STRONGLY LIPSCHITZ STRUCTURE OF THE SINGULAR SET OF DISTANCE FUNCTIONS

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ABSTRACT. For the distance function from any closed subset of any smooth complete Finsler manifold, we prove that the singular set is equal to a countable union of Lipschitz hypersurfaces up to an exceptional set of codimension two. The result is new even in the standard Euclidean space.

1. INTRODUCTION

The distance function from a closed subset emerges in a wide range of fields, in particular in geometry and analysis, not only as a fundamental tool but also a research object in itself. For instance, Euclidean distance functions directly appear as natural weak solutions to the eikonal equation. A central problem on distance or related functions is to understand the general structure of the singular set. This singular set is also closely related to a geometrically important object, the cut locus.

Let $m \geq 2$ and $M^m$ be an $m$-dimensional connected complete smooth Finsler manifold throughout this paper (if not specified). For a nonempty closed subset $N \subset M$ we consider the distance function $d_N : M \setminus N \to (0, \infty)$ from $N$:

\[ d_N(p) := d(N, p) = \inf_{q \in N} d(q, p). \]

We define the singular set of $d_N$ by

\[ \Sigma(N) := \{ p \in M \setminus N \mid d_N \text{ is not differentiable at } p \}. \]

The well-known $\mathcal{H}^{m-1}$-rectifiability of $\Sigma(N)$ gives a sharp structural upper bound. However its lower-bound counterpart is known to be more delicate.

Our main result asserts that a generic part of the singular set admits a finer Lipschitz structure, thus improving the classical $\mathcal{H}^{m-1}$-rectifiability.

**Theorem 1.1.** Let $N \subset M^m$ be a nonempty closed subset. Then the singular set $\Sigma(N)$ is equal to the union of a countable family $\{S_j\}_{j=1}^\infty$ of Lipschitz hypersurfaces $S_j \subset M$ up to an $\mathcal{H}^{m-2}$-rectifiable set $R \subset \Sigma(N)$, that is, $\Sigma(N) \setminus R = \bigcup_{j=1}^\infty S_j$.

Here we call $S \subset M^m$ Lipschitz hypersurface if for any $p \in S$ and any local chart $\varphi : U \to \varphi(U) \subset \mathbb{R}^m$, where $U \subset M$ is a neighborhood of $p$, there is a hyperplane $P \subset \mathbb{R}^m$ such that the orthogonal projection of $\varphi(U \cap S)$ to $P$ is a bi-Lipschitz map from a neighborhood of $\varphi(p)$ onto an open subset of $P$. (See also Definitions 2.1 and 4.1.) In the Euclidean case this means that $S$ can locally be represented by a Lipschitz graph. In addition, we say $\Sigma \subset M$ to be (countably) $\mathcal{H}^r$-rectifiable if $\Sigma$ is...
Theorem 1.1 asserts equality. A simple example describing this difference is a Cantor set that lies in an axis of $\mathbb{R}^2$, which can be covered by a segment (so $H^1$-rectifiable) but not contain any positive-length curve. We also point out that, although distance functions are often regarded as a specific subclass of semi-concave functions, cf. [10], Theorem 1.1 fails for a general semi-concave function; see a counterexample in Section 3.2.

Now we compare our main result with the previous literature. In particular, Theorem 1.1 is new even for the standard Euclidean space $M = \mathbb{R}^m$. Roughly speaking, previously known are a “Lipschitz propagation” of singularities for general semi-concave functions on $\mathbb{R}^m$ of any dimension [1], and a “bi-Lipschitz propagation” for the distance function in the two-dimensional case $M^2$ [27]. Compared to those results, our finer assertion may be called “strongly-Lipschitz propagation”.

The analytical study of the singular set of distance functions has a long history, a part of which we now briefly review focusing on the standard Euclidean case $M = \mathbb{R}^m$. Beforehand we recall that even in Euclidean spaces, Finslerian distance functions directly appear as (semi-concave) viscosity solutions $u : \Omega \to \mathbb{R}$ to first-order Hamilton–Jacobi equations of the form

$$\begin{cases} H(x, \nabla u) = 1 & \text{in } \Omega \subset \mathbb{R}^m, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $H \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ satisfies certain structural assumptions including the uniform convexity of $V_p := \{ H(p, \cdot) < 1 \}$. Then the corresponding setup will be $M = (\mathbb{R}^m, F)$ with metric $F(p, v) := \max \{ v \cdot \xi | \xi \in \partial V_p \}$, and $N = \mathbb{R}^m \setminus \Omega$. For more details, see e.g. the classical book of P.-L. Lions [19] and the celebrated study of Li–Nirenberg [18].

The primitive upper bound that $\Sigma(N) \subset \mathbb{R}^m$ has zero Lebesgue measure follows by Rademacher’s theorem, but it was also geometrically shown by Erdős in 1945 [13]. The $H^{m-1}$-rectifiability of $\Sigma(N)$ is also a part of classical theory of BV-functions, cf. [6]. Finer structural upper bounds are by now well developed. In fact, any distance function is semi-concave, and the singular set $\Sigma(u)$ of any semi-concave function $u : \mathbb{R}^m \to \mathbb{R}$ is well understood. More precisely, letting $\Sigma^k(u) := \{ x \in \mathbb{R}^m | \dim(\partial u(x)) \geq k \}$, where $\partial u$ denotes the supergradient (or generalized gradient), Alberti–Ambrosio–Cannarsa [3] proved the remarkable result that the $k$-th singular set $\Sigma^k(u)$ is countably $H^{m-k}$-rectifiable. (Note that $\Sigma(N) = \Sigma^1(d_N)$.) In particular $\dim_H \Sigma^k(u) \leq m - k$, where $\dim_H$ denotes the Hausdorff dimension. See also [4] for $C^2$ rectifiability.

On the other hand, the structural lower bound is more delicate. This problem is nowadays called propagation of singularities in the context of semi-concave functions or solutions to Hamilton–Jacobi equations. Apart from many important contributions to 1D dynamical propagations (see e.g. the surveys [8] [9] and references therein), a few results are available for higher-dimensional propagations. Ambrosio–Cannarsa–Soner [5] addressed this issue for general semi-concave functions and gave conditions for singularities to have lower bounds of the Hausdorff
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dimension. Albano–Cannarsa proceeded in this direction and obtained a remarkable “Lipschitz propagation” result [1, Theorem 5.2]: Under a topological condition on the generalized gradient (cf. (3.1) below), a singular point propagates along the image of a nontrivial Lipschitz map; see also [11, Theorem 4.2] for more on how it propagates. (Some necessity of this condition is also discussed in [4, Theorem 2].) In particular, by applying the results in [3] to the distance function $d_N$, one can assert that such a Lipschitz propagation occurs at a generic singular point (cf. Section 3.1). This result already implies that either $\dim \Sigma(N) = m - 1$ or $\dim \Sigma(N) \leq m - 2$ holds, and also the aforementioned Cantor-like examples are already ruled out. Compared to Albano–Cannarsa’s, restricting the class of semiconcave functions to distance functions, our result gives a finer structural theorem by proving a more informative “strongly-Lipschitz propagation”. More explicitly, the important novelties here are that the propagation is not only bi-Lipschitz (in particular injective) but also graphical, and that countably many hypersurfaces are shown to be sufficient for covering all the generic part.

On the other hand, from a geometrical point of view, the singular set of the distance function is closely related to the classical notion of cut locus. The cut locus from $N$ is the set of all points whether a minimal geodesic from a closed set $N$ loses its minimality. In classical settings (e.g. for $N$ smooth submanifolds) it is well known that the cut locus is decomposed into two parts; the set of points admitting multiple minimal geodesics to $N$, and the remaining (small) set of points admitting unique minimal geodesics (which are necessarily focal points). The former set is known to be dense in the cut locus and in fact characterized by the singular set $\Sigma(N)$, cf. [27].

The cut locus was first introduced by Poincaré [25] in 1905 in the case of a singleton $N = \{p\}$ in a certain two-dimensional Riemannian manifold $M$, where it is shown that the cut locus is a union of arcs except at finitely many “endpoints”. Since then similar or more precise structural results have been established by many authors for the cut locus from a point in various two-dimensional ambient spaces, see e.g. [22, 23, 32, 14, 16, 17]. The case of a general closed set $N \subset M^m$ is also well understood under the two-dimensionality $m = 2$, namely for Alexandrov surfaces [29] or Finsler surfaces [27] (see also [31] dealing with a wider class of functions). In particular, in [27], a very similar result to Theorem 1.1 is obtained in the two-dimensional case; namely, a generic part of the singular set is covered by a countable family of rectifiable Jordan arcs. Theorem 1.1 not only extends this result to a general dimension, but also gives a stronger assertion even if $m = 2$ in view of graphicality. We also note that our approach is different from the one in [27] that crucially relies on two-dimensionality.

Concerning higher-dimensional ambient spaces $M$, one of the most relevant results would be Hebda’s statement [15, Proposition 1.1], which is strongly based on Ozols’ results on hypersurface properties [24, Propositions 2.3, 2.4]: Let $N = \{p\}$ in a complete Riemannian manifold $M^m$. Let $C_2(N)$ be the set of all cut points that are nonconjugate and admit exactly two minimal segment from $N = \{p\}$ (called normal cut points by Hartman [14], and also cleave points by Hebda [15]). Then $C_2(N)$ is relatively open in the cut locus and consists of smooth hypersurfaces, and also the remaining set has zero $H^{m-1}$-measure. In this regard, our Theorem 1.1 extends those results with respect to the generality of $N$ (as well as $M$). The fact that $N$ is not a singleton yields many substantial difficulties. For example, due to
the possible irregularity of \( N \) one has no canonical notion of (non)conjugacy of cut points. Here we simply do not deal with conjugacy and consider the set \( \Sigma_2(N) \) of all points admitting two minimal geodesics from \( N \). Then the set \( \Sigma_2(N) \) may not be relatively open in general, which is a new technically delicate point. We also mention that from an irregular \( N \) the singular set \( \Sigma(N) \) may be quite pathological, e.g. dense in \( M \setminus N \) even if \( N \) is the outside of a convex region with \( C^1 \)-boundary, cf. [28] and also [21].

For the proof of Theorem 1.1 we develop a proper extension of an implicit function method classically used in differential geometry; among many other earlier works, in [14], [24], [15]. More precisely we focus on a generic subset \( \Sigma_2(N) \subset \Sigma(N) \) and prove that a strongly-Lipschitz propagation occurs from any point of \( \Sigma_2(N) \) by applying an implicit function theorem. To deal with nonsmooth objects we make use of Clarke’s implicit function theorem. In addition, the countability of the covering is delicate because the \( H^{m-1} \)-measure of \( \Sigma(N) \) may not be locally finite, and also because the strongly-Lipschitz propagation from a point in \( \Sigma_2(N) \) may not cleanly “cleave” the singular set, i.e., even after propagation there may remain non-negligible residual points in a neighborhood (cf. an example in [20], or our Example 2.5). To overcome this issue we introduce a new quantitative level of hierarchy and prove a quantitative “cleaving” property. The complement \( \Sigma(N) \setminus \Sigma_2(N) \) has codimension at least two by Albeti–Ambrosio–Cannarsa’s upper bound.

Our method also extends to higher-codimensional propagations, albeit under a non-generic condition. We demonstrate this in the Euclidean case (Theorem 2.6), thus strengthening the assertion of [11] Corollary 6.4] for typical points. This result is however not sufficient to cover all (generic) propagations. A full understanding of the higher-codimensional strongly-Lipschitz structure is remained open; see Problem 2.8.

This paper is organized as follows: To clarify the main ideas, in Section 2 we first prove Theorem 2.2 in the Euclidean case \( M = \mathbb{R}^m \) (Theorem 2.2) and also compare the cases of distance functions and semi-concave functions. We then turn to the Finsler case. In Section 4 we recall some basic notions and prepare basic definitions in terms of Finsler geometry, and then in Section 5 we complete the proof of Theorem 1.1 through a more precise statement, Theorem 5.1.

2. Structure of the singular set: Euclidean case

In this section we prove Theorem 1.1 in the Euclidean case \( M = \mathbb{R}^m \) (with the standard metric) for clarifying the essential techniques, and also for convenience of some of the readers unfamiliar with non-Euclidean arguments.

We first give a precise definition of Lipschitz submanifold of any codimension in the Euclidean case. A map \( f : X \to Y \) between metric spaces \( (X,d_X),(Y,d_Y) \) is called bi-Lipschitz if there is \( L > 0 \) such that \( L^{-1}d_X(x,y) \leq d_Y(f(x),f(y)) \leq Ld_X(x,y) \) holds for any \( x,y \in X \). Recall that any bi-Lipschitz map is a homeomorphism onto its image.

**Definition 2.1** (Lipschitz submanifold: Euclidean case). Let \( 1 \leq d \leq m - 1 \). A subset \( S \subset \mathbb{R}^m \) is called (embedded) \( d \)-dimensional Lipschitz submanifold if for any \( p \in S \) there are an open neighborhood \( U \) of \( p \) in \( \mathbb{R}^m \) and a \( d \)-dimensional affine subspace \( P \) of \( \mathbb{R}^m \) such that if \( \pi : S \to P \) denotes the orthogonal projection of \( S \) to \( P \), then the image \( \pi(S \cap U) \) is open in \( P \) and the restriction \( \pi|_{S \cap U} \) defines a bi-Lipschitz map.
Any Lipschitz submanifold $S$ is by definition locally graphical in the sense that up to a change of basis the set $S$ is locally of the form \{(x', f_1(x'), \ldots, f_{m-d}(x')) \in \mathbb{R}^m \mid x' = (x_1, \ldots, x_d) \in U' \subset \mathbb{R}^d\} with Lipschitz functions $f_1, \ldots, f_{m-d}$. In particular, we call a Lipschitz submanifold of codimension one ($d = m-1$) Lipschitz hypersurface. Note that this definition is slightly different from the one given in the introduction since here we need not choose a local chart. However we will check that those definitions are equivalent, see Section 4.1.

2.1. Lipschitz hypersurface structure. Recall that for $p \in \mathbb{R}^m \setminus N$, $d_N(p) := \inf_{q \in N} |q - p|$. The singular set $\Sigma(N) \subset \mathbb{R}^m \setminus N$ denotes the set of all nondifferentiable points of $d_N$. We also recall the following key characterization: Let $\pi_N : \mathbb{R}^m \setminus N \to 2^N$ be the projection map defined by $\pi_N(p) := \{q \in N \mid d_N(p) = |q - p|\}$. A well-known characterization is that $\Sigma(N) = \{p \in \mathbb{R}^m \setminus N \mid \# \pi_N(p) \geq 2\}$, where $\#$ denotes cardinality (see e.g. [10]). It is also well known and easy to prove that the map $\pi_N$ is set-valued upper semicontinuous, i.e., for any $p_j \to p$ in $M \setminus N$ and $\pi(p_j) \ni q_j \to q$ in $N$, we have $q \in \pi_N(p)$. Finally, for an integer $k \geq 2$ we introduce the subset $\Sigma_k(N) \subset \Sigma(N)$ of points that admit exactly $k$ nearest points to $N$:

\[
\Sigma_k(N) := \{p \in \Sigma(N) \mid \# \pi_N(p) = k\}.
\]

The main goal of this section is to prove a more precise form of Theorem 1.1 in the Euclidean case, given in terms of $\Sigma_2(N)$:

**Theorem 2.2** (Lipschitz hypersurface structure: Euclidean case). Let $m \geq 2$ and $N \subset \mathbb{R}^m$ be a nonempty closed subset. Then there exists an at most countable family of Lipschitz hypersurfaces $S_j \subset \mathbb{R}^m$ such that

- (i) $\Sigma_2(N) \subset S \subset \Sigma(N)$, where $S := \bigcup_{j=1}^\infty S_j$,
- (ii) $\Sigma(N) \setminus S$ is $\mathcal{H}^{m-2}$-rectifiable.

A key tool for Theorem 2.2 is Clarke’s implicit function theorem for Lipschitz maps. To state it we first recall some notions in nonsmooth analysis in the class of locally Lipschitz functions, including all distance functions. For a map $f : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz around a point $x$, let $D^*f(x)$ denote the reachable gradient of $f$ at $x$, which is defined by the set of all vectors that are limits of (classical) gradients $Df$:

\[
D^*f(x) := \left\{ \lim_{j \to \infty} Df(x_j) \right\},
\]

Note that the set $D^*f(x)$ is nonempty by Rademacher’s theorem, bounded since $|Df| \leq L$ holds for the (local) Lipschitz constant $L$, and also closed by its definition using limits. In addition, let $\partial f(x)$ denote the generalized gradient defined by the convex hull of the reachable gradient:

\[
\partial f(x) := \text{conv} \, D^*f(x).
\]

The set $\partial f(x)$ is also compact, cf. [26] Theorem 17.2.]
The following is (a rearranged form of) Clarke’s implicit function theorem [12]
Corollary in p.256:  

**Theorem 2.3** (Clarke’s implicit function theorem). Let $V \subset \mathbb{R}^m$ be an open subset
and $f_1, \ldots, f_k : V \to \mathbb{R}$ be Lipschitz functions, where $1 \leq k \leq m - 1$, and let
$x_0 \in V$. If $f_1(x_0) = \cdots = f_k(x_0) = 0$ and if any choice of elements $(v_1, \ldots, v_k) \in \partial f_1(x_0) \times \cdots \times \partial f_k(x_0)$ is linearly independent, then there exists a neighborhood
$U \subset V$ of $x_0$ in $\mathbb{R}^m$ such that $S := \bigcap_{j=1}^k f_j^{-1}(0) \cap U$ is an $(m-k)$-dimensional
Lipschitz submanifold of $\mathbb{R}^m$.

For Theorem 2.3 we only use the special case of $k = 1$; more precisely, for a
Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$, if $f(x_0) = 0$ and $0 \notin \partial f(x_0)$, then there exist a
neighborhood $U$ of $x_0$ in $\mathbb{R}^m$ such that $S := f^{-1}(0) \cap U$ is a Lipschitz hypersurface.

We also introduce a radius function to quantify singular points. For each $p \in \Sigma(N)$
we define the radius function by

$$
\text{rad}_N(p) := \max_{q_1, q_2 \in \pi_N(p)} |q_1 - q_2|.
$$

In the Euclidean case, $\text{rad}_N(p) > 0$ holds for any $p \in \Sigma(N)$ (although this is not
the case if $M$ is general). Notice that if $p \in \Sigma_2(N)$, then $\text{rad}_N(p)$ denotes exactly
the distance of two points in $\pi_N(p)$.

Now we prove the key fact that from any point in $\Sigma_2(N)$ a Lipschitz hypersurface
propagates along $\Sigma(N)$, with the quantitative cleaving property that the residual
part has relatively small radii.

**Lemma 2.4.** For any $p \in \Sigma_2(N)$, there exist a positive number $\delta(p) > 0$ and a
Lipschitz hypersurface $S_p \subset \mathbb{R}^m$ such that $S_p \subset \Sigma(N)$ and such that $\text{rad}_N(y) \leq \frac{1}{2} \text{rad}_N(p)$ holds for any
$y \in (\Sigma(N) \cap B_{\delta(p)}(p)) \setminus S_p$, where $B_r(p) := \{ x \in \mathbb{R}^m | |p - x| < r \}$.

**Proof.** Let $\pi_N(p) = \{ q_1, q_2 \}$ and $N_j := N \cap \{ y \in \mathbb{R}^m \mid |q_j - y| \leq \frac{1}{4} \text{rad}_N(p) \}$, where
$j = 1, 2$. The sets $N_1$ and $N_2$ are compact. In addition, by $\text{rad}_N(p) = |q_1 - q_2| > 0$, we have

$$
N_1 \cap N_2 = \emptyset,
$$

and also $N_1$ and $N_2$ are (relative) neighborhoods of $q_1$ and $q_2$ in $N$, respectively.

Let $f := d_{N_1} - d_{N_2} = d(N_1, \cdot) - d(N_2, \cdot)$. Since $\pi_N$ is set-valued upper semi-
continuous, there is $r > 0$ such that for any $x \in B_r(p)$ we have $\pi_N(x) \subset N_1 \cup N_2$, and
in particular $d_N(x) = \min\{d_{N_1}(x), d_{N_2}(x)\}$. Hence, if $x \in B_r(p)$ and $f(x) = 0$, then
d_{N_1}(x) = d_{N_2}(x) = d_{N}(x)$ and hence, by (2.4), $\# \pi_N(x) \geq 2$. Therefore, for any
$\delta \in (0, r]$, 

$$
S_p, \delta := f^{-1}(0) \cap B_\delta(p) \subset \Sigma(N).
$$

In addition, if $x \in (\Sigma(N) \cap B_\delta(p)) \setminus S_p, \delta$, then either $\pi_N(x) \subset N_1$ or $\pi_N(x) \subset N_2$
holds (depending on the sign of $f(x)$) and hence $\text{rad}_N(x) \leq \frac{1}{2} \text{rad}_N(p)$ holds by
definition of $N_j$ and the triangle inequality.

We finally prove that for a suitable positive number $\delta := \delta(p) \in (0, r]$ the set
$S_p := S_p, \delta(p)$ is a Lipschitz hypersurface. Since $p$ admits a unique $N_j$-segment
from $q_j$ for $j = 1, 2$, letting $v_j := \frac{x - q_j}{|x - q_j|}$, we have $\partial d_{N_j}(p) = \{ v_j \}$ and hence
$\partial f(p) = \{ v_1 - v_2 \}$. Since $v_1 \neq v_2$ and in particular $0 \notin \partial f(p)$, and since $f(p) = 0$,
Theorem 2.3 with $k = 1$ implies that there is a small positive number $\delta(p) > 0$ such
that $S_p := f^{-1}(0) \cap B_{\delta(p)}(p)$ is a Lipschitz hypersurface. \qed
Lemma 2.4 claims that from any point $p$ of $\Sigma_2(N)$ a Lipschitz hypersurface $S_p$ propagates along $\Sigma(N)$. It looks simple but is somewhat delicate: It says neither $S_p \subset \Sigma_2(N)$ nor that $S_p \cap B_\delta(p) \supset \Sigma_2(N)$. The following example is helpful for understanding these facts, cf. Figure 1.

Example 2.5 (Branched singular set). Define a closed set $N \subset \mathbb{R}^2$ by

$$N := \{(\pm 1, \frac{1}{k}) \in \mathbb{R}^2 \mid k \in \mathbb{Z}_{>0}\} \cup \{(\pm 1,0)\}.$$ 

Then, letting $a_k := \frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})$ and

$$X := \{(x,a_k) \in \mathbb{R}^2 \mid x \in \mathbb{R}, \ k \in \mathbb{Z}_{>0}\}, \quad Y := \{(0,y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\},$$

we can explicitly represent the singular sets $\Sigma(N)$ and $\Sigma_2(N)$:

$$\Sigma(N) = X \cup Y, \quad \Sigma(N) \setminus \Sigma_2(N) = X \cap Y = \{(0,a_k) \in \mathbb{R}^2 \mid k \in \mathbb{Z}_{>0}\}.$$ 

Note that $\text{rad}_N(p) \leq \frac{1}{2}$ for $X \setminus Y$, while $\text{rad}_N(p) \geq 2$ for $p \in Y$. Now we look at the origin $p = (0,0)$. Notice that $p \in \Sigma_2(N)$, and hence a Lipschitz graph $S_p$ propagates along $\Sigma(N)$ by Lemma 2.4. In addition, since $\text{rad}_N(p) = 2$, by the radius estimate in Lemma 2.4 we deduce that $Y \cap B_\delta(p) \subset S_p \cap B_\delta(p)$; by the bijectivity for $S_p$ we thus find that $Y \cap B_\delta(p) = S_p \cap B_\delta(p)$, i.e., the curve $S_p$ must be a vertical segment along $Y$ in $B_\delta(p)$. Therefore, this curve $S_p$ passes through infinitely many points of $\Sigma(N) \setminus \Sigma_2(N) = X \cap Y$ so that $S_p \not\subset \Sigma_2(N)$, and also there remain infinitely many branches $B_\delta(p) \setminus S_p = (X \setminus Y) \cap B_\delta(p) \subset \Sigma_2(N)$ so that $S_p \cap B_\delta(p) \not\subset \Sigma_2(N)$.

\[\text{Figure 1. Branched singular set.}\]

With Lemma 2.4 at hand, we are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. We first prove assertion (i), that is, there is an at most countable union $S$ of Lipschitz hypersurfaces such that $\Sigma_2(N) \subset S \subset \Sigma(N)$, and then (independently) verify that $\Sigma(N) \setminus \Sigma_2(N)$ is countably $H^{m-2}$-rectifiable. These two claims immediately imply assertion (ii) since $\Sigma(N) \setminus S \subset \Sigma(N) \setminus \Sigma_2(N)$.

Step 1: Countable covering by Lipschitz hypersurfaces. We may suppose that $\Sigma_2(N) \neq \emptyset$ without loss of generality since otherwise Step 2 directly completes the proof. Let $K \subset \mathbb{R}^m$ be any compact subset such that $\Sigma_2(N) \cap K \neq \emptyset$. We define

$$\text{rad}_N(K) := \max\{\text{rad}_N(p) \mid p \in \Sigma(N) \cap K\}.$$
For each positive integer $i$,

\begin{equation}
A_i := \left\{ p \in \Sigma_2(N) \cap K \mid \frac{1}{1+i} \text{rad}_N(K) < \text{rad}_N(p) \leq \frac{1}{i} \text{rad}_N(K) \right\}.
\end{equation}

Finally, for each positive integers $i,j$, we define

\[ A'_{ij} := \{ p \in A_i \mid \delta(p) \geq 1/j \}, \]

where $\delta(p)$ is a possible choice in Lemma 2.4.

We prove that for each $i,j$ the set $A'_{ij}$ is covered by an at most finite union of Lipschitz hypersurfaces contained in $\Sigma(N)$. We may assume that $A'_{ij}$ is nonempty since otherwise obvious. Take an arbitrary point $p_1 \in A'_{ij}$, and also a number $\delta(p_1) > 0$ and a Lipschitz hypersurface $S_{p_1} \subset \Sigma(N)$ as in Lemma 2.4. Then, by definition of $A_i$ and by the radius estimate in Lemma 2.4, $(\Sigma(N) \cap B_{\delta(p_1)}(p_1)) \setminus S_{p_1}$ and $A_i$ are disjoint. Therefore,

\[ A'_{ij} \setminus S_{p_1} \subset A_i \setminus S_{p_1} \subset K \setminus B_{\delta(p_1)}(p_1). \]

If $A'_{ij} \setminus S_{p_1}$ is empty, then the proof is complete, while if nonempty, then we can choose a point $p_2 \in A'_{ij} \setminus S_{p_1}$ and a parallel procedure implies that

\[ A'_{ij} \setminus \bigcup_{k=1}^{j} S_{p_k} \subset K \setminus \bigcup_{k=1}^{j} B_{\delta(p_k)}(p_k). \]

If the left-hand side is nonempty, then we choose then $p_3$ and continue this procedure. This inductive procedure terminates in finite steps; indeed, at step $\ell$ we need to choose a point $p_\ell$ included in the set $K \setminus \bigcup_{k=1}^{\ell-1} B_{\delta(p_k)}(p_k)$, which becomes empty in finite steps since $K$ is compact, and since $\delta(p_k) \geq 1/j$ holds for all $k$ by definition of $A'_{ij}$ and also $p_k \in K \setminus \bigcup_{k=1}^{1/j} B_{\delta(p_k)}(p_k) \subset K \setminus \bigcup_{k=1}^{1} B_{1/j}(p_k)$. Therefore, $A'_{ij}$ has the desired finite-covering.

Then, since $A_i = \bigcup_{j=1}^{\infty} A'_{ij}$, the set $A_i$ is covered by an at most countable union of Lipschitz hypersurfaces contained in $\Sigma(N)$. In addition, since $\text{rad}_N(p) > 0$ holds for any $p \in \Sigma(N)$, it is clear that $\Sigma_2(N) \cap K = \bigcup_{i=1}^{\infty} A_i$. Therefore, $\Sigma_2(N) \cap K$ also has the same type of countable-covering. Finally, since $K$ is arbitrary, by taking a countable sequence of closed balls $B_k$ of radius $k$ we deduce that $\Sigma_2(N) = \bigcup_{k=1}^{\infty} \Sigma_2(N) \cap B_k$ also has the desired countable-covering $S$.

Step 2: Codimension-two rectifiability of the residual part. For any $p \in \Sigma(N) \setminus \Sigma_2(N)$ there are at least three distinct points $q_1, q_2, q_3 \in \pi_N(p)$. For each $j = 1, 2, 3$ and any $t \in (0, 1)$ we have $q_j := q_j + t(p - q_j) \notin N \cup \Sigma(N)$, and hence $d_N$ is differentiable at $q_j$ and $\nabla d_N(q_j) = \frac{p - q_j}{|p - q_j|} := v_j$. Letting $t \to 1$ we get $v_1, v_2, v_3 \in \partial d_N(p)$. Since $v_1, v_2, v_3$ are three distinct unit vectors so that $v_3 - v_1$ and $v_2 - v_1$ are linearly independent, the generalized gradient $\partial d_N(p)$ has dimension at least 2. This means that $\Sigma(N) \setminus \Sigma_2(N) \subset \Sigma^2(d_N)$, where we recall that $\Sigma^k(d_N) := \{ p \in \mathbb{R}^m \setminus N \mid \dim(\partial d_N(p)) \geq k \}$. By Lemma 2.4 (1) the set $\Sigma^2(d_N)$ is countably $\mathcal{H}^{m-2}$-rectifiable, and hence so is $\Sigma(N) \setminus \Sigma_2(N)$. 

2.2. Higher codimensional strongly-Lipschitz propagation. In a very parallel way to Lemma 2.4 we can also deduce a (conditional) higher-codimensional strongly-Lipschitz propagation result.

**Theorem 2.6** (Higher codimensional propagation). Let $p \in \Sigma_{k+1}(N)$ with $1 \leq k \leq m$. Let $\pi_N(p) = \{ q_0, \ldots, q_k \}$ and $v_j := \frac{q_j - p}{|q_j - p|}, 0 \leq j \leq k$. Suppose that the unit vectors $v_0, \ldots, v_k$ form a $k$-simplex (i.e., affinely independent). Then there is
an \((m-k)\)-dimensional Lipschitz submanifold \(S\) of \(\mathbb{R}^n\) such that \(p \in S \subset \Sigma(N) \setminus \bigcup_{j=2}^{k} \Sigma_j(N)\).

Proof. Let \(N_i := N \cap \{ y \in \mathbb{R}^m \mid |q_j - y| \leq \frac{1}{2} \min_{i \neq j} |q_i - q_j| \}\). The sets \(N_0, \ldots, N_k\) are compact and, by definition,

\[
N_i \cap N_j = \emptyset \quad \text{for all } 0 \leq i < j \leq k.
\]

Let \(f_j := d_{N_0} - d_{N_j} = d(N_0, \cdot) - d(N_j, \cdot)\) for \(1 \leq j \leq k\). Since \(\pi_N\) is set-valued upper semicontinuous, there is \(r > 0\) such that for any \(x \in B_r(p)\) we have \(\pi_N(x) \subset \bigcup_{j=0}^{k} N_j\). In particular, if \(x \in B_r(p)\), then \(d_N(x) = \min_{0 \leq j \leq k} d_{N_j}(x)\), and hence if in addition \(f_1(x) = \cdots = f_k(x) = 0\), then by (2.7) we have \(d_N(x) = d_{N_0}(x) = \cdots = d_{N_k}(x)\) and \(\#\pi_N(x) \geq k + 1\), that is, \(x \in \Sigma(N) \setminus \bigcup_{j=2}^{k} \Sigma_j(N)\). Hence, for any \(\delta \in (0, r]\),

\[
S_{p, \delta} := \bigcap_{j=1}^{k} f_j^{-1}(0) \cap B_\delta(p) \subset \Sigma(N) \setminus \bigcup_{j=2}^{k} \Sigma_j(N).
\]

Since \(\pi_{N_j}(p) = \{q_j\}\) for each \(j = 0, \ldots, k\), letting \(v_j := \frac{p - q_j}{|p - q_j|}\), we have \(\partial d_{N_j}(p) = \{v_j\}\) and hence \(\partial f_j(p) = \{v_0 - v_j\}\). By the \(k\)-simplicity assumption, the unique element in \(\partial f_1(p) \times \cdots \times \partial f_k(p)\) is linearly independent. Combining this fact with \(f_1(p) = \cdots = f_k(p) = 0\), we can apply Theorem 2.3 to deduce that there is \(\delta(p) > 0\) such that \(S_{p, \delta(p)}\) is an \((m-k)\)-dimensional Lipschitz submanifold.

This result strengthens the assertion of \([1]\) Corollary 6.4] by imposing the stronger assumption of \(k\)-simplicity.

The assumption of \(k\)-simplicity may be removed for \(k < 2\) as it is automatically satisfied; indeed, the case \(k = 1\) is nothing but Lemma 2.3 while for \(k = 2\), thanks to the unit-length property \(|v_0| = |v_1| = |v_2| = 1\) (and \(v_0 \neq v_1 \neq v_2 \neq v_0\) these points cannot be collinear. In particular, from any point in \(\Sigma_3(N)\) a codimension-two Lipschitz submanifold propagates within \(\Sigma(N) \setminus \Sigma_2(N)\). The codimension two is optimal because \(\Sigma(N) \setminus \Sigma_2(N)\) is \(H^{m-2}\)-rectifiable.

On the other hand, Theorem 2.6 is not sufficient for covering all parts of codimension two or higher. For example, there is a simple example of \(N\) such that any point in \(\Sigma(N) \setminus \Sigma_2(N)\) does not satisfy the assumption of Theorem 2.6.

Example 2.7 (Generic non-simplicial singularities). Let \(m = 3\). Let \(k \geq 3\), and \(P \subset \mathbb{R}^2\) be the domain bounded by a regular \((k+1)\)-gon centered at the origin. Consider the closed set \(N := (\mathbb{R}^2 \setminus P) \times \mathbb{R} \subset \mathbb{R}^3\). Then by a simple observation we infer that \(\Sigma(N) \setminus \Sigma_2(N) = \Sigma_{k+1}(N) = \{(0, 0)\} \times \mathbb{R} \subset \mathbb{R}^3\), being a codimension-two submanifold. In this case all points in \(\Sigma(N) \setminus \Sigma_2(N) = \Sigma_{k+1}(N)\) are not \(k\)-simplicial since the vectors \(v_0, \ldots, v_k\) are contained in a plane. (If we replace \(P\) with a disk, then we also notice that the set \(\Sigma_\infty(N) := \{\#\pi_N(p) = \infty\}\) may not be negligible.)

Theorem 2.6 asserts that from a special singular point a higher-codimensional strongly-Lipschitz propagation occurs. However it is not clear if this propagation is generic as in the case of codimension one. In particular, the following problem is remained open:

Problem 2.8. Let \(m \geq 3\) and \(N \subset \mathbb{R}^m\) be a nonempty closed subset. Let \(k\) be an integer such that \(2 \leq k \leq m - 1\). Are there a countable family \(\{S_j\}_{j=1}^{\infty}\) of Lipschitz submanifolds \(S_j \subset \mathbb{R}^m\) up to codimension \(k\) and an \(H^{m-k-1}\)-rectifiable set \(R \subset \Sigma(N)\) such that \(\Sigma(N) \setminus R = \bigcup_{j=1}^{\infty} S_j\)?
3. Semi-concave functions

3.1. Comparison with theory of semi-concave functions. We compare our
Theorem 2.2 for distance functions with general theory of semi-concave functions.

We first quickly review semi-concave functions; for details, see e.g. Cannarsa–
Sinesistri’s book [11]. A continuous function $u : \Omega \to \mathbb{R}$ defined on an open set
$\Omega \subset \mathbb{R}^m$ is locally semi-concave if for any open convex set $\Omega'$ compactly embedded
in $\Omega$ there is $C \in \mathbb{R}$ such that one of the following equivalent conditions hold:

(1) For any $\lambda \in [0, 1]$ and any $x_0, x_1 \in \Omega'$,
$$
\lambda u(x_1) + (1 - \lambda)u(x_0) - u(\lambda x_1 + (1 - \lambda)x_0) \leq C \lambda(1 - \lambda)|x_1 - x_0|^2.
$$

(2) $u(x) - C|x|^2$ is a concave function on $\Omega'$.

(3) $D^2 u \leq 2CI$ on $\Omega'$ in the distributional sense, where $I$ denotes the $m \times m$
identity matrix.

It is well known that any distance function is locally semi-concave. By the second
definition we find that the study of singularities of semi-concave functions is locally
reduced to that of concave functions. Also, any locally semi-convex function $u$ is
locally Lipschitz continuous so that we can define the reachable gradient $D^* u$ as
above. The generalized gradient $\partial u(x) := \text{conv} D^* u(x)$ coincides with the supergra-
dient from convex analysis. As is mentioned in the introduction, general structural
results are known for singular sets of semi-concave functions.

In particular, we precisely recall a closely related result of Albano–Cannarsa [1].
Recall that $\Sigma(u)$ denotes the singular set (where $u$ is not differentiable). Let $\text{bd}[A]$ denote
the topological boundary of a set $A$ in $\mathbb{R}^m$. Let $K_C(x)$ denote the normal
cone of a convex set $C \subset \mathbb{R}^m$ at $x \in C$ defined by $K_C(x) := \{q \in \mathbb{R}^m \mid \langle q, y - x \rangle \leq 0, \forall y \in C\}$. Albano–Cannarsa’s result [1] Theorem 5.1 states that if a locally
semi-concave function $u : \Omega \to \mathbb{R}$ has a point $x_0 \in \Omega$ such that

$$
(3.1) \quad \text{bd}[\partial u(x_0)] \setminus D^* u(x_0) \neq \emptyset,
$$

then for any $p_0 \in \text{bd}[\partial u(x_0)] \setminus D^* u(x_0)$ there exist an open ball $B_{p_0}$ centered at the
origin and a Lipschitz function $F : K_{\partial u(x_0)}(p_0) \cap B_{p_0} \to \Sigma(u)$ of the form

$$
F(q) = x_0 - q + o(|q|) \quad (|q| \to 0).
$$

This “Lipschitz propagation” already implies fine lower bounds such as

$$
\inf \{\text{diam}(F(q)) \mid q \in K_{\partial u(x_0)}(p_0) \cap B_{p_0} \} > 0,
$$

and, for $\nu := \dim K_{\partial u(x_0)}(p_0)$,

$$
\lim_{r \to +0} r^{-\nu} \text{diam}^\nu (F(K_{\partial u(x_0)}(p_0) \cap B_{p_0}) \cap B_r(x_0)) > 0 \quad \Rightarrow \dim_H \Sigma(u) \geq \nu.
$$

Condition (3.1) is somewhat delicate, but by focusing on the distance function $d_N$
one obtains some general consequences. For example, (3.1) holds for $u = d_N$ for any
$x_0 \in \Sigma_2(N)$ since in this case the reachable gradient $D^* d_N(x_0)$ consists of exactly
two points, and hence the set $\partial d_N(x_0) = \text{bd}[\partial d_N(x_0)]$ is a segment. Therefore,
for any point in $\Sigma_2(N)$ the above Lipschitz propagation occurs with dimension
$\nu = m - 1$ (see also [1] Section 6).

It is now clarified that our Lemma 2.4 is quite similar in spirit to Albano–
Cannarsa’s Lipschitz propagation result (with dimension $\nu = m - 1$) and this is
why we call it “strongly-Lipschitz propagation” in the introduction.

Finally we point out two important remarks:
• There are (semi-)concave functions $u : \mathbb{R}^m \to \mathbb{R}$ with singular sets of fractional Hausdorff dimension between $m - 2$ and $m - 1$ (see Example 3.2 below).

• There are (semi-)concave functions $u : \mathbb{R}^m \to \mathbb{R}$ with a singular point at which a Lipschitz propagation occurs but any strongly-Lipschitz propagation does not (see Example 3.4 below).

The first remark ensures that Theorem 1.1 fails for a general (semi-)concave function, since if the same assertion would hold, then we would necessarily have either $\dim_H \Sigma(u) = m - 1$ or $\dim_H \Sigma(u) \leq m - 2$. The second certifies a significant difference between Lipschitz and strongly-Lipschitz propagations at least for general (semi-)concave functions.

3.2. Counterexample: Fractional singular set. In this subsection we construct a concave, or equivalently convex function $u : \mathbb{R}^m \to \mathbb{R}$ whose singular set $\Sigma(u)$ has Hausdorff dimension $\dim_H \Sigma(u) = m - 1 - s$, for any given $s \in (0, 1)$.

To this end it is sufficient to consider the case of $m = 2$. Indeed, once we obtain such a function $u : \mathbb{R}^2 \to \mathbb{R}$ with $\dim_H \Sigma(u) = 1 - s$, then the function $\tilde{u}(x^1, x^2, \ldots, x^m) := u(x^1, x^2)$ defined on $\mathbb{R}^m$ is also convex and has the property that $\Sigma(\tilde{u}) = \Sigma(u) \times \mathbb{R}^{m-2}$ and hence $\dim_H \Sigma(\tilde{u}) = (1 - s) + (m - 2) = m - 1 - s$.

Before constructing a concrete example we state a general result on the attainability of the singular set of a convex function.

Proposition 3.1. For any sequence $\{I_j\}_{j \geq 1}$ of mutually positively-separated open subintervals of $\mathbb{R}$ such that $I_j \subset [0, 1]$, there exists a convex function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $\Sigma(u) = C \times \{0\}$, where $C := [0, 1] \setminus \bigcup_{j=1}^{\infty} I_j$.

Note that the above class of sets $C$ covers not only any Cantor set $C_\sigma$ but also the Smith–Volterra–Cantor set (also known as the fat Cantor set), which is totally disconnected but has positive $H^1$-measure.

Proof of Proposition 3.1. Given a sequence $\{I_j\}_{j \geq 1}$, we define a sequence $\{v_j\}_{j \geq 1}$ of functions in the following way. Let $\nu_0(x, y) := \frac{1}{2} x^2 + |y|$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth even convex function such that $\phi(y) = |y|$ for $|y| \geq 1$; then automatically $\phi > 0$ and $\phi(0) \in (0, 1)$. In addition, writing $I_j = (a_j, b_j)$, we define $r_j : I_j \to \mathbb{R}$ by $r_j(x) := (x - a_j)^2(x - b_j)^2$. We then inductively define

$$v_j(x, y) := v_{j-1}(x, y) + \chi_{I_j}(x) \left(r_j(x)\phi\left(\frac{y}{r_j(x)}\right) - |y|\right),$$

where $\chi_A$ denotes the characteristic function on $A$. Notice that $v_j$ is continuous and $v_j \geq v_{j-1}$ holds on $\mathbb{R}^2$ for all $j$.

We prove that each member $v_j$ is convex by induction. For $j = 0$ it is trivial. Suppose that $v_{j-1}$ is convex and prove that $v_j$ is convex. Note first that $v_j = v_{j-1}$ holds on $\mathbb{R}^2 \setminus D_j$, where $D_j := \{(x, y) \mid x \in I_j, |y| < r_j(x)\}$, and hence $v_j$ is locally convex on $\mathbb{R}^2 \setminus D_j$. Now it suffices to prove the local convexity of $v_j$ on $E_j := D_j \setminus U_j$ with $U_j := I_j \times \mathbb{R}$; indeed, then $v_j \in C(\mathbb{R}^2)$ is locally convex except at the two points $\partial I_j \times \{0\}$ and hence $v_j$ is entirely convex (by approximation of segments). For the local convexity on $E_j$ it suffices to show that the Hessian is positive semidefinite on $U_j$, or equivalently that $\text{tr}(D^2 v_j) \geq 0$ and $\det(D^2 v_j) \geq 0$ on $U_j$. Now the problem is reduced to showing that all $\partial_{xx} v_j, \partial_{yy} v_j$, and $\partial_{xx} v_j \partial_{yy} v_j - (\partial_{xy} v_j)^2$ are nonnegative on $U_j$. By definition of $v_0, \ldots, v_j$ and by the fact that $I_1, \ldots, I_j$ are...
mutually disjoint, the restriction of $v_j$ to $U_j$ is represented by

$$v_j|_{U_j}(x, y) = \frac{1}{2}x^2 + r_j(x)\phi\left(\frac{y}{r_j(x)}\right).$$

This implies that $\partial_{yy}v_j|_{U_j} = \frac{1}{2}r_j''(x) \phi''\left(\frac{y}{r_j(x)}\right) \geq 0$ since $\phi'' \geq 0$ and $r_j > 0$. In addition,

$$\partial_{xx}v_j|_{U_j} = 1 + \frac{y^2 r_j''(x)}{r_j'(x)} \phi''\left(\frac{y}{r_j(x)}\right) + r_j''(x)\left[\phi\left(\frac{y}{r_j(x)}\right) - \frac{y}{r_j(x)}\phi'\left(\frac{y}{r_j(x)}\right)\right].$$

The second term is nonnegative, while the third terms is bounded below by $-1$ since $r_j'' \geq -(b_j - a_j)^2 \geq -1$ and $0 \leq \phi(z) - z\phi'(z) \leq 1$ for any $z \in \mathbb{R}$; the last estimate follows since $(\phi(z) - z\phi'(z))' = -z\phi''(z)$ and hence the maximum $\phi(0) \in (0, 1)$ is taken at $z = 0$ and the minimum 0 on $|z| \geq 1$. Therefore, $\partial_{xx}v_j|_{U_j} \geq 0$.

Finally, we define

$$v_\infty := \sup_{j \geq 0} v_j \ (\leq v_0 + 1).$$

Then $v_\infty$ is convex since it is the supremum of a sequence of convex functions. Let $J := \bigcup_{j=1}^{\infty} I_j$. We confirm that

$$\Sigma(v_\infty) = (\mathbb{R} \setminus J) \times \{0\}.$$

We first note that this $v_\infty$ is not differentiable on $(\mathbb{R} \setminus J) \times \{0\}$ since if $x_0 \not\in J$, then $v_\infty(x_0, y) = v_0(x_0, y) = \frac{1}{2}x_0^2 + |y|$ for any $y$ and hence not differentiable at $y = 0$.

In what follows we argue that $v_\infty$ is differentiable outside $(\mathbb{R} \setminus J) \times \{0\}$. Since the completion of $(\mathbb{R} \setminus J) \times \{0\}$ is the union of $J \times \mathbb{R}$ and $(\mathbb{R} \setminus J) \times (\mathbb{R} \setminus \{0\})$, it is sufficient to prove differentiability on both the sets. Concerning the former set, for any $j$ the function $v_\infty$ is differentiable on the open set $I_j \times \mathbb{R}$ since we have $v_\infty(x, y) = \frac{1}{2}x^2 + r_j(x)\phi\left(\frac{y}{r_j(x)}\right)$, and hence taking the union with respect to $j$ implies that $v_\infty$ is differentiable on $J \times \mathbb{R}$. Concerning the latter, for any point $p_0 = (x_0, y_0) \in (\mathbb{R} \setminus J) \times (\mathbb{R} \setminus \{0\})$, there is $\varepsilon \in (0, |y_0|)$ such that $B_\varepsilon(p_0) \cap \bigcup_{j=1}^{\infty} D_j = \emptyset$; this follows by the fact that the set $\bigcup_{j=1}^{\infty} D_j$ is of the form $\{|y| < f(x)\}$ for a nonnegative continuous function $f : \mathbb{R} \rightarrow [0, \infty)$ such that $\{f = 0\} = \mathbb{R} \setminus J$. Then $v_\infty = v_0 = \frac{1}{2}x^2 \pm y$ holds on $B_\varepsilon(p_0)$, where $\pm$ depends on the sign of $y_0$, so that $v_\infty$ is also differentiable at $p_0$. This implies the desired differentiability of $v_\infty$.

Finally, we define a convex function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\Sigma(u) = C \times \{0\}$ with $C := [0, 1] \setminus J$, by using $v_\infty$. This is easily done by letting $\rho_0(x) := x^2$ and $\rho_1(x) := (x - 1)^2$, and then defining $u$ by

$$u := v_\infty + \chi_{(-\infty, 0)}(x)\left(\rho_0(x)\phi\left(\frac{y}{\rho_0(x)}\right) - |y|\right) + \chi_{(1, \infty)}(x)\left(\rho_1(x)\phi\left(\frac{y}{\rho_1(x)}\right) - |y|\right).$$

The convexity of $u$ and the fact that

$$\Sigma(u) = C \times \{0\}$$

can be confirmed by parallel (or easier) arguments to the above. \qed
Example 3.2 (Cantor-like singular set). For \( \sigma \in (0,1) \) we define the (standard) generalized Cantor set \( C_{\sigma} \subset [0,1] \) by iteratively deleting (at step \( j \)) the middle open interval of length \( \sigma L_{j-1} \) from each of the remaining \( 2^{j-1} \) segments of length \( L_{j-1} = 2^{-j-1}(1-\sigma)^{j-1} \). It is well known (e.g. due to self-similarity) that \( \dim_H C_\sigma = (\log 2)(\log \frac{2}{1-\sigma})^{-1} \). Hence, for any given \( s \in (0,1) \), by choosing \( \sigma := 1 - 2^{-\frac{1}{s-1}} \in (0,1) \), we can apply Proposition 3.1 to deduce that there is a convex function \( u : \mathbb{R}^2 \to \mathbb{R} \) such that \( \dim_H \Sigma(u) = \dim_H (C_\sigma \times \{0\}) = \dim_H C_\sigma = 1 - s \).

We finally discuss the reachable gradient \( D^* u \) of the convex function \( u \) constructed in the proof of Proposition 3.1. After some computations, we deduce that \( D^* u(p) = \{(p_1, y) \in \mathbb{R}^2 \mid \|y\| \leq 1\} \) holds for any \( p = (p_1, p_2) \in \text{bd}[C] \times \{0\} \), where \( \text{bd}[C] \) denotes the topological boundary of the closed set \( C \) in \( \mathbb{R} \). In particular, if \( C \) has empty interior (like a Cantor set), then \( D^* u(p) = \partial u(p) \) for any \( p \in \Sigma(u) \). This implies the fact that no point in \( \Sigma(u) \) satisfies Almog–Cannarsa’s condition (3.1), which is consistent with the fact that no propagation occurs in a Cantor-like singular set.

3.3. Counterexample: Zigzag singular set. Here, again focusing on the planar case for the same reason as above, we construct a convex function \( u : \mathbb{R}^2 \to \mathbb{R} \) that admits a point \( p \in \Sigma(u) \) with the following properties: There is a bi-Lipschitz map \( f : [-\varepsilon, \varepsilon] \to \Sigma(u) \) with \( |f'| \equiv 1 \) a.e., but there is no Lipschitz graph (hypersurface) \( S \subset \mathbb{R}^2 \) such that \( p \in S \subset \mathbb{R}^2 \). The first condition roughly says that \( \Sigma(u) \) itself is a rectifiable Jordan arc, being stronger than a Lipschitz propagation as in Section 3.1. However the second condition shows the invalidity of the strongly-Lipschitz propagation in our sense.

We begin with a general abstract statement. Given two points \( p, q \in \mathbb{R}^2 \) we define \( [p, q] := \{tp + (1-t)q \mid t \in [0,1]\} \). Any set of the form \( [p, q] \) is called a segment. Note that a segment may be a singleton.

Proposition 3.3. Let \( \{K_j\}_{j=1}^\infty \) be a sequence of segments in \( \mathbb{R}^2 \). Suppose that \( K := \bigcup_{j=1}^\infty K_j \) is compact. Then there exists a convex function \( u : \mathbb{R}^2 \to \mathbb{R} \) such that \( \Sigma(u) = K \).

Proof. For each \( j \) the distance function \( u_j := d(K_j, \cdot) \) defines a nonnegative convex function such that \( \Sigma(u_j) = K_j \). We prove that the function defined by

\[
u := \sum_{j=1}^\infty 2^{-j} u_j
\]
gives a desired convex function. By the boundedness of \( K \) there are \( C_1, C_2 > 0 \), depending only on \( K \), such that \( u_j(p) \leq C_1 \|p\| + C_2 \) holds for any \( p \in \mathbb{R}^2 \) and \( j \geq 1 \).

In particular, the series \( \sum 2^{-j} u_j \) converges locally uniformly, so the limit function \( u \) indeed exists and is continuous, and also convex (as it is the supremum of convex functions). Below we prove that \( \Sigma(u) = K \).

We first prove that \( \Sigma(u) \subset K \), i.e., the limit function \( u \) is differentiable on \( \mathbb{R}^2 \setminus K \). Since \( K \) is closed, for any \( p \in \mathbb{R}^2 \setminus K \) there is \( r > 0 \) such that \( p \in B_r(p) \subset \mathbb{R}^2 \setminus K \). For each \( j \), since \( K_j \subset K \), the restriction of \( 2^{-j} u_j \) to \( B_r(p) \) is of class \( C^1 \) and satisfies \( \|\nabla (2^{-j} u_j)\| = 2^{-j} \) there. Hence the series \( \sum 2^{-j} u_j \) converges in the \( C^1 \)-topology locally on \( \mathbb{R}^2 \setminus K \), so that \( u \) is of class \( C^1 \) on \( \mathbb{R}^2 \setminus K \).

Finally we prove that \( K \subset \Sigma(u) \). Fix any \( p \in K \). Let \( j_0 \) be the minimal integer such that \( p \in K_{j_0} \). For notational simplicity we may assume that \( p \) is
the origin and $K_0$ lies in the $x$-axis. Then we have $u_{j_0}(0,y) = |y|$. Since $u$ is of the form $2^{-j_0}u_{j_0} + v_0$, where $v_0 := \sum_{j \neq j_0} 2^{-j}u_j$ is also convex, we have $u(0,y) = 2^{-j_0}|y| + v_0(0,y)$. By convexity of $v_0(0, \cdot)$ the function $u(0, \cdot)$ is not differentiable at $y = 0$. Hence $u$ is also not differentiable at the origin $p$, i.e., $p \in \Sigma(u)$. \hfill \Box

A concrete example is then constructed as follows, cf. Figure 2:

**Example 3.4 (Zigzag singular set).** Let $p_j := \left(\frac{1}{2^j}, 0\right) \in \mathbb{R}^2$, $q_j := \left(\frac{1}{2^j}, \frac{1}{2^j}\right) \in \mathbb{R}^2$, $A_j := [p_{j+1}, q_j]$, and $B_j := [q_j, p_{j+2}]$. Take $K_0 = \{(0,0)\}$ and $K_{4j+1} := A_j$, $K_{4j+2} := RA_j$, $K_{4j+3} := B_j$, $K_{4j+4} := RB_j$ for $j \geq 0$, where $R$ denotes the horizontal reflection matrix $(x, y) \mapsto (-x, y)$. Then the union $K$ of $\{K_j\}_{j \geq 0}$ is compact, so that by Proposition 3.3 there is a convex function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $\Sigma(u) = K$. The singular set $K$ is a zigzag line of finite length thanks to self-similarity. The arclength parameterization of $K$ gives a bi-Lipschitz map from an interval to $K$. However $K$ is so zigzag that any Lipschitz graph passing through the origin cannot be contained in $K$. Indeed, the sets $K \cap \{ \pm x \geq 0 \}$ can be represented by graphs only in directions $(\cos \theta, \sin \theta)$ such that $\pm \tan \theta \in \left(\frac{4}{3}, 2\right)$, respectively, so the whole $K$ is not graphical around the origin.

![Figure 2. Zigzag singular set.](image)

Note that the above $u$ does not satisfy Albano–Cannarsa’s condition (3.1) at the origin. Hence it is not yet clear whether the strongly-Lipschitz propagation always occurs under condition (3.1):

**Problem 3.5.** Let $\Omega \subset \mathbb{R}^m$ be open and $u : \Omega \to \mathbb{R}$ be a semi-concave function. Suppose that condition (3.1) holds at $x_0 \in \Omega$, and also $\dim K_{\partial u(x_0)}(p_0) = m - 1$ holds for some $p_0 \in \partial [\partial u(x_0)] \setminus D^* u(x_0)$. Then, is there a Lipschitz hypersurface $S \subset \mathbb{R}^m$ such that $x_0 \in S \subset \Sigma(u)$?

If not, it is still interesting to find a suitable subclass of semi-concave functions (e.g. solutions to a class of first-order Hamilton–Jacobi equations) for which the above statement holds. The higher-codimensional counterpart is a more general open problem unifying Problems 2.8 and 3.3.

4. **Preliminaries on Finsler geometry**

In what follows we argue on a general Finsler manifold $M$. In this section we review basic definitions, notions and properties in Finsler geometry. We also introduce some terminologies that we use in the sequel.
4.1. **Lipschitz submanifold.** To begin with, we give a definition of Lipschitz submanifold in a general smooth manifold (independent of the metric).

**Definition 4.1** (Lipschitz submanifold: General case). Let $M^m$ be a smooth $m$-dimensional manifold. Let $1 \leq d \leq m - 1$. A subset $S \subset M^m$ is called *(embedded)* $d$-dimensional Lipschitz submanifold if for any $p \in S$ there is a local chart $\varphi : U \to \varphi(U) \subset \mathbb{R}^m$ around $p$ such that $\varphi(S \cap U)$ is a $d$-dimensional Lipschitz submanifold of $\mathbb{R}^m$ in the sense of Definition 2.1.

In fact, the above definition does not depend on the choice of a local chart; in particular this definition is compatible with Definition 2.1 provided that $M$ is Euclidean. Although this fact might be well known for experts, we could not find any explicit argument in the literature so here we verify the following

**Lemma 4.2.** Let $S \subset \mathbb{R}^m$ be a subset. Let $p \in S$ and $U \subset \mathbb{R}^m$ be an open neighborhood of $p$ in $\mathbb{R}^m$. Suppose that there is an affine subspace $P \subset \mathbb{R}^m$ such that the orthogonal projection $\pi : \mathbb{R}^m \to P$ restricted to $S \cap U$ is bi-Lipschitz.

Then, for any neighborhood $V \subset \mathbb{R}^m$ of $p$ in $\mathbb{R}^m$ and any diffeomorphism $\Phi : V \to \Phi(V) \subset \mathbb{R}^m$, there are a neighborhood $V' \subset V$ of $p$ and an affine subspace $P'$ of same dimension as $P$ such that the orthogonal projection $\pi' : \mathbb{R}^m \to P'$ restricted to $\Phi(S \cap V')$ is bi-Lipschitz.

**Proof.** We first note that, since any orthogonal projection is clearly 1-Lipschitz, it is sufficient to verify the estimate of the form $|\pi'(p_1) - \pi'(p_2)| \geq c|p_1 - p_2|$ for any $p_1, p_2 \in \Phi(S \cap V')$, where $c \in (0, 1)$.

 Fix any $p$ in the assumption. Up to a translation we may assume that $P$ is a linear subspace, i.e., $0 \in P$. Hereafter $A^\perp$ denotes the orthogonal complement of $A$. Fix any diffeomorphism $\Phi : V \to \Phi(V) \subset \mathbb{R}^m$. Let $T$ be a linear isomorphism $T := d\Phi_p : \mathbb{R}^m \to \mathbb{R}^m$, and

$$P' := (T(P^\perp))^\perp.$$  

Clearly $P$ and $P'$ have the same dimension. We now prove that the linear map $\pi' \circ T|_P : P \to P'$ is injective. Suppose that $(\pi' \circ T|_P)(v) = 0$ for some $v \in P$. Then $T(v) \in (T(P^\perp)^\perp)^\perp = T(P^\perp) = T(P^\perp)$ and hence $v \in P^\perp$. Therefore, $v \in P \cap P^\perp$, i.e., $v = 0$. This means that the above linear map is injective, and hence there is $c_0 > 0$ such that

$$|\pi'(\Phi(q_1) - \Phi(q_2) - T(q_1 - q_2))| \leq c_0|q_1 - q_2|$$

for $q_1, q_2 \in B_3(p) \cap V$, and hence

$$|\pi'(\Phi(q_1)) - \pi'(\Phi(q_2))| = |\pi'(\Phi(q_1) - \Phi(q_2) - T(q_1 - q_2)) + \pi'(T(q_1 - q_2))|$$

$$\geq |\pi'(T(q_1 - q_2))| - c_0|q_1 - q_2|.$$  

Let $\pi^\perp$ denote the orthogonal projection to $P^\perp$. Then $a = \pi(a) + \pi^\perp(a)$ for any $a \in \mathbb{R}^m$. Taking $a = q_1 - q_2$ and operating the linear map $T$, we obtain

$$T(q_1 - q_2) = T(\pi(q_1 - q_2)) + T(\pi^\perp(q_1 - q_2)),$$

and hence by (4.1),

$$|\pi'(T(q_1 - q_2))| = |(\pi' \circ T)(\pi(q_1 - q_2))| \geq c_0|\pi(q_1) - \pi(q_2)|.$$
In addition, by the bi-Lipschitz property of \(\pi\) there is \(c_1 > 0\) such that
\[
|\pi(q_1) - \pi(q_2)| \geq c_1 |q_1 - q_2| \quad \text{for any } q_1, q_2 \in S \cap U.
\]
Combining (4.2), (4.3), and (4.4), and choosing \(\varepsilon = c_0 c_1 / 2\), we deduce that
\[
|\pi'(\Phi(q_1)) - \pi'(\Phi(q_2))| \geq \varepsilon |q_1 - q_2| \quad \text{for any } q_1, q_2 \in S \cap U \cap B_\delta(p) \cap V.
\]
Taking \(V' := U \cap B_\delta(p) \cap V\) completes the proof. \(\square\)

With this lemma at hand we observe that if \(S \subset \mathbb{R}^m\) is a Lipschitz submanifold in the sense of Definition 4.1 then it also satisfies Definition 2.1. (The converse obviously follows by choosing \(\varphi := \text{id}\).) Indeed, for any \(p \in S\) and the associated local coordinate \(\varphi : U \to \varphi(U) \subset \mathbb{R}^m\) in Definition 4.1, if we apply Lemma 4.2 to the set \(\varphi(S \cap U)\) with \(\Phi := \varphi^{-1} : \varphi(U) \to U\) around \(\varphi(p)\), then we find a desired bi-Lipschitz orthogonal projection of \(S\) around \(p\) in Definition 2.1.

In particular, thanks to this equivalence, Example 3.4 is also not a one-dimensional Lipschitz submanifold in the sense of Definition 4.1.

4.2. Generalized differential. Let us introduce here the notion of the generalized differential \(d^* f(p)\) at a point \(p\) of a locally Lipschitz function \(f\) on a smooth manifold \(M^m\) (again independent of the metric).

Let \(\tilde{f} : U \to \mathbb{R}\) denote a locally Lipschitz function on an open subset \(U\) of \(\mathbb{R}^m\). Define the reachable differential \(d^* \tilde{f}(x)\) of \(\tilde{f}\) at \(x \in U\) by
\[
d^* \tilde{f}(x) := \left\{ \lim_{j \to \infty} d\tilde{f}_{x_j} \big| x_j \to x, \exists d\tilde{f}_{x_j} \right\},
\]
where \(d\tilde{f}_{x_j}\) denotes the (classical) differential of \(\tilde{f}\) at \(x_j\). Notice that the generalized gradient \(\partial \tilde{f}(x)\) defined in (2.2) is linearly isomorphic to \(\text{conv } d^* \tilde{f}(x)\). Then we define the reachable differential \(d^* f(p)\) at a point \(p \in V\) of a locally Lipschitz function \(f\) on an open subset \(V\) of a smooth manifold \(M^m\) by
\[
d^* f(p) := \left\{ \omega \circ d\varphi_p \big| \omega \in d^*(f \circ \varphi^{-1})(\varphi(p)) \right\},
\]
where \(\varphi\) denotes a local chart around the point \(p\). From the chain rule, it is clear that this definition is independent of the choice of a local chart around \(p\). Finally we define the generalized differential \(\partial \tilde{f}(p)\) by
\[
\partial^* f(p) := \text{conv } d^* f(p).
\]

Using this notion we may directly extend Clarke’s implicit function theorem to a general smooth manifold.

Remark 4.3 (Clarke’s implicit function theorem on a manifold). Theorem 2.3 is still true for locally Lipschitz functions \(f_1, \ldots, f_k\) defined on a common open subset \(V\) of an \(m\)-dimensional smooth manifold \(M^m\) if \(\partial f_i(x_0)\) are replaced by \(\partial^* f_i(x_0)\) and if Lipschitz submanifolds are understood in the sense of Definition 4.1.

4.3. Finsler manifold. A smooth \((C^\infty)\) manifold \(M\) is called Finsler manifold if the manifold admits a nonnegative function \(F\) on the tangent bundle \(TM\) of \(M\) such that \(F\) is \(C^\infty\) on \(TM \setminus \{0\}\), where \(0\) denotes the zero section, and the restriction \(F|_{T_p M}\) of \(F\) to each tangent plane \(T_p M\) to \(M\) at \(p \in M\) is a Minkowski norm. Here, \(F\) is called a Minkowski norm if

1. \(F(p, y) \geq 0\) for all \((p, y) \in T_p M\), and \(F(p, y) = 0\) if and only if \(y = 0\);
2. $F$ is positively homogeneous of degree 1, i.e., $F(p, \lambda y) = \lambda F(p, y)$ for all $(p, y) \in T_pM$ and $\lambda \geq 0$;
3. the Hessian of $F^2$ is positive definite on $T_pM \setminus \{0\}$, i.e., for any $y \in T_pM \setminus \{0\}$, the following bilinear symmetric function $g_y$ on $T_pM$ is an inner product,

$$g_y(u, v) := \frac{\partial^2}{\partial s \partial t} \bigg|_{s = t = 0} \left( \frac{1}{2} F^2(p, y + su + tv) \right).$$

On a Finsler manifold $(M, F)$, one can define the length of curves as follows. For a $C^1$-curve $\gamma : [a, b] \to M$, the length $L(\gamma)$ of $\gamma$ is defined by

$$L(\gamma) := \int_a^b F(\gamma(t), \gamma'(t)) dt,$$

where $\gamma'(t)$ denotes the velocity vector of $\gamma$.

One can introduce the Finsler distance on a $C^\infty$-Finsler manifold $(M, F)$. For each pair of points $p, q \in M$, let $\Omega_{pq}$ denote the set of all piecewise $C^1$-curve $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. The distance $d(p, q)$ from $p$ to $q$ is defined by

$$d(p, q) := \inf_{\gamma \in \Omega_{pq}} L(\gamma).$$

Note that the length of a piecewise $C^1$-curve is also defined by (4.5). This distance has the following (quasimetric) properties:

1. $d(p, q) \geq 0$, with equality if and only if $p = q$;
2. $d(p, q) \leq d(p, r) + d(r, q)$ holds for any points $p, q, r$.

Notice carefully that the relation $d$ may not be symmetric due to the anisotropy of the Finsler metric. To avoid the possible asymmetry, we also use the symmetrized distance function

$$d_{\text{max}}(p, q) := \max\{d(p, q), d(q, p)\}.$$ 

This $d_{\text{max}}$ defines a distance function in the standard sense (with symmetry). Note that the topology induced from this distance coincides with the original topology of the manifold $M$.

We recall (cf. [7] p.151]) that a sequence $\{p_i\}^\infty_{i=1}$ of points on a Finsler manifold $(M, F)$ is called backward (resp. forward) Cauchy sequence if for any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that for all $N(\varepsilon) \leq i \leq j$, $d(p_j, p_i) < \varepsilon$ (resp. $d(p_i, p_j) < \varepsilon$).

The Finsler manifold $(M, F)$ is said to be backward (resp. forward) complete if every backward (resp. forward) Cauchy sequence converges. From the Finslerian version of the Hopf-Rinow theorem (see [7] p.168), the Finsler manifold $(M, F)$ is backward (resp. forward) complete if and only if every closed and backward (resp. forward) bounded subset of $(M, F)$ is compact, where a subset $N$ of $(M, F)$ is said to be backward (resp. forward) bounded if there exists a point $p \in M$ and a number $K > 0$ such that $\sup_{x \in N} d(x, p) < \infty$ (resp. $\sup_{x \in N} d(p, x) < \infty$). If a Finsler manifold is backward and forward complete, the manifold is said to be bi-complete, or simply complete in this paper.

4.4. Basic properties of distance functions. Here we exhibit several basic properties of distance functions for later use. From now on we let $d_N := d(N, \cdot) : M \setminus N \to \mathbb{R}$ denote the distance function from a nonempty closed subset $N$ of a connected complete smooth Finsler manifold $M$. 


We first note that by [27, Theorem 2.2] the reachable differential $d^*d_N$ of $d_N$ at a point $p \in M \setminus N$ is exactly given by

\[(4.6) \quad d^*d_N(p) = \{ g_v(v, \cdot) \mid v = \alpha'(d_N(p)), \alpha \text{ is an } N\text{-segment to } p \}. \]

The singular set $\Sigma(N)$ is already defined in (1.1) through non-differentiability, but it can also be geometrically characterized. A curve $\gamma : [0, a] \to M$ is called $N$-segment if $d(N, \gamma(t)) = t$ holds for any $t \in [0, a]$; any $N$-segment is thus a unit-speed geodesic. Then, it is also known in the Finsler case [27, Theorem 2.4] that $p \in M \setminus \Sigma(N)$ if and only if $p$ admits a unique $N$-segment. This fact provides the following characterization:

\[ \Sigma(N) = \{ p \in M \setminus N \mid \#\pi_{UN}(p) \geq 2 \}, \]

where $UN \subset TM$ denotes the (formal) unit normal bundle of $N$ defined by

\[ UN := \{ (\alpha(0), \alpha'(0)) \in TM \mid \alpha \text{ is an } N\text{-segment} \}, \]

and $\pi_{UN} : M \setminus N \to 2^UN$ denotes the projection map defined by

\[ \pi_{UN}(p) := \{ (\alpha(0), \alpha'(0)) \in UN \mid \alpha \text{ is an } N\text{-segment to } p \}. \]

The map $\pi_{UN}$ is also set-valued upper semicontinuous. Notice the difference from the Euclidean case that the image of the projection consists of not only points in $M$ but also tangent directions. This is because in the Finslerian (or even Riemannian) case, common endpoints may admit non-unique $N$-segments (but in this case the tangent vectors need to be different at the initial endpoint). Finally, let $\pi_N : M \setminus N \to 2^N$ be the composition of $\pi_{UN}$ and the canonical projection from $UN$ onto $N$, namely

\[ \pi_N(p) := \{ \alpha(0) \in N \mid \alpha \text{ is an } N\text{-segment to } p \}. \]

This map is again set-valued upper semicontinuous.

5. Structure of the singular set: Finslerian case

Here we complete the proof of Theorem 1.1. All the key ideas and techniques are already given in Section 2. This section is thus mainly devoted to explaining how to extend them to the general Finslerian case.

The set $\Sigma_2(N)$ is defined in the same way as (2.1), namely

\[ \Sigma_2(N) := \{ p \in \Sigma(N) \mid \#\pi_{UN}(p) = 2 \}. \]

Then our precise goal is as follows:

**Theorem 5.1** (Lipschitz hypersurface structure: Finslerian case). Let $m \geq 2$, $M^m$ be an $m$-dimensional smooth complete connected Finsler manifold, and $N \subset M^m$ be a nonempty closed subset. Then there exists an at most countable family of Lipschitz hypersurfaces $S_j \subset M^m$ such that

(i) $\Sigma_2(N) \subset S \subset \Sigma(N)$, where $S := \bigcup_{j=1}^{\infty} S_j$,

(ii) $\Sigma(N) \setminus S$ is $H^{m-2}$-rectifiable.

In addition, for each $p \in \Sigma(N)$, we define the radius function analogously to (2.3):

\[ \text{rad}_N(p) := \max\{ d_{\max}(q_1, q_2) \mid q_1, q_2 \in \pi_N(p) \}. \]

Note that, in contrast to the Euclidean case, the radius may take zero since a single point of $N$ may admit multiple $N$-segments to $p$. This issue is however not essential.
by considering a closed tubular neighborhood of $N$ as in the following lemma, the proof of which is safely omitted.

**Lemma 5.2.** Let $K \subset M \setminus N$ be a compact set. Let $c \in (0, \min_{x \in K} d_N(x))$ and $N_c := \{x \in M \mid d_N(x) \leq c\}$. Then $d_{N_c}(p) + c = d_N(p)$ holds for any $p \in K$. In particular, $\Sigma_2(N) \cap K = \Sigma_2(N_c) \cap K$ and $\text{rad}_{N_c}(p) > 0$ for $p \in K$.

We then obtain the main propagation lemma, a Finslerian version of Lemma 2.4. Although the statement and the proof are similar except for terminological differences, we give a somewhat precise argument for clarity.

**Lemma 5.3.** Let $p \in \Sigma_2(N)$ such that $\text{rad}_N(p) > 0$. Then there exist a positive number $\delta(p) > 0$ and a Lipschitz hypersurface $S_p \subset M$ such that $S_p \subset \Sigma(N)$ and such that $\text{rad}_N(y) \leq \frac{1}{2} \text{rad}_N(p)$ holds for any $y \in (\Sigma(N) \cap B_{\delta(p)}(p)) \setminus S_p$, where $B_r(p) := \{x \in M \mid d_{\text{max}}(p, x) < r\}$.

**Proof.** Let $\pi_N(p) = \{q_1, q_2\}$. Then $q_1 \neq q_2$ by $\text{rad}_N(p) > 0$. Let $N_j := N \cap \{y \in M \mid d_{\text{max}}(q_j, y) \leq \frac{1}{2} \text{rad}_N(p)\}$. The sets $N_1$ and $N_2$ are compact and, since $\text{rad}_N(p) > 0$, they are (relative) neighborhoods of $q_1$ and $q_2$ in $N$. In addition, by $\text{rad}_N(p) = d_{\text{max}}(q_1, q_2)$, we have

$$N_1 \cap N_2 = \emptyset,$$

since otherwise there would be $y \in N_1 \cap N_2$ but then $\text{rad}_N(p) = d_{\text{max}}(q_1, q_2) \leq d_{\text{max}}(q_1, y) + d_{\text{max}}(y, q_2) \leq \frac{1}{2} \text{rad}_N(p)$, which contradicts $\text{rad}_N(p) > 0$.

Let $f := d_{N_1} - d_{N_2} = d(N_1, \cdot) - d(N_2, \cdot)$. Since $\pi_N$ is set-valued upper semi-continuous, there is $r > 0$ such that for any $x \in B_r(p)$ we have $\pi_N(x) \subset N_1 \cup N_2$, and in particular $d_N(x) = \min\{d_{N_1}(x), d_{N_2}(x)\} = d_{\text{max}}(q_1, q_2)$. Hence, if $x \in B_r(p)$ and $f(x) = 0$, then $d_{N_1}(x) = d_{N_2}(x) = d_N(x)$ and hence, by (5.1), $\# \pi_N(x) \geq 2$. Therefore, for any $\delta \in (0, r)$,

$$S_{p, \delta} := f^{-1}(0) \cap B_{\delta}(p) \subset \Sigma(N).$$

In addition, if $x \in (\Sigma(N) \cap B_{\delta}(p)) \setminus S_{p, \delta}$, then either $\pi_N(x) \subset N_1$ or $\pi_N(x) \subset N_2$ holds (depending on the sign of $f(x)$) and hence $\text{rad}_N(x) \leq \frac{1}{2} \text{rad}_N(p)$.

We finally prove that for a suitable positive number $\delta := \delta(p) \in (0, r]$ the set $S_p := S_{p, \delta}(p)$ is a Lipschitz hypersurface. Since $p$ admits a unique $N_j$-segment $\alpha_j$ from $q_j$ for $j = 1, 2$, letting $v_j := \alpha_j'(d_N(p))$ and $\omega_j := g_{v_j}(v_j, \cdot)$, we have $d^* d_N(p) = \{\omega_j\}$ by (4.6), and hence $d^* f(p) = \{\omega_1 - \omega_2\}$. Since $\omega_1(v_1) = 1 > \omega_2(v_1)$ by (1.2.4), (1.2.16), we have $\omega_1 \neq \omega_2$ and in particular $0 \notin d^* f(p)$. Thanks to this fact with $f(p) = 0$, we now deduce by Theorem 2.3 with $k = 1$ in the Finsler case, cf. Remark 4.3, that there is a small positive number $\delta(p) > 0$ such that $S_p = f^{-1}(0) \cap B_{\delta(p)}(p)$ is a Lipschitz hypersurface. 

Next we recall the definition of a semi-concave function on a manifold:

**Definition 5.4** (Semi-concavity on a manifold). A continuous function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset M^m$ is called locally semi-concave if for any local chart $\varphi : \Omega \cap U \to \varphi(U) \subset \mathbb{R}^m$ the function $u \circ \varphi^{-1}$ is locally semi-concave.

This definition does not depend on the choice of the local chart, cf. [20 Proposition 2.6].

The fact that $d_N$ is locally semi-concave is also proven in [20 Proposition 3.4] in the Riemannian case through viscosity solution theory. Their proof is reduced to [19 Theorem 5.3], which might extend to some Finsler setting. Here we give a proof of the general Finsler case by a different and direct argument.
Lemma 5.5. The function $d_N$ is locally semi-concave on $M \setminus N$.

Proof. Let $\varphi : M \setminus N \supset U \to \varphi(U) \subset \mathbb{R}^m$ be any local chart. We prove that

$$u := d_N \circ \varphi^{-1}$$

is locally semi-concave. By locality it is sufficient to prove that for any $z \in \varphi(U)$ there is an open ball $B_\delta(z)$ centered at $z$ such that $B_\delta(z) \subset \varphi(U)$ and $d_N \circ \varphi^{-1}$ is semi-concave on $B_\delta(z)$. Fix any $z_0 \in \varphi(U)$ and take a small radius $\delta_0 > 0$ so that $B_{2\delta_0}(z_0) \subset \varphi(U)$ and the function $d(\varphi^{-1}(\cdot)) : \varphi(U) \times \varphi(U) \to [0, \infty)$ is smooth on $B_{2\delta_0}(z_0) \times B_{2\delta_0}(z_0) \setminus \Delta$, where $\Delta$ denotes the diagonal set. (Such a small radius exists thanks to [30] Theorem 3.1.) Fix any $x_0, x_1 \in B_{\delta_0}(z_0)$ and $\lambda \in [0, 1]$, and let $x_\lambda := \lambda x_0 + (1 - \lambda)x_1$. It now suffices to verify that there is $C > 0$ depending only on $\varphi$, $z_0$, $\delta_0$ such that

$$\lambda u(x_0) + (1 - \lambda)u(x_1) - u(x_\lambda) \leq C\lambda(1 - \lambda)|x_0 - x_1|^2.$$  

(5.2)

Let $p_i := \varphi^{-1}(x_i)$ for $i = 0, \lambda, 1$. Let $\alpha_\lambda : \{0, d_N(p_\lambda)\} \to M$ be an $N$-segment to $p_\lambda$, and choose $t_0 \in [0, d_N(p_\lambda)]$ so that $\alpha_\lambda(t_0) \in U$ and that the point $y_\lambda := \varphi(\alpha_\lambda(t_0))$ satisfies that $|z_0 - y_\lambda| = \frac{3}{2}\delta_0$. By minimality of $\alpha_\lambda$,

$$u(x_\lambda) = d_N(p_\lambda) = d_N(\alpha_\lambda(t_0)) + d(\alpha_\lambda(t_0), p_\lambda) = u(y_\lambda) + d(\varphi^{-1}(y_\lambda), \varphi^{-1}(x_\lambda)),$$

while for $i = 0, 1$, by the triangle inequality,

$$u(x_i) = d_N(p_i) \leq d_N(\alpha_\lambda(t_0)) + d(\alpha_\lambda(t_0), p_i) = u(y_\lambda) + d(\varphi^{-1}(y_\lambda), \varphi^{-1}(x_i)).$$

Using these relations, and letting $v := d(\varphi^{-1}(y_\lambda), \varphi^{-1}(\cdot))$, we deduce that

$$\lambda u(x_0) + (1 - \lambda)u(x_1) - u(x_\lambda) \leq \lambda v(x_0) + (1 - \lambda)v(x_1) - v(x_\lambda).$$

On the other hand, by definition of $\delta_0$ and $|z_0 - y_\lambda| = \frac{3}{2}\delta_0$, the function $v$ is smooth on $B_{3\delta_0}(z_0)$, and hence $|D^2v| \leq C$ holds on $B_{\delta_0}(z_0)$ for a constant $C > 0$ depending only on $(M \text{ and } \varphi, z_0, \delta_0)$. Therefore,

$$\lambda v(x_0) + (1 - \lambda)v(x_1) - v(x_\lambda) \leq C\lambda(1 - \lambda)|x_0 - x_1|^2.$$  

(5.3)

Estimates (5.3) and (5.4) imply (5.2).

We are now in a position to complete the proof of Theorem 1.1 mostly appealing to the proof of Theorem 2.2.

Proof of Theorem 5.1. We mainly follow the proof of Theorem 2.2 with $\mathbb{R}^m$ replaced by $M$, and in particular Lemma 2.4 by Lemma 5.3. In what follows we only emphasize the differences.

Concerning Step 1, the only difference we need to be careful is that for a compact set $K \subset M$, the set $\Sigma_2(N) \cap K$ may not be equal to $\bigcup_{i=1}^\infty A_i$ since $\{p \in \Sigma_2(N) \cap K \mid \text{rad}_N(p) = 0\}$ may not be empty, where $A_i$ is defined in the same way as in (2.6) by using $\text{rad}_N(K)$ in (2.5). To overcome this difference, it is sufficient to just choose a constant $c > 0$ as in Lemma 5.2 (depending on $K$) and to replace $N$ by $N_c$. Then the completely parallel argument implies that for any compact set $K \subset M$ the set $\Sigma_2(N_c) \cap K = \Sigma_2(N) \cap K$ is covered by an at most countably many Lipschitz hypersurfaces. Taking an increasing sequence of $K$ completes the proof.

Concerning Step 2, there is no essential difference but we need to interpret some objects through differentials, so we give a precise argument. Choose any $p \in \Sigma(N) \setminus \Sigma_2(N)$. There exist at least three distinct $N$-segments $\alpha_j : [0, L] \to M$ such that $\alpha_j(L) = p$, where $j = 1, 2, 3$ and $L := d_N(p)$, and there are the 1-forms.
\(\omega_j(\cdot) := g_{v_j}(v_j, \cdot)\) corresponding to the unit vectors \(v_j := \alpha_j'(L) \in T_p M, j = 1, 2, 3\). Now we verify that
\[
(5.5) \quad \text{the three 1-forms } \omega_1, \omega_2, \omega_3 \text{ are not colinear.}
\]
In fact, suppose that they are colinear. We may assume that \(\omega_1 = \lambda \omega_2 + (1 - \lambda) \omega_3\). Since \(\omega_1(v_1) = 1, \omega_2(v_1) < 1, \omega_3(v_1) < 1\) by \([1] (1.2.4), (1.2.16)\), we obtain
\[
1 = \omega_1(v_1) = \lambda \omega_2(v_1) + (1 - \lambda) \omega_3(v_1) < 1,
\]
which is a contradiction. Hence (5.5) holds true.

This together with (5.5) means that
\[
Σ(N) \setminus Σ_2(N) ⊂ Σ^2(d_N), \text{ where } Σ^2(d_N) := \{p ∈ M \mid \dim \partial^* d_N(p) ≥ 2\}.
\]
Thanks to Lemma 5.5 and [3, Theorem 4.1], the \(d_N \circ \varphi^{-1}\) is locally semi-concave for any local coordinate system \((U, \varphi)\), and \(Σ^2(d_N \circ \varphi^{-1})\) is a \(H^{m-2}\)-rectifiable set in \(R^m\). Therefore, the set \(Σ^2(d_N)\) defined above is also \(H^{m-2}\)-rectifiable in \(M\). □

Proof of Theorem 1.1. It is now a direct consequence of Theorem 5.1 □

Remark 5.6. Thanks to the \(C^2\)-rectifiability result of Alberti [4], one can even assert that the residual part \(R\) in Theorem 1.1 is covered by a countable family of \(C^2\)-hypersurfaces up to a negligible set with respect to the \(H^{m-2}\)-measure.

REFERENCES

[1] Paolo Albano and Piermarco Cannarsa, *Structure properties of singularities of semiconcave functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 4, 719–740. MR 1760538
[2] ———, *Propagation of singularities for solutions of nonlinear first order partial differential equations*, Arch. Ration. Mech. Anal. 162 (2002), no. 1, 1–23. MR 1892229
[3] G. Alberti, L. Ambrosio, and P. Cannarsa, *On the singularities of convex functions*, Manuscripta Math. 76 (1992), no. 3-4, 421–435. MR 1185029
[4] Giovanni Alberti, *On the structure of singular sets of convex functions*, Calc. Var. Partial Differential Equations 2 (1994), no. 1, 17–27. MR 1384392
[5] L. Ambrosio, P. Cannarsa, and H. M. Soner, *On the propagation of singularities of semiconcave functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), no. 4, 597–616. MR 1267601
[6] Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000. MR 1857292
[7] D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, Graduate Texts in Mathematics, vol. 200, Springer-Verlag, New York, 2000. MR 1747675
[8] Piermarco Cannarsa and Wei Cheng, *On and beyond propagation of singularities of viscosity solutions*, Proceedings of the International Consortium of Chinese Mathematicians 2017, Int. Press, Boston, MA, 2020, pp. 141–157. MR 4251110
[9] ———, *Singularities of solutions of Hamilton-Jacobi equations*, Milan J. Math. 89 (2021), no. 1, 187–215. MR 4277365
[10] ——— and Carlo Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and their Applications, vol. 58, Birkhäuser Boston, Inc., Boston, MA, 2004. MR 2041617
[11] ——— and Yifeng Yu, *Singular dynamics for semiconcave functions*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 5, 999–1024. MR 2538498
[12] F. H. Clarke, *Optimization and nonsmooth analysis*, second ed., Classics in Applied Mathematics, vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990. MR 1058436
[13] Paul Erdős, *Some remarks on the measurability of certain sets*, Bull. Amer. Math. Soc. 51 (1945), 728–731. MR 13776
[14] Philip Hartman, *Geodesic parallel coordinates in the large*, Amer. J. Math. **86** (1964), 705–727. MR 173222
[15] James J. Hebda, *Parallel translation of curvature along geodesics*, Trans. Amer. Math. Soc. **299** (1987), no. 2, 559–572. MR 869221
[16] ———, *Metric structure of cut loci in surfaces and Ambrose’s problem*, J. Differential Geom. **40** (1994), no. 3, 621–642. MR 1305983
[17] Jin-ichi Itoh, *The length of a cut locus on a surface and Ambrose’s problem*, J. Differential Geom. **43** (1996), no. 3, 642–651. MR 1412679
[18] Yanyan Li and Louis Nirenberg, *The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations*, Comm. Pure Appl. Math. **58** (2005), no. 1, 85–146. MR 2094267
[19] Pierre-Louis Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics, vol. 69, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982. MR 667669
[20] Carlo Mantegazza and Andrea Carlo Mennucci, *Hamilton-Jacobi equations and distance functions on Riemannian manifolds*, Appl. Math. Optim. **47** (2003), no. 1, 1–25. MR 1941909
[21] Tatsuya Miura, *A characterization of cut locus for C^1 hypersurfaces*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 6, Art. 60, 14. MR 3568029
[22] Sumner Byron Myers, *Connections between differential geometry and topology. I. Simply connected surfaces*, Duke Math. J. **1** (1935), no. 3, 376–391. MR 1545884
[23] ———, *Connections between differential geometry and topology II. Closed surfaces*, Duke Math. J. **2** (1936), no. 1, 95–102. MR 1545908
[24] V. Ozols, *Cut loci in Riemannian manifolds*, Tohoku Math. J. (2) **26** (1974), 219–227. MR 390967
[25] Henri Poincaré, *Sur les lignes géodésiques des surfaces convexes*, Trans. Amer. Math. Soc. **6** (1905), no. 3, 237–274. MR 1500710
[26] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR 0274683
[27] Sorin V. Sabau and Minoru Tanaka, *The cut locus and distance function from a closed subset of a Finsler manifold*, Houston J. Math. **42** (2016), no. 4, 1157–1197. MR 3609822
[28] Mario Santilli, *Distance functions with dense singular sets*, Comm. Partial Differential Equations **46** (2021), no. 7, 1319–1325. MR 4279967
[29] Katsuhiro Shiohama and Minoru Tanaka, *Cut loci and distance spheres on Alexandrov surfaces*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 531–559. MR 1427770
[30] Katsuhiro Shiohama and Bankteshwar Tiwari, *The global study of Riemannian-Finsler geometry*, Geometry in history, Springer, Cham, 2019, pp. 581–621. MR 3965775
[31] Minoru Tanaka, *The singular locus of an almost distance function*, Tokyo J. Math. **43** (2020), no. 1, 47–74. MR 4121789
[32] J. H. C. Whitehead, *On the covering of a complete space by the geodesics through a point*, Ann. of Math. (2) **36** (1935), no. 3, 679–704. MR 1503245

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