Simply $sm$-factorizable (para)topological groups and their completions

Li-Hong Xie$^1$ · Mikhail G. Tkachenko$^2$

Received: 12 February 2020 / Accepted: 28 March 2020 / Published online: 7 April 2020
© Springer-Verlag GmbH Austria, part of Springer Nature 2020

Abstract
Let us call a (para)topological group strongly submetrizable if it admits a coarser separable metrizable (para)topological group topology. We present a characterization of simply $sm$-factorizable (para)topological groups by means of continuous real-valued functions. We show that a (para)topological group $G$ is a simply $sm$-factorizable if and only if for each continuous function $f : G \to \mathbb{R}$, one can find a continuous homomorphism $\varphi$ of $G$ onto a strongly submetrizable (para)topological group $H$ and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ \varphi$. This characterization is applied for the study of completions of simply $sm$-factorizable topological groups. We prove that the equalities $\mu G = \varrho \omega G = \nu G$ hold for each Hausdorff simply $sm$-factorizable topological group $G$, where $\nu G$ and $\mu G$ are the realcompactification and Dieudonné completion of $G$, respectively. This result gives a positive answer to a question posed by Arhangel’skii and Tkachenko in 2018. It is also proved that $\nu G$ and $\mu G$ coincide for every regular simply $sm$-factorizable paratopological group $G$ and that $\nu G$ admits the natural structure of paratopological group containing $G$ as a dense subgroup and, furthermore, $\nu G$ is simply $sm$-factorizable. Some results in [Completions of paratopological groups, Monatsh. Math. 183, 699–721 (2017)] are improved or generalized.

Communicated by John S. Wilson.

Li-Hong Xie is supported by NSFC (Nos. 11601393; 11861018).

$\bigstar$ Mikhail G. Tkachenko
mich@xanum.uam.mx

Li-Hong Xie
yunli198282@126.com; xielihong2011@aliyun.com

1 School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, People’s Republic of China

2 Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, CP 09340 Ciudad de México, Mexico
Keywords  Simply $sm$-factorizable · realcompactification · Dieudonné completion · Lindelöf $\Sigma$-space · $\mathbb{R}$-factorizable group

Mathematics Subject Classification  22A05 · 22A30 · 54H11 · 54A25 · 54C30

1 Introduction

A paratopological group $G$ is a group $G$ with a topology such that the multiplication mapping of $G \times G$ to $G$ associating $xy$ to arbitrary $x, y \in G$ is jointly continuous. A paratopological group $G$ is called a topological group if the inversion on $G$ is continuous.

Slightly reformulating the celebrated theorem of Comfort and Ross [5, Theorem 1.2], we can say that the pseudocompact topological groups are exactly the dense $C$-embedded subgroups of compact topological groups. In particular, the Stone-Čech compactification, $\beta G$, the Hewitt-Nachbin completion, $\nu G$, and the Raïkov completion, $\varrho G$, of a pseudocompact topological group $G$ coincide. Hence the Hewitt-Nachbin completion of the group $G$ is again a topological group containing $G$ as a dense $C$-embedded subgroup. Recently, with the idea to study connections between the properties of a Tychonoff space $X$ and its dense $C$-embedded subspace $Y$ homeomorphic to a topological group, the authors of [3] introduced the new notions of $sm$-factorizable, densely $sm$-factorizable and simply $sm$-factorizable (para)topological groups as follows:

Definition 1 (See Definition 5.11 in [3]) A (para)topological group $G$ is called $sm$-factorizable if for each co-zero set $U$ in $G$, there exists a continuous homomorphism $\pi$ of $G$ onto a separable metrizable (para)topological group $H$ such that the set $\pi(U)$ is open in $H$ and $\pi^{-1}(\pi(U)) = U$. Replacing the assumption that $\pi(U)$ is open by the requirement that $\pi(U)$ is dense in some open subset of $H$, we obtain the definition of a densely $sm$-factorizable (para)topological group. Removing the assumption that $\pi(U)$ is open, we obtain the definition of a simply $sm$-factorizable (para)topological group.

It is shown in [3] that the following implications are valid (and none of these implications can be inverted, see [3, Example 5.12, Proposition 5.13]):

$$sm\text{-factorizability} \Rightarrow \text{dense } sm\text{-factorizability} \Rightarrow \text{simple } sm\text{-factorizability.}$$

We recall that a (para)topological group $G$ is called $\mathbb{R}$-factorizable if every continuous real-valued function on $G$ can be factorized through a continuous homomorphism onto a separable metrizable (para)topological group. Arhangel’skii and Tkachenko obtained that a (para)topological group $G$ is $\mathbb{R}$-factorizable if and only if $G$ is $sm$-factorizable [3, Theorem 5.9] (in view of the proof of [3, Theorem 5.9], it is worth noting that it need not any separation axiom on $G$). However, there is a Hausdorff $\omega$-narrow and simply $sm$-factorizable Abelian topological group is not $\mathbb{R}$-factorizable [3, Corollary 5.20]. In [3, Corollary 5.24] it is shown that subgroups of Lindelöf topological groups need not be simply $sm$-factorizable. Since subgroups of Lindelöf
Simply $sm$-factorizable (para)topological groups are $\omega$-narrow, $\omega$-narrow topological groups can fail to be simply $sm$-factorizable. Also, there exists a densely $sm$-factorizable topological Abelian group $G$ such that $ib(G) = 2^\omega$ [3, Example 5.12], so densely $sm$-factorizable topological groups can fail to be $\omega$-narrow. Now we summarize the relations between $\mathbb{R}$-factorizable, $sm$-factorizable, densely $sm$-factorizable, simply $sm$-factorizable and $\omega$-narrow (para)topological groups as follows:

1. $\mathbb{R}$-factorizability $\iff$ $sm$-factorizability;
2. $sm$-factorizability $\Rightarrow$ dense $sm$-factorizability $\Rightarrow$ simple $sm$-factorizability;
3. regularity & $sm$-factorizability $\Rightarrow$ total $\omega$-narrowness (in paratopological groups);
4. regularity & $\omega$-narrowness $\not\Rightarrow$ simple $sm$-factorizability (in topological groups);
5. regularity & dense $sm$-factorizability $\not\Rightarrow$ $\omega$-narrowness (in topological groups).

We denote the Dieudonné completion and Hewitt-Nachbin realcompactification of a completely regular space $X$ by $\mu X$ and $\nu X$, respectively. Recall that a topological group $G$ is called a $PT$-group if the group operations in $G$ can be continuously extended to Dieudonné completion $\mu G$. Since every $\mathbb{R}$-factorizable topological group is a $PT$-group [2, Corollary 8.3.7], the authors of [3] posed the following question:

**Problem 1** (See Problem 7.6 in [3]) Is every simply $sm$-factorizable topological group a $PT$-group? What about densely $sm$-factorizable topological groups?

Continuous open homomorphic images of $\mathbb{R}$-factorizable topological groups are $\mathbb{R}$-factorizable [2, Theorem 8.4.2], but it is unknown whether continuous homomorphisms preserve $\mathbb{R}$-factorizable topological groups [2, Open Problem 8.4.1]. A weaker version of this problem is given below:

**Problem 2** (See Problem 7.8 in [3]) Is every continuous homomorphic image of an $\mathbb{R}$-factorizable topological group $G$ simply $sm$-factorizable?

It is known, however, that continuous homomorphic images of simply $sm$-factorizable topological groups need not be simply $sm$-factorizable [3]. This makes it natural to rise the following question:

**Problem 3** (See Problem 7.9 in [3]) Is every quotient group of a simply $sm$-factorizable topological group $G$ simply $sm$-factorizable? What if, additionally, $G$ is $\omega$-narrow?

In Sect. 2, we present a characterization of simply $sm$-factorizable (para)topological groups in terms of continuous homomorphisms onto strongly submetrizable (para)topological groups. Applying this characterization we give a positive answer to Problem 1 and partially answer Problems 2 and 3.

The article is organized as follows. In Sect. 2, we characterize simply $sm$-factorizable (para)topological groups. We establish the following facts: (1) A (para)topological group $G$ is simply $sm$-factorizable if and only if for each continuous function $f : G \to \mathbb{R}$, one can find a continuous homomorphism $\varphi$ of $G$ onto a strongly submetrizable (para)topological group $H$ and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ \varphi$ (see Theorem 2); (2) an $\omega$-narrow topological group $G$ is simply $sm$-factorizable if and only if for any continuous function $f : G \to \mathbb{R}$, there is an invariant admissible subgroup $N_f$ of $G$ such that $f$ is constant on $gN_f$ for each $g \in G$ (see
Theorem 4). Section 3 contains a positive answer to a question posed by Arhangel’skii and Tkachenko [3, Problem 7.8]. We show that the equalities $\mu G = \varrho_o G = \nu G$ hold for every Hausdorff simply $sm$-factorizable topological group $G$ and, therefore, $G$ is completion friendly (see Theorem 5).

In Sect. 4, we study completions of simply $sm$-factorizable paratopological groups. We prove the following: (1) If $G$ is a regular simply $sm$-factorizable paratopological group, then the realcompactification $\nu G$ of the space $G$ admits a natural structure of paratopological group containing $G$ as a dense subgroup and $\nu G$ is also simply $sm$-factorizable (Theorem 6); (2) If $G$ is a regular paratopological group such that the topological group $G^*$ associated to $G$ is $\omega$-narrow and simply $sm$-factorizable, then the realcompactification $\nu G$ of $G$ admits a natural structure of paratopological group containing $G$ as a dense subgroup and the equality $\nu G = \mu G$ holds (Theorem 8).

Our results are accompanied with several examples and short discussions that outline the limits for their generalizations.

We do not impose any separation restrictions on spaces and (para)topological groups unless the separation axioms are stated explicitly. A space $X$ satisfies the $T_3$-separation axiom if for any point $x \in X$ and a neighborhood $O$ of $x$, there is a neighborhood $U$ of $x$ such that $\overline{U} \subseteq O$. A space $X$ is regular if it is a $T_3$-space satisfying the $T_1$-separation axiom.

Let $X$ be a space with topology $\tau$. Then the family $\{\text{Int}_\tau U : \emptyset \neq U \in \tau\}$ constitutes a base for a coarser topology $\sigma$ on $X$. The space $X_{sr} = (X, \sigma)$ is called the semiregularization of $X$. One of the main results regarding the semiregularization of paratopological groups is the following important theorem proved by A. Ravsky:

**Theorem 1** (See [12]) Let $G$ be an arbitrary paratopological group. Then the space $G_{sr}$ carrying the same group structure is a $T_3$ paratopological group. If $G$ is Hausdorff, then $G_{sr}$ is a regular paratopological group.

The index of narrowness of a paratopological group $G$ is denoted by $ib(G)$ (see [2, Section 5.2]). By definition, $ib(G)$ is the smallest cardinal $\tau \geq \omega$ such that for each open neighborhood $U$ of the identity in $G$, there is a subset $A$ of $G$ satisfying $AU = UA = G$ and $|A| \leq \tau$. If $ib(G) = \omega$, then $G$ is called $\omega$-narrow.

For a paratopological group $G$ with topology $\tau$, one defines the conjugate topology $\tau^{-1}$ on $G$ by $\tau^{-1} = \{U^{-1} : U \in \tau\}$. Then $G^{\prime} = (G, \tau^{-1})$ is also a paratopological group, and the inversion $x \rightarrow x^{-1}$ is a homeomorphism of $G$ onto $G^{\prime}$. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on $G$, and we say that the topological group $G^* = (G, \tau^*)$ is associated to $G$. A paratopological group $G$ is totally $\omega$-narrow if the topological group $G^*$ associated to $G$ is $\omega$-narrow.

A paratopological group $G$ is called $\omega$-balanced if for every neighborhood $U$ of the identity $e$ in $G$ there is a countable family $\{V_n : n \in \omega\}$ of open neighborhoods of $e$ such that for each $g \in G$, some $V_n$ satisfies $gV_n g^{-1} \subseteq U$. In this case we say that the family $\{V_n : n \in \omega\}$ is subordinated to $U$. It is well known that every $\omega$-narrow topological group is $\omega$-balanced [2, Proposition 3.4.10] and every totally $\omega$-narrow paratopological group is $\omega$-balanced [15, Proposition 3.8].

In this paper, $c(X)$ and $\psi(X)$ stand for the cellularity and pseudocellularity of $X$, respectively. The closure of a subset $Y$ of $X$ is denoted by $\overline{Y}$ or $cl_X Y$ if we want to stress that the closure is taken in $X$. The cardinality of the continuum is $c = 2^\omega$.

Springer
2 Characterizations of simply \(sm\)-factorizable (para)topological groups

In this section, we present some characterizations of simply \(sm\)-factorizable topological and paratopological groups via continuous real-valued functions. The following notion plays an important role in this article.

**Definition 2** A (para)topological group \(G\) is **strongly submetrizable** if \(G\) admits a coarser separable metrizable (para)topological group topology or, equivalently, there exists a continuous one-to-one homomorphism of \(G\) onto a separable metrizable (para)topological group.

It is clear that every strongly submetrizable (para)topological group is Hausdorff and has countable pseudocharacter. Furthermore, the identity of a strongly submetrizable paratopological group is the intersection of countably many closed neighborhoods.

The following fact is obvious.

**Proposition 1** Every strongly submetrizable (para)topological group is simply \(sm\)-factorizable.

Every \(\omega\)-narrow Hausdorff topological group of countable pseudocharacter admits a continuous one-to-one homomorphism onto a separable metrizable topological group (see [2, Corollary 3.4.25]). Similarly, a totally \(\omega\)-narrow regular paratopological group of countable pseudocharacter admits a continuous one-to-one homomorphism onto a separable metrizable paratopological group (see [10, Lemma 1.6]). Thus we have:

**Lemma 1** Every \(\omega\)-narrow Hausdorff topological group (totally \(\omega\)-narrow regular paratopological group) of countable pseudocharacter is strongly submetrizable.

Now we give a characterization of simply \(sm\)-factorizable (para)topological groups in terms of continuous homomorphisms to strongly submetrizable (para)topological groups as follows:

**Theorem 2** Let \(G\) be a (para)topological group. Then the following statements are equivalent:

1. \(G\) is simply \(sm\)-factorizable;
2. for each continuous function \(f : G \rightarrow \mathbb{R}\), one can find a continuous homomorphism \(\pi\) of \(G\) onto a strongly submetrizable (para)topological group \(H\) and a continuous function \(g : H \rightarrow \mathbb{R}\) such that \(f = g \circ \pi\).
3. for each continuous function \(f : G \rightarrow \mathbb{R}\), one can find a continuous homomorphism \(\pi\) of \(G\) onto a regular strongly submetrizable (para)topological group \(H\) and a continuous function \(g : H \rightarrow \mathbb{R}\) such that \(f = g \circ \pi\).

**Proof** (1) \(\Rightarrow\) (2). Let \(\mathcal{V}\) be a countable base of \(\mathbb{R}\) consisting co-zero sets and \(f\) be a continuous real-valued function on \(G\). For every \(V \in \mathcal{V}\), let \(U_V = f^{-1}(V)\). Then each element of the family \(\mathcal{U} = \{U_V : V \in \mathcal{V}\}\) is a co-zero set in \(G\). Since \(G\) is simply \(sm\)-factorizable, for each \(V \in \mathcal{V}\) we can find a separable metrizable (para)topological group \(H_V\) and a continuous homomorphism \(\pi_V\) of \(G\) onto \(H_V\) such
that \( U_V = \pi_V^{-1}(\pi_V(U_V)) \). Let \( \pi = \Delta_{V \in \mathcal{V}} \pi_V \) be the diagonal product of the family \( \{ \pi_V : V \in \mathcal{V} \} \) and \( \Pi = \prod_{V \in \mathcal{V}} H_V \) be the topological product of the family \( \{ H_V : V \in \mathcal{V} \} \). Clearly, \( \pi : G \to \pi(G) \subseteq \Pi \) is a continuous homomorphism and \( H' = \pi(G) \) is a separable metrizable (para)topological group. Now let \( H \) have the same group structure as \( H' \) and endow \( H \) with the quotient topology with respect to \( \pi \).

Then clearly \( H \) is strongly submetrizable and \( \pi : G \to H \) is open. Thus it suffices to show that there is a continuous function \( g : H \to \mathbb{R} \) satisfying \( f = g \circ \pi \). Since \( \pi \) is continuous and open, the latter will follow if we show that the equality \( f(g_1) = f(g_2) \) holds for all \( g_1, g_2 \in G \) with \( \pi(g_1) = \pi(g_2) \). Indeed, if \( f(g_1) \neq f(g_2) \), then there is a \( V \in \mathcal{V} \) such that \( f(g_2) \in V \) and \( f(g_1) \notin V \). Thus \( g_2 \in U_V \) and \( g_1 \notin U_V \). Since \( U_V = \pi_V^{-1}(\pi_V(U_V)) \), we have

\[
\pi^{-1}(\pi(g_2)) = \bigcap_{W \in \mathcal{V}} \pi_W^{-1}(\pi_W(g_2)) \subseteq \pi_V^{-1}(\pi_V(U_V)) = U_V.
\]

This implies that \( \pi(g_1) \neq \pi(g_2) \) and completes the proof of the implication.

(2) \( \Rightarrow \) (1). Let \( U \) be a co-zero set in \( G \). Then there is a continuous real-valued function \( f \) on \( G \) such that \( U = f^{-1}(\mathbb{R} \setminus \{0\}) \). By (2), one can find a strongly submetrizable (para)topological group \( H \), a continuous homomorphism \( \pi \) of \( G \) onto \( H \) and a continuous function \( g : H \to \mathbb{R} \) such that \( f = g \circ \pi \). Let \( i : H \to H' \) be a continuous isomorphism onto a separable metrizable (para)topological group \( H' \). Then \( \varphi = i \circ \pi \) is a continuous homomorphism of \( G \) onto the separable metrizable group \( H' \) and \( U = \varphi^{-1}(\varphi(U)) \). This shows that \( G \) is simply sm-factorizable.

Now we show that (2) \( \iff \) (3) for topological groups is obvious.

For paratopological groups, it suffices to show that (2) \( \Rightarrow \) (3). Let \( G \) be a paratopological group satisfying (2). Let also \( f \) be a continuous real-valued function on \( G \). By (2), one can find a continuous homomorphism \( \pi : G \to H \) onto a strongly submetrizable paratopological group \( H \) and a continuous function \( g \) on \( H \) such that \( f = g \circ \pi \). Then \( H \) is a Hausdorff paratopological group. Let \( H_{sr} \) be the semiregularization of \( H \). Then \( H_{sr} \) is a regular paratopological group, by Theorem 1. From the fact that a continuous mapping \( f : X \to Y \) to a regular space \( Y \) remains continuous when considered as a mapping of \( X_{sr} \) to \( Y \) [24, Lemma 3.5], one can easily see that \( H_{sr} \) is a strongly submetrizable paratopological group and \( g : H_{sr} \to \mathbb{R} \) is continuous. Clearly, \( f = g \circ (i \circ \pi) \), where \( i : H \to H_{sr} \) is the identity mapping. The proof is complete.

A space \( X \) is called weakly Lindelöf if for each open cover \( \mathcal{U} \) of \( X \), there exists a countable subfamily \( \mathcal{V} \) of \( \mathcal{U} \) such that \( \bigcup \mathcal{V} \) is dense in \( X \).

**Corollary 1** (See Proposition 5.18 in [3]) Every weakly Lindelöf topological group \( G \) is simply sm-factorizable.

**Proof** Take any continuous function \( f : G \to \mathbb{R} \). According to [2, Theorem 8.1.18], one can find a continuous homomorphism \( \pi : G \to H \) onto a topological group \( H \) of countable pseudocharacter and a continuous real-valued function \( h \) on \( H \) such that \( f = h \circ \pi \). Every weakly Lindelöf topological group is \( \omega \)-narrow [2, Proposition 5.2.8].

\( \square \) Springer
Hence the group $G$ and its continuous homomorphic image $H$ are $\omega$-narrow as well. According to Lemma 1 $H$ is a strongly submetrizable topological group and therefore, $G$ is simply $sm$-factorizable by Theorem 2.

Let $\varphi: G \to H$ be a continuous surjective homomorphism of semitopological groups. The pair $(H, \varphi)$ is called a $T_2$-reflection of $G$ if $H$ is a Hausdorff semitopological group and for every continuous mapping $f: G \to X$ of $G$ to a Hausdorff space $X$, there exists a continuous mapping $h: H \to X$ such that $f = h \circ \varphi$. Abusing of terminology we say that $T_2(G)$ is the $T_2$-reflection of $G$, thus omitting the corresponding homomorphism $\varphi$ (see [21]). The homomorphism $\varphi$ is denoted by $\varphi_{G,2}$ and called the canonical homomorphism of $G$ onto $T_2(G)$.

**Proposition 2** A paratopological group $G$ is simply $sm$-factorizable if and only if so is the $T_2$-reflection $T_2(G)$ of $G$.

**Proof** Let $G$ be simply $sm$-factorizable. Take any continuous function $f: T_2(G) \to \mathbb{R}$. Then $f \circ \varphi_{G,2}$ is continuous real-valued function on $G$, and therefore, we can find a strongly submetrizable paratopological group $H$, a continuous homomorphism $\pi$ of $G$ onto $H$ and a continuous function $g: H \to \mathbb{R}$ such that $f \circ \varphi_{G,2} = g \circ \pi$, by Theorem 2. Since $H$ is a Hausdorff paratopological group, there is a continuous map $p: T_2(G) \to H$ such that $p \circ \varphi_{G,2} = \pi$. Observing that $\varphi_{G,2}$ and $\pi$ are homomorphisms, one can easily show that $p$ is also a homomorphism. Clearly, the subgroup $p(T_2(G))$ of $H$ is strongly submetrizable and $f = g \circ p$, so $T_2(G)$ is simply $sm$-factorizable by Theorem 2.

Let $T_2(G)$ be simply $sm$-factorizable. Take any continuous function $f: G \to \mathbb{R}$. Since $\mathbb{R}$ is a Hausdorff space, there is continuous function $h: T_2(G) \to \mathbb{R}$ such that $h \circ \varphi_{G,2} = f$. Further, combining Theorem 2 and the fact that $T_2(G)$ is simply $sm$-factorizable we see that there are a strongly submetrizable paratopological group $H$, a continuous homomorphism $\pi$ of $T_2(G)$ onto $H$ and a continuous function $g: H \to \mathbb{R}$ such that $h = g \circ \pi$. Thus we have the equality

$$f = h \circ \varphi_{G,2} = g \circ (\pi \circ \varphi_{G,2}).$$

By Theorem 2, this implies that $G$ is simply $sm$-factorizable.

**Proposition 3** A paratopological group $G$ is simply $sm$-factorizable if and only if so is the semiregularization $G_{sr}$ of $G$.

**Proof** Let $G$ be simply $sm$-factorizable. Take any continuous function $f: G_{sr} \to \mathbb{R}$. Clearly, $f$ is continuous on $G$, and applying Theorem 2 we find a regular strongly submetrizable paratopological group $H$, a continuous homomorphism $\pi$ of $G$ onto $H$ and a continuous function $g: H \to \mathbb{R}$ such that $f \circ \varphi_{G,2} = g \circ \pi$. Since $H$ is regular, by [24, Lemma 3.5] $\pi$ is also continuous on $G_{sr}$. Hence $G_{sr}$ is simply $sm$-factorizable according to Theorem 2.

Let $G_{sr}$ be simply $sm$-factorizable. Take any continuous function $f: G \to \mathbb{R}$. Then by [24, Lemma 3.5], $f$ is also continuous on $G_{sr}$. So Theorem 2 implies that there are strongly submetrizable paratopological group $H$, a continuous homomorphism $\pi$...
of $G_{sr}$ onto $H$ and a continuous function $g : H \to \mathbb{R}$ such that $f \circ \varphi_{G,2} = g \circ \pi$. Thus we have that $f = g \circ (\pi \circ i)$, where $i : G \to G_{sr}$ is the identity mapping. By Theorem 2, this implies that $G$ is simply $sm$-factorizable.

Let $f : X \to Y$ be a continuous mapping. Then $f$ is said to be $d$-open if for each open subset $O$ of $X$ there exists an open subset $V$ of $Y$ such that $f(O)$ is a dense subset of $V$ or, equivalently, $f(O)$ is a subset of the interior of the closure of $f(O)$ in $Y$.

We recall that a Hausdorff paratopological group $G$ has countable Hausdorff number, in symbols $H_{s}(G) \leq \omega$, if for each neighborhood $U$ of the identity $e$ in $G$ there is a countable family $\{U_{n} : n \in \omega\}$ of open neighborhoods of $e$ such that $\bigcap_{n \in \omega} U_{n} U_{n}^{-1} \subseteq U$.

**Lemma 2** Let $G$ be a Hausdorff weakly Lindelöf paratopological group with $H_{s}(G) \leq \omega$. If $G$ is $\omega$-balanced, then for any continuous real-valued function $f$ on $G$ one can find a $d$-open homomorphism $\pi : G \to K$ onto a regular paratopological group $K$ of countable pseudocharacter and a continuous function $h : K \to \mathbb{R}$ such that $f = h \circ \pi$.

**Proof** Let $f : G \to \mathbb{R}$ be a continuous function. By [10, Theorem 2.4], we can find an open continuous homomorphism $\pi : G \to K$ of $G$ onto a Hausdorff paratopological group $K$, a continuous function $h$ on $K$ and a sequence $\{V_{n} : n \in \omega\}$ of open neighborhoods of the identity $e_{K}$ in $K$ such that $f = h \circ \pi$ and $\{e_{K}\} = \bigcap_{n \in \omega} \overline{V_{n}}$. Let $K_{sr}$ be the semiregularization of $K$. Since $K$ is a Hausdorff paratopological group, $K_{sr}$ is a regular paratopological group by Theorem 1. Clearly, the elements of the sequence $\{\text{Int}_{K} \text{cl}_{K} V_{n} : n \in \omega\}$ are open neighborhoods of the identity $e_{K}$ in $K_{sr}$ and $\{e_{K}\} = \bigcap_{n \in \omega} \text{Int}_{K} \text{cl}_{K} V_{n}$. This implies that $K_{sr}$ has countable pseudocharacter. By [24, Lemma 3.5], $h$ remains continuous on $K_{sr}$. Since the identity mapping $i : X \to X_{sr}$ is $d$-open and continuous for any space $X$ [25, Lemma 3.2], one can easily check that $i \circ \pi : G \to K_{sr}$ is a $d$-open continuous homomorphism. Clearly, $f = h \circ (i \circ \pi)$. This completes the proof.

**Corollary 2** (See Theorem 2.9 in [10]) Every totally $\omega$-narrow weakly Lindelöf paratopological group $G$ is simply $sm$-factorizable.

**Proof** Clearly weak Lindelöfness and total $\omega$-narrowness are preserved by continuous maps and continuous homomorphisms, respectively. Theorem 1 implies that $(T_{2}(G))_{sr}$ is a regular weakly Lindelöf and totally $\omega$-narrow paratopological group. According to Propositions 2 and 3, $G$ is simply $sm$-factorizable if and if so is $(T_{2}(G))_{sr}$. Hence we can assume that $G$ is regular.

Take any continuous function $f : G \to \mathbb{R}$. Since every regular totally $\omega$-narrow paratopological group $H$ is $\omega$-balanced (see [15, Proposition 3.8]) and satisfies $H_{s}(H) \leq \omega$ ([14, Theorem 2]), we apply Lemma 2 to find a $d$-open continuous homomorphism $\pi : G \to K$ onto a regular paratopological group $K$ with countable pseudocharacter and a continuous function $h : K \to \mathbb{R}$ such that $f = h \circ \pi$. Clearly, $K$ is totally $\omega$-narrow, so $K$ is strongly submetrizable by Lemma 1. Therefore, $G$ is simply $sm$-factorizable by Theorem 2.

It is well known that a subgroup $H$ of an $\mathbb{R}$-factorizable topological group $G$ is $z$-embedded in $G$ if and only if $H$ is $\mathbb{R}$-factorizable. Now we consider the $z$-embedded subgroups of simply $sm$-factorizable (para)topological groups.
Proposition 4 A (para)topological group $G$ is simply $sm$-factorizable if and only if for each continuous function $f : G \to \mathbb{R}^\omega$, there exist a continuous homomorphism $\pi : G \to H$ onto a (regular) strongly submetrizable (para)topological group $H$ and a continuous function $g : H \to \mathbb{R}^\omega$ such that $f = g \circ \pi$.

Proof According to Theorem 2 it suffices to prove the necessity. Take any continuous function $f : G \to \mathbb{R}^\omega$. For every $i \in \omega$, denote by $p_i$ the projection of $\mathbb{R}^\omega$ to the $i$th factor. Then $p_i \circ f : G \to \mathbb{R}$ is a continuous real-valued function. Since $G$ is simply $sm$-factorizable, there are a (regular) strongly submetrizable (para)topological group $H_i$, a continuous homomorphism $\pi_i$ of $G$ onto $H_i$ and a continuous function $g_i : H_i \to \mathbb{R}$ such that $p_i \circ f = g_i \circ \pi_i$, for each $i \in \omega$. Denote by $\pi$ the diagonal product of the family $\{\pi_i : i \in \omega\}$. Then $\pi : G \to \prod_{i \in \omega} H_i$ is a continuous homomorphism and the image $K = \pi(G)$ is a (regular) strongly submetrizable (para)topological group. For each $i \in \omega$, let $q_i : \prod_{j \in \omega} H_j \to H_i$ be the projection. Then $\pi_i = q_i \circ \pi$. Finally, denote by $g^*$ the Cartesian product of the family $\{g_i : i \in \omega\}$. Then $g^* : \prod_{i \in \omega} H_i \to \mathbb{R}^\omega$ is continuous. Let us verify that $f = g^* \circ \pi$. Indeed, for each $i \in \omega$ and each $x \in G$, we have:

$$p_i \circ f = g_i \circ \pi_i = g_i \circ q_i \circ \pi = p_i \circ g^* \circ \pi.$$ 

Hence the function $g = g^*|_K$ satisfies $f = g \circ \pi$. $\square$

Theorem 3 Let $G$ be a simply $sm$-factorizable (para)topological group. If a subgroup $H$ of $G$ is $z$-embedded in $G$, then $H$ is simply $sm$-factorizable.

Proof Consider a continuous function $f : H \to \mathbb{R}$. Let $\mathcal{V}$ be a countable base of $\mathbb{R}$ consisting of co-zero sets and $U_V = f^{-1}(V)$, where $V \in \mathcal{V}$. Then each element of the family $\mathcal{Y} = \{U_V : V \in \mathcal{V}\}$ is co-zero set in $H$. Since $H$ is $z$-embedded in $G$, for each $V \in \mathcal{V}$ there is a continuous function $g_V : G \to \mathbb{R}$ such that $g_V^{-1}(\mathbb{R}\setminus\{0\}) \cap H = U_V$. Denote by $g$ the diagonal product of the family $\{g_V : V \in \mathcal{V}\}$. Then $g : G \to \mathbb{R}^\mathcal{V}$ is continuous. Since $\mathcal{V}$ is countable and $G$ is simply $sm$-factorizable, it follows from Proposition 4 that one can find a continuous homomorphism $\varphi$ of $G$ onto a strongly submetrizable (para)topological group $K$ and a continuous function $h : K \to \mathbb{R}^\mathcal{V}$ such that $g = h \circ \varphi$. Let us verify the following:

Claim. If $x, y \in G$ and $\varphi(x) = \varphi(y)$, then $f(x) = f(y)$.

If $f(x) \neq f(y)$, then there is $V \in \mathcal{V}$ such that $f(x) \in V$ and $f(y) \notin V$. Hence our choice of the function $g_V$ implies that $g_V(x) \neq 0$ and $g_V(y) = 0$. Therefore, $g(x) \neq g(y)$ and, hence, $\varphi(x) \neq \varphi(y)$ since $g = h \circ \varphi$. This proves the claim.

According to the above Claim, there is a function $j : \varphi(H) \to \mathbb{R}$ such that $f = j \circ \varphi|_H$ (j can fail to be continuous when $\varphi(H)$ is endowed with the topology inherited from $K$). Denote by $L$ the group $\varphi(H)$ endowed with the quotient topology with respect to the homomorphism $\varphi$. Then $\varphi|_H : H \to L$ is open, so the function $j : L \to \mathbb{R}$ is continuous. Clearly, the group $L$ is strongly submetrizable. This completes the proof of the theorem. $\square$

Theorem 3 makes it natural to ask the following question. Let $H$ be a subgroup of a topological group $G$. Is $H$ $z$-embedded in $G$ provided both $H$ and $G$ are simply $sm$-factorizable? It turns out that the answer to the question is “No”.

Springer
Example 1 There exists a separable (hence $\omega$-narrow) simply $sm$-factorizable topological group $G$ which contains a simply $sm$-factorizable subgroup $H$ such that $H$ fails to be $z$-embedded in $G$.

Proof Let $H$ be a separable topological group which fails to be $\mathbb{R}$-factorizable. One can take as $H$ the free Abelian topological group over the Sorgenfrey line $[13]$. Then $H$ has countable cellularity, so [3, Corollary 5.19] implies that $H$ is simply $sm$-factorizable. Every separable topological group is $\omega$-narrow, so $H$ embeds as a topological subgroup into a product $G = \prod_{\alpha \in A} G_{\alpha}$ of second-countable topological groups [2, Theorem 3.4.23]. Since the weight of a separable topological group is at most $c = 2^\omega$, we can assume without loss of generality that $|A| \leq c$. Then the product group $G$ is separable, while [2, Corollary 8.1.15] implies that $G$ is $\mathbb{R}$-factorizable. However, the subgroup $H$ of $G$ cannot be $z$-embedded in $G$ — otherwise $H$ would be $\mathbb{R}$-factorizable by [2, Theorem 8.2.6].

In Theorem 4 below we give a characterization of simply $sm$-factorizable topological groups assuming that the groups are $\omega$-narrow. First we recall the notion of admissible subgroup introduced in [19] (see also [2, Section 5.5]).

Definition 3 Let $G$ be a topological group and $\{U_n : n \in \omega\}$ a sequence of open symmetric neighborhoods of the identity in $G$ such that $U_{n+1}^2 \subseteq U_n$, for each $n \in \omega$. Then $N = \bigcap_{n \in \omega} U_n$ is a subgroup of $G$ which is called admissible.

It follows from the above definition that every admissible subgroup of a topological group $G$ is closed and that every neighborhood of the identity in $G$ contains an admissible subgroup [2, Lemma 5.5.2].

Lemma 3 Let $f : G \to H$ be a continuous homomorphism of topological groups. If $H$ has countable pseudocharacter, then $\ker f$ is an invariant admissible subgroup of $G$.

Proof Clearly, $N = \ker f$ is an invariant subgroup of $G$, so it suffices to show that $N$ is admissible. Since $H$ is a topological group with countable pseudocharacter, one can find a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity in $H$ such that $U_{n+1}^2 \subseteq U_n$, for each $n \in \omega$ and $\{e\} = \bigcap_{n \in \omega} U_n$. Let $V_n = f^{-1}(U_n)$, $n \in \omega$. Then $\{V_n : n \in \omega\}$ is a sequence of open symmetric neighborhoods of the identity in $G$ satisfying $V_{n+1}^2 \subseteq V_n$, for each $n \in \omega$ and $N = \bigcap_{n \in \omega} V_n$. This implies that $N$ is an admissible subgroup of $G$.

Theorem 4 The implication (a) $\Rightarrow$ (b) is valid for every topological group $G$, where

(a) $G$ is simply $sm$-factorizable;
(b) for every continuous function $f : G \to \mathbb{R}$, there exists an invariant admissible subgroup $N$ of $G$ such that $f$ is constant on $gN$, for each $g \in G$.

Furthermore, if $G$ is $\omega$-narrow, then (a) and (b) are equivalent.

Proof Let us show that (a) $\Rightarrow$ (b). Assume that $G$ is a simply $sm$-factorizable topological group. Then, according to Theorem 2, one can find a continuous homomorphism

 Springer
\(\pi\) of \(G\) onto a strongly submetrizable topological group \(H\) and a continuous function 
\(g: H \to \mathbb{R}\) such that \(f = g \circ \pi\). Clearly, \(H\) has countable pseudocharacter, so by 
Lemma 3, the kernel \(N = \pi^{-1}(e)\) of \(\pi\) is an invariant admissible subgroup of \(G\). Since 
f = g \circ \pi, one can easily see that \(f\) is constant on \(xN\), for each \(x \in G\). This 
implies (b).

Assume that \(G\) is an \(\omega\)-narrow group satisfying (b), and let \(f: G \to \mathbb{R}\) be a 
continuous function. Then there is an invariant admissible subgroup \(N\) of \(G\) such that 
f is constant on \(xN\) for each \(x \in G\). Let \(G/N\) be the quotient topological group of 
\(G\) and \(\pi: G \to G/N\) be the quotient homomorphism. Since \(G\) is \(\omega\)-narrow and so is 
every quotient group of \(G\), \(G/N\) is an \(\omega\)-narrow group of countable pseudocharacter 
(see [23, Lemma 2.3.6]). By Lemma 1, \(G/N\) is a strongly submetrizable topological 
group. Observing that \(f\) is constant on \(xN\) for each \(x \in G\) and \(\pi: G \to G/N\) is 
open, one can find a continuous function \(h: G/N \to \mathbb{R}\) such that \(f = h \circ \pi\). From 
Theorem 2 it follows that \(G\) is simply \(sm\)-factorizable.

\[\square\]

We have just established the equivalence of items (a) and (b) in Theorem 4 under the 
additional assumption of the \(\omega\)-narrowness of \(G\). Since every discrete abelian group 
of cardinality \(2^\omega\) is simply \(sm\)-factorizable [3, Proposition 5.15], we see that simply 
\(sm\)-factorizable topological groups need not be \(\omega\)-narrow. Therefore, it is natural to 
ask whether every simply \(sm\)-factorizable topological group is \(\omega\)-balanced. In the next 
example we answer this question in the negative.

**Example 2** Let \(H = GL(\mathbb{R}, 2)\) be the group of \(2 \times 2\) invertible matrices with real 
entries. Denote by \(G\) the group \(H^\omega\) endowed with the box topology. Then \(G\) is 
a strongly submetrizable (hence simply \(sm\)-factorizable) group of countable pseudo-

**Proof** Indeed, denote by \(G_\ast\) the group \(H^\omega\) with the usual product topology. Then the 
identity mapping of \(G\) onto \(G_\ast\) is a continuous isomorphism onto a separable metriz-
able topological group, so \(G\) is strongly submetrizable, hence simply \(sm\)-factorizable 
(Proposition 1). However, \(G\) is not \(\omega\)-balanced. To show this we slightly modify the 
argument from [11, Example 2]. It is well known that the group \(H\) contains two 
sequences \(\{a_n : n \in \omega\}\) and \(\{b_n : n \in \omega\}\) and an element \(z_0 \neq e\) such that \(a_nb_n \to e\) 
and \(b_na_n \to z_0\) for \(n \to \infty\), where \(e\) is the identity element of \(H\) [9, 4.24]. Choose an 
open neighborhood \(U_0\) of \(e\) in \(H\) such that \(z_0 \notin U_0\). It is easy to see that for every open 
neighborhood \(V\) of \(e\) in \(H\), there exists \(k \in \omega\) such that \(b_kVb_k^{-1}\setminus U_0 \neq \emptyset\) — otherwise 
b\(k\) \(ak\) = \(b_k(a_kb_k)b_k^{-1} \in b_kVb_k^{-1} \subseteq U_0\) for all sufficiently big \(k \in \omega\), which contradicts 
our choice of the sequences \(\{a_n : n \in \omega\}\), \(\{b_n : n \in \omega\}\) and of the set \(U_0\).

Clearly \(U = U_0^\omega\) is an open neighborhood of the identity in \(G\). Suppose for a 
contradiction that \(G\) is \(\omega\)-balanced. Then there exists a sequence \(\{V_n : n \in \omega\}\) of open 
neighborhoods of the identity in \(G\) subordinated to \(U\). One can assume without loss of 
generality that every \(V_n\) has the form \(\prod_{k \in \omega} V_{n,k}\), where \(V_{n,k}\) is an open neighborhood 
of \(e\) in \(H\) for each \(k \in \omega\). We have just shown that for every \(n \in \omega\), there exists \(k_n \in \omega\) 
such that \(b_{k_n}V_{n,k_n}b_{k_n}^{-1}\setminus U_0 \neq \emptyset\). Let \(x = (x_n)_{n \in \omega}\) be the element of \(G\) defined by 
x\(n\) = \(b_{k_n}\) for each \(n \in \omega\). Then \(xV_nx^{-1}\setminus U \neq \emptyset\) for each \(n \in \omega\) since the projections 
of the sets \(xV_nx^{-1}\) and \(U\) to the \(n\)th factor are \(b_{k_n}V_{n,k_n}b_{k_n}^{-1}\) and \(U_0\), respectively. This
contradicts our choice of the sequence \( \{ V_n : n \in \omega \} \). Therefore, the group \( G \) is not \( \omega \)-balanced.

The following result partially answers Problem 2.

**Corollary 3** Let \( \pi : G \to H \) be a continuous homomorphism of topological groups such that for each invariant admissible subgroup \( K \subseteq G \), the image \( \pi(K) \) contains an invariant admissible subgroup of \( H \). If \( G \) is simply sm-factorizable and \( H \) is \( \omega \)-narrow, then \( H \) is simply sm-factorizable.

**Proof** Let \( f : H \to \mathbb{R} \) be any continuous function. Since \( G \) is simply sm-factorizable, it follows from Theorem 4 that there is an invariant admissible subgroup \( N \) of \( G \) such that for each \( x \in G \), \( f \circ \pi \) is constant on \( xN \). According to our assumption \( \pi(N) \) contains an invariant admissible subgroup \( K \) of \( H \). Since \( \pi \) is a homomorphism and for each \( x \in G \), \( f \circ \pi \) is constant on \( xN \), one can easily verify that for each \( y \in H \), \( f \) is constant on \( yK \). Observing that \( H \) is \( \omega \)-narrow, from Theorem 4 it follows that \( H \) is simply sm-factorizable. \( \square \)

**Remark 1** The condition on the homomorphism \( \pi \) in Corollary 3 is quite strong. It can easily fail, even if the homomorphism \( \pi \) is open. Indeed, according to [2, Theorem 7.6.18], every Hausdorff topological group \( H \) is a quotient group of a topological group \( G \) with countable pseudocharacter. So we can take \( H \) to be a Hausdorff topological group with \( \psi(H) > \omega \) and find an open continuous surjective homomorphism \( \pi : G \to H \), where \( G \) is a topological group of countable pseudocharacter. Then \( K = \{ e \} \) is an invariant admissible subgroup of \( G \), where \( e \) is the identity element of \( G \). Clearly \( \pi(K) = \{ e_H \} \) does not contain any admissible subgroup of \( H \). It also follows from Proposition 5 below that \( G \) cannot be simply sm-factorizable if \( |H| > \aleph_0 \). \( \square \)

As we mentioned after Definition 3, every neighborhood of the identity in a topological group contains an admissible subgroup [2, Lemma 5.5.2]. This conclusion can be strengthened for \( \omega \)-balanced topological groups:

**Lemma 4** Every neighborhood of the identity \( e \) in an \( \omega \)-balanced topological group \( G \) contains an invariant admissible subgroup.

**Proof** Every \( \omega \)-balanced topological group is a subgroup of a topological product of first countable topological groups [2, Theorem 5.1.9], so for each open neighborhood \( U \) of \( e \) in \( G \), one can find a continuous homomorphism \( p \) on \( G \) onto a first countable topological group \( H \) and an open neighborhood \( V \) of the identity \( e_H \) in \( H \) satisfying \( p^{-1}(V) \subseteq U \). Clearly \( K = \{ e_H \} \) is an invariant admissible subgroup of \( H \) satisfying \( K \subseteq V \), so \( p^{-1}(K) \) is an invariant admissible subgroup of \( G \) contained in \( U \).

We recall that \( X \) is a \( P \)-space if every \( G \)-set in \( X \) is open. Similarly, a (para)topological group \( G \) is said to be a \( P \)-group if the underlying space of \( G \) is a \( P \)-space. The following result gives a partial answer to Problem 3.

**Corollary 4** If an \( \omega \)-narrow topological group \( H \) is a quotient group of a simply sm-factorizable \( P \)-group, then \( H \) is \( \mathbb{R} \)-factorizable.
Proof Let $\pi : G \to H$ be a quotient homomorphism and $G$ be a simply $sm$-factorizable $P$-group. Then $H$ is a $P$-group by [2, Lemma 4.4.1 c)]. Since every $\omega$-narrow simply $sm$-factorizable $P$-group is $\mathbb{R}$-factorizable [3, Proposition 5.23], it suffices to show that $H$ is simply $sm$-factorizable. Let $f$ be a continuous real-valued function on $H$. Since $G$ is simply $sm$-factorizable, Theorem 4 implies that there is an invariant admissible subgroup $N$ of $G$ such that for each $x \in G$, $f \circ \pi$ is constant on $xN$. Therefore, $f$ is constant on $y\pi(N)$ for each $y \in H$. Observing that $G$ is a $P$-space and $\pi$ is open, we see that $\pi(N)$ is an open neighborhood of the identity in $H$. Clearly $H$ is $\omega$-balanced, so Lemma 4 implies that $\pi(N)$ contains an invariant admissible subgroup $K$ of $H$. It is clear that $f \circ \pi$ is constant on $yK$ for each $y \in H$, so $H$ is simply $sm$-factorizable by Theorem 4, because $H$ is $\omega$-narrow. \hfill \qed

Proposition 5 Every regular simply $sm$-factorizable (para)topological group $G$ of countable pseudocharacter admits a continuous isomorphic bijection onto a separable metrizable (para)topological group. Therefore, $G$ is strongly submetrizable and satisfies $|G| \leq \mathfrak{c}$.

Proof Every regular paratopological group is completely regular [4]. Let $\{U_n : n \in \omega\}$ be a family of open neighborhoods of the identity $e$ in $G$ such that $\{e\} = \cap_{n \in \omega} U_n$. For every $n \in \omega$, one can find a continuous function $f_n : G \to \mathbb{R}$ such that $f_n(e) = 0$ and $f_n(x) = 1$ for each $x \in G \setminus U_n$. Denote by $f$ the diagonal product of the family $\{f_n : n \in \omega\}$. Then $f : G \to \mathbb{R}^\omega$ is continuous. Since $G$ is simply $sm$-factorizable, Proposition 4 implies that we can find a continuous homomorphism $p : G \to H$ onto a strongly submetrizable (para)topological group $H$ and a continuous map $h : H \to \mathbb{R}^\omega$ such that $f = h \circ p$. Note that $h(e_H) = f(e) = 0 = (0, 0, \ldots)$.

We claim that $p$ is a continuous isomorphic bijection. Indeed, it suffices to show that $\text{ker } p = \{e\}$. Take an element $x \in G \setminus \{e\}$. Then there is $n \in \omega$ such that $x \notin U_n$, so $f_n(x) = 1$. Therefore, $f(x) \neq 0$. Hence $x \notin \text{ker } p$ because $f = h \circ p$ and $h(e_H) = 0$.

Since every separable metrizable space has cardinality at most $\mathfrak{c}$ and $p$ is a bijection, we see that $|G| \leq \mathfrak{c}$. \hfill \qed

3 Completions of simply $sm$-factorizable topological groups

In this section we study the Dieudonné and Hewitt–Nachbin completions of simply $sm$-factorizable topological groups. In Theorem 5 we answer Problem 1 affirmatively. In fact, we prove a stronger result: Every Hausdorff simply $sm$-factorizable topological group is completion friendly.

A subset $Y$ of a space $X$ is $G_\delta$-dense in $X$ if every nonempty $G_\delta$-set in $X$ intersects $Y$. The biggest set $Z \subset X$ which contains $Y$ as a $G_\delta$-dense subset is called the $G_\delta$-closure of $Y$ in $X$. A space $X$ is called Moscow if for each open set $U$ of $X$, the closure $\overline{U}$ of $U$ is the union of a family of $G_\delta$-sets in $X$.

Let $\varrho G$ be the Raïkov completion of a topological group $G$. We denote the $G_\delta$-closure of $G$ in $\varrho G$ by $\varrho_\omega G$. It is easy to verify that $\varrho_\omega G$ is a subgroup of $\varrho G$ (see [2, Section 6.4]).

Proposition 6 Let $H$ be a $G_\delta$-dense simply $sm$-factorizable subgroup of a topological group $G$. Then $H$ is $C$-embedded in $G$. 

 Springer
Proof Take any continuous function $f : H \to \mathbb{R}$. According to Theorem 2, we can find a continuous homomorphism $\pi$ of $H$ onto a strongly submetrizable topological group $F$ and a continuous function $g : F \to \mathbb{R}$ such that $f = g \circ \pi$. Let $\varrho G$ and $\varrho F$ be the Raïkov completions of $G$ and $F$, respectively. Since $H$ is dense in $\varrho G$, $\pi$ extends to a continuous homomorphism $\tilde{\pi} : \varrho G \to \varrho F$. Denote by $\varrho_\omega G$ and $\varrho_\omega F$ the $G_\delta$-closures of $G$ and $F$ in $\varrho G$ and $\varrho F$, respectively. Since $F$ is strongly submetrizable, it has countable pseudocharacter and, hence, $F$ is a Moscow space [2, Corollary 6.4.11(1)].

It is well known that if a Moscow space $Y$ is a $G_\delta$-dense subspace of a homogeneous space $X$, then $X$ is also a Moscow space and $Y$ is $C$-embedded in $X$ [2, Theorem 6.1.8]. Since $\varrho_\omega F$ is a topological group and $F$ is a Moscow space, $F$ is $C$-embedded in $\varrho_\omega F$. Hence $g$ extends to a continuous function $\tilde{g} : \varrho_\omega F \to \mathbb{R}$. Since $H$ is $G_\delta$-dense in $G$, we have the inclusion $\tilde{\pi}(G) \subseteq \varrho_\omega F$. Thus $\tilde{g} \circ \tilde{\pi} | G$ is a continuous extension of $f$. This proves that $H$ is $C$-embedded in $G$. $\square$

A topological group $G$ is called completion friendly if $G$ is $C$-embedded in $\varrho_\omega G$. Every completion friendly group is a $PT$-group [2, Proposition 6.5.17].

The following result gives a positive answer to Problem 2.

Theorem 5 Let $G$ be a Hausdorff simply $sm$-factorizable topological group. Then the equalities $\mu G = \varrho_\omega G = \nu G$ hold and, therefore, $G$ is completion friendly.

Proof Since every simply $sm$-factorizable topological group $H$ satisfies $ib(H) \leq c$ [3, Proposition 5.14], we have that $ib(G) \leq c$. According to [2, Theorem 5.4.10], the inequality $c(H) \leq 2^{ib(H)}$ holds for every topological group $H$. We conclude, therefore, that $c(G) \leq 2^c$. From [2, Theorem 6.2.2] it follows that the cardinal number $2^c$ is Ulam non-measurable, so the cardinality of every discrete family of open sets in $G$ is Ulam non-measurable and hence the equality $\mu G = \nu G$ holds by [2, Lemma 8.3.1].

Note that the space $\varrho_\omega G$ is Dieudonné complete [2, Proposition 6.5.2]. Since $\mu G = \nu G$ and $G$ is $C$-embedded in the Dieudonné complete group $\varrho_\omega G$ by Proposition 6, we conclude that $\mu G = \varrho_\omega G = \nu G$. $\square$

Corollary 5 Every Hausdorff simply $sm$-factorizable topological group $G$ is a $PT$-group, so the Dieudonné completion $\mu G$ of the space $G$ admits a natural structure of topological group containing $G$ as a dense subgroup.

By Corollary 1 and Theorem 5, we obtain the following result:

Corollary 6 (See Proposition 2.4 of [17]) Every Hausdorff weakly Lindelöf topological group $G$ satisfies the equalities $\mu G = \varrho_\omega G = \nu G$ and, therefore, $G$ is completion friendly.

We also present an alternative proof of [3, Theorem 5.21]:

Corollary 7 Let $G$ be a simply $sm$-factorizable topological group which is $C$-embedded in a regular Lindelöf space $X$. Then the group $G$ is $\mathbb{R}$-factorizable and the closure of $G$ in $X$ is a topological group containing $G$ as a dense subgroup.

Proof We can assume without loss of generality that $G$ is dense in $X$. Then $\nu G = X$. Since every $C$-embedded subspace of a regular Lindelöf is pseudo-$\omega_1$-compact [3, Springer]
Corollary 3.3], the space \( G \) is pseudo-\( \omega_1 \)-compact. It now follows from [2, Corollary 8.3.3] that \( X = \nu G = \mu G \), i.e. the Hewitt-Nachbin and Dieudonné completions of \( G \) coincide. By Corollary 5, \( X \) is homeomorphic to a Lindelöf topological group containing \( G \) as a dense subgroup. Hence \( X \) is \( \mathbb{R} \)-factorizable by [2, Theorem 8.1.6]. So \( G \) is \( \mathbb{R} \)-factorizable as a \( C \)-embedded subgroup of the \( \mathbb{R} \)-factorizable topological group \( X \) (see [8, Theorem 3.2]).

4 Completions of simply sm-factorizable paratopological groups

In this section we consider the Dieudonné and Hewitt–Nachbin completions of simply sm-factorizable paratopological groups.

The following result follows from Lemmas 3 and 4 of [18]:

Lemma 5 Let \( G \) be a Hausdorff paratopological group satisfying \( Hs(G) \leq \omega \). Then the \( G_\delta \)-closure of an arbitrary subgroup \( H \) of \( G \) is again a subgroup of \( G \).

Lemma 6 The topological product \( G = \prod_{\alpha \in A} G_\alpha \) of any family of strongly submetrizable paratopological groups satisfies \( Hs(G) \leq \omega \).

Proof According to [20, Proposition 2.3], the class of paratopological groups with countable Hausdorff number is closed under taking arbitrary products and subgroups. Therefore, it suffices to show that any strongly submetrizable paratopological group \( H \) satisfies \( Hs(H) \leq \omega \). Since \( H \) is strongly submetrizable, there exists a continuous isomorphism \( i : H \to H' \) onto a separable metrizable paratopological group \( H' \).

Let \( \{ U_n : n \in \omega \} \) be a local base at the identity \( e' \) in \( H' \). Since \( H' \) is metrizable, we have the equality \( \{ e' \} = \bigcap_{n \in \omega} U_n U_n^{-1} \). Let \( V_n = i^{-1}(U_n), n \in \omega \). For each neighborhood \( V \) of the identity \( e \) in \( H \), we have that \( \{ e \} = \bigcap_{n \in \omega} V_n V_n^{-1} \subseteq V \). This implies that \( Hs(H) \leq \omega \).

Theorem 6 Let \( G \) be a regular simply sm-factorizable paratopological group. Then the realcompactification \( \nu G \) of the space \( G \) admits a natural structure of paratopological group containing \( G \) as a dense subgroup and \( \nu G \) is also simply sm-factorizable.

Proof Since every regular paratopological group is a Tychonoff space [4, Corollary 5], so is \( G \). Let \( \{ f_\alpha : \alpha \in A \} \) be the family of continuous real-valued functions on \( G \). Since \( G \) is simply sm-factorizable, it follows from Theorem 2 that for each \( \alpha \in A \), there exist a continuous homomorphism \( \pi_\alpha \) of \( G \) onto a regular strongly submetrizable paratopological group \( H_\alpha \) and a continuous function \( g_\alpha : H_\alpha \to \mathbb{R} \) such that \( f_\alpha = g_\alpha \circ \pi_\alpha \). Since the family \( \{ f_\alpha : \alpha \in A \} \) separates point and closed sets in \( G \), so does \( \{ \pi_\alpha : \alpha \in A \} \). Therefore, the diagonal product of the family \( \{ \pi_\alpha : \alpha \in A \} \), denoted by \( \pi \), is a topological isomorphism of \( G \) onto the subgroup \( \pi(G) \subseteq \prod_{\alpha \in A} H_\alpha \).

In what follows we identify \( G \) with the subgroup \( \pi(G) \) of \( \Pi \). Then the equality \( f_\alpha = g_\alpha \circ \pi_\alpha \) acquires the form \( f_\alpha = g_\alpha \circ p_\alpha |_G \), where \( p_\alpha \) is the projection of \( \Pi \) to \( H_\alpha \).

By Lemma 6, we have that \( Hs(\Pi) \leq \omega \). Therefore, by Lemma 5, the \( G_\delta \)-closure of \( G \) in \( \Pi \), denoted by \( H \), is a subgroup of \( \Pi \). We claim that the subspace \( H \) of \( \Pi \) is
realcompact. Indeed, for each \( \alpha \in A \), \( H_\alpha \) is a strongly submetrizable paratopological group, so \( H_\alpha \) admits a coarser separable metrizable paratopological group topology. Hence the space \( H_\alpha \) is Dieudonné complete \([2, \text{Proposition 6.10.8}]\) and \( |H_\alpha| \leq c \). In particular, the cellularity of \( H_\alpha \) is less than or equal to \( c \), which is Ulam non-measurable and, therefore, the space \( H_\alpha \) is realcompact by \([2, \text{Proposition 6.5.18}]\). Hence the space \( \Pi = \prod_{\alpha \in A} H_\alpha \) is also realcompact. It also follows from the definition of \( H \) that the complement \( \Pi \setminus H \) is the union of family of \( G_\delta \)-sets in \( \Pi \). Further, every \( G_\delta \)-set in \( \Pi \) is the union of a family of zero-sets in \( \Pi \), and the complement \( \Pi \setminus Z \) is realcompact, for each zero-set \( Z \) in \( \Pi \) (see \([6, \text{Corollary 3.11.8}]\)). By \([6, \text{Corollary 3.11.7}]\), the intersection of a family of realcompact subspaces of a space is realcompact. Therefore, \( H \) is realcompact as the intersection of a family of cozero-sets in \( \Pi \).

Let us show that \( G \) is \( C \)-embedded in \( H \), which implies that \( \nu G = H \) (see \([7, \text{Theorem 8.6}]\)). Indeed, for each continuous real-valued function \( f \) on \( G \), there exists \( \alpha \in A \) such that \( f = f_\alpha \). It follows from \( f_\alpha = g_\alpha \circ p_\alpha |_G \) that \( g_\alpha \circ p_\alpha |_H : H \to \mathbb{R} \) is a continuous extension of \( f \) over \( H \). We have thus proved that the realcompactification \( \nu G \) of \( G \) admits a natural structure of a topological group containing \( G \) as a dense subgroup.

It remains to verify that the paratopological group \( H \) is simply \( sm \)-factorizable. Take any continuous function \( g : H \to \mathbb{R} \) and denote by \( f \) the restriction of \( g \) to \( G \). Then we can find \( \alpha \in A \) such that \( f = f_\alpha \), whence it follows that \( f_\alpha = g_\alpha \circ p_\alpha |_G \). Since \( \pi_\alpha(H) = H_\alpha \) and \( H \) is Hausdorff, we have the equality \( g = g_\alpha \circ p_\alpha |_H \). Thus the continuous homomorphism \( p_\alpha |_H \) of \( H \) to \( H_\alpha \) factorizes the function \( g \). Therefore, by Theorem 2, \( H \) is simply \( sm \)-factorizable.

**Lemma 7** Let \( G \) be a regular simply \( sm \)-factorizable paratopological group. Then the topological group \( G^* \) associated to \( G \) satisfies \( ib(G^*) \leq c \). Therefore, \( c(G) \leq c(G^*) \leq 2^c \) and the equality \( \nu G = \mu G \) is valid.

**Proof** Let \( U \) be an arbitrary neighborhood of the identity \( e \) in the group \( G^* \). It follows from the definition of \( G^* \) that there exists an open neighborhood \( V \) of the identity in \( G \) such that \( V \cap V^{-1} \subseteq U \). Every regular paratopological group is completely regular \([4, \text{Corollary 5}]\), so there exists a cozero set \( W \) in \( G \) satisfying \( e \in W \subseteq V \). Since \( G \) is simply \( sm \)-factorizable, we can find a continuous homomorphism \( \pi : G \to H \) onto a separable metrizable paratopological group \( H \) such that \( W = \pi^{-1}(\pi(W)) \). Hence \( W^{-1} = \pi^{-1}(\pi(W^{-1})) \).

Combining the two equalities we see that

\[
W \cap W^{-1} = \pi^{-1}(\pi(W \cap W^{-1})).
\]

It is clear that \( |H| \leq c \). Choose a subset \( A \) of \( G \) with \( |A| \leq c \) such that \( \pi(A) = H \). Applying (1) and taking into account that \( e \in W \cap W^{-1} \neq \emptyset \) we deduce that \( A \cdot (W \cap W^{-1}) = G = (W \cap W^{-1}) \cdot A \). The latter equalities together with \( e \in W \cap W^{-1} \subseteq V \cap V^{-1} \subseteq U \) imply that \( A \cdot U = G = U \cdot A \). Hence the group \( G^* \) satisfies \( ib(G^*) \leq c \).

We now apply \([2, \text{Theorem 5.4.10}]\) to conclude that \( c(G^*) \leq 2^{ib(G^*)} \leq 2^c \). Since \( G \) is a continuous image of \( G^* \), we also have \( c(G) \leq c(G^*) \leq 2^c \). Finally, the cardinal number \( 2^c \) is Ulam non-measurable by \([2, \text{Theorem 6.2.2}]\). So the cardinality of every
discrete family of open sets in $G$ is Ulam non-measurable and the equality $\mu G = \nu G$ holds by [2, Lemma 8.3.1].

**Corollary 8** Let $G$ be a regular simply $sm$-factorizable paratopological group. Then the equality $\mu G = \nu G$ holds. Furthermore, the space $\mu G$ admits a natural structure of paratopological group containing $G$ as a dense subgroup and $\mu G$ is also simply $sm$-factorizable.

**Proof** Lemma 7 implies that the cardinality of every discrete family of open sets in $G$ is at most $2^c$ and, hence, is Ulam non-measurable. So the equality $\mu G = \nu G$ holds by [2, Lemma 8.3.1]. Therefore, both conclusions of the corollary follow directly from Theorem 6.

**Corollary 9** Let $G$ be a regular totally $\omega$-narrow and weakly Lindelöf paratopological group. Then the equality $\mu G = \nu G$ holds. Furthermore, the Dieudonné completion $\mu G$ of the space $G$ admits a natural structure of paratopological group containing $G$ as a dense subgroup and $\mu G$ is simply $sm$-factorizable.

**Proof** The group $G$ is simply $sm$-factorizable, by Corollary 2. Hence the required conclusions follow from Corollary 8.

**Problem 4** Does Lemma 7 remain valid without the assumption of the regularity of $G$?

One can try to improve one of the conclusions of Lemma 7 as follows:

**Problem 5** Does every Hausdorff (regular) simply $sm$-factorizable paratopological group $G$ satisfy $c(G) \leq c$?

It is worth mentioning that the ‘Hausdorff’ and ‘regular’ versions of Problem 5 are equivalent since every paratopological group $G$ satisfies $c(G) = c(G_{sr})$ (see [22, Proposition 2.2]) and the paratopological group $G_{sr}$ is regular provided $G$ is Hausdorff.

The next result is close to Corollary 7. In it, under stronger assumptions, we extend some properties of topological groups to the wider class of paratopological groups.

**Corollary 10** Let $G$ be a simply $sm$-factorizable paratopological group which is $C$-embedded in a regular space $X$. If $X^2$ is Lindelöf, then the group $G$ is $\mathbb{R}$-factorizable. In addition, if $X$ is a Lindelöf $\Sigma$-space, then all subgroups of $G$ have countable cellularity and for every family $\gamma$ of $G_{\delta}$-sets in $G$, the closure of $\bigcup \gamma$ is a zero-set in $G$.

**Proof** We can assume without loss of generality that $G$ is dense in $X$ because the class of Lindelöf $\Sigma$-spaces is hereditary w.r.t. taking closed subspaces. Then $\nu G = X$. By Corollary 8, $X = \nu G = \mu G$ and $X$ is homeomorphic to a paratopological group containing $G$ as a dense paratopological subgroup. Also, the topological group $X^*$ associated to $X$ is topologically homeomorphic to a closed subspace of the space $X^2$ [1, Lemma 2.2] and, hence, $X^*$ is Lindelöf. Applying [16, Theorem 3.6] we see that $X$ is $\mathbb{R}$-factorizable. So $G$ is $\mathbb{R}$-factorizable as a $C$-embedded subgroup of the $\mathbb{R}$-factorizable paratopological group $X$ (see [26, Theorem 3.2]).
Assume that $X$ is a Lindelöf $\Sigma$-space. Then $X^2$ is also a Lindelöf $\Sigma$-space, so $G$ is $\mathbb{R}$-factorizable. Let $G^*$ be the topological group associated to $G$. It is clear that the identity embedding of $G$ to $X$ is topological isomorphism of $G^*$ onto a subgroup of the topological group $X^*$. Since $X^*$ is homeomorphic to a closed subspace of $X^2$, we deduce that $X^*$ is a Lindelöf $\Sigma$-space. Every subgroup $H$ of $X^*$ has countable cellularity by [2, Corollary 5.3.21]. If $K$ is a subgroup of $G$, then $H = j^{-1}(K)$ is a subgroup of both $G^*$ and $X^*$, where $j$ is the identity mapping of $G^*$ onto $G$. It follows from the continuity of $j$ that $c(K) \leq c(H) \leq \omega$.

Finally, let $\gamma$ be a family of $G_\delta$-sets in $G$. Since $G$ is a Tychonoff space, each element of $\gamma$ is the union of a family of zero-sets. Hence we can assume that $\gamma$ consists of zero-sets in $G$. By our assumptions, $G$ is a dense $C$-embedded subspace of $X$, so the closure in $X$ of every zero-set in $G$ is a zero-set in $X^*$. Let $\gamma^* = \{ \text{cl}_X P : P \in \gamma \}$. Then $\gamma^*$ is a family of zero-sets in $X$ and [15, Theorem 4.2] implies that the closure of $\bigcup \gamma^*$ in $X$ is a zero-set. Since $\bigcup \gamma$ is dense in $\bigcup \gamma^*$, we conclude that $\text{cl}_G \bigcup \gamma$ is a zero-set in $G$.

Remark 2 In Corollary 10, the Lindelöfness of $X^2$ cannot be weakened to the Lindelöf-ness of $X$. Indeed, the Sorgenfrey line $\mathbb{S}$ is a simply $sm$-factorizable paratopological group by Theorem 2. Also, $\mathbb{S}$ is Lindelöf but not $\mathbb{R}$-factorizable [24, Remark 3.22].

Corollary 11 (See Theorem 1 of [18]) Let $G$ be a regular $\mathbb{R}$-factorizable paratopological group. Then the Dieudonné completion of the space $G$, $\mu G$, admits a natural structure of paratopological group containing $G$ as a dense subgroup and $\mu G$ is $\mathbb{R}$-factorizable.

Proof Every regular paratopological group is a Tychonoff space [4, Corollary 5]. Since $G$ is $\mathbb{R}$-factorizable (hence, simply $sm$-factorizable), we can apply Corollary 8 to conclude that both multiplication and inversion on $G$ admit continuous extensions to $\mu G$ making the latter space into a paratopological group. The $\mathbb{R}$-factorizability of the group $\mu G$ can be deduced as in [18].

Our proof of Theorem 7 below requires the next simple fact:

Lemma 8 Let $\mathcal{N}$ be a family of subgroups of a paratopological group $G$ such that every neighborhood of the identity $e$ in $G$ contains an element of $\mathcal{N}$. Then every neighborhood of $e$ in the topological group $G^*$ associated to $G$ also contains an element of $\mathcal{N}$.

Proof Take an arbitrary neighborhood $U$ of $e$ in $G^*$. It follows from the definition of the topology of $G^*$ that there exists an open neighborhood $V$ of $e$ in $G$ such that $V \cap V^{-1} \subset U$. By the assumptions of the lemma, there exists $N \in \mathcal{N}$ satisfying $N \subset V$. Since $N$ is a subgroup of $G$, we have that $N = N^{-1} \subset V^{-1}$. Therefore, $N \subset V \cap V^{-1} \subset U$.

Theorem 7 Let $G$ be a regular paratopological group. If the topological group $G^*$ associated to $G$ is simply $sm$-factorizable and $\omega$-narrow, then so is $G$.

Proof Let $\mathcal{N}$ be the family of closed invariant subgroups $N$ of $G$ such that the quotient paratopological group $G/N$ is strongly submetrizable.
Claim 1. The family $\mathcal{N}$ is closed under countable intersections.

Take any countable subfamily $\mathcal{C} = \{N_k : k \in \omega\}$ of $\mathcal{N}$. For every $k \in \omega$, let $\varphi_k : G \to G/N_k$ be the quotient homomorphism. Then the diagonal product $\psi : G \to \prod_{k \in \omega} G/N_k$ of the family $\{\varphi_k : k \in \omega\}$ is a continuous homomorphism. Since every group $G/N_k$ is strongly submetrizable, so is the countable product $\prod_{k \in \omega} G/N_k$. Further, the subgroup $\psi(G)$ of $\prod_{k \in \omega} G/N_k$ is also strongly submetrizable. Clearly, the kernel of $\psi$ satisfies $\ker \psi = \bigcap \mathcal{C}$. It is easy to see that the quotient group $G/\ker \psi$ is strongly submetrizable and the latter implies that $\bigcap \mathcal{C} = \ker \psi \in \mathcal{N}$. This proves Claim 1.

Let $\mathcal{N} = \{N_i : i \in I\}$ and $\varphi_i : G \to G/N_i$ be the quotient homomorphism, where $i \in I$.

Claim 2. The diagonal product $\varphi : G \to \varphi(G) \subseteq \prod = \prod_{i \in I} G/N_i$ of the family $\{\varphi_i : i \in I\}$ is a topological isomorphism. In particular, every neighborhood of the identity in $G$ contains an element of $\mathcal{N}$.

Since $G$ is regular and totally $\omega$-narrow, it is $\omega$-balanced [15, Proposition 3.8] and satisfies $\text{Ir}(G) \leq \omega$ [14, Theorem 2]. Therefore, it follows from Theorem 3.6 and Lemma 3.7 of [20] that for each open neighborhood $U$ of the identity $e$ in $G$, one can find a continuous homomorphism $p : G \to H$ onto a separable metrizable paratopological group $H$ and an open neighborhood $V$ of the identity in $H$ such that $p^{-1}(V) \subseteq U$. Hence the kernel of $p$ belongs to $\mathcal{N}$ and satisfies $\ker p \subset U$. This implies that the family $\{\varphi_i : i \in I\}$ separates the points and closed sets in $G$ and, therefore, the diagonal product of this family, say, $\varphi$ is a topological isomorphism of $G$ onto $\varphi(G)$, as claimed.

Claim 3. Every $G_\delta$-set $P$ in $G^*$ with $e \in P$ contains some $N \in \mathcal{N}$.

Take a $G_\delta$-set $P$ in $G^*$ containing the identity $e$. Let $\{U_k : k \in \omega\}$ be a family of neighborhoods of the identity in $G^*$ such that $P = \bigcap_{k \in \omega} U_k$. It follows from Lemma 8 and Claim 2 that for every $k \in \omega$, there exists $N_k \in \mathcal{N}$ such that $N_k \subset U_k$. Then by Claim 1, $N = \bigcap_{k \in \omega} N_k$ is an element of $\mathcal{N}$ satisfying $N \subset \bigcap_{k \in \omega} U_k = P$.

Take any continuous function $f : G \to \mathbb{R}$. Then $f$ remains continuous on $G^*$. Since $G^*$ is simply $sm$-factorizable, it follows from item (2) of Theorem 2 that we can find a continuous homomorphism $\pi$ of $G^*$ onto a strongly submetrizable topological group $H$ and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ \pi$. Since $H$ is strongly submetrizable, the identity $e_H$ of $H$ is a $G_\delta$-set in $H$. Therefore, $\ker \pi$ is a $G_\delta$-set in $G^*$. Then Claim 3 implies that there is an $N_i \in \mathcal{N}$ such that $N_i \subset \ker \pi$. Since $f$ is constant on each coset $x \cdot \ker \pi$ and the groups $G$ and $G^*$ share the same underlying set, we see that $f$ is constant on $xN_i$ for each $x \in G$. Therefore, there exists a function $h : G/N_i \to \mathbb{R}$ such that $f = h \circ \varphi_i$, where $G/N_i$ is endowed with the quotient topology and $\varphi_i$ is the quotient homomorphism of $G$ onto $G/N_i$. Since $\varphi_i$ is continuous and open, $h$ is continuous. Clearly, $G/N_i$ is strongly submetrizable, so $G$ is simply $sm$-factorizable by Theorem 2. Finally, $G$ is $\omega$-narrow as a continuous homomorphic image of $G^*$.

Theorem 7 makes it natural to ask the following questions:

Problem 6 Let $G$ be a regular simply $sm$-factorizable paratopological group. Is the topological group $G^*$ associated to $G$ simply $sm$-factorizable? What if $G$ is a regular $\mathbb{R}$-factorizable paratopological group?
**Question 1** Can the requirement of ω-narrowness of $G^{*}$ be dropped in Theorem 7? Does the ω-narrowness of $G$ suffice?

In Example 3 we answer both parts of the above question in the negative. First we present an auxiliary lemma in which $\mathbb{Z}$ stands for the discrete additive group of integers.

**Lemma 9** There exists a countable dense subgroup $S$ of the product $\mathbb{Z}^{c}$ such that for every $x \in S$, every finite set $C \subseteq c$ and every $k \in \mathbb{Z}$, one can find $s \in S$ satisfying $s(\alpha) \leq x(\alpha)$ for each $\alpha \in c \setminus C$ and $s(\alpha) \leq k$ for each $\alpha \in C$.

**Proof** Our argument is a modification of the proof of the Hewitt-Marczewski-Pondiczery theorem as presented in [6, Theorem 2.3.15]. Let $\mu$ be a separable metrizable topology on the index set $c$. Denote by $\mathscr{A}$ a countable base for $(c, \mu)$. Take a countable dense subset $R_{0}$ of the space $\Pi = \mathbb{Z}^{c}$ and let $S_{0} = \langle R_{0} \rangle$. Assume that we have defined countable subgroups $S_{0} \subseteq \cdots \subseteq S_{n}$ of $\Pi$. For every $x \in S_{n}$, every finite disjoint subfamily $v$ of $\mathscr{A}$ and every $k \in \mathbb{Z}$, we define an element $y_{x,v,k} \in \Pi$ by the rule

$$y_{x,v,k}(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha \in c \setminus \bigcup v; \\ \min\{k, x(\alpha)\} & \text{if } \alpha \in \bigcup v. \end{cases}$$

Let $R_{n+1} = \{y_{x,v,k} : x \in S_{n}, v \in [\mathscr{A}]^{<\omega}, k \in \mathbb{Z}\}$. Clearly $R_{n+1}$ is countable, so $S_{n+1} = S_{n} + \langle R_{n+1} \rangle$ is countable as well.

We claim that the subgroup $S = \bigcup_{n \in \omega} S_{n}$ of $\Pi$ is as required. Note that $S$ is countable and dense in $\Pi$ since $S_{0} \subseteq S$. Take an element $x \in S$, a finite subset $C = [\alpha_{1}, \ldots, \alpha_{r}]$ of $c$ and an integer $k$. Then $x \in S_{n}$ for some $n \in \omega$. Since the space $(c, \mu)$ is Hausdorff, we can find pairwise disjoint elements $U_{1}, \ldots, U_{r}$ of $\mathscr{A}$ such that $\alpha_{i} \in U_{i}$ for each $i \leq r$. Let $v = \{U_{1}, \ldots, U_{r}\}$. Then the point $s = y_{x,v,k} \in R_{n+1} \subseteq S_{n+1} \subseteq S$ satisfies $s(\alpha) \leq x(\alpha)$ for each $\alpha \in c \setminus C$ and $s(\alpha) \leq k$ for each $\alpha \in C$. This completes the proof. □

**Example 3** There exists a regular $\omega$-narrow paratopological Abelian group $G$ with $|G| = c$ such that the topological group $G^{*}$ associated to $G$ is discrete (hence simply sm-factorizable by [3, Proposition 5.15]), but $G$ fails to be simply sm-factorizable.

**Proof** We modify the construction described in [20, Example 2.9]. In fact, our group $G$ will be a (dense) subgroup of the paratopological group constructed there.

Let $\mathbb{Z}$ be the discrete additive group of the integers and $\Pi = \mathbb{Z}^{c}$ be the product of $c = 2^{\omega}$ copies of $\mathbb{Z}$. For every $x \in \Pi$, let

$$\text{supp}(x) = \{\alpha \in c : x(\alpha) \neq 0\}.$$ 

Then $\sigma = \{x \in \Pi : |\text{supp}(x)| < \omega\}$ is a subgroup of $\Pi$ which is called the $\sigma$-product of $c$ copies of $\mathbb{Z}$. It is clear that $|\sigma| = c$. Let $S$ be a countable dense subgroup of $\Pi$ as in Lemma 9 ($\Pi$ carries the Tychonoff product topology). Clearly $G = \sigma + S$ is a subgroup of $\Pi$.

For every $A \subseteq c$, we define a subset $U_{A}$ of $G$ by

$$U_{A} = \{x \in G : x(\alpha) = 0 \text{ for each } \alpha \in A \text{ and } x(\alpha) \geq 0 \text{ for each } \alpha \in c\}.$$
Note that each $U_A$ is a subsemigroup of $G$, i.e. $U_A + U_A \subset U_A$. Also, each $U_A$ contains the identity element of $G$. These properties of the sets $U_A$ imply that the family

$$\mathcal{B} = \{ x + U_A : x \in G, \, A \subset e, \, |A| < \omega \}$$

is a base for a paratopological group topology $\tau$ on $G$ and the sets $U_A$, with a finite set $A \subset e$, is a local base at the identity of the paratopological group $(G, \tau)$. It is clear that the topology $\tau$ is (strictly) finer than the topology of $G$ inherited from the Tychonoff product $\mathbb{Z}^\omega$, so the space $(G, \tau)$ is Hausdorff.

**Claim 1.** The group $(G, \tau)$ is $\omega$-narrow.

Consider a basic open neighborhood $U_A$ of the identity in $G$, where $A$ is finite. Denote by $\sigma_A$ the set of all $x \in e$ such that $\text{supp}(x) \subset A$. It is clear that $\sigma_A$ is a countable subgroup of $e$. We claim that $G = S + \sigma_A + U_A$. Indeed, let $y \in G$ be an arbitrary element. Then $y = x + a$ for some $x \in S$ and $a \in \sigma$. If $a$ is the identity element of $G$ (equivalently, $\text{supp}(a) = \emptyset$), then $y = x \in S \subset S + U_A$. Otherwise let $C = \text{supp}(a) \setminus A$ and $k = \min\{y(\alpha) : \alpha \in \text{supp}(a)\}$. According to our choice of $S$, there exists an element $s \in S$ such that $s(\alpha) \leq x(\alpha)$ for each $\alpha \in C$ and $s(\alpha) \leq k$ for each $\alpha \in C$. Since $x$ and $y$ coincide on $C \cap \text{supp}(a)$, our definition of $k$ implies that $s(\alpha) \leq y(\alpha)$ for each $\alpha \in e$. Choose an element $b \in \sigma_A$ such that $y(\alpha) = s(\alpha) + b(\alpha)$ for each $\alpha \in A$. Then $y \in S + b + U_A \subset S + \sigma_A + U_A$. This proves the equality $G = S + \sigma_A + U_A$. Since the subset $S + \sigma_A$ of $G$ is countable, we conclude that the group $(G, \tau)$ is $\omega$-narrow. This proves Claim 1.

**Claim 2.** The set $U_A$ is clopen in $(G, \tau)$, for each finite subset $A$ of $e$.

Indeed, if $x \in G \setminus U_A$, then either $x(\alpha) \neq 0$ for some $\alpha \in A$ or $x(\beta) < 0$ for some $\beta \in A$. In the first case, $x + U_B$ is an open neighborhood of $x$ disjoint from $U_A$, where $B = \{\alpha\}$. In the second case, $x + U_B$ is an open neighborhood of $x$ disjoint from $U_A$, where $B = \{\beta\}$. Therefore, the complement $G \setminus U_A$ is open in $(G, \tau)$ and $U_A$ is clopen. This proves Claim 2.

It follows from Claim 2 that the base $\mathcal{B}$ of $(G, \tau)$ consists of clopen sets and, hence, this space is zero-dimensional. In particular, $(G, \tau)$ is regular.

Let $O = U_\emptyset$. Then $O \cap (-O) = \{\emptyset\}$, where $\emptyset$ is the identity element of $G$. Hence the topological group $(G, \tau)^*$ associated to $(G, \tau)$ is discrete.

**Claim 3.** The group $(G, \tau)$ is not simply $sm$-factorizable.

By Claim 2, the set $O$ is clopen in $(G, \tau)$. Let $f$ be the characteristic function of $O$, i.e. $f(x) = 1$ if $x \in O$ and $f(x) = 0$ otherwise. Then $f$ is continuous. Suppose for a contradiction that there exists a continuous homomorphism $\varphi : (G, \tau) \to H$ to a second-countable paratopological group $H$ such that $O = \varphi^{-1}\varphi(O)$. Let $\{V_n : n \in \omega\}$ be a local base at the identity of $H$. For every $n \in \omega$, take a finite subset $A_n$ of $e$ such that $\varphi(U_{A_n}) \subset V_n$. Then the set $B = \bigcup_{n \in \omega} A_n$ is countable and $U_B \subset \ker \varphi$. Since $\ker \varphi$ is a subgroup of $G$, we see that the subgroup $\langle U_B \rangle$ of $G$ generated by $U_B$ is contained in $\ker \varphi$. An easy verification shows that

$$\langle U_B \rangle = \{ x \in G : x(\alpha) = 0 \text{ for each } \alpha \in B \}.$$

Denote by $p_B$ the natural projection of $G$ to $\mathbb{Z}^B$. Then $\ker p_B = \langle U_B \rangle \subset \ker \varphi$, so our choice of $\varphi$ implies that $O = p_B^{-1}(\ker p_B)$, which is clearly false. Indeed, take an
arbitrary element \( x \in G \) such that \( x(\alpha) = 0 \) for each \( \alpha \in B \) and \( x(\beta) < 0 \) for some \( \beta \in c \setminus B \). Then \( x \notin O \), while \( x \in p_{B}^{-1} p_{B}(O) \). This contradiction proves that the group \((G, \tau)\) fails to be simply \(sm\)-factorizable. \( \square \)

Since every Hausdorff \(R\)-factorizable topological group is \(\omega\)-narrow and simply \(sm\)-factorizable, the next corollary is immediate from Theorem 7.

**Corollary 12** If the topological group \(G^*\) associated to a regular paratopological group \(G\) is \(R\)-factorizable, then \(G\) is simply \(sm\)-factorizable.

**Problem 7** Can one weaken the regularity of \(G\) in Corollary 12 to the Hausdorff separation property?

**Theorem 8** Let \(G\) be a regular paratopological group such that the topological group \(G^*\) associated to \(G\) is \(\omega\)-narrow and simply \(sm\)-factorizable. Then the realcompactification \(\nu G\) of the space \(G\) admits a natural structure of paratopological group containing \(G\) as a dense subgroup and the group \(\nu G\) is simply \(sm\)-factorizable.

**Proof** According to Theorem 7, \(G\) is simply \(sm\)-factorizable. Hence Corollary 8 implies that the space \(\nu G\) admits the structure of paratopological group containing \(G\) as a dense subgroup. By Theorem 6, the group \(\nu G\) is simply \(sm\)-factorizable. \( \square \)

It is well known that every Hausdorff \(R\)-factorizable topological group is \(\omega\)-narrow and simply \(sm\)-factorizable (see [2, Proposition 8.1.3] and [3, Theorem 5.9]). Thus the next corollary follows from Theorem 8.

**Corollary 13** (See Theorem 2 of [18]) Let \(G\) be a regular paratopological group such that the topological group \(G^*\) associated to \(G\) is \(R\)-factorizable. Then the realcompactification \(\nu G\) of the space \(G\) admits a natural structure of paratopological group containing \(G\) as a dense subgroup and the equality \(\nu G = \mu G\) holds.

**Remark 3** Let \(G\) be as in Corollary 13. It is shown in [18, Theorem 2] that the topological groups \((\nu G)^*\) and \(\nu(G^*)\) are topologically isomorphic and \(R\)-factorizable. However, we do not know whether the paratopological group \(G\) is \(R\)-factorizable (see [16, Problem 5.1]). Notice that \(G\) is simply \(sm\)-factorizable, by Theorem 8.

**Remark 4** Recently the authors have succeeded to show that a quotient group of an \(\omega\)-narrow simply \(sm\)-factorizable topological group can fail to be simply \(sm\)-factorizable. This solves Problem 3 in the negative.

**Acknowledgements** This paper is dedicated to Professor Lin Shou on the occasion of his 60th anniversary. He is a distinguished teacher and is one of the founders of the Chinese school of Generalized Metric Spaces Theory. His deep mathematical insight and his warm and sincere personality greatly influenced us.

**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.
References

1. Alas, O.T., Sanchis, M.: Countably compact paratopological groups. Semigroup Forum 74(3), 423–438 (2007)
2. Arhangel’skii, A.V., Tkachenko, M.: Topological Groups and Related Structures. Atlantis Studies in Mathematics, vol. 1, p. 781. Atlantis Press, Paris (2008)
3. Arhangel’skii, A.V., Tkachenko, M.: C-extensions of topological groups. Topol. Appl. 235, 54–72 (2018)
4. Banakh, T., Ravsky, A.: Each regular paratopological group is completely regular. Proc. Am. Math. Soc. 145(3), 1373–1382 (2017)
5. Comfort, W.W., Ross, K.A.: Pseudocompactness and uniform continuity in topological groups. Pacific J. Math. 16, 483–496 (1966)
6. Engelking, R.: General Topology. Heldermann, Berlin (1989)
7. Gillman, L., Jerison, M.: Rings of Continuous Functions. Springer, Berlin (1976)
8. Hernández, S., Sanchis, M., Tkachenko, M.: Bounded sets in spaces and topological groups. Topol. Appl. 157, 800–808 (2010)
9. Peng, L.X., Zhang, P.: R-factorizable, simply sm-factorizable paratopological groups and their quotients. Topol. Appl. 258, 378–391 (2019)
10. Pestov, V.G.: On embeddings and condensations of topological groups. Math. Notes 31, 228–230 (1982)
11. Ravsky, O.V.: Paratopological groups II. Mat. Studii 17, 93–101 (2002)
12. Reznichenko, E.A., Sipacheva, O.V.: The free topological group on the Sorgenfrey line is not R-factorizable. Topol. Appl. 160(11), 1184–1187 (2013)
13. Sanchis, I.: Cardinal invariants of paratopological groups. Topol. Algebra Appl. 1, 37–45 (2013). 102478/taa-2013-0005
14. Sanchis, M., Tkachenko, M.: Totally Lindelöf and totally w-narrow paratopological groups. Topol. Appl. 155, 322–334 (2008)
15. Sanchis, M., Tkachenko, M.: Dieudonné completion and PT-groups. Appl. Categ. Struct. 20(1), 1–20 (2012)
16. Tkachenko, M.: Embedding paratopological groups into topological products. Topol. Appl. 156, 1298–1305 (2009)
17. Tkachenko, M.: Axioms of separation in semitopological groups and related functors. Topol. Appl. 161, 364–376 (2014)
18. Tkachenko, M.: Applications of the reflection functors in paratopological groups. Topol. Appl. 192, 176–187 (2015)
19. Tkachenko, M.: Pseudocompact topological groups, Section 2. In: Hrušák, H., Tamariz-Mascarúa, Á., Tkachenko, M. (eds.) Pseudocompact Topological Spaces. Developments in Mathematics, vol. 55, pp. 39–76. Springer, Cham (2018)
20. Xie, L.H., Lin, S., Tkachenko, M.: Factorization properties of paratopological groups. Topol. Appl. 160, 1902–1917 (2013)
21. Xie, L.H., Yan, P.F.: A note on bounded sets and C-compact sets in paratopological groups. Topol. Appl. 265, 106834 (2019)
22. Xie, L.H., Yan, P.F.: The continuous d-open images and subgroups of R-factorizable paratopological groups (2019). arXiv preprint, arXiv:1905.09577

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.