Equivalent condition for approximately Cohen-Macaulay complexes

Michał Lasoń

Abstract. We give a necessary and sufficient condition for a simplicial complex to be approximately Cohen-Macaulay. Namely it is approximately Cohen-Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear and generated in two consecutive degrees. This completes the result of Herzog and Hibi who proved that a simplicial complex is sequentially Cohen-Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear.

1. Introduction

In [5] Eagon and Reiner proved that a simplicial complex is Cohen-Macaulay if and only if the ideal associated to its Alexander dual has linear resolution. Later Herzog and Hibi [8] generalized it and proved that a simplicial complex is sequentially Cohen-Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear. We use their result to give a similar equivalent condition for a simplicial complex to be approximately Cohen-Macaulay.

We begin with a brief introduction to the topic. When we say that a simplicial complex is Cohen-Macaulay, sequentially Cohen-Macaulay, or approximately Cohen-Macaulay, we always think that its Stanley-Reisner ring has this property.

Definition 1. For a simplicial complex $\Delta$ on the set of vertices $\{1, \ldots, n\}$ and a field $K$, the Stanley-Reisner ring (or face ring) is the ring $K[x_1, \ldots, x_n]/I_\Delta = K[\Delta]$, where $I_\Delta$ is generated by all squarefree monomials $x_{i_1} \cdots x_{i_l}$ for which $\{i_1, \ldots, i_l\}$ is not a face in $\Delta$.

We recall combinatorial description of Cohen-Macaulay complexes:

Definition 2. Let $\sigma$ be a simplex in a simplicial complex $\Delta$. The link of $\sigma$ in $\Delta$, denoted by $lk_{\Delta}\sigma$, is the simplicial complex $\{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$.

Theorem 1. (Reisner [10]) Let $R = K[\Delta]$ be the face ring of $\Delta$. Then the following conditions are equivalent:

1. $R$ is Cohen-Macaulay ring.
2. $\tilde{H}_i(lk_{\Delta}\sigma) = 0$ if $i < \dim(lk_{\Delta}\sigma)$ for all simplices $\sigma \in \Delta$.

Key words and phrases. approximately Cohen-Macaulay ring, Stanley-Reisner ring, Alexander dual complex, sequentially Cohen-Macaulay ring.
For some techniques of counting homology we refer the reader to Section 3.2 of [9]. We need also the following definitions:

**Definition 3.** [7] A non Cohen-Macaulay local ring $A$ is called approximately Cohen-Macaulay if there is an element $a$ in the maximal ideal such that $A/(a^n)$ is Cohen-Macaulay ring of dimension $\dim(A) - 1$ for all $n > 0$.

**Definition 4.** A ring $A$ of dimension $d$ is called sequentially Cohen-Macaulay if there exists a filtration of ideals of $A$:

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = A$$

such that each $D_i/D_{i-1}$ is Cohen-Macaulay and

$$0 < \dim(D_1/D_0) < \dim(D_2/D_1) < \cdots < \dim(D_t/D_{t-1}) = d.$$

**Definition 5.** Let $\Delta$ be a simplicial complex on the set of vertices $V$, we define its Alexander dual to be $\Delta^* = \{V \setminus \sigma : \sigma \notin \Delta\}$.

**Definition 6.** We say that a graded ideal $I \subset A$ is componentwise linear if $I_j$ has linear resolutions for each degree $j$.

There is a nice description of approximately Cohen-Macaulay rings:

**Proposition 1.** [3] Let $A$ be a non Cohen-Macaulay local ring of dimension $d$. Then the following conditions are equivalent:

1. $A$ is an approximately Cohen-Macaulay ring.
2. $A$ is a sequentially Cohen-Macaulay ring with filtration $0 = D_0 \subset D_1 \subset D_2 = A$, where $\dim(D_1) = d - 1$.

**2. Equivalent condition**

We will make use of the following result of Herzog and Hibi [8].

**Theorem 2.** [8] Let $\Delta$ be a simplicial complex. Then Stanley-Reisner ring $K[\Delta]$ is sequentially Cohen-Macaulay if and only if $I_{\Delta^*}$, the ideal associated to its Alexander dual, is componentwise linear.

Our theorem reads as follows.

**Theorem 3.** Let $\Delta$ be a simplicial complex. Then the Stanley-Reisner ring $K[\Delta]$ is approximately Cohen-Macaulay if and only if $I_{\Delta^*}$, ideal associated to its Alexander dual, is componentwise linear and generated in two consecutive degrees.

**Proof.** By Proposition 1 $K[\Delta]$ is approximately Cohen-Macaulay if and only if $K[\Delta]$ is a sequentially Cohen-Macaulay ring with filtration

$$0 = D_0 \subset D_1 \subset D_2 = K[\Delta],$$

where $\dim(D_1) = d - 1$. Due to the Theorem 2 of Herzog and Hibi this is equivalent to componentwise linearity of $I_{\Delta^*}$, and existence of a filtration

$$0 = D_0 \subset D_1 \subset D_2 = K[\Delta],$$

where $\dim(D_1) = d - 1$. From Appendix of [1] we get that if such a filtration exists, then it is unique and coincides with the one given by

$$0 = M_0 \subset \cdots \subset M_{i-1} = I_{\Delta,\Delta^{(j_i-1)}} \subset \cdots \subset K[\Delta].$$
where $I_{\Delta, \Delta^{(j_i-1)}}$ is the ideal in $K[\Delta]$ generated by monomials $x_A$, with $A \in \Delta \setminus \Delta^{(j_i-1)}$. The simplicial complex $\Delta^{(j_i-1)}$ is generated by faces of $\Delta$ of dimension at least $j_i - 1$, where $j_1 - 1 < \ldots < j_s - 1$ are the dimensions of facets of $\Delta$. We have also that $\dim(\Delta^{(j_i-1)}) = j_i - 1$. Hence the desired filtration exists if and only if $\Delta$ has facets of dimension $d$ and $d - 1$. Ideal $I_{\Delta^*}$ is generated by monomials $x_A$ for $A \notin \Delta^*$, that is, for $A = V \setminus \sigma$, where $\sigma \in \Delta$. We have to take all $x_A$ corresponding to facets and they all already generate ideal. Hence the ideal $I_{\Delta^*}$ is generated in two consecutive degrees $v - d$ and $v - (d - 1)$, where $|V| = v$. Since each step of our reasoning was an equivalence, the contrary also holds.

\[\square\]

Acknowledgements

I would like to thank Ralf Fröberg for many inspiring conversations and introduction to this subject. I would also like to thank Jarek Grytczuk for help in preparation of this manuscript. Also I acknowledge a support from Polish Ministry of Science and Higher Education grant MNiSW N N201 413139.

References

[1] A. Björner, M. Wachs, V. Welker, On sequentially Cohen-Macaulay complexes and posets, Israel J. Math. 169 (2009), 295-316.
[2] G. Clements, B. Lindström, A generalization of a combinatorial theorem of Macaulay, J. Combin. Theory 7 (1969), 230-238.
[3] N. Cuong, D. Cuong, On sequentially Cohen-Macaulay Modules, arxiv:math.AC/0507202 v1.
[4] A. Duval, Algebraic Shifting and Sequentially Cohen-Macaulay Simplicial Complexes, Electr. J. Combin. 3 (1996), R21.
[5] J. Eagon, V. Reiner, Resolutions of Stanley-Reisner rings and Alexander duality, J. Pure Appl. Algebra 130 (1998), 265-275.
[6] R. Fröberg, Stanley-Reisner rings and edge ideals, teaching materials.
[7] S. Goto, Approximately Cohen-Macaulay rings, J. Algebra 76 (1982), no. 1, 214-225.
[8] J. Herzog, T. Hibi, Componentwise linear ideals, Nagoya Math. J. 153 (1999), 141-153.
[9] M. Lasoń, M. Michałek, Coll. Math. 62 (2011), no. 3, 275-296.
[10] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Advances in Math. 21 (1976), no. 1, 3049.

Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-956 Warszawa, Poland

Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagellonian University, 30-348 Kraków, Poland

E-mail address: michalason@gmail.com