Non-uniform Braneworld Stars: 
an Exact Solution 

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Abstract

In this paper the first exact interior solution to Einstein’s field 
equations for a static and non-uniform braneworld star with local and 
non-local bulk terms is presented. It is shown that the bulk Weyl 
scalar $\mathcal{U}(r)$ is always negative inside the stellar distribution, in con-
sequence it reduces both the effective density and the effective pres-
sure. It is found that the anisotropy generated by bulk gravity effect 
has an acceptable physical behaviour inside the distribution. Using a 
Reissner-Nördstrom-like exterior solution, the effects of bulk gravity 
on pressure and density are found through matching conditions.

keywords: General Relativity; Braneworld; Exact Solutions.

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1 Introduction

The consequences of the braneworld theory [1] in general relativity have been studied with great interest during the past years [2] (see also a review paper [3] and references therein). Its possible effects on our observable 4D universe have been extensively studied through cosmological scenarios [4] (see also [5] and references therein). The study of the consequences in astrophysics [6] has been mostly limited to the exterior region, even though it is well known that gravitational collapse could produce very high energies in the interior, where the braneworld corrections to general relativity would become significant [7] (see [8] for a recent study of black holes on the brane).

In the astrophysics context, the general fact that a single 5D solution can generate different scenarios in 4D [9], makes the analysis in the exterior region particularly attractive in searching for solutions beyond Schwarzschild’s. Hence several scenarios which might be useful in predicting observable effects from the extra dimension can be considered. On the other hand, the study on internal stellar structure in the braneworld remains unknown so far. The reason is very simple: the internal stellar structure is more complex that any other scenario, thus the high energy and nonlocal corrections to Einstein’s field equation, which in general lead to a complicated indefinite system of equations in the brane, produce an even more complicated system of equations in the interior. Hence the construction and eventual study of internal stellar solutions in the brane is a difficult process, except when uniform distributions are considered[1]. Consequently there are some important questions which remain without answer. For instance, the role played by density gradients as a source of Weyl stresses, and its eventual consequence on the gravitational collapse remain unknown. An internal consistent solution would be useful in order to consider these issues. However, in general, finding a consistent solution in the brane, no matter which scenario is being considered, represents a challenge which final answer require more information on the 5D geometry and the way on how our 4D spacetime is embedded in the bulk.

In the pioneer work of Germani and Maartens [10] exact solutions for a uniform distribution were found, where it can be seen how much more difficult it would be to find a solution except for uniform stellar distributions.
There is a large number of studies on the consistency of braneworld models in the brane. For instance, the approach developed in the series of papers by Mukohyama [11] has been extensively used in calculating gravity perturbations in a braneworld scenario. Early works on consistent linearized gravity [12], tests of consistency beyond linear order [13], stability and energy conservation [14], thus as low energy effective theory in the context of the braneworld [15]-[17], have been extensively used as well. On the other hand, and from the point of view of a brane observer, there is an issue which must be faced in the search of any consistent solution in the brane; namely, the non locality and non closure of the braneworld equations [18]-[22]. This is an open problem for which a solution requires a better understanding of the bulk geometry and proper boundary conditions. It is well known that the source of the non locality and non closure of the braneworld equations is directly related with the projection $E_{\mu\nu}$ of the bulk Weyl tensor on the brane. Several ways have been taken to overcome this problem, most of them based in restrictions on the tensor $E_{\mu\nu}$. For instance a restriction which has proven to be useful consists in discarding the anisotropic stress associated to $E_{\mu\nu}$ [24]. A different and more radical restriction on $E_{\mu\nu}$ consist in imposing the constraint $E_{\mu\nu} = 0$. However this condition, which was initially used in some papers, is incompatible with the Bianchi identity in the brane [25].

In this paper there will be no direct restrictions on $E_{\mu\nu}$. Instead of this approach, the method developed in Ref.[26] shall be used to solve the non closure problem of braneworld equations. This method, based in the fact that any stellar solution on the brane must have the general relativity solution as a limit, is used in this paper to generate the first exact and physically acceptable interior solution for a nonuniform stellar distribution having local and non-local bulk terms on the brane.

This paper is organized as follows. In Section 2 the Einstein field equations and matching conditions in the brane for a spherically symmetric distribution is reminded. In Section 3 a regular and physically acceptable exact internal solution is found. In Section 4 an analysis of the solution is carried out. In the last section the conclusions are presented.

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For a discussion about the role played by the boundary conditions and the gravitational instability on the brane see [23]
2 The field equations and matching conditions

The Einstein field equations on the brane may be written as a modification of the standard field equations [18, 19]

\[ G_{\mu\nu} = -8\pi T_{\mu\nu}^{T} - \Lambda g_{\mu\nu}, \]  

(1)

where \( \Lambda \) is the cosmological constant on the brane. The energy-momentum tensor has new terms carrying bulk effects onto the brane:

\[ T_{\mu\nu} \rightarrow T_{\mu\nu}^{T} = T_{\mu\nu} + \frac{6}{\sigma} S_{\mu\nu} + \frac{1}{8\pi} E_{\mu\nu}, \]

(2)

here \( \sigma \) is the brane tension. The new terms \( S_{\mu\nu} \) and \( E_{\mu\nu} \) are the high-energy corrections and \( KK \) corrections respectively, and are given by

\[ S_{\mu\nu} = \frac{1}{12} T_{\alpha}^{\alpha} T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T_{\nu}^{\alpha} \nu + \frac{1}{24} g_{\mu\nu} \left[ 3 T_{\alpha\beta} T_{\alpha\beta} - (T_{\alpha}^{\alpha})^2 \right], \]

(3)

\[ -8\pi E_{\mu\nu} = -\frac{6}{\sigma} \left[ \mathcal{U}(u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu}) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_{\mu} u_{\nu} \right], \]

(4)

being \( \mathcal{U} \) the bulk Weyl scalar and \( \mathcal{P}_{\mu\nu} \) and \( \mathcal{Q}_{\mu} \) the anisotropic stress and energy flux respectively.

We consider a spherically symmetric static distribution, hence \( Q_{\mu} = 0 \) and

\[ \mathcal{P}_{\mu\nu} = \mathcal{P}(r_{\mu} r_{\nu} + \frac{1}{3} h_{\mu\nu}), \]

(5)

where \( r_{\mu} \) is a unit radial vector and \( h_{\mu\nu} = g_{\mu\nu} - u_{\mu} u_{\nu} \) the projection tensor with 4-velocity \( u^{\mu} \). The line element is given in Schwarzschild-like coordinates by

\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

(6)

where \( \nu \) and \( \lambda \) are functions of \( r \).

The metric (6) has to satisfy (1). In our case with \( \Lambda = 0 \) we have:

\[ -8\pi \left( \rho + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{6}{k^4} \mathcal{U} \right) \right) = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda_1}{r} \right), \]

(7)
\[-8\pi \left( -p - \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) - \frac{4}{k^4} \mathcal{P} \right) \right) = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu_1}{r} \right), \tag{8}\]

\[-8\pi \left( -p - \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) + \frac{2}{k^4} \mathcal{P} \right) = \frac{1}{4} e^{-\lambda} \left[ 2\nu_{11} + \nu_1^2 - \lambda \nu_1 + 2 \left( \nu_1 - \lambda_1 \right) \right], \tag{9}\]

\[p_1 = -\frac{\nu_1}{2} (\rho + p), \tag{10}\]

where \( f_1 \equiv df/dr \) and \( k^2 = 8\pi \). The general relativity is regained when \( \sigma^{-1} \to 0 \) and (10) becomes a linear combination of (7)-(9).

The Israel-Darmois matching conditions at the stellar surface \( \Sigma \) give

\[[G_{\mu\nu} r^\nu]_{\Sigma} = 0 \tag{11}\]

where \([f]_{\Sigma} \equiv f(r) \mid_{R^+} - f(r) \mid_{R^-}\) Using (11) and the field equation (1) with \( \Lambda = 0 \) we have

\[[T_{\mu\nu} r^\nu]_{\Sigma} = 0, \tag{12}\]

which leads to

\[
\left[ \left( p + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) + \frac{4}{k^4} \mathcal{P} \right) \right]_{\Sigma} = 0. \tag{13}\]

This takes the final form

\[p_R + \frac{1}{\sigma} \left( \frac{\rho_R^2}{2} + \rho_R p_R + \frac{2}{k^4} U_R \right) + \frac{4}{k^4} \mathcal{P}_R = \frac{2}{k^4} \frac{U_R^+}{\sigma} + \frac{4}{k^4} \frac{\mathcal{P}_R^+}{\sigma}, \tag{14}\]

where \( f_R \equiv f(r) \mid_{r=R} \). The equation (14) gives the general matching condition for any static spherical braneworld star\(^5\). When \( \sigma^{-1} \to 0 \) we obtain the well known matching condition \( p_R = 0 \). In the particular case of the Schwarzschild exterior solution \( U^+ = P^+ = 0 \), the matching condition (14) becomes:

\[p_R + \frac{1}{\sigma} \left( \frac{\rho_R^2}{2} + \rho_R p_R + \frac{2}{k^4} U_R \right) + \frac{4}{k^4} \frac{\mathcal{P}_R}{\sigma} = 0. \tag{15}\]

\(^5\)The general matching conditions on the brane for a spherically symmetric vacuum region embedded into a cosmological environment can be seen in [20].
Thus the matching conditions do not have a unique solution on the brane.

It is easily seen that the field equations (7)-(9) can be written as

\[ e^{-\lambda} = 1 - \frac{8\pi}{r} \int_0^r r^2 \left[ \rho + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{6}{k^4} \mathcal{U} \right) \right] dr, \]  

(16)

\[ \frac{8\pi \mathcal{P}}{k^4 \sigma} = \frac{1}{6} \left( G_1^1 - G_2^2 \right), \]  

(17)

\[ \frac{6\mathcal{U}}{k^4 \sigma} = -\frac{3}{\sigma} \left( \frac{\rho^2}{2} + \rho p \right) + \frac{1}{8\pi} \left( 2G_2^2 + G_1^1 \right) - 3p \]  

(18)

with

\[ G_1^1 = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu_1}{r} \right), \]  

(19)

\[ G_2^2 = \frac{1}{4} e^{-\lambda} \left[ 2\nu_{11} + \nu_1^2 - \lambda_1 \nu_1 + 2 \left( \frac{\nu_1 - \lambda_1}{r} \right) \right]. \]  

(20)

The equation (16) actually represents an integral differential equation for the geometrical function \( \lambda(r) \), something completely different from the general relativistic case, and a direct consequence of the non locality of the braneworld equations. The only solution known for this equation is given as

\[ e^{-\lambda} = 1 - \frac{8\pi}{r} \int_0^r r^2 \rho dr + e^{-t} \int_0^r \frac{e^t}{(\nu_1^2 + 2)} \left[ H(p, \rho, \nu) + \frac{8\pi}{\sigma} (\rho^2 + 3\rho p) \right] dr, \]  

(21)

with

\[ H(p, \rho, \nu) \equiv 8\pi 3p - \left[ \mu_1 \left( \frac{\nu_1}{2} + \frac{1}{r} \right) + \mu (\nu_{11} + \frac{\nu_1^2}{2} + \frac{2\nu_1}{r} + \frac{1}{r^2}) - \frac{1}{r^2} \right], \]  

(22)

where

\[ I \equiv \int \frac{(\nu_{11} + \frac{\nu_1^2}{2} + \frac{2\nu_1}{r} + \frac{2}{r^2})}{(\frac{\nu_1}{2} + \frac{1}{r})} dr, \]  

(23)

and

\[ \mu \equiv 1 - \frac{8\pi}{r} \int_0^r r^2 \rho dr. \]  

(24)

The function \( H(p, \rho, \nu) \) measures the anisotropic effects due to bulk consequences on \( p, \rho \) and \( \nu \), and its physical meaning will be useful in the searching of exact solutions. This will be addressed in the next section.
3 An exact solution

So far we have the interior Weyl functions $\mathcal{P}$ and $\mathcal{U}$ plus the geometric function $\lambda(r)$, respectively given by (17), (18) and (21), and three unknown functions $\{p(r), \rho(r), \nu(r)\}$ satisfying one equation, namely, the conservation equation (10). Therefore it is necessary to prescribe additional information to close the system. First of all, to ensure the correct limit at low energies, the following constraint is imposed on the brane

$$H(p, \rho, \nu) = 0.$$  \hfill (25)

The constraint (25) has been proven to be useful in finding solutions which possesses general relativity as a limit [26], and has a clear physical interpretation: eventual bulk consequences on $p$, $\rho$ and $\nu$ do not produce anisotropic effects on the brane. This constraint ensures that a braneworld solution be consistent with general relativity, as is shown in Ref. [26]. Unfortunately this constraint is not enough, and one additional condition must be imposed. Therefore, from the point of view of a brane observer, many solutions are possible. However not all of these solutions are of physical interest. Hence the brane observer has to impose a condition in the brane which must lead to a physically acceptable solution, namely, regular at the origin, pressure and density defined positive, well defined mass and radius, monotonic decrease of the density and pressure with increasing radius, etc. All these conditions reduce enormously the possible solutions $(p, \rho, \nu)$ to (10) and (25), even more in the searching of an exact solution.

It might be logical to think of the Schwarzschild condition $e^\nu = e^{-\lambda}$ to close the system. However this would produce a very complicated integral differential equation for $\lambda$, as can be seen through (21). On the other hand, it is clear that to find an exact expression for the geometric function $\lambda(r)$, given by (21), an analytic solution for (23) is needed. Keeping this in mind, a huge and simple family of exact solutions for (23) is considered, given by

$$e^\nu = A(1 + C r^m)^n,$$  \hfill (26)

which is characterized by the constants $A$, $C$, $m$ and $n$. The constants $A$ and $C$ are eventually found through matching conditions, whereas $m$ and $n$
are parameters to be used in the searching of an exact solution for \( p(r) \) and \( \rho(r) \).

Using (26) in (10) the pressure is found in terms of the density

\[
p(r) = \frac{2B - \sqrt{ACm n} \int r^{m-1} (1 + C r^m)^{\frac{n}{2}-1} \rho(r) \, dr}{2 \sqrt{A} (1 + C r^m)^{\frac{n}{2}}}, \tag{27}
\]

with \( B \) a constant of integration.

Now using (26) and (27) in the constraint (25), the following integral equation for the density is obtained

\[
-\frac{(r - 8\pi \int_0^r r^2 \rho(r) \, dr) \left[ 2 + 2C \left( 2 + m n + m^2 n^2 \right) r^m \right] + C^2 \left( 2 + 2 m n + m^2 n^2 \right) r^{2m}}{2 r^3 (1 + C r^m)^2} \nonumber \\
-12\pi \left( -2B + \sqrt{ACm n} \int_0^r r^{-1+m} (1 + C r^m)^{-1+\frac{n}{2}} \rho(r) \, dr \right) \nonumber \\
-\frac{4\pi \left( 2 + C \left( 2 + m n \right) r^m \right) \left( \int_0^r r^2 \rho(r) \, dr - r^3 \rho(r) \right)}{r^3 (1 + C r^m)} + r^{-2} = 0. \tag{28}
\]

Hence \( \rho(r) \) can be found by (28) and then \( p(r) \) through (27). In this stage it is important to stress that the goal of this paper is to find a simple exact braneworld solution, hence we are not interested in the most general solution \( \rho(r) \) to the integral equation (28). Indeed, a particularly useful solution for (27) and (28), leading to an exact braneworld solution through (21), will be constructed. This is shown below.

The first step is to examine the complicated bulk contribution to \( \lambda(r) \) in (21). When the constraint (25) is imposed to ensure general relativity as a limit, the geometric function \( \lambda(r) \) underwent a great simplification, leaving only high energy corrective terms. Then when (26) is used, the complicated integral expression in (21) is reduced even more, leading to

\[
\int_0^r \frac{e^t}{\left( \frac{2t}{3} + \frac{2}{r} \right)} \left( \rho^2 + 3\rho p \right) \, dt = \int_0^r 2r^2 (1 + C r^m)^{n-1} \left[ 4 + C \left( 4 + m n \right) r^m \right]^{\frac{4 + n (m - 3)}{4 + m n}} \left( \rho^2 + 3\rho p \right) \, dr. \tag{29}
\]
Finding a physically acceptable exact solution for (27) and (28), which can produce an exact solution for (29), represents a difficult process. Hence the integral expression (29) has to be simplified as much as possible. Indeed, the integral (29) is tremendously simplified when \( 4 + n (m - 3) = 0 \), leading to

\[
\int_0^r \frac{e^I}{\left( \frac{\nu}{2} + \frac{2}{r} \right)} \left( \rho^2 + 3\rho p \right) dr = \int_0^r 2r^2 \left( 1 + C r^m \right)^{\frac{1}{m}} \left( \rho^2 + 3\rho p \right) dr. \tag{30}
\]

The next step will be to consider in (27) a simple ansatz for \( \rho(r) \) which is capable of producing a simple expression for the pressure, and then to use (28) to fix the parameters of the ansatz. The idea is to obtain a solution for both \( p(r) \) and \( \rho(r) \) as simple as possible such that the integral (30) has an analytic expression. If this is accomplished, the geometric function \( \lambda(r) \), given through (21), and both interior Weyl functions \( P \) and \( U \), given by (17) and (18) respectively, will have exact expressions.

By simple inspection of (27) it is easy to see that a convenient ansatz for \( \rho(r) \) can be written by

\[
\rho(r) = (1 + Cr^m)^{-(n/2+1)} \sum_{s=0} \alpha_s r^s. \tag{31}
\]

On the other hand, since (26) must be regular at the origin \( r = 0 \), \( m \) has to be positive. The constant \( n \), which is given by \( n = \frac{1}{3-m} \), is considered positive to obtain a pressure with a physically acceptable behaviour. Thus in our case \( m \) satisfies \( 0 < m < 3 \). Taking \( m = 2 \) and using (31) in (27), it is found that \( p(r) \) has a relatively simple expression free of special functions when \( 4 \)

\[
p(r) = \frac{(a_0 + a_2r^2 + a_4r^4)}{(1 + Cr^2)^3}. \tag{32}
\]

Hence the following expression for the pressure is found

\[
p(r) = \frac{C_0 + C_1 r^2 + C_2 r^4 + 2 (2 a_4 - a_2 C) (1 + Cr^2) \log(1 + Cr^2)}{C^2 (1 + C r^2)^3}, \tag{33}
\]

\(^4\)Using an algebraic manipulator, it is not complicated to realize that keeping a generic value for \( m \) and \( s \) produce a solution to (27) having special functions. This special functions eventually would make impossible an exact solution to \( \lambda(r) \) through (21).
where
\[ C_0 = 2a_4 - a_2 C + a_0 C^2 + \frac{BC^2}{\sqrt{A}}; \quad C_1 = \frac{BC^3}{\sqrt{A}} - 2a_4 C, \]
\[ C_2 = -2a_4 C^2. \] (34)

The expression (33) will hardly produce an exact function to (30) unless the logarithmic function be removed. Thus,
\[ a_4 = \frac{C}{2} a_2. \] (35)

Now using (32), the constraint (28) is written as
\[ \sqrt{C} \left( -24 C^2 + 66 a_2 \pi + 36 a_0 C \pi + \frac{24 BC \pi}{\sqrt{A}} \right) r + \sqrt{C} \left( -20 C^3 + 70 a_2 C \pi \right) r^3 \]
\[ - (9a_2 - 2a_0 C) \pi \left( 5 + 9 C r^2 \right) \arctan(\sqrt{C}r) = 0, \] (36)

hence it is found that
\[ a_0 = \frac{9C}{7\pi}; \quad a_2 = \frac{2C^2}{7\pi}; \quad a_4 = \frac{C^3}{7\pi}; \quad B = -\frac{12C\sqrt{A}}{7\pi}. \] (37)

Using (34) and (37) in (32) and (33), a simple and physically acceptable expression is found for both the pressure and density. Thus the solution for (10) and (25) is finally written as
\[ e^\nu = A(1 + C r^2)^4 \] (38)

and
\[ \rho(r) = \frac{C \left( 9 + 2Cr^2 + C^2 r^4 \right)}{7\pi (1 + C r^2)^4} \] (39)

and
\[ p(r) = \frac{2C(2 - 7Cr^2 - C^2 r^4)}{7\pi (1 + C r^2)^3}, \quad \text{(40)} \]
leaving \( A \) and \( C \) to be determined by matching conditions.

\[ ^5 \text{The expression for } \nu \text{ and } \rho \text{ were found in [29] for a perfect fluid in the context of } \text{general relativity. However the solution shown there is not physically acceptable due to } G^\nu_\nu \neq G^\rho_\rho. \]
Using \((38)-(40)\) in \((21)\) a regular and well defined solution for \(\lambda(r)\) is obtained
\[
e^{-\lambda(r)} = 1 - \frac{2\tilde{m}(r)}{r},
\]
where the interior mass function \(\tilde{m}\) is given by
\[
\tilde{m}(r) = m(r) - \frac{1}{\sigma} \left( \frac{2}{7} \right)^2 \frac{Cr}{2\pi} \left[ \frac{240 + 589Cr^2 - 25C^2r^4 - 41C^3r^6 - 3C^4r^8}{3(1 + Cr^2)^4(1 + 3Cr^2)} \right] - \frac{80}{(1 + Cr^2)^2 (1 + 3Cr^2) \sqrt{Cr}} \arctg(\sqrt{Cr}) \right],
\]
with \(m(r)\) being the general relativity interior mass function, given by the standard form
\[
m(r) = \int_0^r 4\pi r^2 \rho dr = \frac{4}{7} Cr^3 \frac{(3 + Cr^2)}{(1 + Cr^2)^2},
\]
hence the total general relativity mass is obtained
\[
M \equiv m(r) \bigg|_{r=R} = \frac{4}{7} CR^3 \frac{(3 + CR^2)}{(1 + CR^2)^2},
\]
where \(R\) is the radius of the distribution.

Using \((17)\) and \((18)\) a regular solution for the interior Weyl functions is obtained
\[
\mathcal{P}(r) = \frac{32}{441r^3(1 + Cr^2)^6(1 + 3Cr^2)^2} \left[ Cr \left( 180 + 2040Cr^2 + 8696C^2r^4 + 16533C^3r^6 + 12660C^4r^8 + 146C^5r^{10} - 120C^6r^{12} + 9C^7r^{14} \right) - 60\sqrt{C}(1 + Cr^2)^3(3 + 26Cr^2 + 63C^2r^4)\arctg(\sqrt{Cr}) \right],
\]
\[
\mathcal{U}(r) = \frac{32}{441r(1 + Cr^2)^6(1 + 3Cr^2)^2} \left[ C^2r \left( 795 + 4865Cr^2 + 10044C^2r^4 + 6186C^3r^6 - 373C^4r^8 - 219C^5r^{10} - 18C^6r^{12} \right) - 240C^{3/2}(1 + Cr^2)^3(5 + 9Cr^2)\arctg(\sqrt{Cr}) \right].
\]
The expressions \((38)-(40)\) with \((45)\) and \((46)\) represent an exact analytic solution to the system \((7)-(10)\).
4 Analysis of the solution

As can be seen by figure 1, the scalar function \( U(r) \) is always negative inside the stellar distribution, with a maximum negative value at the origin \( r = 0 \). This situation may be explained through the general expression for \( U(r) \), given by Eq. (18). It shows two "sources" for \( U(r) \): the first kind given by the first two terms in the second hand side of Eq. (18), which are high energy corrections, always negative. The second kind is given by the remaining terms, which clearly represent an anisotropic expression, which is always positive. Hence when the anisotropy projected onto the brane is not high enough, the dominant high energy terms produce a negative scalar function \( U(r) \), which is the case presented here. This negative scalar function reduces both the effective density and the effective pressure, as can be seen by the field equations (7)-(9). On the other hand, the anisotropy inside the braneworld star is shown by the figure 2. It increases until reaches a maximum value, then decreases until \( \mathcal{P} = 0 \) at \( r = 0 \). This is directly connected with the correction for \( \lambda \) proportional to high energy terms shown in (21). This correction is the only bulk effect underwent by the metric when the constraint (25) is imposed, therefore it represents the only source for \( \mathcal{P} \), as can be clearly seen through (17).

The bulk contribution to \( p, \rho \) and \( \nu \) is found by matching conditions, where the assumption of vanishing pressure at the surface will be dropped [27], [28]. As Schwarzschild is not the only possible static exterior solution, we have many scenarios to consider. For instance let us consider the Schwarzschild exterior solution

\[
e^{\nu^+} = e^{-\lambda^+} = 1 - \frac{2M}{r}; \quad U^+ = \mathcal{P}^+ = 0.
\] (47)

The matching condition \([ds^2]_{\Sigma} = 0\) at the stellar surface \( \Sigma \) yields

\[
A = (1 - \frac{2M}{R})(1 + CR^2)^{-4},
\] (48)

\[
\frac{2M}{R} = \frac{2M}{R} - \frac{1}{\sigma} \left( \frac{2}{7} \right)^2 \frac{C}{\pi} \left[ \frac{240 + 589CR^2 - 25C^2R^4 - 41C^3R^6 - 3C^4R^8}{3(1 + CR^2)^4(1 + 3CR^2)} \right]
\]
\[
\frac{-80}{(1 + CR^2)^2 (1 + 3CR^2)\sqrt{CR}} \arctg(\sqrt{CR}) \right]
\]

and using (15) it is found that \( C \) must satisfy the condition

\[
\pi R [168CR^2 + 252(CR^2)^2 - 1848(CR^2)^3 - 4032(CR^2)^4 - 2352(CR^2)^5 - 252(CR^2)^6] \\
+ \frac{1}{\sigma} CR [240 + 2749CR^2 + 5276(CR^2)^2 - 266(CR^2)^3 - 372(CR^2)^4 - 27(CR^2)^5] \\
- \frac{1}{\sigma} 240\sqrt{C\arctg(\sqrt{CR})} (1 + 11CR^2 + 19C^2R^4 + 9C^3R^6) = 0,
\]

hence solving (50) \( C \) is found as a function of the brane tension \( \sigma \).

In order to find the bulk contribution to \( p \) and \( \rho \) we need to find \( C \) satisfying (50). To carry out this the following solution is proposed

\[
C = C_0 + \delta,
\]

where \( C_0 \) is the general relativity value of \( C \), given by

\[
C_0 = \frac{\sqrt{57} - 7}{2R^2},
\]

which is found using the condition \( p(R) = 0 \) in (40). In this sense \( \delta \) represents the ” bulk perturbation” of the general relativity value of \( C \). Using (51) in (50) we have at first order in \( \sigma^{-1} \)

\[
\delta = \frac{-4 \left[ (-236357 + 31281\sqrt{57}) R - 120 (-9235 + 1223\sqrt{57}) \arctan(\sqrt{\frac{7 + \sqrt{57}}{2}}) \right]}{3\pi \sigma R^5 \left(-1261105 + 167083\sqrt{57} \right)}
\]

The pressure can thus be found expanding \( p(C) \) around \( C_0 \)

\[
p(C_0 + \delta) = p(C_0) + \delta \frac{dp}{dC} \bigg|_{C=C_0},
\]

which leads to

\[
p(r) = \frac{2C_0 (2 - 7C_0r^2 - C_0^2r^4)}{7\pi (1 + C_0r^2)^3} + \frac{4}{7\pi} \frac{(1 - 9C_0r^2 + 2C_0^2r^4)}{(1 + C_0r^2)^4} \delta.
\]

By the same way the density is found to be

\[
\rho(r) = \frac{C_0 (9 + 2C_0r^2 + C_0^2r^4)}{7\pi (1 + C_0r^2)^3} + \frac{1}{7\pi} \frac{(9 - 14C_0r^2 + C_0^2r^4)}{(1 + C_0r^2)^4} \delta.
\]
However at the surface and for any arbitrary $R$ always we have

$$p(R) = \frac{4}{7\pi} \frac{(1 - 9C_0R^2 + 2C_0^2R^4)}{(1 + C_0R^2)^4}\delta < 0. \tag{57}$$

Hence the Schwarzschild exterior solution is incompatible with the interior solution found here. Thus a different exterior solution must be considered.

Using now the Reissner-Nördstrom-like solution given in [30]

$$e^{\nu^+} = e^{-\lambda^+} = 1 - \frac{2M}{r} + \frac{q}{r^2}, \tag{58}$$

$$U^+ = -\frac{P^+}{2} = \frac{4}{3} \frac{\pi q\sigma}{r^4}, \tag{59}$$

and considering the matching condition $[ds^2]_\Sigma = 0$ at the stellar surface $\Sigma$, we have

$$A(1 + CR^2)^4 = 1 - \frac{2M}{R} + \frac{q}{R^2}, \tag{60}$$

$$\frac{2M}{R} = 2M - \frac{1}{\sigma} \left( \frac{2}{7} \right)^2 \frac{C}{\pi} \left[ \frac{240 + 589CR^2 - 25C^2R^4 - 41C^3R^6 - 3C^4R^8}{3(1 + CR^2)^4(1 + 3CR^2)} \right. \left. - \frac{80}{(1 + CR^2)^2(1 + 3CR^2)\sqrt{CR}} \right] + \frac{q}{R^2}, \tag{61}$$

and using (14) we obtain

$$q = \frac{-4R}{147(1 + CR^2)^5(1 + 3CR^2)} \left[ C \left( -2 + CR^2 + 22C^2R^4 + 3C^3R^6 \right) \right. \left. 84\pi R^2(1 + CR^2)^2 + \frac{1}{\sigma} \left( -240 - 2749CR^2 - 5276C^2R^4 + 266C^3R^6 \right. \right. \left. + 372C^4R^8 + 27C^5R^{10} \right) \right. \left. + \frac{1}{\sigma} 240\sqrt{C}(1 + CR^2)^2(1 + 9CR^2)\arctg(\sqrt{CR}) \right]. \tag{62}$$

The constants $\mathcal{M}$ and $q$ are given in terms of $C$ through equations (61) and (62) respectively, and $C$ may be determined by (60) if $A$ is kept as a free parameter, which can be used to find a physically acceptable model. However
we have to be aware of the fact that $A$ has a well defined general relativity value, named $A_0$, which is given by (60) at $\sigma^{-1} = 0$

$$A_0(1 + C_0 R^2)^4 = 1 - \frac{2M}{R}.$$  \hfill (63)

In this sense the free parameter associate to $A$, which will be used to obtain an acceptable model, is the "bulk perturbation" of $A$ given through

$$A = A_0 + \varepsilon.$$  \hfill (64)

Using (51) and (64) in (60) we obtain

$$(A_0 + \varepsilon)[1 + (C_0 + \delta) R^2]^4 = 1 - \frac{2M}{R} + \frac{q}{R^2}.$$  \hfill (65)

Evaluating the expressions (61) and (62) at $C = C_0 + \delta$ and keeping linear terms in $\sigma^{-1}$, the equation (65) leads to the explicit form to the bulk perturbation of $C$, which is written as

$$\delta(\sigma) = \frac{7}{4} \left[ (1 + C_0 R^2)^3 [\alpha(\sigma) - (1 + C_0 R^2)^4 \varepsilon] \right],$$  \hfill (66)

with

$$\alpha(\sigma) = \frac{1}{\sigma} \left( \frac{2}{\pi} \right)^2 \frac{C_0}{\pi} \left[ \frac{240 + 589 C_0 R^2 - 25 C_0^2 R^4 - 41 C_0^3 R^6 - 3 C_0^4 R^8}{3 (1 + C_0 R^2)^2 (1 + 3 C_0 R^2)} \right]$$

$$- \frac{80 \arctg(\sqrt{C_0 R})}{(1 + C_0 R^2)^2 (1 + 3 C_0 R^2) \sqrt{C_0 R}}.$$  \hfill (67)

Hence giving by hand the perturbation $\varepsilon$ underwent by $A$ due to the extra dimension, it is possible to obtain $\delta$ and thus the bulk consequences on $p$ and $\rho$ through (55) and (56). The figure 3 shows the behaviour of the pressure in both the general relativity and braneworld case. It can be seen that the bulk gravity effect reduces the pressure deep inside the distribution, but the situation changes for the exterior layers, where the matching conditions lead to $p \neq 0$ at the surface.
Figure 1: The scalar function $U(r)$ for a distribution with $R = 5$. $U(r)$ is always negative in the interior, hence it reduces both the effective density and effective pressure.

Figure 2: Behaviour of the anisotropy $\mathcal{P}(r)$ inside the stellar distribution with $R = 5$. 

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Figure 3: Qualitative comparison of the pressure $p(r)$ in general relativity ($p(R) = 0$) and the braneworld model ($p(R) \neq 0$) with $R = 5$.

5 Conclusions

In the context of the braneworld, a spherically symmetric, static and non-uniform stellar distribution with Weyl stresses was studied. The method developed in Ref. [26], which is based in the fact that any stellar solution on the brane must have the general relativity solution as a limit, was used to overcome the non locality and non closure of the braneworld equations. Hence there was no direct restriction on the projected $\mathcal{E}_{\mu\nu}$ Weyl tensor on the brane, which is a method usually used.

By prescribing the temporal metric component $g_{00}$, the first exact and physically acceptable interior solution to Einstein’s field equations for a static and non-uniform braneworld star was found. It was shown that this solution is incompatible with the Schwarzschild’s exterior metric. Using the Reissner-Nördstrom-like solution given in Ref. [30], the effects of bulk gravity on pressure and density were found through matching conditions, where the assumption of vanishing pressure at the stellar surface was dropped. It was found that the bulk gravity effect reduces the pressure deep inside the distribution, but the situation changes for the exterior layers as a direct consequence of matching conditions, in agreement with the previous study in
It was found that the Weyl scalar function $U(r)$ is always negative inside the stellar distribution. In consequence it reduces both the effective density and the effective pressure. On the other hand, the anisotropy inside the braneworld star, which is directly connected with the deformation underwent by $\lambda$ due to bulk gravity effects, has an acceptable physical behaviour.

The exact solution found in this paper was possible as a direct consequence of the constraint $H(p, \rho, \nu) = 0$. This essentially allows us to simplify the solution for the geometric function $\lambda(r)$ by eliminating some anisotropic effects on the brane. As can be seen by Eq. (17), the source of $\mathcal{P}$ is the deformation underwent by the geometric functions $\lambda(r)$ and $\nu(r)$ due to bulk consequences on the brane. However when the constraint $H(p, \rho, \nu) = 0$ is imposed, the bulk effect on $\nu(r)$ does not produce any anisotropic consequence, leaving thus the deformation underwent by $\lambda(r)$ as the only source of anisotropy on the brane. Furthermore, this constraint reduces the deformation of $\lambda(r)$, leaving a corrective term proportional to high-energy effects of bulk gravity, as can be clearly seen through the equation (21). Since the constraint $H(p, \rho, \nu) = 0$ removes all possible sources of anisotropy except for the corrective term of $\lambda(r)$ proportional to high-energy effects, it follows that the anisotropic effect of the bulk on the brane is reduced to its minimal expression, hence the constraint imposed represents a condition of minimal anisotropy on the brane.

The condition of minimal anisotropy, represented by the constraint $H(p, \rho, \nu) = 0$, is not only a direct path to avoid the loss of the general relativity limit, but also a natural way to reduce the degrees of freedom on the brane. This is an enormous simplification which has proven to be useful in searching physically relevant exact solution on braneworld. Furthermore, this method works equally for analytic as well for numerical methods. Hence the condition of minimal anisotropy might help in the search of not only analytic physically acceptable, but also numerical solutions when a non-uniform distribution is considered. Thus the role played by the density gradients as a source of Weyl stresses in the interior could be studied.

The work developed in this paper represents the point of view of a brane observer. Hence the solution found here, a physically acceptable one, does
not ensure that the bulk eventually constructed will not be plagued with singularities or any other problem. However, since the condition of minimal anisotropy on the brane ensures the correct limit at low energies, it could be used when the bulk configuration is investigated, thus some general features of the five dimensional bulk might be elucidated\textsuperscript{6}. This is currently been investigated.

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