Cosmological Constant as Vacuum Energy Density of Quantum Field Theories on Noncommutative Spacetime

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Abstract

We propose a new approach to understand hierarchy problem for cosmological constant in terms of considering noncommutative nature of space-time. We calculate that vacuum energy density of the noncommutative quantum field theories in nontrivial background, which admits a smaller cosmological constant by introducing an higher noncommutative scale $\mu_{NC} \sim M_p$. The result $\rho_\Lambda \sim 10^{-6} \Lambda_{SUSY}^8 M_p^4 / \mu_{NC}^8$ yields cosmological constant at the order of current observed value for supersymmetry breaking scale at 10TeV. It is the same as Banks’ phenomenological formula for cosmological constant.

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It is great puzzle to understand the hierarchy problem on current observed tiny cosmological constant, $\rho_\Lambda \sim 10^{-12}\text{eV}^4$. The puzzle may be stated by identification between cosmological constant and vacuum energy density of quantum field theories (QFT),

$$\rho_\Lambda \sim \int d^3\vec{k} \omega_{\vec{k}} \sim \int_0^{\Lambda^2} dk^2 k^2 = \Lambda^4.$$  \hspace{1cm} (1)

For $\Lambda = M_p$ with reduced Plank scale $M_p = (8\pi G)^{-1/2} \approx 2.4 \times 10^{18}\text{GeV}$ it exhibits huge hierarchy of $M_p^4/\rho_\Lambda \sim 10^{120}$. Usually one may introduce supersymmetry to improve this hierarchy problem while the identification between cosmological constant and vacuum energy of QFT is insisted. In this scenario the contributions to vacuum energy from bosons and fermions are exactly cancelled each other. Unfortunately, current particle physics experiments reject the supersymmetry at scale below TeV. Then naive treatment $\rho_\Lambda = \Lambda_{SUSY}^4$ with $\Lambda_{SUSY}$ the scale of supersymmetry breaking still produces too large cosmological constant. However, to discuss the cosmological constant problem, we need to bring gravity into the picture. Banks\cite{1}, therefore, suggested a phenomenological formula to link the scale of supersymmetry breaking and the Plank mass to the cosmological constant,

$$\rho_\Lambda \sim \Lambda_{SUSY}^4 \left(\frac{\Lambda_{SUSY}}{M_p}\right)^4.$$  \hspace{1cm} (2)

This formula yields the cosmological constant at the order of current observed value for $\Lambda_{SUSY}$ at TeV. So far, however, it is not well-understood what is theoretical origination of (2). Alternatively one may further introduce a mechanism which greatly suppresses the contribution from high energy in Eq. (1). This mechanism is expected to somehow exhibits some basic properties of quantum gravity or of spacetime. The purpose of this letter is just to propose one of such kind of mechanism by studying that vacuum energy density of QFT at noncommutative spacetime (NCQFT),

$$[x^\mu, x^\nu] = \epsilon^{\mu\nu}/\mu_{NC}^2,$$  \hspace{1cm} (3)

where $\mu_{NC}$ is a scale to specify the noncommutative effect and the components of antisymmetric tensor $\epsilon^{\mu\nu}$ takes the values 1, $-1$ or 0. We shall show that this mechanism together with supersymmetry predicts the formula (2).

The action for field theories on noncommutative spaces can be obtained using the Weyl-Moyal correspondence\cite{2}, according to which, a traditional treatment is to replace the usual product in field theory by the so-called star-product\cite{3, 4}. In this present letter, however,
we will use another more convenient realization of algebra (1) by reordering operators in the plane wave basis [5]

\[ e^{iq \cdot x} e^{ik \cdot x} = e^{-\frac{i}{2} q \times k} e^{i(q+k) \cdot x}, \]

where

\[ q \times k = -k \times q = \frac{1}{\mu_{NC}^2} \epsilon^{\mu \nu} q_\mu k_\nu. \]

Following Eq. (4) we have the properties

\[
\int d^4x e^{iq \cdot x} e^{ik \cdot x} = \delta^{(4)}(q+k),
\]

\[
\int d^4x e^{iq \cdot x} e^{ip \cdot x} e^{ik \cdot x} = \int d^4x e^{ip \cdot x} e^{iq \cdot x} e^{ik \cdot x}.
\]

It has been shown that the most of NCQFTs are renormalizable at one-loop level at least [6, 7]. In this present letter, we consider the simplest model that describes a free scalar particle moving in four-dimensional noncommutative curved spacetime. Its action is specified by

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi \phi \right] + S_{EH}, \]

with \( S_{EH} \) the standard Einstein-Hilbert action. For sake of convenience, we will work in Euclidean signature via substitution \( x^0 \to i x^4 \) and set that \( \epsilon^{12} = \epsilon^{34} = -\epsilon^{21} = -\epsilon^{43} = 1 \) while other components of \( \epsilon^{\mu \nu} \) are set to zero.

The generating functional is obtained by path integral over \( \phi \):

\[ Z[g, m^2] = \mathcal{N} \int [d\phi(x)] e^{-S} = \mathcal{N} \exp \left\{ -\frac{1}{2} \text{Tr} \ln \left( -\partial_\mu (\sqrt{g} g^{\mu \nu} \partial_\nu + m^2 \sqrt{g}) \right) + S_{EH} \right\}, \]

where \( \mathcal{N} \) is an infinite constant from path integral over of canonical momentum of \( \phi \) and “Tr” is a trace over noncommutative space-time. To be precise, Eq. (8) can be rewritten as follows by using Eq. (4) and (6),

\[ \ln Z[g, m^2] - S_{EH} = -\frac{1}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \sqrt{g(x)} e^{-ik \cdot x} \mathcal{D}(x; k) e^{ik \cdot x} \]

\[ = -\frac{1}{2} \int d^4x \frac{d^4y}{(2\pi)^4} \frac{d^4w}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \sqrt{g(w)} \delta^{(4)}(w - x) \]

\[ \cdot e^{-ik \cdot x} \delta^{(4)}(y - x) e^{ik \cdot x} \mathcal{D}(y; k) \]

\[ = -\frac{1}{2} \int d^4w \frac{d^4y}{(2\pi)^4} \int d^4k d^4q \frac{d^4q}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \sqrt{g(w)} e^{i(q-y \cdot w)} e^{-i(q \times k \cdot y)} \mathcal{D}(y; k), \]

[1] Our discussions can be extended to free fields with diverse spin. The results will be same besides that an extra factor \(-1\) is contributed by fermions.
where $D(x; k) = \ln \left\{ \sqrt{g(x)} (g^{\mu\nu}(x) k_\mu k_\nu + m^2) + \partial_\mu (\sqrt{g(x)} g^{\mu\nu}(x)) k_\nu \right\} / \Lambda^2$ with $\Lambda$ the intrinsic UV cut-off of QFT which is defined by $N = \exp \left\{ \frac{1}{2} \ln \Lambda^2 \text{ TrI} \right\}$ with $\text{I}$ identity operator. In Eq. (9) we use the plane wave expansion, $\langle k \mid x \rangle = e^{ik \cdot x}$ and $\delta^{(4)}(x-y) \sim \int d^4q \ e^{iq \cdot (x-y)}$. It is valid, as the argument below, in a region whose size is much smaller than cosmological event horizon since our current observed universe is rather flat.

The “true” vacuum amplitude should be obtained by further integral over metric field $g$, i.e., the degrees of freedom of quantum gravity. So far, unfortunately, we do not know how to construct a consistent theory of quantum gravity in four dimensions and how to perform the integral over its degrees of freedom. Alternatively, we may assume that the spacetime geometry is described by classical solution of Einstein equation at the distance much larger than Plank length, i.e., $|x-y| \gg 1/M_p$ or $q^2 \ll M_p^2$. For example, we can use the de Sitter (static) metric with Euclidean rotation,

$$ds^2 = \left( 1 - \frac{r^2}{L^2} \right) d\tau^2 + \left( 1 - \frac{r^2}{L^2} \right)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (10)$$

The spacetime described by the above metric is very flat in a region whose size is much smaller than the event horizon $L$. Hence Eq. (9) may be split into two parts,

$$\ln Z[g, m^2] = -\frac{1}{2} \int_L^M d^4x \int d^4y \frac{1}{(2\pi)^4} \int_0^{M_p} |q|^3 d|q| \int d^4k \frac{e^{iq \cdot (x-y)}}{(2\pi)^4} e^{-iq \cdot k} \ln \frac{k^2 + m^2}{\Lambda^2} + \Gamma_s[g, m, M_p], \quad (11)$$

where $\int_L^M d^4y$ denotes to cut integral over $y$ at the event horizon and $\Gamma_s[g, m, M_p]$ specifies the short-distance contribution which contains the degrees of freedom of quantum gravity. The UV cut-off at Plank scale in integral over $q$ is significant. It partly reflects effects of quantum gravity, and should be distinguish with cut-off $\Lambda$, which by definition is characteristic scale of QFT, e.g., the supersymmetry breaking scale $\Lambda_{SUSY}$. Then the vacuum amplitude is explicitly specified by

$$Z_{\text{vac}}[m^2, M_p] = \int [dg(x)] Z[g, m^2]. \quad (12)$$

If we ignore the contribution from quantum gravity, the vacuum energy density from NCQFT
reads off² 

\[ \rho_0 = -\frac{1}{(4\pi)^2} \int_L \frac{d^4 y}{(2\pi)^4} \int_0^{M_p} |q^3 d| |q| \int \frac{d^4 k}{(2\pi)^4} e^{i q y} e^{-i q \cdot k} \ln \frac{k^2 + m^2}{\Lambda^2} \]

\[ \rightarrow \int_L \frac{d^4 y}{(2\pi)^4} \int_0^{M_p} |q^3 d| |q| \int \frac{d^4 k}{(2\pi)^4} e^{i q y} \int_0^\infty \frac{dl}{l} \exp \left\{ -l(k^2 + m^2) - \frac{1}{4\Lambda^2} \right\}. \] (13)

To integrate over the \( k \) we obtain

\[ \rho_0 = \int_L \frac{d^4 y}{(2\pi)^4} \int_0^{M_p} |q^3 d| |q| \int \frac{dl}{16\pi^2 l} \times \exp \left\{ -l \left( \frac{q^2}{4\mu_{NC}^4} + \frac{1}{\Lambda^2} \right) - \frac{m^2}{l} + iq \cdot y \right\}. \] (14)

The factor \( \frac{q^2}{4\mu_{NC}^4} + \frac{1}{\Lambda^2} \) is known to denote the UV/IR-mixing phenomenon, which is typical properties of NCQFT². Finally to perform all integrals in Eq. (14) we get:

\[ \rho_0 = \frac{\mu_{NC}^8}{32\pi^4} \int_0^\infty \frac{dl}{l} \int_L \frac{d^4 y}{(2\pi)^4} \left[ 1 - (1 + lM_p^2)e^{-lM_p^2} \right] \exp \left\{ -\frac{4\mu_{NC}^4 l - \frac{m^2}{4\mu_{NC}^4} - \frac{y^2}{4l} \right\} \]

\[ \simeq \frac{\Lambda^4}{512\pi^4} \int_0^\infty dl \left[ 1 - (1 + l\kappa)e^{-ln} \right] \exp \left\{ -l - \frac{m^2}{l\Lambda^2} \right\} \]

\[ = \frac{m^2\Lambda^2}{256\pi^6} \left( K_2 \frac{2m}{\Lambda} - \frac{1}{1 + \kappa}K_2 \frac{2m}{\Lambda} \sqrt{1 + \kappa} - \frac{m}{\Lambda} \frac{\kappa}{(1 + \kappa)^{3/2}}K_3 \frac{2m}{\Lambda} \sqrt{1 + \kappa} \right), \] (15)

where \( K_2, K_3 \) are modified Bessel function of the second kind and \( \kappa = \Lambda^2 M_p^2/(4\mu_{NC}^4) \). For \( \kappa \gg 1 \) we just recover the result of usual QFT, \( \rho_0 \sim \Lambda^4 \) for arbitrary value of \( m \leq \Lambda \). For \( \kappa \ll 1 \), however, from the second line of Eq. (15) we have

\[ \rho_0 \simeq \frac{\Lambda^4}{512\pi^6} \int_0^\infty dl \left[ 1 - (1 + l\kappa)(1 - l\kappa + \frac{l^2\kappa^2}{2} + O(\kappa^3)) \right] \exp \left\{ -l - \frac{m^2}{l\Lambda^2} \right\} \]

\[ \simeq \frac{\kappa^2\Lambda^4}{1024\pi^6} \int_0^\infty dl \frac{l^3}{l^3} \exp \left\{ -l - \frac{m^2}{l\Lambda^2} \right\} + O(\kappa^3) \]

\[ \simeq \frac{6\Lambda^8 M_p^4}{216\pi^6 \mu_{NC}^8} \left( 1 - \frac{m^2}{3\Lambda^2} + O\left( \frac{m^4}{\Lambda^4} \right) \right) + O(\kappa^3). \] (16)

If we believe the noncommutativity is intrinsic property of spacetime, we should expect \( \mu_{NC} \sim M_p \). Then for \( \Lambda = M_p \) we obtain usual result \( \rho_\Lambda \sim M_p^4 \). However, if we expect our world is supersymmetric at higher energy, one has \( \Lambda \sim \Lambda_{SUSY} \) such that

\[ \rho_\Lambda \sim 10^{-6} \frac{\Lambda_{SUSY}^8}{M_p^4}. \] (17)

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² Precisely we should further add an IR cut-off in integral over \( q \), i.e., \( q^2 \geq H^2 \) with \( H \) the Hubble constant, by incorporating the fact that the size of our universe is finite. Since result is proportional to \( M_p^2 - H^2 \) we ignore the Hubble constant in the result.
This is precise formula (2) of Banks’ phenomenological hypothesis on cosmological constant [1, 9]. For lower supersymmetry breaking scale at 10 TeV, Eq. (17) reproduces current observed value of cosmological constant.

Naively the results (16) and (17) should be doubted since our traditional experience tell us that the physics inside very short distance should not exhibit observable effects at low energy. In other words, the results deduced from NCQFT should return to one of commutative QFT at the limit \( \mu_{NC} \to \infty \). However, it is not case of our result. This phenomenon is well-known in NCQFT that \( \mu_{NC} \to \infty \) is not always a smooth limit to commutative case due to the presence of UV/IR-mixing [5, 8]. Technically, it is basic property of so-called non-planar diagrams in NCQFT while planar diagrams reduce to usual results of commutative QFT. In our treatment of Eq. (9), however, we can not consistently introduce “planar”-like terms when metric as external source is presented. In other words, it is by definition “non-planar”-like. The basic physics behind non-planar diagrams is that they exhibit non-local properties of NCQFT, in which the physics inside the very short-distance produces observable effects at long-distance. Therefore, the crucial point hidden in our calculations is the UV/IR-mixing.

In conclusion, in this letter we suggested that the noncommutativity of spacetime may suppress vacuum energy density of QFT to understand the cosmological constant problem. We studied the model describing free particles in the noncommutative spacetime. In our treatment, the spacetime noncommutativity as well as quantum gravity play significant roles. Although the later can not be dealt with in detail, it introduces a characteristic scale, the Plank scale, into our model. It implies that our model has partly included the effect of quantum gravity. It is rather interesting our model reproduces Banks’ phenomenological formula for cosmological constant. For supersymmetry breaking scale at 10 TeV, we obtain current observed value of cosmology constant.

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