Motivic cohomology of fat points in Milnor range via formal and rigid geometries

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Abstract
We present a formal scheme based cycle model for the motivic cohomology of the fat points defined by the truncated polynomial rings $k[t]/(t^m)$ with $m \geq 2$, in one variable over a field $k$. We compute their Milnor range cycle class groups when the field has sufficiently many elements. With some aids from rigid analytic geometry and the Gersten conjecture for the Milnor $K$-theory resolved by M. Kerz, we prove that the resulting cycle class groups are isomorphic to the Milnor $K$-groups of the truncated polynomial rings, generalizing a theorem of Nesterenko-Suslin and Totaro.

Keywords Algebraic cycle · Chow group · Singular scheme · Formal scheme · Tate algebra · Rigid geometry · Motivic cohomology · Milnor $K$-theory · Algebraic de Rham cohomology · De Rham–Witt form

Mathematics subject classification Primary 14C25; Secondary 19D45 · 13F25 · 14B20 · 16W60

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1 Introduction

In this article, we propose a new cycle model through formal geometry for the motivic cohomology of the singular non-reduced scheme $Y := \text{Spec} (k_m)$, where $k_m := k[t]/(tm)$, and we compute its Milnor range for all base fields $k$ except a finite number of finite fields, using some ideas from rigid analytic geometry. The main theorem is summarized in Theorem 1.2.1 below.

1.1 Background and motivation

The motivic cohomology of smooth $k$-schemes is given by higher Chow groups of S. Bloch [5], but the higher Chow complexes fail to give correct models of motivic cohomology when the schemes have singularities.

For such non-smooth schemes, approaches via additive higher Chow groups ([6, 38, 42]) and cycles with modulus ([4]) were proposed as part of potential extensions. Various theoretical advances were made (e.g. [25–27, 32]) in this direction. All of them use the idea of “modulus conditions” imposed on cycles.

Recently in [41], the author and S. Ünver had tried a very different approach for $Y = \text{Spec} (k_m)$. Embedding $\text{Spec} (k_m)$ into the henselian scheme $\text{Spec} (k[[t]])$, the article *ibid.* considered certain cycles on $\text{Spec} (k[[t]]) \times \square^m_k$ subject to a few requirements, where $\square_k := \mathbb{P}^1 \setminus \{1\}$. On the cycles, the *ibid.* puts a “mod $tm$ equivalence relation”, and the homology group of the resulting complex in the Milnor range was identified with the Milnor $K$-group $K^M_n (k_m)$ when $k$ is of characteristic 0.

The author had received a few feedbacks since then. One of them from Fumiharu Kato at a social meeting in Tokyo, Japan in 2019 was that, for future generalizations to general singular schemes, it may be better to use formal schemes. For instance, instead of the scheme $\text{Spec} (k[[t]])$, the formal scheme $\hat{X} = \text{Spf} (k[[t]])$ is the formal neighborhood of the immersion $Y = \text{Spec} (k_m) \hookrightarrow \mathbb{A}^1$, so Kato’s suggestion seems to be naturally consistent with the philosophy of the construction of the algebraic de Rham cohomology in A. Grothendieck [18] and R. Hartshorne [20].

Once one accepts to try this perspective via formal schemes, it becomes necessary to ask what the right notions of (higher Chow) cycles for formal schemes are. To have a higher Chow cycle-style theory, for instance, one should take the fiber product $\text{Spf} (k[[t]]) \times_k \square^m_k$ in the category of formal schemes, instead of the fibre product $\text{Spec} (k[[t]]) \times_k \square^m_k$ of schemes. Here,
one needs to answer what requirements one should impose on the closed formal subschemes to define the right theory.

The first nontrivial test for a potential theory is to see whether its homology in the Milnor range for \( Y \) is the Milnor \( K \)-group \( K^M_n(k_m) \). One far harder test is to show that it extends to a functorial theory on the category of schemes of finite type over \( k \), where on the smooth schemes, it coincides with the classical higher Chow theory.

The objective of this article is to prove that the theory we build here passes the first test in the Milnor range. The above harder test of constructing a functorial theory is studied in the sequel [39]. A related discussion on algebraic \( K \)-theory of singular schemes is studied in [40].

### 1.2 The central result

More specifically, we do the following in this article. Consider the formal scheme \( \hat{X} = \text{Spf}(k[[t]]) \), with the closed immersion \( Y := \text{Spec}(k_m) \hookrightarrow \hat{X} \). Consider the formal scheme \( \hat{X} \times_k \mathbb{P}^n_k \).

We will construct (Definition 2.4.5) a complex of abelian groups

\[
\cdots \to z^q(\hat{X} \text{ mod } Y, n + 1) \overset{\partial}{\to} z^q(\hat{X} \text{ mod } Y, n) \overset{\partial}{\to} z^q(\hat{X} \text{ mod } Y, n - 1) \to \cdots,
\]

where \( z^q(\hat{X} \text{ mod } Y, n) \) is the quotient

\[
z^q(\hat{X} \text{ mod } Y, n) = \frac{z^q(\hat{X}, n)}{\mathcal{M}(\hat{X}, Y, n)}
\]

of the group \( z^q(\hat{X}, n) \) consisting of certain “admissible cycles” of codimension \( q \) on the formal scheme \( \hat{X} \times_k \mathbb{P}^n_k \) modulo a subgroup \( \mathcal{M}(\hat{X}, Y, n) \) defined using the embedding \( Y \hookrightarrow \hat{X} \). The constructions of the groups \( z^q(\hat{X}, n) \) and \( \mathcal{M}(\hat{X}, Y, n) \) are rapidly sketched in Sect. 1.3 below. They are defined in full in Definitions 2.2.2 and 2.4.1. The homology \( H_n(z^q(\hat{X} \text{ mod } Y, \bullet)) \) of the complex (1.2.1) is denoted by any one of

\[
\text{CH}^q(k_m, n) = \text{CH}^q(Y, n) = \text{CH}^q(\hat{X} \text{ mod } Y, n) = H_n(z^q(\hat{X} \text{ mod } Y, \bullet)),
\]

where the first two are written in boldface to distinguish them from the usual higher Chow group [5]. The central result of the article is to show that, under a mild assumption on the field \( k \), we have an isomorphism \( K^M_n(k_m) \cong \text{CH}^q(k_m, n) \). The following (from Proposition 2.4.6 and Theorem 5.2.1) gives more details. This includes a generalization to \( \text{Spec}(k_m) \) of theorems of Nesterenko–Suslin [37] and B. Totaro [47] proven for \( \text{Spec}(k) \):

**Theorem 1.2.1** Let \( m \geq 2, n \geq 1 \) be integers, \( k \) be a field, and \( k_m = k[[t]]/(t^m) \).

1. If \( q > n \), then we have \( \text{CH}^q(\text{Spec}(k_m), n) = 0 \).
2. If \( q = n \), there is a homomorphism, called the graph map

\[
gr_{km} : K^M_n(k_m) \to \text{CH}^n(\text{Spec}(k_m), n),
\]

and it is an isomorphism of abelian groups when the cardinality \( |k| \) is sufficiently large.

The phrase “sufficiently large” in Theorem 1.2.1 means that the cardinality of the field \( k \) is larger than a positive integer \( M_0 \) given by M. Kerz [30, Proposition 10-(5), p.181] so that the Milnor \( K \)-theory \( K^M_n(k[[t]]) \) and the improved Milnor \( K \)-theory \( \tilde{K}_n^M(k[[t]]) \) of Gabber–Kerz coincide. For instance, when \( n = 1 \), this holds for an arbitrary field \( k \).
1.3 The cycles

Recall \( \hat{X} = \text{Spf} \left( k[[t]] \right) \) and \( Y = \text{Spec} \left( k_m \right) \). We sketch the construction of \( z^q \left( \hat{X} \mod Y, n \right) \) of (1.2.2). The formal scheme \( \hat{X} \) has the largest ideal of definition \( \mathcal{I}_0 \) given by \( t \). The reduction by \( \mathcal{I}_0 \) is \( \hat{X}_{\text{red}} = \text{Spec} \left( k \right) \).

1.3.1 The two requirements

The group \( z^q \left( \hat{X}, n \right) \) is (Definition 2.2.2) the free abelian group generated by the integral closed formal subschemes \( Z \subset \hat{X} \times_k \square^n \) of codimension \( q \), satisfying two conditions (GP) and (SF).

The general position property (GP) requires that the intersections of \( Z \) with all faces \( \hat{X} \times F \) are proper, where \( F \subset \square^n \) are given by a finite set of equations of the form \( \{ y_i = \epsilon_i \} \) with \( \epsilon_i \in \{ 0, \infty \} \). The special fiber property (SF) requires that the intersections of \( Z \) with all special fiber faces \( \hat{X}_{\text{red}} \times F = \text{Spec} \left( k \right) \times F \) are proper.

1.3.2 The mod \( Y \)-equivalence

We discuss the subgroup \( M^q \left( \hat{X}, Y, n \right) \subset z^q \left( \hat{X}, n \right) \) of (1.2.2), which defines the mod \( Y \)-equivalence on the cycles.

This subgroup is generated by the cycles of the form

\[
[A_1] - [A_2],
\]

where \( (A_1, A_2) \) runs over all possible pairs of coherent \( \mathcal{O}_{\hat{X} \times \square^n} \)-algebras whose associated cycles belong to \( z^n \left( \hat{X}, n \right) \), such that there is an isomorphism

\[
A_1|_{Y \times \square^n} \simeq A_2|_{Y \times \square^n}
\]

(1.3.1) as \( \mathcal{O}_{Y \times \square^n} \)-algebras (see Definition 2.4.1).

1.4 Benefits of the formal geometry model

One point of using the formal scheme model in studying the motivic cohomology of \( \text{Spec} \left( k[[t]]/ (t^n) \right) \) is its natural generalizability to more general schemes \( Y \) of finite type in \( \text{Sch}_k \). This aspect is studied in [39] in terms of some ideas of the derived algebraic geometry of J. Lurie.

Yet, there is another aspect that we mention, compared to the scheme model in [41]. In \textit{ibid.}, the generating cycles in the Milnor range were integral closed subschemes in \( \text{Spec} \left( k[[t]] \right) \times \square^n \) subject to the three requirements: (i) the proper intersection with the faces, (ii) the proper intersection with the special fiber times faces, and (iii) the extra requirement that each integral cycle is \textit{proper} over \( k[[t]] \), thus finite over \( k[[t]] \).

The first two conditions (i) and (ii) correspond to our requirements (GP) and (SF), respectively. However, to exclude potentially undesirable cycles, the extra condition (iii) was necessary there. One can ask what happens to this condition in our formal scheme model.

One interesting aspect we observe in the formal scheme model is that, in fact, in the Milnor range we \textit{automatically} have the finiteness of the generating integral cycles over \( k[[t]] \). This surprising conclusion in the formal scheme model is deduced in Sect. 3.2, especially in
Corollary 3.2.3. This argument does not work in the scheme model, as we really exploit the situation in the formal geometry.

In connection to this, in Example 2.2.6, we consider an integral closed cycle \(Z_1 \subset \text{Spec} (k[[t]]) \times \mathfrak{p}_{k}^n\) given by two concrete equations, such that it satisfies the two conditions (i), (ii) but not (iii); \(Z_1\) is not finite over \(k[[t]]\). It is forcibly removed in the scheme model by (iii).

In contrast, the corresponding formal scheme \(Z_2 \subset \text{Spf} (k[[t]]) \times \mathfrak{p}_{k}^n\) given by the same equations is empty in the formal scheme model, thanks to the convergence of a sequence in the \((t)\)-adic topology.

This shows the formal scheme model of this article has various stronger points over the scheme model, e.g. of [41].

1.5 Connections to rigid geometry

We explain how and where the rigid analytic geometry enters into the landscape of algebraic cycles.

Using the automorphism \(y \mapsto y/(y-1)\) of \(\mathbb{P}^1\), identify \(\mathfrak{p}_{k}^n\) with \(\mathbb{A}^1\). So for \(y_i' := y_i/(y_i-1)\) where \((y_1', \ldots, y_n') \in \mathfrak{p}_{k}^n\), the ambient space \(\hat{X} \times \mathfrak{p}_{k}^n\) can be seen as the formal spectrum of the ring \(k[[t]](y_1', \ldots, y_n')\) of the \textit{restricted formal power series}, i.e. the formal power series \(\sum_I a_I y^I\) in \(y_1', \ldots, y_n'\) such that the coefficients \(a_I \in k[[t]]\) converge to 0 in the \((t)\)-adic norm as the degree \(|I| \to \infty\) (see Example 2.2.1).

Here, not all prime ideals of \(k[[t]](y_1', \ldots, y_n')\) are suitable for our cycles. In (SF) in Sect. 1.3.1 or Definition 2.2.2, we ruled out those behave badly with respect to the ideals of definition of \(\hat{X} \times_k \mathfrak{p}_{k}^n\). Those ideals give the “special fibers” \(\text{Spec} (k) \times_k F\) for faces \(F \subset \mathfrak{p}_{k}^n\), and the condition (SF) requires the proper intersection of the cycles with such special fibers.

In particular, via localization to the “generic fibers”, our admissible prime ideals of height \(n\) in the Milnor range define prime ideals of height \(n\) of the Tate algebra \(T_n = k((t))\{y_1', \ldots, y_n'\}\) over the non-archimedean complete discrete valued field \(k((t))\).

The Tate algebras were introduced by J. Tate in [46]. Since \(T_n\) has the Krull dimension \(n\), the height \(n\) primes are its maximal ideals. They define finite dimensional Banach algebras over \(k((t))\). One can show that in \(T_n\), all such maximal ideals are complete intersections given by \(n\) polynomials in \(y_1', \ldots, y_n'\). Thus, taking closures, we deduce algebraic generators for the integral admissible cycles in \(z^n(\hat{X}, n)\). This is covered in Sect. 3.1. This allows us to see in Sect. 3.2 that each integral cycle \(\hat{Z}\) can be written as \(\text{Spf} (R)\), where \(R = k[[t]](y_1', \ldots, y_n')/P\) for some prime ideal, and it is finite over \(k[[t]]\).

Another place, where the rigid geometry plays a role, is in the middle of the proof that the graph map (1.2.3) is an isomorphism.

As a first (eventually wrong) attempt, one might try to “construct” its inverse

\[
\text{CH}^n(\text{Spec} (k_m), n) \to K^M_n(k_m),
\]

as follows: let \(\tilde{y}_i\) be the image of \(y_i\) in \(R\). Then one may try to define

\[
\rho : z^n(\text{Spf} (k[[t]]), n) \to K^M_n(k[[t]])
\]

by sending an integral cycle \(\hat{Z}\) to the norm \(N_{R/k[[t]]}(\tilde{y}_1, \ldots, \tilde{y}_n) \in K^M_n(k[[t]])\) of the Milnor symbol \(\{\tilde{y}_1, \ldots, \tilde{y}_n\} \in K^M_n(R)\). Then one may try to show that its composition with the natural surjection \(K^M_n(k[[t]]) \to K^M_n(k[t]/(t^m))\) kills the boundary \(\partial z^n(\text{Spf} (k[[t]]), n + 1)\) as well as the subgroup \(\mathcal{M}^n(\text{Spf} (k[[t]]), Y, n)\). 
Unfortunately, the above proposed story does not work well. The problem is that we do not know whether there is the norm map $N_{R/k[[t]]} : K_n^M(R) \to K_n^M(k[[t]])$ unless $k[[t]] \to R$ is finite étale (see M. Kerz [30, Propositions 3, 10]). In general, its existence is unknown at present as far as the author is aware of.

To get around this difficulty, we take a detour through the rigid geometry via the generic fiber of the cycles, and use the Gersten conjecture for Milnor $K$-theory proven by M. Kerz in *ibid.*

We first (see Sect. 4.2) construct the composite

$$\tilde{\Upsilon} : z^n(Spf(k[[t]]), n) \xrightarrow{\eta} CH^n(Sp(k((t))), n) \to K_n^M(k((t))),$$

where the middle term is a rigid analytic analogue of the higher Chow group (see Definition 4.2.1), $\eta$ is the localization map, and $\tilde{\Upsilon}$ is constructed using the norm map of Bass–Tate [3] and Kato [28] on the Milnor $K$-theory for fields. We check that the image of $\tilde{\Upsilon}$ belongs to the subgroup $K_n^M(k[[t]])$ of $K_n^M(k((t)))$. The rest of the paper is, roughly speaking, about proving that this map descends mod $t^n$, and that $gr_{km}$ in (1.2.3) is an isomorphism.

These offer an interesting new observation: the motivic cohomology of the truncated polynomials can be partially understood through the rigid analytic geometry. Already in [41], the topology given by the non-archimedean $(t)$-adic norm was very convenient. The author hopes that the new door opened here could be useful for the researchers in the field.

### 1.6 The relative parts

Taking the relative parts of the groups of Theorem 1.2.1, we deduce the following result, stated as Theorem 5.2.2:

**Theorem 1.6.1** Let $m \geq 1, n \geq 1$ be integers. Let $k$ be a field of characteristic 0.

Then we have an isomorphism from the group of the big de Rham–Witt forms

$$W_m \Omega_k^{n-1} \sim CH^n((Spec(k_{m+1})), (t)), n)$$

(1.6.1)

to the relative group of the new higher Chow group.

While we guess that the statement of Theorem 1.6.1 may stay valid for any field $k$, at this moment the author was unable to give a proof of it without imposing the assumption on the characteristic. It should be removed in the future. The group of the big de Rham–Witt forms on the left of (1.6.1) is defined by Hesselholt–Madsen [24], while the relative group on the right of (1.6.1) is defined to be (see (5.2.2))

$$\text{ker}(CH^n(Spec(k_{m+1}), (t)), n) \xrightarrow{\text{ev}} CH^n(k, n)),$$

where $CH^n(k, n)$ is the usual higher Chow group of S. Bloch [5].

This Theorem 1.6.1 is an analogue of the theorem of K. Rülling [42] proven in terms of additive higher Chow groups (see also [32] for its higher dimensional generalization). An interesting point is that our theory in this article provides a single unified platform for all of Nesterenko–Suslin [37], Totaro [47] and Rülling [42], and it is also relatively conceptual compared to the approaches through “cycles with modulus” in the literature.

**Conventions** In this paper $k$ is a given arbitrary field, unless said otherwise. For the set inclusion symbol $\subset$, this allows the equality $=$ as well. The strict inclusion is denoted by $\subset$, while we won’t use the symbol $\subseteq$. 

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The notations $k_m$, $Y$, $X$, $\hat{X}$ in this article exclusively mean the following: for an integer $m \geq 1$, we let $k_m := k[t]/(t^m) = k[[t]]/(t^m)$, $Y = \text{Spec} (k_m)$, $Y \hookrightarrow X = \mathbb{A}^1$ is the closed immersion given by $k[t] \to k_m$, and $\hat{X}$ is the completion of $X$ along $Y$, so that $\hat{X} = \text{Spf} (k[[t]])$.

## 2 Some definitions and recollections

In Sect. 2, we discuss the notion of higher Chow cycles on some noetherian affine formal $k$-schemes of finite Krull dimension. Based on these, in Sect. 2.4 we define the main objects of studies in the article, the groups $\text{CH}^q (k_m, n)$. In Sect. 2.5, we also discuss basic definitions and results on Tate algebras, that will be used later to compute the Milnor range $\text{CH}^n (k_m, n)$, when $q = n$.

### 2.1 Noetherian affine formal schemes and cycles

Let $\mathfrak{X} = \text{Spf} (A)$ be a noetherian affine formal scheme of finite Krull dimension.

**Definition 2.1.1** An integral closed formal subscheme $\mathfrak{Y} \subset \mathfrak{X}$ is of the form $\text{Spf} (A/P)$ for a prime ideal $P \subset A$.

The naive group of algebraic cycles $\mathcal{Z}_s (\mathfrak{X})$ on $\mathfrak{X}$ is defined to be the free abelian group on such integral closed formal subschemes. This is equivalent to the group of algebraic cycles on the scheme $\text{Spec} (A)$, that we call the associated scheme of $\mathfrak{X}$. Consequently, we have the same notion of intersections of cycles, and proper intersections of them.

Its subgroup of $d$-dimensional cycles is denoted by $\mathcal{Z}_d (\mathfrak{X})$. In case $\mathfrak{X}$ is equidimensional (i.e. $A$ is equidimensional), we denote the group of the codimension $q$-cycles by $\mathcal{Z}^q (\mathfrak{X})$.

Not all of the cycles in $\mathcal{Z}_s (\mathfrak{X}) = z_s (\text{Spec} (A))$ are suitable for our discussions. A better one is the subgroup $\mathcal{Z}_s (\mathfrak{X}) \subset \mathcal{Z}_s (\mathfrak{X})$ of cycles that intersect properly with $\mathfrak{X}_{\text{red}}$. This $\mathfrak{X}_{\text{red}}$ is the closed subscheme of $\mathfrak{X}$ given by the largest ideal of definition, which exists by EGA I [16, Proposition (10.5.4), p.187].

When $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, we have the associated cycle $[\mathcal{F}] \in \mathcal{Z}_s (\mathfrak{X})$ via minimal associated primes (see e.g. [44, Sect. 02QV]). This is possible because $\mathcal{F} = M^\Delta$ for a finitely generated $A$-module $M$ (see EGA I [16, Proposition (10.10.2), p.201]), and the associated cycle of $M$ on $\text{Spec} (A)$ defines $[\mathcal{F}]$.

### 2.2 Higher Chow cycles over affine formal schemes

Let $\square_k := \mathbb{P}^1_k \setminus \{1\}$. Let $\square^n_k := \text{Spec} (k)$. For $n \geq 1$, let $\square^n_k$ be the $n$-fold fiber product of $\square_k$ over $k$. Let $(y_1, \cdots, y_n) \in \square^n_k$ be the coordinates.

Recall that in the cubical version of higher Chow theory, using the $k$-rational points $\{0, \infty\} \subset \square_k$, we define a face $F \subset \square^n_k$ to be a closed subscheme given by a system of equations of the form $\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}$ for an increasing sequence $1 \leq i_1 < \cdots < i_s \leq n$ of indices and $\epsilon_j \in \{0, \infty\}$. When the set of equations is empty, we let $F = \square^n_k$ by convention. For $s = 1$, we have the codimension $1$ faces given by $\{y_i = \epsilon\}$ for some $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$.

For an equidimensional noetherian affine formal $k$-scheme $\mathfrak{X}$ of finite Krull dimension, consider the fiber product $\mathfrak{X} \times_k \square^n_k$, which we often write $\square^n_k$. The fiber product exists in
the category of formal schemes; see EGA I [16, Proposition (10.7.3), p.193]. A face $F_{\mathcal{X}}$ of $\Box^n_{\mathcal{X}}$ is given by $\mathcal{X} \times_k F$ for a face $F \subset \Box^n_k$.

**Example 2.2.1** Suppose $\mathcal{X} = \text{Spf}(A)$. Using the automorphisms of $\mathbb{P}^1$ given by $y_i \mapsto y'_i := y_i/(y_i - 1)$ for $1 \leq i \leq n$, we may identify $\Box^n_{\mathcal{X}}$ with the integral noetherian affine formal $k$-scheme $\mathcal{X} \times_k k[[y]]$ given by the ring $A[y_1, \ldots, y_n]$ of restricted formal power series.

In case $A = k[[t]]$ with the $(t)$-adic topology, this ring is

$$k[[t]][y'_1, \ldots, y'_n] = \lim_{\longrightarrow} k_m[y'_1, \ldots, y'_n].$$

This ring can also be regarded as the subring of $k[[t]][y_1, \ldots, y_n]]$, whose members are formal power series

$$p(y'_1, \ldots, y'_n) = \sum_I \alpha_I y'^I$$

with $\alpha_I \in k[[t]]$ for multi-indices $I = (i_1, \ldots, i_n)$ with $i_j \geq 0$, such that when $|I| = i_1 + \cdots + i_n \rightarrow \infty$, we have $\alpha_I \rightarrow 0$ in the $(t)$-adic topology.

We define cubical higher Chow cycles over the formal scheme $\mathcal{X}$ (cf. [5]) as follows.

**Definition 2.2.2** Let $\mathcal{X}$ be an equidimensional noetherian affine formal $k$-scheme of finite Krull dimension. Let $q, n \geq 0$ be integers. Let $\mathcal{X}_{\text{red}}$ be the closed subscheme given by the largest ideal $\mathcal{I}_0$ of definition of $\mathcal{X}$.

Let $\mathcal{Z}^q(\mathcal{X}, n) \subset \mathcal{Z}^q(\Box^n_{\mathcal{X}})$ be the subgroup generated by the codimension $q$ integral closed formal subschemes $\mathcal{Z} \subset \Box^n_{\mathcal{X}}$ subject to the following conditions:

**(GP)** (General Position) For each face $F \subset \Box^n_k$, $\mathcal{Z}$ intersects $\mathcal{X} \times F$ properly.

**(SF)** (Special Fiber) For each face $F \subset \Box^n_k$, we have

$$\text{codim}_{\mathcal{X}_{\text{red}} \times F} (\mathcal{Z} \cap (\mathcal{X}_{\text{red}} \times F)) \geq q.$$

A cycle in $\mathcal{Z}^q(\mathcal{X}, n)$ will be called admissible.

We have several remarks on the conditions we introduced in Definition 2.2.2.

**Remark 2.2.3** One notes that in addition to the usual general position condition (the condition (GP)) with respect to the faces for higher Chow cycles, we require the additional condition (SF), which is related to the “special fiber” given by the largest ideal of definition of $\mathcal{X}$. In case $\mathcal{X}$ is a scheme, regarded as a formal scheme, the requirement (SF) holds automatically, so that Definition 2.2.2 is compatible with the definition of the cubical version of the higher Chow cycles of [5], e.g. in [47].

**Remark 2.2.4** Note that by the special fiber condition (SF), the cycles given by open prime ideals of $A[y_1, \ldots, y_n]$ are all excluded from $\mathcal{Z}^q(\mathcal{X}, n)$, where $\mathcal{X} = \text{Spf}(A)$ and $y'_i := y_i/(y_i - 1)$ for $1 \leq i \leq n$ as before.

We make one elementary observation in Lemma 2.2.5 below. In an earlier version of the article, an analogue of this property was considered as part of the requirements in Definition 2.2.2, called the topological support condition (TS). Joseph Ayoub as well as the referee pointed out that, in fact, each integral cycle satisfies it automatically, so that it is redundant.

**Lemma 2.2.5** Let $\mathcal{X}$ be a noetherian formal scheme and let $\mathcal{Z} \subset \mathcal{X}$ be an integral closed formal subscheme. Let $\mathcal{I}$ be an ideal of definition of $\mathcal{X}$ and let $Z$ be the reduction of $\mathcal{Z}$ modulo $\mathcal{I}$. Then we have the equality of the topological supports $|\mathcal{Z}| = |Z|$. 

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**Proof** Let $\mathcal{J} \subset \mathcal{O}_\mathcal{X}$ be the ideal sheaf of $\mathcal{I}$. The ideal sheaf of $Z$ is given by $\mathcal{I} + \mathcal{J}$. However an ideal of definition of $\mathcal{I}$ is also given by $\mathcal{I} + \mathcal{J}$.

**Example 2.2.6** When $\mathcal{X} = \text{Spf} (A)$ for an adic ring $A$, one may ask whether we can consider the scheme $\text{Spec} (A) \times \square^n$ (that uses polynomials) instead of the formal scheme as we do here (that uses restricted formal power series).

Here is an example that shows why our formal scheme based model via $\text{Spf} (k[[t]]) \times \square^n_k$ considered in this article is better than the scheme based model via $\text{Spec} (k[[t]]) \times \square^n_k$, e.g. considered in [41]. The author thanks Joseph Ayoub and the referee for sharing their insights and encouraging him to try to write it down.

In the scheme model, we may have some undesirable cycles such as (here, $n = 2$ and $(y_1, y_2) \in \square^2$)

$\mathcal{Z}_1 : \{ y_1 = 1 + t, \ y_2 = t \}$,

which is not finite over $k[[t]]$. This is part of the reason why in [41], the extra finiteness requirement was imposed to exclude forcibly this kind of cycles.

On the other hand, in the formal scheme model of ours, the corresponding closed formal subscheme $\mathcal{Z}_2 \subset \text{Spf} (k[[t]]) \times \square^2_k$ given by the same defining equations,

$\mathcal{Z}_2 : \{ y_1 = 1 + t, \ y_2 = t \}$,

is empty, thus it vanishes itself.

To see this, applying the automorphism $y \mapsto y/(y - 1)$ of $\mathbb{P}^1$ that sends $(\square, \{ 0, \infty \})$ to $(\mathbb{A}^1, \{ 0, 1 \})$, we have a closed formal subscheme of $\text{Spf} (k[[t]])[y_1, y_2])$ defined by

$$\begin{cases} y_1 y_2 - 1 = 1 + t, & y_2 y_1 - 1 = t \end{cases},$$

which gives

$$\{ 1 - (y_1 - 1)t = 0, \ y_2 = (y_2 - 1)t \}.$$ (2.2.1)

However, the left hand side $1 - (y_1 - 1)t$ of (2.2.1) is a unit in $k[[t]][y_1, y_2]$ because the power series

$$1 + (y_1 - 1)t + (y_1 - 1)^2 t^2 + \cdots + (y_1 - 1)^i t^i + \cdots$$

is convergent in $k[[t]][y_1, y_2]$ and it is the inverse of $1 - (y_1 - 1)t$. In particular, the first equation of (2.2.1) cannot hold. Hence $\mathcal{Z}_2$ in $\text{Spf} (k[[t]]) \times \square^2_k$ is empty, and such $\mathcal{Z}_2$ is automatically eliminated in the formal scheme model, without forcibly imposing an extra condition as in the scheme model of [41].

Furthermore, later we will see that each integral admissible cycle in $\text{Spf} (k[[t]]) \times \square^n_k$ in the Milnor range is always finite over $k[[t]]$ in Corollary 3.2.3.

Coming back to our definition, we want to form a cycle complex:

**Definition 2.2.7** Suppose $n \geq 1$. For each $1 \leq i \leq n$ and each $\epsilon \in \{ 0, \infty \}$, let $\iota^\epsilon_i : \square^{n-1}_{\mathcal{X}} \hookrightarrow \square^n_{\mathcal{X}}$ be the closed immersion given by the single equation $\{ y_i = \epsilon \}$. For each integral cycle $\mathcal{Z} \in z_{\mathcal{X}}^q (\mathcal{X}, n)$, define the face $\partial^\epsilon_i (\mathcal{Z})$ to be the cycle associated to the intersection $\mathcal{Z} \cap \{ y_i = \epsilon \} = [(\iota^\epsilon_i)^* \mathcal{O}_{\mathcal{X}}]$. One defines $\partial := \sum_{i=1}^n (-1)^i (\partial_i^\infty - \partial_i^0)$ and by the usual cubical formalism, we check that $\partial^2 = 0$.

This gives a complex $(z_{\mathcal{X}}^q (\mathcal{X}, \bullet), \partial)$ of abelian groups. The complex $(z_{\mathcal{X}}^q (\mathcal{X}, \bullet), \partial)$ is defined to be the quotient of $(z_{\mathcal{X}}^q (\mathcal{X}, \bullet), \partial)$ by the subcomplex $z_{\mathcal{X}}^q (\mathcal{X}, \bullet)_{\text{deg}}$ of degenerate cycles generated by pull-backs by the projections $\square^n_{\mathcal{X}} \to \square^{n-1}_{\mathcal{X}}$ that drop one of the coordinates. We define $CH^q (\mathcal{X}, n)$ to be the $n$-th homology of the complex $(z_{\mathcal{X}}^q (\mathcal{X}, \bullet), \partial)$.
The group $\text{CH}^q(\mathcal{X}, n)$ is not the primary object of studies of the article. In this article where we mostly consider $\mathcal{X} = \hat{\mathcal{X}} = \text{Spf}(k[[t]])$, we still need to take a quotient of $z^q(\hat{\mathcal{X}}, \bullet)$ by a subcomplex that gives the "mod $Y$-equivalence". It will be given in Definition 2.4.1.

We mention that in the case of $\mathcal{X} = \text{Spf}(k[[t]])$ the following Tor independence property holds (see also Corollary 2.2.10 and Remark 2.4.2):

Lemma 2.2.8 Let $Y = \text{Spec}(k(m))$ and $\hat{\mathcal{X}} = \text{Spf}(k[[t]])$. Let $\mathcal{F}$ be a coherent sheaf on $\square^n_{\mathcal{X}}$, whose associated cycle $[\mathcal{F}]$ belongs to $z^s(\hat{\mathcal{X}}, n)$.

Then we have $\text{Tor}_i^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{F}, \mathcal{O}_{\square^n_Y}) = 0$ for $i > 0$.

Proof As before, for $y'_i := y_i/(y_i - 1)$, note that we have the short exact sequence

$$0 \to k[[t]](y'_1, \ldots, y'_n) \to k[[t]](y'_1, \ldots, y'_n) \to \text{ker}(y'_1, \ldots, y'_n) \to 0,$$

which gives a free $\mathcal{O}_{\square^n_{\mathcal{X}}}$-resolution of $\mathcal{O}_{\square^n_Y}$:

$$0 \to \mathcal{O}_{\square^n_{\mathcal{X}}} \to \mathcal{O}_{\square^n_{\mathcal{X}}} \to \mathcal{O}_{\square^n_Y} \to 0.$$

Hence

$$\mathcal{F} \otimes_{\mathcal{O}_{\square^n_{\mathcal{X}}}} \mathcal{O}_{\square^n_Y} = \mathcal{F} \otimes_{\mathcal{O}_{\square^n_{\mathcal{X}}}} \left(\mathcal{O}_{\square^n_{\mathcal{X}}} \to \mathcal{O}_{\square^n_{\mathcal{X}}}ight) = \left(\mathcal{F} \to \mathcal{F}\right),$$

where the objects in the above complexes are placed in the homological degrees $+1$ and $0$, respectively. This shows that $\text{Tor}_i^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{F}, \mathcal{O}_{\square^n_Y}) = 0$ for $i > 1$, while $\text{Tor}_1^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{F}, \mathcal{O}_{\square^n_Y}) = \text{ker}(\times t^m : \mathcal{F} \to \mathcal{F})$.

It remains to prove the vanishing of $\text{Tor}_1$. There is a finite decreasing filtration of $\mathcal{O}_{\square^n_{\mathcal{X}}}$-submodules

$$0 = \mathcal{F}_r \subset \mathcal{F}_{r-1} \subset \cdots \subset \mathcal{F}_0 = \mathcal{F}$$

such that for each $0 \leq j \leq r - 1$, we have $\mathcal{F}_j/\mathcal{F}_{j+1} \simeq \mathcal{O}_{\mathcal{Z}_j}$, as $\mathcal{O}_{\square^n_{\mathcal{X}}}$-modules, for some integral closed formal subscheme $\mathcal{Z}_j \subset \square^n_{\mathcal{X}}$ (see [44, Lemma 01YF]).

We prove the vanishing of $\text{Tor}_1$ by induction on $r$.

First consider the case when $r = 1$, namely $\mathcal{F} = \mathcal{O}_3$ for an integral cycle $3 \in z^s(\hat{\mathcal{X}}, n)$. Since $3$ is integral and it intersects the closed subscheme $\square^n_Y$ properly by the condition (SF) of Definition 2.2.2, the element $i$ is not a zero-divisor on $\mathcal{O}_3$. In particular, we have $\text{ker}(\times t^m : \mathcal{O}_3 \to \mathcal{O}_3) = 0$, thus,

$$\text{Tor}_1^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{O}_3, \mathcal{O}_{\square^n_Y}) = 0.$$ (2.2.2)

When $r > 1$, suppose we have the vanishing $\text{Tor}_1^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{F}_j, \mathcal{O}_{\square^n_Y}) = 0$ for all $j = 1, \ldots, r$. We want to prove its vanishing for $j = 0$. Indeed, we have the short exact sequence of $\mathcal{O}_{\square^n_{\mathcal{X}}}$-modules

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{O}_3 \to 0$$

from which we deduce part of the Tor long exact sequence

$$\cdots \to \text{Tor}_1^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{F}_1, \mathcal{O}_{\square^n_Y}) \to \text{Tor}_1^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{F}_0, \mathcal{O}_{\square^n_Y}) \to \text{Tor}_1^{\mathcal{O}_{\square^n_{\mathcal{X}}}}(\mathcal{O}_3, \mathcal{O}_{\square^n_Y}) \to \cdots.$$
Here the first term vanishes by the induction hypothesis, and the third term vanishes by (2.2.2). Hence the middle term also vanishes. Since $\mathcal{F}_0 = \mathcal{F}$, this proves the desired assertion of the lemma. \qed

**Remark 2.2.9** As was also remarked by the referee, a bit more general statement than the one given in Lemma 2.2.8 may hold for various regular adic rings $A$. Since we do not need it in this article, we resist the temptation to generalize it.

**Corollary 2.2.10** Let $Y = \text{Spec} (k_m)$ and $\widehat{X} = \text{Spf} (k[[t]])$. Let $A$ be a coherent $O_{\widehat{X}}$-algebra such that $[A] \in z^*(\widehat{X}, n)$.

Then we have an isomorphism of the graded $O_{\widehat{Y}}$-algebras

$$\text{Tor}_n^{O_{\widehat{X}}} (A, O_{\widehat{Y}}) \simeq \text{Tor}_0^{O_{\widehat{X}}} (A, O_{\widehat{Y}}).$$

**Proof** The graded $O_{\widehat{Y}}$-algebra structure on $\text{Tor}_n^{O_{\widehat{X}}} (A, O_{\widehat{Y}})$ is defined in [44, Sect. 068G]. Since all higher Tor’s vanish by Lemma 2.2.8, we deduce the corollary. \qed

### 2.3 Some finite push-forwards

The primary case we are interested in is when $X$ is $\widehat{X} = \text{Spf} (k[[t]])$, but in the middle of our arguments later (Sect. 4.4) we need a bit of push-forwards for cycles on some formal schemes that are finite over $\widehat{X}$. In Sect. 2.3, we minimize our discussions only to those needed in this article.

Recall (Fujiwara–Kato [14, Definition 4.7.1, p.341]) a morphism $f : X \to Y$ of noetherian formal schemes is **proper** if it is separated of finite type ([14, Definition 4.6.2, p.336]) and universally closed ([14, Definition 4.5.5, p.334]). When $X$ and $Y$ are both affine, a proper morphism $f$ between them is necessarily finite; recall ([14, Proposition 4.2.1, Definition 4.2.2, pp.324–325]) that a morphism $f : X \to Y$ of noetherian formal schemes is **finite**, if for any affine open subset $V = \text{Spf} (A)$ of $Y$, $f^{-1} (V)$ is affine of the form $\text{Spf} (B)$ for a ring $B$ that is a finitely generated $A$-module.

**Definition 2.3.1** For a finite morphism $f : X \to Y$ of noetherian affine formal $k$-schemes, define the finite push-forward on the naive groups (Sect. 2.1)

$$f_* : z_d (X) \to z_d (Y)$$

by sending an integral closed formal subscheme $Z \subset X$ of dimension $d$ to the $d$-dimensional cycle $[f_* O_Z]_d$ associated to the coherent sheaf $f_* O_Z$. Since $f$ is proper, the sheaf push-forward $f_* O_Z$ is a coherent $O_Y$-algebra.

Write $X = \text{Spf} (A)$ and $Y = \text{Spf} (B)$. By definition $z_d (X) = z_d (\text{Spec} (A))$ and $z_d (Y) = z_d (\text{Spec} (B))$ and $f$ is finite. Hence $f_*$ is identical to the usual finite push-forward on the cycles on the scheme $\text{Spec} (A)$ via the associated finite morphism $f : \text{Spec} (A) \to \text{Spec} (B)$.

We deduce that the cycle push-forward and the sheaf push-forward are compatible:

**Lemma 2.3.2** Let $f : X \to Y$ be a finite morphism of noetherian affine formal $k$-schemes. Let $\mathcal{F}$ be a coherent $O_X$-module. Then we have the equality of cycles

$$f_* [\mathcal{F}] = [f_* \mathcal{F}] \in z_d (Y),$$

where the left hand side is the push-forward of the cycle $[\mathcal{F}]$, while the right hand side is the cycle $[f_* \mathcal{F}]$ associated to the push-forward sheaf $f_* \mathcal{F}$. \qed
Proof Let $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathcal{Y} = \mathrm{Spf}(B)$ so that the natural homomorphism $B \to A$ is finite. By construction we have $z_s(\mathcal{X}) = z_s(\text{Spec}(A))$ and $z_s(\mathcal{Y}) = z_s(\text{Spec}(B))$, and we are given $[\mathcal{F}] \in z_s(\text{Spec}(A))$. The push-forward $f_*$ on cycles is the push-forward of cycles on the scheme $\text{Spec}(A)$. Hence the statement follows from the classical case of schemes, which is in [44, Lemma 0EP3], for instance. \hfill \Box

The finite push-forwards as in Definition 2.2.1 induce push-forwards on higher Chow cycles on noetherian affine formal $k$-schemes under an additional mild assumption:

Lemma 2.3.3 Let $f: \mathcal{X} \to \mathcal{Y}$ be a finite surjective morphism of equidimensional noetherian affine formal $k$-schemes. Then we have the induced push-forward morphism

$$f_*: z^q(\mathcal{X}, \bullet) \to z^q(\mathcal{Y}, \bullet)$$

of complexes of abelian groups.

Proof Since $f$ is finite surjective, so is the morphism $f \times_k 1: \mathcal{X} \times_k \square^n_k \to \mathcal{Y} \times_k \square^n_k$, which we also denote by $f$. Furthermore, $\dim \mathcal{X} = \dim \mathcal{Y}$ and $\dim \mathcal{X}_{\text{red}} = \dim \mathcal{Y}_{\text{red}}$.

For an integral cycle $\mathcal{Z} \in z^q(\mathcal{X}, n)$, the cycle push-forward $f_*(\mathcal{Z})$ is defined to be $f_*(\mathcal{Z}) := [f_*\mathcal{O}_{\mathcal{Z}}]'$ as in Definition 2.3.1, where $d = \dim \mathcal{Z}$. We need to check that $f_*(\mathcal{Z})$ satisfies the conditions of Definition 2.2.2 over $\mathcal{Y}$. Let $Z = \mathcal{Z} \cap (\mathcal{X}_{\text{red}} \times \square^n_k)$, the reduction by the largest ideal of definition of $\mathcal{X} \times \square^n_k$.

For a face $F \subset \square^n_k$, we have $f_*(\mathcal{Z}) \cap (\mathcal{Y} \times F) = f_*(\mathcal{Z} \cap (\mathcal{X} \times F))$. Thus $\dim (f_*(\mathcal{Z}) \cap (\mathcal{Y} \times F)) \leq \dim (\mathcal{Z} \cap (\mathcal{X} \times F))$.

The general position condition $(\text{GP})$ of Definition 2.2.2 for $\mathcal{Z}$ says

$$\dim (\mathcal{Z} \cap (\mathcal{X} \times F)) \leq \dim \mathcal{X} \times F - q.$$ 

So, we have

$$\dim (f_*(\mathcal{Z}) \cap (\mathcal{Y} \times F)) \leq \dim \mathcal{X} \times F - q = \dim \mathcal{Y} \times F - q, \quad (2.3.2)$$

which is the condition $(\text{GP})$ for $f_*(\mathcal{Z})$.

For the special fiber condition $(\text{SF})$ for $f_*(\mathcal{Z})$, first note that

$$\dim f_*(\mathcal{Z}) \cap (\mathcal{Y}_{\text{red}} \times F) \leq \dim \mathcal{Z} \cap (\mathcal{X}_{f^{-1}(I_0)} \times F), \quad (2.3.3)$$

where $\mathcal{X}_{f^{-1}(I_0)}$ is the scheme associated to the ideal $f^{-1}(I_0)\mathcal{O}_{\mathcal{X}}$. Since $\mathcal{Z} \cap (\mathcal{X}_{\text{red}} \times F)$ and $\mathcal{Z} \cap (\mathcal{X}_{f^{-1}(I_0)} \times F)$ are both noetherian schemes, their dimensions as schemes are equal to the dimensions of their respective underlying noetherian topological spaces. But, their underlying topological spaces are both equal to $|\mathcal{Z} \cap (\mathcal{X} \times F)|$ (Lemma 2.2.5) so that we have

$$\dim \mathcal{Z} \cap (\mathcal{X}_{f^{-1}(I_0)} \times F) = \dim \mathcal{Z} \cap (\mathcal{X}_{\text{red}} \times F). \quad (2.3.4)$$

On the other hand, by the condition $(\text{SF})$ for $\mathcal{Z}$, we have

$$\dim \mathcal{Z} \cap (\mathcal{X}_{\text{red}} \times F) \leq \dim \mathcal{X}_{\text{red}} \times F - q = \dim \mathcal{Y}_{\text{red}} \times F - q. \quad (2.3.5)$$

Combining (2.3.3), (2.3.4), and (2.3.5), we deduce that

$$\dim f_*(\mathcal{Z}) \cap (\mathcal{Y}_{\text{red}} \times F) \leq \dim \mathcal{Y}_{\text{red}} \times F - q,$$

which is the condition $(\text{SF})$ for $f_*(\mathcal{Z})$.

Thus we have checked that $f_*(\mathcal{Z}) \in z^q(\mathcal{Y}, n)$, so that $f_*$ maps $z^q(\mathcal{X}, n)$ to $z^q(\mathcal{Y}, n)$.
For its compatibility with the codimension 1 faces, take the face $F = F_i^e$ for some $1 \leq i \leq n$ and $e \in \{0, \infty\}$ in the proof of the condition $(GP)$ in the above. By (2.3.2), we obtain the square

$$
\begin{array}{ccc}
\mathcal{Z}^q(\mathcal{X}, n) & \xrightarrow{\partial^e_f} & \mathcal{Z}^q(\mathcal{X}, n - 1) \\
\downarrow f_* & & \downarrow f_* \\
\mathcal{Z}^q(\mathcal{Y}, n) & \xrightarrow{\partial^e_f} & \mathcal{Z}^q(\mathcal{Y}, n - 1). \\
\end{array}
$$

That this diagram commutes is checked exactly as done in [5, Proposition (1.3)] using [15, Theorem 6.2(a), p.98]. We omit details. This implies that $f_*$ commutes with $\partial$. This proves the lemma. \hfill \Box

### 2.4 The mod $Y$-equivalence on cycles

Returning back to $Y := \text{Spec}(k_m)$ and $\widehat{X} = \text{Spf}(k[[t]])$, we define the mod $Y$-equivalence on $\mathcal{Z}^q(\widehat{X}, n)$ as follows.

**Definition 2.4.1** Under the above notations:

1. Let $\mathcal{R}^q(\widehat{X}, n)$ be the set of all coherent $\mathcal{O}_{\widehat{X}}$-algebras $A$ such that the associated cycle $[A] \in \mathcal{Z}^q(\widehat{X}, n)$.
2. Let $\mathcal{L}^q(\widehat{X}, Y, n)$ be the set of all pairs $(A_1, A_2)$ of coherent $\mathcal{O}_{\widehat{X}}$-algebras with $A_j \in \mathcal{R}^q(\widehat{X}, n)$ such that there is an isomorphism

$$
A_1 \otimes \mathcal{O}_{\square^n_{\widehat{X}}} \cong A_2 \otimes \mathcal{O}_{\square^n_{\widehat{Y}}} 
$$

of $\mathcal{O}_{\square^n_{\widehat{Y}}}$-algebras.
3. Let $\mathcal{M}^q(\widehat{X}, Y, n) \subset \mathcal{Z}^q(\widehat{X}, n)$ be the subgroup generated by the cycles of the form $[A_1] - [A_2]$ over all pairs $(A_1, A_2) \in \mathcal{L}^q(\widehat{X}, Y, n)$.

We say that two cycles $Z_1$ and $Z_2 \in \mathcal{Z}^q(\widehat{X}, n)$ are mod $Y$-equivalent and write $Z_1 \sim_Y Z_2$, if $Z_1 - Z_2 \in \mathcal{M}^q(\widehat{X}, Y, n)$.

We have two remarks regarding Definition 2.4.1.

**Remark 2.4.2** Corollary 2.2.10 offers a hint on how one may generalize the mod $Y$-equivalence of Definition 2.4.1 to more general $k$-schemes of finite type to be discussed in [39].

For a general formal neighborhood, there seem to be cases where an analogue of Lemma 2.2.8 hold, while it is not clear to the author how far it may generalize. For instance, for this one needs to understand a projective resolution of $\mathcal{O}_{\square^n_Y}$ as an $\mathcal{O}_{\square^n_{\widehat{X}}}$-module. This is relative easier when the closed immersion $Y \hookrightarrow \widehat{X}$ is a local complete intersection. At least, in general the formal scheme $\square^n_{\widehat{X}}$ is regular so that one has a resolution of finite length.

Even if one does not yet have a clearer understanding of the Tor independence, one could still get around this uncertainty. For instance, in the present version of [39], the author uses some ideas from derived algebraic geometry. The isomorphism (2.4.1) is replaced by an isomorphism of a suitable pair of “derived rings”. While it is of a similar form, it is finer than the derived Milnor patching for perfect complexes studied by S. Landsburg [33], which generalizes the classical Milnor patching of J. Milnor [36].
It seems it is worth studying whether one can avoid using this heavy machine of derived algebraic geometry, by improving Lemma 2.2.8 to the fullest possible extent in the future.

Remark 2.4.3 Let $I := (t^m) \subset k[[t]]$. Instead of the above relation $\sim_Y$ on the cycles, one may ask whether it might be better to use the relation $\sim_I$ defined as follows: first declare $\mathcal{I}_1 \sim_I \mathcal{I}_2$ for a pair of integral cycles $\mathcal{I}_1$ and $\mathcal{I}_2 \in z^q(\hat{X}, n)$ when their structure sheaves $(\mathcal{O}_{\mathcal{I}_1}, \mathcal{O}_{\mathcal{I}_2})$ satisfy (2.4.1) of Definition 2.4.1, namely

$$\mathcal{O}_{\mathcal{I}_1} \otimes \mathcal{O}_{\mathcal{I}_2} \simeq \mathcal{O}_{\mathcal{I}_2} \otimes \mathcal{O}_{\mathcal{I}_1}.$$ 

Define the subgroup $\mathcal{N}^q(\hat{X}, Y, n) \subset z^q(\hat{X}, n)$ generated by $[\mathcal{I}_1] - [\mathcal{I}_2]$ for such pairs $(\mathcal{O}_{\mathcal{I}_1}, \mathcal{O}_{\mathcal{I}_2})$. We define $\sim_I$ on $z^q(\hat{X}, n)$ using this subgroup $\mathcal{N}^q(\hat{X}, Y, n)$. It is similar to the version taken in [41], called the “mod $I^m$-equivalence” there.

A benefit of $\sim_I$ is that it is relatively easier to work with; when one wishes to check whether certain operations on cycles respect the equivalence, one can test them just on pairs of integral cycles. A draw-back of this is that, the property of a scheme being integral is not preserved well under base change in general (see, e.g. EGA IV$_2$ [19, Définition (4.6.2), p.68] for the definition of geometric integrality, which does behave well under base change), so that this approach counting on putting relations on pairs integral cycles is unlikely to be generalizable in a functorial way, in general.

However, at least in the Milnor range (i.e. $q = n$), we will show that our definition of $\sim_Y$ via $\mathcal{M}^q(\hat{X}, Y, n)$ in Definition 2.4.1 and $\sim_I$ in the above paragraph via $\mathcal{N}^n(\hat{X}, Y, n)$ are equivalent. The proof requires a few properties of the generating cycles in $z^n(\hat{X}, n)$, so that it is postponed until Lemma 4.3.1. This result will be used in Sects. 4.4 and 5.

Coming back to our construction, we observe:

Lemma 2.4.4 Let $Y = \text{Spec}(k[m])$ and $\hat{X} = \text{Spf}(k[[t]])$. Let $n \geq 1$. For $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$, we have

$$\partial^\epsilon_i \mathcal{M}^q(\hat{X}, Y, n) \subset \mathcal{M}^q(\hat{X}, Y, n - 1).$$

In particular, $\mathcal{M}^q(\hat{X}, Y, \bullet) \subset z^q(\hat{X}, \bullet)$ is a subcomplex.

Proof Let $\iota^\epsilon_i : \square^{n-1} \hookrightarrow \square^n$ be the closed immersion given by $\{y_i = \epsilon\}$.

When $\mathcal{A} \in \mathcal{R}^q(\hat{X}, n)$, we have $[\mathcal{A}] \in z^q(\hat{X}, n)$. Here $[(\iota^\epsilon_i)^*(\mathcal{A})] = \partial^\epsilon_i [\mathcal{A}] \in z^q(\hat{X}, n - 1)$, so that $(\iota^\epsilon_i)^* \mathcal{A} \in \mathcal{R}^q(\hat{X}, n - 1)$.

When $(\mathcal{A}_1, \mathcal{A}_2) \in \mathcal{L}^q(\hat{X}, Y, n)$, we have $\mathcal{A}_j \in \mathcal{R}^q(\hat{X}, n)$ and there is an isomorphism

$$\mathcal{A}_1 \otimes \mathcal{O}_{\square^n} \otimes \mathcal{O}_{\square^{n-1}} \simeq \mathcal{A}_2 \otimes \mathcal{O}_{\square^n} \otimes \mathcal{O}_{\square^{n-1}}$$

(2.4.2)

of $\mathcal{O}_{\square^n}$-algebras. We already saw that $(\iota^\epsilon_i)^* \mathcal{A}_1 \in \mathcal{R}^q(\hat{X}, n - 1)$. Applying $(\iota^\epsilon_i)^*$ to (2.4.2), we deduce an isomorphism

$$(\iota^\epsilon_i)^* \mathcal{A}_1 \otimes \mathcal{O}_{\square^{n-1}} \otimes \mathcal{O}_{\square^{n-1}} \simeq (\iota^\epsilon_i)^* \mathcal{A}_2 \otimes \mathcal{O}_{\square^{n-1}} \otimes \mathcal{O}_{\square^{n-1}}$$

of $\mathcal{O}_{\square^{n-1}}$-algebras. This means $((\iota^\epsilon_i)^* \mathcal{A}_1, (\iota^\epsilon_i)^* \mathcal{A}_2) \in \mathcal{L}^q(\hat{X}, Y, n - 1)$. Thus

$$\partial^\epsilon_i ([\mathcal{A}_1] - [\mathcal{A}_2]) = [(\iota^\epsilon_i)^* \mathcal{A}_1] - [(\iota^\epsilon_i)^* \mathcal{A}_2] \in \mathcal{M}^q(\hat{X}, Y, n - 1).$$

Since the cycles of the form $[\mathcal{A}_1] - [\mathcal{A}_2]$ generate $\mathcal{M}^q(\hat{X}, Y, n)$, this implies that $\partial^\epsilon_i \mathcal{M}^q(\hat{X}, Y, n) \subset \mathcal{M}^q(\hat{X}, Y, n - 1)$ as desired. \qed

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Definition 2.4.5 Let $m \geq 2$. Let $Y = \Spec (k_m)$ and $\widehat{X} = \Spf (k[[t]])$. Define

$$z^q(\widehat{X} \mod Y, \bullet) := \frac{z^q(\widehat{X}, \bullet)}{M^q(\widehat{X}, Y, \bullet)},$$

and define $\CH^q(\widehat{X} \mod Y, n) := H_n(z^q(\widehat{X} \mod Y, \bullet))$. We denote this group also by $\CH^q(k_m, n) = \CH^q(\Spec (k_m), n)$ in the boldface letters, to distinguish it from the higher Chow group of S. Bloch [5].

The following special range is easy to compute. It answers the part (1) of the first main theorem, Theorem 1.2.1:

Proposition 2.4.6 For $q > n$, $z^q(\widehat{X} \mod Y, n) = 0$. In particular, we have

$$\CH^q(\widehat{X} \mod Y, n) = \CH^q(k_m, n) = 0.$$

Proof Note that $\dim \widehat{X} \times \square^n = n + 1$. Thus if $q \geq n + 2$, due to the dimension reason, we have $z^q(\widehat{X}, n) = 0$. Thus $z^q(\widehat{X} \mod Y, n) = 0$ as well.

Suppose $q = n + 1$. Let $\mathfrak{Z} \in z^{n+1}(\widehat{X}, n)$ be an integral cycle. Note that $\widehat{X}_{\text{red}} = \Spec (k)$. By the special fiber condition (SF) of Definition 2.2.2, the intersection $\mathfrak{Z} \cap (\Spec (k) \times \square^n)$ has the codimension $\geq n + 1$ in the space $\Spec (k) \times \square^n$ of dimension $n$. This shows the intersection $\mathfrak{Z} \cap (\Spec (k) \times \square^n)$ is empty. On the other hand, we have

$$|\mathfrak{Z} \cap (\Spec (k) \times \square^n)| \subset |\mathfrak{Z} \cap (\Spec (k_m) \times \square^n)| \subset |\mathfrak{Z} \cap (\widehat{X} \times \square^n)|,$$

and the first and the third topological spaces are equal (Lemma 2.2.5). Thus $\mathfrak{Z} \cap (\Spec (k_m) \times \square^n) = \emptyset$ as well. It means

$$\mathcal{O}_\mathfrak{Z} \otimes \mathcal{O}_{\square^n} \otimes \mathcal{O}_{\square^n} = 0 = \mathcal{O}_\emptyset \otimes \mathcal{O}_{\square^n} \otimes \mathcal{O}_{\square^n},$$

thus the cycle $\mathfrak{Z}$ is mod $Y$-equivalent to the empty scheme $\emptyset$. Thus $[\mathfrak{Z}] \in \mathcal{M}^{n+1}(\widehat{X}, Y, n)$ and the class of $\mathfrak{Z}$ in $z^{n+1}(\widehat{X} \mod Y, n)$ is $0$. This shows $z^{n+1}(\widehat{X} \mod Y, n) = 0$. \hfill $\square$

Remark 2.4.7 If we specialize to the Milnor range $q = n$ in Definition 2.4.5, the group $\CH^n(\widehat{X} \mod Y, n)$ can be expressed as

$$\CH^n(\widehat{X} \mod Y, n) = \frac{z^n(\widehat{X} \mod Y, n)}{\partial(z^n(\widehat{X} \mod Y, n + 1))}$$

(2.4.3)

because $z^n(\widehat{X} \mod Y, n - 1) = 0$ by Proposition 2.4.6 so that

$$\ker(\partial : z^n(\widehat{X} \mod Y, n) \to z^n(\widehat{X} \mod Y, n - 1)) = z^n(\widehat{X} \mod Y, n).$$

Thus we may rewrite (2.4.3) also as

$$\CH^n(\widehat{X} \mod Y, n) := \frac{z^n(\widehat{X}, n)}{\partial(z^n(\widehat{X}, n + 1)) + \mathcal{M}^n(\widehat{X}, Y, n)}.$$  (2.4.4)

One may wonder whether the numerator in (2.4.4) is also equal to $\ker(\partial : z^n(\widehat{X}, n) \to z^n(\widehat{X}, n - 1))$. The answer is yes, and we will see it in Lemma 3.2.5.
2.5 Tate algebras

In Sect. 2.5, we recall some basic definitions and results related to the Tate algebras of J. Tate [46]. We will use them to deal with the “generic fibers” of the cycles in this article. For general references on the Tate algebras, there are a few books in the literature, e.g. Bosch–Güntzer-Remmert [7], Bosch [8], Fresnel–van der Put [13], Fujiwara–Kato [14]. At some places, Tate algebras are also called the standard affinoid algebras or the free affinoid algebras, as well.

For a complete discrete valued field $K$ with a nontrivial non-archimedean norm, such as $k((t))$, the Tate algebra $T_n$ over $K$ with $n$-variables $z_1, \ldots, z_n$ is defined to be the subring $K\{z_1, \ldots, z_n\} \subset K[[z_1, \ldots, z_n]]$ consisting of the restricted formal power series, i.e.

$$p = \sum_{I=(i_1, \ldots, i_n), i_j \geq 0} a_I z^I,$$

with $z^I := z_1^{i_1} \cdots z_n^{i_n}$ such that $|a_I| \to 0$ as $|I| = i_1 + \cdots + i_n \to \infty$. The norm on $K$ extends to the Gauß norm on $T_n$ by taking $|p| := \max |a_I|$, making $T_n$ a Banach algebra over $K$ of at most countable type. See e.g. any one of [7, Sect. 5.1.1, p.192], [8, Sect. 2.2, Definition 2.2, p.13], [13, Sect. 3.1, p.46], or [14, Sect. 9.3(a), p.227].

The following is a compilation of a few facts on Tate algebras, that we need:

**Lemma 2.5.1** Let $K$ be a complete discrete valued field with a nontrivial non-archimedean norm, and let $T_n$ be the Tate algebra over $K$ in $n$-variables. Then:

1. $T_n$ is a noetherian regular UFD of Krull dimension $n$.
2. $T_1$ is a Euclidean domain, in particular a PID.
3. For each maximal ideal $m \subset T_n$, the injective ring homomorphism $K \hookrightarrow T_n/m$ gives a finite extension of fields.
4. For any finite extension $K \subset L$ of fields, there exists a unique extension of the discrete valuation on $K$ to $L$. For this extension, $L$ is again a complete discrete valued field.
5. Every $f \in T_1$ can be written uniquely as $f = ug$, where $u \in T_1^\times$ and $g \in K^0[z_1]$ for the valuation ring $K^0 = \{x \in K \mid |x| \leq 1\}$.

**Proof** (1) is scattered at a few places. Collect them from [7, Sect. 5.2.6, Theorem 1, p.207], [8, Sect. 2.2, Proposition 14, 15, p.20, Proposition 17, p.22], [13, Theorem 3.2.1-(1),(2), p.48], and [14, Exercise 0.9.4, p.23, Proposition 9.3.9, p.229].

(2) : see [8, Sect. 2.2, Corollary 10, p.19].

(3) : see [8, Corollary 12, p.19] or [13, Theorem 3.2.1-(5), p.48].

(4) : see e.g. H. Hasse [23, Ch.12, Prolongation theorem, p.183].

(5) : see [13, Exercises 3.2.2-(1), p.50], as a consequence of the Weierstraß division theorem [13, Theorem 3.1.1, p.46], or e.g. [7, Sect. 5.2.1, Theorem 2, p. 200, Sect. 5.2.2, Theorem 1, p.201], [8, Sect. 2.2, Theorem 8, p.17, Corollary 9, p.18], [14, Exercise 0.A.3, p.236].

3 On the generator cycles

We continue to use the notations $Y = \text{Spec} (k_m)$ and $\widehat{X} = \text{Spf} (k[[t]])$.

In Sect. 3, we study a few general aspects of the cycles in the group $z^n(\widehat{X}, n)$. These cycles generate the groups $\text{CH}^n(\widehat{X}, n)$ and $\text{CH}^n(\text{Spec} (k_m), n)$.

The arguments here are a hybrid of basic commutative algebra, basic rigid analytic geometry, and algebraic geometry of cycles on formal schemes. An essential observation is that
the “generic fiber” of an integral cycle in $z^n(\hat{X}, n)$ induces a maximal ideal of a Tate algebra over the Laurent field $k((t))$. Some results of rudimentary rigid analytic geometry then allow us to obtain “triangular polynomial generators” of the ideal that define the given integral cycle over $\text{Spf}(k[[t]])$ in Sect. 3.3.

### 3.1 Algebraicity and generic fibers

For a complete discrete valuation field $K$ with a nontrivial non-archimedean norm, consider $T_n = K[z_1, \ldots, z_n]$, the Tate algebra in $n$ variables $z_1, \ldots, z_n$. Let’s start with the standard fact that each maximal ideal $m \subset T_n$ of the Tate algebra is generated by $n$ elements (in particular it is a complete intersection); see S. Bosch [8, Sect. 2.2, Proposition 17, p.22]. Here, these $n$ generators can be chosen from the subring $K[z_1, \ldots, z_n] \subset T_n$ of polynomials in $z_1, \ldots, z_n$; see Fresnel–van der Put [13, Exercises 3.2.2-(2), p.50] or Bosch–Güntzer–Remmert [7, Sect. 7.1.1, Proposition 3, p.261].

We improve this fact a bit further as follows. It might be of independent interest:

**Lemma 3.1.1** Let $K$ be a complete discrete valuation field with a nontrivial non-archimedean norm, and $K^*$ be its valuation ring. Let $T_n = K[z_1, \ldots, z_n]$ be the Tate algebra over $K$, and let $m \subset T_n$ be a maximal ideal.

Then we have a sequence of **polynomials** in $z_1, \ldots, z_n$

$$\begin{align*}
p_1(z_1) & \in K^*[z_1], \\
p_2(z_1, z_2) & \in K[z_1, z_2], \\
& \vdots \\
p_n(z_1, \ldots, z_n) & \in K[z_1, \ldots, z_n],
\end{align*}$$

where

1. each $p_i$ is monic in $z_i$ for $1 \leq i \leq n$,
2. for $2 \leq i \leq n$, when $p_i$ is regarded as a polynomial in $z_i$, its coefficients in $K[z_1, \ldots, z_{i-1}]$ have their $z_j$-degrees strictly less than $\deg_{z_j}(p_j)$ for all $1 \leq j < i$, and
3. $(p_1, \ldots, p_n) = (p_1, \ldots, p_n)T_n = m$.

**Proof** The proof is in part based on the argument of S. Bosch [8, Sect. 2.2, Proposition 17, p.22], with some improvements to deduce our stronger statements.

For the sequence of inclusions $T_1 \subset T_2 \subset \cdots \subset T_n$, consider the prime ideals $m_i := T_i \cap m$ for $1 \leq i \leq n$. Here $m_n = m$. These induce injections

$$K \hookrightarrow T_1/m_1 \hookrightarrow \cdots \hookrightarrow T_n/m_n,$$

where $T_n/m_n$ is a finite extension of $K$ of fields (Lemma 2.5.1-(3)). Hence each intermediate $T_i/m_i$ is also a field, thus $m_i \subset T_i$ is a maximal ideal for $1 \leq i \leq n$.

When $n = 1$, the ideal $m_1 \subset T_1$ is maximal, while $T_1$ is a PID by Lemma 2.5.1-(2). Hence we have $(f) = m_1$ for some nonzero $f \in T_1$. By Lemma 2.5.1-(5), we have $f = ug$ for some $u \in T_1^\times$ and a monic polynomial $g \in K^*[z_1]$. Thus $(f) = (g) = m_1$. Taking $p_1 := g$ answers the lemma in this case.

Now suppose $n \geq 2$, and suppose the lemma holds for all positive integers $< n$, so that we have

$$p_1(z_1) \in K^*[z_1], p_2(z_1, z_2) \in K[z_1, z_2], \ldots, p_{n-1}(z_1, \ldots, z_{n-1}) \in K[z_1, \ldots, z_{n-1}]$$
satisfying the conclusions of the lemma for the maximal ideal \( m_{n-1} \subset T_{n-1} \).

Note that (e.g. from \textit{loc.cit}) we have the following commutative diagram of continuous \( K \)-algebra homomorphisms

\[
\begin{array}{ccc}
T_{n-1} \otimes T_{n-1} T_n = T_{n-1}\{z_n\} & \xrightarrow{\pi} & T_n \\
\text{can} \otimes \text{id}_{T_n} = \phi' & \downarrow \phi & \\
(T_{n-1}/m_{n-1}) \otimes T_{n-1} T_n = (T_{n-1}/m_{n-1})\{z_n\} & \xrightarrow{\pi} & T_n/m,
\end{array}
\]

where the vertical maps \( \phi, \phi' \) are given by the reduction mod \( m \) and \( m_{n-1} \), respectively, and \( \pi \) maps \( z_n \) to its residue class in \( T_n/m \). Note that both of the maps \( \phi' \) and \( \pi \) are surjective.

Take \( \ker(\pi) \). Since \( T_n/m \) is a field and \( \pi \) is surjective, by the first isomorphism theorem \( \ker(\pi) \subset (T_{n-1}/m_{n-1})\{z_n\} \) is a maximal ideal. On the other hand, by Lemma 2.5.1-(4), the field \( T_{n-1}/m_{n-1} \) is also a complete discrete valued field with respect to the unique norm extending the norm on \( K \). Hence by the case \( n = 1 \) we proved already, there exists a polynomial \( \bar{p}_n \in (T_{n-1}/m_{n-1})\{z_n\} \) monic in \( z_n \), such that \( (\bar{p}_n) = \ker(\pi) \).

Since \( \phi' \) is surjective, we can choose a lifting \( p_n(z_1, \cdots, z_n) \in T_{n-1}\{z_n\} \subset T_{n-1}\{z_n\} \) of \( \bar{p}_n \), which is still a monic polynomial in \( z_n \).

By the commutativity of the diagram (3.1.1), we have \( (p_1, \cdots, p_n) = m \), but one issue yet to resolve is that, while \( p_n(z_1, \cdots, z_n) \) is in \( T_{n-1}\{z_n\} \), it may not be yet in \( K[z_1, \cdots, z_{n-1}]\{z_n\} \).

To improve this, when \( N := \deg_{z_n}(p_n) \), write

\[
p_n = \sum_{i=0}^{N} \alpha_i z_n^i \in T_{n-1}\{z_n\},
\]

for \( \alpha_i \in T_{n-1} \), \( 0 \leq i \leq N \) with \( \alpha_N = 1 \). By the induction hypothesis \( p_{n-1} \) is a monic polynomial in \( z_{n-1} \). By the Weierstraß division theorem (Fresnel–van der Put [13, Theorem 3.1.1, p.46]), for \( 0 \leq i < N \), we can find \( q_i \in T_{n-1} \) and \( r_i \in T_{n-2}\{z_{n-1}\} \) such that

\[
\alpha_i = q_i p_{n-1} + r_i,
\]

where \( r_i \) is a polynomial in \( z_{n-1} \) with \( \deg_{z_{n-1}}(r_i) < \deg_{z_{n-1}}(p_{n-1}) \). We let \( r_N = 1 \). Then for

\[
p'_n := \sum_{i=1}^{N} r_i z_n^i \in T_{n-2}\{z_{n-1}\}\{z_n\}
\]

we have \( m = (p_1, \cdots, p_{n-1}, p_n) = (p_1, \cdots, p_{n-1}, p'_n) \). Hence we may replace \( p_n \) by \( p'_n \). We can inductively apply the Weierstraß division theorem with the divisions by \( p_{n-2}, p_{n-3}, \cdots, p_1 \), and after these backward inductive replacements, in finite steps we obtain \( p_n \in K[z_1, \cdots, z_n] \). This proves the lemma.

\[\square\]

As an immediate application, we deduce that our cycles in the Milnor range can be seen as separated \( k[[t]]\)-schemes of finite presentation. It will be further improved in the next subsections.
Corollary 3.1.2 Let $\mathfrak{z} \in z^n(\text{Spf}(k[[t]]), n)$ be an integral cycle. Let $y'_i := y_i/(y_i - 1)$ using the automorphism $y_i \mapsto y'_i$ of $\mathbb{P}^1$.

Then there are polynomials $P_i(y'_1, \ldots, y'_n)$ in $k[[t]][y'_1, \ldots, y'_n]$ of the form

$$
\begin{align*}
P_1(y'_1) &\in k[[t]][y'_1], \\
& \vdots \\
P_n(y'_1, \ldots, y'_n) &\in k[[t]][y'_1, \ldots, y'_n],
\end{align*}
$$

such that the ideal $(P_1(y'_1), \ldots, P_n(y'_1, \ldots, y'_n))$ defines $\mathfrak{z}$.

In particular, $\mathfrak{z}$ can be also regarded as a separated $k[[t]]$-scheme of finite presentation.

Proof Let $K := k((t)) = \text{Frac}(k[[t]])$. Write $z_i := y'_i$ for notational simplicity as well as to be consistent with the notation of Lemma 3.1.1.

The cycle $\mathfrak{z}$ is given by a height $n$ prime ideal $I(\mathfrak{z})$ of the Tate algebra $T_n = k[[t]] [z_1, \ldots, z_n]$. So, we have the canonical homomorphisms

$$
k[[t]] \twoheadrightarrow k[[t]][z_1, \ldots, z_n] \to k[[t]][z_1, \ldots, z_n]/I(\mathfrak{z}),
$$

and, we may regard $\mathfrak{z}$ as a scheme over $\text{Spec}(k[[t]])$ as well.

Let $\eta \in \text{Spec}(k[[t]])$ be the generic point. The generic fiber $\mathfrak{z}_\eta$ is obtained by the flat base change via $\eta \to \text{Spec}(k[[t]])$, and it gives a height $n$ prime ideal of the Tate algebra $T_n$. Since the Krull dimension of $T_n$ is $n$ (Lemma 2.5.1-(1)), this height $n$ prime ideal is maximal. Hence the generic fiber $\mathfrak{z}_\eta$ is defined by a maximal ideal $m \subset T_n$.

By Lemma 3.1.1, we then have a set of generators $p_1(z_1) \in K^0[z_1], p_2(z_1, z_2) \in K[z_1, z_2], \ldots, p_n(z_1, \ldots, z_n) \in K[z_1, \ldots, z_n]$ with the properties there.

To obtain the generators for its closure $\mathfrak{z}$, we clear the denominators of the coefficients of the terms of $p_i$ by the l.c.m. in $k[[t]]$. Since $k[[t]]$ is a UFD, this is possible. So obtained elements are denoted by $P_1(z_1) \in k[[t]][z_1], \ldots, P_n(z_1, \ldots, z_n) \in k[[t]][z_1, \ldots, z_n]$.

In Sect. 3.2, after proving the finiteness, we will show that the above can be also stated for a sequence of polynomials in $k[[t]][y_1, \ldots, y_n]$ (not just for $k[[t]][y'_1, \ldots, y'_n]$), where the polynomials $P_i$ can satisfy more requirements.

3.2 Finiteness

In Sect. 3.2, we study the generators of $z^n(\text{Spf}(k[[t]]), n)$ from the algebraic perspective to obtain additional properties.

Lemma 3.2.1 Let $\mathfrak{z} \in z^n(\text{Spf}(k[[t]]), n)$ be an integral cycle.

Then $\mathfrak{z}$ regarded as a scheme over $k[[t]]$, is quasi-finite over $k[[t]]$.

Proof By Corollary 3.1.2, we may regard $\mathfrak{z}$ as a scheme over $k[[t]]$.

We check that for each $p \in \text{Spec}(k[[t]])$, the fiber $\mathfrak{z}_p$ is a finite set.

As before, via a suitable automorphism of $\mathbb{P}^1$, use $y'_i = y_i/(y_i - 1)$ for $1 \leq i \leq n$. The generic fiber $\mathfrak{z}_\eta$ is given by a maximal ideal of the Tate algebra $T_n = k((t))[y'_1, \ldots, y'_n]$. Hence $|\mathfrak{z}_\eta| = 1 < \infty$.

For the closed point $m \in \text{Spec}(k[[t]])$, by the special fiber condition (SF) of Definition 2.2.2, this cycle intersects $m \times \square_k^n$ properly. Hence $\mathfrak{z}_m$ gives a closed subscheme of codimension $\geq n$ in $\square_k^n = \square_k^n$. Since $\dim \square_k^n = n$, we have $\dim (\mathfrak{z}_m) \leq 0$ so that $|\mathfrak{z}_m| < \infty$ again.

Thus we proved that $\mathfrak{z} \to \text{Spec}(k[[t]])$ is quasi-finite, as desired. 

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We will refine Lemma 3.2.1 even further. To do so, recall the following (See Conrad–Stein [10, Lemma 8.3]. See also EGA IV$_4$ [19, Théorème (18.5.11)-(c), p.130]):

**Lemma 3.2.2** Let $A$ be a henselian local ring, and let $G$ be a quasi-finite separated $A$-scheme of finite presentation.

Then there is a unique decomposition $G = G_f \bigsqcup G'$ into disjoint clopen pieces such that $G_f \to \text{Spec}(A)$ is finite, and $G'$ has the empty special fiber.

An interesting and important point is that the nonzero integral cycles in $z^n(X, n)$ are all finite in the Milnor range:

**Corollary 3.2.3** Let $\mathcal{Z} \in z^n(\text{Spf}(k[[t]]), n)$ be a nonzero integral cycle. Then $\mathcal{Z}$ is finite over $k[[t]]$.

**Proof** The given integral cycle $\mathcal{Z}$ is quasi-finite over the henselian local ring $k[[t]]$ by Lemma 3.2.1. Furthermore, it can be regarded as a separated $k[[t]]$-scheme of finite presentation by Corollary 3.1.2.

Hence we may apply Lemma 3.2.2. By this, $\mathcal{Z}$ can be written as the disjoint union of the finite part $\mathcal{Z}_f$, and the part $\mathcal{Z}'$ that has the empty special fiber. But, $\mathcal{Z}$ is integral, so we have either (a) $\mathcal{Z} = \mathcal{Z}_f$, in which case $\mathcal{Z} \to \text{Spec}(k[[t]])$ is finite, or (b) $\mathcal{Z} = \mathcal{Z}'$, in which case $\mathcal{Z} \to \text{Spec}(k[[t]])$ is not finite, and the special fiber is empty.

Since the cycle $\mathcal{Z}$ does have the nonempty special fiber (Lemma 2.2.5), the above case (b) cannot happen. Hence the case (a) is the only possibility and $\mathcal{Z} \to \text{Spec}(k[[t]])$ is finite. $\square$

**Remark 3.2.4** On the first version of the article, Matthew Morrow suggested that there may be a possibly shorter rigid analytic proof of the finiteness in Corollary 3.2.3. Indeed, suppose $\mathcal{Z} = \text{Spf}(B)$ for a $k[[t]]$-algebra $B = k[[t]](y_0', \ldots, y_n')/P$. This gives a homomorphism $\varphi^0 : k[[t]] \to B$. Localizing at the generic point of $k[[t]]$, we obtain the homomorphism $\varphi : k((t)) \to B_0$ of affinoid algebras over $k((t))$. The reduction of $\varphi^0 \mod (t)$ is the homomorphism $\varphi : k \to B/(t)$, and it is finite by the condition (SF). Then by Fresnel-van der Put [13, Theorems 3.5.3, 3.5.6, pp.61–62], this is equivalent to that $\varphi^0$ is finite, i.e. $\mathcal{Z}$ is finite over $k[[t]]$.

While this is short and efficient, after some thoughts, the author decided to keep the previous arguments as they show well how the geometric requirements (GP) and (SF) imply the algebraic implication of finiteness.

Being finite over $k[[t]]$ have a few important consequences. The first is on intersection with the faces:

**Lemma 3.2.5** Let $\mathcal{Z} \in z^n(\text{Spf}(k[[t]]), n)$ be an integral cycle. Then $\mathcal{Z}$ does not intersect any proper face of $\text{Spf}(k[[t]]) \times \Box^n$. In particular, $\partial(\mathcal{Z}) = 0$.

**Proof** We imitate the argument of Krishna–Park [31, Lemma 2.21, p.1005]. Let $\hat{X} = \text{Spf}(k[[t]])$. Let $F \subsetneq \Box^n_k$ be a proper face. We show that $\mathcal{Z} \cap (\hat{X} \times F) = \emptyset$.

Suppose not. As before, we regard $\mathcal{Z}$ as a scheme over $\text{Spec}(k[[t]])$. By Corollary 3.2.3, the morphism $\mathcal{Z} \to \text{Spec}(k[[t]])$ is finite. Consider the composite of finite morphisms

$$
\mathcal{Z} \cap (\hat{X} \times F) \leftrightarrow \mathcal{Z} \to \text{Spec}(k[[t]]).
$$

Its image in $\text{Spec}(k[[t]])$ is closed. Since we supposed that $\mathcal{Z} \cap (\hat{X} \times F)$ is nonempty, the image must intersect the unique closed point $m$ of $\text{Spec}(k[[t]])$. Hence $\mathcal{Z} \cap (m \times F) \neq \emptyset$. $\square$
On the other hand, by the special fiber condition (SF) of Definition 2.2.2 for \(Z\), the codimension of \(Z \cap (m \times F)\) in \(m \times F\) is \(\geq n\). This is equivalent to saying that
\[
\dim (Z \cap (m \times F)) \leq \dim (m \times F) - n = \dim F - n < \dagger 0,
\]
where \(\dagger\) holds because \(F\) is a proper face. But this is a contradiction for a nonempty set. Hence we must have \(Z \cap (\hat{X} \times F) = \emptyset\). This proves the lemma. \(\Box\)

We discuss additional algebraic properties of the cycles in \(z^n(\text{Spf}(k[[t]]), n)\). Recall:

**Lemma 3.2.6** Let \((R, m)\) be a henselian (complete, resp.) local ring and let \(R \hookrightarrow B\) be a finite extension of rings.

Then \(B\) is a finite direct product of henselian (complete, resp.) local rings. Furthermore, there is a bijection between the factors and the maximal ideals of \(B\).

In particular, if \(B\) is an integral domain in addition to the above assumptions, then \(B\) is a henselian (complete, resp.) local domain with a unique maximal ideal.

**Proof** See EGA IV 4 [19, Propositions (18.5.9), (18,5.10), p.130] for the henselian case, and D. Eisenbud [11, Corollary 7.6, p.190] for the complete case. \(\Box\)

**Corollary 3.2.7** Let \(Z \in z^n(\text{Spf}(k[[t]]), n)\) be an integral cycle, and let \(B\) be the integral domain such that \(Z = \text{Spf}(B)\).

Then the canonical homomorphism \(k[[t]] \to B\) of rings is finite, and \(B\) is a complete local integral domain, which is a free \(k[[t]]\)-module of finite rank.

**Proof** By Corollary 3.2.3, the ring homomorphism \(k[[t]] \to B\) is finite. Here \(t\) is a non-zero divisor on \(B\) by the special fiber condition (SF) of Definition 2.2.2 for \(Z\). Since \(B\) is a finitely generated \(k[[t]]\)-module, where \(t\) is not a zero-divisor in \(B\), it is torsion free. Thus, by the fundamental theorem of finitely generated modules over a PID, \(B\) is a free \(k[[t]]\)-module of finite rank.

That \(B\) is a complete local domain follows from Lemma 3.2.6. \(\Box\)

**Definition 3.2.8** For an integral cycle \(Z \in z^n(\text{Spf}(k[[t]]), n)\), let \(k[[t]] \to B\) define \(Z\) as the above. We can define the norm map \(N : B^\times \to k[[t]]^\times\) as follows.

By Corollary 3.2.7, \(B\) is a free \(k[[t]]\)-module of finite rank. Hence for each \(b \in B\), the left multiplication by \(b\)

\[L_b : B \to B, \quad x \mapsto bx\]
is a \(k[[t]]\)-linear endomorphism of the free \(k[[t]]\)-module \(B\) of finite rank. When \(b \in B^\times\), the map \(L_b\) is an automorphism. We define \(N(b)\) to be the determinant \(\det(L_b) \in k[[t]]^\times\).

It is independent of the choice of a \(k[[t]]\)-basis of \(B\), and the map \(N\) is multiplicative, by standard linear algebra.

The finiteness property of \(Z\) over \(k[[t]]\) has another set of consequences. The following is an analogue in the formal setting of the finiteness criterion in [31, Lemma 2.9] :

**Lemma 3.2.9** Let \(Z \subset \hat{X} \times \mathbb{P}^n\) be an integral closed formal subscheme. Consider the open immersion \(\hat{X} \times \mathbb{P}^n \subset \hat{X} \times (\mathbb{P}^1)^n\).

Then the following are equivalent:

1. The morphism \(Z \to \hat{X}\) is finite.
2. The formal scheme \(Z\) is closed in \(\hat{X} \times (\mathbb{P}^1)^n\).
The proof is almost identical to that of [31, Lemma 2.9], except that we use the corresponding properties for formal schemes.

(1) ⇒ (2): Consider the factorization \( \mathcal{Z} \hookrightarrow \hat{X} \times (\mathbb{P}^1)^n \rightarrow \hat{X} \) of the finite morphism. The second morphism is separated of finite type. Thus by Fujiwara–Kato [14, Corollary 4.6.15, p.340], the first morphism is also finite. An immersion is finite only when it is a closed immersion. This implies (2).

(2) ⇒ (1): Since \( \mathcal{Z} \) is closed in \( \hat{X} \times (\mathbb{P}^1)^n \), in particular this inclusion morphism is proper. Thus the composite \( \mathcal{Z} \hookrightarrow \hat{X} \times (\mathbb{P}^1)^n \rightarrow \hat{X} \) is also proper. This implies that the reduction by an ideal of definition of \( \hat{X} \) is also proper ([14, Proposition 4.7.3, p.341]). But both \( \mathcal{Z} \) and \( \hat{X} \) are affine formal schemes, so their reductions are affine schemes. A morphism between affine schemes is proper if and only if it is finite ([21, Exercise II-4.6, p.106]). But this means the morphism \( \mathcal{Z} \rightarrow \hat{X} \) of formal schemes is also finite ([14, Proposition 4.2.3, p.325]). This implies (1).

**Corollary 3.2.10** Let \( \mathcal{Z} \subset z^n(\hat{X}, n) \) be an integral cycle. Consider \( (\mathbb{P}^1)^n \supset \Box^n \), where \( y_1, \ldots, y_n \) are the coordinates.

Then it is closed in \( \hat{X} \times (\mathbb{P}^1)^n \) and it does not intersect any divisor of the form \( \{ y_i = \epsilon \} \), where \( 1 \leq i \leq n \) and \( \epsilon = 0, \infty, 1 \).

In particular, for \( \mathcal{A}_n \subset (\mathbb{P}^1)^n \), with the coordinates \( y_1, \ldots, y_n \), we can regard \( \mathcal{Z} \) as an integral closed formal subscheme of \( \hat{X} \times \mathcal{A}_n \), and there exists a prime ideal \( I \subset k[[r]] \{ y_1, \ldots, y_n \} \) such that

\[
\mathcal{Z} = \operatorname{Spf} (k[[r]] \{ y_1, \ldots, y_n \}/I). \tag{3.2.1}
\]

**Proof** From Corollary 3.2.3, we know that \( \mathcal{Z} \) is finite over \( k[[r]] \), and by Lemma 3.2.9, we know that \( \mathcal{Z} \) is closed in \( \hat{X} \times (\mathbb{P}^1)^n \). Because \( \mathcal{Z} \subset \hat{X} \times \Box^n \) does not intersect the divisors \( \{ y_i = 1 \} \subset (\hat{X} \times (\mathbb{P}^1)^n) \setminus (\hat{X} \times \Box^n) \), in fact \( \mathcal{Z} \subset \hat{X} \times \mathcal{A}_n \), and it is closed. This shows (3.2.1).

The remaining statement that \( \mathcal{Z} \cap \{ y_i = \epsilon \} \) with \( \epsilon = 0, \infty \) follows from Lemma 3.2.5. \( \square \)

Hence, we no longer need to use the ugly coordinate changes \( y_i \mapsto y'_i = y_i/(y_i - 1) \). The generic fibers also enjoy the same property. Namely, Corollary 3.2.10 immediately implies:

**Corollary 3.2.11** Let \( \mathcal{Z}_\eta \in z^n(\hat{X}, n) \) be an integral cycle. Let \( \mathcal{Z}_\eta \) be its generic fiber. Then there exists a maximal ideal \( m \subset k((r)) \{ y_1, \ldots, y_n \} \) such that \( \mathcal{Z}_\eta \) is given by the affinoid algebra \( k((r)) \{ y_1, \ldots, y_n \}/m \).

### 3.3 Triangular generators

For a given integral cycle \( \mathcal{Z} \in z^n(\hat{X}, n) \), in Corollary 3.1.2 we found \( n \) polynomial generators in the variables \( y'_1, \ldots, y'_n \), where \( y'_i = y_i/(y_i - 1) \). While it was enough for our purposes of proving finiteness at that time, we now have obtained the finiteness and some of its consequences. This time, we want to improve it further.

The goal of Sect. 3.3 is to find a nice “triangular” generating set for each integral cycle \( \mathcal{Z} \in z^n(\operatorname{Spf} (k[[r]]), n) \), this time as polynomials in \( y_1, \ldots, y_n \). We will use Corollaries 3.2.10 and 3.2.11 we obtained in Sect. 3.2 as consequences of the finiteness of \( \mathcal{Z} \) over \( k[[r]] \).

**Proposition 3.3.1** Let \( \mathcal{Z} \in z^n(\operatorname{Spf} (k[[r]]), n) \) be an integral cycle.

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Then there are polynomials in $k[[t]][y_1, \ldots, y_n]$ of the form

\[\begin{align*}
P_1(y_1) &\in k[[t]][y_1], \\
&\vdots \\
P_n(y_1, \ldots, y_n) &\in k[[t]][y_1, \ldots, y_n],
\end{align*}\]

such that

1. the ideal $(P_1(y_1), \ldots, P_n(y_1, \ldots, y_n))$ in $k[[t]][y_1, \ldots, y_n]$ defines $\mathfrak{Z}$,
2. for each $i$, the polynomial $P_i$ is monic in $y_i$. When $i = 1$, the constant term of $P_1$ is in $k[[t]]$, while for $2 \leq i \leq n$ and for the image $\bar{P}_i(y_i)$ of $P_i$ in the quotient ring $(k[[t]][y_1, \ldots, y_{i-1}])_{y_i}$, its constant term is a unit in $(k[[t]][y_1, \ldots, y_{i-1}])_{P_1, \ldots, P_{i-1}}$, and
3. for $2 \leq i \leq n$, for $P_i$ regarded as a polynomial in $y_i$, its coefficients in $k[[t]][y_1, \ldots, y_{i-1}]$ have their $y_j$-degrees strictly less than $\deg_{y_j} P_j$ for all $1 \leq j < i$.

**Proof** Let $\eta \in \text{Spec}(k[[t]])$ be the generic point. By Corollary 3.2.11, the generic fiber $\mathfrak{Z}_\eta$ is given by a maximal ideal $m \subset T_n = k((t))[y_1, \ldots, y_n]$.

Applying Lemma 3.1.1 to $m$ with $z_i = y_i$ for $1 \leq i \leq n$, we obtain a triangular shaped set of polynomials $p_1(y_1) \in k[[t]][y_1], p_2(y_1, y_2) \in k((t))[y_1, y_2], \ldots, p_n(y_1, \ldots, y_n) \in k((t))[y_1, \ldots, y_n]$ that generate the maximal ideal $m \subset T_n$, and satisfy the properties there in the lemma.

To obtain the generators for its closure $\mathfrak{Z}$, for each $1 \leq i \leq n$ we clear the denominators of the coefficients in $k((t))$ of the terms of $p_i$ by the l.c.m. of them in $k[[t]]$. We can do it because $k[[t]]$ is a UFD. So obtained elements are denoted by $P_1(y_1) \in k[[t]][y_1], \ldots, P_n(y_1, \ldots, y_n) \in k[[t]][y_1, \ldots, y_n]$, and they satisfy the properties that

(a) the ideal $(P_1, \ldots, P_n)$ generated in $k[[t]][y_1, \ldots, y_n]$ defines $\mathfrak{Z}$,
(b) for each $i$, the highest $y_i$-degree term of $P_i$ involves no other variable, and
(c) for $2 \leq i \leq n$, when $P_i$ is regarded as a polynomial in $y_i$, each of its coefficients in $k[[t]][y_1, \ldots, y_{i-1}]$ has its $y_j$-degree strictly less than $\deg_{y_j} P_j$ for all $1 \leq j < i$.

The conditions (1) and (3) of the proposition are satisfied by the above properties (a) and (c), but the condition (2) is not yet achieved. We show that (2) can be achieved for $\mathfrak{Z}$.

For $1 \leq i \leq n$, write $\mathcal{S}_i := k[[t]][y_1, \ldots, y_i]$. Let $p_i \subset S_i$ be the prime ideal that defines $\mathfrak{Z}$ (by Corollary 3.2.10). For the sequence of injective ring homomorphisms

$$k[[t]] \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_{n-1} \hookrightarrow S_n,$$

consider the prime ideals $p_i := p_n \cap S_i$ for $1 \leq i \leq n-1$. By construction, $p_i = (P_i, \ldots, P_1)$, and

$$k[[t]] \hookrightarrow S_1/p_1 \hookrightarrow \cdots \hookrightarrow S_{n-1}/p_{n-1} \hookrightarrow S_n/p_n.$$

Since $k[[t]] \rightarrow S_i/p_i$ is finite and free (Corollary 3.2.7), and $k[[t]]$ is a PID, all homomorphism $k[[t]] \rightarrow S_i/p_i$ for $1 \leq i \leq n$ are also finite and free because a submodule of a free module of finite rank over a PID is again free by the fundamental theorem on finitely generated modules over a PID.

We let $\mathfrak{Z}^{(i)} \subset \text{Spf}(k[[t]]) \times \mathbb{A}^i$ be the integral closed formal subscheme given by $\text{Spf}(S_i/p_i)$. By construction, its dimension is equal to that of $\mathfrak{Z}$, so that its codimension is $i$. This $\mathfrak{Z}^{(i)}$ is equal to the image of $\mathfrak{Z}$ under the projection $p_{ri} : \text{Spf}(k[[t]]) \times \mathbb{A}^n \rightarrow \text{Spf}(k[[t]]) \times \mathbb{A}^i$, that sends $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i)$.

**Claim:** $\mathfrak{Z}^{(i)} \in z^i(\text{Spf}(k[[t]]), i)$ for $1 \leq i \leq n$. 

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For each codimension 1 face $F = F^e_j \subset k^n$ with $j \leq i$, the intersection $\mathcal{Z}^{(i)} \cap (\tilde{X} \times F)$ is the image of $\mathcal{Z} \cap (\tilde{X} \times F)$ under $pr_j$. But the latter is empty by Lemma 3.2.5, thus so is the former. This means $\mathcal{Z}^{(i)}$ has the empty intersection with all proper faces. In particular, the general position condition (GP) of Definition 2.2.2 for $\mathcal{Z}^{(i)}$ holds trivially.

Since $\mathcal{Z}^{(i)}$ has the empty intersection with all proper faces, to check the condition (SF) for $\mathcal{Z}^{(i)}$, it remains to take $F = k^n$, and we need to look at its special fiber. Since $\mathcal{Z}^{(i)}$ is finite over $k[[r]]$ and the special fiber is given by its unique closed point by Lemma 3.2.6, the condition (SF) holds for $\mathcal{Z}^{(i)}$. Thus $\mathcal{Z}^{(i)} \in \langle z^i \rangle (\text{Spf} (k[[r]]), i)$.

We prove the remaining part (2) of the proposition.

Suppose $i = 1$. Then the intersection of $\mathcal{Z}^{(i)}$ with the faces $\{y_1 = \infty\}$ and $\{y_1 = 0\}$ are empty by Lemma 3.2.5 and Claim. Thus the leading coefficient and the constant term of $P_i(y_1)$ in $y_1$ are both units in $k[[r]]^\times$. After scaling by the leading coefficient of $P_i(y_1)$ in $y_1$, the leading coefficient becomes 1, and the constant term is still a unit in $k[[r]]^\times$, proving (2) for $i = 1$.

Let $i \geq 2$. Among the defining polynomials $P_1, \ldots, P_i$ of $\mathcal{Z}^{(i)}$, the only one that involves the variable $y_i$ is $P_i$. Furthermore, the highest $y_i$-degree term of $P_i$ does not involve any other variables. Hence the empty intersection of $\mathcal{Z}^{(i)}$ with the codimension 1 face $\{y_i = \infty\}$ (Lemma 3.2.5 and the above Claim) means that the leading coefficient of $P_i$ in $y_i$ is a unit in $k[[r]]^\times$. Hence after scaling by the leading coefficient of $P_i$ in $y_i$, the leading coefficient becomes 1, so that we may assume $P_i$ is monic in $y_i$.

Note that scaling by units of $k[[r]]^\times$ does not disturb the pre-existing properties (1), (3).

On the other hand, that the intersection of $\mathcal{Z}^{(i)}$ with $\{y_i = 0\}$ is empty means that the image $\tilde{P}_i$ of $P_i$ in the quotient ring $\left( k[[r]][y_1, \ldots, y_{i-1}] \left/ (y_i) \right. \right)[y_i]$ has a unit constant term, when regarded as a polynomial in $y_i$. Thus we have achieved all of (1), (2), (3). $\square$

4 The graph homomorphisms and the regulators

4.1 Milnor $K$-theory and the graph homomorphism

We discuss the graph homomorphisms from the Milnor $K$-groups. Recall from Elbaz–Vincent–Müller–Stach [12, Lemma 2.1] that for an equidimensional $k$-algebra $R$ essentially of finite type, there is the graph homomorphism

$$gr : K_n^M(R) \rightarrow \text{CH}^n(\text{Spec} (R), n).$$

We have a similar homomorphism for the higher Chow groups of some affine formal schemes. To avoid unnecessary complexities, we consider the relevant case only:

**Lemma 4.1.1** Consider $\tilde{X} = \text{Spf} (k[[r]])$. For each Milnor symbol $\{a_1, \ldots, a_n\} \in K_n^M(k[[r]])$, with $a_i \in k[[r]]^\times$, consider the integral closed formal subscheme $\Gamma_{(a_1, \ldots, a_n)} \subset \tilde{X}$ defined by the system of linear polynomials $y_1 - a_1, \ldots, y_n - a_n$.

Then sending $\{a_1, \ldots, a_n\}$ to the graph cycle $\Gamma_{(a_1, \ldots, a_n)}$ defines the graph homomorphism

$$gr : K_n^M(k[[r]]) \rightarrow \text{CH}^n(\text{Spf} (k[[r]]), n). \quad (4.1.1)$$
Proof One checks immediately that $\Gamma_{(a_1, \ldots, a_n)} \in z^n(Spf (k[[t]]), n)$. By Lemma 3.2.5, this represents a class in $\text{CH}^n(Spf (k[[t]]), n)$.

The rest of the argument is essentially a repetition of [12, Lemma 2.1]. We remark that loc.cit. makes a general assumption that their ring is essentially of finite type, but this is not used there, so we can still follow a large part of the argument.

Firstly, for $a_i \in k[[t]]^\times$ such that $1 - a_i \in k[[t]]^\times$ for $3 \leq i \leq n$, consider the parametrized cycle given by the closure of the graph of the rational map

$$\mathfrak{M}_1 : \square^1_X \rightarrow \square^{n+1}_X, \ x \mapsto \left( x, 1 - x, \frac{a - x}{1 - x}, a_3, \ldots, a_n \right).$$

One checks $\mathfrak{M}_1 \in z^n(Spf (k[[t]]), n + 1)$. By straightforward calculations, one sees that the only nonzero face of $\mathfrak{M}_1$ is

$$\partial^3_0 \mathfrak{M}_1 = \Gamma_{(a_1 - a, a_3, \ldots, a_n)}.$$

In particular, the Steinberg symbol $\{ a, 1 - a, a_3, \ldots, a_n \} \in K^M_n (k[[t]])$ maps to 0 in $\text{CH}^n(Spf (k[[t]]), n)$.

Secondly, for $a, b, an_i \in k[[t]]^\times$ for $2 \leq i \leq n$, consider the parametrized cycle

$$\mathfrak{M}_2 : \square^1_X \rightarrow \square^{n+1}_X, \ x \mapsto \left( x, \frac{ax - ab}{x - ab}, a_2, \ldots, an \right).$$

One checks $\mathfrak{M}_2 \in z^n(Spf (k[[t]]), n + 1)$, and that its only nonzero faces are

$$\begin{align*}
\partial^1_1 \mathfrak{M}_2 &= \Gamma_{(a, a_2, \ldots, a_n)}, \\
\partial^2_0 \mathfrak{M}_2 &= \Gamma_{(b, a_2, \ldots, a_n)}, \\
\partial^2_2 \mathfrak{M}_2 &= \Gamma_{(ab, a_2, \ldots, a_n)}.
\end{align*}$$

Hence killing $\partial \mathfrak{M}_2$ in $\text{CH}^n(Spf (k[[t]]), n)$, we deduce

$$\Gamma_{(ab, a_2, \ldots, a_n)} = \Gamma_{(a, a_2, \ldots, a_n)} + \Gamma_{(b, a_2, \ldots, a_n)} \text{ in } \text{CH}^n(Spf (k[[t]]), n).$$

Considering various permutations of the cycle $\mathfrak{M}_2$, we thus deduce

$$\bigotimes_{i=1}^n k[[t]]^\times \rightarrow \text{CH}^n(Spf (k[[t]]), n),$$

and then using the permutations of $\mathfrak{M}_1$, it further descends to (4.1.1). \qed

Lemma 4.1.2 Let $m \geq 2$. The graph map (4.1.1) induces the homomorphism

$$gr_{km} : K^M_n (k_m) \rightarrow \text{CH}^n(Spec (k_m), n), \quad (4.1.2)$$

where the codomain is the new higher Chow group defined in Definition 2.4.5.

Proof Since $k[[t]]$ is a local ring, the natural homomorphism $k[[t]]^\times \rightarrow k_m^\times$ is surjective (see e.g. Hartshorne–Polini [22, Lemma 5.2]). Thus $K^M_n (k[[t]]) \rightarrow K^M_n (k_m)$ is also surjective. Hence to see that (4.1.1) descends to (4.1.2), it remains to check that $\ker (K^M_n (k[[t]]) \rightarrow K^M_n (k_m))$ is mapped to 0 under the graph map (4.1.1) composed with the natural map $\text{CH}^n(Spf (k[[t]]), n) \rightarrow \text{CH}^n(Spec (k_m), n)$.

Since the above kernel is generated by elements of the form $\{1 + t^m c, c_2, \ldots, c_n\}$, where $c \in k[[t]]$ and $c_i \in k[[t]]^\times$ for $2 \leq i \leq n$ (see Kato-Saito [29, Lemma 1.3.1, p.261]), it is enough to show that

$$\Gamma_{(1 + t^m c, c_2, \ldots, c_n)} = 0 \text{ in } \text{CH}^n(Spec (k_m), n), \quad (4.1.3)$$
where $\Gamma_{(1+t^mc,c_2,\ldots,c_n)}$ is the graph cycle in $z^n(\hat{X}, n)$ of the above Milnor symbol, defined in Lemma 4.1.1.

Before we do so, we first prove:

Claim: Let $a_i, b_i \in k[[t]]^\times$ such that $a_i - b_i \in (t^m)k[[t]]$ for $1 \leq i \leq n$. Then have

$$\Gamma_a - \Gamma_b \in \mathcal{M}^n(\hat{X}, Y, n),$$

where $\Gamma_a := \Gamma_{(a_1, \ldots, a_n)}$ and $\Gamma_b := \Gamma_{(b_1, \ldots, b_n)}$ denote the corresponding integral graph cycles in $z^n(\hat{X}, n)$. In particular, $\Gamma_a \equiv \Gamma_b$ in $CH^n(\text{Spec } (k_m), n)$.

This claim is immediate because we have an isomorphism

$$\mathcal{O}_{\Gamma_a} \otimes \mathcal{O}_{\square^n_{\hat{X}}} \simeq \mathcal{O}_{\Gamma_b} \otimes \mathcal{O}_{\square^n_{\hat{X}}}$$

of $\mathcal{O}_{\square^n_{\hat{X}}}$-algebras.

Going back to the proof of (4.1.3), let $d_i := \tilde{c}_i \in k^\times$ for $2 \leq i \leq n$. Via the obvious injective homomorphism $k^\times \hookrightarrow k[[t]]^\times$, regard $d_i \in k[[t]]^\times$. Then we have

$$1 + t^m c \equiv 1, \quad c_2 \equiv d_2, \quad \ldots, \quad c_n \equiv d_n \mod (t^m)k[[t]].$$

Thus by the Claim, we have

$$\Gamma_{(1+t^mc,c_2,\ldots,c_n)} \equiv \Gamma_{(1,d_2,\ldots,d_n)} \text{ in } CH^n(\text{Spec } (k_m), n). \quad (4.1.4)$$

However, $\square = \mathbb{P}^1 \setminus \{1\}$ so that $\Gamma_{(1,d_2,\ldots,d_n)} = \emptyset$ in $\square^n_{\hat{X}}$. Thus (4.1.4) proves (4.1.3). This induces (4.1.2).

The goal of this article, Theorem 1.2.1, is to prove that (4.1.2) is an isomorphism when $|k| \gg 0$. It will be completed in Theorem 5.2.1.

### 4.2 Regulators via rigid analytic spaces

In Sect. 4.2, we construct a homomorphism

$$\hat{\gamma} : CH^n(\text{Spf } (k[[t]]), n) \rightarrow K^M_n(k((t))).$$

Under the additional assumption that $|k| \gg 0$, the homomorphism $K^M_n(k[[t]]) \rightarrow K^M_n(k((t)))$ is injective (Gersten conjecture for $K^M_n$ proven by M. Kerz [30, Proposition 10, p.181]). We will see that the image of $\hat{\gamma}$ lies in $K^M_n(k[[t]])$.

To facilitate our discussions, we introduce a rigid analytic analogue of higher Chow groups in the Milnor range (see also J. Ayoub [2, Notations 2.4.14, p.297 and beyond]):

**Definition 4.2.1** Let $K$ be a complete discrete valued field with a nontrivial non-archimedean norm. Let $K^\circ$ be its valuation ring and suppose $k \subset K^\circ$. We define the rigid analytic analogue of higher Chow groups for $\text{Sp } (K)$ for the Milnor range, but its generalization to the off-Milnor ranges is similar.

We consider the rigid analytic space $\text{Sp } (K) \times \square^n_k$. (The fiber product exists by Bosch–Güntzer–Remmert [7, Sect. 7.1.4, Proposition 4, p.268]). We define $z^n(\text{Sp } (K), n)$ to be the free abelian group on the codimension $n$ points not lying on $\{y_i = 0, 1, \infty\}$ for $1 \leq i \leq n$, and define $z^n(\text{Sp } (K), n + 1)$ to be the free abelian group on the codimension $n$ integral closed subsets that intersect all faces of $\text{Sp } (K) \times \square^n_k$ properly, where the faces are similarly defined as in Sect. 2.2. We define $z^n(\text{Sp } (K), \ast)$ for $\ast = n, n + 1$ by modding out $z^n(\text{Sp } (K), \ast)$ by the degenerate cycles. Via the closed immersions $i^\varepsilon : \text{Sp } (K) \times \square^n_k \hookrightarrow \text{Sp } (K) \times \square^n_{k^{(1)}}$ for $1 \leq i \leq n + 1$ and $\varepsilon = 0, \infty$, we have the natural boundary map $\partial : z^n(\text{Sp } (K), n + 1) \rightarrow$
$z^n(\text{Sp} (K), n)$ defined by $\partial = \sum_{i=1}^{n+1} (-1)^i (\partial_i^\infty - \partial_i^0)$, where $\partial_i^e = (t_i^e)^n$. Its cokernel is denoted by $\text{CH}^n(\text{Sp} (K), n)$.

**Example 4.2.2** If we take $K = k(t)$ and an integral cycle $\mathcal{Z} \in \text{CH}^n(\text{Sp} (k[[t]]), n)$, then the generic fiber $\mathcal{Z}_\eta$ gives a member of $\text{CH}^n(\text{Sp} (k((t))), n)$. With Corollaries 3.2.10 and 3.2.11 in mind, note that the localization homomorphisms $k[[t]][y_1, \cdots, y_n] \to k((t))[y_1, \cdots, y_n]$ over $n \geq 1$ induce the commutative diagram

$$
\begin{array}{c}
z^n(\text{Sp} (k[[t]]), n + 1) \longrightarrow z^n(\text{Sp} (k((t))), n + 1) \\
\downarrow \partial \quad \quad \quad \downarrow \partial \\
z^n(\text{Sp} (k[[t]]), n) \longrightarrow z^n(\text{Sp} (k((t))), n),
\end{array}
$$

from which we deduce $\eta : \text{CH}^n(\text{Sp} (k[[t]]), n) \to \text{CH}^n(\text{Sp} (k((t))), n)$.

What follows below is essentially an argument in B. Totaro [47, Sect. 3], with some suitable modifications to fit it into the rigid analytic situation being considered.

**Definition 4.2.3** Let $p \in z^n(\text{Sp} (k((t))), n)$ be a point. It gives a maximal ideal $m$ of $T_n = k((t))[y_1, \cdots, y_n]$ (cf. Corollary 3.2.11). By Lemma 2.5.1-(3), the injection $k((t)) \hookrightarrow L := T_n/m$ is a finite extension of fields. This point $p$ defines an $n$-tuple $(x_1, \cdots, x_n)$ whose values lie in $L \setminus \{0, 1\}$, namely $x_i = \bar{y}_i$ in $T_n/m = L$. We consider the Milnor symbol $\{x_1, \cdots, x_n\} \in K_n^M (L)$.

Define

$$
\bar{\Upsilon} (p) := N_{L/k((t))} \{x_1, \cdots, x_n\} \in K_n^M (k((t))) \tag{4.2.2}
$$

via the norm $N_{L/k((t))} : K_n^M (L) \to K_n^M (k((t)))$ of Bass–Tate [3] and K. Kato [28].

**Lemma 4.2.4** The composite

$$
z^n(\text{Sp} (k((t))), n + 1) \overset{\partial}{\longrightarrow} z^n(\text{Sp} (k((t))), n) \overset{\bar{\Upsilon}}{\longrightarrow} K_n^M (k((t)))
$$

is zero. In particular, we have the induced homomorphism

$$
\bar{\Upsilon} : \text{CH}^n(\text{Sp} (k((t))), n) \to K_n^M (k((t))). \tag{4.2.3}
$$

**Proof** We continue to follow B. Totaro [47, Sect. 3] with suitable modifications.

Let $C \in z^n(\text{Sp} (k((t))), n + 1)$ be an integral cycle. Take the normalization $D \to C$; since $C$ is defined by a 1-dimensional noetherian integral domain, it has a unique normalization (see B. Conrad [9] for normalizations of more general rigid analytic spaces). The normalization map is finite because every affinoid $k((t))$-algebra that is an integral domain is a Japanese ring (see Bosch–Güntzer–Remmert [7, Sect. 6.1.2, Proposition 4, p.228] or EGA IV 2 [17, Scholie (7.8.3)-(vi), p.215]). As in [47, Sect. 3], using W. Fulton [15, Example 1.2.3, p.9], it is enough to show that the boundary (defined by pull-back of the faces and push-forward) of $D$ maps to 0 in $K_n^M (k((t)))$.

Let $\overline{D}$ be the unique regular compactification of $D$, which exists as dim $D = 1$. By the rigid analytic GAGA (see e.g. [13, Theorem 4.10.5, p.113]), this $\overline{D}$ can be regarded as a regular projective algebraic curve over $k((t))$. Let $\mathbb{K}$ be the rational function field of $\overline{D}$, which is equal to that of $C$ and $D$. Since $\mathbb{K} \supset k((t))$, it is an infinite field.

The map $D \to \text{Sp} (k((t))) \times \mathbb{A}^{n+1}_{\mathbb{K}}$ is given by $(n + 1)$ rational functions $g_1, \cdots, g_{n+1} \in \mathbb{K}$ on $D$. By the given proper intersection conditions with the faces, no $g_i$ is identically equal
to 0 or $\infty$, and any closed point $w \in D$ with $g_i(w) = 0$ or $\infty$ has $g_j(w) \notin \{0, \infty\}$ for all $j \neq i$.

For each closed point $w \in \overline{D}$, there is its associated discrete valuation on $K$ with the associated boundary map $\partial_w : K^M_{n+1}(K) \to K^M_n(\kappa(w))$ (see J. Milnor [35, Lemma 2.1]), where $\kappa(w)$ is the residue field of $w$.

Since $K$ is infinite, by Suslin reciprocity (see [45]), for the Milnor symbol $\{g_1, \cdots, g_{n+1}\} \in K^M_{n+1}(K)$, we have

$$\sum_{w \in D} N_{\kappa(w)/k(\!(t)\!)} \partial_w \{g_1, \cdots, g_{n+1}\} = 0 \text{ in } K^M_n(k(\!(t)\!)).$$

If $w \in \overline{D} \setminus D$, then one of $g_i$ has $g_i(w) = 1$ so that $\partial_w \{g_1, \cdots, g_{n+1}\} = 0$. Hence the above sum can be written without such points, and

$$\sum_{w \in D} N_{\kappa(w)/k(\!(t)\!)} \partial_w \{g_1, \cdots, g_{n+1}\} = 0 \text{ in } K^M_n(k(\!(t)\!)). \tag{4.2.4}$$

On the other hand, by following the definition of $\partial_w$, we have the commutative diagram

$$
\begin{array}{ccc}
z^n(Sp(k(\!(t)\!)), n + 1) & \xrightarrow{\partial} & z^n(Sp(k(\!(t)\!)), n) \\
\downarrow & & \downarrow \tilde{\gamma} \\
K^M_{n+1}(K) & \xrightarrow{\sum_w N_{\kappa(w)/k(\!(t)\!)} \partial_w} & K^M_n(k(\!(t)\!)),
\end{array}
$$

where the left vertical arrow associates $C$ to the Milnor symbol $\{g_1, \cdots, g_{n+1}\}$ in $K^M_{n+1}(K)$. Thus $\tilde{\gamma} \circ \partial C$ is equal to (4.2.4), which is 0. Thus, we deduce the map (4.2.3) as desired. \[ \square \]

**Definition 4.2.5** We define the map $\tilde{\gamma}$ as the composite

$$\tilde{\gamma} : CH^n(Spf(k[[t]]), n) \xrightarrow{\eta} CH^n(Sp(k(\!(t)\!)), n) \xrightarrow{\tilde{\gamma}} K^M_n(k(\!(t)\!)),$$

where $\eta$ is the localization map (Example 4.2.2) and $\tilde{\gamma}$ is from Definition 4.2.3 and Lemma 4.2.4.

**Lemma 4.2.6** Let $k$ be a field with $|k| \gg 0$ so that $K^M_n(k[[t]]) = \tilde{K}^M_n(k[[t]])$.

Then the image of the composite

$$K^M_n(k[[t]]) \xrightarrow{gr} CH^n(Spf(k[[t]]), n) \xrightarrow{\tilde{\gamma}} K^M_n(k(\!(t)\!)),$$

lands into the subgroup $K^M_n(k[[t]]) \subset K^M_n(k(\!(t)\!))$, and the composite is the identity on $K^M_n(k[[t]])$.

In particular, the map $gr$ is injective and the image of $CH^n(Spf(k[[t]]), n)$ under $\tilde{\gamma}$ is in the subgroup $K^M_n(k[[t]])$ of $K^M_n(k(\!(t)\!))$.

Before the proof, we remark that the condition $K^M_n(k[[t]]) = \tilde{K}^M_n(k[[t]])$ implies that the natural homomorphism $K^M_n(k[[t]]) \to K^M_n(k(\!(t)\!))$ is injective (see M. Kerz [30, Proposition 10, p.181]), thus we can regard $K^M_n(k[[t]]) \subset K^M_n(k(\!(t)\!))$.

When $n = 1$, this always holds for any field $k$. 

Proof Consider the Milnor symbol \( \{a_1, \cdots, a_n\} \in K_n^M(k[[t]]) \) with \( a_i \in k[[t]]^\times \). Its image under the first map \( gr \) is given by the integral graph cycle \( \mathcal{Z} \) defined by the system of polynomials

\[
y_1 - a_1, \cdots, y_n - a_n.
\]

Its coordinates are given by \( a_1, \cdots, a_n \in k[[t]]^\times \subset k((t))^\times \), and they are of degree 1 over \( k((t)) \). Hence the generic fiber \( \mathcal{Z}_n \) gives \( L = k((t)) \) in the notations of Definition 4.2.3, and the norm map in (4.2.2) is the identity map. Hence \( \hat{\Upsilon}(\mathcal{Z}) = \{a_1, \ldots, a_n\} \) and it belongs to the subgroup \( K_n^M(k[[t]]) \). This shows that the composite is the identity map of \( K_n^M(k[[t]]) \). In particular, \( gr \) is injective.

Since the image of the composite lies in \( K_n^M(k[[t]]) \), so does the image of the group \( \text{CH}^n(\text{Spf}(k[[t]]), n) \). \( \square \)

The following is a rigid analytic analogue of [47]:

**Corollary 4.2.7** For any field \( k \), the composite

\[
K_n^M(k((t))) \xrightarrow{gr} \text{CH}^n(\text{Spf}(k((t))), n) \xrightarrow{\hat{\Upsilon}} K_n^M(k((t)))
\]

is the identity map, and all arrows are isomorphisms.

**Proof** For the injectivity of \( gr \), we simply note that the composite \( \hat{\Upsilon} \circ gr \) evaluated at the Milnor symbols \( \{a_1, \cdots, a_n\} \in K_n^M(k((t))) \) for some \( a_i \in (k((t)))^\times \) is again \( \{a_1, \cdots, a_n\} \). Thus \( \hat{\Upsilon} \circ gr = \text{Id} \). In particular, \( gr \) is injective.

For the surjectivity of \( gr \), we may follow B. Totaro [47, Sect. 4]. When \( F \) is a field, to prove that the graph homomorphism \( K_n^M(F) \to \text{CH}^n(\text{Spec}(F), n) \) to the ordinary cubical higher Chow group, is surjective, Totaro used the norm \( N : L^\times \to F^\times \) to relate the class of a given closed point to a cycle given by \( F \)-rational points, where \( L \) is the residue field of the point.

Specializing to the case \( F = k((t)) \) (which is infinite), we can repeat the argument of Totaro *mutatis mutandis*. As this corollary does not play an important role for the rest of the article, we omit details. \( \square \)

### 4.3 A comparison of equivalence relations

We continue to use the notations \( Y = \text{Spec}(k_m) \) and \( \hat{X} = \text{Spf}(k[[t]]) \). The closed immersion \( Y \hookrightarrow \hat{X} \) is given by the ideal \( I := (t^m) \subset k[[t]] \) so that \( k[[t]]/I = k_m \).

Recall from Remark 2.4.3 that, we have the mod \( Y \)-equivalence \( \sim_Y \) on \( z^n(\hat{X}, n) \) given by the subgroup \( \mathcal{M}^n(\hat{X}, Y, n) \), and we have another equivalence \( \sim_I \) given by the subgroup \( \mathcal{N}^n(\hat{X}, Y, n) \) generated by the differences of a pair of integral cycles that are equal after taking the mod \( I \)-reductions. We prove in Lemma 4.3.1 below that these two are equivalent in the Milnor range. It will simplify our discussions from Sect. 4.4.

**Lemma 4.3.1** We have the equality

\[
\mathcal{M}^n(\hat{X}, Y, n) = \mathcal{N}^n(\hat{X}, Y, n)
\]

(4.3.1)

of the subgroups in \( z^n(\hat{X}, n) \).
Proof For each generator \([3_1] - [3_2] \in N^n(\hat{X}, Y, n)\) associated to a pair of integral cycles \(3_1 \text{ and } 3_2 \in z^n(\hat{X}, n)\), by definition we have

\[\mathcal{O}_{3_1} \otimes \mathcal{O}_{\square_x^n} \mathcal{O}_{\square_y^n} \simeq \mathcal{O}_{3_2} \otimes \mathcal{O}_{\square_x^n} \mathcal{O}_{\square_y^n}\]

as \(\mathcal{O}_{\square_y^n}\)-algebras. Hence \([3_1] - [3_2] \in M^n(\hat{X}, Y, n)\), and this shows \(\mathcal{M}^n(\hat{X}, Y, n) \supset N^n(\hat{X}, Y, n)\). It remains to prove the opposite inclusion.

Let \((A_1, A_2) \in L^n(\hat{X}, Y, n)\). We have an isomorphism of \(\mathcal{O}_{\square_y^n}\)-algebras

\[A_1 \otimes \mathcal{O}_{\square_x^n} \mathcal{O}_{\square_y^n} \simeq A_2 \otimes \mathcal{O}_{\square_x^n} \mathcal{O}_{\square_y^n}.\]  \hspace{1cm} \text{(4.3.2)}

Let \(B_j\) be the ring of the global sections of \(A_j\). Since \([A_j] \in z^n(\hat{X}, n)\), by Corollary 3.2.3 we see that \(B_j\) is a \(k[[t]]\)-algebra which is a finite \(k[[t]]\)-module. In particular, it is a semi-local \(k[[t]]\)-algebra.

Having Corollary 3.2.10 in mind, the \(k_m[y_1, \cdots, y_n]\)-algebra isomorphism (4.3.2) in particular implies that we have an isomorphism of Artin rings

\[B_1/(t^m)B_1 \simeq B_2/(t^m)B_2.\]  \hspace{1cm} \text{(4.3.3)}

By the structure theorem for Artin rings (Atiyah-MacDonald [1, Theorem 8.7, p.90] or D. Eisenbud [11, Corollary 2.16, p.76]), each Artin ring decomposes uniquely into a product of Artin local rings, up to isomorphism. So, both sides of (4.3.3) decompose uniquely into products of Artin local rings, and under the isomorphism, there is a 1-1 correspondence between the Artin local factors of both sides.

On the other hand, since \(k[[t]]\) is henselian and complete, by Lemma 3.2.6, each \(B_j\) decomposes into a direct product \(\prod_{i=1}^{j_i} B_{ji}\) of complete local domains, one for each maximal ideal, so that each Artin local factor of \(B_j/(t^m)B_j\) corresponds to a local domain factor of \(B_j\).

The above two paragraphs summarize into the following 1-1 correspondences:

\[
\begin{array}{ccc}
\{\text{Artin local factors of } B_1/(t^m)B_1\} & 1-1 & \{\text{Artin local factors of } B_2/(t^m)B_2\} \\
1-1 & & 1-1 \\
\{\text{local factors of } B_1\} & & \{\text{local factors of } B_2\}
\end{array}
\]

In particular, we have the equality of the number of the local factors \(r_1 = r_2 =: r\).

For each fixed \(j\), after relabeling \(B_{ji}\) over \(i\) if necessary, the above discussion also shows that we have an isomorphism of \(k_m[y_1, \cdots, y_n]\)-algebras

\[\phi_m : B_{1i}/(t^m)B_{1i} \simeq B_{2i}/(t^m)B_{2i}\]  \hspace{1cm} \text{(4.3.4)}

for all \(1 \leq i \leq r\).

Let \(A_{ji}\) be the coherent \(\mathcal{O}_{\square_x^n}\)-algebra corresponding to \(B_{ji}\). Let \(f_{ji} : \text{Spec } (A_{ji}) = \text{Spec } (B_{ji}) \to \text{Spec } (k[[t]][y_1, \cdots, y_n])\) be the corresponding finite morphism (R. Hartshorne [21, Exercise II-5.17-(c),(d), p.128] or more generally Malgoire-Voisin [34, 1.4.2, 1.4.3]). Here we have \(A_{ji} \simeq f_{ji*}\mathcal{O}_{\text{Spec } (B_{ji})}\) so that \([A_{ji}] = f_{ji*}[\text{Spec } (B_{ji})] = d_{ji}[f_{ji*}[\text{Spec } (B_{ji})]]\), where \(d_{ji}\) is the degree of \(f_{ji}\).

Note that \(f_{ji} : \text{Spec } (B_{ji})\) is integral and closed in \(\text{Spec } (k[[t]][y_1, \cdots, y_n])\), and its defining ideal is the kernel of the ring homomorphism \(f_{ji}^z : k[[t]][y_1, \cdots, y_n] \to B_{ji}\).
Let $C_{ji} = \text{Im}(f_{ji}^\sharp)$. By the first isomorphism theorem, this $C_{ji}$ is the coordinate ring of $\mathcal{Z}_{ji}$. We have the natural factorization

$$f_{ji}^\sharp : k[[t]]\{y_1, \ldots, y_n\} \twoheadrightarrow C_{ji} \hookrightarrow B_{ji}.$$  

Going mod $I = (t^m)$, we have a commutative diagram of the induced ring homomorphisms, where $\phi_m$ is the isomorphism of $(4.3.4)$ and $f_{ji}^\sharp$ is the induced homomorphism for $f_{ji}^\sharp$

$$k_m[y_1, \ldots, y_n] \xrightarrow{f_{ji}^\sharp} B_{1i} / (t^m)B_{1i} \xrightarrow{\phi_m} B_{2i} / (t^m)B_{2i}.$$  

As $C_{ji}$ is the image of $f_{ji}^\sharp$, $C_{ji} / (t^m)C_{ji}$ is also the image of $f_{ji}^\sharp$. In particular, under the identification $\phi_m$, we have an isomorphism of $k_m[y_1, \ldots, y_n]$-algebras

$$C_{1i} / (t^m)C_{1i} \simeq C_{2i} / (t^m)C_{2i}. \tag{4.3.5}$$  

This is equivalent to having an $\mathcal{O}_{\mathcal{X}}[\gamma]$-algebra isomorphism

$$\mathcal{O}_{\mathcal{X}_1} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}_2} \simeq \mathcal{O}_{\mathcal{X}_1} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}_2}.$$  

In particular, $[\mathcal{Z}_{1i}] - [\mathcal{Z}_{2i}] \in \mathcal{N}^m(\mathcal{X}, Y, n)$.

On the other hand, $B_{ji}$ is a flat $k[[t]]$-algebra (as there is no $t$-torsion, and $k[[t]]$ is a PID), so that the degree $d_{ji}$ can be computed by specializing $f_{ji}^\sharp$ mod $t$. We have the field extensions $k \hookrightarrow C_{ji} / (t) \hookrightarrow B_{ji} / (t)$, and $d_{ji} = [B_{ji} / (t) : C_{ji} / (t)]$. Since $B_{1j} / (t) \simeq B_{2j} / (t)$ and $C_{1j} / (t) \simeq C_{2j} / (t)$ by $(4.3.4)$ and $(4.3.5)$, plugging in $1$ into $m$, we deduce that $d_{1i} = d_{2i} =: d_i$.

Thus combining the above discussions, we have

$$[A_1] - [A_2] = \sum_{i=1}^r ([A_{1i}] - [A_{2i}]) = \sum_{i=1}^r d_i ([\mathcal{Z}_{1i}] - [\mathcal{Z}_{2i}]) \in \mathcal{N}^m(\mathcal{X}, Y, n),$$  

thus $\mathcal{M}^m(\mathcal{X}, Y, n) \subset \mathcal{N}^m(\mathcal{X}, Y, n)$. This proves the equality $(4.3.1)$. \hfill $\square$

So, for the cycles in the Milnor range over $\text{Spf}(k[[t]])$, we may use the simpler equivalence $\sim_I$, if needed. We do so in §4.4 and beyond.

### 4.4 An argument for pairs of cycles

The goal of Sect. 4.4 is to prove the following:

**Proposition 4.4.1** Let $m \geq 2$. Let $\hat{\mathcal{X}} = \text{Spf}(k[[t]])$ and $I = (t^m) \subset k[[t]]$. Let $\mathcal{Z}_{1}$ and $\mathcal{Z}_{2} \in \mathcal{Z}^n(\hat{\mathcal{X}}, n)$ be integral cycles such that $\mathcal{Z}_{1} \sim_I \mathcal{Z}_{2}$.

Then there exist cycles $E_{\mathcal{Z}_{1}}$ and $E_{\mathcal{Z}_{2}} \in \mathcal{Z}^n(\hat{\mathcal{X}}, n + 1)$ such that

$$\partial E_{\mathcal{Z}_{1}} = 3_\ell - 3'_\ell, \quad \ell = 1, 2$$  

for some positive integer multiples of integral graph cycles, $\mathcal{Z}'_{1}$ and $\mathcal{Z}'_{2} \in \mathcal{Z}^n(\hat{\mathcal{X}}, n)$, satisfying $\mathcal{Z}'_{1} \sim_I \mathcal{Z}'_{2}$. \hfill $\checkmark$
Once proven, we will have two applications. The first is the surjectivity of the graph homomorphism (4.1.1), to be discussed in Sect. 5.1. The second pertains to showing that \( gr_{m} \) is an isomorphism, discussed in Sect. 5.2.

Coming back to the proof of Proposition 4.4.1, let \( \mathfrak{Z} \in z^{n}(\hat{X}, n) \) be an integral cycle. By Corollary 3.2.10, this is given by a prime ideal \( p \subset k[[r]] \) of height \( n \). For \( 1 \leq i \leq n \), we let \( p_{i} \) be the prime ideal \( p \cap k[[r]][y_{1}, \ldots, y_{n}] \). This corresponds to the image \( \mathfrak{Z}_{j}^{(i)} \) of \( \mathfrak{Z} \) under the projection \( \text{Spf}(k[[r]]) \times \prod_{k} \to \text{Spf}(k[[r]]) \times \prod_{k} \) that sends \( (y_{1}, \ldots, y_{n}) \mapsto (y_{1}, \ldots, y_{i}) \). For \( i = 0 \), we let \( \mathfrak{Z}_{j}^{(0)} = \text{Spf}(k[[r]]) \) for our notational conveniences. We have \( \mathfrak{Z}_{j}^{(i)} \in z^{i}(\hat{X}, i) \) for each \( 1 \leq i \leq n \) as seen before, in the Claim in the middle of the proof of Proposition 3.3.1.

**Definition 4.4.2** For an integral \( \mathfrak{Z} \in z^{n}(\hat{X}, n) \), for each pair \( 0 \leq i < j \leq n \) of indices, the induced morphism \( f^{j/i} : \mathfrak{Z}^{(j)} \to \mathfrak{Z}^{(i)} \) is a finite surjective morphism of integral formal schemes. Define \( d^{i/j}(\mathfrak{Z}) := \text{deg}(f^{j/i} : \mathfrak{Z}^{(j)} \to \mathfrak{Z}^{(i)}) \).

Observe the following, whose proof is obvious by definition:

**Lemma 4.4.3** The following are equivalent:

1. \( \mathfrak{Z} \) is a graph cycle.
2. \( d^{n/0}(\mathfrak{Z}) = 1 \)
3. For each \( 1 \leq i \leq n \), we have \( d^{i/(i-1)}(\mathfrak{Z}) = 1 \).

We will show that any integral cycle \( \mathfrak{Z} \in z^{n}(\hat{X}, n) \) can be turned into an integer multiple of an integral graph cycle modulo the boundary of a cycle in \( z^{n}(\hat{X}, n + 1) \), by eventually achieving Lemma 4.4.3-(3). In the process, we will show that each step of this construction respects the mod \( I \)-equivalence. This requires the following discussions for pairs of cycles.

Let \( \mathfrak{Z}_{\ell} \in z^{n}(\hat{X}, n) \) be two integral cycles for \( \ell = 1, 2 \) such that \( \mathfrak{Z}_{1} \sim_{I} \mathfrak{Z}_{2} \). For each \( \ell \), by Proposition 3.3.1, we have a system of polynomials in \( y_{1}, \ldots, y_{n} \) for \( \mathfrak{Z}_{\ell} \)

\[
\begin{align*}
P^{(1)}_{\ell}(y_{1}) &\in k[[r]][y_{1}], \\
P^{(2)}_{\ell}(y_{1}, y_{2}) &\in k[[r]][y_{1}, y_{2}], \\
&\vdots \\
P^{(n)}_{\ell}(y_{1}, \ldots, y_{n}) &\in k[[r]][y_{1}, \ldots, y_{n}],
\end{align*}
\tag{4.4.1}
\]

such that all the properties in Proposition 3.3.1 hold. Note that \( \mathfrak{Z}_{1}^{(i)} \sim_{I} \mathfrak{Z}_{2}^{(i)} \) for each \( 1 \leq i \leq n \) as well. Furthermore we have:

**Lemma 4.4.4** Under the above notations and assumptions, let \( \tilde{P}^{(i)}_{\ell} := P^{(i)}_{\ell} \) and \( R^{(0)}_{\ell} := k[[r]] \), while for \( 2 \leq i \leq n \), let \( \tilde{P}^{(i)}_{\ell} \) be the image of \( P^{(i)}_{\ell}(y_{1}, \ldots, y_{i}) \) in \( R^{(i-1)}_{\ell}[y_{i}] \), where

\[
R^{(i-1)}_{\ell} := k[[r]][y_{1}, \ldots, y_{i-1}]/(P^{(1)}_{\ell}, \ldots, P^{(i-1)}_{\ell}).
\tag{4.4.2}
\]

For each \( 1 \leq i \leq n \) and each \( \ell = 1, 2 \), let \( d^{i}_{\ell} := \text{deg}_{y_{i}} \tilde{P}^{(i)}_{\ell} \). Let \( (-1)^{i}c^{(i)}_{\ell} \in R^{(i-1)}_{\ell} \) be the constant term of \( \tilde{P}^{(i)}_{\ell} \). Let \( \tilde{R}^{(i-1)}_{\ell} = R^{(i-1)}_{\ell}/IR^{(i-1)}_{\ell} \).

Then we have:

1. \( d_{1}^{i} = d_{2}^{i} \).
2. \( \tilde{R}^{(i-1)}_{1} = \tilde{R}^{(i-1)}_{2} \).
3. In the common ring \( \tilde{R}^{(i-1)} \) of the above, we have \( c^{(i)}_{1} \equiv c^{(i)}_{2} \).

\( \square \) Springer
Note that by Proposition 3.3.1-(2), we have \( c^{(i)}_{\ell} \in (R^{(i-1)}_{\ell})^* \).

**Proof** We prove (1), (2), (3) at the same time. By the properties of Proposition 3.3.1 for \( P^{(i)}_{\ell} \), we know that \( d^{(i)}_{\ell} = \deg_{y_i} \tilde{P}^{(i)}_{\ell} \geq 1 \). Going further down mod \( I \), let \( \tilde{P}^{(i)}_{\ell} \) be the image of \( \tilde{P}^{(i)}_{\ell}(y_i) \) in \( \tilde{R}^{(i-1)}_{\ell}[y_i] \).

Note that since \( 3^{(i-1)}_1 \sim_1 3^{(i-1)}_2 \), we have \( \tilde{R}^{(i-1)}_1 = \tilde{R}^{(i-1)}_2 \), proving (2).

Since \( \tilde{P}^{(i)}_{\ell}(y_i) \) is monic in \( y_i \) in the ring \( \tilde{R}^{(i-1)}_{\ell}[y_i] \) by Proposition 3.3.1, its further image \( \tilde{\tilde{P}}^{(i)}_{\ell}(y_i) \) in \( \tilde{R}^{(i-1)}_{\ell}[y_i] \) is also monic in \( y_i \) with the same \( y_i \)-degree. Thus

\[
\deg_{y_i} \tilde{\tilde{P}}^{(i)}_{\ell} = \deg_{y_i} \tilde{P}^{(i)}_{\ell}.
\]

Furthermore, \( 3^{(i)}_1 \mod I = 3^{(i)}_2 \mod I \) as closed subschemes of \( \text{Spec}(\tilde{R}^{(i-1)}[y_i]) \), and they are respectively given by the monic polynomials \( \tilde{P}^{(i)}_{\ell}(y_i) \in \tilde{R}^{(i-1)}[y_i] \). Hence we have the equality

\[
\tilde{P}^{(i)}_{1} = \tilde{P}^{(i)}_{2} \text{ in } \tilde{R}^{(i-1)}[y_i].
\]

In particular,

\[
\deg_{y_i} \tilde{P}^{(i)}_{1} = \deg_{y_i} \tilde{P}^{(i)}_{2}.
\]

Hence, combining \((4.4.3)\) and \((4.4.5)\), we deduce (1).

On the other hand, from the equality \((4.4.4)\), we have the equality of the constant terms \((-1)^{d^{(i)}_{1}c^{(i)}_{1}} = (-1)^{d^{(i)}_{2}c^{(i)}_{2}} \) in \( \tilde{R}^{(i-1)} \). Since \( d^{(i)}_{1} = d^{(i)}_{2} \) by (1), we now have \( c^{(i)}_{1} = c^{(i)}_{2} \) in \( \tilde{R}^{(i-1)} \), proving (3). \( \square \)

**Definition 4.4.5** By Lemma 4.4.4-(1), \( d^{(i)}_{\ell} = \deg_{y_{\ell}} \tilde{P}^{(i)}_{1} = \deg_{y_{\ell}} \tilde{P}^{(i)}_{2} = d^{(i)}_{\ell} \). We now simply call it \( d_{\ell} \).

We remark that by definition, we have \( d_{\ell} = d^{(i-1)}_{(i-1)}(3_{\ell}) \) for \( \ell = 1, 2 \).

The following procedure will be used repeatedly.

**Lemma 4.4.6** Suppose \( 1 \leq i \leq n \). Let \( \ell = 1, 2 \). Let \( 3_{1} \) and \( 3_{2} \in z^n(\hat{X}, n) \) be two integral cycles such that \( 3_{1} \sim_1 3_{2} \), and

(\( \ast \)) they have the generators as in \((4.4.1)\) satisfying the properties of Proposition 3.3.1.

Suppose \( d^{(j-1)}_{(j-1)}(3_{\ell}) = 1 \) for \( j > i \). (We allow \( i = n \) as well, in which case there is no condition.)

Then there exist cycles \( C_{3_{\ell}} \in z^n(\hat{X}, n+1) \) and positive integer multiples of integral cycles, \( 3'_{1} \) and \( 3'_{2} \in z^n(\hat{X}, n) \), such that for \( \ell = 1, 2 \), we have \( 3_{\ell} - 3'_{\ell} = \partial(C_{3_{\ell}}) \), \( 3'_{1} \sim_1 3'_{2} \), the above (\( \ast \)) holds for \( 3'_{\ell} \), and

\[
d^{(j-1)}_{(j-1)}(3'_{\ell}) = \begin{cases} 
\text{not necessarily 1} & \text{for } j \geq i + 1, \\
1 & \text{for } j = i, \\
d^{(j-1)}_{(j-1)}(3_{\ell}) & \text{for } j < i
\end{cases}
\]

**Remark 4.4.7** Before we prove Lemma 4.4.6, let us give an illustration on how one can use it repeatedly to transform an integral cycle \( 3 \) into a positive integer multiple of an integral graph cycle, \( 3' \). For instance, take \( n = 3 \), and suppose that for an integral cycle \( 3 \), we had the “degree vector” \((d^{3/2}, d^{2/1}, d^{1/0})^t = (3, 2, 2)^t \), where the superscript \( t \) denotes the transpose of the matrix.
The following shows how the “degree vectors” change when we repeatedly apply Lemma 4.4.6, and also how to apply it.

The numbers over the arrows are the indices \( i \) for which we apply Lemma 4.4.6, and the entries * are the numbers not known with certainty:

\[
\begin{align*}
\text{3rd} & \rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix} & i = 3 \\
\text{2nd} & \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} & i = 2 \\
\text{1st} & \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} & i = 1
\end{align*}
\]

In general, for \( n \geq 1 \), if we are given \( (d^{n/(n-1)}, \ldots, d^{1/0})^t = (a_n, \ldots, a_1)^t \) with \( a_i \in \mathbb{N} \), then after a finite number of applications of Lemma 4.4.6, we will get to \((1, \ldots, 1)^t\). The upper bound for the number of times this operation is needed to get to \((1, \ldots, 1)^t\) may be computed by solving a recurrence relation. We guess we need at most \( 1 + 2 + \cdots + 2^{n-1} = 2^n - 1 \) times.

**Proof of Lemma 4.4.6** This is a bit technical, although its basic philosophical ideas go back to part of the arguments of [47].

Since \( i \) is fixed here, let \( R_\ell := k[\gamma^{(i-1)}] \) for \( \ell = 1, 2 \). They were previously called \( R_\ell^{(i-1)} \) in Lemma 4.4.4. These are complete local domains finite and free over \( k[[t]] \) (Lemma 3.2.6). Similarly, let \( \tilde{R}_\ell := k[\overline{\gamma}^{(i)}] \). They were previously called \( \tilde{R}_\ell^{(i)} \). Here, \( R_\ell \hookrightarrow \tilde{R}_\ell \) is the finite extension of rings corresponding to the projection \( \tilde{f}_{\ell}^{i/(i-1)} : \tilde{\gamma}^{(i)} \rightarrow \gamma^{(i-1)} \).

By the given assumption that \( d_{j/(j-1)}(\gamma) = 1 \) for \( j > i \), the images \( \tilde{P}_{\ell}^{(j)} \) of the polynomials \( P_{\ell}^{(j)}(y_1, \ldots, y_j) \) in \( \tilde{R}_\ell^{[y_{i+1}, \ldots, y_n]} \) are given by

\[
\tilde{P}_{\ell}^{(j)}(y_{i+1}, \ldots, y_n) = y_j - c_{\ell}^{(j)} \quad \text{for } i < j \leq n, \tag{4.4.6}
\]

where we recall that for \( j > i \), we have \( d_j = 1 \) and \((-1)^{d_j}c_{\ell}^{(j)} = -c_{\ell}^{(i)} \) is the constant term of \( \tilde{P}_{\ell}^{(j)} \).

On the other hand, recall that \((-1)^{d_i}c_{\ell}^{(i)} \) is the constant term of the polynomial \( \tilde{P}_{\ell}^{(i)} \in R_\ell[y_i] \), and we have \( c_{\ell}^{(i)} \in R_\ell^{(i)} \) by Proposition 3.3.1-(2). Let \( (y_i, y'_i) \) be the coordinates of \( \Box^2 \) in \( \text{Spf} \ (R_\ell) \times \Box^2 \), and consider the closed formal subscheme \( C_\ell \subset \text{Spf} \ (R_\ell) \times \Box^2 \) given by the polynomial (cf. [47, Lemma 2])

\[
Q_{3 \ell}(y_i, y'_i) := \tilde{P}_{\ell}^{(i)}(y_i) - (y_i - 1)d_{i-1}^{d_i}(y_i - c_{\ell}^{(i)})y'_i \in R_\ell[y_i, y'_i].
\]

Its codimension 1 faces are

\[
\begin{align*}
C_\ell \cap \{y_i = 0\} &= \{(-1)^{d_i}c_{\ell}^{(i)}(1 - y'_i) = 0\} = \{y'_i = 1\} = \emptyset, \\
C_\ell \cap \{y_i = \infty\} &= \{1 - y'_i = 0\} = \emptyset, \\
C_\ell \cap \{y'_i = 0\} &= \{\tilde{P}_{\ell}^{(i)}(y_i) = 0\}, \\
C_\ell \cap \{y'_i = \infty\} &= \{(y_i - 1)d_{i-1}^{d_i}(y_i - c_{\ell}^{(i)}) = 0\} = \{y_i = c_{\ell}^{(i)}\},
\end{align*}
\tag{4.4.7}
\]

where \( \hat{\ } \) uses that the polynomial \( \tilde{P}_{\ell}^{(i)}(y_i) \in R_\ell[y_i] \) is monic in \( y_i \). Since these faces in (4.4.7) are of codimension \( \geq 1 \) in \( \text{Spf} \ (R_\ell) \times \Box^1 \), where they intersect no additional face (cf. Lemma 3.2.5), we see that \( C_\ell \) intersects no codimension \( \geq 2 \) face. Thus we have the condition (GP) of Definition 2.2.2 for \( C_\ell \). The condition (SF) of Definition 2.2.2 is immediate. Thus we have \( C_\ell \in z^1(\text{Spf} \ (R_\ell), 2) \). Furthermore, from the calculations in (4.4.7), we deduce that

\[
\partial C_\ell = -\{\tilde{P}_{\ell}^{(i)}(y_i) = 0\} + [c_{\ell}^{(i)}] \in z^1(\text{Spf} \ (R_\ell), 1).
\tag{4.4.8}
\]
For $1 \leq j \leq n$, let $\alpha^{(j)}(\ell)$ be the image of $y_j$ in $k[[t]]/I(\mathcal{Z}_\ell).$ For $j > i$, we are given that $d^{(j-i)}(\mathcal{Z}_\ell) = 1$ so that we have
\[
k[[t]]/y_1, \ldots, y_n)/I(\mathcal{Z}_\ell) = \tilde{R}_\ell.
\]
Under this identification, for $j > i$, $\alpha^{(j)}(\ell)$ coincides with the previously defined $c^{(j)}(\ell)$ as used in (4.4.6). However, these may differ for $j \leq i$.

Let $(Y_1, \ldots, Y_{i-1}, Y_i, Y'_i, Y_{i+1}, \ldots, Y_n)$ be the coordinates of $\mathcal{Z}_{n+1}$ in $\text{Spf } (R_\ell) \times \mathcal{Z}^{n+1}$. Define the cycle $\mathcal{C}_{3\ell} \in z^n(\text{Spf } (R_\ell), n + 1)$ given by the system of polynomials
\[
\tilde{C}_{3\ell} : \begin{cases}
P^{(1)}_\ell(Y_1), \\
\vdots \\
P^{(i-1)}_\ell(Y_1, \ldots, Y_{i-1}), \\
Q_{3\ell}(Y_i, Y'_i), \\
P^{(i+1)}_\ell(Y_1, \ldots, Y_{i+1}), \\
\vdots \\
P^{(n)}_\ell(Y_1, \ldots, Y_n)
\end{cases}
\]
in $R_\ell\{Y_1, \ldots, Y_{i-1}, Y_i, Y'_i, Y_{i+1}, \ldots, Y_n\}$. Taking (4.4.6) into account, we use the concatenation notations as in

\[
\mathcal{C}_{3\ell} = 3^{(i-1)} \otimes C_\ell \otimes [c^{(i+1)}] \otimes \cdots \otimes [c^{(n)}].
\]

Define $C_{3\ell} \in z^n(\mathbb{X}, n + 1)$ to be
\[
C_{3\ell} := (-1)^j \pi_\ell, * (\tilde{C}_{3\ell}),
\]
where $\pi_\ell := f^{(i-1)/0}_\ell : \text{Spf } (R_\ell) \to \mathbb{X}$ is the finite surjective morphism, and
\[
\pi_\ell, * : z^n(\text{Spf } (R_\ell), *) \to z^n(\mathbb{X}, *)
\]
is the finite push-forward as in Lemma 2.3.3.

We compute the boundary of $C_{3\ell}$ by looking at the boundary of $\mathcal{C}_{3\ell}$. There, note that both $3^{(i-1)}$ and $[c^{(j)}]_\ell$ for $j > i$ have no proper face (cf. Lemma 3.2.5). Consequently, combined with (4.4.8), we can write
\[
\partial \mathcal{C}_{3\ell} = (-1)^j \partial_\ell 3^{(i-1)} \otimes (\partial C_\ell) \otimes [c^{(i+1)}] \otimes \cdots \otimes [c^{(n)}] \\
= (-1)^j 3^{(i-1)} \otimes [P^{(i)}_\ell = 0] \otimes [c^{(i+1)}] \otimes \cdots \otimes [c^{(n)}] \\
- (\partial_\ell 3^{(i-1)} \otimes [c^{(i)}] \otimes [c^{(i+1)}] \otimes \cdots \otimes [c^{(n)}] \in z^n(\text{Spf } (R_\ell), n).
\]

We analyze the two terms of (4.4.10). Consider the closed formal subscheme $\tilde{Z}_\ell \subset \text{Spf } (\tilde{R}_\ell) \times \mathcal{Z}^n$ given by the system of polynomials in $\tilde{R}_\ell\{Y_1, \ldots, Y_n\}$
\[
\tilde{Z}_\ell : \{Y_1 - \alpha^{(1)}(\ell), \ldots, Y_n - \alpha^{(n)}(\ell)\},
\]
Since $\tilde{R}_\ell$ is an integral $k$-domain and all $\alpha^{(j)}(\ell) \in \tilde{R}_\ell$ for $1 \leq j \leq n$ under the identification (4.4.9), the formal scheme $\tilde{Z}_\ell$ is integral. Then by definition, the first term of (4.4.10) without the sign
\[
3^{(i-1)} \otimes [P^{(i)}_\ell = 0] \otimes [c^{(i+1)}] \otimes \cdots \otimes [c^{(n)}]
is just \( f_{t, \pi}^{i/(i-1)}(\tilde{Z}_\ell) \) because \( \alpha_{\ell}^{(j)} = \epsilon_{\ell}^{(j)} \) for \( j > i \). By definition, we have \( f_{t, \pi}^{i/0}(\tilde{Z}_\ell) = 3\ell \) as well. Since \( f_{t, \pi}^{i/(i-1)} \circ f_{t, \pi}^{i/(i-1)} = f_{t, \pi}^{i/0} \) and \( f_{t, \pi}^{i/(i-1)} = \pi_{t, \pi} \), we thus have \( \pi_{t, \pi}(3\ell^{i-1}) \equiv [\tilde{P}_{\ell}^{(i)} = 0] \otimes [c_{\ell}^{(i+1)}] \otimes \cdots \otimes [c_{\ell}^{(n)}] = 3\ell \).

This time, consider the closed formal subscheme \( \tilde{Z}_\ell' \subset \text{Spf}(\tilde{R}_\ell) \times \mathbb{A}^n \) given by the polynomials in \( \tilde{R}_\ell \{ Y_1, \ldots, Y_n \} \)

\[
\tilde{Z}_\ell' := \{ Y_1 - \alpha_{\ell}^{(1)}, \ldots, Y_{i-1} - \alpha_{\ell}^{(i-1)}, Y_i - \epsilon_{\ell}^{(i)}, Y_{i+1} - \epsilon_{\ell}^{(i+1)}, \ldots, Y_n - \epsilon_{\ell}^{(n)} \}.
\]

Here all \( \alpha_{\ell}^{(j)} \) for \( j < i \), and \( \epsilon_{\ell}^{(j)} \) for \( j > i \) are in \( \tilde{R}_\ell \) so that \( \tilde{Z}_\ell' \) is an integral formal scheme. Note that \( \epsilon_{\ell}^{(i)} \in \tilde{R}_\ell(\subset \tilde{R}_\ell) \), and it may not be equal to \( \alpha_{\ell}^{(i)} \) in \( \tilde{R}_\ell \). By definition

\[
f_{t, \pi}^{i/(i-1)}(\tilde{Z}_\ell') = \tilde{Z}_\ell^{(i-1)} \equiv [\epsilon_{\ell}^{(i)}] \otimes [\epsilon_{\ell}^{(i+1)}] \otimes \cdots \otimes [\epsilon_{\ell}^{(n)}],
\]

which is the second term of \((4.4.10)\) without the sign, so this is an integral cycle. Hence applying \( \pi_{t, \pi} = f_{t, \pi}^{i/(i-1)/0} \), we obtain a cycle \( \pi_{t, \pi}(\tilde{Z}_\ell^{(i-1)}) \equiv [\epsilon_{\ell}^{(i)}] \otimes [\epsilon_{\ell}^{(i+1)}] \otimes \cdots \otimes [\epsilon_{\ell}^{(n)}] \), which is a positive integer multiple of an integral cycle.

Since the boundary operators commute with push-forwards (Lemma 2.3.3), we have

\[
\partial C_{3\ell} = (-1)^j \pi_{t, \pi}(\partial C_{\tilde{3}\ell}) = 3\ell - \pi_{t, \pi}(f_{t, \pi}^{i/(i-1)}(\tilde{3}\ell)) = 3\ell - f_{t, \pi}^{i/0}(\tilde{3}\ell).
\]

So, letting \( 3\ell' := f_{t, \pi}^{i/0}(\tilde{3}\ell) \), we have \( \partial C_{3\ell} = 3\ell - 3\ell' \).

We now claim that this cycle \( 3\ell' \) satisfies the desired properties. This is a positive integer multiple of an integral cycle. We can still define \( d^{i/(i-1)}(3\ell') \), ignoring the integer multiplicity. By definition, for \( j < i \), we have \( (3\ell')^{(j)} = 3\ell^{(j)} \), thus \( d^{j/(j-1)}(3\ell') = d^{j/(j-1)}(3\ell) \).

Since \( \epsilon_{\ell}^{(i)} \in \tilde{R}_\ell \) and \( (3\ell')^{(i)} = 3\ell^{(i-1)} \otimes [\epsilon_{\ell}^{(i)}] \), we have \( k(3\ell^{(i)}) = k(3\ell^{(i-1)}) = R_\ell \).

So, the projection \( (3\ell')^{(i)} \to (3\ell^{(i-1)}) = 3\ell^{(i-1)} \) is of degree 1, i.e. \( d^{i/(i-1)}(3\ell') = 1 \).

On the other hand, we have \( \epsilon_{\ell}^{(j)} \in \tilde{R}_\ell = k(3\ell^{(i)}) \) for \( j > i \), so that \( d^{i+1/(i)}(3\ell') \) may not be equal to 1.

Finally, from that \( 3\ell \sim_1 3\ell \), by Lemma 4.4.4, we know that \( 3\ell^{(i-1)} \sim_1 3\ell^{(i-1)} \) so that \( \epsilon_{\ell}^{(i)} \equiv \epsilon_{\ell}^{(i)} \) in \( R_1/R_1 = R_2/I R_2 \), while we also know that \( 3\ell^{(i)} \sim_1 3\ell^{(i)} \) so that \( \epsilon_{\ell}^{(j)} \equiv \epsilon_{\ell}^{(j)} \) in \( R_1/I \tilde{R}_1 = \tilde{R}_2/I \tilde{R}_2 \). Hence \( 3\ell' \sim_1 3\ell \). This finishes the proof of Lemma 4.4.6.  

\[ \square \]

**Proof of Proposition 4.4.1** We begin to apply Lemma 4.4.6 to a pair of integral cycles \( 3\ell \) in \( z^n(\tilde{X}, n) \) such that \( 3\ell \sim_1 3\ell \). By Proposition 3.3.1, they satisfy the assumptions of Lemma 4.4.6. So we can apply it repeatedly.

As seen in Remark 4.4.7, in finite number of applications of Lemma 4.4.6, we eventually get to a pair of integral cycles with the equal multiplicities, whose degree vectors are \( (d^{i/(n-1)}, \ldots, d^{1/0})^\prime = (1, \ldots, 1)^\prime \). By Lemma 4.4.3, they are positive integer multiples of integral graph cycles in \( z^n(\tilde{X}, n) \).

Take the sums, say \( C_\ell \), of the cycles in \( z^n(\tilde{X}, n+1) \) appearing in repeated applications of Lemma 4.4.6. Taking their boundaries, we have the cancellations of all the intermediate \( 3\ell' \)'s, and we have \( \partial(C_\ell) = 3\ell - 3\ell' \), where \( 3\ell \) is the given cycle in \( z^n(\tilde{X}, n) \), and the resulting \( 3\ell' \) is a positive integer multiple of an integral graph cycle, such that \( 3\ell' \sim_1 3\ell \). This proves Proposition 4.4.1.  

\[ \square \]
5 The proofs of the main theorems

5.1 The isomorphism over \( k[[t]] \)

The first application of Proposition 4.4.1 is the following:

**Lemma 5.1.1** Let \( k \) be a field. Then the graph homomorphism \( \text{gr} : K^M_n(k[[t]]) \to \text{CH}^n(\text{Spf}(k[[t]]), n) \) of (4.1.1) is surjective.

**Proof** Proposition 4.4.1 is stated for pairs of integral cycles that are mod \( I \)-equivalent. We apply it to the identical pair \((Z, Z)\) for each integral cycle \( Z \in \text{zn}(\text{Spf}(k[[t]]), n)\). In particular, for a cycle \( E \in \text{zn}(\text{Spf}(k[[t]]), n+1) \) and a positive integer multiple of a graph cycle, call \( E \) a cycle \( \gamma \) such that \( \partial E = Z - \gamma \). Since \( \gamma \) belongs to the image of the graph map \( \text{gr} \) and \( Z \equiv \gamma \) in \( \text{CH}^n(\text{Spf}(k[[t]]), n) \), this shows that \( \text{gr} \) is surjective. \( \square \)

**Remark 5.1.2** One may give another proof of Lemma 5.1.1 based on arguments of B. Totaro [47, Sect. 4]. We outline what extra modifications are needed in addition to the arguments of loc.cit.

In loc.cit., when \( F \) is a field, to prove that the graph homomorphism \( \text{gr} : K^M_n(F) \to \text{CH}^n(\text{Spec}(F), n) \) is an isomorphism, Totaro used the norm \( N : L^\times \to F^\times \), where \( L \) is the residue field of a given point and \( F \subset L \) is a finite extension of fields.

Specializing to the case \( F = k((t)) \), what we need is that this norm operation for fields is compatible with the finiteness of our cycles over \( k[[t]] \). Indeed if a finite ring extension \( k[[t]] \to B \) describes an integral cycle \( Z \in \text{zn}(\text{Spf}(k[[t]]), n) \), then \( B \) is a free \( k[[t]] \)-module and there is the norm map \( N : B^\times \to k[[t]]^\times \) by Corollary 3.2.7 and Definition 3.2.8. This is compatible with the norm \( N : L^\times \to k((t))^\times \), when \( L = \text{Frac}(B) \), in that the following diagram commutes:

\[
\begin{array}{ccc}
B^\times & \xrightarrow{N} & L^\times \\
\downarrow & & \downarrow \\
k[[t]]^\times & \xrightarrow{N} & k((t))^\times.
\end{array}
\]

So, the norms of elements of \( B^\times \) belong to \( k[[t]]^\times \). Following along the argument of [47, Sect. 4], together with the above, we may prove that the graph map of Lemma 5.1.1 is surjective. We do not attempt to give details as this is redundant.

**Corollary 5.1.3** Let \( k \) be a field with \( |k| \gg 0 \) so that \( K^M_n(k[[t]]) = \hat{K}^M_n(k[[t]]) \).

Then the graph map

\[
gr : K^M_n(k[[t]]) \to \text{CH}^n(\text{Spf}(k[[t]]), n)
\]

is an isomorphism.

**Proof** The map \( gr \) is surjective by Lemma 5.1.1 and injective by Lemma 4.2.6. \( \square \)

We deduce the following formal-rigid geometric analogue of the Gersten conjecture for higher Chow groups (cf. S. Bloch [5, Theorem (10.1)]):

**Corollary 5.1.4** Let \( k \) be a field with \( |k| \gg 0 \). Then the localization map

\[
\text{CH}^n(\text{Spf}(k[[t]]), n) \to \text{CH}^n(\text{Sp}(k((t))), n)
\]

is injective.
This follows from Corollaries 4.2.7 and 5.1.3, and the Gersten conjecture for $K^M_n$ of $k[[t]]$ (see M. Kerz [30, Proposition 10, p. 181]). □

Lemma 5.1.1 for $k[[t]]$ also implies the surjectivity for $k_m$:

**Corollary 5.1.5** Let $m \geq 2$. Let $k$ be a field, $k_m := k[[t]]/(t^m)$, and $Y := \text{Spec} (k_m)$.

Then the graph homomorphism $gr_{k_m} : K^M_n (k_m) \to \text{CH}^n (k_m, n)$ of (4.1.2) is surjective.

**Proof** Recall from (2.4.4) that

$$\text{CH}^n (k_m, n) = \frac{z^n (\text{Spf} (k[[t]]), n)}{\partial (z^n (\text{Spf} (k[[t]]), n + 1)) + M^n (\text{Spf} (k[[t]]), Y, n)}.$$

Thus together with Lemma 3.2.5, we have the natural surjective map

$$\text{CH}^n (\text{Spf} (k[[t]]), n) \to \text{CH}^n (k_m, n),$$

and we deduce the commutative diagram

$$\begin{array}{ccc}
K^M_n (k[[t]]) & \xrightarrow{gr} & \text{CH}^n (\text{Spf} (k[[t]]), n) \\
\downarrow & & \downarrow \\
K^M_n (k_m) & \xrightarrow{gr_{k_m}} & \text{CH}^n (k_m, n).
\end{array}$$

Since the top horizontal arrow is surjective by Lemma 5.1.1, so is the bottom arrow. □

### 5.2 The isomorphism over $k_m$

In Corollary 5.1.5, we saw that $gr_{k_m}$ is surjective. We want to show that it is an isomorphism when $|k| \gg 0$. The following, which is the part (2) of the first main theorem, Theorem 1.2.1, answers it. The author thanks the referee for suggesting this way of simplification of an earlier version.

**Theorem 5.2.1** Let $m \geq 2$. Let $k$ be a field with $|k| \gg 0$ so that $K^M_n (k[[t]]) = \hat{K}^M_n (k[[t]])$.

Then the graph homomorphism

$$gr_{k_m} : K^M_n (k_m) \to \text{CH}^n (k_m, n)$$

is an isomorphism.

**Proof** Consider the following commutative diagram with the exact rows

$$\begin{array}{cccccc}
0 & \to & J_K & \to & K^M_n (k[[t]]) & \to & K^M_n (k_m) & \to & 0 \\
\downarrow & & \downarrow \cong & & \downarrow gr & & \downarrow gr_{k_m} & & \\
0 & \to & J_C & \to & \text{CH}^n (\text{Spf} (k[[t]]), n) & \to & \text{CH}^n (k_m, n) & \to & 0,
\end{array}$$

where $J_K$ and $J_C$ are the respective kernels of the canonical horizontal maps. By Corollary 5.1.3, we know that the middle vertical map is an isomorphism. In particular, the induced left vertical map $J_K \to J_C$ is injective. We claim that this is surjective as well.

Indeed, by definition $J_C$ is generated by the image of $M^n (\hat{X}, Y, n)$ in the middle group $\text{CH}^n (\text{Spf} (k[[t]]), n)$. Since $M^n (\hat{X}, Y, n) = N^n (\hat{X}, Y, n)$ by Lemma 4.3.1, we deduce that

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$J_C$ is generated by the cycles of the form $[3_1] - [3_2]$, where $3_1, 3_2 \in \pi^n(\widehat{X}, n)$ are integral cycles such that $3_1 \sim 3_2$. However, by Proposition 4.4.1, we may further assume that $3_1, 3_2$ are integral graph cycles such that $3_1 \sim 3_2$. Since $gr$ is the graph map, this class lies in the image of a member of $K^M_n(k[[t]])$. By diagram chasing, one notes that this member belongs to $J_K$. Thus $J_K \to J_C$ is surjective as desired.

Once the left and the middle vertical maps are isomorphisms, we deduce that the right vertical map $gr_m$ is also an isomorphism, proving the theorem.

Recall that by the condition (SF) of Definition 2.2.2, we have the Gysin map $ev_{CH} : CH^n((k_{m+1}, n)) \to CH^n((k, n))$ given by the evaluation $t = 0$. This is surjective because $gr_k : K^M_n(k) \to CH^n((k, n))$ is an isomorphism by [37, 47], and the map $ev_0$ in the following commutative diagram is surjective:

$$
\begin{align*}
K^M_n((k_{m+1}, n)) &\xrightarrow{ev_0} K^M_n((k, n)) \\
\simeq gr_m &\xrightarrow{ev_0} gr_k
\end{align*}
$$

(T.5.2.1)

Taking the kernels of the horizontal maps, we define the relative parts

$$
\begin{align*}
\{ K^M_n((k_{m+1}, (t))) := \ker(ev_0) \quad \text{and} \quad CH^n((k_{m+1}, (t)), n) := \ker(ev_{CH}). \}
\end{align*}
$$

(5.2.2)

We deduce the second main theorem, Theorem 1.6.1:

**Theorem 5.2.2** Let $m \geq 1$. Let $k$ be a field of characteristic 0.

Then we have an isomorphism of the big de Rham–Witt forms of $k$ of Hesselholt–Madsen [24] with the relative higher Chow groups

$$
\mathbb{W} m \Omega^n_{k} \simeq CH^n((k_{m+1}, (t)), n).
$$

**Proof** The diagram (5.2.1) with the isomorphic vertical maps induces the isomorphism of the relative parts in (5.2.2),

$$
K^M_n((k_{m+1}, (t))) \simeq CH^n((k_{m+1}, (t)), n).
$$

Here $K^M_n((k_{m+1}, (t))) \simeq \bigoplus_{i=1}^m \Omega^m_{k_{i}}$ (Park-Ünver [41, Proposition 5.4.2]), while $\mathbb{W} m \Omega^n_{k} \simeq \bigoplus_{i=1}^m \Omega^m_{k_{i}}$ (Rülling [42, Remark 1.12]). This proves the theorem. \hfill \square

Theorem 5.2.2 is the counterpart to a result known for additive higher Chow groups of a field by K. Rülling [42]. In an earlier version, it was mistakenly claimed for all fields $k$ using [43], but Matthew Morrow pointed out that the reasoning had a jump. The characteristic $p > 0$ case remains to be checked. The referee has suggested a good way to argue for this remaining case, and it will be discussed in a separate work.

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