The behavior of solutions of an equation with a large spatially distributed control

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Abstract. The paper is devoted to the dynamical properties of the scalar complex equation and system of two equations with spatially distributed parameters. Main assumption is that the coefficient of spatial distribution is sufficiently large. Using asymptotic methods we construct the families of special parabolic equations, which do not contain big and small parameters, which nonlocal dynamics determines the behaviour of solutions of the original equation.

1. Introduction
In this paper we study the behavior of solutions of the two problems with spatial distribution. First one is the complex equation

\[ \dot{u} = \Phi(u) + Ke^{i\varphi} \left( \int_{-\infty}^{\infty} F(s)u(t, x + s)ds - u \right) \]  

with periodic boundary conditions

\[ u(t, x + 2\pi) \equiv u(t, x). \]  

The second problem is a system of two scalar equations Consider also system of equations

\[ \begin{align*}
\dot{u}_1 &= \Phi_1(u_1) + K \left[ \int_{-\infty}^{\infty} F(s)u_2(t, x + s)ds - u_1 \right], \\
\dot{u}_2 &= \Phi_2(u_2) + K \left[ \int_{-\infty}^{\infty} F(s)u_1(t, x + s)ds - u_2 \right]
\end{align*} \]  

with periodic boundary conditions

\[ u_j(t, x + 2\pi) \equiv u_j(t, x) \quad (j = 1, 2). \]  

Here \( \Phi(u), \Phi_1(u) \) and \( \Phi_2(u) \) are sufficiently smooth nonlinear functions; \( u = u(t, x) \) is complex value function, \( u_1 = u(t, x) \) and \( u_2(t, x) \) are real value functions. Parameter \( K \) is real and positive, and \( \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \). Normalization conditions

\[ \int_{-\infty}^{\infty} F(s)ds = 1. \]
are assumed to hold. The dynamics and applications of equations of such kind are studied in numerous works [1, 2, 3, 4, 5]. In particular, in the [4] the function $F(x)$ was defined as $F(x) = \exp(k|x|)$. In this paper, we assume that $F(x)$ is smooth and possibly asymmetric. More specifically,

$$F(x) = \frac{1}{\sqrt{\mu \pi}} \exp \left(-\frac{(x + h)^2}{\mu}\right), \quad (\mu > 0).$$

The research technique is based on the special asymptotic method developed in [6, 7]. In this context, the parameter $\mu$ is assumed to be sufficiently small: $0 < \mu \ll 1$. Note that the corresponding parameter in [4] was approximately equal to 0.2. (The main analytical results in [4] were formulated assuming that the control parameter $K$ is small.) In contrast, we consider the situation when this parameter is sufficiently large:

$$K = \varepsilon^{-1}, \quad 0 < \varepsilon \ll 1.$$

From the problem with two small parameters, it is convenient to pass to a single small parameter. In this context, we set

$$\mu = \kappa \varepsilon^\alpha,$$

where $\kappa$ and $\alpha$ are fixed positive parameters.

Under these conditions, we examine the behavior of all solutions to boundary value problems (1), (2) and (3), (4) with initial conditions from an arbitrary (bounded as $K \to \infty, \mu \to 0$) domain of the phase space $C[0, 2\pi]$ (for (1), (2)) and $C[0, 2\pi] \times C[0, 2\pi]$ (for 3), (4)).

It is convenient to divide the left- and right- hand sides of (1) and (3) by $K$ and make the substitution

$$t = \varepsilon \tau.$$

As a result, we obtain from (1), (2) the problem

$$\frac{\partial u}{\partial \tau} = \varepsilon \Phi(u) + e^{i\varepsilon \tau} \int_{-\infty}^{\infty} F(s)u(\tau, x + s)ds - u], \quad u(\tau, x + 2\pi) \equiv u(\tau, x).$$

The characteristic equation for the linear part of problem (5) for $\varepsilon = 0$ has the form

$$\lambda_m = 1 - e^{i\varepsilon \tau} \exp[-ihm - \kappa \varepsilon m^2] \quad (m \in Z).$$

Dividing the left- and right- hand sides of (3) by $K$ yields the system

$$\frac{\partial u_1}{\partial \tau} = \varepsilon \Phi_1(u_1) + \int_{-\infty}^{\infty} F(s)u_2(\tau, x + s)ds - u_1,$n \quad u_1(t, x + 2\pi) \equiv u_1(t, x).$$

The characteristic equation of system (7) with $\varepsilon = 0$ has the roots

$$\lambda_m = -1 + \exp(-\frac{1}{4}d^2k^2), \quad \tilde{\lambda}_m = -1 - \exp(-\frac{1}{4}d^2k^2) \quad (m \in Z).$$

In the next Section we study the case of symmetric (with respect to zero) $F(x)$. In Section 3, the deviation $h$ is assumed to be asymptotically small; i.e., the function $F(x)$ is close to symmetric. In Section 4, the dynamics of problems (5) and (7) is analysed in the case where $F(x)$ is substantially asymmetric; i.e., $h$ is not small. We show that the dynamic properties of these problems are mainly determined by the nonlocal dynamics of a specially constructed families parabolic boundary value problem.
2. Dynamics in the case of a symmetric function $F(s)$

Let $h = 0$ in this section.

1. First, we consider problem (5) in the case $\alpha = 1$. To formulate the main result introduce parabolic equation

$$\frac{\partial \xi}{\partial t} = \kappa e^{i\varphi} \frac{\partial^2 \xi}{\partial y^2} + \Phi(\xi) \quad (8)$$

with the periodic boundary conditions

$$\xi(t, y + 2\pi) \equiv \xi(t, y). \quad (9)$$

The following theorem holds [5, 8].

**Theorem 1.** Let (8), (9) has a periodic solution $\xi_0(t, y)$. Then the boundary value problem (5) has an asymptotic (in the residual) solution in the form

$$u_0(\tau, x) = \xi_0(\varepsilon \tau, x) + O(\varepsilon).$$

This theorem means that (8), (9) is analogue of normal form for problem (5).

Note that the condition of periodicity $\xi_0$ can be replaced by the condition that $\xi_0$ and its derivatives should be bounded for all $t \geq 0, y \in [0, 2\pi]$.

2. Consider problem (7) for $\alpha = 1$. The normal form in this situation is very similar [9]

$$\frac{\partial \xi}{\partial t} = 1 \frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2}(\Phi_1(\xi) + \Phi_2(\xi)) \quad (10)$$

with boundary conditions (9).

The analogue of Theorem 1 takes place and solution of (7), is given by the formula

$$\left( \begin{array}{c} u_1(\tau, x) \\ u_2(\tau, x) \end{array} \right) = \left( \begin{array}{c} \xi(\varepsilon \tau, x) \\ \xi(\varepsilon \tau, x) \end{array} \right) + O(\varepsilon).$$

3. Consider problems (5) and (7) for $\alpha > 1$. In this situation instead of equations (8) and (10) we have families of equations

$$\frac{\partial \xi}{\partial t} = z^2 \kappa e^{i\varphi} \frac{\partial^2 \xi}{\partial y^2} + \Phi(\xi) \quad (11)$$

and

$$\frac{\partial \xi}{\partial t} = \frac{1}{4} \kappa \frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2}(\Phi_1(\xi) + \Phi_2(\xi)) \quad (12)$$

with periodic boundary conditions (9). Here $z$ is arbitrary real parameter. Let $\theta_z = \theta_z(\varepsilon)$ be value from $[0, 1)$ such that $z\varepsilon((1-\alpha)/2 + \theta_z)$ is integer. The following two theorems hold [8, 9].

**Theorem 2.** Let for some $z$ (11), (9) has a periodic solution $\xi_0(t, y)$. Then the boundary value problem (5) has an asymptotic (in the residual) solution in the form

$$u_0(\tau, x, \varepsilon) = \xi_0(\varepsilon \tau, (z\varepsilon((1-\alpha)/2 + \theta_z)x) + O(\varepsilon).$$

**Theorem 3.** Let for some $z$ (12), (9) has a periodic solution $\xi_0(t, y)$. Then the boundary value problem (7) has an asymptotic (in the residual) solution in the form

$$\left( \begin{array}{c} u_1(\tau, x) \\ u_2(\tau, x) \end{array} \right) = \left( \begin{array}{c} \xi(\varepsilon \tau, (z\varepsilon((1-\alpha)/2 + \theta_z)x) \\ \xi(\varepsilon \tau, (z\varepsilon((1-\alpha)/2 + \theta_z)x) \end{array} \right) + O(\varepsilon).$$

Since $z$ is arbitrary, problem (5) (and (7) too) has unbounded number of solutions with given asymptotic for sufficiently small $\varepsilon$. 

3. Dynamics in the case of a small $h$

In this section, the parameter $h$ is assumed to be sufficiently small:

$$h = \varepsilon^{\alpha/2}h_1.$$  \hspace{1cm} (13)

1. Consider equation (5).

Assuming that $h \neq 0$, $\varphi \neq 0$, and $\varepsilon$ is sufficiently small, characteristic equation (6) has an asymptotically large number of roots with a positive real part bounded away from zero as $\varepsilon \to 0$. Thus, problem (5) cannot have stable steady-state regimes in a bounded domain of the phase space $C_{[0,2\pi]}$. In this context, given a fixed $\varphi_1$, we assume that

$$\varphi = \varepsilon^{\alpha}\varphi_1.$$  

Consider parabolic equation depending on the arbitrary parameter $z$:

$$\frac{\partial \xi}{\partial t} = (z^2\kappa + \frac{1}{2}h_1^2)\frac{\partial^2 \xi}{\partial y^2} + \Phi(\xi).$$  \hspace{1cm} (14)

with the periodic boundary conditions

$$\xi(t, y + 2\pi) \equiv \xi(t, y).$$ \hspace{1cm} (15)

This system plays the role of normal form of (5) in the case (13).

**Theorem 4.** Let the boundary value problem (14), (15) has a periodic solution $\xi_0(t, y)$. Then the problem (5) has an asymptotic (in the residual) solution

$$u_0(\tau, x) = \xi_0(\varepsilon\tau, (z\varepsilon^{1-\alpha}/2 + \theta_2)x + (\varepsilon^{\alpha/2}h_1 + \varepsilon^{\alpha}\varphi_1h_1)t) + O(\varepsilon^{\alpha/2}).$$

2. Now consider problem (7). Let (13) holds, then the normal form is

$$\frac{\partial \xi}{\partial t} = \left(\frac{1}{4}\kappa z^2 + \frac{1}{2}h_1^2\right)\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2}(\Phi_1(\xi) + \Phi_2(\xi))$$

with periodic boundary conditions (15).

The analogue of Theorem 4 takes place and solution of (7), is given by the formula

$$\begin{pmatrix} u_1(\tau, x) \\ u_2(\tau, x) \end{pmatrix} = \begin{pmatrix} \xi(\varepsilon\tau, (z\varepsilon^{1-\alpha}/2 + \theta_2)x - \varepsilon^{\alpha}h_1\tau) \\ \xi(\varepsilon\tau, (z\varepsilon^{1-\alpha}/2 + \theta_2)x - \varepsilon^{\alpha}h_1\tau) \end{pmatrix} + o(1).$$

3. Assume now that $h$ in (7) is small but is considerably larger than $\varepsilon^{\alpha/2}$. More precisely, for some fixed $h_1 > 0$ and $\beta \in (0, \alpha/2)$, let

$$h = h_1\varepsilon^{\beta}.$$  

The most interesting situations occurs in the case [9]

$$\frac{1}{2} \leq \beta \leq (\alpha - 1)/2.$$ \hspace{1cm} (16)

We introduce some notation. Let $z$ and $v$ be arbitrary and fixed real values. Define $\delta$ and $\gamma$ as $\delta = (\alpha - 1)/2 - \beta$ and $\gamma = \beta - 1/2$. Let $\theta_1 = \theta_1(\varepsilon) \in [0, 2)$ be such that the value of $z\varepsilon^{-\delta} + \theta_1$ is an odd integer, and let $\theta_2 = \theta_2(\varepsilon) \in [0, 1)$ and $\theta_3 = \theta_3(\varepsilon) \in (0, 1]$ be such that the values of $\pi h_1^{-1}\varepsilon^{1-\beta}(z\varepsilon^{-\delta} + \theta_1) + \theta_2$ and $v\varepsilon^{-\gamma} + \theta_3$ are integers (for $\gamma = 0$, we set $v = 0$ and $\theta_3 = 1$).
In the considered case the normal form is the system of two parabolic equations in the two-dimensional domain

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= \frac{\kappa \pi^2 z^2}{4h_1^2} \frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2} h_1^2 v^2 \frac{\partial^2 \xi}{\partial w^2} + R_1(\xi, \eta), \\
\frac{\partial \eta}{\partial t} &= \frac{\kappa \pi^2 z^2}{4h_1^2} \frac{\partial^2 \eta}{\partial y^2} + \frac{1}{2} h_1^2 v^2 \frac{\partial^2 \eta}{\partial w^2} + R_2(\xi, \eta)
\end{align*}
\]  

(17)

with boundary conditions

\[
\begin{align*}
\xi(t, y, w + 2\pi) &= \xi(t, y, w), \\
\eta(t, y, w + 2\pi) &= -\eta(t, y + \pi, w).
\end{align*}
\]  

(18)

(19)

Here

\[
R_1(\xi, \eta) = \frac{1}{4} [\Phi_1(\xi + \eta) + \Phi_1(\xi - \eta) + \Phi_2(\xi + \eta) + \Phi_2(\xi - \eta)],
\]

\[
R_2(\xi, \eta) = \frac{1}{4} [\Phi_1(\xi + \eta) - \Phi_1(\xi - \eta) + \Phi_2(\xi + \eta) - \Phi_2(\xi - \eta)].
\]

**Theorem 5.** Let inequalities (16) hold, and let \( z \) and \( v \) be arbitrary fixed parameters. Suppose that the boundary value problem (17)–(19) has a periodic (by \( t \)) solution \( \xi_0(t, y, w), \eta_0(t, y, w) \). Then the problem (7) has an asymptotic (in the residual) solution

\[
\begin{align*}
u_{10}(\tau, x) &= \xi(\varepsilon \tau, y, w) + \eta(\varepsilon \tau, y, w) + o(1), \\
u_{20}(\tau, x) &= \xi(\varepsilon \tau, y, w) + \eta(\varepsilon \tau, y, w) + o(1),
\end{align*}
\]

where \( y = x \left( \pi h_1^{-1} \varepsilon^{-\beta}(\varepsilon^{-\delta} + \theta_1) + \theta_2 \right), w = x (\varepsilon^{-\gamma} + \theta_3) - h_1 \varepsilon^{1/2} \tau. \)

**4. The case of substantially asymmetric \( F(x) \)**

Let \( h \) is not small but is close to a number that is rationally commensurable with \( \pi \). This means that

\[ h = \frac{\pi m_1}{m_2} = \varepsilon^\alpha h_1. \]

where \( m_1 \) and \( m_2 \) are relative prime integers.

1. First, consider problem (5).

Denote \( N = m_2 \) if \( m_1 \) is even and \( N = 2m_2 \) if \( m_1 \) is odd. As in Sections 1, 3, we obtain a boundary value problem similar to (14), (15):

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= N^2 \left( \varepsilon^2 \kappa + \frac{1}{2} h_1^2 \right) \frac{\partial^2 \xi}{\partial y^2} + \Phi(\xi), \\
\xi(t, y + 2\pi) &= \xi(t, y).
\end{align*}
\]

(20)

The solutions of problems (5) and (20) are related by the formula [5, 8]

\[
\begin{align*}
u(\tau, x) &= \xi(\varepsilon \tau, N(\varepsilon^{(1-\alpha)/2} + \theta_3) x - (\varepsilon^{\alpha/2} h_1 + \varepsilon^\alpha \varphi h_1) \tau) + O(\varepsilon^{\alpha/2}).
\end{align*}
\]

2. Results for problem (7) depends on algebraic properties of \( m_1 \). First, consider the case of even \( m_1 \). Then normal form is [9]

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= \left( \varepsilon^2 \kappa + \frac{1}{2} h_1^2 \right) m_2^2 \frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2} (\Phi_1(\xi) + \Phi_2(\xi)), \\
\xi(t, y + 2\pi) &= \xi(t, y),
\end{align*}
\]

(21)

and the solutions of problem (21) are related to the asymptotic (in the residual) solutions of problem (7) by the formula

\[
\begin{align*}
\begin{pmatrix}
u_1(\tau, x) \\
\nu_2(\tau, x)
\end{pmatrix} &= \begin{pmatrix}
\xi(\varepsilon \tau, (\varepsilon^{(1-\alpha)/2} + \theta_3) m_2 x - m_2 \varepsilon^{-\alpha/2} h_1 \tau) \\
\xi(\varepsilon \tau, (\varepsilon^{(1-\alpha)/2} + \theta_3) m_2 x - m_2 \varepsilon^{-\alpha/2} h_1 \tau)
\end{pmatrix} + o(1).
\end{align*}
\]
In the case of odd $m_1$, the situation is more complicated. Instead of (21), we have the two-dimensional boundary value problem

$$ \frac{\partial \xi}{\partial t} = \left( \frac{1}{4} \varepsilon^2 \kappa + \frac{1}{2} h_1^2 \right) m_2 \frac{\partial^2 \xi}{\partial y^2} + R_1(\xi, \eta), \quad \xi(t, y + 2\pi) \equiv \xi(t, y), $$

$$ \frac{\partial \eta}{\partial t} = \left( \frac{1}{4} \varepsilon^2 \kappa + \frac{1}{2} h_1^2 \right) m_2 \frac{\partial^2 \eta}{\partial y^2} + R_2(\xi, \eta), \quad \eta(t, y + \pi) \equiv -\eta(t, y) $$

and

$$ \begin{pmatrix} u_1(\tau, x) \\ u_2(\tau, x) \end{pmatrix} = \begin{pmatrix} \xi(\varepsilon, y) + \eta(\varepsilon, y) \\ \xi(\varepsilon, y) - \eta(\varepsilon, y) \end{pmatrix} + o(1). $$

Here $y = (z\varepsilon^{((1-\alpha)/2} + \theta_2)m_2x - m_2\varepsilon^{\alpha/2}h_1\tau$.

3. If

$$ h = \frac{\pi m_1}{m_2} - \varepsilon^\beta h_1 $$

and condition (16) holds then the dynamics of system (7) for small $\varepsilon$ becomes even more complicated. Following the constructions in [9], we can obtain systems of $2m_2$ parabolic equations, which dynamics determines the behavior of solutions to problem (7). As an example, consider one of the simplest cases when

$$ m_1 = 2, m_2 = 3. $$

We introduce some notation. Let $z$ and $v$ be arbitrary and fixed real values. Define $\delta$ and $\gamma$ as $\delta = (\alpha - 1)/2 - \beta$ and $\gamma = \beta - 1/2$. Let $\theta_1 = \theta_1(\varepsilon) \in [0, 2)$ be such that the value of $\varepsilon\varepsilon^{-\delta} + \theta_1$ is an odd integer, and let $\theta_2 = \theta_2(\varepsilon) \in [0, 1]$ and $\theta_3 = \theta_3(\varepsilon) \in (0, 1]$ be such that the values of $1/2\pi h_1^{-1}\varepsilon^{-\beta}(\varepsilon\varepsilon^{-\delta} + \theta_1) + \theta_2$ and $v\varepsilon^{-\gamma} + \theta_3$ are integers. For $k = 0, 1$ and $j = 1, 2$ define

$$ \Phi_{kj}(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_j(x_1 + x_2 \exp(iks) + x_2 \exp(-iks)) \, ds $$

and let

$$ \begin{align*}
M_{k1}(\xi_1, \eta_1, \xi_2, \eta_2) &= \frac{1}{4} \left[ \Phi_{k1}(\xi_1 + \eta_1, \xi_2 + \eta_2) + \Phi_{k1}(\xi_1 - \eta_1, \xi_2 - \eta_2)
+ \Phi_{k2}(\xi_1 + \eta_1, \xi_2 + \eta_2) + \Phi_{k2}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right], \\
M_{k2}(\xi_1, \eta_1, \xi_2, \eta_2) &= \frac{1}{4} \left[ \Phi_{k1}(\xi_1 + \eta_1, \xi_2 + \eta_2) - \Phi_{k1}(\xi_1 - \eta_1, \xi_2 - \eta_2)
+ \Phi_{k2}(\xi_1 + \eta_1, \xi_2 + \eta_2) - \Phi_{k2}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right].
\end{align*} $$

Consider boundary value problem for two real variables $\xi_1$ and $\eta_1$ and two complex variables $\xi_2$ and $\eta_2$

$$ \begin{align*}
\frac{\partial \xi_1}{\partial t} &= \frac{\varepsilon^2 \kappa^2}{36 h_1^2} \frac{\partial^2 \xi_1}{\partial y^2} + \frac{1}{2} h_1^2 v^2 \frac{\partial^2 \xi_1}{\partial w^2} + M_{01}(\xi_1, \eta_1, \xi_2, \eta_2), \\
\frac{\partial \eta_1}{\partial t} &= \frac{\varepsilon^2 \kappa^2}{36 h_1^2} \frac{\partial^2 \eta_1}{\partial y^2} + \frac{1}{2} h_1^2 v^2 \frac{\partial^2 \eta_1}{\partial w^2} + M_{02}(\xi_1, \eta_1, \xi_2, \eta_2), \\
\frac{\partial \xi_2}{\partial t} &= \frac{\varepsilon^2 \kappa^2}{36 h_1^2} \frac{\partial^2 \xi_2}{\partial y^2} + \frac{1}{2} h_1^2 v^2 \frac{\partial^2 \xi_2}{\partial w^2} + M_{11}(\xi_1, \eta_1, \xi_2, \eta_2), \\
\frac{\partial \eta_2}{\partial t} &= \frac{\varepsilon^2 \kappa^2}{36 h_1^2} \frac{\partial^2 \eta_2}{\partial y^2} + \frac{1}{2} h_1^2 v^2 \frac{\partial^2 \eta_2}{\partial w^2} + M_{12}(\xi_1, \eta_1, \xi_2, \eta_2), \end{align*} $$

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\[ \xi_j(t, y, w) = \xi_j(t, y + 2\pi, w) = \xi_j(t, y, w + 2\pi), \quad j = 1, 2, \]
\[ \eta_j(t, y, w) = \eta_j(t, y + \pi, w) = \eta_j(t, y, w + 2\pi), \quad j = 1, 2. \]

And solution of (7) can be found by formula
\[ u_1(\tau, x) = \xi_1(\varepsilon\tau, y, 3w) + \eta_1(\varepsilon\tau, y, 3w) + [\xi_2(\varepsilon\tau, y, 3w) + \eta_2(\varepsilon\tau, y, 3w)] \exp(-i(2\pi(3h_1\varepsilon^6)^{-1})x) + [\xi_2(\varepsilon\tau, y, 3w) + \eta_2(\varepsilon\tau, y, 3w)] \exp(i(2\pi(3h_1\varepsilon^6)^{-1})x) + o(1), \]
\[ u_2(\tau, x) = \xi_1(\varepsilon\tau, y, 3w) - \eta_1(\varepsilon\tau, y, 3w) + [\xi_2(\varepsilon\tau, y, 3w) - \eta_2(\varepsilon\tau, y, 3w)] \exp(-i(2\pi(3h_1\varepsilon^6)^{-1})x) + [\xi_2(\varepsilon\tau, y, 3w) - \eta_2(\varepsilon\tau, y, 3w)] \exp(i(2\pi(3h_1\varepsilon^6)^{-1})x) + o(1), \]

where \( y = x(\pi(3h_1\varepsilon^6)^{-1}(\varepsilon^6 + 1) + \theta_2), \)
\( w = x(\varepsilon^6 + 1) - h_1\varepsilon^6\tau. \)

5. Conclusions
Special nonlinear families of generally parabolic evolution equations (without small or large parameters) have been constructed to determine the leading terms of the asymptotic representations of solutions to the original boundary value problems. It is well known that equations of this type have rich and complex dynamics. The presence of the continual parameters \( \varepsilon \) and sometimes \( \eta \) in these families suggests that systems (1) and (3) with a large parameter have multistability.

The construction of the leading terms in the asymptotic expansions of solutions to systems (1) and (3) gives the possibility of finding solutions with any prescribed degree of accuracy in \( \varepsilon \). In a number of cases, based on the norm estimates derived in [10] for linear operators, the existence of exact solutions (with the constructed asymptotics) can be proved and their stability can be analyzed. The above results can be extended to more general systems of equations. For example, it is of interest to study new phenomena arising in the case of a two-dimensional domain (see [11]). Note that, for delay equations, we obtain evolution equations with a deviating spatial variable; i.e., the time delay passes into a spatial deviation.

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References
[1] Haken H 1983 Synergetics; An Introducintion, 3rd Revised and Enlarged Edition (Berlin: Springer)
[2] Haken H 2002 Brain dynamics; synchronization and activity patterns in pulse-coupled neural nets with delays and noise (Berlin: Springer)
[3] Kuramoto Y 1984 Chemical oscillations, waves and turbulence (New-York: Springer)
[4] Kuramoto Y and Battogtokh D 2002 Nonlinear phenomena in complex systems 5 380–385
[5] Kashchenko I and Kaschenko S 2016 Communications in Nonlinear Science and Numerical Simulation 34 123–129
[6] Kashchenko I S and Kaschenko S A 2010 Doklady Mathematics 82 850–853
[7] Kashchenko S A 1990 U.S.S.R. Computational Mathematics and Mathematical Physics 30 186–197
[8] Kashchenko I S and Kaschenko S A 2011 Doklady Mathematics 83 408–412 ISSN 1064-5624
[9] Kashchenko I S and Kaschenko S A 2012 Doklady Mathematics 85 129–133 ISSN 1064-5624
[10] Kashchenko S A 1992 Journal of Soviet Mathematics 60 1742–1764
[11] Kashchenko S A 1989 Differential Equations 25 193–199