MODULAR GELFAND PAIRS AND MULTIPLICITY-FREE REPRESENTATIONS

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Abstract. We give a generalization of Gelfand’s criterion on the commutativity of Hecke algebras for Gelfand pairs and multiplicity-free triples over algebraically closed fields of arbitrary characteristic. Using more lenient versions of projectivity and injectivity for modules, we prove a general multiplicity-freeness theorem for finitely-generated modules with commutative endomorphism rings. For representations of finite and profinite groups, Gelfand pairs over the complex numbers are therefore also Gelfand pairs over the algebraic closure of any finite field. Applications include the uniqueness of Whittaker models of modular Gelfand–Graev representations and the uniqueness of modular trilinear forms on irreducible representations of quaternion division algebras over local fields.

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1. Introduction

For representations of a group over the complex numbers, the classical theory of Gelfand pairs considers group–subgroup pairs \((G, H)\) such that the induced representation \(\text{Ind}_{H}^{G}(\text{triv}_{H}) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \text{triv}_{H}\) of the trivial representation of the subgroup \(H\) is multiplicity-free. For finite groups, Gelfand pairs arise as useful tools across disciplines of mathematics, such as in number theory (cf. [Gro91]), combinatorics (cf. [AC12], [Joh19 Chapter 4], [MT09 Section 3]), and probability theory (cf. [Dia88], [Dia90], [CSST08], [CSDD+09]). The fact that \((S_{n}, S_{n-1})\) is a Gelfand pair over \(\mathbb{C}\), for instance, is important in the study of the Gelfand–Tsetlin algebra in the representation theory of \(S_{n}\) (cf. [OV96], [VO04 Section 2]). The
uniqueness of Whittaker models is a fundamental result in the theory of automorphic representations; for the finite group $\text{GL}_n(F_q)$, the key idea of Gelfand–Graev \cite{GG62} is that $(\text{GL}_n(F_q), N)$ is a (twisted) Gelfand pair for the standard nilpotent subgroup $N$ of $\text{GL}_n(F_q)$.

Due to its broad applications, the concept of Gelfand pairs has been generalized in several directions from its original form for complex representations of finite groups and Lie groups with compact subgroups. In particular, the theory of Gelfand pairs has been fruitfully extended by replacing the induced trivial representation $\text{Ind}^G_H(\text{triv}_H)$ with $\text{Ind}^G_H(\eta)$ for an arbitrary irreducible representation $\eta$ of $H$. This generalization is the study of Gelfand triples or multiplicity-free triples $(G,H,\eta)$ (cf. Bump \cite[Section 45]{Bum13}, Ceccherini-Silberstein–Scarabotti–Tolli \cite{CSST20}). Since there is an overlap of terminology with the usage of “Gelfand triple” as a synonym for a rigged Hilbert space in functional analysis, this article will use the less ambiguous “multiplicity-free triple”.

The property of “multiplicity-freeness” for a representation $\rho$ of a group $G$ over a field $F$ here means that irreducible representations occur in $\rho$ at most once, i.e. that for all irreducible representations $\pi$ of $G$ over $F$,

$$\dim_F \text{Hom}_H(\pi, \rho) \leq 1.$$  

Remark 1.1. Note that for non-semisimple categories of representations, this differs from the typical notion of multiplicity-freeness that is defined in terms of unique composition factors; in that sense, this is the multiplicity-freeness of the socle. This property can also be relative to the chosen category of representations and allow for additional conditions, such as requiring that the inequality holds only for smooth or admissible irreducible representations $\pi$ of $G$ over $F$. For much of this article, we will restrict ourselves to algebraically closed $F$, finite or compact $G$, and category of representations $\text{Rep}_F(G)$. When no topology is specified for infinite groups and their representation spaces, we generally assume the discrete topology.

Following the (GP1, GP2, GP3) convention of Aizenbud–Gourevitch–Sayag \cite[Definition 2.2.1]{AGS08}, the following is a definition of a multiplicity-free triple for finite and compact groups over algebraically closed fields.

**Definition 1.2** (Multiplicity-free triple, GT1). Let $F$ be an algebraically closed field, $G$ be a finite or compact group, $H$ be a subgroup of $G$, and $\eta$ be a representation of $H$. The triple $(G,H,\eta)$ is a multiplicity-free triple over $F$ if

$$\dim_F \text{Hom}_G(\pi, \text{Ind}^G_H(\eta)) \leq 1.$$  

for all irreducible representations $\pi$ of $G$.  

Remark 1.3. More generally when $F$ is not algebraically closed, we can modify Definition 1.2 to ask instead that $\text{Hom}_G(\pi, \text{Ind}^G_H(\eta))$ be a free $\text{End}_G(\pi)$-module of rank $\leq 1$. Over algebraically closed $F$, this rank $\leq 1$ condition is equivalent to the dimension condition of Definition 1.2 since the right-hand side is an $\text{End}_G(\pi)$-module, $\text{End}_G(\pi)$ is a division algebra if $\pi$ is irreducible, and $\text{End}_G(\pi) = F$ if and only if $\pi$ is absolutely irreducible. Furthermore, $(G,H,\text{triv}_H)$ being a multiplicity-free triple (GT1) is equivalent to $(G,H)$ being a Gelfand pair (GP1) when $F$ is algebraically closed. Here, moving from the condition of Definition 1.2 to the definition of Gelfand pairs (GP1) of Aizenbud–Gourevitch–Sayag \cite[Definition 2.2.1]{AGS08} is
due to Frobenius reciprocity,
\[
\text{Hom}_H(\text{Res}_G^H(\pi), \text{triv}_H) \cong \text{Hom}_G(\pi, \text{Ind}_G^H(\text{triv}_H)).
\]

Classically, the property of being a Gelfand pair is related to the commutativity of the Hecke algebra of $G$ with respect to $H$ (with $\eta = \text{triv}_H$).

**Definition 1.4.** For a compact group $G$, closed subgroup $H$ of $G$, and representation $\eta$ of $H$, the Hecke algebra $H(G,H,\eta,F)$ over $F$, also written as $\mathcal{H}(G//H,\eta,F)$, is the convolution algebra of continuous functions $\Delta: G \to \text{End}_F(\eta)$ satisfying
\[
\Delta(hgh_1) = \eta(h_2) \circ \Delta(g) \circ \eta(h_1),
\]
for all $g \in G$ and $h_1, h_2 \in H$.

This Hecke algebra is isomorphic to $\text{End}_G(\text{CoInd}_H^G(\eta))$ by Mackey theory (cf. [Bum13, Theorem 45.1]), where coinduction $\text{CoInd}_H^G(\cdot)$ is $\text{Hom}_F[H,G]$, $\cdot$. When $G$ is a finite group with a subgroup $H$, there is the following well-known criterion, essentially due to Gelfand [Gel50] and Gelfand–Graev [GG62], for $(G,H)$ to be a Gelfand pair.

**Proposition 1.5** ([Gel50] [GG62]). Let $G$ be a finite group, $H$ be a subgroup of $G$, and $\text{triv}_H$ be the trivial representation of $H$. If the Hecke algebra $\mathcal{H}(G,H,\text{triv}_H,F)$ is commutative, then $(G,H,\text{triv}_H)$ is a multiplicity-free triple over $F$.

The classical proof of Proposition 1.5 can be trivially generalized to obtain the same result over any algebraically closed field $F$ of characteristic $\ell$ not dividing $|G|$ and for any irreducible representation $\eta$ of $H$. However, the classical proof relies on Maschke’s theorem and Schur’s lemma and therefore does not work for fields of characteristic dividing $|G|$ nor for fields that are not algebraically closed. If the category of representations is semisimple, then one can get around using Schur’s lemma (and therefore algebraic closure) because
\[
\text{End}_G(\text{Ind}_H^G(\text{triv}_H)) \cong F[H\backslash G/H],
\]
but the obstruction remains when $\text{Rep}_F(G)$ is not semisimple.

The motivation of this article is to extend Gelfand’s criterion (Proposition 1.5) to algebraically closed fields $F$ of arbitrary characteristic.

**Theorem 1.6** (Theorem 4.4). Let $F$ be an algebraically closed field, $G$ be a finite group, $H$ be a subgroup of $G$, and $\text{triv}_H$ be the trivial representation of $H$. If $\mathcal{H}(G,H,\text{triv}_H,F)$ is commutative, then $(G,H,\text{triv}_H)$ is a multiplicity-free triple over $F$.

To prove this generalization of Gelfand’s criterion, we establish generalities about relative-projectivity and relative-injectivity (in the sense of Sandomierski and Azumaya), which are more lenient versions of usual projectivity and injectivity. The main input of this article is the following general multiplicity-freeness theorem (stated for more general modules in Theorem 3.3).

**Theorem 1.7** (Theorem 3.3). Let $F$ be an algebraically closed field and $G$ be a finitely-generated group, and $\rho$ be a finite-dimensional representation of $G$. Then $\rho$ is multiplicity-free if both of the following conditions are satisfied:

\begin{enumerate}
\item $\text{End}_G(\rho)$ is commutative;
\item $\rho$ is a self-injective $F[G]$-module.
\end{enumerate}
In the finite group applications of Theorem 1.6 that we highlight, we use the following consequence when $F$ is the algebraic closure of a finite field.

**Corollary 1.8** (Corollary 4.8). Let $F = \mathbb{F}_\ell$ of any positive characteristic $\ell$, $G$ be a finite group, $H$ be a subgroup of $G$, and $\eta$ be a complex multiplicative character of $H$. If $(G,H,\eta)$ is a multiplicity-free triple over $\mathbb{C}$, then $(G,H,\eta)$ is also a multiplicity-free triple over $F$.

As a consequence, the classical multiplicity-freeness theorems for complex representations of finite groups also hold over $F$. This extends to multiplicity-free triples with characters $\eta$ and to totally disconnected compact groups as well. For example, Corollary 1.8 yields the multiplicity freeness of induced characters of the unipotent subgroup in the theory of Gelfand–Graev representations. By Corollary 1.8 we immediately obtain the generalization (already known due to the work of Curtis [Cur65, Cur70], Richen [Ric69], and Steinberg [Ste16]) of the classical Gelfand–Graev [GG62] multiplicity-one theorem without having to use the theory of highest weights and Tits buildings.

**Corollary 1.9** (Corollary 5.4). Let $F = \mathbb{F}_\ell$ of any positive characteristic $\ell$ and let $K$ be any finite field $\mathbb{F}_q$. An irreducible representation of $GL_n(K)$ over $F$ has at most one Whittaker model.

Corollary 1.8 can also be used for certain infinite groups. Since irreducible smooth representations of totally disconnected compact (i.e. profinite) groups factor through finite quotients, we can demonstrate a multiplicity-one theorem for modular trilinear forms of quaternion division algebras, whose complex version is of arithmetic interest due to its role in the non-vanishing of triple product $L$-functions and the development of the Gan–Gross–Prasad conjecture.

**Corollary 1.10** (Corollary 5.6). Let $F = \mathbb{F}_\ell$ of any positive characteristic $\ell$, $k$ be any local field, and $D_k$ be the quaternion division algebra over $k$. If $V_1, V_2, V_3$ are three irreducible smooth representations of $D_k^*$ over $F$, then there exists at most one (up to isomorphism) non-zero $D_k^*$-invariant linear form on $V_1 \otimes V_2 \otimes V_3$ over $F$.

**Remark 1.11.** The criterion given in Theorem 1.7 is stated in sufficient generality for non-compact groups. One area with automorphic applications for future study are totally disconnected locally compact groups; in this setting the Gelfand–Kazhdan criterion for (twisted) Gelfand pairs is also the commutativity of Hecke algebras (cf. [GK75, Hen21]). Multiplicity-free triples for such reductive groups are important objects in areas like the Gan–Gross–Prasad conjecture where Gan–Gross–Prasad triples are known in certain cases to be multiplicity-free over the complex numbers (cf. Beuzart–Plessis [BP20] and Luo [Luo21]). Another potentially interesting direction is to modify Theorem 1.7 for cuspidal Gelfand pairs (cf. [BR08]).

Finally, we explore a multiplicity-freeness question about restrictions of representations of groups that was considered in characteristic 0 by Weyl [Wey39], Kac [Kac80], Howe [How95], Stembridge [Ste03], and more recently by Liebeck–Seitz–Testerman [LST15, LST21]. One consequence of Theorem 1.7 is a Gelfand-like criterion for multiplicity-free restrictions.

**Corollary 1.12** (Corollary 6.3). Let $F$ be an algebraically closed field, $H$ be a subgroup of a discrete finitely generated group $G$ such that $\text{Ind}^G_H$ is an exact functor, and $\rho$ be a finite-dimensional representation of $G$. If $\rho$ is $\text{Ind}^G_H(\text{Res}^G_H(\rho))$-injective and $\text{End}_H(\text{Res}^G_H(\rho))$ is commutative, then $\text{Res}^G_H(\rho)$ is multiplicity-free.
2. Relative projectivity and relative injectivity

We recall some useful characterizations about relatively-projective modules and relatively-injective in the sense of Sandomierski and Azumaya, largely following Wisbauer [Wis91, Chapter 3] (cf. Azumaya [Azu70], Azumaya–Mbuntum–Varadarajan [AMV75], Elliger [Ell75], and Shrikhande [Shr73]). In this section, \( R \) denotes an arbitrary ring with unity.

2.1. Relatively-projective modules.

**Definition 2.1.** Let \( M \) and \( N \) be \( R \)-modules. \( M \) is called \( N \)-projective if and only if for every \( R \)-module \( K \), \( R \)-module homomorphism \( f : M \to K \), and surjective \( R \)-module homomorphism \( g : N \onto K \), there is an \( R \)-module homomorphism \( h : M \to N \) such that the following diagram commutes:

\[
\begin{array}{ccc}
N & \xrightarrow{g} & K \\
\downarrow{h} & & \\
M & \xrightarrow{f} & K
\end{array}
\]

An \( R \)-module \( M \) is called **self-projective** if it is \( M \)-projective.

**Remark 2.2.** What we have defined is an \( R \)-module being projective relative to another \( R \)-module. This notion of relative-projectivity is essentially the same as the one defined by Sandomierski [San64] (cf. de Robert [dR69]), but is slightly different from the notion of relative-projectivity defined by Okuyama [Oku91, Car96]. This is also distinct from the notion an \( F[G] \)-module being projective relative to a subgroup of \( G \).

An \( R \)-module \( M \) is projective if and only if \( \text{Hom}(M, -) \) is an exact functor. Relative projectivity can also be characterized with a similar exactness condition.

**Proposition 2.3.** Let \( M \) and \( N \) be \( R \)-modules. \( M \) is \( N \)-projective if and only if for every exact sequence

\[
0 \longrightarrow L \longrightarrow N \longrightarrow K \longrightarrow 0,
\]

the corresponding sequence of \( R \)-module homomorphism groups is also exact:

\[
0 \longrightarrow \text{Hom}_R(M, L) \longrightarrow \text{Hom}_R(M, N) \longrightarrow \text{Hom}_R(M, K) \longrightarrow 0.
\]

Note that the middle term \( N \) of the exact sequence in Proposition 2.3 is fixed \( a \) priori. If we let \( K = M \), then Proposition 2.3 has the following immediate corollary.

**Corollary 2.4.** Let \( M \) and \( N \) be \( R \)-modules. If \( M \) is \( N \)-projective then every short exact sequence

\[
0 \longrightarrow L \longrightarrow N \longrightarrow M \longrightarrow 0,
\]

splits.

**Corollary 2.5.** Let \( M \) and \( N \) be \( R \)-modules with a surjective homomorphism \( g : N \onto M \). If \( M \) is \( N \)-projective, then \( M \) is a direct summand of \( N \). Furthermore, if \( M \) is \( N \)-projective and \( N \) is projective, then \( M \) is projective.

**Proof.** The surjection \( g \) induces a short exact sequence

\[
0 \longrightarrow \ker(g) \longrightarrow N \xrightarrow{g} M \longrightarrow 0.
\]
Since $M$ is $N$-projective, this short exact sequence splits so

$$M \oplus \ker(g) \cong N.$$ 

Therefore, $M$ is a direct summand of $N$.

If $N$ is projective then it is the direct summand of a free module. Since $M$ is a direct summand of $N$, $M$ is also the direct summand of a free module and therefore projective. \qed

Remark 2.6. The converse of the first claim is false. If

$$\begin{array}{ccc}
N & \xrightarrow{g} & K \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & K
\end{array}$$

is a diagram for which the lifting property fails, then

$$\begin{array}{ccc}
M \oplus N & \xrightarrow{g'} & K \\
\downarrow & & \\
M & \xrightarrow{f} & K
\end{array}$$

with $g'(f, n) := g(n)$ is also a diagram for which the lifting property fails. So for any $R$-module $N$ such that $M$ is not $N$-projective, $M$ is also not $(M \oplus N)$-projective.

Definition 2.7.

(i) For an $R$-module $M$, let $C^p(M)$ denote the class of $R$-modules $N$ such that $M$ is $N$-projective.

(ii) For a finitely generated $R$-module $M$, $C^p(M)$ is also closed under arbitrary direct sums.

(iii) For an $R$-module $N$, the class $C^p(N)$ is closed under taking arbitrary direct sums and direct summands.

2.2. Relatively-injective modules. Similarly, there is the dual notion of relative injectivity in the sense of Sandomierski and Azumaya.

Definition 2.9. Let $M$ and $N$ be $R$-modules. $M$ is called $N$-injective if and only if for every $R$-module $K$, injective $R$-module homomorphism $f : K \to N$, and $R$-module homomorphism $g : K \to M$, there is an $R$-module homomorphism $h : N \to M$ such that the following diagram commutes:

$$\begin{array}{ccc}
K & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M & \xrightarrow{g} & \bullet
\end{array}$$

An $R$-module $M$ is called self-injective if it is $M$-injective.
An $R$-module $M$ is injective if and only if $\text{Hom}(\cdot, M)$ is an exact functor. Relative injectivity can also be characterized with a similar exactness condition.

**Proposition 2.10.** Let $M$ and $N$ be $R$-modules. $M$ is $N$-injective if for every exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0,$$

the corresponding sequence of $R$-module homomorphism groups is also exact:

$$0 \longrightarrow \text{Hom}_R(K, M) \longrightarrow \text{Hom}_R(N, M) \longrightarrow \text{Hom}_R(L, M) \longrightarrow 0.$$

As with Proposition 2.3 for relative projectivity, the middle term $N$ of the exact sequence in Proposition 2.10 is fixed a priori. If we let $K = M$, then Proposition 2.10 has the following immediate corollary.

**Corollary 2.11.** Let $M$ and $N$ be $R$-modules. If $M$ is $N$-injective then every short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0,$$

splits.

**Corollary 2.12.** Let $M$ and $N$ be $R$-modules with an injective homomorphism $f: M \hookrightarrow N$. If $M$ is $N$-injective, then $M$ is a direct summand of $N$.

**Proof.** The surjection $g$ induces a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow \text{coker}(f) \longrightarrow 0.$$

Since $M$ is $N$-projective, this short exact sequence splits so

$$M \oplus \text{coker}(f) \cong N.$$

Therefore, $M$ is a direct summand of $N$. □

**Definition 2.13.**

(i) For an $R$-module $M$, let $C^i(M)$ denote the class of $R$-modules $N$ such that $M$ is $N$-injective.

(ii) For an $R$-module $N$, let $C^i(N)$ denote the class of $R$-modules $M$ such that $M$ is $N$-injective.

The following general properties of relative injectivity are largely due to Azumaya [Azu70] and Shrikande [Shr73].

**Proposition 2.14** ([AMV75, Proposition 1.16], [Wis91, Section 16.2]).

(i) For an $R$-module $M$, the class $C^i(M)$ is closed under taking submodules, arbitrary direct sums, and images of $R$-module homomorphisms.

(ii) For a finitely generated $R$-module $M$, $C^i(M)$ is also closed under arbitrary direct sums.

(iii) For an $R$-module $N$, the class $C^i(N)$ is closed under taking arbitrary direct products and direct factors.
2.3. **Endomorphism rings.** We recall some more definitions from module theory. Let $M$ be an $R$-module. An $R$-submodule $N$ of $M$ is called **maximal** if $M/N$ is a simple $R$-module. The **radical** $\text{rad}(M)$ is defined to be the intersection of all maximal submodules of $M$. The **socle** $\text{soc}(M)$ is defined to be the sum of all simple submodules of $M$. The **cosocle** (also called the **head** or **top**) $\text{cosoc}(M)$ is defined to be the maximal semisimple subquotient of $M$, and is equal to $M/\text{rad}(M)$.

A submodule $N$ of $M$ is called **superfluous** if and only if $K = M$ is the only submodule of $M$ such that $N + K = M$. The dual notion is a submodule $N$ of $M$ being **essential**, which occurs if and only if $K = \{0\}$ is the only submodule of $M$ such that $N \cap K = \{0\}$. Then $\text{rad}(M)$ can be characterized as the sum of all superfluous submodules of $M$, while $\text{soc}(M)$ can be characterized as the intersection of all essential submodules of $M$.

The Jacobson radical of the endomorphism ring of self-projective and self-injective modules can be characterized in terms of superfluous and essential modules respectively.

**Lemma 2.15** ([Wis91, Section 22]). Let $M$ be an $R$-module.

(i) If $M$ is self-projective, then

$$\text{rad}(\text{End}_R(M)) = \{ f \in \text{End}_R(M) \mid \text{im}(f) \text{ is a superfluous submodule of } M \}.$$  

(ii) If $M$ is self-injective, then

$$\text{rad}(\text{End}_R(M)) = \{ f \in \text{End}_R(M) \mid \ker(f) \text{ is an essential submodule of } M \}.$$  

**Proof.** (i): Let $f \in \text{End}_R(M)$ with superfluous image in $M$. Suppose $\text{End}_R(M)f + A = \text{End}_R(M)$ for an ideal $A$ of $\text{End}_R(M)$. Then there is an $s \in \text{End}_R(M)$ and $g \in A$ such that $sf + g = 1$. Then $M = Msf + Mg \subset \text{im}(f) + Mg$, so $Mg = M$ because $\text{im}(f)$ is superfluous in $M$. Using the self-projectivity of $M$ on the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{g} & M \\
h & \downarrow & \\
M & \xrightarrow{1} & M,
\end{array}
$$

we have an $h \in \text{End}_R(M)$ such that $1 = hg \in A$, i.e. $A = \text{End}_R(M)$. So an $f \in \text{End}_R(M)$ whose image is superfluous in $M$ actually satisfies the property that $\text{End}_R(M)f$ is a superfluous submodule of $\text{End}_R(M)$, and such an $f$ is therefore contained in $\text{rad}(\text{End}_R(M))$.

(ii): Let $f \in \text{rad}(\text{End}_R(M))$. Suppose that $K$ is a submodule of $M$ such that $\text{im}(f) + K = M$. Then the composition $M \xrightarrow{f} M \xrightarrow{p} M/K$ is an epimorphism, so we can use the self-projectivity of $M$ to make the following diagram commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{f} & M \\
h & \downarrow & \\
M & \xrightarrow{p} & M/K
\end{array}
$$
From the commutativity of the diagram, $\text{hf}p = p$ so $(1 - \text{hf})p = 0$. Since $f \in \text{rad}(\text{End}_R(M))$, $1 - \text{hf}$ is invertible and hence $p = 0$. Therefore, $K = M$ and $\text{im}(f)$ is superfluous in $M$.

(ii): Take the dual of the arguments in the proof of (i). \qed

Using these characterizations, we can show that endomorphisms of the cosocle of a self-projective module and endomorphisms of the socle of a self-injective module lift to endomorphisms on the original module.

**Theorem 2.16** ([Wis91, Section 22]). Let $M$ be an $R$-module.

(i) If $M$ is self-projective and $\text{rad}(M)$ is a superfluous submodule of $M$, then

$$\text{End}_R(M)/\text{rad}(\text{End}_R(M)) \cong \text{End}_R(M/\text{rad}(M)).$$

(ii) If $M$ is self-injective and $\text{soc}(M)$ is an essential submodule of $M$, then

$$\text{End}_R(M)/\text{rad}(\text{End}_R(M)) \cong \text{End}_R(\text{soc}(M)).$$

**Proof.** (i): From the exact sequence

$$0 \longrightarrow \text{rad}(M) \longrightarrow M \longrightarrow M/\text{rad}(M) \longrightarrow 0,$$

the following is also exact by the self-projectivity of $M$:

$$0 \longrightarrow \text{Hom}_R(M, \text{rad}(M)) \longrightarrow \text{Hom}_R(M, M) \longrightarrow \text{Hom}_R(M, M/\text{rad}(M)) \longrightarrow 0.$$

Notice that $\text{Hom}_R(M, M/\text{rad}(M)) \cong \text{End}_R(M/\text{rad}(M))$ since $\text{rad}(M)$ is necessarily in the kernel of such an $R$-homomorphism. Since $\text{rad}(M)$ is the sum of all superfluous submodules of $M$ and furthermore is itself a superfluous submodule of $M$ (by assumption),

$$\text{Hom}_R(M, \text{rad}(M)) = \{f \in \text{End}_R(M) \mid \text{im}(f) \text{ is a superfluous submodule of } M\}.$$

By Lemma 2.15(i), this is equal to $\text{rad}(\text{End}_R(M))$.

Therefore, we have the exact sequence

$$0 \longrightarrow \text{rad}(\text{End}_R(M)) \longrightarrow \text{End}_R(M) \longrightarrow \text{End}_R(M/\text{rad}(M)) \longrightarrow 0,$$

so $\text{End}_R(M/\text{rad}(M)) \cong \text{End}_R(M)/\text{rad}(\text{End}_R(M))$.

(ii): Consider an endomorphism $f : M \rightarrow M$ such that $\ker(f)$ is an essential submodule of $M$. Such an endomorphism factors through $M/\text{soc}(M)$ since $\text{soc}(M)$ is the intersection of all essential submodules of $M$. By Lemma 2.15(ii),

$$\text{rad}(\text{End}_R(M)) = \{f \in \text{End}_R(M) \mid f \text{ is an essential submodule of } M\}.\text{Hom}_R(M/\text{soc}(M), M).$$

Furthermore, any element $g \in \text{Hom}_R(M/\text{soc}(M), M)$ lifts to an endomorphism $g' \in \text{End}_R(M)$ such that $\text{soc}(M) \subset \ker(g')$. But $\text{soc}(M)$ is an essential submodule of $M$ by assumption, so $\ker(g')$ is also an essential submodule of $M$. Then $g' \in \text{rad}(\text{End}_R(M))$ by Lemma 2.15(ii). \qed

**2.4. Exact functors.** Finally, we briefly consider how exact functors can preserve or transfer relative-projectivity. This will be used later for the study of induction and restriction for representations of finite groups in Section 4.1.

**Proposition 2.17.** Let $R_1$ and $R_2$ be two rings. Suppose that $\mathcal{F} : R_1-\text{Mod} \rightarrow R_2-\text{Mod}$ and $\mathcal{G} : R_2-\text{Mod} \rightarrow R_1-\text{Mod}$ are an exact adjoint pair $\mathcal{F} \dashv \mathcal{G}$.

(i) For $M \in R_1-\text{Mod}$ and $N \in R_2-\text{Mod}$, if $M$ is $\mathcal{G}(N)$-projective, then $\mathcal{F}(M)$ is $N$-projective.
(ii) For $M \in R_2$ and $N \in R_1$, if $M$ is $\mathcal{F}(N)$-injective, then $\mathcal{G}(M)$ is $N$-injective.

*Proof.* (i): Suppose we have an exact sequence of $R_2$-modules

$$0 \longrightarrow L \longrightarrow N \longrightarrow K \longrightarrow 0.$$  

By the exactness of $\mathcal{G}$, this gives the exact sequence

$$0 \longrightarrow \mathcal{G}(L) \longrightarrow \mathcal{G}(N) \longrightarrow \mathcal{G}(K) \longrightarrow 0.$$  

The $\mathcal{G}(N)$-projectivity of $M$ gives the exactness of

$$0 \longrightarrow \text{Hom}_{R_1}(M, \mathcal{G}(L)) \longrightarrow \text{Hom}_{R_1}(M, \mathcal{G}(N)) \longrightarrow \text{Hom}_{R_1}(M, \mathcal{G}(K)) \longrightarrow 0.$$  

The adjunction between $\mathcal{F}$ and $\mathcal{G}$ implies the exactness of

$$0 \longrightarrow \text{Hom}_{R_2}(\mathcal{F}(M), L) \longrightarrow \text{Hom}_{R_2}(\mathcal{F}(M), N) \longrightarrow \text{Hom}_{R_2}(\mathcal{F}(M), K) \longrightarrow 0.$$  

Therefore, $\text{Hom}_{R_2}(\mathcal{F}(M), -)$ is exact.

(ii): Suppose we have an exact sequence of $R_1$-modules

$$0 \longrightarrow L \longrightarrow N \longrightarrow K \longrightarrow 0.$$  

By the exactness of $\mathcal{F}$, this gives the exact sequence

$$0 \longrightarrow \mathcal{F}(L) \longrightarrow \mathcal{F}(N) \longrightarrow \mathcal{F}(K) \longrightarrow 0.$$  

The $\mathcal{F}(N)$-projectivity of $M$ gives the exactness of

$$0 \longrightarrow \text{Hom}_{R_2}(\mathcal{F}(L), M) \longrightarrow \text{Hom}_{R_2}(\mathcal{F}(N), M) \longrightarrow \text{Hom}_{R_2}(\mathcal{F}(K), M) \longrightarrow 0.$$  

The adjunction between $\mathcal{F}$ and $\mathcal{G}$ implies the exactness of

$$0 \longrightarrow \text{Hom}_{R_1}(L, \mathcal{G}(M)) \longrightarrow \text{Hom}_{R_1}(N, \mathcal{G}(M)) \longrightarrow \text{Hom}_{R_1}(K, \mathcal{G}(M)) \longrightarrow 0.$$  

Therefore, $\text{Hom}_{R_1}(-, \mathcal{G}(M))$ is exact. \qed

3. General criteria for multiplicity-freeness

Using the endomorphism ring properties of self-projective and self-injective modules from Section 2, we can modify the proof of the classical Gelfand pair criterion (Proposition 1.5) to prove a multiplicity-freeness theorem in any characteristic. Since the proof still relies on Schur’s lemma for endomorphism rings, $R$ is an algebra over an algebraically closed base field $F$ in this section.

**Theorem 3.1** (Multiplicity-freeness, general version). Let $F$ be an algebraically closed field, $R$ be an algebra over $F$, and $M$ be a finitely-generated $R$-module.

(i) Suppose $M$ is a self-projective $R$-module. If $\text{End}_R(M)$ is commutative, then

$$\dim_F \text{Hom}_R(M, N) \leq 1,$$

for all simple $R$-modules $N$.

(ii) Suppose $M$ is a self-injective and finitely-cogenerated $R$-module. If $\text{End}_R(M)$ is commutative, then

$$\dim_F \text{Hom}_R(N, M) \leq 1,$$

for all simple $R$-modules $N$.  

Proof. If \( M \) is a finitely-generated module, then its radical \( \text{rad}(M) \) is a superfluous submodule of \( M \) ([Wis91, Section 21.6]). So if \( M \) is both finitely-generated and self-projective, then it satisfies the conditions of Theorem 2.16(i). Then by Theorem 2.16(ii), we obtain the isomorphism

\[
\text{End}_R(M)/\text{rad}(\text{End}_R(M)) \cong \text{End}_R(M/\text{rad}(M)).
\]

In particular, \( M/\text{rad}(M) \) is semisimple and finitely-generated (since \( M \) is finitely-generated).

Similarly, if \( M \) is a finitely-cogenerated module, then its socle \( \text{soc}(M) \) is an essential submodule of \( M \). So if \( M \) is both finitely-cogenerated and self-injective, then it satisfies the conditions of Theorem 2.16(ii). Then by Theorem 2.16(ii), we obtain the isomorphism

\[
\text{End}_R(M)/\text{rad}(\text{End}_R(M)) \cong \text{End}_R(\text{soc}(M)).
\]

In particular, \( \text{soc}(M) \) is semisimple and finitely-generated (since \( M \) is finitely-cogenerated [Wis91, Section 21.3]).

Let \( L = M/\text{rad}(M) \) if \( M \) is self-projective, and otherwise let \( L = \text{soc}(M) \) if \( M \) is finitely-cogenerated and self-injective. In both cases, \( L \) is semisimple and finitely-generated, so we may write \( L = \bigoplus d_i N_i \) as a direct sum of distinct simple \( R \)-modules \( N_i \) with positive integer multiplicities \( d_i \). In both cases, we have that

\[
\text{End}_R(L) \cong \bigoplus \text{Mat}(d_i, F),
\]

for division algebras \( D_i \). Since \( \text{End}_R(M) \) is commutative, so is \( \bigoplus \text{Mat}(d_i, F) \). Therefore, the \( D_i \) are actually fields, all \( d_i \leq 1 \), and the semisimple \( R \)-module \( L \) is multiplicity-free. Then if \( L = M/\text{rad}(M) \),

\[
\dim_f \text{Hom}_R(M_1, N) = \dim_f \text{Hom}_R(M/\text{rad}(M), N) \leq 1,
\]

for all simple \( R \)-modules \( N \). If \( L = \text{soc}(M) \), then

\[
\dim_f \text{Hom}_R(N, M) = \dim_f \text{Hom}_R(N, \text{soc}(M)) \leq 1
\]

for all simple \( R \)-modules \( N \).

\[\square\]

Remark 3.2. In the self-injective case of Theorem 3.1(ii), the \( R \)-module \( M \) is required to be both finitely-generated (for Schur’s lemma) and finitely-cogenerated (for \( \text{soc}(M) \) to be an essential submodule of \( M \)). If \( R \) is Artinian (e.g. any group ring of a finite group), then we may ignore the finite-cogeneration condition because all finitely-generated \( R \)-modules are finitely-cogenerated.

For applications of Theorem 3.1 to group representations, \( R \) is the group ring \( F[G] \) of a group \( G \). The formulation of Theorem 1.7 follows directly from Theorem 3.1(ii) by setting \( M \) to be a group representation \( \rho \) of \( G \).

For applications of Theorem 3.1 to group representations, we specialize to the group ring \( R = F[G] \) of a finitely-generated group \( G \) with \( M = \rho \) a finite-dimensional representation. Theorem 3.1(ii) can then be directly reformulated for group representations, but we can strengthen the result in this setting by removing the finite-cogeneration condition through an application of duality to Theorem 3.1(i).

**Theorem 3.3** (Multiplicity-freeness, group representation version). Let \( F \) be an algebraically closed field, \( G \) be a finitely-generated group, and \( \rho \) be a finite-dimensional representation of \( G \). For all irreducible representations \( \pi \) of \( G \),

\[
\dim_f \text{Hom}_G(\pi, \rho) \leq 1,
\]
if both of the following conditions are satisfied:

(i) \( \text{End}_G(\rho) \) is commutative;
(ii) \( \rho \) is self-injective.

\textbf{Proof.} Since \( \rho \) is self-injective, the representation \( \hat{\rho} \) is self-projective by duality. Furthermore, \( \text{End}_G(\hat{\rho}) \cong \text{End}_G(\rho) \) since \( \rho \) is finite-dimensional. Then by Theorem 3.1(i) with \( R = F[G] \) and \( M = \hat{\rho} \),

\[ \dim_F \text{Hom}_{F[G]}(\hat{\rho}, N) \leq 1, \]

for all simple \( F[G] \)-modules \( N \). Viewing the simple \( F[G] \)-modules \( N \) as duals of irreducible representations of \( G \) (using the fact that the dual of a finite-dimensional representation is irreducible if and only if the original representation is irreducible), this means that

\[ \dim_F \text{Hom}_{F[G]}(\hat{\rho}, \pi) \leq 1, \]

for all irreducible representations \( \pi \) of \( G \). By the natural isomorphism

\[ \text{Hom}_{F[G]}(\hat{\rho}, \pi) \cong \text{Hom}_{F[G]}(\pi, \rho), \]

we conclude that

\[ \dim_F \text{Hom}_{F[G]}(\pi, \rho) \leq 1, \]

for all irreducible representations \( \pi \) of \( G \) over \( F \).

Unlike Gelfand’s criterion in the characteristic zero setting, our proof of Theorem 3.1 (and therefore Theorem 3.3) does not provide a converse theorem because in the isomorphism,

\[ \text{End}_R(M)/\text{rad}(\text{End}_R(M)) \cong \bigoplus \text{Mat}(d_i, F), \]

the commutativity of the right-hand side does not necessarily imply the commutativity of \( \text{End}_R(M) \). In fact, the converse is false as illustrated by the following non-example.

\textbf{Non-example 3.4.} Let \( F \) be a field of characteristic \( p \), \( G \) be a finite non-abelian \( p \)-group, \( R = F[G] \), and \( M = F[G] \). Then \( \text{End}_R(M) = F[G] \), which is non-commutative. But the only irreducible representation of \( G \) over \( F \) is the trivial representation, so \( M \) still satisfies the multiplicity-one property despite failing the commutativity condition of Theorem 3.1

The conditions of Theorem 3.1 are also somewhat necessary, as one can otherwise construct a representation with commutative Hecke algebra but higher multiplicity if it is not self-injective.

\textbf{Non-example 3.5.} Let \( \pi \) be an irreducible representation of a group \( G \) over a field \( F \) with two non-split non-isomorphic extensions \( \sigma_1 \) and \( \sigma_2 \):

\[ 0 \longrightarrow \pi \longrightarrow \sigma_1 \longrightarrow \tau_1 \longrightarrow 0, \]

\[ 0 \longrightarrow \pi \longrightarrow \sigma_2 \longrightarrow \tau_2 \longrightarrow 0, \]

where \( \tau_1 \) and \( \tau_2 \) are irreducible representations of \( G \) over \( F \) different from \( \pi \). Such extensions exist, for example, for groups in positive characteristic or even for \( p \)-adic groups in characteristic 0. Then there is the extension \( \rho := \sigma_1 \oplus \sigma_2 \) of \( \pi \oplus \pi \):
Proof.\[0 \longrightarrow \pi \oplus \pi \longrightarrow \rho = \sigma_1 \oplus \sigma_2 \longrightarrow \tau_1 \oplus \tau_2 \longrightarrow 0.\]

Observe that for \(i = 1\) or \(2\), any \(\phi \in \text{End}_G(\rho)\) sends \(\sigma_i\) to itself, and \(\text{End}(\sigma_i)\) sends \(\pi\) to itself since \(\pi\) is different from \(\tau_i\). Also, \(\text{End}_G(\rho) = \text{End}_G(\sigma_1) \oplus \text{End}_G(\sigma_2)\) since \(\sigma_1 \neq \sigma_2\). Then \(\text{End}_G(\rho) \cong F \oplus F\) is commutative, but \(\pi\) has multiplicity greater than 1 in \(\rho\).

4. Finite and compact multiplicity-free triples

Theorem 3.3 is quite general, but when specializing to \(R = F[G]\) for a finite group \(G\), we may use induction and restriction properties of self-projectivity and self-injectivity to remove condition (ii). We also consider the specialization to smooth representations of totally disconnected compact groups.

4.1. Induction and restriction. Let \(G\) be a group and \(H\) be a subgroup of \(G\). Assume that the left and right adjoints of the restriction functor \(\text{Res}_H^G\) exist. Define induction \(\text{Ind}_H^G\) from \(\text{Rep}_H(\mathcal{H})\) to \(\text{Rep}_G(\mathcal{G})\) to be the left adjoint of restriction \(\text{Res}_H^G\) from \(\text{Rep}_F(\mathcal{G})\) to \(\text{Rep}_F(\mathcal{H})\), and define coinduction \(\text{CoInd}_H^G\) to be the right adjoint of \(\text{Res}_H^G\).

Remark 4.1. When \(G\) is discrete, \(\text{Res}_H^G\) has both left-adjoints and right-adjoints (cf. [Hir19]). For our settings of interest, induction and coinduction are often given concretely as \(\text{Ind}_H^G(M) = F[G] \otimes_{F[H]} M\) and \(\text{CoInd}_H^G(M) = \text{Hom}_{F[H]}(F[G], M)\). In the context of representations of locally profinite groups, coinduction \(\text{CoInd}_H^G\) is commonly called “induction” and denoted \(\text{Ind}_H^G\) while induction \(\text{Ind}_H^G\) is commonly called “compact induction” and denoted \(\text{c–Ind}_H^G\) or \(\text{Ind}_H^G\) (cf. [Vig96, I.1.5]).

When specializing the exact adjoint pair \(\mathcal{F} \dashv \mathcal{G}\) to induction-restriction \(\text{Ind}_H^G \dashv \text{Res}_H^G\) and restriction-coinduction \(\text{Res}_H^G \dashv \text{CoInd}_H^G\), Proposition 2.17 has the following consequence.

Corollary 4.2. Let \(F\) be an algebraically closed field, \(G\) be a group, and \(H\) be a subgroup of \(G\).

(i) Let \(M\) be an \(F[G]\)-module and let \(N\) be an \(F[H]\)-module. If \(\text{CoInd}_H^G\) is an exact functor and \(M\) is \(\text{CoInd}_H^G(N)\)-projective, then \(\text{Res}_H^G(M)\) is an \(N\)-projective \(F[H]\)-module.

(ii) Let \(M\) be an \(F[H]\)-module and let \(N\) be an \(F[G]\)-module. If \(\text{Ind}_H^G\) is an exact functor and \(M\) is \(\text{Res}_H^G(N)\)-projective, then \(\text{Ind}_H^G(M)\) is an \(N\)-projective \(F[G]\)-module.

(iii) Let \(M\) be an \(F[G]\)-module and let \(N\) be an \(F[H]\)-module. If \(\text{Ind}_H^G\) is an exact functor and \(M\) is \(\text{Ind}_H^G(N)\)-injective, then \(\text{Res}_H^G(M)\) is an \(N\)-injective \(F[H]\)-module.

(iv) Let \(M\) be an \(F[H]\)-module and let \(N\) be an \(F[G]\)-module. If \(\text{CoInd}_H^G\) is an exact functor and \(M\) is \(\text{Res}_H^G(N)\)-injective, then \(\text{CoInd}_H^G(M)\) is an \(N\)-injective \(F[G]\)-module.

Proof.

(i): Proposition 2.17(i) with \(R_1 = F[G], R_2 = F[H], \mathcal{F} = \text{Res}_H^G,\) and \(\mathcal{G} = \text{CoInd}_H^G\).

(ii): Proposition 2.17(i) with \(R_1 = F[H], R_2 = F[G], \mathcal{F} = \text{Ind}_H^G,\) and \(\mathcal{G} = \text{Res}_H^G\).

(iii): Proposition 2.17(i) with \(R_1 = F[H], R_2 = F[G], \mathcal{F} = \text{Ind}_H^G,\) and \(\mathcal{G} = \text{Res}_H^G\).
where $H$ and $y$ are lifted of any morphism $f$.

If induction and coinduction are exact functors, then we can generate some examples of relatively-projective representations using Corollary 4.2. We will assume that $G$ is finite for simplicity, but exactness of induction and coinduction actually holds in greater generality (cf. [Jan03 Section 8.16] for finite algebraic groups and [Vig96 I.1.5] for locally profinite groups). Induction and coinduction are equal for finite groups, so we will generally only say “induction” and use $\text{Ind}_G^G$ in the finite group case.

**Lemma 4.3.** Let $F$ be an algebraically closed field, $G$ be a finite group, and $H$ be a subgroup of $G$.

(i) If $\rho$ is an irreducible representation of $G$, then $\rho$ is a self-projective $F[G]$-module.

(ii) If $\rho$ is an irreducible representation of $G$, then $\rho$ is a self-injective $F[G]$-module.

(iii) If $\rho$ is a self-projective $F[H]$-module, then the induced representation $\text{Ind}_H^G(\rho)$ is a self-projective $F[G]$-module.

(iv) If $\rho$ is a self-injective $F[H]$-module, then the induced representation $\text{Ind}_H^G(\rho)$ is a self-injective $F[G]$-module.

**Proof.**

(i) A simple $R$-module $M$ is self-projective because a surjective morphism $g : M \to K$ is either zero or an isomorphism. If $g$ is zero, then $f$ is also zero so any endomorphism $h$ suffices. If $g$ is an isomorphism, then $h = f \circ g^{-1}$ is the desired lifting of any morphism $f : M \to K$.

(ii) This follows by duality to (i).

(iii) By Mackey’s restriction formula,

$$\text{Res}_H^G(\text{Ind}_H^G(\rho)) \cong \bigoplus_{x \in H \backslash G / H} \text{Ind}_H^G \rho_x,$$

where $H_s := s H s^{-1} \cap H$ and $\rho_s$ is the representation of $H_s$ defined by $\rho_s(x) := \text{Res}_{H_s}^H(\rho)(s^{-1} xs)$. Note that this is a finite direct sum of elements of $C^G(\rho)$ since

$$\text{Ind}_H^G(\rho_s) = \bigoplus_{y \in H \setminus H} y^{-1}(\rho_s),$$

where $y^{-1}(\rho_s)$ is the image of the action of a representative $y^{-1} \in H$ of $[y]^{-1}$ on the subspace $[1] \otimes \rho_s \subset k[H] \otimes_{k[H_1]} \rho_s = \text{Ind}_H^H \rho_s$.

Since $G$ is finite, Proposition 2.8 implies that $\rho$ is a $\text{Res}_H^G(\text{Ind}_H^G(\rho))$-projective module in two different ways: by Proposition 2.8(i) because $[G : H]$ is finite, and by Proposition 2.8(ii) because dim $\rho$ is finite. Consequently, $\text{Ind}_H^G(\rho)$ is a self-projective $F[G]$-module by Corollary 4.2(ii) with $M = \rho$ and $N = \text{Ind}_H^G(\rho)$.

(iv) This follows by duality to (iii) for coinduction $\text{CoInd}_H^G$ and the equality $\text{Ind}_G^G = \text{ColInd}_G^G$.

**4.2. Commutative Hecke algebra criteria.** Again assuming $G$ to be finite for simplicity, we may use Lemma 4.3 to remove condition (ii) of Theorem 3.3. Recall that the Hecke algebra $H(G, H, \text{triv}_H, \mathbb{C})$ is a convolution algebra that is isomorphic to $\text{End}_G(\text{Ind}_G^G(\text{triv}_H))$. 

\[\]
Theorem 4.4. Let $F$ be an algebraically closed field, $G$ be a finite group, $H$ be a subgroup of $G$, and $\eta$ be an irreducible representation of $H$. If $\mathcal{H}(G,H,\eta,F)$ is commutative, then $(G,H,\eta)$ is a multiplicity-free triple.

Proof. Let $\rho = \text{Ind}_H^G(\eta)$. Since $\dim \eta < \infty$, $\rho$ is finite-dimensional. Furthermore, $\rho$ is a self-injective $F[G]$-module by irreducibility and Lemma 4.3, so condition (ii) of Theorem 3.3 is satisfied. □

Remark 4.5. Only one of the two finiteness assumptions on $[G:H]$ and $\dim \eta$ is used to satisfy condition (ii) of Theorem 3.3.

A common method of proving that a Hecke algebra is commutative is with the criterion known as Gelfand’s trick or Gelfand’s lemma, which works in any characteristic.

Lemma 4.6 (Gelfand’s trick). Let $F$ be an algebraically closed field, $G$ be a finite group, $H$ be a subgroup of $G$, and $\eta$ be an irreducible representation of $H$. If there is an anti-involution $\iota$ such that $f(\iota(g)) = f(g)$ for all $f \in \mathcal{H}(G,H,\eta,F)$ and all $g \in G$, then $\mathcal{H}(G,H,\eta,F)$ is commutative.

For Gelfand pairs (i.e. $\eta = \text{triv}_H$), this condition is equivalent to $\iota$ preserving all double cosets of $H$. Together with Theorem 4.4, this extends Gelfand’s trick to multiplicity-free triples over fields of arbitrary characteristic.

Corollary 4.7. Let $F$ be an algebraically closed field, $G$ be a finite group, $H$ be a subgroup of $G$, and $\eta$ be an irreducible representation of $H$. If there is an anti-involution $\iota$ that preserves all double cosets of $H$, then $(G,H,\eta)$ is a multiplicity-free triple over $F$.

In the case that $F$ is the algebraic closure of a finite field, another consequence of Theorem 4.4 together with Brauer theory is that it is not even necessary to use Gelfand’s trick over $F$ if $(G,H,\eta)$ is already known to be a multiplicity-free triple over $\mathbb{C}$. We briefly outline this process for finite groups following Serre [Ser77, Part III] and Curtis–Reiner [CR81, Section 16C] (for other groups, cf. [Ser68, Vig89, Vig96, Zha23]).

Given a complex representation $(\rho, V)$ of a finite group $G$, there is a notion of reduction modulo $\ell$ for any prime $\ell$, yielding a representation $\overline{\rho}$ of $G$ over a finite extension of $\mathbb{F}_\ell$. Since $G$ is finite, the representation $(\rho, V)$ can be realized over the algebraic integers $\mathcal{O}_\ell$ of a number field $\mathbb{E} \hookrightarrow \overline{\mathbb{Q}}_\ell$; using the embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_\ell$, $\rho$ can then be viewed as a representation $(\rho_K, V)$ over the ring of integers $\mathcal{O}_K$ of a finite extension $K$ of $\mathbb{Q}_\ell$. Let $m_K$ denote the maximal ideal of $\mathcal{O}_K$ and let $k$ denote the residue field $\mathcal{O}_K/m_K \mathcal{O}_K$. Pick a lattice $\Lambda \subset V$ (i.e. a finitely-generated $\mathcal{O}_K$-submodule $\Lambda$ of $V$ that generates $V$ as a $K$-module). Such a lattice exists and can be taken to be $G$-stable (i.e. $g\Lambda \subset \Lambda$ for all $g \in G$) by replacing $\Lambda$ by $\bigoplus_{g \in G} g\Lambda$. Define the reduction,

$$\overline{\Lambda} := \Lambda/m_K \Lambda;$$

this is a $k[G]$-module for a finite extension of $k$ of $\mathbb{F}_\ell$. It is known due to Brauer–Nesbitt [BN37] (cf. [Ser77, Theorem 32] and [CR81, Proposition 16.16]) that this reduction modulo $\ell$ is independent of the choice of lattice $\Lambda$ in the sense that every
such reduction has the same composition factors, so we denote the reduction by $\overline{\mathbb{F}_\ell}$. Take the algebraic closure of the finite field $\mathbb{k}$ to view $\overline{\mathbb{F}_\ell}$ as a representation of $G$ over $F = \overline{\mathbb{F}_\ell}$.

As an immediate consequence of Theorem 4.3, it is entirely sufficient to know that $(G, H, \eta)$ is a multiplicity-free triple over the complex numbers to prove that $(G, H, \overline{\eta})$ is a multiplicity-free triple over such an $F$. We state the following result for a character $\eta$ for simplicity, but a similar statement should also hold for general irreducible complex representations $\eta$.

**Corollary 4.8.** Let $F = \overline{\mathbb{F}_\ell}$ of any positive characteristic $\ell$, $G$ be a finite group, $H$ be a subgroup of $G$, and $\eta$ be a one-dimensional representation of $H$ over $\mathbb{C}$. If $(G, H, \eta)$ is a multiplicity-free triple over $\mathbb{C}$, then $(G, H, \overline{\eta})$ is also a multiplicity-free triple over $F$.

**Proof.** Over $\mathbb{C}$, the converse of Theorem 4.4 is also true by Proposition 4.5. Therefore, the Hecke algebra $\mathcal{H}(G, H, \eta, \mathbb{C})$ is commutative. $G$ is a finite group, so $\eta$ is defined over a finite extension $\mathbb{E}$ of $\mathbb{Q}$ with commutative $\mathcal{H}(G, H, \eta, \mathbb{E})$. This commutativity naturally extends to the Hecke algebra $\mathcal{H}(G, H, \eta, \mathbb{K}, \mathcal{O}_\mathbb{K})$ over the ring of integers of an $\ell$-adic field $\mathbb{K}$.

Since $\text{End}_\mathbb{C}(\eta) \cong \mathbb{C}$ and $\text{End}_F(\eta) \cong F$, the Hecke algebra $\mathcal{H}(G, H, \eta, \mathbb{C})$ (resp. $\mathcal{H}(G, H, \eta, F)$) can be described as the space of continuous $\mathbb{C}$-valued (resp. $F$-valued) functions $\Delta$ on $G$ such that $\Delta(h_2gh_1) = \eta(h_2) \cdot \Delta(g) \cdot \eta(h_1)$ (resp. with $\mathbb{C}$), which has a basis over $\mathbb{C}$ (resp. $F$) corresponding to indicator functions on the finite double coset space $H \backslash G / H$. By the definitions of $\mathbb{E}$, $\mathbb{K}$ and $\overline{\eta}$, the Hecke algebras over $\mathbb{E}$, $\mathbb{K}$, and $\mathbb{k}$ have similar descriptions with coefficients restricted from $\mathbb{C}$ and $F$. In particular, there is a homomorphism of Hecke algebras,

$$
\mathcal{H}(G, H, \eta, \mathcal{O}_\mathbb{K}) \longrightarrow \mathcal{H}(G, H, \overline{\eta}, k)
$$

$$
\sum_{\xi \in H \backslash G / H} a_{\xi} \cdot \Delta_{\xi} \longrightarrow \sum_{\xi \in H \backslash G / H} \pi_{\xi} \cdot \overline{\Delta}_{\xi},
$$

where $\Delta_{\xi} : G \to \mathcal{O}_\mathbb{K}$ is the basis element corresponding to the indicator function of $\xi$, $a_{\xi} \in \mathcal{O}_\mathbb{K}$, and $\pi_{\xi}, \overline{\Delta}_{\xi}$ are their reductions modulo $\mathbb{m}_\mathbb{K}$. Since $a_{\xi}$ is arbitrary and elements of $\mathbb{k}$ lift to $\mathcal{O}_\mathbb{K}$, the homomorphism is surjective and therefore the commutativity of $\mathcal{H}(G, H, \eta, \mathcal{O}_\mathbb{K})$ implies the commutativity of $\mathcal{H}(G, H, \eta, \mathbb{K}, \mathcal{O}_\mathbb{K})$.

Finally, the commutativity of $\mathcal{H}(G, H, \eta, \mathbb{K}, \mathcal{O}_\mathbb{K})$ implies the commutativity of $\mathcal{H}(G, H, \overline{\eta}, F)$ by the flatness of commutative Hecke algebras. Hence $(G, H, \overline{\eta})$ is also a multiplicity-free triple over $F$ by Theorem 4.4. □

**Remark 4.9.** For higher-dimensional $\eta$, one must take care to assume that $\overline{\eta}$ is irreducible over $\overline{\mathbb{F}_\ell}$ in order to use Theorem 4.4. For instance, only the Steinberg representation remains irreducible after equal-characteristic reduction for many finite groups of Lie type (cf. Tiep–Zalesskii [TZ02a], [TZ02b], [TZ04]). Reduction modulo $\ell$ is only well-defined up to composition factors, so Corollary 4.8 applies to all reductions modulo $\ell$ (i.e. all choices of lattices) simultaneously (and all reductions modulo $\ell$ are either simultaneously irreducible or simultaneously reducible).

### 4.3. Compact groups

A totally disconnected compact group $G$ is profinite, so there is an inverse system of finite groups $(G_i)_{i \in \mathbb{N}}$ with compatible homomorphisms
Let \( f_i : G_i \to G_i \) for \( i \leq j \) of which \( G \) is a projective limit,

\[
G = \lim_{i \in \mathbb{N}} G_i.
\]

The kernel of each map \( G \to G_i \) is an open normal subgroup \( U_i \), and any open subgroup \( H \) of \( G \) contains one of these open normal subgroups \( U_i \).

A representation of a totally disconnected group is called \textit{smooth} if the stabilizer subgroup of any vector of the representation is open. In particular, smooth representations of profinite groups factor through finite quotients. In this way, the results for representations of finite groups can be directly applied to smooth representations of compact groups, giving a compact version of Corollary 4.8. Here, we consider coinduction of smooth representations, with \( \text{CoInd}_G^H \) representations of compact groups, giving a compact version of Corollary 4.8.

Here, \( \text{CoInd}_G^H \) is an irreducible smooth representation of any finite dimension over \( \mathbb{C} \). When \( G \) is the complex numbers, there are many well-known examples of Gelfand pairs and tools for finding multiplicity-free triples, such as the generalized Bump–Ginzburg criterion (cf. [Vig96, I.1.5.5]).

\textbf{Corollary 4.10.} Let \( F = \mathbb{F}_\ell \) of any positive characteristic \( \ell \), \( G \) be a totally disconnected compact group, \( H \) be a closed subgroup of \( G \), and \( \eta \) be a one-dimensional smooth representation of \( H \) over \( \mathbb{C} \). If \( (G,H,\eta) \) is a multiplicity-free triple over \( \mathbb{C} \), then \( (G,H,\eta) \) is also a multiplicity-free triple over \( F \).

\textbf{Proof.} Since \( \eta \) is smooth irreducible and \( H \) is a closed subgroup of a profinite \( G \), \( \eta \) is finite-dimensional and there is an open normal subgroup \( U_i \) of \( G \) such that \( U_i' := U_i \cap H \) fixes \( \eta \), i.e. \( \eta \) arises from a representation \( \eta^{U_i'} \) of a finite quotient \( H_i := H/U_i' \). Similarly, \( \eta \) is finite-dimensional and factors through a representation \( \eta^{U_i'} \) of a finite quotient \( H_i \). Since \( \eta \) is invariant by \( U_i' \), \( \eta \) is also invariant by \( U_i' \) and \( \eta^{U_i'} \).

The commutativity of \( \mathcal{H}(G, H, \eta, \mathbb{C}) \) implies the commutativity of \( \mathcal{H}(G_i, H_i, \eta^{U_i'}, \mathbb{C}) \), which in turn gives the commutativity of \( \mathcal{H}(G_j, H_j, \eta^{U_j'}, \mathbb{F}_\ell) = \mathcal{H}(G_i, H_i, \eta^{U_i'}, \mathbb{F}_\ell) \) by Corollary 4.8. This argument works for all \( i \geq j \), so each \( \mathcal{H}(G_i, H_i, \eta, \mathbb{F}_\ell) \) is commutative for \( i \geq j \). The representation \( \text{CoInd}_{H_i}^{G_i}(\eta) \) is a direct limit over \( i \geq j \) of \( U_i' \)-invariant representations \( \text{CoInd}_{H_i}^{G_i}(\eta^{U_i'}) \), which are each finite-dimensional. The Hecke algebra \( \mathcal{H}(G, H, \eta, \mathbb{F}_\ell) \) is a projective limit of the commutative \( \mathcal{H}(G_i, H_i, \eta^{U_i'}, \mathbb{F}_\ell) \); therefore, \( \mathcal{H}(G, H, \eta, \mathbb{F}_\ell) \) is commutative.

\textbf{Remark 4.11.} As with Corollary 4.8 this compact version should also hold if \( \eta \) is an irreducible smooth representation of any finite dimension over \( \mathbb{C} \) such that its modulo-\( \ell \) reduction \( \overline{\eta} \) is irreducible over \( F \). Furthermore, the other results of Section 4.4 for finite groups should also hold for profinite groups. One can use the factorization of smooth representations through finite quotients and apply the theory of finite groups as done immediately above, or modify the proofs using suitable profinite analogues (e.g. the equivalence of induction and coinduction for locally profinite groups when \( H \backslash G \) is compact (cf. [Vig96, I.1.5.2]) and Mackey decomposition for locally profinite groups (cf. [Vig96, I.1.5.5]).

\section{Applications for Finite and Compact Groups}

When \( F \) is the complex numbers, there are many well-known examples of Gelfand pairs and tools for finding multiplicity-free triples, such as the generalized Bump–Ginzburg criterion (cf. [CSST20, Theorem 3.2]), so it is not difficult to find cases to use Corollary 4.8 and Corollary 4.10. In this section, we highlight a few applications of Theorem 5.1 for finite groups and compact groups.
Remark 5.1. There are also many useful applications of multiplicity-free triples for locally compact groups. For instance, the Iwasawa decomposition of Gelfand pairs of locally compact groups with compact subgroups has consequences for the non-commutativity of Hecke algebras of Coxeter groups (cf. [Mon20]). It may be more tractable to apply Theorem 5.1 to the theory of automorphic representations, where the condition of finite generation may not be an obstruction for admissible representations of totally disconnected locally compact groups. Multiplicity-one theorems for p-adic groups, such as the uniqueness of Whittaker models by Jacquet–Langlands [IL70] and Shalika [Sha74] as well as its generalizations (e.g. Vigneras [Vig96 III.1.11]), are key tools in the characteristic 0 and characteristic ℓ ≠ p theory.

5.1. Gelfand–Graev and GL(n, q). For G = GL_n(F_q), there are many well-known examples of Gelfand pairs, such as the infinite families (cf. Bannai–Tanaka [BT03]):

- (GL_n(F_q^2), GL_n(F_q));
- (GL_n(F_q^2), GU_n(F_q));
- (GL_{2n}(F_q), SP_{2n}(F_q));
- (GL_{2n}(F_q), GL_n(F_q^2)).

We recall a particularly important twisted Gelfand pair that was first given by Gelfand–Graev [GG62] (cf. [CSST08 Theorem 14.6.3], Piatetski–Shapiro [PS83 Proposition 10.3], and [Bum97 Section 4.1]).

Let G = GL_2(F_q) for a prime power q and consider the unipotent subgroup,

$$\mathbf{U} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G \mid x \in F_q \right\}.$$ 

The result of Gelfand–Graev, reinterpreted in the language of multiplicity-free triples, is that (G, U, η) is a multiplicity-free triple over C for any nontrivial character η of U by virtue of the commutativity of the Hecke algebra H(G, U, η). This multiplicity-freeness result (for complex Gelfand–Graev representations) is the key ingredient in the proof that there are $$\frac{q(q-1)}{2}$$ cuspidal representations of G and that every cuspidal representation of G has dimension q − 1.

The multiplicity-freeness result of Gelfand–Graev generalizes to G = GL_n(F_q) for n > 2. In this case, there is the unipotent subgroup U ⊆ G of the form

$$\mathbf{U} = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \ldots & x_{1k} \\ 1 & x_{23} & x_{23} & \ldots & x_{2k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ldots & \ldots & \ldots & 1 \end{pmatrix} \in G \mid x_{ij} \in F_q \right\},$$

and given a nontrivial character ψ of F_q, a character η_ψ can be defined on U via

$$\eta_\psi((x_{ij})) := \psi \left( \sum_{i=1}^{n-1} x_{i,i+1} \right).$$

In this setting, (G, U, η_ψ) is a multiplicity-free triple over C for any nontrivial character ψ of F_q. By Corollary 123, the multiplicity-freeness result extends to modular representations over F the algebraic closure of any finite field (even if ℓ := char F divides q or the order of G).
Corollary 5.2. Let \( \mathbb{F} = \mathbb{F}_\ell \) of any positive characteristic \( \ell \) and let \( q \) be any prime power. For any nontrivial character \( \psi \) of \( \mathbb{F}_q \), \( (G, U, \eta_\psi) \) is a multiplicity-free triple over \( \mathbb{F} \). In other words, \( (G, U) \) is a \( \eta_\psi \)-twisted Gelfand pair over \( \mathbb{F} \).

Remark 5.3. In the equal characteristic \((\ell | q)\) setting, the \( \psi = \text{triv}_U \) case can be deduced from classical works that \( (G, U, \eta_\psi) \) is a multiplicity-free triple. A theorem of Steinberg [Ste16, Theorem 44(b)] (largely based on a result of Curtis [Cur65, Theorem 4.1] and later extended from finite Chevalley groups to finite groups with a split (B, N)-pair / Tits building by Richen [Ric69, Theorem 3.9], cf. Curtis [Cur70, Theorem 4.3]) says that every irreducible representation of \( G \) over \( \mathbb{F} \) has a unique fixed vector (up to scalars) by \( U \), where the unique fixed vector corresponds to the highest weight. This implies that \((G, U)\) is a Gelfand pair in equal characteristic without the twist by \( \eta_\psi \) considered by Corollary 5.2. Note that this is the only case in the equal characteristic setting, since there are no nontrivial additive characters \( \psi \) when \( \ell | q \).

Corollary 5.2 directly recovers the uniqueness of Whittaker models over all characteristics.

Corollary 5.4. Let \( \mathbb{F} = \mathbb{F}_\ell \) of any positive characteristic \( \ell \) and let \( q \) be any prime power. An irreducible representation of \( GL_n(\mathbb{F}_q) \) over \( \mathbb{F} \) has at most one Whittaker model for a choice of a nontrivial additive character of \( \mathbb{F}_q \).

There are also known cases of complex multiplicity-free triples for more general representations \( \eta \), which therefore extend to modular multiplicity-free triples by Corollary 1.8. For instance, Ceccherini-Silberstein–Scarabotti–Tolli [CSST20, Section 6] proved that for an odd prime power \( q \), \((GL_2(\mathbb{F}_q), C, \eta)\) is a multiplicity-free triple over \( \mathbb{C} \), where \( C \) is the Cartan subgroup

\[
C = \left\{ \begin{pmatrix} a & \zeta b \\ b & a \end{pmatrix} \in G \mid a, b \in \mathbb{F}_q \setminus \{0\}, \zeta \text{ a generator of } \mathbb{F}_q^* \right\} \cong \mathbb{F}_q^2
\]

and \( \eta \) is a multiplicative character of \( \mathbb{F}_q^2 \).

Remark 5.5. Given the connections between the Gelfand–Graev representation and cuspidal representations, this example hints at interesting questions about how modular multiplicity-freeness results could work in the setting of cuspidal representations. In particular, Baruch–Rallis [BR98] defined a notion of (super)cuspidal Gelfand pairs \((G, H)\) that naturally extends to triples \((G, H, \eta)\). It would be useful to develop general modular techniques in this direction, where for over algebraically closed fields of characteristic coprime to odd \( q \), Sécherre [Sec19, Corollary 2.16] and Zou [Zou22, Theorem 1.2] showed that \((GL_2n(\mathbb{F}_q), GL_n(\mathbb{F}_q) \times GL_n(\mathbb{F}_q))\) is a cuspidal Gelfand pair and \((GL_n(\mathbb{F}_q), (GL_n(\mathbb{F}_q))^\tau)\), where \( \tau \) is a unitary involution, is a supercuspidal Gelfand pair.

5.2. Trilinear forms of quaternion division algebras. For the quaternion division algebra \( D_k \) over a local field \( k \), its multiplicative group modulo center \( D_k^*/Z \) is compact (cf. [Car84, Section 6]). Since any irreducible smooth representation of \( D_k^* \) has a central character, any such representation factors through a finite quotient after a twist.

One application of Corollary 1.8 is that the multiplicity-one theorem for trilinear forms on complex irreducible smooth representations of quaternion division algebras over local fields by Prasad [Pra90, Theorem 1.1] can be extended to all characteristics.
Corollary 5.6. Let $F = \mathbb{F}_\ell$ of any positive characteristic $\ell$, $k$ be any local field, and $D_k$ be the quaternion division algebra over $k$. If $\pi_1, \pi_2, \pi_3$ are three irreducible smooth representations of $D_k^*$ over $F$ for some prime power $q$, then there exists at most one (up to isomorphism) non-zero $D_k^*$-invariant linear form on $\pi_1 \otimes \pi_2 \otimes \pi_3$ over $F$.

Proof. Define the tensor product representation $\pi := \pi_1 \otimes \pi_2 \otimes \pi_3$ of $G := D_k^* \times D_k^* \times D_k^*$. Let $H := \Delta D_k^*$ be the diagonal subgroup of $G$. The number of non-zero $D_k^*$-invariant linear forms on $\pi_1 \otimes \pi_2 \otimes \pi_3$ over $F$ is given by

$$\dim_F \text{Hom}_H \left( \text{Res}_G^H(\pi), \text{triv}_H \right),$$

which is at most one if $(G, H)$ is a Gelfand pair.

Over $\mathbb{C}$, the anti-involution $\iota$ of $G$ defined by $\iota(x, y, z) := (\text{tr}(x) - x, \text{tr}(y) - y, \text{tr}(z) - z)$ preserves $H$-cosets ([Pra90, Proposition 3.3]). Therefore, $(G, H)$ is a Gelfand pair over $\mathbb{C}$. By Corollary 4.10, it is also a Gelfand pair over $F$. □

The original result by Prasad over the complex numbers was one of the ingredients for the proof by Harris–Kudla [HK91, HK04] of the Jacquet conjecture on the non-vanishing of the central value of triple product $L$-functions. With the further understanding of modular multiplicity-free triples of locally compact groups, one could consider modular versions of the work of Prasad and Harris–Kudla to describe modular properties of the central value of triple product $L$-functions.

6. Multiplicity-free restrictions

Let $F$ be an algebraically closed field and let $\rho$ be an irreducible $F[S_n]$ representation, where $S_n$ is the symmetric group on $n$ letters. Consider $S_{n-1}$ as the subgroup of $S_n$ consisting of the permutations of the first $n - 1$ letters. If the characteristic of $F$ is zero then the restriction $\text{Res}_{S_{n-1}}^{S_n}(\rho)$ is completely reducible and multiplicity-free, with its composition factors described by the standard branching theorem. If the characteristic of $F$ is non-zero, then the multiplicity of a composition factor can be arbitrarily large [JS92, Corollary 3.3].

Kleschev [Kle95a, Kle95b, Kle98] proved the conjectures of Benson [Ben87] (for characteristic 2) and Jantzen–Seitz [JS92] (for odd characteristic) surrounding the question of when $\text{Res}_{S_{n-1}}^{S_n}(\rho)$ is irreducible for $p > 0$. One can broaden these conjectures to the following question.

Question 6.1 ([KMT20, Problem 1]). Let $F$ be an algebraically closed field. Classify the triples $(G, H, \rho)$, where $H$ is a subgroup of $G$ and $\rho$ is a representation of $G$ over $F$ of dimension greater than 1 such that the restriction $\text{Res}_H^G(\rho)$ is irreducible.

For maximal subgroups of classical algebraic groups over $F = \mathbb{C}$, the study of this question goes back to Dynkin [Dyn52]. For $G = S_n$ or $A_n$, this question is completely answered over fields of characteristic 0 by Saxl [Sax81], over fields of characteristic greater than 3 by Brundan–Kleschev [BK01] and Kleschev–Sheth [KS02], and over fields of characteristic 2 and 3 by Kleschev–Morotti–Tiep [KMT20].

For general fields, the resolution of Question 6.1 has applications to several problems in representation theory, such as the Aschbacher–Scott program on understanding maximal subgroups of finite classical groups (cf. [KMT20, Section 1]).
We can consider a broader question, allowing the restriction $\text{Res}_H^G(\rho)$ to be reducible but asking when it is composed of unique irreducible representations.

**Question 6.2.** Let $F$ be an algebraically closed field. Classify the triples $(G, H, \rho)$, where $H$ is a subgroup of $G$ and $\rho$ is a representation of $G$ over $F$ of dimension greater than 1 such that the restriction $\text{Res}_H^G(\rho)$ is multiplicity-free.

For irreducible representations of classical algebraic groups and reductive groups over $F = \mathbb{C}$, this problem is closely related to the classical works on $H$-varieties and invariant theory of Kac [Kac80, Theorem 3], Weyl [Wey39], and Howe [How93, Chapters 3-4]. Much of this body of work (including extensions by Brion [Bri85], Benson–Ratcliff [BR96], and Leahy [Lea98], as well as the classification for simple algebraic groups of Krämer [Kra76]) can be interpreted as answering Question 6.2 under the more restrictive condition that $\text{Res}_H^G(\rho)$ is multiplicity-free for all irreducible representations $\rho$ of $G$ (cf. [GW09, Section 5.7, Section 12.2] and [LST21, Chapter 1]).

Recently, Liebeck–Seitz–Testerman [LST15, LST21] gave a classification of irreducible representations $\rho$ with multiplicity-free restrictions when $G$ and $H$ are simple algebraic groups of type $A$ over an algebraically closed field of characteristic 0 and $H$ is an irreducible subgroup of $G$. Their result uses the work of Stembridge [Ste03], who answered Question 6.2 for tensor products $\rho_1 \otimes \rho_2$ of irreducible representations of a simple algebraic group $H$ of type $A$, with $G = \text{GL}(\rho_1) \times \text{GL}(\rho_2)$, over an algebraically closed field of characteristic 0.

Question 6.2 is also related to the notion of strong Gelfand pairs. For finite or compact groups, the pair $(G, H)$ is a strong Gelfand pair if
\[
\dim \text{Hom}_H(\rho \mid_H, \eta) \leq 1
\]
for every irreducible representation $\rho$ of $G$ and every irreducible representation $\eta$ of $H$. This is then equivalent to $(G \times H, \Delta H)$ being a Gelfand pair (here $\Delta H$ is the diagonal of $H$ in $H \times H$) and is therefore characterized by the commutativity of a Hecke algebra.

A naive application of Theorem 3.3 and Corollary 4.2(iii) to Question 6.2 is the following corollary.

**Corollary 6.3.** Let $F$ be an algebraically closed field, $H$ be a subgroup of a discrete finitely generated group $G$ such that $\text{Ind}_H^G$ is an exact functor, and $\rho$ be a finite-dimensional representation of $G$. If $\rho$ is $\text{Ind}_H^G(\text{Res}_H^G(\rho))$-injective and $\text{End}_H(\text{Res}_H^G(\rho))$ is commutative, then $\text{Res}_H^G(\rho)$ is multiplicity-free.

**Proof.** By Corollary 4.2(iii) with $M = \rho$ and $N = \text{Res}_H^G(\rho)$, $\text{Res}_H^G(\rho)$ is self-injective. Then by Theorem 3.3 we have the multiplicity-freeness of $\text{Res}_H^G(\rho)$.

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