New solvable quantum mechanical potentials
by iteration of the free $V = 0$ potential

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Abstract

A huge family of solvable potentials can be generated by systematically exploiting the factorization (Darboux) method. Starting from the free case, a large class of the known solvable families is thus reproduced, together with new ones. We explicitly find and solve several new singular potentials obtained by iteration from the $V = 0$ case; some of them have an $E = 0$ bound state and constant phase shift without being explicitly scale invariant. The new potentials are rational functions, and can be related to rational solutions of the KdV family.

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1 Introduction

The search for solvable potentials in quantum mechanics is an old and large industry. In this paper we use the factorization method of Darboux ([1],[2]) to formally construct infinite families of fully solvable potentials, all related to the free $V(x) = 0$ case.

The essence of the method is the following. We start from a hamiltonian $H_0$ and a particular solution $(E_0, \phi_0)$ ($D \equiv \frac{d}{dx}$ will be used throughout the paper)

$$H_0 = -D^2 + V_0(x); \quad H_0\phi_0(x) = E_0\phi_0(x) \quad (1.1)$$

The solution $\phi_0$ needs not to be physical, i.e. it might blow up at finite or infinite distances. Then we construct the partner potential

$$V_1(x) - E_0 = W''(x) + W''(x), \quad \text{with } \phi_o(x) \equiv \exp(-W(x)) \quad (1.2)$$

such that the new hamiltonian $H_1 = -D^2 + V_1(x)$ has as solutions

$$H_1\psi_k = E_k\psi_k \quad \text{where } \psi_k = A\phi_k, \quad A = D + W'(x) \quad \text{and } H_0\phi_k = E_k\phi_k \quad (1.3)$$

In other words, all the solutions $(\phi_k, E_k)$ of the first problem generate solutions $(A\phi_k, E_k)$ of the new problem. But as $A\phi_0 = 0$, the new solution is $\psi_0 = \phi_o^{-1}$. At times, some solutions have to be excluded by physical reasons.

We shall need also the second solution from the known one $\psi_1$ at the same energy; it is

$$\psi_2 = \psi_1 \int \psi_1^{-2} dx \quad (1.4)$$

These results are fairly well-known; for convenience of the reader we supply simple proofs of the above statements in Appendix I.

Our program is to start with the zero potential (free particle) $V(x) = V_0 = 0$ and iterate new potentials from its solutions. There are four types of different potentials, from four solutions as follows: [3]

$$(E = k^2 > 0, k = 1): \quad \phi = \cos(x), \Rightarrow V_1(x) = +2\sec^2(x)$$

$$(E = 0): \quad \phi = x, \Rightarrow V_1(x) = \frac{2}{x^2}$$
\[ (E = -k^2 < 0, k = 1) : \phi = \cosh(x), \Rightarrow V_1(x) = -2 \cosh^{-2}(x) \]
\[ \phi = \sinh(x), \Rightarrow V_1(x) = +2 \sinh^{-2}(x) \] (1.5)

Notice the last three "solutions" are unphysical; a fifth potential \( V_1(x) = 2 \csc^2(x) \) is just the first one displaced \( \frac{\pi}{2} \).

The complete solutions for these first-step potentials are obtained by pulling from the free solution by the corresponding \( A \) operator, see (1.3). Here we recall only the situation for one of the more important cases, providing the only case with a potential valid in the whole straight line \((-\infty, +\infty)\):

\[ V(x) = -2 \cosh^{-2}(x), \quad W'(x) = -\tanh(x) \] (1.6)

There is a ground state

\[ \psi_0 = \phi_0^{-1}(x) = \cosh^{-1}(x) \quad (unnormalized) \] (1.7)

and scattering solutions

\[ \psi_k(x) = (D - \tanh(x)) \exp(ikx) = (ik - \tanh(x)) \exp(ikx) \] (1.8)

corresponding to a transparent (reflectionless) potential, with pure transmission

\[ t(k) = \text{transmission} = \frac{ik - 1}{ik + 1} \] (1.9)

This potential is critical, having an \( E = 0 \) resonance; in fact, all transparent potentials are critical [4]; our potential (1.6) corresponds to a single soliton.

## 2 Study of a second-step potential

The power of the method can be seen now; as the \( V(x) = 0 \) case is trivially solvable for any energy \( E \), physical or unphysical, we have now four potentials, all fully solvable; and from each solution of each energy of each potential, we can in principle obtain a new, still fully solvable potential. In this paper we shall elaborate only in the families associated to the \( E = 0 \) (intermediate) case.

We start now from the "centrifugal" potential \( V(x) = \frac{2}{x^2} \) and recall the (unphysical!) solutions: one is \( \frac{1}{x} \), as \( \phi_0(x) = x \) is the starting solution for the
Figure 1: The potential (2.10), with two independent parts ($\mu \equiv 1$)

$V(x) = 0$, $E = 0$ case; the other is (cf. (1.4)) $\frac{1}{2} \int x^2 dx \sim x^2$. Although both solutions blow up at $x = 0$ and $x = \infty$ respectively, they are instrumental in obtaining a one-parameter family of bona-fide, physical potentials:

From the general $E = 0$ wavefunction $\phi(x) = \frac{a}{2} + bx^2$, with $W''(x) = -\frac{\phi'(x)}{\phi(x)}$ and $\mu \equiv \frac{a}{b} > 0$ we get the new, second-step potential family

$$V_\mu(x) = W'^2(x) + W''(x) = \frac{6x(x^3 - 2\mu)}{(x^3 + \mu)^2}$$

(2.10)

The new potential(s) is singular: it has a double pole at $x = -c$, $c \equiv \mu^{1/3}$, $0 < \mu < \infty$. For $x > -c$ it has an attractive part, and a repulsive tail; for $x < -c$ is purely repulsive; both interpolate between

$$V(x) \sim \frac{2}{(x + c)^2} \ (x \simeq -c) \ldots V(x) \sim \frac{6}{|x|^2} \ (x \to \pm \infty)$$

(2.11)

See Fig.1

Both $V_1 = \frac{2}{x^2}$ and our potential (2.10) correspond to some rational solutions of the KdV equation [5]; the relation is interesting and we elaborate on it in Appendix II.
Of course, the barrier at \( x = -c \) is impenetrable: we have two different physical problems.

1) **Case** \( x \geq -c \). A *bound state* candidate with \( E = -\kappa^2 < 0 \) would behave like \( \exp(-\kappa x) \) at large \( x \), so we try the \( E < 0 \) (unphysical) solution \( \phi(x) = \exp(-\kappa x) \) of the \( V(x) = 0 \) case and *prolongate it twice*, as explained above. We find

\[
\psi_0(x) = A_2 A_1 \phi(x) = \left( D - \frac{2x^3 - \mu}{x(x^3 + \mu)} \right) \exp(-\kappa x)
\]

\[
= (\kappa^2 + 3x \frac{1 + \kappa x}{\mu + x^3}) \exp(-\kappa x)
\]

For this wavefunction to be physical it has to be zero at the singularity: this leads to the eigenvalue equation \( 1 + \kappa(x = -c) = 0 \). Hence there is a single bound state with energy \( E = -\kappa^2 \), \( \kappa = 1/c = \mu^{-1/3} \), and whose (un-)normalized wavefunction is

\[
\psi_0(x) = (1/c^2 + \frac{3x/c}{x^2 - cx + c^2}) \exp(-x/c) = \frac{(x/c + 1)^2}{x^2 - cx + c^2} \exp(-x/c)
\]

which behaves in the expected way for a ground state: nodeless, normalizable, decaying fast at \( x \to \infty \). By construction, this state is the only bound state.

For \( E = k^2 > 0 \) we have total reflection; we write the wavefunction as

\[
\psi_k(x) = A_2 A_1(a \exp(ikx) + b \exp(-ikx))
\]

and impose \( \psi_k(x = -c) = 0 \); this fixes \( a/b \) as a phase,

\[
a/b = \frac{1 - ikc}{1 + ikc} \exp(2ikc)
\]

From the asymptotic behaviour we extract the S-matrix as usual in scattering in one radial dimension

\[
\psi_k(x >> 0) \equiv \exp(-ik(x + c)) - S(k) \exp(ik(x + c))
\]

and comparing with (2.14), we derive

\[
S(k) = \frac{1 - ikc}{1 + ikc}
\]
Or, for the phase shift $S(k) \equiv \exp(2i\delta(k))$

$$\delta(k) = -\arctan(kc) \mod \pi$$  \hspace{1cm} (2.18)

which has to be interpreted carefully: with centrifugal tails $V(x) \to \frac{A}{x^2}$, for $x \gg 0$ the usual rule $\delta(\infty) = 0$ does not apply. The interpretation of (2.18) is as follows:

At $k = 0$ , the bound state contributes $+\pi$ to the phase shift (Levinson's theorem) and the long tail ($x \gg 0$) of the potential, which is $6/x^2 \equiv l(l+1)/x^2$ ($l = 2$), contributes $-2(\pi/2)$; hence, $\delta(k = 0) = 0$. At very large $k$, the phase shift is dominated only by the short tail, still centrifugal $+2/(x+c)^2$, which should produce a $-\pi/2$ shift. All this is reproduced by (2.18) with the determination $\arctan(0) = 0$.

Notice the S-matrix (2.17) is about the simplest with the pole at the bound state $k = +i/c$ : this is very similar to the forward amplitude for the solitonic scattering (1.9): it seems that the fact that there is a single bound state determines the phase shift, and other features of the potential are somehow irrelevant.

2) Case $x < -c$. Here there is also total reflection, but obviously no bound state, and an analogous calculation gives the S-matrix as inverse of the previous one, and we get

$$\delta(k) = +\arctan(kc) \mod \pi$$ \hspace{1cm} (2.19)

At $k = 0$ the long tail contributes $-2(\pi/2)$, hence we determine $\arctan(0) = -\pi$; as $k \to \infty$, the short tail dominates with $\delta(\infty) = -\pi/2$; of course, the only invariant statement is the difference, that is, the span $\Delta \equiv \delta(0) - \delta(\infty)$.

A surprising property of the potential (2.10) has to do with the golden ratio $\Phi \equiv (1 + \sqrt{5})/2$ : for $x > 0$, the maximum $x_M$ and minimum $x_m$ of $V(x)$ in (2.10) are

$$x^3_m = (2 + 3\Phi), \quad x^3_M = (2 - 3/\Phi) = x^3_m \ (\Phi \to -1/\Phi)$$ \hspace{1cm} (2.20)

and the same happens for the values of the potential:

$$V(x_m) = 2\Phi \left(\frac{2 + 3\Phi}{1 + \Phi}\right)^{1/3}, \quad V(x_M) = V(x_m) \ (\phi \to -1/\Phi) \hspace{1cm} (2.21)$$

\footnote{It is well known, e.g. in 3D scattering, that a purely centrifugal potential $V_{cent} = l(l+1)/x^2$ produces a negative constant phase shift $\delta_{cent}(k) = -l\pi/2$}
While we do not fully understand this relation, we notice the same thing appears in the KdV for two solitons with velocities \( k_1 \) and \( k_2 \): \( \Phi = k_2/k_1 \), separates the overlapping and non-overlapping profiles ([5], p. 190; the discovery seems due to Lax) ; it is another intriguing connection between solitons and special potentials.

3 Some generalizations

For the next step we start with the potential \( V(x) = 6/x^2 \), take the general \( E = 0 \) solution \( \phi(x) = \mu/x^2 + x^3 \) and construct, as before, the new, interesting potential

\[
V(x) = \frac{2}{x^2} \frac{6x^{10} - 18\mu x^5 + \mu^2}{(x^5 + \mu)^2} \tag{3.22}
\]

that we plot in Fig.2

This potential contains three disconnected pieces:
- I- \( x > 0 \) . Attraction plus repulsion.
- II- \( -c < x < 0 \) . A confining potential (\( c \equiv \mu^{1/5} > 0 \))
- III- \( x < -c \) . A repulsive potential.
The solutions are again straightforward but tedious, the procedure to obtain them is as in the previous case, and we just indicate and quote the results:

I- $x > 0$. There is a single bound state with $k = 0$ and (unnormalized) wavefunction

$$\psi_0(x) = \frac{x^2}{\mu + x^5} = \phi^{-1}$$

(3.23)

and total reflection with wavefunction

$$\psi_R(x) = A_3 A_2 \phi_k(x)$$

(3.24)

with

$$A_3 = D + W' = D + \frac{2\mu - 3x^5}{x(\mu + x^5)}$$

(3.25)

and $A_2 = D - 2/x$ , $A_1 = D - 1/x$ . Hence, if $S(k)$ is the S-matrix for the previous $V(x) = 6/x^2$ and $\hat{S}(k)$ the new one,

$$\psi_k(x >> 0) = (D + W'(\infty))\Phi_k(x >> 0) = (D + W'(\infty))(\exp(-ikx) - S(k)\exp(ikx))$$

$$= N(\exp(-ikx) - \hat{S}(k)\exp(ikx)), \text{ hence}$$

$$\hat{S}(k) = S(k)(W'(\infty) + ik)/(W'(\infty) - ik) = (1)(-1) = -1$$

(3.26)

because $W'(\infty) = 0$ and $S(k)$ , due to $2(2 + 1)/x^2$ , is $= +1$. So

$$\hat{S}(k) = 1 \quad \text{or} \quad \delta(k) = \frac{\pi}{2} \text{ mod } \pi!$$

(3.27)

These results are worth commenting: First, the $E = 0$ bound state is obvious, because $A\phi = 0$ in the previous $V = 6/x^2$ potential implies $A^l\phi^{-1} = 0$, and $\phi^{-1}$ zero-less, normalizable. Because of the repulsive tail, it is a bona fide bound state, not an $E = 0$ resonance, so it will contribute $+\pi$ to Levinson’s theorem [7].

The constant phase shift is suspicious of some kind of scale invariance.

In fact, an scale-invariant bound state can exist if at all, at $E = 0$ ; this is our case! The interpretation of the phase shift “span” is this: for $k \rightarrow 0$, the bound state contributes $+\pi$ , the long tail $12/x^2$ gives $-3\pi/2$ : so $\delta(0) = -\pi/2$ , or $S(k = 0) = -1$ . At $k \rightarrow \infty$ , the short tail contributes $-\pi/2$ , so $\delta(k = \infty) = -\pi/2$ , and the total span of $\delta(k)$ is zero (while it was $+\pi/2$ in the previous case).
It is remarkable that a variable (i.e., not purely centrifugal) potential, indeed supporting a \((E = 0!\) bound state is still “conformal” and produces constant phase shift. We offer the following explanation:

The previous potential \(V_0(x) = 6/x^2\) is manifestly scale invariant:

\[
[D, H_0] = -2H_0 \quad \text{where} \quad H_0 = -D^2 + V_0 = A^\dagger A
\]

(3.28)

and \(\hat{D} \equiv x \cdot D\) is a dilatation generator. Now

\[
H_1 = AA^\dagger = A \cdot (A^\dagger \cdot A) \cdot A^{-1} = A \cdot H_0 \cdot A^{-1}
\]

(3.29)

Hence

\[
[D_A, H_1] = -2H_1 \quad \text{with} \quad \hat{D}_A = A \cdot \hat{D} \cdot A^{-1}
\]

(3.30)

(Notice \(A\) is invertible outside the bound state). Now for \(x \gg 0\), \(W'(\infty) = 0\), so \(A = D + W' \to D\), and therefore

\[
\hat{D}_A \to D(x \cdot D)D^{-1} = x \cdot D + 1 = \hat{D} + 1
\]

(3.31)

That is: the traslated symmetry of the new hamiltonian still guarantees constancy of the phase shift.

To the best of our knowledge, this is a first case of a potential, not purely centrifugal, with constant phase shift.

II- \(-c < x < 0\). This confining potential produces of course an uninteresting, infinite ladder of bound states, reminiscent of the potential \(V(x) = -2 \cosh^{-2}(x)\) alluded to in Sect. 1. The eigenvalues are \(E_n = k_n^2\) where

\[
\tan(k_n c) = \frac{3k_n c}{3 - (k_n c)^2}
\]

(3.32)

which is a simple transcendent equation with infinite roots \(0 < k_1 < k_2 < k_3 < ...\) which tend to \(n\pi\) for \(n \gg 1\); hence the spectrum is asymptotically parabolic, as for a particle in an infinite box; this is to be expected, as the potential (also in the \(V(x) = 2 \sec^2(x)\) case) is negligible for higher excited wavefunctions. In fact, the normalizable wavefunctions can be written easily, but we refrain of doing it.

III- \(x < -c\). At the left, a purely repulsive potential produces only total reflection, and the S-matrix and the phase shift are computed to be

\[
S(k) = \frac{(kc)^2 - 3ikc - 3}{-(kc)^2 - 3ikc + 3}, \quad \tan(\delta + \pi/2) = \frac{-3kc}{(kc)^2 - 3}
\]

(3.33)
So the total span $\delta(0) - \delta(\infty)$ is now $= \pi$, and the phase shift is \textit{not} constant, going smoothly from $-3\pi/2$ to $-\pi/2$ in the interval $k = 0 \to k = \infty$. Of course, “conformal” invariance has been lost because the singular point is at $x = -c$, not at $x = 0$.

From the many possible generalizations, we consider in this paper just one more case: the general partner of the $n$-step manifest scale invariant potential $V_0(x) = n(n+1)/x^2$. The two $E = 0$ solutions (both unphysical again) are $x^{n+1}$ and $x^{-n}$; so defining

$$\phi(x) = \mu/x^n + x^{n+1} \quad \mu > 0$$

(3.34)

the partner family is

$$V_\mu(x) = \frac{(n+1)(n+2)x^{2n+2} - 6\mu n(n+1)x^{2n+1} + \mu^2 n(n-1)}{x^2(\mu + x^{2n+1})^2}$$

(3.35)

which again exhibits the three regions as before. In particular

$$x > 0: \text{a partly attractive potential, which supports again just a bound state at zero energy}; \text{ total reflection occurs with (again) \textit{constant} phase shift.}$$

The bound state is $\phi^{-1}$, of course, and it turns out that

$$S(k) = (-1)^{n+1}$$

(3.36)

by the same argument as before, namely $S(k) = -S_n(k)$, where $S_n(k)$ is the S-matrix for $V_0(x) = n(n+1)/x^2$, namely $S_n(k) = (-1)^n$.

We have therefore found and infinite family of “scale” invariant potentials, with a unique $E = 0$ normalizable bound state, and constant (in fact $\pm 1$) S-matrix. The (modified) Levinson theorem applies; namely the span $\delta(0) - \delta(\infty)$ is zero; at low $k$, there is a $+\pi$ contribution from the bound state, and $-n(n+1)\pi/2$ value from the long tail. At large $k$, the short tail takes over, contributing $-(n-1)\pi/2$. The constancy of $\delta(k)$ comes, as before, from the appropriate conjugation of the manifest dilatation symmetry of the previous potential, just as in the worked-out case $n = 2$.

In the confining region $-c < x < 0$, with $c = +\mu^{1/(2n+1)} > 0$, there is a pure point spectrum, with again a limiting parabolic growth in the energy. The spectral equation is a natural generalization of (3.32); we state only the next case, $n = 3$; the transcendental eigenvalue equation is

$$\tan(kc) = \frac{7(kc)^3 - 105(kc)}{42(kc)^2 + 105}$$

(3.37)
Finally, in the pure repulsive part of the potential, \( x < -c \), there is only total reflection with a simply variable phase shift. The total span is

\[
\delta(0) - \delta(\infty) = -(n + 1)\pi/2 - (-\pi/2) = -n\pi/2 \quad (3.38)
\]

because the potential behaves like \(+2/(x + c)^2\) close to the pole. The exact S-matrix can be calculated as before. We just quote the result only again for \( n = 3 \):

\[
S(k) = \frac{42(kc)^2 + 105 - i(7(kc)^3 - 105(kc))}{(\text{complex conjugate})} \quad (3.39)
\]

The general \( S(k) \) starts at \( S(0) = +1 \) for \( n = 3, 5, 7, ... \), and \( S(0) = -1 \) for \( n \) even; it becomes \( S(\infty) = -1 \) after \( n \) half-turns. The phase shift connects smoothly \( -(n + 1)\pi/2 \) at \( k = 0 \) with \( -\pi/2 \) at \( k = \infty \).

We can see also why the first case \( n = 1 \) is special: at right the potential is \(+6/x^2\) for \( n = 2 \), and at \( x = 0 \) is \( V = 0 \), as \( n - 2 = 0 \) so in this case there are only two regions with no confining part.

### 4 Other potentials

Once the general procedure is understood, is a matter of mechanical calculations to find and to solve any other \( V = 0 \)-related potentials. We shall report on a full investigation elsewhere [11].

Here we just report that we can, by our procedure, recover many of the “shape invariant” potentials in the review Infeld-Hull paper [1]; in fact, all the families included in the “A-type” classification of [1]. The other types B...I are in some way degenerate: they include, among others, the oscillators, Kepler and Morse potentials, which are not directly connected to the \( V = 0 \) case, but still are “shape invariant” and solvable. As shown in [11], the Kepler problem is related to the \( V = 0 \) potential in a constant curvature (spherical for bound states) space.

The natural minimal generalization of \( V = 2/x^2 \) is obviously the centrifugal potential

\[
V(x) = \frac{n(n + 1)}{x^2} \quad n = 0, 1, 2... \quad (4.40)
\]

This is obtained from \( V = 0 \) by making use of the solutions \( x, x^2, x^3, ..., x^n \) in each step.
The minimal natural extension of $V = 2 \sec^2(x)$ is

$$V(x) = +n(n + 1) \sec^2(x) \quad n = 0, 1, 2... \quad (4.41)$$

The intertwining superpotential satisfies $W_n'(x) = n \tan(x)$, with $\phi_n(x) = \cos^n(x)$ as the generating wavefunction.; notice the energy scale gets displaced; this confining-potential family contains a pure discrete spectrum, approaching the parabolic infinite-box situation.

Similarly

$$V(x) = -n(n + 1) \cosh^{-2}(x) \quad n = 0, 1, 2... \quad (4.42)$$

comes from $W_n'(x) = -n \tanh(x)$ and $\phi_n(x) = \cosh^n(x)$.

There are $n$ bound states and a $E = 0$ resonance, plus perfect transmission (no reflection); it is the well known “$n$-solitonic” potential, with all the elementary solitons on top of each other at $x = 0$ [5].

The final minimal family is

$$V(x) = +n(n + 1) \sinh^{-2}(x) \quad n = 0, 1, 2,... \quad (4.43)$$

This comes from $W_n'(x) = +n \coth(x)$ and $\phi_n(x) = \sinh^n(x)$. It corresponds to total reflection, with variable phase shifts, and no bound states.

All these four families are still exactly solved even for $n \to \lambda$ noninteger, by “prolongation” (see [1] or the review [8]) ; they are shape-invariant [12] and therefore included in [1]. As they are not related directly with the vacuum $V = 0$ case, we do not discuss them.

The only potential of “A” type of [1] not include so far is

$$V(x) = \frac{a + b \cos(x)}{\sin^2(x)} \quad (4.44)$$

This can still be also obtained in our scheme in the following, indirect way: the potential

$$V(x) = \frac{3/4}{\sin^2(x)} \quad (4.45)$$

is a prolongation of the $V_1(x) = 2 \csc^2(x)$ of § 1, and it admits the unphysical eigenfunction

$$\phi(x) = \sqrt{\sin(x)} \cot(x/2) \quad (4.46)$$
Hence the corresponding partner potential is, with \( W'(x) = -\cot(x/2) + \csc(x) \),

\[
V_1(x) = \frac{7/4 - 2\cos(x)}{\sin^2(x)}
\]  (4.47)

which is of type (4.44). The energy of the unphysical solution \( \phi \) is \(+1/4\), whereas the ground state of (4.47) is \( \sin^{3/2}(x) \), with energy = \(+9/4\). We conclude that the “A” type family of Infeld-Hull [1] can be included also in our scheme of things.

5 Conclusion

The whole set of analytically soluble potentials (not to speak of the quasi-soluble ones [13]) is very, very large. In this paper we have shown how starting with the free case, \( V(x) = 0 \), and just by playing around with the unphysical solutions for \( E = 0 \) only, a large family is obtained; the generic case includes a confining potential defined in a segment of the line, a purely repulsive half-line defined potential, and an also half-line defined potential, supporting a \textit{bona fide} unique \( E = 0 \) bound state with trivial (i.e. constant \( = \pm 1 \)) S-matrix.

The natural generalization of the four different potentials obtained in the first step from \( V(x) = 0 \) includes all the non-degenerate cases in the Infeld-Hull series, if we include prolongations, that is, substituting \( n(n+1) \ n \in \mathbb{N} \), by \( \lambda(\lambda + 1) \) for arbitrary, real positive \( \lambda \). They correspond to solutions of the hypergeometric equation, which is also related to the \( SL(2, R) \) group. ; the degenerate I-H cases B...I (i.e. Coulomb,...) correspond to solutions of the \textit{confluent} hypergeometric equation.

There is still work in progress; we have not exhausted even the \( E = 0 \) family (for example, we can iterate the potential (3.35)!). As stated, we plan to report on other cases in a later publication; see also [11]

The two Appendices explain the Darboux method and elaborate on the KdV connection, as promised.
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A Appendix

We probe the results used in §1. From $H\Psi = (-D^2 + V)\Psi = E\Psi$, define

$$\exp(-W(x)) \equiv \Psi \quad (A.1)$$

$W$ satisfies a Riccati first order equation $V - E = W'^2 - W''$: the scale invariance $\psi \to \lambda \psi$ becomes translation invariance for $W$, hence $W$ itself does not appear in the new equation. $\psi$ needs not to be physical, i.e. it can be singular. But now the hamiltonian factorizes:

$$-D^2 + W'^2(x) - W''(x) = (D + W'(x))(D + W'(x)) = A^\dagger A \quad (A.2)$$

with $A = D + W'(x)$. We obtain

$$A^\dagger A \Psi = (H - E)\Psi \quad (A.3)$$

The partner hamiltonian is defined as $H' = AA^\dagger + E$, so

$$H\phi' = E'\phi' = (A^\dagger A + E)\phi' \quad \Rightarrow \quad (AA^\dagger + E')A\phi' = E'A\phi' \quad (A.4)$$

For each solution $(\phi', E')$ of the former $H$ we obtain a solution $(A\phi', E')$ of the new $H'$. Of course, $\phi'$ might be physically unacceptable. This is the essence of the method.

Now for the second solution. $\phi = \exp(-W)$ implies $A\phi = 0$, $A = D + W'$. Or $(\exp(-W) \cdot D \cdot \exp(+W)) \exp(-w) = 0$, hence $(\exp(+W) \cdot D \cdot \exp(-W))\phi^{-1} = 0$, $A^\dagger \phi^{-1} = 0$ so $\phi^{-1}$ corresponds to $\phi$ for $E' = E$. Now if $\phi'$ is the second solution of the original $H$ with energy $E$, $A^\dagger A\phi' = 0$ but $A\phi' \neq 0$, hence $A\phi'$ is in the kernel of $A^\dagger$, and therefore

$$\exp(-W) \cdot D \cdot \exp(+W)\phi' = \exp(W), \quad \text{or} \quad \phi' = \phi \int \phi^{-2} dx \quad (A.5)$$

as stated.

The use of the second solution to generate new potentials seems to start with [9]; see also the previous work of Abraham and Moses [10].
Appendix

The KdV equation \((u = u(x,t), \ U_{,t} = \partial u / \partial t \text{ etc.})\)

\[
   u_{,t} = 6uu_{,x} - u_{,xxx} \tag{B.1}
\]
is one of the deformation equations associated to the Schrödinger equation, \(H(\mu) = -D^2 + u(x,\mu)\), where \(\mu(= t)\) is the deformation parameter. For this reason some simple solutions of KdV are interesting potentials for the linear problem; we take some results from [5]. The travelling wave solution \(u = u(x - vt)\)

\[
   u(x,t) = -(1/2)v \cosh^{-2}(\sqrt{v}(x - vt - x_0)) \tag{B.2}
\]
corresponds to our first step with \(E < 0\) and \(\phi = \cosh(x)\); it is the solitonic potential. Multisolitonic potentials correspond to iteration from this solution, but these are not considered in this paper.

Rational solutions of KdV are closer to our potentials; for example \(u = 2/x^2\) arises as the simplest t-independent rational solution, and it is our first potential from the \(E = 0\), \(\phi = x\) solution. The natural scaling invariance of KdV \(x \rightarrow \lambda x, \ t \rightarrow \lambda^3 t, \ u \rightarrow \lambda^{-2} u\) leads at once to the rational solution

\[
   u(x,t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2} \tag{B.3}
\]
which is our potential (2.10) (with \(\mu = 12t\)). Similarly the other rational potentials we obtain are connected with rational solutions of higher-order KdV-hierarchy equations; we shall report on a full investigation elsewhere.

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