On the Order of Polynilpotent Multipliers of Some Nilpotent Products of Cyclic $p$-Groups

Behrooz Mashayekhy$^{1,*}$, Fahimeh Mohammadzadeh$^{2}$

$^1$Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran.  
$^2$Department of Mathematics, Payame Noor University, Ahvaz, Iran.

Abstract. In this article we show that if $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, c_2, \ldots, c_s)$, $N_{c_1, c_2, \ldots, c_s}$, and $G \cong \mathbb{Z}_p^{c_1} \ast \mathbb{Z}_p^{c_2} \ast \ldots \ast \mathbb{Z}_p^{c_s}$ is the $n$th nilpotent product of some cyclic $p$-groups, where $c_1 \geq n$, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_s$ and $(q, p) = 1$ for all primes $q$ less than or equal to $n$, then $|N_{c_1, c_2, \ldots, c_s}M(G)| = p^{d_m}$ if and only if $G \cong \mathbb{Z}_p^{n_1} \ast \mathbb{Z}_p^{n_2} \ast \ldots \ast \mathbb{Z}_p^{n_t}$ ($m$-copies), where $m = \sum_{i=1}^{t} \alpha_i$ and $d_m = \chi_{c_1} + 1(\chi_{c_2} + 1(\sum_{j=1}^{n} \chi_{c_3} + j(m)))) \ldots$ . Also, we extend the result to the multiple nilpotent product $G \cong \mathbb{Z}_p^{n_1} \ast \mathbb{Z}_p^{n_2} \ast \ldots \ast \mathbb{Z}_p^{n_t}$, where $c_1 \geq n_1 \geq \ldots \geq n_{t-1}$. Finally a similar result is given for the $c$-nilpotent multiplier of $G \cong \mathbb{Z}_p^{n_1} \ast \mathbb{Z}_p^{n_2} \ast \ldots \ast \mathbb{Z}_p^{n_t}$, with the different conditions $n \geq c$ and $(q, p) = 1$ for all primes $q$ less than or equal to $n + c$.

Keywords: Polynilpotent multiplier; Nilpotent product; Cyclic group; Finite $p$-group; Elementary Abelian $p$-group.

AMS Subject Classifications: 20C25, 20D15, 20E10, 20F18, 20F12.

1 Introduction and motivation

Let $G$ be any group with a presentation $G \cong F/R$, where $F$ is a free group. Then the Baer invariant of $G$ with respect to the variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{|RV \ast F|},$$

*Correspondence to: Behrooz Mashayekhy, Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran. Email: mashaf@math.um.ac.ir

$^1$Received: 29 April 2009, revised: 2 June 2009, accepted: 11 June 2009.

http://www.i-asr.org/jarpm.html 39 c 2009 Institute of Advanced Scientific Research
where $V$ is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of $F$ and
\[
[RV^r F] = \langle v(f_1, \ldots, f_{i-1}, fi r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots, f_n)^{-1} \rangle
\]
for $r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in N$.

One may check that $\mathcal{V}M(G)$ is abelian and independent of the choice of the free presentation of $G$. In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of the group $G$ will be $(R \cap F^r)/[R, F]$, which is isomorphic to the well-known notion the Schur multiplier of $G$, denoted by $M(G)$. If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, \ldots, c_s)$, $\mathcal{N}_{c_1, c_2, \ldots, c_s}$, then the Baer invariant of a group $G$ with respect to this variety, which is called a polynilpotent multiplier of $G$, is as follows:
\[
\mathcal{N}_{c_1, c_2, \ldots, c_s} M(G) = \frac{R \cap \gamma_{c_1+1} \circ \ldots \circ \gamma_{c_s+1}(F)}{[R, c_1 F, c_2 \gamma_{c_1+1}(F), \ldots, c_s \gamma_{c_s+1} \circ \ldots \circ \gamma_{c_1+1}(F)]},
\]
where $\gamma_{c_s+1} \circ \ldots \circ \gamma_{c_1+1}(F) = \gamma_{c_s+1}(\gamma_{c_{s-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))$ are the term of iterated lower central series of $F$. See Hekster [6] for the equality
\[
[R_{N_{c_1, c_2, \ldots, c_s}} F] = [R, c_1 F, c_2 \gamma_{c_1+1}(F), \ldots, c_s \gamma_{c_s+1} \circ \ldots \circ \gamma_{c_1+1}(F)].
\]
In particular, if $s = 1$ and $c_1 = c$, then the Baer invariant of $G$ with respect to the variety $\mathcal{N}_c$, which is called the $c$-nilpotent multiplier of $G$, is
\[
\mathcal{N}_c M(G) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, c F]}.
\]

Historically, Green [4] showed that the order of the Schur multiplier of a finite $p$-group of order $p^n$ is bounded by $p^\frac{n(n-1)}{2}$. Berkovich [2] showed that a finite $p$-group of order $p^n$ is an elementary abelian $p$-group if and only if the order of $M(G)$ is $p^{\frac{n(n-1)}{2}}$. Moghaddam [15,16] presented a bound for the polynilpotent multiplier of a finite $p$-group. He showed that if $\mathcal{V}$ is the variety of polynilpotent groups of a given class row and $G$ is a finite $d$-generator group of order $p^n$, then
\[
|\mathcal{V}M(Z_p^{(d)})| \leq |\mathcal{V}M(G)||V(G)| \leq |\mathcal{V}M(Z_p^{[n]})|,
\]
where $Z_p^{[m]}$ denotes the direct sum of $m$ copies of $Z_p$. As a consequence, using the structure of $\mathcal{V}M(Z_p^{[n]})$ in [12], we can show that the order of the nilpotent multiplier of a finite $p$-group of order $p^n$ is bounded by $p^{\chi_{c+1}(n)}$, where $\chi_{c+1}(n)$ is the number of basic commutators of weight $c+1$ on $n$ letters. The first author and Sanati [13] extended a result of Berkovich to the $c$-nilpotent multiplier of a finite $p$-group. They showed that for an abelian $p$-group $G$, $|\mathcal{N}_c M(G)| = p^{\chi_{c+1}(n)}$ if and only if $G$ is an elementary abelian $p$-group. Putting an additional condition on the kernel of the left natural map of the generalized Stallings-Stammbach five-term exact sequence, they showed that an arbitrary finite $p$-group with the $c$-nilpotent multiplier of maximum order is an elementary abelian $p$-group.
Unfortunately, there is a mistake in the proof of Theorem 3.5 in [13] due to using the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i + 1)$ which is not correct in general. In this paper, first, we give a correct proof for Theorem 3.5 in [13]. Second, we extend the result in different directions. In fact, we show that if $V$ is the variety of polynilpotent groups of class row $(c_1, c_2, ..., c_n)$, and $G \cong \mathbb{Z}_p^{n_1} \ast \mathbb{Z}_p^{n_2} \ast ... \ast \mathbb{Z}_p^{n_l}$ is the $n$th nilpotent product of some cyclic $p$-groups, where $c_1 \geq n_1$, $a_1 \geq c_2 \geq ... \geq c_l$ and $(q, p) = 1$ for all primes $q$ less than or equal to $n$, then $|\mathbb{N}_{c_1},(c_2, ..., c_n)M(G)| = p^d_{m}$ if and only if $G \cong \mathbb{Z}_p^n \ast \mathbb{Z}_p^n \ast ... \ast \mathbb{Z}_p^n (m$-copies), where $m = \sum_{i=1}^{c} \alpha_i$ and $d_m = \chi_{c+1}(...(\chi_{c+1}(\sum_{j=1}^{n} \chi_{c+1}(m)))...).$

Also, we extend the above result to the multiple nilpotent product of cyclic $p$-groups $G \cong \mathbb{Z}_p^{n_1} \ast \mathbb{Z}_p^{n_2} \ast ... \ast \mathbb{Z}_p^{n_l}$, when $c_1 \geq n_1 \geq ... \geq n_l$. As a consequence we show that the polynilpotent multiplier of a finite abelian $p$-group $G$ has maximum order if and only if $G$ is an elementary abelian $p$-group. Finally we give a similar result for the $c$-nilpotent multiplier of $G \cong \mathbb{Z}_p^{n_1} \ast \mathbb{Z}_p^{n_2} \ast ... \ast \mathbb{Z}_p^{n_l}$, with the different conditions $n \geq c$ and $(q, p) = 1$ for all primes $q$ less than or equal to $n + c$.

## 2 Notation and preliminaries

**Definition 2.1.** Let $\{G_i\}_{i \in I}$ be a family of arbitrary groups. The $n$th nilpotent product of the family $\{G_i\}_{i \in I}$ is defined as follows:

$$
\prod_{i \in I}^{*} G_i = \frac{\prod_{i \in I} G_i}{\gamma_{n+1}(\prod_{i \in I} G_i) \cap [G_i]_{i \in I}},
$$

where $\prod_{i \in I}^{*} G_i$ is the free product of the family $\{G_i\}_{i \in I}$, and 

$$
[G_i]_{i \in I} = ([G_i, G_j]|i, j \in I, i \neq j)\prod_{i \in I} G_i
$$

is the cartesian subgroup of the free product $\prod_{i \in I}^{*} G_i$ which is the kernel of the natural homomorphism from $\prod_{i \in I}^{*} G_i$ to the direct product $\prod_{i \in I}^{\times} G_i$. For further properties of the above notation see Neumann [17]. If $\{G_i\}_{i \in I}$ is a family of cyclic groups, then $\gamma_{n+1}(\prod_{i \in I} G_i) \subseteq [G_i]^{*}$ and hence $\prod_{i \in I}^{n} G_i = \prod_{i \in I} G_i/\gamma_{n+1}(\prod_{i \in I} G_i)$.

**Definition 2.2.** A variety $V$ is said to be a Schur-Baer variety if for any group $G$ for which the marginal factor group $G/V^{*}(G)$ is finite, then the verbal subgroup $V(G)$ is also finite and $|V(G)|$ divides a power of $|G/V^{*}(G)|$.

Schur proved that the variety of abelian groups, $A$, is a Schur-Baer variety (see [10]). Also, Baer [1] proved that if $u$ and $v$ have Schur-Baer property, then the variety defined by the word $[u, v]$ has the above property.

The following theorem gives a very important property of Schur-Baer varieties.

**Theorem 2.3.** (Leedham-Green, McKay [11]). The following conditions on a variety $V$ are equivalent:
(i) $\mathcal{V}$ is a Schur-Baer variety.
(ii) For every finite group $G$, its Baer invariant, $\mathcal{V}M(G)$, is of order dividing a power of $|G|$.

In the rest of this section we review some theorems required in the proofs of the main results of the article.

**Theorem 2.4.** (Jones [9]). Let $G$ be a finite $d$-generator group of order $p^n$. Then

$$p^{\frac{1}{2}d(d-1)} \leq |G'| |\mathcal{V}M(G)| \leq p^{\frac{1}{2}n(n-1)}.$$

**Theorem 2.5.** (Berkovich [2]). Let $G$ be a finite group of order $p^n$. Then $|\mathcal{V}M(G)| = p^{\frac{1}{2}n(n-1)}$ if and only if $G$ is an elementary abelian $p$-group.

**Theorem 2.6.** (Moghaddam [15,16]). Let $\mathcal{V}$ be the variety of polynilpotent groups of a given class row. Let $G$ be a finite $d$-generator group of order $p^n$. Then

$$|\mathcal{V}M(\mathbb{Z}_{p}^{(d)})| \leq |\mathcal{V}M(\mathcal{V}M(G))| \leq |\mathcal{V}M(\mathbb{Z}_{p}^{(n)})|.$$

We recall that the number of basic commutators of weight $c$ on $n$ generators, denoted by $\chi_c(n)$, is determined by Witt formula (see [5]).

**Theorem 2.7.** (Moghaddam and Mashayekhy [14]). Let $\mathcal{V}$ be the variety of groups, that is, the above sequence is exact and $N$ is contained in the marginal subgroup of $\mathcal{V}^*(G)$.

The following result is an interesting consequence of Theorems 2.6 and 2.7.

**Corollary 2.8.** Let $G$ be a finite $d$-generator $p$-group of order $p^n$, then

$$p^{\chi_{c+1}(d)} \leq |\mathcal{N}_c M(G)| |\mathcal{V}^*(G)| \leq p^{\chi_{c+1}(n)}.$$

The following theorem is a generalization of Theorem 2.5.

**Theorem 2.9.** (Mashayekhy, Sanati [13]). Let $G$ be an abelian group of order $p^n$. Then $\mathcal{N}_c M(G) = p^{\chi_{c+1}(n)}$ if and only if $G$ is an elementary abelian $p$-group.

Let $1 \to N \to G \to Q \to 1$ be a $\mathcal{V}$-central extension, where $\mathcal{V}$ is any variety of groups, that is, the above sequence is exact and $N$ is contained in the marginal subgroup of $G$, $\mathcal{V}^*(G)$. Then the following five-term exact sequence exists (see Fröhlich [3]):

$$\mathcal{V}M(G) \xrightarrow{\theta} \mathcal{V}M(Q) \to N \to G/\mathcal{V}(G) \to Q/\mathcal{V}(Q) \to 1.$$
Theorem 2.10. (Mashayekhy, Sanati [13]). Let $G$ be a finite $p$-group of order $p^n$. If $N_c M(G) = p^{\alpha_i(n)}$, then

(i) There is an epimorphism $N_c M(G) \rightarrow N_c M(G/G')$ which is obtained from the Fröhlich sequence.

(ii) If $\ker(\theta) = 1$, then $G$ is an elementary abelian $p$-group.

Theorem 2.11. (Mashayekhy, Parvizi [12]). Let

$$G \cong \mathbb{Z}^m \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$$

be a finitely generated abelian group, where $n_{i+1}$ divides $n_i$ for all $1 \leq i \leq k - 1$. Then

$$N_{c_1,c_2,\ldots,c_s} M(G) \cong \mathbb{Z}^{(\beta_m)} \oplus \mathbb{Z}^{(\beta_{m+1} - \beta_m)} \oplus \ldots \oplus \mathbb{Z}^{(\beta_{m+k} - \beta_{m+k-1})},$$

where $\beta_i = \chi_{c_i+1}(\chi_{c_{i-1}+1}(\ldots(\chi_{c_1+1}(i))\ldots))$ for all $m \leq i \leq m + k$.

Theorems 2.6 and 2.11 imply the following useful inequalities.

Corollary 2.12. With the notation of previous theorem let $G$ be a finite $d$-generator $p$-group of order $p^n$. Then

$$p^{\beta_d} \leq |N_{c_1,c_2,\ldots,c_s} M(G)||\gamma_{c_1+1}(\gamma_{c_{s-1}+1}(\ldots(\gamma_{c_1+1}(G))\ldots))| \leq p^{\beta_n}.$$

Theorem 2.13. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong \mathbb{Z}^n \oplus \ldots \oplus \mathbb{Z}^n$ be the $n$th nilpotent product of some cyclic groups, where $r_i$ divides $r_i$ for all $1 \leq i \leq t - 1$. If $c \geq n$ and $(p, r_i) = 1$ for all primes $p$ less than or equal to $n$, then the $c$-nilpotent multiplier of $G$ is isomorphic to

$$\mathbb{Z}^{(\sum_{i=1}^n \chi_{c+i}(m))} \oplus \mathbb{Z}^{(\sum_{i=1}^n (\chi_{c+i}(m+1) - \chi_{c+i}(m)))} \oplus \ldots \oplus \mathbb{Z}^{(\sum_{i=1}^n (\chi_{c+i}(m+t) - \chi_{c+i}(m+t-1)))}.$$

Theorem 2.14. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong \mathbb{Z}^n \oplus \ldots \oplus \mathbb{Z}^n$ be the $n$th nilpotent product of some cyclic groups, where $r_i$ for all $1 \leq i \leq t - 1$. If $(p, r_i) = 1$ for all primes $p$ less than or equal to $n$, then the structure of the polynilpotent multiplier of $G$ is

$$N_{c_1,c_2,\ldots,c_s} M(G) = \mathbb{Z}^{(d_m)} \oplus \mathbb{Z}^{(d_{m+1} - d_m)} \oplus \ldots \oplus \mathbb{Z}^{(d_{m+t} - d_{m+t-1})},$$

where $d_i = \chi_{c_i+1}(\ldots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(i))\ldots))$, for all $c_1 \geq n$ and $c_2, \ldots, c_s \geq 1$ and $m \leq i \leq m + t$.

Theorem 2.15. (Hokmabadi, Mashayekhy [7]). Let $G \cong \mathbb{Z}^n \oplus \ldots \oplus \mathbb{Z}^n \oplus \mathbb{Z}^{r_1} \oplus \ldots \oplus \mathbb{Z}^{r_t}$ be the $n$th nilpotent product of some cyclic groups such that $r_{i+1}$ divides $r_i$ for all $1 \leq i \leq t - 1$. If $(p, r_i) = 1$ for any prime $p$ less than or equal to $n + c$, then

(i) if $n \geq c$, then $N_c M(G) = \mathbb{Z}^{(g_n)} \oplus \mathbb{Z}^{(g_{n-1})} \oplus \ldots \oplus \mathbb{Z}^{(g_{n-c})}$;

(ii) if $c \geq n$, then $N_c M(G) = \mathbb{Z}^{(f_0)} \oplus \mathbb{Z}^{(f_1 - f_0)} \oplus \ldots \oplus \mathbb{Z}^{(f_{c-1} - f_{c-2})}$,

where $f_k = \sum_{i=1}^n \chi_{c+i}(m + k)$ and $g_k = \sum_{i=1}^c \chi_{n+i}(m + k)$ for $0 \leq k \leq t$. 
Theorem 2.16. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let \( G \cong A_1^{n_1} \ast A_2^{n_2} \ast \cdots \ast A_k^{n_k} \) such that \( A_i \cong \mathbb{Z} \) for \( 1 \leq i \leq t \) and \( A_j \cong \mathbb{Z}_{m_j} \) for \( t + 1 \leq j \leq k + 1 \). Let \( c_1 \geq n_1 \geq n_2 \geq \cdots \geq n_k \) and \( m_{k+1}|m_k|\cdots|m_{t+1} \) and \( (p, m_{t+1}) = 1 \) for all primes \( p \leq n_1 \). Then the structure of the polynilpotent multiplier of \( G \) is

\[
N_{c_1,c_2,\ldots,c_s} M(G) = \mathbb{Z}^{(e_0)} \oplus \mathbb{Z}^{(e_1-e_0)}_{\lambda_1+1} \oplus \cdots \oplus \mathbb{Z}^{(e_k-e_{k-1})}_{\lambda_k+1},
\]

where \( e_i = \chi_{c_i+1}((\cdots(\chi_{c_2+1}(u + \sum_{j=1}^i h_j))\cdots) \), for all \( i \leq k \), \( e_0 = \chi_{c_1+1}((\chi_{c_2+1}(u))\cdots) \), \( u = \sum_{j=1}^{n_i} \chi_{c_1+j}(t) + \sum_{j=n_i+1}^{n_j} \chi_{c_1+j}(i+1) \) and \( h_j = \sum_{\lambda=1}^{n_j} (\chi_{c_1+\lambda}(j+1) - \chi_{c_1+\lambda}(1)) \).

3 Main results

As we mentioned before, there is a mistake in the proof of Theorem 2.9. More precisely, in the proof of Theorem 3.5 in [13] it is assumed that \( G \) is a finite abelian \( d \)-generator \( p \)-group of order \( p^n \) and \( |N_c M(G)| = p^{\chi_{c+1}(n)} \). Then using the inequality \( i\chi_{c+1}(i) < \chi_{c+1}(i+1) \) it is proved that \( n = d \) and therefore \( G \) is an elementary abelian \( p \)-group. Unfortunately, the inequality \( i\chi_{c+1}(i) < \chi_{c+1}(i+1) \) is not correct and so the proof is not valid.

In this section, first, we intend to present a new proof for Theorem 2.9 in order to remedy the above mentioned mistake. Second, using this new method, we extend the result to polynilpotent multipliers of nilpotent products of cyclic \( p \)-groups with some conditions.

Proof of Theorem 2.9.

Proof. Let \( G \) be an elementary abelian \( p \)-group of order \( p^n \). Then by Theorem 2.7 we have \( N_c M(G) = \mathbb{Z}_p^{(b_2)} \oplus \mathbb{Z}_p^{(b_3-b_2)} \oplus \cdots \oplus \mathbb{Z}_p^{(b_n-b_{n-1})} \), where \( b_i = \chi_{c+1}(i) \), and hence \( |N_c M(G)| = p^{\chi_{c+1}(n)} \).

Conversely, suppose that \( |N_c M(G)| = p^{\chi_{c+1}(n)} \). Since \( G \) is an abelian \( p \)-group of order \( p^n \), we can consider \( G \) as follows:

\[
G \cong \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_d}},
\]

where \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \) and \( \alpha_1 + \alpha_2 + \cdots + \alpha_d = n \). By Theorem 2.7 \( N_c M(G) = \mathbb{Z}_{p^{\beta_2}}^{(b_2)} \oplus \mathbb{Z}_{p^{\beta_3}}^{(b_3-b_2)} \oplus \cdots \oplus \mathbb{Z}_{p^{\beta_d}}^{(b_n-b_{n-1})} \) and so \( |N_c M(G)| = p^{\beta_2 + \beta_3 + \cdots + \beta_d} \). On the other hand by hypothesis \( |N_c M(G)| = p^{b_n} \). Therefore \( b_n = \alpha_2 b_2 + \alpha_3 (b_3-b_2) + \cdots + \alpha_d (b_d-b_{d-1}) \).

Also \( b_n = (b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \cdots + (b_3 - b_2) + b_2 \). Thus

\[
(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \cdots + (b_3 - b_2) + b_2 = \alpha_2 b_2 + \alpha_3 (b_3-b_2) + \cdots + \alpha_d (b_d-b_{d-1}) = \frac{b_2 + \cdots + b_2 + (b_3 - b_2) + \cdots + (b_3 - b_2) + \cdots + (b_d - b_{d-1})}{\alpha_2 \text{-copies}} + \frac{b_2 + \cdots + b_2 + (b_3 - b_2) + \cdots + (b_3 - b_2) + \cdots + (b_d - b_{d-1})}{\alpha_3 \text{-copies}} + \cdots + \frac{b_2 + \cdots + b_2 + (b_3 - b_2) + \cdots + (b_3 - b_2) + \cdots + (b_d - b_{d-1})}{\alpha_d \text{-copies}}.
\]
So we have the following equality:

\[(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \ldots + (b_{d+1} - b_d) =\]

\[
\underbrace{b_2 + \ldots + b_3}_{\text{\(\alpha_2-1\) copies}} + \underbrace{(b_3 - b_2) + \ldots + (b_3 - b_2)}_{\text{\(\alpha_3-1\) copies}} + \underbrace{(b_4 - b_3) + \ldots + (b_4 - b_3)}_{\text{\(\alpha_4-1\) copies}} (I).\]

One can easily see that for any \(i \geq 1\), \((b_i - b_{i-1})\) is the number of basic commutators of weight \(c + 1\) on \(i\) letters such that \(x_i\) does appear in it. So \((b_j - b_{j-1}) \geq (b_i - b_{i-1})\) whenever \(j \geq i\). Now, assume \(\alpha_1 \geq 2\). Then \(n - 1 > n - \alpha_1\) and so the left-hand side of the above equality has more terms than the right-hand side. Also each term of the left-hand side of the above equality is greater than any term of the right-hand side. These facts imply that the equality \((I)\) does not hold which is a contradiction. Thus we must have \(\alpha_1 = 1\) and hence \(d = n\), \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1\). Therefore the result holds. \(\square\)

The next theorem is a generalization of Theorem 2.9. Note that the nilpotent product of finitely many finite p-groups is also a finite p-group.

**Theorem 3.1.** Let \(G \cong \mathbb{Z}_{p_1}^n \ast \mathbb{Z}_{p_2}^n \ast \ldots \ast \mathbb{Z}_{p_t}^n\) be the \(n\)th nilpotent product of some cyclic groups, where \(\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_t\) and \((q, p) = 1\) for all primes \(q\) less than or equal to \(n\). Let \(N_{c_1, c_2, \ldots, c_s}\) be a variety of polynilpotent groups such that \(c_1 \geq n\). Then

\[|N_{c_1, c_2, \ldots, c_s}M(G)| = p^{dm}\]

if and only if \(G \cong \mathbb{Z}_p^m \ast \mathbb{Z}_p^m \ast \ldots \ast \mathbb{Z}_p^m\), where \(m = \sum_{i=1}^t \alpha_i\) and \(d_m = \chi_{c_s+1}(\ldots(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m)))\ldots)\).

**Proof.** Let \(G = \mathbb{Z}_p^m \ast \mathbb{Z}_p^m \ast \ldots \ast \mathbb{Z}_p^m\) and \((q, p) = 1\) for all primes \(q\) less than or equal to \(n\). Then by Theorem 2.14,

\[N_{c_1, c_2, \ldots, c_s}M(G) = \mathbb{Z}_p^{(d_2)} \oplus \ldots \oplus \mathbb{Z}_p^{(d_m - d_{m-1})},\]

where \(d_i = \chi_{c_i+1}(\ldots(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(i)))\ldots)\), for all \(c_1 \geq n\). Hence

\[|N_{c_1, c_2, \ldots, c_s}M(G)| = p^{dm}.\]

Conversely, suppose that \(|N_{c_1, c_2, \ldots, c_s}M(G)| = p^{dm}\). By the hypothesis \(G = \mathbb{Z}_{p_1}^n \ast \mathbb{Z}_{p_2}^n \ast \ldots \ast \mathbb{Z}_{p_t}^n\) where \(\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_t\) and \(\alpha_1 + \alpha_2 + \ldots + \alpha_t = m\). Now Theorem 2.14 implies that

\[N_{c_1, c_2, \ldots, c_s}M(G) = \mathbb{Z}_{p_2}^{(d_2)} \oplus \mathbb{Z}_{p_3}^{(d_3 - d_2)} \oplus \ldots \oplus \mathbb{Z}_{p_t}^{(d_t - d_{t-1})},\]

where \(d_i = \chi_{c_i+1}(\ldots(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(i)))\ldots)\). Thus

\[|N_{c_1, c_2, \ldots, c_s}M(G)| = p^{\alpha_2 d_2 + \alpha_3 (d_3 - d_2) + \ldots + \alpha_t (d_t - d_{t-1})}.\]
On the other hand by hypothesis \(|N_{c_1, c_2, \ldots, c_s} M(G)| = p^e_m\). Therefore
\[d_m = \alpha_2 d_2 + \alpha_3 (d_3 - d_2) + \ldots + \alpha_t (d_t - d_{t-1})\]
Now applying a similar method to the proof of Theorem 2.9, it is enough to show that if \(j \geq i\), then \((d_j - d_{j-1}) \geq (d_i - d_{i-1})\).
In order to prove this fact consider the following sets:
\[A_1 = \{ \alpha | \alpha \text{ is a basic commutator of weight } c_1 + 1, \ldots, c_1 + n \text{ on } x_1, \ldots, x_i \}\]
and inductively for all \(2 \leq k \leq s\)
\[A_k = \{ \alpha | \alpha \text{ is a basic commutator of weight } c_k + 1 \text{ on } A_{k-1} \}\]
Clearly \(d_i = |A_s|\). It is easy to see that
\[d_i - d_{i-1} = |\{ \alpha | \alpha \text{ is a basic commutator of weight } c_s + 1 \text{ on } A_{s-1} \text{ such that } x_i \text{ does appear in } \alpha \}|\]
Hence the required inequality holds.

Using Theorem 2.16 and a similar proof to the above and noting that \(j \geq i\) implies \(e_j - e_{j-1} \geq e_i - e_{i-1}\), we can state the following theorem.

**Theorem 3.2.** Let \(G \cong Z_{p^{n_1}} \star Z_{p^{n_2}} \star \ldots \star Z_{p^{n_t}}\) be the \(n\)th nilpotent product of some cyclic groups, where \(\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_t\) and \((q, p) = 1\) for all primes \(q\) less than or equal to \(n\). Let \(N_{c_1, c_2, \ldots, c_s} M(G)\) be a variety of polynilpotent groups such that \(c_1 \geq n_t\). Then
\[|N_{c_1, c_2, \ldots, c_s} M(G)| = p^e_m\]
\[e_m = \chi_{c_s+1}(\ldots(\chi_{c_2+1}(\sum_{j=0}^{m-1} h_j)\ldots)), \text{ and } h_j = \sum_{\lambda=1}^{n_j} (\chi_{c_1+\lambda}(j+1) - \chi_{c_1+\lambda}(j))\]

The following result is a consequence of Theorem 2.15 and the above mentioned method with different condition \(n \geq c\).

**Theorem 3.3.** Let \(G \cong Z_{p^{n_1}} \star Z_{p^{n_2}} \star \ldots \star Z_{p^{n_t}}\) be the \(n\)th nilpotent product of some cyclic groups, where \(\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_t\), \(n \geq c\) and \((q, p) = 1\) for all primes \(q\) less than or equal to \(n + c\). Then \(N_{c_1, c_2, \ldots, c_s} M(G) = p^{g_m}\) if and only if \(G \cong Z_{p^{n_1}} \star Z_{p^{n_2}} \star \ldots \star Z_{p^{n_t}}\) where
\[m = \sum_{i=1}^{t} \alpha_i \text{ and } g_m = \sum_{i=1}^{c} \chi_{n+i}(m)\]

With the assumption and notation of Theorem 3.1, let \(n = 1\). Then the \(n\)th nilpotent product of \(Z_{p^{n_1}}\) \((1 \leq i \leq t)\) is the direct product of \(Z_{p^{n_i}}\). So \(G\) is a finite abelian \(p\)-group of order \(p^m\). Also \(d_i\) will be equal to \(\beta_i\) in Theorem 2.12. Therefore the following corollary is a consequence of Theorem 3.1.

**Corollary 3.4.** Let \(G\) be an abelian group of order \(p^m\). Then \(|N_{c_1, c_2, \ldots, c_s} M(G)| = p^{\beta_m}\) if and only if \(G\) is an elementary abelian \(p\)-group, where
\[\beta_m = \chi_{c_s+1}(\ldots(\chi_{c_2+1}(\chi_{c_1+1}(m))\ldots))\]
Note that according to Corollary 2.12 the polynilpotent multiplier of $G$ in the above result has its maximum order. So the above corollary is a vast generalization of Theorem 2.5.

Finally in order to deal with a non-abelian case we present the following theorem. This theorem is a generalization of Theorem 2.10.

**Theorem 3.5.** With the previous notation let $G$ be a finite $p$-group of order $p^m$. If $N_{c_1, c_2, \ldots, c_s}M(G) = p^{\beta m}$, then the following statements holds.

(i) There is an epimorphism $N_{c_1, c_2, \ldots, c_s}M(G) \xrightarrow{\theta} N_{c_1, c_2, \ldots, c_s}M(G/G')$ which is obtained from the Fröhlich sequence.

(ii) If $\ker(\theta) = 1$, then $G$ is an elementary abelian $p$-group.

**Proof.** (i) Let $V$ be the variety of polynilpotent groups of class row $(c_1, c_2, \ldots, c_s)$, $N_{c_1, c_2, \ldots, c_s}$. By Corollary 2.12 we have $|V(G)||V(G)| \leq p^{\beta m}$. Also by the hypothesis $|N_{c_1, c_2, \ldots, c_s}M(G)| = p^{\beta m}$. Therefore $|V(G)| = 1$. Now set $N = G'$ and consider the exact sequence $1 \rightarrow G' \rightarrow G \rightarrow G/G' \rightarrow 1$. Since $|V(G)| = 1$, the above sequence is a $V$-central extension. Therefore by Fröhlich five-term exact sequence we have the following exact sequence:

$$VM(G) \xrightarrow{\theta} VM(G/G') \xrightarrow{\beta} G' \xrightarrow{\alpha} G \rightarrow G/G' \rightarrow 1.$$ 

Clearly $\alpha$ is a monomorphism and so $\text{Im}(\beta) = 1$. This means that $\theta$ is an epimorphism.

(ii) Let $\ker(\theta) = 1$. Then $|V(G/G')| = |V(G)| = p^{\beta m}$ (*)&. Since $|G| = p^m$ then $|G/G'| \leq p^m$. Hence $|G/G'| = p^m$, otherwise, if $|G/G'| = p^k$, where $k < m$, then $|V(G/G')| \leq |V(M(G/G'))|V(G/G')| \leq p^{\beta k} < p^{\beta m}$, which is a contradiction to (*). Hence $|G/G'| = p^m$ which implies that $G$ is an abelian $p$-group. Now by Corollary 3.4 the result holds.

**Acknowledgments**

The authors would like to thank the referee for his/her careful reading.

**References**

[1] R. Baer. Endlichkeitskriterien für kommutatorgruppen. Math. Ann., 1952, 124: 161-177.

[2] Ya. G. Berkovich. On the order of the commutator subgroup and Schur multiplier of a finite $p$-group. J. Algebra, 1991, 144: 269-272.

[3] A. Fröhlich. Baer invariants of algebras. Trans. Amer. Math. Soc., 1963, 109: 221-244.

[4] J. A. Green. On the number of automorphisms of a finite group. Proc. Roy. Soc. London (A), 1956, 237: 574-581.

[5] M. Hall. The Theory of Groups. Macmillian Company, New York, 1959.

[6] N. S. Hekster. Varieties of groups and isologism. J. Austral. Math. Soc. (A), 1989, 46: 22-60.

[7] A. Hokmabadi, B. Mashayekhy. On nilpotent multipliers of some verbal products of groups. J. Algebra, 2008, 320: 3269-3277.
[8] A. Hokmabadi, B. Mashayekhy, F. Mohammadzadeh. Polynilpotent multipliers of some nilpotent products of cyclic groups II. Inter. J. Math. Game Theory Algebra, to appear.

[9] M. R. Jones. Some inequalities for the multiplicator of a finite group. Proc. Amer. Math. Soc., 1973, 39: 450–456.

[10] G. Karpilovsky. The Schur Multiplier. London Math. Soc. Monographs, New Series No. 2, 1987.

[11] C. R. Leedham-Green, S. McKay. Baer invariant, isologism, varietal laws and homology. Acta Math., 1976, 137: 99–150.

[12] B. Mashayekhy, M. Parvizi. Polynilpotent multipliers of finitely generated abelian groups. Inter. J. Math. Game Theory Algebra, 2006, 16:1 93-102.

[13] B. Mashayekhy, M. A. Sanati. On the order of nilpotent multipliers of finite p-groups. Communications in Algebra, 2005, 33: 2079-2087.

[14] M. R. R. Moghaddam, B. Mashayekhy. Higher Schur multiplicator of a finite abelian group. Algebra Colloquium, 1997, 4:3 317-322.

[15] M. R. R. Moghaddam. Some inequalities for the Baer invariant of a finite group. Bull. Iranian Math. Soc., 1981, 9:1 5-10.

[16] M. R. R. Moghaddam. On the Schur-Baer property. J. Austral. Math. Soc. (A), 1981, 31: 343-361.

[17] H. Neumann. Varieties of Groups. Springer-Verlag, Berlin, 1967.