Cumulants of products of Normally distributed random variables

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Abstract
To find moments of various estimators related to Autoregressive models of Statistics, one first needs the cumulants of products of two Normally distributed random variables. The purpose of this article is to derive the corresponding formulas, and extend them to products of three or more such variables.

1 Introduction
The formulas presented in this article are crucial for finding moments (and, subsequently, approximate distributions) of various parameter estimators related to AR(k) models (see [1] and [2]).

1.1 Multivariate Normal distribution
Assume that $X_1, X_2, \ldots$ are centralized (having zero mean), Normally distributed random variables. Their moment-generating function is given by

$$\exp \left( \frac{X^T V X}{2} \right)$$

where $V$ is the corresponding variance/covariance matrix. Based on (1), we can easily find the expected value of a product of any number of such random variables (which defines the corresponding moment), getting zero when this number is odd, and

$$E(X_1 X_2 \ldots X_{2k}) = \mu_{(1,2,\ldots,2k)} = \sum_{i \in P_{2k}} V_{i_1,i_2} V_{i_3,i_4} \ldots V_{i_{2k-1},i_{2k}}$$

when this number is even. The summation is over all possible $\binom{2k}{2,2,\ldots,2}/k!$ ways of pairing the $n$ indices, regardless of the order (both within and between) of the resulting pairs. Thus, for example, $\{1,2\}, \{3,4\}$ and $\{4,3\}, \{2,1\}$ are
considered identical and appear only once, but \{1, 2\}, \{3, 4\} and \{1, 3\}, \{2, 4\} are distinct.

Notationally (and rather symbolically), this has been indicated by

\[ i \in \mathcal{P}_{2k} \]

where \( i \) represents the \{\( i_1, i_2 \}\}, \{\( i_3, i_4 \}\}, \ldots \) indices and \( \mathcal{P}_{2k} \) the set of all such selections.

**Example**

\[
\mu_{\{1,2,3,4\}} = E(X_1X_2X_3X_4) = V_{1,2}V_{3,4} + V_{1,3}V_{2,4} + V_{1,4}V_{2,3} \quad \square \quad (3)
\]

Note that the resulting formulas are fully general; (3) can be read as

\[
\mu_{\{j_1,j_2,j_3,j_4\}} = V_{j_1,j_2}V_{j_3,j_4} + V_{j_1,j_3}V_{j_2,j_4} + V_{j_1,j_4}V_{j_2,j_3}
\]

but using specific integers simplifies the notation; also, the formulas allow any duplication of indices, e.g.

\[ \mu_{\{1,1,2,2\}} = V_{1,1}V_{2,2} + 2V_{1,2}^2 \]

It is important to realize that, for random variables with zero mean (the variables of this section), there is no difference between simple and central moments; our \( \mu \) thus stands for either.

### 1.2 Joint cumulants

Consider a collection of random variables (not necessarily centralized, nor Normally distributed), say \( Y_1, Y_2, \ldots, \) and their joint moment-generating function, defined by

\[
M(t_1, t_2, \ldots) = E[\exp(t_1Y_1 + t_2Y_2 + \ldots)] \quad (4)
\]

The corresponding joint cumulant of \( Y_1, Y_2, \ldots Y_\ell \) is defined by

\[
\kappa_{1,2,\ldots,\ell} \equiv \left. \frac{\partial^\ell}{\partial t_1 \partial t_2 \ldots \partial t_\ell} \ln[M(t_1, t_2, \ldots)] \right|_{t_1=t_2=\ldots=t_\ell=0} \quad (5)
\]

It is well known (and easy to derive) that \( \kappa_i = E(Y_i) \), and that all higher-order cumulants can be expressed in terms of moments (of the \( Y \) variables) thus:

\[
\kappa_{1,2,\ldots,\ell} = \sum_{j_1,j_2,\ldots,j_\ell \in \mathcal{A}(1,2,\ldots,\ell)} (-1)^{i-1}(i-1)!\mu_{j_1}\mu_{j_2}\cdots\mu_{j_\ell} \quad (6)
\]

where \( \mathcal{A}(1,2,\ldots,\ell) \) is the collection of all partitions of the \( 1,2,\ldots,\ell \) indices. A partition is a division of a set into an arbitrary number of non-empty and non-overlapping subsets (these are denoted \( j_1, j_2, \ldots \)). For example, \( \mathcal{A}(1,2,3) \) consists of \( j_1 = \{1,2,3\}, j_1,j_2 = \{1\}, \{2,3\}, j_1,j_2 = \{2\}, \{1,3\}, j_1,j_2 = \{3\}, \{1,2\}, \) and \( j_1,j_2,j_3 = \{1\}, \{2\}, \{3\} \).
There are two points to make about (6):

- the formula is correct regardless of whether the moments are central or simple (from now on, we denote central moments by $\mu_j$ and simple moments by $\tilde{\mu}_j$, respectively),
- using central moments simplifies the RHS substantially - any partition containing a single index can be omitted (the corresponding $\mu_j$ is zero).

We will spell out explicitly the first few of these formulas, first using simple moments

$$\kappa_{1,2} = \tilde{\mu}_{\{1,2\}} - \tilde{\mu}_{\{1\}} \tilde{\mu}_{\{2\}}$$

$$\kappa_{1,2,3} = \tilde{\mu}_{\{1,2,3\}} - \tilde{\mu}_{\{1,2\}} \tilde{\mu}_{\{3\}} - \tilde{\mu}_{\{1,3\}} \tilde{\mu}_{\{2\}} - \tilde{\mu}_{\{2,3\}} \tilde{\mu}_{\{1\}} + 2\tilde{\mu}_{\{1\}} \tilde{\mu}_{\{2\}} \tilde{\mu}_{\{3\}}$$

... 

then, using central moments (note the simplification):

$$\kappa_{1,2} = \mu_{\{1,2\}}$$

$$\kappa_{1,2,3} = \mu_{\{1,2,3\}}$$

$$\kappa_{1,2,3,4} = \mu_{\{1,2,3,4\}} - \mu_{\{1,2\}} \mu_{\{3,4\}} - \mu_{\{1,3\}} \mu_{\{2,4\}} - \mu_{\{1,4\}} \mu_{\{2,3\}}$$

To continue, we consider only the special case of having all indices identical (the corresponding general formulas get too lengthy):

$$\kappa_{1,1,1,1,1,1,1,1,1,1} = \mu_{\{1,1,1,1,1,1,1,1,1,1\}} - 10\mu_{\{1,1,1,1\}} \mu_{\{1,1\}}$$

$$\kappa_{1,1,1,1,1,1,1,1,1,1} = \mu_{\{1,1,1,1,1,1,1,1,1,1\}} - 15\mu_{\{1,1,1,1,1,1\}} \mu_{\{1,1,1\}} - 10\mu_{\{1,1,1,1\}}^2 + 30\mu_{\{1,1\}}^3$$

... 

Similarly to moments, cumulants are fully symmetric in the permutation of indices, e.g. $\kappa_{1,2,3,4}$ is the same as $\kappa_{4,1,3,2}$ etc.

We now proceed to derive explicit formulas for these cumulants when some of the $Y$ variables are centralized, Normally distributed (the $X$’s of the previous section - we call them ‘singlets’), and the others are products of two such $X$’s (doublets). This poses a bit of a notational challenge; we will use simple indices for singlets, two indices in parentheses for doublets. For example, $\kappa_{1,2,(3,4)}$ indicates a third-order cumulant of three random variables, $X_1$, $X_2$ and $X_3X_4$.

## 2 Cumulants involving singlets and/or doublets

It is well known and easy to derive, by combining (11) and (5), that all cumulants involving only singlets are equal to zero, with the exception of

$$\kappa_{1,2} = V_{1,2}$$
For doublets, one can derive (by combining (6) and (2), and using a routine, ‘brute-force’ computation - see the Appendix), that

\[ \kappa_{(1,2), (3,4), \ldots} = V_{1,2} \]
\[ \kappa_{(1,2), (3,4), (5,6)} = V_{1,3}V_{2,4} + V_{1,4}V_{2,3} \]
\[ \kappa_{(1,2), (3,4), (5,6)} = V_{2,3}V_{4,5}V_{1,6} + V_{2,4}V_{3,5}V_{1,6} + V_{2,3}V_{4,6}V_{1,5} + V_{2,4}V_{3,6}V_{1,5} + V_{2,5}V_{3,6}V_{1,4} + V_{2,6}V_{3,5}V_{1,4} + V_{2,5}V_{4,6}V_{1,3} + V_{2,6}V_{4,5}V_{1,3} \]

and

\[ \kappa_{(1,2), \ldots, (2k-1,2k)} = \sum_{i \in \mathcal{P}_{2k}} V_{i_1i_2}V_{i_3i_4}\ldots V_{i_{2k-1}i_{2k}} \]  

in general, where \( \mathcal{P}_{2k} \) indicates the set of all index pairings (the old \( \mathcal{P}_{2k} \)) which do not contain any of the original \( \{1,2\} \) or \( \{3,4\} \) or \( \{2k-1,2k\} \) pairs. There is a simple scheme for building \( \mathcal{P}_{2k} \):

- Start with the original pairs, e.g. \( \{1,2\}, \{3,4\}, \{5,6\} \),
- keeping the first pair fixed, go over all \( (k-1)! \) permutations of the remaining pairs, e.g. \( \{1,2\}, \{5,6\}, \{3,4\} \),
- keeping the first pair fixed, go over all \( 2^{k-1} \) interchanges of indices in the remaining pairs, e.g. \( \{1,2\}, \{6,5\}, \{3,4\} \), etc. (four of them in this case),
- shift each resulting arrangement by one index, e.g. \( \{2,6\}, \{5,3\}, \{4,1\} \).

One can thus see that the number of terms on the RHS of (8) is \((k-1)! \times 2^{k-1}\), which equals to 2, 8, 48 when \( k = 2, 3 \) and 4 respectively (a fast-growing sequence).

### 2.1 Mixed cases

Let us now investigate cumulants with a mixture of singlets and doublets. The rules for computing these prove to be quite simple:

A cumulant involving

- one singlet (regardless of the number of doublets) equals to zero,
- more than two singlets (and any number of doublets) is also equal to zero.
- When a cumulant contains two singlets, it equals to the cumulant in which the singlets are replaced by one doublet, i.e.

\[ \kappa_{1,2, (3,4), \ldots, (2k-1,2k)} = \kappa_{(1,2), (3,4), \ldots, (2k-1,2k)} \]
3 Beyond doublets

The same approach enables us to develop formulas for cumulants which may also involve triplets, quadruplets, etc. These are not needed when dealing with parameter estimation related to AR(k) models, but may have a potential application elsewhere.

Thus, for example, odd-order cumulants involving only triplets are all equal to zero; for even-orders we get

\[ \kappa_{(1,2,3),(4,5,6)} = \mu_{\{1,2,3,4,5,6\}} \]
\[ \kappa_{(1,2,3),(4,5,6),(7,8,9),(10,11,12)} = \mu_{\{1,2,3,4,5,6,7,8,9,10,11,12\}} - \mu_{\{1,2,3,4,5,6\}}\mu_{\{7,8,9,10,11,12\}} - \mu_{\{1,2,3,4,5,7,8,9\}}\mu_{\{6,10,11,12\}} \]
\[ \kappa_{(1,2,3),(4,5,6),(7,8,9),(10,11,12)} = \mu_{\{1,2,3,4,5,6,7,8,9,10,11,12\}} - \mu_{\{1,2,3,4,5,6\}}\mu_{\{7,8,9,10,11,12\}} - \mu_{\{1,2,3,4,5,7,8,9\}}\mu_{\{6,10,11,12\}} - \mu_{\{1,2,3,4,5,8,9,10\}}\mu_{\{6,7,11,12\}} + 2\mu_{\{1,2,3,4\}}\mu_{\{5,6,7,8\}}\mu_{\{9,10,11,12\}} \]

resulting in 3, 96 and 9504 terms, respectively.

One can then deal with ‘mixed’ cumulants in the same manner. There is of course no limit as to their total number; we will give just one example:

\[ \kappa_{\{1,2,3,4,5,6,7,8\}} = \mu_{\{1,2,3,4,5,6,7,8\}} - \mu_{\{1,2,3,4,5,6\}}\mu_{\{7,8\}} - \mu_{\{1,2,3,4,5,7,8\}}\mu_{\{6\}} + 2\mu_{\{1,2,3,4\}}\mu_{\{5,6,7,8\}}\mu_{\{9,10,11,12\}} \]

resulting in a sum of 7848 terms.
4 Appendix

For readers familiar with Mathematica, we now supply a few Mathematica functions to facilitate the computation of cumulants involving centralized, Normally distributed random variables and their products.

4.1 Finding moments

Computing the mean value (MV) of a product of centralized, Normally distributed random variables can be achieved by:

\[
\text{MV} := \text{Module}\{\text{a0 = Sort[Flatten[\{\text{a}\}]}, \text{a1, a2, a3}\},
\text{a1 = Length[a0]; a2 = Table[\{a0[[1]], a0[[i]]\}, \{i, 2, a1\}];}
\text{a3 = Table[Delete[a0, \{\{1\}, \{i\}\}], \{i, 2, a1\}];}
\text{Which[Mod[a1, 2] == 1, 0, a1 == 2, Apply[V, a0], True,}
\text{Apply[Plus, Table[Apply[V, a2[[i]]] MV[a3[[i]]], \{i, a1 - 1\}]]}// \text{Expand}\}
\]

To use it, we type

\[
\text{MV[2, 5, 2, 5, 2, 8]/.V[i, j] -> V[i, j]}
\]

where the integers in square brackets represent the corresponding indices, listed in any order and allowing for any amount of duplication.

This returns the following answer:

\[
3V_{2,2}V_{5,5}V_{2,8} + 6V_{2,0}^2V_{2,8} + 6V_{2,2}V_{2,5}V_{5,8}
\]

4.2 Computing cumulants

This time our goal is a bit more ambitious: we want to build a Mathematica program for computing cumulants of any number of singlets, doublets, triplets, etc. of centralized, Normally distributed random variables.

We start by constructing a few auxiliary Mathematica functions:

\[
\text{aux1[\_\_] := Module[\{\text{b1 = Union[\{\text{a}\}], b2, b3}\},}
\text{b2 = Map[Position[\{\text{a}\}, \#][[1, 1]] &, b1];}
\text{b3 = Length[b2]; Table[\{\text{b[[b2[[i]]]], Delete[b, b2[[i]]]\}, \{i, b3\}\}]]}
\]

\[
\text{aux2[a\_, B\_] := Module[\{\text{b1 = Map[Join[\{\text{a[[1]]}, \#]\} &, B[[1]] - 1]}\],}
\text{KSubsets[Drop[a, 1], B[[1]] - 1]],}
\text{Flatten[Map[aux3[a, \#, B[[2]] \&, b1], 1]]\}]
\]

\[
\text{aux3[a\_, b\_, B\_] := Map[Join[\{\text{b}\}, \#, \&, aux4[Complement[\{\text{a}\}, \text{b}\], B]]}
\]

\[
\text{aux4[a\_, b\_ := Module[\{\text{B = aux1[b]}\}, If[Length[b] == 1, \{\text{a}\},}
\text{Apply[Join, Table[aux2[a, B[[i]]], \{i, Length[B]\}]]]}}
\]
The program to compute a cumulant of any number of products then looks like this:

\[ K[a] := \text{Module}\left\{ \begin{array}{l} a0 = \{a\}, L, q, w, \text{L} = \text{Length}[a0]; \\
q = \text{Map}[\text{Reverse}, \text{Partitions}[\text{L}]]; \\
w = \text{Map}[\text{aux4}[\text{Range}[\text{L}], \#, i, \text{L}]]; \\
\text{Apply}[\text{Plus}, \text{Apply}[\text{Times}, \text{Apply}[\text{MV}, w, \{3\}], \{2\}], \{1\}]. \\
\text{Map}[(-1)^{\text{Length}[\#]-1}(\text{Length}[\#]-1)! & \text{q}] \text{// Expand} \right. \}
\]

To find the fifth-order cumulant of the random variables \(X_3, X_1X_3, X_1X_3, X_1X_2X_3\) and \(X_1X_2X_3^2\), one has to type (to simplify the answer, we have assumed that the \(X\)'s are standardized, i.e. have a mean of zero and the variance equal to 1):

\[ K[3\{1,3\},\{1,3\},\{1,2,3\},\{1,2,3,3\}]//\text{V}[i\_i\_]->1/\text{V}[i\_j\_]->\text{C}_{i,j} \]

resulting in

\[ \begin{align*}
42 + 158C_{1,2}^2 + 438C_{1,3}^2 + 240C_{2,3}^2 + 1784C_{1,2}C_{1,3}C_{2,3} + 1960C_{1,2}C_{1,3}^2C_{2,3} \\
+ 802C_{1,2}C_{1,3} + 1616C_{1,3}C_{2,3}^2 + 240C_{1,3}^2 + 400C_{1,3}C_{2,3}^2
\end{align*} \]

where \(C_{i,j}\) is the correlation coefficient between \(X_i\) and \(X_j\). Note that

\[ K[\{3,1\},\{3,2,1\},3,\{2,3,3,1\},\{1,3\}]//\text{V}[i\_i\_]->1/\text{V}[i\_j\_]->\text{C}_{i,j} \]

would have returned the same answer.

References

[1] VRBIK Jan: "Moments of AR(k) parameter estimators" *Communications in Statistics - Simulation and Computation* 44 (2015) 1239-1252

[2] KALITSI Clarence Deladem: "Approximate Sampling Distributions of the Parameter Estimators in the AR(1) model" *Brock Reports in Mathematics and Statistics* No. 130410 - 01 (May 1, 2013)