Type II hidden symmetries for the homogeneous heat equation in some general classes of Riemannian spaces

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Abstract

We study the reduction of the heat equation in Riemannian spaces which admit a gradient Killing vector, a gradient homothetic vector and in Petrov Type D,N,II and Type III space-times. In each reduction we identify the source of the Type II hidden symmetries. More specifically we find that a) If we reduce the heat equation by the symmetries generated by the gradient KV the reduced equation is a linear heat equation in the nondecomposable space. b) If we reduce the heat equation via the symmetries generated by the gradient HV the reduced equation is a Laplace equation for an appropriate metric. In this case the Type II hidden symmetries are generated from the proper CKVs. c) In the Petrov spacetimes the reduction of the heat equation by the symmetry generated from the nongradient HV gives PDEs which inherit the Lie symmetries hence no Type II hidden symmetries appear. We apply the general results to cases in which the initial metric is specified. We consider the case that the irreducible part of the decomposed space is a space of constant nonvanishing curvature and the case of the spatially flat Friedmann-Robertson-Walker space time used in Cosmology. In each case we give explicitly the Type II hidden symmetries provided they exist.

Keywords: Lie symmetries, Type II hidden symmetries, Heat Equation.

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1 Introduction

Lie point symmetries assist us in the simplification of differential equations (DE) by means of reduction. The reduction is different for ordinary differential equations (ODEs) and partial differential equations (PDEs). In the case of ODEs the use of a Lie point symmetry reduces the order of ODE by one while in the case of PDEs the reduction by a Lie point symmetry reduces by one the number of independent and dependent variables, but not the order of the PDE. A common characteristic in the reduction of both cases is that the Lie point symmetry which is used for the reduction is not admitted as such by the reduced DE, it is "lost".

It has been found that the reduced equation is possible to admit more Lie point symmetries than the ones of the original equation. These new Lie point symmetries have been termed Type II hidden symmetries. Also if one works in the reverse way and either increase the order of an ODE or increase the number of independent
and dependent variables of a PDE then it is possible that the new (the ‘augmented’) DE admits new point symmetries not admitted by the original DE. This type of Lie point symmetries are called Type I hidden symmetries.

The Type I and Type II hidden symmetries have been discussed extensively during the recent years by B Abraham - Shrauner, K S Govinder, P G L Leach and others (see e.g. [1] [2] [3] [4] [5] [6] [7] [8]). In the following we shall consider mainly the Type II hidden symmetries as they are the ones which may be used to reduce the reduced DE further.

The origin of Type II hidden symmetries is different for the ODEs and the PDEs although recently it has been shown [9] that they are nearly the same. For the case of ODEs the inheritance or not of a Lie point symmetry, the \(X_2\) say, by the reduced ODE depends on the commutator of that symmetry with the symmetry used for the reduction, the \(X_1\) say. For example if only two Lie point symmetries \(X_1, X_2\) are admitted by the original equation and the commutator \([X_1, X_2] = cX_2\) where \(c\) may be zero, then reduction by \(X_1\) results in \(X_2\) being a nonlocal symmetry for the reduced ODE while reduction by \(X_2\) results in \(X_1\) being an inherited Lie symmetry of the reduced ODE. In the reduction by \(X_1\) the symmetry \(X_2\) is a Type I hidden symmetry of the original equation relative to the reduced equation. In the case of more than two Lie point symmetries the situation is the same if the Lie bracket gives a third Lie point symmetry, the \(X_3\) say. Then the point like nature of a symmetry is preserved only if reduction is performed using the normal subgroup and \(X_3\) has a certain expression [10].

The above scenario is transferred to PDEs as follows. The reduced PDE loses the symmetry used to reduce the number of variables and it may lose other Lie point symmetries depending on the structure of the associated Lie algebra, that is if the admitted subgroup is normal or not [10]. Again if \(X_1, X_2\) are Lie point symmetries of the original PDE with commutator \([X_1, X_2] = cX_2\) where \(c\) may be zero, then reduction by \(X_2\) results in \(X_1\) being a point symmetry of the reduced PDE while reduction by \(X_1\) results to an expression which has no relevance for the PDE [10].

In addition to that scenario, Abraham - Shrauner and Govinder have proposed a new potential source for the Type II hidden symmetries [5] [11] based on the observation that different PDEs with the same variables which admit different Lie point symmetry algebras may reduce to the same target PDE. Based on that observation they propose that the target PDE inherits Lie point symmetries from all reduced PDEs, which explains why some of the new symmetries are not admitted by the specific PDE used for the reduction. In this context arises the problem of identifying the set of all PDEs which lead to the same reduced PDE after reduction by a Lie point symmetry. In a recent paper [9] it has been shown that this is also the case with the ODEs. That is, it is shown that different differential equations which can be reduced to the same equation provide point sources for each of the Lie point symmetries of the reduced equation even though any particular of the higher order equations may not provide the full complement of Lie point symmetries. Therefore concerning the ODEs the Lie point symmetries of the reduced equation can be viewed as having two sources. Firstly the point and nonlocal symmetries of a given higher order equation and secondly the point symmetries of a variety of higher order ODEs. Finally in a newer paper [11] it has been shown by a counter example that Type II hidden symmetries for PDEs can have a nonpoint origin, i.e. they arise from contact symmetries or even nonlocal symmetries of the original equation. Other approaches may be found in [12] [13].

In the present paper we study the reduction and the consequent existence of Type II hidden symmetries of the heat equation in certain classes of Riemannian spaces.

In a general Riemannian space the heat equation has three Lie point symmetries which give trivial reduced
forms. This implies that if we want to find 'sound' reductions of the heat equation we have to consider Riemannian spaces which admit some type of symmetry(ies) of the metric (these symmetries are not Lie symmetries and are called collineations). Indeed it has been shown [14, 15], that the Lie symmetries of the heat and the Poisson equation in a Riemannian space are generated from the elements of the homothetic algebra and the conformal algebra of the space respectively. Therefore one expects that in spaces with a nonvoid homothetic algebra there will be Lie symmetry vectors which will allow for the reduction of the heat equation and the possibility of the existence of Type II hidden symmetries.

The structure of the paper is as follows. In section 2 we recall results concerning the Lie point symmetry conditions for a general type of second order PDE. For the convenience of the reader we also recall two theorems which relate the Lie point symmetries of the heat equation with the homothetic algebra of the space. The next sections contain the new results. In section 3 we consider a decomposable space - that is a Riemannian space which admits a gradient Killing vector (KV). In section 4 we do the same for a space which admits a gradient Homothetic vector (HV) and show that reduction by the Lie symmetries due to this vector also give rise to Type II hidden symmetries. In section 5 we consider the special cases of the previous section, that is, a decomposable space whose nondecomposable part is a maximally symmetric space of non vanishing curvature and the spatially flat Friedmann Robertson Walker (FRW) space time used in Cosmology.

In section 6 we consider the algebraically special vacuum solutions of Einstein’s equations known as Petrov type N, II, III and D. These space-times belong to space-times which admit a nongradient HV acting simply transitive [16]. In these space-times we consider the reduction of the heat equation using the Lie point symmetries resulting from the homothetic vector and look for Type II hidden symmetries. Finally in section 7 we draw our conclusions.

2 Lie point symmetries and collineations in Riemannian spaces

In a general Riemannian space with metric $g_{ij}$ the heat conduction equation with flux is

$$\Delta u - u_t = q$$  \hspace{1cm} (1)

where $\Delta$ is the Laplace operator $\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right)$ and $q = q(t, x, u)$. Equation (1) can also be written

$$g^{ij} u_{ij} - \Gamma^i u_i - u_t = q$$  \hspace{1cm} (2)

where $\Gamma^i = \Gamma^i_{jk} g^{jk}$ and $\Gamma^i_{jk}$ are the Christofell Symbols of the metric $g_{ij}$.

Equation (2) admits the Lie point symmetries

$$X_t = \partial_t, \ X_u = u \partial_u, \ X_b = b(t, x) \partial_u$$

where $b(t, x)$ is a solution of the heat equation. These symmetries are too general to provide sound reductions and consequently reduced PDEs which can give Type II hidden symmetries. However it has been shown [14, 15] that there is a close relation between the Lie point symmetries of the heat equation (and its special form the Poisson equation) with the collineations of the metric. Specifically it has been shown that the Lie point symmetries of the heat equation are generated from the HV and the KVs of $g_{ij}$ whereas the Lie point symmetries of the Poisson equation form the conformal algebra of the metric. This implies that if we want to have new Lie point symmetries which will allow for sound reductions of the heat equation eqn (2) we have to restrict our
considerations to spaces which admit a homothetic algebra. Our intention is to keep the discussion as general as possible therefore we consider spaces in which the metric $g_{ij}$ can be written in generic form. The spaces we shall consider are:

a. Spaces which admit a gradient KV.

b. Spaces which admit a gradient HV.

The generic form of the metric for these types of spaces has as follows:

1. If an $1 + n$–dimensional Riemannian space admits a gradient KV, the $S^i = \partial x^i (S = x)$ say, then the space is decomposable along $\partial x^i$ and the metric is written as (see e.g. [17])

$$ds^2 = dx^2 + h_{AB} y^A y^B, \ h_{AB} = h_{AB} (y^C)$$

2. If an $n$–dimensional Riemannian space admits a gradient HV, the $H^i = r \partial_r (H = \frac{1}{2} r^2), \ \psi_H = 1$ say, then the metric can be written in the generic form [20]

$$ds^2 = dr^2 + r^2 h_{AB} dy^A dy^B, \ h_{AB} = h_{AB} (y^C)$$

where in both cases $h_{AB}$ is the metric of the $n$– dimensional subspace with coordinates $\{x^A\}$ and the indices $A, B, C = 1, ..., n$.

The Riemannian spaces which admit nongradient proper HV do not have a generic form for their metric. However the spaces on which the HV acts simply transitively are a few and are given together with their homothetic algebra in [16]. A special class of these spaces are the algebraically special vacuum space-times known as Petrov types N, II, III, D whose metric is:

(a) Petrov Type N

$$ds^2 = dx^2 + dy^2 + 2d\rho d\nu - 2\ln (x^2 + y^2) d\rho^2$$

(b) Petrov Type D

$$ds^2 = -dx^2 + x^{-\frac{3}{2}} dy^2 - x^{\frac{1}{2}} (d\rho^2 + dz^2)$$

(c) Petrov Type II

$$ds^2 = \rho^{-\frac{3}{2}} (d\rho^2 + dz^2) - 2\rho dx dy + \rho \ln \rho dy^2$$

(d) Petrov Type III

$$ds^2 = 2d\rho d\nu + \frac{3}{2} \nu d\rho^2 + \frac{\nu^2}{x^3} (dx^2 + dy^2)$$

In what follows all spaces are of dimension $n \geq 2$. The case $n = 1$ although relatively trivial for our approach in general it is not so and has been studied for example in [18, 19].

2.1 Collineations

For the convenience of the reader and the completeness of the paper we present some basic results concerning the conformal algebra. In the following $L_\xi$ denotes Lie derivative with respect to the vector field $\xi^i$.

A vector field $\xi^i$ is Conformal killing vector (CKV) of a metric $g_{ij}$ if $L_\xi g_{ij} = 2\psi g_{ij}$. If $\psi = 0$ then $\xi^i$ is a KV and if $\psi = 1$, $\xi^i$ is a HV. Two metrics $g_{ij}$, $\tilde{g}_{ij}$ are conformally related if $\tilde{g}_{ij} = N^2 g_{ij}$ where the function $N^2$ is the conformal factor.
If $\xi^i$ is a CKV of the metric $\bar{g}_{ij}$ so that $L_{\xi} \bar{g}_{ij} = 2\bar{\psi} \bar{g}_{ij}$ then $\xi^i$ is also a CKV of the metric $g_{ij}$, that is $L_{\xi} g_{ij} = 2\psi g_{ij}$, where the conformal factor

$$\psi = \bar{\psi}N^2 - NN, \xi^i.$$ 

This means that two conformally related metrics have the same conformal algebra but with different conformal factors hence subalgebras. This is an important observation which shall be useful in the following sections.

### 2.2 Lie point symmetry conditions

In [14] it has been shown that the Lie point symmetry conditions for the PDE of the form

$$A^{ij} u_{ij} - B^i(x,u) u_i - f(x,u) = 0 \quad (4)$$

are as follows:

$$A^{ij} (a_{ij} u + b_{ij}) - (a_i u + b_i) B^i - \xi^k f_{,k} - au f_{,u} - bf_{,u} + \lambda f = 0 \quad (5)$$

$$A^{ij} \xi_{,ij} - 2A^{ik} a_{,i} + a B^k + au B^k_{,u} - \xi^i B_k^i - \lambda B^k + b B^k_{,u} = 0 \quad (6)$$

$$L_{\xi^i} A^{ij} = (\lambda - a) A^{ij} - \eta A^{ij}_{,u} \quad (7)$$

$$\eta = a(x^i) u + b(x^i) \quad (8)$$

$$\xi^k = \xi^k(x^i) \quad (9)$$

where the generator of the Lie point symmetry is $X = \xi^i (x^i) \partial_i + (a (x^k) u + b (x^i)) \partial_u$ and $b(x^i)$ is a solution of the PDE.

An immediate conclusion is that if $A^{ij}_{,u} = 0$ then from (7) follows that the Lie symmetries are generated from the CKVs of the ‘metric’ $A^{ij}$.

In the same paper [14] it has also been shown the following result concerning the Lie point symmetries of the inhomogeneous and the homogeneous heat equation.

**Theorem 1** The Lie point symmetries of the heat equation with flux i.e.

$$g^{ij} u_{ij} - \Gamma^{i} u_{i} - u_t = q (t,x,u) \quad (10)$$

in an $n$-dimensional Riemannian space with metric $g_{ij}$ are constructed from the homothetic algebra of the metric as follows:

a. $Y^i$ is a HV/KV.

The Lie point symmetry is

$$X = (2c_2 \psi t + c_1) \partial_t + c_2 Y^i \partial_i + (a (t) u + b (t,x)) \partial_u \quad (11)$$

where the functions $a(t), b(t,x), q(t,x^k,u)$ satisfy the constraint equation

$$-a t + H (b) - (au + b) q_{,u} + aq - (2\psi c_2 qt + c_1 q) \partial_t - c_2 q_{,i} Y^i = 0. \quad (12)$$

b. $Y^i = S^i$ is a gradient HV/KV.

The Lie point symmetry is

$$X = \left(2\psi \int T dt\right) \partial_t + TS^i \partial_i - \left(\frac{1}{2} T S - F (t)\right) u - b (t,x) \quad (13)$$

where

$$T = \frac{1}{2} T_{ij} S - F (t)$$

and

$$F (t) = \int_0^t \psi \partial_s \psi s ds.$$
where the functions \( F(t), T(t), b(t, x^k), q(t, x^k, u) \) satisfy the constraint equation

\[
0 = \left( -\frac{1}{2} T_t \psi + \frac{1}{2} T_t S - F_t \right) u + H(b) + \left( -\frac{1}{2} T_t S + F \right) q_u + \left( -\frac{1}{2} T_t S + F \right) q - \left( 2\psi q \int T dt \right)_t - T q_i S^i. \tag{14}
\]

In the special case in which the heat flux vanishes i.e. \( q(t, x, u) = 0 \), i.e.

\[
g^{ij} u_{ij} - \Gamma^i u_i - u_t = 0 \tag{15}
\]

we have the following result.

**Theorem 2** The Lie point symmetries of the homogeneous heat equation in an \( n \)-dimensional Riemannian space

\[
g^{ij} u_{ij} - \Gamma^i u_i - u_t = 0 \tag{16}
\]

are constructed from the homothetic algebra of the metric \( g_{ij} \) as follows:

a. If \( Y^i \) is a HV/KV of the metric \( g_{ij} \), the Lie point symmetry is

\[
X = (2\psi c_1 t + c_2) \partial_t + c_1 Y^i \partial_i + \left( a_0 u + b(t, x^i) \right) \partial_u \tag{17}
\]

where \( c_1, c_2, a_0 \) are constants and \( b(t, x^i) \) is a solution of the homogeneous heat equation.

b. If \( Y^i = S^i \) is a gradient HV/KV of the metric \( g_{ij} \), the Lie point symmetry is

\[
X = \psi t^2 \partial_t + t S^i \partial_i - \left( \frac{1}{2} S + \frac{1}{2} n \psi t \right) u \partial_u + b(t, x^i) \partial_u \tag{18}
\]

where \( c_3, c_4, c_5 \) are constants and \( b(t, x^i) \) is a solution of the homogeneous heat equation.

In both cases \( \psi = 1 \) for a HV and \( \psi = 0 \) for a KV.

In the following we shall need the Lie point symmetries of Laplace equation \( \Delta u = 0 \) or

\[
g^{ij} u_{ij} - \Gamma^i u_i = 0 \tag{19}
\]

In this context we have the following result

**Theorem 3** \([15]\) The Lie point symmetries of Laplace equation \([19]\) are generated from the CKVs of the metric \( g_{ij} \) as follows

\[
X = \xi^i (x^k) \partial_i + \left[ \left( \frac{2-n}{2} \psi (x^k) + a_0 \right) u + b(x^k) \right] \partial_u \tag{20}
\]

where \( \xi^i (x^k) \) is a CKV with conformal factor \( \psi(x^k) \) and the following conditions hold

\[
g \Delta \psi = 0, \quad g \Delta b = 0 \tag{21}
\]

that is, both the conformal factors and the function \( b \) are solutions of Laplace equation.

In the following sections we apply these theorems in order to reduced the heat equation by the extra Lie point symmetries admitted in the special spaces we considered in section 2.
3 The heat equation in a $1+n$ decomposable space

Without loss of generality we assume the gradient $\nabla V$ to be the $\partial_x$ so that the metric has the generic form

$$ds^2 = dx^2 + h_{AB}dy^Ady^B, \ h_{AB} = h_{AB}(y^C)$$  \hspace{1cm} (22)

where $h_{AB} \ A, B, C = 1, ..., n$ is the metric of the $n-$ dimensional space. For the metric (22) the heat equation (15) takes the form

$$u_{xx} + h_{AB}u_{AB} - \Gamma^A u_B - u_t = 0.$$  \hspace{1cm} (23)

Application of Theorem 2 gives that (23) admits the following extra Lie point symmetries generated by the gradient $\nabla_x$:

$$X_1 = \partial_x, \ X_2 = t\partial_x - \frac{1}{2}xu\partial_u$$

with nonvanishing commutators

$$[X_1, X_2] = X_1, \ [X_2, X_1] = \frac{1}{2}X_u.$$  \hspace{1cm} (24)

We reduce (23) using the zeroth order invariants of the extra Lie point symmetries $X_1, X_2$.

3.1 Reduction by $X_1$

The zeroth order invariants of $X_1$ are

$$\tau = t, \ y^A, \ w = u.$$  

Taking these invariants as new coordinates eqn (23) reduces to

$$\hbar \Delta w - w_t = 0$$  \hspace{1cm} (25)

where $\hbar \Delta$ is the Laplace operator in the $n-$dimensional space with metric $h_{AB}$:

$$\hbar \Delta w = h^{AB}w_{AB} - \Gamma^A w_B.$$  \hspace{1cm} (26)

Equation (25) is the homogeneous heat eqn (15) in the $n-$ dimensional space with metric $h_{AB}$. According to the Theorem 2 the Lie point symmetries of this equation are the homothetic algebra of $h_{AB}$. It is easy to show that the homothetic algebra of the $n$ and the $1+n$ metrics are related as follows [17]:

a. The KVs of the $n-$ metric are identical with those of the $1+n$ metric.

b. The $1+n$ metric admits a HV if the $n$ metric admits one and if $nH^A$ is the HV of the $n-$ metric then the HV of the $1+n$ metric is given by the expression

$$1+nH^\mu = x\delta^\mu_x + nH^A\delta^\mu_A \quad \mu = x, 1, ..., n.$$  \hspace{1cm} (27)

The above imply that equation (25) inherits all symmetries which are generated from the KVs/HV of the $n-$metric $h_{AB}$. Hence we do not have Type II symmetries in this reduction.

3.2 Reduction by $X_2$

The zeroth order invariants of $X_2$ are

$$\tau = t, \ y^A, \ w = ue^{\frac{x^2}{2}}.$$
Taking these invariants as new coordinates eqn (23) reduces to

\[ h^{AB} w_{AB} - \Gamma^A w_B - w_\tau - \frac{1}{2r} w = 0 \]  

(28)
or

\[ h \Delta w - w_\tau = \frac{1}{2r} w. \]

This is the nonhomogeneous heat equation with flux \( q(\tau, y^A, w) = \frac{1}{2r} w \). Application of Theorem 1 gives the following result.

**Proposition 4**  
The Lie point symmetries of the heat equation (28) in an \( n \)-dimensional Riemannian space with metric \( h_{AB} \) are constructed from the homothetic algebra of the metric as follows:

a. \( Y^i \) is a HV/KV.

The Lie point symmetry is

\[ X = (2c_2 \psi T + c_1) \partial_\tau + c_2 Y^i \partial_i + \left( -\frac{c_1}{2r} + a_0 \right) w + b(\tau, x) \partial_w \]  

(29)
b. \( Y^i = S_J^i \) is a gradient HV/KV (the index \( J \) counts gradient KVs).

The Lie point symmetry is

\[ X = (\psi T_0 \tau^2) \partial_\tau + T_0 \tau S_J^i \partial_i - \left( \frac{1}{2} T_0 S_J + T_0 \psi \tau \right) w \partial_w \]  

(30)

where \( b(\tau, x) \) is a solution of the heat equation (28).

We infer that for this reduction we have the Type II hidden symmetry \( \partial_\tau - \frac{1}{2r} w \partial_w \). The rest of the point symmetries are inherited.

4 The heat equation in a space which admits a gradient HV

The metric of an \( n \)-dimensional Riemannian space which admits the gradient HV \( r \partial_r \) has the generic form

\[ ds^2 = dr^2 + r^2 h_{AB} dy^A dy^B. \]

(31)

In this space the heat equation (2) is

\[ u_{rr} + \frac{1}{r^2} h^{AB} u_{AB} + \frac{(n - 1)}{r} u_r - \frac{1}{r^2} \Gamma^A u_A - u_\tau = 0 \]  

(32)

where \( \Gamma^A = \Gamma^A_{BC} h^{BC} \) and \( \Gamma^A_{BC} \) are the connection coefficients of the Riemannian metric \( h_{AB} \) \( (A, B, C = 1, 2, ..., n) \). Application of Theorem 2 gives that (32) admits the following extra Lie point symmetries generated by the gradient homothetic vector

\[ \bar{X}_1 = 2t \partial_t + r \partial_r, \quad \bar{X}_2 = t^2 \partial_t + tr \partial_r - \left( \frac{1}{4} r^2 + \frac{n}{2} t \right) w \partial_u \]  

(33)

with nonzero commutators

\[ [X_t, \bar{X}_1] = 2X_t, \quad [\bar{X}_1, \bar{X}_2] = 2X_t \]

\[ [X_t, \bar{X}_2] = \bar{X}_1 - \frac{n}{2} X_u \]

We consider again the reduction of (32) using the zeroth order invariants of these extra Lie point symmetries.

\[ ^1 \text{The proof is given in Appendix A.} \]
4.1 Reduction by $\bar{X}_1$

The zeroth order invariants of $\bar{X}_1$ are $\phi = \frac{r}{\sqrt{t}}$, $w = u$, $y^A$. We choose $w = w(\phi, y^A)$ as the dependent variable.

Replacing in (32) we find the reduced PDE

$$w_{\phi \phi} + \frac{1}{\phi^2} h^{AB} w_{AB} + \frac{(n-1)}{\phi} w_{\phi} + \frac{\phi}{2} w_{\phi} - \frac{1}{\phi^2} \Gamma^A w_A = 0.$$  \hspace{1cm} (34)

Consider a nonvanishing function $N^2(\phi)$ and divide (34) with $N^2(\phi)$ to get:

$$\frac{1}{N^2} w_{\phi \phi} + \frac{1}{\phi^2 N^2} h^{AB} w_{AB} + \frac{(n-1)}{\phi N^2} w_{\phi} + \frac{\phi}{2N^2} w_{\phi} - \frac{1}{\phi^2 N^2} \Gamma^A w_A = 0.$$  \hspace{1cm} (35)

It follows that (for $n > 2$) equation (35) can be written as

$$\bar{g} \Delta w = 0$$  \hspace{1cm} (36)

where $\bar{g} \Delta$ is the Laplace operator if $N^2(\phi) = \exp\left(\frac{\phi^2}{2(n-2)}\right)$ and $\bar{g}_{ij}$ is the conformally related metric

$$ds^2 = \exp\left(\frac{\phi^2}{2(n-2)}\right) \left(d\phi^2 + \phi^2 h_{AB} dy^A dy^B\right).$$  \hspace{1cm} (37)

According to [15] the Lie point symmetries of (36) are the CKVs of the metric (37) whose conformal factor satisfies the condition $\bar{g} \Delta \psi = 0$. Therefore Type II hidden symmetries will be generated from the proper CKVs. The existence and the number of these vectors depends mainly on the $n-$ metric $h_{AB}$.

4.2 Reduction by $\bar{X}_2$

For $\bar{X}_2$ the zeroth order invariants invariants are $\phi = \frac{r}{\sqrt{t}}$, $w = u \frac{t^2}{r^2} \sqrt{\frac{2}{n-2}}$, $y^A$. We choose $w = w(\phi, y^A)$ as the dependent variable and we have the reduced equation

$$\bar{g} \Delta w = 0$$  \hspace{1cm} (38)

where

$$\bar{g} \Delta w = w_{\phi \phi} + \frac{(n-1)}{\phi} w_{\phi} + \frac{1}{\phi^2} h^{AB} w_{AB} - \frac{1}{\phi^2} \Gamma^A w_A.$$  \hspace{1cm} (39)

Equation (38) is the Laplace equation in the space $(\phi, y^A)$ with metric

$$ds^2 = d\phi^2 + \phi^2 h_{AB} dy^A dy^B.$$  \hspace{1cm} (40)

The Lie symmetries of Laplace equation (38) are given in Theorem 3. As in the last case the existence and the number of these vectors depends mainly on the $n-$ metric $h_{AB}$.

We note that both vectors $\bar{X}_1, \bar{X}_2$ are generated form the gradient HV and in both cases the heat equation has been reduced to Laplace equation. This gives the following

**Proposition 5** The reduction of the heat equation (16) in a space with metric (31) $(n > 2)$ by means of the Lie symmetries generated by the gradient HV leads to Laplace equation $\Delta u = 0$, where $\Delta$ is the Laplace operator for the metric (37) if the reduction is done by $\bar{X}_1$ and for the metric (40) if the reduction is done by $\bar{X}_2$.

5 Applications

In this section we consider applications of the general results of sections 3 and 4.
5.1 The heat equation in a $1+n$ decomposable space where $n$ is a space of constant curvature

Consider the $1+n$ decomposable space

$$ds^2 = dx^2 + N^{-2}(y^C) \delta_{AB} y^A y^B$$

(41)

where $N(y^C) = (1 + \frac{K}{4} y^C y_C)$, that is, the $n$ space is a space of constant non vanishing ($K \neq 0$) curvature. The metric (41) does not admit proper HV. However admits $\frac{n(n-1)}{2}$ nongradient KVs and 1 gradient KV as follows [17]

1 gradient KV: $\partial_x$

$n$ nongradient KVs: $K_V = \frac{1}{N} \left[ (2N - 1) \delta^i_j + \frac{K}{2} N x_i x^j \right] \partial_i$

$\frac{n(n-1)}{2}$ nongradient KVs: $X_{IJ} = \delta^i_j \delta^j_i \partial_i$

In a space with metric (41) the heat equation takes the form

$$u_{xx} + N^2(y^C) \delta^{AB} u_{AB} - \frac{N}{2} K y^A u_A - u_t = 0$$

(42)

which is the homogeneous heat equation. Applying Theorem 2 we find that equation (42) admits the extra Lie point symmetries

$$\partial_x, t \partial_x - \frac{1}{2} xu \partial_x, K_V, X_{IJ}.$$  

(43)

The Lie point symmetries which are generated by the gradient KV are [\partial_x, t \partial_x - \frac{1}{2} xu \partial_x].

Reduction of (42) by means of the gradient KV $\partial_x$ results in the special form of equation (25)

$$\frac{1}{N^2(y^C)} \delta^{AB} w_{AB} - \frac{N}{2} K y^A w_A - w_t = 0.$$  

(44)

This is the homogeneous heat equation in an $n$-dimensional space of constant curvature. The Lie symmetries of this equation have been determined in [14] and are inherited symmetries. Hence in this case we do not have Type II hidden symmetries.

Reduction of (42) with the Lie point symmetry $t \partial_x - \frac{1}{2} xu \partial_x$ gives that the reduced equation (28) is

$$N^2(y^C) \delta^{AB} w_{AB} - \frac{N}{2} K y^A w_A - w_t = \frac{1}{27} w, w = u e^{\frac{x^2}{4}}.$$  

(45)

which is the heat equation with flux. By Proposition 4 the Lie point symmetries of (44) are:

$$X = c_1 \partial_t + (K_V + X_{IJ}) + \left[ \left( \frac{c_1}{27} + a_0 \right) w + b (\tau, y^C) \right] \partial_w.$$  

(46)

where $c_1, a_0$ are constants. From section 3.2 we have that Type II hidden symmetry is the one defined by the constant $c_1$.

\footnote{Here the algebra is the one given in section 3 and a separate algebra is the algebra of the KVs of the space of constant curvature. More specifically the KVs $K_V, X_{IJ}$ commute with all other symmetries but not between themselves.}
5.2 FRW space-time with a gradient HV

Consider the spatially flat FRW metric widely used in Cosmology

\[ ds^2 = d\sigma^2 - \sigma^2 (dx^2 + dy^2 + dz^2) \]  

(47)

where \( \tau \) is the conformal time. This metric admits the gradient HV

\[ H = \sigma \partial_\sigma \quad (\psi_H = 1) \]

and six nongradient KV\s

\[ X_{1-3} = \partial_y^A \quad , \quad X_{4-6} = y^B \partial_A - y^A \partial_B. \]

where \( y^A = (x, y, z) \).

In this space the heat equation takes the form

\[ u_{\sigma\sigma} - \frac{1}{\sigma^2} (u_{xx} + u_{yy} + u_{zz}) + \frac{3}{\sigma} u_\sigma - u_t = 0. \]

(48)

Its Lie point symmetries (48) have been determined in \[14\] and have as follows:

\[ H_1 = 2t \partial_t + \sigma \partial_\sigma \quad , \quad H_2 = t^2 \partial_t + t \sigma \partial_\sigma - \left( \frac{1}{4} \sigma^2 + 2t \right) u \partial_u. \]

The Lie symmetries \( H_1, H_2 \) are produced by the gradient HV therefore we use them to reduce (48). We note that this case is a special case of the one we considered in section 4 for \( h_{AB} = \delta_{AB} \).

Reduction by \( H_1 \) gives that (48) becomes:

\[ w_{\phi\phi} - \frac{1}{\phi^2} (w_{xx} + w_{yy} + w_{zz}) + \left( \frac{3}{\phi} + \frac{\phi}{2} \right) w_\phi = 0 \]

(49)

where \( \phi = \sigma \sqrt{\tau} \), \( w = u \). This is a special form of (51).

Dividing with \( N^2 (\phi) = \exp \left( \frac{\phi^2}{4} \right) \) we find that (49) is written as

\[ \bar{g} \Delta w = 0 \]

(50)

where the metric \( \bar{g}_{ij} \) is the conformally related metric of (47):

\[ ds^2 = e^{\frac{\phi^2}{4}} (d\phi^2 - \phi^2 (dx^2 + dy^2 + dz^2)) \]

(51)

From proposition 3 we have that the Lie point symmetries of (50) are generated from elements of the conformal algebra of the space whose conformal factors satisfy the condition \( \bar{g} \Delta \phi = 0 \). The metric (50) is conformally flat therefore its conformal group is the same with that of the flat space \([17, 21]\), however with different subgroups. We find that these vectors (i.e. the Lie point symmetries) are the vectors

\[ X_{1-3} , X_{4-6} , \partial_t , w \partial_u , b_0 (\phi, y^A) \partial_w. \]

(52)

We conclude that there are no Type II symmetries for this reduction.

Using reduction by \( H_2 \) we find that (48) reduces to :

\[ w_{\phi\phi} - \frac{1}{\phi^2} (w_{xx} + w_{yy} + w_{zz}) + \frac{3}{\phi} w_\phi = 0 \]

(53)
where $\phi = \frac{r}{2}$, $w = ut^2 e^{\frac{r}{2}}$. This is a special form of (38) which is the Laplace equation. In this case the results of [15] apply and we infer that the Lie point symmetries of (53) are:

\[
X_1, X_4 = u \partial_w, b_1 (\phi, y^4) \partial_w
\]
\[
X_7 = \phi \partial_{\phi}, X_8 = \phi y^4 \partial_{\phi} + \ln \phi \partial_A - y^4 w \partial_w.
\]

(54)

The vector $X_7$ is the proper HV of the metric and the vectors $X_8 - 10$ the proper CKVs which are not special CKVs, therefore these vectors are Type II hidden symmetries. A further analysis of (53) can be found in [22].

6 The Heat equation in spaces which admit a nongradient HV

In the previous sections we considered the reduction of the homogeneous heat equation in Riemannian spaces which admit a gradient KV or a gradient HV. In the present section we consider the special class of Petrov type space-times which admit a nongradient HV which acts simply transitively.

6.1 Petrov type N space-time

The metric of the Petrov type N space-time is

\[
ds^2 = dx^2 + x^2 dy^2 + 2d\rho dv + \ln x^2 d\rho^2
\]

(55)

and has the homothetic algebra [16]

\[
K^1 = \partial_\rho, K^2 = \partial_v, K^3 = \partial_y
\]
\[
H = x \partial_x + \rho \partial_\rho + (v - 2\rho) \partial_v \quad (\psi_H = 1)
\]

where $K^{1-3}$ are KVs and $H$ is a nongradient HV.

The heat equation (15) in this space-time is

\[
u_{xx} + \frac{1}{x^2} u_{yy} + 2u_{\rho v} - 2 \ln x^2 u_{vv} + \frac{1}{x} u_x - u_t = 0.
\]

(56)

Application of Theorem 2 gives that the extra Lie point symmetries of (56) are

\[
X_1, X_4 = 2t \partial_t + H
\]

with nonzero commutators

\[
[X_t, X_4] = 2X_t
\]
\[
[X_1, X_4] = X_1 - 2X_2, \quad [X_2, X_4] = X_2.
\]

We use $X_4$ to reduce the PDE because this is the Lie symmetry generated by the HV. The zeroth order invariants of $X_4$ are

\[
\alpha = \frac{x}{\sqrt{t}}, \quad \beta = \frac{\rho}{\sqrt{t}}, \quad \gamma = \frac{v + \rho \ln (t)}{\sqrt{t}}, \quad \delta = y, \quad w = u.
\]

(57)

Choosing $\alpha, \beta, \gamma, \delta$ as the independent variables and $w = w(\alpha, \beta, \gamma, \delta)$ as the dependent variable we find that the reduced PDE is

\[
N \Delta w + \left( \frac{1}{2} \alpha w_\alpha + \frac{1}{2} \beta w_\beta + \left( \frac{1}{2} \gamma - \beta \right) w_\gamma \right) = 0.
\]

(58)
where $N\Delta$ is the Laplace operator for the metric (55).

Equation (58) is of the form (4) with

\[ A_{ij} = g_{ij}(x^k), \quad B_i = \Gamma_i + \frac{1}{2} \alpha \delta_i^k + \frac{1}{2} \beta \delta_i^k (\frac{1}{2} \gamma - \beta) \delta_i^k, \quad f(x^k, u) = 0, \]

where $g_{ij}$ is the metric (55). Replacing in equations (5)-(9) we obtain the Lie symmetry conditions for (58). Because $A_{ij,u} = 0$ it follows from equation (7) that the Lie point symmetries are generated from the CKVs of the metric (55). However taking into consideration the rest of the symmetry conditions we find that the only Lie point symmetry which remains is the one of the KV $X_3$. We conclude that in this reduction we do not have Type II hidden symmetries.

6.2 Petrov type D

The metric of the Petrov type D space-time is

\[ ds^2 = -dx^2 + x^{-\frac{4}{3}}dy^2 - x^{\frac{1}{3}}(d\rho^2 + dz^2) \]  

(59)

with Homothetic algebra

\[ K^1 = \partial_\rho, \quad K^2 = \partial_x, \quad K^3 = \partial_y, \quad K^4 = z \partial_\rho - \rho \partial_z \]

\[ H = x \partial_x + \frac{4}{3} y \partial_y + \frac{2}{3} \partial_x + \frac{2}{3} \partial_\rho (\psi_H = 1) \]

where $K^1-4$ are KVs and $H$ is a nongradient HV.

In this space-time the heat equation (15) takes the form:

\[-u_{xx} + x^{\frac{4}{3}}u_{yy} - x^{-\frac{4}{3}}(u_{\rho \rho} + u_{zz}) - \frac{1}{x}u_x - u_t = 0. \]  

(60)

From Theorem 2 we have that the extra Lie point symmetries are the vectors

\[ X_{1-4} = K_{1-4}, \quad X_5 = 2t \partial_t + H. \]

with nonzero commutators:

\[ [X_t, X_5] = 2X_t \]
\[ [X_1, X_5] = \frac{1}{3} X_1, \quad [X_4, X_1] = -X_2 \]
\[ [X_2, X_4] = X_1, \quad [X_2, X_5] = \frac{1}{3} X_2 \]
\[ [X_3, X_5] = \frac{4}{3} X_3. \]

We use $X_5$ to reduce the PDE because this is the Lie symmetry generated by the HV. The zeroth order invariants of $X_5$ are

\[ \alpha = \frac{x}{t^2}, \quad \beta = \frac{y}{t^2}, \quad \gamma = \frac{\rho}{t^2}, \quad \delta = \frac{z}{t^2}, \quad w = u. \]

We choose $\alpha, \beta, \gamma, \delta$ as the independent variables and $w = w(\alpha, \beta, \gamma, \delta)$ as the dependent variable and we find that the reduced PDE is

\[ D\Delta w + \left( \frac{1}{2} a w_\alpha + \frac{2}{3} \beta w_\beta + \frac{1}{6} \gamma w_\gamma + \frac{1}{6} \delta w_\delta \right) = 0 \]  

(61)

where $D\Delta$ is the Laplace operator with metric (59).

Again working with the Lie symmetry conditions (55)-(59) we find that equation (61) admits as Lie point symmetry only the vector $X_4$ which is an inherited symmetry. Hence we do not have Type II hidden symmetries. Obviously the Lie point symmetries $X_{1-4}$ are Type I hidden symmetries for equation (61) for the reduction by $X_5$.  

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6.3 Petrov type II

The metric of the Petrov type II space-time is
\[ ds^2 = \rho^{-\frac{1}{2}} (d\rho^2 + dz^2) - 2\rho dx dy + \rho \ln \rho dy^2 \]  
(62)

with Homothetic algebra
\[
\begin{align*}
K^1 &= \partial_x, \quad K^2 = \partial_y, \quad K^3 = \partial_z \\
H &= \frac{1}{3} (x + 2y) \partial_z + \frac{1}{3} y \partial_y + \frac{4}{3} z \partial_z + \frac{4}{3} \rho \partial_\rho \quad (\psi_H = 1)
\end{align*}
\]

where \(K^1-4\) are KVs and \(H\) is a nongradient HV.

In this space-time equation (15) takes the form:
\[
\rho^{-\frac{1}{2}} (u_{\rho\rho} + u_{zz}) - \frac{1}{\rho} \varepsilon \ln \rho u_{xx} - \frac{2}{\rho} u_{xy} + \rho^{-\frac{1}{2}} u_{\rho} - u_t = 0.
\]  
(63)

From Theorem 2 we have that the extra Lie point symmetries are the vectors
\[
X_{1-3} = K_{1-3}, \quad X_4 = 2t \partial_t + H
\]

with nonzero commutators:
\[
\begin{align*}
[X_\xi, X_\zeta] &= 2X_\xi \\
[X_1, X_4] &= \frac{1}{3} X_1, \quad [X_3, X_4] = \frac{4}{3} X_3 \\
[X_2, X_4] &= \frac{2}{3} X_1 + \frac{1}{3} X_2.
\end{align*}
\]

We use \(X_4\) to reduce the PDE because this is the Lie symmetry generated by the HV. The zeroth order invariants of \(X_4\) are
\[
\alpha = \frac{\rho}{t^4}, \quad \beta = \frac{z}{t^2}, \quad \gamma = \frac{x - \frac{1}{2} y \ln (t)}{t^3}, \quad \delta = \frac{y}{t^2}, \quad w = u
\]

We choose \(\alpha, \beta, \gamma, \delta\) as the independent variables and \(w = w(\alpha, \beta, \gamma, \delta)\) as the dependent variable and we find that the reduced PDE is
\[
I_1 \Delta w + \frac{2}{3} aw_\alpha + \frac{2}{3} \beta w_\beta + \left( \frac{1}{3} \delta^2 - \frac{1}{6} \gamma \right) w_\gamma + \frac{1}{6} \delta w_\delta = 0
\]  
(64)

where \(I_1 \Delta\) is the Laplace operator for metric (62).

From the Lie symmetry conditions follows that (64) does not admit any Lie point symmetries. Hence we do not have Type II hidden symmetries in this case.

6.4 Petrov type III

The metric of the Petrov type III space-time is
\[ ds^2 = 2d\rho dv + \frac{3}{2} x d\rho^2 + \frac{v^2}{x^3} (dx^2 + dy^2) \]  
(65)

with Homothetic algebra
\[
\begin{align*}
K^1 &= \partial_v, \quad K^2 = \partial_y, \quad K^3 = v \partial_v - \rho \partial_\rho + 2x \partial_x + 2y \partial_y \\
H &= v \partial_v + \rho \partial_\rho \quad (\psi_H = 1)
\end{align*}
\]
where $K^{1-4}$ are KVs and $H$ is a nongradient HV.

In this space-time equation (15) takes the form:

$$-rac{3}{2}xu_{vv} + 2u_{v\rho} + \frac{x^3}{v^2}(u_{xx} + u_{yy}) - 3\frac{x}{v}u_v + \frac{2}{v}u_\rho - u_t = 0.$$  \hspace{1cm} (66)

From Theorem 2 we have that the extra Lie point symmetries are the vectors

$$X_{1-3} = K_{1-3}, \quad X_4 = 2t\partial_t + H$$

with nonzero commutators:

$$[X_1, X_4] = 2X_t, \quad [X_2, X_3] = 2X_2$$

$$[X_3, X_1] = X_1, \quad [X_1, X_4] = X_1$$

We use $X_4$ for reduction because this is the Lie symmetry generated by the HV.

The zeroth order invariants of $X_4$ are

$$\alpha = \frac{v}{\sqrt{t}}, \quad \beta = \frac{\rho}{\sqrt{t}}, \quad \gamma = x, \quad \delta = y, \quad w = u$$

We choose $w = w(\alpha, \beta, \gamma, \delta)$ as the dependent variable and we find that the reduced PDE is

$$III \Delta w + \frac{\alpha}{2} w_\alpha + \frac{\beta}{2} w_\beta = 0$$ \hspace{1cm} (67)

where $III \Delta$ is the Laplace operator for metric (65).

From the Lie symmetry conditions we find that (67) admits only the Lie point symmetries $X_2, X_3$. Therefore we do not have type II hidden symmetries and the symmetries $X_1, X_4$ are Type I hidden symmetries.

7 Conclusion

We have discussed the reduction of the homogeneous heat equation in certain general classes of Riemannian spaces which admit some type of basic symmetry and we have determined in each case the Type II hidden symmetries. These spaces are the spaces which admit a gradient KV or a gradient HV and finally space-times which admit a HV which acts simply and transitively.

In general the reduction of the homogeneous heat equation in these spaces leads to reduced equations in which the second order partial derivatives are within a Laplace operator, that is, the reduced equations are of the form $\Delta u = F(t, x^i, u_i, u_t)$ where $F$ is linear on $u_i, u_t$. This implies that the Lie point symmetries of the reduced equation will be generated from the elements of the conformal algebra of the metric which defines the Laplace operator $\Delta$. It is from these symmetries that the Type II hidden symmetries will emerge. Summarizing we have found the following general geometric results:

- If we reduce the heat equation (15) via the symmetries which are generated by a gradient KV $(S^i)$ the reduced equation is a heat equation in the nondecomposable space. In this case we have the Type II hidden symmetry $\partial_t - \frac{1}{2t}w\partial_w$ provided we reduce the heat equation with the symmetry $tS^i - \frac{1}{2}Su_\partial_u$.

- If we reduce the heat equation (15) via the symmetries which are generated by a gradient HV the reduced equation is Laplace equation for an appropriate metric. In this case the Type II hidden symmetries are generated from the proper CKVs.
• In Petrov spacetimes the reduction of the heat equation (15) via the symmetry generated from the non-gradient HV gives PDEs which inherit the Lie symmetries, hence no Type II hidden symmetries are admitted.

The results we have obtained can be used in many important space-times and help facilitate the solution of the heat equation in these space-times. Finally we mention that the results we have obtained have been checked with the libraries SADE [23] and PDEtools [24] of Maple.

A Appendix

Proof of Corollary 4. Using Theorem 1 and replacing \( q \) we have

For case a)

\[
-a_\tau w + H(b) - \frac{1}{2\tau} (aw + b) + \frac{a}{2\tau} w - \left( \frac{\psi c_2 w + 1}{2\tau} wc_1 \right)_\tau = 0. \tag{68}
\]

\[
-a_\tau w + H(b) - \frac{1}{2\tau} b + \frac{1}{2\tau^2} wc_1 = 0 \tag{69}
\]

\[
\left[ -a_\tau + \frac{c_1}{2\tau^2} \right] w + \left[ H(b) - \frac{1}{2\tau} b \right] = 0 \tag{70}
\]

that is

\[
a = -\frac{c_1}{2\tau} + a_0 , \quad H(b) - \frac{1}{2\tau} b = 0 \tag{71}
\]

For case b)

\[
0 = \left( -\frac{1}{2} T_\tau \psi + \frac{1}{2} T_{\tau\tau} S - F_\tau \right) w - \left( \frac{2\psi q}{\tau} \int T d\tau \right)_\tau - T q_i S^i. \tag{72}
\]

then

\[
0 = \left( -\frac{1}{2} T_\tau \psi + \frac{1}{2} T_{\tau\tau} S - F_\tau \right) w + \frac{\psi}{\tau} \int T d\tau w - \frac{\psi}{\tau} Tw \tag{72}
\]

from here we have

\[
T_{\tau\tau} = 0 \rightarrow T = T_0 \tau + T_1 \tag{73}
\]

and

\[
F = -T_0 \psi. \tag{74}
\]

\[\text{http://www.maplesoft.com/}\]
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