QUASI-LOCAL MASS INTEGRALS AND THE TOTAL MASS

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ABSTRACT. On asymptotically flat and asymptotically hyperbolic manifolds, by evaluating the total mass via the Ricci tensor, we show that the limits of certain Brown-York type and Hawking type quasi-local mass integrals equal the total mass of the manifold in all dimensions.

1. Introduction

In this work, we discuss the relation between certain quasi-local mass integrals and the total mass of asymptotically flat and asymptotically hyperbolic manifolds.

First we introduce some notations and recall some definitions. Let $(M^n, g)$ be a Riemannian manifold of dimension $n \geq 3$. For $\lambda = 0$ or $-1$, let

\begin{equation}
G^g_{\lambda} = \text{Ric}(g) - \frac{1}{2} [S_g - \lambda(n-1)(n-2)] g,
\end{equation}

where $\text{Ric}(g)$ and $S_g$ are the Ricci tensor and the scalar curvature of $g$, respectively.

Throughout the paper, $b_n$ and $c_n$ will always denote the constants:

\begin{equation}
\begin{cases}
  b_n = \frac{1}{2(n-1)\omega_{n-1}}, \\
  c_n = \frac{1}{(n-1)(n-2)\omega_{n-1}},
\end{cases}
\end{equation}

where $\omega_{n-1}$ is the volume of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Definition 1.1. $(M^n, g)$ is an asymptotically flat (AF) manifold if, outside a compact set, $M^n$ is diffeomorphic to $\mathbb{R}^n \setminus \{|x| \leq r_0\}$ for some
$r_0 > 0$ and, with respect to the coordinates $x = (x^1, \ldots, x^n)$ on $\mathbb{R}^n$, the metric components $\{g_{ij}\}$ satisfy

\begin{equation}
    g_{ij} - \delta_{ij} = O(|x|^{-\tau}), \quad \partial g_{ij} = O(|x|^{-1-\tau}), \quad \partial^2 g_{ij} = O(|x|^{-2-\tau})
\end{equation}

for some $\tau > \frac{n-2}{2}$. Here $\partial$ denotes the partial differentiation on $\mathbb{R}^n$. Moreover, one assumes the scalar curvature $\mathcal{S}_g$ is in $L^1(\mathbb{R}^n)$.

On an asymptotically flat $(M^n, g)$, the total mass (or the ADM mass) $m(\mathbb{M})$ is defined as:

\begin{equation}
    m = b_n \lim_{r \to \infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu_e^i d\sigma_e,
\end{equation}

where $S_r = \{ |x| = r \}$, $\nu_e$ is the unit outward normal and $d\sigma_e$ is the area element on $S_r$ both with respect to the Euclidean metric $g_e$.

The definitions of an asymptotically hyperbolic manifold and its mass are more elaborate (cf. [6, 27, 28]). Here we use the most general form given in [6]. Let $\mathbb{H}^n$ denote the standard hyperbolic space. The metric $g_0$ on $\mathbb{H}^n$ can be written as

$$g_0 = d\rho^2 + (\sinh \rho)^2 h_0,$$

where $\rho$ is the $g_0$-distance to a fixed point $o$ and $h_0$ is the standard metric on $\mathbb{S}^{n-1}$. Let $\varepsilon_0 = \partial_\rho$, $\varepsilon_\alpha = \frac{1}{\sinh \rho} f_\alpha$, $\alpha = 1, \ldots, n-1$, where $\{f_\alpha\}_{1 \leq \alpha \leq n-1}$ is a local orthonormal frame on $\mathbb{S}^{n-1}$, then $\{\varepsilon_i\}_{0 \leq i \leq n-1}$ form a local orthonormal frame on $\mathbb{H}^n$. By abuse of notation, we let $\Sigma_\rho$ denote the geodesic sphere in $\mathbb{H}^n$ of radius $\rho$ centered at $o$.

**Definition 1.2.** $(M^n, g)$ is called asymptotically hyperbolic (AH) if, outside a compact set, $M$ is diffeomorphic to the exterior of some geodesic sphere $\Sigma_{\rho_0}$ in $\mathbb{H}^n$ such that the metric components $g_{ij} = g(\varepsilon_i, \varepsilon_j)$, $0 \leq i, j \leq n - 1$, satisfy

\begin{equation}
    |g_{ij} - \delta_{ij}| = O(e^{-\tau \rho}), \quad |\varepsilon_k(g_{ij})| = O(e^{-\tau \rho}), \quad |\varepsilon_k(\varepsilon_l(g_{ij}))| = (e^{-\tau \rho})
\end{equation}

for some $\tau > \frac{n}{4}$. Moreover, one assumes $\mathcal{S}_g + n(n-1)$ is in $L^1(\mathbb{H}^n, e^\rho dv_0)$, where $dv_0$ is the volume element of $g_0$ and $\mathcal{S}_g$ is the scalar curvature of $g$.

On an asymptotically hyperbolic $(M^n, g)$, by the results in [6], its mass $M(g)$ is a linear function on the kernel of $(D\mathcal{S})^*_g$, where $(D\mathcal{S})^*_g$ is the formal adjoint of the linearization of the scalar curvature at $g_0$. Let $\theta = (\theta^1, \ldots, \theta^n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, the functions

\begin{equation}
    V^{(0)} = \cosh \rho, \quad V^{(j)} = \theta^j \sinh \rho, \quad 1 \leq j \leq n
\end{equation}
then form a basis of the kernel of \((DS)^*_g\). \(M(g)\) is defined by

\[
M(V^{(i)}) = b_n \lim_{\rho \to \infty} \int_{\Sigma_\rho} \left[ V^{(i)}(\text{div}_0 h - \text{tr}_0 h) - h(\nabla_0 V^{(i)}, \cdot) + (\text{tr}_0 h) dV^{(i)} \right] \, (\tilde{\nu}) d\sigma_0,
\]

where \(h = g - g_0\), \(\tilde{\nu}\) is the \(g_0\)-unit outward normal to \(\Sigma_\rho\), \(d\sigma_0\) is the volume element on \(\Sigma_\rho\) of the metric induced from \(g_0\), and \(\text{div}, \text{tr}, \nabla_0\) denote the divergence, trace, the covariant derivative with respect to \(g_0\), respectively. We refer readers to [6] (also see [18]) for a well explained motivation behind this definition.

For our purpose in this work, we make use of the following formulae, available in the literature, which compute \(m\) and \(M(g)(V^{(i)})\) in terms of the tensor \(G^g_\lambda\) defined in (1.1).

**Theorem 1.1.** (i) Suppose \((M^n, g)\) is an AF manifold, then

\[
m = -c_n \lim_{r \to \infty} \int_{S_r} G^g_0(X, \nu_g) d\sigma_g.
\]

Here \(\nu_g\) is unit outward normal to \(S_r\) in \((M^n, g)\), \(d\sigma_g\) is the volume element of \(S_r\) with respect to the metric induced by \(g\), \(X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}\) is the conformal radial Killing vector field on \(\mathbb{R}^n\).

(ii) Suppose \((M^n, g)\) is an AH manifold. Then, for each \(0 \leq i \leq n\),

\[
M(g)(V^{(i)}) = -c_n \lim_{\rho \to \infty} \int_{\Sigma_\rho} G^g_{-1}(X^{(i)}, \nu_g) d\sigma_g.
\]

Here \(\{X^{(i)}\}\) are the conformal Killing vector fields on \(\mathbb{H}^n\), given by

\[
X^{(0)} = x^k \frac{\partial}{\partial x^k}, \quad X^{(j)} = \frac{\partial}{\partial x^j}, \quad 1 \leq j \leq n
\]

in the ball model \((B^n, g_0)\) of \(\mathbb{H}^n\) where \(B^n = \{x \in \mathbb{R}^n \mid |x| < 1\}\) and \((g_0)_{ij} = 4 \frac{1}{(1-|x|^2)^2} \delta_{ij}\), \(\{V^{(i)}\}\) are functions on \(\mathbb{H}^n\) defined by (1.6), which on \((B^n, g_0)\) take the form of

\[
V^{(0)} = \frac{1 + r^2}{1 - r^2}, \quad V^{(i)} = \frac{2x^i}{1 - r^2}, \quad 1 \leq i \leq n
\]

with \(r = |x|\), \(\Sigma_\rho\) is the geodesic sphere of radius \(\rho\) centered at \(x = 0\) in \((B^n, g_0)\), \(\nu_g\) is the outward unit normal to \(\Sigma_\rho\) with respect to \(g\), and \(d\sigma_g\) is the volume element on \(\Sigma_\rho\) of the metric induced by \(g\).
We emphasize that Theorem 1.1 (i) is widely known in the relativistic community (cf. [2, 5]) and a proof of it can be found in [12, 17, 10, 25]. Theorem 1.1 (ii) was recently proved by Herzlich [10].

In this paper, we give applications of the formulae in Theorem 1.1. Our main results are the following:

**Theorem 1.2.** Let \((M^n, g)\) be an asymptotically flat manifold. Let \(\{\Sigma_\rho\}\) be a family of nearly round hypersurfaces in \((M^n, g)\) (see Definition 2.1). Then

\[
\lim_{\rho \to \infty} c_n \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \int_{\Sigma_\rho} \left( S^\rho - \frac{n - 2}{n - 1} H^2 \right) d\sigma_\rho = m.
\]

If in addition \(\Sigma_\rho\) can be isometrically embedded in \(\mathbb{R}^n\) when \(\rho\) is sufficiently large, which is automatically satisfied if \(n = 3\), then

\[
\lim_{\rho \to \infty} 2b_n \int_{\Sigma_\rho} (H_0 - H) d\sigma_\rho = m
\]

and

\[
\lim_{\rho \to \infty} c_n \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \int_{\Sigma_\rho} \sum_{\alpha < \beta} (\kappa_\alpha^{(0)} \kappa_\beta^{(0)} - \kappa_\alpha \kappa_\beta) d\sigma_\rho = m.
\]

Here \(|\Sigma_\rho|\), \(S^\rho\), \(d\sigma_\rho\) are the volume, the scalar curvature, the volume element of \(\Sigma_\rho\), respectively; \(H\), \(\{\kappa_\alpha\}\) are the mean curvature, the principal curvatures of \(\Sigma_\rho\) in \((M^n, g)\), respectively; \(H_0\), \(\{\kappa_\alpha^{(0)}\}\) are the mean curvature, the principal curvatures of the isometric embedding of \(\Sigma_\rho\) in \(\mathbb{R}^n\), respectively; and \(m\) is the total mass of \((M^n, g)\).

**Theorem 1.3.** Let \((M^n, g)\) be an asymptotically hyperbolic manifold with suitable decay rate \(\tau\) (specified in Theorem 3.1). Let \(\{\Sigma_\rho\}_{\rho \geq \rho_0}\) be the hypersurfaces in \(M^n\) which are the geodesic spheres of radius \(\rho\) in \(\mathbb{H}^n\) as specified in Definition 1.2. Let \(\{V^{(i)}\}_{0 \leq i \leq n-1}\) be the functions given in (1.6). Then the following are true:

(I) (a) \(M(g)(V^{(0)})\)

\[
= \lim_{\rho \to \infty} \frac{c_n}{2} \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \int_{\Sigma_\rho} \left[ S^\rho - \frac{n - 2}{n - 1} H^2 + (n - 1)(n - 2) \right] d\sigma_\rho.
\]

(b) For \(1 \leq i \leq n\),

\(M(g)(V^{(i)})\)

\[
= \lim_{\rho \to \infty} \frac{c_n}{2} \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \int_{\Sigma_\rho} \frac{x^i}{|x|} \left[ S^\rho - \frac{n - 2}{n - 1} H^2 + (n - 1)(n - 2) \right] d\sigma_\rho,
\]
where \( \{x^i\} \) are the coordinate functions in the ball model \((B^n, g_0)\) of \( \mathbb{H}^n \).

(II) Suppose \( \Sigma_\rho \) can be isometrically embedded in \( \mathbb{H}^n \) for sufficiently large \( \rho \), which is automatically satisfied when \( n = 3 \).

(a) For each \( 0 \leq i \leq n \),

\[
\mathbf{M}(g)(V^{(i)}) = 2b_n \lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H) V^{(i)} \, d\sigma_\rho.
\]

(b) Consider the hyperboloid model of \( \mathbb{H}^n \), i.e.

\[\{(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} | (x^0)^2 - \sum_{i=1}^n (x^i)^2 = 1, x^0 > 0\}\],

where \( \mathbb{R}^{n+1} \) is the \((n + 1)\)-dimensional Minkowski space, there exist isometric embeddings of \( \{\Sigma_\rho\} \) in \( \mathbb{H}^n \) such that

\[
\mathbf{M}(g)(V) = 2b_n \lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H) \mathbf{x} \, d\sigma_\rho,
\]

where \( \mathbf{M}(g)(V) = (\mathbf{M}(g)(V^{(0)}), \ldots, \mathbf{M}(g)(V^{(n)})) \) and \( \mathbf{x} \) is the position vector of the embedding of \( \Sigma_\rho \) in \( \mathbb{H}^n \subset \mathbb{R}^{n+1} \).

Here \( |\Sigma_\rho| \), \( S_\rho \), \( d\sigma_\rho \) are the volume, the scalar curvature, the volume element of \( \Sigma_\rho \), respectively; \( H \), \( \{\kappa_\alpha\} \) are the mean curvature, the principal curvatures of \( \Sigma_\rho \) in \( (M^n, g) \), respectively; \( H_0 \), \( \{\kappa_\alpha^{(0)}\} \) are the mean curvature, the principal curvatures of the isometric embedding of \( \Sigma_\rho \) in \( \mathbb{H}^n \), respectively; and \( \mathbf{M}(g)(\cdot) \) is the mass function of \( (M^n, g) \).

We give some remarks concerning Theorems 1.2 and 1.3.

Remark 1.1. When \( n = 3 \), (1.8) and (1.9) in Theorem 1.2 become

(1.11) \[m = m_H(\Sigma_\rho) + o(1)\]

and

(1.12) \[m = m_{BY}(\Sigma_\rho) + o(1)\]

as \( \rho \to \infty \), respectively, where

\[
m_H(\Sigma_\rho) = \sqrt{|\Sigma_\rho| \over 16\pi} \left[ 1 - {1 \over 16\pi} \int_{\Sigma_\rho} H^2 \, d\sigma_\rho \right]
\]

is the Hawking quasi-local mass [9] of \( \Sigma_\rho \) in \( (M^3, g) \), and

\[
m_{BY}(\Sigma_\rho) = \frac{1}{8\pi} \int_{\Sigma_\rho} (H_0 - H) \, d\sigma_\rho
\]
is the Brown-York quasi-local mass \( (3, 4) \) of \( \Sigma_\rho \) in \( (M^n, g) \). In this case, (1.12) was first proved for coordinate spheres \( \{S_r\} \) in \( \mathbb{R}^n \). Later in [24], (1.11) and (1.12) were proved for nearly round 2-surfaces. (Examples of surfaces which are not nearly round, but along which \( \mathbf{m}_{BY} (\cdot) \) converges to \( \mathbf{m} \), were given in [7].) Both proofs in [8, 24] are via delicate pointwise estimates of \( H \), together with an application of the Minkowski integral formula (cf. [14]) to handle the integral of \( H_0 \). For \( n \geq 3 \), our proof of (1.8) and (1.9) in Theorem 1.2 is to use Theorem 1.1(i) with \( \{S_r\} \) replaced by \( \{\Sigma_\rho\} \). Indeed, it was proved in [17, Theorem 2.1] that Theorem 1.1(i) is still valid if \( \{S_r\} \) is replaced by a family of Lipschitz hypersurfaces \( \{\Sigma_l\} \), which are boundaries of domains in \( \mathbb{R}^n \) so that \( r_l := \inf_{x \in \Sigma_l} |x| \) tends to infinity with the volume \( |\Sigma_l| \) satisfying \( |\Sigma_l| \leq C r_l^{n-1} \) for some constant \( C \) independent of \( l \). In particular, the mass formula in Theorem 1.1(i) is applicable to nearly round hypersurfaces defined in Definition 2.1.

**Remark 1.2.** When \( n \geq 4 \), though (1.9) and (1.10) in Theorem 1.2 are proved under the assumption that \( \Sigma_\rho \) can be isometrically embedded in \( \mathbb{R}^n \) for large \( \rho \), it is directly applicable in certain special cases. For instance, in [22] an asymptotically flat metric \( g = u^2 dr^2 + g_r \) with zero scalar curvature was constructed on the exterior \( E \) of a convex hypersurface \( \Sigma_0 \) in \( \mathbb{R}^n \), where \( r \) is the Euclidean distance to \( \Sigma_0 \) and \( g_r \) represents the induced metric on \( \Sigma_r \) that is the level set of \( r \). In this case, one easily checks that \( \{\Sigma_r\} \) is a family of nearly round hypersurfaces in \( (E, g) \), which are automatically isometrically embedded in \( \mathbb{R}^n \). Hence, (1.9) in Theorem 1.2 implies Theorem 2.1(c) in [22].

**Remark 1.3.** Since \( H_0 = \sum_\alpha \kappa^0_\alpha \), \( H = \sum_\alpha \kappa_\alpha \), and
\[
\sum_{\alpha} \kappa^0_\alpha \kappa^0_\beta, \quad \sum_{\alpha < \beta} \kappa_\alpha \kappa_\beta
\]
are often known as the second order mean curvature of \( \Sigma_\rho \) in \( \mathbb{R}^n \), \( (M^n, g) \) respectively, (1.10) in Theorem 1.2 may be viewed as an analogue of (1.9) in terms of the second order mean curvature.

**Remark 1.4.** When \( n = 3 \), part (I) (a) of Theorem 1.3 becomes
\[
\mathcal{M}(g)(V^{(0)}) = \tilde{\mathbf{m}}_H (\Sigma_\rho) + o(1), \text{ as } \rho \to \infty,
\]
where
\[
\tilde{\mathbf{m}}_H (\Sigma_\rho) = \sqrt{\frac{|\Sigma_\rho|}{16\pi}} \left[ 1 - \frac{1}{16\pi} \int_{\Sigma_\rho} H^2 d\sigma_\rho + \frac{|\Sigma_\rho|}{4\pi} \right]
\]
is known as a hyperbolic analogue of the Hawking mass (cf. [27, 19]). However, part (I) of Theorem 1.3 shows that the limit of \( \tilde{\mathbf{m}}_H (\Sigma_\rho) \) is
only part of the mass function \( M(g)(V) \). Therefore, one may expect a complete definition of the hyperbolic Hawking mass of \( \Sigma_\rho \) to involve all integrals under the limit sign in part (I) (a) and (b) of Theorem 1.3.

Remark 1.5. When \( n = 3 \), the integral

\[
\frac{1}{8\pi} \int_{\Sigma_\rho} (H_0 - H) \, d\sigma_{\rho}
\]

in part (II) (b) of Theorem 1.3 is analogous to the quasi-local mass integral for closed surfaces in 3-manifolds with \( S_g \geq -6 \) considered in [26, 23]. In the context of conformally compact asymptotically hyperbolic 3-manifolds, part (II) (b) of Theorem 1.3 was proved in [15]. For any \( n \geq 3 \), we prove Theorem 1.3 by applying Theorem 1.1(ii).

The paper is organized as follows. In section 2, we prove Theorem 1.2. In section 3, we prove Theorem 1.3.

2. Limits of quasi-local mass integrals in AF manifolds

In an asymptotically flat \((M^n, g)\), given a closed hypersurface \( \Sigma \) that is homologous to a large coordinate sphere at infinity, we let \( H \) and \( A \) be the mean curvature and the second fundamental form of \( \Sigma \) with respect to the infinity pointing unit normal \( \nu \) respectively, and let \( \hat{A} \) be the traceless second fundamental form of \( \Sigma \), i.e. \( \hat{A} = A - \frac{H}{n-1} \sigma \), where \( \sigma \) is the induced metric on \( \Sigma \). We also let \( d\sigma \), \( \nabla \), \( |\Sigma| \) and \( \text{Diam}(\Sigma) \) denote the volume form on \( \Sigma \), the covariant differentiation on \( \Sigma \), the volume and the intrinsic diameter of \( \Sigma \), respectively.

The concept of a 1-parameter family of nearly round surfaces at infinity of an asymptotically flat 3-manifold was used in [24]. Below we give its analogue in general dimensions.

Definition 2.1. Let \( \{ \Sigma_\rho \} \) be a family of closed, connected, embedded hypersurfaces, homeomorphic to the \((n - 1)\)-dimensional sphere, in \((M^n, g)\) with \( \rho \in (\rho_0, \infty) \) for some \( \rho_0 > 0 \). \( \{ \Sigma_\rho \} \) is called nearly round if there exists a constant \( C > 0 \) such that, for all \( \rho \), the following are satisfied:

(i) \( C^{-1} \rho \leq |x| \leq C \rho \) for all \( x \in \Sigma_\rho \);
(ii) \( |\hat{A}| + \rho |\nabla \hat{A}| \leq C \rho^{-1-\tau} \);
(iii) \( |\Sigma_\rho| \leq C \rho^{n-1} \);
(iv) \( \text{Diam}(\Sigma_\rho) \leq C \rho \).

Next, we give a series of lemmas for a family of nearly round hypersurfaces \( \{ \Sigma_\rho \} \) in an asymptotically flat \((M^n, g)\).
Lemma 2.1. There is a constant $C > 0$ independent on $\rho$ such that
\[ |\nabla A| \leq C \rho^{-2-\tau}, \quad |\nabla H| \leq C \rho^{-2-\tau}. \]

Proof. By [13, Lemma 2.2], given any constant $a > 0$,
\[ |\nabla A|^2 \geq \left( \frac{3}{n+1} - a \right) |\nabla H|^2 \]
(2.1)
\[ - \frac{2}{n+1} \left( \frac{2}{n+1} \alpha^{-1} - \frac{n-1}{n-2} \right) |w|^2, \]
where $w$ is the projection of $\text{Ric}(\nu, \cdot)$ on $\Sigma_\rho$ and $\text{Ric}(\cdot, \cdot)$ is the Ricci curvature of $(M^n, g)$. On the other hand, given any constant $\epsilon > 0$,
\[ \frac{1}{n-1} |\nabla H|^2 = |\nabla (A - \hat{A})|^2 \geq (1 - \epsilon) |\nabla A|^2 - C(\epsilon) |\nabla \hat{A}|^2 \]
for some constant $C(\epsilon) > 0$ depending only on $\epsilon$. Since $n \geq 3$, we may choose $a$ and $\epsilon$ such that
\[ \left( \frac{3}{n+1} - a \right) (n-1)(1-\epsilon) > 1. \]
Thus, it follows from (2.1) and (2.2) that
\[ |\nabla A| \leq C \rho^{-2-\tau}, \]
where we also used (1.3) and (i), (ii) in Definition 2.1. The fact $|\nabla A| \leq C \rho^{-2-\tau}$ now follows from (2.1) and (2.3). \[\square\]

In what follows, we let $\hat{g}$ be the background Euclidean metric on the open set of $M^n$ that is diffeomorphic to $\mathbb{R}^n \setminus \{|x| \leq r_0\}$. A symbol with “$\hat{\cdot}$” means the corresponding quantity is computed with respect to $\hat{g}$.

Lemma 2.2. As $\rho \to \infty$,
\[ d\sigma = (1 + O(\rho^{-\tau})) d\hat{\sigma}, \quad \nu - \hat{\nu} = O(\rho^{-\tau}), \]
\[ |A - \hat{A}| = O(\rho^{-\tau}) |\hat{A}| + O(\rho^{-1-\tau}). \]

Proof. At any $p \in \Sigma_\rho$, let $\{e_1, \ldots, e_{n-1}\}$ be a $\hat{g}$-orthonormal frame in $T_p \Sigma_\rho$. Let $e_n = \hat{\nu}$, then $\{e_i \mid i = 1, \ldots, n\}$ form a $\hat{g}$-orthonormal frame in $T_p M^n$. Hence, if $e_i = a^j_i \partial_{x_j}$, then $(a^j_i)_{n \times n}$ is an orthogonal matrix. Thus,
\[ g(e_i, e_j) = a^k_i a^l_j g(\partial_{x_k}, \partial_{x_l}) = \delta_{ij} + O(\rho^{-\tau}), \]
which implies $d\sigma = (1 + O(\rho^{-\tau})) d\hat{\sigma}$. Now assume $\nu = X^i e_i$. By (2.4) and the fact $g(\nu, \nu) = 1$,
\[ 1 = X^i X^j \left[ \delta_{ij} + O(\rho^{-\tau}) \right], \]
which implies $X^i = O(1)$ and hence
\[ \sum_i (X^i)^2 = 1 + O(\rho^{-\tau}). \]
Similarly, for $1 \leq \alpha \leq n - 1$, the fact $g(\nu, e_\alpha) = 0$ and (2.4) imply
\[ 0 = X^\alpha + O(\rho^{-\tau}). \]
Therefore, by (2.6) and (2.7),
\[ \nu - \hat{\nu} = X^\alpha e_\alpha + (X^n - 1)e_n = O(\rho^{-\tau}). \]

To compare $A$ and $\hat{A}$ at $p$, we let $W_\rho$ be a small open set in $\Sigma_\rho$ containing $p$. Let $U_\rho$ be a $\hat{g}$-Gaussian tubular neighborhood of $W_\rho$ in $M^n$ and let \{\(u_1, \ldots, u_{n-1}, t\}\} be local coordinates on $U_\rho$ such that, at $t = 0$, \{\(u_\alpha | \alpha = 1, \ldots, n - 1\)\} are local coordinates on $W_\rho$ satisfying \(\hat{\nabla}_{\partial u_\alpha} \partial u_\beta = 0\) at $p$. In $U_\rho$, suppose $\partial u_\alpha = X^i_\alpha \partial x_i$. Then
\[ \nabla_{\partial u_\alpha} \partial u_\beta = X^i_\alpha \left( X^j_\beta \nabla_{\partial x_i} \partial x_j + \frac{\partial X^j_\beta}{\partial x_i} \partial x_j \right), \]
\[ \hat{\nabla}_{\partial u_\alpha} \partial u_\beta = X^i_\alpha \left( X^j_\beta \hat{\nabla}_{\partial x_i} \partial x_j + \frac{\partial X^j_\beta}{\partial x_i} \partial x_j \right), \]
which shows, at $p$,
\[ \nabla_{\partial u_\alpha} \partial u_\beta = \hat{\nabla}_{\partial u_\alpha} \partial u_\beta + O(\rho^{-1-\tau}). \]
Therefore,
\[ g(\nabla_{\partial u_\alpha} \partial u_\beta, \nu) = g(\hat{\nabla}_{\partial u_\alpha} \partial u_\beta, \nu) + O(\rho^{-1-\tau}) \]
\[ = g(\hat{\nabla}_{\partial u_\alpha} \partial u_\beta, \hat{\nu}) + O(\rho^{-1-\tau})|\hat{A}| + O(\rho^{-1-\tau}) \]
\[ = \hat{g}(\hat{\nabla}_{\partial u_\alpha} \partial u_\beta, \hat{\nu}) + O(\rho^{-1-\tau})|\hat{A}| + O(\rho^{-1-\tau}). \]
From this we conclude
\[ |A - \hat{A}| = O(\rho^{-\tau})|\hat{A}| + O(\rho^{-1-\tau}). \]

\[ \square \]

**Lemma 2.3.** For each large $\rho$, there exists $r(\rho) > 0$ with $r(\rho) \sim \rho$, meaning $C^{-1} \rho < r(\rho) < C\rho$ for some constant $C$ independent on $\rho$, such that
\[ H = \frac{n - 1}{r(\rho)} + O(\rho^{-1-\tau}). \]
Consequently, $|A| = O(\rho^{-1})$, $|\hat{A}| = O(\rho^{-1})$ and
\[ \kappa_\alpha = \frac{1}{r(\rho)} + O(\rho^{-1-\tau}), \quad \hat{\kappa}_\alpha = \frac{1}{r(\rho)} + O(\rho^{-1-\tau}), \]
for all $\alpha = 1, \ldots, n - 1$, where $\kappa_\alpha$ and $\hat{\kappa}_\alpha$ denote the principal curvature of $\Sigma_\rho$ with respect to $g$ and $\hat{g}$ respectively.
Proof. As $\Sigma_\rho$ is homeomorphic to a sphere, for each $\rho$ large, there is an $r_1 = r_1(\rho) > 0$, such that $S_{r_1} = \{|x| = r_1\}$ is the largest coordinate sphere inside $\Sigma_\rho$. Let $x_1$ be a point where $S_{r_1}$ touches $\Sigma_\rho$. It follows from the maximum principle that

\begin{equation}
H(x_1) \leq \bar{H}(x_1)
\end{equation}

where $\bar{H}$ is the mean curvature of $S_{r_1}$ in $(M^n, g)$ with respect to the outward normal. Direct calculation gives

\begin{equation}
\bar{H}(x_1) = \frac{n-1}{r_1} + O(|r_1|^{-1-\tau})
\end{equation}

(cf. Lemma 2.1 in [8] when $n = 3$). By Lemma 2.1 and (iv) in Definition 2.1, $H$ satisfies

\begin{equation}
|H - H(x_1)| \leq C\rho^{-1-\tau}.
\end{equation}

Hence, it follows from (2.10) – (2.12) that

\begin{equation}
H \leq \frac{n-1}{r_1} + O(\rho^{-1-\tau}).
\end{equation}

Similarly, if $r_2 > 0$ is the radius of the smallest coordinate sphere that lies outside of $\Sigma_\rho$, then

\begin{equation}
H \geq \frac{n-1}{r_2} + O(\rho^{-1-\tau}).
\end{equation}

For any fixed point $x_* \in \Sigma_\rho$, these imply

\begin{equation}
\frac{n-1}{r_2} + O(\rho^{-1-\tau}) \leq H(x_*) \leq \frac{n-1}{r_1} + O(\rho^{-1-\tau}).
\end{equation}

By the same reasoning leading to (2.12),

\begin{equation}
|H - H(x_*)| \leq C\rho^{-1-\tau}.
\end{equation}

Hence, choosing $r(\rho) = \frac{n-1}{\bar{H}(x_*)}$, we have

\begin{equation}
H = \frac{n-1}{r(\rho)} + O(\rho^{-1-\tau}),
\end{equation}

where $r(\rho) \sim \rho$ by (2.13). As a result, $A$ satisfies

\begin{equation}
A = \hat{A} + \left[\frac{1}{r(\rho)} + O(\rho^{-1-\tau})\right] \sigma,
\end{equation}

which implies $|A| = O(\rho^{-1})$ and

\begin{equation}
\kappa_\alpha = \frac{1}{r(\rho)} + O(\rho^{-1-\tau}), \ \forall \ \alpha
\end{equation}

by (ii) in Definition 2.1. The claim on $|\hat{A}|$ and $\kappa_\alpha$ now follows from Lemma 2.2.\qed
In the rest of this section, \( r(\rho) \) will always denote the function of \( \rho \) given in Lemma 2.3.

**Lemma 2.4.** Let \( K_1(\rho) \) and \( K_2(\rho) \) be the minimum and maximum of the sectional curvature of \( \Sigma_\rho \) respectively. Then

\[
K_1(\rho) = \frac{1}{r^2(\rho)} + O(\rho^{-2-\tau}), \quad K_2(\rho) = \frac{1}{r^2(\rho)} + O(\rho^{-2-\tau}).
\]

**Proof.** This follows from the Gauss equation, Lemma 2.3 and the fact that the sectional curvature of \((M^n, g)\) at \( x \in \Sigma_\rho \) decays like \( O(\rho^{-2-\tau}) \).

**Lemma 2.5.** Let \( X = (x^1, \ldots, x^n) \) be the position vector given by the coordinates \( \{x^i\} \) near infinity. For each large \( \rho \), there exists a vector \( a(\rho) \in \mathbb{R}^n \) such that \( |a(\rho)| = O(\rho) \) and, on \( \Sigma_{\rho} \),

\[
|X - a(\rho) - r(\rho)\hat{\nu}| = O(\rho^{1-\tau}).
\]

**Proof.** At any \( p \in \Sigma_{\rho} \), let \( \{e_\alpha\} \subset T_p \Sigma_{\rho} \) be a \( \hat{g} \)-orthonormal frame that diagonalizes \( \hat{A} \), i.e. \( A(e_\alpha, e_\beta) = \hat{\kappa}_\alpha \delta_{\alpha\beta} \), then

\[
|\nabla_{e_\alpha}(X - r(\rho)\hat{\nu})| = |e_\alpha - r(\rho)\hat{\kappa}_\alpha e_\alpha| = O(\rho^{-\tau})
\]

by Lemma 2.3. This and (iv) in Definition 2.1 imply

\[
|(X(p) - r(\rho)\hat{\nu}(p)) - (X(p_0) - r(\rho)\hat{\nu}(p_0))| = O(\rho^{1-\tau})
\]

for all \( p \in \Sigma_{\rho} \) and a fixed \( p_0 \in \Sigma_{\rho} \). Let \( a(\rho) = X(p_0) - r(\rho)\hat{\nu}(p_0) \), the lemma follows.

**Lemma 2.6.** For large \( \rho \), the volume \( |\Sigma_{\rho}| \) satisfies

\[
\left( \frac{|\Sigma_{\rho}|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} = r(\rho)(1 + O(\rho^{-\tau})).
\]

**Proof.** Let \( |\Sigma|_{\hat{g}} \) be the volume of \( \Sigma_{\rho} \) with respect to the metric induced from \( \hat{g} \). Since \( d\sigma = (1 + O(\rho^{-\tau}))d\hat{\sigma} \) by Lemma 2.2, we have

\[
(2.14) \quad \frac{|\Sigma_{\rho}|}{|\Sigma_{\rho}|_{\hat{g}}} = 1 + O(\rho^{-\tau}).
\]

Now let \( a(\rho) \) be the vector given in Lemma 2.5, then

\[
|X - a(\rho)| = r(\rho)\left(1 + O(\rho^{-\tau})\right).
\]

Hence, \( \Sigma_{\rho} \) contains a Euclidean sphere \( S_1 \), centered at \( a(\rho) \), of radius \( r_1 \), and is contained in another Euclidean sphere \( S_2 \), centered at \( a(\rho) \), of radius \( r_2 \), such that both \( r_1 \) and \( r_2 \) satisfy

\[
(2.15) \quad r_1 = r(\rho)\left(1 + O(\rho^{-\tau})\right), \quad r_2 = r(\rho)\left(1 + O(\rho^{-\tau})\right).
\]
Let $\eta$ be the Euclidean distance function from $a(\rho)$. Integrating $\Delta_{\hat{g}} \eta$ over the domain bounded between $\Sigma_{\rho}$ and $S_1$, where $\Delta_{\hat{g}}$ denotes the Euclidean Laplacian, we conclude
\[
|\Sigma_{\rho}|_{\hat{g}} \geq |S_1|_{\hat{g}} = \omega_{n-1} r_1^{n-1}.
\]
Hence,
\[
(2.16) \quad \left(\frac{|\Sigma_{\rho}|_{\hat{g}}}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \geq r_1.
\]
Next, let $\zeta$ be the Euclidean distance function from $\Sigma_{\rho}$, defined on the exterior of $\Sigma_{\rho}$. For large $\rho$, $\Sigma_{\rho}$ is strictly convex in $\mathbb{R}^n$ by Lemma 2.4. Therefore, $\zeta$ is smooth and $\Delta_{\hat{g}} \zeta$ equals the mean curvature of the level set of $\zeta$ in $\mathbb{R}^n$ and hence is positive. Integrating $\Delta_{\hat{g}} \zeta$ over the domain bounded by $S_{r_2}$ and $\Sigma_{\rho}$, arguing as before, we conclude
\[
(2.17) \quad \left(\frac{|\Sigma_{\rho}|_{\hat{g}}}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \leq r_2.
\]
Hence, by $(2.16) - (2.17)$,
\[
(2.18) \quad \left(\frac{|\Sigma_{\rho}|_{\hat{g}}}{\omega_{n-1}}\right)^{\frac{1}{n-1}} = r(\rho) \left(1 + O(\rho^{-\tau})\right).
\]
The lemma now follows from $(2.14)$ and $(2.18)$.

When $n = 3$, Lemma 2.4 implies, for large $\rho$, $\Sigma_{\rho}$ can be isometrically embedded in $\mathbb{R}^3$ as a convex surface (cf. [20]). In [24], Shi, Wang and Wu proved that the principal curvatures $\kappa_1^{(0)}, \kappa_2^{(0)}$ of such an embedding satisfy
\[
(2.19) \quad \kappa_1^{(0)} = \frac{1}{r(\rho)} + O(\rho^{-1-\tau}) \quad \text{and} \quad \kappa_2^{(0)} = \frac{1}{r(\rho)} + O(\rho^{-1-\tau})
\]
(see [24, Theorem 4] and its proof). In the next lemma, assuming the isometric embedding exists, we show that such estimates hold in higher dimensions.

**Lemma 2.7.** Suppose $n \geq 4$ and assume $\Sigma_{\rho}$ can be isometrically embedded in $\mathbb{R}^n$ when $\rho$ is sufficiently large. Let $\Sigma_{\rho}^{(0)}$ be the image of the embedding. Let $\kappa_{\alpha}^{(0)}, \alpha = 1, \ldots, n-1$, be the principal curvatures of $\Sigma_{\rho}^{(0)}$ in $\mathbb{R}^n$. Then
\[
\kappa_{\alpha}^{(0)} = \frac{1}{r(\rho)} + O(\rho^{-1-\tau}).
\]
Proof. Let $H_0$ be the mean curvature of $\Sigma_0$ in $\mathbb{R}^n$ with respect to the outward normal. At a point $p_0 \in \Sigma_0$, let $\{e_\alpha\}$ be an orthonormal basis in $T_{p_0} \Sigma_0$ such that $e_\alpha$ points to the principal direction of $\Sigma_0$. Let $R^\rho_{\alpha\beta\gamma\delta}$, $R^\rho_{\alpha\beta}$ and $S^\rho$ be the intrinsic curvature tensor, the Ricci tensor and the scalar curvature of $\Sigma_0$, respectively. For convenience, let $r = r(\rho)$. By Lemma 2.4,

\begin{equation}
R^\rho_{\alpha\alpha} = \frac{n - 2}{r^2} + O(\rho^{-2-\tau}), \quad S^\rho = \frac{(n - 1)(n - 2)}{r^2} + O(\rho^{-2-\tau}).
\end{equation}

Moreover, by the Gauss equation,

\begin{equation}
\kappa_\alpha \kappa_\beta = R^\rho_{\alpha\beta\alpha\beta} = \frac{1}{r^2} + O(\rho^{-2-\tau}), \quad \forall \alpha \neq \beta.
\end{equation}

In particular, for large $\rho$, this implies $\Sigma_0$ is convex, i.e. $\kappa_\alpha > 0$, $\forall \alpha$. As a result, $H_0 > 0$.

Summing over indices in (2.21) repeatedly, we have

\begin{equation}
R^\rho_{\alpha\alpha} = H_0 \kappa_\alpha - (\kappa_\alpha)^2
\end{equation}

and

\begin{equation}
S^\rho = H_0^2 - |\Pi_0|^2
\end{equation}

where $\Pi_0$ is the second fundamental form of $\Sigma_0$. By (2.23) and (2.20),

\begin{equation}
H_0^2 \geq \frac{1}{n - 1} H_0^2 + \frac{(n - 1)(n - 2)}{r^2} + O(\rho^{-2-\tau}),
\end{equation}

which implies

\begin{equation}
H_0 \geq \frac{n - 1}{r} + O(\rho^{-1-\tau}).
\end{equation}

because $H_0 > 0$. On the other hand, by (2.22),

\begin{equation}
\kappa_\alpha = \frac{1}{2} \left[ H_0 \pm (H_0^2 - 4R^\rho_{\alpha\alpha})^{\frac{1}{2}} \right].
\end{equation}

We claim that,

\begin{equation}
\kappa_\alpha = \frac{1}{2} \left[ H_0 - (H_0^2 - 4R^\rho_{\alpha\alpha})^{\frac{1}{2}} \right], \quad \forall \alpha.
\end{equation}

Suppose not, without losing generality, we may assume

\begin{equation}
\kappa_1 = \frac{1}{2} \left[ H_0 + (H_0^2 - 4R^\rho_{11})^{\frac{1}{2}} \right].
\end{equation}

Then $\kappa_1 \geq \frac{1}{2} H_0$. This implies, $\forall \alpha > 1$,

\begin{equation}
\kappa_\alpha = \frac{1}{2} \left[ 4R^\rho_{\alpha\alpha} / (H_0^2 - 4R^\rho_{\alpha\alpha})^{\frac{1}{2}} \right],
\end{equation}
for otherwise we would have \( \kappa^{(0)}_1 + \sum_{\alpha > 1} \kappa^{(0)}_{\beta} > H_0 \) since \( n \geq 4 \) and \( \kappa^{(0)}_\alpha > 0 \), which is a contradiction. By (2.20) and (2.24),

\[
H_0^2 - 4R_{\alpha \alpha}^\rho \geq \frac{(n-1)^2}{r^2} - \frac{4(n-2)}{r^2} + O(\rho^{-2-\tau})
\]

(2.27)

\[
= \frac{(n-3)^2}{r^2} + O(\rho^{-2-\tau}).
\]

Hence, it follows from (2.24), (2.26) and (2.27) that, \( \forall \alpha > 1 \),

\[
\kappa^{(0)}_\alpha \leq \frac{2(n-2)}{r^2} + O(\rho^{-2-\tau})
\]

(2.28)

\[
= \frac{1}{r} + O(\rho^{-1-\tau}).
\]

On the other hand, for any \( \beta, \gamma \) such that \( \beta > 1, \gamma > 1 \) and \( \beta \neq \gamma \),

\[
\kappa^{(0)}_\beta \kappa^{(0)}_\gamma = \frac{1}{r^2} + O(\rho^{-2-\tau})
\]

by (2.21). (As \( n \geq 4 \), such indices \( \beta \) and \( \gamma \) exist.) Hence, (2.28) and (2.29) show

\[
\frac{1}{r^2} + O(\rho^{-2-\tau}) \leq \kappa^{(0)}_\beta \left( \frac{1}{r} + O(\rho^{-1-\tau}) \right),
\]

which gives

\[
\kappa^{(0)}_\beta \geq \frac{1}{r} + O(\rho^{-1-\tau}).
\]

(2.30)

Therefore, by (2.28) and (2.30),

\[
\kappa^{(0)}_\alpha = \frac{1}{r} + O(\rho^{-1-\tau}),
\]

(2.31)

for all \( \alpha > 1 \). But then

\[
\frac{1}{r^2} + O(\rho^{-2-\tau}) = \kappa^{(0)}_1 \kappa^{(0)}_2 \geq \frac{1}{2} H_0 \kappa^{(0)}_2 \geq \frac{n-1}{2r^2} + O(\rho^{-2-\tau}),
\]

which is impossible since \( n \geq 4 \). Therefore, the claim (2.25) holds. Now (2.26) is valid for all \( \alpha \geq 1 \). Repeating the argument leading to (2.31), we conclude (2.31) holds for all \( \alpha \geq 1 \). This completes the proof.

We now recall the statement of Theorem 1.2 and give its proof.
Theorem 2.1. Let \( \{ \Sigma_{\rho} \} \) be a family of nearly round hypersurfaces in an asymptotically flat manifold \((M^n, g)\) of dimension \(n \geq 3\). Then

\[
(2.32) \quad \lim_{\rho \to \infty} \frac{c_n}{2} \left( \frac{|\Sigma_{\rho}|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \int_{\Sigma_{\rho}} \left( S^\rho - \frac{n-2}{n-1} H^2 \right) d\sigma = m.
\]

If in addition \( \Sigma_{\rho} \) can be isometrically embedded in \( \mathbb{R}^n \) when \( \rho \) is sufficiently large, which is automatically satisfied if \( n = 3 \), then

\[
(2.33) \quad \lim_{\rho \to \infty} 2 b_n \int_{\Sigma_{\rho}} (H_0 - H) d\sigma = m
\]

and

\[
(2.34) \quad \lim_{\rho \to \infty} c_n \left( \frac{|\Sigma_{\rho}|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \int_{\Sigma_{\rho}} \sum_{\alpha < \beta} \left( \kappa^{(0)}_\alpha \kappa^{(0)}_\beta - \kappa_\alpha \kappa_\beta \right) d\sigma = m.
\]

Here \(|\Sigma_{\rho}|, S^\rho\) are the volume, the scalar curvature of \( \Sigma_{\rho} \), respectively; \( H, \{ \kappa_* \} \) are the mean curvature, the principal curvatures of \( \Sigma_{\rho} \) in \((M^n, g)\), respectively; \( H_0, \{ \kappa^{(0)}_* \} \) are the mean curvature, the principal curvatures of the isometric embedding of \( \Sigma_{\rho} \) in \( \mathbb{R}^n \), respectively; and \( m \) is the total mass of \((M^n, g)\).

Proof. For simplicity, denote \( G_0^g \) by \( G \) and denote \( r(\rho) \) by \( r \). On \( \Sigma_{\rho} \), by Lemma 2.5 and Lemma 2.2, we have

\[
G(X, \nu) = G(r^\nu, \nu) + G(a(\rho), \nu) + O(\rho^{-1-2\tau})
\]

\[
= rG(\nu, \nu) + G(a(\rho), \nu) + O(\rho^{-1-2\tau}).
\]

We first estimate \( \int_{\Sigma_{\rho}} G(a(\rho), \nu)d\sigma \). To do so, we note the following asymptotic formulae of \( \text{Ric}(g) \) and \( S_g \) (see (2.2) and (2.6) in [17]):

\[
(2.36) \quad 2\text{Ric}(g)_{ij} = g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij} + O(\rho^{-2-2\tau}),
\]

\[
(2.37) \quad S_g = g_{ik,ik} - g_{kk,ii} + O(\rho^{-2-2\tau}).
\]

Here and below, summation is performed over any pair of repeated indices.

Write \( a(\rho) = (a^1, \ldots, a^n) \), which is a constant vector when \( \rho \) is fixed. By Lemma 2.2, Lemma 2.5, (2.36) and (2.37), we have

\[
2G(a(\rho), \nu)d\sigma = \left[ (g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) a^i \hat{\nu}^j 
- (g_{ik,ik} - g_{kk,ii}) a^m \hat{\nu}^m + O(\rho^{-1-2\tau}) \right] d\hat{\sigma}.
\]

In what follows, we arbitrarily extend \( g_{ij} \) as a smooth function on the whole Euclidean space \( \mathbb{R}^n \) such that \( g_{ij} \) remains unchanged on
\( \mathbb{R}^n \setminus \{|x| \leq 2r_0\} \) (see Definition 1.1). We identify \( \Sigma_\rho \) with its image under the diffeomorphism that defines the coordinates \( \{x_i\} \). Let \( D_\rho \subset \mathbb{R}^n \) be the bounded region enclosed by \( \Sigma_\rho \) and let \( d\hat{V} \) be the Euclidean volume form on \( \mathbb{R}^n \). Then

\[
\int_{\Sigma_\rho} \left[ (g_{ki,kj} + g_{kj,ki} - g_{kk,ij}) a^i \hat{\nu}^j \right] d\hat{\sigma} = \int_{D_\rho} (g_{ki,kjj} + g_{kj,kij} - g_{kk,ijj}) a^i d\hat{V} = \int_{\Sigma_\rho} (g_{kj,kij} - g_{kk,ijj}) a^i d\hat{\sigma}.
\]

(2.39)

It follows from (2.38), (2.39) and (iii) in Definition 2.1 that

\[
2 \int_{\Sigma_\rho} G(a(\rho), \nu) d\sigma = o(1).
\]

(2.40)

By (2.35) and (2.40), we conclude

\[
\int_{\Sigma_\rho} G(X, \nu) d\sigma = r \int_{\Sigma_\rho} G(\nu, \nu) d\sigma + o(1).
\]

(2.41)

The Gauss equation implies

\[
G(\nu, \nu) = \frac{1}{2} \left( H^2 - |A|^2 - S^\rho \right) = \sum_{\alpha<\beta} (\kappa_{\alpha} \kappa_{\beta} - \kappa_{\alpha}^{(0)} \kappa_{\beta}^{(0)}).
\]

(2.42)

By Lemma 2.3 and Lemma 2.7 when \( n \geq 4 \),

\[
\kappa_{\alpha} \kappa_{\beta} - \kappa_{\alpha}^{(0)} \kappa_{\beta}^{(0)}
= (\kappa_{\alpha} - \kappa_{\alpha}^{(0)}) \kappa_{\beta} + (\kappa_{\beta} - \kappa_{\beta}^{(0)}) \kappa_{\alpha}^{(0)}
= (\kappa_{\alpha} - \kappa_{\alpha}^{(0)}) \left( \kappa_{\beta} - \frac{1}{r} \right) + (\kappa_{\beta} - \kappa_{\beta}^{(0)}) \left( \kappa_{\alpha}^{(0)} - \frac{1}{r} \right)
+ \frac{1}{r} (\kappa_{\alpha} + \kappa_{\beta} - \kappa_{\alpha}^{(0)} - \kappa_{\beta}^{(0)})
= \frac{1}{r} \left( \kappa_{\alpha} + \kappa_{\beta} - \kappa_{\alpha}^{(0)} - \kappa_{\beta}^{(0)} \right) + O(\rho^{-2-2\tau}).
\]

(2.43)
When \( n = 3 \), (2.43) is also true by Lemma 2.3 and the estimate (2.19). Therefore, by (2.42), (2.43) and the fact \( |A| = O(\rho^{-1-\tau}) \), we have
\[
(2.44) \quad r \int_{\Sigma_\rho} G(\nu, \nu) d\sigma = (n - 2) \int_{\Sigma_\rho} (H - H_0) d\sigma + o(1)
\]
and
\[
(2.45) \quad r \int_{\Sigma_\rho} G(\nu, \nu) d\sigma = r \int_{\Sigma_\rho} \frac{1}{2} \left( \frac{n - 2}{n - 1} \right) d\sigma + o(1).
\]

The theorem now follows readily from (2.41), (2.42), (2.44), (2.45), Lemma 2.6 and Theorem 1.1(i) with \( \{S_r\} \) replaced by \( \{\Sigma_\rho\} \) (see [17, Theorem 2.1]).

3. LIMITS OF QUASI-LOCAL MASS INTEGRALS IN AH MANIFOLDS

Given a compact manifold \( (\Omega^n, g) \) with boundary \( \Sigma \) and assuming \( \Sigma \) can be isometrically embedded in \( \mathbb{H}^n \), motivated by the work [26, 23], we are interested in the vector-valued integral
\[
(3.1) \quad m(\Omega, \Sigma) = \int_{\Sigma} (H_0 - H) x d\sigma,
\]
where \( H \) is the mean curvature of \( \Sigma \) in \( (\Omega^n, g) \), \( H_0 \) is the mean curvature of the embedding of \( \Sigma \) in \( \mathbb{H}^n \), and \( x \) is the position vector of points in \( \mathbb{H}^n \subset \mathbb{R}^{n+1} \), where
\[
(3.2) \quad \mathbb{H}^n = \{ (x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} \mid (x^0)^2 - \sum_{i=1}^{n} (x^i)^2 = 1, x^0 > 0 \}.
\]

When \( n = 3 \), if \( \Sigma \) is homeomorphic to \( \mathbb{S}^2 \) with Gaussian curvature larger than \(-1\), then \( \Sigma \) can be isometrically embedded in \( \mathbb{H}^3 \) and the embedding is unique up to isometries of \( \mathbb{H}^3 \) (cf. [21]). Note that the integral in (3.1) depends also on the embedding of \( \Sigma \).

In this section, we are interested in analyzing the asymptotic behavior of \( m(\Omega_\rho, \Sigma_\rho) \), together with other related geometric quantities, as \( \rho \to \infty \). Here \( \Omega_\rho \) is the bounded domain enclosed by \( \Sigma_\rho \) on an asymptotically hyperbolic manifold \( (M^n, g) \) defined in Definition 1.2. We continue to use \( H, S^\rho \) to denote the mean curvature, the scalar curvature respectively of the hypersurfaces \( \Sigma_\rho \), which are the geodesic spheres in the background \( \mathbb{H}^n \).

Recall that the ball model \( (B^n, g_0) \) of \( \mathbb{H}^n \) is given by
\[
B^n = \{ x \in \mathbb{R}^n \mid |x| < 1 \}, \quad g_0 = \frac{4}{(1 - |x|^2)^2} g_e,
\]
where \( g_e \) is the standard Euclidean metric.
Theorem 3.1. Let \((M^n, g)\) be an asymptotically hyperbolic manifold. Let \(\Sigma_\rho\) be the hypersurface in \(M^n\) corresponding to the geodesic sphere of radius \(\rho\) in \(\mathbb{H}^n\) as specified in Definition 1.2. Let \(\{V^{(i)}\}_{0 \leq i \leq n-1}\) be the functions given in (1.6). Let \(|\Sigma_\rho|\) denote the volume of \(\Sigma_\rho\) and \(d\sigma_\rho\) be the volume element on \(\Sigma_\rho\). Then the following are true

(I) (a)
\[
M(g)(V^{(0)}) = \lim_{\rho \to \infty} \frac{c_n}{2} \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right)^{\frac{n-1}{n-1}} \int_{\Sigma_\rho} \left[ S_\rho^{\alpha} - \frac{n-2}{n-1} H^2 + (n-1)(n-2) \right] d\sigma_\rho.
\]

(b) If \(\tau > n - 1\), then for \(1 \leq i \leq n\),
\[
M(g)(V^{(i)}) = \lim_{\rho \to \infty} \frac{c_n}{2} \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right)^{\frac{n-1}{n-1}} \int_{\Sigma_\rho} \frac{x^i}{|x|} \left[ S_\rho^{\alpha} - \frac{n-2}{n-1} H^2 + (n-1)(n-2) \right] d\sigma_\rho,
\]
where \(\{x^i\}\) are the coordinate functions in the ball model \((B^n, g_0)\) of \(\mathbb{H}^n\).

(II) Let \(g_\rho\) denote the metric on \(\Sigma_\rho\) induced from \(g\). Suppose \((\Sigma_\rho, g_\rho)\) can be isometrically embedded in \(\mathbb{H}^n\) for sufficiently large \(\rho\), which is automatically satisfied when \(n = 3\).

(a) Suppose \(\tau > n - 1\). If \(n = 3\), also assume the decay conditions as in Lemma 3.3. Then, for each \(0 \leq i \leq n\),
\[
M(g)(V^{(i)}) = 2b_n \lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H)V^{(i)} d\sigma_\rho,
\]
where \(H, H_0\) are the mean curvature of \(\Sigma_\rho\) in \((M^n, g), \mathbb{H}^n\) respectively.

(b) Suppose \(\tau > n - 1\) for \(n \geq 4\) and \(\tau > \frac{5}{2}\) if \(n = 3\). If \(n = 3\), also assume the decay conditions as in Lemma 3.3
Suppose \(\mathbb{H}^n\) is embedded in \(\mathbb{R}^{n,1}\) as in (3.2), then there exist isometric embeddings of \((\Sigma_\rho, g_\rho)\) in \(\mathbb{H}^n\) such that
\[
M(g)(V) = 2b_n \lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H)x d\sigma_\rho,
\]
where \(M(g)(V) = (M(g)(V^{(0)}), \ldots, M(g)(V^{(n)}))\) and \(x\) is the position vector of the embedding of \(\Sigma_\rho\) in \(\mathbb{H}^n \subset \mathbb{R}^{n,1}\).

We will prove Theorem 3.1 via a series of lemmas. First, we fix some notations. Let \(h_0\) be the standard metric on the unit sphere \(S^{n-1}\). Choose finitely many geodesic balls of radius 2 in \((S^{n-1}, h_0)\), so that the geodesic balls of radius 1 with the same centers cover \(S^{n-1}\). In each
geodesic ball, fix an orthonormal frame \( \{ f_i \}_{1 \leq i \leq n-1} \). Let \( \{ \varepsilon_j \}_{0 \leq i \leq n-1} \) be given as in Definition 1.2. Let \( \nabla, \tilde{\nabla} \) denote the covariant derivatives with respect to \( g, g_0 \) respectively, where \( g_0 \) is the hyperbolic metric on \( \mathbb{H}^n \). Let \( g_{ij} = g(\varepsilon_i, \varepsilon_j) \), \( g_{0ij} = g_0(\varepsilon_i, \varepsilon_j) \), and define \( \{ \Gamma^k_{ij} \} \) and \( \{ \tilde{\Gamma}^k_{ij} \} \) by:

\[
\nabla_{\varepsilon_i} \varepsilon_j = \Gamma^k_{ij} \varepsilon_k, \quad \tilde{\nabla}_{\varepsilon_i} \varepsilon_j = \tilde{\Gamma}^k_{ij} \varepsilon_k.
\]

Direct computation gives

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\varepsilon_i (g_{jl}) + \varepsilon_j (g_{il}) - \varepsilon_l (g_{ij}))
\]

\[
\tilde{\Gamma}^k_{ij} = \frac{1}{2} \delta^{kl} [- g_0 (\varepsilon_i, \varepsilon_j) - g_0 (\varepsilon_j, \varepsilon_i) + g_0 (\varepsilon_i, \varepsilon_i)]
\]

for \( 0 \leq i, j, k \leq n - 1 \);

\[
[\varepsilon_0, \varepsilon_i] = - \frac{\cosh \rho}{\sinh \rho} \varepsilon_i, \quad [\varepsilon_i, \varepsilon_j] = \frac{1}{\sinh \rho} \sum_{k=1}^{n-1} \lambda^k_{ij} \varepsilon_k,
\]

for \( 1 \leq i, j \leq n - 1 \), where \( \{ \lambda^k_{ij} \} \) are smooth functions in the geodesic balls of radius 2 in \( \mathbb{S}^{n-1} \), and

\[
R(\varepsilon_i, \varepsilon_j, \varepsilon_k, \varepsilon_l) = g_{pl} \left\{ \varepsilon_i (\Gamma^p_{jk}) - \varepsilon_j (\Gamma^p_{ik}) + \Gamma^q_{jk} \Gamma^p_{iq} - \Gamma^q_{ik} \Gamma^p_{jq} - (\Gamma^q_{ij} - \Gamma^q_{ji}) \Gamma^p_{qk} \right\}
\]

\[
\tilde{R}(\varepsilon_i, \varepsilon_j, \varepsilon_k, \varepsilon_l) = \delta_{pl} \left\{ \varepsilon_i (\tilde{\Gamma}^p_{jk}) - \varepsilon_j (\tilde{\Gamma}^p_{ik}) + \tilde{\Gamma}^q_{jk} \tilde{\Gamma}^p_{iq} - \tilde{\Gamma}^q_{ik} \tilde{\Gamma}^p_{jq} - (\tilde{\Gamma}^q_{ij} - \tilde{\Gamma}^q_{ji}) \tilde{\Gamma}^p_{qk} \right\},
\]

where \( R \) and \( \tilde{R} \) are the curvature tensors of \( g \) and \( g_0 \) respectively. It follows from (3.3) - (3.6) that

\[
|\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}| = O(e^{-\rho})
\]

and

\[
|R(X, Y, Z, W) - \tilde{R}(X, Y, Z, W)| \leq O(e^{-\rho}|X|_{g_0} |Y|_{g_0} |Z|_{g_0} |W|_{g_0})
\]

for tangent vectors \( X, Y, Z, W \).

**Remark 3.1.** Because \( \{ \varepsilon_1, \ldots, \varepsilon_{n-1} \} \) are tangential to \( \Sigma_{\rho} \), if we restrict all indices used in (3.3) - (3.8) to the set \( \{ 1, \ldots, n-1 \} \), then we obtain comparison between corresponding quantities on \((\Sigma_{\rho}, g_{\rho})\) and \((\Sigma_{\rho}, g^0_{\rho})\). Here \( g_{\rho}, g^0_{\rho} \) denote the induced metric on \( \Sigma_{\rho} \) from \( g, g_0 \) respectively.

**Lemma 3.1.** Let \( \nu, \tilde{\nu} \) be the unit outward normals to \( \Sigma_{\rho} \) and \( A, \tilde{A} \) be the second fundamental forms of \( \Sigma_{\rho} \) with respect to \( g, g_0 \) respectively. Then
(i) \(|\nu - \tilde{\nu}|_{g_0} = O(e^{-\tau \rho})\).
(ii) \(\tilde{A}(\varepsilon_i, \varepsilon_j) = (\coth \rho) \delta_{ij}\), for \(1 \leq i, j \leq n - 1\).
(iii) \(|A - \tilde{A}|_{g_0} = O(e^{-\tau \rho})\).

**Proof.** (i) This follows from the fact
\[
\nabla \rho = \frac{\nabla \rho}{|\nabla \rho|_{g_0}}
\]
and
\[
\nabla \rho = g_{ij} \varepsilon_j(\rho) \varepsilon_i = \varepsilon_0 + O(e^{-\tau \rho}) = \tilde{\nu} + O(e^{-\tau \rho}).
\]
(ii) For \(1 \leq i \leq n - 1\), by (3.4),
\[
\tilde{A}(\varepsilon_i, \varepsilon_j) = -g_0(\tilde{\nabla}_{\varepsilon_i} \varepsilon_j, \varepsilon_0) = -\tilde{\Gamma}_0^{ij} = (\coth \rho) \delta_{ij}.
\]
(iii) For \(1 \leq i, j \leq n - 1\), by (3.4) and (3.7),
\[
A(\varepsilon_i, \varepsilon_j) - \tilde{A}(\varepsilon_i, \varepsilon_j) = -g(\nabla \varepsilon_i \varepsilon_j, \nu) + g_0(\tilde{\nabla}_{\varepsilon_i} \varepsilon_j, \nu)
\]
\[
= -g(\nabla \varepsilon_i \varepsilon_j - \tilde{\nabla}_{\varepsilon_i} \varepsilon_j, \nu) - (g - g_0)(\tilde{\nabla}_{\varepsilon_i} \varepsilon_j, \nu)
\]
\[
= O(e^{-\tau \rho}).
\]

**Lemma 3.2.** Let \(\{\kappa_i\}_{1 \leq i \leq n-1}\) and \(H\) be the principal curvatures and the mean curvature of \(\Sigma_{\rho}\) in \((M^n, g)\) with respect to \(\nu\), respectively. Let \(K\) be the sectional curvature associated to a tangent plane in \((\Sigma_{\rho}, g_{\rho})\), where \(g_{\rho}\) is the induced metric from \(g\). Then
\[
\begin{align*}
\kappa_i &= \coth \rho + O(e^{-\tau \rho}) \\
H &= (n-1) \coth \rho + O(e^{-\tau \rho}) \\
K &= (\sinh \rho)^{-2} + O(e^{-\tau \rho}).
\end{align*}
\]

**Proof.** This follows from Lemma 3.1, (3.8) and the Gauss equation. (The estimate on \(K\) can also be obtained by applying Remark 3.1.)

**Lemma 3.3.** Suppose \((\Sigma_{\rho}, g_{\rho})\) can be isometrically embedded in \(\mathbb{H}^n\) for large \(\rho\). Let \(\{\kappa_i^{(0)}\}_{1 \leq i \leq n-1}\) be the principal curvatures of \(\Sigma_{\rho} \subset \mathbb{H}^n\) with respect to the outward unit normal \(\nu_0\). Then, for \(n \geq 4\),
\[
\kappa_i^{(0)} = \coth \rho + O(e^{-\tau \rho}).
\]
When \(n = 3\), in addition to the asymptotic conditions specified in Definition 1.2, if for \(1 \leq i, j, k, l, p, q \leq 2\),
\[
(3.9) \quad \varepsilon_p \varepsilon_k(\varepsilon_l(g_{ij})) = O(e^{-(\tau + 1) \rho}), \varepsilon_q(\varepsilon_p(\varepsilon_k(\varepsilon_l(g_{ij})))) = O(e^{-(\tau + 2) \rho}),
\]
then the same conclusion also holds.
Proof. Let $H_0, A_0$ be the mean curvature, the second fundamental form of $\Sigma_\rho \subset \mathbb{H}^n$ with respect to $\nu_0$, respectively. At a point $p$ in $\Sigma_\rho$, let \{e_i\}_{1 \leq i \leq n-1}$ be an orthonormal frame in $T_p \Sigma_\rho$ that diagonalizes $A_0$. Let $R^\rho_{ijkl}, R^\rho_{ij}$ and $S^\rho$ be the intrinsic curvature tensor, the Ricci tensor and the scalar curvature of $(\Sigma_\rho, g_\rho)$. By the Gauss equation and Lemma 3.2:

\begin{equation}
\begin{cases}
\kappa_i^{(0)} \kappa_j^{(0)} = R^\rho_{ijij} + 1 = \coth^2 \rho + O(e^{-\tau \rho}), & i \neq j \\
R^\rho_{ii} = (n-2)(\sinh \rho)^{-2} + O(e^{-\tau \rho}) \\
S^\rho = (n-1)(n-2)(\sinh \rho)^{-2} + O(e^{-\tau \rho})
\end{cases}
\end{equation}

and

$$H_0^2 = (n-1)(n-2) + S^\rho + |A_0|^2$$

$$\geq (n-1)(n-2) \coth^2 \rho + \frac{1}{n-1} H_0^2 + O(e^{-\tau \rho}).$$

In particular, $H_0 \neq 0$ and hence $H_0 > 0$ for large $\rho$ (as $\Sigma_\rho$ is compact). Therefore,

\begin{equation}
H_0 \geq (n-1) \coth \rho + O(e^{-\tau \rho}).
\end{equation}

Next, we consider the case $n \geq 4$ first. The proof is similar to that of Lemma 2.7. By the Gauss equation,

$$R^\rho_{ii} + (n-2) = H_0 \kappa_i^{(0)} - (\kappa_i^{(0)})^2,$$

which shows

$$\kappa_i^{(0)} = \frac{1}{2} \left[ H_0 \pm \sqrt{H_0^2 - 4(R^\rho_{ii} + (n-2))} \right].$$

In particular, $\kappa_i^{(0)} > 0$. We claim

$$\kappa_i^{(0)} = \frac{1}{2} \left[ H_0 - \sqrt{H_0^2 - 4(R^\rho_{ii} + (n-2))} \right].$$

If not, without loss of generality, suppose

$$\kappa_i^{(0)} = \frac{1}{2} \left[ H_0 + \sqrt{H_0^2 - 4(R^\rho_{ii} + (n-2))} \right].$$

By (3.10) and (3.11),

$$H_0^2 - 4(R^\rho_{ii} + (n-2)) \geq (n-1)^2 \coth^2 \rho - 4(n-2) \coth^2 \rho + O(e^{-\tau \rho})$$

$$= (n-3)^2 \coth^2 \rho + O(e^{-\tau \rho}) > 0$$
for large \( \rho \), as \( n \geq 4 \). Hence \( \kappa_1(0) > \frac{1}{2} H_0 \). This implies, for \( i \geq 2 \),

\[
\kappa_i(0) = \frac{1}{2} \left[ H_0 - \sqrt{H_0^2 - 4(R^\rho_{ii} + (n-2))} \right]
\]

(3.12)

\[
\leq \frac{1}{2} \frac{4(R^\rho_{ii} + (n-2))}{H_0 + \sqrt{H_0^2 - 4(R^\rho_{ii} + (n-2))}}
\]

\[
\kappa_i(0) = \coth \rho + O(e^{-\tau \rho}),
\]

for otherwise we would have \( \kappa_1(0) + \kappa_i(0) > H_0 \), for some \( i \neq 1 \), which is impossible.

Now for \( i \neq j \) both larger than 1, by (3.10) and (3.12),

\[
\kappa_i(0) (\coth \rho + O(e^{-\tau \rho})) \geq \kappa_i(0) \kappa_j(0) = \coth \rho + O(e^{-\tau \rho}).
\]

Hence,

\[
\kappa_i(0) \geq \coth \rho + O(e^{-\tau \rho}).
\]

Combining this with (3.12), we have for \( i \geq 2 \),

(3.13)

\[
\kappa_i(0) = \coth \rho + O(e^{-\tau \rho}).
\]

But then it follows from (3.10), (3.11) and (3.13) that, for \( i \geq 2 \),

\[
\coth^2 \rho + O(e^{-\tau \rho}) = \kappa_i(0) \kappa_1(0)
\]

\[
\geq \left( \coth \rho + O(e^{-\tau \rho}) \right) \frac{H_0}{2}
\]

\[
= \frac{(n-1)}{2} \coth^2 \rho + O(e^{-\tau \rho}),
\]

which is impossible since \( n - 1 \geq 3 \). Hence, (3.12) is valid for all \( i \geq 1 \).

Repeating the argument leading to (3.13) for any \( i \neq j \), we conclude that (3.13) holds for all \( i \geq 1 \). This proves the case \( n \geq 4 \).

When \( n = 3 \), using the assumption (3.9) and the fact \( \Gamma^k_{ij} = O(e^{-\rho}) \), for all \( 1 \leq i, j, k \leq 2 \), one checks that

(3.14)

\[
\Delta_{g^\rho} S^\rho = O(e^{-(\tau + 2)\rho}),
\]

where \( \Delta_{g^\rho} \) is the Laplacian on \( (\Sigma_\rho, g^\rho) \). Then by [16, Lemma 2.3] (see also [16]), we have

\[
H_0^2 \leq \max_{\Sigma^\rho} \left( \frac{2(S^\rho + 2)^2 - 4(S^\rho + 2) - \Delta_{g^\rho} S^\rho}{S^\rho} \right)
\]

\[
= 4 \coth^2 \rho + O(e^{-\tau \rho}).
\]

Hence,

\[
H_0 \leq 2 \coth \rho + O(e^{-\tau \rho}),
\]
which together with (3.11) shows

\[ H_0 = 2 \coth \rho + O(e^{-\tau \rho}). \]

Combining (3.15) with (3.10), we conclude

\[ \kappa_\alpha^{(0)} = \coth \rho + O(e^{-\tau \rho}), \quad \forall \alpha = 1, 2. \]

This completes the proof. \qed

On the ball model \((B^n, g_0)\) of \(\mathbb{H}^n\), the vector fields \(X^{(i)}\), \(0 \leq i \leq n\), defined by

\[
X^{(0)} = x^j \frac{\partial}{\partial x^j}, \quad X^{(k)} = \frac{\partial}{\partial x^k}, \quad k = 1, \ldots, n,
\]

are conformal Killing vector fields. Let \(r = |x|\) and define

\[ \rho = \ln \left( \frac{1 + r}{1 - r} \right), \]

then \(g_0 = d\rho^2 + (\sinh \rho)^2 h_0\). In particular, this shows

\[ X^{(0)} = \sinh \rho \frac{\partial}{\partial \rho}, \quad V^{(k)} = (\sinh \rho) r^{-1} x^k, \quad k = 1, \ldots, n, \]

Recall that \(V^{(0)}\) and \(V^{(k)}\) are defined in (1.6).

**Lemma 3.4.** Let \(|\Sigma_\rho|\) denote the volume of \((\Sigma_\rho, g_\rho)\). As \(\rho \to \infty\),

\[
\int_{\Sigma_\rho} G^g_{-1}(X^{(0)}, \nu) d\sigma_\rho = \frac{1}{2} \left( \frac{|\Sigma_\rho|}{\omega_{n-1}} \right) \frac{1}{\omega_{n-1}} \int_{\Sigma_\rho} \left[ \frac{n-2}{n-1} H^2 - S^\rho - (n-1)(n-2) \right] d\sigma_\rho + o(1).
\]

Here \(d\sigma_\rho\) is the volume element of \(g_\rho\) on \(\Sigma_\rho\).

**Proof.** Denote \(G^g_{-1}\) by \(G\). As \((M^n, g)\) is asymptotically hyperbolic,

\[ |\Sigma_\rho| = (\sinh \rho)^{n-1} \omega_{n-1}(1 + O(e^{-\tau \rho})). \]

Using (3.8), Lemma 3.1 and the fact \(\tau > \frac{n}{2}\), as \(\rho \to \infty\), we have

\[
\int_{\Sigma_\rho} G(X^{(0)}, \nu) d\sigma_\rho = \int_{\Sigma_\rho} \sinh \rho \ G(\tilde{\nu}, \nu) d\sigma_\rho = \int_{\Sigma_\rho} \sinh \rho \ G(\nu, \nu) d\sigma_\rho + o(1).
\]
By the Gauss equation,
\[
G(\nu, \nu) = \frac{1}{2} \left[ H^2 - |A|^2 - S^\rho - (n-1)(n-2) \right]
\]
(3.19)
\[
= \frac{1}{2} \left[ \frac{n-2}{n-1} H^2 - S^\rho - (n-1)(n-2) \right] + O(e^{-2\tau \rho}),
\]
where we also have used the fact
\[
|A|^2 = \frac{1}{n-1} H^2 + O(e^{-2\tau \rho})
\]
which follows from Lemma 3.2. Therefore,
\[
\int_{\Sigma^\rho} G(X^{(0)}, \nu) d\sigma^\rho = \frac{1}{2} \sinh \rho \int_{\Sigma^\rho} \left[ \frac{n-2}{n-1} H^2 - S^\rho - (n-1)(n-2) \right] d\sigma^\rho + o(1).
\]
(3.20)

The lemma now follows from (3.17) and (3.20). □

**Lemma 3.5.** For each large \( \rho \), suppose \( \iota_{\rho} : (\Sigma^\rho, g^\rho) \to \mathbb{H}^n \) is an isometric embedding such that its principal curvatures \( \{\kappa_i^{(0)}\}_{1 \leq i \leq n-1} \) satisfy
\[
\kappa_i^{(0)} = \coth \rho + O(e^{-\tau \rho}),
\]
as \( \rho \to \infty \). Then
\[
\int_{\Sigma^\rho} (H_0 - H)V^{(0)} d\sigma^\rho = -\frac{1}{n-2} \int_{\Sigma^\rho} G^g_{-1}(X^{(0)}, \nu) d\sigma^\rho + o(1).
\]
Here \( H_0 \) is the mean curvature of the embedding \( \iota_{\rho} \).

**Proof.** Denote \( G^g_{-1} \) by \( G \). Apply the Gauss equation to \( \iota_{\rho}(\Sigma^\rho) \subset \mathbb{H}^n \), we have
\[
0 = \sum_{i < j} (\kappa_i^{(0)} \kappa_j^{(0)}) - S^\rho - (n-1)(n-2).
\]
(3.21)

Therefore, by (3.18), (3.19) and (3.21),
\[
\int_{\Sigma^\rho} G(X^{(0)}, \nu) d\sigma^\rho = \int_{\Sigma^\rho} \sinh \rho G(\nu, \nu) d\sigma^\rho + o(1)
\]
(3.22)
\[
= \int_{\Sigma^\rho} \sinh \rho \sum_{i < j} (\kappa_i \kappa_j - \kappa_i^{(0)} \kappa_j^{(0)}) d\sigma^\rho + o(1).
\]
Now by the assumption on $\kappa_i^{(0)}$ and by Lemma 3.2,

$$
\kappa_i\kappa_j - \kappa_i^{(0)}\kappa_j^{(0)} = \left(\kappa_i - \kappa_i^{(0)}\right)\kappa_j + \left(\kappa_j - \kappa_j^{(0)}\right)\kappa_i^{(0)} \\
= \left(\kappa_i - \kappa_i^{(0)}\right)\left(\kappa_j - \coth \rho\right) + \left(\kappa_j - \kappa_j^{(0)}\right)\left(\kappa_i^{(0)} - \coth \rho\right) \\
+ \coth \rho \left(\kappa_i + \kappa_j - \kappa_i^{(0)} - \kappa_j^{(0)}\right) \\
= \coth \rho \left(\kappa_i + \kappa_j - \kappa_i^{(0)} - \kappa_j^{(0)}\right) + O(e^{-2\tau \rho}).
$$

Hence,

$$
\sum_{i<j} (\kappa_i\kappa_j - \kappa_i^{(0)}\kappa_j^{(0)}) = (n - 2) \coth \rho (H - H_0) + O(e^{-2\tau \rho}).
$$

Combining this with (3.22), we have

$$
\int_{\Sigma_\rho} G(X^{(0)}, \nu) d\sigma_\rho = -(n - 2) \int_{\Sigma_\rho} \cosh \rho (H_0 - H) d\sigma_\rho + o(1),
$$

which proves the lemma. \(\Box\)

**Lemma 3.6.** With the same assumptions and notation as in Lemma 3.5, suppose in addition $\tau > n - 1$, then

$$
\int_{\Sigma_\rho} (H_0 - H) V^{(k)} d\sigma_\rho = -\frac{1}{n - 2} \int_{\Sigma_\rho} G_{g_0}^{g_0} (X^{(k)}, \nu) d\sigma_\rho + o(1)
$$

as $\rho \to \infty$, $1 \leq k \leq n$.

**Proof.** We still denote $G_{g_0}^{g_0}$ by $G$. For a fixed $k$, denote $X^{(k)}$ by $Y$. Decompose $Y = Y_1 + Z$, where $Y_1 = r^{-2}x^k X^{(0)}$ and $Z$ is normal to $X^{(0)}$ with respect to $g_0$. By the proof of Lemma 3.5, we have

$$
(3.23)
\int_{\Sigma_\rho} G(Y_1, \nu) d\sigma_\rho = \int_{\Sigma_\rho} \frac{x^k}{r^2} G(X^{(0)}, \nu) d\sigma_\rho \\
= -(n - 2) \int_{\Sigma_\rho} \frac{\theta^k}{r} \cosh \rho (H_0 - H) d\sigma_\rho + o(1) \\
= -(n - 2) \int_{\Sigma_\rho} \frac{1 + r^2}{2r^2} \theta^k \sinh \rho (H_0 - H) d\sigma_\rho + o(1) \\
= -(n - 2) \int_{\Sigma_\rho} V^{(k)}(H_0 - H) d\sigma_\rho + o(1)
$$
where we have used the assumption \( \tau > n - 1 \) and the fact

\[
\left( \frac{1 + r^2}{2r^2} - 1 \right) \sinh \rho (H_0 - H) = \frac{2}{r} (H_0 - H) = O(e^{-\tau \rho}),
\]
as \( \rho \to \infty \).

To estimate \( \int_{\Sigma_\rho} G(Z, \nu) d\sigma_\rho \), we adopt an argument in [10]. Let \( \phi \geq 0 \) be a fixed smooth function on \([0, \infty)\) such that \( \phi(s) = 0 \) for \( s \leq \frac{1}{2} \), \( \phi(s) = 1 \) for \( s \geq \frac{3}{4} \) and \( 0 \leq \phi \leq 1 \). For any fixed \( \rho_0 > 0 \) that is sufficiently large, define

\[
\tilde{g} = \left[ 1 - \phi\left( \frac{\rho}{\rho_0} \right) \right] g_0 + \phi\left( \frac{\rho}{\rho_0} \right) g = g_0 + \phi\left( \frac{\rho}{\rho_0} \right) (g - g_0),
\]
which is a metric on \( B^n \) that equals \( g \) outside \( \Omega_{\frac{3}{2}\rho_0} \) and agrees with \( g_0 \) inside \( \Omega_{\frac{1}{4}\rho_0} \). Here \( \Omega_\rho \) denotes the geodesic ball of radius \( \rho \) in \((B^n, g_0)\) centered at the origin. Let \( \tilde{g}_{ij} = \tilde{g}(\varepsilon_i, \varepsilon_j) \), then

\[
|\tilde{g}_{ij} - \delta_{ij}| + |\varepsilon_k(\tilde{g}_{ij})| + |\varepsilon_k(\varepsilon_l(\tilde{g}_{ij}))| \leq C_1 e^{-\tau \rho_0}
\]
on the annulus \( A_{\rho_0} = \Omega_{\rho_0} \setminus \Omega_{\frac{3}{2}\rho_0} \) for some constant \( C_1 \) that is independent on \( \rho_0 \). Hence,

\[
(3.24) \quad |G^{-1}_{\tilde{g}}| \leq C_2 e^{-\tau \rho_0}
\]
on \( A_{\rho_0} \) for some constant \( C_2 \) independent on \( \rho_0 \).

Now let \( \tilde{\beta} \) be the 1-form dual to \( Z \) with respect to \( \tilde{g} \). Let \((\tilde{\beta})^s\) be the symmetric \((0, 2)\) tensor given by \((\tilde{\beta})^s_{ij} = \frac{1}{2} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji})\) where “ ; ” denotes the covariant differentiation on \((B^n, \tilde{g})\). Integrating by parts and using the fact \( G^{-1}_{\tilde{g}} = 0 \) at \( \partial \Omega_{\frac{3}{4}\rho_0} = S_{\frac{3}{4}\rho_0} \), we have

\[
(3.25) \quad \int_{\Sigma_{\rho_0}} G(Z, \nu) d\sigma_{\rho_0} = \int_{A_{\rho_0}} \langle G^{-1}_{\tilde{g}}, (\tilde{\beta})^s \rangle_{\tilde{g}} dV_{\tilde{g}},
\]
where \( dV_{\tilde{g}} \) is the volume element of \( \tilde{g} \).

We next estimate \(|(\tilde{\beta})^s|_{\tilde{g}}\). Note that if we write \( Z = Z^i \varepsilon_i \), then

\[
(3.26) \quad |Z^i| + |\varepsilon^i(Z^j)| \leq C_3 e^\rho
\]
for some constant \( C_3 \) independent on large \( \rho \). Let \( \beta \) be the 1-form dual to \( Z \) with respect to \( g_0 \) and let “ ; ” denote the covariant differentiation on \((B^n, g_0)\). On \( A_{\rho_0} \), we have

\[
(3.27) \quad |\tilde{\beta}_{ij} - \beta_{ij}| \leq |\tilde{\beta}_{ij} - \beta_{ij}| + |(\tilde{\beta} - \beta)_{ij}|
\]
\[
\leq C_4 e^{(-\tau + 1)\rho_0}
\]
for some constant \( C_4 \) independent on \( \rho_0 \), where we have used \((3.20)\) and estimates analogous to \((3.24)\) and \((3.27)\) with \( g \) replaced by \( \tilde{g} \). Let
β^s be the symmetric \((0, 2)\) tensor defined by \(β^s_{ij} = \frac{1}{2}(β_{ij} + β_{ji})\), then direct calculation gives

\[
β^s_{ij} = \frac{1}{2} \left( ε^i (\frac{x^k}{r^2}) δ_{j0} + ε^j (\frac{x^k}{r^2}) δ_{i0} - \frac{x^k}{r^2} \cosh ρ \ δ_{ij} \right)
\]

Therefore, by \((3.27)\) and \((3.28)\), we conclude

\[
|(\tilde{β})^s|_g \leq C_4
\]
on \(A_{ρ_0}\) for some constant \(C_4\) independent on \(ρ_0\). It follows from \((3.24)\), \((3.25)\) and \((3.29)\) that

\[
\int_{S_{ρ_0}} G(Z, ν) dσ_ρ = o(1)
\]
as \(ρ_0 \to ∞\) because \(τ > n - 1\). This together with \((3.23)\) completes the proof. □

**Lemma 3.7.** Suppose \(τ > n - 1\). Then as \(ρ \to ∞\), for \(1 \leq k \leq n\),

\[
\int_{Σ_ρ} G^g_{-1}(X^{(k)}, ν) dσ_ρ
\]

\[
= \frac{1}{2} \left( \frac{|Σ_ρ|}{ω_{n-1}} \right)^{\frac{1}{n-1}} \int_{Σ_ρ} \frac{x^k}{|x|} \left[ \frac{n - 2}{n - 1} H^2 - S^ρ - (n - 1)(n - 2) \right] dσ_ρ + o(1).
\]

Here \(\{x^k\}\) are the coordinate functions in the ball model \((B^n, g_0)\) of \(\mathbb{H}^n\).

**Proof.** Since \(τ > n - 1\), by the proof of Lemma \(3.6\) we have

\[
(3.30) \quad \int_{Σ_ρ} G(X^{(k)}, ν) dσ_ρ = \int_{Σ_ρ} \frac{x^k}{r^2} G(X^{(0)}, ν) dσ_ρ + o(1)
\]
as \(ρ \to ∞\). On the other hand, by \((3.19)\),

\[
\int_{Σ_ρ} \frac{x^k}{r^2} G(X^{(0)}, ν) dσ_ρ
\]

\[
= \sinh ρ \int_{Σ_ρ} \frac{x^k}{r^2} G(ν, ν) dσ_ρ + o(1)
\]

\[
= \frac{1}{2} \sinh ρ \int_{Σ_ρ} \frac{x^k}{r^2} \left[ \frac{n - 2}{n - 1} H^2 - S^ρ - (n - 1)(n - 2) \right] dσ_ρ + o(1).
\]
Since \(\frac{1}{\tau} - \frac{1}{\rho} = O(e^{-\rho})\),
\[
\frac{n-2}{n-1} H^2 - S^\rho - (n-1)(n-2) = O(e^{-\tau\rho}),
\]
and \(\tau > n - 1\), the result follows from (3.17) and (3.30).

In the next Lemma, we normalize the embedding \(\iota_\rho\) used in Lemma 3.5 and 3.6 so that the corresponding position vector in \(\mathbb{R}^n, 1\) can be compared to the vector \((V^{(0)}, V^{(1)}, \ldots, V^{(n)})\) defined at points in \(\Sigma_\rho\).

**Lemma 3.8.** With the same assumptions and notation as in Lemma 3.5, with \(\tau > n - 1\), there exists an isometry \(\Lambda_\rho\) of \(H^n \subset \mathbb{R}^n, 1\) such that
\[
|{(\Lambda_\rho \circ \iota_\rho)^{(i)}(x) - V^{(i)}(x)}| = O(e^{(-\tau+3)\rho})
\]
for \(0 \leq i \leq n\) and for all \(x \in \Sigma_\rho\). Here \((\Lambda_\rho \circ \iota_\rho)^{(i)}(x)\) is the \(i\)-th component of \((\Lambda_\rho \circ \iota_\rho)(x)\) in \(\mathbb{R}^{n,1}\).

**Proof.** We proceed as in [15]. Let \(\coth \varsigma_1 = \max_{x \in \Sigma_\rho, 1 \leq j \leq n-1} \kappa_j^{(0)}(x)\), where as before, \(\kappa_j^{(0)}\) are the principal curvatures of \(\iota_\rho(\Sigma_\rho)\). By Lemma 3.3 for large \(\rho\), we have
\[
\coth \varsigma_1 = \coth \rho + O(e^{-\tau\rho}) > 1
\]
since \(\tau > 2\). Therefore,
\[
\varsigma_1 = \frac{1}{2} \ln \left[ \frac{\coth \rho + O(e^{-\tau\rho}) + 1}{\coth \rho + O(e^{-\tau\rho}) - 1} \right]
= \rho + O(e^{(-\tau+2)\rho}).
\]
Similarly, let \(\coth \varsigma_2 = \min_{x \in \Sigma_\rho, 1 \leq j \leq n-1} \kappa_j^{(0)}(x)\), then \(\coth \varsigma_2 > 1\) and
\[
\varsigma_2 = \rho + O(e^{(-\tau+2)\rho}).
\]

By [11, Theorem 4.5] and [15, Proposition 3.1 and 3.2], we conclude that \(\iota_\rho(\Sigma_\rho)\) lies between two geodesic balls with the same center with radii \(\varsigma_0 \geq \varsigma_1\) so that
\[
(3.31) \quad \varsigma_0, \varsigma_1 = \rho + O(e^{(-\tau+2)\rho}).
\]
Now, first choose an isometry \(\Lambda_\rho\) so that the center of these geodesic balls is at \(o = (1,0,0,0) \in \mathbb{H}^n \subset \mathbb{R}^{n,1}\). Let \(\varsigma\) be the hyperbolic distance function from \(o\). Given \(x \in \Sigma_\rho\), let \(y(x) = \Lambda_\rho \circ \iota_\rho(x)\), then by (3.31),
\[
(3.32) \quad \sinh \varsigma(y(x)) = \sinh \rho + O(e^{(-\tau+3)\rho}),
\]
\[
\cosh \varsigma(y(x)) = \cosh \rho + O(e^{(-\tau+3)\rho}),
\]
as \( \rho \to \infty \). For the simplicity of notation, we still denote \( \Lambda_\rho \circ \iota_\rho \) by \( \iota_\rho \). Using geodesic polar coordinates in \( \mathbb{H}^n \) with respect to \( o = (1, 0, 0, 0) \), for each \( x \in \Sigma_\rho \), we define \( \theta(x), \vartheta(x) \in S^{n-1} \subset T_o \mathbb{H}^n = \mathbb{R}^n \) respectively by

\[
    x = \exp_o(\rho \theta(x)) \quad \text{and} \quad y(x) = \exp_o(\varsigma(y(x))\vartheta(x)).
\]

We want to estimate \( \vartheta(x) - \theta(x) \) as points in \( \mathbb{R}^n \). Let \( d_\rho \) be the intrinsic distance function on \( (\Sigma_\rho, g_\rho) \), \( \tilde{d}_\rho \) be the intrinsic distance of \( \iota_\rho(\Sigma_\rho) \), and let \( d_{S^{n-1}} \) be the distance function on \( S^{n-1} \) with respect to the standard metric \( h_0 \). Given \( x_1, x_2 \in \Sigma_\rho \), we have

\[
    \sinh \rho \; d_{S^{n-1}}(\theta(x_1), \theta(x_2)) = d_\rho(x_1, x_2)
\]

\[
    = \tilde{d}_\rho(y(x_1), y(x_2))
\]

\[
    = [\sinh \rho + O(e^{(-\tau+3)\rho})] d_{S^{n-1}}(\vartheta(x_1), \vartheta(x_2))
\]

by \eqref{3.32}. Hence,

\[
    (3.33) \quad d_{S^{n-1}}(\vartheta(x_1), \vartheta(x_2)) = [1 + O(e^{(-\tau+2)\rho})] d_{S^{n-1}}(\theta(x_1), \theta(x_2)).
\]

Next, let \( e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \cdots, 1) \in S^{n-1} \subset \mathbb{R}^n \). By composing \( \iota_\rho \) with an isometry of \( \mathbb{H}^n \) fixing \( o \) and still denoting the resulting composition by \( \iota_\rho \), we may assume \( \vartheta(\rho e_1) = e_1 \) and, for each \( i > 1 \), \( \vartheta(\rho e_i) \) is a linear combination of \( \{e_1, \ldots, e_i\} \) such that the coefficient of \( e_i \) is nonnegative. For this \( \iota_\rho \), we claim that for \( 1 \leq j \leq n \),

\[
    (3.34) \quad d_{S^{n-1}}(\vartheta(\rho e_j), e_j) = O(e^{(-\tau+2)\rho}).
\]

This is obvious true for \( j = 1 \) because \( \vartheta(\rho e_1) = e_1 \). Suppose \eqref{3.34} is true for \( j = 1, \ldots, i \) where \( i < n \). Write

\[
    \vartheta(\rho e_{i+1}) = \sum_{j=1}^{i+1} a_j e_j
\]

with \( a_{i+1} \geq 0 \). For \( 1 \leq j \leq i \), by the triangle inequality,

\[
    |d_{S^{n-1}}(\vartheta(\rho e_{i+1}), e_j) - d_{S^{n-1}}(\vartheta(\rho e_{i+1}), \vartheta(\rho e_j))| \leq d_{S^{n-1}}(\vartheta(\rho e_{i+1}), \vartheta(\rho e_j), e_j).
\]

Hence, by \eqref{3.33}, we have

\[
    (3.35) \quad d_{S^{n-1}}(\vartheta(\rho e_{i+1}), e_j) = \frac{\pi}{2} + O(e^{(-\tau+2)\rho}),
\]

which shows

\[
    a_j = \cos(d_{S^{n-1}}(\vartheta(\rho e_{i+1}), e_j)) = O(e^{(-\tau+2)\rho})
\]

for all \( j \in \{1, \ldots, i\} \). Thus,

\[
    a_{i+1} = \left(1 - \sum_{j=1}^{2} a_j^2\right)^{\frac{1}{2}} = 1 + O(e^{(-\tau+2)\rho})
\]
because \( a_{i+1} \geq 0 \). Therefore, (3.34) holds for \( j = i + 1 \) and hence it is true for all \( 1 \leq j \leq n \). Now let \( x \in \Sigma_\rho \), by (3.34) and (3.33),

\[
d_{g_{n-1}}(\vartheta(x), e_j) = d_{g_{n-1}}(\vartheta(x), \vartheta(\rho e_j)) + O(e^{-(\tau+2)\rho})
\]

\[
= d_{g_{n-1}}(\theta(x), e_j) + O(e^{-(\tau+2)\rho}).
\]

Hence, this implies

(3.36)

\[
\vartheta(x) - \theta(x) = \sum_{i=1}^{n} \left[ \cos(d_{g_{n-1}}(\vartheta(x), e_i)) - \cos(d_{g_{n-1}}(\theta(x), e_i)) \right] e_i
\]

\[
= O(e^{-(\tau+2)\rho}).
\]

The lemma now follows from (3.32), (3.36) and the fact

\[
V^{(0)}(x) = \cosh\rho, \quad V^{(i)}(x) = \theta^{(i)}(x) \sinh\rho,
\]

\[
(\iota_{\rho})^{(0)}(x) = \cosh\zeta(y(x)), \quad (\iota_{\rho})^{(i)}(x) = \vartheta^{(i)}(x) \sinh\zeta(y(x))
\]

for \( i \geq 1 \).

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. (a) and (b) of Part (I) are direct consequences of Lemma 3.4, 3.7 and Theorem 1.1 (ii).

As for Part (II), (a) follows from Lemma 3.3, 3.5, 3.6 and Theorem 1.1 (ii). To prove (b), by Lemma 3.3, 3.5, 3.6 and 3.8, there exist isometric embeddings \( \iota_{\rho} = (x^0, \ldots, x^n) : (\Sigma_\rho, g_\rho) \to \mathbb{H}^n \subset \mathbb{R}^{n,1} \) such that, for each \( i = 0, \ldots, n \),

\[
\int_{\Sigma_\rho} (H_0 - H)x^i d\sigma_\rho
\]

(3.37)

\[
= \int_{\Sigma_\rho} (H_0 - H)V^{(i)} d\sigma_\rho + \int_{\Sigma_\rho} (H_0 - H)(x^i - V^{(i)}) d\sigma_\rho
\]

\[
= -\frac{1}{n-2} \int_{\Sigma_\rho} G_{-1}^{\rho}(X^{(i)}, \nu) d\sigma_\rho + o(1)
\]

provided \( 2\tau - 3 > n - 1 \). But this is satisfied if \( \tau > n - 1 \) when \( n \geq 4 \) and \( \tau > \frac{n}{2} \) when \( n = 3 \). Hence, (b) follows from (3.37) and Theorem 1.1 (ii).

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