ON THE LENSZ-ISING-ONSAGER PROBLEM IN AN EXTERNAL MAGNETIC FIELD

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Abstract

The Lenz-Ising-Onsager (LIO) problem in an external magnetic field in the second quantization representation is the subject of consideration of the paper. It is shown that the operator $V_h$ in the second quantization representation corresponding to Ising spins interaction with the external magnetic field $H$ can be represented in terms of single-subscript creation and annihilation Fermi operators in such a form that the operator $V_h$ commutes with the operator $\hat{P} \equiv (-1)^{\hat{S}}$, where $\hat{S} = \sum_m \beta_m^\dagger \beta_m$ is the operator of a total number of Fermions. The possible consequences of such representation with its relation to the LIO is discussed. In particular, the constructive proof of the Lee-Yang theorem on the absence of phase transition for Ising model in nonzero magnetic field ($\Re h \neq 0$) is demonstrated.

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I. INTRODUCTION

As is known, up to now there is no solution for the 2D Ising model (IM) in an external magnetic field \( H \) (Lenz-Ising-Onsager problem - LIO) still. The algebraic, topological and graph-theoretical difficulties of taking into account the external magnetic field in LIO problem are well known and widely discussed \([1–9]\). As is known \([3]\), the statistical sum for the 2D Ising model in external field \((H)\) in the representation of second quantization can be written in the form:

\[
Z_{2D}(h) = Tr(V)^N = Tr(V_1 V_2 V_h)^N,
\]

where the operators \( V_i \), expressed in terms of the Fermi creation and annihilation operators \( (c_m^\dagger, c_m) \), are of the form:

\[
\begin{align*}
V_1 &= (2\sinh 2K_1)^{M/2} \exp \left[-2K_1^* \sum_{m=1}^M (c_m^\dagger c_m - 1/2)\right], \\
V_2 &= \exp \left\{ -K_2 \left[\sum_{m=1}^{M-1} (c_m^\dagger - c_m) (c_{m+1}^\dagger + c_{m+1}) - (-1)^M (c_M^\dagger - c_M) (c_1^\dagger + c_1)\right]\right\}, \\
V_h &= \exp \left\{ i\pi \sum_{p=1}^{M-1} (c_p^\dagger c_p) (c_m^\dagger + c_m)\right\},
\end{align*}
\]

where \( K_j = \beta J_j \) (\( j = 1, 2, \) ) and \( h = \beta H \) (\( \beta = 1/k_B T \) - inverse temperature, \( H \) - external magnetic field), and \( M = \sum_M c_m^\dagger c_m \) is the operator of the total number of particles and \( K_1^\dagger \) and \( K_1 \) are connected by the following formulae:

\[
\tanh(K_1) = \exp(-2K_1^*), \quad \text{or} \quad \sinh 2K_1 \sinh 2K_1^* = 1.
\]

One can see that the operator \( V_h \) in the second quantization representation, that describes interaction of the spins with external magnetic field, has rather complicated structure. It is easy to see that this operator does not commute with the operator \( P \equiv (-1)^M \). As a result the operator \( V_2 \) has also not a very tractable form, i.e. it has not the needed translational symmetry \((2)\). More exactly, although the operators \( V_1 \) and \( V_2 \) commute with the operator \( P \), the operator \( V \) \((1)\) does not commute with the operator \( P \), i.e. \([P, V]_\pm \neq 0\), because \([P, V_h]_\pm \neq 0\). Therefore, we can not divide all states of the operator \( V = V_1 V_2 V_h \) into eigenstates of the operator \( P \) with eigenvalues \( \lambda = \pm 1 \), and this leads to nonconservation of the states with even and odd numbers of fermions (for details see \([3]\)). Namely this is the fundamental reason which stops solving the problem under consideration within this formalism.

Coming back to the difficulties mentioned above which are connected with the operator \( V_h \), \((2)\), it is now clear that to overcome the troubles within the approach \([3]\), one should find an appropriate method of substituting the operator \( V_h \) \((2)\) with another one which would be equivalent to the former in the sense of correct counting of the interaction of external magnetic field with the spins of the system. Namely, as it could be easily seen, the only contribution to \( Z_{2D} \) \((1)\) from the operator \( V_h \) comes, in the representation of second quantization, from the ”even” part with respect to operators \( c_m^\dagger, c_m \) of the operator \( V_h \).

It was shown in the author’s paper \([4]\) how this difficulty can be overcome by transition to the space of higher dimensionality and supposing then one of the interaction constants to be equal to zero. Then all the operators \( V_j \) in the second quantization representation are expressed in terms of two-subscript creation and annihilation Fermi operators in the finite-dimensional Fock space \( 2^{NM} \) in the particle number representation. These operators \( V_j \)
commute with the operator $\hat{P} = (-1)^{\hat{S}}$, where $\hat{S} = \sum_{nm} \alpha_{nm}^\dagger \alpha_{nm}$ is the operator of total number of fermions. Here we discuss briefly how the analogous result could be achieved in the space of single-subscript Fermi operators, that is, in the finite-dimensional Fock space $2^M$. In particular, I'm intend to sketch briefly that the partition function for the 2D Ising model in an external magnetic field can be expressed in the form:

$$Z_{2D}(h) = Tr(V)^N = Tr(V_1 V_2 V_h)^N,$$

$$V_1 = (2 \sinh 2K_1)^{M/2} \exp \left[ -2K_1^* \sum_{m=1}^{M} (\beta_{m}^\dagger \beta_{m} - 1/2) \right],$$

$$V_2 = \exp \left\{ K_2 \left[ \sum_{m=1}^{M} (\beta_{m}^\dagger - \beta_{m})(\beta_{m+1}^\dagger + \beta_{m+1}) \right] \right\},$$

$$V_h = (\cosh h)^M \prod_{m=1}^{M} \prod_{k=1}^{M-m} \left[ 1 + \tanh^2 h(\beta_{m}^\dagger - \beta_{m})(\beta_{m+1}^\dagger + \beta_{m+1})(-1)^{\sum_{p=m+1}^{m+k} \beta_{p}^\dagger \beta_{p}} \right].$$

It is quite obvious that the operator $V_h$ (5) commutes with the operator $\hat{P} = (-1)^{\hat{S}}$, where $\hat{S} = \sum_{m=1}^{M} \beta_{m}^\dagger \beta_{m}$ is the operator of total number of fermions, and hence, the operator $V_2$ (4) also can be expressed in that form, just as it is done here, that is without term $\sim (-1)^{\hat{S}}$

unlike the representation (2).

**II. THE PARTITION FUNCTION**

The main idea of representation (4)-(5) is to consider the 2D Ising model in an external magnetic field in terms of second quantization representation, in the space of two-subscript creation and anihilation Fermi operators and after that to factorize the corresponding operators $T_{1,2}$ (see below). Namely, it was shown in the author’s [9] that the partition function of 2D IM in an external magnetic field is of the form:

$$Z_{2D}(h) = (2 \cosh h)^{NM} \left( \prod_{0<q,p<\pi} A_1^2(q) \right) \left( \prod_{0<q,p<\pi} A_2^2(p,h) \right) \langle 0 | T_2(h) T_1 | 0 \rangle,$$

where

$$T_1 = \exp \left[ \sum_{n=1}^{N} \sum_{l=1}^{N-n} \sum_{m=1}^{M} a(l) \beta_{nm}^\dagger \beta_{n+l,m}^\dagger \right], \quad a(l) = \frac{1}{N} \sum_{0<q<\pi} 2B_1(q) \sin(lq);$$

$$T_2(h) = \exp \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{M-m} b(k) \alpha_{n,m+k} \alpha_{nm} \right], \quad b(k) = \frac{1}{M} \sum_{0<p<\pi} 2B_2(p) \sin(kp),$$

and

$$A_2(p,h) = \cosh 2K_2 - \sinh 2K_2 \cos p + \alpha(h) \sinh 2K_2 \sin p,$$

$$A_1(q) = \cosh 2K_1 - \sinh 2K_1 \cos q, \quad \alpha(h) = \tanh^2 h \frac{1+\cosh 2p}{\sinh p},$$

$$B_2(p) = \frac{\alpha(h)[\cosh 2K_2 + \sinh 2K_2 \cos p] + \sinh 2K_2 \sin p}{A_2(p,h)}, \quad B_1(q) = \frac{\sinh 2K_1 \sin q}{A_1(q)}.$$  

Here $(\alpha_{nm}^\dagger, \alpha_{nm})$ and $(\beta_{nm}^\dagger, \beta_{nm})$ are the two-subscript Fermi operators, $|0\rangle$ is the fermionic vacuum function in the finite-dimensional Fock space of $2^{NM}$ dimensions in the occupation number representation. Fermi operators $\alpha$- and $\beta$- are related to each other by canonical unitary transformations $(\alpha_{nm}^\dagger \alpha_{nm} = \beta_{nm}^\dagger \beta_{nm})$.
\[ \alpha_{nm}^* (\alpha_{nm}) = \exp (i \pi \phi_{nm}) \beta_{nm} (\beta_{nm}), \]
\[ \phi_{nm} = \left[ \sum_{k=n+1}^{N} \sum_{p=1}^{M} \sum_{m=1}^{N} - \sum_{m=p+m+1}^{M-1} \right] \alpha_{kp} \alpha_{kp} \beta_{kp} \beta_{kp}. \]  
(9)

In order to transform the representation (6)-(8) into (4)-(5), one should make the following: to rewrite the operator \( T_1 \) (7) in terms of Pauli operators \((\tau_{nm}^\pm)\), to apply the well-known representation

\[ \exp(\hat{C}^2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-\xi^2 + 2\hat{C}\xi]d\xi, \]

(which is correct for the totally bounded operators), to turn Pauli operators \((\tau_{nm}^\pm)\) into to the Fermi ones \((\alpha_{nm}^\dagger, \alpha_{nm})\), then to ”drag” the emerging phase factors through \([0]\), to introduce Ising-type variables \((\mu_{nm} = \pm 1)\), to apply for the second time transfer-matrix method and finally to use the one-dimensional Jordan-Wigner transformation. Omitting the great number of of intermediate calculations, for the partition function (6) we have:

\[ Z_{2D}(h) = (\cosh h)^{NM} \cdot Tr(V_1 V_2 V_h^*)^N, \]
(10)

where operators \( V_{1,2} \) is defined by formulæ (4) and trace is calculated by means of \( \beta \)-operators, where operator \( V_h^* \) is of the form:

\[ V_h^* = \langle 0 | \exp(\sum_{m=1}^{M} \sum_{k=1}^{M-m} \alpha_{m+k} \alpha_{m} \prod_{m=1}^{M} [1 + (\alpha_{m}^\dagger - \alpha_{m})(\alpha_{m+1}^\dagger + \alpha_{m+1}) \cdot (\beta_{m}^\dagger - \beta_{m})(\beta_{m+1}^\dagger + \beta_{m+1})]) | 0 \rangle \alpha. \]
(11)

Here the single-subscript \( \alpha \) - and \( \beta \)-Fermi operators are absolutely independent ones (as they are introduced), commuting or anti-commuting with each other (the final result does not depend on the fact, as it should be expected) and the vacuum matrix element is calculated by means of \( \alpha \)-operators. Using Wick theorem, one can easily calculate the vacuum matrix element in (11) and obtain the final expression (5) for the operator \( V_h \), taking into account the factor \((\cosh h)^{NM}\) at the \( Tr(...) \) in (10). One can say, using the figure of speech that the operator \( \exp(\sum_{m=1}^{M} \sum_{k=1}^{M-m} \alpha_{m+k} \alpha_{m} \prod_{m=1}^{M} [1 + (\alpha_{m}^\dagger - \alpha_{m})(\alpha_{m+1}^\dagger + \alpha_{m+1}) \cdot (\beta_{m}^\dagger - \beta_{m})(\beta_{m+1}^\dagger + \beta_{m+1})]) \) within the brackets \( \langle 0 | ... | 0 \rangle \) acts on the operator \( \prod_{m} [...] \) just like the ”Ockham razor” does, cutting off, due to Wick theorem [10], all the superfluous terms.

From (4)-(5) one can derive some consequences, in particular, the constructive proof of well-known Lee-Yang theorem [11][12] on the absence of phase transition for the Ising model in nonzero magnetic field \( \Re h \neq 0 \). Indeed, it can be easily shown from (4)-(5) that at \( h = 0 \) we have Onsager solution and supposing \( K_1 \) to be zero \((K_1 = 0)\), we have next representation for the partition function:

\[ Z_{1D} = (2 \cosh h)^{M} \langle 0 | (V_2 V_h^*) | 0 \rangle, \]
\[ V_h^* = \exp \left[ \tanh h \sum_{m=1}^{M} \sum_{p=1}^{M-m} \beta_{m}^\dagger \beta_{m+p} \right], \]

from which we can derive exactly the classic Ising result [13].

But it is not trivial to have by means of (4)-(5) Ising solution supposing \( K_2 \) to be zero \((K_2 = 0)\). For this case operator \( V_2(K_2 = 0) = \hat{I} \) and the problem is reduced to the calculation of trace with the operator \((V_1 V_h)^N\), which is not easy. However, as it follows
from (4)-(5) by it’s construction, we have here the Ising solution also. The last one allows to prove constructively the Lee-Yang theorem for Ising model [11, 12].

First of all, it is easy to demonstrate that at small but finite external magnetic field \( (h \sim \varepsilon \ll 1) \), the expression for operator \( V_h, (5) \) can be represented as:

\[
V_h \cong \exp \left[ h^2 \sum_{m=1}^{M} \sum_{k=1}^{M-m} (\beta_m^\dagger - \beta_m)(\beta_{m+k}^\dagger + \beta_{m+k})(-1)^{\sum_{p=m+1}^k \beta_p^\dagger \beta_p} \right],
\]

with the accuracy up to \( \sim \varepsilon^4 \). It is obvious that the next Hamiltonian:

\[
\mathcal{H} = - \sum_{n=1,m=1}^{N,M} \left( J_1 \sigma_{nm} \sigma_{n+1,m} + J_2 \sigma_{nm} \sigma_{n,m+1} \right) - \frac{H^2}{k_B T} \sum_{n=1,m=1}^{N,M} \sum_{k=1}^{M-m} \sigma_{nm} \sigma_{n,m+k},
\]

corresponds to the partition function (4) with the operator \( V_h (12) \), with the accuracy up to unessential constant \( (\sim H^2/2k_B T) \), where \( T \) denotes temperature and \( k_B \) the Boltzmann constant and \( J_{1,2} \) are the interaction constants. Although the model (13) looks like asymmetric with respect to the second subscript \( m \), actually this asymmetry does not influence on the final result. It is obvious, that at \( J_1 = 0 \) the Hamiltonian (13) describes 1D Ising model at small magnetic field. However, the situation is quite different, if \( J_2 = 0 \) and we have two-dimensional Ising model with the interaction constants \( J_1 \) and \( J_2^* = H^2/k_B T \). For this model as it is stressed above, there is no phase transition at finite temperature because this model actually describes 1D Ising model in small magnetic field \( H \). It is obvious that the including an additional nearest neighbours interaction \( J_2 \neq 0 \) corresponding to 2D Ising model in a small magnetic field \( (H \sim \varepsilon \ll 1) \), does not lead to the phase transition also. The last one can be derived directly from the Hamiltonian (13). The same reasoning are also valid for the 3D Ising model in a small magnetic field; here it is also possible to introduce Hamiltonian, just like to (13) with the field-square term, using for this case the representation:

\[
Z_3(h) = (2 \cosh h)^{N M K} \langle 0 \| T_3 T_2 T_1 T_h \| 0 \rangle,
\]

\[
T_1 = \exp \left[ K_1 \sum_{n,m,k=1}^{N,M,K} (\alpha_{nmk}^\dagger - \alpha_{nmk})(\alpha_{n+1,mk}^\dagger + \alpha_{n+1,mk}) \right],
\]

\[
T_2 = \exp \left[ K_2 \sum_{n,m,k=1}^{N,M,K} (\beta_{nmk}^\dagger - \beta_{nmk})(\beta_{n,m+1,k}^\dagger + \beta_{n,m+1,k}) \right],
\]

\[
T_3 = \exp \left[ K_3 \sum_{n,m,k=1}^{N,M,K} (\theta_{nmk}^\dagger - \theta_{nmk})(\theta_{nm+k+1}^\dagger + \theta_{nm+k+1}) \right],
\]

and \( (\mu \equiv \tanh^2(h)) \)

\[
T_h = \exp \left\{ \mu \left[ \sum_{nmk}^{N,M,K} \sum_{s=1}^{N-n} \alpha_{nmk}^\dagger \alpha_{n+s,mk} + \sum_{nm'km}^{N,M,K} \sum_{l=1}^{M-m} \alpha_{nmk}^\dagger \alpha_{n',m,t,k} + \sum_{nm'm'km'}^{N,M,K} \sum_{l=1}^{K-k} \alpha_{nmk}^\dagger \alpha_{n',m',k+l} \right] \right\},
\]

for the 3D Ising model in an external magnetic field [13], where \( \alpha_{nmk}, \ldots \) are Fermi operators.

### III. CONCLUSIONS

Despite the three representations (1)-(3), (4)-(5) and (6)-(8) for the partition function of the Ising model in an external magnetic field include "unpleasant" phase factors, the
representation (4)-(5) has some advantages comparing to the (1)-(3) and might be to (6)-(8) ones. To the author’s mind, the application of well-known approximate methods (analytical as well as numerical) to the Hamiltonian (13) can lead to the much better results than it took place in case of standard Hamiltonian of Ising model in an external magnetic field (with linear-field term). It is also possible that the application of the direct Onsager method [1, 2] to the operator $V ≡ V_1V_2V_h$ (4)-(5) could allow to find out the eigenfunctions and eigenvalues of that operator.

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