Entanglement growth in diffusive systems

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(Dated: January 22, 2020)

We study the influence of conservation laws on entanglement growth. Focusing on systems with \(U(1)\) symmetry, i.e., conservation of charge or magnetization, that exhibits diffusive dynamics, we theoretically predict the growth of entanglement, as quantified by the Rény entropy, in lattice systems in any spatial dimension \(d\) and for any local Hilbert space dimension \(q\) (qudits). We find that the growth depends both on \(d\) and \(q\), and is in generic case first linear in time, similarly as for generic systems without any conservation laws. Exception to this rule are chains of 2-level systems where the dependence is a square-root of time. All predictions are numerically verified by simulations of diffusive Clifford circuits with up to \(10^3\) qudits. Such efficiently simulable circuits should be a useful tool for other many-body problems.

**Introduction.**— Entanglement is one of crucial quantum resources responsible for the emerging 2nd quantum revolution – exploiting quantumness to perform tasks not possible by classical means, for instance, quantum computation, teleportation, or secure communication [1]. Even if not easily measurable [2], it is an extremely powerful theoretical concept. This was further underlined by another discovery from ’80, from a seemingly unrelated field, namely the quantum Hall effect [3]. It gradually brought to light the fact that there can be phases of matter that have topological order which goes beyond the Landau’s paradigm of classifying all phases of matter just by local order parameters. Today we understand that such topological order is connected to certain patterns of entanglement [4]. A modern view in fact uses entanglement to distinguishing different phases of matter [4][5]. Entanglement though plays a role also beyond the equilibrium phases. An example is for instance a putative non-thermal many-body-localized phase [6], one of the distinguishing features of which is slow logarithmic-in-time growth of entanglement [8].

Conservation laws and the associated symmetries are one of the most important properties of laws of physics. On the smallest scale, the elementary particles differ by their symmetries, and on the large scale, as well, the most violent objects we know – black holes – are believed to be defined only by their conserved quantities, charge, mass and angular momentum [9]. Furthermore, the symmetry to translations in time and its associated generator is the very object that governs dynamics. In short, symmetries are crucially responsible for the simplicity of nature at its core.

An important question is what role do conservation laws play on the dynamics of entanglement? Its growth with time is important also from a practical point of view. Namely, if it is small then efficient classical simulation of such systems is possible [10]. For generic local systems and initial states one expects that dynamics explores the whole available Hilbert space and therefore entanglement grows linearly with time. This holds true even for integrable systems, see e.g. [11][12]. Because symmetries are about constraints, and because entanglement is given essentially by the number of degrees of freedom (two-level systems) involved, one might argue that symmetries will certainly affect entanglement growth. However, in the thermodynamic limit (TDL) conservation of a single e.g. charge should not matter much in a large Hilbert space. Therefore it was surprising and interesting when it was presented [13] (focusing on diffusive 1D systems with conserved charge) that the entanglement, as quantified by the Rény entropy

\[
S_r(t) := \frac{\log_2(\text{tr}_A \rho_A^r)}{1 - r}, \tag{1}
\]

grows in fact as \(S_2 \sim \sqrt{t}\) instead of the “expected” \(S_2 \sim t\), starting from a generic separable initial state, see also [14][15]. This finding, if holding for generic systems, would have many consequences. For instance, one could argue that simple charge conservation causes the “Rény complexity” \(\sim 2^{S_2}\) to grow only as \(\sim b\sqrt{t}\), i.e. slower than exponentially (though still super-polynomially). A system with *diffusive conserved charge* would seem to be a less powerful quantum information resource than one without it.

We address the question of the Rény entropy growth in local systems in any spatial dimension \(d\) and for any local Hilbert space dimension \(q\). Theoretical predictions, summarized in Table [I], are numerically verified on

| \(d\) | \(q\) | \(S_2(t < t_1)\) | \(t_1\) | \(S_2(t_1 < t < t_\infty)\) | \(t_\infty\) |
|---|---|---|---|---|---|
| 1D | 2 | \(cL\) | \(t\) | \(O(L^2)\) | \(O(L^2)\) |
| 2D | 2 | \(\frac{1}{\sqrt{2}}t\) | \(O(L)\) | \(\sim \sqrt{L}\) | \(O(L^2)\) |
| 3D | 2 | \(\frac{1}{3L^2}t\) | \(O(L^2)\) | \(\sim \sqrt{L}\) | \(O(L^4)\) |
| \(d\) \(\geq 3\) | \(\sim t\) | \(t_1 = t_\infty\) | \(O(L^d)\) |
large systems, with the total number of qubits up to e.g. 252 × 252 ≈ 6 · 10^4 in 2D, and 48 × 48 × 48 ≈ 10^5 in 3D. Perhaps the most interesting finding is that the entanglement growth is generically linear in time. While for diffusive qubit (spin-1/2) systems the asymptotic growth is still ∼ √t, it starts in d > 1 only at a late time when the entropy becomes extensively large, S_2 ∼ L^{d-1}. In other words, in the TDL the S_2 grows linearly with time at any finite value of entropy. As a side result, the presented new class of efficiently simulable systems with nontrivial dynamics could be useful in addressing other many-body questions.

**Theoretical prediction.** A class of systems that we study are lattice systems with local nearest-neighbor (n.-n.) interactions in d spatial dimensions and with q-dimensional local Hilbert space, whose dynamics has a nontrivial conservation of the total particle number or the total spin in z-direction (i.e., a U(1) symmetry). The dynamics of that conserved degree of freedom is assumed to be diffusive, while the rest of dynamics is generic (we exclude integrable systems). Specifically, the influence of possible non-U(1) symmetries is left for future. Linear dimension is denoted by L with the total number of qubits n := L^d.

We shall discuss the entropy growth as quantified by the Rényi entropy S_r, (integer index r > 1) starting from a pure product initial state. We prefer S_r over von Neumann entropy S_{r=1} due to its analytical simplicity. We remark that in generic systems all S_r, including S_1, are expected to behave in the same way, whereas for diffusive systems the S_1 (which we don’t discuss) can perhaps behave differently [13]. We will mostly focus on S_2 as a representative case of S_{r>1}, and will use a bipartition to regions A and B. We remark that sometimes S_2, rather e.g. S_1, is the more relevant quantity [16], and is furthermore also easier to measure [17]. The reduced density operator is ρ_A(t) := tr_B[ψ(t)]⟨ψ(t)⟩. The size of region A will be extensive, |A| ∼ L^d, as otherwise in the TDL one can get strong effects simply due to typicality – tracing a random state over |B| ≫ |A| results in a spectrum of ρ_A whose relative deviation from a flat one is ~ q^{-(|B|−|A|)}/2 [18] and therefore negligible. To ease comparison of different d and L we will measure t in such units that one will generate a unit of entanglement in a unit of time, S_2(1) ∼ O(1), i.e., in the language of quantum circuits ∼ O(1) gates connecting regions A and B are applied per unit of time (compared to a Hamiltonian evolution this means a rescaling of time by L^{d−1}).

Let us first argue why and how conservation of magnetization (charge) matters for the long-time behavior of S_2. As we will see, in the TDL S_2 is self-averaging and we will for simplicity focus on the purity I(t) := 2^{-S_2(t)} = tr(ρ^2_A(t)). A non-rigorous intuitive meaning of the entropy is that it measures the effective number of the explored degrees of freedom needed to describe ρ_A(t). For purity one can write I ∼ 1/N_{eff}, where N_{eff} ∼ 2^{L_{eff}} is the effective Hilbert space size on which ρ_A is supported, while S_2 ∼ log_2 N_{eff} ∼ L_{eff}. Averaging I over all computational initial states one gets

$$\bar{I}(t) = \frac{1}{q^n} \sum_e \text{tr}[D^{(E)}_A(t)^2],$$

where $D_{k}^{(E)} := |c_k⟩⟨c_k|$ is a basis of diagonal matrices with $c_k \in \mathbb{Z}^d$ labeling the local computational state, and $D^{(E)}_A(t) = \text{tr}_B[U^{(1)} D^{(c_1)}_A(1) \otimes \cdots \otimes D^{(c_n)}_A(n) U^{-1}]$. We see that at large times what matters is the spreading of diagonal operators $D^{(E)}_A(t)$. Provided the dynamics of all diagonal operators is diffusive, one expects $L_{eff} \sim \sqrt{t}$, resulting in $S_2 \sim \sqrt{t}$. For qubits, $q = 2$, there are just two local diagonal operators, 1 and σ₂, and therefore if σ₂ is diffusive one expects a long-time asymptotic growth $S_2 \sim \sqrt{t}$. However, for higher dimensional qudits, $q \geq 3$, the diagonal basis is spanned by $q$ linearly independent diagonal operators, only one of which is the conserved operator (magnetization). While one might think that diffusive modes that contribute to purity decay as $e^{-\sqrt{t}}$ will due to their slow decay still dominate over non-diffusive ones, which decay as $e^{-t}$, this will not be the case in general. In a system of $n$ qubits the number of diagonal operators that are products of only diffusive magnetization $σ_k$ and the identity $1_k$ is $2^n$, while the number of all other non-diagonal ones is $(q^n−2^n)$. Simple counting then suggests that $I \sim 2^n e^{-\sqrt{t}} + (q^n−2^n)e^{-t}$, so that in the TDL only the exponential term survives. For generic $q \geq 3$ with only one conserved charge one expects the asymptotic linear growth $S_2 \sim t$.

How about the short-time behavior of $S_2(t)$? We shall argue that it is, instead, always linear in time, even for qubits. Let us limit our discussion to qubits, $q = 2$, as for $q \geq 3$ one anyway has linear growth $S_2 \sim t$ even at long times. For short-time behavior it is crucial to account for correlations spreading in a direction transversal to the boundary of dimension $d−1$ and area $A^{d−1}$ between regions A and B (in 2D $A^{d−1}$ is the circumference l, in 3D a true two-dimensional area A). In 1D the number of boundaries c, Table I). Starting from a product initial state the dynamics tries to generate entanglement across the boundary. For local (n.-n.) interaction the natural first candidate sites to be entangled are all ~ $L^{d−1}$ n.n. pairs lying on the boundary between A and B. Only after all those qubits are entangled can a slowing down due to diffusion in a transversal direction kick in. Let us be more specific, with a view on numerical demonstration. In our random quantum circuits we will apply $L$ gates between random n.-n. qubits per unit of time. Such scaling is in line with the mentioned units of time − probability that such random n.-n. gate connects A and B is ~ $L^{-d−1} = \frac{1}{L}$, and therefore applying $L$ of them means we will have ~ $1$ gates connecting regions A and B, and therefore, at least initially, generate one bit of entanglement in a unit of time. More precisely, the probability that a random gate connects A and B is $A^{d−1}_{L_{eff}}$, where the denominator $dL_{eff}$ is the number of all nearest-neighbor bonds on a d dimensional square lattice. The initial growth of entanglement
is therefore expected to be

\[ S_2 \approx \frac{A^{(d-1)}}{dL^{d-1}t}. \] (3)

We expect this linear growth to hold for any \( S_r \), including \( r = 1 \). Such linear growth will continue until the time \( t_1 \sim L^{d-1} \) at which \( S_2(t_1) \sim A^{(d-1)} \). After that one will crossover into the asymptotic diffusive growth \( S_2(t_\infty) \sim \sqrt{t} \), until at \( t_\infty \sim L^{d+1} \) a finite-size saturation value \( S_2(t_\infty) \sim L^d \) is reached.

We see that in higher spatial dimensions the region of diffusive growth is parametrically small, it lasts from \( t_1 \sim L^{d-1} \) till \( t_\infty \sim L^{d+1} \). Furthermore, in the TDL it is pushed to infinitely large values of entropy \( L^{d-1} \lesssim S_2 \lesssim L^d \) and will be hard to observe. Qubits in \( d = 1 \) are rather special because the linear growth ends at \( S_2 \sim L^0 = 1 \) (i.e., at short time \( t_1 \sim 1 \)) and one gets \( S_2 \sim \sqrt{t} \) in the whole range of \( S_2 \) (and \( t \)). In short, in \( d = 1 \) the asymptotic \( \sim \sqrt{t} \) growth is “easy” to observe, while in \( d > 1 \) it is hard because it appears in the TDL only at infinitely large values of \( S_2 \). Therefore the generic behavior after a quench from a product state is in \( d > 1 \) the linear growth (which is as fast as allowed by the Lieb-Robinson bound \[19\]). Table 1 summarizes these findings.

**Clifford circuits.**– It is always useful to take the simplest model, analytically or numerically, that displays the physics one wants to explore. A setting for which one can get exact results for the entanglement dynamics are so-called random quantum circuits \[21\] composed of a series of (random) 2-site unitaries. Random circuits can be thought of as handy toy models of many-body physics but also as a useful theoretical concept called a unitary designs \[21\]. One of the first exact results was obtained by rewriting the dynamics of purity on average as a classical Markov process \[22\], mapping it to a solvable quantum spin chain and getting an exact expression for the gap \( \Delta \) or the decay rate \[23\], i.e., entanglement speed \[21\] \( v_E \) in modern language. For instance, for a circuit composed of a random 2-site unitaries applied to a random n.n. pair of qubits in a chain with \( L \) sites, one gets \[23\] \( v_E = (1 - \frac{2}{\cos \frac{\pi}{L}}) \approx \frac{1}{2} \). If one would instead take a regular brick-wall pattern of applied gates, like in \[20\], one instead has to calculate the gap of a product of Markovian matrices, obtaining a “multiplicative” form \( v_E = 2 \ln \frac{5}{4 \cos \frac{\pi}{L}} \), going in the TDL to \( v_E \sim 2 \ln \frac{1}{2} \), as also calculated in \[25\] \[26\]. Studies of random circuits have expanded in recent years, including \( U(1) \) conserving ones \[25\] \[26\], with many nice exact results, see e.g. \[24\] \[30\]. They have been also notably used in a race towards quantum supremacy \[31\].

Let us check the above predictions for \( S_2 \) by numerical simulations of random circuits. In order to be able to simulate large systems we resort to the so-called Clifford circuits. For a \( q \) level system the local generalized Pauli operators \( X \) and \( Z \) are defined \[32\] \[33\] as

\[ X|j\rangle = |j \oplus 1\rangle, \quad Z|j\rangle = \omega^j|j\rangle, \quad j = 0, \ldots, q - 1, \] (4)

where \( \omega := e^{2\pi i/q} \), and all additions are modulo \( q \) (sign \( \oplus \)). Generators of the local Pauli group are all \( q^2 \) products \( X^n Z^m \), with \( n, m = 0, \ldots, q - 1 \). The generalized Pauli group (GPG) on \( n \) sites is then formed by the tensor product of \( q^2 \) local Paulis, allowing also for all overall phases \( \omega^j \). Due to \( ZX = \omega XZ \), a product of two members of the GPG is again in the GPG. The action of such Pauli operators on the computational basis states is simple, for instance, \( X^i Z^j \otimes \cdots \otimes X^s Z^w|a\rangle = \omega^{x \cdot a}|a \oplus x\rangle \).

Evolution of states is however not done by updating each computational basis state – that would be inefficient for highly entangled states – but rather by a stabilizer formalism \[34\]. A state \( |\psi\rangle \) on \( n \) qubits is called a stabilizer state if it is a unique joint eigenstate with eigenvalue 1 of \( n \) independent stabilizer generators \( g_j \) from the GPG. For qubits one can obtain it as a product of projectors, \( |\psi\rangle\langle\psi| = \Pi_j (1 + g_j) / 2 \), for qudits one has \( |\psi\rangle\langle\psi| = \Pi_j (1 + g_j + q^2 g_j^2) / 3 \). A Clifford circuit is a series of Clifford gates \( U \), each of which preserves the GPG. That is, \( U_j \) acting nontrivially on sites \( j \) and \( k \) maps a member of the GPG to another member of the GPG (instead of to a superposition of GPGs as for generic \( U \)). And here lies the advantage of Clifford circuits. Instead of updating the state \( |\psi\rangle \) one instead updates each generator \( g_j \), whose number is always \( n \) and which will remain elements of the GPG \[32\] \[33\]. Performing one gate, i.e., updating all stabilizers, takes \( O(n^2) \) operations. Entanglement and state overlaps can also be calculated efficiently \[35\] \[40\].

A common choice of Clifford gates are the phase gate \( P|j\rangle = \omega^{j(2 - j)}|j\rangle \), the Hadamard gate \( H|j\rangle = \frac{1}{\sqrt{q}} \sum_k \omega^{kj}|k\rangle \), and a 2-qudit controlled-NOT gate \( \text{CNOT}_{12}|j,k\rangle = |j, k \oplus j\rangle \). The dynamics of Clifford circuits therefore boils down to modular arithmetic \[37\]. So far such circuits have been extensively studied in quantum information, but not so much in condensed matter or statistical physics. The reason being that their dynamics is typically either ballistic or localized \[38\] (fluctuations though can exhibit interesting behavior \[24\]). We shall

![FIG. 1. (Color online) Diffusive melting of a domain wall in a 1D Clifford circuit with \( L = 1000 \) sites. The inset shows the domain wall profile, while the main plot show a diffusive growth of transferred magnetization across the domain wall (averaged over \( 10^3 \) circuit realizations).](image-url)
study a new class of random Clifford circuits that conserve magnetization and whose dynamics is diffusive. By looking at random circuits we are also able to focus exclusively on the role of the $U(1)$ symmetry without any stray effects caused by other conservation laws (e.g., conservation of energy).

**Numerical verification.**— Let us first focus on qubits. For qubits the elements of the local GPG are just the ordinary Pauli matrices $\{\sigma^x, \sigma^y, \sigma^z, \mathbb{1}\}$. To preserve the total magnetization our Clifford circuit consists of applying the XY gate $U_{XY} = \exp(-i\frac{\pi}{2}(\sigma^x_k \sigma^z_k + \sigma^z_k \sigma^x_k))$. To a randomly selected n- n. pair of sites on a $d$ dimensional square lattice. It is easy to verify that $U_{XY} \mathbb{1}_1 \sigma^x_k U_{XY} = \sigma^x_k \mathbb{1}_k$ and $U_{XY} \sigma^z_k \sigma^z_k U_{XY} = \mathbb{1}_k \sigma^z_k$, and therefore the total magnetization $\sigma^z_0$ is conserved. It also implies that a pair of oppositely polarized spins is exchanged, $U_{XY}|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$. Because the pair $(j,k)$ is chosen at random, it is also immediately clear that the dynamics of magnetization is diffusive, e.g., starting from a domain wall initial state $|\downarrow\ldots\downarrow\uparrow\ldots\uparrow\rangle$ the average profile at time $t$ can be expressed exactly in terms of binomial probabilities, that can be approximated in the large-$t$ limit by the error function (see Fig. 1 for an explicit numerical demonstration).

Starting with the initial state $|\psi\rangle \sim (|\uparrow\rangle + |\downarrow\rangle)^\otimes n$ stabilized by $g_j = \sigma^x_1$, we can simulate our Clifford circuit for thousands of qubits up to very long times, despite the entanglement eventually being a volume-law thousands of qubits up-to very long times, despite the fact that a pair of oppositely polarized spins is exchanged, $U_{XY}|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$. Because the pair $(j,k)$ is chosen at random, it is also immediately clear that the dynamics of magnetization is diffusive, e.g., starting from a domain wall initial state $|\downarrow\ldots\downarrow\uparrow\ldots\uparrow\rangle$ the average profile at time $t$ can be expressed exactly in terms of binomial probabilities, that can be approximated in the large-$t$ limit by the error function (see Fig. 1 for an explicit numerical demonstration).

We also observe that at short times $t \lesssim 10$ the growth is a bit faster than diffusive. This means that in small systems $L \sim 30$ (being a typical maximal size amenable to other methods) it would be very difficult to see the true asymptotic growth over a significant range of times. In 2D we show data only for the case where the region A is the middle-$\frac{1}{4}$ part of the full square lattice with $n = L \times L$ qubits, as this bipartition gives a clearer transition between linear and diffusive growth (see Appendix A for the half-cut). Numerics confirms the short-time growth given by Eq. (3) without any additional prefactors ($A(1) = 4L/3$). We also note that to see the asymptotic growth $\sim \sqrt{L}$ one needs fairly large systems; even for $L = 252$ one can see only about one decade in time of $S_2^t \sim \sqrt{t}$, while on the other hand three decades of $S_2 \sim t$. In 3D the situation is even less favorable for slow asymptotic diffusive growth. Nevertheless, in the more favorable half-cut bipartition we can see a transition from the short-time $S_2 = \frac{A(2)}{3L^2} t$ to the long-time $S_2 \sim \sqrt{L^2 t}$. Again, for $t < t_1$ the linear growth $\frac{A}{L^2}$ has no additional prefactors (for a half-cut $A(2) = L^2$, for the middle-$\frac{1}{4}$ cut $A(2) = \frac{2}{3} L^2$). For the middle-$\frac{1}{4}$ cut, despite a large number of qubits, $n = 48^3 = 110592$, one can barely hint the eventual $\sim \sqrt{t}$ growth. Finally, we show entanglement profiles (Fig. 3), i.e. $S_2$ for a bipartite cut with region A being the first $k$ spins. Also shown are the fluctuations of $S_2$ between different circuit realizations, showing that the relative fluctuations scale as $\sigma(S_2)/S_2 \sim 1/\sqrt{S_2}$, and therefore in the TDL at large times dynamics is self-averaging. It suffices to look at a single random circuit realization.

We also check the case of qutrits with $q > 2$, also studied in [13]. To that end we simulate a qudit chain ($q = 3$, i.e. spin-1 particles) where the local diagonal basis is spanned by $\{Z_k, Z_k^2 = Z_k^{-1}, Z_1^3 = 1\}$, and we take the initial state stabilized by generators $g_j = X_j$. Taking a Clifford circuit with the n- n gate being $U_D = H_2 \text{CNOT}_{21} \text{CNOT}_{12} \text{CNOT}_{12} H_1$, which gives rise to diffusive conservative dynamics of both diagonal matrices, $U^D_{12} \mathbb{1}_j Z_k U_D = Z_j Z_k, U^D_{12} \mathbb{1}_j Z_k^2 U_D = Z_j Z_k, U^D_{12} \mathbb{1}_j Z_k^3 U_D = 1_j Z_k$, and $U^D_{12} \mathbb{1}_j Z_k^2 U_D = Z_j Z_k^2, U^D_{12} \mathbb{1}_j Z_k^3 U_D = 1_j Z_k$, we observe the expected diffusive $S_2 \sim \sqrt{t}$ (Fig. 4). For further qudit numerics see Appendix B.
preserving $U_{XY}$ on a random bond in the upper leg, or a non-conserving $U_G = \text{CNOT}_{12} H_1 \exp (i \frac{1}{2} \sigma^x_1)$ on a random bond of the lower leg. Magnetization is conserved only in the upper leg and one therefore expects generic behavior with $S_2 \sim t$. This is indeed observed in Fig. 4.

We can also see that for times less than $\approx 10^2$ a slower growth is observed in which diffusive dynamics competes with increasingly dominating non-diffusive dynamics of other operators, and therefore, once again, one needs large systems with $L \gtrsim 100$ in order to see the true linear asymptotic growth. We contrast this linear growth with a special case: if we instead of $U_G$ apply $U_{XY}$ also on the lower leg, such that magnetization on both legs is conserved, one again gets a non-generic $S_2 \sim \sqrt{t}$. We remark that this latter case of using $U_{XY}$ on both legs corresponds to a Trotterized dynamics of the Hubbard chain. Using Jordan-Wigner transformation the upper leg represents spin-up fermions, the lower spin-down, while $U_{ZZ}$ is hopping, where $U_{ZZ}$ is the on-site interaction. In Appendix C we show further ladder examples.

**Conclusion.**—We have presented a theory of the Rényi entropy growth in lattice systems that conserve the total magnetization due to $U(1)$ symmetry. We show that in general in qubit systems the entanglement grows linearly in time until at area-law values of $S_2$ a (late-time) crossover to slower square-root growth happens. In 1D qubit system the diffusive $\sqrt{t}$ growth is generic because the crossover happens already at small values of entanglement $S_2 \sim 1$, while in higher dimensions the regime of such growth is pushed to infinitely large values of $S_2$ in the thermodynamic limit. For lattice systems with more than 2 local levels (spin-$s$ particles, $s \geq 1$) and a single conserved charge one will generically observe only linear growth, irrespective of diffusion, the exception being a situation where the dynamics of all diagonal operators is diffusive. Entanglement growth can therefore distinguish both the spatial dimensionality as well as the size of the local Hilbert space.

For other transport types of a conserved quantity and qubits one can conjecture that asymptotically $S_{r>1} \sim t^{1/z}$, with $z$ being the dynamical exponent. An interesting question is the influence of other symmetries, and the behavior of $S_1$. A promising direction is also employing the presented nontrivial Clifford circuits to further explore the many-body physics.

M.Ž. is supported by Grants No. J1-1698 and No. P1-0402 from the Slovenian Research Agency.

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[1] I. L. Chuang and M. A. Nielsen, *Quantum Computation and Quantum Information*, (CUP, 2000).

[2] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. Rispoli, and M. Greiner, *Measuring entanglement entropy in a quantum many-body system*, Nature **528**, 77 (2015).

[3] K. von Klitzing, G. Dorda, and M. Pepper, *New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance*, Phys. Rev. Lett. **45**, 494 (1980).

[4] B. Zheng, X. Chen, D.-L. Zhou, and X.-G. Wen, *Quantum Information Meets Quantum Matter*, (Springer, 2019).

[5] X. Chen, Z.-C. Gu, and Z.-G. Wen, *Classification of gapped symmetric phases in one-dimensional spin systems*, Phys. Rev. B **83**, 035107 (2011).
FIG. 5. (Color online) Entanglement growth for 2D Clifford qubit system and the half-cut bipartition.

Appendix A: Half-cut bipartition for 2D Clifford system

In Fig. 5 we show data for the same 2D diffusive qubit Clifford circuit utilizing $U_{XY}$ gate as in Fig. 2b, but for a half-cut bipartition. We can see that the agreement with the theoretical short-time as well as long-time prediction (Table 1) is good. The short-time growth is $S_2 \approx 0.5t$, where $0.5 = \frac{1}{2L}$ with $l = L$, whereas the asymptotic growth goes into $S_2 \propto \sqrt{2Lt}$ (no fitting parameters).

Appendix B: Additional data for qutrit systems (spin $S = 1$)

In the main text we demonstrated that the dynamics given by the qutrit gate $U_D$, which conserves both the non-trivial diagonal operators $Z_1 + Z_2$ and $Z_1^2 + Z_2^2$, results in a diffusive asymptotic growth of $S_2$ (Fig. 6). Instead of the two “Clifford”-basis diagonal operators $Z_j$ and $Z_j^2$ we can also use the language of spin $S = 1$ particles: there two non-trivial diagonal operators are $S_j^x = \text{diag}(1, 0, -1)$ and $S_j^z = 3(S_j^x)^2 - 2 \cdot 1_j = \text{diag}(1, -2, 1)$. The gate $U_D$ of course also conserves those, $U_D^\dagger(S_j^x + S_j^z)U_D = S_j^x + S_j^z$, $U_D^\dagger(S_j^x + S_j^z)U_D = S_j^x + S_j^z$. Note that $U_D$ is up-to phases equal to the spin-1 SWAP gate $U_{\text{SWAP}} = \exp(-i\frac{\pi}{2}[(S_1 \cdot S_2)^2 + (S_1 \cdot S_2) + 2 \cdot 1])$, $U_DU_{\text{SWAP}}^\dagger = \text{diag}(1, 1, 1, 1, \omega^2, \omega, \omega^2, \omega)$.

We are going to show additional data for a number of spin-1 quantum circuits that are not of the Clifford type. We will always use a half-cut bipartition and the same initial state as used for Clifford circuits, that is a uniform superposition of all computational states, $\psi(0) \sim \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle + |2\rangle)$ $\otimes L$. The shown $S_2$ is the average over between $10^3$ (small $L$) and $20$ (for largest $L = 20$) circuit realizations. The aim is to further shed light on the fact that we expect the asymptotic $S_2 \propto \sqrt{t}$ growth only if all diagonal operators are conserved and diffusive. For $q = 3$ this means that both $S^x$ and $S^z$ should be conserved (alternatively, both $Z$ and $Z^2$).

We first check the evolution using a 2-site gate $U_1$ that is a concatenation of the diffusive $U_D$ we already used in the main text and the isotropic gate, that is $U_1 = U_{\text{ISO}}U_D$, where $U_{\text{ISO}} = \exp(-i\frac{\pi}{\sqrt{2}}S_1 \cdot S_2)$. The gate $U_{\text{ISO}}$ conserves $S^z$, $U_{\text{ISO}}^\dagger(S_1^z + S_2^z)U_{\text{ISO}} = S_1^z + S_2^z$, but not $S_1^x + S_2^x$. The gate $U_1$ therefore conserves only $S_1^z + S_2^z$, the dynamics of which is diffusive due to the spatial randomness (a gate is applied to a random n.n. bond), and we expect the asymptotic growth of $S_2$ to be linear despite diffusive total magnetization. The results are shown in Fig. 6. We immediately have to remark that a drawback of non-Clifford circuits is that only very small systems can be simulated, and correspondingly the reachable times are far from the asymptotic ones. For comparison we also show data for the Clifford circuit with $U_D$ and $L = 2000$ (the same data as in Fig. 4), as well as $L = 10$ and $L = 20$. Because the initial rates of the entropy production are different for $U_D$ and $U_1$ we multiply the times for the $U_D$ data by 1.4 so that the curves overlap at short times. We can see that up-to times $t \approx 6$ the two evolutions result in the same growth of $S_2$ (one could fit $S_2 \approx 0.59t^{0.76}$); for instance, for $L = 10$ or $L = 20$ it is hard to claim any difference between $U_D$ and $U_1$.

After $t \approx 10$ though deviations start to appear: the Clifford case that conserves both diagonal operators starts to converge to slower $S_2 \propto \sqrt{t}$ growth, whereas $U_1$ that conserves only the total $S^z$ starts to grow faster. While from such short-time data it is impossible to reliably infer the asymptotic growth, what we observe is compatible with the asymptotics $S_2 \sim t$ for $U_1$. If the Clifford case, where we can simulate large systems, is any indication of the required sizes necessary to reach the asymptotics, we can say that likely about 5 times larger systems would be required to really see the asymptotic linear growth for $U_1$ (for the Clifford data in Fig. 4 we can see that one converges to $S_2 \sim \sqrt{t}$ only at $t \approx 10^3$ where $S_2 \approx 10^2$).

In Fig. 5 we show data for further non-Clifford random circuits. We show results for $U_{XX2} =$
exp \(-i \frac{\pi}{3\sqrt{2}} [S_1^y S_2^x + S_1^x S_2^y]\) that conserves only $S_1^y + S_2^x$, but not $S_1^x + S_2^z$. Data in frame (a) is compatible with the linear asymptotic growth. If we on the other hand change the gate to $U_{XX} = \exp \left( -i \frac{\pi}{\sqrt{2}} [S_1^x S_2^x + S_1^y S_2^y] \right)$, which conserves both $S_1^x + S_2^x$ and $S_1^y + S_2^y$, we see that the growth is much slower, like $S_2 \sim t^{0.1}$ at short times. While it is hard to make any asymptotic claims about $S_2 \sim \sqrt{t}$ based on such numerics (exact numerics for larger systems get hampered by memory requirements; the Hilbert space size for $L = 21$ qubits is about the same as for $\approx 33$ qubits), what is very distinct is that the exponent is very different in (a) and (b) despite a very similar 2-site gate; the only difference between the two is the conservation of $S_1^x + S_2^x$. Finally, as a third example we show the anisotropic XXZ-like gate $U_{XXZ} = \exp \left( -i \frac{\pi}{2} [S_1^x S_2^y + S_1^y S_2^z + 1.5S_1^z S_2^x] \right)$ that again conserves only the total magnetization, and therefore one has $S_2 \sim t$ visible already at short times.

![FIG. 7. (Color online) $S_2(t)$ for three non-Clifford spin-1 random circuits. Top frame (a) is for $U_{XXZ}$, frame (b) for $U_{XX}$, and frame (c) for $U_{XXZ}$ (see text for details). The logarithm in the definition of $S_2$ is always base-3.](image)

FIG. 7. (Color online) $S_2(t)$ for three non-Clifford spin-1 random circuits. Top frame (a) is for $U_{XXZ}$, frame (b) for $U_{XX}$, and frame (c) for $U_{XXZ}$ (see text for details). The logarithm in the definition of $S_2$ is always base-3.

**Appendix C: Data for ladder systems ($q = 4$)**

Here we show further data supporting the claim that for ladders the growth of $S_2$ is generically linear in time. We simulate Clifford ladders with a large number of rungs $L = 10000$ using different 2-site gates. On legs, upper or lower, we apply either the already seen $U_{XY} = \exp \left( -i \frac{\pi}{4} (\sigma_1^x \sigma_2^x + \sigma_2^y \sigma_1^y) \right)$, or $U_G = \text{CNOT}_{12} H_1 \exp \left( i \frac{\pi}{2} \sigma_2^z \right)$. On the rungs we use either $U_{ZZ} = i \exp \left( -i \frac{\pi}{2} \sigma_1^z \sigma_2^z \right)$, or $U_{SXY} = \exp \left( -i \frac{\pi}{4} \sigma_1^z \right) \exp \left( -i \frac{\pi}{4} \sigma_2^z \right)$. The protocol is always the same: at each step we apply one of the leg gates on a random bond on either the upper or the lower leg, and a rung gate on an independent random rung. The type of the gate applied on rungs as well as on the upper and lower leg is held fixed, so that one can get 8 different protocols out of the mentioned 4 gates.

The gate $U_{XY}$ conserves the total magnetization on the respective leg on which it acts, while $U_G$ does not. Namely, $U_G^j \sigma_k^x U_G = \sigma_k^x \sigma_j^x$, and $U_G^j \sigma_k^y U_G = \sigma_k^y \sigma_j^y$, so that one has $U_G^j (\sigma_1^z + \sigma_j^z) U_G = \sigma_j^z (\sigma_1^z + \sigma_j^z)$. The rung gate $U_{ZZ}$ does not break conservation of $\sigma_1^z + \sigma_2^z$, nor of $\tau_1^z + \tau_2^z$ because one has $U_{ZZ}^j \tau_1^z \tau_2^z U_{ZZ} = \tau_1^z \tau_2^z$, and $U_{ZZ}^j 1 \tau_1^z U_{ZZ} = 1 \tau_1^z \tau_2^z$ (as well as $U_{ZZ}^j \sigma_1^y \sigma_2^y U_{ZZ} = -\sigma_1^y \sigma_2^y$, $U_{ZZ}^j \sigma_1^x \sigma_2^x U_{ZZ} = -\sigma_1^x \sigma_2^x$). In other words, the gate $U_{ZZ}$ introduces non-trivial phases $\pm 1$ in the dynamics of Pauli $x$ and $y$ matrices. The gate $U_{SXY}$ on the other hand preserves conservation of magnetization only on the upper leg, $U_{SXY}^j \sigma_k^z U_{SXY} = \sigma_k^z$, and $U_{SXY}^j 1 \sigma_1^z U_{SXY} = -\sigma_1^z 1$, while it breaks conservation on the lower leg, $U_{SXY}^j 1 \tau_2^z U_{SXY} = 1 \tau_2^z \tau_1^z$, and $U_{SXY}^j 1 \tau_1^z U_{SXY} = 1 \tau_1^z \tau_2^z$.

In Fig. 8 we show results of numerical simulation for different protocols. Taking the $(XY)-(ZZ)-(XY)$ protocol where the leg gates $U_{XY}$ as well as the rung gates $U_{ZZ}$ conserve magnetization on the upper and the lower leg, dynamics of all diagonal operators is diffusive and
one has $S_2 \approx \sqrt{t}$. The same data for $L = 4000$ has been already shown in Fig. [3]. We can break conservation of magnetization on the lower leg by using $U_G$, which as we can see results in the asymptotic growth $S_2 \approx t$ (red curve in Fig. [3]), the same data as in Fig. [4]. We can however break the conservation on the lower leg also by changing the rung gate to $U_{Sxy}$. This is illustrated by the protocol $(XY) - (Sxy) - (XY)$, which again results in $S_2 \approx t$. Note that here the dynamics along the two rungs is purely diffusive – the gate $U_{XY}$ is used on both legs – it is only the non-trivial rung dynamics that breaks one $U(1)$ symmetry and causes the asymptotic linear growth of $S_2$ (similar results would be obtained also if at each step of the protocol the $U_{XY}$ gate would be applied simultaneously to a pair of upper- and lower-leg bonds forming a local plaquette with two rungs). Finally, using conservation breaking $U_G$ on the lower leg and well as the rung $U_{Sxy}$, one again has $S_2 \approx t.$