On the volume conjecture for classical spin networks

Abdelmalek Abdesselam

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Department of Mathematics, P. O. Box 400137, University of Virginia, Charlottesville, VA 22904-4137, USA
email: malek@virginia.edu

In memoriam Pierre Leroux

Abstract. We prove an upper bound for the evaluation of all classical $SU_2$ spin networks conjectured by Garoufalidis and van der Veen. This implies one half of the analogue of the volume conjecture which they proposed for classical spin networks. We are also able to obtain the other half, namely, an exact determination of the spectral radius, for the special class of generalized drum graphs. Our proof uses a version of Feynman diagram calculus which we developed as a tool for the interpretation of the symbolic method of classical invariant theory, in a manner which is rigorous yet true to the spirit of the classical literature.

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1 Introduction

1.1 Motivation

The volume conjecture [55, 67] is one of the most important open problems in low-dimensional topology, with vast ramifications in many active areas of mathematics and theoretical physics (see, e.g., [77, 83, 33] and references therein). The conjecture relates the exponential growth rate of the colored Jones polynomial of a hyperbolic knot evaluated at a root of unity to the hyperbolic volume of the knot complement. There is a close connection between this problem and that of analysing asymptotics of $q$-deformed or quantum spin networks [77, 68, 84, 26, 27, 86]. Perhaps as a simpler setting for studying such asymptotics, a vigorous program for the systematic study of large angular momentum asymptotics of classical spin networks (CSN), with precisely formulated conjectures, was presented in [44]. It is the main source of inspiration for the present article. As a further piece of motivation for the study of CSN’s, one can note that spin networks feature in a great variety of topics, e.g., quantum computation [64], Tyurin’s approach to nonabelian theta functions [85], shell models of turbulence [38], to only cite some of the perhaps less well known.
1.2 History

The introduction of CSN’s is usually attributed to Roger Penrose [69, 70] who used them in an attempt to combinatorially quantize gravity. This is in similar spirit to Regge’s calculus [75]. Indeed, by analysing the asymptotics of CSN’s when all angular momenta are large, Ponzano and Regge [73] made a strong case for the emergence of the Regge action of 3d gravity from this semiclassical regime. This is based on a precise asymptotic formula for the 6-j symbol conjectured in [73] but proved much later by Roberts in [76].

Recently spin networks and their generalizations such as the Barrett-Crane model [12] and spin foams [71] have become a staple food at the table of loop quantum gravity (see [78] and references therein). A central issue in this approach to quantum gravity is the understanding of this semiclassical regime (see, e.g., [9, 13, 40, 50]).

Earlier, CSN’s essentially appeared in the works of the Lithuanian School of quantum angular momentum theory (QAMT) [88]. Indeed some sort of graphical notation becomes indispensable when calculating complicated 3n-j symbols. However, the beginnings of QAMT and the theory of CSN’s go much further back to the classical invariant theory (CIT) of binary forms, although there is some effort in mathematical translation needed in order to see the connection. The ideal tool for this translation is Feynman diagram calculus (FDC), namely a diagrammatic representation of contractions of tensors as in the remarkable book [30]. Enough elements of such a translation were presented in [3, 4, 5] to suit the needs of these articles. More is required here in order to translate the modern CSN formalism into the framework of CIT, and this is the object of [2]. A piece of data which appears naturally in the CIT/FDC picture is the notion of smooth orientations of a cubic graph, according to the terminology of [60]. This is key to our solution of some of the questions raised in [44].

There is a huge physical literature on QAMT where such objects as Clebsch-Gordan, Clebsch-Gordon (sic), Clebsh-Gordon (sic), . . . coefficients are ubiquitous. Yet, with only a few exceptions such as [32], this literature shows almost no sign of awareness or acknowledgement of the work of Alfred Clebsch and Paul Gordan. Aside from introducing some foundational material needed in the subsequent proofs, we attempt to correct this injustice in [2]. Our hope is that by allowing the users of QAMT to tap into the vast and most often perfectly rigorous 19th century literature on CIT, and through cross-fertilization with modern theories in mathematics and physics, new ideas and unexpected connections will emerge.
1.3 Definitions and statement of results

We will follow the definitions of [44] where a CSN is defined as a pair \((\Gamma, \gamma)\) consisting of an abstract cubic ribbon graph \(\Gamma\) together with a decoration \(\gamma\) of the edges by natural integers. By cubic ribbon graph we mean a trivalent regular graph equipped with a cyclic ordering of the edges at each vertex. These are also called rotation systems, or fat graphs, although one has to be careful as definitions may vary in the literature [16, §2.1]. The notion we use here is that of pure rotation systems as in [49, Chap. 3]. It is well known that such a ribbon graph defines (up to orientation preserving equivalence of imbeddings) a unique imbedding of the underlying abstract graph into a compact orientable Riemann surface [49, Thm. 3.2.3]. Note that we allow multiple edges and loops, i.e., edges with both ends attached to the same vertex. We also allow \(\Gamma\) to be disconnected. Finally, although this means a slight arm twisting on the usual definition of a graph, we allow trivial components without vertices which are made of a single edge closing upon itself.

\[\text{a loop} \quad \text{, a trivial component}\]

Note that the decorations \(\gamma = (\gamma(e))_{e \in E(\Gamma)} \in \mathbb{N}^{E(\Gamma)}\) where \(E(\Gamma)\) is the edge set of \(\Gamma\) must satisfy the following admissibility conditions. For every vertex, the three integers \(a, b, c\) associated to the incident edges are such that \(a + b + c\) is even and the triangle inequality \(|a - b| \leq c \leq a + b\) holds. Note that for a loop vertex

\[\text{the decoration } a \text{ is counted twice so the constraint reduces to: } b \text{ is even and } 0 \leq b \leq 2a.\]

For a trivial component

\[\text{the decoration } a \text{ can be any nonnegative integer.}\]
We can now define as in [44] the Penrose evaluation \( \langle \Gamma, \gamma \rangle^P \) of such a spin network:

1. Use the imbedding into the surface \( \Sigma \) to thicken the vertices into discs and the edges into bands.

2. On an edge carrying the decoration \( a \), draw \( a \) parallel strands and a perpendicular bar \( \equiv \) which we call a Penrose bar:

\[
\begin{array}{c}
\text{a=3} \\
\hline
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{P} \\
\end{array}
\]

3. Connect the strands at each vertex as in the picture seen from outside \( \Sigma \). This connection is made possible by the admissibility constraints. It is also essentially unique since the number of strands connecting the \( a \) edge to the \( b \) edge is \( \frac{a+b-c}{2} \), etc.

4. Replace each Penrose bar \( \frac{a}{p} \) by an alternating sum over permutations \( \sigma \in \mathfrak{S}_a \) as in

\[
\begin{array}{c}
a=3 \\
\hline
\end{array} \quad \rightarrow \quad \begin{array}{c}
- \quad - \quad + \quad - \quad + \\
\end{array}
\]

5. Finally each term in the sum over permutations will produce a collection of closed curves drawn on the surface \( \Sigma \). One associates a factor \((-2)\) to each such curve.

6. The Penrose evaluation \( \langle \Gamma, \gamma \rangle^P \) is the result of summing the corresponding \((-2)\) to the power of the number of curves times \((-1)\) to the power of the number of crossings, over all possible states or connection schemes specified by the permutations \( \sigma \) at each edge.
Example:

\[
\langle \begin{array}{c}
2 \\
2
\end{array} \rangle_P = \langle \begin{array}{c}
P \\
P
\end{array} \rangle_P
\]

\[
= \langle \begin{array}{c}
2 \\
2
\end{array} \rangle_P - \langle \begin{array}{c}
2 \\
2
\end{array} \rangle_P - \langle \begin{array}{c}
2 \\
2
\end{array} \rangle_P + \langle \begin{array}{c}
2 \\
2
\end{array} \rangle_P
\]

\[
= (-2)^3 - (-2)^2 - (-2)^2 + (-2)
\]

\[
- (-2)^2 + (-2) + (-2) - (-2)
\]

\[
= -24.
\]

Note that we used the counterclockwise cyclic ordering at the two vertices and the drawing on the plane of the written page, or equivalently on the sphere.

The article \cite{44} also considers the standard evaluation \( \langle \Gamma, \gamma \rangle^S \) of a spin network

\[
\langle \Gamma, \gamma \rangle^S = \langle \Gamma, \gamma \rangle^P \times \prod_{v \in V(\Gamma)} \left\{ \left( \frac{a_v + b_v - c_v}{2} \right)! \left( \frac{a_v + c_v - b_v}{2} \right)! \left( \frac{b_v + c_v - a_v}{2} \right)! \right\}^{-1}
\]

(1)

where \( V(\Gamma) \) is the vertex set of \( \Gamma \) and the \( a_v, b_v, c_v \) denote the decorations of the edges incident to vertex \( v \). One again counts a decoration twice in the case of a loop vertex. One also defines the unitary evaluation \( \langle \Gamma, \gamma \rangle^U \) by

\[
\langle \Gamma, \gamma \rangle^U = \langle \Gamma, \gamma \rangle^S \times \prod_{v \in V(\Gamma)} \Theta(a_v, b_v, c_v)^{-\frac{1}{2}}
\]

(2)

where

\[
\Theta(a, b, c) = \frac{(a+b+c+1)!}{\left( \frac{a+b-c}{2} \right)! \left( \frac{a+c-b}{2} \right)! \left( \frac{b+c-a}{2} \right)!}.
\]
Remark 1 In [44, Def. 9.4], due to a typo, \( \langle \Gamma, \gamma \rangle^P \) is incorrectly stated with \( \langle \Gamma, \gamma \rangle^S \).

In [44, Lem. 6.1] the following property is proved.

**Lemma 1** Changing the cyclic orientations at the vertices modifies any of the previous evaluations by a sign. More precisely, if one changes the cyclic ordering at a vertex with decorations \( a, b, c \), the resulting sign factor is

\[
(-1)^{a(a-1)+b(b-1)+c(c-1)}.
\]

The case of trivial components, which is a good warm-up exercise on Penrose evaluations, is dispensed with in the next easy lemma.

**Lemma 2**

\[
\langle a \bigcirc \rangle^P = \langle a \bigcirc \rangle^S = \langle a \bigcirc \rangle^U = (-1)^a (a+1)!
\]

**Proof:** One can use the chromatic method of Penrose and Moussouris and a recursion such as [30, Eq. 6.20], but we prefer to use the definition. With self-explanatory notations:

\[
\langle a \bigcirc \rangle^P = \sum_{\sigma \in \mathfrak{S}_a} \text{sign}(\sigma)
\]

where \( c(\sigma) \) is the number of cycles of the permutation \( \sigma \). Therefore,

\[
\langle a \bigcirc \rangle^P = (-1)^a \sum_{k=0}^{a} 2^k c(a,k)
\]

where \( c(a,k) \) is the number of permutations in \( \mathfrak{S}_a \) with exactly \( k \) cycles. It is related to the Stirling number of the first kind \( s(a,k) \) by \( c(a,k) = (-1)^{a-k} s(a,k) \). By [81, Prop. 1.3.4] one has

\[
\sum_{k=0}^{a} 2^k c(a,k) = 2(2+1) \cdots (2+a-1) = (a+1)!
\]

Another trivial consequence of the previous definitions is the following.
Lemma 3 The $\langle \cdots \rangle^P$, $\langle \cdots \rangle^S$ and $\langle \cdots \rangle^U$ evaluations factorize over the connected components of $\Gamma$.

Given an admissible spin network $(\Gamma, \gamma)$, for any $n \in \mathbb{N}$, the dilation $(\Gamma, n\gamma)$ where each decoration gets multiplied by $n$ is also admissible. The main problem addressed in [44] and which goes back to [73] is the study of the asymptotics of evaluations of $(\Gamma, n\gamma)$ as $n$ goes to infinity. To this end, Garoufalidis and van der Veen introduced the power series

$$F_{\Gamma, \gamma}(z) = \sum_{n=0}^{\infty} \langle \Gamma, n\gamma \rangle^S z^n$$

and defined the spectral radius $\rho_{\Gamma, \gamma} \in [0, \infty]$ of the spin network as the inverse of the radius of convergence of the series $F_{\Gamma, \gamma}$. Among the challenge problems mentioned in [44], Problem 2 therein is the statement that $\langle \Gamma, \gamma \rangle^U$ is bounded in absolute value by one. Problem 3 therein is the statement that for uniform decorations $\gamma(e) = 2$ for every edge $e$, the spectral radius is exactly $3^{\frac{\text{3V}(\Gamma)}{2}}$ where we used notation $|\cdot|$ for the cardinality of finite sets. This is a hopefully easier analogue for CSN’s of the volume conjecture in knot theory [55, 67]. Indeed the decorations play a role similar to that of the colors of the colored Jones polynomial. We prefer the “decoration” terminology rather than the “coloring” one used in [44].

Clearly, trivial components violate both statements completely and must be excluded. Another interesting practice example is the dumbbell which evaluates to

$$\langle a \bigcirc \bigcirc b \rangle^U = \delta_{c,0}(-1)^{a+b} \sqrt{(a+1)(b+1)},$$

see Remark [11] More generally, loops which carry sufficiently large decorations can cause the failure of the bound $|\langle \Gamma, \gamma \rangle^U| \leq 1$. Bridges also spoil the conjecture for the value of the spectral radius because of the Kronecker delta factor as in (3).

Our main result is the solution of Problem 2 in [44].

**Theorem 1** For any spin network $(\Gamma, \gamma)$ without trivial components and without loops, one has

$$|\langle \Gamma, \gamma \rangle^U| \leq 1.$$

For the case of graphs with loops one has the following weaker statement.

**Theorem 2** For any spin network $(\Gamma, \gamma)$ without trivial components, $|\langle \Gamma, n\gamma \rangle^U|$ grows at most polynomially with $n$. 8
As to Problem 3 in [44], we can state the following corollary which via Stirling’s formula is an easy consequence of Theorem 2.

**Corollary 1** For any spin network \((\Gamma, \gamma)\) without trivial components, the spectral radius satisfies the bound

\[
\rho_{\Gamma, \gamma} \leq \prod_{v \in V(\Gamma)} \sqrt{\beta(a_v, b_v, c_v)}
\]

where

\[
\beta(a, b, c) = \frac{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}\right)}{\left(\frac{a+b-c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{b+c-a}{2}\right)}
\]

and using the convention \(0^0 = 1\) so that degenerate cases are covered as well.

Note that \(\beta(2, 2, 2) = 9\) so in the case \(\gamma \equiv 2\) we have ‘half’ of the volume conjecture for CSN’s.

**Corollary 2** For any spin network \((\Gamma, \gamma)\) without trivial components,

\[
\rho_{\Gamma, \gamma \equiv 2} \leq \frac{3^{\frac{3|V(\Gamma)|}{2}}}{2}.
\]

The full conjecture should be as follows.

**Conjecture 1** For any spin network \((\Gamma, \gamma)\) without trivial components, and without bridges, i.e., which is 2-edge-connected

\[
\rho_{\Gamma, \gamma \equiv 2} = \frac{3^{\frac{3|V(\Gamma)|}{2}}}{2}.
\]

We were able to prove the equality for graphs of the form
with \( s \) edges between the two circles. These are called generalized drum graphs and are denoted by \( \text{Drum}_s \) in [43].

**Theorem 3** For any \( s \geq 2 \),

\[
\rho_{\text{Drum}_s, \gamma} = 3^{3s}.
\]

This of course covers the case of the cube \((s = 4)\) which is new.

## 2 Clebsch-Gordan networks and a brief tour of classical invariant theory

Let \( G \) be a cubic graph, possibly disconnected, which may contain multiple edges and loops. However, we exclude trivial vertex-less components in this section. A smooth orientation \( \mathcal{O} \) of \( G \) is an orientation of the edges of \( G \) such that the resulting digraph only has two types of vertices:

![Gate Signage](image)

A gate signage \( \tau \) corresponds to an ordering for every vertex of the two edges (or rather half-edges in order to cover the loop case as well) which share the same direction. Such pair of half-edges is called a gate. The ordering is indicated by a small curved arrow as in:

![Gate Signage](image)

We also consider a decoration \( \gamma \) of the edges by nonnegative integers satisfying the same admissibility conditions as in [1, 3]. To the data \((G, \mathcal{O}, \tau, \gamma)\) which we call a Clebsch-Gordan or CG network we will associate a number \( \langle G, \mathcal{O}, \tau, \gamma \rangle_{\text{CG}} \) using FDC, i.e., a graphical encoding of tensor contractions (see, e.g., [2, 3]). These tensors typically belong to spaces of the form \( V \otimes V \otimes V^* \otimes V^* \otimes V \cdots \) where the fundamental vector space is \( V = \mathbb{C}^2 \). The FDC has no need for a ‘coordinate free’ approach, since it is to modern tensor algebra what matrix algebra is to abstract linear algebra. We think of tensors simply as arrays or ‘matrices’ of numbers \( T_{i_1 i_2 \cdots} \) with any number of indices taking their values in \( \{1, 2\} \). One can of course extend the scalars and allow entries which are polynomials in formal letters, but here we will mostly work with complex numbers.
Remark 2  Note that the FDC used in quantum field theory is concerned with the situation where \( V \) is infinite-dimensional say \( V = L^2(\mathbb{R}^d) \), tensors become integral kernels and sums over indices become integration over \( \mathbb{R}^d \). Since the kernels involved are singular, this poses a nontrivial problem of mathematical analysis which is the object of renormalization theory. It is a far-reaching extension of Schwartz’s kernel theory which would be enough if the ‘sums over indices’ or rather \( L^2(\mathbb{R}^d) \) pairings were always between \( S(\mathbb{R}^d) \) and \( S'(\mathbb{R}^d) \) tensor factors.

A vector \( \mathbf{x} = (x_1, x_2) \) in \( V \) is denoted graphically by

\[
x_i = \includegraphics{vector_x_i}
\]

A binary form \( F \) of order \( r \) is a polynomial in \( \mathbf{x} \), homogeneous of degree \( r \), which following Cayley we write using binomial coefficients as:

\[
F(\mathbf{x}) = \sum_{p=0}^{r} \binom{r}{p} f_p x_1^{r-p} x_2^p.
\]

Equivalently,

\[
F(\mathbf{x}) = \sum_{i_1, \ldots, i_r = 1}^{2} F_{i_1 \ldots i_r} x_{i_1} \ldots x_{i_r}
\]

where the tensor \( (F_{i_1 \ldots i_r})_{1 \leq i_1, \ldots, i_r \leq 2} \) is completely symmetric in its \( r \) indices and is related to the previous description by \( F_{i_1 \ldots i_r} = f_p \) where \( p \) is the number of indices which happen to be equal to 2. Introduce the graphical notation

\[
\includegraphics{binary_form_graph}
\]

Then

\[
F(\mathbf{x}) = \includegraphics{binary_form_diagram}
\]

using the main recipe of FDC:

The evaluation of a diagram obtained from basic pieces say \( \includegraphics{basic_piece_1} \) and \( \includegraphics{basic_piece_2} \) by a gluing of the half-lines (or legs) is the result of
assigning an index to each pair of glued half-lines, taking the product of corresponding tensor elements and summing over indices.

We need more pieces to continue playing this game. The Kronecker delta is

\[ i \rightarrow j = \delta_{ij} \]

The epsilon or Levi-Civita tensor is

\[ i \rightarrow j = \epsilon_{ij} \]

where \( \epsilon = (\epsilon_{ij})_{1 \leq i,j \leq 2} \) is the antisymmetric matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The symmetrizer is

\[ \frac{1}{a!} \sum_{\sigma \in S_a} \delta_{i_1j_{\sigma(1)}} \cdots \delta_{i_aj_{\sigma(a)}} \]

For example

\[ \begin{pmatrix} l m n \\ i j k \end{pmatrix} = \frac{1}{6} \begin{bmatrix} + & + & + & + & + \\ i j k & i j k & i j k & i j k & i j k \end{bmatrix} \]

A 2 \times 2 matrix \( g = (g_{ij})_{1 \leq i,j \leq 2} \) is represented by

\[ i \rightarrow g \rightarrow j = g_{ij} \]

One can also use this graphical representation for differential operators

\[ \frac{\partial}{\partial x_i} \]

the only difference is that one needs to indicate a direction of reading, since the \( \frac{\partial}{\partial x} \)'s do not commute with the \( x \)'s. For instance one has the identity

\[ \frac{1}{r^d} \delta_0 \cdots \delta_{d-1} = \frac{1}{r} x \cdots x \] (4)
where the formula on the left-hand side is read from left to right. The familiar Liebnitz’s rule can be interpreted by saying that each \(-\partial a\) selects an \(a\) to annihilate and glues whatever it is attached to together with what the selected \(a\) is attached to. One also needs to sum over all possible ways of making these selections. See [11] for a nice presentation along this line of FDC and Feynman diagram expansions intuitively seen as the result of a ‘chemical reaction’ between \(-\partial a\) graphs and \(a\) graphs. The previous identity says nothing more than the trivial fact

\[
\frac{1}{r!} \sum_{i_1, \ldots, i_r=1}^2 F_{i_1 \ldots i_r} \prod_{i=1}^2 \frac{\partial}{\partial a_{i_i}} \sum_{j_1, \ldots, j_r=1}^2 a_{j_1} x_{j_1} \cdots a_{j_r} x_{j_r} = F(x)
\]

However, it is the basis of a correct interpretation of the very powerful classical symbolic method, as we will shortly see.

**Remark 3** In our opinion, the most pedagogically useful way of introducing the so-called ‘Wick Theorem’ which generates Feynman diagram expansions from integration with respect to a Gaussian measure with covariance \(C\), is by following the same philosophy as above. Namely, integration amounts in this case to applying a differential operator

\[
\exp \left( \frac{1}{2} \mathcal{C} \right)
\]

albeit of infinite order. The immediate follow up is that one does not need a Gaussian measure in order to generate such a diagram expansion, and the differential operator in the exponential needs not be of second order. One does not need to generate graphs but could also generate higher-dimensional objects as in random matrix theory. As is well-known, traditional Feynman diagram expansions provide the explicit form of the stationary phase asymptotic series in the presence of nondegenerate critical points. Trying to generalize this to degenerate critical points [8] is largely open and is related to the problem addressed in [37]. We believe this investigation would most likely benefit from the classical invariant theoretic point of view. Also note that generating graphs by applying differential operators in this way, was already known to Arthur Cayley, one of the first developers of CIT [21]. Also, Wick’s Theorem which expresses the moments of a Gaussian measure was discovered by Leon Isserlis [52] who in addition to his expertise in statistics was well versed in the techniques of CIT [53, 51].
The fundamental property of $SL_2(\mathbb{C})$ invariance of diagrams built with the previous pieces is the identity

$$
\begin{array}{c}
\begin{array}{c}
\triangle \quad i \\
\triangle \quad j
\end{array}
\end{array}
= (\det g)
\begin{array}{c}
\begin{array}{c}
\triangle \quad i \\
\triangle \quad j
\end{array}
\end{array}
\ .
\tag{5}
\end{equation}

Indeed, if we decide that $GL_2(\mathbb{C})$ acts on vectors by

$$
\begin{array}{c}
\begin{array}{c}
\{x\}
\end{array}
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{c}
\{\mathbf{x}g\}
\end{array}
\end{array}
= \begin{array}{c}
\{x\}g
\end{array}
\ ,
\tag{6}
\end{equation}

then it will also act on binary forms by

$$
\begin{array}{c}
\begin{array}{c}
\{F\}
\end{array}
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{c}
\{gF\}
\end{array}
\end{array}
= \begin{array}{c}
\{F\}g^{-1}
\end{array}
\ .
\tag{7}
\end{equation}

A (homogeneous) classical invariant of degree $d$ of a binary form $F$ of order $r$ is a polynomial $I(F) = I(f_0, f_1, \ldots, f_r)$ which is homogeneous of degree $d$ in the coefficients of $F$ and satisfies

$$
I(gF) = I(F) \times (\det g)^{-w}
$$

for any $g \in GL_2(\mathbb{C})$. The power $w$ is called the weight of the invariant and it is given by $w = \frac{dr}{2}$. More generally, a classical covariant is a polynomial

$$
C(F, \mathbf{x}) = C(f_0, f_1, \ldots, f_r; x_1, x_2)
$$

separately homogeneous in $F$ and $\mathbf{x}$ which satisfies

$$
C(gF, g\mathbf{x}) = C(F, \mathbf{x}) \times (\det g)^{-w}
$$

where the weight is $w = \frac{dr+n}{2}$. Here $d$ is the degree of the covariant, i.e., the degree in the coefficients of $F$. Whereas $n$ is the order of the covariant, i.e., the degree in $\mathbf{x}$. Invariants are covariants of order 0.

The so-called First Fundamental Theorem of CIT for $SL_2$ says:
1. Consider Feynman diagrams obtained by assembling pieces taken among \( F \), \( x \), and \( \epsilon \), with the condition that direct connections between two \( F \)'s or two \( x \)'s are forbidden, and that \( \epsilon \) arrows must join two \( F \) vertices or ‘blobs’. The evaluation of such a diagram is a covariant. The degree is the number of \( F \) blobs, the order is the number of \( x \) blobs, and the weight is the number of \( \epsilon \) arrows.

2. Any covariant is a linear combination of such diagrams (or evaluations thereof).

Part 1) as well as the conceptual framework in which expressing this statement was made possible are due to Cayley [20]. This property is a consequence of \([ 45 \])\). Note that the meaning of the word “hyperdeterminant” used by Cayley in [20] is somewhat different from that in some of his earlier work and the modern understanding \([ 45 \]). It refers to any polynomial one can obtain by application of the rules in Part 1). The easy proof of Part 1) is best seen on an example. If \( F \) is a binary quintic, its canonisant is the polynomial

\[
C(F, x) = \ldots \quad (8)
\]

Now, applying the definitions \([ 45 \]) and \([ 47 \])

\[
C(gF, gx) = \ldots
\]
where “+” refers to the matrix $g$ and “−” to the matrix $g^{-1}$. Therefore

$$C(gF, gx) = [\det (g^{-1})]^6 \times C(F, x)$$

because of (5) and \(\oplus\), i.e., $g^{-1}g = \text{Id}$. Note in passing the use of ‘substitutions of diagrams into blobs’. This is related to plethysm and was exploited extensively by the classics.

The proof of Part 2), for $SL_n$, is due to Clebsch [23 §3]. With the hindsight provided by the FDC devised by Michael Creutz for $SU_n$ integration [28 29], i.e., the misnamed Reynolds operator, one can paraphrase Clebsch’s original proof, in the simpler case of invariants, as follows. An invariant $I$ also has its blob:

$$I(F) = \begin{array}{c}
I \\
E \\
E \\
\. \\
\. \\
F \\
F \\
\end{array}$$

(9)

except that indices run over the set with $r+1$ elements which labels a basis of $\text{Sym}^d(V^*)$. These are the equivalence classes of tuples $(i_1, \ldots, i_r) \in \{1, 2\}^r$ under permutations. One can revert back to the description in terms of binary indices and rewrite (9) ‘microscopically’ as

$$I(F) = \begin{array}{c}
I \\
\end{array}$$

(10)

with indices in $\{1, 2\}$. This is an example of “categorification” in the sense of [10], since we sum over tuples instead of equivalence classes. The tensor $I$ has $dr$ indices, arranged in $d$ groups of $r$, taking values in $\{1, 2\}$. The tensor, or corresponding blob, must be symmetric with respect to permutation of the groups as well as permutations of indices within groups. In other words, $I$ lives in $\text{Sym}^d(\text{Sym}^r(V))$.

By $SL_2(\mathbb{C})$ and therefore $SU_2(\mathbb{C})$ invariance, $I(F) = I(g^{-1}F)$ for any $g \in SU_2(\mathbb{C})$. Thus, denoting the normalized Haar measure on $SU_2(\mathbb{C})$ by
\[ d\mu, \]

\[ I(F) = \int_{SU_2(\mathbb{C})} d\mu(g) I(g^{-1}F) = \text{Cst} \times \det \left( \frac{\partial}{\partial g} \right) \frac{dg}{2} I(g^{-1}F) \]

\[ = \text{Cst} \times \begin{array}{c}
\begin{array}{ccc}
\partial g & \partial g & \ldots \\
\partial g & \partial g & \partial g
\end{array}
\end{array} \ 	imes \text{times} \ 	imes \frac{dg}{2} \times \text{times} \]

Now the reader who understood (4) sees what is happening. Each \( \partial g \) square selects, in all possible ways, a pair of \( g \)'s to contract to as in

\[ \begin{array}{c}
\begin{array}{ccc}
\partial g & \partial g & \ldots \\
\partial g & \partial g & \partial g
\end{array}
\end{array} \ 	imes \text{times} \ 	imes \frac{dg}{2} \times \text{times} \]

One has severed the line of communication between the \( I \) blob and the \( F \)'s and one obtains a sum of diagrams as described in Part 1) times pure scalars (or ‘reduced tensor elements’) of the form

\[ \begin{array}{c}
\begin{array}{ccc}
\partial g & \partial g & \ldots \\
\partial g & \partial g & \partial g
\end{array}
\end{array} \ 	imes \text{times} \ 	imes \frac{dg}{2} \times \text{times} \]

where all legs are contracted by \( \epsilon \) arrows. This is essentially what physicists call the Wigner-Eckart Theorem, or rather a microscopic version of it (see [30, §5.3]).

At this point one could object to this rewriting of the history of 19th century invariant theory using post-WW II Feynman diagrams, and ask where are these graphs to be found in the classical literature? One can answer: in [21, 25, 83, 18, 58, 59], see also [79]. A more important point
is that classics went beyond graphs and devised a formalism in order to encode graphs by compact algebraic expressions: the symbolic method. The availability of this algebraic formalism is another possible explanation to be added to those listed in [30 §4.9] as per the relative rarity of graphs in the printed CIT literature.

**Conjecture 2** *(For historians of mathematics)* If one could get a hold of handwritten notes and papers by Paul Gordan, especially from the time when he wrote [46], one should find many ‘birdtracks’ in the sense of [30].

Going back to the example of the canonisant of a binary quintic (8), one can rewrite it using (4) as

\[
\frac{1}{5!^3} \times
\]

\[
F
\]

\[
D S
\]

where \(D\) is the differential operator

\[
\frac{1}{5!^3} F \left( \frac{\partial}{\partial a} \right) F \left( \frac{\partial}{\partial b} \right) F \left( \frac{\partial}{\partial c} \right)
\]

acting on the symbolic expression

\[
S = (ab)^2(ac)^2(bc)^2a_xb_yc_x.
\]

Here a bracket factor \((ab)\) is shorthand for \(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\), and \(a_x = a_1x_1 + a_2x_2\), etc. The symbolic expression \(S\) is a bonafide polynomial involving
auxiliary variables $a, b, \ldots$ which play the same role as dummy variables of integration \[4, \S 4.3\]. To make things harder for the modern reader, the classics did not bother writing the operator $D$ and wrote instead an equality between a covariant and its symbolic expression

$$C(F, x) = (ab)^2(ac)^2(bc)^2a_xb_yc_z.$$  

with the provision that the right-hand side must ‘interpreted’ according to the recipe:

Expand the right-hand side as a polynomial in $a, b, c, x$. Keep the $x$’s as they are. Turn $a_1^{5-p}b_2^p$ into the coefficient $f_p$ of $F$, and likewise for $b_1^{5-p}b_2^p$ and $c_1^{5-p}c_2^p$.

For instance, the expansion of the symbolic expression for the canonisant produces terms such as

$$-a_1b_2a_2b_1a_1c_2b_1c_2b_2a_1x_1b_2x_2c_2x_2$$

which is to be interpreted as $-f_1f_2f_5 x_1x_2^2$. Note that the weight is $6 = (1 + 2 + 5) - 2$, i.e., the sum of the $f$ subscripts minus the number of $x_2$’s.

In particular, invariants are isobaric.

This symbolic recipe can be made precise using the umbral calculus of \[61\]. We believe however that the interpretation using differential operators $D$ (or, even better, using the integral notation of \[4, \S 4.3\]) is conceptually much simpler and also more powerful: one can treat some variables as symbolic, others as ‘actual’, one can ‘iterate’ the symbolic representation as when substituting diagrams into blobs, etc.

Let $H_n = \text{Sym}^n(V^*)$ be the space of binary forms of order $n$. It has a natural finite dimensional Hilbert space structure given by the inner product

$$\langle F | G \rangle = \begin{array}{c} \begin{array}{c} \prod \end{array} \end{array} \begin{array}{c} \begin{array}{c} \prod \end{array} \end{array}$$  

where $\prod$ is the complex conjugate of $\prod$. This is the spin $\frac{n}{2}$ irreducible representation of $SU_2(\mathbb{C})$ where the group action is that of \[7\]. The tensor product $H_m \otimes H_n$ can be seen as the space of bihomogeneous forms $B(x, y)$ of degree $m$ in $x = (x_1, x_2)$ and degree $n$ in $y = (y_1, y_2)$. 

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Graphically

\[ B(x, y) = \begin{array}{c}
\text{Diagram}
\end{array} \]

The blob of such a form \( B \) is symmetric in its first \( m \) indices and in its last \( n \) indices. One also has a natural inner product on \( \mathcal{H}_m \otimes \mathcal{H}_n \) defined as in [10].

For any integer \( k, 0 \leq k \leq \min(m, n) \), one has a natural equivariant map \( \mathcal{H}_m \otimes \mathcal{H}_n \to \mathcal{H}_{m+n-2k} \) given by

\[
\begin{array}{c}
\text{Diagram}
\end{array} \rightarrow \begin{array}{c}
\text{Diagram}
\end{array}
\]

where \( k \) epsilon arrows as well as one symmetrizer are used. This applied to a decomposable element of the form \( F \otimes G \) is called the \( k \)-th transvectant (or Ueberschubung) of the binary forms \( F(x) \) and \( G(x) \), denoted by

\[
(F, G)_k = \begin{array}{c}
\text{Diagram}
\end{array}
\]

or \( (ab)^k a_x^{m-k} b_x^{n-k} \) in classical symbolic notation, where \( a, b \) are symbolic letters for \( F, G \) respectively. The beginning of \( SU_2 \) recoupling theory for quantum angular momentum is the discovery by Paul Gordan and Alfred Clebsch of the fundamental identity

\[
\begin{array}{c}
\text{Diagram}
\end{array} = \sum_{k=0}^{\min(m, n)} \binom{m}{k} \binom{n}{k} \binom{m+n-k+1}{k} \begin{array}{c}
\text{Diagram}
\end{array}
\]

or \( (12) \)
which holds for any assignment of the $2m + 2n$ indices attached to the legs of the diagrams, consistently on both sides of the equation. The existence of such an identity with undetermined numerical coefficients was proved by Gordan in [46, §2]. Gordan’s argument is very elegant and based on a sort of Taylor expansion with respect to the diagonal $x = y$ of $\mathbb{P}^1 \times \mathbb{P}^1$ (see also [87]). The explicit formula with binomial coefficients seems to be due to Clebsch [24]. The key to the determination of these coefficients is the identity

\begin{equation}
\begin{align*}
m n & \equiv \delta_{pq} \frac{k! \; (m + n - k + 1)! \; (m - k)! \; (n - k)!}{m! \; n! \; (m + n - 2k + 1)!} \\
\end{align*}
\end{equation}

with $p = m + n - 2k$ and $q = m + n - 2l$. Indeed,
\[
\frac{(m-l)! (n-l)!}{m! n!} \left\{ \Omega_{xy} (xy)^k a_x^{m-k} a_y^{n-k} \right\}
\]

in classical notation where

\[
\Omega_{xy} = \left| \begin{array}{cc}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\
\frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2}
\end{array} \right|
\]

is Cayley’s Omega Operator. Formula (13) is thus an immediate consequence of the easy identity

\[
\Omega_{xy} \left[ (xy)^k a_x^{m-k} a_y^{n-k} \right] = \begin{cases} 
0 & \text{if } k = 0, \\
k(m + n - k + 1) (xy)^{k-1} a_x^{m-k} a_y^{n-k} & \text{if } k \geq 1.
\end{cases}
\]

Without the \(a\)’s this is the simplest nontrivial case of the so-called Cayley Identity which is nowhere to be found in Cayley’s work [7].

**Remark 4** We are not aware of a higher-dimensional generalization of the identity with the \(a\)’s. This is related to the study of 3-j symbols (the Theta graph rather than the 3-jm Wigner symbols) for the group \(U_n\) [31]. The classical references related to this issue can be found in [63]. Note that even the admissibility condition becomes nontrivial since it is given by the Littlewood-Richardson rule.

The existence of identity (12) was proved by Gordan in order to show that every covariant of a binary form \(F\) is a linear combination of compounded transvectants of the form \((\cdots ((F, F)_{k_1}, F)_{k_2}, \ldots, F)_{k_p}\). These can be depicted graphically using the FDC introduced above. For instance if \(F\) is of order \(d\):

\[
(((F, F)_{k_1}, F)_{k_2}, F)_{k_3} = \cdots.
\]
This can be represented ‘macroscopically’ or in shorthand notation as

\[
x F F \quad d \quad d \quad 2d - 2k_1 \quad d \quad 3d - 2k_1 - 2k_2 \quad d \quad 4d - 2(k_1 + k_2 + k_3)
\]

In other words, classical covariants of binary forms expressed using iterated transvectants are examples of open-ended spin networks as considered in [69] where external legs here correspond to the \( F \) blobs or a collection of \( x \) blobs. The orientation of edges keeps track of the distinction between \( V \) and \( V^* \), even though they are the same as \( SL_2 \) representations, which is special to the 2d situation. Here we will mainly be concerned with vacuum diagrams without external legs or what we earlier called a CG network.

An example of such a structure \((G, O, \tau, \gamma)\) is the Theta network:

where \( G = \) is the underlying cubic graph, \( O \) is the given orientation of the edges of \( G \). The gate signage \( \tau \) has been indicated by the two small curved arrows. Finally the decoration \( \gamma \) is indicated by the nonnegative integers \( a, b, c \) which satisfy \( a + b + c \in 2\mathbb{N} \) and \( |a - b| \leq c \leq a + b \). Now, by definition, what we call the Clebsch-Gordan evaluation of this CG network
Here $a, b, c$ indicate the number of Kronecker delta strands to be used. The number of $\epsilon$ arrows at each trivalent vertex is $k = \frac{b+c-a}{2}$. The right-hand side is meant as the FDC evaluation of the given diagram made of $\delta$’s, $\epsilon$’s and symmetrizers as explained above. Using the identity (13) and the idempotence of symmetrizers one immediately obtains

$$\langle CG \rangle = \frac{(b+c-a)!}{b!c!(a+1)!} \times \frac{(a+b+c+1)!}{a!b!c!} \frac{(a+b-c)!}{a!b!c!} \frac{(a+c-b)!}{a!b!c!} \frac{(a+c-b)!}{a!b!c!}.$$ 

Using the same calculation and notation as in Lemma 2 one can write

$$\langle CG \rangle = \frac{1}{a!} \sum_{\sigma \in S_a} 2^{c(\sigma)} = \frac{1}{a!} \sum_{k=0}^{a} 2^k c(a, k) = a + 1 \quad (14)$$

as it should be. Indeed, this is the trace of the identity operator on $\text{Sym}^a(V^*)$, i.e., the dimension of this representation. Finally, the evaluation of the Theta graph in the CG formalism is

$$\langle CG \rangle = \frac{(a+b+c+1)!}{a!b!c!} \frac{(a+b-c)!}{a!b!c!} \frac{(a+c-b)!}{a!b!c!} \frac{(a+c-b)!}{a!b!c!}.$$ 

More generally, we have the following definition.

**Definition 1** The CG evaluation $\langle G, O, \tau, \gamma \rangle_{CG}$ of a CG network $(G, O, \tau, \gamma)$ is obtained by:

1. replacing each edge $e$ carrying a decoration $\gamma(e)$ by a number $\gamma(e)$ of Kronecker delta strands,
2. replacing a vertex

3. evaluating the resulting ‘microscopic’ Feynman diagram using the rules of FDC, with indices summed over the range \{1, 2\}.

Remark 5 Nothing, in this formalism, relies on how the graph is drawn on the page or any kind of surface imbedding. This is the main difference with Penrose’s binor calculus as explained for instance in [70, 56] or [78, App. A].

The precise correspondence between CSN’s of \$\text{§1.3} and CG networks is the object of \$4. We will conclude this all too brief tour of CIT by a few comments. It has been said that the main goal of CIT is to prove finite generation of rings of invariants for reductive groups. Anyone who would make the effort of reading an original source which either uses the English Omega Operator and hyperdeterminant formalism, or the German symbolic method, will see that the goal of CIT is much more ambitious. It is to develop explicit algebraic geometry (not just effective algebraic geometry), i.e., nonlinear algebra in the sense of [35]. Of course one wants an explicit description of a minimal system of generators for invariant rings, but this is only a means to an end which is to understand invariants which detect a specific geometrical event. The most important perhaps are the multidimensional resultants which detect when \( n \) algebraic hypersurfaces in \( \mathbb{P}^{n-1} \) have a common intersection, and multidimensional discriminants which detect when a hypersurface is singular [45]. Apart from their theoretical interest these often are in practice the most efficient tools for elimination, indeed faster than Gröbner basis methods (see, e.g., [82, 17]). Note in passing that one of the first nontrivial examples of Gröbner bases is also due to Gordan [47]. A long time after Hilbert is said to have killed CIT, Gordan was still working on the problem of symbolic forms of resultants. In one of his last articles [48], he succeeded in finding an explicit symbolic expression for the resultant of two binary forms \( F, G \) of the same order \( d \). His formula amounts to a cycle expansion of the Bezout determinant. When translated in macroscopic
FDC, it is an explicit linear combination of products of ‘wheels’

These essentially are spin networks with external $F$ and $G$ legs. Finding similar symbolic or microscopic FDC formulae for the multidimensional case is a very old and very open problem. It was briefly hinted at in §VIII]. The reader who would like to learn more about the CIT of binary forms must consult the works of classical masters such as Clebsch, von Gall, Gordan, Stroh, Young, etc. The introduction via FDC provided in this section should make the reading much easier.

3 Clebsch-Gordan networks with external legs and $SL_2$ invariance

In this section, we will slightly generalize the notion of CG networks $(G, \mathcal{O}, \tau, \gamma)$ by allowing $G$ to have 1-valent vertices. So now $G$ is any graph with vertices of degree 1 or 3. The orientation $\mathcal{O}$ of the edges is smooth in the sense that 3-valent vertices must have (indegree, outdegree) equal to $(1, 2)$ or $(2, 1)$. The gate signage $\tau$ is an ordering of the gates at each 3-valent vertex and $\gamma$ is, as before, a decoration of the edges by an admissible collection of non-negative integers. The 1-valent vertices impose no new constraints as far as admissibility goes. We will use the expression ‘external leg’ of the network indiscriminately for a 1-valent vertex or for the unique edge incident to it. A
1-valent vertex \(\rightarrow\) of outdegree 1 is called an entry vertex. A 1-valent vertex \(\leftarrow\) of indegree 1 is called an exit vertex. We denote by \(V_{\text{in}}(G), V_{\text{out}}(G)\) the set of entry, exit vertices respectively. We will commit a slight abuse of notation by writing \(\gamma(v)\) for the decoration of the unique edge incident to a 1-valent vertex \(v\).

We will now define as in Def. 1 the Clebsch-Gordan evaluation \(\langle G, O, \tau, \gamma \rangle^{CG}\) of such a network which will be a tensor living in

\[
\left( \bigotimes_{v \in V_{\text{in}}(G)} H_{\gamma(v)}^* \right) \otimes \left( \bigotimes_{v \in V_{\text{out}}(G)} H_{\gamma(v)} \right)
\]

where \(H_a = \text{Sym}^a(V^*), H_a^*\) is the dual \(\text{Sym}^a(V)\), and \(V = \mathbb{C}^2\) as in §2. The rules for constructing \(\langle G, O, \tau, \gamma \rangle^{CG}\) are the same as before except that a 1-valent vertex gives

\[
\begin{align*}
\begin{array}{c}
\node{a} \\
\node{o}
\end{array}
& \quad \longrightarrow \\
\begin{array}{c}
\node{a} \\
\node{o}
\end{array}
\end{align*}
\]

external indices .

For instance identity (13) is

\[
\langle m \, n \rangle^{CG} = \delta_{pq} \frac{k! \, (m + n - k + 1)! \, (m - k)! \, (n - k)!}{m! \, n! \, (m + n - 2k + 1)!} \langle p \rangle^{CG}
\]

and the Gordan series (12) is

\[
\langle m \, n \rangle^{CG} = \sum_{k=0}^{\min(m,n)} \frac{\binom{m}{k} \binom{n}{k}}{\binom{m + n - k}{k + 1}} \langle m \, n \rangle^{CG}
\]

Classical transvectants as in (11) are obtained by plugging the ‘holes’ corresponding to the \(H_{\gamma(v)}^*\)’s by blobs \(\begin{array}{c}
\node{a} \\
\node{o}
\end{array}\) of binary forms, and those corresponding to \(H_{\gamma(v)}\)’s by \(\begin{array}{c}
\node{a} \\
\node{o}
\end{array}\) or by other series of binary variables \(y, z, \ldots\)
There are many natural operations one can do with CG networks \((G, \mathcal{O}, \tau, \gamma)\).

One can cut an edge

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{cut.png}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{cut.png}
\end{array}
\end{array}
\] (17)

and create two new 1-valent vertices. The shaded areas represent the rest of the graph which does not necessarily have to be connected. Conversely, for a pair \(v_{\text{in}} \in V_{\text{in}}(G), v_{\text{out}} \in V_{\text{out}}(G)\) with \(\gamma(v_{\text{in}}) = \gamma(v_{\text{out}})\) one can glue them

Now a matrix \(g \in GL_2(\mathbb{C})\) naturally acts on the tensor \(\langle G, \mathcal{O}, \tau, \gamma \rangle_{CG}\) following (6) and (7). The resulting tensor \(g \cdot \langle G, \mathcal{O}, \tau, \gamma \rangle_{CG}\) can be obtained using FDC as before, except that an entry leg contributes

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{entry.png}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{entry.png}
\end{array}
\end{array}
\] and for an exit leg

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{exit.png}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{exit.png}
\end{array}
\end{array}
\]

An important property of CG networks is \(SL_2\) invariance.

**Proposition 1** For any \(g \in SL_2(\mathbb{C})\), \(g \cdot \langle G, \mathcal{O}, \tau, \gamma \rangle_{CG} = \langle G, \mathcal{O}, \tau, \gamma \rangle_{CG}\).

**Proof:** Consider the microscopic Feynman diagram for \(g \cdot \langle G, \mathcal{O}, \tau, \gamma \rangle_{CG}\). For any edge between 3-valent vertices (eventually the same in case of a loop edge) insert the identity

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{identity.png}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{identity.png}
\end{array}
\end{array}
\]
with due care for the indicated orientation. Now the vicinity of every 3-valent vertex becomes

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\rightarrow
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\text{macroscopic}
\end{align*}
\]

or

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\rightarrow
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\text{microscopic}
\end{align*}
\]

One can easily see that one can push the matrices through the symmetrizers

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 M
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 M
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
= \begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 M
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 M
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\text{.} \quad \tag{18}
\]

29
Therefore, using $[5]$, $\det(g) = 1$ and $g \cdot g^{-1} = \text{Id}$ one obtains the original blown-up vertex from Rule 2) in Def. $[1]$.

Now identity $[15]$ simply is an explicit instance of Schur’s Lemma.

**Proposition 2** More generally for CG networks with one entry and one exit leg we have

$$
\langle \begin{array}{c} a \\ b \end{array} \rangle_{\text{CG}} = \frac{\delta_{a,b}}{a+1} \langle a \rangle_{\text{CG}} + \langle b \rangle_{\text{CG}}.
$$

**Proof:** Schur’s Lemma tells us that the left hand side vanishes unless $b = a$, i.e., the two irreducible representations are the same. Besides, if $a = b$, then the left-hand side is a multiple of the identity with a proportionality constant which can be determined as the ratio of the traces of these two operators.

In general, a tensor $\langle G, O, \tau, \gamma \rangle_{\text{CG}}$ can be seen as an $SL_2$-equivariant map from $\bigotimes_{v \in V_{\text{in}}(G)} \mathcal{H}_{\gamma(v)}$ to $\bigotimes_{v \in V_{\text{out}}(G)} \mathcal{H}_{\gamma(v)}$. For further use, we state the following trivial consequence.

**Corollary 3** If $(G, O, \tau, \gamma)$ is a CG network with only one external leg $v$, its evaluation $\langle G, O, \tau, \gamma \rangle_{\text{CG}}$ vanishes unless $\gamma(v) = 0$.

Classically this amounts to the fact that there are no nonzero linear invariants for a binary form. This follows from the First Fundamental Theorem stated earlier.

An important remark is that the Feynman diagram piece used to compute the contribution of a 3-valent vertex by Rule 2) of Def. $[1]$ is the same regardless of the two possible orientations of the edges incident to the vertex. Besides, this contribution is real for any values of the $a + b + c$ indices. As a result, the two equivariant maps

$$
\Pi_{\text{CG}} : \mathcal{H}_m \otimes \mathcal{H}_n \to \mathcal{H}_{m+n-2k} \quad \text{given by}
$$

and

$$
I_{\text{CG}} : \mathcal{H}_{m+n-2k} \to \mathcal{H}_m \otimes \mathcal{H}_n \quad \text{given by}
$$
are Hermitian conjugates. Indeed, for a binary form \( F \in \mathcal{H}_{m+n-2k} \) and a bihomogeneous form \( B \in \mathcal{H}_m \otimes \mathcal{H}_n \) we have

\[
\langle F| \Pi_{CG}(B) \rangle = \langle B| I_{CG}(F) \rangle = \langle I_{CG}(F)|B \rangle.
\]

Using identities (15) and (16) and similar graphical computations of inner products, one can easily see that the map \( \iota_k : \mathcal{H}_{m+n-2k} \to \mathcal{H}_m \otimes \mathcal{H}_n \) defined by

\[
\iota_k = \sqrt{\frac{m! \ n! (m+n-2k+1)!}{k! (m+n-k+1)! (m-k)! (n-k)!}} \times I_{CG}
\]

and the map \( \pi_k : \mathcal{H}_m \otimes \mathcal{H}_n \to \mathcal{H}_{m+n-2k} \) defined by

\[
\pi_k = \sqrt{\frac{m! \ n! (m+n-2k+1)!}{k! (m+n-k+1)! (m-k)! (n-k)!}} \times \Pi_{CG}
\]

satisfy the following properties.

**Proposition 3**

1. \( \iota_k \) is an isometric injection.
2. \( \pi_k \) is the Hermitian conjugate of \( \iota_k \).
3. \( \pi_k \circ \iota_k = \text{Id}_{\mathcal{H}_{m+n-2k}} \).

4. The images of the \( \iota_k \), \( 0 \leq k \leq \min(m, n) \), are orthogonal.

5. One has the decomposition of the identity

\[
\sum_{k=0}^{\min(m, n)} \iota_k \circ \pi_k = \text{Id}_{\mathcal{H}_m \otimes H_n}.
\]

These are the standard normalizations for the maps underlying the QAMT in the physics literature. See [5, §7] for a precise dictionary with the CIT approach, and formulae for 6-j and 9-j symbols.

**Remark 6** In some instances one might need to allow trivial components \( \bigcirc^a \) in a CG network even though the underlying graph \( G \) does not satisfy the usual definitions of a graph. One can decide that the evaluation of such a component is

\[
\langle \bigcirc^a \rangle_{CG} = \bigcirc^{a+1}
\]

from [14].

### 4 The negative dimensionality theorem

We now turn to the precise relationship between spin networks of §1.3 and CG networks. Let \((\Gamma, \gamma)\) be a spin network as in §1.3. For more precision let us write \( \Gamma = (G, R) \) where \( G \) is the underlying cubic graph and \( R \) is the extra structure corresponding to the pure rotation system needed in order to define the imbedding of \( \Gamma \) in an orientable compact Riemann surface \( \Sigma \). Suppose one also has a smooth orientation \( \mathcal{O} \) of the edges of \( G \) and a gate signage \( \tau \) as in §2. Then one has the following correspondence which is reminiscent of the negative dimensionality theorem of [31] (see also [30, Ch. 13]).

**Theorem 4** One has

\[
(\Gamma, \gamma)^P = \mu \times \left( \prod_{e \in E(G)} \gamma(e)! \right) \times \langle G, \mathcal{O}, \tau, \gamma \rangle_{CG}
\]

where \( \mu = \pm \) is a global sign which depends on the given combinatorial data.
Proof: Because of the factorization property of Lemma 3 which also holds for CG evaluations of CG networks, it is enough to consider connected graphs. The case of trivial components is dealt with by the comparison of Lemma 2 and Remark 6. We now only consider nontrivial connected cubic graphs which may have loops and multiple edges. Define \( H(G) \) the set of half-edges of \( G \) obtained by cutting the edges of \( G \) according to

\[
\begin{align*}
\text{e} & \quad \rightarrow \quad h_1 \quad h_2 \\
\circ \text{e} & \quad \rightarrow \quad h_1 \bigcup h_2
\end{align*}
\]

where \( e \in E(G) \) and \( h_1 \neq h_2 \) belong to \( H(G) \). Two half-edges are called edge-adjacent if they are distinct and come from the same edge. Two half-edges are called vertex-adjacent if they are distinct and attached to the same vertex.

A directed path is a sequence

\[
v_0, h_1, h'_1, v_1, h_2, h'_2, \ldots v_{n-1}, h_n, h'_n, v_n
\]

where \( n \geq 0 \) and where the \( v_i \) are vertices and the \( h_i, h'_i \) are half-edges. We also impose the condition that \( h_i \) is incident to \( v_{i-1} \), for \( 1 \leq i \leq n \), that \( h'_i \) is incident to \( v_i \), for \( 1 \leq i \leq n \), and that \( h_i, h'_i \) are edge-adjacent for \( 1 \leq i \leq n \). Finally we enforce a no backtracking clause \( h'_i \neq h_{i+1} \) for any \( i, 1 \leq i < n \).

Example: On the graph

the path

is allowed and corresponds to the sequence

\[
v_2, h_4, h_3, v_1, h_1, h_2, v_1, h_1, h_2, v_1, h_3, h_4, v_2, h_5, v_2, h_5, v_2, h_4, h_3, v_1.
\]
An undirected path is an equivalence class of directed paths with respect to reversal
\[ v_0, h_1, h'_1, \ldots, h'_n, v_n \rightarrow v_n, h'_n, h_n, \ldots, h_1, v_0. \]

A pointed closed directed path is a directed path with \( n \geq 1 \) such that \( v_n = v_0 \) and \( h_1 \neq h'_n \). Note that on the same graph as before violates the last condition and is not allowed. A closed directed path is an equivalence class of pointed closed directed paths under cyclic transformation
\[ v_0, h_1, h'_1, \ldots, h'_n, v_n \rightarrow v_i, h_{i+1}, h'_{i+1}, \ldots, h'_n, v_n, h_1, h'_1, v_1, \ldots h_i, h'_i, v_i \]
where \( v_0 \) has been deleted and \( v_i \) has been repeated. A closed path is an equivalence class of pointed directed paths under both cyclic transformations and reversal.

When expanding the state sum over all connecting permutations in the definition of \( \langle \Gamma, \gamma \rangle^E \), each continuous curve drawn on \( \Sigma \) defines in an unambiguous manner a closed path in the previous sense. Each term contributes \( (-1)^{C(\bar{\sigma})} (-2)^{N(\bar{\sigma})} \) where \( \bar{\sigma} = (\sigma_e)_{e \in E(G)} \in \prod_{e \in E(G)} S_{\gamma(e)} \) is the given configuration or state, \( C(\bar{\sigma}) \) denotes the total number of strand crossings and \( N(\bar{\sigma}) \) is the number of closed curves. Consider such a curve \( \mathcal{C} \) or rather its corresponding closed path \( [v_0, h_1, h'_1, \ldots, h'_n, v_n] \) where brackets mean equivalence class. To such data one can associate the number \( B(\mathcal{C}) \) of gates crossed. Indeed, the half-edges inherit the \( O \) orientation of their parent edge. Gates correspond to vertex-adjacent pairs of half-edges with opposite orientations
\[ \backslash/ \] or \[ /\ \].

A gate crossing is an index \( i, 1 \leq i \leq n \), such that either \( i < n \) and \( h'_i, h_{i+1} \) form a gate or \( i = n \) and \( h'_n, h_1 \) form a gate. We now make the following observation.

**Key observation:** The number \( B(\mathcal{C}) \) of gates crossed must be even.

Indeed, this is the number of changes of direction (relative to the edge orientation \( O \)) encountered as one travels along the closed path starting and ending at the same half-edge say \( h_1 \).
Now choose a closed directed path structure for the closed path of $C$. This amounts to picking a direction of travel. One then defines a good gate crossing as a gate crossing of the form

\[ \text{direction of travel} \]

or

which is a notion relative to the gate signage $\tau$. The other crossings of the form

or

are bad gate crossings. Let $B_+(C)$ be the number of good gate crossings and $B_-(C)$ that of bad gate crossings. Obviously $B(C) = B_+(C) + B_-(C)$ and choosing the other direction of travel merely exchanges $B_+(C)$ and $B_-(C)$. Because of the key observation that $B(C)$ is even, the sign $(-1)^{B_+(C)} = (-1)^{B_-(C)}$ is independent of the direction of travel. Let $B_+(\bar{\sigma})$ be the sum, over all curves $C$ present in a given configuration produced by $\bar{\sigma}$, of the $B_+(C)$’s. The main fact needed to prove the theorem is the following statement.

**Claim:** The sign $\tilde{\mu} = (-1)^{C(\bar{\sigma}) + N(\bar{\sigma}) + B_+(\bar{\sigma})}$ is independent of $\bar{\sigma}$.

Since all configurations $\bar{\sigma}$ are related by sequences of transpositions it is enough to show the invariance of this sign with respect to any such transposition of the strands at some edge of the cubic graph. Of course $(-1)^{C(\bar{\sigma})}$ changes to the opposite sign under a transposition and we need to show that so does $(-1)^{N(\bar{\sigma}) + B_+(\bar{\sigma})}$. For a strand transposition

\[ || \quad \rightarrow \quad \chi \]

depending on the configuration of curves $C$ to which these strands belong, there are three cases to consider (and not two as in [30 Ch. 13]):

[Diagram of cases I, II, III]
where we highlighted the strands along the edge of the transposition.

**Case I:**

This fuses two curves or closed paths into one. Choose a direction of travel on the resulting closed path and deduce from it a direction of travel on both initial curves as in

\[ C_1 \rightarrow C_2 \rightarrow C \]

We have that \( N(\vec{\sigma}) \) dropped to \( N(\vec{\sigma}) - 1 \). On the other hand, one clearly has \( B_+(C) = B_+(C_1) + B_+(C_2) \) therefore \( B_+(\vec{\sigma}) \) remains unchanged.

**Case II:**

Choose a direction of travel on \( C_1 \) and \( C_2 \). These will coincide on one of the two portions of the path (say the bottom one on the picture) and differ on the other portion.
By the same argument as in the key observation one easily sees that the number of gate crossings in the upper portion is odd. Therefore the number of good and bad crossings on this portion add up to an odd number and therefore have different parity. The reversal of direction of travel on the upper portion will exchange these two numbers and thus \((-1)^{B_+(C_2)} = -(-1)^{B_+(C_1)}\) while \(N(\tilde{\sigma})\) remains unchanged.

**Case III:**

This is the undoing of case I. One can again choose a direction of travel on \(C\) and deduce from it directions of travel on the two resulting curves \(C_1\) and \(C_2\).

One then has \(N(\tilde{\sigma}) \to N(\tilde{\sigma}) + 1\) and \((-1)^{B_+(\tilde{\sigma})}\) remains unchanged. This concludes the proof of the claim.

As a result, one has

\[
\langle \Gamma, \gamma \rangle^P = \sum_{\tilde{\sigma}} (-1)^{C(\tilde{\sigma})} (-2)^{N(\tilde{\sigma})}
\]

\[
= \tilde{\mu} \sum_{\tilde{\sigma}} (-1)^{B_+(\tilde{\sigma})} 2^{N(\tilde{\sigma})} .
\]

Now by the rules of Def. 1

\[
\langle G, \mathcal{O}, \tau, \gamma \rangle^{CG} = \left( \prod_{e \in E(G)} \gamma(e)! \right)^{-1} \times \sum_{\sigma} \prod_{C} \text{tr } C
\]
where the contribution $\text{tr} \mathcal{C}$ of each curve is the trace of a product of $2 \times 2$ matrices taken among the commuting matrices $I, \epsilon, -\epsilon$. More precisely

$$
\text{tr} \mathcal{C} = \text{tr} \left[ \epsilon^{B_+}(-\epsilon)^{B_-} \mathcal{C} \right]
$$

$$
= (-1)^{B_-} \text{tr} \left[ \epsilon^{B} \mathcal{C} \right] = (-1)^{B_+} \text{tr} \left[ (-I)^{B} \mathcal{C} \right]
$$

since $B(\mathcal{C})$ is even by the key observation. Therefore $\text{tr} \mathcal{C} = (-1)^{B_+} \times 2 \times (-1)^{\frac{B}{2}}$ and

$$
\langle \Gamma, \gamma \rangle^P = \mu \times \left( \prod_{e \in E(G)} \gamma(e)! \right) \times \langle G, \mathcal{O}, \tau, \gamma \rangle^{CG}
$$

with $\mu = \tilde{\mu} \times (-1)^{\frac{k}{2}}$ where $k$ is the sum over curves $\mathcal{C}$ of the number of gate crossings $B(\mathcal{C})$. This is the same as the total number of epsilon arrows in the microscopic Feynman diagram used to evaluate the CG network. 

**Remark 7** A byproduct of this proof is that the number $k$ of $\epsilon$’s is even. Another way to see this is to write $k = k_\pi + k_i$ where $k_\pi$ is the total number of epsilons coming from vertices

\[ \begin{array}{c}
\text{a} \\
\text{c} \\
\text{b}
\end{array} \]

and $k_i$ is the total number of epsilons coming from vertices

\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \]

By using the $\frac{a+b+c}{2}$ formula to count the $\epsilon$’s at these vertices one easily finds that $k_\pi - k_i = 0$. Indeed, an edge decoration is counted positively at the vertex of destination and negatively at the vertex of origin, with respect to the orientation $\mathcal{O}$.

**Remark 8** One can use the same parity arguments on the number of gate crossings in order to prove Corollary 3 in a purely diagrammatic way, without the invocation of Schur’s Lemma and the irreducibility of the $\mathcal{H}_d$’s as $SU_2$ representations. For a graph with only one external leg
any strand attached to an external index which goes in must come out. Besides the number of gates crossed, i.e., the number of \( \epsilon \) arrows along the way must be odd, as in Case II above. Therefore, the contribution \( \pm \epsilon^{2p+1} = \pm \epsilon \) is antisymmetric and is killed by the symmetrization over the indices \( i_1, \ldots, i_a \).

Note that one can also show Schur’s Lemma in the same manner since a strand that comes in one way must come out the other way.

We will later show in §5 that there always exist smooth orientations \( \mathcal{O} \) and, consequently, that any CSN is amenable to the evaluation of a CG network. In order to make the dictionary between CSN’s and CG networks more satisfactory, it would be desirable to have a canonical way to associate a smooth orientation to a surface imbedding \( \Gamma \xrightarrow{R} \Sigma \). Perhaps the methods of [66] could help towards that goal. By contrast, a possible canonical choice for the gate signage \( \tau \) is trivially obtained by orienting the small curved arrows counterclockwise along the faces of the map as in

\[
\text{It would also be nice to be able to compute the sign } \mu \text{ for such canonical prescriptions.}
\]

**Remark 9** Note that an alternative to the proof of Theorem 4 is that sketched in [76, §2.2]. Up to a harmless global prefactor, the setting of [76, Def. 3] corresponds to a CG network evaluated by FDC as in Def. 1 but using \( \epsilon \) arrows and no Kronecker delta’s. One can connect Roberts definition to ours by pushing the edge epsilons into the vertices, in the direction indicated by the orientation \( \mathcal{O} \), and eliminating those which can be eliminated by the relation \( \epsilon^2 = -\text{Id} \).

We now define an alternative evaluation for CG networks which is closely related to the unitary evaluation of CSN’s. Given a CG network \( (G, \mathcal{O}, \tau, \gamma) \),
possibly with external legs, we let the $\pi&i$ evaluation $\langle G, O, \tau, \gamma \rangle^{\pi_i}$ be defined by the same rules as in §3 except that the contribution of a vertex

\[ \text{or} \]

gets multiplied by the factor

\[ \sqrt{\frac{a! \ b! \ (c + 1)!}{(a+b+c+1)! \ (a+b-c)! \ (a+c-b)! \ (b+c-a)!}} \]

dictated by the considerations preceding Proposition 3. For such a vertex $v$ we define $\dim(v) = c + 1$. Now an immediate corollary of Theorem 4 is as follows.

**Corollary 4** With the same hypotheses and notations as in Theorem 4 one has

\[ \langle \Gamma, \gamma \rangle^U = \mu \times \left( \prod_{v \in V(G)} \frac{1}{\sqrt{\dim(v)}} \right) \times \langle G, O, \tau, \gamma \rangle^{\pi_i} . \]

### 5 The existence of smooth orientations

The following proposition guarantees that one can always use the CG formalism in order to evaluate a spin network.

**Proposition 4** For any spin network $(\Gamma, \gamma)$ without trivial components, the underlying graph has a smooth orientation.

We can of course assume $\Gamma = (G,R)$ where $G$ is a connected cubic graph, possibly containing loops and multiple edges. For good measure, we will provide several proofs.

**1st proof for the case of bridgeless graphs:** (Indicated to us by Bill Jackson and Gordon Royle) If $g$ is bridgeless or 2-edge-connected, one can use Petersen’s famous CIT-inspired 1-factor Theorem [72] (see also [72]). Namely, there exists a 1-factor of $G$, i.e., a set of edges $E_1 \subset E(G)$ such that every vertex $v \in V(G)$ has degree 1 in the spanning subgraph given by $E_1$. The edge complement is therefore a collection of cycles. For each such cycle, choose a coherent orientation, i.e., an orientation which follows a
direction of travel along the cycle. Finally, take any orientation of the edges in $E_1$. The obtained orientation $\mathcal{O}$ is smooth.

2nd proof in the general case: Choose a spanning tree $T$ in the connected graph $G$. Choose $v_0$ among the leafs of the tree, i.e., vertices of degree 1 relatively to $T$. Choose $v_0$ as a root for $T$, and orient all the edges of $T$ towards the root $v_0$. For example:

![Diagram of a graph and its orientation](image)

where the edges of $T$ are indicated by thicker lines. Because we have chosen a leaf to be the root, the vertices of degree 3 in the tree are of the form which is acceptable. The vertices of degree 2 in $T$ must be of the form . Clearly, the remaining edges can be organized into a collection of paths and cycles (which can be reduced to a loop). For the cycles which bounce around vertices of degree 1 in $T$ and then close upon themselves, choose a coherent orientation following a direction of travel along the cycle. For a path which must start and end at two vertices of degree 2 in $T$, and, in between, may bounce around vertices of degree 1 in $T$, again choose a coherent orientation along the path. The resulting orientation $\mathcal{O}$ is easily seen to be smooth.

For the previous example a possible outcome of this procedure is

![Another diagram](image)
3rd proof in the general case: (Indicated to us by Bill Jackson) We will show that all graphs $G$, and not only cubic graphs, have an orientation such that, for any vertex, the absolute value of the difference between indegree and outdegree is most one. The proof is by induction on the number of edges. If some edge $e$ is incident to a vertex of odd degree, then by the induction hypothesis $G - e$ (i.e., the graph with edge $e$ removed) has such an orientation $O'$. If $e$ is a loop, then any orientation of $e$ will do. Otherwise, $e$ is incident to two distinct vertices $v_1, v_2$

such that say $v_1$ is of odd degree in $G$. The orientations indicated are the ones provided by $O'$. Since $|a - b| \leq 1$ and $a + b + 1$ is odd, we must have $a = b$. If $c = d$, any orientation of $e$ will do. If $c = d - 1$, then orient $e$ from $v_2$ to $v_1$. If $c = d + 1$, then orient $e$ from $v_1$ to $v_2$. The resulting orientation is then acceptable.

If no edge is incident to a vertex of odd degree, then each connected component of $G$ is Eulerian and one can direct the edges around an Euler tour of each connected component. The resulting orientation will have indegree=outdegree at each vertex.

Remark 10 Note that one can find a blend of Proof 1 and Proof 3 in [60, Prop 3.4], concerning the notion of ‘NS orientations’. These are synonymous with smooth orientations, as well as orientations without sources and sinks, in the case of cubic graphs.

6 The bridge reduction

Let $(\Gamma, \gamma)$ be a connected CSN which is not reduced to a trivial component. Let us also assume there is a cut-edge or bridge $e_0$, i.e., an edge which when removed makes the graph disconnected.

Lemma 4 If $\gamma(e_0) \neq 0$ then $(\Gamma, \gamma)^P = 0$.

Proof: Choose a smooth orientation $O$ on the underlying graph $G$ by Proposition[4]. Choose a gate signage $\tau$. By Theorem[3] we only need to show that $(G, O, \tau, \gamma)^{CG} = 0$. Let $i_1, \ldots, i_d$, with $d = \gamma(e_0) \neq 0$, be the indices summed
over at the strands of the Feynman diagram used to compute $\langle G, O, \tau, \gamma \rangle^{CG}$. By Corollary 3, the contribution of the CG networks with one external leg on either side of $e_0$ must vanish, for any choice of the indices $i_1, \ldots, i_d$ in $\{1, 2\}$.

We now assume $\gamma(e_0) = 0$. This forces the decorations at the endpoints of $e_0$ to be of the form

$$
\begin{array}{c}
\gamma(0) = a \\
\gamma(a) = b
\end{array}
$$

by the admissibility condition. Let $\Gamma_1, \Gamma_2$ be the connected ribbon graphs one obtains by removing $e_0$ and erasing the vertices $v_1, v_2$ that $e_0$ was incident to.

Of course, $\Gamma_1, \Gamma_2$ inherit decorations $\gamma_1, \gamma_2$ coming from $\gamma$.

**Proposition 5**

$$
\langle \Gamma, \gamma \rangle^U = \langle \Gamma_1, \gamma_1 \rangle^U \langle \Gamma_2, \gamma_2 \rangle^U \delta(v_1)\delta(v_2)
$$

where $\delta(v)$ for a vertex of the form

$$
\begin{array}{c}
\gamma(0) = a \\
\gamma(a) = 0
\end{array}
$$

is $\delta(v) = \frac{1}{a!\sqrt{a+1}}$ if $v$ is a loop vertex $\bigcirc^a$ and $\delta(v) = \frac{1}{\sqrt{a+1}}$ otherwise.
**Proof:** First let us cut the edge $e_0$ of the ribbon graph $\Gamma$ by introducing two 1-valent vertices $v'_1$ and $v'_2$:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{proof.png}
\end{array}
\]

One therefore obtains two (nonregular) ribbon graphs $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ together with their imbeddings in surfaces $\Sigma_1$ and $\Sigma_2$ respectively:

\[
\Sigma_i
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{proof.png}
\end{array}
\]

The shaded area represents the rest of the graph $\hat{\Gamma}_i$ which can wrap around the surface $\Sigma_i$. Then cut a small disc $D_i$ around $v'_i$.

\[
\Sigma_i
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{proof.png}
\end{array}
\]

Clearly the imbedding $\Gamma \to \Sigma$ can be obtained by gluing $\Sigma_1$ and $\Sigma_2$ along the boundaries $\partial D_1$, $\partial D_2$ of the removed discs. Since the decorations of $v_i$ are of the form $v_i^a_i$, there are no strands joining the $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ parts when applying the rules of Penrose evaluation in §1.3. Also the application of Rule 3) to $v_i$ produces the same strand structure as if there was no vertex:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{proof.png}
\end{array}
\]

As a result of the previous considerations, the Penrose evaluation $\langle \Gamma, \gamma \rangle^P$ factors into $\langle \Gamma_1, \gamma_1 \rangle^P \times \langle \Gamma_2, \gamma_2 \rangle^P$ except for one subtlety of the vertex erasure
and its effect on the Penrose bars of Rule 2). If \( v_i \) is not a loop vertex, its contribution is

\[
\begin{array}{c}
\text{\( a_i \)} \\
v_i \\
0
\end{array}
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\rightarrow
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\]

namely, \( a_i! \) times the contribution

\[
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\rightarrow
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\]

one would get after erasing \( v_i \) and fusing the two edges carrying the decoration \( a_i \). If \( v_i \) is a loop vertex, then both calculations produce the same result

\[
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\rightarrow
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\rightarrow
\begin{array}{c}
\text{\( a_i \)}
\end{array}
\]

From the definitions (1) and (2), the formula relating the Penrose and the unitary evaluations is

\[
\langle \Gamma, \gamma \rangle_U = \langle \Gamma, \gamma \rangle_P \times \prod_{v \in V(\Gamma)} \left\{ \left( \frac{a_v + b_v + c_v}{2} \right)! \right\}^{-\frac{1}{2}}
\]

Since the factors \([(a_i + 1)! a_i!]^{-\frac{1}{2}} \) corresponding to \( v_1, v_2 \) are missing in the \( \langle \Gamma_i, \gamma_i \rangle_U \), the proposition follows.

Remark 11  
From the proposition and the evaluation of trivial components in Lemma 2, the evaluation 3 of the dumbell graph follows.

Proposition 6  
Theorem 7 implies Theorem 2.

Proof: By induction on the number of vertices. By the factorization property of Lemma 3, it is enough to consider graphs which are connected. If
(Γ, γ) has no loops, then |⟨Γ, γ⟩U| ≤ 1 by Theorem 1 which is assumed to hold. Otherwise, if there is a loop, then let e₀ be the bridge which connects it to the rest of the graph. Using the notations and setting of Proposition 5 let v₁ be the vertex in the given loop. Now if the other vertex v₂ is also a loop vertex, then by connectedness (Γ, γ) must be a dumbell graph and, for n ≥ 1, we have

\[ \left\langle \begin{array}{c} \bullet \circ \bullet \\ n_a & n_e & n_b \end{array} \right\rangle^U = \delta_{c,0} \sqrt{(n a + 1)(n b + 1)}. \]

This provides the desired polynomial bound in n. Else, if v₂ is not a loop vertex then by Proposition 5 and Lemma 2

\[ |⟨Γ, nγ⟩^U| = \sqrt{na_1 + 1} \times \frac{1}{\sqrt{na_2 + 1}} \times |⟨Γ_2, nγ_2⟩^U| \]

and |⟨Γ_2, nγ_2⟩^U| is polynomially bounded by the induction hypothesis. Therefore so is |⟨Γ, nγ⟩^U|.

7 Estimates on special spin networks

7.1 The tetrahedron or 6-j symbol

In this well-known case one has a precise n → ∞ asymptotic: the Ponzano-Regge formula. This will provide both upper and lower bounds on \( \sqrt{⟨Γ, nγ⟩^U} \) as n → ∞. However, we will derive an upper bound using soft analysis: a simple Cauchy-Schwarz inequality. This will provide a gentle introduction to the proof of our main theorem in §8.

7.1.1 Preparation

Consider the spin network (Γ, γ) given by

\[ \begin{array}{c}
  e \\
  a \\
  b \\
  c \\
  d \\
  f 
\end{array} \]
The cyclic orientations at the vertices are counterclockwise, and the imbedding is planar. By Corollary 4:

$$\left| \left\langle \Gamma, \gamma \right\rangle^U \right| = \frac{1}{(e + 1) \sqrt{(c + 1)(f + 1)}} \times \left| \left\langle e \right\rangle^\pi_\iota \right| \ .$$

As recalled in [5, §7.6] the corresponding standard Wigner 6-j symbol is

$$\left\{ \begin{array}{ccc} a & b & c \\ \frac{d}{2} & \frac{e}{2} & \frac{f}{2} \end{array} \right\} = \frac{(-1)^{a+b+d+e}}{\sqrt{(c+1)(f+1)}} \times \alpha$$

where \( \alpha \) is the proportionality constant in the explicit instance of Schur’s Lemma

$$\left\langle a \ b \ c \ d \ e \ f \right\rangle^\pi_\iota = \alpha \times \left\langle e \right\rangle^\pi_\iota \ .$$

By taking the trace in \( \mathcal{H}_e^* \otimes \mathcal{H}_e \), one has

$$\alpha = \frac{1}{e + 1} \times \left| \left\langle e \right\rangle^\pi_\iota \left\langle a \ b \ c \ d \ f \right\rangle^\pi_\iota \right|$$

and therefore

$$\left| \left\langle \Gamma, \gamma \right\rangle^U \right| = \left| \left\{ \begin{array}{ccc} a & b & c \\ \frac{d}{2} & \frac{e}{2} & \frac{f}{2} \end{array} \right\} \right| \ . \quad (21)$$

### 7.1.2 The upper bound

Notice that by introducing a splitting of the graph and the associated Hilbert space \( \mathcal{H} = \mathcal{H}_e^* \otimes \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \) one has

$$\left\langle e \right\rangle^\pi_\iota \left\langle a \ b \ c \ d \ f \right\rangle^\pi_\iota = \left\langle B | A \right\rangle_{\mathcal{H}}$$
where

\[ A = \langle \begin{array}{c}
\pi^i
\end{array} \rangle_{\pi^i} \in \mathcal{H} \]

and

\[ B = \langle \begin{array}{c}
\pi^i
\end{array} \rangle_{\pi^i} \in \mathcal{H} \]

and the inner product is the natural one as in (10) and (20), using FDC. Note the reversal of orientations in B.

By the Cauchy-Schwarz inequality

\[
\left| \langle \Gamma, \gamma \rangle_U \right| \leq \frac{1}{(e + 1) \sqrt{(c + 1)(f + 1)}} \times \sqrt{\langle A | A \rangle_{\mathcal{H}} \langle B | B \rangle_{\mathcal{H}}}. 
\]

However,

\[
\langle A | A \rangle_{\mathcal{H}} = \langle e \bigg| e \rangle_{\pi^i} = e + 1
\]

by using part 3) of Proposition 3 twice. Likewise, \( \langle B | B \rangle_{\mathcal{H}} = e + 1 \). Therefore

\[
\left| \langle \Gamma, \gamma \rangle_U \right| = \left| \left\{ \begin{array}{c}
a \\
b \\
c \\
d \\
e \\
f \\
\end{array} \right\} \right| \leq \frac{1}{\sqrt{(c + 1)(f + 1)}} \leq 1
\]

which explicitly shows that the tetrahedral spin network satisfies the statement in Theorem 11. Of course, one can use the symmetries of the the 6-j symbol in order to obtain analogous bounds by \( \frac{1}{\sqrt{(a+1)(d+1)}} \), etc. Similar bounds for the 6-j symbol are well-known (see, e.g., [41, App. D]). Also note that similar considerations of inner products in tensor spaces were used in [6, §3.1] in order to prove the nonvanishing of some combinations of CG networks (one can also prove [5, Lemma 2.3] in the same way). A thematically similar use of the Cauchy-Schwarz inequality for 3-manifold invariants or chromatic polynomials of planar graphs can also be found in [42, Thm. 2.2] and [39, §7].
7.1.3 The lower bound

Lower bounds seem much more difficult to obtain. In particular, we have not been able to find one for the 6-j symbol by trying to quantitatively analyse how far the Cauchy-Schwarz inequality is from an equality. One difficulty towards this goal is that Wigner symbols can have accidental zeros which are quite poorly understood (see [74] and references therein). We will rely instead on the Ponzano-Regge asymptotic formula. It was conjectured in [73] based on numerical evidence and some very clever consistency checks, and it was rigorously proved in [76]. Other work related to this asymptotic formula can be found in [80, 13, 40, 22, 50, 44]. Note that [40, 13] consider the square of the 6-j symbol which is also suitable for the needs of this section.

We will consider the case where the original decorations are uniform \( \gamma \equiv 2 \) and then the rescaled network \( \gamma \rightarrow n\gamma \).

**Lemma 5**

\[
\limsup_{n \to \infty} \left| \langle \begin{array}{ccc}
\gamma & 2
\end{array}, n\gamma \rangle \right|^{\frac{1}{n}} = 1 .
\]

**Proof:** By (21)

\[
\left| \langle \begin{array}{ccc}
\gamma & 2
\end{array}, 2n \rangle \right| = \left| \left\{ \begin{array}{ccc}
n & n & n \\
n & n & n
\end{array} \right\} \right| .
\]

By the Ponzano-Regge formula for the case of a regular (Euclidean) tetrahedron (see, e.g., [44 §10]), we have:

\[
\left\{ \begin{array}{ccc}
n & n & n \\
n & n & n
\end{array} \right\} = \frac{-1}{2 + \frac{\pi}{2} n} \cos \left( 6 \left( n + \frac{1}{2} \right) \frac{\omega - \pi}{4} \right) + O \left( n^{-\frac{5}{2}} \right)
\]

when \( n \to \infty \), and where \( \omega = \arccos \left( \frac{1}{3} \right) \). Therefore

\[
\limsup_{n \to \infty} \left| \langle \begin{array}{ccc}
\gamma & 2
\end{array}, n\gamma \rangle \right|^{\frac{1}{n}} = \limsup_{n \to \infty} \left| \cos \left( 6 \left( n + \frac{1}{2} \right) \frac{\omega - \pi}{4} \right) + \frac{\alpha_n}{n} \right|^{\frac{1}{n}}
\]

for some bounded sequence \((\alpha_n)_{n \geq 1}\). Trivially, the limsup is \( \leq 1 \). On the other hand, \( \frac{\omega}{\pi} - 1 = 0.1754 \ldots \in (0, \frac{1}{3}) \), hence one can extract a subsequence for which the angle is in \( \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \, \text{mod} \, 2\pi \) and the cosine is \( \geq \frac{1}{2} \). As a result the limsup is \( \geq 1 \).

This proves the statement in Conjecture 1 for the tetrahedron graph, which was already known [44].

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7.2 The generalized drum

As in [44], we consider the graph Drum_s:

with s parallel edges between the two circles.

7.2.1 The upper bound

We consider symmetric decorations γ of the form

\[
\begin{array}{cccccc}
  & a_1 & a_2 & \cdots & a_s & \\
\hline
b_1 & b_2 & b_3 & \cdots & b_s & \\
  & a_1 & a_2 & \cdots & a_s & \\
\end{array}
\]

Lemma 6 For any \( s \geq 1 \), and admissible decorations γ as above

\[
\left| \langle \text{Drum}_s, \gamma \rangle^U \right| \leq \frac{\left( \min(a_1, \ldots, a_s) + 1 \right)^2}{(a_1 + 1) \cdots (a_s + 1)}.
\]

In particular, the conclusion of Theorem 4 holds in this case as soon as \( s \geq 2 \).

Proof: By Corollary 4 and the indicated choice of smooth orientation

\[
\left| \langle \text{Drum}_s, \gamma \rangle^U \right| = \prod_{i=1}^{s} \frac{1}{a_i + 1} \times \left| \langle \text{Drum}_s, \gamma \rangle^U \right| \quad . \quad \text{(23)}
\]
We now perform for every \( i, 1 \leq i \leq s \), the following FDC calculation

\[
\langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{CG} = \langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{\pi_i} = \langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{\pi_i}.
\]

Then, we insert the identity (12)

\[
\langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{CG} = \sum_{k_i=0}^{a_i} \left( \frac{a_i}{k_i} \right)^2 \frac{2a_i - k_i + 1}{k_i}
\]

by pushing the outside \( \epsilon \) arrows through the facing symmetrizers as in (18).

We also let \( c_i = 2a_i - k_i \). Thus

\[
\langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{\pi_i} = \langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{CG} = \sum_{c_i=0}^{a_i} \frac{c_i + 1}{a_i + 1} \langle \begin{array}{c}
{a_i} \\
{c_i}
\end{array} \rangle_{\pi_i}.
\]

We now insert this in (23) and get

\[
\langle \text{Drum}_s, \gamma \rangle^U = \prod_{i=1}^{s} \frac{1}{(a_i + 1)^2} \times \sum_{c_i=0}^{a_i} \prod_{i=1}^{s} (c_i + 1) \times
\]

\[
\langle \begin{array}{c}
{a_i} \\
{a_i}
\end{array} \rangle_{\pi_i}.
\]

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By Schur’s Lemma, namely identity (19), the $c_i$ have to be the same and one gets

$$\left| \langle \text{Drum}_s, \gamma \rangle^U \right| = \prod_{i=1}^{s} \frac{1}{(a_i + 1)^2} \times \sum_{c=0}^{2\min(a_1, \ldots, a_s)} (c + 1) \times$$

$$\times \prod_{i=1}^{s} \left| \langle a_i \rangle \right|$$

with the convention $a_0 = a_s$. Therefore, by Corollary 4

$$\left| \langle \text{Drum}_s, \gamma \rangle^U \right| \leq \prod_{i=1}^{s} \frac{1}{(a_i + 1)^2} \times \sum_{c=0}^{2\min(a_1, \ldots, a_s)} (c + 1) \times \prod_{i=1}^{s}$$

$$\left[ (a_i-1 + 1)(a_i + 1) \right]$$

$$\leq \sum_{c=0}^{2\min(a_1, \ldots, a_s)} (c + 1) \times \prod_{i=1}^{s} \frac{1}{\sqrt{(a_i-1)(a_i + 1)}} = \frac{[\min(a_1, \ldots, a_s) + 1]^2}{(a_1 + 1) \cdots (a_s + 1)}$$

by (22) with the bound corresponding to the highlighted pair of opposite edges. □

7.2.2 The lower bound

We now consider the uniform decoration by $2n$ for $\text{Drum}_s$, $s \geq 2$.

**Proposition 7** For $s \geq 2$

$$\limsup_{n \to \infty} \left| \langle \text{Drum}_s, 2n \rangle^U \right|^{\frac{1}{n}} = 1.$$
Proof: The limsup is $\leq 1$ by Lemma 6. By (23) and the same argument as in §7.2.2 we have

$$\left| \langle \text{Drum}_s, 2n \rangle^U \right| = \frac{1}{(2n + 1)^s} \times \langle A|A \rangle_{\mathcal{H}}$$

where $\mathcal{H} = \mathcal{H}_{2n}^s$ and

$$A = \left\langle \begin{array}{ccc}
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& & \\n\end{array} \right\rangle \in \mathcal{H}$$

with decorations by $2n$ everywhere. By the Cauchy-Schwarz inequality

$$\langle A|A \rangle_{\mathcal{H}} \geq \frac{\langle A|B \rangle_{\mathcal{H}}^2}{\langle B|B \rangle_{\mathcal{H}}}$$

for any nonzero $B \in \mathcal{H}$. We will obtain adequate lower bounds by making a judicious ansatz for $B$. For this, we need to distinguish two cases.

**Case 1 where $s$ is even:** Take $B$ to be given by the microscopic FDC formula

$$B = \underbrace{\text{Drum}_s}^{2n} \underbrace{\text{Drum}_s}^{2n} \ldots$$

Since by pushing the $\varepsilon$ arrows as indicated

$$= (-1)^n \times$$

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one has

\[ \langle A|B\rangle_{\mathcal{H}} = (-1)^{\pi_i} \times \langle \pi_i \rangle \]

with $2n$ everywhere. By Part 3) of Proposition 5,

\[ |\langle A|B\rangle_{\mathcal{H}}| = \langle 2n \rangle_{\pi_i} = 2n + 1. \]

On the other hand,

\[ \langle B|B\rangle_{\mathcal{H}} = \left[ \begin{array}{c} 2n \\ \vdots \end{array} \right] \hat{\pi} = (2n + 1)^{\hat{n}} > 0. \]

Finally,

\[ \left| \langle \text{Drum}_s, 2n\rangle_U \right| \geq \frac{1}{(2n + 1)^s} \times \frac{(2n + 1)^2}{(2n + 1) \hat{n}} = (2n + 1)^{-\frac{\hat{n}}{s} + 2} \]

and $\limsup_{n \to \infty} \left| \langle \text{Drum}_s, 2n\rangle_U \right|^{\frac{1}{n}} = 1$ follows.

**Case 2 where $s$ is odd:** Since one must have $s \geq 3$, we will take
Then, by pushing the epsilons, we have

\[ \langle A | B \rangle_H = (-1)^{n(n-3)/2} \langle \cdots \rangle \]

Hence
with $2n$ everywhere. By Part 3) of Proposition \ref{prop:3} and Corollary \ref{cor:4}

\[
\langle A|B \rangle^2_{\mathcal{H}} = \left[ \begin{array}{ccc}
2n & 2n & 2n \\
2n & 2n & 2n \\
2n & 2n & 2n \\
\end{array} \right]^{\pi t} \]

\[
= (2n + 1)^4 \times \left| \left( \frac{2n}{\dim(\pi t)} \right) \right|^2.
\]

Whereas

\[
\langle B|B \rangle_{\mathcal{H}} = (2n + 1)^{\frac{3}{2}} \times \left| \left( \frac{2n}{\dim(\pi t)} \pi t \right) \right|^2.
\]

\[
= (2n + 1)^{s - 1} > 0.
\]

Therefore

\[
\left| \langle \text{Drum}_s, 2n \rangle^U \right| \geq \frac{1}{(2n + 1)^s} \times (2n + 1)^4 \times \left| \left( \frac{2n}{\dim(\pi t)} \right) \right|^2 \times \frac{1}{(2n + 1)^{\frac{s-1}{2}}}
\]

and \( \limsup_{n \to \infty} \left| \langle \text{Drum}_s, 2n \rangle^U \right|^\frac{1}{n} \geq 1 \) follows from Lemma \ref{lem:5} \hfill \blacksquare

This completes the proof of Theorem \ref{thm:3}

\section{Proof of the main theorem}

We now tackle the proof of Theorem \ref{thm:4} using an elaboration of the Cauchy-Schwarz argument of \ref{sec:7.1.2}. The key ingredient is the smooth orientation \( \mathcal{O} \). It is enough to consider connected spin networks \((\Gamma, \gamma)\) without loops and which are not reduced to a trivial component. Let \( G \) be the underlying cubic graph and choose a smooth orientation \( \mathcal{O} \) of \( G \) by Proposition \ref{prop:4} as well as a gate signage \( \tau \). By Corollary \ref{cor:4}

\[
\left| \langle \Gamma, \gamma \rangle^U \right| = \left( \prod_{v \in \Gamma(G)} \frac{1}{\sqrt{\dim(v)}} \right) \times \left| \langle G, \mathcal{O}, \tau, \gamma \rangle^{\pi t} \right|.
\]
Now let $V_{\pi}(G) \subset V(G)$ be the set of vertices with indegree 2 and outdegree 1. Let $V_{\iota}(G) \subset V(G)$ be the complement, i.e., the set of vertices with indegree 1 and outdegree 2. Since there are as many incoming half-edges as there are outgoing half-edges
\[ 2|V_{\pi}(G)| + |V_{\iota}(G)| = |V_{\pi}(G)| + 2|V_{\iota}(G)| \]
and thus
\[ |V_{\pi}(G)| = |V_{\iota}(G)| = \frac{|V(G)|}{2}. \]

By pulling the vertices of $V_{\pi}(G)$ on one side and those of $V_{\iota}(G)$ on the other, we have
\[
\left| \langle \Gamma, \gamma \rangle \right|^U = \left( \prod_{v \in V(G)} \frac{1}{\sqrt{\dim(v)}} \right) \times \left| \langle \tilde{V}_{\iota}(G) \rangle \right| \times \left| \langle \tilde{V}_{\pi}(G) \rangle \right|^{\pi_{\iota}}
\]
where we omitted the orientations of the crossing edges in the picture, since some edges go up and others go down. Note that the $V_{\pi}(G)$ and $V_{\iota}(G)$ induced subgraphs need not be connected. There is a Hilbert space $\mathcal{H}$ associated to the splitting indicated by the dotted line. One can therefore use the Cauchy-Schwarz inequality as in §7.1.2, with the effect that
\[
\left| \langle \Gamma, \gamma \rangle \right|^U \leq \left( \prod_{v \in V(G)} \frac{1}{\sqrt{\dim(v)}} \right) \times \left\{ \left| \langle \tilde{V}_{\iota}(G) \rangle \right| \times \left| \langle \tilde{V}_{\pi}(G) \rangle \right|^{\pi_{\iota}} \right\}^{\frac{1}{2}}
\]
where $\tilde{V}_{\iota}(G)$ subgraph is the mirror image of the $V_{\iota}(G)$, and likewise for $\tilde{V}_{\pi}(G)$. This means that the decorations and gate signage are preserved, but the edge orientations are reversed. Now consider a graph such as
obtained by cutting the crossing edges as in (17) and filling the ends by 1-valent vertices. Such a possibly disconnected directed graph only has three types of vertices:

A (very short) moment of thought will convince the reader of the following key observation.

**Key fact:** A connected component for such a digraph must either be a (binary) tree coherently oriented towards the root such as

![Type I](image1)

or a collection of such trees attached to a unique coherently oriented central cycle of length at least 2, such as:

![Type II](image2)

Indeed, such combinatorial structures are well-known to the practitioners of the theory of species [14]. They correspond to the species of endofunctions where each element has at most two preimages. Previous experience with
this type of graphs \[1\] \[2\] was very useful in making this observation. Note that central cycles have length at least two since, otherwise, the \(V_{\pi}(G)\) subgraph would have a loop and therefore \(G\) also which is forbidden by the hypotheses.

Let \(k_I\) be the number of tree components of type I in \(V_{\pi}(G)\). Let \(k_{II}\) be the number of tree components of type II in \(V_{\pi}(G)\). For \(i, 1 \leq i \leq k_I\), let us denote by \(c_i\) the decoration of the root edge of type I component number \(i\) as in

\[
v_i \quad c_i = \dim(v_i) - 1
\]

For \(j, 1 \leq j \leq k_{II}\), let \(s_j \geq 2\) be the length of the central cycle of type II component number \(j\), and let \(a_{ij}, b_{ij}, 1 \leq i \leq s_j\), be the decorations around the cycle and on the edges incident to the cycle as in

\[
\begin{array}{c}
b_{1j} \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
a_{1j} \\
a_{3j}
\end{array} \\
\begin{array}{c}
\vdots \\
\end{array}
\]

Also note the labelling of the vertices \(v_{ij}\), so that \(\dim(v_{ij}) = a_{ij} + 1\). Then, by repeated application of Part 3) of Proposition \[3\] we have

\[
\langle \cdots \rangle_{\pi I} = \prod_{i=1}^{k_I} (c_i + 1) \times \prod_{j=1}^{k_{II}} \langle \cdots \rangle_{\pi I}
\]

with a positive sign in front. Indeed, all \(\langle \cdots \rangle_{\pi I}\) involved are squares of norms. Therefore, by a coarse application of Lemma \[6\] where we throw away the
denominator,

\[
\langle \tilde{V}_{\pi}(G) \rangle_{\pi} \leq \prod_{i=1}^{k_1} (c_i + 1) \times \prod_{j=1}^{k_{II}} \left[ \min_{1 \leq i \leq s_j} (a_{ij}) + 1 \right]^2
\]

because \( s_j \geq 2 \) for any \( j, 1 \leq j \leq k_{II} \). As a result,

\[
\langle V_{\pi}(G) \rangle_{\pi} \leq \prod_{i=1}^{k_1} \dim(v_i) \times \prod_{j=1}^{k_{II}} \left[ \prod_{i=1}^{s_j} \dim(v_{ij}) \right]
\]

\[
\leq \prod_{v \in V_{\pi}(G)} \dim(v)
\]

where the last bound is interpreted in the context of the original unsplit CG network \((G, \mathcal{O}, \tau, \gamma)\). Exactly the same reasoning shows

\[
\langle V_{\iota}(G) \rangle_{\pi} \leq \prod_{v \in V_{\iota}(G)} \dim(v)
\]

and thus \( |\langle \Gamma, \gamma \rangle_U| \leq 1 \) since \( V_{\pi}(G) \) and \( V_{\iota}(G) \) form a partition of \( V(G) \).

This concludes the proof of Theorem \( \Box \) Note that an interesting question raised by the proof is that of optimizing the choice of orientation \( \mathcal{O} \). If all decorations are equal to \( 2n \) as in the setting of Conjecture \( \Box \) and if there is only one component of type I in \( V_{\pi}(G) \) and also in \( V_{\iota}(G) \) one can easily
see that our method provides a bound $|\langle \Gamma, 2n \rangle_U| \leq (2n + 1)^{1 - \frac{1}{2}|V(G)|}$. This is the case for standard 3n-j symbols obtained by comparing two binary coupling schemes. Not all cubic graphs can be decomposed in this way [36]. This is related to an old conjecture by François Jaeger [54]. A tantalizing question which is left for future investigations is whether one can transpose the methods of this article to quantum spin networks and the colored Jones polynomial of knots, with the hope of improving known upper bounds for the volume conjecture [43]. We expect this investigation to benefit from the development of a quantum version of CIT for binary forms by Frank Leitenberger [62] which could usefully complement the more widely known techniques of [57, 65, 19].

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