Essentially And Weakly Essentially Coretrectable Modules

Shukur Neamah Al-aeshi¹, Farhan Dakhil Shyaa², Inaam Mohammed Ali Hadi³

¹Department of Urban Planning, College of Physical Planning, University Of Kufa, IRAQ
²Department of Mathematics, College of Education, University of Al-Qadisiyah, IRAQ
³ Department Of Mathematics, College of Education for Pure Sciences (Ibn-Al-Haitham), University of Baghdad, IRAQ
Corresponding author’s e-mail: shukur.mobred@uokufa.edu.iq

Abstract In this paper, we introduce two new notions related with coretrectable modules which are called essentially coretrectable and weakly essentially coretrectable modules. Some basic properties of these classes are investigated and some relationships between these modules and other related concepts are given.

Keywords: Coretrectable module, Strongly Coretrectable module, Essentially Coretrectable module, Weakly Essentially Coretrectable module.

Introduction

Throughout all rings have identity and all modules are unitary. "A module M is coretrectable if Hom(M/X, M)≠0 for each proper submodule X of M.[3, 6].Many types related with coretrectable modules are introduced and studied such as: epi-coretrectable, strongly coretrectable, C-coretrectable, Y-coretrectable and P-coretrectable module, see [2], [9], [10], [11], [12], [13], [14] and [15]. Recall that "A submodule X of M is essential (briefly, X≤M) if it have nonzero intersection with other nonzero submodules". This paper consists of three sections, in section one, we shall recall some important basic concepts and results which are needed in the next sections. In section two, we study the concept of essentially coretrectable modules where an module M is called essentially coretrectable if for each proper submodule X of M, there is nonzerohomomorphism g:X→M such that Img≤M. It is clear that the class of essentially coretrectable module contains in the class of epi-coretrectable, also it is contained in the class of coretrectable module. That means we have the following:

epi-coretrectable modules⇒essentially coretrectable modules⇒coretrectable modules

The converse implications are not true in general (see parts (3, 7) of Examples and Remarks(2.2)), we give many results concerned with essentially coretrectable modules, for example, any direct summand of essentially coretrectable is essentially coretrectable (see Corollary(2.7)), but the direct sum of two essentially coretrectable modules need not essentially coretrectable (see Examples and Remarks(2.2)(9)). Also, we introduce the concept of weakly essentially. Several relationships between this class module and other known modules are given. Also we get that the direct sum of two weakly essentially coretrectable modules M₁ and M₂ is weakly essentially coretrectable (see Theorem(3.8)) and the direct summand of weakly essentially coretrectable module is weakly essentially coretrectable (see Corollary(3.7)).

Preliminaries

Definition(1.1):[9,10]"A module M is called strongly coretrectable if for each proper submodule X of M, there exists g∈Hom(M/X,M) such that g≠0 and Img+X=M. Equivalently, M is a strongly coretrectable R-module iff for each proper submodule X of M, there exists 0≠g∈End(M), Img+X=M and g(X)=0.
Definition(1.2):[2] "A module M is called epi-coretrectable if for each proper submodule X of M, there exists an epimorphism g\in\text{Hom}(M/X,M). Some authors called it co compressible module" [2].

Proposition(1.3):[9] "Let M be hollow module. If M is strongly coretectable. Then M is epi-coretrectable", where a module M is hollow if all its submodules are small and a proper submodule X of M is small if X+W=M implies W=M for all W\subseteq M".

Definition(1.4):[17] "A module M is called quasi-Dedekind if for each proper submodule X of M \text{Hom}(M/X,M)=0. Equivalently, M is quasi-Dedekind module if for each nonzero endomorphism is monomorphism. A submodule X of M is called coquasi-invertible if \text{Hom}(M, X)= 0 and a non-zero module M is coquasi-Dedekind if each proper submodule is coquasi-invertible [19, P.32]. Equivalently, M is coquasi-Dedekind module if for each g \in \text{End}(M), g \neq 0, g is epimorphism"[19, Theorem(2.1.4), P.33].

Definition(1.5):[14] "A module M is called C-coretrectable if \text{Hom}(M/X,M)\neq 0 for each proper closed submodule X of M, where a submodule X of M is closed if X has no proper essential extension in M".

Definition(1.6):[11] "A module M is called Y-coretrectable if \text{Hom}(M/X,M) \neq 0 for each proper y-closed submodule X of M, where a submodule X of M is called y-closed if M/X is nonsingular module and every y-closed submodule is closed but not conversely. For example, <\overline{2}> in \mathbb{Z}_6 is closed, but not y-closed submodule".

Definition(1.7):[12] "A module M is called P-coretrectable if \text{Hom}(M/X,M)\neq 0 for each proper pure submodule X of M. A submodule X of M is called pure if IM\cap X=IX for all finitely generated ideal I in R".

Definition(1.8):"A submodule X of M is called fully invariant if g(X)\subseteq X for each endomorphism g of M and M is called duo module if all submodules are fully invariant".

Definition(1.9):[7]"A module M is hopfian if each g \in \text{End}(M) such that g is an epimorphism, then g is monomorphism. And M is antihopfian module if M/X\cong M for each submodule X of M. Clear that antihopfian module implies epi-coretrectable.

Proposition(1.10): Any hopfian epi-coretrectable module is antihopfian.

Proof: Let X \leq M, so there exists g:M/X\rightarrow M epimorphism, hence g\pi\in\text{End}(M) where \pi:M\rightarrow M/X is the natural epimorphism and so g\pi is epimorphism. But M is hopfian module, so that g\pi is monomorphism, hence it is isomorphism. Thus g is monomorphism and hence g is isomorphism, so M/X\cong M. Thus M is antihopfian.

Corollary(1.11): Any Noetherian epi-coretrectable module is antihopfian".

Proof: It is clear since every Noetherian module is hopfian.

Recall that "an R-module M is called comultiplication if for each submodule X of M, there exists an ideal I of R such that X=\text{ann}_R(I)= \{ m \in M: ml=0 \}. Equivalently, a module M is called comultiplication if for each submodule X of M, X=\text{ann}_R\cap \text{ann}_R X" [4]. Recall that "an R-module M is called coprime if \text{ann}_R(M)=\text{ann}_R(M/X) for each proper submodule X of M"[19].

Under the class of comultiplication coprime modules, the coretectable equivalent to epi-coretrectable.

Proposition(1.12): If M is comultiplication coprime module, then M is coretectable iff M is epi-coretrectable.
**Proof:** ($\Rightarrow$) Let $X \leq M$, $M$ is coretractable, so $\exists g: M/X \to M, g \neq 0$. Hence by 1st Fundamental isomorphism Theorem $(M/X)/\ker g \cong \text{Img}$. Put $\ker g = U/X$ for some $U < M$, so $M/U \cong \text{Img}$ and hence $\operatorname{ann}_R(M/U) = \operatorname{ann}_R(\text{Img})$, but $\operatorname{ann}_R(M/U) = \operatorname{ann}_R(M)$ since $M$ is coprime module, hence $\operatorname{ann}_R(M) = \operatorname{ann}_R(\text{Img})$ and so that $\operatorname{ann}_R(M) = \operatorname{ann}_R(\text{Img})$, which implies that $M = \text{Img}$, since $M$ is a comultiplication module. Thus $g$ is epimorphism and so $M$ is epi-coretractable module.

($\Leftarrow$) It is clear.

2. **Essentially Coretractable Module**

**Definition 2.1:** A module $M$ is called essentially coretractable (briefly, e-coretractable) if for each proper submodule $X$ of $M$, there exists $g \in \text{Hom}(M/X, M)$ such that $\text{Img} \leq \text{e} M$. A ring $R$ is called essentially coretractable ring if $R$ is essentially coretractable $R$-module.

**Examples and Remarks 2.2:**

1. A module $M$ is essentially coretractable iff $\forall X < M, \exists g \in \text{End}_k(M), g \neq 0$ such that $\text{Img} \leq \text{e} M$ and $g(X) = 0$.

   **Proof:** ($\Rightarrow$) Since $M$ is essentially coretractable module, so $\forall X < M, \exists \neq f \in \text{Hom}(M/X, M)$, where $\text{Im} f \leq \text{e} M$. Let $g = f \circ \pi$, $\pi$ is the natural epimorphism from $M$ into $M/X$, we can easily find $\text{Img} \leq \text{e} M$ and $g(X) = 0$.

   ($\Leftarrow$) Let $X < M$, by hypothesis $\exists g \in \text{End}_k(M), g \neq 0$ such that $\text{Img} \leq \text{e} M$ and $g(X) = 0$. Define $f : M/X \to M$ as $f(m + X) = g(m) \forall m \in M$, clear that $f$ is well-defined and homomorphism moreover $\text{Im} f = \text{Img} \leq \text{e} M$. Thus $M$ is essentially coretractable module.

2. Every antihopfian module is essentially coretractable, but the converse is not true as $Z_4$ as $Z$-module.

   **Proof:** Let $M$ be antihopfian module, hence $M/X \cong M \forall X < M$, so $\exists g: M/X \to \text{Misomorphism}$.

   It follows that $g(M/X) \leq \text{e} M$ and hence $M$ is essentially coretractable module.

3. Every essentially coretractable module is coretractable, but the converse is not true in general, see $Z_4$ as $Z$-module is coretractable, but not essentially coretractable. The converse become true under the condition uniform $R$-module, where a module is uniform if every submodule is essential submodule. In particular $Z$ as $Z$-module is not coretractable and hence not essentially coretractable.

4. An essentially coretractable module may be not strongly coretractable. See $Z_4$ as $Z$-module is not strongly coretractable, but $Z_4$ is essentially coretractable.

5. A semisimple module may be not essentially coretractable, for example, $Z_6$ as $Z$-module.

6. It is clear that an e-coretractable module is essentially coretractable.

7. The converse of part (6) is not true in general for example, $Z_4$ as $Z$-module is essentially coretractable but not e-coretractable.

8. If $M$ is semisimple module. Then $M$ is epi-coretractable iff essentially coretractable.

   **Proof:** ($\Rightarrow$) By part (6).

   ($\Leftarrow$) Let $\forall X < M$ and $M$ is essentially coretractable module, so $\exists \neq f \in \text{Hom}(M/X, M)$ and $\text{Im} f \leq \text{e} M$. But $M$ is semisimple, so $\text{Im} f = M$ which implies $f$ is epimorphism. Therefore $M$ is epi-coretractable module.

9. The direct sum of two essentially coretractable modules need not be essentially coretractable. For example, each of the $Z$-modules $Z_2$ and $Z_3$ are essentially coretractable but $Z_2 \oplus Z_3 \cong Z_6$ is not essentially coretractable module.

10. If $M$ is coquasi-Dedekind module. Then $M$ is essentially coretractable iff $M$ is epi-coretractable.

11. A module $M$ is essentially coretractable $R$-module iff $M$ is essentially coretractable $R$-module (where $R = R/\text{ann} M$).

**Proposition 2.3:** Let $M \cong M$ and $M$ be essentially coretractable module, then $M$ is also essentially coretractable.
Proof: Since $M \cong M'$, so $\exists g: M \rightarrow M'$ isomorphism. Let $W < M'$. Then $W = g(X)$ for some $X < M$. Since $M$ is essentially coretractable, so $\exists \epsilon \in \text{End}(M)$ such that $f(M) \leq M$. It follows that $g(f(M)) \leq M$ (since $g$ is monomorphism), so $g^*(f(M)) \leq M$. Let $h = g^*f \circ g^{-1}$, then $h(M) = g^*f(g^{-1}(M)) = g^*(f(\epsilon(M))) = g^*(f(M)) \leq M$. Thus $h(M) \leq M$ and hence $M'$ is essentially coretractable module.

Corollary (2.4): If $R$ is essentially coretractable ring and $M$ is faithful cyclic module. Then $M$ is essentially coretractable module.

Proposition (2.5): If $M$ is essentially coretractable module, then $M/X$ is also essentially coretractable for each proper closed submodule $X$ in $M$.

Proof: Let $W/X < M/X$, so $W \neq M$. We must find $0 \neq f:(M/X)/(W/X) \rightarrow M/X$ such that $\text{Im}f \leq M/X$. Since $M$ is essentially coretractable, then $\exists g: M/W \rightarrow M, g \neq 0$ and $\text{Im}g \leq M$. But $(M/X)/(W/X) \cong M/W$. So we can get the result.

Corollary (2.6): If $f: M \rightarrow M'$ is epimorphism with ker$f$ is closed submodule of $M$ and $M$ is essentially coretractable, then $M'$ is essentially coretractable too.

Proof: From the 1st isomorphism Theorem, $M/\text{ker}f \cong M'$. Since $0 \neq \text{ker}f < M$, so $M/\text{ker}f$ is essentially coretractable by Proposition (2.5). Then $M'$ is a essentially coretractable by Proposition (2.3).

Corollary (2.7): Any direct summand of essentially coretractable module is essentially coretractable.

Proof: Let $X$ be a direct summand of $M$. Then $M = X \oplus W$ for some submodule $W$ of $M$, hence $W$ is closed submodule in $M$, so $M/W$ is essentially coretractable by Prop. (2.5). But $M/W \cong X$. Then $X$ is essentially coretractable module by Proposition (2.3).

We shall call for a module $M$ is essentially $C$-coretractable if for all proper closed submodule $X$ of $M$, $\exists 0 \neq f \in \text{Hom}(M/X, M)$ and $\text{Im}f \leq M$. So we can get the result:

Proposition (2.8): Let $M$ be a module such that for each proper submodule $K$ of $M$, there exists a proper closed submodule $X$ of $M$ such that $K \subseteq X \subseteq M$. Then $M$ is essentially coretractable iff essentially $C$-coretractable.

Proof: $(\Rightarrow)$ It is clear.

$(\Leftarrow)$ Let $K < M$. By hypothesis, there exists a proper closed submodule $X$ of $M$ such that $K \subseteq X$. Also there exists $g \in \text{Hom}(M/X, M)$, $g \neq 0$ and $\text{Im}g \leq X$. Define $f: M/K \rightarrow M/X$ by $f(m + K) = m + X$ for each $m \in M$. $f$ is well-defined epimorphism. It follows that $g^*f \in \text{Hom}(M/X, M)$ and $g^*f \neq 0$ moreover $g^*(f(M/X)) = g(M/X) \leq M$. Thus $M$ is essentially coretractable module.

Proposition (2.9): Let $M$ be hollow and strongly coretractable module then $M$ is essentially coretractable.

Proof: By Prop. (1.3), $M$ is epi-coretractable and hence by Ex. and Rem. (2.2(6)), $M$ is essentially coretractable.

Consequently, we can get every strongly coretractable module is essentially coretractable under the class Artinian couniform module because every Artinian couniform module is hollow by [8, Prop. (2.1)], where "a nonzero module $M$ is called couniform, if every proper submodule $X$ of $M$ is either zero or there exists a proper submodule $L$ of $X$ such that $X/L$ small submodule in $M/L"$ [8].

Remark (2.10): The condition of hollow module in Proposition (2.9) is necessary. Consider $Z_6$ as $Z$-module is strongly coretractable, but it is not hollow and it is not essentially coretractable.
Remark(2.11): Let M be a module. If M is essentially coretectable, then M is C-coretectable and also Y-coretectable. One can see the converse is not true in general. For example, \( \mathbb{Z}_6 \) as \( \mathbb{Z} \)-module.

Proof: By Examples and Remarks(2.2(3)), M is coretectable module and hence M is P-coretectable and Y-coretectable module by [11],[12].

Proposition(2.12): Let M be a comultiplication coprime module. Then the following statements are equivalent:

(i) M is essentially coretectable;
(ii) M is coretectable;
(iii) M is epi-coretectable.

Proof: (ii) \( \Leftrightarrow \) (iii) It follows by Proposition(1.13).

(i) \( \Rightarrow \) (ii) It follows by Remark(2.2(3)).

(iii) \( \Rightarrow \) (i) It follows by Remark(2.2(6)).

Proposition(2.13): A module M is essentially coretectable iff for each proper submodule X of M, there exists a proper submodule W of M such that X \( \subseteq \) W, M/W \( \cong \) U \( \leq \) M for some U \( \leq \) M.

Proof: (\( \Rightarrow \)) Let X \( \leq \) M. Since M is essentially coretectable, so \( \exists 0 \neq f \in \text{Hom(M/X, M)} \) such that Imf \( \leq_e \) M. But (M/X)/kerf \( \cong \) f(M/X) by the 1st fundamental Theorem. Put kerf=W/X for some X \( \leq \) W \( < \) M and f(M/X)=U. Then M/W \( \cong \) U=f(M/X) \( \leq \) M.

(\( \Leftarrow \)) Let X \( \leq \) M. By hypothesis \( \exists \) W \( < \) M and X \( \leq \) W such that M/W \( \cong \) U for some U \( \leq \) M, hence \( \exists f:M/W\rightarrow U \) isomorphism. Define g:M/X\rightarrow M/W by g(m+X)=m+W for each m \( \in \) M, g is epimorphism, hence h=i\circ g \( \in \) Hom(M/X, M), h\( \neq \)0, i is the inclusion mapping from U into M. Thus h(M/X)= U \( \leq \) M and hence M is essentially coretectable.

Recall "A module M is scalar if for all f \( \in \) End(M), there exists 0\( \neq \)r \( \in \mathbb{R} \), f(m)=mr \( \forall \) m \( \in \) M.

Proposition(2.14): If M is essentially coretectable scalar module. Then there exists 0\( \neq \)r \( \in \mathbb{R} \), Mr \( \subseteq \) M.

Proof: Let X \( \leq \) M. Since M is essentially coretectable, so \( \exists 0 \neq g \in \text{End}(M) \), Img \( \subseteq \) M and g(X)=0. But M is scalar module, so \( \exists 0 \neq r \in \mathbb{R} \), g(m)=mr \( \forall \) m \( \in \) M. Thus g(M)=Mr \( \subseteq \) M.

Corollary(2.15): If R is essentially coretectable ring. Then there exists 0\( \neq \)r \( \in \mathbb{R} \) such that \( < r > \subseteq \mathbb{R} \).

Recall "A module M is called dual Rickart if for each g \( \in \text{End}(M) \), Img is direct summand in M [16].

Proposition(2.16): If M is dual Rickart module. Then M is essentially coretectable iff M is epi-coretectable.

Proof: (\( \Rightarrow \)) Let X \( \leq \) M, since M is essentially coretectable, so \( \exists g \in \text{End}(M) \), g\( \neq \)0 and Img \( \subseteq \) M. But M is dual Rickart hence Img is direct summand in M, so Img=Mr and so g is epimorphism. Thus M is epi-coretectable.

(\( \Leftarrow \)) It is clear.

3.Weakly Essentially Coretectable Module

Definition(3.1): A module M is called weakly essentially coretectable (briefly, we-coretectable) if for each proper submodule X of M, there exists g \( \in \text{Hom}(M/X, M) \) such that Img \( + \) X \( \subseteq \) M. A ring R is called weakly essentially coretectable ring if R is weakly essentially coretectable R-module.
Examples and Remarks(3.2):

1. M is weakly essentially corectractable module iff for each proper submodule X of M, there exists \( \theta \neq g \in \text{End}(M) \), \( \text{Img } + X \leq e \cdot M \) and \( g(X) = 0 \).

**Proof:** It clear by the same way of Remark((2.2(1))).

2. Clear that every essentially corectractable module implies to weakly essentially corectractable but the converse is not true in general. Consider the \( \mathbb{Z} \)-modules \( Z_6 \) and \( Z_31 \)are weakly essentially corectractable, but they are not essentially corectractable \( \mathbb{Z} \)-modules.

3. Clear that every strongly corectractable is weakly essentially corectractable, but the converse is not true in general. Consider \( Z_4 \) as \( \mathbb{Z} \)-module is weakly essentially corectractable, but it is not strongly corectractable.

4. Any semisimple(simple) module is weakly essentially corectractable but not conversely, see\( Z_4 \) as \( \mathbb{Z} \)-module.

5. If \( R \) is semisimple ring, then every \( R \)-module is weakly essentially corectractable ( by part(4) ).

6. Every weakly essentially corectractable module is corectractable. But under the condition uniform module we can get easily the converse is true.

7. Every epi-corectractable module is weakly essentially corectractable.

8. Clearly every weakly essentially corectractable is \( C \)-corectractable and also \( Y \)-corectractable.

9. A module \( M \) is weakly essentially corectractable iff \( M \) is weakly essentially corectractable \( \overline{R} \)-module (where \( \overline{R} = R/\text{ann } M \)).

10. Every antihopfian module is weakly essentially corectractable.

**Proposition (3.3):** Let \( M \) and \( M' \) be \( R \)-modules such that \( M \cong M' \). Then \( M \) is weakly essentially corectractable iff \( M' \) is weakly essentially corectractable.

**Proof:** As \( M \cong M' \), so there exists \( f : M \rightarrow M' \) an isomorphism. Let \( W \subset M \), hence \( W = f(X) \) for some \( X \subset M \). But \( M \) is weakly essentially corectractable, so there exists \( g \in \text{End}(M) \), \( g(M) + X \subset e \cdot M \). It follows that \( f(g(M) + X) \subset e \cdot M \) (since \( f \) is monomorphism). Thus \( f(g(M) + X) \subset e \cdot M \). Let \( h = f \circ g \circ f^{-1} \), then \( h(M) + W = f \circ g \circ f^{-1}(M) + f(X) = f(g(f^{-1}(M)) + f(X) = f(g(M)) + f(X) \subset e \cdot M \). Thus \( h(M) + W \subset e \cdot M \). Therefore, \( M \) is weakly essentially corectractable module.

Similarity, the reverse direction.

**Proposition (3.4):** If \( M \) is weakly essentially corectractable module. Then \( M/X \) is weakly essentially corectractable for each proper closed submodule \( X \) of \( M \).

**Proof:** Let \( W/X < M/X \), so \( W \neq M \). Since \( M \) is weakly essentially corectractable. Then \( \exists g : M/W \rightarrow M \), \( \text{Img } + W \subset M \). But \( (M/X)/(W/X) \cong M/W \) that means \( \exists h : (M/X)/(W/X) \rightarrow M/W \) isomorphism. Let \( f = \pi \circ g \circ h \) where \( \pi \) is the natural epimorphism from \( M \) into \( M/W \).

\[
\text{Img} + \frac{W}{X} = \pi \circ g(h(M + W/X)) = \pi \circ g(M/W) + \frac{g(M/W) + \frac{W}{X}}{X} = \frac{g(M/W) + W}{X}.
\]

But \( g(M/W) + W \subset e \cdot M \) and \( X \subset e \cdot M \), so \( \frac{g(M/W) + W}{X} \leq e \cdot \frac{M}{X} \) and hence \( \text{Img} + \frac{W}{X} \leq e \cdot \frac{M}{X} \).

Then \( M/X \) is weakly essentially corectractable module.

**Corollary (3.5):** Let \( f : M \rightarrow M' \) be an \( R \)-epimorphism module and \( \text{ker } f \) be a closed submodule of \( M \). If \( M \) is weakly essentially corectractable, then \( M' \) is weakly essentially corectractable.

**Proof:** By the 1st isomorphism Theorem, \( M/\text{ker } f \subset M' \). Since \( 0 \neq \text{ker } f < M \), and hence \( M/\text{ker } f \) is weakly essentially corectractable by Proposition(3.4). Therefore \( M' \) is weakly essentially corectractable module by Proposition(3.3).
Recall "A module $M$ is called C-Rickart if ker $f$ is closed submodule of $M$ for all $f \in \text{End}(M)$" [14]. The following result is satisfied directly by Corollary(3.5).

**Corollary(3.6):** If $M$ is C-Rickart. Then the epimorphism image of weakly essentially coretrectable module $M$ is weakly essentially coretrectable too.

**Corollary(3.7):** Every direct summand of weakly essentially coretrectable module is weakly essentially coretrectable.

**Proof:** Let $X$ be a direct summand of $M$. Then $M = X \oplus W$ for some submodule $W$ of $M$. $M/W$ is weakly essentially coretrectable by Proposition(3.4). But $M/W \cong X$. Then $X$ is weakly essentially coretrectable by Proposition(3.3).

**Theorem(3.8):** Let $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ be $R$-modules and $\text{ann}M_1 + \text{ann}M_2 = R$ (or $M$ is duo $R$-module or distributive). Then $M$ is weakly essentially coretrectable iff $M_1$ and $M_2$ are weakly essentially coretrectable modules.

**Proof:** ($\Rightarrow$) It follows directly by Corollary (3.7).

($\Leftarrow$) Let $X < M$. Since $\text{ann}M_1 + \text{ann}M_2 = R$ (or $M$ is duo $R$-module or distributive module ), then $X = X_1 \oplus X_2$ for some $X_1 \leq M_1$ and $X_2 \leq M_2$.

Case(1): $X_1 < M_1$ and $X_2 < M_2$. Since $M_1$ and $M_2$ are weakly essentially coretrectable modules, there exists $f: M_1/X_1 \rightarrow M_1$ and $g: M_2/X_2 \rightarrow M_2$ such that $f(M_1/X_1) + X_1 \leq M_1$ and $g(M_2/X_2) + X_2 \leq M_2$ respectively.

Define $h:M/X \rightarrow M_1$; that is $h:(M_1 \oplus M_2)/(X_1 \oplus X_2) \rightarrow M_1 \oplus M_2$ by $h((m_1 + X_1, m_2 + X_2)) = (f(m_1 + X_1), g(m_2 + X_2))$ for each $m_1 \in M_1$ and $m_2 \in M_2$. $h$ is well-defined and $h(M/X) + X = (f(M_1/X_1) + X_1) \oplus (g(M_2/X_2) + X_2) \leq M_1 \oplus M_2$.

Case(2): $X_1 = M_1$ and $X_2$ is a nonzero proper submodule of $M_2$.

Consider $(M_1 \oplus M_2)/(X_1 \oplus X_2) \rightarrow (M_1 \oplus M_2)/X_2 \cong M_2/X_2$. Since $M_1$ is weakly essentially coretrectable, there exists $f: M_1/X_1 \rightarrow M_2$ such that $f(M_1/X_1) + X_2 \leq M_2$. But $X_1 = M_1 \leq M_1$. Therefore $f(M_1/X_1) \leq M_2$.

Case(3): $X_1$ is a nonzero proper submodule of $M_1$ and $X_2 = M_2$, then by a similar proof of case(2), we can get the result

Case(4): $X_1 = 0$ and $X_2 = M_2$.

Consider $M/X \cong (M_1 \oplus M_2)/(0 \oplus M_2) \cong M_1$. Let $i$ is the inclusion mapping from $M_1$ into $M_1 \oplus M_2$. Hence $i(M_1/X) + X = i(M_1) + (0 \oplus M_2) = (M_1 \oplus 0) \oplus (0 \oplus M_2) = M \leq M$.

Case(5): $X_1 = M_1$ and $X_2 = 0$, then the proof is similar to case(4).

**Proposition(3.9):** Let $M$ be a quasi-Dedekind module. Then $M$ is weakly essentially coretrectable iff $M$ is essentially coretrectable.

**Proof:** ($\Rightarrow$) Let $M$ be a weakly essentially coretrectable module and $X < M$. Then there $f \in \text{End}(M)$ with $f(M) + X \leq M$. $f(M) = 0$. But $M$ is quasi-Dedekind, hence $f$ is monomorphism which implies $X = 0$ and so $f(M) \leq M$. Thus $M$ is essentially coretrectable module.

($\Leftarrow$) It is clear by Examples and Remarks(3.2)(2).

**Corollary(3.10):** If $M$ is nonsingular uniform module. Then $M$ is weakly essentially coretrectable iff $M$ is essentially coretrectable.
Proof: Since M is nonsingular uniform module, so M is quasi-Dedekind module by [13, Prop.(1.5)], and hence the result is hold by Prop.(3.9).

Corollary(3.11): Let M be a indecomposable Rickart module. Then M is weakly essentially coretrectable iff M is essentially coretrectable.

Proof: Since M is indecomposable Rickart, so it is easy to prove that M is quasi-Dedekind and hence the result is hold by Proposition(3.9).

Proposition(3.12): If M is coquasi-Dedekind. Then M is weakly essentially coretrectable iff M is epicoaretctable.

Proof: Let M be a weakly essentially coretrectable, and hence M is coretrectable. But M is coquasi-Dedekind, hence M is epi-coretrectable by [9, Proposition(2.8)].

(⇐) It is clear by Examples and Remarks(3.2(6)).

Proposition(3.13): If M is semisimple. Then M is coretrectable iff M is weakly essentially coretrectable.

Proof: Let X<M, since M is semisimple, so ∃W<M such that X⊕W=M, but M is coretrectable, then ∃#f∈Hom(M/X,M).Since M/X≅W, so Imf=W which implies Imf+X=W+X=M ≤M. Thus M is weakly essentially coretrectable.

(⇐) It is clear by Remark(3.2(4)).

One can easy to prove the following results since that implies M is semisimple:

Proposition(3.14): Let M be a module and S=End(M) be a von Neumann regular ring. Then M is strongly coretrectable iff M is weakly essentially coretrectable.

Proposition(3.15): If M is scalar faithful module over commutative regular ring. Then the following are equivalent:

(i) M is coretrectable;
(ii) M is semisimple;
(iii) M is strongly coretrectable;
(iv) M is weakly essentially coretrectable.

Proof: (i)⇔(ii)⇔(iii) By [9, Corollary(2.3)].

As M is scalar faithful module, hence by [18, Lemma(6.1)], End₆(M) ≅R, thus End(M) is regular ring. Therefore (iii)⇔(iv) by Proposition(3.14).

Proposition(3.16): Let M be an R-module and S=End₆(M) be a von Neumann regular ring. Then the following statements are equivalent:

(i) M is coretrectable;
(ii) M is semisimple.
(iii) M is strongly coretrectable;
(iv) M is weakly essentially coretrectable.

Proof: (i)⇔(ii)⇔(iii) It follows by [9, Proposition(2.1)].

(iii)⇔(iv) It is satisfied by Proposition(3.14).

One can easy to see the Proposition(3.16) is satisfied whenever an R-module is finitely generated multiplication over commutative regular ring R by using Lemma(2.7) in [10]. Where " a module M is called multiplication if for each submodule X of M, there exists a right ideal in R such that MI=X "[5].

8
Corollary (3.17): Let M be a finitely generated projective R-module over von Neumann regular ring. Then the following statements are equivalent:

(i) M is coretractable;
(ii) M is semisimple;
(iii) M is strongly coretractable;
(iv) M is weakly essentially coretractable.

Proof: From [16, Corollary (2.2.22) and Corollary (2.2.20), P.41], we get S=End(M) is a von Neumann regular and hence the result follows by Proposition (3.14).

Proposition (3.18): If M is weakly essentially coretractable R-module with End(M) is von Neumann regular ring. Then every submodule is weakly essentially coretractable.

Proof: Let X<M. Since M is weakly essentially coretractable and End(M) is von Neumann regular ring, then M is semisimple, and so every submodule is direct summand of M, therefore by Coro. (3.7), X is weakly essentially coretractable.

Corollary (3.19): Let M be a finitely generated multiplication module over commutative regular ring R. If M is weakly essentially coretractable, then every submodule is weakly essentially coretractable.

Proof: Since M is finitely generated multiplication over commutative regular, then End(M) is von Neumann regular ring by [10, Lemma (2.7)], and hence by Prop. (2.18), we get every submodule is weakly essentially coretractable module.

Proposition (3.20): Let M be a prime module. If M̅ (the quasi-injective hull of M) is weakly essentially coretractable, then M is also weakly essentially coretractable.

Proof: Since M is prime module implies that End(M̅) is regular by [17, Proposition (3.7), P.36], but M̅ is weakly essentially coretractable, so by [3, Proposition (4.4)], M̅ is semisimple and hence M is semisimple. Thus M is weakly essentially coretractable by Remark (3.2(3)).

Proposition (3.21): If M is nonsingular and the injective hull of M (E(M)) is weakly essentially coretractable, then M is also weakly essentially coretractable.

Proof: M is nonsingular implies E(M) is nonsingular by [24, P.247], hence E(M) is semisimple hence M is semisimple. Thus M is weakly essentially coretractable by Remark (3.2(3)).

Proposition (3.22): If M is multiplication module with ann(M) is prime ideal. Then the following are equivalent:

(i) M is coretractable;
(ii) M is weakly essentially coretractable;
(iii) M is semisimple module.

Proof: (i)⇒(iii) By [13, Proposition (2.9)].

(iii)⇒(ii) It follows by Remark (3.2(3)).
(ii)⇒(i) It follows by Remark (3.2(6)).

Proposition (3.23): Let M be a nonsingular finitely generated multiplication module. Then the following statements are equivalent:

(i) M is Kasch;
(ii) M is semisimple;
(iii) M is coretractable;
(iv) M is strongly coretractable;
(v) M is weakly essentially coretractable;
**Proof:** (i) $\iff$ (ii) Since $M$ is nonsingular, so it follows by [1].

(ii) $\iff$ (iii) $\iff$ (iv) It follows by [10, Corollary(2.8)].

(ii) $\iff$ (v) It is clear since $M$ is nonsingular.

The following diagram explains relation between some classes of coretectable module which are studied and discussed.

```
Essentially coretectable

| Weakly essentially coretectable | Coretectable module |
|--------------------------------|---------------------|
|                                 | C-coretectable      |
|                                 | P-coretectable      |
|                                 | Y-coretectable      |

Strongly coretectable

Epi-coretectable
```

**References**

[1] Albu T and Wisbauer R 1997 Kasch Modules In: Jain S K and Rizvi S T (eds) Advances in Ring Theory. Trends in Mathematics. Birkhäuser, Boston, MA.

[2] Al-Hosainy A M A and Kadhim H 2014 A Co-compressible Module International Research Journal of Scientific Findings 1(6).

[3] Amini B., Ershad M. and Sharif H.. 2009 Coretectable modules. J. of Australian Mathematics Society. 86 pp 289–304.

[4] Ansari-Toroghy H and Farshadifar F 2007 The Dual Notion Of Multiplication Modules. Taiwanese J. Of Math. 11 4 pp 1189-1201.

[5] Bernard A 1981 Multiplication Modules. Journal of Algebra 71 1 pp 174-178.

[6] Clark J, Lomp C, Vanaja N and Wisbauer R 2006 Lifting Modules. Frontiers In Mathematics, Birkhäuser.

[7] Gang Y and Zhong-kui L 2007 On Hopfian and Co-Hopfian Modules Vietnam Journal of Mathematics 35 1 pp 73–80.

[8] Hadi I M and Ahmed M A 2013 Couniform Modules Journal of Baghdad for Science 10 1 pp 243-250.

[9] Hadi I M and Al-aaeashi S N 2017 Strongly Coretectable modules Iraqi journal of sciences 58 2C pp 1069-1075.
[10] Hadi I M and Al-aeashi S N 2016 Strongly Pretractable modules and Some Related Concepts. Journal of advances in Mathematics 12 12 pp 6881-6888.
[11] Hadi I M and Al-aeashi S N 2017 Y-Coretractable and Strongly Y-Coretractable modules Asian Journal of Applied Sciences 5 2 pp 427-433.
[12] Hadi I M and Al-aeashi S N 2017 P-Coretractable and Strongly P-Coretractable modules Asian Journal of Applied Sciences 5 2 pp 477-482.
[13] Hadi I M and Al-aeashi S N 2017 Some Results About Coretractable modules Journal of AL-Qadisiyah for computer science and mathematics 9 2 pp 40-48.
[14] Hadi I M and Al-aeashi S N 2017 C-Coretractable and Strongly C-Coretractable modules 1st Scientific International Conference, College of Science, Al-Nahrain University Part II pp 65-72.
[15] Hadi I M and Al-aeashi S N 2018 Weakly Coretractable modules. J. Phys.: Conf. Ser. 1003 012061.
[16] Lee G 2010 Theory Of Rickart Modules. Ph.D. Thesis, the Ohio State University, U.S.A.
[17] Mijbass A S 1997 Quasi-Dedekind Modules Ph.D. Thesis, University of Baghdad, Iraq.
[18] Mohammad E A 2006 On Ikeda-Nakayama Modules Ph.D Thesis University of Baghdad Iraq.
[19] Yaseen S M 2003 Coquasi-Dedekind Modules Ph.D. Thesis University Of Baghdad, Iraq.