COMPUTING BORCHERDS PRODUCTS

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Abstract. We present an algorithm for computing Borcherds products, which has polynomial runtime. This improves on the currently common method, which has exponential runtime. An implementation of the new algorithm shows that it is superior also in practice.

1. Introduction

Product expansions are a utilie construction in the world of automorphic forms. For example, writing $\tau$ and $\tau'$ for two variables in the Poincaré upper half plane, we can express the famous $j$-invariant as follows:

$$j(\tau) - j(\tau') = \left( e^{-2\pi i \tau'} - e^{-2\pi i \tau} \right) \prod_{m,n=1}^{\infty} \left( 1 - e^{2\pi i (m\tau + n\tau')} \right)^{c_{mn}}$$

for certain integral coefficients $c_{mn}$. This formula holds for sufficiently large imaginary parts of $\tau$ and $\tau'$. Several facts can be deduced from it. For instance, one sees immediately that close to infinity, the $j$-function is one-to-one on the fundamental domain.

It turns out that product expansions exist for a much larger class of automorphic forms, namely orthogonal modular forms for $O_{2,n}(\mathbb{R})$. Thanks to the Koecher principle, such product expansions allow for a direct investigation of all zeros of such an automorphic form, regardless of the limited domain of convergence they have.

The convergence of these more general product expansions remained an open problem for a long time. When Borcherds presented his first work on product expansions for orthogonal groups [Bor98] (see also [Bor95, HM98, Kon97]), he started a revolution. Product expansions, which at that time played a role in the classical theory of elliptic modular forms and Jacobi forms only, became important to other areas in mathematics. One branch of research aimed at Kac-Moody algebras which could be constructed by means of lattice theory [Gri95, GN98, Sch04, Sch08]. As a natural generalization of finite dimensional Lie algebras, they attracted the physicists’ attention. On the automorphic forms side, research in this area focuses on orthogonal groups for rather big lattices. A second, equally popular branch of research originating in Borcherds’ work is concerned with the structure of divisor...
class groups and Chern classes \cite{Bor99, Bor00, Bru02, BO10, CG08}. It is founded on the fact that one can easily read off the divisor of an automorphic form given by a product expansion. Lattices with particular nice structure, which are typically of moderate size, are in the focus of mathematicians working in this direction. The Fourier expansion of Borcherds products is a particular aspect that is interesting in this context. Among other things, it can be used to understand the structure of graded rings of modular forms \cite{DK04, Kri05, May10}, or to check for relations like the ones coming from Saito-Kurokawa-Maaß-Gritsenko lifts \cite{HM10}. Most importantly, the multiplicative version of the degeneracy of BPS dyons, occurring in string theory, are Borcherds product of this type (see, for example, \cite{HM98, JS05, DM11}). The last named author will return to this subject in the sequel.

Large scale computational approaches to Siegel modular forms were pioneered by Skoruppa in \cite{Sko92}. His considerations became famous, in particular, because in the course of his studies, he discovered two exceptional Hecke splittings in weight 24 and 26, which until today remain unexplained. Equally important computations that led to deep insights have been performed by Ibukiyama \cite{Ibu02}, and Poor and Yuen \cite{PY00, OPY08}. A recent, striking result was obtained in \cite{PY09}, where strong evidence for the Paramodular Conjecture \cite{Yos80, Yos07}, which parallels the Shimura-Taniyama Conjecture, was provided using paramodular forms, a variation of Siegel modular forms.

The purpose of this work is to provide an algorithm for explicit computations with Borcherds products that runs in polynomial time. Such an algorithm is vastly faster than the direct evaluation of Borcherds’ formula, which requires repeated multiplication of increasingly large polynomials, leading to an exponential runtime. The presented algorithm is based on two ideas: to make the problem as linear as possible and to split off parts that behave distinctly. In order to achieve the first objective, we apply $\exp \circ \log$ to Borcherds’ original formula, and we evaluate the logarithmic term first. This avoids the repeated multiplication of polynomials. To achieve the second objective, we analyze the structure of Fourier indices that are positive with respect to a fixed Weyl chamber (see (3.4) to (3.8)). This allows us to reduce the number of terms taken into consideration.

We demonstrate the actual superiority of our approach by providing an implementation for Hermitian modular forms over $\mathbb{Q}(\sqrt{-3})$ in Sage \cite{S+11b, DGPS10}. We compare it to a usual implementation: for moderate precision (see Section 3 for a discussion of precisions) the new algorithm is already faster by a factor between 2 and 4, depending on the memory architecture of the used hardware. The coefficient of the leading term of the new implementation’s runtime is seemingly large. This prevents it from improving even more on the runtime of the naïve implementation. It originates, however, in the not optimal memory management of Sage, and it will reduce drastically for a reimplementation in C or C++.

In Section 2 we review the basic theory of orthogonal modular forms and Borcherds products. Section 3 deals with the algorithm to compute Fourier expansions of Borcherds products. The implementation for Hermitian modular forms
is discussed in Section [H]. In particular, this section contains a comparison of run-
time tests.

2. Preliminaries

2.1. Orthogonal modular forms. Throughout the paper, we fix an even lattice
$L$ of signature $(2, n)$ with $n \geq 3$. The quadratic form attached to $L$ is denoted by
$q$, while we write $(\mu, \lambda)$ for the bilinear form $q(\mu + \lambda) - q(\mu) - q(\lambda)$. If $\lambda$ is a vector
in a lattice with fixed and obvious embedding into $L \otimes \mathbb{Q}$, we understand that $q(\lambda)$
is the quadratic length of $\lambda$ therein. The dual lattice of $L$ is

$$L^\# = \{ \lambda \in L \otimes \mathbb{Q} : (\mu, \lambda) \in \mathbb{Z} \text{ for all } \mu \in L \}. $$

Clearly, $L \subseteq L^\#$. Thus we can construct the $\mathbb{Z}$-module $L^\#/L$, that is equipped with
the quadratic form $\bar{q}$ taking values in $\mathbb{Q}/\mathbb{Z}$. The pair $(L^\#/L, \bar{q})$ is a finite quadratic
module of order $|L|$. It is called the discriminant module of $L$, and we denote it by
disc $L$.

We write $O(L)$ for the orthogonal group associated to $L$. It is the group of all
$\mathbb{Z}$-linear bijections of $L$ that leave the quadratic form $q$ invariant. Any such trans-
formation acts also on $L^\#$ and disc $L$. The stable orthogonal group $O(L)[\text{disc } L]$
consists of all transformations fixing disc $L$ elementwise.

We consider a fixed connected component $\mathcal{D}^+_\mathfrak{e}$ of

$$\mathcal{D}_\mathfrak{e} := \{ \lambda \in L \otimes \mathbb{C} : q(\lambda) = 0 \land (\lambda, \overline{\lambda}) > 0 \}. $$

The projectification $\mathcal{D}^+ = \mathbb{P}(\mathcal{D}^+_\mathfrak{e})$, which is a Hermitian locally symmetric domain
of type IV, is the natural domain of definition for orthogonal modular forms. We
deﬁne them using the $-k$-homogeneous pullback to $\mathcal{D}_\mathfrak{e}$. A function $f : \mathcal{D}_\mathfrak{e} \to \mathbb{C}$ is called
$-k$-homogeneous if $f(s\lambda) = s^{-k} f(\lambda)$ for all $s \in \mathbb{C}$ and all $\lambda \in \mathcal{D}_\mathfrak{e}$.

**Definition 2.1.** An orthogonal modular form of weight $k$ for a finite index subgroup
$\Gamma \leq O(L)$ is a holomorphic, $-k$-homogeneous function $f : \mathcal{D}_\mathfrak{e} \to \mathbb{C}$ satisfying the following conditions:

1. For all $\gamma \in \Gamma$ we have $f \circ \gamma = f$.
2. For every isotropic vector $\lambda \in iL \cap \mathcal{D}^+_\mathfrak{e}$ the function $f(y\lambda)$ is bounded as $y \to \infty$.

After fixing a section $\mathcal{D}^+ \to \mathcal{D}_\mathfrak{e}$, we freely identify any orthogonal modular form
with its restriction to $\mathcal{D}^+$. The Borcherds products that we are interested in are arithmetic orthogonal modular
forms. Throughout the paper, we assume that $L = U \oplus U \oplus (-1)L_0$, where $U$
is the unimodular hyperbolic lattice of rank 2 and $L_0$ is a fixed even positive def-
definite lattice. Notice that disc $L = \text{disc } (-1)L_0$. Under this assumption, there is a
canonical section $\mathcal{D}^+ \to \mathcal{D}_\mathfrak{e}$ whose image has typical elements $(1, q(z) - \tau \tau', \tau, \tau', z)$,
$\tau, \tau' \in \mathbb{C}$, $z \in L_0 \otimes \mathbb{C}$. In this setting, any orthogonal modular form has a Fourier expansion

$$\sum \alpha([a, b, c]) e(a\tau + c\tau' + (b, z)),$$

where $e(x) := e^{2\pi ix}$. The sum ranges over triples $[a, b, c]$ with $a, b \in \mathbb{Q}$ and $b \in L_0 \otimes \mathbb{Q}$. We write $\mathcal{O}_\mathbb{Q}$ for the additive group of such triples. The submonoid of integral
indices $[a, b, c] \in \mathbb{Q}_0$ that satisfy $a, c \in \mathbb{Z}$ and $b \in \mathbb{L}_0^\#$, is denoted by $\mathbb{Q}$. All Fourier expansions that we will deal with satisfy $\alpha([a, b, c]) = 0$, if $[a, b, c] \not\in \mathbb{Q}$.

We call an index positive definite or semi-definite if $\text{disc}([a, b, c]) := ab - q(b) > 0$ and $a > 0$, or $\text{disc}([a, b, c]) \geq 0$ and $a, c \geq 0$. We write $[a, b, c] > 0$ if $[a, b, c]$ is positive definite, and $[a, b, c] \geq 0$ if it is positive semi-definite. The monoids of integral positive semi-definite indices is denoted by $\mathbb{Q}^+$.

The group $M := O(U \oplus (-1)L_0) \subset O(U \oplus U \oplus (-1)L_0)$ gives rise to symmetries of the Fourier expansion. That is, we have $\alpha([a, b, c]) = \chi(m) \alpha(m[a, b, c])$ for all $m \in M$ and a fixed character $\chi$ of $M$.

**Remark 2.2.** For the time being, drop the condition $L = U \oplus U \oplus (-1)L_0$, that we have imposed on $L$. If $n \geq 5$, it is always possible to find a subgroup $L' \subset L$ of finite index that splits: $L' = u_1U \oplus u_2U \oplus (-1)L_0$ with positive constants $u_1$ and $u_2$ (see, e.g., [Bor98 Section 8]). Since the orthogonal groups of $L$ and $L'$ are commensurable, the considerations in this paper apply to arbitrary Borcherds products, if only $n \geq 5$, even though the details are tedious to work out.

### 2.2. Borcherds products

We revise the construction of Borcherds products, which dates back to [Bor95] and [Bor98]. A more accessible discussion, valid in some interesting special cases, can be found in [Gri95, GN98]. Given an elliptic vector-valued modular form, we construct an orthogonal modular form.

Let $\mathbb{H} := \{ \tau \in \mathbb{C} : \text{Im}(\tau) > 0 \}$ denote the Poincaré upper half-plane. Since we deal with half-integral weight, the next definition involves the metaplectic cover $Mp_2(\mathbb{Z})$ of $SL_2(\mathbb{Z})$. It is the preimage of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$, the connected double cover of $SL_2(\mathbb{R})$. The primages of $\ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}))$ in $Mp_2(\mathbb{Z})$, where $N$ runs through $\mathbb{N}$, will be called principal congruence subgroups.

Write $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ for a typical element of $SL_2(\mathbb{R})$. The elements of $Mp_2(\mathbb{R})$ can be written $(g, \tau \mapsto \sqrt{ct + d})$, where the first component is an element of $SL_2(\mathbb{R})$ and the second is a holomorphic function on $\mathbb{H}$. Since there are two branches of the square root, this yields indeed a double cover of $SL_2(\mathbb{R})$.

Given a representation $(\rho, V_\rho)$ of $Mp_2(\mathbb{Z})$, $k \in \frac{1}{2}\mathbb{Z}$ and $F : \mathbb{H} \to V_\rho$, we define

$$F|_{k, \rho}(g, \sqrt{c \cdot \tau + d}) = F(g, \sqrt{c \cdot \tau + d}) \rho(g, \sqrt{c \cdot \tau + d}).$$

**Definition 2.3.** Let $(\rho, V_\rho)$ be a finite-dimensional representation of $Mp_2(\mathbb{Z})$. A weakly holomorphic vector-valued modular form of type $\rho$ and weight $k \in \frac{1}{2}\mathbb{Z}$ is a holomorphic function $F : \mathbb{H} \to V_\rho$ such that the following conditions are satisfied:

1. For all $g \in Mp_2(\mathbb{Z})$ we have $F|_{k, \rho}g = F$.
2. The Fourier expansion of $F$ has the form

$$\sum_{-\infty < n \in \mathbb{Q}} f(n)e(n\tau), \quad f(n) \in V_\rho.$$

**Remark 2.4.** If the weight $k$ is integral and $\rho$ is trivial on some principal congruence subgroup, the representation $\rho$ and the factor of automorphy factor through $SL_2(\mathbb{Z})$.

The most relevant case to us is $\rho = \rho_\mathbb{L}$, where $\rho_\mathbb{L}$ is the Weyl representation associated to the finite quadratic module $\text{disc} L$. The space $V_\rho$ has a canonical
basis index by \( \text{disc } L \). We refer the reader to [Bor98, Sko08] for a definition and more details. Suppose that \( F \) is an elliptic modular form of type \( \rho_L \). The Fourier expansion of its components \( F_\lambda \) is

\[
F_\lambda(\tau) = \sum_{-\infty < n} f(\lambda, n) e(n\tau)
\]

with \( \lambda \in \text{disc } L \). By \( f(\lambda, n) \) with \( \lambda \in L^\# \) we mean \( f(\overline{\lambda}, n) \).

With a fixed basis \( w_1, \ldots, \overline{w}_{n-2} \) of \( L_0 \otimes \mathbb{R} \) given, we write \( b > 0 \) if there is \( 1 \leq j \leq n - 2 \) such that \( (b, \overline{w}_j) > 0 \) and \( (b, \overline{w}_j') = 0 \) for all \( j' < j \). A triple \( [a, b, c] \) is called positive, \( [a, b, c] > 0 \), if \( c > 0 \), or \( c = 0 \) and \( a > 0 \), or \( a = c = 0 \) and \( b > 0 \). This definition is motivated by Weyl chambers discussed in [Bor98, Section 6]. We can find a Weyl chamber \( W \) for \( U \oplus (-1) L_0 \) such that \( ([a, b, c], W) > 0 \) if and only if \( [a, b, c] > 0 \), where the \( w_j \) are rational vectors.

Borcherds also defined a Weyl vector \( W_F \) attached to \( F \) and \( W \). In our setting (see also [Der01]), we have:

\[
a_W := \frac{1}{2} \sum_{b \in L_0^\#} f(b, -\text{disc}(b)), \quad b_W := \frac{1}{2} \sum_{0 < b \in L_0^\#} f(b, -\text{disc}(b)) \cdot b, \quad c_W := a_W - \sum_n \sigma_1(n) \sum_{b \in L_0^\#} f(b, -n - \text{disc}(b)),
\]

\[
W_F := e(a_W \tau + c_W \tau' + (b_W, z)).
\]

With this notation at hand, we reformulate [Bor98, Theorem 13.3]. Note that we suppress a root of unity in the definition of \( \Psi_F \).

**Theorem 2.5.** Fix a weakly-holomorphic vector-valued elliptic modular form \( F \) of type \( \rho_L \) and weight \((2 - n)/2\) with Fourier expansion (2.1). Then the following product converges and is an orthogonal modular form for \( O(L)[\text{disc } L] \):

\[
\Psi_F := W_F \prod_{[a, b, c] > 0} \left( 1 - e(a \tau + c \tau' + (b, z)) \right)^{f(b, \text{disc}([a, b, c]))}.
\]

Crucial to our approach is the function

\[
\Phi_F = \log \Psi_F = \log W_F + \sum_{[a, b, c] > 0} f(b, [a, b, c]) \log \left( 1 - e(a \tau + c \tau' + (b, z)) \right),
\]

which is well-defined and holomorphic in the Weyl chamber \( W \). We have \( \Psi_F = \exp(\Phi_F) \).

### 3. An algorithm for Fourier expansions of Borcherds products

We write \( \mathbb{Q}[Q] \) for the direct product of 1-dimensional \( \mathbb{Q} \)-vector spaces indexed by \( Q \). Any element will be written as a possibly infinite sum of multiples of \( e^{[a, b, c]} \) for \( [a, b, c] \in Q \).

Fix a vector-valued elliptic modular form \( F \) of type \( \rho_L \) and weight \((2 - n)/2\). Define

\[
\overline{\Phi}_F := -\sum_{[a, b, c] > 0} f(b, \text{disc}([a, b, c])) \sum_{n=1}^\infty e^{n[a, b, c]} n \in \mathbb{Q}[Q].
\]
Clearly, \( \tilde{\Phi}_F \to \Phi_F - \log W_F \) as \( e^{[a,b,c]} \) is mapped to \( e(a\tau + c\tau' + (b, z)) \).

Even though \( \bigotimes Q \) carries no algebra structure, we can define powers of \( \tilde{\Phi}_F \). Write \( \phi_F([a,b,c]) \) for the coefficient of \( e^{[a,b,c]} \). We set

\[
\tilde{\Phi}_F := \sum_{\{a,b,c\}} \left( \sum_{\Sigma_{i=1}^{\infty} \{a,b,c\} = \{a,b,c\}} \prod_{i} \phi_F([a_i,b_i,c_i]) \right) e^{[a,b,c]},
\]

where the \([a_i,b_i,c_i]\)'s are elements of \( Q \). By the next lemma, the inner sum is finite.

**Lemma 3.1.** Given \( l \in \mathbb{N} \), the set

\[
\left\{ ([a_i,b_i,c_i])_{i=1}^{\ldots,n} \in Q^l : \sum_{i=1}^{l} [a_i,b_i,c_i] = [a,b,c], \phi_F([a_i,b_i,c_i]) \neq 0 \text{ for all } i \right\}
\]

is finite.

Proof. For reasons of symmetry, it suffices to prove that there are only finitely many \([a_1,b_1,c_1]\) that possibly contribute, regardless of the \([a_i,b_i,c_i]\)'s with \( i \neq 1 \). Assume that the tuple \(([a_i,b_i,c_i])_{i=1}^{\ldots,n} \in Q^l \) is an element of the above set.

Since \( F \) is a weakly holomorphic modular form, there is a lower bound \( d \) such that \( f(b,D) = 0 \), whenever \( D \leq d \). By definition of the relation > and of \( \tilde{\Phi}_F \), only indices \([a_1,b_1,c_1]\) satisfying \( c_1 \geq 0 \) contribute. In particular, we can find an upper bound on \( c_1 \) for those \([a_1,b_1,c_1]\) that contribute. Consider those with \( c_1 > 0 \). If \( \phi_F([a_1,b_1,c_1]) \neq 0 \), we can write \([a_1,b_1,c_1] = m \tilde{a}_1 \tilde{b}_1 \tilde{c}_1 \) with \( m \in \mathbb{N} \) and \( \text{disc}([a_1,b_1,c_1]) > d^2 \). This gives rise to the inequality \( \text{disc}([a_1,b_1,c_1]) > c^2 d \). From this, we deduce lower and upper bounds \( A_1 \) and \( A_0 \) on \( a_1 \) for those \([a_1,b_1,c_1]\) with \( c_1 > 0 \) that contribute. For every fixed \( a_1 \), the same inequality excludes all but finitely many \( b_1 \), since \( L_0 \) is definite.

In the case of \( c_1 = 0 \), only indices \([a_1,b_1,0]\) satisfying \( a_1 \geq 0 \) contribute. By what we observed in the case \( c_1 > 0 \), we have \( a_1 \leq - (l-1) \min\{0, A_1\} + a \). That is, we have an upper bound on \( a_1 \) for all \([a_1,b_1,c_1]\) that contribute.

Assume that \( a_1 > 0 \). Using the factorization \([a_1,b_1,0] = m_1 \tilde{a}_1 \tilde{b}_1 \tilde{0} \) analogous to the one above, we conclude that there are only finitely many \( b_1 \)'s that can occur.

Consider the case \( a_1 = c_1 = 0 \). Then \( b_1 > 0 \). Recall the rational vectors \( w_1, \ldots, w_{n-2} \in L_0 \otimes \mathbb{Q} \) discussed in Section 2. Choose \( j \) between 1 and \( n-2 \) such that \((w_j,b_1) > 0\) and \((w_{j'},b_1) = 0\) for all \( 1 \leq j' < j \). Since \( w_j \) is rational, the set of possible \((w_j,b_1)\) for \( b_1 > 0 \) is discrete in \( \mathbb{R} \). Because \( \sum_i (w_j,b_1) = (w_j,b) \) is fixed and, in particular, independent of \( b_1 \), there is only a finite number of \( b_1 \)'s that can occur by the same argument as above.

The following theorem is used implicitly, when computing Borcherds products, whether or not Borcherds’ formula is evaluated only naively. Nevertheless, no proof seems to be available in the literature.

**Theorem 3.2** (Formal Borcherds convergence theorem). The exponential

\[
\tilde{\Psi}_F := W_F \exp(\tilde{\Phi}_F) = W_F \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Phi}_F^n
\]

is well-defined in the sense that, given \([a,b,c]\), the coefficient of \( e^{[a,b,c]} \) in \( \tilde{\Phi}_F^n \) is nonzero for only finitely many \( n \).
Moreover, for any $[a, b, c]$ the coefficient of $e^{[a,b,c]}$ in $\tilde{\Phi}_F$ equals the Fourier coefficient of $e(a\tau + c\tau' + (b, z))$ in $\Phi_F$.

**Proof.** The first part follows by analyzing the proof of Lemma 3.1 in the light of the new statement. The second part follows from the first part, since by Theorem 2.5, due to Borcherds, the product expansion for $\Phi_F$ converges locally uniformly absolutely. $\square$

We split $\tilde{\Phi}_F$ into five pieces:

- $\mathfrak{A} := \sum_{[a,b,c]>0, a,c>0} -f(b, \text{disc}([a,b,c])) \sum_{n=1}^{\infty} \frac{e^{n[a,b,c]}}{n}$.

- $\mathfrak{B} := \sum_{[a,b,c]>0, a>0, c>0} -f(b, \text{disc}([a,b,c])) \sum_{n=1}^{\infty} \frac{e^{n[a,b,c]}}{n}$.

- $\mathfrak{C} := \sum_{[a,b,c]: a\leq 0, c>0} -f(b, \text{disc}([a,b,c])) \sum_{n=1}^{\infty} \frac{e^{n[a,b,c]}}{n}$.

- $\mathfrak{D} := \sum_{[a,b,c]: a>0, c=0} -f(b, \text{disc}([a,b,c])) \sum_{n=1}^{\infty} \frac{e^{n[a,b,c]}}{n}$.

- $\mathfrak{E} := \sum_{[a,b,c]: a=c=0, b>0} -f(b, \text{disc}([a,b,c])) \sum_{n=1}^{\infty} \frac{e^{n[a,b,c]}}{n}$.

We can further split $\mathfrak{E} = \sum_{j=1}^{n-1} \mathfrak{E}_j$:

- $\mathfrak{E}_j := \sum_{[a,b,c]: a=c=0, b>0} -f(b, \text{disc}([a,b,c])) \sum_{n=1}^{\infty} \frac{e^{n[a,b,c]}}{n}$

where the sum only ranges over those $b$ satisfying $(w_j, b) > 0$ and $(w_{j'}, b) = 0$ for all $1 \leq j' < j$. Note that $\mathfrak{A}$ is $M$-invariant, while the four other pieces are not.

We use capital letters to denote elements of $\mathbb{Q}[\mathbb{Q}]$ and lower case letters to denote their coefficients. For example, we write $\xi([a,b,c])$ for the coefficient of $e^{[a,b,c]}$ in $\Xi \in \mathbb{Q}[\mathbb{Q}]$.

In the last step of the next algorithm, the multinomial coefficients

$$\binom{n_1 + \cdots + n_l}{n_1, \ldots, n_l} := \frac{(\sum_{j=1}^{l} n_j)!}{\prod_{j=1}^{l} n_j!}$$

for $n_1, \ldots, n_l \in \mathbb{N}$ will occur.

**Algorithm 3.3.** The following algorithm computes the Fourier expansion $\tilde{\Phi}_F$ for all $e^{[a,b,c]}$ with $a, c < B \in \mathbb{N}$.
1: Set \( d \leftarrow \max \{ d : f(b, d') = 0 \text{ for all } d' \leq d \text{ and all } b \} \).

2: Set \( a_{\text{neg}} \leftarrow -(B - c_W - 1) d \).

3: For \( 0 \leq \beta < \min \{ B + a_{\text{neg}} - a_W - \alpha, B - c_W - \alpha \} \) and \( 0 \geq \gamma < B - c_W - \alpha \) do Step 4 to 6 and then jump to Step 13.

4: Compute \( \Xi_{\beta, \gamma} \leftarrow \mathfrak{A}^\beta \mathfrak{C}^\gamma \).

5: Set \( a_{\text{min}} \leftarrow \min \{ a : \exists[a, b, c] : \xi([a, b, c]) \neq 0 \} \).

6: For all \( \delta < B - a_{\text{min}} \) do Step 7 and 8.

7: Set \( \Xi_{\beta, \gamma, \delta} \leftarrow \Xi_{\beta, \gamma} \mathfrak{A}^\delta \) and \( \Xi \leftarrow \Xi_{\beta, \gamma, \delta} \) (for abbreviation only).

8: For \( 1 < j < n - 2 \) recursively do Step 9 to Step 12 and, at any leaf of the associated decision tree, set \( \Xi_{\beta, \gamma, \delta, \eta} \leftarrow \Xi \) with \( \eta = (\eta_1, \ldots, \eta_{n-2}) \).

9: Set \( b_{\text{min}, j} \leftarrow \min \{ (w_j, b) : \exists[a, b, c] : \xi([a, b, c]) \neq 0 \} \).

10: Set \( b_{\text{max}, j} \leftarrow \max \{ (w_j, b) : \exists[a, \tilde{b}, c] : \xi([a, \tilde{b}, c]) \neq 0 : [a, b, c] > 0 \} \).

11: For all \( 0 \leq \eta_j < (-b_{\text{min}, j} + b_{\text{max}, j}) / \min \{ (w_j, b) > 0 : b > 0 \} \) do Step 13.

12: Set \( \Xi \leftarrow \Xi \mathfrak{C}_j \).

13: Compute \( \sum \left( \frac{1}{\alpha, \beta, \gamma, \delta, \eta} \right) (\alpha + \beta + \gamma + \delta + \eta)^{-1} \mathfrak{A}^\alpha \Xi_{\beta, \gamma, \delta, \eta} \).

The last sum ranges over \( 0 \leq \alpha \leq B - a_w \) and all \( \Xi_{\beta, \gamma, \delta, \eta} \) that were defined.

**Proof of correctness.** The correctness follows immediately, when making the bounds used in the proof of Lemma 3.1 explicit. \( \square \)

**Remark 3.4.** Considering each step separately, one easily sees that Algorithm 3.3 has polynomial runtime in \( B \). Evaluating formula (2.2) directly has exponential runtime, since the coefficients of a weakly holomorphic modular forms grow exponentially.

## 4. Borcherds products for the Hermitian modular group

The authors provide an implementation of Algorithm 3.3 for Hermitian modular forms over the imaginary quadratic fields \( \mathbb{Q}(\sqrt{D}) \). It is written in Sage \([\text{Sage}^{*}11b]\), and the authors think of it as a proof of concept. It is equally based on another implementation presented in \([\text{Rau}11b]\) by the last named author, providing a model for general Fourier expansions. This dependency results in restricted functionality for \( D \neq -3 \); For all these cases the framework currently provides only limited support. Both implementations are available on the authors’ homepage \([\text{Rau}11a]\) and are intended for prompt integration into Purple Sage \([\text{Sage}^{*}11a]\).

As a basis for the maximal order \( \mathcal{O} \subseteq \mathbb{Q}(\sqrt{D}) \) we use \( 1 / \sqrt{D}, (1 + \sqrt{D}) / 2 \), writing \( b = b_1 / \sqrt{D} + b_2(1 + \sqrt{D}) / 2 \). With an appropriate Weyl chamber chosen, the condition \( b > 0 \), described in Section 2.2, is equivalent to \( b_2 < 0 \lor (b_2 = 0 \land b_1 < 0) \); A condition easy to test. When computing the powers of \( \mathfrak{A} \), the implementation makes use of its \( \mathrm{GL}_2(\mathcal{O}) \)-invariance. Crucial to the implementation is the fact that \( \widetilde{\Phi}_f \) is the Fourier expansion of a Hermitian modular form. This allows for a significant reduction of the number of coefficients that need to be calculated. For more details on the implementation, the reader it referred to comments to be found in the source code.
To demonstrate the value of our approach, we compare our implementation with the common, naive method. The example that the authors computed is $\phi_{45}$, the weight 45 Hermitian modular form for $\mathbb{Q}(\sqrt{-3})$ (see, e.g., [Der01]). For the sake of clearness, the authors have implemented the algorithm in Sage. The memory footprint of Sage, however, is larger by a factor exceeding 40 compared to native implementations, and the memory access is not optimal. For this reason the provided implementation suffers from cache misses and an enormous memory consumption for higher precisions. In Table 1 the impact of Sage’s memory consumption can be observed. For precision $8$ and $9$ the performance gain realized up to then increases much slower than expected. The reason is that for example in the case of precision $9$, the memory management of Sage (presumably due to non-efficient iteration over large lists) allocates 100 GB of memory, while theoretically the new algorithm does not consume more memory than the naive evaluation does. Nonetheless, the runtimes for precisions 7 to 9 show that the new algorithm is not only asymptotically faster, but for moderate precisions yields a clear advantage over the naive one. Note that the examples in Table 1 were computed using a Sun server of the Lehrstuhl A für Mathematik at the RWTH Aachen University. The relative increase in performance was even stronger when using other systems, based on AMD chipsets, with more efficient memory architecture. For lack of memory these could not be used to run all tests.

We conclude that an implementation of the presented algorithm aiming at real applications must be implemented in C, C++ or a language alike. This will be a comparatively easy task with the provided Sage code at hand.

### References

[BO10] J. Brunier and K. Ono, *Heegner divisors, $l$-functions and harmonic weak maass forms*, Annals of Math. **172** (2010), 2135–2181.

[Bor95] R. Borcherds, *Automorphic forms on $O_{n+2,2}(R)$ and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213.

[Bor98] ——*, Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562.

[Bor99] ——*, The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), no. 2, 219–233.

[Bor00] ——*, Correction to: “The Gross-Kohnen-Zagier theorem in higher dimensions”, Duke Math. J. **105** (2000), no. 1, 183–184.

[Bru02] J. Bruinier, *Borcherds products on $O(2, 1)$ and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, vol. 1780, Springer-Verlag, Berlin, 2002.

[CG08] F. Clery and V. Gritsenko, *The Siegel modular forms of genus 2 with the simplest divisor*, arXiv:0812.3962v1 [math.NT], 2008.
[Der01] T. Dern, *Hermische Modulformen zweiten Grades*, Ph.D. thesis, RWTH Aachen University, Germany, 2001.

[DGPS10] W. Decker, G. Greuel, G. Pfister, and H. Schönemann, *Singular 3-1-1 — A computer algebra system for polynomial computations*, 2010, http://www.singular.uni-kl.de.

[DK04] T. Dern and A. Krieg, The graded ring of Hermitian modular forms of degree 2 over $\mathbb{Q}(\sqrt{-2})$, J. Number Theory 107 (2004), no. 2, 241–265.

[DM11] A. Dabholkar and D. Murthy, S. and Zagier, Quantum black holes, wall crossing, and mock modular forms, 2011, preprint.

[GN98] V. Gritsenko and V. Nikulin, Automorphic forms and Lorentzian Kac-Moody algebras II, Internat. J. Math. 9 (1998), no. 2, 201–275.

[Gre05] V. Gritsenko, Arithmetical lifting and its applications, Séminaire de théorie des nombres de Paris 1999-2000 (D. Sinnou, ed.), Cambridge University Press, 2001, pp. 103–126.

[HM98] J. Harvey and G. Moore, On the algebras of BPS states, Comm. Math. Phys. 197 (1998), no. 3, 489–519.

[HM10] B. Heim and A. Murase, Borcherds lifts on $\text{Sp}_2(\mathbb{Z})$, preprint, 2010.

[Ibu02] Tomoyoshi Ibukiyama, Vector-valued Siegel modular forms of $\text{Sym}(4)$ and $\text{Sym}(6)$, Sūrikaisekikenkyūshō Kōkyūroku (2002), no. 1281, 126 –140 (Japanese).

[JS05] D. Jatkar and A. Sen, Dyson spectrum in CHL models, Journal of High Energy Physics 2006 (2005), no. 04, 38.

[Kon97] M. Kontsevich, Product formulas for modular forms on $O(2, n)$ (after R. Borcherds), Astérisque (1997), no. 245, Exp. No. 821, 3, 41–56, Séminaire Bourbaki, Vol. 1996/97.

[Kri05] A. Krieg, The graded ring of quaternionic modular forms of degree 2, Math. Z. 251 (2005), no. 4, 929–942.

[May10] S. Mayer, Calculation of Hilbert Borcherds products, Experiment. Math. 19 (2010), no. 2, 243–256.

[OPY08] M. Oura, C. Poor, and D. Yuen, Towards the Siegel ring in genus four, Int. J. Number Theory 4 (2008), no. 4, 563–586.

[PY00] C. Poor and D. Yuen, Linear dependence among Siegel modular forms, Math. Ann. 318 (2000), no. 2, 205–234.

[PY09] ———, Paramodular cusp forms, arXiv:0912.0049 [math.NT], 2009.

[PY09b] C. Poor and D. Yuen, Linear dependence among Siegel modular forms, Math. Ann. 318 (2000), no. 2, 205–234.

[S*11a] W. Stein et al., Purple Sage, 2011, http://purple.sagemath.org/.

[S*11b] ———, Sage Mathematics Software (Version 4.6.1), The Sage Development Team, 2011, http://www.sagemath.org.

[Sch04] N. Scheithauer, Generalized Kac-Moody algebras, automorphic forms and Conway’s group. I, Adv. Math. 183 (2004), no. 2, 240–270.

[Sch08] ———, Generalized Kac-Moody algebras, automorphic forms and Conway’s group. II, J. Reine Angew. Math. 625 (2008), 125–154.

[Sko92] N. Skoruppa, Computations of Siegel modular forms of genus two, Math. Comp. 58 (1992), no. 197, 381–398.

[Sko08] ———, Jacobi forms of critical weight and Weil representations, Modular Forms on Schiermonnikoog (B. Edixhoven, G. v. d. Geer, and B. Moonen, eds.), Cambridge University Press, Cambridge, 2008.

[Yos07] H. Yoshida, Siegel’s modular forms and the arithmetic of quadratic forms, Invent. Math. 60 (1990), no. 3, 193–248.

[Yos07] ———, On generalization of the Shimura-Taniyama conjecture I and II, Proceedings of the 4-th Spring Conference on Modular Forms and Related Topics: Siegel modular forms and abelian varieties (T. Ibukiyama, ed.), Ryushido, 2007, pp. 1–26.
COMPUTING BORCHERDS PRODUCTS

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