A dynamic analytic method for risk-aware controlled martingale problems

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Abstract

We present a new, tractable method for solving and analyzing risk-aware control problems over finite and infinite, discounted time-horizons where the dynamics of the controlled process are described as a martingale problem. Supposing general Polish state and action spaces, and using generalized, relaxed controls, we state a risk-aware dynamic optimal control problem of minimizing risk of costs described by a generic risk function. We then construct an alternative formulation that takes the form of a nonlinear programming problem, constrained by the dynamic, i.e. time-dependent, and linear Kolmogorov forward equation describing the distribution of the state and accumulated costs. We show that the formulations are equivalent, and that the optimal control process can be taken to be Markov in the controlled process state, running costs, and time. We further prove that under additional conditions, the optimal value is attained. An example numeric problem is presented and solved.

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1 Introduction

We consider the risk-aware optimization of controlled stochastic processes over a finite $T = [0, T)$, $T > 0$, or infinite time-horizon $T = [0, \infty)$ on general Polish state and action spaces $X$ and $A$. That is, for a filtered probability space $(\Omega, \Sigma, \mathcal{F}, \mathbb{P})$, we solve

$$\inf_a \rho \left( \int_0^\infty e^{-at} c(x_t, a_t, t) \, dt \right) \quad (T = [0, \infty)) \quad \text{or}$$

$$\inf_a \rho \left( \int_0^T c(x_t, a_t, t) \, dt + v(x_T) \right) \quad (T = [0, T]),$$

where $\rho : \mathcal{L}(\Omega ; \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is a risk function, $\alpha > 0$ is the discount rate, $c : X \times A \times T \to \mathbb{R}$, and $v : X \to \mathbb{R}$ are given cost rate and terminal cost functions, and the infima run over sets of admissible of control processes $a$ while $x$ are the controlled stochastic processes. Here, the controlled processes shall be determined by the martingale formulation, and we will consider generalized, relaxed controls.

The introduction of the risk function $\rho$ sets our control problem apart from the classical, or risk-neutral problem where $\rho$ is the expectation, denoted $\mathbb{E}$. The expectation judges events of high probability and low cost with the same standard as unlikely events but high costs, and this may often be undesirable. Risks matter, and our intuitions immediately weigh minor recurring adversities differently from major catastrophes. The role of the risk function is to describe the controllers preferences that may feature e.g. tail-risk avoidance, loss aversion, or even risk-seeking tendencies. Practical risk-management applications need proper models of risk and risk preferences, and various risk functions such as the conditional value-at-risk [51], or classes of risk functions [5, 18] have become an import tool in risk modeling [47].

The motivation of this work is constructing a tractable and readily generalizable method to solving problems of the form of Problem (1.1). Approaches for risk-neutral dynamic control do not trivially generalize to the risk-aware setup. The standard approaches can be broadly binned into three categories: (i) dynamic programming, (ii) probabilistic methods, and (iii) the convex analytic approach. All may be applicable to risk-aware problems, but none are without notable issues. Dynamic programming methods [38] [39] [17] form arguably the most well-known and most frequently used approach in the risk-neutral case. However, the derivation of the dynamic programming
equations hinges on the properties of the expectation that are not always shared by risk functions. Overcoming this generally requires considering dynamic risk measures that impose nontrivial additional structure on the risk functions [1].

The second, probabilistic group of methods include most notably various formulations of the stochastic Pontryagin’s minimum principle, see e.g. [42] [63] [61]. In this context, solutions to the optimal control problem are found from stochastic equations, hence the descriptor “probabilistic.” These methods have been amenable to risk-aware problems, and risk-aware specific approaches have been successfully developed, see e.g. [41] [45] [46], though these too are constrained to specific forms of dynamic risk measures. Recently, an alternative formulation for generic (not necessarily dynamic) risk functions was also found [30]. The solution of the probabilistic formulations nonetheless involves the nontrivial task of solving systems of forward-backward systems of stochastic dynamic equations.

Convex analytic methods recast the dynamic control problem to a static problem of optimizing over distributions, often called occupation measures. In the risk-neutral case, this approach conventionally yields linear programming problems, e.g. in the discounted infinite time-horizon setup without explicit time-dependence.

\[
\begin{equation}
\inf_{\mu \geq 0} \int_{\mathbb{X} \times \mathbb{A}} c(x,a) \mu(dx \times da) \\
\text{s.t. } L(\mu) = \alpha \nu_0.
\end{equation}
\]

Here, the measure \(\mu\) represents the (discounted) likelihood of the state-control pair visiting a given point in the state-action space, and \(L(\mu) = \alpha \nu_0\), the adjoint equation, linear in \(\mu\), encodes a set of constraints that determine the occupation measure. The measure \(\nu_0\) is the initial distribution for the controlled process. Proving the equivalence of Problem (1.2) and Problem (1.1) with \(\rho = \mathbb{E}\) requires showing that a solution of one of the problems yields a solution to the other.

The convex analytic approach extends to the risk-aware case more readily than dynamic programming methods, as risk functions can evaluate risks from the cost distributions, and derivations of the convex analytic problem do not heavily rely on the properties of the expectation. In [23], a state space augmentation scheme similar to that of [6] was used to derive a risk-aware convex analytic formulation in discrete time. However, as convex analytic methods construct the occupation measures from long-run, discounted visitation frequencies, recovering the full cost distribution from the adjoint equation \(L(\mu) = \alpha \nu_0\) becomes technically awkward.

Here, we take a different approach that nonetheless bears some similarity to the convex analytic method, in that we obtain a linearly constrained nonlinear programming problem that is equivalent to a generalization of Problem (1.1). We formulate the problem as a “dynamic” analytic problem, in the sense that the static adjoint equations of the convex analytic method are replaced by a time-dependent equation, the Kolmogorov forward equation. The forward equation yields the joint, time-dependent distribution of the state of the controlled process and the associated cumulative costs. This distribution is then in turn used to evaluate the risk-aware objective that can now feature generic risk functions. The dynamic formulation is natural to the risk-aware problem: Risk-awareness generally requires in some way tracking running costs, or future risks, given the information available to the controller at any given time, see e.g. [30], where we showed that; Peng’s nonlinear expectations [43] also introduce an additional process, modeling the controller’s risks.

1.1 Related literature

Risk measures There is a substantial body of work on risk measures in the static setting, such as [53] [37] [19]. This work focuses on axiomatic foundations for modeling preferences, as well as for tractable risk-aware optimization schemes. Dynamic risk functions are discussed in [1]. Nonlinear expectations form a subset of dynamic risk functions, and are considered in [44] [52].

The convex analytic approach The convex analytic method (or the linear programming method, in the case that the problem is risk-neutral) is closest in spirit to the approach we take in this paper. It has featured heavily in the study of Markov decision processes (MDPs) and controlled stochastic processes. In the discrete time setting, the linear programming approach for MDPs is pioneered in [40] and further developed in [31]. An early survey of this technique is found in [4]. The main idea is that some MDPs can be written as linear programming problems in terms of appropriate occupation measures. A rigorous theory of the convex analytic approach for MDPs with general Borel state and action spaces is developed in the works [11] [20] [12] [27]. Detailed monographs on Markov decision processes are found in [28] [29] [50].

The convex analytic approach has also been well studied for continuous time controlled Markov processes. Occupation measures for controlled Markov processes in continuous time and state and action spaces were first
introduced in [38, 57], where the process dynamics were stated as a martingale problem and long term average costs were considered. The theory was extended to discounted and finite-horizon problems in the closely related papers [7] and [34], which also proved the optimality of feedback controls (i.e. controls that depend only on the current state). Convex analytic methods for controlled stochastic differential equations are considered in [10]. Singular controls (see e.g. [54] for an introduction) have subsequently been analyzed within the convex analytic framework in [59] for diffusion processes with discounted costs. Martingale problems with singular dynamics and controls, with ergodic and discounted costs, were studied in [35] and the constrained case was studied in [33]. The martingale formulation of the problem and convex analytic methods were used in the study of optimal stopping problems in [14], and in [25] where also singular dynamics and controls were included. Constrained continuous time MDPs are solved using convex analytic techniques in [22], where the process dynamics are described by a transition kernel rather than the generator. More recently, a similar occupation measure approach for controlled Markov jump processes is developed in [19, 48]. A survey of optimal control methods for diffusion processes in particular can be found in [13].

1.2 Contributions

The main contribution of this paper is developing a dynamic analytic formulation of a generic risk-aware control problem. In particular, (i) we firstly state risk-aware control problems where the controlled processes are described by martingale problems. We allow for generic, Polish state and action space which makes our results applicable for a broad family of types of stochastic processes; continuous-time Markov decision processes and controlled Lévy processes are examples of these. We require a number of rather technical assumptions that are nonetheless often satisfied. (ii) Additionally, we introduce a number of regularity conditions that ensure that the solutions of the martingale problem are sufficiently well-behaved, e.g. in the sense that the solutions never “explode” by diverging to some infinity point. (iii) We then derive our dynamic analytic formulation, and prove its equivalence with the original martingale problem. This is based on a state space augmentation scheme, similar to the one in [6, 24], that allows for the Kolmogorov forward equation to also capture the distribution of costs. We additionally provide conditions under which the optimal value is attained.

This paper is organized as follows. We begin in the next section by introducing standard notation and describing the control model we consider. This section defines our risk-aware problem, and states the main assumptions. Section 3 contains our main results, where we show that Problem (1.1) is equivalent to a static optimization problem over measure-valued functions of time satisfying a linear constraint (namely, the forward equation). In Section 4 we present a simple application of the results. Section 5 gives a short summary of the results. Some of the proofs and frequently used auxiliary results are given in the Appendix.

2 Model

Basic definitions. Let $T := \mathbb{R}_{\geq 0}$ or $[0, T]$ for some $T \in \mathbb{R}_{\geq 0}$ be the set of time indices, and let $\mathbb{R}_{\infty} := \mathbb{R} \cup \{\infty\}$. We shall cover both finite and infinite time-horizon problems; which one we consider is determined whether $T$ is compact or not.

For any topological space $U$, we denote the Borel $\sigma$-algebra on $U$ by $\mathcal{B}(U)$. Finite Borel (probability) measures on $U$ are denoted $\mathcal{M}(U)$ ($\mathcal{P}(U)$). The space of probability measures default to the topology of weak convergence, and for separable metric space $U$, this topology is metrizable using the Prokhorov metric, denoted $d_P$ [56, Section 3.1]. Weak convergence of $(\mu_n)_{n \in \mathbb{N}} \in \mathcal{P}(U)^{\mathbb{N}}$ to a $\mu \in \mathcal{P}(U)$ is denoted $\mu_n \rightharpoonup \mu$. Given topological spaces $U_1$ and $U_2$, we say that a Borel measurable mapping $\pi : U_2 \rightarrow \mathcal{P}(U_1)$ is a transition function from $U_2$ to $U_1$, and denote the set of transition functions from $U_2$ to $U_1$ by $\mathcal{P}(U_1 | U_2)$.

For a given probability space $(\Omega, \Sigma, \mathbb{P})$, we denote the set of all $(\Omega, \mathcal{B}(U))-\text{valued}$ random variables by $\mathcal{L}(\Omega, \Sigma, \mathbb{P}; U)$ or $\mathcal{L}(\Omega; U)$ for short. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. The law of a random variable $X \in \mathcal{L}(\Omega, \Sigma, \mathbb{P}; U)$ is denoted $\mathcal{L}(X) := \mathbb{P} \circ X^{-1}$. For a Banach space $(U, \| \cdot \|)$, by $\mathcal{L}^p(\Omega, \Sigma, \mathbb{P}; U)$ or simply $\mathcal{L}^p(\Omega; U), p \in [1, \infty)$, we mean the set of $X \in \mathcal{L}(\Omega; U)$ such that $\mathbb{E}[\|X\|^p] < \infty$. The norms on the spaces $\mathcal{L}^p(\Omega; \mathbb{R}), p \in [1, \infty]$, are denoted $\| \cdot \|_p$. For every $p \in [1, \infty]$ and Polish $(U, d)$, we use $\mathcal{P}(U)$ to denote the probability measures such that for all $\mu \in \mathcal{P}(U)$, for some $u_0 \in U$, $\int d(u, u_0)^p d\mu(du) < \infty$. We assign $\mathcal{P}(U)$ the $p$-Wasserstein metric [60, Definition 6.1], denoted $W^p$.

For a pair of measurable spaces $U$ and $V$, measurable functions from $U$ to $V$ are denoted $M(U, V)$, $B(U, V)$ if they are bounded and $V$ is metric. Continuous functions shall be the set $C(U, V)$ which is by default assigned the compact-open topology. If $V = \mathbb{R}$, the $V$ argument is omitted. Bounded and continuous, and compactly supported continuous $\mathbb{R}$-valued functions are denoted $C_b(U)$ and $C_c(U)$, respectively, and these are assigned the supremum norm, denoted $\| \cdot \|$. If $(U, d)$ is a metric space, bounded Lipschitz functions are denoted $C_l(U)$, and are defined so
that $C_{0d}(U) := \{ f \in C_b(U) \mid \| f \|_{M} < \infty \}$, where $\| \cdot \|_{M} := \| \cdot \| + \| \cdot \|_{t}$ and $\| f \|_{t} := \sup_{u \neq u'} | f(u') - f(u) | / d(u', u)$ for all $f \in C_{b}(U)$.

Càdlàg, or left-continuous with limits from the right, functions from $\mathbb{T}$ to a Polish $U$ are denoted $D(\mathbb{T}, U)$. For $U = \mathbb{R}^{n}$, $n \in \mathbb{N}$ we use $C^{(k_{1}, \ldots, k_{n})}(U)$ to denote functions that can be differentiated $k_{i}$ times with respect to the $i$th argument, $i \in \{1, \ldots, n\}$, with all the first derivatives in $C(U)$, and similarly for the function spaces $C_{b}$ and $C_{c}$.

Let $U_{1}$ and $U_{2}$ be Polish spaces. For all $\mu \in \mathcal{P}(U_{1} \times U_{2})$ we denote the $U_{1}$, $U_{2}$ marginals of $\mu$ by $\mu^{U_{1}}$ and $\mu^{U_{2}}$, respectively. The regular conditional probabilities on $U_{1}$ given $u_{2} \in U_{2}$ are denoted $\mu^{U_{1}}|_{U_{2}} \in \mathcal{P}(U_{1} | U_{2})$ so that for all $f \in M(U_{1} \times U_{2})$,

$$
\int_{U_{1} \times U_{2}} f(u_{1}, u_{2}) \mu(du_{1}, du_{2}) = \int_{U_{2}} \left[ \int_{U_{1}} f(u_{1}, u_{2}) \mu^{U_{1}}|_{U_{2}}(du_{1} | u_{2}) \right] \mu^{U_{2}}(du_{2}).
$$

We will frequently need to separate measures into their marginal and conditional parts, and hence we abbreviate equalities of the above form to $\mu(du_{1} \times du_{2}) = \mu^{U_{1}}|_{U_{2}}(du_{1} | u_{2}) \mu^{U_{2}}(du_{2})$.

Evaluation of a function $f$ defined on $\mathbb{T}$ at a point $t \in \mathbb{T}$ is denoted $f_{t}$.

We introduce a weak topology for functions $\mu \in M(\mathbb{T}, \mathcal{P}(U))$, where $U$ is Polish. We say that $(\mu^{(n)})_{n \in \mathbb{N}} \in M(\mathbb{T}, \mathcal{P}(U))^{\mathbb{N}}$ converges weakly to a $\mu \in M(\mathbb{T}, \mathcal{P}(U))$ and denote $\mu^{(n)} \rightharpoonup \mu$, if and only if for all $h \in C_{b}(\mathbb{U} \times \mathbb{T})$ such that the support of $h$ is contained in a set $\mathbb{U} \times [0, t_{h}]$, $t_{h} \in \mathbb{T}$, we have that $\int_{\mathbb{T}} \int_{\mathbb{U}} h(u, t) \mu^{(n)}(du \, dt) \to \int_{\mathbb{T}} \int_{\mathbb{U}} h(u, t) \mu(du \, dt)$; this is used in e.g. [35]. For $C(\mathbb{T}, \mathcal{P}(U))$, we assume the (metrizable) topology of uniform convergence on compacts, and denote $\mu^{(n)} \xrightarrow{w}\mu$ when a sequence $(\mu^{(n)})_{n \in \mathbb{N}} \in C(\mathbb{T}, \mathcal{P}(U))^{\mathbb{N}}$ converges to a $\mu \in C(\mathbb{T}, \mathcal{P}(U))$. Additionally, for any Polish $U$ and $\mu \in M(\mathbb{T}, \mathcal{P}(\mathbb{U} \times \mathbb{V}))$, we denote $\mu^{U} := (\mu^{U}_{t})_{t \in \mathbb{T}} \in M(\mathbb{T}, \mathcal{P}(U))$.

The Dirac measure centered at $u \in \mathbb{U}$, $U$ a measurable space, is denoted by $\delta_{u}$.

### 2.1 Martingale formulation of the control problem

In the following, $\mathbb{X}$ and $\mathbb{A}$ shall represent the state and action spaces, both assumed Polish. We will also need to consider processes on other (Polish) state spaces, and so, when appropriate we state our definitions for a generic state space $\mathbb{U}$.

The dynamics of the control problem are determined by the generator of the process and an initial distribution. The following definition formalizes these terms and introduces the notion of a solution that we shall be using to describe the dynamics of our controlled processes.

**Definition 2.1.** (Relaxed controlled martingale problem) Let $U$ and $\mathbb{A}$ be Polish spaces, and let $A : \mathcal{D}(A) \supset C_{b}(U) \rightarrow \mathcal{A}(A) \subset C(U \times \mathbb{A} \times \mathbb{T})$ and $\nu_{0} \in \mathcal{P}(U)$ be given.

We call the pair $(A, \nu_{0})$ a relaxed controlled martingale problem, where $A$ is the generator of the processes considered, and $\nu_{0}$ is the initial distribution.

**Solution to a relaxed controlled martingale problem** Let $(A, \nu_{0})$ be a relaxed controlled martingale problem. A solution to the relaxed controlled martingale problem $(A, \nu_{0})$ consists of a filtered probability space $(\Omega, \Sigma, \mathcal{F} = (\mathcal{F}_{t})_{t \in \mathbb{T}}, \mathbb{P})$ and a $U \times \mathcal{A}$-valued stochastic process $(u, \pi) = (u_{t}, \pi_{t})_{t \in \mathbb{T}}$ defined on $(\Omega, \Sigma, \mathcal{F}, \mathbb{P})$ such that: (i) The process $(u, \pi)$ is progressively measurable with respect to the filtration $\mathcal{F}$; (ii) the distribution of $u_{0}$ equals $\nu_{0}$; and (iii) for all $f \in \mathcal{D}(A)$, the process $(m_{t}^{f})_{t \in \mathbb{T}}$,

$$m_{t}^{f} := f(u_{t}) - f(u_{0}) - \int_{0}^{t} \int_{\mathbb{A}} A(f(u_{s}, a, s) \pi_{s}(da) \, ds \quad \forall f \in \mathcal{D}(A), t \in \mathbb{T},$$

is an $\mathcal{F}$-martingale for all $f \in \mathcal{D}(A)$. We denote the set of relaxed controlled solutions by $\mathcal{R}(A, \nu_{0})$, and for brevity, we shall identify a solution by its control component, i.e. write $\pi \in \mathcal{R}(A, \nu_{0})$ to mean $(\Omega, \Sigma, \mathcal{F}, \mathbb{P}, u_{\pi})$.

**Càdlàg solution to a relaxed controlled martingale problem** A solution $\pi \in \mathcal{R}(A, \nu_{0})$ is a càdlàg solution to the relaxed controlled martingale problem if additionally $u \in D(\mathbb{T}, U)$, $\mathbb{P}$-almost surely. The subset of càdlàg solutions shall be denoted $\mathcal{D}(A, \mathbb{K}, \nu_{0})$.

We allow constraints on the relaxed controlled solutions that only depend on the finite dimensional distributions of controls and states.

**Definition 2.2.** Let $(A, \nu_{0})$ be a relaxed controlled problem and $K \subset M(\mathbb{T}, \mathcal{P}(\mathbb{U} \times \mathbb{A}))$. A relaxed controlled solution $\pi \in \mathcal{R}(A, \nu_{0})$ is admissible (given $K$) if $\mu \in K$, where $\mu$ is defined

$$\int_{\mathbb{U} \times \mathbb{A}} h(u, a) \mu_{t}(da \times du) = \mathbb{E} \left[ \int_{\mathbb{A}} h(u_{t}, a) \pi_{t}(da) \right] \quad \forall h \in C_{b}(\mathbb{U} \times \mathbb{A}), t \in \mathbb{T}.$$
We emphasize that each relaxed controlled solution \( \pi \in \mathcal{R}(A, \nu_0) \) comes in general with its own filtered probability space. When appropriate, we label the objects forming the solution as \( (\Omega^\pi, \Sigma^\pi, \mathcal{F}^\pi, \mathbb{P}^\pi, (u^\pi, \pi)) \) to make this point explicit. In the following, we shall consider almost exclusively càdlàg solutions.

**Baseline assumptions on the relaxed controlled problem** First, we introduce a few technical definitions that are necessary to state our main assumptions. We recall the notion of pre-generators, used to characterize the operators that are sufficiently regular to correspond to generators of Markov processes [35].

**Definition 2.3.** Let \( \mathbb{U} \) be a Polish space. An operator \( A : M(\mathbb{U}) \to M(\mathbb{U}) \) is a *pre-generator* if it is: (i) dissipative, i.e. for all \( \lambda > 0 \) and all \( f \in \mathcal{D}(A) \), \( \| (\lambda - A)f \| \geq \lambda \| f \| \), and (ii) there are sequences of measure valued functions \( (\mu_n)_{n \in \mathbb{N}} \) with \( \mu_n : \mathbb{U} \to \mathcal{P}(\mathbb{U}) \) and \( (\lambda_n)_{n \in \mathbb{N}} \) with \( \lambda_n : \mathbb{U} \to \mathbb{R}_{\geq 0} \), for all \( n \in \mathbb{N} \), such that \( h(u) = \lim_{n \to \infty} \lambda_n(u) \int_E (f(u) - f(u')) \mu_n(u)(du') \) for all \( u \in \mathbb{U} \) and for every \( f \in \mathcal{D}(A) \), \( h \in \mathcal{A}(\mathbb{U}) \) such that \( A^*_f = h \).

We also utilize the notion of bounded point-wise limit and strong separability of points, see e.g. [50] Chapter 3.4.

**Definition 2.4.** Let \( U \) be a metric space. (i) A sequence of functions \( (f_k)_{k \in \mathbb{N}} \subset B(\mathbb{U}) \) converges *boundedly and point-wise* to a function \( f \in B(\mathbb{U}) \) if \( \sup_{k \in \mathbb{N}} \| f_k \| < \infty \) and \( \lim_{k \to \infty} f_k(u) = f(u) \) for all \( u \in \mathbb{U} \). Denote this \( \text{bp-lim}_{k \to \infty} f_k = f \). (ii) A set \( M \subset B(\mathbb{U}) \) is said to be bp-closed, if for all \( (f_k)_{k \in \mathbb{N}} \subset M \), \( \text{bp-lim}_{k \to \infty} f_k = f \in B(\mathbb{U}) \) implies \( f \in M \). (iii) The bp-closure of a set \( M \subset B(\mathbb{U}) \) is the smallest bp-closed set that contains \( M \). (iv) A set of functions \( \mathcal{A} \subset C_b(\mathbb{U}) \) is said to *strongly separate points* if for every \( u \in \mathbb{U} \) and a neighborhood \( \mathcal{U} \) of \( u \), there is a finite \( \mathcal{A}(u) \subset \mathcal{A} \) such that \( \inf_{u' \in \mathcal{U}} \max_{f \in \mathcal{A}(u) \cup \mathcal{U)} |f(u') - f(u)| > 0 \).

The following assumption, adapted from [35] [33], is used to guarantee existence of relaxed solutions to controlled martingale problems, as stated below in Theorem 3.7.

**Assumption 2.5.** Let \( \mathbb{U} \) and \( \mathcal{A} \) be Polish spaces, and let \( A : C_b(\mathbb{U}) \subset \mathcal{D}(A) \to \mathcal{D}(A) \subset C(\mathbb{U} \times \mathcal{A} \times T) \). The tuple \( (\mathbb{U}, \mathcal{A}, A) \) satisfies the following conditions:

(i) The constant function \( 1 \in C_b(\mathbb{U}) \) is in \( \mathcal{D}(A) \) and \( A1 = 0 \).

(ii) The operator \( A_u : \mathcal{D}(A) \to \mathcal{D}(A) \) defined as \( A_u f(u, a, t) := Af(u, a, t) \) for all \( f \in \mathcal{D}(A) \) and \( a \in A \) is a pre-generator.

(iii) The domain of \( A, \mathcal{D}(A) \), is an algebra that strongly separates points.

(iv) There is a function \( \psi \in C(\mathbb{U} \times \mathcal{A} \times T) \), \( \psi \geq 1 \), such that for each \( f \in \mathcal{D}(A) \) there is a constant \( a_f \) satisfying \( |Af(u, a, t)| \leq a_f \psi(u, a, t) \) for all \( (u, a, t) \in \mathbb{U} \times \mathcal{A} \times T \).

(v) The set \( \mathcal{A}_0 := \{(f, \psi^{-1}Af) \mid f \in \mathcal{D}(A)\} \) is such that there exists \( \{f_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(A) \) for which \( A_0 \) is contained in the bp-closure of the linear span of \( \{f_k, A_0 f_k\}_{k \in \mathbb{N}} \).

Parts (i)-(iii) in Assumption 2.5 amount to basic requirements for the martingale problem and its associated forward equation to have solutions (compare to the standard, though stronger assumptions of Theorem 4.5.4 and Theorem 4.9.19 in [50] in the uncontrollable case, with a locally compact state space \( \mathbb{U} \)). The requirement that \( A \) is a pre-generator is a relaxation of the assumption that \( A \) satisfies the positive maximum principle. Part (iv) of Assumption 2.5 allows for construction of an operator, specifically \( \psi^{-1}A \), that takes values on bounded continuous functions, and which is used in weak convergence arguments. Part (v) is used in [35] to construct a compact Polish space \( \hat{\mathbb{U}} \) along with a continuous mapping \( \Gamma : \mathbb{U} \to \hat{\mathbb{U}} \) with a measurable inverse that allows extending of results assuming a compact state space to the case where \( \mathbb{U} \) is not compact or locally compact. This condition was earlier applied in [5] for the same purpose in the context of uncontrollable martingale problems and in [7] for controlled problems. Additional discussion and examples can be found in [35] [33]. Returning to part (iii), we note that typically it is assumed that \( \mathcal{D}(A) \) only separates points. Here, we assume strong separation of points, and this is to ensure that the above mapping \( \Gamma \) is in fact a homeomorphism (that is, its inverse is also continuous) [9] Lemma 1. A convenient characterization of sets that strongly separate points is given in [9] Lemma 4. We also recall that sets that strongly separate points are convergence determining [50] Theorem 3.4.5(b).

In order to establish the equivalence of control problems stated in terms of relaxed controlled solutions and those formulated using analytic methods, we will require additional constraints on the generator \( A \).

**Definition 2.6.** Suppose \( (\mathbb{U}, \mathcal{A}, A) \) satisfies Assumption 2.5 We say the martingale problem \( (A, K, \nu_0) \) is regular, if there exists constants \( L_1, \psi_U, L_\mathcal{A} > 0 \), \( \beta_1 > 1 \), and \( \Lambda_1, \Lambda_\mathcal{A} > 0 \), non-negative functions \( \phi = (\phi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A) \), and \( \psi_U \in C(\mathbb{U}) \) and \( \psi_\mathcal{A} \in C(\mathcal{A}) \) such that (i) \( |A\phi_n(u, a, t)| \leq \Lambda_1 (1 + \psi_U(u) + \psi_\mathcal{A}(a)) \) and \( \psi(u, a, t)^{\beta_1} \leq L_1 (1 + \psi_U(u) + \psi_\mathcal{A}(a)) \)
ψ_κ(a)) for all (u, a, t) ∈ U × A × T and n ∈ N; (ii) the sequence (φ_n)_{n∈N} is increasing and converges pointwise to ψ_U; (iii) ψ_U and ψ_κ are inf-compact; (iv) the initial distribution satisfies
\[ \int \psi_U(x) \nu_0(dx) \leq L_U; \] (2.2)
(v) for all μ ∈ K,
\[ \int \psi_κ(a) \mu_κ(da) \leq L_κ e^{\Lambda_κ t} \quad \forall t \in T; \] (2.3)
and (vi) K is closed in the weak topology.

As a notational aside, we use the symbol Λ for quantities representing exponential growth rates, L for bounds and constants of proportionality, and β for powers that control relative magnitudes and scaling rates between different quantities; naturally, the Λ’s are the most important, while the L’s tend to be the least significant.

The condition that a relaxed controlled martingale problem is regular can be viewed as a generalization of growth bounds on e.g. the solutions of stochastic differential equations, for which it is common to assume that the drift and diffusion coefficients have at most linear growth. The following example illustrates this.

**Example 2.7.** Consider a stochastic differential equations driven by orthogonal martingale measures, see e.g. [15], on X = R^d_A, A = R^d_a, d_x, d_a ∈ N, characterized by drift and diffusion functions b ∈ C(X × A × T, X) and σ ∈ C(X × A × T, X × X). Suppose b and σ have bounded growth in the sense that |b(x, a, t)|, |σ(z, a, t)| ≤ L(1 + |x| + |a|^q), q ∈ R_≥0, for some L > 0 and all (x, a, t) ∈ X × A × T (|·| stands for the Frobenius norm for matrices). The corresponding generator reads
\[ Gf(x, a, t) := b(x, a, t)^T \nabla f(x) + \frac{1}{2} \text{tr} \left\{ \sigma \sigma^T (x, a, t) \nabla^T \nabla f(x) \right\} \]
\[ \forall f \in C_c^{(2)}(X), (x, a, t) \in X × A × T, \]
and where \( \nabla \) and \( \nabla^T \nabla \) stand for the gradient and Hessian operators, respectively. The domain of G can be taken to be \( D(G) = \{ f + f_0 \mid f \in C_c^{(2)}(X), f_0 \in R \} \). The regularity conditions are satisfied e.g. with the choices \( \psi(x, a, t) = 1 + |a|^{2q}, \psi_U(x) = |x|^2 \), \( \psi_κ(a) = |a|^{2\beta_1} \) for all \( (x, a, t) \in X \times A \times T \), and where \( \beta_1 > 1 \) can be arbitrarily small. Additionally, we can take
\[ a_f = \bar{a} \left\{ \left\| (1 + |·|) \nabla f(·) \right\| + \left\| (1 + |·|^2) \nabla^T \nabla f(·) \right\| \right\} \quad \forall f \in D(G), \]
where \( \bar{a} \) is a constant independent of f.

The regularity assumptions guarantee that, almost surely, a càdlàg solution never explodes in the sense that, almost surely, \( \psi_U(u_t) \) is finite for all \( t \in T \).

**Proposition 2.8.** Suppose \( (A, K, \nu_0) \) is regular, with \( \psi_U \) and \( \psi_κ \) as in Definition 2.6. Then for all \( \pi \in D(A, K, \nu_0) \), \( \int_A (1 + \psi_U(u_t) + \psi_κ(a)) \pi(da) < \infty \) for all \( t \in T \), \( \pi \)-almost surely.

The regularity requirement is important, as it constrains the problems we consider to those with well-behaved trajectories. While weaker assumptions were used in the treatment of risk-neutral problems in e.g. [17][18], our approach describes the costs associated with each relaxed controlled solution via their distributions as given by the forward equation, and for validity of this approach, a higher degree of regularity is necessary.

**Remark 2.9.** We note the difference between the functions \( \psi \), as given in Assumption 2.5 and (\( \psi_U, \psi_κ \)), given in Definition 2.6. The former describes how large the functions in the range of the generator may be, while the latter characterize how large values the solutions themselves may take, cf. the bound given by Proposition 2.8.

### 2.2 Risk-aware objectives

Given a relaxed controlled problem \( (G, K, \nu) \) on Polish state and action spaces X and A, we then suppose we are also provided a cost rate function \( c \in C(X \times A \times T) \), and in the case of finite-horizon problems, a terminal cost function \( v \in C(X) \). In addition, we suppose we are given a risk function \( \rho: \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}_\infty \) that is defined on some reference probability space \( (\Omega, \Sigma, \mathbb{P}) \). Since relaxed controlled solutions in general come with their own probability spaces, we make the restriction to law-invariant risk functions, so that the problem is well-defined.
Definition 2.10. Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space. A mapping \(\rho : \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}_\infty\) is law invariant if there exists a function \(\tilde{\rho} : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_\infty\) such that \(\rho(X) = \tilde{\rho}(\mathcal{L}(X))\) for all \(X \in \mathcal{L}(\Omega; \mathbb{R})\).

The requirement that the risk functions are law invariant is very mild, and is in practice essentially always satisfied.

Any law invariant risk function \(\rho\) defined on random variables of some fixed probability space \((\Omega, \Sigma, \mathbb{P})\) can be used to evaluate the risk of random variables on any other \((\Omega', \Sigma', \mathbb{P}')\) by setting \(\rho(X') := \tilde{\rho}(\mathcal{L}(X'))\) for all \(X' \in \mathcal{L}(\Omega', \Sigma', \mathbb{P}; \mathbb{R})\). For law invariant risk functions we can then define the risk-aware problem, Problem \(\mathcal{P}_L\), as

\[
\inf_{\pi \in \mathcal{D}(G,K,\nu)} \limsup_{t \to \infty} \rho \left( \int_0^t e^{-\alpha s} \int_A c(x^s_\pi, a, s) \pi_s(d a) \, d s \right) \quad (T = \mathbb{R}_{\geq 0}),
\]

\[
\inf_{\pi \in \mathcal{D}(G,K,\nu)} \rho \left( \int_0^T c(x^s_\pi, a, s) \pi_s(d a) \, d s + v(x^T_\pi) \right) \quad (T = [0, T]).
\]

(2.4)

By Definition 2.10 a law invariant risk function \(\rho : \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}_\infty\) can be equivalently expressed using a functional \(\tilde{\rho} : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_\infty\). Since our dynamic analytic formulation constructs directly the distribution of the input random variable representing total costs, it will sometimes be more natural to consider the risk function as a functional on distributions rather than random variables. We note that the literature on risk functions typically favors the picture of a risk function as functional on random variables. Indeed, properties of risk functions such as coherence and convexity, important from both practical applications and theoretical analysis points of view \([5, 18, 20]\), are conventionally defined for \(\rho\) viewed as mappings from \(\mathcal{L}(\Omega; \mathbb{R})\) to \(\mathbb{R}_\infty\). Analogous properties can be defined for risk functions on probability measures, or equivalently, for \(\tilde{\rho}\), but in general, e.g. the convexity properties of \(\rho\) and \(\tilde{\rho}\) can be very different. In fact, convex risk functions generally have representations on measures that are concave \([2]\). Here, we shall not consider questions such as the uniqueness of solutions, and we do not require convexity of the risk functions.

Our baseline assumptions are then as follows.

Assumption 2.11. Let \(X\) and \(A\) be given Polish state and action spaces, with \(d\) denoting the metric on \(X\). (i) The generator \(G : C_b(X) \supset \mathcal{D}(G) \to \mathcal{L}(\mathbb{R}) \subset C(X \times A \times T)\), admissible solutions \(K\), and the initial distribution \(\nu \in \mathcal{P}(X)\) are such that the relaxed controlled martingale problem \((G, K, \nu)\) is regular; (ii) the cost rate function \(c \in C(X \times A \times T)\) is non-negative, and there are \(L_c > 0\) and \(\beta_c \geq \beta_1 > 1\) such that \(c^{\beta_c} \leq L_c(1 + \psi_X + \psi_A)\); (iii) if a finite time-horizon problem is considered, then we have a terminal cost function \(v \in C(X)\) that is non-negative, else we are given a discount rate \(\alpha > 0\); (iv) the function \(\rho\) is law invariant.

We will later require continuity of the risk functions, and in particular, continuity of its representation on measures. The following shows that if a risk function is continuous on random variables, then it is continuous on measures, and similarly for lower semicontinuity.

Proposition 2.12. Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space, \(\rho : \mathcal{L}^p(\Omega; \mathbb{R}) \to \mathbb{R}, p \in [1, \infty]\), and let \(\tilde{\rho} : \mathcal{P}(\mathbb{R}) \to \mathbb{R}\) be such that \(\rho(X) = \tilde{\rho}(\mathcal{L}(X))\) for all \(X \in \mathcal{L}^p(\Omega; \mathbb{R})\). If \(\rho\) is continuous (respectively lower semicontinuous) in the strong, \(\| \cdot \|_p\)-norm topology, then \(\tilde{\rho}\) is continuous (respectively lower semicontinuous) in the topology induced by the \(p\)-Wasserstein metric.

Continuity holds for many common risk functions. Indeed, convex risk functions \(\rho : \mathcal{L}^p(\Omega; \mathbb{R}) \to \mathbb{R}\) are norm-continuous \([3, 3]\), and hence their representations in terms of functionals over measures are also continuous.

Example 2.13. Returning to the problem of Example 2.7, we can now consider cost rate functions that satisfy Assumption 2.11. In particular, the cost rate function \(c(x, a, t) := 1 + |x|^q + |a|^{q_2}\), or anything by this, for all \((x, a, t) \in X \times A \times T\) is admissible, if \(q_1 \leq 2/\beta_c, q_2 \leq 2q_1\beta_1/\beta_c\) for some \(\beta_c \geq \beta_1 > 1\). As examples of law invariant risk functions, we mention here the entropic risk function \(\rho^{\text{Ent}} : \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}_\infty\), and the mean semi-deviation risk function \(\rho^{\text{MD+}} : \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}_\infty\). For an arbitrary reference probability space \((\Omega, \Sigma, \mathbb{P})\), these are defined for any \(X \in \mathcal{L}(\Omega; \mathbb{R})\) as

\[
\rho^{\text{Ent}}(X) := \frac{1}{\beta} \ln \mathbb{E} \left[ e^{\beta X} \right],
\]

\[
\rho^{\text{MD+}}(X) := \mathbb{E} [X] + \beta \mathbb{E} [(X - \mathbb{E} [X])_+],
\]
and where $\theta \in (0, \infty)$ and $\beta \in [0,1]$ are parameters. These have the following representations as functions on probability measures: For all $\mu \in \mathcal{P}(\mathbb{R})$,

$$\tilde{\rho}^{\text{Ent}}(\mu) := \frac{1}{\theta} \ln \left( \int e^{\theta x} \mu(dx) \right),$$

$$\tilde{\rho}^{\text{MD}+}(\mu) := \int x \mu(dx) + \beta \left( x - \int x' \mu(dx) \right) \mu(dx).$$

Other examples would include e.g. mean-variance risk functions, and the conditional value-at-risk.

### 3 Dynamic analytic formulation

We can now construct our dynamic analytic formulation of the problem. The first step is to find evolution equations for the joint distribution of the controlled processes state and accumulated costs.

**Forward equation and time-dependent distributions** The main tool for finding the time-dependent distribution of a stochastic process is the Kolmogorov forward equation, which we shall discuss next.

**Definition 3.1.** We say that $\mu \in M(\mathbb{T}, \mathcal{P}(\mathbb{U} \times \mathbb{A}))$ satisfies the forward equation for initial condition $\nu_0 \in \mathcal{P}(\mathbb{U})$ and generator $A : C_b(\mathbb{U}) \supset \mathcal{D}(A) \to \mathcal{R}(A) \subset C(\mathbb{U} \times \mathbb{A} \times \mathbb{T})$ if (we recall our notation where superscripts on measures indicate taking marginals)

$$\int_\mathbb{U} f(u) \mu_t^U(du) - \int_\mathbb{U} f(u) \nu_0(du) = \int_0^t \int_\mathbb{U} \mathcal{A} f(u, a, s) \mu_s(du \times da) \, ds,$$

for all $f \in \mathcal{D}(A)$ and $t \in \mathbb{T}$. We use $\mathcal{F}(\mathbb{A}, \nu_0) \subset M(\mathbb{T}, \mathcal{P}(\mathbb{U} \times \mathbb{A}))$ to denote the set of solutions of Eq. (3.1) and constrained solutions of Eq. (3.1) are defined analogously to Definition 2.2: $\mathcal{F}(\mathbb{H}, \nu_0) := \mathcal{F}(\mathbb{A}, \nu_0) \cap K$, where $K \subset M(\mathbb{T}, \mathcal{P}(\mathbb{U} \times \mathbb{A}))$ is again the set of admissible solutions.

**Cost distribution** To evaluate a law invariant risk function appearing in the objective, we need means for finding the distribution of the costs appearing in the objective, and the conditional value-at-risk. This is done by introducing an extended forward equation corresponding to a given martingale problem $(G, K, \nu)$ that gives the joint distribution of the state and running costs, that is, cost accumulated up to a given time $t \in \mathbb{T}$.

We define $\mathbb{Y} := \mathbb{R}_{\geq 0}$ to stand for the state space of the running costs, and consider the original state and the running costs in parallel on the space $\mathbb{X} \times \mathbb{Y}$. The equation for the joint distribution of states and costs shall be the forward equation corresponding to a new generator $H$, describing the joint evolution of the states variables, and the same equation can be co-opted to additionally yield the cost distribution. This is done by introducing an extended forward equation corresponding to a given martingale problem $(G, K, \nu)$ that gives the joint distribution of the state and running costs, that is, cost accumulated up to a given time $t \in \mathbb{T}$.

For each $\pi \in \mathcal{R}(\mathbb{H}, K, \nu)$, we associate a real-valued running costs process $y^\pi = (y^\pi_t)_{t \in \mathbb{T}}$, defined

$$y^{\pi}_t := \int_0^t e^{-\alpha s} \int_\mathbb{A} c(x^\pi_s, a, s) \pi_s(da) \, ds \quad \forall t \in \mathbb{T},$$

where $\alpha = 0$ if $\mathbb{T} = [0, T]$. The following theorem states that under our baseline assumptions, considering càdlàg relaxed controlled solutions $\mathcal{D}(\mathbb{H}, K, \nu)$ together with costs $y^\pi$ as defined in Eq. (3.3), is equivalent to considering solutions to the forward equation for joint, time-dependent state-cost distributions, $\mathcal{F}(\mathbb{H}, K, \nu)$. That is, a solution
for either (i) the martingale problem with running costs or (ii) the extended forward equation problem, can be used to construct a solution for the other problem type. For brevity, we are using formally the same set of admissible solutions $K \subset M(T, \mathcal{P}(X \times \Lambda))$ for both problems; in $\mathcal{F}(H, K, v)$ the constraints are assumed to hold for the $X \times \Lambda$-marginals of the $M(T, \mathcal{P}(X \times Y \times \Lambda))$ solutions.

**Theorem 3.2.** Suppose Assumption 2.11 holds, so that $(G, K, \nu)$ is a regular controlled martingale problem on the state-action space $X \times \Lambda$.

(i) If $\mu \in \mathcal{F}(H, K, \nu)$, then there exists a càdlàg relaxed controlled solution $\pi \in \mathcal{D}(G, K, \nu)$ and a cost process $y^\pi$ defined by Eq. (3.3), such that the finite dimensional distributions of $(x^\pi, y^\pi)$ are given by $\mu^{|X \times Y}$, and with the control process satisfying $\pi_t = \mu_t^{|X \times Y}(\cdot \mid x_t, y_t)$ for all $t \in T$.

(ii) If $\pi \in \mathcal{D}(G, K, \nu)$ and $y^\pi$ is the associated costs process of Eq. (3.3), then $\mu_t \in M(T, \mathcal{P}(X \times Y \times \Lambda))$ defined

$$\int_{X \times Y \times \Lambda} h(x, y, a) \mu_t(dx \times dy \times da) = \mathbb{E} \left[ \int_{\Lambda} h(x_t, y_t, a_\pi) \pi_t(da) \right],$$

for all $h \in C_b(X \times Y \times \Lambda)$ and $t \in T$ is a solution $\mu \in \mathcal{F}(H, K, \nu)$.

The proof is deferred to the second half of this section. We can now move on to state our main results.

**Main results** We define the dynamic analytic problem, Problem $\mathcal{P}_p$ as

$$\inf_{\mu \in \mathcal{F}(H, K, \nu)} \limsup_{t \to \infty} \hat{\rho}(\mu_t^{
abla}) \quad (T = \mathbb{R}_\geq 0),$$

$$\inf_{\mu \in \mathcal{F}(H, K, \nu)} \hat{\rho}(\mu_t^{|X \times Y} \circ \Theta^{-1}) \quad (T = [0, T]),$$

where $\Theta(x, y) := y + v(x)$ for all $(x, y) \in X \times \mathcal{Y}$, $\hat{\rho} : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is such that for the given risk function $\rho : \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}$, $\rho(X) = \hat{\rho}(\mathbb{L}(X))$ for all $X \in \mathcal{L}(\Omega; \mathbb{R})$.

Two theorems comprise our main results. The first states that under our baseline assumptions, Problems $\mathcal{P}_\mathcal{L}$ and $\mathcal{P}_p$ are equivalent, and optimal controls are Markov in the state, running costs, and time.

**Theorem 3.3.** If Assumption 2.11 holds, then the optimal values of Problems $\mathcal{P}_\mathcal{L}$ and $\mathcal{P}_p$ are equal. If there is a $\mathcal{P}_p$-optimal $\mu \in \mathcal{F}(H, K, \nu)$, then there exists a $\mathcal{P}_\mathcal{L}$-optimal $\pi \in \mathcal{D}(G, K, \nu)$ such that $\pi_t = \mu_t^{|X \times Y}(\cdot \mid x_t, y_t)$ for all $(t, x, y) \in T \times X \times \mathcal{Y}$. That is, the control $\pi$ is Markov, depending only on time, state, and running costs.

While Theorem 3.3 guarantees that the forward equation formulation, Problem $\mathcal{P}_p$, yields the same optimal value as solutions of Problem $\mathcal{P}_\mathcal{L}$, it does not establish the existence of solutions. The following theorem and our second main result gives sufficient conditions for there to be a $\mu \in \mathcal{F}(H, K, \nu)$ that attains the optimal value, provided the next assumptions hold.

**Assumption 3.4.** Assumption 2.11 holds, and additionally: (i) $\mathcal{D}(G) \subset C_b(\mathcal{X})$, and if $a_f$ is as in Assumption 2.7(iv), then $f \to a_f$ defines a seminorm on $\mathcal{D}(G)$ and $a_f \geq ||f||$, the Lipschitz constant, for all $f \in \mathcal{D}(G)$; (ii) in the infinite time-horizon case, the discount rate satisfies $\alpha > (\Lambda_1 + \Lambda_6)/\beta$; (iii) either (a), the risk function $\rho : \mathcal{L}(\Omega; \mathbb{R}) \to \mathbb{R}_\infty$ is bounded from below, and continuous and coercive on $L^p(\Omega; \mathbb{R})$ for some $p \in [1, \infty)$, that is, $\|X\|_p \to \infty$ implies $\rho(X) \to \infty$, or (b), the cost rate function, and the terminal cost function if $T = [0, T]$, are bounded, the risk function $\rho$ is finite for compactly supported random variables, and its representation $\hat{\rho}$ on measures is continuous in the topology of weak convergence.

If a finite time horizon is considered, then the continuity of $\rho$ or $\hat{\rho}$ may be replaced by lower semicontinuity.

The condition that $\mathcal{D}(G)$ is a subset of bounded Lipschitz functions and that $a_f$ is a seminorm bounded by $||\cdot||$ is used to construct a metric on probability measures that allows us to prove uniform convergence and equicontinuity of families of solutions to the forward equation. Note that e.g. the $a_f$ obtained in Example 2.7 is indeed a seminorm bounded from below by the Lipschitz constants. The lower bound on the discount rate $\alpha$ is needed to ensure that the cost distributions become stationary as time tends to infinity. Intuitively, the exponent $\beta$ describes how fast the cost rate $c$ grows relative to the growth of the solutions, represented by $\psi$, cf. Assumption 2.11. The part $\Lambda_1 + \Lambda_6$ in turn gives the growth rate of $\psi_X + \psi_\Lambda$, as given by Definition 2.6(i). Hence, the inequality describes the balance between the growth of costs and the rate of discounting; satisfying it guarantees that the risks converge rather than oscillate as time tends to infinity. Continuity or lower semicontinuity of the risk function is naturally necessary, as we will be taking limits of minimizing sequences. Part (iii) of the assumption is split into (a) and (b) alternatives and the latter case is included to accommodate risk functions defined on essentially bounded random variables, that is, the case where $p = \infty$. 

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Theorem 3.5. Suppose Assumption 2.4 holds, and let $\rho^* \in \mathbb{R}_\infty$ be the $\mathcal{P}_\rho$-optimal value. If $\mathcal{F}(H, K, \nu) \neq \emptyset$, then there is a $\mu \in \mathcal{F}(H, K, \nu)$ that attains $\rho^*$.

The result of Theorem 3.5 immediately implies that the corresponding relaxed controlled martingale problem, $(G, K, \nu)$, has an optimal solution for which the control process is Markov in time, state, and running costs.

Proofs of main results. We begin with the proof of Theorem 3.2. To this end, we first give a pair of auxiliary results, first one stating that càdlàg relaxed controlled solutions to processes, are in a sense equivalent to càdlàg relaxed controlled solutions to the augmented problem, $(H, K, \nu)$. Some proofs are deferred to the Appendix.

Proposition 3.6. Suppose Assumption 2.4 holds and $\pi$ is a càdlàg relaxed controlled solution to the problem $(H, K, \nu)$, then there is a function $\tilde{y}_t^\pi := \int_0^t e^{-as} \int c(x_s, a, s)\pi(a)\mu(da)\,ds$ for all $t \in \mathbb{T}$, we have that of $y^\pi$ and $\tilde{y}^\pi$ are indistinguishable. Conversely, if $\pi$ is a càdlàg relaxed controlled solution to the problem $(G, K, \nu)$, and $y^\pi$ the corresponding running costs process, then $(\Omega^\pi, \Sigma^\pi, \mathbb{F}^\pi, \mathbb{P}^\pi, (x^\pi, y^\pi), \pi) \in \mathcal{D}(H, K, \nu)$.

Proof of Proposition 3.6 is given in Appendix A.2.

Proposition 3.9. Suppose $(A, K, \nu_0)$ is a regular relaxed controlled martingale problem, and $\psi$ satisfy Assumption 2.5. If $\mu \in \mathcal{F}(A, \nu_0)$ is a solution of the forward equation satisfying
\[
\int_0^t \int_{\mathcal{U} \times \Lambda} \psi(u, a, s) \mu(du \times da)\,ds < \infty \quad \forall t \in \mathbb{T},
\]
then there exists a relaxed controlled solution $(u^\pi_t, \pi_t)_{t \in \mathbb{T}} \in \mathcal{R}(A, \nu_0)$ such that $\mathcal{L}(u_t) = \mu_t^U$ and $\pi_t = \mu_t^A(\cdot | u_t)$ for all $t \in \mathbb{T}$.

Remark 3.8. In the given reference, this result is stated as applying to uncontrolled problems, in particular, to an uncontrolled generator $\hat{A} : C_0(\mathcal{U}) \supseteq \mathcal{P}(\hat{A}) \to \mathcal{R}(\hat{A}) \subset M(\mathcal{U})$. We note however that the generator $\hat{A}$ is constructed from a controlled generator $\hat{A}$ satisfying an equivalent of our Assumption 2.5 by integrating it over a transition function $\eta \in \mathcal{P}(\hat{A} | \mathcal{U})$. Re-writing the theorem in terms of the controlled generator $\hat{A}$ recovers the result stated above. The utility of constructing an uncontrolled generator in this way is in the fact that after integrating over a control, the resulting generator needs to satisfy notably weaker conditions than the original, controlled generator, in particular, the generator $\hat{A}$ can have its range extend to discontinuous measurable functions.

An essential step in applying Theorem 3.7 is showing that Eq. (3.5) holds. A similar condition was already shown to be true for càdlàg relaxed controlled solution in Proposition 2.8 and the following proposition can be viewed as an analogue of that result for solutions of the forward equation.

Proposition 3.9. Suppose $(A, K, \nu_0)$ is a regular relaxed controlled martingale problem, and $\mu \in \mathcal{F}(A, K, \nu_0)$. Then
\[
\int_{\mathcal{U} \times \Lambda} (1 + \psi_0(u) + \psi_0(a)) \mu_t(du \times da) \leq \mathcal{L}_\psi e^{\Lambda_\psi t} \quad \forall t \in \mathbb{T},
\]
where $\Lambda_\psi := \Lambda_1 + \Lambda_\Lambda$ and $\mathcal{L}_\psi \in \mathbb{R}_{>0}$ is independent of $\mu$. Eq. (3.5) holds, and $\mu_t^U \in C(\mathbb{T}, \mathcal{P}(\mathcal{U}))$. Moreover, for all $\epsilon > 0$ and $t \in \mathbb{T}$, there exists $\delta_{\epsilon, t} > 0$ such that
\[
\left| \int_{\Lambda} f(u)\mu_t^U(du) - \int_{\Lambda} f(u)\mu_0^U(du) \right| < a_f\epsilon \quad \forall|t - s| < \delta_{\epsilon, t}, \ f \in \mathcal{P}(A),
\]
where $a_f$ is as in Assumption 2.5 iv) and $\delta_{\epsilon, t}$ does not depend on $f$ or $\mu$.

Equipped with the above results, we can move on to the proof of Theorem 3.2.

Proof of Theorem 3.2. By Proposition 3.6 we may consider càdlàg relaxed controlled solutions $\mathcal{D}(H, K, \nu)$ instead of solutions $\mathcal{D}(G, K, \nu)$ together with their associated costs processes.

(i) Let then $\mu \in \mathcal{F}(H, K, \nu)$. It is straight-forward to verify that $\mu^{X \times \Lambda} := (\mu^{X \times \Lambda}_t)_{t \in \mathbb{T}}$ is in $\mathcal{F}(G, K, \nu)$, and so by Proposition 3.9
\[
\int_0^t \int_{\mathcal{X} \times \Lambda} \psi(x, a, s) \mu^{X \times \Lambda}_t(dx \times da)\,ds < \infty \quad \forall t \in \mathbb{T},
\]
and Theorem 3.7 yields a relaxed controlled solution $\pi \in \mathcal{R}(G, K, \nu)$.

By Remark 3.9, we can suppose the solution $\pi$ obtained from Theorem 3.7 has the form $x^\pi_t = \Gamma^{-1}(\xi^\pi_t)$, where $\xi^\pi$ is an adapted càdlàg process, and $\hat{X}$ and $G : \hat{X} \to \mathcal{X}$ are defined

$$\hat{X} := \left\{ -\|f_1\|, +\|f_1\| \times -\|f_2\|, +\|f_2\| \times \ldots \right\},$$

$$\Gamma(x) := (f_1(x), f_2(x), \ldots) \quad \forall x \in \mathcal{X},$$

with the functions $(f_k)_{k \in \mathbb{N}}$ being as in Assumption 2.5(v). The space $\hat{X}$ is compact and the mapping $\Gamma$ is continuous with a measurable inverse $\Gamma^{-1} : \Gamma(\hat{X}) \to \hat{X}$. Since we require in Assumption 2.5(iii) that $\mathcal{D}(G)$ strongly separates points, then by Lemma 1, $\Gamma^{-1}$ is continuous. It then follows that $x^\pi = (\xi^\pi_t = \Gamma^{-1}(\xi^\pi_t))$ is also càdlàg up to the first time $t$ such that $\lim_{s \to t} \xi^\pi_s \notin \Gamma(\hat{X})$ for all $t \in \mathbb{T}$. Clearly, $\xi^\pi$ exits the image of $\hat{X}$ when $\lim_{s \to t} \psi_\xi(x^\pi_s) = \infty$. We can now use the argument of Proposition 2.8 to estimate the first time at which the càdlàg process $x^\pi$ reaches infinity, and conclude that almost surely this never happens. Therefore, the solution is càdlàg for all $t \in \mathbb{T}$, almost surely.

For part (ii), suppose $\pi \in \mathcal{D}(H, K, \nu)$ and let $\mu$ be as in Eq. (3.4). Letting $fg \in \mathcal{D}(H)$ be arbitrary, and taking the expectation of Eq. (2.1), one finds that $\mu \in \mathfrak{F}(H, K, \nu)$.

It remains for us to provide the short proof of Theorem 3.3.

**Proof of Theorem 3.3**. The equality of $\mathcal{P}_L$ and $\mathcal{P}_P$-optimal values follows now from Theorem 3.2 and the law invariance of the risk function. In addition, if $\pi \in \mathcal{D}(G, K, \nu)$ is $\mathcal{P}_L$-optimal, then there exists a $\mathcal{P}_P$-optimal $\mu \in \mathfrak{F}(H, K, \nu)$ with the same optimal value. Again, by Theorem 3.2, there exists a $\tilde{\pi} \in \mathcal{D}(G, K, \nu)$ constructed from $\mu$ such that the control process has the form given in the statement of the theorem.

For the proof of Theorem 3.3, we introduce a family of metrics, whose members each induce a topology at least as fine that of weak convergence.

**Lemma 3.10**. Suppose $\mathcal{U}$ is Polish, $\mathcal{G} \subset C_b(\mathcal{U})$ is an algebra that strongly separates points, and $\| \cdot \|_{\mathcal{G}} : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a seminorm. We define $d_{\mathcal{G}} : \mathcal{P}(\mathcal{U}) \times \mathcal{P}(\mathcal{U}) \to \mathbb{R}_{\geq 0}$ via

$$d_{\mathcal{G}}(\mu, \nu) := \sup \left\{ \left\| \frac{\int f(u)\mu(du)}{\|f\|_{\mathcal{G}}} - \frac{\int f(u)\nu(du)}{\|f\|_{\mathcal{G}}} \right\| : f \in \mathcal{G}, f \neq 0 \right\}, \quad \forall \mu, \nu \in \mathcal{P}(\mathcal{U}),$$

which is equivalent to the definition $d_{\mathcal{G}}(\mu, \nu) := \sup_{f \in \mathcal{G}_1} \left\{ \left\| \frac{\int f(u)\mu(du)}{\|f\|_{\mathcal{G}}} - \frac{\int f(u)\nu(du)}{\|f\|_{\mathcal{G}}} \right\| : \|f\|_{\mathcal{G}} \leq 1 \right\}$, where $\mathcal{G}_1 := \{ f \in \mathcal{G} \mid \|f\|_{\mathcal{G}} \leq 1 \}$ for all $\mu, \nu \in \mathcal{P}(\mathcal{U})$.

(i) The mapping $d_{\mathcal{G}}$ is a metric, and convergence in the topology induced by $d_{\mathcal{G}}$ implies weak convergence. (ii) If $\mathcal{G}'$ is a subalgebra of $\mathcal{G}$, and there is a seminorm $\| \cdot \|_{\mathcal{G}'} : \mathcal{G} \to \mathbb{R}_{\geq 0}$ such that $\| \cdot \|_{\mathcal{G}} \leq \| \cdot \|_{\mathcal{G}'}$, then

$$d_{\mathcal{G}'}(\mu, \nu) \leq d_{\mathcal{G}}(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{P}(\mathcal{U}).$$

The $d_{\mathcal{G}}$-metrics defined above include the bounded Lipschitz metric, $d_{bl}$ defined below, as a special case. As a consequence of Lemma 3.10, we obtain the following comparison result.

**Corollary 3.11**. Suppose $\mathcal{U}$ is Polish, and define the bounded Lipschitz metric $d_{bl} : \mathcal{P}(\mathcal{U}) \times \mathcal{P}(\mathcal{U}) \to \mathbb{R}_{\geq 0}$ as in [13, Section 11.3],

$$d_{bl}(\mu, \nu) := \sup \left\{ \left\| \int f(u)\mu(du) - \int f(u)\nu(du) \right\| : f \in C_b(\mathcal{U}), \|f\|_{bl} \leq 1 \right\},$$

for all $\mu, \nu \in \mathcal{P}(\mathcal{U})$. Then, if $(\mathcal{G}, \| \cdot \|_{\mathcal{G}})$ is as in the statement Lemma 3.10 and $\mathcal{G} \subset C_b(\mathcal{U})$ and $\| \cdot \|_{\mathcal{G}} \geq \| \cdot \|_l$ on $\mathcal{G}$, then $d_{\mathcal{G}}(\mu, \nu) \leq d_{bl}(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}(\mathcal{U})$ and the topology induced by $d_{\mathcal{G}}$ is equivalent to the topology of weak convergence.

**Proof**. The set $C_b(\mathcal{U})$ is an algebra that strongly separates points, and the Lipschitz constant $\| \cdot \|_l$ is a seminorm on $C_b(\mathcal{U})$. Hence the definition of $d_{bl}$ is a special case of the metrics defined in Lemma 3.10. From there, it follows that $d_{\mathcal{G}}(\mu, \nu) \leq d_{bl}(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}(\mathcal{U})$, and therefore the topology induced by $d_{bl}$ is finer than that of $d_{\mathcal{G}}$, which itself is finer than the topology of weak convergence. However, by [13, Theorem 11.3.3], $d_{bl}$ and $d_P$ yield equivalent topologies, and so also $d_{\mathcal{G}}$ induces the same topology.

\[\square\]
Note that tightness is a topological property: If \( U \) is Polish, then the tightness of a set \( M \subset \mathcal{P}(U) \) implies convergence in any metric that is equivalent to the Prokhorov metric \( d_p \). Since sequential compactness implies compactness in metric spaces, tightness of \( M \) further implies it has compact closure, regardless of which (equivalent) metric is used. Continuing this line of reasoning, we get the following result.

**Corollary 3.12.** Suppose the assumptions of Corollary 3.11 hold. Then the metric \( d_G \) is complete.

*Proof.* We may simply follow the proof of [15, Theorem 11.5.4]; the details are omitted here. As argued above, tightness of a \( M \subset \mathcal{P}(U) \) implies that \( M \) has compact closure in any metric equivalent to \( d_p \). Hence, \( M \) is totally bounded in any of the metrics \( d_M, d_G, \) or \( d_p \). Total boundedness of a \( M \subset \mathcal{P}(U) \) can subsequently be shown to imply tightness, and hence convergence of a subsequence in any of the metrics. It then suffices to note, as in [15, Corollary 11.5.5] that Cauchy sequences are totally bounded.

**Remark 3.13.** In the light of the above discussion, it is clear that in Assumption 3.4(i) and Corollary 3.11 \((C_b(U), \| \cdot \|) \) can be replaced by any \((G^*, \| \cdot \|_{G^*})\) satisfying assumptions of Lemma 3.10 if \( d_M \leq d_{G^*} \) and \( d_{G^*} \) is topologically equivalent to \( d_p \). Note also that at one extreme, we can choose \( G^{**} = C_b(U) \) and \( \| \cdot \|_{G^{**}} = 0 \). We then obtain a metric \( d_{G^{**}} \) such that \( d_G \leq d_{G^{**}} \) for all other metrics \( d_G \) obtained from Lemma 3.10. The metric \( d_{G^{**}} \) coincides with the total variation norm of signed measures, see e.g. [23, Section 29], and it is therefore not topologically equivalent to \( d_p \).

We shall also need the following basic statement regarding sets that strongly separate points. We omit the proof, as it is a straightforward application of [9, Lemma 4].

**Proposition 3.14.** Let \( U \) and \( \mathcal{V} \) be Polish spaces, and suppose \( \mathcal{G} \subset C_b(U) \) and \( \mathcal{H} \subset C_b(\mathcal{V}) \) strongly separate points. Then \( \mathcal{J} := \{ f g \mid f \in \mathcal{G}, g \in \mathcal{H} \} \subset C_b(U \times \mathcal{V}) \) strongly separates points.

As the first step towards proving Theorem 3.5 we give a compactness result for families of solutions to the forward equation.

**Lemma 3.15.** Suppose Assumption 3.4 holds, and \( \{ \mu^{(n)} \}_{n \in \mathbb{N}} \subset \mathcal{F}(H,K,\nu) \) is such that there are \( Y \geq 0 \) and \( N \in \mathbb{N} \) for which

\[
\sup_{n > N} \limsup_{t \to \infty} \int y^p \mu_T^{(n)}(Y) \leq Y \quad (T = \mathbb{R}_{\geq 0}) \quad \text{or} \quad 
\sup_{n > N} \int y^p \mu_T^{(n)}(Y) \leq Y \quad (T = [0,T]).
\]

Then there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) and a \( \mu \in M(T, \mathcal{P}(X \times \mathcal{Y} \times \mathcal{A})) \) such that \( \mu^{(n_k)} \overset{w}{\to} \mu, \mu^{X \times \mathcal{Y}} \in C(T, \mathcal{P}(X \times \mathcal{Y})) \), and \( \mu^{(n_k)} = \mu^{X \times \mathcal{Y}} \) for all \( t \in T \) and the limit \( \mu \) satisfies the forward equation, \( \mu \in \mathcal{F}(H,K,\nu) \).

*Proof.* We first note that the sets \( \{ \mu_T^{(n)} \}_{n \in \mathbb{N}} \) are tight for each \( t \in T \). The functions \( \{ \mu_T^{X \times \mathcal{Y}} \}_{n \in \mathbb{N}} \) are solutions to the forward equation corresponding to \( G_t \), and the assumptions of Proposition 3.9 hold. By Eq. (3.6), inf-compactness of \( \psi_X \) and \( \psi_A \), and Proposition A.1(iii), for every \( t \in T \) the set of measures \( \{ \mu_T^{X \times \mathcal{Y}} \}_{n \in \mathbb{N}} \) is tight. From the forward equation for \( H_t \), by considering non-negative functions \( g_{\ell} \in \mathcal{D}(G) \), \( \ell \in \mathbb{N} \), that are constant on \( X \) and increasing towards \((-p)^{P} \), and using the monotone convergence theorem, we find

\[
\int y^p \mu_T^{(n)}(Y) \, (dy) = \int_0^t e^{-\alpha s} \int_X x^p qy^{p-1} \mu_T^{X \times \mathcal{Y}}(dx \times dy \times d\alpha) \, ds \quad \forall t \in T, \, n > N.
\]

Therefore, \( t \to \int y^p \mu_T^{(n)}(Y) \, (dy) \) is increasing and bounded by \( Y \) for all \( t \in T \) for at least all \( n > N \). Proposition A.1(ii) asserts the tightness of \( \{ \mu_T^{X \times \mathcal{Y}} \}_{n \in \mathbb{N}} \), so that \( \{ \mu_T^{(n)} \}_{n \in \mathbb{N}} \) is tight for all \( t \in T \).

We next consider the continuity of the solutions \( \{ \mu^{(n)} \}_{n \in \mathbb{N}} \). Let \( \mathcal{H} \) be the linear span of \( \mathcal{D}(H) \) and define

\[
\| h \|_{\mathcal{H}} := \sup_{y \in \mathcal{Y}} \| a_{h(y)} \| + \| \partial_y h \| \quad \forall h \in \mathcal{H},
\]

where \( \partial_y \) is the partial derivative along the \( \mathcal{Y} \)-space. By using the assumption that \( a_{\cdot} \) is a seminorm, it follows that \( \| \cdot \|_{\mathcal{H}} \) is also a seminorm. The set \( \mathcal{H} \) is closed under multiplications, is therefore an algebra, and by Proposition 3.14...
strongly separates points. Moreover, \(\|h\|_H \geq \|h\|_I\) for all \(h \in H\), which can be shown using elementary estimates:

For any \(h \in H\),

\[
\|h\|_H \geq \sup_{y \in Y} \|h(\cdot, y)\| + \sup_{(x, y) \in X \times Y} |\partial_y h(x, y)|
\]

\[
\geq \sup_{y \in Y} \|h(\cdot, y)\| + \|h(x, \cdot)\|_I
\]

\[
= \sup_{y \in Y} \sup_{x \neq x'} \frac{|h(x', y) - h(x, y)|}{d(x', x)} + \sup_{y \in Y} \sup_{x \neq x'} \frac{|h(x, y') - h(x, y)|}{|y' - y|},
\]

\[
\|h\|_I = \sup_{(x,y) \neq (x',y')} \frac{|h(x', y') - h(x, y)|}{d(x', x) \vee |y' - y|}
\]

\[
\leq \sup_{(x,y) \neq (x',y')} \frac{|h(x', y) - h(x, y)|}{d(x', x) \vee |y' - y|} + \sup_{(x,y) \neq (x',y')} \frac{|h(x', y') - h(x, y)|}{d(x', x) \vee |y' - y|}
\]

\[
\leq \sup_{x \neq x' \neq x'' \in Y} \frac{|h(x, y'') - h(x, y')|}{d(x', x) \vee |y' - y|} + \sup_{y \neq y' \neq y'' \in X} \frac{|h(x', y'') - h(x', y')|}{|y' - y|}.
\]

From this, we get \(\|h\|_H \geq \|h\|_I\) for all \(h \in H\). Selecting \(G = H\) in Lemma 3.10, we obtain a metric \(d_H\) on \(P(X \times Y)\), and by Corollary 3.11 \(d_H\) induces the topology of weak convergence.

We now use the metric \(d_H\) to estimate the distances between \(\mu_i^{(n)} X \times Y\) and \(\mu_s^{(n)} X \times Y\), \(n \in \mathbb{N}\) and \(t, s \in T\). Proposition 3.9 is not directly applicable to the augmented problem \((H, K, v)\), as this has not been established to be regular, but an analogue of Eq. 3.7 nonetheless holds. A straightforward calculation yields that

\[
\left| \int h(x, y)\mu^{(n)} X \times Y (dx \times dy) - \int h(x, y)\mu^{(n)} X \times Y (dx \times dy) \right|
\]

\[
\leq \sup_{y \in Y} \int_{-t}^{t} \int_{-t}^{t} \psi(x, a, r) \mu_r(dx \times dy \times da) dr
\]

\[
+ \|\partial_y h\| \int_{-t}^{t} e^{-\alpha r} \int c(x, a, r) \mu_r(dx \times dy \times da) dr
\]

\[
\leq \sup_{y \in Y} \int_{-t}^{t} \int_{-t}^{t} \left(1 + \psi_V(u) + \psi_V(a)\right)^{1/\beta_i} \mu_r(dx \times dy \times da) dr
\]

\[
+ \|\partial_y h\| \int_{-t}^{t} e^{-\alpha r} \int \left(1 + \psi_V(u) + \psi_V(a)\right)^{1/\beta_i} \mu_r(dx \times dy \times da) dr
\]

\[
\leq \|h\|_H \left(\int_{-t}^{t} \int_{-t}^{t} \left(1 + \psi_V(u) + \psi_V(a)\right) \mu_r(dx \times dy \times da) dr\right),
\]

for all \(n \in \mathbb{N}\), \(t, s \in T\) and \(h \in H\). Estimating as in Eq. (4.4), we find that for all \(\epsilon > 0\) and \(t \in T\), there exists a \(\delta_{\epsilon, t} > 0\) such that

\[
\sup_{n \in \mathbb{N}} \int h(x, y)\mu^{(n)} X \times Y (dx \times dy) - \int h(x, y)\mu^{(n)} X \times Y (dx \times dy) \leq \|h\|_H \epsilon,
\]

for all \(|t - s| < \delta_{\epsilon, t}\) and \(h \in H\). From the definition of \(d_H\) and Eq. (3.9), we get

\[
d_H \left(\mu^{(n)} X \times Y, \mu_s^{(n)} X \times Y\right) \leq \epsilon \quad \forall \ |t - s| < \delta_{\epsilon, t}.
\]

Therefore, the set \(\{\mu^{(n)} X \times Y\}_{n \in \mathbb{N}}\) is pointwise equicontinuous when considered as a family of mappings from \((T, \cdot, |\cdot|)\) to \((P(X \times Y), d_H)\). Above, we already showed that \(\{\mu^{(n)} X \times Y\}_{n \in \mathbb{N}}\) are tight for each \(t \in T\), and hence have compact closure. By the Arzelà-Ascoli theorem [32, Theorem 4.17], \(\{\mu^{(n)} X \times Y\}_{n \in \mathbb{N}}\) has compact closure, and consequently there is a subsequence \((n_k)_{k \in \mathbb{N}}\) such that \((\mu^{(n_k)} X \times Y)_{k \in \mathbb{N}} \in C(T, (P(X \times Y), d_H))^{\mathbb{N}}\) converges to a limit \(\mu^{X \times Y} \in C(T, (P(X \times Y), d_H))\). Since \(d_H\) and \(d_P\) are topologically equivalent, the limit \(\mu^{X \times Y} \in C(T, (P(X \times Y), d_P))\) and \(\mu^{(n_k)} X \times Y \Rightarrow \mu^{X \times Y}\) for all \(t \in T\). For simplicity, we suppose the whole sequence converges.

We remark that albeit \(d_P\) and \(d_H\) are topologically equivalent, there appears to be no easy way of replacing the latter by the former in the above argument. This is because equicontinuity depends on the properties of the metric rather than that of the topology generated by it. The same applies for uniform convergence.
Next, we want to show that $\mu^{(n)} \xrightarrow{w} \mu$. Let $L_\Psi$, $\Lambda_\Psi$ be as in the statement of Proposition 3.9 and define $k^{(n)} \in P(X \times Y \times \mathbb{A} \times \mathbb{T})$ via

$$
k^{(n)}(dx \times dy \times da \times dt) := (\Lambda_\psi + 1)e^{-((\Lambda_\psi + 1) t)\kappa_{\psi}}(dx \times dy \times da) dt,
$$

$$
\Psi(x, a) := (1 + \psi(x) + \psi(a))^{1/\beta_1} \quad \forall (x, a) \in X \times \mathbb{A},
$$

$$
\mathcal{N}_t^{(n)} := \int \Psi(x, a)\mu_t^{(n)}(dx \times dy \times da)
\leq L_\psi^{1/\beta_1} e^{\frac{\Lambda_\psi}{\beta_1}} \quad \forall n \in \mathbb{N}, t \in \mathbb{T}.
$$

Since $\Psi \geq 1$, $\mathcal{N}_t^{(n)} \leq 1$ for all $n \in \mathbb{N}$ and $t \in \mathbb{T}$. We prove that $\{\kappa^{(n)}\}_{n \in \mathbb{N}}$ is tight, which we do by showing that its marginals are tight, and use Proposition A.1(i). The tightness of the time-marginals is trivial, since they are all the same. For the $X \times \mathbb{A}$-marginal, we estimate

$$
\int (1 + \psi(x) + \psi(a))^{1-1/\beta_1}\kappa^{(n)}(dx \times da) = \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t}\mathcal{N}_t^{(n)} \times \int (1 + \psi(x) + \psi(a))\mu_t^{(n)}(dx \times dy \times da) dt
\leq \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t}L_\psi e^{\Lambda_\psi t} dt
\leq L_\psi(\Lambda_\psi + 1),
$$

and so Proposition A.1(iii) implies that $\{\kappa^{(n)}\}_{n \in \mathbb{N}}$ is tight. For the $Y$-marginal, we use Young’s inequality to get

$$
\int y^\frac{\beta_2 - 1}{\beta_1} \kappa^{(n)}(dy) \leq \int \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} y^\frac{\beta_2 - 1}{\beta_1} \Psi(x, a)\mu_t^{(n)}(dx \times dy \times da) dt
\leq \frac{1}{\beta_1} \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} \left[ (\beta_2 - 1) y^\beta + \Psi(x, a)\beta_1 \right] \mu_t^{(n)}(dx \times dy \times da) dt
= \frac{\beta_1 - 1}{\beta_1} \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} \int y^\beta \mu_t^{(n)}(dx \times dy \times da) dt
+ \frac{1}{\beta_1} \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} (1 + \psi(x) + \psi(a))\mu_t^{(n)}(dx \times dy \times da) dt
\leq \frac{\beta_1 - 1}{\beta_1} \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} \frac{1}{\beta_1} \int (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} L_\psi e^{\Lambda_\psi t} dt
\leq \frac{\beta_1 - 1}{\beta_1} Y + (\Lambda_\psi + 1)L_\psi,
$$

for all $n > N$, and thus $\{\kappa^{(n)}\}_{n \in \mathbb{N}}$ is tight. We conclude that $\{\kappa^{(n)}\}_{n \in \mathbb{N}}$ is tight, and hence contains a convergent subsequence with a limit $\kappa \in P(X \times \mathbb{Y} \times \mathbb{A} \times \mathbb{T})$. Clearly, the $\mathbb{T}$-marginal of $\kappa$ must be $(\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t} dt$, since again, this is the marginal of all $\kappa^{(n)}$, $n \in \mathbb{N}$. Moreover, as multiplication of measures by bounded continuous functions, in particular by $\Psi^{-1} \leq 1$, preserves weak convergence, we find that the limit has the form

$$
\kappa(dx \times dy \times da \times dt) = (\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t}\mathcal{N}(dx \times dy \times da) dt,
$$

where $\mathcal{N}$ is a normalization coefficient. From this, the convergence $\mu^{(n)} \xrightarrow{w} \mu$, follows. Note that for this convergence result, showing that $\{(\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t}\mu_t^{(n)}(dx \times dy \times da) dt\}_{n \in \mathbb{N}}$ is tight would have sufficed, however, in the following we need the original definition of $\kappa^{(n)}$ that additionally features the $\Psi$ coefficient. We again assume the whole sequence converges.

We can now show that the limit $\mu$ satisfies the forward equation. First note that since $\mu_t^{(n)} \Rightarrow \mu_t^{X \times Y}$ for all $t \in \mathbb{T}$, we have that

$$
\lim_{n \to \infty} \int f(x)g(y)\mu_t^{(n)X \times Y}(dx \times dy) = \int f(x)g(y)\mu_t^{X \times Y}(dx \times dy)
$$

where $\mathcal{N}$ is a normalization coefficient. From this, the convergence $\mu^{(n)} \xrightarrow{w} \mu$, follows. Note that for this convergence result, showing that $\{(\Lambda_\psi + 1)e^{-(\Lambda_\psi + 1) t}\mu_t^{(n)}(dx \times dy \times da) dt\}_{n \in \mathbb{N}}$ is tight would have sufficed, however, in the following we need the original definition of $\kappa^{(n)}$ that additionally features the $\Psi$ coefficient. We again assume the whole sequence converges.
for all $t \in \mathbb{T}$ and $fg \in \mathcal{D}(H)$. To show that

$$\int_0^t \int Hfg(x, y, a, t)\mu_{t}^{(n)}(dx \times dy \times da) \, ds$$

$$\quad \rightarrow \int_0^t \int Hfg(x, y, a, t)\mu_{s}(dx \times dy \times da) \, ds$$

for all $fg \in \mathcal{D}(H)$ and $t \in \mathbb{T}$, it suffices now to show that

$$\int \int h(t)Hfg(x, y, a, t)\mu_{t}^{(n)}(dx \times dy \times da) \, ds$$

$$\quad \rightarrow \int \int h(t)Hfg(x, y, a, t)\mu_{s}(dx \times dy \times da) \, ds$$

for all $h \in C_c(\mathbb{T})$. We note that for all $fg \in \mathcal{D}(H)$ and $h \in C_c(\mathbb{T})$, we have

$$\int \int h(t)g(y)Gf(x, a, t)\mu_{t}^{(n)}(dx \times dy \times da) \, dt$$

$$\quad = \int \int h(t)g(y)Gf(x, a, t)\mu_{t}(dx \times dy \times da) \, dt$$

Recalling that $\psi \leq L_c^{1/\beta_1}\Psi$, and $|Gf| \leq a_f\psi$ for all $f \in \mathcal{D}(G)$, the integrand on the right-hand side is bounded, and continuous, and by the weak convergence of the sequence $(\kappa_{t}^{(n)})_{n \in \mathbb{N}}$, we have that

$$\lim_{n \to \infty} \int \int h(t)g(y)Gf(x, a, t)\mu_{t}^{(n)}(dx \times dy \times da) \, dt$$

$$\quad = \int \int h(t)g(y)Gf(x, a, t)\mu_{t}(dx \times dy \times da) \, dt.$$ 

Finally, again for all $fg \in \mathcal{D}(H)$ and $h \in C_c(\mathbb{T})$,

$$\int \int h(t)e^{-\alpha t}f(x)g'(y)c(x, a, t)\mu_{t}^{(n)}(dx \times dy \times da) \, dt$$

$$\quad = \int \int \frac{h(t)e^{-\alpha t}}{(\Lambda_\psi + 1)e^{-(\Lambda_\psi+1)t}} \frac{f(x)g'(y)c(x, a, t)}{\Psi(x, a, t)}\kappa_{t}^{(n)}(dx \times dy \times da \times dt).$$

As above, the integrand is bounded since by Assumption 2.11 $c \leq L_c^{1/\beta_1}\Psi^{\beta_1/\beta_c}$ and $\beta_c \geq \beta_1$, and weak convergence implies

$$\lim_{n \to \infty} \int \int h(t)e^{-\alpha t}f(x)g'(y)c(x, a, t)\mu_{t}^{(n)}(dx \times dy \times da) \, dt$$

$$\quad = \int \int h(t)e^{-\alpha t}f(x)g'(y)c(x, a, t)\mu_{t}(dx \times dy \times da) \, dt.$$ 

Combining the above, we have that $\mu \in \mathcal{F}(H, K, v)$. The constraints are satisfied, as we assume $K$ to be closed in the weak topology.}

We can now give the proof of Theorem 3.5.

**Proof of Theorem 3.5.** We give the proof for the case where Assumption 3.4 (iii) holds; in the (b) case where the cost rate $c$ and terminal cost $v$ are bounded we may take the space of costs $\mathcal{Y}$ to be compact, and skip tightness arguments that are otherwise necessary.

Let $\rho^* \in \mathbb{R} \cup \{\infty\}$ be the optimal value of the problem. If $\rho^*$ is infinite, then every solution to the forward equation is optimal, and we are done. We suppose then that $\rho^* < \infty$, and let $(\mu_{t}^{(n)}) \in \mathcal{F}(H, K, v)^{\mathbb{N}}$ be a minimizing sequence.

We first show that the $p$th moments of the running costs are bounded. If $\mathbb{T} = \mathbb{R}_{\geq 0}$, for a sufficiently large $N \in \mathbb{N}$, we have that

$$\limsup_{t \to \infty} \rho(\mu_{t}^{(n)}_{t}^{\mathcal{Y}}) \leq \rho^* + 1 \quad \forall n > N,$$
and, since $\rho$ is coercive, there is a $Y \geq 0$ such that
\[
\limsup_{t \to \infty} \int y^p \mu_{t}^{(n)} \leq Y \quad \forall n > N.
\]

If instead $T = [0, T]$, then
\[
\int (y + v(x))^p \mu_{T}^{(n)} \leq Y \quad \forall n > N.
\]

The assumptions of Lemma 3.15 now hold, and we obtain a subsequence $(\mu_{t_k}^{(n)})_{n \in \mathbb{N}} \in \mathcal{M}(T, \mathcal{P}(\mathbb{X} \times \mathbb{Y} \times \mathbb{A}))$ with a limit $\mu = \mu^{(\infty)} \in \mathcal{M}(T, \mathcal{P}(\mathbb{X} \times \mathbb{Y} \times \mathbb{A}))$. We shall, as usual, suppose the whole sequence converges.

As the next step, we prove that for each $\mu^{(n)}$, $n \in \mathbb{N} \cup \{\infty\}$, the cost marginal distributions become stationary when $T = \mathbb{R}_{\geq 0}$, and that the infinite time limit is obtained uniformly, that is, at rates independent of $n$. We use much the same methods as above when proving that the $\mathbb{X} \times \mathbb{Y}$-marginals are continuous. Here, we need to only focus on the $\mathbb{Y}$-marginals. We define $\mathcal{C} := \{f + f_0 \mid f \in C_c^{(1)}(\mathbb{Y}), f_0 \in \mathbb{R}\}$, $\|f\|_\mathcal{C} := \|f'\|$ for all $f \in \mathcal{C}$. Assumptions of Lemma 3.10 and Corollaries 3.11 and 3.12 hold, and we obtain a complete metric $d_\mathcal{C}$ defined on $\mathcal{P}(\mathbb{Y})$.

From the forward equation, for all $g \in C_c^{(1)}(\mathbb{Y})$ and $n \in \mathbb{N} \cup \{\infty\}$,
\[
\left| \int g(y) \mu_{t}^{(n)}(dy) - \int g(y) \mu_{s}^{(n)}(dy) \right| = \left| \int_s^t e^{-\alpha r} \int g'(y) e(x, a, r) \mu_{t}^{(n)}(dx \times dy \times da) \, dr \right|
\leq \|g'\| \int_s^t e^{-\alpha r} \int L_{\psi}^{1/\beta_c} (1 + \psi(x) + \psi(a))^1/\beta_c \mu_{t}^{(n)}(dx \times dy \times da) \, dr
\leq L_{\psi}^{1/\beta_c} \|g'\| \int_s^t e^{-\alpha r} \left( (1 + \psi(x) + \psi(a)) \mu_{t}^{(n)}(dx \times dy \times da) \right)^{1/\beta_c} dr
\leq L_{\psi}^{1/\beta_c} \|g'\| \int_s^t e^{-\alpha r} (L_{\psi} e^{A r})^{1/\beta_c} \, dr
\leq L_{\psi}^{1/\beta_c} \|g'\| \int_s^t e^{(\frac{A}{\beta}) r - \frac{\alpha}{\beta} s} \, dr
\leq L_{\psi}^{1/\beta_c} L_{\psi}^{1/\beta_c} \|g'\| \left( e^{\frac{A}{\beta} s} - e^{\frac{\alpha}{\beta} s} \right)
\leq \|g'\| \eta(s, t).
\]

Recalling the bound on $\alpha$ given in Assumption 3.4, we now have that $\eta(s, t) = \eta(t, s)$ and $\eta(t, t) = 0$ for all $s, t \in T$, and $\eta(t, t + h) \to 0$ for all $h \geq 0$ as $t \to \infty$. Thus, for any $\epsilon > 0$, we can find $T_\epsilon \in \mathbb{R}$ so that $\eta(t, s) \leq \epsilon$ for all $t, s \geq T_\epsilon$. This implies that
\[
d_\mathcal{C} \left( \mu_{t}^{(n)} \right) \leq \epsilon \quad \forall t, s \geq T_\epsilon,
\]
and so by using the completeness of $d_\mathcal{C}$, there is a $\mu^{(n)} \in \mathcal{P}(\mathbb{Y})$ such that $\mu_{t}^{(n)} \Rightarrow \mu^{(n)} \in \mathcal{P}(\mathbb{Y})$ as $t \to \infty$. This is to say, the cost distributions converge to stationary distributions $\mu^{(n)}$, $n \in \mathbb{N} \cup \{\infty\}$ as time tends to infinity. In addition, a similar argument as in the proof of Lemma 3.15 shows that $\mu^{(n)} \xrightarrow{uc} \mu^{(\infty)}$ (with the metric $d_\mathcal{C}$ assigned to probability measures on $\mathcal{P}(\mathbb{Y})$).

Let $\epsilon > 0$ be arbitrary. For every $t \in \mathbb{R}$ there now is an $N_{c, t} \in \mathbb{N}$ such that $d_\mathcal{C}(\mu_{t}^{(n)}, \mu^{(\infty)}) \leq \epsilon$ for all $n \geq N_{c, t}$. Therefore, for all $t \geq T_{\epsilon}/3$ and $n \geq N_{c, t}$, we have
\[
d_\mathcal{C} \left( \mu_{t}^{(n)}, \mu^{(\infty)} \right) \leq d_\mathcal{C} \left( \mu_{t}^{(n)}, \mu_{t}^{(\infty)} \right) + d_\mathcal{C} \left( \mu_{t}^{(n)}, \mu^{(\infty)} \right) \leq \epsilon,
\]
which implies $\mu^{(n)} \Rightarrow \mu^{(\infty)}$. Since $t \to \int y^p \mu_{t}^{(n)}(dy)$ is bounded by $Y$ and increasing for every $n \in \mathbb{N} \cup \{\infty\}$, each of these function converges for every $n \in \mathbb{N} \cup \{\infty\}$ to a limit as $t \to \infty$. Then, using Skorokhod’s representation theorem [60, Theorem 3.1.8], the uniform boundedness of the $p$th moments, and dominated convergence theorem, we have that $\int y^p \mu^{(\infty)}(dy) \to \int y^p \mu^{(\infty)}(dy)$. Convergence of the moments together with weak convergence implies convergence in the $p$-Wasserstein metric $W^p$ [60, Theorem 6.9].
If $T = [0, T]$, we can directly use the convergence $\mu_T^{(n)}X \times Y \Rightarrow \mu_T^{(\infty)}X \times Y$ and the continuity of $\Theta$ to conclude using the continuous mapping theorem [53 Corollary 3.1.9] that $\mu_T^{(n)}X \times Y \circ \Theta^{-1} \Rightarrow \mu_T^{(\infty)}X \times Y \circ \Theta^{-1}$. Since the $p$th moments of $(x, y) \rightarrow \Theta(x, y) = y + v(x)$ with respect to $\mu_T^{(n)}X \times Y$ are uniformly bounded over $n > N$, the same argument as above shows that the sequence converges also in the $p$-Wasserstein metric.

We can now complete the proof. For $T = \mathbb{R}_{\geq 0}$, by continuity of $\tilde{\rho} : (P^p(\mathbb{R}), W^p) \rightarrow (\mathbb{R}, | \cdot |)$ and the asymptotic time convergence of the cost distributions, for all $n \in \mathbb{N}$,

$$
\limsup_{t \rightarrow \infty} \tilde{\rho}\left(\mu_t^{(n)}X \times Y\right) = \tilde{\rho}\left(\mu_\infty^{X \times Y}\right)
$$

and taking the limit $n \rightarrow \infty$,

$$
\rho^\ast = \lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \tilde{\rho}\left(\mu_t^{(n)}X \times Y\right) = \tilde{\rho}\left(\mu_\infty^{X \times Y}\right),
$$

and similarly for the case of $T = [0, T]$, assuming lower semicontinuity of $\tilde{\rho}$,

$$
\rho^\ast \geq \liminf_{n \rightarrow \infty} \tilde{\rho}\left(\mu_T^{(n)}X \times Y \circ \Theta^{-1}\right) \geq \tilde{\rho}\left(\mu_T^{(\infty)}X \times Y \circ \Theta^{-1}\right).
$$

Therefore the limits are optimal, and the proof is complete. \hfill \square

4 Numerical example

Here, as a proof of concept, we present and solve a particularly simple risk-aware optimization problem. The state and action spaces are compact and the processes one-dimensional, but we emphasize that our general setup allows for non-compact state and action spaces, and infinite dimensional state and action spaces.

We consider a follower problem on a circle: The setup consists of an uncontrolled stochastic process $(b_t)_{t \in T}$ (the target) being pursued by a controlled process $(v_t)_{t \in T}$ (the pursuer) whose objective is to minimize the distance between itself and the target by choosing the direction and speed at which the pursuer moves. We assume the pursuer’s cost rate is the sum of the distance between itself and the target (as we define later), and the velocity squared.

To formalize the problem, let the state space $X = \mathbb{R}/2\pi\mathbb{N}$ (we interpret $X$ as being formed from copies of the interval $[0, 2\pi]$), equipped with the metric $d(\theta, \phi) = |1 - \cos(\theta - \phi)|^{1/2}$ for all $\theta, \phi \in X$, and the action space $A = [a, \bar{a}]$ where $-\infty < a < \bar{a} < \infty$. The pursuer’s action is interpreted as its velocity along the circle: The pursuer’s position is represented by a process $(\phi_t)_{t \in T}$, $\phi_t \in X$ for all $t \in T$, such that $d\phi_t = a_t dt$ where $a_t \in A$ is the action at time $t \in T$. We take the target’s position to be the process $(\sigma w_t)_{t \in T}$ where $(w_t)_{t \in T}$ is a Wiener process on $\mathbb{R}$, mapped onto $X$, and $\sigma > 0$ is a constant representing the magnitude of randomness in its motion. Let the difference between the pursuer’s and target’s positions $x_t = \phi_t - w_t$ be the state process $(x_t)_{t \in T}$, $x_t \in X$ for all $t \in T$. This follows the stochastic differential equation

$$
\text{d}x_t = a_t dt - dw_t,
$$

with an initial value $x_0 = 0$. This process has the generator $G : C^2(X) \rightarrow C(X \times A)$, such that for all $f \in \mathcal{D}(G)$,

$$
Gf(x, a) = a \frac{\partial f}{\partial x}(x) + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2}(x).
$$

We choose the cost rate function

$$
c(x, a) = d(x, 0)^2 + \gamma a^2 = 1 - \cos x + \gamma a^2,
$$

for all $x, a \in \mathbb{R}$ and where $\gamma \geq 0$ is a parameter representing the magnitude of the pursuer’s cost of moving. We consider discounted costs with a discount rate $\alpha > 0$. The space $Y$ of values of accumulated costs is $Y = \mathbb{R}$, with $y := 2 + \gamma a^2$.

As our risk function, we use the entropic risk measure: Let $\theta \in \mathbb{R}$, and define $\rho_\theta \in \mathcal{P}(Y) \rightarrow \mathbb{R}$ as

$$
\rho_\theta(\lambda) := \begin{cases}
\frac{1}{\theta} \ln \left[ \int_Y e^{\theta y} \lambda(dy) \right], & \theta \neq 0, \\
\int_Y y \lambda(dy), & \theta = 0,
\end{cases}
$$

(4.3)
Figure 4.1: Marginal distributions obtained from an optimal solution of the follower problem on a circle. Panel (a): the $X_n$-marginal distribution as a function of time, logarithmic scale is used to enhance visibility of features; (b) the $Y_n$-marginal distribution as a function of time. Both distributions are normalized on the discrete space, with the coordinate axes showing the corresponding $X$ and $Y$ space values.

for all $\lambda \in \mathcal{P}(\mathcal{Y})$. The parameter $\theta$ represents risk preferences: Positive values translate to risk-averse objectives, and the larger $\theta$ is, the more the risks are weighted in assessing risk. Conversely, negative values of $\theta$ imply risk seeking preferences. Note that for $\theta$ close to zero, $\rho_\theta(\lambda) = \int_\mathcal{Y} [y + y^2/(2\theta)]\lambda(dy) + O(\theta^2)$, so that $\rho_\theta$ approximates linear-quadratic costs as a special case. The risk function $\rho_\theta$ is not convex, but since the logarithm is strictly increasing, we can just as well consider the equivalent risk function

$$\hat{\rho}_\theta(\lambda) := \int_\mathcal{Y} e^{\theta y}\lambda(dy),$$

(4.4)

for all $\lambda \in \mathcal{P}(\mathcal{Y})$, which is linear. We also restrict $\theta$ to non-negative values.

Given the compactness of $X$ and $\mathcal{A}$, Assumptions 2.11 and 3.4 are readily verified.

For the numerical solution of the problem, we discretize the forward equation using standard methods, cf. [21, Chapter 2.2], so that the discretized equation corresponds to a forward equation on a discrete space. As the discretized system corresponds to a finite state continuous-time controlled Markov chain, the weak convergence results of [36, Chapter 10] apply. The time axis is truncated to a maximum time of $T^\ast \in \mathbb{T}$, and the $X$, $Y$, $\mathcal{A}$, and $\mathbb{T}$ axis are discretized to $n$ equidistant samples each; the discretized spaces are labeled with an underscore $n$.

The details of the construction are omitted, and proving the convergence of this approach in the risk-aware case is beyond the scope of this paper. The objective is linear in the cost distribution, and we may use linear programming methods to solve the problem.

We have numerically solved the problem for the following values of the parameters:

$$n = 21, \ T^\ast = 25,$$

$$a = -0.5, \bar{a} = 0.5,$$

$$\sigma = 1, \gamma = 2, \alpha = 0.25, \theta = 1,$$

$$\nu = \delta_0.$$
Figure 4.2: Visualization of the optimal control at time point $T^*/2$, obtained by plotting non-zero points of the $X_n \times X_n$-marginal distribution. As for each $x$ there appears only one action $a$ where the marginal distribution is zero, the control appears to be strict.

a single action for a given state. To further validate our results, we independently solved the risk-neutral version of the problem using dynamic programming methods. Comparing the results to the risk-aware problem with a small value of $\theta$, we found very good agreement.

5 Conclusions

We have presented a dynamic analytic formulation of a generic risk-aware optimal control problem, with Polish, possibly infinite-dimensional state and action spaces, and where the underlying dynamics are given in the martingale formulation. Our primary goal was constructing a practical method for solving risk-aware relaxed controlled martingale problems, which we accomplished by providing an equivalent formulation that takes the form of a nonlinear programming problem with a linear constraint determining the controlled processes joint state and cost distributions. The ability to obtain the full cost distribution is characteristic to the dynamic analytic method, and is the reason why it is well-suited for risk-aware problems: In this context, evaluating the objective function requires knowledge of the joint state-cost distribution. Contrast this to convex analytic methods, which only yield the occupation measures which can be seen as one dimensional projections of the joint state and cost distributions.

The dynamic analytic method was also capable of providing analytic insight into the control problem. In particular, we found that the optimal control processes can be taken to be Markov in time and the system state and running costs. Significantly, we were also able to prove the existence of optimal Markov controls under additional conditions. We also provided a rather simple but instructive example of how the method can be used to numerically solve risk-aware optimal control problems.

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A Proofs and auxiliary results

A.1 Proofs for Section 2

Proof of Proposition 2.8. Let $\pi \in \mathcal{D}(A, K, \nu_0)$, $K_n := \{u \in \mathbb{U} \mid \psi_U(u) \leq n\}$ and $\tau_n := \inf\{t \in T \mid \psi_U(u_t) \notin K_n\}$ for all $n \in \mathbb{N}$. By the inf-compactness of $\psi_U$, $K_n$ is compact for all $n \in \mathbb{N}$, so that $\tau_n$ is a stopping time for all $n \in \mathbb{N}$,
see e.g. [56] Proposition 2.1.5. From Eq. (2.1), for all \( k, n \in \mathbb{N} \) and \( t \in T, s \leq t, \)
\[
m_{t,\tau_n}^{k} - m_{s,\tau_n}^{k} = \phi_k(u_{t,\tau_n}^{\pi}) - \phi_k(u_{s,\tau_n}^{\pi}) - \int_{s}^{t} \int_{\mathcal{A}} A \phi_k(u_r^{\pi},a) \pi_r(da) \, dr.
\]
By the optional stopping theorem [41] Theorem 1.62, \( \mathbb{E}[m_{t,\tau_n}^{k} - m_{s,\tau_n}^{k} \mid \mathcal{F}_{s,\tau_n}] = 0 \) for all \( n, k \in \mathbb{N} \), and using the boundedness of \( A \phi_k \) by \( \psi_0 \) and \( \psi_1 \),
\[
\mathbb{E}\left[ \phi_k(u_{t,\tau_n}^{\pi}) \mid \mathcal{F}_{s,\tau_n} \right] = \mathbb{E}\left[ \phi_k(u_{s,\tau_n}^{\pi}) + \int_{s}^{t} \int_{\mathcal{A}} A \phi_k(u_r^{\pi},a) \pi_r(da) \, dr \mid \mathcal{F}_{s,\tau_n} \right]
\leq \phi_k(u_{s,\tau_n}^{\pi}) + \mathbb{E}\left[ \int_{s}^{t} \Lambda_1 \left( 1 + \psi_0(u_r^{\pi}) + \int_{\mathcal{A}} \psi_1(a) \pi_r(da) \right) \, dr \mid \mathcal{F}_{s,\tau_n} \right]
\forall t \in T, s \leq t.
\]
Letting \( k \to \infty \) and using the dominated convergence theorem and Grönwall’s inequality [41] Corollary 6.60, we obtain
\[
\mathbb{E}\left[ \psi_0(u_{t,\tau_n}^{\pi}) \mid \mathcal{F}_{s,\tau_n} \right] \leq \left( \psi_0(u_{s,\tau_n}^{\pi}) + \Lambda_1(t-s) + \Lambda_1 \mathbb{E}\left[ \int_{s}^{t} \psi_1(a) \pi_r(da) \, dr \mid \mathcal{F}_{s,\tau_n} \right] \right) e^{\Lambda_1(t-s)},
\]
for all \( t \in T, s \leq t \). Setting
\[
b_t := \Lambda_1 \mathbb{E}\left[ t + \int_{0}^{t} \int_{\mathcal{A}} \psi_1(a) \pi_r(da) \, dr \right],
\]
from Eq. (A.1), we find that
\[
\psi_0(u_{0}^{\pi}) \geq e^{-\Lambda_1 t} \mathbb{E}\left[ \psi_0(u_{t,\tau_n}^{\pi}) \right] - b_t \\
\geq e^{-\Lambda_1 t} \mathbb{E}\left[ \| \tau_n - \infty \|_t \psi_0(u_{t,\tau_n}^{\pi}) \right] - b_t \\
\geq e^{-\Lambda_1 t} n \mathbb{E}\left[ \| \tau_n - \infty \|_t \right] - b_t,
\]
and hence,
\[
\mathbb{P}[\tau_n \leq t] \leq e^{\Lambda_1 t} \frac{\psi_0(u_{0}^{\pi}) + b_t}{n} < \infty \quad \forall t \in T, n \in \mathbb{N},
\]
where the finiteness follows from the regularity conditions given in Definition 2.6 (iv, v).

Defining \( \tau_{\infty} := \limsup_{n \to \infty} \tau_n \), the time for reaching infinity, we then get
\[
\mathbb{P}[\tau_{\infty} \leq t] = \lim_{n \to \infty} \mathbb{P}[\tau_n \leq t] = 0 \quad \forall t \in T,
\]
and so the process \( \psi_0(u_{\cdot}^{\pi}) \) is almost surely finite for all \( t \in T \). The finiteness of \( \int_{\mathcal{A}} (1 + \psi_0(u_{\cdot}^{\pi}) + \psi_1(a)) \pi_r(da) \) is then a direct consequence of the above, and the regularity of \( (A, K, \nu_0) \).

**Proof of Proposition 2.12.** We prove the statement for the case of lower semicontinuity, for continuity the argument is identical. Let \( (\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}^p(\mathbb{R}) \) be an arbitrary sequence converging to some \( \mu \in \mathcal{P}^p(\mathbb{R}) \) in the \( p \)-Wasserstein metric. This implies that \( \mu_n \Rightarrow \mu \), and \( \int |x|^p \mu_n(dx) \to \int |x|^p \mu(dx) \) as \( n \to \infty \). [60] Theorem 6.9. By Skorokhod’s representation theorem [56] Theorem 3.1.8, there exists a probability space \( (\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mathbb{P}}) \) and random variables \( (\tilde{X}_n)_{n \in \mathbb{N}} \subset \mathcal{L}(\tilde{\Omega}; \mathbb{R}), \tilde{X} \in \mathcal{L}(\tilde{\Omega}; \mathbb{R}) \) such that \( \mathcal{L}(\tilde{X}_n) = \mu_n \) for all \( n \in \mathbb{N} \) and \( \mathcal{L}(\tilde{X}) = \mu \), and \( \tilde{X}_n \to \tilde{X} \) almost surely. Since \( \mu_n \in \mathcal{P}^p(\mathbb{R}), \mu \in \mathcal{P}^p(\mathbb{R}) \), we also have that \( \tilde{X}_n \in \mathcal{L}^p(\tilde{\Omega}; \mathbb{R}) \) for all \( n \in \mathbb{N} \), and similarly for \( \tilde{X} \). An application of the dominated convergence theorem shows that \( \| X_n - \tilde{X} \|_p \to 0 \), and so by the law invariance and lower semicontinuity of \( \rho \), we have that
\[
\liminf_{n \to \infty} \tilde{\rho}(\mu_n) = \liminf_{n \to \infty} \rho(\tilde{X}_n) \geq \rho(\tilde{X}) = \tilde{\rho}(\mu),
\]
and therefore \( \tilde{\rho} \) is lower semicontinuous.
A.2 Proofs for Section 3

Proof of Proposition A.3. Let \( \pi \in \mathcal{D}(H, K, \nu) \) and let \( \hat{y}^\pi = (\hat{y}^\pi_t)_{t \in \mathbb{T}} \) as in the statement. It is straightforward to see that \((\Omega^\pi, \Sigma^\pi, \mathbb{F}^\pi, \mathbb{P}^\pi, x^\pi, \pi)\) is a càdlàg relaxed controlled solution to \((G, K, \nu)\): It suffices to consider Eq. (2.1) of the definition of a solution, and select functions \( f \in \mathcal{D}(H) \) such that only depend on the \( \mathbb{X}\)-component. Regularity of \((G, K, \nu)\), Assumption 2.11 and Proposition 2.8 imply that \( \psi(x^\pi) \), \( \int_H \alpha e^{-\alpha \cdot c(x^\pi, a, \cdot)} \pi(\text{da}) \), and \( \hat{y}^\pi \) are all almost surely finite for all \( t \in \mathbb{T} \).

We first note that \( y^\pi \) is continuous. Let \( \mathcal{D}_0 := \{g + g_0 \mid g \in C_1^\infty(\mathbb{Y}), g_0 \in \mathbb{R}\} \). By the martingale property of the solutions,

\[
m^g_t := g(y^\pi_t) - g(y^\pi_n) - \int_0^t \int_H e^{-\alpha \cdot c(x^\pi_r, a, r)} g'(y^\pi_r) \pi_r(\text{da}) \, dr \quad \forall t \in \mathbb{T}
\]

is a martingale for all \( g \in \mathcal{D}_0 \). Thus, for an arbitrary \( g \in \mathcal{D}_0 \),

\[
m^g_t = g(y^\pi_t)^2 - g(y^\pi_n)^2 - \int_0^t \int_H e^{-\alpha \cdot c(x^\pi_r, a, r)} g'(y^\pi_r) g(y^\pi_r) \pi_r(\text{da}) \, dr \quad \forall t \in \mathbb{T}
\]

is also a martingale. Using the above two equalities,

\[
(g(y^\pi_t) - g(y^\pi_n))^2 = g(y^\pi_t)^2 - 2g(y^\pi_t)g(y^\pi_n) + g(y^\pi_n)^2 = m^g_t - 2g(y^\pi_n) m^g_t + 2 \int_0^t \int_H e^{-\alpha \cdot c(x^\pi_r, a, r)} g'(y^\pi_r) g(y^\pi_r) \pi_r(\text{da}) \, dr
\]

so that

\[
\mathbb{E} \left[(g(y^\pi_t) - g(y^\pi_n))^2 \right] = \mathbb{E} \left[ \left( \int_0^t \int_H e^{-\alpha \cdot c(x^\pi_r, a, r)} \pi_r(\text{da}) \right) 2 \left( g(y^\pi_t) - g(y^\pi_n) \right) g'(y^\pi_t) \, dr \right]^2.
\]

From this, in then follows that the (optional) quadratic variation of \( g(y^\pi) \) is almost surely zero: If \( P_t = (t_1, t_2, \ldots, t_n) \) is an arbitrary partition of \([0, t]\), \( t \in \mathbb{T}\), and \( |P_t| := \max_{i \in \{1, \ldots, n-1\}} |t_{i+1} - t_i| \), then

\[
\lim_{|P_t| \to 0} \mathbb{E} \left[ (g(y^\pi_t) - g(y^\pi_n))^2 \right] = 0 \quad \forall t \in \mathbb{T},
\]

at least almost surely for every \( g \in \mathcal{D}_0 \). This implies that the quadratic variation of \( m^g \) is zero, and therefore \( m^g \) is itself zero. So being, \( y^\pi \) is continuous, and

\[
g(y^\pi_t) = g(y^\pi_0) + \int_0^t \int_H e^{-\alpha \cdot s} c(x^\pi, a, s) \pi_s(\text{da}) g'(y^\pi_t) \, ds
\]

for all \( g \in \mathcal{D}_0 \), almost surely. Consider then any sequence of functions \((g_n)_{n \in \mathbb{N}} \in \mathcal{D}_0^\infty\) of the form \( g_n(y) = y \) for all \( y \leq n \), and for which \( n \to g'_n(y) \) is non-decreasing for all \( y \in \mathbb{Y} \). From the above, it then follows using monotone convergence theorem that

\[
y^\pi_t = \int_0^t \int_H e^{-\alpha \cdot s} c(x^\pi_s, a, s) \pi_s(\text{da}) \, ds \quad \forall t \in \mathbb{T},
\]

almost surely, and so \( y^\pi \) is indistinguishable from \( \hat{y}^\pi \).

For the converse part of the Proposition, let \( \pi \in \mathcal{D}(G, K, \nu) \) and set \( y^\pi \) to be the corresponding running costs, defined as the integral in Eq. (3.3). Our goal is to show that for all \( f, g \in \mathcal{D}(H) \), the process

\[
m^f,g_t := f(x^\pi_t) g(y^\pi_t) - f(x^\pi_0) g(0) - \int_0^t \int_H H g(x^\pi_s, y^\pi_s, a, s) \pi_s(\text{da}) \, ds \quad \forall t \in \mathbb{T},
\]

is a martingale, which is sufficient to establish that \((\Omega^\pi, \Sigma^\pi, \mathbb{F}^\pi, (x^\pi, y^\pi), \pi) \in \mathcal{D}(H, K, \nu)\). The proof of this follows closely that of [56] Lemma 4.3.4(a)], however, the problem here does not quite satisfy the boundedness conditions of that result.
Suppose \( fg \in \mathcal{D}(H) \) is arbitrary. Note first that \( \mathbb{E}[|m_{t}^{f,g}|] < \infty \) for all \( t \in T \); this follows from the bounds on \( Gf \) and the cost rate \( c \) given in Assumption 2.11. Let \( s, t \in T \), \( s \leq t \), and let \( (t_{i})_{i \in \{1, \ldots, n\}} \) be an arbitrary partition of \( [s, t] \), \( t_{1} = s \), \( t_{n} = t \). Then, using the martingale property for the \( x^{\pi} \) process and the differentiability of \( g \\
\begin{align*}
\mathcal{M}_{t,s}^{f} &= f(x_{s}^{\pi}) - f(x_{s}^{\pi}) \int_{s}^{t} \int_{\mathcal{A}} Gf(x_{r}^{\pi}, a, \pi_{r})(da) \, dr,
0 &= g(y_{s}^{\pi}) - g(y_{s}^{\pi}) \int_{s}^{t} e^{-\alpha r} \int_{\mathcal{A}} c(x_{r}^{\pi}, a, \pi_{r})g'(y_{r}^{\pi})\pi_{r}(da) \, dr,
\end{align*}
\] 
\( m_{t,s}^{f} \) is a martingale, and defining for brevity, for all \( t' \in [s, t] \\
\begin{align*}
W_{t'} &:= \int_{\mathcal{A}} Gf(x_{t'}^{\pi}, a, t')\pi_{t'}(da),
V_{t'} &:= e^{-\alpha t'} \int_{\mathcal{A}} c(x_{t'}^{\pi}, a, t')g'(y_{t'}^{\pi})\pi_{t'}(da),
\end{align*}
we find \\
\begin{align*}
\mathbb{E} \left[ f(x_{t}^{\pi})g(y_{t}^{\pi}) - f(x_{s}^{\pi})g(y_{s}^{\pi}) \mid \mathcal{F}_{s} \right] &= \sum_{k=1}^{n} \mathbb{E} \left[ f(x_{t_{k+1}}^{\pi})g(y_{t_{k+1}}^{\pi}) - f(x_{t_{k}}^{\pi})g(y_{t_{k}}^{\pi}) \mid \mathcal{F}_{s} \right] \\
&= \sum_{k=1}^{n} \mathbb{E} \left[ f(x_{t_{k+1}}^{\pi})g(y_{t_{k+1}}^{\pi}) - g(y_{t_{k}}^{\pi}) \right.
+ \left. f(x_{t_{k+1}}^{\pi}) - f(x_{t_{k}}^{\pi}) \right] \mathcal{F}_{s} \\
&= \sum_{k=1}^{n} \mathbb{E} \left[ \int_{t_{k}}^{t_{k+1}} \left[ f(x_{r}^{\pi}) + f(x_{r}^{\pi}) - f(x_{r}^{\pi}) \right] V_{r} \\
+ \left. \left[ g(y_{r}^{\pi}) + g(y_{r}^{\pi}) - g(y_{r}^{\pi}) \right] W_{r} \right] \mathcal{F}_{s} \\
&= \mathbb{E} \left[ \int_{s}^{t} f(x_{r}^{\pi})V_{r} + g(y_{r}^{\pi})W_{r} \mid \mathcal{F}_{s} \right] + R_{t,s},
\end{align*}
where \\
\begin{align*}
R_{t,s} &:= \int_{s}^{t} \int_{t_{k}}^{t_{k+1}} \left[ f(x_{r}^{\pi}) - f(x_{r}^{\pi}) \right] V_{r} + \left[ g(y_{r}^{\pi}) - g(y_{r}^{\pi}) \right] W_{r} \mathcal{F}_{s}.
\end{align*}
Estimating the above using H"{o}lder’s inequality, we have for the first term \\
\begin{align*}
\left| \mathbb{E} \left[ \int_{t_{k}}^{t_{k+1}} f(x_{r}^{\pi}) - f(x_{r}^{\pi}) \right] V_{r} \mathcal{F}_{s} \right| \\
&\leq \mathbb{E} \left[ \int_{t_{k}}^{t_{k+1}} |f(x_{r}^{\pi}) - f(x_{r}^{\pi})|^{1 + \frac{1}{\beta_{1}}} \mathcal{F}_{s} \right]^{\frac{\beta_{1}}{1 + \frac{\beta_{1}}{\beta_{1}}}}
\times \mathbb{E} \left[ \int_{t_{k}}^{t_{k+1}} |V_{r}|^{\beta_{1}} \mathcal{F}_{s} \right]^{\frac{1}{\beta_{1}}}
\forall k \in \{1, \ldots, n - 1\}.
\end{align*}
This is \( o(|t_{k+1} - t_{k}|) \), since \( x^{\pi} \) is càdlàg, \( f \), \( g \), and \( g' \) are continuous and bounded, and by Assumption 2.11 and
The right-hand side vanishes as and the coefficients of properties of hold.

Proof of Lemma 3.10. Proof of Proposition 3.9.

Proposition 2.8, can be treated similarly, using the continuity of \( y^T \). Letting \( \max_{k \in \{1, \ldots, n\}} |t_{k+1} - t_k| \to 0 \), we have \( R_{t,s} \to 0 \) and the claim follows.

Proof of Proposition 3.9. Substituting \( \phi_n \) into the forward equation, Eq. (3.1), rearranging, and by using the properties of \((\psi, \psi_h, \phi, L_1, L_U, L_\beta, \beta_1, \Lambda_1, \Lambda_\beta)\) given in Definition 2.6, we get for all \( n \in \mathbb{N} \) and \( t \in \mathbb{T} \),

\[
\int_U \phi_n(u) \mu_t^U(du) = \int_U \psi(u) \nu_0(du) + \int_0^t \int_{U \times A} A \phi_n(u, a, s) \mu_s(du \times da) \, ds \\
+ \int_U [\phi_n(u) - \psi(u)] \nu(du)
\leq \int_U \psi(u) \nu_0(du) + \int_0^t \int_{U \times A} A \phi_n(u, a, s) \mu_s(du \times da) \, ds \\
\leq \int_U \psi(u) \nu_0(du) + \int_0^t \int_{U \times A} \Lambda_1 (1 + \psi(u) + \psi_h(a)) \mu_s(du \times da) \, ds \\
\leq L_U + \int_0^t \int_{U \times A} \Lambda_1 \psi(u) \mu_s(du) \, ds + \Lambda_1 t + \Lambda_1 L_\Lambda \frac{1}{\Lambda_\beta} (e^\Lambda_t - 1).
\]

Applying the monotone convergence theorem, we have that

\[
\int_U \psi(u) \mu_t^U(du) \leq L_U + \int_0^t \int_{U \times A} \Lambda_1 \psi(u) \mu_s(du) \, ds + \Lambda_1 t + \Lambda_1 L_\Lambda \frac{1}{\Lambda_\beta} (e^\Lambda_t - 1),
\]

for all \( t \in \mathbb{T} \). By using Grönwall’s inequality,

\[
\int_U \psi(u) \mu_t^U(du) \leq \left( L_U + \Lambda_1 t + \Lambda_1 L_\Lambda \frac{1}{\Lambda_\beta} (e^\Lambda_t - 1) \right) e^{\Lambda_1 t},
\]

(A.3)
and the coefficients \( L_\psi, \Lambda_\psi \) satisfying Eq. (3.6) can be readily found.

Recalling the bound of \( \psi \) in terms of \( \psi_U \) and \( \psi_A \), see Definition 2.6 (i), it is clear that Eq. (3.5) holds.

To show \( \mu^U \) is continuous, we return to the forward equation, and estimate for an arbitrary \( f \in \mathcal{D}(G) \),

\[
\left| \int_U f(u) \mu_t^U(du) - \int_U f(u) \mu_t^U(du) \right| \leq \int_0^t \int_{U \times A} a_f \psi(u, a) \mu_s(du \times da) \, ds \\
\leq a_f (t-s)L_1 \left( \frac{e^{\Lambda_\psi t} - e^{\Lambda_\psi s}}{t-s} \right) \frac{\pi_1}{\Lambda_\psi}.
\]

(A.4)

The right-hand side vanishes as \( t \to s \), and since \( \mathcal{D}(G) \) is convergence determining (see e.g. [59] Theorem 3.4.5), \( \mu^U \) is continuous. Finally, as the right-hand side of Eq. (A.4) is also independent of \( \mu \) and \( f \), it is clear that Eq. (3.7) holds.

Proof of Lemma 3.10. We first show that the two given definitions are indeed equivalent. We set

\[
d_\phi (\mu, \nu) := \sup \left\{ \left| \int f(u) \mu(du) - \int f(u) \nu(du) \right| \mid f \in \mathcal{G}, \| f \| + \| f \|_{\phi} \leq 1 \} \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{U}),
\]

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and show $d_G = d'_G$. This follows from the set $G_1 := \{ f \in G \mid \|f\| + \|f\|_G \leq 1 \}$ being absorbing, see e.g. [62, Proposition I.2]. Because of this, for arbitrary $\mu, \nu \in P(U)$,

$$d_G(\mu, \nu) = \sup \left\{ \left| \int f(u) \mu(du) - \int f(u) \nu(du) \right| \mid f_1 \in G_1, s > 0, f_1 \neq 0 \right\}$$

and so $d_G(\mu, \nu) = d'_G(\mu, \nu)$ for all $\mu, \nu \in P(U)$.

Turning to the proof of part (i), the mapping $d_G$ is clearly finite and symmetric, and satisfies the triangle inequality. It is also apparent that if $\mu = \nu$, then $d_G(\mu, \nu) = 0$ for all $\mu, \nu \in G$. Conversely, if $d_G(\mu, \nu) = 0$, then

$$\left| \int f(u) \mu(du) - \int f(u) \nu(du) \right| = 0 \quad \forall f \in G,$$

By [56, Theorem 3.4.5(a)], this implies that $\mu = \nu$. Therefore, $d_G$ is a metric. If $\{\mu_n\}_{n \in N} \cup \mu \subset P(U)$ and $d(\mu_n, \mu) \to 0$ as $n \to \infty$, then [56, Theorem 3.4.5(b)] implies that $\mu_n \Rightarrow \mu$ as $n \to \infty$.

(ii) First note that

$$\left| \int f(u) \mu(du) - \int f(u) \nu(du) \right| \geq \left| \int f(u) \mu(du) - \int f(u) \nu(du) \right| \frac{\|f\| + \|f\|_G}{\|f\| + \|f\|_G} \quad \forall f \in G'.$$

Taking the supremum over $G'$, and using basic estimates, we then get

$$\sup_{f \in G'} \left\{ \frac{\left| \int f(u) \mu(du) - \int f(u) \nu(du) \right|}{\|f\| + \|f\|_G} \right\} \geq \sup_{f \in G'} \left\{ \frac{\left| \int f(u) \mu(du) - \int f(u) \nu(du) \right|}{\|f\| + \|f\|_G} \right\},$$

and the claim follows.

A.3 Other proofs

The following proposition asserts some basic properties regarding tightness of sets of measures. This is used frequently in weak convergence arguments.

**Proposition A.1.** (i) Let $U_1$ and $U_2$ be topological spaces, $\{\mu_n\}_{n \in N} \subset M(U_1 \times U_2)$, and let $\mu_1^I := \mu_1^{U_1} = \mu_1(\cdot \times U_2)$ and $\mu_2^I := \mu_2^{U_2} = \mu_2(U_1 \times \cdot)$ be respectively the $U_1$ and $U_2$ marginals of $\mu_n$ for every $n \in N$. If $\{\mu_n\}_{n \in N}$ and $\{\mu_n^I\}_{n \in N}$ are both tight, then $\{\mu_n\}_{n \in N}$ is tight. By extension, this statement holds for all finite Cartesian products of topological spaces. (ii) Let $\{\mu_i\}_{i \in I} \subset M(\mathbb{R}^+)$, where $I$ is a (possibly uncountable) index set. Suppose $\phi$ is a non-decreasing non-negative measurable function and that there is a $b > 0$ such that $\int \phi(x) \mu_i(dx) < b$ for all $i \in I \setminus F$ where $F$ is finite. Then $\{\mu_i\}_{i \in I}$ is tight. (iii) Let $U$ be Polish, $\{\mu_i\}_{i \in I} \subset M(U)$, and $\phi : U \to \mathbb{R}$ be inf-compact. If $\int \phi(x) \mu_i(dx) < b$ for all $i \in I \setminus F$ where $F$ is finite, then $\{\mu_i\}_{i \in I}$ is tight.
Proof. (i) By tightness of the marginals, for each $\epsilon > 0$ we can find compact $K_1^\epsilon \subset U_1$ and $K_2^\epsilon \subset U_2$ such that $\mu_n^1(K_1^\epsilon) < \epsilon/2$ and $\mu_n^2(K_2^\epsilon) < \epsilon/2$ for all $n \in \mathbb{N}$. Let $K := K_1 \times K_2$. As a product of compact sets, Tychonoff’s theorem [3, Theorem 2.57] states that $K$ is compact. Noting that $K_n^c \subset (K_1^\epsilon \times U_2) \cup (U_1 \times K_2^\epsilon)$, we have that $\mu_n(K_n^c) \leq \mu_n(K_1^\epsilon \times U_2) + \mu_n(U_1 \times K_2^\epsilon) = \mu_n^1(K_1^\epsilon) + \mu_n^2(K_2^\epsilon) < \epsilon$, demonstrating that $\{\mu_n\}_{n \in \mathbb{N}}$ is tight.

(ii) It suffices to show that $\{\mu_i\}_{i \in I,F}$ is tight; we may always add a finite collection of measures into it and maintain tightness. Suppose this set is not tight. Then we can find an $\epsilon > 0$ such that for all compact $K \subset U$, there is a measure $\mu_i$ for which $\mu_i(K^c) \geq \epsilon$. Let $K = \{u \in U \mid \phi(u) \leq b/\epsilon\}$, and select $\mu_i$ so that $\mu_i(K^c) \geq \epsilon$. Then,

$$\int \phi(u)\mu_i(du) = \int_K \phi(u)\mu_i(du) + \int_{K^c} \phi(u)\mu_i(du) \geq b,$$

a contradiction. Proof of (ii) uses the same idea and is omitted. \hfill \blacksquare

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