Flows and invariance for elliptic operators
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Abstract

Let $S$ be the submarkovian semigroup on $L_2(\mathbb{R}^d)$ generated by a self-adjoint, second-order, divergence-form, elliptic operator $H$ with $W^{1,\infty}$ coefficients $c_{kl}$. Further let $\Omega$ be an open subset of $\mathbb{R}^d$. Under mild conditions we prove that $S$ leaves $L_2(\Omega)$ invariant if, and only if, it is invariant under the flows generated by the vector fields $\sum_{l=1}^{d} c_{kl} \partial_l$ for all $k$.

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1 Introduction

Let $S$ be a submarkovian semigroup on $L_2(\mathbb{R}^d)$ generated by a self-adjoint second-order elliptic operator $H$ in divergence form. If the operator is strongly elliptic then $S$ acts ergodically, i.e. there are no non-trivial $S$-invariant subspaces of $L_2(\mathbb{R}^d)$. Nevertheless there are many examples of degenerate elliptic operators for which there are subspaces $L_2(\Omega)$ invariant under the action of $S$ (see, for example, [ERSZ2] [ERSZ1] [RoS1] [ElR1]).

Our aim is to examine operators with coefficients which are Lipschitz continuous and characterize the $S$-invariance of $L_2(\Omega)$ by the invariance under a family of associated flows. In order to formulate our main result we need some further notation.

First define the positive symmetric operator $H_0$ with domain $D(H_0) = C^\infty_c(\mathbb{R}^d)$ and action

$$H_0 \varphi = -\sum_{k,l=1}^{d} \partial_k c_{kl} \partial_l \varphi$$

where the coefficients $c_{kl} = c_{lk} \in W^{1,\infty}(\mathbb{R}^d)$ are real and $C = (c_{kl})$ is a positive-definite matrix over $\mathbb{R}^d$. Then the corresponding quadratic form $h_0$ given by

$$h_0(\varphi) = \sum_{k,l=1}^{d} (\partial_k \varphi, c_{kl} \partial_l \varphi)$$

with domain $D(h_0) = C^\infty_c(\mathbb{R}^d)$ is closable. The closure $h = \overline{h_0}$ determines in a canonical manner a positive self-adjoint extension $H$ of $H_0$, the Friedrichs’ extension [Fri] (see, for example, [RSN], §124, or [Kat], Chapter VI). The closed form $h$ is a Dirichlet form and the self-adjoint semigroup $S$ generated by $H$ is automatically submarkovian (for details on Dirichlet forms and submarkovian semigroups see [FOT] or [BoH]). We call $H$ the degenerate elliptic operator with coefficients $(c_{kl})$.

Secondly, if $b_1, \ldots, b_d \in W^{1,\infty}(\mathbb{R}^d)$ then the first-order partial differential operator

$$\varphi \mapsto \sum_{k=1}^{d} b_k \partial_k \varphi - \frac{1}{2} \sum_{k=1}^{d} (\partial_k b_k) I$$

with domain $C^\infty_c(\mathbb{R}^d)$ is essentially skew-adjoint (see, for example, [Rob1], Theorem 3.1). Therefore the principal part is closable and generates a positive, continuous, one-parameter group on $L_2(\mathbb{R}^d)$. We refer to such a group as flows. Specifically we are interested in the flows associated with the coefficients $(c_{kl})$ of $H$. For all $k \in \{1, \ldots, d\}$ let $Y_k$ denote the $L_2$-closures of the first-order partial differential operator

$$\varphi \mapsto \sum_{l=1}^{d} c_{kl} \partial_l \varphi$$

with domain $C^\infty_c(\mathbb{R}^d)$. Then denote by $T^{(k)}$ the flows generated by the $Y_k$. The operators $Y_k$ were used by Oleinik and Radkevič [OIR] to analyze hypoellipticity and subellipticity properties of degenerate elliptic operators $H$ with $C^\infty$-coefficients $c_{kl}$ (see [JeS] for a review of these and related results). We, however, use the flows to characterize the invariant subspaces of the semigroup generated by $H$.

Theorem 1.1 Let $\Omega$ be a measurable subset of $\mathbb{R}^d$. Consider the following conditions.
I. \( S_t L_2(\Omega) \subseteq L_2(\Omega) \) for all \( t > 0 \).

II. \( T_t^{(k)} L_2(\Omega) = L_2(\Omega) \) for all \( k \in \{1, \ldots, d\} \) and \( t \in \mathbb{R} \).

Then I\(\Rightarrow\)II. Moreover, if \( C_c^\infty(\mathbb{R}^d) \) is a core for \( H \), or, if \( \Omega \) is open and the boundary \( \partial \Omega \) of \( \Omega \) is (locally) Lipschitz then I\(\Rightarrow\)II.

Recall that the open set \( \Omega \) is defined to have a (locally) Lipschitz boundary if for every \( y \in \partial \Omega \) there exist an isometry \( \Psi: \mathbb{R}^d \to \mathbb{R}^d \), a real function \( \tau \in W^{1,\infty}(\mathbb{R}^{d-1}) \) and an \( r > 0 \) such that

\[
\Omega \cap B_y(r) = \{ \Psi(x_1, x') : (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, \tau(x') < x_1 \} \cap B_y(r)
\]

where \( B_y(r) = \{ x \in \mathbb{R}^d : \|x - y\| < r \} \). Thus in a neighbourhood of \( y \) the boundary \( \partial \Omega \) of \( \Omega \) is the graph of a Lipschitz function \( \tau \), up to an isometry \( \Psi \).

There are two variations of the theorem which will be established in the course of its proof.

First, for all \( \psi \in C_c^\infty(\mathbb{R}^d) \) define \( Y_\psi \) as the \( L_2 \)-closure of the first-order partial differential operator

\[
\varphi \mapsto \sum_{k,l=1}^d (\partial_k \psi) c_{kl} \partial_l \varphi
\]

with domain \( C_c^\infty(\mathbb{R}^d) \) and let \( T^\psi \) be the associated flow. Then invariance of \( L_2(\Omega) \) under the \( T^\psi \) is equivalent to invariance under the family of flows \( T^\psi \). More precisely one has the following.

**Proposition 1.2** Let \( \Omega \) be a measurable subset of \( \mathbb{R}^d \). The following conditions are equivalent:

I. \( T_t^\psi L_2(\Omega) = L_2(\Omega) \) for all \( \psi \in C_c^\infty(\mathbb{R}^d) \) and \( t \in \mathbb{R} \).

II. \( T_t^{(k)} L_2(\Omega) = L_2(\Omega) \) for all \( k \in \{1, \ldots, d\} \) and \( t \in \mathbb{R} \).

This will be established in Section 2.

Secondly, the condition that \( C_c^\infty(\mathbb{R}^d) \) is a core for \( H \) does not follow in general from the assumption that the coefficients are in \( W^{1,\infty}(\mathbb{R}^d) \). The one-dimensional example considered in [ERSZ2], Section 5 gives a counterexample. Specifically, let \( \delta \in [1/2, \infty) \) and \( H = -d c d \) with \( c(x) = |x|^{2\delta}(1 + x^2)^{-\delta} \). Then \( c \in W^{1,\infty}(\mathbb{R}) \) but \( C_c^\infty(\mathbb{R}) \) is a core of \( H \) if and only if \( \delta \geq 3/4 \) by the arguments in [CMP], Proposition 3.5. (See also [RoS2], Section 6, or [ELR2] Proposition 2.3). Moreover, the core condition can be derived from weaker smoothness assumptions on the \( c_{kl} \) (see Section 4).

2 Flows

In this section we derive some properties of the flows defined in Section 1 and prove Proposition 1.2. Although we deal primarily with the flows on \( L_2(\mathbb{R}^d) \) we will need, in Section 3 some properties of their extensions to \( L_\infty(\mathbb{R}^d) \). Therefore we begin by summarizing some general features of the flows.
Let $b_1, \ldots, b_d \in W^{1,\infty}(\mathbb{R}^d)$ and define $Y$ as the $L_2$-closure of the first-order differential operator $\phi \mapsto \sum_{k=1}^d b_k \partial_k \phi$ and domain $W^{1,2}(\mathbb{R}^d)$. Further let $T$ denote the flow generated by $Y$. Then for all $p \in [1, \infty]$ the group $T$ leaves the subspace $L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ of $L_2(\mathbb{R}^d)$ invariant and $T$ extends from $L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ to a flow $T[p]$ on $L_p(\mathbb{R}^d)$ such that $T[p]$ is strongly continuous if $p \in [1, \infty)$ and $T[\infty]$ is weakly* continuous. The groups act in a consistent and compatible manner on the $L_p$-spaces. Moreover, $T[\infty]$ is a group of automorphisms of $L_\infty(\mathbb{R}^d)$, i.e. $T_t[\infty](\psi \varphi) = (T_t[\infty] \psi) (T_t[\infty] \varphi)$ for all $\psi, \varphi \in L_\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$. Then since the $L_\infty$-functions are multipliers on the $L_p$-spaces one deduces that

$$
T_t[p] (\tau \varphi) = (T_t[p][\tau]) (T_t[p] \varphi)
$$

for all $\tau \in L_\infty(\mathbb{R}^d)$, $\varphi \in L_p(\mathbb{R}^d)$, $p \in [1, \infty]$ and $t \in \mathbb{R}$. If $Y[p]$ is the generator of $T[p]$ then $W^{1,p}(\mathbb{R}^d) \subset D(Y[p])$ and $Y[p] \varphi = \sum_{k=1}^d b_k \partial_k \varphi$ for all $\varphi \in W^{1,p}(\mathbb{R}^d)$.

These properties depend critically on the fact that $Y$ is a first-order partial differential operator with coefficients $b_k \in W^{1,\infty}(\mathbb{R}^d)$. They can be verified either by general arguments of functional analysis (see, for example, [Rob2], Theorem V.4.1) or by methods of ordinary differential equations. The crucial observation in the latter context is that if $\varphi \in C_c^\infty(\mathbb{R}^d)$ then $(T_t \varphi)(x) = \varphi(\omega_t(x))$ where $t \mapsto \omega_t(x)$ is the unique solution of the differential equation $(d/dt) \omega_t(x) = b(\omega_t(x))$, with initial value $\omega_0(x) = x$ (see, for example, [Hil], Chapters 2 and 3).

Our first result is an approximation result which will be needed on $L_2(\mathbb{R}^d)$ but whose proof extends to the $L_p$-spaces.

**Proposition 2.1** Let $p \in [1, \infty]$. Let $Y[p]$ denote the generator of the flow $T[p]$ on $L_p(\mathbb{R}^d)$. Further let $\tau \in C_c^\infty(\mathbb{R}^d)$ with $\int \tau = 1$ and for all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R}^d)$ by $\tau_n(x) = n^d \tau(n x)$.

Then $\lim_{n \to \infty} Y[p] (\tau_n \varphi) = Y[p] \varphi$ in $L_p(\mathbb{R}^d)$ for all $\varphi \in D(Y[p])$ if $p < \infty$. If $p = \infty$ then $\lim_{n \to \infty} Y[p] (\tau_n \varphi) = Y[p] \varphi$ weakly* in $L_\infty(\mathbb{R}^d)$ for all $\varphi \in D(Y[\infty])$.

**Proof** First, for all $n \in \mathbb{N}$ define the bounded operator $B_n : L_p \to L_p$ by

$$
B_n \varphi = \sum_{k=1}^d \tau_n * ((\partial_k b_k) \varphi) + \sum_{k=1}^d \int dy (\partial_k \tau_n)(y) \left( (I - L_y) b_k \right) (L_y \varphi)
$$

where $L$ denotes the left regular representation of $\mathbb{R}^d$, i.e. $(L_y \psi)(x) = \psi(x - y)$. Secondly, if $\varphi \in C_c^\infty$ and $n \in \mathbb{N}$ then

$$
Y[p] (\tau_n \varphi) = \sum_{k=1}^d b_k \int dy \tau_n(y) L_y \partial_k \varphi
$$

$$
= \sum_{k=1}^d \int dy \tau_n(y) (b_k - L_y b_k) L_y \partial_k \varphi + \sum_{k=1}^d b_k \int dy \tau_n(y) L_y (b_k \partial_k \varphi)
$$

The second term equals $\tau_n * Y[p] \varphi$. For the first term use $L_y \partial_k \varphi = -\partial_y \partial_k L_y \varphi$. Therefore integration by parts gives

$$
Y[p] (\tau_n \varphi) - \tau_n * Y[p] \varphi = \sum_{k=1}^d \int dy \frac{\partial}{\partial y_k} (\tau_n(y) (b_k - L_y b_k))(L_y \varphi) = B_n \varphi
$$


Since $B_n$ is bounded one deduces by density that
\[ Y_{[p]}(\tau_n \ast \varphi) - \tau_n \ast Y_{[p]} \varphi = B_n \varphi \] (3)
for all $n \in \mathbb{N}$ and $\varphi \in D(Y_{[p]})$.

Thirdly, it follows from the definition of $B_n$ that
\[
\|B_n \varphi\|_p \leq \sum_{k=1}^{d} \left( \| (\partial_k b_k) \varphi \|_p + \int dy \| (\partial_k \tau_n)(y) \| \left( (I - Ly)b_k \right) \| (L_y \varphi) \|_p \right) \\
\leq \sum_{k=1}^{d} \| b_k \|_{W^{1,\infty}} \| \varphi \|_p + \sum_{k=1}^{d} \int dy \| (\partial_k \tau_n)(y) \| \left( (I - Ly)b_k \right) \| \varphi \|_p
\]
for all $n \in \mathbb{N}$ and $\varphi \in L_p$. But $\| (I - Ly)b_k \|_{W^{1,\infty}} \leq |y| \| b_k \|_{W^{1,\infty}}$ and $\int dy \| (\partial_k \tau_n)(y) \| \| \varphi \|_p = \int dy \| (\partial_k \tau)(y) \| |y|$.
Therefore $\|B_n \varphi\|_p \leq M \| \varphi \|_p$ uniformly for all $n \in \mathbb{N}$ and $\varphi \in L_p$, where
\[ M = \sum_{k=1}^{d} (1 + \int dy \| (\partial_k \tau)(y) \| |y|) \| b_k \|_{W^{1,\infty}} \].
The conclusion holds for all $p \in [1, \infty)$. So $B_1, B_2, \ldots$ are equicontinuous.

Next assume $p < \infty$. If $\varphi \in W^{1,p}$ then $\lim_{n \to \infty} \tau_n \ast \varphi = \varphi$ in $W^{1,p}$. Consequently,
\[ \lim_{n \to \infty} Y_{[p]}(\tau_n \ast \varphi) = Y_{[p]} \varphi \] strongly in $L_p$. Moreover, $\lim_{n \to \infty} \tau_n \ast (Y_{[p]} \varphi) = Y_{[p]} \varphi$ strongly in $L_p$.
Therefore $\lim_{n \to \infty} B_n \varphi = 0$ in $L_p$ for all $\varphi \in W^{1,p}$ by (3). Since $W^{1,p}$ is strongly dense in $L_p$ and $B_1, B_2, \ldots$ are equicontinuous it follows that $\lim_{n \to \infty} B_n \varphi = 0$ in $L_p$ for all $\varphi \in L_p$.
Finally, let $\varphi \in D(Y_{[p]})$. Then one establishes from (3) that $\lim_{n \to \infty} Y_{[p]}(\tau_n \ast \varphi) = \lim_{n \to \infty} (\tau_n \ast Y_{[p]} \varphi + B_n \varphi) = Y_{[p]} \varphi$ in $L_p$.

The argument for $p = \infty$ is very similar. If $\varphi \in W^{1,\infty}$ then $\lim_{n \to \infty} \tau_n \ast \varphi = \varphi$ and $\lim \partial_k \tau_n \ast \varphi = \partial_k \varphi$ weakly*. Therefore $\lim Y_{[\infty]}(\tau_n \ast \varphi) = Y_{[\infty]} \varphi$ weak* on $L_\infty$. Then since $W^{1,\infty}$ is weakly* dense in $L_\infty$ and $B_1, B_2, \ldots$ are equicontinuous the desired conclusion follows as before.

Now we return to consideration of the vector fields $Y_1, \ldots, Y_d$ defined in Section 1 acting on $L_2(\mathbb{R}^d)$.

**Corollary 2.2** Let $\tau$ and $\tau_n$ be as in Proposition 2.1. Then for all $\varphi \in \bigcap_{k=1}^{d} D(Y_k)$ one has $\lim_{n \to \infty} Y_k(\tau_n \ast \varphi) = Y_k \varphi$ for all $k \in \{1, \ldots, d\}$.

Note that convolution with $\tau_n$ maps $L_2(\mathbb{R}^d)$ into $W^{\infty,2}(\mathbb{R}^d)$ so the corollary establishes that $W^{\infty,2}(\mathbb{R}^d)$ is a simultaneous core for the $Y_1, \ldots, Y_d$.

Now we turn to the proof of Proposition 1.2. Note that if $T$ is a flow with generator $Y$ then $T$-invariance of $L_2(\Omega)$ is equivalent to the the commutation of $Y$ and the operator of multiplication with $1_\Omega$, i.e. if $\varphi \in D(Y)$ then $1_\Omega \varphi \in D(Y)$ and $Y(1_\Omega \varphi) = 1_\Omega Y \varphi$.

**Proof of Proposition 1.2** Let $k \in \{1, \ldots, d\}$ and $U \subset \mathbb{R}^d$ a bounded open subset. There exist $\chi, \psi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi|_{U} = 1$ and $\psi(x) = x_k$ for all $x \in \text{supp } \chi$.
Then $Y_k(\chi \varphi) = Y_k(\chi \varphi)$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Since $\varphi \mapsto \chi \varphi$ is continuous on $D(Y_k)$ and on $D(Y_\psi)$, with the graph norm, it follows from Proposition 2.1 that $\chi \varphi \in D(Y_k)$ for all $\varphi \in D(Y_\psi)$. In particular, if $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } \varphi \subset U$ then $1_\Omega \varphi \in D(Y_k)$ and therefore $1_\Omega \varphi = \chi 1_\Omega \varphi \in D(Y_k)$. Moreover, $Y_k(1_\Omega \varphi) = Y_k(1_\Omega \varphi) = Y_k 1_\Omega \varphi = 1_\Omega Y_k \varphi$. Then it follows by continuity that $1_\Omega \varphi \in D(Y_k)$ and $Y_k(1_\Omega \varphi) = 1_\Omega Y_k \varphi$ for all $\varphi \in D(Y_k)$.
Therefore Condition II is valid.
It follows from Condition I that \( \mathbb{1}_\Omega \varphi \in D(Y_k) \) and \( Y_k(\mathbb{1}_\Omega \varphi) = \mathbb{1}_\Omega Y_k \varphi \) for all \( \varphi \in D(Y_k) \). Let \( \psi \in C^\infty_c(\mathbb{R}^d) \). Then \( Y_\psi \varphi = \sum_{k=1}^d (\partial_k \psi) Y_k \varphi \) for all \( \varphi \in C^\infty_c(\mathbb{R}^d) \). Since the coefficients \( c_{kl} \) are in \( W^{1,\infty}(\mathbb{R}^d) \) it follows from Corollary 2.2 that \( \varphi \in D(Y_\psi) \) and \( Y_\psi \varphi = \sum_{k=1}^d (\partial_k \psi) Y_k \varphi \) for all \( \varphi \in \bigcap_{k=1}^d D(Y_k) \). Hence if \( \varphi \in C^\infty_c(\mathbb{R}^d) \) then \( \mathbb{1}_\Omega \varphi \in D(Y_\psi) \) and \( Y_\psi(\mathbb{1}_\Omega \varphi) = \mathbb{1}_\Omega Y_\psi \varphi \). By density the latter extends to all \( \varphi \in D(Y_\psi) \) and therefore Condition II is valid.

Finally we note that the flows \( T_\psi \) can be defined for all \( \psi \in W^{2,\infty}(\mathbb{R}^d) \) and the conditions of Proposition 1.2 are equivalent to invariance of \( L_2(\Omega) \) for all \( T_\psi^t \) with \( \psi \in W^{2,\infty}(\mathbb{R}^d) \) and \( t > 0 \). This follows from the arguments of the foregoing proof.

### 3 Semigroup invariance

In this section we prove Theorem 1.1. First, however, we observe that Condition I of the theorem, the invariance of \( L_2(\Omega) \) under the flows \( T^{(k)} \) is equivalent to \( T_\psi \)-invariance of \( L_2(\Omega) \) for all \( \psi \in C^\infty_c(\mathbb{R}^d) \). This is a direct consequence of Proposition 1.2 which was established in the previous section. Therefore in the subsequent discussion we will consider the \( T_\psi \)-invariance condition.

**Proof of Theorem 1.1** \( \Box \rightarrow \Box \). It suffices, by the foregoing observation, to prove the \( T_\psi \)-invariance of \( L_2(\Omega) \) for all \( \psi \in C^\infty_c(\mathbb{R}^d) \).

First, it follows from the density of \( C^\infty_c(\mathbb{R}^d) \) in \( D(h) \) that there exists a unique bilinear map \( \Gamma: D(h) \times D(h) \rightarrow L_1 \), the carré du champ, such that

\[
\Gamma(\psi, \varphi) = \sum_{k,l=1}^d c_{kl} (\partial_k \psi)(\partial_l \varphi)
\]

for all \( \psi, \varphi \in W^{1,2}(\mathbb{R}^d) \). Then \( \|\Gamma(\psi, \varphi)\|_1 \leq h(\psi)^{1/2} h(\varphi)^{1/2} \) for all \( \psi, \varphi \in D(h) \) by the Cauchy–Schwarz inequality. Moreover,

\[
\int \tau \Gamma(\psi, \varphi) = \frac{1}{2} \left( h(\tau \psi, \varphi) + h(\psi, \tau \varphi) - h(\tau, \psi \varphi) \right)
\]

for all \( \tau, \psi, \varphi \in C^\infty_c(\mathbb{R}^d) \). But \( \Box \) then extends to all \( \tau, \psi, \varphi \in D(h) \cap L_\infty \) by density.

Secondly, the form \( h \) is local in the sense that \( h(\psi, \varphi) = 0 \) for all \( \psi, \varphi \in D(h) \) with \( \psi \varphi = 0 \) (see [Sch]). Therefore it follows from \( \Box \) that \( \Gamma \) is local in the same sense.

Thirdly, since \( L_2(\Omega) \) is \( S \)-invariant the operation of multiplication by \( \mathbb{1}_\Omega \) maps \( D(h) \) into itself. Therefore if \( \psi, \varphi, \tau \in D(h) \cap L_\infty \) then \( \mathbb{1}_\Omega \varphi, \mathbb{1}_\Omega \tau \in D(h) \cap L_\infty \). By locality of \( h \) one deduces from \( \Box \) that

\[
\int \tau \Gamma(\psi, \mathbb{1}_\Omega \varphi) = \frac{1}{2} \left( h(\tau \psi, \mathbb{1}_\Omega \varphi) + h(\psi, \tau \mathbb{1}_\Omega \varphi) - h(\tau, \psi \mathbb{1}_\Omega \varphi) \right)
\]

\[
= \frac{1}{2} \left( h(\mathbb{1}_\Omega \tau \psi, \varphi) + h(\psi, \mathbb{1}_\Omega \tau \varphi) - h(\mathbb{1}_\Omega \tau, \psi \varphi) \right) = \int \mathbb{1}_\Omega \tau \Gamma(\psi, \varphi)
\]

Hence \( \Gamma(\psi, \mathbb{1}_\Omega \varphi) = \mathbb{1}_\Omega \Gamma(\psi, \varphi) \). But \( D(h) \cap L_\infty \) is dense in \( D(h) \). Therefore \( \Gamma(\psi, \mathbb{1}_\Omega \varphi) = \mathbb{1}_\Omega \Gamma(\psi, \varphi) \) for all \( \psi, \varphi \in D(h) \).
Now fix $\psi \in C_c^\infty(\mathbf{R}^d)$. Let $\tau \in C_c^\infty(\mathbf{R}^d)$. Then

$$( (Y_\psi)^* \tau, \eta ) = (\tau, Y_\psi \eta ) = (\tau, \Gamma(\psi, \eta))$$

for all $\eta \in C_c^\infty(\mathbf{R}^d)$. Since $C_c^\infty(\mathbf{R}^d)$ is dense in $D(h)$ one deduces that $((Y_\psi)^* \tau, \eta ) = (\tau, \Gamma(\psi, \eta))$ for all $\eta \in D(h)$. Choosing $\eta = 1_{\Omega} \varphi$ it follows that

$$( (Y_\psi)^* \tau, 1_{\Omega} \varphi ) = (\tau, \Gamma(\psi, 1_{\Omega} \varphi)) = (1_{\Omega} \tau, \Gamma(\psi, \varphi)) = (1_{\Omega} \tau, Y_\psi \varphi) = (\tau, 1_{\Omega} Y_\psi \varphi) .$$

Since $C_c^\infty(\mathbf{R}^d)$ is a core for $(Y_\psi)^*$ one deduces that $1_{\Omega} \varphi \in D(Y_\psi)$ and $Y_\psi(1_{\Omega} \varphi) = 1_{\Omega} Y_\psi \varphi$. This conclusion then extends to all $\varphi \in D(Y_\psi)$ by density. Therefore $L_2(\Omega)$ is invariant under $T^\psi$.

The converse implication $\Rightarrow$ consists of two special cases.

Case 1. $C_c^\infty(\mathbf{R}^d)$ is a core for $H$.

Condition $\Rightarrow$ is equivalent to $T^\psi$ invariance of $L_2(\Omega)$ for all $\psi \in C_c^\infty(\mathbf{R}^d)$ by Proposition $\Rightarrow$. Therefore we assume the latter condition.

Let $\psi, \tau \in C_c^\infty(\mathbf{R}^d)$. Then

$$( H \psi, \tau \varphi ) = h(\psi, \tau \varphi) = \int \Gamma(\psi, \tau \varphi) = \int \tau \Gamma(\psi, \varphi) + \varphi \Gamma(\psi, \tau) = (\tau, Y_\psi \varphi) + (\varphi, Y_\psi \tau)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$. Since $C_c^\infty(\mathbf{R}^d)$ is dense in $D(Y_\psi)$ one deduces that

$$( H \psi, \tau \varphi ) = (\tau, Y_\psi \varphi) + (\varphi, Y_\psi \tau)$$

for all $\varphi \in D(Y_\psi)$.

Now let $\psi, \tau, \varphi \in C_c^\infty(\mathbf{R}^d)$. Then by $T^\psi$-invariance of $L_2(\Omega)$ and (5) one deduces that $1_{\Omega} \varphi \in D(Y_\psi)$ and

$$( H \psi, \tau 1_{\Omega} \varphi ) = (\tau, Y_\psi (1_{\Omega} \varphi)) + (1_{\Omega} \varphi, Y_\psi \tau)$$

$$= (1_{\Omega} \tau, Y_\psi \varphi) + (1_{\Omega} \varphi, Y_\psi \tau) = (1_{\Omega} \tau, \Gamma(\psi, \varphi)) + (1_{\Omega} \varphi, \Gamma(\psi, \tau)) .$$

Therefore

$$( H \psi, \tau 1_{\Omega} \varphi ) \leq \| 1_{\Omega} \tau \| \| \Gamma(\psi, \varphi) \|_1 + \| 1_{\Omega} \varphi \| \| \Gamma(\psi, \tau) \|_1 \leq c(\psi)^{1/2} \leq c \| (I + H)^{1/2} \|_2$$

where $c = \| \tau \| \| h(\varphi)^{1/2} \| \| \varphi \| \| h(\tau)^{1/2} \|$. This estimate is uniform for all $\psi \in C_c^\infty(\mathbf{R}^d)$. Since by assumption the space $C_c^\infty(\mathbf{R}^d)$ is a core for $D(H)$ it follows that $1_{\Omega} \tau \varphi \in D(Y_\psi)$ and $Y_\psi(1_{\Omega} \varphi) = 1_{\Omega} Y_\psi \varphi$. This completes the proof of the first case in the proof of $\Rightarrow$.

Case 2. $\partial \Omega$ is (locally) Lipschitz.

Let $P_\Omega$ be the orthogonal projection of $L_2(\mathbf{R}^d)$ onto $L_2(\Omega)$. By assumption $T^\psi$ leaves $L_2(\Omega)$ invariant for all $\psi \in C_c^\infty(\mathbf{R}^d)$. Hence

$$T^\psi_t P_\Omega = P_\Omega T^\psi_t P_\Omega$$

for all $t \in \mathbf{R}$. Let $B$ denote multiplication by the bounded function $\sum_{k,l=1}^d (\partial_k \psi)(\partial_l c_{kl})$ and set $M_t = e^{-tB}$ for $t \in \mathbf{R}$. Clearly each $M_t$ leaves $L_2(\Omega)$ invariant. Therefore $(T^\psi_{-t/n} M_{-t/n})^n$
leaves $L_2(\Omega)$ invariant for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. But $(Y_\psi)^* = -Y_\psi - B$. Then the Trotter product formula establishes that $(T^\psi_t)^*$ is the strong limit of $(T^\psi_{t/n} M_{t/n})^n$ as $n \to \infty$. So $(T^\psi_t)^*$ leaves $L_2(\Omega)$ invariant. Hence $(T^\psi_t)^* P_\Omega = P_\Omega (T^\psi_t)^* P_\Omega$ for all $t \in \mathbb{R}$. Therefore $P_\Omega T^\psi_t = P_\Omega T^\psi_t P_\Omega$ and by (9) it follows that $T^\psi_t P_\Omega = P_\Omega T^\psi_t$ for all $t \in \mathbb{R}$. Then

$$1_\Omega T^\psi_t \varphi = P_\Omega T^\psi_t \varphi = T^\psi_t P_\Omega \varphi = T^\psi_t (1_\Omega \varphi) = (T^\psi_\infty 1_\Omega) (T^\psi_t \varphi)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$ where $T^{\psi,\infty}$ denotes the extension of the flow $T^\psi$ to $L_\infty(\mathbb{R}^d)$ (see Section 2) and we have used (2). Since $T^\psi_t(C_c^\infty(\mathbb{R}^d))$ is dense in $L_2(\mathbb{R}^d)$ one deduces that $T^\psi_{t\infty} 1_\Omega = 1_\Omega$ for all $t \in \mathbb{R}$.

Next let $\varphi \in C^\infty_c(\mathbb{R}^d)$. Then $(Y_\psi)^* \varphi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, so $(Y_\psi^{(\infty)})^* \varphi = (Y_\psi)^* \varphi$, where $Y_\psi^{(\infty)}$ is the generator of $T^{\psi,\infty}$. Since $((T^\psi_t)^*) (\varphi, 1_\Omega) = (\varphi, T^\psi_t 1_\Omega) = (\varphi, 1_\Omega)$ for all $t \in \mathbb{R}$ it follows by differentiation that $((Y_\psi)^* \varphi, 1_\Omega) = 0$. Therefore setting $\Phi_k = \sum_{l=1}^d c_{kl} \partial_l \psi$ for $k \in \{1, \ldots, d\}$ one has

$$\int_\Omega \text{div}(\varphi \Phi) = (\langle (Y_\psi)^* \varphi, 1_\Omega \rangle) = 0 \quad .$$

At this point we use the (local) Lipschitz continuity of $\partial \Omega$.

The Gauss–Green theorem is valid for open sets $\Omega$ with a (locally) Lipschitz boundary (see, for example, [EvG] page 209). It states that

$$\int_\Omega \text{div} \Psi = \int_{\partial \Omega} dS \langle n, \Psi \rangle$$

for all $\Psi \in W^{1,\infty}(\mathbb{R}^d)$ with compact support where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$, $dS$ is the Euclidean measure on $\partial \Omega$ and $n$ is the unit outward normal to $\partial \Omega$. The normal is defined $dS$-almost everywhere. Thus if one sets $\Phi = \varphi \Phi$ with $\varphi \in C^\infty_c(\mathbb{R}^d)$ one has

$$\int_\Omega \text{div}(\varphi \Phi) = \int_{\partial \Omega} dS \varphi \langle n, \Phi \rangle = 0$$

where the last equality uses (7). Since this is valid for all $\varphi \in C^\infty_c(\mathbb{R}^d)$ it follows that $\langle n, \Phi \rangle = 0$ almost everywhere on $\partial \Omega$. Therefore $\langle \langle \nabla \psi \rangle(x), C(x) n_x \rangle = 0$ for almost every $x \in \partial \Omega$. But this is also valid for all $\psi \in C^\infty_c(\mathbb{R}^d)$. Hence one must have $C(x) n_x = 0$ for almost every $x \in \partial \Omega$. This corresponds to the condition of zero flux across the boundary as defined in [RoS1] and then the $S$-invariance of $L_2(\Omega)$ follows from Theorem 1.2 of this reference.

The argument in [RoS1] that zero flux implies invariance is somewhat indirect as it first proves that the capacity of $\partial \Omega$ with respect to $h$ is zero and then uses this to deduce the $S$-invariance of $L_2(\Omega)$. Nevertheless, the same reasoning can be adapted to give a direct proof of the invariance since the proof can be reduced to a local estimate as in [RoS1]. (The latter proof and this proof are an adaption of the argument used to prove Proposition 6.5 in [ERSZ2].)

First, it suffices to prove that if $\varphi \in C^\infty_c(\mathbb{R}^d)$ then $1_\Omega \varphi \in D(h)$. This is a consequence of [BRI1] Proposition 2.1 and locality of $h$. But this is obvious if the support of $\varphi$ and the boundary are disjoint. Therefore it suffices to consider $\varphi$ with support close to the boundary $\partial \Omega$. Then, however, one can use a decomposition of the identity to reduce to the case $\text{supp} \varphi \subset B_y(r)$ with $y \in \partial \Omega$ and $r > 0$ small.
and only if $W$.

If one of the following two conditions is valid then Theorem 4.1

Theorem 4.1 If one of the following two conditions is valid then $C_c^\infty(\mathbb{R}^d)$ is a core for $H$:

I. $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ for all $k, l \in \{1, \ldots, d\}$,

II. the matrix $(c_{kl}(x))$ is invertible for all $x \in \mathbb{R}^d$. 

4 Core properties

In this section we examine conditions which ensure that $C_c^\infty(\mathbb{R}^d)$ is a core for the degenerate elliptic operator $H$ with coefficients $(c_{kl})$ in $W^{1,\infty}$. Obviously $C_c^\infty(\mathbb{R}^d)$ is a core for $H$ if and only if $W^{2,\infty}(\mathbb{R}^d)$ is a core for $H$.

First, we recall two known core criteria.

Secondly, let $\tau, \Psi$ be as in (I). Without loss of generality we may assume that $\Psi(x) = x$ for all $x \in \mathbb{R}^d$. For all $n \in \mathbb{N}$ define $\psi_n : \mathbb{R}^d \to \mathbb{R}$ by $\psi_n(x) = \chi_n(x_1 - \tau(x'))$, where $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $\chi_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi_n(t) = \begin{cases} 0 & \text{if } t \leq 1/n, \\ \log(tn) / \log n & \text{if } 1/n < t < 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

Then $\lim(\psi_n \varphi) = 1_{\Omega} \varphi$ in $L_2(\mathbb{R}^d)$. Thus to establish that $1_{\Omega} \varphi \in D(h)$ it suffices to prove that $\{h(\psi_n \varphi) : n \in \mathbb{N}\}$ is bounded. But

$$h(\psi_n \varphi) \leq 2 h(\varphi) + 2 \int |\varphi|^2 \sum_{k,l=1}^d c_{kl} (\partial_k \psi_n) (\partial_l \psi_n)$$

$$\leq 2 h(\varphi) + 2 (\log n)^{-2} \int_{\mathbb{R}^{d-1}} dx' \int_{\tau(x') + 1/n}^{\tau(x') + 1} dx_1 |\varphi(x)|^2 \frac{\langle \nu_x, C(x) \nu_x \rangle}{(x_1 - \tau(x'))^2}$$

for all $n \in \mathbb{N}$ where $\nu_x = (1, -(\nabla \tau)(x'))$. Since the coefficients $c_{kl}$ are in $W^{1,\infty}(\mathbb{R}^d)$ there exists an $M > 0$ such that $|\langle \xi, C(x) \xi \rangle - \langle \xi, C(z) \xi \rangle| \leq M \|\xi\|^2$ for all $x, z, \xi \in \mathbb{R}^d$. If $x = (x_1, x') \in B_y(r)$, the function $\tau$ is differentiable at $x'$ and $x_1 = \tau(x')$ then

$$\langle \nu_x, C(\tau(x'), x') \nu_x \rangle = (1 + |(\nabla \tau)(x')|^2) \langle n_x, C(\tau(x'), x') n_x \rangle = 0$$

by the zero flux condition. Hence $\langle \nu_x, C(x_1, x') \nu_x \rangle \leq M_1 |x_1 - \tau(x')|$ for all $x = (x_1, x') \in B_y(r)$ with $\tau$ differentiable at $x'$, where $M_1 = M(1 + \|\nabla \tau\|_\infty^2)$. It follows that

$$(\log n)^{-2} \int_{\mathbb{R}^{d-1}} dx' \int_{\tau(x') + 1/n}^{\tau(x') + 1} dx_1 |\varphi(x_1, x')|^2 \frac{\langle \nu_x, C(x_1, x') \nu_x \rangle}{(x_1 - \tau(x'))^2}$$

$$\leq M_1 (\log n)^{-2} \int_{\mathbb{R}^{d-1}} dx' \int_{\tau(x') + 1/n}^{\tau(x') + 1} dx_1 \frac{|\varphi(x_1, x')|^2}{(x_1 - \tau(x'))} \leq M_1 (\log n)^{-1} |\varphi|_\infty^2 |K'|$$

uniformly for all $n \in \mathbb{N}$, where $K' \subset \mathbb{R}^{d-1}$ is a compact set such that $\text{supp} \varphi \subset \mathbb{R} \times K'$. So $\{h(\psi_n \varphi) : n \in \mathbb{N}\}$ is bounded, as required. In fact a slightly more detailed argument establishes that $\lim h(\psi_n \varphi - 1_{\Omega} \varphi) = 0$. 

4 Core properties

In this section we examine conditions which ensure that $C_c^\infty(\mathbb{R}^d)$ is a core for the degenerate elliptic operator $H$ with coefficients $(c_{kl})$ in $W^{1,\infty}$. Obviously $C_c^\infty(\mathbb{R}^d)$ is a core for $H$ if and only if $W^{2,\infty}(\mathbb{R}^d)$ is a core for $H$.

First, we recall two known core criteria.
Proposition 2.3, or by an adaption of the proof of Proposition 2.1. If Condition II is valid then \( C^\infty_c(\mathbb{R}^d) \) is a core by [Rob1] Section 6, or [ER2] Proposition 2.3, or by an adaption of the proof of Proposition 2.1. If Condition II is valid then \( C^\infty_c(\mathbb{R}^d) \) is a core by the arguments in [Dav] Theorem 3.1. Davies requires that the coefficients are smooth, but if the coefficients are bounded the smoothness condition can be relaxed to \( W^{1,\infty} \).

We shall prove a core theorem with a mixture of the two conditions of Theorem [4.1] in Corollary [4.5].

Lemma 4.2 If \( \chi \in W^{2,\infty}(\mathbb{R}^d) \) and \( \varphi \in D(H) \) then \( \chi \varphi \in D(H) \).

Fix \( \chi \in W^{2,\infty}(\mathbb{R}^d) \). Then it follows from Lemma 3.4 in [ERSZ1] that \( \chi \varphi \in D(h) \) and \( h(\chi \varphi)^{1/2} \leq \| \chi \|_\infty h(\varphi)^{1/2} + \| \Gamma(\chi) \|^{1/2}_d \| \varphi \|_2 \) for all \( \varphi \in D(h) \), where we define \( \Gamma(\chi) = \sum_{k,l=1}^d c_{kl} (\partial_k \chi) (\partial_l \chi) \in L_\infty \). If \( \varphi, \psi \in C^\infty_c \) then

\[
h(\psi, \chi \varphi) = h(\chi \psi, \varphi) - \sum_{k,l=1}^d \int \psi \varphi (\partial_k c_{kl} \partial_l \chi) - 2 \sum_{k,l=1}^d \int c_{kl} (\partial_k \varphi) (\partial_l \chi) \psi.
\]

So

\[
|h(\psi, \chi \varphi)| \leq |h(\chi \psi, \varphi)| + a \| \psi \|_2 \| \varphi \|_2 + 2h(\varphi)^{1/2} \| \Gamma(\chi) \|^{1/2}_d \| \psi \|_2,
\]

where \( a = \sum \partial_k c_{kl} \partial_l \chi \|_\infty \). Then by continuity (8) is valid for all \( \psi, \varphi \in D(h) \). Finally, if \( \varphi \in D(H) \) then \( |h(\chi \psi, \varphi)| = |(\chi \psi, H \varphi)| \leq \| H \varphi \|_2 \| \chi \|_\infty \| \psi \|_2 \) for all \( \psi \in D(h) \). Using (8) it follows that there exists a \( c > 0 \) such that \( |h(\psi, \chi \varphi)| \leq c \| \psi \|_2 \) for all \( \psi \in D(h) \). Therefore \( \chi \varphi \in D(H) \).

If \( A \subset \mathbb{R}^d \) with \( A \neq \emptyset \) and \( \delta > 0 \) define the open set \( A_\delta \subset \mathbb{R}^d \) by \( A_\delta = \{ x \in \mathbb{R}^d : d(x, A) < \delta \} \).

Lemma 4.3 Let \( H_1 \) and \( H_2 \) be degenerate elliptic operators with \( W^{1,\infty} \)-coefficients \((c^{(1)}_{kl})\) and \((c^{(2)}_{kl})\) and let \( h^{(1)} \) and \( h^{(2)} \) be the corresponding quadratic forms. Let \( U \subset \mathbb{R}^d \) be an open set and suppose that \( c^{(1)}_{kl} |_{\overline{U}} = c^{(2)}_{kl} |_{\overline{U}} \) for all \( k, l \in \{1, \ldots, d\} \). Let \( \varphi \in L_2(\mathbb{R}^d) \setminus \{0\} \) and suppose that \( (\text{supp } \varphi) \subset U \).

Then \( \varphi \in D(h^{(1)}) \) if and only if \( \varphi \in D(h^{(2)}) \) and \( h^{(1)}(\varphi) = h^{(2)}(\varphi) \). Similarly, \( \varphi \in D(H_1) \) if and only if \( \varphi \in D(H_2) \) and then \( H_1 \varphi = H_2 \varphi \). Moreover, \( \text{supp } H_1 \varphi \subseteq \text{supp } \varphi \).

Proof There exists a \( \chi \in W^{2,\infty}(\mathbb{R}^d) \) such that \( \chi|_{\text{supp } \varphi} = 1 \) and \( \text{supp } \chi \subset U \). Suppose \( \varphi \in D(h^{(1)}) \). Then there exists a sequence \( \varphi_1, \varphi_2, \ldots \in W^{1,2}(\mathbb{R}^d) \) such that \( \lim \varphi_n = \varphi \) in \( D(h^{(1)}) \). Then \( \lim \varphi_n = \varphi \) in \( L_2(\mathbb{R}^d) \). But \( h^{(1)}(\chi \varphi_n) = h^{(2)}(\chi \varphi_n) \) and \( h^{(1)}(\chi \varphi_n - \chi \varphi_m) = h^{(2)}(\chi \varphi_n - \chi \varphi_m) \) for all \( n, m \in \mathbb{N} \). Therefore \( \chi \varphi_1, \chi \varphi_2 \) is a Cauchy sequence in \( D(h^{(2)}) \). Since \( \lim \chi \varphi_n = \varphi \) in \( L_2(\mathbb{R}^d) \) one deduces that \( \varphi \in D(h^{(2)}) \) and \( h^{(2)}(\varphi) = h^{(1)}(\varphi) \).

Finally suppose that \( \varphi \in D(H_1) \). If \( \psi \in C^\infty_c(\mathbb{R}^d) \) with \( \text{supp } \psi \subset (\text{supp } \varphi)^c \) then \( (H_1 \varphi, \psi) = h^{(1)}(\varphi, \psi) = 0 \) by locality. Therefore \( \text{supp } H_1 \varphi \subseteq \text{supp } \varphi \). Clearly \( \varphi \in D(h^{(1)}) \) and by the first part, also \( \varphi \in D(h^{(2)}) \). Let \( \psi \in D(h^{(2)}) \). Then \( \chi \psi \in D(h^{(2)}) \) and \( \text{supp } \chi \psi \subset U \). Therefore \( \chi \psi \in D(h^{(1)}) \). Then by locality one deduces that \( h^{(2)}(\varphi, \psi) = h^{(2)}(\varphi, \chi \psi) = h^{(2)}(\varphi, (1 - \chi) \psi) = h^{(2)}(\varphi, (1 - \chi) \psi) = h^{(1)}(\varphi, \chi \psi) \). So \( |h^{(2)}(\varphi, \psi)| = |h^{(1)}(\varphi, \chi \psi)| = \| H_1 \varphi, \psi \|_{L_2(\mathbb{R}^d)} \| \chi \|_\infty \| \psi \|_2 \). Therefore \( \varphi \in D(H_2) \). If \( \psi \in C^\infty_c(U) \) then \( (H_1 \varphi, \psi) = (\varphi, H_2 \psi) = (H_2 \varphi, \psi) \). Since \( \text{supp } H_1 \varphi \subseteq U \) and \( \text{supp } H_2 \varphi \subseteq U \) it follows that \( H_1 \varphi = H_2 \varphi \).
Proposition 4.4  Let $A \subset \mathbb{R}^d$, $\delta > 0$, let $H_1$ and $H_2$ be degenerate elliptic operators with $W^{1,\infty}$-coefficients $(c_k^{(1)})$ and $(c_k^{(2)})$. Suppose $\emptyset \neq A \neq \mathbb{R}^d$, $c_k^{(1)}|_{A_\delta} = c_k^{(2)}|_{A^c} = c_k|_{A^c}$ for all $k, l \in \{1, \ldots, d\}$ and $C_c^\infty(\mathbb{R}^d)$ is a core for both $H_1$ and $H_2$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for $H$.

Proof Let $\tau \in C_c^\infty(\mathbb{R}^d)$ be such that $\int \tau = 1$ and $\tau(x) = 0$ for all $x \in \mathbb{R}^d$ with $|x| > \frac{\delta}{4}$. Let $\chi = \tau \ast 1_{A_{3\delta/4}}$. Then $\chi \in W^{2,\infty}(\mathbb{R}^d)$, $\chi|_{A_{\delta/4}} = 1$ and supp $\chi \subset A_{3\delta/4}$. Moreover, supp$(1 - \chi) \subset (A_{\delta/4})^c \subset A^c$. There exist $\chi_1, \chi_2 \in W^{2,\infty}(\mathbb{R}^d)$ such that $\chi_1|_{A_{3\delta/4}} = 1$, supp $\chi_1 \subset A_{\delta}$, $\chi_2|_{A^c} = 1$ and supp $\chi_2 \subset (A^c)^\delta$.

Let $\varphi \in D(H)$. It follows from Lemma 4.2 that $\chi \varphi \in D(H)$ and $(1 - \chi)\varphi \in D(H)$. We shall show that we can approximate both elements by $C_c^\infty$-functions. We may assume that $\chi \varphi \neq 0$ and $(1 - \chi)\varphi$. Since supp$(\chi \varphi) \subset A_{3\delta/4}$ one deduces from Lemma 4.3 that $\chi \varphi \in D(H_1)$ and $H_1(\chi \varphi) = H(\chi \varphi)$. By assumption there exist $\varphi_1, \varphi_2, \ldots \in C_c^\infty(\mathbb{R}^d)$ such that lim $\varphi_n = \chi \varphi$ in $D(H_1)$. Then lim $\chi_1 \varphi_n = \chi_1 \chi \varphi = \chi \varphi$ in $D(H_1)$ by Lemma 4.2. But $\chi_1 \varphi_n \in C_c^\infty(\mathbb{R}^d)$ and supp $\chi_1 \varphi_n \subset A_{\delta}$ for all $n \in \mathbb{N}$. Therefore $\chi_1 \varphi_n \in D(H)$ and $H_1(\chi_1 \varphi_n) = H(\chi_1 \varphi_n)$, again by Lemma 4.3. So lim $\chi_1 \varphi_n = \chi \varphi$ in $D(H)$. Similarly, using $H_2$ and $\chi_2$ there exists a sequence $\psi_1, \psi_2, \ldots \in C_c^\infty(\mathbb{R}^d)$ such that lim $\chi_2 \psi_n = (1 - \chi)\varphi$ in $D(H)$. Then lim $\chi_1 \varphi_n + \chi_2 \psi_n = \varphi$ in $D(H)$. Since $\chi_1 \varphi_n + \chi_2 \psi_n \in C_c^\infty(\mathbb{R}^d)$ the proposition follows. 

Corollary 4.5  Suppose there exist a set $A$ and $\delta > 0$ such that $\emptyset \neq A \neq \mathbb{R}^d$, the matrix $(c_k(x))$ is invertible for all $x \in (A^c)^\delta$ and $c_k|_{A_\delta} \in W^{2,\infty}(A_\delta)$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for $H$.

Proof There exists a $\chi_1 \in W^{2,\infty}(\mathbb{R}^d)$ such that $\chi_1|_{A_{\delta/2}} = 1$ and supp $\chi_1 \subset A_{\delta}$. Define $c_k^{(1)} = \chi_1 c_k \in W^{2,\infty}(\mathbb{R}^d)$. Then $c_k^{(1)}|_{A_{\delta/2}} = c_k|_{A_{\delta/2}}$.

There exists a $\chi_2 \in W^{1,\infty}(\mathbb{R}^d)$ such that $\chi_2|_{(A^c)^{\delta/2}} = 1$ and supp $\chi_2 \subset (A^c)^\delta$. Define $c_k^{(2)} = \chi_2 c_k + (1 - \chi_2)\delta_k \in W^{1,\infty}(\mathbb{R}^d)$. Let $H_1$ and $H_2$ be the degenerate elliptic operator with coefficients $(c_k^{(1)})$ and $(c_k^{(2)})$. Now apply Theorem 4.11 to $H_1$, Theorem 4.11 to $H_2$ and use Proposition 4.4.

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