OPTIMIZED STATISTICAL APPROACH FOR COMBINING MULTI-MESSENGER DATA FOR NEUTRON STAR EQUATION OF STATE INFERENCE

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ABSTRACT

The neutron star equation of state (EOS) is now being constrained from a diverse set of multi-messenger data, including gravitational waves from binary neutron star mergers, X-ray observations of the neutron star radius, and many types of laboratory nuclear experiments. These measurements are typically mapped to a common domain – either to a corresponding radius or to a parametrized EOS using a Bayesian inference scheme – for comparison with one another. Using the first two binary neutron star mergers as an example, we show that a uniform prior in the tidal deformability can produce artificial evidence for large radii, which the data do not support. We present a new prescription for defining Bayesian priors in each domain of measurement, that will allow for minimally-biased constraints in the domain of comparison. Finally, using this new prescription, we provide a status update on multi-messenger EOS constraints on the neutron star radius.

1. INTRODUCTION

We are now in an era of true multi-messenger constraints on the neutron star equation of state (EOS), with a wealth of new results coming in from electromagnetic observations of astrophysical sources, gravitational wave detections of binary systems, and laboratory-based nuclear experiments.

On the astrophysical side, X-ray observations of surface emission from neutron stars in low-mass X-ray binaries (LMXBs) have constrained the radii of at least a dozen sources (Özel et al. 2009; Güver et al. 2010; Guillot & Rutledge 2014; Heinke et al. 2014; Nättilä et al. 2016; Özel et al. 2016; Bogdanov et al. 2016) for a recent review, see Özel & Freire (2016). Under the assumption that all neutron stars have a common radius, these measurements combine to yield a narrowly-constrained radius of \( R = 10.3 \pm 0.5 \) km (Özel et al. 2016). Additionally, the NICER collaboration recently reported the first radius constraint for an isolated X-ray pulsar (Bogdanov et al. 2019), which is quite broad but seems to favor relatively large radii, \( R = 12.71^{+1.14}_{-1.19} \) km, for a multi-component, phenomenological set of pulse-profile models (Riley et al. 2019). The LIGO-Virgo collaboration has also now detected two likely binary neutron star mergers. The first event, GW170817, provided strong constraints on the effective tidal deformability of the binary neutron star system, \( \Lambda = 300^{+130}_{-220} \) (Abbott et al. 2017, 2019). While there was no strong detection of tidal effects in the second event, GW190425, the masses from this event render it likely to be a second binary neutron star system, which some studies have already used in placing new, multi-messenger constraints on the neutron star EOS (The LIGO Scientific Collaboration et al. 2020; Dietrich et al. 2020; Landry et al. 2020). \( ^{[1]} \)

In addition to these astrophysical measurements, a wide variety of nuclear experiments have placed complementary constraints on the low-density portion of the EOS. For example, the two-body potential can be constrained from nucleon-nucleon scattering data at energies below 350 MeV and from the properties of light nuclei, which directly informs the EOS at densities near the nuclear saturation density, \( n_{\text{sat}} \) (Akmal et al. 1998; Morales et al. 2002). Experimental constraints are also often expressed in terms of the nuclear symmetry energy, which characterizes the difference in energy between pure neutron matter and symmetric nuclear matter. The value of the nuclear symmetry energy at \( n_{\text{sat}} \) and its slope, \( L_{0} \), have been constrained by fits to nuclear masses, by measurements of the neutron skin thickness, the giant dipole resonance, and electric dipole polarizability of \(^{208}\text{Pb}\), and by observations of isospin diffusion or multifragmentation in heavy ion collisions (e.g., Danielewicz 2003; Centelles et al. 2009; Roca-Maza et al. 2013; Tamii et al. 2011; Tsang et al. 2012; Oertel et al. 2017, for a recent review).

With this diversity of data, the question then arises of how one might robustly compare the results. Whether the astrophysical results are mapped to the EOS-domain for comparison using a Bayesian inference scheme, or the nuclear results are mapped to astrophysical quantities using approximate universal relations, the resulting comparisons will be slightly different. As we will demonstrate in this paper, each transformation of a measured quantity into a different domain has the potential to bias the resulting comparison. In particular, what may be a non-
informative assumption for the Bayesian prior in one domain may transform to be highly informative in another domain. The issue becomes more pressing when the data are weak, as was the case with the second binary neutron star merger, in which case the resulting constraints are dominated by the Bayesian prior.

The approach we are describing here – in which constraints on a particular quantity are made and then, in a second and independent step, those constraints are transformed to a different domain for comparison – is closely related to the so-called “exterior-prior paradigm” of Riley et al. (2018). Using example measurements of the neutron star radius to infer the underlying EOS parameters, Raaijmakers et al. (2018) showed that the two-step process of the exterior-prior paradigm can distort the resulting posterior, depending on whether the Bayesian prior is defined in the interior (e.g., EOS) or exterior (e.g., radius) domain. The authors of both works conclude that the interior-prior paradigm, in which the EOS parameters are inferred directly from the experimental data, is perhaps more statistically robust, as it side-steps the intermediate process of inferring, e.g., the radius. However, given the current landscape in which different research groups publish constraints only on these intermediate, observable features, the exterior-prior paradigm remains widely used.

In this paper, we will focus on issues that occur within this exterior-prior paradigm. However, we will take a different approach than that of Riley et al. (2018). We will demonstrate that, regardless of the end domain – i.e., whether the second step of the analysis is to infer EOS parameters or to transform to an independent observable feature, such as $R$ or $\Lambda$ – the priors can be distorted in ways that significantly affect the resulting comparisons. This is true whether the second mapping is a full Bayesian inference scheme to constrain the pressures of a parametrized EOS or whether the second step makes use of approximate universal relations to analytically map to other domains. In this paper, we will show that even mappings to other external domains can lead to a distortion of the priors and, thus, of the resulting posterior distribution.

Our goal is to derive a minimally-biased method for comparing constraints on the neutron star EOS from different types of experimental data that does not depend on the domain of comparison. We will specifically focus on recent constraints from X-ray observations of the neutron star radius, gravitational waves constraints on $\Lambda$, and nuclear experiments constraining $L_0$. We will introduce a general framework for defining Bayesian priors that are minimally-informative in the domain of comparison. These priors, once defined, can then be transformed to the domain of each measurement. As we will show, this procedure of choosing the domain of comparison a priori is critical for ensuring unbiased constraints.

We start with a brief review of Bayesian statistics, in order to define the issues that arise when the domain of measurement differs from the domain of comparison. In §3, we also introduce Bayesian priors for the different domains of comparison that are relevant for EOS inference. In §4 we derive a set of analytic transformation equations that facilitate the mapping between any two domains. In particular, we make use of previously-published mappings between the nuclear symmetry energy and the radius, as well as between the radius and the binary tidal deformability. We additionally introduce a new, simplified transformation function between the neutron star radius and the pressure at roughly twice the nuclear saturation density, which is approximate but helpful for illustrative purposes. In §4 we apply the newly-derived priors to the concrete example of the measured tidal deformability from GW170817 and GW190425. We find that the choice of priors strongly dominates for the weakly-informative GW190425, but that for both events, the choice of a uniform prior in the tidal deformability artificially inflates the evidence for larger radii. By defining a less informative prior that is uniform in the radius, the evidence points to slightly smaller radii. Finally, in §5 we combine the composite set of data from X-ray observations, both gravitational wave events, and a recent study using heavy-ion collisions and we present summary constraints on the neutron star radius.

2. BAYESIAN PRIORS

We start with a general review of Bayesian statistics, in order to illustrate the problems that arise when the domain of measurement differs from the domain of comparison. Bayes’ theorem states that, when modeling some collection of data with a set of parameters $\theta$, the posterior distribution on $\theta$ is given by

$$ P(\theta|\text{data}) = P_{\text{pr}}(\theta) \mathcal{L}(\text{data}|\theta), $$

where $P_{\text{pr}}(\theta)$ represents the Bayesian prior on $\theta$ and $\mathcal{L}(\text{data}|\theta)$ represents the likelihood of observing the measured data given a particular set of values for $\theta$.

We can transform this measurement of $\theta$ to a new set of parameters, $\phi$, with a simple transformation of variables,

$$ P(\phi|\text{data}) = P(\theta|\text{data}) \mathcal{J} \left( \begin{array}{c} \theta \\ \phi \end{array} \right), $$

where $\mathcal{J}$ represents the Jacobian of transformations. In the case that $\theta$ and $\phi$ are both single parameters, the Jacobian is simply $|\partial \theta / \partial \phi|$. Equations (2) shows that, depending on the nature of this Jacobian, even a broad posterior on $\theta$ can potentially lead to stringent constraints on $\phi$, simply by the transformation of variables.

As an example, let us imagine a scenario in which we measure weak constraints on $\theta$, given a noisy data set. Suppose we have limited prior knowledge of what $\theta$ should be and thus define a minimally-informative Bayesian prior that is a simple boxcar function over $\theta$. For this noisy measurement, the Bayesian evidence will be small, we will essentially recover our flat prior distribution, and we will safely conclude that no new knowledge of $\theta$ was measured. However, if $\theta$ depends strongly on $\phi$, then eq. (2) will imply that the data strongly constrain $\phi$, even though the actual measurement was uninformative. In §4 we will show that this is exactly what has happened with EOS constraints from GW190425.

In order to avoid this problem, it is important to decide, a priori, what parameter we are most interested in and then define a prior that is minimally informative in that domain. For the purposes of this paper, we consider
three different types of experimental measurements: nuclear experiments, of which we will focus on those that constrain \(L_0\); X-ray observations, which constrain \(R\); and gravitational waves, which constrain \(\Lambda\).

There is no unique choice for the domain in which to compare these experimental results. Arguably, constraining the parameters of the dense-matter EOS is the ultimate goal of this line of research. If we consider some fiducial EOS pressure, \(P_0\), to be the fundamental variable that we are interested in comparing, then one can define a set of minimally-informative priors in that domain. X-ray observations quite directly measure the neutron star radius. Furthermore, there exist strong correlations between the neutron star radius and the tidal deformability inferred from a gravitational wave event, as well as between \(R\) and the slope of the nuclear symmetry energy. This allows for a simple one-step transformation from either the nuclear or gravitational wave data to radius constraints, which is convenient, though not necessary.

If we choose the radius, \(R\), as the fundamental variable, then the gravitational wave and nuclear experimental constraints will be mapped to the radius domain for comparison with the X-ray results, as has been done in nearly all cross-domain comparisons to date. Our goal is to again define priors on \(\Lambda\) and \(L_0\) that are consistent with a prior that is minimally-informative in \(R\). We can define this self-consistent set of priors as

\[
P_{pr; \Lambda}(L_0) = P_{pr}(\Lambda) \left| \frac{\partial R}{\partial L_0} \right|^{-1}
\]

(3a)

\[
P_{pr; \Lambda}(R) = P_{pr}(\Lambda) \left| \frac{\partial R}{\partial P_0} \right|^{-1}
\]

(3b)

\[
P_{pr; \Lambda}(\tilde{\Lambda}) = P_{pr}(\Lambda) \left| \frac{\partial \tilde{\Lambda}}{\partial P_0} \right|^{-1},
\]

(3c)

where we have simply applied different transformations of variables. In these equations, we have introduced a short-hand notation for the prior, \(P_{pr; X}(Y)\), which indicates a Bayesian prior on the measurement of a variable \(Y\) that is defined with respect to the desired domain of comparison \(X\). In defining the transformation of variables, we have chosen to expand the derivatives so that we ultimately have only three derivatives to calculate: \(\partial R/\partial P_0, \partial R/\partial L_0, \text{ and } \partial \Lambda/\partial R\). This choice is particularly convenient because functions for \(R(L_0)\) and \(\Lambda(R)\) have been previously reported in other works, as we will review in 3. In 3 we will further introduce a new approximation for \(R(P_0)\), which allows for the priors in eq. 3 to be calculated fully analytically, which is convenient for illustrative purposes.

Even though it is true that the EOS parameters are closely related to the experimentally-measured nuclear symmetry energy, they cannot be directly probed astrophysically. Moreover, the parametric inference schemes that are often used to invert astrophysical data to constraints on the EOS can be sensitive to the choice of parametrization or priors (e.g., Steiner et al. 2016; Raithel et al. 2017; Carney et al. 2018; Raaijmakers et al. 2018; Riley et al. 2019). Thus, perhaps a more natural domain of comparison is in the radius domain. X-ray observations quite directly measure the neutron star radius. Furthermore, there exist strong correlations between the neutron star radius and the tidal deformability inferred from a gravitational wave event, as well as between \(R\) and the slope of the nuclear symmetry energy. This allows for a simple one-step transformation from either the nuclear or gravitational wave data to radius constraints, which is convenient, though not necessary.

3. TRANSFORMATION FUNCTIONS

We now turn to deriving the transformation functions needed to calculate the priors in eqs. 3-5. We will start at the microscopic level, with a new approximation for \(R(P_0)\). We will then connect \(R\) with \(L_0\) and, finally, \(R\) with \(\Lambda\) using the results of previous works.

3.1. From \(P_0\) to the neutron star radius

To start, our goal is to derive a simple mapping between the neutron star radius and a fiducial pressure of the EOS. This relationship, while approximate, will allow us to cleanly illustrate how the priors transform between different domains.

In general, in order to compute the radius for a star with a given central density, the TOV equations must be solved using the full EOS. The inverse problem— that is, determining the EOS given a radius— is only solvable with data that span the full mass-radius relation (Lindblom 1992). In the absence of perfect data and due to the fact that neutron stars are not expected to form with birth masses below \(\sim 1 \ M_\odot\), the inversion is inexact. As a result, many Bayesian statistical inference schemes have been developed to facilitate the mapping from neutron star observables to the EOS (Steiner et al. 2010; 2016; Özel et al. 2016; Raithel et al. 2017). However, for radii in particular, some simplifications to this problem have been identified. Lattimer & Prakash (2001) first showed that the neutron star radius is primarily determined by the pressure at \(1 - 2 \ \text{times the...}
nuclear saturation density. Özel & Psaltis (2009) later found that the pressure near 1.85 \( n_{\text{sat}} \) is highly correlated with the resulting radius. Furthermore, a large family of nucleonic EOS predict that all neutron stars across a wide range of masses will have the same radius, corresponding to “vertical” mass-radius relations (see, e.g., the middle panel of Fig. 1). In other words, for nucleonic EOS, we expect that nearly all neutron stars will have identical radii, the value of which is set by the pressure at \( \sim 1.85 n_{\text{sat}} \). Thus, in the following analysis, our goal is to create a simple mapping between the stellar radius and \( P_0 \approx P(1.85n_{\text{sat}}) \).

We start by constructing a large number of vertical mass-radius relations, using a sequence of piecewise polytropic EOS. We fix the low-density portion of each EOS to that of WFF1,\(^1\) which is then smoothly connected to the high-density EOS with a polytropic index of \( \Gamma = 3.7 \). Many previous studies have found evidence of strong correlations between these parameters (e.g., Lattimer & Prakash 2001; Steiner et al. 2013; Alam et al. 2016). Here, we use the approximate nuclear symmetry energy, \( L \), in Tews et al. (2017). In order to vary the radius of each EOS, we vary the value of \( P_0 \), which is then smoothly connected to the low-density EOS with a polytropic index of \( \Gamma = 3 \), as is approximately consistent with most realistic EOS reported in Read et al. (2009). For densities \( n > 1.85 n_{\text{sat}} \), we stiffen the EOS to have a polytropic index of 3.7, in order to ensure that there is enough pressure at high densities to reach a maximum mass of at least 2 \( M_\odot \). If this construction results in an EOS with a superluminal sound speed at high densities, we limit the polytropic index to the causal value. That is, for \( n > 1.85 n_{\text{sat}} \), we adopt \( \Gamma = \min[3.7, \Gamma_{\text{causal}}] \). We integrate each EOS with the standard TOV equations to construct a mass-radius sequence. The EOS and their resulting mass-radius relations are shown in Fig. 1 for a wide range of values of \( P_0 \).

The right panel of Fig. 1 also shows the relation between \( P_0 \) and the radius of a 1.4 \( M_\odot \) star, \( R_{1.4} \). The colorful symbols correspond to each EOS constructed in the left two panels, while the gray dashed line shows a best fit model, which we find to be

\[
R_{1.4} = 3.519 \left( \frac{P_0}{\text{MeV/fm}^3} \right)^{−0.355} + 5.047 \left( \frac{P_0}{\text{MeV/fm}^3} \right)^{0.277} \text{ km}, \tag{6}
\]

with a Bayesian information criteria strongly favoring this model over either a linear (\( \Delta \text{ BIC} = 190 \)) or single power-law model (\( \Delta \text{ BIC} = 100 \)). The derivative of this analytic function is then simply

\[
\frac{\partial R}{\partial P_0} = −1.248 \left( \frac{P_0}{\text{MeV/fm}^3} \right)^{−1.355} + 1.397 \left( \frac{P_0}{\text{MeV/fm}^3} \right)^{−0.723} \text{ km/MeV/fm}^3, \tag{7}
\]

where we have assumed \( R \approx R_{1.4} \).

3.2. From the nuclear symmetry energy to the neutron star radius

In order to map directly from nuclear constrains to \( P_0 \) in eqs. (3)-(5), we also need a transformation equation between the neutron star radius, \( R \), and the slope of the nuclear symmetry energy, \( L_0 \). Many previous studies have found evidence of strong correlations between these parameters (e.g., Lattimer & Prakash 2001; Steiner et al. 2013; Alam et al. 2016). Here, we use the approximate relation

\[
R_{1.4} \approx (4.51 \pm 0.26) \left( \frac{L_0}{\text{MeV}} \right)^{1/4} \text{ km}, \tag{8}
\]

which was calculated as a function of pressure for a sample of realistic EOS in Lattimer & Lim (2013) and later translated to be a function of \( L_0 \) in Tews et al. (2017). The derivative is then simply

\[
\frac{\partial R}{\partial L_0} \approx (1.128 \pm 0.065) \left( \frac{L_0}{\text{MeV}} \right)^{−3/4} \text{ km/MeV}, \tag{9}
\]

where we have again assumed \( R \approx R_{1.4} \), as is reasonable for EOS with vertical mass-radius relations.

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\(^2\) We note that by assuming a common radius for all neutron stars, we are implicitly assuming that the EOS is nucleonic for densities \( n \lesssim 2 n_{\text{sat}} \). Any significant softening or stiffening of the EOS, corresponding to the emergence of new particles or interactions in the matter, would cause a deviation from a vertical mass-radius relation. The following analysis should be modified, if a strong phase transition is suspected at densities below \( \sim 2 n_{\text{sat}} \).
3.3. From tidal deformability to the neutron star radius

Finally, we turn to the relationship between the radius and the effective tidal deformability measured from a gravitational wave event. Several studies have shown that $\tilde{\Lambda}$ is effectively a mono-parametric function of the neutron star radius (De et al. 2018; Raithel et al. 2018; Raithel 2019), which scales quite strongly as $\tilde{\Lambda} \sim R^{3-6}$, where the exponent varies according to the slightly different assumptions made in these analyses. We use the formalism of Raithel et al. (2018) to exactly calculate $\partial \tilde{\Lambda} / \partial R$.

In that study, we defined a quasi-Newtonian framework for calculating $\tilde{\Lambda}$, in which

$$\tilde{\Lambda} \approx \tilde{\Lambda}_0 [1 + \delta_0 (1 - q)^2] + \mathcal{O} ((1 - q)^3),$$

where

$$\tilde{\Lambda}_0 = \frac{15 - \pi^2}{3\pi^2} \xi^{-5} (1 - 2\xi)^{5/2},$$

$$\delta_0 = \frac{3}{104} (1 - 2\xi)^{-2} (10 - 94\xi - 83\xi^2),$$

and $\xi$ was introduced as an effective compactness, defined as

$$\xi \equiv \frac{2^{1/3} G M_c}{R c^2}.$$  \hspace{1cm} (13)

In these equations, $M_c$ is the chirp mass, $q$ is the mass ratio of the binary (defined such that $q \leq 1$), $G$ is the gravitational constant, and $c$ is the speed of light. Combining these results, one finds that the radius-dependence of the binary tidal deformability scales approximately as $\tilde{\Lambda} \sim R^{6}$.

In this framework, the derivative of $\tilde{\Lambda}$ is then given by

$$\frac{\partial \tilde{\Lambda}}{\partial R} \approx \frac{\partial \tilde{\Lambda}_0}{\partial R} \left[ 1 + \left( \delta_0 + \tilde{\Lambda}_0 \left( \frac{\partial \delta_0}{\partial R} \right) \left( \frac{\partial \tilde{\Lambda}_0}{\partial R} \right)^{-1} \right) (1 - q)^2 \right],$$

where we have neglected the higher-order terms and where the auxiliary derivatives are given by

$$\frac{\partial \delta_0}{\partial R} = -\frac{\delta_0}{R} \left[ \frac{54 + 22\xi}{-10 + 114\xi - 271\xi^2 + 166\xi^3} \right]$$

and

$$\frac{\partial \tilde{\Lambda}_0}{\partial R} = \frac{5\tilde{\Lambda}_0 \xi}{R} \left( \frac{1}{\xi} + \frac{1}{1 - 2\xi} \right).$$

The importance of the 2nd-order correction term in eq. (14) increases with $M_c$ and the mass asymmetry of the binary. That is, larger values of $M_c$ and smaller values of $q$ will both act to increase the coefficient of the 2nd-order term. However, even for a very large $M_c = 1.44 M_\odot$, as was measured for GW190425 and for $q = 0.7$, as was the lower limit for both GW170817 and GW190425, the correction term is at most 4%. Thus, we neglect the 2nd-order correction term and simply approximate

$$\frac{\partial \tilde{\Lambda}}{\partial R} \approx \frac{\partial \tilde{\Lambda}_0}{\partial R},$$

which scales roughly as $R^5$.

3.4. Summary of transformations

We now apply all of these transformation functions to compute the priors in eqs. (3)-(5). For each fundamental variable, we assume a bounded uniform distribution. We bound the uniform prior on $\tilde{\Lambda}$ to be positive and less than 1200, which is well above the limits that were derived for either GW170817 (with an adjusted chirp mass of $M_c = 1.44 M_\odot$) or GW190425. We bound the uniform prior on $R$ to be between 9 and 16 km, in order to broadly encompass all current measurements of X-ray radii. Finally, we bound the uniform prior on the pressure such that $P_0 \in [5, 50]$ MeV/fm$^{-3}$. We choose a lower bound of 5 MeV/fm$^{-3}$, to be roughly consistent with state-of-the-art chiral effective field theory calculations for matter in $\beta$-equilibrium (Lonardoni et al. 2019). We choose a very large upper limit on $P_0$, in order to be as unrestricted as possible, but we note that this upper limit is.

\[\text{Collaboration et al. (2020)}.\]
limit is much larger than what is allowed within the chiral effective field theory calculation of Lonardoni et al. (2019).

We show the resulting transformations of these priors in Fig. 2. In blue, we show the original case of a uniform prior on $\Lambda$, as was used by the LIGO-Virgo collaboration for both GW170817 and GW190425. The middle panel shows how the flat prior in $\Lambda$ maps to a highly informative prior in $R$, which is biased towards large radii. The right panel shows that a flat prior in $\Lambda$ is moderately biased towards larger values of $P_0$. Figure 2 also shows how a uniform prior in $R$ or $P_0$ transforms to the other domains, in orange and purple lines, respectively. Clearly, a “non-informative” prior in one domain can be highly informative in a different domain.

4. EXAMPLE APPLICATION TO GRAVITATIONAL WAVE DATA

With these transformation functions now in hand, we turn to a concrete example. In this section, we will calculate posteriors for $\Lambda$ using priors that are minimally informative in either $\Lambda$, $R$, or $P_0$. We will then map each set of posteriors to constraints on $R$ and $P_0$, in order to illustrate the sensitivity of the resulting constraints to the particular choice of priors.

We start with the measurement of $\Lambda$ from GW170817. The original posteriors utilized a flat prior in $\Lambda$ (Abbott et al. 2017). These posteriors are shown in blue in the left panel of Fig. 3, for an adjusted chirp mass of $M_c = 1.44\ M_\odot$. We then modify the published posterior to calculate the posterior that would have been inferred had the prior been uniform in radius (shown in orange) or uniform in $P_0$ (shown in purple). We calculate these new posteriors as

$$P(\Lambda|\text{data}) = P(\Lambda|\text{data}) \left[ \frac{P_{\text{pr, new}}(\Lambda)}{P_{\text{pr, old}}(\Lambda)} \right],$$

where $P_{\text{pr, new}}(\Lambda)$ indicates the new prior, which is given by eq. (3c) for the case of a uniform prior in $P_0$ or by eq. (4c) for the case of a uniform prior in $R$. Here, $P_{\text{pr, old}}(\Lambda)$ represents the original, uniform prior on $\Lambda$ and $P_{\text{old}}(\Lambda|\text{data})$ represents the original, published posterior. By dividing the reported posterior by the old prior, we essentially recover the original likelihood.

For each of the three, new posteriors on $\Lambda$, we then transform to find the corresponding constraints on $R$, according to

$$P(R|\text{data}) = P(\Lambda|\text{data}) \left| \frac{\partial \Lambda}{\partial R} \right|, \quad (19)$$
We similarly transform the posteriors on $\tilde{\Lambda}$ to constraints on $P_0$, according to

$$P(P_0|\text{data}) = P(\tilde{\Lambda}|\text{data}) \left| \frac{\partial \tilde{\Lambda}}{\partial R} \right| \left| \frac{\partial R}{\partial P_0} \right|. \quad (20)$$

The inferred constraints on $R$ and $P_0$ are shown in the middle and right panels of Fig. 3 respectively. At 68% confidence (highest-posterior density), the radius is constrained to $R = 10.9_{-0.8}^{+3.8}$ km for uniform priors in $P_0$, $R = 10.9_{-0.7}^{+3.8}$ km for uniform priors in $R$, and $R = 11.1_{-0.8}^{+3.8}$ km for uniform priors in $\tilde{\Lambda}$. There is a small difference between the inferred constraints, depending on which choice of prior is used. In particular, assuming a flat prior in $\tilde{\Lambda}$ or $P_0$ leads to evidence for slightly larger radii compared to the radii that are inferred when a flat prior distribution in $R$ is assumed. However, the data for this event are constraining enough that the overall effect of the prior remains small.

In contrast, Fig. 4 shows that the constraints inferred from $\tilde{\Lambda}$ for GW190425 are much more sensitive to the choice of the prior. As for GW170817, the LIGO-Virgo collaboration reported posteriors on $\tilde{\Lambda}$ assuming a uniform prior distribution on $\tilde{\Lambda}$ ([The LIGO Scientific Collaboration et al. 2020]). However, unlike GW170817, the resulting posteriors for GW190425 essentially represent a non-detection: the authors state that they lack the requisite sensitivity to detect matter effects for this system ([The LIGO Scientific Collaboration et al. 2020]). Nevertheless, they report constraints on $\tilde{\Lambda}$, the neutron star EOS, and $R$, assuming that GW190425 is indeed a binary neutron star system based on its component masses. Following suit, we re-weight the reported posteriors on $\tilde{\Lambda}$ to determine the posteriors that would have been inferred had a uniform prior in $R$ or $P_0$ instead been used, according to eq. (15). The resulting posteriors, and their transformations to $R$ and $P_0$, are shown in Fig. 4.

We find that the choice of prior strongly influences the resulting constraints on $R$ and $P_0$ for GW190425. In particular, the assumption of a flat prior in $\tilde{\Lambda}$ leads to the inference of quite large radii, $R = 3.3_{-1.7}^{+1.1}$ km (68% credibility interval), even though no significant matter effects were detected in the actual measurement. The inference of large radii is purely an artifact of the transformation of variables. If we instead use a uniform prior in the radius, then the corresponding constraints on $R$ are also relatively uniform, such that it does not make sense to report a 68% credibility interval. We find that the constraints on $R$ are essentially flat across the range of 9-13 km, with values of $R \gtrsim 13$ km disfavored. Figure 4 thus demonstrates that the prior outweighs the actual data for this event. Moreover, Fig. 4 demonstrates that comparing in the radius domain, when the measurement and original prior were in the $\tilde{\Lambda}$ domain, produces artificial evidence for large radii, even in the absence of a measured signal.

The conclusion that the prior outweighs the data for GW190425 may be obvious when the posteriors are examined in the domain in which they are made. In this case, the relatively flat posterior measured for $\tilde{\Lambda}$ is clearly mostly consistent with the flat prior that was assumed, and we can conclude that the event was not very informative. The picture becomes less clear, however, when transforming to a different domain and then making comparisons in that domain. In fact, several studies are already using GW190425, in conjunction with other observations, to place constraints on the EOS (e.g., [Dietrich et al. 2020] and [Landry et al. 2020]). However, our results suggest that any such inferences will be biased towards larger radii by the uniform prior in $\tilde{\Lambda}$ that was assumed in the published LIGO-Virgo posteriors (as was used by Landry et al. 2020) or with the choice of a uniform prior in the component tidal deformabilities (as was assumed in the re-analysis performed in Dietrich et al. 2020, and which approaches the uniform prior for $\Lambda$ for $q \rightarrow 1$).

The two gravitational wave events that have been detected so far are relatively straightforward to identify as “strongly” and “weakly” constraining events. However, in the coming years, it is likely that the LIGO-Virgo collaboration will measure many events whose constraints on $\tilde{\Lambda}$ fall in the more intermediate category of constraining power. Thus, in general, we argue that the optimal way to avoid the pitfalls of overly-informative priors is to use a flat prior in the domain in which the constraint is actually being made. That is, if the goal is to compare $\tilde{\Lambda}$ with X-ray constraints on the neutron star radius, one should use priors that are minimally-informative in $R$. Alternatively, if the goal is to compare $\tilde{\Lambda}$ with nuclear constraints on $P_0$, one should use priors that are minimally-informative in $P_0$. A flat prior on $\tilde{\Lambda}$ is the natural choice only when comparing to other measurements of the tidal deformability.

5. COMPOSITE CONSTRAINTS ON THE NEUTRON STAR RADIUS

With the new prescription for defining priors introduced in this paper, we now present summary constraints on the neutron star radius, using the latest results from X-ray data, gravitational waves, and nuclear constraints on $L_0$.

These results are summarized in Figure 3 for two choices of priors. In this figure, we include constraints from GW170817 (in blue: [Abbott et al. 2017]) and GW190425 (in green: [The LIGO Scientific Collaboration et al. 2020]). X-ray constraints on the radii of 12 neutron stars in LMXBs (in red: [Ozel et al. 2016]), the X-ray timing results from NICER for PSR J0030+0451 (in orange: [Riley et al. 2019]), and a recent constraint on $L_0$ from an analysis of single and double ratios of neutron and proton spectra from heavy-ion collisions (in purple: [Morfouace et al. 2019]). While we only include a single constraint on $L_0$, we note that this posterior ($L_0 = 49.6 \pm 13.7$ MeV, with values below 32 MeV or above 120 MeV forbidden) is consistent with the results of a recent meta-analysis of several dozen studies that determined $L_0 = 58.7 \pm 28.1$ MeV ([Oertel et al. 2017]). Thus, we include the [Morfouace et al. 2019] results in...
Fig. 5.—Top: Constraints on the neutron star radius from X-ray observations, gravitational wave inference, and nuclear experimental data, assuming a uniform prior in each of the measured quantities (i.e., $\tilde{\Lambda}$, $R$, and $L_0$). Bottom: Constraints on the neutron star radius from the same data, but now assuming a uniform prior in the radius. We find that using prior distributions that are chosen to be minimal-informative in the radius results in more evidence for smaller radii.

Fig. 6.—Joint posterior distribution on the radius, determined by various combinations of experimental data. Orange lines correspond to any combination of experimental results that include the 12 LMXB sources. Blue lines indicate combinations that include the NICER source, PSR J0030+0451, as the only X-ray data. The purple line shows the constraints inferred from only gravitational wave and nuclear constraints; i.e., with no X-ray data.

We find that the data from the 12 LMXB sources are the most constraining measurement included in this paper. Any joint posteriors that contain these data point to $R \sim 10 - 11.5$ km. Moreover, small radii are supported by any combination of results that exclude the NICER data, including the combination of gravitational wave and $L_0$ constraints alone. In contrast, if the NICER source is included as the only X-ray data, then the resulting radii are much larger, $R \sim 12 - 13$ km. Currently, the NICER collaboration has published radius constraints for just a single source, PSR J0030+0451, using a multi-component, phenomenological pulse-profile model to fit the data. As more physical pulse-profile models are developed and more sources are included in the analysis, it will be interesting to see whether this systematic offset persists.

As the community works towards ever-more stringent constraints on the neutron star radius, these joint posteriors can be helpful for understanding the relative constraining power of each additional measurement. They can also help to identify systematic offsets between different types of measurements. Finally, we note that regardless of which data are included in any meta-analysis, defining the prior to be minimally-informative in the domain of comparison is an important step towards getting unbiased constraints.

6. CONCLUSIONS

With the recent flood of multi-messenger constraints on the neutron star EOS, it is important to start iden-
Identifying the statistical biases that enter into comparisons of these diverse data sets. In this paper, we have highlighted the importance of defining minimally-informative priors directly in the domain of comparison. We introduced a general prescription for calculating such priors and derived the relevant transformation functions so that these priors can be properly mapped back to and used in the measurement domain.

Using the example of GW170817 and GW190425, we showed that assuming a Bayesian prior that is “non-informative” in \( \Lambda \) leads to a highly-informative constraint on \( R \), even in the absence of a measured signal. In particular, a flat prior in \( \Lambda \) biases the resulting constraint on \( R \) to large values that are unsupported by the data. If we instead define a prior on \( \Lambda \) that is minimally-informative in the radius, it becomes clear that the gravitational wave data provide evidence for slightly smaller radii.

As the community continues to collect more and higher quality data, the relative importance of the priors should diminish. We have already shown this for the case of radius constraints inferred from \( \Lambda \) for GW170817, for which the choice of prior does not strongly affect the resulting posterior. However, for gravitational waves in particular, we may see far more low-significance events than we do GW170817-like events. Thus, if we hope to use the future constraints on \( \Lambda \) to compare with other radius measurements, it is important to account for the role of the assumed priors.

As new events – gravitational and otherwise – continue to be observed, the general prescription introduced in this paper will facilitate increasingly stringent, and statistically robust, constraints on the neutron star EOS.

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REFERENCES

Abbott, B. P., Abbott, R., Abbott, T. D., et al. 2017, Physical Review Letters, 119, 161101
—. 2019, Physical Review X, 9, 011001
Akmal, A., Pandharipande, V. R., & Ravenhall, D. G. 1998, Phys. Rev. C, 58, 1804
Alam, N., Agrawal, B. K., Fortin, M., et al. 2016, Phys. Rev. C, 94, 052801
Bogdanov, S., Heinke, C. O., Özel, F., & Güver, T. 2016, ArXiv e-prints, arXiv:1605.01630
Bogdanov, S., Guillot, S., Ray, P. S., et al. 2019, ApJ, 887, L25
Carney, M. F., Wade, L. E., & Irwin, B. S. 2018, Phys. Rev. D, 98, 063004
Centelles, M., Roca-Maza, X., Viñas, X., & Warda, M. 2009, Physical Review Letters, 102, 122502
Danielewicz, P. 2003, Nuclear Physics A, 727, 233
De, S., Finstad, D., Lattimer, J. M., et al. 2018, Physical Review Letters, 121, 091102
Dietrich, T., Coughlin, M. W., Pang, P. T. H., et al. 2020, arXiv e-prints, arXiv:2002.11355
Greif, S. K., Raaijmakers, G., Hebel, K., Schwenk, A., & Watts, A. L. 2019, MNRAS, 485, 5363
Guillot, S., & Rutledge, R. E. 2014, ApJ, 796, L3
Guillot, S., Servillat, M., Webs, N. A., & Rutledge, R. E. 2013, ApJ, 772, 7
Güver, T., Özel, F., Cabrera-Lavers, A., & Wroblewski, P. 2010, ApJ, 712, 964
Heinke, C. O., Cohn, H. N., Lugger, P. M., et al. 2014, MNRAS, 444, 443
Landry, P., Essick, R., & Chatziioannou, K. 2020, arXiv e-prints, arXiv:2003.04880
Lattimer, J. M., & Lim, Y. 2013, ApJ, 771, 51
Lattimer, J. M., & Prakash, M. 2001, ApJ, 550, 426
Lindblom, L. 1992, ApJ, 398, 569
Lonardoni, D., Tesi, I., Gandolfi, S., & Carlson, J. 2019, arXiv e-prints, arXiv:1912.09411
Morałes, J., Pandharipande, V. R., & Ravenhall, D. G. 2002, Phys. Rev. C, 66, 054308
Morfouace, P., Tsang, C. Y., Zhang, Y., et al. 2019, Physics Letters B, 799, 135045
Nättilä, J., Steiner, A. W., Kajava, J. J. E., Suleimanov, V. F., & Poutanen, J. 2016, A&A, 591, A25
Oertel, M., Hempel, M., Klähn, T., & Typel, S. 2017, Reviews of Modern Physics, 89, 015007
Özel, F., & Freire, P. 2016, ARA&A, 54, 401
Özel, F., Güver, T., & Psaltis, D. 2009, ApJ, 693, 1775
Özel, F., & Psaltis, D. 2009, Phys. Rev. D, 80, 103003
Özel, F., Psaltis, D., Güver, T., et al. 2016, ApJ, 820, 28
Raaijmakers, G., Riley, T. E., & Watts, A. L. 2018, MNRAS, 478, 2177
Raiithel, C. A. 2019, European Physical Journal A, 55, 80
Raiithel, C. A., Özel, F., & Psaltis, D. 2017, ApJ, 844, 156
—. 2018, ApJ, 857, L23
Read, J. S., Lackey, B. D., Owen, B. J., & Friedman, J. L. 2009, Phys. Rev. D, 79, 124032
Riley, T. E., Raaijmakers, G., & Watts, A. L. 2018, MNRAS, 478, 1093
Riley, T. E., Watts, A. L., Bogdanov, S., et al. 2019, ApJ, 887, L21
Roca-Maza, X., Brenna, M., Agrawal, B. K., et al. 2013, Phys. Rev. C, 87, 034301
Steiner, A. W., Lattimer, J. M., & Brown, E. F. 2010, ApJ, 722, 478, 1093
—. 2018, ApJ, 822, 33
—. 2013, ApJ, 765, L5
—. 2016, European Physical Journal A, 52, 18
Tamil, A., Poltoratska, I., von Neumann-Cosel, P., et al. 2011, Physical Review Letters, 107, 062502
Tews, I., Lattimer, J. M., Ohnishi, A., & Kolomeitsev, E. E. 2017, ApJ, 848, 105
The LIGO Scientific Collaboration, the Virgo Collaboration, Abbott, B. P., et al. 2020, arXiv e-prints, arXiv:2001.01761
Tsang, M. B., Stone, J. R., Camera, F., et al. 2012, Phys. Rev. C, 86, 015803
Wiringa, R. B., Fiks, V., & Fabrocini, A. 1988, Phys. Rev. C, 38, 1010