RIGIDITY IN HIGHER REPRESENTATION THEORY

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Abstract. We describe a categorical $g$ action, called a $(g, \theta)$ action, which is easier to check in practice. Most categorical $g$ actions can be shown to be of this form. The main result is that a $(g, \theta)$ action carries actions of quiver Hecke algebras (KLR algebras). We discuss applications of this fact to categorical vertex operators, affine Grassmannians (or Nakajima quiver varieties) and to homological knot invariants.

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1. Introduction

The higher representation theory of Kac-Moody Lie algebra $g$ involves the action of $U_q(g)$ on categories. This means that to each weight $\lambda$ of $g$ one assigns an additive (graded) category $C(\lambda)$ and to generators $E_i$ and $F_i$ of $U_q(g)$ one assigns functors $E_i : C(\lambda) \to C(\lambda + \alpha_i)$ and $F_i : C(\lambda + \alpha_i) \to C(\lambda)$. These functors are then required to satisfy certain relations analogous to those in $U_q(g)$. For example, the relation $[E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}}$ becomes

(1) $E_iF_i|_{C(\lambda)} \cong F_iE_i|_{C(\lambda)} \bigoplus_{[\lambda, \alpha_i]} \text{id}_{C(\lambda)}$ if $\langle \lambda, \alpha_i \rangle \geq 0$

(2) $F_iE_i|_{C(\lambda)} \cong E_iF_i|_{C(\lambda)} \bigoplus_{[-\lambda, \alpha_i]} \text{id}_{C(\lambda)}$ if $\langle \lambda, \alpha_i \rangle \leq 0$. 

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Since such categorical actions involve functors one would like to understand their natural transformations. For instance, one might wonder about \( \text{End}(E_i E_j) \) or \( \text{End}(E_i E_j) \) or any other composition of \( E \)'s and \( F \)'s. Moreover, these natural transformations should induce the isomorphisms in (1) and (2).

One answer to this question is given by Khovanov-Lauda [KL1, KL2, KL3] and Chuang-Rouquier [CR, R]. They suggest that one should require the endomorphism algebras of composition of \( E \)'s to be certain Hecke quiver algebras (or KLR algebras). More precisely, Khovanov-Lauda show that these algebras, together with several other explicit natural transformations relating \( E \)'s and \( F \)'s, determine all the necessary isomorphisms of functors such as the ones in (1) and (2). In this sense this gives a categorification of quantum groups by replacing isomorphisms of functors with equalities of natural transformations.

On the other hand, suppose one is given a “naive” categorical action of \( g \). This means that we have categories \( \mathcal{C}(\lambda) \) and functors \( E_i, F_i \) together with the knowledge that certain compositions of functors, such as those in (1) or (2), are isomorphic. There are many examples of such actions (see section 1.2 for a couple). In such cases there is little reason to expect the quiver Hecke algebras to act. In other words, one would expect the space of natural transformations, between say compositions of \( E \)'s, to depend on the choice of categories \( \mathcal{C}(\lambda) \).

The purpose of this paper is to show that, under some mild conditions, a naive categorical action of \( g \) carries an action of the quiver Hecke algebras by natural transformations. This is a rigidity result for the higher representation theory of Kac-Moody Lie algebras \( g \) because it implies that the spaces of natural transformations are determined by \( g \).

We do not know if such rigidity phenomena are common. For example, it is not clear the extent to which similar results hold for the higher representation theory of the Heisenberg algebras from [CLl1].

1.1. Main results. In section 2.2 we introduce the concept of a \((g, \theta)\) action. Roughly this is a naive categorical \( g \) action together with another piece of data \( \theta \). The main result of this paper, Theorem 2.2, states the following.

**Theorem 1.1.** A \((g, \theta)\) action carries an action of a quiver Hecke algebra (modulo transient maps).

**Remark 1.2.** Transient maps are certain negligible 2-morphisms as defined in section 7. If \( g = \mathfrak{sl}_n \) then this result holds without having to mod out by transient maps (section 12).

The definition of a \((g, \theta)\) action is designed to be as minimal as possible so that it is easier to check in examples. For example, we do not require the existence of divided powers \( E_i^{(r)} \) and \( F_i^{(r)} \). These only appear \textit{a posteriori} as a consequence of the affine nilHecke relations. We also do not require the Serre relation nor the commutativity relation \( E_i E_j \cong E_j E_i \) when \( (\alpha_i, \alpha_j) = 0 \).

The proof of Theorem 2.2 is largely based on a sequence of Hom-space calculations. For example, one can show by using adjunction together with isomorphisms (1) and (2) that \( \text{Hom}(E_i E_j, E_j E_i) \) is one-dimensional if \( (\alpha_i, \alpha_j) = 0 \). Thus, up to rescaling, this gives us a map \( T_{ij} : E_i E_j \rightarrow E_j E_i \) which subsequently turns out to be the isomorphism \( E_i E_j \cong E_j E_i \). These Hom-space calculations are performed in a series of Lemmas in the appendix. They reflect a certain rigidity of categorical \( g \) actions.

Finally, Theorem 2.2 together with the main result of [CLa] implies the following.

**Corollary 1.3.** A \((g, \theta)\) action induces a 2-representation in the sense of Khovanov-Lauda (modulo transient maps).

**Remark 1.4.** As before, if \( g = \mathfrak{sl}_n \) then the condition on transient maps is not necessary.

Recall that a 2-representation in this sense is a 2-functor \( \mathcal{U}_Q(g) \rightarrow \mathcal{K} \). Here \( \mathcal{U}_Q(g) \) is the 2-category defined by Khovanov-Lauda which categorifies \( g \) while \( \mathcal{K} \) is the target category.
1.2. Applications. In section 14 we discuss three applications of Theorem 2.2.

The first application (section 14.1) is to categorical vertex operators. In [CL12] we construct a $(\mathfrak{g}, \theta)$ action on the homotopy category of a categorification $\mathcal{F}_T$ of the Fock space for a particular Heisenberg algebra $\mathfrak{h}_T$. Corollary 1.3 implies that this action can be lifted to a 2-representation of $\hat{\mathcal{U}}_q(\mathfrak{g})$ (in this case there are no transient maps to worry about).

It is interesting to note that in this case it is very difficult to work with (or even define) divided powers $E_i^{(k)}$ and $F_i^{(k)}$. Likewise it is very difficult to check the Serre relation. Subsequently the computations in [CL12] are made tractable because a $(\mathfrak{g}, \theta)$ action does not require checking either of these relations.

A second application is to geometric categorical $\mathfrak{g}$ actions (section 14.2). We show that such a geometric action, defined in [CK3], induces a $(\mathfrak{g}, \theta)$ action (essentially, a geometric categorical $\mathfrak{g}$ action is a geometric way to define a $(\mathfrak{g}, \theta)$ action). This implies that the quiver Hecke algebras act on the 2-category $\mathcal{K}^n_{Gr,m}$ constructed in [CKL1, C] using coherent sheaves on certain varieties $Y(\lambda)$. These $Y(\lambda)$ are convolution varieties obtained from the affine Grassmannian of $\text{PGL}_m$. As in the previous example, this action of the quiver Hecke algebras is difficult to see directly.

For our final application (section 14.3) we discuss how the rigidity of categorical $\mathfrak{g}$ actions implies a rigidity for knot homologies. More precisely, in [C] we explained how to use categorical $(\mathfrak{sl}_\infty, \theta)$ actions to define homological knot invariants. By Corollary 1.3 we know any such action lifts to a 2-representation of $\hat{\mathcal{U}}_q(\mathfrak{sl}_\infty)$. Subsequently this knot invariant does not depend on the particular 2-representation we choose. In particular, this means that various homological knot invariants defined by very different means (coherent sheaves, category $\mathcal{O}$, matrix factorizations, foams, etc.) must be isomorphic.

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2. $(\mathfrak{g}, \theta)$ actions

2.1. Notation. Fix a base field $k$, which is not assumed to be of characteristic 0, nor algebraically closed. Let $\Gamma$ be a connected graph without multiple edges or loops and with finite vertex set $I$ (i.e. simply laced Dynkin diagram). In addition, fix the following data:

- a free $\mathbb{Z}$ module $X$ (the weight lattice),
- for $i \in I$ an element $\alpha_i \in X$ (simple roots),
- for $i \in I$ an element $\Lambda_i \in X$ (fundamental weights),
- a symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $X$.

These data should satisfy:

- the set $\{\alpha_i\}_{i \in I}$ is linearly independent,
- $C_{i,j} = \langle \alpha_i, \alpha_j \rangle$ (the Cartan matrix) so that $\langle \alpha_i, \alpha_i \rangle = 2$ and for $i \neq j$, $\langle \alpha_i, \alpha_j \rangle \in \{0, -1\}$ depending on whether or not $i, j \in I$ are joined by an edge,
- $\langle \Lambda_i, \alpha_j \rangle = \delta_{i,j}$ for all $i, j \in I$.

We will often abbreviate $\langle \lambda, \alpha_i \rangle = \lambda_i$ and $\langle \alpha_i, \alpha_j \rangle = \langle i, j \rangle$. The root lattice will be denoted $Y$ and $Y_k := Y \otimes_{\mathbb{Z}} k$. Associated to a Cartan datum we fix a choice of scalars $Q$ consisting of $t_{ij} \in k^\times$ for all $i \neq j \in I$ such that $t_{ij} = t_{ji}$ if $\langle i, j \rangle = 0$.

2.2. Definition of $(\mathfrak{g}, \theta)$ actions. Associated to $\Gamma$ we have a Kac-Moody Lie algebra $\mathfrak{g}$. A $(\mathfrak{g}, \theta)$ action consists of a target graded, additive, $k$-linear idempotent complete 2-category $\mathcal{K}$ where the objects (0-morphisms) are indexed by $\lambda \in X$ and equipped with

(i) 1-morphisms: $E_i \mathbf{1}_\lambda = \mathbf{1}_{\lambda + \alpha_i} E_i$ and $F_i \mathbf{1}_{\lambda + \alpha_i} = \mathbf{1}_{\lambda} F_i$ where $\mathbf{1}_\lambda$ is the identity 1-morphism of $\lambda$.
(ii) 2-morphisms: for each $\lambda \in X$, a linear map $Y_k \rightarrow \text{End}^2(\mathbf{1}_\lambda)$. 
Remark 2.1. We will abuse notation and denote by \( \theta \in \text{End}^2(1_{\lambda}) \) the image of \( \theta \in Y_k \) under the linear maps above.

By a graded 2-category we mean a 2-category whose 1-morphisms are equipped with an auto-equivalence \( \langle 1 \rangle \). \( \mathcal{K} \) is idempotent complete if for any 2-morphism \( f \) with \( f^2 = f \) the image of \( f \) is contained in \( \mathcal{K} \). On this data we impose the following conditions.

(i) \( \text{Hom}(1_{\lambda}, 1_{\lambda} \langle l \rangle) \) is zero if \( l < 0 \) and one-dimensional if \( l = 0 \) and \( 1_{\lambda} \neq 0 \). Moreover, the space of maps between any two 1-morphisms is finite dimensional.

(ii) \( E_i \) and \( F_i \) are left and right adjoints of each other up to specified shifts. More precisely

(a) \( \langle E_i 1_{\lambda} \rangle_R \cong 1_{\lambda} F_i \langle \lambda_i + 1 \rangle \)

(b) \( \langle E_i 1_{\lambda} \rangle_L \cong 1_{\lambda} F_i \langle -\lambda_i - 1 \rangle \).

(iii) We have

\[
E_i F_j 1_{\lambda} \cong F_j E_i 1_{\lambda} \bigoplus_{\lambda_1} 1_{\lambda} \quad \text{if } \lambda_i \geq 0
\]

\[
F_i E_i 1_{\lambda} \cong E_i F_i 1_{\lambda} \bigoplus_{\lambda_1} 1_{\lambda} \quad \text{if } \lambda_i \leq 0
\]

(iv) If \( i \neq j \) then \( F_j E_i 1_{\lambda} \cong E_i F_j 1_{\lambda} \).

(v) If \( \lambda_i \geq 0 \) then map \( (1 \theta I) : \text{End}^2(E_i 1_{\lambda} F_i) \) induces an isomorphism between \( \lambda_i + 1 \) (resp. zero) of the \( \lambda_i + 2 \) summands \( 1_{\lambda+\alpha_i} \), when \( \langle \theta, \alpha_i \rangle \neq 0 \) (resp. \( \langle \theta, \alpha_i \rangle = 0 \)). If \( \lambda_i \leq 0 \) then the analogous result holds for \( (1 \theta I) \in \text{End}^2(F_i 1_{\lambda} E_i) \).

(vi) If \( \alpha = \alpha_i = \alpha_j + \alpha_k \) for some \( i, j \in I \) with \( \langle i, j \rangle = -1 \) then \( 1_{\lambda+r\alpha} = 0 \) for \( r \gg 0 \) or \( r \ll 0 \).

(vii) If \( \delta = \alpha_i + \alpha_j + \alpha_k \) with \( i, j, k \in I \) forming a square then \( 1_{\lambda+r\delta} = 0 \) for \( r \gg 0 \) and \( \langle \lambda, \delta \rangle > 0 \) if \( 1_{\lambda} \neq 0 \).

(viii) Suppose \( i \neq j \) and \( \lambda \in X \). If \( 1_{\lambda+\alpha_i} \) and \( 1_{\lambda+\alpha_j} \) are nonzero then \( 1_{\lambda} \) and \( 1_{\lambda+\alpha_i+\alpha_j} \) are also nonzero.

One thing to notice is that there are no divided powers \( E_i^{(r)} \) or \( F_j^{(r)} \) mentioned in the definition above. This is because their existence follows \textit{a posteriori}. In certain examples (such as [CLi2]) these divided powers are complicated and it is helpful to not have to deal with them. Here are a few more remarks about the conditions above.

- The condition \( 1_{\lambda} = 0 \) (or \( 1_{\lambda} \neq 0 \)) means that the object in \( \mathcal{K} \) indexed by \( \lambda \) is zero (or nonzero).
- Condition (v) is necessary in order to avoid “degenerate” examples of categorical \( g \) actions (c.f. [CR, Remark 5.19]).
- Condition (vii) is only used in the proofs of Lemmas 11.3 and A.12.
- Condition (viii) is only used in a few places to shorten the argument (mostly in section 6). It is not used, for instance, to prove the Serre relation. It is possible to remove this condition but it would make several arguments more cumbersome and lengthy.

2.3. The quiver Hecke algebra action. Given a \((g, \theta)\) action on \( \mathcal{K} \), an action of the quiver Hecke algebra (a.k.a. KLR algebra) \( R_Q \) on \( \mathcal{K} \) consists of a choice of 2-morphisms:

(i) \( X_i : E_i 1_{\lambda} \to E_i 1_{\lambda} \langle 2 \rangle \) for each \( i \in I, \lambda \in X \),

(ii) \( T_{ij} : E_i E_j 1_{\lambda} \to E_j E_i 1_{\lambda} \langle -\langle i, j \rangle \rangle \) for each \( i, j \in I, \lambda \in X \).

These 2-morphisms must satisfy the following relations:

(i) \( T_{ij} \) and \( X_i \) satisfy the affine nilHecke relations

\[
T_{ii}(X_i I) = (IX_i)T_{ii} + II \quad \text{and} \quad (X_i I)T_{ii} = T_{ii}(IX_i) + II \in \text{End}(E_i E_i),
\]

\[
T_{ii}^2 = 0 \in \text{End}^{-1}(E_i E_i) \quad \text{and} \quad (T_{ii} I)(IT_{ii})(T_{ii} I) = (IT_{ii})(T_{ii} I)(IT_{ii}) \in \text{End}^{-6}(E_i E_i E_i).
\]
(ii) If $i \neq j \in I$ then

$$(IX_i)T_{ij} = T_{ij}(X_iI) \text{ and } (X_jI)T_{ij} = T_{ij}(IX_j) \in \text{Hom}(E_iE_j, E_jE_i(-\langle i, j \rangle + 2)).$$

(iii) If $i, j \in I$ with $\langle i, j \rangle = -1$ then

$$(T_{ji})(IT_{ii})(T_{ij}) = (IT_{ij})(T_{ii})I + t_iT_{ij}(III) \in \text{End}(E_iE_jE_i)$$

$$(T_{ji})(I_{ji}) = T_{ij}(X_iI) + t_iT_{ij}(X_jI) \in \text{End}(E_iE_j).$$

(iv) If $i, j \in I$ with $\langle i, j \rangle = 0$ then $(T_{ji})(T_{ij}) = t_iT_{ij}(II) \in \text{End}(E_iE_j).$

(v) If $i, j, k \in I$ with $i \neq k$ then

$$(T_{jk})(IT_{ik})(T_{ij}) = (IT_{ij})(T_{ik})(IT_{jk}) : E_iE_jE_k \to E_kE_iE_i(-\ell_{ijk})$$

where $\ell_{ijk} = \langle i, j \rangle + \langle i, k \rangle + \langle j, k \rangle.$

(vi) Far apart maps that do not interact with each other commute. For instance,

$$(X_iI)(IX_j) = (IX_j)(X_iI) \text{ and } (T_{ij})(IIX_k) = (IIX_k)(T_{ij}).$$

The following is the main result in this paper.

**Theorem 2.2.** Modulo transients, a $(g, \theta)$ action carries an action of a quiver Hecke algebra $R_Q$.

**Remark 2.3.** Transients are certain negligible 2-morphisms defined in section 7. However, if $g = \mathfrak{sl}_n$, then we show in section 12 that Theorem 2.2 holds without having to mod out by transients.

Unfortunately, in Theorem 2.2 we cannot say for what choice of $Q$ the algebra $R_Q$ acts. One exception is when $\Gamma$ is a tree in which case any two choices of $Q$ are equivalent (so there is no ambiguity).

2.4. **Outline of proof.** Many of the arguments are based on a series of Hom-space calculations. These calculations only involve properties in the definition of a $(g, \theta)$ action. In particular, they do not use results proved in the main body of the paper. For this reason we have separated them and placed them in the appendix. The common argument used is to prove these lemmas involves repeatedly applying adjunction (condition ii) and then using the commutator relation between $E$'s and $F$'s (conditions iii and iv) to simplify.

The proof of Theorem 2.2 follows a sequence of 10 steps where we gradually prove the quiver Hecke algebra relations. The order of these steps is important as we often use earlier results in later proofs. We now briefly describe these 10 steps.

- **Step #0.** Define 2-morphisms $T_{ij}, u_{ij}, v_{ij}$ and adjunctions $adj_j, adj_i$. The maps $T_{ij}$ will be subsequently rescaled but nonetheless they can be used to define a pairing

$$(\cdot, \cdot)_{\lambda} : \text{End}^2(1_{\lambda}) \otimes_k Y_k \to k.$$

- **Step #1.** Study the structure of $\text{End}(E_iE_j).$ In the process show that $T_{iii}'$ and $T_{iij}'$ are nonzero which allows one to prove a preliminary version of the affine nilHecke relations (Proposition 4.7). Proposition 4.7 also implies that $E_iE_j \cong \oplus_{\langle i, j \rangle} E_{iij}^{(2)}$ which is used later.

- **Step #2.** Show that $T_{iii}', T_{ijj}', T_{jii}', T_{jij}$ are all nonzero. Show that $E_iE_j \cong E_iE_j$ if $\langle i, j \rangle = 0.$

- **Step #3.** Rescale $T_{ii}$ so that $T_{iii} = T_{iij}, T_{ijj} = T_{jii}$ and $T_{jij} = T_{jii}.$

- **Step #4.** Define transient maps (from here until step #9 in section 12 we will work modulo transients). Rescale $\alpha_i \in \text{End}^2(1_{\lambda})$ so that $\langle \alpha_i, \alpha_i \rangle_{\lambda} = 2.$ Use this to define $X_i \in \text{End}^2(E_i)$.

- **Step #5.** Prove the affine nilHecke relations.

- **Step #6.** Prove the Serre relation $E_iE_jE_i \cong E_{iij}^{(2)}E_jE_{iij}^{(2)}$ if $\langle i, j \rangle = -1.$
• Step #7. If $\langle i, j \rangle = -1$ rescale $T_{ij}$ so that, for some $t_{ij}, t_{ji} \in \mathbb{k}^\times$, we have
\[ T_{iji} = T'_{iji} + t_{ij}(III) \in \text{End}(E_iE_jE_i) \quad \text{and} \quad T_{ji} = T'_{ji} + t_{ji}(III) \in \text{End}(E_jE_iE_j) \]
$T_{ji}T_{ij} = t_{ij}(X_iI) + t_{ji}(I X_j) \in \text{End}^2(E_jE_i)$ and $T_{ij}T_{ji} = t_{ij}(I X_i) + t_{ji}(X_j I) \in \text{End}^2(E_jE_i)$

• Step #8. Rescale maps $T_{ij}$ again so that $T_{ijk} = T'_{ijk}$ when $i, j, k$ are distinct.
• Step #9. Show that if $g = s_\lambda$ one can redefine the $X$’s so that the quiver Hecke algebra relations hold on the nose and not just modulo transients.

In the sequence of steps above we tried to prove as many of the relations involving $T$’s as possible before moving onto relations involving $T$’s. The former are usually easier to prove because the spaces of maps involved are smaller. For example, $\dim \text{End}^6(\mathbf{E}E_\mathbf{E}_i) \leq 1$ so the relation $T_{iii} = T'_{iii}$ is almost immediate once we know that $T_{ii} = T'_{ii}$ and $T'_{iii}$ are both nonzero. On the other hand, the relation $T_{ji}T_{ij} = t_{ij}(X_iI) + t_{ji}(I X_j)$ is much more difficult because $\dim \text{End}^2(\mathbf{E}E_j)$ is relatively large.

3. Step #0 – Preliminary definitions and properties

3.1. Notation and assumptions. From now on we will assume that if $\lambda, \mu \in \mathcal{X}$ correspond to nonzero weights spaces inside $\mathcal{K}$ then $\lambda - \mu$ belongs to the root lattice $Y$ (in other words, $\lambda$ and $\mu$ belong to the same coset in $X/Y$). This assumption is only for notational convenience and not an additional condition because $E$’s and $F$’s only go between weights in the same coset of $X/Y$.

For a weight $\lambda$ we will denote by $1_\lambda$ the identity 1-morphism of $\lambda$ in $\mathcal{K}$. The identity 2-morphism of $1_\lambda$ will be denoted $I_\lambda$. Sometimes, to shorten notation, we will omit $I_\lambda$.

Given two 2-morphisms $f, g \in \mathcal{K}$ we write $f \sim g$ to mean that $f$ equals some nonzero multiple of $g$. If $A$ and $B$ are 1-morphisms in $\mathcal{K}$ then $\text{End}^d(A)$ will be short hand for $\text{Hom}(A, A(d))$ and likewise $\text{Hom}^d(A, B)$ for $\text{Hom}(A, B(d))$.

The fact that the space of maps between any two 1-morphisms in $\mathcal{K}$ is finite dimensional means that the Krull-Schmidt property holds. This means that any 1-morphism has a unique direct sum decomposition (see section 2.2 of [Ri]). In particular, this means that if $A, B, C$ are morphisms and $V$ is a $\mathbb{Z}$-graded vector space then we have the following cancellation laws (see section 4 of [CK3]):

\[ A \oplus B \cong A \oplus C \Rightarrow B \cong C \]
\[ A \otimes_k V \cong B \otimes_k V \Rightarrow A \cong B. \]

Suppose that $A$ is a 1-morphism in $\mathcal{K}$ with $\text{End}^d(A) \cong \mathbb{k}$ and that $X, Y$ are arbitrary 1-morphisms. Then a 2-morphism $f : X \to Y$ gives rise to a bilinear pairing $\text{Hom}(A, X) \times \text{Hom}(Y, A) \to \text{Hom}(A, A) \cong \mathbb{k}$. We define the $A$-rank of $f$ to be the rank of this bilinear pairing.

We can also define $A$-rank as follows. Choose (non-canonical) direct sum decompositions $X = A \otimes_k V \oplus B$ and $Y = A \otimes_k V' \oplus B'$ where $V, V'$ are $\mathbb{k}$ vector spaces and $B, B'$ do not contain $A$ as a direct summand. Then one of the matrix coefficients of $f$ is a map $A \otimes_k V \to A \otimes_k V'$, which (since $\text{End}^d(A) \cong \mathbb{k}$) is equivalent to a linear map $V \to V'$. The $A$-rank of $f$ equals the rank of this linear map. We define the total $A$-rank of $f$ as the sum of all the $A(d)$-ranks as $d$ varies over $\mathbb{Z}$. In this paper this will always turn out to be finite.

For $n \geq 1$ we denote by $[n]$ the quantum integer $q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$. By convention $[-n] = -[n]$ and $[0] = 0$. More generally,

\[ \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n] \cdots [1]}{([n-k] \cdots [1])([k] \cdots [1])}. \]

If $f = f_A q^a \in \mathbb{N}[q, q^{-1}]$ and $A$ is a 1-morphism in $\mathcal{K}$ we write $\oplus fA$ for the direct sum $\oplus_{a \in \mathbb{Z}} A^{\oplus f_a}(a)$. For example, $\oplus_{[n]} A = \oplus_{k=0}^{n-1} A(n - 1 - 2k)$. 

3.2. Definition of $T_{ij}$. Using Lemmas A.2, A.4 and A.5 we can fix nonzero maps

$$T_{ij} : E_i E_j 1_\lambda \rightarrow E_j E_i 1_\lambda (-\langle i, j \rangle)$$

for any $\lambda \in X$, $i, j \in I$. This can be done uniquely up to a nonzero multiple. For the moment we choose this multiple arbitrarily.

To shorten notation, for $i, j, k \in I$ (not necessarily distinct) we will denote

$$T_{ijk} := (T_{ik} I)(IT_{ik})(T_{ij} I), T_{ijk} := (IT_{ij})(IT_{ik}) I \in \text{Hom}(E_i E_j E_k, E_k E_j E_i (-\ell_{ijk}))$$

where $\ell_{ijk} := \langle i, j \rangle + \langle i, k \rangle + \langle j, k \rangle$.

3.3. Definition of $u_{ij}$ and $v_{ij}$. Using Lemma A.6 we fix maps the following morphisms which span the corresponding one-dimensional spaces:

$$u_{ij} : F_j E_i 1_\lambda \rightarrow E_i F_j 1_\lambda \quad \text{and} \quad v_{ij} : E_i F_j 1_\lambda \rightarrow F_j E_i 1_\lambda$$

$$\text{adj}_j : F_i E_i 1_\lambda \rightarrow 1_\lambda \langle \lambda_i + 1 \rangle \quad \text{and} \quad \text{adj}^j : 1_\lambda \rightarrow F_i E_i 1_\lambda \langle \lambda_i + 1 \rangle$$

These are uniquely defined up to a nonzero multiple. Note that if $i \neq j$ then $u_{ij}$ and $v_{ij}$ are isomorphisms since $E_i F_j \cong F_j E_i$.

**Lemma 3.1.** The compositions $(v_{ij})(u_{ij})1_\lambda \in \text{End}(F_i E_i 1_\lambda)$ and $(u_{ij})(v_{ij})1_\lambda \in \text{End}(E_i F_i 1_\lambda)$ are equal to some nonzero multiple of the identity if $\lambda_i \geq 0$ and $\lambda_i \leq 0$ respectively.

**Proof.** If $\lambda_i \geq 0$ then $E_i F_i 1_\lambda \cong F_i E_i 1_\lambda \oplus [\lambda_i] 1_\lambda$ where by adjunction

$$\text{Hom}(F_i E_i 1_\lambda, \oplus [\lambda_i] 1_\lambda) \cong \text{Hom}(E_i E_i 1_\lambda, \oplus [\lambda_i] 1_\lambda \langle -\lambda_i - 1 \rangle)$$

$$\cong \bigoplus_{r=0}^{\lambda_i-1} \text{Hom}(E_i E_i 1_\lambda, 1_\lambda \langle -2 - 2r \rangle).$$

This is zero by Lemma A.1. Thus the map $u_{ii} I_\lambda$ must be the inclusion $F_i E_i 1_\lambda \rightarrow E_i F_i 1_\lambda$. Similarly, the map $v_{ii} I_\lambda$ must be the projection $E_i F_i 1_\lambda \rightarrow F_i E_i 1_\lambda$. Subsequently, the composition $(v_{ij})(u_{ij}) I_\lambda$ is some nonzero multiple of the identity map on $F_i E_i 1_\lambda$. The case of $(u_{ii})(v_{ii}) I_\lambda$ when $\lambda_i \leq 0$ is proved similarly. \qed

3.4. The “pitchfork” relations.

**Lemma 3.2.** For $i, j \in I$ the following pairs of compositions are equal up to a nonzero multiple

$$\begin{align*}
(3) \quad & E_i E_j 1_\lambda F_i \xrightarrow{T_{ij} I} E_i E_j 1_\lambda (-d_{ij}) \xrightarrow{\text{adj}_j} E_i 1_{\lambda + \alpha_j} (-\lambda_i - 1 - d_{ij}) \\
& E_i E_j 1_\lambda F_i \xrightarrow{I v_{ij}} E_i F_j 1_{\lambda + \alpha_i + \alpha_j} E_j \xrightarrow{\text{adj}_j I} E_j 1_{\lambda + \alpha_i} (-\lambda_i - 1 - d_{ij})
\end{align*}$$

$$\begin{align*}
(4) \quad & E_j 1_\lambda \xrightarrow{\text{adj}_j'} F_i E_j 1_\lambda \langle \lambda_i + 1 \rangle \xrightarrow{v_{ij} I} F_i E_i E_j 1_\lambda \langle \lambda_i + 1 \rangle \\
& E_j 1_\lambda \xrightarrow{\text{adj}_j} F_i E_j 1_\lambda \langle \lambda_i + d_{ij} + 1 \rangle \xrightarrow{IT_{ij}} E_i F_i E_j 1_\lambda \langle \lambda_i + 1 \rangle
\end{align*}$$

$$\begin{align*}
(5) \quad & F_i E_j E_i 1_\lambda \xrightarrow{u_{ij} I} E_j F_i E_i 1_\lambda \xrightarrow{\text{adj}_j} E_i 1_\lambda \langle \lambda_i + 1 \rangle \\
& F_i E_j E_i 1_\lambda \xrightarrow{IT_{ij}} F_i E_i 1_{\lambda + \alpha_j} E_j \langle -d_{ij} \rangle \xrightarrow{\text{adj}_j I} E_j 1_\lambda \langle \lambda_i + 1 \rangle
\end{align*}$$

$$\begin{align*}
(6) \quad & E_j 1_{\lambda + \alpha_j} \xrightarrow{\text{adj}_j'} F_i E_j 1_{\lambda + \alpha_j} (-\lambda_i - 1) \xrightarrow{T_{ij} I} E_j E_i E_j 1_{\lambda + \alpha_i} (-\lambda_i - 1 - d_{ij}) \\
& E_j 1_{\lambda + \alpha_j} \xrightarrow{\text{adj}_j I} E_j E_i E_j 1_{\lambda + \alpha_i} (-\lambda_i - 1 - d_{ij}) \xrightarrow{I u_{ij}} E_i E_j E_j 1_{\lambda + \alpha_i} (-\lambda_i - 1 - d_{ij})
\end{align*}$$

where $d_{ij} := \langle i, j \rangle$. Moreover, each one these compositions is nonzero if and only if $1_\lambda, 1_{\lambda + \alpha_i}, 1_{\lambda + \alpha_j}$ and $1_{\lambda + \alpha_i + \alpha_j}$ are all nonzero.
Proof. We prove that the two compositions in (3) are equal (the proof of the others is the same). First, note that
\[
\text{Hom}(E_iE_j1_{\lambda+i}, E_j1_{\lambda+i+\alpha_j}(-\lambda_i - 1 - d_{ij})) \equiv \text{Hom}(E_iE_j1_{\lambda}, E_j(1_{\lambda+i})_L(-\lambda_i - 1 - d_{ij})) \\
\equiv \text{Hom}(E_iE_j1_{\lambda}, E_j1_{\lambda}(-d_{ij}))
\]
which, by Lemmas A.2, A.4 and A.5, is at most one-dimensional. Moreover, it is nonzero if and only if \(1_{\lambda}, 1_{\lambda+\alpha_i}, 1_{\lambda+\alpha_j} \) are all nonzero. So it remains to show that our two compositions in (3) are nonzero. But, by adjunction, these two compositions are equivalent to \(T_{ij} \in \text{Hom}(E_iE_j1_{\lambda}, E_j1_{\lambda}(-d_{ij})) \) and \(v_{ji} \in \text{Hom}(E_jF_i1_{\lambda+i}, F_jE_j1_{\lambda+i}). \) Both of these are nonzero if \(1_{\lambda}, 1_{\lambda+\alpha_i}, 1_{\lambda+\alpha_j} \) are nonzero.

\[ \square \]

Corollary 3.3. For \(i, j, k \in I \) not necessarily distinct and \(\gamma \in \text{End}^d(E_jE_k1_{\lambda+\alpha_i}) \) the compositions
\[
(\gamma I)(IT_{ik})(T_{ij}I) \in \text{Hom}^{d-d_{ij}-d_{ik}}(E_jE_k1_{\lambda}, E_kE_j1_{\lambda}) \quad \text{and}
\]
\[
(I\gamma)(v_{ji}I)(Tv_{ki}) \in \text{Hom}^{d}(E_jE_k1_{\lambda+i}, F_jE_jE_k1_{\lambda+\alpha_i})
\]
are either both zero or both nonzero. Similarly for
\[
(I\gamma)(T_{ji}I)(IT_{ki}) \in \text{Hom}^{d-d_{ij}-d_{ik}}(E_jE_k1_{\lambda+i}, E_jE_kE_k1_{\lambda+i}) \quad \text{and}
\]
\[
(\gamma I)(u_{ik}I)(u_{jk}) \in \text{Hom}^{d}(F_jE_jE_k1_{\lambda}, E_jE_kE_k1_{\lambda}).
\]

Proof. The composition in (7) is zero if and only if the composition
\[
E_jE_k1_{\lambda+i+\alpha_i} \xrightarrow{\text{adj II}} E_jE_kF_i1_{\lambda+i+\alpha_i} \xrightarrow{I[(\gamma I)(IT_{ik})(T_{ij}I)]} E_jE_kE_jE_k1_{\lambda+\alpha_i} \xrightarrow{I\text{adj} I} E_jE_k1_{\lambda+\alpha_i},
\]
is zero (we omit the shifts to simplify the notation). Using Lemma 3.2 twice, we find that this composition is (up to a nonzero multiple) equal to
\[
E_jE_k1_{\lambda+i+\alpha_i} \xrightarrow{\text{IIadj I}} E_jE_kF_i1_{\lambda+i+\alpha_i} \xrightarrow{I\text{adj} I} E_jE_kE_jF_i1_{\lambda+i+\alpha_i} \xrightarrow{I\text{adj} I} E_jE_kE_k1_{\lambda+i},
\]
which (up to a multiple) is the same as (8). Thus (7) and (8) are either both zero or both nonzero. The equivalence of the second pair of compositions follows similarly. \[ \square \]

3.5. Some properties of \(\theta\)'s.

Lemma 3.4. Suppose \(i \in I \) and \(\theta \in Y_k \) so that \(\langle \theta, \alpha_i \rangle \neq 0. \) Then the compositions
\[
1_{\lambda} \xrightarrow{\text{adj} I} E_i1_{\lambda-\alpha_i}F_i(-\lambda_i + 1) \xrightarrow{I\text{adj} -\alpha_i I} E_i1_{\lambda-\alpha_i}F_i(-\lambda_i - 1) \xrightarrow{\text{adj} I} 1_{\lambda} \quad \text{if} \quad \lambda_i \geq 1
\]
\[
1_{\lambda} \xrightarrow{\text{adj} I} F_i1_{\lambda+\alpha_i}E_i(-\lambda_i + 1) \xrightarrow{I\text{adj} -\alpha_i I} F_i1_{\lambda+\alpha_i}E_i(-\lambda_i - 1) \xrightarrow{\text{adj} I} 1_{\lambda} \quad \text{if} \quad \lambda_i \leq -1
\]
are both to a nonzero multiple of the identity map in \(\text{End}(1_{\lambda}). \)

Proof. If \(\lambda_i \geq 0 \) then \(1_{\lambda}(-\lambda_i) \xrightarrow{\text{adj} I} E_i1_{\lambda-\alpha_i}F_i(-\lambda_i + 1) \xrightarrow{I\text{adj} -\alpha_i I} E_i1_{\lambda-\alpha_i}F_i(-\lambda_i - 1) \xrightarrow{\text{adj} I} 1_{\lambda} \) is the inclusion of \(1_{\lambda}(-\lambda_i - 1) \) into the lowest degree summand \(1_{\lambda} \) on the right side. By assumption (v) on \(\theta, \) the map \(I\theta^{\lambda_i-1}I \in \text{End}^{2(-\lambda_i-1)}(E_i1_{\lambda-\alpha_i}F_i) \) induces an isomorphism between the lowest degree and highest degree summands of \(1_{\lambda} \) inside \(E_iF_i1_{\lambda}. \)

Finally, \(F_iE_i1_{\lambda} \xrightarrow{\text{adj} I} 1_{\lambda}(-\lambda_i - 1) \) is the projection from the highest degree summand of \(1_{\lambda} \) in \(E_iF_i1_{\lambda} \) onto \(1_{\lambda}(-\lambda_i - 1). \) Thus composing the three maps gives an isomorphism \(1_{\lambda} \xrightarrow{\text{adj} I} 1_{\lambda} \) which, by condition (i), must be a multiple of the identity. The case \(\lambda_i \leq 0 \) is similar. \[ \square \]
Lemma 3.5. Suppose $i \in I$ and $\theta \in Y_k$ so that $\langle \theta, \alpha_i \rangle \neq 0$. Then the maps

$$u_{i_i} \overset{\lambda_i - 1}{\bigoplus}_{r=0}^{\lambda_i - 1} [(I\theta^r I) \circ \text{adj}^i] : F_i E_i 1_\lambda \bigoplus_{[\lambda_i]} 1_\lambda \xrightarrow{\sim} E_i 1_{\lambda - \alpha_i} F_i \quad \text{if } \lambda_i \geq 0$$

and

$$v_{i_i} \overset{-\lambda_i - 1}{\bigoplus}_{r=0}^{\lambda_i - 1} [(I\theta^r I) \circ \text{adj}^i] : E_i F_i 1_\lambda \bigoplus_{[-\lambda_i]} 1_\lambda \xrightarrow{\sim} F_i 1_{\lambda + \alpha_i} E_i \quad \text{if } \lambda_i \leq 0$$

are isomorphisms.

Proof. Suppose $\lambda_i \geq 0$ (the case $\lambda_i \leq 0$ is similar). By Lemma 3.1 the map $u_{i_i}$ is the inclusion of the summand $F_i E_i 1_\lambda$ into $E_i F_i 1_\lambda$.

On the other hand, $\text{adj}^i : 1_\lambda (\lambda_i - 1) \to E_i F_i 1_\lambda$ is the inclusion of $1_\lambda (\lambda_i - 1)$ as the lowest degree summand $1_\lambda$ inside $E_i F_i 1_\lambda$. This means that, by condition (v), the composition

$$1_\lambda (\lambda_i - 1 - 2r) \xrightarrow{\text{adj}^i} E_i 1_{\lambda - \alpha_i} F_i (-2r) \xrightarrow{I\theta^r I} E_i 1_{\lambda - \alpha_i} F_i$$

is an isomorphism between $1_\lambda$ and the degree $-\lambda_i + 1 + 2r$ summand of $1_\lambda$ inside $E_i 1_{\lambda - \alpha_i} F_i$ (for $r = 0, \ldots, \lambda_i - 1$). Since $\text{End}^l(1_\lambda) = 0$ for $l < 0$ we get that the composition

$$\bigoplus_{r=0}^{\lambda_i - 1} [(I\theta^r I) \circ \text{adj}^i] : \bigoplus_{[\lambda_i]} 1_\lambda \to E_i 1_{\lambda - \alpha_i} F_i \cong F_i E_i 1_\lambda \bigoplus_{[\lambda_i]} 1_\lambda \to \bigoplus_{[\lambda_i]} 1_\lambda$$

(where the rightmost map is a projection) must be an upper triangular matrix with isomorphisms on the diagonal. The result follows. \qed

Next we have the following description of $\text{End}^2(E_i)$.

Lemma 3.6. Choose $\theta \in Y_k$ so that $\langle \theta, \alpha_i \rangle \neq 0$. Then the space $\text{End}^2(1_{\lambda + \alpha_i} E_i 1_\lambda)$ is spanned by

(i) $(I\theta I)$ and elements $(\gamma^2 I)$ for some $\gamma \in \text{End}^2(1_{\lambda + \alpha_i})$, if $\lambda_i \geq -1$,

(ii) $(\theta I^2)$ and elements $(I^2 \gamma)$ for some $\gamma \in \text{End}^2(1_\lambda)$, if $\lambda_i \leq -1$.

Proof. Suppose $\lambda_i \geq 0$. First we have

$$\text{Hom}(E_i 1_\lambda, E_i 1_\lambda \langle 2 \rangle) \cong \text{Hom}(1_{\lambda + \alpha_i}, E_i 1_\lambda L(2))$$

$$\cong \text{Hom}(1_{\lambda + \alpha_i}, E_i F_i 1_{\lambda + \alpha_i} \langle -\lambda_i + 1 \rangle)$$

$$\cong \text{Hom}(1_{\lambda + \alpha_i}, \bigoplus_{r=0}^{\lambda_i + 1} 1_{\lambda + \alpha_i} \langle 2 - 2r \rangle \bigoplus F_i E_i 1_{\lambda + \alpha_i} \langle -\lambda_i + 1 \rangle)$$

$$\cong \bigoplus_{r=0}^{\lambda_i + 1} \text{Hom}(1_{\lambda + \alpha_i}, 1_{\lambda + \alpha_i} \langle 2 - 2r \rangle) \bigoplus \text{Hom}((1_{\lambda + \alpha_i} F_i) L_i, E_i 1_{\lambda + \alpha_i} \langle -\lambda_i + 1 \rangle)$$

$$\cong \text{End}(1_{\lambda + \alpha_i}) \bigoplus \text{End}^2(1_{\lambda + \alpha_i}) \bigoplus \text{Hom}(E_i 1_{\lambda + \alpha_i}, E_i 1_{\lambda + \alpha_i} \langle -2\lambda_i - 2 \rangle).$$

The right hand term above is zero by Lemma A.1. If $f \in \text{End}^2(1_{\lambda_i})$ then we denote the map induced by adjunction $f' \in \text{Hom}(1_{\lambda + \alpha_i}, E_i F_i 1_{\lambda + \alpha_i} \langle -\lambda_i + 1 \rangle)$ and the induced maps (via the isomorphisms above) in $\text{End}(1_{\lambda + \alpha_i})$ and $\text{End}^2(1_{\lambda + \alpha_i})$ by $f_0$ and $f_1$ respectively.

Using the isomorphism

$$E_i F_i 1_{\lambda + \alpha_i} \xrightarrow{\sim} F_i E_i 1_{\lambda + \alpha_i} \bigoplus_{[\lambda_i + 2]} 1_{\lambda + \alpha_i}$$
from Lemma 3.5 we see that \( f' \) is a sum of compositions

\[
1_{\lambda + \alpha_i} \xrightarrow{f_{i-r}} 1_{\lambda + \alpha_i} (2 - 2r) \xrightarrow{adj_i} E_i 1_{\lambda} F_i (-\lambda_i + 1 - 2r) \xrightarrow{10r} E_i 1_{\lambda} F_i (-\lambda_i + 1).
\]

where \( r = 0, 1 \). Consequently, by adjunction, \( f \) is spanned by compositions of the form

\[
1_{\lambda + \alpha_i} E_i 1_{\lambda} \xrightarrow{f_{i-r} II} 1_{\lambda + \alpha_i} E_i 1_{\lambda} (2 - 2r) \xrightarrow{10r} 1_{\lambda + \alpha_i} E_i 1_{\lambda} (2).
\]

Finally, \( f_0 \) must be some multiple of the identity by Lemma A.1 and the result follows.

The case \( \lambda_i = -1 \) follows as above except that (9) is now \( \text{End}(1_{\lambda + \alpha_i}) \oplus \text{End}(E_i 1_{\lambda + \alpha_i}) \). The case \( \lambda_i \leq -1 \) is similar except that the first step is \( \text{Hom}(E_i 1_{\lambda}, E_i 1_{\lambda} (2)) \cong \text{Hom}((E_i 1_{\lambda})_L E_i 1_{\lambda}, 1_{\lambda} (2)) \). □

3.6. A natural pairing. For any \( i \in I \) and \( \theta \in \text{End}^2(1_{\lambda}) \) we have \( T_{i\theta}(II) T_{ii} = c T_{ii} \in \text{End}^{-2}(E_i 1_{\lambda} E_i) \) for some \( c \in k \) (this is simply because \( \dim \text{End}^{-2}(E_i 1_{\lambda} E_i) \leq 1 \) by Lemma A.1).

**Definition:** For each \( \lambda \in X \) we define the pairing \( (\cdot, \cdot)_\lambda : \text{End}^2(1_{\lambda}) \otimes_k Y_k \to k \) by \( (\theta, \alpha_i)_\lambda := c \) which we extend linearly.

Remark 3.7. The definition of \( (\cdot, \cdot)_\lambda \) depends on \( T_{ii} \) but this dependence is mild. For example, rescaling \( T_{ii} \) (as we do in section 6) does not changes whether or not \( (\theta, \alpha_i)_\lambda \) is zero.

4. Step #1 – The structure of \( \text{End}(E_i E_i) \)

We begin by studying the structure of \( \text{End}(E_i E_i) \) and ultimately show that there exist \( E_i^{(2)} \) such that \( E_i^2 \cong \oplus_{[2]} E_i^{(2)} \).

4.1. Some technical Lemmas. For the remainder of this section we fix \( i \in I \) and \( \theta \in Y_k \) such that \( \langle \theta, \alpha_i \rangle \neq 0 \). We also use the convention that a claim such as “\( f \in \text{End}(A) \)” is nonzero assumes the obviously necessary condition that \( A \) is nonzero. We begin with a few technical results.

**Lemma 4.1.** If \( \theta \in Y_k \) with \( \langle \theta, \alpha_i \rangle \neq 0 \) then \( T_{ii}(II) T_{ii} \in \text{End}^{2|\lambda_i|+2}(E_i 1_{\lambda} E_i) \) is nonzero.

**Proof.** Let us suppose \( \lambda_i \geq 0 \) (the case \( \lambda_i < 0 \) is the same). Consider the following composition

\[
E_i 1_{\lambda} \xrightarrow{10adj_i} E_i 1_{\lambda} F_i (2i) (T_{ii} II) \xrightarrow{10adj_i} E_i 1_{\lambda} F_i (T_{ii} II) \xrightarrow{10adj_i} E_i 1_{\lambda} F_i (II) \xrightarrow{adj_i} E_i 1_{\lambda}.
\]

where we omit the shifts for convenience. We can use Lemma 3.2 to rewrite this composition as

\[
E_i 1_{\lambda} \xrightarrow{(adj) II} E_i 1_{\lambda} F_i (II) \xrightarrow{(adj) II} E_i 1_{\lambda} F_i (II) \xrightarrow{(adj) II} E_i 1_{\lambda}.
\]

Now, since \( \lambda_i \geq 0 \) we have \( v_{ii} u_{ii} \sim id \) and so this simplifies to give

\[
E_i 1_{\lambda} \xrightarrow{(adj) II} E_i 1_{\lambda} F_i E_i (II) \xrightarrow{(adj) II} E_i 1_{\lambda} F_i E_i (II) \xrightarrow{(adj) II} E_i 1_{\lambda}.
\]

By Lemma 3.4 this is (up to rescaling) equal to the identity map on \( E_i 1_{\lambda} \). □

**Lemma 4.2.** If \( \lambda_i \leq -2 \) then \( \text{End}(E_i E_i 1_{\lambda}) \) is spanned by \( T_{ii}(II) \rho \) where \( \rho \in \text{End}^2(1_{\lambda}) \) together with another 3-dimensional space. Likewise, if \( \lambda_i \geq 2 \) then \( \text{End}(1_{\lambda} E_i E_i) \) is spanned by \( T_{ii}(\rho II) \) where \( \rho \in \text{End}^2(1_{\lambda}) \) together with another 3-dimensional space.

**Proof.** Let us suppose we are in the second case where \( \lambda_i \geq 2 \) (the first case is the same). The same adjunction argument as in the proof of Lemma A.2 shows that

\[
\text{End}(1_{\lambda} E_i E_i) \cong \text{Hom}(1_{\lambda + \alpha_i} E_i E_i, 1_{\lambda + \alpha_i} E_i E_i(-2\lambda_i + 2)) \oplus \text{Hom}(1_{\lambda} E_i, \bigoplus_{[2]\lambda} 1_{\lambda} E_i(-\lambda_i + 3)).
\]
If $\lambda_i \geq 3$ then the middle term above is zero while all but three terms in the direct sum on the right are zero. The surviving three terms are $\text{End}^2(1, \lambda E_i) \oplus \text{End}(1, \lambda E_i) \oplus \mathbb{k}$. The term $\text{End}^2(1, \lambda E_i) \oplus \mathbb{k}$ is spanned by $\theta I\theta$ for a choice of $\theta \in Y_k$ with $\langle \theta, \alpha_i \rangle \neq 0$ and by $\rho I\rho$ where $\rho \in \text{End}^2(1, \lambda E_i)$ is arbitrary. If one traces back through the isomorphisms the maps $\rho I\rho$ correspond to maps $T_{ii}(\rho I\rho) \in \text{End}(1, \lambda E_i)$. The result follows.

If $\lambda_i = 2$ then the middle term above is one-dimensional and the direct sum on the right is just

$$\text{Hom}(1, \lambda E_i) \oplus \mathbb{k} = \text{End}^2(1, \lambda E_i) \oplus \text{End}(1, \lambda E_i).$$

The same argument as above now applies. \hfill \qed

4.2. Some non-vanishing results.

**Lemma 4.3.** If $\lambda_i \geq -1$ then $T_{iii} I_{\lambda-\alpha_i}$ is nonzero while if $\lambda_i \leq -1$ then $T_{iiii} I_{\lambda-\alpha_i}$ is nonzero.

**Remark 4.4.** The adjunction argument below does not tell us whether $T_{iii} I_{\lambda-\alpha_i}$ and $T_{iiii} I_{\lambda-\alpha_i}$ are nonzero if $\lambda_i < -1$ and $\lambda_i > -1$ respectively.

**Proof.** To show that $T'_{iiii} I_{\lambda-\alpha_i} \neq 0$ it suffices to show that

$$T'_{iiii} I_{\lambda-\alpha_i} \neq 0$$

is nonzero. This is a consequence of Corollary 3.3 where we take $i = j = k$ and $\gamma = T_{ii}$. Suppose $\lambda_i \leq -2$ (the case $\lambda_i = -1$ is a little different). Precomposing with $(v_{ii} I)(v_{ii} I)$ and using Lemma 3.1 we obtain the map $T_{iiii} I \in \text{End}^{-2}(1, \lambda E_i)$ (up to a nonzero multiple). This map is clearly nonzero since we can precompose it with $E_i$ and simplify to obtain several copies of $T_{iiii} I_{\lambda-\alpha_i}$, which are nonzero.

If $\lambda_i = -1$ we have

$$E_i 1 \lambda \oplus F_i E_i 1 \lambda \cong E_i F_i 1 \lambda \cong E_i 1 \lambda \oplus E_i F_i 1 \lambda \cong E_i 1 \lambda \oplus E_i F_i 1 \lambda$$

which means that $F_i E_i 1 \lambda \cong E_i F_i 1 \lambda$. Then the same argument as in Lemma 3.1 shows that the composition $(v_{ii} I)(v_{ii} I)$ is an isomorphism. The argument used above when $\lambda_i \leq -2$ now applies.

This concludes the proof that $T'_{iiii} I_{\lambda-\alpha_i} \neq 0$ if $\lambda_i \leq -1$. The nonvanishing of $T_{iii} I_{\lambda-\alpha_i}$ if $\lambda_i \geq -1$ is proved similarly. \hfill \qed

**Lemma 4.5.** Suppose $\lambda_i \geq -1$. If

$$id = a_1((IT_{ii} I)(I\theta II)) + a_2((IT_{ii} I)(I\theta II)) + (IT_{ii} I)(\gamma) \in \text{End}(1, \lambda E_1, 1 \lambda E_1, 1 \lambda-\alpha_i)$$

where $\gamma = a_3(I\theta II) + (I\theta II)$ for some $\rho \in \text{End}^2(1, \lambda E_1)$ and $a_1, a_2, a_3 \in \mathbb{k}$, then $T_{iiii} I_{\lambda-\alpha_i}$ and $T_{iiii} I_{\lambda-\alpha_i}$ are both nonzero. Conversely, if $T_{iiii} I_{\lambda-\alpha_i}$ and $T_{iiii} I_{\lambda-\alpha_i}$ are both nonzero then (10) holds.

Similarly, suppose $\lambda_i \leq 1$. If

$$id = a_1((IT_{ii} I)(I\theta II)) + a_2((IT_{ii} I)(I\theta II)) + (IT_{ii} I)(\gamma) \in \text{End}(1, \lambda E_1, 1 \lambda E_1, 1 \lambda-\alpha_i)$$

where $\gamma = a_3(I\theta III) + (I\theta III)$ for some $\rho \in \text{End}^2(1, \lambda E_1)$, then $T_{iiii} I_{\lambda-\alpha_i}$ and $I_{\lambda+\alpha_i}$ are both nonzero. Conversely, if $T_{iiii} I_{\lambda-\alpha_i}$ and $I_{\lambda+\alpha_i}$ are both nonzero then (11) holds.

**Proof.** Suppose $\lambda_i = 0$ (the case $\lambda_i \leq 0$ is the same). Composing (10) on the right with $(IT_{ii} I)$ we get $(IT_{ii} I)_{\alpha_1}((IT_{ii} I)(I\theta II))((IT_{ii} I) I\theta II)$ which implies that $a_1 \neq 0$ and $(IT_{ii} I) I\theta II \in \text{End}^{-2}(1, \lambda E_1)$ is nonzero. Note that we also get $a_2 \neq 0$ by multiplying on the left with $(IT_{ii} I)$.

Now suppose $T_{iiii} I_{\lambda-\alpha_i} = 0$ and consider

$$T_{iiii} I_{\lambda-\alpha_i} = (IT_{ii} I)(T_{ii} I)(IT_{ii} I)(I\theta II) \in \text{End}^{-4}(1, \lambda E_1).$$
On the one hand this is zero while on the other we can use (10) to rewrite it as
\[ a_1^{-1}(IT_{ii})(T_{ii}I) - a_2a_1^{-1}(IT_{ii})(II\theta I)(IT_{ii}) = a_1^{-1}(IT_{ii})(T_{ii}I) - a_2a_1^{-1}(IT_{ii})(II\theta I)(T_{ii}I)(IT_{ii}) \]
where we use that \( T_{ii}'I_{\lambda - \alpha_i} = 0 \) on several occasions to simplify above. Thus we get that
\[ a_1(IT_{ii})(T_{ii}I) = a_2^2(T_{ii}I)(IT_{ii}) \in \text{End}^{-4}(E_iE_iE_i1_{\lambda - \alpha_i}). \]
Multiplying by \( (T_{ii}I) \) on the left gives \( a_1T_{ii}I_{\lambda - \alpha_i} = 0 \) which is a contradiction by Lemma 4.3.

To prove the converse suppose we have
\[ c_1(IT_{ii})(II\theta II) + c_2(II\theta II)(IT_{ii}I) + c_3(IT_{ii}I)(III\theta I) + (IT_{ii}I)(\rho II) = 0 \]
inside \( \text{End}(1_{\lambda + \alpha_i}E_i1_{\lambda - \alpha_i}) \) for some \( \rho \in \text{End}^2(1_{\lambda + \alpha_i}) \) and \( c_1, c_2, c_3 \in \mathbb{k} \). Let \( e \geq 1 \) be the smallest integer such that \( T_{ii}(\theta^eI)T_{ii} \in \text{End}^{2e-4}(E_i1_iE_i) \) is nonzero (such an \( e \) exists by Lemma 4.1). Then composing (12) on the right with \( (II\theta^{e-1}I)(IT_{ii}I) \) all but the first terms vanish and we get \( c_1 = 0 \).

Similarly, composing on the left gives \( c_2 = 0 \).

Next, consider the composition
\[ E_iE_i1_{\lambda - \alpha_i}E_i \xrightarrow{(IT_{ii})(II\theta II)} E_iE_i1_{\lambda - \alpha_i}E_i \xrightarrow{T_{ii}I} E_iE_i1_{\lambda - \alpha_i}E_i. \]
On the one hand, since \( T_{ii}'I_{\lambda - 2\alpha_i} \neq 0 \) it follows that \( T_{ii}'I_{\lambda - \alpha_i} \sim T_{ii}I_{\lambda - \alpha_i} \) and hence the composition above equals (up to rescaling)
\[ E_iE_i1_{\lambda - \alpha_i}E_i \xrightarrow{(IT_{ii})(II\theta II)(IT_{ii}I)} E_iE_i1_{\lambda - \alpha_i}E_i \xrightarrow{(-2)(IT_{ii})} E_iE_i1_{\lambda - \alpha_i}E_i. \]
Now, since \( T_{ii}(\theta^eI)T_{ii} \in \text{End}^{-2}(E_i1_{\lambda - \alpha_i}E_i) \) is nonzero, it must be equal to \( T_{ii}I_{\lambda - 2\alpha_i} \) which is nonzero.

On the other hand, if \( c_3 \neq 0 \) then we can rewrite the composition as
\[ 1_{\lambda + \alpha_i}E_iE_i \xrightarrow{c_3^{-1}(IT_{ii})(\rho II)} 1_{\lambda + \alpha_i}E_iE_i \xrightarrow{T_{ii}I} 1_{\lambda + \alpha_i}E_iE_iE_i \xrightarrow{(-6)}. \]
which equals zero. We conclude that \( c_3 \) must be zero.

This means that modulo maps of the form \( (IT_{ii})(\rho II) \in \text{End}(1_{\lambda + \alpha_i}E_i1_{\lambda - \alpha_i}) \) the first three maps in (12) are linearly independent. It follows by Lemma 4.2 that \( id \in \text{End}(E_i1_iE_i) \) must be a linear combination as in (10). This concludes the proof.

**Corollary 4.6.** The maps \( T_{ii}I_{\lambda} \) and \( T_{ii}'I_{\lambda} \) as well as \( T_{ii}(\theta^eI)T_{ii} \in \text{End}^{-2}(E_i1_iE_i) \) are all nonzero.

**Proof.** Suppose \( \lambda_i \geq 0 \) (the case \( \lambda_i \leq 0 \) is the same). By Lemma 4.5 it suffices to show that \( T_{ii}'I_{\lambda - \alpha_i} \) and \( T_{ii}(\theta^eI)T_{ii} \in \text{End}^{-2}(E_i1_iE_i) \) are both nonzero when \( \lambda_i = -1 \) and \( \lambda_i = 0 \).

Suppose \( \lambda_i = -1 \). We know \( T_{ii}'I_{\lambda - \alpha_i} \neq 0 \) by Lemma 4.3. Now suppose \( T_{ii}(\theta^eI)T_{ii} \in \text{End}^{-2}(E_i1_iE_i) \) is zero. By Lemma 4.2 we know that there exists a nontrivial linear relation
\[ b_0 \cdot id + b_1(IT_{ii}I)(II\theta II) + b_2(II\theta II)(IT_{ii}I) + b_3(IT_{ii}I)(\theta III I) + (IT_{ii}I)(III\theta I) \]
inside \( \text{End}(1_{\lambda + \alpha_i}E_i1_{\lambda - \alpha_i}) \) for some \( b_0, b_1, b_2, b_3 \in \mathbb{k} \). Composing on the left by \( (IT_{ii}I)(II\theta II) \) and on the right by \( (III\theta I)(IT_{ii}I) \) all but the first terms vanish and we get \( b_0(IT_{ii}I)(II\theta^2II)(IT_{ii}I) = 0 \).

By Lemma 4.1 this means \( b_0 = 0 \).

Next, composing on the right with \( (II\theta II)(IT_{ii}I) \) we get \( b_1(IT_{ii}I)(III\theta I)(IT_{ii}I) = 0 \) which means \( b_1 = 0 \). Likewise, composing on the left gives \( b_2 = 0 \). Subsequently, we are left with
\[ b_3(IT_{ii}I)(\theta III I) + (IT_{ii}I)(III\theta I) = 0 \in \text{End}(1_{\lambda + \alpha_i}E_i1_{\lambda - \alpha_i}). \]

To see this cannot be the case consider the composition
\[ E_i1_{\lambda + \alpha_i}E_i \xrightarrow{(IT_{ii}I)} E_i1_{\lambda + \alpha_i}E_i \xrightarrow{(-2)(II\theta II)} E_i1_{\lambda + \alpha_i}E_i \xrightarrow{(2)} \xrightarrow{T_{ii}I} E_i1_{\lambda + \alpha_i}E_i \xrightarrow{(-4)}. \]
On the one hand this is equal to
\[ E_1\lambda_{\alpha_{\lambda}}E_1E_1 \xrightarrow{(T_{ii})^2} E_1\lambda_{\alpha_{\lambda}}E_1E_1 \xrightarrow{(T_{ii})^2} E_1\lambda_{\alpha_{\lambda}}E_1E_1. \]
By the same argument as in the proof of Lemma 4.3 this composition is equivalent to (after composing with the appropriate adjunction maps and applying Lemma 3.2)
\[ (IT_{ii})(\theta I^2 T)(IT_{ii}) \in \text{End}(E_1E_1E_1). \]
Composing on the left with \( E_1 \) and simplifying gives 3 copies of \((E_1\lambda_{\alpha_{\lambda}}E_1E_1)\) which is nonzero by Lemma 4.1. The composition in (14) is nonzero.

On the other hand, (14) is equal to (up to rescaling)
\[ (T_{iii})^2(\theta I^2 T)(T_{iii})^2 = b_{1+2}(T_{iii})^2(\theta I^2 T)(T_{iii})^2 = 0 \in \text{End}^3(E_1E_1E_1E_1). \]
Thus we arrive at a contradiction which means that \((T_{ii})(\theta I^2 T)T_{ii} \in \text{End}^2(E_1E_1E_1)\) must be nonzero.

Next, suppose \( \lambda_i = 0 \). A more careful look at the argument used to prove Lemma 4.2 reveals that if you follow through the adjunctions when \( \lambda_i > 0 \) then the 3-dimensional subspace of \( \text{End}(E_1E_1E_1) \) when \( \lambda_i > 0 \) is spanned by \((T_{ii})(\theta I^2 T)I_{ii} \) and \( id \). More precisely, the first step in the adjunction gives
\[ \text{Hom}(E_1E_1E_1, \text{End}(E_1E_1E_1)) \cong \text{Hom}(E_1E_1E_1, \text{End}(\lambda_{\alpha_{\lambda}}E_1E_1E_1)). \]
and the identity map \( id \) corresponds to the copy of \( E_1 \) in the highest degree (i.e. in degree zero) in the right hand summation. However, if \( \lambda_i = 0 \) the direct sum vanishes. In this case the 3-dimensional space is spanned by \( T_{ii}(\theta I^2 T)I_{ii} \) as before but instead of \( id \) one finds a map \( T_{ii}(\beta I_{ii}) \) for some \( \beta \in \text{End}^2(E_1E_1E_1) \). We could make \( \beta \) explicit but we do not need to. The only thing to notice now is that we must have a relation as in (10) except that
\[ \gamma = (IT_{ii})^2(\theta I^2 T)T_{ii} + (\rho I I I) \in \text{End}(E_1E_1E_1E_1) \]
However, it is easy to check that the rest of the argument in Lemma 4.5 goes through exactly as before to show that \( T_{ii}(\theta I^2 T)T_{ii} \in \text{End}^2(E_1E_1E_1) \) and \( T_{iii}I_{\lambda_{\alpha_{\lambda}}} \) are still nonzero. \( \square \)

4.3. Divided powers.

**Proposition 4.7.** We have
\[ id = a \cdot (\theta I^2 T)T_{ii} + a \cdot T_{ii}(\theta I^2 T) + T_{ii}(\tau) \in \text{End}(E_1E_1E_1) \]
where \( a \neq 0 \) and
\[ (11) \]
\[ \tau = \begin{cases} (\rho I I) + (1 I I) \in \text{End}^2(\lambda_{\alpha_{\lambda}}E_1E_1E_1) & \text{if } \lambda_i \leq 0 \\ (\rho I I) + (1 I I) \in \text{End}^2(\lambda_{\alpha_{\lambda}}E_1E_1E_1) & \text{if } \lambda_i \geq 0 \end{cases} \]
for some \( b \in \mathbb{k} \) and \( \rho \in \text{End}^2(\lambda_{\alpha_{\lambda}}) \) or \( \rho \in \text{End}^2(\lambda_{\alpha_{\lambda}}) \) respectively.

**Proof.** By combining Lemma 4.5 and Corollary 4.6 we know that
\[ id = a_1 \cdot (\theta I I)T_{ii} + a_2 \cdot T_{ii}(\theta I I) + T_{ii}(\tau) \]
for some nonzero \( a_1, a_2 \in \mathbb{k} \). Composing with \( T_{ii} \) on the left one gets \( T_{ii} = a_1 \cdot T_{ii}(\theta I I)T_{ii} \) while composing on the right gives \( T_{ii} = a_2 \cdot T_{ii}(\theta I I)T_{ii} \). Thus \( a_1 = a_2 \) and the result follows. \( \square \)

**Corollary 4.8.** There exists \( E_2 \) \( \lambda_{\alpha_{\lambda}} \), so that \( E_2 = \oplus_2 \mathbb{E}_i \lambda_{\alpha_{\lambda}} \), and likewise for \( E_i \).
holds. It is now a standard fact that $E_i1_\lambda E_i \cong E_i^{(2)}e_{1-\alpha_i,1}(1) \oplus E_i^{(3)}e_{0,1}(1)$. More precisely, the two summands are given by orthogonal idempotents $T_{ii}(X_iI)$ and $-(IX_i)T_{ii}$. The isomorphism between them is given by $T_{ii}$ and $(X_iI)$ (which explains why they lie in different degrees). The same result for $F_i$’s follows by adjunction.

**Remark 4.9.** The definitions of $X_i$ above are not the ones we will use later. They are only used here in order to prove the decomposition of $E_i E_i$.

5. **Step #2 – Nonvanishing of $T_{iij}$**

The next step is to show that $T_{iij}$ as well as $T'_{iij}$ are all nonzero.

**Proposition 5.1.** Suppose $i, j \in I$ are distinct. Then $T_{iij}1_\lambda, T'_{iij}1_\lambda, T_{iij}1_\lambda$ and $T'_{iij}1_\lambda$ are all nonzero if and only if $1_{\lambda+\epsilon_i+\epsilon_j} \neq 0$ for $\epsilon_i \in \{0, 1, 2\}$ and $\epsilon_j \in \{0, 1\}$.

**Proof.** One direction is immediate since one can easily check that the weights $\lambda + \epsilon_i + \epsilon_j$ all appear in every one of the compositions $T_{iij}1_\lambda, T'_{iij}1_\lambda, T_{iij}1_\lambda$ or $T'_{iij}1_\lambda$.

For the converse we prove the case of $T_{iij}1_\lambda$ (the other cases are the same). If $\langle i, j \rangle = 0$ then $T_{ij}$ and $T_{ji}$ are invertible and the claim is obvious. If $\langle i, j \rangle = -1$ then using Corollary 3.3 it suffices to prove that the composition

$$E_i E_i F_j 1_{\lambda+\epsilon_j} T_{iij} E_i F_j 1_{\lambda+\epsilon_j} T_{iij}$$

is nonzero. Since $T_{iij}$ is an isomorphism it remains to show that $E_i E_i F_j 1_{\lambda+\epsilon_j} \neq 0$. If $\lambda_i \geq -2$ then we compose on the left with $F_i F_j$ and simplify to get several copies of $F_j 1_{\lambda+\epsilon_j}$. This is nonzero since $E_i 1_\lambda \neq 0$. Similarly, if $\lambda_i \leq -1$ then we compose $E_i E_i F_j 1_{\lambda+\epsilon_j} \cong F_j 1_{\lambda+\epsilon_j}2_\alpha E_i E_i$ on the right with $F_i F_j$ to obtain several copies of $1_{\lambda+2_\alpha}$ which are again nonzero. □

**Corollary 5.2.** For $i, j \in I$ we have $T_{iij} \sim T'_{iij}$ and $T_{jii} \sim T'_{jii}$.

**Proof.** To prove that $T_{iij} \sim T'_{iij}$ suppose $\langle i, j \rangle = -1$ (the case $\langle i, j \rangle = 0$ is clear). By Lemma A.9 we know that $\dim \text{Hom}(E_j, E_i^{(2)}1_\lambda, E_i^{(2)}1_\lambda \langle d \rangle)$ is zero if $d < 2$ and at most one-dimensional if $d = 2$. From this it follows that $\dim \text{End}(E_j, E_i^{(2)}1_\lambda, E_i^{(2)}1_\lambda) \leq 1$. On the other hand, we know that $T_{iij}1_\lambda, T'_{jjii}1_\lambda \in \text{Hom}(E_j, E_i^{(2)}1_\lambda, E_i^{(2)}1_\lambda)$ are nonzero by Lemma 5.1. So we must have $T_{iij}1_\lambda \sim T'_{jjii}1_\lambda$. The proof that $T_{jii}1_\lambda \sim T'_{jjjj}1_\lambda$ is the same. □

**Corollary 5.3.** Suppose $i, j \in I$ and $\theta, \gamma \in Y_k$ so that $\langle \theta, \alpha_j \rangle = 0$ and $\langle \theta, \alpha_i \rangle \neq 0$.

(i) If $\lambda_j \geq -1$ then $(\gamma II) = (\gamma II) \in \text{End}(1_{\lambda+\alpha_j}1_i1_\lambda)$.

Moreover, if $E_i E_i 1_{\lambda+\alpha_j} \neq 0$ then $(\gamma, \alpha_i)1_{\lambda+\alpha_j} \neq 0$.

(ii) If $\lambda_j \leq -1$ then $1_{\lambda+\alpha_j}1_i1_\lambda$.

Moreover, if $1_{\lambda+\alpha_j}1_i1_\lambda \neq 0$ then $\langle \gamma, \alpha_i \rangle 1_{\lambda+\alpha_j} \neq 0$.

**Proof.** Suppose $\lambda_i \geq -1$. The fact that $(\gamma II) = (\gamma II)$ for some $\gamma \in \text{End}(1_{\lambda+\alpha_j})$ follows from Lemma 3.6. It remains to show that $(\gamma, \alpha_i) 1_{\lambda+\alpha_j} \neq 0$. To do this consider the following composition

$$E_i E_i E_i \xrightarrow{(IT_{ii})} E_i E_i 1_\lambda E_i \xrightarrow{(II)} E_i E_i 1_\lambda E_i \xrightarrow{(IT_{ii})} E_i E_i E_i$$

where we omit the grading shifts for convenience. On the one hand this is equal to

$$T_{iij} \sim (\gamma II)(T_{iij}) \sim (\gamma II)(T_{jjii}) \sim (\gamma II)(T_{jjjj}) \sim (\gamma II)(T_{jjjj})(T_{jjjj}) \sim (\gamma II)(T_{jjjj})(T_{jjjj}) \sim \langle \theta, \alpha_i \rangle 1_{\lambda+\alpha_j}.$$
Notice that $T_{iij}I_{\lambda-\alpha_i} \sim T_{iij}I_{\lambda-\alpha_i}$ are nonzero since $E_iE_jE_i1_{\lambda-\alpha_i} \neq 0$ by assumption and $E_i1_{\lambda}E_i \neq 0$ because $(\theta, \alpha_i) \neq 0$.

On the other hand, the composition in (17) is equal to

$E_iE_iE_j \xrightarrow{(IT_{ij})(IT_{ij})} E_i1_{\lambda+\alpha_i}E_jE_i \xrightarrow{I_{\lambda+\alpha_i}} E_i1_{\lambda+\alpha_i}E_i E_i \xrightarrow{(IT_{ij})(IT_{ij})} E_iE_iE_i$ 

which we can simplify as above to get $(\ell, \alpha_i) \lambda \neq 0$: Since $(\theta, \alpha_i) \lambda \neq 0$ we get $(\gamma, \alpha_i) \lambda \neq 0$. The case $\lambda_i \leq 1$ is proved in the same way. \hfill \Box

As a final consequence of Corollary 5.3 we have the following result which will be used on several occasions later (particularly in the proof of the Serre relation).

**Lemma 5.4.** Fix $i, j \in I$ with $\langle i, j \rangle = -1$. If $\lambda_i \leq -1$ then $E_j$-ranks of

$F_iE_iE_j1_{\lambda} \xrightarrow{IT_{ij}} F_iE_jE_i1_{\lambda}(1)$ and $F_iE_iE_j1_{\lambda} \xrightarrow{IT_{ij}} F_iE_jE_i1_{\lambda}(1)$

are both $-\lambda_i$. Similarly, if $\alpha_i \leq 0$ then the $E_j$-ranks of

$E_iE_j1_{\lambda}F_i \xrightarrow{T_{ij}} E_jE_i1_{\lambda}F_i(1)$ and $E_iE_j1_{\lambda}F_i \xrightarrow{T_{ij}} E_jE_i1_{\lambda}F_i(1)$

are both $\lambda_i + 1$.

**Proof.** Let us consider the first map

(18) $IT_{ij} : F_iE_iE_j1_{\lambda} \rightarrow F_iE_jE_i1_{\lambda}$

(we will omit the grading shifts for convenience). Fix $\theta \in Y_k$ so that $\langle \theta, \alpha_i \rangle \neq 0$ but $\langle \theta, \alpha_j \rangle = 0$. The left side of (18) is isomorphic to $E_iE_jF_i1_{\lambda} \oplus [-\lambda_i, -1]E_j1_{\lambda}$ where, by Corollary 3.5, the isomorphism involving the $E_j1_{\lambda}$ summands is given by

(19) $E_j1_{\lambda} \xrightarrow{\text{adj}^i} F_i1_{\mu}E_iE_j \xrightarrow{I_{\mu}^0} F_i1_{\mu}E_iE_j$

where $\mu = \lambda + \alpha_i + \alpha_j$ and $\ell_1 \in \{0, 1, \ldots, -\lambda_i\}$.

Now, suppose $\mu_j \leq 1$ (the case $\mu_j \geq -1$ is similar). Then by Corollary 5.3 we have

$(\theta^0)(\theta^1) \in \text{End}^2(I_{\mu}E_j1_{\mu-\alpha_i})$

for some $\gamma \in \text{End}^2(I_{\mu-\alpha_i})$ with $\langle \gamma, \alpha_i \rangle \mu_{-\alpha_i} \neq 0$.

Now, the right side of (18) is isomorphic to $E_iE_jF_i1_{\lambda} \oplus [-\lambda_i, -1]E_j1_{\lambda}$ where the isomorphism involving $E_j1_{\lambda}$ summands is given by

(20) $F_iE_iE_j1_{\lambda} \xrightarrow{u_{ij}} E_jF_i1_{\lambda+\alpha_i}E_i \xrightarrow{I_{\lambda+\alpha_i}} E_jF_i1_{\lambda+\alpha_i}E_i \xrightarrow{\text{adj}^i} E_j1_{\lambda}$

where $\ell_2 \in \{0, 1, \ldots, -\lambda_i - 1\}$. The composition of (19) and (20) is given by

$(\text{adj}^i)(\theta^0^1)(\theta^0^1^1)(u_{ij})(IT_{ij})(\theta^0^1^1^1)(\theta^0^1^1^1^1)| = (\text{adj}^i)(\theta^0^1^1^1^1)(\theta^0^1^1^1^1|)(u_{ij})(IT_{ij})(\theta^0^1^1^1^1)|$

where we used the second relation in Lemma 3.2 to get the second line. This last map is the composition

$E_j1_{\lambda} \xrightarrow{\text{adj}^i} E_jF_i1_{\lambda+\alpha_i}E_i \xrightarrow{I_{\lambda+\alpha_i}} E_jF_i1_{\lambda+\alpha_i}E_i \xrightarrow{\text{adj}^i} E_j1_{\lambda}$

By degree reasons this composition is zero if $\ell := \ell_1 + \ell_2 < -\lambda_i - 1$ and, by Lemma 3.4, a nonzero multiple of the identity if $\ell = -\lambda_i - 1$. Now, $IT_{ij}$ from (18) induces a map between summands $E_j1_{\lambda}$ on both sides which is represented by a matrix of size $(-\lambda_i + 1) \times (-\lambda_i)$ whose $(\ell_1, \ell_2)$-entry is the composition of (19) and (20). By the
calculation above this matrix is upper triangular and carries invertible maps on the (almost) diagonal. Thus it induces a map whose $E_j$-rank is $-\lambda_i$. This proves that the map

$$F_i E_j 1_\lambda \overset{IT_i I}{\longrightarrow} F_i E_j 1_\lambda (1)$$

has $E_j$-rank $-\lambda_i$. The $E_j$-ranks of the other three maps are calculated similarly. \hfill \Box

**Lemma 5.5.** Fix $i, j \in I$ with $\langle i, j \rangle = 0$. If $\lambda_i \leq -1$ then $E_j$-ranks of

$$F_i E_j 1_\lambda \overset{IT_i I}{\longrightarrow} F_i E_j 1_\lambda$$

are both $-\lambda_i$. Similarly, if $\lambda_i \geq -1$ then the $E_j$-ranks of

$$E_i E_j 1_\lambda F_i \overset{T_i II}{\longrightarrow} E_i E_j 1_\lambda F_i$$

are both $\lambda_i + 2$.

**Proof.** The proof is similar to that of Lemma 5.4 (the only difference is keeping track of the gradings correctly given that $\langle i, j \rangle = 0$). \hfill \Box

**Corollary 5.6.** If $\langle i, j \rangle = 0$ then $T_{ij} : E_j 1_\lambda \rightarrow E_i E_j 1_\lambda$ is an isomorphism.

**Proof.** Suppose $\lambda_i \geq -1$ (the case $\lambda_i \leq 1$ is the same). By Lemma 5.5 the map $T_{ij} II : E_j 1_\lambda F_i \rightarrow E_i E_j 1_\lambda F_i$ induces an isomorphism between all $\lambda_i + 2 > 0$ summands $E_i$ on either side. Likewise for $T_{ji} II : E_i 1_\lambda F_i \rightarrow E_i E_j 1_\lambda F_i$. Thus the composition $T_{ji} T_{ij} \in \text{End}(E_i E_j 1_\lambda)$ is nonzero. Since $\dim \text{End}(E_j 1_\lambda) \leq 1$ by Lemma A.5 it follows $T_{ji} T_{ij}$ is some multiple of the identity. Similarly we can argue that $T_{ij} T_{ji}$ is also some multiple of the identity. It follows that $T_{ij}$ must be an isomorphism. \hfill \Box

6. **Step #3 – Rescaling $T_{ii}$**

Fix $i \in I$. We will now explain how to rescale each $T_{ii} I_\lambda$ so that for any $j \neq i$ we have

(21) \hspace{1em} (T_{ii} I) (IT_{ii}) (T_{ii} I) I_\lambda = (IT_{ii}) (T_{ii} I) (IT_{ii}) I_\lambda \in \text{End}^{-6} (E_i E_j 1_\lambda)

(22) \hspace{1em} (IT_{ii}) (T_{ii} I) I_\lambda = (T_{ii} I) (IT_{ii}) (T_{ii} I) I_\lambda \in \text{Hom}^{-2-2\langle i, j \rangle} (E_j E_i 1_\lambda, E_i E_j 1_\lambda)

(23) \hspace{1em} (T_{ii} I) (IT_{ii}) (T_{ii} I) I_\lambda = (IT_{ii}) (T_{ii} I) (IT_{ii}) I_\lambda \in \text{Hom}^{-2-2\langle i, j \rangle} (E_i E_j 1_\lambda, E_i E_j 1_\lambda)

The key observation that allows us to do this is that given $T_{ii} I_\lambda$ there is a unique way of rescaling $T_{ii} I_{\lambda + c_k}$ so that (21) holds if $i = j$ and (22), (23) hold if $i \neq j$. We denote this rescaling $T_{ii} I_{\lambda + c_k} \sim T_{ii} I_\lambda$. Similarly one also has the rescaling $T_{ii} I_{\lambda + c_k} \sim T_{ii} I_\lambda$.

**Proposition 6.1.** One can rescale all $T_{ii} I_\lambda$ so that relations (21), (22) and (23) hold.

**Proof.** Consider a sequence of rescalings

(24) \hspace{1em} T_{ii} I_\lambda \sim T_{ii} I_{\lambda + c_k} \sim T_{ii} I_{\lambda + c_k + c_{k+1}} \sim \cdots \sim T_{ii} I_{\lambda + \sum_c c_k k} = T_{ii} I_\lambda

where $c_k = \pm 1$ and $\sum c_k k = 0$. We need to show that this sequence ends up trivially rescaling $T_{ii} I_\lambda$ (multiplication by one). Let us encode the sequence of rescalings from (24) as $c = (c_1 k_1, \ldots, c_m k_m)$. There are two operations that one can perform on such a sequence.

- The switch operation $S_a$ given by

$$(c_1 k_1, \ldots, c_a k_a, c_{a+1} k_{a+1}, \ldots, c_m k_m) \mapsto (c_1 k_1, \ldots, c_{a+1} k_{a+1}, c_a k_a, \ldots, c_m k_m)$$

if $c_a + c_{a+1} = 0$ and $k_a \neq k_{a+1}$. 

- The addition operation $A_a$ given by

$$(c_1 k_1, \ldots, c_a k_a, \ldots, c_m k_m) \mapsto (c_1 k_1, \ldots, c_{a+1} k_{a+1}, \ldots, c_m k_m)$$

if $c_a = c_{a+1}$.
The drop operation $D_a$ given by

$$(c_1 k_1, \ldots, c_a k_a, c_{a+1} k_{a+1}, \ldots, c_m k_m) \mapsto (c_1 k_1, \ldots, c_{a-1} k_{a-1}, c_{a+2} k_{a+2}, \ldots, c_m k_m)$$

if $c_a k_a + c_{a+1} k_{a+1} = 0$.

The way $\ck$ and $S_a \cdot \ck$ ends up rescaling $T_{ii}I_\lambda$ are the same due to Corollary 6.6. The way $\ck$ and $D_a \cdot \ck$ rescales $T_{ii}I_\lambda$ are also the same (essentially by definition). Finally, it is easy to see that the actions of $S_a$ and $D_a$ can be used to transform any sequence $\alpha$ into the trivial sequence of length zero. This completes the proof. \hfill \Box

**Proposition 6.2.** Suppose $i, j, k \in I$ are distinct. Then $T_{ijk}I_\lambda$ and $T'_{ijk}I_\lambda$ are nonzero if and only if $1_{\lambda+\epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_k \alpha_k} \neq 0$ for $\epsilon_i, \epsilon_j, \epsilon_k \in \{0,1\}$.

**Proof.** One direction is immediate since one can easily check that the weights $\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_k \alpha_k$ all appear in both $T_{ijk}I_\lambda$ and $T'_{ijk}I_\lambda$. For the converse there are various cases to consider depending on whether $i, j, k \in I$ are joined by an edge. We will consider the most difficult case when $\langle i, j \rangle = \langle i, k \rangle = \langle j, k \rangle = -1$ as the other cases are similar but easier. Note that $\lambda_i + \lambda_j + \lambda_k > 0$ so at least one of $\lambda_i, \lambda_j, \lambda_k$ must be positive.

To show $T_{ijk}I_\lambda \neq 0$ it suffices to show, using Corollary 3.3, that the following composition is nonzero

$$E_j E_k F_i 1_{\lambda+\alpha_i} \xrightarrow{\nu_{kj}} E_j E_k E_i 1_{\lambda+\alpha_j} \xrightarrow{v_{ji}^I} F_i E_j E_k 1_{\lambda+\alpha_j} \xrightarrow{IT_{jk}} F_i E_k E_j 1_{\lambda+\alpha_j}.$$ 

Since $\nu_{kj}$ and $v_{ji}$ are isomorphisms we conclude that

$$T_{ijk}I_\lambda \neq 0 \iff IT_{jk} : F_i E_j E_k 1_{\lambda+\alpha_j} \to F_i E_k E_j 1_{\lambda+\alpha_j} \langle 1 \rangle$$

is nonzero. If $\langle \lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_k \alpha_k \rangle \geq 1 \iff \lambda_i \geq 1$ then we can compose with $E_i$ and simplify to get that $IT_{jk}$ is nonzero if $T_{jk}$ is nonzero. This is nonzero since $1_{\lambda+\alpha_i}, 1_{\lambda+\alpha_j}, 1_{\lambda+\alpha_k}, 1_{\alpha_i+\alpha_j}, 1_{\alpha_i+\alpha_k}$ are all nonzero.

On the other hand, if $\langle \lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j, \alpha_j \rangle \geq 2 \iff \lambda_j \geq 1$ we can compose with $F_j$ on the right to get

$$IT_{jk} \circ E_j E_k F_i 1_{\lambda+\alpha_i+\alpha_j} \to F_i E_k F_j 1_{\lambda+\alpha_i+\alpha_j}.$$

By Lemma 5.4 the $F_i E_k$-rank of this map is $\lambda_j > 0$ which proves it is nonzero. Finally, if $\lambda_k \geq 1$ we compose with $F_k$ instead of $F_j$ on the right and find that the $F_i E_j$-rank is positive. The result follows. \hfill \Box

**Corollary 6.3.** For distinct $i, j, k \in I$ we have $T_{ijk} \sim T'_{ijk}$.

**Proof.** This follows from Proposition 6.2 together with Lemma A.12 which states that

$$\dim \text{Hom}(E_i E_j E_k 1_{\lambda}, E_k E_j E_i 1_{\lambda}(\ell_{ijk})) \leq 1.$$ 

\hfill \Box

**Lemma 6.4.** Suppose $i, a, b \in I$ with $a \neq b$ and $b \neq i$ and let $\ell_{iab} = 2 \langle i, a + b \rangle + 2$. Then

$$\nu_{ia} I I T_{ia}(I I T_{ia}) (I I T_{ia}) (I I T_{ia}) (I I T_{ia}) (I I T_{ia}) (I I T_{ia})(I I T_{ia}) (I I T_{ia}) : E_i E_a E_b 1_{\lambda} \to E_b E_a E_i 1_{\lambda} \langle -\ell_{iab} \rangle$$

is nonzero if and only if $1_{\lambda+\epsilon_i \alpha_i + \epsilon_a \alpha_a + \epsilon_b \alpha_b}$ for $\epsilon_i \in \{0,1,2\}$ and $\epsilon_a, \epsilon_b \in \{0,1\}$.

**Proof.** One direction is immediate since one can easily check that the weights $\lambda + \epsilon_i \alpha_i + \epsilon_a \alpha_a + \epsilon_b \alpha_b$ all appear in the composition (26) (after possibly having to use the relations $T_{ia} \sim T'_{ia}, T_{ib} \sim T'_{ib}$ and $T_{ab} \sim T'_{ab}$).

Let us suppose $a \neq i$ (the case $a = i$ is similar but easier). For the reverse consider the composition

$$E_i E_a E_b F_i \xrightarrow{\text{adj} \ I I I I} F_i E_i E_a E_b F_i \xrightarrow{I(2\alpha)} F_i E_b E_a E_i E_i \xrightarrow{I I I (\alpha)} F_i E_b E_a E_i E_i$$
where we omit the grading shift for convenience. It suffices to show this is nonzero. By using Lemma 3.2 we can rewrite it as

\[ E_i E_a E_b F_i \xrightarrow{T_{\alpha b}x} E_b E_a E_b F_i \xrightarrow{\mu \nu \alpha} E_b E_a E_i F_i \xrightarrow{v a} \mu H \rightarrow F_i E_b E_a E_i. \]

This is just an analogue of Corollary 3.3.

Since \( v a \) and \( v b \) are invertible we are left with showing that

\[ E_a E_b F_i,1_{\lambda + \alpha} T_{\alpha b}^{\text{adj}} \rightarrow E_b E_a E_b F_i,1_{\lambda + \alpha} H v a \rightarrow E_b E_a E_i F_i \xrightarrow{v a} \mu H \rightarrow F_i E_b E_a F_i \]

is nonzero. To do this we apply the same argument again, namely we consider the composition

\[ E_a E_b F_i \xrightarrow{\text{adj}^\dagger H} F_i E_a E_b F_i \xrightarrow{\mu (\text{adj})} F_i E_a E_b F_i E_a F_i \xrightarrow{\text{adj}^\dagger H} F_i E_b E_a F_i \]

and show it is nonzero. Using the same methods as before this composition is equal to (up to rescaling)

\[ E_a E_b F_i \xrightarrow{H T_{\alpha b}} E_a E_b F_i \xrightarrow{T_{\alpha b}^{\text{adj}} \rightarrow E_b E_a F_i \xrightarrow{v a (H \rightarrow F_i E_a E_i F_i \xrightarrow{\text{adj}^\dagger H} F_i E_b E_a F_i \]

where the first map \( T_{\alpha b} \in \text{End}^{-2}(F_i F_i) \) is the adjoint of \( T_{\alpha b} \in \text{End}^{-2}(F_i F_i) \) (we abuse notation a bit). Note that to show this we need one more relation not included in Lemma 3.2, namely that

\[ E_i F_i F_i \xrightarrow{\mu I} F_i E_i F_i \xrightarrow{\text{adj}^\dagger H} F_i \quad \text{and} \quad E_i F_i F_i \xrightarrow{H T_{\alpha b}} E_i F_i F_i \xrightarrow{\text{adj}^\dagger H} F_i \]

are equal to each other (up to rescaling). However, this is easy to prove as in Lemma 3.2 since the space of maps is one-dimensional and by adjunction neither composition is zero. Finally, since \( v b \) and \( v a \) are invertible, to show that (27) is nonzero it suffices to show that \( T_{a b} T_{\alpha b} : E_a E_b 1_{\lambda F_i F_i} \rightarrow E_b E_a 1_{\lambda F_i F_i} \)

is nonzero or equivalently that

\[ T_{a b} T_{\alpha b} : E_a E_b 1_{\lambda F_i F_i} \rightarrow E_b E_a 1_{\lambda F_i F_i} \]

is nonzero. If \( \lambda \leq -2 \) then we can compose (29) on the right with \( E_i^{(2)} \) to get several copies of \( T_{a b} : E_b E_a 1_{\lambda} \rightarrow E_a E_b 1_{\lambda} \) which is nonzero. In fact, this argument can be extended to the case \( \lambda = -1 \) by composing with \( E_i^{(3)} \) on the right to obtain the summand \( T_{a b} T_{\alpha b} : E_a E_b 1_{\lambda E_i} \rightarrow E_b E_a 1_{\lambda E_i} \) which is nonzero using the same arguments used to show \( T_{a b} \) is nonzero.

Finally, if \( \lambda \geq 0 \) then an analogous argument as above reduces the nonvanishing of (26) to showing that

\[ T_{a b} T_{\alpha b} : E_a E_b 1_{\lambda F_i F_i} \rightarrow E_b E_a 1_{\lambda F_i F_i} \]

is nonzero, where \( \mu = \lambda + 2\alpha_i + \alpha_a + \alpha_b \). Composing on the left this time with \( E_i^{(2)} \) we get several copies of \( T_{a b} : E_a E_b \rightarrow E_a E_b \) (which is straight-forward to check is nonzero) as long as

\[ (\mu, \alpha_i) \geq 2 \Leftrightarrow \lambda_i \geq -2 - \langle a + b, i \rangle. \]

This holds if \( \lambda_i \geq 0 \) which concludes the proof that (26) is nonzero. \( \square \)

**Lemma 6.5.** Suppose \( i, a, b \in I \) so that \( T_{i i} I_{\lambda + \epsilon \alpha_a + \epsilon \alpha_b} \) are all nonzero for \( \epsilon \alpha_a, \epsilon \alpha_b \in \{0, 1\} \). Then the two sequences

\[ T_{i i} I_{\lambda} \rightarrow T_{i i} I_{\lambda + \alpha_a} \rightarrow T_{i i} I_{\lambda + \alpha_a + \alpha_b} \quad \text{and} \quad T_{i i} I_{\lambda} \rightarrow T_{i i} I_{\lambda + \alpha_b} \rightarrow T_{i i} I_{\lambda + \alpha_a + \alpha_b} \]

lead to the same rescaling of \( T_{i i} I_{\lambda + \alpha_a + \alpha_b} \).

**Proof.** Assume \( a \neq b \) (otherwise the claim is tautological). Suppose \( a, b \neq i \) and consider

\[ (IIT_{ii})(IIT_{\alpha a})(IIT_{\alpha b})(IIT_{ab})(IIT_{ab}) : E_i E_i E_a E_b 1_{\lambda} \rightarrow E_b E_a E_i E_1 \lambda \]
where we omit the grading shifts for convenience. On the one hand, we can write this as

\[
(IT_{iab})^2(IT_{iab})(IT_{iab}) = (IT_{iab})(IT_{iab})(IT_{iab})(IT_{iab})
\]

where we rescaled \(I_i I \lambda \sim T_i I \lambda + \alpha \sim T_i I \lambda + \alpha + \alpha \) in order to get the first and fourth equalities.

On the other hand, by Corollary 6.3, we know that \(T'_i I \lambda = c_1 T_i I \lambda \) and \(T'_i I \lambda + \alpha_i = c_2 T_i I \lambda + \alpha_i \) for some \(c_1, c_2 \in \mathbb{K} \). Thus, one can rewrite (32) as

\[
c_1 c_2 (T_i I)(IT_{iab})(IT_{iab})(IT_{iab})(IT_{iab})
\]

where we slide the \(T_{ab} \) from the far right to the far left. Then the same sequence of equalities as above shows that this is equal to

\[
c_1 c_2 (T_i I)(IT_{iab})(IT_{iab})(IT_{iab})(IT_{iab})
\]

where this time we had to rescale \(T_i I \lambda \sim T_i I \lambda + \alpha \sim T_i I \lambda + \alpha + \alpha \). Finally, sliding back the \((T_i I)(IT_{iab})\) we find that (34) is equal to (33). Since by Lemma 6.4 the composition in (32) is nonzero it follows that \(c_1 c_2 = 1 \) and so the two sequences of rescalings must be the same.

Finally, suppose one of \(a, b \) is equal to \(i \) (we can assume \(a = i \)). The same argument as above works since, by Corollary 5.2, we have \(T'_i I \lambda = c_1 T_i I \lambda \) and \(T'_i I \lambda + \alpha_i = c_2 T_i I \lambda + \alpha_i \) for some \(c_1, c_2 \in \mathbb{K} \).

**Corollary 6.6.** Suppose \(i, a, b \in 1 \) so that \(T_i I \lambda \) and \(T_i I \lambda + \alpha - \alpha \) are nonzero. Then the two sequences

\[
T_i I \lambda \sim T_i I \lambda + \alpha \sim T_i I \lambda + \alpha - \alpha \quad \text{and} \quad T_i I \lambda \sim T_i I \lambda - \alpha \sim T_i I \lambda + \alpha - \alpha
\]

lead to the same rescaling of \(T_i I \lambda + \alpha - \alpha \).

**Proof.** This is an immediate consequence of Lemma 6.5 as long as we can show that \(T_i I \lambda + \alpha \) and \(T_i I \lambda - \alpha \) are nonzero. By assumption we know \(E_i E_i I \lambda \) and \(E_i E_i I \lambda + \alpha - \alpha \) are nonzero. This implies that \(1_{\lambda + \ell \alpha_i} \) and \(1_{\lambda + \alpha - \alpha + \ell \alpha_i} \) are all nonzero for \(\ell \in \{0, 1, 2\} \). By condition viii this means that

\[
1_{\lambda + \alpha + \ell \alpha_i} \neq 0 \quad \text{for} \quad \ell \in \{0, 1, 2\} \Rightarrow E_i E_i 1_{\lambda + \alpha} \neq 0 \Rightarrow T_i I \lambda + \alpha \neq 0
\]

and likewise \(T_i I \lambda - \alpha \neq 0 \).

**7. Step #4 - The definition of \(X_i \)**

We will call \(\gamma \in \text{End}^2(1_\lambda)\) a transient map if \(\langle \gamma, \alpha_i \rangle_\lambda = 0 \) for all \(i \in I \). The reason for this terminology is that by Lemma 3.6 such maps can be moved past any \(E_i \) or \(F_i \). From hereon until section 12 we will work modulo transient maps. In particular, results will hold modulo transient maps unless explicitly indicated otherwise.

**Lemma 7.1.** For \(\theta \in Y_k \), if \(\langle \theta, \alpha_i \rangle = 0 \) then \(\langle \theta, \alpha_i \rangle_\lambda = 0 \).

**Proof.** Suppose \(\lambda_i \geq 0 \) (the case \(\lambda_i \leq 0 \) is the same). We need to show that \(T_i (\theta I) T_i = 0 \). This is equivalent to showing that the map \(\theta I \in \text{End}^2(E_i E_i E_i)\) induces zero between the summands \(E_i^{(2)} 1_{\lambda} \) on either side. To see this suppose it is nonzero and consider the map \(\theta II \in \text{End}^2(E_i E_i F_i)\) then the \(E_i\)-rank of this map is at least the number of copies of \(E_i \in E_i^{(2)} F_i 1_{\lambda}\) which is \(\lambda_i + 1 > 0 \).

On the other hand, the \(E_i\)-rank of this map is the same as that of \(\theta II \in \text{End}^2(E_i 1_{\lambda} F_i E_i)\) which is zero since \(\langle \theta, \alpha_i \rangle = 0 \) (condition (vi)). This is a contradiction.

**Definition.** For each \(i \in I, \lambda \in X \) rescale the definition of \(\alpha_i \in \text{End}^2(1_\lambda)\) so that \(\langle \alpha_i, \alpha_i \rangle_\lambda = 2 \).
Lemma 7.2. For \( \theta \in Y_k, i \in I \) and \( \lambda \in X \) we have \( \langle \theta, \alpha_i \rangle_\lambda = \langle \theta, \alpha_i \rangle \).

Proof. It suffices to show this for \( \theta = \alpha_j \) where \( j \in I \). First, if \( \langle j, i \rangle = 0 \) then by Lemma 7.1 we have \( \langle \alpha_j, \alpha_i \rangle_\lambda = 0 \). On the other hand, if \( \langle j, i \rangle = -1 \) then \( \langle i + 2j, i \rangle = 0 \) which, by Lemma 7.1 again, means that \( \langle \alpha_i + 2\alpha_j, \alpha_i \rangle_\lambda = 0 \Rightarrow \langle \alpha_j, \alpha_i \rangle_\lambda = -1 \).

Remark 7.3. If \( \theta \in Y_k \) belongs to the radical \( R_k \subset Y_k \) of the bilinear form \( \langle \cdot, \cdot \rangle \) then, by Lemma 7.2, \( \theta \in \text{End}^2(1_\lambda) \) is transient.

Proposition 7.4. If \( \langle \theta, \alpha_i \rangle = 0 \) then \( II\theta = \theta II \in \text{End}^2(1_\lambda) \).

Proof. Suppose \( \lambda_i \geq -1 \) (the case \( \lambda_i \leq -1 \) is similar). Then by Corollary 5.3 we know \( (II\theta) = (\gamma II) \) for some \( \gamma \in \text{End}^2(1_\lambda) \). Since we are working modulo transient maps it suffices to show that \( (\gamma - \theta, \alpha_j)_{\lambda + \alpha_i} = 0 \) for all \( j \in I \).

To see this one follows the same argument as in Corollary 5.3 (the main difference being that now we know \( T_{jji} = T_{jji}^I \) rather than \( T_{jji} \sim T_{jji}^I \)). More precisely, we consider the composition

\[
E_j E_j E_j \xrightarrow{(T_{jji}(T_{jji}))} E_j E_j 1_\lambda E_j \xrightarrow{II\theta} E_j E_j 1_\lambda E_j \xrightarrow{(T_{jji}(T_{jji}))} E_j E_j E_j
\]

where we omit the shifts for convenience. Then, using the same argument as in Corollary 5.3, this composition equals \( \langle \theta, \alpha_j \rangle_{T_{jji}, I} \) while on the other hand it equals \( \langle \gamma - \theta, \alpha_j \rangle_{\lambda + \alpha_i} \). Since \( \langle \theta, \alpha_j \rangle_\lambda = \langle \theta, \alpha_j \rangle_{\lambda + \alpha_i} \), we get \( \langle \gamma - \theta, \alpha_j \rangle_{\lambda + \alpha_i} = 0 \) as long as \( T_{jji} 1_{\lambda - \alpha_j} \neq 0 \).

If \( T_{jji} 1_{\lambda - \alpha_j} = 0 \) then either \( E_j E_j 1_{\lambda + \alpha_i - \alpha_j} = 0 \) or \( E_j E_j 1_{\lambda - \alpha_j} = 0 \). In the first case we have \( (\phi, \alpha_j)_{\lambda + \alpha_i} \neq 0 \) for any \( \phi \in \text{End}^2(\lambda + \alpha_i) \) and so, in particular, \( (\theta - \gamma, \alpha_j)_{\lambda + \alpha_i} = 0 \). Otherwise we have \( 1_{\lambda + \alpha_i - \alpha_j} \neq 0 \) for \( \epsilon_j \in \{-1, 0, 1\} \) and \( 1_\lambda \neq 0 \) which implies \( 1_{\lambda - \alpha_j} \neq 0 \) by condition viii. Using that \( E_j E_j 1_{\lambda - \alpha_j} = 0 \) we get \( 1_{\lambda + \alpha_i - \alpha_j} = 0 \). However, \( 1_{\lambda + \alpha_i - \alpha_j} \neq 0 \) which leads to a contradiction if \( \lambda_i \geq -1 \).

Thus we are done unless \( \lambda_i = -1 \) and \( E_j E_j 1_{\lambda - \alpha_j} = 0 \). This last case is taken care by the analogous argument for \( \lambda_i \leq 1 \) which argues in the same way as above starting with the fact that \( (\theta II) = (II\gamma) \in \text{End}^2(1_{\lambda + \alpha_i} E_j 1_{\lambda}) \) for some \( \gamma \in \text{End}^2(1_{\lambda + \alpha_i}) \).

Definition: Choose \( \theta \in Y_k \) so that \( \langle \theta, \alpha_i \rangle = 1 \) and let \( X_i := -(\theta II) + (II\theta) \in \text{End}^2(1_{\lambda + \alpha_i} E_j 1_\lambda) \).

Corollary 7.5. If \( i \neq j \) then we have

\[
(IX_i)T_{ij} = T_{ij}(X_i) \in \text{Hom}(E_j E_j 1_\lambda E_j E_j 1_\lambda \langle \langle i, j \rangle + 2 \rangle),
\]

\[
(X_i T_{ij}) = T_{ij}(IX_i) \in \text{Hom}(E_j E_j 1_\lambda E_j E_j 1_\lambda \langle \langle i, j \rangle + 2 \rangle).
\]

Proof. Choose \( \gamma \in Y_k \) so that \( \langle \gamma, \alpha_i \rangle = 1 \) and \( \langle \gamma, \alpha_j \rangle = 0 \). Then \( \langle \theta - \gamma, \alpha_i \rangle = 0 \) and so by Proposition 7.4 we have \( (\theta II) - (\gamma II) = (II\theta) - (II\gamma) \in \text{End}(1_{\lambda + \alpha_i} E_j 1_\lambda) \). This means that \( X_i = -(\theta II) + (II\gamma) \). But then, since \( \langle \gamma, \alpha_j \rangle = 0 \), we have

\[
(\gamma II) = (III\gamma) \in \text{End}^2(1_{\lambda + \alpha_i} E_j 1_\lambda)
\]

\[
(\gamma III) = (IIII\gamma) \in \text{End}^2(1_{\lambda + \alpha_i + \alpha_j} E_j 1_{\lambda + \alpha_i} E_j)
\]

from which we get \( T_{ij}(X_i) = (IX_i)T_{ij} \). The second relation follows similarly.

Remark 7.6. The argument in Corollary 7.5 shows that the definition of \( X_i \) does not depend on our choice of \( \theta \).
8. Step #5 – Action of the affine nilHecke algebra

Fix $i \in I$. We will now prove that the affine nilHecke algebra on products of $E_i$’s. This argument is essentially a simplification and strengthening of the one in [CKL2].

**Proposition 8.1.** Inside $\text{End}(E_iE_i)$ we have (modulo transient maps)

$$T_{ii}(X_iI) = (IX_iT_{ii} + II \quad \text{and} \quad (X_iI)T_{ii} = T_{ii}(IX_i) + II.$$ \hfill (36)

**Proof.** Let us suppose we are trying to prove this for $\text{End}(E_i1_\lambda E_i)$ where $\lambda_i \geq 0$ (the case $\lambda_i \leq 0$ is the same). By Lemma 4.7 we have a

$$id = a \cdot (I\theta I)T_{ii} + a \cdot T_{ii}(I\theta I) + T_{ii}(\tau) \in \text{End}(E_i1_\lambda E_i)$$

where $\tau = (\rho III) + b \cdot (III\theta) \in \text{End}^2(1_{\lambda+\alpha_i}E_i1_{\lambda-\alpha_i})$. Composing on the left with $T_{ii}$ and using that $T_{ii}(I\theta I)T_{ii} = (\theta, \alpha_i)\lambda T_{ii} = T_{ii}$ we get $T_{ii} = aT_{ii}$ which means $a = 1$. Thus, we can rewrite (36) as

$$T_{ii}(X_iI)T_{ii} = T_{ii}(IX_i) + II + T_{ii}(\tau') \in \text{End}(E_iE_i)$$

where $\tau' = (\gamma' III) + (III\gamma) \in \text{End}(1_{\lambda+\alpha_i}E_i1_{\lambda-\alpha_i})$. It remains to show that $(\gamma, \alpha_j)_{\lambda-\alpha_i} = 0 = (\gamma', \alpha_j)_{\lambda+\alpha_i}$ for all $j \in I$.

First we show that $(\gamma, \alpha_i)_{\lambda-\alpha_i} = 0$. To do this consider

$$(X_iII)(T_{ii}I) = (T_{ii}I)(IX_iI) + III + (T_{ii}I)(\tau'I) \in \text{End}(1_{\lambda+\alpha_i}E_iE_iE_i)$$

and compose on the left with $(T_{ii}I)(IT_{ii})$ and on the right with $(IT_{ii})$. This gives us

$$(T_{ii}I)(IT_{ii})(X_iII)(T_{ii}I)(IT_{ii}) = T_{ii}(IX_iI)(IT_{ii}I) + (T_{ii}I)(IT_{ii}I)(\tau'I)(IT_{ii}I)$$

$$= T_{ii}(IX_iI)(IT_{ii}) + (T_{ii}I)(IT_{ii})(\tau'I)(IT_{ii}I)$$

$$= T_{ii}(IX_iI)(IT_{ii}) + T_{ii}(\tau'I)(IT_{ii}I)$$

$$= T_{ii}(I)(X_iII)(T_{ii}I)(IT_{ii}) = T_{ii}T_{ii} + (T_{ii}I)(IT_{ii})(\tau'I)(IT_{ii})$$

which we use $T_{ii}(X_iI)T_{ii} = T_{ii}$ and that

$$(IT_{ii})(\tau')(IT_{ii}) = (IT_{ii})(\gamma' IIII) + (III\gamma I)(IT_{ii}) = 0 + (\gamma, \alpha_i)_{\lambda-\alpha_i}T_{ii}$$

to obtain the last two equalities. From this it follows that $(\gamma, \alpha_i)_{\lambda-\alpha_i} = 0$. The argument that $(\gamma', \alpha_i)_{\lambda+\alpha_i} = 0$ is similar.

Next we show that $(\gamma, \alpha_j)_{\lambda-\alpha_i} = 0$ for $j \neq i$. To do this we first compose (37) with $F_j$ on either sides. The term involving $\gamma$ then becomes

$$h := (IT_{ii}II)(III\gamma I) \in \text{End}(E_iE_iE_i1_{\lambda-\alpha_i}F_j).$$

Now consider the composition

$$(38) \quad E_iE_iF_j1_{\lambda-\alpha_i}F_j \xrightarrow{(\psi j II)(\psi j II)} F_jE_iE_i1_{\lambda-\alpha_i}F_j \xrightarrow{h} F_jE_iE_i1_{\lambda-\alpha_i}F_j \xrightarrow{(\psi j II)(\psi j II)} E_iE_iF_j1_{\lambda-\alpha_i}F_j.$$

We have

$$F_jE_iE_i1_{\lambda-\alpha_i}F_j : E_iE_iF_j1_{\lambda-\alpha_i} \rightarrow E_iE_iF_j1_{\lambda-\alpha_i}$$

which means that the composition in (38) is (up to a multiple) equal to

$$(T_{ii}III)(T_{ii}II)(u_{ji}III)(u_{ji}III) : F_jE_iE_i1_{\lambda-\alpha_i} \rightarrow E_iE_iF_j1_{\lambda-\alpha_i}$$

which means that the composition in (38) is (up to a multiple) equal to

$$(T_{ii}III)(u_{ji}III)(u_{ji}III)(u_{ji}III)(IX_jI) \sim (T_{ii}III)(III\gamma I) \in \text{End}(E_iE_iF_j1_{\lambda-\alpha_i}F_j).$$

Finally, composing on the left and right with $II T_{jj}'$ where $T_{jj}' \in \text{End}^{-2}(F_jF_j)$ is the unique map we get (up to a multiple)

$$(\gamma, \alpha_j)_{\lambda-\alpha_i} \sim (T_{jj}III)(III' T_{jj}') \in \text{End}(E_iE_iF_jF_j).$$

On the other hand, performing the same manipulations with the others terms in (37) ends up giving us zero. It follows that $(\gamma, \alpha_j)_{\lambda-\alpha_i} = 0$. A similar argument also shows $(\gamma', \alpha_j)_{\lambda+\alpha_i} = 0$. □
Thus, Proposition 8.1 and Corollary 7.5 gives us the following relations (modulo transient maps):

\[(39) \quad T_{ii}(X,I) = (IX_I)_iT_{ii} + II \quad \text{and} \quad (X,I)T_{ii} = T_{ii}(IX_I) + II \in \text{End}(E_iE_i),\]

\[(40) \quad (IX_I)T_{ij} = T_{ij}(X,I) \in \text{Hom}(E_iE_j, E_jE_i(-\langle i,j \rangle + 2)) \text{ if } i \neq j,\]

\[(41) \quad (X,I)T_{ji} = T_{ji}(IX_i) \in \text{Hom}(E_iE_i, E_jE_i(-\langle i,j \rangle + 2)) \text{ if } i \neq j.\]

**Corollary 8.2.** For any \(i \in I\) the compositions

\[1_{\lambda} \xrightarrow{\text{adj}} E_iF_i1_{\lambda}(-\lambda_i + 1) \xrightarrow{X_{\lambda_i}^{-1}I} E_iF_i1_{\lambda}(\lambda_i - 1) \xrightarrow{\text{adj}} 1_{\lambda} \quad \text{if } \lambda_i \geq 0 \quad \text{and} \]

\[1_{\lambda} \xrightarrow{\text{adj}} F_iE_i1_{\lambda}(\lambda_i + 1) \xrightarrow{IX_{\lambda_i}^{-1}I} F_iE_i1_{\lambda}(-\lambda_i - 1) \xrightarrow{\text{adj}} 1_{\lambda} \quad \text{if } \lambda_i \leq 0\]

are equal to a nonzero multiple of the identity map in \(\text{End}(1_{\lambda})\).

**Proof.** This is an immediate consequence of Lemma 3.4, the fact that \(\text{End}^\ell(1_{\lambda}) = 0\) if \(\ell < 0\) and the definition of \(X_i\) from section 7. \(\square\)

9. Step #6 - The Serre relation

In this section we fix \(i, j \in I\) with \(\langle i,j \rangle = -1\) and prove the Serre relation (Corollary 9.6). The idea behind the proof is to first show that by adjunction there exist unique maps (up to rescaling)

\[E_i^{(2)}E_j \oplus E_j^{(2)}E_i \rightarrow E_iE_i \rightarrow E_i^{(2)}E_j \oplus E_j^{(2)}E_i\]

whose composition is the identity (this last step is the hardest part). Separately, by using adjunction we know that \(\dim \text{End}(E_iE_jE_i) \leq 2\) from which the Serre relation follows.

**Lemma 9.1.** Suppose \(i, j \in I\) with \(\langle i,j \rangle = -1\). If \(E_iE_i1_{\lambda} = 0\) then \(E_j^{(2)}E_i1_{\lambda} = 0\). Likewise, if \(E_iE_i1_{\lambda + \alpha_i} = 0\) then \(E_i^{(2)}E_j1_{\lambda} = 0\).

**Proof.** We prove the first assertion (the second assertion is similar). If \(1_{\lambda}, 1_{\lambda + \alpha_i}, 1_{\lambda + 2\alpha_i}\) or \(1_{\lambda + \alpha_i + \alpha_j}\) are zero then we are done. So suppose they are all nonzero.

Since \(E_jE_i1_{\lambda} = 0\) this means that \(1_{\mu} = 0\) where \(\mu := \lambda + \alpha_i + \alpha_j\). Since \(1_{\nu + \alpha_i} \neq 0\) we must have \(\langle \mu, \alpha_i, \alpha_i \rangle \leq 0 \Leftrightarrow \lambda_i \leq -3\) because otherwise \(1_{\mu + \alpha_i}\) is a direct summand of \(E_i1_{\mu}F_i = 0\).

Finally, if \(\lambda_i \leq -3\) then \(E_i^{(2)}E_i1_{\lambda}\) is a direct summand of

\[E_jF_i(-\lambda_i - 2)_11_{\lambda} \cong F_i(-\lambda_i - 2)_11_{s_i\nu}E_jE_i(-\lambda_i)_11_{\lambda}\]

Since, in general, \(1_{\nu}\) if and only if \(1_{s_i\nu} = 0\) we find that the right side above is zero and hence \(E_jE_i^{(2)}1_{\lambda} = 0\). \(\square\)

**Proposition 9.2.** If \(E_jE_i^{(2)}1_{\lambda} \neq 0\) then \((T_{ji}I)(IT_{ii})(T_{jj}I) \in \text{End}(E_iE_jE_i1_{\lambda})\) is nonzero. Likewise, if \(E_i^{(2)}E_j1_{\lambda} \neq 0\) then \((IT_{ii})(T_{ij}I)(II) \in \text{End}(E_iE_jE_i1_{\lambda})\) is nonzero. Moreover, these two maps are linearly independent.

**Proof.** Suppose \((T_{ji}I)(IT_{ii})(T_{jj}I) \in \text{End}(E_iE_jE_i1_{\lambda})\) is zero. Then composing with \((XII)\) we get that

\[(T_{ji}I)(IT_{ii})(T_{jj}I)(XII) = (T_{ji}I)(IT_{ii})(IX_I)(T_{ij}I)\]

\[= (T_{ji}I)(IX_I)(T_{ij}I) + (T_{ji}I)(T_{ij}I)\]

must be zero, which implies that \((T_{ji}I)(T_{ij}I) \in \text{End}^2(E_iE_jE_i1_{\lambda})\) is zero.
Now, if \( \langle \lambda + \alpha_i, \alpha_i \rangle \leq -1 \) (meaning \( \lambda_i \leq -3 \)) then we compose on the left with \( F_i \) and by Lemma 5.4 find that the \( E_j E_i \lambda \)-rank of \( (T_{ij} I)(T_{ij} I) \) is \( -\lambda_i - 2 \geq 1 \). Since by Lemma 9.1 \( E_j E_i^{(2)} \lambda \neq 0 \Rightarrow E_j E_i \lambda \neq 0 \) this means that \( (T_{ij} I)(T_{ij} I) \) cannot be zero (contradiction).

Likewise, if \( \langle \lambda + \alpha_i, \alpha_i \rangle \geq 0 \) (meaning \( \lambda_i \geq -2 \)) then we compose with \( F_i \) on the right and by Lemma 5.4 find that the \( E_j E_i \lambda \)-rank of \( (T_{ij} II)(T_{ij} II) \) is at least \( \lambda_i + 3 \geq 1 \) so once again \( (T_{ij} I)(T_{ij} I) \) cannot be zero (contradiction).

The case of \( (IT_{ij})(IT_{ij}) \) is proved similarly.

Finally, suppose \( (T_{ij} I)(IT_{ii})(T_{ij} I) = c(II)(IT_{ii})(IT_{ij}) \) for some \( c \in k^\times \). Composing with \( (X_{iII}) \) as above and simplifying we get that
\[
(T_{ij} I)(T_{ij} I) = c(II)(IT_{ij}) \in \text{End}(E_j E_i \lambda).
\]

If \( \lambda_i \leq -3 \) then we compose with \( F_i \) on the left and find that the left hand side has positive \( E_j E_i \lambda \)-rank. However, the right hand side factors through \( F_i E_j E_i \lambda \) and hence must have zero \( E_j E_i \lambda \)-rank. \( \square \)

**Corollary 9.3.** There exist \( m_\lambda^{ij}, n_\lambda^{ij} \in k^\times \) so that
\[
m_\lambda^{ij}(T_{ij} I)(IT_{ii})(T_{ij} I) + n_\lambda^{ij}(T_{ij} I)(IT_{ii})(IT_{ij}) = (III) \in \text{End}(E_j E_i \lambda).
\]

**Remark 9.4.** This result is true without having to mod out by transient maps. This is because the key result that \( \dim \text{End}(E_j E_i \lambda) \leq 2 \) holds without having to mod out.

**Proof.** Let us suppose \( E_j^{(2)} \lambda \) and \( E_j E_i^{(2)} \lambda \) are both nonzero (the other cases are strictly easier). Recall that \( T_{ij} = (T_{ij} I)(IT_{ii})(T_{ij} I) \) and \( T_{ij}' = (IT_{ii})(T_{ij} I)(IT_{ij}) \). By Corollary 9.2 we know \( T_{ij} I \lambda \) and \( T_{ij}' I \lambda \) are nonzero and linearly independent. On the other hand, by Lemma A.11, we have \( \dim \text{End}(E_j E_i E_i \lambda) \leq 2 \). Thus \( T_{ij} I \lambda \) and \( T_{ij}' I \lambda \) span this space and the identity morphism must be a linear combination of these two.

Finally, to show that \( m_\lambda^{ij} \) and \( n_\lambda^{ij} \) are nonzero note that by Lemma 9.5 we have \( m_\lambda^{ij} T_{ij}^2 I \lambda = T_{ij} I \lambda \) and, in particular, \( m_\lambda^{ij} \neq 0 \). Likewise, we also get \( n_\lambda^{ij} \neq 0 \).

**Lemma 9.5.** We have \( T_{ij} T_{ij}' I \lambda = 0 \) while \( m_\lambda^{ij} T_{ij}^2 I \lambda = T_{ij} I \lambda \) and \( n_\lambda^{ij} T_{ij}'^2 I \lambda = T_{ij}' I \lambda \).

**Proof.** We have
\[
T_{ij} T_{ij}' = (T_{ij} I)(IT_{ii})(T_{ij} I)(IT_{ii})(T_{ij} I)(IT_{ii}) = (T_{ij} I)(IT_{ii})(T_{ij} I)(IT_{ji})(T_{ij} I)(IT_{ji}) = 0
\]
where we used relation (23) to rewrite the part in the brackets. Multiplying (42) by \( T_{ij} \) we get \( m_\lambda^{ij} T_{ij}^2 I \lambda = T_{ij} I \lambda \) and likewise for the last relation. \( \square \)

**Corollary 9.6.** We have \( E_i E_j E_i \cong E_i^{(2)} E_j \oplus E_j E_i^{(2)} \).

**Proof.** Suppose \( E_j E_i^{(2)} \lambda \) and \( E_j E_i^{(2)} \lambda \) are nonzero (the cases when one or both are zero is strictly easier). First consider the composition
\[
E_j E_i^{(2)} \lambda \xrightarrow{(IT_{ii})(T_{ij})} E_j E_i E_i \lambda \xrightarrow{(IT_{ij})(T_{ij})} E_j E_i^{(2)} \lambda
\]
where \( \iota : E_i^{(2)} \to E_i E_i \langle -1 \rangle \) is the natural inclusion and \( \pi : E_i E_i \to E_i^{(2)} \langle -1 \rangle \) the natural projection.

Claim: the composition in (43) is nonzero. Note that \( \iota \circ \pi \in \text{End}^2(E_i E_i) \) is equal to \( T_{ii} \) up to rescaling. So it suffices to show that the following composition is nonzero
\[
E_j E_i E_i \lambda \xrightarrow{(IT_{ii})} E_j E_i E_i \lambda \xrightarrow{(IT_{ij})(T_{ij})} E_j E_i E_i \lambda \xrightarrow{(IT_{ii})} E_j E_i E_i \lambda
\]
where we omit shifts for convenience. This composition appears as a factor inside \( T_{ij} T_{ij} I \lambda \) so it suffices to show \( T_{ij} T_{ij} I \lambda \neq 0 \). This follows from Lemma 9.5 since \( m_\lambda^{ij} T_{ij} T_{ij} I \lambda = T_{ij} I \lambda \) and completes the proof of the claim.
Since (43) is nonzero it must be some multiple of the identity (by Lemma A.9). Thus $E_i E_j E_k^{1/2} 1\lambda$ is a direct summand of $E_i E_j E_k 1\lambda$. Likewise, one can also prove that $E_i^{1/2} E_j 1\lambda$ is a direct summand of $E_i E_j E_k 1\lambda$. Thus

$$E_i E_j E_k 1\lambda \cong E_i E_j E_k^{1/2} 1\lambda \oplus E_i^{1/2} E_j 1\lambda \oplus R 1\lambda$$

for some 1-morphism $R$. But by Lemma A.11 we know $\dim \text{End}(E_i E_j E_k 1\lambda) \leq 2$ which means $R 1\lambda = 0$ and the result follows. \hfill $\square$

10. Step #7 – The $T_{iji}$ Relation

In this section we consider $i, j \in I$ such that $\langle i, j \rangle = -1$. We fix a subset $S_{ij} \subset I \setminus \{i, j\}$ such that \{\$\alpha_k : k \in S_{ij}\} \cup \{\alpha_i, \alpha_j\$ give a basis of $Y_k/R_k$ where $R_k \subset Y_k$ denotes the radical. By construction this means that $\langle \cdot, \cdot \rangle$ is nondegenerate on the subspace of $Y_k$ spanned by \{\$\alpha_k : k \in S_{ij}\} \cup \{\alpha_i, \alpha_j\$.

**Lemma 10.1.** We have

$$T_{ij} T_{ij} I = (\phi^{ij}_{\lambda} III) + (III \theta^{ij}_{\lambda}) : 1_{\lambda} + \alpha_i + \alpha_j \mapsto 1_{\lambda} + \alpha_i + \alpha_j \mapsto 1_{\lambda} \langle 2$$

where $\theta^{ij}_{\lambda}$ and $\phi^{ij}_{\lambda}$ satisfy $\theta^{ij}_{\lambda} (\alpha_k) + (\phi^{ij}_{\lambda} (\alpha_k) + \alpha_i + \alpha_j = 0$ for any $k \in S_{ij}$.

**Proof.** Any map $1_{\lambda} + \alpha_i + \alpha_j \mapsto 1_{\lambda} + \alpha_i + \alpha_j \mapsto 1_{\lambda} \langle 2$ is of the form $\phi^{ij}_{\lambda} III + III \theta^{ij}_{\lambda}$ for some $\theta^{ij}_{\lambda} \in \text{End}^2(1_{\lambda})$ and $\phi^{ij}_{\lambda} \in \text{End}^2(1_{\lambda} + \alpha_i + \alpha_j)$. Now consider $k \in S_{ij}$. By nondegeneracy of $\langle \cdot, \cdot \rangle$ on $Y_k/R_k$ we can find $\gamma \in Y_k$ so that $\langle \gamma, \alpha_k \rangle = 1$ while $\langle \gamma, \alpha_i \rangle = \langle \gamma, \alpha_j \rangle = 0$.

Now consider the composition

$$E_i E_j F_k F_k (v_{ik} II) (v_{jk} II) = F_k E_i E_j F_k (v_{ik} II) (v_{jk} II) = F_k E_i E_j F_k (v_{ik} II) (v_{jk} II)$$

On the other hand, since $k \neq i, j$ we can apply Corollary A.13 to conclude that

$$\langle IT_{ij} I (v_{ik} II) (v_{jk} II) \sim (v_{ik} II) (v_{jk} II) (IT_{ij} II) \quad \text{and} \quad (I u_{kj} I) (u_{ki} II) (u_{ki} II) (u_{ki} II)$$

This means that, up to a multiple, the composition in (44) is equal to

$$E_i E_j F_k F_k (v_{ik} II) (v_{jk} II) = F_k E_i E_j F_k (v_{ik} II) (v_{jk} II) = F_k E_i E_j F_k (v_{ik} II) (v_{jk} II)$$

Since $u_{kj}$ and $v_{jk}$ are inverses (up to a multiple) and likewise for $u_{ki}$ and $v_{ik}$ this composition is (up to a multiple) equal to $(T_{ij} II')(T_{ij} II) \in \text{End}^2(E_i E_j F_k F_k)$. In particular, this means that composing (44) on the left and right with $(IT_{kk} II)$ we get zero.

On the other hand, we can rewrite (44) as

$$E_i E_j F_k F_k (v_{ik} II) (v_{jk} II) = F_k 1_{\mu} E_i E_j F_k (I u_{kj} I) (v_{jk} II) = F_k 1_{\mu} E_i E_j F_k (I u_{kj} I) (v_{jk} II)$$

where $\mu = \lambda + \alpha_i + \alpha_j$. Now $(\phi^{ij}_{\lambda} II) \in \text{End}^2(1_{\mu} F_k 1_{\lambda + \alpha_k})$ can be rewritten as $(II \rho) + a(II)$ for some $\rho \in \text{End}^2(1_{\lambda + \alpha_k})$ and where $a := (\phi^{ij}_{\lambda} (\alpha_k) \lambda$. Likewise, $(II \phi^{ij}_{\lambda}) \in \text{End}^2(1_{\mu - \alpha_k} F_k 1_{\mu})$ can be rewritten as $(\rho II) + b(II \gamma)$ where $b := (\phi^{ij}_{\lambda} (\alpha_k) \lambda + \alpha_k + \alpha_j$. Also, since $\langle \gamma, \alpha_i \rangle = \langle \gamma, \alpha_j \rangle = 0$ the map $b(II \gamma) \in \text{End}^2(F_k 1_{\lambda + \alpha_k + \alpha_j} E_j)$ is equal to $b(II \gamma) \in \text{End}^2(F_k 1_{\lambda + \alpha_k} 1\lambda)$. Thus (45) is equal to the composition

$$E_i E_j F_k F_k (v_{ik} II) (v_{jk} II) = F_k E_i E_j F_k (I u_{kj} I) (v_{jk} II) = F_k E_i E_j F_k (I u_{kj} I) (v_{jk} II)$$

where $h$ is the map

$$1_{\mu - \alpha_k} F_k E_i E_j 1_{\lambda} F_k 1_{\lambda + \alpha_k} \sim (\rho II II) + (II II II) + (a + b)(II II II) \sim 1_{\mu - \alpha_k} F_k E_i E_j 1_{\lambda} F_k 1_{\lambda + \alpha_k} 2.$$
Up to rescaling this is equal to
\[ 1_{\mu-\alpha_i} E_i E_j F_k 1_{\lambda+\alpha_i} \cdot (\phi^{ij}_{\lambda+\alpha_i})^{(I)(II)(III)} + (\phi^{ij}_{\lambda+\alpha_i})^{(II)(II)(II)} + (\phi^{ij}_{\lambda+\alpha_i})^{(III)(II)} + (\phi^{ij}_{\lambda+\alpha_i})^{(II)(I)(I)} \to 1_{\mu-\alpha_i} E_i E_j F_k 1_{\lambda+\alpha_i} \cdot (\phi^{ij}_{\lambda+\alpha_i})^{(I)(II)(III)} + (\phi^{ij}_{\lambda+\alpha_i})^{(II)(II)(II)} + (\phi^{ij}_{\lambda+\alpha_i})^{(III)(II)} + (\phi^{ij}_{\lambda+\alpha_i})^{(II)(I)(I)}. \]

Composing on both sides with \( IIT_{kk} \in \text{End}^{-2}(E_i E_j F_k) \) we find that we get zero if and only if \( \alpha = 0 \). This concludes the proof. \( \square \)

**Lemma 10.2.** Using the notation from Corollary 9.3 and Lemma 10.1 we have

(i) \( n_{ij}^{\lambda} \cdot (\phi^{ij}_{\lambda}, \alpha_i)_{\lambda+\alpha_i} = 1 \) if \( E_i^{(i)} E_j \neq 0 \) and

(ii) \( m_{ij}^{\lambda-\alpha_i} \cdot (\phi^{ij}_{\lambda}, \alpha_j)_{\lambda} = 1 \) if \( E_i^{(i)} E_j \neq 0 \).

**Proof.** On the one hand we have \( n_{ij}^{\lambda}(T'_{ij})^2 I_\lambda = T'_{ij} I_\lambda \) by Lemma 9.5. On the other hand,

\[ (T'_{ij})^2 I_\lambda = (IT_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) I_\lambda \]

which is equal to the composition

\[ E_i E_j 1_{\lambda} \xrightarrow{(IT_{ij})(IT_{ij})} E_i 1_{\mu} E_j 1_{\lambda} (1) \xrightarrow{(IT_{ij})(IT_{ij})} E_i E_j E_1. \]

where \( \mu = \lambda + \alpha_i + \alpha_j \). Here we used Lemma 10.1 to rewrite the part in the brackets.

Now \( (T_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) = 0 \) whereas \( (T_{ii})(IT_{ij})(IT_{ij})(IT_{ij}) = \phi_{\lambda}^{ij}, \alpha_i + \alpha_j \). Thus

\[ (n_{ij}^{\lambda})^{-1} T_{ij} I_\lambda = (\phi^{ij}_{\lambda}, \alpha_i)_{\lambda+\alpha_i+\alpha_j} \]

and hence \( n_{ij}^{\lambda}(\phi^{ij}_{\lambda}, \alpha_i)_{\lambda+\alpha_i+\alpha_j} = 1 \). The second relation follows by considering \( T_{ij} \) instead of \( T'_{ij} \). \( \square \)

**Lemma 10.3.** Using the notation from Corollary 9.3 and Lemma 10.1 we have

(i) \( m_{ij}^{\lambda+\alpha_i} \cdot (\phi^{ij}_{\lambda}, \alpha_i)_{\lambda+\alpha_i} = -1 \) assuming both terms are nonzero and

(ii) \( n_{ij}^{\lambda} \cdot (\theta^{ij}_{\lambda}, \alpha_i) = 1 \) assuming both terms are nonzero.

**Proof.** First note that

\[ (T_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) = (T_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) \]

and similarly

\[ (IT_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) = (IT_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) \]

Using that \( m_{ij}^{\lambda} \cdot (T_{ij})(IT_{ij})(IT_{ij})(IT_{ij}) I_\lambda = (III)(I)(I)(I) I_\lambda \) and simplifying we arrive at the relation

\[ m_{ij}^{\lambda}(T_{ij})(IT_{ij})(IT_{ij}) I_\lambda + n_{ij}^{\lambda}(IT_{ij})(IT_{ij}) I_\lambda = (X)(I)(I)(I) I_\lambda - (II)(I)(I)(I) I_\lambda \in \text{End}^2(E_i E_j E_1). \]

Now, the left side is equal to

\[ m_{ij}^{\lambda}[(\phi^{ij}_{\lambda+\alpha_i})^{(I)(I)(I)(I)} + (III)(I)(I)(I)] + n_{ij}^{\lambda}[(II)(I)(I)(I)(I)(I)(I)] \]

as an element in \( \text{End}^2(E_i E_j E_1) \), where \( \mu = \lambda + \alpha_i + \alpha_j \). On the other hand, the right side is equal to

\[ - (\theta(I)(I)(I)(I)(I)(I)(I)) \]

for some \( \theta \in Y_k \) with \( \langle \theta, \alpha_i \rangle = 1 \). Comparing (46) and (47) we find that

\[ m_{ij}^{\lambda}[(\phi^{ij}_{\lambda+\alpha_i}, \alpha_i)^{(I)(I)(I)(I)}] = - (\theta, \alpha_i)^{(I)(I)(I)(I)(I)} \]

and

\[ n_{ij}^{\lambda}[(\theta^{ij}_{\lambda}, \alpha_i)(I)(I)(I)(I)(I)] = - (\theta, \alpha_i)^{(I)(I)(I)(I)(I)}. \]

This concludes the proof. \( \square \)

**Lemma 10.4.** Using the notation from Lemma 10.1 we have
(i) \((\theta^i_j, \alpha_i)_\lambda = (\theta^j_i, \alpha_j)_\lambda\) if \(E^{(2)}_i E_j 1_{\lambda - \alpha_i} \neq 0\) and \(E^{(2)}_j E_i 1_{\lambda - \alpha_j} \neq 0\),

(ii) \((\phi^i_j, \alpha_j)_\lambda+\alpha_i+\alpha_j = (\phi^j_i, \alpha_j)_\lambda+\alpha_i+\alpha_j\) if \(E^{(2)}_j E_i 1_{\lambda} \neq 0\) and \(E^{(2)}_i E_j 1_{\lambda} \neq 0\).

**Proof.** Consider the following composition

\[ (48) \quad E_j E_i 1_{\lambda} E_i \xrightarrow{(IT_{ii})} E_j E_i 1_{\lambda} E_i \xrightarrow{(T_{ji})} E_j E_i 1_{\lambda} E_i \xrightarrow{(T_{ii})} E_j E_i 1_{\lambda} E_i \xrightarrow{(T_{ii})} E_j E_i E_j \]

where we omit the grading shifts for convenience. On the one hand, this is equal to the composition

\[ E_j E_i 1_{\lambda} E_i \xrightarrow{(IT_{ii})} E_j E_i 1_{\lambda} E_i \xrightarrow{(\phi^j_i II\theta^j_i I + II\theta^j_i I)} E_j E_i 1_{\lambda} E_i \xrightarrow{T_{ji}} E_j E_i E_j \]

where \(\mu = \lambda + \alpha_i + \alpha_j\). Now \(T_{ji} = T_{ji} = (IT_{ji})(T_{ji})(IT_{ii})\) which means that

\[ T_{ji}(II\theta^j_i I)(IT_{ii}) = (IT_{ji})(T_{ji})(II\theta^j_i I)(IT_{ii}) = \phi^j_i II\theta^j_i I \]

while \(T_{ji}(II\theta^j_i I)(IT_{ii}) = 0\). In particular we get that the composition in (48) is equal to \((\theta^j_i, \alpha_j)_T T_{ji}\).

On the other hand, we can also write the composition in (48) as

\[ E_j E_i 1_{\lambda} E_i \xrightarrow{(IT_{ii})} E_j E_i 1_{\lambda} E_i \xrightarrow{(\phi^j_i II\theta^j_i I + II\theta^j_i I)} E_j E_i 1_{\lambda} E_i \xrightarrow{T'_{ji}} E_j E_i E_j \]

The same argument as above simplifies this composition to give \((\theta^j_i, \alpha_j)_T T_{ji}\). Thus must have \((\theta^j_i, \alpha_j)_T = (\theta^j_i, \alpha_j)_T\). The second relation follows similarly. \(\square\)

**Proposition 10.5.** One can rescale the maps \(T_{ij}\) so that \(m_{\lambda ij}^\mu = r_i^{-1}\) and \(n_{ij}^\lambda = -t_{ij}^{-1}\) for all \(\lambda\).

**Proof.** First notice that rescaling some \(T_{ij} 1_{\lambda}\) does not affect the relations in (22), (23), (40) and (41).

This is because each \(T_{ij}\) occurs the same number of times on both sides of these relations.

Choose \(\lambda\) so that \(1_{\lambda} \neq 0\) whereas \(1_{\lambda - \alpha_i} = 0\). We will now rescale all \(T_{ij} I\mu\) where \(\mu = \lambda + r\alpha_i\) for some \(r \in \mathbb{Z}\). We assume that \(1_{\lambda + \alpha_i} = 0\) and proceed by increasing induction on \(\lambda_i\). If this is not the case then instead of \(\lambda\) we choose \(s_1 \cdot \lambda := \lambda - \lambda_1 \alpha_i\) where \(1_{s_1 \cdot \lambda} \neq 0\) and \(1_{s_1 \cdot \lambda + \alpha_i} = 0\) and proceed by decreasing induction on \(\lambda_i\).

**The base case.** Since \(1_{\lambda + \alpha_i} = 0\) we have \(E^{(2)}_i E_j 1_{\lambda} = 0\) and hence \((IT_{ij})(IT_{ii}) I_{\lambda} = 0\). Thus, by Corollary 9.3, \(m_{\lambda ij}^\mu (T_{ij} I)(IT_{ii}) I_{\lambda} = (III) I_{\lambda}\). So we can rescale \(T_{ij} I_{\lambda} + \alpha_i\) so that \(m_{\lambda ij}^\mu = t_{ij}^{-1}\). This proves the base case.

**The induction step.** Consider now

\[ m_{\lambda ij}^\mu (T_{ij} I)(IT_{ii}) I_{\lambda} + n_{\lambda ij}^\mu (IT_{ji}) (IT_{ii}) I_{\lambda} = (III) I_{\mu} \in \text{End}(E_j E_i E_j 1_{\mu}) \]

where \(\mu = \lambda + r\alpha_i\) for some \(r > 0\). By induction we have \(E^{(2)}_i E_j 1_{\mu - \alpha_i} \neq 0\) and we have rescaled \(T_{ij} I_{\mu}\) so that \(m_{\lambda ij}^\mu = t_{ij}^{-1}\). We claim that this implies \(n_{\lambda ij}^\mu = -t_{ij}^{-1}\).

To see this note that by Lemma 10.2 we have \(\lambda \mu = \mu\mu = t_{ij}\). Now, if \(E^{(2)}_i E_j 1_{\mu} = 0\) there is nothing to prove. If it is nonzero then by Lemma 10.3 we have \(n_{\lambda ij}^\mu \cdot (\theta^j_i, \alpha_j)_\mu = -1\) which proves the claim.

So now we can just rescale \(T_{ij} I_{\lambda} + \alpha_i\), so that \(m_{\lambda ij}^\mu = t_{ij}^{-1}\). This completes the induction. \(\square\)

At this point we know the following relations:

\[ (49) \quad (\theta^i_j, \alpha_k)_\lambda = -(\phi^i_j, \alpha_j)_\lambda + \alpha_i + \alpha_j \quad \text{if} \; k \in S_{ij}, \]

\[ (50) \quad (\theta^i_j, \alpha_i)_\lambda = t_{ij} = -(\phi^i_j, \alpha_j)_\lambda + \alpha_i + \alpha_j \quad \text{and} \]

\[ (51) \quad (\theta^i_j, \alpha_j)_\lambda = -(\phi^i_j, \alpha_j)_\lambda + \alpha_i + \alpha_j. \]

To see the last relation above notice that switching the roles of \(i\) and \(j\) in Lemma 10.2 equation (i) gives \(n_{\lambda ij}^\mu (\phi^j_i, \alpha_j)_\lambda + \alpha_i + \alpha_j = 1\) which together with Lemma 10.4 equation (ii) gives \(n_{\lambda ij}^\mu (\phi^j_i, \alpha_j)_\lambda + \alpha_i + \alpha_j = 1\). On the other hand, switching the roles of \(i\) and \(j\) in Lemma 10.3 equation (ii) gives \(n_{\lambda ij}^\mu (\phi^j_i, \alpha_j)_\lambda = -1\). Putting these two relations together gives \(\theta^j_i, \alpha_j)_\lambda = -(\phi^j_i, \alpha_j)_\lambda + \alpha_i + \alpha_j\).
Corollary 10.6. There exists $t_{ji}^\lambda \in k^\times$ such that
\begin{align}
\text{(52)} & \quad (T_{ji})(T_{ij})I_\lambda = t_{ij}(X_{ij}I_\lambda) + t_{ji}^\lambda(I X_{ij})I_\lambda \in \text{End}^2(E_jE_j1_\lambda) \\
\text{(53)} & \quad (T_{ij})(T_{ji})I_\lambda = t_{ij}(X_{ij}I_\lambda) + t_{ji}^\lambda(X_{ij}I_\lambda) \in \text{End}^2(E_jE_j1_\lambda).
\end{align}

Proof. We have
\[(T_{ji})(T_{ij})I_\lambda = (\phi^j_\lambda IIII) + (III\phi^i_\lambda I) \in \text{End}^2(E_j1_\mu E_j1_\lambda + 1_\mu E_j1_\lambda + E_j1_\mu).
\]
where $\mu = \lambda + \alpha_i + \alpha_j$. Since $(\phi^j_\lambda, \alpha_i)\mu = -t_{ij}$ we can write
\[
(\phi^j_\lambda IIII) = -t_{ij}(\phi IIII) + t_{ij}(II\phi I) + (II\phi^i_\lambda I) = t_{ij}(X_{ij}I_\lambda) + (II\phi^i_\lambda I)
\]
for some $\theta$ satisfying $(\theta, \alpha_i) = 1$ and some $\sigma \in \text{End}^2(1_{\lambda + \alpha_j})$ satisfying $(\sigma, \alpha_k)_{\lambda + \alpha_j} = (\theta^i_\lambda, \alpha_k)_{\mu}$ for all $k \in S_{ij}$. Similarly we get
\[
(III\phi^i_\lambda I) = -t_{ji}^\lambda(III\rho I) + t_{ji}^\lambda(II\rho I) + (II\phi^i_\lambda I) = t_{ji}^\lambda(X_{ij}I_\lambda) + (II\phi^i_\lambda I)
\]
for some $\rho$ satisfying $(\rho, \alpha_j) = 1$ and where $t_{ji}^\lambda := -(\phi^i_\lambda, \alpha_j)\lambda$ and $\tau \in \text{End}^2(1_{\lambda + \alpha_j})$ satisfies $(\tau, \alpha_k)_{\lambda + \alpha_j} = (\theta^i_\lambda, \alpha_k)_{\mu}$ for all $k \in S_{ij}$. Using relation (49), (50) and (51) it is straightforward to check that $(\sigma + \tau, \alpha_k)_{\lambda + \alpha_j} = 0$ for all $k \in S_{ij}$. This completes the proof of (52).

Relation (53) now also follows since by switching the roles of $i$ and $j$ in Lemma 10.4 we have
\[
(\theta^j_\lambda, \alpha_k)_{\lambda} = (\theta^i_\lambda, \alpha_k)_{\lambda} \quad \text{and} \quad (\phi^j_\lambda, \alpha_k)_{\lambda + \alpha_i + \alpha_j} = (\phi^i_\lambda, \alpha_k)_{\lambda + \alpha_i + \alpha_j} \quad \text{for} \quad k = i, j.
\]

\square

Proposition 10.7. The value of $t_{ji}^\lambda$ is independent of $\lambda$.

Proof. We break up the argument into three claims which together give the result.

Claim 1: $t_{ji}^\lambda = t_{ji}^{\lambda + \alpha_i}$. Consider the composition
\[E_jE_j1_\lambda \xrightarrow{(IT_{ji})(IT_{ij})} E_jE_j1_\lambda \xrightarrow{T_{ij}I_j} E_jE_j1_\lambda \xrightarrow{IT_{ij}} 1_\mu E_jE_j1_\lambda \]
where we omit the grading shifts for convenience. On the one hand, this composition equals
\begin{align}
\text{(54)} & \quad (IT_{ij})(T_{ij}I)[t_{ij}(X_{ij}I) + t_{ji}^\lambda(I X_{ij})] = [t_{ij}(X_{ij}I) + t_{ji}^\lambda(I X_{ij})](IT_{ij})(T_{ij}I) - t_{ij}(IT_{ij}) \nonumber
\end{align}
while on the other hand it is equal to
\begin{align}
\text{(55)} & \quad (T'_{ji})(IT_{ij}) = (T_{ji})(IT_{ij}) - t_{ij}(IT_{ij}) \\& = (T_{ji}I)(IT_{ij})(IT_{ij}) - t_{ij}(IT_{ij}) \\& = (T_{ji}I)(T_{ij}I)(IT_{ij})(T_{ij}I) - t_{ij}(IT_{ij}) \\& = t_{ij}(X_{ij}I)(IT_{ij})(T_{ij}I) + t_{ji}^{\lambda + \alpha_i}(I X_{ij}I)(IT_{ij})(T_{ij}I) - t_{ij}(IT_{ij}).
\end{align}

Comparing (54) and (55) we get that $t_{ji}^\lambda = t_{ji}^{\lambda + \alpha_i}$.

Claim 2: $t_{ji}^\lambda = t_{ji}^{\lambda + \alpha_j}$. Consider the composition
\[E_jE_j1_\lambda \xrightarrow{(IT_{ji})(IT_{ij})} E_jE_j1_\lambda \xrightarrow{(T_{ji}I)(IT_{ij})} E_jE_j1_\lambda \xrightarrow{IT_{ij}} E_jE_j1_\lambda \]
where we again have ignored the grading shifts. On the one hand this composition equals
\begin{align}
\text{(56)} & \quad (IT_{ij})(T_{ij}I)[t_{ij}(I X_{ij}) + t_{ji}^\lambda(I I X_{ij})] = [t_{ij}(X_{ij}I) + t_{ji}^\lambda(I X_{ij})](IT_{ij})(T_{ij}I) - t_{ij}^{\lambda + \alpha_j}(T_{ij}I) \nonumber
\end{align}
On the other hand it is equal to
\begin{align}
\text{(57)} & \quad (T_{ij}I)T'_{ji} = -m_{ji}^\lambda(n_{ji}^\lambda)^{-1}(T_{ji}I)T'_{ji} + (n_{ji}^\lambda)^{-1}(T_{ji}I) \\& = -m_{ji}^\lambda(n_{ji}^\lambda)^{-1}[t_{ij}(X_{ij}I) + t_{ji}^{\lambda + \alpha_j}(I X_{ij}I)](IT_{ij})(T_{ij}I) + (n_{ji}^\lambda)^{-1}(T_{ji}I).
\end{align}
Now, by Lemma 10.3 relation (i) we have that
\[ m_{ijj}^\lambda \cdot (\rho^j_{ij}, \alpha_j)_{\lambda + \alpha_j} = -1 \Rightarrow m_{ijj}^\lambda \cdot t_{ijj}^{\lambda + \alpha_j} = -1 \]
and likewise, from Lemma 10.3 relation (ii) we have \( n_{ijj}^\lambda \cdot t_{ijj}^\lambda = -1 \). Thus the last two terms in (56) and (57) are equal, leaving us with
\[ t_{ijj}(X, I)(IT_{ijj})(T_{ijj}) = -m_{ijj}^\lambda (n_{ijj}^\lambda)^{-1} t_{ijj}(X, II)(IT_{ijj})(T_{ijj}). \]
Thus we get \( m_{ijj}^\lambda = -n_{ijj}^\lambda \) which means that they must be equal to each other up to a multiple.

Claim 3: \( t_{ji}^\lambda = t_{ji}^{\lambda + \alpha_k} \) when \( k \neq i, j \). Consider the composition
\[
(58) \quad E_k E_j E_j 1_\lambda \xrightarrow{(IT_{ijj})(IT_{ij})} E_k E_j E_j 1_\lambda \xrightarrow{(IT_{ij})} E_k E_j E_j 1_\lambda \xrightarrow{(IT_{ij})} E_k E_j E_k 1_\lambda.
\]
On the one hand this composition equals
\[
(59) \quad (IT_{kj})(T_{ki})[t_{ij}(IX, I) + t_{ji}(IIX, I)] = [t_{ij}(X, III) + t_{ji}(IX, I)][IT_{kj}](T_{ki}).
\]
On the other hand, we know that
\[
(60) \quad c_1 t_{ij}(T_{kj})(IT_{ij})(T_{kj})(T_{ki}) = c_2 (T_{ij})(IT_{kj})(T_{ki})
\]
for some constants \( c_1, c_2 \). So the composition (58) equals
\[
(61) \quad (T_{ijj})(IT_{ijj})(T_{ijj}) = \text{End}(E_j E_j 1_\lambda).
\]
(62) \quad \text{End}(E_j E_j 1_\lambda).
(63) \quad \text{End}(E_j E_j 1_\lambda).
(64) \quad \text{End}(E_j E_j 1_\lambda).

Finally, if \( \langle j, k \rangle = 0 \) then \( T_{kj} T_{jk} 1_\lambda \in \text{End}(E_j E_k 1_\lambda) \) is some nonzero multiple of the identity. So one can rescale each \( T_{jk} 1_\lambda \) so that
\[
(65) \quad (T_{kj})(T_{jk}) = t_{jk}(II) \in \text{End}(E_j E_k 1_\lambda).
\]
It is easy to see that this implies \( T_{jk}(T_{kj}) = t_{jk}(II) \in \text{End}(E_k E_j) \) meaning that \( t_{jk} = t_{kj} \).

Remark 10.8. Equalities (61), (62) and (65) hold even without modding out by transient maps.

11. Step #8 – The \( T_{ijk} \) relation

In this section we will show how to rescale maps so that
\[
T_{ijk} I_X = T_{ijk}' I_\lambda \in \text{Hom}(E_i E_j E_k 1_\lambda, E_k E_j E_i 1_\lambda(-\ell_{ijk})).
\]
whenever \( i, j, k \in I \) are all distinct. Assuming the Hom space above is nonzero we know by Lemma A.12 that it is one-dimensional. By Proposition 6.2, we also have that \( T_{ijk} I_\lambda \) and \( T_{ijk}' I_\lambda \) are nonzero which means that they must be equal to each other up to a multiple.
It remains to find a consistent way to rescale $T$'s so that $T_{ijk} = T'_{ijk}$. To do this we fix a total ordering on $< \, \text{on} \, I$. By decreasing induction (with respect to $< \, \text{on} \, I$) we will rescale $T_{ij}$'s so that $T_{ijk} = T'_{ijk}$ if $i < j < k$. More precisely, we can already rescaled all $T_{ik}$ and $T_{jk}$ where $i, j < k$ as well as all $T_{ki,k_2}$ with $i < k_1 < k_2$ so that $T_{ik_1,k_2} = T'_{ik_1,k_2}$ and $T_{kj_1,k_2} = T'_{kj_1,k_2}$. Then, given $T_{ij}I_\alpha$ one can uniquely rescale $T_{ij}I_{\lambda + \alpha}$ so that $T_{ij}I_\alpha = T'_{ij}I_\lambda$.

**Proposition 11.1.** This rescaling algorithm for $T_{ij}I_\lambda$ is well defined.

*Proof.* The proof is analogous to that of Proposition 6.1 with $T_{ii}$ replaced by $T_{ij}$. More precisely, we consider a sequence of rescalings

$$ T_{ij}I_\lambda \rightsquigarrow T_{ij}I_{\lambda + c_1\alpha_k} \rightsquigarrow T_{ij}I_{\lambda + c_1\alpha_k + c_2\alpha_k} \rightsquigarrow \cdots \rightsquigarrow T_{ij}I_{\lambda + \sum c_\ell\alpha_k} = T_{ij}I_\lambda $$

where $c_\ell = \pm 1$, $i, j < k_\ell$ for all $\ell$ and $\sum \ell c_\ell\alpha_k = 0$ which we encode as $c_k = (c_1k_1, \ldots, c_\alpha k_m)$. To show that such a sequence does not rescale $T_{ij}I_\alpha$ we consider the two operators $S_a$ and $D_a$ acting on such sequences as in Proposition 6.1. Then the way $c_k$ and $S_a$, $c_k$ ends up rescaling $T_{ij}I_\lambda$ is the same due to Corollary 11.4 while the way $c_k$ and $D_a$, $c_k$ rescale $T_{ij}I_\lambda$ is also the same (essentially by definition). The result now follows since any sequence $c_k$ can be transformed into the trivial sequence by repeatedly applying these moves.

Having rescaled $T_{ij}I_\lambda$ as above we rescale $T_{ij}I_\lambda$ by the inverse. This has the effect of preserving all the other relations such as those in (63) and (64). So we maintain all our prior results while adding the following relation:

$$ T_{ijk}I_\lambda = T'_{ijk}I_\lambda : E_iE_jE_k1_\lambda \rightarrow E_kE_jE_i1_\lambda(-\ell_{ij,k}) \quad \text{for all} \quad i, j, k \in I \text{ with } i < j < k. $$

**Lemma 11.2.** Suppose $i, j, a, b \in I$ with $i < j < a, b$ and that $1_{\lambda + c_i\alpha + c_j\alpha + a, a + b} \neq 0$ for $c_i, c_j, c_a, c_b \in \{0, 1\}$. If $T_{ab} = T'_{ab}$ and $T_{jb} = T'_{jb}$ then the two sequences

$$ T_{ij}I_\lambda \rightsquigarrow T_{ij}I_{\lambda + \alpha} \rightsquigarrow T_{ij}I_{\lambda + \alpha + \alpha} \quad \text{and} \quad T_{ij}I_\lambda \rightsquigarrow T_{ij}I_{\lambda + \alpha} \rightsquigarrow T_{ij}I_{\lambda + \alpha + \alpha} $$

lead to the same rescaling of $T_{ij}I_{\lambda + \alpha + \alpha}$.

*Proof.* Consider the composition

$$ (IT_{ij})(IT_{ia}I)(IT_{ja})(IT_{ib}I)(IT_{ab}I) : E_iE_jE_aE_b1_\lambda \rightarrow E_iE_jE_aE_b1_\lambda. $$

On the one hand one can write this as

$$ (IT_{ij}')(IT_{ib}I)(IT_{jb}I)(IT_{ab}) = (IT_{ja})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

$$ = (IT_{ja})(IT_{ia})(IT_{id}I)(IT_{jd}I)(IT_{ab}) $$

where we rescaled $T_{ij}I_\lambda \rightsquigarrow T_{ij}I_{\lambda + \alpha} \rightsquigarrow T_{ij}I_{\lambda + \alpha + \alpha}$. In order to get the first and fourth equalities.

On the other hand, using that $T_{ib} = T'_{ib}$ and $T_{jb} = T'_{jb}$ one can rewrite (69) as

$$ (IT_{ij}')(IT_{ib}I)(IT_{jb}I)(IT_{ia}I)(IT_{ab}) $$

where we use that we know $T_{ib} = T'_{ib}$ and $T_{jb} = T'_{jb}$ to slide the $T_{ab}$ from the far right to the far left. Then the same sequence of equalities as above shows that this is equal to

$$ (IT_{ij}')(IT_{ib}I)(IT_{jb}I)(IT_{ia}I)(IT_{ab}) $$

where this time we had to rescale $T_{ij}I_\lambda \rightsquigarrow T_{ij}I_{\lambda + \alpha} \rightsquigarrow T_{ij}I_{\lambda + \alpha + \alpha}$. Finally, sliding back the $(IT_{ab}I)$ we find that (71) is equal to (70). Since by Lemma 11.3 the composition in (69) is nonzero it follows that the two sequences of rescalings must be the same.  

\[\square\]
Lemma 11.3. Suppose $i, j, a, b \in I$ are distinct and let $\ell_{iab} := \langle i, j + a + b \rangle + \langle j, a + b \rangle + \langle a, b \rangle$. Then (72)

\[(II T_{ij})(II T_{ja})(II T_{jb})(II T_{ab}) : E_i E_j E_a E_b 1_\lambda \to E_b E_a E_j E_i 1_{\lambda - \ell_{iab}}\]

is nonzero if and only if $1_{\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_a \alpha_a + \epsilon_b \alpha_b} \neq 0$ for $\epsilon_i, \epsilon_j, \epsilon_a, \epsilon_b \in \{0, 1\}$.

Proof. This result is the analogue of Lemma 6.4 which dealt with the case $i = j$. The proof is also very similar. One direction is immediate since one can check that the weights $\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_a \alpha_a + \epsilon_b \alpha_b$ all appear somewhere in the composition (72) (after possibly having to use relations such as $T_{ij} \sim T_{ij}'$).

For the converse one can argue as in Lemma 6.4 that (72) is nonzero if either (73)

\[T_{ab} T_{ij} : E_a E_b 1_\lambda F_i F_j \to E_b E_a 1_\lambda F_i F_j \quad \text{or} \quad T_{ij} T_{ab} : F_i F_j 1_\mu E_a E_b \to F_j F_i 1_\mu E_a E_b\]

are nonzero, where $\mu := \lambda + \alpha_i + \alpha_j + \alpha_a + \alpha_b$ and we abuse notation by writing $T_{ij} \in \text{Hom}^{-(i, j)}(F_i F_j, F_j F_i)$ for the analogue of $T_{ij} \in \text{Hom}^{-(i, j)}(E_i E_j, E_j E_i)$. These two maps are the analogues of (29) and (30).

Now, compose the left map in (73) with $F_a$ on the left. If $\langle a, b \rangle = -1$ and $\lambda_a \leq -1$ then by Lemma 5.4 the $E_b$-rank of $T_{ab} T_{ij} : E_a E_b 1_\lambda \to F_a E_b 1_\lambda (1)$ is $-\lambda_a > 0$ so we reduce to showing $II T_{ij} : E_b 1_\lambda F_i F_j \to E_i 1_\lambda F_i F_j$. Composing with invertible maps $v_{ij}$ and $v_{bi}$ leaves us with the composition

\[E_b F_i F_j \xrightarrow{T_{ij}} E_b F_j F_i \xrightarrow{v_{ij}} F_j E_b F_i \xrightarrow{v_{bi}} F_j F_i E_b .\]

This is adjoint to $T_{ij} E_b$ and hence nonzero by Proposition 6.2. Thus $\lambda_a \leq -1 \Rightarrow (72)$ is nonzero.

A similar argument except composing the right map in (73) with $F_a$ on the right show that $\lambda_a \geq -1 - \langle a, i + j + b \rangle \Rightarrow (72)$ is nonzero.

Hence we are finished unless

\[\langle a, i + j + b \rangle = -2 \quad \text{and} \quad \lambda_a = 0 \quad \text{or} \quad \langle a, i + j + b \rangle = -3 \quad \text{and} \quad \lambda_a = 0 \text{ or } 1 .\]

Similar arguments with $b, i$ and $j$ give us that we are also finished unless

\[\langle b, i + j + a \rangle = -2 \quad \text{and} \quad \lambda_b = 0 \quad \text{or} \quad \langle b, i + j + a \rangle = -3 \quad \text{and} \quad \lambda_b = 0 \text{ or } 1 \quad \text{or} \quad \langle b, i + j + a \rangle = -4 \quad \text{and} \quad \lambda_b = 0 \text{ or } 1 .\]

We now argue case by case that these four conditions cannot all hold. There are three cases depending on whether the Dynkin diagram containing $i, j, a, b$ is a square, contains 5 edges or contains 6 edges (i.e. a complete graph).

In the first case it follows by condition (vii) that $\langle \lambda, i + j + a + b \rangle > 0$ since $1_\lambda \neq 0$. On the other hand, $\lambda_i = \lambda_j = \lambda_a = \lambda_b = 0$ which contradicts this.

In the second case suppose (without loss of generality) that $i, j, a$ generates one of the two triangles. Then as above we must have $\langle \lambda + b, i + j + a \rangle > 0 \Rightarrow \lambda_i + \lambda_j + \lambda_a \geq 3$ since vertex $b$ is connected to exactly two of $i, j, a$. But this is impossible since at least one of $\lambda_i, \lambda_j, \lambda_a$ must be zero (and the other two either zero or one).

In the third case $i, j, b$ form a triangle which means $\langle \lambda + a, i + j + b \rangle \geq 1 \Rightarrow \lambda_i + \lambda_j + \lambda_b \geq 4$. This is not possible since $\lambda_i, \lambda_j, \lambda_b \leq 1$.

\[ \square \]

Corollary 11.4. Suppose $i, j, a, b \in I$ with $i \not< j \not< a, b$ and that $T_{ij} I_\lambda$ and $T_{ij} I_{\lambda + \alpha_a - \alpha_b}$ are nonzero. If $T_{iab} = T_{iab}'$ and $T_{jab} = T_{jab}'$ then the two sequences

\[ T_{ij} I_\lambda \sim T_{ij} I_{\lambda + \alpha_a - \alpha_b} \quad \text{and} \quad T_{ij} I_\lambda \sim T_{ij} I_{\lambda - \alpha_a + \alpha_b} \sim T_{ij} I_{\lambda + \alpha_a - \alpha_b} \]

lead to the same rescaling of $T_{ij} I_{\lambda + \alpha_a - \alpha_b}$.

Proof. Since $T_{ij} I_\lambda \neq 0$ and $T_{ij} I_{\lambda + \alpha_a - \alpha_b} \neq 0$ we get that $1_{\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j} \neq 0$ and $1_{\lambda + \alpha_a - \alpha_b + \epsilon_i \alpha_i + \epsilon_j \alpha_j} \neq 0$, where $\epsilon_i, \epsilon_j \in \{0, 1\}$. By condition (viii) this implies that all $1_{\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_a \alpha_a + \epsilon_b \alpha_b}$ are nonzero and the result follows from Lemma 11.2. \[ \square \]
Finally, given (67) it remains to check that $T_{ijk}I_\lambda = T'_{ijk}I_\lambda$ for any distinct $i, j, k \in I$ (not just when $i < j < k$). This follows from the following Lemma.

Lemma 11.5. For distinct $a, b, c \in I$, if $T_{abc}I_\lambda = T'_{abc}I_\lambda$ for all $\lambda$ then $T_{a'b'c'}I_\lambda = T'_{a'b'c'}I_\lambda$ for any permutation $a', b', c'$ of $a, b, c$.

Proof. We prove the case when $(a', b', c') = (b, a, c)$ (the general case follows similarly). There are various cases to consider depending on whether $a, b, c$ are connected in the Dynkin diagram. We will deal with the most difficult case when $(a, b) = (a, c) = (b, c) = -1$ (the other cases are proved in the same way but involve simpler computations).

By Proposition 6.2 we know that $T_{bac}I_\lambda = sT_{bac}I_\lambda$ for some $s \in \mathbb{k}^\times$. We need to show that in fact $s = 1$. To do this consider the composition

$$(74) \quad E_aE_bE_c1_\lambda \xrightarrow{T_{abc}I} E_bE_aE_c1_\lambda \xrightarrow{T_{bac}I} E_aE_bE_c1_\lambda \xrightarrow{IT_{abc}} E_aE_bE_c1_\lambda \xrightarrow{T_{bac}I} E_aE_bE_c1_\lambda.$$

On the one hand, this is equal to

$$(T_{bac})(T_{abc}) = s(T_{abc})(T_{bac}) = s(ITT_{ba})(T_{abc})(ITT_{ba}) = s(ITT_{ba})(T_{abc}).$$

Comparing (75) with (76) we get $s = 1$ as long as

$$[t_{ab}(IX_a\lambda) + t_{ba}(II\lambdaX_b)](T_{abc})(IT_{bc}) = E_aE_bE_c1_\lambda,$$

is nonzero. To show this note that this map is adjoint to

$$(77) \quad t_{ab}(X_a \lambda II) + t_{ba}(IX_b\lambda) : E_aE_b1_\lambda F_c \to E_aE_b1_\lambda F_c.$$

Now if $\lambda_c \leq -1$ then you can compose (77) on the right with $E_c$. Simplifying one finds that one of the summands is

$$(78) \quad t_{ab}(X_a \lambda II) + t_{ba}(IX_b\lambda) : E_aE_b1_\lambda \to E_aE_b1_\lambda.$$

Now, if $\lambda_b \geq 0$ then one can compose (78) on the right with $F_b$ and consider the composition

$$(E_a1_{\lambda + \alpha_b} \xrightarrow{T_{ab}b} E_aE_bF_b1_{\lambda + \alpha_b} \xrightarrow{(78)} E_aE_bF_b1_{\lambda + \alpha_b} \xrightarrow{T_{adj}} E_a1_{\lambda + \alpha_b},)$$

which, by Corollary 8.2, is equal to (a nonzero multiple of) the identity map on $E_a1_{\lambda + \alpha_b}$. Finally, $E_a1_{\lambda + \alpha_b} \neq 0$ by Lemma A.7 since $1_{\lambda + \epsilon_a\alpha_a + \epsilon_b\alpha_b + \epsilon_c\alpha_c} \neq 0$ for $\epsilon_a, \epsilon_b, \epsilon_c \in \{0, 1\}$. Thus (78) is nonzero in this case. The case $\lambda_b \leq -2$ is similar by composing with $F_b$ on the left instead. Thus we are left with the case $\lambda_b = -1$.

A similar argument with $a$ instead of $b$ leaves us with the case $\lambda_a = 0$. To take care of this possibility that $\lambda_a = 0$ and $\lambda_b = -1$ we note that we could have switched the roles of $a, b$ in the original equation (74). Then the same argument above would take care of this possibility too.

This completes the case when $\lambda_c \leq -1$. One has a similar argument when $\lambda_c \geq 1$ by composing (77) with $E_c$ on the left to reduce to showing that $t_{ab}(X_a \lambda II) + t_{ba}(IX_b\lambda) \in \text{End}(E_aE_b1_{\lambda + \alpha_c})$ is nonzero. This is done in the same way as above.

Finally, if $\lambda_c = 0$ one cannot simplify as above. On the other hand, one can still argue as above to reduce to the case when $\lambda_a = 0, 1$ and $\lambda_b = 0, 1$. Since $\lambda_a + \lambda_b + \lambda_c \geq 1$ we must have that $\lambda_a = 1$ and $\lambda_b = 0$. This final possibility that $\lambda_a = 1, \lambda_b = 0$ and $\lambda_c = 0$ can be taken care of again by reversing the roles of $a$ and $b$ in (74). This concludes the proof that the map in (77) is nonzero. \qed
So we arrive at the following relation, which holds without having to mod out by transient maps.

\( T_{ijk} = T'_{ijk} \in \text{Hom}(E_iE_jE_k, E_kE_jE_i\langle -\ell_{ijk} \rangle) \) for all distinct \( i, j, k \in I \).

12. Step #9 – Transients

At this point several of the relations above hold only modulo transient maps, namely relations (39), (40), (41), (63) and (64). In many cases this may be good enough since transient maps are usually negligible. For example, in applications to knot homology (section 14.3), the homology of any link involves a computation whose output is an endomorphism of the highest weight space where there are no transient maps. Even better, in our application to vertex operators (section 14.1), there are no transient maps at all (i.e. they are all zero).

Nevertheless, it would be nice to know whether all relations hold on the nose. In this section we prove this is the case when \( g = sl_n \). We expect this also holds when the graph \( \Gamma \) associated to \( g \) is a tree and perhaps more generally.

To understand the general argument it is worth first considering the case \( g = sl_2 \). In this case one needs to prove the affine nilHecke relations or, equivalently, that

\[
\text{End}(E_1\lambda E) \ni T[I\theta I] + (I\theta IT) = ([IIT][III\phi] + (\theta I\lambda II)) + id \in \text{End}(E_1\lambda + \alpha EE_1\lambda - \alpha).
\]

We know this relation holds modulo transient maps. On the other hand, a transient map belonging to \( \text{End}^2(1_{\lambda + \alpha} EE_1\lambda - \alpha) \) is of the form \( (III\phi) \) if \( \lambda \leq 0 \) and \( (\phi III) \) if \( \lambda \geq 0 \). This means that, if \( \lambda \geq 0 \) (resp. \( \lambda \leq 0 \)) then one can redefine \( \theta \in \text{End}^2(1_{\lambda + \alpha}) \) (resp. \( \theta \in \text{End}^2(1_{\lambda - \alpha}) \)) by adding some transient map so that (80) holds. Thus, starting in the middle (i.e. at weight space \( \lambda = 0 \) or \( \lambda = -1 \)) we can go out in both directions and redefine the \( \theta \)'s so that (80) holds on the nose.

When \( g = sl_n \) (and \( n > 2 \)) things are more complicated because there are further relations to check. Moreover, it is not so clear what is the analogue of the middle weight \( \lambda = 0 \) or \( \lambda = -1 \). However, the general idea is the same: we will start at a “middle weight” (see section 12.1) and work our way outwards to redefine all \( \theta \)'s by adding to them appropriate transient maps.

More precisely, we will proceed as follows (from hereon \( g = sl_n \)). Fix an orientation of \( \Gamma \) so that from each vertex there is at most one arrow leaving. Also, fix \( \theta_i \in Y_\lambda \) so that \( \langle \theta_i, \alpha_j \rangle = \delta_{ij} \).

(i) Fix a middle weight \( \mu \) (see section 12.1). Show that having fixed \( \theta_i \in \text{End}^2(1_\mu) \) and \( \theta_i \in \text{End}^2(1_{\mu - \alpha_i}) \) one can then redefine the remaining \( \theta_i \)'s so that

\[
\text{II} \theta_i = \theta_i \text{II} \in \text{End}^2(1_{\lambda + \alpha_j} E_j E_\lambda) \quad \text{for any } i \neq j
\]

(ii) For each edge \( i \to j \) in \( \Gamma \), redefine \( \theta_i \in \text{End}^2(1_\mu) \) and \( \theta_i \in \text{End}^2(1_{\mu - \alpha_i}) \) so that

\[
(T_{ij})(T_{ij}) = t_{ij}(X_i I) + t_{ji}(I X_j) \in \text{End}^2(E_i E_\lambda E_j)
\]

holds when \( \lambda = \mu - \alpha_i \). Then use step 1 to redefine all other \( \theta_i \)'s and show that this implies (83) for all \( \lambda \).

(iii) Show that

\[
(T_{ij})(T_{ji}) = t_{ij}(I X_i) + t_{ji}(X_j I) \in \text{End}^2(E_j E_\lambda E_i)
\]

follows as a consequence of (81), (82) and (83).

Remark 12.1. If, following section 7, we define \( X_i \in \text{End}^2(1_{\lambda + \alpha_j} E_\lambda) \) as \( -\langle \theta_i, I I \rangle \) then relations (81) and (82) imply (39), (40) and (41) while (83) and (84) are the same as (63) and (64).
12.1. The middle weight. For each $i \in I$ we define the following valid slides

$$\lambda \rightsquigarrow \lambda + \alpha_i \text{ if } \lambda_i \geq -1 \quad \text{and} \quad \lambda \rightsquigarrow \lambda - \alpha_i \text{ if } \lambda_i \leq 1.$$ 

We say $\lambda \in X$ is a middle weight of $\mathcal{K}$ if you can reach any other nonzero weight space of $X$ from $\lambda$ by using a sequence of valid slides (such a sequence is called a path).

Proposition 12.2. If $g = \mathfrak{sl}_n$ then $\mathcal{K}$ has a middle weight.

Proof. For $\mathfrak{sl}_n$, the cosets $X/Y$ are indexed by $\omega \in \{0, \Lambda_1, \ldots, \Lambda_{n-1}\}$. We will show that these are all middle weights. To simplify notation suppose $\omega = 0$ (the other cases are exactly the same). Consider $\lambda = \sum_{j=1}^{n-1} a_j \alpha_j$ for some $a_j \in \mathbb{Z}$ with $1_{\lambda} \neq 0$.

Claim: if $\lambda \neq 0$ then there exists an $i$ so that either $a_i, \lambda_i \geq 1$ or $a_i, \lambda_i \leq -1$.

If $a_1 = 0$ then this claim is reduced to $\mathfrak{sl}_{n-1}$ and we proceed by induction. If $a_1 \geq 1$ then $\lambda_1 \leq 0$ or otherwise we are done. Thus

$$\lambda_1 = 2a_1 - a_2 \leq 0 \Rightarrow a_2 \geq 2a_1 \geq 2.$$

But now $a_2 \geq 2$ means $\lambda_2 \leq 0$ or otherwise we are done. Then

$$\lambda_2 = 2a_2 - a_1 - a_3 \leq 0 \Rightarrow a_3 \geq 2a_2 - a_1 \geq a_2 + a_1 \geq 3$$

meaning $\lambda_3 \leq 0$. Repeating this arguments gives the claim. The case $a_1 \leq -1$ is proved similarly.

The result now follows from the claim. More precisely, if $\lambda = 0$ we are done. Otherwise suppose $a_i, \lambda_i \geq 1$ (the case $a_i, \lambda_i \leq -1$ is similar). Then we have a slide $\lambda - \alpha_i \rightsquigarrow \lambda$. Note that $1_{\lambda - \alpha_i} \neq 0$ since $e_i 1_{\lambda - \alpha_i} F_i$ contains at least one copy of $1_{\lambda} \neq 0$ (using that $\lambda_i \geq 1$). Finally, by induction on $\sum_j |a_j|$, there exists a path $0 \rightsquigarrow \cdots \rightsquigarrow \lambda - \alpha_i$. Together with the slide $\lambda - \alpha_i \rightsquigarrow \lambda$ this gives a path $0 \rightsquigarrow \cdots \rightsquigarrow \lambda$.

From now on we fix a middle weight $\mu$. The proof above gives us a canonical path from $\mu$ to any weight space $\lambda$. More precisely, we start at $\lambda$ and word backwards. Using the claim in the Lemma we choose the smallest $i$ so that $a_i, \lambda_i \geq 1$ or $a_i, \lambda_i \leq -1$. Take the last slide in the path to be $\lambda - \alpha_i \rightsquigarrow \lambda$ or $\lambda + \alpha_i \rightsquigarrow \lambda$ respectively, and then repeat. We refer to this sequence of slides as the canonical path from $\mu$ to $\lambda$.

Remark 12.3. For convenience, if $\mu = \Lambda_i$ for some $i \in I$ then we choose the orientation on $\Gamma$ so that all edges incident on $i$ are oriented away from $i$.

12.2. Slide equivalences. Consider a path $\mu \rightsquigarrow \mu + c_1 \alpha_k \rightsquigarrow \cdots \rightsquigarrow \mu + \sum c_i \alpha_k = \lambda$ where $c_i = \pm 1$. We will encode it as $(ck) = (c_1 k_1, \ldots, c_m k_m)$ and denote by its length by $|ck|$ (the length equals $m$). We denote the canonical path from $\mu$ to $\lambda$ by $(ck)$. Note that the canonical path has the smallest possible length among all paths, namely $\sum_j |a_j|$ if $\lambda = \mu + \sum_j a_j \alpha_j$.

We have two operations we can perform on a path $(ck)$, namely

$$S_a \cdot (c_1 k_1, \ldots, c_a k_a, c_{a+1} k_{a+1}, \ldots, c_m k_m) = (c_1 k_1, \ldots, c_{a+1} k_{a+1}, c_a k_a, \ldots, c_m k_m)$$

$$D_a \cdot (c_1 k_1, \ldots, c_a k_a, c_{a+1} k_{a+1}, \ldots, c_m k_m) = (c_1 k_1, \ldots, c_{a-1} k_{a-1}, c_a k_{a+2}, \ldots, c_m k_m)$$

where $S_a$ is defined as long as the right hand side is a valid path and $D_a$ is defined when $c_a k_a = -c_{a+1} k_{a+1}$. If two paths $(ck)$ and $(ck')$ are equivalent via a sequence of such operations we write $(ck) \sim (ck')$ and say they are slide equivalent.

Lemma 12.4. Suppose $\lambda = \mu + \sum_j a_j \lambda_j$. If $a_i \leq -1$ then $(ck \lambda, i) \sim (ck')$ where $|ck| = |ck'| - 1$.

Similarly, if $a_i \geq 1$ then $(ck \lambda, -i) \sim (ck')$ where $|ck| = |ck'| - 1$.

Proof. We prove the case $a_i \leq -1$ (the case $a_i \geq 1$ is the same). Suppose $(ck \lambda, i)$ looks like

$$\mu \rightsquigarrow \cdots \rightsquigarrow \rho + \alpha_i \rightsquigarrow \rho \rightsquigarrow \cdots \rightsquigarrow \lambda - c_m \alpha_k \rightsquigarrow \lambda \rightsquigarrow \lambda + \alpha_i$$

where the path from $\rho$ to $\lambda$ contains no $\alpha_i$. We will prove the result by induction on the length $\ell$ of this path. Notice that if $\ell = 1$ then we have $\ldots \sim \rho \sim \rho - \alpha_i = \lambda \sim \lambda + \alpha_i$ and the result follows by applying $D_m$.

If $\ell > 1$ then $k_m \neq i$. If $(i, k_m) = 0$ then the path $\lambda - c_m \alpha_{k_m} \sim \lambda - c_m \alpha_{k_m} + \alpha_i \sim \lambda + \alpha_i$ are valid slides which means $(ck, i) = (\ldots, c_m k_m, i) \sim (\ldots, i, c_m k_m)$ and the result follows by induction. So we are left with the cases $k_m = i - 1$ or $k_m = i + 1$.

If $k_m = i + 1$ then since $ck$ is canonical it means $c_m = -1$. Now, if $\lambda_i \geq 0$ then

$$\lambda + \alpha_{i+1} \sim \lambda + \alpha_{i+1} + \alpha_i \sim \lambda + \alpha_i$$

are valid slides and hence $(ck, i) = (\ldots, k_m, i) \sim (\ldots, i, -k_m)$ and we are done by induction. Otherwise, since $\lambda \sim \lambda + \alpha_i$ is a valid slide, we must have $\lambda_i \geq 1$ which leaves us with $\lambda_i = -1$. But then $\lambda + \alpha_i \sim \lambda$ is a valid slide which means $ck = (\ldots, i)$ (contradiction).

If $k_m = i - 1$ then all the $\alpha_a$ which appear between $\rho$ and $\lambda$ satisfy $\alpha < i$. If $c_m = 1$ then

$$\lambda = \rho + c \alpha_{i-1} + \nu$$

where $c \geq 1$ and $\langle \nu, i \rangle = 0$. This means $\lambda_i = \rho_i - c \leq \rho_i - 1 \leq -2$ since $\rho_i \leq -1$. But this is a contradiction since $\lambda \sim \lambda + \alpha_i$ is a valid slide. On the other hand, if $c_m = -1$ then $(ck, i) = (\ldots, -i, c_j k_j, \ldots, c_m k_m, i)$ where for each $a = j, \ldots, m$ either $k_a \leq i - 2$ or $k_a = i - 1$ and $c_a = -1$. In either case one can check that $\tau \sim \tau + c_a \alpha_{k_a} \sim \tau + c_a \alpha_{k_a} - \alpha_i$ are valid slides where $\tau = \rho + \alpha_i + c_j \alpha_k + \ldots + c_{a-1} \alpha_{k_{a-1}}$. This is because $\langle \tau + c_a \alpha_{k_a}, i \rangle \leq \rho_i + 2 \leq 1$. Thus we can repeatedly slide the $-i$ to the right to obtain

$$(ck, i) \sim (\ldots, c_{j-2} k_{j-2}, c_j k_j, \ldots, c_m k_m, -i, i)$$

and we are done once again. This concludes the proof. \qed

**Proposition 12.5.** Any two minimal length paths between $\mu$ and $\lambda$ are slide equivalent.

**Proof.** Consider an arbitrary minimal length path from $\mu$ to $\lambda$. Let us write it as $(ck, cj)$. It suffices to show that $(ck, cj) \sim (ck)$. The proof is by induction on $|ck|$. Notice that by induction we can assume $(ck) \sim (ck)$. If $c = -1$ and $(ck) = (ck, i)$ for some $i$ (the other cases when $c = 1$ or $(ck) = (ck) - \alpha_i$ are the same). Since $\lambda - \alpha_i \sim \lambda$ is a valid slide we have $\lambda_i \geq 1$. If $\lambda_i = 1$ then by induction we have $(ck, i) \sim (ck, cj, -i)$ which means that

$$(ck) \sim (ck, -\alpha_i, i) \sim (ck, cj, -i, i) \sim (ck, cj)$$

and we are done.

If $\lambda_i \geq 2$ then $\lambda - \alpha_i + \alpha_j \sim \lambda + \alpha_j \sim \lambda$ are valid slides (the first slide can only fail to be valid if $\lambda_i = 1$ and $\langle i, j \rangle = -1$). Hence we get

$$(ck) \sim (ck, -\alpha_i, i) \sim (ck, -\alpha_i + \alpha_j, -j, i) \sim (ck, -\alpha_i + \alpha_j, i, -j) \sim (ck, -j)$$

where the second and last equivalences follows by induction. This completes the proof. \qed

12.3. **Step (i).**

**Proposition 12.6.** Having fixed $\theta_i \in \End^2(1_{\mu})$ and $\theta_i \in \End^2(1_{\mu - \alpha_i})$ one can redefine the remaining $\theta_i \in \End^2(1_{\lambda})$ so that relations (81) and (82) hold.

**Proof.** We prove the case $\mu_i = 0$ (the case $\mu_i = 1$ is the same). Let us fix $\theta_i I_{\mu}$ and $\theta_i I_{\mu + \alpha_i}$. We redefine the remaining $\theta_i I_\lambda$ in three steps.

Step A. First we redefine each $\theta_i I_{\lambda}$ with $\langle \lambda, \Lambda_i \rangle = 0$. To do this we use Part I of Proposition 12.7 to redefine each $\theta_i$ along the canonical path $\mu \sim \lambda$ (the condition $\langle \lambda, \Lambda_i \rangle = 0$ means that this path does not involve any slides along $\alpha_i$).
Step B. Next we redefine each \( \theta_i I_\lambda \) with \( \langle \lambda, \Lambda_i \rangle = -1 \). In this case, the proof of Proposition 12.2 also gives a canonical path from \( \mu - \alpha_i \) to \( \lambda \) (without any slides along \( \alpha_i \)). Once again we redefine each \( \theta_i \), using Part I of Proposition 12.7, along this canonical path \( \mu - \alpha_i \bowtie \lambda \).

Step C. Finally, if \( \langle \lambda, \Lambda_i \rangle > 0 \) (resp. \( \langle \lambda, \Lambda_i \rangle < -1 \)) then we redefine \( \theta_i \) along the canonical path from \( \mu \) (resp. \( \mu - \alpha_i \)) to \( \lambda \) using Parts I and II of Proposition 12.7. Notice that if, along the path, we wish to redefine \( \theta_i I_{\rho+\alpha_i} \) (resp. \( \theta_i I_{\rho-\alpha_i} \)) then we can use Part II of 12.7 because now \( \theta_i I_{\rho} \) and \( \theta_i I_{\rho-\alpha_i} \) (resp. \( \theta_i I_{\rho+\alpha_i} \)) have already been fixed.

It remains to show that relations \((81)\) and \((82)\) hold. To see \((81)\) suppose \( \lambda = \mu + \sum_j a_j \alpha_j \) with \( a_i \geq 0 \) (the case \( a_i \leq -1 \) is the same). Suppose \( a_j \geq 0 \) (the case \( a_j \leq 0 \) is the same). If \( \lambda_j \geq -1 \) then \((ck_\lambda, j)\) is a path of minimal length and, by Proposition 12.5, \((ck_\lambda, j) \sim (ck_\lambda, j)\). By applying Lemma 12.8 repeatedly these two paths give the same rescaling of \( \theta_i I_{\lambda+\alpha_j} \). But redefining via the path \((ck_\lambda, j)\) implies, by construction, that \((81)\) holds.

Similarly, if \( \lambda_j \leq -1 \) then by Lemma 12.4 \((ck_\lambda) \sim (ck_\lambda, -j)\). But redefining via the path \((ck_\lambda, -j)\) again, by construction, that \((81)\) holds.

To see \((82)\) suppose again \( a_i \geq 0 \) (the case \( a_i \leq -1 \) is the same). The paths \((ck_{\lambda - \alpha_i})\) and \((ck_{\lambda, -i})\) are minimal so if \( \lambda_i \leq 0 \) then \((ck_{\lambda - \alpha_i}) \sim (ck_{\lambda, -i})\) by Proposition 12.5. By applying Lemma 12.8 repeatedly these paths give the same rescaling of \( \theta_i I_{\lambda - \alpha_i} \). But rescaling via the path \((ck_{\lambda, -i})\) implies, by construction, that \((82)\) holds.

Similarly, by Lemma 12.4 we have \((ck_{\lambda + \alpha_j}) \sim (ck_{\lambda}, i)\) so if \( \lambda_i \geq 0 \) then rescaling \( \theta_i I_{\lambda+\alpha_i} \) via the path \((ck_{\lambda}, i)\) implies, by construction again, that \((82)\) holds.

Proposition 12.7. There are two parts to redefining \( \theta_i \in \text{End}^2(1_\lambda) \).

Part I: For \( i \neq j \in I \), if \( \lambda_j \leq -1 \) (resp. \( \lambda_j \geq 1 \)) one can redefine \( \theta_i \in \text{End}^2(1_\lambda) \) by adding some transient map so that \((81)\) holds inside \( \text{End}^2(E_j 1_\lambda) \) (resp. inside \( \text{End}^2(1_\lambda E_j) \)).

Part II: For \( i = j \in I \), if \( \lambda_i \leq -2 \) (resp. \( \lambda_i \geq 2 \)) one can redefine \( \theta_i \in \text{End}^2(1_\lambda) \) by adding some transient map so that \((82)\) holds inside \( \text{End}(E_i E_j 1_\lambda) \) (resp. inside \( \text{End}(1_\lambda E_i E_j) \)).

Proof. To prove the first claim suppose \( \lambda_j \leq -1 \) (the case \( \lambda_j \geq 1 \) is the same). We know that relation \((81)\) holds modulo transient maps. By Lemma 3.6 it follows that

\[
(II\theta_i) - (\theta_i II) = (II\gamma) \in \text{End}^2(1_{\lambda+\alpha_i} E_i 1_\lambda)
\]

for some transient \( \gamma \in \text{End}^2(1_\lambda) \). The result now follows if we redefine \( \theta_i \in \text{End}^2(1_\lambda) \) as \( \theta_i - \gamma \).

To prove the second claim suppose \( \lambda_i \leq -2 \) (the case \( \lambda_i \geq 2 \) is the same). From the proof of Proposition 8.1 We know that relation \((82)\) holds modulo maps of the form \((T_{ij} II) (II\gamma) \in \text{End}^2(E_i E_j 1_\lambda) \) where \( \gamma \in \text{End}^2(1_\lambda) \) is transient. Thus, if we redefine \( \theta_i \in \text{End}^2(1_\lambda) \) as \( \theta_i - \gamma \) then \((82)\) will hold on the nose.

Lemma 12.8. Using Proposition 12.7 there are two ways of redefining \( \theta_i \in \text{End}^2(1_{\lambda\pm\alpha_i\pm\alpha_b}) \) given \( \theta_i \in \text{End}^2(1_\lambda) \), namely

\[
\theta_i I_{\lambda} \sim \theta_i I_{\lambda\pm\alpha_a} \sim \theta_i I_{\lambda\pm\alpha_a\pm\alpha_b} \quad \text{or} \quad \theta_i I_{\lambda} \sim \theta_i I_{\lambda\pm\alpha_b} \sim \theta_i I_{\lambda\pm\alpha_a\pm\alpha_b}.
\]

However, these yield the same \( \theta_i I_{\lambda\pm\alpha_a\pm\alpha_b} \).

Proof. We prove the case of \( \theta_i I_{\lambda+a_i \pm \alpha_b} \) (the other cases are the same). For simplicity we assume \( \langle a, b \rangle = -1 \) (if \( a = b \) there is nothing to prove and the case \( \langle a, b \rangle = 0 \) is strictly easier). Let us denote by \( \phi \in \text{End}^2(1_{\lambda+a_i \pm \alpha_b}) \) the difference between the two redefinitions. We need to show that \( \phi = 0 \).

By considering \((T_{ab} II) (II\theta_i) \in \text{Hom}^{-1}(E_a E_b 1_{\lambda}, E_a E_b 1_{\lambda}) \) we find that

\[
(\phi II)(IT_{ab}) \in \text{Hom}^{-1}(1_{\lambda+a_i \pm \alpha_a E_b 1_{\lambda} + \alpha_b E_a 1_{\lambda}} E_a E_b)
\]

must be zero. Now we know that \( \langle \lambda + \alpha_a, \alpha_b \rangle \geq -1 \Rightarrow \lambda_a \geq 0 \) and likewise \( \lambda_b \geq 0 \). Then by Lemma 5.4 we know that the \( E_b \)-rank of \( T_{ab} II : E_a E_b 1_{\lambda} F_a \to E_b E_a 1_{\lambda} F_a \) is positive. Thus \( (\phi II)(IT_{ab}) = 0 \) implies...
$(\phi I) \in \text{End}^2(1_{\lambda + \alpha_i + \alpha_j} E_b)$ is zero. But then $1_{\lambda + \alpha_i + \alpha_j} E_b$ contains $\lambda_b + 1 \geq 1$ copies of $1_{\lambda + \alpha_i + \alpha_j}$, which means that $\phi \in \text{End}^2(1_{\lambda + \alpha_i + \alpha_j})$ must also be zero. This completes the proof. □

12.4. **Step (ii).** By Remark 12.3 we can assume $\mu_j = 0$. Then if we take $\lambda = \mu - \alpha_i$ we have $\lambda_i = -2$ and $\lambda_j = 1$. This means that a transient map in $\text{End}^2(E_i 1_{\lambda} E_j)$ is of the form $I \phi I$ for some $\phi \in \text{End}^2(1_{\lambda})$. Thus, by redefining $\theta_i I_{\mu - \alpha_i}$, we can fix it so that (83) holds when $\lambda = \mu - \alpha_i$.

Next, having fixed $\theta_i I_{\mu}$ and $\theta_i I_{\mu - \alpha_i}$, we can redefine the remaining $\theta_i I_{\lambda}$ so that relations (81) and (82) hold (Proposition 12.6). Finally, since (83) holds for $\lambda = \mu - \alpha_i$ it holds for all $\lambda$ by Corollary 12.10. This completes Step (ii).

**Proposition 12.9.** Suppose (81) and (82) both hold for an edge $i \rightarrow j$ in $\Gamma$ and all $\lambda$. Assuming that (83) holds when $\lambda = \nu$ then it also holds for $\lambda = \nu + \alpha_k$ if $\langle \nu + \alpha_i, \alpha_k \rangle \geq -1$ and for $\lambda = \nu - \alpha_k$ if $\langle \nu + \alpha_i, \alpha_k \rangle \leq 1$.

**Proof.** We assume $\langle \nu + \alpha_i, \alpha_k \rangle \geq -1$ (the case $\langle \nu + \alpha_i, \alpha_k \rangle \leq 1$ is the same). There are three cases depending on whether $k = i, k = j$ or $k \neq i, j$. All three are proven in the same manner so we only prove the more complicated case $k = j$.

Consider the composition

$$
E_j E_i E_j 1_{\nu - \alpha_j} (IT_{ji})(IT_{ji}) \rightarrow E_j E_i E_j 1_{\nu - \alpha_j} (IT_{ji}) \rightarrow E_j E_i E_j 1_{\nu - \alpha_j} (1).
$$

On the one hand, using (83) when $\lambda = \nu$ this is equal to

$$(IT_{j}) (T_{ji}) (IT_{ji}) (T_{ji}) (IT_{ji}) (T_{ji}) = (T_{ji}) (T_{ji}) (T_{ji}) - t_{ji} (T_{ji}) (T_{ji}) - t_{ji} (T_{ji})$$

where we used the affine nilHecke relation to commute the $X_j$. On the other hand, this composition equals

$$(T_{jji})(IT_{ji}) = (T_{jji})(IT_{ji}) = (T_{ji})(T_{ji})(T_{ji}) - t_{ji} (T_{ji}) (T_{ji}) - t_{ji} (T_{ji})$$

for some transient map $\gamma \in \text{End}^2(E_i E_j 1_{\nu})$. Comparing expressions we get $(\gamma I)(IT_{ji})(T_{ji}) = 0$. It remains to show $\gamma = 0$.

By Corollary 3.3 we get $(I \gamma)(v_{ji} I) I (v_{ji}) I = 0$. Now $(I v_{ji})(I v_{ji}) \sim id$ and $(v_{ji}) (u_{ji}) \sim id$ since $\nu_j \geq 0$. So composing on the right with $(I v_{ji}) (u_{ji})$ we get that $(\gamma I) \in \text{End}^2(E_i E_j 1_{\nu})$ is zero. If we compose with $E_j$ on the left and simplify we get several copies of $\gamma \in \text{End}^2(E_i E_j 1_{\nu + \alpha_j})$ (this uses that $\langle \nu + \alpha_i, \alpha_j \rangle \geq 1$). It follows that $\gamma = 0$ and we are done.

**Corollary 12.10.** Suppose (81) and (82) hold for an edge $i \rightarrow j$ in $\Gamma$ and all $\lambda$. If (83) holds for $\lambda = \mu - \alpha_i$ then it holds for all $\lambda$.

**Proof.** Proposition 12.9 states that for a valid slide $\nu \rightarrow \nu \pm \alpha_k$ relation (83) holds for $\lambda = \nu - \alpha_i$ assuming it holds for $\lambda = \nu - \alpha_i$. The result follows since we can reach any weight via a sequence of valid slides starting from $\mu$. □

12.5. **Step (iii).** We will show that (84) holds when $\lambda = \mu - \alpha_j$. It then follows by Corollary 12.10 that (84) holds for all $\lambda$.

We know that $(T_{ij})(T_{ji}) = t_{ij}(X_i I) + t_{ji}(X_j I) + \phi \in \text{End}^2(E_j 1_{\mu - \alpha_j} E_j)$ for some transient map $\phi$. Now consider the composition $(T_{ij})(T_{ji})(T_{ij}) \in \text{Hom}^3(E_i 1_{\mu - \alpha_i} E_j, E_j 1_{\mu - \alpha_j} E_i)$. On the one hand, using relation (83), this equals

$$(T_{ij})(t_{ij}(X_i I) + t_{ji}(X_j I)) = (t_{ij}(X_i I) + t_{ji}(X_j I))(T_{ij}).$$

On the other hand, the composition equals $(t_{ij}(X_i I) + t_{ji}(X_j I) + \phi)(T_{ij})$. It follows that $\phi(T_{ij}) = 0$. 

It remains to show that $\phi(T_{ij}) = 0 \Rightarrow \phi = 0$. We consider two cases depending on whether $\mu_j = 0$ or $\mu_j = 1$. If $\mu_j = 0$ then $\phi$ is of the form $(\gamma I) \in \text{End}^2(E_j 1_{\mu - \alpha_j}, E_i)$ where $\gamma \in \text{End}^2(1_{\mu - \alpha_j})$ is transient. Composing on the left with $F_j$ we get that

$$F_j E_j 1_{\mu - \alpha_j} E_j \xrightarrow{(I\theta I\gamma)} F_j E_j 1_{\mu - \alpha_j} E_j \langle 1 \rangle \xrightarrow{(I\theta I\gamma)} F_j E_j 1_{\mu - \alpha_j} E_i \langle 3 \rangle$$

is zero. Since $\mu_j = 0$ the first map induces an isomorphism between one copy of $1_{\mu - \alpha_j} E_i$. This means that $(\gamma I) \in \text{End}^2(1_{\mu - \alpha_j}, E_i)$ is zero and hence $\phi = 0$.

Similarly, if $\mu_j = 1$ then $\phi$ is of the form $(\gamma II) \in \text{End}^2(1_{\mu} E_i E_i)$. This time we compose with $F_j$ on the right and find that

$$1_{\mu} E_i E_i F_j \xrightarrow{(IT_{ij})} 1_{\mu} E_i E_i F_j \langle 1 \rangle \xrightarrow{(\gamma III)} 1_{\mu} E_i E_i F_j \langle 3 \rangle$$

is zero. Since $\mu_j = 1$ the first map again induces an isomorphism between one copy of $1_{\mu} E_i$ and hence $(\gamma I) \in \text{End}^2(1_{\mu} E_i)$ is zero. So again we find $\phi = 0$ which completes the argument.

13. AN ALTERNATIVE DEFINITION OF A $(g, \theta)$ ACTION

An equivalent definition of a $(g, \theta)$ action involves replacing condition $(v)$ with the following.

The composition $E_i E_i$ decomposes as $E_i^{(2)} (-1) \oplus E_i^{(2)} (1)$ for some 1-morphism $E_i^{(2)}$. Moreover, if $\theta \in Y_\kappa$ where $(\theta, \alpha_i) \neq 0$ (resp. $(\theta, \alpha_i) = 0$) then $I\theta I \in \text{End}^2(E_i 1_{\lambda} E_i)$ induces a nonzero map (resp. the zero map) between the summands $E_i^{(2)} (1)$ on either side.

**Lemma 13.1.** The condition above implies condition $(v)$ from section 2.2.

**Proof.** A very similar version of this result appears as [CLa, Lemma 3.6]. We prove the case $\lambda_i \geq 0$ (the case $\lambda_i \leq 0$ is the same). First, we know that

$$(I\theta II) : E_i 1_{\lambda} E_i F_i \rightarrow E_i 1_{\lambda} E_i F_i \langle 2 \rangle$$

induces an isomorphism between the summands $E_i^{(2)} (1) 1_{\lambda} 1_{\lambda} (1)$ on either side. Since

$$E_i^{(2)} 1_{\lambda} 1_{\lambda} \cong F_i E_i^{(2)} 1_{\lambda} \bigoplus_{[\lambda_i + 1]} E_i 1_{\lambda}$$

this means that the total $E_i 1_{\lambda}$-rank of the map in (85) is at least $\lambda_i + 1$.

On the other hand, the map in (85) induces a map

$$(I\theta II) \bigoplus_{[\lambda_i]} (I\theta) : E_i 1_{\lambda} F_i E_i \bigoplus_{[\lambda_i]} E_i 1_{\lambda} \rightarrow E_i 1_{\lambda} F_i E_i \langle 2 \rangle \bigoplus_{[\lambda_i]} E_i 1_{\lambda} (2).$$

Thus the total $E_i 1_{\lambda}$-rank of the map in (85) is equal to the total $E_i 1_{\lambda}$-rank of $(I\theta II) \in \text{End}^2(E_i 1_{\lambda} F_i E_i)$. Since $E_i 1_{\lambda} F_i E_i \cong F_i E_i 1_{\lambda} \bigoplus_{[\lambda_i + 2]} E_i 1_{\lambda}$ where $F_i E_i 1_{\lambda}$ contains no summands $E_i 1_{\lambda}$ we get that $(I\theta I) \in \text{End}^2(E_i 1_{\lambda} F_i)$ has total $1_{\lambda + \alpha_i}$-rank at least $\lambda_i + 1$. The result follows because by degree reasons the $1_{\lambda + \alpha_i}$-rank of $(I\theta I) \in \text{End}^2(E_i 1_{\lambda} F_i)$ cannot be any larger than $\lambda_i + 1$. 

This alternative condition is similar to the condition present in the definition of a geometric categorical $\mathfrak{gl}_n$ action from [CK3]. In that geometric setup the existence of $E_i^{(2)}$ is obvious and checking the condition above is easier than checking condition $(v)$. On the other hand, condition $(v)$ is easier to check in other setups such as [CLi2] where even the existence of divided powers $E_i^{(r)}$ is very difficult.
14. Applications

14.1. Categorical vertex operators. In [CL11] we explained how, starting with the zig-zag algebra $A_F$ associated to the Dynkin diagram $\Gamma$ of a finite type Lie algebra $\mathfrak{g}$, one obtains a Heisenberg algebra $\mathfrak{h}_F$ and a 2-category $H_F$ which categorifies it. In the process we also categorified the Fock space of $\mathfrak{h}_F$ using a 2-category $\mathcal{F}_F$.

In the subsequent paper [CL2] we showed how to define a $(\hat{\mathfrak{g}}, \theta)$ action on the 2-category $\mathcal{K}_{F, \Gamma} := \text{Kom}(\mathcal{F}_F) \otimes \mathbb{Z}[Y]$ where $\text{Kom}(\cdot)$ denotes the homotopy category and $(\cdot) \otimes \mathbb{Z}[Y]$ means that we have one copy of $(\cdot)$ for each element of the root lattice $Y$.

This construction categorifies the Frenkel-Kac-Segal vertex operator construction [FK, Se] of the basic representation of $\hat{\mathfrak{g}}$. One subtlety here is that $\hat{\mathfrak{g}}$ is the affine Lie algebra of $\mathfrak{g}$ in its loop presentation. We will show in future work that such a $(\hat{\mathfrak{g}}, \theta)$ action is equivalent to a $(\hat{\mathfrak{g}}, \theta)$ action where $\hat{\mathfrak{g}}$ is in its Kac-Moody presentation. Moreover, in the category $\mathcal{K}_{F, \Gamma}$ all transient maps are zero so we do not need to worry about them. Thus Theorem 2.2 implies the following.

**Theorem 14.1.** The quiver Hecke algebras associated to $\hat{\mathfrak{g}}$ act on $\mathcal{K}_{F, \Gamma}$.

It seems very difficult to explicitly construct this action on $\mathcal{K}_{F, \Gamma}$. Indeed, this result is surprising because it is even difficult to show by hand that $E_i^2 \cong \oplus_{[2]} E_i^{(2)}$ holds in $\mathcal{K}_{F, \Gamma}$. In fact, an explicit form of $E_i^{(k)}$ for $k > 2$ exists only conjecturally. An immediate Corollary of Theorem 14.1 and [CLa] is the following.

**Corollary 14.2.** The 2-category $\mathcal{K}_{F, \Gamma}$ is a 2-representation of $\hat{\mathfrak{u}}_Q(\hat{\mathfrak{g}})$ in the sense of Khovanov-Lauda.

**Remark 14.3.** Unfortunately, we do not know for what choice of $Q$ this is a 2-representation. If $\mathfrak{g}$ is of type $A$ then the space of such $Q$’s is parametrized by $k^\times$.

14.2. The affine Grassmannian and geometric categorical $\mathfrak{g}$ actions. In [CK3, section 2.2] we introduced the idea of a geometric categorical $\mathfrak{g}$ action. This definition was helpful because it was easier to check in geometric situations. Briefly, such an action consists of the following data.

(i) A collection of smooth varieties $Y(\lambda)$ for $\lambda \in X$.

(ii) A collection of kernels

$$E_i^{(r)}(\lambda) \in D(Y(\lambda) \times Y(\lambda + r\alpha_i))$$

and $F_i^{(r)}(\lambda) \in D(Y(\lambda + r\alpha_i) \times Y(\lambda))$

where $D(Y)$ denotes the derived category of coherent sheaves on $Y$.

(iii) For each $Y(\lambda)$ a flat deformation $\tilde{Y}(\lambda) \rightarrow Y_k$.

From this data we obtain a 2-category $\mathcal{K}$ where the objects are $D(Y(\lambda))$, the 1-morphisms are kernels and 2-morphisms are maps between kernels. The extra data of the deformation $\tilde{Y}(\lambda)$ can be used to obtain a linear map $Y_k \rightarrow \text{End}^2(1_\lambda)$ as follows. From the standard short exact sequence

$$0 \rightarrow T_Y(\lambda) \rightarrow T_{\tilde{Y}(\lambda)} \rightarrow \mathcal{O}_Y(\lambda) \{2\} \otimes_k Y_k \rightarrow 0$$

one obtains the Kodaira-Spencer map $Y_k \rightarrow H^1(T_Y(\lambda)\{-2\})$ (the $\{2\}$ is a grading shift corresponding to a $\mathbb{C}^\times$ action on $\tilde{Y}(\lambda)$ which acts on $Y_k$ with weight 2). On the other hand, the Hochschild-Kostant-Rosenberg isomorphism states that

$$\text{End}^2(1_\lambda) = HH^2(Y(\lambda)) \cong H^0(\lambda^2 T_Y(\lambda)) \oplus H^1(T_Y(\lambda)) \oplus H^2(\mathcal{O}_Y(\lambda))$$

and so we get a linear map $Y_k \rightarrow HH^2(Y(\lambda)\{-2\})$. Thus, identifying the shift (1) with $[1]\{-1\}$, we obtain $Y_k \rightarrow \text{End}^2(1_\lambda)$.

**Proposition 14.4.** A geometric categorical $\mathfrak{g}$ action induces a $(\mathfrak{g}, \theta)$ action, assuming $\mathcal{K}$ satisfies conditions (vi), (vii) and (viii).
Proof. Conditions (i) – (iv) are immediate. Most naturally we obtain a \((\mathfrak{g}, \theta)\) action in its alternative description from section 13. The alternative condition (v) follows from conditions [CK3, Sect. 2.2: (vi)+(x)]. □

Remark 14.5. With hindsight of Theorem 2.2 the definition of a geometric categorical \(g\) action can be simplified. For instance, one does not need to require the existence of divide powers \(E_i^{(r)}\) for \(r > 1\).

An example of a geometric categorical \(g = \mathfrak{sl}_n\) action categorifying \(\Lambda_q^n(\mathbb{C}^m \otimes \mathbb{C}^n)\) was defined in [CKL1, C]. More precisely, having fixed \(N \in \mathbb{N}\) we take

\[
Y(\lambda) := \{ \mathbb{C}[z]^m = L_0 \subset L_1 \subset \ldots \subset L_n \subset \mathbb{C}(z)^m : zL_i \subset L_{i-1}, \dim(L_i/L_{i-1}) = k_i \}
\]

where the \(L_i\) are complex vector subspaces, \(N = \sum k_i\) with \(0 \leq k_i \leq m\), and \(\lambda\) is determined by \(\lambda_i = k_{i+1} - k_i\). These varieties are obtained from the affine Grassmannian of \(\text{PGL}_m\) via the convolution product. Namely, \(Y(\lambda) = \Gr^{\lambda_{k1}} \times \ldots \times \Gr^{\lambda_{kn}}\) where \(\times\) denotes the convolution product.

The kernels \(\mathcal{K}_i^{(r)}\) and \(\mathcal{F}_i^{(r)}\) are then defined by certain Hecke correspondence (see [C, section 8.3]). We denote the resulting 2-category \(K^{\mathfrak{sl}_n}_{\Gr,m}\).

Theorem 14.6. There exists an \((\mathfrak{sl}_n, \theta)\) action on \(K^{\mathfrak{sl}_n}_{\Gr,m}\).

Proof. The fact that there exists a geometric categorical \(\mathfrak{sl}_n\) action on \(K^{\mathfrak{sl}_n}_{\Gr,m}\) was proved in [CKL1] (with some of the details appearing also in [CK3, C]). Since \(K^{\mathfrak{sl}_n}_{\Gr,m}\) categorifies a finite dimensional representation of \(\mathfrak{sl}_n\), conditions (vi) and (vii) are immediate. Finally, condition (viii) is an easy consequence of the particular representation being categorified (namely, \(\Lambda_q^N(\mathbb{C}^m \otimes \mathbb{C}^n)\)). □

Corollary 14.7. \(K^{\mathfrak{sl}_n}_{\Gr,m}\) is a 2-representation of \(\hat{U}_Q(\mathfrak{sl}_n)\) in the sense of Khovanov-Lauda.

Note that in this case, since \(g = \mathfrak{sl}_n\), this result holds without having to mod out by transient maps. Also, all choices of parameters \(Q\) are equivalent so there is no ambiguity.

Remark 14.8. In [CKL3] we constructed a geometric categorical \(g\) action on Nakajima quiver varieties which lifted Nakajima’s action on K-theory [N]. For a dominant weight \(\Lambda\) one can define a 2-category \(K^\Lambda_Q\) consisting of derived categories of coherent sheaves on Nakajima quiver varieties with highest weight \(\Lambda\). Subsequently, Theorem 14.6 and Corollary 14.7 also hold if we replace \(K^{\mathfrak{sl}_n}_{\Gr,m}\) with \(K^\Lambda_Q\) (except that we have to mod out by transient maps if \(g \neq \mathfrak{sl}_n\)).

14.3. Rigidity of homological knot invariants. In [C] we explained that given a categorification of the \(U_q(\mathfrak{sl}_\infty)\) representation \(\Lambda^\infty(\mathbb{C}^m \otimes \mathbb{C}^\infty) = \lim_{N \to \infty} \Lambda_q^N(\mathbb{C}^m \otimes \mathbb{C}^{2N})\) one obtains a homological knot invariant categorifying the Reshetikhin-Turaev knot invariants of type \(\mathfrak{sl}_m\). In fact, we explained that one only needs an action whose nonzero weight spaces are the same as those of \(\Lambda^\infty(\mathbb{C}^m \otimes \mathbb{C}^\infty)\).

In principle, the homology you get will depend on the specific categorification. However, if the categorification is given by a 2-representation in the sense of Khovanov-Lauda then it can be calculated entirely from this information. Since such a 2-representation is induced by a \((\mathfrak{sl}_\infty, \theta)\) action we obtain the following.

Theorem 14.9. Any two \((\mathfrak{sl}_\infty, \theta)\) actions whose nonzero weight spaces are the same as those of \(\Lambda^\infty(\mathbb{C}^m \otimes \mathbb{C}^\infty)\) yield isomorphic homological knot invariants.

A variety of methods have been used to define homological knot invariants over the last few years. These include: derived categories of coherent sheaves [CK1, CK2, C], category \(\mathcal{O}\) [MS, Siu], matrix factorizations [KR, W, Y] and foams [MSV, LQR, QR]. Since all these invariants fit within the framework described above they define equivalent homologies. Thus Theorem 2.2 also implies a certain rigidity for homological knot invariants of Reshetikhin-Turaev type \(\Lambda\).
APPENDIX A. SPACES OF MORPHISMS

In this section we collect a series of calculations of the dimension of spaces of maps between various 1-morphisms. All these computations are basically performed in the same way, namely by repeatedly applying adjunction and simplifying until one ends up with \( \text{End}^d(\mathbf{1}_\lambda) \) which we have assumed to be zero if \( i < 0 \) and one-dimensional if \( i = 0 \). All the proofs are independent of other results in the main body of this paper, meaning that they only use the definition of a \((g, \theta)\) action from section 2.2.

A.1. Spaces not involving divided powers.

**Lemma A.1.** We have \( \dim \text{End}^d(\mathbf{E}, \mathbf{1}_\lambda) \leq \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d < 0. \end{cases} \)

**Proof.** Suppose \( \lambda_i \leq 0 \) (the case \( \lambda_i \geq 0 \) is the same). In this case we have

\[
\text{Hom}(\mathbf{E}, \mathbf{1}_\lambda, \mathbf{1}_\lambda(\langle d \rangle)) \cong \text{Hom}(\mathbf{F}_i \mathbf{E}, \mathbf{1}_\lambda, \mathbf{1}_\lambda(\langle d + \lambda_i + 1 \rangle))
\]

\[
\cong \text{Hom}(\mathbf{E}_i \mathbf{F}_i \mathbf{1}_\lambda \bigoplus \mathbf{1}_\lambda, \mathbf{1}_\lambda(\langle d + \lambda_i + 1 \rangle)).
\]

On the one hand, if \( \lambda_i < 0 \) then

\[
\dim \text{Hom}(\mathbf{1}_\lambda, \bigoplus \mathbf{1}_\lambda(\langle d + \lambda_i + 1 \rangle)) = \sum_{r=0}^{-\lambda_i-1} \dim \text{Hom}(\mathbf{1}_\lambda, \mathbf{1}_\lambda(\langle d - 2r \rangle)) \leq \begin{cases} 1 & \text{if } d = 0 < 0 \\ 0 & \text{if } d < 0 \end{cases}
\]

while on the other hand

\[
\dim \text{Hom}(\mathbf{E}_i \mathbf{F}_i \mathbf{1}_\lambda, \mathbf{1}_\lambda(\langle d + \lambda_i + 1 \rangle)) = \dim \text{Hom}(\mathbf{E}_i \mathbf{1}_{\lambda-\alpha_i}, \mathbf{1}_{\lambda-\alpha_i}(\langle d + 2\lambda_i \rangle)) = 0.
\]

where the last equality follows by induction on \( \lambda_i \) (this is where we use the condition that \( \mathbf{1}_{\lambda-\alpha_i} = 0 \) for \( r > 0 \)). The result follows. In the special case \( \lambda_i = 0 \) the first term vanishes while the second is \( \text{End}^d(\mathbf{E}_i \mathbf{1}_{-\alpha_i}) \) and the result follows again by induction.

**Lemma A.2.** We have \( \dim \text{End}^{-2}(\mathbf{E}_i \mathbf{E}_i \mathbf{1}_\mu) \leq 1 \) with equality if \( \mathbf{E}_i \mathbf{E}_i \mathbf{1}_\mu \neq 0 \).

**Proof.** By applying the commutation relation twice one finds that

\[
\mathbf{E}_i \mathbf{F}_i \mathbf{1}_\mu \cong \mathbf{F}_i \mathbf{E}_i \mathbf{1}_\nu \bigoplus \mathbf{E}_i \mathbf{1}_\nu \quad \text{if } \nu_i \geq -1
\]

\[
\mathbf{F}_i \mathbf{E}_i \mathbf{1}_\nu \cong \mathbf{E}_i \mathbf{F}_i \mathbf{1}_\nu \bigoplus \mathbf{E}_i \mathbf{1}_\nu \quad \text{if } \nu_i \leq -1.
\]

Thus, if \( \mu_i \geq -2 \) then we get

\[
\text{End}^{-2}(\mathbf{E}_i \mathbf{E}_i \mathbf{1}_\mu)
\]

\[
\cong \text{Hom}(\mathbf{E}_i \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}, \mathbf{E}_i \mathbf{F}_i \mathbf{1}_{\mu+\alpha_i}(\langle -\mu_i - 3 \rangle))
\]

\[
\cong \text{Hom}(\mathbf{E}_i \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}, \mathbf{F}_i \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}(\langle -\mu_i - 3 \rangle) \bigoplus \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}(\langle -\mu_i - 3 \rangle))
\]

\[
\cong \text{Hom}(\mathbf{E}_i \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}, \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}(\langle 2\mu_i - 8 \rangle) \oplus \text{Hom}(\mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}, \bigoplus \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}(\langle -\mu_i - 3 \rangle)).
\]

The left term vanishes by induction while, using Lemma A.1, the only surviving term in the right hand sum is \( \text{Hom}(\mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}, \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i}) \cong k \). Thus \( \text{End}^{-2}(\mathbf{E}_i \mathbf{E}_i \mathbf{1}_\mu) \cong k \) (as long as \( \mathbf{E}_i \mathbf{1}_{\mu+\alpha_i} \neq 0 \)). The case \( \mu_i \leq -2 \) is similar.

**Lemma A.3.** We have \( \dim \text{End}^{-6}(\mathbf{E}_i \mathbf{E}_i \mathbf{E}_i \mathbf{1}_\mu) \leq 1 \) with equality if \( \mathbf{E}_i \mathbf{E}_i \mathbf{E}_i \mathbf{1}_\mu \neq 0 \).
Proof. By applying the commutation relation three times one finds that
\[
E_iE_jE_k1_\nu \cong F_iE_jE_k1_\nu \bigoplus [3\nu+2] E_iE_j1_\nu \quad \text{if } \nu_i \geq -2
\]
\[
F_iE_jE_k1_\nu \cong E_iE_jE_k1_\nu \bigoplus [3\nu-2] E_iE_j1_\nu \quad \text{if } \nu_i \leq -2.
\]
Thus, if \(\mu_i \geq -3\) then we get
\[
\text{End}^{-6}(E_iE_j1_\mu) \\
\cong \text{Hom}(E_iE_j1_{\mu+\alpha_i}, E_iE_jF_i1_{\mu+\alpha_i}(-\mu_i - 7)) \\
\cong \text{Hom}(E_iE_j1_{\mu+\alpha_i}, F_iE_jE_k1_{\mu+\alpha_i}(-\mu_i - 7) \bigoplus E_iE_j1_{\mu+\alpha_i}(-\mu_i - 7)) \\
\cong \text{Hom}(E_iE_j1_{\mu+\alpha_i}, E_iE_jE_k1_{\mu+\alpha_i}(\alpha_j - \mu_i - 14)) \oplus \text{Hom}(E_iE_j1_{\mu+\alpha_i}, E_iE_j1_{\mu+\alpha_i}(-\mu_i - 7)).
\]
The left term vanishes by induction while, using Lemma A.2, the only surviving term in the right hand sum is \(\text{Hom}(E_iE_j1_{\mu+\alpha_i}, E_iE_j1_{\mu+\alpha_i}(-2))\). Thus, by Lemma A.2, \(\dim \text{End}^{-6}(E_iE_j1_\mu) \leq 1\) and equality holds if \(E_iE_j1_{\mu+\alpha_i} \neq 0\). The case \(\mu_i \leq -3\) is similar. \(\square\)

**Lemma A.4.** Suppose \(i,j \in I\) with \(\langle i,j \rangle = -1\). Then
\[
\dim \text{Hom}(E_iE_j1_\lambda, E_iE_j1_\lambda \langle d \rangle) \leq \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } d < 1 \end{cases}
\]
\[
\dim \text{Hom}(E_iE_j1_\lambda, E_iE_j1_\lambda \langle d \rangle) \leq \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d < 0. \end{cases}
\]
In (86) equality holds when \(d = 1\) if \(E_iE_j1_\lambda\) and \(E_iE_j1_\lambda\) are both nonzero. Likewise in (87) equality holds when \(d = 0\) if \(E_iE_j1_\lambda\) is nonzero.

Proof. Suppose \(\mu_i + \mu_j \leq 0\) (the case \(\mu_i + \mu_j \geq 0\) is the same). This means that either \(\mu_i \leq 0\) or \(\mu_j \leq 0\). We assume \(\mu_j \leq 0\) as the other case is similar.

The proof is by induction on \(\mu_i + \mu_j\). The base case follows from condition (vi) where we take \(\alpha = \alpha_i + \alpha_j\). This is the only time we use this case of condition (vi) in this paper.

To obtain the induction step we first note that
\[
\text{Hom}(E_iE_j1_\mu, E_iE_j1_\mu \langle d \rangle) \cong \text{Hom}(F_jE_iE_j1_\mu(-\mu_j), E_i1_\mu \langle d \rangle)
\]
\[
\cong \text{Hom}(E_iE_jF_j1_\mu, E_i1_\mu \langle \mu_j + d \rangle) \bigoplus_{r=0}^{-\mu_j-1} \text{Hom}(E_i1_\mu, E_i1_\mu(-1 - 2r + d)).
\]
Now, if \(d \leq 1\) then by Lemma A.1 every term in the right hand sum is zero unless \(r = 0\), \(d = 1\) and \(E_i1_\mu \neq 0\) (and \(\mu_i < 0\)) in which case it is one-dimensional. Meanwhile, the left hand term is equal to
\[
\text{Hom}(E_iE_j1_{\mu-\alpha_j}, E_i(1_{\mu-\alpha_j}F_j) \langle \mu_j + d \rangle) \cong \text{Hom}(E_iE_j1_{\mu-\alpha_j}, E_iE_j1_{\mu-\alpha_j}(2\mu_j + d - 1)).
\]
By (87) this is zero unless \(d = 1\), \(\mu_j = 0\) and \(E_iE_j1_{\mu-\alpha_j} \neq 0\) in which case it is one-dimensional. Thus (86) when \(\lambda = \mu\) follows from (87) when \(\lambda = \mu - \alpha_j\).

Now, if we study (87) we have
\[
\text{Hom}(E_iE_j1_\mu, E_iE_j1_\mu \langle d \rangle) \cong \text{Hom}(1_\mu, F_jE_iE_j1_\mu \langle \mu_i + \mu_j + 1 + d \rangle).
\]
If \( \mu_i \leq 1 \) then we can simplify \( F_j F_i E_j 1_\mu \) directly (this is the easier case). We assume the more difficult situation that \( \mu_i \geq 1 \) in which case we have

\[
F_j F_i E_j 1_\mu \cong F_j F_i E_i 1_\mu \bigoplus F_j E_j 1_\mu \bigoplus F_j E_j 1_\mu
\]

while the left hand side of the equation above equals

\[
E_i F_j E_j 1_\mu \cong E_i F_j F_i 1_\mu \bigoplus E_i F_i 1_\mu \bigoplus E_i F_i 1_\mu
\]

Now, if \( a, b \in \mathbb{N} \) with \( b \geq a \) then \( [a][b-1] = [a-1][b] + [b-a] \). Thus, taking \( a = \mu_i, b = -\mu_j \) and using that morphisms have a unique decomposition we get that

(89)

\[
F_j F_i E_i 1_\mu \bigoplus E_j F_j 1_\mu \cong E_i E_j 1_\mu \bigoplus F_i E_1 1_\mu \bigoplus 1_\mu
\]

Now

\[
\text{Hom}(1_\mu, E_j F_j 1_\mu(s)) \cong \text{Hom}((1_{\mu-\alpha_j} F_j)_L, E_j 1_{\mu-\alpha_j}(s)) \cong \text{Hom}(1_\mu E_j, 1_\mu E_j(s + \mu_j - 1))
\]

so by Lemma A.1 this vanishes if \( s \leq -\mu_j \). In particular, this means that

\[
\text{Hom}(1_\mu, \bigoplus_{[\mu_i-1]} E_j F_j 1_\mu(\mu_i + \mu_j + 1 + d)) = 0 \quad \text{if} \quad d \leq 0.
\]

Similarly, one finds that \( \text{Hom}(1_\mu, \bigoplus_{[\mu_j-1]} F_i E_i 1_\mu(\mu_i + \mu_j + 1 + d)) = 0 \) if \( d \leq 0 \). Finally,

\[
\text{Hom}(1_\mu, E_i E_j F_j 1_\mu(\mu_i + \mu_j + 1 + d)) \cong \text{Hom}((1_{\mu-\alpha_i} F_i)_R, E_i E_j 1_\nu(\mu_i + \mu_j + 1 + d))
\]

(90)

\[
\cong \text{Hom}(E_i E_j 1_\mu, E_i E_j 1_\nu(2(\mu_i + \mu_j) + d))
\]

where \( \nu = \mu - \alpha_i - \alpha_j \). Now, \( \lambda_{\nu, \alpha_i + \alpha_j} = (\mu, \alpha_j + \alpha_i) - 2 \) so by induction this is zero if \( d \leq 0 \) (unless \( \mu_i + \mu_j = 0 \) in which case it is one-dimensional). Thus we get that

\[
\text{Hom}(E_i E_j 1_\mu, E_i E_j 1_\mu(d)) \cong \text{Hom}(1_\mu, F_j F_i E_i 1_\mu(\mu_i + \mu_j + 1 + d))
\]

\[
\cong \text{Hom}(1_\mu, \bigoplus_{[\mu_i-\mu_j-1]} 1_\mu(\mu_i + \mu_j + 1 + d))
\]

\[
\cong \bigoplus_{r=0} \text{Hom}(1_\mu, 1_\mu(d - 2r)).
\]

Each term in the last direct sum above is zero if \( d < 0 \) or \( r > 0 \) and one-dimensional if \( d = 0 = r \) (unless \( d = \mu_i + \mu_j = 0 \) in which case the sum vanishes but then (90) contributes 1 to the dimension). In conclusion, equation (87) when \( \lambda = \mu \) follows from (86) when \( \lambda = \mu - \alpha_i - \alpha_j \). This completes the induction. \( \square \)
Lemma A.5. Suppose \( i,j \in I \) with \( \langle i,j \rangle = 0 \). Then

\[
\dim \text{Hom}(E_iE_j1_\lambda, E_jE_i1_\lambda(d)) \leq \begin{cases} 
1 & \text{if } d = 0 \\
0 & \text{if } d < 0
\end{cases}
\]

\[
\dim \text{Hom}(E_iE_j1_\lambda, E_iE_j1_\lambda(d)) \leq \begin{cases} 
1 & \text{if } d = 0 \\
0 & \text{if } d < 0.
\end{cases}
\]

In (91) equality holds when \( d = 0 \) if \( E_iE_j1_\lambda \) and \( E_jE_i1_\lambda \) are both nonzero. Likewise in (92) equality holds when \( d = 0 \) if \( E_jE_i1_\lambda \) is nonzero.

Proof. The argument here is a much simpler version of that in the proof of Lemma A.4. The main difference here is that \( E_iE_j \cong E_jE_i \) so one only needs to mimic the first part of the proof of Lemma A.4. We leave the details to the reader. \( \Box \)

Lemma A.6. If \( i \neq j \in I \) then

\[
\dim \text{Hom}(F_iE_i1_\lambda, E_jF_j1_\lambda(d)) = \dim \text{Hom}(E_iF_i1_\lambda, F_jE_j1_\lambda(d)) = \begin{cases} 
1 & \text{if } d = 0 \text{ and } F_jE_i1_\lambda \neq 0 \\
0 & \text{if } d < 0
\end{cases}
\]

\[
\dim \text{Hom}(F_iE_j1_\lambda, 1_\lambda(d)) = \dim \text{Hom}(1_\lambda, F_iE_j1_\lambda(d)) = \begin{cases} 
1 & \text{if } d = \lambda_i + 1 \text{ and } E_i1_\lambda \neq 0 \\
0 & \text{if } d < \lambda_i + 1
\end{cases}
\]

\[
\dim \text{Hom}(E_iF_i1_\lambda, 1_\lambda(d)) = \dim \text{Hom}(1_\lambda, E_iF_i1_\lambda(d)) = \begin{cases} 
1 & \text{if } d = -\lambda_i + 1 \text{ and } F_i1_\lambda \neq 0 \\
0 & \text{if } d < -\lambda_i + 1.
\end{cases}
\]

Proof. There are three cases to consider in proving the first equality. If \( i \neq j \) with \( \langle i,j \rangle = -1 \) then

\[
\text{Hom}(F_jE_i1_\lambda, E_jF_j1_\lambda(d)) \cong \text{Hom}(E_i(1_\lambda-\alpha_iF_j)_R(1_\lambda+\alpha_i-\alpha_jF_j)_R^dE_i(1_\lambda)) \cong \text{Hom}(E_iE_j, E_jE_i(d+1))
\]

and the result follows from Lemma A.4. Likewise, if \( i \neq j \) with \( \langle i,j \rangle = 0 \) the result follows from Lemma A.5. Finally, if \( i = j \) then \( \text{Hom}(F_iE_i1_\lambda, E_iF_i1_\lambda(d)) \cong \text{Hom}(E_i(E_i1_\lambda-\alpha_i), E_iE_i1_\lambda(d-2)) \) and the result follows from Lemma A.1. The second equality follows similarly.

The second and third pairs of equalities follow directly by adjunction together with Lemma A.1. \( \Box \)

Lemma A.7. For any \( i,j \in I \) we have:

(i) \( E_i1_\lambda \neq 0 \) if and only if \( 1_\lambda \) and \( 1_{\lambda+\alpha_i} \) are both nonzero.

(ii) \( E_jE_i1_\lambda \neq 0 \) if and only if \( 1_\lambda, 1_{\lambda+\alpha_i}, \) and \( 1_{\lambda+\alpha_i+\alpha_j} \) are all nonzero.

(iii) \( E_i1_\lambda F_j \neq 0 \) if and only if \( 1_\lambda, 1_{\lambda+\alpha_i}, 1_{\lambda+\alpha_j}, \) and \( 1_{\lambda+\alpha_i+\alpha_j} \) are all nonzero.

(iv) \( E_i1_\lambda E_jF_j \neq 0 \) if and only if \( 1_{\lambda+r\alpha_i}, \) and \( 1_{\lambda+\alpha_i+r\alpha_j} \) are nonzero for \(-1 \leq r \leq 1\).

Proof. Clearly if \( 1_\lambda \neq 0 \) or \( 1_{\lambda+\alpha_i} = 0 \) then \( E_i1_\lambda = 0 \). Conversely, suppose \( 1_\lambda \) and \( 1_{\lambda+\alpha_i} \) are nonzero. If \( \lambda_i \geq -1 \) then

\[
E_i1_\lambda F_i \cong F_iE_i1_{\lambda+\alpha_i} \bigoplus_{[\lambda_i+2]} 1_{\lambda+\alpha_i}
\]

which is nonzero since \( 1_{\lambda+\alpha_i} \neq 0 \). Thus \( E_i1_\lambda \neq 0 \). Similarly, if \( \lambda_i \leq -1 \) then \( F_iE_i1_\lambda \neq 0 \) so \( E_i1_\lambda \neq 0 \).

This proves (i). The argument for (ii) is the same. Namely, if \( \lambda_i \geq -1 \) then

\[
E_jE_i1_\lambda F_i \cong E_jF_iE_i1_{\lambda+\alpha_i} \bigoplus_{[\lambda_i+2]} E_j1_{\lambda+\alpha_i}
\]

which is nonzero since \( 1_{\lambda+\alpha_i} \) and \( 1_{\lambda+\alpha_i+\alpha_j} \) are nonzero (and similarly if \( \lambda_i \leq -1 \)).
We now prove (iii). Since $E_i F_j \cong F_j E_i$ it is clear that $1_\lambda, 1_{\lambda + \alpha_i}, 1_{\lambda + \alpha_j}, 1_{\lambda + \alpha_i + \alpha_j}$ are all nonzero if $E_i F_j \neq 0$. Conversely, suppose first that $\lambda_i \leq -1$. Then we compose with $F_i$ to get

$$F_i E_i 1_\lambda F_j \cong E_i F_i 1_\lambda F_j \bigoplus_{[-\lambda_i]} 1_\lambda F_j$$

which is nonzero since $1_\lambda$ and $1_{\lambda + \alpha_i}$ are nonzero. If $\lambda_i \geq 0$ then we precompose with $F_i$ to get

$${E_i 1_\lambda F_i F_j} \cong {F_j E_i 1_{\lambda + \alpha_i + \alpha_j}} \bigoplus_{\lambda_i=2+(i,j)} {F_j 1_{\lambda + \alpha_i + \alpha_j}}$$

which is nonzero since $1_{\lambda + \alpha_i}$ and $1_{\lambda + \alpha_i + \alpha_j}$ are nonzero (here we use that $\lambda_i + 2 + (i,j) \geq 1$).

The proof of (iv) is similar to that of (iii). The main difference is that we compose with $F_i F_j$ if $\lambda_i \leq -2$ and precompose with $F_i F_i$ if $\lambda_i \geq -1$. \qed

A.2. Spaces involving divided powers. In this section the proofs are still independent of the other results in this paper with one exception, we assume that we know $E_i^2 \cong E_i^{[2]} [1] \oplus E_i^{[2]} [-1]$.

**Lemma A.8.** We have $\dim \text{End}^d(E_i^{[2]} 1_\lambda) \leq \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d < 0. \end{cases}$

**Proof.** The proof is analogous to the one computing $\text{End}^d(E_i 1_\lambda)$ from Lemma A.1. \qed

**Lemma A.9.** Suppose $i, j \in I$ with $\langle i, j \rangle = -1$. Then

(93) $\dim \text{Hom}(E_i E_j^{(2)} 1_\lambda, E_j^{(2)} E_i 1_\lambda \langle d \rangle) \leq \begin{cases} 1 & \text{if } d = 2 \\ 0 & \text{if } d < 2. \end{cases}$

(94) $\dim \text{Hom}(E_j^{(2)} E_i 1_\lambda, E_i^{(2)} E_j 1_\lambda \langle d \rangle) \leq \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d < 0. \end{cases}$

In (93) equality holds when $d = 2$ if and only if $E_i E_j^{(2)} 1_\lambda$ and $E_j^{(2)} E_i 1_\lambda$ are both nonzero. Likewise in (94) equality holds when $d = 0$ if and only if $E_j^{(2)} E_i 1_\lambda$ is nonzero.

**Proof.** The proof is similar to that of Lemma A.4. For example, suppose $\mu_i + \mu_j \leq 0$ with $\mu_i \leq 0$ (the case $\mu_i \geq 0$ is similar). Then by adjunction we have

$$\text{Hom}(E_i E_j^{(2)} 1_\mu, E_j^{(2)} E_i 1_\mu \langle d \rangle)$$

$$\cong \text{Hom}(E_j^{(2)} 1_\mu, (E_i 1_{\lambda + 2\alpha_i})_R E_j^{(2)} E_i 1_\mu \langle d \rangle)$$

$$\cong \text{Hom}(E_j^{(2)} 1_\mu, E_j^{(2)} E_i 1_\mu \langle d + \mu_i - 1 \rangle)$$

$$\cong \text{Hom}(E_j^{(2)} 1_\mu, E_j^{(2)} E_i 1_\mu \langle -\mu_i \rangle) \bigoplus_{r=0}^{\mu_i - 1} \text{Hom}(E_j^{(2)} 1_\mu, E_j^{(2)} E_i 1_\mu \langle d - 2r - 2 \rangle)$$

$$\cong \text{Hom}(E_j^{(2)} E_i 1_{\mu - \alpha_i} \langle -\mu_i + 1 \rangle, E_j^{(2)} E_i 1_{\mu - \alpha_i} \langle -\mu_i \rangle) \bigoplus_{r=0}^{\mu_i - 1} \text{Hom}(E_j^{(2)} 1_\mu, E_j^{(2)} 1_\mu \langle d - 2r - 2 \rangle).$$

The terms in the direct sum on the right side are all zero by Lemma A.1, unless $r = 0$ and $d = 2$ in which case the term is one dimensional. The left hand term is zero by (94) with $\lambda = \mu - \alpha_i$. Thus (93) when $\lambda = \mu$ follows from (94) when $\lambda = \mu - \alpha_i$. Likewise one can show that (94) follows from (93) as in the proof of Lemma A.4. \qed
Lemma A.10. Suppose $i, j \in I$ with $\langle i, j \rangle = -1$. Then assuming $E_i^{(2)} E_j 1_\lambda, E_j E_i^{(2)} 1_\lambda$ and $E_i E_j E_i 1_\lambda$ are nonzero, the spaces

$$\text{Hom}^d(E_i^{(2)} E_j 1_\lambda, E_i E_j E_i 1_\lambda) \quad \text{and} \quad \text{Hom}^d(E_j E_i^{(2)} 1_\lambda, E_i E_j E_i 1_\lambda)$$

$$\text{Hom}^d(E_i E_j E_i 1_\lambda, E_j^{(2)} E_j 1_\lambda) \quad \text{and} \quad \text{Hom}^d(E_i E_j E_i 1_\lambda, E_j E_i^{(2)} 1_\lambda)$$

are all zero if $d < 0$ and one-dimensional if $d = 0$.

Proof. This follows by the usual adjunction formalism. We will always assume $d \leq 0$. First, if $\lambda_i \leq 0$ then we have

$$\text{Hom}^d(E_i^{(2)} E_j 1_\lambda, E_i E_j E_i 1_\lambda)$$

$$\cong \text{Hom}^d((E_i 1_{\lambda+\alpha_\lambda})_i E_i^{(2)} E_j, E_j E_i 1_\lambda)$$

$$\cong \text{Hom}^d(F_i E_i^{(2)} E_j 1_\lambda, -\langle \lambda_i - 2 \rangle, E_j E_i 1_\lambda)$$

$$\cong \text{Hom}^d(E_i^{(2)} E_j F_i 1_\lambda, E_j E_i 1_\lambda \langle \lambda_i + 2 \rangle) \bigoplus \text{Hom}^d(E_j E_i 1_\lambda, E_j E_i 1_\lambda (\lambda_i + 2))$$

$$\cong \text{Hom}^d(E_j^{(2)} E_j E_j 1_\lambda, E_j E_i 1_\lambda (2 \lambda_i + 1)) \bigoplus \text{Hom}^d(E_i E_j 1_\lambda, E_j E_i 1_\lambda (1 - 2r)).$$

By Lemma A.4 all the terms in the sum on the right are zero except when $d = 0$ and $r = 0$ where the space is one-dimensional. Moreover, by Lemma A.9, the left term is zero if $\lambda_i < 0$ or $d < 0$. The case $\lambda_i = 0$ is special since the sum on the right disappears. But now the left hand term is equal to

$$\text{Hom}^d(E_i^{(2)} E_j 1_\lambda, E_j E_i^{(2)} 1_\lambda \langle 2 \rangle \oplus E_j E_i^{(2)} 1_\lambda) \cong \begin{cases} k & \text{if } d = 0 \\ 0 & \text{if } d < 0. \end{cases}$$

Thus, in both cases we get $\dim \text{Hom}^d(E_i^{(2)} E_j 1_\lambda, E_j E_i 1_\lambda) = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d < 0. \end{cases}$

If $\lambda_i \geq 0$ the argument is similar except that the first step is

$$\text{Hom}^d(E_i^{(2)} E_j 1_\lambda, E_i E_j E_i 1_\lambda) \cong \text{Hom}^d(E_i^{(2)} E_j (E_i 1_\lambda) R, E_i E_j 1_{\lambda+\alpha_\lambda}).$$

The other three Hom-space calculations are the same. \qed

Lemma A.11. If $i, j \in I$ with $\langle i, j \rangle = -1$ then

$$\dim \text{End}(E_i E_j E_i 1_\lambda) \leq \dim \text{End}(E_i^{(2)} E_j 1_\lambda) + \dim \text{End}(E_i E_i^{(2)} 1_\lambda).$$

Proof. Part 1. Let us first assume $E_i^{(2)} E_j 1_\lambda$ and $E_j E_i^{(2)} 1_\lambda$ are both nonzero. By Lemma A.9 we need to show that $\dim \text{End}(E_i E_j E_i 1_\lambda) \leq 2$.

Case 1: $\lambda_i < -1$. By adjunction we have

$$\text{End}(E_i E_j E_i 1_\lambda) \cong \text{Hom}((E_i 1_{\lambda+\alpha_\lambda})_i E_i E_i E_i 1_\lambda)$$

$$\cong \text{Hom}(F_i E_i 1_{\lambda+\alpha_\lambda} E_j E_i (-\langle \lambda_i - 2 \rangle), E_j E_i 1_\lambda)$$

$$\cong \text{Hom}(E_j E_j F_i 1_\lambda, E_j E_i 1_\lambda \langle \lambda_i + 2 \rangle) \bigoplus \text{Hom}(E_j E_i 1_\lambda, E_j E_i 1_\lambda (\lambda_i + 2)).$$

Then the right hand sum is equal to $\bigoplus_{r=0}^{-\lambda_i - 2} \text{Hom}(E_j E_j 1_\lambda, E_j E_i 1_\lambda (-2r))$. By Lemma A.4, all these terms are zero except when $r = 0$ where it is at most one-dimensional.
On the other hand, the left hand term in (95) equals

\[
\text{Hom}(E_i E_j E_i(1_{\lambda - \alpha_i} F_i), E_j E_i(1_{\lambda}(\lambda_i + 2))) \bigoplus \text{Hom}(E_i E_j E_i(1_{\lambda}(\lambda_i + 2)))
\]

![Math notation](image)

(96) \[ \cong \text{Hom}(E_i E_j E_i(1_{\lambda - \alpha_i}, E_j E_i(1_{\lambda}(2\lambda_i + 1))) \bigoplus \text{Hom}(E_i E_j E_i, E_j E_i(1_{\lambda}(1 - 2r))). \]

Then by Lemma A.10 the left hand term is zero while, by Lemma A.4, all the terms in the right hand sum are zero except when \( r = 0 \) when it is at most one-dimensional. Thus \( \dim \text{End}(E_i E_j E_i(1_{\lambda})) \leq 2 \).

Case 2: \( \lambda_i = -1 \). This is a special case of the argument above. The right hand sum in (95) disappears but now the left hand term in (96) is

\[
\dim \text{Hom}(E_i E_j E_i(1_{\lambda}, E_j E_i(1_{\lambda}(2\lambda_i + 1)) \bigoplus E_j E_i(1_{\lambda})) \leq 1
\]

where we use Lemmas A.9 and A.10 to get this isomorphism. So once again \( \dim \text{End}(E_i E_j E_i(1_{\lambda})) \leq 2 \).

Case 3: \( \lambda_i > -1 \). The argument is the same as in Case 1 except that the first step is

\[
\text{End}(E_i E_j E_i(1_{\lambda})) \cong \text{Hom}(E_i E_j, E_i E_j(1_{\lambda}(1 + \alpha_i)). \]

**Part 2.** It remains to consider the situation when \( E_i^{(2)} E_j 1_{\lambda} \) or \( E_j E_i^{(2)} 1_{\lambda} \) are zero. Suppose \( E_i^{(2)} E_j 1_{\lambda} = 0 \) and \( E_j E_i^{(2)} 1_{\lambda} \neq 0 \) with \( \lambda_i < -1 \) (the other cases are similar). Then the second term in (95) still contributes at most one to the dimension. On the other hand, the second term in (96) is now entirely zero because \( E_i E_j 1_{\lambda} = 0 \). To see that \( E_i E_j 1_{\lambda} = 0 \) we use that \( E_i^{(2)} E_j 1_{\lambda} = 0 \) which means

\[
0 = F_i E_i E_j E_i(1_{\lambda} \cong E_i F_i E_j E_i(1_{\lambda} \bigoplus E_j E_i(1_{\lambda}).
\]

Thus \( \dim \text{End}(E_i E_j E_i(1_{\lambda})) \leq 1 \) which is what we wanted to show. \( \square \)

**Lemma A.12.** Let \( i, j, k \in I \) be distinct. Then

\[
\dim \text{Hom}(E_i E_j E_k 1_{\lambda}, E_k E_j E_i(1_{\lambda}(d))) \leq \begin{cases} 1 & \text{if } d = -\ell_{ijk} \\ 0 & \text{if } d < -\ell_{ijk} \end{cases}
\]

If \( d = -\ell_{ijk} \) then equality holds if and only if \( 1_{\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j + \epsilon_k \alpha_k} \neq 0 \) for \( \epsilon_i, \epsilon_j, \epsilon_k \in \{0, 1\} \).

**Proof.** This computation depends on whether \( i, j, k \in I \) are joined by an edge. The most difficult case is when they are all joined by an edge, meaning \( \langle i, j, k \rangle = 0 \) (this is because in this case \( E_i, E_j \) and \( E_k \) do not commute among each other). We will only deal with this case as the general case is essentially the same (and in fact a bit easier).

By condition (vii) we have \( \ell := \lambda_i + \lambda_j + \lambda_k > 0 \). There are several cases to consider depending on whether \( \lambda_i, \lambda_j, \lambda_k \) are positive or negative.

**Case 1:** \( (\lambda_i, \lambda_j, \lambda_k) = (+, +, -) \) (meaning that \( \lambda_i, \lambda_j \geq 0 \) and \( \lambda_k \leq 0 \)). First, we have

\[
\text{Hom}(E_i E_j E_k 1_{\lambda}, E_k E_j E_i(1_{\lambda}(3))) \cong \text{Hom}(1_{\mu}, E_k E_j E_i(1_{\lambda}(E_k 1_{\lambda} E_j(1_{\lambda}(1 + \alpha_k)))) E_i(1_{\alpha_j + \alpha_k} L(3)))
\]

\[ \cong \text{Hom}(1_{\mu}, E_k E_j E_i F_j F_i(1_{\lambda}(1 + \alpha_i + \alpha_j + \alpha_k))). \]

where \( \mu := \lambda + \alpha_i + \alpha_j + \alpha_k \). Next, we can simplify the right hand term as follows

\[
E_k E_j E_i F_j F_i(1_{\mu}) \cong E_k F_k E_j F_j E_i(1_{\mu}) \bigoplus E_k F_k E_j F_i E_i(1_{\mu}) \bigoplus E_k F_k E_j F_i(1_{\mu}) \bigoplus E_j F_k E_i(1_{\mu}).
\]
Now
\[ \text{Hom}(1_\mu, E_k F_k F_j E_j 1_\mu(-\ell + 3)) \cong \text{Hom}((E_k 1_{\mu-\alpha_k})_L(E_j 1_\mu)_R, F_k F_j 1_{\mu+\alpha_j}(-\ell + 3)) \]
\[ \cong \text{Hom}(F_k F_j 1_{\mu+\alpha_j}, F_k F_j 1_{\mu+\alpha_j}(-\lambda_i - 2\lambda_j + 1)) \]
so by Lemmas A.7 and A.4 we have
\[
\dim \text{Hom}(1_\mu, \bigoplus_{[\lambda_i]} E_k F_k F_j E_j 1_\mu(-\ell + 3)) = \sum_{d=0}^{\lambda_i-1} \dim \text{Hom}(F_k F_j 1_{\mu+\alpha_j}, F_k F_j 1_{\mu+\alpha_j}(-2\lambda_j - 2d)) \\
= \begin{cases} 
1 & \text{if } \lambda_j = 0 \text{ and } 1_{\mu+\alpha_j}, 1_\mu, 1_{\mu-\alpha_k} \text{ are nonzero} \\
0 & \text{if } \lambda_j > 0 \text{ or } \lambda_i = 0.
\end{cases}
\]
(97)

Notice that if \( \lambda_j = 0 \) then \( \mu_j = 0 \) so \( 1_{\mu+\alpha_j} \neq 0 \Leftrightarrow 1_{\mu-\alpha_j} \neq 0 \). Likewise one can show that
\[
\dim \text{Hom}(1_\mu, \bigoplus_{[\lambda_j]} E_k F_k F_j E_j 1_\mu(-\ell + 3)) = \begin{cases} 
1 & \text{if } \lambda_i = 0 \text{ and } 1_{\mu-\alpha_i}, 1_\mu, 1_{\mu-\alpha_k} \text{ are nonzero} \\
0 & \text{if } \lambda_i > 0 \text{ or } \lambda_j = 0.
\end{cases}
\]
(98)

Finally, to compute \( \text{Hom}(1_\mu, E_k F_k F_j F_j E_j E_j 1_\mu(-\ell + 3)) \) we use that
\[
F_k E_k F_k F_j F_j E_j 1_\mu \cong E_k F_k F_k F_j F_j E_j 1_\mu \bigoplus_{[-\lambda_k]} F_j F_j E_j E_j 1_\mu.
\]
Looking at the left hand side we have
\[
\text{Hom}(1_\mu, F_k E_k F_j F_j E_j E_j 1_\mu(-\ell + 3)) \cong \text{Hom}(1_\mu, F_k F_k F_j F_j E_j E_j 1_\mu(-\ell + 3)) \\
\cong \text{Hom}(E_j E_j E_j 1_\mu, E_k E_k E_j E_j 1_\mu(-\ell + 3))
\]
which, by induction, is zero since (by condition (vii)) \( \mu_i + \mu_j + \mu_k = \ell > 0 \) and \( 1_{\lambda_i+\alpha_j+\alpha_k} = 0 \) for \( r > 0 \). This means that \( \text{Hom}(1_\mu, E_k F_k F_j F_j E_j E_j 1_\mu(-\ell + 3)) = 0 \).

Finally, (97), (98) and (99) together end up contributing a total of 1 to the dimension (note that there are several subcases to consider here depending on whether or not \( \lambda_i = 0 \) and \( \lambda_j = 0 \)). Thus \( \dim \text{Hom}(E_j E_j E_k 1_{\lambda}, E_k E_j E_j 1_{\lambda}(3)) \leq 1 \) with equality holding if \( 1_\mu, 1_{\mu-\alpha_i}, 1_{\mu-\alpha_j}, 1_{\mu-\alpha_k} \) are all nonzero. This is precisely what we needed to prove.

**Case 2:** \( (\lambda_i, \lambda_j, \lambda_k) = (-, +, -) \). Arguing as above we have
\[
\text{Hom}(E_j E_j E_k 1_{\lambda}, E_k E_j E_j 1_{\lambda}(3)) \cong \text{Hom}(1_\mu, E_k E_j E_j F_k F_j F_j F_j 1_\mu(-\ell + 3)) \\
\cong \text{Hom}(1_\mu, E_k F_k F_j F_j F_j E_j 1_\mu(-\ell + 3) \bigoplus_{[\mu_j]} E_k F_k E_j E_j 1_\mu(-\ell + 3)).
\]

On the one hand, by adjunction and Lemma A.4 it follows that
\[
\dim \text{Hom}(1_\mu, \bigoplus_{[\mu_j]} E_k F_k E_j E_j 1_\mu(-\ell + 3)) = \begin{cases} 
1 & \text{if } \lambda_j > 0 \text{ and } 1_\mu, 1_{\mu-\alpha_i}, 1_{\mu-\alpha_j}, 1_{\mu-\alpha_k} \text{ are nonzero} \\
0 & \text{if } \lambda_j = 0.
\end{cases}
\]
On the other hand, \( E_k F_k F_j E_j E_j 1_\mu \) is a direct summand of \( F_k E_k F_j F_j E_j E_j 1_\mu \) and
\[
\text{Hom}(1_\mu, F_k E_k F_j F_j E_j E_j 1_\mu(-\ell + 3)) \cong \text{Hom}(E_j E_j E_k 1_{\lambda}, E_k E_j E_j 1_{\lambda}(3)) \cong \text{Hom}(E_j E_j E_k 1_{\lambda}, E_k E_j E_j 1_{\lambda}(3)).
\]
By induction this is zero, which means that $\text{Hom}(1_\mu, E_k F_k E_j F_j E_i F_i 1_\mu (-\ell + 3)) = 0$. It follows that $\dim \text{Hom}(E_i E_j E_k 1_\lambda, E_k E_j E_i 1_\lambda(3)) \leq 1$ with equality holding if $1_\mu, 1_{\mu - \alpha_i}, 1_{\mu - \alpha_j}, 1_{\mu - \alpha_k}$ are nonzero. This is again precisely what we wanted to prove.

**Case 3**: $(\lambda_i, \lambda_j, \lambda_k) = (+, -, +)$. First we have

$$E_k E_j E_F F_j F_i 1_\mu \cong F_k E_k E_j F_j F_i E_i 1_\mu \bigoplus_{[\lambda_k]} E_j F_j E_i 1_\mu \bigoplus_{[\lambda_i]} F_k E_k E_j F_j F_i 1_\mu \bigoplus_{[\lambda_i][\lambda_k]} E_j F_j 1_\mu.$$ 

Then, arguing as in case 1, we get

$$\dim \text{Hom}(1_\mu, F_k E_k E_j F_j F_i 1_\mu (-\ell + 3)) = 0$$

and

$$\dim \text{Hom}(1_\mu, \bigoplus_{[\lambda_k]} E_j F_j E_i 1_\mu (-\ell + 3)) = \begin{cases} 1 & \text{if } \lambda_i = 0 \text{ and } 1_{\mu - \alpha_i}, 1_{\mu - \alpha_j}, 1_{\mu - \alpha_k} \text{ are nonzero} \\
0 & \text{if } \lambda_i > 0 \text{ or } \lambda_k = 0 \end{cases}$$

$$\dim \text{Hom}(1_\mu, \bigoplus_{[\lambda_i][\lambda_k]} E_j F_j 1_\mu (-\ell + 3)) = \begin{cases} 1 & \text{if } \lambda_i, \lambda_k > 0 \text{ and } 1_{\mu - \alpha_j} \text{ are nonzero} \\
0 & \text{if } \lambda_i = 0 \text{ or } \lambda_k = 0. \end{cases}$$

Thus we get that

$$\dim \text{Hom}(E_i E_j E_k 1_\lambda, E_k E_j E_i 1_\lambda(3)) = \dim \text{Hom}(1_\mu, E_k E_j E_i F_k F_j F_i 1_\mu (-\ell + 3)) \leq 1$$

with equality holding if $1_\mu, 1_{\mu - \alpha_i}, 1_{\mu - \alpha_j}, 1_{\mu - \alpha_k}$ are nonzero.

**Other cases.** There are four other cases, namely $(\lambda_i, \lambda_j, \lambda_k)$ equal to $(-, +, +), (+, -, +), (-, -, +)$ and $(+, +, +)$. The first is a consequence of Case 1 by symmetry while the others follow using the same arguments as above.

**The converse.** Suppose $\text{Hom}(E_i E_j E_k 1_\lambda, E_k E_j E_i 1_\lambda) \neq 0$. Then $E_i E_j E_k 1_\lambda \neq 0$ which means $1_\lambda, 1_{\lambda + \alpha_k}, 1_{\lambda + \alpha_j + \alpha_k}, 1_{\lambda + \alpha_i + \alpha_j + \alpha_k}$ are all nonzero. Similarly, $E_k E_j E_i 1_\lambda \neq 0$ means that $1_{\lambda + \alpha_i}, 1_{\lambda + \alpha_i + \alpha_j}$ are also nonzero. It remains to show that $1_{\lambda + \alpha_j}$ and $1_{\lambda + \alpha_i + \alpha_k}$ are nonzero. This follows from condition (viii).

**Corollary A.13.** Let $i, j, k \in I$ be distinct. Then

$$\text{Hom}(F_k E_i E_j 1_\lambda, E_j E_i F_k 1_\lambda(3)) \leq \begin{cases} 1 & \text{if } d = -\langle i, j \rangle \\
0 & \text{if } d < -\langle i, j \rangle \end{cases}$$

and when $d = -\langle i, j \rangle$ equality holds if and only if $1_{\lambda + \epsilon_i \alpha_i + \epsilon_j \alpha_j - \epsilon_k \alpha_k}$ for $\epsilon_i, \epsilon_j, \epsilon_k \in \{0, 1\}$.

**Proof.** By adjunction

$$\text{Hom}(F_k E_i E_j 1_\lambda, E_j E_i F_k 1_\lambda(3)) \cong \text{Hom}(E_i E_j (1_{\lambda - \alpha_k} F_k) R, (1_{\lambda + \alpha_i + \alpha_j - \alpha_k} F_k) R E_j E_i(3))$$

$$\cong \text{Hom}(E_i E_j E_k 1_{\lambda - \alpha_k} (-\lambda_k + 2 - 1), E_k E_j E_i 1_{\lambda - \alpha_k}(d - \lambda_k - \langle i, k \rangle - \langle j, k \rangle + 2 - 1)).$$

The result now follows from Lemma A.12.

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