Counting solutions for the CDMA multiuser MAP demodulator

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We evaluate the average number of locally minimal solutions for maximum-a-posteriori (MAP) demodulation in code-division multiple-access (CDMA) systems. For this purpose, we use a sophisticated method to investigate the ground state properties for the Sherrington-Kirkpatrick-type (i.e. fully connected) spin glasses established by Tanaka and Edwards in 1980. We derive the number of locally minimal solutions as a function of several parameters which specify the CDMA multiuser MAP demodulator. We also calculate the distribution function of the normalized-energies for the locally minimum states. We find that for a small number of chip intervals (or equivalently a large number of users) and large noise level at the base station, the number of local minimum solutions becomes larger than that of the SK model. This provides us with useful information about the computational complexity of the MAP demodulator.

KEYWORDS: CDMA, Statistical Mechanics, Spin Glasses, Metastable States, Bayesian Statistics, Tanaka-Edwards Theory, Replica method

1. Introduction

Recently, statistical-mechanical analysis has revealed many important aspects of probabilistic information processing. Among them, many studies addressing the code-division multiple-access (CDMA) communication problem succeeded not only in investigating the statistical properties of demodulators but also constructing iterative algorithms based on so-called belief-propagation and examining the dynamics of decoding algorithms. Within the framework of Bayesian inference, marginal-posterior-mode (MPM) demodulation provides the best possible performance in the sense that its bit-error rate is minimized under a specific condition, namely, the so-called Nishimori condition. However, it is also possible for us to choose another strategy, that is, the maximum-a-posteriori (MAP) demodulator, in order to estimate the original information bit for each user from the received signals at the base station. The MAP demodulator attempts to achieve the maximum in the posterior distribution of the sent bits. In practice this means that we choose as our estimate of the original information bits the ground state of the Hamiltonian, which is defined by minus the logarithm

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of the posterior distribution. Therefore, it is quite important for us to be able to evaluate how many local minimum (metastable) solutions exist around the ground state, as when we attempt to minimize the Hamiltonian to obtain the ground state many local search algorithms will get trapped in these metastable states. However, there are few studies to investigate such a computational complexity aspect of the CDMA multiuser demodulator. For this kind of problem, Tanaka and Edwards\textsuperscript{8,9} established a general theory to count the number of locally minimum energy states for the model class of the Sherrington-Kirkpatrick spin glasses.\textsuperscript{10} They showed that the average number of the local minimum states scales (to leading order in $N$) as $\sim e^{0.19923N} = 2^{0.28743N}$ for the SK-type long-range mean-field model. As there exists a close relationship between the CDMA multiuser demodulator and the Hopfield model with extensive number of patterns, which is itself strongly disordered in a similar way to the SK model, their method appeared to be useful for us to investigate the ground state properties of the CDMA multiuser MAP demodulator. With the assistance of the Tanaka-Edwards theory, in this paper, we evaluate the number of the locally minimum normalized-energy solutions for CDMA multiuser MAP demodulation problems. We also calculate the distribution of the normalized-energies of the local minimum states and discuss how often we obtain the deep local minimum energy level of the solution for a given parameter set, namely, the number of users and the noise level at the base station.

This paper is organized as follows. In the next section, following the scheme of Tanaka,\textsuperscript{2,3} we introduce a model system for the CDMA multiuser demodulator. The MAP demodulator is formulated in the context of Bayesian statistics. Then, the energy function to be minimized is introduced naturally. In the same section, we define the local minimum solution of the MAP demodulator as a fixed point of the zero-temperature dynamics of the CDMA multiuser demodulation problem. In section 3, we evaluate the number of local minimum solutions for the MAP demodulator by using Tanaka-Edwards theory.\textsuperscript{8,9} In the following section, we calculate the distribution of the local minimum normalized-energies and make it clear how often the deep locally minimum state appears for a given number of users and noise level at the base station. It is well-known that the Tanaka-Edwards theory is based on the annealed calculation for the average of the macroscopic quantities of the system. Therefore, it cannot take into account the effect of the quenched disorder correctly. In order to evaluate the number of local minimum states in the proper way in section 5 we attempt to recalculate the number of the locally minimum solutions for the MAP demodulator modeling it as a quenched system with assistance of the replica method. Then, we compare the result of the annealed calculation with that calculated by the replica method. The last section is devoted to a summary of our results.
2. Bayesian inference for a model CDMA system

We consider the direct-sequence binary phase-shift-keying (DS/BPSK) CDMA code where we have $K$ users, $s_0^0, \ldots, s_0^K$ which use a spreading code $b_1^1, \ldots, b_N^N$, where we define $N/K = \alpha$.

The received signal is given by

$$y^\mu = \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b_k^\mu s_k^0 + \nu^\mu,$$

where $\nu^\mu \sim N(0, 1/\beta_s)$ is Gaussian noise and the label $\mu$ takes the values $\mu = 1, \cdots, N$. $N$ is the number of components of spreading codes for each user. The CDMA demodulation problem is to estimate the original information for each user $s_0^1, \ldots, s_0^K$ given that the output $y^\ell$ and the spreading code for each user $b_k^1, \ldots, b_k^N$ is known. We model the spreading code sequences $\{b_k^\mu\}$ as sequences of independent identically distributed binary random variables with $\text{Prob}[b_k^\mu = \pm 1] = 1/2$.

For this problem, we introduce a model of the system (1):

$$y^\mu = \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b_k^\mu s_k + \tilde{\nu}^\mu,$$

where $s_1, \ldots, s_K$ are estimates of the corresponding original information bit $s_0^1, \ldots, s_0^K$ for each user and $\tilde{\nu}^\ell$ is a model of the noise at the base station which follows a Gaussian distribution with variance $\beta^{-1}$.

Within the context of Bayesian statistics, we consider the posterior distribution which is given by

$$P(\{s_k\}|\{y^\mu\}, \{b_k^\mu\}) = \frac{\beta}{\sqrt{2\pi}} \exp \left[ -\frac{\beta}{2} \sum_{\mu=1}^{N} \left( y^\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b_k^\mu s_k \right)^2 \right]$$

$$= \frac{\beta}{\sqrt{2\pi}} \exp \left[ -\frac{\beta}{2} \sum_{\mu=1}^{N} (y^\mu)^2 - \beta H(s) \right],$$

where we choose the uniform distribution $P(\{s_k\}) = 2^{-K}$ as a prior. Using these definitions we are working with a system which has an effective Hamiltonian given by

$$H(s) = \frac{1}{2} \sum_{i,j} s_i J_{ij} s_j - \sum_{i} f_i s_i$$

$$J_{ij} = \frac{1}{K} \sum_{\mu=1}^{N} b_i^\mu b_j^\mu \quad f_k = \frac{1}{\sqrt{K}} \sum_{\mu=1}^{N} y^\mu b_k^\mu,$$

which is constructed so that maximizing the posterior distribution corresponds to minimizing this Hamiltonian (4). The Hamiltonian gives rise to local fields acting on each spin $s_k$

$$h_k(s) = f_k - \sum_{j \neq k} J_{kj} s_j$$
so that the zero temperature dynamics, which attempts to find the maximum-a-posteriori (MAP) demodulator solution, is given via

\[ s_k = \text{sgn}(h_k(s)). \] (7)

Additionally, this means that a given state \( s \) is a stable fixed point of the dynamics if, and only if, \( h_k(s) = \lambda_k s_k, \lambda_k \geq 0 \forall k \). We would like to comment that the above estimate becomes the so-called conventional demodulator (CD) if we neglect the second term of the right hand side of equation (6) (i.e. we ignore the interactions between different bits in the received message).

The dynamics (7) actually minimizes the Hamiltonian (4) (i.e. finds its global minimum) if there is no local minimum in the energy landscape. However, due to the quenched disorder \( \{b_{ik}\} \) in (4), which manifests itself in the interactions \( J_{ij} \) and fields \( f_k \), there exists many local minima and the dynamics (7) may well become trapped in one of the locally minimum states. Therefore, it is quite important for us to evaluate how many locally minimum states exist around the globally minimum state. Obviously, the number of solutions depends on the number of users \( K \) and the noise level at the base station \( \beta_s \). Our main goal in this study is to make this point clear in a quantitative manner.

### 3. The average number of local minimum states : annealed calculation

In this section, following the method developed by Tanaka and Edwards\(^8,9\) (especially, by their formulation for the Ising case\(^8\)), we calculate the number of solutions to the dynamical equations (7), which are attempting to construct the MAP demodulator for a given solution \( s \) of the dynamics (7). We first define the local energy \( \epsilon_i \) by

\[ \epsilon_i = -s_i f_i + s_i \sum_{j \neq i} J_{ij} s_j. \] (8)

Then, according to Tanaka and Edwards\(^8\), we assume that each bit asynchronously updates and the energy difference due to the bit flip \( s_i \rightarrow -s_i \) is given by

\[ \Delta \epsilon_i = \epsilon'_i - \epsilon_i = 2s_i \left( f_i - \sum_{j \neq i} J_{ij} s_j \right). \] (9)

If we define the parameter \( \lambda_i \) in terms of the local field \( h_i(s) \) via

\[ h_i(s) = f_i - \sum_{j \neq i} J_{ij} s_j = \lambda_i s_i, \] (10)

then for a given realization of disorder \( \{J, f\} \), the condition for the solution \( s \) to be one of the locally minimum states is given as \( \Delta \epsilon_i > 0 \forall i \), namely,

\[ \Delta \epsilon_i = 2s_i \lambda_i s_i = 2\lambda_i > 0. \] (11)

Of course, we might modify the condition (11) to investigate the stability against a cluster spin flip, however, the analysis for such cases is beyond the scope of our present abilities.
Therefore, we can calculate the average number of locally minimum states, \( \langle g_0 \rangle \), through

\[
\langle g_0 \rangle = \left\langle \sum_{\mathbf{s}} \prod_i \Theta(\lambda_i) \right\rangle_{\{J,f\}} = \left\langle \sum_{\mathbf{s}} \prod_i \left[ \int_0^\infty d\lambda_i \delta \left( f_i - \sum_{j \neq i} J_{ij} s_j - \lambda_i s_i \right) \right] \right\rangle_{\{J,f\}}
\]

(12)

where \( \Theta(\cdots) \) denotes the step function. The \( f_k \) are defined in terms of the output of the Gaussian channel \( y^\mu \), the measure of which is given by the probability measure

\[
Z(y) = 2^{-K} \left( \frac{\beta_s}{2\pi} \right)^{\frac{N}{2}} \sum_{\mathbf{s}^0} \exp \left[ -\beta_s \frac{N}{2} \sum_{\mu=1}^K \left( y^\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b_k^\mu s_k^0 \right)^2 \right].
\]

(13)

Thus we may write the average number of fixed points of the dynamics (7) as

\[
\langle g_0 \rangle = 2^{-K} \left( \frac{\beta_s}{2\pi} \right)^{\frac{N}{2}} \sum_{\mathbf{s}^0} \prod_{\mu=1}^K dy^\mu \exp \left[ -\beta_s \frac{N}{2} \sum_{\mu=1}^K \left( y^\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b_k^\mu s_k^0 \right)^2 \right]
\]

\[
2^{-NK} \sum_{\mathbf{s},\mathbf{b},\ldots,\mathbf{b}^N} \prod_i \int_0^\infty d\lambda_i \int_{-\infty}^\infty \frac{d\lambda_i}{2\pi i} \exp \left[ \lambda_i \left( \frac{1}{\sqrt{K}} \sum_{\mu} y^\mu b_i^\mu - \frac{1}{K} \sum_{j \neq \mu} b_j^\mu b_i^\mu s_j - \lambda_i s_i \right) \right]
\]

(14)

under the assumption that the system is well-approximated as an annealed system. By using the saddle point method in the limit of \( K \to \infty \), the average \( \langle g_0 \rangle \) under the annealed approximation is given by

\[
\langle g_0 \rangle = \int \frac{dt d\dot{t}}{2\pi i} \frac{dud\dot{u}}{2\pi i} \frac{dwd\dot{w}}{2\pi i} \frac{dq \dot{q}}{2\pi i} \frac{q \dot{q}}{K} e^{K \Phi(t,u,w,q,i,i,\dot{u},\dot{w},\dot{q})}
\]

\[
\Phi = \dot{t} + \dot{u} + \dot{w} + \dot{q} - \frac{\alpha}{2} \log \left\{ u[1 + 2\beta_s(1 - q)] + \beta_s(1 + t - w)^2 \right\}
\]

\[
+ \log \left\{ \cosh(q) + \frac{1}{2} \left[ e^{-q} \text{Erf} \left( \frac{\alpha - \dot{t} - \dot{w}}{2\sqrt{u}} \right) + e^q \text{Erf} \left( \frac{\alpha - \dot{t} + \dot{w}}{2\sqrt{u}} \right) \right] \right\} + \frac{\alpha}{2} \log \beta_s.
\]

(15)

The details of the derivation is shown in Appendix A.

We now have to vary this saddle point surface to find its extremum. The definition we have used for the error function is \( \text{Erf}(z) = (2/\sqrt{\pi}) \int_0^z dt e^{-t^2} \) so that

\[
\frac{d}{dz} \text{Erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}.
\]

(16)

Varying with respect to the true parameters gives

\[
\dot{t} = \frac{\alpha \beta_s(1 + t - w)}{u[1 + 2\beta_s(1 - q)] + \beta_s(1 + t - w)^2}
\]

(17a)

\[
\dot{w} = -\dot{t}
\]

(17b)

\[
\dot{u} = \frac{\alpha[1 + 2\beta_s(1 - q)]}{2 u[1 + 2\beta_s(1 - q)] + \beta_s(1 + t - w)^2}
\]

(17c)

\[
\dot{q} = \frac{-\alpha u \beta_s}{u[1 + 2\beta_s(1 - q)] + \beta_s(1 + t - w)^2}
\]

(17d)
while varying the conjugate parameters leads to

\[
q = \frac{1}{2} \left[ e^{-\hat{q}} \text{Erf} \left( \frac{\alpha - \hat{t} - \hat{w}}{2\sqrt{u}} \right) - e^{\hat{q}} \text{Erf} \left( \frac{\alpha - \hat{t} + \hat{w}}{2\sqrt{u}} \right) \right] - \sinh(\hat{q})
\]

(18a)

\[
t = \frac{1}{2\sqrt{u}} \left[ e^{-\hat{q}} - \left| \frac{\alpha - \hat{t} - \hat{w}}{2\sqrt{u}} \right|^2 + e^{\hat{q}} - \left| \frac{\alpha - \hat{t} + \hat{w}}{2\sqrt{u}} \right|^2 \right]
\]

(18b)

\[
w = \frac{1}{2\sqrt{u}} \left[ e^{-\hat{q}} - \left| \frac{\alpha - \hat{t} - \hat{w}}{2\sqrt{u}} \right|^2 + e^{\hat{q}} - \left| \frac{\alpha - \hat{t} + \hat{w}}{2\sqrt{u}} \right|^2 \right]
\]

(18c)

\[
u = \frac{1}{4\sqrt{u}} \left[ (\alpha - \hat{t} - \hat{w}) e^{-\frac{(\alpha - \hat{t} - \hat{w})^2}{2\sqrt{u}}} + (\alpha - \hat{t} + \hat{w}) e^{\frac{(\alpha - \hat{t} + \hat{w})^2}{2\sqrt{u}}} \right]
\]

(18d)

We see that we can reduce the complexity slightly by introducing \( s \equiv t - w \) and \( \hat{s} \equiv \hat{t} = -\hat{w} \) and then eliminate \( \{t, w, \hat{t}, \hat{w}\} \) in favour of \( \{s, \hat{s}\} \). This gives us the (slightly reduced) saddle point surface as

\[
\Phi = \hat{s}s + \hat{u}u + \hat{q}q - \frac{\alpha}{2} \log \left\{ u[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2 \right\}
\]

\[
+ \log \left\{ \cosh(\hat{q}) + \frac{1}{2} \left[ e^{-\hat{q}} \text{Erf} \left( \frac{\alpha}{2\sqrt{u}} \right) + e^{\hat{q}} \text{Erf} \left( \frac{\alpha - 2\hat{s}}{2\sqrt{u}} \right) \right] \right\} + \frac{\alpha}{2} \log \beta_s
\]

and the saddle point equations as:

\[
\hat{s} = \frac{\alpha \beta_s(1 + s)}{u[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2} \quad (19a)
\]

\[
\hat{u} = \frac{1}{2} \frac{\alpha[1 + 2\beta_s(1 - q)]}{u[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2} \quad (19b)
\]

\[
\hat{q} = \frac{-\alpha u \beta_s}{u[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2} \quad (19c)
\]

\[
q = \frac{1}{2} \left[ e^{-\hat{q}} \text{Erf} \left( \frac{\alpha}{2\sqrt{u}} \right) - e^{\hat{q}} \text{Erf} \left( \frac{\alpha - 2\hat{s}}{2\sqrt{u}} \right) \right] - \sinh(\hat{q})
\]

(19d)

\[
s = \frac{1}{2\sqrt{u}} \left[ e^{-\hat{q}} - \left| \frac{\alpha - 2\hat{s}}{2\sqrt{u}} \right|^2 \right]
\]

(19e)

\[
u = \frac{1}{4\sqrt{u}} \left[ \alpha e^{-\frac{\alpha - 2\hat{s}^2}{2\sqrt{u}}} + (\alpha - 2\hat{s}) e^{\frac{\alpha - 2\hat{s}^2}{2\sqrt{u}}} \right]
\]

(19f)

We solve the equations (19a)-(19f) numerically for given values of the parameters \( (\beta_s, \alpha) \). In Fig. 1, we plot the logarithm of the number, namely, the function \( \Phi = \log(g_0)/K \) at the saddle point \( (\hat{s}, \hat{u}, \hat{q}, q, s, u) \). From this figure, we find that the number of metastable solutions
Fig. 1. The logarithm of the number of solution Φ as a function of α for several values of β_s. The inset is β_s-dependence of α_{SK} at which Φ takes the same value as that of the SK model Φ_{SK} = 0.19923.

for the zero-temperature dynamics (7) is larger than that of the SK model (Φ_{SK} ≃ 0.19923) for large numbers of users and small values of the Gaussian noise. As the number of users K = N/α decreases (α increases), the saddle point surface Φ tends to zero so the number of local minimum solutions rapidly decreases. However, it remains finite for finite K.

From Fig. 1, we also see that the number of locally minimum solutions decreases as the parameter β_s decreases. At a first glance, this seems to be rather counter-intuitive because β_s is the inverse of the variance of the Gaussian noise which is defined by (13). However, it might be possible for us to show that this fact can be naturally understood. We should notice that the Hamiltonian (4) can be rewritten as

$$H(s) = \frac{1}{2} \sum_{ij} s_i J_{ij} s_j - \sum_i h_i s_i$$

(20)

where $h_i$ is defined by

$$h_i = \frac{1}{K} \sum_{\mu=1}^{N} \sum_{k=1}^{K} b_{k}^{\mu} b_{i}^{\mu} s_k^{0} + \beta_s^{-1/2} \sum_{\mu=1}^{N} \eta_{\mu} b_{i}^{\mu}.$$  

(21)

and where $\eta^\mu$ is a Gaussian variable with zero mean and unit variance. The random field $h_i$ appearing in the above Hamiltonian (20) has zero mean and the variance $\overline{h_i^2} = \alpha (1 + \beta_s^{-1})$. Obviously, if $\beta_s \ll 1$, the second term of (20) becomes dominant. Therefore, the best way to minimize the Hamiltonian $H(s)$ is to minimize the second term of the Hamiltonian (20). In other words, making each bit $s_i$ in the same direction as the random field $h_i$ is the best possible strategy to minimize the Hamiltonian. As the result, the frustration, which mainly comes from the first term of the Hamiltonian, is weakened. This is a reason why the number of locally stable solutions decreases as the parameter β_s decreases.
We also should notice that the anomalous discontinuity of the $\Phi$-$\alpha$ curve for $\beta_s = 0.1$. These demodulation problems as an annealed system have two locally stable solutions for $\Phi$ with large and small overlap at the fixed point of the dynamics (7). This result seems to imply that there exists a close relationship between the discontinuity of the $\Phi$-$\alpha$ curve and the spinodal observed in the bit-error rate. However, from the pioneering study by Tanaka, one naturally expects the spinodal more likely to be observed when inter-user interference effects are more significant, namely, when the parameter $\beta_s$ is large. This counter-intuitive result might be caused due to our rough evaluation of the $\Phi$-$\alpha$ curve by annealed calculation. In fact, as we shall see later, the discontinuity disappears when we treat the problem as a quenched system.

4. Distribution of the local minimum energies

Our next problem is to calculate the distribution of the energies of these local minimum states. The local energy $-\epsilon_0$ is given by

$$-\epsilon_0 = \frac{1}{2} \sum_{\mu=1}^{N} \left( y^\mu - \frac{1}{\sqrt{K}} \sum_i b^\mu_i s_i \right)^2. \tag{22}$$

Then, the distribution of local energies is given by

$$\mathcal{N}(\epsilon_0) = \left\langle \sum_i \prod_s \left[ \int_0^{\infty} \frac{d\lambda_i}{g_0(b^\mu_i, z^\mu)} \right] \delta \left( f_i - \sum_{j \neq i} J_{ij}s_j - \lambda_i \sigma_i \right) \right\rangle \times \delta \left[ \epsilon_0 - \frac{1}{2} \sum_{\mu=1}^{N} \left( y^\mu - \frac{1}{\sqrt{K}} \sum_i b^\mu_i s_i \right)^2 \right] \delta \left\{ \{b^\mu_i\}, \{y^\mu_i\} \right\}. \tag{23}$$

This is rather challenging to calculate directly, so we assume rather that $g_0(\{b^\mu_i, y^\mu_i\})$ is self-averaging (following Tanaka-Edwards), or at least a slowly varying function of the disorder and make the annealed approximation for this variable. We then look at $\mathcal{P}(\epsilon_0) = \langle g_0 \rangle \mathcal{N}(\epsilon_0)$.

$$\mathcal{P}(\epsilon_0) = \left\langle \sum_i \prod_s \left[ \int_0^{\infty} d\lambda_i \delta \left( f_i - \sum_{j \neq i} J_{ij}s_j - \lambda_i \sigma_i \right) \right] \right\rangle \times \delta \left[ \epsilon_0 - \frac{1}{2} \sum_{\mu=1}^{N} \left( y^\mu - \frac{1}{\sqrt{K}} \sum_i b^\mu_i s_i \right)^2 \right] \delta \left\{ \{b^\mu_i\}, \{y^\mu_i\} \right\}. \tag{23}$$

Now we can write the delta function constraining the energy in Fourier representation as

$$\delta \left[ \epsilon_0 - \frac{1}{2} \sum_{\mu=1}^{N} \left( y^\mu - \frac{1}{\sqrt{K}} \sum_i b^\mu_i s_i \right)^2 \right] = \int \frac{d\epsilon_0}{2\pi} \exp \left[ i\epsilon_0 \epsilon_0 - \frac{i\epsilon_0}{2} \sum_{\mu=1}^{N} \left( y^\mu - \frac{1}{\sqrt{K}} \sum_i b^\mu_i s_i \right)^2 \right]. \tag{23}$$
Then the calculation proceeds much as the previous one did, leading to a saddle point surface given by:

\[
\langle g_0 \rangle = \left( \frac{\beta_s}{2\pi} \right)^\frac{N}{2} \int \frac{dt d\dot{u} du d\dot{w} dw dq d\dot{q}}{2\pi/K} \frac{d\dot{\epsilon}_0}{2\pi} e^{iK(\dot{t} + \dot{u} + \dot{w} + \dot{q}) + i\epsilon_0 \epsilon_0} 
\times \exp \left[ \alpha K \log \int dq \frac{d\dot{q} d\dot{q}}{2\pi} e^{-\frac{\beta_s}{2}(y - v)^2 + i(\dot{q}_0 v^0 - \dot{v}) - \frac{1}{2} \dot{q}^2 - \frac{1}{2}(\dot{v})^2 - q\dot{v} - \frac{1}{2}(v - y)^2} \right. 
\left. \times \exp \left[ K \log \frac{1}{2} \sum_{s,s'} \int_0^\infty \frac{d\lambda}{2\pi} \int_{-\infty}^\infty \frac{d\lambda'}{2\pi} e^{i\lambda(\alpha - \lambda) - i(s\lambda s' + \dot{u} + \dot{w} - \dot{q})} \right] \right]
\]

From here we rotate \( \dot{\epsilon}_0 \to -i\epsilon_0 \) and then rescale \( \epsilon_0 \to \epsilon_0/K \). Hence, from now on, we call the \( \epsilon_0 \) as "normalized-energy". It is relatively straightforward to read off the final result:

\[
\mathcal{P}(\epsilon_0) = \int \frac{dt d\dot{u} du d\dot{w} dw dq d\dot{q}}{2\pi/K} \frac{d\dot{\epsilon}_0}{2\pi} e^{K\Psi(\tilde{t}, u, w, q, \tilde{u}, \tilde{w}, \tilde{q}, \epsilon_0, \dot{\epsilon}_0)} 
\Psi = \tilde{t} t + \tilde{u} u + \tilde{w} w + \tilde{q} q + \epsilon_0 \epsilon_0 - \frac{\alpha}{2} \log \left\{ (u + \epsilon_0)[1 + 2\beta_s(1 - q)] + \beta_s(1 + t - w)^2 \right\} 
+ \log \left\{ \cosh(\dot{q}) + \frac{1}{2} \left[ e^{-\dot{q}} \text{Erf} \left( \frac{\alpha - \dot{t} - \tilde{w}}{2\sqrt{u}} \right) + e^{\dot{q}} \text{Erf} \left( \frac{\alpha - \dot{t} + \tilde{w}}{2\sqrt{u}} \right) \right] \right\} + \frac{\alpha}{2} \log \beta_s 
\]

From the following equations we see that the saddle point equations will be very similar to those we obtained for the calculation of the number of solutions exp\[N\Phi]\] only now with \( u \to u + \epsilon_0 \). Varying with respect to \( \epsilon_0 \) leads to the conclusion that we must have \( \epsilon_0 = \dot{u} \). We can make the same substitutions as before with \( s \) and \( s' \) to find

\[
\Psi = \ddot{s} s + \ddot{u} u + \ddot{q} q + \dot{\epsilon}_0 \epsilon_0 - \frac{\alpha}{2} \log \left\{ (u + \dot{\epsilon}_0)[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2 \right\} 
+ \log \left\{ \cosh(\dot{q}) + \frac{1}{2} \left[ e^{-\dot{q}} \text{Erf} \left( \frac{\alpha - \dot{t} - \tilde{w}}{2\sqrt{u}} \right) + e^{\dot{q}} \text{Erf} \left( \frac{\alpha - \dot{t} + \tilde{w}}{2\sqrt{u}} \right) \right] \right\} + \frac{\alpha}{2} \log \beta_s. 
\]

Thus, the saddle point equations are given by

\[
\epsilon_0 = \frac{1}{2} \frac{\alpha[1 + 2\beta_s(1 - q)]}{(u + \epsilon_0)[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2} 
\dot{s} = -\frac{\alpha \beta_s(1 + s)}{(u + \epsilon_0)[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2} 
\dot{u} = \frac{1}{2} \frac{\alpha[1 + 2\beta_s(1 - q)]}{(u + \epsilon_0)[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2} = \epsilon_0 
\dot{q} = \frac{1}{2} \left[ e^{-\dot{q}} \text{Erf} \left( \frac{\alpha}{2\sqrt{u}} \right) - e^{\dot{q}} \text{Erf} \left( \frac{\alpha - 2\dot{s}}{2\sqrt{u}} \right) \right] - \sinh(\dot{q}) 
q = \frac{1}{2} \left[ e^{-\dot{q}} \text{Erf} \left( \frac{\alpha}{2\sqrt{u}} \right) + e^{\dot{q}} \text{Erf} \left( \frac{\alpha - 2\dot{s}}{2\sqrt{u}} \right) \right] 
\]

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\[ s = \frac{1}{\sqrt{\pi u}} \left[ e^{\hat{q} - \frac{[\alpha - 2\hat{s}]^2}{2\sqrt{u}}} \right] \cosh(\hat{q}) + \frac{1}{2} \left[ e^{-\hat{q}} \text{Erf}\left(\frac{\alpha}{2\sqrt{u}}\right) + e^{\hat{q}} \text{Erf}\left(\frac{\alpha - 2\hat{s}}{2\sqrt{u}}\right) \right] \] (24f)

\[ u = \frac{1}{4\sqrt{\pi \hat{q}}} \left[ \alpha e^{\hat{q} - \frac{[\alpha - 2\hat{s}]^2}{2\sqrt{u}}} + (\alpha - 2\hat{s}) e^{-\hat{q}} \right] \cosh(\hat{q}) + \frac{1}{2} \left[ e^{-\hat{q}} \text{Erf}\left(\frac{\alpha}{2\sqrt{u}}\right) + e^{\hat{q}} \text{Erf}\left(\frac{\alpha - 2\hat{s}}{2\sqrt{u}}\right) \right] \] (24g)

In order to solve these saddle point equations, we evaluate the fixed points for a given value of \( \hat{\epsilon}_0 \), then calculate the value of the normalized-energy \( \epsilon_0 \), and finally read off the value of \( n(\epsilon_0) \) for the scaling form of the distribution \( N(\epsilon_0) \approx \exp[K(\Psi - \Phi)] = [n(\epsilon_0)]^K \), \( \log n(\epsilon_0) \equiv \Psi - \Phi \) and then change \( \hat{\epsilon}_0 \) and repeat. This is more straightforward than trying to find the correct \( \hat{\epsilon}_0 \) for a given \( \epsilon_0 \). Thus, for the solution of the saddle point equations (24a)-(24g), the distribution we seek to obtain is given by

\[
\log n(\epsilon_0) = \hat{\epsilon}_0 \epsilon_0 - \frac{\alpha}{2} \log \left\{ (u + \hat{\epsilon}_0)[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2 \right\} \\
+ \frac{\alpha}{2} \log \left\{ u[1 + 2\beta_s(1 - q)] + \beta_s(1 + s)^2 \right\}.
\] (25)

where the variables on the right hand side obviously take their saddle point values. In Fig. 2,

![Graphs showing the distribution of the minimum normalized-energy states.](image)

we plot the distribution \( n(\epsilon_0) \). From the left panel of this figure, we find that the probability for deep normalized-energy state \( \epsilon_0 \gg 1 \) is almost zero for all values of the number of users \( \alpha \), however, relatively higher normalized-energy states appear much more frequently as the number of users increases (\( \alpha \) decreases). On the other hand, the right panel tells us that the deep energy states frequently appear in the case of large variance \( \beta_s^{-1} \) of the Gaussian. The reason why the deep energy states frequently appear as the variance of the Gaussian noise \( \beta_s^{-1} \) increases is the same reason as we explained for the \( \beta_s \)-dependence of the locally minimum
solutions. For large value of $\beta_s^{-1}$, some of the bits $b_i$ take their direction so as to take the same sign as that of the local field $h_i$ whose strength is estimated as $\alpha(1 + \beta_s^{-1})$. As a result, such a bit becomes free from frustration effects and hence, the deep energy state appears more frequently. However, if $\beta_s^{-1}$ decreases, the effect of the random field term $\sum_i h_i s_i$ is weakened and the frustration has more effect on some of the bits. Therefore, the $\beta_s^{-1}$-dependence of the distribution shown in the right panel of Fig. 2 is clearly understood, although we should note that of course this does not necessarily mean that a better decoding result is possible at higher noise levels, just that lower energy solutions are more accessible.

5. Beyond the annealed approach

In section 3, we considered the quantity $g_0 \sim e^{K\Phi(\{J,f\})}$ for a given realization of disorder $\{J,f\}(= \{b_i^\mu, y_i^\mu\})$ and assumed that the value $g_0$ is identical to the average $\langle g_0 \rangle$ in thermodynamic limit. However, for a specific choice of the disorder $\{J,f\}$, the $g_0$ might take an extremely large value of exponential order of $K$ and it is difficult for us to confirm that such an extreme value coincides with the average $\langle g_0 \rangle$. For this reason, we should take the quantity $\log g_0$ instead of $g_0$ and assume that $\log g_0 = K\Phi(\{J,f\})$ for a given $\{J,f\}$ should be equal to the average $\langle \log g_0 \rangle$ in thermodynamic limit. Then, the $\log g_0$ does not take extremely large values in comparison with the typical value of $\log g_0$ even if we choose a specific choice of the disorder $\{J,f\}$.

However, this will, unfortunately, require more technology and a more involved calculation as we will have to introduce replica theory. In some ways the previous sections can be viewed as an introductory calculation. Now,

$$g_0(\{b_i^\mu, y_i^\mu\}) = \sum_s \prod_i \left[ \int_0^\infty d\lambda_i \delta \left( f_i - \sum_{j \neq i} J_{ij} s_j - \lambda_i \sigma_i \right) \right]$$

and we wish to calculate

$$\langle \log g_0(\{b_i^\mu, y_i^\mu\}) \rangle_{\{\psi_i^\mu, y_i^\mu\}} = \lim_{n \to 0} \frac{1}{n} \langle \log g_0(\{b_i^\mu, y_i^\mu\}) \rangle_{\{\psi_i^\mu, y_i^\mu\}} - 1.$$  (27)

using the powerful replica approach. Then, following the usual algebra (similar in many respects to the earlier sections), we have

$$\langle g_0^n \rangle = \int \mathcal{D}(m, w, r, q, u) e^{K\Phi(\{m, w, q, u, \tilde{m}, \tilde{w}, \tilde{q}, \tilde{u}\})}$$

(28)

with

$$\Phi = i \sum_{\alpha > 0} (\tilde{m}_\alpha m^\alpha + \tilde{w}_\alpha w^\alpha) + i \sum_{\alpha < \beta} \tilde{q}^{\alpha \beta} q^{\alpha \beta} + i \sum_{\alpha \leq \beta} \tilde{u}^{\alpha \beta} u^{\alpha \beta} + i \sum_{\alpha \beta} \tilde{r}^{\alpha \beta} r^{\alpha \beta} + \frac{\alpha}{2} \log \frac{\beta_s}{2\pi}$$

$$+ \log \frac{1}{2} \sum_{s^0, s^1, \ldots, s^n} \prod_{\alpha} \left[ \int \frac{d\lambda^\alpha d\tilde{\lambda}^\alpha}{2\pi} \right] \exp \left[ i \sum_{\alpha} \tilde{\lambda}^\alpha (\alpha - \lambda^\alpha) - is_0 \sum_{\alpha} (\tilde{m}_\alpha s^\alpha + \tilde{w}_\alpha s^\alpha \tilde{\lambda}^\alpha) \right]$$

$$- i \sum_{\alpha < \beta} s^{\alpha \beta} \tilde{q}^{\alpha \beta} - i \sum_{\alpha \leq \beta} \tilde{u}^{\alpha \beta} s^\alpha s^\beta \tilde{\lambda} - i \sum_{\alpha \beta} \tilde{r}^{\alpha \beta} s^\alpha s^\beta \tilde{\lambda}^\beta$$
\[ + \alpha \log \int \left[ \frac{dy}{2\pi} \prod_{\alpha=0}^{n} \frac{dv_{\alpha}d\hat{v}_{\alpha}}{2\pi} \right] \exp \left[ \frac{-\beta}{2} (y - v^0)^2 + i \sum_{\alpha=0}^{n} \hat{v}_{\alpha}v_{\alpha} \right] \]
\[ \times \exp \left[ - \frac{1}{2} (\hat{v}^0)^2 + \sum_{\alpha, \beta > 0} [d_{\alpha \beta} \hat{v}_{\alpha} \hat{v}_{\beta} + 2r_{\alpha \beta} \hat{v}_{\alpha} (v_{\beta} - y) + u_{\alpha \beta} (v_{\alpha} - y) (v_{\beta} - y)] \right] \]
\[ + 2v^0 \sum_{\alpha > 0} [m_{\alpha} \hat{v}_{\alpha} + u_{\alpha} (v_{\alpha} - y)] \]

(29)

where we defined \( m_{\alpha}, w_{\alpha}, q_{\alpha \beta}, u_{\alpha \beta}, r_{\alpha \beta} \) as
\[ m_{\alpha} = \frac{1}{K} \sum_{k} s_{k}^0 s_{k}^0 \]
\[ w_{\alpha} = \frac{1}{K} \sum_{k} s_{k}^0 \hat{\lambda}_{k}^0 \]
\[ q_{\alpha \beta} = \frac{1}{K} \sum_{k} s_{k}^0 s_{k}^0 \hat{\lambda}_{k}^0 \hat{\lambda}_{k}^0 \]
\[ u_{\alpha \beta} = \frac{1}{K} \sum_{k} s_{k}^0 s_{k}^0 \hat{\lambda}_{k}^0 \hat{\lambda}_{k}^0 \]
\[ r_{\alpha \beta} = \frac{1}{K} \sum_{k} s_{k}^0 s_{k}^0 \hat{\lambda}_{k}^0 \hat{\lambda}_{k}^0 \]

and their conjugates : \( \hat{m}_{\alpha}, \hat{w}_{\alpha}, \hat{q}_{\alpha \beta}, \hat{u}_{\alpha \beta}, \hat{r}_{\alpha \beta} \) by introducing the definitions (30a)-(30e) via integral representation of delta function as e.g.
\[ \int \prod_{1 \leq \alpha \leq n} \left[ \frac{dm_{\alpha}d\hat{m}_{\alpha}}{2\pi/K} \right] \exp \left[ iK \sum_{\alpha} \left[ \hat{m}_{\alpha}^{\alpha} \left( m_{\alpha} - \frac{1}{K} \sum_{k} s_{k}^0 s_{k}^0 \right) \right] \right] = 1. \] (31)

We also introduced the shorthand \( D(m, w, r, q, u) \) to indicate integral over these saddle point variables.

Then, the replica symmetric ansatz simplifies the saddle point defining \( \langle g_{0}^{n} \rangle \). After a relatively involved calculation, we obtain

\[ \frac{1}{n} \Phi_{RS} = \hat{m}m + \hat{w}w + \frac{1}{2} \hat{q}q + \frac{1}{2} \hat{u}u_d - \frac{1}{2} \hat{u}u - \hat{r}r_d + \hat{r}r \]
\[ - \frac{\hat{q}}{2} + \frac{1}{2} \sum_{s^0} \int Dz_1 Dz_2 Dz_3 \log \left\{ \frac{1}{2} \sum_{s} e^{-\beta s_{s^0} + z_1 \sqrt{q - r_d + r_d} + z_3 \sqrt{r_d}} \left[ 1 + \text{Erf}\left( \frac{\alpha + \hat{r}d - \hat{r} - s_{s^0} \hat{w} + z_2 \sqrt{u - r_d + z_3 \sqrt{r_d}}}{\sqrt{2(u_d - \hat{u})}} \right) \right] \right\} \]
\[ + \alpha \left\{ - \frac{1}{2} \log [(1 + r_d - r)^2 + (u_d - u)(1 - q)] \right\} \]
\[ + \frac{2w - r)(1 + r_d - r) - u(1 - q) - (1 + \beta^{-1} + q - 2m)(u_d - u)}{2((1 + r_d - r)^2 + (u_d - u)(1 - q))} \]

(32)

The details of the derivation is explained in Appendix B.
By varying the order parameters themselves, we have

\[ \langle \ldots \rangle_1 \equiv \frac{1}{2} \sum_{s^0} \int Dz_1 Dz_2 Dz_3 \frac{\sum_s \cdots e^{-s^0 s \hat{m} + z_1 \sqrt{q - \hat{r}} + z_3 \sqrt{\hat{r}}} \left[ 1 + \text{Erf} \left( \frac{\alpha + \hat{r} - \hat{r} - s^0 \hat{w} + z_2 \sqrt{u - \hat{r}} + z_3 \sqrt{\hat{r}}}{\sqrt{2(u_d - \hat{u})}} \right) \right]}{\sum_s e^{-s^0 s \hat{m} + z_1 \sqrt{q - \hat{r}} + z_3 \sqrt{\hat{r}}} \left[ 1 + \text{Erf} \left( \frac{\alpha + \hat{r} - \hat{r} - s^0 \hat{w} + z_2 \sqrt{u - \hat{r}} + z_3 \sqrt{\hat{r}}}{\sqrt{2(u_d - \hat{u})}} \right) \right]} \]

\[ \langle \ldots \rangle_2 \equiv \frac{1}{\sqrt{\pi}} \sum_{s^0} \int Dz_1 Dz_2 Dz_3 \frac{\sum_s \cdots e^{-s^0 s \hat{m} + z_1 \sqrt{q - \hat{r}} + z_3 \sqrt{\hat{r}}} \left[ 1 + \text{Erf} \left( \frac{\alpha + \hat{r} - \hat{r} - s^0 \hat{w} + z_2 \sqrt{u - \hat{r}} + z_3 \sqrt{\hat{r}}}{\sqrt{2(u_d - \hat{u})}} \right) \right]}{\sum_s e^{-s^0 s \hat{m} + z_1 \sqrt{q - \hat{r}} + z_3 \sqrt{\hat{r}}} \left[ 1 + \text{Erf} \left( \frac{\alpha + \hat{r} - \hat{r} - s^0 \hat{w} + z_2 \sqrt{u - \hat{r}} + z_3 \sqrt{\hat{r}}}{\sqrt{2(u_d - \hat{u})}} \right) \right]} \]

and varying our conjugate order parameters, we obtain

\[ q = 1 - \left\langle \frac{z_1 s}{\sqrt{q - r}} \right\rangle_1 \]

\[ m = \langle ss_0 \rangle_1 \]

\[ w = \left\langle \frac{ss_0}{\sqrt{2(u_d - \hat{u})}} \right\rangle_2 \]

\[ u = \left\langle \frac{z_2 s}{\sqrt{2(u_d - \hat{u})(u - \hat{r})}} \right\rangle_2 + 2 \left\langle \frac{\alpha + \hat{r} - \hat{r} - s^0 \hat{w} + z_2 \sqrt{u - \hat{r}} + z_3 \sqrt{\hat{r}}}{[2(u_d - \hat{u})]^{3/2}} \right\rangle_2 \]

\[ u_d = \left\langle \frac{\alpha + \hat{r} - \hat{r} - s^0 \hat{w} + z_2 \sqrt{u - \hat{r}} + z_3 \sqrt{\hat{r}}}{[2(u_d - \hat{u})]^{3/2}} \right\rangle_2 \]

\[ r = \left\langle \frac{z_1 s}{2\sqrt{q - r}} - \frac{z_3 s}{2\sqrt{\hat{r}}} \right\rangle_1 + \left\langle \frac{z_2 s}{2\sqrt{2(u_d - \hat{u})(u - \hat{r})}} - \frac{z_3 s}{2\sqrt{2(u_d - \hat{u})^3}} \right\rangle_2 + \frac{1}{\sqrt{2(u_d - \hat{u})}} \]

\[ r_d = \left\langle \frac{1}{\sqrt{2(u_d - \hat{u})}} \right\rangle_2 \]

We introduce a further minor shorthand:

\[ A \equiv (1 + r_d - r)^2 + (u_d - u)(1 - q) \]

\[ B \equiv 2(w - r)(1 + r_d - r) - u(1 - q) - (1 + \beta_s^{-1} + q - 2m)(u_d - u) \]

By varying the order parameters themselves, we have

\[ \hat{m} = -\frac{\alpha(u_d - u)}{A} \]

\[ \hat{w} = -\frac{\alpha(1 + r_d - r)}{A} \]

\[ \hat{q} = -\frac{\alpha u}{A} - \frac{\alpha B(u_d - u)}{A^2} \]

\[ \hat{u}_d = \frac{\alpha(2 + \beta_s^{-1} - 2m)}{A} + \frac{\alpha B(1 - q)}{A^2} \]
\begin{align}
\hat{u} &= \frac{\alpha (1 + \beta_s^{-1} + q - 2m)}{A} + \frac{\alpha B (1 - q)}{A^2} \\
\hat{r}_d &= \frac{\alpha (w - 1 - r_d)}{A} - \frac{\alpha B (1 + r_d - r)}{A^2} \\
\hat{r} &= \frac{\alpha (w - r)}{A} - \frac{\alpha B (1 + r_d - r)}{A^2}
\end{align}

We solve these fourteen equations (33a)-(33g) and (35b)-(35g) numerically to obtain the average number of locally minimum states of the CDMA multiuser MAP demodulator treated as a quenched system. In Fig.3, we plot $\Phi$, that is, the logarithm of the average number of locally minimum solutions evaluated by the quenched calculation as a function of $\alpha$ for $\beta_s = 0.1$.

![Fig. 3](image)

From this figure, we find that the number of solutions decreases in comparison with the results found in the annealed calculation. This result can be confirmed by the following argument. Convexity of the logarithm gives

$$\langle \log g_0 \rangle \leq \log \langle g_0 \rangle$$

and the logarithm of the average number of the MAP solutions evaluated by the annealed approximation should be larger than the result of the quenched calculation. Taking into account this fact, the result shown in Fig. 3 is quite natural. It should be noted that the discontinuity observed in the annealed calculation disappears for the quenched evaluation. From these results we conclude that within a much more precise treatments than the annealed calculation in the sense that self-averaging quantity here is not $g_0$ but $\log g_0$, the average number of locally minimum solutions of the CDMA multiuser MAP demodulator continuously decreases as $\alpha$.
increases, however, it is still of exponential order. The result we obtained here might provide useful information about the computational complexity which could help in constructing sophisticated algorithms to obtain the solution for the CDMA multiuser MAP demodulator.

6. Summary

In this paper, we investigated the ground state properties of the CDMA multiuser demodulator, in particular, the number of locally minimum solutions of the zero-temperature dynamics of the MAP demodulator by both annealed and replica symmetric calculations. Moreover, we evaluated the distribution of the local minimum normalized-energy. We found that the number of locally stable solutions of the MAP demodulator is larger than that of the SK model for a large number of users and small values of the deviation of the Gaussian noise. We also found that when the number of the users $K$ decreases, the saddle point surface also decreases. However, it never reaches zero for finite $K$ and as a result, the number of solutions turns out to be exponential order. From these results, we might have useful information when we attempt to construct an algorithm to search for the ground state of the CDMA Hamiltonian (4). From the evaluation of the distribution of the local minimum normalized-energies, we found that the probability for deep normalized-energy state $\epsilon_0 \gg 1$ is almost zero for all values of the number of the users $\alpha$, however, relatively higher energy states appear much more frequently as the number of the users increases ($\alpha$ decreases). The analysis also told us that the deep energy states frequently appears for the case of large variance $\beta^{-1}$ of the Gaussian.

We hope that our analysis here provides a useful guide for the engineers to construct the MAP demodulator for CDMA systems.

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Appendix A: Derivation of the average number of solutions by annealed calculation

In this appendix, we explain the details of the derivation of the average number of locally minimum solutions from the definition (14). First of all, we introduce new variables $v^0_\mu$ and $v_\mu$ by inserting the equations:

$$1 = \prod_\mu \int dv^0_\mu \delta \left[ v^0_\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b^\mu_k s_k \right] = \int \prod_\mu \left[ \frac{dv^0_\mu dv^0_\mu}{2\pi} \right] \exp \left[ i \sum_\mu \hat{v}^0_\mu \left( v^0_\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b^\mu_k s_k \right) \right] (A-1)$$

and

$$1 = \int \prod_\mu \left[ \frac{dv_\mu dv_\mu}{2\pi} \right] \exp \left[ i \sum_\mu \hat{v}_\mu \left( v_\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b^\mu_k s_k \right) \right]. \quad (A-2)$$

Then, equation (14) is rewritten as

$$\langle g_0 \rangle = 2^{-K} \left( \frac{\beta_s}{2\pi} \right)^N \sum_{s^0} \int \prod_{\mu=1}^{N} \left[ \frac{dy^\mu dy^\mu}{2\pi} \right] \exp \left[ -\frac{\beta_s}{2} \sum_{\mu=1}^{N} \left( y^\mu - v^0_\mu \right)^2 \right] 2^{-NK}$$

$$\times \sum_{s,b^1 \ldots b^N} \prod_i \int_{-\infty}^{\infty} d\lambda_i \int_{0}^{\infty} d\hat{\lambda}_i \exp \left[ \lambda_i \left\{ \frac{1}{\sqrt{K}} \sum_{\mu} y^\mu b^\mu_k - \frac{1}{\sqrt{K}} \sum_{\mu} b^\mu_k \left( v_\mu - \frac{1}{\sqrt{K}} b^\mu_k s_k \right) - \lambda s_i \right\} \right]$$

$$\times \exp \left[ i \sum_\mu \hat{v}_\mu \left( v_\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b^\mu_k s_k \right) + i \sum_\mu \hat{v}_\mu \left( v^0_\mu - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} b^0_k s_k \right) \right]$$

where we used $(b^\mu_k)^2 = 1$. We next focus on the average in the last line in the limit of $K \to \infty$:

$$2^{-NK} \sum_{b^1 \ldots b^N} \exp \left[ -\frac{i}{\sqrt{K}} \sum_{\mu} \sum_{k=1}^{K} b^\mu_k \left( \hat{v}_\mu s_k + \hat{v}^0_\mu s_k + \hat{\lambda}_k v_\mu - \hat{\lambda}_k y^\mu \right) \right]$$

$$= \prod_{\mu k} \exp \left[ \log \cosh \left[ -\frac{i}{\sqrt{K}} \left( \hat{v}_\mu s_k + \hat{v}^0_\mu s_k + \hat{\lambda}_k v_\mu - \hat{\lambda}_k y^\mu \right) \right] \right]$$

$$= \exp \left[ -\frac{1}{2K} \sum_{\mu k} \left( \hat{v}_\mu s_k + \hat{v}^0_\mu s_k + \hat{\lambda}_k v_\mu - \hat{\lambda}_k y^\mu \right)^2 \right] \quad (A-3)$$

and this reads

$$\langle g_0 \rangle = 2^{-K} \left( \frac{\beta_s}{2\pi} \right)^N \int \prod_{\mu=1}^{N} \left[ \frac{dy^\mu dy^\mu}{2\pi} \right] \exp \left[ -\frac{\beta_s}{2} \sum_{\mu=1}^{N} \left( y^\mu - v^0_\mu \right)^2 \right]$$

$$\times \exp \left[ -\frac{\beta_s}{2} \sum_{\mu=1}^{N} \left( y^\mu - v^0_\mu \right)^2 + i \sum_{\mu} \left( \hat{v}^0_\mu \hat{v}_\mu + \hat{v}_\mu v_\mu \right) \right]$$

$$\times \sum_{s,s^0} \prod_i \int_{0}^{\infty} \frac{d\lambda_i}{2\pi} \int_{-\infty}^{\infty} \frac{d\hat{\lambda}_i}{2\pi} \exp \left[ i \sum_i \hat{\lambda}_i (\alpha - \lambda_i) s_i \right]$$
\[
\times \exp\left[ -\frac{1}{2K} \sum_{\mu k} \lambda_k^2 (\nu^\mu - y^\mu)^2 + \hat{v}^2 + (\hat{\nu}^0)^2 + 2\hat{v}_\mu \hat{v}_\mu^0 s_k s_k^0 \right.
\]
\[
+\left. 2\hat{v}_\mu \hat{\lambda}_k (\nu^\mu - y^\mu) s_k + 2\hat{\nu}_\mu^0 \hat{\lambda}_k (\nu^\mu - y^\mu) s_k^0 \right] .
\] (A-4)

By introducing the following order parameters:
\[
q = \frac{1}{K} \sum_k s_k s_k^0 \] (A-5a)
\[
t = \frac{1}{K} \sum_k \hat{\lambda}_k s_k \] (A-5b)
\[
u = \frac{1}{K} \sum_k \hat{\lambda}_k^2 \] (A-5c)
\[
w = \frac{1}{K} \sum_k \hat{\lambda}_k s_k^0 \] (A-5d)

via
\[
1 = \int \frac{dq dq}{2\pi/K} \exp\left[ iK\dot{q} \left( q - \frac{1}{K} \sum_k s_k s_k^0 \right) \right] \] (A-6a)
\[
1 = \int \frac{dt dt}{2\pi/K} \exp\left[ iK\dot{t} \left( t - \frac{1}{K} \sum_k \hat{\lambda}_k s_k \right) \right] \] (A-6b)
\[
1 = \int \frac{du du}{2\pi/K} \exp\left[ iK\dot{u} \left( u - \frac{1}{K} \sum_k \hat{\lambda}_k^2 \right) \right] \] (A-6c)
\[
1 = \int \frac{dw dw}{2\pi/K} \exp\left[ iK\dot{w} \left( w - \frac{1}{K} \sum_k \hat{\lambda}_k s_k^0 \right) \right] \] (A-6d)

we obtain
\[
\langle g_0 \rangle = \frac{\beta_s}{(2\pi)^{N/2}} \int \frac{dt dt}{2\pi/K} \frac{du du}{2\pi/K} \frac{dw dw}{2\pi/K} \frac{dq dq}{2\pi/K} \exp\left[ iK(\dot{t} + \dot{u} + \dot{w} + \dot{q}) \right]
\]
\[
\times \exp\left[ \int dy \frac{dy}{2\pi} \frac{dy}{2\pi} \frac{dy}{2\pi} \frac{dy}{2\pi} \exp\left[ \frac{\beta_s}{4}(y-y^0)^2 + i(\dot{\nu} + \dot{\nu}) - \frac{1}{2} \dot{\nu}^2 - \frac{1}{2} (\dot{\nu})^2 - \frac{1}{2} (\dot{\nu})^2 - \frac{1}{2} (\dot{\nu})^2 \right] \right]
\]
\[
\times \exp\left[ K \log \frac{1}{2} \sum_{s, s^0} \int_0^\infty d\lambda \int_{-\infty}^\infty d\lambda \exp\left[ i\lambda (\alpha - \lambda) s - i(\dot{q} s + \dot{u} \dot{\lambda} + i\dot{w} \dot{\lambda}) \right] \right] .
\] (A-7)

Now, we rotate the variables \( \dot{q}, \dot{u} \) to \(-i\dot{q}, -i\dot{u} \), and focus on the part:
\[
I = \frac{1}{2} \sum_{s, s^0} \int_0^\infty d\lambda \int_{-\infty}^\infty d\lambda \exp\left[ i\lambda (\alpha - \lambda) s - i(\dot{q} s + \dot{u} \dot{\lambda} + i\dot{w} \dot{\lambda}) \right]
\]
\[ \frac{1}{2} \sum_{s,s'} e^{-\tilde{q} s \tilde{q}} \int_0^\infty d\lambda \int_{-\infty}^\infty \frac{d\tilde{\lambda}}{2\pi} e^{-\tilde{\lambda}^2 + i\lambda(\alpha - \tilde{t} - \tilde{w}s - \tilde{w}s')} \]

By changing the variable \( \hat{\lambda} \to \hat{\lambda}/\sqrt{2u} \), we obtain

\[ I = \cosh(\tilde{q}) + \frac{1}{2} \left[ e^{-\tilde{q}Erf} \left( \frac{\alpha - \tilde{t} - \tilde{w}}{2\sqrt{u}} \right) + e^{\tilde{q}Erf} \left( \frac{\alpha - \tilde{t} + \tilde{w}}{2\sqrt{u}} \right) \right] . \]  

(A-8)

where we used \( \int D\tilde{z} e^{\phi} = e^{\varphi/2} \) and Erf(\( z \)) = \( (2/\sqrt{\pi}) \int_0^z dt e^{-t^2} \). We also calculate the other integral in equation (A-7), that is,

\[ \int dy \frac{dy}{{\sqrt{2\pi}}(1 + \beta_s(1 - q^2))} e^{-\frac{\beta_s}{2}(y-v^0)^2 + i(\tilde{t}v^0 + \tilde{w}v) - \frac{\beta_s}{2}v^2 - \frac{\beta_s}{2}q^2 - \frac{\beta_s}{2}(v-y)^2 - tv(v-y) - wv^0(v-y)} \]

\[ = \int dy v \left[ \frac{\beta_s}{2}[1 + \beta_s(1 - q^2)] - \frac{(q-t)^2}{2[1 + \beta_s(1 - q^2)]} \right] \]

\[ \times e^{-\frac{\beta_s}{2}(y-v^0)^2 + i(\tilde{t}v^0 + \tilde{w}v) - \frac{\beta_s}{2}v^2 - \frac{\beta_s}{2}q^2 - \frac{\beta_s}{2}(v-y)^2 - tv(v-y) - wv^0(v-y)} \]

where we rotated the variables \( t, w \) to \(-it, -iw\). In this expression, the coefficients of \( y^2, v^2 \) and \( yv \) are given by, respectively,

\[ A = \frac{\beta_s[1 + qt - w]^2}{[1 + \beta_s(1 - q^2)]} + u + t^2 \]  

(A-9a)

\[ B = \frac{\beta_s[q(1 + t) - w]^2}{[1 + \beta_s(1 - q^2)]} + u + (1 + t)^2 \]  

(A-9b)

\[ C = \frac{(1 + qt - w)[q + qt - w]\beta_s}{[1 + \beta_s(1 - q^2)]} + u + t(t + 1) \]  

(A-9c)

Thus, our integral to be calculated is now written as

\[ \sqrt{2\pi} \int \frac{dy \frac{dy}{2\pi\sqrt{1 + \beta_s(1 - q^2)}} e^{-\frac{\beta_s}{2}y^2 - \frac{\beta_s}{2}v^2 + Cyv} = \frac{1}{\sqrt{(BA - C^2)[1 + \beta_s(1 - q^2)]}} = \sqrt{\frac{2\pi}{a[1 + 2\beta_s(1 - q)] + \beta_s(1 + t + w)^2}} . \]

So the average number of locally minimum solutions is given by

\[ \langle g_0 \rangle = \int \frac{dt \frac{dt}{2\pi/K}}{2\pi/K} \frac{du \frac{du}{2\pi/K}}{2\pi/K} \frac{dw \frac{dw}{2\pi/K}}{2\pi/K} \frac{dq \frac{dq}{2\pi/K}}{2\pi/K} e^{K\Phi(t,u,w,q,t,u,w,q)} \]

\[ \Phi = \hat{t}t + \hat{u}u + \hat{w}w + \hat{q}q - \frac{\alpha}{2} log \left\{ u[1 + 2\beta_s(1 - q)] + \beta_s(1 + t - w)^2 \right\} + \log \left\{ \cosh(\tilde{q}) + \frac{1}{2} \left[ e^{-\tilde{q}Erf} \left( \frac{\alpha - \tilde{t} - \tilde{w}}{2\sqrt{u}} \right) + e^{\tilde{q}Erf} \left( \frac{\alpha - \tilde{t} + \tilde{w}}{2\sqrt{u}} \right) \right] \right\} + \frac{\alpha}{2} log \beta_s \]

which is just the expression given in (15).
Appendix B: Evaluation of the replica symmetric saddle point surface

In this appendix, we explain the derivation of the replica symmetric saddle point, namely, (32) from (29) with (28). By using the replica symmetric ansatz:

\[ m_\alpha = m, \quad \dot{m}_\alpha = \dot{m} \]  
\[ w_\alpha = w, \quad \dot{w}_\alpha = \dot{w} \]  
\[ q_{\alpha\beta} = q, \quad \dot{q}_{\alpha\beta} = \dot{q} \]  
\[ u_\alpha = u, \quad \dot{u}_\alpha = \dot{u} \]

we can rewrite the saddle point surface \( \Phi \) as

\[
\Phi_{RS} = \ln(m\dot{m} + \dot{w}w + \dot{u}_d u_d + \dot{r}_d r_d) + i\frac{n(n-1)}{2}(\ddot{q}q + \ddot{u}u + 2\ddot{r}r) + \frac{\alpha}{2} \log \frac{\beta_s}{2\pi}
\]

\[ + \log \frac{1}{2} \sum_{s^0, s^1, \ldots, s^n} \prod_{\alpha} \left[ \int \frac{d\lambda^\alpha d\dot{\lambda}^\alpha}{2\pi} \right] \exp \left[ i \sum_{\alpha} \lambda^\alpha (\alpha - \lambda^\alpha) - i s^0 \sum_{\alpha} (\dot{m}s^\alpha + \dot{w}s^\alpha \dot{\lambda}^\alpha) \right] \exp \left[ -i\dot{q} \sum_{\alpha < \beta} s^\alpha s^\beta - i\dot{u}_d \sum_{\alpha} \dot{\lambda}^\alpha - i\ddot{u}_d \sum_{\alpha} \ddot{\lambda}^\alpha - i\ddot{r}_d \sum_{\alpha} \ddot{\lambda}^\alpha - i\dddot{r}_d \sum_{\alpha \neq \beta} \dddot{\lambda}_s \right] \exp \left[ -r_d \sum_{\alpha} \ddot{v}_\alpha (v_\alpha - y) - r \sum_{\alpha \neq \beta} \ddot{v}_\alpha (v_\beta - y) - \frac{1}{2} u_d \sum_{\alpha} (v_\alpha - y)^2 \right]
\]

\[ - \frac{1}{2} u \sum_{\alpha \neq \beta} (v_\alpha - y)(v_\beta - y) - \ddot{v}_0 m \sum_{\alpha} \ddot{v}_\alpha - \ddot{v}_0 w \sum_{\alpha} (v_\alpha - y) \]

\[ = n(m\dot{m} + i\dot{w}w + \dot{u}_d u_d - i\ddot{r}_d r_d) + \frac{n(n-1)}{2}(\dot{u}u - \dot{q}q - 2\ddot{r}r) + \frac{\alpha}{2} \log \frac{\beta_s}{2\pi}
\]

+ \log I_1 + \alpha \log I_2. \quad (B.2)

where we made the rotations: i\dot{m} \to \dot{m}, i\dot{w} \to \dot{w}, i\dot{q} \to -\dot{q} and i\ddot{r} \to \dddot{r}. We now focus on the first integral, \( I_1 \),

\[
I_1 = \frac{1}{2} \sum_{s^0, s^1, \ldots, s^n} \prod_{\alpha} \left[ \int \frac{d\lambda^\alpha d\dot{\lambda}^\alpha}{2\pi} \right] \exp \left[ i \sum_{\alpha} \lambda^\alpha (\alpha - \lambda^\alpha) - i s^0 \sum_{\alpha} (\dot{m}s^\alpha + \dot{w}s^\alpha \dot{\lambda}^\alpha) \right] \times \exp \left[ -i\dot{q} \left( \sum_{\alpha} s^\alpha \right)^2 + i\ddot{q}n \left( \sum_{\alpha} \dot{\lambda}^\alpha \right) \sum_{\alpha} \dot{\lambda}^\alpha - i\dddot{r}_d \sum_{\alpha \neq \beta} \dddot{\lambda}^\alpha \right]
\]

\[ \times \exp \left[ -i\dot{r}_d \sum_{\alpha} \dddot{\lambda}^\alpha - i\ddot{r}_d \sum_{\alpha \neq \beta} \dddot{\lambda}_s \right]. \quad (B.3) \]
We consider the identity:

\[
i \sum_{\alpha \beta} s^\alpha s^\beta \lambda^\beta = \frac{1}{2} \left\{ \sum_{\alpha} (s^\alpha + is^\beta \lambda^\beta) \right\} ^2 - \frac{1}{2} \left( \sum_{\alpha} s^\alpha \right) ^2 + \frac{1}{2} \left( \sum_{\alpha} s^\alpha \lambda^\alpha \right) ^2
\]

(B-4)

Then, we have

\[
I_1 = \frac{1}{2} \sum_{s^0, s^1, \ldots, s^n} \exp \left[ -s^0 \sum_{\alpha} s^\alpha m \right] \prod_{\alpha} \left[ \int \frac{d\lambda^\alpha d\lambda^\alpha}{2\pi} \right] \exp \left[ i \sum_{\alpha} \lambda^\alpha (\alpha - \hat{r}_d - \hat{r} - s^0 w - \lambda^\alpha) \right]
\]

\[
\times \exp \left[ \frac{\hat{q} - \hat{r}}{2} \left( \sum_{\alpha} s^\alpha \right)^2 - \frac{\hat{q} m}{2} - \left( \hat{u}_d - \hat{u} \right) \sum_{\alpha} \lambda^\alpha - \frac{\hat{u} - \hat{r}}{2} \left( \sum_{\alpha} s^\alpha \lambda^\alpha \right)^2 \right]
\]

\[
+ \frac{1}{2} \left\{ \sum_{\alpha} (s^\alpha + is^\alpha \lambda^\alpha) \right\} ^2
\]

\[
e^{-\frac{\hat{q} m}{2}} \frac{1}{2} \sum_{s^0} \int Dz_1 Dz_2 Dz_3 \left\{ \frac{1}{2} \sum_{s} e^{-s^0 \sum_{\alpha} \sqrt{q - r s + z_3 \sqrt{r s}}} \right\} ^n
\]

where we made the transformation: \( \hat{u}_d \to \hat{u}_d/2 \) and performed the Hubbard-Stratonovich transformations for \( \exp[(\hat{q} - \hat{r}) (\sum_{\alpha} s^\alpha)^2/2], \exp[-(\hat{u} - \hat{r})(\sum_{\alpha} s^\alpha \lambda^\alpha)^2/2] \) and \( \exp[\hat{r}(\sum_{\alpha} (s^\alpha + is^\alpha \lambda^\alpha))^2/2] \). We now turn to the second integral:

\[
I_2 = \int \left[ dy \prod_{\alpha=0}^{n} \frac{dv^\alpha d\hat{v}^\alpha}{2\pi} \right]
\]

\[
\times \exp \left[ -\frac{\beta_s}{2} (y - v^0)^2 + i \sum_{\alpha=0}^{n} \hat{v}^\alpha v^\alpha - \frac{1}{2} \hat{v}^2 - \frac{1}{2} \sum_{\alpha=1}^{n} \hat{v}^\alpha - \frac{1}{2} q \sum_{\alpha \neq \beta} \hat{v}^\alpha \hat{v}^\beta \right]
\]

\[
\times \exp \left[ -r_d \sum_{\alpha} \hat{v}_\alpha (v_\alpha - y) - r \sum_{\alpha \neq \beta} \hat{v}_\alpha (v_\beta - y) - \frac{1}{2} u_d \sum_{\alpha} (v_\alpha - y)^2 \right.
\]

\[
- \frac{u}{2} \sum_{\alpha \neq \beta} (v_\alpha - y)(v_\beta - y) - \hat{v}_0 m \sum_{\alpha} \hat{v}_\alpha - \hat{v}_0 w \sum_{\alpha} (v_\alpha - y)
\]

\[
= \int \frac{dy dv^0 d\hat{v}^0}{2\pi} \exp \left[ -\frac{\beta_s}{2} (y - v^0)^2 + iv^0 \hat{v}^0 - \frac{1}{2} \hat{v}^0 \right] \int \left[ \prod_{\alpha=1}^{n} \frac{dv^\alpha d\hat{v}^\alpha}{2\pi} \right]
\]

\[
\times \exp \left[ i \sum_{\alpha} v^\alpha \hat{v}^\alpha - v^0 m \sum_{\alpha} \hat{v}_\alpha - \hat{v}^0 \sum_{\alpha} w (v_\alpha - y) \right]
\]

\[
\times \exp \left[ -\frac{q - r}{2} \left( \sum_{\alpha} \hat{v}^\alpha \right)^2 + \frac{q - 1}{2} \sum_{\alpha} \hat{v}^2 - \frac{u - r}{2} \left\{ \sum_{\alpha} (v_\alpha - y) \right\} ^2 \right.
\]

\[
+ \frac{1}{2} \left( u - u_d \right) \sum_{\alpha} (v_\alpha - y)^2 + (r - r_d) \sum_{\alpha} \hat{v}_\alpha (v_\alpha - y) - \frac{r}{2} \left\{ \sum_{\alpha} \hat{v}^\alpha + v_\alpha - y \right\} ^2 \right].
\]
The Hubbard-Stratonovich transformations with respect to the factors: \(\exp[-(q-r)(\sum \alpha \tilde{v}_\alpha)/2], \exp[-(u-r)(\sum \alpha (v_\alpha - u))/2] \) and \(\exp[-r\{\sum \alpha (\tilde{v}_\alpha + v_\alpha - u)/2\}] \) give

\[
I_2 = \int Dz_1 Dz_2 Dz_3 \frac{dy dv_0 d\tilde{v}_0}{2\pi} \mu^{2\alpha(y-v_0)^2 + iv_0 \tilde{v}_0 - \frac{1}{2} \tilde{v}_0^2} \left\{ \int \frac{dvd\tilde{v}}{2\pi} e^{i\tilde{v}_0 \tilde{v} - \tilde{v}_0 w(v-y)} \right\} e^{z_1 \sqrt{-(q-r)\tilde{v} + \frac{1}{2} \tilde{v}^2} + 2z_2 \sqrt{-(u-r)(v-y)} + \frac{1}{2} (u-d)(v-y)^2 + (r-r_d)\tilde{v}(v-y) + z_3 \sqrt{-(\tilde{v} + v-y)}} \right\}^n.
\]

We should notice that \(I_2\) now has the form:

\[
I_2 = \langle \langle 1 + n \log X \rangle_I \rangle_z \tag{B-5}
\]

with the averages over the disorder variables \((z_1, z_2, z_3)\) given by \(\langle \cdots \rangle_z\) and over the measure on \(y, v_0, \tilde{v}_0\) given by \(\langle \cdots \rangle_I\). We focus on the inner integral \(X\). By making the change of variable \(v \to v + y\), we have

\[
X = \frac{e^{-(y+iv_0 m + z_1 \sqrt{q-r} + z_3 \sqrt{7})^2}}{\sqrt{(1 + ir_d - ir)^2 + (u_d - u)(1 - q)}} \times e^{2(1 + ir_d - ir)^2 + (u_d - u)(1 - q)} \left\{ \frac{1 - q}{2(1 + ir_d - ir)^2 + (u_d - u)(1 - q)} \right\} \left\{ -u + 2 \left( \frac{1 + ir_d - ir}{1 - q} \right) \right\} \left( 1 + \beta_s^{-1} + q - 2m \right) \right\} \right\}
\]

where we used the fact:

\[
\langle 1 \rangle_I = \langle v_0^2 \rangle_I = \langle yv_0 \rangle_I = \sqrt{\frac{2\pi}{\beta_s}} \quad \langle y^2 \rangle_I = \sqrt{\frac{2\pi}{\beta_s}} (1 + \beta_s^{-1}) \quad \langle y \rangle_I = \langle \tilde{v}_0 \rangle_I = \langle y_0 \rangle_I = \langle v_0 \rangle_I = \langle v_0^2 \rangle_I = 0 \tag{B-8}
\]

with the definition:

\[
\langle \cdots \rangle_I = \int \frac{dy dv_0 d\tilde{v}_0}{2\pi} e^{-\frac{\beta_s}{2}(y-v_0)^2 - \frac{1}{2} \tilde{v}_0^2 + iv_0 \tilde{v}_0} (\cdots).
\]

By rotating the variables \(iw \to w, ir \to r\), we obtain

\[
\langle \langle \log X \rangle_I \rangle_z = \sqrt{\frac{2\pi}{\beta_s}} \left\{ -\frac{1}{2} \log[(1 + r_d - r)^2 + (u_d - u)(1 - q)] \right\}
\]

\[
+ \frac{2(w - r)(1 + r_d - r) - u(1 - q) - (1 + \beta_s^{-1} + q - 2m)(u_d - u)}{2(1 + r_d - r)^2 + (u_d - u)(1 - q)} \right\}.
\]

\]

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Putting $I_1$ and $I_2$ into (B·2), we obtain the replica symmetric saddle point surface:

$$\frac{1}{n} \Phi_{RS} = \hat{m}m + \hat{w}w + \frac{1}{2} \hat{q}q + \frac{1}{2} \hat{u}u + \frac{1}{2} \hat{d}d - \frac{1}{2} \hat{u}u - \hat{r}r + \hat{r}r$$

$$- \frac{\hat{q}}{2} + \frac{1}{2} \sum_{s=0} Dz_1 Dz_2 Dz_3 \log \left\{ \frac{1}{2} \sum_s e^{-s^0 s \hat{m} + z_1 \sqrt{q - r} + z_3 \sqrt{r}} \right\}$$

$$+ \text{Erf} \left( \frac{\alpha + \hat{r} - ss^0 \hat{w} + z_2 \sqrt{u - r} + z_3 \sqrt{r}}{\sqrt{2(u - \hat{u})}} \right)$$

$$+ \alpha \left\{ - \frac{1}{2} \log \left[ (1 + r_d - r)^2 + (u_d - u)(1 - q) \right] \right\}$$

$$+ \frac{2(w - r)(1 + r_d - r) - u(1 - q) - (1 + \beta^{-1}_s + q - 2m)(u_d - u)}{2\{1 + r_d - r)^2 + (u_d - u)(1 - q)\}} \right\}.$$
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