Logical errors on proving theorem

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Abstract. In tertiary level, students of mathematics education department attend some abstract
courses, such as Introduction to Real Analysis which needs an ability to prove mathematical
statements almost all the time. In fact, many students have not mastered this ability appropriately.
In their Introduction to Real Analysis tests, even though they completed their proof of theorems,
they achieved an unsatisfactory score. They thought that they succeeded, but their proof was not
valid. In this study, a qualitative research was conducted to describe logical errors that students
made in proving the theorem of cluster point. The theorem was given to 54 students.

1. Introduction

When students enroll tertiary level, especially in mathematics field, they are faced with abstract
mathematics. They increasingly encounter new cases related to the proof of characteristics and theorems,
as in the Introduction to Real Analysis course. The course studies abstract concepts, such as
characteristics of real numbers and their proof, sequences of real numbers and characteristics of its
convergence. Proving mathematical statements and the proof they face at this level are different from
what they used in high school level [1]. The ability to construct proof is crucial for them [2][3]. They
need it to explain the truth of mathematical statements [4][3][5][6], to discover new theorems and to
arrange mathematics systematically [4][3][5]. Even though it is important, students often have problems in
constructing proof [7][8][9]. In fact, the proof of mathematical statements is not unique. For instance, a
theorem could be validated through various proofs. The new proofs of a theorem have evolved to enrich
mathematics [10]. Therefore, students have more choices to use effective ways to prove mathematical
statements. However, according to Selden and Selden [11], when students face proving problems, they
cannot complete it because they get confused in using definitions or implementing mathematical
concepts.

Proving involves thinking about a new concept, focusing on important aspects, using existing
knowledge, looking at existing relationships, defining new things (if necessary), and making valid
arguments [12]. A formal proof is a sequence of formulations that forms, consists of formed axioms or
other conclusions, and ends up with a proven theorem [10]. Basically, proofs consist of two aspects,
namely the formal-rhetorical part and the problem-centered part. The first part is like proof framework,
it consists of statements given by the theorem, definitions, earlier results, or other statements which do
not need a deep understanding. The latter is a core statement of the problem which needs deep
understanding of concepts and intuitions. Weber [2] categorizes students’ proofs into four categories: 1)
correct proof, 2) fail to assemble the syntactic knowledge, 3) lack sufficient syntactic knowledge as the
basis of proof, and 4) logical error. The first is valid proof, others are incorrect. When students fail to use their syntactic knowledge, they write incoherent statements because they cannot compile their possessed concepts. Meanwhile, students lack sufficient syntactic knowledge if they do not know the definitions, theorems, or other mathematical statements. Therefore, they are not able to complete their proof. The latter, students complete the proof, but their proof is incorrect. Students usually do an incorrect proof by assuming statement that would prove to construct its proof, generating specific example to be generalized into a general statement, doing one way of implication on proving bi-implication statements [13].

Considering that most of the contents of Introduction to Real Analysis are concepts and theorems, reasoning on proving is absolutely necessary. Various students’ proofs have to be known, especially logical error on proving, a proof that is usually considered a correct proof. This study aims to identify the proof produced by the students in solving Introduction to Real Analysis problem, especially invalid proofs. Furthermore, the lecturer could give action in the teaching and learning process so that the invalid proofs will be no longer conducted by the students.

2. Methods

A qualitative research was conducted to identify the students’ proofs in proving Introduction to Real Analysis case. In this study, the subject of the study was determined by purposive sampling. The subject consisted of 54 students of Introduction to Real Analysis courses. The main data were the result of writing test, while the supporting data were collected by observations of teaching and learning activities and the results of structured tasks. An investigator triangulation was employed to check the validity of data, i.e. checking by some observers or experts [14].

3. Results and Discussion

3.1. General Result

A case of proof related to cluster point was adopted from the book by Bartle and Sherbert [15]. The case asked the students to prove a part of the theorem:

Given a set \( A \subseteq R \) and \( c \in R \). Show that if there is a sequence of \( (x_n) \subseteq A \), \( x_n \neq c \) for all \( n \in N \) and \( x_n \rightarrow c \), then \( c \) is a cluster point of \( A \).

In the case, \( R \) symbolizes the real number system. The case was presented to the 54 students who had attended Introduction to Real Analysis course. As many as thirteen students left the case without responses and the rest tried to prove the case in various ways. The way to prove the theorem is not unique. Proofs of a theorem still appear with wide variations, even more efficient than when the proof is introduced at first [10].

Furthermore, a variety of proof generated by 41 students is discussed further. Figure 1 shows an outline of the proofs.

| The number of students | A  | B  | C  | D  |
|------------------------|----|----|----|----|
|                        | 4  | 1  | 19 | 17 |

Notes:
A: correct proof
B: failure to invoke syntactic knowledge
C: insufficient syntactic knowledge
D: proof with logical errors

Figure 1. Students’ proof on cluster point theorem
Almost a half of number of students who tried to prove the theorem are not able to complete it because they lack syntactic knowledge, such as definitions, lemmas, or other theorems that could be used to prove the theorem. Several students construct proof but their proof is invalid (41.4%). They make errors in understanding the target of proving, proving by examples, or making errors in understanding concepts related to cluster point. There are five students who are proving by using a specific example. In fact, the theorem is a general statement. More detail about proof by using specific example could be seen in Sari et al [16].

3.2 Logical errors
A total of six students seem to get confused concerning a cluster point of \( A \subseteq R \) with the limit of function and or the limit of sequence on \( R \). One of them is Eko. Eko’s proof could be seen in Figure 2.

![Figure 2](image)

**Figure 2.** The beginning of Eko’s proof

Based on Figure 2, Eko considers that proving a cluster point is same as proving the limit of sequence \( f(x_n) \), for a \( (x_n) \subseteq A \). Even though both of them discuss the limit, these concepts could not be equated. In the given theorem, function \( f \) does not appear at all. Moreover, he did not introduce function \( f \) at first before he used it. After he stated his target of proof, Eko wrote as shown in the Figure 3.

![Figure 3](image)

**Figure 3.** Eko’s belief about definition of convergence of \( f(x_n) \)

The definition of convergence of \( f(x_n) \) written by Eko is incorrect. His definition was the definition of the limit of function \( f \) if \( x \) tends to \( c \). This incorrect definition often occurred when students solved proving problems[13]. As the study of Alcock and Simpson [17], only 29% students used the definition correctly. Furthermore, in carrying the target of proof out, Eko wrote as shown in Figure 4 below.

![Figure 4](image)

**Figure 4.** Eko’s first step in carrying the target of proof out.

Eko wrote “dipilih”, this means “chosen”, instead of “diketahui”, this means “given”, or other words to state that mathematical object (sequence \( (x_n) \subseteq A \) where \( x_n \neq c \) for all \( n \in N \) and \( x_n \to c \)) is given on the question. Eko did not write the formal-theoretical part correctly. This leads Eko to failure in constructing proof.

Another logical error was done by Wati. A part of Wati’s proof could be seen in Figure 5 below.
At a glance, Wati’s proof framework led to the proof of limit of function generally, started by taking any $\varepsilon > 0$, showed the existence of $\delta > 0$, then ended up with $|x - c| < \varepsilon$. In her proof, Wati was seen to use the definition of the limit of function on limit of sequence. This part also indicates that Wati do not understand about $\varepsilon$ and $\delta$. Even though they are just symbols, mathematicians usually use these symbols in the definition of the limit of function. It is noticeable that Wati played $x$ until the end of her proof without introducing $x$ first. After she wrote $x \rightarrow c$, she concluded that $c$ was cluster point of $A$. The arrow symbol ($\rightarrow$) is associated with convergence. It is like a symbol $x_n \rightarrow c$ that states sequence $(x_n)$ converges to $c$, this means $c$ is the limit of sequence $(x_n)$ [15]. Nevertheless, the statement $x \rightarrow c$ that Wati wrote, has no reason to say that $c$ is cluster point of $A$. According to Stavrou [13], students usually do mistakes on proving because of misconception of understanding the definition.

A complete proof is written by Anto. The last part of Anto’s proof could be seen in Figure 6.

At the end of his proof, Anto wrote, “Because $\forall (x_n) \subseteq A, x_n \neq c, x_n \rightarrow c$ then it is clear that $c$ is a cluster point of $A$”. In fact, it was what he intended to prove. The theorem given to him to prove, it was not clear. He needed some arguments to ensure it. It is noticeable that his proof framework is incorrect. Moreover, in the beginning of his proof, he wrote as seen in the Figure 7.

At first, Anto wrote an implication argument. However, it was not a valid argument. In the given theorem, no symbol “$f$” was given. Moreover, the theorem did not address the function $f$. He also never declared about $f$. If other people read his proof, they as if reading a paragraph with incoherent sentences. It was challenging to understand.
It is interesting to note that Anto inscribed an invalid definition as shown in the Figure 8.

![An invalid definition of limit of sequence](image)

**Figure 8.** An invalid definition of limit of sequence

At a glance, his definition of the limit of sequence \((x_n \to c)\) is true. In point of fact, it is incorrect. Saying “there exists \(n_0 \in N\) such that for all \(n \geq n_0\) satisfy \(|x_n - c| < \varepsilon, \forall \varepsilon > 0\)” is different from saying “For all \(\varepsilon > 0\), there exists \(n \in N\) such that for all \(n \geq n_0\) satisfies \(|x_n - c| < \varepsilon\)”. On the first statement, the existence of a number \(n_0 \in N\) is true for all \(\varepsilon > 0\). The latter means that the existence of \(n_0 \in N\) depends on the given \(\varepsilon > 0\). In a definition or theorem, the order of writing is crucial. Students were not able to reverse the order at will. They often ignore the importance of the order of writing definition or theorem, as is in the Davis and Vinner’s paper summary in [18], students neglect the important role of the order when they face the concept of a limit of sequence or series. Moreover, Anto possible lack understanding about \(\varepsilon\) and \(\delta\) in his definition. We are not able to conclude this, but his proof indicates this.

4. Conclusion

Based on data analysis, there are some misconceptions in understanding definitions. Students make errors in using the definition of cluster point, the limit of function, and the limit of sequence, but they feel their understanding is correct. Besides, they seem to get confused about each definitions. Moreover, they use some routine symbols, such as symbol “\(f\)” refers to function, though the theorem does not contain it.

Even though the understanding of definition is not enough to construct a valid proof, it is necessary for students to be encouraged in writing correct proof [11]. A lecturer has to put more emphasis on students’ understanding of definitions, how definitions work, and what are the examples and non-examples of the definitions. Besides, for some concepts that have similar “terms” (e.g. convergence of sequence and convergence of function), the lecturer has to explain the similarities and differences. Furthermore, lecturers do not use a routine symbol to state mathematical object. For example, the choice of symbol \(f\) as a function could be interspersed with other alphabets. By these ways, students might experience many kinds of concepts and mathematical objects and statements. Their understanding could improves when solving proving problems which demand various concepts. In addition, the lecturer should familiarize students with writing a proof framework before they go on next steps of proving. By writing it, students always lead their steps to the target of proof. Therefore, their final proof is in accordance with the formal-rhetorical part. According to Selden and Selden [19], by writing the formal-rhetorical part of proof, students could improve their writing proof, reveal the characteristics of the problem, and focus on the problem. Besides, it enables the lecture to help students become more targeted and effective. A further study which investigates how they prove theorems and why they do their proving steps can be conducted by task-based interview or think aloud method so that we understand every steps that they do.
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