KINEMATICAL STRUCTURAL STABILITY

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Abstract. This paper gives an overview of our results obtained from 2009 until 2014 about paradoxical stability properties of non conservative systems which lead to the concept of Kinematical Structural Stability (Ki.s.s.). Due to Fischer-Courant results, this ki.s.s. is universal for conservative systems whereas new interesting situations may arise for non conservative ones. A remarkable algebraic property of the symmetric part of linear operators may generalize this result for divergence stability but leading only to a conditional ki.s.s. By duality, the concept of geometric degree of nonconservativity is highlighting. Paradigmatic examples of Ziegler systems illustrate the general results and their effectiveness.

1. Introduction. This paper deals with the so-called kinematical structural stability (ki.s.s.). This concept recently arises from a set of issues concerning stability of circulatory (and more generally nonconservative) systems (see [1] for a recent clarification about the “realistic” follower forces). It is well-known since for example Bolotin works ([2] for example) that such mechanical systems may exhibit significant “paradoxical” behaviors like the so-called damping destabilizing effect: as a well chosen value of damping is introduced in a stable undamped system (meaning that the load value $p$ is lower than the critical load of stability $p^*$), the damped system becomes unstable without changing the loading value. Some recent publications like [7] give a topological description of the threshold stability in terms of the Whitney’s umbrella giving through the catastrophe theory a modern explanation of this “paradox”.

However, circulatory systems may exhibit, in comparison with conservative systems, another paradoxical behavior that has been little emphasized in the past: a

2010 Mathematics Subject Classification. Primary: 34D30, 37C20; Secondary: 47A20.
Key words and phrases. Kinematical structural stability, second-order work criterion, geometric degree of nonconservativity, compression of linear operators.

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(linear) stable mechanical system may become (linear) unstable as a family of kinematic constraints is applied to the system. A paper of J.M.T Thompson of 1982 ([11]) mentioned this paradoxical effect through a well chosen example and specified that it is a signature of nonconservativity as well. For a couple of years, a group of french researchers systematically investigated this paradox through several papers. This set of investigations lead to the concept of kinematical structural stability of a mechanical (or a physical) system. This paper, written for a tribute to Alain Cimetière’s carrier, focus on the recent developments on this topic and proposes a structural overview of these recent developments. In a first paragraph, as a motivation for the following, we illustrate the paradoxical phenomenon on the usual Ziegler system and the issue of k.s.s. is set up. This example is in the same vein than this used by J.M.T Thompson in the paper cited above. The second section gives the main result concerning the divergence instability and the relationship with the old second order work criterion introduced in a complete another framework by Hill since 1958 (see [5] or [6] for example). The third last and short paragraph is devoted to another new concept introduced during these investigations: the geometric degree of nonconservativity of a mechanical system. Some open problems are mentioned to conclude this paper. Due to the context, neither proof nor expanded calculation are given in this paper.

2. A motivating sequence of examples. Let Σ be a discrete holonomic linear elastic undamped mechanical system, \( q = (q_1, \ldots, q_n) \) a coordinate system of Σ in a neighborhood of 0 equilibrium position, \( p \geq 0 \) a loading parameter. The linearization of the dynamic equations leads to:

\[
M \ddot{X} + K(p)X = 0
\]

where \( M, K(p) \in \mathcal{M}_n(\mathbb{R}) \) are respectively the mass and stiffness matrices (at 0) of Σ and \( X = q^T \in \mathcal{M}_n(\mathbb{R}) \). We suppose that for \( p = 0 \), Σ is conservative stable and that the domain of stability of this equilibrium 0 is \([0, p^*]\) with eventually \( p^* = +\infty \).

A family of \( m \) linear kinematic constraints is a family \( C = (C_1, \ldots, C_m) \) of linear forms identified by the scalar product with column vectors so that these constraints are equivalent to a matrix \( C = \text{mat}(C_1, \ldots, C_m) \in \mathcal{M}_{nm}(\mathbb{R}) \). The constrained system will be denoted by \( \Sigma_C \) and 0 is supposed to be still an equilibrium position of \( \Sigma_C \) (we do not discuss here situations like those investigated in [10] where the system is elastic but where the constraints change the equilibrium configuration(s)). Analogously, \([0, p^*_C]\) is the stability interval of the configuration 0 for \( \Sigma_C \).

The main question reads: what is the relationship between \( p^* \) and \( p^*_C \)? More accurately, do we have \( p^* \leq p^*_C \) as it always happens for conservative systems? If not, can we explain why and can we find (or build) a family of constraints \( C \) so that \( p^*_C < p^* \).

**Definition 2.1.** If \( p^* \leq p^*_C \quad \forall C \) we say that the system (more accurately the equilibrium position of the system) \( \Sigma \) is universally kinematically structurally stable or that its kinematical structural stability (k.s.s) is universal. If there is a value \( p^*_C (< p^*) \) such that \( p^*_C \leq p^*_C \quad \forall C \), the k.s.s. is only conditional.

Since the nineteenth century and the works of Rayleigh and Fischer-Courant, it is indeed well-known that, if \( K \) is symmetric (meaning if the system is conservative), adding kinematic constraints moves the spectrum on the right and is favorable to
the stability and the safety of the structure: universal ki.s.s is the rule for conservative mechanical systems. Observe now on the contrary what happens for a nonconservative system Σ.

Let us consider now the well-known two degrees of freedom Ziegler system Σ built by two bars \( OA, AB \) of same length \( ℓ \) with a follower force \( \vec{F} = -P \frac{\vec{AB}}{||\vec{AB}||} \) \((P > 0)\) acting on \( AB \) at \( B \). Suppose that two torsional linear springs with same stiffness \( k \) are acting at \( 0 \) and \( A \) (see figure 1).

![Figure 1. 2 d.o.f. Ziegler System](image)

The mass distribution is supposed uniform and the total mass of each bar is \( m \). The chosen parametrization of the system is \( \theta = (Oy, \vec{OA}) \) and \( \phi = (Oy, \vec{AB}) \) so that the equilibrium position whose stability is investigated is \((0,0)\). Put \( p = \frac{Pℓ}{k} \) the dimensionless load parameter and \( \Omega = \sqrt{\frac{6k}{mℓ^2}} \) the natural pulsation of the system. The linearization of the dynamic equations at the equilibrium leads to the linear dynamical system (1) with

\[
M = \frac{1}{Ω^2} \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}
\]

and

\[
K(p) = \begin{pmatrix} 2 - p & -1 + p \\ -1 & 1 \end{pmatrix}
\]

Because \( \det(K(p)) = 1 \) for all \( p \), the matrix \( K(p) \) may not become singular for any value of \( p \). The non symmetry of \( K(p) \) signs the nonconservativity of \( Σ \) and is the source of the interesting following arising issues. Two distinct kinds of linear instability could occur as the load parameter \( p \) increases. The first one is the divergence instability that means that the smallest eigenfrequency pass through 0 (the corresponding critical load is \( p_{\text{div}}^* \)) and the second one is the flutter instability that means that two eigenfrequencies are confusing before going complex (the corresponding critical load is \( p_{\text{fl}}^* \)). Straightforward calculations show that:

- \( p_{\text{div}}^* = +∞ \)
\[ p_{\text{fl}}^* = \frac{18 - 2\sqrt{7}}{5} \approx 2.54 \]

Because \( n = 2 \) we may consider only \( m = 1 \) constraint. Several cases are investigated:

- for \( C = C_{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) meaning that the constraint reads \( \theta = 0 \) then \( p_{\text{div},C_{1,0}}^* = +\infty \) and \( p_{\text{fl},C_{1,0}}^* = +\infty \)
- for \( C = C_{0,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) meaning that the constraint reads \( \phi = 0 \) then \( p_{\text{div},C_{0,1}}^* = 2 \) and \( p_{\text{fl},C_{0,1}}^* = +\infty \)
- for \( C = C_{1,2} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \) meaning that the constraint reads \( \theta + 2\phi = 0 \) then
  \[
  p_{\text{div},C_{1,2}}^* = \frac{5}{2}, \quad \text{and} \quad p_{\text{fl},C_{1,2}}^* = +\infty
  \]
- for \( C = C_{-1,1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) meaning that the constraint reads \( -\theta + \phi = 0 \) then
  \[
  p_{\text{div},C_{-1,1}}^* = +\infty \quad \text{and} \quad p_{\text{fl},C_{-1,1}}^* = +\infty
  \]
- for \( C = C_{\alpha,1} = \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \) meaning that the constraint reads \( \alpha\theta + \phi = 0 \) then
  \[
  p_{\text{div},C_{\alpha,1}}^* = \frac{\alpha^2 + 2}{\alpha + 1} \quad \text{and} \quad p_{\text{fl},C_{\alpha,1}}^* = +\infty
  \]
- It is interesting to evaluate the minimum of the divergence critical load on the set of all constraints. Straightforward calculations give:
  \[
  \min_{\alpha} p_{\text{div},C_{\alpha,1}}^* = 2
  \]

This last result means that no constraint may destabilize by divergence \( \Sigma \) as long as \( p < 2 \) (whereas \( p_{\text{div}}^* = +\infty \) means that \( \Sigma \) cannot be destabilized by divergence).

But for \( p = 2 \), there is at least one constraint (for example \( C_{0,1} \)) such that the constrained system \( \Sigma_{C_{0,1}} \) is divergence unstable. It should be noted that \( p = 2 \) is also the positive root of \( \det(K_s(p)) = 1 - \frac{p^2}{4} \). This actually absolutely general result is investigated in the following section. To conclude this section, let us also remark that flutter never occurs for 1 d.o.f. system explaining why in this case \( p_{\text{fl},C}^* = +\infty \) for any constraint \( C \). For investigating the k.i.s.s. issue for flutter, we would have to start from a 3 d.o.f. system.

3. **Conditional k.i.s.s. For divergence and second order work criterion.**

From now on, for this section, we only focus on divergence instability of a mechanical system \( \Sigma \).

**Definition 3.1.** We say that the second order work criterion (s.o.w.c.) holds for a value \( p > 0 \) of the loading if \( K_s(p) \) is a symmetric positive definite matrix \( (X^T K_s(p) X = X^T K(p) X > 0 \text{ if } X \neq 0) \). The critical value \( p_{\text{sw}}^* \) is the lowest positive value of \( p \) (eventually \( = +\infty \)) such that \( K_s(p) \) ceases to be positive definite (for \( p = 0 \), \( K_s(0) = K(0) \) is supposed positive definite).

It is easy to show that, because of the supposed monotony increasing load path, \( p_{\text{sw}}^* \) is also the the lowest positive root of \( \det(K_s(p)) = 0 \). We also may observe that, thanks to an usual result on the determinants, \( p_{\text{sw}}^* < p_{\text{div}}^* \).

The first result linking closely the k.i.s.s. and the s.o.w.c. may be find in [3] for only one constraint. By use of the Schur complement formula on the equations with
one Lagrange multiplier, it had been proved that the previous result observed for
the 2 degree of freedom Ziegler system is valid for any mechanical system:

**Proposition 1.** as long as \( 0 \leq p < p^*_{sw} \), no kinematic constraint may destabilize
the system \( \Sigma \). For \( p = p^*_{sw} \), there is a constraint \( C = (U) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \) such that \( \Sigma_C \)
is divergence unstable (the constraint then reads \( a_1 x_1 + \ldots + a_n x_n = 0 \)). Moreover,
the vector \( U \) has to be chosen on the invert image by \( K(p^*_{sw}) \) of the isotropic cone
of \( K_s(p^*_{sw}) \) (because of \( p^*_{sw} < p^*_{div} \), \( K(p^*_{sw}) \) is invertible).

Obviously the issue of generalizing this result to any number \( m \) of constraints
(\( 1 \leq m \leq n - 1 \)) is natural. The paper [8] answers this issue by introducing the
concept of an \( m \)-definite matrix. Indeed, nice algebraic transformations pass from
the divergence instability of the dynamical system of the constrained system \( \Sigma_C \)
$$
\begin{cases}
M\ddot{X} + K(p)X = 0 \\
 CX = 0
\end{cases}
$$
to the invertibility of a matrix \( A^T K(p)A \) with a convenient expression of \( A \) as a
function of \( K(p) \) and \( C \). Properties linked with the increasing loading path leads to put

**Definition 3.2.** Let \( 1 \leq m \leq n - 1 \) be an integer. \( K(p) \) is said \( m \)-positive definite if
$$
\det(A^T KA) > 0 \ \forall A \in G_{nm}
$$
where \( G_{nm} \) is the set of matrices of \( M_{nm}(\mathbb{R}) \) with a maximal rank \( m \).

It may be proved that

**Theorem 3.3.** If \( K(p) \) is positive definite, \( K(p) \) is \( m \)-positive definite for all \( 1 \leq m \leq n - 1 \).

So that a similar conclusion holds for the divergence ki.s.s for a family of \( m \)
constraints than for only one constraint. That allows to answer to the divergence
ki.s.s issue:

**Proposition 2.** The divergence ki.s.s. is universal for conservative systems but
only conditional for nonconservative ones. For the latter, the condition is given by
the s.o.w.c.: \( p^*_{k,div} = p^*_{sw} \)

We illustrate now these results with the following 3 degree of freedom Ziegler
system (figure 2) constrained by two kinematic constraints. That also shows that the
mathematical result is constructive and gives an algorithm to build the convenient
constraints. Calculations give:

\[
K(p) = \begin{pmatrix}
2 - p & -1 & p \\
-1 & 2 - p & -1 + p \\
0 & -1 & 1
\end{pmatrix}
\]

\[
\det(K(p)) = 1,
\]
then the free system is divergence stable for any value of \( p \).
As it is nonconservative, constraints may exist that destabilize it.
the roots of \( \det(K_s(p)) = 0 \) are \( \{1 - \sqrt{3}, 1, 1 + \sqrt{3}\} \) so that \( p_{sw} = 1 \). The
algorithm to build a destabilizing family of constraints is the following:
Figure 2. 3 d.o.f. Ziegler System

\[ K_s(1) = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \quad K_a(1) = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \]

\[ \ker(K_s(1)) = \text{Vec}(X_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{Vec}(K_a(1)(X_1)) = \text{Vec}(\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}) \]

Let be \( X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \text{Vec}(K_a(1)(X_1))^\perp \). Thus \( A = (X_1 \quad X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \)

is such that:

\[ A^T K(1) A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

Finally, \( C = K(1) A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \)

is the wanted matrix of constraints.

Thus, for \( p = 1 \), \( \Sigma \) constrained by \( \theta_1 - \theta_2 = 0 \) and \( \theta_3 = 0 \) is divergence unstable.

4. Geometric degree of nonconservativity. By duality, instead to build a family of constraints that destabilizes (by divergence) a nonconservative system, we may try to build a family of constraints that restores the conservativity. That leads to the present paragraph about the geometric degree of nonconservativity of \( \Sigma \) which sums up the paper [9].

Definition 4.1. The geometric degree or index of nonconservativity of \( \Sigma \) is the smallest integer \( d = \text{card} ~ C \) of kinematic constraints family \( C = (C^1, \ldots, C^d) \) that allows to restore \( \Sigma_C \) as a conservative system.

The issue is to calculate \( d \) and to build \( C = (C^1, \ldots, C^d) \). It should be noted that the divergence k.i.s.s. or the s.o.w. criteria involves the load parameter \( p \) and the symmetric part \( K_s \) of \( K \) but, according to the previous theorem, it does not involve the number of constraints. On the contrary, for the geometric degree of nonconservativity of \( \Sigma \), \( p \) is not involved whereas a geometric object linked to the skew-symmetric part \( K_a \) of \( K \) will determine the degree \( d \):
Theorem 4.2. \( d = \frac{1}{2} \text{rank}(K_a(p)) \), each constraint \( C^i \) must be chosen in each 2-dimensional eigenspace of the symmetric matrix \( K^2_a(p) \) that has \( d \) such 2-dimensional eigenspaces. Except to certain values \( 0 = p_0 < p_1 < p_2, \ldots \) the rank of \( K_a(p) \) remains constant showing that \( d \) does not depend on \( p \) on each interval \([p_i, p_{i+1}]\).

To illustrate this result, we use again the same 3-dimensional Ziegler system as previously:

- one finds
  \[
  K_a(p) = \frac{p}{2} \begin{pmatrix}
  0 & 0 & -1 \\
  0 & 0 & -1 \\
  1 & 1 & 0
  \end{pmatrix},
  K^2_a(p) = -\frac{p^2}{4} \begin{pmatrix}
  1 & 1 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 2
  \end{pmatrix}
  \]
- \( \text{rang}(K_a(p)) = 2 \) except for \( p_0 = 0 \) without interest; thus, \( p_1 = +\infty \) and the geometric degree of nonconservativity is \( d = 1 \) independently of the degree of freedom \( n \)
- \( -\mu^2 = -\frac{p^2}{2} \) is the unique not nil double eigenvalue of \( K^2_a(p) \). The calculation gives:
  \[
  E_{-\frac{p^2}{2}}(K^2_a(p)) = \text{Vec}\{\begin{pmatrix}
  0 \\
  0 \\
  1
  \end{pmatrix}, \begin{pmatrix}
  -1 \\
  -1 \\
  0
  \end{pmatrix}\}
  \]
meaning that \( \theta_3 = 0 \) or \( \theta_1 + \theta_2 = 0 \) are (a basis of) constraints that make \( \Sigma \) conservative.

5. Some open problems. The divergence ki.s.s being solved in the framework of discrete dynamical systems, we come back, in this section, to the first whole issue about the ki.s.s. of a mechanical system and we suggest a short list of open problems and possible generalizations.

- the first remaining issue to be resolved deals with the second kind of linear instability namely the flutter ki.s.s. This is doubtless a more difficult problem because the \( \mathbb{R} \)-diagonalizibility of a matrix is much more difficult to specify that its invertibility. The solution for \( n = 3 \) has been recently submitted and needs tools of differential geometry like Grassmannian manifolds.
- the second challenge would be the generalization of these reasonings and results to continuous mechanical systems. Obviously the infinite dimension create new challenges like topological challenges. However, the first difficulty is to have the good language to tackle the ki.s.s. issue so that this language could be generalized in infinite dimension. This aspect is solved through the concept of compression of an operator of an Hilbert space. This concept is well-known in the topic of the numerical range of operators (see [4] for example) and, in fact, it may be proved that adding kinematic constraints is a concrete realization of the compression of the stiffness operator for divergence.
- a further natural generalization of these ideas might concern the extension to the non linear framework. For example, it will be interesting to show the integrability of the vector field defining the constraints allowing to transform a nonconservative system to a conservative one. If the geometric degree of nonconservativity is \( d = 1 \), it is obvious and the Cauchy Lipschitz theorem gives a positive answer to the (local) existence of a non linear constraint solution of the problem. For \( d \geq 2 \) the issue is much more delicate and needs the calculations of the brackets of the vector fields (Frobenius theorem).
finally, it would be significant to build a similar framework for the evolution of non associated plasticity materials (which are the source of the s.o.w. criterion) and for circulatory elastic systems that both exhibit non symmetric tangent stiffness matrices

6. Conclusion. In this paper, we give an overview of the recent developments about the ki.s.s. issue that arises in studying stability of circulatory and more generally nonconservative systems. The solution for the divergence ki.s.s. is given by use of the s.o.w. criterion and the concept of m-positive definite matrices. A dual problematic dealing with the transformation of a nonconservative system into a conservative one by adding a well-chosen family of kinematic constraints is also solved through the concept of the geometric degree of nonconservativity. All these results have been published during the four last years and some open issues are suggested to conclude this review.

REFERENCES

[1] D. Bigoni and G. Noselli, Experimental evidence of flutter and divergence instabilities induced by dry friction, *Journal of the Mechanics and Physics of Solids*, 59 (2011), 2208–2226.
[2] V. V. Bolotin, *Non-conservative Problems of the Theory of Elastic Stability*, Pergamon Press, 1963.
[3] N. Challamel, F. Nicot, J. Lerbet and F. Darve, Stability of non-conservative elastic structures under additional kinematics constraints, *Engineering Structures*, 32 (2010), 3086–3092.
[4] K. E. Gustafson and D. K. M. Rao, *Numerical Range. The field of Values of Linear Operators and Matrices*, Universitext, Springer, 1997.
[5] R. Hill, A general theory of uniqueness and stability in elastic-plastic solids, *Journal of the Mechanics and Physics of Solids*, 6 (1958), 236–249.
[6] R. Hill, Some basic principles in the mechanics of solids without a natural time, *J. Mech. Phys. Solids*, 7 (1959), 209–225.
[7] O. N. Kirillov and F. Verhulst, Paradoxes of dissipation-induced destabilization or who opened Whitney’s umbrella?, *Z. Angew. Math. Mech.*, 90 (2010), 462–488.
[8] J. Lerbet, M. Aldowaji, N. Challamel, F. Nicot, F. Prunier and F. Darve, P-positive definite matrices and stability of nonconservative systems, *Z. Angew. Math. Mech.*, 92 (2012), 409–422.
[9] J. Lerbet, M. Aldowaji, N. Challamel, F. Nicot, O. Kirillov and F. Darve, Geometric degree of nonconservativity, *Math. and Mech. of Complex Systems*, 2 (2014), 123–139.
[10] T. Tarnai, Paradoxical behaviour of vibrating systems challenging Rayleigh’s theorem, 21st International Congress of Theoretical and Applied Mechanics, Warsaw, 2004.
[11] J. M. T. Thompson, ‘Paradoxical’ mechanics under fluid flow, *Nature*, 296 (1982), 135–137.

Received November 2014; revised November 2015.

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