GRÖBNER STRATA IN THE HILBERT SCHEME OF POINTS

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Abstract. The present paper shall provide a framework for working with Gröbner bases over arbitrary rings \( k \) with a prescribed finite standard set \( \delta \). We show that the functor associating to a \( k \)-algebra \( B \) the set of all reduced Gröbner bases with standard set \( \delta \) is representable and that the representing scheme is a locally closed stratum in the Hilbert scheme of points. We cover the Hilbert scheme of points by open affine subschemes which represent the functor associating to a \( k \)-algebra \( B \) the set of all border bases with standard set \( \delta \) and give reasonably small sets of equations defining these schemes. We show that the schemes parametrizing Gröbner bases are connected; give a connectedness criterion for the schemes parametrizing border bases; and prove that the decomposition of the Hilbert scheme of points into the locally closed strata parametrizing Gröbner bases is not a stratification.

1. Introduction

Let \( k \) be a ring and \( S = k[x_1, \ldots, x_n] \) be the polynomial ring over \( k \). There are various notions of Gröbner bases, and of reduced Gröbner bases, of an ideal \( I \subset S \) (see [Pau07] for an overview). We use that notion of a reduced Gröbner basis which is entirely analogous to definition in the case where \( k \) is a field. The definition will be given in Section 2. The same notion of a reduced Gröbner is used in [Wib07], a paper which was a significant source of inspiration for the work presented here. However, not every ideal \( I \) has a reduced Gröbner basis in this sense; a reduced Gröbner basis exists if, and only if, \( I \) is a monic ideal. Attached to a monic ideal is its standard set, which is the set of those elements of \( \mathbb{N}^n \) which do not occur as the multidegree of an element of \( I \).

If \( B \) is a \( k \)-algebra and \( \delta \) is a finite standard set, we attach to \( B \) the set of all monic ideals \( I \subset B[x] = B[x_1, \ldots, x_n] \) with standard set \( \delta \). As the reduced Gröbner basis of a monic ideal is unique, we may equivalently attach
to $B$ the set of all reduced Gröbner bases in $B[x]$ with standard set $\delta$. It turns out that this map is functorial in $B$. We denote the functor by $\mathcal{Hilb}_{S/k}^\delta$. The notation is motivated by the fact that $\mathcal{Hilb}_{S/k}^\delta$ is a subfunctor of the Hilbert functor of points $\mathcal{Hilb}_{S/k}^r$. The Hilbert functor of points has been widely studied (see [Iar77], [Hui06], [GLS07], [Ber08] and references therein). In particular, it is well-known that this functor is represented by a scheme $\text{Hilb}_{S/k}^r$. The notions of Hilbert functor and Hilbert scheme were introduced by Grothendieck in [Gro95]; see [Nit05] for an accessible account of the subject. In our paper we will show that $\mathcal{Hilb}_{S/k}^\delta$ is a locally closed subfunctor of $\mathcal{Hilb}_{S/k}^r$, hence representable by a locally closed subscheme of the Hilbert scheme. We will study an intermediate functor $\mathcal{Hilb}_{S/k}^\delta$, which is also representable, such that in the chain of representing objects

$$\mathcal{Hilb}_{S/k}^\delta \subset \mathcal{Hilb}_{S/k}^r \subset \mathcal{Hilb}_{S/k}^r,$$

the first inclusion is a closed immersion and the second inclusion is an open immersion.

The moduli spaces $\text{Hilb}_{S/k}^\delta$ and $\text{Hilb}_{S/k}^r$ have been studied by numerous authors, at least in the case where $k$ is a field. In the article [KR08], the scheme $\text{Hilb}_{S/k}^\delta$ is called border basis scheme, and in the article [Rob09], the scheme $\text{Hilb}_{S/k}^\delta$ is called Gröbner basis scheme. In the cited papers, and in [KK05], [KK06], [KKR05], [KR05], a theory of border bases, which generalizes the theory of Gröbner basis, is developed. In the present paper we use border bases as well, in studying the functor $\text{Hilb}_{S/k}^\delta$. Some of the results of the cited papers are parallel to those of our article here. Each time we state one such result, we will indicate its relation to the cited papers. However, our treatment is more general than that of the cited papers, as our $k$ is an arbitrary ring. (This motivates the use of the functorial language, which is not necessary if $k$ is a field.) Moreover, several of our results are novel, or stronger than previous results, even if we specialize to the case where $k$ is a field.

Another line of work is to study strata analogous to ours in Grothendieck’s classical Hilbert scheme $\text{Hilb}_{\mathbb{P}^n_k}^{p(z)}$, where $p(z)$ is a polynomial. The paper [NS00] is devoted to this project; the equations defining the strata are derived from Buchberger’s $S$-pair criterion. However, the cited paper contains a few inconsistencies, which are being corrected in [Rob09] and [LR09]. In particular, in the latter paper, the embedding of the strata in $\text{Hilb}_{\mathbb{P}^n_k}^{p(z)}$ is elaborated upon with care. Also, it appears to be the first paper in which the term Gröbner stratum is used. Other papers in which related ideas appear are [CF88] and [RT08].
As was mentioned above, the research presented here was largely inspired by the paper [Wib07]. In that paper, \( k \) is a noetherian ring. Wibmer considers an arbitrary ideal \( I \subset S \). The canonical map \( k \to S/I \) corresponds to a morphism of affine schemes \( \phi : \text{Spec} \, S/I \to \text{Spec} \, k \). The main theorem of [Wib07] (Theorem 11) states the existence of a unique decomposition of \( \text{Spec} \, k \) into a finite number of locally closed strata such that on each stratum the reduction of \( I \) to each point of the stratum has a reduced Gröbner basis of a prescribed shape. It is striking to note the analogy of that theorem to Theorem 2 of our paper here. However, in Wibner’s setting \( k \) has to be noetherian, whereas our setting requires no restriction on \( k \).

Our article is organized as follows. In Section 2, we introduce the basic notions of monic ideals, reduced Gröbner bases and standard sets. In Section 3, we define the Hilbert functor \( \text{Hilb}_S^r/k \) and the open subfunctors \( \text{Hilb}^\delta_S/k \), where \( \delta \) runs through all standard sets of size \( r \). In Section 4, we thoroughly prove that these subfunctors cover the whole functor \( \text{Hilb}^r_S/k \). That gives us the key to defining the subfunctor \( \text{Hilb}^\delta_S/k \) of \( \text{Hilb}_S^r/k \) in Section 5. In Section 6, we show that from representability of \( \text{Hilb}^\delta_S/k \), representability of \( \text{Hilb}^\delta_S/k \) follows. In Section 7, we show that the Hilbert scheme \( \text{Hilb}^r_S/k \) is the disjoint union of the representing schemes \( \text{Hilb}^\delta_S/k \). So far the techniques we use are non-explicit in the sense that we use abstract representability criteria for functors rather than explicit descriptions of representing schemes. Once functoriality is proved, we can turn to more concrete questions. In Section 8, we write down explicitly a set of equations defining the affine schemes \( \text{Hilb}^\delta_S/k \) and \( \text{Hilb}^\delta_S/k \). In Section 9, we study a few examples and improve the result of the previous section in shrinking the set of equations defining the affine schemes. In Section 10, we write down the universal objects of the functors \( \text{Hilb}^\delta_S/k \) and \( \text{Hilb}^\delta_S/k \), which are affine schemes over \( \text{Hilb}^\delta_S/k \) and \( \text{Hilb}^\delta_S/k \), resp. In Section 11, we present a homogeneous variant of what we have done so far, show that \( \text{Hilb}^\delta_S/k \) is connected and give a connectedness criterion for \( \text{Hilb}^\delta_S/k \). In Section 12, we explore the transition maps between \( \text{Hilb}^\delta_S/k \) and \( \text{Hilb}^\delta_S/k \). In Section 13, we use result of the previous section for tracking \( \text{Hilb}^\delta_S/k \) in \( \text{Hilb}^\delta_S/k \) and show that the decomposition of Theorem 2 in general is not a stratification.

Let us conclude the introduction with a summary of philosophy of this article:

- A good notion of Gröbner basis over an arbitrary ring is that of the reduced Gröbner basis of a monic ideal.
- The reducedness property guarantees functoriality of Gröbner bases.
- Functoriality guarantees the existence of a moduli space.
Some familiar techniques for Gröbner bases over fields can be carried
over to the situation over rings.

S-pair criteria as such are not needed.

2. Notation

We start by collecting the relevant definitions and facts concerning elements
and ideals in $S$. Throughout, a monomial order $\prec$ on $S$ will be fixed. This
is, in particular, a total order on the set of monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$,
where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Let $f \in S$, then the monomial order gives the
well-known definitions of

- coefficient $\text{coef}(f, x^\alpha)$ of $f$ at $x^\alpha$;
- support $\text{supp}(f)$, which is the set of all $x^\alpha$ such that $\text{coef}(f, x^\alpha) \neq 0$;
- leading monomial $\text{LM}(f)$;
- leading coefficient $\text{LC}(f)$;
- leading term $\text{LT}(f)$, which equals $\text{LC}(f)\text{LM}(f)$;
- leading exponent (or multidegree) $\text{LE}(f)$, for which $\text{LM}(f) = x^{\text{LE}(f)}$;
- the non-leading exponents, which are those $\alpha$ such that $x^\alpha$ lies in $\text{supp}(f)$ but does not equal $\text{LM}(f)$.

If $I \subseteq S$ is an ideal, we let $\text{LM}(I)$ be the set of all $\text{LM}(f)$, where $f$ runs
through $I$. This set is closed with respect to multiplication by arbitrary
monomials. Analogously, we let $\text{LT}(I)$ be the set of all $\text{LT}(f)$, where $f$ runs
through $I$. This set is also closed with respect to multiplication by arbitrary
monomials. However, if $k$ is not a domain, it may not be closed with respect
to multiplication by arbitrary terms. Clearly $\text{LT}(I)$ in general carries more
information than $\text{LM}(I)$.

If $I$ is an ideal in $S$ and $x^\alpha$ is a monomial, the set

$$\text{LC}(I, x^\alpha) = \{\text{LC}(f); f \in I, f \neq 0, \text{LM}(f) = x^\alpha\} \cup \{0\}$$

is an ideal in $k$. An ideal $I$ is called monic (see [Pau92] or [Wib07]) if the
following equivalent conditions are satisfied:

- For all monomials $x^\alpha$, the ideal $\text{LC}(I, x^\alpha)$ is either the zero ideal or
  the unit ideal;
- each element of $\text{LM}(I)$ arises as the leading monomial of a monic
  $f \in I$;
- $\text{LT}(I)$ is a monomial ideal;
- the sets $\text{LM}(I)$ and $\text{LT}(I)$ carry the same information.

We will mostly be working with leading exponents, more precisely, with the
set $\text{LE}(I)$, which is the set of all $\text{LE}(f)$, where $f$ runs through $I$. Clearly
$\text{LE}(I)$ carries the same information as $\text{LM}(I)$. Therefore it carries the same
information as \( \text{LT}(I) \) if, and only if, \( I \) is monic. In fact, we will not be working with \( \text{LE}(I) \) itself but rather with its complement in \( \mathbb{N}^n \):

**Definition 1.**

- A **standard set** (or **staircase**, or **Gröbner escalier**) in \( \mathbb{N}^n \) is a subset \( \delta \subset \mathbb{N}^n \) such that its complement in \( \mathbb{N}^n \) is closed with respect to addition with elements of \( \mathbb{N}^n \). (Equivalently, standard sets are precisely the complements of the sets \( \text{LE}(I) \), where \( I \) runs through the ideals in \( S \).)
- If for a given \( I \) we have \( \text{LE}(I) = \mathbb{N}^n - \delta \), we say that \( \delta \) is the **standard set attached to** \( I \).
- The set \( \mathcal{C}(\delta) \) of **corners** of \( \delta \) is the set of all \( \alpha \in \mathbb{N}^n - \delta \) such that for all \( i \), \( \alpha - e_i \notin \mathbb{N}^n - \delta \), where \( e_i \) is the \( i \)-standard basis vector.
- The **border** of \( \delta \) is the set \( \mathcal{B}(\delta) = \bigcup_{i=1}^{n}(\delta + e_i) - \delta \).
- An **edge point** of \( \delta \) is an \( \alpha \in \delta \) such that \( \alpha \in \mathbb{N}e_i \oplus \mathbb{N}e_j \) for some \( i \neq j \) (or equivalently, only the \( i \)-th and the \( j \)-coordinate of \( \alpha \) does not vanish), and \( \alpha + e_i \) and \( \alpha + e_j \) both do not lie in \( \delta \).

An example is depicted in Figure 1, in which the standard set is dawn with thick lines, the corners are marked by bullets, all other points in the border are marked by circles, and the edge points are marked with boxes.

![Figure 1](image)

**Figure 1.** A standard set, ◦ its border, • its corners, and □ its edge points.

A **Gröbner basis** of an ideal \( I \) is a finite subset \( G \) of \( I \) such that the ideals in \( S \) generated by \( \text{LT}(I) \) and \( \text{LG}(g) \), for \( g \in G \), agree. Note that not every ideal \( I \) necessarily admits a Gröbner basis, since \( k \) was not assumed to be noetherian. A Gröbner basis \( G \) is called **reduced** if \( \{ \text{LE}(g); g \in G \} = \mathcal{C}(\delta) \), where \( \text{LE}(I) = \mathbb{N}^n - \delta \); each \( g \in G \) is monic; and all non-leading exponents of \( g \) lie in the standard set attached to \( I \). An ideal \( I \) admits a reduced Gröbner basis if, and only if, \( I \) is monic. (See [Pau92], [Asc05], [Wib07],)
3. The Hilbert functor of points and its standard subfunctors

Fix a positive integer \( r \). We consider the Hilbert functor of points
\[ \mathcal{Hilb}^r_{S/k} : (k\text{-Alg}) \to (\text{Sets}) \]
(2)
\[ B \mapsto \{ \phi : B[x] \to Q \} / \sim, \]
which associates to a \( k \)-algebra \( B \) the set of all equivalence classes of surjective \( B \)-algebra homomorphisms \( \phi : B[x] \to Q \), where \( Q \) is a \( B \)-algebra and locally free as a \( B \)-module. We say that \( \phi : B[x] \to Q \) and \( \phi' : B[x] \to Q' \) are equivalent if there exists a \( B \)-algebra isomorphism \( \psi : Q \to Q' \) such that the following diagram commutes:
\[
\begin{array}{ccc}
B[x] & \xrightarrow{\phi} & Q \\
\downarrow{id} & & \downarrow{\psi} \\
B[x] & \xrightarrow{\phi'} & Q'.
\end{array}
\]
Therefore \( \phi \) and \( \phi' \) are equivalent if, and only if, their kernels agree. In this sense, the functor \( \mathcal{Hilb}^r_{S/k} \) parametrizes all ideals in the polynomial ring \( S \) which are locally free of codimension \( r \).

At this point a remark on local freeness is in order. In the literature, one can find at least two definitions of when a \( B \)-module is locally free (see [Eis95], p.137). The first is to demand that for each prime ideal \( p \subset B \), the localized module \( M_p \) is free over the localized ring \( B_p \). (Equivalently, one can also demand that for each maximal ideal \( m \subset B \), \( M_m \) is free over \( B_m \).) The second is to demand that there exist \( f_1, \ldots, f_t \in B \) generating the unit ideal such that each localization \( M[f_i^{-1}] \) is a free \( R[f_i^{-1}] \)-module. The second definition (which is used e.g. in [HS04]) is stronger. However, if the module \( M \) is locally free of a finite rank \( r \), both definitions agree.

Our first goal is to cover the functor \( \mathcal{Hilb}^r_{S/k} \) by a finite collection of open subfunctors, indexed by all standard sets of size \( r \). We shall now define these subfunctors. Given a standard set \( \delta \), we use the shorthand notation \( x^\delta \) for the family \((x^\beta)_{\beta \in \delta}\) and \( kx^\delta = \bigoplus_{\beta \in \delta} kx^\beta \). We consider the inclusion
\[
i_{\delta} : kx^\delta \to S \\
x^\beta \mapsto x^\beta.
\]

**Definition 2.** Let \( \delta \) be a standard set of size \( r \). We define \( \mathcal{Hilb}^\delta_{S/k} \) to be the subfunctor of \( \mathcal{Hilb}^r_{S/k} \) which associates to each \( k \)-algebra \( B \) the set of equivalence classes of all \( \phi : B[x] \to Q \) as in (2) such that the composition
\[
Bx^\delta = B \otimes_k kx^\delta \xrightarrow{id \otimes \iota_{\delta}} B[x] = B \otimes_k S \xrightarrow{\phi} Q
\]
is surjective, and therefore an isomorphism.
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(The same functors are considered also in [Hui06].) In particular, all $Q$ appearing in $\mathcal{H}ilb_{S/k}^\delta(B)$ are free $B$-modules of rank $r$. Evidently there are the following alternative descriptions of the subfunctor,

$$\mathcal{H}ilb_{S/k}^\delta(B) = \{ \text{classes of surjective } \phi : B[x] \to Q \text{ s.t. } (x^\beta + \ker \phi)_{\beta \in \delta} \text{ is a } B\text{-basis of } B[x]/\ker \phi \},$$

and also

$$\mathcal{H}ilb_{S/k}^\delta(B) = \{ \text{classes of surjective } \phi : B[x] \to Q \text{ s.t. } (\phi(x^\beta))_{\beta \in \delta} \text{ is a } B\text{-basis of } Q \}. $$

Since $Q$ is free of rank $r$, that module must be isomorphic to $Bx^\delta$. Upon fixing an isomorphism $Q = Bx^\delta$ and requiring that $\phi \circ (\text{id} \otimes \iota_\delta) = \text{id}$, we can rephrase the functor $\mathcal{H}ilb_{S/k}^\delta$ as follows:

$$\mathcal{H}ilb_{S/k}^\delta = \{ \phi : B[x] \to Bx^\delta; \phi \text{ is a } k\text{-algebra homomorphism s.t. } \phi \circ (\text{id} \otimes \iota_\delta) = \text{id} \}. $$

(The multiplicative structure on $Bx^\delta$, making this module a $B$-algebra, is induced by that on $B[x]$ by the Homomorphism Theorem.) In what follows, we will shift freely between the descriptions (3), (4) and (5).

One can replace the homomorphism $\iota_\delta$ by an arbitrary $k$-module homomorphism $\phi : k^r \to S$ and define a functor $\mathcal{H}ilb_{S/k}^\phi$ analogous to the above. Such functors have been used in [GLS07]. The authors of that paper state that $\mathcal{H}ilb_{S/k}^\phi$ is an open subfunctor of $\mathcal{H}ilb_{S/k}^r$ and give a sketch of proof for this. For preparing the ground for the next sections, we carefully prove openness of the subfunctor $\mathcal{H}ilb_{S/k}^\delta$ of $\mathcal{H}ilb_{S/k}^r$ here. The proof for a functor $\mathcal{H}ilb_{S/k}^\phi$ as in [GLS07] is entirely analogous.

Given a $k$-scheme $X$, we denote by $h_X$ the Hom functor which sends a $k$-scheme $Y$ to the set $\text{Mor}_k(Y, X)$. We want to show that the canonical inclusion $i : \mathcal{H}ilb_{S/k}^\delta \to \mathcal{H}ilb_{S/k}^r$ makes the first functor an open subfunctor of the second. By [EH00], Definition VI-5, we have to check that for each $k$-algebra $B$ and each morphism of functors $\psi : h_{\text{Spec } B} \to \mathcal{H}ilb_{S/k}^r$, the above horizontal arrow in the cartesian diagram

$$\xymatrix{ \mathcal{G} & h_{\text{Spec } B} \ar[d]^\psi \\
\mathcal{H}ilb_{S/k}^\delta \ar[r]^i & \mathcal{H}ilb_{S/k}^r }$$

is isomorphic to the the inclusion of functors $h_U \to h_{\text{Spec } B}$ induced by the inclusion of schemes $U \to \text{Spec } B$ where $U$ is an open subscheme of $\text{Spec } B$.

So let an arrow $\psi : h_{\text{Spec } B} \to \mathcal{H}ilb_{S/k}^r$ be given. By Yoneda’s Lemma this is an element of $\mathcal{H}ilb_{S/k}^r(B)$, therefore the equivalence class of a surjective
$\phi : B[x] \to Q$. After localizing in $B$ at $f_1, \ldots, f_s \in B$ generating the unit ideal, we may assume that $Q$ is a free $B$-module of rank $r$. Further, let $\rho : k \to B$ be the structure morphism of the $k$-algebra $B$. The functor $\mathcal{G}$ in the cartesian diagram (6) associates to each $k$-algebra $A$ the set of all pairs $(g, h)$ in $h_{\text{Spec} B}(\text{Spec} A) \times \text{Hilb}^B_{S/k}(A)$ such that $\psi(g) = i(h)$ in $\text{Hilb}^B_{S/k}$. However, $g$ is nothing but a $k$-algebra homomorphism $\gamma : B \to A$, and $h$ is nothing but (the equivalence class of) a $k$-algebra homomorphism $\eta : A[x] \to Q'$. Therefore, the condition $\psi(g) = i(h)$ says that the morphisms

$$\phi \otimes \gamma : A[x] \otimes_B A = B[x] \to Q \otimes_B A$$

and

$$\eta : A[x] \to Q'$$

are in the same equivalence class. After localizing also at certain elements of $A$, we may assume that $Q'$ is free of rank $r$. We now fix isomorphisms $Q \otimes_B A = Ax^\delta$ and $Q' = Ax^\delta$ and accordingly demand that $\phi \otimes \gamma = \eta$. Then the condition making the diagram cartesian is that $\eta$ lies in $\text{Hilb}^B_{S/k}(A)$. In other words, we have reformulated the functor $\mathcal{G}$ as follows: $\mathcal{G}(A)$ is the set of all $\gamma : B \to A$ such that $\phi \otimes \gamma : A[x] \to Ax^\delta$ is an $A$-algebra homomorphism and

$$(\phi \otimes \gamma) \circ (\iota_\delta \otimes (\gamma \circ \rho)) : Ax^\delta \to A[x] \to Ax^\delta$$

is an isomorphism. Consider the special case $B = B$, $\gamma = \text{id} : B \to B$ and the composition

$$\phi \circ (\iota_\delta \otimes \rho) : Bx^\delta \to B[x] \to Bx^\delta.$$ 

Let $M$ be the matrix of this $B$-module homomorphism, and $J \subset B$ be the ideal generated by $\det(M)$. Then clearly, for any $\gamma : B \to A$, the composition (7) is an isomorphism if, and only if, $A = A\gamma(J)$. By exercise VI-6 of [EH00], we are done.

4. The standard covering

In Section 5.2 of [GLS07], the authors show with a very quick argument that their functors $\text{Hilb}^B_{S/k}$, where $\phi$ runs through all homomorphisms $B^r \to B[x]$, form an open cover of the functor $\text{Hilb}^r_{S/k}$, and also that there exists a finite set of such subfunctors which covers $\text{Hilb}^r_{S/k}$. We now show that our subfunctors $\text{Hilb}^\delta_{S/k}$, which are also finite in number, suffice to cover $\text{Hilb}^r_{S/k}$.

**Proposition 1.** The functors $\text{Hilb}^\delta_{S/k}$, where $\delta$ runs through all standard sets of size $r$, form an open cover of the functor $\text{Hilb}^r_{S/k}$. Moreover, this cover is minimal in the sense that when removing any member of it, the result is no longer a cover.
Proof. Let $B$ be a $k$-algebra and $\phi : B[x] \to Q$ be a $B$-algebra homomorphism representing an element of $\mathcal{H}ilb^r_{\mathcal{O}_{/k}}(B)$, and let $m \subset B$ be a maximal ideal. We use the localization $B_m$ and its residue field $\kappa = B_m/mB_m$. Upon tensoring $\phi$ with $B_m$ and $\kappa$, respectively, we obtain the extensions
\[
\phi_m : B_m[x] \to Q_m \quad \text{and} \quad \phi_\kappa : \kappa[x] \to Q_\kappa.
\]
By assumption, $Q$ is locally free of rank $r$, i.e., there exist an $f \in B - m$ such that $Q_f = \bigoplus_{i=1}^r B_f \epsilon_i$. Localizing further, we get $Q_m = \bigoplus_{j=1}^r B_m \epsilon_j$. Taking residue classes, we get $Q_\kappa = \bigoplus_{j=1}^r \kappa \epsilon_j$. Local freeness of $Q$ and surjectivity of $\phi$ imply that both maps in (8) are surjective. Since $\kappa$ is a field, the ideal $\ker \phi_\kappa$ has a Gröbner basis w.r.t. $\prec$, with a standard set $\delta$ attached to it.

As $Q_\kappa$ has dimension $r$, the standard set has size $r$. The family $x^\beta + \ker \phi_\kappa$, where $\beta$ runs through $\delta$, is a $\kappa$-basis of $\kappa[x]/\ker \phi_\kappa$. Therefore the family $\phi_\kappa(x^\beta)$, where $\beta$ runs through $\delta$, is a $\kappa$-basis of $Q_\kappa$. From the commutative diagram
\[
\begin{array}{ccc}
B_m[x] & \xrightarrow{\phi_m} & Q_m \\
\downarrow \text{can} & & \downarrow \text{can} \\
\kappa[x] & \xrightarrow{\phi_\kappa} & Q_\kappa,
\end{array}
\]
where the vertical arrows are the canonical map, we see that $\phi_m(x^\beta)$ is a lift of $\phi_\kappa(x^\beta)$ w.r.t. the canonical map. Nakayama’s Lemma implies that the family $\phi_m(x^\beta)$, where $\beta$ runs through $\delta$, generates the $B_m$-module $Q_m$. As the rank of $Q_m$ is $r = \# \delta$, this family is even a $B_m$-basis.

Therefore the composition
\[
\phi_m \circ \iota_\delta : B_m x^\delta \to B_m[x] \to Q_m = \bigoplus_{i=1}^r B_m \epsilon_i
\]
is an isomorphism. Going from left to right, we write the image of the basis element $x^\gamma$ under the composition as
\[
(\phi_m \circ \iota_\delta)(x^\beta) = \sum_{i=1}^r c_{\beta,i} \epsilon_i.
\]
Going from right to left, we write the image of the basis element $\epsilon_i$ as
\[
(\phi_m \circ \iota_\delta)^{-1}(\epsilon_i) = \sum_{\beta \in \delta} d_{i,\beta} x^\beta.
\]
Here all $g_{\beta,i}$ and all $h_{i,\beta}$ lie in $B - m$. We set
\[
h = \left( \prod_{\beta \in \delta} \prod_{i=1}^r g_{\beta,i} \right) \cdot \left( \prod_{i=1}^r \prod_{\beta \in \delta} h_{i,\beta} \right)
\]
and $g = fh$. (Remember that $f$ is the element of $B - m$ with respect to which we localized earlier.) Then $B_g = (B_f)h$ and therefore $Q_g = \bigoplus_{i=1}^r B_g \epsilon_i$. 

\[
\begin{array}{ccc}
\phi_m : B_m[x] & \to Q_m \\
\phi_\kappa : \kappa[x] & \to Q_\kappa,
\end{array}
\]
Formulas (9) and (10) define homomorphisms
\[ B_m x^\delta \to \bigoplus_{i=1}^r B_m \epsilon_i \]
and
\[ \bigoplus_{i=1}^r B_m \epsilon_i \to B_m x^\delta, \]
resp., which are obviously inverses of each other.

We have shown that for all \( B \in (k\text{-}\text{Alg}) \), for all maximal ideals \( m \subset B \) and for all \( \phi \in \text{Hilb}_{S/k}(B) \) there exist a \( g \in B - m \) and a standard set \( \delta \) of size \( r \) such that the localization
\[ (\phi \otimes \text{id}_{B_f}) \circ (t_\delta \otimes \text{id}_{B_f}) : B_g x^\delta \to B_g[x] \to Q_g \]
is an isomorphism. Therefore the various \( \text{Hilb}_{S/k}^\delta \) cover \( \text{Hilb}_{S/k}^r \).

5. Gröbner bases in the standard subfunctors

Let us further investigate the functor \( \text{Hilb}_{S/k}^\delta \). Let \( B \) be a \( k \)-algebra, \( m \subset B \) a maximal ideal and \( \phi \in \text{Hilb}_{S/k}^\delta(B) \). In the course of the proof of Proposition 1, we made use of polynomials lying in the ideal \( \ker \phi_\kappa \). Since \( \delta \) is the standard set attached to the ideal \( \ker \phi_\kappa \), each element of the reduced Gröbner basis of \( \ker \phi_\kappa \) can be expressed as
\[ f_\alpha = x^\alpha + \sum_{\beta \in \delta} c_{\alpha,\beta} x^\beta, \text{ where } c_{\alpha,\beta} = 0 \text{ if } \alpha \prec \beta. \]

The latter condition guarantees that \( \text{LE}(f_\alpha) = \alpha \). A priori a polynomial as in (11) exists only for all \( \alpha \in \mathcal{C}(\delta) \); the collection \( \{ f_\alpha; \alpha \in \mathcal{C}(\delta) \} \), is the reduced Gröbner basis, which is unique; therefore the polynomial of (11) is unique for all \( \alpha \in \mathcal{C}(\delta) \). The following lemma (applied to \( R = \kappa, I = \ker \phi_\kappa \)) implies that a polynomial as in (11) exists and is unique for all \( \alpha \in \mathbb{N}^n - \delta \).

**Lemma 1.** Let \( R \) be a ring and \( \delta \) a standard set. Assume that for all \( \alpha \in \mathcal{C}(\delta) \), there exists a monic \( f_\alpha \in R[x] \) such that \( \text{LE}(f_\alpha) = \alpha \) and all non-leading exponents of \( f_\alpha \) lie in \( \delta \). Define \( I \) to be the ideal \( (f_\alpha; \alpha \in \mathcal{C}(\delta)) \) in \( R[x] \). Then the following statements hold.

(i) For all \( \alpha \in \mathbb{N}^n - \delta \), there exists a unique \( f_\alpha \in I \) such that \( \text{LE}(f_\alpha) = \alpha \) and all non-leading exponents of \( f_\alpha \) lie in \( \delta \).
(ii) All coefficients of all \( f_\alpha \) are polynomial expressions with coefficients in \( \mathbb{Z} \) of \( \text{coef}(f_\alpha, x^\beta) \), for \( \alpha \in \mathcal{C}(\delta) \), \( x^\beta \in \text{supp}(f_\alpha) \).

(iii) If \( \text{LE}(I) = \delta \), then \( I \) is monic with reduced Gröbner basis \( (f_\alpha)_{\alpha \in \mathcal{C}(\delta)} \).

Moreover, the family \( (f_\alpha)_{\alpha \in \mathbb{N}^n - \delta} \) is an \( R \)-basis of the module \( I \).

This lemma is apparently well-known, at least in the case where \( R \) is field. However, as was mentioned in [Led09], it is hard to find reference for it in the literature. We will need the inductive construction of the polynomials \( f_\alpha \) of (i) in the proof of Proposition 5 below. (That construction is essentially the whole proof of the lemma.)

We have seen that by Nakayama’s Lemma the family of all \( x^\beta \), where \( \beta \) runs through \( \delta \), is a \( B_m \)-basis of \( B_m[x]/\ker \phi_m \). Therefore each polynomial \( f_\alpha \in \ker \phi_\kappa \) as in (11), for \( \alpha \in \mathbb{N} - \delta \), has a unique lift to an element

\[
\tilde{f}_\alpha = x^\alpha + \sum_{\beta \in \delta} c_{\alpha, \beta} x^\beta
\]

of \( \ker \phi_m \). However, though \( c_{\alpha, \beta} = 0 \) for \( \alpha < \beta \), the coefficients \( \tilde{c}_{\alpha, \beta} \) need not be zero for \( \alpha < \beta \).

**Proposition 2.** The ideal \( \ker \phi_m \) is monic with Gröbner basis \( \tilde{f}_\alpha \), for \( \alpha \in \mathcal{C}(\delta) \), if, and only if, \( \tilde{c}_{\alpha, \beta} = 0 \) for all \( \alpha \in \mathcal{C}(\delta) \) and for all \( \beta \in \delta \) such that \( \alpha < \beta \).

**Proof.** This is a consequence of Lemma 1. \( \square \)

In the complementary case, the set \( \{ f_\alpha; \alpha \in \mathcal{B}(\delta) \} \) is still the *border basis* of \( \ker \phi_m \), in the terminology of [KR05]. In our context, border bases are best described as follows. Take a \( k \)-algebra \( B \) and a \( \phi : B[x] \to Q \) in \( \text{Hilb}^S_{S/k} \). Let \( \alpha \in \mathbb{N}^n - \delta \), then by (3), there exist unique \( d_{\alpha, \beta} \in B \), for \( \beta \in \delta \), such that

\[
x^\alpha + \sum_{\beta \in \delta} d_{\alpha, \beta} x^\beta = 0 \in B[x]/\ker \phi,
\]

or equivalently,

\[
f_\alpha = x^\alpha + \sum_{\beta \in \delta} d_{\alpha, \beta} x^\beta \in \ker \phi.
\]

The collection \( \{ f_\alpha; \alpha \in \mathcal{B}(\delta) \} \) is the border basis of \( \ker \phi \). If in addition \( \ker \phi \) is monic with standard set \( \delta \), then Lemma 1 implies that the collection \( \{ f_\alpha; \alpha \in \mathcal{C}(\delta) \} \) is the reduced Gröbner basis of \( \ker \phi \). In this sense the notion of border bases is a generalization of the notion of Gröbner bases. The goal of the next section is to exhibit that observation in the language of Hilbert functors.
6. The Gröbner subfunctors

In [Hui06], [GLS07] and [Ber08], the authors show that the functor \( \mathcal{H} \text{ilb}^{\delta}_{S/k} \) is representable by an affine scheme. We make use of this fact in this section, denoting by \( \mathcal{H} \text{ilb}^{\delta}_{S/k} \) the representing scheme. (We will give explicit descriptions of the coordinate ring of this scheme in Sections 8 and 9.) Proposition 2 suggests to consider the following elements of \( \mathcal{H} \text{ilb}^{\delta}_{S/k}(B) \):

**Definition 3.** For each \( k \)-algebra \( B \), let \( \mathcal{H} \text{ilb}^{\delta}_{S/k}(B) \) be the set of equivalence classes of surjective \( B \)-algebra homomorphisms \( \phi : B[x] \to Q \) such that \( \ker \phi \) has a reduced Gröbner basis of the form

\[
f_\alpha = x^\alpha + \sum_{\beta \in \delta, \beta < \alpha} d_{\alpha, \beta} x^\beta,
\]

where \( \alpha \) runs through \( \mathcal{C}(\delta) \).

As was mentioned in Section 2, an ideal admits a reduced Gröbner basis if, and only if, it is monic. Moreover, the equivalence class of a surjective \( B \)-algebra homomorphism \( \phi : B[x] \to Q \) is determined by its kernel, and as a monic ideal is determined by its reduced Gröbner basis. This gives us the following alternative characterizations of \( \mathcal{H} \text{ilb}^{\delta}_{S/k}(B) \) as:

- the set of equivalence classes of surjective \( \phi : B[x] \to Q \) such that \( \ker \phi \) is a monic ideal with standard set \( \delta \).
- the set of all monic ideals in \( B[x] \) with standard set \( \delta \).
- the set of all reduced Gröbner bases in \( B[x] \) with standard set \( \delta \).

**Lemma 2.** \( \mathcal{H} \text{ilb}^{\delta}_{S/k} \) is a subfunctor of \( \mathcal{H} \text{ilb}^{\delta}_{S/k} \).

**Proof.** Let \( \phi : B[x] \to Q \) be an element of \( \mathcal{H} \text{ilb}^{\delta}_{S/k}(B) \). The division algorithm (see [CLO97], Section 2, §3) shows that the family \( (x^\beta + \ker \phi) \), where \( \beta \) runs through \( \delta \), is a \( B \)-basis of \( B[x]/\ker \phi \). By the Homomorphism Theorem, the family \( \phi(x^\beta) \), where \( \beta \) runs through \( \delta \), is a \( B \)-basis of \( Q \). Hence \( \phi : B[x] \to Q \) is also an element of \( \mathcal{H} \text{ilb}^{\delta}_{S/k}(B) \). In particular, we may assume that \( Q = Bx^\delta \).

We show that \( \mathcal{H} \text{ilb}^{\delta}_{S/k} \) is a functor. Let

\[
\phi : B[x] \to Bx^\delta
\]

be an element of \( \mathcal{H} \text{ilb}^{\delta}_{S/k}(B) \) and \( \psi : B \to A \) be a \( k \)-algebra homomorphism. Tensoring is right exact, hence a surjective homomorphism

\[
\phi \otimes \text{id} : A[x] \to Ax^\delta.
\]
We have to show that \( \ker \phi \otimes \text{id} \) is monic with standard set \( \delta \). For this, we write the elements of the reduced Gröbner basis of \( \ker \phi \) as in formula (13). We define
\[
g_\alpha = x_\alpha + \sum_{\beta \in \delta, \beta < \alpha} \psi(d_{\alpha, \beta})x_\beta,
\]
for all \( \alpha \in \mathcal{C}(\delta) \). Then clearly all \( g_\alpha \) lie in \( \ker (\phi \otimes \text{id}) \). By Lemma 1 (i), we get a unique polynomial of the form
\[
g_\alpha = x_\alpha + \sum_{\beta \in \delta, \beta < \alpha} e_{\alpha, \beta}x_\beta
\]
even for all \( \alpha \in \mathbb{N}^n - \delta \), and in particular, all these \( g_\alpha \) lie in \( \ker (\phi \otimes \text{id}) \). Now let \( g \) be an arbitrary element of \( \ker (\phi \otimes \text{id}) \). Denote the leading term of \( g \) by \( cx_\mu \). We have to show that \( \mu \) lies in \( \mathbb{N}^n - \delta \), as in this case, Lemma 1 (iii) guarantees that \( \ker (\phi \otimes \text{id}) \) is monic with standard set \( \delta \). Consider the polynomial
\[
g' = g - \sum_{\beta \in \mathbb{N}^n - \delta, \beta < \mu} \text{coef}(g, x_\beta)g_\beta.
\]
Then \( g' \) lies in \( \ker (\phi \otimes \text{id}) \); its support is contained in \( \delta \cup \{ \mu \} \); and its leading term is \( cx_\mu \). However, as \( \phi \otimes \text{id} \) lies in \( \text{Hilb}_{S/k}^\delta (A) \), we know that the family \( x_\beta + \ker (\phi \otimes \text{id}) \), where \( \beta \) runs through \( \delta \), is a basis of \( A[x]/\ker (\phi \otimes \text{id}) \). This shows that if \( \mu \in \delta \), then \( c = 0 \), a contradiction. Hence \( \mu \in \mathbb{N}^n - \delta \).  

**Lemma 3.** \( \text{Hilb}_{S/k}^\delta \) is a Zariski sheaf.

*Proof.* Let \( B \) be a \( k \)-Algebra, \( (U_i = \text{Spec } B_{g_i})_{i \in I} \) an open cover of \( \text{Spec } B \) by distinguished open sets and \( \phi_i \in \text{Hilb}_{S/k}^\delta (B_{g_i}) \) such that for all \( i, j \),
\[
\phi_i \otimes \text{id} : B_{g_i} \otimes_{B_{g_i}} B_{g_j} \to Q_i \otimes_{B_{g_i}} B_{g_j}
\]
and
\[
\phi_j \otimes \text{id} : B_{g_j} \otimes_{B_{g_j}} B_{g_i} \to Q_j \otimes_{B_{g_j}} B_{g_i}
\]
agree, i.e., define the same map
\[
\phi_{ij} : B_{g_i g_j} \to Q_{ij} = B_{g_i g_j} x_\delta.
\]
We write the elements of the reduced Gröbner basis of \( \ker \phi_i \) and \( \ker \phi_j \), resp., as
\[
f^{(i)}_\alpha = x_\alpha + \sum_{\beta \in \delta, \beta < \alpha} d^{(i)}_{\alpha, \beta}x_\beta,
\]
\[
f^{(j)}_\alpha = x_\alpha + \sum_{\beta \in \delta, \beta < \alpha} d^{(j)}_{\alpha, \beta}x_\beta,
\]
respectively, where \( \alpha \) runs through \( \mathcal{C}(\delta) \). From Lemma 2 we know that \( \ker \phi_{ij} = \ker \phi_i \otimes \text{id} = \ker \phi_j \otimes \text{id} \) is monic with standard set \( \delta \). The images of \( f^{(i)}_\alpha \) and \( f^{(j)}_\alpha \), respectively, in \( B_{g_i g_j}[x] \) have the following properties:
• They lie in \( \ker \phi_{ij} \).
• Their leading exponent is \( \alpha \).
• Their non-leading exponents lie in \( \delta \).

Therefore they are the reduced Gröbner basis of \( \ker \phi_{ij} \). In particular, \( f^{(i)}_\alpha \) and \( f^{(j)}_\alpha \) agree on \( \text{Spec } B_{g_i g_j} \). The sheaf axiom for the quasi-coherent sheaf \( B[x]^- \) on \( \text{Spec } B \) provides a polynomial \( f_\alpha \in B[x] \) whose image in \( B_{g_i}[x] \) is \( f^{(i)}_\alpha \) for all \( i \). It is clear that this polynomial takes the shape (13). Upon defining \( I = (f_\alpha; \alpha \in \mathcal{C}(\delta)) \) and \( \phi : S \to Q = S/I \) to be the canonical map, we have lifted the various homomorphisms \( \phi_i \) to a homomorphism \( \phi \). The same line of arguments as at the end of the proof of Lemma 2 shows that \( I \) is monic with Gröbner basis \( f_\alpha \), where \( \alpha \) runs through \( \mathcal{C}(\delta) \). Therefore \( \phi \) lies in \( \text{Hilb}^\delta_{S/k}(B) \).

\[ \square \]

**Theorem 1.** \( \text{Hilb}^\delta_{S/k} \) is represented by a closed subscheme \( \text{Hilb}^\delta_{S/k} \) of \( \text{Hilb}^\delta_{S/k} \).

**Proof.** We prove this by applying Proposition 2.9 of [HS04]. For this we adopt two items of the terminology of the cited paper.

- Let \( B \) be an object of \((k\text{-Alg})\), and let a condition on morphisms \( \psi : B \to A \) in \((k\text{-Alg})\) be given. We say that the condition is closed if there exists an ideal \( J \subset B \) such that \( \psi : B \to A \) satisfies the condition if, and only if, \( \psi \) factors through the canonical map \( B \to B/J \).
- Let \( B \) be an object of \((k\text{-Alg})\) and the \( B \)-algebra homomorphism \( \phi : B[x] \to Q \) be an object of \( \text{Hilb}^\delta_{S/k}(B) \). We say that a morphism \( \psi : B \to A \) in \((k\text{-Alg})\) satisfies \( V_{B,\phi} \) if the \( A \)-algebra homomorphism \( \text{Hilb}^\delta_{S/k}(\psi)(\phi) \), which is an element of \( \text{Hilb}^\delta_{S/k}(A) \), lies in \( \text{Hilb}^\delta_{S/k}(A) \).

By Proposition 2.9 of [HS04], the functor \( \text{Hilb}^\delta_{S/k} \) (which is a Zariski sheaf by Lemma 3) is represented by a closed subscheme of \( \text{Hilb}^\delta_{S/k} \) if, and only if, for all \( B \) in \((k\text{-Alg})\) and all \( \phi : B[x] \to Q \) in \( \text{Hilb}^\delta_{S/k}(B) \), the condition \( V_{B,\phi} \) is closed.

Let \( B \) and \( \phi \) as above be given. Then the family \( (x^\beta + \ker \phi)_{\beta \in \mathcal{C}(\delta)} \) is a \( B \)-basis of \( B[x]/\ker \phi \). By Lemma 1, there is a unique polynomial of the form

\[ f_\alpha = x^\alpha + \sum_{\beta \in \delta} d_{\alpha, \beta} x^\beta \in \ker \phi \]

for all \( \alpha \in \mathbb{N}^n - \delta \). Define \( J \subset B \) be the ideal generated by all \( d_{\alpha, \beta} \), where \( \alpha \) runs through \( \mathbb{N}^n - \delta \) and \( \beta \) runs through all elements of \( \delta \) such that \( \alpha < \beta \).
Given a morphism $\psi : B \to A$ in $(k\text{-Alg})$, the homomorphism $\text{Hilb}_{S/k}^r(\psi) : A[x] \to Q \otimes_B A$. The polynomial

$$\psi(f_\alpha) = x^\alpha + \sum_{\beta \in \delta} \psi(d_{\alpha,\beta})x^\beta$$

is the unique element of $\ker(\phi \otimes \text{id})$ such that its leading exponent is $\alpha$ and all non-leading exponents lie in $\delta$. Now $\psi : B \to A$ factors through $B \to B/J$ if, and only if, for all $\alpha \in s(\delta)$ and all $\beta \in \delta$ such that $\alpha < \beta$, we have $\psi(d_{\alpha,\beta}) = 0$. This is is equivalent to the ideal $(\psi(f_\alpha); s(\delta)) \subset A[x]$ being monic with reduced Gröbner basis $\{\psi(f_\alpha); \alpha \in s(\delta)\}$. Therefore $\psi : B \to A$ factors through $B \to B/J$ if, and only if, $\ker(\phi \otimes \text{id})$ is monic with Gröbner basis $\psi(f_\alpha)$, where $\alpha$ runs through $s(\delta)$. We have proved that $V_{B,\phi}$ is a closed condition.

\section{The Gröbner strata}

We have proved that in the chain of inclusion (1) the first inclusion is a closed immersion and the second inclusion is an open immersion.

\textbf{Definition 4.} We call the locally closed subscheme $\text{Hilb}_{S/k}^r$ of $\text{Hilb}_{S/k}^r$ the Gröbner stratum attached to the standard set $\delta$.

Gröbner strata and related objects have been studied by many authors, see [AS05], [Eva02], or [Eva04]. The cited authors refer to these schemes as \textit{Schubert schemes}, or \textit{Schubert cells}. Their terminology is motivated by the analogy of the inclusion $\text{Hilb}_{S/k}^r \subset \text{Hilb}_{S/k}^r$ to the inclusion of a Schubert cell in the Grassmannian in the case where $\delta$ is a subset of the standard basis $\{e_1, \ldots, e_n\} \subset \mathbb{N}^n$, augmented by $0 \in \mathbb{N}^n$. One interesting thing about Gröbner strata is the following statement.

\textbf{Theorem 2.} As a topological space, the scheme $\text{Hilb}_{S/k}^r$ decomposes into locally closed strata as follows,

$$\text{Hilb}_{S/k}^r = \bigsqcup_{\delta} \text{Hilb}_{S/k}^r,$$

where the disjoint union goes over all standard sets $\delta \subset \mathbb{N}^n$ of size $r$.

\textit{Proof.} We have to show that each closed point of $\text{Hilb}_{S/k}^r$ lies in precisely one stratum $\text{Hilb}_{S/k}^r$. Let $x : \text{Spec } F \to \text{Hilb}_{S/k}^r$, be a closed point, $F$ a field. We interpret this as an element of of $\text{Hilb}_{S/k}(F)$, i.e., as a surjective $F$-algebra homomorphism $\phi : F[x] \to Q$. The kernel of this homomorphism has a well defined reduced Gröbner basis, and a well defined standard set $\delta$. Therefore $x$ lies in $\text{Hilb}_{S/k}^r$.\hfill \square
For all $F$-valued points $x \in \text{Hilb}^\delta_{S/k}(F)$, the kernel of the corresponding $\phi : F[x] \to Q$ is monic with standard set $\delta$. Therefore an $F$-valued point $x$ of $\text{Hilb}^\delta_{S/k}$ cannot lie in both $\text{Hilb}^\delta_{S/k}$ and $\text{Hilb}^\epsilon_{S/k}$ if $\delta \neq \epsilon$. $\square$

Note that in general a non-closed point of $\text{Hilb}^r_{S/k}$ does not lie in any stratum $\text{Hilb}^\delta_{S/k}$. Indeed, a non-closed point of $\text{Hilb}^r_{S/k}$ corresponds to an element of $\text{Hilb}^r_{S/k}(B)$, where $B$ is a ring rather than a field. That element is a $k$-algebra homomorphism $\phi : B[x] \to Q$ satisfying the conditions of Definition 2. Let us assume that $B$ is a local ring with maximal ideal $m$, and let us write $\kappa = B/m$. In the proof of Proposition 1, we we tensored with $\kappa$ and thus obtained $\phi_\kappa : \kappa[x] \to Q_\kappa$. The reduced Gröbner basis of $\ker \phi_\kappa$ led to the $\kappa$-basis $(\phi_\kappa(x^\gamma))_{\gamma \in \delta}$ of $Q_\kappa = \kappa[x]/\ker \phi_\kappa$. We lifted that family to the $B$-basis $(\phi(x^\gamma))_{\gamma \in \delta}$ of $Q = B[x]/\ker \phi$. However, the ideal $\ker \phi$ need not be monic. Equivalently, a lift of the reduced Gröbner basis of $\ker \phi_\kappa$ need not be a Gröbner basis of $\ker \phi$ at all. It is a one if, and only if, our non-closed point lies in $\text{Hilb}^\delta_{S/k}$.

8. Representing the functors

We start this section by briefly reviewing the construction of the affine scheme $\text{Hilb}^\delta_{S/k}$ which is given in [GLS07] and [Ber08]. $\text{Hilb}^\delta_{S/k}(B)$ is the set of equivalence classes of $B$-algebra homomorphisms $\phi : B[x] \to Q$ such that the composition $Bx^\delta \to B[x] \to Q$ is an isomorphism. Each equivalence class of $\phi : B[x] \to Q$ corresponds to precisely one $B$-algebra structure on the $B$-module $Bx^\delta$. Therefore $\text{Hilb}^\delta_{S/k}(B)$ is reinterpreted as the set of all $B$-algebra homomorphisms $\phi : B[x] \to Bx^\delta$ such that $\phi \circ (\iota_\delta \otimes \text{id}) : Bx^\delta \to Bx^\delta$ is the identity map. Now that we have free modules with bases, we identify $\phi$ with its matrix $(a_{\alpha\beta})_{\alpha \in \mathbb{N}^n, \beta \in \delta}$, which is given by

$$
\phi(x^\alpha) = \sum_{\beta \in \delta} a_{\alpha\beta} x^\beta, \text{ for all } \alpha \in \mathbb{N}^n.
$$

The condition $\phi \circ (\iota_\delta \otimes \text{id}) = \text{id}$ says that

$$
a_{\alpha\beta} = \delta_{\alpha\beta}, \text{ for all } \alpha \in \mathbb{N}^n \text{ and for all } \beta \in \delta.
$$

The $B$-module homomorphism $\phi$ is a $B$-algebra homomorphism if, and only if, is is multiplicative. This characterization will be used in the proof of Proposition 3 below. Here we use the another characterization, derived from the Homomorphism Theorem: The $B$-module homomorphism $\phi$ is a $B$-algebra homomorphism if, and only if, its kernel is an ideal in $B[x]$. It is easy to check that the family $x^\alpha - (\iota_\delta \otimes \text{id}) \circ \phi(x^\alpha)$, where $x^\alpha$ runs through all monomials in $B[x]$, generates the $B$-module $\ker \phi$. Therefore $\ker \phi$ is an
ideal in $B[x]$ if, and only if,
\[
\phi(x^\lambda(x^\alpha - (\iota_\delta \otimes \text{id}) \circ \phi(x^\alpha))) = 0, \text{ for all } \lambda, \alpha \in \mathbb{N}^n.
\]
Upon expressing $\phi$ by its matrix and using the fact that $\iota_\delta$ is the canonical inclusion, this condition reads as follows:
\[
\sum_{\beta \in \delta}(a_{\lambda+\alpha,\beta} - \sum_{\gamma \in \delta}a_{\alpha,\gamma}a_{\lambda+\gamma,\beta})x^\beta = 0, \text{ for all } \lambda, \alpha \in \mathbb{N}^n.
\]
Since $Bx^\delta$ is free with basis $x^\delta$, this means that
\[
a_{\lambda+\alpha,\beta} - \sum_{\gamma \in \delta}a_{\alpha,\gamma}a_{\lambda+\gamma,\beta} = 0, \text{ for all } \lambda, \alpha \in \mathbb{N}^n \text{ and for all } \beta \in \delta.
\]
Clearly it suffices to let $x^\lambda$ run only through $x_1, \ldots, x_n$. Therefore the functor $\text{Hilb}^\delta_{S/k}$ is represented by the affine scheme
\[
\text{Hilb}^\delta_{S/k} = \text{Spec } R/I,
\]
where $I$ is the ideal
\[
I = (T_{\alpha,\beta} - \delta_{\alpha,\beta}; \alpha \in \mathbb{N}^n, \beta \in \delta)
\]
\[
+ (T_{\lambda+\alpha,\beta} - \sum_{\gamma \in \delta}T_{\alpha,\gamma}T_{\lambda+\gamma,\beta}; \alpha \in \mathbb{N}^n, \lambda \in \{e_1, \ldots, e_n\}, \beta \in \delta)
\]
in the polynomial ring $R = k[T_{\alpha,\beta}; \alpha \in \mathbb{N}^n, \beta \in \delta]$.

The heart of the above described method for obtaining the coordinate ring of the scheme $\text{Hilb}^\delta_{S/k}$ is the system of equations (15). These are the structural equations defining the multiplicative structure on the $B$-algebra $Bx^\delta$. In contrast to this approach to the coordinate ring of $\text{Hilb}^\delta_{S/k}$, the same ring is obtained by using a border basis variant of Buchberger’s S-pair criterion in the articles [Hui06], [KK05], [KK06], [KKR05], [KR08] and [Rob09]. In those articles, finite presentations of the coordinate ring of $\text{Hilb}^\delta_{S/k}$ are given. At the moment our approach seems weaker, as the presentation of (16) uses infinitely many generators and relations. In the next section we will see that our approach is in fact stronger. First we give a finite presentation of that coordinate ring.

For this, we introduce the following notation. If $N \subset \mathbb{N}^n$ is a standard set, we write $N^{(1)} = \mathcal{B}(N)$ and, for all $i \geq 1$, $N^{(i+1)} = \mathcal{B}(N \cup N^{(1)} \cup \ldots \cup N^{(i)})$.

**Proposition 3.** Let $N$ be a standard set in $\mathbb{N}^n$ containing $\delta$. Then the functor $\text{Hilb}^\delta_{S/k}$ is represented by the affine scheme $\text{Hilb}^\delta_{S/k} = \text{Spec } R^\delta$, where
\[
R^\delta = R/I^\delta.
\]
and $I^\delta = I^\delta_1 + I^\delta_2 + I^\delta_3$ is the sum of the ideals

\[
I^\delta_1 = (T_{\alpha,\beta} - \delta_{\alpha,\beta}; \alpha \in N, \beta \in \delta),
\]

\[
I^\delta_2 = (T_{\alpha,\lambda + \beta} - \sum_{\gamma \in \delta} T_{\alpha,\gamma} T_{\gamma + \lambda,\beta}; \alpha \in N \cup N^{(1)} \text{ s.t. } \alpha + \lambda \in N \cup N^{(1)}), \lambda \in \{e_1, \ldots, e_n\}\]

\[
I^\delta_3 = \left( \sum_{\gamma \in \delta} T_{\alpha,\gamma} T_{\gamma + \lambda,\beta} - \sum_{\gamma \in \delta} T_{\alpha',\gamma} T_{\gamma + \lambda',\beta}; \alpha, \alpha' \in N^{(1)}, \lambda, \lambda' \in \{e_1, \ldots, e_n\} \text{ s.t. } \alpha + \lambda = \alpha' + \lambda' \in N^{(2)}, \beta \in \delta \right)
\]

in the polynomial ring $R = k[T_{\alpha,\beta}; \alpha \in N \cup N^{(1)}, \beta \in \delta]$.

**Proof.** Again we start with a $B$-module homomorphism $\phi : B[x] \to Bx^\delta$, represented by its matrix $(a_{\alpha,\beta})$ as in (14). Our goal is to find constraints on the coefficients $a_{\alpha,\beta}$ which guarantee that $Bx^\delta$ has a multiplicative structure such that $\phi$ is a $B$-algebra homomorphism. As was mentioned above, $\phi$ is a $B$-algebra homomorphism if, and only if, $\phi$ is multiplicative. By linearity of $\phi$, this is equivalent to $\phi(x^{\alpha+\beta}) = \phi(x^\alpha)\phi(x^\beta)$ for all $\alpha, \beta \in \mathbb{N}^n$, and by an easy induction argument, the latter condition is equivalent to

\[
\phi(x^{\alpha+\lambda}) = \phi(x^\alpha)\phi(x^\lambda)
\]

for all $\alpha \in \mathbb{N}^n$ and all $\lambda \in \{e_1, \ldots, e_n\}$. Therefore, our goal is to find constraints on the coefficients $a_{\alpha,\beta}$ which guarantee that $Bx^\delta$ has a multiplicative structure and the multiplicativity condition (17) holds true for all $\alpha \in \mathbb{N}^n$ and all $\lambda \in \{e_1, \ldots, e_n\}$. We will see in the course of the proof that we do not need the full matrix $(a_{\alpha,\beta})_{\alpha \in \mathbb{N}^n, \beta \in \delta}$, but rather only the rows indexed by $\alpha \in N \cup N^{(1)}$.

**Step 1.** As above, the condition $\phi \circ (\iota_\delta \otimes \text{id}) = \text{id}$ translates into the following constraints on the coefficients $a_{\alpha,\beta}$:

\[
\forall \alpha \in N, \forall \beta \in \delta : a_{\alpha,\beta} = \delta_{\alpha,\beta}.
\]

**Step 2.** We impose the multiplicativity condition (17) on all $\alpha$ and $\alpha + \lambda$ which lie in $N \cup N^{(1)}$. Let us translate this into equations for the coefficients $a_{\alpha,\beta}$. The left hand side of (17) is

\[
\phi(x^{\alpha+\lambda}) = \sum_{\beta \in \delta} a_{\alpha+\lambda,\beta} x^\beta.
\]

The right hand side of (17) is a priori not defined before we have the multiplicative structure of $Bx^\delta$ at hand. However, upon assuming that (17) holds
true for elements of $N \cup N^{(1)}$, we can surmount that obstacle by a trick:

$$\phi(x^\alpha)\phi(x^\lambda) = \sum_{\gamma \in \delta} a_{\alpha,\gamma} x^\gamma \phi(x^\lambda) = \sum_{\gamma \in \delta} a_{\alpha,\gamma} \phi(x^\gamma) \phi(x^\lambda)$$

$$= \sum_{\gamma \in \delta} a_{\alpha,\gamma} \phi(x^{\gamma+\lambda}) = \sum_{\beta,\gamma \in \delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} x^\beta.$$  

Here we used the fact that $x^\gamma = \phi(x^\gamma)$ if $\gamma \in \delta$, and the multiplicativity condition (17) for $\gamma$ and $\gamma + \lambda$ lying in $N \cup N^{(1)}$. The two expressions have to coincide, hence the following constraints on the coefficients $a_{\alpha,\beta}$:

$$\forall \alpha \in N \cup N^{(1)}, \forall \lambda \in \{e_1, \ldots, e_n\} \text{ s.t. } \alpha + \lambda \in N \cup N^{(1)}, \forall \beta \in \delta :$$

$$a_{\alpha+\lambda,\beta} = \sum_{\gamma \in \delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} \tag{19}$$

Note that these are just (some of) the structural equations (15).

At this point **multiplicativity holds true within** $N \cup N^{(1)}$ in the sense that (17) holds true if $\alpha, \alpha + \lambda \in N \cup N^{(1)}$.

**Step 3.** We define more values of $\phi$ by means of the equation (17). More precisely, we take $\alpha \in N^{(1)}$ and $\lambda \in \{e_1, \ldots, e_n\}$ such that $\alpha + \lambda \in N^{(2)}$ and define

$$\phi(x^{\alpha+\lambda}) = \sum_{\beta,\gamma \in \delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} x^\beta.$$  

Then the multiplicativity condition (17) holds true for these values of $\alpha$ and $\lambda$, as the right hand side of the last equation is

$$\sum_{\beta,\gamma \in \delta} a_{\alpha,\gamma} \phi(x^{\gamma+\lambda}) = \sum_{\beta,\gamma \in \delta} a_{\alpha,\gamma} \phi(x^\gamma) \phi(x^\lambda)$$

$$= \sum_{\beta,\gamma \in \delta} a_{\alpha,\gamma} x^\gamma \phi(x^\lambda) = \phi(x^\alpha) \phi(x^\lambda).$$

At this point we have to make sure that the definition just given is unambiguous. This means that if $\alpha' \in N^{(1)}$ and $\lambda' \in \{e_1, \ldots, e_n\}$ are such that $\alpha + \lambda = \alpha' + \lambda'$, the definitions of $\phi(x^{\alpha+\lambda})$ and of $\phi(x^{\alpha'+\lambda'})$ coincide. This translates into the following constraints on the coefficients $a_{\alpha,\beta}$:

$$\forall \alpha, \alpha' \in N^{(1)}, \forall \lambda, \lambda' \in \{e_1, \ldots, e_n\} \text{ s.t. } \alpha + \lambda = \alpha' + \lambda' \in N^{(2)},$$

$$\forall \beta \in \delta : \sum_{\gamma \in \delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} = \sum_{\gamma \in \delta} a_{\alpha',\gamma} a_{\gamma+\lambda',\beta}. \tag{20}$$

At this point **multiplicativity also holds true when passing from** $N^{(1)}$ to $N^{(2)}$ in the sense that (17) holds true if $\alpha \in N^{(1)}$ and $\alpha + \lambda \in N^{(2)}$.

**Step 4.** We claim that **multiplicativity holds true within** $N \cup N^{(1)} \cup N^{(2)}$ in the sense that (17) holds true if $\alpha, \alpha + \lambda \in N \cup N^{(1)} \cup N^{(2)}$.  

We only have to check that if both $\alpha$ and $\alpha + \lambda$ lie in $N^{(2)}$. In this case there exists some $\nu \in \{e_1, \ldots, e_n\}$ such that $\alpha + \lambda - \nu \in N^{(1)}$. In particular, $\nu \neq \lambda$. This implies that also $\alpha - \nu$ lies in $\mathbb{N}^n$, as that element arises from $\alpha + \lambda$ by subtracting two different standard basis elements, $\lambda$ and $\nu$. Therefore $\alpha - \nu$ in fact lies in $N^{(1)}$. We obtain
\[
\phi(x^{\alpha+\lambda}) = \phi(x^{\alpha+\lambda-\nu})\phi(x^\nu) = \phi(x^{\alpha-\nu})\phi(x^\lambda) = \phi(x^\alpha)\phi(x^\lambda),
\]
as desired. Here we used the fact that multiplicativity holds true when passing from $N^{(1)}$ to $N^{(2)}$ for the outer two equalities and the fact that multiplicativity holds true within $N \cup N^{(1)}$ for the inner equality.

**Step 5.** We define more values of $\phi$ in analogy to Step 3: We take $\alpha \in N^{(1)}$ and $\lambda, \mu \in \{e_1, \ldots, e_n\}$ such that $\alpha + \lambda \in N^{(2)}$ and $\alpha + \lambda + \mu \in N^{(3)}$ and define
\[
\phi(x^{\alpha+\lambda+\mu}) = \sum_{\beta, \gamma, \gamma' \in \delta} a_{\alpha, \gamma}a_{\gamma' + \lambda, \gamma}a_{\gamma + \mu, \beta}x^\beta.
\]
This definition makes the identity
\[
(21) \quad \phi(x^{\alpha+\lambda+\mu}) = \phi(x^{\alpha+\lambda})\phi(x^\mu)
\]
hold true. We claim that the definition just given is unambiguous, i.e. that if $\alpha' \in N^{(1)}$ and $\lambda', \mu' \in \{e_1, \ldots, e_n\}$ are such that $\alpha + \lambda + \mu = \alpha' + \lambda' + \mu'$, the corresponding definitions of $\phi(x^{\alpha+\lambda})$ coincide. For verifying this, we first note that we may assume that $\mu \neq \mu'$, since otherwise unambiguity is trivial. By the same argument as in Step 4, we see that $\alpha + \lambda - \mu' = \alpha' + \lambda' - \mu$ lies in $\mathbb{N}^n$. Therefore $\alpha + \lambda - \mu'$ in fact lies in $N \cup N^{(1)} \cup N^{(2)}$. Now we distinguish three cases.

**Case a.** $\lambda \neq \mu'$. Then by the same argument once more, $\alpha - \mu'$ lies in $\mathbb{N}^n$. It follows that $\alpha - \mu'$ lies in $N \cup N^{(1)} \cup N^{(2)}$. Therefore
\[
\phi(x^{\alpha+\lambda+\mu}) = \phi(x^{\alpha+\lambda})\phi(x^\mu) = \phi(x^\alpha)\phi(x^\lambda)\phi(x^\mu)
\]
\[
= \phi(x^{\alpha-\mu'})\phi(x^{\mu'})\phi(x^\lambda) = \phi(x^{\alpha-\mu'+\lambda})\phi(x^{\mu'})\phi(x^\lambda)
\]
\[
= \phi(x^{\alpha-\mu'+\lambda'})\phi(x^{\mu'}) = \phi(x^{\alpha'+\lambda'})\phi(x^{\mu'}) = \phi(x^{\alpha'+\lambda'+\mu'}).
\]
Here we used (21) for the outer equalities and the fact that multiplicativity holds true within $N \cup N^{(1)} \cup N^{(2)}$ for all other equalities.

**Case b.** $\lambda' \neq \mu$. This is the same as the previous with the roles of primed and non-primed elements interchanged.

**Case c.** $\lambda = \mu'$ and $\lambda' = \mu$. Then $\alpha = \alpha'$. As $\alpha \in N^{(1)}$, there exists a $\nu \in \{e_1, \ldots, e_n\}$ such that $\alpha - \nu \in N$. Again we get a chain of equalities:
\[
\phi(x^{\alpha+\lambda+\mu}) = \phi(x^{\alpha+\lambda'})\phi(x^\nu) = \phi(x^\alpha)\phi(x^\lambda)\phi(x^\nu)
\]
\[
= \phi(x^{\alpha-\nu})\phi(x^{\mu'})\phi(x^{\lambda'}) = \phi(x^{\alpha+\lambda'-\nu})\phi(x^{\nu})\phi(x^{\lambda})
\]
\[
= \phi(x^{\alpha+\lambda'})\phi(x^\lambda) = \phi(x^{\alpha+\lambda'+\mu'})
\]
Again we used (21) for the outer equalities and the fact that multiplicativity holds true within $N \cup N^{(1)} \cup N^{(2)}$ for all other equalities.

At this point multiplicativity also holds true when passing from $N^{(2)}$ to $N^{(3)}$ in the obvious sense.

**Induction Step.** Note that the proof of multiplicativity given in Step 4 was completely formal, only using the fact that multiplicativity holds true within $N \cup N^{(1)}$ and when passing from $N^{(1)}$ to $N^{(2)}$. Therefore, now that we know that multiplicativity holds true within $N \cup N^{(1)} \cup N^{(2)}$ and when passing from $N^{(2)}$ to $N^{(3)}$, we can imitate Step 4 and prove that multiplicativity holds true within $N \cup N^{(1)} \cup N^{(2)} \cup N^{(3)}$. Analogously, Step 5 was completely formal and can be imitated for proving that multiplicativity holds true when passing from $N^{(3)}$ to $N^{(4)}$. Then we imitate Step 4 again and prove that multiplicativity holds true within $N \cup N^{(1)} \cup N^{(2)} \cup N^{(3)} \cup N^{(4)}$. Then we imitate Step 4 again and prove that multiplicativity holds true when passing from $N^{(4)}$ to $N^{(5)}$, and so on. This proves that the multiplicativity condition (17) holds true for all $\alpha \in \mathbb{N}^n$ and all $\lambda \in \{e_1, \ldots, e_n\}$.

**End of proof.** We see that the $B$-module homomorphism $\phi : B[x] \to Bx^\delta$ defines a $B$-algebra homomorphism if, and only if, the coefficients $a_{\alpha,\beta}$, where $\alpha \in N \cup N^{(1)}$ and $\beta \in \delta$, satisfy the three conditions (18), (19) and (20) of Steps 1, 2 and 3, resp. Therefore, an element $\phi$ of $\text{Hilb}^\delta_{S/k}(B)$ is uniquely determined by the choice of elements $a_{\alpha,\beta} \in B$, for all $\alpha \in N \cup N^{(1)}$ and all $\beta \in \delta$, such that (18), (19) and (20) hold true. That choice corresponds to the choice of a $k$-algebra homomorphism $R^\delta \to B$. □

The set $N$ of Proposition 3 can be chosen finite. Therefore the scheme $\text{Hilb}^\delta_{S/k}$ is of finite type over $k$, embedded as the closed subscheme corresponding to $I^\delta$ in affine space with coordinates $T_{\alpha,\beta}$, for $\alpha \in N$, $\beta \in \delta$. In view of the summand $I^\delta_1$ of $I^\delta$, we see that we only need the coordinates $T_{\alpha,\beta}$, for $\alpha \in N - \delta$, $\beta \in \delta$, for the ambient space. The smallest possible $N$ is $\delta$, hence a closed immersion of $\text{Hilb}^\delta_{S/k}$ into affine space of dimension $\#\mathcal{B}(\delta)\#\delta$. The same immersion is studied in [Hui06], [KK05], [KK06], [KKR05], [KR08] and [Rob09]. However, these articles do not use the matrices of $\phi(x^\alpha)$ but rather the polynomials

$$f_\alpha = x^\alpha + \sum_{\beta \in \delta} d_{\alpha,\beta} x^\beta \in \ker \phi$$

(cf. (13)). These polynomials carry the same information as our matrix, as

$$a_{\alpha,\beta} = \begin{cases} \delta_{\alpha,\beta} & \text{if } \alpha \in \delta, \\ -d_{\alpha,\beta} & \text{if } \alpha \in \mathbb{N}^n - \delta, \end{cases}$$

(22)
The work with the polynomials $f_{\alpha}$ makes syzygy criteria necessary in the cited articles. The two summands $I_{2}^{\delta}$ and $I_{3}^{\delta}$ in our ideal $I^{\delta}$ correspond to the concepts of next-door-neighbors and across-the-street neighbors, resp., in [KK05] and [KR08]. We will say more about that in the next section, in which we dispose of many of the generators of $I_{3}^{\delta}$. Now we focus on the Gröbner functors.

**Corollary 1.** Let $N$, $R$, and $I^{\delta}$ be as in Proposition 3. Then the functor $\text{Hilb}_{S/k}^{\delta}$ is represented by the affine scheme $\text{Hilb}_{S/k}^{\delta} = \text{Spec } R^{\delta}$, where

\[ R^{\delta} = R/I^{\delta} \]

and

\[ I^{\delta} = I^{\delta} + (T_{\alpha,\beta}; \alpha \in N \cup N^{(1)}, \beta \in \delta, \alpha < \beta). \]

**Proof.** The additional conditions defining $I^{\delta}$ express the constraints on the subfunctor $\text{Hilb}_{S/k}^{\delta}$ of $\text{Hilb}_{S/k}^{\delta}$ which we discussed in the proof of Theorem 1, in terms of the variables $T_{\alpha,\beta}$. □

Equivalently, the scheme $\text{Hilb}_{S/k}^{\delta}$ is the closed subscheme of $\text{Hilb}_{S/k}^{\delta}$ defined by the ideal in $R^{\delta}$ generated by the images of all $T_{\alpha,\beta}$, for $\alpha \in N \cup N^{(1)}$ and $\beta \in \delta$ such that $\alpha < \beta$. Note that for the ideal defining $\text{Hilb}_{S/k}^{\delta}$, the identity

\[ I^{\delta} = I^{\delta} + (T_{\alpha,\beta}; \alpha \in \mathcal{C}(\delta), \beta \in \delta, \alpha < \beta) \]

holds true. In other words, many of the additional conditions defining $I^{\delta}$ follow from a few basic ones. (This is a consequence of Lemma 1, applied to the ring $R^{\delta}$ of Proposition 3.)

By Corollary 1, $\text{Hilb}_{S/k}^{\delta}$ is a closed subscheme of dimension $\# \mathcal{B}(\delta) \# \delta$. We now cut down further the dimension of the ambient space. For this we consider the polynomial ring $R_{m} = k[T_{\alpha,\beta}; \alpha \in \mathcal{C}(\delta), \beta \in \delta, \alpha \succ \beta]$. For all $\alpha \in \mathcal{B}(\delta) - \mathcal{C}(\delta)$ and all $\beta \in \delta$ such that $\alpha \succ \beta$, we define elements $T_{\alpha,\beta}$ of $R_{m}$ by recursion over $\alpha$ as follows:

- We start with $\Gamma = \mathcal{B}(\delta) - \mathcal{C}(\delta)$.
- While $\Gamma \neq \emptyset$, we take the minimal element $\alpha$ of $\Gamma$, we find some $\nu \in \{e_{1}, \ldots, e_{n}\}$ such that $\alpha - \nu \in \mathcal{B}(\delta)$, we define

\[ T_{\alpha,\beta} = \sum_{\gamma \in \delta, \alpha \succ \gamma + \nu \succ \beta} T_{\alpha - \nu,\gamma} T_{\gamma + \nu,\beta}, \]

where $T_{\gamma + \nu,\beta} = \delta_{\gamma + \nu,\beta}$ if $\gamma + \nu \in \delta$, and we replace $\Gamma$ by $\Gamma - \{\alpha\}$.

Moreover, we define an ideal $I_{m}^{\delta} \subset R_{m}$ by the same formulas as the ideal $I^{\delta} \subset R$ in Proposition 3, applied in the case where $N = \delta$, with the following modification: In all summands of all generators of $I^{\delta}$, we replace all $T_{\alpha,\beta}$ such that $\alpha \in \delta$, by $\delta_{\alpha,\beta}$, and delete all $T_{\alpha,\beta}$ such that $\alpha \succ \beta$. 
Corollary 2. With the above notation, we have $\text{Hilb}^\delta_{S/k} = \text{Spec} R^\delta_m$, where

$$R^\delta_m = R_m/I^\delta_m.$$ 

Proof. We first check that our recursion is well-defined. Indeed, for all $\alpha \in \mathcal{B}(\delta) - \mathcal{C}(\delta)$, the existence of a standard basis element $\nu$ such that $\alpha - \nu \in \mathcal{C}(\delta)$ follows directly from the definitions. If $\alpha$ is the object of consideration in one particular step of the recursion, the element $\alpha - \nu$ either lies in $\mathcal{C}(\delta)$ or has been the object of consideration in an earlier stage of the recursion, as $\alpha \simeq \alpha - \nu$. In both cases $T_{\alpha - \nu, \gamma}$ is a well-defined element of $R$. Moreover, in the sum (23) we only consider those $\gamma \in \delta$ for which $\alpha - \nu \simeq \gamma$, or equivalently, $\alpha \simeq \gamma + \nu$. Therefore the element $\gamma + \nu$ either lies in $\delta$ or has been the object of consideration in an earlier stage of the recursion. In both cases $T_{\gamma + \nu, \beta}$ is a well-defined element of $R$.

Now we return to the notation of Corollary 1 in the case where $N = \delta$ and consider the rings $R$ and $R^\delta = R/I^\delta$ defined there. For simplicity we denote the images of the variables $T_{\alpha, \beta} \in R$ in the quotient $R^\delta$ by $T_{\alpha, \beta}$. In particular, $T_{\alpha, \beta} = 0$ for all $\alpha \in \mathcal{B}(\delta)$ and all $\beta \in \delta$ such that $\alpha \prec \beta$. Furthermore, the presence of the summand $I^\delta_2$ in the ideal $I^\delta$ implies that whenever $\alpha - \nu$ and $\alpha$ lie in $\mathcal{B}(\delta)$, the identity

$$T_{\alpha, \beta} = \sum_{\gamma \in \delta} T_{\alpha - \nu, \gamma} T_{\gamma + \nu, \beta}$$

holds true in $R^\delta$. However, only those $\gamma \in \delta$ for which $\alpha - \nu \simeq \gamma$ and $\gamma + \nu \simeq \beta$ make a contribution to that sum. This explains the definition of $T_{\alpha, \beta}$ given in (23). As for the definition of $I^\delta_m$, the replacement $T_{\alpha, \beta} = \delta_{\alpha, \beta}$ for all $\alpha \in \mathcal{B}(\delta)$ is clear from the presence of the summand $I^\delta_2$ in the ideal $I^\delta$; and deleting all $T_{\alpha, \beta}$ such that $\alpha \simeq \beta$ stems from the equality $T_{\alpha, \beta} = 0$ in $R^\delta$. The assertion follows from Corollary 1. 

The subscript in the ideal $I^\delta_m$ stands for minimal, as the dimension of the ambient space of $\text{Hilb}^\delta_{S/k}$ constructed in Corollary 2 is the smallest which one can reach. The dimension of the ambient space of $\text{Hilb}^\delta_{S/k}$ constructed in Proposition 3 for the case where $N = \delta$ equals $\# \mathcal{B}(\delta) \# \delta$. It is interesting to ask wether that dimension is also the smallest which one can reach for a generic $\delta$. This is certainly not the case for special shapes of $\delta$: In [Hui02], Huibregtse shows that for $n = 2$ and a certain class of $\delta$, $\text{Hilb}^\delta_{S/k}$ is an affine space of dimension $2r$.

If $N$ is strictly larger than $\delta$, the dimension of the ambient space of $\text{Hilb}^\delta_{S/k}$ given in Proposition 3 is far from minimal, thus seeming unnecessarily large. Alas also that presentation is useful, as it leads to a compact formula for
the coordinate change between two charts \( \text{Hilb}^δ_{S/k} \) and \( \text{Hilb}^ε_{S/k} \) of \( \text{Hilb}^ε_{S/k} \). We will carry this out in Section 12 below.

9. A SMALLER SET OF GENERATORS

For illustrating the presentation of \( \text{Hilb}^δ_{S/k} \) given in the last section, we go through a few examples. These will also serve as a motivation for Theorem 3 below, which is a substantial improvement of Proposition 3. In all examples we only study \( \text{Hilb}^δ_{S/k} \), and not \( \text{Hilb}^ε_{S/k} \), as the latter arises from the former by simply setting \( T_{α,β} = 0 \) for all \( α ≺ β \).

Example 1. Consider the following standard set \( δ \) and its borders:

\[
δ = \{(0,0),(1,0),(0,1),(1,1)\},
δ^{(1)} = \{(2,0),(0,2),(2,1),(1,2)\},
δ^{(2)} = \{(3,0),(3,1),(2,2),(1,3),(0,4)\}.
\]

Figure 2 shows \( δ \), drawn in thick lines; \( δ^{(1)} \), drawn in thin lines; and \( δ^{(2)} \), marked by circles.

In view of the presence of \( I^δ_1 \) in the ideal \( I^δ \) of Proposition 3, we replace the polynomial ring \( k[T_{α,β}; α ∈ δ \cup δ^{(1)}, β ∈ δ] \) of that theorem by \( R = k[T_{α,β}; α ∈ δ^{(1)}, β ∈ δ] \). Then

\[
\text{Hilb}^δ_{S/k} = \text{Spec } R/I^δ,
\]

where \( I^δ \) is the sum of the three ideals

\[
I^δ_{2,1} = (T_{(1,2),(0,0)} - T_{(0,2),(1,0)}T_{(2,0),(0,0)} - T_{(0,2),(1,1)}T_{(2,1),(0,0)}),
T_{(1,2),(1,0)} - T_{(0,2),(1,0)}T_{(2,0),(1,0)} - T_{(0,2),(1,1)}T_{(2,1),(1,0)} - T_{(0,2),(0,0)},
T_{(1,2),(0,1)} - T_{(0,2),(1,0)}T_{(2,0),(0,1)} - T_{(0,2),(1,1)}T_{(2,1),(0,1)},
T_{(1,2),(1,1)} - T_{(0,2),(1,0)}T_{(2,0),(1,1)} - T_{(0,2),(1,1)}T_{(2,1),(1,1)} - T_{(0,2),(0,1)}),
\]
Example 2. Consider the standard set \( \delta \) along with its borders \( \delta^{(1)} \) and \( \delta^{(2)} \), as depicted in Figure 3. We define the polynomial ring \( R \) by the same formula as in the previous example. Then

\[
\text{Hilb}_{S/k}^\delta = \text{Spec } R/I^\delta,
\]

where

\[
I^\delta = I^\delta_{2,1} + \ldots + I^\delta_{2,6} + I^\delta_{3,1} + I^\delta_{3,2}.
\]

We do not write down the summands of \( I \) explicitly, but rather describe them as follows: The ideals \( I^\delta_{2,j} \) and \( I^\delta_{3,j} \) correspond to the equalities derived in Step 2 and Step 3, resp., of the proof of Proposition 3, according to the following values of \( \alpha, \lambda, \) and \( \alpha', \lambda' \), resp.
The interesting observation here is that the summands $I_{3,1}$ and $I_{3,2}$ correspond to the two edge points $(1,4)$ and $(4,1)$ of $\delta$.

**Example 3.** Consider the standard set $\delta$, along with its borders $\delta^{(1)}$ and $\delta^{(2)}$, as depicted in Figure 4. We define the polynomial ring $R$ by the same formula as in the previous two examples. Then

$$\text{Hilb}_{S/k} = \text{Spec } R / I^\delta,$$

where

$$I^\delta = I_{\delta,1} + \ldots + I_{\delta,30} + I_{\delta,3,1} + I_{\delta,3,2} + I_{\delta,3,3}.$$

The summands of $I^\delta$ have analogous descriptions as in the previous example. The ideals $I_{\delta,i}$ correspond to the 30 possible values of $\alpha, \lambda$ such that $\alpha$ and $\alpha + \lambda$ both lie in $\delta^{(1)}$. The ideals $I_{\delta,i}$ correspond to the following values of $\alpha, \lambda$ and $\alpha', \lambda'$, resp.
Figure 4. A standard set $\delta$ together with $\delta^{(1)}$ and three elements of $\delta^{(2)}$.

| $I^3_{3,1}$ | $\alpha = (1, 2, 0)$ | $\lambda = (1, 0, 0)$ | $\alpha' = (2, 1, 0)$ | $\lambda' = (0, 1, 0)$ |
| $I^3_{3,2}$ | $\alpha = (1, 0, 5)$ | $\lambda = (1, 0, 0)$ | $\alpha' = (2, 0, 4)$ | $\lambda' = (0, 0, 1)$ |
| $I^3_{3,3}$ | $\alpha = (0, 4, 2)$ | $\lambda = (0, 1, 0)$ | $\alpha' = (0, 5, 1)$ | $\lambda' = (0, 0, 1)$ |

Again the summands $I^3_{3,1}$, $I^3_{3,2}$ and $I^3_{3,3}$ correspond to the three edge points $(1, 1, 0)$, $(1, 0, 4)$ and $(0, 4, 1)$ of $\delta$. The following theorem explains this fact.

**Theorem 3.** Let $R^\delta = R/I^\delta$ be the coordinate ring of $\text{Hilb}^\delta_{S/k}$ as presented in Proposition 3, where $N = \delta$. Then the summand $I^\delta_3$ of $I^\delta$ can be replaced by the ideal

$$I^\delta_{3,e} = \left( \sum_{\gamma \in \delta} T_{\alpha + \lambda', \gamma} T_{\gamma + \lambda, \beta} - \sum_{\gamma \in \delta} T_{\alpha + \lambda, \gamma} T_{\gamma + \lambda', \beta}; \right.$$  

$$\epsilon \in N \lambda \oplus N \lambda' \text{ is an edge point of } \delta;$$  

$$\lambda \neq \lambda' \in \{e_1, \ldots, e_n\}, \beta \in \delta \right).$$

in the polynomial ring $R$.

**Proof.** We define $\Gamma_0$ to be the set of all sums $\alpha + \lambda = \alpha' + \lambda' \in \delta^{(2)}$ where $\alpha \neq \alpha' \in \delta^{(1)}$ and $\lambda \neq \lambda' \in \{e_1, \ldots, e_n\}$. Remember that the generators of the ideal $I^\delta_3$ correspond to equations

(24) \quad \phi(x^\alpha)\phi(x^\lambda) = \phi(x^{\alpha'})\phi(x^{\lambda'}),

where $\alpha + \lambda = \alpha' + \lambda' \in \Gamma_0$, in the following way: Equation (24) translates into the system of equations

$$\forall \beta \in \delta: \sum_{\gamma \in \delta} a_{\alpha, \gamma} a_{\gamma + \lambda, \beta} = a_{\alpha', \gamma} a_{\gamma + \lambda', \beta},$$
which system is then replaced by the generators
\[ \sum_{\gamma \in \delta} T_{\alpha,\gamma} T_{\gamma + \lambda, \beta} - T_{\alpha', \gamma} T_{\gamma + \lambda', \beta}, \beta \in \delta \]
of \( I_3^\delta \). Analogously, the generators of the ideal \( I_2^\delta \) correspond to equations
\[ (25) \quad \phi(x^{\alpha + \lambda}) = \phi(x^\alpha)\phi(x^\lambda), \]
where \( \alpha, \alpha + \lambda \in \delta^{(1)} \). In the theorem we only modify \( I_3^\delta \) and leave \( I_1^\delta \) and \( I_2^\delta \) untouched. Therefore we may assume that (25) holds true for all \( \alpha, \alpha + \lambda \in \delta^{(1)} \).

Our first claim is that if (24) holds true for all \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathcal{C}(\delta \cup \delta^{(1)}) \), then (24) automatically holds true for all \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \). For this we define \( \Gamma = \Gamma_0 - \mathcal{C}(\delta \cup \delta^{(1)}) \) and prove the claim by induction over \( \Gamma \). Take an arbitrary \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma \). First we observe that there exists some \( \nu \in \{e_1, \ldots, e_n\} \) such that \( \alpha + \lambda - \nu \) lies in \( \delta^{(2)} \), since if all \( \alpha + \lambda - \nu \) were to lie either in \( \delta^{(1)} \) or outside \( \mathbb{N}^\nu \), then \( \alpha + \lambda \) would lie in \( \mathcal{C}(\delta \cup \delta^{(1)}) \), a contradiction. As \( \alpha + \lambda - \nu = \alpha' + \lambda' - \nu \) lies in \( \delta^{(2)} \) and not in \( \delta^{(1)} \), the element \( \nu \) equals neither \( \lambda \) nor \( \lambda' \). Therefore both \( \alpha - \nu \) and \( \alpha' - \nu \) lie in \( \mathbb{N}^\nu \), hence in \( \delta^{(1)} \). It follows that \( (\alpha - \nu) + \lambda = (\alpha' - \nu) + \lambda' \) lies in \( \Gamma_0 \). If \( (\alpha - \nu) + \lambda \) lies in \( \Gamma_0 \cap \mathcal{C}(\delta \cup \delta^{(1)}) \), then (24) holds true for \( (\alpha - \nu) + \lambda = (\alpha' - \nu) + \lambda' \) by assumption. If \( (\alpha - \nu) + \lambda \) lies in the complement, then (24) holds true for \( (\alpha - \nu) + \lambda = (\alpha' - \nu) + \lambda' \) by our induction hypothesis, as \( (\alpha - \nu) + \lambda < \alpha + \lambda \).

In both cases we obtain
\[ \phi(x^\alpha)\phi(x^\lambda) = \phi(x^{\alpha - \nu})\phi(x^\nu)\phi(x^\lambda) = \phi(x^{\alpha' - \nu})\phi(x^\nu)\phi(x^\lambda) = \phi(x^{\alpha'})\phi(x^{\lambda'}). \]
Here we used (25) for the first and the last equality. Therefore (24) holds true for \( \alpha + \lambda = \alpha' + \lambda' \), and the first claim is proved.

Our second claim is that if (24) holds true for all \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathcal{C}(\delta \cup \delta^{(1)}) \) which lie in a plane \( \mathbb{N}e_i \oplus \mathbb{N}e_j \), for some \( i \neq j \), then (24) automatically holds true for all \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathcal{C}(\delta \cup \delta^{(1)}) \) which do not lie in a plane \( \mathbb{N}e_i \oplus \mathbb{N}e_j \). Indeed, if such an \( \alpha + \lambda \) is not contained in any of the planes, then there exists some \( \nu \in \{e_1, \ldots, e_n\} \) such that \( \nu \neq \lambda, \lambda' \) and \( \alpha + \lambda - \nu \in \mathbb{N}^\nu \). As \( \alpha + \lambda \in \mathcal{C}(\delta \cup \delta^{(1)}) \), it follows that \( \alpha + \lambda - \nu \in \delta \cup \delta^{(1)} \). We obtain
\[ \phi(x^\alpha)\phi(x^\lambda) = \phi(x^{\alpha - \nu})\phi(x^\nu)\phi(x^\lambda) = \phi(x^{\alpha' + \lambda - \nu})\phi(x^\nu) \]
\[ = \phi(x^{\alpha' + \lambda' - \nu})\phi(x^\nu) = \phi(x^{\alpha' - \nu})\phi(x^\nu)\phi(x^\lambda) = \phi(x^{\alpha'})\phi(x^{\lambda'}), \]
for which we used (25) again. The second claim is proved.

Our third claim is that if (24) holds true for all \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathcal{C}(\delta \cup \delta^{(1)}) \) lying in a plane \( \mathbb{N}e_i \oplus \mathbb{N}e_j \) such that \( \alpha - \lambda' = \alpha' - \lambda \) lies in \( \delta \), then (24) holds true for all \( \alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathcal{C}(\delta \cup \delta^{(1)}) \) lying in a plane \( \mathbb{N}e_i \oplus \mathbb{N}e_j \) with no additional restriction. Indeed, from \( \lambda \neq \lambda' \) it follows that
\( \alpha - \lambda = \alpha' - \lambda \) lies in \( \mathbb{N}^n \), and therefore either in \( \delta^{(1)} \) or in \( \delta \). In the former case we compute

\[
\phi(x^\alpha) \phi(x^{\lambda'}) = \phi(x^{\alpha-\lambda'}) \phi(x^{\lambda'}) = \phi(x^{\alpha'-\lambda}) \phi(x^{\lambda'}) = \phi(x^{\alpha'}) \phi(x^{\lambda'}),
\]

the second case is the one where (24) holds true by assumption.

Each of the remaining \( \alpha + \lambda = \alpha' + \lambda' \) lies in the plane \( \mathbb{N} \lambda \oplus \mathbb{N} \lambda' \) and has the property that \( \epsilon = \alpha - \lambda' = \alpha' - \lambda \) lies in \( \delta \). As \( \epsilon + \lambda + \lambda' \) lies in \( \mathcal{C}(\delta \cup \delta^{(1)}) \), neither \( \epsilon + \lambda \) nor \( \epsilon + \lambda' \) lies in \( \delta \). Therefore \( \epsilon \) is an edge point of \( \delta \). We have shown that all generators of \( I_3^\delta \) can be expressed by those generators of \( I_3^\delta \) which correspond to equations (24) where \( \alpha = \epsilon + \lambda' \) and \( \alpha' = \epsilon + \lambda \), for an edge point \( \epsilon \in \mathbb{N} \lambda \oplus \mathbb{N} \lambda' \). The theorem follows.

Note that though \( I_3^\delta + I_2^\delta + I_1^\delta = I_3^\delta + I_2^\delta + I_3^\delta_{3,e} \), the ideal \( I_3^\delta_{3,e} \) is strictly smaller than \( I_3^\delta \). Of course the presentations of the coordinate ring of \( \text{Hilb}^{\delta}_{S/k} \) given in Corollaries 1 and 2 can be reformulated in the spirit of Theorem 3.

Example 2 illustrates the third claim in the proof, and Example 3 illustrates the first claim. The theorem we just proved is a substantial improvement of Proposition 3, as it makes the number of generators needed for \( I^\delta \) much smaller. The concept of across-the-corner neighbors of [KR08] uses the same idea which we used in the proof of the third claim here. Yet our set of generators for \( I^\delta \) of Theorem 3 is much smaller than the set of generators of the cited article.

**Example 4.** For \( \delta = \{0, \ldots, (r-1)e_1\} \subset \mathbb{N}^n \), the scheme \( \text{Hilb}^{\delta}_{S/k} \) is the \( rn \)-dimensional affine space with coordinates \( T_{r1,\beta}, T_{e2,\beta}, \ldots, T_{en,\beta} \), for \( \beta \in \delta \).

We just give a hint for the proof of this: First consider the case \( n = 2 \). For \( a = 0, \ldots, r - 2 \), the equations \( \phi(x^{(a+1,1)}) = \phi(x^{(a,1)}) \phi(x^{(1,0)}) \) give explicit formulas for all \( T_{(a+1,1),\beta} \) as polynomials in \( T_{(0,1),\gamma} \) and \( T_{(r,0),\gamma} \), \( \gamma \in \delta \). In the system of polynomial equations corresponding to the equation \( \phi(x^{(r-1,1)}) \phi(x^{(1,0)}) = \phi(x^{(r,0)}) \phi(x^{(0,1)}) \), replace each \( T_{(a+1,1),\beta} \), for \( a = 0, \ldots, r - 2 \), by the polynomial expression from above. Then it turns out that the above system of equations is trivial. The assertion follows in the case \( n = 2 \). For larger \( n \), fix an \( i \in \{2, \ldots, n\} \) and derive analogous formulas for \( T_{(a+1),e_i,\beta} \) as polynomials in \( T_{e_i,\gamma} \) and \( T_{(r,0),\gamma} \), \( \gamma \in \delta \). The system of polynomial equations corresponding to the equation \( \phi(x^{(r-1,0)e_i + e_1)} \phi(x^{e_1}) = \phi(x^{re_i}) \phi(x^{e_i}) \) is trivial again. By Theorem 3, these equations, for \( i = 2, \ldots, n \), are all we have to study. The assertion follows for all \( n \).
Equation (22) describes the transition between the matrix \((a_{\alpha\gamma})\) of a homomorphism \(\phi \in \text{Hilb}^\delta_{S/k}(B)\) and the elements \(f_\alpha\) of the kernel of \(\phi\). Together with Corollary 1 and Proposition 3, this enables us to directly write down the universal objects.

**Proposition 4.**  
(i) Let \(R^\delta\) be the coordinate ring of \(\text{Hilb}^\delta_{S/k}\) as presented in Proposition 3 or Theorem 3. Then the universal object of the representable functor \(\text{Hilb}^\delta_{S/k}\) is the affine scheme

\[
U^\delta = \text{Spec } R^\delta[x]/(x^\alpha - \sum_{\beta \in \delta} T_{\alpha,\beta}x^\beta; \alpha \in N \cup N^{(1)})
\]

over \(\text{Hilb}^\delta_{S/k} = \text{Spec } R^\delta\).

(ii) Let \(R'^\delta\) be the coordinate ring of \(\text{Hilb}'^\delta_{S/k}\) as presented in Corollaries 1 or 2. Then the universal object of the representable functor \(\text{Hilb}'^\delta_{S/k}\) is the affine scheme

\[
U'^\delta = \text{Spec } R'^\delta[x]/(x^\alpha - \sum_{\beta \in \delta, \beta \prec \alpha} T_{\alpha,\beta}x^\beta; \alpha \in \mathcal{C}(\delta))
\]

over \(\text{Hilb}'^\delta_{S/k} = \text{Spec } R'^\delta\).

**Proof.** (i) is clear. As for (ii), the only thing we have to prove is that for generating the ideal

\[
(x^\alpha - \sum_{\beta \in \delta, \beta \prec \alpha} T_{\alpha,\beta}x^\beta; \alpha \in N \cup N^{(1)})
\]

it suffices take all \(\alpha \in \mathcal{C}(\delta)\). However, in \(R'^\delta\) we have \(T_{\alpha,\beta} = 0\) whenever \(\alpha < \beta\). Therefore, the assertion follows from Lemma 1 (i). \(\square\)

Note that the theorem gives us

\[
U^\delta \hookrightarrow \mathbb{A}^n_{\text{Hilb}^\delta_{S/k}} \quad \text{and} \quad U'^\delta \hookrightarrow \mathbb{A}^n_{\text{Hilb}'^\delta_{S/k}}, \text{ resp.,}
\]

as closed subschemes. Moreover, the coordinate rings of \(U^\delta\) and \(U'^\delta\), resp. are free over \(R^\delta\) and \(R'^\delta\), resp. by definition of the functor \(\text{Hilb}^\delta_{S/k}\). In particular, the morphisms \(U^\delta \to \text{Hilb}^\delta_{S/k}\) and \(U'^\delta \to \text{Hilb}'^\delta_{S/k}\) are automatically flat.

Proposition 4 makes the statement precise that the scheme \(\text{Hilb}'^\delta_{S/k}\) is the parametrizing space of all reduced Gröbner bases in \(S\) with standard sets \(\delta\): A point \(\text{Spec } B \to \text{Hilb}'^\delta_{S/k}\) is a homomorphism \(R'^\delta \to B\). In other words,
we assign to the variables $T_{\alpha,\beta}$ values $a_{\alpha,\beta} \in B$ which satisfy the structural equations (15). Then we define

$$f_\alpha = x^\alpha - \sum_{\beta \in \delta, \beta \prec \alpha} a_{\alpha,\beta} x^\beta.$$  

Geometrically this means that we consider the cartesian diagram

\[
\begin{array}{c}
\text{Spec } S/(f_\alpha; \alpha \in \mathcal{C}(\delta)) \\
\downarrow \\
\text{Spec } B \\
\downarrow \\
\text{Hilb}^{\delta}_{\mathbb{A}^1_k} \\
\end{array}
\]

The structural equations guarantee that the polynomials $f_\alpha$ are a reduced Gröbner basis. Equivalently, by Buchberger’s $S$-pair criterion (see [CLO97], Section 2, §6), that the $S$-pairs of the various $f_\alpha$ reduce to zero modulo all $f_\alpha$.

11. Connectedness

We revisit the well-known construction of curves in Hilbert schemes (see [Eis95], Section 15.8).

**Proposition 5.** There exists a morphism

$$g : \mathbb{A}^1_k \times \text{Spec } k \text{ Hilb}^{\delta}_{\mathbb{A}^1_k} \to \text{Hilb}^{\delta}_{\mathbb{A}^1_k}$$

such that

- the restriction of $g$ to $(t-1) \times \text{Spec } k \text{ Hilb}^{\delta}_{\mathbb{A}^1_k}$ is the identity; and
- for each point $p$ of $\text{Hilb}^{\delta}_{\mathbb{A}^1_k}$, defined by a monic ideal $I \subset B[x]$ over some $k$-algebra $B$, the restriction of $g$ to $\mathbb{A}^1_k \times \text{Spec } k p$ is a curve on $\text{Hilb}^{\delta}_{\mathbb{A}^1_k}$ which connects $p$ with the point defined by the monomial ideal $(x^\alpha; \alpha \in \mathbb{N}^n - \delta) \subset S$.

In particular, if Spec $k$ is connected, then Hilb$^{\delta}_{\mathbb{A}^1_k}$ is connected as well.

**Proof.** We use the polynomial ring $R_m$ and its quotient $R_m^\delta = R_m/I_m^\delta$ of Corollary 2. Consider the ideal $J = (g_\alpha; \alpha \in \mathcal{C}(\delta)) \subset R_m^\delta[x]$ of Proposition 4 (ii), where

$$g_\alpha = x^\alpha - \sum_{\beta \in \delta, \beta \prec \alpha} T_{\alpha,\beta} x^\beta.$$  

Let $S$ be the set of pairs of exponents $(\alpha, \beta)$ such that $\alpha \in \delta^{(1)}$, $\beta \in \delta$ and $T_{\alpha,\beta} \neq 0$ in the ring $R_m^\delta$. By Exercise 15.12 of [Eis95] or Chapter 1, §1 of [Bay82], there exists a linear map $\ell : \mathbb{Z}^n \to \mathbb{Z}$ such that $\ell(\alpha) > \ell(\beta)$ for all $(\alpha, \beta) \in S$. The idea is to use $\ell$ for defining the homogeneity of the variables.
$T_{\alpha, \beta}$ to be $\ell(\alpha) - \ell(\beta)$. This makes the polynomials generating the ideal $I_3$ (cf. Proposition 3) homogeneous of a positive weight, therefore they be deformed to 0.

More precisely, we introduce a new variable $t$ and define, for all $\alpha \in \mathcal{G}(\delta)$, a new polynomial

$$\tilde{g}_\alpha = x^\alpha - \sum_{\beta \in \delta, \beta < \alpha} t^{\ell(\alpha) - \ell(\beta)} T_{\alpha, \beta} x^\beta \in (k[t] \otimes_k R'_m)[x],$$

and consider the ideal $\tilde{J} = (\tilde{g}_\alpha; \alpha \in \mathcal{G}(\delta))$ in $(k[t] \otimes_k R'_m)[x]$. By Lemma 1, for all $\alpha \in \delta(1)$, there exist unique polynomials $g_\alpha \in J$ and $h_\alpha \in \tilde{J}$ with leading exponents $\alpha$ and non-leading exponents in $\delta$. We denote these polynomials by

$$g_\alpha = x^\alpha - \sum_{\beta \in \delta, \beta < \alpha} a_{\alpha, \beta} x^\beta \quad \text{and} \quad h_\alpha = x^\alpha - \sum_{\beta \in \delta, \beta < \alpha} b_{\alpha, \beta} x^\beta,$$

resp., and claim that for all $\alpha \in \delta(1)$, the identity

$$b_{\alpha, \beta} = t^{\ell(\alpha) - \ell(\beta)} a_{\alpha, \beta}$$

holds true. For $\alpha \in \mathcal{G}(\delta)$, this is true by definition, as $a_{\alpha, \beta} = T_{\alpha, \beta}$ and $b_{\alpha, \beta} = t^{\ell(\alpha) - \ell(\beta)} T_{\alpha, \beta}$. Otherwise, we show the claim by induction over $\alpha \in \delta(1)$. For this we use the inductive construction of $g_\alpha$ and $h_\alpha$. As $\alpha \notin \mathcal{G}(\delta)$, there exists some $\nu \in \{e_1, \ldots, e_n\}$ such that $\alpha - \nu \in \delta(1)$. We define

$$\Gamma = \{ \gamma + \nu; \gamma \in \delta, \gamma < \alpha - \nu \} \cap \delta(1).$$

Then the polynomials

$$x^\nu g_{\alpha - \nu} + \sum_{\gamma + \nu \in \Gamma} a_{\alpha - \nu, \gamma} g_{\gamma + \nu} \quad \text{and} \quad x^\nu h_{\alpha - \nu} + \sum_{\gamma + \nu \in \Gamma} b_{\alpha - \nu, \gamma} h_{\gamma + \nu}$$

are well-defined, as $\alpha - \nu < \alpha$ and all $\gamma + \nu < \alpha$, and therefore, $g_{\alpha - \nu}$, all $g_{\gamma + \nu}$, $h_{\alpha - \nu}$ and all $h_{\gamma + \nu}$ are well-defined by induction hypothesis. Moreover, both polynomials have leading exponent $\alpha$ and non-leading exponents in $\delta$. It follows that the polynomials equal $g_\alpha$ and $h_\alpha$, resp. Upon writing the second polynomial as

$$h_\alpha = x^\alpha - \sum_{\beta' \in \delta, \beta' < \alpha - \nu} b_{\alpha - \nu, \beta'} x^{\beta'+\nu}$$

$$+ \sum_{\gamma + \nu \in \Gamma} b_{\alpha - \nu, \gamma} (x^{\gamma + \nu} - \sum_{\beta'' \in \delta, \beta'' < \gamma + \nu} b_{\gamma + \nu, \beta''} x^{\beta''}),$$
we see that
\[ e_{\alpha,\beta} = e_{\alpha - \nu,\beta - \nu} + \sum_{\gamma + \nu \in \Gamma, \beta < \gamma + \nu} e_{\alpha - \nu,\gamma} e_{\gamma + \nu,\beta}. \]

An analogous formula holds for the coefficients \( a_{\alpha,\beta} \) of \( g_{\alpha} \). From our induction hypothesis we conclude that
\[ b_{\alpha,\beta} = t^{\ell(\alpha) - \ell(\nu) - \ell(\beta) + \ell(\nu)} a_{\alpha - \nu,\beta - \nu} + \sum_{\gamma + \nu \in \Gamma, \beta < \gamma + \nu} t^{\ell(\alpha) - \ell(\nu) - \ell(\gamma)} a_{\alpha - \nu,\gamma} t^{\ell(\gamma) + \ell(\nu) - \ell(\beta)} a_{\gamma + \nu,\beta}. \]

and the claim is proved.

Next we claim that the polynomials \( \tilde{g}_{\alpha} \), for \( \alpha \in \mathcal{G}(\delta) \), are the reduced Gröbner basis of \( \tilde{J} \). For this we only have to check that the coefficients of these polynomials satisfy the equations defining the ideal \( I_{\delta} \) of Corollary 1. As for the equations defining the ideal \( I_{\delta}^2 \), we compute
\[ t^{\ell(\alpha) - \ell(\beta)} (a_{\alpha - \nu,\beta} - \sum_{\gamma + \nu \in \Gamma, \beta < \gamma + \nu} a_{\alpha - \nu,\gamma} a_{\gamma + \nu,\beta}) = t^{\ell(\alpha) - \ell(\beta)} a_{\alpha,\beta}, \]
and analogously for the equations defining the ideal \( I_{\delta}^3 \). The universal property of \( \text{Hilb}_{S/k}^{\delta} \) provides a unique \( g \) making the following diagram cartesian:

\[
\begin{array}{ccc}
\text{Spec} (k[t] \otimes_k R^t_m / \tilde{J}) & \longrightarrow & \text{U}^{\delta} \\
\downarrow & & \downarrow \\
\text{Spec} (k[t] \otimes_k R^t_m) & \longrightarrow & \text{Hilb}_{S/k}^{\delta}.
\end{array}
\]

The morphism \( g \) corresponds to the ring homomorphism which sends all \( T_{\alpha,\beta} \) to \( t^{\ell(\alpha) - \ell(\beta)} T_{\alpha,\beta} \). Therefrom follow the two properties of \( g \) as stated in the Proposition: When specializing to \( t = 1 \), the ring homomorphism becomes the identity, and when specializing to \( t = 0 \), the ring homomorphism kills all \( T_{\alpha,\beta} \). The connectedness assertion is then immediate. \( \square \)

We fix a linear map \( \ell : \mathbb{Z}^n \to \mathbb{Z} \) and consider the functor
\[ \text{Hilb}_{S/k}^{\delta,\ell} : (k\text{-Alg}) \to (\text{Sets}) \]
\[ B \mapsto \{ \phi : B[x] \to Q \text{ in } \text{Hilb}_{S/k}^{\delta} \text{ s.t.} \]
\[ \ker \phi \text{ is homogeneous w.r.t. } \ell \} \]
(A homogeneous ideal is one generated by polynomials of the form \( f = \sum \ell(\beta) = a_{\beta} x^\beta \)). Upon using the notation of Proposition 3 or Theorem 3, we
see that $\text{Hilb}_{S/k}^{\delta,\ell}$ is representable by the affine subscheme $\text{Hilb}_{S/k}^{\delta,\ell}$ of $\text{Hilb}_{S/k}^{\delta}$ defined by the ideal $(T_{\alpha,\beta}; \alpha \in \delta \cup \delta^{(1)}, \beta \in \delta, \ell(\alpha) \neq \ell(\beta))$ in the coordinate ring $R^{\delta}$. (In an analogous way, one can define a functor $\text{Hilb}_{S/k}^{\delta,\ell}$ and represent that by a closed subscheme of $\text{Hilb}_{S/k}^{\delta,\ell}$.)

**Proposition 6.** Assume that there exists a linear map $\ell : \mathbb{Z}^n \to \mathbb{Z}$ such that

- $\ell(\alpha) \geq \ell(\beta)$ for all $\alpha \in \mathbb{N}^n - \delta$ and all $\beta \in \delta$; and
- $\ell(e_i) > 0$ for all $i$.

Then the following statements hold true:

1. $\text{Hilb}_{S/k}^{\delta,\ell}$ is an affine space.
2. $\text{Hilb}_{S/k}^{\delta,\ell}$ is connected.

**Proof.** We leave the proof of (i) as an exercise. (The same statement is proved in [KR08].) As for (ii), we can imitate the proof of Proposition 5 and get a morphism $g : \text{Spec } (k[t] \otimes_k R^{\delta}) \to \text{Hilb}_{S/k}^{\delta,\ell}$ corresponding to the ring homomorphism $R^{\delta} \to k[t] \otimes_k R^{\delta}$ which sends all $T_{\alpha,\beta}$ to $t^{\ell(\alpha) - \ell(\beta)}T_{\alpha,\beta}$. If $p$ is an arbitrary point in $\text{Hilb}_{S/k}^{\delta,\ell}$, then $g((t), p)$ is a point in the closed subscheme $\text{Hilb}_{S/k}^{\delta,\ell}$ of $\text{Hilb}_{S/k}^{\delta,\ell}$. Then the assertion follows from (i). \[\square\]

If we replace the first condition on $\ell$ by the stronger condition that $\ell(\alpha) > \ell(\beta)$ for all $\alpha \in \mathbb{N}^n - \delta$ and all $\beta \in \delta$, then connectedness of $\text{Hilb}_{S/k}^{\delta,\ell}$ follows from Proposition 5. Indeed, we use an arbitrary term order $\prec_\ell$ as a tie-breaker in the usual sense, defining a term order $\prec_\ell$ by setting $\alpha \prec_\ell \beta$ if either $\ell(\alpha) < \ell(\beta)$ or $\ell(\alpha) = \ell(\beta)$ and $\alpha < \beta$. Then $\text{Hilb}_{S/k}^{\delta,\ell} = \text{Hilb}_{S/k}^{\delta,\ell}$ for the term order $\prec_\ell$. However, the two conditions are not the same:

**Example 5.** Remove from the set $\{\alpha \in \mathbb{N}^n; |\alpha| \leq d\}$ any subset $S$ of $\{\alpha \in \mathbb{N}^n; |\alpha| = d\}$, then the remaining set $\delta$ is a standard set. $\delta$ satisfies the conditions of Proposition 6, but not the stronger conditions described above.

12. Changing the charts

In Section 8, we studied equations defining the affine scheme $\text{Hilb}_{S/k}^{\delta}$. Now we determine, for all standard sets $\delta$ and $\epsilon$,

- the equations defining the open subscheme $\text{Hilb}_{S/k}^{\delta,\ell} \cap \text{Hilb}_{S/k}^{\epsilon,\ell}$ of the affine scheme $\text{Hilb}_{S/k}^{\delta,\ell}$;
the gluing morphism $\psi_{\delta,\epsilon}$ which identifies the intersection $\text{Hilb}^\delta_{S/k} \cap \text{Hilb}^\epsilon_{S/k}$ as an open subscheme of $\text{Hilb}^\delta_{S/k}$ with an open subscheme of $\text{Hilb}^\epsilon_{S/k}$.

Take a $k$-algebra $B$ and a homomorphism which lies in both $\text{Hilb}^\delta_{S/k}(B)$ and $\text{Hilb}^\epsilon_{S/k}(B)$. This homomorphism is represented by two surjections $\phi$ and $\phi'$, respectively, such that there exists an isomorphism $\Psi$ making the following diagram commutative:

$$
\begin{array}{ccc}
Bx^\epsilon & \longrightarrow & B[x] \\
\Psi \downarrow \cong & & \downarrow \cong \\
Bx^\delta & \longrightarrow & B[x]
\end{array}
$$

For all $\alpha \in \epsilon - \delta$, consider the elements $f_\alpha \in \ker \phi$ of equation (12). From the commutative diagram above it follows that

$$
\Psi(x^\alpha) = \begin{cases} 
x^\alpha & \text{if } \alpha \in \epsilon \cap \delta, \\
-\sum_{\gamma \in \epsilon} d_{\alpha,\gamma} x^\gamma & \text{if } \alpha \in \epsilon - \delta.
\end{cases}
$$

Indeed, the first line is immediate; as for the second line, if $\alpha \in \epsilon - \delta$, then $\phi(x^\alpha + \sum_{\gamma \in \delta} d_{\alpha,\gamma} x^\gamma) = 0$, i.e.

$$
\Psi(x^\alpha) = \Psi(\phi'(x^\alpha)) = \phi(x^\alpha) = -\sum_{\gamma \in \delta} \phi(x^\gamma) = -\sum_{\gamma \in \delta} d_{\alpha,\gamma} x^\gamma.
$$

As for the inverse of $\Psi$, we define the polynomials

$$
g_\alpha = x^\alpha + \sum_{\gamma \in \epsilon} e_{\alpha,\gamma} x^\gamma \in \ker \phi'
$$

in analogy to (12) and obtain

$$
\Psi^{-1}(x^\alpha) = \begin{cases} 
x^\alpha & \text{if } \alpha \in \epsilon \cap \delta, \\
-\sum_{\gamma \in \delta} e_{\alpha,\gamma} x^\gamma & \text{if } \alpha \in \delta - \epsilon.
\end{cases}
$$

**Proposition 7.** Let $\delta \subset N$ and $\epsilon \subset M$ be standard sets. Then $\text{Hilb}^\delta_{S/k} \cap \text{Hilb}^\epsilon_{S/k}$ is the open subscheme

$$
\text{Spec} \, k[T_{\alpha,\beta}; \alpha \in N \cup M \cup (N \cup M)^{(1)}, \beta \in \delta]/I^\delta - \mathbb{V}(J^\delta,\epsilon)
$$

of $\text{Hilb}^\delta_{S/k}$, where $I^\delta$ is defined as in Proposition 3 (with $N$ replaced by $N \cup M$) and

$$
J^\delta,\epsilon = \langle \det(T_{\alpha,\beta})_{\alpha \in \epsilon - \delta, \beta \in \delta - \epsilon} \rangle.
$$

**Proof.** Note that $N \cup M$ is standard set. A point of $\text{Hilb}^\delta_{S/k}$ also lies in $\text{Hilb}^\epsilon_{S/k}$ if, and only if, the linear map (26) is invertible. From the explicit
descriptions of $\Psi$ and $\Psi^{-1}$ given above, we see that this condition is equivalent to the matrix

$$(d_{\alpha, \beta})_{\alpha \in \epsilon - \delta, \beta \in \delta - \epsilon}$$

being invertible. □

Upon replacing the matrix $T = (T_{\alpha, \beta})$ of indeterminates by a matrix

$$U = (U_{\alpha, \xi})_{\alpha \in \mathbb{N} \cup \mathbb{M} \cup (\mathbb{N} \cup \mathbb{M})^{(1)}, \xi \in \epsilon},$$

and swapping the roles of $\delta$ and $\epsilon$, Proposition 7 also explicitly gives $\text{Hilb}_{S/k}^{\delta} \cap \text{Hilb}_{S/k}^{\epsilon}$ as an open subscheme of $\text{Hilb}_{S/k}^{\epsilon}$. We decompose the indexing set of the rows as follows:

$$N \cup M \cup (N \cup M)^{(1)} = (\delta \cap \epsilon) \coprod (\delta - \epsilon) \coprod (\epsilon - \delta) \coprod \rho.$$

Accordingly, we decompose the two matrices of indeterminates into blocks

$$T = \begin{pmatrix} E & 0 \\ 0 & E \\ T_{31} & T_{32} \\ T_{41} & T_{42} \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} E & 0 \\ U_{21} & U_{22} \\ 0 & E \\ U_{41} & U_{42} \end{pmatrix},$$

where $E$ is the identity matrix. The lines in the uppermost blocks (i.e., $T_{11}$ and $T_{12}$ of $T$, and $U_{11}$ and $U_{12}$ of $U$, respectively) are indexed by $\delta \cap \epsilon$; the lines in the second sub-matrices by $\delta - \epsilon$; the next lines by $\epsilon - \delta$; the last lines by $\rho$. The columns in the left blocks of $T$ and $U$ are indexed by $\delta \cap \epsilon$; those in the right blocks of $T$ by $\delta - \epsilon$; those in the right blocks of $U$ by $\epsilon - \delta$.

**Proposition 8.** Let the intersection $\text{Hilb}_{S/k}^{\delta} \cap \text{Hilb}_{S/k}^{\epsilon}$ be given as in Proposition 7, by the matrix $T$ of coordinates on $\text{Hilb}_{S/k}^{\delta}$, and by the matrix $U$ of coordinates on $\text{Hilb}_{S/k}^{\epsilon}$. We decompose the matrices as in (??). Then the gluing morphism $\psi_{\delta, \epsilon}$ which identifies the intersection as an open subscheme of $\text{Hilb}_{S/k}^{\delta}$ with an open subscheme of $\text{Hilb}_{S/k}^{\epsilon}$ is given by the homomorphism

$$U \mapsto T = U \cdot \begin{pmatrix} E & 0 \\ T_{31} & T_{32} \end{pmatrix}$$

between the coordinate rings.

**Proof.** This is proved by the same token as equation (26). □

Note that therefrom follows that the matrices

$$T^{\Box} = \begin{pmatrix} E & 0 \\ T_{31} & T_{32} \end{pmatrix} \quad \text{and} \quad U^{\Box} = \begin{pmatrix} E & 0 \\ U_{21} & U_{22} \end{pmatrix}$$

are inverse to each other on the intersection $\text{Hilb}_{S/k}^{\delta} \cap \text{Hilb}_{S/k}^{\epsilon}$. 

13. A Gröbner stratum in different charts

Let $R^\delta$ be the coordinate ring of $\text{Hilb}^\delta_{S/k}$ as presented in Proposition 3, where $\alpha$ runs through the set $M \cup N \cup (M \cup N)^{(1)}$ of Proposition 8. Consider the matrix of indeterminates $T = (T_{\alpha,\beta})$ from the last section. The rows and columns of that matrix are indexed by elements of $\mathbb{N}^n$, which are ordered by the term order $\prec$. By Corollary 1, $\text{Hilb}^{\delta'}_{S/k}$ is the closed subscheme of $\text{Spec} R$ on which $T$ is a lower triangular matrix w.r.t. $\prec$.

Let $R^\epsilon$ be the coordinate ring of $\text{Hilb}^\epsilon_{S/k}$ as presented in Proposition 3, where $\alpha$ runs through the same set $M \cup N \cup (M \cup N)^{(1)}$, and let $U = (U_{\alpha,\xi})$ be the matrix of indeterminates from the last section. By Proposition 8, the intersection $\text{Hilb}^\delta_{S/k} \cap \text{Hilb}^\epsilon_{S/k}$ is the open subscheme of $\text{Hilb}^\epsilon_{S/k}$ on which the identity $T \square U \square = E$ holds true. We obtain the following:

**Proposition 9.** The intersection $\text{Hilb}^{\delta'}_{S/k} \cap \text{Hilb}^{\epsilon'}_{S/k}$ is the locally closed subscheme of $\text{Hilb}^\epsilon_{S/k}$ in which

- $U \square$ is the inverse of $T \square$, and
- $T$ is a lower triangular matrix w.r.t. $\prec$.

It is thus tempting to suspect that the boundary in $\text{Hilb}^\epsilon_{S/k}$ of $\text{Hilb}^{\delta'}_{S/k} \cap \text{Hilb}^{\epsilon'}_{S/k}$ is just $\text{Hilb}^{\epsilon'}_{S/k}$. Indeed, “the inverse of a lower triangular matrix is a lower triangular matrix”, hence from $T \square U \square = E$, we might deduce that $U \square$ is lower triangular. Moreover, “the product of two lower triangular matrices is lower triangular”, hence $U$, which by Proposition 8 can be written as $U = TU \square$, must be lower triangular.

However, the first quoted assertion only holds for such square matrices in which both rows and columns are indexed by the same totally ordered set. This is not the case for $T \square$ and $U \square$, whose rows and columns are indexed by $\delta$ and $\epsilon$, resp. (The second quoted assertion is true for all products $AB$ of matrices $A$ and $B$ indexed by subsets of a totally ordered set such that the indexing set of the columns of $A$ equals the indexing set of the rows of $B$.) Moreover, if $T$ and $U$ were both lower triangular in a point $p \in \text{Spec} k$, that point would lie in both $\text{Hilb}^{\delta'}_{S/k}$ and $\text{Hilb}^{\epsilon'}_{S/k}$, in contradiction to Theorem 2.

The most we can hope for is that

$$\partial(\text{Hilb}^{\delta'}_{S/k} \cap \text{Hilb}^{\epsilon'}_{S/k}) = \text{Hilb}^{\epsilon'}_{S/k},$$

as indicated above. If this was true for all $\delta$ and $\epsilon$, the decomposition of Theorem 2 would be a stratification. Indeed, denote by $\mathcal{S}$ the set of all standard sets of size $r$. Then the decomposition of Theorem 2 is a stratification if, and only if, for all $\delta \in \mathcal{S}$, there exists a subset $\mathcal{S}(\delta) \subset \mathcal{S}$
such that
\[(28) \quad \text{Hilb}_{S/k}^{r,0} = \prod_{\epsilon \in S(\delta)} \text{Hilb}_{S/k}^{r,\epsilon}. \]

If this is true, then
\[
S(\delta) = \{ \epsilon \in S; \text{Hilb}_{S/k}^{r,\delta} \cap \text{Hilb}_{S/k}^{r,\epsilon} \neq \emptyset \}
= \{ \epsilon \in S; \text{Hilb}_{S/k}^{r,\delta} \cap \text{Hilb}_{S/k}^{r,\epsilon} \neq \emptyset \}
= \{ \epsilon \in S; \text{Hilb}_{S/k}^{r,\delta} \cap \text{Hilb}_{S/k}^{r,\epsilon} \neq \emptyset \}. \]

Therefore it is clear that (27) holds true for all $\delta, \epsilon \in S$ if, and only if, for all $\delta \in S$ there exists an $S(\delta) \subset S$ such that (28) holds true. Here is a negative result on the question whether or not we have a stratification here.

**Proposition 10.** If $\prec$ is the lexicographic order on $S$ and $n \geq 3$, $r \geq 8$, the decomposition of Theorem 2 is not a stratification.

**Proof.** We order the variables such that $x_1 \succ \ldots \succ x_n$ and consider the standard set $\delta = \{0, \ldots, (r-1)e_n\} \subset \mathbb{N}^n$. Remember that by Example 4, $\text{Hilb}_{S/k}^{r,\delta}$ has dimension $rn$. Furthermore, $\text{Hilb}_{S/k}^{r,\delta} = \text{Hilb}_{S/k}^{r,\delta}$, as $\alpha \succ \beta$ for all $\alpha \in \mathbb{N}^n - \delta$ and all $\beta \in \delta$ in the lexicographic order. We claim that
\[(29) \quad \forall \epsilon \in S: \text{Hilb}_{S/k}^{r,\delta} \cap \text{Hilb}_{S/k}^{r,\epsilon} \neq \emptyset. \]

For proving that, we use a few schemes introduced and discussed in Section 5 of [?]. The first is
$$\text{Hilb}_{S/k}^{r,0} = ((\mathbb{A}^n)^r - \Lambda)/S_r,$$
where for $i = 1, \ldots, r$, we denote by $(x_1^{(i)}, \ldots, x_n^{(i)})$ the coordinates on the $i$-th copy of $\mathbb{A}^n$ in the product; where $\Lambda = \bigcup_{i \neq j} \mathbb{V}(x_1^{(i)} - x_1^{(j)}, \ldots, x_n^{(i)} - x_n^{(j)})$ is the *large diagonal* in the product; and where $S_r$ is the symmetric group acting on the product in the obvious way. $\text{Hilb}_{S/k}^{r,0}$ is an open subscheme of $\text{Hilb}_{S/k}^{r,\delta}$, and the functor associated to the scheme $\text{Hilb}_{S/k}^{r,0}$ sends a $k$-algebra $B$ to the set of all closed subschemes $Z \subset \mathbb{A}^n_B$ such that the restriction of the projection $p = \mathbb{A}^n_B \to \text{Spec } B$ is finite étale of degree $r$. (In contrast to that, the functor associated to $\text{Hilb}_{S/k}^{r,\delta}$ sends a $k$-algebra $B$ to the set of all closed subschemes $Z \subset \mathbb{A}^n_B$ such that the restriction of the projection $p = \mathbb{A}^n_B \to \text{Spec } B$ is finite flat of degree $r$. Finite flatness of $p: Z = \text{Spec } B[x]/I \to \text{Spec } B$ translates to local freeness of the $B$-algebra $B[x]/I$. Therefore, the additional requirement in $\text{Hilb}_{S/k}^{r,0}$ is *unramifiedness.*) Moreover, we use the subscheme
$$\text{Hilb}_{S/k}^{r,\delta,0} = \text{Hilb}_{S/k}^{r,\delta} \cap \text{Hilb}_{S/k}^{r,0}$$
of $\text{Hilb}_{S/k}^{r,0}$ and its analogue for $\epsilon$ instead of $\delta$. The functor associated to that scheme sends a $k$-algebra $B$ to the set of all étale $p: Z = \text{Spec } B[x]/I \to \text{Spec } B$ such that $I \subset B[x]$ is monic with standard set $\delta$. (As $\text{Hilb}_{S/k}^{r,\delta}$ =
Hilb$_{S/k}^{r}$, the functor equivalently sends a $k$-algebra $B$ to the set of all étale $p : Z = \text{Spec } B[x]/I \to \text{Spec } B$ such that $B[x]/I$ is free with basis $x^\delta$.)

We fix a homomorphism from $k$ to a field $k'$ having at least $r$ elements and a bijection $\{0, \ldots, r - 1\} \to D$, where $D \subset k'$ has $r$ elements. That induces a bijection $\{0, \ldots, r - 1\}^n \to D^n$, whose restriction to $\epsilon$ induces a bijection $\epsilon \to E$, where $E \subset D^n \subset (k')^n$ has $r$ elements. Let $I \subset k'[x]$ be the ideal defining $E$. Then the main theorem of [Led08] says that the ideal $I$ is monic with standard set $\epsilon$. Therefore $\xi = k'[x]/I$ is a $k'$-rational closed point of $\text{Hilb}_{S/k}^{r}$. We shall prove that $\xi$ lies in the closure in $\text{Hilb}_{S/k}^{r_0} \subset \text{Hilb}_{S/k}^{r_0}$. Then $\xi$ will also lie in the closure in $\text{Hilb}_{S/k}^{r_0}$ of $\text{Hilb}_{S/k}^{r_0}$. Then (29) will be proved.

For this we denote by

$$\pi : (\mathbb{A}_k^n)^r - \Lambda \rightarrow \text{Hilb}_{S/k}^{r_0}$$

the canonical morphism. As $\text{Hilb}_{S/k}^{r_0}$, and therefore $\text{Hilb}_{S/k}^{r_0}$, it follows that $\pi^{-1}(\text{Hilb}_{S/k}^{r_0}) = (\mathbb{A}_k^n)^r - \Lambda - A$ for some closed $A \subset (\mathbb{A}_k^n)^r$. (In fact, $A$ is the scheme associated to the ideal $(x_i - x_j^n; i \neq j)$, but we will not need that here.) Remember that we have to show that for all open $U \subset \text{Hilb}_{S/k}^{r_0}$ such that $\eta \in U$, we have $U \cap \text{Hilb}_{S/k}^{r_0} \neq \emptyset$. Let $\eta$ be an element of $\pi^{-1}(\xi)$. (The choice of $\eta$ corresponds to the choice of a labeling of the elements of $E$. ) It clearly suffices to show that for all open $V \subset ((\mathbb{A}_k^n)^r - \Lambda$ such that $\eta \in V$, we have $V \cap \{(\mathbb{A}_k^n)^r - \Lambda - A\} \neq \emptyset$. Upon writing $V = (\mathbb{A}_k^n)^r - \Lambda - A'$, for some closed $A' \subset (\mathbb{A}_k^n)^r$, it follows that $V \cap \{(\mathbb{A}_k^n)^r - \Lambda - A\} = (\mathbb{A}_k^n)^r - \Lambda - \Lambda'$. That intersection is empty if, and only if, $\Lambda \cup A \cup \Lambda' = (\mathbb{A}_k^n)^r$. But this is impossible as all three summands are closed subschemes of $(\mathbb{A}_k^n)^r$ and strictly smaller than the ambient scheme.

Now that (29) is proved, we conclude as follows: If the decomposition in question was a stratification, (29) would say that $S(\delta) = S$. Moreover, each $\text{Hilb}_{S/k}^{r_0}$, being a subscheme of the closure of $\text{Hilb}_{S/k}^{r_0}$, would have a dimension smaller than $rn$. Therefore by Theorem 2 and (28), the dimension of

$$\text{Hilb}_{S/k}^{r} = \bigsqcup_{\iota \in S} \text{Hilb}_{S/k}^{r_\iota} = \bigsqcup_{\iota \in S(\delta)} \text{Hilb}_{S/k}^{r_\iota} = \text{Hilb}_{S/k}^{r_\delta}.$$

would be $rn$. But from [CEVV09], we know that as $n \geq 3$, $r \geq 8$, the dimension of $\text{Hilb}_{S/k}^{r}$ is strictly larger than $rn$, a contradiction.  

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