Aggregation of weakly dependent doubly stochastic processes

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Abstract

The aim of this paper is to extend the aggregation convergence results given in [1, 2] to doubly stochastic linear and nonlinear processes with weakly dependent innovations. First, we introduce a weak dependence notion for doubly stochastic processes, based in the weak dependence definition given by Doukhan and Louhichi, [6], and we exhibit several models satisfying this notion, such as: doubly stochastic Volterra processes and doubly stochastic Bernoulli scheme with weakly dependent innovations. Afterwards we derive a central limit theorem for the partial aggregation sequence considering weakly dependent doubly stochastic processes. Finally, show a new SLLN for the covariance function of the partial aggregation process in the case of doubly stochastic Volterra processes with interactive innovations.

Keywords: Aggregation, weak dependence, doubly stochastic processes, Volterra processes, Bernoulli shift, TCL, SLLN.

1 Introduction

The aggregation of doubly stochastic processes with interactive innovations has been little studied, because dependent innovations induce a dependency structure for the doubly stochastic processes which can be difficult to study, especially in the case of nonlinear processes.

In the literature for aggregation of linear process, see [12, 13, 19, 20, 28], usually introduces interaction between individual innovations $\varepsilon_{it}$ considering that this can be decomposed into a common innovation plus an independent innovation, i.e. $\varepsilon_{it} = u_{it} + \xi_{it}$. So, under the presence of common innovation, the aggregation process keeps only the structure given by the common component, whereas the aggregation of independent components are washed out by aggregation, i.e. the convergence results obtained are the same that considering only the common innovation component.

The immediate question is: it is possible to consider another kind of interactive innovation that allows to obtain different results to the cases of independent innovations and common innovation? We answer this question affirmatively, in [1] we have studied the aggregation of doubly stochastic gaussian processes and we have approached the problem in a different way. We introduce interaction between elementary processes “living at i” starting from interaction between innovations as $\mathbb{E}[\varepsilon_{it}'\varepsilon_{jt}'] = \chi(i-j)$, where $\chi$ is a given covariance function. Thus, the common innovation case is given by $\chi(j) = 1$, for all $j$, and the independent innovations case by $\chi(j) = 0$, for $j \neq 0$ and $\chi(0) = 1$. The procedure includes naturally the aforementioned case. Hence we obtain some interesting qualitative behavior of the aggregation process.

Some specific results already exist for aggregation of nonlinear processes. Several authors have investigated the aggregation of ARCH and GARCH processes, see [4, 15, 18, 23, 29]. All these works have been developed considering exclusively common innovation or independent innovations. As far as we know does not exist literature about the aggregation of nonlinear processes with interactive innovations.

Our purpose is to extend the convergence results of the aggregation procedure, given in [1], for gaussian elementary processes with interactive innovations to case of nongaussian processes with
weakly dependent innovations, our results include elementary doubly stochastic processes such as: nongaussian linear processes, nonlinear Bernoulli shifts, Volterra processes, ARCH, GARCH, LARCH and bilinear models.

We consider a sequence of stationary doubly stochastic processes $Z = \{Z^i : i \in \mathbb{N}\}$ on $(\Omega, \mathcal{A}, P) \times (\Omega', \mathcal{A}', P')$ in $\mathbb{R}^2$ by

$$Z^i = \{Z^i_t (y^i(\omega), \epsilon^i(\omega')) : t \in \mathbb{Z}\},$$

where $Y = \{y^i : i \in \mathbb{Z}\}$ a sequence of random variables defined on $(\Omega, \mathcal{A}, P)$, with distribution $\mu$ on $\mathbb{R}^s$ and $\epsilon = \{\epsilon^i : i, t \in \mathbb{Z}\}$ be a doubly indexed process defined on $(\Omega', \mathcal{A}', P')$ satisfying the following assumption

**Assumption A1:**

1. $\mathcal{E}$ is an array of strong white noises, i.e. for each $i$, $\epsilon^i = \{\epsilon^i_t\}$ is an i.i.d sequence.
2. $Y$ is an i.i.d sequence with distribution $\nu = \mu^{\otimes \mathbb{N}}$.
3. $Y$ is independent of $\mathcal{E}$.

Here, the sequence $Y$ is considered as the random environment model and $\mathcal{E} = \{\epsilon^i : i \in \mathbb{Z}\}$ is the sequence of innovations $\epsilon^i = \{\epsilon^i_t : t \in \mathbb{Z}\}$. For simplicity, throughout the work $Z^i_t := Z_i(y^i_t, \epsilon^i_t)$, and we use $Z_t(y^i, \epsilon^i)$ only if we want to state explicitly the dependence on $y^i$ and $\epsilon^i$.

For every trajectory fixed $Y$, we define $X^N(Y) = \{X^N_t(Y) : t \in \mathbb{Z}\}$ as the partial aggregation of the elementary processes $\{Z^i\}$ by

$$X^N_t(Y) = \frac{1}{B_N} \sum_{i=1}^N Z_t(y^i_t, \epsilon^i_t),$$

where $B_N$ is an appropriate normalization sequence.

We study, for almost every fixed trajectory $Y$, the convergence of $X^N(Y)$ to some process $X$, called the aggregation process.

We introduce a new notion of weak dependence, for a sequence of doubly stochastic processes. This notion is based in the weak dependence definition given by Doukhan and Louhichi, see [6].

We give several models of doubly stochastic processes satisfying the new notion of weak dependence. To do that we consider innovations defined in such a form that, for each $t$, $\{\epsilon^i_t : i \in \mathbb{Z}\}$ is a weakly dependent stationary sequence in the index $i$, see [9]. The notion of weak dependence for the innovations sequence $\{\epsilon^i\}$ makes explicit the asymptotic independence between $(\epsilon^i_{t_1}, \ldots, \epsilon^i_{t_u})$ and $(\epsilon^j_{i_1}, \ldots, \epsilon^j_{i_v})$ when $i_1 < \ldots < i_u < i_1 + r \leq j_1 < \ldots < j_v$ and $r$ tends to infinity.

Considering several types of weak dependence already used in the literature, we show that the weak dependence property of innovations is transferred to the sequence $\{Z^i\}$ of doubly stochastic elementary processes. In fact we will prove that $Z = \{Z^i\}$ satisfies the new notion of weak dependence. This result will be proved for orthogonal expansion Volterra processes such as: linear processes, ARCH, GARCH, LARCH and bilinear models. We will also show this transference property for doubly stochastic uniform Lipschitz Bernoulli shift processes. For instance, Lipschitz functions of linear processes.

Let us suppose that elementary process $Z^i$ satisfies the following moment condition

$$K_{2\delta} \quad \mathbb{E}[|Z^i|^2] < \infty, \text{ with } \delta > 0.$$

This conditions yields the existence of $Z^i$ in $L^2(\Omega \times \Omega')$. We prove that under the weak dependence property, the interaction $\chi$, defined by $\mathbb{E}[\epsilon^i_t \epsilon^j_t] = \chi(i - j)$, is always a weak interaction in $\ell^1$. So $\chi$ has a finite limit in the Cesaro sense.

In general, we assume that the covariance function $\Gamma^N(Y)$ of partial aggregation process $X^N(Y)$ satisfies

$$K_4 \quad \Gamma^N(t, Y) \text{ converges } \nu - a.s. \text{ to } \Gamma(t), \text{ for all } t \in \mathbb{Z}.$$
Thus, assuming that the doubly stochastic processes satisfy this new notion of weak dependence and moreover the conditions K2 and K4, we prove a Central Limit Theorem $\nu - a.s.$ for the partial aggregation $X^N(Y)$. This proof is done directly on the structure of elementary doubly stochastic processes and not as in $[2]$, for the aggregation of doubly stochastic linear process, which is done on the structure of innovations. This result includes some cases that have not been addressed in the literature. We can cite to illustrate the cases of nonlinear processes with weakly dependent innovations, such as general doubly stochastic orthogonal expansion Volterra processes and doubly stochastic uniform Lipchitz Bernoulli shift processes.

Finally, we show a new SLLN for the covariance function $\Gamma^N(Y)$ in the case of doubly stochastic orthogonal expansion Volterra processes, $DSV^*$ processes, with interactive innovations.

This paper is organized as follows. In Section 2 we present some doubly stochastic Bernoulli shift processes and we give the necessary and sufficient condition to obtain their existence in $L^2(\Omega \times \Omega')$. In Section 3 we introduce the notion of weak dependence for doubly stochastic processes. In Section 4 we present several examples of weakly dependent innovations and we show that considering different weakly dependent innovation models we can obtain weakly dependent doubly stochastic models. In Section 5 we give the CLT $\nu - a.s.$ for general sequence of weakly dependent doubly stochastic processes. In Section 6 we give a SLLN for $\Gamma^N$ in the case of $DSV^*$ processes. Proofs are given in Section 7, we prove the results relating to the transfer of weak dependence property to $Z$, we develop the standard Linderberg method with Bernstein’s block and yield the CLT $\nu - a.s.$.

Finally, we prove the SLLN for $\Gamma^N(Y)$.

# 2 Some doubly stochastic Bernoulli shift processes

In this section, we describe doubly stochastic versions of several models used in statistics, econometrics and finance. We consider some stationary doubly stochastic elementary processes generated by a random parameter $y$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $\mu$ on $\mathbb{R}^s$, and a strong white noise $\varepsilon = \{\varepsilon_t : t \in \mathbb{Z}\}$ on $(\Omega', \mathcal{A}', \mathbb{P}')$; i.e. doubly stochastic processes defined on $(\Omega, \mathcal{A}, \mathbb{P}) \times (\Omega', \mathcal{A}', \mathbb{P}')$ in $\mathbb{R}^2$ of the form [1]

We introduce the doubly stochastic Bernoulli shifts, $DSBS$, a very broad class of models that contains the major part of processes derived from a stationary sequence. We define the $DSBS$ process in the following way, see [3].

**Definition 1.** Let $\mathcal{H} : \mathbb{R}^s \times \mathbb{R}^Z \to \mathbb{R}$ be a measurable function. We define a $DSBS$ process $Z(y, \varepsilon)$ with random parameter $y$ in $\mathbb{R}^s$ and innovation $\varepsilon = \{\varepsilon_k : k \in \mathbb{Z}\}$ by

$$Z_t(y, \varepsilon) = \mathcal{H}(y, \{\varepsilon_{t-k}\}_{k \in \mathbb{Z}}).$$

(3)

Since $\mathcal{H}$ is a function depending on an infinite number of arguments, it is generally given in term of a series defined in $L^2(\Omega \otimes \Omega')$. In order to define $[3]$ in a general setting, we denote for any subset $J \subset \mathbb{Z}$:

$$\mathcal{H}_J := \mathcal{H}(y, \{\varepsilon_j\}_{j \in J}) = \mathcal{H}(y, \{\varepsilon_j 1_{J}(j)\}_{j \in \mathbb{Z}}).$$

For finite subsets $J$ this expression is generally well defined and simple to handle. In order to define such model in $L^2(\Omega \otimes \Omega')$ we assume that the function $\mathcal{H}$ is such that

$$E[|\mathcal{H}(y, \varepsilon)|^2] < \infty.$$  

(4)

The $DSBS$ process $Z(y, \varepsilon) = \{Z_t(y, \varepsilon)\}$ is stationary, $\mu - a.s.$ Then, condition (4) implies that $Z(y, \varepsilon)$ is on $L^2(\Omega \otimes \Omega')$, which implies its existence, $\mu - a.s.$, in $L^2(\Omega')$.

As we will see in the following, most of the usually used stochastic processes can be represented as Bernoulli shifts.

As an example we consider doubly stochastic uniform Lipschitz Bernoulli shift ($DSULBS$) processes. We also study doubly stochastic Volterra processes, which correspond to random parameters Volterra processes and we detail the doubly stochastic bilinear models given afterwards.
their expression as a Volterra’s expansion. Finally, we present several well-known doubly stochastic bilinear models such as LARCH, GARCH and ARCH processes with random coefficients.

We will discuss the necessary and sufficient conditions for the existence a.s. in $L^2(\Omega \otimes \Omega')$ of these models. Those conditions will be very important for studying the convergence of the aggregation procedure.

### 2.1 Doubly stochastic uniform Lipschitz Bernoulli shifts

Let $\mathcal{H} : \mathbb{R}^s \times \mathbb{R}^z \to \mathbb{R}$ be a function such that if $\varepsilon, \varepsilon' \in \mathbb{R}^2$ coincide for all indexes but one, let say $k_0 \in \mathbb{Z}$, then there exists $a_{k_0}(y)$ $\mu$-a.s. such that

$$|\mathcal{H}(y, \{\varepsilon_k\}_{k \in \mathbb{Z}}) - \mathcal{H}(y, \{\varepsilon'_k\}_{k \in \mathbb{Z}})| \leq |a_{k_0}(y)||\varepsilon_{k_0} - \varepsilon'_{k_0}|.$$  \hspace{1cm} (5)

Thus, there exists a sequence $a(y) = \{a_k(y) : k \in \mathbb{Z}\}$, such that $\mathcal{H}$ satisfies $\mu$-a.s., for all $\varepsilon, \varepsilon' \in \mathbb{R}^2$, the following regularity condition

$$|\mathcal{H}(y, \{\varepsilon_k\}_{k \in \mathbb{Z}}) - \mathcal{H}(y, \{\varepsilon'_k\}_{k \in \mathbb{Z}})| \leq \sum_{k \in \mathbb{Z}} |a_k(y)||\varepsilon_k - \varepsilon'_k|.$$ \hspace{1cm} (6)

We say that $\mathcal{H}$ is a uniform Lipschitz Bernoulli shift with respect the coordinates $\varepsilon$.

Moreover supposing the following

$$\int \|a(y)\|\ell_1 \mu(dy) < \infty,$$ \hspace{1cm} (7)

where $\|a(y)\|\ell_1 = \sum_{k |a_k(y)|}$. Condition (7) implies that $a(y) \in \ell_1$, $\mu$-a.s.

Then, $Z(y, \varepsilon) = \{Z_t(y, \varepsilon) : t \in \mathbb{Z}\}$ given by

$$Z_t = \mathcal{H}(y^i, \{\varepsilon_{t-k}\}_{k \in \mathbb{Z}}).$$

is defined as a doubly stochastic uniform Lipschitz Bernoulli shifts (DSULBS) process.

Simple examples of this situation is Lipschitz function of linear processes. Another example of this situation is the following stationary doubly stochastic process

$$Z_t(y, \varepsilon) = \varepsilon_t \left( c_0(y) + \sum_{k \neq 0} c_k(y)\varepsilon_{t-k} \right),$$

where the innovation $\varepsilon_t$ is bounded and $c(y) = \{c_k(y) : k \in \mathbb{Z}\}$ is such that $\mathbb{E}[\|c(y)\|\ell_1] < \infty$. In this case, condition (7) also holds with $|a_k(y)| = 2\|\varepsilon_0\|\ell_\infty^2|c_k(y)|$.

### 2.2 Doubly stochastic Volterra processes

A doubly stochastic Volterra (DSV) process is defined through a convergent Volterra expansion with random coefficients

$$Z_t(y, \varepsilon) = \sum_{k=0}^{\infty} \sum_{l_1, \ldots, l_k \in \mathbb{Z}^k} c_{k:l_1, \ldots, l_k}(y)\varepsilon_{t-l_1} \cdots \varepsilon_{t-l_k},$$ \hspace{1cm} (8)

where $\varepsilon$ is a sequence of innovation and $c(y) = \{c_{k:l_1, \ldots, l_k}(y) : (l_1, \ldots, l_k) \in \mathbb{Z}^k, k \in \mathbb{Z}\}$ is a sequence of random coefficients.

We assume that $\varepsilon$ is a strong white noise such that, for all $k \in \mathbb{N}$, $\mathbb{E}[|\varepsilon_t|^{2k}] < \infty$. Let us note $\ell_\varepsilon$ the space of sequence $c = \{c_{k:l_1, \ldots, l_k} : (l_1, \ldots, l_k) \in \mathbb{Z}^k, k \in \mathbb{Z}\}$ such that

$$\|c\|\ell_\varepsilon = \sum_{k=0}^{\infty} \mathbb{E}[|\varepsilon_t|^{2k}] \sum_{l_1, \ldots, l_k} |c_{k:l_1, \ldots, l_k}|^2 < \infty.$$ \hspace{1cm} (9)
If $c(y)$ satisfies condition $\mathcal{I}$, $\mu - a.s.$, this entails the $\mu - a.s.$ convergence in $L^2(\Omega')$ of the series $\mathfrak{S}$. Since $\varepsilon$ is a strong white noise a sufficient condition for the convergence in $L^2(\Omega \times \Omega')$ of the series $\mathfrak{S}$ is

\[ \mathbf{C2} \quad \mathbb{E}[|c(y)|^2_\xi] < \infty. \]

In order to obtain condition C2 it is necessary the existence of all moments of $\varepsilon$, hence is sufficient the existence of Laplace transform of distribution $\mathbb{P}_\varepsilon$. This holds, for instance, if $\varepsilon$ is a sub-gaussian process.

Remark 1. We note that the DSV process correspond to a DSBS process in the case of chaotic expansion associated with the discrete chaos generated by the innovation $\varepsilon$, $[20, 22]$. We can consider that, for each $y$ fixed, $\mathcal{H}(y, \cdot)$ is expanded, in $L^2$ sense, into this chaos as

\[ \mathcal{H}(y, \varepsilon) = \sum_{k=0}^{\infty} \mathcal{H}^{(k)}(y, \varepsilon), \]

where $\mathcal{H}^{(k)}(y, \varepsilon)$ denotes the $k$-th order chaotic contribution, $\mathcal{H}^{(0)} = c_0(y)$ and

\[ \mathcal{H}^{(k)}(y, \varepsilon) = \sum_{l_1, \ldots, l_k \in \mathbb{Z}^k} c_{k;1, \ldots, l_k}(y) \varepsilon_{l_1} \cdots \varepsilon_{l_k}. \]

The process $\mathcal{H}^{(k)}(y, \{\varepsilon_{l-j}\}_{j \in \mathbb{Z}})$ is called the DSV process of order $k$.

The most simple DSV process are the linear processes with random coefficients, defined by

\[ Z_t(y, \varepsilon) = \sum_{k \in \mathbb{Z}} c_k(y) \varepsilon_{t-k}. \tag{10} \]

These linear processes are DSV processes of order one.

An interesting particular case of Volterra processes is obtained when the chaotic expansion is given by

\[ Z_t(y, \varepsilon) = \sum_{k=0}^{\infty} \sum_{l_1, \ldots, l_k \in \mathbb{Z}^k} c_{k;1, \ldots, l_k}(y) \varepsilon_{l_1-1} \cdots \varepsilon_{l_k-1}. \tag{11} \]

The corresponding $k$-th order Volterra processes are pairwise orthogonal. In the following, we will refereer this type of doubly stochastic Volterra processes with orthogonal expansion by $\text{DSV}^*$ processes. Since $Z_t(y, \varepsilon)$ is a sum of orthogonal terms, if the innovations satisfies $\mathbb{E}[|\varepsilon_t|^2] = 1$, the series $\mathbf{II}$ is $\mu - a.s.$ convergent in $L^2(\Omega')$ if and only if

\[ \|c(y)\|^2_2 = \sum_{k=0}^{\infty} \sum_{l_1, \ldots, l_k \in \mathbb{Z}^k} |c_{k;1, \ldots, l_k}(y)|^2 < \infty, \quad \mu - a.s. \tag{12} \]

In this case, we put $\ell_\ast = \ell_2$, the space of sequence $c$ such that $\|c\|_{\ell_\ast} = \|c\|_2 < \infty$. Then, we have that $Z(y, \varepsilon)$ belongs to $L^2(\Omega \otimes \Omega')$ if and only if condition C2 is satisfied.

The difference with the general case is that the indexes in the product are all different. So that, in the general case, we do not get the orthogonality property, which is a disadvantage, see $[5]$.

Nevertheless, the i.i.d sequence $\varepsilon = \{\varepsilon_t : t \in \mathbb{Z}\}$ can be replaced by a vector-valued i.i.d sequence $\{\left(\varepsilon_t^{(1,k)}, \ldots, \varepsilon_t^{(k,k)}\right) : t \in \mathbb{Z}\}$, such that $\{\varepsilon^{(k,l)} : l \leq k\}$ are mutually orthogonal. This is obtained by replacing each power of an innovation variable with its decomposition on the Appell polynomials of the distribution of $\varepsilon_0$, see e.g. $[5]$.

Then we can rewrite the process, under condition C2, as a sum of orthogonal terms given by

\[ Z_t(y, \varepsilon) = \sum_{k=0}^{\infty} \sum_{l_1, \ldots, l_k \in \mathbb{Z}^k} c_{k;1, \ldots, l_k}(y) \varepsilon_{l_1-1}^{(1,1)} \cdots \varepsilon_{l_k-1}^{(k,k)}. \tag{13} \]

In the following section, we present some examples of $\text{DSV}^*$ processes. These models are a natural extension of models that are well known in the literature and have many sorts of applications.
2.3 Doubly stochastic bilinear models

A vast literature is devoted in order to study the conditionally heteroscedastic models. The heteroscedasticity property is characterized by the fact that the conditional variance \( \text{var}(Z_t|I_{t-1}) \), given an information set \( I_{t-1} \), is not a constant.

The bilinear models allows unify the treatment of heteroscedastic models. These models are defined by the relation

\[
Z_t(y, \varepsilon) = \left( a_0(y) + \sum_{k=1}^{\infty} a_k(y)Z_{t-k} \right) + \left( b_0(y) + \sum_{k=1}^{\infty} b_k(y)Z_{t-k} \right) \varepsilon_t. \tag{14}
\]

where \( \varepsilon = \{ \varepsilon_t \} \) is a sequence of i.i.d. centered random variables such that \( \mathbb{E}[|\varepsilon_t|^2] = 1 \), and \( a(y) = \{a_k(y)\} \), \( b(y) = \{b_k(y)\} \) are real random coefficients, not necessary nonnegative.

This models appears naturally when studying the class of processes with the property that the conditional mean \( m_t(y) = \mathbb{E}[y|Z_t|Z_k, \ k < t] \) is a linear combination of \( Z_k \), \( k < t \), and the conditional variance \( \sigma^2_t(y) = \text{var}(Z_t|Z_k, \ k < t) \) is the square of a linear combinations of \( Z_k \), for \( k < t \), as it is in the case of \( \text{ARCH} \).

\[
\sigma^2_t(y) = \left( b_0(y) + \sum_{k=1}^{\infty} b_k(y)Z_{t-k} \right)^2 \quad \text{and} \quad m_t(y) = \left( a_0(y) + \sum_{k=1}^{\infty} a_k(y)Z_{t-k} \right). \tag{15}
\]

Let us take

\[
a(s, y) = \sum_{k=1}^{\infty} a_k(y)s^k, \quad g(s, y) = (1 - b(s, y))^{-1} = \sum_{k=0}^{\infty} g_k(y)s^k, \quad h(s, y) = a(s, y)(1 - b(s, y))^{-1} = \sum_{k=0}^{\infty} h_k(y)s^k.
\]

If we suppose that \( H(y) = \sum_{l=1}^{\infty} h_l^2(y) < 1 \mu-a.s. \), then there is, \( \mu-a.s. \), a unique second order stationary solution given by

\[
Z_t(y, \varepsilon) = b_0(y) + \sum_{k=0}^{\infty} \sum_{l_1, \ldots, l_k \leq t \leq l_1, \ldots, l_k} g_{l_1}(y)h_{l_2-l_1}(y) \ldots h_{l_{k}-l_{k-1}-1}(y)\varepsilon_{l-1} \ldots \varepsilon_{l-k}. \tag{16}
\]

Then, \( Z(y, \varepsilon) \) is a \( \text{DSV}^* \) process with random coefficients

\[\begin{align*}
c_0(y) &= b_0(y) \quad \text{and} \quad c_{k; l_1, \ldots, l_k}(y) = g_{l_1}(y)h_{l_2-l_1}(y) \ldots h_{l_{k}-l_{k-1}-1}(y).
\end{align*}\]

We present the necessary and sufficient conditions for the existence in \( L^2 \) of process \( Z(y, \varepsilon) \) defined by (14), see (14).

Let \( G(y) = \sum_{l=0}^{\infty} g_l^2(y) \). If \( b_0(y) < \infty \) and \( H(y) < 1 \mu-a.s. \) we obtain

\[
\|c(y)\|^2 = b_0^2(y) \sum_{k=1}^{\infty} \sum_{0 \leq l_1 < \ldots < l_k} g_{l_1}^2(y)h_{l_2-l_1}^2(y) \ldots h_{l_{k}-l_{k-1}-1}^2(y) = \frac{b_0^2(y)G(y)}{1 - H(y)}. \tag{16}
\]

Then, from (16), we obtain that the following conditions

\[
H(y) < 1 \quad \mu-a.s. \quad \text{and} \quad \mathbb{E} \left[ \frac{b_0^2(y)G(y)}{1 - H(y)} \right] < \infty. \tag{17}
\]

are necessary and sufficient for the existence of a stationary \( L^2 \)-solution to (14). This solution \( Z(y, \varepsilon) \) is given by (15).

Formally the classes \( \text{AR}, \text{ARMA}, \text{ARCH}, \text{GARCH}, \text{LARCH} \) with random coefficients all belong to the class of doubly stochastic bilinear models. In the follows we details the necessary and sufficient condition for the existence of \( L^2 \)-solutions for some of those classes.
2.3.1 Doubly stochastic LARCH(∞) processes

We consider the doubly stochastic LARCH(∞) models given by

\[ Z_t = \left( b_0(y) + \sum_{k=1}^\infty b_k(y)Z_{t-k} \right) \varepsilon_t. \]  

(18)

with real value random coefficients, not necessary nonnegatives, and \( \varepsilon = \{\varepsilon_t\} \) a white noise. The LARCH model with deterministic coefficients has been introduced by Robinson, see [25]. For a vectorial version of this model, see [8].

Let us denote \( B_2(y) = \sum_{k=1}^\infty b_k^2(y) \) and \( \mathbb{E}[|\varepsilon|^2] = 1 \). Following the same way that for bilinear models we obtain that conditions

\[ B_2(y) < 1 \quad \mu - a.s. \quad \text{and} \quad \mathbb{E} \left[ \frac{b_0^2(y)}{1 - B_2(y)} \right] < \infty \]

(19)

are necessary and sufficient for the existence of a stationary \( L^2 \)-solution to (18).

The stationary solution of equation (18) has a orthogonal Volterra expansion given by equation

\[ Z_t(y, \varepsilon) = \sum_{k=0}^\infty \sum_{0 < l_1 < \ldots < l_k} b_0(y)b_{l_1}(y)b_{l_2-1}(y) \ldots b_{l_k-l_{k-1}}(y)\varepsilon_t\varepsilon_{t-l_1} \ldots \varepsilon_{t-l_k}. \]

(20)

2.3.2 Doubly stochastic ARCH(∞) processes

Here we consider the random coefficients nonnegative ARCH model defined as follow. A process \( Z(y, \varepsilon) = \{Z_t(y, \varepsilon) : t \in \mathbb{Z}\} \) is said to satisfy random coefficient ARCH(∞) equations if there exist a nonnegative i.i.d. innovation sequence \( \varepsilon = \{\varepsilon_t : t \in \mathbb{Z}\} \) and nonnegative random variables \( \{b_k(y)\} \) independent of \( \varepsilon \) such that

\[ Z_t = \left( b_0(y) + \sum_{k=1}^\infty b_k(y)Z_{t-k} \right) \varepsilon_t \quad a.s. \]

(21)

Note that the random coefficient ARCH process given by equation (21) is a nonergodic process.

Classically, ARCH process we mean the model where the returns \( r_t \) admit a representation of the form

\[ r_t = \sigma_t \xi_t, \quad \sigma_t^2 = b_0 + \sum_{k=1}^\infty b_k r_{t-k}^2. \]

(22)

where \( \xi = \{\xi_t\} \) is a sequence of i.i.d random variables with zero mean and finite variance, and \( \sigma_t^2 \) is a linear combination of the squares of past returns. The GARCH(\( p,q \)) model is defined by

\[ r_t = \sigma_t \xi_t, \quad \sigma_t^2 = \alpha_0 + \sum_{k=1}^p \beta_k \sigma_{t-k}^2 + \sum_{k=1}^q \alpha_k r_{t-k}^2. \]

(23)

Under some restrictions on the polynomials \( \alpha(z) = \sum_{k=1}^q \alpha_k z^k \) and \( \beta(z) = \sum_{k=1}^p \beta_k z^k \), the model can be rewritten in the form (22).

Then, denoting \( Z_t = r_t^2 \) and \( \xi_t = \xi_t^2 \), we represent this model in the ARCH form (21), see [25].

In the case of nonrandom coefficients, those models are introduced by [25] and subsequently studied in [10, 14, 16]. For the case of random coefficients see [14, 17].

Let \( \lambda_1 = \mathbb{E}[\varepsilon_1], \lambda_2 = \mathbb{E}[\varepsilon_1^2] \) and \( B(y) = \sum_{k=1}^\infty b_k(y) \). The recursion relation (21) yields the following Volterra series expansion of \( Z(y, \varepsilon) \):

\[ Z_t(y, \varepsilon) = \sum_{k=0}^\infty \sum_{0 < l_1 < \ldots < l_k} b_0(y)b_{l_1}(y)b_{l_2-1}(y) \ldots b_{l_k-l_{k-1}}(y)\varepsilon_t\varepsilon_{t-l_1} \ldots \varepsilon_{t-l_k}. \]

(24)
In order to obtain the necessary and sufficient conditions for the existence in $L^2$ of process $Z(y, \varepsilon)$ defined by (21), one needs to study orthogonal Volterra representation of $Z(y, \varepsilon)$. This orthogonal representation is obtained by replacing the $\varepsilon_t$’s by $\varepsilon_t = \lambda_1(\kappa \tilde{\varepsilon}_t + 1)$, where $\kappa^2 = (\lambda_2 - \lambda_1^2)/\lambda_1^2$ and $\tilde{\varepsilon}_t$ have zero mean and unit variance. We can rewrite (21) of the bilinear form, see [11],

$$Z_t(y, \varepsilon) = \left( \lambda_1 b_0(y) + \sum_{k=1}^{\infty} \lambda_1 b_k(y) Z_{t-k} \right) + \left( \kappa \lambda_1 b_0(y) + \sum_{k=1}^{\infty} \kappa \lambda_1 b_k(y) Z_{t-k} \right) \tilde{\varepsilon}_t.$$ 

Then, similarly to case of bilinear models, we obtain that $Z(y, \varepsilon)$ the following DSV* representation where

$$Z_t(y, \varepsilon) = \frac{\lambda_1 b_0(y)}{1 - \lambda_1 B(y)} \left( 1 + \sum_{k=0}^{\infty} \sum_{0 \leq t_1 < \ldots < t_k} h_{l_1}(y) h_{l_2-t_1} \ldots h_{l_k-l_{k-1}}(y) \tilde{\varepsilon}_{t-l_1} \ldots \tilde{\varepsilon}_{t-l_k} \right),$$

with $g_0 = 1,$

$$\sum_{k=0}^{\infty} g_k(y) s^k = \left( 1 - \lambda_1 \sum_{k=1}^{\infty} b_k(y) s^k \right)^{-1},$$

and $h_k(y) = \kappa g_k(y)$, for $k \geq 1$. More explicitly,

$$g_l(y) = \sum_{k=1}^{l} \lambda_1^k \sum_{0 \leq j_1 < \ldots < j_{k-1} < l} b_{j_1}(y) b_{j_2-j_1} \ldots b_{l-j_{k-1}}(y).$$

Let $H(y) = \sum_{l=1}^{\infty} h_l^2(y).$ Thus, we obtain that the following conditions

$$\lambda_1 B(y) < 1, \quad H(y) < 1 \quad \mu - a.s. \quad (25)$$

$$E \left[ \frac{b_0^2(y)}{(1 - \lambda_1 B(y))^2(1 - H(y))} \right] < \infty. \quad (26)$$

are necessary and sufficient for the existence of a stationary $L^2$-solution to (21).

Now, we present an examples of ARCH models.

**Example 1** (Doubly stochastic GARCH(1,1) processes). Let us consider the random parameters GARCH$(1,1)$ model given by

$$r_t(y) = \sigma_t(y) \xi_t, \quad \sigma_t^2(y) = \alpha_0(y) + \alpha(y) \sigma_{t-1}(y) + \beta(y) r_{t-1}^2(y),$$

where $\alpha_0(y), \alpha(y)$ and $\beta(y)$ are nonnegative random variables independent of $\{\xi_t\}$. We note that the corresponding ARCH$(\infty)$ equation (22) is obtained taking the random parameters as

$$b_0(y) = \frac{\alpha_0(y)}{1 - \beta(y)} \quad \text{and} \quad b_k(y) = \alpha(y) \beta^{k-1}(y).$$

Let $g(y) = \lambda_1 \alpha(y) + \beta(y)$. In this case $h_k(y) = \lambda_1 \kappa \alpha(y) \beta^{k-1}$, then

$$B(y) = \frac{\alpha(y)}{1 - \beta(y)} \quad \text{and} \quad H(y) = \frac{\lambda_1^2 \kappa^2 \alpha^2(y)}{1 - g^2(y)}.$$ 

So, $Z(y, \varepsilon)$ exist in $L^2$ if and only if

$$g^2(y) + \lambda_1^2 \kappa^2 \alpha^2(y) < 1 \quad \mu - a.s. \quad (27)$$

$$E \left[ \frac{\alpha_0^2(y)}{(1 - g^2)(1 - g^2(y) - \lambda_1^2 \kappa^2 \alpha(y)^2)} \right] < \infty. \quad (28)$$
Example 2 (Doubly stochastic ARCH(1) processes). We consider the random parameters ARCH(1) model given by
\[ r_i(y) = \kappa_i(y)\xi_i, \quad \kappa_i^2(y) = \alpha_0(y) + \alpha(y)\kappa_{i-1}^2(y), \]
where \( \alpha_0(y) \) and \( \alpha(y) \) are nonnegative random variables independent of \( \{\xi_i\} \). We note that the corresponding ARCH(1) process is a particular case of the GARCH(1,1) process, when we take the parameter \( \beta(y) = 0 \). Thus, \( b_0(y) = \alpha_0(y) \), \( b_1(y) = \alpha(y) \) and \( b_k(y) = 0 \) for \( k > 1 \). In this case \( h_k(y) = \kappa(\lambda_1\lambda_0)(y)^{k-1} \). Then \( Z(y,\varepsilon) \) exists in \( L^2 \) if and only if
\[ \sqrt{\lambda_2\alpha(y)} < 1 \quad \mu - \text{a.s.} \quad \text{E} \left[ \frac{\alpha_0^2(y)}{\lambda_1\alpha(y)^2(1 - \lambda_2\alpha^2(y))} \right] < \infty. \]

3 Weak dependence for doubly stochastic processes

In this section we introduce a notion of weak dependence for sequence of doubly stochastic processes following the definition of weak dependence given by Doukhan and Louhichi, see \[6\].

Let \( \Delta^{(k)} \) be the set of bounded Lipschitz functions \( f \) defined on \( \mathbb{R}^k \) such that for all \( (z_1, \ldots, z_k) \) and \( (z_1', \ldots, z_k') \) in \( \mathbb{R}^k \)
\[ |f(z_1, \ldots, z_k) - f(z_1', \ldots, z_k')| \leq \text{Lip}(f) \sum_{m=1}^{k} |z_m - z_m'|. \]

Above we denote the Lipschitz constant of \( f \) by \( \text{Lip}(f) \). Let \( \Delta^{(k)}_1 \) be the set of function \( f \) in \( \Delta^{(k)} \) such that \( \|f\|_\infty \leq 1 \).

Let \( \{Z_t\} \) be a real-valued stationary process.

Definition 2 (Doukhan and Louhichi). The process \( \{Z_t : i \in \mathbb{Z}\} \) is \((\epsilon, \psi)\)-weakly dependent if there exist a sequence \( \epsilon(r) \) decreasing to zero at infinity and a function \( \psi \) from \( \mathbb{N}^2 \times (\mathbb{R}^+)^2 \) to \( \mathbb{R}^+ \) such that
\[ |\text{cov}(f(Z_{i_1}, \ldots, Z_{i_u}), g(Z_{j_1}, \ldots, Z_{j_v}))| \leq \psi(u, v, \text{Lip}(f), \text{Lip}(g))\epsilon(r), \]
for any \( r \geq 0 \) and any \((u + v)\)-tuples such that \( i_1 < \ldots < i_u \leq i_u + r < j_1 < \ldots < j_v \), where \( (f, g) \in \Delta_1^{(u)} \times \Delta_1^{(v)} \).

We introduce next \( \epsilon \) the dependence coefficient.

Remark 2. The \( \epsilon \)-coefficients depend on the form of function \( \psi \). Let \( \mathcal{J}(u, v, r) \) the set of \((u + v)\)-tuples such that \( i_1 < \ldots < i_u \leq i_u + r < j_1 < \ldots < j_v \). Given a function \( \psi \) from \( \mathbb{N}^2 \times (\mathbb{R}^+)^2 \) to \( \mathbb{R}^+ \), the \( \epsilon \)-coefficient associated to \( \psi \) is defined by
\[ \epsilon(r) = \sup_{u, v} \sup_{f \in \mathcal{J}(u, v, r)} \frac{\sup_{(f, g) \in \Delta_1^{(u)} \times \Delta_1^{(v)}} |\text{cov}(f(Z_{i_1}, \ldots, Z_{i_u}), g(Z_{j_1}, \ldots, Z_{j_v}))|}{\psi(u, v, \text{Lip}(f), \text{Lip}(g))}. \]

Specific functions \( \psi \) yield notion of weak dependence appropriated to describe different models. In the following, we will consider different types of \( \psi \) which are used in the current bibliography of weak dependence, see \[3, 6, 9\].

- \( \lambda \)-weak dependence: the \( \lambda \)-coefficient corresponds to \( \psi(u, v, a, b) = au + bv + abuv \), in this case we simply denote \( \epsilon(r) = \lambda(r) \).
- \( \eta \)-weak dependence: the \( \eta \)-coefficient corresponds to \( \psi(u, v, a, b) = au + bv \), we denote \( \epsilon(r) = \eta(r) \).
- \( \theta \)-weak dependence: \( \theta \)-coefficient corresponds to \( \psi(u, v, a, b) = bv \), in this case we write \( \epsilon(r) = \theta(r) \). This is the causal counterpart of \( \eta \)-coefficients.
• \( \kappa \)-weak dependence: in this case the \( \kappa \)-coefficient corresponds to \( \psi(u, v, a, b) = abuv \), and we write \( \epsilon(r) = \kappa(r) \).

• \( \kappa' \)-weak dependence: the \( \kappa' \)-coefficient correspond to \( \psi(u, v, a, b) = abv \), in this case we denote \( \epsilon(r) = \kappa'(r) \). This is the causal counterpart of \( \kappa \)-coefficient.

We now extend the notion of weak dependence to a sequence of doubly stochastic processes. We consider \( Z = \{ Z^i : i \in \mathbb{Z} \} \) a stationary sequence of stochastic processes, \( Z^i = \{ Z^i_t : t \in \mathbb{Z} \} \).

**Definition 3.** We say that \( Z = \{ Z^i : i \in \mathbb{Z} \} \) is \( (\epsilon, \psi, Y) \)-weakly dependent if, conditionally to \( Y \) and for almost all trajectory \( Y \), there exists a sequence \( \epsilon(r) \) decreasing to zero at infinity, a function \( \psi \) from \( \mathbb{R}^2 \times (\mathbb{R}^+)^2 \) to \( \mathbb{R}^+ \) and a positive random variable \( V(y) \) such that,

\[
|\text{cov} \left( f(Z^i_{t_1}, \ldots, Z^i_{t_u}), g(Z^j_{t'_1}, \ldots, Z^j_{t'_v}) \right) | \leq \psi(d_{i_u}(Y), d_{j_v}(Y), \text{Lip}(f), \text{Lip}(g)) \epsilon(r),
\]

for all \( (t_1, \ldots, t_u) \in \mathbb{Z}^u \) and \( (t'_1, \ldots, t'_v) \in \mathbb{Z}^v \), for any \( r \geq 0 \) and for any \( (u, v) \)-tuples such that \( i_1 < \ldots < i_u \leq i_0 + r < j_1 < \ldots < j_v \), where \( (f, g) \in \Delta_1^{(u)} \times \Delta_2^{(v)} \), \( i_u = (i_1, \ldots, i_u) \), \( j_v = (j_1, \ldots, j_v) \) and

\[
d_{i_u} = \sum_{m=1}^u V(y^{i_m}), \quad d_{j_v} = \sum_{m=1}^v V(y^{j_m})
\]

and \( \mathbb{E}_Y[||Z^i||^2] \leq V^2(y) < \infty, \mu.a.s. \). 

**Remark 3.** When measure \( \mu \) is degenerate, i.e when \( y^i = y \) for all \( i \), then if the stationary process \( Z = \{ Z_t(\epsilon^i) : i, t \in \mathbb{Z} \} \) satisfies Definition 3 we simply say that \( Z \) is \( \epsilon \)-weakly dependent.

We introduce the following condition:

**K5.** \( \mathbb{E}[V(y)] < \infty. \)

This condition implies that \( V(y) < \infty, \mu.a.s. \).

**Remark 4.** In definition 3 the random variable \( V^2(y) \) establish a control of the \( y \)-conditional variance of \( Z_t(y, \epsilon) \). We will see that in some case \( \mathbb{E}[||Z_t(y, \epsilon)||^2] = V^2(y) \), for instance for DSV* processes.

## 4 Transference of the weak dependence property to doubly stochastic models

We show that for different weakly dependent innovation models we can obtain weakly dependent doubly stochastic models.

Let us consider doubly stochastic Bernoulli shift processes defined, as in Section 2, by

\[
Z_t(y^i, \epsilon) = \mathcal{H} \left( y^i, \{ \epsilon^i_{t+k} \}_{k \in \mathbb{Z}} \right).
\]

where \( \mathcal{H} : \mathbb{R}^s \times \mathbb{R}^\mathcal{Y} \rightarrow \mathbb{R} \) be a measurable function in \( L^2(\Omega \otimes \mathcal{Y}^t) \) and we suppose that, for each \( t, \{ \epsilon^i_t : i \in \mathbb{Z} \} \) is weakly dependent. We show that such property of weak dependence for the innovations is transferred to the doubly stochastic processes. We present some examples, already known in the literature, of weakly dependent innovations.

### 4.1 Examples of weakly dependent innovations

In the following, we give examples of weakly dependent innovations, for more details to respect of theses innovation models see [3, 9].

**Remark 5.** In this work, we denote by \( \mathcal{H}_c \) to Bernoulli shift innovation and simply by \( \mathcal{H} \) to doubly stochastic Bernoulli shift.
4.1.1 Bernoulli shift innovations

Let us take $\varepsilon^t_1 = H_\varepsilon(\{\xi_t^{-1}\}_{t \in \mathbb{Z}})$ a Bernoulli shift with input $\{\xi_t^i\}$. We assume that, for all $t$ fixed, the shift $H_\varepsilon$ satisfies the following condition

$$\sum_{n=1}^{\infty} w_n < \infty. \quad (32)$$

where

$$w_n = E\left[\left|H_\varepsilon\left(\{\xi_t^{-1}\}_{|t| \leq n}\right) - H_\varepsilon\left(\{\xi_t^{-1}\}_{|t| < n}\right)\right|^2\right]^{\frac{1}{2}}.$$  

This condition indeed proves that the sequence $\{H_\varepsilon(\{\xi_t^i\}_{|t| \leq n})\}$ has the Cauchy property in the space $L^2(\Omega^\prime)$ and so its convergent. Let $\delta_r = \sum_{n \geq r} w_n$ then $\delta_r = \sum_{r \in \mathbb{N}} w_n$ converges to zero as $r \to \infty$.

Doukhan et al. prove the following results, for each $t$ fixed, see Lemma 3.1, Lemma 3.2 and Lemma 3.3 in [3].

**Theorem 1** (Doukhan et al.).

- **Noncausal shift innovations with independent inputs**
  - If $\{\xi_t^1\}$ a sequence of i.i.d. random variables then the innovations $\{\varepsilon_t^i : i \in \mathbb{Z}\}$ are $\eta$-weakly dependent with $\eta(r) \leq 2\delta_{r/2}$.

- **Noncausal shift innovations with dependent inputs**
  - If $\{\xi_t^i : l \in \mathbb{Z}\}$ is $\eta_k$-weakly dependence then the innovations $\{\varepsilon_t^i : i \in \mathbb{Z}\}$ are $\eta$-weakly dependent.
  - If $\{\xi_t^i\}$ is $\lambda_k$-weakly dependent then $\{\varepsilon_t^i : i \in \mathbb{Z}\}$ are neither $\kappa$ nor $\eta$-weakly dependent.

- **Causal shift innovations with independent inputs**
  Let us take $\varepsilon_t^i = H_\varepsilon(\{\xi_t^{-1}\}_{t \in \mathbb{N}})$ a causal Bernoulli shift with input $\{\xi_t^i\}$ a sequence of i.i.d. random variables. Then, the innovations $\{\varepsilon_t^i : i \in \mathbb{Z}\}$ are $\theta$-weakly dependent with $\theta(r) \leq 2\delta_r$ and $\delta_r = \sum_{n \geq r} w_n$.

**Example 3** (Linear innovations). If $\varepsilon_t^1 = \sum_{l \in \mathbb{Z}} \beta_l \xi_t^{-l}$, $\|\beta\|_2^2 = \sum_l \beta_l^2 < \infty$ with $\{\xi_t^i\}$ a sequence of i.i.d. random variables, then $\varepsilon_t^i$ is $\eta$-weakly dependent with $\eta(r) \leq \left(\sum_{|l| \geq r} \beta_l^2\right)^{\frac{1}{2}}$.

**Example 4** (Volterra innovations). If we suppose that $\varepsilon_t^1$ is defined by the following Volterra expansion

$$\varepsilon_t^i = \sum_{k=0}^{\infty} \sum_{l_1 < \ldots < l_k} \beta_{l_1, \ldots, l_k} \xi_t^{-l_1} \ldots \xi_t^{-l_k} \quad \text{where } \{\beta_{l_1, \ldots, l_k} : (l_1, \ldots, l_k) \in \mathbb{Z}^k\} \text{ is a sequence of real numbers and } \{\xi_t^i\} \text{ is a sequence of i.i.d. random variables. This expression converges in } L^2(\Omega^\prime) \text{ provided that } E[|\xi_t^i|^2] < \infty \text{ and }$$

$$\|\beta\|_2^2 = \sum_{k=0}^{\infty} \sum_{l_1 < \ldots < l_k} |\beta_{l_1, \ldots, l_k}|^2 < \infty.$$  

In this case, we can verify that $\varepsilon_t^i$ is $\eta$-weakly dependent.

$$\eta(r)^2 \leq \|\beta\|_2^2 - \sum_{k=1}^{2r-1} \sum_{-r<l_1<\ldots<l_k<r} |\beta_{l_1, \ldots, l_k}|^2.$$  

Condition $\|\beta\|_2 < \infty$ implies that $\eta(r)$ converges to zero as $r \to \infty$.
In the general case of Volterra innovation

\[ \varepsilon_t^i = \sum_{k=0}^{\infty} \sum_{l_k} \beta_{k;l_1,\ldots,l_k} \varepsilon_{t-l_1}^i \cdots \varepsilon_{t-l_k}^i, \]

we have that this expansion converge in \( L^2(\Omega') \), whenever

\[ \sum_{k=0}^{\infty} \mathbb{E} \left[ |\varepsilon_t^i|^{2k} \right] \sum_{l_1,\ldots,l_k} |\beta_{k;l_1,\ldots,l_k}|^2 < \infty. \] (33)

Then,

\[ w_n^2 = \sum_{k=1}^{2n+1} \mathbb{E} \left[ |\varepsilon_t^i|^{2k} \right] \sum_{-n<\cdots<l_k=n} |\beta_{k;l_1,\ldots,l_k}|^2, \]

\[ \eta(r)^2 \leq \sum_{k=0}^{\infty} \mathbb{E} \left[ |\varepsilon_t^i|^{2k} \right] \sum_{l_1<\cdots<l_k<r} |\beta_{k;l_1,\ldots,l_k}|^2 + \sum_{k=0}^{\infty} \mathbb{E} \left[ |\varepsilon_t^i|^{2k} \right] \sum_{l_1<\cdots<l_k<r} |\beta_{k;l_1,\ldots,l_k}|^2. \]

So, under condition (33), we can also obtain that \( \eta(r) \) converges to zero as \( r \to \infty \).

This case include noncausal LARCH, ARCH, GARCH and bilinear models.

Example 5 (Causal LARCH(\( \infty \)) innovations).

general causal LARCH(\( \infty \)) models are \( \theta \)-weakly dependents. For instance, linear innovations, ARCH, GARCH and bilinear models, see [7].

4.1.2 Associated innovations

A process \( \{\varepsilon^i\} \) is associated if

\[ \text{cov} \left( f \left( \varepsilon_{i_1}, \ldots, \varepsilon_{i_n} \right) , g \left( \varepsilon_{i_1}, \ldots, \varepsilon_{i_n} \right) \right) \geq 0, \]

for any coordinate wise non-decreasing function \( f, g : \mathbb{R}^n \to \mathbb{R} \). Associated processes or Gaussian stationary processes are \( \kappa \)-weakly dependents with

\[ \kappa(r) = \sup_{j \geq r} \left| \text{cov} \left( \varepsilon^i, \varepsilon^{i+j} \right) \right|. \]

For instance, independent sequence are associated and gaussian processes with non-negative covariance are also associated. This models are classically built from i.i.d sequence, see [21].

4.2 Uniform Lipschitz Bernoulli Shifts with weakly dependent innovations

Here, we consider the DSULBS processes \( Z^i = \{ Z_t^i : t \in \mathbb{Z} \} \) given by

\[ Z_t^i = \mathcal{H} \left( y^i, \{ \varepsilon_{t-k}^i \}_{k \in \mathbb{Z}} \right). \]

where \( \mathcal{H} : \mathbb{R}^s \times \mathbb{R}^2 \to \mathbb{R} \) be a measurable function satisfying conditions (6) and (7).

A simple example of this situation is the doubly stochastic linear process. Another examples are obtained considering Lipschitz function of linear processes.

In the following, we will consider that, for each \( t \), \( \{ \varepsilon_t^i : i \in \mathbb{Z} \} \) is \( \epsilon \)-weakly dependent. Then, we prove that the sequence \( Z \) of DSULBS processes is in general a \( (\epsilon, Y) \)-weakly dependent doubly stochastic processes. This is a new result and is obtained when the \( \epsilon \)-coefficient is taking in the cases of \( \lambda, \eta, \theta, \kappa \) or \( \kappa' \)-weak dependence.
Theorem 2. Under conditions (43) and (44), the DSULBS processes $Z$ with $η$-weakly dependent innovation is $(η, Y)$-weakly dependent, with $V(y') = \|a(y')\|_1$.

We will give the proof in Section 7.

Remark 6. If condition $E_{2δ} \mathbb{E}[\|ε_{1}^{2+δ}\|] < \infty$ for some $δ > 0$, and condition (32) holds then condition $K_{2δ}$ is satisfied for the DSULBS process.

Remark 7. Condition $K_5$ is satisfied by this model, since $V(y) = \|a(y)\|_1$ and so $K_5$ is implied by condition (32).

4.3 Doubly stochastic Volterra processes with weakly dependent innovations

We will consider $DVS^*$ processes defined by equation (11); i.e. we consider the Bernoulli shift

$$H(y, ε) = \sum_{k=0}^{∞} \sum_{j_1 < ... < j_k} c_{k,j_1,...,j_k}(y)ε_{j_1} \ldots ε_{j_k}.$$ 

Then, for all $ε, ε' \in \mathbb{R}^Z$, we have that

$$H(y, ε) - H(y, ε') = \sum_{l \in \mathbb{Z}} \sum_{k=1}^{∞} \sum_{j_1 < ... < j_k} c_{k,j_1,...,j_k}(y)ε_{j_1} \ldots ε_{j_k} - \sum_{l \in \mathbb{Z}} \sum_{k=1}^{∞} \sum_{j_1 < ... < j_k} c_{k,j_1,...,j_k}(y)ε'_{j_1} \ldots ε'_{j_k}. \quad (34)$$

We suppose that the $DVS^*$ processes $Z_{i} = H(y, \{ε_{i-k}\}_{k \in \mathbb{Z}})$ are such that the sequence $c(y) = \{c_{k,j_1,...,j_k}(y) : k \in \mathbb{Z}, (j_1, ..., j_k) \in \mathbb{Z}^k\}$ satisfies condition $C_2$, i.e.

$C_2\quad \mathbb{E}[\|c(y)\|_2^2] < \infty.$

Let us take $ε_{i} = H_{ε} (\{ε_{i-l}\}_{l \in \mathbb{Z}})$ a Bernoulli shift with independent inputs $\{ε_{i}\}$. We assume, in the same way of Section 4.1 that the Bernoulli shift $H_{ε}$ satisfies condition (32). Then, $δ_r = \sum_{n \geq r} w_n$ converges to zero as $r \rightarrow \infty$.

In this section we assume the following condition

$$\mathbb{E} \left[ \left| ε_{t}^{i(s)} \right|^2 \right] \leq \mathbb{E} \left[ \left| ε_{t}^{i} \right|^2 \right] = 1. \quad (35)$$

An adaptation of the proof of Lemma 3.1 and Lemma 3.2 given in [3], allows us to extend easily these results for doubly stochastic Volterra processes with Bernoulli shift innovations.

Theorem 3. Under conditions (32), (35) and condition $C_2$, the sequence $Z$ of $DVS^*$ processes with noncausal Bernoulli shift innovations is $(η, Y)$-weakly dependent, with $η(r) \leq 2δ_{r}/2$, and $V(y') = \|c(y')\|_2$.

Theorem 4. Under conditions (32), (35) and condition $C_2$, the sequence $Z$ of $DVS^*$ processes with causal Bernoulli shift innovations is $(η, Y)$-weakly dependent, with $η(r) \leq 2δ_{r}$ and $V(y') = \|c(y')\|_2$.

The proofs are deferred to Section 7.

From Theorem 4 we have that the noncausal Bernoulli shift innovation with independent inputs is $η$-weakly dependent and the causal Bernoulli shift innovation with independent inputs is $θ$-weakly dependent. Thus, from Theorem 3 and Theorem 4 we can confirmed that the weak dependence property of innovations is transferred to the doubly stochastic process.
Remark 8. Note that condition (32) and (34) are satisfied in the case of linear innovations or Volterra innovations, see Example 5 and Example 6. So, considering DSV* processes $Z$ such that the innovations $\{\varepsilon^i\}$ are Volterra models with independent inputs, we have that $Z$ is a sequence of weakly dependent doubly stochastic processes. For instance, this is the case of GARCH, ARCH($\infty$), LARCH($\infty$) and bilinear doubly stochastic processes.

Remark 9. Condition K5 is satisfied by this type of doubly stochastic Volterra models, in this case $V(y) = \|c(y)\|_2$. So K5 is implied by condition C2.

5 Aggregation convergence results

We consider that $Z = \{Z^i\}$ is a weakly dependent stationary sequence of doubly stochastic centered processes. Here, we consider that conditions K2, K4 and K5 hold.

We now present our main results: the CLT, $\nu - a.s.$, for the sequence $\{X^N(Y)\}$ in the case of $(\lambda, Y)$-weak dependence and $(\kappa, Y)$-weak dependence.

Theorem 5 (CLT: $(\lambda, Y)$-weak dependence). We assume that $Z = \{Z^i\}$ is $(\lambda, Y)$-weakly dependent satisfying conditions K2, K4, K5 and such that $\lambda(r) = O(r^{-\lambda})$, as $r \to \infty$, for $\lambda > 2 + \frac{3}{2}$. Then, $X^N(Y)$ converges in distribution, $\nu - a.s.$, to a Gaussian process $X$ with covariance function $\Gamma$.

Theorem 6 (CLT: $(\kappa, Y)$-weak dependence). We assume that $Z = \{Z^i\}$ is $(\kappa, Y)$-weakly dependent satisfying conditions K2, K4, K5 and such that $\kappa(r) = O(r^{-\kappa})$, as $r \to \infty$, for $\kappa > 2 + \frac{3}{2}$. Then, $X^N(Y)$ converges in distribution, $\nu - a.s.$, to a Gaussian process $X$ with covariance function $\Gamma$.

Remark 10.

- The result for $(\lambda, Y)$-weak dependence implies those for $(\eta, Y)$ or $(\theta, Y)$-weak dependence. In both cases, we do not achieve the better results, in the sense that the bound for the dependence parameters is also $2 + \frac{3}{2}$.
- The result for $(\kappa, Y)$-weak dependence is implied by case of $(\kappa, Y)$-weak dependence. In this case the bound for these dependence parameters is $2 + \frac{3}{2}$.

The proofs of these CLT are extensions for the case doubly stochastic processes of the proof of the CLT for weakly dependent sequences given in (Section 7.1, [3]). The proofs are deferred to Section 7.

On the other hand, we prove the following lemma, which implies that under weak dependence property and condition E2, given in Remark 9, $\chi$ is a weak interaction in $\ell_1$.

Lemma 1. Under condition E2:
If $\{\varepsilon^i\}$ is $\lambda$-weakly dependent then $|\chi(r)| \leq O(\lambda(r)^{\frac{1}{\lambda+2}})$.
If $\{\varepsilon^i\}$ is $\kappa$-weakly dependent then $|\chi(r)| \leq O(\kappa(r))$.

The proof of Lemma 11 will be given in Section 7. This result is essentially of technical character and it allows us to get the SLLN for the covariance function $\Gamma^N(Y)$.

Remark 11. In the case of $\lambda$-weak dependence, the Lemma 11 implies that $\chi(r) = O(r^{-\frac{1}{\lambda+2}})$. Since, we suppose that $\lambda > 2 + \frac{3}{2}$ then $\frac{1}{\lambda+2} > 1$, so $\chi \in \ell_1$.

In the cases of $\eta$, $\theta$-weak dependence we obtain a similar result that for the case of $\lambda$ weak dependence. Nevertheless, in the case of $(\kappa, Y)$-weak dependence the result is similar to case $\kappa$-weak dependence.

Remark 12. In the case of elementary linear processes $Z^i$ with interactive linear innovation $\varepsilon^i$ given by $\varepsilon^i = \sum_{t \in \mathbb{Z}} \beta_t \varepsilon^t$, where $\|\beta\|_2 < \infty$ and $\{\varepsilon^i\}$ a sequence of i.i.d. random variables we have the following results.

As we have mentioned in Example 5, $\varepsilon^i$ is $\eta$-weakly dependent with $\eta(r) \leq \left(\sum_{|t| \geq r} \beta_i^2\right)^{\frac{1}{2}}$. 

From Theorem 4 and Remark 5 we have that $Z = \{Z^i\}$ is $(\eta, Y)$-weakly dependent. Furthermore, from Remark 9 condition $K5$ holds.

On the other hand, from Remark 12 we have that if condition $K2_\delta$ hold and $\eta(r) = \mathcal{O}(r^{-\eta})$ for $\eta > 2 + \frac{3}{\nu}$ then $\chi \in \ell_1$. So, from the SLLN given in [1] condition $K4$ holds.

Therefore, if condition $K2_\delta$ holds and $\eta(r) = \mathcal{O}(r^{-\eta})$ for $\eta > 2 + \frac{3}{\nu}$ then Theorem 5 implies the $nu_{-a.s.}$ weak convergence of $X^N(Y)$ to a Gaussian process $X$ with covariance function $\Gamma$.

6 An SLLN for $\Gamma^N(Y)$ in the case of DSV* processes

Now we give a SLLN for the covariance function $\Gamma^N(Y)$ of $X^N(Y)$, in the case of DSV* elementary processes defined by equation (11). We consider the case of interactive innovations, i.e. $E[\varepsilon_i \varepsilon_j^t] = \chi(i - j)$. In this case, the quadratic form $\Gamma^N(Y)$ is defined by

$$\Gamma^N(\tau, Y) = \frac{1}{B_N} \sum_{i=1}^{N} \Psi_\tau(y^i, y^i) + \frac{1}{B_N} \sum_{1 \leq i < j \leq N} \Psi_\tau(y^i, y^j).$$

where

$$\Psi_\tau(y^i, y^j) = \sum_{k=1}^{\infty} \Psi_{\tau,k}(y^i, y^j) \chi^k(i - j),$$

with

$$\Psi_{\tau,k}(y^i, y^j) = E[Y[Z^{(k)}_{i+\tau} Z^{(k)}_{i+\tau}]] = \sum_{l_1 < \cdots < l_k} c_{l_1, \ldots, l_k}(y^i)c_{l_1+\tau, \ldots, l_k+\tau}(y^j).$$

Let $\gamma_k(\tau) = E[\Psi_{\tau,k}(y^i, y^i)]$ and $\phi_k(\tau) = E[\Psi_{\tau,k}(y^i, y^j)]$. Condition C2 implies that

$$\gamma(\tau) = \sum_{k=1}^{\infty} \gamma_k(\tau) < \infty \text{ and } \phi(\tau) = \sum_{k=1}^{\infty} \phi_k(\tau) < \infty.$$

We denote

$$[\chi^k]_{N,1} = \sum_{1 \leq i < j \leq N} \chi^k(i - j), \quad [\chi]_{N,1} = \sum_{1 \leq i < j \leq N} |\chi(i - j)|,$$

where the function $\chi$ denotes the interaction between the innovations. If $s_{i,k} = \sum_{j=1}^{i-1} \chi^k(j)$ then

$$[\chi^k]_{N,1} = 2 \sum_{i=1}^{N} s_{i,k}.$$

Given that, for all $k \geq 1$, $\chi^k$ is positive definite, we have that $N\chi^k(0) + [\chi^k]_{N,1} \geq 0$. Hence

$$\frac{1}{N} [\chi^k]_{N,1} = \frac{2}{N} \sum_{i=1}^{N} s_{i,k} \geq -1.$$

If $\chi \in \ell_1$ then, for all $k \geq 1$, $\chi^k \in \ell_1$ and

$$\frac{2}{N} \sum_{i=1}^{N} s_{i,k} = 2 \sum_{i=1}^{N} \left(1 - \frac{|i|}{N}\right) \chi^k(i) \xrightarrow{N \to \infty} 2 \sum_{i=1}^{\infty} \chi^k(i) := s_k.$$

So, $\{s_{i,k} : i \in \mathbb{N}\}$ converges in the Cesaro sense to $\frac{2}{N} s_k$ with $-1 \leq s_k < \infty$.

Theorem 7 (SLLN for $\Gamma^N(\tau)$: Volterra processes case). If $\chi \in \ell_1$ and condition C2 holds, then taking $B_N = \sqrt{N}$ we have that $\Gamma^N(Y)$ converge $\nu_{-a.s.}$ and in $L^1(\nu)$ to $\Gamma$ given by

$$\Gamma(\tau) = \sum_{k=1}^{\infty} \gamma_k(\tau) + \sum_{k=1}^{\infty} \phi_k(\tau)s_k.$$

The proof of this theorem is an extension of SLLN given in [1] for the case of linear processes. The details are given in Section 4.
7 Proof of the main results

7.1 Proof of Theorem 2

Without loss of generality we give the proof in the case of $\lambda$-weakly dependent innovations, since in the other case the proof is similar.

Proof. Let $i_1 \leq \ldots \leq i_u \leq i_u + r < j_1 \leq \ldots \leq j_v$ any $(u + v)$-tuples and $r > 0$. Let us consider $Z^i_t = \mathcal{H} \left(y_t, \\{\varepsilon_{t(k)}^i\}_{k \in \mathbb{Z}}\right)$, with $\{t(k) : k \in \mathbb{Z}\}$ any sequence of indexes in $\mathbb{Z}$ and let us take

$$\varepsilon_{t(k)}^i = \varepsilon_t^i \mathbb{I}(|k| \leq s), \quad Z_t^i = \mathcal{H} \left(y_t, \\{\varepsilon_{t(k)}^i\}_{k \in \mathbb{Z}}\right).$$

For sake of simplicity, for all $f \in \Delta_1^{(u)}$ and $g \in \Delta_1^{(v)}$, we denote $F = f \left(Z_{t_1}^{i_1}, \ldots, Z_{t_u}^{i_u}\right)$, $G = g \left(Z_{t_1'}^{j_1}, \ldots, Z_{t_v'}^{j_v}\right)$ and

$$F^{(s)} = F^{(s)} \left(\{\varepsilon_{t_1(k)}^{i_1}\}_{k \leq s}, \ldots, \{\varepsilon_{t_u(k)}^{i_u}\}_{k \leq s}\right) = f \left(Z_{t_1}^{i_1(s)}, \ldots, Z_{t_u}^{i_u(s)}\right),$$

$$G^{(s)} = G^{(s)} \left(\{\varepsilon_{t_1'(k)}^{j_1}\}_{k \leq s}, \ldots, \{\varepsilon_{t_v'(k)}^{j_v}\}_{k \leq s}\right) = g \left(Z_{t_1'}^{j_1(s)}, \ldots, Z_{t_v'}^{j_v(s)}\right).$$

(36)

It is easy to verify that

$$\text{Lip} \left(F^{(s)}\right) \leq d_{i_u}^{(s)}(Y)\text{Lip}(f), \quad \text{Lip} \left(G^{(s)}\right) \leq d_{i_v}^{(s)}(Y)\text{Lip}(g),$$

(37)

where

$$d_{i_u}^{(s)}(Y) = \sum_{m=1}^{u} \sum_{|k| \leq s} |a_k(y^m)| \quad \text{and} \quad d_{i_v}^{(s)}(Y) = \sum_{m=1}^{v} \sum_{|k| \leq s} |a_k(y^m)|.$$

The proof proceeds in three parts:

Part a: First, we want to prove by means of an inductive procedure the following result: considering $F^{(s)}$ and $G^{(s)}$ functions of types (36) such that equations (37) holds, then

$$\left|\text{cov}(F^{(s)}, G^{(s)})|Y\right| \leq \psi \left(d_{i_u}^{(s)}(Y), d_{i_v}^{(s)}(Y), \text{Lip}(f), \text{Lip}(g)\right) \lambda(r),$$

where $\psi(d_{i_u}^{(s)}, d_{i_v}^{(s)}, a, b) = ad_{i_u}^{(s)} + bd_{i_v}^{(s)} + abd_{i_u}^{(s)}d_{i_v}^{(s)}$.

The proof of this part contains three steps.

Step 1: We verify the inductive hypothesis for $s = 0$.

Let us $\mathcal{B}_0^0 = \sigma\{0\}$ and $\mathcal{B}_0^m = \sigma \left(\{\varepsilon_{t_l}^{(s)} : 1 \leq l \leq m\}\right)$ for $1 \leq m \leq u \wedge v$. If we fix $\varepsilon_{t_l}^{(s)} = \varepsilon_{t_l}^{(0)} = \varepsilon_{t_l}^{(0)}$ for $1 \leq l \leq m$, then we can write $F^{(0)}$ and $G^{(0)}$ respectively as

$$\tilde{F}^{(0,m)} := \tilde{F}^{(0,m)} \left(\varepsilon_{t_{m+1}}^{(0)}, \ldots, \varepsilon_{t_{u}}^{(0)}\right) = F^{(0)} \left(\varepsilon_{0}^{(0)}, \ldots, \varepsilon_{0}^{(0)}, \varepsilon_{t_{m+1}}^{(0)}, \ldots, \varepsilon_{t_{u}}^{(0)}\right),$$

$$\tilde{G}^{(0,m)} := \tilde{G}^{(0,m)} \left(\varepsilon_{t_{m+1}}^{(0)}, \ldots, \varepsilon_{t_{u}}^{(0)}\right) = G^{(0)} \left(\varepsilon_{0}^{(0)}, \ldots, \varepsilon_{0}^{(0)}, \varepsilon_{t_{m+1}}^{(0)}, \ldots, \varepsilon_{t_{u}}^{(0)}\right).$$

Now, we introduce the following notation

$$F^{(0,1)} := F^{(0,1)} \left(\varepsilon_{t_1}^{(0)}\right) := \mathbb{E} \left[F^{(0)} \mid \mathcal{B}_0^1\right],$$

$$G^{(0,1)} := G^{(0,1)} \left(\varepsilon_{t_1}^{(0)}\right) := \mathbb{E} \left[G^{(0)} \mid \mathcal{B}_0^1\right],$$

and
and for $m > 1$
\[
\tilde{F}^{(0,m)} = \tilde{F}^{(0,m)} \left( \epsilon_{m,\infty}^{i} \right) := \mathbb{E} \left[ \tilde{F}^{(0,m-1)} \mid \mathcal{B}_{m}^{0} \right],
\]
\[
\tilde{G}^{(0,m)} = \tilde{G}^{(0,m)} \left( \epsilon_{m,\infty}^{i} \right) := \mathbb{E} \left[ \tilde{G}^{(0,m-1)} \mid \mathcal{B}_{m}^{0} \right].
\]

By using the expression for the conditional covariance given $\mathcal{B}_{0}^{m}$, for $Y$ fixed, we have that
\[
\begin{align*}
|\text{cov} (F^{(0)}, G^{(0)} \mid Y) | & \leq \mathbb{E} \left[ |\text{cov} (F^{(0)}, G^{(0)} \mid Y, \mathcal{B}_{0}^{1}) | \right] + |\text{cov} (\mathbb{E} Y [F^{(0)} \mid \mathcal{B}_{0}^{1}], \mathbb{E} Y [G^{(0)} \mid \mathcal{B}_{0}^{1}] \mid Y) | \\
& = \mathbb{E} \left[ |\text{cov} (\tilde{F}^{(0,1)}, \tilde{G}^{(0,1)} \mid Y, \mathcal{B}_{0}^{1}) | \right] + |\text{cov} (F^{(0,1)}, G^{(0,1)} \mid Y) | \\
& \leq \mathbb{E} \left[ |\text{cov} (\tilde{F}^{(0,m)}, \tilde{G}^{(0,m)} \mid Y, \mathcal{B}_{0}^{m-1}) | \right] + \sum_{l=1}^{m} \mathbb{E} \left[ |\text{cov} (F^{(0,l)}, G^{(0,l)} \mid Y, \mathcal{B}_{0}^{l-1}) | \right].
\end{align*}
\]

Following this procedure inductively until $m = u \land v$, we have
\[
|\text{cov} (F^{(0)}, G^{(0)} \mid Y) | \leq \sum_{m=1}^{u \land v} \mathbb{E} \left[ |\text{cov} (\tilde{F}^{(0,m)}, \tilde{G}^{(0,m)} \mid Y, \mathcal{B}_{0}^{m-1}) | \right].
\] (38)

Since, for all $t$, $\{z_{i}^{t} : i \in \mathbb{Z}\}$ is $\lambda$-weakly dependent then
\[
|\text{cov} (\tilde{F}^{(0,m)}, \tilde{G}^{(0,m)} \mid Y, \mathcal{B}_{0}^{m-1}) | \leq \psi \left( 1, 1, \text{Lip} (\tilde{F}^{(0,m)}), \text{Lip} (\tilde{G}^{(0,m)}) \right) \lambda(r). \] (39)

Furthermore, for $Y$ fixed, we can verify
\[
\text{Lip} (\tilde{F}^{(0,m)}) \leq \text{Lip}(f) \left| a_{0} \left( y^{m} \right) \right|, \quad \| \tilde{F}^{(0,m)} \|_{\infty} \leq \| f \|_{\infty} \leq 1, \]
\[
\text{Lip} (\tilde{G}^{(0,m)}) \leq \text{Lip}(g) \left| a_{0} \left( y^{m} \right) \right|, \quad \| \tilde{G}^{(0,m)} \|_{\infty} \leq \| g \|_{\infty} \leq 1. \] (40)

Thus, from equations (38), (39) and (40), we prove that
\[
|\text{cov} (F^{(0)}, G^{(0)} \mid Y) | \leq \sum_{m=1}^{u \land v} \psi \left( 1, 1, \text{Lip}(F^{(0,m)}), \text{Lip}(G^{(0,m)}) \right) \lambda(r) \]
\[
\leq \psi \left( d_{u}^{(0)}, d_{v}^{(0)}, \text{Lip}(f), \text{Lip}(g) \right) \lambda(r). \]

**Step 2:** We suppose the inductive hypothesis satisfied for $0 \leq s < n$, i.e. if $F^{(s)}$ and $G^{(s)}$ are functions defined in (56) satisfying equations (57) then
\[
|\text{cov} (F^{(s)}, G^{(s)} \mid Y) | \leq \psi \left( d_{u}^{(s)}(Y), d_{v}^{(s)}(Y), \text{Lip}(f), \text{Lip}(g) \right) \lambda(r).
\]

**Step 3:** Now we will prove the inductive hypothesis for $s = n$.

Let $\mathcal{B}_{n} = \sigma \left( \left\{ \epsilon_{m,\infty}^{i}, \mid k \mid = n, m = 1 \ldots u \right\} \cup \left\{ \epsilon_{m,\infty}^{j}, \mid k \mid = n, m = 1 \ldots v \right\} \right)$. If we fix, for $|k| = n$, $\epsilon_{m,\infty}^{i} = \epsilon_{m,\infty}^{i}$ with $m = 1 \ldots u$ and $\epsilon_{m,\infty}^{j} = \epsilon_{m,\infty}^{j}$ with $m = 1 \ldots v$ then we can write $F^{(n)}$ and $G^{(n)}$ respectively as
\[
\tilde{F}^{(n-1)} := \tilde{F}^{(n-1)} \left( \epsilon_{m,\infty}^{i}, \mid k \mid \leq n-1, \ldots, \epsilon_{m,\infty}^{i}, \mid k \mid \leq n-1 \right) = f \left( \tilde{Z}_{m}^{(n-1)} \right),
\]
\[
\tilde{G}^{(n-1)} := \tilde{G}^{(n-1)} \left( \epsilon_{m,\infty}^{i}, \mid k \mid \leq n-1, \ldots, \epsilon_{m,\infty}^{i}, \mid k \mid \leq n-1 \right) = g \left( \tilde{Z}_{m}^{(n-1)} \right),
\]
where $\tilde{Z}_{t}^{i} = \mathcal{H} \left( y^{i}, \ldots, 0, \epsilon_{t}^{i}, \epsilon_{t+1}^{i}, \ldots, \epsilon_{t}^{i}, 0, \ldots \right)$. 

Let us denote
\[
F_n = F_n \left( \{ \varepsilon_{t_{1}(k)}^{i_{1}} | k | = n, \ldots, \varepsilon_{t_{u}(k)}^{i_{u}} | k | = n \} \right) := \mathbb{E} \left[ F^{(n)} \middle| B_n \right],
\]
\[
G_n = G_n \left( \{ \varepsilon_{t_{1}'(k)}^{j_{1}} | k | = n, \ldots, \varepsilon_{t_{v}'(k)}^{j_{v}} | k | = n \} \right) := \mathbb{E} \left[ G^{(n)} \middle| B_n \right].
\]

By using the expression for the conditional covariance given \( B_n \), for \( Y \) fixed, we have
\[
\begin{align*}
\text{cov} \left( F^{(n)}, G^{(n)} \middle| Y \right) & \leq \mathbb{E}^Y \left[ \text{cov} \left( F^{(n)}, G^{(n)} \middle| Y, B_n \right) \right] + \text{cov} \left( \mathbb{E}^Y \left[ F^{(n)} \middle| B_n \right], \mathbb{E}^Y \left[ G^{(n)} \middle| B_n \right] \right) Y \\
& = \mathbb{E}^Y \left[ \text{cov} \left( \tilde{F}^{(n-1)}, \tilde{G}^{(n-1)} \middle| Y, B_n \right) \right] + \text{cov} \left( F_n, G_n \middle| Y \right).
\end{align*}
\]

(41)

It is easy to verify that
\[
\begin{align*}
\text{Lip}(\tilde{F}^{(n-1)}) & \leq \text{Lip}(f) \sum_{m=1}^{n} \sum_{|k| \leq n-1} |a_k(y^{m})| = \text{Lip}(f)d_{a_n}^{(n-1)}(Y) \\
\text{Lip}(\tilde{G}^{(n-1)}) & \leq \text{Lip}(g) \sum_{m=1}^{n} \sum_{|k| \leq n-1} |a_k(y^{m})| = \text{Lip}(g)d_{b_n}^{(n-1)}(Y).
\end{align*}
\]

Thus, applying the inductive hypothesis we obtain
\[
\mathbb{E}^Y \left[ \text{cov} \left( \tilde{F}^{(n-1)}, \tilde{G}^{(n-1)} \middle| Y, B_n \right) \right] \leq \psi \left( d_{a_n}^{(n-1)}(Y), d_{b_n}^{(n-1)}(Y), \text{Lip}(f), \text{Lip}(g) \right) \lambda(r). \quad (42)
\]

On the other hand, let us \( B_{n}^{+} = \sigma \left( \{ \varepsilon_{t_{m}(n)}^{i_{m}} : m = 1 \ldots u \} \cup \{ \varepsilon_{t_{m}(n)}^{j_{m}} : m = 1 \ldots v \} \right) \). Now, if we fix \( \varepsilon_{t_{m}(n)}^{i_{m}} = e_{n}^{i_{m}} \) for \( m = 1 \ldots u \) and \( \varepsilon_{t_{m}(n)}^{j_{m}} = e_{n}^{j_{m}} \) for \( m = 1 \ldots v \), we can write \( F_n \) and \( G_n \) respectively as
\[
F_{0,n} = F_{n}^{(0)} \left( \varepsilon_{t_{1}(n)}^{i_{1}}, \ldots, \varepsilon_{t_{u}(n)}^{i_{u}} \right) := F_n \left( \{ \varepsilon_{t_{1}(n)}^{i_{1}}, \varepsilon_{t_{1}(n)}^{i_{1}}, \ldots, \varepsilon_{t_{u}(n)}^{i_{u}}, e_{n}^{i_{u}} \} \right),
\]
\[
G_{0,n} = G_{n}^{(0)} \left( \varepsilon_{t_{1}'(n)}^{j_{1}}, \ldots, \varepsilon_{t_{v}'(n)}^{j_{v}} \right) := G_n \left( \{ \varepsilon_{t_{1}'(n)}^{j_{1}}, \varepsilon_{t_{1}'(n)}^{j_{1}}, \ldots, \varepsilon_{t_{v}'(n)}^{j_{v}}, e_{n}^{j_{v}} \} \right).
\]

Using the notation
\[
\begin{align*}
F^{(0)}_n & = F_{n}^{(0)} \left( \varepsilon_{t_{1}(n)}^{i_{1}}, \ldots, \varepsilon_{t_{u}(n)}^{i_{u}} \right) := \mathbb{E} \left[ F_{n} \middle| B_{n}^{+} \right], \\
G^{(0)}_n & = G_{n}^{(0)} \left( \varepsilon_{t_{1}'(n)}^{j_{1}}, \ldots, \varepsilon_{t_{v}'(n)}^{j_{v}} \right) := \mathbb{E} \left[ G_{n} \middle| B_{n}^{+} \right].
\end{align*}
\]

We have
\[
\begin{align*}
\text{cov} \left( F_n, G_n \right) & \leq \mathbb{E}^Y \left[ \text{cov} \left( F_n, G_n \middle| Y, B_n^{+} \right) \right] + \text{cov} \left( \mathbb{E}^Y \left[ F_n \middle| B_n^{+} \right], \mathbb{E}^Y \left[ G_n \middle| B_n^{+} \right] \right) Y \\
& = \mathbb{E}^Y \left[ \text{cov} \left( \tilde{F}^{(0,n)}, \tilde{G}^{(0,n)} \middle| Y, B_n^{+} \right) \right] + \text{cov} \left( F_{0,n}, G_{0,n} \middle| Y \right).
\end{align*}
\]

(43)

Let us \( B_{k}^{0} = \sigma \{ 0 \} \) and \( B_{k}^{m} = \sigma \left( \{ \varepsilon_{t_{l}(k)}^{i_{l}}, \varepsilon_{t_{l}'(k)}^{j_{l}} : 1 \leq l \leq m \} \right) \), for \( 1 \leq m \leq u \) and \( |k| = n \). If we fix \( \varepsilon_{t_{l}(k)}^{i_{l}} = e_{k}^{i_{l}}, \varepsilon_{t_{l}'(k)}^{j_{l}} = e_{k}^{j_{l}} \) for \( 1 \leq l \leq m \), then we can write \( F_{k}^{(0)} \) and \( G_{k}^{(0)} \) respectively as
\[
\begin{align*}
\tilde{F}_{k}^{(0,m)} & := \tilde{F}_{k}^{(0,m)} \left( \varepsilon_{t_{m+1}(k)}^{i_{m+1}}, \ldots, \varepsilon_{t_{u}(k)}^{i_{u}} \right) = F_{k}^{(0)} \left( \varepsilon_{k}^{i_{1}}, \ldots, \varepsilon_{k}^{i_{m}}, \varepsilon_{t_{m+1}(k)}^{i_{m+1}}, \ldots, \varepsilon_{t_{u}(k)}^{i_{u}} \right), \\
\tilde{G}_{k}^{(0,m)} & := \tilde{G}_{k}^{(0,m)} \left( \varepsilon_{t_{m+1}(k)}^{j_{m+1}}, \ldots, \varepsilon_{t_{v}(k)}^{j_{v}} \right) = G_{k}^{(0)} \left( \varepsilon_{k}^{j_{1}}, \ldots, \varepsilon_{k}^{j_{m}}, \varepsilon_{t_{m+1}(k)}^{j_{m+1}}, \ldots, \varepsilon_{t_{v}(k)}^{j_{v}} \right).
\end{align*}
\]
Let us denote
\[ \tilde{F}_k^{(0,1)} = \tilde{F}_k^{(0,1)}(\varepsilon_{t_1(k)}) := E Y \left[ F_k^{(0)} | B_k^1 \right], \]
\[ \tilde{G}_k^{(0,1)} = \tilde{G}_k^{(0,1)}(\varepsilon_{t_1(k)}) := E Y \left[ G_k^{(0)} | B_k^1 \right], \]
and for \( m > 1 \)
\[ \tilde{F}_k^{(0,m)} = \tilde{F}_k^{(0,m)}(\varepsilon_{t_m(k)}) := E Y \left[ \tilde{F}_k^{(0,m-1)} | B_k^m \right], \]
\[ \tilde{G}_k^{(0,m)} = \tilde{G}_k^{(0,m)}(\varepsilon_{t_m(k)}) := E Y \left[ \tilde{G}_k^{(0,m-1)} | B_k^m \right]. \]

In the same way to Step 1, for \( Y \) fixed, is obtain
\[ Lip \left( \tilde{F}_k^{(0,m)} \right) \leq Lip(f) \left| a_k (y^{\infty}) \right|, \quad \| \tilde{F}_k^{(0,m)} \| \leq \| f \| \leq 1, \]
\[ Lip \left( \tilde{G}_k^{(0,m)} \right) \leq Lip(g) \left| a_k (y^{\infty}) \right|, \quad \| \tilde{G}_k^{(0,m)} \| \leq \| g \| \leq 1, \]
and
\[ |cov \left( F^{(0)}, G^{(0)} \right) | Y, B_\infty^+ \leq \psi \left( \left| d^{(0,-n)}_i (Y), d^{(0,-n)}_j (Y), Lip(f), Lip(g) \right| \lambda(r) \right), \]
\[ |cov \left( F^{(0)}, G^{(0)} \right) | Y \leq \psi \left( \left| d^{(n)}_i (Y), d^{(n)}_j (Y), Lip(f), Lip(g) \right| \lambda(r) \right), \]
where \( d^{(0,k)}_i (Y) = \sum_{m=1}^n \| a_k (y^{\infty}) \| \) and \( d^{(n,k)}_i (Y) = \sum_{m=1}^n \| a_k (y^{\infty}) \| \).

Getting from equations (13) and (14),
\[ |cov \left( F_n, G_n \right) | Y \leq \psi \left( \left| d^{(0,n)}_i - d^{(0,n-1)}_i , d^{(n,n)}_j - d^{(n-1,n)}_j , Lip(f), Lip(g) \right| \lambda(r) \right). \]

Finally, from (11), (12) and (15), we have that
\[ |cov \left( F^{(n)}, G^{(n)} \right) | Y \leq \psi \left( \left| d^{(n)}_i , d^{(n)}_j , Lip(f), Lip(g) \right| \lambda(r) \right). \]

Part b: Taking \( n \to \infty \) yields
\[ |cov(F, G|Y) \leq \psi (d_\infty (Y), d_\infty (Y), Lip(f), Lip(g))\lambda(r), \]
where
\[ d_\infty (Y) = \sum_{m=1}^n \| a(y^{\infty}) \|, \quad \text{and} \quad d_\infty (Y) = \sum_{m=1}^n \| a_k (y^{\infty}) \|. \]

Part c: For \( f, g \in \Delta^{(u)} \times \Delta^{(v)} \) and \( Z_i = H \left( y^i, \{ \varepsilon_{\iota(k)} \} \right) \) we denote
\[ f = f \left( Z^{i_1}_1, \ldots, Z^{i_k}_k \right), \quad \text{and} \quad g = g \left( Z^{i_1}_1, \ldots, Z^{i_k}_k \right). \]

Then, reindexing the sequences \( \{ \varepsilon_{\iota(k)} \} \) by \( \{ \varepsilon_{\iota(k)} \} \) it holds
\[ |cov(f, g|Y) \leq \psi (d_\infty (Y), d_\infty (Y), Lip(f), Lip(g))\lambda(r). \]

Furthermore, we can verify that \( E^Y \left[ ||Z_i||^2 \right] \leq \| a(y^i) \| = V(y^i) \). Therefore, \( Z \) is a \( (\lambda, Y) \)-weakly dependent doubly stochastic process. Analogously, the result can be proved for the cases of \( \eta, \theta, \kappa \) or \( \kappa' \)-weak dependence.
7.2 Proof of Theorem 3

This proof is an adaptation for the case doubly stochastic of the proof of Lemma 3.1 given in [3].

Proof. Let \( \xi_k^{i-1,(s)} = \xi_k^{i-1} \mathbb{1}_{|i|<s} \) and \( \xi_k^{i,(s)} = \mathcal{H}_{\varepsilon} \left( \left\{ \xi_k^{i-1,(s)} \right\}_{i \in \mathbb{Z}} \right) \), and \( Z_t^{i,(s)} = \mathcal{H} \left( y, \left\{ \xi_k^{i,(s)} \right\}_{k \in \mathbb{Z}} \right) \).

Let \( f \in \Delta_1^{(u)} \) and \( g \in \Delta_1^{(v)} \) with \( u, v \in \mathbb{N} \), and \( i_1 \leq \ldots \leq i_u \leq i_u + r < j_1 \leq \ldots \leq j_v \) with \( r > 2s \). We denote

\[
\begin{align*}
 f &= f(Z_{i_1}^{j_1}, \ldots, Z_{i_u}^{j_u}), \\
g &= g(Z_{i_1}^{j_1}, \ldots, Z_{i_v}^{j_v}).
\end{align*}
\]

The sequences \( \{ \xi_k^{i,(s)} \}_{i \leq t} \) and \( \{ \xi_k^{i,(s)} \}_{i \geq t+r} \) are independent if \( r > 2s \). Then, we have that \( f^{(s)} \) and \( g^{(s)} \) are independent and thus

\[
\begin{align*}
|\text{cov}(f, g|Y)| & \leq |\text{cov}(f - f^{(s)}, g|Y)| + |\text{cov}(f^{(s)}, g - g^{(s)}|Y)| \\
& \leq 2\|g\|_{\infty} E^y \left[ |f - f^{(s)}| \right] + 2\|f\|_{\infty} E^y \left[ |g - g^{(s)}| \right] \\
& \leq 2Lip(f) \sum_{m=1}^{u} E^y \left[ |Z_{im}^{t_m} - Z_{im}^{t_m,(s)}| \right] + 2Lip(g) \sum_{m=1}^{v} E^y \left[ |Z_{im}^{t_m} - Z_{im}^{t_m,(s)}| \right].
\end{align*}
\]

We can verify that, for almost all \( y^i \), \( E^{y^i} \left[ |Z_t^{i,(s)}|^{\frac{1}{2}} \right] = \|c(y^{i})\|_{2} \) and conditions (32), (35) imply

\[
E^{y^i} \left[ |Z_t^{i,(s)} - Z_t^{i,(s)}|^{\frac{1}{2}} \right] \leq \left( \sum_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} \sum_{i_{1}=1}^{\infty} \sum_{j_{1}=1}^{\infty} \sum_{j_{2}=1}^{\infty} \ldots \sum_{j_{k}=1}^{\infty} c_{k}^{2} \right)^{\frac{1}{2}} \|c(y^{i})\|_{2} \delta_{m}. \tag{46}
\]

From condition C2 we have that \( \|c(y^{i})\|_{2} < \infty \mu \) a.s. Therefore, for \( \eta(r) \leq 2\delta_{r/2} \) and \( V(y) = \|c(y)|_{2} \) the result follows.

7.3 Proof of Theorem 4

This proof is an adaptation for the case doubly stochastic of the proof of Lemma 3.2 given in [3].

Proof. Let \( \xi_k^{i-1,(r)} = \xi_k^{i-1} \mathbb{1}_{|i|<r} \), \( \xi_k^{i,(r)} = \mathcal{H}_{\varepsilon} \left( \left\{ \xi_k^{i-1,(r)} \right\}_{i \in \mathbb{Z}} \right) \), and \( Z_t^{i,(r)} = \mathcal{H} \left( y, \left\{ \xi_k^{i,(r)} \right\}_{k \in \mathbb{Z}} \right) \).

Let \( f \in \Delta_1^{(u)} \) and \( g \in \Delta_1^{(v)} \) with \( u, v \in \mathbb{N} \), and \( i_1 \leq \ldots \leq i_u \leq i_u + r < j_1 \leq \ldots \leq j_v \) with \( r > 0 \). We denote

\[
\begin{align*}
 f &= f(Z_{i_1}^{j_1}, \ldots, Z_{i_u}^{j_u}), \\
g &= g(Z_{i_1}^{j_1}, \ldots, Z_{i_v}^{j_v}).
\end{align*}
\]

\[
\begin{align*}
 g^{(r)} &= g(Z_{i_1}^{j_1,(r)}, \ldots, Z_{i_v}^{j_v,(r)}).
\end{align*}
\]
The sequences \( \{\xi_i^r\}_{i \leq r} \) and \( \{\xi_k^{(r)}\}_{i > r} \) are independent if \( r > 0 \). Then, \( f \) and \( g^{(r)} \) are independent and so

\[
|\text{cov} (f, g | Y) | = |\text{cov} (f, g^{(r)} | Y) |
\leq 2 \| f \|_\infty \text{Lip}(g) \sum_{m=1}^{v} \mathbb{E}^Y \| Z_{i_m}^r - Z_{i_m}^{j^{(r)}} \|.
\]

On the other hand, conditions (32), (35) imply \( \mathbb{E}^y \| Z_t^r - Z_t^{j^{(s)}} \| \leq \| c(y') \|_2 2 \delta_r \). Furthermore, from condition C2, we can verify that \( \mathbb{E}^y \| Z_t^r - Z_t^{j^{(s)}} \| \leq 2 \| c(y') \|_2 < \infty \) \( a.s. \). Therefore, we obtain the result for \( \theta(r) \leq 2 \delta_r \) and \( V(y) = \| c(y) \|_2 \).

\[ \square \]

### 7.4 Proof of Central Limit Theorems 5 and 6

In this section we proof the Central Limit Theorems 5 and 6 for the aggregation of weakly dependent doubly stochastic processes under condition K2, K4 and K5. The CLT are obtained by using Berstein’s blocks arguments. This proof is similar to proof of CLT given in (Section 7.1, [3]) but it is adapted in the context of sequence weakly dependent of doubly stochastic processes.

Let \( f(z) = \exp^{-izx} \), \( f \in C^1(\mathbb{R}) \) with bounded derivatives up to order 3. In the following, for \( x, t \) fixed, we prove that

\[
|\Delta_N| = |\mathbb{E}^y [f(X_t^N(Y)) - f(X_t)]| \xrightarrow{N \to \infty} 0,
\]

where \( X_t^N(Y) \) is the partial aggregation process and \( X \) is a gaussian process with covariance function \( \Gamma(t) \).

Let us consider three sequences of positives integers \( p = \{p(N)\}_{N \in \mathbb{N}}, \ q = \{q(N)\}_{N \in \mathbb{N}} \) and \( r = \{r(N)\}_{N \in \mathbb{N}} \) such that:

- \( \lim_{N \to \infty} \frac{p(N)}{N} = \lim_{N \to \infty} \frac{q(N)}{p(N)} = 0 \).
- \( r(N) = \frac{N}{p(N) + q(N)} \), thus \( \lim_{N \to \infty} r(N) = \infty \).

These sequences are chosen to form the Berstein’s blocks \( I_1, ..., I_r \) and \( J_1, ..., J_r \) defined by:

\[
I_m = \{(m-1)(p(N) + q(N)) + 1,..., (m-1)(p(N) + q(N)) + p(N)\}.
J_m = \{(m-1)(p(N) + q(N)) + p(N) + 1,..., m(p(N) + q(N))\}.
J_r = \{r(p(N) + q(N)) + 1,..., N\}.
\]

Let \( I = \bigcup_{m=1}^{r} I_m, \ J = \bigcup_{m=1}^{r} J_m \) and \( U_m = \sum_{i \in I_m} Z_i^r \). Let \( \mathcal{N}, \mathcal{N}_1, ..., \mathcal{N}_r \) be zero-mean Gaussian r.v. independent of the innovations \( \{\varepsilon_i\} \) and such that \( \mathbb{E}[\|\mathcal{N}_m\|^2] = \mathbb{E}[\|U_m\|^2] \). We also consider a sequence \( U_1^*, ..., U_r^* \) of mutually independent r.v. such that, given \( Y, U_m^* \) has the same distribution as \( U_m \). We take \( B_N = \sqrt{N} \), then \( \Delta_N \) is decomposed as

\[
\Delta_N(Y) = \sum_{i=1}^{4} \Delta_{i,N}(Y), \quad (47)
\]
where

\[
\begin{align*}
\Delta_{1,N}(Y) &= E^V \left[ f \left( X_i^N(Y) \right) - f \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{r} U_j \right) \right], \\
\Delta_{2,N}(Y) &= E^V \left[ f \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{r} U_j \right) - f \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{r} U_j^* \right) \right], \\
\Delta_{3,N}(Y) &= E^V \left[ f \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{r} U_j^* \right) - f \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{r} N_j \right) \right], \\
\Delta_{4,N}(Y) &= E^V \left[ f \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{r} N_j \right) - f(\Gamma(0,N)) \right].
\end{align*}
\]

We now define a truncation procedure in order to be able to use the previous weak dependence condition and applying the Lindenberg method.

For \( T > 1 \), define \( f_T(z) = (z \vee -T) \wedge T, z \in \mathbb{R} \). Then \( \text{lip}(f_T) = 1 \) and \( \|f_T\|_\infty = T \).

**Lemma 2.** \( E[|Z_i - f_T(Z_i)|^k] \leq 2E[|Z_i|^m]T^{-(m-k)}, \) for \( k \in \mathbb{N} \) and \( m > 0 \).

**Proof.** Applying the Holder’s inequalities yield

\[
E[|Z_i - f_T(Z_i)|^k] \leq E[|Z_i^k + T^k| I_{|Z_i| \geq T}]
\]
\[
\leq 2E[|Z_i|^k I_{|Z_i| \geq T}]
\]
\[
\leq 2E[|Z_i|^m]^\frac{k}{m} P(|Z_i| \geq T)^{1-\frac{k}{m}}
\]
\[
\leq 2E[|Z_i|^m]^\frac{k}{m} \left( \frac{E[|Z_i|^m]}{T^m} \right)^{1-\frac{k}{m}}
\]
\[
\leq 2E[|Z_i|^m]T^{-(m-k)}.
\]

**Lemma 3.**

If \( Z \) is \((\lambda, Y)\)-weakly dependent then

\[
|\text{cov}(Z_i^1, Z_i^2) \leq (6E[|Z_i^1|^2 + \delta] + 2(E[V(y)] + E[V(y)]^2))\lambda(i-j)^{1-\frac{k}{m}}.
\]

If \( Z \) is \((\kappa, Y)\)-weakly dependent then

\[
|\text{cov}(Z_i^1, Z_i^2) \leq (6E[|Z_i^1|^2 + \delta] + E[V(y)]^2)\kappa(i-j).
\]

**Proof.**

\[
|\text{cov}(Z_i^1, Z_i^2) \leq |\text{cov}(Z_i^1 - f_T(Z_i^1), Z_i^2)| + |\text{cov}(f_T(Z_i^1), Z_i^2 - f_T(Z_i^2))| + |\text{cov}(f_T(Z_i^1), f_T(Z_i^2))|.
\]

Let \( m = 2 + \delta \) and \( q = \frac{m}{m-1} \), then \( \frac{1}{m} + \frac{1}{q} = 1 \). The term (i) is bounded applying Lemma 2 and Holder’s inequality

\[
|\text{cov}(Z_i^1 - f_T(Z_i^1), Z_i^2)| \leq E[|Z_i^1 - f_T(Z_i^1)||Z_i^2|]
\]
\[
\leq E[|Z_i^1 - f_T(Z_i^1)|^q]^{\frac{1}{q}} E[|Z_i^1|^m]^{\frac{1}{m}}
\]
\[
\leq (2E[|Z_i|^m]T^{m-q})^{\frac{k}{m}} E[|Z_i|^m]^{\frac{k}{m}}
\]
\[
\leq 2E[|Z_i|^m]T^{-(m-2)}.
\]
For the term (ii), from Lemma 2, we have that
\[
|\text{cov}(f_T(Z^i_1), Z^j_1 - f_T(Z^j_1))| \leq 2TE[|Z^i_1 - f_T(Z^j_1)|] \\
\leq 4E[|Z^i_1|^m] T^{-m-2}.
\]

We now consider a trajectory \( Y \) fixed, then if \( Z \) is \((\lambda, Y)\)-weakly dependent
\[
|\text{cov}(f_T(Z^i_1), f_T(Z^j_1)|Y|) \leq (TV(y^i) + TV(y^j) + V(y^i)V(y^j)) \lambda(i - j).
\]
Taking expectation with respect to \( Y \) and applying Holder’s and Jensen’s inequalities we can bound the (iii) term by
\[
|\text{cov}(f_T(Z^i_1), f_T(Z^j_1)|Y|) \leq (2T\mathbb{E}[V(y)] + \mathbb{E}[V(y)]^2)\lambda(i - j) \\
\leq 2T(\mathbb{E}[V(y)] + \mathbb{E}[V(y)]^2)\lambda(i - j).
\]
Finally, taking \( T = \lambda(i - j)^{-\frac{m+2}{m-2}} \)
\[
|\text{cov}(Z^i_1, Z^j_1)| \leq (6E[|Z^i_1|^m] T^{-m-2} + 2T\mathbb{E}[V(y)] + \mathbb{E}[V(y)]^2))\lambda(i - j) \\
\leq (6E[|Z^i_1|^{2+\delta}] + 2(\mathbb{E}[V(y)] + \mathbb{E}[V(y)]^2))\lambda(i - j) \tag{\text{\ref*{lem}}.}
\]
In the case of \((\kappa, Y)\)-weakly dependent, we have
\[
|\text{cov}(f_T(Z^i_1), f_T(Z^j_1)|Y|) \leq V(y^i)V(y^j)\kappa(i - j).
\]
Then, if we take \( T = \kappa(i - j)^{-\frac{m+2}{m-2}} \), in a similar way to the case of \((\lambda, Y)\)-weakly dependent it yields
\[
|\text{cov}(Z^i_1, Z^j_1)| \leq 6E[|Z^i_1|^m] T^{-m-2} + \mathbb{E}[V(y)]^2\kappa(i - j) \\
\leq (6E[|Z^i_1|^{2+\delta}] + \mathbb{E}[V(y)]^2)\kappa(i - j).
\]

\[\square\]

**Lemma 4.** \(|\Delta_{1,N}(Y)| = \mathcal{O}\left(\left(\frac{N^{-p_r}}{N}\right)^{\frac{1}{2}}\right)\) \(\nu - a.s.\)

**Proof.** Using Taylor’s expansion up to the first order, we obtain
\[
|\Delta_{1,N}(Y)| \leq \|f'|\|_\infty \frac{1}{\sqrt{N}} \mathbb{E}^Y \left[ \sum_{i \in J} Z^i_t \right] \\
\leq \|f'|\|_\infty \frac{1}{\sqrt{N}} \mathbb{E}^Y \left[ \sum_{i,j \in J} Z^i_t Z^j_t \right]^{\frac{1}{2}} \\
= \|f'|\|_\infty \left(\frac{N - p_r}{N}\right)^{\frac{1}{2}} \left(\frac{1}{N - p_r} \sum_{i,j \in J} \Psi_0(y^i, y^j)\right)^{\frac{1}{2}}.
\]
On the other hand, since \( Y \) is stationary then condition K4 implies
\[
\frac{1}{N - p_r} \sum_{i,j \in J} \Psi_0(y^i, y^j) \xrightarrow{\nu-a.s.} \Gamma(0).
\]
Therefore, \(\frac{1}{N - p_r} \sum_{i,j \in J} \Psi_0(y^i, y^j) = \mathcal{O}(1)\ \nu - a.s.,\) and the result holds. \(\square\)
Lemma 5. \(|\Delta_{4,N}| = o(1) \nu - a.s.\)

Proof. Taylor’s expansion up to the second order entails

\[
|\Delta_{4,N}| \leq \frac{1}{2} \|f''\|_{\infty} \frac{rp}{N} \left| \frac{1}{rp} \sum_{j=1}^{r} \mathbb{E}Y[|U_j|^2] - \Gamma(0) \right| + \frac{1}{2} \|f''\|_{\infty} \frac{N - rp}{N} \Gamma(0).
\]

From condition K4 we have

\[
\frac{1}{rp} \sum_{j=1}^{r} \mathbb{E}Y[|U_j|^2] = \frac{1}{rp} \sum_{j=1}^{r} \sum_{i_1, i_2 \in I_j} \Psi_0(y_{i_1}^j, y_{i_2}^j) \overset{\nu - a.s.}{\to} \frac{N}{N} \Gamma(0).
\]

Furthermore, \(\frac{N - rp}{N} \to 0\) as \(N \to \infty\), so that \(|\Delta_{4,N}| = o(1)\). \qed

Lemma 6.

In the \((\lambda, Y)\)-weak dependence setting,

\[
|\Delta_{2,N}| \leq \left( \frac{r^2 p^2}{\sqrt{N}} + \frac{rp}{\sqrt{N}} + \frac{r^2 p^2}{N} \right) \lambda(r) \nu - a.s.
\]

Moreover in the case of \((\kappa, Y)\)-weak dependence

\[
|\Delta_{2,N}| \leq \left( \frac{r^2 p^2}{N} \right) \kappa(r) \nu - a.s.
\]

Proof. Let define, for \(j = 1 \ldots r\),

\[
\Lambda_j = \mathbb{E}Y[f(W_j + u_j) - f(W_j + u_j^*)],
\]

where \(u_j = \frac{1}{\sqrt{N}} U_j, u_j^* = \frac{1}{\sqrt{N}} U_j^*\), \(W_j = \omega_j + \sum_{i > j} u_i^*\) and \(\omega_j = \sum_{i < j} u_i\).

Using the properties of the exponential function \(f(z) = \exp^{-ixz}\) and the independence properties of the variables \(\{U_j^*\}\) we have

\[
\Lambda_j = \left( \mathbb{E}Y[f(\omega_j)f(u_j)] - \mathbb{E}Y[f(\omega_j)]\mathbb{E}Y[f(u_j^*)] \right) \mathbb{E}Y \left[ f \left( \sum_{i > j} u_i^* \right) \right].
\]

Then, we have

\[
|\Delta_{2,N}| \leq \sum_{j=1}^{r} |\Lambda_j|.
\]

From equation (48) and applying the definition of \((\lambda, Y)\)-weak dependence, for \(Y\) fixed, it follows that

\[
|\Lambda_j| \leq \|f\|_{\infty} \left| \text{cov} \left( f \left( \frac{1}{\sqrt{N}} \sum_{m < j} \sum_{i \in I_m} Z_i^l \right), f \left( \frac{1}{\sqrt{N}} \sum_{i \in I_j} Z_i^l \right) \right) \right|
\]

\[
\leq \|f\|_{\infty} \text{lip}(f) \|f\|_{\infty} \frac{1}{\sqrt{N}} \sum_{m < j} \sum_{i_1, i_2 \in I_m} V(y_{i_1}^j) + \text{lip}(f) \|f\|_{\infty} \frac{1}{\sqrt{N}} \sum_{i_2 \in I_j} V(y_{i_2}^j)
\]

\[
+ \text{lip}(f) \text{lip}(f) \frac{1}{\sqrt{N}} \sum_{m < j} \sum_{i_1 \in I_m} V(y_{i_1}^j) \frac{1}{\sqrt{N}} \sum_{i_2 \in I_j} V(y_{i_2}^j) \lambda(q).
\]

Furthermore, by the SLLN we have that condition K5 implies

\[
\sum_{m < j} \sum_{i_1 \in I_m} V(y_{i_1}^j) = O(rp), \quad \text{and} \quad \sum_{i_2 \in I_j} V(y_{i_2}^j) = O(p).
\]

Then, from equation (49), the results follows for \((\lambda, Y)\)-weak dependence. In the case of \((\kappa, Y)\)-weak dependence the proof is similar. \qed
Lemma 7. \(|\Delta_{3,N}| \leq \frac{r^2 \delta^2}{N^{1+\frac{\alpha}{2}}}, \nu - a.s.\)

Proof. Let define, for \(j = 1 \ldots r\),

\[ A_j = \mathbb{E}^Y [f(W_j + u_j^*) - f(W_j)] \]

where \(u_j^* = \frac{1}{\sqrt{N}} U_j^*, \ u_j^* = \frac{1}{\sqrt{N}} N_j, \ W_j = \sum i < j u_i^* + \sum_{i > j} u_i'\).

Using the properties of exponential function \(f(z) = \exp^{-izz}\) and the independence properties of the variables \(\{N_j^*\}\), we have

\[ A_j = \mathbb{E}^Y [f(u_j^*) - f(u_j')] \mathbb{E}^Y [f(W_j)]. \tag{50} \]

Thus,

\[ |\Delta_{3,N}| \leq \sum_{j=1}^r |A_j|. \tag{51} \]

As \(\mathbb{E}^Y [u_j^*] = \mathbb{E}^Y [u_j'] = 0\) and \(\mathbb{E}^Y [|u_j|^2] = \mathbb{E}^Y [|u_j^*|^2]\), then taking Taylor’s expansion up to order 2 or 3 respectively yield:

\[ \mathbb{E}^Y [f(u_j^*) - f(u_j')] = \frac{1}{2} \mathbb{E}^Y \left[ f^{(2)}(\theta^*) (u_j^*)^2 - f^{(2)}(\theta^*) (u_j')^2 \right], \]

\[ \mathbb{E}^Y [f(u_j') - f(u_j')] = \frac{1}{6} \mathbb{E}^Y \left[ f^{(3)}(\theta^*) (u_j^*)^3 - f^{(3)}(\theta^*) (u_j')^3 \right]. \]

This implies that

\[ |A_j| \leq \left( \frac{1}{2} \|f^{(2)}\|_\infty \mathbb{E}^Y [(u_j^*)^2 + (u_j')^2] \right) \land \left( \frac{1}{6} \|f^{(3)}\|_\infty \mathbb{E}^Y [(u_j^*)^3 + (u_j')^3] \right) \leq \frac{1}{\Delta} \|f\|_\infty \left( \mathbb{E}^Y [U_j^{2+\delta}] + \mathbb{E}^Y [|N_j^*|^{2+\delta}] \right). \tag{52} \]

Since \(\{N_j^*\}\) is a sequence of gaussian r.v.

\[ \mathbb{E}^Y [|N_j^*|^{2+\delta}] \leq 3^{\frac{2+\delta}{2}} \mathbb{E}^Y [U_j^2]^{\frac{2+\delta}{2}}. \]

On the other hand, applying Jensen’s and Minkowski’s inequalities, we get

\[ \mathbb{E}^Y [U_j^2]^{\frac{2+\delta}{2}} \leq \mathbb{E}^Y [U_j^{2+\delta}] \leq \left( \sum_{i \in I_j} \mathbb{E}^Y [Z_i (y^i)^{2+\delta}] \right)^{\frac{2+\delta}{4}}. \]

From condition K2\(\delta\) and applying the SLLN it comes

\[ \mathbb{E}^Y [U_j^{2+\delta}] + \mathbb{E}^Y [|N_j^*|^{2+\delta}] = O(p^{2+\delta}). \]

So, from (50) and (51) the result holds.

Let us assume that there exists \(\alpha, \beta\) with \(0 < \beta < \alpha < 1\) such that for \(N \in \mathbb{N}\)

\[ p(N) = N^\alpha, \quad \text{and} \quad q(N) = N^\beta. \]

In the case of \((\lambda,Y)\)-weak dependence: from Lemmas 1, 2, 3 and 7 we obtain that

\[ \Delta_{1,N} = O \left( N^{\frac{2+\alpha}{\alpha}} + N^{\frac{2+\beta}{\beta}} \right), \]

\[ \Delta_{2,N} = O \left( N^{-\alpha-\lambda\beta+\frac{1}{2}} + N^{-\lambda\beta+\frac{1}{2}} + N^{-\lambda\beta+1} \right), \]

\[ \Delta_{3,N} = O \left( N^{\alpha(1+\delta)-\frac{1}{2}} \right), \]

\[ \Delta_{4,N} = o(1). \]
Then, there exists \( \theta > 0 \) such that \( |\Delta_N| \leq N^{-\theta} \) if and only if
\[
0 < \beta < \alpha < \frac{\delta}{2(1 + \delta)} \quad \text{and} \quad \frac{3}{2} - \lambda \beta < \alpha. \tag{53}
\]

We can determine the values of \( \alpha \) and \( \beta \) so that condition \( (53) \) holds, whenever \( \lambda > 2 + \frac{2}{\delta} \). Moreover, condition K2 and K4 are satisfied for \( \kappa \), \( \sigma \delta \)-dependent. Further, condition K3 and K5 are satisfied for \( \kappa = 2 \), \( \sigma \delta \)-dependent. Moreover, condition K2, K3 and K5 are satisfied for \( \kappa > 2 + \frac{2}{\delta} \), \( \sigma \delta \)-dependent.

This allows us to prove that, for all \( t \) fixed, \( \{X_t^N(Y)\} \) converge in distribution to a mean zero Gaussian random variable with variance \( \Gamma(0) \).

We can write \( \hat{X}_N(Y) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{Z}^i \), where \( \hat{Z}^i = \sum_{k=1}^{d} \alpha_k Z^i_k \).

Using a similar technique to the one applied in the proof of Theorem 2, we can verify easily that if \( Z = \{Z^i\} \) is a \((\epsilon, Y)\)-weakly dependent doubly stochastic processes then \( \{\hat{Z}^i\} \) is also \((\epsilon, Y)\)-weakly dependent. Moreover, condition K2, K4 are satisfied for \( \hat{Z}^i \). Furthermore, the variance \( \sigma_N^2(Y) \) of \( \hat{X}_N(Y) \) is such that
\[
\sigma_N^2(Y) = \sum_{k,l=1}^{d} \sigma_k \Gamma(t_k - t_l) \alpha_l \frac{\nu-a.s.}{\sum_{k,l=1}^{d} \sigma_k} \Gamma(t_k - t_l) \alpha_l = \sigma^2.
\]

Then, in the same way as before we show that \( \hat{X}_N(Y) \) converge weakly, \( \nu - a.s. \), to a centered Gaussian random variable with variance \( \sigma^2 \). Therefore, we obtain that the process \( X^N(Y) = \{X^N(Y) : t \in \mathbb{Z}\} \) converges in distribution, \( \nu - a.s. \), to a centered Gaussian process \( X = \{X_t : t \in \mathbb{Z}\} \) with covariance function \( \Gamma \).

### 7.5 Proof of Lemma 1

We prove, under weak dependence property and condition E2, that there exist \( d > 1 \) such that \( |\chi(r)| \leq O(r^{-d}) \), so \( \chi \) is a weak interaction in \( \ell^1 \).

**Proof.** Let \( T > 1 \), we define \( f_T(z) = (z \vee -T) \wedge T, \ z \in \mathbb{R} \), \lip(f_T) = 1 \) and \( \|f_T\|_\infty = T \). By Lemma 2, for \( k \in \mathbb{N} \), we have
\[
\mathbb{E}[|\varepsilon^i_k - f_T(\varepsilon^i_k)|^d] \leq 2 \mathbb{E}[|\varepsilon^i_k|^2 + T(2 + \delta - k)]^d.
\]

Then if \( \{\varepsilon^i_k\} \) is \( \lambda \)-weakly dependent, following the proof of Lemma 3, we have
\[
|\chi(i - j)| \leq |\text{cov}(\varepsilon^i_k - f_T(\varepsilon^i_k), \varepsilon^i_j)|
+ |\text{cov}(f_T(\varepsilon^i_k), \varepsilon^i_k - f_T(\varepsilon^i_j))|
+ |\text{cov}(f_T(\varepsilon^i_k), f_T(\varepsilon^i_j))|
\leq 6 \mathbb{E}[|\varepsilon^i_k|^2 + T(2 + \delta - k)]^d + (2 + 1)\lambda(i - j)
\leq (6 \vee \mathbb{E}[|\varepsilon^i_k|^2 + T(2 + \delta - k)]^d)(2 + \delta - k).
\]

Finally, taking \( T = \lambda(i - j)^{\frac{1}{1 + \delta}} \), we obtain that \( |\chi(i - j)| \leq O(\lambda(i - j)^{\frac{1}{1 + \delta}}) \).

Otherwise in the case of \( \kappa \)-weakly dependent, we obtain
\[
|\chi(i - j)| \leq 6 \mathbb{E}[|\varepsilon^i_k|^2 + T(2 + \delta - k)]^d + \kappa(i - j).
\]

Then, by choosing \( T \) such that \( \kappa(i - j) = 6 \mathbb{E}[|\varepsilon^i_k|^2 + T(2 + \delta - k)]^d \), it follows \( |\chi(i - j)| \leq O(\kappa(i - j)) \).
7.6 Proof of the SLLN for $\Gamma^N(Y)$: case of DSV* processes.

The following proof goes to the same lines of the one given in [1] for SLLN in the case of linear processes.

**Proof.** The proof of theorem will be presented in three parts.

**Part 1:** Since $\chi \in \ell_1$, then $\frac{[x]^N}{N} \leq \|\chi\|_1$ and $\frac{[x]^N}{N}$ converge to $s_k$, for each $k$, with $|s_k| \leq \|\chi\|_1$. On the other hand, condition C2 implies $\sum_{k=1}^{\infty} \phi_k(\tau) < \infty$. Then, for all $\tau \in \mathbb{Z}$, we have

$$\sum_{k=1}^{\infty} \phi_k(\tau) \left[\frac{[x]^N}{N}\right] \xrightarrow{N \to \infty} \sum_{k=1}^{\infty} \phi_k(\tau) s_k < \infty.$$  

Therefore,

$$R_N(\tau) := \mathbb{E}[\Gamma^N(\tau, Y)] = \sum_{k=1}^{\infty} \gamma_k(\tau) + \sum_{k=1}^{\infty} \phi_k(\tau) \left[\frac{[x]^N}{N}\right] \xrightarrow{N \to \infty} \Gamma(\tau).$$

For any values of sequence $\{s_k\}$, $R_N(\tau)$ has a non-zero limit.

**Part 2:** Let $M_{\tau,k}(y^i) = \mathbb{E}[\Psi_{\tau,k}(y^i, y^j)]$, for $i \neq j$, where $\mathbb{E}[\cdot]$ is the conditional expectation with respect to $y^i$. Then, $\{M_{\tau,k}(y^i) : i \in \mathbb{N}\}$ is an i.i.d. sequence with $\mathbb{E}[M_{\tau,k}(y^i)] = \phi_k(\tau)$.

Let $\mathbb{H}$ be the Hilbert space generated by $\{\Psi_{\tau,k}(y^i, y^j) : k \in \mathbb{N}, 1 \leq i \neq j \leq N\}$, $\mathbb{H}_1$ the linear space generated by $\{\Psi_{\tau,k}(y^i, y^j) - M_{\tau,k}(y^i) - M_{\tau,k}(y^j) + \phi_k(\tau) : k \in \mathbb{N}, 1 \leq i < j \leq N\}$, $\mathbb{H}_2$ the linear space generated by $\{M_{\tau,k}(y^i) + M_{\tau,k}(y^j) \leq \phi_k(\tau) : k \in \mathbb{N}, 1 \leq i < j \leq N\}$ and $C$ the space of constants, then $\mathbb{H}_1, \mathbb{H}_2, C$ form an orthogonal decomposition of $\mathbb{H}$; i.e. $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus C$. This can be checked by realizing that $\mathbb{E}[\Psi_{\tau,k}(y^i, y^j)] = 0 \mu - a.s.$ for $i \neq j$. We define

$$T_N(\tau, Y) = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{\infty} \left(\Psi_{\tau,k}(y^i, y^i) - \gamma_k(\tau)\right),$$

$$Q_N(\tau, Y) = \frac{1}{N} \sum_{1 \leq i \neq j \leq N} \sum_{k=1}^{\infty} \left(M_{\tau,k}(y^i) + M_{\tau,k}(y^j) - 2\phi_k(\tau) \chi^k(i - j)\right),$$

$$U_N(\tau, Y) = \frac{1}{N} \sum_{1 \leq i \neq j \leq N} \sum_{k=1}^{\infty} \left(\Psi_{\tau,k}(y^i, y^j) - M_{\tau,k}(y^i) - M_{\tau,k}(y^j) + \phi_k(\tau)\right) \chi^k(i - j).$$

$\mathbb{H}$’s orthogonal decomposition applied to $\Gamma^N(\tau, Y) - T_N(\tau, Y)$ gives the following orthogonal decomposition

$$\Gamma^N(\tau, Y) - T_N(\tau, Y) = R_N(\tau, Y) + Q_N(\tau, Y) + U_N(\tau, Y).$$

In what follows, we will show that $T_N(\tau, Y), Q_N(\tau, Y)$ and $U_N(\tau, Y)$ converge to zero $\nu - a.s.$ and in $L^1(\nu)$ under some given conditions. If the limits $\nu - a.s.$ and in $L^1(\nu)$ of $\Gamma^N$ exist, then they must be the same.

**Step 1:** ($T_N$’s convergence to zero).

Under condition C2 we have

$$\mathbb{E} \left[\sum_{k=1}^{\infty} \Psi_{\tau,k}(y^i, y^i)\right] \leq \mathbb{E} \left[\|c(y^i)\|_2^2\right] < \infty.$$

Then, for each $\tau \in \mathbb{Z}$, the SLLN implies that $T_N(\tau, Y)$ converges in $L^1(\nu)$ and $\nu - a.s.$ to zero.
Step 2: \((Q_N)'s convergence to zero). We can write
\[
Q_N(\tau, Y) = \frac{2}{N} \sum_{i=1}^{N} \sum_{k=1}^{\infty} \left( M_{\tau,k}(y^i) - \phi_k(\tau) \right) (s_{N+1-i}(k) + s_i(k)),
\]
So that \(\mathbb{E}[Q_N(\tau, Y)] = 0\) and
\[
\mathbb{E}[|Q_N(\tau, Y)|^2] = \frac{\Delta_N}{N^2},
\]
where
\[
\Delta_N = 4 \sum_{i=1}^{\infty} \sum_{k,l=1}^{\infty} \text{cov} \left( M_{\tau,k}(y^i), M_{\tau,l}(y^j) \right) (s_{N-i}(k) + s_i(k)) (s_{N-i}(l) + s_i(l))
\]
\[
\leq 4 \sum_{i=1}^{\infty} \sum_{k,l=1}^{\infty} A_{\tau,k} A_{\tau,l} |s_{N-i}(k) + s_i(k)| |s_{N-i}(l) + s_i(l)|,
\]
with \(A_{\tau,k} = \mathbb{E} \left[ |M_{\tau,k}(y^i) - \phi_k(\tau)|^2 \right]^{\frac{1}{2}}\). Furthermore,
\[
|s_{N-i}(l) + s_i(l)| \leq \sum_{j=1}^{N-i-1} |\chi'(j)| + \sum_{j=1}^{i} |\chi'(j)| \leq 2\|\chi\|_1 \leq 2\|\chi\|
\]
\[
\frac{1}{N^2} \sum_{i=1}^{N} |s_{N-i}(k) + s_i(k)| \leq \frac{\|\chi\|_{N,1}}{N^2}.
\]
Moreover, from condition C2
\[
\sum_{k=1}^{\infty} A_{\tau,k} \leq \sum_{k=1}^{\infty} \mathbb{E} \left[ |\Psi_{\tau,k}(y^i, y^j) - \phi_k(\tau)|^2 \right]^{\frac{1}{2}}
\]
\[
\leq \sum_{k=1}^{\infty} \mathbb{E} \left[ |\Psi_{\tau,k}(y^i, y^j)|^2 \right]^{\frac{1}{2}}
\]
\[
\leq \sum_{k=0}^{\infty} \sum_{l_1 < \ldots < l_k} \mathbb{E}[|c_{k,l_1,\ldots,l_k}(y)|^2]^{\frac{1}{2}} \mathbb{E}[|c_{k,l_1+\tau,\ldots,l_k+\tau}(y)|^2]^{\frac{1}{2}}
\]
\[
\leq \mathbb{E} \left[ \|c(y)\|_2^2 \right] < \infty.
\]
Therefore,
\[
\Delta_N \leq 8\|\chi\|_1 \left( \sum_{k=1}^{\infty} A_{\tau,k} \right)^2 \left( \|\chi\|_{N,1} \right) \text{ and } \mathbb{E}[|Q_N(\tau, Y)|^2] \leq O \left( \frac{\|\chi\|_{N,1}}{N^2} \right).
\]

Thus, we obtain that \(\sum_N \mathbb{E}[|Q_N(\tau, Y)|^2] < \infty\). Since \(\|\chi\|_{N,1} = O(N)\) then \(Q_N(\tau, Y)\) converges to zero in \(L^2(\nu)\).

We have two cases:
- Case 1: \(\Delta_N\) converges to a finite limit, \(\Delta_N = O(1)\).
- Case 2: \(\Delta_N\) converges to +\(\infty\).
In the case 1, we have \( \mathbb{E}[|Q_N(\tau,Y)|^2] = \mathcal{O}(N^{-2}) \), so from Borel-Cantelli’s lemma we derive the \( \nu - a.s. \) convergence to zero of \( Q_N(\tau,Y) \).

In the case 2, to prove \( \lim_{N \to \infty} Q_N(\tau,Y) = 0 \) \( \nu - a.s. \), we can apply Petrov’s Theorem ([24], p 222): let \( Q_N^* = \sum_{i=1}^N \xi_i \) be a sum of independent centered random variables such that its variance \( \Delta_N \) diverge to infinity. Then \( Q_N^* = o(\sqrt{\Delta_N \Theta(\Delta_N)}) \) for all function \( \Theta \) such that \( \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty \).

In our case, \( Q_N = Q_N^*/N \), so it is sufficient to find \( \Theta \) such that

\[
\Delta_N \Theta(\Delta_N) = \mathcal{O}(N^2).
\]

We have from [54] that \( \Delta_N = \mathcal{O}([|\chi|]_{N,1}) \) and \( [|\chi|]_{N,1} = \mathcal{O}(N) \). Then, taking \( \Theta(n) = n \) we obtain

\[
\Delta_N \Theta(\Delta_N) = \Delta_N^2 = \mathcal{O}([|\chi|]^2_{N,1}) = \mathcal{O}(N^2).
\]

Consequently \( Q_N(\tau,Y) \) converges \( \nu - a.s. \) to zero.

**Step 3:** \( (U_N)’s \) convergence to zero.

We consider the kernel defined, for \( i \neq j \), by

\[
\Phi_\tau(y^i, y^j) = \sum_{k=1}^\infty \Phi_{\tau,k}(y^i, y^j)\chi_k(i-j),
\]

where

\[
\Phi_{\tau,k}(y^i, y^j) = \Psi_{\tau,k}(y^i, y^j) - M_{\tau,k}(y^i) - M_{\tau,k}(y^j) + \phi_k(\tau).
\]

This kernel is symmetric and degenerated, i.e., \( \mathbb{E}[\Phi_\tau(y^i, y^j)] = 0 \) \( \mu - a.s. \) Whence we have that \( \mathbb{E}[\Phi_\tau(y^i, y^j)] = 0, \mathbb{E}[\Phi_\tau(y^i, y^j)\Phi_\tau(y^m, y^n)] = 0 \) for \( (i,j) \neq (m,n) \).

In the same way as we bound \( \sum_k A_{\tau,k} \) in the Step 2, we can verify that condition C2 implies

\[
\sigma_\tau = \sum_{k=1}^\infty \mathbb{E}[|\Phi_{\tau,k}(y^i, y^j)|^2]^{1/2} < \infty.
\]

Then writing

\[
U_N(\tau,Y) = \frac{1}{N} \sum_{1 \leq i \neq j \leq N} \Phi_\tau(y^i, y^j),
\]

we obtain

\[
\mathbb{E}[|U_N(\tau,Y)|^2] = \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \sum_{k,l=1}^{\infty} \mathbb{E}[|\Phi_{\tau,k}(y^i, y^j)\Phi_{\tau,l}(y^i, y^j)|^2] \chi_k(i-j)^k \chi_l(i-j)^l \leq \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \sum_{k,l=1}^{\infty} \mathbb{E}[|\Phi_{\tau,k}(y^i, y^j)|^2]^{1/2} \mathbb{E}[|\Phi_{\tau,l}(y^i, y^j)|^2]^{1/2} \chi_k(i-j)^k \chi_l(i-j)^l \leq \left( \sum_{l=1}^{\infty} \mathbb{E}[|\Phi_{\tau,l}(y^i, y^j)|^2]^{1/2} \left( \frac{1}{N^2} [\chi^2]_{N,1} \right)^{1/2} \right)^2 \leq \sigma_\tau^2 \frac{[|\chi|]_{N,1}}{N^2}.
\]

Hence \( \sum_N \mathbb{E}[|U_N(\tau,Y)|^2] < \infty \). Since \( [|\chi|]_{N,1} = \mathcal{O}(N) \) then \( Q_N(\tau,Y) \) converges to zero in \( L^2(\nu) \).

Let us now prove the \( \nu - a.s. \) convergence of \( U_N \). We prove the \( \nu - a.s. \) convergence of \( U_N \) to zero, following the scheme of the classical proof for the SLLN in the case of i.i.d., [24]. See [27] for the Central Limit Theorem.
Let $a > 0$,

$$
P\left(\max_{n \geq N} |U_n(\tau, Y)| \geq 2a\right) \leq \sum_{k=\lceil \sqrt{N} \rceil}^{\infty} P\left(|U_k(\tau, Y)| \geq a\right) + \sum_{k=\lceil \sqrt{N} \rceil}^{\infty} P\left(\max_{n \leq (k+1)^2} |U_n(\tau, Y) - U_k(\tau, Y)| \geq a\right).
$$

From estimation of $E[|U_N(\tau, Y)|^2]$ and applying Tchebychev’s inequality, we have

$$
P(|U_k(\tau, Y)| \geq a) \leq \frac{\sigma^2_k \|\chi\|k^2}{a^2k^4} \leq \frac{\sigma^2_k r_k}{a^2k^2},
$$

where $r_n = \sum_{j=1}^{n} |\chi(j)| < \|\chi\|\ell_1 < \infty$. Then,

$$
\sum_{k=1}^{\infty} \frac{r_k^2}{k^2} < \sum_{k=1}^{\infty} \frac{\|\chi\|\ell_1}{k^2} < \infty.
$$

So, the series $\sum_{k=1}^{\infty} \mathbb{P}(|U_k(\tau, Y)| \geq a)$ converges.

On the other hand,

$$
U_n(\tau, Y) - U_k(\tau, Y) = \frac{1}{n} \sum_{A(n,k^2)} \Phi_\tau(y^i, y^j) + \left(\frac{1}{n} - \frac{1}{k^2}\right) \sum_{1 \leq i < j \leq k^2} \Phi_\tau(y^i, y^j),
$$

where $A(n, k^2) = \{i, j : 1 \leq i < j, k^2 < j \leq n\} \cup \{i, j : 1 \leq j < i, k^2 < i \leq n\}$. So

$$
\max_{k^2 \leq n \leq (k+1)^2} |U_n(\tau, Y) - U_k(\tau, Y)| \leq a_k + b_k,
$$

with

$$
a_k = \max_{k^2 \leq n \leq (k+1)^2} \left|\frac{1}{n} \sum_{A(n,k^2)} \Phi_\tau(y^i, y^j)\right|,
$$

$$
b_k = \max_{k^2 \leq n \leq (k+1)^2} \left|\frac{1}{n} - \frac{1}{k^2}\right| \sum_{1 \leq i < j \leq k^2} \Phi_\tau(y^i, y^j).
$$

Since $a_k \leq \frac{1}{k^4} \sum_{A(k^2,(k+1)^2)} |\Phi_\tau(y^i, y^j)|$, then

$$
P(a_k \geq a) \leq \frac{1}{a^2k^4} \mathbb{E}\left[\left(\sum_{A(k^2,(k+1)^2)} |\Phi_\tau(y^i, y^j)|\right)^2\right] \leq \frac{\sigma^2_k}{a^2k^4} \left(\sum_{A(k^2,(k+1)^2)} |\chi(i - j)|\right)^2 \leq \frac{\sigma^2_k}{a^2k^4} ((k+1)^2 - k^2)^2 r_{k^2}^2.\]
Since $\chi$ is a stationary interaction it holds

$$(k + 1)^2 r_{(k+1)^2} - k^2 r_{k^2} = \sum_{j=k^2+1}^{(k+1)^2} \left( ((k + 1)^2 - j) |\chi(j)| + \sum_{j=1}^{k^2} ((k + 1)^2 - k^2) |\chi(j)| \right)$$

$$\leq (2k + 1) \left( \sum_{j=1}^{(k+1)^2} |\chi(j)| \right)$$

$$\leq 2(k + 1)r_{(k+1)^2}.$$ 

So that

$$\mathbb{P}(a_k \geq a) \leq \frac{4\sigma^2 (k + 1)^2 r_{(k+1)^2}}{a^2 k^4}$$

$$\leq \frac{4\sigma^2 \|\chi\|_{l_1}^2 (k + 1)^2}{a^2 k^4}.$$ 

In the same way, $b_k \leq \frac{2}{\mathbb{P}} \left| \sum_{1 \leq i < j \leq k^2} \Phi_{\tau}(y^i, y^j) \right|$ so

$$\mathbb{P}(b_k \geq a) \leq \frac{4\sigma^2 k^2 \|\chi\|_{k^2, 2}}{a^2 k^6} \leq \frac{4\sigma^2 \|\chi\|_{l_1}^2}{a^2 k^4}.$$ 

Since $\mathbb{P}(\max_{k^2 \leq n \leq (k+1)^2} |U_n(\tau, Y) - U_{k^2}(\tau, Y)| \geq a) \leq \mathbb{P}(a_k \geq a) + \mathbb{P}(b_k \geq a)$ then

$$\sum_{k=1}^{\lfloor \sqrt{N} \rfloor} \mathbb{P} \left( \max_{k^2 \leq n \leq (k+1)^2} |U_n(\tau, Y) - U_{k^2}(\tau, Y)| \geq a \right) < \infty.$$ 

Finally, from Borel-Cantelli’s lemma we derive the $\nu$–a.s. convergence to zero of $U_N(\tau, Y)$. This proves the convergence $\nu$–a.s.

**Part 3:** ($\Gamma^N$’s convergence).

We have proved, in Part 2, that $T_N$, $Q_N$ and $U_N$ converge $\nu$–a.s. and in $L_1(\nu)$ to zero. Then, following Part 3 in the proof of Theorem 1 in [1], we obtain from orthogonal decomposition

$$\Gamma^N(\tau, Y) - R_N(\tau) = T_N(\tau, Y) + Q_N(\tau, Y) + U_N(\tau, Y),$$

that $\Gamma^N(\tau, Y)$ converge in $L^1(\nu)$ and $\nu$–a.s. to $\Gamma(\tau)$.

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