The spatially homogeneous Boltzmann equation for massless particles in an FLRW background

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Abstract

We study the spatially homogeneous relativistic Boltzmann equation for massless particles in an FLRW background with scattering kernels in a certain range of soft and hard potentials. We obtain the future global existence of small solutions in a weighted $L^1 \cap L^\infty$ space.

1 Introduction

The relativistic Boltzmann equation describes the time evolution of the distribution function for fast-moving particles undergoing binary collisions. It was first considered in general relativity [6, 7, 10], but detailed analysis of the collision operator was initiated in special relativity [13, 14, 17, 18]. There have been many works on the relativistic Boltzmann equation [1, 2, 16, 19, 20, 27, 32, 33, 34], but still many open problems remain to be investigated. For basic information about the relativistic Boltzmann equation we refer to [12, 15].

In this paper, we are interested in the relativistic Boltzmann equation, but it will be studied in an FLRW1 background. To be consistent with the FLRW geometry, we assume that the distribution function is also spatially homogeneous. There have been only a few results concerning the spatially homogeneous relativistic Boltzmann equation in an FLRW spacetime. Global existence of small solutions was obtained in [21], but a certain restriction was imposed on the angular part of the scattering kernel. The restriction was removed in [23], but the argument applies only to the case of the scattering kernel for Israel particles. These results have been extended to the Bianchi cases [22, 24, 25], and we also refer to [28, 30, 31] for a different approach. The purpose of this paper is to study the global existence of small solutions to the spatially homogeneous relativistic Boltzmann equation in an FLRW background, and the results of the paper will be an improvement of [21, 23], in the sense that the unphysical restriction of [21] will be removed, and the global existence will be proved for a wider class of scattering kernels.

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On the other hand, we will consider the Boltzmann equation for massless particles, which differs from the massive cases of [21, 23]. In this paper, the spatially homogeneous relativistic Boltzmann equation for massless particles will be referred to as the massless Boltzmann equation, for simplicity. The massless Boltzmann equation has recently been studied in [8, 9], where an analytic solution has been found, and in [26], where a local existence was obtained. The main interest in [26] was to study the isotropic singularity problem, for which one needs to establish a well-posed Cauchy problem with data at $t = 0$, where the initial singularity is located (see [3, 4, 5, 35, 36] for more details). In this paper, we also study the Cauchy problem for the massless Boltzmann equation, but (a) data will be given at a finite time after the initial singularity, say $t = t_0 > 0$, (b) we will obtain the global existence, and (c) the scattering kernel in this paper will differ from the one in [26], which was the type of soft potentials:

$$
\sigma(h, \omega) = h^{-b} \quad (1 < b < 2),
$$

but in this paper it will be assumed to be of the following type:

$$
\sigma(h, \omega) = \begin{cases} h^{-b} & (0 < b < 1), \\
\hbar^a & (0 \leq a < 2), \end{cases}
$$

which covers a wide range of soft and hard potentials. Unfortunately, we were not able to obtain the result for $1 < b < 2$, which could have lead to a global existence result with data at the initial singularity. We note that the distribution function in [26] was assumed to be spatially homogeneous and isotropic, i.e., $f = f(t, |p|)$, but the isotropy assumption will be removed in this paper so that we only have the spatial homogeneity on the distribution function. Hence, we expect that the results of this paper can be extended to the Bianchi cases. We also expect that the idea of this paper can be used to extend [26] to the Bianchi cases.

The strategy of this paper is as follows. We first consider the arguments of [27], where the global existence for general initial data was obtained in the massive case. Applying [27] to the massless case, we encounter a singularity in the collision operator (see (11) and (14)–(15)), but it will be shown that the singularity can be controlled by using the singular weights $|p|^r$ and $|p|v^w$ (see (23)–(24)). In the case of soft potentials, we estimate the $L^1_{1,2}$ norm to obtain the existence in $L^1_{1,1}$. The $L^1_{1,2}$ norm will be estimated by using the $L^\infty_w$ norm, and the boundedness of the $L^\infty_w$ norm will be obtained by assuming small initial data and using the expansion of the universe. In this paper, the scale factor in the FLRW metric (see (9)) will be assumed to be given by

$$
R = C(t + t_0)^{\frac{1}{2}},
$$

for some constant $C > 0$, so that initial data will be given at $t = 0$, and the initial singularity will be located at $t = -t_0$ (see page 4 of [26] for more details). Similar arguments will be given in the case of hard potentials, and the following are the main results of this paper.

\textbf{Theorem 1.} Let $f_0$ be an initial data of the massless Boltzmann equation (14) satisfying $0 \leq f_0 \in L^1_{1,2}(\mathbb{R}^3) \cap L^\infty_w(\mathbb{R}^3)$. Then, there exists $\varepsilon > 0$ such that for
any \(\|f_0\|_{L^\infty_w} < \varepsilon\), the massless Boltzmann equation has a unique non-negative solution \(f \in C^1([0, \infty); L^1(\mathbb{R}^3) \cap L^1_1(\mathbb{R}^3))\) satisfying

\[
\sup_{0 \leq t < \infty} \|f(t)\|_{L^\infty_w} \leq C\varepsilon.
\]

**Theorem 2.** Let \(f_0\) be an initial data of the massless Boltzmann equation \([15]\) satisfying \(0 \leq f_0 \in L^\infty_w(\mathbb{R}^3)\). Then, there exists \(\varepsilon > 0\) such that for any \(\|f_0\|_{L^\infty_w} < \varepsilon\), the massless Boltzmann equation has a unique non-negative solution \(f \in C^1([0, \infty); L^1(\mathbb{R}^3) \cap L^1_1(\mathbb{R}^3))\) satisfying

\[
\sup_{0 \leq t < \infty} \|f(t)\|_{L^\infty_w} \leq C\varepsilon.
\]

The plan of this paper is as follows. In Section 2, we introduce the massless Boltzmann equation in an FLRW background and collect some basic lemmas. In Section 3, we prove the theorems. In the case of soft potentials, we estimate the \(L^1_{-2}\) norm in Proposition \([1]\) where the estimate of the \(L^\infty_w\) norm will be crucially used, and prove the global existence in \(L^1(\mathbb{R}^3) \cap L^1_1(\mathbb{R}^3)\) in Section 3.1. In the case of hard potentials, we need to estimate the \(L^1_2\) norm, but the estimate of the \(L^\infty_w\) norm will be enough to obtain the global existence in \(L^1(\mathbb{R}^3) \cap L^1_1(\mathbb{R}^3)\). This will be given in Proposition \([2]\) and Section 3.2.

## 2 Preliminaries

### 2.1 Boltzmann equation

The Boltzmann equation describes the time evolution of the distribution function \(f = f(t, x, p)\), which is the density function in the phase space. Let \(p^\alpha\) be the four-momentum of a particle with rest mass \(m \geq 0\). By the mass shell condition:

\[
p^0 p^0 = -m^2,
\]

we have \(p^0\) as a function of \(p\). In the Minkowski case we have

\[
p^0 = \sqrt{m^2 + |p|^2}.
\]

We consider only binary collisions and assume that the total energy and momentum is conserved. Let \(p^\alpha, q^\alpha, p'^\alpha,\) and \(q'^\alpha\) denote the pre-collision and the post-collision momenta of two colliding particles. Then, we have

\[
p'^\alpha + q'^\alpha = p^\alpha + q^\alpha.
\]

The relative momentum \(h\) and the total energy \(s\) are defined by

\[
h = \sqrt{(p^\alpha - q^\alpha)(p'^\alpha - q'^\alpha)} = \sqrt{-2m^2 - 2p_\alpha q^\alpha},
\]

\[
s = -(p^\alpha + q^\alpha)(p'^\alpha + q'^\alpha) = 2m^2 - 2p_\alpha q^\alpha,
\]

and the energy-momentum conservation shows that they are collisional invariants. In the Minkowski case the Boltzmann equation is written as follows:

\[
\partial_t f + \frac{p}{p^0} \cdot \nabla_x f = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{h\sqrt{s}}{p^0 q'^\alpha} \tilde{\sigma}(h, \omega)(f(p')f(q') - f(p)f(q))d\omega dq.
\]
The quantity $\sigma$ is the scattering kernel, and the post-collision momentum $p'^{\alpha}$ and $q'^{\alpha}$ can be parametrized by

$$p'^0 = \frac{p^0 + q^0}{2} + \frac{h(p + q) \cdot \omega}{2\sqrt{s}},$$  
(5) $p'^0_{\text{ortho}}$  

$$q'^0 = \frac{p^0 + q^0}{2} - \frac{h(p + q) \cdot \omega}{2\sqrt{s}},$$  
(6) $q'^0_{\text{ortho}}$

and

$$p' = \frac{p + q}{2} + \frac{h}{2} \left( \omega + \frac{(p + q) \cdot \omega (p + q)}{\sqrt{s}(p^0 + q^0 + \sqrt{s})} \right),$$  
(7) $p'_{\text{ortho}}$

$$q' = \frac{p + q}{2} - \frac{h}{2} \left( \omega + \frac{(p + q) \cdot \omega (p + q)}{\sqrt{s}(p^0 + q^0 + \sqrt{s})} \right).$$  
(8) $q'_{\text{ortho}}$

Several different ways to parametrize the post-collision momentum are known [17, 22, 23], but the expressions (5)–(8) are the ones of [32, 33]. We refer to [12] for more details about the relativistic Boltzmann equation.

In this paper we are interested in the FLRW spacetime. We will assume that the metric is given by

$$g = -dt^2 + R^2((dx^1)^2 + (dx^2)^2 + (dx^3)^2)$$  
(9) $\text{metric}$

and the scale factor $R$ satisfies the assumption [2]. Indices are now raised and lowered via the metric $g$ so that we have

$$p^0 = -p_0, \quad p_i = R^2 p^i, \quad i = 1, 2, 3.$$  

In this paper, we will consider the spatially homogeneous case, and the distribution function will be assumed to be a function of $t$ and $p_i$. We will write

$$p = (p_1, p_2, p_3)$$

so that the distribution function can be written as

$$f = f(t, p).$$

Let us define

$$|p| := \sqrt{\sum_{i=1}^{3} (p_i)^2}.$$  

Then, we obtain from the mass shell condition

$$p^0 = \sqrt{m^2 + R^{-2}|p|^2}.$$  

The Boltzmann equation in an FLRW background is now written as follows:

$$\partial_t f = R^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{h^2}{p^0 q^0} \sigma(h, \omega)(f(p') f(q') - f(p) f(q)) d\omega dq,$$  
(10) $B$

The notations should not be confused with the ones in [26], where $p$ was used to denote a three dimensional vector, and the modulus of $p$ was denoted by $|p|$.  

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where $R^{-3}$ corresponds to $(-\det g)^{-1/2}$, $dq = dq_1 dq_2 dq_3$, and $\omega = (\omega_1, \omega_2, \omega_3)$ is a unit vector such that $|\omega| = 1$. The quantities (3) and (4) are now given by

$$h = \sqrt{-2m^2 - 2p_0 q^0} = \sqrt{-2m^2 + 2p^0 q^0 - 2R^{-2}(p \cdot q)},$$

$$s = 2m^2 - 2p_0 q^0 = 2m^2 + 2p^0 q^0 - 2R^{-2}(p \cdot q),$$

where $p \cdot q := p_1 q_1 + p_2 q_2 + p_3 q_3$.

The expressions of post-collision momentum can be obtained by considering an orthonormal frame. For instance, we may choose $e_0 = dt = dx^0$ and $e_i = R dx^i$ to obtain (5)–(8) with respect to $\{\epsilon^\alpha\}$. Hence, we obtain with respect to $\{dx^\alpha\}$ the following:

$$p_0' = p_0 + \frac{h(p + q) \cdot \omega}{2R\sqrt{s}},$$

$$q_0' = p_0 + \frac{h(p + q) \cdot \omega}{2R\sqrt{s}},$$

and

$$p' = \frac{p + q}{2} + \frac{h}{2} \left( R\omega + \frac{(p + q) \cdot \omega(p + q)}{R\sqrt{s}(p^0 + q^0 + \sqrt{s})} \right),$$

$$q' = \frac{p + q}{2} - \frac{h}{2} \left( R\omega + \frac{(p + q) \cdot \omega(p + q)}{R\sqrt{s}(p^0 + q^0 + \sqrt{s})} \right).$$

We note that the above expressions are the same with the expression (2.5) of [21], where the Boltzmann equation was studied for massive particles in a given FLRW spacetime.

In this paper we will consider massless particles:

$$m = 0.$$

By the mass shell condition we have

$$p^0 = R^{-1}|p|. \quad (11)$$

Let us define

$$\varrho := \sqrt{2(|p||q| - p \cdot q)}, \quad (12)$$

then we obtain

$$h = \sqrt{s} = R^{-1}\varrho. \quad (13)$$

Now, applying the assumption on the scattering kernel (1) to the equation (10) together with (11)–(13), we obtain the following:

$$\partial_t f = R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} \frac{\varrho^{2-b}}{|p||q|} (f(p') f(q') - f(p) f(q)) d\omega dq \quad (14)$$

in the soft potential case, and

$$\partial_t f = R^{-3-a} \int_{\mathbb{R}^3} \int_{S^2} \frac{\varrho^{2+a}}{|p||q|} (f(p') f(q') - f(p) f(q)) d\omega dq \quad (15)$$

in the hard potential case.
in the hard potential case, where $0 < b < 1$ and $0 \leq a < 2$. The post-collision momentum can be written as

\begin{align*}
|p'| &= \frac{|p| + |q|}{2} + \frac{(p + q) \cdot \omega}{2}, \\
|q'| &= \frac{|p| + |q|}{2} - \frac{(p + q) \cdot \omega}{2},
\end{align*}

and

\begin{align*}
p' &= \frac{p + q}{2} + \frac{\rho}{2} \left( \omega + \frac{(p + q) \cdot \omega)(p + q)}{\rho(|p| + |q| + \rho)} \right), \\
q' &= \frac{p + q}{2} - \frac{\rho}{2} \left( \omega + \frac{(p + q) \cdot \omega)(p + q)}{\rho(|p| + |q| + \rho)} \right).
\end{align*}

Moreover, by the energy conservation we have

\begin{equation}
|p'| + |q'| = |p| + |q|,
\end{equation}

and the change of variables between $(p, q)$ and $(p', q')$ is given by

\begin{equation}
\frac{1}{|p||q|} dp dq = \frac{1}{|p'||q'|} dp' dq'.
\end{equation}

In this paper the massless Boltzmann equation will refer to the equation (14) or (15). Note that the equation (14) is the same with the equation (21) of [26], where a different time coordinate was used so that the factor $R^{-3+b}$ does not appear. In the present paper we will make use of the integrability of $R^{-3+b}$ or $R^{-3-a}$, so we do not need to redefine the time coordinate.

### 2.2 Basic lemmas

In this part we collect basic lemmas. They are almost the same with the lemmas in [26], but we present them for the reader’s convenience.

**Lemma 1.** The quantity $\varrho$ defined by (12) satisfies the following:

\begin{align*}
\varrho^2 &= 4|p||q| \sin^2 \frac{\varphi}{2}, \\
\varrho^2 &\leq 4 \min\{|p||q|, |p'||q'|\},
\end{align*}

where $\varphi$ is the angle between the three-dimensional vectors $p$ and $q$.

**Proof.** By the definition (12) we have

\begin{equation}
\varrho^2 = 2(|p||q| - p \cdot q) = 2|p||q|(1 - \cos \varphi) = 4|p||q| \sin^2 \frac{\varphi}{2}.
\end{equation}

The inequality is clear by the fact that $\varrho$ is a collisional invariant.

**Lemma 2.** The post-collision momenta satisfy for any $\delta > 0$ the following:

\begin{equation}
\int_{S^2} \frac{1}{|p'|} d\omega = \int_{S^2} \frac{1}{|q'|} d\omega \leq \frac{C}{\varrho^3(|p| + |q|)^{1-\delta}},
\end{equation}

where $C$ is a positive constant depending on $\delta$. 

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Proof. For simplicity let us write
\[ \nu := |p| + |q|, \quad n := p + q, \]
so that we can write
\[ |p'| = \frac{\nu + n \cdot \omega}{2}, \quad |q'| = \frac{\nu - n \cdot \omega}{2}. \]
We use \( \mathcal{P}_2 = \nu^2 - |n|^2 \) to obtain the following:
\[
\int_{S^2} \frac{1}{|p'|} d\omega = \int_{S^2} \frac{2}{\nu + n \cdot \omega} d\omega \\
= 4\pi \int_0^\pi \sin \theta \frac{\nu + |n| \cos \theta}{\nu - |n|} d\theta \\
= \left( \frac{\nu + |n|}{|n|} \right) \ln \left( \frac{|n|}{g} \left[ 1 + \sqrt{1 + \frac{|n|^2}{g^2}} \right] \right) \\
= \frac{8\pi}{\mathcal{P}_2} \beta^{1-\delta} \left( 1 + \frac{|n|^2}{\mathcal{P}_2} \right)^{\frac{1}{2-\delta}} \ln \left( \frac{|n|}{g} + \sqrt{1 + \frac{|n|^2}{g^2}} \right). 
\]
Note that for any \( \delta > 0 \) the following is bounded:
\[
\left( 1 + x^2 \right)^{1-\delta} \ln \left( |x| + \sqrt{1 + x^2} \right) \frac{1}{|x|}. 
\]
Hence, we obtain the desired result:
\[
\int_{S^2} \frac{1}{|p'|} d\omega \leq \frac{C}{\mathcal{P}_2} \beta^{1-\delta}. 
\]
The calculation for \( q' \) is the same, and this completes the proof. \( \Box \)

Lemma 3. The post-collision momenta satisfy the following:
\[
\int_{S^2} \frac{1}{|p'|^2} d\omega = \int_{S^2} \frac{1}{|q'|^2} d\omega = \frac{16\pi}{\mathcal{P}_2}. 
\]
Proof. By a direct calculation we obtain
\[
\int_{S^2} \frac{1}{|p'|^2} d\omega = \int_{S^2} \frac{4}{(\nu + n \cdot \omega)^2} d\omega \\
= 8\pi \int_0^\pi \sin \theta \frac{\nu + |n| \cos \theta}{\nu^2} d\theta \\
= \frac{16\pi}{\mathcal{P}_2}, 
\]
where \( \nu \) and \( n \) are the same as in the proof of the previous lemma. The calculation for \( q' \) is the same, and this completes the proof. \( \Box \)
3 Existence of solutions

We prove the global existence of solutions to the massless Boltzmann equation. The strategy of proving the global existence is to follow the standard arguments, for instance see [11, 28], but the arguments will be successfully applied to the massless case. We first consider the following modified equation:

\[ \partial_t f = Q_k(f, f), \]  

(22) \textbf{Bmod}

where \( Q_k \) is the collision operator with cutoff defined as follows:

\[ Q_k(f, f) := R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} \mathbb{1}_{\{ |p| \leq |q| \leq k^{-1} \}} \frac{\partial^{2-b}}{|p||q|} (f(p') f(q') - f(p) f(q)) d\omega dq, \]  

(23) \textbf{Qmod}

in the soft potential case, and

\[ Q_k(f, f) := R^{-3-a} \int_{\mathbb{R}^3} \int_{S^2} \mathbb{1}_{\{ |p| \leq k \}} \frac{\partial^{2+a}}{|p||q|} (f(p') f(q') - f(p) f(q)) d\omega dq, \]  

(24) \textbf{Qmodh}

in the hard potential case. Notice that the kernels are bounded in both cases by Lemma 1. The quantities \( R^{-3+b} \) and \( R^{-3-a} \) are decreasing, so it is easy to follow the arguments of [26] to obtain the global existence of solutions to the modified equation.

Next, we need to consider weighted norms in order to remove the cutoffs. Let \( L^1_1(\mathbb{R}^3) \) and \( L^\infty_w(\mathbb{R}^3) \) denote the spaces of functions equipped with the following norms:

\[ \| f \|_{L^1_1} := \int_{\mathbb{R}^3} |f(p)| |p|^r dp, \]  

(25) \textbf{norm1}

\[ \| f \|_{L^\infty_w} := \sup_{p \in \mathbb{R}^3} |w f(p)|, \quad w := |p|^e, \]  

(26) \textbf{norm2}

Note that \( \| \cdot \|_{L^1_1} \) is the usual \( L^1 \)-norm, in which case we will write \( \| \cdot \|_{L^1} \) for simplicity.

In the following we obtain the global existence and uniform boundedness of solutions to the modified equation. We study the soft potential case in Proposition 1 and the hard potential case in Proposition 2.

**Proposition 1.** Let \( k > 0 \) be given. For any initial data \( 0 \leq f_0 \in L^1_1(\mathbb{R}^3) \) the modified equation (22) with (23) has a unique non-negative solution \( f \in C^1((0, \infty); L^1(\mathbb{R}^3)) \). If, in addition, \( f_0 \in L^1(\mathbb{R}^3) \cap L^\infty_w(\mathbb{R}^3) \), then there exists \( \varepsilon > 0 \) such that for any \( \| f_0 \|_{L^\infty_w} < \varepsilon \), the corresponding solution satisfies the following:

\[ \sup_{0 \leq t < \infty} \| f(t) \|_{L^\infty_w} \leq C \varepsilon, \]

\[ \sup_{0 \leq t < \infty} \| f(t) \|_{L^1_1} \leq C, \]

where the constants \( C \) are independent of \( k \).

**Proof.** Because of the cutoff the kernel is bounded. Hence, the existence in \( L^1 \) is obtained by following the same arguments as in [25], and we skip the proof.
We suppose that \( f_0 \in L^1_{\text{loc}}(\mathbb{R}^3) \cap L^\infty_{\text{loc}}(\mathbb{R}^3) \), and let \( f \in C^1([0, \infty); L^1(\mathbb{R}^3)) \) be the unique non-negative solution to the modified equation (22) with (23). Multiplying the equation (22) with (23) by \(|p|^r\), integrating it over \( p \), and applying (21), we obtain the following:

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p)|p|^r dp = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^3} \int_{S^2} \mathbf{1}_{\{p \geq k-1\}} \frac{\theta^{2-b}}{|p|} f(p) f(q) (|p'|^r + |q'|^r - |p|^r - |q|^r) d\omega dp dq.
\]

We immediately obtain for \( r = 0 \),

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p) dp = 0.
\]

Since \( f \) is non-negative, we obtain for all \( t \geq 0 \),

\[
\|f(t)\|_{L^1} = \|f_0\|_{L^1}.
\]

In order to estimate the case \( r = -2 \), we first need to estimate the \( L^\infty_{\text{loc}} \) norm. Multiplying the equation (22) with (23) by \( w \), we obtain

\[
\frac{\partial (wf)}{\partial t} = wQ_k(f, f)
\]

\[
\leq R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} \mathbf{1}_{\{p \geq k-1\}} \frac{\theta^{2-b}}{|p|} |\epsilon| |p| f(p') f(q') d\omega dq
\]

\[
= R^{-3+b} \|f\|_{L^\infty}^2 \int_{\mathbb{R}^3} \int_{S^2} \frac{\theta^{2-b}}{|p|} \frac{1}{|p'|} e^{-|p'|} \frac{1}{|q'|} e^{-|q'|} d\omega dq
\]

\[
= R^{-3+b} \|f\|_{L^\infty}^2 \int_{\mathbb{R}^3} \int_{S^2} \frac{\theta^{2-b}}{|p|} \frac{1}{|p'| |q'|} d\omega dq,
\]

where we used (20). The integration on \( S^2 \) is as follows:

\[
\int_{S^2} \frac{d\omega}{|p||q|} = \int_{S^2} \frac{4d\omega}{\nu^2 - (n \cdot \omega)^2} = \int_0^\pi \frac{8\pi \sin \theta d\theta}{\nu^2 - |n|^2 \cos^2 \theta}
\]

where \( \nu \) and \( n \) are the same as in the proof of Lemma 2. Then, we have

\[
\int_0^\pi \frac{8\pi \sin \theta d\theta}{\nu^2 - |n|^2 \cos^2 \theta} = \frac{8\pi}{\nu |n|} \ln \left( \frac{\nu + |n|}{\nu - |n|} \right) = \frac{16\pi}{\nu |n|} \ln \left( \frac{\nu + |n|}{\nu |n|} \right).
\]

The last quantity can be estimated as in the proof of Lemma 2 for any \( \delta > 0 \) we have

\[
\frac{16\pi}{\nu |n|} \ln \left( \frac{\nu + |n|}{\nu |n|} \right) \leq \frac{C}{\nu^{9-b} |n|^\delta}.
\]

Since \( 0 < b < 1 \), we can choose \( \delta = 2 - b \) to obtain

\[
\int_{S^2} \frac{d\omega}{|p||q|} \leq \frac{C}{\nu^{9-b} |n|^\delta} \leq \frac{C}{\nu^{9-b} |q|^\delta}.
\]
where we used the fact that $b > 0$ in the last inequality. Now, we have
\[
\frac{\partial (wf)}{\partial t} \leq CR^{-3+b}\|f\|_{L_\infty^w}^2 \int_{\mathbb{R}^3} \frac{1}{|q|^{1+\epsilon}} e^{-|q|} dq
\]

where the integral above is finite since $0 < b < 1$. Then, we obtain
\[
\frac{d}{dt}\|f\|_{L_\infty^w} \leq CR^{-3+b}\|f\|_{L_\infty^w}^2.
\]

Since $R^{-3+b}$ is integrable, we conclude that there exists $\varepsilon > 0$ such that if
\[
\|f_0\|_{L_\infty^w} \leq \varepsilon,
\]

then
\[
\sup_{0 \leq t < \infty} \|f(t)\|_{L_\infty^w} \leq C\varepsilon. \tag{31}
\]

We now estimate the expression (27) in the case $r = -2$ as follows:
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p) \frac{1}{|p|^2} dp \leq R^{-3+b} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} 1_{\{\varphi \geq b^{-1}\}} \frac{\varrho^{2-b}}{|p||q|} f(p)f(q) \left( \frac{1}{|p|^2} + \frac{1}{|q|^2} \right) d\omega dp dq
\]

\[
\leq CR^{-3+b} \int_{\mathbb{R}^3} \frac{\varrho^{2-b}}{|p||q|} f(p)f(q) dp dq
\]

\[
\leq CR^{-3+b} \int_{\mathbb{R}^3} \frac{f(p)f(q)}{|p|^{1+\frac{b}{2}}|q|^{1+\frac{b}{2}} \sin^b(\varphi/2)} dp dq,
\]

where we used Lemma 3 and Lemma 1. Let us consider the integration over $q$ on the right hand side. We use (31) to obtain the following:
\[
\int_{\mathbb{R}^3} f(q) \frac{1}{|q|^{1+\frac{b}{2}} \sin^b(\varphi/2)} dq \leq C\varepsilon \int_{\mathbb{R}^3} \frac{e^{-|q|}}{|q|^{1+\frac{b}{2}} \sin^b(\varphi/2)} dq
\]

\[
\leq C\varepsilon \int_0^\infty \int_0^{\pi} e^{-|q|} \sin^b(\varphi/2) \sin \varphi d\varphi dq
\]

\[
\leq C\varepsilon \int_0^\infty \frac{e^{-|q|}}{|q|^{1+\frac{b}{2}}} dq,
\]

where the last integral is finite, since $0 < b < 1$. Hence, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p) \frac{1}{|p|^2} dp \leq CR^{-3+b} \int_{\mathbb{R}^3} \frac{f(p)}{|p|^{1+\frac{b}{2}}} dp
\]

\[
\leq C\varepsilon R^{-3+b} \left( \|f\|_{L^1} + \|f\|_{L_{\infty}^w} \right).
\]

Since $R^{-3+b}$ is integrable, we obtain the desired result by Grönwall’s inequality together with (28). \qed

**Remark 1.** In [23], the authors studied the massless Boltzmann equation in a different situation, but it was also necessary to estimate the $L_{1-2}^1$ norm. In Proposition 1, we could estimate the $L_{1-2}^1$ norm by using the $L_{\infty}^w$ norm. On the
other hand, in [26], the distribution function $f$ was assumed to be isotropic, i.e.,
$f(p) = f(|p|)$, so that the $L^{1,2}$ norm could be estimated without the $L^\infty_w$ norm.
One might expect that by using the $L^\infty_w$ norm it should be possible to extend the
result of [26] to the Bianchi case.

**Remark 2.** Note that the first result of Proposition 1 shows that
$$
\sup_{0 \leq t < \infty} \|f(t)\|_{L^1_w} \leq C\varepsilon,
$$
for any $-2 < r \leq 0$.

**Proposition 2.** Let $k > 0$ be given. For any initial data $0 \leq f_0 \in L^1(\mathbb{R}^3)$
the modified equation (22) with (24) has a unique non-negative solution $f \in C^1((0,\infty);L^1(\mathbb{R}^3))$. If, in addition, $f_0 \in L^\infty_w(\mathbb{R}^3)$, then there exists $\varepsilon > 0$ such
that for any $\|f_0\|_{L^\infty_w} < \varepsilon$, the corresponding solution satisfies the following:
$$
\sup_{0 \leq t < \infty} \|f(t)\|_{L^\infty_w} \leq C\varepsilon,
$$
where the constant $C$ is independent of $k$.

**Proof.** As in the Proposition 1 one can easily obtain the existence in $L^1$. We
now assume that $f_0 \in L^\infty_w(\mathbb{R}^3)$. Multiplying the equation (22) with (24) by $w$,
we obtain
$$
\frac{\partial(wf)}{\partial t} = wQ_k(f,f)
$$
$$
\leq R^{-3-a} \int_{\mathbb{R}^3} \int_{S^2} \mathbf{1}_{\{e \leq k\}} |p| |q| \frac{a^{2+a}}{|p'|} |f(p')f(q')| d\omega dq
$$
$$
\leq R^{-3-a} \|f\|_{L^2_w}^2 \int_{\mathbb{R}^3} \int_{S^2} \frac{a^{2+a}}{|q|} |p| e^{-|q'||q'|} \frac{1}{|p'|} e^{-|p'||p'|} d\omega dq
$$
$$
= R^{-3-a} \|f\|_{L^2_w}^2 \int_{\mathbb{R}^3} \int_{S^2} \frac{a^{2+a}}{|q|} e^{-|q'|} \frac{1}{|p'|} e^{-|q'||p'|} d\omega dq,
$$
where we used (20). By the same calculation as in the Proposition 1 we obtain
for any $\delta > 0$,
$$
\int_{S^2} \frac{d\omega}{|p'| |q'|} \leq \frac{C}{\nu^{2-a-\delta}}, \quad (32)
$$
Here, if we choose $\delta = 2 + a$ as in the Proposition 1 then the estimate (32)
shows that
$$
\int_{S^2} \frac{d\omega}{|p'| |q'|} \leq \frac{C\nu^a}{\omega^{2+a}},
$$
so that the dependence on $p$ remains in $\nu$ (see (31) in the soft potential case).
Instead, we choose $0 < \delta < 2$ such that
$$
a + \delta < 2.
$$
Then, we apply Young’s inequality as follows:
$$
\nu \geq c |p|^{\frac{2+a-\delta}{2+a}} |q|^{\frac{2-a-\delta}{2+a}}, \quad (33)
$$
where $c$ is a positive constant depending on $a$ and $\delta$. Hence, the $L^\infty_w$ norm can be estimated together with (62) and (63) as follows:

$$\frac{\partial (wf)}{\partial t} \leq CR^{-3-a}\|f\|_{L^\infty_w}^2 \int_{\mathbb{R}^3} \frac{\rho^{2+a-\delta}}{|q|} e^{-|q|} \frac{1}{|p|^{2+b-\delta} |q|^{2+b-\delta}} dq$$

$$\leq CR^{-3-a}\|f\|_{L^\infty_w}^2 \int_{\mathbb{R}^3} |q|^{a-1} e^{-|q|} dq$$

$$\leq CR^{-3-a}\|f\|_{L^\infty_w}^2,$$

where we used Lemma 1 and the last integral is finite since $0 \leq a < 2$. Therefore, we conclude that there exists $\varepsilon > 0$ such that if $\|f_0\|_{L^\infty_w} \leq \varepsilon$, then

$$\sup_{0 \leq t < \infty} \|f(t)\|_{L^\infty_w} \leq C\varepsilon,$$

which completes the proof.

Remark 3. Note that the above result shows that for any $r \geq 0$,

$$\sup_{0 \leq t < \infty} \|f(t)\|_{L^r_w} \leq C\varepsilon.$$

On the other hand, in a similar way to the soft potential case we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(q)|p|^r dp$$

$$= \frac{R^{-3-a}}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \mathbf{1}_{|q| \geq \varepsilon} \frac{\rho^{2+a}}{|p||q|} f(p)f(q)(|p|^r + |q|^r - |p|^r - |q|^r) d\omega dp dq.$$

Hence, we obtain $\|f(t)\|_{L^r_w} = \|f_0\|_{L^r_w}$ for all $t \geq 0$ in the cases $r = 0, 1$.

3.1 Proof of Theorem 1

We are now ready to remove the cutoff. We first consider Theorem 1 for the soft potential case. Note that $f_0 \in L^\infty_w(\mathbb{R}^3)$ implies $f_0 \in L^1(\mathbb{R}^3)$. Hence, we can apply Proposition 1 to obtain a sequence $\{f_k\}_{k=1}^\infty$, which are the solutions to the modified equation (22) with (23):

$$\partial_t f_k = Q_k(f_k, f_k), \quad f_k(0) = f_0 \geq 0.$$

Below, we will show that the sequence $\{f_k\}_{k=1}^\infty$ converges in $L^1(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$.

For $m < n$, we have

$$\partial_t f_m - \partial_t f_n = Q_m(f_m, f_m) - Q_m(f_n, f_n) + Q_m(f_n, f_n) - Q_n(f_n, f_n),$$

where

$$Q_m(f_m, f_m) - Q_m(f_n, f_n)$$

$$= \frac{R^{-3+b}}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \mathbf{1}_{|q| \geq \varepsilon} \frac{\rho^{2-b}}{|p||q|}$$

$$\times \left\{ (f_m + f_n)(p')(f_m - f_n)(q') + (f_m + f_n)(q')(f_m - f_n)(p') - (f_m + f_n)(p')(f_n - f_n)(q) - (f_m + f_n)(q)(f_n - f_n)(p) \right\} d\omega dq,$$
and

\[ Q_m(f_n, f_n) - Q_n(f_n, f_n) \]

\[ = -R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{n^{-1} \leq \rho \leq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} (f_n(p')f_n(q') - f_n(p)f_n(q)) d\omega dq. \]

Multiplying the above equation by \( \text{sgn}(f_m - f_n)(p) \) we obtain

\[ \partial_t |f_m - f_n|(p) \]

\[ \leq \frac{R^{-3+b}}{2} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{\rho \geq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} \]

\[ \times \left\{ (f_m + f_n)(p')|f_m - f_n|(q') + (f_m + f_n)(q')|f_m - f_n|(p') \right. \]

\[ + (f_m + f_n)(p)|f_m - f_n|(q) - (f_m + f_n)(q)|f_m - f_n|(p) \}

\[ + R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{n^{-1} \leq \rho \leq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} (f_n(p')f_n(q') + f_n(p)f_n(q)) d\omega dq \]

where we used the fact that the solutions are non-negative. Then, multiplying the above by \( |p|^r \) and integrating it over \( \mathbb{R}^3 \), we obtain the following:

\[ \frac{d}{dt} \|f_m - f_n\|_{L^1} \]

\[ \leq \frac{R^{-3+b}}{2} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{\rho \geq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} \]

\[ \times \left\{ (f_m + f_n)(p')|f_m - f_n|(q') + (f_m + f_n)(q')|f_m - f_n|(p') \right. \]

\[ + (f_m + f_n)(p)|f_m - f_n|(q) - (f_m + f_n)(q)|f_m - f_n|(p) \}

\[ + R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{n^{-1} \leq \rho \leq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} \left\{ f_n(p')f_n(q') + f_n(p)f_n(q) \right\} |p|^r d\omega dq dp \]

\[ = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{\rho \geq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} \]

\[ \times (f_m + f_n)(p)|f_m - f_n|(q) \left\{ |p'|^r + |q'|^r + |p|^r - |q|^r \right\} d\omega dq dp \]

\[ + R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{n^{-1} \leq \rho \leq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} f_n(p)f_n(q) \left\{ |p'|^r + |p|^r \right\} d\omega dq dp. \]

For \( r = 0 \) we have

\[ \frac{d}{dt} \|f_m - f_n\|_{L^1} \leq I_1 + I_2, \]

where

\[ I_1 = R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} \frac{\rho^{2-b}}{|p| |q|} (f_m + f_n)(p)|f_m - f_n|(q) d\omega dq dp, \]

\[ I_2 = 2R^{-3+b} \int_{\mathbb{R}^3} \int_{S^2} 1_{\{\rho \geq m^{-1}\}} \frac{\rho^{2-b}}{|p| |q|} f_n(p)f_n(q) d\omega dq dp. \]
The integrals $I_1$ and $I_2$ are estimated as follows:

$$I_1 \leq CR^{-3+b}\int_{\mathbb{R}^6} \frac{1}{|p| |q|} (f_m + f_n)(p)|f_m - f_n|(q)dqdp$$

$$\leq CR^{-3+b} \sup_k \|f_k\|_{L_{1/4}^1} \|f_m - f_n\|_{L_{1/4}^1},$$

(34)  \hspace{1cm} \text{sI.1}

and

$$I_2 \leq CR^{-3+b}m^{-2+b}\int_{\mathbb{R}^6} \frac{1}{|p| |q|} f_n(p)f_n(q)dqdp$$

$$\leq CR^{-3+b}m^{-2+b} \sup_k \|f_k\|_{L_{1/4}^1}^2.$$  

(35)  \hspace{1cm} \text{sI.2}

For $r = -1$, we have

$$\frac{d}{dt} \|f_m - f_n\|_{L_{1/4}^1} \leq J_1 + J_2 + J_3 + J_4 + J_5,$$

where

$$J_1 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_m + f_n)(p)|f_m - f_n|(q) \frac{1}{|p|} d\omega dq dp,$$

$$J_2 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_m + f_n)(p)|f_m - f_n|(q) \frac{1}{|q|} d\omega dq dp,$$

$$J_3 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_m + f_n)(p)|f_m - f_n|(q) \frac{1}{|p|} d\omega dq dp,$$

$$J_4 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_n(p)f_n(q) \frac{1}{|p|} d\omega dq dp,$$

$$J_5 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_n(p)f_n(q) \frac{1}{|p|} d\omega dq dp.$$

We use Lemma 2 to estimate $J_1$ as follows:

$$J_1 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_m + f_n)(p)|f_m - f_n|(q) \frac{1}{|p|} d\omega dq dp$$

$$\leq CR^{-3+b} \int_{\mathbb{R}^6} \frac{\theta^{1-b}}{|p||q|} (f_m + f_n)(p)|f_m - f_n|(q)dqdp$$

$$\leq CR^{-3+b} \int_{\mathbb{R}^6} \frac{1}{|p||q|^{1-b}} (f_m + f_n)(p)|f_m - f_n|(q)dqdp$$

$$\leq CR^{-3+b} \sup_k \|f_k\|_{L_{1/4}^1} \|f_m - f_n\|_{L_{1/4}^1/(1+b)/2},$$

(36)  \hspace{1cm} \text{sJ.1}

The estimate of $J_2$ is exactly the same with that of $J_1$:

$$J_2 \leq CR^{-3+b} \sup_k \|f_k\|_{L_{1/(1+b)/2}^1} \|f_m - f_n\|_{L_{1/(1+b)/2}^1},$$

(37)  \hspace{1cm} \text{sJ.2}

The integral $J_3$ is estimated as follows:

$$J_3 = \frac{R^{-3+b}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{\theta^{2-b}}{|p||q|} (f_m + f_n)(p)|f_m - f_n|(q) \frac{1}{|p|} d\omega dq dp$$

$$\leq CR^{-3+b} \int_{\mathbb{R}^3} (f_m + f_n)(p) \frac{1}{|p|^{1+b}} dp \int_{\mathbb{R}^3} |f_m - f_n|(q) \frac{1}{|q|^{1-b}} dq$$

$$\leq CR^{-3+b} \sup_k \|f_k\|_{L_{1-1/4}^1} \|f_m - f_n\|_{L_{1/4}^1}.$$  

(38)  \hspace{1cm} \text{sJ.3}
For the integral $J_4$, since $0 < b < 1$, we have the following:

$$J_4 = R^{-3+b} \int_{\mathbb{R}^6} \int_{\mathbb{R}^2} \mathbb{1}_{\{0 \leq m-1\}} \frac{q^{2-b}}{|p|^b} f_n(p)f_n(q) \frac{1}{|p|} d\omega dq dp$$

$$\leq CR^{-3+b} \int_{\mathbb{R}^6} \int_{\mathbb{R}^2} \mathbb{1}_{\{0 \leq m-1\}} \frac{q^{1-b}}{|p| |q|^b} f_n(p)f_n(q) d\omega dq dp$$

$$\leq CR^{-3+b} m^{-1+b} \int_{\mathbb{R}^6} \frac{1}{|p| |q|^b} f_n(p)f_n(q) d\omega dq dp$$

$$\leq CR^{-3+b} m^{-1+b} \sup_k \|f_k\|_{L^2_{p,q}}^2. \tag{39}$$

Similarly, $J_5$ is estimate as follows:

$$J_5 = R^{-3+b} \int_{\mathbb{R}^6} \int_{\mathbb{R}^2} \mathbb{1}_{\{0 \leq m-1\}} \frac{q^{2-b}}{|p|^b} f_n(p)f_n(q) \frac{1}{|p|} d\omega dq dp$$

$$\leq CR^{-3+b} m^{-2+b} \int_{\mathbb{R}^6} f_n(p) \frac{1}{|p|^2} d\omega dq dp$$

$$\leq CR^{-3+b} m^{-2+b} \sup_k \|f_k\|_{L^2_{p,q}} \sup_k \|f_k\|_{L^1_{p,q}}. \tag{40}$$

We now apply Proposition [1] with Remark [2] to obtain from (39)–(40) that

$$\frac{d}{dt} \left( \|f_m - f_n\|_{L^1} + \|f_m - f_n\|_{L^1_{p,q}} \right)$$

$$\leq C \varepsilon R^{-3+b} \left( m^{-1+b} + \|f_m - f_n\|_{L^1} + \|f_m - f_n\|_{L^1_{p,q}} \right).$$

Since $R^{-3+b}$ is integrable and $f_m(0) = f_n(0) = f_0$, we obtain

$$\|f_m - f_n\|_{L^1} + \|f_m - f_n\|_{L^1_{p,q}} \leq C m^{-1+b},$$

which shows that the sequence converges in $L^1(\mathbb{R}^3) \cap L^1_{p,q}(\mathbb{R}^3)$ as $m \to \infty$. Hence, we obtain the existence part of Theorem 1. To obtain the boundedness of $\|f\|_{L^\infty}$ we multiply the equation (14) by $w$ and follow the proof of Proposition [1]. Note that the estimate (29) still holds for the original equation (13) without the cutoff. Therefore, we obtain again the estimate (31), and this completes the proof of Theorem 1.

### 3.2 Proof of Theorem 2

We now consider the hard potential case. The strategy is basically the same as in the soft potential case, but we consider the $L^1_{p,q}$ norm instead. By the same arguments we obtain for $m < n$:

$$\frac{d}{dt} \|f_m - f_n\|_{L^1_{p,q}}$$

$$\leq \frac{R^{-3-a}}{2} \int_{\mathbb{R}^6} \int_{\mathbb{R}^2} \mathbb{1}_{\{0 \leq m\}} \frac{q^{2+a}}{|p| |q|^a} \times (f_m + f_n)(p)|f_m - f_n|(q) \left\{ |p'|^\gamma + |q'|^\gamma + |p|^\gamma - |q|^\gamma \right\} d\omega dq dp$$

$$+ R^{-3-a} \int_{\mathbb{R}^6} \int_{\mathbb{R}^2} \mathbb{1}_{\{m \leq n\}} \frac{q^{2+a}}{|p| |q|^a} f_n(p)f_n(q) \left\{ |p'|^\gamma + |p|^\gamma \right\} d\omega dq dp.$$
where $f_m$ and $f_n$ are the solutions of (22) with (24). Then, for $r = 0$ we have
\[
\frac{d}{dt} \|f_m - f_n\|_{L^1} \leq I_1 + I_2,
\]
where
\[
I_1 = R^{-3-a} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{a^{2+a}}{|p||q|} (f_m + f_n)(p) |f_m - f_n(q)| d\omega dp dq,
\]
\[
I_2 = 2R^{-3-a} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \mathbf{1}_{\{q \geq m\}} \frac{a^{2+a}}{|p||q|} f_n(p) f_n(q) d\omega dp dq.
\]
The integrals $I_1$ and $I_2$ are estimated as follows:
\[
I_1 \leq CR^{-3-a} \int_{\mathbb{R}^6} |p|^\frac{2}{1+a} |q|^\frac{2}{1+a} (f_m + f_n)(p) |f_m - f_n(q)| d\omega dp dq
\leq CR^{-3-a} \sup_k \|f_k\|_{L^1_{\frac{3}{1+a}}} \|f_m - f_n\|_{L^1_{\frac{1}{1+a}}},
\] (41) \hfill \text{(I.1)}
and since $0 \leq a < 2$,
\[
I_2 \leq CR^{-3-a} m^{-2+a} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \mathbf{1}_{\{q \geq m\}} \frac{a^4}{|p||q|} f_n(p) f_n(q) d\omega dp dq
\leq CR^{-3-a} m^{-2+a} \sup_k \|f_k\|_{L^1_{\frac{3}{2}}}.
\] (42) \hfill \text{(I.2)}
For $r = 1$, we have
\[
\frac{d}{dt} \|f_m - f_n\|_{L^1} \leq J_1 + J_2,
\]
where
\[
J_1 = R^{-3-a} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \frac{a^{2+a}}{|p||q|} (f_m + f_n)(p) |f_m - f_n(q)| |p| d\omega dp dq,
\]
\[
J_2 = R^{-3-a} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \mathbf{1}_{\{q \geq m\}} \frac{a^{2+a}}{|p||q|} f_n(p) f_n(q) \{|p'| + |p|\} d\omega dp dq,
\]
where we used (20) for $J_1$. We estimate $J_1$ as follows:
\[
J_1 \leq CR^{-3-a} \int_{\mathbb{R}^6} |p|^{1+\frac{2}{1+a}} f_m + f_n)(p) |q|^{\frac{2}{1+a}} |f_m - f_n(q)| d\omega dp dq
\leq CR^{-3-a} \sup_k \|f_k\|_{L^1_{\frac{1}{1+a}}} \|f_m - f_n\|_{L^1_{\frac{1}{1+a}}},
\] \hfill \text{(I.1)}
By (20) and the symmetry, $J_2$ can be estimated as
\[
J_2 \leq CR^{-3-a} m^{-2+a} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \mathbf{1}_{\{q \geq m\}} \frac{a^{2+a}}{|p||q|} f_n(p) f_n(q) |p| d\omega dp dq
\leq CR^{-3-a} m^{-2+a} \sup_k \|f_k\|_{L^1_{\frac{3}{2}}} \sup_k \|f_k\|_{L^1_{\frac{3}{2}}},
\] (44) \hfill \text{(I.2)
Applying Proposition 2 and the estimates of Remark 3, we obtain from (41)–(44) that

\[
\frac{d}{dt} \left( \|f_m - f_n\|_{L^1} + \|f_m - f_n\|_{L^1} \right) \leq C \varepsilon R^{3-\alpha} \left( m^{-2+\alpha} + \|f_m - f_n\|_{L^1} + \|f_m - f_n\|_{L^1} \right).
\]

Since \(0 \leq \alpha < 2\), one can prove the existence of solutions in \(L^1(\mathbb{R}^3) \cap L_1^1(\mathbb{R}^3)\) as in the proof of Theorem 1. The boundedness of \(\|f\|_{L^\infty}\) can be obtained by the same calculation as in Proposition 2. This completes the proof of Theorem 2.

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