About economic qubit cloning

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In this paper we establish a deep connection between the 3 qubit one-to-two phase-covariant quantum cloning network of Fuchs et al. [C. Fuchs, N. Gisin, R.B. Griffiths, C.S. Niu, and A. Peres, Phys. Rev. A 56 n°4, 1163 (1997)], and its economic 2 qubit counterpart due to Niu and Griffiths [Phys.Rev. A 60 n°4, 2764 (1999)]. A general, necessary and sufficient criterion is derived in order to characterize the reducibility of 3 qubit cloners to 2 qubit cloners. When this criterion is fulfilled, economic cloning is possible. We show that the optimal isotropic or universal 3 qubit cloning machine is not reducible to a 2 qubit cloner.

I. INTRODUCTION

Quantum Information emerged from the fruitful cross-fertilization of quantum mechanics and information technology. In the last decade, particularly promising applications such as quantum cryptography, quantum cloning, quantum teleportation, quantum games and quantum computers were implemented experimentally, with more or less success [1, 2, 3, 4, 5]. Although it is not certain that these progresses will lead to a practical quantum computer [6], because of the difficulties inherent to decoherence, quantum cryptography is already a grown up, user friendly and efficient technology [1, 2, 7]. Traditionally, it is implemented with two-level quantum systems, known as qubits. The inviolability of the quantum key distribution protocols such as the BB84 protocol [8] is guaranteed by the no-cloning theorem [3, 10] that states that perfect copying (cloning) of a set of input states that contains at least two non-orthogonal states is impossible. It is however possible to realize approximate quantum cloning, a concept that was introduced in a seminal paper of Buzek and Hillery [11], where a universal (or state-independent) and symmetric one-to-two cloning transformation was introduced for qubits. Here we are rather interested in state-dependent, symmetric or asymmetric one-to-two quantum cloning, in particular in the so-called phase-covariant qubit transformation [12, 13, 14] that optimally copies (with the same fidelity) all the pure states of the form $\sqrt{1/2}(|0\rangle^Z + e^{i\phi}|1\rangle^Z)$ (where $\phi \in [0, 2\pi]$) while $|0\rangle^Z$ and $|1\rangle^Z$ represent up and down spin states along a conventional direction $Z$). Such “equatorial” states are located on the intersection of the $XY$ plane with the Bloch sphere. It is important in the context of quantum cryptography to study the performances of such cloners because they condition the security of quantum cryptographic protocols. For instance the optimal phase-covariant cloner is presently believed to provide the most dangerous eavesdropping strategy for the BB84 quantum cryptographic protocol [8]. This justifies the interest of implementing experimentally the qubit one-to-two phase covariant cloners that were theoretically proposed in the past. The first theoretic proposal, that we shall from now on denote the FGGNP proposal, required the use of 3 qubits [12], two in addition to the one carrying the original signal. Unfortunately, recent NMR experiments that were realized in order to implement the (3 qubit) universal qubit cloner [2] showed that a substantial loss occurred, due to inhomogeneities of the magnetic field and decoherence. It was difficult during that experiment to avoid such effects because the corresponding cloning network contained no less than ten single qubit gates and five 2 qubit gates. Moreover, it required to control with enough precision the two-by-two entanglement between three qubits which is not a simple task at all. A quick analysis of the results of [12] shows that an experimental implementation of their proposal for phase-covariant cloning would certainly face the same problems. Fortunately, in the case of phase-covariant cloning, there exists in theory a simplified, “cheap” or “economic” 2 qubit network due to Niu and Griffiths [13], that we shall from now on denote the NG proposal, in which no external ancilla is required and that exhibits the same cloning properties as the FGGNP proposal. This proposal is quite simpler to implement experimentally, it requires only two single qubit gates and one 2 qubit gate, and it requires to control the entanglement of a pair of qubits only. It is thus likely to be quite less noisy than its 3 qubit counterpart. The goal of the present paper is to elucidate the connections that exist between 2 and 3 qubit cloners. It can be seen as a first step towards a study of the possibility to replace $3$ qubit cloners by (economic) 2 qubit cloners, a problem that presents a real interest in connection with the security of quantum cryptographic protocols.

In the second section, we introduce some useful formal tools and establish a deep relation between the FGGNP transformation and the NG transformation, in the sense that we prove that the FGGNP transformation is equivalent to a symmetrised version of the NG transformation. In the third section, we derive a general theorem that defines precisely under which conditions a $3$ qubit cloner is reducible to a $2$ qubit cloner. It allows us to prove the non-existence of such a relation in the case of universal cloning.
A summary of our results, conclusions, and a brief discussion of some open problems are presented in the last section.

II. 3 QUBIT AND 2 QUBIT PHASE-COVARIANT CLONERS, AND CONNECTIONS BETWEEN THEM.

A. (A strictly covariant generalisation of) Cerf’s formalism for 3 qubit phase-covariant cloners

Before we introduce Cerf’s formalism for cloning machines it is useful to recall the properties of the so-called Bell states. The four Bell states are defined as follows:

$$|B_{m,n}\rangle_{1,2} = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} (-)^{k-n} |k\rangle_{1}^{Z} |k+m\rangle_{2}^{Z}$$

(1)

where $m,n \in \{0,1\}$. Consequently:

$$|B_{0,0}\rangle_{1,2} = \frac{1}{\sqrt{2}} \{|0\rangle_{1}^{Z} |0\rangle_{2}^{Z} + |1\rangle_{1}^{Z} |1\rangle_{2}^{Z}\}, \quad |B_{0,1}\rangle = \frac{1}{\sqrt{2}} \{|0\rangle_{1}^{Z} |0\rangle_{2}^{Z} - |1\rangle_{1}^{Z} |1\rangle_{2}^{Z}\}$$

(2)

$$|B_{1,0}\rangle = \frac{1}{\sqrt{2}} \{|0\rangle_{1}^{Z} |1\rangle_{2}^{Z} + |1\rangle_{1}^{Z} |0\rangle_{2}^{Z}\}, \quad |B_{1,1}\rangle = \frac{1}{\sqrt{2}} \{|0\rangle_{1}^{Z} |1\rangle_{2}^{Z} - |1\rangle_{1}^{Z} |0\rangle_{2}^{Z}\}$$

(3)

where $|0(1)\rangle_{1(2)}$ represents a spin up (down) state of the qubit system $1$ ($2$) along a conventional direction (here the Z direction). They are maximally entangled states and form an orthonormal basis of the 4-dimensional Hilbert space spanned by the qubit states 1 and 2.

Let us now consider the following situation: Alice sends to Bob qubits that are either spin up or spin down along Z with 50-50 probability. Note that this is equivalent to a situation during which Alice and Bob share the maximally entangled state $|B_{0,0}\rangle_{A,B}$, while Alice measures the spin of her qubit along Z. N. Cerf proposed in Refs. [15, 16] a general characterization of asymmetric and state-dependent 1 → 2 cloning transformations for 2-level systems which is, roughly speaking, summarised as follows. Eve copies the state $|B_{0,0}\rangle_{A,B}$ by replacing it by the cloning state, which is assumed to be a 4 qubit state (one qubit for Alice, one for Bob, one for Eve and one ancilla). We shall from now on denote it $|\Psi\rangle_{A,B,E,M}$ where the indices are representative of the reference qubit possessed by Alice (A), or the two output clones (B for Bob and E for Eve), and of the (2-dimensional) ancilla or cloning machine (M). According to Cerf’s ansatz, the cloning state is biorthogonal in the Bell bases, which imposes that

$$|\Psi\rangle_{A,B,E,M} = \sum_{m,n=0}^{1} a_{m,n} |B_{m,n}\rangle_{A,B} |B_{m,n}\rangle_{E,M}$$

(4)

where $a_{m,n}$ is a (normalised) 2x2 matrix. The specification of the amplitudes $a_{m,n}$ defines the cloning transformation. Remark that such a state can be obtained by letting work on the initial state $|B_{0,0}\rangle_{A,B} |B_{0,0}\rangle_{E,M}$ a unitary transformation of the type $I_{A} \otimes U_{BE} \otimes I_{M}$ that affects neither Alice’s qubit, nor the ancilla (although the preparation of the initial state requires that Eve entangles the clone and the ancilla).

The deep reason therefore is that Bell states $|B_{m,n}\rangle_{1,2}$ can be generated from an initial state prepared along $|B_{0,0}\rangle_{1,2}^{Z}$ via local transformations. For instance, we have $1_{1} \otimes \sigma_{z}^{Z} |B_{0,0}\rangle_{1,2}^{Z} = |B_{1,0}\rangle_{1,2}^{Z}$, $1_{1} \otimes \sigma_{x}^{Z} |B_{0,0}\rangle_{1,2}^{Z} = i |B_{1,1}\rangle_{1,2}^{Z}$, $1_{1} \otimes \sigma_{y}^{Z} |B_{0,0}\rangle_{1,2}^{Z} = |B_{0,1}\rangle_{1,2}^{Z}$ where the $\sigma$’s are the Pauli matrices. In virtue of this property, it is not absolutely necessary that Alice and Bob share a maximally entangled state to begin with: we can as well consider the situation in which Alice sends directly a qubit to Bob. Nevertheless, it is convenient to consider directly the cloning state in a 4 qubit space because as we shall see now there exists a covariant generalization of Cerf’s formalism in which a “mirror” relation exists between Alice’s qubit and the cloning machine at one side, and between Bob’s clone and Eve’s clone at the other side.

Before describing this generalization, it is useful to introduce and to motivate the concept of strict covariance. In order to do so, let us consider the $Z$‘ basis defined as follows: $|0\rangle^{Z} = |0\rangle$, $|1\rangle^{Z} = i |1\rangle$. Obviously the following identities are satisfied: $|B_{0,0}\rangle_{1,2}^{Z} = |B_{1,0}\rangle_{1,2}^{Z}$, $|B_{0,1}\rangle_{1,2}^{Z} = |B_{0,0}\rangle_{1,2}^{Z}$, $|B_{1,0}\rangle_{1,2}^{Z} = i |B_{1,0}\rangle_{1,2}^{Z}$, and $|B_{1,1}\rangle_{1,2}^{Z} = i |B_{1,1}\rangle_{1,2}^{Z}$. Because of this, the amplitudes of the cloning state $|\Psi\rangle_{A,B,E,M}^{Z}$ are not necessarily...
the same in the primed basis. For instance, \( \langle Z'|A\rangle(1|Z|^2_0)\langle0|Z|^2_1\rangle_\psi \mathcal{X}_{AB,EM} = (-)\langle Z'|A\rangle(1|Z|^2_0)\langle0|Z|^2_1\rangle_\psi \mathcal{X}_{AB,EM} \). We shall say that \( |\Psi\rangle_{AB,EM}^Z \) is not strictly covariant when we pass from the \( Z \) to the \( Z' \) basis. Nevertheless, all the expectation values, of the type \( \langle ijkl|\mathcal{Z}_{AB,EM}'(\psi')_Z \rangle_{AB,EM} \psi \mathcal{X}_{AB,EM}|ijkl|\mathcal{Z}_{Z'} \rangle_{AB,EM} \), which are the diagonal coefficients of the density matrix considered in the product basis assigned to the local detectors labelled by \( A, B, E, M \), are the same in both bases. As these are the single quantities that are of physical interest in the present context, we can say that although the state \( |\Psi\rangle_{AB,EM}^Z \) is not strictly covariant when we pass from the \( Z \) to the \( Z' \) basis it is covariant FAPP.

Now, let us recall that the theory of cloning machines was developed for estimating the safety of quantum cryptographie protocols. In such protocols, the information is encoded in at least two non-commuting bases, which guarantees that a perfect cloning process is impossible, in virtue of the no-cloning theorem. Because of this limitation, the best that Eve can do for eavesdropping the signal is approximate cloning, and optimal approximate cloning corresponds, as far as we know, to the most dangerous (imperfect) eavesdropping strategy that Eve is able to resort to. It is usually assumed, conservatively, that Eve has perfect technology at her disposal (perfect transmission lines, perfect quantum devices and so on), and that she dissipates the presence of her imperfect under the noise that would be otherwise attributed to an imperfect transmission lines. Usually, transmission lines are isotropic, which implies that the error rate or transmission noise is the same in all encryption bases so that the cloners must fulfill a fundamental constraint: they must exhibit the same fidelities in all encryption bases (the fidelity \( F \) is defined for a dichotomic signal as follows: \( F=1-e \) where \( e \) is the error rate). In the following, we shall be interested in cloners that (i) satisfy the Cerf ansatz, and (ii) that are strictly covariant when we pass from one encryption basis to the other. This constraint is certainly exaggerated because in principle FAPP covariance (even less, FAPP covariance in Alice and Bob’s encryption/decryption bases only) is a sufficient constraint in order that the cloner provides an unwarable eavesdropping strategy. Nevertheless, it can be checked that all the interesting cloners in the literature, without exception, satisfy the conditions (i) and (ii). It is not our goal in the present paper to motivate why these conditions seem to be so natural in the study of cloning machines (see for instance the fourth section of Ref.\[17\] for a tentative explanation of the generality of Cerf’s ansatz). We shall show in the rest of this section that once they are accepted, it is very easy to establish a deep connection between FGGNP’s 3 qubit phase-covariant cloner and NG’s 2 qubit phase covariant cloner.

Let us come back to the \( Z-Z' \) transformation previously introduced. As we noted already, the first Bell state \( |B_{0,0}\rangle \) is not invariant: \( |B_{0,1}\rangle \mathcal{Z}_{1,2} = |B_{0,0}\rangle \mathcal{Z}_{1,2} \neq |B_{0,0}\rangle \mathcal{Z}_{1,2} = |B_{0,1}\rangle \mathcal{Z}_{1,2} \). There is another way to show this dissymetry between the \( Z \) and the \( Z' \) bases: formally, if Alice and Bob share the maximally entangled state \( |B_{0,0}\rangle \mathcal{Z}_{A,B} \), and that Alice wants to transmit to Bob states that are encrypted in the primed basis, she must encode her own qubits into the conjugate basis \( Z^* \) defined as follows: \( |0^*\rangle \mathcal{Z} = |0\rangle \mathcal{Z}, \ 1^* \mathcal{Z} = -i|1\rangle \mathcal{Z} \). Indeed, it is easy to check the identity

\[
|B_{0,0}\rangle \mathcal{Z}_{A,B} = \frac{1}{\sqrt{2}}(|0\rangle \mathcal{Z}|0\rangle + |1\rangle \mathcal{Z}|1\rangle) = \frac{1}{\sqrt{2}}(0|+|1\rangle \mathcal{Z}^*|1\rangle \mathcal{Z})
\]

This identity is the special case of a very general property: let us consider an arbitrary basis \( (|\psi_0\rangle, |\psi_1\rangle) \) that we denote the \( \psi \) basis (with \( \langle i|\psi_j\rangle = U_{ij} \)). If Alice and Bob share the joint state \( |B_{0,0}\rangle \mathcal{Z}_{A,B} \) and that Alice wishes to encode the signal in the \( \psi \) basis, she must project her component of \( |B_{0,0}\rangle \mathcal{Z}_{A,B} \) into the (conjugate) \( \psi^* \) basis defined as follows: \( \langle i|\psi_j^*\rangle = U_{ij}^* \). This property is obvious if we note that, in virtue of the unitarity of \( U_{ij} \),

\[
|B_{0,0}\rangle \mathcal{Z}_{A,B} = \frac{1}{\sqrt{2}}(|0\rangle \mathcal{Z}|0\rangle + |1\rangle \mathcal{Z}|1\rangle) = \frac{1}{\sqrt{2}} \sum_{k,l,m=0} |\psi_{kl}^*\rangle_A \langle \psi_{kl}^*|k\rangle_A \otimes |\psi_{lm}^*\rangle_B \langle \psi_{lm}^*|k\rangle_B
\]

\[
= \frac{1}{\sqrt{2}} \sum_{k,l,m=0} |\psi_{kl}^*\rangle_A U_{kl} \langle \psi_{lm}^*\rangle_B U_{lm}^* = \frac{1}{\sqrt{2}} \sum_{k,l,m=0} |\psi_{kl}^*\rangle_A \langle \psi_{lm}^*\rangle_B \delta_{ml} = \frac{1}{\sqrt{2}} \sum_{k=0} |\psi_k^*\rangle_A \langle \psi_k|B\rangle
\]

It would be nice to modify slightly Cerf’s ansatz in order that this basic invariance property is respected. As it was shown by one of us [17, 18, 19], it is possible to achieve this goal thanks to a redefinition of Bell states. These generalized Bell states can be shown to obey the following definition [17, 19]:

\[
|B_{m,n}^\psi\rangle_{AB} = \frac{1}{\sqrt{2}} \sum_{k=0} (-)^{k,n} |\psi_k\rangle_A \langle \psi_{k+m}|B
\]

Now, Eve’s clone (labelled by \( E \)) is assumed to “mirror” Bob’s qubit, and as it is shown in [12] the formalism is simplified if we consider that the ancilla (\( M \)) “mirrors” Alice’s qubit, which motivates the following definition:
\[ |B_{m,n}^{\psi}\rangle_{E,M} = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} (-)^{k,n} |\psi_k\rangle_E |\psi_{k+m}\rangle_M \]

Note that when the \( \psi \) basis is real, we recover the usual definition:
\[ |B_{m,n}^{\psi}\rangle_{A,B} = |B_{m,n}^{\psi}\rangle_{A,B} = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} (-)^{k,n} |\psi_k\rangle_A |\psi_{k+m}\rangle_B = |B_{m,n}^{\psi}\rangle_{A,B} \]

Accordingly, we shall postulate in the following the (modified) Cerf ansatz: the cloning state is assumed to take the following form:

\[ |\Psi\rangle_{A,B,E,M}^{\psi} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\psi}\rangle_{A,B} |B_{m,n}^{\psi}\rangle_{E,M} \] (8)

On the basis of these definitions, it is natural to define strict covariance as follows:

Definition:
The (generalised) Cerf cloner is said to be strictly covariant in the \( \psi \) basis and in the \( \tilde{\psi} \) basis if and only if

\[ |\Psi\rangle_{A,B,E,M}^{\psi} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\psi}\rangle_{A,B} |B_{m,n}^{\psi}\rangle_{E,M} = |\Psi\rangle_{A,B,E,M}^{\tilde{\psi}} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\tilde{\psi}}\rangle_{A,B} |B_{m,n}^{\tilde{\psi}}\rangle_{E,M} \] (9)

In [10], the following theorem is shown:

Theorem:
The (generalised) Cerf cloner is strictly covariant in the \( \psi \) basis and in the \( \tilde{\psi} \) basis if and only if whenever
\( A,B \langle B_{m,n}^{\psi}|B_{m',n'}^{\psi}\rangle_{A,B} \neq 0 \) then \( a_{m,n} = a_{m',n'} \).

This theorem provides an operational approach in order to build strictly covariant (generalised) Cerf states: it is sufficient to compute the in-products between the generalised Bell states evaluated in the different bases: whenever this in-product differs from zero, the corresponding elements of the \( a_{m,n} \) matrix that defines the cloning transformation are equal.

Note that according to the identity [8] for any pair of bases \( \{ \psi, \tilde{\psi} \} \), \( A,B \langle B_{0,0}^{\psi}|B_{m',n'}^{\psi}\rangle_{A,B} = \delta_{m',0} \delta_{n',0} \) so that \( a_{0,0} \) is a free and independent parameter for all cloning machines.

B. 3 qubit phase-covariant cloner (FGGNP cloner).

We shall now apply these results in order to derive the 3 qubit phase-covariant cloner. This cloner is aimed at providing to Eve a strategy for eavesdropping the signal sent by Alice to Bob during the completion of the BB84 protocol. During this protocol, Eve encrypts Bob’s signal (a fresh cryptographic key) either into a state of the \( X \) basis, or into a state of the \( Y \) basis. The 3 qubit phase-covariant cloner is thus necessarily covariant in the \( X \) basis and in the \( Y \) basis. We fix the phases of the \( X \) and \( Y \) basis states as follows:

\[ |+\rangle^X = \frac{1}{\sqrt{2}} (|0\rangle^Z + |1\rangle^Z), \quad |\rangle^X = \frac{1}{\sqrt{2}} (|0\rangle^Z - |1\rangle^Z) \]
\[ |+\rangle^Y = \frac{1}{\sqrt{2}} (|0\rangle^Z + i|1\rangle^Z), \quad |\rangle^Y = \frac{1}{\sqrt{2}} (|0\rangle^Z - i|1\rangle^Z) \]

Henceforth, it is easy to show the following equalities:

\[ |+^*\rangle^X = |+\rangle^X, \quad |\rangle^X = |\rangle^X \]
\[ |+^*\rangle^Y = |\rangle^Y, \quad |+^*\rangle^Y = |+\rangle^Y \] (12)

and also

\[ |0\rangle^Z = \frac{1}{\sqrt{2}} (|+\rangle^X + |\rangle^X), \quad |1\rangle^Z = \frac{1}{\sqrt{2}} (|+\rangle^X - |\rangle^X) \]
\[ |0\rangle^Z = \frac{1}{\sqrt{2}} (|+^*\rangle^Y + |\rangle^Y), \quad |1\rangle^Z = \frac{(-i)}{\sqrt{2}} (|+\rangle^Y - |\rangle^Y) \]

\[ |0\rangle^Z = \frac{1}{\sqrt{2}} (|+^*\rangle^X + |\rangle^X), \quad |1\rangle^Z = \frac{i}{\sqrt{2}} (|+\rangle^Y - |\rangle^Y) \] (14)
From now on we shall omit to write the conjugation marks (\*) when we work in the Z basis or in the X basis because, as all the states of those bases have real amplitudes relatively to the reference basis states (Z). In order to improve the clarity, we shall also always use the notations + and - for labelling the spin up and down states on the equator, and reserve the notations 0 and 1 for the spins polarized along the Z direction because this axis plays a special role in the whole treatment. One can establish the following equalities by direct computation:

\[
\begin{align*}
|B_{0,0}^Z\rangle_{A,B} &= |B_{0,0}^X\rangle_{A,B} = |B_{0,0}^Y\rangle_{A,B} = |B_{0,0}^Y\rangle_{A,B} \\
|B_{0,0}^\phi\rangle_{A,B} &= |B_{0,0}^\phi\rangle_{A,B} = |B_{0,0}^\phi\rangle_{A,B} = |B_{0,0}^\phi\rangle_{A,B} \\
|B_{1,0}^Z\rangle_{A,B} &= i|B_{-1,0}^Y\rangle_{A,B} = (-i) k e t |B_{1,0}^Y\rangle_{A,B} \\
|B_{1,0}^\phi\rangle_{A,B} &= -|B_{-1,0}^\phi\rangle_{A,B} = (-i)|B_{0,0}^\phi\rangle_{A,B} = i|B_{0,1}^\phi\rangle_{A,B} \\
\end{align*}
\]

(17)

In virtue of the aforementioned theorem, the state \(|\Psi\rangle^Z_{A,B,E,M}\) is strictly covariant in the X and the Y bases if and only if \(a_{1,0} = a_{1,1}\). In the following, we shall parametrize the matrix \(a_{m,n}\) of the phase-covariant cloner as follows: \(a_{0,0} = v, a_{0,1} = y, a_{1,0} = a_{1,1} = x\).

Then, in virtue of Eq. (4), the cloning state of the phase-covariant cloner obeys the following equation:

\[
|\Psi\rangle^X_{A,B,E,M} = v|B_{0,0}^Z\rangle_{A,B}|B_{0,0}^Z\rangle_{E,M} + y|B_{0,1}^Z\rangle_{A,B}|B_{0,1}^Z\rangle_{E,M} + x(|B_{1,0}^Z\rangle_{A,B}|B_{1,0}^Z\rangle_{E,M} + |B_{1,1}^Z\rangle_{A,B}|B_{1,1}^Z\rangle_{E,M})
\]

(18)

It is shown in [17] that the information gained by Eve is optimal when the phases of the amplitudes \(a_{m,n}\) are all the same. Physically, this condition can be shown to ensure that constructive interferences occur in certain detectors of Eve, and destructive interference in others, in such a way that Eve maximizes her information over Alice’s key. Consistently, we shall thus systematically assume in what follows that the matrix \((a_{m,n})\) is a real matrix, with real positive coefficients. Normalisation of the phase-covariant cloner imposes that \(2x^2+y^2+z^2=1\).

We can always define an “equatorial” basis that makes an angle \(\phi\) with the X basis on the Bloch sphere through the expression \(|+\rangle^\phi = \frac{1}{\sqrt{2}}(|0\rangle^Z + e^{i\phi}|1\rangle^Z|), -|\rangle^\phi = \frac{1}{\sqrt{2}}(|0\rangle^Z - e^{i\phi}|1\rangle^Z).\) By a straightforward computation, it is easy to check that the following relations are valid:

\[
\begin{align*}
|B_{0,0}^Z\rangle_{A,B} &= |B_{0,0}^\phi\rangle_{A,B} = |B_{0,0}^\phi\rangle_{A,B}, \\
|B_{0,1}^Z\rangle_{A,B} &= |B_{0,0}^\phi\rangle_{A,B} = |B_{0,1}^\phi\rangle_{A,B}, \\
|B_{1,0}^Z\rangle_{A,B} &= \cos\phi|B_{1,0}^\phi\rangle_{A,B} + i\sin\phi|B_{1,0}^\phi\rangle_{A,B}, \\
|B_{1,1}^Z\rangle_{A,B} &= (-\cos\phi|B_{1,1}^\phi\rangle_{A,B} - i\sin\phi|B_{1,1}^\phi\rangle_{A,B}
\end{align*}
\]

On the basis of these relations and on their complex conjugates, it is easy to establish the strict covariance of the cloner that satisfies \(a_{1,0} = a_{1,1}\) on the whole equator. This justifies the name “phase-covariant” or “equatorial” cloner sometimes met in the literature.

It is worth noting that, although in the seminal paper of FGGNP [12] the generalized Bell states did not play any special role, the phase-covariant cloner that they proposed can be shown to be equivalent to the phase-covariant cloner derived in this section. Therefore, in the rest of the paper we shall systematically refer to the phase-covariant cloner as the 3 qubit FGGNP cloner: from now on, we shall denote the phase-covariant state \(|\Psi\rangle^X_{A,B,E,M}\) defined in Eq. (13) as \(|\Psi\rangle^{FGGNP}_{A,B,E,M}\).

Actually, if we consider a cloner that clones equally well the Z and the X bases, then the constraint is \(a_{1,0} = a_{0,1}\), and the cloner that satisfies this constraint is strictly covariant on the meridian that passes through the poles and the X basis states on the Bloch sphere (the big circle constituted by the intersection of the XZ plane and the Bloch sphere). We could say that it is a “Greenwich-covariant” cloner. Moreover, as all the states on this meridian are purely real, the generalized Cerf formalism reduces to the Cerf formalism in this case. The “Greenwich-covariant” and the phase-covariant cloner obviously exhibit similar properties. This explains why the equivalence between the Greenwich-covariant cloner a la Cerf and the phase-covariant FGGNP cloner was already established by N. Cerf in the past (see for instance the appendix of reference [21]).

C. 2 qubit phase-covariant cloner (NG cloner), and connections with the 3 qubit phase-covariant cloner (FGGNP cloner).

As before, let us consider that Alice sends a fresh cryptographic key according to the BB84 protocol, which means that she encrypts the signal along a state \(|\psi\rangle_B\) chosen at random among one of the four following states: \(|+\rangle^X_B, |-\rangle^X_B, |+\rangle^Y_B\) or \(|-\rangle^Y_B\).
The NG copying machine \[13\] works as follows: Eve lets work on the initial state \(|\psi\rangle_B|0\rangle_E\) a unitary transformation of the type \(U_{BE}\) that is conceived in such a way that \(U_{BE}|0\rangle_B|0\rangle_E = |0\rangle_B|0\rangle_E\) and \(U_{BE}|1\rangle_B|0\rangle_E = \cos \alpha |1\rangle_B|0\rangle_E + \sin \alpha |0\rangle_B|1\rangle_E\).

Remark that no ancilla \((M)\) is required.

In order to implement BB84 protocol, Alice and Bob could as well share the maximally entangled states \(|B_{0,0}\rangle_{A,B}\), while Alice would measure the spin of her qubit at random along \(X\) or along \(Y\). Then, the NG proposal can be formulated as follows: Eve copies the state \(|B_{0,0}\rangle_{A,B}\) by replacing it by the cloning state, which is assumed to be a 3-qubit state (one qubit for Alice, one for Bob, one for Eve and no ancilla). We shall from now on denote this state \(|\Psi\rangle_{A,B,E}^{NG}\) as:

\[
|\Psi\rangle_{A,B,E}^{NG} = \frac{1}{\sqrt{2}}(|0\rangle_B U_{BE}|0\rangle_B|0\rangle_E + |1\rangle_B U_{BE}|1\rangle_B|0\rangle_E) = \frac{1}{\sqrt{2}}(|000\rangle_{ABE} + \cos \alpha |110\rangle_{ABE} + \sin \alpha |101\rangle_{ABE})
\]

By direct computation we obtain the following expressions for the NG cloning state \(|\Psi\rangle_{A,B,E}^{NG}\) in the X basis:

\[
|\Psi\rangle_{A,B,E}^{NG} = \frac{1}{\sqrt{2}} \{(|+\rangle_A \mp |\rangle_B)(|+\rangle_E + |\rangle_B) + \cos \alpha (|+\rangle_A \mp |\rangle_B)(|+\rangle_E + |\rangle_B) + \sin \alpha (|+\rangle_A \mp |\rangle_B)(|+\rangle_E + |\rangle_B)\}
\]

In the Y basis, we get:

\[
|\Psi\rangle_{A,B,E}^{NG} = \frac{1}{\sqrt{2}} \{(|+\rangle_A \pm |\rangle_B)(|+\rangle_E \pm |\rangle_E) + \cos \alpha (|+\rangle_A \pm |\rangle_B)(|+\rangle_E \pm |\rangle_E) + \sin \alpha (|+\rangle_A \pm |\rangle_B)(|+\rangle_E \pm |\rangle_E)\}
\]

This proves the strict covariance of the NG cloning state in the \(X\) and \(Y\) bases. It is legitimate to investigate whether a connection could exist between the 2 qubit NG cloning state defined in Eq. \(21\) and the 3 qubit FGGNP cloning state defined in Eq. \(13\). Many differences are manifest between this cloning state and the NG state: as we noted from the beginning, the FGGNP is a 3 qubit state (actually three + 1 if we also take account of Alice’s qubit) while the NG state is a 2 qubit state (actually two + 1), but we have also that the FGGNP cloning state is symmetric under the spin flip along \(Z\) (which exchanges \(|0\rangle\) and \(|1\rangle\)) or under the spin flips along \(X\) or \(Y\), which is not true for the FGGNP cloning state. Actually, this property is a very general property of the Bell states that can be generalised to arbitrary dimensions \(17, 19\).

Another difference is that, according to Eq. \(21\) the NG states form a family of states that are parametrized by one parameter only \((\alpha)\), while, taking account of the normalisation, two parameters are necessary for specifying the FGGNP state defined in Eq. \(13\).

Nevertheless, the optimal state derived in \(12\) and \(13\) exhibits the same properties: the fidelity is then the same for Alice and Bob’s clones and is equal to \(\frac{1}{2} + \frac{v}{\sqrt{8}}\). This corresponds to the parameters choices \(\alpha = \frac{\pi}{4}\), \(v = \frac{1}{2} + \frac{1}{\sqrt{8}}\), \(y = \frac{1}{2} - \frac{1}{\sqrt{8}}\), and \(x = \frac{1}{\sqrt{8}}\).

In general (this is for instance the case with the universal cloning machine \(21\)), if Eve suppresses the ancilla, she loses some useful and relevant information, but this is not true in the present (optimal) case as is also shown in \(21\).

We shall now show that a subclass of the set of FGGNP states, for which the ancilla can be dropped without losing information, reduces to a one-parameter class of states that “looks like” the NG state after elimination of the ancilla (the qubit \(M\)). In order to do so, let us for instance consider that after the FGGNP cloning transformation, Alice and Bob measure their respective qubit in the X basis. Naturally, Eve, who is assumed to listen to their public communication gets informed about their choice of basis and decides to measure her qubit and the ancilla in the \(X\) basis too. Let us assume that Alice’s measurement reveals that the state of the \(A\) qubit is up \((|+\rangle_A\) (as the full state is symmetric under the exchange of \(|+\rangle\) and \(|-\rangle\), the treatment would be entirely similar when she measures a spin down along \(X\)). Then, we obtain by direct computation that the probabilities that Eve’s qubit is polarised along \(+X\) \((−X)\) while the ancilla is polarised along \(+X\) \((−X)\), denoted \(P_{EM}(+X,−X), +X(−X)\) are distributed as follows: \(P_{EM}(+X, +X) = \frac{1}{4}(v + x)^2, P_{EM}(−X, +X) = \frac{1}{4}(v − x)^2, P_{EM}(+X, −X) = \frac{1}{4}(v + x)^2, P_{EM}(−X, −X) = \frac{1}{4}(v − x)^2\). Now, as it is shown in \(21, 22\), the extra-information that is present in the ancilla is optimally exploited if Eve conditions the result of the measurement on the qubit \(E\) on its equality \(24\) (inequality) with the result of the measurement on the ancilla \(M\). We shall now omit the heuristic hypothesis according to which it is possible to drop the ancilla and thus to replace the 3 qubit cloning state by a 2 qubit state when this can be done without losing information so to say when the statistics of Eve’s results is invariant when we condition them on their equality (inequality) with the values of the spin of the ancilla. The validity of this hypothesis will be discussed in detail in the next section.

The ancilla can be dropped without losing information whenever \(P_{EM}(+X, +X) = P_{EM}(−X, +X)\). Under this condition the probabilities of firing of Eve’s detectors are the same, when they are conditioned on their equality (inequality).
with the polarisations of the ancilla. These constraints are fulfilled either when

\[
\frac{v + x}{x + y} = \frac{v - x}{x - y} \tag{22}
\]

or when

\[
\frac{v + x}{x + y} = -\frac{v - x}{x - y} \tag{23}
\]

In addition of the normalisation condition, each of these constraints defines a one-parameter class of phase-covariant cloners, but the optimal cloner fulfills only the first equation. Therefore, from now on we shall focus only on the family defined by the first equation. We shall elucidate the real meaning of this constraint in the next subsection. Taking account of the definition of the Bell states (Eq. (1)), the FGGNP state can be rewritten as follows:

\[
|\Psi\rangle_{A,B,E,M}^{FGGNP} = \frac{1}{\sqrt{2}} \left[ (|000\rangle_A|000\rangle_B|000\rangle_E + |111\rangle_A|111\rangle_B|111\rangle_E) - \sqrt{2} |A,B,E\rangle_{ABE} |\alpha\rangle_{ABE} \right]
\]

If \(\frac{v + x}{x + y} = \frac{v - x}{x - y}\), then \(x^2 = vy\). Beside, \(2x^2 + y^2 = 1\) by normalisation so that \((v + y)^2 = v^2 + 2vy + y^2 = v^2 + 2x^2 + y^2 = 1\). As \(v\) and \(y\) are real positive parameters, we have that \(v + y = 1\). The normalisation condition can also be rewritten as follows: \((v - y)^2 + 4x^2 = 1\) so that we are free to redefine \(v - y\) and \(2x\) as follows: \(v - y = \cos \alpha\) and \(2x = \sin \alpha\), where \(\alpha \in [0, \pi]\). Taking account of these reparametrisations, and of the definition of Bell states, we can finally express the FGGNP cloning state as follows:

\[
|\Psi\rangle_{A,B,E,M}^{FGGNP} = \frac{1}{\sqrt{2}} \left[ (|000\rangle_A|000\rangle_B|000\rangle_E + \cos \alpha |101\rangle_{ABE} + \sin \alpha |010\rangle_{ABE} ) - |A,B,E\rangle_{ABE} \right]
\]

This means that

\[
|\Psi\rangle_{A,B,E,M}^{FGGNP} = \frac{1}{\sqrt{2}} (|\Psi\rangle_{A,B,E}^{NG} |0\rangle_M + |\Psi\rangle_{A,B,E}^{NGflip} |1\rangle_M)
\]

where \(|\Psi\rangle_{A,B,E}^{NG}\) is obtained by inverting the north and south poles (0 and 1) inside the expression of \(|\Psi\rangle_{A,B,E}^{NG}\). In other words, when we trace over the ancilla during the FGGNP cloning process, everything happens as if we realised the NG attack. For instance, in the optimal case, the error rate of the NG cloning process is equal to 50 % along the north pole and to 0 along the north pole. In average it is thus equal to 25 % during the symmetrized NG process, the same as for the optimal FGGNP process, as it must.

### III. About the Reducibility of 3 Qubit Cloners to 2 Qubit Cloners.

In the previous chapter, we showed that, when Eve does not gain more information when she conditions the results of the measurements performed on the clone onto their equality with those obtained from the ancilla, the 3 qubit cloner can be reduced to a 2 qubit cloner. This condition was introduced heuristically. It is interesting to understand why it is so, and also to investigate the generality of this condition: for instance it is legitimate to understand whether or not it is a necessary condition, or a sufficient one.

Before doing so, it is worth recalling that in the present approach we are only interested in 3 qubit cloning states that fulfill Cerf’s ansatz. This means that the cloning state is biorthogonal in the Bell bases, which imposes that the Eq. (1) is fulfilled, where \(a_{m,n}\) is a (normalised) 2x2 matrix and where the Bell states are defined by Eq. (1), relatively to a reference basis, say the \((Z)\) basis. The specification of the amplitudes \(a_{m,n}\) defines the cloning transformation.

The following definition that was inspired by the equality \(\Psi\) helps us to precise what we mean in general by reducibility of 3 qubits cloners to 2 qubit cloners.

**Main definition:**
A 3 qubit cloner that fulfills Cerf’s ansatz (in the reference (Z) basis) is said to be reducible to a 2 qubit cloner in the Z basis iff it is possible to find a qubit basis \{(|0\rangle, |1\rangle)\}, a positive real number \(p\) comprised between 0 and 1 and two unitary two-qubit transformations \(U_{BE}\) and \(V_{BE}\) such that

\[
|\Psi\rangle_{A,B,E,M} = \sum_{m,n=0}^{1} a_{m,n}|B_{m,n}\rangle_{A,B}|B_{m,n}\rangle_{E,M} = \sqrt{p} |\Psi\rangle_{A,B,E,0} + \sqrt{1-p} |\Psi\rangle_{A,B,E,1} \tag{27}
\]

where \(|\Psi\rangle_{A,B,E} = 1_A \otimes U_{BE} |B_{0,0}^Z\rangle_{AB} |0\rangle^Z_E\) and \(|\Psi\rangle_{A,B,E} = 1_A \otimes V_{BE} |B_{0,0}^Z\rangle_{AB} |0\rangle^Z_E\.

It is very easy to check that the following corollaries are valid:

Corollary 1: a 3 qubit cloner is reducible to a qubit cloner iff it is possible to find a qubit basis \{(|0\rangle, |1\rangle)\}, and two unitary two-qubit transformations \(U_{BE}\) and \(V_{BE}\) such that

\[
|\Psi\rangle_{A,B,E,M}^C = \sqrt{p} |\Psi\rangle_{A,B,E,0} + \sqrt{1-p} |\Psi\rangle_{A,B,E,1} \tag{28}
\]

where \(|\Psi\rangle_{A,B,E} = 1_A \otimes U_{BE} |B_{0,0}^Z\rangle_{AB} |0\rangle^Z_E\) and \(|\Psi\rangle_{A,B,E} = 1_A \otimes V_{BE} |B_{0,0}^Z\rangle_{AB} |0\rangle^Z_E\), where the \(Z'\) basis is arbitrary. This is a direct consequence of the identity \(\mathcal{P}_{ABE} = \text{Trace}_M |\Psi\rangle_{A,B,E,M}^C \langle \Psi|_{A,B,E,M}^C = p |\Psi\rangle_{A,B,E} \langle \Psi|_{A,B,E} + (1-p) |\Psi\rangle_{A,B,E} \langle \Psi|_{A,B,E}.\)

Moreover, \(|\Psi\rangle_{A,B,E}^C\) and \(|\Psi\rangle_{A,B,E}\) can be generated by interactions that do not involve Alice’s qubit, and no ancilla \(M\) is required, which represents a serious simplification of the cloning process.

We shall now prove the following theorem:

Main theorem:

A) Necessary condition:

When a 3 qubit cloner that fulfills Cerf’s ansatz (in the reference (Z) basis) is reducible to a 2 qubit cloner in the Z basis and when the matrix \(a_{m,n}\) is purely real, then:

i) \(a_{0,0}a_{0,1} = a_{1,0}a_{1,1}\)

or

ii) \(a_{0,0}a_{1,0} = a_{0,1}a_{1,1}\)

or

iii) \(a_{0,0}a_{1,1} = a_{0,1}a_{1,0}\)

B) Sufficient condition:

i) When \(a_{0,0}a_{1,1} = a_{1,0}a_{1,1}\), then the Eq. (27) is fulfilled with the qubit basis \{(|0\rangle, |1\rangle)\} equal to the Z basis.

ii) When \(a_{0,0}a_{1,0} = a_{0,1}a_{1,1}\), then the Eq. (27) is fulfilled with the qubit basis \{(|0\rangle, |1\rangle)\} equal to the X basis.

iii) When \(a_{0,0}a_{1,1} = a_{0,1}a_{1,0}\), then the Eq. (27) is fulfilled with the qubit basis \{(|0\rangle, |1\rangle)\} equal to the Y basis

Proof of the main theorem:

A) Proof of the necessary condition.

Without loss of generality, we can parametrize \{(|0\rangle, |1\rangle)\} as follows: \{(|0\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle, |1\rangle = \sin \frac{\theta}{2}|0\rangle - e^{i\phi} \cos \frac{\theta}{2}|1\rangle\}. Then, in virtue of Eq. (11) and Eq. (27), we have that

\[
|\Psi\rangle_{A,B,E,M} = \sum_{m,n=0}^{1} a_{m,n}|B_{m,n}\rangle_{A,B}|B_{m,n}\rangle_{E,M} =
\]

\[
\frac{1}{2} |0\rangle_{A}((a_{0,0} + a_{1,1})|00\rangle_{BE} + (a_{1,0} - a_{1,1})|11\rangle_{BE})|0\rangle_{M} +
\]

\[
|0\rangle_{A}((a_{0,0} - a_{1,1})|01\rangle_{BE} + (a_{1,0} + a_{1,1})|10\rangle_{BE})|1\rangle_{M} +
\]

\[
|1\rangle_{A}((a_{0,0} + a_{1,1})|00\rangle_{BE} + (a_{1,0} - a_{1,1})|11\rangle_{BE})|0\rangle_{M} +
\]

\[
|1\rangle_{A}((a_{0,0} - a_{1,1})|01\rangle_{BE} + (a_{1,0} + a_{1,1})|10\rangle_{BE})|1\rangle_{M} =
\]

\[
\frac{\sqrt{p}}{\sqrt{2}} |0\rangle_{A}U_{BE}|00\rangle_{BE} + |1\rangle_{A}U_{BE}|10\rangle_{BE}(|\cos \frac{\theta}{2}|0\rangle_{M} + e^{i\phi} \sin \frac{\theta}{2}|1\rangle_{M}) +
\]

\[
\frac{\sqrt{1-p}}{\sqrt{2}} |0\rangle_{A}V_{BE}|00\rangle_{BE} + |1\rangle_{A}V_{BE}|10\rangle_{BE}(|\sin \frac{\theta}{2}|0\rangle_{M} - e^{i\phi} \cos \frac{\theta}{2}|1\rangle_{M})
\]

(29)

Unitarity of \(U\) imposes that \(U_{BE}|00\rangle_{BE}\) and \(U_{BE}|10\rangle_{BE}\) are mutually orthogonal and normalised. A similar constraint holds for \(V\). Now, projecting the equality (29) onto the basis \(|ij\rangle_{AM}\) (where \(i,j \in \{0,1\}\)), we obtain four
identities:

\[
\frac{1}{2}((a_{0,0} + a_{0,1})|00\rangle_{BE} + (a_{1,0} - a_{1,1})|11\rangle_{BE}) = \frac{\sqrt{p}}{\sqrt{2}} \cos \frac{\theta}{2} U_{BE}|00\rangle_{BE} + \frac{\sqrt{1-p}}{\sqrt{2}} \sin \frac{\theta}{2} V_{BE}|00\rangle_{BE}
\]  

(30)

\[
\frac{1}{2}((a_{0,0} - a_{0,1})|01\rangle_{BE} + (a_{1,0} + a_{1,1})|10\rangle_{BE}) = \frac{\sqrt{p}}{\sqrt{2}} e^{i\phi} \sin \frac{\theta}{2} U_{BE}|00\rangle_{BE} + \frac{\sqrt{1-p}}{\sqrt{2}} (-e^{i\phi} \cos \frac{\theta}{2} V_{BE}|00\rangle_{BE}
\]  

(31)

\[
\frac{1}{2}((a_{0,0} - a_{0,1})|10\rangle_{BE} + (a_{1,0} + a_{1,1})|01\rangle_{BE}) = \frac{\sqrt{p}}{\sqrt{2}} \cos \frac{\theta}{2} U_{BE}|10\rangle_{BE} + \frac{\sqrt{1-p}}{\sqrt{2}} \sin \frac{\theta}{2} V_{BE}|10\rangle_{BE}
\]  

(32)

\[
\frac{1}{2}((a_{0,0} + a_{0,1})|11\rangle_{BE} + (a_{1,0} - a_{1,1})|00\rangle_{BE}) = \frac{\sqrt{p}}{\sqrt{2}} e^{i\phi} \sin \frac{\theta}{2} U_{BE}|10\rangle_{BE} + \frac{\sqrt{1-p}}{\sqrt{2}} (-e^{i\phi} \cos \frac{\theta}{2} V_{BE}|10\rangle_{BE}
\]  

(33)

By elementary algebra, it is easy to check that:

\[
\sqrt{2p} e^{i\phi} U_{BE}|00\rangle_{BE} = e^{i\phi} \cos \frac{\theta}{2} ((a_{0,0} + a_{0,1})|00\rangle_{BE} + (a_{1,0} - a_{1,1})|11\rangle_{BE}) + \sin \frac{\theta}{2} ((a_{0,0} - a_{0,1})|01\rangle_{BE} + (a_{1,0} + a_{1,1})|10\rangle_{BE})
\]  

(34)

\[
\sqrt{2(1-p)} e^{i\phi} V_{BE}|00\rangle_{BE} = e^{i\phi} \sin \frac{\theta}{2} ((a_{0,0} + a_{0,1})|00\rangle_{BE} + (a_{1,0} - a_{1,1})|11\rangle_{BE}) - \cos \frac{\theta}{2} ((a_{0,0} - a_{0,1})|01\rangle_{BE} + (a_{1,0} + a_{1,1})|10\rangle_{BE})
\]  

(35)

\[
\sqrt{2p} e^{i\phi} U_{BE}|10\rangle_{BE} = e^{i\phi} \cos \frac{\theta}{2} ((a_{0,0} - a_{0,1})|10\rangle_{BE} + (a_{1,0} + a_{1,1})|01\rangle_{BE}) + \sin \frac{\theta}{2} ((a_{0,0} + a_{0,1})|11\rangle_{BE} + (a_{1,0} - a_{1,1})|00\rangle_{BE})
\]  

(36)

\[
\sqrt{2(1-p)} e^{i\phi} V_{BE}|10\rangle_{BE} = e^{i\phi} \sin \frac{\theta}{2} ((a_{0,0} - a_{0,1})|10\rangle_{BE} + (a_{1,0} + a_{1,1})|01\rangle_{BE}) - \cos \frac{\theta}{2} ((a_{0,0} + a_{0,1})|11\rangle_{BE} + (a_{1,0} - a_{1,1})|00\rangle_{BE})
\]  

(37)

When \(a_{m,n}\) is a purely real matrix, the orthogonality of \(U_{BE}|00\rangle_{BE}\) and \(U_{BE}|10\rangle_{BE}\) imposes that \(e^{-i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2} (a_{0,0} + a_{0,1})(a_{1,0} - a_{1,1}) + e^{i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2} (a_{0,0} - a_{0,1})(a_{1,0} + a_{1,1}) = 0\). This equation has two groups of solutions: either \(\cos \frac{\theta}{2} \sin \frac{\theta}{2} = 0\), or \((a_{0,0} + a_{0,1})(a_{1,0} - a_{1,1}) = (-e^{2i\phi}(a_{0,0} - a_{0,1})(a_{1,0} + a_{1,1}) = 0\).

When \(\cos \frac{\theta}{2} \sin \frac{\theta}{2} = 0\), the basis \(\{|0\rangle, |1\rangle\}\) is the Z basis, and, because \(||U_{BE}|00\rangle_{BE}|| = ||U_{BE}|10\rangle_{BE}||\) by unitarity, we get, taking account of the reality of the matrix \(a_{m,n}\), that \((a_{0,0} + a_{0,1})^2 + (a_{1,0} - a_{1,1})^2 = (a_{0,0} - a_{0,1})^2 + (a_{1,0} + a_{1,1})^2\), so that we must impose that (i) \(a_{0,0}a_{0,1} = a_{1,0}a_{1,1}\).

When \((a_{0,0} + a_{0,1})(a_{1,0} - a_{1,1}) = (-e^{2i\phi}(a_{0,0} - a_{0,1})(a_{1,0} + a_{1,1})\), taking account of the reality of the matrix \(a_{m,n}\), either \(\phi = 0\) and (ii) \(a_{0,0}a_{1,1} = a_{1,0}a_{1,1}\) or \(\phi = \frac{\pi}{2}\) and (iii) \(a_{0,0}a_{1,1} = a_{1,0}a_{1,1}\). It is easy to check that when the conditions i, ii, or iii are fulfilled the other constraints imposed by the unitarity of \(U\) and \(V\) are automatically satisfied.

B. Proof of the sufficient condition.

Let us firstly prove the sufficient condition (ii). If \(a_{0,0}a_{1,1} = a_{1,0}a_{1,1}\), then \(a_{0,0}a_{1,1}(a_{1,0} - a_{1,1}) = (-a_{0,0} - a_{0,1})(a_{1,0} + a_{1,1})\) and we can, without loss of generality, assume that \(a_{0,0}a_{1,1} = \alpha, \alpha(0,0 - a_{0,1}) = \beta, (a_{1,0} + a_{1,1}) = -r\alpha\), and \(a_{1,0} + a_{1,1} = r\beta\) with \(r, \alpha\) and \(\beta\) real (in the special cases where \(\alpha\) and \(\beta\) would be equal to 0, we must be careful and consider the limit in which \(r\) would go to infinity, but it does not invalidate the reasoning). We have then, taking account of Eq. (10) and Eq. (11), after substitution, that

\[
|\Psi\rangle_{A,B,E,M}^{\text{Cerf}} =
\]

\[
\frac{1}{\sqrt{2}} (|0\rangle_{A}((|00\rangle_{BE} + r\beta|11\rangle_{BE})|0\rangle_{Z} + (|01\rangle_{BE} - r\alpha|10\rangle_{BE})|1\rangle_{Z}) +
\]

\[
|1\rangle_{A}((|01\rangle_{BE} + r\alpha|00\rangle_{BE})|0\rangle_{Z} + (|11\rangle_{BE} + r\beta|00\rangle_{BE})|1\rangle_{Z}) =
\]

\[
\]

\[
+ |1\rangle_{A}(\alpha|00\rangle_{BE} + r\beta|11\rangle_{BE} + \beta|01\rangle_{BE} - r\alpha|10\rangle_{BE})\frac{1}{\sqrt{2}}(|0\rangle_{Z} + |1\rangle_{Z})
\]

\[
+ |0\rangle_{A}(\alpha|00\rangle_{BE} + r\beta|11\rangle_{BE} - \beta|01\rangle_{BE} + r\alpha|10\rangle_{BE})\frac{1}{\sqrt{2}}(|0\rangle_{Z} - |1\rangle_{Z})
\]

\[
+ |1\rangle_{A}(-\alpha|00\rangle_{BE} - r\beta|11\rangle_{BE} + \beta|01\rangle_{BE} - r\alpha|10\rangle_{BE})\frac{1}{\sqrt{2}}(|0\rangle_{Z} - |1\rangle_{Z})
\]

(38)
Let us consider the transformations $U$ and $B$ defined as follows:

\[
\begin{align*}
U_{BE}|00\rangle_{BE} &= \frac{1}{\sqrt{2}}(\alpha|00\rangle^Z_{BE} + r\beta|11\rangle^Z_{BE} + \beta|01\rangle^Z_{BE} - r\alpha|10\rangle^Z_{BE}) \\
U_{BE}|10\rangle_{BE} &= \frac{1}{\sqrt{2}}(\alpha|01\rangle^Z_{BE} + r\beta|00\rangle^Z_{BE} + \beta|11\rangle^Z_{BE} - r\alpha|10\rangle^Z_{BE}) \\
V_{BE}|00\rangle_{BE} &= \frac{1}{\sqrt{2}}(\alpha|00\rangle^Z_{BE} + r\beta|11\rangle^Z_{BE} - \beta|01\rangle^Z_{BE} + r\alpha|10\rangle^Z_{BE}) \\
V_{BE}|00\rangle_{BE} &= \frac{1}{\sqrt{2}}(-\alpha|11\rangle^Z_{BE} - r\beta|00\rangle^Z_{BE} + \beta|10\rangle^Z_{BE} - r\alpha|01\rangle^Z_{BE})
\end{align*}
\]  

(39)

It is easy to check that $U$ and $V$ are unitary, taking account of the normalisation of $|\Psi\rangle^{Cerf}_{A,B,E,M}$ and of the fact that $r$ is real. Substituting Eq. (39) into Eq. (33) ends the proof of the sufficient condition (iii).

The proof of the sufficient condition (iii) is entirely similar. For proving the sufficient condition (i) it is enough to make use of the identities [14]. Then, in virtue of the sufficient condition (ii), we have that when $a_{0,0}a_{0,1} = a_{1,0}a_{1,1}$,

\[
|\Psi\rangle^{Cerf}_{A,B,E,M} = \sqrt{p}\left|\Psi^U\rangle_{A,B,E}|\tilde{0}\rangle_M + \frac{\sqrt{1-p}}{\sqrt{2}}|\Psi^V\rangle_{A,B,E}|\tilde{1}\rangle_M\right.
\]

(40)

where $|\tilde{0}\rangle_M = \frac{1}{\sqrt{2}}(|+\rangle^X + |-\rangle^X) = |0\rangle^Z_M$ and $|\tilde{1}\rangle_M = \frac{1}{\sqrt{2}}(|+\rangle^X - |-\rangle^X) = |1\rangle^Z_M$, and where $|\Psi^U\rangle_{A,B,E} = 1_A \otimes U_{BE}|B_{0,0}^XAB\rangle_{EB} + |+\rangle^Z_E = 1_A \otimes V_{BE}|B_{0,0}^X_{AB}\rangle_{EB} + |+\rangle^Z_E$ and $|\Psi^V\rangle_{A,B,E} = 1_A \otimes V_{BE}|\tilde{A}_{0,0}^Z\rangle_{EB} + |+\rangle^Z_E = 1_A \otimes V_{BE}|\tilde{A}_{0,0}^Z\rangle_{EB} + |+\rangle^Z_E$ with $U$, $U'$, $V$, $V'$ unitary, so that:

\[
|\Psi\rangle^{Cerf}_{A,B,E,M} = \sqrt{p}\left|\Psi^U\rangle_{A,B,E}|0\rangle^Z_M + \frac{\sqrt{1-p}}{\sqrt{2}}|\Psi^V\rangle_{A,B,E}|1\rangle^Z_M\right.
\]

(41)

which ends the proof. Note that in all the cases the symmetry of $|\Psi\rangle^{Cerf}_{A,B,E,M}$ under permutation of the basis states of the $X$, $Y$, and $Z$ bases imposes that $p=1 - p = \frac{1}{2}$.

Let us now reconsider the heuristic hypothesis according to which it is possible to drop the ancilla and thus to replace the 3 qubit cloning state by a 2 qubit state when the statistics of Eve’s results is invariant when we condition them on their equality (inequality) with the values of the spin of the ancilla. According to this hypothesis, the ancilla can be dropped without losing information whenever $P_{m,n}(+z,+z) = P_{m,n}(-z,+z)$. When the matrix $a_{m,n}$ is purely real, this means that either $a_{0,0}a_{1,0} = a_{1,1}a_{0,1}$ or $a_{0,0}a_{1,0} = a_{1,1}a_{0,1}$, which corresponds to the conditions ii and iii. Obviously the generality of this hypothesis is envalidated by the sufficient condition $i$.

Note that the class of cloning machines that was considered in the section [14] corresponds to the condition $ii$, excepted that the reference basis is then the $X$ basis (or the $Y$ basis in virtue of the covariance). The condition $i$ must be rewritten after permutation of $a_{1,0}$ and $a_{0,1}$, which is equivalent to the condition $i$ in terms of the $Z$ basis, in accordance with the identities [14].

On the basis of the main theorem, it is easy to prove that the (optimal) universal cloner is not reducible to a 2 qubit cloner. Indeed, for such a cloner, $a_{0,1} = a_{1,0} = a_{1,1}$ and $a_{0,0} = a_{0,1}$ so that none of the conditions $ii$ $iii$ can be fulfilled.

**IV. CONCLUSIONS AND COMMENTS.**

In summary, we analyzed the possibility to reduce 3 qubit cloning to 2 qubit cloning. From the point of view of practical realisability, it is advantageous to economize one qubit in the cloning process. Therefore it is interesting to know precisely when this reduction is possible, and this was the goal of the main theorem presented in the section [14].

It is worth noting that there are two distinct cases of reducibility that enter into the scope of this theorem: when the condition $ii$ or $iii$ is fulfilled, it is equivalent for Eve to replace the 3 qubit cloning state by its 2 qubit version, because she does not lose any information by doing so, but this is not true when the condition $i$ is fulfilled.

Beside, the extent of validity of our main theorem is per se limited: it could happen that a 3 qubit cloning machine can be reduced to a mixture of more than two 2 qubit cloning machines, and it is not possible, on the basis of our
main theorem to determine whether or not this is the case. It is also limited to qubits, and can be considered as a first step towards a theory of economic quantum cloners.

Moreover, it could happen that 3 qubit cloning states are well approximated by mixtures of 2 qubit cloning states, and our main theorem is silent about this situation. It could also happen that mixtures of 2 qubit cloners provide a good approximation of a non-reducible cloner along certain directions of the Hilbert space only. For instance a mixture of 3 phase covariant cloners, each of which being covariant along a great circle orthogonal to one of the three directions $X, Y$, and $Z$, exhibits the same fidelity in the $X, Y$ and $Z$ bases. It provides a good candidate for cloning the cryptographic key exchanged between Alice and Bob during the 6 states protocol in which each state of encryption is chosen at random in one of these bases \[\text{[22]}.\]

It is not as performant as the symmetric universal qubit cloner of Buzek and Hillery, for which a fidelity of $5/6 \approx 0.8333\%$ is achieved \[\text{[11]},\] and constitutes a less dangerous attack than the optimal (asymmetric) isotropic cloner of threshold fidelity $\approx 0.8436$ described in \[\text{[21, 22, 23]}\]. Nevertheless, it provides in principle a fidelity equal to $2/3 (\frac{1}{\sqrt{2}}) + 1/3.3.3/4 \approx 0.8190\%$, and it can be realised, in virtue of the results of the second section during an attack that consists of a mixture of six 2 qubit cloning processes, it is thus reducible to a 2 qubit attack. Remark that the fidelity of this attack is not the same along all directions on the Bloch sphere but is maximal along the three canonical bases.

Finally, one could object that the main theorem concerns only cloning states that are pure and fulfill Eq. \[\text{[8]},\] with purely real amplitudes $a_{n,m}$ but it can be checked that all the interesting cloners considered in the literature are of this type. A partial elucidation of why it is so can be found in \[\text{[15]}\].

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The forementioned result according to which the information of Eve is maximized, in arbitrary dimension, when the phases of the amplitudes $a_{m,n}$ are real, concerns actually Eve’s information conditioned on the difference between results of measurements performed on the clone $E$ and results of measurements performed on the ancilla $M$).

One of us (T.D) was informed by N. Cerf (private communication), after the completion of this work, that he and N. Gisin already derived a similar result in the past but never published it.