SECONDARY FANS AND TROPICAL SEVERI VARIETIES

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Abstract. This article studies the relationship between tropical Severi varieties and secondary fans. In the case when tropical Severi varieties are hypersurfaces this relationship is very well known; specifically, in this case, a tropical Severi variety of codimension 1 is a subfan of the corresponding secondary fan. It was expected for some time that this continues to hold more generally, but Katz found a counterexample in codimension 2, showing that this relationship is more subtle. The two main results in this paper are as follows. The first theorem finds a simple condition under which a tropical Severi variety cannot be a subfan of the corresponding secondary fan. The second theorem provides a partial converse, namely, we find conditions under which a cone of the secondary fan is fully contained in the tropical Severi variety. As a first application of these results, we also find a combinatorial formula for the tropical intersection multiplicities for secondary fans.

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1. Introduction

Severi varieties are classical objects in algebraic geometry which go back to F. Enriques [6] and F. Severi [15]. They are projective varieties which parameterize nodal curves on toric surfaces. Recent developments of tropical geometry suggest that these classical algebro-geometric objects can be viewed with a different perspective, namely the tropicalizations of Severi varieties (tropical Severi varieties). These are polyhedral objects on which combinatorial tools can be used. In [17] the author found certain descriptions of tropical Severi varieties and used them to provide a tropical intersection theoretic-computation of the degrees of Severi varieties. In particular, to each point on tropical Severi varieties some subdivisions of polygons can be assigned and the tropical intersection multiplicities appearing in the computation are all described in terms of these simple objects, namely, subdivisions of polygons (in fact, only triangles and parallelograms are involved.)
On the other hand, there are very well-known polyhedral fans which parametrize subdivisions of polygons, called secondary fans. While the constructions of tropical Severi varieties are algebro-geometric, the constructions of secondary fans are purely combinatorial, although secondary fans were introduced to study the discriminantal varieties in [8] and also have rich connections to algebraic geometry [4, 8].

So there is a very natural question: how are secondary fans and tropical Severi varieties related? Initially it had been expected for a while that a tropical Severi variety for a toric surface $X_\Delta$ from a nondegenerate lattice polygon $\Delta$ is a subfan of the secondary fan of $\mathcal{A} = \Delta \cap \mathbb{Z}^2$, the set of all lattice points on $\Delta$. In fact, it is straightforward to show that this statement holds true when the tropical Severi variety is a hypersurface (that is, of codimension 1) (Remark 3.2). Also more explicit descriptions of tropical Severi varieties of codimension 1 were studied in [3, 14]. However, Katz [10] found a counterexample showing that there is a tropical Severi variety of codimension 2 on which there does not exist any subfan structure of the corresponding secondary fan.

This paper provides some answers to the question above for general tropical Severi varieties. In Theorem 3.1, we find a simple sufficient condition under which tropical Severi variety cannot be a subfan of the corresponding secondary fan. This implies that the combinatorial object, tropical Severi variety, contains a certain data which comes from algebro-geometric properties of Severi variety. Theorem 3.3 addresses the opposite direction. Namely, it describes when a cone in the secondary fan is fully contained in the tropical Severi variety.

As a first application of this understanding of the relationship between tropical Severi varieties and secondary fans, in Theorem 3.5 we obtain a combinatorial formula for the tropical intersections of secondary fans with tropical linear spaces. The proof uses a result on tropical intersections obtained in the author’s previous paper [17].

## 2. Secondary fans and tropical Severi varieties

### 2.1. Secondary Fan

We review on the study of secondary fans. The main references for this subsection is [8, §7]. Simply speaking, the secondary fan of a finite subset $\mathcal{A}$ of the lattice $\mathbb{Z}^k$ is a fan in $\mathbb{R}^{|A|}$ whose cones parameterize the coherent marked subdivisions of $(\Delta, \mathcal{A})$, where $\Delta$ is the convex hull of $\mathcal{A}$. In this paper, we only consider the case of $k = 2$ and $\mathcal{A}$ is the set of all lattice points on a non-degenerate convex lattice polygon $\Delta$. (that is, the dimension of $\Delta$ is 2.) Let us fix the precise definitions for the notions which we will use in this paper.

A marked polygon is a pair $(\Delta, \mathcal{A})$ where $\Delta \subset \mathbb{R}^2$ is a convex lattice polygon and $\mathcal{A} \subset \mathbb{Z}^2$ is a finite subset of $\Delta$ containing all the vertices of $\Delta$ so that the convex hull of $\mathcal{A}$ coincides with $\Delta$. We always assume that $\Delta$ is non-degenerate, that is, $\dim(\Delta) = 2$.

Now let $\mathcal{A} = \Delta \cap \mathbb{Z}^2$, the set of all lattice points in $\Delta$. Then a (marked) subdivision $S$ of $\Delta$ is a collection of marked polygons

\[ \{(\Delta_i, \mathcal{A}_i) : i = 1, \ldots, m\}, \quad m \in \mathbb{Z}_{>0} \]  

such that

1. each $\mathcal{A}_i$ is a subset of $\mathcal{A}$ and each $\Delta_i$ is non-degenerate;
2. any intersection $\Delta_i \cap \Delta_j$ is a face (possibly empty) of both $\Delta_i$ and $\Delta_j$, and $\mathcal{A}_i \cap (\Delta_i \cap \Delta_j) = \mathcal{A}_j \cap (\Delta_i \cap \Delta_j)$;
A subdivision $S$ of $\Delta$ is called coherent if it can be constructed from a function $\psi \in \mathbb{R}^A$ as follows: Let $G_\psi \subset \mathbb{R}^3$ be the convex hull of the set

$$\{(a, y) : y \leq \psi(a), \ a \in A, \ y \in \mathbb{R}\}. \quad (2.3)$$

The upper boundary of $G_\psi$ is the graph of a concave piecewise-linear function which we call the concave hull of $\psi$ and denote by $cc(\psi)$,

$$cc(\psi) : \Delta \rightarrow \mathbb{R}, \quad x \mapsto \max\{y : (x, y) \in G_\psi\}. \quad (2.4)$$

(The upper boundary of $G_\psi$ is by definition the union of faces of $G_\psi$ which do not contain vertical half lines.) Then $S$ coincides with

$$\Delta_\psi := \{(\Delta_i, A_i) : i = 1, \ldots, m\}, \quad m \in \mathbb{Z}_{>0}, \quad (2.5)$$

where $\Delta_i \subset \Delta$ are the domains of linearity of $cc(\psi)$ and $A_i \subset \Delta_i$ consists of all $a \in A \cap \Delta_i$ such that $cc(\psi)(a) = \psi(a)$ (i.e., the point $(a, \psi(a))$ lies on the upper boundary of $G_\psi$ and thus it is "visible" (or "marked").)

Given a coherent subdivision $S$ of $\Delta$, let $C(S) \subset \mathbb{R}^A$ be the set of all functions $\psi$ such that $S$ refines $\Delta_\psi$.

**Proposition 2.1.** [5 [7] For any coherent subdivision $S$ of a non-degenerate convex lattice polygon $\Delta$, the set $C(S) \subset \mathbb{R}^A$ is a closed convex polyhedral cone so that the set of all such cones is a complete fan in $\mathbb{R}^A$. Furthermore, $C(S) = \overline{C(S)}^\circ$, where the relative interior $C(S)\circ$ is the set of all $\psi \in \mathbb{R}^A$ such that $S$ coincides with $\Delta_\psi$.

Note that any cone $C(S)$ in the Proposition above contains the line $\mathbb{R} \cdot 1$ consisting of the constant functions. The induced complete fan in the quotient space $\mathbb{R}^A / \mathbb{R} \cdot 1$ is called the secondary fan of $\Delta$ (recall that $A = \Delta \cap \mathbb{Z}^2$ and denoted by $\Sigma(\Delta)$). For simplicity we will use the same notation $C(S)$ for the induced cone in the quotient space $\mathbb{R}^A / \mathbb{R} \cdot 1$.

**Example 2.2.** The figure on the right illustrates the case when $\Delta$ is the segment $\text{Conv}\{0, 1, 2, 3, 4\}$, the convex hull of the points 0, 1, 2, 3, 4. The (marked coherent) subdivision $\Delta_\psi$ consists of $(\Delta_1, \{0, 2\})$ and $(\Delta_2, \{2, 3, 4\})$. Note that the point 1 is not marked.

2.2. Tropical plane curves and connection to secondary fan. In this subsection, we review the definition of a tropical plane curve and find an explicit connection to a secondary fan.
As before let $\Delta$ be a non-degenerate convex lattice polygon in $\mathbb{R}^2$ and let $\mathcal{A} = \Delta \cap \mathbb{Z}^2$. Given a function $\psi \in \mathbb{R}^\mathcal{A}$, the \textit{tropical plane curve} $\tau_\psi$ with \textit{degree} $\Delta$ is the corner locus of the piecewise-linear function
\[ \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \alpha \mapsto \max_{a \in \mathcal{A}} \{ a \cdot \alpha + \psi(a) \}. \] (2.6)

Note that the map $\psi \mapsto \tau_\psi$ is not injective. In fact, the tropical plane curve $\tau_\psi$ is uniquely determined by $\text{cc}(\psi)_{\mathcal{A}}$, the concave hull of $\psi$ (for definition see \S 2.1) restricted on $\mathcal{A}$. Also it is known that the tropical plane curve $\tau_\psi = \tau_{\text{cc}(\psi)_{\mathcal{A}}}$ is dual to the subdivision $\Delta_{\text{cc}(\psi)}$ of $\Delta$, as stated in the following Proposition. Note that the subdivision induced by a \textit{concave} function, $\Delta_{\text{cc}(\psi)}$, has the property that every lattice point is marked (or visible), that is, $\mathcal{A}_i = \Delta_i \cap \mathbb{Z}^2$ for every $i$. We introduce several notions to describe this type of subdivision in the Definition below.

**Proposition 2.3.** (\cite{[9], \S 2.5.1}) The coherent subdivision $\Delta_{\text{cc}(\psi)}$ of $\Delta$ is dual to the tropical curve $\tau_\psi$ in the following sense:

- the components of $\mathbb{R}^2 \setminus \tau_\psi$ are in 1-to-1 correspondence with $\text{Vert}(\Delta_{\text{cc}(\psi)})$;
- the edges of $\tau_\psi$ are in 1-to-1 correspondence with $\text{Edges}(\Delta_{\text{cc}(\psi)})$ so that an edge $e$ of $\tau_\psi$ is dual to an edge of $\Delta_{\text{cc}(\psi)}$ which is orthogonal to $e$ with the lattice length equal to the weight of $e$. (For the definition of the weight of an edge of a tropical curve, see \cite{[9], \S 2.5.1});
- the vertices of $\tau_\psi$ are in 1-to-1 correspondence with the 2-dimensional faces of $\Delta_{\text{cc}(\psi)}$ so that the valency of a vertex of $\tau_\psi$ is equal to the number of sides of the dual face.

**Definitions 2.4.**

1. We say $\psi \in \mathbb{R}^\mathcal{A}$ is \textit{effective} if $\psi = \text{cc}(\psi)_{\mathcal{A}}$.
2. A subdivision $S$ of $\Delta$ is called \textit{effective} if it is the coherent subdivision $\Delta_{\psi}$ for some effective element $\psi \in \mathbb{R}^\mathcal{A}$. Equivalently, an effective subdivision of $\Delta$ is a coherent subdivision such that every lattice point is marked, that is, for every $i$, $\mathcal{A}_i = \Delta_i \cap \mathbb{Z}^2$.
3. For a $\psi \in \mathbb{R}^\mathcal{A}$, define the \textit{rank} of $\psi$ to be the dimension of the cone $C(\Delta_{\text{cc}(\psi)})$ in the secondary fan $\Sigma(\Delta)$ of the effective division $\Delta_{\text{cc}(\psi)}$.

Therefore, given an effective subdivision $S$ of $\Delta$ we can identify the cone $C(S)^\circ$ in the secondary fan $\Sigma(\Delta)$ with the set of all tropical plane curves which are dual to $S$. Thus the \textit{effective part} of $\Sigma(\Delta)$, the union of such cones for all effective subdivisions, can be identified with the set of all tropical plane curves with degree $\Delta$.

**Remark 2.5.** In \cite{[14]} similar notions are introduced. The \textit{type} of the subdivision $\Delta_{\psi}$ ("forgetting all lattice points") is essentially same as the effective subdivision $\Delta_{\text{cc}(\psi)}$ ("marking all lattice points"). The rank of $\psi$ is equal to the \textit{type dimension} of the type of $\Delta_{\psi}$. Also the subdivision $\Delta_{\psi}$ is effective if and only if it is of \textit{maximal dimensional type}.

2.3. \textbf{Tropical plane curves and tropical Severi varieties.} In this subsection we review and collect some known results on tropical Severi varieties, which are needed to prove the main theorems in the next section. Also in this subsection we show the connection between points in tropical Severi varieties and tropical plane curves.
2.3.1. Severi variety. As before, let $\Delta$ be a non-degenerate convex lattice polygon in $\mathbb{R}^2$. Denote by $X_{\Delta}$ the projective toric surface constructed from $\Delta$, $\mathbb{P}_{\Delta}$ the tautological linear system on $X_{\Delta}$ and $T_{\Delta}$ the big open torus of $\mathbb{P}_{\Delta}$. That is, $\mathbb{P}_{\Delta}$ is the projective space parameterizing curves on the surface $X_{\Delta}$ defined by polynomials of the following form,

$$f = \sum_{a \in A} c_ax^a,$$

where $A = \Delta \cap \mathbb{Z}^2$, $x^a$ is the term $x_1^{a_1}x_2^{a_2}$, and the coefficients $c_a$ are taken from the base field. By ordering the elements of $A$ we can identify $\mathbb{P}_{\Delta}$ with $\mathbb{P}^{n-1}$, where $n = |A|$. By definition, the Severi variety $\text{Sev}(\Delta, \delta)$ is the (Zariski) closure of the subset of $\mathbb{P}_{\Delta} = \mathbb{P}^{n-1}$ consisting of curves with exactly $\delta$ nodes (ordinary double points) as their only singularities. It is known that the dimension of $\text{Sev}(\Delta, \delta)$ is equal to $n - 1 - \delta$, when the nonnegative integer $\delta$ is at most the number of interior lattice points of $\Delta$. In particular, when $\delta = 1$, $\text{Sev}(\Delta, 1)$ is a hypersurface in $\mathbb{P}_{\Delta}$ which is known as $A$-discriminantal variety, where $A = \Delta \cap \mathbb{Z}^2$.

2.3.2. Tropical Severi variety. Now we consider the tropicalization of $X = \text{Sev}(\Delta, \delta)$, which we call the tropical Severi variety and denote by $\text{Trop}(X)$. First, we fix the base field equipped with a non-Archimedean valuation. In this paper, we use $K$, the field of locally convergent Puiseux series over $\mathbb{C}$, that is, the element of $K$ are power series of the form

$$c(t) = \sum_{\tau \in R} t^\tau,$$

where $R \subset \mathbb{Q}$ is contained in an arithmetic progression bounded from above, $c_\tau \in \mathbb{C}$ and $\sum_{\tau \in R} |c_\tau|t^\tau < \infty$ for sufficiently large positive $t$. This is an algebraically closed field of characteristic zero with a non-Archimedean valuation

$$\text{Val}(c(t)) := \max\{\tau \in R : c_\tau \neq 0\}.$$

Remark 2.6. The definition of $K$ given in this paper is different from the standard one in literature, which can be obtained by the change of variable $t \mapsto t^{-1}$. We choose this definition not to have the minus sign in the definition of $\text{Val}(c(t))$.

Now, by definition $\text{Trop}(X)$ is the closure of the image of the following map (Refer to [2, 9, 12, 13] for more details about tropicalization):

$$X \cap T_{\Delta} \to \mathbb{R}^A/\mathbb{R}\cdot 1, \quad [c_a(t)]_{a \in A} \mapsto [a \mapsto \text{Val}(c_a(t))],$$

where we identify a point in $X \cap T_{\Delta}$ with the curve defined by the polynomial $f = \sum_{a \in A} c_a(t)x^a$ up to scalar multiplications and $\mathbb{R}\cdot 1$ is the subspace of $\mathbb{R}^A$ consisting of constant functions. For simplicity, denote the image of $[c_a(t)]_{a \in A}$ under this map by $\text{Val}(f)$.

2.3.3. Tropical plane curves from tropical Severi variety. Now we can attach to $\text{Val}(f)$ a tropical plane curve and the corresponding effective subdivision of $\Delta$,

$$\tau_f := \tau_{\text{Val}(f)}, \quad \Delta_f := \Delta_{\text{cc}(\text{Val}(f))}$$

(Note that any representative in $\mathbb{R}^A$ of $\text{Val}(f)$ gives rise to a unique tropical plane curve.)
Shustin found very nice combinatorial as well as geometric results in this process of tropicalization. To summarize his results we need one more data, namely (tropical) degenerations of the curve defined by $f$, which we describe below: First, we write each $c_a(t)(a \in A)$ as follows

$$c_a(t) = c_a^0 \cdot \text{cc(Val}(f)(a)) + \text{L.o.t.},$$  \hspace{1cm} (2.12)

where $c_a^0$ is some complex number which is zero if $\text{cc(Val}(f))(a) > \text{Val}(f)(a)$, and $\text{L.o.t.}$ stands for “lower order terms”. For the effective subdivision $\Delta_f : \Delta_1 \cup \cdots \cup \Delta_m$, we have the following collection of complex polynomials (equations for the tropical degenerations):

$$f_i := \sum_{a \in \Delta_i \cap \mathbb{Z}^2} c_a^0 x^a, \quad (i = 1, \ldots, m).$$  \hspace{1cm} (2.13)

Note that the Newton polygon for each $f_i$ (i.e., the convex hull of $a$ such that $c_a^0 \neq 0$ in $f_i$) is equal to $\Delta_i$.

**Proposition 2.7.**

(1) [16, §3.3] (tropicalization) Suppose that $\text{rank}(\tau_f) \geq \dim(X)$ for $f \in X = \text{Sev}(\Delta, \delta)$. Then the corresponding effective subdivision $\Delta_f$ of $\Delta$ has the following properties:

(a) (simple) Every boundary lattice point of $\Delta$ is a vertex of some subpolygon $\Delta_i, (i = 1, \ldots, m)$.

(b) (nodal) Every $\Delta_i(i = 1, \ldots, m)$ is either a triangle or a parallelogram.

(In fact, it is known that the rank of any tropical plane curve $\tau_f$ from $f \in X$ is at most $\dim(X)$. Thus the hypothesis of this statement is equivalent to saying that $\tau_f$ has the maximal rank.)

Also, the complex polynomials $f_1, \ldots, f_m$ have the following properties ($\ast$):

(a) the Newton polygon of $f_i$ is equal to $\Delta_i$.

(b) (▲) if $\Delta_i$ is a triangle, the curve defined by $f_i$ is rational and meets the union of toric divisors $X_{\partial \Delta_i}$ at exactly three points, where it is unibranch.

(c) (■) if $\Delta_i$ is a parallelogram, the polynomial $f_i$ has the form

$$x^k y^l (\alpha x^a + \beta y^b)^p (\gamma x^c + \delta y^d)^q$$  \hspace{1cm} (2.14)

with $\text{gcd}(a, b) = \text{gcd}(c, d) = 1, (a : b) \neq (c : d), \alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \{0\}$. ($\text{gcd}$ stands for the greatest common divisor)

(d) for any common edge $\sigma = \Delta_i \cap \Delta_j$ the truncations $f_i^\sigma$ and $f_j^\sigma$ coincide.

(2) [16] §5 (patchworking) Let $\Delta_\psi : \Delta_1 \cup \cdots \cup \Delta_m$ be an effective simple nodal subdivision of $\Delta$ with rank equal to $\dim(X)$. Suppose we have a collection of complex polynomials $F = \{f_1, \ldots, f_m\}$ which satisfies the conditions ($\ast$) above. Then there exists $f \in X$ with $\text{cc(Val}(f)) = \psi$.

### 2.4. Tropical intersection and weighted counts of tropical plane curves.

As the last topic of this section, we summarize some results presented in [17] which will be used for the last theorem in the next section. In the tropical intersection theory [11, 12, 17], there is an important intersection multiplicity (which was called an extrinsic intersection multiplicity in [17]). Let us recall it. Suppose $\psi$ is a transversal intersection point of two tropical varieties $T_1, T_2$ of complementary
Definition 2.8. [9, Definition 2.4] Multiplicities. (For details, see [17].) The intersection points of these two spaces correspond to tropical plane curves passing through the given points counted with certain multiplicities. (For details, see [17].)

Lemma 2.9. Let \( \psi \) be an effective subdivision of \( \Delta \). We say that the distinct points \( x_1, \ldots, x_\zeta \in \mathbb{Q}^2 \) are in \( S \)-general position, if the condition for tropical curves with degree \( \Delta \) to pass through \( x_1, \ldots, x_\zeta \) (“base-point-condition”) cuts out the cone \( C(S) \) either the empty set, or a polyhedron of codimension \( \zeta \).

Lemma 2.10. Let \( p = \{p_1, \ldots, p_\zeta\} \subset (\mathbb{K}^*)^2 \) be a finite set of points in \( (\mathbb{K}^*)^2 \). Define \( L(p) \subset \mathbb{P}_\Delta \) to be the parameter space of algebraic curves on the toric surface \( X_\Delta \) passing through all the points in \( p \). (Remember that the base field is \( \mathbb{K} \).) This parameter space \( L(p) \) is the complete intersection of hyperplanes \( H_{p_j} \subset \mathbb{P}_\Delta \) defined by the condition of passing through the point \( p_j \), \( j = 1, \ldots, \zeta \).

Now for \( r = \dim(\mathcal{X}) \) let \( p = \{p_1, \ldots, p_r\} \subset (\mathbb{K}^*)^2 \) be a configuration of \( r \) generic points in \( (\mathbb{K}^*)^2 \) so that \( \text{Val}(p) := \{\text{Val}(p_1), \ldots, \text{Val}(p_r)\} \subset \mathbb{Q}^2 \) is in \( \Delta \)-general position and \( \text{Trop}(L(p)) \cap \text{Trop}(\mathcal{X}) \) is a transversal intersection. (For \( p_i = (p_{i1}, p_{i2}), \text{Val}(p_i) := \text{Val}(p_{i1}), \text{Val}(p_{i2}) \) Then for any intersection point \( \psi \in \text{Trop}(L(p)) \cap \text{Trop}(\mathcal{X}) \), the tropical plane curve \( \tau_\psi \) passes through all the points in \( \text{Val}(p) \). Since \( \text{rank}(\psi) = r \), these points lie on \( r \) distinct edges of \( \tau_\psi \) which correspond to some \( r \) edges of the subdivision \( \Delta_\psi \). If \( \sigma_i \in \text{Edges}(\Delta_\psi) \) correspond to a point \( \text{Val}(p_i) \) and \( a_i, a'_i \) are the endpoints of \( \sigma_i \), \( 1 \leq i \leq r \), then we have the following linear conditions on \( \psi(a_i) \) and \( \psi(a'_i) \):

\[
\psi(a_i) - \psi(a'_i) = (a'_i - a_i) \cdot \text{Val}(p_i), \quad (i = 1, \ldots, r) \quad (2.15)
\]

We also suppose that this linear system is independent.

Proposition 2.11. [17, Theorem 4.14] With the assumptions as given above, we can compute the multiplicity as follows:

\[
\xi(\psi; \text{Trop}(L(p)), \text{Trop}(\mathcal{X})) = \frac{\prod 2\text{area(Triangles)}}{l(\mathbb{V}_{S_\psi}) \cdot \prod \text{length(Edges)}}, \quad (2.16)
\]

where

(1) \( \prod 2\text{area(Triangles)} \) is the product of twice the (Euclidean) area of each triangle in \( \Delta_\psi \).
(2) \( \prod \text{length}(\text{Edges}) \) is the product of the lattice lengths of the edges which are representatives of each equivalence class in \( \text{Edges}(\Delta_\psi) \), where we define an equivalence relation as follows: let \( e \sim e' \) if \( e \) and \( e' \) are the parallel edges of a parallelogram in \( \Delta_\psi \) and extend it by transitivity.

(3) \( l(\mathcal{V}_{\Delta_\psi}) \) is the number of components of the algebraic set \( \mathcal{V}_{\Delta_\psi} \) defined below.

**Definition 2.12.** Suppose that \( S := \Delta_1 \cup \cdots \cup \Delta_m \) is a nodal effective subdivision of \( \Delta \) that is, every sub-polygon is either a triangle or a parallelogram. Define \( \mathcal{V}_S \) to be the set of all \( f \in P_\Delta \) with the following properties,

- (\( \blacklozenge \)) For every triangle \( \Delta_i \), the curve defined by \( f_{\Delta_i} \) (the truncation of \( f \) along \( \Delta_i \)) is rational and meets the union of toric divisors \( X_{\partial \Delta} \) at exactly three points, where it is unibranch;
- (\( \blacksquare \)) For every parallelogram \( \Delta_j \), the polynomial \( f_{\Delta_j} \) has the form

\[
x^k y^l (\alpha x^a + \beta y^b)^p (\gamma x^c + \delta y^d)^q \tag{2.17}
\]

with \( \gcd(a,b) = \gcd(c,d) = 1 \), \( (a : b) \neq (c : d) \), \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \{0\} \).

The set \( \mathcal{V}_S \) is an algebraic set in a certain torus and its number of components can be computed easily using linear algebra. More details can be found in [17, §3].

3. Main results

In this last section, we state and prove the main results of this paper, that is, answers to the question: **how are secondary fans and tropical Severi varieties related?** We use the notations defined in the previous sections.

The first theorem provides a simple sufficient condition under which the tropical Severi variety, \( \text{Trop}(\mathcal{X}) \), cannot be a subfan of the Secondary fan \( \Sigma(\Delta) \). That is, we cannot find a fan structure on \( \text{Trop}(\mathcal{X}) \) such that each cone of \( \text{Trop}(\mathcal{X}) \) is the union of some cones of \( \Sigma(\Delta) \).

**Theorem 3.1.** Suppose that there exists a non-effective \( \psi \in \text{Trop}(\mathcal{X}) \) with the maximal rank, \( \text{rank}(\psi) = \dim(\mathcal{X}) \). Then there is no fan structure on \( \text{Trop}(\mathcal{X}) \) which makes \( \text{Trop}(\mathcal{X}) \) to be a subfan of the secondary fan \( \Sigma(\Delta) \).

**Proof.** We consider the following two cones in the secondary fan \( \Sigma(\Delta) \):

- the cone of the subdivision induced by \( \psi \), \( C(\Delta_\psi) \);
- the effective cone of the subdivision induced by \( \text{cc}(\psi) \) (the concave hull of \( \psi \)), \( C(\Delta_{\text{cc}(\psi)}) \).

Since \( \psi \) is non-effective, \( \psi \) is contained in the relative interior of the cone \( C(\Delta_\psi) \) while the cone \( C(\Delta_{\text{cc}(\psi)}) \) lies on the boundary of the cone \( C(\Delta_\psi) \). By the definition of rank, we know that the dimension of \( C(\Delta_{\text{cc}(\psi)}) \) is equal to \( r = \dim(\mathcal{X}) \). Thus \( \psi \) lies in the relative interior of the cone \( C(\Delta_\psi) \) whose dimension is strictly larger than \( r \). It is known that any fan structure on \( \text{Trop}(\mathcal{X}) \) is of dimension \( r \). Thus no fan structure on \( \text{Trop}(\mathcal{X}) \) can be a subfan of \( \Sigma(\Delta) \).

\( \square \)

**Remark 3.2.** Let us consider the case of \( \delta = 1 \) for \( \mathcal{X} = \text{Sev}(\Delta, \delta) \). As we mentioned before, in this case \( \mathcal{X} \) is the hypersurface of \( \mathbb{P}_\Delta \cong \mathbb{P}^{n-1} \) defined by a polynomial called \( \mathcal{A} \)-discriminant, where \( \mathcal{A} = \Delta \cap \mathbb{Z}^2 \). Thus, \( \text{Trop}(\mathcal{X}) \) is the codimension one skeleton of the normal fan \( \mathcal{F} \) of the Newton polytope of \( \mathcal{A} \)-discriminant. In [8] it is shown that \( \mathcal{A} \)-discriminant is a divisor of another polynomial called the principal
and the normal fan of the Newton polytope of the principal $A$-determinant is equal to the secondary fan $\Sigma(\Delta)$. Therefore $F$ is a subfan of the secondary fan $\Sigma(\Delta)$. As the codimension one skeleton of $F$, $\text{Trop}(\mathcal{X})$ is also a subfan of the secondary fan $\Sigma(\Delta)$. From the theorem above, we can induce that there is no non-effective $\psi$ in $\text{Trop}(\mathcal{X})$ with the maximal rank. Indeed, this fact is easily seen as follows: Suppose $\psi \in \text{Trop}(\mathcal{X})$ has the maximal rank, $\text{rank}(\psi) := \dim(C(\text{cc}(\psi))) = \dim(X) = n - 2$. However $\dim(C(\text{cc}(\psi))) \leq \dim(C(\Delta)) \leq n - 2$. Therefore $C(\text{cc}(\psi)) = C(\Delta)$ and so $\psi$ is effective.

The following theorem provides another relation between $\text{Trop}(\mathcal{X})$ and $\Sigma(\Delta)$. It is in an opposite direction of the previous theorem in a sense.

**Theorem 3.3.** Suppose that there exists an effective cone $C$ in $\Sigma(\Delta)$ such that $\dim(C) = \dim(X)$ and its relative interior $C^\circ$ intersects with $\text{Trop}(\mathcal{X})$ at non-zero vectors. Then $C^\circ$ is fully contained in $\text{Trop}(\mathcal{X})$.

**Proof.** Let $\eta \in C \cap \mathbb{Q}^A$. It is enough to show that in$_\eta\mathcal{X}$ (the initial scheme of $\mathcal{X}$ with respect to $\eta$) is not empty, which is equivalent to $\eta \in \text{Trop}(\mathcal{X})$.

Now by the hypothesis given above, we can choose a non-zero rational vector $\psi$ contained in $\text{Trop}(\mathcal{X}) \cap C^\circ$. Since $\psi \in \text{Trop}(\mathcal{X})$, we have a function of the form

$$f(x) = \sum_{a \in A} c_a(t)x^a; \quad c_a(t) = \bar{c}_a t^{\psi(a)} + \text{l.o.t.}, \quad \bar{c}_a \in \mathbb{C}^*.$$  \hspace{1cm} (3.1)

Since $\psi$ induces a concave function, the tropicalization of $f$ records all $\bar{c}_a$ for $a \in A$. Now by Proposition 2.7 (2) (patchworking) we can obtain a $g \in \mathcal{X}$ of the form

$$g(x) = \sum_{a \in A} c'_a(t)x^a; \quad c'_a(t) = \bar{c}_a t^{\eta(a)} + \text{l.o.t.}.$$  \hspace{1cm} (3.2)

Thus $(\bar{c}_a)_{a \in A} \in \text{in}_\eta\mathcal{X}$ and so $\text{in}_\eta\mathcal{X}$ is not empty. \hfill \Box

Furthermore, by Proposition 2.7 we can obtain the following corollary.

**Corollary 3.4.** The cone $C(S)^\circ$ of an effective subdivision $S$ with $\dim(C(S)) = \dim(X)$ is fully contained in $\text{Trop}(\mathcal{X})$ if and only if $S$ satisfies the following conditions ($\star \star$):

1. it is simple;
2. it is nodal;
3. any parallelogram in $S$ has no special points. (Special points in a parallelogram are lattice points in the parallelogram which are not in the lattice generated by the primitive vectors along the sides of the parallelogram.)

As an application of the results above and the result on the intersections of tropical Severi varieties [2.4], we find a combinatorial formula in the following for the intersections of secondary fans with tropical linear spaces. (Note that the codimension one skeleton of the secondary fan of $\mathcal{A}$ is equal to the tropicalization of the hypersurface defined by the principal $\mathcal{A}$-determinant. (see remark 3.2.)

**Theorem 3.5.** Let $S$ be an effective subdivision of $\Delta$ with dimension $m$ which satisfies the three conditions ($\star \star$) in Corollary 3.4. Let $H_{q_1}, \ldots, H_{q_m}$ be the point-condition tropical hyperplanes for a generic configuration of points $\{q_1, \ldots, q_m\} \subset \Delta$. Then the intersection of the secondary fan $\Sigma(S)$ with the tropical hyperplanes is given by

$$\text{Trop}^{-1}(\bigcap_{q_i \in \mathcal{X}} H_{q_i}) = \text{Trop}(\mathcal{X}) \cap \bigcap_{i=1}^m C(S)_{q_i}^\circ$$
The condition of being generic is given in the proof. Then any point $\psi \in C(S)^\circ \cap H_{q_1} \cap \cdots \cap H_{q_m}$ occurs with multiplicity equal to

$$\prod 2\text{area(Triangles)} \prod \text{area(Parallelograms)}$$

$$\frac{l(V_S)}{l(V_S) \cdot \prod \text{length(Edges)}} \quad (3.3)$$

Proof. Let $\delta = |A| - 1 - m \geq 0$ so that $\dim (\mathcal{X}) = m$, where $\mathcal{X} = \text{Sev}(\Delta, \delta)$. Note that $C(S)^\circ \subset \text{Trop}(X)$. Now we choose $p = \{p_1, \ldots, p_m\} \subset (\mathbb{K}^*)^2$ with $\text{Val}(p) = \{q_1, \ldots, q_m\}$, which satisfies the hypothesis in Proposition 2.11. Then the intersection multiplicity at $\psi$ is equal to

$$\xi(\psi; \text{Trop}(L(p)), \text{Trop}(X)) = \prod 2\text{area(Triangles)}$$

$$\frac{l(V_S) \cdot \prod \text{length(Edges)}}{l(V_S) \cdot \prod \text{length(Edges)}} \quad (3.4)$$

Since the subdivision $S$ has no special points,

$$\prod 2\text{area(Triangles)} \prod \text{area(Parallelograms)}$$

$$\frac{l(V_S) \cdot \prod \text{length(Edges)}}{l(V_S) \cdot \prod \text{length(Edges)}} \quad (3.5)$$

Thus we completed the proof. □

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