Increased Sensitivity of Higher-Order Laser Beams to Mode Mismatches

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1. INTRODUCTION

Optical higher order modes have a wide range of uses, for example driving micro-machines [1, 2], manipulation of cold atoms [3] and telecommunications [4]. In precision metrology, the performance of current and future gravitational-wave detectors is limited by self-noise of the detectors, which is dominated over a wide frequency band by the quantum noise of the interrogating light field and the thermal noise of the optics. The introduction of non-classical light (also called squeezing) into advanced gravitational-wave detectors [5–7], leaves thermal noise as the fundamentally limiting noise in the detectors’ most sensitive frequency range [8, 9].

There are proposals to use a spatial Higher-Order Mode (HOM) as the carrier mode in the interferometer to mitigate thermal noise [10–12] in gravitational-wave detectors. This technique may also be of interest to other thermal-noise limited optical cavities [13]. Sorazu et al. studied the use of a Laguerre-Gauss 3,3 (LG33) mode in a 10 m suspended optical resonator [14] and noted that astigmatism caused the break up of the LG33 mode into component Hermite-Gauss (HG) modes with similar, but not equal round trip Gouy phase, which resulted in distorted control signals and a poor power coupling into the resonator.

*In-situ* astigmatism actuation has been demonstrated to increased power transfer from the LG33 mode into the cavity mode [15]. This could be combined with recent work on high spatial-order sensors such as scanning, lock in and Spatial Light Modulator (SLM) based phase cameras [16–18] and direct mode analyzers [19]. In addition, matrix heaters in gravitational-wave detectors [20] offer another actuation option.

However, it is well known that the transfer of squeezed light into the interferometer is exceptionally sensitive to optical losses [21]. Mode mismatch can be a dominant source of squeezing loss [22] and a 98% mode-matching target is achievable with Advanced LIGO [23].

This paper revisits the subject of higher-order mode to resonator matching, in the context of HG modes, and derives an increased sensitivity to mode mismatch which scales monotonically with mode index. Hermite-Gauss modes are naturally astigmatic, therefore, they may be more compatible with the long baseline optical resonators used in gravitational-wave detectors [13]. The HG55 has been discussed [13] and in this case, the losses due to waist-size mismatch would be 31 times worse than for the fundamental mode. Our results are derived using a computer algebra system [24] and shown to match numeric integration. A higher-order astigmatic beam passing though the LIGO Output Mode Cleaner is considered as an example of applying these coefficients.

2. ANALYTICAL MODEL

The mode-coupling coefficients were derived in the general case in 1984 by Bayer-Helms [25]. Consider two mode bases, the first with waist \( w_0 \) at \( z_0 \) (typically the mode basis of the incoming light) and the second with waist \( w_1 \) at \( z_1 \) (typically the resonator mode basis). Then, the amplitude coupling of a mode with indices \( n_m \) in the first basis, to mode \( \Pi \) in the second basis is described by, \( k_{n_m,\Pi} \), which is in general complex. In this work, all parameters with an overline correspond to the resonator basis.

For Hermite-Gauss modes, this coupling coefficient is separable [25],

\[
  k_{n_m,\Pi} = k_{n,\Pi} k_{m,\Pi}.
\]
Fig. 1. 1D mode mismatch parameter, \(k_{n,\pi}\), for a waist size only mismatch between the incoming beam and the 1 mm resonator waist size. Solid lines show numerical solutions to Equation 2 and dotted lines show the approximate analytic solutions in Equation 5. See Supplement 1 for an analogous waist-position mismatch.

Reduced to [25],

\[
k_{n,\pi} = \int_{-\infty}^{\infty} u_n(x',z) \hat{\pi}_\pi(x',z) \, dx'.
\] (2)

Considering only, \(w_0 \neq 0, 2z_0 = z_0\) (See Supplement 1 for \(w_0 = 2z_0, 2z_0 \neq z_0\), then evaluating both beams at the waist \(z = z_0\), the beam size is \(w(z) = w_0\) and the radius of curvature is \(R_\pi = \infty\). Additionally, the Gouy phase of the resonator mode at the waist is zero \(\Psi(z_0) = 0\). Then, by rescaling \(x = x'/w_0\), the spatial properties of the resonator eigenmodes are [26],

\[
\hat{\pi}_\pi(x,z) = \left(\frac{2}{\pi}\right)^{1/2} \sqrt{\frac{1}{2^m \pi^{2m}}} \exp\left(\frac{i(2n+1)\Psi(z)}{2}\right) \frac{H_n\left(\sqrt{x}w\right)}{w} \exp\left(-\frac{x^2}{w^2}\right).
\] (3)

Defining the fractional waist size mismatch, \(w = w_0/w_0\), the distribution of the incoming light is,

\[
u_n(x,z) = \left(\frac{2}{\pi}\right)^{1/2} \sqrt{\frac{1}{2^m \pi^{2m}}} \exp\left(\frac{i(2n+1)\Psi(z)}{2}\right) \frac{H_n\left(\sqrt{x}w\right)}{w} \exp\left(-\frac{x^2}{w^2}\right).
\] (4)

where \(\Psi\) is free to describe some accumulated Gouy phase. After substitution of Equations 4 & 3, Equation 2 is difficult to solve. However, the integral function from SymPy [24] v1.3, can solve this for a specific \(n\), which may then be expanded with the series method. For the first 10 orders, the coupling constant between the same mode in each basis \((n = \pi)\) is,

\[
k_{n,n} \approx \exp\left(\frac{i(2n+1)\Psi(z)}{2}\right) \left(1 - \frac{C_n}{4} \left((w - 1)^2 + (w - 1)^3\right) + \mathcal{O}\left((w - 1)^4\right)\right),
\] (5)

where,

\[
C_0 = 1, \quad C_1 = 3, \quad C_2 = 7, \quad C_3 = 13, \quad C_4 = 21, \quad C_5 = 31, \\
C_6 = 43, \quad C_7 = 57, \quad C_8 = 73, \quad C_9 = 91, \quad C_{10} = 111,
\] (6)

and Supplement 2 can be used to compute additional values of \(C_n\). Figure 1 shows a numerical solution to Equation 2 using PyKat [27] against Equation 5 expanded to order \((w - 1)^3\). For a waist size mismatch less than 5% there is good agreement between the analytic solution and the numerical ones.

When considering a resonator, it is sometimes preferable to think about the power coupling efficiency, \(k_{n,m,m}k_{\pi,n,\pi}\), with reference to the losses of the fundamental (TEM00) mode, typically referred to as mode matching losses. Defining the horizontal losses to be,

\[
W_x \equiv \frac{(w - 1)^2 + (w - 1)^3}{4} \approx 1 - |k_{0,0}|
\] (7)

and likewise for the vertical losses, \(W_y\), the full 2D coupling coefficient is,

\[
k_{n,m,m} \approx \exp\{i(n+m+1)\Psi(z)\}(1 - C_nW_x - C_mW_y + C_nC_mW_xW_y).
\] (8)

For an almost matched beam in \(x\) and \(y\), the last term may be safely ignored. The power coupling coefficient is then,

\[
k_{n,m,m}k_{\pi,n,\pi} \approx 1 - 2C_nW_x - 2C_mW_y,
\] (9)

where terms of order \(W_x^2, W_y^2, W_xW_y\) have been neglected. This result conclusively shows that, higher-order modes are more susceptible to mode mismatching losses when coupling into cavities. A similar result is also obtained for waist position mismatching, see Supplement 1.

3. EXAMPLE - POWER THROUGHPUT OF THE ADVANCED LIGO OUTPUT MODE CLEANER

Advanced LIGO operates with a high degree of mode matching to ensure power couples efficiently between the resonators; however, some degree of mismatch is always present. The HG55 mode has been proposed as a possible option for revisiting a higher order mode carrier, to reduce thermal noise [13].

Within the core interferometer an increased sensitivity to mode mismatch will likely cause an increased contrast defect. In addition, since the core interferometer is dual recycled and has focusing elements within the recycling cavities, an increased sensitivity to mode mismatch may lead to challenges in defining an operating point for the resonators [28].
Table 1. Mode mismatch induced power losses through the OMC for an astigmatic input beam with \( w_{0x} = 0.98w_{0y} \) and \( w_{0y} = 0.96 w_{0y} \). The analytic response is determined from Equation 11 and the simulated response is determined from the F~\text{inesse}~cavity scan in Figure 2.

| Input Mode | Analytic (x) | Analytic (y) | Analytic (Total) | Simulation | Difference |
|------------|-------------|-------------|-----------------|------------|------------|
| HG00       | 204.0 ppm   | 832.6 ppm   | 1036.7 ppm      | 1036.5 ppm | 0.2 ppm    |
| HG30       | 2652.5 ppm  | 832.6 ppm   | 3485.1 ppm      | 3472.8 ppm | 12.3 ppm   |
| HG50       | 6325.1 ppm  | 832.6 ppm   | 7157.8 ppm      | 7131.6 ppm | 26.1 ppm   |
| HG33       | 2652.5 ppm  | 10824.1 ppm | 13476.6 ppm     | 13389.8 ppm| 86.8 ppm   |

In the case of the Input and Output Mode Cleaners (IMC and OMC), modes which are not resonant are reflected and dumped. Therefore, the effect of the mode mismatch is a reduced power transmission through the resonator. In the case of the IMC, small mismatches can be compensated for by increasing laser power. In the case of the OMC the mode mismatch directly causes a loss of signal and loss for squeezed light injection.

A F~\text{inesse}~model [28, 32] of the Advanced LIGO OMC was produced and the transmission efficiency was studied for a range of input modes, results are shown in Figure 2. The input power was chosen such that a mode-matched beam produced 1 W of power on transmission, when the resonator was tuned and was constant for all simulations. This power scaling means that the power on transmission is equal to the OMC power coupling efficiency. The input beam was astigmatic with \( w_{0x} = 0.98w_{0y} \) and \( w_{0y} = 0.96 w_{0y} \). This astigmatism was chosen to highlight the differing losses for modes with \( m \neq n \). The tuning range was measured from the expected resonance position. Simulation modes \( n', m' \), up to \( n' + m' \leq n + m + 4 \), for input mode \( n, m \) were enabled.

The parameter \( 2W_c \) was determined by running an additional simulation with TEM00 input and \( w_{0y} = \pi y \) and \( w_{0x} = 0.98w_{0x} \), then

\[
2W_c = 1 - \frac{P_{Tc}}{P_{T}} ,
\]

(10)

where \( P_{Tc} \) is the power measured on transmission and \( P_T \) is the transmitted power for no mismatch \( (w_{0x} = \pi x , w_{0y} = \pi y) \). In this work, the input power scaling meant \( P_T = 1 \). The parameter \( 2W_c \) was obtained similarly. The analytically determined OMC power coupling efficiency for mode HGnm is then,

\[
|k_{n,n,n,m}|^2 = 1 - C_n \left( 1 - \frac{P_{Tc}}{P_T} \right) - C_m \left( 1 - \frac{P_{Tc}}{P_{Tc}} \right) ,
\]

(11)

which is shown by the dotted lines in Figure 2. This general method also works as an experimental procedure and can be used to estimate losses in switching to a higher-order mode.

Mode-mismatch-induced power losses in the OMC correspond directly to a loss of signal and increased quantum noise. Changing to an equivalently stable higher-order spatial mode will reduce thermal noise, however, unless the higher-order mode matching is improved compared to the TEM00 mode matching, the mode mismatching induced signal degradation will be 13 times worse for a HG33 and 31 times worse for a HG55 carrier mode.

4. CONCLUSIONS

The two fundamentally limiting noise sources in the most sensitive region of advanced detectors are thermal noise and quantum noise. Recent developments in adaptive astigmatism control may be combined with improved modal readout methods to permit increasing the spatial frequency of the carrier mode in the HG basis.

However, mitigation of quantum noise requires extremely high levels of mode matching [23]. This work shows that as the spatial carrier frequency is increased, the sensitivity to a fractional waist size mode mismatch is also increased. The results are analytically derived for a small mismatch and shown to be valid against numerical solutions. These results are consistent with the evidence discussed in [14], the decreased mode purity and power observed in [13] and experimental observations [31].

This paper highlights an increased sensitivity to mode mismatch and a scaling is provided for single resonator cavities. The core interferometer within a gravitational-wave detector is composed of multiple coupled cavities where mode matching is more complex [28, 32]. While the general principles outlined in this paper still apply, the detailed requirements for a higher order mode carrier in such systems should be derived using numerical models of the full optical setup.

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DISCLOSURES

The authors declare no conflicts of interest.

See Supplement 1 for supporting content.

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1. INTRODUCTION
The focus of the main text is the determination of the increased sensitivity of a higher-order laser beam to mode mismatches. The main text uses a waist size mismatch as an example and derives coefficients describing the increase in sensitivity to mode mismatch. However, similar coefficients can also be derived for waist position mismatches. Beam misalignment has not been considered as gravitational wave detectors use resonant wavefront sensing to maintain the alignment [1].

2. WAIST POSITION MISMATCH FORMULATION
First, we define a normalized waist position mismatch parameter (which is equal to $K_0$ in [2]),

$$f = \frac{z_0 - z_0^0}{z_R}.$$  
(S1)

Without loss of generality, the resonator waist position is defined to be $(z_0 = 0)$ and the resonator Gouy phase is zero at the waist. The waist sizes are assumed to be the same, therefore $z_R = z_R^0$. Then, the incoming beam radius and radius of curvature at $z_0^0$ may then be rewritten in terms of the mismatch parameter $f$,

$$R_C(f) = z_R (f + 1/f),$$  
(S2)

$$w^2(f) = \frac{2z_R}{k} \left( 1 + f^2 \right).$$  
(S3)

Substitution of these parameters into,

$$k_{n,\pi} = \int_{-\infty}^{\infty} u_n(x, z) \pi_{\pi}(x, z) \, dx.$$  
(S4)

then yields the coupling coefficients. We found that a series expansion needed to be taken prior to integrating Equation S4, this series expansion was taken to order $f^7$. Solving using a symbolic library then yields,

$$k_{n,\pi} \approx \exp \left( \frac{i(2n + 1)\Psi(z)}{2} \right) \left( 1 + \frac{iC_n f}{4} - \frac{C_n f^2}{32} \right) + O(f^3),$$  
(S5)

where, for the first 10 orders, the linear coefficients are just $C_n = 2n + 1$ and the quadratic coefficients are,

$$C_0^L = 3, \quad C_1^L = 15, \quad C_2^L = 39, \quad C_3^L = 75, \quad C_4^L = 123, \quad C_5^L = 183, \quad C_6^L = 255, \quad C_7^L = 339, \quad C_8^L = 435, \quad C_9^L = 543, \quad C_{10}^L = 663.$$  
(S6)

3. NUMERICAL COMPARISON
Figure S1 shows a numerical solution to Equation S4 using PyKat[3]. For a normalized waist position mismatch less than 5% there is good agreement between the analytics and the numerics for all modes. As the mode index increases, more terms are required to model a given position mismatch.

4. CONCLUSIONS
Both waist position and waist size mismatches result in increasing mode mismatching losses as the mode order is increased.
Fig. S1. 1D mode mismatch parameter, $k_{n,\pi}$ for a waist position only mismatch between the incoming beam. Both beams have $z_R = 1$ m. Solid lines show numerical solutions to Equation S4 and dotted lines show the approximate analytic solutions in Equation S5. This result is generalizable for other waist radii by normalizing the mismatch by the resonator Rayleigh range.

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