★-Cohomology, Connes-Chern Characters, and Anomalies in General Translation-Invariant Noncommutative Yang-Mills

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Abstract: Topological structure of translation-invariant noncommutative Yang-Mills theories are studied by means of a cohomology theory, so called ★-cohomology, which plays an intermediate role between de Rham and cyclic (co)homology theory for noncommutative algebras and gives rise to a cohomological formulation comparable to Seiberg-Witten map.

Keywords: Translation-Invariant Star Product, Noncommutative Yang-Mills, Spectral Triple, Chern Character, Connes-Chern Character, Family Index Theory, Topological Anomaly, BRST.

I. INTRODUCTION

Noncommutative geometry is one of the most prominent topics in theoretical physics. Through with last three decades it was extensively believed that the fundamental forces of the nature could be interpreted with more success via the machinery of noncommutative geometry and its different viewpoints [1–4]. Moyal noncommutative fields, inspired by fascinating formulations of strings, were proposed as the emergence of this idea in order to ramify the singular behaviors of quantum field theories, especially the quantum gravity.1 Appearing UV/IR mixing as a pathological feature of the Moyal quantum fields led to a concrete generalization of Moyal product as general translation-invariant noncommutative star products.2

However, the topology and the geometry of noncommutative field theories with general translation-invariant star products have not been studied thoroughly yet. Actually, on the one hand it is a problem in noncommutative geometry, and on the other hand it is a physical behavior correlated to the topology and the geometry of the underlying spacetime and corresponding fiber bundles. But, in contrast to commutative algebras the correlation of noncommutative geometric machineries, such as cyclic (co)homology, and those of the ordinary differential structures of commutative geometry, such as the underlying spacetime, is not clear for noncommutative algebras. The former demands a point-free formulation, while the later one is based on ordinary topology and differentiation.

In this article we try to fill this gap by introducing a new cohomology theory, so called ★-cohomology, which can play an intermediate role between noncommutative and commutative differential geometry for noncommutative algebras. In the following we give a brief introduction to basic algebraic structures

1 See [5] for a proper overview to this topic.

2 See [6] and the references therein for a complete list of references and a short history on this topic.
of $\ast$-cohomology and in next sections we will develop it to study and formulate the topology of general translation-invariant noncommutative Yang-Mills theories.

Suppose that $X = Y \times Z$ is a $2n$-dimensional closed spin manifold with $\dim Y = D$ and $Z = \mathbb{T}^{2m}$. Assume that there is a dense contractible open set, say $U \subset X$, which defines a chart with coordinate functions $x^\mu$, $\mu = 0, \cdots, 2n - 1$. Let $x^\mu$s split $X$, that is $x^\mu$ belongs to $Y$ for $\mu = 0, \cdots, D - 1$, and $x^i$ is a torus canonical coordinate for $D \leq i \leq 2n - 1$. Here $x^0 = t$ is the time parameter which varies from $-\infty$ to $+\infty$ on $U$. We also put a metric on $X$ unit diagonal for $x^\mu$s. Now consider a general translation-invariant noncommutative star product on $Z$, say $\ast$, defined for 2-cocycle $\alpha$ as \cite{6}:

$$(f \ast g)(x) = \sum_{p,q \in \mathbb{Z}^{2m}} \tilde{f}(p)\tilde{g}(q) \ e^{\alpha(p+q,p)} \ e^{i(p+q).x/R}, \quad f, g \in C^\infty (Z),$$

for Fourier transformation

$$\tilde{f}(p) = \frac{1}{(2\pi R)^{2m}} \int_Z f(x) \ e^{-p(x)} , \quad f(x) = \sum_{p \in \mathbb{Z}^{2m}} \tilde{f}(p) \ e_p(x) ,$$

with the Fourier basis $e_p(x) = e^{ip.x/R}$; $p \in \mathbb{Z}^{2m}$. The integration in (1.2) is taken for the Riemannian volume form of $Z$. Due to Hodge decomposition theorem in $\alpha^*$-cohomology \cite{6} one readily finds $\alpha = \alpha_M + \beta$, where $\alpha_M$ is a Moyal 2-cocycle providing a Moyal star product $\ast_M$, and $\beta$ is a 1-cocycle $(\beta(0) = 0$ and $\beta(p) = \beta(-p))$ with $\partial \beta(p,q) = \beta(p-q) - \beta(p) + \beta(q)$. In fact, $f' \ast_M g' = (f \ast g)'$ where

$$f'(x) = \sum_{p \in \mathbb{Z}^{2m}} \tilde{f}(p) \ e^{\beta(p)} \ e_p(x) , \quad f \in C^\infty (Z).$$

Assume $H$ is a separable Hilbert space with coordinate function with orthonormal basis $\{|p\rangle\}_{p \in \mathbb{Z}^{2m}}$. Then, it can be checked that $\pi : C^\infty (Z) \rightarrow \mathcal{L}(H)$ with

$$\pi(f) := \hat{f} := \sum_{p \in \mathbb{Z}^{2m}} \tilde{f}(p) \ \hat{e}_p , \quad (\hat{e}_p)_{r,s} = e^{\alpha_M(r,s)} \delta_{p,r-s} ,$$

gives rise to a representation of $C^\infty (Z)$ on $H$, i.e. $\hat{f} \ast \hat{g} = \hat{f} \ast \hat{g}$, $f, g \in C^\infty (Z)$. This representation can be similarly extended to smooth functions on $X$. In this case for any $f \in C^\infty (X)$ the mapped element $\hat{f}$ is an operator valued function on $Y$. Let $\mathfrak{C}_0$ be the algebra generated with $\hat{f}$, $f \in C^\infty (X)$. Then $\mathfrak{C}_0$ is isomorphic to $C^\infty_\mathfrak{g}(X)$ via $\pi$. This leads to definition of noncommutative polynomial. If $P(x_1, \cdots, x_k)$ is a polynomial of probably noncommutative variables $x_1, \cdots, x_k$, the noncommutative version of $P$, denoted by $P_\ast$ is defined as:

$$P_\ast(\hat{f}_1, \cdots, \hat{f}_k) = \pi^{-1}(P(\hat{f}_1, \cdots, \hat{f}_k)) .$$

In fact, $\mathfrak{C}_0$ is a unital $\ast$-algebra for $1 = \hat{1}$ and $\hat{f}^* = \hat{f}^\dagger = \overline{\hat{f}}$. The domain of $\pi$ can be simply extended to vector and matrix valued smooth functions on $X$ with $v = (v_i) \rightarrow \hat{v} = (\hat{v}_i)$ and $g = (g_{ij}) \rightarrow \hat{g} = (\hat{g}_{ij})$. Moreover, if $\{e^a\}$ is a basis for Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$-valued smooth function $f = f^a e^a$ is mapped to $\hat{f} = \hat{f}^a e^a$ via $\pi$.

II. STAR PRODUCT, $\ast$-COHOMOLOGY AND CHERN CHARACTERS

The representation map $\pi$ can be simply defined for differential forms on $X$. Let us denote the image of $k$-forms by $\mathfrak{C}_k$ and set $\mathfrak{C} = \oplus_{k=0}^{2n} \mathfrak{C}_k$. Actually, $\mathfrak{C}_k$ is the space of operator valued $k$-forms on
Y with extra dimensions on Z. Moreover, \( \mathfrak{C} \) is obviously a graded algebra with operator and wedge product. We define an exterior derivative operator on \( \mathfrak{C} \) as;
\[
d\hat{f} = \overline{\partial}_\mu f \, dx^\mu, 
\]
(II.1)
and extend it to \( \mathfrak{C} \). Then, \((\mathfrak{C}, d)\) is a differential graded algebra. The corresponding cohomology of \((\mathfrak{C}, d)\) is referred to as *-cohomology and we denote its groups with \( H_k^* (X, \mathfrak{C}) \), \( 0 \leq k \leq 2n \). Since \( \pi \) is an isomorphism and \( C^\infty (X) = C^\infty (X) \) as sets we readily find that \( \pi : \Omega_k (X) \rightarrow \mathfrak{C}_k \) is a bijective map for any \( k \geq 0 \), where here \( \Omega_k (X) \) is the space of differential \( k \)-forms on \( X \). Thus any element of \( \mathfrak{C} \) is simply shown with an overal symbol of \( \hat{\cdot} \). Moreover, we readily see that \( \pi \circ d_X = d \circ \pi \) for \( d_X \) the exterior differential operator on \( X \). Therefore, the following diagram commutes
\[
\begin{array}{cccccccc}
0 & \rightarrow & \Omega_0 (X) & \rightarrow & \Omega_1 (X) & \rightarrow & \cdots & \rightarrow & \Omega_k (X) & \rightarrow & \Omega_{k+1} (X) & \rightarrow & \cdots \\
\pi & \downarrow & \pi & \downarrow & \cdots & \pi & \downarrow & \pi & \downarrow & \cdots & ,
\end{array}
\]
(II.2)
wherein the upper arrows stand for \( d_X \) and the lower ones indicate \( d \). This provides isomorphisms \( \pi_* : H_k^\text{dr} (X, \mathfrak{C}) \rightarrow H_k^* (X, \mathfrak{C}) \). We are mostly interested in integral group \( H_k^* (X, \mathbb{Z}) \). To define it an integration structure is put on \( \mathfrak{C} \) as
\[
\int \hat{\omega} = \left\{ \begin{array}{ll}
\int_X f, & \hat{\omega} = \hat{f} \, d^{2n}x \in \mathfrak{C}_{2n} \\
0, & \text{otherwise}
\end{array} \right.
\]
(II.3)
where the integral is taken for Riemann volume form of \( X \). It is easily seen that (II.3) defines a graded closed trace over \((\mathfrak{C}, d)\) for \( \int d\hat{\omega} = 0 \) and \( \int \hat{\omega_1} \hat{\omega_2} = (-1)^{||\omega_1|| \cdot ||\omega_2||} \int \hat{\omega_2} \hat{\omega_1} \). The space of classes \([\hat{\omega}] \in H_{2n}^\text{dr} (X, \mathfrak{C}) \) with \( \int \hat{\omega} \in \mathbb{Z} \) is referred to as \( 2n \)th integral *-cohomology group and is denoted with \( H_{2n}^* (X, \mathbb{Z}) \). It can be seen that \( \pi_* : H_{2n}^\text{dr} (X, \mathbb{Z}) \rightarrow H_{2n}^* (X, \mathbb{Z}) \) is also an isomorphism. Therefore, due to Chern-Weil theory \( H_{2n}^* (X, \mathbb{Z}) \) is generated with the \( \pi \) image of \( n \)th Chern character \( \hat{\text{ch}}_n (E) \) for vector bundles \( E \rightarrow X \).

The noncommutativity via \( \pi \) lets us to define three types of Chern characters. The first type Chern character is simply \( \hat{\text{ch}}_n (E) = \pi (\text{ch}_n (E)) \), which as mentioned above represents integral class in \( H_{2n}^* (X, \mathbb{Z}) \). Let us denote it with \( \hat{\text{ch}}_n^{(1)} (E) \) to stress on its type. Second type Chern character is\(^3\)
\[
\hat{\text{ch}}_n^{(2)} (E) = \frac{1}{(4\pi)^n n!} \text{Tr} \left\{ \hat{F}^n \right\}
\]
(II.4)
wherein \( \nabla^2 = -\frac{i}{2} F \) is the curvature of connection \( \nabla \) on \( \mathbb{C}^k \rightarrow E \rightarrow X \) and \( \text{Tr} \) is over \( \mathbb{C}^k \). Hence;
\[
\hat{\text{ch}}_n^{(2)} (E) = \frac{1}{(4\pi)^n n!} \epsilon_{\mu_1 \nu_1 \cdots \mu_n \nu_n} \pi (\text{Tr} \{ F_{\mu_1 \nu_1} \cdots \ast F_{\mu_n \nu_n} \} ) \, d^{2n}x.
\]
It is obvious that if \( n = 1 \) then \( \hat{\text{ch}}_n^{(2)} (E) \) and \( \hat{\text{ch}}_n^{(1)} (E) \) coincide and hence the cohomology class of \( \hat{\text{ch}}_n^{(2)} (E) \) is independent of connection \( \nabla \). But, however, it is not the case in general;\(^4\)

\(^3\) From now on the notation \( \text{Tr} \) is used for trace on vector bundle dimensions or gauge group colors.

\(^4\) Similar results for Moyal product are worked out in [10].
**Theorem 1:** Suppose $\star$ is a general translation-invariant noncommutative star product on $X$ with $n \geq 2$. Then, the cohomology class of $\hat{\text{ch}}_{n}^{(2)}(E)$ is independent of connection $\nabla$ if and only if $\star$ is a Moyal star product. Hence after;

a) $\hat{\text{ch}}_{n}^{(2)}(E)$ defines an integral cohomology class in $H^{2n}_{\star}(X, \mathbb{C})$.

b) $\hat{\text{ch}}_{n}^{(2)}(E)$ and $\hat{\text{ch}}_{n}^{(1)}(E)$ are cohomologous. That is;

$$\int \hat{\text{ch}}_{n}^{(2)}(E) = \int \hat{\text{ch}}_{n}^{(1)}(E) = \int_X \text{ch}_{n}(E) \in \mathbb{Z} \ .$$ (II.5)

**Hint to the Proof:** Actually, according to Chern-Weil theory it is enough to prove equality (II.5). The integral on the far left consists of the following integration according to (I.3);

$$\int_X \epsilon^{\mu_1\nu_1 \cdots \mu_n\nu_n} \text{Tr} \left\{ F_{\mu_1\nu_1} \star_M \cdots \star_M F_{\mu_n\nu_n} \right\} ,$$

where $\star_M$ is the Moyal product cohomologous to $\star$ via Hodge decomposition \[8\]. It is also equal to

$$\int_X \epsilon^{\mu_1\nu_1 \cdots \mu_n\nu_n} \text{Tr} \left\{ F_{\mu_1\nu_1}' \cdots F_{\mu_n\nu_n}' \right\} ,$$

since Moyal product can be replaced by the ordinary product for integration of symmetric polynomials \[10\]. This integral is independent of connection if and only if there exists some $C \in \mathbb{C}$ so that;

$$\int_X \epsilon^{\mu_1\nu_1 \cdots \mu_n\nu_n} \text{Tr} \left\{ F_{\mu_1\nu_1}' \cdots F_{\mu_n\nu_n}' \right\} = C \int_X \epsilon^{\mu_1\nu_1 \cdots \mu_n\nu_n} \text{Tr} \left\{ F_{\mu_1\nu_1} \cdots F_{\mu_n\nu_n} \right\} .$$

But it is easily seen that the above equation holds if and only if 1-cocycle $\beta$ is linear and $C = 1$. That is $\star = \star_M$. The rest of the proof is due to Chern-Weil theory. This finishes the theorem. Q.E.D

To define the third type Chern character we represent vector bundle $\mathbb{C}^{k} \to E \to X$ by $\pi$. Set

$$\hat{E} = \{ \hat{V}; \ V \in C^\infty(E) \} .$$ (II.6)

Then, the connection $\nabla = d_{X} + A$, for $A \in \Omega_{1}(X) \otimes M_{k}(\mathbb{C})$, is mapped to $\hat{\nabla} = d + \hat{A}$ on $\hat{E}$. Then, its curvature is an element of $\mathfrak{e}_{2} \otimes M_{k}(\mathbb{C})$ and is equal to $\hat{\nabla}^{2} = d\hat{A} + \hat{A}^{2} = -\frac{1}{2} \hat{F}_{\star}$. The third type Chern character is then defined with $\hat{\nabla}^{2}$ as;

$$\hat{\text{ch}}_{n}^{(3)}(E) = \frac{1}{(4\pi)^n n!} \text{Tr} \left\{ \hat{F}_{\star}^n \right\} = \frac{1}{(4\pi)^n n!} \epsilon^{\mu_1\nu_1 \cdots \mu_n\nu_n} \pi (\text{Tr} \{ F_{\mu_1\nu_1} \star \cdots \star F_{\mu_n\nu_n} \}) \ d^{2n}x .$$ (II.7)

We show $\hat{\text{ch}}_{n}^{(3)}(E)$ with $\hat{\text{ch}}_{n}(E)$ for simplicity. The next theorem is significant to our formalism.

**Theorem 2:** For any general translation-invariant noncommutative star product $\star$ on $X$ we have;

a) The cohomology class of $\hat{\text{ch}}_{n}(E)$ in $H^{2n}_{\star}(X, \mathbb{C})$ is independent of connection $\nabla$ ($\hat{\nabla}$).

b) $\hat{\text{ch}}_{n}(E)$ represents an integral cohomology class in $H^{2n}_{\star}(X, \mathbb{Z})$.

c) $\hat{\text{ch}}_{n}(E)$ and $\hat{\text{ch}}_{n}^{(1)}(E)$ are cohomologous. That is;

$$\int \hat{\text{ch}}_{n}(E) = \int \hat{\text{ch}}_{n}^{(1)}(E) = \int_X \text{ch}_{n}(E) \in \mathbb{Z} .$$ (II.8)
Hint to the Proof: For connection $\nabla = d_X + A$ we obtain;

$$\int \check{ch}_n(E, A; \ast) = \int \check{ch}_n(E, A'; \ast_M) = \int_X \check{ch}_n(E, A') = \int_X \check{ch}_n(E),$$

where the first equality is due to (I.3), the second one is the replacement of $\ast_M$ with the ordinary product for integration of symmetric polynomials, and the last equation is an immediate consequence of Chern-Weil theory. This proves (I.8) and thus the theorem follows. Q.E.D

III. INDEX THEOREM IN $\ast$-COHOMOLOGY AND ABELIAN ANOMALY

Assume vector bundle $\mathbb{C}^N \rightarrow E \rightarrow X$ with structure group $U(N)$ for some $N$, in fundamental representation. The generators of $u(N)$, say $t^a$'s, are supposed to be Hermitian with totally anti-symmetric structure group $if^{abc}$, anti-commutator $\{t^a, t^b\} = -\epsilon^{abc}t^c$ and normalization condition $\text{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. We show the space of $\mathfrak{g}$-valued differential forms on $X$ by $\tilde{\Omega}(X) = \oplus_{k=0}^{2n} \tilde{\Omega}_k(X)$, where $\mathfrak{g} = u(N)$. Similarly the $\mathfrak{c}_0 \otimes \mathfrak{g}$-valued differential forms on $Y$ with extra dimensions on $Z$ are denoted as $\tilde{\mathfrak{c}}$. We have the natural grading for $\tilde{\mathfrak{c}}$ as $\tilde{\mathfrak{c}} = \oplus_{k=0}^{2n} \tilde{\mathfrak{c}}_k$ which for $d$ is defined accordingly. Thus, $(\tilde{\mathfrak{c}}, d)$ is a differential graded algebra on which $\text{Tr} \otimes \int$ defines an integration structure or a closed graded trace. It is easy to see that $\tilde{\mathfrak{c}}_0$ is a unital $\ast$-algebra with $1 = \hat{1}I$ and $(\hat{f}^a t^a)^\ast = \hat{f}^{\dagger a} t^{\dagger a} = \hat{f}^{\ast a} t^{\ast a}$. We refer to it with $\mathcal{A}$.

The involution $\ast$ is naturally extended to $\tilde{\mathfrak{c}}_k$ for any $k \geq 0$. Conventionally, $\sigma \in \tilde{\mathfrak{c}}$ is said to be Hermitian if $\sigma^\ast = \sigma$, and it is anti-Hermitian when $\sigma^\ast = -\sigma$. The set of anti-Hermitian elements of $\mathcal{A}$, denoted by $\tilde{\mathfrak{g}}$, is in fact a Lie algebra. Each element of $\tilde{\mathfrak{g}}$ is also known as infinitesimal gauge transformation. The Lie group generated by exponential of elements of $\tilde{\mathfrak{g}}$, indicated with symbol $\tilde{G}$, is the gauge transformation group. Accordingly, the space of anti-Hermitian elements in $\tilde{\mathfrak{c}}_1$, shown with $\gamma$, is called the connection space. Any element of $\gamma$, say $\tilde{A} = -i\hat{A}^a \mu a dx^\mu$, for real functions $A^a_\mu \in C^\infty(X)$, is a connection form on mapped vector bundle $\hat{E}$ for connection $\hat{\nabla} = d + \hat{A}$ due to (II.6).

The gauge transformation group $\tilde{G}$ (resp. $\tilde{\mathfrak{g}}$) acts on $\gamma$ form right:

$$A \triangleright g = g^{-1}dg + g^{-1}Ag, \quad A \in \Gamma, \quad g \in \tilde{G} \quad (\text{resp. } A \triangleright \alpha = d\alpha + [A, \alpha], \quad \alpha \in \tilde{\mathfrak{g}}). \quad \text{(III.1)}$$

The Yang-Mills theory of vector bundle $SE := S(X) \otimes E \rightarrow X$ is mapped to that of $\hat{SE}$ which by definition is equipped with connection $\hat{\nabla} = d - i\hat{A}^a \mu a dx^\mu$ and curvature $\hat{\nabla}^2 = -\frac{i}{2} \hat{F}^a_{\mu \nu} t^a dx^\mu dx^\nu$. The Lagrangian and the action of the $U(N)$-Yang-Mills theory is then given by:

$$L_{Y-M}(\hat{SE}, \hat{\nabla}) = \text{Tr} \left\{ \hat{\nabla}^2 \ast \hat{\nabla}^2 \right\} \in \hat{\mathcal{C}}_{2n}, \quad S_{Y-M} = \int L_{Y-M}(\hat{SE}, \hat{\nabla}), \quad \text{(III.2)}$$

for $\ast$ the Hodge star. Vector bundle $\hat{SE}$, the space of spinors, is subject to Dirac operator $\mathcal{D}_0$ as

$$\mathcal{D}_0 \hat{\psi} = i\gamma^\mu \partial_\mu \hat{\psi}, \quad \text{(III.3)}$$

\footnote{For $U(1)$ gauge theory one needs an overal factor of $\frac{i}{2}$. This is mandatory to compensate the normalization condition $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$ for $N > 1$.}
on $U$ for Dirac matrices $\gamma^\mu$. Dirac operator $D_0$ is usually perturbed to $D_A$, for $A = -i\hat{A}_\mu^a t^a \, dx^\mu \in \Gamma$; 

$$D_A \hat{\psi} = i\gamma^\mu \partial_\mu \hat{\psi} + \gamma^\mu t^a \hat{A}_\mu^a \hat{\psi} .$$

(III.4)

The relevant Lagrangian and action are respectively 

$$L_{D_0}(SE) = \bar{\psi} D_0 \hat{\psi} \, d^{2n}x = \bar{\psi}^t \gamma^0 D_0 \hat{\psi} \, d^{2n}x \in \mathfrak{c}_{2n} , \quad S_{D_0} = \int L_{D_0}(SE) ,$$

(III.5)

$$L_{\text{int}}(SE, \hat{\nabla}) = \bar{\psi} \gamma^\mu \hat{A}_\mu^a \hat{\psi} \, d^{2n}x \in \mathfrak{c}_{2n} , \quad S_{\text{int}} = \int L_{\text{int}}(SE, \hat{\nabla}) .$$

(III.6)

The total action then defines the well-known noncommutative $U(N)$-Yang-Mills theory; 

$$S_{\text{total}} = S_{Y-M} + S_{D_0} + S_{\text{int}} = -\frac{1}{4} \int_X F_{\mu\nu} F_{\mu\nu}^* + i \int_X \bar{\psi} \gamma^\mu \partial_\mu \psi + \int_X j_a^a A_\mu^a ,$$

(III.7)

for $j_a^a = \psi_{\beta,j} \star \psi_{\alpha,i} \gamma_{ij}^a \gamma_{ai}$. Replacing $t^a$ and $\gamma^\mu$ respectively with $I$ and $\gamma^\mu P_{\pm}$, for $P_{\pm} = \frac{1 \pm \gamma^5}{2}$ and 

$$\gamma = \gamma_{2n+1} := i^{-n-1} \gamma^0 \cdots \gamma^{2n-1} ,$$

(III.8)

leads to chiral singlet currents $j_{s \pm}^\mu$, which obey the classical equation 

$$\partial_\mu j_{s \pm}^\mu + i [A_\mu^a, j_{s \pm}^a]_s = 0 .$$

(III.9)

Let this equation be anomalously broken at quantum levels with appearing $A_{s \pm}$ on the right hand side. Then, since any translation-invariant noncommutative star product $\star$ is cyclic under integration the corresponding charges $Q_{s \pm}(t)$ receive variations from $t = -\infty$ to $t = +\infty$ as; 

$$\Delta Q_{s \pm} = Q_{s \pm}(+\infty) - Q_{s \pm}(-\infty) = \int_{-\infty}^{+\infty} \frac{d}{dt} Q_{s \pm}(t) = \int_X A_{s \pm} = \int \hat{A}_{s \pm} .$$

(III.10)

Now let us make an ansatz here. By this ansatz, so called physical consistency, we assume that despite to appearance of consistent anomaly $A_{s \pm}$ the charge variation $\Delta Q_{s \pm}$ respects the theory so that it is equal to an integer times the unit charge of it. Actually, the results achieved through with various methods of anomaly derivation in Moyal noncommutative gauge theories, such as noncommutative calculus [11], perturbative loop calculations [12–14], Seiberg-Witten map [15] and matrix model [16], confirm the reasonability of the ansatz for the Moyal case. However, the idea for generalizing the results derived for Moyal product to general translation-invariant star products comes in principal from quantum equivalence theorem introduced in [8, 9] which asserts that the whole quantum behaviors of a noncommutative quantum field theory with an arbitrary translation-invariant star product coincide precisely with those of its Moyal product case of the same $\alpha^s$-cohomology class.

Thus, according to physical consistency ansatz, we demand the results of integrals in (III.10) to be integers for any given translation-invariant noncommutative star product $\star$. Hence, for a homotopy of 2-cocyles as $s^a$ with $s \in [0, 1]$ and corresponding star product $\star_s$ we obtain $\int_X A_{s \pm} \in \mathbb{Z}$, which leads to 

$$\frac{d}{ds} \int_X A_{s \pm} = 0 \text{ due to continuity. Furthermore, for the commutative fields, i.e. } s = 0, \text{ we have;}$$

\footnote{We should emphasize that the results of [16] is in fact for general translation-invariant star products. See also [10] and the references therein for a more complete list of such papers.}
\[ \mathcal{A}_\pm = \mp \text{ch}_n(SE) \]. Therefore, \( \hat{A}_* \pm \) and \( \mp \hat{\text{ch}}_n(SE) \) must be cohomologous in \( H^2_{\text{dR}}(X, \mathbb{Z}) \). On the other hand, the left hand side of (III.9) transforms covariantly under infinitesimal gauge transformations via (III.1) and \( \hat{\psi} \to \alpha \circ \hat{\psi} := -i\hat{\alpha}^a t^a \psi \) for \( \alpha = -i\hat{\alpha}^a t^a \in \hat{\mathfrak{g}} \). Hence, \( \hat{A}_* \pm \) is an equivariant form. That is, \( \hat{\omega} = \hat{A}_* \pm \pm \hat{\text{ch}}_n(SE) \) is an equivariant form and thus it vanishes on \( U \) for triviality of \( E \) over it. Since \( \hat{\omega} \) represents the null integral cohomology class in \( H^2_{\text{dR}}(X, \mathbb{Z}) \) we readily conclude \( \hat{\omega} = 0 \). Thus, we have already established the following theorem.

**Theorem 3:** The chiral Abelian anomaly in noncommutative Yang-Mills theories with general translation-invariant noncommutative star product \( \ast \) is given by \( \hat{A}_* \pm = \mp \hat{\text{ch}}_n(SE) \).

It is well-known that the Dirac operator in the ordinary commutative case, i.e. \( D_A = i\gamma^\mu \nabla_\mu \), \( \nabla = d_X + A \), via \( \mathbb{Z}_2 \)-grading of \( SE \to X \) due to \( \gamma \) of (III.8), is given as

\[
\gamma = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad D_A = \begin{pmatrix}
0 & D_A^- \\
D_A^+ & 0
\end{pmatrix}, \quad \gamma D_A = -D_A \gamma, \quad D_A^{+\dagger} = D_A^-.
\]

(III.11)

Following the main approach of Atiyah-Singer index theorem [17] for \( \ast \)-cohomology and also employing Theorem 3 we readily find an index formula for any translation-invariant noncommutative anomalies via the machinery of \( \ast \)-cohomology. Show \( \pi(D_A^{\mp}) \) with \( D_A^{\mp} \).

**Theorem 4:** For any noncommutative \( U(N) \)-Yang-Mills theory with general translation-invariant noncommutative star product \( \ast \) the topological index of \( D_A^{\mp} \) is given by \( \hat{\text{ch}}_n(SE) \) as;

\[
\int_X \mathcal{A}_* \pm = \mp \int_X \hat{\text{ch}}_n(SE) = \text{Index}(D_A^{\mp}).
\]

(III.12)

The topological index is given for de Rham integral class in \( H^2_{\text{dR}}(X, \mathbb{Z}) \) with \( \text{ch}_n(SE) \) via (I.3);

\[
\text{Index}(D_A^{\mp}) = \int_X \mathcal{A}_* \pm = \mp \int_X \hat{\text{ch}}_n(SE).
\]

**IV. ANOMALIES, \( \ast \)-COHOMOLOGY AND THE CONNES-CHERN CHARACTERS**

In previous section we established an intimate correlation between \( \ast \)- and de Rham cohomology to figure out the topological structure in the background of a translation-invariant noncommutative Yang-Mills theory. In this section we try to demonstrate a similar relation for cyclic (co)homology and the corresponding Connes-Chern character.\(^8\) One should remember that since \( \mathcal{A} \) (or \( C^\infty_*(X) \)) is a noncommutative algebra there is no definite coincidence for de Rham and cyclic (co)homology (in the sense of \([19]\) via Hochschild-Kostant-Rosenberg formula\(^9\)). Therefore, topological interpretation of noncommutative anomalies need some breakthrough between cyclic (co)homology of \( \mathcal{A} \) on the one hand and de Rham cohomology of the topologically commutative underlying spacetime \( X \) on the other.

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\(^7\) See also the approach of [18].

\(^8\) See [11] for the special case of Moyal star product.

\(^9\) See also [20, 21].
hand. In this section we proved that this subjective is properly achieved by applying the machinery of $\star$-cohomology.

Consider a trivial vector bundle over $X$, say $\mathbb{C}^{N'} \to E' = \mathbb{C}^{N'} \times X \to X$, for some large enough $N'$, so that $E \to X$ is embedded in via the image of an idemponent $e \in \mathbb{M}_{N'}(\mathcal{C}^\infty(X))$, i.e.;

$$C^\infty(E) = \{\sigma \in C^\infty(E'); e.\sigma = \sigma\}.$$ 

Then, $SE' := S(X) \otimes E' \to X$ is subject to Dirac operator $D = i\gamma^\mu \partial_\mu$ on $U$. Actually, $\widetilde{SE}'$ could be completed to a Hilbert space, $\mathcal{H}$, with ordinary inner product $\langle \rho(\psi_1) | \rho(\psi_2) \rangle = \int_X \overline{\psi}_1 \psi_2$. We see that $\pi$ and the Dirac operator $D$ commute, i.e. $\pi \circ D = D \circ \pi$ for $D = i\gamma^\mu \partial_\mu$, and therefore, the spectrum of $D$ coincides with that of $D$. Hence, Dirac operator $D$ is a densely defined unbounded Hermitian operator on $\mathcal{H}$. Thus $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. The action of unital $\ast$-algebra $\mathcal{A}$ on $\mathcal{H}$ is also given as $(\hat{\alpha}^a t^a) \triangleright \hat{\psi} = \hat{\alpha}^a t^a \hat{\psi}$. In addition, we have a $\mathbb{Z}_2$-grading of $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with;

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad \gamma D = -D \gamma, \quad D^+ D^- = D^- D^+, \quad (IV.1)$$

for $\gamma$ of $\text{III.11}$. It is also clear that $\gamma \alpha = \alpha \gamma$, $\alpha \in \mathcal{A}$. Now we employ the homotopy of last section for star products $\ast_s$. Then, it can be seen that for any $\hat{a} \in \mathcal{A}$, the operator $[D, \hat{a}]$ is homotopic to $[D, a]$ and thus is a densely defined operator which could be extended to a bounded operator on $\mathcal{H}$. Also for any $p > 2n$ we have $(1 + D^2)^{-1} \in \mathcal{L}^{0/2}(\mathcal{H})$ since $\text{spec}(D) = \text{spec}(D)^\mathbb{C}$. Set $F = D/|D|$. Actually, $F$ is a bounded operator with $\gamma F = -F \gamma$ and $F^2 = 1$. Hence, $(\mathcal{H}, F, \gamma)$ is an even $p$-summable Fredholm module over $\mathcal{A}$, and an element of $K$-homology group $K^0(\mathcal{A})$.

The vector bundle $\widetilde{SE}$ is in fact a dual element with respect to $(\mathcal{H}, F, \gamma)$ in $K$-theory group $K_0(\mathcal{A})$. Assume an idemponent $e \in \mathbb{M}_{N'}(\mathcal{A})$ so that $\widetilde{SE}$ embeds in $\widetilde{SE}'$ as $\widetilde{SE} = \{\hat{\psi} \in \widetilde{SE}' ; e\hat{\psi} = \hat{\psi}\}$. The connection on $\widetilde{SE}$ is canonically defined with $\hat{\nabla} \hat{\psi} = e.d \hat{\psi}$. However, it is seen that if for a local basis over $U$ we define $\hat{\nabla} = d + A = d - i\hat{\alpha}^a t^a dx^\mu$, then; $\text{Index}(F^+_e) = \text{Index}(D^+_A)$, in which

$$F = \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix},$$

for the $\mathbb{Z}_2$-grading of $\gamma$ and $F^+_e = e F^+ e : e\mathcal{H}^+ \to e\mathcal{H}^-$. Hence, according to noncommutative index theorem and Theorem 4 we find;

**Theorem 5;** The integral cohomology class of the third type Chern character $\hat{\chi}_n(\mathcal{S}E)$ in $H^{2n}(X, \mathbb{Z})$ (and the corresponding topological index of chiral Abelian anomaly) is given by the pairing of Connes-Chern characters of $2n^{th}$ cyclic (co)homology groups due to $K$-theory and $K$-cohomology. That is,

$$\int \hat{\chi}_n(\mathcal{S}E) = \langle \text{Ch}^{2n}(\mathcal{H}, F, \gamma), \text{Ch}_{2n}[e] \rangle = \frac{(-1)^n}{2} \text{Trace}\{\gamma F[e][F, e] \cdots [F, e]\}, \quad (IV.2)$$

for $2n + 1$ copies of $e$ and for Trace the trace of operators.

---

10 The notation Trace is used for trace of operators on the corresponding Hilbert space of Dirac operator.
We remember that $\text{Ch}^{2n}(\mathcal{H}, F, \gamma)$ in (IV.2) is the Connes-Chern character of $(\mathcal{H}, F, \gamma) \in K^0(\mathcal{A})$ as

\[
\text{Ch}^{2n}(\mathcal{H}, \mathcal{D}, \gamma)(a_0, a_1, \cdots, a_{2n}) = (-1)^n \left(\frac{n!}{2}\right) \text{Tr}\{\gamma F[a_0][F, a_1] \cdots [F, a_{2n}]\}, \tag{IV.3}
\]

for $a_0, a_1, \cdots, a_{2n} \in \mathcal{A}$, and

\[
\text{Ch}_{2n}[e] = \sum_{k=0}^{n} (-1)^k \left(\frac{2k)!}{k!}\right) \text{tr}\left\{\left(e - \frac{1}{2}\right) \otimes e^{2k} \otimes \right\}, \tag{IV.4}
\]

wherein $e \in \mathbb{M}_N(A)$ represents the $K$-theory class $[e]$ in $K_0(A)$ and $\text{tr}$ is the trace of $\mathbb{C}^N$.

As we mentioned above, *-cohomology plays an intermediate role between de Rham and cyclic (co)homologies for any general translation-invariant noncommutative star product $\ast$. In previous sections we established an intimate correlation of $\ast$- and de Rham cohomology theories. Now, by employing the special abilities of $\ast$-cohomology, due to its partly commutative geometric structures, we can also prove a geometric correspondence between $\ast$- and cyclic (co)homology. We emphasize that this relation must be implemented in the setting we just apply for demonstrating that of de Rham and cyclic (co)homologies for commutative algebras. By means of $\ast$-cohomology we find the same arguments even for noncommutative algebra $\mathcal{A}$ (or $\mathcal{C}_0$).

This correlation is easy to see within familiar concepts of noncommutative geometry. Actually, the well-known Hochschild-Kostant-Rosenberg map $\alpha : HH_k(\mathcal{A}) \rightarrow \Omega_k(X)$, for $HH_*(\mathcal{A})$ the Hochschild homology group, leads to an isomorphism, similarly denoted by $\alpha$, between cyclic homology group $HC_{2n}(\mathcal{A})$ and $\oplus_{j=0}^n H^{2j}_*(X, \mathbb{C})$ due to commutativity of $d \circ \pi = \pi \circ d_X$. It is not hard to see that $\alpha$ produces the third type Chern character $\text{ch}_n(SE)$ from Connes-Chern character $\text{Ch}_{2n}[e] \in HC_{2n}(\mathcal{A})$.

In fact, the composition of isomorphism $\alpha : HC_{2n}(\mathcal{A}) \rightarrow \oplus_{j=0}^n H^{2j}_*(X, \mathbb{C})$ and the canonical projection $\pi_{2n} : \oplus_{j=0}^n H^{2j}_*(X, \mathbb{C}) \rightarrow H^{2n}_*(X, \mathbb{C})$ leads to the following result;

\[
\Xi = \left(\frac{1}{2i\pi}\right)^n \sum_{k=0}^{n} \frac{1}{(2n)!} \pi_{2n} \circ \alpha : HC_{2n}(\mathcal{A}) \rightarrow H^{2n}_*(X, \mathbb{C}), \quad \Xi([\text{Ch}_{2n}[e]]) = [\text{ch}_n(SE)], \tag{IV.5}
\]

for corresponding cohomology classes $[\text{Ch}_{2n}[e]]$ and $[\text{ch}_n(SE)]$. The homomorphism $\Xi$ can be derived with more detailed formalism. To see this we note that for connection $\hat{\nabla} = e.d = d - iA, \mu^\alpha dx^\mu\hat{\nabla}^2$ is actually $e.d.e.d = -\frac{1}{2}\hat{F}_{\mu\nu} dx^\mu dx^\nu$. However, with $(e.d.e.d)^n = e.(d.e)^2n$ we readily find $\alpha(\text{Ch}_{2n}[e]) = (2i\pi)^n (2n)! \text{ch}_n(SE) + \Delta$, where $\Delta$ is a direct summation of an exact $2n$-form and closed forms of lower even orders in $\mathcal{C}$. Now let $\Omega^*(\mathcal{A}) = \oplus_{k=0}^\infty \mathcal{A}^{(k+1)}$ and consider $\xi : \Omega^*(\mathcal{A}) \rightarrow \mathcal{C}$, with $\xi = 0$ on $\mathcal{A}^{(k+1)}$ for $k > 2n$, and

\[
\xi(a_0 \otimes \cdots \otimes a_k) = -\left(\frac{-1}{2\pi}\right)^n \frac{i}{2n(2n)!} \text{Tr}\{\gamma a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k]\} d^{2n}x, \tag{IV.6}
\]

for $k \leq 2n$. Here $\text{Tr}$ is the trace on Dirac matrices and on the colors. Thereby, we readily find that $\xi(\text{Ch}_{2n}[e]) = \text{ch}_n(SE) + d\phi$ for some $\phi \in \mathcal{C}_{2n-1}$. We have the following lemma.

**Lemma 1:** The linear map $\xi$ of (IV.4) leads to a surjection from cyclic cohomology group $HC_{2n}(\mathcal{A})$ to $H^{2n}_*(X, \mathbb{C})$, i.e. $\xi : HC_{2n}(\mathcal{A}) \rightarrow H^{2n}_*(X, \mathbb{C})$ for any general translation-invariant noncommutative star product $\ast$. Moreover, $\xi_\ast$ is in fact a redefinition for Hochschild-Kostant-Rosenberg map due to
\( (IV.5) \) with \( \xi_s = \Xi \).

**Proof:** First note that \( [D, a] = i \gamma^\mu \partial_\mu a, \ a \in A \). Then since
\[
\text{Tr} (\gamma \gamma^\mu_1 \cdots \gamma^\mu_{2n}) = (-i)^{n-1} 2^n \epsilon^{\mu_1 \cdots \mu_{2n}}, \quad \text{Tr} (\gamma \gamma^\mu_1 \cdots \gamma^\mu_k) = 0, \quad k < 2n,
\]
it is seen that \( \xi \) vanishes on \( A^{(k+1)\otimes} \) for all \( k \neq 2n \). On the other hand, \( \phi := \int \circ \xi \) is a closed graded trace on \( \Omega(A) \) with support in \( A^{(2n+1)\otimes} \). Thus, it defines a cyclic cohomology class in \( HC_{2n}(A) \).

This proves that the Connes’ \((b, B)\)-bicomplex is compatible with the de Rham complex via \( \xi \) so that \( \xi \circ b = 0 \) and \( \xi \circ B \in d\mathcal{C} \). Thus \( \xi \) is reduced to a well-defined map \( \xi_s : HC_{2n}(A) \to H_{2n}(X, \mathbb{C}) \). Direct calculation shows that \( \xi_s = \Xi \). This finishes the lemma. **Q.E.D**

Therefore, we have already established the following theorem.

**Theorem 6:** For any general translation-invariant noncommutative star product \( \star \) on \( X \) the integral cohomology class of the third type Chern character \( \hat{\text{ch}}_n(SE) \) in \( H_{2n}^2(X, \mathbb{Z}) \) is the image of that of \( \text{Ch}_{2n}[e] \) in cyclic cohomology group \( HC_{2n}(A) \) via \( \xi \) due to \( (IV.6) \).

**Corollary 1:** The topological index of chiral Abelian anomaly for any general translation-invariant noncommutative U(\( N \))-Yang-Mills theory is given by Connes-Chern character of the corresponding vector bundle \( \hat{SE} \). That is;
\[
\text{Index} (D^+_A) = \int \hat{A}_s = \mp \int \xi (\text{Ch}_{2n}[e]) .
\]

**V. FAMILY INDEX AND HOMOTOPY CLASS OF TOPOLOGICAL ANOMALY**

Topological or consistent anomalies in noncommutative field theories have been considered by several authors \cite{11, 12, 15, 16}, but however, the topological/geometric meaning of the solutions and of the corresponding formulations remained unclear especially for the case of general translation-invariant noncommutative star products. Thus, in this section the subjective is to studying topological anomalies of translation-invariant noncommutative Yang-Mills theories via homotopy classes due to Bismut-Freed determinant bundle and the family index approach \cite{23, 25}.

Let us assume the action of \( \hat{G} \) on \( \Gamma \) of \( (III.1) \) is free so that \( \Gamma \to \Gamma/\hat{G} \) provides a principal \( \hat{G} \)-bundle. We also suppose that \( \hat{X} := \Gamma/\hat{G} \) is a smooth manifold. Remember that for any \( A \in \Gamma \), the Dirac operator is perturbed to unbounded Hermitian operator \( D_A := D + A \). Thus, \( (\mathcal{H}, D_A, \gamma) \) is also regarded as an even \( p \)-summable Fredholm module homotopic to \( (\mathcal{H}, D, \gamma) \). Also
\[
D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix},
\]

\footnote{In \cite{11, 12, 15} the authors considered the Moyal product and in \cite{16} general translation-invariant noncommutative star product has been considered via some matrix model.}
according to $Z_2$-grading of $\gamma$. Thus, $\text{Ker}(D_A^+) \subset \mathcal{H}^+$ and $\text{Ker}(D_A^-) \subset \mathcal{H}^-$ are finite dimensional. Consider the Bismut-Freed determinant line bundle \cite{23,24}

$$\text{Det} := \text{Det}(\mathcal{D}, \delta : \gamma) := \det(\text{Ker}(D_A^+)) \otimes \det(\text{Ker}(D_A^-)),$$  \hspace{1cm} (V.1)

which has a natural metric and unitary connection, say $\nabla^{\text{Det}} = d_\Gamma - 2\pi i\Pi(A)$, where $\Pi(A)$ is a one-form on $\Gamma$, and $A \in \Gamma$. Here, $d_\Gamma$ is the exterior differential operator on $\Gamma$. Actually $\Pi(A)$ is a closed form when it is restricted to orbits of $G$ \cite{23}. Therefore, the connection of $\nabla^{\text{Det}}$ is flat. That is;

$$2\pi i \Pi(A) = \delta W(A), \quad \delta \Pi(A) = 0,$$  \hspace{1cm} (V.2)

where $\delta$ is the exterior differential operator on $G$, the BRST operator. Due to Bismut-Freed results $\Pi$ represents a non-trivial cohomology class in $H^1_{\text{DR}}(\tilde{G}, \mathbb{C})$, since in fact $W(A)$, the quantum action, is not in general a smooth function on $\Gamma$. The BRST closedness of $\Pi$, the Wess-Zumino consistency condition, is then an immediate consequence of (V.2). Actually, $\mathfrak{S}(A) := 2\pi i \Pi(A)$ is the topological anomaly. The flatness of $\nabla^{\text{Det}}$ on orbits of $\tilde{G}$ implies that the parallelism structure on $\text{Det}$ provides a covering space for Lie group $\tilde{G}$. Thus, if $i : S^1 \to \tilde{G}$ defines a smooth map then;\footnote{Actually, for commutative $\text{SU}(N)$-Yang-Mills theories $\mathbb{Q}$ can be replaced with $\mathbb{Z}$. But for the noncommutative case the gauge group $\text{SU}(N)$ has to be replaced with $U(N)$, since $F_s$ and the Lagrangian contain anti-commutators of the corresponding Lie algebra elements. This, leads to a non-trivial fundamental group for the gauge transformation group $\tilde{G}$ \cite{24}. Thus, despite to ordinary Yang-Mills theories, for those with noncommutative star products we should deal with rational cohomology $H^1_{\text{DR}}(\tilde{G}, \mathbb{Q})$ instead of $H^1_{\text{DR}}(\tilde{G}, \mathbb{Z})$ for topological anomalies. This is in fact due to $U(1)$ component in $U(N) = U(1) \times SU(N)$. The only finite covering of $U(1)$ is $U(1)$ itself. Therefore, the result of (V.3) belongs to $\frac{1}{l}\mathbb{Z}$ for $l$-covering. For more details see \cite{22,23,24}.

$$\int_{S^1} i^*(\Pi) \in \mathbb{Q}.$$  \hspace{1cm} (V.3)

Therefore, we readily find that $\Pi \in H^1_{\text{DR}}(\tilde{G}, \mathbb{Q})$. In principal, the equality of (V.3) should be invariant with respect to homotopy of 2-cocycles $s_i$ for $s \in [0, 1]$ and its corresponding translation-invariant noncommutative star producs $\ast_s$, with $\ast_1 = \ast$ and $\ast_0$ the ordinary product. To see this let $\tilde{G}_s$ be the gauge transformation group defined with $\ast_s$. That is $\tilde{G} = \tilde{G}_1$ and $\tilde{G}_0$ is the gauge transformation group in commutative $U(N)$-Yang-Mills theory. Then $M = \{\tilde{G}_s; \; s \in I = [0, 1]\}$ is topologically a cylindrical space with an end $\tilde{G}$ and $\tilde{G}_0$. Actually, $M$ is topologically equivalent to $I \times \tilde{G}$ and therefore it deformation retracts on both $\tilde{G}$ and $\tilde{G}_0$ \cite{23,26}. Put a smooth structure on $M$ with $d_M = \delta + ds \otimes \partial / \partial s$. Note that here $\delta$ is the exterior derivative operator on $\tilde{G}_s$ for any $s \in [0, 1]$. Now, $\Pi$ can be assumed as a one-form on $M$. By homotopy invariance of (V.3) we see that, there is some $\Phi \in C^\infty(M)$ so that $\frac{d}{ds}\Pi = \delta \Phi$. In other words, if $\Pi_0$ be the Bismut-Freed connection of commutative Yang-Mills theory;

$$\Pi - \Pi_0 = \delta \left( \int_0^1 \Phi \; ds \right).$$  \hspace{1cm} (V.4)

Moreover, $\tilde{G}$ and $\tilde{G}_0$ are homotopic equivalent. Let $f : \tilde{G} \to \tilde{G}_0$ defines this equivalence relation. Then (V.4) leads to the following theorem.

**Theorem 7;** Let $\ast$ be a general translation-invariant noncommutative star product on $X$. Also suppose that $[\mathfrak{S}_0(A)]$ (resp. $[\mathfrak{S}(A)]$) is the cohomology class of topological anomaly of commutative
\((\text{resp. } \star\text{-noncommutative}) \text{ U}(N)\text{-Yang-Mills theory in } H^1_{\text{dR}}(\tilde{G}_0, \mathbb{C}) \text{ (resp. } H^1_{\text{dR}}(\tilde{G}, \mathbb{C})\text{). Then, we have; } f^*([\mathcal{E}(0)]) = [\mathcal{E}(A)]. \text{ Moreover, } \mathcal{E}_0(A)/2\pi i \text{ (resp. } \mathcal{E}(A)/2\pi i\text{) defines a rational cohomology class in } H^1_{\text{dR}}(\tilde{G}_0, \mathbb{Q}) \text{ (resp. } H^1_{\text{dR}}(\tilde{G}, \mathbb{Q})\text{).}

We should explain } f \text{ with more details. Let } \tilde{g}_s \text{ be the infinitesimal gauge transformation group as the Lie algebra of } \tilde{G}_s. \text{ Moreover, let } \pi_s \text{ be the corresponding representation map for star product } \star_s. \text{ Therefore, } \{\pi_s(e_p)t^a\} \text{ for } p \in \mathbb{Z}^{2m} \text{ and } a = 0, \ldots, N^2 - 1 \text{ provides a basis for } \tilde{g}_s. \text{ Set } df : \tilde{g} \to \tilde{g}_0 \text{ with } df(\pi(e_p))t^a = \pi_0(e_p)t^a. \text{ Then, it is seen that its integral provides a group isomorphism } f : \tilde{G} \to \tilde{G}_0.\text{ Actually, } f^* \text{ replaces the ordinary product with translation-invariant noncommutative star product } \star \text{ within differential forms. This leads us to the following corollary.}

**Corollary 2:** Assume that the topological anomaly of commutative U(N)-Yang-Mills gauge theory, \(\mathcal{E}_0(A)\), is given by integration of polynomial \(P(A, C)\) over spacetime \(X\) as; \(\mathcal{E}_0(A) = \int_X P(A, C)\), for \(C\) the ghost field. Then, the topological anomaly of noncommutative U(N)-Yang-Mills theory for any translation-invariant noncommutative star product \(\star\) is;

\[
\mathcal{E}(A) = \int_X P_\star(A, C) .
\]

wherein \(P_\star\) is the noncommutative polynomial of \(P\) due to \([15]\).

Note that according to **Theorem 7** we know that

\[
\mathcal{E}(A) = \int_X P_\star(A, C) + \delta \Phi(A)
\]

for BRST exact term \(\delta \Phi(A)\). Adding \(-\Phi(A)\) to the quantum action \(W(A)\) as a counter term, i.e. \(W'(A) = W(A) - \Phi(A)\), to renormalize the theory accordingly, then we obtain the topological anomaly \(\mathcal{E}(A)\) as described in \([V.5]\). In principal, the topological anomaly of noncommutative U(N)-Yang-Mills gauge theory on 4-dimensional spacetime \(X\) and for any translation-invariant noncommutative star product \(\star\) is given for Weyl fermions as;\textsuperscript{14}

\[
\mathcal{E}(A) = \frac{1}{24\pi^2} \int \text{tr}\{C \star d(A \star dA + \frac{1}{2} A^3)\} .
\]

Up to now we studied the topological anomaly within homotopy classes in \(H^1_{\text{dR}}(\tilde{G}, \mathbb{Q})\) via family index theory due to Bismut-Freed determinant line bundle. However, it can also be regarded as a noncommutative geometric problem through with the machinery of cyclic (co)homology \([30, 32]\).

To see this we note that the inclusion \(\tilde{g}_A : \tilde{G} \to \Gamma\) as an orbit which passes \(A \in \Gamma\) is an invertible element of \(C^\infty(\Gamma) \otimes \mathcal{A}\), and hence defines a class in \(K^\text{alg}_1(C^\infty(\Gamma) \otimes \mathcal{A})\). Its Chern character in odd periodic cyclic homology \(HP_{\text{odd}}(C^\infty(\tilde{G} \otimes \mathcal{A})\text{ is in fact an element of differential graded algebra } (\Lambda', \tilde{d})\text{ with } \Lambda' := \oplus_{k \geq 0} \Lambda^k_{\text{alg}}(C^\infty(\tilde{G}) \otimes \mathcal{A})\text{ and } \tilde{d} = \delta + d. \text{ For the Maurer-Cartan form } \omega_A = \tilde{g}_A^{-1}d\tilde{g}_A \text{ we read}

\[
\text{Ch}^1_{\tilde{g}_A} = \sum_{k \geq 0} (-1)^k \frac{k!}{(2k + 1)!} \omega_A^{2k+1} \in HP_{\text{odd}}(C^\infty(\tilde{G}) \otimes \mathcal{A}) .
\]

\textsuperscript{13} For more details of this proof see \([24]\). In \([24]\) we also proved similar results for the special case of Moyal product.

\textsuperscript{14} Here the notation tr is used for trace on both group colors and matrix representation due to \(\pi\).
for \( \text{Ch}^1_{\ast}[^g \mathcal{A}] \) the corresponding Connes-Chern character in odd periodic cyclic homology for \( ^g \mathcal{A} \). An important formula is \([30, 33]\):

\[
\text{CS}(\mathcal{H}, \mathcal{D}, \gamma) = \langle \text{Ch}^1_{\ast}[^g \mathcal{A}], \text{Ch}^1_{\ast}(\mathcal{H}, \mathcal{D}, \gamma) \rangle ,
\]

wherein \( \text{Ch}^1_{\ast}(\mathcal{H}, \mathcal{D}, \gamma) \) is the class of Connes-Chern character in odd periodic cyclic cohomology \( HP^{\text{odd}}(C^\infty(\tilde{G}) \otimes \mathcal{A}) \) via \([I.3]\) and \( \text{CS}(\mathcal{H}, \mathcal{D}, \gamma) \) is the corresponding Chern-Simons form. The pairing \((V.8)\) is in fact taken place via cup product

\[
\cup : H_k(\tilde{G}, \mathbb{C}) \otimes HC^j(\mathcal{A}) \to HC^{k+j}(C^\infty(\tilde{G}) \otimes \mathcal{A}) ,
\]

and hence, it represents de Rham cohomology classes in \( H^{\text{odd}}_{\text{dR}}(\tilde{G}, \mathbb{C}) \). The main result is that the component in \( H^1_{\text{dR}}(\tilde{G}, \mathbb{C}) \) of pairing \((V.8)\) coincides with \( \Pi(\mathcal{A}) \) up to some exact form on \( \tilde{G} \). Therefore, if \( i : S^1 \to \tilde{G} \) is a smooth map, then we find;

\[
\int_{S^1} i^*(\mathcal{G}(A)) = 2\pi i \langle \text{Ch}^1_{\ast}[^g \mathcal{A}], [S^1] \cup \text{Ch}^2_{\ast}(\mathcal{H}, \mathcal{D}, \gamma) \rangle \in 2\pi i \mathbb{Q} ,
\]

(V.9)

where \( i : S^1 \to \tilde{G} \) is considered to represent a class in \( H_1(\tilde{G}, \mathbb{C}) \), say \([S^1]\). Calculating \((V.9)\) is actually accomplished by using Connes-Moscovici local index formula as a residue of some zeta function.\(^{15}\)

**VI. SUMMARY AND CONCLUSIONS**

Through this paper we introduced a cohomology theory, so called \( \ast \)-cohomology, with cohomology groups \( H^k_{\ast}(X, \mathbb{C}) \), on spacetime manifold \( X \), to describe general translation-invariant noncommutative quantum field theories by means of both commutative and noncommutative geometric structures. In fact, \( \ast \)-cohomology plays an intermediate role between de Rham and cyclic (co)homology theories for noncommutative algebras. It provides a breakthrough between commutative and noncommutative quantum field theories and thus is comparable to the Seiberg-Witten map.

Employing this framework for Chern-Weil theory we introduced three types of Chern characters so that the third type, \( \tilde{\text{ch}}_{\ast} \), is shown to belong to \( H^k_{\ast}(X, \mathbb{Z}) \) and has intimate correlation to Connes-Chern characters in cyclic (co)homology groups. On the other hand, \( \tilde{\text{ch}}_{\ast} \) induces integral classes in \( H^1_{\text{dR}}(X, \mathbb{Z}) \) which include significant information about topology of translation-invariant noncommutative Yang-Mills theories over \( X \). Therefore, the topology of Abelian and topological anomalies of translation-invariant noncommutative Yang-Mills theories were studied thoroughly with correlation to Connes-Chern characters in cyclic (co)homology groups and the machinery of noncommutative geometry.

**VII. ACKNOWLEDGMENTS**

The author says his gratitude to S. Ziaee who was the main reason for appearing this article. Moreover, the author wishes to dedicate this work to Mohammad Reza Shajarian for all he has done

\(^{15}\) See \([30]\) for more details.
to Iranian art and culture along last fifty years. Finally, my special thanks and highest regards go to
the esteemed referee and the respectable editor of ROMP for all their honest considerations.

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