Quantum Corrections to Nucleation Rates

G. C. Marques*

Texas A & M University, Dept. of Physics
College Station, Texas 77853, USA

Rudnei O. Ramos

Dartmouth College, Dept. of Physics and Astronomy,
Hanover, NH 03755, USA

September 1992

Abstract

In this paper we show how to compute in a consistent way nucleation rates in field theory at finite temperatures, with metastable vacuum. Using the semiclassical approach in field theory at finite temperature we show that the prefactor term can be calculated explicitly (in the thin-wall approximation) and that the same provides exponential finite temperature quantum corrections to nucleation rates, when fluctuations around the bubble field configuration are considered.

PACS number(s): 98.80.Cq, 64.60.Qb .

*On leave from Instituto de Física, Universidade de São Paulo, C.P. 20516, São Paulo, SP 01498, Brazil.
1 Introduction

The study of first-order phase transitions in cosmology has received increased interest recently due to their possible relevance to the physics of the early Universe. Indeed, phase transitions have been invoked in the context of the generation of large scale structure of the Universe, the flatness and horizon problems\cite{1} and, more recently, the electroweak phase transition, which could be responsible for the generation of the cosmological baryon asymmetry\cite{2}.

In a first-order phase transition the Universe is believed to be in a metastable phase (the false-vacuum) that decays to a stable phase (the true vacuum) as the Universe expands and cools. The decay process of the metastable phase to the stable one happens by the nucleation in the system of droplets (or bubbles)\cite{3} of the stable phase whose dynamics will determine the completion of the phase transition\cite{4}. Of fundamental relevance for the understanding of the development of the phase transition is the determination of the bubble nucleation rate per unit volume.

In this paper we deal with the problem of determination, using the semiclassical approach, of the nucleation rate in field theory at finite temperature, which includes its definition from a droplet model field theoretical point of view\cite{3} and the evaluation of the prefactor term appearing in the nucleation rate expression. We show that the prefactor term provides an exponential finite temperature quantum correction to the nucleation rate which has not been properly discussed in most of the studies involving this quantity\cite{1,5}. Our method of evaluation is self-consistent and we show that the prefactor can be evaluated directly by computing the eigenvalues of the determinant terms, in the case of a convenient bubble field configuration, or by developing an appropriate field theoretical expansion for the determinant ratio.
The usual expression for the nucleation rate per unit volume used in the literature is given by\[^{[5]}\]

\[
\Gamma = T \left( \frac{S_3(\bar{\phi}, T)}{2\pi T} \right)^{\frac{3}{2}} \left\{ \frac{\det[-\nabla^2 + V''(\phi_f, T)]}{\det'[\nabla^2 + V''(\bar{\phi}, T)]} \right\}^{\frac{1}{2}} e^{-\frac{S_3(\bar{\phi}, T)}{T}},
\]

(1)

where \( \bar{\phi} \) stands for the (static) bounce field configuration describing the bubble, \( \phi_f \) is the false vacuum (or metastable) field configuration, \( S_3(\bar{\phi}, T) \) is the three-dimensional Euclidean action, at finite temperature, of the non-trivial field configuration \( \bar{\phi} \),

\[
S_3(\bar{\phi}, T) = \int d^3x \left[ \frac{1}{2} (\nabla \bar{\phi})^2 + V_{eff}(\bar{\phi}, T) \right],
\]

(2)

where \( V_{eff}(\bar{\phi}, T) \) is the effective potential at finite temperature.

What is usually done to compute the determinant ratio in (1) is the use of dimensional analysis to approximate it by a preexponential factor of order \( T^4 \), for example, \( T_c^4 \) (the critical temperature) or \( m^4(\phi_f, T) \) (\( m^2(\phi_f, T) = \frac{d^2V_{eff}(\phi, T)}{d\phi^2}\big|_{\phi=\phi_f} \)). This way of defining the nucleation rate presupposes that the determinant ratio in (1) will not contribute to the term in the exponential and that all the quantum corrections are included, from the beginning, in the form of \( S_3(\bar{\phi}, T) \) in the exponential.

We shall show that our way of defining the nucleation rate \( \Gamma \) results in an expression quite different from that given by (1) and that the usual approximation for the overall preexponential factor is not a simple function of the temperature, especially near \( T_c \), as observed recently by Csernai and Kapusta\[^{[6]}\], that have computed the preexponential factor in the case of a QCD phase transition.

The paper is organized in the following way: in Sec. II we review the droplet model in field theory at finite temperature for a generic model of a scalar field \( \phi \), which could be coupled with other fields, gauge or other scalar fields, for example. If the effective potential at finite \( T \), of the scalar field \( \phi \), admits a first-order phase transition, then
the nucleation rate can be defined as being the imaginary part of the free energy, in the approximation of a dilute gas, of droplets of the stable phase inside the metastable phase\(^3\). This imaginary free energy arises due to the existence of a negative eigenvalue, associated with the instability of the critical bubble. In Sec. III we deal with the evaluation of the prefactor appearing in the nucleation rate and show that the same is of fundamental importance for the proper determination of the finite temperature corrections to the exponential factor term in the nucleation rate. In Sec. IV we study the model of a scalar field theory coupled with fermion fields, computing the additional determinant factor due to the integration over the fermion fields. In Sec. V we apply the method by choosing a simple potential which exhibits metastability, at \(T \neq 0\). Conclusions are presented in Sec. VI.

2 Droplet Model in Field Theory

Let us consider a system described by a scalar field \(\phi\) and by a set of fields \(\chi\) (bosonic or fermionic fields), that can be coupled to \(\phi\). The partition function \(Z\) of this system, at \(T \neq 0\), is

\[
Z = \int D\phi D\chi e^{-S_{\text{Eucl}}(\phi, \chi)}, \tag{3}
\]

where \(S_{\text{Eucl}}(\phi, \chi)\) is the Euclidean action of the system and the functional integrations are carried over field configurations subject to periodic (anti-periodic) conditions: \(\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)\) for bosons and \(\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)\) for fermions. \(\beta\) is the inverse of the temperature, \(\beta = T^{-1}\).

If one integrates out the \(\chi\) fields in (3) and makes the expansion \(\phi(\vec{x}, \tau) \to \phi_c(\vec{x}, \tau) + \eta(\vec{x}, \tau)\), where \(\phi_c(\vec{x}, \tau)\) is a field configuration that extremizes the Euclidean action \(\bar{S}_{\text{Eucl}}(\phi)\)
(after integrating out the $\chi$ fields) for the scalar field $\phi$ and $\eta(\vec{x}, \tau)$ are small perturbations around $\phi_c(\vec{x}, \tau)$, then the effective action $\Gamma_{eff}(\phi_c)$ for the field configuration $\phi_c$, can be written as

$$\Gamma_{eff}(\phi_c) = \ln \int D\eta e^{-\bar{S}_{Eucl}(\phi)} ,$$

or using an expansion in terms of $\phi_c$ and its derivatives,

$$\Gamma_{eff}(\phi_c) = \int_0^\beta d\tau \int d^3x \left[ -V_{eff}(\phi_c, T) + \frac{1}{2}(\partial_\mu \phi_c)^2 Z(\phi_c) + \ldots \right] . \tag{4}$$

In (4), for constant field configurations, $\phi_c(\vec{x}, \tau) = \phi_c$, $V_{eff}(\phi_c, T)$ defines the effective potential, at finite $T$. In principle, from $V_{eff}(\phi, T)$ we can determine the order of the phase transition. If above a certain critical temperature $T_c$ the system is in a symmetric phase $\phi_f$ and below $T_c$ $V_{eff}(\phi, T)$ develops an energetically favorable phase $\phi_t$, then the phase $\phi_f$ becomes a metastable phase and we have a first-order phase transition in which the metastable phase $\phi_f$ decays to the new favorable phase $\phi_t$. $T_c$ is defined by the condition $V_{eff}(\phi_f, T_c) = V_{eff}(\phi_t, T_c)$.

The process of the decay of the metastable vacuum can be seen as happening by the nucleation of bubbles (or droplets) of the new phase $\phi_t$ in the system, that in a successful phase transition will grow and coalesce completing the phase transition.

Let us write the partition function (3), after integrating out the $\chi$ fields as

$$Z = \oint D\phi \exp \left\{ -\bar{S}_{Eucl}(\phi) \right\} , \tag{5}$$

where $\bar{S}_{Eucl}(\phi)$ is an effective Euclidean action for the scalar field $\phi$.

If the system defined by (5) admits an effective potential like that one described above, then one can imagine the transition from the metastable phase $\phi_f$ to the stable phase $\phi_t$ as being preceded by the formation of small nuclei of the phase $\phi_t$ inside the phase $\phi_f$. 5
Note that $\phi_f$ and $\phi_t$ are functions of the temperature, $\phi_{f(t)} \equiv \phi_{f(t)}(T)$, given by the minima of $V_{\text{eff}}(\phi, T)$, $\frac{dV_{\text{eff}}(\phi, T)}{d\phi}|_{\phi=\phi_{f(t)}} = 0$. Following Langer\cite{3}, in a dilute gas of droplets of the phase $\phi_t$, one can infer the thermodynamics of the system from the knowledge of the partition function of a single bubble. Therefore, in a dilute gas approximation, the principal contribution for the partition function (4) for the system can be written as

$$ Z \simeq Z(\phi_f) + Z(\phi_b) $$

and the partition function $Z$ of this gas of bubbles can be approximated to\cite{3}

$$ Z \simeq Z(\phi_f) \left[ 1 + \frac{Z(\phi_b)}{Z(\phi_f)} + \ldots \right] \simeq Z(\phi_f) \exp \left[ \frac{Z(\phi_b)}{Z(\phi_f)} \right], \quad (6) $$

where in the expressions above $Z(\phi_f)$ is the partition function for the metastable phase and $Z(\phi_b)$ is the partition function for the bubble in the system, or in a field theory language, $Z(\phi_b)$ is the partition function of the system in the presence of a bubble field configuration $\phi_b$ \cite{7}.

For $S_{\text{Eucl}}(\phi) = \int_0^\beta d\tau \int d^3x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right]$, the bubble field configuration is a non-trivial solution of the Euler-Lagrange equation:

$$ \left( \frac{\partial^2}{\partial \tau^2} + \nabla^2 \right) \phi - V'(\phi) = 0 \quad (7) $$

and therefore, in the semiclassical approach, the field configuration $\phi_b$ is the one that extremizes the effective action

$$ \left. \frac{\delta \Gamma_{\text{eff}}(\phi)}{\delta \phi} \right|_{\phi=\phi_b} = 0. \quad (8) $$

For static field configurations, from Eq. (7), $\phi_b(r)$ is a static bounce solution of the differential equation
\[
\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = V'(\phi)
\]  
(9)

with boundary conditions, \(\lim_{r \to +\infty} \phi = \phi_f\) and \(\frac{d\phi}{dr}|_{r=0} = 0\).

One can then describe \(\phi_b(r)\) by the following spherically symmetric field configuration

\[
\phi_b(r) = \begin{cases} 
\phi_t, & 0 < r < R - \Delta R \\
\phi_{\text{wall}}, & R - \Delta R < r < R + \Delta R \\
\phi_f, & r > R + \Delta R 
\end{cases}
\]

(10)

that is, \(\phi_b(r)\) describes a bubble of radius \(R\), of the nucleating phase \(\phi_t\) embedded in the metastable phase \(\phi_f\) with the bubble wall described by a field configuration \(\phi_{\text{wall}}\), that separates the two phases (for example, one could imagine \(\phi_{\text{wall}}\) as a kink-like field configuration). \(\Delta R\) is the bubble wall thickness. The solution (10) is a very good approximation for \(\phi_b\) when \(R >> \Delta R\), or in the thin-wall approximation. The simplest assumption for the validity of the thin-wall approximation is to consider that the bubble nucleation happens at temperatures close to \(T_c\), that is, with relatively small supercooling\(^5\). In the rest of this work we will concentrate on this assumption, where we can approximate the bubble field configuration quite well by \(\phi_b(r)\) given by (10).

Let us now compute the partition function (6) using a semiclassical approach. Within the semiclassical approach one expands the Lagrangean field \(\phi\) in (6) as \(\phi \to \eta(\vec{x}, t) + \phi_b(\vec{x})\) for \(Z(\phi_b)\) and \(\phi \to \zeta(\vec{x}, t) + \phi_f\) for \(Z(\phi_f)\). \(\eta(\vec{x}, t)\) and \(\zeta(\vec{x}, t)\) are small perturbations around the classical field configurations \(\phi_b(\vec{x})\) and \(\phi_f\) respectively. Up to 1–loop order one keeps the quadratic terms in the fields \(\eta(\vec{x}, t)\) and \(\zeta(\vec{x}, t)\) in these expansions. In this way one can write the following expressions for \(Z(\phi_b)\) and \(Z(\phi_f)\), respectively,

\[
Z(\phi_b)_{1-\text{loop order}} \simeq e^{-\bar{S}_{\text{Eucl}}(\phi_b)} \int D\eta \exp \left\{ - \int_0^\beta d\tau \int d^3x \frac{1}{2} \eta \left[ -\Box_{\text{Eucl}} + V''(\phi_b) \right] \eta \right\} 
\]

(11)
and

\[
Z(\phi_f)^{1-\text{loop order}} \approx e^{-S_{\text{Eucl}}(\phi_f)} \oint D\zeta \exp \left\{ - \int_0^\beta d\tau \int d^3x \frac{1}{2} \zeta \left[ -\Box_{\text{Eucl}} + V''(\phi_f) \right] \zeta \right\}, \quad (12)
\]

where \( S_{\text{Eucl}}(\phi) = \int_0^\beta d\tau \int d^3x [\frac{1}{2} (\partial \mu \phi)^2 + V(\phi)] \), \( V''(\phi) = \frac{d^2V}{d\phi^2} \) and \( \Box_{\text{Eucl}} = \frac{\partial^2}{\partial \tau^2} + \vec{\nabla}^2 \).

Performing the functional gaussian integrals in (11) and (12) one gets the following expression for the ratio between the partition functions, \( \frac{Z(\phi_b)}{Z(\phi_f)} \), appearing in (6):

\[
\frac{Z(\phi_b)}{Z(\phi_f)^{1-\text{loop order}}} \approx \left[ \frac{\det(-\Box_{\text{Eucl}} + V''(\phi_b))}{\det(-\Box_{\text{Eucl}} + V''(\phi_f))} \right]^{-\frac{1}{2}} e^{-\Delta S}, \quad (13)
\]

where \( [\det(M)]^{-\frac{1}{2}} \equiv \oint D\eta \exp \left\{ - \int_0^\beta d\tau \int d^3x \frac{1}{2} \eta [M] \eta \right\} \) and \( \Delta S = \bar{S}_{\text{Eucl}}(\phi_b) - \bar{S}_{\text{Eucl}}(\phi_f) \) is the difference between the Euclidean actions for the field configurations \( \phi_b \) and \( \phi_f \).

The free energy of the system, \( F = -\beta^{-1} \ln Z \), up to 1-loop approximation, from (6) and (13), can be written as

\[
F = -T \left[ \frac{\det(-\Box_{\text{Eucl}} + V''(\phi_b))}{\det(-\Box_{\text{Eucl}} + V''(\phi_f))} \right]^{-\frac{1}{2}} e^{-\Delta S}. \quad (14)
\]

It is well known that the determinant for the bubble field configuration in (14) has a negative eigenvalue, that signals the presence of a metastable state, and that it has also three zero eigenvalues related with the translational invariance of the bubble in the three dimensional space. Because of the negative eigenvalue, the free energy \( F \) is then imaginary. However the imaginary of \( F \) can be related exactly with the nucleation rate of bubbles of the phase \( \phi_t \) (the stable vacuum) inside the metastable phase \( \phi_f \), as shown by Langer\[3\]. At finite temperatures, Affleck, ref. \[8\], showed that the nucleation rate \( \Gamma \) is given by

\[
\Gamma \equiv \left| \frac{\omega_*}{\pi} \frac{Im F}{\beta^{-1}} \right|, \quad (15)
\]
where $|\omega_-|$ is the frequency of the unstable mode.

### 3 Evaluation of the Determinants

Let us now compute the ratio of the determinants appearing in the nucleation rate, Eq. (14), and show that the same provides a finite temperature correction to the exponential term $\Delta S$.

One remembers that $\Delta S$ is given by

$$
\Delta S = \beta \int d^3x \left[ \bar{\mathcal{L}}_{\text{Eucl}}(\phi_b) - \bar{\mathcal{L}}_{\text{Eucl}}(\phi_f) \right] = \frac{\Delta E}{T}. \tag{16}
$$

If one uses the bubble field configuration $\phi_b(r)$ as given by (10), then in the thin-wall approximation one obtains the following expression for $\Delta E$ in (16):

$$
\Delta E = -\frac{4\pi R^3}{3} \Delta V + 4\pi R^2 \sigma_0, \tag{17}
$$

where $\Delta V = V(\phi_f) - V(\phi_t)$ is the potential difference between the false and true vacua (the metastable and stable vacua, respectively) and $\sigma_0$ is the surface tension of the bubble wall (with no corrections due fluctuations around the bubble wall field configuration $\phi_{\text{wall}}$).

$$
\sigma_0 \simeq \int_{-\Delta R}^{+\Delta R} dr \left[ \bar{\mathcal{L}}_{\text{Eucl}}(\phi_{\text{wall}}) - \bar{\mathcal{L}}_{\text{Eucl}}(\phi_f) \right]. \tag{18}
$$

Remind that in the above equations and in $\phi_b(r)$, given by (10), $\phi_{f(t)} \equiv \phi_{f(t)}(T)$, are the minima of the finite temperature effective potential, that is, $\phi_{f(t)}$ are given by

---

1We are supposing that $\bar{\mathcal{L}}_{\text{Eucl}}(\phi)$ denotes an “effective Euclidean action” for the field $\phi$, where possible other fields coupled with $\phi$ has been integrated out. In this way $V(\phi)$ includes not only the classical potential for $\phi$ but also can include corrections ($T \neq 0$) coming from the integration of that fields.
\[
\frac{dV_{\text{eff}}(\phi, T)}{d\phi}_{\phi = \phi_{f(i)}} = 0. \tag{19}
\]

From (17) one can see that \( \Delta E \) can be associated with the activation energy of one bubble of radius \( R \). What we are going to show is that the determinant ratio in (14) will give a finite temperature correction to this bubble activation energy, where this correction comes exactly from fluctuations around the bubble field configuration \( \phi_b \).

The computation of the ratio of determinants in (14) can be done by using two approaches. The first one involves obtaining directly the eigenvalues of the determinants in Eq. (14). The second one consists in developing a consistent expansion for the determinant ratio.

### 3.1 Evaluation of the prefactor in terms of the eigenvalues

The evaluation of the determinants in (14) can be done by computing directly, if possible, the eigenvalues of the differential equations

\[
[-\Box_{\text{Eucl}} + V''(\phi_f)] \varphi_f(i) = \varepsilon_f^2(i) \varphi_f(i) \tag{20}
\]

and

\[
[-\Box_{\text{Eucl}} + V''(\phi_b)] \varphi_b(i) = \varepsilon_b^2(i) \varphi_b(i). \tag{21}
\]

In momentum space one writes, \( \varepsilon^2 = \omega_n^2 + E^2 \), where \( \omega_n = \frac{2\pi n}{\beta} \), \( n = 0, \pm 1, \pm 2, \ldots \), for bosons (for fermion fields \( \omega_n = \frac{(2n+1)\pi}{\beta} \)). From (20) and (21) one can write the determinant ratio in (14) as

\[
\left[ \frac{\det(-\Box_{\text{Eucl}} + V''(\phi_f))_{\beta}}{\det(-\Box_{\text{Eucl}} + V''(\phi_b))_{\beta}} \right]^{\frac{1}{2}} = \exp \left\{ \frac{1}{2} \ln \left[ \frac{\det(-\Box_{\text{Eucl}} + V''(\phi_f))_{\beta}}{\det(-\Box_{\text{Eucl}} + V''(\phi_b))_{\beta}} \right] \right\} = \]
\[
\begin{align*}
&= \exp \left\{ \frac{1}{2} \ln \left[ \prod_{n=-\infty}^{\infty} \prod_{i} \left( \omega_{n}^2 + E_{f}^2(i) \right) \right] \right\}. \tag{22}
\end{align*}
\]

Using the identity:
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{n^2} \right) = \frac{\sinh(\pi z)}{\pi z} \tag{23}
\]
and taking into account that we have in (22) a negative and three zero eigenvalues, one obtains for (22) the expression:
\[
\begin{align*}
\left[ \frac{\det(-\Box_{\text{Eucl}} + V''(\phi_f))_{\beta}}{\det(-\Box_{\text{Eucl}} + V''(\phi_b))_{\beta}} \right]^{\frac{1}{2}}
&= \frac{T^4}{i|E_-|} \frac{\beta |E_-|}{\sin \left( \beta \frac{|E_-|}{2} \right)} \left[ \Delta S \right]^{\frac{1}{2}} \\
&\times \exp \left\{ \sum_{i} \left[ \frac{\beta}{2} E_{f}(i) + \ln \left( 1 - e^{-\beta E_{f}(i)} \right) \right] \right. \\
&\left. - \sum_{j} \prime \left[ \frac{\beta}{2} E_{b}(j) + \ln \left( 1 - e^{-\beta E_{b}(j)} \right) \right] \right\}. \tag{24}
\end{align*}
\]

The factor \( \left[ \frac{\Delta S}{2\pi} \right]^{\frac{3}{2}} \) in the right hand side of (24) comes from the contribution of the zero eigenvalues\(^9\). The prime in \( \sum_{j} \) is a reminder that we have excluded the negative and the three zero eigenvalues from the sum. \( E_{-}^2 \) denotes the negative eigenvalue (|\( \omega_{-} \)| = |\( E_{-} \)|, in (15)).

Substituting (24) in (14) and using the bubble field configuration \( \phi_{b}(r) \), given by (10), one obtains the following expression for the nucleation rate \( \Gamma \), given by (15):
\[
\Gamma = \mathcal{A} T^4 \exp \left[ - \frac{\Delta F(T)}{T} \right], \tag{25}
\]
where we have denoted by \( \mathcal{A} \) the adimensional factor \( \alpha \left[ \frac{\Delta E_{-}}{2\pi T} \right]^{\frac{3}{2}} \) with \( \alpha \) given by
\[
\alpha = \frac{1}{\pi} \frac{|E_{-}|}{\sin \left( \frac{|E_{-}|}{2T} \right)} \tag{26}
\]
and $\Delta E$ is given by (17). $\Delta F(T)$ in (25) is given by

$$\Delta F(T) = -\frac{4\pi R^3}{3} \Delta U(T) + 4\pi R^2 \sigma(T),$$

(27)

where

$$\Delta U(T) = V(\phi_f) - V(\phi_t) + T \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 - e^{-\beta \sqrt{k^2 + m^2(\phi_f)}} \right] - T \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 - e^{-\beta \sqrt{k^2 + m^2(\phi_t)}} \right],$$

(28)

and

$$\sigma(T) = \sigma_0 + \frac{T}{4\pi R^2} \left\{ \sum_j \left[ \frac{\beta}{2} E_{wall}(j) + \ln \left( 1 - e^{-\beta E_{wall}(j)} \right) \right] - \sum_i \left[ \frac{\beta}{2} E_f(i) + \ln \left( 1 - e^{-\beta E_f(i)} \right) \right] \right\}.$$  

(29)

In (27) we have used again the thin-wall approximation. In (28) we have substituted the discrete sums by integrals over momenta (for the constant field configurations $\phi_f$ and $\phi_t$, we have the continuum eigenvalues, $E_f^2 = \vec{k}^2 + m^2(\phi_f)$ and $E_t^2 = \vec{k}^2 + m^2(\phi_t)$, respectively, with $m^2(\phi_f) = \frac{d^2V(\phi)}{d\phi^2}|_{\phi=\phi_f}$ and $m^2(\phi_t) = \frac{d^2V(\phi)}{d\phi^2}|_{\phi=\phi_t}$). The terms, like $\int d^3k \sqrt{\vec{k}^2 + m^2(\phi)}$, that should appear in (28) are ultraviolet divergent and can be subtracted from the theory by the introduction of the usuals counterterms of renormalization that render the theory finite[10].

In (29) $E_{wall}(j)$ are the eigenvalues related with the bubble wall field configuration $\phi_{wall}$. The problem then is reduced to the computation of these eigenvalues for a field configuration describing the bubble wall.

It is easy to see that $\Delta U(T)$ as given by (28) is exactly the finite temperature effective potential difference between the false and true vacua, as expected[11]. The second term in
the right hand side in (29) clearly represents the finite temperature contribution for the surface tension $\sigma_0$, coming from the 1-loop finite temperature quantum corrections due to fluctuations around the bubble wall field configuration $\phi_{wall}$.

### 3.2 A field theoretical expansion for the determinants

The second approach for the computation of the determinants in (14) consists in developing a simple field theoretical expansion for it. Let us write for the ratio of the determinants the following expression

$$
\frac{\det(-\Box_{Euc} + V''(\phi_f))}{\det'(-\Box_{Euc} + V''(\phi_b))} \frac{1}{2} = \exp \left\{ \frac{1}{2} Tr \ln [-\Box_{Euc} + V''(\phi_f)] - \frac{1}{2} Tr' \ln [-\Box_{Euc} + V''(\phi_b)] \right\},
$$

(30)

where we have used in (30) the identity $\ln \det \hat{M} = \text{Tr} \ln \hat{M}$ and the prime in both sides denote that the negative and the zero modes have been omitted.

Formally one can write (30) as

$$
\frac{\det(-\Box_{Euc} + V''(\phi_f))}{\det'(-\Box_{Euc} + V''(\phi_b))} \frac{1}{2} = \exp \left\{ -\frac{1}{2} Tr \ln \left[ 1 + G_\beta(\phi_f) [V''(\phi_b) - V''(\phi_f)] \right] \right\},
$$

(31)

where

$$
G_\beta(\phi_f) = \frac{1}{-\Box_{Euc} + m^2(\phi_f)}
$$

(32)

is just the free propagator, at finite temperature, for the scalar field $\phi$, with mass squared given by $m^2(\phi_f) = V''(\phi_f)$.

If we expand the natural logarithm in (31) in powers of $G_\beta(\phi_f) [V''(\phi_b) - V''(\phi_f)]$, we get formally
\[ \text{Tr} \ln \{1 + G_\beta(\phi_f) [V''(\phi_b) - V''(\phi_f)] \} = \]

\[ = + \ldots, \quad (33) \]

where the dashed lines correspond to the background-like field \([V''(\phi_b) - V''(\phi_f)]\) and the internal lines denote the propagator \(G_\beta(\phi_f)\). The expression (33) can be written as

\[ \text{Tr} \ln \{1 + G_\beta(\phi_f) [V''(\phi_b) - V''(\phi_f)] \} = \sum_{m=1}^{+\infty} \frac{(-1)^{m+1}}{m} \int d^3x \int d^3k \frac{1}{(2\pi)^3} \ln \left[ 1 + \frac{V''(\phi_b) - V''(\phi_f)}{\omega_n^2 + \vec{k}^2 + m^2(\phi_f)} \right]^m. \quad (34) \]

The sum in \(m\) in (34) can be formally performed and one obtains

\[ \text{Tr} \ln \{1 + G_\beta(\phi_f) [V''(\phi_b) - V''(\phi_f)] \} = \int d^3x \sum_{m=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 + \frac{V''(\phi_b) - V''(\phi_f)}{\omega_n^2 + \vec{k}^2 + m^2(\phi_f)} \right]. \quad (35) \]

Substituting (35) in (31) and again taking into account that we have eliminated the negative and zero modes from the determinant in (14), which provides us with that factor multiplying the exponential in (24), one obtains an expression for the nucleation rate \(\Gamma\) like (25), but now with \(\Delta F(T)\) given by (using (10) as a solution for \(\phi_b\) and the thin-wall approximation)

\[ \Delta F(T) = -\frac{4\pi R^3}{3} \Delta V_{\text{eff}}(T) + 4\pi R^2 \sigma(T), \quad (36) \]
where

\[
\Delta V_{\text{eff}}(T) = V(\phi_f) - V(\phi_t) - \frac{1}{2\beta} \sum_{n=\pm\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 + \frac{m^2(\phi_t) - m^2(\phi_f)}{\omega_n^2 + \vec{k}^2 + m^2(\phi_f)} \right]
\]  

(37)

and

\[
\sigma(T) = \sigma_0 + \frac{1}{4\pi R^2} \int d^3x \frac{1}{2\beta} \sum_{n=\pm\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 + \frac{V''(\phi_{\text{wall}}) - V''(\phi_f)}{\omega_n^2 + \vec{k}^2 + m^2(\phi_f)} \right].
\]  

(38)

Expression (37) can easily be identified with \( \Delta U(T) \), given by (28), if one performs the sum in \( n \) in (37) by using the identity

\[
\sum_{n=\pm\infty}^{+\infty} \ln \left[ \frac{\omega_n^2 + E_t^2(\vec{k})}{\omega_n^2 + E_f^2(\vec{k})} \right] = \beta E_t(\vec{k}) + 2 \ln \left( 1 - e^{-\beta E_t(\vec{k})} \right) - \beta E_f(\vec{k}) - 2 \ln \left( 1 - e^{-\beta E_f(\vec{k})} \right)
\]  

(39)

and as it was done in (28), eliminating the ultraviolet divergent terms, we obtain

\[
\Delta V_{\text{eff}}^{\text{Ren}}(T) = V_{\text{eff}}^{\text{Ren}}(\phi_f, T) - V_{\text{eff}}^{\text{Ren}}(\phi_t, T),
\]  

(40)

where

\[
V_{\text{eff}}^{\text{Ren}}(\phi, T) = V(\phi) + T \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - e^{-\beta \sqrt{k^2 + m^2(\phi)}} \right)
\]  

(41)

is the renormalized effective potential at finite temperature.

Expression (38) for \( \sigma(T) \) can be easily identified as being the surface tension at finite temperature for the bubble wall. If one writes (38) as

\[
\sigma(T) = \frac{1}{4\pi R^2} \left[ \Gamma_{\text{eff}}(\phi_{\text{wall}}, T) - \Gamma_{\text{eff}}(\phi_f, T) \right],
\]  

(42)
where $\Gamma_{\text{eff}}(\phi, T)$ is given by

$$
\Gamma_{\text{eff}}(\phi, T) = \int_0^\beta d\tau \int d^3x \left\{ \tilde{\mathcal{L}}_{\text{Eucl}}(\phi) + \frac{1}{2\beta} \sum_{n=0}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ \omega_n^2 + \vec{k}^2 + m^2(\phi) \right] \right\},
$$

(43)

$\Gamma_{\text{eff}}(\phi, T)$, as given by (43), is the effective action at finite temperature for a field configuration $\phi$ at 1-loop order$^{[12]}$. $\sigma(T)$, given by (42), is the definition of the surface tension in field theory$^{[12]}$.

It is easily shown$^{[7]}$ that in a high temperature limit ($\beta \to 0, T \to \infty$) the terms in the expansion (33) that have higher superficial degree of divergence will contribute with a leading power in $T$. This contribution is just the first graph in (33) ($m = 1$ in (34)) such that at high temperatures one can consider only the following simple term of the graphic expansion in (33):

$$
\Gamma_{\text{eff}}(\phi, T) \approx \int d^3x \left[ V''(\phi_b) - V''(\phi_f) \right] \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + k^2 + m^2(\phi_f)},
$$

(44)

making the computation of $\Delta F(T)$ and then of $\Gamma$ very simple with a bubble field configuration $\phi_b$ as given by (10).

4 A Scalar Field $\phi$ Coupled to Fermion Fields

Let us now study a model of a scalar field $\phi$ coupled with massless fermion fields $\psi$, with Lagrangian density given by

$$
\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) - g\phi \bar{\psi} \psi + i\bar{\psi} \gamma^\mu \partial_\mu \psi.
$$

(45)

$V(\phi)$ in (45) is such that the finite temperature effective potential of the scalar field $\phi$ admits a first-order phase transition. For example with $V(\phi)$ given by Eq. (67), Sec. V.
The partition functions $Z(\phi_b)$ and $Z(\phi_f)$ in (6) are now defined by the field expansions

\[ \phi(\vec{x},t) \rightarrow \phi_b(\vec{x}) + \eta(\vec{x},t) \] and

\[ \psi(\vec{x},t) \rightarrow \psi(\vec{x},t) \] for $Z(\phi_b)$ and

\[ \phi(\vec{x},t) \rightarrow \phi_f + \zeta(\vec{x},t) \] and

\[ \psi(\vec{x},t) \rightarrow \psi(\vec{x},t) \] for $Z(\phi_f)$. Again, $\phi_{f(t)} \equiv \phi_{f(t)}(T)$ are determined by the minima of the effective potential $V_{\text{eff}}(\phi,T)$. Up to 1-loop order one gets the following expressions for $Z(\phi_b)$ and $Z(\phi_f)$, respectively,

\begin{equation}
Z(\phi_b) = e^{-S_{\text{Eucl}}(\phi_b)} \oint D\eta \exp \left\{ - \int_0^\beta d\tau \int d^3x \frac{1}{2} \eta [-\Box_{\text{Eucl}} + V''(\phi_b)] \eta \right\} \times \oint D\bar{\psi}D\psi \exp \left\{ - \int_0^\beta d\tau \int d^3x \bar{\psi} \left[ - \partial - ig\phi_b \right] \psi \right\} ,
\end{equation}

and

\begin{equation}
Z(\phi_f) = e^{-S_{\text{Eucl}}(\phi_f)} \oint D\zeta \exp \left\{ - \int_0^\beta d\tau \int d^3x \frac{1}{2} \zeta [-\Box_{\text{Eucl}} + V''(\phi_f)] \zeta \right\} \times \oint D\bar{\psi}D\psi \exp \left\{ - \int_0^\beta d\tau \int d^3x \bar{\psi} \left[ - \partial - ig\phi_f \right] \psi \right\} ,
\end{equation}

where the functional integration over the bosonic (fermionic) fields satisfy periodic (anti-periodic) boundary conditions in Euclidean time.

As in (14) the free energy of the system can be written now as

\begin{equation}
\mathcal{F} = -T \left[ \frac{\det(-\Box_{\text{Eucl}} + V''(\phi_f))}{\det(-\Box_{\text{Eucl}} + V''(\phi_b))} \right]^{\frac{1}{2}} \left[ \frac{\det(-\partial - ig\phi_b)}{\det(-\partial - ig\phi_f)} \right] e^{-\Delta S} ,
\end{equation}

where $\Delta S = S_{\text{Eucl}}(\phi_b) - S_{\text{Eucl}}(\phi_f)$, with, from (45),

\[ S_{\text{Eucl}}(\phi) = \int_0^\beta \int d^3x \left[ \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right] . \]

The determinant ratio of the bosonic part in (48) can be computed in exactly same way as before and we get the following expression for the nucleation rate $\Gamma$ for the model (45),

\begin{equation}
\Gamma = \mathcal{A}T^4 \left[ \frac{\det(-\partial - ig\phi_b)}{\det(-\partial - ig\phi_f)} \right] \exp \left[ - \frac{\Delta \mathcal{F}(T)}{T} \right] ,
\end{equation}

17
where $A$ and $\Delta F(T)$ have analogous identification with the terms appearing in (25). $\Delta F(T)$ can be given by both expressions (27) or (36) obtained by the eigenvalue method or by the graphic expansion, respectively.

The fermionic determinant ratio appearing in (49) can be calculated by the same way as the bosonic one. We shall see that it provides also a correction to the exponential term of the nucleation rate $\Gamma$, that is, a fermionic correction to the factor $\Delta F(T)$ in eq. (49).

If one uses the identity (which follows from charge-conjugation invariance):

$$
[\det(-\not\partial - ig\phi)]^2 = \det(-\not\partial - ig\phi) \cdot \det(-\not\partial + ig\phi) = \\
= \det \left[ (-\Box_{\text{Eucl}} + g^2\phi^2)1_{4\times4} - ig\gamma^\mu_{\text{Eucl}} \partial_\mu \phi \right],
$$

(50)

where $1_{4\times4}$ is the $4 \times 4$ unit matrix. From (50) the denominator of the fermionic determinant ratio in (49) can be written as ($\phi_f$ is the metastable vacuum field configuration)

$$
\det(-\not\partial - ig\phi_f)_\beta = \left[ \det(-\Box_{\text{Eucl}} + g^2\phi_f^2) \right]^{1/2}. 
$$

(51)

For the determinant involving the bubble field configuration one can again use a configuration like (10) for $\phi_b$, which describes a radially symmetric bubble solution. Making the following choice for the Dirac matrix in the radial direction

$$
\gamma_r = i \left( \begin{array}{cc} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{array} \right), 
$$

(52)

where $1_{2\times2}$ denotes a $2 \times 2$ unit matrix, one can write $\det(-\not\partial - ig\phi_b(r))_\beta$ as

$$
\det(-\not\partial - ig\phi_b(r))_\beta = \det \hat{\Omega}^{(+)}(\phi_b). \det \hat{\Omega}^{(-)}(\phi_b),
$$

(53)

where
\[ \hat{\Omega}^{(\pm)}(\phi_b) = -\Box_{\text{Eucl}} + g^2 \phi_b^2 \pm g \frac{\partial \phi_b}{\partial r}. \]  

(54)

The fermionic determinant ratio in (49) can be written then as (using again that \[ \ln \det \hat{M} = Tr \ln \hat{M} \])

\[
\frac{\det(-\bar{\partial} - ig\phi_b)_\beta}{\det(-\bar{\partial} - ig\phi_f)_\beta} = \exp \left\{ Tr \ln \left[ -\Box_{\text{Eucl}} + g^2 \phi_b^2 + g \frac{\partial \phi_b}{\partial r} \right] \right\} = \\
= \exp \left\{ Tr \ln \left[ -\Box_{\text{Eucl}} + g^2 \phi_b^2 + g \frac{\partial \phi_b}{\partial r} \right] + Tr \ln \left[ -\Box_{\text{Eucl}} + g^2 \phi_b^2 - g \frac{\partial \phi_b}{\partial r} \right] - \\
- 2Tr \ln \left[ -\Box_{\text{Eucl}} + g^2 \phi_f^2 \right] \right\}. 
\]

(55)

As in (31) one can write (55) in the form

\[
\frac{\det(-\bar{\partial} - ig\phi_b)_\beta}{\det(-\bar{\partial} - ig\phi_f)_\beta} = \exp \left\{ Tr \ln \left[ 1 + S_\beta(\phi_f) \left[ g^2 (\phi_b^2 - \phi_f^2) + \frac{\partial \phi_b}{\partial r} \right] \right] + \\
+ Tr \ln \left[ 1 + S_\beta(\phi_f) \left[ g^2 (\phi_b^2 - \phi_f^2) - \frac{\partial \phi_b}{\partial r} \right] \right] \right\}, 
\]

(56)

where

\[ S_\beta(\phi_f) = \frac{1}{-\Box_{\text{Eucl}} + m_f^2(\phi_f)} \]  

(57)

is the fermionic analogous of \( G_\beta(\phi_f) \) given by (32) and \( m_f^2(\phi_f) = g^2 \phi_f^2 \). The expression in the exponent in (56) can be written as a series expansion like (34):

\[
Tr \ln \left[ 1 + S_\beta(\phi_f) \left[ g^2 (\phi_b^2 - \phi_f^2) \pm \frac{\partial \phi_b}{\partial r} \right] \right] = \sum_{m=1}^{+\infty} \frac{(-1)^{m+1}}{m} \int d^3x \left[ g^2 (\phi_b^2 - \phi_f^2) \pm \frac{\partial \phi_b}{\partial r} \right]^m \\
\times \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{\omega}_n^2 + \vec{k}^2 + m_f^2(\phi_f)^m}, 
\]

(58)
where $\bar{\omega}_n = \frac{(2n+1)\pi}{\beta}$, for fermionic fields. As before, (58) can be expressed as a graphic expansion similar to (33) with the propagators $G_\beta(\phi_f)$ exchanged now by $S_\beta(\phi_f)$ and the external lines given by $g^2(\phi_b^2 - \phi_f^2) + g\partial\phi_b/\partial r$ or $g^2(\phi_b^2 - \phi_f^2) - g\partial\phi_b/\partial r$.

When (56) is substituted in (49) and using the bubble field configuration $\phi_b$ as given by (10), in the thin-wall approximation one obtains

$$\Gamma = \mathcal{A}T^4 \exp \left[ -\frac{\Delta F^{B+F}(T)}{T} \right], \quad (59)$$

where $\Delta F^{B+F}(T)$ is given by

$$\Delta F^{B+F}(T) = -\frac{4\pi R^3}{3} \Delta V^{B+F}_{eff}(T) + 4\pi R^2 \sigma^{B+F}(T), \quad (60)$$

where $\Delta V^{B+F}_{eff}(T) = V^{B+F}_{eff}(\phi_f, T) - V^{B+F}_{eff}(\phi_t, T)$ is the effective potential difference between the false and true vacua for the model (45). Up to 1-loop order one obtains,

$$V^{B+F}_{eff}(\phi, T) = V(\phi) + T \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - e^{-\beta\sqrt{k^2 + m^2_B(\phi)}} \right) -$$

$$- 4T \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 + e^{-\beta\sqrt{k^2 + m^2_F(\phi)}} \right), \quad (61)$$

where $m^2_B(\phi) = V''(\phi)$ and $m^2_F(\phi) = g^2\phi^2$ are the boson and fermion effective masses, respectively, in the background field $\phi$. $\sigma^{B+F}(T)$ in (60) is given by

$$\sigma^{B+F}(T) = \frac{1}{4\pi R^2\beta} \left[ \Gamma^{B+F}_{eff}(\phi_{wall}, T) - \Gamma^{B+F}_{eff}(\phi_f, T) \right] \quad (62)$$

with

$$\Gamma^{B+F}_{eff}(\phi, T) = \int_0^\beta d\tau \int d^3x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) + \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ \omega_n^2 + \vec{k}^2 + m^2_B(\phi) \right] \right. \right.$$

$$- \left. \frac{2}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ \bar{\omega}_n^2 + \vec{k}^2 + m^2_F(\phi) + g|\vec{\nabla}\phi| \right] \right\}. \quad (63)$$
Γ_{eff}^{B+F}(ϕ, T) is the effective action, up to 1-loop order, for the model (45). As in (42), (62) defines the surface tension for the bubble wall.

The determinat ratio in (55) can also be derived by the eigenvalue method if one knows how to compute the eigenvalues of the differential equation for the field configuration φ_{wall} describing the bubble wall,

\[ \left[ -\Box_{Euc} + g^2 Φ_{wall}^2 ± g \frac{∂φ_{wall}}{∂r} \right] ϕ^{(±)}_{wall}(j) = [ε_{wall}^{(±)}(j)]^2 ϕ^{(±)}_{wall}(j), \]

where, in momentum space, \[ [ε^{(±)}_{wall}(j)]^2 = \tilde{ω}^2_n + [E^{(±)}_{wall}(j)]^2. \] For the vacuum fields φ_{f} and φ_{t} we have the continuum eigenvalues (in momentum space)

\[ [ε^{F}_{f}(k)]^2 = \tilde{ω}^2_n + [E^{F}_{f}(k)]^2, \]
\[ [E^{F}_{f}(k)]^2 = k^2 + g^2 ϕ_{f}^2 \]

\[ [ε^{F}_{t}(k)]^2 = \tilde{ω}^2_n + [E^{F}_{t}(k)]^2, \]
\[ [E^{F}_{t}(k)]^2 = k^2 + g^2 ϕ_{t}^2 \]

for the false and true vacua, respectively.

Using these eigenvalues in (55) one obtains an expression for (49) like (59), with ΔF^{B+F}(T) given again by (60) and as it was shown in the bosonic case, from (65), the expression for ΔV_{eff}^{B+F}(T) remains the same as that one given in (60). The expression for σ^{B+F}(T) is now rewritten in terms of the eigenvalues for the bubble wall field configuration φ_{wall}, such that, from (64) and taking into account the bosonic part of σ^{B+F}(T) given by (29), one can write

\[ \sigma^{B+F}(T) = \sigma^B(T) - \frac{1}{4πR^2β} \left\{ \sum_{j(+)j(-)} \left[ βE^{(±)}_{wall}(j_{(±)}) + 2 ln \left( 1 + e^{-βE^{(±)}_{wall}(j_{(±)})} \right) \right] - \sum_{i} \left[ 2βE^{F}_{f}(i) + 4 ln \left( 1 + e^{-βE^{F}_{f}(i)} \right) \right] \right\}, \]

where σ^{B}(T) is given by (29) and \( E^{F}_{f}(i) = \sqrt{k^2 + m^2_F(ϕ_f)}. \)
5 Simple Example

Let us now illustrate our method for a simple model that exhibits metastability and for which we can estimate the eigenvalues related with a bubble wall field configuration $\phi_{\text{wall}}$.

Consider the model of a scalar field $\phi$ coupled to fermion fields $\psi$ with Lagrangian density given by (45) and potential $V(\phi)$ given by

$$V(\phi, h) = -\frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 - h\phi,$$

where $h$ is assumed to be a constant external current.

It is well known that the potential (67) exhibits metastability when one varies the sign of $h$\cite{3,13}. $V(\phi, h)$ has two minima, $\phi_f(h)$ and $\phi_t(h)$. For $h > 0$ $\phi_t$ describes the stable phase and $\phi_f$ the metastable phase. For $h < 0$ the rules of $\phi_f(h)$ and $\phi_t(h)$ are reversed. At finite temperatures, we can always find values for the Yukawa coupling $g$ and for the constant $h$, such that the condition of existence of metastability is satisfied\cite{14}, with $\phi_{f(t)}(h, T)$ describing the metastable (stable) phases, respectively. In the limit $h \to 0^\pm$, the vacua solutions $\phi_{f,t}(h, T)$ tend to the limit $\pm \phi_0(T) = \pm \left(\frac{6m^2(T)}{\lambda}\right)^{\frac{1}{2}}$, where $m^2(T)$ is the finite temperature effective mass for the scalar field $\phi$ (in the high-temperature limit, $m^2(T) \simeq m^2 - \frac{\lambda T^2}{24} - \frac{g T^2}{6}$.) and one of the non-trivial static solutions of the Euler-Lagrange equation for the field $\phi$ in model (45), with potential given by (67), is

$$\phi_{\text{kink}}(x_L) = \phi_0(T) \tanh \left(\frac{m(T)}{\sqrt{2}} x_L\right),$$

which describes a planar interface between the two vacua $\pm \phi_0(T)$ ($h = 0$). $x_L$ is the longitudinal component of the spatial vector ($\vec{x} = (x_L, \vec{x}_T)$).

For sufficiently small values of $h$, the bubble field configuration $\phi_b(r)$ is well approximated by the radially symmetric solution\cite{13}:
\[ \phi_b(r) \simeq \frac{1}{2}(\phi_f + \phi_t) + \frac{1}{2}(\phi_f - \phi_t) \tanh \left[ \frac{m(T)}{\sqrt{2}}(r - R) \right], \] (69)

which describes a bubble of radius \( R \) of the nucleating vacuum \( \phi_t \equiv \phi_t(T) \) (the true vacuum) embedded in the metastable vacuum \( \phi_f \equiv \phi_f(T) \) (the false vacuum). From (69), the thin-wall approximation is equivalent to considering \( R >> m^{-1}(T) \), where \( \Delta R \), the bubble wall thickness, is proportional to \( m^{-1}(T) \).

From (29) and (66) we see that we have to compute the eigenvalues of the differential equations (approximating for small values of \( h \))

\[
\left[ \mathbf{- \nabla^2} + \frac{\lambda}{6} \left( 3\phi_{wall}^2 - \frac{6m^2}{\lambda} \right) \right] \varphi_{wall}^B = E_{wall} \varphi_{wall}^B
\]

(70)

\[
\left[ \mathbf{- \nabla^2} + g^2\phi_{wall}^2 \pm g|\nabla \phi_{wall}| \right] \varphi_{wall}^{(\pm)} = \left[ E_{wall}^{(\pm)} \right]^2 \varphi_{wall}^{(\pm)}
\]

for the bosonic and fermionic terms, respectively.

If one writes \( \varphi_{wall} = \Psi_{n,l}(r)\Phi_{l,m}(\theta, \varphi) \), and as \( \phi_{wall} \) can be written as a radially symmetric solution, (70) is given by

\[
\left[ -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{l(l+1)}{r^2} + \frac{\lambda}{6} \left( 3\phi_{wall}^2 - \frac{6m^2}{\lambda} \right) \right] \Psi_{n,l}(r) = E_{n,l}^2 \Psi_{n,l}(r)
\]

(71)

\[
\left[ -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{l(l+1)}{r^2} + g^2\phi_{wall}^2 \pm g\frac{d\phi_{wall}}{dr} \right] \Psi_{n,l}^{(\pm)}(r) = \left[ E_{n,l}^{(\pm)} \right]^2 \Psi_{n,l}^{(\pm)}(r)
\]

for the bosonic and fermionic terms, respectively.

For small enough \( h \) one can consider the bubble wall field configuration \( \phi_{wall} \) as given basically by a field configuration like (68), that is, a kink-like field configuration and (71) is reduced to
\[
\left[- \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{l(l+1)}{r^2} - m^2 + 3m^2(T) - 3m^2(T) \text{sech}^2 \left( \frac{m(T)}{\sqrt{2}} r \right) \right] \Psi_{n,l}(r) = E_{n,l}^2 \Psi_{n,l}(r)
\]

(72)

\[
\left[- \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{l(l+1)}{r^2} + \frac{m^2(T)}{2} \left[ S^2 - S(S \pm 1) \text{sech}^2 \left( \frac{m(T)}{\sqrt{2}} r \right) \right] \right] \Psi_{n,l}^{(\pm)}(r) = \left[ E_{n,l}^{(\pm)} \right]^2 \Psi_{n,l}^{(\pm)}(r).
\]

In (72) \(S^2 = \frac{12g^2}{\lambda}\). As \(g\) and \(\lambda\) are arbitrary constants in the model, \(S\) is an arbitrary positive constant. Choosing \(S\) as a positive integer number, the eigenvalues of (72), for low temperatures, can be easily estimated (the differential equations in (72) are equivalent to the Schrödinger equation for a Posch-Teller potential). The eigenvalues \(E_{n,l}^2\) and \(\left[ E_{n,l}^{(\pm)} \right]^2\) are given by (see ref. [15] for instance)

\[
E_{n,l}^2 = \begin{cases} \frac{l(l+1)-2}{R^2}, & n = 0 \\ \frac{l(l+1)}{R^2} + \frac{3}{2}m^2 + \mathcal{O}(\frac{T^2}{m^2 R^2}), & n = 1 \\ k^2 + 2m^2 + \frac{l(l+1)}{R^2} + \mathcal{O}(\frac{T^2}{m^2 R^2}), & n \rightarrow k \text{ (continuum)} \end{cases}
\]

(73)

and

\[
\left[ E_{n,l}^{(\pm)} \right]^2 = \begin{cases} \frac{l(l+1)-2}{R^2} + m^2 p(S - \frac{p}{2}) + \mathcal{O}(\frac{T^2}{m^2 R^2}), & p = \begin{cases} 0, 1, 2, ..., S - 1, & \text{for (+)} \\ 1, 2, ..., S - 1, & \text{for (-)} \end{cases} \\ k^2 + \frac{l(l+1)}{R^2} + \frac{1}{2}m^2 S^2 + \mathcal{O}(\frac{T^2}{m^2 R^2}) \end{cases}
\]

(74)

where, in the equations above, \(R\) is the bubble radius.

For the bosonic eigenvalues we have \(E_{bos}^2 = \left( \frac{2\pi j}{\beta} \right)^2 + E_{n,l}^2\) (at \(T \neq 0\)). Therefore, for \(j = 0\), \(E_{0,0}^2\) corresponds to a negative mode, which is associated with the instability of the critical bubble. For \(j = 0\), \(E_{0,1}^2\) \((m = 0, \pm 1)\) represents the three translational modes related to the space translation invariance of the bubble. At high temperatures
we still may associate the negative and the zero eigenvalues with \( \frac{l(l+1)-2}{R^2} \), for \( l = 0 \) and \( l = 1 \), respectively. Note that, at \( T = T_c \), the critical radius of the bubble must go to infinity, \( R_{cr}(T_c) \to \infty \), and therefore the negative eigenvalue \( E_{0,0}^2 = E_{-}^2 = -\frac{2}{R^2} \) vanishes, as expected.

Taking the limit \( h \to 0^\pm \) is equivalent to taking the limit of infinite bubble radius, \( R \to \infty \). In this case the bubble wall solution \( \phi_{wall} \) tends to the planar wall interface solution \( \phi_{kink} \), eq. (68), and the eigenvalues (73) and (74) become that of the planar wall solution. (In (73) and (74), in the limit \( h \to 0^\pm \) \( (R \to \infty) \), the relevant contribution of the \( l \)–dependence comes from large values of \( l \) such that the sum over \( l \), of the eigenvalues, \( \sum_{l=0}^{\pm \infty} (2l + 1) \), can be replaced by a continuum integration in the two-dimensional momenta parallel to the wall’s surface, \( \int d^2k_\parallel \). )

The evaluation of expression (62), for the surface tension, in terms of the eigenvalues, of course, is not an easily task. However, as shown in the last two sections, we still can use the expansion for the determinants, Eqs. (34) and (58). For example, in the limit \( h \to 0^\pm \), we can easily estimate the extra contribution to the surface tension, coming from fluctuations around the wall field configuration \( \phi_{wall} \). From (34) and (58) one gets, in leading order in \( T \) (that is, from the tadpole graphs, like the one in (44) ), that these extra contribution for the surface tension \( \sigma_0 \), given by (18), is given by (approximating \( \phi_{wall}(r) \) by (68), \( h \to 0^\pm \))

\[
\sigma_T = \sigma_T^B + \sigma_T^F ,
\]

where
\[
\sigma_B^T = \frac{1}{2} \int_{-\infty}^{+\infty} dL \left[ V''(\phi_{kink}(x_L)) - V''(\phi_0) \right] \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m_B^2(\phi_0)}} \left( e^{\beta \sqrt{k^2 + m_B^2(\phi_0)}} - 1 \right)
\]

(76)

for the bosonic contribution and

\[
\sigma_F^T = 2 \int_{-\infty}^{+\infty} dL \left[ g^2 \phi_{kink}^2(x_L) - g^2 \phi_0^2 \right] \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m_F^2(\phi_0)}} \left( e^{\beta \sqrt{k^2 + m_F^2(\phi_0)}} + 1 \right)
\]

(77)

is the fermionic contribution. In (76) and (77), \( \phi_{kink}(x_L) \) is given by (68), \( \phi_0^2 = \frac{6m^2(T)}{\lambda} \), \( m_B^2(\phi_0) = \frac{dV(\phi)}{d\phi^2} |_{\phi=\phi_0(T)} \) and \( m_F^2(\phi_0) = g^2 \phi_0^2(T) = \frac{6m^2(T)g^2}{\lambda} \).

In the high-temperature limit, \( T >> m (\lambda \sim g^2, \lambda << 1) \), we obtain that

\[
\sigma_B^T \simeq -\frac{\sqrt{2}m(T)T^2}{4} + O(T),
\]

(78)

\[
\sigma_F^T \simeq -\frac{\sqrt{2}m(T)g^2T^2}{\lambda} + O(T).
\]

Using that \( S^2 = \frac{12a^2}{\lambda} \), we obtain

\[
\sigma_T \simeq -\frac{\sqrt{2}m(T)T^2}{4} \left[ 1 + \frac{S^2}{3} \right] + O(T).
\]

(79)

where \( m^2(T) \), in leading order in \( T \), is given by \( m^2(T) \simeq m^2 - \frac{\lambda T^2}{24} - \frac{gT^2}{6} \).

6 Conclusions

In this paper we have shown how to compute the nucleation rate in field theory at finite temperature and also we have shown how the prefactor term (the ratio of the determinant)
contribute to the exponential term in the classical nucleation rate, showing that it provides a finite temperature quantum correction to the exponential term.

The procedure we have used is completely consistent and it is based on the usual description of the decay of metastable states developed first by Langer\textsuperscript{[3]}, translated here for finite temperature field theories. Actually, the final expressions that we have obtained for the nucleation rate, Eq. (25) or Eq. (59), are similar the one proposed by Langer, in which the droplet energy $\Delta E$ is replaced by the droplet free energy $\Delta F(T)$, when one takes into account the prefactor term.

For the evaluation of the prefactor term (the determinant ratio) we have used two methods. The first method is based on the explicit evaluation of the eigenvalues. The second procedure is based on a field theoretic expansion for the determinant ratio, such that we avoid the task of computing the eigenvalues of the differential equations. Both methods depend on the knowledge, at least in an approximate way, of a field configuration describing the bubble wall.

We have shown (using the thin-wall approximation) that the nucleation rate has a form that is much like the “classical” one (without taking into account the determinants). In the general expression for the nucleation rate $\Gamma$, we have shown that $\Delta F(T)$ is given by

$$\Delta F(T) = -\frac{4\pi}{3}R^3 \Delta V_{eff}(T) + 4\pi R^2 \sigma(T),$$

where $\Delta V_{eff}(T)$ is just the effective potential contribution that contains the “classical” result plus a quantum correction that is calculable in field theory. The surface term $\sigma(T)$ is a sum of two contributions: a “classical” term plus a temperature dependent quantum correction. The general structure of $\sigma(T)$ is given by Eq. (66).

Due to the similarity between $\Delta F(T)$ and the free energy of the bubble, there is appar-
ently no distinction between our results and analogous ones presented in the literature\cite{1,5}. This will be true as long as one computes $\Delta V_{\text{eff}}(T)$ and $\sigma(T)$ in a consistent way. We have given in this paper examples of how to carry on such a calculation consistently.

It is interesting to contrast the expression that we get for the nucleation rate with expression (1). We have shown that all quantum corrections, due to fluctuations around the bubble solution, to $\Delta S$ in the exponential term in (14) are in the determinant ratio. For the nucleation rate associated to a critical bubble (when $\Delta F(T)$ assumes its maximum value), from (25) and (26) we get the following complete expression for $\Gamma$:

$$\Gamma = I_0(R_{\text{cr}})e^{-\frac{16\pi^3\sigma^3(T)}{3T\Delta V_{\text{eff}}(T)}}$$  \hspace{1cm} (80)

with the preexponential term $I_0(R_{\text{cr}})$ given by

$$I_0(R_{\text{cr}}) = \frac{1}{\pi} \frac{1}{\sqrt{2T R_{\text{cr}}(T)}} \sin \left[ \frac{1}{\sqrt{2T R_{\text{cr}}(T)}} \right] \frac{\Delta E(R_{\text{cr}})}{2\pi T} \frac{1}{T^4}.$$  \hspace{1cm} (81)

In (80) we have used (from the expression for $\Delta F(T)$) that

$$R_{\text{cr}}(T) = \frac{2\sigma(T)}{\Delta V_{\text{eff}}(T)}$$

and in (81) we have used, from Eq. (73), that the negative eigenvalue is given by $E_{-}^2 = \frac{2}{R_{\text{cr}}(T)^2}$, for the model (45) with potential given by (67).

We remark also that in ref. [6] the preexponential factor $I_0(R)$ for critical bubbles was estimated, in the case that $\Gamma$ represents the probability per unit volume per unit time to nucleate a critical bubble of the hadronic phase inside the quark phase, to be\cite{6}

$$I_0(R_{\text{cr}}) = \frac{4}{\pi} \left( \frac{\sigma}{3T} \right)^{\frac{3}{2}} \frac{\sigma(\zeta_q + \frac{4\eta_q}{3})R_{\text{cr}}}{\xi_q^4(\Delta \omega)^2},$$  \hspace{1cm} (82)

where $\zeta_q$ and $\eta_q$ are viscosity coefficients, $\xi_q$ is the correlation length in the quark-gluon phase, $\Delta \omega$ is the difference of enthalpy density between the quark-gluon phase and $\sigma$ is
the surface tension, assumed there to be temperature independent. The authors in [6] have pointed out then that the usual approximation for the preexponential term, like $T^4$ or $T_c^4$, can not be a good approximation, since, from (82), the preexponential is a complicate function of the temperature, diverging as $T \to T_c$. The same is true here, where the preexponential factor is given by Eq. (81).

Despite the fact that we have carried out our analysis by using a thin-wall approximation, that is only valid for nucleation happening at temperatures close to $T_c$, or equivalently, for small supercooling, it is possible to extent the method for other cases of interest, for example, thick-wall bubbles\cite{5,16}, provided that we can find an approximate solution $\phi_0(r)$, describing the bubble field configuration.

\textbf{Acknowledgements}

We thank M.Gleiser for useful discussions and for a critical reading of the manuscript and A. Linde for discussions. We also thank the kind hospitality of the Institute for Theoretical Physics (ITP) in Santa Barbara, CA, where, during the last month of the Cosmological Phase Transitions research program, this work began. This work was supported in part by a National Science Foundation Grant n. PHY89-04035 at ITP and by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq (Brazil).

\textbf{References}

[1] E. W. Kolb and M. S. Turner, \textit{The Early Universe} (Addison-Wesley, Redwood, CA, 1990), chapter 7;
[2] M. E. Shaposhnikov, *Nucl. Phys.* **B287**, 757 (1987); P. Arnold and L. McLerran, *Phys. Rev.* **D36**, 581 (1987); M. Dine, P. Huet, R. S. Singleton Jr., and L. Susskind, *Phys. Lett.* **B257**, 351 (1991); M. E. Schaposhnikov, *Phys. Lett.* **B277**, 324 (1992);

[3] J. S. Langer, *Ann. Phys. (NY)* **41**, 108 (1967); **54**, 258 (1969); J. D. Gunton, M. San Miguel and P. S. Sahni, in *Phase Transitions and Critical Phenomena*, Vol. 8, Ed. C. Domb and J. L. Lebowitz (Academic Press, London, 1983);

[4] M. Gleiser, E. W. Kolb and R. Watkins, *Nucl. Phys.* **B364**, 411 (1991); M. S. Turner, E. J. Weinberg and L. M. Widrow, *Phys. Rev.* **D46**, 2384 (1992);

[5] A. D. Linde, *Nucl. Phys.* **B216**, 421 (1983); [Erratum: **B223**, 544 (1983)];

[6] L. P. Csernai and J. I. Kapusta, *Phys. Rev.* **D46**, 1379 (1992);

[7] G. C. Marques and R. O. Ramos, *Phys. Rev.* **D45**, 4400 (1992);

[8] I. Affleck, *Phys. Rev. Lett.* **46**, 388 (1981);

[9] S. Coleman, *Phys. Rev.* **D15**, 2929 (1977); C. Callan and S. Coleman, *Phys. Rev.* **D16**, 1762 (1977);

[10] L. Dolan and R. Jackiw, *Phys. Rev.* **D9**, 3320 (1974); M. B. Kislinger and P. D. Morley, *Phys. Rev.* **D13**, 2771 (1976);

[11] refs. [1] and [5] above and M. Gleiser et. al. in ref. [4];

[12] C. A. Carvalho, G. C. Marques, A. J. Silva and I. Ventura, *Nucl. Phys.* **B265**, 45 (1986); C. A. Carvalho, D. Bazeia, O. J. P. Eboli, G. C. Marques, A. J. Silva and I. Ventura *Phys. Rev.* **D31**, 1411 (1985);
[13] N. J. Günther, D. A. Nicole and D. J. Wallace, *J. Phys. A: Math. Gen.* **13**, 1755 (1980); K. Binder, *Rep. Prog. Phys.* **50**, 783 (1987);

[14] M. Gleiser, *Phys. Rev.* **D42**, 3350 (1990);

[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill, NY, 1953); R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982);

[16] M. Dine, R. G. Leigh, P. Huet, A. Linde and D. Linde, *Phys. Rev.* **D46**, 550 (1992).