Abstract

Obvious view of distribution function of Markovian random evolution is found in terms of Bessel functions of $n + 1$-th order.

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1 Introduction.

Markovian random evolutions in $\mathbb{R}^n$ were studied in the work [1], where the connection between the equation for the functional of the evolution and Bessel equation of high order was found.

Bessel equations of high order and their solutions - Bessel functions of high order - were studied by Turbin and Plotkin in [2].

In the works [3,4] Orsingher and Sommella managed to combine these two results and to receive the obvious view for the distribution functions of random evolutions in $\mathbb{R}^2$ and $\mathbb{R}^3$. In [4] they also made a conjecture about the view of the distribution function of the random evolution in $\mathbb{R}^n$.

In this paper we generalize the results of Orsingher and Sommella and prove their conjecture.

Let us start with the definition of Markovian random evolution. To do this we have to define a regular $n + 1$-hedron inscribed into a unit sphere in $\mathbb{R}^n$.

Definition 1.1: We call figure in $\mathbb{R}^n$ a regular $n + 1$-hedron inscribed into a unit sphere if the following conditions are true:
1. The figure has \( n + 1 \) vertices \( \tau_0, \ldots, \tau_n \) situated at the unit sphere with the center in \((0, \ldots, 0)\).

2. The center of masses of the figure is situated in \((0, \ldots, 0)\).

3. The \( i \)-th vertex is situated into \( i + 1 \)-dimensional subspace, determined by the first \( i + 1 \) coordinates of \( R^n \).

Having this definition, we may easily find the coordinates of the vertices. The components of \( \tau_i, i = 0, n \) are equal to
\[
\tau_i^j = \begin{cases} 
-\frac{1}{n} \sqrt{\frac{n(n+1)}{(n-j+1)(n-j+2)}}, & j < i + 1 \\
\sqrt{\frac{(n+1)(n-i)}{n(n-i+1)}}, & j = i + 1 \\
0, & j > i + 1
\end{cases}, j = 1, n.
\]

Really, the first condition is true, because the distance from every vertex to the zero point is equal to 1.

The second condition is true, because the sum of first components of all coordinates is equal to 0, the same for second components, etc.

At last, for the \( i \)-th vertex \( i + 2, i + 3, \ldots \) components of coordinates are equal to 0. So the third condition is also true.

We are now ready to define Markovian random evolution in \( R^n \) from [1].

**Definition 1.2:** We call a random process \( \overline{S}(t) \) Markovian random evolution in \( R^n \) if:

1. The process begins its motion at the point \( \overline{x} = (x_1, \ldots, x_n) \).

2. The initial direction of the motion is \( \overline{\tau}_i, i = 0, n \), where the components of coordinates of \( \overline{\tau}_i \) are presented in (1).

3. The time of the motion at some direction is distributed like \( e^{-\lambda t} \).

4. The \( k \)-th direction is followed by the \( k + 1 \)-th direction (the \( n \)-th one is followed by 0).

5. The velocity of the particle’s motion is equal to \( v \).

The system of backward Kolmogorov equations and corresponding \( n+1 \)-dimensional equation for the functionals (here \( i = 0, n \) is the start direction)
\[
u_i(\overline{x}, t) = E_i(f(\overline{S}(t))
\]
were also studied in [1].

Using a well-known result from the theory of Markovian processes we may receive the system of differential equations for the distribution functions
\[
f_j(x_1, \ldots, x_n, t)dx_1 \ldots dx_n = P\{S_1(t) \in dx_1, \ldots, S_n(t) \in dx_n, D(t) = j\},
\]
where \( D(t) \) is the direction at time \( t \).

The system of equations for the distribution functions (3) is adjoint to the system for the functionals (2) (see [1]):
\[
\begin{align*}
\frac{\partial}{\partial t} f_0(\overline{x}, t) &= -\lambda f_0(\overline{x}, t) - v(\overline{x}_0, \nabla) f_0(\overline{x}, t) + \lambda f_n(\overline{x}, t) \\
\frac{\partial}{\partial t} f_1(\overline{x}, t) &= -\lambda f_1(\overline{x}, t) - v(\overline{x}_1, \nabla) f_1(\overline{x}, t) + \lambda f_0(\overline{x}, t) \\
&\vdots \\
\frac{\partial}{\partial t} f_n(\overline{x}, t) &= -\lambda f_n(\overline{x}, t) - v(\overline{x}_n, \nabla) f_n(\overline{x}, t) + \lambda f_{n-1}(\overline{x}, t)
\end{align*}
\]
The corresponding $n + 1$-dimensional equation is:

$$
\prod_{i=0}^{n} \left( \frac{\partial}{\partial t} + \lambda + v(\tau_i, \nabla) \right) f(\tau, t) = \lambda^{n+1} f(\tau, t),
$$

where $f$ is any of functions $f_0, \ldots, f_n$.

To describe the support of the distribution of random evolution we use the notion of regular $n + 1$-hedron from Definition 1.1.

One of the most important characteristics of random evolution that differs it from Brownian motion is that evolution with probability 1 moves into a regular $n + 1$-hedron

$$
T_{vt} = \left\{ x_1, \ldots, x_n : \frac{-vt}{n} < x_1 < vt, \frac{1}{n-k+1} \left[ \frac{(n-1) \ldots (n-k+1)}{(n+1) \ldots (n-k+3)} x_1 + \ldots + \sqrt{\frac{(n-k+1)}{(n-k+3)}} x_{k-1} - \sqrt{\frac{(n-1) \ldots (n-k+1)}{(n+1) \ldots (n-k+3)}} x_k < \ldots \right] < x_k < \ldots < x_n \right\}.
$$

The probability that evolution is on the edge $\partial T_{vt}$ of $n + 1$-hedron is equal to:

$$
P\{S(t) \in \partial T_{vt}\} = e^{-\lambda t} + \lambda t e^{-\lambda t} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}.
$$

Here $e^{-\lambda t}$ is the probability of being at time $t$ on the vertices of $T_{vt}$, $\frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}, k = 1, \ldots, n - 1$ is the probability of being on $k$-dimensional edge of $T_{vt}$.

So, the continuous part of the distribution we are to find has the property:

$$
\int_{T_{vt}} \ldots \int_{T_{vt}} f(x_1, \ldots, x_n; t) dx_1 \ldots dx_n = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} - \ldots - \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}.
$$

In other words, we have to find a non-negative continuous function that satisfies system (4) and condition (7).

## 2 Derivation of the equations.

Making exponential transformation $f_j = e^{-\lambda t} g_j, j = 0, \ldots, n$ we receive from (4):

$$
\begin{align*}
\frac{\partial}{\partial t} g_0(\tau, t) &= -v(\tau_0, \nabla) g_0(\tau, t) + \lambda g_n(\tau, t) \\
\frac{\partial}{\partial t} g_1(\tau, t) &= -v(\tau_1, \nabla) g_1(\tau, t) + \lambda g_0(\tau, t) \\
&\vdots \\
\frac{\partial}{\partial t} g_n(\tau, t) &= -v(\tau_n, \nabla) g_n(\tau, t) + \lambda g_{n-1}(\tau, t)
\end{align*}
$$
Theorem 2.1 The functions \( g_j, j = 0, n \) are solutions of the equation
\[
\frac{\partial g}{\partial y_1 \ldots \partial y_{n+1}} = \left( \frac{\lambda}{v} \right)^{n+1} \left( \frac{\sqrt{n}}{\sqrt{n+1}} \right)^{n+1} \left( \frac{1}{2^{n+2} n} \right)^{n+1} g, \tag{9}
\]
where
\[
y_1 = \frac{vt}{n} + x_1, \]
\[
y_k = \frac{1}{n-k+1} \left[ \sqrt{(n-1) \ldots (n-k+1)} vt - \sqrt{(n-1) \ldots (n-k+3)} x_1 - \ldots - \sqrt{n-k+1} x_{k-1} \right] + x_k, k = 2, n, \]
\[
y_{n+1} = \sqrt{(n-1) \ldots 1} \left[ vt - \sqrt{(n-1) \ldots 3} x_1 - \ldots - \sqrt{3} x_{n-1} - x_n. \tag{10}\right.
\]

Proof. By applying the transformation (10) to the system (8) we receive:
\[
\begin{aligned}
\frac{n+1}{n} v \frac{\partial g_0}{\partial y_0} &= \lambda g_n \\
\sqrt{(n+1)(n-k+1)} v \frac{\partial g_k}{\partial y_{k+1}} &= \lambda g_{k-1}, k = 1, n-1 \\
\sqrt{2(n+1)} v \frac{\partial g_n}{\partial y_{n+1}} &= \lambda g_{n-1}.
\end{aligned}
\]
By differentiation we easily have (9).

Theorem proved.

Theorem 2.2 The transformation \( z = \sqrt[n+1]{y_1 \ldots y_{n+1}} \) converts the equation (9) into \( n+1 \)-dimensional Bessel equation:
\[
\left( z \frac{\partial}{\partial z} \right)^{n+1} g = \left( \frac{\lambda}{v} \right)^{n+1} \left( \sqrt{n(n+1)} \right)^{n+1} \left( \frac{1}{2^{n+2} n} \right)^{n+1} z^{n+1} g, \tag{11}\]
where \( g \) is any of the functions \( g_j, j = 0, n \).

Proof. We don’t show the proof since plain calculations are involved.

The solutions of equation (11) were found in [2]. We use here one of the functions that is solution of (11):
\[
I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v \, 2^{n+2} n + 2} z \right) = \sum_{k=0}^{\infty} \left( \frac{\lambda \sqrt{n(n+1)}}{v \, 2^{n+2} n + 2} \frac{1}{n+1} \right)^{n+1} \frac{1}{(k!)^{n+1}} . \tag{12}\]

Remark 2.1 The proofs that this function is solution of (11) in case of \( n = 2, 3 \) may also be found in [3,4].

Making the backward change of variables in (12) and exponential transformation we receive the solution of system (4).
3 Probability distribution.

Now we are ready to formulate the main result of the article:

**Theorem 3.1** The absolutely continuous component of the distribution of Markovian random evolution in $\mathbb{R}^n$ is:

$$f(x_1, \ldots, x_n; t) = \frac{\left(\frac{\sqrt{n}}{v} e^{-\lambda t}\right)^n}{(\sqrt{n} + 1)^{n+1} \rho^n} \left[ \lambda^n + \lambda^{n-1} \frac{\partial}{\partial t} + \ldots + \right.$$

$$+ \frac{\partial^n}{\partial t^n} \right] I_{0,n} \left( \frac{\lambda}{v} \frac{\sqrt{n(n+1)}}{2n+2} \sqrt{y_1 \ldots y_{n+1}} \right),$$

(13)

here and later $y_i, i = 1, n + 1$ are the linear combinations of $t, x_i, i = 1, n$ pointed in (10).

*Proof.* The fact that (13) is a non-negative function may be easily seen from the view of the series (12).

The function (13) satisfies the equation (5). Really, $I_{0,n} \left( \frac{\lambda}{v} \frac{\sqrt{n(n+1)}}{2n+2} \times \sqrt{y_1 \ldots y_{n+1}} \right)$, where $y_i, i = 1, n + 1$ are pointed in (10) satisfies equation (9). Any linear combination of derivatives of the last function with the coefficient \(\frac{\left(\frac{\sqrt{n}}{v}\right)^n}{(\sqrt{n}+1)^{n+1} \rho^n}\) also satisfies (9). So, if we make backward exponential transformation, we receive a solution of (5).

The last problem is to prove the condition (7) for the function $f(x_1, \ldots, x_n; t)$.

We first note that

$$\int \ldots \int_{T_{vt}} I_{0,n} \left( \frac{\lambda}{v} \frac{\sqrt{n(n+1)}}{2n+2} \sqrt{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n =$$

$$= \sum_{k=0}^{\infty} \left( \frac{\lambda}{(n+1)v} \right)^{(n+1)k} \frac{(\sqrt{n(n+1)})^{n+1} k}{(2n+2)^k (k!)^{n+1}} \int_{-vt}^{vt} \frac{1}{n^k [vt + nx_1]^k dx_1 \ldots}$$

$$- \left[ \frac{\sqrt{(n-1) \ldots 2}}{(n+1) \ldots 4} x_1 + \ldots + \sqrt{(n-1) \ldots 2} x_{n-2} - \sqrt{(n-1) \ldots 2} vt \right]$$

$$\frac{1}{2^k} \left[ - \frac{(n-1) \ldots 2}{(n+1) \ldots 4} x_1 - \ldots - \right.$$

$$\left[ \frac{2}{4} x_{n-2} + \sqrt{(n-1) \ldots 4} vt + 2x_{n-1} \right]^k$$

$$\times \quad \frac{1}{3 \sqrt{(n-1) \ldots 3} x_1 + \ldots + \sqrt{(n-1) \ldots 3} x_{n-1} - \sqrt{(n-1) \ldots 3} vt} \left[ - \frac{(n-1) \ldots 1}{(n+1) \ldots 3} x_1 - \right.$$
\[-\cdots - \sqrt{\frac{1}{3}} x_{n-1} + \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}}^2 - x_n^2\right]^k \, dx_n. \quad (14)\]

The inner, first, integral could be found using the change of variables \(x_n = -\sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}} x_1 - \cdots - \sqrt{\frac{1}{3}} x_{n-1} + \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}}\) \(z\). We have:

\[-\left[\sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}} x_1 + \cdots + \sqrt{\frac{1}{3}} x_{n-1} - \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}}\right] \int \left(\sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}} x_1 - \cdots - \sqrt{\frac{1}{3}} x_{n-1} + \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}}\right)^2 \, dx_n = \left(-\sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}} x_1 - \cdots - \sqrt{\frac{1}{3}} x_{n-1} + \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}}\right)^{2k+1} \frac{2^{2k+1}(k)!^2}{(2k+1)!}.\]

In the following, second, integral

\[-\int \frac{1}{2k} \left[-\sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 4 vt}} x_1 - \cdots - \sqrt{\frac{2}{4}} x_{n-2} + \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 4 vt}}\right]^{k} \left[-\sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}} x_1 - \cdots - \sqrt{\frac{1}{3}} x_{n-1} + \sqrt{\frac{\left(\prod_{i=1}^{n-1}(1)\ldots 1\right)}{(n+1)\ldots 3 vt}}\right]^{2k+1} \frac{2^{2k+1}(k)!^2}{(2k+1)!}.\]
The following integrals could be found in the same way. For the $m$-th integral we have:

\[
\left( \frac{m - 1}{m + 1} \right)^{m(k + 1) - 1} \Gamma(m(k + 1)) \Gamma(k + 1) \left( \frac{m + 1}{m} \right)^{(m+1)(k+1)-1} \Gamma((m + 1)(k + 1)) \times
\]
\[
\times \left( -\sqrt{\frac{(n-1)\ldots m}{(n+1)\ldots (m+2)}} x_1 - \ldots - \sqrt{\frac{m}{m+2}} x_{n-m} + \frac{\sqrt{(n-1)\ldots m}}{\sqrt{(n+1)\ldots (m+2)}} v \right)^{(m+1)(k+1)-1}.
\]

Substituting the integrals found into (14) and performing simple calculations we receive:

\[
\int \ldots \int_{\tau_{vt}} I_{0,n} \left( \frac{\lambda}{v} \sqrt{\frac{n(n+1)}{2n+2}} \sqrt{\frac{y_1\ldots y_{n+1}}{\lambda^2}} \right) = \left( \frac{v}{\lambda} \right)^n \left( \frac{\sqrt{n+1}}{(\sqrt{n})^n} \right) x_1 \ldots x_n, m = \frac{1}{n}.
\]

To do this let us find

\[
\frac{\partial}{\partial t} \int \ldots \int_{\tau_{vt}} I_{0,n} \left( \frac{\lambda}{v} \sqrt{\frac{n(n+1)}{2n+2}} \sqrt{\frac{y_1\ldots y_{n+1}}{\lambda^2}} \right) dx_1 \ldots dx_n = \frac{\partial}{\partial t} \int_{-\frac{v}{\lambda}}^{vt} \ldots
\]

\[
- \left[ \sqrt{\frac{(n-1)\ldots m}{(n+1)\ldots (m+2)}} x_1 - \ldots - \sqrt{\frac{(n-1)\ldots m}{(n+1)\ldots (m+2)}} v \right] - \left[ \sqrt{\frac{(n-1)\ldots m}{(n+1)\ldots (m+2)}} x_1 - \ldots - \sqrt{\frac{(n-1)\ldots m}{(n+1)\ldots (m+2)}} v \right] \times \frac{\lambda}{v} x_1 \ldots x_n - dx_1 \ldots dx_n.
\]

The main rule for such integrals is: we find the derivative of the above limit and multiply it on the integral in which we substitute the above limit instead of \(x_1\), then we subtract the derivative of the down limit multiplied on the integral in which we substitute the down limit instead of \(x_1\). On so on for every integral.

But in our case when we substitute the above limit into the next integral it becomes equal to 0 - this may be easily seen using (6). Otherwise, when we substitute the down limit the argument of Bessel function becomes equal to 0 - this may be seen from (6) and (10). But it’s clear that \(I_{0,n}(0) = 1\), so the integrand will be always equal to 1.

In other words, we have the following equality:

\[
\frac{\partial}{\partial t} \int \ldots \int_{\tau_{vt}} I_{0,n} \left( \frac{\lambda}{v} \sqrt{\frac{n(n+1)}{2n+2}} \sqrt{\frac{y_1\ldots y_{n+1}}{\lambda^2}} \right) dx_1 \ldots dx_n =
\]
\[
\frac{\partial}{\partial t} \int_{T_{vt}} \ldots \int \frac{\partial}{\partial t} I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n.
\]

But the first integral is the derivative of \( VolT_{vt} \). We may easily find that \( VolT_{vt} = \frac{(\sqrt{n+1})^{n+1}}{(\sqrt{n})^n(n-1)!} v^n t^{-1} \).

So,

\[
\frac{\partial}{\partial t} \int_{T_{vt}} \ldots \int I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n =
\]

\[
= \frac{(\sqrt{n+1})^{n+1}}{(\sqrt{n})^n(n-1)!} v^n t^{-1} + \int_{T_{vt}} \ldots \int \frac{\partial}{\partial t} I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n.
\]

For the next derivative we have:

\[
\frac{\partial^2}{\partial t^2} \int_{T_{vt}} \ldots \int I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n =
\]

\[
= \frac{(\sqrt{n+1})^{n+1}}{(\sqrt{n})^n(n-2)!} v^n t^{-2} + \int_{T_{vt}} \ldots \int \frac{\partial}{\partial t} I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n.
\]

We’ll find the last integral by the rule we mentioned above, but here the above limit makes the following integral equal to 0, and when we substitute the down limit into the integrand we have \( \frac{\partial}{\partial t} I_{0,n}(0) = 0 \). So,

\[
\frac{\partial^2}{\partial t^2} \int_{T_{vt}} \ldots \int I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n =
\]

\[
= \frac{(\sqrt{n+1})^{n+1}}{(\sqrt{n})^n(n-2)!} v^n t^{-2} + \int_{T_{vt}} \ldots \int \frac{\partial^2}{\partial t^2} I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n.
\]

By analogy, for the \( k \)-th derivative we have \( (k = 3, n) \):

\[
\frac{\partial^k}{\partial t^k} \int_{T_{vt}} \ldots \int I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n =
\]

\[
= \frac{(\sqrt{n+1})^{n+1}}{(\sqrt{n})^n(n-k)!} v^n t^{-k} + \int_{T_{vt}} \ldots \int \frac{\partial^k}{\partial t^k} I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n.
\]

Using the formulas found, we receive:

\[
\int_{T_{vt}} \ldots \int \left[ \lambda^n + \lambda^{n-1} \frac{\partial}{\partial t} + \ldots + \frac{\partial^n}{\partial t^n} \right] I_{0,n} \left( \frac{\lambda \sqrt{n(n+1)}}{v^{2n+2}\sqrt{2n+2}} \sqrt[n]{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n =
\]
\[
= v^n \frac{\left(\sqrt{n} + 1\right)^{n+1}}{\left(\sqrt{n}\right)^n} \left[ e^{-\lambda t} - 1 - \lambda t - \ldots - \frac{(\lambda t)^{n-1}}{(n-1)!} \right].
\]

And finally,
\[
\frac{(\sqrt{n})^n}{(\sqrt{n} + 1)^{n+1} v^n} e^{-\lambda t} \int \ldots \int_{T_{vt}} \left[ \lambda^n + \lambda^{n-1} \frac{\partial}{\partial t} + \ldots + \frac{\partial^n}{\partial t^n} \right] I_{0,n} \left( \frac{\lambda}{v} \times \frac{n(n+1)}{2n+\sqrt{2n} + 2} \sqrt{y_1 \ldots y_{n+1}} \right) dx_1 \ldots dx_n = 1 - e^{-\lambda t} - \ldots - \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}.
\]

Theorem is proved.

Remark 3.1 In the work [4] Orsingher and Sommella conjectured that the normalizing constant in (13) should be equal to \( \frac{(\sqrt{n})^n}{(\sqrt{n} + 1)^{n+1} v^n} = \frac{v^n}{Vol T_{vt}} \). As we’ve seen it is not surprising because this constant appears when we find the integral \( \int \ldots \int_{T_{vt}} 1 dx_1 \ldots dx_n = Vol T_{vt} \).

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