Deconfined Quantum Criticality at the Quantum Phase Transition from Antiferromagnetism to Algebraic Spin Liquid

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We investigate the quantum phase transition from antiferromagnetism (AF) to algebraic spin liquid (ASL). We propose that spin 1/2 fermionic spinons in the ASL fractionalize into spin 1/2 bosonic spinons and spinless fermions at the quantum critical point (QCP) between the AF and the ASL. Condensation of the bosonic spinons leads to the AF, where the condensed bosonic spinons are confined with the spinless fermions to form the fermionic spinons. These fermionic spinons are also confined to make antiferromagnons as elementary excitations in the AF. Approaching the QCP from the AF, spin 1 critical antiferromagnetic fluctuations are expected to break up into spin 1/2 critical bosonic spinons. Then, these bosonic spinons hybridize with spin 1/2 fermionic spinons, making spinless fermions. As a result the fermionic spinons decay into the bosonic spinons and the spinless fermions. But, the spinless fermions are confined and thus, only the bosonic spinons emerge at the QCP. This coincides with the recent studies of deconfined quantum criticality.

When the bosonic spinons are gapped, the ASL is realized. The bosonic spinons are confined with the spinless fermions to form the fermionic spinons. These fermionic spinons are deconfined to describe the ASL.

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I. INTRODUCTION

Nature of quantum criticality is one of the central interests in modern condensed matter physics. Especially, deconfined quantum criticality has been proposed in various strongly correlated electron systems such as low dimensional quantum antiferromagnetism on square lattices, frustrated lattices, and heavy fermion liquids. In the present paper we focus our attention on the quantum antiferromagnetism in two dimensional square lattices. Starting from the antiferromagnetic Heisenberg model, one can derive the O(3) nonlinear σ model (NLσM) as an effective field theory. Utilizing the CP^1 representation of the NLσM, one can show that although the appropriate off-critical elementary degrees of freedom are given by either spin 1 excitons (gapped paramagnons) in the paramagnetism or spin 1 antiferromagnons in the antiferromagnetism, at the quantum critical point (QCP) such excitations break up into more elementary spin 1/2 excitations called spinons. This is the precise meaning of the deconfined quantum criticality in the context of quantum antiferromagnetism.

In the present paper we investigate the quantum phase transition from antiferromagnetism (AF) to algebraic spin liquid (ASL) based on the deconfined quantum criticality of the O(3) NLσM. According to one possible scenario, the pseudogap phase in high Tc cuprates is proposed to be the ASL for spin degrees of freedom. In this respect it is considerable to understand how the ASL emerges from the AF, i.e., the quantum phase transition between the AF and the ASL. The ASL is the state described by quantum electrodynamics in two space and one time dimensions (QED_3) in terms of massless Dirac spinons interacting via U(1) gauge fluctuations. On the other hand, low energy physics of the AF is well understood by the O(3) NLσM in terms of Neel order parameter fields. One can say that the AF results from the ASL via spontaneous chiral symmetry breaking (SχSB). The central question in this paper is the nature of the QCP, where the SχSB occurs. Concretely speaking, we examine how the Neel order parameter fields in the NLσM can be smoothly connected to the fermionic spinons in the QED_3 by studying the QCP. We apply the deconfined quantum criticality of the O(3) NLσM to the QCP between the AF and the ASL. Approaching the QCP from the AF, critical antiferromagnetic spin fluctuations in the NLσM are conjectured to break up into critical bosonic spinons, consistent with the previous studies. Based on this conjecture we propose that the critical bosonic spinons hybridize with the fermionic spinons in the QED_3 via Kondo-like couplings, screening out spin degrees of freedom of the fermionic spinons and thus, making spinless fermions. In other words, it can be stated that the fermionic spinons in the ASL fractionalize into the bosonic spinons and the spinless fermions at the QCP. In this paper we construct a critical field theory at the QCP and discuss how the AF and the ASL recover from the critical field theory.

In the language of high energy physics the present study examines the chiral phase transition in QED_3. In this respect we would like to point out an interesting previous study of the chiral phase transition. The authors calculated a fermion-antifermion scattering amplitude in order to find light scalar mesons (fermion-antifermion composites) showing up as poles in the scattering amplitude. Approaching the QCP from the symmetric phase (ASL in the context of quantum
antiferromagnetism), the authors could not find the poles in the scattering amplitude. This leads them to the conclusion that there are no light mesons corresponding to order parameter fluctuations. Based on this result they proposed that the chiral phase transition is different from usual second order phase transitions described by Landau-Ginzburg-Wilson type theories of order parameters. Our present scenario of the deconfined QCP for the chiral phase transition seems to be consistent with this previous study since the deconfined QCP is not described by fluctuations of fermion-antifermion composite order parameters owing to spin fractionalization.

II. FERMIONIC NONLINEAR $\sigma$ MODEL AS AN EFFECTIVE FIELD THEORY FOR QUANTUM ANTIFERROMAGNETISM

We consider the following effective action called fermionic nonlinear $\sigma$ model

$$Z = \int D\bar{\psi}_0 D\psi_0 e^{-S_{ASL} - S_M - S_{NL\sigma M}},$$

$$S_{ASL} = \int d^4x [\bar{\psi}_0 \gamma_\mu (\partial_\mu - ia_\mu) \psi_0 + \frac{1}{2e^2} |\partial \times a|^2],$$

$$S_M = \int d^4x m_\psi \bar{\psi}_0 \bar{\sigma}_\sigma \psi_0 \cdot \vec{n},$$

$$S_{NL\sigma M} = \int d^4x \left[ \frac{N m_\psi}{4\pi} |\partial \vec{n}|^2 - i\lambda (|\vec{n}|^2 - 1) \right].$$

In $S_{ASL}$, $\psi_0$ is a massless Dirac spinor with a flavor index $\sigma = 1, ..., N$ associated with SU(N) spin symmetry. $a_\mu$ is a compact U(1) gauge field mediating long range interactions between Dirac spinors. $e$ is an internal electric charge of the Dirac spinon. $S_{ASL}$ in Eq. (1) was proposed to be an effective field theory for one possible quantum paramagnetism (ASL) of SU(N) quantum antiferromagnets on two dimensional square lattices. However, the stability of the ASL has been suspected owing to instanton excitations of compact U(1) gauge fields. Condensation of instantons (magnetic monopoles) is believed to cause confinement of charged particles, here Dirac spinons. Recently, Hermele et al. showed that the ASL can be stable against instanton excitations. Ignoring the compactness of U(1) gauge fields, one can show that the $S_{ASL}$ has a nontrivial charged fixed point in two space and one time dimensions $[(2 + 1)D]$ in the limit of large flavors, identified with the ASL. Hermele et al. showed that the charged critical point in the case of noncompact U(1) gauge fields can be stable against instanton excitations of compact U(1) gauge fields in the limit of large flavors. Condensation of instantons can be forbidden at the stable charged fixed point owing to critical fluctuations of Dirac fermions. The $S_{ASL}$ in Eq. (1) is a critical field theory at the charged critical point, where correlation functions exhibit power law behaviors with anomalous critical exponents resulting from long range gauge interactions. This is the reason why the state described by the $S_{ASL}$ is called the ASL. In appendix A we briefly discuss how the effective $QED_3$, $S_{ASL}$ in Eq. (1) can be derived from the antiferromagnetic Heisenberg model.

However, we should remember that the ASL criticality holds only for large flavors of critical Dirac spinons. If the flavor number is not sufficiently large, the internal charge $e$ is not screened out satisfactorily by critical Dirac fermions. Then, gauge interactions can make bound states of Dirac fermions, resulting in massive Dirac spinons. This is known to be spontaneous chiral symmetry breaking ($S\chi SB$). In the case of physical SU(2) antiferromagnets it is not clear if the ASL criticality remains stable owing to the $S\chi SB$ causing antiferromagnetism. It is believed that there exists the critical flavor number $N_c$ associated with $S\chi SB$ in $QED_3$. But, the precise value of the critical number is far from consensus. If the critical value is larger than 2, the $S\chi SB$ is expected to occur for the physical $N = 2$ case, resulting in massive spinons. On the other hand, in the case of $N_c < 2$ the ASL criticality remains stable against the $S\chi SB$. Experimentally, an antiferromagnetic long range order is observed in the SU(2) antiferromagnet. This leads us to consider the $S\chi SB$ in the ASL. In Eq. (1) $S_M$ represents the contribution of a fermion mass due to the $S\chi SB$. $m_\psi$ is a mass parameter corresponding to staggered magnetization. $\vec{n}$ represents fluctuations of Neel order parameter fields regarded as Goldstone bosons in the $S\chi SB$. $\bar{\sigma}$ is Pauli matrix acting on the spin (flavor) space of Dirac spinons. The mass parameter $m_\psi$ can be determined by a self-consistent gap equation, given by $m_\psi \approx e^2 \exp[-2\pi/\sqrt{N_c/N} - 1]$ in the $1/N$ approximation. Varying the flavor number $N$ controls the mass parameter $m_\psi$. In the present paper we use the mass $m_\psi$ as a controlling parameter of the quantum phase transition. This is analogous to a quantum phase transition in the $NL\sigma M$, where the quantum phase transition is realized by varying the spin stiffness parameter associated with the value of spin in the system. For completeness of this paper we briefly sketch the derivation of dynamical mass generation in appendix B. There are additional fermion bilinears connected with the Neel state by "chiral" transformations because the $S_{ASL}$ in Eq. (1) has more symmetries than those of the Heisenberg model. These order parameters are associated with valance bond orders. But, in the present paper we consider only the Neel order parameter in order to take account of the AF described by the O(3) $NL\sigma M$.

Contributions of high energy spinons in $S_{ASL} + S_M$ lead to the $S_{NL\sigma M}$ in the gradient expansion. This effective action is nothing but the $O(3) NL\sigma M$ describing a collinear spin order. In the $S_{NL\sigma M}$ we introduced a Lagrange multiplier field $\lambda$ to impose the rigid rotor constraint $|\vec{n}|^2 = 1$. In appendix C we give a detailed derivation of the $NL\sigma M$. The total effective action $S_{ASL} + S_M + S_{NL\sigma M}$ called fermionic $NL\sigma M$ in Eq. (1) naturally describes the quantum $AF$ resulting from the
ASL via $S_{SB}$ at half filling. Based on this fermionic O(3) $NLoM$ we investigate the QCP of the AF to ASL transition.

In Eq. (1) we concentrate on the Kondo-like spin coupling term $\vec{n} \cdot \vec{\psi} \vec{\tau} \psi$ between antiferromagnetic spin fluctuations $\vec{n}$ and Dirac fermions $\psi$. This term shows that an antiferromagnetic excitation of spin 1 can fractionalize into two fermionic spinons of spin $1/2$. But, long range gauge interactions prohibit antiferromagnetic spin fluctuations from decaying into spinons. Because massive Dirac spinons in the $S_{SB}$ can generate only the Maxwell kinetic energy for the gauge field $a_{\mu}$ via particle-hole polarizations, they should be confined to form spin 1 antiferromagnetic fluctuations owing to the effect of instantons. Remember that the Maxwell gauge theory shows confinement of charged matter fields in $(2 + 1)D$ owing to the condensation of instantons $^{27}$. As a result spinon-antispinon composites appear to be identified with spin 1 antiferromagnetic fluctuations, i.e., $\vec{n} = (\vec{\psi} \vec{\tau} \psi)$, where $\langle ... \rangle$ denotes a vacuum expectation value. A resulting effective field theory for this antiferromagnet is obtained to be

$$ Z_{AF} = \int D\bar{\psi} e^{-S},$$

$$ S_\pi = \int d^3x \frac{Nm_\psi}{4\pi} \left[ |\partial_\mu \pi|^2 + \left( \vec{\pi} \cdot \vec{\partial}_\mu \vec{\pi} \right)^2 \right]$$

$$ \approx \int d^3x \frac{Nm_\psi}{4\pi} \left[ |\partial_\mu \pi|^2 + (\vec{\pi} \cdot \vec{\partial}_\mu \vec{\pi})^2 \right], \quad (2) $$

where the Neel vector is represented as $\vec{n} = (\pi, n^3)$ with $n^3 = \sqrt{1 - |\vec{\pi}|^2}$. In the last line we obtained an effective field theory for small fluctuations of $\pi$ fields around the Neel axis $n^3$ by using $n^3 \approx 1 - (1/2)|\vec{\pi}|^2$. The $\pi$ fields are nothing but antiferromagnons, considered to be spinon-antispinon composites $\pi = \langle \vec{\psi} \vec{\tau} \psi \rangle = \pi_1 \pm i\pi_2$ with the relativistic spectrum of $\omega = k$ in the low energy limit. The last term in the last line represents interactions between antiferromagnons. Low energy physics in this conventional quantum antiferromagnet is well described by the interacting antiferromagnons $^{16}$. 

III. QUANTUM PHASE TRANSITION FROM ANTIFERROMAGNETISM TO ALGEBRAIC SPIN LIQUID

A. Effective Field Theory

Reducing the mass parameter $m_\psi$ in Eq. (1) results in the QCP of the AF to ASL transition. It is important to realize that although the Dirac spinons are confined to form antiferromagnons in the AF, they should be deconfined to emerge in the ASL. In order to investigate their confinement to deconfinement transition, Dirac spinons should be explicitly introduced in the effective action. Especially, it needs much care to treat the $S_M$ in Eq. (1) representing spin fractionalization. In the AF the spinon mass can be considered to be infinite owing to the confinement of spinons, allowing us to ignore the process of spinon fractionalization. This permits us to integrate over the spinon degrees of freedom completely and obtain the effective $NLoM$ Eq. (2) in terms of only the Neel order parameter fields. On the other hand, near the QCP this full integration of spinons is difficult to be justified. Because the mass parameter $m_\psi$ becomes small near the critical point, the spin fractionalization cannot be ignored and thus, should be taken into account appropriately. Neel fields and Dirac spinons should be treated on equal footing. In order to solve the Kondo-like spin coupling term $S_M$ in Eq. (1), we utilize the $CP^1$ representation $\bar{n} = \frac{1}{2}z_\sigma^+ \tau_{\sigma \sigma'} z_{\sigma'}$ or equally, $\bar{n} \cdot \vec{\tau} = U\tau^3 U^\dagger$, where $z_\sigma$ is a bosonic spinon and $U$, its corresponding SU(2) matrix $U = \begin{pmatrix} z_\uparrow \cdot \tau \cdot \uparrow & z_\downarrow \cdot \tau \cdot \downarrow \\ z_\downarrow \cdot \tau \cdot \uparrow & z_\uparrow \cdot \tau \cdot \downarrow \end{pmatrix}$. Inserting this $CP^1$ representation into Eq. (1), the Kondo coupling term of $m_\psi \bar{\psi}_\sigma \tau_{\sigma \sigma'} \psi_{\sigma'} \bar{n}$ is written to be $L_K = m_\psi \bar{\psi}_\sigma \tau_{\sigma \sigma'} \psi_{\sigma'}$. Now this Kondo coupling term can be easily solved by the following gauge transformation

$$ \Psi_\sigma = U_{\sigma \sigma'}^{\dagger} \psi_{\sigma'}, \quad (3) $$

resulting in $L_K = m_\psi \bar{\Psi}_\sigma \tau_{\sigma \sigma'} \psi_{\sigma'}$. Physical meaning of this gauge transformation is clear. At the QCP critical bosonic spinons $U_{\sigma \sigma'}$ are expected to hybridize with the fermionic spinons $\psi_{\sigma'}$ via the Kondo coupling, thus screening out their spin degrees of freedom and resulting in the spinless fermions $\Psi_\sigma$.

Representing Eq. (1) in terms of the fractionalized fields $U_{\sigma \sigma'}$ and $\Psi_\sigma$, we already mentioned that the Kondo coupling term is completely solved to be $m_\psi \bar{\Psi}_\sigma \tau_{\sigma \sigma'} \psi_{\sigma'}$. In this representation the scattering effect of fermionic spinons $\psi_{\sigma'}$ by spin fluctuations $\vec{n}$ appears in the kinetic energy of spinons as $\bar{\Psi}_\sigma \gamma_\mu (U^{\dagger} \partial_\mu U)_{\sigma \sigma'} \Psi_{\sigma'}$, where currents of the spinless fermions couple to those of the bosonic spinons,

$$ Z = \int D\Psi \bar{D} a_\mu e^{-S},$$

$$ S = \int d^3x \left[ \bar{\Psi}_\sigma \gamma_\mu (\partial_\mu - ia_\mu) \psi_{\sigma} + (\bar{U} U)_{\sigma \sigma'} \right] \Psi_{\sigma'},$$

$$ +m_\psi \bar{\Psi}_\sigma \tau_{\sigma \sigma'} \psi_{\sigma'} + \frac{1}{2e^2} |\vec{\partial} \times a|^2$$

$$ + Tr \left( \frac{Nm_\psi}{8\pi} |\partial_\mu (U\tau^3 U^\dagger)|^2 \right) \right]. \quad (4) $$

This action has $U_A(1) \otimes U_c(1)$ local gauge symmetry. The former $U_A(1)$ guarantees the invariance of the action Eq. (4) under the gauge transformations of

$$ \psi'_{\sigma} = e^{i\theta} \psi_{\sigma}, \quad \psi'_{\sigma'} = e^{i\theta} \Psi_{\sigma},$$

$$ U'_{\sigma \sigma'} = U_{\sigma \sigma'}, \quad a'_c = a_c + \partial_\mu \theta. \quad (5) $$

This is nothing but the gauge symmetry of the $U(1)$ slave boson representation. On the other hand, the latter $U_c(1)$
This new local gauge symmetry originates from the \( CP^1 \) representation and implies that there should be a \( U(1) \) gauge field corresponding to the gauge symmetry. Indeed, performing some standard algebra such as the Hubbard-Stratonovich transformation, we can reach the following effective action

\[
Z = \int DUDc_\mu D\Psi_d Da_\mu e^{-S},
\]

\[
S = \int d^3x \left[ \bar{\Psi} \gamma_\mu \left( \partial_\mu - i\eta_{\mu}} \right) \bar{\sigma} \sigma^\alpha_\mu \psi_{\sigma^\alpha_\mu} \right] + \frac{\mu_\Psi}{N} \left( \bar{\Psi} \gamma_\mu \sigma^3_\mu \bar{\sigma} \sigma^\alpha_\mu \psi_{\sigma^\alpha_\mu} \right)^2 + \frac{1}{2c^2} \left( \partial \times \epsilon \right)^2,
\]

\[
\begin{align*}
&+ m_\psi \left( \bar{\Psi} \sigma^3_\mu \bar{\sigma} \sigma^\alpha_\mu \psi_{\sigma^\alpha_\mu} \right)^2 + \frac{1}{2c^2} \left( \partial \times \epsilon \right)^2,
\end{align*}
\]

Here \( c_\mu \) is a new compact \( U(1) \) gauge field guaranteeing the \( U(1) \) local gauge symmetry, under the gauge field \( c_\mu \) is transformed into \( c_\mu = c_\mu + \partial_\mu \eta \). This is the reason why we use the term, "based on the conjecture of the deconfined (3 + 1)-dimensional \( O(3) \) gauge symmetry, there should exist the deconfined \( \text{AF} \) between the spinons because the Maxwell gauge action is irrelevant in \( (2 + 1)D \) in the renormalization group (RG) sense and thus, does not affect the physics of the QCP.

\[
|\Delta| \text{ and } \phi \text{ are the amplitude and phase of Cooper pair fields, analogous to } m_\psi \text{ and } \eta, \text{ respectively.}
\]

The gauge transformation Eq. (3) introduced for solving the Kondo coupling term in Eq. (1) may be still suspected to be unnatural although similar decoupling schemes have been utilized in the context of the Kondo lattice model\[13, 14, 15\]. In this respect it is necessary to understand the present methodology more deeply by comparing this with other well studied ones. A good example is a \( d-wave \) superconductivity, the coupling term of \( \Delta |e^{i\theta}c_\mu c_\mu^\dagger| \) between Cooper pairs and electrons plays the same role as the Kondo coupling term of \( m_\psi \eta \cdot \bar{c}_\sigma \sigma^\alpha_\mu \psi_{\sigma^\alpha_\mu} \) between spin fluctuations and Dirac spinons in the context of antiferromagnetism. Here \( |\Delta| \) and \( \phi \) are the amplitude and phase of Cooper pair fields, analogous to \( m_\psi \) and \( \eta \), respectively. In order to solve this coupling term several kinds of gauge transformations are introduced\[25, 36\].

In these decoupling schemes critical phase fluctuations of Cooper pairs screen out charge degrees of freedom of electrons, causing electrically neutral but spinful electrons called ”spinons”. As a result the phase factor disappears in the coupling term when it is rewritten in terms of spinons. Instead, this coupling effect appears as current-current interactions of neutral spinons and phase fields of Cooper pairs in the kinetic energy of electrons.

In order to convince critical physicists of the effective action Eq. (7), in appendix D we show that Eq. (7) can be derived directly from the antiferromagnetic Heisenberg Hamiltonian in a standard fashion. We would like to suggest one good reference\[37\] which derives a similar \( \text{ASL} \) to that of spin SU(2) symmetry in the context of superconductivity to that of spin SU(2) symmetry in the context of antiferromagnetism.

B. Quantum Phase Transition from Antiferromagnetism to Algebraic Spin Liquid

Now we discuss the quantum phase transition between the \( \text{AF} \) and the \( \text{ASL} \) based on Eq. (7). The critical value of the mass parameter \( m_\psi \) for the antiferromagnetic para- magnetic transition is assumed to be \( m_\psi^c \). In the case of \( m_\psi > m_\psi^c \), condensation of bosonic spinons occurs, \( U_\sigma^\alpha \neq 0 \) (\( \eta > 0 \)), causing the \( \text{AF} \). The spinon condensation leads the gauge field \( c_\mu \) to be massive (Anderson-Higgs mechanism).

In the context of gauge theories this phase corresponds to the Higgs-confinement phase\[28, 29\], where the matter fields with internal charge \( g \) are confined to form gauge singlets. The condensed bosonic spinons \( U_{\sigma^\alpha_\mu} \) are confined with the QED3 with the massless Dirac spinons and slave boson \( U(1) \) gauge fields recover from Eq. (7) in the \( \text{AF} \) and in the \( \text{ASL} \), respectively.
the spinless fermions $\bar{\Psi}_\sigma$, and other bosonic spinons $U_{\sigma \sigma'}^\dagger$, to make the fermionic spinons $\psi_\sigma$ and the Neel order parameter fields $\vec{n}$, respectively. Integrating over the massive $U(1)$ gauge field $c_\mu$ in Eq. (7), we obtain the following Lagrangian of $\mathcal{L} = \bar{\psi}_\sigma \gamma_\mu (\partial_\mu - i a_\mu) \psi_\sigma + \frac{1}{2} m_\psi \bar{\psi}_\sigma \bar{\tau}_{\sigma \sigma'} \psi_{\sigma'} \cdot \vec{n} + \frac{N_{m_\psi}}{4\pi} |\partial_\mu \vec{n}|^2$, where the local interactions between spinless fermions are cancelled. In the AF the Neel vectors align to the $z$ direction. Admitting transverse fluctuations of the Neel vectors, we obtain an effective field theory, $\mathcal{L}_{AF} = \bar{\psi}_\sigma \gamma_\mu (\partial_\mu - i a_\mu) \psi_\sigma + \frac{1}{2} m_\psi \bar{\psi}_\sigma \bar{\tau}_{\sigma \sigma'} \psi_{\sigma'} \cdot \vec{n} + \frac{N_{m_\psi}}{4\pi} |\partial_\mu \vec{n}|^2 + m_\psi \bar{\psi}_n \bar{\tau}_{nm} \psi_m \cdot \vec{\pi} + \frac{N_{m_\psi}}{4\pi} (|\partial_\mu \vec{\pi}|^2 + (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2)$, where we used $n^3 = \sqrt{1 - |\vec{\pi}|^2} \approx 1 - \frac{1}{2} |\vec{\pi}|^2$ as Eq. (2). The massive Dirac spinons should be confined to form the antiferromagnons $\pi^\pm$. As a result the conventional AF described by Eq. (2) recovers from Eq. (7).

In our recent study\cite{34},\cite{35},\cite{36},\cite{37} we proposed an interesting possibility that massive Dirac spinons can appear to make broad continuum spectrum at high energies in inelastic neutron scattering\cite{38}. The mechanism of spinon deconfinement results from the existence of fermion zero modes in single instanton potentials. Neel vectors can make a skyrmion configuration around an instanton of the compact $U(1)$ gauge field $a_\mu$. Remarkably, in the instanton-skyrmion composite potential the Dirac spinon is shown to have a zero mode. The emergence of the fermion zero mode forbids the condensation of instantons, resulting in the deconfinement of Dirac spinons in the AF. Notice that the Dirac spinons are massive owing to $S \chi SB$.

Approaching the QCP ($m_\psi \rightarrow m_\psi^c$), the critical bosonic spinons $z_\sigma (U_{\sigma \sigma'}^\dagger)$ would be deconfined\cite{1,2,3,4} to exhibit broad continuum spectrum in spin susceptibility. In Ref. 2 Berry phase is shown to play a crucial role for the deconfinement of bosonic spinons at the QCP. Berry phase gives rise to destructive interference for instanton excitations of the $c_\mu$ gauge fields. As a result only quadrupled instanton excitations contribute to a partition function for the $NL\sigma M$. These quadrupled instanton excitations are known to be irrelevant at the QCP\cite{2}. On the other hand, in Refs. 1, 3, 4 the spinon deconfinement is claimed to occur even in the absence of Berry phase. In these studies\cite{1,3,4} the crucial point is the existence of quantum criticality itself. Internal electric charges of spinons can be sufficiently screened by critical fluctuations of spinons. The smaller internal electric charges, the larger internal magnetic charges. As a result instanton excitations can be irrelevant at the QCP. Especially, in Ref. 3 the present author pointed out that the $CP^1$ action of the $NL\sigma M$ in the easy plane limit has two kinds of fixed points owing to an additional vortex gauge field in the dual vortex representation. One is the charged fixed point (inverted XY fixed point) and the other, the neutral fixed point (XY fixed point). In Ref. 4 the inverted XY fixed point is identified with the QCP studied by Senthil et al.\cite{2}. The inverted XY fixed point is shown to be unstable against instanton excitations and the instanton excitations are proliferated. Since condensation of instantons does not allow spinon unbinding, this is consistent with the conclusion of Senthil and coworkers, that is, with the absence of spinon deconfinement without a Berry phase term. At the inverted XY critical point the spinon deconfinement can occur only in the presence of Berry phase. On the other hand, the XY fixed point is shown to remain stable against instanton excitations. From the RG analysis in Ref. 8 one sees that although off criticality the instantons are relevant everywhere, they become irrelevant at the XY QCP. This allows spinon deconfinement, which is different from the conclusion by Senthil et al.\cite{2}, since this novel critical point does not coincide with their one. The XY fixed point is, instead, identified with the QCP studied by Bernevig et al.\cite{1}, thus showing that the deconfinement of spinons takes place even without the Berry phase. See Ref. 8 for details. We would like to mention the result of Monte-Carlo simulation\cite{4} supporting the existence of deconfined spinons at the QCP of the $O(3)$ $NL\sigma M$ without Berry phase. They claimed that critical fluctuations of bosonic spinons result in a nonlocal action of $U(1)$ gauge fields and this contribution causes the deconfinement of spinons\cite{4}.

Although the internal charge $g$ is deconfined owing to critical fluctuations of the bosonic spinons, the spinless fermions $\bar{\Psi}_\sigma$ would not appear at the QCP. Remember that the $\Psi_\sigma$ carries the internal charge $e$. The spinless fermions would feel confining interactions mediated by $U(1)$ gauge fluctuations $a_\mu$ owing to their finite mass $m_\psi^c$ at the QCP. Ignoring antiferromagnetic spin fluctuations, i.e., $\vec{n} = \vec{z}$ in the mean field level, one can find that the antiferro- to para- magnetic transition is determined by the self-consistent gap equation in Ref. 2\cite{4} (appendix B). In this case the spinon mass $m_\psi$ is zero at the critical transitions. 

### Table I: Quantum phase transition from antiferromagnetism to algebraic spin liquid

| Antiferromagnetism | Quantum Critical Point | Algebraic Spin Liquid |
|--------------------|------------------------|-----------------------|
| $m_\psi > m_\psi^c$ | $m_\psi = m_\psi^c$   | $m_\psi < m_\psi^c$ |
| Order Parameter    | $< U_{\sigma \sigma'}^\dagger > \neq 0$ ($< z_\sigma \neq 0$) | $< U_{\sigma \sigma'}^\dagger > = 0$ ($< z_\sigma > = 0$) |
| $g$                | $\bar{U}_{\sigma \sigma'} \Psi_\sigma \rightarrow \psi_\sigma$ | $\Psi_\sigma \Psi_\sigma \rightarrow \psi_\sigma$ |
| $e$                | $\psi_\sigma \tau_{\sigma \sigma'} \psi_{\sigma'} \rightarrow \pi^\pm$ | $\psi_\sigma \Psi_\sigma \rightarrow \psi_\sigma$ |
| Elementary Excitations | $\pi^\pm = U_{\sigma \sigma'}^\dagger (z_\sigma)$, $c_\mu$ | $\psi_\sigma \Psi_\sigma \rightarrow \psi_\sigma$ |

**Order Parameter**

- $< U_{\sigma \sigma'}^\dagger > \neq 0$ ($< z_\sigma > \neq 0$)
- $< U_{\sigma \sigma'}^\dagger > = 0$ ($< z_\sigma > = 0$)

**Excitations**

- $\pi^\pm = U_{\sigma \sigma'}^\dagger (z_\sigma)$, $c_\mu$

**Critical Points**

- $m_\psi = m_\psi^c$
- $m_\psi < m_\psi^c$
point. However, allowing spin fluctuations beyond the mean field level, the critical value of the spinon mass for the magnetic transition would not be zero. This can be justified by the RG analysis of the O(3) NLoM. Using the standard poor man’s scaling for the $S_{NLσM}$ in Eq. (1), one obtains the well known RG equation for the spin stiffness $m_\psi$ in $(2 + 1)D$:

$$\frac{dm_\psi^{-1}}{dt} = -m_\psi^{-1} + \lambda m_\psi^{-12},$$

where $l$ is a scaling parameter and $\lambda$, a positive numerical constant. This RG equation yields three fixed points: $m_\psi \rightarrow \infty$, $m_\psi \rightarrow 0$, and $m_\psi \rightarrow m_\psi^c = \lambda$, corresponding to AF, quantum disordered paramagnetism, and QCP, respectively. The nonzero critical coupling in $m_\psi^c$ at the QCP seems to be consistent with that of the Kondo lattice model[14, 15, 16] which has a similar structure with Eq. (7). Integration over the massive fermions generates the Maxwell kinetic energy of the gauge field $c_\mu$. In this case the well known classic result by Polyakov[27] can be applied, leading to confinement of the charge $c$. The resulting critical field theory is obtained from Eq. (7) to be $L_{QCP} = \frac{N m_\psi^c}{2} (\partial_\mu - ic_\mu) z_\sigma |^2 + \frac{1}{4g^2} |\partial \times c|^2$, where $c_\mu$ is a noncompact $U(1)$ gauge field. In appendix E, in order to confirm this critical field theory we derive the similar critical theory in a different way, where the Neel fields are not introduced.

In the case of $m_\psi < m_\psi^c$ the bosonic spinons are gapped, $< U_{\sigma\sigma}^\dagger, >= 0 (< -\sigma_0 > = 0)$. In order to correctly describe the ASL, the massive bosonic spinons should be confined with the spinless fermions to form the fermionic spinons. In other words, the internal charge $c$ should be confined via gauge fluctuations $c_\mu$. As the RG Eq. (8) shows $m_\psi \rightarrow 0$, the spinless fermions $Ψ_\sigma$ would become massless in the quantum disordered paramagnetism. One can think that the vanishing mass makes the $Ψ_\sigma$ field critical, resulting in deconfinement of the internal charge $c$ owing to critical fluctuations of these fermions. However, this guess is not correct. The vanishing fermion mass gives rise to one problem that the strength of local interactions goes to infinity in Eq. (7). In order to treat the infinitely strong local interactions, we perform the Hubbard-Stratonovich transformation in Eq. (7) to obtain $S = \int d^3x \left[ \bar{Ψ}_\sigma \gamma_\mu (\partial_\mu - ia_\mu) \delta_{\sigma\sigma'} - ic_\mu T_{\sigma\sigma'} \right] Ψ_\sigma' + \frac{1}{2g^2} |\partial \times c|^2$, where the gapped bosonic spinons are integrated out to produce the Maxwell kinetic energy of the gauge field $c_\mu$ with a renormalized coupling strength $g$, and $α_\mu$ is an auxiliary field to impose the local interactions. In the above we utilized $m_\psi \rightarrow 0$. Integration over the auxiliary field $α_\mu$ results in the local constraint, $Ψ_\sigma \gamma_\mu T_{\sigma\sigma'} Ψ_\sigma' = 0$, indicating that the $Ψ_\sigma$ fields do not screen out the internal charge $c$. Only the Maxwell kinetic energy is available, causing confinement of the charge $q$.[24] The gapped bosonic spinons should be confined with the spinless fermions to make the Dirac spinons. Since the effective field theory Eq. (7) is unstable in the parameter range of $m_\psi < m_\psi^c$, a new effective theory necessarily results. Inserting $Ψ_\sigma = U_{σσ'} Ψ_\sigma'$ into Eq. (7) under the constraint of $Ψ_\sigma \gamma_μ T_{σσ'} Ψ_\sigma' = 0$ and integrating over the gapped bosonic spinons $U_{σσ'}^\dagger$, we can obtain the following Lagrangian of $L_{ASL} = \bar{Ψ}_\sigma \gamma_μ (\partial_μ - ia_μ) Ψ_σ + \frac{1}{2g^2} |\partial \times a|^2$. Local interactions of spinon currents $|\bar{Ψ}_\sigma \gamma_μ Ψ_σ|^2$ would appear in the above effective Lagrangian, but this term does not affect low energy physics because it is irrelevant in $(2 + 1)D$ in the RG sense. As discussed earlier, the Dirac spinons with the internal charge $e$ are deconfined to emerge at the charged fixed point of the $L_{ASL}$. We summarize the quantum phase transition from the AF to the ASL in Table I.

IV. SUMMARY

In summary, we proposed the fractionalization of the Dirac spinons into the bosonic spinons and spinless fermions near the quantum critical point between the antiferromagnetism and the algebraic spin liquid (Table I). Based on this conjecture we constructed the ”mother” critical field theory Eq. (7) and showed that the antiferromagnetism and the algebraic spin liquid can successfully recover from Eq. (7).

V. ACKNOWLEDGEMENT

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APPENDIX A

In appendix A we briefly sketch how to obtain the $S_{ASL}$ in Eq. (1) from the antiferromagnetic Heisenberg model on two dimensional square lattices, $H_J = J \sum_{<i,j>} \bar{S}_i \cdot \bar{S}_j$ with $J > 0$. Inserting the following spinon representation for spin, $\bar{S}_i = \frac{1}{2} f^\dagger_{iσ} τ_{σσ'} f_{iσ'}$ into the above Heisenberg model, and performing the standard Hubbard-Stratonovich transformation for an exchange interaction channel, we obtain an effective one body Hamiltonian for fermions coupled to an order parameter, $H_{eff} = -J \sum_{<i,j>} f^\dagger_{iσ} χ_{ij} f_{jσ} - h.c.$ Here $f_{iσ}$ is a fermionic spinon with spin $σ = Â, Â'$, and $χ_{ij}$ is an auxiliary field called a hopping order parameter. Notice that the hopping order parameter $χ_{ij}$ is a complex number defined on links $ij$. Thus, it can be decomposed into $χ_{ij} = |χ_{ij}| e^{iθ_{ij}}$, where $|χ_{ij}|$ and $θ_{ij}$ are the amplitude and phase of the hopping order parameter, respectively. Inserting this representation for the
\( \chi_{ij} \) into the effective Hamiltonian, we obtain \( H_{\text{eff}} = -J \sum_{i,j} \langle \chi_{ij} \rangle f_{ij}^\dagger e^{i\theta_{ij}} f_{ij} - h.c. \). We can easily see that this effective Hamiltonian has an internal U(1) gauge symmetry, \( H_{\text{eff}}(f_{ij}^\dagger, \theta_{ij}) = H_{\text{eff}}(f_{ij}, \theta_{ij}) \) under the following U(1) phase transformations, \( f_{ij}^\dagger = e^{i\phi_i} f_{ij}^\dagger \) and \( \theta_{ij}' = \theta_{ij} + \phi_i - \phi_j \). This implies that the phase field \( \theta_{ij} \) of the hopping order parameter plays the same role as a U(1) gauge field \( a_{ij} \). When a spinon hops on lattices, it obtains an Aharonov-Bohm phase owing to the U(1) gauge field \( a_{ij} \). It is known that a stable mean field phase is a \( \pi \) flux state if an antiferromagnetic order is not taken into account. This means that a spinon gains the phase of \( \pi \) when it turns around one plaquette. In the \( \pi \) flux phase low energy elementary excitations are massless Dirac spinons near nodal points showing gapless Dirac spectrum and U(1) gauge fluctuations. In the low energy limit the amplitude \( |\chi_{ij}| \) is frozen to \( |\chi_{ij}| = |J| < f_{ij}^\dagger f_{ij}^\dagger | \). A resulting effective field theory for this possible quantum paramagnetism of the antiferromagnetic Heisenberg model is \( QED_3 \) in terms of massless Dirac spinons interacting via compact U(1) gauge fields, \( S_{\text{ASL}} \) in Eq. (1).

In the \( S_{\text{ASL}} \) \( \psi_{\sigma} = \begin{pmatrix} \chi_+^\sigma \\ \chi_-^\sigma \end{pmatrix} \) is a four component massless Dirac fermion, where \( \sigma = 1,2 \) represents its SU(2) spin \( (\uparrow, \downarrow) \) and \( \pm \) denote the nodal points of \( (\pi/2, \pm \pi/2) \) in momentum space. Usually, SU(N) quantum antiferromagnets are considered by generalizing the spin components \( \sigma = 1,2 \) into \( \sigma = 1,2, ..., N \). The two component spinors \( \chi_{\pm}^\sigma \) are given by \( \chi_1^+ = \begin{pmatrix} f_{11e}^\dagger \\ f_{11o}^\dagger \end{pmatrix} \), \( \chi_1^- = \begin{pmatrix} f_{12o}^\dagger \\ f_{12e}^\dagger \end{pmatrix} \), \( \chi_2^+ = \begin{pmatrix} f_{11e}^\dagger \\ f_{11o}^\dagger \end{pmatrix} \), and \( \chi_2^- = \begin{pmatrix} f_{12o}^\dagger \\ f_{12e}^\dagger \end{pmatrix} \), respectively. In the spinon field \( f_{abc} \) \( a = \uparrow, \downarrow \) represents its SU(2) spin, \( b = 1,2 \), the nodal points \( (\pi_1, \cdots, \pi_N) \), and \( c = \sigma, \sigma' \) even and odd sites, respectively. Dirac matrices \( \gamma_\mu \) are given by \( \gamma_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), \( \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), and \( \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), respectively, where they satisfy the Clifford algebra \( [\gamma_\mu, \gamma_\nu] = 2 \delta_{\mu\nu} \).

APPENDIX B

In appendix B, for completeness of this paper we briefly sketch how we obtain the dynamically generated spinon mass \( m_\psi \) in \( S_M \) in Eq. (1). A single spinon propagator is given by \( G^{-1}(k) = G_0^{-1}(k) - \Sigma(k) \), where \( G_0^{-1}(k) = i\gamma_\mu k_\mu \) is the inverse of a bare spinon propagator, and \( \Sigma(k) \), a spinon self-energy resulting from long range gauge interactions. The spinon self-energy is determined by the self-consistent gap equation, \( \Sigma(k) = \int \frac{d^4k}{(2\pi)^4} Tr[\gamma_\mu G(k-q)\gamma_\mu D_{\mu\nu}(q)] \), where \( D_{\mu\nu}(q) \) is a renormalized propagator of the U(1) gauge field \( a_\mu \) due to particle-hole excitations of massless Dirac fermions. The self-energy can be written as \( \Sigma(k) = -m_\psi(k)\pi^3 \) for staggered magnetization.

\begin{equation}
S_{\text{ASL}} = -\ln \left[ \int D\psi \exp \left\{ -\int d^3x \left( \bar{\psi}_{\sigma} (\partial_\mu - ia_\mu) \psi_{\sigma} + m_\psi \bar{\psi}_{\sigma} \bar{\psi}_{\sigma} \right) \right\} \right]
\end{equation}

\begin{equation}
= -N \ln \left[ \left| \gamma_\mu (\partial_\mu - ia_\mu) + m_\psi \bar{n} \cdot \vec{r} \right| \right]
\approx -N \ln \left( -\partial^2 + m_\psi^2 \right)
\end{equation}

APPENDIX C

In appendix C we discuss how the O(3) NLσM can be derived from the effective action \( S_{\text{ASL}} + S_M \) in Eq. (1). Integration over the Dirac fermions in \( S_{\text{ASL}} + S_M \) results in the following expression,

\begin{equation}
S_{\text{NLσM}} = -\ln \left[ \int D\psi \exp \left\{ -\int d^3x \left( \bar{\psi}_{\sigma} (\partial_\mu - ia_\mu) \psi_{\sigma} + m_\psi \bar{\psi}_{\sigma} \bar{\psi}_{\sigma} \right) \right\} \right]
\end{equation}

\begin{equation}
= -N \ln \left| \gamma_\mu (\partial_\mu - ia_\mu) + m_\psi \bar{n} \cdot \vec{r} \right|
\approx -N \ln \left( -\partial^2 + m_\psi^2 \right)
\end{equation}

\begin{equation}
+ \int d^3x \left( \frac{N m_\psi}{4\pi} |\partial_\mu \bar{n}|^2 + \frac{N}{12\pi m_\psi} |\partial \times \bar{n}|^2 \right).
\end{equation}
The second term in the last line is well derived in Ref. 24,

\[-\frac{N}{2} \ln \det \left[1 - \frac{m_0 \gamma_\mu \partial_\mu (\vec{n} \cdot \vec{\tau})}{-\partial^2 + m_\psi^2}\right] = -\frac{N}{2} \left\langle \frac{d^3 x}{(2\pi)^3} \int d^3 x \left[ \ln \left[1 - \frac{m_0 \gamma_\mu \partial_\mu (\vec{n} \cdot \vec{\tau})}{-\partial^2 + m_\psi^2}\right] \right] x\right\rangle = -\frac{N}{2} \int d^3 x \int d^3 k \left(\frac{2\pi)^3}{2}\right)e^{-ikx} Tr \ln \left[1 - \frac{m_0 \gamma_\mu \partial_\mu (\vec{n} \cdot \vec{\tau})}{-\partial^2 + m_\psi^2}\right] e^{ikx} = -\frac{N}{2} \int d^3 x \int d^3 k \left(\frac{2\pi)^3}{2}\right) Tr \ln \left[1 - \frac{m_0 \gamma_\mu \partial_\mu (\vec{n} \cdot \vec{\tau})}{-\partial^2 + m_\psi^2}\right] \approx \int d^3 x \frac{Nm_\psi}{4\pi} |\partial_\mu \vec{n}|^2. \tag{C2}\]

In the above Tr stands for not a functional but a usual matrix trace for both flavor (spin) and spinor indices. In going from the third to the fourth line we have dragged the factor $e^{ikx}$ through the operator, thus shifting all differential operators $\partial_\mu \rightarrow \partial_\mu + ik_\mu$.\]24.\] Expanding the argument of the logarithmic term in powers of $\partial_\mu \vec{n}$ and of $2ik_\mu \partial_\mu + \partial^2$, one can easily obtain the expression in the fifth line. Performing the momentum integration, we obtain an effective spin stiffness proportional to the mass parameter $m_\psi$. This implies that the rigidity of fluctuations in the Neel field is controlled by the mass parameter $m_\psi$ of Dirac spinons. Eq. (C2) is nothing but the $O(3)$ $NLoM$ describing a collinear spin order. Note that higher order derivative terms in the gradient expansion are irrelevant in $(2+1)D$ in the RG sense.

Next, we sketch the derivation of the Maxwell gauge action. Expanding the argument of the logarithmic term in Eq. (C1) to the second order of the gauge field $a_\mu$, we obtain $S_{\text{gauge}} = \int d^4x \frac{1}{2} a_\mu(q) \Pi_{\mu\nu}(q) a_\nu(-q)$, where the fermion polarization function $\Pi_{\mu\nu}(q)$ is given by $\Pi_{\mu\nu}(q) = -N \int d^4k Tr G(k + q) i\gamma_\nu G(k)$ with the single spinon propagator $G(k) = [\gamma_\mu k_\mu + m_\psi \vec{n} \cdot \vec{\tau}]^{-1}$. Utilizing the Feynman identity and trace identity for Dirac gamma matrices, one can obtain the following expression for the polarization function, $\Pi_{\mu\nu}(q) = 2N (Tr \Gamma_{(2-D/2)}(q^2 \delta_{\mu\nu} - q_\mu q_\nu)) \int_0^1 dx(1-x)x(m_\psi^2 + q^2 x(1-x))^{D/2-2} =\frac{(TrI) N}{4\pi} (q^2 \delta_{\mu\nu} - q_\mu q_\nu) \left(\frac{m_\psi^2}{q^2} + \frac{q^2 - 4m_\psi^2}{4q^4} \sin^{-1}\left(\sqrt{\frac{4m_\psi^2}{q^2}}\right)\right)$[22]. This leads to the Maxwell gauge action in Eq. (C1).

We should comment the reason why imaginary terms do not arise in the present $NLoM$ because some previous studies have shown the emergence of imaginary terms[40,41]. Following the evaluations in Ref. 41, one can obtain two imaginary terms in the irreducible representation of gamma matrices; one is a coupling term $i\alpha_\mu J_\mu$, between topologically nontrivial fermionic currents $J_\mu = \frac{1}{\sqrt{2}}\epsilon_{\mu\nu\lambda} \epsilon_{\alpha\beta\gamma} n^\alpha \partial_\nu n^\beta \partial_\lambda n^\gamma$ and U(1) gauge fields $a_\mu$, and the other, a geometrical phase $iN\Gamma \vec{m}[40,41]$. The key point is the representation of Dirac gamma matrices[42]. Here, we utilized four-by-four Dirac matrices by combining the two nodal points. Note that the signs of Pauli matrices in the Dirac gamma matrices are opposite for the nodal points $\pm$. This fact results in cancellation of the imaginary terms. The first imaginary term can be considered from a variation of the logarithmic term in Eq. (C1) with respect to the gauge field $a_\mu$

\[iN Tr \left[ \gamma_\mu \delta a_\mu \gamma_\mu(\partial_\mu - ia_\mu) + m_\psi(\vec{n} \cdot \vec{\tau}) \right] = iN Tr \left[ \gamma_\mu \delta a_\mu \gamma_\mu(\partial_\mu - ia_\mu) + m_\psi(\vec{n} \cdot \vec{\tau}) \right] \approx iN Tr \left[ \gamma_\mu \delta a_\mu \gamma_\mu(\partial_\mu + ia_\mu) + m_\psi(\vec{n} \cdot \vec{\tau}) \right] \approx iN Tr \left[ \gamma_\mu \delta a_\mu \gamma_\mu(\partial_\mu + ia_\mu) + m_\psi(\vec{n} \cdot \vec{\tau}) \right]. \tag{C3}\]

In this expression the key point is the triple product of Dirac matrices, $\gamma_\mu \gamma_\nu \gamma_\lambda$ in the last line. In the irreducible representation of gamma matrices, i.e., Pauli matrices, this contribution is nonzero, leading to $\epsilon_{\mu\nu\lambda}$. As a result the imaginary term of $\gamma_\mu \gamma_\nu \gamma_\lambda$ can be obtained[40,41]. Another $\epsilon_{\alpha\beta\gamma}$, associated with the Neel vectors appears from the triple product of Pauli matrices, $\tau_\alpha \tau_\beta \tau_\gamma$. On the other hand, in the present representation of Dirac matrices the contribution of the + nodal point leads to $+\epsilon_{\mu\nu\lambda}$ while that of the $-$ nodal point, $-\epsilon_{\mu\nu\lambda}$. Thus, these two contributions are exactly cancelled. The geometrical phase term, considered from a variation of the logarithmic term in Eq. (C1) with respect to the Neel field $\vec{n}$[40,41],

\[-N Tr \left[ m_\psi \delta a_\mu \gamma_\mu(\partial_\mu + m_\psi(\vec{n} \cdot \vec{\tau}) \right] \approx -N Im Tr \left[ m_\psi \delta a_\mu \gamma_\mu(\partial_\mu + m_\psi(\vec{n} \cdot \vec{\tau}) \right] \approx N Tr \left[ m_\psi \delta a_\mu \gamma_\mu(\partial_\mu + m_\psi(\vec{n} \cdot \vec{\tau}) \right] \approx N Tr \left[ m_\psi \delta a_\mu \gamma_\mu(\partial_\mu + m_\psi(\vec{n} \cdot \vec{\tau}) \right]. \tag{C4}\]

is also exactly zero owing to the same reason. Another way to say this is that the signs of mass terms for the Dirac fermions ($\chi_+^\dagger$ and $\chi_-^\dagger$) at the two Dirac nodes ($+$ and $-$) are opposite, resulting in cancellation of the parity anomaly[12]. If we fix the Neel vector in the $z$ direction ($\vec{n} = \vec{z}$), we can see the opposite signs explicitly from $m_\psi \chi_-^\dagger \tau_+^\dagger \chi_- \phi_+ \chi_-^\dagger \tau_+ \chi_- - m_\psi \chi_-^\dagger \tau_+ \chi_- \phi_- \chi_-^\dagger \tau_+ \chi_-$. Both massive Dirac fermions ($\chi_{\pm}^\dagger$) contribute to the imaginary terms, respectively. However, the signs of the imaginary terms are opposite and thus, the cancellation occurs. As a result the imaginary terms do not appear in the present $NLoM$. This was already discussed in Refs. 43 44.
It is possible that the mass terms have the same signs. Considering the two gamma matrices of \( \gamma_4 = 
abla \) and \( \gamma_5 = \left( \begin{array}{c} 0 & I \\ I & 0 \end{array} \right) \), we can obtain the following mass terms with the same signs, \( \hat{L}_M = m_x + \alpha t \bar{\chi} \hat{n} \). These mass terms can arise from the \( S_{\text{ASL}} \) in Eq. (1) via \( S_{\chi S} \) because the \( S_{\text{ASL}} \) has the enlarged symmetry \( \mathbb{T}_A \), as discussed earlier. This does not occur and thus, the imaginary terms necessarily arise. This \( \mathbb{T}_A \) would not be conventional since it breaks not only time reversal symmetry but also parity symmetry. When this \( \mathbb{T}_A \) disappears via strong quantum fluctuations, its corresponding quantum disordered paramagnet is expected to be the chiral spin liquid \( \mathbb{T}_A \). In the present paper we did not discuss the quantum phase transition from this anomalous antiferromagnetism to the chiral spin liquid.

**APPENDIX D**

In appendix D we briefly sketch a direct derivation of Eq. (7) from the antiferromagnetic Heisenberg Hamiltonian \( H_J = \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j \) at half filling. Inserting the unit (1) slave boson representation of the spin operator \( \bar{S}_i = \frac{1}{2} f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \) into the Heisenberg Hamiltonian, we obtain

\[
\hat{H}_J = \frac{1}{2} \sum_{<i,j>} \left( f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \right) \cdot \left( f_{\rho \beta}^\dagger \bar{\tau}_{\rho \beta} f_{\rho \beta} \right). 
\]

These two body interactions of spinons can be decoupled into the Hartree (direct), Fock (exchange), Bogoliubov (pairing) channels respectively, \( \frac{1}{2} \left( f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \right) \cdot \left( f_{\rho \beta}^\dagger \bar{\tau}_{\rho \beta} f_{\rho \beta} \right) = -\frac{1}{2} \left( f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \right) \cdot \left( f_{\rho \beta}^\dagger \bar{\tau}_{\rho \beta} f_{\rho \beta} \right) + \left( f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \right) \left( f_{\rho \beta}^\dagger \bar{\tau}_{\rho \beta} f_{\rho \beta} \right) - \frac{1}{2} \left( f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \right) \left( f_{\rho \beta}^\dagger \bar{\tau}_{\rho \beta} f_{\rho \beta} \right) + \frac{1}{2} \left( f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\sigma \alpha} \right) \left( f_{\rho \beta}^\dagger \bar{\tau}_{\rho \beta} f_{\rho \beta} \right)
\]

Hence, performing the Hubbard-Stratonovich transformation for each interaction channel, we obtain the following effective Lagrangian

\[
L_{\text{eff}} = \sum_i \left( f_{\sigma \alpha}^\dagger \left( \partial_\tau - i a_{\sigma \tau} \right) f_{\sigma \alpha} - \frac{1}{2} \left( f_{\sigma \alpha}^\dagger \chi_{ij} f_{\rho \beta} + h.c. \right) \right)
\]

Here \( \chi_{ij} \) is a hopping parameter, \( \bar{m}_i \), a spin density wave order parameter, \( \rho_i \), a charge density wave order parameter, and \( \Delta_{ij} \), a pairing order parameter. \( a_{\sigma \tau} \) is the Lagrange multiplier field to impose the single occupancy constraint of \( f_{\sigma \alpha}^\dagger f_{\sigma \alpha} = 1 \), and \( \mu \), a spinon chemical potential formally introduced. In the following we do not consider the pairing channel. Instead, we focus our attention on the spinon exchange and magnetization associated with the \( \text{ASL} \) and \( \mathbb{T}_A \), respectively. The ansatz of an antiferromagnetic order leads us to the saddle point equations of the three order parameters, \( \chi_{ij} = J \langle f_{\sigma \alpha}^\dagger f_{\rho \beta} \rangle, \bar{m}_i = \frac{1}{\alpha} \langle f_{\sigma \alpha}^\dagger \bar{\tau}_{\sigma \alpha} f_{\rho \beta} \rangle, \) and \( \rho_i = \frac{1}{\alpha} \langle f_{\sigma \alpha}^\dagger f_{\rho \beta} \rangle \). At half filling the saddle point value of the charge density wave order parameter is given by \( \rho_i = J/4 \). Shifting the \( a_{\sigma \tau} \) field into \( i a_{\sigma \tau} = -i a_{\sigma \tau} - \mu - J/4 \) in Eq. (D1) leads to the following effective Lagrangian

\[
L_{\text{eff}} = \sum_i f_{\sigma \alpha}^\dagger \left( \partial_\tau - i a_{\sigma \tau} \right) f_{\sigma \alpha} - \chi_0 \sum_{<i,j>} \left( f_{\sigma \alpha}^\dagger e^{i a_{\sigma \tau}} f_{\rho \beta} + h.c. \right)
\]

\[
-\mu \sum_{<i,j>} f_{\sigma \alpha} \bar{m}_i \bar{\tau}_{\sigma \alpha} f_{\rho \beta}. 
\]

Here \( \chi_0 = |\chi_{ij}| \) and \( m = \bar{m}_i \) are amplitudes of the order parameters, assumed to be uniform in space and time. \( a_{\sigma \tau} \) is the phase of \( \chi_{ij} \) and \( \bar{m}_i \), that of \( \bar{m}_i \).

In order to derive Eq. (7) from the above effective Lagrangian Eq. (D2), one can use the Haldane mapping of \( \bar{m}_i \) and \( \bar{m}_i \) into the O(3) effective action with an emergent U(1) gauge field \( \Omega \). This procedure is well derived in Ref. [37]. Combining the spinless fermions with this O(3) NLm, one can finally obtain the O(3) NLmM. This procedure is well derived in Ref. [37]. Combining the spinless fermions with this O(3) NLmM in order to examine the magnetic QCP, and performing the Hubbard-Stratonovich transformation used in section III-A, one can find an effective action with an emergent U(1) gauge field \( c_\mu \) similar to Eq. (7) in the continuum limit.

We would like to comment how the present study can be expanded to the case of geometrically frustrated lattices. Remember that in the present paper a collinear spin order is considered to result in the O(3) NLmM in Eq. (1). In the frustrated antiferromagnets non-collinear spin orders are expected and thus, the present NLm cannot be applied. In this case the unitary vector \( \Omega \) in Eq. (D2) should be generalized to impose non-collinear spin orders. This requires a different kind of Haldane mapping from the above. In this case a paramagnetic phase resulting from each non-collinear spin order is an interesting problem. Remember that in appendix C we mentioned an exciting quantum phase transition from an...
anomalous antiferromagnetism to the chiral spin liquid. These two interesting studies remain as future works.

APPENDIX E

In order to confirm the present description of the QCP, we derive the similar critical field theory, but in a totally different way where the Neel order parameter fields are not introduced. We consider the following effective Hamiltonian (appendix A and D)

\[
H_{\text{eff}} = -\chi_0 \sum_{<i,j>} \left( f_{i\sigma}^\dagger e^{ia_{i,j}} f_{j\sigma} + \text{h.c.} \right) - m \sum_{<i,j>} \left( -1 \right)^{j} f_{i\sigma}^\dagger f_{j\sigma} - m \sum_{<i,j>} \left( -1 \right)^{j} f_{i\sigma}^\dagger f_{j\sigma}^\dagger,
\]

(E1)

describing staggered magnetization in the flux phase of the Heisenberg model \cite{46}. Because the flux phase results in massless Dirac fermions near the nodal points, the QED with \( S_x S_B \), \( \mathcal{L}_{\text{QED}} = \bar{\psi}_x \gamma_{\mu} (\partial_{\mu} - ia_{\mu}) \psi_x + m_{\psi} \bar{\psi}_x \psi_x + \frac{g^2}{4} (\partial \times a)^2 \) can be derived from Eq. (E1). We perform bosonization by attaching a flux to the fermionic spinon, \( f_{i\sigma} = f_{i\sigma}^\dagger f_{i\sigma} \), where \( b_{i\sigma} \) is a statistically transmuted bosonic spinon with \( n_{i\sigma} = f_{i\sigma}^\dagger f_{i\sigma} = b_{i\sigma}^\dagger b_{i\sigma} \). Our strategy is to rewrite Eq. (E1) in terms of the newly introduced boson fields \( b_{i\sigma} \),

\[
H_{\text{eff}} = -\chi_0 \sum_{<i,j>} \left( b_{i\sigma}^\dagger e^{ia_{i,j}} e^{ia_{i,j}} b_{j\sigma} + \text{h.c.} \right) - m \sum_{<i,j>} \left( -1 \right)^{j} b_{i\sigma}^\dagger b_{j\sigma} - m \sum_{<i,j>} \left( -1 \right)^{j} b_{i\sigma}^\dagger b_{j\sigma}^\dagger,
\]

(E2)

with the constraint of \( 2\theta b_{i\sigma} b_{i\sigma}^\dagger = (\nabla \times \vec{a}_{i\sigma}) \cdot \vec{z} \), arising from the flux attachment. Here \( a_{i\sigma}^\dagger = \bar{a}_{i\sigma}^\dagger \) is a Chern-Simons gauge field with a statistical angle \( \theta = \pi \), guaranteeing the statistical transmutation of the fermionic spinons. Eq. (E2) can be written to be in a general gauge in the continuum limit

\[
\mathcal{L} = \left| (\partial_{\mu} - ia_{\mu} - ia_{\mu}^c) \phi_{\sigma} \right|^2 + m_{\phi}^2 \left| \phi_{\sigma} \right|^2 + \frac{u_{\phi}}{2} \left| \phi_{\sigma} \right|^4 + \frac{1}{2g^2} |\partial \times a|^2 + \frac{i}{4\theta} \epsilon_{\mu\nu\lambda} a_{\mu}^c a_{\nu} a_{\lambda}^c,
\]

(E3)

where the boson field \( b_{i\sigma} \) is replaced with a coarse grained field \( \phi_{\sigma} \) in the continuum limit. Comparing Eq. (E3) with the QED with \( S_x S_B \), the Dirac Lagrangian in the QED is replaced with the Klein-Gordon one in the same dispersion relation owing to the bosonic statistics of the \( \phi_{\sigma} \) field in the flux phase. \( m_{\phi} \) is a phenomenological boson mass associated with the magnetization \( m \) in Eq. (E1) and \( u_{\phi} \), strength of local interactions. \( m_{\phi} \) is assumed to be a function of the fermion flavor number \( N \). As \( N \) approaches the critical value \( N_c \), \( m_{\phi} \) goes to zero. The above bosonization procedure can be found in Ref. \cite{15}, but in an opposite direction (boson to fermion instead of fermion to boson). The main point is that at the QCP (\( m_{\phi} \rightarrow 0 \)) the Chern-Simons gauge field does not play any role\cite{15}. Shifting the gauge field \( a_{\mu} \) to \( c_{\mu} = a_{\mu} + a_{\mu}^c \) and integrating over the Chern-Simons gauge field, we obtain a higher order derivative of the gauge field \( c_{\mu} \), \( (\partial \times \partial \times a)^2 \). This higher order derivative term is irrelevant at the QCP in the RG sense. The resulting effective field theory is obtained to be \( \mathcal{L}_{\text{QCP}} = |(\partial_{\mu} - ic_{\mu}) \phi_{\sigma}|^2 + m_{\phi}^2 |\phi_{\sigma}|^2 + \frac{u_{\phi}}{2} |\phi_{\sigma}|^4 + \frac{1}{2g^2} |\partial \times a|^2 \).

Remarkably, this critical field theory has the same form as that derived in a different way earlier \cite{12}. It is important to realize that this effective Lagrangian can be physically meaningful only at the QCP because the compactness of the gauge field \( c_{\mu} \) can be irrelevant only at the QCP. Remember that away from the QCP this effective field theory becomes unstable in the RG sense owing to spinon condensation in the case of \( m_{\phi}^2 < 0 \) and instanton condensation in the case of \( m_{\phi}^2 > 0 \), respectively. Especially, the case of \( m_{\phi}^2 > 0 \) corresponds to the ASL in the fermion representation. This discussion shows that the statistics of spinons is fermionic in the ASL.

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