A Remark on Random Vectors and Irreducible Representations

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Abstract
It was observed in [1] that the expectation of a squared scalar product of two random independent unit vectors that are uniformly distributed on the unit sphere in $\mathbb{R}^n$ is equal to $1/n$. It is shown below that this is a characteristic property of random unit vectors defined on invariant probability subspaces of irreducible real representations of compact Lie groups.

1 Introduction

Suppose that a compact Lie group $G$ acts on real vector space $V$. We will always assume that scalar product $\langle , \rangle$ on $V$ is $G$-invariant (cf. e.g [2]).

We will identify elements of $G$ with linear operators on $V$, denoting an action of $g \in G$ on $v \in V$ as $g \circ v$ or just $gv$.

Let $\mathcal{U} \subset V$ be a $G$-invariant subset. Assume that there is a probability (Radon) measure $\mu$ defined on $\mathcal{U}$. The measure $\mu$ is called $G$-invariant if $\mu(g\mathcal{U}') = \mu(\mathcal{U}')$ for any measurable $\mathcal{U}' \subset \mathcal{U}$ and any $g \in G$ (cf. e.g. [3], [4]). Such a subset $\mathcal{U}$ will be called an invariant probability subspace of $V$.

We will sometimes omit mentioning of a group $G$ and just say that a measure $\mu$ and a set $\mathcal{U}$ are invariant. A random vector $x$ uniformly distributed on $\mathcal{U}$ according to the invariant measure $\mu$ will thus be called $G$-invariant random vector.

The expectation of a random vector $x$ defined on a probability space $\mathcal{U} \subset V$ will be denoted by $\mathbb{E}_\mu(x)$ or just $\mathbb{E}(x)$. It is clear from definitions that for invariant random vectors $\mathbb{E}(gx) = \mathbb{E}(x)$. 
Let $\chi$ be a normalized Haar measure on $G$, in other words, assume that $\int_G d\chi(g) \equiv \chi(G) = 1$ (cf. e.g. [3]). An invariant projector $P^G : V \to V^G$ onto the subspace of $G$-fixed points in $V$ can be thus written as

$$P^G(v) = \int_G gvd\chi(g)$$

for any $v \in V$ (cf. e.g. [2]).

The following statement is probably well known.

**Lemma 1** If $x$ is a random vector defined on any invariant probability subspace of a real representation $V$ of a compact (Lie) group $G$, then

$$E(x) = E(P^G(x))$$

Proof. By invariance of $x$ and by linearity of expectation, we have

$$E(P^G(x)) = E(\int_G gx d\chi(g)) = \int_G E(gx) d\chi(g) = \int_G E(x) d\chi(g) = E(x)$$

Although we will not need it, we formulate an obvious

**Corollary 1** If a real representation of a compact Lie group has no non-trivial fixed points then the expectation of an invariant random vector defined on any invariant probability subspace of $V$ is zero.

## 2 Main Result

Let $G$ be a compact Lie group and let $S(V)$ be a unit sphere in a space of a real $G$-representation $V$ of dimension $n$.

**Theorem 1**

(i) For independent invariant random vectors $x, y$ defined on any invariant probability subspace $U \subset S(V)$

$$E(<x, y>^2) \geq 1/n$$

(ii) If the representation of $G$ in $V$ is irreducible then for independent invariant random vectors $x, y$ defined on any invariant probability subspace $U \subset S(V)$

$$E(<x, y>^2) = 1/n$$
(iii) If there is a non-trivial invariant subspace \( W \subset V \) then there is an invariant probability subspace \( U \subset S(V) \) and independent invariant random vectors \( x, y \) defined on \( U \) such that
\[
\mathbb{E}(< x, y >^2) > 1/n
\]

Therefore, a real representation \( V \) of a compact Lie group \( G \) is irreducible if and only if
\[
\mathbb{E}(< x, y >^2) = 1/n
\]
for independent invariant random vectors \( x, y \) defined on any probability subspace \( U \subset S(V) \)

We precede the proof of Theorem 1 by some well known (cf. e.g. [2]) general observations.

\section{2.1 Preliminaries}

Let \( V^* \) be a space of real linear functions on \( V \). This is a space of a conjugate \( G \)-representation, i.e. by definition
\[
gv^*(w) = v^*(g^{-1}w), \ v^* \in V^*, \ w \in V
\]

Consequently \( G \) acts on \( V \otimes V^* \) and due to isomorphisms
\[
V \otimes V^* \approx Hom(V, V) \approx Mat_n(V) \equiv M_n \tag{1}
\]

this action can be identified with the action of \( G \) on algebra \( M_n = M_n(\mathbb{R}) \) of \( n \times n \) real matrices, defined by the rule
\[
g \circ a = gag^{-1}, \ a \in M_n \tag{2}
\]

Using \( G \)-invariant scalar product, we can identify real vector spaces \( V \) and \( V^* \) (in orthogonal coordinates, \( v \in V \) is represented by a column vector and corresponding \( v^* \in V^* \) is represented by a transposed row vector \( v^T \)). We further notice that for \( u_1, u_2 \in V \) and \( v_1, v_2 \in V^* \) we have \((u_1 \otimes v_1)(u_2 \otimes v_2) = v_1(u_2)u_1 \otimes v_2 = < v_1, u_2 > u_1 \otimes v_2 \) and it easy to see that isomorphisms (1) are isomorphisms of algebras.

The scalar product in \( V \otimes V^* \approx M_n \) is given by
\[
< u_1 \otimes v_1, u_2 \otimes v_2 > = < u_1, u_2 > < v_1, v_2 > \tag{3}
\]
and for matrices \( a, b \in M_n \) the formula (3) is equivalent to
\[
< a, b > = Tr(ab^T)
\]
where \( Tr(.) \) stands for the trace of a matrix. This scalar product is clearly \( G \)-invariant.

The following well known lemma can be verified by a direct computation.

**Lemma 2** An orthogonal projection of a matrix \( b \in M_n \) onto the one-dimensional subspace spawned by a matrix \( a \in M_n \) is given by
\[
\frac{Tr(ab^T)}{Tr(aa^T)} a
\]
In particular, an orthogonal projection of a matrix \( b \) onto the one-dimensional subspace spawned by a unity matrix \( I_n \) is equal to
\[
\frac{Tr(b)}{n} I_n
\]

There is a \( G \)-equivariant diagonal map
\[
\pi : V \rightarrow M_n, \quad \pi(v) = v \otimes v^*, \quad v \in V
\]
that takes the unit sphere \( S(V) \) into a unit sphere in \( S(M_n) \) as it is identified with \( S(V \otimes V^*) \) by (2). If \( U \subset V \) is an invariant probability subspace in \( S(V) \) with probability measure \( \mu \) then its image \( \pi(U) \subset S(V \otimes V^*) \) is a \( G \)-invariant probability subspace in \( S(M_n) \) with a \( G \)-invariant push-forward probability measure \( \hat{\mu} = \mu \circ \pi^{-1} \) (cf. [3]). Thus we can say that

**Lemma 3** If a random unit vector \( x \) is invariant then the random unit vector \( x \otimes x^* \in \pi(U) \subset S(V \otimes V^*) \) is also invariant.

### 2.2 Proof of Theorem 1

Let \( \mu \) be an invariant probability measure defined on \( U \subset S(V) \). By (3), Lemma 3 and independence of \( x \) and \( y \)
\[
\mathbb{E}_{\mu}(< x, y >^2) = \mathbb{E}_{\hat{\mu}}(< x \otimes x^*, y \otimes y^* >) = < \mathbb{E}_{\hat{\mu}}(x \otimes x^*), \mathbb{E}_{\hat{\mu}}(y \otimes y^*) >
\]
where as above, \( \hat{\mu} = \mu \circ \pi^{-1} \) is a push-forward probability measure. It follows from (1) and (2) that the unity matrix \( I_n \) belongs to the linear subspace of fixed points \( M^G_n \). Let \( \mathcal{I} \subset M^G_n \) be a one-dimensional subspace of \( M^G_n \) spawned by \( I_n \). We have an orthogonal decomposition \( M^G_n = \mathcal{I} \oplus M' \) where \( M' \) is some
(possibly zero and obviously invariant) linear subspace of \( M_n^G \). It is clear that an orthogonal (and hence invariant) projection of the random vector \( x \otimes x^* \) onto \( M_n^G \) is a sum of its orthogonal projection onto \( \mathcal{I} \) and its orthogonal projection \( x' \) onto \( M' \). Since \( x \) is a unit vector, \( \text{Tr}(x \otimes x^*) = 1 \) and applying lemmas 1 and 2 we get

\[
E_{\hat{\mu}}(y \otimes y^*) = E_{\hat{\mu}}(x \otimes x^*) = E_{\hat{\mu}}\left(\frac{\text{Tr}(x \otimes x^*)}{n}\right)I_n + E_{\hat{\mu}}(x') = \frac{I_n}{n} + z \tag{6}
\]

where \( z \in M' \). Statement (i) now follows from (5) and (6):

\[
E_{\mu}(< x, y >^2) = < \frac{I_n}{n}, \frac{I_n}{n} > + < z, z > = 1/n + < z, z > \geq 1/n \tag{7}
\]

Before going further, observe that statement (i) and its proof are valid even for trivial \( G \)-representations. Below, we will make use of the group-invariant proof of the statement (i). For now, we have in any case

**Corollary 2** If \( x, y \) are independent random unit vectors defined on any probability subspace in \( S(\mathbb{R}^n) \) then \( E(< x, y >^2) \geq 1/n \).

To prove statement (ii), recall that if \( V \) is an irreducible representation of \( G \) then the linear subspace of fixed points \( M_n^G \) of the \( G \)-action (2), is isomorphic to either real field \( \mathbb{R} \) or to the complex field \( \mathbb{C} \) or to the field of quaternions \( \mathbb{H} \) (cf. e.g [2]). In the first case, the fixed point space is exactly \( \mathcal{I} \) and (ii) follows from (7) where one can set \( z = 0 \). Something similar happens in other two cases. We will consider only the case of the field of quaternions as it will become obvious that the case of complex numbers is covered by the quaternionic case. Hence, suppose that \( M_n^G \) is isomorphic to a field of quaternions \( \mathbb{H} \) and therefore, there is a linear basis of \( M_n^G \) that consists of four real matrices \( I_n, i, j, k \in M_n^G \subset M_n \) that satisfy all the relations of the algebra of quaternions, in particular \( i^2 = j^2 = k^2 = -I_n \). It is not hard to figure out that matrices \( i, j, k \) are skew-symmetric and hence (cf. (4)) are orthogonal to \( I_n \) and to each other (e.g. \( \text{Tr}(ij^T) = -\text{Tr}(k) = 0 \)). On the other hand, the matrix \( x \otimes x^* \) is symmetric and therefore its projection onto a linear space of skew-symmetric matrices is zero. The conclusion is, that for irreducible representations, the extra term \( z \) in (6-7) vanishes in all cases.

The proof of (iii) is very simple. Let \( m < n \) be the dimension of invariant subspace \( W \) and let \( W' \) be its orthogonal complement in \( V \). Take, for example, an obviously invariant random vector \( x \in S(V) \) that has zero projection on \( W' \) and is uniformly distributed on \( S(W) \). Then, by (i) the expectation of \( x \) is no less than \( 1/m > 1/n \).
Corollary 3 (Cf. [1]). In the conditions of Theorem 1 (ii), let Euclidean coordinates of a random unit vector \( x \in \mathcal{U} \subset S(V) \) be \( x_1, x_2, \ldots, x_n \). Then \( \mathbb{E}(x_i x_j) = 0 \) if \( i \neq j \) and \( \mathbb{E}(x_i^2) = 1/n \) for all \( i,j = 1,2,\ldots,n \).

Proof. From the proof of Theorem 1 (ii), we have
\[
\mathbb{E}_\mu(x \otimes x^*) = \mathbb{E}_\mu \left( \frac{\text{Tr}(x \otimes x^*)}{n} \right) I_n = \frac{I_n}{n}.
\]
Now recall that matrix elements of \( x \otimes x^* \) are \( x_i x_j, i,j = 1,2,\ldots,n \).

2.3 An Example

We will see shortly that for \( G \)-orbits, the statement (ii) of Theorem 1 is just a special case of orthogonality relations for matrix coefficients of irreducible group representations (see e.g. [6]). Let \( V \) be a real \( n \)-dimensional representation of a compact Lie group \( G \). Let \( v \in S(V) \) and let \( H = G_v \subset G \) be a stationary subgroup of \( v \). The Haar measure on \( G \) induces unique up to a scalar multiplier \( G \)-invariant measure on a (left) factor-space \( G/H \) and hence on \( G \)-orbit \( \mathcal{U} = Gv \subset S(V) \) (cf. e.g. [4]). Normalizing thus defined measure, we get a probability measure on \( \mathcal{U} \) that will be called below orbital (probability) measure. For orbits in irreducible representations the statement (ii) of Theorem 1 can be read as

Corollary 4 Let \( V \) be a real \( n \)-dimensional irreducible representation of a compact Lie group \( G \). Take an orbit of \( \mathcal{U} = Gv \subset S(V) \) of a unit vector \( v \in S(V) \) and let \( \mu \) be the orbital probability measure on \( \mathcal{U} \). Then
\[
\int_{\mathcal{U}} \int_{\mathcal{U}} < x, y >^2 d\mu(x) d\mu(y) = 1/n \tag{8}
\]
which leads to

Corollary 5 (Cf. [7], [6]). Under conditions of Corollary 4
\[
\int_{\mathcal{U}} < x, v >^2 d\mu(x) = 1/n \tag{9}
\]

Proof. Setting in (8) \( y = g_y v \), for some (depending on \( y \)) \( g_y \in G \) and using invariance of the orbital measure, we get
\[
\frac{1}{n} = \int_{\mathcal{U}} \int_{\mathcal{U}} < x, y >^2 d\mu(x) d\mu(y) = \int_{\mathcal{U}} \int_{\mathcal{U}} < g_y^{-1}x, v >^2 d\mu(x) d\mu(y) = \int_{\mathcal{U}} \int_{\mathcal{U}} < x, v >^2 d\mu(x) d\mu(y) = \int_{\mathcal{U}} < x, v >^2 d\mu(x) \tag{10}
\]
Remark 1 A unit vector $v$ can be viewed as an element of some orthogonal basis. Hence, (9) is an orthogonality relation for the corresponding matrix coefficient $c_{v,v}(g) \equiv < gv, v >$, $g \in G$ (cf. e.g. [6]). Moreover, following the sequence of equations (10) from right to left one can deduce (8) from (9). Hence, when an invariant probability subspace is a $G$-orbit, Theorem 1 (ii) is a special case of orthogonality relations for matrix coefficients of irreducible representations.

Remark 2 A unit sphere in $\mathbb{R}^n$ is an orbit of the full orthogonal group. Hence, an already mentioned example from [3] is also a special case of orthogonality relations (cf. Corollary 5).

It is probably worth mentioning a discrete version of Corollary 5.

**Corollary 6** Let $V$ be a space of a real irreducible exact $n$-dimensional representation of a finite group $G$. Then for any $v \in S(V)$

$$\sum_{k \in G} < kv, v >^2 = \frac{|G|}{n}$$

Proof. The Haar measure on a finite group $G$ is a point-mass measure that assigns equal probabilities $1/|G|$, to all $g \in G$, where $|G|$ denotes the order of the group. Hence, if a stationary subgroup $G_v$ of a unit vector $v \in S(V)$ is trivial, the corresponding orbital measure $\mu$ is given by $\mu(gv) = 1/|G|$ for all $g \in G$. Since our representation is exact, its space $V$ contains a principal orbit of length $|G|$, actually, the set of principal orbits is everywhere dense in $V$ (cf. e.g. [5]). If a principal orbit of $G$ passes through $v \in S(V)$ then the above formula is precisely a discrete analogue of Corollary 5. Moreover, the same formula should be valid for orbits of any type, because any orbit is a limit of a sequence of principal orbits.

We conclude with a concrete numerical example.

### 2.4 Minimal Irreducible Representation of $S_n$

The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permuting Euclidean coordinates, that is, if $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\sigma \in S_n$ then $\sigma x = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. This representation of $S_n$ has a one dimensional invariant subspace $D$ of vectors with equal coordinates. An orthogonal complement of $D$ in $\mathbb{R}^n$ is an invariant subspace $A$ defined by equation $\sum_{i=1}^n x_i = 0$. It is well known that thus defined representation of $S_n$ in $A$ is irreducible and exact. Let $\cos(x, y)$ denote the cosine of an angle between vectors $x, y \in \mathbb{R}^n$ and let, as usual, $\|x\|$ be the Euclidean norm of a vector $x \in \mathbb{R}^n$. Interpretation of Corollary 6 in this setting yields...
Corollary 7  Let $n \geq 2$. If Euclidean coordinates of a vector $x \in \mathbb{R}^n$ add up to zero, then

$$\sum_{\sigma \in S_n} \cos^2(x, \sigma x) \equiv \frac{1}{\|x\|^4} \sum_{\sigma \in S_n} <x, \sigma x>^2 = \frac{n!}{(n - 1)}$$

References

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