QUASI-HOPF ALGEBRAS ASSOCIATED WITH SEMISIMPLE LIE ALGEBRAS AND COMPLEX CURVES

B. ENRIQUEZ

Abstract. We construct quasi-Hopf algebras associated with a semisimple Lie algebra, a complex curve and a rational differential. This generalizes our previous joint work with V. Rubtsov (Israel J. Math. (1999) and q-alg/9608005).

1. Introduction

1.1. In this paper, we construct quasi-Hopf algebras associated with the data of a semisimple Lie algebra \( a \) and the triple \((C, \omega, S)\) of a complex curve \( C \), a rational differential \( \omega \) and a finite set \( S \) of points of \( C \), containing all zeroes and poles of \( \omega \).

These quasi-Hopf algebras are quantizations of a Manin pair associated to \((a, C, S, \omega)\). Recall that a Manin pair is a triple \((u, \langle \cdot, \cdot \rangle_u, v)\) formed by a Lie algebra \( u \), a nondegenerate invariant pairing \( \langle \cdot, \cdot \rangle_u \) on \( u \), and a Lagrangian (i.e., maximal isotropic) subalgebra \( v \) of \( u \). The Manin pair associated with \((a, C, S, \omega)\) is \((g, \langle \cdot, \cdot \rangle_g, g^{out})\), where \( g \) is a double extension of \( a \otimes K \), \( K = \bigoplus_{s \in S} K_s \) is the direct sum of the local fields of \( C \) at the points of \( S \), \( \langle \cdot, \cdot \rangle_g \) is constructed using \( \omega \), and \( g^{out} \) is an extension of \( a \otimes R \), where \( R \) is the ring of regular functions on \( C - S \).

The nonextended versions of these Manin pairs are all the untwisted Manin pairs introduced by Drinfeld in [3].

To construct these quasi-Hopf algebras, we follow the strategy we used in [9] in the case \( a = \mathfrak{sl}_2 \). We first construct Manin triples \((g, g_+, g_-)\) and \((\tilde{g}, \tilde{g}_+, \tilde{g}_-)\). After we construct the suitable Serre relations, we define Hopf algebras \((U_h g, \Delta)\) and \((U_h \tilde{g}, \tilde{\Delta})\) associated to these triples. When \( C = \mathbb{C}P^1 \) and \( \omega = dz \) or \( \frac{dz}{z} \), the defining relations of \( U_h g \) coincide with the “new realizations” presentation of the double Yangians and of the quantum affine algebras. We then show that \((U_h g, \Delta)\) and \((U_h \tilde{g}, \tilde{\Delta})\) are quantizations of \((g, g_+, g_-)\) and \((\tilde{g}, \tilde{g}_+, \tilde{g}_-)\). For this, we show the Poincaré-Birkhoff-Witt (PBW) result for \( U_h g \) by comparing its positive part \( U_h L_{n_+} \) with an analogue of the Feigin-Odesskii algebras and by using Lie bialgebra duality argument (see [4]). We then construct an element \( F \) in a completion of \( U_h g^{\otimes 2} \), conjugating the coproducts \( \Delta \) and \( \tilde{\Delta} \). We also construct a subalgebra \( U_h g^{out} \) of \( U_h g \), which is a flat deformation the enveloping algebra \( U g^{out} \). We show that \( U_h g^{out} \) is a left coideal of \( U_h g \) for \( \Delta \), and a right coideal for \( \tilde{\Delta} \). We express \( F \) as a product \( \tilde{F}_2 F_1 \), with \( F_1 \) (resp., \( \tilde{F}_2 \)) in completions of \( U_h g \otimes U_h g^{out} \).
B. ENRIQUEZ

We show that \( F_1 \Delta F_1^{-1} \) defines a quasi-triangular quasi-Hopf algebra structure on \( U_\hbar g \), for which \( U_\hbar g^{\text{out}} \) is a sub-quasi-Hopf algebra. This pair \((U_\hbar g, U_\hbar g^{\text{out}})\) of quasi-Hopf algebras is the solution to our quantization problem.

Here are the new ingredients of this paper with respect to \([9]\). The construction of the Serre relations for \( U_\hbar g \) is new, as are the PBW results for \( U_\hbar g \) and \( U_\hbar g^{\text{out}} \).

Our approach to constructing \( F \) is also new. It relies on duality results for a Hopf pairing in \( U_\hbar g \) (Section 7), which are based on the construction of quantized formal series Hopf algebras inside quantized enveloping algebras (\([1]\)).

This paper is organized as follows. In Section 2, we introduce the Manin pairs and triples \((g, g^{\text{out}})\) and \((g, g^+, g^-), (g, \bar{g}^+, \bar{g}^-)\). In Section 3, we construct the Serre relations for \( U_\hbar g \). In Section 4, we construct the Hopf algebras \((U_\hbar g, \Delta)\) and \((U_\hbar g, \bar{\Delta})\). In Section 5, we define the subalgebra \( U_\hbar g^{\text{out}} \) of \( U_\hbar g \) and show PBW results for these algebras. In Section 6, we construct a Hopf pairing inside \( U_\hbar g \), and we prove a duality result about this pairing in Section 7. We are then ready to construct \( F \) in Section 8 and the quasi-Hopf structures on \( U_\hbar g \) and \( U_\hbar g^{\text{out}} \) in Section 9.

Let us now say some words about the motivation of this work. We applied the construction of our paper \([4]\) to a) the construction of realizations of the elliptic quantum groups in terms of quantum currents (\([5]\)), and b) the construction of a new family of commuting difference operators, associated with \((C, \omega, S)\) (see \([7]\)). The present work could lead to higher rank generalizations of these works. In our work \([10]\), we showed that quasi-Hopf algebras naturally lead to the construction of quantum homogeneous spaces. This is another possible application of the present paper.

1.2. Part of this work was done while I was visiting Kyoto university in May 1999, ESI (Vienna) in August 1999 and university of Roma I in December 1999; I would like to thank respectively M. Jimbo, A. Alekseev, P. Piazza and C. De Concini for their kind invitations.

2. Manin pairs and triples

2.1. Lie algebras and bilinear forms. Recall that \( K_s \) denotes the local field at a point \( s \) of \( S \), and \( K \) is the direct sum \( \oplus_{s \in S} K_s \). The ring \( R \) of regular functions on \( C - S \) is viewed as a subring of \( K \) by associating to a function of \( R \), the collection of its Laurent expansions at each point of \( S \).

Let us equip \( K \) with the bilinear form \( \langle f, g \rangle_K = \sum_{s \in S} \text{res}_s(f g \omega) \). Then \( \langle \ , \ \rangle_K \) and \( R \) is a Lagrangian subring of \( K \). Let us set, for \( f \in K \), \( \partial f = \frac{df}{\omega} \). Then \( \partial \) is a derivation of \( K \), which preserves \( R \). The bilinear form \( \langle \ , \ \rangle_K \) is \( \partial \)-invariant.

Let \( \langle \ , \ \rangle_a \) be a nondegenerate invariant bilinear form on \( a \). Let us equip

\[ g = (a \otimes K) \oplus \mathbb{C} D \oplus \mathbb{C} K \]
with the bracket \[ ([a \otimes f, \lambda D, \mu K], (a' \otimes f', \lambda' D, \mu' K)] = ([a, a'] \otimes f f' + \lambda a' \otimes \partial f' - \lambda' a \otimes \partial f, 0, \mu + \mu' + \langle a, a' \rangle_\alpha \langle \partial f, f' \rangle_K). \] Then \( g \) is a Lie algebra. Let us set \( g^{\text{out}} = (a \otimes R) \oplus CD. \) Then \( g^{\text{out}} \) is a Lie subalgebra of \( g. \)

Let us set \( \langle (a \otimes f, \lambda D, \mu K), (a' \otimes f', \lambda' D, \mu' K) \rangle_g = \langle a, a' \rangle_\alpha \langle f, f' \rangle_K + \lambda \mu' + \lambda' \mu. \) Then \( \langle , \rangle_g \) is a nondegenerate invariant bilinear form on \( g. \)

2.2. **Manin pairs.** \( g^{\text{out}} \) is a maximal isotropic subalgebra of \( g. \) Therefore, \( (g, g^{\text{out}}) \) is a Manin pair (see \( \mathcal{R} \)).

Let \( O_s \) be ring of integers of \( K_s \) and let us fix a Lagrangian complement \( \Lambda \) of \( R \) in \( K, \) commensurable with \( \bigoplus_{s \in S} O_s. \) Let us set \( g_\Lambda = (a \otimes \Lambda) \oplus CK. \) Then \( g_\Lambda \) is a Lagrangian complement of \( g^{\text{out}} \) in \( g. \) The triple \( (g, g^{\text{out}}, g_\Lambda) \) is sometimes called a pointed Manin pair. Let us describe the quasi-Lie bialgebra structures on \( g \) and \( g^{\text{out}} \) associated with \( (g, g^{\text{out}}, g_\Lambda). \)

Let \( m_r \) be the maximal ideal of \( O_s. \) For \( N \) integer, let us set \( i_N = a \otimes (\prod_{s \in S} m_r^N). \) Then the restriction of \( \langle , \rangle_g \) to \( g^{\text{out}} \times g^{\text{out}} \) defines a canonical element \( r_{\text{out}, \Lambda} \) in \( \lim_{\rightarrow -N} g^{\text{out}} \otimes (g_\Lambda / (g_\Lambda \cap i_N)). \) There is a unique map \( \delta^{\text{out}} : g \rightarrow \wedge^2 g, \) such that for any \( x \in g, \) \( \delta^{\text{out}}(x) = [r_{\text{out}, \Lambda}, x \otimes 1 + 1 \otimes x]. \) Let us set \( \varphi = [r_{\text{out}, \Lambda}^{(12)} + r_{\text{out}, \Lambda}^{(13)} + r_{\text{out}, \Lambda}^{(23)}]. \) Then \( \varphi \) belongs to \( \wedge^3 g^{\text{out}}, \) and \( (g, \delta^{\text{out}}, \varphi) \) is a quasi-Lie bialgebra.

Moreover, \( \varphi \) belongs to \( \wedge^3 g^{\text{out}}, \) and \( \delta^{\text{out}}(g^{\text{out}}) \) is contained in \( \wedge^2 g^{\text{out}}, \) therefore \( (g^{\text{out}}, \delta^{\text{out}}, \varphi) \) is also a quasi-Lie bialgebra.

2.3. **Manin triples.** Let us fix a Cartan decomposition \( a = n_+ \oplus h \oplus n_- \) of \( a. \) Let us set
\[
g_+ = (h \otimes R) \oplus (n_+ \otimes \mathcal{K}) \oplus CD, \quad g_- = (h \otimes \Lambda) \oplus (n_- \otimes \mathcal{K}) \oplus CK,
\]
and
\[
g_+ = (h \otimes R) \oplus (n_- \otimes \mathcal{K}) \oplus CD, \quad g_- = (h \otimes \Lambda) \oplus (n_+ \otimes \mathcal{K}) \oplus CK.
\]
\( g_+ \) and \( g_- \) supplementary Lagrangian subalgebras of \( g; \) the same is true for \( g_+ \) and \( g_-, \) therefore \( (g, g_+, g_-) \) and \( (g, g_-, g_-) \) are Manin triples. Let us describe the corresponding Lie bialgebra structures.

The restriction of \( \langle , \rangle_g \) to \( g_+ \times g_- \) (resp., to \( g_+ \times g_- \)) defines a canonical element \( r_{g_+, g_-} \) (resp., \( r_{g_+, g_-} \)) of \( \lim_{\rightarrow -N} (g / i_N) \otimes (g / i_N) \) (resp., of \( \lim_{\rightarrow -N} (g / i_N) \otimes g). \) There are unique maps \( \delta \) and \( \tilde{\delta} \) from \( g \) to \( \lim_{\rightarrow -N} \wedge^2 (g / i_N), \) such that for any \( x \in g, \) \( \delta(x) = [r_{g_+, g_-}, x \otimes 1 + 1 \otimes x] \) and \( \tilde{\delta}(x) = [r_{g_+, g_-}, x \otimes 1 + 1 \otimes x]. \)

\( \delta \) and \( \tilde{\delta} \) satisfy topological versions of the Lie bialgebra axioms. In the next two sections, we are going to construct topological Hopf algebras \((U_{k^2} g, \Delta)\) and \((U_k g, \Delta)\), quantizing \((g, \delta)\) and \((g, \tilde{\delta}).\)

3. **Construction of Serre relations**

In this section, we construct functions on products of \( C \) with itself, which will serve as coefficients for the Serre relations of \( U_k g. \)
3.1. Notation. We introduce a formal variable $h$ and set $q = e^h$. For each $s$ in $S$, we choose a local coordinate $z_s$ of $C$ at $s$.

3.1.1. Functions. For any complex number $\sigma$, $q^{\sigma \partial}$ is an automorphism of $\mathcal{K}[\![h]\!]$, preserving $R[\![h]\!]$. If $f$ belongs to $\mathcal{K}[\![h]\!]$, then $f = (f_s(z_s))_{s \in S}$, for $f_s$ in $\mathcal{C}(\!(z_s)\!)[\![h]\!]$. We have then $q^{\sigma \partial} f = (f_s(q^{\sigma \partial} z_s))_{s \in S}$. We will simply write $f = f(z)$, and $q^{\sigma \partial} f = f(q^{\sigma \partial} z)$.

For $V$ a vector space, we set $V((z)) = \prod_{s \in S} V[[z_s]][z_s^{-1}]$. This is a completion of $V \otimes \mathcal{K}$. If $V$ is a ring, $V((z))$ is also equipped with a ring structure. The ring $\prod_{(s,t) \in S \times S} \mathbb{C}[[z_s, w_t]][z_s^{-1}, w_t^{-1}]$ is a completion of $\mathcal{K} \otimes \mathcal{K}$. We denote it $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$. For $f$ an element of $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$, and $s, t$ elements of $S$, we denote by $f_{st}$ the component of $f$ in $\mathbb{C}[[z_s, w_t]][z_s^{-1}, w_t^{-1}]$.

We denote by $f \mapsto f^{(21)}$ the permutation of factors in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$. For $f$ in $R \otimes R$, to written as $\sum_i f_i' \otimes f_i''$, we set $f(z, w) = \sum_i f_i'(z) f_i''(w)$; this is a complex function on $(C - S) \times (C - S)$. Then $f^{(21)}(z, w) = f(w, z)$.

3.1.2. $\mathbb{C}[[h]]$-modules. For $M$ a module over $\mathbb{C}[[h]]$, we write $M/(h^k)$ for the quotient $M/h^k M$. A topologically free $\mathbb{C}[[h]]$-module is a module of the form $V[[h]]$, where $V$ is a complex vector space. When $M$ and $N$ are $\mathbb{C}[[h]]$-modules, we will denote by $M \otimes N$ their tensor product over $\mathbb{C}[[h]]$.

3.2. Results on kernels. Let $(r^\alpha)_{\alpha \geq 0}, (\lambda_\alpha)_{\alpha \geq 0}$ be dual bases of $R$ and $\Lambda$, such that $\lambda_\alpha$ tends to zero in the topology of $\mathcal{K}$ when $\alpha$ tends to infinity. Let us set $G(z, w) = \sum_{\alpha \geq 0} r^\alpha(z) \lambda_\alpha(w)$. This is an element of $\mathbb{C}((z))((w))$. We have

\[(\partial \otimes \text{id}) G(z, w) = G(z, w)^2 + \gamma,\]

for some $\gamma \in R \otimes R$ ($\gamma$ is the $\gamma$ of $\mathbb{S}$).

Let $\phi, \psi$ be the elements of $h \mathbb{C}[\gamma_0, \gamma_1, \ldots][[h]]$ defined as the solutions of the differential equations

\[
\partial_h \psi = D \psi - 1 - \gamma_0 \psi^2, \quad \partial_h \phi = D \phi - \gamma_0 \psi,
\]

where $D = \sum_{i \geq 0} \gamma_{i+1} \frac{\partial}{\partial \gamma_i}$. We have

\[
\psi(h, \gamma, \partial_h \gamma, \cdots) = -h + o(h), \quad \phi(h, \gamma, \partial_h \gamma, \cdots) = \frac{1}{2} h^2 \gamma + o(h^2).
\]

For $\sigma$ a complex number, we will denote by $\phi(\sigma h)$ and $\psi(\sigma h)$ the elements $\phi(\sigma h, \gamma, \partial_h \gamma, \cdots)$ and $\psi(\sigma h, \gamma, \partial_h \gamma, \cdots)$ of $R^{\otimes 2}[[h]]$. Set $G^{(21)}(z, w) = G(w, z)$. It follows from identity (3.11) of $\mathbb{S}$ that

\[
\sum_{\alpha} q^{\sigma \partial} - \frac{1}{\partial} \lambda_\alpha(z) r^\alpha(w) = (\phi(h) + \ln(1 - G^{(21)} \psi(h))) (z, w).
\]

For $f$ an element of $\mathcal{K}$, we denote by $f_R$ (resp., $f_\Lambda$) the projection of $f$ on $R$ parallel to $\Lambda$ (resp., on $\Lambda$ parallel to $R$). For any integer $\sigma$, $\sum_{i \geq 0} r^i \left( \frac{q^{\sigma \partial/2} - q^{-\sigma \partial/2}}{\partial} \lambda_i \right)_R$
is a symmetric element of $R^\otimes 2[[h]]$. We fill fix an element $\tau_\sigma$ of $R^\otimes 2[[h]]$ such that

$$\tau_\sigma + \tau_\sigma^{(21)} + \sum_{i\geq 0} r^i \otimes \left( \frac{q^{i\partial/2} - q^{-i\partial/2}}{\partial} \lambda_i \right)_R = 0.$$ 

We will set

$$q_\sigma(z, w) = \exp \left( \sum_{\alpha} \left( \frac{q^{\alpha\partial/2} - q^{-\alpha\partial/2}}{\partial} \lambda_\alpha \right) \otimes r^\alpha \right) \exp(\tau_\sigma)(z, w).$$

Then $q_\sigma(z, w)$ belongs to $\mathbb{C}((z))((w))[[h]]$. It may be expressed as the product of an element $\rho_\sigma$ of $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[h]]$ and the expansion, for $z$ close to $S$, of an element $\rho'_\sigma$ of the ring $\mathbb{C}(C \times C)_{S \times C, C \times S, diag}$ of rational functions on $C \times C$, with only possible poles on $S \times C, C \times S$ and the diagonal. We may then define the “analytic prolongation” $q_\sigma(z, w)_{w \ll z}$ as the element of $\mathbb{C}((z))((w))[[h]]$ equal to the product of $\rho_\sigma$ and the expansion of $\rho'_\sigma$ when $w$ is close to $S$. We have then

$$q_\sigma(z, w) q_\sigma(w, z)_{z \ll w} = 1.$$ 

The “singularities” of $q_\sigma(z, w)$ may be described as follows. If $s$ and $t$ are elements of $S$ such that $s \neq t$, $(q_\sigma)_{st}(z, t)$ belongs to $1 + h \mathbb{C}[[z_s, w_t]][z_s^{-1}, w_t^{-1}][[h]]$; and for any element $s$ of $S$, there exists an element $i_\sigma(z_s, w_s) \in 1 + h \mathbb{C}[[z_s, w_s]][z_s^{-1}, w_s^{-1}][[h]]$ such that the equality

$$(q_\sigma)_{ss}(z_s, w_s) = (i_\sigma)_{ss}(z_s, w_s)$$(w_s - q^{\alpha\partial/2} w_s - z_s)

holds in $\mathbb{C}((w_s))((z_s))[[h]]$ (see [7]).

**Example.** Assume that $C = \mathbb{C}P^1$, $\omega = dz$ and $S = \{ \infty \}$. Then the local coordinate is $z_\infty = z^{-1}$. We have $R = \mathbb{C}[[z]]$ and may choose $\Lambda = z^{-1} \mathbb{C}[[z^{-1}]]$. Then $G(z, w)$ and $q_\sigma(z, w)$ coincide with the expansions of $-\frac{1}{w - z}$ for $z \ll w$ and of $\frac{z - w + \alpha\partial/2}{z - w - \alpha\partial/2}$ for $w \ll z$.

### 3.3. Construction of Serre relations.

Let $m$ be an integer in $\{1, 2, 3\}$. Let $(\epsilon_\alpha)_{\alpha \in \mathbb{Z}}$ be a basis of $\mathcal{K}$, with dual basis $(\epsilon^\alpha)_{\alpha \in \mathbb{Z}}$, such that both $\epsilon_\alpha$ and $\epsilon^\alpha$ tend to zero when $\alpha$ tends to infinity (we may take $\epsilon_\alpha = r_\alpha, \epsilon^- = \lambda_\alpha$ for $\alpha \geq 0$).

Let us consider an algebra with generators $a_\alpha, b_\alpha, \alpha \in \mathbb{Z}$, and relations

$$(q^{m\partial} w_s - z_s)a(z)a(w) = i_{2m, s}(z, w)(w_s - q^{m\partial} z_s)a(w)a(z), a(z)_s a(w)_t = (q_{2m})_{st}(z, w)a(w)_t a(z)_s, (q^{\partial} w_s - z_s)b(z)b(w) = i_{2, s}(z, w)(w_s - q^{\partial} z_s)b(w)b(z), b(z)_s b(w)_t = (q_{2})_{st}(z, w)b(w)_t b(z)_s, (q^{-m\partial/2} w_s - z_s)a(z)b(w) = i_{-m, s}(z, w)(w_s - q^{-m\partial/2} z_s)b(w)a(z), a(z)_s b(w)_t = (q_{-m})_{st}(z, w)b(w)_t a(z)_s,)$$

for any elements $s, t$ of $S$, such that $s \neq t$, where we set

$$a(z) = \sum_{\alpha \in \mathbb{Z}} a_\alpha \epsilon_\alpha(z), b(z) = \sum_{\alpha \in \mathbb{Z}} b_\alpha \epsilon_\alpha(z),$$
and \( a(z) \) is the \( s \)th component of \( a(z) \).

As we will see (Prop. 3.1), the Serre identities compatible with these relations are

\[
\sum_{k=0}^{m+1} \sum_{\sigma \in S_{m+1}} A_{k,\sigma}(w, z_1, \ldots, z_{m+1}) a(z_{\sigma(1)}) \cdots a(z_{\sigma(k)}) b(w) a(z_{\sigma(k+1)}) \cdots a(z_{\sigma(m+1)}) = 0,
\]

(2)

and (2) with \( a \) and \( b \) exchanged, and \( m \) replaced by 1, where the functions

\[
A_{k,\sigma}(w, z_1, \ldots, z_{m+1}) \text{ belong to}
\]

\[
(-1)^k \binom{m+1}{k} + hR^{\otimes m+1}[[h]]
\]

and should satisfy the identity

\[
\sum_{k=1}^{m+1} \sum_{\sigma \in S_{m+1}} A_{k,\sigma}(z, w_1, \ldots, w_{m+1}) \prod_{i>k} q_{\sigma(i)}(z, w_{\sigma(i)}) \prod_{i<j, \sigma(i)<\sigma(j)} q_2(w_i, w_j) = 0.
\]

(3)

**Theorem 3.1.** (*existence of Serre identities*) There exist functions \( A_{k,\sigma}(z, w_1, \ldots, w_{m+1}) \) in

\[
(-1)^k \binom{m+1}{k} + hR^{\otimes m+1}[[h]],
\]

satisfying identity (3).

We will prove Thm. 3.1 in the case \( m = 1 \); the proof is similar in the general case.

**Theorem 3.2.** (*Thm. 3.1 for \( m = 1 \)*) There exist functions \( \alpha, \ldots, \gamma' \) in \( R^{\otimes 3}[[h]] \), such that \( \alpha, \gamma, \alpha', \gamma' \in 1 + hR^{\otimes 3}[[h]] \), \( \beta, \beta' \in -2 + hR^{\otimes 3}[[h]] \), and

\[
\begin{align*}
\alpha(z, w_1, w_2) &\cdot q_{-1}(z, w_1) q_{-1}(z, w_2) q_2(w_1, w_2) + \beta(z, w_1, w_2) q_{-1}(z, w_2) q_2(w_1, w_2) \\
&+ \gamma(z, w_1, w_2) q_2(w_1, w_2) + \alpha'(z, w_1, w_2) q_{-1}(z, w_1) q_2(w_1, w_2) \\
&+ \beta'(z, w_1, w_2) q_{-1}(z, w_1) + \gamma'(z, w_1, w_2) = 0.
\end{align*}
\]

(4)

**Proof of Thm 3.2.** Let us explain our strategy. We first give some conditions on \((\alpha, \ldots, \beta')\) in \((R^{\otimes 3}[[h]])\), which guarantee that they determine a system \((\alpha, \ldots, \gamma') \in (R^{\otimes 3}[[h]])^6\) satisfying the requirements of the Theorem (Prop. 3.1). We then show the existence of a system \((\alpha, \ldots, \beta')\) fulfilling these conditions (Prop. 3.2).

**Proposition 3.1.** If \((\alpha, \ldots, \beta')\) in \((R^{\otimes 3}[[h]])\) satisfy

\[
\begin{align*}
\alpha, \alpha', \gamma &\in 1 + hR^{\otimes 3}[[h]], \quad \beta, \beta' \in -2 + hR^{\otimes 3}[[h]],
\end{align*}
\]

(5)

and

\[
\begin{align*}
\alpha/\beta'(q^{-\theta} w_1, w_1, w_2) = \exp \left( -\tau_2 - (q^{-\theta} \otimes id) \tau_1 + \phi(2h) \right) \frac{\psi(-2h)}{\psi(2h) - \psi(-2h)}(w_1, w_2),
\end{align*}
\]

(6)
\[ \alpha'/\beta'(q^{-\theta}w_1, w_1, w_2) = \exp \left(-\left(q^{-\theta} \otimes \text{id}\right)\tau_1 + \phi(-2\mathcal{h})\right) \frac{\psi(2\mathcal{h})}{\psi(-2\mathcal{h}) - \psi(2\mathcal{h})}(w_1, w_2), \]  

\[ \alpha/\beta(q^{-\theta}w_2, w_1, w_2) = \exp \left(-\left(q^{-\theta} \otimes \text{id}\right)(\tau_1) + \phi(-2\mathcal{h})\right) \frac{\psi(2\mathcal{h})}{\psi(-2\mathcal{h}) - \psi(2\mathcal{h})}(w_1, w_2), \]  

\[ \alpha'/\beta(q^{-\theta}w_2, w_1, w_2) = \exp \left(-\tau_2 - \left(q^{-\theta} \otimes \text{id}\right)(\tau_1) + \phi(2\mathcal{h})\right) \frac{\psi(-2\mathcal{h})}{\psi(2\mathcal{h}) - \psi(-2\mathcal{h})}(w_1, w_2). \]  

\[ \alpha/\beta(z, q^{3\theta}w_2, q^\theta w_2) = -\exp \left((q^{3\theta} \otimes \text{id})(\tau_1) + \phi(4\mathcal{h}) - \phi(2\mathcal{h})\right) \frac{\psi(2\mathcal{h})}{\psi(4\mathcal{h})}(z, w_2), \]  

\[ \gamma/\beta(z, q^{3\theta}w_2, q^\theta w_2) = \exp \left(-\left(q^\theta \otimes \text{id}\right)\tau_1 - \phi(2\mathcal{h})\right) \frac{\psi(2\mathcal{h}) - \psi(4\mathcal{h})}{\psi(4\mathcal{h})}(w_2, z), \]

then \( \gamma' = -\left(q_2(z, w_1)q_2(z, w_2)q_4(w_1, w_2) + \beta q_{-2}(z, w_2)q_4(w_1, w_2) + \gamma q_4(w_1, w_2) + \alpha q_2(z, w_1)q_2(z, w_2) + \beta q_{-2}(z, w_1) \right) \) belongs to \( 1+R^\otimes[h] \). Therefore \( (\alpha, \ldots, \beta') \) uniquely determines a system \( (\alpha, \ldots, \gamma') \) satisfying the conditions of Thm. 3.2 (with \( \mathcal{h} \) replaced by \( 2\mathcal{h} \)).

Proof of Prop. Let \( (\alpha, \ldots, \beta') \) be arbitrary elements of \( (R^\otimes[h])^5 \). Set

\[ \gamma' = -\left(q_2(z, w_1)q_2(z, w_2)q_4(w_1, w_2) + \beta q_{-2}(z, w_2)q_4(w_1, w_2) + \gamma q_4(w_1, w_2) + \alpha q_2(z, w_1)q_2(z, w_2) + \beta q_{-2}(z, w_1) \right) . \]

\( \gamma'(z, w_1, z_2) \) belongs to \( \mathbb{C}((w_2))((w_1))((z))[[\mathcal{h}]] \). Moreover, for any \( s \) and \( t \) in \( S \), the products

\[ (z_s - q^{-\theta}(w_1)_s)\gamma'(z, w_1, w_2), (z_s - q^{-\theta}(w_2)_s)\gamma'(z, w_1, w_2) \text{ and } ((w_1)_s - q^{2\theta}(w_2)_s)\gamma'(z, w_1, w_2) \]

belong respectively to \( \mathbb{C}[[z_s, (w_1)_s][z_s^{-1}, (w_1)_s^{-1}]][((w_2))[[\mathcal{h}]] \), \( \mathbb{C}[[z_s, (w_2)_s][z_s^{-1}, (w_2)_s^{-1}]]((w_1))[[\mathcal{h}]] \) and \( \mathbb{C}[[z_s, (w_1)_s][z_s^{-1}, (w_1)_s^{-1}]][((w_2)_s^{-1}, (w_2)_s)][((z))[[\mathcal{h}]] \].

Lemma 3.1. Assume moreover that \( (\alpha, \ldots, \beta') \) satisfy conditions (4)-(13). Then the products (12) vanish when we substitute respectively \( z_s = q^{-\theta}(w_1)_s, z_s = q^{-\theta}(w_2)_s \) and \( (w_1)_s = q^{2\theta}(w_2)_s \) (in other words, these conditions are sufficient for the “poles” of \( \gamma' \) at these points to vanish).

Proof of Lemma. See Appendix A. \( \square \)

End of proof of Prop. 3.3. Recall the following fact:
Lemma 3.2. If \( f \) belongs to \( \mathbb{C}[[z_s, w_s]][z_s^{-1}, w_s^{-1}] \) and vanishes when we substitute \( w_s = z_s \), then there exists \( g \) in \( \mathbb{C}[[z_s, w_s]][z_s^{-1}, w_s^{-1}] \) such that \( g(z_s, w_s) = (z_s - w_s)f(z_s, w_s) \).

Assume that \((\alpha, \ldots, \beta')\) satisfy conditions (3)-(11). Recall that we have set

\[
\gamma' = -(q_2(z, w_1)q_2(z, w_2)q_4(w_1, w_2) + \beta q_2(z, w_2)q_4(w_1, w_2) + \gamma q_4(w_1, w_2) + \alpha q_2(z, w_1)q_2(z, w_2) + \beta' q_2(z, w_1)) \quad (15)
\]

Lemma 3.3 then implies that \( (15) \) and \( (16) \) then imply that \( \gamma' \) belongs to \( \mathbb{C}[[z, w]][z^{-1}, w^{-1}]((w_2))[[h]] \).

Replacing in this argument \( (z_s - q^{-\partial}(w_1)_s)\gamma'(z, w_1, w_2) \) by \( (z_s - q^{-\partial}(w_1)_s)\gamma'(z, w_1, w_2) \) and \( ((w_1)_s - q^{\partial}(w_2)_s)\gamma'(z, w_1, w_2) \), we find

\[
\gamma'(z, w_1, w_2) \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}]((w_1))[[h]], \gamma'(z, w_1, w_2) \in \mathbb{C}[[w_1, w_2]][w_1^{-1}, w_2^{-1}]((z))[[h]]. \quad (14)
\]

(13) and (14) imply that

\[
\gamma'(z, w_1, w_2) \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}, w_1^{-1}, w_2^{-1}][[h]]. \quad (15)
\]

Moreover, \( q_\sigma(z, w) \) belongs to \( 1 + h\mathbb{C}((w))((z))[[h]] \), therefore relations (3) imply that

\[
\gamma' \in 1 + h\mathbb{C}((w_2))((w_1))[[h]]. \quad (16)
\]

(13) and (16) then imply that

\[
\gamma' \in 1 + h\mathbb{C}[[z, w]][z^{-1}, w^{-1}, w_1^{-1}, w_2^{-1}][[h]]. \quad (17)
\]

Let us now show that \( \gamma' \) belongs to \( R \otimes 3[[h]] \). We will need the following statements.

Lemma 3.3. If \( f \) belongs to \( \mathbb{C}[[z, w]][z^{-1}, w^{-1}] \) and is such that for any \( \alpha \) in \( R \), \( (\alpha(z) - \alpha(w))f(z, w) \) belongs to \( R \otimes R \), then \( f \) belongs to \( R \otimes R \).

Proof of Lemma. For any \( \alpha \) in \( C \), let \( f_\alpha \) be the element \( (\alpha(z) - \alpha(w))f(z, w) \) of \( R \otimes R \). Let \( \alpha \) be any nonconstant element of \( C \). Then the function \( \phi : (P, Q) \mapsto \frac{f_\alpha(P, Q)}{\alpha(P) - \alpha(Q)} \) is a rational function on \( C \times C \), with poles when \( P \) or \( Q \) meet \( S \) and on the divisor \{\( (P, Q) \in C \times C | \alpha(P) = \alpha(Q) \}\}, which contains the diagonal \( C_{\text{diag}} \) of \( C \).

Let then \( P \) and \( Q \) be any pair of different points of \( C - S \). As \( C \) is smooth, \( R \) separates the points of \( C \). Let \( \alpha_{PQ} \) be a function of \( R \) such that \( \alpha_{PQ}(P) \neq \alpha_{PQ}(Q) \). Since \( \phi \) is equal to the function \( (P', Q') \mapsto \frac{f_{\alpha_{P,Q}(P')}(P', Q')}{\alpha_{P, Q}(P') - \alpha_{P, Q}(Q')} \), \( \phi \) has no
poles at \((P, Q)\). Therefore the only possible poles of \(\phi\) are on \(C_{\text{diag}} \cup [C \times S] \cup [S \times C]\).

Moreover, at each point \(P\) such that \(da(P)\) is nonzero, the possible pole of \(\phi\) at \((P, P)\) is simple; since the set of these points forms an open subset of \(C\), the possible pole of \(\phi\) at the diagonal is simple. The coefficient of this pole, which is a rational function on \(C\), is given by the substitution \(z = w\) in \((\alpha(z) - \alpha(w))\phi(z, w)\). The image in \(K\) of this function is the Taylor expansion of \(f_\alpha(P, P)\) for \(P\) in \(S\), which is zero, therefore \(\phi\) has no pole at the diagonal of \(C\) and belongs to \(R \otimes R\).

Since the image of \(\phi\) in \(\mathbb{C}[[z, w]][z^{-1}, w^{-1}]\) coincides with \(f\), \(f\) belongs to \(R \otimes R\).

We also have

**Lemma 3.4.** 1) (see also [2]) For any \(\alpha\) in \(R\), \((\alpha(z) - \alpha(w))G^{(21)}(z, w)\) belongs to \(R \otimes R\).

2) For \(\sigma\) a complex number and any \(\alpha\) in \(R\), \((\alpha(q^{-\sigma}z) - \alpha(w))q_{2\sigma}(z, w)\) belongs to \(R^{\otimes 2}[[h]]\).

**Proof.** 1) The delta-function of \(K\) is \(\delta(z, w)dw = \sum_\alpha e^\alpha(z)\omega_\alpha(w)\), where \((e^\alpha)\) and \((\omega_\alpha)\) are dual bases of \(K\) and its module of one-forms \(\Omega_K\). \(G + G^{(21)}\) is then the ratio \(\delta(z, w)dw/\omega(w)\), therefore \((\alpha(z) - \alpha(w))G^{(21)}(z, w) = -(\alpha(z) - \alpha(w))G(z, w)\). Since \((\alpha(z) - \alpha(w))G(z, w) = \sum_\gamma (\alpha r^\gamma)(z)\lambda_\gamma(w) - r^\gamma(z)(\alpha \lambda_\gamma)(w)\), the product \((\alpha(z) - \alpha(w))G(z, w)\) belongs to \(R((w))\), where \(z\) is attached to the factor \(R\). In the same way, \((\alpha(z) - \alpha(w))G^{(21)}(z, w)\) belongs to \(R((z))\), where \(w\) is attached to the factor \(R\). It follows that \((\alpha(z) - \alpha(w))G(z, w)\) belongs to \(R \otimes R\).

2) We have

\[
q_{2\sigma}(q^{\sigma\partial}z, w) = \exp \left( (q^{\sigma\partial} \otimes id)(\tau_{2\sigma} - \phi(2\sigma h)) \right) \left( 1 - G^{(21)}(q^{\sigma\partial} \otimes id)(\psi(2\sigma h)) \right)(z, w).
\]

It follows then from 1) that \([\alpha(z) - \alpha(w)]q_{2\sigma}(q^{\sigma\partial}z, w)\) belongs to \(R^{\otimes 2}[[h]]\). Since \(q^{-\sigma\partial}\) preserves \(R[[h]]\), \((\alpha(q^{-\sigma\partial}z) - \alpha(w))q_{2\sigma}(z, w)\) belongs to \((R \otimes R)[[h]]\).  

Assume now that \((\alpha, \ldots, \beta')\) satisfy conditions \([\text{i}]-[\text{iv}]\) and let us show that \(\gamma'\) belongs to \(R^{\otimes 3}[[h]]\). It follows from Lemma 3.4 that for any \(\alpha\) in \(R\), \((\alpha(z) - \alpha(q^{-\theta}w_1))\gamma'(z, w_1, w_2)\) belongs to \(R \otimes R((w_2))[[h]]\) (variables \(z\) and \(w_1\) correspond to the first and second factors of \(R \otimes R\)). Lemma 3.3 then implies that

\[
\gamma' \text{ belongs to } R \otimes R((w_1))[[h]]. \quad (18)
\]

In the same way, Lemma 3.4 implies that \((\alpha(z) - \alpha(q^{-\theta}w_2))\gamma'(z, w_1, w_2)\) belongs to \(R \otimes R((w_1))[[h]]\), where variables \(z\) and \(w_2\) correspond to the first and second factors of \(R \otimes R\). Lemma 3.3 then implies that

\[
\gamma' \text{ belongs to } R \otimes R((w_2))[[h]]. \quad (19)
\]

(18) and (19) then imply that \(\gamma'\) belongs to \(R^{\otimes 3}[[h]]\). Together with (17), this implies that \(\gamma'\) belongs to \(1 + hR^{\otimes 3}[[h]]\). This proves Prop. 3.4.  

\[\square\]
Proposition 3.2. There exists a family $(\alpha, \ldots, \beta')$ of $(R^{\otimes 3}[[h]])^5$, satisfying the conditions of Prop. 3.1.

Proof. We will use the following fact:

Lemma 3.5. Let $f(z, w)$ and $g(z, w)$ be two functions in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[h]]$, and let $\sigma, \sigma'$ be two complex numbers. There exists a function $h(z_1, z_2, z_3)$ in $\mathbb{C}[[z_1, z_2, z_3]][z_1^{-1}, z_2^{-1}, z_3^{-1}][[h]]$ such that $h(z, q^{\sigma_0}z, w) = f(z, w)$ and $h(z, w, q^{\sigma_0}w) = g(z, w)$, if the functions $f(z, q^{\sigma_0}z)$ and $g(z, q^{\sigma_0}z)$ coincide. If moreover $f, g$ belong to $R^{\otimes 2}[[h]]$, then $h$ may be chosen in $R^{\otimes 3}[[h]]$.

Proof of Lemma. Replacing $h(z_1, z_2, z_3)$ by $h(z_1, q^{-\sigma}z_2, q^{-\sigma}z_3)$, we may assume that $\sigma = \sigma' = 0$. One sets then $h(z_1, z_2, z_3) = g(z_1, z_3) + f(z_2, z_3) - g(z_2, z_3)$.

Let us first set $\beta(z, w_1, w_2) = -2$, and $\gamma(z, w_1, w_2) = -2 \times$ (right side of (11)). Then $\gamma$ belongs to $1 + h R^{\otimes 3}[[h]]$.

Let us determine $\alpha(z, w_1, w_2)$ satisfying conditions (8) and (10). Both equations should give the same values to

$$\alpha/\beta(w_2, q^{-\sigma_0}w_2, q^{\sigma_0}w_2).$$

This means that

$$- \exp \left( (q^{3\sigma} \otimes id)_{r-1} \right) \exp \left( \phi(4h) - \phi(2h) \right) \frac{\psi(2h)}{\psi(4h)} (w_2, w_2)$$

(20)

$$= \frac{1}{u \psi(-2h) - \psi(2h)} (q^\sigma w_2, q^{3\sigma} w_2)$$

$$= \exp \left( -r_{-1}(w_2, q^{3\sigma} w_2) \right) \exp \left( \phi(-2h) \right) (q^\sigma w_2, q^{3\sigma} w_2).$$

$$\cdot \frac{\psi(-2h)}{\psi(-2h) - \psi(2h)} (q^\sigma w_2, q^{3\sigma} w_2).$$

Let us show (20). $\exp \left( -\phi(2h) \right) \psi(2h)(w_2, w_2)$ is the residue at $z_s = (w_2)$ of $\exp \left( \sum_{\alpha \geq 0} \frac{q^{-\sigma}}{\partial} \lambda_\alpha \otimes r^\alpha \right)_{ss} (z, w_2)$, and $\exp \left( -\phi(4h) \right) \psi(4h)(w_2, w_2)$ is the residue at the same point of $\exp \left( \sum_{\alpha \geq 0} \frac{q^{-2\sigma}}{\partial} \lambda_\alpha \otimes r^\alpha \right)_{ss} (z, w_2)$, therefore

$$\exp \left( -\phi(2h) \right) \psi(2h)(w_2, w_2)$$

is the value at $z = w_2$ of $\exp \left[ \sum_{\alpha \geq 0} \frac{q^{-2\sigma} - q^{4\sigma}}{\partial} \lambda_\alpha \otimes r^\alpha \right] (z, w_2)$. Therefore the left side of (20) is

$$- \exp \left( (q^{3\sigma} \otimes id)_{r-1} \right) \exp \left( \sum_{\alpha \geq 0} \frac{q^{2\sigma} - q^{4\sigma}}{\partial} \lambda_\alpha \otimes r^\alpha \right) (w_2, w_2),$$

which is $-q_{-2}(q^{3\sigma} w_2, w_2)$. 
On the other hand, \( q_2(q^{-\theta}w_2, w_1)q_1(w_1, w_2)^{-1} \) vanishes when \( w_1 = q^{2\theta}w_2 \), therefore any functions \( \alpha_0, \beta_0, \gamma_0 \) satisfying (73) are such that

\[
\alpha_0(q^{-\theta}w_2, q^{2\theta}w_2, w_2)q_2(q^{-\theta}w_2, q^{2\theta}w_2) + \beta_0(q^{-\theta}w_2, q^{2\theta}w_2, w_2) = 0,
\]

which means that these functions verify

\[
\alpha_0(w_2, q^{2\theta}w_2, q^{\theta}w_2)q_2(w_2, q^{3\theta}w_2) + \beta_0(w_2, q^{3\theta}w_2, q^{\theta}w_2) = 0.
\]

The right side of (20) is equal to the ratio \( \alpha/\beta(w_2, q^{3\theta}w_2, q^{\theta}w_2) \) for \( \alpha, \beta \) as in (8), which are part of a system \( (\alpha, \beta, \gamma) \) of functions satisfying (73). Therefore this ratio is equal to \( -q_2(w_2, q^{3\theta}w_2)^{-1} \). It follows that the right side of (20) is also equal to \( -q_2(w_2, q^{3\theta}w_2)^{-1} \).

It follows that both sides of (20) are equal. Moreover, the right sides of (8) and (10) are of the form \( \alpha(z, w_2)q^{3\theta}w_2, q^{\theta}w_2 \) applying Lemma 3.5 to the system of equations

\[
\begin{align*}
\left(\begin{array}{c}
\partial \\
\partial w_2
\end{array}
\right) \qquad \left(\begin{array}{c}
\partial \\
\partial w_2
\end{array}
\right)
\end{align*}
\]

The system is equivalent to (8), (7) and (9). This system is equivalent to (8),

\[
\begin{align*}
\alpha'/\alpha(q^{-\theta}w_2, w_1, w_2) &= -\frac{u}{\exp[\tau_2 + (q^{-\theta} \otimes id)\tau_{-1}]\exp[-\phi(2\hbar)]} \frac{\psi(-2\hbar)}{\psi(2\hbar)} (w_2, w_1), \\
\alpha'/\alpha(q^{-\theta}w_1, w_1, w_2) &= -\frac{\psi(2\hbar)}{\psi(-2\hbar)} \frac{\exp[\tau_2 + (q^{-\theta} \otimes id)\tau_{-1}]\exp[-\phi(2\hbar)]}{u} (w_1, w_2).
\end{align*}
\]

If a solution \( \alpha' \) to (21) and (22) exists, both equations should give the same value to \( \alpha'/\alpha(q^{-\theta}w_1, w_1, w_1) \). If we set

\[
t = -\frac{\psi(2\hbar)}{\psi(-2\hbar)} \frac{\exp[\tau_2 + (q^{-\theta} \otimes id)\tau_{-1}]\exp[-\phi(2\hbar)]}{u},
\]

this means that

\[
t(z, z)^{-1} = t(z, z);
\]

in other terms, \( t(z, z)^2 = 1 \). We have

\[
t(z, z) = -\frac{\psi(2\hbar)\exp[-\phi(2\hbar)]}{\psi(-2\hbar)\exp[-\phi(-2\hbar)]} \exp[\tau_2](z, z).
\]
ψ(±h) exp[−φ(±h)] is the residue at \( w = z \) of \( \exp(\sum_{\alpha \geq 0} \frac{q^{±2\alpha} - 1}{\partial} \lambda_\alpha \otimes r^\alpha)(z, w); \) more precisely, we have

\[
G(z, w)(z-w)_{w=z} \psi(\pm 2h) \exp[-\phi(\pm 2h)](z, z) = \exp(\sum_{\alpha \geq 0} \frac{q^{±2\alpha} - 1}{\partial} \lambda_\alpha \otimes r^\alpha)(z, w)(z-w)_{w=z}.
\]

Let us set

\[
\tilde{q}^+_2(z, w) = \exp[r_2] \exp[\sum_{\alpha \geq 0} \frac{q^{2\alpha} - 1}{\partial} \lambda_\alpha \otimes r^\alpha](z, w),
\]

\[
\tilde{q}^-_2(z, w) = \exp[\sum_{\alpha \geq 0} \frac{q^{-2\alpha} - 1}{\partial} \lambda_\alpha \otimes r^\alpha](z, w).
\]

We have then

\[
(z - w)\tilde{q}^+_2(z, w)_{w=z} t(z, z) = (z - w)\tilde{q}^+_2(z, w)_{w=z}.
\]

It follows that

\[
(z - w)\tilde{q}^-_2(w, z)_{z\ll w} t(z, z) = (z - w)\tilde{q}^-_2(w, z)_{z\ll w}.
\]

Since we have

\[
\tilde{q}^-_2(z, w)\tilde{q}^-_2(w, z)_{z\ll w} = \tilde{q}^-_2(z, w)\tilde{q}^-_2(w, z)_{z\ll w},
\]

we get \( t(z, z)^2 = 1 \), as wanted (one can actually show that \( t(z, z) = 1 \)).

The right sides of (21) and (22) belong to \( 1 + hR^{\otimes 2}[[h]] \), let us denote them as \( 1 + h\tilde{s} \) and \( 1 + h\tilde{t} \). We have seen that \( s(z, z) = i(z, z) \), therefore by Lemma 3.3, the system of equations (21) and (22) has a solution \( \alpha' \) in \( 1 + hR^{\otimes 3}[[h]] \). We then set \( \beta' = \alpha'/ \mathrm{right \ side \ of \ (6)} \).□

This ends the proof of Thm. 3.2.□

4. The Hopf algebras \((U_hg, \Delta)\) and \((U_hg, \tilde{\Delta})\)

4.1. The algebras \(U_hL_n\). Let \( A = (a_{ij})_{i,j=1,\ldots,n} \) be a Cartan matrix of finite type. Let \( V = \bigoplus C e_i \) be a vector space of dimension \( n \). Let \( T(V \otimes K) \) be the tensor algebra of \( V \otimes K \) and let \( T(V \otimes K)[[h]] \) be its \( h \)-adic completion. Let us set \( e_i[\phi] = e_i \otimes \phi \) and \( e_i(z) = \sum_{r} e_i[r^\alpha] \lambda_\alpha(z) + e_i[\lambda_\alpha] r^\alpha(z) \).

We will define \( U_hL_n \) as the quotient of \( T(V \otimes K)[[h]] \) by the \( h \)-adically closed two-sided ideal generated by the coefficients of monomials in the identities

\[
(q^{d_{ij}a_{ij}}w_s - z_s)(e_i)_{s}(z)(e_j)_{s}(w) = i_{d_{ij}a_{ij},s}(z, w)w_s(q^{d_{ij}a_{ij}}z_s)(e_j)_{s}(w)(e_i)_{s}(z),
\]

\[
e_i(z)s_{e_j}(w)t = (q^{d_{ij}a_{ij}})_{st}(z, w)e_j(w)s_{e_i}(z),
\]

\[
(q^{d_{ij}a_{ij}}w_s - z_s)(e_i)_{s}(z)(e_j)_{s}(w) = i_{d_{ij}a_{ij},s}(z, w)w_s(q^{d_{ij}a_{ij}}z_s)(e_j)_{s}(w)(e_i)_{s}(z),
\]

\[
e_i(z)s_{e_j}(w)t = (q^{d_{ij}a_{ij}})_{st}(z, w)e_j(w)s_{e_i}(z),
\]

\[
(q^{d_{ij}a_{ij}}w_s - z_s)(e_i)_{s}(z)(e_j)_{s}(w) = i_{d_{ij}a_{ij},s}(z, w)w_s(q^{d_{ij}a_{ij}}z_s)(e_j)_{s}(w)(e_i)_{s}(z),
\]

\[
e_i(z)s_{e_j}(w)t = (q^{d_{ij}a_{ij}})_{st}(z, w)e_j(w)s_{e_i}(z),
\]
and
\[
\sum_{k=0}^{1-a_{ij}} \sum_{\sigma \in S_{1-a_{ij}}} A_{k,\sigma}(z, w_1, \ldots, w_{1-a_{ij}}) e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(k)}) e_j(w) e_i(z_{\sigma(k+1)}) \cdots e_i(z_{\sigma(1-a_{ij})}) = 0
\]
(26)

for any \(i, j = 1, \ldots, n\) and any elements \(s, t\) of \(S\) such that \(s \neq t\). As before \(e_i(z)_s\) is the \(s\)th component of \(e_i(z)\).

We will also define \(U_h L_n^-\) as follows. Let \(V_- = \bigoplus_{i=1}^n \mathbb{C} f_i\) be a complex vector space of dimension \(n\). \(U_h L_n^-\) will be the quotient of \(T(V_- \otimes \mathcal{K})[[\hbar]]\) by the \(h\)-adically closed two-sided ideal generated by the relations
\[
(q^{d_{\alpha i j}^\partial} w_s - z_s)(f_j)_s(w)(f_i)_s(z) = i_{d_{\alpha i j}, s}(z, w)(w_s - q^{d_{\alpha i j}^\partial} z_s)(f_i)_s(z)(f_j)_s(w),
\]
(27)
\[
f_i(z)_s f_j(w)_t = (q_{d_{\alpha i j}}^{-1})_{st}(z, w) f_j(w)_t f_i(z)_s,
\]
(28)
and
\[
\sum_{k=0}^{1-a_{ij}} \sum_{\sigma \in S_{1-a_{ij}}} A_{k,\sigma}(z, w_1, \ldots, w_{1-a_{ij}}) f_i(z_{\sigma(1-a_{ij})}) \cdots f_i(z_{\sigma(k+1)}) f_j(w)f_i(z_{\sigma(k)}) \cdots f_i(z_{\sigma(1)}) = 0
\]
(29)

for any \(i, j = 1, \ldots, n\) and any pair of different elements \(s, t\) of \(S\), where \(f_i[\phi] = f_i \otimes \phi\) and \(f_i(z) = \sum_{a \in \mathbb{Z}} f_i[r^a] \lambda_a(z) + f_i[\lambda_a] r^a(z)\). \(U_h L_n^-\) is therefore isomorphic with the opposite algebra of \(U_h L_n^+\).

4.2. The algebra \(U_h L_h\). Let \(H = \bigoplus_{i=1}^n \mathbb{C} h_i\) be a complex vector space of dimension \(n\). Let us first define some generating series in \(T((H \otimes \mathcal{K}) \oplus \mathbb{C} D \oplus \mathbb{C} K)[[\hbar]]\).

Define \(T_\sigma : \mathcal{K} \to \mathcal{K}\) by
\[
T_\sigma(\epsilon) = \frac{q^{\sigma \partial/2} - q^{-\sigma \partial/2}}{h\partial} \epsilon + \frac{1}{h}(\tau_\sigma, id \otimes \epsilon)_\mathcal{K},
\]
and set \(T_{ij} = T_{d_{\alpha i j}}\), for \(i, j = 1, \ldots, n\).

**Lemma 4.1.** Let \(T\) be the endomorphism of \(R^n[[\hbar]]\) defined by \(T(r_i)_{i=1,\ldots,n} = (\sum_{k=1}^n T_{ki} r_k)_{i=1,\ldots,n}\). Then \(T\) is invertible.

**Proof.** The reduction of \(T\) modulo \(\hbar\) coincides with the action of the symmetrized Cartan matrix \((d_{\alpha i j})_{i,j=1,\ldots,n}\), which is invertible (because \(\mathfrak{a}\) was assumed semisimple). \(\square\)

Let \(A_\sigma\) be the linear operator from \(\Lambda\) to \(R\) defined by
\[
A_\sigma(\lambda) = \langle \lambda \otimes id, \frac{1}{2}(\partial_z + \partial_w) \ln q_\sigma(z, w) \rangle_\mathcal{K},
\]
and \(A_{ij} = A_{d_{\alpha i j}}\), for \(i, j = 1, \ldots, n\).
Let us define $U_\sigma : \Lambda \to R$ by $U_\sigma(\lambda) = -\frac{1}{\hbar}\langle \tau_\sigma, \text{id} \otimes \lambda \rangle_\mathcal{K}$. Let us set $U_{ij} = U_{d_{\ast}a_{ij}}$. It follows from Lemma 4.1 that there exist unique linear maps $\rho_{ij} : \Lambda \to R$, $i, j = 1, \ldots, n$, such that

$$U_{ij} = \sum_{k=1}^{n} T_{kj} \circ \rho_{ik}.$$ 

It follows also from Lemma 4.1 that there exist unique linear operators $C_{ij} : \Lambda \to R$, such that $\sum_{k=1}^{n} T_{kj} \circ C_{ik} = A_{ij}$, for $i, j = 1, \ldots, n$.

We set then, for $\lambda \in \Lambda$ and $i = 1, \ldots, n$, $\tilde{h}_i[\lambda] = \sum_{j=1}^{n} h_j[\rho_{ij}(\lambda)]$, and

$$h_i^+(z) = \sum_{\alpha} h_i[r^\alpha] \lambda_\alpha(z) + \sum_{\alpha} \tilde{h}_i[\lambda_\alpha] r^\alpha(z), \quad h_i^-(z) = \sum_{\alpha} h_i[\lambda_\alpha] r^\alpha(z),$$

$$K_i^+(z) = q h_i^+(z) \quad K_i^-(z) = q h_i^-(z),$$

and for $\epsilon \in \mathcal{K}$,

$$K_i^+[\epsilon] = \sum_{s \in S} \text{res}_{z=s}(K_i^+(z) \epsilon(z) \omega_z), \quad (K_i^-)^{-1}[\epsilon] = \sum_{s \in S} \text{res}_s((K_i^-)^{-1}(z) \epsilon(z) \omega_z)$$

(so $(K_i^-)^{-1}[r] = 0$ for any $r$ in $R$). We also set

$$H_i[\lambda] = \sum_{j=1}^{n} h_j[C_{ij}(\lambda)], \quad H_i(z) = \sum_{\alpha \geq 0} H_i[\lambda_\alpha] r^\alpha(z).$$

Define $U_\hbar L \hbar$ as the quotient of $T((H \otimes \mathcal{K}) \oplus \mathcal{C}D \oplus \mathcal{C}K)[[\hbar]]$ by the $\hbar$-adically closed two-sided ideal generated by the relations

$$[K, h_i[\phi]] = 0, \quad [K, D] = 0,$$

$$[h_i[r], h_j[r']] = 0, \quad [h_i[r], h_j[\lambda]] = \frac{1}{\hbar}((1 - q^{-K \partial})T_{ij} r, \lambda)_\mathcal{K}, \quad (30)$$

$$[h_i[\lambda], h_j[\lambda']] = \frac{1}{\hbar}((q^{-K \partial} \otimes q^{-K \partial} - 1) \sum_{\alpha} T_{ij} \lambda_\alpha \otimes r^\alpha, \lambda \otimes \lambda')_{\mathcal{K} \otimes \mathcal{K}} \quad (31)$$

$$[D, h_i[r]] = h_i[\partial_r], \text{ for each } r \in R, \quad (32)$$

$$[D, (K_i^-)^{-1}[\lambda]] = (K_i^-)^{-1}[\partial \lambda] + \sum_{s \in S} \text{res}_s((H_i(z) + H_i(q^{-K \partial} z) + \psi_i(z))K_i^-(z)^{-1}\lambda(z)\omega_z), \quad (33)$$

for $r, r'$ in $R$, $\lambda, \lambda'$ in $\Lambda$ and $i, j = 1, \ldots, n$, where we set $h_i[\phi] = h_i \otimes \phi$, for $i = 1, \ldots, n$ and $\phi$ in $\mathcal{K}$. We also set

$$\psi_i(z) = \frac{1}{2}(\partial_{r'} + \partial_{r}) \ln q_{ii}(z', z)|_{z'=q^{-K \partial} z}.$$ 

We denote by $\langle , \rangle_{\mathcal{K} \otimes \mathcal{K}}$ the tensor square of $\langle , \rangle_{\mathcal{K}}$; it is a bilinear form on $\mathcal{K} \otimes \mathcal{K}$. We also write, if $\phi = \sum_i \phi_i' \otimes \phi_i''$, $\langle \phi, \text{id} \otimes x \rangle_{\mathcal{K}} = \sum_i \phi_i'(\phi_i'', x)_{\mathcal{K}}$, and $\langle \phi, x \otimes \text{id} \rangle_{\mathcal{K}} = \sum_i \phi_i'(\phi_i', x)_{\mathcal{K}}$. 


4.3. The algebra $U_h\mathfrak{g}$. Let $W = \oplus_{x=1}^{2n}(\mathbb{C}e_i \oplus \mathbb{C}h_i \oplus \mathbb{C}f_i)$ be a complex vector space of dimension $3n$. For $i = 1, \ldots, n$, $\phi$ in $K$ and $x = e, f, h$, we again denote by $x_i[\phi]$ the element $x_i \otimes \phi$ of the tensor algebra $T((W \otimes K) \oplus CD \oplus CK)[[h]]$.

Let us define $H_i[\lambda]$ and $K_i^\pm(z)$ by the morphism $T((W \otimes K) \oplus CD \oplus CK)[[h]] \to T((H \otimes K) \oplus CD \oplus CK)[[h]]$, $h_i[e] \mapsto h_i[e]$, $D \mapsto D$, $K \mapsto K$.

Define $U_{h\mathfrak{g}}$ as the quotient of $T((W \otimes K) \oplus CD \oplus CK)[[h]]$ by the $h$-adically closed ideal generated by relations (24), (25), (26), (27), (28), (29), (31), (32), (33) and relations

$$[K, e_i[e]] = [K, f_i[e]] = 0,$$

$[h_i[r], e_j[e]] = e_j[T_{ij}(r)e], \quad [h_i[\lambda], e_j[e]] = e_j[(q^{-K_0}T_{ij})(\lambda)e], \quad (34)$

$[h_i[e], f_j[e']] = -f_j[T_{ij}(e)e'], \quad (35)$

$$[e_i[e], f_j[e']] = \left(\frac{\delta_{ij}}{h} (K_i^+e\epsilon' - (K_i^-)^{-1}[(q^{-K_0}\partial\epsilon')e'])\right), \quad (36)$$

$$[D, e_i^\pm] = e_i^\pm[\partial e] + \sum_\beta H_i[(\epsilon\epsilon^\beta)\lambda]e_i^\pm[e_\beta], \quad (37)$$

for $\epsilon, \epsilon'$ in $K$, $r$ in $R$ and $\lambda$ in $\Lambda$, where $\sum_\beta \epsilon^\beta \otimes \epsilon_\beta = \sum_\alpha r^\alpha \otimes \lambda_\alpha + \lambda_\alpha \otimes r^\alpha$.

**Remark 1.** Generating series. Let us set $q_{ij}(z, w) = q_{da_{ij}}(z, w)$. For $(a(z), b(z))$ series in $U_{h\mathfrak{g}}[[z, z^{-1}]]$, and $\epsilon(z, w)$ an element of $\mathbb{Q}((z))((w))[[h]]$ or $\mathbb{Q}((w))((z))[[h]]$, let us write the equality $a(z)b(w)a(z)^{-1} = \epsilon(z, w)b(w)$ as $(a(z), b(w)) = \epsilon(z, w)$. Then we set $e_i(z) = \sum_\beta \epsilon^\beta \epsilon_\beta(z), f_i(z) = \sum_\beta f^\beta \epsilon_\beta(z)$, relations (31), (32), (34) and (33) are expressed as

$$(K_i^+(z), K_i^+(w)) = 1, \quad (K_i^+(z), K_i^-(w)) = \frac{q_{ij}(z, w)}{q_{ij}(z, q^{-K_0}w)},$$

$$(K_i^-(z), K_i^-(w)) = \frac{q_{ij}(q^{-K_0}z, q^{-K_0}w)}{q_{ij}(z, w)}.$$

and

$$(K_i^+(z), e_j(w)) = q_{ij}(z, w), \quad (K_i^-(z), e_j(w)) = q_{ij}(w, q^{-K_0}z),$$

$$(K_i^+(z), f_j(w)) = q_{ij}(z, w)^{-1}, \quad (K_i^-(z), f_j(w)) = q_{ij}(w, z)^{-1}.$$
Lemma 4.2. There exist unique elements $i, j$ for any $\alpha$ identities $\sum_i \sum_j$, we have

$$[D, K_i^+ (z)] = -\partial_z K_i^+ (z) + 2H_i(z)K_i^+ (z), \quad [D, e_i^+(z)] = (-\partial_ze_i^+ + H_ie_i^+) (z),$$

$$[D, K_i^- (z)^{-1}] = -\partial_z K_i^- (z)^{-1} + (H_i(z) + H_i(q^{-K_0}z) + \psi_i(z)) K_i^- (z)^{-1};$$

$H_i(z)$ also satisfies the relations

$$[H_i(z), e_j^+(w)] = \pm \frac{1}{2} (\partial_z + \partial_w) \ln q_{ij}(z, w)e_j^+(w),$$

Remark 2. As we will see in Prop. 5.2, there are natural embeddings of $U_h L_n \pm$ and of $U_h L_\hbar$ in $U_h g$; this justifies a posteriori that we denote elements of these algebras the same way as their images in $U_h g$.

4.4. Hopf algebra structures on $U_h g$. Let us set, for $i, j = 1, \ldots, n$, $c^{ij} = \sum_{\alpha \geq 0} C_{ij}(\lambda_\alpha) \otimes r^\alpha$. Then $c^{ij}$ is an element of $(R \otimes R)[[h]]$.

Lemma 4.2. There exist unique elements $(r^{ij})_{i, j = 1, \ldots, n}$ of $(R \otimes R)[[h]]$, such that for any $i, j$, we have $\sum_{\alpha \geq 0} C_{ij}(\lambda_\alpha) \otimes r^\alpha = c^{ij}$. The $r^{ij}$ also satisfy $\sum_{\alpha \geq 0} C_{ij}(\lambda_\alpha) \otimes r^\alpha = -\alpha^{ij} (21)$, for any $i, j = 1, \ldots, n$.

Proof. The existence and uniqueness of the $r^{ij}$ follows from Lemma 4.2. Let us set $\alpha^{ij} = \sum_{\alpha \geq 0} A_{ij}(\lambda_\alpha) \otimes r^\alpha$. The $c^{ij}$ are uniquely determined by the identities $\sum_{k=1}^n (T_{kj} \otimes id)(c^{ik}) = \alpha^{ij}$, for any $i, j = 1, \ldots, n$. On the other hand, $[-\sum_{\alpha \geq 0} C_{ij}(\lambda_\alpha) \otimes r^\alpha] (21)$ satisfies the same identities, because we have $\alpha^{ij} = -\alpha^{ij} (21)$.

We define $r^{ij}_{\alpha \beta}$ as the elements of $C[[h]]$ such that $r^{ij}_{\alpha \beta} = \sum_{\alpha, \beta \geq 0} r^{ij}_{\alpha \beta} r^\alpha \otimes r^\beta$.

Define completions of tensor powers of $U_h g$ as follows. For $N$ an integer, let $I_N$ be the left ideal of $U_h g$ generated by the $x[z^p], x \in \{e_i, h_i, f_i, i = 1, \ldots, n\}$, where $p \geq N$. Let us set, for $k$ integer,

$$U_h g \otimes^k = \lim_{l \rightarrow N} \lim_{k \rightarrow \infty} U_h g \otimes^k / \left( \sum_{p=0}^{k-2} U_h g \otimes^p \otimes I_N \otimes U_h g \otimes^{k-1-p} + h^l U_h g \otimes^k, \right)$$

and

$$U_h g \otimes^k = \lim_{l \rightarrow N} \lim_{k \rightarrow \infty} U_h g \otimes^k / \left( \sum_{p=1}^{k-1} U_h g \otimes^p \otimes I_N \otimes U_h g \otimes^{k-1-p} + h^l U_h g \otimes^k, \right),$$

where all tensor products are over $C[[h]]$.

For any $x$ in $U_h g$ and any integers $N$ and $l \geq 0$, there exists an integer $N'(x, N, l)$ such that $I_{N'(x, N, l)} x \subset I_N + h^l U_h g$. It follows that the above tensor products are endowed with algebra structures.
Proposition 4.1. There exists a unique algebra morphism $\Delta$ from $U_h\mathfrak{g}$ to $U_h\mathfrak{g} \otimes <U_h\mathfrak{g}$, such that

$$
\Delta(K) = K \otimes 1 + 1 \otimes K, \quad \Delta(D) = D \otimes 1 + 1 \otimes D + \sum_{i,j=1,\ldots,n,\alpha,\beta \geq 0} r_{ij}^{\alpha\beta} h_i[r_{ij}^{\alpha}] \otimes h_j[r_{ij}^{\beta}],
$$

$$
\Delta(h_i[r]) = h_i[r] \otimes 1 + 1 \otimes h_i[r], \quad \Delta(h_i[\lambda]) = h_i[\lambda] \otimes 1 + 1 \otimes h_i[(q^{K_i} \alpha) \Lambda]
$$

for $r \in R, \lambda \in \Lambda$, $\Delta(e_i[\epsilon]) = \sum_{\beta} e_i[\epsilon^\beta] \otimes K_i^+ [\epsilon_{\beta}] + 1 \otimes e_i[\epsilon], \Delta(f_i[\epsilon]) = f_i[\epsilon] \otimes 1 + \sum_{\beta} (K_i^-)^{-1} [\epsilon e^\beta] \otimes f_i[q^{-K_i} \alpha \epsilon_{\beta}]

for $\epsilon \in \mathcal{K}$. We set $\sum_{\beta} \epsilon^\beta \otimes \epsilon_{\beta} = \sum_{\alpha \geq 0} r_{ij}^{\alpha \beta} \Lambda + \sum_{\alpha \geq 0} \Lambda_{ij} \otimes r_{ij}^{\alpha}$, and $K_1 = K \otimes 1$.

Moreover, for each integers $N$ and $p \geq 0$, there exists an integer $N'(N,p)$ such that $\Delta(I_{N'(N,p)})$ is contained in the completion of $h^p U_h\mathfrak{g} \oplus + I_N \otimes U_h\mathfrak{g}$.

Proof. For $\lambda$ in $\Lambda$, $\Delta(h_i[\lambda])$ belongs to the $h$-adic completion of $U_h\mathfrak{g} \otimes \mathbb{C}[\hbar]] U_h\mathfrak{g}$. On the other hand, for any $\epsilon$ in $\mathcal{K}$, both $K_i^+ [\epsilon \Lambda_{\alpha}]$ and $(K_i^-)^{-1} [\epsilon r_{\alpha}]$ tend to zero (in the $h$-adic topology) in $U_h\mathfrak{g}$ when $\alpha$ tends to infinity, and $e_i[\Lambda_{\alpha}]$ and $(K_i^-)^{-1} [\epsilon \Lambda_{\alpha}]$ tend to zero in the topology defined by the $I_N$, so that $\Delta(e_i[\epsilon])$ and $\Delta(f_i[\epsilon])$ both converge in $U_h\mathfrak{g} \otimes <U_h\mathfrak{g}$.

After we write $\Delta$ in terms of generating series as

$$
\Delta(K_i^+(z)) = K_i^+(z) \otimes K_i^+(z), \quad \Delta(K_i^-(z)) = K_i^-(z) \otimes K_i^-(q^{-K_i} \alpha z),
$$

$$
\Delta(e_i(z)) = e_i(z) \otimes K_i^+(z) + 1 \otimes e_i(z),
\Delta(f_i(z)) = f_i(z) \otimes 1 + K_i^-(z)^{-1} \otimes f_i(q^{-K_i} \alpha z),
$$

it is easy to check that the extension of $\Delta$ to the free algebra $T((W \otimes \mathcal{K}) \otimes \mathbb{C}D \otimes \mathbb{C}K)[[\hbar]]$ maps all quadratic relations of $U_h\mathfrak{g}$ relations to zero; in the case of the Serre relations, this follows from the identities (3). The statement on $\Delta(I_N)$ is immediate. 

There is a unique algebra morphism $\epsilon : U_h\mathfrak{g} \to \mathbb{C}[[\hbar]]$, such that $\epsilon(x[\epsilon]) = \epsilon(K) = \epsilon(D) = 0$, for $x = h_i, e_i, f_i$ and $\epsilon \in \mathcal{K}$. There is also a unique algebra morphism $S : U_h\mathfrak{g} \to \lim_{\leftarrow p} \lim_{\leftarrow N} U_h\mathfrak{g}/(I_N + h^p U_h\mathfrak{g})$, such that

$$
S(K) = -K, \quad S(D) = -D + \sum_{i,j,\alpha,\beta} r_{ij}^{\alpha\beta} h_i[r_{ij}^{\alpha}] h_j[r_{ij}^{\beta}],
$$

$$
S(h_i[r]) = -h_i[r], \quad S(h_i[\lambda]) = -h_i[(q^{-K_i} \alpha) \Lambda],
S(e_i[\epsilon]) = -\sum_{\beta} e_i[\epsilon \epsilon^\beta] (K_i^+)^{-1} [\epsilon_{\beta}], \quad S(f_i[\epsilon]) = -\sum_{\beta} K_i^- [\epsilon \epsilon^\beta] f_i[(q^{-K_i} \alpha) \epsilon_{\beta}],
$$

where we set

$$(K_i^+)^{-1} [\epsilon] = \sum_{s \in S} \text{res}_{z=s} (K_i^+(z)^{-1} \epsilon(z) \omega(z)), \quad K_i^- [\epsilon] = \sum_{s \in S} \text{res}_{z=s} (K_i^-(z) \epsilon(z) \omega(z)).$$

$S$ is continuous in the topology defined by the $I_N$ and has therefore a unique extension to an algebra automorphism of $\lim_{\leftarrow p} \lim_{\leftarrow N} U_h\mathfrak{g}/(I_N + h^p U_h\mathfrak{g})$. 


Proposition 4.2. \((U_h\mathfrak{g}, \Delta, \varepsilon, S)\) is a topological Hopf algebra.

Here “topological” should be understood in the sense that in the Hopf algebra axioms, tensor powers should be replaced by their completions \(U_h\mathfrak{g} \otimes^\mathbb{C} \), and one factor \(U_h\mathfrak{g}\) should be replaced by \(\lim_{\leftarrow N} U_h\mathfrak{g}/I_N\) in each of the two antipode axioms.

\((U_h\mathfrak{g}, \Delta, \varepsilon, S)\) also induces a topological Hopf algebra structure on \(\lim_{\leftarrow p} \lim_{\leftarrow N} U_h\mathfrak{g}/(I_N + h^p U_h\mathfrak{g})\).

Proposition 4.3. There exists a unique algebra morphism \(\bar{\Delta}\) from \(U_h\mathfrak{g}\) to \(U_h\mathfrak{g} \otimes_{U_h\mathfrak{g}} U_h\mathfrak{g}\) such that

\[
\bar{\Delta}(K) = \Delta(K), \quad \bar{\Delta}(D) = \Delta(D), \quad \bar{\Delta}(h_i[r]) = \Delta(h_i[r])
\]

for \(r \in R\),

\[
\bar{\Delta}(h_i[\lambda]) = h_i[(q^{-K^\partial} \lambda)\Lambda] \otimes 1 + 1 \otimes h_i[\lambda]
\]

for \(\lambda \in \Lambda\), and

\[
\bar{\Delta}(e_i[\epsilon]) = e_i[\epsilon] \otimes 1 + \sum_{\beta} (K_i^-)^{-1} [(q^{-K_i^\partial} \epsilon) \otimes e_i[\epsilon]], \quad \bar{\Delta}(f_i[\epsilon]) = \sum_{\beta} f_i[\epsilon] \otimes K_i^+ [\epsilon \epsilon^\beta] + 1 \otimes f_i[\epsilon],
\]

where we set \(K_2 = 1 \otimes K\). For any integers \(N\) and \(p \geq 0\), there exists an integer \(N'\) such that \(\Delta(I_{N'(N,p)})\) is contained in the completion of \(h^p U_h\mathfrak{g} \otimes_{U_h\mathfrak{g}} I_N\).

There is a unique algebra morphism \(\bar{S} : U_h\mathfrak{g} \to \lim_{\leftarrow p} \lim_{\leftarrow N} U_h\mathfrak{g}/(I_N + h^p U_h\mathfrak{g})\), such that

\[
\bar{S}(K) = -K, \quad \bar{S}(D) = S(D), \quad \bar{S}(h_i[r]) = -h_i[r]
\]

for \(r \in R\), and

\[
\bar{S}(h_i[\lambda]) = -h_i[(q^{-K^\partial} \lambda)\Lambda],
\]

\[
\bar{S}(e_i[\epsilon]) = -\sum_{\beta} K_i^- [\epsilon^\beta] e_i[(q^{-K^\partial}\epsilon) \{\epsilon\}], \quad \bar{S}(f_i[\epsilon]) = -\sum_{\beta} f_i[\epsilon \epsilon^\beta] (K_i^+)\Lambda [\epsilon^\beta].
\]

Proposition 4.4. \((U_h\mathfrak{g}, \bar{\Delta}, \bar{\varepsilon}, \bar{S})\) is a topological Hopf algebra.

Remark 3. The formulas defining \(S, \bar{\Delta}\) and \(\bar{S}\) can be rewritten as

\[
S(e_i(z)) = -e_i(z)K_i^+(z)^{-1}, \quad S(f_i(z)) = -K_i^-(z)f_i(q^{-K^\partial}z), \quad S(K_i^-(z)) = K_i^- (q^{-K^\partial}z)^{-1},
\]

\[
\bar{\Delta}(e_i(z)) = e_i(z) \otimes 1 + K_i^- (q^{-K^\partial}z)^{-1} \otimes e_i(q^{-K^\partial}z), \quad \bar{\Delta}(f_i(z)) = f_i(z) \otimes K_i^+(z) + 1 \otimes f_i(z),
\]

\[
\bar{\Delta}(K_i^-) = K_i^- (q^{-K^\partial}z) \otimes K_i^-(z)
\]

and

\[
\bar{S}(e_i(z)) = -K_i^- (q^{-K^\partial}z) e_i(q^{-K^\partial}z), \quad \bar{S}(f_i(z)) = -f_i(z) K_i^+(z), \quad \bar{S}(K_i^-) = K_i^- (q^{-K^\partial}z)^{-1}.
\]
Remark 4. Let $J_N$ be the left ideal generated by the $e_i[z_i^\ast]$, $x \in \{e, f, h\}$, $i \in \{1, \ldots, n\}$, $s \in S$, $l \geq N$. The formulas defining $\Delta, S, \tilde{\Delta}$ and $\tilde{S}$ also define topological Hopf algebra structures $(U_\hbar g, \Delta(J_N))$, $(U_\hbar g, \Delta(J_N))$, where the tensor powers of $U_\hbar g$ are completed using the family $J_N$ instead of $I_N$.

In Theorem 9.1, we are going to construct a quasi-Hopf algebra structure on an algebra $U_\hbar g^{out}$, starting from $\Delta$ and $\tilde{\Delta}$. One could also construct a quasi-Hopf structure starting from $\Delta(J_N)$ and $\Delta(J_N)$. This quasi-Hopf algebra structure is probably equivalent to that obtained in Theorem 9.1.

5. PBW theorems

The proofs of the PBW theorems for $U_\hbar L\mathfrak{n}_+$ and $U_\hbar \mathfrak{g}$ will follow the proofs of [1], which rely on Lie bialgebra duality.

5.1. The case of $U_\hbar L\mathfrak{n}_+$. Let us denote by $(\bar{e}_i, \bar{h}_i, \bar{f}_i)_{i=1,\ldots,n}$ the Chevalley generators of $\mathfrak{a}$.

Theorem 5.1. $U_\hbar L\mathfrak{n}_+$ is a topologically free algebra over $\mathbb{C}[[\hbar]]$, and the map $e_i[\epsilon] \mapsto \bar{e}_i \otimes \epsilon$ induces an isomorphism from $U_\hbar L\mathfrak{n}_+ / hU_\hbar L\mathfrak{n}_+$ to $UL\mathfrak{n}_+$.

Proof of Thm. 5.1. Following Feigin and Odesskii ([11]), define a functional shuffle algebra as follows. Set $FO = \oplus_{k \in \mathbb{N}} FO_k$, where $FO_k$ is the subspace of $\mathbb{C}((t_1)) \cdots ((t_N))[[\hbar]]$ formed of the series of the form

$$f(t_1, \ldots, t_N) \prod_{1 \leq i < j \leq N, \alpha(i) \neq \alpha(j)}(t_i - t_j),$$

where $N = \sum_{\sigma=1}^n k_\sigma$, $f$ belongs to $\mathbb{C}[[t_1, \ldots, t_N]][[t_1^{-1}, \ldots, t_N^{-1}]]$ and is symmetric in each group of variables $(t_1^\sigma)^{1 \leq j \leq k_\sigma}$, we set $t_k^\sigma = t_{k_1 + \cdots + k_{\sigma - 1} + k}$ for $k = 1, \ldots, k_\sigma$, $\alpha(k_1 + \cdots + k_{\sigma - 1} + k) = \alpha_\sigma$ for $k = 1, \ldots, k_\sigma$; by convention, $\frac{a}{a-b} = \sum_{a \geq 0} a^{-1} b^i$.

For any integer $\sigma$, $\tau_\sigma + \sum_{\alpha \geq 0}(\frac{1 - q^{-\sigma\partial/2}}{\partial} \lambda_\alpha) R \otimes r^\alpha$ is an antisymmetric element of $h(R \otimes R)[[\hbar]]$. Let us fix $\alpha_\sigma$ in $h(R \otimes R)[[\hbar]]$ such that

$$\alpha_\sigma - \alpha_\sigma^{(21)} = \tau_\sigma + \sum_{\alpha \geq 0}(\frac{1 - q^{-\sigma\partial/2}}{\partial} \lambda_\alpha) R \otimes r^\alpha;$$

for example, we may set $\alpha_\sigma = \frac{1}{2}(\tau_\sigma + \sum_{\alpha \geq 0}(\frac{1 - q^{-\sigma\partial/2}}{\partial} \lambda_\alpha) R \otimes r^\alpha)$.

Let us set

$$q_\sigma[z, w] = \exp \left( \sum_{\alpha \geq 0} \frac{q^{\sigma/2} \partial}{\partial} \lambda_\alpha(z) r^\alpha(w) \right) \exp(\alpha_\sigma)(z, w);$$

we have then

$$q_\sigma[0, w] / q_\sigma[0, z] \mid_{w \ll z} = q_\sigma(z, w).$$
Define a composition law \( FO_k \times FO_l \to FO_{k+l} \) by
\[
(f \ast g)(t^{(i)}_j) = \text{Sym}_{t_1^{(i)}, \ldots, t_{k_1+i}^{(i)}} \cdots \text{Sym}_{t_1^{(n)}, \ldots, t_{k_n+l}^{(n)}}
\]
\[
\{ \prod_{i=1}^{N} \prod_{j=N+1}^{N+M} q_{\alpha(i),\alpha(j)}^{(i)}(t_i, t_j) f(t_1, \ldots, t_N) g(t_{N+1}, \ldots, t_{N+M}) \}
\]
where \( N = \sum_i k_i, M = \sum_i l_i, \alpha(k_1 + \cdots + k_{a-1} + i) = \alpha(N + l_1 + \cdots + l_{a-1} + j) = \delta_a, \)
for \( i = 1, \ldots, k_a \) and \( j = 1, \ldots, l_a, \langle \delta_a, \delta_t \rangle = d_a a_{\sigma_t} \) and
\[
t_{k_1+\cdots+k_{a-1}+i} = t_{i}^{(a)}, \quad t_{N+l_1+\cdots+l_{a-1}+j} = t_{l_a+j}^{(a)}
\]
for \( i = 1, \ldots, k_a \) and \( j = 1, \ldots, l_a \). Here \( \delta_1, \ldots, \delta_n \) are the basis vectors of \( \mathbb{N}^n \).

One checks directly that \((FO, \ast)\) is an associative algebra.

**Proposition 5.1.** There is a unique algebra morphism from \( U_h \mathfrak{g}_\mathcal{K} \) to \( FO \), sending \( e_i [\phi] \) to \( \phi \in FO_{\alpha_i} \), for any \( \phi \) in \( \mathcal{K} \) and \( i = 1, \ldots, n \). Here \( \alpha_i \) is the \( i \)th basis vector of \( \mathbb{N}^n \).

**Proof.** The fact that the vertex relations are sent to zero is immediate; the fact that the quantum Serre relations are sent to zero follows from the identities (26).

Define \( \langle LV \rangle \) as the \( h \)-adically complete subalgebra of \( FO \) generated by the \( FO_{\alpha_i}, i = 1, \ldots, n \).

For \( \lambda \) in \( \Lambda \) and \( i \) in \( \{1, \ldots, n\} \), define endomorphisms \( \delta_i[\lambda] \) of \( FO \) as follows: for \( f \in FO_k \), \( \delta_i[\lambda](f) \) belongs to \( FO_k \) and
\[
(\delta_i[\lambda] f)(t_1, \ldots, t_N) = \left( \sum_{j=1}^{n} \sum_{k=1}^{l} (T_{ij}\lambda)(t_k^{(j)}) \right) f(t_1, \ldots, t_N).
\]
The \( \delta_i[\lambda] \) define commuting derivations of \( FO \), which preserve \( \langle LV \rangle \).

Define \( \mathcal{V} \) and \( \mathcal{S} \) as the semidirect products of \( \langle LV \rangle \) and of \( FO \) by this commuting family of derivations. Explicitly, we have
\[
\mathcal{V} = \lim_{\leftarrow \mathcal{N}} \langle LV \rangle \otimes \mathbb{C}[h_i[\lambda_\alpha]^S], \quad i = 1, \ldots, n, \alpha \geq 0 \}/(h^N),
\]
\[
\mathcal{S} = \lim_{\leftarrow \mathcal{N}} FO \otimes \mathbb{C}[h_i[\lambda_\alpha]^S], \quad i = 1, \ldots, n, \alpha \geq 0 \}/(h^N),
\]
and the product maps are defined in \( \mathcal{V} \) and \( \mathcal{S} \) by
\[
\left( \sum_{n(i,a) \geq 0} \phi_{n(i,a)} \otimes \prod_i \prod_\alpha (h_i[\lambda_\alpha]^S)^{n(i,a)} \right) \left( \sum_{m(i,a) \geq 0} \psi_{m(i,a)} \otimes \prod_i \prod_\alpha (h_i[\lambda_\alpha]^S)^{m(i,a)} \right)
\]
\[
= \sum_{n(i,a),m(i,a) \geq 0} \phi_{n(i,a)} \prod_i \prod_\alpha \delta_i[\lambda_\alpha]^{n(i,a)} (\psi_{m(i,a)}) \otimes \prod_i \prod_\alpha (h_i[\lambda_\alpha]^S)^{n(i,a)+m(i,a)}.
\]
\( \mathcal{V} \) is then a subalgebra of \( \mathcal{S} \).

Define \( \mathcal{I}_N \) as the complete left ideal of \( \mathcal{S} \) generated by the \( h_i[\lambda_\alpha], i = 1, \ldots, n, \alpha \geq N \).

Define a topological Hopf algebra structure on \( \mathcal{S} \) as follows. Let us set \( K_i^- (z)^\mathcal{S} = \exp(h \sum_\alpha h_i[\lambda_\alpha]^\mathcal{S} r^\alpha(z)) \), and for \( \epsilon \) in \( \mathcal{K} \), let us set
\[
(K_i^-)^{-1}[\epsilon]^\mathcal{S} = \sum_{s \in S} \text{res}_{z=s} ([K_i^- (z)^\mathcal{S}]^{-1} \epsilon(z) \omega(z)).
\]

There is a unique algebra morphism \( \Delta_\mathcal{S} \) from \( \mathcal{S} \) to \( \text{lim}_{N \to \infty} (\mathcal{S} \otimes \mathcal{S}) / (\mathcal{S} \otimes \mathcal{I}_N + h^N \mathcal{S} \otimes \mathcal{S}) \), such that
\[
\Delta_\mathcal{S}(h_i[\lambda]^\mathcal{S}) = h_i[\lambda]^\mathcal{S} \otimes 1 + 1 \otimes h_i[\lambda]^\mathcal{S}
\]
for \( \lambda \in \Lambda \), and for \( P \in FO_\mathcal{K} \), \( \Delta_\mathcal{S}(P) = \sum_{k', k'' = k} \Delta_\mathcal{S}^{k', k''}(P) \), where
\[
\Delta_\mathcal{S}^{k', k''}(P) = \sum_{\nu, \nu_1, \ldots, \nu_N \in \mathbb{Z}} \left( \prod_{i=1}^{N'} \epsilon_{\nu_i}(u_i) P_{\nu}(u_1, \ldots, u_{N'}) \right) \otimes \left( P''(u_{N'+1}, \ldots, u_N) \prod_{i=1}^{N'} (K_i^-)^{-1}[\epsilon^\mathcal{S}] \right);
\]
we set
\[
N' = \sum_{i=1}^{n} (k'_l - k'_l - 1) n'_l + t'_l(\alpha) \quad \text{for} \quad l = 1, \ldots, k'_\sigma; \quad u_{N'+1} = N' = \sum_{j=1}^{k''_l} k''_l + 1 + t''_l(\alpha) \quad \text{for} \quad l = 1, \ldots, k''_\sigma,
\]
where the \( t'_l(\alpha) \) and \( t''_l(\alpha) \) are the analogues of the variables \( t_l(\alpha) \) for the first and second copy of \( \mathcal{S} \), and we set
\[
\sum_{\alpha} P_{\nu}(v_1, \ldots, v_{N'}) P''_{\nu}(u_{N'+1}, \ldots, u_N)
= P(t_1, \ldots, t_N) \prod_{i=1, \ldots, N', j=N'+1, \ldots, N} q^+_{(\alpha(i), \alpha(j))}(v_i, v_j)^{-1},
\]
where we recall that \( t_{\sum_{j=1}^{\sigma} k'_j + l} = t_l(\alpha) \) for \( l = 1, \ldots, k'_\sigma \) and we set \( v_{N'+1} = \sum_{j=1}^{k''_l} k''_l + 1 + t''_l(\alpha) \) for \( l = 1, \ldots, k''_\sigma \). \( v_{N'+1} = N' + \sum_{j=1}^{k', k''} k''_l + l = \alpha(N' + \sum_{j=1}^{k''_l} k''_l + l') = \alpha_{\sigma} \) for \( l = 1, \ldots, k', k''_l + l' = 1, \ldots, k''_\sigma \), and \( \alpha_{i, j} = d_{ij} \).

\( \Delta_\mathcal{S} \) defines a topological Hopf algebra structure on \( \mathcal{S} \); \( \mathcal{V} \) is then a Hopf subalgebra of \( \mathcal{S} \).

Let us define \( U_h \tilde{\mathfrak{g}}_+ \) as the quotient of the subalgebra of \( U_h \mathfrak{g} \) generated by \( K \), the \( h_i[\lambda], \lambda \in \Lambda \) and \( e_i[\epsilon], \epsilon \in \mathcal{K} \), by the ideal generated by \( K \). The opposite coproduct \( \tilde{\Delta} \) to \( \Delta \) induces a topological Hopf algebra structure on \( U_h \tilde{\mathfrak{g}}_+ \). The map \( x_{i, \alpha} \mapsto h_i[\lambda_\alpha] \) composed with the product map defines a topological \( \mathbb{C}[[h]] \)-module isomorphism between \( \text{lim}_{N \to \infty} \{ \mathbb{C}[x_{i, \alpha}, i = 1, \ldots, n, \alpha \geq 0] \otimes U_h \mathfrak{L}_+ \}/(h^N) \) and \( U_h \tilde{\mathfrak{g}}_+ \).
Moreover, the map \( p : U_h\tilde{g}_+ \to \mathcal{V} \) defined by \( p(h_\lambda[\lambda]) = h_\lambda[\lambda]^S \), \( p(c_i[\epsilon]) = \epsilon \in FO_{\alpha_i} \) is a morphism of topological Hopf algebras from \((U_h\tilde{g}_+, \Delta')\) to \((\mathcal{V}, \Delta_\mathcal{V})\).

Let us denote by \( \tilde{g}_+ \) the Lie subalgebra of \( \mathfrak{a} \otimes \mathcal{K} \) equal to \((\mathfrak{h} \otimes \Lambda) \oplus (\mathfrak{n}_+ \otimes \mathfrak{h})\).

The specialization \( \hbar = 0 \) in the quantum Serre relations (26) yields the usual Serre relations, therefore \( U_h\tilde{g}_+/hU_h\tilde{g}_+ \) is isomorphic to the cocommutative Hopf algebra \( U\tilde{g}_+ \). Moreover, the grading of \( \mathfrak{a} \) by the lattice of roots induces a grading of \( \tilde{g}_+ \) by the same lattice.

\( \mathcal{V} \) is a torsion-free, \( \hbar \)-adically complete \( \mathbb{C}[[\hbar]] \)-module, it is therefore topologically free over \( \mathbb{C}[[\hbar]] \). Moreover, \( \mathcal{V}_0 = \mathcal{V}/\mathcal{V}\mathcal{h} \) is a cocommutative Hopf algebra, and \( p \) induces a Hopf co-Poisson algebra map \( p_0 \) from \( U\tilde{g}_+ \) to \( \mathcal{V}_0 \). Let \( i \) be the intersection \( \text{Ker}(p_0) \cap \tilde{g}_+ \). \( i \) is a graded Lie algebra ideal of \( \tilde{g}_+ \), so \( \tilde{g}_+/i \) has a graded Lie algebra structure and \( \mathcal{V}_0 \) is isomorphic to the enveloping algebra \( U(\tilde{g}_+/i) \). One also checks that the intersection of \( i \) with the homogeneous parts of \( \tilde{g}_+ \) of principal degrees zero and one, are zero.

Moreover, \( p_{0|i\tilde{g}_+} \) induces also a Lie bialgebra map from \( \tilde{g}_+ \) to \( \tilde{g}_+/i \). It follows that the dual map \( p_{0|i\tilde{g}_+}^* \), to \( p_{0|i\tilde{g}_+} \) induces a Lie algebra map from \( i^\perp \) to \( \tilde{g}_+^\perp \). But \( \tilde{g}_+^\perp \) is isomorphic to the opposite Lie algebra to \( \tilde{g}_- = (\mathfrak{h} \otimes R) \oplus (\mathfrak{n}_- \otimes \mathcal{K}) \). \( \tilde{g}_- \) is generated by its homogeneous parts of principal degrees zero and one, and the restriction of \( p_{0|i\tilde{g}_+}^* \) to these degrees is an isomorphism, therefore the image of \( p_{0|i\tilde{g}_+}^* \) contains these homogeneous parts of \( \mathfrak{g}_- \). It follows that \( p_{0|i\tilde{g}_+}^* \) is surjective, so \( p_{0|i\tilde{g}_+} \) is injective. Since \( p_{0|i\tilde{g}_+} \) is obviously surjective, it is an isomorphism. Therefore \( i = 0 \), \( \mathcal{V}_0 \) is isomorphic to \( U\tilde{g}_+ \) and \( p_0 \) is an isomorphism. Now \( p \) is a morphism from a \( \hbar \)-adically complete \( \mathbb{C}[[\hbar]] \)-module to a topologically free \( \mathbb{C}[[\hbar]] \)-module, which induces an isomorphism between the associated \( \mathbb{C} \)-vector spaces, therefore \( p \) is an isomorphism.

\[ \square \]

5.2. PBW theorem for \( U_h\mathfrak{g} \).

**Proposition 5.2.** There are unique algebra maps \( i_+, i_0 \) and \( i_- \) from \( U_hL\mathfrak{n}_+ \), \( U_h\mathfrak{h} \) and \( U_hL\mathfrak{n}_- \) to \( U_h\mathfrak{g} \), such that

\[
i_+(e_i[\phi]) = e_i[\phi], \quad i_0(h_i[\phi]) = h_i[\phi], \quad i_0(D) = D, \quad i_0(K) = K, \quad i_+(f_i[\phi]) = f_i[\phi].
\]

The composition of the tensor product \( i_+ \otimes i_0 \otimes i_- \) of these maps with the product map in \( U_h\mathfrak{g} \) is an isomorphism of \( \mathbb{C}[[\hbar]] \)-modules from \( \text{lim}_-N(U_hL\mathfrak{n}_+ \otimes U_h\mathfrak{h} \otimes U_hL\mathfrak{n}_-)/(\hbar^N) \) to \( U_h\mathfrak{g} \).

**Proof.** Let \( U_h\mathfrak{g}' \) and \( U_hL\mathfrak{h}' \) the analogues of the algebras with the same generators and relations as \( U_h\mathfrak{g} \) and \( U_h\mathfrak{h} \), except for generator \( D \). Assume that we have proved the analogue of the statement for \( U_h\mathfrak{g}' \). Then one checks that there is a unique derivation \( \tilde{D} \) of \( U_h\mathfrak{g}' \), such that \( \tilde{D}(K) = 0, \tilde{D}(h_i[r]) = h_i[\partial r], \tilde{D}(\partial r^\perp[\lambda]) = \text{the right side of (13),} \tilde{D}(e_i^\perp[\epsilon]) = \text{the right side of (14)}, \) for \( i = 1, \ldots, n, \) \( r \in R, \lambda \in \Lambda, \epsilon \in \mathcal{K} \). Then \( U_h\mathfrak{g} \) is isomorphic to the semidirect product of \( U_h\mathfrak{g}' \) with \( \tilde{D} \); this implies the triangular decomposition for \( U_h\mathfrak{g} \).
Let us prove triangular decomposition for $U_h\mathfrak{g}'$. Let us denote by $c$ the composition of the algebra maps $U_h\mathfrak{L}_+ \to U_h\mathfrak{g}'$ and $U_h\mathfrak{L}_- \to U_h\mathfrak{g}'$ with the product map. Using relations (36), one can reorder any monomial in the generators of $U_h\mathfrak{g}$ as a sum of monomials in the image of $c$, therefore $c$ is surjective.

Let us construct a left Verma module $V_+$ and a right Verma module $V_-$ over $U_h\mathfrak{g}'$ as follows. Define $U_h\mathfrak{L}_+$ as the algebra with generators $K_i, e_i, h_i$, $i = 1, \ldots, n$, $e, h \in K$, and the relations of $U_h\mathfrak{L}_-$, those of $U_h\mathfrak{L}_+$, (34) and $[K, e_i] = 0$. The composition of the obvious morphisms with product in $U_h\mathfrak{L}_+$ defines an algebra isomorphism of $\lim_{\leftarrow} U_h\mathfrak{L}_+ \otimes U_h\mathfrak{L}_-/ (h^N)$ with $U_h\mathfrak{L}_+$.

As a $\mathbb{C}[\hbar]$-module, $V_+$ is isomorphic to $U_h\mathfrak{L}_+$. The action of generators $e_i, h_i$ and $K$ of $U_h\mathfrak{g}'$ on $V_+$ is the same as left multiplication in $U_h\mathfrak{L}_+$. The action of $f_i$ on the element $K^0 \prod_{s=1}^l h_i^{[\epsilon_s]} \prod_{t=1}^m e_{j_t}[\eta_t]$ is given by

$$f_i[\epsilon] \left( K^0 \prod_{s=1}^l h_i^{[\epsilon_s]} \prod_{t=1}^m e_{j_t}[\eta_t] \right) = \sum_{J \in \{1, \ldots, l\}} \sum_{t=1}^l K^0 \prod_{s=1}^l h_i^{[\epsilon_s]} \frac{\delta_{jj_t}}{\hbar} \prod_{t'=1}^m e_{j_{t'}}[\epsilon_{t'}];$$

one checks that this is a well-defined endomorphism of $V_+$ (i.e. the endomorphisms of the free algebra defined by the same formulas preserve the ideal defining $U_h\mathfrak{L}_+$) and that it makes $V_+$ a left $U_h\mathfrak{g}'$-module.

As a $\mathbb{C}[\hbar]$-module, $V_-$ is isomorphic to $U_h\mathfrak{L}_-$. The action of the generators $f_i$ of $U_h\mathfrak{g}'$ coincide with right multiplication in $U_h\mathfrak{L}_-$. The right actions of $K, h_i, e_i$ are given by $xK = 0$ for $x \in V_-$,

$$\left( \prod_{\sigma=1}^l f_{i_{\sigma}}[\epsilon_{\sigma}] \right) h_i[\epsilon] = \sum_{\sigma=1}^{l-1} \left( \prod_{\sigma'=1}^l f_{i_{\sigma'}}[\epsilon_{\sigma'}] \right) f_{i_{\sigma}}[\epsilon_{\tau} T_{i_{\sigma_{\tau}}}(\epsilon)] \left( \prod_{\sigma'=1}^l f_{i_{\sigma'}}[\epsilon_{\sigma'}] \right),$$

$$\left( \prod_{\sigma=1}^l f_{i_{\sigma}}[\epsilon_{\sigma}] \right) e_i[\epsilon] = \sum_{\sigma=1}^{l-1} \frac{\delta_{i_{\sigma}}}{\hbar} \res_{z_1 \in S} \res_{z_1 \in S} \cdots \res_{z_{\sigma-1} \in S} \left( \prod_{k=1}^l f_{i_k}(z_k) \right).$$

where we use the notation $\res_{z \in S}$ for $\sum_{z \in S} \res_{z \in S}$.

Define $1_{\pm}$ as the vectors of $V_+$ corresponding to $1$ in $U_h\mathfrak{L}_+$ and $U_h\mathfrak{L}_-$. Consider now the sequence of maps

$$\lim_{\leftarrow} U_h\mathfrak{L}_+ \otimes U_h\mathfrak{L}_- \otimes (h^N) \to U_h\mathfrak{g}' \to \lim_{\leftarrow} U_h\mathfrak{g}' \otimes U_h\mathfrak{g}' / U_h\mathfrak{g}' \otimes I_N \to \lim_{\leftarrow} \End(V_+) \otimes \End(V_-)^{opp} / (h^N) \to \lim_{\leftarrow} V_+ \otimes V_- / (h^N) \to \lim_{\leftarrow} U_h\mathfrak{L}_+ \otimes U_h\mathfrak{L}_- / (h^N),$$

QLAI23
where the maps are $c$, the coproduct $\Delta$, the tensor product of the module structures on $V_+$ and on $V_-$, the action on $1_+ \otimes 1_-$ and the isomorphism maps $V_+ \to U_h Lb_+$ and $V_- \to U_h Ln_-$. The composition of these maps sends $x_+ \otimes x_0 \otimes x_-$ to $x_+ x_0 \otimes x_-$ and is therefore injective. It follows that $c$ is also injective. □

5.3. Regular subalgebras. Define $Ln^+_+^+$ as the Lie subalgebra of $Ln_+$ equal to $n_+ \otimes R$.

We will define $U_h Ln^+_+^+$ as the $h$-adically complete subalgebra of $U_h Ln_+$ generated by the $e_i[r]$, $i = 1, \ldots, n$, $r \in R$.

Proposition 5.3. $U_h Ln^+_+^+$ is a divisible subalgebra of $U_h Ln_+$, and $U_h Ln^+_+^+ / hU_h Ln^+_+^+$ is isomorphic to $ULn^+_+^+$.

Proof. Let $\Delta_+$ be the set of positive roots of $n_+$. For any $\alpha \in \Delta_+$, let $e_\alpha$ be a nonzero vector of $n_+$ corresponding to $\alpha$. We may assume that when $\alpha$ is a simple root $\alpha_i$, $e_\alpha$ coincides with the generator $e_i$, and that when $\beta$ is arbitrary, $e_\beta$ has the form $[\bar{e}_{i_1}, [\bar{e}_{i_2}, \ldots, [\bar{e}_{i_{b(\beta)}-1}, e_{b(\beta)}]],$ for some integer $b(\beta)$ and some sequence $(i_1, \ldots, i_{b(\beta)})$ in $\{1, \ldots, n\}$, depending on $\beta$.

Then there are numbers $N_{\beta, \beta'}$ such that $[\bar{e}_\beta, \bar{e}_{\beta'}] = N_{\beta, \beta'} \bar{e}_{\beta + \beta'}$ if $\beta + \beta' \in \Delta_+$ and $[\bar{e}_\beta, \bar{e}_{\beta'}] = 0$ else.

For $\beta$ a non-simple root of $\Delta_+$ and $r \in R$, define $e_\beta[r]$ as the element of $U_h Ln^+_+^+$ given by

$$e_\beta[r] = [e_{i_1}[1], [e_{i_2}[1], \ldots, [e_{i_{b(\beta)}-1}[1], e_{i_{b(\beta)}}[r]]].$$

(40)

Lemma 5.1. The $e_\beta[r]$ defined by (40) are such that for any $\beta, \beta' \in \Delta_+$ and $r, r' \in R$, we have

$$[e_\beta[r], e_{\beta'}[r']] \in N_{\beta, \beta'} e_{\beta + \beta'}[rr'] + hU_h Ln^+_+^+$$

(41)

if $\beta + \beta' \in \Delta_+$,

$$[e_\beta[r], e_{\beta'}[r']] \in hU_h Ln^+_+^+$$

(42)

else.

Proof of Lemma. We will show this in the case $a = \mathfrak{sl}_3$, the general case being similar. Assume that we define $e_{\alpha_1 + \alpha_2}$ as $[\bar{e}_1, \bar{e}_2]$, so

$$e_{\alpha_1 + \alpha_2}[r] = [e_1[1], e_2[r]].$$

(43)

We define the following order on the positive roots of $\mathfrak{sl}_3$: $\alpha_1 < \alpha_1 + \alpha_2 < \alpha_2$. It is clearly enough to prove (11), (12) in the case $\beta \leq \beta'$.

For $r, r' \in R$, $[r(z)r'(w) - r'(z)r(w)]q_2^+(w, z)$ is an element of $(r \otimes r' - r' \otimes r) + h(R \otimes R)[[h]]$. Let us express it as $(r \otimes r' - r' \otimes r) + \sum_{n \geq 0} A_{n; \alpha, \alpha'}(r, r') r_{\alpha} \otimes r_{\alpha'}$, where the maps $(r, r') \mapsto \sum A_{n; \alpha, \alpha'}(r, r') r_{\alpha} \otimes r_{\alpha'}$ are bilinear maps from $R \times R$ to $R \otimes R$. Equation (10) implies that for any element $\phi(z, w)$ of $R \otimes R$, vanishing on the diagonal of $C \times C$, and $i = 1, 2$, we have $\phi(z, w)q_2^+(w, z)e_i(z) = \phi(z, w)
\(\phi(z, w)q^+_1(z, w)e_i(z)e_i(z)\). Set \(\phi = r \otimes r' - r' \otimes r\) and pair the resulting equality with \(1 \otimes 1\) (for \(\langle , \rangle_{\mathcal{K} \otimes \mathcal{K}}\)). We get

\[
[e_i[r], e_i[r']] = - \sum_{n \geq 1, \alpha, \alpha' \geq 0} \hbar^n A_{n, \alpha, \beta}(r, r')e_i[r_\alpha]e_i[r_{\alpha'}].
\]

(44)

This proves (42) in the case \(\beta = \beta' = \alpha_i, i = 1, 2\).

There exist two sequences of bilinear maps \(B_n\) and \(C_n\ (n \geq 1)\) from \(R \times R\) to \(R \otimes R\), such that

\[
[r(z)r'(w) - rr'(w)]q^+_1(w, z) = [r(z)r'(w) - rr'(w)] + \sum_{n \geq 1} \hbar^n B_n(r, r'),
\]

\[
[r(z)r'(w) - rr'(w)]q^+_1(z, w) = [r(z)r'(w) - rr'(w)] + \sum_{n \geq 1} \hbar^n C_n(r, r').
\]

We will write \(B_n(r, r') = \sum_{\alpha, \alpha' \geq 0} B_{n, \alpha, \alpha'}(r, r')r_\alpha \otimes r_{\alpha'}\), \(C_n(r, r') = \sum_{\alpha, \alpha' \geq 0} C_{n, \alpha, \alpha'}(r, r')r_\alpha \otimes r_{\alpha'}\). The identities (38) imply that for \(\phi\) a function of \(R \otimes R\) vanishing on the diagonal of \(C\), we have \(\phi(z, w)q^+_1(w, z)e_1(z)e_2(w) = \phi(z, w)q^+_1(z, w)e_2(w)e_1(z)\). Set \(\phi(z, w) = r(z)r'(w) - rr'(w)\) and pair the resulting equality with \(1 \otimes 1\); we obtain

\[
[e_1[r], e_2[r']] = [e_1[1], e_2[rr']] + \sum_{n \geq 1, \alpha, \alpha' \geq 0} \hbar^n \{C_{n, \alpha, \alpha'}e_2[r_{\alpha'}]e_1[r_\alpha] - B_{n, \alpha, \alpha'}e_1[r_\alpha]e_2[r_{\alpha'}]\}
\]

\[
\quad = e_{\alpha_1 + \alpha_2}[rr'] + \sum_{n \geq 1, \alpha, \alpha' \geq 0} \hbar^n \{C_{n, \alpha, \alpha'}e_2[r_{\alpha'}]e_1[r_\alpha] - B_{n, \alpha, \alpha'}e_1[r_\alpha]e_2[r_{\alpha'}]\},
\]

by (33), which proves (11) in the case \(\beta = \alpha_1, \beta' = \alpha_2\).

One of the quantum Serre relations is written

\[
Ae_1(z_1)e_1(z_2)e_2(w) + Be_1(z_1)e_2(w)e_1(z_2) + Ce_2(w)e_1(z_1)e_1(z_2)
\]

\[
+ A'e_1(z_2)e_1(z_1)e_2(w) + B'e_1(z_2)e_2(w)e_1(z_1) + C'e_2(w)e_1(z_2)e_1(z_1) = 0,
\]

where \(A, \ldots, C'\) are functions of \((z_1, z_2, w)\), \(A, C, A', C'\) belong to \(1 + \hbar R^{\otimes 3}[[\hbar]]\) and \(B, B'\) belong to \(-2 + \hbar R^{\otimes 3}[[\hbar]]\). Let us expand \(A, \ldots, C'\) as \(A = 1 + \sum_{n \geq 1} \hbar^n A_n\), etc., with \(A_n \in R^{\otimes 3}\), and write \(A_n = \sum_{\alpha, \alpha', \alpha'' \geq 0} A_{n, \alpha, \alpha', \alpha''}r_\alpha \otimes r_{\alpha'} \otimes r_{\alpha''}\). Pairing (45) with \(r(z_1)r'(w)\), we get

\[
[e_1[r], [e_1[1], e_2[r']]] = - \frac{1}{2}[[e_1[1], e_1[r]], e_2[r']] - \frac{1}{2} \sum_{n \geq 1, \alpha, \alpha', \alpha'' \geq 0} \hbar^n
\]

\[
(A_{n, \alpha, \alpha', \alpha''}e_1[r_{\alpha'}]e_1[r_\alpha]e_2[r'_{\alpha''}] + B_{n, \alpha, \alpha', \alpha''}e_1[r_{\alpha''}]e_2[r'_{\alpha'}]e_1[r_\alpha])
\]

\[
+ C_{n, \alpha, \alpha', \alpha''}e_2[r'_{\alpha'}]e_1[r_{\alpha''}]e_1[r_\alpha] + A'_{n, \alpha, \alpha', \alpha''}e_1[r_\alpha]e_1[r_{\alpha''}]e_2[r'_{\alpha'}]
\]

\[
+ B'_{n, \alpha, \alpha', \alpha''}e_1[r_{\alpha'}]e_2[r'_{\alpha''}]e_1[r_\alpha] + C'_{n, \alpha, \alpha', \alpha''}e_2[r'_{\alpha''}]e_1[r_{\alpha'}]e_1[r_\alpha]\}.
\]

In view of (14) for \(i = 1\), this proves (42) when \(\beta = \alpha_1\) and \(\beta' = \alpha_1 + \alpha_2\). Using the other Serre relation, one shows that

\[
[e_{\alpha_1 + \alpha_2}[r], e_2[r']] \in \hbar U_h L_n^{out},
\]

(47)
that is (12) when $\beta = \alpha_1 + \alpha_2$ and $\beta' = \alpha_2$.

Applying $[e_1[1], \cdot]$ to (17), we find that $[e_{\alpha_1+\alpha_2}[r], e_{\alpha_1+\alpha_2}[r']]+[[e_1[1], e_{\alpha_1+\alpha_2}[r]], e_2[r']] \in \hU_Ln_+^{\text{out}}$. (46) then implies that $[e_{\alpha_1+\alpha_2}[r], e_{\alpha_1+\alpha_2}[r']] \in \hU_Ln_+^{\text{out}}$, that is (12) for $\beta = \beta' = \alpha_1 + \alpha_2$.

End of proof of Prop. 5.3. Let us denote by $M$ the set of maps $n$ from $\Delta_+ \times \mathbb{N}$ to $\mathbb{N}$, which are zero except on a finite subset of $\Delta_+ \times \mathbb{N}$. Let us fix an order on $\Delta_+ \times \mathbb{N}$ and set

$$ (e_n)_{n \in M} = \prod_{\beta \in \Delta_+, \alpha \geq 0} e_{\beta [r_{\alpha}]} n^{(\beta, \alpha)} [\sum_{\alpha} M] $$

and show that (48) topologically spans $U_Ln_+^{\text{out}}$.

For this, start from a monomial in the $e_i[r], i = 1, \ldots, n, r \in R$ and operate in the same way as one does for $h = 0$ (transforming non-well ordered monomials using commutation relations). By Lemma 5.1, the result will be the sum of a linear combination (with coefficients in $\mathbb{C}$) of elements of the family (48), and of an element of $hU_Ln_+^{\text{out}}$. Decomposing this element as a combination of monomials and repeating this procedure, we find that any element of $U_Ln_+^{\text{out}}$ is the sum of a series $\sum_{n \geq 0} h^n \sum_{n \in M} \theta_{n,a} e_n$, where for fixed $n$, all the $(\theta_{n,a})_{a \in M}$ are zero, except for a finite number of them.

On the other hand, it follows from Thm. 5.1 that (48) is a topologically free family, therefore (48) is a topological basis of $U_Ln_+$. This proves Prop. 5.3.

We defined the Lie algebra $g$ is section 2.

**Proposition 5.4.** $U_hg$ is a topologically free, complete $\mathbb{C}[[h]]$-algebra, and $U_hg/\hU_hg$ is isomorphic to $Ug$.

**Proof.** Let $M'$ (resp., $P'$) be the set of maps from $\Delta_+ \times \mathbb{Z}$ (resp., $\{1, \ldots, n\} \times \mathbb{Z}$) to $\mathbb{N}$, which are zero except on a finite subset. Define, for $e \in K$ and $\lambda, \eta \in \Delta_+$,

$$ e_{\beta [e]} = [e_1[1], e_{i_2[1]}, \ldots, e_{i_{s-1}[1]}, e_{i_s[e]}], \quad f_{\beta [e]} = [f_{i_1[1]}, f_{i_2[1]}, \ldots, f_{i_{s-1}[1]}, f_{i_s[e]}] $$

(see Lemma 5.1) and fix orders in $\Delta_+ \times \mathbb{Z}$ and $\{1, \ldots, n\} \times \mathbb{Z}$.

The analogue of Thm. 5.1 for $U_Ln_-^{\text{out}}$ and Prop. 5.2 imply that

$$ \left( \prod_{\beta \in \Delta_+, \alpha \in \mathbb{Z}} e_{\beta [\epsilon_{\alpha}]} n^{(\beta, \alpha)} K^a D^b \prod_{i=1, \ldots, n, a \in \mathbb{Z}} h_i[\epsilon_{\alpha}] n^{(i, \alpha)} \prod_{\beta \in \Delta_+, \alpha \in \mathbb{Z}} f_{\beta [\epsilon_{\alpha}]} n^{(\beta, \alpha)} \right)_{n, m \in M', p \in P, a, b \in \mathbb{N}} $$

is a topological basis of $U_hg$. This implies the first statement. The second statement follows from the fact that the specialization $h = 0$ in the relations defining $U_hg$ yields the relations defining $Ug$. \hfill \Box

Recall that $g^{\text{out}}$ is the Lie subalgebra of $\mathfrak{g}$ equal to $(\mathfrak{a} \otimes R) \oplus \mathbb{C}D$.

Define now $U_hg^{\text{out}}$ as the $h$-adically complete subalgebra of $U_hg$ generated by $D$ and the $e_i[r], f_i[r]$ and $h_i[r], i = 1, \ldots, n, r \in R$. 
Proposition 5.5. \( U_{\mathfrak{g}^{\text{out}}} \) is a divisible subalgebra of \( U_{\mathfrak{g}} \), and \( U_{\mathfrak{g}^{\text{out}}}/hU_{\mathfrak{g}^{\text{out}}} \) is isomorphic to \( U_{\mathfrak{g}^{\text{out}}} \).

Proof. Let us denote by \( P \) the set of maps from \( \{1, \ldots, n\} \times \mathbb{N} \) to \( \mathbb{N} \), which are zero except on a finite subset. Let us first show that

\[
\left( \prod_{\beta \in \Delta, \alpha \geq 0} e_\beta [r_\alpha]^{\mu(\beta, \alpha)} D^\alpha \prod_{\beta \in \Delta, i \geq 0} h_i [r_\alpha]^{\mu(i, \alpha)} \prod_{\beta \in \Delta, i \geq 0} f_\beta [r_\alpha]^{\mu(\beta, \alpha)} \right),
\]

where \( \Delta \) is a coproduct \( \Delta \) with the permutation of factors. In Section 2, we prove that \( \Delta \) allows to express any ordered monomial of the form (50) as a series in \( \hbar \), whose coefficients are linear combinations of ordered monomials of the form

\[
\prod_{s=1}^{p} e_i [r_s] D^a \prod_{t=1}^{p'} h_j [r'] D^a \prod_{l=1}^{p''} f_k [r''] D^a.
\]

(50)

Now Prop. 5.3, the analogous statement to this Proposition for the subalgebra \( U_{\mathfrak{g}} \) generated by the \( f_i [r], r \in \mathbb{R} \) and the relations \( [h_i [r], h_j [r']] = 0 \) allow to express any ordered monomial of the form (50) as a series in \( \hbar \), whose coefficients are linear combinations of ordered monomials of the form

\[
\prod_{s=1}^{p} e_i [r_s] D^a \prod_{t=1}^{p'} h_j [r'] D^a \prod_{l=1}^{p''} f_k [r''] D^a.
\]

Now Prop. 5.3, the analogous statement to this Proposition for the subalgebra \( U_{\mathfrak{g}} \) generated by the \( f_i [r], r \in \mathbb{R} \) and the relations \( [h_i [r], h_j [r']] = 0 \) allow to express any ordered monomial of the form (50) as a series in \( \hbar \), whose coefficients are linear combinations of ordered monomials of the form

\[
\prod_{s=1}^{p} e_i [r_s] D^a \prod_{t=1}^{p'} h_j [r'] D^a \prod_{l=1}^{p''} f_k [r''] D^a.
\]

(50)

Now Prop. 5.3, the analogous statement to this Proposition for the subalgebra \( U_{\mathfrak{g}} \) generated by the \( f_i [r], r \in \mathbb{R} \) and the relations \( [h_i [r], h_j [r']] = 0 \) allow to express any ordered monomial of the form (50) as a series in \( \hbar \), whose coefficients are linear combinations of monomials (50). This shows that (50) topologically spans \( U_{\mathfrak{g}} \).

Moreover, since (50) can be completed to a topological basis of \( U_{\mathfrak{g}} \), it is a topological basis of \( U_{\mathfrak{g}^{\text{out}}} \), and \( U_{\mathfrak{g}^{\text{out}}} \) is divisible in \( U_{\mathfrak{g}} \).

It follows that \( U_{\mathfrak{g}^{\text{out}}}/hU_{\mathfrak{g}^{\text{out}}} \) is equal to the image of \( U_{\mathfrak{g}^{\text{out}}} \) by the projection map \( U_{\mathfrak{g}} \to U_{\mathfrak{g}}/hU_{\mathfrak{g}} = U_{\mathfrak{g}} \), which is equal to \( U_{\mathfrak{g}^{\text{out}}} \).

\[\square\]

6. The pairings \( \langle \cdot, \cdot \rangle_{U_{\mathfrak{g}} \pm} \) and \( \langle \cdot, \cdot \rangle_{U_{\mathfrak{g}}} \pm \)

Define \( U_{\mathfrak{g}}^{\pm}, U_{\mathfrak{g}}^{\pm} \) as the subalgebras of \( U_{\mathfrak{g}} \) generated by \( D \), the \( h_i [r], r \in \mathbb{R} \) and the \( e_i^\pm [\epsilon], i = 1, \ldots, n, \epsilon \in \mathcal{K} \); and define \( U_{\mathfrak{g}}^{\pm}, U_{\mathfrak{g}}^{\pm} \) as the subalgebras of \( U_{\mathfrak{g}} \) generated by \( K \), the \( h_i [\lambda], i = 1, \ldots, n, \lambda \in \mathbb{R} \) and the \( e_i^\pm [\epsilon], i = 1, \ldots, n, \epsilon \in \mathcal{K} \).

Recall that a Hopf pairing between two Hopf algebras \( (B, \Delta_B) \) and \( (C, \Delta_C) \) is bilinear map \( \langle \cdot, \cdot \rangle_{B \otimes C} \) from \( B \times C \) to a commutative ring over their base ring, such that for any \( b, c \in B \),

\[
\langle b, c \rangle_{B \otimes C} = \sum \langle b^{(1)}, c \rangle_{B \times C} \langle b^{(2)}, c \rangle_{B \times C}.
\]

(51)

where \( \Delta_B (b) = \sum b^{(1)} \otimes b^{(2)} \) and \( \Delta_C (c) = \sum c^{(1)} \otimes c^{(2)} \). We denote by \( \Delta' \) the composition of a coproduct \( \Delta \) with the permutation of factors. In Section 5, we defined two pairs of supplementary subalgebras \( (\mathfrak{g}_+, \mathfrak{g}_-) \) and \( (\mathfrak{g}_+, \mathfrak{g}_-) \) of \( \mathfrak{g} \).
Proposition 6.1. \( U_{\hbar}\mathfrak{g}_\pm \) and \( U_{\hbar}\hat{\mathfrak{g}}_\pm \) are divisible subalgebras of \( U_{\hbar}\mathfrak{g} \), and the quotients \( U_{\hbar}\mathfrak{g}_\pm/hU_{\hbar}\mathfrak{g}_\pm \) and \( U_{\hbar}\hat{\mathfrak{g}}_\pm/hU_{\hbar}\hat{\mathfrak{g}}_\pm \) are isomorphic to \( U_{\hbar}\mathfrak{g}_\pm \) and \( U_{\hbar}\hat{\mathfrak{g}}_\pm \).

Moreover, \((U_{\hbar}\mathfrak{g}_+, \Delta)\) and \((U_{\hbar}\mathfrak{g}_-, \Delta)\) are topological Hopf subalgebras of \((U_{\hbar}\mathfrak{g}, \Delta)\). There is a unique Hopf algebra pairing \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{g}_\pm} : U_{\hbar}\mathfrak{g}_+ \times U_{\hbar}\mathfrak{g}_- \to \mathbb{C}((\hbar)) \) between \((U_{\hbar}\mathfrak{g}_+, \Delta)\) and \((U_{\hbar}\mathfrak{g}_-, \Delta')\), such that
\[
\langle e_i[\epsilon], e_j[\eta] \rangle_{U_{\hbar}\mathfrak{g}_\pm} = \frac{\delta_{ij}}{\hbar} \langle \epsilon, \eta \rangle_{\Delta}, \quad \langle h_i[r], h_j[\lambda] \rangle_{U_{\hbar}\mathfrak{g}_\pm} = \frac{1}{\hbar} \langle T_{ij}(r), \lambda \rangle_{\Delta}, \quad \langle D, K \rangle_{U_{\hbar}\mathfrak{g}_\pm} = \frac{1}{\hbar},
\]
for \( \epsilon, \eta \in \mathcal{K}, r \in R, \lambda \in \Lambda, i, j = 1, \ldots, n \), and the other pairings between the generators of \( U_{\hbar}\mathfrak{g}_+ \) and \( U_{\hbar}\mathfrak{g}_- \) are zero.

In the same way, \((U_{\hbar}\hat{\mathfrak{g}}_+, \tilde{\Delta})\) and \((U_{\hbar}\hat{\mathfrak{g}}_-, \tilde{\Delta})\) are topological Hopf subalgebras of \((U_{\hbar}\hat{\mathfrak{g}}, \tilde{\Delta})\), and there is a unique Hopf algebra pairing \( \langle \cdot, \cdot \rangle_{U_{\hbar}\hat{\mathfrak{g}}_\pm} : U_{\hbar}\hat{\mathfrak{g}}_+ \times U_{\hbar}\hat{\mathfrak{g}}_- \to \mathbb{C}((\hbar)) \) between \((U_{\hbar}\hat{\mathfrak{g}}_+, \tilde{\Delta})\) and \((U_{\hbar}\hat{\mathfrak{g}}_-, \tilde{\Delta}')\), such that
\[
\langle f_i[\epsilon], e_j[\eta] \rangle_{U_{\hbar}\hat{\mathfrak{g}}_\pm} = \frac{\delta_{ij}}{\hbar} \langle \epsilon, \eta \rangle_{\tilde{\Delta}}, \quad \langle h_i[r], h_j[\lambda] \rangle_{U_{\hbar}\hat{\mathfrak{g}}_\pm} = \frac{1}{\hbar} \langle T_{ij}(r), \lambda \rangle_{\tilde{\Delta}}, \quad \langle D, K \rangle_{U_{\hbar}\hat{\mathfrak{g}}_\pm} = \frac{1}{\hbar},
\]
é, \( \eta \in \mathcal{K}, r \in R, \lambda \in \Lambda, i, j = 1, \ldots, n \), and the other pairings between the generators of \( U_{\hbar}\hat{\mathfrak{g}}_+ \) and \( U_{\hbar}\hat{\mathfrak{g}}_- \) are zero.

Proof. Let us denote by \( U_{\hbar}\tilde{\mathfrak{g}} \) the free algebra with the same generators as \( U_{\hbar}\mathfrak{g} \), and coproduct \( \tilde{\Delta} \) given by the formulas defining \( \Delta \). Define \( U_{\hbar}\tilde{\mathfrak{g}}_\pm \) as the subalgebras of \( U_{\hbar}\tilde{\mathfrak{g}} \) with the same generators as \( U_{\hbar}\mathfrak{g}_\pm \). \( U_{\hbar}\tilde{\mathfrak{g}}_\pm \) are Hopf subalgebras of \( (U_{\hbar}\tilde{\mathfrak{g}}, \tilde{\Delta}) \) and there is a unique Hopf pairing between \((U_{\hbar}\tilde{\mathfrak{g}}_+, \tilde{\Delta})\) and \((U_{\hbar}\tilde{\mathfrak{g}}_-, \tilde{\Delta}')\), defined by formulas (53). Computation shows that the ideals generated by the relations defining \( U_{\hbar}\mathfrak{g}_\pm \) are contained in the kernel of this pairing. The argument is the same in the case of \( U_{\hbar}\hat{\mathfrak{g}}_\pm \).

Proposition 6.2. The pairings \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{g}_\pm} \) and \( \langle \cdot, \cdot \rangle_{U_{\hbar}\hat{\mathfrak{g}}_\pm} \) are nondegenerate (i.e., \( (U_{\hbar}\mathfrak{g}_\pm)^\perp = 0 \) and \( (U_{\hbar}\hat{\mathfrak{g}}_\pm)^\perp = 0 \)).

Proof. Define \( U_{\hbar}\mathfrak{h}_+ \) as the \( \hbar \)-adically complete subalgebra of \( U_{\hbar}\mathfrak{g}_+ \) generated by \( D \) and the \( h_i[r], r \in R, i = 1, \ldots, n \), and by \( U_{\hbar}\mathfrak{h}_- \) as the \( \hbar \)-adically complete subalgebra of \( U_{\hbar}\mathfrak{g}_- \) generated by \( K \) and the \( h_i[\lambda], \lambda \in \Lambda, i = 1, \ldots, n \).

Inclusion followed by multiplication induces isomorphisms between the completed tensor products \( \lim_{\leftarrow} N U_{\hbar}\mathfrak{h}_{\pm} \otimes U_{\hbar}\mathfrak{h}_{\pm}/(\hbar^N) \) and \( U_{\hbar}\mathfrak{g}_{\pm} \). Moreover, the image of \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{g}_{\pm}} \) by the product of these isomorphisms is the tensor product of its restrictions \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{h}_{\pm}} \) and \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{h}_+ \mathfrak{h}_-} \) to \( U_{\hbar}\mathfrak{h}_{\pm} \otimes U_{\hbar}\mathfrak{h}_{\pm} = U_{\hbar}\mathfrak{h}_+ \otimes U_{\hbar}\mathfrak{h}_- \).

It is easy to see that the pairing \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{h}_{\pm}} \) is nondegenerate. Let \( I_{\hbar,k} \) be the left ideal of \( U_{\hbar}\mathfrak{h}_- \) generated by the \( h_i[\zeta^\alpha] \geq k \). The pairing \( \langle \cdot, \cdot \rangle_{U_{\hbar}\mathfrak{h}_{\pm}} \) defines a canonical element of \( \lim_{\leftarrow} N U_{\hbar}\mathfrak{h}_+ \otimes U_{\hbar}\mathfrak{h}_-/(U_{\hbar}\mathfrak{h}_+ \otimes I_{\hbar,k} + \hbar^N U_{\hbar}\mathfrak{h}_+ \otimes U_{\hbar}\mathfrak{h}_-) \) equal to
\[
R_{\hbar} = q^{D \otimes K} \exp \left( \hbar \sum_{i,j=1}^n \sum_{\alpha \geq 0} h_i[T_{ij}^\alpha r^\alpha] \otimes h_j[\lambda^\alpha] \right).
\]
Let us show that the pairing \( \langle \ , \rangle_{\mathcal{F}O \times U_-} : \mathcal{F}O \times U_- \to \mathbb{C}(\mathfrak{h}) \), such that \\
\[
\langle P, f_1[\epsilon_1](\text{free}) \cdots f_n[\epsilon_N](\text{free}) \rangle_{U_+ \times \mathcal{F}O_-} = \delta_{k,\sum_{j=1}^{N} \delta_{ij}} \text{res}_{u_i \in S} \cdots \text{res}_{u_1 \in S} 
\]
where we set \( t_{i_1+\cdots+i_{a-1}+j} = t_{j}^{(a)} \) for \( a = 1, \ldots, n \) and \( j = 1, \ldots, i_{a} \), and \( u_s = t_{i_{a+1}}^{(i_{a})} \), where \( u_s \) is the number of indices \( t \) such that \( t < s \) and \( i_t = i_s \).

The two-sided ideal \( I_- \) generated by the crossed vertex relations (27) and (28), and the Serre relations (29) is contained in the kernel of this pairing, so \( \langle \ , \rangle_{\mathcal{F}O \times U_-} \) induces a pairing \( \langle \ , \rangle_{\mathcal{F}O \times U_-} \). On the other hand, the proof of Thm. 1 implies that the morphism from \( U_h \mathfrak{L} \mathfrak{n}_+ \) to \( \mathcal{F}O \), sending \( e_i[\epsilon] \) to \( \epsilon \in \mathcal{F}O_{\alpha_i} \), is an isomorphism. Via this isomorphism, \( \langle \ , \rangle_{\mathcal{F}O \times U_-} \) identifies then with \( \langle \ , \rangle_{U_h \mathfrak{L} \mathfrak{n}_+} \).

On the other hand, if \( f \) is a formal series in \( \mathbb{C}(\langle t_1 \rangle) \cdots (\langle t_N \rangle) \) such that for any \( \omega_1, \ldots, \omega_n \in \mathbb{C}(\langle t \rangle) dt, \text{res}_{t_N \in S} \cdots \text{res}_{t_1 \in S} (f \omega_1(t_1) \cdots \omega_N(t_N)) = 0 \), then \( f \) is zero. It follows that the annihilator of \( U_- \) for \( \langle \ , \rangle_{\mathcal{F}O \times U_-} \) is zero. So the annihilator of \( U_h \mathfrak{L} \mathfrak{n}_- \) for \( \langle \ , \rangle_{U_h \mathfrak{L} \mathfrak{n}_+} \) is also zero. In the same way, one proves that the annihilator of \( U_h \mathfrak{L} \mathfrak{n}_+ \) for \( \langle \ , \rangle_{U_h \mathfrak{L} \mathfrak{n}_+} \) is zero, therefore \( \langle \ , \rangle_{U_h \mathfrak{L} \mathfrak{n}_+} \) is nondegenerate.

\[ \square \]

7. The annihilator of \( U_h \mathfrak{L} \mathfrak{n}_+^{\text{out}} \)

**Lemma 7.1.** The restrictions to \( U_h \mathfrak{L} \mathfrak{n}_+ \otimes U_h \mathfrak{L} \mathfrak{n}_- \) of \( \langle \ , \rangle_{U_h \mathfrak{g}_\pm} \) and \( \langle \ , \rangle_{U_h \mathfrak{g}_\pm} \) coincide.

We will denote by \( \langle \ , \rangle_{U_h \mathfrak{L} \mathfrak{n}_\pm} \) the restriction of any of these pairings to \( U_h \mathfrak{L} \mathfrak{n}_+ \times U_h \mathfrak{L} \mathfrak{n}_- \).

The aim of this section is to compute the annihilator of \( U_h \mathfrak{L} \mathfrak{n}_+^{\text{out}} \) for this pairing. Let us set, for \( x \in U_h \mathfrak{g}_+^\pm \),
\[
\delta_1(x) = x, \quad \delta_2(x) = \Delta(x) - x \otimes 1 - 1 \otimes x + \varepsilon(x)1,
\]
and \( \delta_n = (\delta_2 \otimes id_{U_h \mathfrak{g}_+^\pm}) \circ \delta_{n-1} \). \( \delta_n \) is a linear map from \( U_h \mathfrak{g}_+^\pm \) to \( U_h \mathfrak{g}_+^\pm \), and its restriction to \( U_h \mathfrak{L} \mathfrak{n}_+ \) maps to \( U_h \mathfrak{L} \mathfrak{n}_+ \).\( \hat{U}_h \mathfrak{g}_+^{\otimes n-1} \).

Let us define \( U_{QSFH} \) as \( \{ x \in U_h \mathfrak{g}_+^\pm | \forall n \geq 0, \delta_n(x) \in h^n U_h \mathfrak{g}_+^\pm \} \), and \( U_{QSFH} \) as \( U_{QSFH} \cap U_h \mathfrak{L} \mathfrak{n}_+ \) (see [4], Sect. 6). Let us define \( \mathfrak{f} \), resp., \( \mathfrak{g} \) as the \( h \)-adic completions of the \( \mathbb{C}[[h]] \)-Lie subalgebras of \( U_h \mathfrak{g}_+ \) generated by the \( e_i[\phi], i = 1, \ldots, n, \phi \in \mathcal{K} \), resp., by \( \mathfrak{f} \) and the \( h_i[r], r \in R \). The map \( \mod h : U_h \mathfrak{g}_+^\pm \to U_h \mathfrak{g}_+^\pm \) induces surjective \( \mathbb{C} \)-Lie algebra maps from \( \mathfrak{g} \) to \( \mathfrak{g}_+^\pm \) and from \( \mathfrak{f} \) to \( \mathfrak{L} \mathfrak{n}_+^\pm \). Let \( \overline{\alpha}_\cdot \) be an element of \( \mathfrak{n}_+ \) associated with the root \( \alpha \). Let us fix a \( \mathbb{C} \)-linear section \( \sigma \).
of the first map, such that for \( \alpha \) in \( \Delta_+ \), \( r \) in \( R \), \( \sigma (\bar{e}_\alpha \otimes r) \) belongs to \( U_h L^{0}_{n_+} \) and denote by \( \bar{L}_{n_+} \) and \( \bar{g}_+ \) the \( h \)-adic completions of the \( \mathbb{C}[[h]] \)-submodules of \( U_h L^{0}_{n_+} \) generated by \( \sigma (f) \) and \( \sigma (g) \).

Let us denote by \( A_0 \) and \( A \) the \( h \)-adic completions of the subalgebras of \( U_h L^{0}_{n_+} \) generated by \( h L^{0}_{n_+} \) and \( h f \). Denote by \( \bar{A}_0 \) and \( \bar{A} \) the \( h \)-adic completions of the subalgebras of \( U_h \bar{g}_+ \) generated by \( h \bar{g}_+ \) and \( h \bar{g} \).

**Lemma 7.2.** We have \( A_0 = A = U^{QFS H} \), and \( \bar{A}_0 = \bar{A} = U^{QFS H} \).

**Proof.** We will prove the first chain of equalities; the proof of the second one is similar. For this, we will show the inclusions \( A_0 \subset A \subset U^{QFS H} \subset A_0 \).

The inclusion \( A_0 \subset A \) is clear. Let us show that

\[
\delta_2 (f) \subset \left( \sum_{i+j>1} h^{i+j-1} f^{\leq i} \otimes g^{\leq j} \right)^{\text{compl}}. \tag{52}
\]

(52) means that for any \( x \) in \( f \), we have

\[
\Delta (x) \in x \otimes 1 + 1 \otimes x + \left( \sum_{i+j>1} h^{i+j-1} f^{\leq i} \otimes g^{\leq j} \right)^{\text{compl}}. \tag{53}
\]

It suffices to show (53) for \( x \) a Lie expression in the \( e_i [\phi] \). We then work by induction on the length of \( x \). (53) is obvious for \( x = e_i [\phi] \). Assume that (53) is true for \( x \) and \( y \), then \( \Delta ([x, y]) = [\Delta (x), \Delta (y)] \) is contained in the space

\[
[x \otimes 1 + 1 \otimes x + \left( \sum_{i+j>1} h^{i+j-1} f^{\leq i} \otimes g^{\leq j} \right)^{\text{compl}}, y \otimes 1 + 1 \otimes y + \left( \sum_{i+j>1} h^{i+j-1} f^{\leq i} \otimes g^{\leq j} \right)^{\text{compl}}]. \tag{54}
\]

Since \( [x, f^{\leq i}] \subset f^{\leq i} \), \( [x, g^{\leq j}] \subset g^{\leq j} \), \( [f^{\leq i}, f^{\leq j}] \subset f^{\leq i+j-1} \), \( [g^{\leq i}, g^{\leq j}] \subset g^{\leq i+j-1} \), the space (54) is contained in

\[
[x, y] \otimes 1 + 1 \otimes [x, y] + \left( \sum_{i+j>1} h^{i+j-1} f^{\leq i} \otimes g^{\leq j} \right)^{\text{compl}}.
\]

This implies (53) and therefore (52).

It follows then from (53) that for any \( n \geq 1 \), we have

\[
\delta_2 (f^{\leq n}) \subset \left( \sum_{i+j\geq n} h^{i+j-n} f^{\leq i} \otimes g^{\leq j} \right)^{\text{compl}}. \tag{55}
\]

It follows then from (55) that we have, if \( 1 \leq k \leq n - 1 \),

\[
\delta_k (f^{\leq n}) \subset \left( \sum_{i\geq 0} h^i f^{\leq n-k+i} \otimes U_h \hat{g}_+^{k-1} \right)^{\text{compl}},
\]

and then that if \( k \geq 0 \)

\[
\delta_{n+k} (f^{\leq n}) \subset \left( \sum_{i\geq 0} h^k f^{\leq i} \otimes U_h \hat{g}_+^{k-1} \right)^{\text{compl}}. \tag{56}
\]

The inclusion

\[
\delta_k (h^n f^{\leq n}) \subset h^k U_h \hat{g}_+^{k}
\]
is evident for $k \leq n$, and it is also true for $k \geq n$, by (56). It follows that $h^a f^{\leq n}$ is contained in $U^{QFSH}$, therefore $A \subset U^{QFSH}$.

Finally, let $x$ belong to $U^{QFSH}$. Assume $x$ is nonzero. Let $k$ be the $h$-adic valuation of $x$. Let us set $\bar{x} = h^{-k} x \pmod{h}$; $\bar{x}$ belongs then to $UL_n$. Let us denote by $(UL_n)_p$ the subspace of $UL_n$ spanned by the products of $\leq p$ elements of $L_n$. There is a unique integer $p$ such that $\bar{x}$ belongs to $(UL_n)_p \setminus (UL_n)_{p-1}$. Let $\delta_n^{(0)}$ be the classical limits of the maps $\delta_n$. $\delta_n^{(0)}$ are maps from $UL_n$ to $(UL_n)^{\otimes n}$, defined by $\delta_1 = id_{UL_n}$, $\delta_2^{(0)}(x) = \Delta_{UL_n}(x) - x \otimes 1 - 1 \otimes x + \varepsilon(x)1$, $\delta_n^{(0)} = (\delta_2^{(0)} \otimes id_{UL_n^{n-2}}) \circ \delta_n^{(0)}$. We have $\text{Ker}(\delta_n^{(0)}) = (UL_n)_{\leq p-1}$, and the restriction of $\delta_n^{(0)}$ to $(UL_n)_{\leq p}/(UL_n)_{\leq p-1}$ is injective.

Therefore, $\delta_n^{(0)}(\bar{x})$ is a nonzero element of $(UL_n)_p$. The fact that $x$ belongs to $U^{QFSH}$ then implies that $k \geq p$.

Let us now show that $U^{QFSH}$ is contained in $(\sum_{p \geq 0} h^p \widehat{L_n}^{\leq p})^{\text{compl}}$. For $x$ as above, let $y$ belong to $\widehat{L_n}^{\leq p}$ be such that $y \pmod{h} = \bar{x}$. Then $x - h^k y$ belongs to $U^{QFSH}$ and has $h$-adic valuation $> k$. Repeating this reasoning, we express $x$ as a formal series in $(\sum_{p \geq 0} h^p \widehat{L_n}^{\leq p})^{\text{compl}}$. Therefore $U^{QFSH} \subset A_0$. It follows that $A_0 = A = U^{QFSH}$.

**Proposition 7.1.** $U^{QFSH}$ is a $h$-adically complete topologically free subalgebra of $U_h L_n$. The smallest divisible submodule of $U_h L_n$ containing $U^{QFSH}$ is $U_h L_n$ itself.

Let $S[[L_n]]$ be the completion of the symmetric algebra $S(L_n)$ with respect to the topology defined by the ideal $\bigoplus_{i > 0} S_i(L_n)$; $S[[L_n]]$ is equal to the direct product $\prod_{i \geq 0} S_i(L_n)$. Set

$$e_\alpha[\phi]^{FSH} = \text{the class of } h\sigma(e_\alpha \otimes \phi) \text{ in } U^{QFSH}/hU^{QFSH},$$

for any $\alpha$ in $\Delta_+$ and $\phi$ in $K$. There is a unique algebra isomorphism $i_\sigma : S[[L_n]] \rightarrow U^{QFSH}/hU^{QFSH}$ sending $e_\alpha \otimes \phi$ to $e_\alpha[\phi]^{FSH}$.

**Proof:** $U_h L_n$ is a topologically free, countably generated $\mathbb{C}[[h]]$-module, and $U^{QFSH}$ is a complete $\mathbb{C}[[h]]$-submodule of $U_h L_n$. It follows from e.g. [4], Lemma A.2 that $U^{QFSH}$ topologically free and countably generated. The second statement follows from Thm. 5.1.

We have $[f^{\leq n}, f^{\leq m}] \subset f^{\leq n+m-1}$. Therefore, $[A, A] \subset hA$. It follows that $A/hA$ is commutative.

Therefore, there is an algebra morphism $j$ from $S(L_n)$ to $U^{QFSH}/hU^{QFSH}$ defined by $j(e_\alpha \otimes \phi) = \text{the class of } h\sigma(e_\alpha \otimes \phi)$. Since $U^{QFSH}$ is $h$-adically complete, $j$ can be prolonged to an algebra morphism $\bar{j}$ from $S[[L_n]]$ to $U^{QFSH}/hU^{QFSH}$.

It follows from Thm. 5.3 that a topological basis of $A_0/hA_0$ is formed by the products $\{\prod_{\alpha \in \Delta_+, i \in \mathbb{Z}} (e_\alpha[\phi])^{FSH})^{n(\alpha,i), n(\alpha,i) \geq 0}\}$. Since $A_0 = U^{QFSH}$, $\bar{j}$ is an isomorphism. □
Lemma 7.3. The restriction \( \langle \cdot, \cdot \rangle_{U_0 U_h L_n} \) of \( \langle \cdot, \cdot \rangle_{U_0 U_h L_n} \) to \( U_0 U_h L_n \) has values in \( \mathbb{C}[[h]] \).

Proof. For any \( i = 1, \ldots, n \) and \( \phi \) in \( K \), we have \( \langle e_i[\phi], U_0 U_h L_n \rangle \subset \frac{1}{h} \mathbb{C}[[h]] \). On the other hand, if \( x \) and \( y \) in \( U_0 U_h L_n \) are such that

\[
\langle x, U_0 U_h L_n \rangle \subset \frac{1}{h} \mathbb{C}[[h]] \quad \text{and} \quad \langle y, U_0 U_h L_n \rangle \subset \frac{1}{h} \mathbb{C}[[h]],
\]

then for any \( z \) in \( U_0 U_h L_n \), \( \langle x, y, z \rangle U_0 U_h L_n \) is equal to \( \langle x \otimes y, (\Delta - \Delta')(z) \rangle U_0 U_h L_n \), and since \( (\Delta - \Delta')(U_0 U_h L_n) \subset h(U_0 U_h L_n) \), \( \langle x, y, z \rangle U_0 U_h L_n \) is in \( \frac{1}{h} \mathbb{C}[[h]] \).

It follows that \( \langle f, U_0 U_h L_n \rangle \subset \mathbb{C}[[h]] \). Since \( U_0 U_h L_n = A \), we get

\[
\langle U_0 U_h L_n, U_0 U_h L_n \rangle \subset \mathbb{C}[[h]].
\]

\( \Box \)

Let \( m_{QFSH} \) and \( m'_{QFSH} \) be the maximal ideals of \( U_{QFSH} \) and \( U'_{QFSH} \). We have \( m_{QFSH} = (\sum_{n>0} h^n f_{\leq n})^{\text{compl}} \) and \( m'_{QFSH} = (\sum_{n>0} h^n g_{\leq n})^{\text{compl}} \).

Let \( i_{QFSH} \) and \( i'_{QFSH} \) be the right ideals of \( U_{QFSH} \) and \( U'_{QFSH} \) generated by the \( h e_\alpha[\phi] \), with \( \alpha \in \Delta_+, \phi \in z^N \mathbb{C}[[z]] \).

Then (55) implies that the restriction \( \Delta_{U_0 U_h L_n} \) of \( \Delta \) to \( U_0 U_h L_n \) induces a map \( \Delta_{U_{QFSH}} : U_{QFSH} \to \lim_{n} U_{QFSH} \otimes U'_{QFSH} / (\sum_{p,q>n} m_{QFSH}^p \otimes m_{QFSH}^q + i_n \otimes U'_{QFSH} + U_{QFSH} \otimes i'_n) \).

Let us denote by \( m_0 \) and \( m'_0 \) the maximal ideals of \( S[[L_n]] \) and \( S[[g_n]] \), and by \( i_n \) and \( i'_n \) the ideals of \( S[[L_n]] \) and \( S[[g_n]] \) generated by the \( e_\alpha[\phi] \), \( \phi \in z^n \mathbb{C}[[z]] \), then \( \Delta_{U_{QFSH}} \) induces an algebra morphism \( \Delta_{FSH} \) from \( S[[L_n]] \) to \( \lim_{n} S[[L_n]] \otimes S[[g_n]] / (\sum_{p,q>n} m_0^p \otimes m_0^q + i_n \otimes S[[L_n]] + S[[g_n]] \otimes i'_n) \).

Define \( \Delta_{FSH}^{(0)} \) as the composition \( (id \otimes \pi) \circ \Delta_{FSH} \), where \( \pi \) is the morphism from \( U_{QFSH} \otimes U'_{QFSH} \) to \( U_{QFSH} / h U_{QFSH} \) sending each \( e_\alpha[\phi] \) to \( e_\alpha[\phi] \) and \( h_i[\phi] \) to 0.

Then \( \Delta_{FSH}^{(0)} \) induces the structure of ring of a topological formal group on \( S[[L_n]] \) (again topological means that the tensor powers of \( S[[L_n]] \) are completed with respect to the topology defined by the \( i_n \)).

We have then

\[
\Delta_{FSH}^{(0)}(e_\alpha[t^n]_{FSH}) = e_\alpha[t^n]_{FSH} \otimes 1 + 1 \otimes e_\alpha[t^n]_{FSH}
\]

\[
+ \sum_{q>1, \beta_i \in \Delta_+} \sum_{\sum \beta_i = \alpha, k_i \in \mathbb{Z}} \lambda((\beta_i), n, k_i) e_{\alpha_1}[t^k_1]_{FSH} \cdots e_{\alpha_p}[t^k_p]_{FSH}
\]

\[
\otimes e_{\alpha_{p+1}}[t^k_{p+1}]_{FSH} \cdots e_{\alpha_q}[t^k_q]_{FSH}.
\]

(57)
It follows from the identities $\langle x, fh[g] \rangle_{U^{QFSH} \times U^{Ln}} = 0$ when $x \in U^{QFSH}$ and $f \in U^{g}$ that $\langle \ , \ \rangle_{U^{QFSH} \times U^{Ln}}$ satisfies the Hopf pairing rules

$$\langle xx', f \rangle_{U^{QFSH} \times U^{Ln}} = \sum \langle x, f^{(1)} \rangle_{U^{QFSH} \times U^{Ln}} \langle x', f^{(2)} \rangle_{U^{QFSH} \times U^{Ln}};$$

$$\langle x, fg \rangle_{U^{QFSH} \times U^{Ln}} = \sum \langle x^{(1)}, f \rangle_{U^{QFSH} \times U^{Ln}} \langle x^{(2)}, g \rangle_{U^{QFSH} \times U^{Ln}}. \quad (58)$$

(57) and (58) then imply that

$$\langle S^n[[L^n_+]], (U^{Ln}_--)_{\leq n-1} \rangle_{U^{QFSH} \times U^{Ln}} = 0.$$ 

A topological basis of $U^{Ln}$ is the family $(\prod_{\alpha \in \Delta_+} \ell \in \mathbb{Z} f_{\alpha} [\epsilon_i]^{n_{\alpha,i}})$ where all but a finite number of $n_{\alpha,i}$ are zero. Let us set $S^{\geq k}[[L^n_+]] = \prod_{i \geq k} S[[L^n_+]], S^{\geq k}[[L^n_+]] = \prod_{i \geq k} S_0[[L^n_+]]$. There exist $\phi_{(n_{\alpha,i})}$ in $S^{\geq \sum n_{\alpha,i}}[[L^n_+]]$, such that the dual basis of $(\prod_{\alpha \in \Delta_+} \ell \in \mathbb{Z} f_{\alpha} [\epsilon_i]^{n_{\alpha,i}})$ is $(\prod_{\alpha \in \Delta_+} \ell \in \mathbb{Z} (e_{\alpha} [\epsilon_i]^{FS})^{n_{\alpha,i}} + \phi_{(n_{\alpha,i})})$.

It follows that the annihilator of $U^{Ln}$ is the span of all $\prod_{\alpha \in \Delta_+} \ell \in \mathbb{Z} (e_{\alpha} [\epsilon_i]^{FS})^{n_{\alpha,i}} + \phi_{(n_{\alpha,i})}$ where $n_{\alpha,i}$ is nonzero for at least one $i < 0$.

On the other hand, the Hopf pairing rules imply that $\sum_{1 \leq i \leq n,r \in R} e_i[r] U^{Ln}$ is contained in the annihilator of $U^{Ln}$ for $\langle \ , \ \rangle_{U^{Ln}}$. The former space is also $\sum_{\alpha \in \Delta_+} \ell \in \mathbb{Z} \sigma(e_{\alpha} \otimes r) U^{Ln}$, therefore $\sum_{\alpha \in \Delta_+} \ell \in \mathbb{Z} e_{\alpha} [\epsilon_i]^{FS} S[[L^n_+]]$ is contained in the annihilator of $U^{Ln}$.

**Proposition 7.2.** The spaces $U^{Ln}$ and $\sum_{1 \leq i \leq n,r \in R} e_i[r] U^{Ln}$ are each other’s annihilators for the pairing $\langle \ , \ \rangle_{U^{Ln}}$.

**Proof.** Let us denote by $(U^{Ln})^\perp$ the annihilator of $U^{Ln}$ in $U^{Ln}$. We want to show that it is equal to $\sum_{1 \leq i \leq n,r \in R} e_i[r] U^{Ln}$.

Let us fix $x$ in $(U^{Ln})^\perp$. Assume that $x$ is nonzero and let $j$ be the integer such that $h^j x$ lies in $U^{QFSH} \setminus h U^{QFSH}$. Then $h^j x$ mod $h$ lies in $S[[L^n_+]]$ and is the annihilator of $U^{Ln}$. Let $p$ be the smallest integer such that $h^j x$ lies in $S^{\geq p}[[L^n_+]]$. Since $(h^j x \mod h)$ mod $S^{\geq p}[[L^n_+]]$ is also in the annihilator of $U^{Ln}$, it is a linear combination of classes modulo $S^{\geq p}[[L^n_+]]$ of products of the form $\prod_{i=-\infty}^{\infty} \prod_{\alpha \in \Delta_+} (e_{\alpha} [\epsilon_i]^{FS})^{n_{\alpha,i}}$ where $n_{\alpha,i}$ is nonzero for at least one $i < 0$, and the sum of all $n_{\alpha,i}$ is $p$. Let us use the same coefficients of the products $\prod_{i=-\infty}^{\infty} \prod_{\alpha \in \Delta_+} (e_{\alpha} [\epsilon_i]^{FS})^{n_{\alpha,i}}$ to $h^j x$, and call the resulting element $x_1$. Then $(x_1 \mod h)$ belongs to $S^{\geq p}[[L^n_+]]$. We can repeat this procedure with $x_1$; the number of steps is finite, because all elements of the sequence $(x_i)$ have the same degree (in the root lattice).

This shows that for any $x$ in $(U^{Ln})^\perp$, we can find $y \in U^{Ln}$ and an integer $k \geq 0$ such that $h^k y \in \sum_{1 \leq i \leq n,r \in R} e_i[r] U^{Ln}$ and $x - y$ belongs to $h(U^{Ln})^\perp$. It follows that $(U^{Ln})^\perp$ is equal to space of elements $x$ of $U^{Ln}$ such that for some integer $l \geq 0$, $h^l x$ belongs to $\sum_{1 \leq i \leq n,r \in R} e_i[r] U^{Ln}$; in other words, $x$ belongs to the smallest divisible submodule of $U^{Ln}$ containing $\sum_{1 \leq i \leq n,r \in R} e_i[r] U^{Ln}$.
Now, \( \sum_{1 \leq i \leq n, r \in R} e_i[r] U_h L_{n+} \) is equal to \( \sum_{\alpha \in \Delta_+} \alpha \in R} e_\alpha[r] U_h L_{n+} \), which is a flat deformation of \( \sum_{\alpha \in \Delta_+} \alpha \in R} e_\alpha[r] L_{n+} \) by Thm. [4.7]. It follows that \( \sum_{1 \leq i \leq n, r \in R} e_i[r] U_h L_{n+} \) is a divisible submodule of \( U_h L_{n+} \), and is therefore equal to \( (U_h L_{n+})^\perp = U_h L_{n^+} \).

The same argument shows that \( (\sum_{1 \leq i \leq n, r \in R} e_i[r] U_h L_{n+})^\perp = U_h L_{n^-} \). □

**Corollary 7.1.** The spaces \( U_h L_{n^+} \) and \( \sum_{1 \leq i \leq n, r \in R} U_h L_{n-} f_i[r] \) are each other’s annihilators for \( \langle \ , \ \rangle_{U_h L_{n\pm}} \).

8. **The Universal \( R \)-matrices of \( (U_h g, \Delta) \) and \( (U_h \bar{g}, \bar{\Delta}) \)**

In this section, we construct the universal \( R \)-matrices of \( (U_h g, \Delta) \) and \( (U_h \bar{g}, \bar{\Delta}) \). These \( R \)-matrices are products of the Cartan \( R \)-matrix \( \mathcal{R}_h \) with the canonical element \( F \) associated with the pairing between \( U_h L_{n+} \) and \( U_h L_{n-} \). \( F \) belongs to a completion of \( U_h L_{n+} \otimes U_h L_{n-} \). We construct \( F \) in Section 8.1, and derive its properties and the \( R \)-matrices in Section 8.2.

8.1. **Construction of \( F \).** In this section, we construct the element \( F \). Let us set \( A = U_h L_{n+}, B = U_h L_{n-}, A_{out} = U_h L_{n+}^\perp = U_h L_{n^+} \). Then \( F \) is a product \( F_2 F_{int} F_1 \), where \( F_1 \) and \( F_2 \) are “semiinfinite” elements in completions of \( \Lambda \otimes B_2 \), and of \( A_1 \otimes \Lambda \), and \( F_{int} \) is an “intermediate” element in \( A_2 \otimes B_1 \). The elements \( F_1, F_{int} \) and \( F_2 \) are determined by lifts \( \tau_A \) and \( \tau_B \) of the canonical maps \( A \to A_2 = A^{in} \) and \( B \to B_1 = B^{in} \), where \( A^{in} = \mathbb{C}[[h]] \otimes A^{out} A \) and \( B^{in} = B \otimes B^{out} \mathbb{C}[[h]] \), enjoying certain properties with respect to the coproduct.

This section is organized as follows. We first define the completions in which we will work (Section 8.1.1). In Section 8.1.2, we construct canonical elements of completions of \( A^{in} \otimes B_1 \) and \( A_2 \otimes B^{in} \). In Section 8.1.3, we construct the maps \( \tau_A \) and \( \tau_B \). In Section 8.1.4, we construct \( F_1, F_{int} \) and \( F_2 \). Finally (Section 8.1.5), we show that the product \( F = F_2 F_{int} F_1 \) is the canonical element for the pairing between \( A \) and \( B \).

8.1.1. **Completions.**

Let \( I_N^{(A)} \) (resp., \( I_N^{(B)} \)) denote the left ideal of \( A \) (resp., \( B \)) generated by \( e_i[z_i^t], i = 1, \ldots, n, l \geq N \) (resp., by \( f_i[z_i^t], i = 1, \ldots, n, l \geq N \)).

Define \( A^{in} \) as the tensor product \( \lim_{\to \mathbb{C}[[h]] \otimes A^{out} A} \mathbb{C}[[h]] \otimes A^{out} A \), where the left \( A^{out} \)-module structure on \( \mathbb{C}[[h]] \) is provided by the counit map. In the same way, define \( B^{in} \) as the tensor product \( \lim_{\to \mathbb{C}[[h]] \otimes B^{out} \mathbb{C}[[h]]} \mathbb{C}[[h]] \otimes B^{out} \mathbb{C}[[h]] \). Let \( p^{(A)}_N \) and \( p^{(B)}_N \) be the canonical projection maps from \( A \) to \( A^{in} \) and from \( B \) to \( B^{in} \). Let us set

\[
I_N^{(A^{in})} = p^{(A)}_N (I_N^{(A)}), \quad I_N^{(B^{in})} = p^{(B)}_N (I_N^{(B)}).
\]

As are \( U_h L_{n \pm} \), the modules \( A^{out}, B^{in} \), etc., are graded by \( \mathbb{N}^n \) (recall that the degree of \( e_i[z_i^t] \) is \( \alpha_i \) and the degree of \( f_i[z_i^t] \) is \( -\alpha_i \)). Moreover, \( \langle \ , \ \rangle_{U_h L_{n \pm}} \) is a graded pairing. For \( M \) a \((\pm \mathbb{N})^n\)-graded module, and for \( \alpha \in (\pm \mathbb{N})^n \), we denote by \( M[\alpha] \) the homogeneous component of \( M \) of degree \( \alpha \).
Lemma 8.1. For \( \tilde{\beta} \in \Delta_+ \) expressed as \( \beta = \sum_{i=1}^{n} n_i \beta_i \), let us set \( \deg(\beta) = \sum_{i=1}^{n} n_i \beta_i \), and \([\alpha]\) is the integral part of \( \alpha \). For a rational number, let \([x]\) denote the integral part of \( x \).

Proof. We first prove:

Lemma 8.1. For \( \beta \in \Delta_+ \) expressed as \( \beta = \sum_{i=1}^{n} n_i \beta_i \), let us set \( \deg(\beta) = \sum_{i=1}^{n} n_i \beta_i \), and \([\alpha]\) is the integral part of \( \alpha \). For a rational number, let \([x]\) denote the integral part of \( x \).

There exists an integer \( K \) and a family \((\tilde{e}_\beta[z_i^l])_{\beta \in \Delta_+, s \in S, l \in \mathbb{Z}} \) (resp., \((\tilde{f}_\beta[z_i^l])_{\beta \in \Delta_+, s \in S, l \in \mathbb{Z}} \)) of elements of \( A \) (resp., \( B \)) lifting \((\bar{e}_\beta \otimes z_i^l)_{\beta \in \Delta_+, s \in S, l \in \mathbb{Z}} \) (resp., \((\bar{f}_\beta \otimes z_i^l)_{\beta \in \Delta_+, s \in S, l \in \mathbb{Z}} \)), such that \( \tilde{e}_\beta[z_i^l] \) (resp., \( \tilde{f}_\beta[z_i^l] \)) belongs to \( I_{[l/\deg(\beta)]-K}^{(A)} \) (resp., \( I_{[l/\deg(\beta)]-K}^{(B)} \)).

Proof. The family \((\tilde{e}_\beta[z_i^l])_{\beta \in \Delta_+, s \in S, l \in \mathbb{Z}} \) may be constructed as follows. For each \( \beta \) in \( \Delta_+ \), fix an expression \( \bar{e}_\beta[\phi] = e_{\alpha_1} \ldots , e_{\alpha_n}[\phi] \). Let us denote by \( 1_s \) the element of \( K \) whose \( t \)-th component is \( \delta_{st}1 \). Then if \( 0 \leq N < |\beta| \), set \( \tilde{e}_\beta[z_i^N] = [e_{\alpha_1}[1_s], \ldots , e_{\alpha_n}[z_i^N]] \). Define \( Z_s \) as the endomorphism of \( U_h L_{\lambda_s} \) such that \( Z_s(e_{\alpha_i}[\phi]) = e_{\alpha_i}[z_i^N] \). Then \( \tilde{e}_\beta[z_i^N] \) may be defined by the condition that \( \tilde{e}_\beta[z_i^{N+|\beta|}] = Z_s(\tilde{e}_\beta[z_i^N]) \), for any \( N \in \mathbb{Z} \). The family \((\tilde{e}_\beta[z_i^l])_{\beta \in \Delta_+, s \in S, l \in \mathbb{Z}} \) is constructed in the same way.

End of proof of the proposition. Let \( N_0 \) be an integer such that \( \prod_{s \in S} z_s^{N_0} \mathbb{C}[[z_s]] \subset \Lambda \) and let \( \lambda_1, \ldots , \lambda_k \) be elements of \( \Lambda \) such that their class in \( \Lambda / \prod_{s \in S} z_s^{N_0} \mathbb{C}[[z_s]] \) is a basis of this space. Let \( \tilde{e}_\beta[\lambda_i] \) (resp., \( \tilde{f}_\beta[\lambda_i] \)) be lifts to \( A \) (resp., \( B \)) of the \( e_\beta \otimes \lambda_i \) (resp., \( f_\beta \otimes \lambda_i \)).

Let us define \( \tilde{I}_M^{(A)} \) (resp., \( \tilde{I}_M^{(B)} \)) as the submodule of \( A \) (resp., \( B \)) spanned by the products
\[
\prod_{i=1}^{k} \prod_{\beta \in \Delta_+} p_{i,n}^{(A)}(\tilde{e}_\beta[\lambda_i])^{k(i,\beta)}, \quad \prod_{i=1}^{k} \prod_{\beta \in \Delta_+} p_{i,n}^{(B)}(\tilde{f}_\beta[\lambda_i])^{k(i,\beta)},
\]
resp.,
\[
\prod_{i=1}^{k} \prod_{\beta \in \Delta_+} p_{i,n}^{(B)}(\tilde{f}_\beta[\lambda_i])^{k(i,\beta)}, \quad \prod_{i=1}^{k} \prod_{\beta \in \Delta_+} p_{i,n}^{(B)}(\tilde{f}_\beta[\lambda_i])^{k(i,\beta)},
\]
where almost all exponents are zero and at least one of the \( k(l,s,\beta) \) is nonzero when \( l \geq M \).

It follows from (10) that \( A \) (resp., \( B \)) is a topologically free \( \mathbb{C}[[\hbar]] \)-module, such that \( A/hA = U L_{n_+} \otimes_{U L_{out}} \mathbb{C} \) (resp., \( B/hB = U L_{n_+} \otimes_{U L_{out}} \mathbb{C} \)). Therefore, the families \((59) \) (resp., \((60) \)), where almost all exponents are zero, is a topological basis of \( A \) (resp., \( B \)). Therefore \( (A/hA) \) [\( \tilde{I}_M^{(A)} \)] and \( (B/hB) \) [\( \tilde{I}_M^{(B)} \)] are finitely generated \( \mathbb{C}[[\hbar]] \)-modules. Since for any \( N \), there exists \( M \) such that \( I_N^{(A)} \) \( \supseteq \tilde{I}_M^{(A)} \) and \( I_N^{(B)} \) \( \supseteq \tilde{I}_M^{(B)} \), \( A/hA \) [\( \tilde{I}_M^{(A)} \)] and \( B/hB \) [\( \tilde{I}_M^{(B)} \)] are finitely generated \( \mathbb{C}[[\hbar]] \)-modules.

\[ \square \]
Proposition 8.2. For any integers \( N \) and \( k \geq 0 \), and any \( \alpha \in \mathbb{N}^n \), there exists an integer \( M(N, k, \alpha) \) such that \( \tilde{I}_{M(N, k, \alpha)}[\alpha] \subset I_N^{(A^{in})}[\alpha] + \hbar^k A \), and \( \tilde{I}_{M(N, k, \alpha)}[-\alpha] \subset I_N^{(B^{in})}[-\alpha] + \hbar^k B \).

Proof. The proposition follows from the following statement. Let \( (\lambda_n) \) be the sequence of elements of \( A \) given by \( \lambda_1, \ldots, \lambda_k, (z_s^{N_s})_{s \in S}, (z_s^{N_s+1})_{s \in S}, \) etc. It follows from their construction that the \( \tilde{e}_\beta[\lambda_\alpha] \) satisfy the rules

\[
[\tilde{e}_\beta[\lambda_\alpha], \tilde{e}_\gamma[\lambda_{\alpha'}]] = \sum_{p \geq 0} \hbar^p A^{out} \left( \prod_{\varepsilon \in \Delta_+} \prod_{\alpha'' \geq k(\alpha, \alpha', p)} \tilde{e}_\varepsilon[\lambda_{\alpha''}]^{l(\epsilon, i)} \right)
\]

and

\[
[f_\beta[\lambda_\alpha], f_\gamma[\lambda_{\alpha'}]] = \sum_{p \geq 0} \hbar^p \left( \prod_{\alpha'' \geq k(\alpha, \alpha', p)} \prod_{\varepsilon \in \Delta_+} f_\varepsilon[\lambda_{\alpha''}]^{l(\epsilon, i)} \right) B^{out},
\]

where the indices \( k(\alpha, \alpha', p) \) are such that for \( \alpha \) and \( p \) fixed, the functions \( \alpha' \mapsto k(\alpha, \alpha', p) \) and \( \alpha' \mapsto k(\alpha', \alpha, p) \) tend to infinity with \( \alpha' \).

8.1.2. Construction of \( F_{in, out} \) and \( F_{out, in} \). It follows from Proposition 7.2 and Corollary 7.1 that \( \langle \cdot, \cdot \rangle_{U_\lambda L_{\lambda+}} \) induces nondegenerate pairings

\[ \langle \cdot, \cdot \rangle_{in, out} : A^{out} \otimes B^{in} \to \mathbb{C}((\hbar)) \text{ and } \langle \cdot, \cdot \rangle_{in, out} : A^{in} \otimes B^{out} \to \mathbb{C}((\hbar)). \]

Recall that \( B^{in} \) is a topologically free \( \mathbb{C}[[\hbar]] \)-module. Let us denote by \( (B^{in})^{QFSH} \) the image of \( B^{QFSH} \) by the projection from \( B \) to \( B^{in} \). Then \( (B^{in})^{QFSH} \) is a topologically free \( \mathbb{C}[[\hbar]] \) module, such that \( (B^{in})^{QFSH} / h(B^{in})^{QFSH} \) is isomorphic to the dual \( O_{LN_{out}} \otimes\mathbb{C}[[\hbar]] \).

Moreover, \( \langle \cdot, \cdot \rangle_{out, in} \) induces a pairing \( A^{out} \times (B^{in})^{QFSH} \to \mathbb{C}[[\hbar]] \), whose reduction modulo \( \hbar \) is the duality pairing between \( U L_{\lambda+} \) and its dual. We can then modify the \( \mathbb{C}[[\hbar]] \)-module isomorphism of \( (B^{in})^{QFSH} \) with \( (UL_{\lambda+}^{out})^{\ast}[[\hbar]] \), so that the pairing between \( A^{out} \) and \( B^{in} \) is transported to the canonical pairing between \( UL_{\lambda+}^{out}[[\hbar]] \) and its dual.

The fact that it is contained in some \( (I^{(B^{in})}_{M} \cap (B^{in})^{QFSH})[[\hbar]] \) shows that \( (I^{(B^{in})}_{M} \cap (B^{in})^{QFSH})[[\hbar]] \) is a cofinite submodule in \( (B^{in})^{QFSH}[[\hbar]] \) (which means that the corresponding quotient is a finitely generated \( \mathbb{C}[[\hbar]] \)-module).

Lemma 8.2. Let \( V \) be a vector space with countable basis. There exists a unique element \( F_V \in \lim_{\to W} V[[h]] \otimes (V^*[h]/W) \), where the inverse limit is over all cofinite submodules of \( V^*[h] \), such that for any \( \xi \in V^* \), \( \langle F_V, \xi \otimes id \rangle \) is equal to the class of \( \xi \) in \( \lim_{\to W}(V^*[h]/W) \) and for any \( v \in V[[h]] \), \( \langle F_V, id \otimes v \rangle \) (which is well-defined because the \( \hbar \)-adic valuation of \( v \) tends to infinity) is equal to \( v \).
Proof. Let $W$ be a cofinite submodule of $V^*[[\hbar]]$. Set $W_0 = W \mod \hbar$. Then $W_0$ is a finite-codimensional vector subspace of $V^*$. It follows that $W_0^\perp$ is a finite-dimensional subspace of $V$, such that the pairing between $W_0^\perp$ and $V^*/W_0$ is nondegenerate. Then the class of $F_V$ in $V^*[[\hbar]] \otimes (V^*[[\hbar]]/W)$ is the image of the corresponding canonical element in $W_0^{\perp} \otimes (V^*/W_0)$. \hfill \Box

Let us denote by $F_{\text{out,in}}[\alpha]$ the canonical element of

$$\lim_{\longleftarrow N} A^{\text{out}}[\alpha] \otimes (B^{\text{in}}/I^{(B^{\text{in}})}_N)[-\alpha]$$

defined by the pairing $\langle \cdot , \cdot \rangle_{\text{out,in}}$. Let also $F_{\text{in,out}}$ be the canonical element of

$$\prod_{\alpha \in \mathbb{N}^n} \lim_{\longleftarrow N} (A^{\text{in}}/I^{(A^{\text{in}})}_N)[\alpha] \otimes B^{\text{out}}[-\alpha]$$

associated with $\langle \cdot , \cdot \rangle_{\text{in,out}}$.

8.1.3. Construction of $\tau_A$ and $\tau_B$.

Lemma 8.3. There exists sections $\sigma_A : A^{\text{in}} \to A$ of the projection $p^{(A)}_{\text{in}} : A \to A^{\text{in}}$, and $\sigma_A : B^{\text{in}} \to B$ of $p^{(B)}_{\text{in}}$, such that for any integers $N$ and $k \geq 0$ and any $\alpha \in \mathbb{N}^n$, there exists an integer $M(N,k,\alpha)$ such that $\sigma_A^{-1}(I^{(A)}_N[\alpha]) \subset I^{(A^{\text{in}})}_{M(N,k,\alpha)} + \hbar^k A$, and $\sigma_B^{-1}(I^{(B)}_N[\alpha]) \subset I^{(B^{\text{in}})}_{M(N,k,\alpha)} + \hbar^k B$.

Proof. The family $p^{(A)}_{\text{in}}(\prod_{\alpha=0}^\infty \prod_{\beta \in \Delta_+} \tilde{e}_\beta[\lambda]^{n(i,\beta)})$ is a topological basis of $A^{\text{in}}$. Set

$$\sigma_A(p^{(A)}_{\text{in}}(\prod_{\alpha=0}^\infty \prod_{\beta \in \Delta_+} \tilde{e}_\beta[\lambda]^{n(i,\beta)}) = \prod_{i=0}^\infty \prod_{\beta \in \Delta_+} \tilde{e}_\beta[\lambda]^{n(i,\beta)}.$$

In the same way, set

$$\sigma_B(p^{(B)}_{\text{in}}(\prod_{i=\infty}^0 \prod_{\beta \in \Delta_+} \tilde{e}_\beta[\lambda]^{n(i,\beta)}) = (\prod_{i=0}^\infty \prod_{\beta \in \Delta_+} \tilde{e}_\beta[\lambda]^{n(i,\beta)})$$

$\sigma_A$ and $\sigma_B$ are then lifts of $p^{(A)}_{\text{in}}$ and $p^{(B)}_{\text{in}}$, and their continuity properties follows from Proposition 8.2. \hfill \Box

Inclusion followed by multiplication induces isomorphisms $i_A : \sigma_A(A^{\text{in}}) \otimes A^{\text{out}} \to A$ and $i_B : B^{\text{out}} \otimes \sigma_B(B^{\text{in}}) \to B$. Define linear maps $p^{(A)}_{\text{out}} : A \to A^{\text{out}}$ and $p^{(B)}_{\text{out}} : B \to B^{\text{out}}$ by

$$p^{(A)}_{\text{out}} = (\varepsilon \otimes \text{id}) \circ i_A^{-1} \quad \text{and} \quad p^{(B)}_{\text{out}} = (\text{id} \otimes \varepsilon) \circ i_B^{-1}.$$

Lemma 8.4. $p^{(A)}_{\text{out}}$ (resp., $p^{(B)}_{\text{out}}$) is a right (resp., left) $A^{\text{out}}$-module (resp., $B^{\text{out}}$-module) map, such that $p^{(A)}_{\text{out}}(1) = 1$ (resp., $p^{(B)}_{\text{out}}(1) = 1$). Moreover, for any integer $k \geq 0$ and $\alpha \in \mathbb{N}^n$, there exists an integer $N(k,\alpha)$ such that $(p^{(A)}_{\text{out}})^{-1}(\hbar^k A^{\text{out}}) \supset I^{(A)}_{N(k,\alpha)}$ and $(p^{(B)}_{\text{out}})^{-1}(\hbar^k B^{\text{out}}) \supset I^{(B)}_{N(k,\alpha)}.$
Proof. The first part of the lemma is clear. The continuity statement follows from estimates \([\text{(1)}]\) and \([\text{(2)}]\).

Our next step is the construction of maps \(\tau_A\) and \(\tau_B\). In order to state their properties, we introduce maps \(\Delta_A\) and \(\Delta_B\), which are modifications of the restrictions of the coproducts \(\Delta\) and \(\Delta'\) to \(A\) and \(B\).

Let us denote by \(U_h\mathfrak{h}_+\) (resp., \(U_h\mathfrak{g}_-\)) the subalgebra of \(U_h\mathfrak{g}_+\) (resp., \(U_h\mathfrak{g}_-\)) generated by \(D\) and the \(h_i[r], i = 1, \ldots, n, r \in R\) (resp., by \(K\) and the \(h_i[l], i = 1, \ldots, n, l \in \Lambda\)). Inclusion followed by multiplication induces a \(\mathbb{C}[\{h\}]\)-module isomorphism \(i_+ : U_h\mathfrak{h}_+ \otimes A \to U_h\mathfrak{g}_+\) (resp., \(i_- : U_h\mathfrak{g}_- \otimes B \to U_h\mathfrak{g}_-\)). Let us denote by \(\pi_A : U_h\mathfrak{g}_+ \to A\) (resp., \(\pi_B : U_h\mathfrak{g}_- \to B\)) the composition \((\varepsilon \otimes id) \circ i_+^{-1}\) (resp., \((\varepsilon \otimes id) \circ i_-^{-1}\)).

Define
\[
\Delta_A : A \to \lim_{\leftarrow k} \lim_{\leftarrow N} \left( A/I_N^{(A)} \right) \otimes (A/h^k)
\]
and
\[
\Delta_B : B \to \lim_{\leftarrow k} \lim_{\leftarrow N} \left( B/I_N^{(B)} \right) \otimes (B/h^k)
\]
as the compositions \((\pi_A \otimes id) \circ \Delta_A\) and \((\pi_B \otimes id) \circ \Delta_B\). For \(I\) a subset of \(\{1, \ldots, n\}\), let \(I\) denote its complement \(\{1, \ldots, n\} - I\). We have then
\[
\Delta_A \left( \prod_{j=1}^N e_{i_j}(z_j) \right) = \sum_{I \subset \{1, \ldots, N\}} \left( \prod_{j \in I, j' \in I, j \geq j'} q_{i_j,i_j'}(z_j, z_{j'})^{-1} \right) \prod_{j \in I} e_{i_j}(z_j) \otimes \prod_{j' \in I} e_{i_{j'}}(z_{j'}),
\]
and
\[
\Delta_B \left( \prod_{j=1}^N f_{i_j}(z_j) \right) = \sum_{I \subset \{1, \ldots, N\}} \left( \prod_{j \in I, j' \in I, j \geq j'} q_{i_j,i_j'}(z_j, z_{j'})^{-1} \right) \prod_{j \in I} f_{i_j}(z_j) \otimes \prod_{j' \in I} f_{i_{j'}}(z_{j'}). \]

There are unique automorphisms \(S_A\) of \(\lim_{\leftarrow N} (A/I_N^{(A)})/(h^k)\) and \(S_B\) of \(\lim_{\leftarrow k} \lim_{\leftarrow N} (B/I_N^{(B)})/(h^k)\), such that for any \(a \in A\), \(\sum a^{(1)} S_A(a^{(2)}) = \sum S_A(a^{(1)}) a^{(2)} = \varepsilon(a) 1_A\) and for any \(b \in B\), \(\sum b^{(1)} S_B(b^{(2)}) = \sum S_B(b^{(1)}) b^{(2)} = \varepsilon(b) 1_B\). If \(\alpha = \sum_i n_i \alpha_i \in \mathbb{N}^n\), we have \((S_A)_{A[a]} = (-1)^{\sum_i n_i} id_{A[a]}\) and \((S_B)_{B[-a]} = (-1)^{\sum_i n_i} id_{B[-a]}\).

The maps \(\Delta_A\) and \(S_A\) (resp., \(\Delta_B\) and \(S_B\)) are continuous with respect to the topology defined by the \(I_N^{(A)}\) (resp., \(I_N^{(B)}\)).

The Hopf pairing rules then yield
\[
\langle aa', b \rangle_{U_h\mathfrak{g}_+} = \langle a \otimes a', \Delta_B(b) \rangle_{U_h\mathfrak{g}_+^{\otimes 2}}, \quad \text{and} \quad \langle a, bb' \rangle_{U_h\mathfrak{g}_+} = \langle \Delta_A(a), b \otimes b' \rangle_{U_h\mathfrak{g}_+^{\otimes 2}},
\]
for any \(a, a' \in A\) and \(b, b' \in B\).

For \(C\) any augmented algebra, we denote by \(C_0\) the kernel of its augmentation.

It follows from the Hopf pairing rules and the fact the \(A^{out}\) (resp., \(B^{out}\)) is a subalgebra of \(A\) (resp., \(B\)) that \(\Delta_A(A^{out} A)\) (resp., \(\Delta_B(B^{out} B)\)) is contained in the completion of \(A^{out} A/L + A \otimes A^{out} A\) (resp., of \(B^{out} B \otimes B + A \otimes B^{out} B\)). It follows
that $\Delta_A$ (resp., $\Delta_B$) induces a map $\Delta^{A_{\text{in}}} : A^{\text{in}} \to \lim_{\leftarrow k} \lim_{\leftarrow N} (A^{\text{in}}/I_N^{(A^{\text{in}})}) \otimes A^{\text{in}}/(\hbar^k)$ (resp., $\Delta_{B^{\text{in}}} : B^{\text{in}} \to \lim_{\leftarrow k} \lim_{\leftarrow N} B^{\text{in}} \otimes (B^{\text{in}}/I_N^{(B^{\text{in}})})/(\hbar^k)$).

**Proposition 8.3.** There exists a $\mathbb{C}[\hbar]$-linear map $\tau_A$ (resp., $\tau_B$) from $A^{\text{in}}$ to $A$ (resp., from $B^{\text{in}}$ to $B$), which is a section of the canonical projection $p_{\text{in}}^{(A)} : A \to A^{\text{in}}$ (resp., $p_{\text{in}}^{(B)} : B \to B^{\text{in}}$), such that for any $N,k,\alpha$, there exists $M(N,k,\alpha)$ such that $\tau_A^{-1}((I_N^{(A)}+\hbar^k A)[\alpha]) \supset (I_M^{(A)}+\hbar^k A^{\text{in}})[\alpha]$, and $\tau_B^{-1}((I_N^{(B)}+\hbar^k B)[-\alpha]) \supset (I_M^{(B)}+\hbar^k B^{\text{in}})[-\alpha]$, and satisfying

$$
\varepsilon(\tau_A(a^{\text{in}})) = \varepsilon(a^{\text{in}}), \quad (\id \otimes p_{\text{in}}^{(A)})\Delta_A(\tau_A(a^{\text{in}})) = (\tau_A \otimes \id)(\Delta^{A_{\text{in}}}(a^{\text{in}}))
$$

(64)

for any $a^{\text{in}} \in A^{\text{in}}$, and

$$
\varepsilon(\tau_B(b^{\text{in}})) = \varepsilon(b^{\text{in}}), \quad (p_{\text{in}}^{(B)} \otimes \id)\Delta_B(\tau_B(b^{\text{in}})) = (\id \otimes \tau_B)(\Delta_{B^{\text{in}}}(b^{\text{in}}))
$$

for any $b^{\text{in}} \in B^{\text{in}}$.

**Proof.** Define $\tau_A$ as follows. Let $j_A$ be the composition $(p_{\text{out}}^{(A)} \otimes p_{\text{in}}^{(A)}) \circ \Delta_A$; $j_A$ induces a $\mathbb{C}[\hbar]$-linear map from $\lim_{\leftarrow k} \lim_{\leftarrow N} A/(I_N^{(A)}+\hbar^k A)$ to $\lim_{\leftarrow k} \lim_{\leftarrow N} A^{\text{out}} \otimes (A^{\text{in}}/I_N^{(A^{\text{in}})})/(\hbar^k)$.

For any $a^{\text{in}} \in A^{\text{in}}$, let us set $\varpi(a^{\text{in}}) = \sum p_{\text{out}}^{(A)}(a^{\text{in}(1)})(\sigma_A \circ p_{\text{in}}^{(A)})(a^{\text{in}(2)})$, where $\Delta_A(a) = \sum a^{(1)} \otimes a^{(2)}$. $p_{\text{in}}^{(A)}(\varpi(a^{\text{in}})) = p_{\text{in}}^{(A)}(a^{\text{in}})$, therefore $\varpi$ is injective, so there is a unique bicontinuous isomorphism $\mu : A^{\text{out}} \otimes A^{\text{in}} \to A$ such that $\mu \varpi(a^{\text{out}} \otimes a^{\text{in}}) = a^{\text{out}} \varpi(a^{\text{in}})$. On the other hand, $\Delta(A^{\text{out}})$ is contained in the completion of $A \otimes A^{\text{out}}$, so for any $a^{\text{out}} \in A^{\text{out}}$ and $a \in A$, we have $(\id \otimes \varpi^{(A)})(a^{\text{out}} a) = (a^{\text{out}} \otimes 1)\Delta_A(a^{\text{in}})$. All this implies that the map $j_A$ has a continuous inverse, which is the unique map $j_A^{-1}$ such that $j_A^{-1}(a^{\text{out}} \otimes a^{\text{in}}) = \mu(1 \otimes \sigma_A)\mu^{-1}(a^{\text{out}} a^{\text{in}})$, where $\mu$ is the product map in $A$.

For $a^{\text{in}}$ in $A^{\text{in}}$, let us set

$$
\tau_A(a^{\text{in}}) = j_A^{-1}(1 \otimes a^{\text{in}}).
$$

In the same way, define $j_B : \lim_{\leftarrow k} \lim_{\leftarrow N} (B/I_N^{(B)}+\hbar^k B) \to \lim_{\leftarrow k} \lim_{\leftarrow N} (B^{\text{in}}/I_N^{(B^{\text{in}})}) \otimes B^{\text{out}}/(\hbar^k)$ as $(p_{\text{in}}^{(B)} \otimes p_{\text{out}}^{(B)}) \circ \Delta_B$, and for $b^{\text{in}}$ in $B^{\text{in}}$, set

$$
\tau_B(b^{\text{in}}) = j_B^{-1}(b^{\text{in}} \otimes 1).
$$

The first identity of (64) is clear. The coassociativity of $\Delta_A$ implies that $(\id \otimes \Delta_{A^{\text{in}}}) \circ j_A = (j_A \otimes p_{\text{in}}^{(A)}) \circ \Delta_A$. Therefore, $(j_A^{-1} \otimes \id)(\id \otimes \Delta_{A^{\text{in}}}) = (\id \otimes p_{\text{in}}^{(A)}) \circ \Delta_A \circ j_A^{-1}$. Restricting this identity to $1 \otimes A^{\text{in}}$ yields the second identity of (64). □
8.1.4. Construction of $F_1, F_{\text{int}}$ and $F_2$. Let us set

$$F_1 = (\tau_A \otimes \text{id})(F_{\text{in, out}}), \quad F_2 = (\text{id} \otimes \tau_B)(F_{\text{out, in}}).$$

Then

$$F_1 \in \lim \lim_{\leftarrow k \leftarrow N} (A/I_N^{(A)}) \otimes B^\text{out}/(h^k) \quad \text{and} \quad F_2 \in \lim \lim_{\leftarrow k \leftarrow N} A^\text{out} \otimes (B/I_N^{(B)})/(h^k).$$

Lemma 8.5. When $b \in B$, the valuation of $\langle I_N^{(A)}, b \rangle_{U_h L_n}$ tends to infinity with $N$. Therefore $\langle F_1, b \otimes \text{id} \rangle_{U_h L_n}$ is a well-defined element of $B^\text{out}$.

Let us set $\Pi_B(b) = \langle F_1, b \otimes \text{id} \rangle_{U_h L_n}$. Then $\Pi_B$ is a linear map from $B$ to $B^\text{out}$, and it is a right $B^\text{out}$-module map. We have $\Pi_B(1) = 1$.

In the same way, if we set for $a \in A$, $\Pi_A(a) = \langle F_2, \text{id} \otimes a \rangle_{U_h L_n}$, then $\Pi_A$ is a linear map from $A$ to $A^\text{out}$, which is a right $A^\text{out}$-module map such that $\Pi_A(1) = 1$.

Proof. This follows from the pairing rules (8.3) and the coproduct properties of $\tau_A$ and $\tau_B$ proved in Proposition 8.3. □

Let us define

$$\Delta^{(2)}_A : A \to \lim \lim_{\leftarrow k \leftarrow N} \left((A/I_N^{(A)}) \otimes (A/I_N^{(A)}) \otimes A/(h^k)\right),$$

$$\Delta^{(2)}_B : B \to \lim \lim_{\leftarrow k \leftarrow N} \left(B \otimes (B/I_N^{(B)}) \otimes (B/I_N^{(B)})/(h^k)\right)$$

as the compositions $(\pi_A \otimes \pi_A \otimes \text{id}) \circ (\text{id} \otimes \Delta_{iA}) \circ \Delta_i A$ and $(\text{id} \otimes \pi_B \otimes \pi_B) \circ (\text{id} \otimes \Delta'_{iB}) \circ \Delta'_i B$. We have $\Delta^{(2)}_A = (\Delta_i \otimes \text{id}) \circ \Delta_A = (\text{id} \otimes \Delta_A) \circ \Delta_A$ and $\Delta^{(2)}_B = (\Delta_B \otimes \text{id}) \circ \Delta_B = (\text{id} \otimes \Delta_B) \circ \Delta_B$.

For $b$ in $B$, let us set

$$\sigma_{\text{int}}(b) = \sum \langle F_1, b^{(1)} \otimes \text{id} \rangle_{U_h L_n} S_B(b^{(2)}) \langle F_2, b^{(3)} \otimes \text{id} \rangle_{U_h L_n},$$

where $\sum b^{(1)} \otimes b^{(2)} \otimes b^{(3)}$ is $\Delta^{(2)}_B(b)$. Then $\sigma_{\text{int}}$ is a linear map from $B$ to $\lim_{\leftarrow k \leftarrow N} \lim_{\leftarrow N} B/(I_N^{(B)} + h^k B)$.

Lemma 8.6. $(A^\text{out})^\perp$ is contained in the kernel of $\sigma_{\text{int}}$.

Proof. Let us fix $b$ in $(A^\text{out})^\perp$. It follows from Cor. 8.3 that $b$ is a sum of elements of the form $b' f_i[r]$, $b' \in B$, $r \in R$ and $i \in \{1, \ldots, n\}$. One checks that $\Delta_B(b' f_i[r])$ can be written as a series $\Delta_B(b')(f_i[r] \otimes 1) + \sum_{j=1}^n \sum_{\alpha' \in R} \sum_{\alpha'' \in R} \gamma_{\alpha'}(b'_{\alpha'} \otimes b'_{\alpha''}) (1 \otimes f_i[r'^{\alpha''}])$, where $b'_{\alpha'}, b'_{\alpha''} \in B$. The pairing of $F_2$ with the second factor of the latter sum is zero. Therefore, if we set $\Delta_B(b') = \sum b^{(1)} \otimes b^{(2)}$, and since $\langle F_1, b \otimes \text{id} \rangle_{U_h L_n} = \Pi_B(b)$,

$$\sigma_{\text{int}}(b) = \sum \Pi_B((b^{(1)}) f_i[r])^{(1)} S_B((b^{(1)}) f_i[r])^{(2)} \langle F_2, b^{(2)} \otimes \text{id} \rangle_{U_h L_n},$$

it follows from the explicit form of $\Delta_B$ and $S_B$ that $(\text{id} \otimes S_B) \circ \Delta_B(b^{(1)} f_i[r]) = \sum (b^{(1)} f_i[r])^{(1)} \otimes S_B((b^{(1)} f_i[r])^{(2)})$ is the sum of a series $\sum_{i \in \{1, \ldots, n\}} \sum_{\alpha} a_{\alpha i} f_i[r_{\alpha}] \otimes b_{\alpha} - a_{\alpha i} \otimes f_i[r_{\alpha}] b_{\alpha}$, where $a_{\alpha i}$ and $b_{\alpha}$ belong to $B$. Since $\Pi_B$ is a right $B^\text{out}$-module map, $\sigma_{\text{int}}(b) = 0$. □
Lemma 8.7. $B^{\text{out}}$ injects into $\varprojlim_k \varprojlim_N B/(I_N^{(B)} + \hbar k B)$. The annihilator of $\sum_{i=1}^n \sum_{r \in R} Ae_i[r]$ in $\varprojlim_k \varprojlim_N B/(I_N^{(B)} + \hbar k B)$ for the pairing induced by $\langle , \rangle_{\text{U}_L \text{N}_L}$ is $B^{\text{out}}$.

Proof. More generally, let us show that $B$ injects into $\varprojlim_k \varprojlim_N B/(I_N^{(B)} + \hbar k B)$. This means that $\cap_{k,N}(I_N^{(B)} + \hbar k B) = 0$. Let $x$ belong to this intersection. For any $a \in A$, the $\hbar$-adic valuation of $\langle a, I_N^{(B)} + \hbar k B \rangle_{\text{U}_L \text{N}_L}$ tends to infinity with $k$ and $N$. Therefore $\langle a, x \rangle_{\text{U}_L \text{N}_L}$ vanishes, and since $\langle , \rangle_{\text{U}_L \text{N}_L}$ is nondegenerate, $x$ is zero.

The inverse limit $\varprojlim_k \varprojlim_N B/(I_N^{(B)} + \hbar k B)$ injects into $\text{Hom}_{\mathbb{C}[[h]]}(A, \mathbb{C}((h)))$, therefore it is torsion-free. Therefore it is a topologically free module, with associated vector space $\varprojlim_N \text{U}_L \text{N}_L / \sum_{i=1}^n \sum_{s \in S, k \geq N} \text{U}_L \text{N}_L f_i[z_i^k]$. The annihilator of $\mathcal{O}_{L^{\text{out}}}$ in this space is $\text{U}_L \text{N}_L^{\text{out}}$. The lemma follows. \hfill \□

Lemma 8.8. The image of $\sigma^{\text{int}}$ is contained in $B^{\text{out}}$.

Proof. Let us fix $a, b$ in $A$ and $B$, $i$ in $\{1, \ldots, n\}$ and $r$ in $R$. As in Lemma 8.6, the fact that $\Pi_A$ is a left $A^{\text{out}}$-module map implies that $\langle ae_i[r], \sigma^{\text{int}}(b) \rangle_{\text{U}_L \text{N}_L}$ is zero. So the image of $\sigma^{\text{int}}$ is contained in the annihilator of $\sum_{i=1}^n \sum_{r \in R} Ae_i[r]$. The lemma then follows from Lemma 8.7. \hfill \□

It follows from Lemmas 8.6 and 8.8 that $\sigma^{\text{int}}$ induces a map $\tilde{\sigma}^{\text{int}}$ from $B^{\text{in}}$ to $B^{\text{out}}$. Moreover, $\tilde{\sigma}^{\text{int}}$ is continuous in the following sense: for any integer $k \geq 0$ and any $\alpha$ in $\mathbb{N}$, there exists an integer $N(k, \alpha)$ such that $\tilde{\sigma}^{-1}((h\hbar)B^{\text{out}}) \supseteq I_{N(k, \alpha)}^{(A^{\text{in}})}[-\alpha]$. It follows that if we set

$$F'_{\text{int}} = (id \otimes \tilde{\sigma}^{\text{int}})(F_{\text{out}, \text{in}}),$$

$F'_{\text{int}}$ belongs to $\prod_{\alpha \in \mathbb{N}} \varprojlim_k (A^{\text{out}}[\alpha] \otimes B^{\text{out}}[-\alpha])/(h\hbar (A^{\text{out}}[\alpha] \otimes B^{\text{out}}[-\alpha]))$. Moreover, the bidegree $(0, 0)$ component of $F'_{\text{int}}$ is equal to $1 \otimes 1$. Since $A^{\text{out}}$ and $B^{\text{out}}$ are graded algebras, the series $F_{\text{int}} = \sum_{i \geq 0} (-1)^i(F'_{\text{int}} - 1 \otimes 1)^i$ belongs to $\prod_{\alpha \in \mathbb{N}} \varprojlim_k (A^{\text{out}}[\alpha] \otimes B^{\text{out}}[-\alpha])/(h\hbar (A^{\text{out}}[\alpha] \otimes B^{\text{out}}[-\alpha]))$. We have $F_{\text{int}}F'_{\text{int}} = F'_{\text{int}}F_{\text{int}} = 1$.

8.1.5. Definition and pairing properties of $F$. For any homogeneous elements $a, b$ in $A$ and $B$ of degrees $|a|, |b|$, for any integers $N$ and $k \geq 0$, and for any $\alpha$ in $\mathbb{N}$, there exists integers $M(N, k, a, \alpha)$ and $M(N, k, b, \alpha)$ such that

$$I_{M(N, k, a, \alpha)}^{(A)}[\alpha - |a|]a \subset I_N^{(A)}[\alpha]\hbar k A, \quad \text{and} \quad I_{M(N, k, b, \alpha)}^{(B)}[-\alpha - |b|]b \subset I_N^{(B)}[-\alpha]\hbar k B.$$

It follows that

$$\prod_{\alpha \in \mathbb{N}} \varprojlim (-k \leftarrow N)(A/I_N^{(A)})[\alpha] \otimes (B/I_N^{(B)})[-\alpha]/(\hbar k)$$

(65)
has an algebra structure. Moreover, \( \prod_{\alpha \in \mathbb{N}} \lim_{k \to \infty} (A^\text{out}[\alpha] \otimes B^\text{out}[-|\alpha|])/(h^k) \) is a subalgebra of \( \mathfrak{L} \). Let us set

\[ F = F_2 F_{\text{int}} F_1. \]

Then \( F \) belongs to the algebra \( \mathfrak{L} \).

For any element \( \phi \) of the algebra \( \mathfrak{L} \), and any elements \( a, b \) of \( A \) and \( B \), \( \langle \phi, b \otimes id \rangle_{\mathfrak{L}} \) is a well-defined element of \( \lim_{k \to \infty} h^{-\deg(b)} B/(I_N^{(B)} + h^k B)[|b|] \), and \( \langle \phi, id \otimes a \rangle_{\mathfrak{L}} \) is a well-defined element of \( \lim_{k \to \infty} h^{\deg(a)} A/(I_N^{(A)} + h^k A)[|a|] \). (\( \deg(a) \) and \( \deg(b) \) are the principal degrees of \( a \) and \( b \), defined as \( \deg(a) = \sum_i n_i \) if \( |a| = \sum_i n_i \alpha_i \) and \( \deg(b) = -\sum_i m_i \) if \( |b| = -\sum_i m_i \alpha_i \)).

**Proposition 8.4.** For any elements \( a \) of \( A \) and \( b \) of \( B \), we have

\[ \langle F, id \otimes a \rangle_{\mathfrak{L}} = a \quad \text{and} \quad \langle F, b \otimes id \rangle_{\mathfrak{L}} = b. \]

**Proof.** Let us prove the second equality. Since \( S_B \) is the only endomorphism of \( B \) satisfying the identity \( \sum b^{(1)} S_B(b^{(2)}) = \varepsilon(b) \), and by the Hopf pairing rules, this equality is equivalent to

\[ \forall b \in B, \quad \langle F^{-1}, b \otimes id \rangle_{\mathfrak{L}} = S_B(b). \]

\( \langle F^{-1}, b \otimes id \rangle_{\mathfrak{L}} \) is equal to \( \sum \langle F^{-1}_{F_1}, b^{(1)} \otimes id \rangle_{\mathfrak{L}} \), \( \langle F^{-1}_{F_{\text{int}}}, b^{(2)} \otimes id \rangle_{\mathfrak{L}} \), \( \langle F^{-1}_{F_2}, b^{(3)} \otimes id \rangle_{\mathfrak{L}} \). Since \( \langle F^{-1}_{F_{\text{int}}}, b \otimes id \rangle_{\mathfrak{L}} \) is equal to \( \sigma_{\text{int}}(b) \), the definition of \( \sigma_{\text{int}} \) and the pairing rules \( \mathfrak{L} \) imply that this is \( S_B(b) \). The proof of the first identity is similar. \( \square \)

**Remark 5.** Assume that \( \Lambda \) is a \( \vartheta \)-invariant subalgebra of \( \mathcal{K} \). This is the case if \( C = \mathbb{C} P^1 \), \( \omega = dz \) and \( S = S_0 \cup \{ \infty \} \), where \( S_0 \) is a finite subset of \( \mathbb{C} \). Then if we set \( z_s = z - s \) for \( s \in S_0 \) and \( z_\infty = z^{-1} \), so \( \mathcal{K} = \prod_{s \in S_0} \mathbb{C}((z_s)) \times \mathbb{C}((z_\infty)) \), \( R = \mathbb{C}[z, \frac{1}{z_s}, s \in S_0] \) and we may set \( \Lambda = \prod_{s \in S_0} \mathbb{C}[[z_s]] \times z_\infty \mathbb{C}[[z_\infty]] \).

Then \( n_\pm \otimes \Lambda \) is a Lie subalgebra of \( L_n \). Let us denote by \( A_\Lambda \) the subalgebra of \( A \) (resp., of \( B_\Lambda \) the subalgebra of \( B \)) generated by the \( e_i[\lambda], i \in \{1, \ldots, n\} \), \( \lambda \in \Lambda \) (resp., the \( f_i[\lambda], i \in \{1, \ldots, n\}, \lambda \in \Lambda \)). Then \( A_\Lambda \subset A \) (resp., \( B_\Lambda \subset B \)) is a flat deformation of the inclusion \( (n_\pm \otimes \Lambda) \subset U L_n \).

The restriction of \( p_\text{in}^{(A)} \) to \( A_\Lambda \) (resp., of \( p_\text{in}^{(B)} \) to \( B_\Lambda \)) induces an isomorphism from \( A_\Lambda \) to \( A_\text{in} \) (resp., from \( B_\Lambda \) to \( B_\text{in} \)). We may choose \( \sigma_A \) and \( \sigma_B \) to be the corresponding inverse maps.

Then \( F_{\text{int}} \) equals 1, so \( F = F_2 F_1 \). In that case, \( p_\text{out}^{(A)} = \Pi_A \) and \( p_\text{out}^{(B)} = \Pi_B \). So \( F_1 \) is the Hopf twist relating Drinfeld’s coproduct and the usual coproduct of the Yangian algebra (the latter coproduct is defined in terms of \( L \)-operators, in the case \( A = sl_n \)). This can be proved using the arguments of \( \mathfrak{L} \) (there we treated the case \( A = sl_2 \) and \( S = \{ \infty \} \)).

**Remark 6.** In [12], Khoroshkin and Tolstoy expressed \( F_1 \) and \( F_2 \) in terms of the generators of the algebras \( U L_n \), in the case \( C = \mathbb{C} P^1, \omega = \frac{dz}{z} \). In the
particular case $a = \mathfrak{sl}_2$, Khoroshkin and Pakuliak also showed the commutativity of the families $I_n^+ = \text{res}_{z=0}(e^+(z) \otimes f^-(z)) n \frac{dz}{z}$ and $I_n^- = \text{res}_{z=0}(e^-(z) \otimes f^+(z)) n \frac{dz}{z}$, where $x^+(z) = \sum_{n \geq 0} x[n] z^{-n}$ and $x^-(z) = \sum_{n < 0} x[n] z^{-n}$ for $x \in \{e, f\}$, and they expressed $F_1$ and $F_2$ as series in the $I_n^\pm$.

8.2. The $R$-matrices. In this section, we express the $R$-matrices $\mathcal{R}$ and $\overline{\mathcal{R}}$ of $(U_q \mathfrak{g}, \Delta)$ and $(U_q \mathfrak{g}, \overline{\Delta})$. We then show properties on the $h$-adic valuation of $F, F_1, F_2$ and prove that $F$ satisfies a Hopf cocycle property.

Let us set $U = U_q \mathfrak{g}$ and define $U_+$ (resp., $U_-$) as the $h$-adically complete subalgebra of $U$ generated by $K, D, h_i[e], e_i[e]$ (resp., $K, D, h_i[e], f_i[e]$), $i \in \{1, \ldots, n\}, e \in \mathcal{K}$.

Recall that $\mathcal{R} = \mathcal{R}_h F$ is a well-defined element of $\mathcal{R}$. In fact, $\mathcal{R}$ even belongs to

$$\lim_{\leftarrow k} \prod_{\alpha \in \mathbb{N}^n} \lim_{\leftarrow N} \frac{(U_+/U_+ \cap I_N)[\alpha] \otimes (U_-/U_- \cap I_N)[\alpha]}{(h^k)};$$

multiplication induces on the structure of a left module over $\lim_{\leftarrow k} \lim_{\leftarrow N} (U_+ \otimes (U_+/U_- \cap I_N)[0])/(h^k)$, therefore the product

$$\mathcal{R} = \mathcal{R}_h F$$

is a well-defined element of $\mathcal{R}$. In fact, $\mathcal{R}$ even belongs to

$$\lim_{\leftarrow k} \prod_{\alpha \in \mathbb{N}^n} \lim_{\leftarrow N} U_q \mathfrak{g}_+/(U_q \mathfrak{g}_+ \cap I_N)[\alpha] \otimes U_q \mathfrak{g}_-/(U_q \mathfrak{g}_- \cap I_N)[\alpha] / (h^k),$$

and the Hopf pairing rules imply that it satisfies the identities

$$\langle \mathcal{R}, \text{id} \otimes a \rangle_{U_q \mathfrak{g}_\pm} = a, \quad \langle \mathcal{R}, b \otimes \text{id} \rangle_{U_q \mathfrak{g}_\pm} = b$$

for $a$ in $U_q \mathfrak{g}_+$ and $b$ in $U_q \mathfrak{g}_-$. Recall that $\Delta$ and $\Delta'$ both map $U_\pm$ to

$$\lim_{\leftarrow k} \lim_{\leftarrow N} U_\pm/(U_\pm \cap I_N) \otimes U_\pm/(U_\pm \cap I_N) / (h^k).$$

On the other hand, the multiplication map of $U_+ \otimes U$ induces an algebra structure on

$$\lim_{\leftarrow k} \bigoplus_{\alpha \in \mathbb{N}^n} \prod_{\beta \in \mathbb{N}^n} U_+/(U_+ \cap I_N)[\beta] \otimes U/I_N[\alpha - \beta] / (h^k).$$

Then the identities (67) imply that for $x \in U_+$, the equality

$$\mathcal{R} \Delta(x) = \Delta'(x) \mathcal{R}$$

takes place in (68). The identities (67) also imply that the equalities

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{(13)} \mathcal{R}^{(23)}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{(13)} \mathcal{R}^{(12)}$$

hold in

$$\lim_{\leftarrow k} \prod_{\alpha \in \mathbb{N}^n} \bigoplus_{\beta, \gamma \in \mathbb{N}^n, \beta + \gamma = \alpha} \lim_{\leftarrow N} (U_+/(U_+ \cap I_N)[\alpha] \otimes (U_-/U_- \cap I_N)[\beta] \otimes (U_-/U_- \cap I_N)[\gamma] / (h^k)$$
Corollary 8.1. Then it follows from Proposition 8.5 that \( \lim_{\to k} U/I \cap N \) \( \lim_{\to N}(U_+/U_+ \cap I_N)[\beta] \otimes (U_+/U_+ \cap I_N)[\gamma] \otimes (U_-/U_- \cap I_N)[-\alpha]/(h^k) \).

In [4], Section 2.1, we proved a statement of Drinfeld on the form of the \( R \)-matrix for quantized Kac-Moody algebras ([1]). The same argument, together with (69) and (70), implies that \( F \) has the following form.

**Proposition 8.5.** For any \( \alpha \) in \( \mathbb{N}^r \), let us denote by \( F_\alpha \) the bidegree \( (\alpha, -\alpha) \) part of \( F \). There is a unique integer \( k \) such that \( \alpha \) belongs to \( k(\Delta_+ \cup \{0\}) \setminus (k-1)(\Delta_+ \cup \{0\}) \), which we denote \( \ell(\alpha) \). Then \( F_\alpha \) belongs to the completion of \( h^{\ell(\alpha)}(A[\alpha] \otimes B[-\alpha]) \), and \( h^{-\ell(\alpha)}F_\alpha \mod h \) is equal to

\[
\frac{1}{\ell(\alpha)!} \left( \sum_{\beta \in \Delta_+} \sum_{\ell \in \mathbb{Z}} \epsilon_{\beta}[\ell] \otimes f_{\beta}[\ell] \right)^{\ell(\alpha)}.
\]

The properties of \( I_N \) imply that \( \lim_{\to k} U/I \cap N \otimes (U/I_N)/(h^k) \) has an algebra structure. Then it follows from Proposition 8.5 that

**Corollary 8.1.** \( F \) and \( F^{-1} \) belong to \( \lim_{\to k} \lim_{\to N}(A/I_N^{(A)}) \otimes (B/I_N^{(B)})/(h^k) \). Therefore (as does \( R_h \)), \( F \), \( R \) and their inverses belong to \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (U/I_N)/(h^k) \).

It follows that the identities (70) take place in \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (h^k) \), and the identity \( \Delta'(x) = R \Delta(x) R^{-1} \) holds in \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (h^k) \), for any \( x \in U \).

Moreover, one checks that \( R_h \) is a Hopf twist connecting \( \Delta \) and \( \Delta' \); more precisely, we have the identities

\[
\Delta'(x) = R_h \Delta(x) R_h^{-1} \quad \text{and} \quad R_h^{(12)}(\Delta \otimes id)(R_h) = R_h^{(23)}(id \otimes \Delta)(R_h)
\]

in \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (h^k) \) and \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (h^k) \), for any \( x \in U \). Since the quasitriangular identities mean that \( R \) is a Hopf twist connecting \( \Delta \) and \( \Delta' \), we get (as in [4]):

**Proposition 8.6.** \( F \) is a Hopf twist connecting \( \Delta \) and \( \Delta \), which means that the identities

\[
\Delta(x) = F \Delta(x) F^{-1} \quad \text{and} \quad F^{(12)}(\Delta \otimes id)(F) = F^{(23)}(id \otimes \Delta)(F)
\]

hold in \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (h^k) \) and in \( \lim_{\to k} \lim_{\to N}(U/I_N) \otimes (h^k) \), for any \( x \in U \).

**Corollary 8.2.** \( F_1 \) and \( F_1^{-1} \) belong to \( \lim_{\to k} \lim_{\to N}(A/I_N^{(A)}) \otimes B^{out}/(h^k) \); \( F_2 \) and \( F_2^{-1} \) belong to \( \lim_{\to k} \lim_{\to N} A^{out} \otimes (B/I_N^{(B)})/(h^k) \).
Proposition 9.2. \[
\lim_{\alpha \in \mathbb{N}} A^\text{out}[\alpha] \otimes B^\text{out}[-\alpha]/(\mathfrak{h}^k) = (\text{id} \otimes p_{\text{in}}^{(B)})(F) \text{ and } F_{\text{int},\text{out}} = (p_{\text{in}}^{(A)} \otimes \text{id})(F). \]
Since \( p_{\text{in}}^{(A)} \) preserves the degree, the \( \mathfrak{h} \)-adic valuation of the bidegree \((\alpha, -\alpha)\) part of \( F_{\text{int},\text{out}} \) tends to infinity with \( \alpha \). It follows that the same is true for \( F_2 \), therefore \( F_2 \) belongs to \( \lim_{\alpha \in \mathbb{N}} A^\text{out} \otimes (B/I_N^{(B)})/(\mathfrak{h}^k) \). Since the bidegree \((\alpha, -\alpha)\) part of \( F_2 \) is 1 1 if \( \alpha = 0 \) and has positive valuation else, \( F_2^{-1} \) belongs to the same completion. The argument is the same in the case of \( F_1 \). \( \square \)

Remark 7. The \( R \)-matrix of \((U_{h\mathfrak{g}}, \Delta)\) is then \( \mathcal{R} = F^{(21)} \mathcal{R}_b \).

9. QUASI-HOPF STRUCTURES ON \( U_{h\mathfrak{g}} \) AND \( U_{h\mathfrak{g}}^\text{out} \)

In this section, we will denote \( U_{h\mathfrak{g}}^\text{out} \) by \( U^\text{out} \). Let us set \( \tilde{F}_2 = F_2 F_{\text{int}} \). Then \( F = \tilde{F}_2 F_1 \),

with \( F_1, F_1^{-1} \in \lim_{\leftarrow k \leftarrow N} (U/I_N) \otimes U^\text{out}/(\mathfrak{h}^k) \), and \( F_2, F_2^{-1} \in \lim_{\leftarrow k \leftarrow N} (U^\text{out} \otimes U/I_N)/(\mathfrak{h}^k) \).

Let us set, for \( x \) in \( U \),

\[
\Delta^\text{out}(x) = F_1 \Delta(x) F_1^{-1}.
\]

Proposition 9.1. \( \Delta^\text{out} \) is an algebra morphism from \( U \) to \( \lim_{\leftarrow k}(U \otimes U)/(\mathfrak{h}^k) \).

Proof. Since \( \Delta \) maps \( U \) to \( U \otimes U = \lim_{\leftarrow k \leftarrow N} (U/I_N) \otimes U/(\mathfrak{h}^k) \), \( \Delta^\text{out} \) is an algebra morphism from \( U \) to \( U \otimes U \). For any integer \( k \), the intersection \( \cap_{N \geq 0} (I_N + \mathfrak{h}^k U) \) is reduced to \( \mathfrak{h}^k U \), therefore \( U \otimes U \) is a subalgebra of \( \lim_{\leftarrow k \leftarrow N} (U/I_N) \otimes (U/I_N)/(\mathfrak{h}^k) \). Moreover, for any \( x \) in \( U \), the identity \( \Delta^\text{out}(x) = \tilde{F}_2^{-1} \Delta(x) \tilde{F}_2 \) takes place in the latter algebra. Since the right side of this identity belongs to \( U \otimes U = \lim_{\leftarrow k \leftarrow N} (U/I_N) \otimes U/(\mathfrak{h}^k) \), \( \Delta^\text{out} \) takes values in the intersection \( (U \otimes U) \cap (U \otimes U) \); since for any integer \( k \), the intersection \( \cap_{N \geq 0} (I_N + \mathfrak{h}^k U) \) is reduced to \( \mathfrak{h}^k U \), this intersection is \( \lim_{\leftarrow k}(U \otimes U)/(\mathfrak{h}^k) \). \( \square \)

Proposition 9.2. \( \Delta^\text{out}(U^\text{out}) \) is contained in \( \lim_{\leftarrow k}(U^\text{out} \otimes U^\text{out})/(\mathfrak{h}^k) \), therefore \( \Delta^\text{out} \) induces an algebra morphism from \( U^\text{out} \) to \( \lim_{\leftarrow k}(U^\text{out} \otimes U^\text{out})/(\mathfrak{h}^k) \).

Proof. We have \( \Delta(U^\text{out}) \subset \lim_{\leftarrow k \leftarrow N} (U/I_N) \otimes U^\text{out}/(\mathfrak{h}^k) \), and \( \Delta^\text{out}(U^\text{out}) \subset \lim_{\leftarrow k \leftarrow N} (U^\text{out} \otimes (U/I_N))/(\mathfrak{h}^k) \). Therefore, \( \Delta^\text{out}(U^\text{out}) \) is contained in the intersection of \( \lim_{\leftarrow k \leftarrow N} (U/I_N) \otimes U^\text{out}/(\mathfrak{h}^k) \) and \( \lim_{\leftarrow k \leftarrow N} U^\text{out} \otimes (U/I_N)/(\mathfrak{h}^k) \), which is \( \lim_{\leftarrow k}(U^\text{out} \otimes U^\text{out})/(\mathfrak{h}^k) \). \( \square \)

Let us set \( \Phi = F_1^{(23)}(\text{id} \otimes \Delta)(F_1) \left( F_1^{(12)}(\Delta \otimes \text{id})(F_1) \right)^{-1} \).
Proposition 9.3. $\Phi$ belongs to $\lim_{\to k}(U^\text{out})\otimes^3/(h^k)$, and even to $\lim_{\to k} A^\text{out} \otimes U^\text{out} \otimes B^\text{out}/(h^k)$.

Proof. The argument is the same as in [6]. By its definition, $\Phi$ belongs to $\lim_{\to k} \lim_{\to N}(U/I_N)\otimes^2 \otimes B^\text{out}/(h^k)$. Since $F$ satisfies the cocycle identity (74), we have the equality

$$\Phi = (F_2^{-1})^{(23)}(id \otimes \tilde{\Delta})(F_2^{-1}) \left( (\tilde{F}_2^{-1})^{(12)}(\tilde{\Delta} \otimes id)(\tilde{F}_2^{-1}) \right)^{-1}$$

in $\lim_{\to k} \lim_{\to N}(U/I_N)\otimes^3/(h^k)$. Therefore $\Phi$ belongs to $\lim_{\to k} \lim_{\to N} A^\text{out} \otimes (U/I_N)\otimes^2/(h^k)$. We can again write $\Phi$ as

$$\Phi = ((id \otimes \Delta_{\text{out}})(F_1)) (F_2^{-1})^{(23)}(\Delta \otimes id)(F^{-1})(\Delta \otimes id)(F_2)(F_1^{-1})^{(12)},$$

which shows that it belongs to $\lim_{\to k} \lim_{\to N}(U/I_N)\otimes(U^\text{out}(U_h\mathfrak{g}+//(U_h\mathfrak{g}+ \cap I_N))U^\text{out} \otimes (U/I_N))/(h^k)$, and as

$$\Phi = (F_2^{-1})^{(23)}(id \otimes \tilde{\Delta})(F_1)(id \otimes \tilde{\Delta})(F^{-1})(\Delta \otimes id)(F_2)(F_1^{-1})^{(12)},$$

which shows that it belongs to $\lim_{\to k} \lim_{\to N}(U/I_N)\otimes(U^\text{out}(U_h\mathfrak{g}-/(U_h\mathfrak{g}- \cap I_N))U^\text{out} \otimes (U/I_N))/(h^k)$. The result now follows from the fact that the intersection of $\lim_{\to k} \lim_{\to N} U^\text{out}(U_h\mathfrak{g}+//(U_h\mathfrak{g}+ \cap I_N))U^\text{out}/(h^k)$ and $\lim_{\to k} \lim_{\to N} U^\text{out}(U_h\mathfrak{g}-/(U_h\mathfrak{g}- \cap I_N))U^\text{out}/(h^k)$ is reduced to $U^\text{out}$. 

Let us set $u_{\text{out}} = m(id \otimes S)(F)$, and $S_{\text{out}}(x) = u_{\text{out}}S(x)u_{\text{out}}^{-1}$. Then $S_{\text{out}}$ is an algebra morphism from $U$ to $\lim_{\to k} \lim_{\to N}(U/I_N)/(h^k)$. The proof of [6], Theorem 6.1, shows that $S_{\text{out}}$ is an algebra automorphism of $U$, which restricts to an algebra automorphism of $U^\text{out}$. Then

Theorem 9.1. The algebra $U$, endowed with the coproduct $\Delta_{\text{out}}$, the associator $\Phi_{\text{out}}$, the counit $\varepsilon$, the antipode $S_{\text{out}}$ and the $R$-matrix

$$\mathcal{R}_{\text{out}} = (F_1^{-1})^{(21)}\mathcal{R}_h \tilde{F}_2,$$

is a quasitriangular quasi-Hopf algebra. $U^\text{out}$ is a sub-quasi-Hopf algebra of $U$. Moreover, $\mathcal{R}_{\text{out}}$ belongs to $\lim_{\to k} \lim_{\to N} U^\text{out} \otimes (U/I_N)/(h^k)$.

Appendix A. Proof of Lemma 3.1

To show Lemma 3.1 we will prove the following statements:
1) if $(\alpha, \ldots, \beta')$ satisfies conditions (3) and (6), then the first product of (12) vanishes if we substitute $z = q^{-\partial w_1}$;
2) if $(\alpha, \ldots, \beta')$ satisfies conditions (8) and (9), then the second product of (12) vanishes when we substitute $z = q^{-\partial w_2}$;
3) if $(\alpha, \ldots, \beta')$ satisfies conditions (10) and (11), then the third product of (12) vanishes when we substitute $w_1 = q^{2\partial} w_2$. 
A.0.1. **Proof of 1) (sufficient conditions for regularity at** \( z = q^{-\theta}w_1 \)). 1) means that

\[
\alpha(q^{-\theta}w_1, w_1, w_2)q_{-2}(q^{-\theta}w_1, w_2)q_{4}(w_1, w_2) + \alpha'(q^{-\theta}w_1, w_1, w_2)q_{-2}(q^{-\theta}w_1, w_2) + \beta'(q^{-\theta}w_1, w_1, w_2) = 0.
\]

We have

\[
q_{-2}(q^{-\theta}z, w) = \exp \left( \sum_{\alpha \geq 0} \frac{q^{-2\theta} - q^{-\theta}}{\partial} q^{-\theta} \lambda_\alpha \otimes r^\alpha \right) \exp \left( (q^{-\theta} \otimes \text{id}) \tau_{-1} \right)
\]

\[
= \exp \left( \sum_{\alpha \geq 0} \frac{q^{-2\theta} - 1}{\partial} \lambda_\alpha \otimes r^\alpha \right) \exp \left( (q^{-\theta} \otimes \text{id}) \tau_{-1} \right),
\]

therefore

\[
q_{-2}(q^{-\theta}z, w) = u(z, w) + v(z, w)G^{(21)}(z, w),
\]

where

\[
u(z, w) = -u(z, w)\psi(-2h, \gamma, \partial_z)(z, w).
\]

On the other hand,

\[
q_{-2}(q^{-\theta}z, w)q_{4}(z, w)
\]

\[
= \exp \left( \sum_{\alpha \geq 0} \frac{q^{-2\theta} - 1}{\partial} \lambda_\alpha \otimes r^\alpha \right) \exp \left( (q^{-\theta} \otimes \text{id}) \tau_{-1} \right) \exp \left( \sum_{\alpha \geq 0} \frac{q^{-2\theta} - q^{-\theta}}{\partial} \lambda_\alpha \otimes r^\alpha \right)
\]

\[
= \exp \left( \sum_{\alpha \geq 0} \frac{q^{-2\theta} - 1}{\partial} \lambda_\alpha \otimes r^\alpha \right) \exp \left( \tau_2 + (q^{-\theta} \otimes \text{id}) \tau_{-1} \right)(z, w),
\]

therefore

\[
q_{-2}(q^{-\theta}z, w)q_{4}(z, w) = (u' + v'G^{(21)})(z, w),
\]

where

\[
u' = \exp \left( \tau_2 + (q^{-\theta} \otimes \text{id}) \tau_{-1} \right) \exp \left( -\phi(2h, \gamma, \partial_z \gamma, \ldots) \right), u' = -u'\psi(2h, \gamma, \partial_z \gamma, \ldots).
\]

To satisfy (72), we impose the conditions

\[
\alpha/\beta'(q^{-\theta}w_1, w_1, w_2)u'(w_1, w_2) + \alpha'/\beta'(q^{-\theta}w_1, w_1, w_2)u(w_1, w_2) + 1 = 0,
\]

\[
\alpha/\beta'(q^{-\theta}w_1, w_1, w_2)v'(w_1, w_2) + \alpha'/\beta'(q^{-\theta}w_1, w_1, w_2)v(w_1, w_2) = 0,
\]

which give

\[
\alpha/\beta'(q^{-\theta}w_1, w_1, w_2) = -\frac{v}{u'u - uv'}(w_1, w_2), \quad \alpha'/\beta'(q^{-\theta}w_1, w_1, w_2) = -\frac{v'}{u'v - uv'}(w_1, w_2),
\]

that is (3) and (7).
A.0.2. Proof of 2) (regularity at $z = q^{-\theta}w_2$). 2) means that
\[
\alpha(q^{-\theta}w_2, w_1, w_2)q_2(q^{-\theta}w_2, w_1)q_4(w_1, w_2) + \beta(q^{-\theta}w_2, w_1, w_2)q_4(w_1, w_2)
+ \alpha'(q^{-\theta}w_2, w_1, w_2)q_2(q^{-\theta}w_2, w_1) = 0,
\]
in other terms
\[
\alpha(q^{-\theta}w_2, w_1, w_2)q_2(q^{-\theta}w_2, w_1) + \beta(q^{-\theta}w_2, w_1, w_2)\quad(73)
+ \alpha'(q^{-\theta}w_2, w_1, w_2)q_2(q^{-\theta}w_2, w_1)q_4(w_1, w_2)^{-1} = 0.
\]
We have
\[
q_2(q^{-\theta}w_2, w_1) = u(w_2, w_1) + v(w_2, w_1)G(w_1, w_2),
\]
and
\[
q_2(q^{-\theta}w_2, w_1)q_4(w_1, w_2)^{-1} = q_2(q^{-\theta}w_2, w_1)q_4(w_2, w_1) = \\
\exp\left(\sum_{\alpha \geq 0} \frac{q^{-\theta} - q^{-\theta}\lambda_\alpha \otimes r^\alpha}{\partial} \right) \exp\left((q^{-\theta} \otimes id)\tau_{-1}\right) \exp\left(\sum_{\alpha \geq 0} \frac{q^{2\theta} - q^{-2\theta}}{\partial} \lambda_\alpha \otimes r^\alpha\right) \\
\exp(\tau_2)(w_2, w_1)
\]
\[
= \exp(\tau_2 + (q^{-\theta} \otimes id)\tau_{-1}) \exp\left(\sum_{\alpha \geq 0} \frac{q^{2\theta} - 1}{\partial} \lambda_\alpha \otimes r^\alpha\right) (w_2, w_1)
\]
\[
\exp(\tau_2 + (q^{-\theta} \otimes id)\tau_{-1}) \exp(-\phi(2\hbar)) \left(1 - G^{(21)} \psi(2\hbar)\right) (w_2, w_1).
\]
We have therefore
\[
q_2(q^{-\theta}w_2, w_1) = l(w_1, w_2) + m(w_1, w_2)G^{(21)}(w_1, w_2),
q_2(q^{-\theta}w_2, w_1)q_4(w_1, w_2)^{-1} = l'(w_1, w_2) + m'(w_1, w_2)G^{(21)}(w_1, w_2),
\]
with
\[
l(w_1, w_2) = u(w_2, w_1), \quad m(w_1, w_2) = -v(w_2, w_1) = -l\psi(-2\hbar)^{(21)}(w_1, w_2),
\]
and
\[
l'(w_1, w_2) = \exp(\tau_2 + (q^{-\theta} \otimes id)\tau_{-1}) \exp(-\phi(2\hbar))(w_2, w_1),
\]
\[
m'(w_1, w_2) \equiv l'(w_1, w_2)\psi(2\hbar)(w_2, w_1).
\]
Then (73) is satisfied if we impose
\[
\alpha(q^{-\theta}w_2, w_1, w_2)l(w_1, w_2) + \alpha'(q^{-\theta}w_2, w_1, w_2)l'(w_1, w_2) + \beta(q^{-\theta}w_2, w_1, w_2) = 0,
\]
\[
\alpha(q^{-\theta}w_2, w_1, w_2)m(w_1, w_2) + \alpha'(q^{-\theta}w_2, w_1, w_2)m'(w_1, w_2) = 0,
\]
so that
\[
\alpha/\beta(q^{-\theta}w_2, w_1, w_2) = \frac{m'}{m'l' - lm'}(w_1, w_2) = \frac{1}{l} \frac{\psi(2\hbar)^{(21)}}{\psi(-2\hbar)^{(21)} - \psi(2\hbar)^{(21)}}(w_1, w_2),
\]

\[
\alpha' / \beta(q^{-\theta}w_2, w_1, w_2) = -\frac{m}{ml' - lm'}(w_1, w_2)
\]
\[
= \frac{1}{l'} \psi(-2\hbar, \gamma, \cdots)^{(21)}(w_1, w_2)\psi(2\hbar, \gamma, \cdots)^{(21)} - \psi(-2\hbar, \gamma, \cdots)^{(21)}(w_1, w_2),
\]
that is (8) and (9).

A.0.3. Regularity at \( w_1 = q^{2\theta}w_2 \). 3) means that
\[
\alpha(z, q^{2\theta}w_2, w_2)q_{-2}(z, q^{2\theta}w_2)q_{-2}(z, w_2) + \beta(z, q^{2\theta}w_2, w_2)q_{-2}(z, w_2) + \gamma(z, q^{2\theta}w_2, w_2) = 0,
\]
which we write as
\[
\alpha(z, q^{2\theta}w_2, q^{\theta}w_2)q_{-2}(z, q^{2\theta}w_2)q_{-2}(z, q^{\theta}w_2) + \beta(z, q^{3\theta}w_2, q^{\theta}w_2)q_{-2}(z, q^{\theta}w_2) + \gamma(z, q^{3\theta}w_2, q^{\theta}w_2) = 0.
\]

We have
\[
q_{-2}(z, q^{\theta}w_2) = q_{-2}(q^{\theta}w_2, z)^{-1}
\]
\[
= \exp\left(\sum_{\alpha \geq 0} \frac{q^{2\theta} - 1}{\partial} \lambda_{\alpha} \otimes r^{\alpha}\right) \exp \left(- (q^{\theta} \otimes id)(\tau_{-1})\right) (w_2, z)
\]
\[
= \exp\left(- (q^{\theta} \otimes id)(\tau_{-1})\right) \exp(-\phi(2\hbar))(1 - G^{(21)}\psi(2\hbar))(w_2, z),
\]
and
\[
q_{-2}(z, q^{2\theta}w_2)q_{-2}(z, q^{\theta}w_2) = q_{-2}(q^{3\theta}w_2, z)^{-1}q_{-2}(q^{\theta}w_2, z)^{-1}
\]
\[
= \exp\left(\sum_{\alpha \geq 0} \frac{q^{2\theta} - 1}{\partial} \lambda_{\alpha} \otimes r^{\alpha}\right) \exp \left(- (q^{3\theta} + q^{\theta}) \otimes id(\tau_{-1})\right) (w_2, z)
\]
\[
= \exp\left(- (q^{3\theta} + q^{\theta}) \otimes id(\tau_{-1})\right) \exp(-\phi(4\hbar))(1 - G^{(21)}\psi(4\hbar))(w_2, z).
\]

Therefore
\[
q_{-2}(z, q^{\theta}w_2) = r(z, w_2) + s(z, w_2)G(z, w_2),
\]
\[
q_{-2}(z, q^{2\theta}w_2)q_{-2}(z, q^{\theta}w_2) = r'(z, w_2) + s'(z, w_2)G(z, w_2),
\]
with
\[
r(z, w_2) = \exp \left(- (q^{\theta} \otimes id)(\tau_{-1})\right) \exp(-\phi(2\hbar))(w_2, z),
\]
\[
r'(z, w_2) = \exp \left(- (q^{3\theta} + q^{\theta}) \otimes id(\tau_{-1})\right) \exp(-\phi(4\hbar))(w_2, z),
\]
\[
s(z, w_2) = r\psi(2\hbar)^{(21)}(z, w_2), \quad s' = r'\psi(4\hbar)^{(21)}(z, w_2),
\]
so (74) is fulfilled if \( \alpha, \beta \) and \( \gamma \) satisfy the following conditions:
\[
\alpha(z, q^{3\theta}w_2, q^{\theta}w_2)r'(z, w_2) + \beta(z, q^{3\theta}w_2, q^{\theta}w_2)r(z, w_2) + \gamma(z, q^{3\theta}w_2, q^{\theta}w_2) = 0,
\]
\[
\alpha(z, q^{3\theta}w_2, q^{\theta}w_2)s'(z, w_2) + \beta(z, q^{3\theta}w_2, q^{\theta}w_2)s(z, w_2) = 0,
\]
so that
\[
\alpha / \beta(z, q^{3\theta}w_2, q^{\theta}w_2) = -s / s'(z, w_2)
\]
and
\[
\gamma / \beta (z, q^3 w_2, q^9 w_2) = \frac{r's - rs'}{s'}(z, w_2),
\]
that is (10) and (11). This ends the proof of Lemma 3.1.

References

[1] V. Drinfeld, Quantum groups, Proc. ICM-86, Berkeley, eds. AMS, 798-820.
[2] V. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36:2 (1988), 212-6.
[3] V. Drinfeld, Quasi-Hopf algebras, Leningrad Math. J. 1:6 (1990) 1419-57.
[4] B. Enriquez, PBW and duality theorems for quantum groups and quantum current algebras, math/9904113.
[5] B. Enriquez, G. Felder, Elliptic quantum group \( E_{\tau, \eta} (\mathfrak{sl}_2) \) and quasi-Hopf algebras, Commun. Math. Phys. 195 (1998), 651-89.
[6] B. Enriquez, G. Felder, A construction of Hopf algebra cocycles for double Yangians, q-alg/9703012. J. Phys. A: Math. Gen. 31 (1998), 2401-13.
[7] B. Enriquez, G. Felder, Commuting differential and difference operators associated with complex curves II, math/9812152.
[8] B. Enriquez, V. Rubtsov, Quantum groups in higher genus and Drinfeld’s new realizations method (\( \mathfrak{sl}_2 \) case), q-alg/9601022, Ann. Sci. Ec. Norm. Sup. 30, sér. 4 (1997), 821-46.
[9] B. Enriquez, V. Rubtsov, Quasi-Hopf algebras associated with \( \mathfrak{sl}_2 \) and complex curves, q-alg/9608005. Israel J. of Math. 112 (1999), 61-108.
[10] B. Enriquez, Y. Kosmann-Schwarzbach, Quantum homogeneous spaces and quasi-Hopf algebras, math/9912243.
[11] B. Feigin, A. Odesskii, A family of elliptic algebras, Int. Math. Res. Notices, 11 (1997), 531-9.
[12] S. Khoroshkin, V. Tolstoy, On Drinfeld’s realization of quantum affine algebras, J. Geom. Phys. 11 (1993), 445-52.