CENTRAL ELEMENTS OF THE ELLIPTIC YANG–BAXTER
ALGEBRA AT ROOTS OF UNITY

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Abstract. We give central elements of the Yang-Baxter algebra for the 
$R$-matrix of the eight-vertex model, in the case when the crossing parameter is a rational multiple 
of one of the periods.

1. As usual, let $R(u)$ be an $R$-matrix acting on $\mathbb{C}^n \otimes \mathbb{C}^n$ and satisfy the Yang–Baxter
equation

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2),$$

(1)

where $R_{12}(u) = R(u) \otimes I$, $R_{23} = I \otimes R(u)$, etc.. Here and after, we indicate by suffix
the tensor components on which operators act nontrivially. By a Yang-Baxter algebra
$A$, we mean the algebra generated by the entries of an $n \times n$ matrix $L(u)$, subject to
the defining relations

$$R_{12}(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u - v),$$

(2)

where $L_1(u) = L(u) \otimes I$ and $L_2(u) = I \otimes L(u)$.

The simplest and best-known solutions of (1) arise for $n = 2$ and correspond to six-
and eight-vertex models [1]. In the latter case, the elements of the $R$-matrix are written
in terms of the Jacobi elliptic theta functions with the half-periods $K$ and $K'$ as

$$R_{\varepsilon \varepsilon'}^\varepsilon(u) = \rho \Theta(\lambda \Theta(\lambda u)H(\lambda(u + 1)),$$

$$R_{\varepsilon \varepsilon'}^\varepsilon(u) = \rho \Theta(\lambda H(\lambda u)\Theta(\lambda(u + 1)),$$

$$R_{\varepsilon \varepsilon'}^{\varepsilon}(u) = \rho H(\lambda)H(\lambda u)H(\lambda(u + 1)),$$

$$R_{\varepsilon \varepsilon'}^{\varepsilon}(u) = \rho H(\lambda)\Theta(\lambda u)\Theta(\lambda(u + 1)),$$

(3)

where $\varepsilon, \varepsilon' = \pm (\varepsilon \neq \varepsilon')$ are the labels of basis vectors $e_{\pm}$ of $V = \mathbb{C}^2$. The crossing
parameter $\lambda$ plays an important role in the structure of $A$, its representations, and
the integrable models connected with $A$. As can be seen from (2), the parameter $\rho$ is
inessential.

In this paper, we focus attention to the center of the Yang–Baxter algebra for the
eight-vertex model when $p = \lambda/2K$ is a rational number. We refer to this situation as
the ‘root of unity’ case. We shall consider the case $p = 2m/N$ where $m, N$ are coprime
positive integers with $N \geq 3$ being odd.

In the trigonometric degeneration $K' = 0$, and for a general value of $\lambda$, it is well
known that the center of $A$ is generated by the quantum determinant. It is also well
known that at roots of unity the center is enlarged. In this case, Tarasov [2] gave an
explicit expression for the additional central elements. In terms of the entries $L_{\varepsilon\varepsilon'}(u)$ of $L(u)$ it reads

\begin{equation}
\langle L_{\varepsilon\varepsilon'} \rangle(u) = L_{\varepsilon\varepsilon'}(u + N - 1)L_{\varepsilon\varepsilon'}(u + N - 2) \cdots L_{\varepsilon\varepsilon'}(u).
\end{equation}

Formula (4) has the following elliptic counterpart. Fix $u_0 \in \mathbb{C}\setminus\mathbb{Z}$ where $\mathbb{Z} = \mathbb{Q}K + \mathbb{Q}K'$, and set for $a, b \in \mathbb{Z}$

\[ L_{\varepsilon\varepsilon';a,b}(u; u_0) = \phi_{a,a+\varepsilon}(u_0)L_{\varepsilon\varepsilon'}(u)\phi_{b,b+\varepsilon'}(u_0), \]

where $\phi_{a,b}(u)$, $\phi_{a,b}^*(u)$ are Baxter's intertwining vectors given in (20), (21) below. Then the elements

\begin{equation}
\Lambda_{\varepsilon\varepsilon'}(u; u_0) = \frac{1}{N} \sum_{a=0}^{N-1} L_{\varepsilon\varepsilon';a,b}(u + N - 1; u_0)\Lambda_{\varepsilon\varepsilon';a,b}(u + N - 2; u_0) \cdots \\
\times L_{\varepsilon\varepsilon';a+(N-1),b+(N-1)\varepsilon'}(u; u_0)
\end{equation}

do not depend on $b \in \mathbb{Z}$, and are central in $\mathcal{A}$. For different choices of the parameter $u_0$, the matrices $\Lambda(u; u_0) = (\Lambda_{\varepsilon\varepsilon'}(u; u_0))$ and $\Lambda(u; u'_0)$ are related by conjugation by a numerical matrix.

Formula (3) is a straightforward extension of Tarasov's formula (4), and we have no intention to claim any sort of originality. As we have not been able to find (3) in the literature, we give its derivation in this note.

2. From now on we specialize the crossing parameter as

\begin{equation}
\lambda = \frac{4m}{N}K,
\end{equation}

where $m, N$ are mutually prime positive integers and $N \geq 3$ is odd. We fix a basis $e_+, e_-$ of $V = \mathbb{C}^2$ and $e^+_+, e^-_+$ of the dual space $V^*$ with the coupling $e^+_+ e^-_+ = \delta_{\varepsilon\varepsilon'}$.

The method of constructing central elements is a modification of the one in [2]. The outline is as follows. Let $V_i (i = 0, \cdots, N)$ stand for copies of $V$. Define

\begin{equation}
L_{1\cdots N}(u) = L_1(u + N - 1) \cdots L_N(u), \\
R_{1\cdots N,0}(u) = R_{10}(u + N - 1) \cdots R_{N0}(u).
\end{equation}

Both $L_{1\cdots N}(u)$ and $R_{1\cdots N,0}(u)$ are operators on $V_1 \otimes \cdots \otimes V_N$, whose entries belong to $\mathcal{A}$ and $\text{End}(V_0)$ respectively. From the form of the $R$-matrix (3), together with (1) and (2), it follows for any value of $\lambda$ that $R_{1\cdots N,0}(u)$ and $L_{1\cdots N}(u)$ leave invariant a subspace $W_0 \subset V_1 \otimes \cdots \otimes V_N$ of codimension $N + 1$. Moreover, if $\lambda$ satisfies (3), then $W_0$ is contained in a larger invariant subspace $W$, such that the quotient space $\overline{W} = W/W_0$ is 2-dimensional.

Consider the restriction of $L_{1\cdots N}(u)$ and $R_{1\cdots N,0}(u)$ to $\overline{W}$ and denote the resulting operators by $\Lambda(u) \in \text{End}(\overline{W}) \otimes \mathcal{A}$ and $\overline{R}(u) \in \text{End}(\overline{W}) \otimes \text{End}(V_0)$. We have

\begin{equation}
\overline{R}(u - v)\Lambda(u)L_0(v) = L_0(v)\Lambda(u)\overline{R}(u - v).
\end{equation}

We then use the results in [3] to show that

\begin{equation}
\overline{R}(u) = f(u)I \otimes I,
\end{equation}

where $f(u)$ is a polynomial in $u$.
where $f(u)$ is a non-vanishing scalar factor. From (9), we therefore have
\begin{equation}
[\Lambda(u), L_0(v)] = 0,
\end{equation}
which shows that the entries of $\Lambda(u)$ are central elements of $A$. The space $\overline{W}$ has a basis $\{\Psi_\varepsilon(u_0)\}_{\varepsilon=\pm}$, which are obtained by $N$-fold fusion of the intertwining vectors. Writing $\Lambda(u)$ in this basis we arrive at (3).

3. In this and the next subsection, we recall some aspects of the standard fusion procedure.

From (1) and (2), we have
\begin{equation}
R_{i,i+1}(1)R_{1 \ldots N,0}(u) = R_{1 \ldots N,0}(u)R_{i,i+1}(1),
\end{equation}
with some operators $R_{i,i+1}^{i+1}(u)$ and $L^{i+1}(u)$ which differ from $R_{1 \ldots N,0}(u)$ and $L_{1 \ldots N}(u)$ by permutation of the $i$-th and $(i+1)$-th factors. Taking $u = 1$ in (3), we see that $R_{i,i+1}(1)$ has a kernel spanned by the vectors
\[ e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{i-1}} \otimes (e_+ \otimes e_- - e_- \otimes e_+) \otimes e_{\alpha_{i+1}} \otimes \cdots \otimes e_{\alpha_N}, \]
where $\alpha_k = \pm 1$. The subspace
\begin{equation}
W_0 = \sum_{i=1}^{N-1} \ker R_{i,i+1}(1) \subset V_1 \otimes \cdots \otimes V_N
\end{equation}
is invariant under the action of $R_{1 \ldots N,0}(u)$ and $L_{1 \ldots N}(u)$. The subspace of symmetric tensors in $V_1 \otimes \cdots \otimes V_N$ is isomorphic to $V_1 \otimes \cdots \otimes V_N/W_0$. We have $Pv \equiv v \mod W_0$ ($v \in V_1 \otimes \cdots \otimes V_N$) for any permutation $P$ of the tensor components.

Introduce the operator
\begin{equation}
S = S^{(1)}S^{(2)} \cdots S^{(N-1)}, \quad S^{(j)} = R_{j,j+1}(1) \cdots R_{23}(2)R_{12}(1).
\end{equation}
It follows from (1) and (3) that
\begin{equation}
SR_{1 \ldots N,0}(u) = R_{N \ldots 1,0}(u)S,
\end{equation}
\begin{equation}
SL_{1 \ldots N}(u) = L_{N \ldots 1}(u)S,
\end{equation}
where $R_{N \ldots 1,0}(u)$ and $L_{N \ldots 1}(u)$ differ from $R_{1 \ldots N,0}(u)$ and $L_{1 \ldots N}(u)$ by permutation of all factors to the opposite order. Therefore
\begin{equation}
W = \ker S
\end{equation}
is also invariant under $R_{1 \ldots N,0}(u)$ and $L_{1 \ldots N}(u)$. By using (3), for any given $i$ the factors of $S$ can be reordered in such a way that $R_{i,i+1}(1)$ comes to the rightmost position. Hence we have $W_0 \subset W$.

4. Let us show that $\overline{W} = W/W_0$ is 2-dimensional if the elliptic modulus is generic.

In the trigonometric limit, it is verified as follows. With an appropriate base change, the operator $S$ commutes with an action of $U_q(sl_2)$ where $q$ is a primitive $N$-th root of unity. One checks that $\text{Im } S$ is a proper nontrivial submodule of the specialization of the standard $N + 1$ dimensional representation. From the representation theory of $U_q(sl_2)$ at $q^N = 1$, we conclude that $\dim \text{Im } S = N - 1$, and hence $\dim \ker S/W_0 = 2$. 
From the above consideration, we have $\dim W/W_0 \leq 2$ in the elliptic case if the modulus is generic. We show the equality by constructing two linearly independent elements of $W$. For this purpose we use Baxter’s intertwining vectors $\phi_{a,b}(u) \in V$ and their duals $\phi_{a,b}^*(u) \in V^*$ given in [20],[21]. Their main properties are summarized in the appendix.

Set
\begin{align}
\Psi_{\varepsilon,a}(u) &= \phi_{a,a+\varepsilon}(u+N-1) \otimes \phi_{a+\varepsilon,a+2\varepsilon}(u+N-2) \otimes \cdots \otimes \phi_{a+(N-1)\varepsilon,a+N\varepsilon}(u), \\
\Psi_{\varepsilon,a}^*(u) &= \phi_{a,a+\varepsilon}^*(u+N-1) \otimes \phi_{a+\varepsilon,a+2\varepsilon}^*(u+N-2) \otimes \cdots \otimes \phi_{a+(N-1)\varepsilon,a+N\varepsilon}^*(u).
\end{align}

We will use the following properties:
\begin{enumerate}
  \item $S\Psi_{\varepsilon,a}(u) = 0$.
  \item $\Psi_{\varepsilon,a}^*(u)\Psi_{\varepsilon',b}(u) = \delta_{\varepsilon,\varepsilon'}$ \quad ($a,b \in \mathbb{Z}, \varepsilon, \varepsilon' = \pm$).
  \item $\Psi_{\varepsilon,a}(u) \equiv \Psi_{\varepsilon,b}(u) \mod W_0$ \quad ($a,b \in \mathbb{Z}, \varepsilon, \varepsilon' = \pm$).
  \item The vector
  \[ \Psi_{\varepsilon}^*(u) = \frac{1}{N} \sum_{a=0}^{N-1} \Psi_{\varepsilon,a}^*(u) \]
  is orthogonal to $W_0$.
\end{enumerate}

Assertion (i) is a consequence of the vertex-face correspondence (22). The orthogonality (ii) follows from (23)–(24). To see (iii), we use the expression (20), $\phi_{a,a+\varepsilon}(u) = \phi(a-\varepsilon u)$, to find
\[ \Psi_{\varepsilon,a+2\varepsilon}(u) = C\Psi_{\varepsilon,a}(u) \equiv \Psi_{\varepsilon,a}(u) \mod W_0, \]
where $C$ is a cyclic permutation of the tensor components. Since $N$ is odd, (iii) follows. One can verify (iv) in a similar manner.

From (ii) and (iv), it is clear that the two vectors in $W = W/W_0$
\[ \Psi_{\varepsilon}(u) = \Psi_{\varepsilon,a}(u) \mod W_0 \quad (\varepsilon = \pm) \]
are linearly independent for any $u \in \mathbb{C}\setminus\mathbb{Z}$.

5. Let $\overline{R}(u)$ and $\Lambda(u)$ be restrictions of $R_{1\ldots N,0}(u)$ and $L_{1\ldots N}(u)$ on $\overline{W}$. Fixing $u_0 \in \mathbb{C}\setminus\mathbb{Z}$, we have in the basis $\{\Psi_{\varepsilon}(u_0)\}_{\varepsilon = \pm}$
\[ L_{1\ldots N}(u)\Psi_{\varepsilon'}(u_0) = \sum_{\varepsilon = \pm} \Psi_{\varepsilon}(u_0)\Lambda_{\varepsilon\varepsilon'}(u;u_0) \mod W_0, \]
where
\[ \Lambda_{\varepsilon\varepsilon'}(u;u_0) = \Psi_{\varepsilon}^*(u_0)L_{1\ldots N}(u)\Psi_{\varepsilon'}(u_0). \]
The right hand side yields the formula (3).

To complete the proof of the centrality (10), it remains to verify (9). For this, it suffices to show the relation
\begin{align}
R_{1\ldots N,0}(u - v)\Psi_{\varepsilon,a}(u) \otimes \phi_{a,a+\varepsilon'}(v) &= f(u - v)\Psi_{\varepsilon,a}(u) \otimes \phi_{a,a+\varepsilon'}(v) \mod W_0.
\end{align}
In the present notation, the fused intertwining vectors \( \phi_{N,a,a+N\varepsilon}(u) \) of (2.3.8) in [3] is \( P_{1 \ldots N} \Psi_{\varepsilon,a} \), where \( P_{1 \ldots N} \) denotes the complete symmetrizer in \( V_1 \otimes \cdots \otimes V_N \). Noting \( R_{1 \ldots N,0}(u)P_{1 \ldots N} = R_{1 \ldots N,0}(u) \), we have from Theorem 2.3.3 in [3]

\[
R_{1 \ldots N,0}(u-v)\Psi_{\varepsilon,a}(u) \otimes \phi_{a,a+\varepsilon}(v)
= P_{1 \ldots N} R_{1 \ldots N,0}(u-v) \phi_{N,a,a+N\varepsilon}(u) \otimes \phi_{a,a+\varepsilon}(v)
= \tilde{f}(u-v) \sum_{\varepsilon''=\pm} W_{N1}(a+\varepsilon'',a,\varepsilon',a,u|u-v)
\times \phi_{N,a-(N-1)\varepsilon'+\varepsilon'',a+\varepsilon'}(u) \otimes \phi_{a,a+\varepsilon'}(v) \mod W_0,
\]
where \( \tilde{f}(u) \) is a non-vanishing scalar function (see the end of p.45, [3]), and \( W_{N1}(a, b, c, d|u) \) are the Boltzmann weights of the fusion SOS models. We have used its periodicity by \( N \) with respect to \( a, b, c, d \). From the explicit formula (2.1.20) in [3] we have

\[
W_{N1}(a+\varepsilon'',a,\varepsilon',a,a|u) = \delta_{\varepsilon'\varepsilon''} g(u),
\]
where \( g(u) \) is another non-zero scalar.

Returning to the notation \( \Psi_{\varepsilon,a}(u) \) and setting \( f(u) = \tilde{f}(u)g(u) \), we find

\[
R_{1 \ldots N,0}(u-v)\Psi_{\varepsilon,a}(u) \otimes \phi_{a,a+\varepsilon}(v) = f(u-v)\Psi_{\varepsilon',a}(v) \otimes \phi_{a,a+\varepsilon}(v) \mod W_0.
\]

In view of (19), we arrive at (19).

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**Appendix.** Here we collect known formulas concerning intertwining vectors.

Following the notation of [3], pp.46–47, we choose \( s^+ = s^- = \xi + K/\lambda \), where \( \lambda \) is specified as in [3], and \( \xi \) is a generic complex parameter. For \( a \in \mathbb{Z} \) and \( \varepsilon = \pm 1 \), Baxter’s intertwining vectors are given by

\[
\begin{align*}
\phi_{a,a+\varepsilon}(u) &= \phi(a - \varepsilon u), \\
\phi(u) &= H_1(\lambda(\xi + u))e_+ + \Theta_1(\lambda(\xi + u))e_-,
\end{align*}
\]
where \( H_1(u) = H(u+K) \), \( \Theta_1(u) = \Theta(u+K) \). The dual intertwining vectors are defined by \( \phi^*_{a+\varepsilon,a}(u)\phi^*_{a+\varepsilon',a}(u) = \delta_{\varepsilon\varepsilon'} \). Explicitly we have

\[
\begin{align*}
\phi^*_{a+\varepsilon,a}(u) &= \frac{\varepsilon}{\Delta_a(u)} \phi^*(a - \varepsilon u), \\
\phi^*(u) &= -\Theta_1(\lambda(\xi + u))e^+_+ + H_1(\lambda(\xi + u))e^*_+.
\end{align*}
\]

Here

\[
\Delta_a(u) = [1 + u \frac{\xi + a + \frac{2K}{\lambda}}{\xi^{\frac{2}{\lambda}}}], \\
[u] = H(\lambda u)\Theta(\lambda u),
\]
and \( \zeta \) is a non-zero complex number depending only on \( K'/K \). Notice that \( [u+N] = [u], \) \( [N] = 0. \) We have
\[
\phi(u + N) = \phi(u), \quad \phi^*(u + N) = \phi^*(u).
\]
We set \( \phi_{a,b}(u) = \phi_{a,b}^*(u) = 0 \) unless \( b = a \pm 1 \) mod \( N \). In other words, we consider a cyclic SOS model rather than a restricted SOS model.

We have the vertex-face correspondence
\[
R(u - v)\phi_{d,c}(u) \otimes \phi_{c,b}(v) = \sum_d W_{11}(a, b, c, d|u - v)\phi_{a,b}(u) \otimes \phi_{d,a}(v),
\]
where \( W_{11}(a, b, c, d|u) \) denotes the Boltzmann weights of the SOS model as given in (2.1.4a)–(2.1.4c), [3].

We have also the relations
\[
\phi_{a,\pm \epsilon, a}^*(u)\phi_{b, \pm \epsilon, b}(u) = \frac{\zeta + \frac{a+b}{2}}{\zeta + a} \frac{[u + 1 \mp \frac{a-b}{2}]}{[u + 1]},
\]
(23)
\[
\phi_{a,\pm \epsilon, a}^*(u)\phi_{b, \pm \epsilon, b}(u) = \frac{\frac{a-b}{2}}{\zeta + a} \frac{[u + 1 \pm (\xi + \frac{a+b}{2})]}{[u + 1]}.
\]
(24)

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