The Uniformization of Certain Algebraic Hypergeometric Functions

Robert S. Maier

Depts. of Mathematics and Physics, University of Arizona, Tucson, AZ 85721, USA

Abstract

The hypergeometric functions $\,_{n}F_{n-1}$ are higher transcendental functions, but for certain parameter values they become algebraic. This occurs, e.g., if the defining hypergeometric differential equation has irreducible but imprimitive monodromy. It is shown that many algebraic $\,_{n}F_{n-1}$’s of this type can be represented as combinations of certain explicitly algebraic functions of a single variable, i.e., the roots of trinomial equations. This generalizes a result of Birkeland. Any tuple of roots of a trinomial equation traces out a projective algebraic curve, and it is determined when this curve is of genus zero, i.e., admits a rational parametrization. Any such parametrization yields a hypergeometric identity that explicitly uniformizes a family of algebraic $\,_{n}F_{n-1}$’s. Even if the governing curve is of positive genus, it is shown how it may be possible to construct single-valued or multivalued parametrizations of individual algebraic $\,_{n}F_{n-1}$’s, by computation in rings of symmetric polynomials.

Key words: hypergeometric function, algebraic function, imprimitive monodromy, trinomial equation, binomial coefficient identity, algebraic geometry, projective algebraic curve, Belyi cover, uniformization, symmetric function, symmetric polynomial

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1. Introduction

The generalized hypergeometric functions $\,_{n}F_{n-1}(\zeta), n \geq 1$, are parametrized special functions of fundamental importance. Each $\,_{n}F_{n-1}(\zeta)$ is a function of a single complex variable, and in general, is a higher transcendental function. It is parametrized by complex numbers $a_1, \ldots, a_n; b_1, \ldots, b_{n-1}$, and is written as $\,_{n}F_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; \zeta)$. It is analytic on $|\zeta| < 1$, with definition

$$\,_{n}F_{n-1}\left( \begin{array}{c} a_1, \ldots, a_n \\ b_1, \ldots, b_{n-1} \end{array} \mid \zeta \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_n)_k}{(b_1)_k \cdots (b_{n-1})_k (1)_k} \zeta^k. \quad (1.1)$$
Here \((c)_k := c(c+1)\cdots(c+k-1)\), and the lower parameters \(b_1, \ldots, b_{n-1}\) may not be non-positive integers. The \(n=1\) function \(1F_0(a_1; \zeta)\) equals \((1-\zeta)^{-a_1}\), and the \(n=2\) function \(2F_1(a_1, a_2; b_1; \zeta)\) is the Gauss hypergeometric function.

If the parameters are suitably chosen, then \(nF_{n-1}(\zeta)\) will become an algebraic function of \(\zeta\). Equivalently, if one regards \(nF_{n-1}\) as a single-valued function on a Riemann surface, defined by continuing from the open unit disk, then the Riemann surface will become compact. If \(n=1\), this phenomenon will occur when \(a_1 \in \mathbb{Q}\). In the less trivial case \(n=2\), the characterization of the triples \((a_1, a_2; b_1)\) for which \(2F_1(a_1, a_2; b_1; \zeta)\) is algebraic is a classical result of Schwarz. There is a finite list of possible normalized triples (the famous ‘Schwarz list’), and \(2F_1\) is algebraic iff \((a_1, a_2; b_1)\) is a denormalized version of a triple on the list. Denormalization may involve integer displacements of the parameters. For specifics, see [6, §2.7.2].

More recently, Beukers and Heckman [3] treated \(n \geq 3\), and obtained a complete characterization of the parameters \((a_1, \ldots, a_n; b_1, \ldots, b_{n-1})\) that yield algebraicity. Like the \(n=2\) result of Schwarz, their result was based on the fact that the function \(nF_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; \zeta)\) satisfies an order-\(n\) differential equation on the Riemann sphere \(\mathbb{P}^1_\zeta\), called \(E_n(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}, 1; \zeta)\) here. In modern language, \(E_n\) specifies a flat connection on an \(n\)-dimensional vector bundle over \(\mathbb{P}^1_\zeta\). It has singular points at \(\zeta = 0, 1, \infty\), and its (projective) monodromy group is generated by loops about these three points. The monodromy, resp. projective monodromy group is a subgroup of \(GL_n(\mathbb{C})\), resp. \(PGL_n(\mathbb{C})\), and algebraicity occurs iff the monodromy is finite. Schwarz exploited the classification of the finite subgroups of \(PGL_2(\mathbb{C})\). In a tour de force, Beukers and Heckman handled the \(n \geq 3\) case by exploiting the Shephard–Todd classification of the finite subgroups of \(GL_n(\mathbb{C})\) generated by complex reflections. Their characterization result, however, is non-constructive: it supplies an algorithm for determining whether a given function \(nF_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; \zeta)\) is algebraic in \(\zeta\), but it does not yield a polynomial equation (with coefficients polynomial in \(\zeta\)) satisfied by the function.

In this article, we take a new approach to the problem of constructing equations satisfied by algebraic \(nF_{n-1}\)’s. We succeed in making explicit several classes of generalized hypergeometric function, known to be algebraic by Beukers–Heckman, by uniformizing them. Recall that an algebraic function \(F\) may be of genus zero, i.e., may have a ‘uniformization,’ or parametrization, by rational functions. That is, one may have \(F(R_1(t)) = R_2(t)\), for certain rational functions \(R_1, R_2\); which says that the Riemann surface of \(F = F(\zeta)\), on which \(\zeta\) and \(F\) are single-valued meromorphic functions, is isomorphic to the Riemann sphere \(\mathbb{P}^1_\zeta\). A sample result of ours is the following. Let \(n = p+2\), where \(p \geq 1\) is odd. Then

\[
nF_{n-1} \left( \begin{array}{c}
\frac{a_1}{p+1}, \ldots, \frac{a+(n-1)}{p+1} \\
\frac{a_1}{p+1}, \ldots, \frac{a_{p+1}}{p+1}
\end{array} \right) \mid \frac{4n^n t^2(1-t^2)^{2p} [(1+t)^p + (1-t)^p]^p}{[(1+t)^n + (1-t)^n]^n} = \frac{1}{2} \left[ (1+t)^{-2a} + (1-t)^{-2a} \right] \left[ (1+t)^n + (1-t)^n \right]^a / (1+t)^p + (1-t)^p). \tag{1.2}
\]

2
Eq. (1.2) holds as an equality for all \( t \) in a neighborhood of \( t = 0 \); or equivalently, if one expands about \( t = 0 \), as an equality between power series. It is a hypergeometric identity, which is more general than a uniformization of a fixed \( F \). This is because \( a \in \mathbb{C} \) is arbitrary. (If any lower hypergeometric parameter is a non-positive integer, the identity must be interpreted in a limiting sense.) If \( a \in \mathbb{Q} \), it follows that \( nF_{n-1}(\frac{a}{n}, \ldots, \frac{a+(n-1)}{n}, \frac{a+1}{p}, \ldots, \frac{a+p}{p}, \frac{1}{2}; \zeta) \) is algebraic in \( \zeta \). If, moreover, \( a \in \mathbb{Z} \), in which case (1.2) is a uniformization, then it must be of genus zero.

Most of the algebraic \( nF_{n-1} \)'s that we uniformize are of the following type. The \( nF_{n-1} \) in (1.2) is simply the \( q = 2 \) case of

\[
nF_{n-1} \left( \frac{\alpha}{n+1}, \ldots, \frac{\alpha+(n-1)}{n+1}, \frac{\alpha+1}{p}, \ldots, \frac{\alpha+p}{p}, \frac{1}{q}, \ldots, \frac{q-1}{q} \; \zeta \right),
\]

where \( n = p+q \). The order-\( n \) differential equation \( E_n \) on \( \mathbb{P}_1(\zeta) \) associated to (1.3),

\[
E_n \left( \frac{\alpha}{n+1}, \ldots, \frac{\alpha+(n-1)}{n+1}, \frac{\alpha+1}{p}, \ldots, \frac{\alpha+p}{p}, \frac{1}{q}, \ldots, \frac{q-1}{q}, 1 \right) \zeta,
\]

plays a role in the Beukers–Heckman analysis of algebraicity (see §2). Our treatment of (1.3) and (1.4) is connected to the following fact. If \( \gcd(p, q) = 1 \), the solution space of (1.4) is spanned by \( x_1^{\gamma}(\zeta), \ldots, x_n^{\gamma}(\zeta) \), where \( \gamma := -qa \) and the \( n \) algebraic functions \( x_1(\zeta), \ldots, x_n(\zeta) \) are the roots of the trinomial equation

\[
x^n - g x^p - \beta = 0,
\]

in which \( g \neq 0 \) is fixed and \( \beta \) is determined implicitly by

\[
\zeta = (-)^q \frac{n^n \beta^q}{p^p q^q \; g^n}.
\]

(This assumes \( \gamma \notin \mathbb{Z} \).) Formulas relating such hypergeometric functions as (1.3) to the solutions of trinomial equations have been known since the eighteenth century, and can be traced to a 1758 result of Lambert [2, p. 72]. They can be derived with the aid of Lagrange inversion, and especially useful formulas were developed by Birkeland in the 1920s (see [4], and [1, 19] for general reviews). He expressed the function (1.3) as a combination of only \( q \) of the \( n \) functions \( x_1^{\gamma}(\zeta), \ldots, x_n^{\gamma}(\zeta) \). Using not Lagrange inversion (explicitly) but rather the Vandermonde convolution transform of Gould [10], we extend Birkeland's formulas to handle modified versions of (1.3), in which the so-called parametric excess \( S = \sum_{i=1}^{n-1} b_i - \sum_{i=1}^n a_i \) does not equal \( 1/2 \), as it does in (1.3), but rather, is an arbitrary half-odd-integer.

The heart of this article is a study of the trinomial equation (1.5), focused on basic algebraic geometry rather than on controlling or bounding the location of its roots. (For the latter, see [7] and papers cited therein.) Following Kato and Noumi [14], for coprime \( p, q \geq 1 \) (with \( n := p + q \)) we define a complex projective curve \( C_{p, q}^{(n)} \subseteq \mathbb{P}^{n-1} \) comprising all \( [x_1: \ldots: x_n] \in \mathbb{P}^{n-1} \) such that
$x_1, \ldots, x_n$ are the roots of some equation of the form (1.5). That is, $C_{p,q}^n$ is the common zero set of the $n - 2$ elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{q-1}$ and $\sigma_{q+1}, \ldots, \sigma_{n-1}$ in $x_1, \ldots, x_n$. Clearly $g = (-)^{n-1} \sigma_q$ and $\beta = (-)^{n-1} \sigma_n$, giving a formula for $\zeta$; so one has a degree-$n$ covering $\phi_{p,q}^n \rightarrow P_1(\zeta)$. We call $\phi_{p,q}^n$ a Schwarz curve. It is a key result [13, Cor. 4.7] that this curve is irreducible.

For each $k, n-1 \geq k \geq 2$, we define a subsidiary Schwarz curve $C_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, also irreducible, that includes all $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ such that $x_1, \ldots, x_k$ are $k$ of the $n$ roots of some equation of the form (1.5). The curve $C_{p,q}^{(2)}$ is $\mathbb{P}^1$ itself, and one can introduce a final curve $\phi_{p,q}^{(1)}$, also isomorphic to $\mathbb{P}^1$. These curves are joined by maps $\phi_{p,q}^{(k)}: C_{p,q}^{(k)} \rightarrow C_{p,q}^{(k-1)}$ with $\deg \phi_{p,q}^{(k)} = n - k + 1$, i.e.,

$$\phi_{p,q}^{(n)} \phi_{x_1}^{(n-1)} \phi_{x_2}^{(n-2)} \cdots \phi_{x_{k-1}}^{(1)} \phi_{p,q}^{(1)} \rightarrow \mathbb{P}^1(\zeta). \tag{1.7}$$

Each covering $\phi_{p,q}^{(k)} \rightarrow \mathbb{P}^1(\zeta)$ is a Bely˘ı map, i.e., is ramified only over $\zeta = 0, 1, \infty$.

The Schwarz curves are not, in general, of genus zero, but $\phi_{p,q}^{(2)}$ and $\phi_{p,q}^{(1)}$ always are. One can write $\phi_{p,q}^{(2)} \simeq \mathbb{P}^1(t)$ and $\phi_{p,q}^{(1)} \simeq \mathbb{P}^1(s)$, where $t$ and $s$ are rational parameters. It follows immediately from the Birkeland formulas that the $nF_{n-1}(\zeta)$ of (1.3) can be parametrized by $t$ or $s$, when $q = 2$ or $q = 1$ respectively; and $\zeta$ is respectively a degree-$[n(n - 1)]$ rational function of $t$, and a degree-$n$ rational function of $s$. The former is, in fact, how the sample identity (1.2) arises. Its right side is the sum of two terms, proportional to $(1 + t)^{-2a}, (1 - t)^{-2a}$, and these come from $x_1^2, x_2^2$.

Using this technique and a supplementary one, namely computation in rings of symmetric polynomials, we can parametrize many algebraic $nF_{n-1}$’s, including ones governed by Schwarz curves of positive genus. We give many examples.

This article is arranged as follows. In §2 the differential equations $E_n$ are introduced and their monodromy is discussed. In §3 and 4 formulas of Birkeland type for various algebraic $nF_{n-1}$’s are derived with the aid of the Vandermonde convolution transform. The key results are Thms. 4.4 and 4.6 which extend Birkeland’s representations. In §5 and 6 the Schwarz curves are introduced and studied. Their ramification structures and genera are computed, and the curves of genus zero are determined. (For instance, Thm. 5.3 generalizes a recent theorem of [13, 4].) In §7, many hypergeometric identities with a free parameter $a \in \mathbb{C}$ are derived, including uniformizations of algebraic $nF_{n-1}$’s of genus zero. A typical result is Thm. 7.3 of which Eq. 1.2 above is a special case. In §8, the just-mentioned supplementary technique is introduced and applied, yielding many parametrizations in which $a \in \mathbb{Q}$ is fixed. Some of these are multivalued parametrizations with radicals.

2. Monodromy and degeneracy

The parametrized functions $nF_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; \zeta)$ are locally defined by Eq. (1.1). Two sorts of degeneracy are of interest, for the functions themselves, and for the equations that they satisfy. The first is when an upper
and a lower parameter equal each other. If so, they may be ‘cancelled’, reducing the \( a F_{n-1} \) to an \( n-1 F_{n-2} \). The second arises when one of the lower parameters is taken to an integer \(-m\), and one of the upper ones to an integer \(-m'\), with \(-m \leq -m' \leq 0\). Let \([\zeta^{m'}]F\) be the sum of the terms proportional to \( \zeta^k \), \( k > m' \), in the series for \( F \); and let \((a)\) and \((b)\) signify \((a_1, \ldots, a_{n-1})\) and \((b_1, \ldots, b_{n-2})\). The following lemma permits such hypergeometric identities as \( \text{(1.2)} \) to be interpreted in a limiting sense.

**Lemma 2.1.** Let \( a_n \to -m', b_{n-1} \to -m \), with \( (a_n + m')/(b_{n-1} + m) \to \alpha \). Then

\[
[\zeta^{m'}]_{n} F_{n-1} \left( \frac{(a)}{(b), b_{n-1}} \bigg| \zeta \right) \longrightarrow \\
\alpha (-)^{m-m'} \binom{m}{m'}^{-1} \frac{(a)_{m+1}}{(b)_{m+1}(1)_{m+1}} \zeta^{m+1} n F_{n-1} \left( \frac{(a) + m + 1, m - m' + 1}{(b) + m + 1, m + 2} \bigg| \zeta \right).
\]

If \( m = m' \), this limit simplifies to

\[
\alpha \cdot [\zeta^{m'}]_{n-1} F_{n-2} \left( \frac{(a)}{(b)} \bigg| \zeta \right).
\]

**Proof.** By taking the term-by-term limit of the series in \((1.1)\). \( \square \)

It is well known that the function \( n F_{n-1} (a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; \zeta) \), defined on the disk \( |\zeta| < 1 \), satisfies the \( b_n = 1 \) case of the order-\( n \) equation \( D_n F = 0 \), where

\[
D_n = D_n (a_1, \ldots, a_n; b_1, \ldots, b_n) \\
= (\zeta D_{\zeta} + b_1 - 1) \cdots (\zeta D_{\zeta} + b_n - 1) - \zeta (\zeta D_{\zeta} + a_1) \cdots (\zeta D_{\zeta} + a_n).
\]

The equation \( D_n F = 0 \) is denoted by \( E_n (a_1, \ldots, a_n; b_1, \ldots, b_n) \) here; a subscript \( \zeta \) will be added as appropriate to indicate the independent variable. This equation has three regular singular points on \( \mathbb{P}^1_\zeta \), namely \( \zeta = 0, 1, \infty \), with respective characteristic exponents \( 1 - b_1, 1 - b_n \); and \( 0, 1, \ldots, n - 2, S; \) and \( a_1, \ldots, a_n \). In this, \( S = \left( \sum_{i=1}^{n} b_i \right) - \left( \sum_{i=1}^{n} a_i \right) - 1 \)

(2.2)

is the ‘parametric excess,’ which reduces to \( \sum_{i=1}^{n-1} b_i - \sum_{i=1}^{n} a_i \) for \( n F_{n-1} \). The canonical solution \( n F_{n-1} \), defined if none of \( b_1, \ldots, b_{n-1} \) is a non-positive integer, is analytic at \( \zeta = 0 \) and belongs to the exponent \( 1 - b_n = 0 \). Allowing \( b_n \) to differ from unity in \( E_n \) is not a major matter, since adding \( \delta \) to each parameter of \( E_n \) merely multiplies all solutions by \( \zeta^{-\delta} \). The following is a standard fact.

**Lemma 2.2.** If \( b_j - b_j' \notin \mathbb{Z}, \forall j, j' \), so that the singular point \( \zeta = 0 \) is non-logarithmic, then the solution space of \( E_n \) at \( \zeta = 0 \) is spanned by the functions

\[
[\zeta^{1-b_j}]_{n} F_{n-1} \left( \frac{a_1 - b_j + 1, \ldots, a_n - b_j + 1}{b_1 - b_j + 1, \ldots, b_j - b_j + 1, \ldots, b_n - b_j + 1} \bigg| \zeta \right), \\
j = 1, \ldots, n.
\]
The equation $E_n$ on $\mathbb{P}^1_\mathbb{C}$ is the canonical order-$n$ one with three regular singular points, the monodromy at one of them being special. This is seen as follows. Associated to any $E_n$ is a monodromy representation of $\pi_1(X, \zeta_0)$, the fundamental group of the triply punctured sphere $X = \mathbb{P}^1_\mathbb{C} \setminus \{0, 1, \infty\}$. (The base point $\zeta_0$ plays no role, up to a similarity transformation.) The image $H$ of $\pi_1(X, \zeta_0)$ in $GL_n(\mathbb{C})$ is the monodromy group of the specified $E_n$. It is generated by $h_0, h_1, h_\infty$, the monodromy matrices around $\zeta = 0, 1, \infty$, which satisfy $h_\infty h_1 h_0 = I$. Their eigenvalues are exponentiated characteristic exponents, so they have characteristic polynomials $p, q$ prime

$$\alpha = e^{2\pi i\alpha_j}; \beta_j = e^{2\pi i\beta_j}.$$ Moreover, if $S \notin \mathbb{Z}$ then $h_1$ is diagonalizable. If exactly one upper and one lower parameter of $n F_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; \zeta)$ are equal, permitting cancellation, then the corresponding $E_n$ will have reducible monodromy. One can show that in this case, the differential operator $\mathcal{D}_n$ will have an order-$(n - 1)$ right factor, which comes from the $n F_{n-1}$ to which the $n F_{n-1}$ reduces. In the theory of the Gauss function $2 F_1$, the reducible case, when $a_1$ or $a_2$ differs by an integer from $b_1$ or $b_2 = 1$, is often called the ‘degenerate case’ [6, § 2.2]. Reducibility of $E_2$ facilitates the explicit construction of solutions [14].

A monodromy group $H < GL_n(\mathbb{C})$ of an equation $E_n$, if irreducible, is said to be imprimitive if there is a direct sum decomposition $V_1 \oplus \cdots \oplus V_d$ of $\mathbb{C}^n$ with $\dim V_j \geq 1$ and $d \geq 2$, such that each element of $H$ permutes the spaces $V_j$. The following is a key theorem of Beukers and Heckman. It refers to the complex reflection subgroup $H_r$ of $H$, which is generated by the elements $h_k^+ \cdot h^- \cdot h^-; k \in \mathbb{Z}$. A ‘scalar shift’ by $\delta \in \mathbb{C}$ means the addition of $\delta$ to each parameter.

**Theorem 2.3** (cf. [3], Thm. 5.8). Suppose that $H, H_r$, computed from $E_n$, are irreducible in $GL_n(\mathbb{C})$. Then $H$ will be imprimitive iff there exist relatively prime $p, q \geq 1$ with $p + q = n$, and $a \in \mathbb{C}$, such that $E_n$ takes on one of the two forms

$$E_n \left( \frac{a}{n+1}, \ldots, \frac{a+(a-1)}{p}, \frac{1}{q}, \ldots, \frac{q}{q} \right), \quad E_n \left( \frac{-a}{p}, \ldots, \frac{-a-(p-1)}{p}, \frac{a}{q}, \ldots, \frac{a+(q-1)}{q} \right),$$

up to integer displacements of parameters, and up to a scalar shift by some $\delta \in \mathbb{C}$. (For irreducibility, one must have that $qa \notin \mathbb{Z}$, resp. $na \notin \mathbb{Z}$; this
condition ensures that \(a_j - b_j, j \in \mathbb{Z}, \forall j, j'.\) Moreover, if \(a \in \mathbb{Q}\), then \(H\) will be finite.

This theorem yields a class of algebraic \(n\) 's, including (1.2) and (1.3). One must choose the shift \(\delta\) so that one of the lower parameters equals 1; e.g., \(\delta = 0\). Beukers and Heckman do not compute the monodromy group \(H\) of the two \(E_n\)'s in the theorem, but it follows readily from their proof that if, e.g., \(a = -1/mq\), resp. \(a = -1/nn\), with \(m \geq 2\), then \(H\) is of order \(m^{n-1}n!\), and is isomorphic to the ‘symmetric’ index-\(m\) subgroup of the wreath product \(C_m \wr \mathfrak{S}_n\), where \(C_m\) and \(\mathfrak{S}_n\) are the usual cyclic and symmetric groups. So is the corresponding projective monodromy group \(\mathfrak{P} < \text{PGL}_n(\mathbb{C})\) (see [13, Cor. 4.6]).

If \(a \in \mathbb{Z}\) then Thm. 2.3 does not apply, due to reducibility: in each \(E_n\) an upper and a lower parameter will differ by an integer. But one has the following.

**Theorem 2.4** (cf. [3], Prop. 5.9). If there are equal upper and lower parameters in either \(E_n\) of Theorem 2.3, and \(a \in \mathbb{Z}\) (e.g., if \(a = \pm 1\)), then the \(E_{n-1}\) obtained by cancelling them will have monodromy group \(H < \text{GL}_{n-1}(\mathbb{C})\) isomorphic to \(\mathfrak{S}_n\).

This yields a class of algebraic \(n-1\)'s. The hypergeometric functions appearing in the following sections will include algebraic \(n\)'s of the type arising from Thm. 2.3 but for certain choices of parameter, they will reduce to \(n-1\)'s. On the power series level, Lemma 2.1 will perform some of these reductions.

### 3. Binomial coefficient identities

In this section and §4, we show that the solution spaces of hypergeometric equations \(E_n\) of the types introduced in Thm. 2.3 with imprimitive monodromy, include algebraic functions that are solutions of trinomial equations. Explicit expressions for the associated \(n\)'s as combinations of algebraic functions will be derived.

The natural tool is the Vandermonde convolution transform of Gould [9, 10], from which one can derive many binomial coefficient identities. In what follows, \(\binom{a}{r} := \frac{(a - r + 1)}{r!}\) (3.1) extends the binomial coefficient to the case of arbitrary upper parameter, and
\[
(n)_r := \begin{cases} 
(a) \cdots (a + r - 1), & r \geq 0; \\
[(a - s) \cdots (a - 1)]^{-1}, & r = -s \leq 0,
\end{cases}
\]
(3.2) extends the usual rising factorial, so that \((a)_r = [(a + r)_r]^{-1}\) for all \(r \in \mathbb{Z}\). The standard forward finite difference operator \(\Delta_{A,B}\), which acts on functions of \(A\), is defined by \(\Delta_{A,B}[h](A) = h(A + B) - h(A)\). Its \(n\)'th power \(\Delta^n_{A,B}\) satisfies
\[
\Delta^n_{A,B}[h](A) = \sum_{k=0}^{n} (-)^{n-k} \binom{n}{k} h(A + kB).
\]
(3.3)
Theorem 3.1 ([10]). Let $A, B \in \mathbb{C}$ be given. Let $f(k), k \geq 0$, be an infinite sequence of complex numbers, and let $\hat{f}(n), n \geq 0$, be defined by

$$\hat{f}(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{A + Bk}{n} f(k).$$  

(3.4)

In a neighborhood of $z = 0$, let $y$ near 1 be defined implicitly by $y - 1 - zy^B = 0$. Then

$$\sum_{k=0}^{\infty} \binom{A + Bk}{k} f(k) z^k = y^A \sum_{n=0}^{\infty} \hat{f}(n) \left(\frac{1 - y}{y}\right)^n$$

(3.5)

holds as an equality between power series in $z$; and hence, if either has a positive radius of convergence, as an equality in a neighborhood of $z = 0$.

If the theorem applies, $\hat{f}$ is said to be the $(A, B)$-Vandermonde convolution transform of $f$. If $f(0) = 1$, then $\hat{f}(0) = 1$ also.

Definition 3.2. For each $\ell \in \mathbb{Z}$, the sequence $f_{\ell}(A, B; k), k \geq 0$, is defined by

$$f_{\ell}(A, B; k) := \frac{(A + Bk + 1)_{\ell-1}}{(A + 1)_{\ell-1}} = \frac{(A + \ell - 1)}{(A + Bk + \ell)_{1-\ell}}.$$  

(3.6)

A normalization factor is included, so that $f_{\ell}(A, B; 0) = 1$.

The $(A, B)$-Vandermonde convolution transform of $f_{\ell}(A, B; k)$, defined by Eq. (3.4), has the representation

$$\hat{f}_{\ell}(A, B; n) = \frac{(-)^n}{(A + 1)_{\ell-1}} \Delta_{A,B}^{n} \left[\binom{A}{n} (A + 1)_{\ell-1}\right]$$

$$= \frac{(-)^n}{n!(A + 1)_{\ell-1}} \Delta_{A,B}^{n} [(A - n + 1)_{n+\ell-1}]$$

(3.7)

Here and in subsequent equations and identities, parameters such as $A, B$ are constrained, sometimes tacitly, so that division by zero occurs on neither side.

Theorem 3.3. (i) For each $\ell = -m \leq 0$, $\hat{f}_{\ell}(A, B; n)$ is rational in $A, B$ for each $n$, and is nonzero only if $n = 0, \ldots, m$. (ii) For each $\ell \geq 1$, $\hat{f}_{\ell}(A, B; n)$ equals a degree-$(\ell - 1)$ polynomial in $n$ with coefficients polynomial in $A, B$, multiplied by the function $(n + 1)_{\ell-1}(-B)^n/(A + 1)_{\ell-1}$.

Proof. Part (i) follows from the fact that if $n \geq m + 1$, then $(A - n + 1)_{n-m-1}$, appearing in (3.7), is a polynomial in $A$ of degree $n - m - 1$; hence its $n$th finite difference must equal zero. Part (ii) comes from an elementary finite difference computation (cf. [11, §76]). Explicitly, $\hat{f}_{\ell}$ equals $(-B)^n/(A + 1)_{\ell-1}$ times

$$\sum_{i=0}^{\ell-1} \sum_{j=0}^{i} s(n + \ell - 1, n + i) S(n + i, n + j) (n + 1)_{j} B^{i} \binom{(A + \ell - 1)/B}{j},$$

(3.8)

where $s, S$ are the Stirling numbers of the first and second kinds. But $s(\nu, \nu - \rho)$, $S(\nu, \nu - \rho)$ are degree-$2\rho$ polynomials in $\nu$ that are divisible by $(\nu - \rho + 1)_\rho$, so each summand is of degree at most $2(\ell - 1)$ in $n$ and is divisible by $(n + 1)_{\ell-1}$. □
As examples of transformed sequences \( \hat{f}_\ell(A, B; n) \), one has

\[
\begin{align*}
\hat{f}_{-1}(A, B; n) &= \delta_{n,0} + \frac{AB}{A + B - 1} \delta_{n,1}, \\
\hat{f}_{0}(A, B; n) &= \delta_{n,0}, \\
\hat{f}_{1}(A, B; n) &= (-B)^n, \\
\hat{f}_{2}(A, B; n) &= [(A + 1) + \frac{1}{2}(B - 1)n] \frac{(n+1)(-B)^n}{A + 1}. 
\end{align*}
\] (3.9a)

These follow readily from the formula (3.7). For later reference, note that

\[
\hat{f}_2(A, B; n) + B \hat{f}_2(A, B; n - 1) = \left[ (A + 1) + \frac{1}{2}(B - 1)n \right] \frac{(n+1)(-B)^n}{A + 1}. 
\] (3.10)

**Theorem 3.4.** For each \( \ell \in \mathbb{Z} \), there is a rational function of \( y, F_\ell(A, B; y) \), with coefficients polynomial in \( A, B \), such that

\[
y^A F_\ell(A, B; y) = 1 + \sum_{k=1}^{\infty} \frac{(A + Bk + 1)_{\ell-1}}{(A + 1)_{\ell-1}} \binom{A + Bk}{k} z^k 
\] (3.11)

in a neighborhood of \( z = 0 \), if \( y \) near 1 is defined by \( y - 1 - z y^B = 0 \). Specifically,

\[
F_\ell(A, B; y) = \sum_{n=0}^{\infty} \hat{f}_\ell(A, B; n) \left( \frac{1-y}{y} \right)^n. 
\] (3.12)

\( F_\ell \) has at most one pole on \( \mathbb{P}_1^1 \). If \( \ell = -m < 0 \), it is at \( y = 0 \) and is of order \( m \); and if \( \ell > 0 \), it is at \( y = B/(B - 1) \) and is of order \( 2\ell - 1 \). In the latter case, \( F_\ell \) equals \( (y^2 D_y)^{\ell-1} \tilde{F}_\ell \), where \( \tilde{F}_\ell \) is rational with a pole at \( y = B/(B - 1) \) of order \( \ell \).

**Proof.** The formula (3.12) is an instance of (3.5). By Thm. 3.3, \( F_\ell \) is a degree-\( m \) polynomial in \( v := (1-y)/y \) if \( \ell = -m \leq 0 \), and if \( \ell > 0 \), it is rational on \( \mathbb{P}_v^1 \) with a unique pole of order \( 2\ell - 1 \) at \( v = (-B)^{-1} \), i.e., at \( y = B/(B - 1) \). As \( (n+1)_{\ell-1} \mid \hat{f}_\ell \), the function \( F_\ell \) on \( \mathbb{P}_v^1 \) is the \((\ell - 1)\)th derivative of some rational function with a unique pole at finite \( v \) located at \( v = (-B)^{-1} \), of order \( \ell \); and \( D_v = -y^2 D_y \).

As examples of the use of formula (3.12), the first four rational functions \( F_\ell \) are listed in Table 1. They come from the sequences \( \hat{f}_\ell \) of (3.9).
Table 1: The first few rational functions $F_{\ell}$. (Here, $S := \frac{1}{2} - \ell$.)

| $\ell$ | $S$ | $F_{\ell}(A, B; y)$ |
|-------|-----|---------------------|
| $-1$  | $\frac{A-B-(A-1)(B-1)y}{(A+B-1)y}$ |
| $0$   | $\frac{1}{2}$ | $1$ |
| $1$   | $\frac{y}{2(B-1)y+B}$ |
| $2$   | $\frac{y^2[(B-1)(B-A-1)y-B(B-A-2)]}{(A+1)((1-B)y+B)^3}$ |

defined by $y - 1 - z y^B = 0$. 

$$y^A \left[ \frac{AB - (A-1)(B-1)y}{(A+B-1)y} \right] = 1 + \sum_{k=1}^{\infty} \frac{A-1}{A+Bk-1} \frac{A}{A+Bk} \binom{A+Bk}{k} z^k,$$

(3.13a)

$$y^A = 1 + \sum_{k=1}^{\infty} \frac{A}{A+Bk} \binom{A+Bk}{k} z^k,$$

(3.13b)

$$y^A \left[ \frac{y}{1 - (B)y + B} \right] = 1 + \sum_{k=1}^{\infty} \binom{A+Bk}{k} z^k,$$

(3.13c)

$$y^A \left\{ \frac{y^2 [(B-1)(B-A-1)y-B(B-A-2)]}{(A+1)[(1-B)y+B]^3} \right\} = 1 + \sum_{k=1}^{\infty} \frac{A+Bk+1}{A+1} \frac{A+Bk}{k} z^k,$$

(3.13d)

The $\ell = 0$ identity (3.13b) and the $\ell = 1$ identity (3.13c) are well known. The former is due to Lambert in 1758 (and also to Ramanujan [2, p. 72]), and the latter to Pólya [20]. They can be proved by Lagrange inversion [17, Thm. 2.1], and imply each other [21, Lem. 1]. However, the identities (3.13a), (3.13d) may be new.

Each identity in this family can be obtained by differentiating the preceding. This is formalized in the following recurrence. By starting with $F_0 \equiv 1$ and iterating, one can generate all $F_{\ell}$, $\ell > 0$. By doing the reverse, i.e., by integrating (and exploiting the fact that $F_{\ell}(1) = 1$, $\forall \ell \in \mathbb{Z}$), one can generate all $F_{\ell}$, $\ell < 0$.

**Theorem 3.5.** For all $\ell \in \mathbb{Z}$,

$$F_{\ell+1}(A, B; y) = \frac{(A-B+2)^{\ell-1}}{(A+1)^{\ell}} \frac{y^{-A+B+1}}{(1-B)y+B} D_y \left[ y^{A-B+1} F_{\ell}(A - B + 1, B; y) \right].$$

**Proof.** Apply $D_z$ to both sides of (3.11), using $D_z = \frac{y^{2\mu}}{y^\mu - B(y-1)y^{\mu-\tau}} D_y$, which follows from $y - 1 - z y^B = 0$. 

10
Theorem 3.6. The rational function $F_\ell(A, B; y)$ is of the form

$$\Pi_m(A, B; y) = \frac{(A + B - m)_m(A + 2B - m)_{m-1} \cdots (A + mB - m)_1 y^m}{\Pi_{\ell}(A, B; y)}$$

(3.14a) \quad \ell = -m \leq 0;

(3.14b) \quad \ell \geq 1.

where $\Pi_m$, $\Pi_\ell$ are polynomials.

Proof. By iterating Thm. 3.5 toward negative and positive $\ell$. \hfill \square

It should be noticed that in the identities with $\ell \geq 1$, the poles in $A$ that are evident in (3.12a), located at $A = -1, \ldots, -\ell + 1$, are removable. The denominator factor $(A + 1)_{\ell-1}$ is present on the right as well as the left side of (3.11), and can simply be cancelled. The poles in (3.14a), though, are less easily removed.

This family of identities can be generalized by modifying the initial sequences $f_\ell(A, B, k)$ of (3.10), i.e., by introducing one or more free parameters. A pair of such generalizations, involving the functions $2F_1$ and $3F_2$, will prove useful. Let

$$g_\ell(A, B, C; k) := \frac{A + C + \ell}{A + C + Bk + \ell} f_{\ell+1}(A, B; k), \quad k \geq 0,$$

(3.15)

so that $g_\ell$ reduces to $f_\ell$ if $C = 0$ and to $f_{\ell+1}$ as $C \to \infty$. The $(A, B)$-transforms of $g_0, g_1$ are

$$\hat{g}_0(A, B, C; n) = (-)^n (A + C)^n \Delta^n_{A,B} \left[ \binom{A}{n} \left( \frac{1}{A + C} \right) \right] (A),$$

(3.16a)

$$\hat{g}_1(A, B, C; n) = (-)^n \left( \frac{A + C + 1}{A + 1} \right)^n \Delta^n_{A,B} \left[ \binom{A}{n} \left( \frac{A + 1}{A + C + 1} \right) \right] (A)$$

(3.16b)

By a finite difference computation or an expansion in partial fractions (cf. [9, § 6]),

$$\hat{g}_0(A, B, C; n) = (-)^n \frac{(C)_n}{\left( \frac{A + C}{B} + 1 \right)_n},$$

(3.17a)

$$\hat{g}_1(A, B, C; n) + B \hat{g}_1(A, B, C; n - 1) = (-)^n \frac{(C)_n \left( \frac{A + 1}{B} + 1 \right)_n}{\left( \frac{A + C + 1}{B} + 1 \right)_n \left( \frac{A + 1}{B - 1} \right)_n},$$

(3.17b)
for each $n \geq 0$, resp. $n \geq 1$, showing by comparison with (3.9) and (3.10) that $g_0(A, B, C; n)$ reduces to $f_0(A, B; n) = (-B)^n$ if $C = 0$ and to $f_1(A, B; n) = \delta_{n, 0}$ as $C \to \infty$; and that $g_1(A, B, C; n)$ reduces to $f_1(A, B; n) = 1$ if $C = 0$, and to $f_2(A, B; n)$, given in (3.9a), as $C \to \infty$. To avoid division by zero in (3.17), one may assume that $B \neq 0, 1$ and that $(A + C)/B$ is not a negative integer.

The right sides of (3.17a), (3.17b) have the form of coefficients of ${}_2F_1$ and ${}_3F_2$ series. Therefore, applying Thm. 3.1 yields a pair of hypergeometric identities.

**Theorem 3.7.** Under the above assumptions, one has the interpolating identities

$$y^A \, {}_2F_1 \left( \frac{A + C}{B}; \frac{1}{y} \left| \frac{A + C}{B} \right. \right) = 1 + \sum_{k=1}^{\infty} \frac{A + C}{A + C + Bk} \left( \frac{A + Bk}{B} \right)^k,$$

$$y^A \left[ \frac{y}{(1 - B)y + B} \right] \, {}_3F_2 \left( \frac{A + C + 1, 1}{B}; \frac{1}{y} \left| \frac{A + C + 1}{B} \right. \right) = 1 + \sum_{k=1}^{\infty} \frac{A + C + 1}{A + C + Bk + 1} \left( \frac{A + Bk + 1}{B} \right)^k,$$

which are valid near $z = 0$, if $y$ near 1 is defined by $y - 1 - z y^B = 0$.

Of these two identities the first was found by Gould [3, §6], but the second appears to be new. The parametrized functions multiplied by $y^A$ on their left sides will be denoted $G_\ell(A, B, C; y)$, $\ell = 0, 1$, respectively.

The first identity reduces if $C = 0$ to (3.13a), the $\ell = 0$ identity of Lambert and Ramanujan, and as $C \to \infty$ to (3.13c), the $\ell = 1$ identity of Pólya. The function $G_0(A, B, C; y)$ reduces to $F_0(A, B; y) \equiv 1$ and $F_1(A, B; y) = y/[(1 - B)y + B]$, respectively. Similarly, the second identity reduces if $C = 0$ to Pólya’s identity, and as $C \to \infty$ to the $\ell = 2$ identity (3.13a). The function $G_1(A, B, C; y)$ reduces respectively to $F_1(A, B; y)$ and $F_2(A, B; y)$.

### 4. Algebraic and hypergeometric functions

In this section we express certain algebraic functions, namely solutions of trinomial equations, in terms of ${}_nF_{n-1}$’s that are solutions of $E_n$’s with imprimitiv monodromy. By ‘inverting’ these representations, in Thm. 4.4 which is the main result of this section, we express the ${}_nF_{n-1}$’s in terms of algebraic functions. If the parameter $\ell$ in Thm. 4.4 is set to zero, then our representations reduce to those of Birkeland [3]. Theorem 4.6 is a partial extension of Thm. 4.4 that is less algebraic, with an extra free parameter.

The standardized trinomial equation $y - 1 - z y^B = 0$, yielding a solution $y$ that is near 1 for $z$ near 0, was considered in §3. The general trinomial equation

$$x^n - g x^p - \beta = 0 \quad (4.1)$$

is now the focus. Here $n = p + q$, for integers $p, q \geq 1$, and $g, \beta \in \mathbb{C}$ with at most one of $g, \beta$ equal to zero. The condition $\gcd(p, q) = 1$ will be added later.
Let the $n$ solutions of (4.1), with multiplicity, be denoted $x_1, \ldots, x_n$. In the limit $\beta \to 0$ with fixed $g > 0$, one may choose

$$x_j = \begin{cases} \varepsilon^{-j(q-1)g^{1/q}} & j = 1, \ldots, q, \\ 0 & j = q+1, \ldots, n. \end{cases} \quad (4.2)$$

In the limit $g \to 0$ with fixed $\beta > 0$, one may choose

$$x_j = \varepsilon^{-(j-1)\beta^{1/n}} \quad (j = 1, \ldots, n). \quad (4.3)$$

Here and below, $\varepsilon := \exp(2\pi i/r)$ signifies a primitive $r$‘th root of unity.

To reduce the first case of (4.1), i.e., that of $\beta$ near zero, let

$$y = g^{-1} x^q, \quad z = \varepsilon^{(j-1)n} g^{-n/q} \beta, \quad B = -p/q. \quad (4.4)$$

To reduce the second case, i.e., that of $g$ near zero, let

$$y = \beta^{-1} x^n, \quad z = \varepsilon^{(j-1)\beta^{-q/n}} g, \quad B = p/n. \quad (4.5)$$

By undoing these two reductions, and letting $A = \gamma/q$, resp. $A = \gamma/n$, one obtains the following from Thm. 3.4, in which the rational functions $F_{\ell}(A, B; y)$, $\ell \in \mathbb{Z}$, were defined.

**Theorem 4.1.** The following hold for $\ell \in \mathbb{Z}$ and $\gamma \in \mathbb{C}$.

(i) In a neighborhood of $\beta = 0$ with fixed $g > 0$, and $x_j$ near $\varepsilon^{-j-1}g^{1/q}$ defined by (4.1), (4.2),

$$[\varepsilon^{(j-1)g^{1/q}} x_j^\gamma] F_{\ell}(\gamma/q, -p/q; g^{-1} x_j^q) = 1 + \sum_{k=1}^{\infty} \left( \frac{\gamma/q - pk/q + 1}{\gamma/q + 1} \right)^{\ell-1} \left( \frac{\gamma/q - pk/q}{k} \right) \left[\varepsilon^{(j-1)n} g^{-n/q} \beta\right]^k,$$

for $j = 1, \ldots, q$.

(ii) In a neighborhood of $g = 0$ with fixed $\beta > 0$, and $x_j$ near $\varepsilon^{-(j-1)\beta^{1/n}}$ defined by (4.1), (4.3),

$$[\varepsilon^{(j-1)\beta^{1/n}} x_j^\gamma] F_{\ell}(\gamma/n, p/n; \beta^{-1} x_j^n) = 1 + \sum_{k=1}^{\infty} \left( \frac{\gamma/n + pk/n + 1}{\gamma/n + 1} \right)^{\ell-1} \left( \frac{\gamma/n + pk/n}{k} \right) \left[\varepsilon^{(j-1)q} \beta^{-q/n} g\right]^k,$$

for $j = 1, \ldots, n$.

In (i) and (ii), $\gamma \in \mathbb{C}$ is constrained so that no division by zero occurs in the evaluation of either the rational function $F_{\ell}$ or the first factor of the summand.

The connection to hypergeometric equations $E_n$ with imprimitive monodromy can now be made. Henceforth, let a Riemann sphere $\mathbb{P}_1^\beta$ be parametrized by

$$\zeta := (-)^q n^{\beta^q} p^{\beta^q} q^{\beta^q} g^{\beta^q}. \quad (4.6)$$
Proof. Partition the series of Thm. 4.1 into residue classes of \(g\) integer. \(g\) chosen so that the result equals \(F\) near \(t\) in a neighborhood of \(t\) and for each \(n\), that is, let \(k = \kappa + i \cdot q\), resp. \(k = \kappa + i \cdot n\), and for each \(\kappa\), sum over \(i = 0, 1, \ldots\), using the identity

\[
(t)^{i+r} = r^r \left( \frac{t}{r} \right)^i \left( \frac{t+1}{r} \right)^i \ldots \left( \frac{t+r-1}{r} \right)^i. \tag{4.7}
\]

Also, let \(\gamma = -qa\), resp. \(\gamma = -na\). (That is, \(A = -a\).)

\[\square\]

Corollary 4.3. The following hold for \(\ell \in \mathbb{Z}\) and \(a \in \mathbb{C}\).

(i) Near \(\zeta = 0\) on \(\mathbb{P}^1\), with \(x_j\) near \(\varepsilon_q^{-(j-1)} q^{1/q}\) defined by (4.1), (4.2), the coefficient \(g > 0\) being fixed and arbitrary and the coefficient \(\beta\) being defined in terms of \(\zeta\) by (4.6), the quantities

\[
x_j^{-qa} F\left(-a, -p/q; g^{-1} x_j^q\right), \quad j = 1, \ldots, q,
\]
regarded as functions of $\zeta$, lie in the solution space of the differential equation
\[
E_n\left(\frac{a}{n}, \ldots, \frac{a+(n-1)}{n}, \frac{a-\ell+1}{p}; \frac{1}{q}, \ldots, \frac{1}{q}\right)\zeta.
\]

(ii) Near $\zeta = \infty$ on $\mathbb{P}_1^1$, with $x_j$ near $\varepsilon_n^{(j-1)}\beta^{1/n}$ defined by (4.1), (4.3), the coefficient $\beta > 0$ being fixed and arbitrary and the coefficient $g$ being defined in terms of $\zeta$ by (4.6), the quantities
\[
x_j^{-na} F_{\ell}(-a, p/n; \beta^{-1}x_j^n), \quad j = 1, \ldots, n,
\]
regarded as functions of the reciprocal variable $\tilde{\zeta} := \zeta^{-1}$, lie in the solution space near $\tilde{\zeta} = 0$ of the differential equation
\[
E_n\left(\frac{-a+\ell}{n}, \ldots, \frac{-a+\ell+(p-1)}{p}; \frac{a}{q}, \ldots, \frac{a+(q-1)}{q}\right)\tilde{\zeta}.
\]

In (i), it is assumed that $a \in \mathbb{C}$ is such that no pair of lower parameters differs by an integer; i.e., that the singular point $\zeta = 0$ of the $E_n$ is non-logarithmic.

Proof. Immediate by Lemma 2.2, applied to Thm. 4.2. □

If in either $E_n$ of Corollary 4.3, $a$ is chosen so that no upper parameter differs by an integer from a lower one, then the monodromy group of the $E_n$ will be irreducible; and moreover, if gcd($p, q$) = 1 then the $E_n$ will be of the imprimitive irreducible type characterized in Thm. 2.3. The latter fact is obvious if $\ell = 0$, though if $\ell \in \mathbb{Z} \setminus \{0\}$, integer displacements of parameters are needed.

**Theorem 4.4.** If gcd($p, q$) = 1, the following hold for each $\ell \in \mathbb{Z}$ and $a \in \mathbb{C}$.

(i) Near $\zeta = 0$ on $\mathbb{P}_1^1$, for arbitrary fixed $g > 0$ and for $\kappa = 0, \ldots, q-1$,
\[
\zeta^{\kappa/q} F_{n-1}\left(\frac{a}{n}, \ldots, \frac{a+(n-1)}{n}, \frac{a-\ell+1}{p}; \frac{1}{q}, \ldots, \frac{1}{q}\zeta\right) = \frac{(-)^{\kappa}(1)_n(a+n\kappa/q)_{1-\kappa}}{(a)_{1-\ell}} (-1)^{\kappa} n^{\kappa/q/p^q q^q} \times q^{-1} \sum_{j=1}^{q} (\varepsilon_q^{-\kappa})^{(j-1)\kappa} [\varepsilon_q^{(j-1)} g^{1/q} x_j]^{-\kappa} F_{\ell}(-a, -p/q; -g^{-1} x_j^n),
\]
in which $x_j$ near $\varepsilon_q^{(j-1)} g^{1/q}$, $j = 1, \ldots, q$, is defined by (4.1), (4.2), with the coefficient $\beta$ determined by $\zeta$ according to (4.6)
Proof. Each of the two formulas in Thm. 4.2 has the form of a linear transformation expressed as a matrix–vector product, i.e., $u = \sum_k \kappa M_{jk} v_k$. Here, $M = (M_{jk}) = (\varepsilon^{(j-1)\kappa})$ is a $q \times q$, resp. $n \times n$ matrix, with $\varepsilon := \varepsilon_n^a$, resp. $\varepsilon_n^\kappa$. If $\gcd(p, q) = 1$, or equivalently $\gcd(q, n) = 1$, then the root of unity $\varepsilon$ will be a primitive $q'$th root, resp. $n'$th root, and $M$ will be nonsingular. The $v_k$ are expressed in terms of the $u_j$ by inverting $M$. But $M^{-1} = q^{-1}M^*$, resp. $M^{-1} = n^{-1}M^*$. Elementwise multiplication by $M^*$ yields the claimed summation formulas. \( \Box \)

Corollary 4.5. If $\gcd(p, q) = 1$, then the following holds for $a \in \mathbb{C}$ with $na \notin \mathbb{Z}$.

Near $\zeta = \infty$ on $\mathbb{P}_\zeta^1$, if $x_j \nearrow \varepsilon_n^{(j-1)\beta} 1/n$, $j = 1, \ldots, n$, is defined by (4.1), (4.3), with the coefficient $q$ determined by $\zeta$ according to (4.6), then the exponentiated roots

$$x_j^{-na}, \quad j = 1, \ldots, n,$$

in which $x_j$ near $\varepsilon_n^{(j-1)\beta} 1/n$, $j = 1, \ldots, n$, is defined by (4.1), (4.3), with the coefficient $q$ determined by $\zeta$ according to (4.6).

Remark 4.4.1. A convention for choosing branches of fractional powers must be adhered to, in interpreting these formulas. In (i), there are $q$ choices for each of

$$\zeta^{1/q}, \quad \left(\frac{(-q)n^a}{p^aq^q}\right)^{1/q}, \quad \beta.$$  

(The last is evident from (4.6).) Any choices will work, so long as the formal $q'$th root of (4.6), i.e.,

$$\zeta^{1/q} = \left(\frac{(-q)n^a}{p^aq^q}\right)^{1/q} \cdot g^{-n/q} \cdot \beta,$$  

is satisfied. This removes one degree of freedom. Part (ii) is similarly interpreted.
regarded as functions of $\tilde{\zeta} := \zeta^{-1}$, span the solution space near $\tilde{\zeta} = 0$ of the differential equation

$$E_n \left( \frac{-a}{p}, \ldots, \frac{-a+(p-1)}{p}, \frac{a}{q}, \ldots, \frac{a+(q-1)}{q} \right).$$

**Proof.** Immediate by Lemma 2.2 applied to the $\ell = 0$ case of Thm. 4.4(ii). (The condition $na \notin \mathbb{Z}$ ensures linear independence of the exponentiated roots.)

The expansions of certain $_nF_{n-1}$’s given in Thm. 4.4 in terms of the solutions of trinomial equations, are the point of departure for the rest of this article. In each expanded $_nF_{n-1}$, the parametric excess $S$ (the sum of the lower parameters, minus the sum of the upper ones) equals $\ell$, noted, the $\ell$ case, when the $F_\ell$ factor on the right-hand side degenerates to unity, was previously treated by Birkeland. It is remarkable that many classical hypergeometric identities, such as cubic transformations of $\binom{1}{3}$ unity, was previously treated by Birkeland. It is remarkable that many classical hypergeometric identities, such as cubic transformations of $\binom{3}{3}$ found by Bailey, are restricted to $_nF_{n-1}$’s with $S = \frac{1}{2}$. Sometimes there are ‘companion’ identities that are satisfied by hypergeometric functions with $S = -\frac{1}{2}$. (For Bailey’s identities, see [8], (5.35.4) and (5.65.7).) But in the present context, the cases $S = \frac{1}{2}$ and $-\frac{1}{2}$, i.e., $\ell = 0$ and 1, are merely the most easily treated steps on an infinite ladder of possibilities.

The preceding theorems relating hypergeometric and algebraic functions, including Thm. 4.4 came from Thm. 3.3 which applied the Vandermonde convolution transform to the sequences $f_\ell$ of [8]. One could also start with Thm. 3.7 i.e., with the sequences $g_0, g_1$ parametrized by $C$, which reduce to $f_0, f_1$ if $C = 0$ and to $f_1, f_2$ as $C \to \infty$. Deriving the following theorem, which interpolates between the $\ell = 0, 1$ and $1, 2$ cases of Thm. 4.4 is straightforward. (In it, $c$ stands for $-C$.)

**Theorem 4.6.** If $\gcd(p, q) = 1$, the following hold for $\ell = 0, 1$ and $a \in \mathbb{C}$, with

$$\begin{align*}
G_0(A, B, C; y) &:= \binom{C}{A+B+1} \left( \frac{y - 1}{y} \right), \\
G_1(A, B, C; y) &:= \binom{y}{(1-B)y + B} \binom{C}{A+B+1} \left( \frac{1}{y} \right). 
\end{align*}$$

(i) Near $\zeta = 0$ on $\mathbb{P}^1_\zeta$, for arbitrary fixed $g > 0$ and for $\kappa = 0, \ldots, q - 1$,

$$\zeta^{\kappa/q} \left( \frac{\binom{a + \kappa}{p} + \binom{a + \kappa}{q} + \binom{a + \kappa}{\infty}}{p \cdot q \cdot \infty} \left( \frac{a + \kappa - \ell}{p} + \frac{a + \kappa}{q} \right) \right) \frac{(-y)^{\kappa/q} \Gamma(a + \kappa/q)}{p^q q^q} \left( \frac{\binom{\kappa}{p} + \binom{\kappa}{q} + \binom{\kappa}{\infty}}{p \cdot q \cdot \infty} \right)$$

$$\times q^{-1} \sum_{j=1}^{\infty} \binom{\kappa}{q} \left( \frac{j^{-1}}{x_j} \right)^{-\kappa/q} \binom{-a, -p/q, -c}{g^{-1/q} x_j} G_\ell(-a, -p/q, -c; g^{-1/q} x_j).$$

17
in which \( x_j \), near \( \varepsilon_q^{-(j-1)} \), \( j = 1, \ldots, q \), is defined by (4.1), (4.2), with the coefficient \( \beta \) determined by \( \zeta \) according to (1.6).

(ii) Near \( \zeta = \infty \) on \( \mathbb{P}^1 \), for arbitrary fixed \( \beta > 0 \) and for \( \kappa = 0, \ldots, n - 1 \),

\[
\zeta - \kappa/n_n+1F_n \left( \begin{array}{cccc}
\frac{1}{n} + \frac{\ell}{p} & \frac{1}{n} + \frac{\ell}{p} & \cdots & \frac{1}{n} + \frac{\ell}{p} \\
\frac{1}{n} + \frac{\ell}{p} & \frac{1}{n} + \frac{\ell}{p} & \cdots & \frac{1}{n} + \frac{\ell}{p} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} + \frac{\ell}{p} & \frac{1}{n} + \frac{\ell}{p} & \cdots & \frac{1}{n} + \frac{\ell}{p}
\end{array} \right) \zeta^{-1}
\]

\[
= \left( -\gamma \right)^{(1)(a + qk/n - \ell - \kappa(a + c - \ell - pq/n)) \left[ (-\gamma)^{\kappa/n} \right]^{-\kappa/n} (a + c - \ell) \right. \\
\times n^{-1} \sum_{j=1}^{n} \left( \varepsilon_q \right)^{(j-1)\kappa} \left( \varepsilon_q \right)^{(j-1)/n x_j} \varepsilon_q^{-\kappa/n} G_{\ell}(-a, p/n, c; \beta^{-1} x_j^n),
\]

in which \( x_j \) near \( \varepsilon_q^{-(j-1)} \), \( j = 1, \ldots, n \), is defined by (4.1), (4.3), with the coefficient \( g \) determined by \( \zeta \) according to (1.6).

In (i) and (ii), \( a, c \in \mathbb{C} \) are constrained so that no division by zero occurs, and so that no lower parameter of any \( n+1F_n \), for any \( \kappa \), is a non-positive integer.

The \( \ell = 0, 1 \) representations of \( n+1F_n \) given in Thm. (4.6) reduce if \( c = 0 \) to the \( \ell = 0, 1 \) representations of \( nF_{n-1} \) given in Thm. (4.4) by cancelling equal upper and lower parameters, and as \( c \to \infty \) to the \( \ell = 1, 2 \) ones. It should be noted that the \( E_{n+1} \)'s parametrized by \( c \), of which the left-hand functions in Thm. (4.6) are solutions, do not generally have imprimitive monodromy.

5. Schwarz curves: Generalities

This section introduces what we shall call Schwarz curves, which are projective algebraic curves that parametrize ordered \( k \)-tuples of roots (with multiplicity) of

\[
x^n - gx^p - \beta = 0,
\]

the degree-\( n \) trinomial equation. As in §4, it is assumed that \( n = p + q \) for relatively prime integers \( p, q \geq 1 \), and that \( g, \beta \in \mathbb{C} \) with at most one of \( g, \beta \) equaling zero. For each \( k \), any ordered \( k \)-tuple of roots will trace out a Schwarz curve as

\[
\zeta = (-\gamma)^{\frac{n}{n/pq^q}} \frac{\beta^q}{g^n}
\]

(5.2)

varies. Any Schwarz curve is a projective curve, as multiplying each of the roots by any \( \alpha \neq 0 \) will multiply \( g, \beta \) by \( \alpha^q, \alpha^n \), but leave \( \zeta \) invariant. The dependence on \( \zeta \), which is really a projectivized (and normalized) version of the discriminant of (5.1), will be interpreted as specifying a covering of \( \mathbb{P}^1 \) by the Schwarz curve, the genus of which will be calculated. Determining the dependence on \( \zeta \) will be of value when uniformizing the solutions of \( E_n \)'s with imprimitive monodromy, since Thm. (4.4(i),(ii)) express many such solutions in terms of \( q \)-tuples, resp. \( n \)-tuples of roots of (5.1).

Let \( \sigma_l = \sigma_l(x_1, \ldots, x_n) \) denote the \( l \)'th elementary symmetric polynomial in the roots \( x_1, \ldots, x_n \), so that \( \sigma_0 = 1, \sigma_1 = \sum_{i=1}^{n} x_i \), etc. From (5.1),

\[
\beta = (-)^{n-1} \sigma_n, \quad g = (-)^{q-1} \sigma_q;
\]

(5.3)
and $\sigma_l = 0$ for $l = 1, \ldots, q - 1$ and $q + 1, \ldots, n - 1$. Parametrizing tuples of roots, given $\zeta \in \mathbb{P}^1$, means parametrizing them given the ratio $[\sigma_n^q : \sigma_n^q]$.

**Definition 5.1.** The top Schwarz curve $\mathcal{C}_{p,q}^{(n)} \subset \mathbb{P}^{n-1}$ is the algebraic curve comprising all points $[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}$ such that $x_1, \ldots, x_n$ satisfy the $n - 2$ homogeneous equations

$$\sigma_l(x_1, \ldots, x_n) = 0, \quad l = 1, \ldots, q - 1 \text{ and } q + 1, \ldots, n - 1. \quad (5.4)$$

That is, $\mathcal{C}_{p,q}^{(n)}$ comprises all $[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}$ such that $x_1, \ldots, x_n$ are the $n$ roots (with multiplicity) of some trinomial equation of the form $\tau x^k + \sigma x + \rho = 0$. An associated degree-$n$! covering map $\pi_{p,q}^{(n)} : \mathcal{C}_{p,q}^{(n)} \rightarrow \mathbb{P}^1$ is defined by

$$\zeta = \left( \frac{q}{p^\ell q^n} \right)^n \frac{\sigma}{\sigma_n^n}. \quad (5.5)$$

**Proposition 5.2** ([13], Cor. 4.7). The curve $\mathcal{C}_{p,q}^{(n)}$ is irreducible.

**Definition 5.3.** For each $k$ with $n > k > 2$, the subsidiary Schwarz curve $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, also irreducible, is the image under the map $[x_1 : \ldots : x_n] \mapsto [x_1 : \ldots : x_k]$ of $\mathcal{C}_{p,q}^{(n)}$. A more concrete definition of $\mathcal{C}_{p,q}^{(k)}$ is the following. It is the closure in $\mathbb{P}^{k-1}$ of the set of all points $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ such that $x_1, \ldots, x_k$ are $k$ of the $n$ roots (with multiplicity) of some trinomial equation of the form (5.1).

For each $k$ with $n \geq k > 2$, a degree-$(n - k + 1)$ map $\phi_{p,q}^{(k)} : \mathcal{C}_{p,q}^{(k)} \rightarrow \mathcal{C}_{p,q}^{(k-1)}$ is defined by $\phi_{p,q}^{(k)}([x_1 : \ldots : x_k]) = [x_1 : \ldots : x_{k-1}]$.

**Remark 5.3.1.** As will be explained in §5 defining $\mathcal{C}_{p,q}^{(k)}$ as a closure appends at most a finite number of points to it. Namely, if $k \leq p$, it appends each point $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ in which the $x_i, 1 \leq i \leq k$, are distinct $p^\ell$th roots of unity. These $(p - k + 1)k_{k-1}$ limit points are associated to trinomial equations with $\beta = 0$.

**Remark 5.3.2.** The projection maps $\phi_{p,q}^{(k)}$ are really only partial maps, being undefined at a finite number of singular points, as is typical of maps between algebraic curves. (The symbol $\rightarrow$ could be used instead of $\rightarrow$.) Specifically, if $x_1 = \ldots = x_{k-1} = 0$ then $\phi_{p,q}^{(k)}([x_1 : \ldots : x_k])$ is undefined. The problems with zero tuples, associated to trinomial equations with $\beta = 0$, will be dealt with in §6 where each curve $\mathcal{C}_{p,q}^{(k)}$ will be lifted to a desingularized curve $\tilde{\mathcal{C}}_{p,q}^{(k)}$.

Each subsidiary curve $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, $n > k > 2$, is the solution set of a system of $k - 2$ homogeneous equations in $x_1, \ldots, x_k$, obtained by eliminating $x_{k+1}, \ldots, x_n$ from the system (5.3); the details of this will be given shortly.

It is evident that $\mathcal{C}_{p,q}^{(n)} \cong \mathcal{C}_{p,q}^{(n-1)}$ and that $\mathcal{C}_{p,q}^{(2)} \cong \mathbb{P}^1$, where birational equivalence is meant. Here $t$ is any homogeneous degree-1 rational function of $x_1, x_2$. Henceforth the choice $t = (x_1 + x_2)/(x_1 - x_2)$ will be made; so
\[x_1 : x_2 = [t + 1 : t - 1] \text{ will be a uniformization of } \mathcal{C}_{p,q}^{(2)} \text{ by the rational parameter } t. \] It will also prove useful to define \( \mathcal{C}_{p,q}^{(1)} := \mathbb{P}^1 \). This will make it possible to define \( \phi_{p,q}^{(2)} : \mathcal{C}_{p,q}^{(2)} \to \mathcal{C}_{p,q}^{(1)} \) and \( \phi_{p,q}^{(1)} : \mathcal{C}_{p,q}^{(2)} \to \mathbb{P}_\zeta^1 \) in such a way that

\[
\pi_{p,q}^{(n)} = \phi_{p,q}^{(1)} \circ \cdots \circ \phi_{p,q}^{(n)},
\]

in the sense of being an equality on a cofinite domain. (See (5.17) and (5.18) below.) For each \( k \) with \( n > k \geq 1 \),

\[
\pi_{p,q}^{(k)} = \phi_{p,q}^{(1)} \circ \cdots \circ \phi_{p,q}^{(k)}
\]

will then define a subsidiary (partial) map \( \pi_{p,q}^{(k)} : \mathcal{C}_{p,q}^{(k)} \to \mathbb{P}_\zeta^1 \) of degree \((n-k+1)_k\).

The reason for the term ‘Schwarz curve’ is this. The ratio of any \( n \) independent solutions \( y_1, \ldots, y_n \) of an order-\( n \) differential equation on \( \mathbb{P}_\zeta^1 \) defines a (multivalued) Schwarz map from \( \mathbb{P}_\zeta^1 \) to \( \mathbb{P}^{n-1} \). Its image is a curve in \( \mathbb{P}^{n-1} \), and in many cases the inverse map from the image is single-valued, i.e., supplies a covering of \( \mathbb{P}_\zeta^1 \) by the curve. Studying Schwarz maps is a standard way of computing the monodromy groups of differential equations [12, 13], and goes back to Schwarz’s classical work on \( E_2 \), the Gauss hypergeometric equation. In the present article, solutions of \( E_n \)’s with imprimitive monodromy have been expressed in terms of tuples of solutions of trinomial equations, which are algebraic in \( \zeta \); so a purely algebraic use of the Schwarz map and curve concepts seems warranted. The top Schwarz curve \( \mathcal{C}_{p,q}^{(n)} \) was introduced and used by Kato and Noumi [13], though not under that name; the subsidiary Schwarz curves seem not to have been treated or exploited before.

One can derive a system of polynomial equations for each \( \mathcal{C}_{p,q}^{(k)} \), \( n > k > 2 \), in terms of \( x_1, \ldots, x_k \), by using resultants to eliminate \( x_{k+1}, \ldots, x_n \). But one can also eliminate them by hand in the following way, which will incidentally indicate how best to define the curve \( \mathcal{C}_{p,q}^{(1)} \). For any \( k \), \( 0 < k < n \), let \( \sigma_m, \hat{\sigma}_m \) denote the \( m \)th symmetric polynomial in \( x_1, \ldots, x_k \), resp. \( x_{k+1}, \ldots, x_n \). Then

\[
\sigma_l = \sum_{m=0}^{l} \sigma_m \hat{\sigma}_{l-m}, \quad 0 \leq l \leq n,
\]

it being understood that \( \hat{\sigma}_m = 0 \) if \( m \notin \{0, \ldots, k\} \), and similarly, that \( \hat{\sigma}_m = 0 \) if \( m \notin \{0, \ldots, n-k\} \), with \( \sigma_0 = \hat{\sigma}_0 = 1 \). The defining equations of the top curve \( \mathcal{C}_{p,q}^{(n)} \), by Defn. [5.11]

\[
\begin{align*}
1 = \sigma_0 &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_0 - m; \\
0 = \sigma_1 &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_1 - m; \quad \ldots, \quad 0 = \sigma_{q-1} &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_{(q-1)} - m; \\
\sigma_q &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_q - m; \\
0 = \sigma_{q+1} &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_{(q+1)} - m; \quad \ldots, \quad 0 = \sigma_{n-1} &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_{(n-1)} - m; \\
\sigma_n &= \sum_{m=0}^{n} \sigma_m \hat{\sigma}_n - m.
\end{align*}
\]

(5.9)
Lemma 5.4. \( \{ \hat{\sigma}_m \}_{m=0}^{n-k} \) can be expressed in terms of \( x_1, \ldots, x_k \) (and \( \sigma_n \)) by

\[
\hat{\sigma}_m = \begin{cases} 
(-)^m \sum_{j=1}^{k} \frac{x_j^{m+k-1}}{\prod_l(x_j - x_l)}, & m = 0, \ldots, \min(q-1, n-k); \\
(-)^{m-(n-k)} \sigma_n \sum_{j=1}^{k} \frac{x_j^{m-(n-k)-1}}{\prod_l(x_l - x_j)}, & m = \max(q-k+1, 0), \ldots, n-k;
\end{cases}
\]

in which \( \prod'_l \) signifies a product over \( l = 1, \ldots, k \), with \( l = j \) omitted. Note that the \( m \)-ranges of validity of these two formulas, namely \( 0 \leq m \leq \min(q-1, n-k) \) and \( \max(q-k+1, 0) \leq m \leq n-k \), may overlap.

Proof. For any \( l \) in the range \( 0 \leq l \leq n \), the \( l \)'th equation in (5.9), i.e., the one with \( \sigma_l \) on its left side, contains a term \( \bar{\sigma}_l \sigma_0 = \hat{\sigma}_l \) on its right side and can therefore be solved for \( \hat{\sigma}_l \). This is how the two claimed formulas are derived.

To obtain the first formula, notice that it is equivalent to

\[
\hat{\sigma}_m = (-)^m \sum_{m_1 + \cdots + m_k = m} x_1^{m_1} \cdots x_k^{m_k}, \quad m = 0, \ldots, \min(q-1, n-k). \tag{5.10}
\]

Formula (5.10) is proved by induction: one solves the zeroth equation in (5.9) for \( \hat{\sigma}_0 \), the first for \( \hat{\sigma}_1 \), etc.; necessarily ending with the \( \min(q-1, n-k) \)'th, since the next one may involve \( \sigma_q \), which is not known.

The second formula, in a sense dual to the first, is verified ‘downwards.’ One solves the \( (n-k) \)'th equation in (5.9) for \( \hat{\sigma}_{n-k} \), the \( (n-k-1) \)'th equation for \( \hat{\sigma}_{n-k-1} \), etc.; necessarily ending with the \( \max(q-k+1, 0) \)'th, since the next one may involve \( \sigma_q \), which is not known.

The formulas of the lemma are accompanied by constraints. First, there are the implicit constraints coming from the equivalence between the alternative expressions given in the lemma for \( \hat{\sigma}_m \), for each \( m \) in the doubly covered range

\[
\max(q-k+1, 0) \leq m \leq \min(q-1, n-k). \tag{5.11}
\]

Second, there are the equations \( \sigma_l = 0 \) in (5.9), for each \( l \) in the ranges

\[
\min(n-k+1, q) \leq l \leq q-1, \quad q+1 \leq l \leq \max(k-1, q), \tag{5.12}
\]

which have not yet been exploited. In all, there are \( k-1 \) constraint equations. They serve (if \( k \geq 2 \)) to do two things: (i) they yield an expression for \( \sigma_n \) as a rational function of \( \hat{\sigma}_1, \ldots, \hat{\sigma}_{n-k} \), and hence as a rational function of \( x_1, \ldots, x_k \), homogeneous of degree \( n \); and (ii) they impose \( k-2 \) homogeneous conditions on \( x_1, \ldots, x_k \), which are the desired defining equations of the curve \( C_{p,q}^{(k)} \subset \mathbb{P}^{k-1} \).

The formula for \( \sigma_q \) in (5.9), as yet unused, yields an expression for \( \sigma_q \) as a rational function of \( x_1, \ldots, x_k \), homogeneous of degree \( q \). Therefore, \( \zeta \propto \sigma_n q / \sigma_q n \) is homogeneous of degree zero and is in the function field of \( C_{p,q}^{(k)} \), and

\[
\mathbb{P}^{k-1} \ni C_{p,q}^{(k)} \ni [x_1 : \ldots : x_k] \mapsto \zeta \in \mathbb{P}^{1} \tag{5.13}
\]

yields a formula for the degree-\((n-k+1)k\) (partial) map \( \pi_{p,q}^{(k)} : C_{p,q}^{(k)} \to \mathbb{P}^{1} \).
Lemma 5.5. Applying the just-sketched elimination procedure to the case \( k = 2 \) yields the formulas

\[
\sigma_n = (-1)^n (x_1 x_2) x_1^n - x_2^n, \quad \sigma_q = (-1)^{q-1} x_1^n - x_2^n,
\]

\[
\zeta = (-1)^n \frac{n^n \sigma_q}{p^q q^q} = \frac{n^n}{p^q q^q} (x_1 x_2)^p q (x_1^n - x_2^n)^q
\]

the last of which defines a degree-[\( n(n-1) \)] covering \( \mathcal{C}^{(2)}_{p,q} : \mathbb{P}^1_{\zeta} \). 

Proof. By elementary algebra. The \( k = 2 \) case is trivial: there are no conditions on \( [x_1 : \ldots : x_k] \in \mathbb{P}^1 \), i.e., on \( [x_1 : x_2] \), and the curve \( \mathcal{C}^{(2)}_{p,q} \) is \( \mathbb{P}^1 \) itself. To see that \( \deg \mathcal{C}^{(2)}_{p,q} = n(n-1) \), note that the numerator and denominator of the quotient in the formula for \( \zeta \) have \( (x_1 - x_2)^n \) as a common factor. If one uses \( [x_1 : x_2] = [t+1 : t-1] \), one can write \( \zeta \) as a degree-[\( n(n-1) \)] rational function of \( t \). (And similarly, \( \sigma_n \) and \( \sigma_q \) as rational functions of \( t \), times \( x_1^n \) and \( x_2^n \), respectively.) \( \square \)

Remark 5.5.1. If one defines \( \pi^{(k)}_{p,q} : \mathcal{C}^{(k)}_{p,q} \to \mathbb{P}^1_{\zeta} \) as the composition of the two rational maps \( [x_1 : \ldots : x_k] \mapsto [x_1 : x_2] \) and \( \pi^{(2)}_{p,q} \), then it follows immediately from Lemma 5.5 that \( \zeta \) is in the function field of \( \mathcal{C}^{(k)}_{p,q} \), even in the absence of an explicit formula for \( \zeta = \zeta(x_1, \ldots, x_k) \).

The case \( k = 1 \) obviously requires special treatment, as one cannot express \( \sigma_n, \sigma_q \) in terms of \( x_1 \) alone. If \( k = 1 \), the system (5.9) reduces to

\[
\begin{align*}
0 &= \sigma_1 = x_1 + \sigma_1, & 0 &= \sigma_{q-1} = x_1 \sigma_{q-2} + \sigma_{q-1}; \\
\sigma_q &= x_1 \sigma_{q-1} + \sigma_q; \\
0 &= \sigma_{q+1} = x_1 \sigma_q + \sigma_{q+1}, & 0 &= \sigma_{n-1} = x_1 \sigma_{n-2} + \sigma_{n-1}; \\
\sigma_n &= x_1 \sigma_{n-1},
\end{align*}
\]

the solution of which can be written as

\[
\sigma_n = (-1)^{n-q-1} x_1^{n-q} \sigma_q, \quad \sigma_q = \sigma_q + (-1)^{q-1} x_1^q.
\]

This suggests focusing on \( s \), the element of the function field of \( \mathcal{C}^{(n)}_{p,q} \) defined by

\[
\begin{align*}
s := (-1)^{n-1} \sigma_n / x_1^n, & \quad 1 - s := (-1)^{q-1} \sigma_q / x_1^q \\
= \beta / x_1^n, & \quad = g / x_1^q
\end{align*}
\]

(the two definitions being equivalent), and defining the special \( k = 1 \) Schwarz curve \( \mathcal{C}^{(1)}_{p,q} \) to be \( \mathbb{P}^1_{s} \). There is a degree-[\( n-1 \)] map \( \phi^{(2)}_{p,q} : \mathcal{C}^{(2)}_{p,q} \to \mathcal{C}^{(1)}_{p,q} \) given by the map \( t = (x_1 + x_2)/(x_1 - x_2) \mapsto s \), since both \( \sigma_n, \sigma_q \) are rational functions of \( x_1, x_2 \). By comparing the expressions for \( \sigma_n, \sigma_q \) in Lemma 5.5 with those for
In Sec. 5.4, one finds that this degree-$(n-1)$ map $\phi^{(2)}_{p,q}$ is given by

\[
\begin{aligned}
\sigma_n/x_1^n, \sigma_q/x_2^q \text{ in (5.10), one finds that this degree-} (n-1) \text{ map } \phi^{(2)}_{p,q} \text{ is given by } \\
s &= \frac{x_1^p x_2^q - x_1^q}{x_1^p x_2^q - x_2^q} = 1 - \frac{1}{1 - \frac{(t+1)^n}{(t+1)^p - (t-1)^p}}. \\
&= 1 - \frac{(t-1)^p (t+1)^q - (t-1)^q}{(t+1)^q (t+1)^p - (t-1)^p} = 1 - \frac{1}{1 - \frac{(t+1)^n}{(t+1)^p - (t-1)^p}}.
\end{aligned}
\]

Moreover, there is a final degree-$n$ map $\phi^{(1)}_{p,q}: \mathbb{C}^{(1)}_{p,q} \cong \mathbb{P}^1 \to \mathbb{P}^1$, given by

\[
\zeta = (-)^n n^n \sigma_q^{q^n} = (-)^n \frac{n^n}{p^p q^q} \frac{s^n}{(1-s)^n},
\]

which completes the sequence of maps leading from the top curve $\mathbb{C}^{(n)}_{p,q}$ down to $\mathbb{P}^1$.

Pulling everything together, one has the following theorem, which summarizes the results of this section.

**Theorem 5.6.** For any pair of relatively prime integers $p, q \geq 1$ with $n = p+q$, there is a sequence of algebraic curves and (partial) maps

\[
\mathbb{C}^{(n)}_{p,q} \to \mathbb{C}^{(n-1)}_{p,q} \to \cdots \to \mathbb{C}^{(1)}_{p,q} \to \mathbb{C}^{(0)}_{p,q} \cong \mathbb{P}^1,
\]

in which $\deg \phi^{(k)}_{p,q} = n-k+1$. For $k = n, n-1, \ldots, 2$, the curve $\mathbb{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$, prior to closure, comprises all $[x_1 : \ldots : x_k]$ in which $x_1, \ldots, x_k$ is a nonzero ordered $k$-tuple of roots (with multiplicity) of some trinomial equation of the form (5.1). The partial maps $\phi^{(k)}_{p,q}, n \geq k > 2$, take $[x_1 : \ldots : x_k]$, where at least one of $x_1, \ldots, x_{k-1}$ is nonzero, to $[x_1 : \ldots : x_{k-1}]$. One writes $\pi^{(k)}_{p,q}$ for $\phi^{(1)}_{p,q} \circ \cdots \circ \phi^{(k)}_{p,q}$, which is of degree $(n-k+1)$. Any $[x_1 : \ldots : x_k]$ in the domain of $\pi^{(k)}_{p,q}$ is taken by it to $\zeta$, computed from the associated trinomial equation by Eq. (5.2).

The final two curves $\mathbb{C}^{(k)}_{p,q}, k = 2, 1$, are of genus zero; i.e., $\mathbb{C}^{(2)}_{p,q} \cong \mathbb{P}^1$ and $\mathbb{C}^{(1)}_{p,q} \cong \mathbb{P}^1$, where $t := (x_1 + x_2)/(x_1 - x_2)$ and $s$ are rational parameters. The final two maps $\phi^{(2)}_{p,q}, \phi^{(1)}_{p,q}$ are given by

\[
\begin{aligned}
s &= \phi^{(2)}_{p,q}(t) = \frac{(t-1)^p (t+1)^q - (t-1)^q}{(t+1)^q (t+1)^p - (t-1)^p}, \\
\zeta &= \phi^{(1)}_{p,q}(s) = (-)^n \frac{n^n}{p^p q^q} \frac{s^n}{(1-s)^n},
\end{aligned}
\]

and their composition $\pi^{(2)}_{p,q} = \phi^{(1)}_{p,q} \circ \phi^{(2)}_{p,q}$ by

\[
\zeta = \pi^{(2)}_{p,q}(t) = \phi^{(1)}_{p,q}(\phi^{(2)}_{p,q}(t)) = \frac{n^n}{p^p q^q} (t^2 - 1)^p [t(t+1)^p - (t-1)^p]^q / [t(t+1)^q - (t-1)^q]^q.
\]

\[
\sum_{n=0}^{\infty} \frac{(-)^n}{p^n q^n} \frac{n^n}{(1-s)^n},
\]
6. Schwarz curves: Ramifications and genera

The Schwarz curves \( C_{p,q}(k) \) of \( \S 5 \) in particular \( C_{p,q}(q), C_{p,q}(n) \), will be used to uniformize the solutions of \( E_n \)'s with imprimitive monodromy. The formulas of Thm. 5.6 will be especially useful. They parametrize \( C_{p,q}(2), C_{p,q}(1) \) by \( \mathbb{P}^1 \)-valued rational parameters \( t, s \), which is possible because \( C_{p,q}(2), C_{p,q}(1) \) are of genus zero.

The question arises whether 'higher' \( C_{p,q}(k) \), such as the family of plane projective curves \( C_{p,q}(3) \subset \mathbb{P}^2 \), are ever of genus zero. In principle, the genus of each \( C_{p,q}(k) \subset \mathbb{P}^{k-1} \), \( n \geq k > 2 \), can be calculated from the Hurwitz formula applied to the map \( \pi_{p,q}(k) : C_{p,q}(k) \rightarrow \mathbb{P}^k_1 \). But some care is needed, since in general, \( C_{p,q}(k) \) is a singular projective curve, and not being smooth, is not a Riemann surface. By resolving singularities, one must first construct a desingularization (or 'normalization') \( C_{p,q}(k) \), which is smooth and covers \( C_{p,q}(k) \). (Up to birational equivalence, \( C_{p,q}(k) \) is unique.) The Hurwitz formula can then be applied to the lifted map \( \tilde{\pi}_{p,q}(k) : \tilde{C}_{p,q}(k) \rightarrow \mathbb{P}^k_1 \), which is a degree-\((n-k+1)k\) map of Riemann surfaces. Sections 6.1 and 6.2 indicate the construction of \( \tilde{\pi}_{p,q}(k) \rightarrow C_{p,q}(k) \) and determine the ramification structure of \( \tilde{\pi}_{p,q}(k) \). Section 6.3 parametrizes several plane curves \( C_{p,q}(3) \subset \mathbb{P}^2 \), which according to the resulting genus formula (Thm. 6.3) are of genus zero.

6.1. Desingularization

Recall that \( C_{p,q}(k) \subset \mathbb{P}^{k-1} \), \( k \geq 2 \), is an algebraic curve including each of the points \( [x_1 : \ldots : x_k] \in \mathbb{P}^{k-1} \) in which \( x_1, \ldots, x_k \) are \( k \) of the \( n \) roots (with multiplicity) of some trinomial equation

\[
x^n - g x^p - \beta = 0
\]

(with \( n = p + q, \gcd(p,q) = 1 \), and at most one of \( g, \beta \) equaling zero). The formula

\[
\zeta := (-)^q \frac{n^n \beta^q}{p^p q^q g^n}
\]

gives the covering map \( \tilde{\pi}_{p,q}(k) : \tilde{C}_{p,q}(k) \rightarrow \mathbb{P}^k_1 \). ‘Cremona inversion’ in \( \mathbb{P}^{k-1} \), i.e., the substitution \( x_i = 1/x'_i \), \( i = 1, \ldots, k \), induces a birational equivalence \( C_{p,q}(k) \cong C_{q,p}(k) \).

The case \( k = n \), in which the closure in \( \mathbb{P}^{k-1} \) need not be taken, will be treated first. To determine the singular points of the curve \( C_{p,q}(n) \subset \mathbb{P}^{n-1} \), one must compute the tangent vector to each point on it, and determine whether the tangent is multiply defined.

At any point coming from a trinomial with \( g \neq 0 \), the tangent can be computed from the effects of varying \( \beta \) on the roots \( \{x_j\}_{j=1}^n \) (with multiplicity) of \( P(x) = x^n - g x^p - \beta \). It is easy to see that if the roots are distinct, \( \delta_j := \frac{dx_j}{d\beta} \) equals \( \left[ \frac{dP}{dx_j} \right]_{x=x_j}^{-1} \), and the tangent vector to \( C_{p,q}(n) \) at \( [x_1 : \ldots : x_n] \) is unique and equals \( [\delta_1 : \ldots : \delta_n] \). The case when \( g = 0 \) (and \( \zeta = \infty \)) also leads to distinct roots and a unique tangent.
There remains the case of coincident roots. It is a standard fact (see below) that if \( \beta \neq 0 \), at most two of the roots of the trinomial can coincide (in fact such coincidences occur only when \( \zeta = 1 \)). Let \( e_j \in \mathbb{C}^n \) denote the \( j \)’th unit vector. At a point \( [x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}_p \) with \( x_{j_1} = x_{j_2} \), the tangent vector to \( \mathbb{E}_p^{(n)} \) will alternatively (by symmetry) be \( e_{j_1} - e_{j_2} \) or \( e_{j_2} - e_{j_1} \); but these two vectors are equal in \( \mathbb{P}^{n-1} \).

So, nonunique tangent vectors to \( \mathbb{E}_p^{(n)} \subset \mathbb{P}^{n-1} \) are possible only at points associated to trinomials with \( \beta = 0 \) (and hence \( \zeta = 0 \)), at which higher-order coincidences of roots can occur. There are \( n!/plq \) such points. One (cf. (4.2)) is \( p_0 = (x_1, \ldots, x_n) \) with

\[
x_j = \begin{cases} 
\epsilon_q^{-(j-1)} g^{1/q}, & j = 1, \ldots, q, \\
0, & j = q + 1, \ldots, n,
\end{cases}
(6.3)
\]

and the others come from permuting the coordinates \( \{x_j\}_{j=1}^n \). Near \( p_0 \), each \( x_j \) has a Puiseux expansion of which \( (6.3) \) is the zeroth term. To leading order, this expansion is

\[
x_j \sim \begin{cases}
\epsilon_q^{-(j-1)} g^{1/q}, & j = 1, \ldots, q, \\
(-\beta/g)^{j/p}, & j = q + 1, \ldots, n,
\end{cases}
(6.4)
\]

as \( \beta \to 0 \) with \( g \) fixed. Here \( \sim \) means that on each branch, the difference between \( x_j \) and the right side will be of smaller magnitude than a higher power of \( \beta \). The presence of multiple branches is due to the fact that in \( (6.4) \), the \( p' \)th roots of \( -\beta/g \) may take on any of \( p \) values, the ordering of the \( p \) distinct values among \( x_{q+1}, \ldots, x_n \) not being uniquely determined. So, a tangent vector to \( \mathbb{E}_p^{(n)} \) at \( p_0 \) may be \( (0, \ldots, 0; \epsilon_0^{\chi(0)}, \ldots, \epsilon_0^{\chi(p-1)}) \), where \( \chi(0), \ldots, \chi(p-1) \) is any permutation of \( 0, \ldots, p-1 \). Up to multiplication by nonzero scalars (i.e., by powers of \( \epsilon_p \)), there are \( (p-1)! \) distinct tangent vectors of this sort.

It follows that \( p_0 \) is a \((p-1)!\)-fold ordinary multiple point: in a neighborhood of \( p_0 \), the curve \( \mathbb{E}_p^{(n)} \) is homeomorphic to \((p-1)!\)-distinct disks with their centers identified. The same is true of each of the \( n!/plq \) points on \( \mathbb{E}_p^{(n)} \) coming from \( \beta = 0 \). Each such point \( p \), including \( p_0 \), will lift to \((p-1)!\) points on the desingularization \( \tilde{\mathbb{E}}_p^{(n)} \), all other points on \( \mathbb{E}_p^{(n)} \) lifting to a single point. The local behavior of the map \( \tilde{\mathbb{E}}_p^{(n)} \): \( \mathbb{E}_p^{(n)} \to \mathbb{P}^1_\zeta \) and its proper (lifted) counterpart \( \tilde{\pi}_p^{(n)} : \tilde{\mathbb{E}}_p^{(n)} \to \mathbb{P}^1_\zeta \) follow from the abovementioned Puiseux expansion. Each root \( (-\beta/g)^{j/p} \) is proportional to \( \zeta^{1/pq} \) as \( \zeta \to 0 \); so, \( \tilde{\pi}_p^{(n)} \) maps each lifted point to \( \zeta = 0 \) with multiplicity \( pq \). The desingularization of a general curve \( \mathbb{E}_p^{(k)} \) is similar. As \( \beta \to 0 \) with \( g \) fixed, any \( k \)-tuple of roots \( x_1, \ldots, x_k \) of \( (6.1) \), up to ordering, will have asymptotic behavior

\[
x_j \sim \begin{cases} 
\epsilon_q^\nu g^{1/q}, & j = 1, \ldots, \nu, \\
\epsilon_p^\nu (-\beta/g)^{1/p}, & j = \nu + 1, \ldots, k,
\end{cases}
(6.5)
\]
The case $\epsilon$ the point $[\alpha]$ the closure when defining $\beta$. When $\nu$ for some $0 \leq \nu < k$, $k - \nu \leq p$, and for certain exponents $\alpha_j \in \{0, \ldots, q-1\}$ and $\alpha_j \in \{0, \ldots, p-1\}$, distinct but otherwise unconstrained. In the $\epsilon \rightarrow 0$ limit, the final $k - \nu$ coordinates of $(x_1, \ldots, x_k)$ will tend to zero. The case $\nu = 0$ is special, as the limit is then $(0, \ldots, 0)$; but it exists in $\mathbb{P}^k$ as the point $[\varepsilon^0_{p, q} : \ldots : \varepsilon^0_{p, q}]$. This explains Remark 5.3.1 on the need for taking the closure when defining $\bar{\gamma}_{p, q}^{(k)}$. There are $(p-k+1)_{k-1}$ limit points of this $\nu = 0$ type.

An analysis similar to the above treatment of the case $(k, \nu) = (n, q)$ yields the data in Table 2. $N^{T}_{p, q}(k, \nu)$ is the number of points on $C_{p, q}$ associated to trinomial equations with $\beta = 0$, which are classified as being ordinary multiple points of the type indexed by $\nu$. For each such point, $N^{T}_{p, q}(k, \nu)$ is the number of distinct tangents, i.e., the number of points to which it lifts on the desingularization $\tilde{\gamma}_{p, q}^{(k)}$. The final quantity $M_{p, q}(k, \nu)$ is the multiplicity with which each lifted point on $C_{p, q}$ is mapped to $\mathbb{P}^1_{\zeta}$ by $\tilde{\gamma}_{p, q}^{(k)}$. This is usually $pq$, as in the case $(k, \nu) = (n, q)$, but in the special cases $\nu = 0$, resp. $\nu = k$, by examination it is $p$, resp. $q$. Note that

$$\sum_{\nu = \max(0, k-p)}^{\min(q, k)} N^{T}_{p, q}(k, \nu) \cdot N^{T}_{p, q}(k, \nu) \cdot M_{p, q}(k, \nu) = (n - k + 1)_{k}, \quad (6.6)$$

as the left side equals the number of points (with multiplicity) on the fibre of $\bar{\gamma}_{p, q}^{(k)}$ above $\zeta = 0$, i.e., $\deg \bar{\gamma}_{p, q}^{(k)} = (n - k + 1)_{k}$. This is a binomial identity.

6.2. Genera

For each $k \geq 2$, the ramifications of the degree-$(n - k + 1)_{k}$ map of Riemann surfaces $\tilde{\pi}_{p, q}^{(k)} : C_{p, q}^{(k)} \to \mathbb{P}^1_{\zeta}$ can now be determined. Any $\bar{p} \in C_{p, q}^{(k)}$ can be mapped with nontrivial multiplicity to $\mathbb{P}^1_{\zeta}$ only if (i) it is obtained by lifting from some $p \in C_{p, q}$ coming from $\beta = 0$, in which case $\zeta = 0$; or (ii) its image $p \in C_{p, q}$ is mapped with nontrivial multiplicity by $\pi_{p, q}^{(k)}$ to $\mathbb{P}^1_{\zeta}$. Case (ii) can occur only if $\zeta = \pi_{p, q}^{(k)}(p)$ is a critical value of $\pi_{p, q}^{(k)}$, i.e., only if $\pi_{p, q}^{(k-1)} \zeta$ comprises fewer than $(n - k + 1)_{k}$ points on $C_{p, q}$; i.e., only if $\pi_{p, q}^{(n-k)} \zeta$ comprises fewer than $n!$ points on $C_{p, q}$. Case (ii) can accordingly occur only if two of the $n$ roots (with multiplicity) of the trinomial equation $(6.1)$ coincide; or, if the roots are distinct but are
proportional to \( \{\zeta^n\}_{m=0}^{n-1} \), the \( n \)’th roots of unity, so that there are only \((n-1)!\) distinct ordered \( n \)-tuples \([x_1: \ldots : x_n]\) \( \in \mathbb{P}^{n-1} \). By computing the discriminant of the trinomial (see [15]), one finds that when \( \beta \neq 0 \), the first of these two subcases occurs only if
\[
(pg/n)^n - (-p\beta/q)^q = 0,
\]
i.e., only if \( \zeta = 1 \); in which case exactly two of the \( n \) roots coincide. (For a description of the monodromy of the roots around \( \zeta = 1 \), see Mikhailkin [15, §2].) The second subcase occurs only if \( g = 0 \), i.e., \( \zeta = \infty \). One concludes that \( \tilde{\pi}_{p,q}^{(k)}: \tilde{\mathcal{C}}_{p,q}^{(k)} \to \mathbb{P}^1_\zeta \) is ramified only over the three points \( \zeta = 0, 1, \infty \). It is a so-called Belyǐ cover.

If \( \zeta = 1 \), i.e., Eq. (6.7) holds, then without loss of generality one can take \( g = n/p \), \( \beta = -q/p \), in which case the trinomial is \( px^n - nx^p + q \), with a single doubled root at \( x = 1 \). There are \( n/2 \) distinct points \([x_1: \ldots : x_n]\) \( \in \mathcal{C}_{p,q}^{(n)} \) over \( \zeta = 1 \), namely, permutations of the roots of \( px^n - nx^p + q \), including the single doubled root.

**Definition 6.1.** For each \( p, q \geq 1 \) with \( \gcd(p, q) = 1 \), a degree-\((p + q - 2)\) polynomial with simple roots \( T_{p,q}(x) = \sum_{j=0}^{p+q-2} t_j x^j \), satisfying
\[
p x^{p+q} - (p+q)x^p + q = (x-1)^2 T_{p,q},
\]
is defined by
\[
t_j = \begin{cases} (j+1)q, & 0 \leq j \leq p - 1, \\ (p+q-1-j)p, & p - 1 \leq j \leq p + q - 2. \end{cases}
\]
Its roots will be denoted \( x^*_{p,q,\alpha}, 1 \leq \alpha \leq p + q - 2 \).

**Theorem 6.2.** For each \( k, n \geq k \geq 2 \), the degree-\((n-k+1)k\) map of Riemann surfaces \( \tilde{\pi}_{p,q}^{(k)}: \tilde{\mathcal{C}}_{p,q}^{(k)} \to \mathbb{P}^1_\zeta \) is a Belyǐ cover: it is ramified only over \( \zeta = 0, 1, \infty \).

1. The fibre over \( \zeta = 0 \) contains \((p-k+1)_{k-1}\) points of multiplicity \( p \), resp. \((q-k+1)_{k-1}\) points of multiplicity \( q \), all other points being of multiplicity \( pq \).
2. The fibre over \( \zeta = 1 \) contains \((n-k-1)k\) points of unit multiplicity, all other points being of multiplicity \( 2 \).
3. The fibre over \( \zeta = \infty \) contains \((n-k+1)_{k-1}\) points of multiplicity \( n \).

**Remark 6.2.1.** The case \( k = n \) of this theorem, dealing with the top Schwarz curve \( \mathcal{C}_{p,q}^{(n)} \), was proved by Kato and Noumi [13]. Note that the fibre over \( \zeta = 0 \) contains no points of multiplicity \( p \) if \( k > p \), and none of multiplicity \( q \) if \( k > q \); and that the fibre over \( \zeta = 1 \) contains no points of unit multiplicity if \( k \geq n - 1 \).

**Proof.** The facts about the fibre over \( \zeta = 0 \) can be read off from Table 2. Above \( \zeta = 1 \), the points \([x_1: \ldots : x_k]\) of unit multiplicity are those in which \( x_i = x^*_{p,q,\chi(i)}, \) where \( \chi(1), \ldots , \chi(k) \) are distinct integers selected from \( 1, \ldots , n - 2 \); and the points of multiplicity \( 2 \) are those in which one or two of the coordinates
are equal instead to the double root 1. As already remarked, in any point \([x_1: \ldots : x_k]\) above \(\zeta = \infty\) each coordinate \(x_i\) must be a distinct \(n'\)th root of unity; but in \(\mathbb{P}^{k-1}\), there are \((n - k + 1)_{k-1} = (n - k + 1)/n\) rather than \((n - k + 1)\). 

**Theorem 6.3.** For each \(n \geq k \geq 1\), the genus of \(\mathcal{C}_{p,q}^{(k)}\) as an algebraic curve, and the topological genus of the Riemann surface \(\tilde{\mathcal{C}}_{p,q}^{(k)}\), are equal to

\[
1 + \left[\frac{(k-1)(2n-k-2)}{4(n-1)} - \frac{n}{2pq}\right] (n-k+1)_{k-1} - \frac{q-1}{2q} (p-k+1)_{k-1} - \frac{p-1}{2p} (q-k+1)_{k-1}.
\]

**Proof.** Apply the Hurwitz genus formula to the data given in Thm. 6.2 (which trivially extends to \(k = 1\)). The genus is stable under \(p \leftrightarrow q\), since \(\mathcal{C}_{p,q}^{(k)} \cong \mathcal{C}_{q,p}^{(k)}\), even though the singular points (i.e., their number and type) are not.

**Corollary 6.4.** (i) The following Schwarz curves, and only these, are rational, i.e., of genus zero.

- The trivial curves \(\mathcal{C}_{p,q}^{(1)} \cong \mathbb{P}^1_s\) and \(\mathcal{C}_{p,q}^{(2)} \cong \mathbb{P}^1_t\), for all coprime \(p, q \geq 1\).
- \(\mathcal{C}_{p,q}^{(3)}\) for \(\{p, q\} = \{1, 2\}\) and \(\{1, 3\}\).
- \(\mathcal{C}_{p,q}^{(4)}\) for \(\{p, q\} = \{1, 3\}\).

In consequence, \(\mathcal{C}_{p,q}^{(q)}\) can be rationally parametrized only if \(q = 1\), \(q = 2\), or \((p, q) = (1, 3)\); and the top curve \(\mathcal{C}_{p,q}^{(\alpha)}\) only if \(\{p, q\} = \{1, 1\}, \{1, 2\}, \text{ or } \{1, 3\}\).

(ii) The following Schwarz curves, and only these, are elliptic, i.e., of genus 1.

- \(\mathcal{C}_{p,q}^{(3)}\) for \(\{p, q\} = \{1, 4\}\).

**Proof.** That these curves are of genus 0 and genus 1, as claimed, follows by direct computation. The ‘only these’ part remains to be proved.

It follows from Thm. 6.2 that for each \(k\) with \(n > k \geq 2\), \(\tilde{\mathcal{C}}_{p,q}^{(k)} \to \tilde{\mathcal{C}}_{p,q}^{(k-1)}\) is a covering of degree \(\geq 1\) with nontrivial branching. Hence, the genus of \(\mathcal{C}_{p,q}^{(k)}\) is strictly greater than that of \(\mathcal{C}_{p,q}^{(k-1)}\), by the Hurwitz formula. To determine which Schwarz curves with \(k \geq 3\) are of genus 0 or genus 1, it suffices to consider the case \(k = 3\). Substituting \(k = 3\) and \(n = p + q\) into Thm. 6.3 yields

\[
g(\mathcal{C}_{p,q}^{(3)}) = \left[(p^2 + 4pq + q^2) - 9(p + q) + 14\right] / 2. \tag{6.8}
\]

By examination, this equals 0 only if \(\{p, q\} = \{1, 2\}\) or \(\{1, 3\}\), and equals 1 only if \(\{p, q\} = \{1, 4\}\). 

**6.3. Plane projective curves**

The curves \(\mathcal{C}_{p,q}^{(3)} \subset \mathbb{P}^2\) are plane projective curves and include the first Schwarz curves of positive genus. The following theorem makes each \(\mathcal{C}_{p,q}^{(3)}\) and the corresponding covering map \(\pi_{p,q}^{(3)}: \mathcal{C}_{p,q}^{(3)} \to \mathbb{P}^1_s\) quite concrete.

28
Theorem 6.5. For all coprime \( p, q \geq 1 \) with at least one of \( p, q \) greater than 1 and \( n := p + q \), the curve \( \mathcal{C}_{p,q}^{(3)} \subset \mathbb{P}^2 \) has defining equation
\[
\frac{x_1^n (x_2^3 - x_3^3) + x_2^n (x_3^3 - x_1^3) + x_3^n (x_1^3 - x_2^3)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)} = 0,
\] (6.9)
where the left side is a symmetric homogeneous polynomial in \( x_1, x_2, x_3 \), which is of degree \( n + p - 3 \) and is of degree \( n - 2 \) in any single variable. The curve \( \mathcal{C}_{p,q}^{(3)} \) has singular points if and only if \( p \geq 3 \), in which case there are exactly three: the fundamental points \([1 : 0 : 0],[0 : 1 : 0],[0 : 0 : 1]\), each of which is an ordinary \((p - 1)\)-fold multiple point of \( \mathcal{C}_{p,q}^{(3)} \). The (partial) covering map \( \pi_{p,q}^{(3)} : \mathcal{C}_{p,q}^{(3)} \to \mathbb{P}_\mathbb{C}^1 \) is given by
\[
\sigma_n = \begin{cases} 
(-1)^{n-1} x_1^p x_2^p x_3^p & \frac{x_1 (x_2^{p+1} - x_3^{p+1}) + \text{cycl.}}{x_1 (x_2^{p+1} - x_2^{p+1} x_3^{p+1}) + \text{cycl.}}, & p > 1, \\
(-1)^{n-1} x_1^p x_2^p x_3^p & \frac{x_1 (x_3^p - x_2^p) + \text{cycl.}}{x_1 (x_2^{p+1} - x_3^{p+1}) + \text{cycl.}}, & q > 1;
\end{cases}
\]
\[
\sigma_q = \begin{cases} 
(-1)^{q-1} & \frac{x_1 (x_2^p x_3^{n+1} - x_2^{n+1} x_3^p) + \text{cycl.}}{x_1 (x_2^p x_3^{n+1} - x_2^{n+1} x_3^p) + \text{cycl.}}, & p > 1, \\
(-1)^{q-1} & \frac{x_1^{p+1} (x_2^q - x_2^q) + \text{cycl.}}{x_1^{p+1} (x_2^q - x_2^q) + \text{cycl.}}, & q > 1;
\end{cases}
\]
\[
\zeta = (-1)^n n^n \frac{\sigma_n q^n}{p^n q^n};
\]
and is of degree \( n(n - 1)(n - 2) \).

Proof. To get (6.9), substitute the formulas for \( \beta = (-1)^{n-1} \sigma_n, \ g = (-1)^{q-1} \sigma_q \) provided by Lemma 5.5 into the trinomial equation (6.1), and relabel \( x \) as \( x_3 \). To get the other formulas, apply the elimination procedure sketched in §5. (Cf. the proof of Lemma 5.4, the details are tedious and are omitted.) The number of singular points (0 or 3) comes from Table 2 and their location is easily checked.

Parametrizing the plane curve \( \mathcal{C}_{p,q}^{(3)} \) is a key step leading to a hypergeometric identity; especially, if \( q \) or \( n = p + q \) equals 3. The following examples are illustrative.

Example 6.6. \( \{p, q\} = \{1, 2\} \), the simplest case. The top curves \( \mathcal{C}_{1,2}^{(3)}, \mathcal{C}_{2,1}^{(3)} \) are of genus zero (see Cor. 6.4). They are respectively a line and a conic, with defining equations \( x_1 + x_2 + x_3 = 0 \) and \( x_1 x_2 + x_2 x_3 + x_3 x_1 = 0 \). In general, \( \mathcal{C}_{p,q}^{(n)} \cong \mathcal{C}_{p,q}^{(n-1)} \), which is reflected in the fact that \( \phi_{p,q}^{(n)} : \mathcal{C}_{p,q}^{(n)} \to \mathcal{C}_{p,q}^{(n-1)} \) is of degree 1. The parameter \( t := (x_1 + x_2)/(x_1 - x_2) \) that was used in the uniformization of \( \mathcal{C}_{p,q}^{(2)} \) could accordingly be used as a parameter for \( \mathcal{C}_{1,2}^{(3)}, \mathcal{C}_{2,1}^{(3)} \), but it is convenient to use
an alternative rational parameter \( \tilde{t} \) (related to \( t \) by a Möbius transformation), thus:

\[
[x_1 : x_2 : x_3] = \begin{cases} 
[1 + \tilde{t} : \tilde{\omega} + \omega \tilde{t} : \omega + \tilde{\omega} \tilde{t}], & (p, q) = (1, 2); \\
[1/(1 + \tilde{t}) : 1/(\omega + \tilde{\omega} \tilde{t}) : 1/(\omega + \tilde{\omega} \tilde{t})], & (p, q) = (2, 1),
\end{cases}
\]

(6.10)

where \( \omega := \varepsilon_3 \) and \( \tilde{\omega} := \varepsilon_3^2 \). (Note that at \( \tilde{t} = 0 \), \([x_1 : x_2 : x_3] = [1 : \tilde{\omega} : \omega] \).

Substitution into Thm. 6.5 yields a degree-6 rational map \( t \mapsto \zeta \), namely the Belyi map

\[
\zeta = \frac{(1 + \tilde{t})^2}{4s^3} = \begin{cases} 
\frac{27}{4} \frac{s^2}{(1 - s)^3} \cdot \frac{1 - \tilde{t} + t^2}{(1 + t)^2}, & (p, q) = (1, 2); \\
\frac{27}{4} \frac{s}{(1 - s)^3} \cdot \frac{(1 + \tilde{t})^2}{1 - t + t^2}, & (p, q) = (2, 1),
\end{cases}
\]

(6.11)

as the coverings \( \pi_{1,2}^{(3)} : \mathcal{C}_{1,2}^{(3)} \cong \mathcal{C}_{1,2} \to \mathbb{P}^1_\zeta \) and \( \pi_{2,1}^{(3)} : \mathcal{C}_{2,1}^{(3)} \cong \mathcal{C}_{2,1} \to \mathbb{P}^1_\zeta \). The alternative ‘decomposed’ representations in (6.11), motivated by the formula given in (6.18) for \( \phi_{1,2}^{(1)} \), make explicit the compositions \( \pi_{1,2}^{(2)} = \phi_{1,2}^{(1)} \circ \phi_{1,2}^{(2)} \) and \( \pi_{2,1}^{(2)} = \phi_{2,1}^{(1)} \circ \phi_{2,1}^{(2)} \). One can obtain an equivalent degree-6 rational map \( t \mapsto \zeta \) by combining (5.17) with (5.18).

**Example 6.7.** \( \{p, q\} = \{1, 3\} \). The curves \( \mathcal{C}_{1,3}^{(3)}, \mathcal{C}_{3,1}^{(3)} \) are of genus zero (see Cor. 6.4). Consider first the case \( (p, q) = (1, 3) \). The equation of \( \mathcal{C}_{1,3}^{(3)} \subset \mathbb{P}^2 \) is

\[
x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1 = 0,
\]

(6.12)

which follows from Thm. 6.5 or more directly, by eliminating \( x_4 \) from the equations \( \sigma_1 = 0, \sigma_2 = 0 \). The conic (6.12) can be parametrized by inspection, with rational parameter \( u \in \mathbb{P}^1 \), as

\[
[x_1 : x_2 : x_3] = [(1 - u)(1 + 2u) : \tilde{\omega}(1 - \omega u)(1 + 2\omega u) : \omega(1 - \tilde{\omega} u)(1 + 2\tilde{\omega} u)],
\]

(6.13)

where \( \omega := \varepsilon_3 \). (Note that at \( u = 0 \), \([x_1 : x_2 : x_3] = [1 : \tilde{\omega} : \omega] \).

Substitution into Thm. 6.5 yields a degree-24 rational map \( u \mapsto \zeta \), namely the Belyi map

\[
\zeta = \frac{-256}{(1 - 20u^3 - 8u^6)^4} \frac{u^3(1 - u^3)^3(1 + 8u^3)^3}{(1 - 20u^3 - 8u^6)^4} \frac{1 - 2u - 2u^2}{1 + 4u - 2u^2},
\]

(6.14a)

as the covering \( \pi_{1,3}^{(3)} : \mathcal{C}_{1,3}^{(3)} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta \). The decomposed representation (6.14b) comes from comparing (6.14a) with (5.17) and (5.18). It makes explicit the composition \( \pi_{1,3}^{(3)} = \phi_{1,3}^{(1)} \circ \phi_{1,3}^{(2)} \circ \phi_{1,3}^{(3)} \). The degree-2 map \( \phi_{1,3}^{(2)} : \mathcal{C}_{1,3}^{(3)} \to \mathcal{C}_{1,3}^{(2)} \), i.e., \( \mathbb{P}^1_u \to \mathbb{P}^1_\zeta \), was found by inspection. For the quadratic map \( u \mapsto t \), to ensure consistency with the preceding, one should really use the more complicated expression for \( t = (x_1 + x_2)/(x_1 - x_2) \) that follows from (6.13).

30
The top Schwarz curve \( C_{1,3}^{(4)} \subset \mathbb{P}^3 \) is birationally equivalent to \( C_{1,3}^{(3)} \), as the map \( \phi^{(3)} : C_{1,3}^{(4)} \to C_{1,3}^{(3)} \) is of degree 1; so it can also be parametrized by \( u \). Solving the equation \( \sigma_1 = 0 \) for \( x_4 = x_4(u) \) yields \( x_4 = -3u \), hence

\[
[x_1 : x_2 : x_3 : x_4] = [(1 - u)(1 + 2u) : \bar{\omega}(1 - \omega u)(1 + 2\omega u) : \omega(1 - \bar{\omega} u)(1 + 2\bar{\omega} u) : -3u] \quad (6.15)
\]
is an (asymmetric) rational parametrization of \( C_{1,3}^{(4)} \). Equation \( (6.14a) \) can equally well be viewed as defining the degree-24 covering \( \pi_{1,3}^{(4)} : C_{1,3}^{(4)} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta \).

The treatment of \((p, q) = (3, 1)\) is similar. The genus-zero quartic \( C_{3,1}^{(3)} \subset \mathbb{P}^2 \) can be obtained from \( C_{1,3}^{(3)} \subset \mathbb{P}^2 \) by a Cremona inversion \((x_i = 1/x'_i)\), or equivalently by the standard quadratic transformation \((x_i = x_jx_k)\). It has defining equation

\[
x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_3x_3 + x_2^2x_3x_1 + x_3^2x_1x_2 = 0 \quad (6.16)
\]
and is trinodal, with ordinary double points at \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\).

A rational parametrization of \( C_{3,1}^{(3)} \) by \( u \) can be obtained from \( (6.13) \) by undoing the quadratic transformation, and an asymmetric rational parametrization of the top curve \( C_{3,1}^{(4)} \) is similarly obtained from \( (6.15) \). The degree-24 rational map \( u \mapsto \zeta \) given in \( (6.14a) \) can be used as both \( \pi_{3,1}^{(3)} : C_{3,1}^{(3)} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta \) and \( \pi_{3,1}^{(4)} : C_{3,1}^{(4)} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta \).

**Example 6.8.** \( \{p, q\} = \{1, 4\} \). A discussion of the case \( (p, q) = (1, 4) \) will suffice. The equation of \( C_{1,4}^{(3)} \subset \mathbb{P}^2 \) is

\[
x_1^3 + x_2^3 + x_3^3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_1x_1^2 + x_3x_1^2 + x_3x_2^2 + x_1x_2x_3 = 0 \quad (6.17)
\]
which follows from Thm. \( 6.5 \) or more directly, by eliminating \( x_4, x_5 \) from the equations \( \sigma_1 = 0, \sigma_2 = 0, \sigma_3 = 0 \). The cubic \( 6.17 \) has no singular points and is elliptic, i.e., of genus 1, with Klein–Weber invariant \( j = -5^2/2 \). It can be uniformized with the aid of elliptic functions, but it is easier to construct a multivalued parametrization. As usual, let \( t = (x_1 + x_2)/(x_1 - x_2) \), so that \( x_1 : x_2 \) equals \([t + 1 : t - 1]\), and notice that as Thm \( 6.5 \) predicts, Eq. \( 6.17 \) is of degree \( n - 2 = 3 \) in \( x_3 \). By symmetry, \( x_3, x_4, x_5 \) are the three roots, and they are computable in terms of radicals from \( t \) by Cardano’s formula. It follows that each of \( C_{1,4}^{(3)}, C_{1,4}^{(4)}, C_{1,4}^{(5)} \) has a multivalued parametrization with radicals in terms of \( t \), respectively 3, 6, and 6-valued.

The technique of the last example immediately yields the following theorem.

**Theorem 6.9.** For all coprime \( p \geq 1, q \geq 2 \) with \( n := p + q \leq 6 \), one can construct multivalued parametrizations with radicals for the subsidiary curve \( C_{p,q}^{(q)} \subset \mathbb{P}^{q-1} \) and the top curve \( C_{p,q}^{(n)} \subset \mathbb{P}^{n-1} \), respectively \((p + 1)_{q-2}\)-valued and \((n - 2)!\)-valued.
7. Identities with free parameters

We now put the results of §§5 and 6 to use by deriving an interesting collection of parametrized hypergeometric identities. The source of many is Thm. 4.4, which expressed certain \( nF_n - 1 \)'s in terms of algebraic functions. Parts (i) and (ii) of that theorem involve the roots \( x_1, \ldots, x_q \), resp. \( x_1, \ldots, x_n \), of the trinomial equation

\[
x^n - gx^p - \beta = 0,
\]

with \( n := p + q \) and \( \gcd(p, q) = 1 \). It follows that any uniformization of the Schwarz curve \( \mathcal{C}_{p,q}^{(q)} \), resp. \( \mathcal{C}_{p,q}^{(n)} \), will yield a hypergeometric identity. The curves \( \mathcal{C}_{p,q}^{(q)} \), \( \mathcal{C}_{p,q}^{(n)} \) of genus zero were classified in Cor. 6.4, and in the following subsections we give the identities correspondingly uniformized by \( s, t, u \). These are the rational parameters for any \( \mathcal{C}_{1,3}^{(1)} \), any \( \mathcal{C}_{3,1}^{(2)} \), and (by Example 6.7) the plane curves \( \mathcal{C}_{1,3}^{(3)} \), \( \mathcal{C}_{3,1}^{(3)} \).

Each identity derived from Thm. 4.4 in this section, for \( nF_n - 1 \), depends on a discrete parameter \( \ell \in \mathbb{Z} \) and involves a rational function \( F_\ell = F_\ell(A, B; y) \). The functions \( F_\ell \) were defined in §3 (see Table 1 and Thm. 3.5). The reader will recall that in particular, \( F_0 \equiv 1 \) and \( F_1(A, B; y) = y/[(1-B)y+B] \).

Each identity involving \( nF_n - 1 \), derived from Thm. 4.4, also depends on a parameter \( a \in \mathbb{C} \). If \( a \) is chosen so that no upper parameter of the corresponding differential equation \( E_n \) differs by an integer from a lower one, the monodromy group of the \( E_n \) will be of the imprimitive irreducible type characterized in Thm. 2.3, and if \( a \in \mathbb{Q} \), the group will be finite. (The case of equal upper and lower parameters, permitting ‘cancellation,’ is possible only for a finite number of choices of \( a \), such as \( a = \pm 1 \) when \( \ell = 0 \); it is mentioned in Thm. 2.4.) One must treat with care the possibility that one of the lower parameters may be a non-positive integer, in which case \( nF_n - 1 \) is undefined (though it may still be possible to interpret the identity in a limiting sense; cf. Lemma 2.1). It is assumed for simplicity in this section that \( a \in \mathbb{C} \) is chosen so that this does not occur, and so that no division by zero occurs.

The several identities derived below from Thm. 4.4 rather than Thm. 4.3 for \( n_+1F_n \) rather than \( nF_n - 1 \), involve instead of the rational functions \( F_\ell \), \( \ell \in \mathbb{Z} \), the interpolating functions \( G_0, G_1 \) that were defined in Thm. 4.6 in terms of \( 2F_1, 3F_2 \). Each of these has a second free parameter \( c \in \mathbb{C} \), and a similar caveat applies.

7.1. Uniformizations by \( s \)

The rational parametrization of the curve \( \mathcal{C}_{p,q}^{(q)} = \mathcal{C}_{p,1}^{(1)} \) by \( s \in \mathbb{P}^1 \) given in Eqs. 5.16 and 5.18, when substituted into part (i) of each of Thms. 4.4 and 4.6 yields the following.

**Theorem 7.1.** For each \( p \geq 1 \), with \( n := p + 1 \), define a degree-\( n \) Belyi map \( \mathbb{P}_1^1 \rightarrow \mathbb{P}_1^1 \) by

\[
\zeta = \zeta_{p,1}(s) := \frac{n^n}{p^p} \frac{s}{(1-s)^n}.
\]
Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, one has that near $s = 0$,
\[
nF_{n-1} \left( \frac{a, \ldots, a+(n-1)}{p, \ldots, a+(n-1)} \bigg| \zeta, p \right) = (1 - s)^a F_\ell(-a, -p; (1 - s)^{-1}) .
\]

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0, 1$, one has that near $s = 0$,
\[
n+1F_n \left( \frac{a, \ldots, a+(n-1)}{p, \ldots, a+c+(p-1)} \bigg| \zeta, s \right) = (1 - s)^a G_\ell(a, -p, -c; (1 - s)^{-1}) .
\]

**Proof.** Straightforward, as $g^{-1}x_1 = (1 - s)^{-1}$ by [5.16].

**Remark 7.1.1.** For each of $\ell = 0, 1$, the second identity reduces to the first when $c = 0$; and as $c \to \infty$, it reduces to a version of the first in which $\ell$ is incremented by 1. Interpolation of this sort is familiar from Thm. [4.6], and will be seen repeatedly.

Similarly, the parametrization of the top curve $e(n)_{p,q} = e^{(2)}_{1,1} \equiv e^{(1)}_{1,1}$ by $s \in \mathbb{P}^1$, substituted into part (ii) of each of Thms. [4.4] and [4.6] yields the following.

**Theorem 7.2.** As in Theorem [7.1] let
\[
\zeta_{1,1}(s) := -\frac{4s}{(1 - s)^2} .
\]

Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, and $\kappa = 0, 1$, one has that near $s = 1$,
\[
\begin{align*}
\quad & 2F_1 \left( \frac{-a + \ell + \frac{\kappa}{2}, a - \frac{\kappa}{2}}{\frac{\ell}{2} + \kappa} \bigg| \frac{1}{\zeta_{1,1}(s)} \right) \\
= & \frac{(-)^{\kappa} (a + \kappa/2)^{1-\ell+\kappa} \left[-\zeta_{1,1}(s)/4\right]^{\kappa/2}}{(a)_{1-\ell}} \\
& \times \frac{1}{2} \left[s^a F_\ell(-a, \frac{1}{2}; s^{-1}) + (-)^\kappa s^{-a} F_\ell(-a, \frac{1}{2}; s) \right].
\end{align*}
\]

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0, 1$, and $\kappa = 0, 1$, one has that near $s = 1$,
\[
\begin{align*}
\quad & 3F_2 \left( \frac{-a + \ell + 1 + \frac{\kappa}{2}, a + \frac{\kappa}{2}, -a - c + \ell + \frac{\kappa}{2}}{\frac{\ell}{2} + \kappa} \bigg| \frac{1}{\zeta_{1,1}(s)} \right) \\
= & \frac{(-)^{\kappa} (a + \kappa/2)^{-\ell-\kappa} (a + c - \ell - \kappa/2)}{(a)_{-\ell}(a + c - \ell)} \left[-\zeta_{1,1}(s)/4\right]^{\kappa/2} \\
& \times \frac{1}{2} \left[s^a G_\ell(-a, \frac{1}{2}, -c; s^{-1}) + (-)^\kappa s^{-a} G_\ell(-a, \frac{1}{2}, -c; s) \right].
\end{align*}
\]
Proof. Straightforward, as \( \beta^{-1/2}x_1 = s^{-1/2} \) and \( \beta^{-1/2}x_2 = -s^{1/2} \).

The cases \( \ell = 0,1 \) of the identities of Thm. 7.1 were previously obtained by Gessel and Stanton using series manipulations [8]. (See their Eqs. (5.10)–(5.13),(5.15).) When \( p = 2 \), the \( \ell = 0,1 \) cases of the first identity become one-parameter specializations of Bailey’s first cubic transformation of \( 3F_2 \), and its companion.

By comparison, the two identities of Thm. 7.2 are relatively elementary. It should be possible to derive them, or at least the first, by using contiguous function relations or other classical hypergeometric techniques.

7.2. Uniformizations by \( t \)

The rational parametrization of the curve \( \phi_{p,q}^{(2)} = \phi_{p,2}^{(2)} \) by \( t \in \mathbb{P}^1 \) given in Eq. (5.3) and Lemma 5.5 and the fact that \( (5.13),(5.15).) \) When \( p = 2 \), the \( \ell = 0,1 \) cases of the first identity become one-parameter specializations of Bailey’s first cubic transformation of \( 3F_2 \), and its companion.

For each odd \( p \geq 1 \), with \( n := p + 2 \), define a map \( \mathbb{P}^1_\ell \to \mathbb{P}^1_\ell \) by

\[
\zeta = \zeta_{p,2}(t) := \frac{4^n n!}{p^n} \frac{t^2(1 - t)^2p \left[(1 + t)^p + (1 - t)^p\right]}{\left[(1 + t)^n + (1 - t)^n\right]^n}.
\]

Then for all \( a \in \mathbb{C} \) and \( \ell \in \mathbb{Z} \), and \( \kappa = 0,1 \), one has that near \( t = 0 \),

\[
n_{n-1} F_n \left( \left. \frac{a + \kappa}{p}, \ldots, \frac{a + (n-1) + \kappa}{p} \right| \frac{4p^n}{n!} \zeta_{p,2}(t) \right) = (-)^a (a + n\kappa/2)_{1-\ell-\kappa} \left[ \frac{4p^n}{n!} \zeta_{p,2}(t) \right]^{-\kappa/2} \times \frac{1}{2} \left[ (1 + t)^{-2a} F_1 \left( \left. -a, p/2 \right| \frac{4p^n}{n!} \zeta_{p,2}(t) \right) \right]
\]

\[
+ (n-2a) (1 - t)^{-2a} F_1 \left( \left. -a, -p/2 \right| \frac{4p^n}{n!} \zeta_{p,2}(t) \right) \times \left[ \frac{(1 + t)^n + (1 - t)^n}{(1 + t)^p + (1 - t)^p} \right]^{-\kappa/2}.
\]

Moreover, for all \( a \in \mathbb{C} \), \( c \in \mathbb{C} \) and \( \ell = 0,1 \), and \( \kappa = 0,1 \), one has that near \( t = 0 \),

\[
n_{n+1} F_n \left( \left. \frac{a + \kappa}{p}, \ldots, \frac{a + (n+1) + \kappa}{p} \right| \frac{4p^n}{n!} \zeta_{p,2}(t) \right) = (-)^a (a + n\kappa/2)_{1-\ell-\kappa} (a + c - \ell + pk/2) \left[ \frac{4p^n}{n!} \zeta_{p,2}(t) \right]^{-\kappa/2} \times \frac{1}{2} \left[ (1 + t)^{-2a} G_1 \left( \left. -a, p/2, -c ; (1 + t)^2\right| \frac{4p^n}{n!} \zeta_{p,2}(t) \right) \right]
\]

\[
+ (n-2a) (1 - t)^{-2a} G_1 \left( \left. -a, -p/2, -c ; (1 + t)^2\right| \frac{4p^n}{n!} \zeta_{p,2}(t) \right) \times \left[ \frac{(1 + t)^n + (1 - t)^n}{(1 + t)^p + (1 - t)^p} \right]^{-\kappa/2}.
\]

34
Remark 7.3.1. The even function $\zeta = \zeta_{p,2}(t)$ is a degree-$[n(n-1)]$ Belyi map. The case $\ell = 0$, $\kappa = 0$ of the first identity appeared in §1 as the sample result (1.2). If $\kappa = 0$ then there is simplification; the right-hand prefactor becomes unity.

Similarly, the rational parametrization of the top curve $e^{(n)}_{\nu,\beta} = e^{(3)}_{1,2} \cong e^{(2)}_{1,2}$ given in (6.10) by $\tilde{t} \in \mathbb{P}^1$ (related to $t$ by a Möbius transformation), substituted into part (ii) of each of Thms. 4.1 and 4.6 yields the following.

**Theorem 7.4.** Define a degree-6 Belyi map $\mathbb{P}^1_\tilde{t} \to \mathbb{P}^1_\zeta$ by

$$\zeta = \zeta_{1,2}(\tilde{t}) := \frac{(1 + \tilde{t}^3)^2}{4\tilde{t}^3} = 1 + \frac{(1 - \tilde{t}^3)^2}{4\tilde{t}^3}.$$  

Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, and $\kappa = 0,1,2$, one has that near $\tilde{t} = 0$,

$$3F_2\left(\begin{array}{c}-a + \ell + \tfrac{4}{3}; 1 + \tfrac{4}{3}, 2 + \tfrac{4}{3}, \tfrac{a+3}{2} + \tfrac{4}{3} \\ b_1(\kappa), b_2(\kappa)\end{array}\bigg| \frac{1}{\zeta_{1,2}(\tilde{t})}\right) = \left(\frac{-1}{\kappa}(1, a+2\kappa/3)_{1-\kappa}ight) \left(\frac{4}{27} \zeta_{1,2}(\tilde{t})\right)^{\kappa/3}$$

$$\times \left[\frac{(1 + \tilde{t}^3)^{-a}}{1 + \tilde{t}^3}\right] F_\ell\left(-a, \frac{1}{3}; \frac{1 + \tilde{t}^3}{1 + t^3}\right) + \omega^n \left[\frac{(\omega + \omega\tilde{t})^3}{1 + \tilde{t}^3}\right]^{-a} F_\ell\left(-a, \frac{1}{3}; \frac{\omega + \omega\tilde{t}}{1 + \tilde{t}^3}\right) - \omega^n \left[\frac{(\omega + \omega\tilde{t})^3}{1 + \tilde{t}^3}\right]^{-a} F_\ell\left(-a, \frac{1}{3}; \frac{\omega + \omega\tilde{t}}{1 + \tilde{t}^3}\right) \bigg\frac{1}{\zeta_{1,2}(\tilde{t})}\bigg\right.$$  

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0,1,2$, and $\kappa = 0,1$, one has that near $\tilde{t} = 0$,

$$4F_3\left(\begin{array}{c}-a + \ell + 1 + \tfrac{5}{3}; 1 + \tfrac{5}{3}, a + \tfrac{5}{3}; -a + c + \ell + \tfrac{4}{3} \\ b_1(\kappa), b_2(\kappa)\end{array}\bigg| \frac{1}{\zeta_{1,2}(\tilde{t})}\right) = \left(\frac{-1}{\kappa}(1, a+2\kappa/3)_{1-\kappa}(a + c - \ell - \kappa/3)\right) \left(\frac{4}{27} \zeta_{1,2}(\tilde{t})\right)^{\kappa/3}$$

$$\times \left[\frac{(1 + \tilde{t}^3)^{-a}}{1 + \tilde{t}^3}\right] G_\ell\left(-a, \frac{1}{3}; c; \frac{1 + \tilde{t}^3}{1 + t^3}\right) + \omega^n \left[\frac{(\omega + \omega\tilde{t})^3}{1 + \tilde{t}^3}\right]^{-a} G_\ell\left(-a, \frac{1}{3}; c; \frac{\omega + \omega\tilde{t}}{1 + \tilde{t}^3}\right)$$

$$\omega^n \left[\frac{(\omega + \omega\tilde{t})^3}{1 + \tilde{t}^3}\right]^{-a} G_\ell\left(-a, \frac{1}{3}; c; \frac{\omega + \omega\tilde{t}}{1 + \tilde{t}^3}\right) \bigg\frac{1}{\zeta_{1,2}(\tilde{t})}\bigg\right.$$  

In these two identities, the lower parameters $b_1, b_2$ are $\frac{1}{3}, \frac{2}{3}$; or $\frac{2}{3}, \frac{4}{3}$; or $\frac{4}{3}, \frac{5}{3}$, depending on whether $\kappa = 0, 1, 2$. Also, $\omega := \varepsilon_3 = \exp (2\pi i/3)$ and $\bar{\omega} := \varepsilon_3^2$.

**Proof.** Straighforward, as $\beta = \sigma_3 = x_1, x_2, x_3 = 1 + \tilde{t}^3$ by (6.10).

**Remark 7.4.1.** In the final factor on the right of each identity, each of the three square-bracketed rational functions of $\tilde{t}$ equals unity at $\tilde{t} = 0$, determining the branch to be used when exponentiating.

One can derive identities from the parametrization by $\tilde{t} \in \mathbb{P}^1$ of the birationally equivalent top curve $e^{(n)}_{2,1} \equiv e^{(2)}_{1,2}$, also given in (6.10). Details are left to the reader.  

35
7.3. Uniformizations by \( u \)

The rational parametrization of the curve \( C_{p,q} = C_{1,3} \) by \( u \in \mathbb{P}^1 \) given in \([6.13]\) in Example 6.7 when substituted into part (i) of each of Thms. 4.4 and 4.6 yields the following.

**Theorem 7.5.** Define a degree-24 Bely\'ı map \( \mathbb{P}^1 \to \mathbb{P}^1 \) by

\[
\zeta = \zeta_{1,3}(u) := -256 \frac{u^3(1-u^3)^3 + 8u^3}{(1-20u^3 - 8u^6)^4} = 1 - \frac{(1 + 8u^6)^2(1 + 88u^3 - 8u^6)^2}{(1 - 20u^3 - 8u^6)^4}.
\]

Then for all \( a \in \mathbb{C} \) and \( \ell \in \mathbb{Z} \), and \( \kappa = 0, 1, 2 \), one has that near \( u = 0 \),

\[
4F_3 \begin{pmatrix} \frac{a}{3} + \frac{\kappa}{3}, \ldots, \frac{a + 3}{3} + \frac{\kappa}{3} \end{pmatrix}; \frac{b_1(\kappa)}{3}, b_2(\kappa) \mid \zeta_{1,3}(u) \\
\begin{aligned}
&= \frac{(-)^\kappa(a + 4\kappa/3)}{(a + 6\kappa/3)} \frac{1}{(1-20u^3 - 8u^6)} G_\ell \left( a, -\frac{1}{3}; \frac{(1-u^3)(1+2u^3)}{1-20u^3 - 8u^6} \right) \\
&\quad + \bar{\omega} \frac{(1-\omega u^3)(1+2\omega u^3)^{3-\alpha}}{1-20u^3 - 8u^6} G_\ell \left( a, -\frac{1}{3}; \frac{(1-\omega u^3)(1+2\omega u^3)}{1-20u^3 - 8u^6} \right)
\end{aligned}
\]

Moreover, for all \( a \in \mathbb{C} \), \( c \in \mathbb{C} \) and \( \ell = 0, 1, \) and \( \kappa = 0, 1, 2 \), one has that near \( u = 0 \),

\[
5F_4 \begin{pmatrix} \frac{\alpha}{3} + \frac{\kappa}{3}, \ldots, \frac{\alpha + 4\kappa/3}{3} + \frac{\kappa}{3} \end{pmatrix}; \frac{\alpha + c - \ell/3}{3}, \frac{\alpha + c + 1 + \kappa/3}{3} \mid \zeta_{1,3}(u) \\
\begin{aligned}
&= \frac{(-)^\kappa(a + 4\kappa/3)}{(a + 6\kappa/3)} \frac{1}{(1-20u^3 - 8u^6)} G_\ell \left( a, -\frac{1}{3}; -c; \frac{(1-u^3)(1+2u^3)}{1-20u^3 - 8u^6} \right) \\
&\quad + \bar{\omega} \frac{(1-\omega u^3)(1+2\omega u^3)^{3-\alpha}}{1-20u^3 - 8u^6} G_\ell \left( a, -\frac{1}{3}; -c; \frac{(1-\omega u^3)(1+2\omega u^3)}{1-20u^3 - 8u^6} \right)
\end{aligned}
\]

In both identities, the lower parameters \( b_1, b_2 \) are defined as in the previous theorem.

**Proof.** Straightforward, as \( g = \sigma_3 = 1 - 20u^3 - 8u^6 \) by \([6.13]\). \( \square \)

One can similarly derive identities from the rational parametrizations of the pair of top curves \( C_{1,3} \cong C_{1,3} \) and \( C_{3,1} \cong C_{3,1} \) given in Example 6.7 by substituting them into Thms. 4.4(ii) and 4.6(ii). The resulting identities also involve \( 4F_3, 5F_4 \).
8. Identities without free parameters

In this final section we sketch another approach to the parametrizing of \( nF_{n-1} \)'s, based on computation in rings of symmetric polynomials. The need for this is indicated by the previous results. Each identity in \( \mathbb{Z} \) had a free parameter \( a \in \mathbb{C} \) and was based on a (rational) parametrization of a Schwarz curve, either \( \mathcal{C}_{p,q}^{(q)} \) or \( \mathcal{C}_{p,q}^{(n)} \). But not many such curves are of zero or low genus, making it hard to derive general identities, even if multivalued parametrizations with radicals are allowed. We now show that if \( a \in \mathbb{Z} \), the relevant curve is a quotient curve \( \mathcal{C}_{p,q}^{(q,\text{symm})} \) or \( \mathcal{C}_{p,q}^{(n,\text{symm})} \); and if \( qa \in \mathbb{Z} \), resp. \( na \in \mathbb{Z} \), it is a quotient \( \mathcal{C}_{p,q}^{(q,\text{cycl})} \) resp. \( \mathcal{C}_{p,q}^{(n,\text{cycl})} \). This facilitates parametrization. The examples in \( \mathbb{Z} \) illustrate this, though they deal with algebraic \( nF_{n-1} \)'s arising from \( E_n \)'s that are not of the imprimitive irreducible type characterized in Thm. 2.3, on account of reducible monodromy and/or a lower hypergeometric parameter being a non-positive integer.

8.1. Theory

The following is a restatement of the \( \ell = 0 \) case of Thm. 2.3 emphasizing the role played by symmetric polynomials. As usual, \( \sigma_1 = \sigma_1(x_1, \ldots, x_n) \) denotes the \( l \)th elementary symmetric polynomial, and \( \mathcal{G}^{(n)}_{p,q} \subset \mathbb{P}^{n-1} \) comprises all points \( [x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} \) such that \( \sigma_1, \ldots, \sigma_{q-1} \) and \( \sigma_{q+1}, \ldots, \sigma_{n-1} \) equal zero.

**Definition 8.1.** For each \( p, q \geq 1 \) with \( \gcd(p, q) = 1 \) and \( n := p + q \), let

\[
\mathcal{F}_{p,q}^{(q,n)}(a; \zeta) := nF_{n-1} \left( \frac{\zeta}{q}, \frac{\zeta}{q}, \ldots, \frac{\zeta}{q}, \frac{\zeta}{q}, \frac{\zeta}{q} \right)
\]

for \( \zeta = 0, \ldots, q - 1 \), and

\[
\mathcal{F}_{p,q}^{(q,n)}(a; \zeta) := nF_{n-1} \left( \frac{\zeta}{q}, \frac{\zeta}{q}, \ldots, \frac{\zeta}{q}, \frac{\zeta}{q} \right)
\]

for \( \zeta = 0, \ldots, n - 1 \). For all \( a \in \mathbb{C} \), each is defined and analytic in a neighborhood of \( \zeta = 0 \), unless (in the first) a lower parameter is a non-positive integer. If exactly one lower parameter equals such an integer, \( -m \), and an upper parameter equals an integer \( -m' \) with \( -m \leq -m' \leq 0 \), then the function can be defined as a limit, as in Lemma 2.4.

By a congruential argument, the differential equation \( E_n \) corresponding to each of these \( nF_{n-1} \)'s will have reducible monodromy, i.e., have an upper and a lower parameter that differ by an integer, if \( qa \in \mathbb{Z} \), resp. \( na \in \mathbb{Z} \). In fact, the \( E_n \) will have \( \zeta = 0 \), resp. \( \zeta = \infty \) as a logarithmic point, i.e., have a pair of lower resp. upper parameters that differ by an integer, if \( qa \in \mathbb{Z} \), resp. \( na \in \mathbb{Z} \).

**Theorem 8.1.** (i) Near the point \( [x_1 : \ldots : x_n] \) on \( \mathcal{G}_{p,q}^{(n)} \) with

\[
x_j = \begin{cases} 
\varepsilon_q^{-(j-1)}, & j = 1, \ldots, q, \\
0, & j = q + 1, \ldots, n,
\end{cases}
\]

37
at which $(-)^{q-1}\sigma_q = 1$ and $\sigma_n = 0$, for each $\kappa = 0, \ldots, q - 1$ one has

$$
\mathcal{F}_{p,q}^{(q;\kappa)}(a;\zeta) = \frac{(-)^{\kappa}(1)\varepsilon(a + n\kappa/q)_{1-\kappa}}{(a)_1} 
\times \left[ (-)^{q-1}\sigma_\zeta \right]^{-\kappa/q} \left[ (-)^{q-1}\sigma_q \right]^a \left[ \frac{1}{q} \sum_{1 \leq j \leq q} (\varepsilon_q^{q-n_j})^{(j-1)\kappa} (\varepsilon_q^{j-1})^{-\kappa} \right].
$$

(ii) Near the point $[x_1 : \ldots : x_n]$ on $\mathcal{O}_{p,q}^{(n)}$ with

$$
x_j = \varepsilon_n^{(j-1)}, \quad j = 1, \ldots, n,
$$

at which $\sigma_q = 0$ and $(-)^{q-1}\sigma_n = 1$, for each $\kappa = 0, \ldots, n - 1$ one has

$$
\mathcal{F}_{p,q}^{(n;\kappa)}(a;\zeta^{-1}) = \frac{(-)^{\kappa}(1)\varepsilon(a + q\kappa/n)_{1-\kappa}}{(a)_1} 
\times \left[ (-)^{q-1}\sigma_\zeta \right]^{-\kappa/n} \left[ (-)^{q-1}\sigma_n \right]^a \left[ \frac{1}{n} \sum_{1 \leq j \leq n} (\varepsilon_n^{q-n_j})^{(j-1)\kappa} (\varepsilon_n^{j-1})^{-\kappa} \right].
$$

In both (i) and (ii),

$$
\zeta := (-)^n \frac{n^n \sigma_n^q}{p^n q^n \sigma_q^{q-1}}; \quad (-)^q \frac{p^n q^n}{n^n} \zeta = (-)^q \frac{\sigma_n^q}{\sigma_q^{q-1}}.
$$

so that $\zeta$ is near $0$, resp. $\infty$; and it is assumed that $a \in \mathbb{C}$ is such that the left-hand function is defined. When $\kappa > 0$, branches must be chosen appropriately.

The case when $a \in \mathbb{Z}$, in particular when $a$ is a negative integer, is especially easy to parametrize. If $a \in \mathbb{Z}$ and $\kappa = 0$, the identities of Thm. 8.2 reduce to

$$
\mathcal{F}_{p,q}^{(q;0)}(a;\zeta) = \left[ (-)^{q-1}\sigma_q \right]^a \cdot \frac{1}{q} \sum_{j = 1}^q x_j^{-q\zeta}, \quad (8.1a)
$$

$$
\mathcal{F}_{p,q}^{(n;0)}(a;\zeta^{-1}) = \left[ (-)^{q-1}\sigma_n \right]^a \cdot \frac{1}{n} \sum_{j = 1}^n x_j^{-n\zeta}. \quad (8.1b)
$$

There is permutation symmetry under $\mathcal{G}_q$, resp. $\mathcal{G}_n$. This suggests focusing, for $k = 2, \ldots, n$, on the action of $\mathcal{G}_k$ on $\mathcal{O}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$. It was shown in §5 that $\mathcal{O}_{p,q}^{(k)}$ is defined by a system of $k - 2$ equations, each of which sets to zero a homogeneous polynomial in the invariants $\bar{\sigma}_1, \ldots, \bar{\sigma}_k$, the elementary symmetric polynomials in $x_1, \ldots, x_k$. The function field of $\mathcal{O}_{p,q}^{(k)}$ is the field of rational functions in $x_1, \ldots, x_k$ that are homogeneous of degree zero, with these defining equations quotiented out.

The function field of $\mathcal{O}_{p,q}^{(k)}$ contains a subfield of $\mathcal{G}_k$-stable functions, comprising all rational functions in $\bar{\sigma}_1, \ldots, \bar{\sigma}_k$ that are homogeneous (in $x_1, \ldots, x_k$) of degree zero; again, with the defining equations quotiented out. One element
of this subfield is the function $\zeta$, which gives the degree-$(n - k + 1)k$ covering $\pi_{p,q}^{(k)}: \mathcal{C}^{(k)}_{p,q} \to \mathbb{P}^{1}_{\zeta}$. Up to birational equivalence, define a quotiented curve $\mathcal{C}^{(k;\text{symm})}_{p,q} := \mathcal{C}^{(k)}_{p,q}/S_{n}$ that has this subfield of $\mathcal{S}_{n}$-stable functions as its function field, so that

$$\mathcal{C}^{(k)}_{p,q} \longrightarrow \mathcal{C}^{(k;\text{symm})}_{p,q} \longrightarrow \mathbb{P}^{1}_{\zeta} \quad (8.2)$$

is a decomposition of $\pi_{p,q}^{(k)}$ into maps of respective degrees $k!, \binom{n}{q}$. For instance, one has $\mathcal{C}^{(2)}_{p,q} \cong \mathbb{P}^{1}_{1}$ and $\mathcal{C}^{(2;\text{symm})}_{p,q} \cong \mathbb{P}^{1}_{v}$, where $v := t^{2}$. This is because the only nontrivial element of $\mathcal{S}_{2}$ is the involution $x_{1} \leftrightarrow x_{2}$, which on account of $[x_{1} : x_{2}] = [t + 1 : t - 1]$ is the involution $t \mapsto -t$; so the function field of $\mathcal{C}^{(2;\text{symm})}_{p,q}$ comprises only even functions. By convention, $\mathcal{C}^{(1;\text{symm})}_{p,q} \cong \mathcal{C}^{(1)}_{p,q} \cong \mathbb{P}^{1}_{s}$ and $\mathcal{C}^{(0;\text{symm})}_{p,q} \cong \mathbb{C}_{p,q} \cong \mathbb{P}^{1}_{\zeta}$.

**Lemma 8.3.** $\mathcal{C}^{(k;\text{symm})}_{p,q} \cong \mathcal{C}^{(n-k;\text{symm})}_{p,q}$ for $k = 0, \ldots, n$, due to the respective function fields being isomorphic. Here $n := p + q$, as usual.

**Proof.** To prove the case $k = n$ (or $k = 0$), observe that the function field of $\mathcal{C}^{(n;\text{symm})}_{p,q}$ comprises all rational functions of $\sigma_{n}/\sigma_{q} \propto \zeta$. (Also, $\mathcal{S}_{n}$ is the covering group of $\pi_{p,q}^{(n)}: \mathcal{C}^{(n)}_{p,q} \to \mathbb{P}^{1}_{\zeta}$.) If $0 < k < n$, use the fact that any element of $\mathcal{C}^{(n-k;\text{symm})}_{p,q}$ can be viewed as a quotient of two symmetric polynomials in $x_{k+1}, \ldots, x_{n}$ that are homogeneous and of the same degree. Any homogeneous symmetric polynomial in $x_{k+1}, \ldots, x_{n}$ can be expressed as a quotient of two such polynomials in $x_{1}, \ldots, x_{k}$ by the elimination procedure of §5 cf. Lemma 5.1.

**Theorem 8.4.** For each $a \in \mathbb{Z}$,

(i) the algebraic function $\zeta \mapsto \mathcal{F}^{(q;0)}_{p,q}(a; \zeta)$ can be uniformized by $\mathcal{C}^{(q;\text{symm})}_{p,q}$.

(ii) the algebraic function $\zeta \mapsto \mathcal{F}^{(n;0)}_{p,q}(a; \zeta)$ can be uniformized by $\mathcal{C}^{(n;\text{symm})}_{p,q}$.

**Proof.** Immediate, by the permutation symmetry of $\mathcal{F}^{(q;0)}_{p,q}$, $\mathcal{F}^{(n;0)}_{p,q}$. But note that when uniformizing, one should really use smooth models of $\mathcal{C}^{(q;\text{symm})}_{p,q}$, $\mathcal{C}^{(n;\text{symm})}_{p,q}$. Arbitrary models may have multiple points, as the unquotiented $\mathcal{C}^{(k)}_{p,q}$, $\mathcal{C}^{(k)}_{p,q}$ may.

The following is a specialization of Thm. 8.4 with a constructive proof.

**Theorem 8.5.** For each $a \in \mathbb{Z}$,

(i) the algebraic function $\zeta \mapsto \mathcal{F}^{(q;0)}_{p,q}(a; \zeta)$ can be uniformized by $\mathcal{C}^{(1)}_{p,q}$, resp. $\mathcal{C}^{(2)}_{p,q}$, i.e., rationally parametrized by the rational parameter $s$, resp. $t$, if $p = 1$ or $q = 1$, resp. $p = 2$ or $q = 2$.

(ii) the algebraic function $\zeta \mapsto \mathcal{F}^{(n;0)}_{p,q}(a; \zeta)$ is in fact rational, i.e., is uniformized by $\mathcal{C}^{(0)}_{p,q} := \mathbb{P}^{1}_{\zeta}$.
In part (i), the rational parametrization of the argument, whether $ζ = π_{p,q}^{(1)}(s)$ or $ζ = π_{p,q}^{(2)}(t)$, was given in Theorem 5.6.

Proof. Part (ii) is trivial, as $x_{p,q}^{(a,0)}(a; ζ)$ will be a polynomial in $ζ$ if $a ∈ ℤ$, as one of the upper hypergeometric parameters is then a nonpositive integer (see Defn. 5.1). Also, the cases $q = 1, 2$ of part (i) are familiar from §5. If $q = 1$ then the right side of (5.1a) equals $(1 − s)^a$, and if $q = 2$ then one can use the parametrization $[x_1 : x_2] = [t + 1 : t − 1]$ and the formula for $σ_q = σ_q(x_1, x_2)$ given in Lemma 5.3 to express the right side of (5.1a) as a rational function of the parameter $t$.

The cases $p = n − q = 1, 2$ of part (i) are more interesting, and can be viewed as consequences of Lemma 5.3. If the integer $γ = −qa$ is positive, write

$$
\sum_{j=1}^q x_j^γ = \begin{cases} pγ - x_j^n, & p = 1, \\
               pγ - x_j^n - x_j^{n−1}, & p = 2,
\end{cases}
$$

where $p_γ := \sum_{j=1}^n x_j^γ$. This is a symmetric polynomial in $x_1, . . . , x_n$, the so-called $γ$'th power-sum symmetric polynomial. The elementary symmetric polynomials $\{σ_l\}_{l=1}^n$ in $x_1, . . . , x_n$ form an algebraic basis for the ring of symmetric ones; so the $\{p_γ\}_{γ≥1}$ can be expressed as polynomials in the $\{σ_l\}_{l=1}^n$, e.g., by inverting or otherwise exploiting the Newton–Girard formula [16, §1.2]

$$
σ_l = l^{-1} \sum_{γ=1}^l (−)^γ p_γ σ_{l−γ}.
$$

This formula holds for all $l ≥ 1$, it being understood that $σ_l = 0$ if $l > n$.

Now, introduce an alternative sequence of subsidiary Schwarz curves

$$
euc_{p,q}^{(n)} → euc_{p,q}^{(n−1)} → \cdots → euc_{p,q}^{(2)} → euc_{p,q}^{(1)} → \mathbb{P}^1_{ζ},
$$

based on the successive elimination of $x_1, . . . , x_n$ rather than of $x_n, . . . , x_1$, so that $euc_{p,q}^{(1)} ≅ \mathbb{P}^1_{s'}$, where

$$
σ_q = (−)^q−1 x_q^n · (1 − s'), \quad σ_n = (−)^n−1 x_n · s',
$$

(see (5.10)), and $euc_{p,q}^{(2)} ≅ \mathbb{P}^1_{t'}$ is parametrized by $t' := (x_n + x_{n−1})/(x_n − x_{n−1})$, so that $[x_n : x_{n−1}] = [t' + 1 : t' − 1]$. By using (5.4) and (5.6), one can write the right side of (5.1a) as a rational function of $s'$, resp. $t'$; and the expressions for $ζ$ in terms of $s', t'$ are the same as those for $ζ$ in terms of $s, t$.

The proof when $γ = −qa < 0$ is an easy modification of the preceding. □

Remark 8.5.1. The cases $q = 1, 2$ of part (i) were proved in §5. That rational uniformization is also possible when $p = 1, 2$ is attributable to $euc_{p,q}^{(k)} ≅ euc_{q,p}^{(k)}$.

Remark 8.5.2. Examples of Thm. 8.5.1, showing how the algorithm in its proof is applied, are given in §8.2 below. They also show how Lemma 2.4 is applied, to deal with lower hypergeometric parameters that are non-positive integers.
Now consider what the two identities of Thm. 8.7 amount to, when \( a \) is not an integer, but nonetheless \( qa \in \mathbb{Z} \), resp. \( na \in \mathbb{Z} \). If \( a = \frac{m}{q} \), resp. \( a = \frac{m}{n} \), with \( m \in \mathbb{Z} \), and also \( \kappa = 0 \), they reduce to

\[
\mathcal{F}_{p,q}^{(q;0)}(\frac{m}{q}; \zeta) = \left[ (-)^{q-1} \sigma_q \right]^{m/q} \frac{1}{q} \sum_{j=1}^{q} \varepsilon_q^{-(j-1)m} x_j^{-m}, \tag{8.7a}
\]

\[
\mathcal{F}_{p,q}^{(n;0)}(\frac{m}{n}; \zeta^{-1}) = \left[ (-)^{n-1} \sigma_n \right]^{m/n} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_n^{-(j-1)m} x_j^{-n}. \tag{8.7b}
\]

It follows that the power \( \left[ \mathcal{F}_{p,q}^{(q;0)}(\frac{m}{q}; \zeta) \right]^q \), resp. \( \left[ \mathcal{F}_{p,q}^{(n;0)}(\frac{m}{n}; \zeta^{-1}) \right]^n \), is a rational function of \( x_1, \ldots, x_q \), resp. \( x_1, \ldots, x_n \), i.e., is in the function field of \( \mathcal{E}_{p,q}^{(q)} \) resp. \( \mathcal{E}_{p,q}^{(n)} \). But these rational functions are not stable under the action of \( \mathcal{E}_q \) on \( x_1, \ldots, x_q \), resp. \( \mathcal{E}_n \) on \( x_1, \ldots, x_n \). Rather, they are invariant under the associated subgroups of cyclic permutations. The group of cyclic permutations of \( x_1, \ldots, x_k \) will be denoted by \( \mathcal{E}_k \), and the group of dihedral ones (if \( k > 2 \)) by \( \mathcal{D}_k \). The relevant order and subgroup indices are \( |\mathcal{E}_k| = k \) and \( (\mathcal{D}_k : \mathcal{E}_k) = 2 \), \( (\mathcal{S}_k : \mathcal{D}_k) = (k-1)!/2 \).

**Definition 8.6.** The quotient curves \( \mathcal{E}_{p,q}^{(k;\text{cycl})} \), \( \mathcal{E}_{p,q}^{(k;\text{dihedr})} \), \( \mathcal{E}_{p,q}^{(k;\text{symm})} \) are defined to be \( \mathcal{E}_{p,q}^{(k)} / \Gamma \) with \( \Gamma = \mathcal{E}_k \), \( \mathcal{D}_k \), \( \mathcal{S}_k \), so that if \( k > 2 \) one has the sequence of coverings

\[
\mathcal{E}_{p,q}^{(k)} \rightarrow \mathcal{E}_{p,q}^{(k;\text{cycl})} \rightarrow \mathcal{E}_{p,q}^{(k;\text{dihedr})} \rightarrow \mathcal{E}_{p,q}^{(k;\text{symm})} \rightarrow \mathbb{P}^1_{\zeta},
\]

which have respective degrees \( k, (k-1)!/2, \) and \( (\begin{array}{c} n \vspace{0.5em} \end{array} k) \), as a decomposition of the degree-\((n-k+1)k\) covering map \( \pi_{p,q}^{(k)} : \mathcal{E}_{p,q}^{(k)} \rightarrow \mathbb{P}^1_{\zeta} \). If \( k = 1, 2 \) then there is no dihedral curve, but \( \mathcal{E}_{p,q}^{(k;\text{cycl})} \rightarrow \mathcal{E}_{p,q}^{(k;\text{symm})} \) has degree \((k-1)!\) in all cases.

**Theorem 8.7.** For each \( m \in \mathbb{Z} \),

(i) the algebraic function \( \zeta \mapsto \left[ \mathcal{F}_{p,q}^{(q;0)}(\frac{m}{q}; \zeta) \right]^q \) can be uniformized by \( \mathcal{E}_{p,q}^{(q;\text{cycl})} \).

(ii) the algebraic function \( \zeta \mapsto \left[ \mathcal{F}_{p,q}^{(n;0)}(\frac{m}{n}; \zeta^{-1}) \right]^n \) can be uniformized by \( \mathcal{E}_{p,q}^{(n;\text{cycl})} \).

**Proof.** Immediate, by the permutation symmetry of \( (8.7a),(8.7b) \). \( \square \)

The following is a specialization of Theorem 8.7 with a constructive proof.

**Theorem 8.8.** For each \( m = -1, -2, \ldots \),

(i) the algebraic function \( \zeta \mapsto \left[ \mathcal{F}_{1,q}^{(q;0)}(\frac{m}{q}; \zeta) \right]^q \), for each \( q \geq 1 \), has a \((q-1)!\)-valued parametrization by \( s \), the rational parameter of \( \mathcal{E}_{1,q}^{(1)} \cong \mathbb{P}^1_s \), i.e., it satisfies a degree-\((q-1)!\) polynomial equation with coefficients in \( \mathbb{Z}[s] \).
(ii) the algebraic function \( \zeta \mapsto \left[ \mathcal{F}_{\frac{m}{q}}^{(q,0)}(\frac{m}{q}; \zeta) \right]^q \), for each odd \( q \geq 1 \), has a \((q-1)!\)-valued parametrization by \( v = t^2 \), the square of the rational parameter of \( \mathcal{C}_{1,q}^{(2)} \cong \mathbb{P}^1 \), i.e., it satisfies a degree-\((q-1)!\) polynomial equation with coefficients in \( \mathbb{Z}[v] \).

The rational parametrizations \( \zeta = \pi_{1,q}^{(1)}(s) \) and \( \zeta = \pi_{2,q}^{(2)}(t) \) of the argument of the algebraic function, for (i),(ii), were given in Theorem 5.6 the latter is even in \( t \).

**Proof.** The rings of symmetric polynomials in \( x_1, \ldots, x_n \) and in \( x_1, \ldots, x_n \) have algebraic bases \( \{r_i^{(n)}\}_{i=1}^q \) or \( \{p_i^{(n)}\}_{i=1}^q \), resp. \( \{\sigma_i^{(n)}\}_{i=1}^n \) or \( \{\rho_i^{(n)}\}_{i=1}^n \), of elementary or power-sum symmetric polynomials. Consider

\[
G_{q,-m}(y) := \prod_{\chi} \left\{ y - \left[ \sum_{j=1}^q \varepsilon_q^{-j-1} m x_{\chi(j)}^{-m} \right]^q \right\}, \quad (8.8)
\]

the product being taken over one representative \( \chi \) of each of the \((q-1)!\) cosets of \( \mathcal{C}_q \) in \( \mathfrak{S}_q \), acting on \( x_1, \ldots, x_q \). By Eq. (8.7),

\[
\left[ \mathcal{F}_{p,q}^{(m)}(\frac{m}{q}; \zeta) \right]^q = q^{-q} \left[ (-)^q \sigma_q \right]^m y_0, \quad (8.9)
\]

where \( y_0 \) is a certain root of \( G_{q,-m}(y) \). Each coefficient of the degree-\((q-1)!\) polynomial \( G_{q,-m}(y) \) is stable under the action of \( \mathfrak{S}_q \) on \( x_1, \ldots, x_q \) and can therefore be expressed as a polynomial in the \( \{p_i^{(q)}\}_{i=1}^q \). But (cf. Lemma 5.4 once again),

\[
p_{\gamma} = \sum_{j=1}^q x_j^\gamma = \begin{cases} p_n^{(n)} - x_n^\gamma, & p = 1, \\ p_n^{(n)} - x_n^\gamma, & p = 2. \end{cases} \quad (8.10)
\]

Hence each coefficient of \( G_{q,-m}(y) \) is expressible in terms of the \( \{\sigma_i^{(n)}\}_{i=1}^n \) and \( x_n \), resp. \( x_n, x_{n-1} \). But on \( \mathcal{C}_q \), each of the \( \{\sigma_i^{(n)}\}_{i=1}^n \) equals zero, except for \( \sigma_q^{(n)}, \sigma_n^{(n)} \). By introducing the alternative sequence \( \mathcal{C}_{p,q}^{(n)} \) of subsidiary Schwarz curves, one can express \( \sigma_q^{(n)}, \sigma_n^{(n)} \) and \( x_n, x_{n-1} \), in terms of \( s' \) resp. \( t' \), the rational parameter of \( \mathcal{C}_{1,p}^{(1)} \) resp. \( \mathcal{C}_{2,p}^{(2)} \). And the expressions for \( \zeta \) in terms of \( s', t' \) are the same as those for \( \zeta \) in terms of \( s, t \). Note that each rational function of \( t' \) encountered is an even one, a function of \( v' := (t')^2 \), by invariance under \( x_n \leftrightarrow x_{n-1} \).

The theorems in the following two subsections, \( \S \S 8.2 \) and \( \S 8.3 \) show how the approach of this section can parametrize an individual algebraic \( \mathfrak{a}_n F_{n-1} \), even when it is difficult to derive a general hypergeometric identity with a free parameter \( a \in \mathbb{C} \). Computations in rings of symmetric polynomials, as in the proofs of Thms. 8.3 and 8.8 are the key. The point is that the quotiented Schwarz curves \( \mathcal{C}_{p,q}^{(q,\text{symm})} \) and \( \mathcal{C}_{p,q}^{(n,\text{symm})} \), resp. \( \mathcal{C}_{p,q}^{(q,\text{cycl})} \) and \( \mathcal{C}_{p,q}^{(n,\text{cycl})} \), can be
used for uniformization. The examples in §8.3 show that even if the relevant curve is of positive genus, one can sometimes obtain a useful (multivalued) parametrization with radicals.

8.2. Parametrizations with \( a \in \mathbb{Z} \)

The two theorems below are sample applications of the algorithm embedded in the proof of Thm. 8.5. They show how it is possible to parametrize rationally many algebraic \( n \)’s with \( a \in \mathbb{Z} \), due to their being uniformized by quotient curves \( \mathcal{C}_{p,q}^{(q,\text{symm.})} \) that are of genus zero. These examples also illustrate how lower parameters that are non-positive integers can be handled with the aid of Lemma 2.1.

The first theorem is relatively simple, as the governing curve is \( \mathcal{C}_{p,q}^{(q)} = \mathcal{C}_{n-1,1}^{(1)} \), which is of genus zero without quotienting; though Lemma 2.1 is used.

**Theorem 8.9.** For each \( n \geq 2 \),

\[
S_{n-1}^{-2} \left( \frac{-1}{n}, \frac{1}{n}, \ldots, \frac{n-2}{n} \right) - \frac{n^a}{(n-1)^{n-1}} \left( \frac{s}{(1-s)^n} \right) = \frac{(n-1) + s}{(n-1)(1-s)}
\]

in a neighborhood of \( s = 0 \).

**Proof.** As mentioned in the proof of Thm. 8.5, \( \mathcal{C}_{p,q}^{(q)}(a; \zeta) \) equals \( (1-s)^a \), where

\[
\zeta = (-q)^a \frac{n^a}{p^aq^q} \frac{s^q}{(1-s)^n} \quad (8.11)
\]

is the map \( \phi^{(1)}: \mathcal{C}_{p,q}^{(1)} \cong \mathbb{P}^1_s \to \mathbb{P}^1_{\zeta} \). Hence, \( \mathcal{C}_{n-1,1}^{-1}(1; \zeta) \) equals \( (1-s)^{-1} \).

Formally,

\[
\mathcal{C}_{n-1,1}^{-1}(1; \zeta) = \lim_{a \to -1} nF_{n-1} \left( \frac{-1}{n}, \frac{a+1}{a+n}, \frac{a+2}{a+n}, \ldots, \frac{a+(n-1)}{n} \right) \zeta \quad (8.12)
\]

but the right side is undefined, on account of the presence of a non-positive integer \( -m = 0 \) as a lower parameter (accompanied, fortunately, by a matching upper parameter, \( -m' = 0 \)). One must interpret this in a limiting sense, i.e.,

\[
\mathcal{C}_{n-1,1}^{-1}(1; \zeta) = \lim_{a \to -1} nF_{n-1} \left( \frac{a+1}{n}, \frac{a+2}{n}, \ldots, \frac{a+(n-1)}{n} \right) \zeta \quad (8.13)
\]

By Lemma 2.1 this limit equals

\[
\frac{1}{n} + n-A^{-1} \cdot n^{-1}F_{n-2} \left( \frac{-1}{n}, \frac{1}{n}, \ldots, \frac{n-2}{n} \right) \zeta \quad (8.14)
\]

and the theorem now follows by a bit of algebra. \( \square \)

**Remark 8.9.1.** Theorem 8.9 can also be viewed as a special, degenerate case of Thm. 2.1. Explicit representations for this algebraic \( n \)’s, for low \( n \), were recently given by Dominici [5]. The differential equation \( E_{n-1} \) of which this \( n^{-1}F_{n-2} \) is a solution has monodromy group \( H < GL_{n-1}(\mathbb{C}) \) isomorphic to \( \mathfrak{S}_n \) by Thm. 2.1.
The following formula, of Girard type, facilitates computation in the function fields of any top Schwarz curve and its subsidiaries.

**Lemma 8.10.** On the top curve $\mathcal{C}_{p,q}^{(n)} \subset \mathbb{P}^{n-1}$ defined by the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{q-1}$ and $\sigma_{q+1}, \ldots, \sigma_{n-1}$ equaling zero, one can express any power-sum symmetric polynomial $p_\gamma = \sum_{j=1}^n x_j^\gamma$, $\gamma \geq 1$, in terms of $\sigma_q$ and $\sigma_n$ by

$$p_\gamma = \sum c_{m,q,m_n} \sigma_q^{m_q} \sigma_n^{m_n},$$

where the sum is over all $m_q, m_n \geq 0$ with $m_q q + m_n n = \gamma$, and

$$c_{m_q,m_n} = (-1)^{q(m_q + \chi(n)m_n)} q^2 \left( \binom{m_q + m_n - 1}{m_n} + m_n \binom{m_q + m_n - 1}{m_q} \right),$$

with $\chi(l) := 1, 0$ if $l \equiv 0, 1 \pmod{2}$.

**Proof.** This comes, e.g., from the determinantal formula

$$p_\gamma = \begin{vmatrix}
\sigma_1 & 1 & 0 & \cdots & 0 \\
2\sigma_2 & \sigma_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma \sigma \gamma & \sigma \gamma - 1 & \sigma \gamma - 2 & \cdots & \sigma_1 \\
\end{vmatrix}, \quad (8.15)$$

with $\sigma_l = 0$ if $l > n$, which is inverse to the Newton–Girard formula [16, §1.2].

Each hypergeometric function in the following theorem, with $a \in \mathbb{Z}$, is algebraic with governing curve $\mathcal{C}_{p,q}^{(p,q,\text{symm})} = \mathcal{C}_{2,3}^{(3,\text{symm})}$. This quotient curve is of genus zero, even though the unquotiented curve $\mathcal{C}_{p,q}^{(q)} = \mathcal{C}_{2,3}^{(3)}$ is of genus 3 by the formula of Thm. [6,3]. The point of the following theorem is that if $a \in \mathbb{Z}$, a rational parametrization is still possible.

**Theorem 8.11.** For each integer $a \leq -1$,

$$F_{2,3}^{(3;0)}(a; \zeta(t)) = \frac{1}{3} [\mathfrak{s}_3(t)]^a [P_a (\mathfrak{s}_3(t), \mathfrak{s}_5(t)) - (t + 1)^{-3a} - (t - 1)^{-3a}]$$

in a neighborhood of $t = 0$, where

$$\mathfrak{s}_3(t) = \frac{1 + 10t^2 + 5t^4}{2t}, \quad \mathfrak{s}_5(t) = \frac{(1 - t^2)^2 (1 + 3t^2)}{2t^2},$$

$$\zeta(t) = -\frac{5^5}{2^2 3^3} \frac{\mathfrak{s}_5^3(t)}{\mathfrak{s}_3^5(t)} = \frac{5^5}{3^3} \frac{t^2 (1 - t^2)^6 (1 + 3t^2)^3}{(1 + 10t^2 + 5t^4)^5},$$

and $P_a \in \mathbb{Z}[\mathfrak{s}_3, \mathfrak{s}_5]$, e.g.,

$$P_a(\mathfrak{s}_3, \mathfrak{s}_5) = \begin{cases} 3 \mathfrak{s}_3^{-a}, & a = -1, -2, -3, -4; \\
3 \mathfrak{s}_3^{-5} + 5 \mathfrak{s}_5^{-3}, & a = -5. \\
\end{cases}$$
For each \(a\), the special function \(F_{2,3}^{(3,0)}(a;\zeta)\) may be expressed in terms of non-degenerate hypergeometric functions by Lemma 2.1. E.g., for \(s\) of the alternative Schwarz curve \(S\) to \(\zeta\)

\[
F_{2,3}^{(3,0)}(-1; \zeta) = 1 - \frac{2^2 3^3}{5^5} \zeta \cdot {}_5F_4\left(\begin{array}{c} \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{8}{3} \end{array} \right| \zeta) ,
\]

\[= \frac{3}{5} + \frac{2}{5} \cdot {}_4F_3\left(\begin{array}{c} -1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \\ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \end{array} \right| \zeta),
\]

\[
F_{2,3}^{(3,0)}(-5; \zeta) = 1 - \frac{2^2 3^2}{5^4} \zeta - \frac{2^6 3^9}{5^7} \zeta^3 \cdot {}_5F_4\left(\begin{array}{c} \frac{11}{5}, \frac{13}{3}, \frac{13}{3}, \frac{14}{3}, 2 \\ \frac{1}{5}, \frac{3}{5}, \frac{5}{3}, \frac{3}{5}, \frac{2}{3} \end{array} \right| \zeta).
\]

**Proof.** This is the \((p,q) = (2,3)\) case of Thm. 8.5(i). The rational functions \(s_3(t), s_5(t)\) are \(\sigma_3, \sigma_5\), i.e., \(\sigma_q, \sigma_n\), expressed in terms of \(t\), the rational parameter of the alternative Schwarz curve \(E_{p,q} = E_{2,3}\). The given expressions and the formula for \(\zeta = x_p(t)\) come from Lemma 5.5 and Thm. 5.6, though \(x_1, x_2\) are to be replaced by \(x_3, x_4\), and \(t'\) is to be understood as \((x_3 + x_4)/(x_5 - x_4)\), not \((x_1 + x_2)/(x_1 - x_2)\), so that \([x_5 : x_4] = [t + 1 : t - 1]\). For each integer \(a \leq -1\), the quantity \(p_a\) is the power-sum symmetric polynomial \(p_a\), i.e.,

\[
p_a = \sum_{j=1}^{n} x_j^a = \sum_{j=1}^{5} x_j^a, \quad \gamma := -qa = -3a, \quad (8.16)
\]

which is expressed as a polynomial in \(\sigma_3, \sigma_5\) by the Girard formula of Lemma 8.10.

The removal from \(F_{2,3}^{(3,0)}(a;\zeta)\), which is a \(5F_4(\zeta)\), of lower hypergeometric parameters that are non-positive integers, is a straightforward application of Lemma 2.1. E.g., for \(a = -1, -2, -3, -4, -5\), the relevant lower/upper parameters \((-m, -m')\) are respectively \((0, 0), (0, 0), (-1, 0), (-1, 0), (-2, -1)\). Only if \(a = -1\) or \(-2\) does \(-m = -m'\), permitting the \(5F_4\) to be reduced to a \(4F_3\).

**Remark 8.11.1.** In the two cases \(a = -1, -2\) in which an upper and a lower parameter of \(F_{2,3}^{(3,0)}(a;\zeta)\) can be cancelled, reducing it to a \(4F_3(\zeta)\), the \(E_4\) of which this \(4F_3\) is a solution has monodromy group \(H < GL_4(\mathbb{C})\) isomorphic to \(E_5\) by Thm 2.4.

If \(a \leq -3\) then \(F_{2,3}^{(3,0)}(a;\zeta)\) is essentially a \(5F_4(\zeta)\), which is the solution of an \(E_5\) that has reducible monodromy, due to an upper and a lower parameter differing by an integer. (See the typical case \(a = -5\), below.)

**Corollary 8.12.** Define a degree-10 Belyı map \(\mathbb{P}_v^1 \rightarrow \mathbb{P}_\zeta^1\) by

\[
\zeta = \zeta_{2,3}(v) := \frac{5^5 v(1-v)^6(1+3v^3)^3}{3^3 (1+10v+5v^2)^5} = 1 - \frac{(1-35v-125v^2-225v^3)(27+115v+25v^2+25v^3)}{27(1+10v+5v^2)^5}.
\]
Then in a neighborhood of \( v = 0 \),

\[
4F_3\left( \begin{array}{c}
-\frac{1}{3}; \frac{2}{3}, \frac{8}{3}, \frac{1}{3} \\
\frac{1}{3}, \frac{4}{3}, \frac{5}{3}
\end{array} \right| \zeta_{2,3}(v) \right) = \frac{3 + 5v^2}{3(1 + 10v + 5v^2)},
\]

\[
5F_4\left( \begin{array}{c}
\frac{11}{5}, \frac{12}{5}, \frac{11}{3}, \frac{4}{3}, \frac{2}{3}
\end{array} \right| \zeta_{2,3}(v) \right) = \frac{(3 + v)(5 + 10v + v^2)(1 + 28v + 134v^2 + 92v^3 + v^4)(1 + 10v + 5v^2)^{10}}{15(1 - v)^{18}(1 + 3v)^9}.
\]

**Proof.** These are the rational parametrizations of the nondegenerate hypergeometric functions obtained in the theorem (coming from \( a = -1, -5 \)). The parametrizations are even in \( t \) and expressible in terms of \( v := t^2 \), as expected.

**Remark 8.12.1.** In the given uniformizations of these two algebraic hypergeometric functions, the uniformizing parameter \( v \) is the rational parameter of the genus-zero quotient curve \( C_{p,q} \). The reason why this \( 4F_3(\zeta) \) and \( 5F_4(\zeta) \) are 10-branched functions of \( \zeta \) is that \( C_{2,3} \cap \mathbb{Q} \) is a 10-sheeted covering, with the sheetedness coming from \( \binom{n}{q} = \binom{5}{3} = 10 \).

### 8.3. Parametrizations with \( qa \in \mathbb{Z} \)

The theorems below are sample applications of the algorithm embedded in the proof of Thm. 8.8 to \( (p, q) = (2, 3) \) and \( (1, 4) \), with \( a = -\frac{1}{q} \) in both cases. They show how one can construct parametrizations of many algebraic \( nF_{n-1} \)'s with \( qa \in \mathbb{Z} \), due to their being uniformized by quotient curves \( C_{p,q}^{(n)} \) that if not rational, are at least low-degree covers of rational ones. In these theorems the rational lower curves will be \( C_{p,q}^{(cyl)} \) and \( C_{p,q}^{(dihedral)} \), respectively, and the parametrizations of the \( nF_{n-1} \) will for the first time involve radicals.

**Theorem 8.13.** Define a degree-10 Belyi map \( \mathbb{P}^1_v \to \mathbb{P}_\zeta^1 \) by the rational formula for \( \zeta = \zeta_{2,3}(v) \) given in Corollary 8.12. Then in a neighborhood of \( v = 0 \),

\[
(1 + 10v + 5v^2)^{1/3} 4F_3\left( \begin{array}{c}
-\frac{1}{15}, \frac{2}{15}, \frac{8}{15}, \frac{11}{15}
\end{array} \right| \zeta_{2,3}(v) \right) = \left\{ \frac{1}{2} + \frac{5}{3}v - \frac{5}{27}v^2 + \frac{1}{2} \left[ (1 + 3v)(1 + 11\frac{15}{27}v + 25\frac{27}{15}v^2 + 25\frac{27}{25}v^3) \right]^{1/2} \right\}^{1/3}.
\]

**Proof.** This is the \( (p, q) = (2, 3) \) case of Thm. 8.8 (ii) with \( m = -1 \), i.e., \( a = -1/3 \) and \( qa = -1 \). The relevant top and subsidiary Schwarz curves are \( C_{p,q}^{(cyl)} = C_{2,3}^{(3)} \) and \( C_{p,q}^{(q)} = C_{2,3}^{(3)} \), and the complementary subsidiary curve is \( C_{p,q}^{(p)} = C_{2,3}^{(2)} \). The coordinates on \( C_{2,3}^{(3)} \subset \mathbb{P}^2 \) are \( x_1, x_2, x_3 \) and those on \( C_{2,3}^{(2)} = \mathbb{P}_2^1 \) are \( x_4, x_5 \). The
rational parameter on the latter curve will be \( t := (x_5 + x_4)/(x_5 - x_4) \), and
\[
\sigma_3 = \frac{x_5^3 - x_4^3}{x_5 - x_4} = \frac{1 + 10t^2 + 5t^4}{2t}, \quad \sigma_5 = -(x_5x_4)^2 \frac{x_3^3 - x_4^3}{x_5 - x_4} = -\frac{(1 - t^2)^2(1 + 3t^2)}{2t},
\]
\[
\zeta = -\frac{5}{2} \sigma_3^2 \sigma_5^3 = \frac{5}{2} \sigma_3^2 \sigma_5^3 (x_5x_4)^6 \frac{(x_5^2 - x_4^2)^2(x_5^2 - x_4^3)^3}{(x_5^2 - x_4^3)^5} = \frac{5}{3} t^2(1 - t^2)^6(1 + 3t^2)^3}
\]

follow from Lemma 5.5 (with \( x_1, x_2 \) relabelled as \( x_3, x_4 \)).

By the formula in Eq. (8.7a),
\[
\mathcal{G}_{2,3}^{(3,0)} \left( -\frac{1}{3}; \zeta \right) := F_4 \left( -\frac{1}{3}, \frac{1}{9}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3} \right) = 4F_3 \left( -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3} \right) \quad (8.17)
\]

(an upper and a lower parameter being cancelled) has the representation
\[
\mathcal{G}_{2,3}^{(3,0)} \left( -\frac{1}{3}; \zeta \right) = \frac{1}{9} \sigma_3^{-1/3} \left[ x_1 + \varepsilon_3 x_2 + \varepsilon_3^2 x_3 \right], \quad (8.18a)
\]
\[
\left[ \mathcal{G}_{2,3}^{(3,0)} \left( -\frac{1}{3}; \zeta \right) \right]^3 = \frac{1}{81} \sigma_3^{-1} \left[ x_1 + \varepsilon_3 x_2 + \varepsilon_3^2 x_3 \right]^3. \quad (8.18b)
\]

Define a polynomial \( G_{3,1}(y) \), symmetric in \( x_1, x_2, x_3 \), one of the roots of which is the
final factor in (8.18a), as a product over the two cosets of \( \mathfrak{S}_3 \) in \( \mathfrak{S}_5 \), i.e.,
\[
G_{3,1}(y) = \left\{ y - \left[ x_1 + \varepsilon_3 x_2 + \varepsilon_3^2 x_3 \right]^3 \right\} \left\{ y - \left[ x_1 + \varepsilon_3^2 x_2 + \varepsilon_3 x_3 \right]^3 \right\}
\]
\[
y^2 + \left[ -2\sigma_3^3 + 9\sigma_3 \sigma_5 - 27\sigma_3 \right] y + \left[ 3\sigma_3^2 - 3\sigma_5 \right]^3,
\]

as one finds by a bit of computation. Here \( \{\bar{\sigma}_i\}_{i=1}^3 \) are the elementary symmetric
polynomials in \( x_1, x_2, x_3 \); the ones in \( x_4, x_5 \) will be denoted by \( \{\bar{\sigma}_i\}_{i=1}^5 \).

Each of the coefficients of \( G_{3,1}(y) \) can be expressed rationally and symmetrically
in terms of \( x_4, x_5 \), as the function fields of \( \mathfrak{c}_{2,3}^{(3,\text{symm.})} \), \( \mathfrak{c}_{2,3}^{(2,\text{symm.})} \) are isomorphic.
One way to do this is to exploit the formula given in Lemma 5.4. Another
uses the structure of the ring of symmetric polynomials in \( x_1, \ldots, x_5 \), and was
sketched in the proof of Thm. 8.3. One can express \( \{\bar{\sigma}_i\}_{i=1}^3 \) in terms of the
power-sum symmetric polynomials \( \{\bar{p}_i\}_{i=1}^3 \), which can be expressed in terms of
the overall power-sum symmetric polynomials \( \{p_i\}_{i=1}^5 \) in \( x_1, x_2, x_3, x_4, x_5 \),
together with \( x_4, x_5 \). But the \( \{p_i\}_{i=1}^5 \) can be expressed in terms of \( \{\sigma_i\}_{i=1}^5 \).
Of these, the only nonzero ones are \( \sigma_3, \sigma_5 \), which were expressed above in terms
of \( x_4, x_5 \).

Regardless of which technique one uses, one finds that
\[
G_{3,1}(y) = y^2 + \left[ -\frac{7\sigma_1^4 + 36\sigma_3^2 \sigma_2 - 27\sigma_3^2}{\sigma_1} \right] y + \left[ -2\sigma_1^2 + 3\sigma_2 \right]^3. \quad (8.20)
\]

Let \( F_3 \) denote the left side of Eq. (8.18a). It follows from (8.20) and the formula
for \( \sigma_3 \) in terms of \( t \), and the fact that \( [x_5 : x_4] = [t + 1 : t - 1] \), that \( F_3 \) satisfies
\[
F_3^2 - \frac{27 + 90v - 5v^2}{27(1 + 10v + 5v^2)} F_3 - \frac{4v(3 + 5v)^3}{729(1 + 10v + 5v^2)^2} = 0, \quad (8.21)
\]
where \( v = t^2 \). The theorem now follows by applying the quadratic formula. \( \Box \)
Remark 8.13.1. The cube of $\mathcal{F}_{2,3}^{(3,0)}(-\frac{1}{3};\zeta)$, i.e., of the $4F_3$ in the theorem, is uniformized by $\mathcal{C}_{2,3}^{(3,\text{cycl})}$, which doubly covers $\mathcal{C}_{2,3}^{(3,\text{symm})} \cong \mathbb{P}^1$. By examining the statement of the theorem, one sees that it contains a plane model of $\mathcal{C}_{2,3}^{(3,\text{cycl})}$, i.e.,

$$w^2 = (1 + 3v) \left(1 + \frac{115}{27}v + \frac{22}{27}v^2 + \frac{25}{27}v^3\right).$$

(8.22)

This affine quartic is elliptic (of genus 1), with Klein–Weber invariant $j = -2^{12}5^2/3$. It is triply covered by the unquotiented Schwarz curve $\mathcal{C}_{2,3}^{1}$, of genus 3.

Theorem 8.14. Define a degree-15 Belyï map $\mathbb{P}^1_x \to \mathbb{P}^1_\zeta$ by

$$\zeta = \zeta_{1,4}(x) = \frac{1}{4}x(1 - x)(5 + 6x + 5x^2)^2 \left(\frac{55}{48}x^3 - \frac{11}{2}x^2 - \frac{5}{4}x - \frac{5}{16} \right) \frac{5}{4} \frac{5}{2} \frac{5}{3} \frac{11}{20} \frac{5}{2} \frac{5}{3} \frac{11}{20}

= 1 - \frac{(4 - 5x - 10x^2 - 5x^3)(1 - 55x - 5x^2 - 5x^3)}{4(1 - x)^5(1 + 10x - 5x^2)^5}.

Then in a neighborhood of $x = 0$,

$$(1 - x)^{1/4}(1 + 10x + 5x^2)^{1/4} 4F_3 \left(-\frac{1}{2}, \frac{5}{16}, \frac{5}{2}, \frac{5}{3}, \frac{11}{20}, 1, 2, 1 \right) \frac{1}{2} \frac{5}{16} x - \frac{5}{2} x^2 - \frac{5}{3} x^3 + \frac{1}{2} \frac{1}{2} \frac{5}{16} \frac{1}{2} \frac{5}{2} \frac{5}{3} \frac{11}{20} \frac{5}{2} \frac{5}{3} \frac{11}{20} \right)^{1/4}.

Proof. This is the $(p, q) = (1, 4)$ case of Thm. 8.5 with $m = -1$, i.e., $\alpha = -1/4$ and $qa = -1$. The relevant top and subsidiary Schwarz curves are $\mathcal{C}_{1,4}^{(3)}$ and $\mathcal{C}_{1,4}^{(4)}$, and the complementary subsidiary curve is $\mathcal{C}_{1,4}^{(1)} \cong \mathbb{P}^1_\zeta$.

The computations resemble those in the proof of Thm. 8.13 and will therefore only be sketched. One defines a degree-6 polynomial $G_{4,1}$, symmetric in $x_1, x_2, x_3, x_4$, by a product over the six cosets of $\mathfrak{S}_4$. Each coefficient can be expressed rationally in $x_5$ and the rational parameter $s$. This leads to a degree-6 polynomial equation for $F_4$, defined as the fourth power of $\mathcal{F}_{1,4}^{(4,0)}(-\frac{1}{4};\zeta)$, i.e., of the $4F_3$ in the theorem. The coefficients of the degree-6 polynomial are polynomial in $s$. If one substitutes

$$s = s(x) = \frac{-(1 - 5x)(5 + 6x + 5x^2)}{64x},\tag{8.23}$$

this polynomial will factor, and only one quadratic factor will be relevant, namely

$$F_4^2 - \frac{8 - 5x - 10x^2 - 5x^3}{8(1 - x)(1 + 10x + 5x^2)} F_4 + \frac{25x^2(1 + x)^4}{256(1 - x)^2(1 + 10x + 5x^2)} = 0.\tag{8.24}$$

The theorem follows from Eq. (8.23) by applying the quadratic formula. □

Remark 8.14.1. The fourth power of $\mathcal{F}_{1,4}^{(4,0)}(-\frac{1}{4};\zeta)$, i.e., of the $4F_3$ in the theorem, is uniformized by $\mathcal{C}_{1,4}^{(4,\text{cycl})}$, which doubly covers $\mathcal{C}_{1,4}^{(4,\text{dihedral})}$, which in turn, triply
covers \( c^{(4;\text{symm})}_{1,4} \cong c^{(1;\text{symm})}_{1,4} \cong \mathbb{P}^1_4 \). Hence, \( x \) can be identified as the rational parameter of the genus-0 curve \( c^{(4;\text{dihedr})}_{1,4} \). The theorem contains a plane model of \( c^{(4;\text{cycl})}_{1,4} \),

\[
w^2 = 1 - \frac{5}{4}x - \frac{5}{2}x^2 - \frac{5}{4}x^3.
\]  

(8.25)

This affine cubic is elliptic (of genus 1) with Klein–Weber invariant \( j = -\frac{5^2}{2} \), like \( c^{(3)}_{1,4} \). It is quadruply covered by the top curve \( c^{(5)}_{1,4} \cong c^{(4)}_{1,4} \) of genus 4.

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