Rate of convergence of random polarizations

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Abstract

After \( n \) random polarizations on \( \mathbb{S}^d \), the expected symmetric difference of a Borel set from a polar cap is bounded by \( C_d n^{-1} \), where the constant \( C_d \) depends on the dimension [1]. We show here that this power law is best possible and that necessarily \( C_d \geq d \).

1 Introduction

Let \( A \) be a subset of the \( d \)-dimensional sphere \( \mathbb{S}^d \) (viewed as the unit sphere in \( \mathbb{R}^{d+1} \)), and let \( \sigma : x \mapsto \bar{x} \) be a reflection at a great circle that does not pass through the north pole. The polarization of \( A \) with respect to \( \sigma \) is defined by

\[
x \in SA \Leftrightarrow \begin{cases} x \in A \text{ or } \bar{x} \in A, & \text{if } \delta(x, O) \leq \delta(\bar{x}, O), \\ x \in A \text{ and } \bar{x} \in A, & \text{if } \delta(x, O) \geq \delta(\bar{x}, O). \end{cases}
\]

Here, \( \delta(x, y) \) denotes the geodesic distance on \( \mathbb{S}^d \) given by the angle enclosed between \( x \) and \( y \), and \( O \) denotes the north pole. Since reflections preserve the uniform probability measure \( m(\cdot) \) on the sphere, so do polarizations, and

\[
m(SA \cap SB) - m(A \cap B) = \int_{\mathbb{S}^d} I_{A\setminus B}(x)I_{B\setminus A}(\bar{x}) \, dx \geq 0. \tag{1}
\]

We parametrize the reflections on \( \mathbb{S}^d \) by \( u \in \Omega = \mathbb{S}^d/\pm, \) setting

\[
\sigma_u(x) = x - 2(u \cdot x)u,
\]

and we denote the corresponding polarization by \( S_u \). A random polarization \( S_U \) is polarization in the direction of a uniformly distributed random variable \( U \) on \( \Omega \). We consider sequences of random polarizations \( S_{U_1 \ldots U_n} = S_{U_n} \circ \cdots \circ S_{U_1} \), where the \( \{U_i\}_{i \geq 1} \) are independent. Van Schaftingen has shown that almost surely, for every Borel set \( A \) the sequence \( S_{U_1 \ldots U_n} A \) converges to the polar cap \( A^* \) of the same volume [5, Theorem 3.13]. The convergence occurs in symmetric difference if \( A \) is measurable, and in Hausdorff metric if \( A \) is compact [1 Corollary 4].

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Subject of this note is the rate of convergence. In prior work, we have shown that under a similar sequence of random polarizations on $\mathbb{R}^d$,

$$\mathbb{E}
\left[
m(S_{U_1...U_n}A \triangle A^*)
\right]
\leq C_d n^{-1}.
(2)$$

There, $A$ is a Borel measurable subset of the unit ball, $m$ is Lebesgue measure (normalized so that the unit ball has measure one), and $C_d = d 2^{d+1}$ [1, Proposition 4.1]. This rate of convergence is much slower than what is known for other symmetrizations. Klartag has proved that a sequence of $3d$ carefully chosen Steiner symmetrizations in $\mathbb{R}^d$ followed by a random sequence where each step consists of $d$ orthogonal Steiner symmetrizations converges faster than every polynomial. The leading constant depends only on the dimension and grows at most polynomially [4, Theorem 1.5]. Although Klartag’s result applies only to convex bodies, it raises the question whether the power law in Eq. (2) can be improved. For random polarizations, the answer is negative:

**Proposition.** For random polarizations of a Borel set $A \subset \mathbb{S}^d$, Eq. (2) holds with $C_d = 2^d$. The $n^{-1}$ power law is optimal, and the sharp constant satisfies $C_d \geq d$.

The proof of the proposition has two parts. Eq. (2) and the upper bound on $C_d$ are obtained by simply adjusting Proposition 4.1 of [1] to the sphere. For the lower bound on $C_d$ and to prove the optimality of the power law, we analyze how spherical caps move under polarization. If $A$ is a hemisphere, we compare the difference of $S_{U_1...U_n}A$ from $A^*$ with with the order statistics of the uniform distribution, and relate the limiting distribution of $n \cdot m(S_{U_1...U_n}A \triangle A)$ to a Gamma distribution. We work on the sphere rather than on $\mathbb{R}^d$, because the additional symmetry simplifies the calculations. It will be clear from the proofs that similar bounds hold for the polarization of balls in $\mathbb{R}^d$. Other questions remain open: How quickly do the sharp constants grow with the dimension? What is the impact of the distribution of $U$? Can one speed up the convergence by adapting the sequence to $A$?

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## 2 The upper bound on the sharp constant

In this section, we show that Eq. (2) holds on the sphere with $C_d = 2^d$. For a single random polarization we have by the identity (1) and Fubini’s theorem,

$$m(A \triangle A^*) - \mathbb{E}[m(S_U A \triangle A^*)] = 2 \int_{A^* \setminus A} P(\sigma_U(x) \in A \setminus A^*) \ dm(x).$$

We compute the probability under the integral as an average over the hemisphere where $u \cdot x > 0$, and change variables to $z = \sigma_u(x)$. The inverse of the map $u \mapsto z$ and its Jacobian on the tangent space of $\mathbb{S}^d$ are given by

$$u(z; x) = \frac{x - z}{|x - z|}, \quad j(z; x) = \left(2|x - z|^{d-1}\right)^{-1},$$

and

$$2 \int_{A^* \setminus A} P(\sigma_U(x) \in A \setminus A^*) \ dm(x).$$


where $|x - z| = 2\sin \frac{\delta(x,z)}{2}$ is the Euclidean distance between $x$ and $z$ in $\mathbb{R}^{d+1}$. We obtain

$$m(A \triangle A^*) - \mathbb{E}[m(S_U A \triangle A^*)] = 2\int_{A^* \setminus A} \int_{A \setminus A^*} |x - z|^{-(d+1)} \, dm(z) \, dm(x) \geq 2^{-d}(m(A \triangle A^*))^2.$$ 

For a random sequence $S_{U_1 \ldots U_n}$, we take expectations again and apply Jensen’s inequality to see that

$$\mathbb{E}\left[m(S_{U_1 \ldots U_{n-1}} A \triangle A^*) - m(S_{U_1 \ldots U_n} A \triangle A^*)\right] \geq 2^{-d}\left(\mathbb{E}[m(S_{U_1 \ldots U_{n-1}} A \triangle A^*)]\right)^2.$$ 

It follows that $z_n = 2^{-d}\mathbb{E}[m(S_{U_1 \ldots U_n} A \triangle A^*)]$ satisfies $z_n^{-1} \geq z_{n-1}^{-1} + 1$, proving Eq. (2) with constant $C_d = 2^d$. $\square$

### 3 Random compressions

Let $A$ be a spherical cap centered at a point $a$. Polarization with respect to a reflection $\sigma : x \mapsto \bar{x}$ transforms $A$ into the spherical cap of the same volume centered at $\tau(a)$, where

$$\tau(x) = \begin{cases} 
  x, & \delta(x, O) \leq \delta(\bar{x}, O), \\
  \bar{x}, & \text{otherwise}.
\end{cases}$$

We will refer to $\tau$ as the **compression** associated with $\sigma$. The compression associated with a random reflection $\sigma_U$ will be denoted by $\tau_U$. The following lemma describes the distribution of the distance of $\tau_U(x)$ from the north pole.

**Lemma.** If $U$ is uniformly distributed on $\Omega$, then for every point $x \in S^d$ with $\delta(x, O) = \xi$

$$P(\delta(\tau_U(x), O) > \beta) = I_{\xi > \beta} \left\{ 1 - \frac{1}{\pi} \int_0^\beta \left( \frac{\cos \theta - \cos \beta}{\cos \theta - \cos \xi} \right)^{(d-1)/2} \, d\theta \right\}, \quad \beta \in [0, \pi]. \quad (3)$$

**Proof.** By definition of the compression,

$$P(\delta(\tau_U(x), O) > \beta) = I_{\xi > \beta} P(\delta(\sigma_U(x), O) > \beta).$$

For $\xi \leq \beta$, there is nothing more to show. For $\xi > \beta$, we set $t = \cos \beta$ and calculate the spherical average as an expectation with respect to the standard normal probability measure on $\mathbb{R}^{d+1}$, see [2, Exercise 63 on p.80]. We use the coordinate system $u = (r \cos \theta, r \sin \theta, \bar{u}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}$, which we rotate into a position where $x = (\cos \frac{\xi}{2}, \sin \frac{\xi}{2}, 0)$ and $O = (\cos \frac{\xi}{2}, -\sin \frac{\xi}{2}, 0)$. Then

$$(u \cdot x)(u \cdot O) = \frac{r^2}{2} (\cos 2\theta + \cos \xi),$$

and $\delta(\sigma_u(x), O) \leq \beta$ if and only if $-r^2(\cos 2\theta + \cos \xi) \geq (|\bar{u}|^2 + r^2)(t - \cos \xi)$. This results in

$$P(\delta(\tau_U(x), O) \leq \beta) = \int_{\mathbb{S}^d} I_{\delta(\sigma_u(x), O) \geq t} \, dm(u)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} \int_0^\infty I_{-r^2(\cos 2\theta + t) \geq |\bar{u}|^2(t - \cos \xi)} 2re^{-r^2} \, dr \, d\gamma(\bar{u}) \, d\theta,$$
where $\gamma$ is the standard normal probability measure on $\mathbb{R}^{d-1}$. We integrate explicitly over $r$ and evaluate the remaining Gaussian integral by rescaling $\hat{v} = \left(1 - \frac{t - \cos \xi}{t + \cos 2\theta}\right)^{1/2} \hat{u}$.

$$P(\delta(\tau_U(x), O) \leq \beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} I_{\cos 2\theta + t < 0} \left\{ \int_{\mathbb{R}^{d-1}} \frac{e^{t - \cos \xi}}{t + \cos 2\theta} d\gamma(\hat{u}) \right\} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} I_{\cos 2\theta + t < 0} \left( \frac{-\cos 2\theta - \cos \beta}{-\cos 2\theta - \cos \xi} \right)^{(d-1)/2} d\theta.$$

The claim follows after restricting the integral to a half-period and changing variables $2\theta \to \pi - \theta$. □

For $d = 1$, the reflected point $\sigma_U(x)$ is uniformly distributed on $S^1$, and Eq. (3) reduces to

$$P(\delta(\tau_U(x), O) > \beta) = I_{\delta(x,O) > \beta} \left(1 - \frac{\beta}{\pi}\right), \quad \beta \in [0, \pi].$$

As $d$ increases, $\sigma_U(x)$ concentrates in a ball of radius comparable to $d^{-1/2}$ about $x$, its distance from the north pole concentrates in an interval of length comparable to $d^{-1}$ about $\xi$, and the integral in Eq. (3) goes to zero. For all $d \geq 1$ and $0 \leq \beta \leq \xi \leq \pi$, we have the bound

$$\frac{1}{\pi} \int_0^\beta \left( \frac{\cos \theta - \cos \beta}{\cos \theta - \cos \xi} \right)^{(d-1)/2} d\theta \leq \frac{\beta}{\pi} \left( \frac{1 - \cos \beta}{1 - \cos \xi} \right)^{(d-1)/2} = \left(1 + \mathcal{O}(\beta^2)\right) \frac{2 \sin \frac{\beta}{2}}{\pi} \left( \frac{\sin \frac{\xi}{2}}{\sin \frac{\beta}{2}} \right)^{d-1}.$$

If $\{U_i\}_{i \geq 1}$ is a sequence of independent uniformly distributed random variables in $\Omega$, it follows that the Euclidean distance $Y_n = |\tau_{U_1 \ldots U_n}(x) - O|$ satisfies the recursion

$$P(Y_{n+1} > \eta | Y_n) \geq I_{Y_n > \eta} \left\{ 1 - \frac{\eta}{\ell} \left( \frac{\eta}{Y_n} \right)^{d-1} \right\}, \quad \eta \in [0, \ell]$$

(4)

with initial value $Y_0 = |x - O| = 2 \sin \frac{\xi}{2}$ and with $\ell = \pi - \mathcal{O}(\xi^2)$.

### 4 The lower bound on the sharp constant

Let $A$ be the hemisphere centered at a point $a \neq O$, and set $\alpha = \delta(a, O)$. We claim that

$$\liminf_{n \to \infty} n \mathbb{E}[m(S_{U_1 \ldots U_n} A \triangle A^*)] \geq (1 - \mathcal{O}(\alpha^2)) d.$$  

(5)

Taking $\alpha \to 0$, we see that the sharp constant satisfies $C_d \geq d$, completing the proof of the proposition.

To prove the claim, consider a sequence of random points $\{V_i\}_{i \geq 1}$ that are distributed independently and uniformly on an interval $[0, \ell]$, and let $\tilde{Y}_n$ be the $d$-th lowest point among $V_1, \ldots, V_{n+d}$. The random variable $\tilde{Y}_n$ is called the $d$-th order statistic of $V_1, \ldots, V_{n+d}$. The sequence $\{\tilde{Y}_n\}_{n \geq 0}$ solves Eq. (4) with equality, because conditioned on $\tilde{Y}_n = y$, the $d-1$ points among $V_1, \ldots, V_{n+d}$ to the left of $y$ are independent and uniformly distributed on $[0, y]$, and $V_{n+d+1}$ is independent and uniformly distributed on $[0, \ell]$. The joint distribution of the order statistics can be written explicitly in terms of binomial random variables $B(n, p)$, see [3] Exercises 21-25 on p. 142. We have

$$P(\tilde{Y}_n > \eta | \tilde{Y}_0 = y) = I_{Y > \eta} \sum_{j+k<d} P(B(d-1, \frac{\eta}{y}) = j) \cdot P(B(n, \frac{\eta}{\ell}) = k),$$

4
where the first factor in the sum accounts for the points among \( V_1, \ldots, V_d \) that fall to the left of \( \eta \), while the second factor accounts for such points among \( V_{d+1}, \ldots, V_{n+d} \). By Stirling’s formula,

\[
P(n\bar{Y}_n > \eta \mid \bar{Y}_0 = y) \rightarrow P\left(\Gamma(d) > \frac{y}{\eta}\right) \quad (n \to \infty)
\]

for each \( y \in (0, \ell] \), where \( \Gamma(d) \) is a Gamma random variable that describes the \( d \)-th point in a Poisson process of intensity one [3, Exercise 24 (b) on p.142]. In particular, \( E[\bar{Y}_n \mid \bar{Y}_0 = y] \to \ell d \).

The center of \( S_{U_1 \ldots U_n} A \) is given by \( \tau_{U_1 \ldots U_n}(a) \). We have shown in Section 3 that \( Y_n = |\tau_{U_1 \ldots U_n}(a) - O| \) satisfies Eq. (4). Since the right hand side of this recursion increases with \( Y_n \) and the geodesic distance on the sphere exceeds the Euclidean distance,

\[
P(\delta(\tau_{U_1 \ldots U_n}(a), O) > \eta) \geq P(\bar{Y}_n > \eta \mid \bar{Y}_0 = \alpha)
\]

for all \( n \geq 0 \) with \( \ell = \pi - O(\alpha^2) \). For the mean, this implies that

\[
\liminf_{n \to \infty} n E[\delta(\tau_{U_1 \ldots U_n}(a), O)] \geq (\pi - O(\alpha^2)) d.
\]

Eq. (5) follows because the symmetric difference between two hemispheres is just the distance of their centers, expressed as a fraction of \( \pi \). □

**Remark.** A slightly more careful analysis of Eq. (3) shows that for \( a \neq O \),

\[
n \delta(\tau_{U_1 \ldots U_n}(a), O) \to \pi \Gamma(d) \quad (n \to \infty)
\]

in distribution, and hence

\[
\lim_{n \to \infty} n E[n(S_{U_1 \ldots U_n} A \triangle A^*)] = d \text{ for the hemisphere } A \text{ centered at } a.
\]

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