Magneto-elastic modes and lifetime of magnons in thin yttrium-iron garnet films

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We calculate the effects of the spin-lattice coupling on the magnon spectrum of thin ferromagnetic films consisting of the magnetic insulator yttrium-iron garnet. The magnon-phonon hybridisation generates a characteristic minimum in the spin dynamic structure factor which quantitatively agrees with recent Brillouin light scattering experiments. We also show that at room temperature the phonon contribution to the magnon damping exhibits a rather complicated momentum dependence: in the exchange regime the magnon damping is dominated by Cherenkov type scattering processes, while in the long-wavelength dipolar regime these processes are subdominant and the magnon damping is two orders of magnitude smaller. We supplement our calculations by actual measurements of the magnon relaxation in the dipolar regime. Our theory provides a simple explanation of a recent experiment probing the different temperatures of the magnon and phonon gases in yttrium-iron garnet.

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I. INTRODUCTION

Although the spin-lattice interactions in magnetic insulators can often be ignored, in some cases the coupling between the spin degrees of freedom and the lattice vibrations (phonons) plays a crucial role. For example, in ultrasound experiments one uses the spin-lattice coupling to study the properties of the spin degrees of freedom from the measurement of the propagation and the attenuation of sound waves. The theory of magneto-elastic effects in magnetic insulators has been developed more than half a century ago by Abrahams and Kittel and by Kaganov and Tsukernik. While in the past decades a few theoretical studies of magneto-elastic effects have appeared, recent experimental progress in the field of spintronics has revived the interest in the interactions between spin and lattice degrees of freedom.

The present work is motivated by an experiment where the coupling between magnons and phonons in the magnetic insulator yttrium iron garnet (YIG) was probed via a spatially resolved measurement of the magnon temperature $T_m$ in the presence of a thermal gradient. In the short wavelength exchange part of the magnon spectrum $T_m$ was found to agree almost perfectly with the temperature $T_p$ of the phonon bath. In order to reconcile this finding with earlier studies of the spin Seebeck effect (which relies on the difference between $T_m$ and $T_p$), the authors of Ref. speculated that in the long-wavelength dipolar part of the spectrum the magnon temperature significantly differs from $T_p$, suggesting a rather weak coupling between magnons and phonons in this regime. In this work we offer a microscopic explanation for such a momentum dependence of the magnon temperature; we show that the lifetime $\tau(k)$ of magnons due to coupling to the phonons in YIG is strongly momentum dependent; in particular $\tau(k)$ exhibits a pronounced minimum in the exchange regime and is two orders of magnitude larger in the dipolar regime. Since in the dipolar part of the spectrum the magnons have a longer lifetime, in this regime phonons are less effective to thermalize the magnons and the differences between $T_m$ and $T_p$ can persist for longer times.

We also present experimental results for the magnon damping in the dipolar regime, which have been obtained by means of time- and wave-vector-resolved Brillouin light scattering (BLS) spectroscopy. The experimentally determined damping rate is an order of magnitude larger than our theoretical prediction, although the qualitative behavior as a function of in-plane momentum is similar. One should keep in mind, however, that in our calculations only the relaxation due to magnon-phonon interactions has been taken into account. Apparently, in the long-wavelength dipolar regime other scattering channels leading to momentum relaxation are dominant, such as elastic magnon-impurity scattering.

The rest of this work is organized as follows: In Sec. we briefly review the calculation of the spin wave spectrum of a thin YIG stripe and fix the experimentally relevant parameters. In Sec. we carefully derive the magnon-phonon interaction by quantizing the phenomenological classical magneto-elastic energy. The calculation of the magneto-elastic modes due to the magnon-phonon hybridisation is presented in Sec. III. We also calculate the resulting spectral function of the magnons and the transverse dynamic structure factor which is proportional to the BLS cross section.
we are interested in the magnon dynamics, we derive the effective action of the magnons by integrating over the phonon degrees of freedom. Using this effective action, we proceed in Sec. [V] to calculate the damping of the magnons due to the coupling to the phonons. We also present new experimental results for the magnon damping in the long-wavelength dipolar regime, and compare them to our calculations. Finally, in Sec. [VI] we present our conclusions. To make contact with previous work on magnon-phonon interactions in YIG, we present in the appendix an alternative derivation of the dispersion of the magneto-elastic modes using the equations of motion.

II. MAGNONS IN YIG

It is generally established that the magnetic properties of YIG at room temperature can be obtained from the following effective quantum spin Hamiltonian\[15\-17\]

$$\mathcal{H} = \frac{1}{2} \sum_{ij} \left( J_{ij} \delta_{\alpha\beta} + D_{ij} \delta_{\alpha\beta} \right) S_i^\alpha S_j^\beta - h \sum_i S_i^z, \quad (2.1)$$

where the spin operators $S_i = S(R_i)$ are localized at the sites $R_i$ of a cubic lattice with lattice spacing $a \approx 12.376$ Å, the exchange couplings $J_{ij} = J(R_i - R_j)$ and the dipole-dipole interaction as

$$J_{ij} = \frac{\mu^2}{|R_i - R_j|^3} \left[ 3 R_i \cdot R_{ij} - \delta_{\alpha\beta} \right], \quad (2.2)$$

where $R_{ij} = R_i - R_j$ and $|R_{ij}|$. Within the framework of the usual expansion in inverse powers of the spin $S$ the low-energy magnon spectrum of YIG can be quantitatively described if one chooses $J \approx 3.19$ K and $S = M_s a^3 / \mu \approx 14.2$, where the saturation magnetization of YIG is given by $4\pi M_s = 1750$ G. Due to the large value of the effective spin $S$ the spin operators $S_i$ and $b_i$ and expand the square roots,

$$S_i^+ = \sqrt{2S} \left( 1 - \frac{b_i^\dagger b_i}{2S} \right) \approx \sqrt{2S} b_i - \frac{b_i^\dagger b_i}{4S}, \quad (2.3a)$$

$$S_i^- = \sqrt{2S} b_i^\dagger \left( 1 - \frac{b_i b_i^\dagger}{2S} \right) \approx \sqrt{2S} b_i^\dagger - \frac{b_i b_i^\dagger}{4S}, \quad (2.3b)$$

$$S_i^z = S - b_i^\dagger b_i, \quad (2.3c)$$

where $S_i^+ = S_i^x + iS_i^y$ and $S_i^- = S_i^x - iS_i^y$, and we have assumed that the magnetization of the system is uniform and points along the $z$-axis. To describe a thin stripe we can work with an effective two-dimensional model, as explained in Ref.\[17\].

![FIG. 1. Thin YIG stripe of thickness $d$ in a magnetic field $\mathbf{H} = H_\| \mathbf{e}_z$ along the direction $\mathbf{e}_z$ of the long axis. We consider magnons with wave vector $\mathbf{k} = |\mathbf{k}| \cos \theta_k \mathbf{e}_z + |\mathbf{k}| \sin \theta_k \mathbf{e}_y$ in the stripe-plane.]

The in-plane magnon wave vectors are then of the form

$$\mathbf{k} = k_y \mathbf{e}_y + k_z \mathbf{e}_z = |\mathbf{k}| \cos \theta_k \mathbf{e}_z + |\mathbf{k}| \sin \theta_k \mathbf{e}_y. \quad (2.4)$$

where $N$ denotes the number of lattice sites in the $yz$-plane, and retaining only quadratic terms in the bosons, we obtain the bosonized Hamiltonian for YIG,

$$\mathcal{H}^{(2)}_m = \sum_k \left[ A_k b_k^\dagger b_k + \frac{B_k}{2} \left( b_k^\dagger b_{-k} + b_{-k} b_k \right) \right]. \quad (2.5)$$

Here the energies $A_k$ and $B_k$ can be expressed in terms of the Fourier transforms $J_k$ and $D_{ij}^{\alpha\beta}$ of the exchange couplings and the dipole-dipole interaction as

$$A_k = h + S(J_0 - J_k) + S \left( D_{xx}^z - \frac{D_{xx}^z + D_{yy}^z}{2} \right), \quad (2.6a)$$

$$B_k = -S \frac{D_{xx}^z - D_{yy}^z}{2}. \quad (2.6b)$$

The Hamiltonian\[25\] is easily diagonalized by means of a Bogoliubov transformation,

$$\begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \end{pmatrix} \begin{pmatrix} \beta_k \\ \beta_k^\dagger \end{pmatrix}, \quad (2.7)$$

with

$$u_k = \sqrt{A_k + E_k} / 2E_k, \quad v_k = B_k / (B_k) \sqrt{A_k - E_k} / 2E_k, \quad (2.8)$$

and

$$E_k = \sqrt{A_k^2 - B_k^2}. \quad (2.9)$$

In terms of the magnon quasi-particle operators $\beta_k$ the Hamiltonian is diagonal,

$$\mathcal{H}^{(2)}_m = \sum_k \left[ E_k \beta_k^\dagger \beta_k + \frac{A_k - E_k}{2E_k} \right]. \quad (2.10)$$
For a thin stripe with thickness $d$ the magnon dispersion can be approximated for small wave vectors by
\[ E_k = \left[ h + \rho_s k^2 + \Delta(1 - f_k) \sin^2 \theta_k \right]^{1/2} \]
\[ \times \left[ h + \rho_s k^2 + \Delta f_k \right]^{1/2}, \] (2.11)
where $\theta_k$ is the angle between the magnetic field and the wave vector $k$ and the form factor can be approximated by
\[ f_k \approx \frac{1 - e^{-|k|d}}{|k|d}. \] (2.12)

For a thin YIG stripe the exchange spin stiffness $\rho_s$ and the dipolar energy scale $\Delta$ have the values
\[ \rho_s = 2\mu_B \times 5.17 \times 10^{-9} \text{Oe cm}^2 \]
\[ \approx 945 \text{GHz} \times a^2 \times 2\pi h, \] (2.13)
\[ \Delta = 3SD_{0}^{1.5} = 2\mu_B \times 4\pi M_s = 2\mu_B \times 1750 \text{G} \]
\[ \approx 0.235 K \times k_B = 4.89 \text{GHz} \times 2\pi h. \] (2.14)

Note that in units where $k_B = 1$ and $h = 2\pi h = 1$ we have $1 K \approx 20.8$ GHz.

III. MAGNON-PHONON HAMILTONIAN OF THIN YIG-FILMS

One source of the spin-phonon coupling is the dependence of the true positions $\mathbf{r}_i = \mathbf{R}_i + \mathbf{X}_i$ of the spins on the phonon displacements $\mathbf{X}_i = \mathbf{X}(\mathbf{R}_i)$ at the lattice sites $\mathbf{R}_i$. The resulting magnon phonon interaction can be derived from the effective spin-model by expanding the exchange couplings $J_{ij} = J(\mathbf{R}_i - \mathbf{R}_j + \mathbf{X}_i - \mathbf{X}_j)$ in powers of the phonon displacements. However, in collinear magnets such a procedure does not take into account the dominant source of the magnon-phonon interaction, which is generated by relativistic effects such as dipole-dipole interactions and spin-orbit coupling. These effects involve the charge degrees of freedom so that they cannot be simply included in our effective spin model. To derive the proper quantized interaction between magnons and phonons in YIG, we therefore follow the semi-phenomenological approach pioneered by Abrahams and Kittel, which relies on the quantization of the phenomenological expression for the classical magneto-elastic energy.

A. Classical magneto-elastic energy in YIG

To second order in the spin-density field $\mathbf{M}(\mathbf{r})$ and to first order in the displacement field $\mathbf{X}(\mathbf{r})$ associated with long-wavelength acoustic phonons the phenomenological expression for the magneto-elastic energy is
\[ E_{me}[\mathbf{M}, \mathbf{X}] = \frac{n}{M_s} \int \frac{d^3r}{(2\pi)^3} \sum_{\alpha\beta} \left[ B_{\alpha\beta} M_\alpha(r) M_\beta(r) \right] \]
\[ + B'_{\alpha\beta} \frac{\partial M_\alpha(r)}{\partial r_\beta} \cdot \frac{\partial M_\beta(r)}{\partial r_\alpha} X_{\alpha\beta}(r), \] (3.1)
where $\mathbf{M}(\mathbf{r})$ is the local magnetization (i.e., the magnetic moment per unit volume at position $\mathbf{r}$), $M_s$ is the saturation magnetization, $n = N/V$ is the number density of the magnetic particles in the system, and
\[ X_{\alpha\beta}(\mathbf{r}) = \frac{1}{2} \left[ \frac{\partial X_\alpha(r)}{\partial r_\beta} + \frac{\partial X_\beta(r)}{\partial r_\alpha} \right] \] (3.2)
is the symmetric strain tensor. Here $X_\alpha(r) = e_\alpha \cdot \mathbf{X}(\mathbf{r})$ are the projections of the phonon displacement field $\mathbf{X}(\mathbf{r})$ onto the cartesian unit vectors $e_\alpha$, where $\alpha = x, y, z$. The first term on the right-hand side of Eq. (3.1) involving the couplings $B_{\alpha\beta}$ is due to relativistic effects such as dipole-dipole interactions and spin-orbit coupling, while the gradient term involving $B'_{\alpha\beta}$ is generated by the dependence of the exchange interaction on the phonon coordinates. For a cubic lattice the phenomenological coupling tensors $B_{\alpha\beta}$ and $B'_{\alpha\beta}$ have the structure
\[ B_{\alpha\beta} = \delta_{\alpha\beta} B_\parallel + (1 - \delta_{\alpha\beta}) B_\perp, \] (3.3a)
\[ B'_{\alpha\beta} = \delta_{\alpha\beta} B'_\parallel + (1 - \delta_{\alpha\beta}) B'_\perp. \] (3.3b)
The prefactor of $n/M_s^2$ in Eq. (3.1) is introduced such that $B_\parallel$ and $B'_\parallel$ have units of energy. Transforming to wave vector space,
\[ M(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{M}(\mathbf{k}), \] (3.4a)
\[ \mathbf{X}(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{X}(\mathbf{k}), \] (3.4b)
\[ X_{\alpha\beta}(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} X_{\alpha\beta}(\mathbf{k}), \] (3.4c)
where $\int_k = \int \frac{d^3k}{(2\pi)^3}$, the matrix elements of the strain tensor in Fourier space are
\[ X_{\alpha\beta}(\mathbf{k}) = \frac{i}{2} \left[ X_\alpha(\mathbf{k}) k_\beta + X_\beta(\mathbf{k}) k_\alpha \right], \] (3.5)
and the magneto-elastic energy can be written as
\[ E_{me}[\mathbf{M}, \mathbf{X}] = \frac{n}{M_s} \int_k \int_{q} \int_{q'} (2\pi)^3 \delta(k + k' + q) \]
\[ \times \sum_{\alpha\beta} \left[ B_{\alpha\beta} M_\alpha(\mathbf{k}) M_\beta(\mathbf{k'}) \right] X_{\alpha\beta}(\mathbf{q}). \] (3.6)

At long wavelength the dominant contribution to the magneto-elastic coupling is due to spin-orbit coupling, so that in this work we shall neglect the exchange contribution, setting $B'_{\alpha\beta} = 0$. For YIG at room temperature the numerical values for the magneto-elastic coupling constants are
\[ nB_\parallel = 3.48 \times 10^6 \text{erg/cm}^3, \quad nB_\perp = 6.96 \times 10^6 \text{erg/cm}^3. \] (3.7)
The number density of the magnetic ions in YIG is
\[ n = 1/a^3, \quad a = 12.376 \text{Å}. \] (3.8)
so that in units where the Boltzmann constant and $2\pi\hbar$ are set equal to unity

$$B_\parallel = 47.8 \text{ K} = 994 \text{ GHz}, \quad B_\perp = 95.6 \text{ K} = 1998 \text{ GHz}.$$  

(3.9)

### B. Quantized magnon-phonon interaction in the Holstein-Primakoff basis

Let us now quantize the magneto-elastic energy. For the phonon field we adopt the usual strategy of expressing the displacement field by the bosonic creation and annihilation operators $a_{k\lambda}^\dagger$ and $a_{k\lambda}$ of the phonon eigenmodes with momentum $\mathbf{k}$ and polarization $\mathbf{e}_{k\lambda}$,

$$X(k) \rightarrow \frac{V}{\sqrt{N}} X_k,$$  

(3.10)

$$X_k = \sum_\lambda X_{k\lambda} \mathbf{e}_{k\lambda} = \sum_\lambda \frac{a_{k\lambda}^\dagger + a_{k\lambda}}{\sqrt{2m\omega_{\lambda}}} \mathbf{e}_{k\lambda},$$  

(3.11)

where $\lambda = 1, 2, 3$ labels the three acoustic phonon branches, $\omega_{\lambda \mathbf{k}} = c_\lambda |\mathbf{k}|$ are the phonon dispersions, $m$ is the effective ionic mass in a unit cell, $V$ is the volume of the system, and $N$ is the number of unit cells. Note that in general the polarization vectors satisfy the orthogonality relations $\mathbf{e}_{k\lambda}^\dagger \cdot \mathbf{e}_{k\lambda'} = \delta_{\lambda\lambda'}$ and the completeness relation $\sum_\lambda \mathbf{e}_{k\lambda} \mathbf{e}_{k\lambda}^\dagger = 1$, so that we may identify

$$X_{k\lambda} = (\mathbf{e}_{k\lambda}^\dagger \cdot \mathbf{X}_k) \frac{a_{k\lambda}^\dagger + a_{k\lambda}}{\sqrt{2m\omega_{\lambda}}}.$$  

(3.12)

Below we shall denote the Fourier transform of the quantized strain tensor by

$$X_{k\alpha\beta} = \frac{i}{2} \mathbf{k}_{\alpha\beta} \cdot \mathbf{X}_k,$$  

(3.13)

We may also choose the phases of the unit vectors such that $\mathbf{e}_{k\lambda}^\alpha = \mathbf{e}_{-k\lambda}$. For a quasi two-dimensional YIG stripe in the $yz$-plane we parametrize the in-plane wave vectors as $\mathbf{k} = |\mathbf{k}| (\cos \theta_k \mathbf{e}_x + \sin \theta_k \mathbf{e}_y)$, as shown in Fig. [1].

A convenient choice for the polarization vectors of the longitudinal and transverse phonons is then

$$\mathbf{e}_{k\alpha} = i(e_z \cos \theta_k + e_y \sin \theta_k),$$  

(3.14a)

$$\mathbf{e}_{k,1\alpha} = i(e_z \sin \theta_k - e_y \cos \theta_k),$$  

(3.14b)

$$\mathbf{e}_{k,2\alpha} = e_x.$$  

(3.14c)

For YIG the relevant value of the effective ionic mass appearing in Eqs. (3.11)–(3.12) is

$$m = \rho a^3, \quad \rho = 5.17 \text{ g/cm}^3,$$  

(3.15)

and the longitudinal and transverse phonon velocities are

$c_{||} = 7.209 \times 10^5 \text{ cm/s}, \quad c_\perp = 3.843 \times 10^5 \text{ cm/s}$.  

(3.16)

To quantize the magnetic sector, we follow Refs. [2–4] and replace the magnetization $\mathbf{M}(\mathbf{r})$ at lattice site $\mathbf{r} = \mathbf{R}_i$ by the corresponding spin-operator $\mathbf{S}_i$ according to the prescription

$$\mathbf{M}(\mathbf{R}_i) \rightarrow 2\mu_B n \mathbf{S}_i,$$  

(3.17)

where $n = N/V = 1/a^3$ is the number density. In Fourier space Eq. (3.17) corresponds to

$$\mathbf{M}(\mathbf{k}) \rightarrow g\mu_B \sqrt{N} \mathbf{S}_k = g\mu_B \sum_i e^{-ik_{\mathbf{R}}_i} \mathbf{S}_i,$$  

(3.18)

Assuming that the macroscopic magnetization points along the $z$ axis, we may express the components of the spin operators at lattice site $\mathbf{R}_i$ in terms of Holstein-Primakoff bosons ($\text{magnons}$) $b_i$ and $b_i^\dagger$ as usual with Eqs. (2.3a–2.3c). In momentum space we have

$$S_k^z = \sqrt{N} \delta_{k,0} - \frac{1}{\sqrt{N}} \sum_q b_q^\dagger b_{q+k},$$  

(3.19)

and to leading order in an expansion in powers of $1/S$,

$$S_k^z \approx \frac{\sqrt{2} S}{2} (b_k + b_k^\dagger),$$  

(3.20)

$$S_k^\perp \approx \frac{\sqrt{2} S}{2i} (b_k - b_k^\dagger).$$  

(3.21)

For large $S$ it is reasonable to retain only terms up to quadratic order in the magnons, so that the resulting magneto-elastic Hamiltonian can be written as

$$\mathcal{H}_{\text{me}} = \mathcal{H}^{(2)}_{\text{me}} + \mathcal{H}^{(3)}_{\text{me}},$$  

(3.22)

where the superscript indicates the number of operators involved. The quadratic term $\mathcal{H}^{(2)}_{\text{me}}$ is

$$\mathcal{H}^{(2)}_{\text{me}} = B_\perp \sqrt{\frac{S}{2}} \sum_k \left[ \mathbf{X}^z_{-k} (b_k + b_k^\dagger) - i \mathbf{X}^z_{-k} (b_k - b_k^\dagger) \right]$$

$$= \sum_k X_{-k} \cdot \mathbf{\Gamma}_k b_k + \mathbf{\Gamma}^\star_{-k} b^\dagger_{-k},$$  

(3.23)

where we have used the fact that the $\mathbf{X}_k$ are the Fourier components of a hermitian operator so that

$$X_{-k} = X_k^\star.$$  

(3.24)

and the vector vertex $\mathbf{\Gamma}_k$ is defined by

$$\mathbf{\Gamma}_k = -\frac{B_\perp}{\sqrt{2} S} (k_y i + k_x z)$$

$$= -\frac{B_\perp}{\sqrt{2} S} [(k_y + ik_x) e_z + k_z (e_y + ie_x)].$$  

(3.25)

Expanding

$$\mathbf{X}_k = \sum_\lambda X_{k\lambda} \mathbf{e}_{k\lambda}, \quad \mathbf{\Gamma}_k = \sum_\lambda \mathbf{\Gamma}_{k\lambda} \mathbf{e}_{k\lambda},$$  

(3.26)
with
\[ X_{k\lambda} = e_{k\lambda}^* \cdot X_k, \quad \Gamma_{k\lambda} = e_{k\lambda}^* \cdot \Gamma_k, \] (3.27)
the Hamiltonian (3.23) can be written as
\[ \mathcal{H}_{\text{me}}^{(2)} = \sum_{k\lambda} \left[ \Gamma_{k\lambda} X_{k\lambda}^* b_k + \Gamma_{k\lambda}^* X_{k\lambda} b_k \right] = \sum_{k\lambda} X_{-k\lambda} \left[ \Gamma_{-k\lambda} b_k + \Gamma_{-k\lambda}^* b_k^\dagger \right]. \] (3.28)

This part of the magneto-elastic Hamiltonian describes the hybridisation between the phonon and magnon degrees of freedom. With the choice \( \text{3.14a} \) of the phonon basis we obtain for the projections of the hybridisation vertex
\[ \Gamma_{k\parallel} = \frac{i B_{\perp}}{\sqrt{2} S} k_x k_z = \frac{i B_{\perp}}{2 S} |k| \sin(2\theta_k), \] (3.29a)
\[ \Gamma_{k\perp} = \frac{i B_{\perp}}{\sqrt{2} S} \left| k \right| - \frac{i B_{\parallel}}{2 S} |k| \cos(2\theta_k), \] (3.29b)
\[ \Gamma_{k\perp} = -\frac{i B_{\perp}}{\sqrt{2} S} k_z = -\frac{i B_{\parallel}}{2 S} |k| \cos \theta_k. \] (3.29c)

Before discussing the spectrum of the magneto-elastic modes, let us write down the cubic term \( \mathcal{H}_{\text{me}}^{(3)} \) of the magneto-elastic Hamiltonian in the form\(^{[21]}\)
\[ \mathcal{H}_{\text{me}}^{(3)} = \frac{1}{\sqrt{N}} \sum_{kk'k''} \left[ \Gamma_{k,k'}^b \cdot X_{k-k'}^b b_{k''}^b b_{k''}^\dagger \right] = \frac{1}{\sqrt{N}} \sum_{kq} X_{k-q} \left[ U_{q-b} b_{k+q} + V_{q-b}^* b_{k-q}^\dagger \right], \] (3.30)
where the vector vertices are given by
\[ \Gamma_{k,k'}^b = U_{k-k'}, \] (3.31a)
\[ \Gamma_{k,k'}^b = V_{k-k'}, \] (3.31b)
\[ \Gamma_{k,k'}^b = V_{k-k'}, \] (3.31c)
with
\[ U_q = \frac{i B_{\parallel}}{S} (q_x e_x + q_y e_y - 2 q_z e_z), \] (3.32a)
\[ V_q = \frac{i B_{\parallel}}{S} (q_x e_y - q_y e_x) + \frac{B_{\parallel}}{S} q_{xy} = \frac{i B_{\parallel}}{S} (q_x e_x - q_y e_y) + \frac{B_{\parallel}}{S} (q_x e_y + q_y e_x). \] (3.32b)

C. Magneto-elastic interaction vertices in the quasi-particle basis

For the calculation of the spectral function of the magnons it is useful to express the magneto-elastic interaction in terms of the magnon quasi-particle operators \( \beta_k \) and \( \beta_k^\dagger \) which are related to the Holstein-Primakoff bosons \( b_k \) and \( b_k^\dagger \) via the Bogoliubov transformation \( \text{2.7} \). Substituting this transformation into Eqs. \( \text{3.23} \) and \( \text{3.30} \) we obtain for the hybridisation part
\[ \mathcal{H}_{\text{me}}^{(2)} = \sum_{k\lambda} X_{-k\lambda} \left[ \Gamma_{k\lambda}^\beta \beta_k + \Gamma_{k\lambda}^\beta^\dagger \right], \] (3.33)
with
\[ \Gamma_{k\lambda}^\beta = u_k \Gamma_{k\lambda} - v_k^* \bar{e}_{-k\lambda}, \] (3.34a)
\[ \Gamma_{k\lambda}^\beta = u_k \Gamma_{k\lambda} - v_k \bar{e}_{k\lambda} = (\Gamma_{k\lambda}^\beta)^*. \] (3.34b)
The magnon-phonon interaction defined in Eq. \( \text{3.30} \) can be written as
\[ \mathcal{H}_{\text{me}}^{(3)} = \frac{1}{\sqrt{N}} \sum_{kk'k''} \left[ \Gamma_{k,k'}^b \cdot X_{k-k'}^b \beta_{k''}^\beta \right] = \frac{1}{2!} \left( \Gamma_{k,k'}^b \cdot X_{k-k'}^b \beta_{k''}^\beta + \Gamma_{k,k'}^b \cdot X_{k+k'}^b \beta_{k''}^\beta \right) \] (3.36)
with
\[ \Gamma_{k,k'}^b = u_k u_{k'} \Gamma_{k,k'}^b + v_k v_{k'}^* \Gamma_{k,k'}^b - v_k u_{k'} \Gamma_{k-k'}^b - u_k v_{k'} \Gamma_{k-k'}^b = (u_k u_{k'} + v_k v_{k'}) U_{k-k'} - u_k v_{k'} \Gamma_{k-k'}^b \] (3.37a)
\[ \Gamma_{k,k'}^b = u_k u_{k'} \Gamma_{k,k'}^b + v_k v_{k'}^* \Gamma_{k,k'}^b - v_k u_{k'} \Gamma_{k-k'}^b - u_k v_{k'} \Gamma_{k-k'}^b = u_k u_{k'} \Gamma_{k,k'}^b + v_k v_{k'} \Gamma_{k-k'}^b \] (3.37b)
\[ \Gamma_{k,k'}^b = u_k u_{k'} \Gamma_{k,k'}^b + v_k v_{k'} \Gamma_{k-k'}^b - u_k u_{k'} \Gamma_{k-k'}^b \] (3.37c)
Note that the hermiticity of the Hamiltonian implies
\[ \Gamma_{k,k'}^b = (\Gamma_{k,k'}^b)^*, \quad \Gamma_{k,k'}^b = (\Gamma_{k,k'}^b)^*. \] (3.38)
Below we shall need these interaction vertices to calculate the damping of the magnons in YIG due to the coupling to the phonons. In fact, we shall need the projections of the three-legged vertices onto the phonon basis, which we define by
\[ \Gamma_{k,k',\lambda}^{b\beta} = e_{k',-k\lambda} \cdot \Gamma_{k,k'}^{b\beta}, \] (3.39a)
\[ \Gamma_{k,k',\lambda}^{b\beta} = e_{k',-k\lambda} \cdot \Gamma_{k,k'}^{b\beta}, \] (3.39b)
or in terms of shifted labels,
\[ \Gamma_{k,k',q+\lambda}^{b\beta} = e_{q+\lambda} \cdot \Gamma_{k,k'+q}^{b\beta}, \] (3.40a)
\[ \Gamma_{k,-k+q,\lambda} = e_{q+\lambda} \cdot \Gamma_{k,-k+q}^{b\beta}. \] (3.40b)
These vertices should also be useful in microscopic calculations of the non-equilibrium dynamics of magnons in YIG. Note that in Ref. [25] only the Cherenkov type of process described by the vertex $\Gamma^{\beta\beta\beta}_{k_{1}k_{2}k_{3}}$ has been taken into account (however, using a simple phenomenological expression for this vertex). It should be interesting to repeat the calculations of Ref. [25] for the non-equilibrium magnon dynamics in YIG using the more realistic magnon-phonon vertices given above.

**IV. MAGNETO-ELASTIC MODES IN YIG**

To calculate the energy dispersion of the magneto-elastic modes, it is sufficient to retain only terms which are quadratic in the magnon and phonon operators. The Hamiltonian of the coupled magnon-phonon system can then be approximated by

$$\mathcal{H}^{(2)} = \mathcal{H}^{(2)}_{m} + \mathcal{H}^{(2)}_{e} + \mathcal{H}^{(2)}_{\text{me}}$$

(4.1)

where the quadratic spin wave part $\mathcal{H}^{(2)}_{m}$ is given in Eqs. (2.5) and (2.10), the pure phonon part can be written as

$$\mathcal{H}^{(2)}_{e} = \sum_{k\lambda} \left[ \frac{P_{-k\lambda}P_{k\lambda}}{2m} + \frac{m}{2} \omega_{k\lambda} X_{-k\lambda} X_{k\lambda} \right]$$

$$= \sum_{k\lambda} \omega_{k\lambda} \left[ a_{k\lambda}^{\dagger} a_{k\lambda} + \frac{1}{2} \right]$$

(4.2)

and the magnon-phonon hybridization $\mathcal{H}^{(2)}_{\text{me}}$ is given in Eqs. (3.28) and (3.33).

**A. Effective magnon action and magnon self-energies**

To study the effect of the lattice vibrations on the spin excitations, it is convenient to use a functional integral formulation of the coupled magnon-phonon system and integrate over the phonon degrees of freedom, which in our approximation can be done exactly because we have truncated the expansion (3.4) of the magneto-elastic energy at the linear order in the phonon coordinates. The magnon operators $\beta_{k}$ and $\beta_{k}^{\dagger}$ in the quasi-particle basis should then be represented by complex fields $\beta_{K}$ and $\bar{\beta}_{K}$ depending on momentum $k$ and bosonic Matsubara frequency $i\omega$, which we collect into the label $K = (k, \omega)$. The resulting Euclidean effective action of the magnons is of the form

$$S_{\beta, \bar{\beta}} = S_{2}[\bar{\beta}, \beta] + S_{3}[\bar{\beta}, \beta] + S_{4}[\bar{\beta}, \beta]$$

(4.3)

where the Gaussian part is given by

$$S_{2}[\bar{\beta}, \beta] = -\frac{1}{T} \sum_{K} \left[ i\omega - E_{k} - \Sigma_{1}(K) \right] \bar{\beta}_{K} \beta_{K}$$

$$- \frac{1}{2} \left[ \Pi_{1}(K) \beta_{-K} \beta_{K} + \Pi_{1}^{*}(K) \bar{\beta}_{-K} \bar{\beta}_{K} \right]$$

(4.4)

Here the normal and anomalous self-energies to first order in the small parameter $1/S$ are given by

$$\Sigma_{1}(K) = -\sum_{\lambda} \frac{|\Gamma^{\beta}_{k\lambda}|^{2}}{m(\omega^{2} + \omega_{k\lambda}^{2})} = -\sum_{\lambda} |\Gamma^{\beta}_{k\lambda}|^{2} F_{0}(K\lambda),$$

$$\Pi_{1}(K) = -\sum_{\lambda} \frac{\Gamma^{\beta}_{k\lambda} \Gamma^{\beta\lambda}_{-k\lambda}}{m(\omega^{2} + \omega_{k\lambda}^{2})} = -\sum_{\lambda} \frac{\Gamma^{\beta}_{k\lambda} \Gamma^{\beta\lambda}_{-k\lambda} F_{0}(K\lambda)}{m(\omega^{2} + \omega_{k\lambda}^{2})}$$

(4.5)

(4.6)

where we have introduced the symmetric phonon propagator

$$F_{0}(K\lambda) = \frac{1}{m(\omega^{2} + \omega_{k\lambda}^{2})}.$$  

(4.7)

The interference of the magnon-phonon hybridisation with the cubic term of the magneto-elastic coupling yields a cubic contribution to the effective magnon-magnon interaction,

$$S_{3}[\bar{\beta}, \beta] = -\frac{1}{T\sqrt{N}} \sum_{K_{1}K_{2}K_{3}} \delta K_{1}K_{2}K_{3} \left[ \right.$$  

$$\left. \frac{1}{2} \Gamma^{\beta\beta\beta}_{123} \beta_{-1} \beta_{2} \beta_{3} + \frac{1}{2} \Gamma^{\beta\beta\beta}_{123} ^{\dagger} \beta_{-1}^{\dagger} \beta_{2} \beta_{3}^{\dagger} \right.$$  

$$+ \frac{1}{3} \Gamma^{\beta\beta\beta}_{123} \beta_{1} \beta_{2} \beta_{3} + \frac{1}{3} \Gamma^{\beta\beta\beta}_{123} ^{\dagger} \beta_{1}^{\dagger} \beta_{2}^{\dagger} \beta_{3}^{\dagger}, \right.$$  

(4.8)

Here $\delta_{K,K'} = \delta_{k,k'} \delta_{\omega,\omega'}$ and for simplicity we have abbreviated $\beta_{1} \equiv \beta_{K_{1}}$ and $\Gamma^{\beta\beta\beta}_{123} \equiv \Gamma^{\beta\beta\beta}_{123}$, and similarly for the other labels. Introducing the notation $F_{0}(K\lambda) = F_{1,\lambda}$, the properly symmetrized cubic interaction vertices can be written as
Finally, the quartic magnon-magnon interaction is generated from the square of the cubic magneto-elastic coupling via the exchange of a virtual phonon,

\[ S_4(\bar{\beta}, \beta) = -\frac{1}{TN} \sum_{K_1...K_4} \delta_{K_1+...+K_4,0} \left[ \frac{1}{(2!)^2} \Gamma_{1234}^{\bar{\beta}\beta\beta\beta} \Gamma_{-1234}^{\bar{\beta}\beta\beta\beta} + \frac{1}{3!} (\Gamma_{1234}^{\bar{\beta}\beta\beta\beta} \Gamma_{-1234}^{\bar{\beta}\beta\beta\beta} - \Gamma_{1234}^{\bar{\beta}\beta\beta\beta} \Gamma_{-1234}^{\bar{\beta}\beta\beta\beta}) \beta_1 \beta_2 \beta_3 \beta_4 + \frac{1}{4!} (\Gamma_{1234}^{\bar{\beta}\beta\beta\beta} \Gamma_{-1234}^{\bar{\beta}\beta\beta\beta} - \Gamma_{1234}^{\bar{\beta}\beta\beta\beta} \Gamma_{-1234}^{\bar{\beta}\beta\beta\beta}) \beta_1 \beta_2 \beta_3 \beta_4 \right]. \]  

The symmetrized quartic vertices are

\[
\begin{align*}
\Gamma_{1234}^{\bar{\beta}\beta\beta\beta} &= \sum_{\lambda} \left[ F_{1+2,\lambda} \Gamma_{-1,2,\lambda}^{\bar{\beta}\beta\beta\beta} + F_{2+3,\lambda} \Gamma_{-2,3,\lambda}^{\bar{\beta}\beta\beta\beta} + F_{3+1,\lambda} \Gamma_{-3,4,\lambda}^{\bar{\beta}\beta\beta\beta} \right], \\
\Gamma_{1234}^{\bar{\beta}\beta\beta\beta} &= \sum_{\lambda} \left[ F_{1+2,\lambda} \Gamma_{-1,2,\lambda}^{\bar{\beta}\beta\beta\beta} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right], \\
\Gamma_{1234}^{\bar{\beta}\beta\beta\beta} &= \sum_{\lambda} \left[ F_{1+2,\lambda} \Gamma_{-1,2,\lambda}^{\bar{\beta}\beta\beta\beta} + (2 \leftrightarrow 3) + (1 \leftrightarrow 3) \right], \\
\Gamma_{1234}^{\bar{\beta}\beta\beta\beta} &= \sum_{\lambda} \left[ F_{1+2,\lambda} \Gamma_{-1,2,\lambda}^{\bar{\beta}\beta\beta\beta} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right].
\end{align*}
\]

To leading order in \(1/S\), the damping of the magnons is determined by the \(1/S^2\) correction to the normal component of the magnon self-energy, which can be written as

\[
\Sigma_2(K) = \frac{T}{N} \sum_{K'} G_0(K') \Pi(K', K) G_0(K).
\]

\[
= \frac{T}{N} \sum_{K',\lambda} \left| \Gamma_{k,k',\lambda}^{\bar{\beta}\beta} \right|^2 F_0(K - K', \lambda) + \left| \Gamma_{k,k',\lambda}^{\bar{\beta}\beta} \right|^2 F_0(K + K', \lambda) \right] G_0(K'),
\]

where

\[
G_0(K) = \frac{1}{i\omega - E_k}.
\]

is the non-interacting magnon Green function.

**B. Magnon spectral function and dynamic structure factor**

Due to the off-diagonal self-energy \(\Pi(K)\) generated by the magnon-phonon interaction, the magnon Green function has also an off-diagonal component, so that we should consider the normal and anomalous propagators. In terms of the normal irreducible self-energies the normal magnon propagator can be written as

\[
G(K) = -T \langle \beta_K \beta_K \rangle = -\frac{i\omega + E_k + \Sigma_0(-K)}{D(K)},
\]

while the anomalous magnon propagator is

\[
P(K) = \frac{-T \langle \beta_K \beta_{-K} \rangle}{D(K)} = \frac{\Pi^*(K)}{D(K)}.
\]

Here

\[
D(K) = -\left[ i\omega - \Sigma_0(K) \right] + |E_k + \Sigma_0(K)|^2 - |\Pi(K)|^2,
\]

as

\[
G_0(K) = \frac{1}{i\omega - E_k}
\]

can be identified with the determinant of the inverse matrix Green function, and

$$\Sigma_\pm(K) = \frac{1}{2} [\Sigma(K) \pm \Sigma(-K)].$$  \hspace{1cm}  (4.17)

The spectrum of the magneto-elastic modes can be obtained from the roots of the equation

$$D(k, \omega) = 0,$$  \hspace{1cm}  (4.18)

with infinitesimal positive \( \eta \). Hence, the magneto-elastic modes are determined by

$$\begin{align*}
&[\omega - \Sigma_-(k, \omega)]^2 - E_k^2 - 2E_k\Sigma_+(k, \omega) \\
&= [\Sigma_+(k, \omega)]^2 - |\Pi(k, \omega)|^2.
\end{align*}$$  \hspace{1cm}  (4.19)

For large effective spin \( S \) we may approximate the self-energies by the first order corrections given in Eqs. (4.5) and (4.6). Using the explicit polarization basis given in Eqs. (3.14a), (3.14c), it is easy to see that \( \Sigma_-(K) = 0 \) in this approximation, so that we may identify \( \Sigma_+(K) = \Sigma(K) \). Still, Eq. (4.19) amounts in general to finding the solutions of a sixth order polynomial. However, the last two terms on the right hand side involving the square of the self-energies are of order \( 1/S^2 \) and can be neglected; we have checked numerically that these terms do not have any significant effect for the parameters relevant to YIG.

The equation for the magneto-elastic modes then reduces to

$$\omega^2 - E_k^2 = 2E_k\Sigma_1(k, \omega),$$  \hspace{1cm}  (4.20)

where \( \Sigma_1(k, \omega) \) is defined in Eq. (4.5). To further simplify this equation let us assume that either the energy of the longitudinal phonon mode or the energy of the transverse phonon modes is close to the magnon energy \( E_k \). In the first case we may approximate

$$\Sigma_1(k, \omega) \approx \frac{|\Gamma^\beta_{k||}|^2}{m(\omega^2 - \omega_{k\perp}^2)},$$  \hspace{1cm}  (4.21)

while in the second case

$$\Sigma_1(k, \omega) \approx \frac{|\Gamma^\beta_{k\perp}||^2}{m(\omega^2 - \omega_{k\perp}^2)},$$  \hspace{1cm}  (4.22)

where

$$|\Gamma^\beta_{k\perp}||^2 = |\Gamma^\beta_{k\perp,1}|^2 + |\Gamma^\beta_{k\perp,2}|^2.$$  \hspace{1cm}  (4.23)

Eq. (4.20) is then bi-quadratic and has the solutions

$$\begin{align*}
\Omega_{k\lambda\pm}^4 &= \frac{\omega_{k\lambda}^2 + E_k^2}{2} \pm \sqrt{\left(\frac{\omega_{k\lambda}^2 - E_k^2}{2}\right)^2 + \Delta_{k\lambda}^4},
\end{align*}$$  \hspace{1cm}  (4.24)

where

$$\Delta_{k\lambda}^4 = 2\frac{E_k}{m}|\Gamma^\beta_{k\lambda}|^2.$$  \hspace{1cm}  (4.25)

The energy dispersion of these modes is shown graphically in Fig. 2 for \( k = ke_\perp \) parallel to the in-plane magnetic field. Note that for this propagation direction the longitudinal phonon does not hybridise with the magnon dispersion because for \( k_y = 0 \) (corresponding to \( \theta_k = 0 \)) the relevant hybridisation function \( \Gamma_{k\parallel} \) given in Eq. (3.29a) vanishes.

The normal component of the magnon Green function is in this approximation

$$G(k, i\omega) = \frac{i\omega + E_k + \Sigma_1(k, i\omega)}{(i\omega)^2 - E_k^2 - 2E_k\Sigma_1(k, i\omega)}.$$  \hspace{1cm}  (4.26)

If \( i\omega \) is close to \( \omega_{k\lambda} \) this can be approximated by

$$G(k, i\omega) \approx \frac{|\omega + E_k|[(i\omega)^2 - \omega_{k\lambda}^2] + \frac{|\Gamma^\beta_{k\lambda}|^2}{m}}{|(i\omega)^2 - E_k^2|[(i\omega)^2 - \omega_{k\lambda}^2] - 2E_k|\Gamma^\beta_{k\lambda}|^2}.$$  \hspace{1cm}  (4.27)

After analytic continuation to the real frequency axis we obtain for the corresponding spectral function

$$A(k, \omega) = -\frac{1}{\pi} \text{Im} G(k, \omega + i0)$$

$$\approx Z_{k\lambda}(\omega) \sum_{s=\pm} s[\delta(\omega - \Omega_{k\lambda s}) + \delta(\omega + \Omega_{k\lambda s})],$$  \hspace{1cm}  (4.28)

where

$$Z_{k\lambda}(\omega) = \frac{[\omega + E_k][\omega^2 - \omega_{k\lambda}^2] + \frac{|\Gamma^\beta_{k\lambda}|^2}{m}}{2\omega|\Omega_{k\lambda}^2 - \Omega_{k\lambda}^2|}.$$  \hspace{1cm}  (4.29)

An intensity plot of the magnon spectral function for YIG is shown in Fig. 3. One clearly sees the transfer of spectral weight between the magnon and the phonon branch at the crossing point. Actually, the Brillouin light scattering intensity is proportional to the transverse spin structure factor:

$$S_{k\perp}(k, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle S_{-k}(0)S_{k}(t) + S_{-k}(0)S_{k}(t) \rangle.$$  \hspace{1cm}  (4.30)
where the Fourier transform of the spin-operator is defined in Eq. (3.18). To leading order in spin wave theory we obtain

$$S_{\perp}(k, \omega) = \frac{S}{1 - e^{-\omega/T}} \times \left\{ (u_k^2 + |v_k|^2) \left[ A(-k, \omega) - A(k, -\omega) \right] - 2u_kv_kB(k, \omega) - 2u_kv_kB^*(k, \omega) \right\},$$

(4.31)

where

$$B(k, \omega) = -\frac{1}{\pi} \text{Im} P(k, \omega + i0)$$

(4.32)

is the spectral function of the anomalous magnon Green function $P(k, i\omega)$, which for imaginary frequencies is defined by Eq. (4.15). We obtain

$$B(k, \omega) \approx Y_{k\lambda}(\omega) \sum_{s=\pm} s\delta(\omega - \Omega_{k\lambda s}) + \delta(\omega + \Omega_{k\lambda s}),$$

(4.33)

where

$$Y_{k\lambda}(\omega) = -\frac{\Gamma_{-k\perp,1}^{\lambda} \Gamma_{k\perp,2}^{\lambda} + \Gamma_{-k\perp,2}^{\lambda} \Gamma_{k\perp,1}^{\lambda}}{2m\omega[\Omega_{k\lambda s}^2 - \Omega_{k\lambda s}^2]}.$$  

(4.34)

An intensity plot of the transverse dynamic structure factor is shown in Fig. 4. Note that the qualitative behavior is very similar to the behavior of the magnon spectral function, which is due to the fact that the Bogoliubov transformation has only a small effect because of the smallness of the Bogoliubov coefficient $v_k$ in the entire Brillouin zone.

**V. MAGNON DAMPING**

In this section we calculate the decay rate (i.e., the damping) of the magnons due to the magnon-phonon interaction in YIG at room temperature. The damping $\gamma(k)$ of magnons with wave vector $k$ and energy $E_k$ can be obtained from the imaginary part of the self-energy $\Sigma(K) = \Sigma(k, i\omega)$ after analytic continuation to the real frequency axis,

$$\gamma(k) = -\text{Im} \Sigma(k, i\omega \rightarrow E_k + i\eta).$$

(5.1)

To leading order in $1/S$ the damping is determined by the second order self-energy given in Eq. (4.12); after carrying out the frequency sum we obtain
Here \( b(\omega) = 1/(e^{\omega/T} - 1) \) is the Bose function. The corresponding damping function off resonance is

\[
\gamma_2(k, \omega) = -\text{Im}\Sigma_2(k, \omega + i\eta)
\]

\[
= \left(1 - e^{-\omega/T}\right) \frac{\pi}{N} \sum_{k'\lambda} \left\{ \frac{|\Gamma_{k,k',\lambda}^\beta|^2}{2m\omega_{k-k'}} \left[ \delta(E_{k'} - \omega_{k-k'})(1 + b(E_{k'}))(b(\omega_{k-k'})) 
\right.
\]

\[
+ \delta(E_{k'} - \omega_{k-k'})(1 + b(E_{k'}))(1 + b(\omega_{k-k'})) 
\]

\[
+ \frac{|\Gamma_{k,k',\lambda}^\beta|^2}{2m\omega_{k-k'}} \delta(E_{k} + E_{k'} - \omega_{k+k'}) \right\}
\]

\[
(5.2)
\]

Since the experiments of interest to us are performed at room temperature which is large compared with all other energy scales, we may use the high temperature expansion of the Bose functions, \( b(\omega) \approx T/\omega \). Setting now \( \omega = E_k \) we obtain for the magnon damping on resonance at high temperatures,

\[
\gamma_2(k) = \frac{\pi T E_k}{2mN} \sum_{k'\lambda} \left\{ \frac{|\Gamma_{k,k',\lambda}^\beta|^2}{E_{k'}\omega_{k-k'}} \delta(E_k - E_{k'} + \omega_{k-k'}) 
\right.
\]

\[
+ \delta(E_k - E_{k'} - \omega_{k-k'}) \left( 1 + b(E_k)(1 + b(\omega_{k-k'})) \right) 
\]

\[
+ \frac{|\Gamma_{k,k',\lambda}^\beta|^2}{E_{k'}\omega_{k+k'}} \delta(E_k + E_{k'} - \omega_{k+k'}) \right\}
\]

\[
= \gamma_{2a}^{\text{Che}}(k) + \gamma_{2b}^{\text{Che}}(k) + \gamma_{2c}^{\text{con}}(k),
\]

\[
(5.3)
\]

where

\[
\gamma_{2a}^{\text{Che}}(k) = \frac{\pi T E_k}{2mN} \sum_{q\lambda} \left\{ \frac{|\Gamma_{k,k+q,\lambda}^\beta|^2}{E_{k+q}\omega_{k+q\lambda}} \delta(E_k - E_{k+q} + \omega_{q\lambda}) \right\}
\]

\[
(5.5)
\]

\[
\gamma_{2b}^{\text{Che}}(k) = \frac{\pi T E_k}{2mN} \sum_{q\lambda} \left\{ \frac{|\Gamma_{k,k-q,\lambda}^\beta|^2}{E_{k-q}\omega_{k-q\lambda}} \delta(E_k - E_{k-q} - \omega_{q\lambda}) \right\}
\]

\[
(5.6)
\]

\[
\gamma_{2c}^{\text{con}}(k) = \frac{\pi T E_k}{2mN} \sum_{q\lambda} \left\{ \frac{|\Gamma_{k,k+q,\lambda}^\beta|^2}{E_{k+q}\omega_{k+q\lambda}} \delta(E_k + E_{k+q} - \omega_{q\lambda}) \right\}
\]

\[
(5.7)
\]

The contributions \( \gamma_{2a}^{\text{Che}}(k) \) and \( \gamma_{2b}^{\text{Che}}(k) \) are due to the Cherenkov type process where a magnon with energy \( E_k \) emits or absorbs a phonon with energy \( \omega_q \) and decays into a magnon with energy \( E_{k+q} \). The last contribution \( \gamma_{2c}^{\text{con}}(k) \) describes a confluent scattering process where two magnons with energies \( E_k \) and \( E_{k+q} \) decay into a phonon with energy \( \omega_q \). The corresponding Feynman diagrams are shown in Fig. 5. Taking into account that for YIG the interaction vertices in the quasi-particle basis can be approximated by the corresponding interaction vertices in the Holstein-Primakoff basis, we may approximate the squared matrix elements in the above expressions by

\[
|\Gamma_{k,k+q,\lambda}^\beta|^2 \approx |e_{q\lambda}^* \cdot \Gamma_{k,k+q,\lambda}^\beta|^2 \approx |e_{q\lambda}^* | \Gamma_{k,k+q,\lambda}^\beta|^2
\]

\[
= |e_{q\lambda}^* | U_{\lambda}^2 |^2 \equiv |q|^2 U_{\lambda}^2(q),
\]

\[
(5.8)
\]

\[
|\Gamma_{k,k-q,\lambda}^\beta|^2 \approx |e_{q\lambda}^* \cdot \Gamma_{k,k-q,\lambda}^\beta|^2 \approx |e_{q\lambda}^* | \Gamma_{k,k-q,\lambda}^\beta|^2
\]

\[
= |e_{q\lambda}^* | V_{\lambda}^2 |^2 \equiv |q|^2 V_{\lambda}^2(q),
\]

\[
(5.9)
\]
Here \( \hat{q}_a = q_a / |q| \) are the components of the unit vector in the direction of \( q \). Using the phonon basis in Eqs. (3.14a-3.14c) we obtain for \( q_x = 0 \),

\[
U_x(\hat{q}) = \frac{B_y^2}{S^2} (\hat{q}_y^2 - 2\hat{q}_z^2)^2 = \frac{B_y^2}{S^2} (1 - 3 \cos^2 \theta_q)^2, \\
V_x(\hat{q}) = \frac{B_x^2}{S^2} (\hat{q}_x^2 - \hat{q}_y^2)^2 = \frac{B_x^2}{S^2} \frac{9}{4} \sin^2(2\theta_q), \\
U_x(\hat{q}) = 0,
\]

and

\[
V_\| (\hat{q}) = \frac{B_y^2}{S^2} \hat{q}_y^4 = \frac{B_y^2}{S^2} \sin^4 \theta_q, \\
V_{\perp}(\hat{q}) = \frac{B_y^2}{S^2} (\hat{q}_y^2 - 2\hat{q}_z^2)^2 = \frac{B_y^2}{S^2} \frac{1}{4} \sin^2(2\theta_q), \\
V_{\perp}(\hat{q}) = \frac{B_y^2}{S^2} \hat{q}_y^2 = \frac{B_y^2}{S^2} \sin^2 \theta_q,
\]

where we have set \( \hat{q}_z = \cos \theta_q \) and \( \hat{q}_y = \sin \theta_q \).

### A. Dipolar regime: Theory

In the long-wavelength regime \(|k| \lesssim \sqrt{\Delta/\rho_s}\) the behavior of the magnon damping (3.4) strongly depends on the size \( \nu(k) = |v(k)| \) of the group velocity \( v(k) = \nabla_k E_k \) of the magnons in comparison with the phonon velocities. In Fig. 6 we show the momentum range where the magnon velocity \( v(k) \) exceeds the phonon velocities \( c_\lambda \). In regime around the minima of the dispersion, the velocity \( v(k) \) is small compared with the phonon velocities, while at very small wave vectors \( v(k) > c_\lambda \). In this regime around the minima of the dispersion the decay rate the magnons is dominated by the confluent process given in Eq. (5.7) because the Cherenkov processes are kinematically suppressed. In fact, in a substantial region around dispersion minima the quasi-particle velocity is small compared with the phonon velocities, so that we may approximate \( E_{-k+q} \approx E_k - v(k) \cdot q \) and expand the decay rate in powers of \( v(k)/c_\lambda \). The momentum integration in Eq. (5.7) can then be carried out and we obtain

\[
\gamma^\text{con}_2(k) = \frac{T E_k a^2}{4S^2 m} \left[ \frac{B_y^2}{c_A^2} \left( 1 + \frac{3 v_y^2(k)}{4 c_A^2} + \frac{v_z^2(k)}{4 c_A^2} \right) \\
+ \frac{B_z^2}{4 c_A^2} \left( 1 + \frac{v_z^2(k)}{2 c_A^2} + \frac{v_y^2(k)}{2 c_A^2} \right) \\
+ \frac{3}{4 c_A^2} \left( 1 + \frac{5 v_y^2(k)}{6 c_A^2} + \frac{v_z^2(k)}{6 c_A^2} \right) \right].
\]

(5.14)

The confluent contribution to the high-temperature damping rate in the dipolar regime is shown graphically as the thin dotted line in Fig. 7. On the other hand, the Cherenkov-type process contributes only for very small \( k \), when \( v(k) \gtrsim c_\lambda \), see Fig. 6. However, for those momenta it is no longer valid to linearize the dispersion due to the strong effect of the Bogoliubov transformation, and no analytical approximation can be obtained. Therefore we only present the numerical solution as dashed line in Fig. 6. Obviously, apart from an enhancement for \( k \lesssim 2 \times 10^4 \text{ cm}^{-1} \) the damping exhibits a rather weak dependence on the wave vector in this regime.

### B. Dipolar regime: Experiment

For a comparison of our calculation with experiments one should keep in mind that we have only considered the contribution from the magnon-phonon interactions on the damping of the magnons. Of course, in the real system there are other sources leading to magnon decay, such as magnon-magnon interactions or the elastic
scattering of magnons by impurities. We therefore expect that the magnon damping due to magnon-phonon interactions is a lower limit to the experimentally observed magnon damping rate. In fact, our experimental data presented below show that in the dipolar regime magnon-phonon interactions are not the dominant source of magnon damping.

In order to determine the relaxation time of different groups of magnons, a measurements of the spectral distribution of magnon gas densities as a function of the frequency and wave vector using time- and wave-vector-resolved BLS spectroscopy \[^{31–34}\] has been performed. Due to technical reasons only in-plane wave vectors from 0 up to the \(k_{\text{max}} = 11 \times 10^4 \text{cm}^{-1}\) are accessible within our apparatus. The measurements were performed using a YIG film with thickness 6.7 \(\mu\text{m}\), which was liquid-phase epitaxially grown on a 500 \(\mu\text{m}\) thick gallium gadolinium garnet substrate.

The magnon spectrum was populated by intensive thermalization\[^{25–28,29}\] of magnons, which were injected by the parallel parametric pumping technique\[^{30}\] at half of the pumping frequency \(f_p = 13.62 \text{GHz}\). The bias magnetic field of \(H = 1710 \text{Oe}\) was tuned to provide the excitation of parametric magnons at the ferromagnetic resonance frequency. In this case the direct transition of magnons to the bottom of spin wave spectrum is prohibited by conservation laws. This ensures high efficiency of multi-stage four-magnon scattering which is necessary for thermalization and thus population of the spectrum.

We have measured the redistribution of thermalized magnons along the fundamental backward-volume magnetostrictive spin wave mode as a function of time and wave vector. After switching off the pumping, the magnons are allowed to relax freely. By fitting the relaxation times for different groups of thermalized magnons we were able to extract the damping rates. The obtained dependence of measured total damping rate \(\gamma_{\text{tot}}\) on the in-plane wave vector for dipolar-exchange spin waves is shown in Fig. 8. Obviously, the value of relaxation rate is roughly an order of magnitude larger than our calculated relaxation rate due to magnon-phonon interactions shown Fig. 7. We thus conclude that in the long-wavelength dipolar regime other relaxation channels (in particular two-magnon scattering processes\[^{31–34}\]) dominate the magnon damping. The rather irregular behavior of the measured damping rate in Fig. 8 suggests that elastic scattering of magnons by impurities might play an important role in this regime. Note that within the tolerance limits of the experiment the measured relaxation rate has a rather weak dependence on the in-plane wave vectors in the entire accessible range of wave vectors. In this respect the experimental results agree with our prediction of a momentum-independent damping rate in this regime.

Unfortunately, microscopic calculations of the magnon decay rates at room temperature, taking magnon-impurity and magnon-magnon interactions into account, are not available in the momentum range relevant for our experiment. One should keep in mind, however, that magnon-impurity scattering can only explain the momentum-relaxation of the magnon gas; for the equilibration of the different temperatures of the magnon and the phonon systems magnon-phonon interactions are essential.

C. Exchange regime

For wave vectors in the regime where the exchange energy \(\rho_0 k^2\) exceeds the characteristic dipolar energy \(\Delta\) we may ignore the dipole-dipole interaction in the magnon
FIG. 9. (Color online) Numerical evaluation of our result for the damping rate of magnons in a thin YIG stripe at temperature \( T = 300 \) K, in the exchange momentum regime. The plot is for a thin stripe of thickness \( d = 6.7 \mu m \) in an external magnetic field \( H = 1710 \) Oe, for \( k = k_z \) parallel to the in-plane magnetic field. Solid lines correspond to the total damping rate, while the dashed and the dotted lines are the contribution from the Cherenkov and the confluent processes, respectively. The corresponding thin lines are the approximations in the exchange momentum regime, see Eqs. (5.15) and (5.17).

dispersion \( \gamma^2 \) and approximate the long-wavelength magnon dispersion by \( E_k \approx \hbar + \rho_s k^2 \). Then the evaluation of the integrals in Eqs. (5.5 – 5.7) simplifies. We obtain for the Cherenkov part

\[
\gamma^2_2(k) = \gamma^2_2(k) + \gamma^2_2(k) = \frac{TE_k m_s}{2m} \sum_{\lambda} a^2 \frac{U^2(q_\varphi)}{2 \pi E_k + 2m_s c_\lambda (c_\lambda - v(k) \cos \varphi)}.
\]

(5.15)

where \( v(k) \equiv |k|/m_s \), the mass \( m_s \) is defined via \( \rho_s = 1/(2m_s) \), and the angle-dependent unit vector \( q_\varphi \) is defined by

\[
q_\varphi = [\hat{v}_y(k) \cos \varphi - \hat{v}_z(k) \sin \varphi] e_y + [\hat{v}_z(k) \cos \varphi + \hat{v}_y(k) \sin \varphi] e_z.
\]

(5.16)

Here \( \hat{v}_u(k) \) are the components of the unit vector in the direction of the magnon velocity \( v(k) = \nabla_k E_k = k/m_s \). Although the angular integration in Eq. (5.15) can be done analytically, the result is not very transparent so that we omit it here. A graph of the Cherenkov contribution to the high-temperature damping rate in the exchange regime is shown as the thin dashed line in Fig. [9].

Next, consider the contribution \( \gamma^2_2 \) of the confluent scattering process to the damping rate in the exchange regime. Setting again \( E_k \approx \hbar + \rho_s k^2 \) and carrying out the integration over \( |q| \) we obtain

\[
\gamma^2_2(k) = \frac{TE_k a^2}{4m} \sum_{\lambda} \int_0^{2\pi} d\varphi \frac{U^2(q_\varphi)}{2 \pi c_\lambda^2} \Theta(c_\lambda + v(k) \cos \varphi - \sqrt{4E_k/m_s}) \times \sqrt{(c_\lambda + v(k) \cos \varphi)^2 - 4E_k/m_s} \times \left[ \frac{c_\lambda q_+ - E_k}{c_\lambda q_-} + \frac{c_\lambda q_- - E_k}{c_\lambda q_+} \right],
\]

(5.17)

where

\[
q_\pm = m_s(c_\lambda + v(k) \cos \varphi) \pm m_s \sqrt{(c_\lambda + v(k) \cos \varphi)^2 - 4E_k/m_s}.
\]

(5.18)

The confluent contribution to the high-temperature damping rate in the exchange regime is shown graphically as the thin dotted line in Fig. [9]. As one can see, the magnon damping is strongly \( k \) dependent. In particular, it exhibits peaks at the crossing points of magnon and phonon dispersions as well as velocities. It also increases by two orders of magnitude between dipolar and exchange regimes, which are dominated by confluence and Cherenkov processes respectively. For the magnon lifetime \( \tau(k) = 1/(2\gamma^2(k)) \), this implies values of the order of microseconds in the dipolar momentum range, while it can be as low as tens of nanoseconds for exchange momenta.

VI. SUMMARY AND CONCLUSIONS

In this work we have studied magneto-elastic interactions in experimentally relevant thin films of the magnetic insulator YIG. As the dominant sources of magneto-elastic interactions are due to relativistic effects\(^{[9]}\), which cannot be taken into account within an effective model containing only spin degrees of freedom, we have used a semi-phenomenological approach\(^{[2]}\), which relies on the quantization of a suitable phenomenological expression for the magneto-elastic energy. For the quantized theory we have then carefully derived the momentum dependence of the magneto-elastic interaction vertices within the framework of the conventional 1/S-expansion for ordered quantum spin systems. Using these vertices, we have explicitly calculated the leading contributions to the hybridisation between magnon and phonon modes, as well as the damping of the magnons due to spin-lattice coupling. The hybridisation has been shown to give rise to a characteristic minimum in the spin dynamic structure factor at the crossing point of magnon and transversal phonon dispersions, where the spectral weight is transferred from the magnons to the transverse phonon mode. The position of this minimum quantitatively agrees with the recent experimental observation of the magneto-elastic mode\(^{[15]}\).
The damping at room temperature has been shown to be strongly momentum dependent. In the long-wavelength dipolar regime it is rather flat and almost exclusively driven by confluent magnon-phonon scattering processes where two magnons decay into a phonon or vice versa. In this regime, we have also presented new experimental results for the magnon damping obtained by wave-vector-resolved Brillouin light scattering spectroscopy. The fact that the experimental results for the magnon damping are roughly an order of magnitude larger than our theoretical results indicate that in the dipolar regime magnon-phonon interactions are not the dominant source of magnon damping in our samples at room temperature. We suspect that in this regime the magnon damping is dominated by elastic scattering of magnons from impurities. On the other hand, in the short-wavelength exchange regime the damping is due to magnon-phonon scattering processes of the Cherenkov type and is two orders of magnitude larger than in the dipolar regime. The damping rate exhibits pronounced peaks at the crossing points of magnon and phonon dispersions and velocities. This agrees very well with the conclusions of the experiment [12], where the authors suggested that the spin-lattice relaxation in the dipolar regime should be much slower than in the exchange regime. Note, however, that recently Chernyshev has shown in Fig. 8, which is an order of magnitude larger than the damping due to magnon-phonon interactions in the dipolar regime magnon-phonon interactions are not the dominant source of magnon damping in our samples at room temperature. We suspect that in this regime the magnon damping is dominated by elastic scattering of magnons from impurities. On the other hand, in the short-wavelength exchange regime the damping is due to magnon-phonon scattering processes of the Cherenkov type and is two orders of magnitude larger than in the dipolar regime. The damping rate exhibits pronounced peaks at the crossing points of magnon and phonon dispersions and velocities. This agrees very well with the conclusions of the experiment [12], where the authors suggested that the spin-lattice relaxation in the dipolar regime should be much slower than in the exchange regime.

It is useful to express the equations of motion for the magneto-elastic modes from the linearized dispersions and velocities. This agrees very well with the conclusions of the experiment [12], where the authors suggested that the spin-lattice relaxation in the dipolar regime should be much slower than in the exchange regime.

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In this appendix we show how to obtain the energy dispersions of the eigenmodes we write down the Heisenberg equations of motion of both the Holstein-Primakoff bosons and the phonon operators. Within the linear approximation we have

\[
\begin{align*}
\dot{b}_k &= A_k b_k + B_k b_k^\dagger + \Gamma_k^\perp \cdot X_k, \\
\dot{b}_k^\dagger &= A_k^\dagger b_k^\dagger + B_k^\dagger b_k + \Gamma_{-k} \cdot X_k,
\end{align*}
\]

This appendix is more in the spirit of previous work using classical equations of motion.

To obtain the energy dispersions of the eigenmodes we write down the Heisenberg equations of motion of both the Holstein-Primakoff bosons and the phonon operators. Within the linear approximation we have

\[
\begin{align*}
\dot{X}_{\lambda \lambda} &= \frac{P_{\lambda \lambda}}{m}, \\
\dot{P}_{\lambda \lambda} &= -m\omega_{\lambda}^2 X_{\lambda \lambda} - \Gamma_{\lambda} b_{-\lambda} - \Gamma_{-\lambda}^\dagger b_{-\lambda}. 
\end{align*}
\]

The equations of motion for the magneto-elastic modes can be identified with

\[
\begin{align*}
S_k^\perp &= \frac{\sqrt{2S}}{2} \left( b_k + b_k^\dagger \right), \\
S_k^\parallel &= \frac{\sqrt{2S}}{2i} \left( b_k - b_k^\dagger \right).
\end{align*}
\]

Then Eqs. (A1a) and (A1b) can be written as

\[
\begin{align*}
\dot{S}_k^\perp &= (A_k - B_k) S_k^\perp + i B_k^\dagger k_{yz} \cdot X_k, \\
\dot{S}_k^\parallel &= -(A_k + B_k) S_k^\parallel - i B_k k_{xz} \cdot X_k,
\end{align*}
\]

while the phonon momenta satisfy

\[
\begin{align*}
\dot{P}_{\lambda \lambda} &= -m\omega_{\lambda}^2 X_{\lambda \lambda} + i B_{\lambda} e_{\lambda}^* \cdot (k_{xz} S_k^\perp + k_{yz} S_k^\parallel), \\
\end{align*}
\]

implying

\[
\dot{X}_{\lambda \lambda} + \omega_{\lambda}^2 X_{\lambda \lambda} = i \frac{B_{\lambda}}{m} e_{\lambda}^* \cdot (k_{xz} S_k^\perp + k_{yz} S_k^\parallel).
\]

If we ignore the magnon-phonon coupling, we obtain from Eqs. (A5a, A5b),

\[
\dot{S}_k^\alpha + E_{k,k\lambda}^\alpha S_k^\alpha = 0, \quad \alpha = x, y,
\]

where the magnon dispersion in the absence of phonons is given in Eq. (2.9). With finite magnon-phonon hybridization we obtain the energies of the magneto-elastic modes from the roots of the secular determinant of the above equations of motion. For simplicity, let is assume that the energy of only one particular phonon mode \(\omega_{\lambda}^0\) is close to \(E_{k}\). To calculate the energy of the magneto-elastic mode close to the crossing point, it is then sufficient to approximate \(X_k \approx X_{k\lambda} e_{k\lambda}\) in the above equations of motion. Then we obtain the energies of the magneto-elastic modes from the roots of the following quartic secular equation,

\[
\begin{align*}
(\omega^2 - \omega_{k\lambda}^2)(\omega^2 - E_k^2) \\
= B_k^2 \left\{ (A_k + B_k) |k_{yz} \cdot e_{k\lambda}|^2 + (A_k - B_k) |k_{xz} \cdot e_{k\lambda}|^2 \\
+ 2\omega \text{Im} \left[ (k_{xz} \cdot e_{k\lambda})(k_{yz} \cdot e_{k\lambda}) \right] \right\}.
\end{align*}
\]
For the phonons in a thin YIG stripe the basis vectors $e_{k\lambda}$ can always be chosen such that the last term in Eq. (A9) vanishes, so that the secular equation is bi-quadratic and can be explicitly solved. The square of the energies of the magneto-elastic modes in a thin YIG stripe are therefore

$$
(\Omega_{k\lambda}^\pm)^2 = \frac{\omega_{k\lambda}^2 + E_k^2}{2} \pm \sqrt{\frac{(\omega_{k\lambda}^2 - E_k^2)^2}{4} + \Delta_{k\lambda}^4.}
$$

(A10)

If we approximate $B_k \approx 0$ (which corresponds to neglecting quantum fluctuations generated by the dipolar interaction) Eq. (A10) agrees with the result obtained via the classical equations of motion.[9]

where

$$
\Delta_{k\lambda}^4 = B_k^2 \left[ (A_k + B_k) \frac{|k_{yz} \cdot e_{k\lambda}|^2}{m} + (A_k - B_k) \frac{|k_{xz} \cdot e_{k\lambda}|^2}{m} \right].
$$

(A11)

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