HIGHER GALOIS THEORY

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Abstract. We generalize toposic Galois theory to higher topoi. We show that locally constant sheaves in a locally \((n - 1)\)-connected \(n\)-topos are equivalent to representations of its fundamental pro-\(n\)-groupoid, and that the latter can be described in terms of Galois torsors. We also show that finite locally constant sheaves in an arbitrary \(\infty\)-topos are equivalent to finite representations of its fundamental pro-\(\infty\)-groupoid. Finally, we relate the fundamental pro-\(\infty\)-groupoid of 1-topoi to the construction of Artin and Mazur and, in the case of the étale topos of a scheme, to its refinement by Friedlander.

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The goal of this paper is to generalize Galois theory, as it is understood in the context of Grothendieck topos, to higher topos.

In §1, we review the notion of shape, or fundamental pro-\(\infty\)-groupoid, of an \(\infty\)-topos. For other equally terse accounts, we refer the reader to Toën and Vezzosi [TV03, §5.3] and to Lurie [Lur09, §7.1.6].

In §2, we prove our generalization of (infinite) Galois theory to \(n\)-topoi, for \(0 \leq n \leq \infty\). Specializing to \(n = 1\) recovers classical results of Moerdijk [Moe89] and of Kennison [Ken91], but our proofs are quite different as they make essential use of higher topos theory even in that case (specifically, of the theory of stacks in groupoids). The main result is that locally constant sheaves in a locally \((n - 1)\)-connected \(n\)-topos are equivalent to representations of its fundamental pro-\(n\)-groupoid. For locally \(n\)-connected (and 1-localic) \(n\)-topoi, this statement already appears in Grothendieck’s famous letter to Breen [Gro75, (C)], but most of our work deals with the weakening of the local connectedness assumption, which is necessary to recover for instance the Galois theory of fields. As consequences, we obtain a model for the fundamental pro-\(n\)-groupoid as a cofiltered diagram indexed by Galois torsors, and we deduce the following recognition principle: a locally \((n - 1)\)-connected \(n\)-topos that is generated by its locally constant sheaves is equivalent to the classifying \(n\)-topos of a pro-\(n\)-groupoid with \((n - 1)\)-connected transition maps.

In §3, we generalize finite Galois theory to arbitrary \(\infty\)-topoi: we show that finite locally constant sheaves in an \(\infty\)-topos are equivalent to finite representations of its fundamental pro-\(\infty\)-groupoid.

In §4, we observe that if \(\mathcal{C}\) is a Grothendieck site with a final object, then a coherent refinement of the construction of Artin and Mazur [AM69] is a model for the shape of the \(\infty\)-topos of hypercomplete sheaves on \(\mathcal{C}\). Specializing to the étale site of a locally connected scheme \(X\), we show that the étale topological type of \(X\) defined by Friedlander [Fri82] is a model for the shape of the \(\infty\)-topos of hypercomplete étale sheaves on \(X\).

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Notation.
\[ ∞ \, (∞, 1) \]
\[ S \, \text{∞-category of small } ∞\text{-groupoids} \]
\[ Cat \, (\text{possibly large }) ∞\text{-categories} \]
\[ Top \, ∞\text{-topoi and geometric morphisms} \]
\[ 1 \, \text{final object in an } ∞\text{-topos} \]
\[ \text{Fun}(C, D) \, \text{∞-category of functors from } C \text{ to } D \]
\[ \text{Map}(X, Y) \, \text{∞-groupoid of maps from } X \text{ to } Y \text{ in an } ∞\text{-category} \]
\[ \mathcal{E}/X, \mathcal{E}/_X \, \text{overcategory, undercategory} \]
\[ \mathcal{E}_{≤n} \, \text{subcategory of } n\text{-truncated objects} \]
\[ h\mathcal{C} \, \text{homotopy category} \]
\[ \text{Set}_Δ \, \text{category of simplicial sets} \]
\[ \text{Map}_Δ(X, Y) \, \text{simplicial set of maps from } X \text{ to } Y \text{ in a } \text{Set}_Δ\text{-enriched category} \]

1. Preliminaries on shapes

Let \( C \) be an \( ∞ \)-category. The \( ∞ \)-category \( \text{Pro}(C) \) of pro-objects in \( C \) and the Yoneda embedding \( j : C \to \text{Pro}(C) \) are defined by the following universal property:

1. The category \( \text{Pro}(C) \) admits small cofiltered limits;
2. Let \( D \) be an \( ∞ \)-category which admits small cofiltered limits, and let \( \text{Fun}(\text{Pro}(C), D) \) be the full subcategory of \( \text{Fun}(\text{Pro}(C), D) \) consisting of functors preserving small cofiltered limits. Then \( j \) induces an equivalence \( \text{Fun}(\text{Pro}(C), D) \simeq \text{Fun}(C, D) \).

If \( C \) is small, it is clear that \( \text{Pro}(C) \) can be identified with the smallest full subcategory of \( \text{Fun}(C, S)^{op} \) containing representables and closed under cofiltered limits. If \( C \) is accessible and admits finite limits, then \( \text{Pro}(C) \) can be identified with the category of left exact accessible functors in \( \text{Fun}(C, S)^{op} \) [Lur09, Proposition 3.1.6]. The universal property is then explicitly realized as follows. Recall that \( S \) is the base of the universal left fibration \( u : S_*/ \to S \). Thus, any functor \( f : C \to S \) classifies a left fibration \( \mathcal{E}/_f \to C \) given by the cartesian square

\[
\begin{array}{ccc}
\mathcal{E}/_f & \rightarrow & S_*/ \\
\downarrow & & \downarrow u \\
\mathcal{E} & \rightarrow & S.
\end{array}
\]

The condition that \( f \) is left exact is then equivalent to the condition that \( \mathcal{E}/_f \) is cofiltered [Lur09, Remark 5.3.2.11], and the condition that \( f \) is accessible implies that \( \mathcal{E}/_f \) is accessible [Lur09, Proposition 5.4.6.6]. Accessible \( ∞ \)-categories have small left cofinal subcategories since for \( A \) small, \( A \subset \text{Ind}_*(A) \) is left cofinal. Thus, diagrams indexed by \( \mathcal{E}/_f \) will have a limit in any category \( D \) that admits small cofiltered limits. In this way any functor \( C \to D \) lifts to a functor \( \text{Pro}(C) \to D \), sending \( f \) to the limit of the composition \( \mathcal{E}/_f \to C \to D \).

Note that the diagram \( u : \mathcal{E}/_f \to C \) corepresents the pro-object \( f \):

\[ f(K) \simeq \colim_{X \in \mathcal{E}/_f} \text{Map}(uX, K). \]

By [Lur09, Proposition 5.3.1.16], any pro-object can be further corepresented by a diagram \( J \to C \) where \( J \) is a small cofiltered poset. Using this fact one can show that, if \( C \) is the underlying \( ∞ \)-category of a proper model category \( M \), then \( \text{Pro}(C) \) is the underlying \( ∞ \)-category of the strict model structure on \( \text{Pro}(M) \) defined in [Isa07].

Let \( G : D \to C \) be an accessible functor between presentable \( ∞ \)-categories, and let \( F : C \to \text{Fun}(D, S)^{op} \) be the “formal” left adjoint to \( G \). By the adjoint functor theorem, \( F \) factors through \( D \) if and only if \( G \) preserves small limits. Clearly, \( F \) factors through \( \text{Pro}(D) \) if and only if \( G \) preserves finite limits. In this case the functor \( F : C \to \text{Pro}(D) \) is called the pro-left adjoint to \( G \). Its extension \( \text{Pro}(C) \to \text{Pro}(D) \) is a genuine left adjoint to \( \text{Pro}(G) \).

If \( f : y \to x \) is a geometric morphism of \( ∞ \)-topoi, we will write \( f_* : y \to x \) for the direct image functor and \( f^* : x \to y \) for the left adjoint of the latter. Since \( f^* \) preserves finite limits, it admits a pro-left adjoint
If $X$ is an $\infty$-topos, we will usually denote by $\pi: X \to S$ the unique geometric morphism to $S$, given informally by $\pi_s(X) = \text{Map}(1, X)$. We define

$$\Pi_\infty X = \pi_1.$$

The object $\Pi_\infty X \in \text{Pro}(S)$ is called the fundamental pro-$\infty$-groupoid or the shape of the $\infty$-topos $X$. As a left exact functor $S \to S$, $\Pi_\infty X$ is the composition $\pi_\pi$, i.e., it sends an $\infty$-groupoid to the global sections of the associated constant sheaf. Note that $\pi_\pi X \simeq \Pi_\infty(\chi/\chi)$ since $X \simeq \rho_1$, where $\rho: \chi/\chi \to \chi$ is the canonical geometric morphism. In other words, the functor $\Pi_\infty: \text{Top} \to \text{Pro}(S)$ simultaneously extends the functors $\pi_\pi: X \to \text{Pro}(S)$ for all $\infty$-topoi $X$.

**Example 1.1.** If $T$ is a topological space homeomorphism equivalent to a CW complex, the shape of $\text{Shv}(T)$ is the weak homotopy type of $T$ [Lur14, Remarks A.1.4 and A.4.7]. As we will see in Proposition 1.11 below, this is a refinement of the fact that the singular cohomology of $T$ with coefficients in a local system coincides with its sheaf cohomology with coefficients in the corresponding locally constant sheaf.

A torsor $(A, \chi)$ in $X$ is an $\infty$-groupoid $A \in S$ together with a map $\chi: 1 \to \pi^* A$ in $X$. We denote by

$$\text{Tors}(X) = \chi_1/ \times \chi S$$

the $\infty$-category of torsors in $X$ and by

$$\text{Tors}(X, A) = \text{Tors}(X) \times_S \{A\} \simeq \text{Map}_X(1, \pi^* A)$$

the $\infty$-groupoid of $A$-torsors. By descent, an $A$-torsor is equivalently an action $P: A \to X$ of the $\infty$-groupoid $A$ in $X$ which is principal in the sense that $\text{colim}_{x \in A} P_x \simeq 1$.

**Proposition 1.2.** Let $f: \mathcal{Y} \to X$ be a geometric morphism of $\infty$-topoi. The following conditions are equivalent:

1. $f$ is a shape equivalence, i.e., $\Pi_\infty(f)$ is an equivalence in $\text{Pro}(S)$.
2. $f^*: \text{Tors}(X) \to \text{Tors}(\mathcal{Y})$ is an equivalence of $\infty$-categories.
3. For every $A \in S$, $f^*: \text{Tors}(X, A) \to \text{Tors}(\mathcal{Y}, A)$ is an equivalence of $\infty$-groupoids.

**Proof.** As we recalled above, the Grothendieck construction provides an equivalence between accessible functors $S \to S$ and accessible left fibrations over $S$, under which $\Pi_\infty X$ corresponds to $\text{Tors}(X)$. Explicitly, we have natural equivalences

$$\Pi_\infty X \simeq \lim_{(A, \chi) \in \text{Tors}(X)} j(A) \quad \text{and} \quad \text{Tors}(X) \simeq S_{\Pi_\infty X}/.$$

This proves (1) $\iff$ (2). A morphism of left fibrations is an equivalence if and only if it is a fiberwise equivalence, so (2) $\iff$ (3). $\square$

We now construct a right adjoint to the functor $\Pi_\infty$. Recall that, for any $\infty$-topos $X$, there is a fully faithful functor

$$X \hookrightarrow \text{Top}_{/X}, \quad U \mapsto X/U.$$

When $X = S$, we denote this functor by

$$\iota: S \hookrightarrow \text{Top},$$

and we call $\iota A = S/A \simeq \text{Fun}(A, S)$ the classifying $\infty$-topos of the $\infty$-groupoid $A$. The $\infty$-topos $\iota A$ classifies $A$-torsors, meaning that there is an equivalence of $\infty$-categories

$$(1.3) \quad \text{Fun}_{\text{Top}}(X, \iota A) \simeq \text{Tors}(X, A),$$

for every $\infty$-topos $X$ (this is a special case of [Lur09, Corollary 6.3.5.6]). The diagonal map $\delta: A \to A \times A$ in $S/A$ is the universal $A$-torsor.

Since the $\infty$-category $\text{Top}$ admits small cofiltered limits, $\iota$ extends to a functor

$$\iota: \text{Pro}(S) \to \text{Top}.$$

Explicitly, let $X \in \text{Pro}(S)$ be a pro-$\infty$-groupoid given in the form of a cofiltered diagram $X: \mathcal{I} \to S$. The classifying $\infty$-topos $\iota(X)$ is then the limit of the cofiltered diagram

$$\iota \circ X: \mathcal{I} \to \text{Top}.$$
Recall that limits of cofiltered diagrams in $\text{Top}$ are created by the forgetful functor $\text{Top} \to \text{Set}$ [Lur09, Theorem 6.3.3.1]. Thus, an object $L \in \iota(X)$ is a family of objects $L_i \in \iota(X_i)$ together with coherent equivalences $f_i, L_i \simeq L_j$ for all arrows $f : i \to j$ in $\mathcal{I}$.

**Remark 1.4.** While the functor $\iota$ is fully faithful and colimit-preserving on $\mathcal{S}$, its extension to $\text{Pro}(\mathcal{S})$ does not have either property: if $X$ is a pro-set whose limit is empty, it is clear that $\iota(X)$ is the empty $\infty$-topos, and if $\iota$ preserved pushouts of pro-$\infty$-groupoids, it would imply that cofiltered limits preserve pushouts in $\mathcal{S}$. Nevertheless, one can show that $\iota$ preserves arbitrary coproducts of pro-$\infty$-groupoids (and, more generally, colimits of “levelwise” diagrams indexed by $\infty$-groupoids).

**Remark 1.5.** For $X \in \text{Pro}(\mathcal{S})$, the $\infty$-topos $\iota(X)$ is typically not hypercomplete. For example, it is shown in [Lur09, Warning 7.2.2.31] that, for $p$ prime, the classifying $\infty$-topos $\iota(B\mathbb{Z}_p)$ is not hypercomplete.

If $X : \mathcal{I} \to \mathcal{S}$ is a pro-$\infty$-groupoid and $\mathcal{C}$ is an $\infty$-category, it is convenient to write

$$\text{Fun}(X, \mathcal{C}) = \colim_{i \in \mathcal{I}} \text{Fun}(X_i, \mathcal{C})$$

—in other words, we view both $X$ and $\mathcal{C}$ as pro-$\infty$-categories. The inverse image functors $\iota(X_i) \to \iota(X)$ then induce a canonical functor

$$\text{Fun}(X, \mathcal{S}) \to \iota(X);$$

an object $L \in \iota(X)$ is called split if it belongs to its essential image, and if $L$ is the image of $K \in \iota(X_i)$ we say that $L$ is represented by $K$. An arbitrary $L \in \iota(X)$ is the filtered colimit of the split objects represented by $L_i$ for $i \in \mathcal{I}$ [Lur09, Lemma 6.3.3.6].

**Definition 1.6.** If $\mathcal{X}$ is an $\infty$-topos, $\iota(\Pi_\infty \mathcal{X})$ is the $\infty$-topos of local systems on $\mathcal{X}$.

Since $\Pi_\infty \mathcal{X}$ is corepresented by the forgetful functor $\text{Tors}(\mathcal{X}) \to \mathcal{S}$, a local system on $\mathcal{X}$ is a family of objects $L_{(X,x)} \in \iota(X)$ indexed by pairs $(X,x)$ where $X \in \mathcal{S}$ and $x : 1 \to \pi^* X$ is an $X$-torsor in $\mathcal{X}$, together with coherent equivalences $f_i L_{(X,x)} \simeq L_{(Y,y)}$ for all morphisms of torsors $f : (X,x) \to (Y,y)$.

It follows at once from (1.3) that we have an adjunction

$$\text{Top} \underbrace{}_{\iota} \xrightarrow{\Pi \mathcal{S}} \text{Pro}(\mathcal{S});$$

we will denote by $\varphi : \mathcal{X} \to \iota(\Pi_\infty \mathcal{X})$ its unit. If $L$ is a local system on $\mathcal{X}$, the object $\varphi^* L \in \mathcal{X}$ will be called the underlying sheaf of $L$. We wish to describe $\varphi^* L$ more explicitly. Since a general local system is a colimit of split local systems, it suffices to describe the underlying sheaves of the latter. Let $(X,x : 1 \to \pi^* X) \in \text{Tors}(\mathcal{X})$ be a torsor. This determines a geometric morphism $f : \mathcal{X} \to \iota(X)$. If $L$ is represented by $K \in \iota(X)$, we therefore have $\varphi^* L \simeq f^* K$. This means that $\varphi^* L$ fits in a cartesian square

$$\varphi^* L \longrightarrow \pi^* K \quad \text{with} \quad 1 \longrightarrow \pi^* X.$$

In addition to preserving colimits (being left adjoint), the functor $\Pi_\infty : \text{Top} \to \text{Pro}(\mathcal{S})$ also preserves some interesting limits:

**Proposition 1.7.**

1. If $\mathcal{X}$ is the limit of a cofiltered diagram of proper $\infty$-topoi $(\mathcal{X}_i)$ with proper transition morphisms, then $\Pi_\infty \mathcal{X} \simeq \lim_i \Pi_\infty \mathcal{X}_i$.
2. If $\mathcal{X}$ and $\mathcal{Y}$ are proper $\infty$-topoi, then $\Pi_\infty (\mathcal{X} \times \mathcal{Y}) \simeq \Pi_\infty \mathcal{X} \times \Pi_\infty \mathcal{Y}$.

**Proof.** (1) Let $\pi : \mathcal{X} \to \mathcal{S}$ and $\pi_i : \mathcal{X}_i \to \mathcal{S}$ be the unique geometric morphisms. Since proper geometric morphisms preserve filtered colimits [Lur09, Remark 7.3.1.5], the canonical map

$$\text{colim} \ i_\ast \pi_i^* \to \pi^*$$

is an equivalence. This exactly says that $\Pi_\infty \mathcal{X} \simeq \lim_i \Pi_\infty \mathcal{X}_i$.

(2) The properness of $\mathcal{Y}$ implies, by proper base change, that

$$\Pi_\infty (\mathcal{X} \times \mathcal{Y}) \simeq \Pi_\infty \mathcal{X} \circ \Pi_\infty \mathcal{Y}.$$
The properness of $\mathcal{X}$ implies that $\Pi_\infty \mathcal{X}$ preserves filtered colimits. To conclude, note that if $X, Y \in \text{Pro}(\mathcal{S})$ and $X$ preserves filtered colimits, then $X \circ Y \simeq X \times Y$. 

**Example 1.8.** If a topological space $T$ is the limit of a cofiltered diagram $(T_i)$ of compact Hausdorff spaces, then

$$\Pi_\infty \text{Shv}(T) \simeq \varprojlim \Pi_\infty \text{Shv}(T_i).$$

This is a consequence of Proposition 1.7 (1) and the following facts: passing to locales preserves the limit of the diagram $(T_i)$, and maps between compact Hausdorff spaces induce proper morphisms of $\infty$-topoi [Lur09, Theorem 7.3.1.16]. Together with Example 1.1, we deduce for example that, if $T$ is a compact subset of the Hilbert cube, then the shape of $\text{Shv}(T)$ is corepresented by the diagram $\mathbb{R}_{>0} \to \mathcal{S}$ sending $\epsilon$ to the weak homotopy type of the $\epsilon$-neighborhood of $T$.

**Remark 1.9.** Since any compact Hausdorff space is a cofiltered limit of compact metrizable ANRs, which have the homotopy type of CW complexes, Examples 1.1 and 1.8 show that the shape theory of $\infty$-topoi completely subsumes the classical shape theory of compact Hausdorff spaces [MS71].

For $n \geq -2$, consider the adjunction

$$\text{Pro}(\mathcal{S}) \overset{\tau_{\leq n}}{\leftrightarrow} \text{Pro}(\mathcal{S}_{\leq n}).$$

We denote by $\Pi_n : \text{Top} \to \text{Pro}(\mathcal{S}_{\leq n})$ the composition $\tau_{\leq n} \circ \Pi_\infty$. Note that $\Pi_n \mathcal{X}$ is corepresented by the forgetful functor $\text{Tors}_{\leq n}(\mathcal{X}) \to \mathcal{S}_{\leq n}$, where $\text{Tors}_{\leq n}(\mathcal{X}) = \text{Tors}(\mathcal{X}) \times_{A} \mathcal{S}_{\leq n}$. The description of torsors in terms of principal actions shows that the functor $\Pi_n$ factors through the reflective subcategory $\mathcal{T}_n \subset \mathcal{Top}$ of $n$-localic $\infty$-topoi. Moreover, if $X \in \text{Pro}(\mathcal{S}_{\leq n}) \subset \text{Pro}(\mathcal{S})$, then $\iota X$ is a limit of $n$-localic $\infty$-topoi and hence is $n$-localic. Thus, we obtain an induced adjunction

$$\Pi_n : \mathcal{T}_n \rightleftarrows \text{Pro}(\mathcal{S}_{\leq n}) : \iota$$

between $n$-topoi and pro-$n$-groupoids.

**Remark 1.10.** If $\mathcal{X}$ is a 1-topos, $\Pi_1 \mathcal{X}$ is equivalent to the fundamental pro-groupoid defined by Dubuc in [Dub10, §8], since both classify torsors. Earlier constructions of $\Pi_1 \mathcal{X}$ were given in special cases, notably by Kennison [Ken91] (for $\mathcal{X}$ connected), by Bunge [Bun92] (for $\mathcal{X}$ locally connected), and by Barr and Diaconescu [BD81] (for $\mathcal{X}$ locally simply connected).

By definition, $\Pi_\infty \mathcal{X}$ corepresents the cohomology of $\mathcal{X}$ with constant coefficients. Our next observation is that this can be extended to cohomology with coefficients in the underlying sheaf of a split local system. In §2 we will show that there are many such sheaves: for example, if $\mathcal{X}$ is locally connected, $A$ is a locally constant sheaf of abelian groups, and $n \geq 0$, then the Eilenberg–Mac Lane sheaf $K(A, n)$ is the underlying sheaf of a split local system.

**Proposition 1.11.** Let $\mathcal{X}$ be an $\infty$-topos and let $L \in \text{Fun}(\Pi_\infty \mathcal{X}, \mathcal{S})$. Denote also by $L$ the image of $L$ in $\iota(\Pi_\infty \mathcal{X})$. Then $\varphi^*$ induces an equivalence

$$\text{Map}_{\text{Fun}(\Pi_\infty \mathcal{X}, \mathcal{S})}(*, L) \simeq \text{Map}_\mathcal{X}(1, \varphi^* L).$$

**Proof.** Suppose that $L$ comes from the object $K \in \mathcal{S}_{/X}$ labeled by the torsor $(X, x)$ in $\mathcal{X}$. The proposition follows by comparing the two cartesian squares

$$\begin{array}{ccc}
\text{Map}_{\text{Fun}(\Pi_\infty \mathcal{X}, \mathcal{S})}(*, L) & \longrightarrow & \text{Map}(\Pi_\infty \mathcal{X}, K) \\
\downarrow & & \downarrow \\
\text{Map}(\Pi_\infty \mathcal{X}, X) & \longrightarrow & \pi_* \pi^* \mathcal{X}
\end{array}$$

in which the lower maps are induced by $x$. 

□
2. Galois theory

Classical Galois theory states that the étale topos $\mathcal{X}$ of a field $k$ is equivalent to the classifying topos of the absolute Galois group of $k$. More precisely:

(1) For any separable closure $k^s$ of $k$, there is a canonical equivalence of pro-groupoids

$$BGal(k^s/k) \simeq \Pi_1\mathcal{X}.$$ 

(2) The geometric morphism $\varphi : \mathcal{X} \rightarrow \iota(\Pi_1\mathcal{X})$ is an equivalence of topoi and identifies locally constant sheaves with split local systems.

Statement (2) is true more generally of any locally connected topos $\mathcal{X}$ generated by its locally constant object. There is also an analog of statement (1) for any locally connected topos. In this section we prove the expected $n$-toposic generalizations of these results for $0 \leq n \leq \infty$. The case $n = \infty$ is treated in [Lur14, §A.1], but the case of finite $n$ is more complicated.

Let $\mathcal{X}$ be an $\infty$-topos. An object $X \in \mathcal{X}$ is called locally constant if there exists an effective epimorphism $\coprod \alpha U_\alpha \rightarrow 1$ such that $X$ is constant over each $U_\alpha$, i.e., such that $X \times U_\alpha \simeq \pi^*X_\alpha \times U_\alpha$ for some $X_\alpha \in S$.

Let $-2 \leq n \leq \infty$. A geometric morphism $f : Y \rightarrow \mathcal{X}$ is called $n$-connected if $f^*$ is fully faithful on $n$-truncated objects. An $\infty$-topos $\mathcal{X}$ is called locally $n$-connected if $\pi^* : S_{\leq n} \rightarrow \mathcal{X}_{\leq n}$ preserves infinite products, or, equivalently, if its pro-left adjoint is a genuine left adjoint. Note that every $\infty$-topos is locally $(-1)$-connected, since $S_{\leq -1} = \{\emptyset \rightarrow \ast\}$.

**Proposition 2.1.** Let $\mathcal{X}$ be a locally $\infty$-connected $\infty$-topos. Then $\varphi : \mathcal{X} \rightarrow \iota(\Pi_1\mathcal{X})$ is $\infty$-connected and its image is the subcategory of locally constant objects in $\mathcal{X}$.

**Proof.** This is [Lur14, Theorem A.1.15]. \qed

If we were to repeat the proof of Proposition 2.1 in the world of $(n + 1)$-topoi, it would only show that, in a locally $n$-connected $\infty$-topos, $\varphi^*$ identifies local systems of $(n - 1)$-groupoids (which are always split) with locally constant $(n - 1)$-truncated objects. To treat the edge case of local systems of $n$-groupoids, which need not be split, new arguments are needed.

The proof of the following result is the same as the first half of the proof of [Lur14, Theorem A.1.15].

**Proposition 2.2.** Let $\mathcal{X}$ be an $\infty$-topos and $L$ a split local system on $\mathcal{X}$. Then the underlying sheaf $\varphi^*L$ is locally constant.

**Proof.** Let $L$ be represented by $K \in \iota(X)$ for some torsor $(X, x)$ in $\mathcal{X}$, so that $\varphi^*L$ is given by the cartesian square

$$\begin{array}{ccc}
\varphi^*L & \rightarrow & \pi^*K \\
\downarrow & & \downarrow \\
1 & \rightarrow & \pi^*X.
\end{array}$$

Let $\coprod \alpha U_\alpha \rightarrow X$ be an effective epimorphism in $\mathcal{S}$ where each $U_\alpha$ is contractible. Then $\coprod \alpha \varphi^*U_\alpha \rightarrow 1$ is an effective epimorphism. There is a commutative diagram in $\mathcal{S}$

$$\begin{array}{ccc}
\mathcal{S}/X & \rightarrow & \mathcal{X}/\pi^*X \\
\downarrow & & \downarrow \\
\mathcal{S}/U_\alpha & \rightarrow & \mathcal{X}/\varphi^*U_\alpha
\end{array}$$

such that $K$ in the top left corner goes to $\varphi^*L \times \varphi^*U_\alpha$ in the bottom right corner. Since $\mathcal{S}/U_\alpha \simeq \mathcal{S}$, this shows that $\varphi^*L \times \varphi^*U_\alpha$ is constant over $\varphi^*U_\alpha$. Thus, $\varphi^*L$ is locally constant. \qed

**Lemma 2.3.** Let $\mathcal{X}$ be a locally $n$-connected $\infty$-topos. Then $\pi^* : \mathcal{S} \rightarrow \mathcal{X}$ preserves the limits of cofiltered diagrams with $n$-truncated transition maps.
Proof. Let $K: J \to S$ be a cofiltered diagram with $n$-truncated transition maps. Assume without loss of generality that $J$ has a final object $0$. We then have a commutative square

$$
\begin{array}{ccc}
S_{/K(0)} & \xrightarrow{\pi^*} & \mathcal{X}_{/\pi^*K(0)} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi^*} & \mathcal{X}
\end{array}
$$

where the vertical arrows are the forgetful functors. Since the latter preserve and reflect cofiltered limits, it will suffice to show that $\pi^*: (S/K(0))_{\leq n} \to (\mathcal{X}/\pi^*K(0))_{\leq n}$ preserves limits. By descent, this functor can be identified with the functor

$$\text{Fun}(K(0), S_{\leq n}) \to \text{Fun}(K(0), \mathcal{X}_{\leq n})$$

given objectwise by $\pi^*$. This functor preserves limits since they are computed objectwise. \hfill $\square$

Recall that a morphism of $\infty$-groupoids $f: X \to Y$ is $n$-connected if its fibers are $n$-connected. This is the case if and only if the induced geometric morphism $f_\ast: \iota(X) \to \iota(Y)$ is $n$-connected. We call a pro-$\infty$-groupoid $n$-strict if it can be corepresented by a cofiltered diagram in which the transition maps are $n$-connected.

**Lemma 2.4.** Let $\mathcal{X}$ be a locally $n$-connected $\infty$-topos. Then there exists a coreflective subcategory of $\text{Tors}(\mathcal{X})$ in which all morphisms are $n$-connected.

**Proof.** Let $(A, \chi)$ be a torsor. For every morphism of torsors $(B, \psi) \to (A, \chi)$, consider the unique factorization

$$B \to e(B) \to A$$

where $B \to e(B)$ is $n$-connected and $e(B) \to A$ is $n$-truncated. Let $\hat{A}$ be the limit of the cofiltered diagram

$$\text{Tors}(\mathcal{X})/((A, \chi)) \to S, \quad (B, \psi) \mapsto e(B).$$

By construction, this is a diagram with $n$-truncated transition maps. Hence, by Lemma 2.3, $\pi^*$ preserves the limit of this diagram. In particular, there is an $A$-torsor $\hat{\chi}: 1 \to \pi^*\hat{A}$ which is the limit of the torsors $1 \to \pi^*e(B)$. One verifies easily that $(A, \chi) \mapsto (\hat{A}, \hat{\chi})$ is a coreflector and that, for every morphism of torsors $(B, \psi) \to (A, \chi)$, $\hat{B} \to \hat{A}$ is $n$-connected. \hfill $\square$

**Proposition 2.5.** Let $\mathcal{X}$ be an $\infty$-topos, let $-2 \leq n \leq \infty$, and let $\{X_\alpha\}$ be a family of objects generating $\mathcal{X}$ under colimits. The following conditions are equivalent:

1. $\mathcal{X}$ is locally $n$-connected.
2. For every $\alpha$, the pro-$n$-groupoid $\tau_{\leq n} \pi_!(X_\alpha)$ is constant.
3. For every $\alpha$, the pro-$\infty$-groupoid $\pi_!(X_\alpha)$ is $n$-strict.

**Proof.** By definition, $\mathcal{X}$ is locally $n$-connected if and only if the composition

$$\mathcal{X} \xrightarrow{\pi_!} \text{Pro}(S) \xrightarrow{\tau_{\leq n}} \text{Pro}(S_{\leq n})$$

factors through the Yoneda embedding $S_{\leq n} \hookrightarrow \text{Pro}(S_{\leq n})$. Since the latter preserves colimits, we see that (1) $\iff$ (2). The implication (3) $\Rightarrow$ (2) is obvious. Let us prove (1) $\Rightarrow$ (3). It is clear that $X_{/U}$ is locally $n$-connected for every $U \in \mathcal{X}$, since $X_{/U} \to \mathcal{X}$ is étale, so it will suffice to show that $\Pi_\infty \mathcal{X}$ is $n$-strict. Let $J \subset \text{Tors}(\mathcal{X})$ be a coreflective subcategory as in Lemma 2.4. By Joyal’s criterion [Lur09, Theorem 4.1.3.1], $J$ is a left cofinal subcategory. Hence, $\Pi_\infty \mathcal{X}$ is corepresented by the forgetful functor $J \to S$ and in particular is $n$-strict. \hfill $\square$

For $n = -1$, Proposition 2.5 says that the fundamental pro-$\infty$-groupoid of any $\infty$-topos $\mathcal{X}$ can be corepresented by a cofiltered diagram whose transition maps are effective epimorphisms.

**Lemma 2.6.** If $X \in \text{Pro}(S)$ is $n$-strict, the canonical functor

$$\text{Fun}(X, S_{\leq n}) \to \iota(X)_{\leq n}$$

is fully faithful.
Proof. Let $X : \mathcal{I} \to S$ be a corepresentation with $n$-connected transition maps. We claim that each functor
\[ \pi_i^* : \text{Fun}(X_i, S) \to \iota(X) \]
is fully faithful on $n$-truncated objects. Since $\mathcal{I}/i \to \mathcal{I}$ is left cofinal, we may replace $\mathcal{I}$ by $\mathcal{I}/i$ and assume that $i$ is a final object. Let $p : \mathcal{E} \to \mathcal{I}^{op}$ be the topos fibration associated with the functor $\iota \circ X : \mathcal{I} \to \text{Top}$. By the construction of cofiltered limits of $\infty$-topoi, $\iota(X)$ is the $\infty$-category of cartesian sections of $p$. Let $\theta(X)$ be the $\infty$-topos of all sections of $p$, and factor $\pi_i$ as $\rho_i \circ \sigma$ where $\sigma_* : \iota(X) \to \theta(X)$ is the inclusion and $\rho_i^* : \theta(X) \to \text{Fun}(X_i, S)$ is evaluation at $i$. Since $\rho_i^*$ is fully faithful, it will suffice to show that, for $F \in \text{Fun}(X_i, S)_{\leq n}$, the section $\rho_i^* F$ is already cartesian. Note that, since $i \in \mathcal{I}$ is final, we have $(\rho_i^* F)_j = f^* F$ where $f : j \to i$ is the unique map. Given $f : j \to i$, $g : k \to j$, and $x \in X_j$, the component at $x$ of the natural transformation $(\rho_i^* F)_j \to g^* (\rho_i^* F)_k$ in $\text{Fun}(X_j, S)$ is the canonical map $F(f(x)) \to \lim_{y \in g^{-1}(x)} F(f(y)) \simeq \text{Map}(g^{-1}(x), F(f(x)))$. This is an equivalence since $g^{-1}(x)$ is $n$-connected and $F(f(x))$ is $n$-truncated. □

Lemma 2.7. Let $\mathcal{X}$ be an $\infty$-topos and let $E : \mathcal{I} \times \mathcal{J} \to \mathcal{X}$ be a diagram where $\mathcal{I}$ has a final object $e$. Suppose that, for every $i \in \mathcal{I}$ and every $j \to k$ in $\mathcal{J}$, the square
\[
\begin{array}{ccc}
E(i, j) & \longrightarrow & E(i, k) \\
\downarrow & & \downarrow \\
E(e, j) & \longrightarrow & E(e, k)
\end{array}
\]
is cartesian. Then the canonical map
\[
\colim_i \lim_j E(i, j) \rightarrow \lim_i \colim_j E(i, j)
\]
is an equivalence. Proof. By descent, the square
\[
\begin{array}{ccc}
E(i, j) & \longrightarrow & \colim_j E(i, j) \\
\downarrow & & \downarrow \\
E(e, j) & \longrightarrow & \colim_j E(e, j)
\end{array}
\]
is cartesian for every $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Taking the limit over $i$, we obtain a cartesian square
\[
\begin{array}{ccc}
\lim_i E(i, j) & \longrightarrow & \lim_i \colim_j E(i, j) \\
\downarrow & & \downarrow \\
E(e, j) & \longrightarrow & \colim_j E(e, j)
\end{array}
\]
By universality of colimits, we get the desired equivalence by taking the colimit over $j$ of the left column. □

Lemma 2.8. Let $\mathcal{X}$ be an $\infty$-topos, $X$ a pro-$\infty$-groupoid, $\varphi : \mathcal{X} \to \iota(X)$ a geometric morphism, and $n \geq -2$ an integer. Suppose that
(1) $X$ is $n$-strict.
(2) The composition $\text{Fun}(X, S_{\leq n}) \to \iota(X)_{\leq n} \xrightarrow{\varphi^*} X_{\leq n}$ is fully faithful.
Then $\varphi$ is $n$-connected.
Theorem 2.10. Let \(-2 \leq n \leq \infty\) and let \(X\) be a locally \(n\)-connected \(\infty\)-topos. Then the geometric morphism \(\varphi : X \to \iota(\Pi_{\infty}X)\) is \(n\)-connected and identifies split local systems of \(n\)-groupoids with locally constant \(n\)-truncated sheaves on \(X\).
Proof. The case \( n = \infty \) is Proposition 2.1 (but the following proof also works for \( n = \infty \)). We already know from Proposition 2.2 that \( \varphi^* \) sends split local systems to locally constant objects. By Proposition 2.5, \( \Pi_\infty \mathcal{X} \) is \( n \)-strict. By Lemma 2.8, it therefore remains to prove that the composition

\[
(2.11) \quad \text{Fun}(\Pi_\infty \mathcal{X}, S_{\leq n}) \to \iota(\Pi_\infty \mathcal{X})_{\leq n} \xrightarrow{\varphi^*} \mathcal{X}_{\leq n}
\]

is fully faithful and that every locally constant object is in its image.

First we establish a crucial preliminary result. By the assumption of local \( n \)-connectedness, the functor \( \pi^*: S_{\leq n} \to \mathcal{X}_{\leq n} \) preserves limits. It follows that for \( n \)-groupoids \( X \) and \( Y \),

\[
\pi^* \text{Map}(X, Y) \simeq \pi^*(\lim X) \simeq \lim \pi^* Y \simeq \text{Hom}(\pi^* X, \pi^* Y),
\]

where \( \text{Hom} \) is the internal mapping object of \( \mathcal{X} \). Taking global sections, we get, for \( n \)-groupoids \( X \) and \( Y \),

\[
(2.12) \quad \text{Map}_{\text{Pro}(\mathcal{S})}(\Pi_\infty \mathcal{X}, \text{Map}(X, Y)) \simeq \text{Map}_{\mathcal{X}}(\pi^* X, \pi^* Y).
\]

Let \( \mathcal{O}: \mathcal{X}^{\text{op}} \to \text{Cat} \) be the functor \( U \mapsto (\mathcal{X}/U)_{\leq n} \), and let \( \mathcal{L}: \mathcal{X}^{\text{op}} \to \text{Cat} \) be the functor \( U \mapsto \text{Fun}(\pi_1 U, S_{\leq n}) \). The morphism \( \varphi \), being natural in \( \mathcal{X} \), gives rise to a natural transformation \( \varphi^*: \mathcal{L} \to \mathcal{O} \) whose components are the instances of (2.11) for all étale \( \mathcal{X} \)-topoi. Both \( \mathcal{O} \) and \( \mathcal{L} \) are sheaves on \( \mathcal{X} \), i.e., limit-preserving functors, the former by descent and the latter because \( \pi_1 \) is left adjoint. It will therefore suffice to prove the following:

For every \( U \in \mathcal{X} \):

(a) For every \( L, M \in \mathcal{L}(U) \), there exists a diagram \( P: A \to \mathcal{X} \) with colimit \( U \) such that

\[
\text{Map}_{\mathcal{L}(P_{\alpha})}(L|P_{\alpha}, M|P_{\alpha}) \to \text{Map}_{\mathcal{O}(P_{\alpha})}(\varphi^* L|P_{\alpha}, \varphi^* M|P_{\alpha})
\]

is an equivalence, for all \( \alpha \in A \).

(b) For every \( X \in \mathcal{O}(U) \) locally constant, there exists an effective epimorphism \( \Pi_\alpha U_{\alpha} \to U \) such that \( X|U_{\alpha} \) belongs to the essential image of \( \mathcal{L}(U_{\alpha}) \) \( \to \mathcal{O}(U_{\alpha}) \), for all \( \alpha \).

Assertion (b) is obvious since any constant sheaf belongs to the image of \( \varphi^* \). To prove (a), we choose a torsor \((A, \chi) \in \text{Tors}((\mathcal{X}/U))\) such that \( L \) and \( M \) are represented by functors \( A \to S_{\leq n} \). Let \( P: A \to \mathcal{X}/U \) be the associated principal action, whose colimit is \( U \). Given \( \alpha \in A \), the cartesian square

\[
\begin{array}{ccc}
P_{\alpha} & \to & \pi^*(\ast) \\
\downarrow & & \downarrow \\
U & \xrightarrow{\chi} & \pi^*(A)
\end{array}
\]

shows that \( L|P_{\alpha} \) and \( M|P_{\alpha} \) are constant functors \( \pi_1 P_{\alpha} \to S_{\leq n} \). Replacing \( \mathcal{X} \) by \( \mathcal{X}/P_{\alpha} \), which is also locally \( n \)-connected, we are reduced to proving the following statement: for every \( X, Y \in S_{\leq n} \), the map

\[
\text{Map}_{\text{Fun}(\mathcal{S}),\mathcal{X}}(X, Y) \to \text{Map}_{\mathcal{X}}(\pi^* X, \pi^* Y)
\]

is an equivalence. This is exactly our preliminary result (2.12). \( \square \)

Remark 2.13. An example of a 1-topos for which \( \varphi \) is not connected was constructed by Kennison in [Ken91, Example 4.16], showing that the local connectedness assumption in Theorem 2.10 cannot be dropped. On the other hand, \( \varphi \) may be \( n \)-connected even if \( \Pi_\alpha \mathcal{X} \) is not constant: if \( \mathcal{X} = \text{Shv}(\mathbb{Q}) \), then \( \varphi \) is an equivalence of \( \infty \)-topoi (but (2.11) fails to be fully faithful for \( n = 0 \)).

If \( \mathcal{X} \) is an \( \infty \)-topos and \( n \geq 0 \), denote by \( \mathcal{E}M_n(\mathcal{X}) \) the category of Eilenberg–Mac Lane objects of degree \( n \) in \( \mathcal{X} \).

Corollary 2.14. Let \( \mathcal{X} \) be a locally connected \( \infty \)-topos and let \( n \geq 0 \). Then the functor

\[
\varphi^*: \mathcal{E}M_n(\iota(\Pi_\infty \mathcal{X})) \to \mathcal{E}M_n(\mathcal{X})
\]

is fully faithful and identifies split local systems with locally constant objects. If moreover \( \mathcal{X} \) is locally simply connected, then all local systems of Eilenberg–Mac Lane objects are split.
Proof. Say $n \geq 2$. Then we have a commutative square
\[
\begin{array}{ccc}
\mathcal{EM}_n(\iota(\Pi_\infty \mathcal{X})) & \xrightarrow{\varphi^*} & \mathcal{EM}_n(\mathcal{X}) \\
\pi_n & & \pi_n \\
\mathbb{Ab}(\iota(\Pi_\infty \mathcal{X})_{\leq 0}) & \xrightarrow{\psi^*} & \mathbb{Ab}(\mathcal{X}_{\leq 0})
\end{array}
\]
in which the vertical functors are equivalences \cite[Proposition 7.2.2.12]{Lur09} and the lower row has the desired properties.

Let $\mathcal{X}$ be an $\infty$-topos. A torsor $(A, \chi)$ in $\mathcal{X}$ is called Galois if the associated principal action $P: A \to \mathcal{X}$ is a fully faithful functor. In other words, a Galois torsor is a full sub-$\infty$-groupoid of $\mathcal{X}$ whose colimit is a final object. For example, if $A \in \mathcal{S}$, the universal $A$-torsor is Galois since it corresponds to the Yoneda embedding $A \hookrightarrow \iota A$. If $\mathcal{X}$ is the étale topos of a field $k$, then a Galois torsor in $\mathcal{X}$ is precisely a finite Galois extension of $k$. In this case, the following corollary shows that the absolute Galois group of $k$ computes $\Pi_1 \mathcal{X}$.

**Corollary 2.15.** Let $\mathcal{X}$ be a locally $n$-connected $\infty$-topos and let $\mathcal{J} \subset \operatorname{Tors}_{\leq n+1}(\mathcal{X})$ be the full subcategory spanned by the Galois torsors. Then the inclusion $\mathcal{J} \subset \operatorname{Tors}_{\leq n+1}(\mathcal{X})$ is left cofinal.

**Proof.** By Lemma 2.4, there exists a left cofinal subcategory $\mathcal{J} \subset \operatorname{Tors}_{\leq n+1}(\mathcal{X})$ in which all morphisms are $n$-connected. It will suffice to show that $\mathcal{J} \subset \mathcal{J}$. If $(A, \chi)$ is a torsor in $\mathcal{J}$, the associated principal action is the composition
\[
A \hookrightarrow \iota(A)_{\leq n} \to \iota(\Pi_{n+1} \mathcal{X})_{\leq n} \xrightarrow{\varphi^*} \mathcal{X}_{\leq n}.
\]
The second arrow is fully faithful by Lemma 2.6 and the third by Theorem 2.10.

An $\infty$-topos $\mathcal{X}$ is called $n$-Galois if it is $n$-localic, locally $(n-1)$-connected, and generated under colimits by the images of its Galois torsors $A \hookrightarrow \mathcal{X}$ with $A \in \mathcal{S}_{\leq n}$. For $n = 1$, this recovers the usual notion of Galois topos \cite[§3]{Moe89}, except that we do not require connectedness. We then have the following generalization of \cite[Theorem 3.2]{Moe89}:

**Corollary 2.16.** Let $\mathcal{X}$ be an $n$-localic $\infty$-topos, $-1 \leq n \leq \infty$. The following are equivalent:

1. $\mathcal{X}$ is $n$-Galois;

2. $\mathcal{X}$ is locally $(n-1)$-connected and generated under colimits by its locally constant objects;

3. $\mathcal{X}$ is locally $(n-1)$-connected and generated under colimits and finite limits by its locally constant objects;

4. $\mathcal{X}$ is locally $(n-1)$-connected and $\varphi: \mathcal{X} \to \iota(\Pi_n \mathcal{X})$ is an equivalence of $\infty$-topoi;

5. There exist an $(n-1)$-strict pro-$n$-groupoid $\mathcal{X}$ and an equivalence $\mathcal{X} \simeq \iota(\mathcal{X})$.

**Proof.** (1) $\Rightarrow$ (2). It suffices to note that, if $P: A \to \mathcal{X}$ is a torsor, then $P_\alpha$ is locally constant for all $\alpha \in A$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). Follows from Theorem 2.10.

(4) $\Rightarrow$ (5). Follows from Proposition 2.5.

(5) $\Rightarrow$ (1). Let $X: \mathcal{I} \to \mathcal{S}_{\leq n}$ be a cofiltered diagram with $(n-1)$-connected transition maps. It is clear that $\iota(X)$ is generated under colimits by the images of the torsors
\[
X_i \hookrightarrow \iota(X_i) \to \iota(X).
\]
These are Galois torsors by Lemma 2.6. That $\iota(X)$ is locally $(n-1)$-connected was verified at the beginning of the proof of Lemma 2.8.

### 3. Finite Galois Theory

In this short section, we give a classification of finite locally constant sheaves in an $\infty$-topos. Compared to the classification of $n$-truncated locally constant sheaves established in §2, the main difference is that we do not need any assumption of local connectedness.

Recall from \cite[§2.3]{Lur11} that an $\infty$-groupoid is finitely constructible if it is truncated and coherent, i.e., has finite homotopy groups. We denote by $\mathcal{S}^{\text{fc}} \subset \mathcal{S}$ the full subcategory of finitely constructible $\infty$-groupoids. If $\mathcal{X}$ is an $\infty$-topos, an object $X \in \mathcal{X}$ is finite locally constant if there exists an effective epimorphism $\coprod_\alpha U_\alpha \to 1$ and finitely constructible $\infty$-groupoids $X_\alpha \in \mathcal{S}^{\text{fc}}$ such that $X \times U_\alpha \simeq \pi^* X_\alpha \times U_\alpha$ in $\mathcal{X}/U_\alpha$, for all $\alpha$. 

Let $\mathcal{S}_{<\infty} \subseteq \mathcal{S}$ be the full subcategory of truncated $\infty$-groupoids. Since it is stable under finite limits, we have an adjunction

$$\text{Pro}(\mathcal{S}) \xRightarrow{\eta_{<\infty}} \text{Pro}(\mathcal{S}_{<\infty}).$$

**Lemma 3.1.** Let $\mathcal{X}$ be an $\infty$-topos and let $A \in \mathcal{S}$ be coherent. Then the canonical map

$$\Pi_{\infty}(\mathcal{X}_{/\pi^*A}) \to A \times \Pi_{\infty}\mathcal{X}$$

is a $\tau_{<\infty}$-equivalence.

**Proof.** By descent, we have

$$\Pi_{\infty}(\mathcal{X}_{/\pi^*A}) \simeq \colim_{a \in A} \Pi_{\infty}\mathcal{X}.$$ To conclude, we observe that for $X \in \text{Pro}(\mathcal{S})$ and $A \in \mathcal{S}$ coherent, the canonical map $\colim_a A \to A \times X$ is a $\tau_{<\infty}$-equivalence. Indeed, by [Lur11, Proposition 2.3.9], $\text{Map}(A, -) : \mathcal{S}_{\leq n} \to \mathcal{S}$ preserves filtered colimits for all finite $n$. If $Y \in \mathcal{S}_{<\infty}$, we therefore have

$$\text{Map}(\colim A, Y) \simeq \text{Map}(A, \colim_M (X_i, Y)) \simeq \colim_M \text{Map}(A \times X_i, Y) \simeq \text{Map}(A \times X, Y),$$

which proves the result. \(\square\)

**Theorem 3.2.** Let $\mathcal{X}$ be an $\infty$-topos. Then the geometric morphism $\varphi : \mathcal{X} \to \iota(\Pi_{\infty}\mathcal{X})$ induces a fully faithful functor

$$\text{Fun}(\Pi_{\infty}\mathcal{X}, S^\text{fc}) \hookrightarrow \mathcal{X}$$

whose essential image is the subcategory of finite locally constant sheaves on $\mathcal{X}$.

**Proof.** By the proof of Proposition 2.2, this functor lands in the subcategory of finite locally constant sheaves. Since the functors $U \mapsto \mathcal{X}_{/U}$ and $U \mapsto \text{Fun}(\pi_*U, S^\text{fc})$ are sheaves on $\mathcal{X}$, we are reduced as in the proof of Theorem 2.10 to proving the following statement: for every $A, B \in S^\text{fc}$, the map

$$\text{Map}_{\text{Fun}(\pi_*\mathcal{X}, S)}(A, B) \to \text{Map}_{\mathcal{X}}(\pi_*A, \pi_*B)$$

is an equivalence. Replacing $\mathcal{X}$ by $\mathcal{X}_{/\pi^*A}$ and using Lemma 3.1, we can assume that $A = *$. The result then follows from Proposition 1.11. \(\square\)

4. The étale homotopy type of Artin–Mazur–Friedlander

Let $\mathcal{C}$ be a small Grothendieck site with a final object. We denote by $\text{Shv}(\mathcal{C})$ the $\infty$-topos of sheaves of $\infty$-groupoids on $\mathcal{C}$ and by $\text{Shv}(\mathcal{C})^\wedge$ its hypercompletion. Let $HC(\mathcal{C})$ be the category of hypercovers of $\mathcal{C}$, in the sense of [DHI04, Definition 4.2]. This is a full subcategory of the category $\text{Fun}(\mathcal{C}^\text{op}, \text{Set}_\Delta)$ of simplicial presheaves on $\mathcal{C}$, and as such it is enriched in simplicial sets. We let $\pi HC(\mathcal{C})$ be the category obtained from $HC(\mathcal{C})$ by identifying simplicially homotopic morphisms.

The constant simplicial presheaf functor $\text{Set}_\Delta \to \text{Fun}(\mathcal{C}^\text{op}, \text{Set}_\Delta)$ has a (simplicially enriched) left adjoint, and we denote by

$$\Pi \mathcal{C} : HC(\mathcal{C}) \to \text{Set}_\Delta$$

its restriction to $HC(\mathcal{C})$. We would like to regard $\Pi \mathcal{C}$ as a cofiltered diagram of $\infty$-groupoids. The issue that usually arises at this point is that the category $HC(\mathcal{C})$ is not cofiltered, and while $\pi HC(\mathcal{C})$ is cofiltered, $\Pi \mathcal{C}$ does not identify simplicially homotopic morphisms. This led Artin and Mazur [AM69, §9] to consider instead the induced functor $\pi HC(\mathcal{C}) \to hS$, which is a cofiltered diagram in the homotopy category $hS$. However, $HC(\mathcal{C})$ is cofiltered as a *simplicially enriched* category, in the following sense:

- Any finite collection of hypercovers admits a common refinement.
- For any inclusion of finite simplicial sets $K \subseteq L$ and any map $K \to \text{Map}_\Delta(V, U)$, there exists a refinement $W \to V$ of $V$ such that the induced map $K \to \text{Map}_\Delta(W, U)$ extends to $L$.

Both statements follow from [DHI04, Proposition 5.1]. They imply that the $\infty$-category associated with $HC(\mathcal{C})$ is cofiltered (by [Lur09, Proposition 5.3.1.13] and an easy argument using Kan’s $\text{Ex}^\infty$ functor).

**Proposition 4.1.** Let $\mathcal{C}$ be a small site with a final object. Then $\Pi_{\infty} \text{Shv}(\mathcal{C})^\wedge$ is corepresented by the simplicially enriched cofiltered diagram $\Pi \mathcal{C} : HC(\mathcal{C}) \to \text{Set}_\Delta$. 


Proof. This is a straightforward consequence of the generalized Verdier hypercovering theorem [DHI04, Theorem 8.6], applied to a constant simplicial presheaf. □

Remark 4.2. The pro-∞-groupoid $\Pi_\infty \text{Shv}(\mathcal{C})$ does not admit such an explicit model in general. Note however that, for any ∞-topos $\mathcal{X}$, the map $\Pi_\infty \mathcal{X}^\wedge \to \Pi_\infty \mathcal{X}$ is a $\tau_{<\infty}$-equivalence, since truncated objects are hypercomplete. In particular, $\Pi_n \mathcal{X}^\wedge \simeq \Pi_n \mathcal{X}$ for any finite $n$.

Lemma 4.3. Let $\mathcal{I}$ and $\mathcal{J}$ be cofiltered ∞-categories and let $f: \mathcal{J} \to \mathcal{I}$ be a functor. Then $f$ is left cofinal if and only if $hf: h\mathcal{J} \to h\mathcal{I}$ is left 1-cofinal.

Proof. Assume that $hf$ is left 1-cofinal, and let $p: \mathcal{J}^{\text{op}} \to S$ be a diagram. We must show that the map

$$\operatorname{colim}_{j \in \mathcal{J}} p(f(j)) \to \operatorname{colim}_{i \in \mathcal{I}} p(i)$$

is an equivalence, i.e., induces isomorphisms on homotopy groups. This follows from the assumption and the fact that homotopy groups

$$\pi_n: S_{/j} \to S_{/i}$$

preserve filtered colimits. The other implication is obvious. □

Corollary 4.4. Let $\mathcal{C}$ be a small site with a final object, $\mathcal{I}$ a cofiltered category, and $f: \mathcal{I} \to \Pi_\infty \text{Shv}(\mathcal{C})$ a functor such that the composition

$$\mathcal{I} \xrightarrow{f} \Pi_\infty \text{Shv}(\mathcal{C}) \to \pi_0 \Pi_\infty \text{Shv}(\mathcal{C})$$

is left 1-cofinal. Then the cofiltered diagram $\Pi_\mathcal{C} \circ f$ corepresents $\Pi_\infty \text{Shv}(\mathcal{C})^\wedge$.

Proof. By Lemma 4.3, $f$ induces a left cofinal functor of ∞-categories. The result now follows from Proposition 4.1. □

We now turn to the étale site $\text{Et}_X$ of a scheme $X$, i.e., the category of étale $X$-schemes equipped with the étale topology. Let us not dwell on the fact that $\text{Et}_X$ is not small: suffice it to say, this can safely be ignored. We denote by $X_{\text{ét}}$ the ∞-topos of sheaves of ∞-groupoids on $\text{Et}_X$ and by $X_{\text{ét}}^\wedge$ its hypercompletion. The abelian cohomology of either ∞-topos is the étale cohomology of $X$.

If $X_{\text{Zar}}$ denotes the ∞-topos of sheaves on the small Zariski site of $X$, there is an obvious geometric morphism $X_{\text{ét}} \to X_{\text{Zar}}$ which induces an equivalence on $(-1)$-truncated objects. In particular, $\Pi_0(X_{\text{ét}}) \simeq \Pi_0(X_{\text{Zar}})$. By Proposition 2.5, we deduce that $X_{\text{ét}}$ is locally connected if and only if the underlying topological space of $X$ is locally connected (as this property persists after étale extension). Note however that $X_{\text{ét}}$ is rarely locally simply connected, since the étale fundamental pro-groupoid is often nonconstant. If $X$ is locally connected, then:

- Every locally constant Eilenberg–Mac Lane sheaf on $\text{Et}_X$ is the underlying sheaf of a unique Eilenberg–Mac Lane local system (Corollary 2.14).
- If $A$ is a locally constant sheaf of abelian groups on $\text{Et}_X$ and $L$ is the corresponding split local system, then $H^*(X_{\text{ét}}, A) \simeq H^*(\Pi_\infty X_{\text{ét}}, L)$ (Proposition 1.11).

On the other hand, with no assumptions on the scheme $X$, we have:

- Finite locally constant sheaves on $\text{Et}_X$ are classified by functors $\Pi_\infty X_{\text{ét}} \to S^\mathbb{C}$ (Theorem 3.2).

In [Fri82, §4], Friedlander defines the étale topological type of a locally connected scheme $X$: it is a pro-simplicial set given by a diagram

$$\mathcal{I} \to \text{HC}'(\text{Et}_X) \subset \text{HC}(\text{Et}_X) \overset{\Pi_\mathcal{I}}{\to} S_{\text{Set}_\Delta},$$

where $\mathcal{I}$ is some small cofiltered poset (whose definition is specific to the étale site) and $\text{HC}'(\text{Et}_X) \subset \text{HC}(\text{Et}_X)$ is the full subcategory spanned by the basal hypercovers [DHI04, §9] that are moreover degreewise coproducts of connected étale $X$-schemes. By [DHI04, Theorem 9.6], any hypercover of $\text{Et}_X$ is refined by a basal hypercover, and since $X$ is locally connected, it is clear that any such is further refined by a hypercover in $\text{HC}'(\text{Et}_X)$. In particular, the inclusion $\pi \text{HC}'(\text{Et}_X) \subset \pi \text{HC}(\text{Et}_X)$ is left 1-cofinal. The key property of Friedlander’s construction is that the composite functor

$$\mathcal{I} \to \text{HC}'(\text{Et}_X) \to \pi \text{HC}'(\text{Et}_X)$$

is left 1-cofinal [Fri82, p. 38]. Combining these facts and applying Corollary 4.4, we deduce the following:
Corollary 4.5. Let $X$ be a locally connected scheme. Then the étale topological type of $X$, as defined by Friedlander, corepresents the shape of $X^X_{\text{ét}}$.

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