The definition of $SP_k$ is given in Section 2 and the simple path $SP_4$ is depicted in Figure 1 (m).) For a positive integer $k$, $k \geq 2$, we denote by $C_k$ the class of graphs $G$ that admit a circular ordering $c(\leq)$ such that $SP_k \nRightarrow (G, c(\leq))$ (as circularly ordered graphs). We proceed to characterize these classes in terms of the circular chromatic number, and we begin with the following observation.

**Observation 15.** For any positive integer $k$, $k \geq 2$, the class $C_k$ is closed under homomorphic pre-images. That is, if a graph $G$ belongs to $C_k$, then for any graph $H$ such that $H \rightarrow G$ we have that $H \in C_k$.

**Proof.** Let $\varphi : H \rightarrow G$ be a homomorphism. It suffices to order vertices if $H$ in any way such that for every $x \in V(G)$ the vertices of $H$ in $\varphi^{-1}(x)$ are contiguous in the circular ordering.

It is not hard to observe that for every positive integer $k$, $k \geq 2$, there is a finite set $H_k$ such that a graph belongs to $C_k$ if and only if it admits an $H_k$-free circular ordering. Indeed, $H_k$ can be constructed by first considering the family of all circularly ordered graphs that are homomorphic images of $SP_k$, then obtaining $H_k$ as the antichain of minimal circularly ordered graphs (with respect to the order of induced circularly ordered graphs) in this family. For instance, $H_4$ consists of the triangle, the simple $C_4$ and $SP_4$. These circularly ordered graphs are depicted in Figure 7.

![Figure 7: An illustration of the circularly ordered graphs in $H_4$.](image-url)

We proceed to show that for any positive integer $k$, $k \geq 2$, a graph $G$ with $\chi_c(G) < k$ must satisfy $G \in C_{k+1}$. In fact, we can immediately show that when $k = 2$, this condition is not only sufficient, but also necessary.

**Observation 16.** A graph $G$ belongs to $C_3$ if and only if $\chi_c(G) < 2$.

**Proof.** On one hand, $SP_3$ maps homomorphically to the unique circular ordering of $K_2$. So $G \in C_3$ if and only if $G$ has no edges. On the other hand, if $r/q < 2$ then $K_{r/q}$ is an edgeless graph. So by Proposition 2, $\chi_c(G) < 2$ if and only if $G$ had no edges.

Now, for every positive integer $k$, $k \geq 3$, we construct a sequence of graphs $\{H_n^k\}_{n \geq 1}$ such that $\chi_c(H_n^k) < k$ and $H_n^k \in C_{k+1}$ for every $n \geq 1$. Consider a pair of positive integers $n$ and $k$, $k \geq 3$, the graph $H_n^k$ is defined as the rational complete graph $K_{(kn - 1)/n}$. In particular, $H_1^k \cong K_{k-1}$ and $H_2^k \cong \overline{C}_{2k-1}$. In
Figure 8 we depict $H_2^3$ and $H_3^3$. Note that $H_3^3 \cong M_8$, where $M_8$ is the Möbius ladder on eight vertices.

Figure 8: The graphs $H_2^3$ (left) and $H_3^3$ (right).

**Lemma 17.** Let $G$ be a graph and $k$ a positive integer, $k \geq 3$. If $\chi_c(G) < k$ then there is a positive integer $m$ such that $G \rightarrow H_m^k$.

**Proof.** It is not hard to observe that \( \{\frac{kn-1}{n}\}_{n \in \mathbb{Z}^+} \) is an increasing sequence that converges to $k$. So for any rational number $r$ such that $r < k$, there is a positive integer $m$ such that $r \leq \frac{km-1}{m}$. Consider a graph $G$ such that $\chi_c(G) < k$. By Proposition 2 there is a rational number $p/q$ such that $G \rightarrow K_{p/q}$ and $p/q < k$. Let $m$ be a positive integer such that $p/q \leq \frac{(km-1)/m}{m}$, which concludes the proof since $G \rightarrow K_{p/q}$ and $H_m^k = K_{(km-1)/m}$.

**Proposition 18.** Let $G$ be a graph and $k$ a positive integer, $k \geq 3$. If $\chi_c(G) < k$ then $G \in C_{k+1}$.

**Proof.** By Lemma 17 if the circular chromatic number of a graph $G$ is strictly less than $k$, then there is a positive integer $m$ such that $G \rightarrow H_m^k$. Also, recall that, by Observation 16 for every positive integer $k$ the class of graphs $C_{k+1}$ is closed under homomorphic preimages. Hence, if $H_m^k \in C_{k+1}$ for every $m \geq 1$, then any graph with circular chromatic number strictly less than $k$ belongs to $C_k$. So we proceed to prove that for every positive integer $m$ the graph $H_m^k$ belongs to $C_{k+1}$. To do so, we consider the canonical circular ordering $c(\leq)$ of the vertices of $H_m^k$, i.e., the circular closure of $0 \leq 1 \leq \cdots \leq km - 2$. We want to prove that $SP_{k+1} \not\rightarrow (H_m^k, c(\leq))$; we proceed by contradiction.

Suppose there is a homomorphism $\varphi: SP_{k+1} \rightarrow (H_m^k, c(\leq))$. Since $(H_m^k, c(\leq))$ is a vertex-transitive circularly ordered graph, we can assume that $\varphi(v_1) = 0$. Let $u_i$ be the image of $v_i$ and note that the only indices for which $u_i$ might be equal to $u_j$ are $i = 1$ and $j = k + 1$; the remaining pairs of vertices $u_i$ and $u_j$ must be different if $i \neq j$. So there are $k$ different vertices $0 = u_1 < u_2 < \cdots < u_k < km - 1$ such that $u_iu_{i+1} \in E(H_m^k)$ for every $i \in \{1, \ldots, k-1\}$, and a
vertex \( u_{k+1} \in \{u_k + 1, u_k + 2, \ldots, km - 2, 0\} \) such that \( u_k u_{k+1} \in E(G) \). The existence of \( u_{k+1} \) will yield the contradiction. Recall that \( H_m^k = K_{(km - 1)/m} \), so there is an edge \( rs \in E(H_m^k) \) if and only if the circular distance between \( r \) and \( s \) is at least \( m \). Since \( 0 = u_1 < \cdots < u_k \) is an increasing sequence in \( \{0, 1, \ldots, km - 1\} \), then \( u_{i+1} - u_i \geq m \), so \( u_k \geq (k - 1)m \). Therefore, since \( u_{k+1} \in \{u_k + 1, u_k + 2, \ldots, km - 2, 0\} \), then the circular distance between \( u_k \) and \( u_{k+1} \) is at most \( m \), which is \( km - 1 - (k - 1)m \) which equals \( m - 1 \). Thus, \( u_k \) and \( u_{k+1} \) are not adjacent vertices, which contradicts the fact that \( v_k v_{k+1} \in E(SP_{k+1}) \) and \( \varphi(v_k) = u_k \) and \( \varphi(v_{k+1}) = u_{k+1} \). Hence, \( SP_{k+1} \not\rightarrow (H_m^k, c(\leq)) \), so \( H_m^k \in C_{k+1} \).

For every positive integer \( k \), \( k \geq 4 \), we construct a set of linearly ordered graphs, \( \mathcal{PH}_k \) as follows. The straight cycle on \( k \) vertices \( StC_k \), consists of the \( k \)-cycle, \( v_1 \cdots v_k v_1 \), where \( v_i \preceq v_j \) if and only if \( i \preceq j \). The shifted straight path on \( k \) vertices, \( sSt_k \), consists of the path on \( k \) vertices, \( v_1 \cdots v_k \), where \( v_k \preceq v_i \) for every \( i \in \{1, \ldots, k\} \), and \( v_i \preceq v_j \) for every \( i, j \in \{1, \ldots, k - 1\} \). Define \( \mathcal{PH}_k \) as the set generated by all linearly ordered spanning supergraphs of \( \{St_k, sSt_k, StC_k, StC_{k-1}\} \). In Figure 9 we depict these four generating linearly ordered graphs.

![Figure 9: The four generating linearly ordered graphs of \( \mathcal{PH}_k \).](image)

**Proposition 19.** Let \( G \) be a graph. If \( G \) admits an \( H_k \)-free circular ordering then \( G \) admits a \( \mathcal{PH}_k \)-free linear ordering.

**Proof.** Recall that \( L \) is the function that “linearizes” a set of circularly ordered graphs. Observation \( \Box \) asserts that if \( F \) is a set of circularly ordered graphs that describes a property by forbidden circularly ordered graphs, then \( L(F) \) describes the same property by forbidden linearly ordered graphs. So the statement of this proposition follows by this observation and the fact that \( \mathcal{PH}_k \subseteq L(H_k) \). \( \Box \)

The following statement is a simple technical lemma that will be useful to prove our main result of this section.

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17
Lemma 20. Let $k$ be a positive integer, $k \geq 4$, let $G'$ be an acyclic oriented graph with no directed path on $k$ arcs, and let $W$ be an oriented path in $G'$. The following two assertions hold:

1. If $|A(W)| = kn$ for some positive integer $n$ then, $|W^+| \leq (k - 1)n$ and $|W^-| \geq n$.

2. If the final arc of $W$ is a backward arc, then there is a positive integer $n$ such that $|W^+| \leq (k - 1)n$ and $|W^-| \geq n$ (regardless of $|A(W)|$).

Proof. The first statement follows easily by induction on $n$. To prove the second statement we will assume that $|A(W)| = km + l$ for some integer $l$, $0 \leq l \leq k - 1$. If $l = 0$ the second statement is a particular case of the first one. Suppose $1 \leq l \leq k - 1$ and let $W'$ be the subpath of $W$ obtained by removing the final $l$ vertices. Clearly, $|A(W')| = km$ and so by the first statement $|W'^+| \leq (k - 1)m$ and $|W'^-| \geq m$. Since the final arc of $W'$ is a backward arc, then $|W^-| \geq m + 1$ and $|W'^+| \leq (k - 1)m + l \leq (k - 1)m + k - 1 \leq (k - 1)(m + 1)$. By letting $n = m + 1$ the statement follows.

Theorem 21. For any graph $G$ and a positive integer $k$, $k \geq 3$, the following statements are equivalent:

- $G$ admits a circular ordering $c(\leq)$ such that $SP_{k+1} \not\rightarrow (G,c(\leq))$,
- $G$ admits a $\mathcal{PH}_{k+1}$-free linear ordering, and
- the circular chromatic number of $G$ is strictly less than $k$.

Proof. Proposition 19 shows that the first item implies the second one, while Proposition 18 asserts that the third one implies the first item. We now prove that the second statement implies the third one. To do so, let $(G, c(\leq))$ be a $\mathcal{PH}_{k+1}$-free linear ordering of a graph $G$, and consider the orientation $G'$ of $G$ obtained by orienting every edge $xy \in E(G)$ from $x$ to $y$ if $x \leq y$. This is clearly an acyclic orientation of $G$. We will show that for every cycle $C$ of $G$ the strict inequality $\frac{|C^+|}{|C^-|} < k - 1$ holds, and thus, by Theorem 3 we conclude that $\chi_c(G) \leq 1 + \frac{|C^+|}{|C^-|} < k$.

Since the straight path on $k + 1$ vertices belongs to $\mathcal{PH}_{k+1}$, then $G'$ has no directed path on $k$ arcs. Let $C = c_1 \cdots c_mc_1$ be a cycle of $G$ and without loss of generality assume that $c_1$ is the first vertex of $C$ with respect to $\leq$. Moreover, we will assume that $m \geq k - 1$; otherwise it is immediate that $\frac{|C^+|}{|C^-|} < k - 1$. We begin by first considering the case when $c_m \leq c_{m-1}$. In this case, $(c_m, c_{m-1}) \in A(G')$ and so the oriented path $W = c_1Cc_m$ ends with a backward arc. Since $G'$ has no directed path on $k$ arcs, by Lemma 20 there is an integer $l$ such that $|W^+| \leq (k - 1)l$ and $|W^-| \geq l$. The only remaining arc in $C$ that does not belong to $W$ is the arc $(c_1, c_m)$ and it is a backward arc in the direction we are traversing $C$. Thus $|C^-| = |W^-| + 1 \geq l + 1$ and $|C^+| = |W^+| \leq (k - 1)l$, so $\frac{|C^+|}{|C^-|} \leq \frac{(k - 1)l}{l + 1} < k - 1$. Now suppose that $c_{m-1} < c_m$. Let $s$ be the maximum integer such that $c_{m-i} \leq c_{m+i+1}$ for every $i \in \{1, \ldots, s\}$. 

18
Claim 1. The strict inequality \( s < k - 1 \) holds.

Indeed, if \( s \geq k - 1 \) then we have the following structure: \( c_1 \leq c_{m-(k-1)} \leq c_{m-(k-2)} \leq \cdots \leq c_m \) where \( c_i c_{m-i} \in E(G) \) and \( c_{m-i} c_{m-i+1} \in E(G) \) for every \( i \in \{1, \ldots, k-1\} \). Regardless of whether \( c_1 = c_{m-(k-1)} \) or \( c_1 \neq c_{m-(k-1)} \), we can find either \( StC_k \) or \( sSt_{k+1} \) as a linearly ordered subgraph of \((G, \leq)\) which contradicts the fact that \((G, \leq)\) is \( \mathcal{PH}_{k+1}-\text{free} \). This concludes the proof of Claim 1.

By definition of \( s \), we know that \( c_{m-s} \leq c_{m-(s+1)} \) so \((c_{m-s}, c_{m-(s+1)}) \in A(G')\). Hence, \((c_{m-s}, c_{m-(s+1)})\) is a backward arc in the oriented path \( W = c_1 Cc_{m-s} \). Again, by Lemma [20] there is a positive integer \( l \) such that \(|W^+| \leq (k-1)l\) and \(|W^-| \geq l\). Thus, \(|C^+] = |W^+| + s \leq (k-1)l + s\) and \(|C^-| = |W^-| + 1 \geq l + 1\), so \(|C^+] - |C^-| \leq \frac{(k-1)l + s - 1}{l+1}\). By Claim [4] we know that \( s < k - 1 \) and therefore \(|C^+] - |C^-| < \frac{(k-1)l + k - 1}{l+1} = k - 1\). This shows that the orientation \( G'\) satisfies that for every cycle \( C \) of \( G \) the strict inequality \( \frac{|C^+] - |C^-|}{C} < k - 1 \) holds. So by Theorem [3] we conclude that \( \chi_c(G) \leq 1 + \frac{|C^+] - |C^-|}{C} < k \).

Recall that \( \mathcal{H}_{k+1} \) is a finite set of circularly ordered graphs such that a graph belongs to \( \mathcal{C}_k \) if and only if \( G \) admits a \( \mathcal{H}_{k+1} \)-free circular ordering.

Corollary 22. Let \( k \) be a positive integer \( k, k \geq 2 \), and let \( G \) be a graph. Then, \( \chi_c(G) < k \) if and only if \( G \) admits an \( \mathcal{H}_{k+1} \)-free circular ordering.

5 Complexity issues

Now we look at the problem of determining whether an input graph admits an \( F \)-free circular ordering, where \( F \) is a fixed finite set of circularly ordered graphs. We call this problem the \( F \)-free circular ordering problem.

Recall that for any set of linearly ordered patterns on three vertices there is a polynomial time algorithm that determines whether an input graph admits an \( F \)-free linear ordering of its vertices or not [7]. When it comes to circular orderings this observation is trivial since for any graph on three vertices \( G \) there is a unique circular ordering of \( G \). Thus, for any set \( F \) of circularly ordered graph on three vertices, the \( F \)-free circular ordering problem can be solved in polynomial time. What about forbidding larger circularly ordered graphs? The little evidence gathered at this point, suggest that if \( F \) is a set of circularly ordered graphs on four vertices, then the \( F \)-free circular ordering problem could be polynomial time solvable. Before looking at this case, we show that there is a set of circularly ordered graphs \( F \) on five vertices such that the \( F \)-free circular ordering problem is \( NP \)-complete. Thus, for any positive integer \( k, k \geq 5 \), there is a set \( F \) of circularly ordered graphs on \( k \) vertices such that the \( F \)-free circular ordering problem is \( NP \)-complete.

In [6] Hatami and Tusserkani consider the following decision problem. The input is a graph \( G \) together with its chromatic number \( k \), and one must decide if \( \chi_c(G) < k \). Their main result asserts that this problem is \( NP \)-hard. By
reading their proof one can notice that they actually show that this problem is \(NP\)-hard even when restricted to 4-chromatic graphs. For the sake of clarity we state this result as follows.

**Proposition 23.** [6] Given a 4-chromatic graph \(G\), the problem of determining if \(\chi_c(G) < 4\) is \(NP\)-hard.

Now note that the problem stated in Proposition 23 is also a particular case of the problem of determining if an arbitrary graph \(G\) satisfies \(\chi_c(G) < 4\). Since this problem belongs to \(NP\), then the following statement directly follows from Proposition 23.

**Theorem 24.** Given a graph \(G\), the problem of determining if \(\chi_c(G) < 4\) is \(NP\)-complete.

Recall that by Corollary 22, \(H_5\) is a set of circularly ordered graphs such that a graph \(G\) admits an \(H_5\)-free circular ordering if and only if \(\chi_c(G) < 4\). Thus as a consequence of this observation and Theorem 24 we obtain the following corollary.

**Corollary 25.** For every positive integer \(k, \ k \geq 5\), there is a set \(F\) of circularly ordered graphs on \(k\) vertices such that the \(F\)-free circular ordering problem is \(NP\)-complete.

**Proof.** By the arguments preceding this statement, \(H_5\) is a set of circular ordered graphs on five vertices such that the \(H_5\)-free circular ordering problem is \(NP\)-complete. Moreover, given a set \(F\) of circular ordered graphs on \(k\)-vertices such that the \(F\)-free circular ordering problem is \(NP\)-complete, it is not hard to construct a set \(F'\) of circularly ordered graphs on \((k+1)\)-vertices such that the \(F'\)-free circular ordering is \(NP\)-complete as well. Indeed, simply let \(F'\) be the set of all circularly ordered supergraphs on \((k+1)\)-vertices of circularly ordered graphs in \(F\).

To conclude this section we construct a set \(F_{CO}\) of circularly ordered graphs such that the \(F_{CO}\)-free circular ordering problem nicely relates to the cyclic ordering problem. The cyclic ordering problem takes as an input a set of ordered triples \(R\) of some finite set \(A\) and asks if the triples of \(R\) are generated by some circular ordering of \(A\). This problem was proved to be \(NP\)-complete in [5].

Consider the graph \(G_{aco}\) with vertex set \(\{v_1, v_2, v_3, v_4, v_5, v_6\}\) where \(\{v_3, v_4, v_5, v_6\}\) induce a clique and we add the edges \(v_1v_6, v_2v_5\) and \(v_2v_6\). We define the circularly ordered graph \(ACO\) as \(G_{aco}\) with the circular closure of \(v_1 < v_2 < v_3 < v_4 < v_5 < v_6\). We depict this circularly ordered graph in Figure 10. Denote by \(F_{CO}\) the set of all circular orderings of \(G_{aco}\) that are not isomorphic (as circularly ordered graphs) to \(ACO\). \(F_{CO}\) is not an empty set, for instance, consider \(G\) with the circular ordering closure of \(v_2 < v_1 < v_5 < v_3 < v_4 < v_6\).

**Remark 26.** Note that there are only two automorphisms of \(ACO\): the identity, and the transposition of \(v_3\) with \(v_4\) leaving every other vertex fixed. In any of these two cases, the induced cyclic order in \(\{v_1, v_2, v_3\}\) is the circular closure of \(v_1 < v_2 < v_3\).
Consider an instance \((A, R)\) of the cyclic ordering problem. We construct the input graph \(G(A, R)\) for the \(F_{\text{CO}}\)-free circular ordering problem as follows.

The vertex set \(V\) of \(G(A, R)\), is the union \(A \cup \mathcal{R}\), where \(\mathcal{R} = \{r_x: r \in R\} \cup \{r_y: r \in R\} \cup \{r_z: r \in R\}\), i.e., \(\mathcal{R}\) contains three vertices for every \(r \in R\). The set \(A \subseteq V\) is an independent set and for every \(r \in R\) the vertices \(r_x, r_y\) and \(r_z\) induce a triangle. Finally, we add the following edges between \(A\) and \(\mathcal{R}\): for every \(r = (a, b, c) \in R\) we add the set of edges \(\{r_xa, r_zb, r_zc, r_yb, r_yb, r_zc\}\). In other words, for \(r \in R\), if \(r = (a, b, c)\), the vertices \(\{a, b, c, r_x, r_y, r_z\}\) induce a copy of \(G_{\text{ACO}}\). A simple calculation shows that \(|V| = |A| + 3|R|\), and \(|E| = 8|R|\), so this construction can be done in linear time. Before showing that \(G(A, R)\) is a yes-instance of the \(F_{\text{CO}}\)-free circular ordering problem if and only if \((A, R)\) is yes-instance of the cyclic ordering problem, we prove the following claim.

**Claim 2.** If \(F\) is a set of vertices of \(G(A, R)\) that induces a copy of \(G_{\text{ACO}}\), then there is an element \(r \in R\), \(r = (a, b, c)\), such that \(F = \{a, b, c, r_x, r_y, r_z\}\).

We will show that if \(F\) is such a set then \(F = \{r_z\} \cup N r_z\) for some \(r \in R\); it should be clear that \(N r_z = \{a, b, c, r_x, r_y\}\), where \(r = (a, b, c)\). Since \(G_{\text{ACO}}\) has a unique universal vertex, then there is a unique vertex \(v_F \in F\) such that \(F = \{v_F\} \cup N'\) where \(N' \subseteq N(v_F)\). In particular, the degree of \(v_F\) in \(G(A, R)\) is at least five. It is not hard to observe that for every \(r \in R\) the degrees of \(r_x\), \(r_y\) and \(r_z\) are 3, 4 and 5 respectively. Hence, \(v_F = r_z\) for some \(r \in R\) or \(v_F \in A\). By construction of \(G(A, R)\), the graph induced by \(\mathcal{R}\) is a disjoint union of triangles, so every connected subgraph of \(G(A, R)[\mathcal{R}]\) contains at most three vertices. Now note that the neighborhood of \(v_F\) in \(G(A, R)[F]\) consists of an isolated vertex and a connected component on four vertices. Since \(A\) is an independent set, for every \(a \in A\), \(N(a) \subseteq \mathcal{R}\), so the neighborhood of \(a\) cannot contain a connected component on four vertices. Therefore \(v_F \not\in A\), and thus \(v_F = r_z\) for some \(r \in R\). As mentioned at the beginning of the paragraph, this concludes the proof of Claim 2.

Now we show that the proposed reduction translates yes-instances to yes-instances and no-instances to no-instances of the corresponding problems. Sup-

![Figure 10: On the left, a representation of ACO. On the right, a circular ordered graph not isomorphic to ACO, but with the same underlying graph.](image)
pose that $G(A, R)$ admits an $F_{CO}$-free circular ordering $G'$. Since the only admissible circular orderings of $G_{ACO}$ are isomorphic to $ACO$, by Remark 26, for every ordered triple $(a, b, c) \in R$, the circular ordering of $\{a, b, c\}$ in $G'$ is $a < b < c < a$. So by considering the circular ordering of $A$ inherited from $G'$, we obtain a circular ordering $R'$ of $A$, such that $R \subseteq R'$. Conversely, suppose that $(A, R)$ is a yes-instance for the cyclic ordering problem, and let $a_1 < \cdots < a_n < a_1$ be the corresponding cyclic ordering of $A$. We want to extend this ordering to a circular ordering of $V$. For every $r \in R$, $r = (a_i, a_j, a_k)$, include $r_x, r_y, r_z$ in any way such that $a_i < a_j < a_k < r_x < r_y < r_z < a_{k+1} < a_i$. In other words, the circularly ordered graph induced by these vertices $(a_i, a_j, a_k, r_x, r_y, r_z)$ is a copy of $ACO$. Once we have extended the circular ordering of $A$ to $V$ in this manner, call the resulting circularly ordered graph $G'$. To see that $G'$ is an $F_{CO}$-free circularly ordered graph, consider a set of six vertices $F \subseteq V$. If $F$ does not induce a copy of $G_{ACO}$ in $G(A, R)$, then $F$ cannot induce a copy of any circularly ordered graph in $F_{CO}$. If $F$ does induce a copy of $G_{ACO}$ in $G(A, R)$ then, by Claim 2, there is an element $r \in R$, $r = (a, b, c)$, such that $F = \{a, b, c, r_x, r_y, r_z\}$. Hence, by how we extended the circular ordering of $A$ to $V$, $F$ induces a copy of $ACO$ in $G'$, and thus it does not induce any circularly ordered graph of $F_{CO}$. Therefore, $G'$ is an $F_{CO}$-free circular ordering of $G(A, R)$.

6 Conclusions and open problems

It is now well-known that equipping a graph with a linear ordering of its vertex set, or an orientation of its arc set, leads to characterizations of some hereditary families of graphs in terms of finitely many forbidden induced linearly ordered subgraphs or induced oriented subgraphs, respectively, in cases where forbidding infinitely many induced subgraphs (without additional structure) is needed. In this work we show that similar results can be obtained when we equip a graph with a circular ordering of its vertex set. In this type of problems, it is natural to ask for the limitations of the proposed framework, in particular, we think that the following problem is interesting.

Problem 28. Find a (relatively well-known) hereditary property that cannot be described by a finite set of forbidden circularly ordered graphs.

In this work we show that if a graph family can be described by finitely many forbidden circularly ordered graphs, then it can be described by finitely many forbidden linearly ordered graphs. So it is natural to ask if the converse implication is also true. Since we do not think that it is true, we propose this question in a negative way.

Question 29. Is there a hereditary property described by finitely many forbidden linearly ordered graphs that does not admit a characterization by finitely many forbidden circularly ordered graphs?
In particular, we believe that the classes of $k$-colourable graphs are possible candidates to answer the previous question in the negative, but finding any such a class seems to be an interesting problem.

**Question 30.** For which positive integer $k$ the class of $k$-colourable graphs can be described by finitely many forbidden circularly ordered graphs? In particular, is there a finite set of circularly ordered graphs that describes the class of bipartite graphs?

There are a couple of characterizations of graphs with circular chromatic number at least 3 by certain unavoidable structures in every maximal triangle free super graph. They read as follows.

**Theorem 31.** [2] Let $H$ be the graph obtained from the Petersen graph by deleting one vertex. A graph $G$ has circular chromatic number at least 3 if and only if every maximal triangle-free supergraph $G_0$ of $G$ contains $H$ as a subgraph.

**Theorem 32.** [9] A graph $G$ has circular chromatic number at least 3 if and only if every maximal triangle-free supergraph $G_0$ of $G$ has an independent set whose elements do not have a common neighbour.

Even though they are nice characterizations, they yield no information on how to generalize it for larger circular chromatic numbers. Theorem 21 provides a nice characterization through unavoidable structures in circular orderings of graphs with circular chromatic number at least $k$ for any $k \geq 3$.

**Corollary 33.** Let $k$ be a positive integer, $k \geq 3$. A graph $G$ satisfies $\chi_c(G) \geq k$ if and only if for every circular ordering of $V(G)$, $v_1 \leq v_2 \leq \cdots \leq v_n \leq v_1$, there are $k$ vertices $u_1 \leq u_2 \leq \cdots \leq u_k$ such that for every $i \in \{1, \cdots, k-1\}$ there is an edge $u_i u_{i+1} \in E(G)$ (note that $u_1$ might be $u_k$).

Theorem 21 also provides a characterization of the class of graphs such that $\chi_c(G) = \chi(G)$ in terms of linear orderings of the vertex set.

**Theorem 34.** A graph $G$ satisfies that $\chi(G) = \chi_c(G) = k$ if and only if every ordering $\leq$ of $V(G)$ that avoids the structure $v_1 \leq v_2 \leq \cdots \leq v_{k+1}$ where $v_i v_{i+1} \in E(G)$ for $i \in \{1, \cdots, k\}$, contains the structure $u_1 \leq u_2 \leq \cdots \leq u_{k+1}$ where $u_1 u_{k+1} \in E(G)$ and $u_i u_{i+1} \in E(G)$ for $i \in \{2, \cdots, k\}$ (note that $u_1$ might be $u_2$).

**Corollary 35.** Consider a graph $G$ with $\chi(G) = k$. If $G$ admits a $k$-coloring, $c : V(G) \to \{1, \cdots, k\}$, with no (possibly closed) walk $v_1 v_2 \cdots v_{k+1}$ such that $c(v_i) = i$ for $i \in \{1, \cdots, k\}$ and $c(v_{k+1}) = 1$, then $\chi_c(G) < \chi(G)$.

We dealt with complexity issues of the $F$-free circular ordering problem – we show that for every positive integer $k$, $k \geq 5$, there is a set $F$ of circularly ordered graphs on $k$ vertices such that the $F$-free circular ordering problem is $NP$-complete. Moreover, we already discussed that for every set $F$ of circularly ordered graphs on 3 vertices, the $F$-free circular ordering problem can be solved in polynomial time, so we ask:
Question 36. Is there a set $F$ of circularly ordered graphs on 4 vertices such that the $F$-free circular ordering problem is $NP$-complete?

We showed that the $H_5$-free circular ordering problem is $NP$-complete as a consequence of Theorem 24, which states that determining if $\chi_c(G) < 4$ for an arbitrary graph $G$ is an $NP$-complete problem as well. On the other hand, deciding if a graph $G$ satisfies that $\chi_c(G) < 2$ can be (trivially) done in polynomial time. It is only natural to ask the following question.

Question 37. Given a graph $G$, is the problem of determining if $\chi_c(G) < 3$ an $NP$-complete problem?

Note that by Corollary 22, Question 37 is a particular instance of Question 36 since a graph $G$ admits an $H_4$-free circular ordering if and only if $\chi_c(G) < 3$, and $H_4$ consists of the triangle and circularly ordered graphs on 4 vertices.

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