Generalized Hyperkähler Geometry and Supersymmetry

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Abstract

We propose the definition of (twisted) generalized hyperkähler geometry and its relation to supersymmetric non-linear sigma models. We also construct the corresponding twistor space.

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1 Introduction

It is well-known that supersymmetry has a deep relation to geometry [32, 1]. In the context of sigma models the geometry of the target space is restricted by the amount of supersymmetry on the worldsheet and its dimension. For example, for a two-dimensional worldsheet with manifest $N = (1,1)$ supersymmetry the sigma model has $N = (2,2)$ supersymmetry if the target manifold is bi-hermitian [9]. For around twenty years now the different possible geometries have been studied [15, 5, 12, 16, 17, 19]. For an overview over the different possibilities, see also [22]. Although this classification is known, it was only recently that an elegant mathematical framework to deal with these geometries was developed. Hitchin introduced the notion of generalized complex geometry in the context of generalized Calabi-Yau manifolds with fluxes [13]. Later, it was clarified by Gualtieri [11]. The main idea is to replace the tangent bundle $TM$ of a manifold $M$ with the sum of the tangent and the cotangent bundle $TM \oplus T^*M$. This puts metrics and two-forms of the sigma model on an equal footing and unifies the concepts of complex and symplectic geometry. It has been shown that bi-hermitian geometry corresponds to a subset of generalized complex geometry called generalized Kähler geometry. The obvious question is how generalized complex geometries arise in the sigma model context. A lot of work has been done in this direction [21, 2, 31, 22, 3, 29, 6] and by now for the generalized Kähler geometry, the picture is rather clear [24, 30]. In the Lagrangian formulation of the sigma model generalized Kähler geometry is completely specified by the definition of a Lagrangian in a manifest $N = (2,2)$ supersymmetric formulation. The Lagrangian is the generalized Kähler potential [23]. On the other hand it has been shown that it can be physically derived as the geometry admitting $N = (2,2)$ supersymmetry in the phase space formulation of the sigma model [4]. In this letter, we elaborate further in this second direction and derive the conditions for $N = 4$ and $N = (4,4)$ supersymmetry. We show how generalized hypercomplex and generalized hyperkähler geometry arise in the phase space formulation and construct the twistor space of generalized complex structures that is associated with the $N = (4,4)$ supersymmetry.

This letter is organized as follows: In section 2, we review extended supersymmetry of the $N = (1,1)$ supersymmetric sigma model and the relevant target space geometries. In section 3, we give an overview on the results for the phase space formulation of the $N = (1,1)$ sigma model. In section 4 we show generalized hypercomplex geometry arises in this notion and in section 5 we combine these results with those of a previous paper [4] and give the derive generalized hyperkähler geometry from $N = (4,4)$ supersymmetry. In section 6 we comment shortly on how this result relates to the Lagrangian formulation of the sigma model. In sections 7 and 8 we define the twistor space for the generalized hyperkähler geometry before concluding with a short discussion in section 9.
2 \textit{N = (1, 1) sigma model and extended supersymmetry}

In this section, we review extended supersymmetry of the \textit{N = (1, 1)} supersymmetric sigma model and introduce the necessary notation. The action for the \textit{N = (1, 1)} supersymmetric sigma model is given by

\[ S = \frac{1}{2} \int d^2 \sigma d \theta^+ d \theta^- D_+ \Phi^\mu D_- \Phi^\nu (G_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)). \]  

(1)

Here, \( D_\pm \) are spinorial derivatives with

\[ D_\pm = \partial \theta^\pm + i \theta^\pm \partial \mp, \quad D_\pm^2 = i \partial \mp, \quad \{D_+, D_-\} = 0. \]  

(2)

The action is invariant under a manifest supersymmetry that is given by

\[ \delta_0(\epsilon) \Phi^\mu = -i(\epsilon^+ Q_+ + \epsilon^- Q_-) \Phi^\mu, \]  

(3)

where \( Q_\pm = i D_\pm + 2 \theta^\pm \partial \mp \) are the supersymmetry charges. The action admits an extension to \( N = (2, 2) \) supersymmetry if the target space geometry is bi-hermitian \[9\]. The additional supersymmetry transformation is of the form

\[ \delta_1(\epsilon) \Phi^\mu = \epsilon^+ D_+ \Phi^\nu J^\mu_{\nu} + \epsilon^- D_- \Phi^\nu J^\mu_{-\nu}, \]  

(4)

where \( J_\pm \) are complex structures that satisfy

\[ J^\rho_{\pm \nu} G_{\rho \sigma} J^\sigma_{\pm \nu} = G_{\mu \nu}, \quad \nabla^{(\pm)}_{\rho} J^\mu_{\pm \nu} = 0. \]  

(5)

The connections for the covariant derivatives are given by

\[ \Gamma^{(\pm)}_{\nu \rho} = \Gamma^{(0)}_{\nu \rho} \pm G^{\mu \sigma} H_{\sigma \nu \rho}, \]  

(6)

where \( \Gamma^{(0)} \) is the Levi-Civita connection for the metric \( G_{\mu \nu} \) and \( H = dB \) is the torsion three-form. Explicitly, it is given by

\[ H_{\mu \nu \rho} = \frac{1}{2} (B_{\mu \nu, \rho} + B_{\nu \rho, \mu} + B_{\rho \mu, \nu}). \]  

(7)

Indices separated by a comma define derivatives with respect to the corresponding space-time direction \( B_{\mu \nu, \rho} = \partial_{\rho} B_{\mu \nu} \). We denote the Kähler forms as \( \omega_{\pm \mu \nu} = G_{\mu \rho} J^\rho_{\pm \nu} \). In general, the non-manifest supersymmetry transformation only closes on-shell

\[ [\delta_1(\epsilon), \delta_1(\tilde{\epsilon})] \Phi^\mu = 2 \epsilon^+ \tilde{\epsilon}^+ \partial_\mu \Phi^\mu + 2 \epsilon^- \tilde{\epsilon}^- \partial_\pm \Phi^\mu. \]  

(8)

The restrictions on the target manifold geometry increases if we consider \( N = (4, 4) \) supersymmetry. In that case, there are three additional supersymmetries of the form \( \delta_i(\epsilon) \Phi^\mu = \epsilon^+ D_+ \Phi^\nu J^\mu_{+i \nu} + \epsilon^- D_- \Phi^\nu J^\mu_{-i \nu}, \quad i = 1, 2, 3. \)  

(9)
As an implication of the previous discussion, these are (on-shell) supersymmetry transformations if the six tensors $J_{±i}$ are complex structures that satisfy (5). This is a consequence of the fact that each of the additional supersymmetry transformations commute with the manifest supersymmetry (3). In addition, the transformations have to commute among themselves in order to satisfy the supersymmetry algebra. Altogether, we have

$$[\delta_i(\epsilon), \delta_j(\tilde{\epsilon})]\Phi^\mu = 2\delta_{ij}(\epsilon^+ \tilde{\epsilon}^+ \partial_+ \Phi^\mu + \epsilon^- \tilde{\epsilon}^- \partial_- \Phi^\mu), \quad i, j = 0, 1, 2, 3. \quad (10)$$

We do not discuss the possibility of central charges here. These relations are satisfied on-shell if the left- and the right-going complex structures anticommute among themselves

$$\{J_{±i}, J_{±j}\} = -2\delta_{ij}. \quad (11)$$

They form two hypercomplex structures. The target space geometry is bi-hypercomplex [9]. We collect the requirements needed for $N = (4, 4)$ supersymmetry of the action (1):

$$J^p_{±i\mu} G_{ρσ} J^ρ_{±iν} = G_{μν}, \quad \nabla_ρ^{(±)} J^ρ_{±ν} = 0, \quad J^μ_{±iρ} J^p_{±jρ} + J^μ_{±iρ} J^ρ_{±jν} = -2\delta_{ij}\delta^μ_κ. \quad (12)$$

One way to achieve off-shell supersymmetry is if the left- and right-going complex structures in (4) or (9) commute. A particular case is when the two complex structures are equal and the extended supersymmetry transformations are of the form

$$\delta_i(\epsilon)\Phi^μ = (\epsilon^+ D_+ \Phi^ν + \epsilon^- D_- \Phi^ν)J^μ_{iν}. \quad (13)$$

The conditions for supersymmetry imply that this is only possible if the torsion $H$ is zero.

In the case of $N = (2, 2)$ supersymmetry, the target space geometry is Kähler, while the action (1) admits $N = (4, 4)$ supersymmetry on a hyperkähler manifold.

3 $N = 2$ extended supersymmetry in phase space

In [29] $N = 1$ supersymmetric phase space was introduced as the cotangent bundle $\Pi T^*\mathcal{L}M$ (with parity reversed) of the superloop space $\mathcal{L}M = \{\phi : S^{1,1} \rightarrow M\}$. Here, $S^{1,1}$ is a supercircle, a circle with an additional Grassmann direction $θ$. $ϕ^μ$ is a superfield that embeds the supercircle into the manifold. The conjugate momenta $S_μ$ are spinorial fields. Therefore, the cotangent bundle has reversed parity on its fibers. The superfields have the following expansions in the odd coordinate

$$ϕ^μ(\sigma, θ) = X^μ(σ) + θλ^μ(σ), \quad S_μ(σ, θ) = ψ_μ(σ) + θp_μ(σ), \quad (14)$$

where $p_μ$ is the momentum conjugate to $X^μ$. We use the notation of [4]. The phase space is equipped with a symplectic structure

$$ω = 1 \int dσdθ δS_μ ∧ δϕ^μ, \quad (15)$$

3
where $\delta$ is the de Rham differential on the manifold. The convention for $\omega$ is chosen in such a way that its purely bosonic part is equal to the usual definition of the canonical symplectic structure in bosonic phase space

$$\omega|_{\text{bos}} = \int d\sigma \delta X^\mu \wedge \delta p_\mu. \quad (16)$$

The symplectic structure yields a superPoisson bracket

$$\{F, G\} = i \int d\sigma d\theta F \left( \delta S_\mu \wedge \delta \phi^\mu - \frac{1}{2} H_{\mu \nu \rho} D\phi^\mu \delta \phi^\nu \wedge \delta \phi^\rho \right) G. \quad (17)$$

$\omega$ can be twisted by a three-form $H$

$$\omega_H = i \int d\sigma d\theta \left( \delta S_\mu \wedge \delta \phi^\mu - \frac{1}{2} H_{\mu \nu \rho} D\phi^\mu \delta \phi^\nu \wedge \delta \phi^\rho \right). \quad (18)$$

Here, $\partial$ is the derivative with respect to $\sigma$. Consequently, this twists the Poisson bracket by $H$. To keep things simple, we here work with the case $H = 0$ and only comment on the changes for $H \neq 0$. The calculations in that case work out exactly in the same way.

The phase space has two natural operations

$$D = \partial_\theta + i \theta \partial, \quad Q = \partial_\theta - i \theta \partial. \quad (19)$$

They satisfy the algebra

$$D^2 = i \partial, \quad Q^2 = -i \partial, \quad \{D, Q\} = 0. \quad (20)$$

With this, the generator for manifest supersymmetry is given by

$$Q_0(\epsilon) = -\int d\sigma d\theta \epsilon S_\mu Q \phi^\mu, \quad (21)$$

where $\epsilon$ is an odd parameter. It acts on the fields through the Poisson bracket

$$\delta(\epsilon) \phi^\mu = \{\phi^\mu, Q_0(\epsilon)\} = -i \epsilon Q \phi^\mu, \quad \delta(\epsilon) S_\mu = \{S_\mu, Q_0(\epsilon)\} = -i \epsilon S_\mu. \quad (22)$$

Being a supersymmetry generator, it satisfies the supersymmetry algebra

$$\{Q_0(\epsilon), Q_0(\tilde{\epsilon})\} = P(2\epsilon \tilde{\epsilon}), \quad P(a) = \int d\sigma d\theta a S_\mu \partial \phi^\mu, \quad (23)$$

where $P(a)$ is the generator of $\sigma$-translations. Any additional supersymmetry that is generated by some $Q_1(\epsilon)$ has to satisfy the brackets

$$\{Q_0(\epsilon), Q_1(\tilde{\epsilon})\} = 0, \quad \{Q_1(\epsilon), Q_1(\tilde{\epsilon})\} = P(2\epsilon \tilde{\epsilon}). \quad (24)$$

The condition for which these conditions are satisfied was found in \[29\]. The form of $Q_1(\epsilon)$ is determined by dimensional arguments

$$Q_1(\epsilon) = -\frac{1}{2} \int d\sigma d\theta \epsilon \left( 2 D\phi^\mu S_\nu J^\nu_\mu + D\phi^\mu D\phi^\nu L_{\mu \nu} + S_\mu S_\nu P^{\mu \nu} \right). \quad (25)$$
where the tensors can be conveniently combined into a map $\mathcal{J} : T \oplus T^* \rightarrow T \oplus T^*$ given by

$$\mathcal{J} = \begin{pmatrix} -J & P \\ L & J^t \end{pmatrix}.$$  \hspace{1cm} (26)

$Q_1(\epsilon)$ is the generator of a supersymmetry transformation if the target space is generalized complex and $\mathcal{J}$ is a generalized complex structure.

For $H \neq 0$, we can construct the generators of supersymmetry in a similar way. With $H = dB$, this can be achieved by replacing

$$S_\mu \rightarrow S_\mu - B_{\mu\nu} D\phi^\nu$$  \hspace{1cm} (27)

in the definitions of the supersymmetry generators, for example

$$Q_0 = -\int d\sigma d\theta \epsilon (S_\mu - B_{\mu\nu} D\phi^\nu) Q_\phi^\nu.$$  \hspace{1cm} (28)

Since we are forced to use the twisted version of the Poisson bracket as well, the transformation on the fields remains unchanged. Concerning the additional supersymmetry, we may afterwards rename the tensors in such a way that $Q_1$ remains in the form (25). $\mathcal{J}$ is then a twisted generalized complex structure.

4 $N = 4$ extended supersymmetry in phase space

In the previous section we reviewed the steps that lead to the condition that $N = 2$ extended supersymmetry in phase space is possible if the target space is a generalized complex manifold. The generator for the extended supersymmetry is given in (25). Here, we discuss the necessary conditions for to have two such extended supersymmetries with generators $Q_1(\epsilon)$ and $Q_2(\epsilon)$ of the form (25).

We show that the target space geometry has to be generalized hypercomplex. We define generalized hypercomplex geometry by three generalized complex structures that satisfy the algebra of quaternions

$$\mathcal{J}_3 = \mathcal{J}_1 \mathcal{J}_2.$$  \hspace{1cm} (29)

Following the discussion of the previous section, $Q_1(\epsilon)$ and $Q_2(\epsilon)$ are generators of supersymmetry if we can relate them to two generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. In addition to (23) and (24), the Poisson bracket of $Q_1(\epsilon)$ with $Q_2(\epsilon)$ has to vanish

$$\{Q_1(\epsilon), Q_2(\tilde{\epsilon})\} = 0.$$  \hspace{1cm} (30)

We can collect all the Poisson brackets in the following way:

$$\{Q_i(\epsilon), Q_j(\tilde{\epsilon})\} = \delta_{ij} P(2\epsilon\tilde{\epsilon}), \quad i = 0, 1, 2.$$  \hspace{1cm} (31)
The calculation of the bracket (30) is tedious but results in the two conditions for the generalized complex structures

\[ \{ \mathcal{J}_1, \mathcal{J}_2 \} = 0, \quad N(\mathcal{J}_1, \mathcal{J}_2) = 0. \] (32)

Here, \( N(\mathcal{J}_1, \mathcal{J}_2) \) is the (generalized) Nijenhuis concomitant of \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \). It is given by

\[ N(\mathcal{J}_1, \mathcal{J}_2) = [\mathcal{J}_1(u + \xi), \mathcal{J}_2(v + \eta)] - \mathcal{J}_1[\mathcal{J}_1(u + \xi), \mathcal{J}_2(v + \eta)] - \mathcal{J}_2[\mathcal{J}_1(u + \xi), v + \eta] + \mathcal{J}_1\mathcal{J}_2[u + \xi, v + \eta] - (\mathcal{J}_1 \leftrightarrow \mathcal{J}_2), \] (33)

where the bracket is the Courant bracket and \( u + \xi, v + \eta \) are sections of \( TM \oplus T^*M \).

It follows that \( \mathcal{J}_3 = \mathcal{J}_1\mathcal{J}_2 \) is the third generalized complex structure. Its integrability is guaranteed by the vanishing of the (generalized) Nijenhuis concomitant and integrability of \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \). This proves that the target manifold is generalized hypercomplex.

For the case \( H \neq 0 \) the calculations remain true but we have to replace the Poisson bracket by its twisted version. The target space geometry is then twisted generalized hypercomplex.

## 5 Generalized Hyperkähler Geometry

The previous discussion was completely model independent. In this section, we combine the results with those of a previous paper [4]. There, it was shown that from the sigma model point of view, the relation between generalized Kähler and bi-hermitian geometry follows from the equivalence of the Hamilton and the Lagrange description of the sigma model. We start with a short recapitulation of those results. The sigma model Hamiltonian for (1) is obtained by performing one of the \( d\theta \)-integrations. To this extend, we introduce new Grassmann coordinates \( \theta^0, \theta^1 \) such that

\[ \theta^{0,1} = \frac{1}{\sqrt{2}}(\theta^+ \mp \sqrt{i} \theta^-), \quad D_{0,1} = \frac{1}{\sqrt{2}}(D_+ \pm \sqrt{i} D_-), \quad Q_{0,1} = \frac{1}{\sqrt{2}}(Q_+ \pm \sqrt{i} Q_-). \] (34)

We define the \( N = 1 \) component fields of \( \Phi^\mu \) by

\[ \phi^\mu = \Phi^\mu|_{\theta^0=0}, \quad S_\mu = G_{\mu\nu}D_0\Phi^\mu|_{\theta^0=0} \] (35)

and denote \( G_{\mu\nu}(\phi) = G_{\mu\nu}(\Phi)| \), \( D = D_1 \) and \( \partial = \partial_\sigma \). Performing the \( d\theta^0 \) integral in the usual way by replacing \( \int d\theta^0 \to D_0 \) and taking the \( \theta^0 = 0 \) component we obtain the action (11) in terms of the \( N = 1 \) superfields

\[ S = \int d^2\sigma d\theta \left( iS_\mu \partial_0 \phi^\mu - \frac{1}{2}(i\partial_\phi^\mu D\phi^\nu + S_\mu DS_\nu G^{\mu\nu} + S_\mu D\phi^\nu S_\rho G^\rho\Gamma_{\mu\nu}\Gamma_{\mu\nu}) \right). \] (36)

1See also [25] for a mathematically more rigid derivation.
Here, we focus on the case $B_{\mu\nu} = 0$. The action has the typical form of a Legendre transformation where the first term says that $S_\mu$ is the conjugate momentum for $\phi^\mu$ and the second term yields the Hamiltonian

$$ H = \frac{1}{2} \int d\sigma d\theta (i \partial \phi^\mu D\phi^\nu + S_\mu DS_\nu G^{\mu\nu} + S_\mu D\phi^\nu S_\rho G^{\sigma\rho} \Gamma^\mu_{\nu\sigma}).$$

(37)

$H$ is invariant under the manifest supersymmetry transformation (21) with $Q \equiv Q_1$. The second manifest supersymmetry of the original action (1) is non-manifest in this formulation. It is given by

$$ \tilde{\delta}_0 (\epsilon) \phi^\mu = \epsilon G^{\mu\nu} S_\nu, \quad \tilde{\delta}_0 (\epsilon) S_\mu = i \epsilon G_{\mu\nu} \partial \phi^\nu + \epsilon S_\nu S_\rho G^{\sigma\rho} \Gamma^{\mu}_{\nu\sigma}. $$

(38)

In [4] it is shown that $H$ is invariant under the additional supersymmetry if the target space geometry is generalized Kähler:

$$ \{Q_1(\epsilon), H\} = 0 $$

(39)

implies that $J$ commutes with the generalized metric

$$ G = \begin{pmatrix} G & G^{-1} \\ G^{-1} & -G \end{pmatrix}. $$

(40)

As a consequence, $\tilde{J} = GJ$ is an additional generalized complex structure with supersymmetry generator $\tilde{Q}_1(\tilde{\epsilon})$ of the form (25) such that

$$ [J, \tilde{J}] = 0, \quad \{Q_1(\epsilon), \tilde{Q}_1(\tilde{\epsilon})\} = 2i \epsilon \tilde{\epsilon} H. $$

(41)

We now show that the Hamiltonian has $N = (4,4)$ supersymmetry if the geometry is generalized hyperkähler. We call a manifold generalized hyperkähler if it admits six generalized complex structures $J_i, \tilde{J}_i$ and a generalized metric $G$ that satisfy the algebra of bi-quaternions $Cl_{2,1}(\mathbb{R})$

$$ J_i J_j = -\delta_{ij} 1_{2d} + \epsilon_{ijk} J_k, \quad \tilde{J}_i \tilde{J}_j = -\delta_{ij} 1_{2d} + \epsilon_{ijk} \tilde{J}_k $$

and

$$ J_i \tilde{J}_j = -\delta_{ij} G + \epsilon_{ijk} \tilde{J}_k, \quad \tilde{J}_i J_j = -\delta_{ij} G + \epsilon_{ijk} J_k. $$

(42)

This definition coincides with [18, 10]. However, we derive this definition from the sigma model.

The Hamiltonian is invariant under the three additional supersymmetries of the previous sections if it satisfies

$$ \{Q_i(\epsilon), H\} = 0, \quad i = 1, 2, 3. $$

(43)

As in (39), this is the case if the generalized complex structures $J_i$ commute with $G$

$$ [J_i, G] = 0. $$

(44)
According to the above, this induces three generalized complex structures \( \tilde{\mathcal{J}}_i = \mathcal{G}J_i \). Each of the triples \( \{ \mathcal{J}_i, \mathcal{G}, \tilde{\mathcal{J}}_i \} \) for \( i = 1, 2, 3 \) form a generalized Kähler structure. The three generalized complex structures \( \tilde{\mathcal{J}}_i \) are associated to three additional supersymmetry generators \( \tilde{Q}_i \). The generators satisfy the algebra

\[
\{ Q_i(\epsilon), Q_j(\tilde{\epsilon}) \} = 2i \delta_{ij} \epsilon \tilde{\epsilon} H,
\]

\[
\{ Q_i(\epsilon), Q_j(\tilde{\epsilon}) \} = \delta_{ij} P(2\epsilon \tilde{\epsilon}).
\]

(45)

A straightforward calculation shows that these brackets are equivalent to the relations (42) and integrability of \( J_i \) and \( \tilde{\mathcal{J}}_i \). We conclude that the sigma model Hamiltonian admits \( N = (4,4) \) supersymmetry if the target space is generalized hyperkähler.

For \( H \neq 0 \), the Hamiltonian is given by

\[
H = \frac{1}{2} \int d\sigma d\theta \left( i \partial \phi^\mu D \phi^\nu + S_\mu D S_\nu G^{\mu\nu} + S_\mu D \phi^\nu S_\rho G^{\sigma\rho} \Gamma_{\mu\nu} \right.
\]

\[
- \frac{1}{3} H^{\mu\rho\sigma} S_\mu S_\nu S_\rho + D \phi^\mu D \phi^\nu S_\rho H_{\mu\rho} \left. \right) \quad (46)
\]

and admits \( N = (4,4) \) supersymmetry for a twisted generalized hyperkähler geometry which is defined in analogy to generalized hyperkähler geometry but with twisted generalized complex structures.

### 6 Relation to the Lagrangian formulation

The three different generalized Kähler structures \( \{ \mathcal{J}_i, \mathcal{G}, \tilde{\mathcal{J}}_i \} \) correspond to bi-hermitian geometries, where the metric is read off from (40) and the complex structures are given via the relation [11]

\[
\mathcal{J}_i = \frac{1}{2} \begin{pmatrix}
-(J_{+i} + J_{-i}) & -(\omega_{+i}^{-1} - \omega_{-i}^{-1}) \\
\omega_{+i} - \omega_{-i} & (J_{+i} + J_{-i})^t
\end{pmatrix},
\]

\[
\tilde{\mathcal{J}}_i = \frac{1}{2} \begin{pmatrix}
-(J_{+i} - J_{-i}) & -(\omega_{+i}^{-1} + \omega_{-i}^{-1}) \\
\omega_{+i} + \omega_{-i} & (J_{+i} - J_{-i})^t
\end{pmatrix},
\]

(47)

where \( \omega_{\pm i} = GJ_{\pm i} \) are the Kähler forms. From the bi-quaternion algebra it is easy to see that \( J_{+i} \) and \( J_{-i} \) form two independent hypercomplex structures with

\[
\{ J_{+1}, J_{+2} \} = 0, \quad \{ J_{-1}, J_{-2} \} = 0
\]

but with nothing implied for the commutation relations of \( J_{+i} \) and \( J_{-j} \). For the case \( H \neq 0 \) we obtain in addition the relations of section 2. We conclude that (twisted) generalized hyperkähler geometry is the phase space equivalent to bi-hypercomplex geometry.

### 7 Twistor Space of Generalized Complex Structures

In this section, we define the twistor space of generalized complex structures that is associated to the \( N = (4,4) \) supersymmetry of the sigma model Hamiltonian. The idea
of a twistor space is to encode the geometric properties of the target manifold \( M \) in the holomorphic structure of a larger manifold, the twistor space. The original idea goes back to Penrose \cite{Penrose} and Salamon \cite{Salamon1, Salamon2}. We here follow the same approach as in the definition of the twistor space for hyperkähler geometry \cite{Gualtieri}. Twistor spaces of generalized complex structures and generalized Kähler structure are also discussed in \cite{Gualtieri1, Gualtieri2} in order to find examples of generalized complex and generalized Kähler structures that are not induced by complex, symplectic and Kähler structures. Before discussing the twistor space for the generalized hyperkähler geometry, we first review the results for hyperkähler geometry.

Given a hypercomplex structure \( J_1, J_2, J_3 \) the linear combination
\[
K = c_1 J_1 + c_2 J_2 + c_3 J_3
\]
is a complex structure if \( c \) lies on the unit sphere: \( c^2 = 1 \). This sphere can be identified with \( \mathbb{C}P^1 \). \( \mathbb{C}P^1 \) is usually represented as \( \mathbb{C}^2 \) with coordinates \((\zeta, \bar{\zeta})\) and the identification \((\zeta, 1) \simeq (\lambda \zeta, \lambda)\) for \( \lambda \neq 0 \). Therefore, we can cover it with two sheets of coordinates \((\zeta, 1)\) and \((1, \bar{\zeta})\) such that \( \bar{\zeta} = \zeta^{-1} \) in the overlapping region. In these coordinates,
\[
K = \frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_1 + \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_2 + \frac{1 - \zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_3.
\]

The twistor space of complex structures is the product space \( M \times S^2 \), such that at any point \( p \in M \), \( S^2 \) parametrized the space of complex structures on \( T_p M \). A complex structure for the whole manifold is then given by the pair
\[
K_\zeta = \left( \frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_1 + \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_2 + \frac{1 - \zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_3, I \right),
\]
where \( I \) is the is the standard complex structure on the sphere. This construction allows to define hyperkähler geometry in terms of an abstract parameter space.

We now define the twistor space of generalized complex structures in a completely analogous way. Given the six generalized complex structures \( J_i \) and \( \tilde{J}_i \) of the previous section, we find that the linear combinations that define generalized complex structures are given by the relation
\[
K = \frac{1}{2}(c^i + d^i)J_i + \frac{1}{2}(c^i - d^i)\tilde{J}_i,
\]
\[
c^2 = \tilde{d}^2 = 1.
\]
The space of generalized complex structures for a generalized hyperkähler structure is parametrized by \( S^2 \times S^2 \). In \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) coordinates \( z, w \), the vectors \( \vec{c} \) and \( \vec{d} \) are given by
\[
\vec{c} = \left( \frac{1 - z \bar{z}}{1 + z \bar{z}}, \frac{z + \bar{z}}{1 + z \bar{z}}, \frac{1(z - \bar{z})}{1 + z \bar{z}} \right), \quad \vec{d} = \left( \frac{1 - w \bar{w}}{1 + w \bar{w}}, \frac{w + \bar{w}}{1 + w \bar{w}}, \frac{1(w - \bar{w})}{1 + w \bar{w}} \right).
\]

Since the generalized complex structures \( J_i, \tilde{J}_i \) are a realization of the bi-quaternionic algebra, it follows that \( K^2 = -1 \) and \( \tilde{K} = G K \) where \( G \) is the generalized metric. The generalized metric \( G \) acts on the parameter space by letting \( \vec{d} \rightarrow -\vec{d} \). In the \( \mathbb{C}P^1 \) coordinate \( w \), this corresponds to the anti-podal map
\[
\tau_w : w \rightarrow -\bar{w}^{-1}
\]
that changes the orientation of the $w$-sphere. The ordinary complex structures for the
two spheres $I_z$ and $I_w$ define a complex structure $J_S$ for $S^2 \times S^2$. This complex structure
induces a generalized complex structure on $T(S^2 \times S^2) \oplus T^*(S^2 \times S^2)$ by

$$J_S = \begin{pmatrix} -J_S & 0 \\
0 & J_S^t \end{pmatrix}. \quad (55)$$

A generalized complex structure for the combined space $M \times S^2 \times S^2$ is then given by

$$J = (K(z, w), J_S). \quad (56)$$

The proof that $J$ is integrable is presented in the next section using the formulation of
generalized complex structures in terms of pure spinor lines. It is an interesting question,
if $I$ can be chosen in a more general way in this context. Generalized complex structures
for $S^2 \times S^2$ were explicitly defined in [14].

The triples $\{K, G, \tilde{K} = GK\}$ form different generalized Kähler structures. The two spheres
parametrize the space of ordinary left- and right-complex structures on $TM$. We can
clarify this by introducing

$$J_i^{(\pm)} = \frac{1}{2}(J_i \pm \tilde{J}_i) = \frac{1}{2}(1 \pm G)J_i. \quad (57)$$

These are the projections of the generalized complex structures on the $\pm$ eigenspaces of
$G$. Explicitly and with relation (17), they are given by

$$J_i^{(\pm)} = \frac{1}{2} \begin{pmatrix} -J_{\pm i} & -\omega_{\pm i}^{-1} \\
\omega_{\pm i} & J_{\pm i}^t \end{pmatrix}. \quad (58)$$

With this, (52) becomes

$$K = c^i J_i^{(+)} + d^i \tilde{J}_i^{(-)}. \quad (59)$$

We indeed find that $c$ and $d$ parametrize the two sets of complex structures $J_{+i}$ and $J_{-i}$.

## 8 Pure spinors

It remains to show that $J$ as defined in (56) is indeed a generalized complex structure. In order to see this, we reformulate the previous discussion in the pure spinor language. Since $TM \oplus T^*M$ always admits a $Spin(d, d)$ structure that is isomorphic to the exterior algebra $\wedge T^*M$, we can associate the $+i$ eigenspace $L$ of a generalized complex structure $J_1$ with the annihilation space of a spinor $\varphi$ such that for the sections $u + \xi$ of $L$,

$$(u + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi = 0. \quad (60)$$

The spinor can in general only be defined locally. This is suitable for our purposes. More
generally, we associate $L$ with a pure spinor line $U$ such that $\varphi$ is locally a representative
of $\mathcal{U}$. The spinor satisfies $J_1 \cdot \varphi = m\varphi$, where $n$ is the complex dimension of the manifold and the multiplication is given by the action

$$J_1 \cdot \varphi = -L \wedge \varphi + ip\varphi - J^* \varphi + \frac{1}{2}\text{tr}(J)\varphi. \quad (61)$$

Here, $J, P, L$ are the components of $J$ as given in (26). Since $J$ is integrable, the spinor is pure and satisfies

$$d\varphi = (u + \xi) \cdot \varphi \quad (62)$$

for some section $u + \xi$ of $TM \oplus T^*M$. If $J_1$ was a twisted generalized complex structure then this equation would have been modified incorporating $H$:

$$d_H \varphi = (d + H \wedge)\varphi = (u + \xi) \cdot \varphi. \quad (63)$$

Given that $\varphi$ is a pure spinor for the $+i$ eigenspace of $J_1$, then

$$\phi = (1 + \frac{1}{2}zJ_3^{(+)}) + \frac{1}{2}wJ_3^{(-)} \cdot \varphi \quad (64)$$

is a pure spinor for $\mathcal{K}$. Since $J_i$ and $\tilde{J}_i$ are integrable by assumption, $\mathcal{K}$ is integrable as well. This follows from the fact that the Nijenhuis concomitants vanish. Especially, for fix $z, w$,

$$d\phi|_{z,w} = (u + \xi) \cdot \phi \quad (65)$$

for some $u + \xi \in \Gamma(TM \oplus T^*M)$. The bar indicates that the derivative is taken for fixed values of $z$ and $w$. The generalized complex structure $J_S$ is integrable by construction. We can associate to it a pure spinor $\eta$ such that $(A + b) \cdot \eta = 0$ for sections $A + b$ of $T(S^2 \times S^2) \oplus T^*(S^2 \times S^2)$. Explicitly, $\eta$ is the top-holomorphic form $\eta = dz \wedge dw$. Since $\phi$ is holomorphic in $z, w$, the spinor $\rho = \phi \wedge \eta$ satisfies

$$d(\phi \wedge \eta) = d\phi|_{z,w} \wedge \eta + (-1)^{\frac{1}{2}|\phi|} \phi \wedge d\eta + dz \wedge \nabla_{\partial_z} \phi \wedge \eta + dw \wedge \nabla_{\partial_w} \phi \wedge \eta. \quad (66)$$

$\rho$ is a spinor. It is an element of the exterior algebra $\wedge^* (M \times S^2 \times S^2) = (\wedge^* M) \wedge (\wedge^* S^2) \wedge (\wedge^* S^2)$. By construction the last two terms in (66) vanish such that

$$d\rho = (X + \xi) \cdot \phi \wedge \eta + (-1)^{|\phi|} \phi \wedge (A + b) \cdot \eta,$n\rho = (X + \xi + A + b) \cdot \rho. \quad (67)$$

$\rho$ is a pure spinor for the almost generalized complex structure $J = (\mathcal{K}(z, w), J_S)$ and we conclude that $J$ is integrable.

## 9 Discussion

In this short note we showed how generalized hypercomplex geometry emerges as the target space geometry for the $N = 4$ supersymmetric sigma model phase space. We applied
this result to the Hamilton formulation of the \( N = (1, 1) \) supersymmetric sigma model and combined it with the results of [11] to show that the Hamiltonian admits \( N = (4, 4) \) supersymmetry if the target space is generalized hyperkähler. We defined the twistor space of generalized complex structures and clarified why the two parameterizing two-spheres do not parametrize the generalized complex structures but the supersymmetries in the Lagrangian formulation. Our results fit the discussion in [18]. We also discussed the twistor space construction in terms of pure spinors. This construction should be related to the deformation complex of generalized complex structures [11]. Recently, it has been shown that generalized Kähler geometry corresponds to a manifest formulation of the \( N = (2, 2) \) supersymmetric sigma model [23]. It would be interesting to relate our results to the harmonic superspace formulation of \( N = (4, 4) \) supersymmetry [20] in the same way. Since hyperkähler geometry is always Calabi-Yau, another interesting open question is how generalized hyperkähler geometry relates to generalized Calabi-Yau geometry.

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A Generalized Complex Geometry

The notation of generalized complex geometry was first introduced by Hitchin [13] and later clarified by Gualtieri [11].

The vector bundle \( T \oplus T^* \) on a complex \( d \)-dimensional manifold \( M \) has a natural pairing

\[
\langle X + \xi, Y + \eta \rangle = \frac{1}{2} (i_Y \xi + i_X \eta).
\] (68)

The smooth sections of \( T \oplus T^* \) have a natural bracket, called the Courant bracket

\[
[X + \xi, Y + \eta]_c = [X, Y] + L_X \eta - L_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi).
\] (69)

It is a natural extension of the Lie-bracket \([\cdot, \cdot]\) on the tangent bundle onto \( T \oplus T^* \). Here, \( L_X \) is the Lie derivative with respect to \( X \). This bracket has non-trivial automorphism parametrized by a closed two-form \( b \in \Omega^2_{\text{closed}}(M) \) acting on the sections as

\[
e^b(X + \xi) = X + (\xi + i_X b).
\] (70)

This transformation is called a \( b \)-transform and it acts on Courant bracket as

\[
[e^b(X + \xi), e^b(Y + \eta)]_c = e^b[X + \xi, Y + \eta]_c.
\] (71)

A generalized complex structure is the complex version of two complementary Dirac structures with \((T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L}\). We can define it as a map \( \mathcal{J} : (T \oplus T^*) \otimes \mathbb{C} \rightarrow (T \oplus T^*) \otimes \mathbb{C} \) satisfying

\[
\mathcal{J}^4 \mathcal{J} = \mathcal{I}, \quad \mathcal{J}^2 = -1_{2d}, \quad \Pi_{\pm} [\Pi_{\pm}(X + \xi), \Pi_{\pm}(Y + \eta)]_c = 0,
\] (72)

where \( \Pi_{\pm} = \frac{1}{2} (1_{2d} \pm \mathcal{J}) \) are projectors on \( L \) and \( \bar{L} \).
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