1. Introduction

It was first realized in [FK] and [S1] that the basic representation $V$ of an affine Lie algebra $\hat{g}$ of ADE type can be constructed from the root lattice $Q$ of the corresponding finite dimensional Lie algebra $g$ as follows:

$$V = S(Q \otimes \mathbb{C}[t^{-1}t^{-1}]) \otimes \mathbb{C}[Q],$$

where the first factor is a symmetric algebra and the second one is a group algebra. The affine algebra $\hat{g}$ contains a Heisenberg algebra $\hat{h}$. One can define the so-called vertex operators $X(\alpha, z)$ associated to $\alpha \in Q$ acting on $V$ essentially using the Heisenberg algebra $\hat{h}$. The representation of $\hat{g}$ on $V$ is then obtained from the action of the Heisenberg algebra $\hat{h}$ and the vertex operators $X(\alpha, z)$ associated to $\alpha$ in the root system of $g$.

This construction was extended in [F2] to more general lattices to provide vertex representations of affinization of Kac-Moody Lie algebras. The special case of affinization of affine Lie algebras called toroidal Lie algebras was discussed further in [MRY].

Another important special case of vertex representations is given by the standard lattice $\mathbb{Z}^N$. The vertex operators corresponding to the unit vectors in $\mathbb{Z}^N$ give rise to a representation of an infinite-dimensional Clifford algebra [F3], the relation known as boson-fermion correspondence. In the special case $N = 1$, the transition matrix between the monomial bases for representations of Heisenberg and Clifford algebras yields the character tables of symmetric groups $S_n$ for all $n$ [F3] (see [J1]).

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Vertex operators have appeared and played an important role in many diversified fields of mathematics and physics (cf. [FLM] and references therein). Surprisingly, they also appeared in recent works of Nakajima (cf. [N] and references therein) and independently of Goro-jnowski [Gr] on the (homology groups of) Hilbert schemes of points on a surface \(X\). In particular when \(X\) is a minimal resolution \(\mathbb{C}^2/\Gamma\) of the simple quotient singularity of \(\mathbb{C}^2\) by a finite subgroup \(\Gamma\) of \(SU_2\), they were able to realize geometrically the vertex representations of affine Lie algebras.

Vertex operators showed up in equivariant K-theory as well. Partly motivated by a footnote in [Gr], Segal [S2] outlined the \(S_n\)-equivariant K-theory of the \(n\)-th direct product \(X^n\) of \(X\). Generalizing [S2], the third-named author [W] studied in detail the \(\Gamma_n\)-equivariant K-theory of \(X^n\) for a \(\Gamma\)-space \(X\), where \(\Gamma_n\) is the so-called wreath product which is the semi-direct product of the symmetric group \(S_n\) and the \(n\)-th direct product of a finite group \(\Gamma\). Among other results a construction of vertex representations was proposed in [W] in terms of representations of wreath products; in particular when \(\Gamma\) is a subgroup of \(SU_2\), a realization of affine algebras thus obtained can be regarded as a new form of McKay correspondence [Mc].

The goal of this paper is to establish firmly such a finite group theoretic approach of the vertex representations corresponding to a very general class of lattices, including the (affine) root lattices of ADE type and the standard lattice \(\mathbb{Z}^N\) as special cases. It was conjectured [W] that there exists a natural isomorphism from the representation ring of \(\Gamma_n\) for a finite subgroup \(\Gamma\) of \(SU_2\) to the homology group of the Hilbert scheme of \(n\) points on \(\mathbb{C}^2/\Gamma\) after the dimensions of these two spaces were shown to be the same. In this way, the group theoretic realization of vertex representations for affine algebras in this paper might be regarded as a counterpart of the geometric construction given in [Gr, N] for the surface \(\mathbb{C}^2/\Gamma\).

Let us explain the contents of this paper in more detail. Let \(\Gamma\) be an arbitrary finite group. Denote by \(R_{\mathbb{Z}}(\Gamma_n)\) the representation ring of the wreath product \(\Gamma_n\) and let \(R(\Gamma_n) = R_{\mathbb{Z}}(\Gamma_n) \otimes_{\mathbb{Z}} \mathbb{C}\). We set

\[ R_{\Gamma} = \bigoplus_{n \geq 0} R(\Gamma_n). \]

The representation theory of wreath products was developed by Specht (cf. [M2, Z]). Given a self-dual virtual character \(\xi\) in \(R(\Gamma)\), we introduce a \textit{weighted} bilinear form on \(R(\Gamma_n)\) and an induced one on \(R_{\Gamma}\).
denoted by $\langle \cdot, \cdot \rangle_\xi$. When $\xi$ is trivial, it reduces to the standard bilinear form on the representation ring of a finite group.

It turns out that the spaces constructed from the representation theory of $\Gamma_n$ reproduce those appearing in vertex representations (see Eqn. [1.1]). In fact one can construct the symmetric algebra $S_\Gamma$ associated to the lattice $R_\mathbb{Z}(\Gamma)$. We define a natural bilinear form $\langle \cdot, \cdot \rangle_\xi'$ induced from the $\xi$-weighted bilinear form on $R_\Gamma$. We define a characteristic map $\text{ch}$ from $(R_\Gamma, \langle \cdot, \cdot \rangle_\xi)$ to $(S_\Gamma, \langle \cdot, \cdot \rangle_\xi')$ and show that it is an isometry. The map $\text{ch}$ for $\xi$ being a trivial character was considered in [M1, M2].

We introduce an infinite-dimensional Heisenberg algebra $\hat{h}_{\Gamma,\xi}$ acting on $S_\Gamma$ associated to $\Gamma$ and $\xi$, and thus regard $S_\Gamma$ as the Fock space of $\hat{h}_{\Gamma,\xi}$. We give a group theoretic realization [W] of this Heisenberg algebra acting on $R_\Gamma$ which is compatible with the one on $S_\Gamma$ via the characteristic map $\text{ch}$.

Denote by $\mathcal{F}_\Gamma$ the tensor product of $R_\Gamma$ and the group algebra of $R_\mathbb{Z}(\Gamma)$ which is the full space of a vertex representation. Given $\gamma \in R_\mathbb{Z}(\Gamma)$, we define a generating function $X(\gamma, z)$ of group theoretic operators acting on $R_\Gamma$. We prove that $X(\gamma, z)$ thus defined is precisely the vertex operator associated to $\gamma$ which can be essentially constructed in terms of the Heisenberg algebra $\hat{h}_{\Gamma,\xi}$. We then calculate the product of two vertex operators of the form $X(\gamma, z), \gamma \in R_\mathbb{Z}(\Gamma)$, which are equivalent to the commutation relations among the components of $X(\gamma, z)$.

Now we specialize to the important case when $\Gamma$ is a finite group of $SU_2$ and $\xi$ is twice the trivial character minus the character of a two-dimensional natural representation of $\Gamma$. According to McKay [Mc], the lattice $(R_\mathbb{Z}(\Gamma), \langle \cdot, \cdot \rangle_\xi)$ is positive semi-definite and its radical is one-dimensional with a generator $\delta$ given by the regular character of $\Gamma$. Furthermore $(R_\mathbb{Z}(\Gamma), \langle \cdot, \cdot \rangle_\xi)$ can be identified with the root lattice of the affine Lie algebra associated to a simple Lie algebra $\tilde{\mathfrak{g}}$ of ADE type. Our general construction when specialized to this case gives a group theoretic realization of the toroidal Lie algebra associated to $\tilde{\mathfrak{g}}$.

The sublattice $\overline{R}_\mathbb{Z}(\Gamma)$ of $R_\mathbb{Z}(\Gamma)$ generated by non-trivial irreducible characters of $\Gamma$ can be identified with the root lattice of $\tilde{\mathfrak{g}}$. It follows that the lattice $R_\mathbb{Z}(\Gamma)$ is a direct sum of $\overline{R}_\mathbb{Z}(\Gamma)$ and the lattice $\mathbb{Z}\delta$ with zero bilinear form generated by $\delta$. Therefore $\mathcal{F}_\Gamma$ can be decomposed as a tensor product of the space $\mathcal{F}_\Gamma$ associated to the lattice $\overline{R}_\mathbb{Z}(\Gamma)$ with the space associated to the lattice $\mathbb{Z}\delta$. This gives a new form of “McKay correspondence”: starting with a finite subgroup $\Gamma \subset SU_2$ we
have been able to construct the basic representation on $\mathcal{F}_\Gamma$ of the affine Lie algebra $\widehat{\mathfrak{g}}$ with the Dynkin diagram corresponding to $\Gamma$.

If we specialize $\xi$ to be the trivial character of $\Gamma$, then the lattice $(R\mathbb{Z}(\Gamma), \langle \cdot, \cdot \rangle_{\xi})$ is just the standard lattice $\mathbb{Z}^N$ where $N$ is the number of irreducible characters of $\Gamma$. The vertex operators corresponding to irreducible characters of $\Gamma$ generate an infinite-dimensional Clifford algebra. In this case the transition matrix between the monomial bases of Heisenberg and Clifford algebras yields the character table of the wreath product $\Gamma_n$ for all $n$, generalizing the symmetric group picture.

We announce some further directions we are currently pursuing. One is on the $q$-deformation of our construction while another is on the generalized McKay correspondence and affine Lie algebras of non-simply-laced type. We will also generalize the results in [11, 12] to the wreath product setting. In the end we pose the very important problem to give a group theoretic construction of the whole vertex algebra structure on $\mathcal{F}_\Gamma$ [13, FLM].

The paper is organized as follows. In Sect. 2 we review the theory of wreath products. In Sect. 3 we introduce the weighted bilinear form on $R\Gamma$. In Sect. 4 we define the Heisenberg algebra $\mathfrak{h}_{\Gamma,\xi}$ and its Fock space together with some natural basis. In Sect. 5 we establish the isometry between $R\Gamma$ and $S\Gamma$. In Sect. 6 we define the vertex operators acting on $R\Gamma$. In Sect. 7 we calculate products of vertex operators and establish a new McKay correspondence. In Sect. 8 we derive the character tables of $\Gamma_n$ from the vertex operator approach.

2. Preliminaries on wreath products

2.1. The wreath product $\Gamma_n$. Let $\Gamma$ be a finite group with $r + 1$ conjugacy classes. We denote by $\Gamma^*$ the set of complex irreducible characters and by $\Gamma_*$ the set of conjugacy classes. The character value $\gamma(c)$ of $\gamma \in \Gamma^*$ at a conjugacy class $c \in \Gamma_*$ yields the character table $\{\gamma(c)\}$ of $\Gamma$.

The space of class functions on $\Gamma$ is given by

$$R(\Gamma) = \bigoplus_{i=0}^{r} \mathbb{C}\gamma_i,$$

where $\gamma_i$ are all the inequivalent irreducible characters of $\Gamma$ and in particular $\gamma_0$ is the trivial character. We denote by $R\mathbb{Z}(\Gamma)$ the integral combination of irreducible characters of $\Gamma$. Denote by $c^i, i = 0, \ldots, r$ the distinct conjugacy classes in $\Gamma_*$, in particular $c^0$ is the identity conjugacy class.
For \( c \in \Gamma_* \) let \( \zeta_c \) be the order of the centralizer of an element in the class \( c \), so the order of the class is then \( |c| = |\Gamma|/\zeta_c \). The usual bilinear form on \( R(\Gamma) \) is defined as follows:

\[
\langle f, g \rangle_{\Gamma} = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x)g(x^{-1}) = \sum_{c \in \Gamma_*} \zeta_c^{-1} f(c)g(c^{-1}),
\]

where \( c^{-1} \) denotes the conjugacy class \( \{x^{-1}, x \in c\} \). Clearly \( \zeta_c = \zeta_{c^{-1}} \). We will refer to this bilinear form as the standard one on \( R(\Gamma) \). We will often write \( \langle , \rangle \) for \( \langle , \rangle_{\Gamma} \) when no ambiguity may arise. It is well known that

\[
\langle \gamma_i, \gamma_j \rangle = \delta_{ij},
\]

\[
\sum_{\gamma \in \Gamma_*} \gamma(c')\gamma(c^{-1}) = \delta_{c,c'} \zeta_c, \quad c, c' \in \Gamma_*.
\]

Thus \( R_{\mathbb{Z}}(\Gamma) \) endowed with this bilinear form becomes an integral lattice in \( R(\Gamma) \).

**Remark 2.1.** Besides the standard bilinear form on a finite group one can introduce a standard Hermitian form which is also useful in our constructions. Throughout the paper we have chosen to use bilinear forms instead of Hermitian forms.

Given a positive integer \( n \), let \( \Gamma^n = \Gamma \times \cdots \times \Gamma \) be the \( n \)-th direct product of \( \Gamma \), and let \( \Gamma^0 \) be the trivial group. The symmetric group \( S_n \) acts on \( \Gamma^n \) by permutations: \( \sigma(g_1, \cdots, g_n) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)}) \). The wreath product of \( \Gamma \) with \( S_n \) is defined to be the semi-direct product

\[
\Gamma_n = \{(g, \sigma)| g = (g_1, \cdots, g_n) \in \Gamma^n, \sigma \in S_n \}
\]

with the multiplication

\[
(g, \sigma) \cdot (h, \tau) = (g \sigma(h), \sigma \tau).
\]

Note that \( \Gamma_n = S_n \) when \( \Gamma = 1 \), and \( \Gamma_n \) is the hyperoctahedral group of rank \( n \) when \( \Gamma = \mathbb{Z}/2\mathbb{Z} \).

**2.2. Conjugacy classes of \( \Gamma_n \).** Let \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \) be a partition of integer \( |\lambda| = \lambda_1 + \cdots + \lambda_l \), where \( \lambda_1 \geq \cdots \geq \lambda_l \geq 1 \). The integer \( l \) is called the length of the partition \( \lambda \) and is denoted by \( l(\lambda) \). We will identify the partition \( (\lambda_1, \lambda_2, \cdots, \lambda_l) \) with \( (\lambda_1, \lambda_2, \cdots, \lambda_l, 0, \cdots, 0) \). We will also make use of another notation for partitions:

\[
\lambda = (1^{m_1}2^{m_2} \cdots),
\]

where \( m_i \) is the number of parts in \( \lambda \) equal to \( i \).
We will use partitions indexed by $\Gamma_*$ and $\Gamma^*$. For a finite set $X$ and $\rho = (\rho(x))_{x \in X}$ a family of partitions indexed by $X$, we write
\[
\|\rho\| = \sum_{x \in X} |\rho(x)|.
\]

Sometimes it is convenient to regard $\rho = (\rho(x))_{x \in X}$ as a partition-valued function on $X$. We denote by $\mathcal{P}(X)$ the set of all partitions indexed by $X$ and by $\mathcal{P}_n(X)$ the set of all partitions in $\mathcal{P}(X)$ such that $\|\rho\| = n$.

The conjugacy classes of $\Gamma_n$ can be described in the following way. Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \cdots, g_n) \in \Gamma^n_*$, $\sigma \in S_n$. The permutation $\sigma$ is written as a product of disjoint cycles. For each such cycle $y = (i_1i_2\cdots i_k)$ the element $g_{i_k}g_{i_{k-1}}\cdots g_{i_1} \in \Gamma$ is determined up to conjugacy in $\Gamma$ by $g$ and $y$, and will be called the cycle-product of $x$ corresponding to the cycle $y$. For any conjugacy class $c$ and each integer $i \geq 1$, the number of $i$-cycles in $\sigma$ whose cycle-product lies in $c$ will be denoted by $m_i(c)$. Denote by $\rho(c)$ the partition $(1^{m_1(c)}2^{m_2(c)}\cdots)$, $c \in \Gamma_*$. Then each element $x = (g, \sigma) \in \Gamma_n$ gives rise to a partition-valued function $\rho(c)_{c \in \Gamma_*} \in \mathcal{P}(\Gamma_*)$ such that $\sum_{i,c} im_i(c) = n$. The partition-valued function $\rho = (\rho(c))_{c \in \Gamma_*}$ is called the type of $x$. It is known (cf. [M2]) that any two elements of $\Gamma_n$ are conjugate in $\Gamma_n$ if and only if they have the same type.

Given a partition $\lambda = (1^{m_1}2^{m_2}\cdots)$, we define
\[
z_\lambda = \prod_{i \geq 1} i^{m_i}m_i!.
\]

We note that $z_\lambda$ is the order of the centralizer of an element of cycle-type $\lambda$ in $S_{|\lambda|}$. The order of the centralizer of an element $x = (g, \sigma) \in \Gamma_n$ of the type $\rho = (\rho(c))_{c \in \Gamma_*}$ is
\[
Z_\rho = \prod_{c \in \Gamma_*} z_{\rho(c)} \zeta_c(\rho(c)).
\]

2.3. Hopf algebra structure on $R_\Gamma$. We define
\[
R_\Gamma = \bigoplus_{n \geq 0} R(\Gamma_n).
\]

$R_\Gamma$ carries a natural Hopf algebra structure (cf. e.g. [M1, Z, W]) with multiplication defined by the composition
\[
m : R(\Gamma_n) \otimes R(\Gamma_m) \xrightarrow{\cong} R(\Gamma_n \times \Gamma_m) \xrightarrow{Ind} R(\Gamma_{n+m}),
\]
and comultiplication defined by the composition

\[ \Delta : R(\Gamma_n) \xrightarrow{\text{Res}} \bigoplus_{m=0}^{n} R(\Gamma_{n-m} \times \Gamma_m) \xrightarrow{\otimes} \bigoplus_{m=0}^{n} R(\Gamma_{n-m}) \otimes R(\Gamma_m). \]

Here \( \text{Ind} : R(\Gamma_n \times \Gamma_m) \rightarrow R(\Gamma_{n+m}) \) is the induction functor and \( \text{Res} : R(\Gamma_n) \rightarrow R(\Gamma_{n-m} \times \Gamma_m) \) is the restriction functor.

The standard bilinear form in \( R(\Gamma) \) is defined in terms of those on \( R(\Gamma_n) \) as follows:

\[ \langle u, v \rangle = \sum_{n \geq 0} \langle u_n, v_n \rangle \Gamma_n, \]

where \( u = \sum_n u_n \) and \( v = \sum_n v_n \) with \( u_n, v_n \in \Gamma_n \).

3. Weighted bilinear forms on \( R(\Gamma) \) and \( R(\Gamma_n) \)

3.1. A weighted bilinear form on \( R(\Gamma) \). Let us fix a class function \( \xi \in R(\Gamma) \). The multiplication in \( R(\Gamma) \) corresponding to the tensor product of two representations will be denoted by \(*\).

We denote by \( a_{ij} \in \mathbb{C} \) the (virtual) multiplicities of \( \gamma_j \) in \( \xi * \gamma_i \). In other words, we have the following decomposition

\[ \xi * \gamma_i = \sum_{j=0}^{r} a_{ij} \gamma_j. \]

We denote by \( A \) the \((r+1) \times (r+1)\) matrix \((a_{ij})_{0 \leq i,j \leq r}\).

We introduce the following weighted bilinear form

\[ \langle f, g \rangle_{\xi} = \langle \xi * f, g \rangle_{\Gamma}, \quad f, g \in R(\Gamma). \]

We also have an alternative formula:

\[ \langle f, g \rangle_{\xi} = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} \xi(x)f(x)g(x^{-1}) \]

\[ = \sum_{c \in \Gamma^*} \zeta_c^{-1} \xi(c)f(c)g(c^{-1}). \]

In other words, \( \langle \cdot, \cdot \rangle_{\Gamma} \) is the average of the character \( \xi * f * \overline{\gamma} \).

In particular, Eqn. (3.1) can be reformulated as

\[ \langle \gamma_i, \gamma_j \rangle_{\xi} = a_{ij}. \]

Throughout this paper we will always assume that \( \xi \) is a self-dual, i.e. \( \overline{\xi} = \xi \), or equivalently \( \xi(x) = \xi(x^{-1}), x \in \Gamma \). The self-duality of \( \xi \) together with (3.2) implies that

\[ a_{ij} = a_{ji}, \]

i.e. \( A \) is a symmetric matrix. An equivalent formula for \( \langle \cdot, \cdot \rangle_{\xi} \) is
(3.3) \[ \langle f, g \rangle_\xi = \sum_{c \in \Gamma^*} \zeta_c^{-1} \xi(c) f(c^{-1}) g(c). \]

**Remark 3.1.** If \( \xi \) is the trivial character \( \gamma_0 \), then the weighted bilinear form becomes the standard one on \( R(\Gamma) \).

3.2. **A weighted bilinear form on \( R(\Gamma_n) \).** Given a representation \( V \) of \( \Gamma \) with character \( \gamma \in R(\Gamma) \), the \( n \)-th outer tensor product \( V^{\otimes n} \) of \( V \) can be regarded naturally as a representation of the wreath product \( \Gamma_n \) whose character will be denoted by \( \eta_n(\gamma) \): the direct product \( \Gamma^n \) acts on \( \gamma^{\otimes n} \) factor by factor while \( S_n \) by permuting the \( n \) factors. Denote by \( \varepsilon_n \) the (1-dimensional) sign representation of \( \Gamma_n \) on which \( \Gamma_n \) acts trivially while \( S_n \) acts as sign representation. We denote by \( \varepsilon_n(\gamma) \in R(\Gamma_n) \) the character of the tensor product of \( \varepsilon_n \) and \( V^{\otimes n} \).

We may extend naturally \( \eta_n \) to a map from \( R(\Gamma) \) to \( R(\Gamma_n) \) (cf. [W]). In particular, if \( \beta \) and \( \gamma \) are characters of representations \( V \) and \( W \) of \( \Gamma \) respectively, then

\[
\eta_n(\beta - \gamma) = \sum_{m=0}^{n} (-1)^m \text{Ind}_{\Gamma_{n-m} \times \Gamma_m}^{\Gamma_n} [\eta_{n-m}(\beta) \otimes \varepsilon_m(\gamma)].
\]

We introduce a **weighted bilinear form** on \( R(\Gamma_n) \) by letting

\[
\langle f, g \rangle_{\xi, \Gamma_n} = \langle \eta_n(\xi) \ast f, g \rangle_{\Gamma_n}, \quad f, g \in R(\Gamma_n).
\]

We shall see in Corollary 5.3 that \( \eta_n(\xi) \) is self-dual. It follows that the bilinear form \( \langle \ , \ \rangle_\xi \) is symmetric.

**Remark 3.2.** When \( n = 1 \), this weighted bilinear form obviously reduces to the weighted bilinear form defined on \( R(\Gamma) \).

On \( R_\Gamma = \bigoplus_n R(\Gamma_n) \) a symmetric bilinear form is given by

\[
\langle u, v \rangle_\xi = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\xi, \Gamma_n},
\]

where \( u = \sum_n u_n \) and \( v = \sum_n v_n \) with \( u_n, v_n \in \Gamma_n \).

3.3. **A McKay specialization.** Denote by \( d_i = \gamma_i(c^0) \) the dimension of the irreducible representation of \( \Gamma \) corresponding to the character \( \gamma_i \). The following proposition appeared in [St] which was motivated by a special important case observed first by McKay [Mc].

**Proposition 3.1.** The column vector

\[
v_i = (\gamma_0(c^i), \gamma_1(c^i), \ldots, \gamma_r(c^i))^t \quad (i = 0, \ldots, r)
\]

is an eigenvector of the matrix \( A \) with eigenvalue \( \xi(c^i) \). In particular \( (d_0, d_1, \ldots, d_r)^t \) is an eigenvector of \( A \) with eigenvalue \( \xi(c^0) \).
Denote by $E$ the character matrix $[v_0, v_1, \ldots, v_r]$ and $D$ the diagonal matrix $\text{diag}(\xi(c^0), \ldots, \xi(c^r))$. The proposition above can be reformulated as

$$AE = ED.$$  

We fix an irreducible faithful complex representation $\pi$ of $\Gamma$ of dimension $d$. We still assume that $\pi$ is self-dual. We set

$$\xi = d\gamma_0 - \pi.$$  

In this case, the weighted bilinear form $\langle \cdot, \cdot \rangle_\xi$ on $R_\Gamma$ become semi-positive definite. The radical of this bilinear form is one-dimensional and spanned by the character of the regular representation of $\Gamma$

$$\delta = \sum_{i=0}^r d_i \gamma_i.$$  

We further specialize to the case in which $\pi$ embeds $\Gamma$ into $SU_2$, i.e. $d = 2$. The classification of finite subgroups of $SU_2$ is well known. The following is a complete list of them: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups. We recall the following well-known facts [Mc]: if $\Gamma \neq \mathbb{Z}/2\mathbb{Z}$ and $i \neq j$ then $a_{ij} = 0$ or $-1$. If $\Gamma \equiv \mathbb{Z}/2\mathbb{Z}$ then $a_{01} = -2$.

We associate a diagram with vertices corresponding to elements $\gamma_i$ in $\Gamma^\ast$. We draw one edge (resp. two edges) between the $i$-th and $j$-th vertices if $a_{ij} = -1$ (resp. $-2$). According to McKay [Mc], the associated diagram can be identified with affine Dynkin diagram of ADE type and the matrix $A$ is the corresponding affine Cartan matrix.

4. Heisenberg algebras and $\Gamma_n$

4.1. Heisenberg algebra $\hat{h}_{\Gamma, \xi}$. Let $\hat{h}_{\Gamma, \xi}$ be the infinite dimensional Heisenberg algebra over $\mathbb{C}$, associated with $\Gamma$ and $\xi$, with generators $a_m(\gamma), m \in \mathbb{Z}, \gamma \in \Gamma^\ast$ and a central element $C$ subject to the following commutation relations:

$$[a_m(\gamma), a_n(\gamma')] = m\delta_{m, -n}\langle \gamma, \gamma' \rangle_\xi C, \quad m, n \in \mathbb{Z}, \gamma, \gamma' \in \Gamma^\ast.$$  

It is convenient to extend $a_m(\gamma)$ to all $\gamma = \sum_{i=0}^r s_i \gamma_i \in R(\Gamma)$ ($s_i \in \mathbb{C}$) by linearity: $a_m(\gamma) = \sum_i s_i a_m(\gamma_i)$.

Denote by $R_0$ the radical of the bilinear form $\langle \cdot, \cdot \rangle_\xi$ in $R(\Gamma)$. We note that the Heisenberg algebra may contain a large center since the bilinear form $\langle \cdot, \cdot \rangle_\xi$ may degenerate. In this sense we have abused the notion of Heisenberg algebra. The center of $\hat{h}_{\Gamma, \xi}$ is spanned by $C$ together with $a_m(\gamma), \gamma \in R_0, m \in \mathbb{Z}$.
For \( m \in \mathbb{Z}, c \in \Gamma_* \) we define
\[
a_m(c) = \sum_{\gamma \in \Gamma^*} \gamma(c^{-1})a_m(\gamma).
\]

From the orthogonality of the irreducible characters of \( \Gamma \) (2.1) it follows that
\[
a_m(\gamma) = \sum_{c \in \Gamma_*} \zeta_c^{-1}\gamma(c)a_m(c).
\]

Thus \( a_n(c) \ (n \in \mathbb{Z}, c \in \Gamma_*) \) and \( C \) form a new basis for the Heisenberg algebra \( \hat{h}_{\Gamma,\xi} \).

**Proposition 4.1.** The commutation relations among the new basis for \( \hat{h}_{\Gamma,\xi} \) are given by
\[
[a_m(c'^{-1}), a_n(c)] = m\delta_{m,-n}\delta_{c,c'}\zeta_c\xi(c)C, \quad c, c' \in \Gamma_*.
\]

**Proof.** We calculate by using Eqns. (4.1), (3.3) and (2.1) that
\[
[a_m(c'^{-1}), a_n(c)] = \sum_{\gamma', \gamma \in \Gamma^*} \gamma'(c')\gamma(c^{-1})[a_m(\gamma'), a_n(\gamma)]
\]
\[
= m\delta_{m,-n}\sum_{\gamma', \gamma \in \Gamma^*} \gamma'(c')\gamma(c^{-1})\langle \gamma', \gamma \rangle \zeta_c\xi(c)C
\]
\[
= m\delta_{m,-n}\sum_{\gamma', \gamma \in \Gamma^*} \gamma'(c')\gamma(c^{-1})\left(\sum_{\nu \in \Gamma_*} \zeta_\nu^{-1}\xi(\nu)\gamma(\nu)\gamma'(\nu^{-1})\right)C
\]
\[
= m\delta_{m,-n}\sum_{\nu \in \Gamma_*} \zeta_\nu^{-1}\xi(\nu)\left(\sum_{\gamma' \in \Gamma^*} \gamma'(c')\gamma'(\nu^{-1})\right)\left(\sum_{\gamma \in \Gamma^*} \gamma(c^{-1})\gamma(\nu)\right)C
\]
\[
= m\delta_{m,-n}\sum_{\nu \in \Gamma_*} \zeta_\nu^{-1}\xi(\nu)\delta_{c,c'}\zeta_c\xi(c)C
\]
\[
= m\delta_{m,-n}\delta_{c,c'}\zeta_c\xi(c)C.
\]

\[
\square
\]

4.2. **Action of** \( \hat{h}_{\Gamma,\xi} \) **on** \( S_\Gamma \) **and** \( \overline{S}_\Gamma \). Denote by \( S_\Gamma \) the symmetric algebra generated by \( a_{-n}(\gamma), n \in \mathbb{N}, \gamma \in \Gamma_* \). There is a natural degree operator on \( S_\Gamma \)
\[
\deg(a_{-n}(\gamma)) = n,
\]
which makes \( S_\Gamma \) into a \( \mathbb{Z}_+ \)-graded space.

We define an action of \( \hat{h}_{\Gamma,\xi} \) on \( S_\Gamma \) as follows: \( a_{-n}(\gamma), n > 0 \) acts as multiplication operator on \( S_\Gamma \) and \( C \) as the identity operator; \( a_n(\gamma), \)
$n \geq 0$ acts as a derivation of algebra

$$a_n(\gamma) a_{-n_1}(\alpha_1) a_{-n_2}(\alpha_2) \ldots a_{-n_k}(\alpha_k)$$

$$= \sum_{i=1}^{k} \delta_{n,n_i} \langle \gamma, \alpha_i \rangle \xi a_{-n_1}(\alpha_1) a_{-n_2}(\alpha_2) \ldots a_{-n_i}(\alpha_i) \ldots a_{-n_k}(\alpha_k).$$

Here $n_i > 0$, $\alpha_i \in R(\Gamma)$ for $i = 1, \ldots, k$, and $a_{-n_i}(\alpha_i)$ means the very term is deleted. In other word, the operator $a_n(\gamma)$, $n > 0$, $\gamma \in R^0$ acts as 0, and $a_n(\gamma)$, $n > 0$, $\gamma \in R(\Gamma) - R^0$ acts as certain non-zero differential operator. Note that $S_\Gamma$ is not an irreducible representation over $\hat{h}_\Gamma, \xi$ in general since the bilinear form $\langle , \rangle_\xi$ may be degenerate.

Denote by $S_0^\Gamma$ the ideal in the symmetric algebra $S_\Gamma$ generated by $a_{-n}(\gamma)$, $n \in \mathbb{N}$, $\gamma \in R_0$. Denote by $S^\Gamma$ the quotient $S_\Gamma/S_0^\Gamma$. It follows from the definition that $S_0^\Gamma$ is a subrepresentation of $S_\Gamma$ over the Heisenberg algebra $\hat{h}_\Gamma, \xi$. In particular, this induces a Heisenberg algebra action on $S^\Gamma$ which is irreducible.

We denote by 1 the unit in the symmetric algebra $S_\Gamma$. By abuse of notation we will denote by 1 its image in the quotient $S^\Gamma$. The element 1 is the highest weight vector in $S^\Gamma$.

4.3. **The bilinear form on $S_\Gamma$.** The space $S_\Gamma$ admits a bilinear form $\langle , \rangle_\xi$ characterized by

$$\langle 1, 1 \rangle_\xi' = 1, \quad a_n(\gamma)^* = a_{-n}(\gamma), \quad n \in \mathbb{Z}\setminus 0.$$  

Here $a_n(\gamma)^*$ denotes the adjoint of $a_n(\gamma)$.

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\gamma \in \Gamma^*$, we define

$$a_{-\lambda}(\gamma) = a_{-\lambda_1}(\gamma) a_{-\lambda_2}(\gamma) \ldots.$$  

For $\rho = (\rho(\gamma))_{\gamma \in \Gamma^*} \in \mathcal{P}(\Gamma^*)$, we define

$$a_{-\rho} = \prod_{\gamma \in \Gamma^*} a_{-\rho(\gamma)}(\gamma).$$

It is clear that $a_{-\rho}, \rho \in \mathcal{P}(\Gamma^*)$ consist of a $\mathbb{C}$-basis for $S_\Gamma$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $c \in \Gamma_*$, we define

$$a_{-\lambda}(c) = a_{-\lambda_1}(c) a_{-\lambda_2}(c) \ldots.$$  

For any $\rho = (\rho(c))_{c \in \Gamma_*} \in \mathcal{P}(\Gamma_*), \rho \in \mathcal{P}(\Gamma_*), \rho \in \mathcal{P}(\Gamma_*),$, we further define

$$a'_{-\rho} = \prod_{c \in \Gamma_*} a_{-\rho(c)}(c).$$

The elements $a'_{-\rho}, \rho \in \mathcal{P}(\Gamma_*)$ provide a new $\mathbb{C}$-basis for $S_\Gamma$. 

We define \( \overline{\rho} \in \mathcal{P}(\Gamma_\ast) \) by assigning to \( c \in \Gamma_\ast \) the partition \( \rho(c^{-1}) \), which is the composition of \( \rho \) with the involution on \( \Gamma_\ast \) given by \( c \mapsto c^{-1} \). It follows from Proposition 4.1 that

\[
\langle a_{-\rho}, a'_{-\overline{\rho}} \rangle_{\xi} = \delta_{\rho, \rho} \prod_{c \in \Gamma_\ast} \xi(c)^{l(\rho(c))}, \quad \rho, \rho' \in \mathcal{P}(\Gamma_\ast).
\]

**Remark 4.1.** \( S^0 \) can be characterized as the radical of the bilinear form \( \langle \, , \rangle_\xi \) in \( S_\Gamma \). Thus the bilinear form \( \langle \, , \rangle_\xi \) descends to \( \overline{S}_\Gamma \).

## 5. Isometry between \( R_\Gamma \) and \( S_\Gamma \)

### 5.1. The characteristic map \( \text{ch} \)

Let \( \Psi : \Gamma_\ast \to S_\Gamma \) be the map defined by \( \Psi(x) = a'_\rho \) if \( x \in \Gamma_\ast \) is of type \( \rho \). Given \( y \in \Gamma_m \) of type \( \rho' \), we may regard \( x \times y \in \Gamma_n \times \Gamma_m \) to be in \( \Gamma_{n+m} \) of type \( \rho \cup \rho' \) so that

\[
\Psi(x \times y) = \Psi(x)\Psi(y).
\]

We define a \( \mathbb{C} \)-linear map \( \text{ch} : R_\Gamma \to S_\Gamma \) by letting

\[
\text{ch}(f) = \langle f, \Psi \rangle_{\Gamma_n} = \sum_{\rho \in \mathcal{P}(\Gamma_\ast)} Z_{\rho}^{-1} f_{\rho} a'_{-\rho},
\]

where \( f_{\rho} \) is the value of \( f \) at elements of type \( \rho \). The map \( \text{ch} \) is called the characteristic map.

We may think of \( a_{-n}(\gamma), n > 0, \gamma \in \Gamma_\ast \) as the \( n \)-th power sum in a sequence of variables \( y_\gamma = (y_{i\gamma})_{i \geq 1} \). In this way we identify the space \( S_\Gamma \) with the space \( \Lambda_\Gamma \) of symmetric functions indexed by \( \Gamma_\ast \) (cf. [M2]). In particular given a partition \( \lambda \) we denote by \( s_\lambda(\gamma) \) the Schur function associated to \( y_\gamma \). By abuse of notation, we denote by \( s_\lambda(\gamma) \) the corresponding element in \( S_\Gamma \) by the identification of \( S_\Gamma \) and \( \Lambda_\Gamma \). For \( \lambda \in \mathcal{P}(\Gamma_\ast) \), we denote

\[
s_\lambda = \prod_{\gamma \in \Gamma_\ast} s_{\lambda(\gamma)}(\gamma) \in S_\Gamma.
\]

Then \( s_\lambda \) is the image under the isometry \( \text{ch} \) of the character of an irreducible representation \( \chi_\lambda \) of \( \Gamma_n \) (cf. [M2]).

Denote by \( c_n(c \in \Gamma_\ast) \) the conjugacy class in \( \Gamma_n \) of elements \( (x, s) \in \Gamma_n \) such that \( s \) is an \( n \)-cycle and the cycle product of \( x \) is \( c \). Denote by \( \sigma_n(c) \) the class function on \( \Gamma_n \) which takes value \( n \zeta_c \) (i.e. the order of the centralizer of an element in the class \( c_n \)) on elements in the class \( c_n \) and 0 elsewhere. For \( \rho = \{m_r(c)\}_{r \geq 1, c \in \Gamma_\ast} \in \mathcal{P}(\Gamma_\ast) \), \( \sigma_\rho = \prod_{r \geq 1, c \in \Gamma_\ast} \sigma_r(c)^{m_r(c)} \) is the class function on \( \Gamma_n \) which takes value \( Z_{\rho} \) on the conjugacy class of type \( \rho \) and 0 elsewhere. Given \( \gamma \in R(\Gamma) \), we
Lemma 5.1. The map \( ch \) sends \( \sigma_{\rho} \) to \( \alpha'_{-\rho} \). In particular, it sends \( \sigma_n(c) \) to \( a_{-n}(c) \) in \( S_\Gamma \) while sending \( \sigma_n(\gamma) \) to \( a_{-n}(\gamma) \).

5.2. The characters \( \eta_n(\gamma) \) and \( \varepsilon_n(\gamma) \). As we have remarked, the map from \( \gamma \in \Gamma^* \) to \( \eta_n(\gamma) \) can be extended to be a map \( \rho \) from \( R(\Gamma) \) to \( R(\Gamma_n) \). We will need the following proposition which appeared in [W] in a more general setting. Eqn. (5.2) for \( \gamma \in \Gamma^* \) is well known (cf. [M2]). We present a complete proof here as we will need some formulas appearing in the proof later on.

Proposition 5.1. For any \( \gamma \in R(\Gamma) \), we have

\[
\sum_{n \geq 0} ch(\eta_n(\gamma)) z^n = \exp \left( \sum_{n \geq 1} \frac{1}{n} a_{-n}(\gamma) z^n \right),
\]

(5.2)

\[
\sum_{n \geq 0} ch(\varepsilon_n(\gamma)) z^n = \exp \left( \sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} a_{-n}(\gamma) z^n \right).
\]

(5.3)

Proof. First assume that \( \gamma \in R(\Gamma) \) is the character of a representation \( V_\gamma \) of \( \Gamma \). We calculate the character value of \( \eta_n(\gamma) \) at \( c_n(e_i) \in \Gamma_* \) first. Take \( (g, s) \in \Gamma_n \), where \( g = (g_1, \ldots, g_n) \in \Gamma^\circ \) and \( s \) is an \( n \)-cycle, say \( s = (12\ldots n) \). Denote by \( e_1, \ldots, e_k \) a basis of \( V_\gamma \), and we write \( ge_j = \sum_i a_{ij}(g)e_i, a_{ij}(g) \in \mathbb{C} \). It follows that

\[
(g, s)(e_{j_1} \otimes \ldots \otimes e_{j_n}) = g_1(e_{j_n}) \otimes g_2(e_{j_1}) \otimes \ldots \otimes g_n(e_{j_{n-1}}),
\]

in which the coefficient of \( e_{j_1} \otimes \ldots \otimes e_{j_n} \) is

\[
a_{j_1\ldots j_n}(g_1)a_{j_2j_1}(g_2)\ldots a_{j_{n-1}j_1}(g_n).
\]

Thus we obtain

\[
\eta_n(\gamma)(c_n) = \text{trace } a(g_n)a(g_{n-1})\ldots a(g_1)
\]

\[
= \text{trace } a(g_n g_{n-1} \ldots g_1) = \gamma(c)
\]

since \( (g, s) \) is in the conjugacy class \( c_n \) which means \( g_n g_{n-1} \ldots g_1 \) lies in \( c \in \Gamma_* \). Since the sign character of \( \Gamma_n \) takes value \( (-1)^{n-1} \) at \( c_n \), we have

\[
\varepsilon_n(\gamma)(c_n) = (-1)^{n-1} \gamma(c).
\]

Given \( x \times y \in \Gamma_n \), where \( x \in \Gamma \) and \( y \in \Gamma_{n-r} \), we clearly have \( \eta_n(\gamma)(x \times y) = \eta_n(\gamma)(x) \eta_n(\gamma)(y) \). Thus it follows that if \( x \in \Gamma_n \) is of
type $\rho$, then

$$\eta_n(\gamma)(x) = \prod_{c \in \Gamma_*} \gamma(c)^{l(\rho(c))}, \quad (5.4)$$

$$\varepsilon_n(\gamma)(x) = (-1)^n \prod_{c \in \Gamma_*} (-\gamma(c))^{l(\rho(c))}, \quad (5.5)$$

where $||\rho|| = n$. Putting (5.4) into a generating function, we have

$$\sum_{n \geq 0} \text{ch}(\eta_n(\gamma))z^n = \sum_{\rho} Z_{\rho}^{-1} \prod_{c \in \Gamma_*} \gamma(c)^{l(\rho(c))} a'_{-\rho(c)} z^{||\rho||}$$

$$= \prod_{c \in \Gamma_*} \left( \sum_{\lambda} (\zeta_c^{-1} \gamma(c))^{l(\lambda)} z^{-\lambda} a_{-\lambda}(c) z^{||\lambda||} \right)$$

$$= \exp \left( \sum_{n \geq 1} \frac{1}{n} \sum_{c \in \Gamma_*} \zeta_c^{-1} \gamma(c) a_{-n}(c) z^n \right)$$

$$= \exp \left( \sum_{n \geq 1} \frac{1}{n} a_{-n}(\gamma) z^n \right).$$

In a similar manner we can prove (5.3) by using (5.5).

Note that (5.3) can be obtained from (5.2) by substituting $\gamma$ with $-\gamma$ and $z$ with $-z$. Let $\beta, \gamma$ be the characters of two representations of $\Gamma$. It follows from (3.4) that

$$\sum_{n \geq 0} \text{ch}(\eta_n(\beta - \gamma))z^n$$

$$= \left( \sum_{n \geq 0} \text{ch}(\eta_n(\beta))z^n \right) \cdot \left( \sum_{n \geq 0} -\text{ch}(\varepsilon_n(\gamma))(-z)^n \right)$$

$$= \exp \left( \sum_{n \geq 1} \frac{1}{n} a_{-n}(\beta) z^n \right) \cdot \exp \left( \sum_{n \geq 1} \frac{1}{n} a_{-n}(-\gamma) z^n \right)$$

$$= \exp \left( \sum_{n \geq 1} \frac{1}{n} a_{-n}(\beta - \gamma) z^n \right).$$

Therefore the proposition holds for $\beta - \gamma$, and so for any element $\gamma \in R(\Gamma)$. \hfill \square

Remark 5.2. Formulas (5.2-5.3) are equivalent, since (5.3) can be obtained (5.3) from by substituting $\gamma$ by $-\gamma$ and $z$ by $-z$. 


Corollary 5.3. The formula (5.4) holds for any $\gamma \in R(\Gamma)$. In particular $\eta_n(\xi)$ is self-dual.

Componentwise, we obtain

$$\text{ch}(\eta_n(\gamma)) = \sum_{\lambda} \frac{1}{z_\lambda} a_{-\lambda}(\gamma),$$

$$\text{ch}(\varepsilon_n(\gamma)) = \sum_{\lambda} \frac{1}{z_\lambda} (-1)^{|\lambda| - l(\lambda)} a_{-\lambda}(\gamma),$$

where the sum runs over all the partitions $\lambda$ of $n$.

5.3. Isometry between $R_\Gamma$ and $S_\Gamma$. It is well known that there exists a natural Hopf algebra structure on the symmetric algebra $S_\Gamma$ with the usual multiplication and a comultiplication $\Delta$ characterized by

$$\Delta(a_n(\gamma)) = a_n(\gamma) \otimes 1 + 1 \otimes a_n(\gamma).$$

Recall that we have defined a Hopf algebra structure on $R_\Gamma$ in Sect. 2. The following proposition is easy to check.

Proposition 5.2. The characteristic map $\text{ch} : R_\Gamma \rightarrow S_\Gamma$ is an isomorphism of Hopf algebras.

Recall that we have defined a bilinear form $\langle \ , \ \rangle_\xi$ on $R_\Gamma$ and a bilinear form on $S_\Gamma$ denoted by $\langle \ , \ \rangle'_\xi$. The lemma below follows from our definition of $\langle \ , \ \rangle'_\xi$ and the comultiplication $\Delta$.

Lemma 5.4. The bilinear form $\langle \ , \ \rangle'_\xi$ on $S_\Gamma$ can be characterized by the following two properties:

1. $\langle a_{-\gamma}(\beta), a_{-\gamma}(\gamma) \rangle'_\xi = \delta_{n,m} \langle \beta, \gamma \rangle'_\xi, \quad \beta, \gamma \in \Gamma^*.$

2. $\langle fg, h \rangle'_\xi = \langle f \otimes g, \Delta h \rangle'_\xi$, where $f, g, h \in S_\Gamma$, and the bilinear form on the r.h.s of 2), which is defined on $S_\Gamma \otimes S_\Gamma$, is induced from $\langle \ , \ \rangle'_\xi$ on $S_\Gamma$.

Theorem 5.5. The characteristic map $\text{ch}$ is an isometry from the space $(R_\Gamma, \langle \ , \ \rangle_\xi)$ to $(S_\Gamma, \langle \ , \ \rangle'_\xi)$.

Proof. By Corollary 5.3, the character value of $\eta_n(\xi)$ at an element $x$ of type $\rho$ is

$$\eta_n(\xi)(x) = \prod_{c \in \Gamma^*} \xi(c)^{l(\rho(c))}.$$
Thus it follows by definition of the weighted bilinear form
\[
\langle \sigma_{\rho'}, \sigma_{\rho} \rangle_{\xi} = \sum_{\mu \in P_n(\Gamma')} Z_{\mu, \rho}^{-1} \xi(c_{\mu}) \sigma_{\rho'}(c_{\mu}) \sigma_{\rho}(c_{\mu}^{-1})
\]
\[
= Z_{\rho'}^{-1} \xi(c_{\rho}) \delta_{\mu, \rho} Z_{\rho} \delta_{\mu, \rho} Z_{\rho}
\]
\[
= \delta_{\mu, \rho} Z_{\rho} \prod_{c \in \Gamma'} \xi(c)^{l(\rho(c))}.
\]
Here \(c_{\mu}\) denotes the conjugacy class in \(\Gamma_n\) of type \(\mu\). By Lemma 5.1 and the formula (4.3), we see that
\[
\langle \sigma_{\rho'}, \sigma_{\rho} \rangle_{\xi} = \langle a_{-\rho'}, a_{-\rho} \rangle'_{\xi} = \langle \text{ch}(\sigma_{\rho'}), \text{ch}(\sigma_{\rho}) \rangle'_{\xi}.
\]
Since \(\sigma_{\rho'}, \rho \in \mathcal{P}(\Gamma')\) consist a \(\mathbb{C}\)-basis of \(R_{\Gamma}\), we have established that 
the characteristic map \(\text{ch} : R_{\Gamma} \rightarrow S_{\Gamma}\) is an isometry.

Thanks to the isometry established above, we will write \(\langle , \rangle_{\xi}\) for \(\langle , \rangle'_{\xi}\) on \(S_{\Gamma}\) from now on.

**Remark 5.6.** In the special case when \(\xi\) is trivial, Theorem 5.5 was established in [M1, M2] once we identify \(S_{\Gamma}\) with the space of symmetric functions parameterized by \(\Gamma^*\).

**Remark 5.7.** The standard Hermitian form on \(R(\Gamma_n)\) and therefore on \(R_{\Gamma}\) is compatible via the characteristic map \(\text{ch}\) with the hermitian form characterized by (4.2) on \(S_{\Gamma}\).

### 6. Vertex operators and \(R_{\Gamma}\)

#### 6.1. A 2-cocycle on an integral lattice.

Let \(Q\) be an integral lattice with a symmetric bilinear form \(\langle , \rangle\). One can easily check that the map from \(Q\) to \(\mathbb{Z}/2\mathbb{Z}\) given by \(\alpha \mapsto (-1)^{(\alpha, \alpha)}\) is a group homomorphism. This homomorphism naturally induces a \(\mathbb{Z}/2\mathbb{Z}\)-gradation on \(Q\).

Let \(\epsilon : Q \times Q \rightarrow \mathbb{C}^\times\) be such that

\[
(6.1) \quad \epsilon(\gamma, 0) = \epsilon(0, \gamma) = 1,
\]
\[
(6.2) \quad \epsilon(\alpha, \beta) \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma) \epsilon(\beta, \gamma).
\]
i.e. \(\epsilon\) is a 2-cocycle of the group \(Q\) with values in \(\mathbb{C}^\times\). Another 2-cocycle \(\epsilon'\) is equivalent to \(\epsilon\) if and only if there exists \(\epsilon_{\alpha}(\alpha \in R_{\mathbb{Z}}(\Gamma))\) such that
\[
\epsilon'(\alpha, \beta) = \epsilon_{\alpha} \epsilon_{\beta} \epsilon_{\alpha + \beta}^{-1} \epsilon(\alpha, \beta).
\]
The group of equivalent classes of 2-cocycles is the second cohomology group \(H^2(Q, \mathbb{C}^\times)\).
We introduce
\[ B_\epsilon(\alpha, \beta) = \epsilon(\alpha, \beta)\epsilon(\beta, \alpha)^{-1}. \]

Clearly \( B_\epsilon \) is skew-symmetric: \( B_\epsilon(\alpha, \beta) = B_\epsilon(\beta, \alpha)^{-1} \). It follows from Eqns. (6.1) and (6.2) that \( B_\epsilon \) is bimultiplicative:
\begin{align*}
B_\epsilon(\alpha + \beta, \gamma) &= B_\epsilon(\alpha, \gamma)B_\epsilon(\beta, \gamma), \\
B_\epsilon(\alpha, \beta + \gamma) &= B_\epsilon(\alpha, \beta)B_\epsilon(\alpha, \gamma).
\end{align*}

The map \( \epsilon \mapsto B_\epsilon \) gives rise to a homomorphism from \( H^2(Q, \mathbb{C}^\times) \) to the group of bimultiplicative skew-symmetric functions on \( Q \times Q \). It is easy to show that the homomorphism \( h \) is indeed an isomorphism.

**Proposition 6.1.** [FK] There exists a unique 2-cocyle \( \epsilon : Q \times Q \longrightarrow \mathbb{C}^\times \) up to equivalence such that
\[ B_\epsilon(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle\langle \beta, \beta \rangle}. \]

### 6.2. Vertex Operators \( X(\gamma, z) \)

Endowed with the weighted bilinear form \( \langle \cdot, \cdot \rangle_\xi \), the lattice \( R_\mathbb{Z}(\Gamma) \) is an integral lattice. We will always associate a 2-cocycle \( \epsilon \) as in Proposition 6.1 to the integral lattice \( (R_\mathbb{Z}(\Gamma), \langle \cdot, \cdot \rangle_\xi) \) (and its sublattices). Denote by \( \mathbb{C}[R_\mathbb{Z}(\Gamma)] \) the group algebra generated by \( e^\gamma, \gamma \in R_\mathbb{Z}(\Gamma) \). We introduce two special operators acting on \( \mathbb{C}[R_\mathbb{Z}(\Gamma)] \), namely a \( (\epsilon\text{-twisted}) \) multiplication operator \( e^\alpha \):
\[ e^\alpha e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}, \quad \alpha, \beta \in R_\mathbb{Z}(\Gamma), \]
and a differentiation operator \( \partial_\gamma \):
\[ \partial_\gamma e^\beta = \langle \gamma, \beta \rangle_\xi e^\beta, \quad \alpha, \beta \in R_\mathbb{Z}(\Gamma). \]

We extend these two operators acting on \( \mathbb{C}[R_\mathbb{Z}(\Gamma)] \) to the following space
\[ \mathcal{F}_\Gamma = R_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\Gamma)], \]
by letting them act on the \( R_\Gamma \) part trivially.

Introduce the operators \( H_{\pm n}(\gamma), E_{\pm n}(\gamma), \gamma \in R(\Gamma), n > 0 \) as the following compositions of maps:
\begin{align*}
H_{-n}(\gamma) : &\quad R(\Gamma_m) \xrightarrow{\eta_n(\gamma)} R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{\text{Ind}} R(\Gamma_{n+m}), \\
E_{-n}(\gamma) : &\quad R(\Gamma_m) \xrightarrow{\epsilon_n(\gamma)\xi} R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{\text{Ind}} R(\Gamma_{n+m}), \\
E_n(\gamma) : &\quad R(\Gamma_m) \xrightarrow{\text{Res}} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \xrightarrow{\langle \eta_n(\gamma), \cdot \rangle_\xi} R(\Gamma_{m-n}), \\
H_n(\gamma) : &\quad R(\Gamma_m) \xrightarrow{\text{Res}} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \xrightarrow{\langle \eta_n(\gamma), \cdot \rangle_\xi} R(\Gamma_{m-n}).
\end{align*}
Define
\[ H_+(\gamma, z) = \sum_{n > 0} H_{-n}(\gamma) z^n, \quad E_+(\gamma, z) = \sum_{n > 0} E_{-n}(\gamma)(-z)^n, \]
\[ E_-(\gamma, z) = \sum_{n > 0} E_n(\gamma)(-z)^{-n}, \quad H_-(\gamma, z) = \sum_{n > 0} H_n(\gamma) z^{-n}. \]

Further define operators \( X_n(\gamma), n \in \mathbb{Z} \langle \gamma, \gamma \rangle \xi/2 \) by the following generating functions:
\[ X_+(\gamma, z) \equiv X(\gamma, z) = \sum_{n \in \mathbb{Z} \langle \gamma, \gamma \rangle \xi/2} X_n(\gamma) z^{-n} - \langle \gamma, \gamma \rangle \xi/2, \]
\[ E_+(\gamma, z) = E(\gamma, z) e^\gamma z \partial_z. \]

Sometimes we denote
\[ X_-(\gamma, z) \equiv X(-\gamma, z) = \sum_{n \in \mathbb{Z} \langle \gamma, \gamma \rangle \xi/2} X_{-n}(\gamma) z^{-n} - \langle \gamma, \gamma \rangle \xi/2. \]

One easily sees that the operators \( X_n(\gamma) \) are well-defined operators acting on the space \( F_\Gamma \). We extend the bilinear form \( \langle \ , \rangle_\xi \) on \( R_\Gamma \) to \( F_\Gamma \) by letting
\[ \langle a.e^\alpha, b.e^\beta \rangle_\xi = \langle a, b \rangle_\xi \delta_{\alpha,\beta}, \quad a, b \in R_\Gamma, \alpha, \beta \in R_{\mathbb{Z}}(\Gamma). \]

We extend the \( \mathbb{Z}_+ \)-gradation on \( R_\Gamma \) to a \( \frac{1}{2} \mathbb{Z}_+ \)-gradation on \( F_\Gamma \) by letting
\[ \deg a_{-n}(\gamma) = n, \quad \deg e^\gamma = \frac{1}{2} \langle \gamma, \gamma \rangle \xi. \]

Similarly we extend the bilinear form \( \langle \ , \rangle_\xi \) to the space
\[ V_\Gamma = S_\Gamma \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\Gamma)] \]
and extend the \( \mathbb{Z}_+ \)-gradation on \( S_\Gamma \) to a \( \frac{1}{2} \mathbb{Z}_+ \)-gradation on \( V_\Gamma \).

We extend the characteristic map \( \chi \) to an isometry \( \iota \) from \( F_\Gamma \) to \( V_\Gamma \) by the identity operator on the factor \( \mathbb{C}[R_{\mathbb{Z}}(\Gamma)] \), which is denoted again by \( \chi \).

6.3. Heisenberg algebra and \( R_\Gamma \). We define \( \tilde{a}_{-n}(\gamma), n > 0 \) to be a map \( \iota \) from \( R_\Gamma \) to itself by the following composition
\[ R(\Gamma_m) \xrightarrow{\sigma_n(\gamma) \otimes} R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{Ind} R(\Gamma_{n+m}). \]
We also define $\tilde{a}_n(\gamma), n > 0$ to be a map from $R_\Gamma$ to itself as the composition

$$R(\Gamma_m) \xrightarrow{Res} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \xrightarrow{(\sigma_n(\gamma), \cdot)} R(\Gamma_{m-n}).$$

We denote by $R_0^{\Gamma}$ the radical of the bilinear form $\langle \cdot, \cdot \rangle_\xi$ in $R_\Gamma$ and denote by $\overline{R}_\Gamma$ the quotient $R_\Gamma/R_0^{\Gamma}$. $\overline{R}_\Gamma$ inherits the bilinear form $\langle \cdot, \cdot \rangle_\xi$ from $R_\Gamma$.

**Theorem 6.1.** $R_\Gamma$ is a representation of the Heisenberg algebra $\hat{h}_{\Gamma,\xi}$ by letting $a_n(\gamma)$ ($n \in \mathbb{Z}\setminus0$) act as $\tilde{a}_n(\gamma)$, $a_0(\gamma)$ as 0 and $C$ as 1. $R_0^{\Gamma}$ is a subrepresentation of $R_\Gamma$ over $\hat{h}_{\Gamma,\xi}$ and the quotient $\overline{R}_\Gamma$ is irreducible. The characteristic map $ch$ is an isomorphism of $R_\Gamma$ (resp. $R_0^{\Gamma}$, $\overline{R}_\Gamma$) and $S_\Gamma$ (resp. $S_0^{\Gamma}$, $\overline{S}_\Gamma$) as representations over $\hat{h}_{\Gamma,\xi}$.

**Remark 6.2.** For a $\Gamma$-space $X$, the wreath product $\Gamma_n$ acts on the $n$-th direct product $X^n$. A Heisenberg algebra was defined in [W] to act on a direct sum of equivariant K-theory $\bigoplus_{n \geq 0} K_{\Gamma_n}(X^n) \otimes \mathbb{C}$. In view of Corollary 5.3, our Heisenberg algebra here is a special case of the Heisenberg algebra constructed in [W] when $X$ is a point, cf. [W] for a proof.

6.4. **Vertex operators and Heisenberg algebra $\hat{h}_{\Gamma,\xi}$.** The relation between the vertex operators defined in (6.3) and the Heisenberg algebra $\hat{h}_{\Gamma,\xi}$ is revealed in the following theorem.

**Theorem 6.3.** For any $\gamma \in R(\Gamma)$, we have

$$ch(H_+(\gamma, z)) = \exp \left( \sum_{n \geq 1} \frac{1}{n} a_{-n}(\gamma) z^n \right),$$

$$ch(E_+(\gamma, z)) = \exp \left( - \sum_{n \geq 1} -\frac{1}{n} a_{-n}(\gamma) z^n \right),$$

$$ch(H_-(\gamma, z)) = \exp \left( \sum_{n \geq 1} \frac{1}{n} a_n(\gamma) z^{-n} \right),$$

$$ch(E_-(\gamma, z)) = \exp \left( - \sum_{n \geq 1} -\frac{1}{n} a_n(\gamma) z^{-n} \right).$$

**Proof.** The first and second identities were essentially established in Proposition 5.1 together with Lemma 5.1. The only minor difference is that the components appearing in these two identities are regarded as operators acting on $R_\Gamma$ and $S_\Gamma$. 

Note that by definition the adjoints of $E_+(\gamma, z)$ and $H_+(\gamma, z)$ with respect to the bilinear form $\langle \ , \ \rangle_\xi$ are $E_-(\gamma, z^{-1})$ and $H_-(\gamma, z^{-1})$, respectively. The third and fourth identities are obtained by applying the adjoint functor to the first two identities.

Putting all pieces together, we have

$$\text{ch}(X(\gamma, z)) = \exp\left(\sum_{n \geq 1} \frac{1}{n} a_{-n}(\gamma) z^n \right) \exp\left(-\sum_{n \geq 1} \frac{1}{n} a_n(\gamma) z^{-n} \right) e^{z^\partial_\gamma}.$$  

7. **Vertex representations and the McKay correspondence**

7.1. **Product of two vertex operators.** We first note that

$$X_n(\gamma)^* = X_{-n}(-\gamma), \quad n \in \mathbb{Z} + \langle \gamma, \gamma \rangle_\xi/2.$$  

The normal ordered product $: X(\alpha, z)X(\beta, w) :$, $\alpha, \beta \in R(\Gamma)$ of two vertex operators is defined as follows:

$$: X(\alpha, z)X(\beta, w) : = \exp\left(\sum_{n \geq 1} \frac{1}{n} \left( a_{-n}(\alpha) z^n + a_{-n}(\beta) w^n \right) \right) \cdot \exp\left(-\sum_{n \geq 1} \frac{1}{n} \left( a_n(\gamma) z^{-n} + a_n(\beta) w^{-n} \right) \right) e^{\alpha + \beta} z^{\partial_\alpha} w^{\partial_\beta}.$$  

The term $(z - w)^{(\alpha, \beta)}_\xi$ in the following theorem is understood as the formal series expansion (cf. [FLM])

$$z^{(\alpha, \beta)}_\xi \sum_{k \geq 0} \left( \frac{\langle \alpha, \beta \rangle_\xi}{k!} \right)(-z^{-1}w)^k.$$  

This applies to similar expressions in similar contexts below.

**Theorem 7.1.** For $\alpha, \beta \in R_{\mathbb{Z}}(\Gamma)$, we have the following identity for a product of two vertex operators:

$$X(\alpha, z)X(\beta, w) = \epsilon(\alpha, \beta) : X(\alpha, z)X(\beta, w) : (z - w)^{(\alpha, \beta)}_\xi.$$
Proof. We first compute

\[ E_-(\alpha, z)H_+(\beta, w) = \exp \left( -\sum_{n \geq 1} \frac{1}{n} a_n(\alpha)z^{-n} \right) \exp \left( \sum_{n \geq 1} \frac{1}{n} a_n(\beta)w^n \right) \]

\[ = \exp \left( \sum_{n \geq 1} \frac{1}{n} a_n(\beta)w^n \right) \exp \left( -\sum_{n \geq 1} \frac{1}{n} a_n(\alpha)z^{-n} \right) \cdot \exp \left( -\langle \alpha, \beta \rangle \xi \sum_{n \geq 1} \frac{1}{n} z^{-n}w^n \right) \]

\[ = \exp \left( \sum_{n \geq 1} \frac{1}{n} a_n(\beta)w^n \right) \exp \left( -\sum_{n \geq 1} \frac{1}{n} a_n(\alpha)z^{-n} \right) \cdot \exp \left( -\langle \alpha, \beta \rangle \xi (1 - z^{-1}w)^{(\alpha, \beta)\xi} \right). \]

The second identity above uses the formula

\[ \exp(A) \exp(B) = \exp(B) \exp(A) \exp([A, B]) \]

since \([A, B]\) here commutes with \(A\) and \(B\).

On the other hand, we have

\[ z^{\partial_{\alpha_i}}e^{\gamma_j} = z^{a_{ij}}e^{\gamma_j}z^{\partial_{\alpha_i}}. \]

Combining with Eqns. (6.1) and (6.2) we compute

\[ X(\alpha, z)X(\beta, w) = H_+(\alpha, z)E_-(\alpha, z)e^\alpha z^{\partial_{\alpha_i}}H_+(\beta, w)E_-(\beta, w)e^\beta w^{\partial_{\beta}} \]

\[ = \epsilon(\alpha, \beta)H_+(\alpha, z)H_+(\beta, w)E_-^{\alpha}(\alpha, z)E_-^{\beta}(\beta, w) \]

\[ \cdot e^{\alpha + \beta}z^{\langle \alpha, \beta \rangle \xi} z^{\partial_{\alpha_i}}w^{\partial_{\beta}} (1 - w)^{(\alpha, \beta)\xi} \]

\[ = \epsilon(\alpha, \beta) : X(\alpha, z)X(\beta, w) : (z - w)^{(\alpha, \beta)\xi}. \]

\[ \square \]

The following proposition is easy to check.

**Proposition 7.1.** Given \(\alpha \in R(\Gamma), \beta \in R_Z(\Gamma),\) we have

\[ [a_n(\alpha), X(\beta, z)] = \langle \alpha, \beta \rangle \xi X(\beta, z)z^n. \]

7.2. **Affine Lie algebra \(\widehat{\mathfrak{g}}\) and toroidal Lie algebra \(\widehat{\mathfrak{g}}\).** Let \(\mathfrak{g}\) be a complex simple Lie algebra of ADE type of rank \(r\), with a root system \(\Delta\) and a set of simple roots \(\alpha_1, \ldots, \alpha_r\). Denote by \(\theta\) the highest root. Denote by \(e_\alpha, f_\alpha, h_\alpha, \alpha \in \Delta\) the Chevalley generators. Sometimes we will write \(e_i, f_i\) and \(h_i\) for \(e_{\alpha_i}, f_{\alpha_i}\) and \(h_{\alpha_i}\) respectively. Denote by \(\langle , \rangle\) the invariant bilinear form on \(\mathfrak{g}\) normalized by letting \(\langle \theta, \theta \rangle = 2\). Let \(\mathbb{C}[t, t^{-1}]\) be the space of Laurent polynomials in an indeterminate \(t\).
The affine algebra \( \hat{g} \) is the universal central extension of \( g \otimes \mathbb{C}[t, t^{-1}] \) by a one-dimensional center with a generator \( C \). As a vector space, 
\[ \hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C. \]

We denote by 
\[ a(n) = a \otimes t^n, \quad a \in g, n \in \mathbb{Z}. \]

The commutation relations in \( \hat{g} \) are given as follows:
\[
[a(n), b(m)] = [a, b](n + m) + n\delta_{n, -m}(a, b)C,
\]
\[
[C, a(n)] = 0, \quad a, b \in g, n, m \in \mathbb{Z}.
\]

We denote by \( A = (a_{ij})_{0 \leq i, j \leq r} \) the affine Cartan matrix associated to \( \hat{g} \). The matrix \( (a_{ij})_{1 \leq i, j \leq r} \) obtained by deleting the first row and column of \( A \) is the Cartan matrix of \( g \). We fix a 2-cocycle \( \varepsilon \) on the affine root lattice given in Proposition 6.1.

The so-called basic representation \( V \) of \( \hat{g} \) is an irreducible highest weight representation generated by a highest weight vector which is annihilated by \( a(n), n \geq 0, a \in g; C \) acts on \( V \) as the identity operator.

We denote by \( \hat{g} \) the toroidal Lie algebra over \( \mathbb{C} \) (associated to \( g \)) with the following presentation \([MRY]\): generators are
\[ C, h_i(n), x_n(\pm \alpha_i), n \in \mathbb{Z}, i = 0, \ldots, r; \]

relations are given by: \( C \) is central, and
\[
[h_i(n), h_j(m)] = n a_{ij} \delta_{n, -m} C,
\]
\[
[h_i(n), x_m(\pm \alpha_j)] = \pm a_{ij} x_{n+m}(\pm \alpha_j),
\]
\[
[x_n(\alpha_i), x_m(-\alpha_j)] = \delta_{ij} \varepsilon(\alpha_i, -\alpha_i)\{h_i(n + m) + n\delta_{n, -m} C\},
\]
\[
[x_n(\pm \alpha_i), x_m(\pm \alpha_j)] = 0,
\]
\[
(ad x_0(\pm \alpha_i))^{1-a_{ij}} x_m(\pm \alpha_j) = 0, \quad (i \neq j),
\]
where \( n, m \in \mathbb{Z}, i, j = 0, 1, \ldots, r \).

There exists a surjective homomorphism from the toroidal algebra \( \hat{g} \) to the double loop algebra \( g \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}] \) given by:
\[ C \mapsto 0, \quad h_i(k) \mapsto h_i \otimes s^k, \quad i = 0, \ldots, r; \]
\[ x_n(\alpha_i) \mapsto e_i \otimes s^n, \quad x_n(-\alpha_i) \mapsto f_i \otimes s^n, \quad i = 1, \ldots, r; \]
\[ x_n(\alpha_0) \mapsto f_0 \otimes s^n t, \quad x_n(-\alpha_0) \mapsto e_0 \otimes s^n t^{-1}, \quad n \in \mathbb{Z}. \]

7.3. A new form of McKay correspondence. In this subsection we set \( \Gamma \) to be a finite subgroup of \( SU_2 \) and \( \xi \) to be the virtual character \( 2\gamma_0 - \pi \) of \( \Gamma \), where \( \pi \) is the character of the two-dimensional natural representation in which \( \Gamma \) embeds in \( SU_2 \). We recall that the matrix \( A = (a_{ij})_{0 \leq i, j \leq r} \) in Sect. B.3 is the Cartan matrix for the corresponding affine Lie algebra \( \hat{g} \).
Theorem 7.2. A vertex representation of the toroidal Lie algebra \( \hat{g} \) is defined on the space \( \mathcal{F}_\Gamma \) by letting
\[
x_n(\alpha_i) \mapsto X_n(\gamma_i), \quad x_n(-\alpha_i) \mapsto X_n(-\gamma_i),
\]
\[
h_i(n) \mapsto a_n(\gamma_i), \quad C \mapsto 1,
\]
where \( n \in \mathbb{Z}, 0 \leq i \leq r \).

Proof. The identity for the product of two vertex operators in Theorem 7.1 implies the commutation relations between the components of vertex operators either by formal calculus or the Cauchy residue formula (see [FLM]). Taking into account Proposition 7.1 one deduces that the correspondence defined in Theorem 7.2 indeed realizes a representation of the toroidal algebra \( \hat{g} \), (cf. [F2, MRY]). \( \square \)

Recall that \( \delta = \sum_{i=0}^r d_i \gamma_i \) generates the one-dimensional radical \( R^0_\mathbb{Z} \) of the bilinear form \( \langle , \rangle_\xi \) in \( R_\mathbb{Z}(\Gamma) \). The lattice \( R_\mathbb{Z}(\Gamma) \) in this case can be identified with the root lattice for the corresponding affine Lie algebra. The quotient lattice \( R_\mathbb{Z}(\Gamma) / R^0_\mathbb{Z} \) inherits a positive definite integral bilinear form. Denote by \( \Gamma^* \) the set of non-trivial irreducible characters of \( \Gamma \):
\[
\Gamma^* = \{ \gamma_1, \gamma_2, \ldots, \gamma_r \}.
\]
Denote by \( \overline{\Gamma}_\mathbb{Z}(\Gamma) \) the sublattice in \( R_\mathbb{Z}(\Gamma) \) generated by \( \overline{\Gamma} \). The subset of elements in \( R_\mathbb{Z}(\Gamma) \) of length \( \sqrt{2} \) can be identified with the finite root system \( \overline{\Delta} \). Equipped with bilinear form \( \langle , \rangle_\xi \), \( \overline{\Gamma}_\mathbb{Z}(\Gamma) \) is obviously isomorphic to \( R_\mathbb{Z}(\Gamma) / R^0_\mathbb{Z} \), which is in turn identified with the root lattice of Lie algebra \( g \). Furthermore \( R_\mathbb{Z}(\Gamma) \) can be written as a direct sum of two integral lattices \( \overline{\Gamma}_\mathbb{Z}(\Gamma) \oplus \mathbb{Z}\delta \).

Denote by \( \text{Sym}(\overline{\Gamma}) \) the symmetric algebra generated by \( a_{-n}(\gamma_i), n > 0, i = 1, \ldots, r \). Equipped with the bilinear form \( \langle , \rangle_\xi \), \( \text{Sym}(\overline{\Gamma}) \) is isometric to \( \mathbb{S}_\Gamma \) which is in turn isometric to \( \overline{\Gamma}_\mathbb{R} \) as well.

We define
\[
\overline{V}_\Gamma = \mathbb{S}_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\Gamma)/R^0_\mathbb{Z}] \cong \text{Sym}(\overline{\Gamma}) \bigotimes \mathbb{C}[\overline{\Gamma}_\mathbb{Z}(\Gamma)],
\]
\[
\mathcal{F}_\Gamma = \overline{\Gamma}_\mathbb{R} \bigotimes \mathbb{C}[R_\mathbb{Z}(\Gamma)] \cong \overline{V}_\Gamma.
\]

The space \( V_\Gamma \) associated to the lattice \( R_\mathbb{Z}(\Gamma) \) is isomorphic to the tensor product of the space \( \overline{V}_\Gamma \) associated to \( \overline{\Gamma}_\mathbb{Z}(\Gamma) \) and the space associated to the rank 1 lattice \( \mathbb{Z}\delta \) equipped with the zero bilinear form.

The identity for a product of vertex operators \( X(\gamma, z) \) associated to \( \gamma \in \Delta \) (cf. Theorem 7.2) imply that \( \overline{V}_\Gamma \) affords a realization of
generate a Clifford algebra: the following theorem provides a direct link from the finite group $\Gamma \in SU_2$ to the affine Lie algebra $\hat{\mathfrak{g}}$ and thus it can be regarded as a new form of McKay correspondence.

**Theorem 7.3.** The operators $X_n(\gamma), \gamma \in \Delta, a_n(\gamma_i), i = 1, 2, \ldots, r, n \in \mathbb{Z}$ define an irreducible representation of the affine Lie algebra $\hat{\mathfrak{g}}$ on $\mathfrak{f}_\Gamma$ isomorphic to the basic representation. In particular the operators $X_0(\gamma), \gamma \in \Delta, a_0(\gamma_i), i = 1, 2, \ldots, r$ define a representation of the simple Lie algebra $\mathfrak{g}$. $\mathfrak{v}_\Gamma$ carries a distinguished basis $\chi^\rho \otimes e^\gamma, \rho \in \mathcal{P}(\Gamma), \gamma \in \mathfrak{f}_\Gamma(\Gamma).

8. Vertex operators and irreducible characters of $\Gamma_n$

In this section we specialize $\xi$ to be the trivial character $\gamma_0$ of $\Gamma$.

8.1. Algebra of vertex operators for $\xi = \gamma_0$. In this case the weighted bilinear form reduces to the standard one $\langle \cdot, \cdot \rangle$ and $\mathcal{R}_\mathbb{Z}(\Gamma)$ is isomorphic to the lattice $\mathbb{Z}^n$ with the standard integral bilinear form. Recall that $\langle \gamma_i, \gamma_j \rangle = \delta_{ij}$. The 2-cocycle $\epsilon$ can be chosen by letting $\epsilon(\gamma_i, \gamma_j) = 1$ if $i \leq j$ and $-1$ if $i > j$. It follows by standard argument that the identity for product of vertex operators given in Theorem 7.1 implies the following anti-commutation relations (cf. [F1]).

**Theorem 8.1.** The operators $X_n^+(\gamma_i), X_n^- (\gamma_i) \ (n \in \mathbb{Z} + \frac{1}{2}, 0 \leq i \leq r)$ generate a Clifford algebra:

\[
\{X_m^+(\gamma_i), X_n^+(\gamma_j)\} = 0,
\]

\[
\{X_m^- (\gamma_i), X_n^- (\gamma_j)\} = 0,
\]

\[
\{X_m^+(\gamma_i), X_n^- (\gamma_i)\} = \delta_{ij}\delta_{m,-n}.
\]

8.2. Character tables of $\Gamma_n$ and vertex operators. We now construct a special orthonormal basis in $V_\Gamma$ and then interpret them as irreducible characters of $\Gamma_n$. First we have the following simple lemma, cf. [K3] for a proof.

**Lemma 8.2.** For $m \in \mathbb{Z} + 1/2, \alpha \in R(\Gamma)$ and $\gamma \in \Gamma^*$, we have

\[
X_m(\gamma)e^\alpha = \delta_{m,-\langle \alpha, \gamma \rangle-1/2}e^{\alpha+\gamma}, \quad m \geq -\langle \gamma, \alpha \rangle - 1/2,
\]

\[
X_m^- (\gamma)e^\alpha = \delta_{m,\langle \alpha, \gamma \rangle-1/2}e^{\alpha-\gamma}, \quad m \geq \langle \gamma, \alpha \rangle - 1/2.
\]

Consequently we have

\[
X_{-m-\langle \gamma, \alpha \rangle+1/2}(\gamma)X_{-m-\langle \gamma, \alpha \rangle+3/2}(\gamma) \cdots X_{-\langle \gamma, \alpha \rangle-1/2}(\gamma)e^\alpha = e^{\alpha+m\gamma},
\]

\[
X_{-m+\langle \gamma, \alpha \rangle+1/2}(\gamma)X_{-m+\langle \gamma, \alpha \rangle+3/2}(\gamma) \cdots X_{\langle \gamma, \alpha \rangle-1/2}(\gamma)e^\alpha = e^{\alpha-m\gamma}.
\]
For a \( m \)-tuple index \( \phi = (\phi_1, \cdots, \phi_m) \in (\mathbb{Z} + 1/2)^m \) we denote
\[
X_\phi(\gamma) = X_{\phi_1}(\gamma) \cdots X_{\phi_m}(\gamma)
\]
We will use similar notations for other vertex operators.

Let \( \delta_l = (l - 1, l - 2, \cdots, 1, 0) \) be the special partition of length \( l \). We often omit the subscript if the meaning is clear from the context.

Given \( \lambda \in \mathcal{P}(\Gamma^*) \), we define
\[
\omega(\lambda) = \sum_{\gamma \in \Gamma^*} l(\lambda(\gamma)) \gamma \in \mathbb{R}^\mathbb{Z}(\Gamma),
\]
and
\[
s_{\lambda, \alpha} = \prod_{\gamma \in \Gamma^*} X_{-\lambda(\gamma) - \delta - ((\gamma, \alpha) + \gamma / 2)}(\gamma)e^\alpha,
\]
where the order of \( \prod_{\gamma \in \Gamma^*} \) is understood as a product over \( \gamma_0, \gamma_1, \cdots, \gamma_r \) from the left to the right.

Recall that \( s_\lambda \in S_\Gamma \) for \( \lambda \in \mathcal{P}(\Gamma^*) \) is defined in Eq. (5.1).

**Theorem 8.3.** The vectors \( s_{\lambda, \alpha} \) for \( \lambda = (\lambda(\gamma))_{\gamma \in \Gamma^*} \in \mathcal{P}(\Gamma^*), \alpha \in \mathbb{R}^\mathbb{Z}(\Gamma) \) form an orthonormal basis in the vector space \( \mathcal{F}_\Gamma \), where \( 1 = (1, \cdots, 1) \) and \( \delta \) are both of length \( l = l(\lambda(\gamma)) \).

The isometry \( \text{ch} \) maps the basis vector \( s_{\lambda, \alpha} \) to \( s_\lambda e^{\alpha + \omega(\lambda)} \in V_\Gamma \).

**Proof.** The Clifford algebra structure \( \mathcal{S} \) implies that the nonzero elements
\[
\prod_{\gamma \in \Gamma^*} X_{-n_1}(\gamma) \cdots X_{-n_l}(\gamma)e^\alpha
\]
of distinct indices generate a spanning set in the space \( V_\Gamma \). To see that they correspond to Schur functions we compute in a way similar to Theorem 7.1 that
\[
X(\gamma, z_1) \cdots X(\gamma, z_l)e^\alpha = X(\gamma, z_1) \cdots X(\gamma, z_l) : \prod_{i<j}(1 - z_jz_i^{-1}) z^\delta(\lambda(\gamma)) \cdot e^\alpha.
\]

The method in \[\text{[J1, J2, J3]}\] implies that this is exactly the generating function of Schur functions under the isomorphism \( \text{ch} \). Since the vertex operators corresponding to different \( \gamma \in \Gamma^* \) commute up to a sign it follows that the isometry \( \text{ch} \) maps the basis vector \( s_{\lambda, \alpha} \) to \( s_\lambda e^{\alpha + \omega(\lambda)} \in V_\Gamma \).

The orthonormality follows readily from the Clifford algebra commutation relations in Theorem 5.1. \( \square \)

It is a routine computation either by symmetric functions or vertex operator calculus (cf. \[\text{[J1]}\]) that the complete character table of \( \Gamma_n \) is given by matrix coefficients of the vertex operators \( X(\gamma, z) \).
Theorem 8.4. Given partition-valued functions $\lambda \in \mathcal{P}(\Gamma^*)$ and $\mu \in \mathcal{P}(\Gamma_n^*)$, the matrix coefficient
$$\langle s_{\lambda,-\omega(\lambda)}, a'_{-\mu} \rangle$$
is equal to the character value of the irreducible character of $\chi^\lambda$ associated to $\lambda$ at the conjugacy class of type $\mu$ in $\Gamma_n$.

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