MONADS AND COMONADS IN MODULE CATEGORIES

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Abstract. Let $A$ be a ring and $\mathcal{M}_A$ the category of $A$-modules. It is well known in module theory that for any $A$-bimodule $B$, $B$ is an $A$-ring if and only if the functor $- \otimes_A B : \mathcal{M}_A \to \mathcal{M}_A$ is a monad (or triple). Similarly, an $A$-bimodule $C$ is an $A$-coring provided the functor $- \otimes_A C : \mathcal{M}_A \to \mathcal{M}_A$ is a comonad (or cotriple). The related categories of modules (or algebras) of $- \otimes_A B$ and comodules (or coalgebras) of $- \otimes_A C$ are well studied in the literature. On the other hand, the right adjoint endofunctors $\text{Hom}_A(B, -)$ and $\text{Hom}_A(C, -)$ are a comonad and a monad, respectively, but the corresponding (co)module categories did not find much attention so far. The category of $\text{Hom}_A(B, -)$-comodules is isomorphic to the category of $B$-modules, while the category of $\text{Hom}_A(C, -)$-modules (called $C$-contramodules by Eilenberg and Moore) need not be equivalent to the category of $C$-comodules.

The purpose of this paper is to investigate these categories and their relationships based on some observations of the categorical background. This leads to a deeper understanding and characterisations of algebraic structures such as corings, bialgebras and Hopf algebras. For example, it turns out that the categories of $C$-comodules and $\text{Hom}_A(C, -)$-modules are equivalent provided $C$ is a coseparable coring. Furthermore, a bialgebra $H$ over a commutative ring $R$ is a Hopf algebra if and only if $\text{Hom}_R(H, -)$ is a Hopf bimonad on $\mathcal{M}_R$ and in this case the categories of $H$-Hopf modules and mixed $\text{Hom}_R(H, -)$-bimodules are both equivalent to $\mathcal{M}_R$.

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1. Introduction

The purpose of this paper is to present a categorical framework for studying problems in the theories of rings and modules, corings and comodules, bialgebras and (mixed) bimodules and Hopf algebras and Hopf modules. The usefulness of this
framework is illustrated by analysing the structure of the category of contramodules and the bearing of this structure on the properties of corings and bialgebras.

It is well-known that for a right module \( V \) over an \( R \)-algebra \( A \), the dual \( R \)-module \( V^* = \text{Hom}_R(V, R) \) is a left module over \( A \). It is equally well-known that for a right comodule \( V \) of an \( R \)-coalgebra \( C \), in general \( V^* \) is not a \( C \)-comodule (left or right). It has already been realised in [11, Chapter IV.5] that to a coalgebra \( C \) two (different) representation categories can be associated: the familiar category of \( C \)-comodules and the category of \( C \)-contramodules introduced therein. If \( V \) is a \( C \)-comodule, then \( V^* \) is a \( C \)-contramodule.

While comodules of coalgebras (and corings) have been intensively studied, contramodules seem to have been rather neglected. Yet the category of contramodules is as fundamental as that of comodules, and both categories are complementary to each other. To substantiate this claim, one needs to resort to the categorical point of view on corings. An \( A \)-coring can be defined as an \( A \)-bimodule \( C \) such that the tensor endofunctor \(- \otimes_A C\) on the category of right \( A \)-modules \( \mathbb{M}_A \) is a comonad or a cotriple. Right \( C \)-comodules are the same as comodules (or coalgebras in category theory terminology) of the comonad \(- \otimes_A C\). On the other hand, the tensor functor \(- \otimes_A C\) has a right adjoint, the Hom-functor \( \text{Hom}_A(C, -) \). By purely categorical arguments (see Eilenberg and Moore [12, Proposition 3.1]), the functor \(- \otimes_A C\) is a comonad if and only if its right adjoint \( \text{Hom}_A(C, -) \) is a monad. Thus, \( C \) is an \( A \)-coring if and only if \( \text{Hom}_A(C, -) \) is a monad on \( \mathbb{M}_A \); right \( C \)-contramodules are simply modules (or algebras in category theory terminology) of this monad. This categorical interpretation explains the way in which contramodules complement comodules. For example, since \( C \)-comodules are comodules of a comonad on an abelian category, their category has cokernels but not necessarily kernels. On the other hand, since \( C \)-contramodules are modules of a monad, their category has kernels but not necessarily cokernels. Thus one category provides the structure which the other one misses.

Again purely categorical considerations (see [12]) explain that, while there are two categories of representations of a coring, there is only one category of representations of a ring – the familiar category of modules. More precisely, a ring morphism \( A \to B \) can be equivalently described as the monad structure of the tensor functor \(- \otimes_A B\) on \( \mathbb{M}_A \) associated to an \( A \)-bimodule \( B \). With this interpretation, right \( B \)-modules are simply modules of the monad \(- \otimes_A B\). The right adjoint functor \( \text{Hom}_A(B, -) \) is a comonad on \( \mathbb{M}_A \) and the category of comodules of \( \text{Hom}_A(B, -) \) is isomorphic to the category of modules of the monad \(- \otimes_A B\). Consequently, there is only one type of representation categories for rings – the category of right (or left) modules over a ring. Since modules of a ring are thus both algebras and coalgebras of respective monads and comonads, the category of modules inherits both kernels and cokernels from the category of abelian groups.

The above comments illustrate how the categorical point of view can give significant insight into algebraic structures. There are many constructions developed in category theory that are directly applicable to ring theoretic situations but they seem not to be sufficiently explored. Contramodules of a coring are a good example of this. On one hand, from the category point of view, they are as natural as comodules, on the other hand, their structure was not analysed properly until very recently, when their important role in semi-infinite homology was outlined by Positselski [25]. The main motivation of our paper is a study of contramodules of corings. This aim is achieved
by placing it in a broader context: we revisit category theory, more specifically the theory of adjoint comonad-monad pairs, in the context of rings and modules.

We begin by summarising the categorical framework, and then apply it first to rings in module categories, next to corings. In the latter case, we concentrate on properties of the less-known category of contramodules, and derive consequences of the categorical formulation in this context. We analyse functors between categories of comodules and contramodules, and study equivalences between such categories involving a Galois theory for bicomodules. We also derive the characterisation of entwining structures as liftings of Hom-functors to module and contramodule categories.

Finally, we study contramodules of corings associated to bialgebras and provide new extensions of the Fundamental Theorem of Hopf algebras (see 8.11). First we observe that an $R$-module $B$ over a commutative ring $R$ is a bialgebra if and only if $\text{Hom}_R(B, -)$ is a bimonad, that is, a monad and a comonad on $\mathbb{M}_R$ satisfying some compatibility conditions (see 8.10). The Fundamental Theorem says that a bialgebra $B$ is a Hopf algebra if and only if $- \otimes_R B$ induces an equivalence between $\mathbb{M}_R$ and the category $\mathbb{M}_B^R$ of Hopf modules. This can also be formulated as $B$ being a Galois comodule of associated corings. Here we add that a Hopf algebra $B$ is characterised by a bimonad $\text{Hom}_R(B, -)$ inducing an equivalence between $\mathbb{M}_R$ and the category $\mathbb{M}_{[B, -]}^R$ of certain $\text{Hom}_R(B, -)$-bimodules (Hopf contramodules). Again this can be seen as $B$ being a Galois comodule with respect to the Hom-functors of the associated corings.

2. Categorical framework

Our main concern is to apply abstract categorical notions to special situations in module categories. We begin by recalling some basic definitions and properties (e.g. from [12]) to fix notation, and then develop a categorical framework which is later applied to categories of (co)modules.

Throughout, the composition of functors is denoted by juxtaposition, and the usual composition symbol $\circ$ is reserved for natural transformations and morphisms. Given functors $F, G$ and a natural transformation $\varphi$, $F \varphi G$ denotes the natural transformation, which, evaluated at an object $X$ gives a morphism obtained by applying $F$ to a morphism provided by the natural transformation $\varphi$ evaluated at the object $GX$.

By $\mathbb{A} \cong \mathbb{B}$ we denote equivalences between categories and $\mathbb{A} \cong \mathbb{B}$ is written for their isomorphisms. The symbol $\cong$ is also used to denote isomorphisms between objects in any category, in particular isomorphisms of modules and (natural) isomorphisms of functors.

2.1. Adjoint functors. A pair $(L, R)$ of functors $L: \mathbb{A} \rightarrow \mathbb{B}$ and $R: \mathbb{B} \rightarrow \mathbb{A}$ between categories $\mathbb{A}, \mathbb{B}$ is called an adjoint pair if there is a natural isomorphism

$$\text{Mor}_\mathbb{B}(L(-), -) \cong \text{Mor}_\mathbb{A}(-, R(-)).$$

This can be described by natural transformations \textit{unit} $\eta: I_\mathbb{A} \rightarrow RL$ and \textit{counit} $\varepsilon: LR \rightarrow I_\mathbb{B}$ satisfying the \textit{triangular identities} $\varepsilon L \circ L \eta = I_L$ and $R \varepsilon \circ \eta R = I_R$.

Recall the properties of an adjoint pair $(L, R)$:

(1) $L$ is full and faithful if and only if $\varepsilon: LR \rightarrow I_\mathbb{B}$ is an isomorphism.

(2) $L$ is full and faithful if and only if $\eta: I_\mathbb{A} \rightarrow RL$ is an isomorphism.
(3) \( L \) is an equivalence if and only if \( \varepsilon \) and \( \eta \) are isomorphisms.

2.2. **Natural transformations for adjoints.** For two adjunctions \((L, R)\) and \((\tilde{L}, \tilde{R})\) between \( \mathbb{A} \) and \( \mathbb{B} \), with respective units \( \eta, \tilde{\eta} \) and counits \( \varepsilon, \tilde{\varepsilon} \), there is an isomorphism between the natural transformations (cf. [18], [22])

\[
Nat(L, \tilde{L}) \rightarrow Nat(\tilde{R}, R), \quad f \mapsto \tilde{f} := R\tilde{\varepsilon} \circ Rf \circ \tilde{R} \circ \eta \tilde{R},
\]

with the inverse map

\[
Nat(\tilde{R}, R) \rightarrow Nat(L, \tilde{L}), \quad \tilde{f} \mapsto f := \varepsilon \tilde{L} \circ L\tilde{f} \circ L\eta.
\]

We say that \( f \) and \( \tilde{f} \) are mates under the adjunctions \((L, R)\) and \((\tilde{L}, \tilde{R})\). For natural transformations \( f : L_1 \rightarrow L_2 \) and \( g : L_2 \rightarrow L_3 \) between left adjoint functors, naturality and the triangle identities imply \( g \circ f = \tilde{f} \circ \tilde{g} \). In particular, \( f \) is a natural isomorphism if and only if its mate \( \tilde{f} \) is a natural isomorphism. Moreover, if for an adjunction \((L, R)\), the composites \( LL_1 \) (and hence \( LL_2 \)) are meaningful, then \( L\tilde{f} = fR \). Similarly, if the composites \( L_1L \) (and thus \( L_2L \)) are meaningful then \( Lf = \tilde{f}R \).

2.3. **Monads on \( \mathbb{A} \).** A monad on the category \( \mathbb{A} \) is a triple \( F = (F, m, i) \), where \( F : \mathbb{A} \rightarrow \mathbb{A} \) is a functor with natural transformations \( m : FF \rightarrow F \) and \( i : I_\mathbb{A} \rightarrow F \) satisfying associativity and unitality conditions. A morphism of monads \((F, m, i) \rightarrow (F', m', i')\) is a natural transformation \( \varphi : F \rightarrow F' \) such that \( m' \circ \varphi F' \circ F\varphi = \varphi \circ m \) and \( \varphi \circ i = i' \).

An \( F \)-module is a pair consisting of \( A \in \text{Obj}(\mathbb{A}) \) and a morphism \( g_A : FA \rightarrow A \) such that the following diagrams

\[
\begin{array}{ccc}
FFA & \xrightarrow{mA} & FA \\
\downarrow^F \varphi_A & & \downarrow^\varphi_A \\
FA & \xrightarrow{g_A} & A
\end{array}
\quad \quad
\begin{array}{ccc}
A & \xrightarrow{iA} & FA \\
\downarrow^I_A & & \downarrow^\varphi_A \\
A & & A
\end{array}
\]

are commutative.

Morphisms between \( F \)-modules \( f : A \rightarrow A' \) are morphisms in \( \mathbb{A} \) with \( g_{A'} \circ Ff = f \circ g_A \) and the Eilenberg-Moore category of \( F \)-modules is denoted by \( \mathbb{A}_F \).

For any object \( A \) of \( \mathbb{A} \), \( FA \) is an \( F \)-module and this yields the free functor

\[
\phi_F : \mathbb{A} \rightarrow \mathbb{A}_F, \quad A \mapsto (FA, mA),
\]

which is left adjoint to the forgetful functor \( U_F : \mathbb{A}_F \rightarrow \mathbb{A} \) by the isomorphism

\[
\text{Mor}_{\mathbb{A}_F}(\phi_F A, B) \rightarrow \text{Mor}_{\mathbb{A}}(A, U_F B), \quad f \mapsto f \circ iA.
\]

The full subcategory of \( \mathbb{A}_F \) consisting of all free \( F \)-modules (i.e. the full subcategory of \( \mathbb{A}_F \) generated by the image of \( \phi_F \)) is called the Kleisli category of \( F \) and is denoted by \( \hat{\mathbb{A}}_F \).

2.4. **Comonads on \( \mathbb{A} \).** A comonad on \( \mathbb{A} \) is a triple \( G = (G, d, e) \), where \( G : \mathbb{A} \rightarrow \mathbb{A} \) is a functor with natural transformations \( d : G \rightarrow GG \) and \( e : G \rightarrow I_\mathbb{A} \) satisfying coassociativity and counitality conditions. A morphism of comonads is a natural transformation that is compatible with the coproduct and counit.

A \( G \)-comodule is an object \( A \in \mathbb{A} \) with a morphism \( g^A : A \rightarrow GA \) compatible with \( d \) and \( e \). Morphisms between \( G \)-comodules \( g : A \rightarrow A' \) are morphisms in \( \mathbb{A} \) with \( g^{A'} \circ g = Gg \circ g^A \) and the Eilenberg-Moore category of \( G \)-comodules is denoted by \( \mathbb{A}^G \).
For any $A \in \mathbb{A}$, $GA$ is a $G$-comodule yielding the (co)free functor
\[ \phi^G : \mathbb{A} \rightarrow \mathbb{A}^G, \ A \mapsto (GA, dA) \]
which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \rightarrow \mathbb{A}$ by the isomorphism
\[ \text{Mor}_{\mathbb{A}^G}(B, \phi^G A) \rightarrow \text{Mor}_{\mathbb{A}}(U^G B, A), \ f \mapsto eA \circ f. \]
The full subcategory of $\mathbb{A}^G$ consisting of all (co)free $G$-comodules (i.e. the full subcategory of $\mathbb{A}^G$ generated by the image of $\phi^G$) is called the Kleisli category of $G$ and is denoted by $\tilde{\mathbb{A}}^G$.

2.5. (Co)monads related to adjoints. Let $L : \mathbb{A} \rightarrow \mathbb{B}$ and $R : \mathbb{B} \rightarrow \mathbb{A}$ be an adjoint pair of functors with unit $\eta : I_{\mathbb{A}} \rightarrow RL$ and counit $\varepsilon : LR \rightarrow I_{\mathbb{B}}$. Then
\[ \text{F} = (RL, RLRL \xrightarrow{RL} RL, I_{\mathbb{A}} \xrightarrow{\eta} RL) \]
is a monad on $\mathbb{A}$. Similarly a comonad on $\mathbb{B}$ is defined by
\[ \text{G} = (LR, LR \xrightarrow{LR} LRLR, LR \xrightarrow{\varepsilon} I_{\mathbb{B}}). \]

As already observed by Eilenberg and Moore in [12], the monad structure of an endofunctor induces a comonad structure on any adjoint endofunctor.

2.6. Adjoints of monads and comonads. Let $L : \mathbb{A} \rightarrow \mathbb{A}$ and $R : \mathbb{A} \rightarrow \mathbb{A}$ be an adjoint pair of functors with unit $\eta : I_{\mathbb{A}} \rightarrow RL$ and counit $\varepsilon : LR \rightarrow I_{\mathbb{A}}$.

(1) The following are equivalent:
(a) $L$ is a monad;
(b) $R$ is a comonad.

In this case the Eilenberg-Moore categories $\mathbb{A}_L$ and $\mathbb{A}_R$ are isomorphic to each other.

(2) The following are equivalent:
(a) $L$ is a comonad;
(b) $R$ is a monad.

In this case the Kleisli categories $\tilde{\mathbb{A}}^L$ and $\tilde{\mathbb{A}}^R$ are isomorphic to each other.

Proof. (1) This is shown in [12, Proposition 3.1] and here it follows from 2.2 (see also [22]).

The isomorphism of $\mathbb{A}_L$ and $\mathbb{A}_R$ is mentioned in [29, p. 3935]. For convenience we explain the relevant functor leaving objects and morphisms unchanged but turning $L$-module structure maps to $R$-comodule structure maps and vice versa. An $L$-module $\varrho_A : LA \rightarrow A$ induces a morphism
\[ A \xrightarrow{\eta A} RLA \xrightarrow{R\varrho_A} RA, \]
making $A$ an $R$-comodule. Similarly, a comodule $\varrho^A : A \rightarrow RA$ induces
\[ LA \xrightarrow{L\varrho^A} LRA \xrightarrow{\varepsilon^A} A, \]
defining an $L$-module structure on $A$. In a word, $\varrho_A$ and $\varrho^A$ are mates under the adjunctions $(L, R)$ and $(I_{\mathbb{A}}, I_{\mathbb{A}})$. 

An $L$-module morphism $f : A' \to A$ yields a commutative diagram

\[
\begin{array}{ccc}
LA' & \xrightarrow{Lf} & LA \\
\downarrow{\varepsilon A'} & & \downarrow{\varepsilon A} \\
A' & \xrightarrow{f} & A,
\end{array}
\]

from which we obtain the commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{f} & A \\
\downarrow{\eta A'} & & \downarrow{\eta A} \\
RLA' & \xrightarrow{RLf} & RLA \\
\downarrow{R\varepsilon A'} & & \downarrow{R\varepsilon A} \\
RA' & \xrightarrow{Rf} & RA.
\end{array}
\]

Commutativity of the outer rectangle shows that $f$ is also an $R$-comodule morphism. Similarly one can prove that $R$-comodule morphisms are also $L$-module morphisms.

(2) The equivalence of (a) and (b) is proved similarly to (1).

The isomorphism of the Kleisli categories $\tilde{A}_L$ and $\tilde{A}_R$ was observed in [19] and is also mentioned in [29, p. 3935]. It is provided by the canonical isomorphisms for $A, A' \in A$,

\[
\text{Mor}_{\tilde{A}_L}(\phi^L A, \phi^L A') \cong \text{Mor}_A(LA, A') \\
\cong \text{Mor}_A(A, RA') \cong \text{Mor}_{\tilde{A}_R}(\phi^R A, \phi^R A').
\]

\[
\square
\]

2.7. **Relative projectivity and injectivity.** An object $A$ of a category $\mathsf{A}$ is said to be *projective relative to a functor* $F : \mathsf{A} \to \mathsf{B}$ (or *$F$-projective* in short) if $\text{Hom}_\mathsf{A}(A, f) : \text{Hom}_\mathsf{A}(A, X) \to \text{Hom}_\mathsf{A}(A, Y)$ is surjective for all those morphisms $f$ in $\mathsf{A}$, for which $Ff$ is a split epimorphism in $\mathsf{B}$. Dually, $A \in \mathsf{A}$ is said to be *injective relative to $F$* (or *$F$-injective*) if $\text{Hom}_\mathsf{A}(f, A) : \text{Hom}_\mathsf{A}(Y, A) \to \text{Hom}_\mathsf{A}(X, A)$ is surjective for all such morphisms $f$ in $\mathsf{A}$, for which $Ff$ is a split monomorphism in $\mathsf{B}$.

For an adjunction $(L : \mathsf{A} \to \mathsf{B}, R : \mathsf{B} \to \mathsf{A})$, with unit $\eta$ and counit $\varepsilon$, an object $A \in \mathsf{A}$ is $L$-injective if and only if $\eta A$ is a split monomorphism in $\mathsf{A}$. Dually, $B \in \mathsf{B}$ is $R$-projective if and only if $\varepsilon B$ is a split epimorphism in $\mathsf{B}$.

Recall (e.g. from [5, Section 6.5] or [21, A.1, p. 62]) that the *Cauchy completion*, also called *Karoubian closure*, of any category $\mathsf{A}$ is the smallest category $\overline{\mathsf{A}}$ that contains $\mathsf{A}$ as a subcategory and in which idempotent morphisms split (i.e. can be written as a composite of an epimorphism and its section). The Karoubian closure is unique up to equivalence and can be constructed as follows. Objects of $\overline{\mathsf{A}}$ are pairs $(A, a)$, where $A$ is an object in $\mathsf{A}$ and $a : A \to A$ is an idempotent morphism (i.e. $a \circ a = a$). Morphisms $(A, a) \to (A', a')$ are morphisms $f : A \to A'$ in $\mathsf{A}$, such that $a' \circ f = f \circ a$. If $F : \mathsf{A} \to \mathsf{B}$ is an isomorphism, then an isomorphism $\overline{\mathsf{A}} \to \overline{\mathsf{B}}$ is given by the object map $(A, a) \mapsto (FA, Fa)$ and the morphism map $f \mapsto Ff$. 
2.8. Equivalence of Karoubian closures. Let \( \mathbb{A} \) be a category in which idempotent morphisms split. Then for any comonad \((L, d, e)\) on \( \mathbb{A} \) with right adjoint monad \((R, m, i)\), there is an equivalence

\[
E : \mathbb{A}^L_{\text{inj}} \to \mathbb{A}^R_{\text{proj}},
\]

where \( \mathbb{A}^L_{\text{inj}} \) denotes the full subcategory of \( \mathbb{A}^L \) whose objects are injective relative to the forgetful functor \( U_L : \mathbb{A}^L \to \mathbb{A} \) and \( \mathbb{A}^R_{\text{proj}} \) denotes the full subcategory of \( \mathbb{A}^R \) whose objects are projective relative to the forgetful functor \( U_R : \mathbb{A}^R \to \mathbb{A} \).

Explicitly, for \((A, \varrho^A) \in \mathbb{A}^L_{\text{inj}}\), the object \( E(A, \varrho^A) \) is given by the equaliser of the parallel morphisms \( R\varrho^A \) and \( \omega := mL \eta A \circ R\eta A : RA \to RLA \), where \( \eta \) is the unit of the adjunction \((L, R)\).

Proof. By 2.6(2), the Kleisli categories \( \tilde{\mathbb{A}}^L \) and \( \tilde{\mathbb{A}}^R \) are isomorphic. As recalled above, this isomorphism extends to their Karoubian closures. The Karoubian closure of \( \tilde{\mathbb{A}}^R \) is equivalent to the full subcategory of \( U_R \)-projective objects of \( \mathbb{A}^R \) (see [16], [27, Theorem 2.5]). Dually, the Karoubian closure of \( \tilde{\mathbb{A}}^L \) is equivalent to the full subcategory of \( U_L \)-injective objects of \( \mathbb{A}^L \). This proves the equivalence \( \mathbb{A}^L_{\text{inj}} \simeq \mathbb{A}^R_{\text{proj}} \).

The explicit form of the equivalence functor is obtained by computing the composite of the isomorphism between the Karoubian closures of the Kleisli categories with the equivalences in [27, Theorem 2.5]. This (straightforward) computation yields the equaliser \( E(A, \varrho^A) \to RA \) of the identity morphism \( I_{RA} \) and the idempotent morphism \( R\nu^A \circ \omega : RA \to RA \), where \( \nu^A \) is a retraction of \( \eta(A, \varrho^A) = \varrho^A \) in \( \mathbb{A}^L \). This equaliser exists by the assumption that idempotents split in \( \mathbb{A} \). Since

\[
\omega \circ R\nu^A \circ \omega = R\varrho^A \circ R\nu^A \circ \omega \quad \text{and} \quad R\nu^A \circ R\varrho^A = I_{RA};
\]

\( E(A, \varrho^A) \to RA \) is also an equaliser of \( \omega \) and \( R\varrho^A \). \( \square \)

Recall from [24], [26] that a functor \( F : \mathbb{B} \to \mathbb{A} \) is said to be separable if and only if the transformation \( \text{Mor}_{\mathbb{B}}(-, -) \to \text{Mor}_{\mathbb{A}}(F(-), F(-)), f \mapsto Ff \), is a split natural monomorphism. Separable functors reflect split epimorphisms and split monomorphisms.

Questions related to 2.9(1) are also discussed in [6, Proposition 6.3].

2.9. Separable monads and comonads. Let \( \mathbb{A} \) be a category.

(1) For a monad \((R, m, i)\) on \( \mathbb{A} \), the following are equivalent:

(a) \( m \) has a natural section \( \hat{m} \) such that

\[
Rm \circ \hat{m} R = \hat{m} \circ m = m R \circ R \hat{m};
\]

(b) the forgetful functor \( U_R : \mathbb{A}^R \to \mathbb{A} \) is separable.

(2) For a comonad \((L, d, e)\) on \( \mathbb{A} \), the following are equivalent:

(a) \( d \) has a natural retraction \( \hat{d} \) such that

\[
\hat{d}L \circ Ld = d \circ \hat{d} = Ld \circ dL;
\]

(b) the forgetful functor \( U^L : \mathbb{A}^L \to \mathbb{A} \) is separable.
Proof. (1) By Rafael’s theorem [26, Theorem 1.2], $U_R$ is separable if and only if the counit $\varepsilon_R$ of the adjunction $(\phi_R, U_R)$ (see [23]) is a split natural epimorphism.

(1) (a)⇒(b). A section $\nu : I_{\hat{A}_R} \to \phi_R U_R$ of $\varepsilon_R$ is given by a morphism

$$\nu(X, \varrho_X) : X \xrightarrow{iX} RX \xrightarrow{\hat{m}X} RRX \xrightarrow{R\varrho_X} RX,$$

for any $(X, \varrho_X)$ in $\hat{A}_R$. By naturality and the properties in 2.9(2)(a). Thus the claim follows by 2.9.

(2) The proof is symmetric to (1). □

2.10. Separability of adjoints. Let $L : \hat{A} \to \hat{A}$ and $R : \hat{A} \to \hat{A}$ be an adjoint pair of functors with unit $\eta : I_{\hat{A}} \to RL$ and counit $\varepsilon : LR \to I_{\hat{A}}$.

If $(L, d, e)$ is a comonad with corresponding monad $(R, m, i)$, then there are pairs of adjoint (free and forgetful) functors (see [23, 2.4]):

$$\begin{align*}
\hat{A} & \xrightarrow{\phi_R} \hat{A}_R, & \hat{A}_R & \xrightarrow{UR} \hat{A}, & \text{with unit } \eta_R \text{ and counit } \varepsilon_R, \text{ and} \\
\hat{A} & \xrightarrow{UL} \hat{A}_L, & \hat{A} & \xrightarrow{\phi^L} \hat{A}_L, & \text{with unit } \eta^L \text{ and counit } \varepsilon^L.
\end{align*}$$

(1) $\phi^L$ is separable if and only if $\phi_R$ is separable.

(2) $U^L$ is separable if and only if $U_R$ is separable.

If the properties in part (2) hold, then any object of $\hat{A}_R$ is injective relative to $U^L$ and every object of $\hat{A}_R$ is projective relative to $U_R$.

Proof. (1) By Rafael’s theorem [26, Theorem 1.2], $\phi^L$ is separable if and only if $\varepsilon^L = e$ is a split natural epimorphism, while $\phi_R$ is separable if and only if $\eta_R = i$ is a split natural monomorphism. By construction, $i$ and $e$ are mates under the adjunction $(L, R)$ and the trivial adjunction $(I_{\hat{A}}, I_{\hat{A}})$. That is, $e = \varepsilon \circ L i$ equivalently, $i = R e \circ \eta$. Hence a natural transformation $\hat{\gamma} : R \to I_{\hat{A}}$ is a retraction of $i$ if and only if its mate $\hat{\gamma} := \hat{\gamma} L \circ \eta$ under the adjunctions $(I_{\hat{A}}, I_{\hat{A}})$ and $(L, R)$ is a section of $e$.

(2) Since $d$ and $m$ are mates under the adjunctions $(L, R)$ and $(LL, RR)$, a natural transformation $\hat{m}$ satisfies the properties in [23, 1)(a) if and only if its mate $\hat{d}$ satisfies the properties in [23, 2)(a). Thus the claim follows by [23].

It remains to prove the final claims. Following [27] an $L$-comodule $(A, \varrho^A)$ is $U^L$-injective if and only if $\eta^L(A, \varrho^A) = \varrho^A$ is a split monomorphism in $\hat{A}_L$. Since $\varrho^A$ is split in $\hat{A}$ (by $eA$) and $U^L$, being separable, reflects split monomorphisms, any $(A, \varrho^A) \in \hat{A}_L$ is $U^L$-injective. $U_R$-projectivity of every object of $\hat{A}_R$ is proven by a symmetrical reasoning. □
2.11. Lifting of functors. Let \( F : \mathcal{A} \to \mathcal{A}, G : \mathcal{B} \to \mathcal{B} \) and \( T : \mathcal{A} \to \mathcal{B} \) be functors on the categories \( \mathcal{A}, \mathcal{B} \). If \( F, G \) are monads or comonads, we may consider the diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{T} & B \\
\downarrow{U_F} & & \downarrow{U_G} \\
\mathcal{A}' & \xrightarrow{T} & \mathcal{B},
\end{array}
\begin{array}{ccc}
A & \xrightarrow{T} & B \\
\downarrow{U_F} & & \downarrow{U_G} \\
\mathcal{A}' & \xrightarrow{T} & \mathcal{B},
\end{array}
\]

where the \( U \)'s denote the forgetful functors. If there exist \( T \) or \( \hat{T} \) making the corresponding diagram commutative, they are called liftings of \( T \). Their existence as well as their properties depend on (the existence of) natural transformations between combinations of the functors involved. These are called distributive laws.

The following theorem is proved by Johnstone in [17, Lemma 1 and Theorem 2] (see also [2, Section 3.3], [34, 3.3]).

2.12. Lifting for monads. Let \( F = (F, m, i) \) and \( G = (G, m', i') \) be monads on the categories \( \mathcal{A} \) and \( \mathcal{B} \), respectively, and let \( T : \mathcal{A} \to \mathcal{B} \) be a functor.

(1) The liftings \( \hat{T} : \mathcal{A}_F \to \mathcal{B}_G \) of \( T \) are in bijective correspondence with the natural transformations \( \lambda : GT \to TF \) inducing commutative diagrams

\[
\begin{array}{ccc}
GGT & \xrightarrow{m'T} & GT \\
\downarrow{G\lambda} & & \downarrow{\lambda} \\
GTF & \xrightarrow{\lambda F} & TF
\end{array},
\begin{array}{ccc}
G\lambda \\
\downarrow{G\lambda} & & \downarrow{\lambda} \\
GTF & \xrightarrow{\lambda F} & TF
\end{array}
\]

Such a pair \((T, \lambda)\) is called a monad morphism in [20].

(2) If \( T \) has a left adjoint and \( \mathcal{A} \) has coequalisers of reflexive \( TU_F \)-contractible coequaliser pairs, then any lifting \( \hat{T} \) has a left adjoint.

For endofunctors the preceding diagrams simplify and we consider

2.13. Lifting of endofunctors. For a monad \( F \), a comonad \( G \) and an endofunctor \( T \) on the category \( \mathcal{A} \), consider the diagrams

\[
\begin{array}{ccc}
\mathcal{A}_F & \xrightarrow{T} & \mathcal{A}_F \\
\downarrow{U_F} & & \downarrow{U_F} \\
\mathcal{A} & \xrightarrow{T} & \mathcal{A},
\end{array}
\begin{array}{ccc}
\mathcal{A}_G & \xrightarrow{\hat{T}} & \mathcal{A}_G \\
\downarrow{U_G} & & \downarrow{U_G} \\
\mathcal{A} & \xrightarrow{T} & \mathcal{A},
\end{array}
\]

where the \( U \) denote the forgetful functors.

2.14. Monad distributive laws. If \( T \) is also a monad, a natural transformation \( \lambda : FT \to TF \) is called a monad distributive law provided \( T \) can be lifted to a monad \( \hat{T} \) (in 2.13). These conditions can be formulated by some commutative diagrams (e.g. [34, 4.4]). In this case \( \lambda : FT \to TF \) induces a canonical monad structure on \( TF \).

\( TF \)-modules are equivalent to \( \hat{T} \)-modules, i.e. \( T \)-modules whose structure map is a morphism of \( F \)-modules. This means \( F \)-modules \( \alpha : FA \to A \) as well as \( T \)-modules
$\beta: TA \to A$ inducing commutativity of the diagram

\[
\begin{array}{c}
FTA \xrightarrow{\lambda A} TFA \\
\downarrow F\beta & \downarrow T\alpha \\
FA \xrightarrow{\alpha} A \xleftarrow{\beta} TA.
\end{array}
\]

2.15. **Comonad distributive laws.** If $T$ is a comonad, a natural transformation $\varphi: TG \to GT$ is called a **comonad distributive law** provided $T$ can be lifted to a comonad $\hat{T}$ (in \[\ref{2.13}\]). The properties can be expressed by some commutative diagrams (e.g. \[\ref{34}, 4.9\]). In this case $\varphi: TG \to GT$ induces a canonical comonad structure on $TG$. $TG$-comodules are equivalent to $\hat{T}$-comodules, i.e. objects $A$ that are $G$-comodules $\gamma: A \to GA$ as well as $T$-comodules $\delta: A \to TA$ inducing commutativity of the diagram

\[
\begin{array}{c}
TA \xrightarrow{\delta} A \xleftarrow{\gamma} GA \\
\downarrow T\gamma & \downarrow G\delta \\
TGA \xrightarrow{\varphi A} GTA.
\end{array}
\]

2.16. **Mixed distributive laws.** If $T$ is a comonad, a natural transformation $\lambda: FT \to TF$ is called a **mixed distributive law** or an **entwining** provided the functor $T$ can be lifted to a comonad on the module category $A_F$ (equivalently, $F$ can be lifted to a monad on the comodule category $A_T$). Again this can be characterised by some commutative diagrams (e.g. \[\ref{34}, 5.3\]).

**Mixed bimodules** or $\lambda$-**bimodules** are defined as those $A \in \text{Obj}(\mathbb{A})$ with morphisms

\[
FA \xrightarrow{h} A \xleftarrow{k} TA
\]

such that $(A, h)$ is an $F$-module and $(A, k)$ is a $T$-comodule satisfying the pentagonal law

\[
\begin{array}{c}
FA \xrightarrow{h} A \xleftarrow{k} TA \\
\downarrow Fk & \downarrow Th \\
FTA \xrightarrow{\lambda A} TFA.
\end{array}
\]

Morphisms between two $\lambda$-bimodules, called **bimodule morphisms**, are both $F$-module and $T$-comodule morphisms.

These notions yield the category of $\lambda$-bimodules which we denote by $\mathbb{A}_F^\lambda$. It can also be considered as the category of $\hat{T}$-comodules for the comonad $\hat{T}$ or as the category of $\hat{F}$-modules for the monad $\hat{F}$.

In \[\ref{22}, 2.2\] also entwinings of the type $GF \to FG$ are considered, for a monad $F$ and a comonad $G$. While the compatibility conditions can be written formally symmetrically to mixed distributive laws in \[\ref{2.16}\] such entwinings have no interpretation in terms of liftings.

2.17. **Distributive laws for adjoint functors.** Let $(L, R)$ be an adjoint pair of endofunctors on a category $\mathbb{A}$ with unit $\eta$ and counit $\varepsilon$, and $F$ be an endofunctor on $\mathbb{A}$. Consider a natural transformation $\psi: LF \to FL$ and set

\[
\tilde{\psi} = RF\varepsilon \circ R\psi R \circ \eta FR : FR \to RF.
\]
(1) If \( L \) and \( F \) are monads, then \( \psi \) is a monad distributive law if and only if \( \tilde{\psi} \) is an entwining (mixed distributive law).

(2) If \( L \) is a monad and \( F \) a comonad, then \( \psi \) is an entwining (mixed distributive law) if and only if \( \tilde{\psi} \) is a comonad distributive law.

(3) If \( L \) is a comonad and \( F \) a monad, then \( \psi \) is an entwining if and only if \( \tilde{\psi} \) is a monad distributive law.

(4) If \( L \) and \( F \) are comonads, then \( \psi \) is a comonad distributive law if and only if \( \tilde{\psi} \) is an entwining.

**Proof.** All these claims are easily checked by using that the structure maps of the adjoint monad-comonad (or comonad-monad) pair \((L, R)\) are mates under adjunctions, together with naturality and the triangle identities. Details are left to the reader. □

Combining the correspondences in 2.17(1) and (2) with the isomorphism of module and comodule categories in 2.6(1), further isomorphisms, between categories of mixed bimodules, can be derived. Note, however, that since the entwinings occurring in parts (3) and (4) of 2.17 can not be translated to liftings, these claims do not lead to similar conclusions.

2.18. Modules and distributive laws. Let \( L \) be a monad with right adjoint comonad \( R \) on a category \( A \).

(1) Let \( G \) be a comonad with a mixed distributive law \( \lambda : LG \to GL \). Then the category of mixed \((L, G)\)-bimodules \( A^G_L \) is isomorphic to the category of \((R, G)\)-bicomodules \( A^{GR} \) (see 2.17) defined by the associated comonad distributive law \( \hat{\lambda} : GR \to RG \) (see 2.17(2)).

(2) Let \( F \) be a monad with a mixed distributive law \( \tau : FR \to RF \). Then the category of mixed \((F, R)\)-bimodules \( A^R_F \) is isomorphic to the category of \((L, F)\)-bimodules \( A_{FL} \) defined by the monad distributive law \( \hat{\tau} : LF \to FL \) (see 2.17(1)).

**Proof.** (1) The mixed distributive law \( \lambda : LG \to GL \) yields a lifting of \( G \) to a comonad \( \hat{G} \) on the category \( A_L \) of \( L \)-modules. Moreover, it determines a comonad distributive law \( \hat{\lambda} : GR \to RG \) which is equivalent to a lifting of \( G \) to a comonad \( \hat{G} \) on the category \( A^R \) of \( R \)-comodules. By 2.6(1), \( A_L \) and \( A^R \) are isomorphic, and this isomorphism obviously ‘intertwines’ the comonads \( \hat{G} \) and \( \hat{G} \). Thus the isomorphism \( A_L \cong A^R \) lifts to an isomorphism between the categories of \( \hat{G} \)-comodules and \( G \)-comodules. By characterisation of \( \hat{G} \)-comodules as comodules for the composite comonad \( GR \), and characterisation of \( G \)-comodules as mixed \((L, G)\)-bimodules, we obtain the isomorphism claimed.

In fact, it can also be computed directly that a mixed bimodule

\[
\begin{array}{ccc}
LA & \xrightarrow{h} & A \\
\downarrow{Lk} & & \downarrow{Gh} \\
LGA & \xrightarrow{\lambda} & GLA,
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{k} & GA \\
\end{array}
\]

\[
\begin{array}{ccc}
LA & \xrightarrow{h} & A \\
\downarrow{Lk} & & \downarrow{Gh} \\
LGA & \xrightarrow{\lambda} & GLA,
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{k} & GA
\end{array}
\]
transforms to an \((R,G)\)-bicomodule with commutative diagram

\[
\begin{array}{ccc}
RA & \xleftarrow{h'} & GA \\
\downarrow{rk} & & \downarrow{gh'} \\
RGA & \xleftarrow{\lambda_A} & GRA,
\end{array}
\]

where \(h'\) is the mate of \(h\) under the adjunction \((L,R)\), cf. 2.6.

(2) is shown similarly to (1). \(\square\)

2.19. \textbf{F-actions on functors.} Let \(A\) and \(B\) be categories. Given a monad \(F = (F,m,i)\) on \(A\), any functor \(R : B \to A\) is called a \textit{(left) \(F\)-module} if there exists a natural transformation \(\varphi : FR \to R\) satisfying associativity and unitality conditions (corresponding to those required for objects, see \([22, 3.1]\)). Clearly, for any functor \(R : B \to A\), \((FR,mR)\) is an \(F\)-module functor.

2.20. \textbf{F-Galois functors.} For a monad \(F\) on a category \(A\) and any functor \(R : B \to A\), consider the diagram

\[
\begin{array}{ccc}
A_F & \xrightarrow{\mathcal{L}} & B & \xrightarrow{R} & A_F \\
\phi_F & \downarrow & \| & \downarrow \mathcal{U}_F & \\
A & \xleftarrow{L} & B & \xrightarrow{R} & A,
\end{array}
\]

As a particular instance of \([2.12]\) there exists some functor \(\overline{R}\) making the right square commutative if and only if \(R\) has an \(F\)-module structure \(\varphi : FR \to R\) (see \([2.19]\)).

If \(R\) has a left adjoint \(L : A \to B\), with unit \(\eta\) and counit \(\varepsilon\) of the adjunction, then there is a monad morphism

\[
\text{can} : F \xrightarrow{F\eta} FRL \xrightarrow{\varphi L} RL.
\]

We call an \(F\)-module functor \(R\) an \textit{\(F\)-Galois functor} if it has a left adjoint and \(\text{can}\) is an isomorphism.

Consider an \(F\)-module functor \(R : B \to A\) with \(F\)-action \(\varphi\), a left adjoint \(L\), unit \(\eta\) and counit \(\varepsilon\) of the adjunction. If \(B\) admits coequalisers of the parallel morphisms \(L\varphi_X \circ \varepsilon LX \circ L \varphi LX \circ LF \eta_X : LF X \to LX\), for any object \((X, \varrho_X)\) in \(A_F\), then this coequaliser yields the left adjoint \(\overline{\mathcal{L}}(X, \varrho_X)\) of \(\overline{R}\) (see left square). By uniqueness of the adjoint, \(\overline{\mathcal{L}}\varphi_F \cong L\) (see \([212]\)). Denoting the coequaliser natural epimorphism \(LU_F \to \overline{\mathcal{L}}\) by \(p\), the unit of the adjunction \((\overline{\mathcal{L}}, \overline{R})\) is the unique natural morphism \(\overline{\eta} : I_{A_F} \to \overline{R}\overline{L}\) such that \(U_F \overline{\eta} = R p \circ \eta U_F\). The counit is the unique natural morphism \(\overline{\varepsilon} : \overline{L}\overline{R} \to I_B\), such that \(\overline{\varepsilon} \circ p \overline{R} = \varepsilon\).

If \(B\) has coequalisers of all parallel morphisms, then the following are equivalent (dual to \([22, \text{Theorem 3.15}]\)):

(a) \(R\) is an \(F\)-Galois functor;
(b) the unit of \((\overline{\mathcal{L}}, \overline{R})\) is an isomorphism for

(i) all free \(F\)-modules (i.e. modules in the Kleisli category of \(F\)), or
(ii) all \(U_F\)-projective \(F\)-modules.

From \([10]\) and \([2, 3.3 \text{Theorem 10}]\) we recall a result of central importance in our setting in the form it can be found in \([15, \text{Theorem 1.7}]\).
2.21. Beck’s theorem. Consider a monad $F$ on a category $\mathcal{A}$ and an $F$-module functor $R: \mathcal{B} \to \mathcal{A}$. Then the induced lifting $\overline{R}: \mathcal{B} \to \mathcal{A}_F$ in 2.20 is an equivalence if and only if the following hold:

(i) $R$ is an $F$-Galois functor,
(ii) $R$ reflects isomorphisms,
(iii) $\mathcal{B}$ has coequalisers of $R$-contractible coequaliser pairs and $R$ preserves them.

The Galois property also transfers to adjoint functors.

2.22. Proposition. Consider an adjoint pair $(F, G)$ of endofunctors on a category $\mathcal{A}$. Let $T: \mathcal{B} \to \mathcal{A}$ be a functor which has both a left adjoint $L$ and a right adjoint $R$.

(1) If $F$ is a monad (equivalently, $G$ is a comonad) then $T$ is an $F$-Galois functor as in 2.20 if and only if it is a $G$-Galois functor (in the sense of [22, Definition 3.5]).

(2) If $F$ is a comonad (equivalently, $G$ is a monad) then $R$ is a $G$-Galois functor as in 2.20 if and only if $L$ is an $F$-Galois functor (in the sense of [22, Definition 3.5]).

Proof. Denote the unit of the adjunction $(F, G)$ by $\eta$ and its counit by $\varepsilon$. Denote furthermore the unit and counit of the adjunction $(L, T)$ by $\eta_L$ and $\varepsilon_L$, respectively, and for the unit and counit of the adjunction $(T, R)$ write $\eta_R$ and $\varepsilon_R$, respectively.

(1) A bijective correspondence between $F$-actions $\varphi_T$ and $G$-coactions $\varphi^T$ on $T$ is given by $\varphi^T := G\varphi_T \circ \eta T$. The canonical comonad morphism corresponding to $\varphi^T$ comes out as

$$\text{can} : TR \xrightarrow{\eta T} GFT \xrightarrow{G\varphi_T} GTR \xrightarrow{\varepsilon_R} G.$$

Comparing it with the canonical monad morphism $\text{can} : F \to TL$ in 2.20, they are easily seen to be mates under the adjunctions $(F, G)$ and $(TL, TR)$. That is,

$$\text{can} = G\varepsilon_R \circ GT\varepsilon_L \circ G\text{can} T R \circ \eta T R.$$

Thus $\text{can}$ is an isomorphism if and only if can is an isomorphism.

(2) A bijective correspondence between $G$-actions $\varphi_R : GR \to R$ and $F$-coactions $\varphi^L : L \to FL$ is given by

$$\varphi^L := \varepsilon_L FL \circ L \varepsilon_R T FL \circ LG\varphi_R T FL \circ LTG\eta_R FL \circ LT \varepsilon_L L \circ L \eta_L.$$

The canonical comonad morphism $\overline{\text{can}} : LT \to F$ corresponding to $\varphi^L$, and the canonical monad morphism $\overline{\text{can}} : G \to RT$ corresponding to $\varphi_R$, turn out to be mates under the adjunctions $(LT, RT)$ and $(F, G)$. That is,

$$\overline{\text{can}} = \varepsilon_L F \circ L \varepsilon_R T F \circ LT \text{can} F \circ LT \eta.$$

This proves that can is a natural isomorphism if and only if $\text{can}$ is a natural isomorphism, as stated. □

3. Rings in module categories

Let $A$ be an associative ring with unit. In this section we study the relationship between ring extensions of $A$ and monads on the category $\mathcal{M}_A$ of right $A$-modules.
3.1. \textit{A-rings.} A ring $B$ is said to be an \textit{A-ring} provided there is a ring morphism $\iota : A \to B$. This is equivalent to saying that $B$ is an $A$-bimodule with $A$-bilinear multiplication and unit, 
\[ \mu : B \otimes_A B \to B, \quad \iota : A \to B, \]
inducing commutative diagrams for associativity and unitality.

A right $B$-\textit{module} is a right $A$-module $M$ with an $A$-linear map 
\[ \varrho_M : M \otimes_A B \to M, \quad m \otimes b \mapsto mb, \]
satisfying the associativity and unitality conditions. $B$-module morphisms $f : M \to N$ are $A$-linear maps and $f \circ \varrho_M = \varrho_N \circ (f \otimes_A I_B)$.

The category of right $B$-modules is denoted by $\mathcal{M}_B$. It is isomorphic to the module category over the ring $B$ and thus is an abelian category with $B$ as a projective generator.

3.2. \textbf{Adjointness of $- \otimes_A B$ and $\text{Hom}_A(B, -)$.} As an endofunctor on $\mathcal{M}_A$, $- \otimes_A B$ is left adjoint to the endofunctor $\text{Hom}_A(B, -)$ with unit and counit 
\[ \eta_X : X \to \text{Hom}_A(B, X \otimes_A B), \quad x \mapsto [b \mapsto x \otimes b], \]
\[ \varepsilon_N : \text{Hom}_A(B, N) \otimes_A B \to N, \quad f \otimes b \mapsto f(b). \]

3.3. \textbf{Remark.} Since $A$ is a generator in $\mathcal{M}_A$ and $- \otimes_A B$ preserves direct sums and epimorphisms, the functor $- \otimes_A B : \mathcal{M}_A \to \mathcal{M}_A$ is fully determined by the value at $A$, that is by $A \otimes_A B \cong B$. Similarly, a natural transformation $\varphi : - \otimes_A B \to - \otimes_A B'$ between such ‘tensor functors’ is equal to $- \otimes_A \varphi_A$, where $\varphi_A : B \to B'$ is an $A$-bimodule map.

In general, $\text{Hom}_A(B, -)$ is not determined by $B^* = \text{Hom}_A(B, A)$, unless it preserves direct sums and epimorphisms, that is, unless $B$ is a finitely generated and projective right $A$-module. However, $\text{Hom}_A(B, -)$ is determined by $\text{Hom}_A(B, Q)$ for any cogenerator $Q \in \mathcal{M}_A$ since it is left exact and preserves direct products. For a natural transformation $\varphi : \text{Hom}_A(B, -) \to \text{Hom}_A(B', -)$ between ‘Hom functors’, it follows by the Yoneda Lemma that $\varphi = \text{Hom}_A(\varphi_B(I_B), -)$, where $\varphi_B(I_B) : B' \to B$ is an $A$-bimodule map.

3.4. \textbf{Monad-comonad.} For an $A$-bimodule $B$ the following are equivalent:

(a) $(B, \mu, \iota)$ is an $A$-ring;

(b) $- \otimes_A B : \mathcal{M}_A \to \mathcal{M}_A$ is a monad;

(c) $\text{Hom}_A(B, -) : \mathcal{M}_A \to \mathcal{M}_A$ is a comonad.

\textbf{Proof.} (a)$\iff$(b). An $A$-ring $(B, \mu, \iota)$ determines a monad $(- \otimes_A B, - \otimes_A \mu, - \otimes_A \iota)$ on $\mathcal{M}_A$ and this is a bijective correspondence in light of Remark 3.3. The equivalence (b)$\iff$(c) follows by 2.6.

Explicitly, for an $A$-ring $(B, \mu, \iota)$, $\text{Hom}_A(B, -)$ is a comonad by coproduct and counit 
\[ \text{Hom}_A(B, -) \xrightarrow{\text{Hom}_A(\mu, -)} \text{Hom}_A(B \otimes_A B, -) \xrightarrow{\cong} \text{Hom}_A(B, \text{Hom}_A(B, -)), \]
\[ \text{Hom}_A(B, -) \xrightarrow{\text{Hom}_A(\iota, -)} \text{Hom}_A(A, -) \xrightarrow{\cong} I_{\mathcal{M}_A}. \]
\[ \square \]
Adjointness of the free and forgetful functors for the monad \(- \otimes A B\) is just the isomorphism

\[ \text{Hom}_B(- \otimes A B, N) \to \text{Hom}_A(-, N), \quad f \mapsto f \circ (- \otimes A \iota). \]

We write \([B, -] = \text{Hom}_A(B, -)\) for short. Comultiplication and counit of the comonad \([B, -]\) in 3.4(c) are denoted by \([\mu, -]\) and \([\iota, -]\), respectively.

The following was pointed out in [2, page 141].

3.5. \textit{B-modules are \textit{Hom\(_A\)}(\(B, -\)-comodules}. For any \(A\)-ring \(B\), the category of right \(B\)-modules is isomorphic to the category of \(\text{Hom\(_A\)}(B, -)\)-comodules, that is, there exists an isomorphism

\[ \mathbb{M}_B \cong [B, -]. \]

\textit{Proof.} This is a special case of 2.6(1). Here the isomorphism has the following form. Given a \(B\)-module \(\varrho_N : N \otimes A B \to N\), applying \(\text{Hom\(_A\)}(B, -)\) and composing with the unit of the adjunction yields

\[ N \xrightarrow{\eta_N} \text{Hom\(_A\)}(B, N \otimes A B) \xrightarrow{\text{Hom\(_(B, \varrho_N)\)}} \text{Hom\(_A\)}(B, N), \]

imposing a \(\text{Hom\(_A\)}(B, -)\)-comodule structure on \(N\).

Conversely, a comodule structure map \(\varrho^N : N \to \text{Hom\(_A\)}(B, N)\) induces

\[ N \otimes A B \xrightarrow{\varrho^N \otimes A B} \text{Hom\(_A\)}(B, N) \otimes A B \xrightarrow{\epsilon_N} N, \]

defining a \(B\)-module structure on \(N\). \(\Box\)

4. \textbf{Corings in module categories}

Again \(A\) denotes an associative ring with unit. To any \(A\)-coring \(C\) we can associate a comonad \(- \otimes A C\) and a monad \(\text{Hom\(_A\)}(C, -)\) on the category \(\mathbb{M}_A\) of right \(A\)-modules. Here we consider the relationship between \(- \otimes A C\)-comodules (i.e. \(C\)-comodules) and \(\text{Hom\(_A\)}(C, -)\)-modules (i.e. \(C\)-contramodules).

4.1. \textit{A-Corings}. An \(A\)-coring is an \(A\)-bimodule \(C\) with \(A\)-bilinear maps

\[ \Delta : C \to C \otimes A C, \quad \varepsilon : C \to A \]

satisfying coassociativity and counitality conditions.

Similar to the characterisation of \(A\)-rings in 3.4, 2.6 implies the following characterisation of corings:

4.2. \textit{Corings}. For an \(A\)-bimodule \(C\), the following are equivalent:

(a) \((C, \Delta, \varepsilon)\) is an \(A\)-coring;
(b) \(- \otimes A C : \mathbb{M}_A \to \mathbb{M}_A\) induces a comonad;
(c) \(\text{Hom\(_A\)}(C, -) : \mathbb{M}_A \to \mathbb{M}_A\) induces a monad.

Writing \(\text{Hom\(_A\)}(C, -) = [C, -]\), the related monad is \(([C, -], [\Delta, -], [\varepsilon, -])\).

In the rest of this section \(C\) will be an \(A\)-coring. We first recall properties of the category of comodules for the related comonad (see [9]).

4.3. \textbf{The category \(\mathbb{M}^C\)}. The comodules for the comonad \(- \otimes A C : \mathbb{M}_A \to \mathbb{M}_A\) are called right \(C\)-comodules and their category is denoted by \(\mathbb{M}^C\).
(1) $\mathcal{M}^C$ is an additive category with coproducts and cokernels.

(2) The (co)free functor $- \otimes_A C$ is right adjoint to the forgetful functor by the isomorphism
$$\text{Hom}^C(M, X \otimes_A C) \rightarrow \text{Hom}_A(M, X), \quad f \mapsto (I_X \otimes_A \varepsilon) \circ f.$$ 

(3) For any generator $P \in \mathcal{M}_A$, $P \otimes_A C$ is a subgenerator in $\mathcal{M}^C$, in particular, $C$ is a subgenerator in $\mathcal{M}^C$.

(4) For any cogenerator $Q \in \mathcal{M}_A$, $Q \otimes_A C$ is a cogenerator in $\mathcal{M}^C$.

(5) For any injective $X \in \mathcal{M}_A$, $X \otimes_A C$ is injective in $\mathcal{M}^C$.

(6) For any monomorphism $f : X \rightarrow Y$ in $\mathcal{M}_A$, $f \otimes_A I_C : X \otimes_A C \rightarrow Y \otimes_A C$ is a monomorphism in $\mathcal{M}^C$.

(7) For any family $X_\lambda$ of $A$-modules, $\prod_A X_\lambda \otimes_A C$ is the product of the $C$-comodules $X_\lambda \otimes_A C$.

(8) $C$ is a flat left $A$-module if and only if monomorphisms in $\mathcal{M}^C$ are injective maps.

Left comodules of an $A$-coring $C$ are defined symmetrically to the right comodules in $\mathcal{M}^C$ as comodules of the comonad $\mathcal{C} \otimes_A -$ on the category of left $A$-modules. Furthermore, if $C$ is an $A$-coring and $D$ is a $B$-coring, then we can consider the (composite) comonad $\mathcal{C} \otimes_A D$ on the category of $(A, B)$-bimodules. Its comodules are called $(\mathcal{C}, D)$-bicomodules. Equivalently, a $(\mathcal{C}, D)$-bicomodule is a left $C$-comodule and a right $D$-comodule such that the right $D$-coaction is a left $C$-comodule map, cf. 2.15. The category of $(\mathcal{C}, D)$-bicomodules is denoted by $\mathcal{M}^{\mathcal{C} \otimes_A D}$.

While $C$-comodules are well studied in the literature (e.g. [9]), $(\mathcal{C}, [-])$-modules have not attracted so much attention so far. They were addressed by Eilenberg-Moore in [12] as $\mathcal{C}$-contramodules and reconsidered recently by Positselski [26] in the context of semi-infinite cohomology.

4.4. The category $\mathcal{M}_{[\mathcal{C}, -]}$. The modules for the monad $[\mathcal{C}, -] : \mathcal{M}_A \rightarrow \mathcal{M}_A$ are right $A$-modules $N$ with some $A$-linear map $[\mathcal{C}, N] \rightarrow N$ subject to associativity and unitality conditions. Their category is denoted by $\mathcal{M}_{[\mathcal{C}, -]}$.

(1) $\mathcal{M}_{[\mathcal{C}, -]}$ is an additive category with products and kernels.

(2) The (free) functor $[\mathcal{C}, -] : \mathcal{M}_A \rightarrow \mathcal{M}_{[\mathcal{C}, -]}$ is left adjoint to the forgetful functor by the isomorphism
$$\text{Hom}_{[\mathcal{C}, -]}([\mathcal{C}, X], M) \rightarrow \text{Hom}_A(X, M), \quad f \mapsto f \circ [\varepsilon, X].$$

(3) For any generator $P \in \mathcal{M}_A$, $[\mathcal{C}, P]$ is a generator in $\mathcal{M}_{[\mathcal{C}, -]}$; in particular, $\mathcal{C}^* = [\mathcal{C}, A]$ is a generator in $\mathcal{M}_{[\mathcal{C}, -]}$ and $\text{Hom}_{[\mathcal{C}, -]}([\mathcal{C}, A], M) \cong M$.

(4) For any cogenerator $Q \in \mathcal{M}_A$, $[\mathcal{C}, Q]$ is a weak subgenerator in $\mathcal{M}_{[\mathcal{C}, -]}$, that is, every object of $\mathcal{M}_{[\mathcal{C}, -]}$ is a subfactor of some product of copies of $[\mathcal{C}, Q]$ (compare [32, 1.6]).

(5) For any projective $Y \in \mathcal{M}_A$, $[\mathcal{C}, Y]$ is projective in $\mathcal{M}_{[\mathcal{C}, -]}$.

(6) For any epimorphism $h : X \rightarrow Y$ in $\mathcal{M}_A$, $[\mathcal{C}, h] : [\mathcal{C}, X] \rightarrow [\mathcal{C}, Y]$ is an epimorphism (not necessarily surjective) in $\mathcal{M}_{[\mathcal{C}, -]}$.

(7) For any family $X_\lambda$ of $A$-modules, $[\mathcal{C}, \bigoplus \lambda X_\lambda]$ is the coproduct of the $[\mathcal{C}, -]$-modules $[\mathcal{C}, X_\lambda]$. 
(8) \( C \) is a projective right \( A \)-module if and only if epimorphisms in \( \mathbb{M}_{\mathcal{C},-} \) are surjective maps.

**Proof.** The proof is similar to the comodule case. Some of the assertions can also be found in [25]. □

4.5. **Right and left contramodules.** In [4.4] modules of the monad \( [\mathcal{C},-] \equiv \text{Hom}_{\mathcal{C}}(\mathcal{C},- \cdot) \) on the category \( \mathbb{M}_{A} \) of right \( A \)-modules are considered. Symmetrically, an \( A \)-coring \( \mathcal{C} \) determines a monad \( \text{Hom}_{A}(\mathcal{C},- \cdot) \) also on the category \( \mathcal{A}\mathbb{M} \) of left \( A \)-modules. In situations when both monads on the categories of right and left \( A \)-modules occur at the same time, we use the following terminology to distinguish between their modules. Modules for the monad \( [\mathcal{C},-] \equiv \text{Hom}_{\mathcal{C}}(\mathcal{C},- \cdot) \) on \( \mathbb{M}_{A} \) are called *right \( \mathcal{C} \)-contramodules*, and modules for the monad \( \text{Hom}_{A}(\mathcal{C},- \cdot) \) on \( \mathcal{A}\mathbb{M} \) are called *left \( \mathcal{C} \)-contramodules*. If not specified otherwise, we mean by contramodules right contramodules, throughout.

We saw in [3.5] that for any \( A \)-ring \( B \), the categories \( \mathbb{M}_{\mathcal{B}} \) and \( \mathbb{M}^{[\mathcal{B},-]} \) are isomorphic (see also [2.6(1)]). In view of the asymmetry of assertions (1) and (2) in [2.6], the corresponding statement for corings is no longer true and we will come back to this question in [5.6]. So far we know from [2.6(2)]:

4.6. **Related Kleisli categories.** For any \( A \)-coring \( \mathcal{C} \), the Kleisli categories of \( -\otimes_{\mathcal{C}} \) and \( [\mathcal{C},-] \) are isomorphic by the isomorphisms for \( X,Y \in \mathbb{M}_{A} \),

\[
\text{Hom}(X \otimes_{\mathcal{C}} Y \otimes_{\mathcal{C}} \mathcal{C}) \cong \text{Hom}_{A}(X \otimes_{\mathcal{C}} Y \otimes_{\mathcal{C}} \mathcal{C}),
\]

\[
\cong \text{Hom}_{A}(A, [\mathcal{C}, Y]),
\]

\[
\cong \text{Hom}_{[\mathcal{C},-]}([\mathcal{C}, X], [\mathcal{C}, Y]).
\]

Recall that for any \( A \)-coring \( \mathcal{C} \), the right dual \( \mathcal{C}^{*} = \text{Hom}_{\mathcal{A}}(\mathcal{C}, A) \) has a ring structure by the convolution product for \( f, g \in \mathcal{C}^{*} \), \( f \ast g = f \circ (g \otimes_{\mathcal{A}} I_{\mathcal{C}}) \circ \Delta \) (convention opposite to [9, 17.8]). Similarly a product is defined for the left dual \( ^{*}\mathcal{C} \).

The relation between \( \mathcal{C} \)-comodules and modules over the dual ring of \( \mathcal{C} \) is well studied (see e.g. [9, Section 19]).

4.7. **The comonads \(-\otimes_{\mathcal{C}} \) and \(^{*}\mathcal{C},-\).** The comonad morphism

\[
\alpha : -\otimes_{\mathcal{C}} \to \text{Hom}_{A}(^{*}\mathcal{C},-), \quad -\otimes_{\mathcal{C}} \mapsto [f \mapsto -f(c)],
\]

yields a faithful functor

\[
\alpha : \mathbb{M}_{\mathcal{C}} \to \mathbb{M}^{[\mathcal{C},-]} \cong \mathbb{M}^{*\mathcal{C}},
\]

\[
(N \xrightarrow{\alpha_{N}} N \otimes_{\mathcal{C}} \mathcal{C}) \longmapsto (N \xrightarrow{\alpha_{N}} N \otimes_{\mathcal{C}} \mathcal{C} \xrightarrow{\alpha_{N}} \text{Hom}_{A}(^{*}\mathcal{C}, N))
\]

and the following are equivalent:

(a) \( \alpha_{N} \) is injective for each \( N \in \mathbb{M}_{A} \);

(b) \( \alpha_{A} \) is a full functor;

(c) \( \mathcal{C} \) is a locally projective left \( A \)-module.

If these conditions are satisfied, \( \mathbb{M}_{\mathcal{C}} \) is equal to \( \sigma[\mathcal{C}_{\mathcal{C}}] \), the full subcategory of \( \mathbb{M}^{*\mathcal{C}} \) subgenerated by \( \mathcal{C} \).

Similar to [4.7] \( \mathcal{C} \)-contramodules can be related to \( ^{*}\mathcal{C} \)-modules.
4.8. The monads $[\mathcal{C}, -]$ and $- \otimes_A \mathcal{C}^*$. The monad morphism
\[
\beta : - \otimes_A \mathcal{C}^* \rightarrow \text{Hom}_A(\mathcal{C}, -), \quad - \otimes f \mapsto [c \mapsto -f(c)],
\]
yields a faithful functor
\[
F_\beta : \mathcal{M}_{[\mathcal{C}, -]} \rightarrow \mathcal{M}_{\mathcal{C}^*},
\]
\[
(\text{Hom}_A(\mathcal{C}, M) \xrightarrow{\beta_M} M) \mapsto (M \otimes_A \mathcal{C}^* \xrightarrow{\beta_M} \text{Hom}_A(\mathcal{C}, M) \xrightarrow{\beta_M} M),
\]
associating to a $[\mathcal{C}, -]$-module $M$ the same object with a $\mathcal{C}^*$-module structure. The following are equivalent:
(a) $\beta$ is surjective for all $M \in \mathcal{M}_A$;
(b) $F_\beta$ is a full functor;
(c) $\mathcal{C}$ is a finitely generated and projective right $A$-module;
(d) $F_\beta$ is an isomorphism.

In general $\mathcal{C}$ is not a $[\mathcal{C}, -]$-module and $[\mathcal{C}, A]$ is not a $\mathcal{C}$-comodule. In fact, $[\mathcal{C}, A] \in \mathcal{C}\mathcal{M}$ holds provided $\mathcal{C}$ is finitely generated and projective as a right $A$-module.

5. Functors between co- and contramodules

Categories of comodules and contramodules have complementary features. Therefore, it is of interest to find $A$-corings $\mathcal{C}$ and $B$-corings $\mathcal{D}$ (over possibly different base rings) such that the category of $\mathcal{D}$-comodules and that of $[\mathcal{C}, -]$-modules are equivalent. As we will see in Section 5.4, functors between these categories are provided by bicomodules. It turns out that the question, when they provide an equivalence, fits the standard problem in (categorical) descent theory.

Since comodules for the trivial $B$-coring $B$ are simply $B$-modules, our considerations include the particular case when the category of $[\mathcal{C}, -]$-modules is equivalent to the category of $B$-modules. Dually, when the coring $\mathcal{C}$ is trivial (i.e. equal to $A$), the problem reduces to a study of equivalences between $A$-module and $\mathcal{D}$-comodule categories. This question is already discussed in the literature, see e.g. [30], [15].

Throughout this section $\mathcal{C}$ is an $A$-coring and $\mathcal{D}$ a $B$-coring for rings $A$ and $B$. The following observation was made in [25], 5.1.2.

5.1. $[\mathcal{C}, -]$-modules induced by $\mathcal{C}$-comodules. Let $N$ be a $(\mathcal{C}, \mathcal{D})$-bicomodule with left $\mathcal{C}$-coaction $N_{\mathcal{C}}$. For any $Q \in \mathcal{M}^D$, there is an isomorphism
\[
\varphi : \text{Hom}_A(\mathcal{C}, \text{Hom}^D(N, Q)) \rightarrow \text{Hom}^D(\mathcal{C} \otimes_A N, Q), \quad h \mapsto [c \otimes m \mapsto h(c)(m)],
\]
(see e.g. [9], 18.11). Then the right $A$-module $N^Q := \text{Hom}^D(N, Q)$ is a $[\mathcal{C}, -]$-module by $\alpha_{NQ}$:
\[
\text{Hom}_A(\mathcal{C}, \text{Hom}^D(N, Q)) \xrightarrow{\varphi} \text{Hom}^D(\mathcal{C} \otimes_A N, Q) \xrightarrow{\text{Hom}^D(N_{\mathcal{C}})} \text{Hom}^D(N, Q).
\]
Thus there is a bifunctor $\text{Hom}^D(-, -) : (\mathcal{C}\mathcal{M}^D)^{op} \times \mathcal{M}^D \rightarrow \mathcal{M}_{[\mathcal{C}, -]}$,
\[
(N, Q) \mapsto (N^Q, \alpha_{NQ}), \quad (f, g) \mapsto \text{Hom}^D(f, g).
\]

Proof. The identification of $\text{Hom}_A(\mathcal{C}, \text{Hom}^D(N, Q))$ with $\text{Hom}^D(\mathcal{C} \otimes_A N, Q)$, yields $\alpha_{NQ} = \text{Hom}^D(N_{\mathcal{C}}, Q)$. Since the left $\mathcal{C}$-coaction $N_{\mathcal{C}}$ of a $(\mathcal{C}, \mathcal{D})$-bicomodule $N$ is right $\mathcal{D}$-colinear, $\alpha_{NQ}(f)$ is right $\mathcal{D}$-colinear, for all $f \in \text{Hom}^D(\mathcal{C} \otimes_A N, Q)$. Hence $\alpha_{NQ}$
is well-defined. The left $A$-linearity of $N\varphi$ implies that $\alpha_{NQ}$ is right $A$-linear. By coassociativity of $N\varphi$, one concludes on associativity of $\alpha_{NQ}$. Similarly, using the counituality of $N\varphi$, one proves the commutativity of the triangle diagram in [2,3] for $\alpha_{NQ}$. Again a similar computation yields that for a morphism $f : N \to N'$ of $(C, D)$-bicomodules, $\text{Hom}^D(f, Q)$ is a morphism of $[C, -]$-modules. If $g : Q \to P$ is a morphism of right $D$-comodules, then $\text{Hom}^D(N, g)$ is a morphism of $[C, -]$-modules since $\alpha_{NQ} = \text{Hom}^D(N\varphi, Q)$ implies that $\alpha_{NQ}$ is natural in $Q$. □

In a symmetric way, for any left $D$-comodule $Q$ and a $(D, C)$-bicomodule $N$ (with $C$-coaction $\varrho^N : N \to N \otimes_A C$), $\text{Hom}^D(N, Q)$ is a left $C$-contramodule by $\text{Hom}^D(\varrho^N, Q)$.

If $N$ is just a left $C$-comodule we tacitly assume $D = B = \text{End}^C(N)$ to apply the preceding notions and results.

5.2. **Corollary.**

(1) Let $N$ be a left $C$-comodule with $B = \text{End}^C(N)$. For any subring $B' \subset B$ and $Q \in \mathcal{M}_{B'}$, $\text{Hom}_{B'}(N, Q)$ is a $[C, -]$-module.

(2) For any $Q \in \mathcal{M}_C$, $\text{Hom}^C(C, Q)$ is a $[C, -]$-module.

5.3. **Contratensor product.** For any $(C, D)$-bicomodule $N$, the construction in 5.1 yields a functor

$$\overline{\text{Hom}}^D(N, -) : \mathcal{M}^D \to \mathcal{M}_{[C, -]},$$
inducing the commutative diagram of (right adjoint) functors

$$
\begin{array}{ccc}
\mathcal{M}^D & \xrightarrow{\overline{\text{Hom}}^D(N, -)} & \mathcal{M}_{[C, -]} \\
\downarrow & & \downarrow_{U_{[C, -]}} \\
\mathcal{M}^D & \xleftarrow{\text{Hom}^D(N, -)} & \mathcal{M}_A.
\end{array}
$$

Since $\text{Hom}^D(N, -)$ has the left adjoint $- \otimes_A N$ and $\mathcal{M}^D$ has coequalisers, it follows from [2,20] that $\overline{\text{Hom}}^D(N, -)$ also has a left adjoint which comes out as follows (see [25]).

For any $(C, D)$-bicomodule $(N, N\varphi, \varrho^N)$ and $[C, -]$-module $(M, \alpha_M)$, the _contratensor product_, $M \otimes_{[C, -]} N$ is defined as the coequaliser

$$\text{Hom}_A(C, M) \otimes_A N \rightrightarrows M \otimes_A N \to M \otimes_{[C, -]} N,$$

where the coequalised maps are $f \otimes n \mapsto (f \otimes_A I_N) \circ N\varphi(n)$ and $\alpha_M \otimes_A I_N$. Projection of an element $m \otimes n$ to $M \otimes_{[C, -]} N$ is denoted by $m \otimes n$.

As a coequaliser of right $D$-comodule maps, $M \otimes_{[C, -]} N$ is a right $D$-comodule, and thus defines a functor $- \otimes_{[C, -]} N : \mathcal{M}_{[C, -]} \to \mathcal{M}^D$. Note that this coequaliser splits in $\mathcal{M}^D$ provided $(M, \alpha_M)$ is $U_{[C, -]}$-projective.

5.4. **Functors between comodules and contramodules.** Any $(C, D)$-bicomodule $N$ induces an adjoint pair of functors

$$- \otimes_{[C, -]} N : \mathcal{M}_{[C, -]} \to \mathcal{M}^D, \quad \text{Hom}^D(N, -) : \mathcal{M}^D \to \mathcal{M}_{[C, -]},$$

that is, for $M \in \mathcal{M}_{[C, -]}$ and $P \in \mathcal{M}^D$, there is an isomorphism

$$\text{Hom}^D(M \otimes_{[C, -]} N, P) \cong \text{Hom}_{[C, -]}(M, \text{Hom}^D(N, P)).$$
Conversely, any right adjoint functor $F : \mathcal{M}^D \to \mathcal{M}_{[\mathcal{C},-]}$ is naturally isomorphic to $\text{Hom}^D(N, -)$, for an appropriate $(\mathcal{C}, \mathcal{D})$-bicomodule $N$.

Proof. In view of the discussion in [2.20] for $[\mathcal{C}, -]$-modules $M$, the unit of the adjunction is given by,

$$\eta_M : M \to \text{Hom}^D(N, M \otimes_{[\mathcal{C},-]} N), \; m \mapsto [n \mapsto m \otimes_{[\mathcal{C},-]} n].$$

Also by [2.20] the counit of the adjunction comes out (and is in particular well defined) as

$$\varepsilon_Q : \text{Hom}^D(N, Q) \otimes_{[\mathcal{C},-]} N \to Q, \; f \otimes_{[\mathcal{C},-]} n \mapsto f(n),$$

for all right $\mathcal{D}$-comodules $Q$.

Conversely, assume that $F : \mathcal{M}^D \to \mathcal{M}_{[\mathcal{C},-]}$ has a left adjoint. Then so does the composite $F' := U_{[\mathcal{C},-]} \circ F : \mathcal{M}^D \to \mathcal{M}_A$, in light of [1.4]. Hence it follows by [30] Theorem 3.2] that there exists an $(A, \mathcal{D})$-bicomodule $N$ such that $F'$ is naturally isomorphic to $\text{Hom}^D(N, -)$. Moreover, by construction, for any $Q \in \mathcal{M}^D$, $\text{Hom}^D(N, Q)$ is a $[\mathcal{C}, -]$-module via some action $\kappa_Q : \text{Hom}_A(C, \text{Hom}^D(N, Q)) \to \text{Hom}^D(N, Q)$, and for $q \in \text{Hom}^D(Q, Q')$, $\text{Hom}^D(N, q)$ is a morphism of $[\mathcal{C}, -]$-modules. This amounts to saying that $\kappa(-)$ is a natural transformation $\text{Hom}^D(C \otimes_A N, -) \to \text{Hom}^D(N, -)$. Therefore, it follows by the Yoneda Lemma that there is an $(A, \mathcal{D})$-bicomodule map $\tau : N \to C \otimes_A N$, such that $\kappa_Q = \text{Hom}^D(\tau, Q)$, for $Q \in \mathcal{M}^D$. Unitality and associativity of the action $\kappa_Q$, for any $Q \in \mathcal{M}^D$, imply counitality and coassociativity of the left $\mathcal{C}$-coaction $\tau$, respectively. \qed

Consider a $(\mathcal{C}, \mathcal{D})$-bicomodule $N$, over an $A$-coring $\mathcal{C}$ and a $B$-coring $\mathcal{D}$. A $[\mathcal{C}, -]$-module map $g : (L, \alpha_L) \to (M, \alpha_M)$ is said to be $[\mathcal{C}, -]N$-pure provided the sequence

$$0 \longrightarrow \ker g \otimes_{[\mathcal{C},-]} N \longrightarrow L \otimes_{[\mathcal{C},-]} N \xrightarrow{g \otimes_{[\mathcal{C},-]} I_N} M \otimes_{[\mathcal{C},-]} N,$$

is exact (in $\mathcal{M}_B$).

5.5. Some tensor relations. Let $(N, \gamma_N)$ be a left $\mathcal{C}$-comodule. Then:

1. For any right $A$-module $X$, $\text{Hom}_A(C, X) \otimes_{[\mathcal{C},-]} N \cong X \otimes_A N$.

2. If $(M, g^M)$ is right $\mathcal{C}$-comodule for which the map

$$\gamma : \text{Hom}_A(C, M) \to \text{Hom}_A(C, M \otimes_A C), \; f \mapsto g^M \circ f - (f \otimes_A I_C) \circ \Delta,$$

is $[\mathcal{C}, -]N$-pure, then $\text{Hom}^C(C, M) \otimes_{[\mathcal{C},-]} N$ is isomorphic to the cotensor product $M \otimes^C N$.

Proof. (1) This is mentioned in [25] 5.1.1]. It is a special case of a natural isomorphism of left adjoint functors, recalled in [2.20]. Explicitly, we may put $M = \text{Hom}_A(C, X)$ and $\mathcal{D} = B = \text{End}^C(N)$ in the adjointness isomorphism to get for $P \in \mathcal{M}_B$,

$$\text{Hom}_B(\text{Hom}_A(C, X) \otimes_{[\mathcal{C},-]} N, P) \cong \text{Hom}_{[\mathcal{C},-]}(\text{Hom}_A(C, X), \text{Hom}_B(N, P)) \cong \text{Hom}_B(\text{Hom}_A(X, N), \text{Hom}_B(N, P)) \cong \text{Hom}_B(X \otimes_A N, P).$$

By the Yoneda Lemma this implies the isomorphism claimed.
(2) Consider the commutative diagram in $\mathbb{M}_B$ (for $B = \text{End}^C(N)$).

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}^C(C, M) & \otimes_{[C,-]} & N & \longrightarrow & \text{Hom}_A(C, M) & \otimes_{[C,-]} & N \\
& & \downarrow \vartheta & & \downarrow \gamma & & \downarrow \gamma \otimes_{[C,-]} I_N & \cong & \text{Hom}_A(C, M \otimes_A C) & \otimes_{[C,-]} & N \\
0 & \longrightarrow & M & \otimes^C N & \longrightarrow & M \otimes_A N & \longrightarrow & M \otimes_A C \otimes_A N,
\end{array}
\]

Since $\gamma$ is $[C,-]N$-pure, and $\text{Hom}^C(C, M) = \ker \gamma$, the top row is exact. The bottom row is the defining exact sequence of the cotensor product (see e.g. [9, 21.1]). The vertical isomorphisms are obtained from part (1). Thus there is an isomorphism $\vartheta : \text{Hom}^C(C, M) \otimes_{[C,-]} N \rightarrow M \otimes^C N$ extending the diagram commutatively. $\square$

From previous considerations we obtain the following result by Positselski.

5.6. Correspondence of categories.

1. For any $A$-coring $C$, there is an adjoint pair of functors

$$- \otimes_{[C,-]} C : \mathbb{M}_{[C,-]} \rightarrow \mathbb{M}^C, \quad \text{Hom}^C(C, -) : \mathbb{M}^C \rightarrow \mathbb{M}_{[C,-]}.$$

2. For any $X \in \mathbb{M}_A$,

$$X \otimes_A C \mapsto \text{Hom}^C(C, X \otimes_A C) \cong \text{Hom}_A(C, X),$$

$$\text{Hom}_A(C, X) \mapsto \text{Hom}_A(C, X) \otimes_{[C,-]} C \cong X \otimes_A C.$$

Thus the functors in part (1) restrict to inverse isomorphisms between the Kleisli subcategories of $\mathbb{M}^C$ and $\mathbb{M}_{[C,-]}$.

3. There is an equivalence

$$\text{Hom}^C(C, -) : \mathbb{M}^C_{inj} \rightarrow \mathbb{M}_{proj}^{[C,-]},$$

where $\mathbb{M}^C_{inj}$ denotes the full subcategory of $\mathbb{M}^C$ of objects relative injective to the forgetful functor $\mathbb{M}^C \rightarrow \mathbb{M}_A$, and $\mathbb{M}^{proj}_{[C,-]}$ the full subcategory of $\mathbb{M}_{[C,-]}$ of objects relative projective to the forgetful functor $\mathbb{M}_{[C,-]} \rightarrow \mathbb{M}_A$.

Proof. This is shown in [23, Theorem in 5.3]. Here (1) follows by putting in [5.3] $D = C$ and considering $C$ as a $(C, C)$-bimodule, see [7.2] Claim (2) (cf. [1.6]) is obtained by applying [2.6] (2) to the adjoint comonad-monad functor pair $(- \otimes_A C, \text{Hom}_A(C, -))$. Part (3) follows from [2.8]. Note that the equaliser in the more general situation of [2.8] yields here the equivalence functor $E(M, \varrho^M) = \text{Hom}^C(C, M)$, for any $(M, \varrho^M) \in \mathbb{M}^C_{inj}$, as stated.

Recall that an $A$-coring $C$ is said to be a coseparable coring if its coproduct is a split monomorphism of $C$-bicomodules. Equivalently, there is an $A$-bimodule map $\delta : C \otimes_A C \rightarrow A$ such that $\delta \circ \Delta = \varepsilon$ and

$$(I_C \otimes_A \delta) \circ (\Delta \otimes_A I_C) = (\delta \otimes_A I_C) \circ (I_C \otimes_A \Delta).$$

Such a map $\delta$ is called a cointegral (e.g. [9, 26.1]). Equivalently, coseparable corings can be described by separable functors as follows.

5.7. Coseparable corings. For $C$ the following are equivalent.

1. $C$ is a coseparable coring;
2. the forgetful functor $U^C : \mathbb{M}^C \rightarrow \mathbb{M}_A$ is separable;
(c) the forgetful functor \( U_{\mathcal{C},-} : \mathbb{M}_{\mathcal{C},-} \to \mathbb{M}_A \) is separable.

If these assertions hold then, in particular, any \( \mathcal{C} \)-comodule is \( U^\mathcal{C} \)-injective and any \( [\mathcal{C},-] \)-module is \( U_{\mathcal{C},-} \)-projective.

Proof. Equivalence (a) \( \iff \) (b) is quoted from [9, 26.1]. It can be derived alternatively from 2.9(2). Equivalence (b) \( \iff \) (c) and the final claims follow by 2.10(2). \( \square \)

Combining 5.7 with 5.6 we obtain:

5.8. Comodules and contramodules of coseparable corings. For a coseparable coring \( \mathcal{C} \), the category \( \mathbb{M}_{\mathcal{C} inj} \) coincides with \( \mathbb{M}_{\mathcal{C} proj} \) and \( \mathbb{M}_{[\mathcal{C},-]} \) is equal to \( \mathbb{M}_{[\mathcal{C},-]} \). Thus there is an equivalence

\[ \text{Hom}^\mathcal{C}(\mathcal{C},-) : \mathbb{M}^\mathcal{C} \to \mathbb{M}_{[\mathcal{C},-]} \].

This equivalence between comodules and contramodules for coseparable corings plays an important role in the characterisation of categories of Hopf (contra)modules in 8.8.

6. Galois bicomodules

In this section we analyse, when a comodule category is equivalent to a contramodule category. Any such equivalence is necessarily given by functors associated to a bicomodule. The latter must possess additional properties.

6.1. \([\mathcal{C},-]\)-Galois bicomodules. For a \((\mathcal{C},\mathcal{D})\)-bicomodule \( (N,N^\mathcal{D},\varrho^N) \), the commutative diagram in 5.3 yields a canonical monad morphism (by 2.20)

\[ \text{can}^N : \text{Hom}_A(\mathcal{C},-) \to \text{Hom}_D(N,- \otimes_A N), \quad f \mapsto (f \otimes_A I_N) \circ N^\mathcal{D}. \]

Let \( \eta \) denote the unit of the adjunction \( (- \otimes_{[\mathcal{C},-]} N, \text{Hom}_D(N,-)) \) in 5.4.

The following statements are equivalent:

(a) The natural transformation \( \text{can}^N \) is an isomorphism;
(b) \( \eta_{\text{Hom}_A(\mathcal{C},Q)} \) is an isomorphism, for all \( Q \in \mathbb{M}_A \);
(c) \( \eta_M \) is an isomorphism, for all \( U_{[\mathcal{C},-]} \)-projective \( M \in \mathbb{M}_{[\mathcal{C},-]} \).

If these conditions hold, then \( \text{Hom}_D(N,-) \) is a \([\mathcal{C},-]\)-Galois functor and we call \( N \in \mathcal{C} \mathbb{M}^D \) a \([\mathcal{C},-]\)-Galois bicomodule.

If \( N \) is just a left \( \mathcal{C} \)-comodule we tacitly take \( \mathcal{D} = B = \text{End}^\mathcal{C}(N) \) and call \( N \) a \([\mathcal{C},-]\)-Galois left comodule.

Symmetrically to the above considerations, any right adjoint functor from the category of left comodules of a \( \mathcal{B} \)-coring \( \mathcal{D} \) to the category of left contramodules of an \( \mathcal{A} \)-coring \( \mathcal{C} \) is naturally isomorphic to \( \text{Hom}_D(N,-) \), for some \((\mathcal{D},\mathcal{C})\)-bicomodule \( N \). In analogy with 6.1 also \( \text{Hom}_{A,-}(\mathcal{C},-) \)-Galois \((\mathcal{D},\mathcal{C})\)-bicomodules and in particular \( \text{Hom}_{A,-}(\mathcal{C},-) \)-Galois right \( \mathcal{C} \)-comodules can be defined.

Studying \([\mathcal{C},-]\)-Galois bicomodules we are on the one side interested in there own structural properties and on the other side also in conditions which make the related functors fully faithful.

6.2. \(- \otimes_{[\mathcal{C},-]} N\) fully faithful. Let \( N \) be a \((\mathcal{C},\mathcal{D})\)-bicomodule. Then the functor

\[ - \otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^D \] is fully faithful if and only if

(i) \( N \) is a \([\mathcal{C},-]\)-Galois bicomodule and
(ii) for any $[C, -]$-module $M$, the functor $\text{Hom}^P(N, -) : \mathbb{M}^D \to \mathbb{M}_A$ preserves the coequaliser
\[ \text{Hom}_A(C, M) \otimes_A N \xrightarrow{\cong} M \otimes_A N \xrightarrow{\cong} M \otimes_{[C, -]} N, \]

defining the contrator product (cf. 5.3).

Proof. Since in $\mathbb{M}^D$ any parallel pair of morphisms has a coequaliser, the claim follows by (the dual version of) [15, Theorem 2.6]. □

6.3. Corollary. Let $N \in \mathbb{C} \mathbb{M}^D$ be a $[C, -]$-Galois bicomodule. If the functor $\text{Hom}^P(N, -) : \mathbb{M}^D \to \mathbb{M}_A$ preserves coequalisers, then $- \otimes_{[C, -]} N : \mathbb{M}_A \to \mathbb{M}^D$ is fully faithful and $C$ is a projective right $A$-module.

Proof. It follows immediately by (6.2) that $- \otimes_{[C, -]} N : \mathbb{M}_A \to \mathbb{M}^D$ is fully faithful. The left adjoint functor $- \otimes_A N$ always preserves cokernels and $\text{Hom}^P(N, -)$ does so by hypothesis. Thus their composite $\text{Hom}^P(N, - \otimes_A N) : \mathbb{M}_A \to \mathbb{M}_A$ preserves cokernels, i.e. epimorphisms. Since $\text{can}^N$ in (6.1) is assumed to be an isomorphism, we conclude that also the functor $\text{Hom}_A(C, -)$ preserves epimorphisms, i.e. $C$ is projective as a right $A$-module. □

For our investigation it is of interest to extend the notion of Galois comodules from [33, 4.1] to bicomodules.

6.4. $\mathcal{D}$-Galois bicomodules. For any $(C, \mathcal{D})$-bicomodule $N$, the left adjoint functor $- \otimes_{[C, -]} N : \mathbb{M}_A \to \mathbb{M}_B$ is a left $- \otimes_B \mathcal{D}$-comodule functor (in the sense of [22, 3.3]) by the coaction
\[ - \otimes_{[C, -]} \varrho_N : - \otimes_{[C, -]} N \to - \otimes_{[C, -]} N \otimes_B \mathcal{D}. \]

We call $N$ a $\mathcal{D}$-Galois bicomodule if $- \otimes_{[C, -]} N : \mathbb{M}_A \to \mathbb{M}_B$ is a $- \otimes_B \mathcal{D}$-Galois functor (in the sense of [22, 3.3]), that is, if the comonad morphism
\[ \text{Hom}_B(N, -) \otimes_{[C, -]} N \xrightarrow{I_{\text{Hom}_B(N, -) \otimes_{[C, -]} \varrho_N}} \text{Hom}_B(N, -) \otimes_{[C, -]} N \otimes_B \mathcal{D} \xrightarrow{\varrho^N \circ I_D} - \otimes_B \mathcal{D} \]
is an isomorphism.

For a right $\mathcal{D}$-comodule $N$, one can put $C = A = \text{End}^D(N)$. In this way we re-obtain the usual notion of a $\mathcal{D}$-Galois right comodule in [33, 4.1].

The $\mathcal{D}$-Galois property of a $(\mathcal{D}, C)$-bicomodule is defined symmetrically by the Galois property of the induced functor between the category of left $\mathcal{D}$-comodules and the category of left $C$-contramodules. In the particular case of a left $\mathcal{D}$-comodule $N$, it reduces to the usual notion of a $\mathcal{D}$-Galois left comodule in [9] by putting $A = C = \text{End}^D(N)$.

6.5. $\text{Hom}^P(N, -)$ fully faithful. Let $N$ be a $(C, \mathcal{D})$-bicomodule. Then the functor $\text{Hom}^P(N, -) : \mathbb{M}^D \to \mathbb{M}_A$ is fully faithful if and only if

(i) $N$ is a $\mathcal{D}$-Galois bicomodule and

(ii) the functor $- \otimes_{[C, -]} N : \mathbb{M}_A \to \mathbb{M}_B$ preserves the equaliser
\[ \text{Hom}^P(N, Q) \xrightarrow{\text{Hom}_B(N, \varrho^Q)} \text{Hom}_B(N, Q \otimes_B \mathcal{D}), \]

for any right $\mathcal{D}$-comodule $(Q, \varrho^Q)$, where $\omega(f) = (f \otimes_B I_D) \circ \varrho^N$. 

Proof. This follows again by [15, Theorem 2.6]. \qed

6.6. Corollary. Let \( N \in \mathcal{C}\mathcal{M}^D \) be a \( \mathcal{D} \)-Galois bicomodule. If the functor \( - \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B \) preserves equalisers, then \( \text{Hom}^D(N,-) : \mathcal{M}^D \to \mathcal{M}_{[\mathcal{C},-]} \) is fully faithful and \( \mathcal{D} \) is a flat left \( B \)-module.

Proof. (Compare with [33, 4.8]). The first assertion follows immediately from 6.5. In particular, this means that \( N \) is a generator in \( \mathcal{M}^D \). Moreover, there is a natural isomorphism \( - \otimes_B \mathcal{D} \cong \text{Hom}_B(N,-) \otimes_{[\mathcal{C},-]} N \), where the right adjoint functor \( \text{Hom}_B(N,-) \) always preserves kernels and by assumption so does \( - \otimes_{[\mathcal{C},-]} N \). This implies that \( - \otimes_B \mathcal{D} : \mathcal{M}_B \to \mathcal{M}_B \) preserves kernels, i.e. monomorphisms, hence \( \mathcal{D} \) is flat as a left \( B \)-module. \qed

Recall from [9, 19.19] that, for a left \( \mathcal{B} \)-module \( (N, \rho) \) which is finitely generated and projective as a left \( \mathcal{A} \)-module, the left dual \( *N = \text{Hom}_{\mathcal{A}}(N, \mathcal{A}) \) carries a canonical right \( \mathcal{C} \)-comodule structure, via

\[
*N \longrightarrow \text{Hom}_{\mathcal{A}}(N, \mathcal{C}) \cong *N \otimes_{\mathcal{A}} \mathcal{C}, \quad g \mapsto (I_{\mathcal{C}} \otimes_{\mathcal{A}} g) \circ N \rho.
\]

In what follows, \( \mathcal{C}, - \)-Galois and \( \mathcal{C} \)-Galois properties of a finitely generated projective comodule are compared.

6.7. \( \mathcal{C}, - \)-Galois comodules and \( \mathcal{C} \)-Galois comodules. Let \( N \) be a left \( \mathcal{C} \)-comodule finitely generated and projective as a left \( \mathcal{A} \)-module. The following assertions are equivalent.

(a) \( N \) is a \( \text{Hom}_{-\mathcal{A}}(\mathcal{C},-) \)-Galois left comodule;
(b) \( N \) is a \( \mathcal{C} \)-Galois left comodule;
(c) \( *N \) is a \( \text{Hom}_{-\mathcal{A}}(\mathcal{C},-) \)-Galois right comodule;
(d) \( *N \) is a \( \mathcal{C} \)-Galois right comodule.

Proof. (b) \( \Leftrightarrow \) (d) is proven in [33, p 514].

(a) \( \Leftrightarrow \) (d). Put \( B = \text{End}^\mathcal{C}(N) \) and consider the \( (A, B) \)-bimodule \( N \) and the \( (B, A) \)-bimodule \( *N \). The stated equivalence follows by applying [2.22(2)] to the adjoint comonad-monad pair \( - \otimes_A \mathcal{C}, \text{Hom}_{\mathcal{A}}(\mathcal{C},-) \) and the functor \( - \otimes_A N \cong \text{Hom}_{\mathcal{A}}(*N,-) : \mathcal{M}_A \to \mathcal{M}_B \), possessing the right adjoint \( \text{Hom}_B(N,-) \) and the left adjoint \( - \otimes_B *N \).

(b) \( \Leftrightarrow \) (c) is proven similarly to (a) \( \Leftrightarrow \) (d). \qed

Sufficient and necessary conditions for the equivalence between a comodule and a contramodule category are obtained by applying Beck’s theorem; see [2.21.

6.8. Equivalences. For an \( \mathcal{A} \)-coring \( \mathcal{C} \) and a \( \mathcal{B} \)-coring \( \mathcal{D} \), the following assertions are equivalent.

(a) The categories \( \mathcal{M}_{[\mathcal{C},-]} \) and \( \mathcal{M}^D \) are equivalent;
(b) there exists a \( (\mathcal{C}, \mathcal{D}) \)-bicomodule \( N \) with the properties:

(i) \( N \) is a \( [\mathcal{C},-] \)-Galois bicomodule,
(ii) the functor \( \text{Hom}^D(N,-) : \mathcal{M}^D \to \mathcal{M}_A \) reflects isomorphisms,
(iii) the functor \( \text{Hom}^D(N,-) : \mathcal{M}^D \to \mathcal{M}_A \) preserves \( \text{Hom}^D(N,-) \)-contractible coequalizers.
(c) there exists a \( (\mathcal{C}, \mathcal{D}) \)-bicomodule \( N \) with the properties:
(i) $N$ is a $\mathcal{D}$-Galois bicomodule,

(ii) the functor $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B$ reflects isomorphisms,

(iii) the functor $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B$ preserves $- \otimes_{[\mathcal{C},-]} N$-contractible equalisers.

Proof. (a)$\Rightarrow$(b). By Lemma 6.8 any equivalence functor $M^\mathcal{D} \to \mathcal{M}_{[\mathcal{C},-]}$ is naturally isomorphic to $\text{Hom}^\mathcal{D}(N,-)$, for some $(\mathcal{C},\mathcal{D})$-bicomodule $N$. By Beck’s theorem $\text{Hom}^\mathcal{D}(N,-) : M^\mathcal{D} \to \mathcal{M}_{[\mathcal{C},-]}$ is an equivalence if and only if the conditions in part (b) hold.

(a)$\Rightarrow$(c) is shown with similar arguments. □

6.9. Equivalence for abelian categories. For a $(\mathcal{C},\mathcal{D})$-bicomodule $N$, the following are equivalent.

(a) $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \mathcal{M}_{[\mathcal{C},-]}$ is an equivalence, $\mathcal{C}$ is a projective right $A$-module and $\mathcal{D}$ is a flat left $B$-module;

(b) $\mathcal{D}$ is flat as a left $B$-module and $N$ is a $[\mathcal{C},-]$-Galois bicomodule and a projective generator in $\mathcal{M}^\mathcal{D}$;

(c) $\mathcal{C}$ is projective as a right $A$-module and $N$ is a $\mathcal{D}$-Galois bicomodule and the functor $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B$ is left exact and faithful.

Proof. (a)$\Rightarrow$(b). By Theorem 6.8 $N$ is a $[\mathcal{C},-]$-Galois bicomodule. Being an equivalence, $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \mathcal{M}_{[\mathcal{C},-]}$ is faithful. Since the forgetful functor from $\mathcal{M}_{[\mathcal{C},-]}$ to $\mathcal{M}_A$ (or to $\mathcal{M}_Z$ or $\text{Set}$) is faithful, so is the composite $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \text{Set}$. This proves that $N$ is a generator in $\mathcal{M}^\mathcal{D}$. Finally, $U_{[\mathcal{C},-]} : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_A$ is right exact by 4.4 (8). Since $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \mathcal{M}_{[\mathcal{C},-]}$ is an equivalence, this implies that also $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \mathcal{M}_A$ is right exact, by commutativity of the diagram in 5.3. By flatness of $\mathcal{D}$ as a left $B$-module, this implies projectivity of $N$ (cf. [9, 18.20]).

(b)$\Rightarrow$(a). By the hypothesis, the functor $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \mathcal{M}_A$ preserves etc. (cf. [9, 18.20]).

(a)$\Rightarrow$(c). If $\text{Hom}^\mathcal{D}(N,-) : \mathcal{M}^\mathcal{D} \to \mathcal{M}_{[\mathcal{C},-]}$ is an equivalence, then so is its left adjoint $- \otimes_{[\mathcal{C},-]} N$. Thus $N$ is a $\mathcal{D}$-Galois bicomodule by 6.8. The functor $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B$ is equal to the composite of the equivalence $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}^\mathcal{D}$ and the forgetful functor $\mathcal{M}^\mathcal{D} \to \mathcal{M}_B$. The forgetful functor is faithful and also left exact by the flatness of the left $B$-module $\mathcal{D}$. Thus the functor $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B$ is also faithful and left exact.

(c)$\Rightarrow$(a). Since $\mathcal{C}$ is a projective right $A$-module, $\mathcal{M}_{[\mathcal{C},-]}$ is abelian. Hence faithfulness of $- \otimes_{[\mathcal{C},-]} N : \mathcal{M}_{[\mathcal{C},-]} \to \mathcal{M}_B$ implies that it reflects isomorphisms. Since it also preserves etc. (cf. [9, 18.20]).

□

In the rest of the section we study the particular case of a trivial $B$-coring $\mathcal{D} = B$. That is, the situation when the category of contramodules of a coring $\mathcal{C}$ is equivalent to that of modules over a ring $B$.

6.10. Lemma. [13] Proposition 2.5] Let $N$ be an $(A,B)$-bimodule which is finitely generated and projective as an $A$-module. Consider the comatrix coring $\mathcal{C} := N \otimes_B * N$
and denote by $T$ the ring of endomorphisms of $N$ as a left $C$-comodule. Then $N \cong N \otimes_B T$ via the right $T$-action on $N$.

The next result may be seen as a counterpart to the Galois comodule structure theorem \cite{9} 18.27.

6.11. **Theorem.** Let $N \in \mathcal{C}M$ be a $[\mathcal{C}, -]$-Galois comodule over an $A$-coring $\mathcal{C}$, put $T = \text{End}^C(N)$ and $B \subset T$ be a subring. Assume that $N$ is a projective generator of right $B$-modules. Then the following hold.

1. $- \otimes_{[\mathcal{C}, -]} N : M_{[\mathcal{C}, -]} \to M_B$ is an equivalence.
2. $\mathcal{C}$ is a projective right $A$-module.
3. $N$ is a finitely generated and projective left $A$-module.
4. $\mathcal{C}$ is isomorphic to the comatrix $A$-coring $N \otimes_B ^*N$.
5. $B$ is isomorphic to $T$.
6. If, in addition, $\mathcal{C}$ is a generator of right $A$-modules, then $N$ is a faithfully flat left $A$-module.

**Proof.** Assertions (1) and (2) are immediate by \cite{3, 9}. (3) Since $- \otimes_{[\mathcal{C}, -]} N$ is an equivalence, it has a left adjoint $\text{Hom}_B(N, -) : M_B \to M_{[\mathcal{C}, -]}$. The free functor $\text{Hom}_A(\mathcal{C}, -)$ has a left adjoint $- \otimes_{[\mathcal{C}, -]} \mathcal{C} : M_{[\mathcal{C}, -]} \to M_A$ by \cite{5, 3} Hence also the composite functor, that is naturally isomorphic to $- \otimes_A N : M_A \to M_B$ by \cite{5, 5} (1), has a left adjoint. This proves that $N$ is a finitely generated and projective left $A$-module.

4. By part (3),

$$\text{Hom}_B(N, - \otimes_A N) \cong \text{Hom}_B(N, \text{Hom}_A(^*N, -)) \cong \text{Hom}_A(N \otimes_B ^*N, -).$$

Composing this natural isomorphism with the canonical monad morphism $\text{can}^N$ (at $D = B$), it yields a monad isomorphism $\text{Hom}_A(\mathcal{C}, -) \cong \text{Hom}_A(N \otimes_B ^*N, -)$. By Yoneda’s Lemma this proves $\mathcal{C} \cong N \otimes_B ^*N$.

5. The composite of the forgetful functor $\text{Hom}_T(T, -) : M_T \to M_B$ and $\text{Hom}_B(N, -) : M_B \to M_A$ is naturally isomorphic to

$$\text{Hom}_B(N, \text{Hom}_T(T, -)) \cong \text{Hom}_T(N \otimes_B T, -) \cong \text{Hom}_T(N, -),$$

where the last isomorphism follows by part (4) and Lemma \cite{6, 10}. The forgetful functor $M_T \to M_B$ reflects isomorphisms. Since $N$ is a generator in $M_B$ by assumption, the (fully faithful) functor $\text{Hom}_B(N, -) : M_B \to M_A$ reflects isomorphisms too. Hence also the composite $\text{Hom}_T(N, -) : M_T \to M_A$ reflects isomorphisms. The forgetful functor $M_T \to M_B$ has a right adjoint (the coinduction functor $\text{Hom}_B(T, -)$) hence it preserves coequalisers. Since $N$ is a projective right $B$-module by assumption, $\text{Hom}_B(N, -) : M_B \to M_A$ preserves coequalisers too. Hence also the composite $\text{Hom}_T(N, -) : M_T \to M_A$ preserves coequalisers. The equivalence functor $- \otimes_{[\mathcal{C}, -]} N : M_{[\mathcal{C}, -]} \to M_B$ factorises through $- \otimes_{[\mathcal{C}, -]} : M_{[\mathcal{C}, -]} \to M_T$ and the forgetful functor $M_T \to M_B$. Thus the forgetful functor is full (and obviously faithful). This implies that $- \otimes_{[\mathcal{C}, -]} N : M_{[\mathcal{C}, -]} \to M_T$ is fully faithful, hence the corresponding canonical monad morphism

$$\text{Hom}_A(\mathcal{C}, -) \to \text{Hom}_T(N, - \otimes_A N), \quad f \mapsto [n \mapsto (f \otimes_A I_N) \circ N g(n)],$$
is a natural isomorphism by Theorem 6.2. So we conclude by Theorem 6.8 that
− ⊗ : M_A → M_B is an equivalence and so is the forgetful functor M_T → M_B.
This proves the isomorphism of algebras T ≅ B.

(6) N is a flat left A-module by part (3). Hence it suffices to show that, under
the assumptions made, − ⊗ A N : M_A → M_B is a faithful functor, so it reflects both
monomorphisms and epimorphisms. Recall that, by 5.5(1), − ⊗ A N : M_A → M_B is
naturally isomorphic to the composite of the free functor Hom_A(C, −) : M_A → M[C, −]
and the equivalence − ⊗ : M[−] → M.B. By assumption, Hom_A(C, −) : M_A → M_A is faithful. Then also Hom_A(C, −) : M_A → M[C, −] is faithful, what completes the
proof. □

Note that C is a generator of right A-modules as in Theorem 6.11 (6) in various
situations. For example, whenever the counit of C is an epimorphism (e.g. because
there exists a grouplike element in C or C is faithfully flat as a left or right A-module).

7. Contramodules and entwining structures

As recalled in 2.13 lifting of a monad F on a category A to a monad on the category
A A for a comonad G, or lifting of a comonad G to a comonad on the category A
for a monad F, are both equivalent to the existence of a mixed distributive law
(entwining) between F and G. Combining this general fact with properties of module
categories, we obtain a description of entwinings between A-rings and A-corings (A
is an associative ring with unit). Recall that a (left) entwining map between an
A-ring B and an A-coring C is an A-bimodule morphism ψ : B ⊗ A C → C ⊗ A B
which respects (co)multiplications and (co)units. Similarly, (right) entwining maps
λ : C ⊗ A B → B ⊗ A C are defined (e.g. [9 Chapter 5]).

7.1. Entwining maps. For all A-rings B and A-corings C, the following assertions
are equivalent.

(a) There is an entwining map ψ : B ⊗ A C → C ⊗ A B;
(b) the monad B ⊗ A − on A M has a lifting to a monad on C M;
(c) the comonad C ⊗ A − on A M has a lifting to a comonad on B M;
(d) the monad Hom_A(C, −) on M_A has a lifting to a monad on M_B;
(e) the comonad Hom_A(B, −) on M_A has a lifting to a comonad on M[C, −].

Proof. (a) ⇔ (b) and (a) ⇔ (c). An entwining map ψ determines a mixed distributive
law Ψ := ψ ⊗ A : B ⊗ A C ⊗ A B. Conversely, if Ψ : B ⊗ A C ⊗ A B → C ⊗ A B ⊗ A −
is a mixed distributive law, then ψ := Ψ_A is an entwining map.

(a) ⇔ (d) and (a) ⇔ (e). An entwining map ψ determines a mixed distributive law
Ψ:
Hom_A(C, Hom_A(B, −)) ≅ Hom_A(C ⊗ A B, −) \xrightarrow{\text{Hom}_A(\psi, −)} Hom_A(B ⊗ A C, −).

On the other hand, by the Yoneda Lemma, any mixed distributive law Ψ : Hom_A(C ⊗ A
B, −) → Hom_A(B ⊗ A C, −) is of this form. □

Under the equivalent conditions of 7.1 C ⊗ A B is a B-coring, cf. [9, 32.6]. Its
contramodules can be described as follows.
7.2. $[\mathcal{C} \otimes_A B, -]$-modules. Let $B$ be an $A$-ring and $\mathcal{C}$ an $A$-coring with an entwining map $\psi : B \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A B$. Then the following structures on a right $B$-module $M$ are equivalent.

(a) A module structure map $\varrho_M : \text{Hom}_B(\mathcal{C} \otimes_A B, M) \to M$;

(b) a $B$-linear module structure map $\alpha_M : \text{Hom}_A(\mathcal{C}, M) \to M$ (where $fb = \sum f(-)^{\psi}b^{\psi}$, for $f \in \text{Hom}(\mathcal{C}, M)$, $b \in B$, hence $B$-linearity means $\alpha_M(f)b = \sum \alpha_M(f(-)^{\psi})b^{\psi}$), with notation $\psi(b \otimes_A c) = \sum c^{\psi} \otimes_A b^{\psi}$;

(c) a module structure for the monad $\text{Hom}_A(\mathcal{C}, -)$ on $\mathsf{Mod}_B$;

(d) a comodule structure for the comonad $\text{Hom}_A(B, -)$ on $\mathsf{Mod}_{[\mathcal{C}, -]}$.

Proof. (a)$\Leftrightarrow$(b). The isomorphism $\text{Hom}_B(\mathcal{C} \otimes_A B, M) \cong \text{Hom}_A(\mathcal{C}, M)$ of right $A$-modules induces an isomorphism

$$\xi : \text{Hom}_A(\text{Hom}_A(\mathcal{C}, M), M) \to \text{Hom}_A(\text{Hom}_B(\mathcal{C} \otimes_A B, M), M).$$

As easily checked, $\xi(\alpha_M)$ belongs to $\text{Hom}_B(\text{Hom}_B(\mathcal{C} \otimes_A B, M), M)$ if and only if $\alpha_M$ satisfies the $B$-linearity condition in (b). Associativity and unitality of a $[\mathcal{C} \otimes_A B, -]$-action $\xi(\alpha_M)$ are equivalent to analogous properties of the $[\mathcal{C}, -]$-action $\alpha_M$.

Equivalences (b)$\Leftrightarrow$(c) and (b)$\Leftrightarrow$(d) follow by 2.16 (cf. 3.4.5.7).

In light of 7.1, the following describes a special case of 2.17 and 2.18.

7.3. Distributive laws for rings and corings. Let $B$ be an $A$-ring and $\mathcal{C}$ an $A$-coring over any ring $A$.

1. $\lambda : \mathcal{C} \otimes_A B \to B \otimes_A \mathcal{C}$ is an entwining map if and only if

$$\tilde{\lambda} : \text{Hom}_A(B, -) \otimes_A \mathcal{C} \to \text{Hom}_A(B, - \otimes_A \mathcal{C}), \quad f \otimes c \mapsto (f \otimes_A I_{\mathcal{C}}) \circ \lambda(c \otimes -),$$

is a comonad distributive law. In this case, $\text{Hom}_A(B, -) \otimes_A \mathcal{C}$ is a comonad on $\mathsf{Mod}_A$ and the category of its comodules is isomorphic to the category of $- \otimes_A \lambda$-bimodules (i.e. usual entwined modules), cf. 2.16.

2. $\psi : B \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A B$ is an entwining map if and only if

$$\tilde{\psi} : \text{Hom}_A(\mathcal{C}, -) \otimes_A B \to \text{Hom}_A(\mathcal{C}, - \otimes_A B), \quad g \otimes a \mapsto (g \otimes_A I_{B}) \circ \psi(a \otimes -),$$

is a monad distributive law. In this case, $\text{Hom}_A(\mathcal{C}, - \otimes_A B)$ is a monad on $\mathsf{Mod}_A$ and the category of its modules is isomorphic to the category of $\text{Hom}_A(\psi, -)$-bimodules (cf. 2.16).

Note that for a commutative ring $R$, any $R$-algebra $A$ and $R$-coalgebra $C$ are entwined by the twist maps $C \otimes_R A \to A \otimes R C$ and $A \otimes_R C \to C \otimes_R A$. Applying 7.3 to these particular entwinings, we conclude that the canonical natural transformations

$$\text{Hom}_R(A, -) \otimes_R C \to \text{Hom}_R(A, - \otimes_R C), \quad f \otimes c \mapsto f(-) \otimes c, \quad \text{and}$$

$$\text{Hom}_R(C, -) \otimes_R A \to \text{Hom}_R(C, - \otimes_R A), \quad g \otimes a \mapsto g(-) \otimes a,$$

yield a comonad distributive law and a monad distributive law, respectively.
8. Bialgebras and bimodules

There are many equivalent characterisations of bialgebras and Hopf algebras. A bialgebra over a commutative ring \( R \) can be seen as an \( R \)-module that is both an algebra and a coalgebra entwined in a certain way. In category theory terms, a bialgebra is defined as an \( R \)-module such that the tensor functor \(- \otimes_R R\) is a bimonad on \( \mathbb{M}_R \). Associated to a bialgebra \( B \), there is a category of Hopf modules, whose objects are \( B \)-modules with a compatible \( B \)-comodule structure. A Hopf algebra can be characterised as a bialgebra \( B \) such that the functor \(- \otimes_R B\) is an equivalence between the categories of \( R \)-modules and Hopf \( B \)-modules. In this section we supplement this description of bialgebras and Hopf algebras by the equivalent description in terms of properties of the Hom-functor \( [B, -] \), and hence in terms of contramodules.

Throughout, \( R \) is a commutative ring. The unit element of a (bi)algebra \( B \) is denoted by \( 1_B \). For the coproduct \( \Delta \) of a bialgebra \( B \), if applied to an element \( b \in B \), we use Sweedler’s index notation \( \Delta(b) = b_1 \otimes b_2 \), where implicit summation is understood.

### 8.1. Bialgebras

Let \( B \) be an \( R \)-module which is both

- an \( R \)-algebra \( \mu : B \otimes R B \to B \), \( \iota : R \to B \); and
- an \( R \)-coalgebra \( \Delta : B \to B \otimes R B \), \( \varepsilon : B \to R \).

Based on the canonical twist \( \text{tw} : B \otimes R B \to B \otimes R B \), we obtain the following \( R \)-module maps

\[
\psi_r = (I_B \otimes R \mu) \circ (\text{tw} \otimes R I_B) \circ (I_B \otimes R \Delta) : B \otimes R B \to B \otimes R B,
\]

\[
\psi_l = (\mu \otimes R I_B) \circ (I_B \otimes R \text{tw}) \circ (\Delta \otimes R I_B) : B \otimes R B \to B \otimes R B.
\]

Evaluated on elements, \( \psi_r(a \otimes b) = b_1 \otimes a b_2 \) and \( \psi_l(a \otimes b) = a_1 b \otimes a_2 \).

To make \( B \) a bialgebra, \( \mu \) and \( \iota \) must be coalgebra maps (equivalently, \( \Delta \) and \( \varepsilon \) are to be algebra maps) with respect to the obvious product and coproduct on \( B \otimes R B \) (induced by \( \text{tw} \)). The compatibility between multiplication and comultiplication can be expressed by commutativity of the diagram

\[
\begin{array}{ccc}
B \otimes R B & \xrightarrow{\mu} & B \\
\downarrow{\Delta \otimes R I_B} & & \downarrow{\mu \otimes R I_B} \\
B \otimes R B \otimes R B & \xrightarrow{I_B \otimes R \psi_r} & B \otimes R B \otimes R B.
\end{array}
\]

For a bialgebra \( B \), both maps \( \psi_r \) and \( \psi_l \) are (right, respectively, left) entwining maps between the algebra \( B \) and the coalgebra \( B \). Going to the functor level it turns out that \( \psi_r \) yields a mixed distributive law for the monads and comonads \( - \otimes_R B \), while \( \psi_l \) is related to the endofunctors \( B \otimes_R - \).

Given an \( R \)-bialgebra \( B \), it will sometimes help to write \( \overline{B} \) when we focus on the algebra structure and \( \overline{B} \) when focussing on the coalgebra part. From [7.1] we know:

### 8.2. Entwining maps for bialgebras

Consider an \( R \)-module \( B \) which is an \( R \)-algebra and an \( R \)-coalgebra. The following assertions are equivalent.

(a) There is an entwining map \( \psi : \overline{B} \otimes_R \overline{B} \to \overline{B} \otimes_R \overline{B} \);

(b) the monad \( \overline{B} \otimes_R - \) on \( \mathbb{M}_R \) has a lifting to a monad on \( \overline{\mathbb{M}}_R \);

(c) the comonad \( \overline{B} \otimes_R - \) on \( \mathbb{M}_R \) has a lifting to a comonad on \( \overline{\mathbb{M}}_R \);
(d) the monad $\text{Hom}_R(\overline{B}, -)$ on $\mathbb{M}_R$ has a lifting to a monad on $\mathbb{M}_B$;
(e) the comonad $\text{Hom}_R(\overline{B}, -)$ on $\mathbb{M}_R$ has a lifting to a comonad on $\mathbb{M}_{[B,-]}$.

A symmetric form of Theorem 8.2 can be obtained by interchanging left and right (co)module structures.

8.3. Entwinings and corings. Let $B$ be an $R$-bialgebra with an entwining map $\psi : \overline{B} \otimes_R \overline{B} \rightarrow \overline{B} \otimes_R \overline{B}$. Then $B \otimes_R B$ is a $\overline{B}$-coring with structure maps

$$I_B \otimes_R \Delta : B \otimes_R B \rightarrow (B \otimes_R B) \otimes_B (B \otimes_R B), \quad I_B \otimes_R \varepsilon : B \otimes_R B \rightarrow B,$$

and $\overline{B}$-actions $d \cdot (a \otimes b) \cdot c = d\psi(b \otimes c)$.

Symmetrically, an entwining map $\psi : \overline{B} \otimes_R \overline{B} \rightarrow \overline{B} \otimes_R \overline{B}$ determines a $\overline{B}$-coring $B \otimes_R B$, with structure maps

$$\Delta \otimes_R I_B : B \otimes_R B \rightarrow (B \otimes_R B) \otimes_B (B \otimes_R B), \quad \varepsilon \otimes_R I_B : B \otimes_R B \rightarrow B,$$

and $\overline{B}$-actions $d \cdot (a \otimes b) \cdot c = \psi(d \otimes a)bc$.

In particular, the entwining maps $\psi_r : \overline{B} \otimes_R B \rightarrow \overline{B} \otimes_R B$ and $\psi_l : B \otimes_R \overline{B} \rightarrow \overline{B} \otimes_R B$ in 8.1 determine $\overline{B}$-corings $B \otimes_R B$, denoted by $B \otimes_R^l B$ and $B \otimes_R^r B$, respectively.

8.4. $B$-Hopf modules. Let $B$ be an $R$-bialgebra and consider the $\overline{B}$-coring $B \otimes_R B$ in 8.3. The following structures on a right $\overline{B}$-module $M$ are equivalent:

(a) A right $B \otimes_R^l B$-comodule structure map $\varrho^M : M \rightarrow M \otimes_B (B \otimes_R B)$;

(b) a right $\overline{B}$-linear $B$-comodule structure map $\alpha^M : M \rightarrow M \otimes_B B$, (where $\overline{B}$-linearity means commutativity of the diagram

\[
\begin{array}{ccc}
M \otimes_B B & \xrightarrow{\alpha^M} & M \\
\downarrow{\alpha^M \otimes_R I_B} & & \downarrow{\alpha^M \otimes_R I_B} \\
M \otimes_B B \otimes_R B & \xrightarrow{I_M \otimes_R \varrho_r} & M \otimes_B B \otimes_R B,
\end{array}
\]

where $\alpha^M : M \otimes_B B \rightarrow M$ denotes the $\overline{B}$-action on $M$);

(c) a comodule structure for the comonad $- \otimes_R B$ on $\mathbb{M}_B$;

(d) a module structure for the monad $- \otimes_R B$ on $\mathbb{M}_B$.

A right $\overline{B}$-module $M$ with these equivalent properties is called a $B$-Hopf module. Morphisms of $B$-Hopf modules are $B \otimes_R^l B$-comodule maps. Equivalently, they are $B$-module as well as $\overline{B}$-comodule maps. The category of right $B$-Hopf modules is denoted by $\mathbb{M}_{B}^{\overline{B}}$. By the above considerations, it is isomorphic to $\mathbb{M}_{[B,-]}^{\overline{B}}$.

Based on $\varrho^M$, left $B$-Hopf modules are defined in a symmetric way. Note that a bialgebra $B$ is both a left and a right $B$-Hopf module.

From 7.2 we obtain:

8.5. $[B, -]$-Hopf modules. Let $B$ be an $R$-bialgebra and consider the $B$-coring $B \otimes_R^l B$ in 8.3. Then the following structures on a right $\overline{B}$-module $M$ are equivalent.

(a) A $[B \otimes_R^l B, -]$-module structure map $\varrho^M : \text{Hom}_B(B \otimes_R^l B, M) \rightarrow M$;

(b) a $\overline{B}$-linear $[\overline{B}, -]$-module structure map $\alpha^M : \text{Hom}_R(B, M) \rightarrow M$

(i.e. $\alpha^M(f)b = \sum \alpha^M(f(b_1))b_2$ for $f \in \text{Hom}_R(B, M), b \in B$);

(c) a module structure for the monad $\text{Hom}_R(B, -)$ on $\mathbb{M}_B$;
(d) a comodule structure for the comonad $\text{Hom}_R(B, -)$ on $\mathcal{M}_{[B,-]}$.

A right $B$-module $M$ with these equivalent properties is called a $[B,-]$-Hopf module or right Hopf contramodule for $B$. Morphisms of $[B,-]$-Hopf modules are $B \otimes_R B$-contramodule maps. Equivalently, they are $B$-module as well as $\overline{B}$-contramodule maps. The category of $[B,-]$-Hopf modules is denoted by $\mathcal{M}_{[B,-]}$. By the above considerations, it is isomorphic to $\mathcal{M}_{[B \otimes_R B,-]}$.

Based on $\psi_r$, left Hopf contramodules modules for $B$ are defined in a symmetric way.

Applying 7.3, the following alternative description of Hopf modules is obtained.

8.6. Distributive laws for bialgebras. Let $B$ be an $R$-bialgebra. Then:

1. The entwining $\psi_r$ in 8.1 induces a comonad distributive law

$$\text{Hom}_R(B, -) \otimes_R \overline{B} \to \text{Hom}_R(B, - \otimes_R \overline{B}), \quad f \otimes b \mapsto \sum f((-)_1) \otimes b((-)_2).$$

Hence $\text{Hom}_R(B, -) \otimes_R \overline{B}$ is a comonad on $\mathcal{M}_R$. The category of its comodules is isomorphic to the category of $B$-Hopf modules.

2. The entwining $\psi_l$ in 8.1 induces a monad distributive law

$$\text{Hom}_R(\overline{B}, -) \otimes_R B \to \text{Hom}_R(\overline{B}, - \otimes_R B), \quad f \otimes b \mapsto \sum f(b_2-) \otimes b_2.$$

Hence $\text{Hom}_R(\overline{B}, - \otimes_R B)$ is a monad on $\mathcal{M}_R$. The category of its modules is isomorphic to the category of $[B,-]$-Hopf modules.

8.7. Hopf algebras. An $R$-bialgebra $(H, \mu, \iota, \Delta, \varepsilon)$ is said to be a Hopf algebra if there is an $R$-module map $S: H \to H$, called the antipode, such that

$$\mu \circ (I_H \otimes_R S) \circ \Delta = \iota \circ \varepsilon = \mu \circ (S \otimes_R I_H) \circ \Delta.$$

If the antipode exists, then it is unique and it is an anti-algebra and anti-coalgebra map.

For an $R$-Hopf algebra $H$, the $H$-corings $H \otimes^r_R H$ and $H \otimes^l_R H$ in 8.3 are isomorphic via the mutually inverse maps

$$H \otimes^r_R H \to H \otimes^l_R H, \quad a \otimes b \mapsto \sum a_1 S(b_3) \otimes a_2 S(b_1) b_2;$$

$$H \otimes^l_R H \to H \otimes^r_R H, \quad a \otimes b \mapsto \sum a_1 S(a_3) b_1 \otimes S(a_2) b_2.$$

8.8. Hopf algebras and coseparability. Let $H$ be an $R$-Hopf algebra.

1. The $H$-coring $H \otimes^r_R H$ is coseparable.

2. The following functor is an equivalence:

$$\text{Hom}^{H \otimes^r_R H}(H \otimes^r_R H, -): \mathcal{M}_{H \otimes^r_R H} \to \mathcal{M}_{[H \otimes^r_R H,-]}.$$

3. The category of $H$-Hopf modules (in 8.4) and the category of $[H, -]$-Hopf modules (in 8.3) are equivalent.

Proof. (1) A cointegral is given by

$$(H \otimes^r_R H) \otimes_H (H \otimes^r_R H) \to H, \quad (a \otimes b) \otimes_H (1_H \otimes c) \mapsto a S(b) c.$$

(2) In view of (1), this is a special case of 5.8.
(3) The category of $H$-Hopf modules is isomorphic to $\mathcal{M}^{H \otimes_R H}$ and the category of $[H, -]$-Hopf modules is isomorphic to $\mathcal{M}^{H \otimes'_R H}$. So the claim follows by corestriction isomorphism $H \otimes_R H \cong H \otimes'_R H$ in \textbf{S.1} and part (2).

The final aim of this section is to characterise Hopf algebras via their induced (co)monads. The following notions were introduced in \cite{34} and \cite{22}. Note that these terms have different meanings in Moerdijk \cite{23} and Bruguières-Virelizier \cite{6}.

8.9. Bimonads and Hopf monads. A bimonad on a category $\mathcal{A}$ is a functor $F : \mathcal{A} \to \mathcal{A}$ with a monad structure $F = (F, m, i)$ and a comonad structure $\overline{F} = (F, d, e)$ subject to the compatibility conditions

(i) $e$ is a monad morphism $F \to I_\mathcal{A}$;
(ii) $i$ is a comonad morphism $I_\mathcal{A} \to F$;
(iii) there is a mixed distributive law $\Psi : FF \to FF$, satisfying $d \circ m = Fm \circ \Psi F \circFd$.

A bimonad $(F, m, i, d, e)$ is called a Hopf monad if there exists a natural transformation $S : F \to F$, called the antipode, such that $m \circ SF \circFd = i \circ e = m \circ FS \circ d$.

By \cite{22} Proposition 6.3, a $\tau$-bimonad $F$ is a bimonad with respect to the mixed distributive law $\Psi := mF \circ F \tau \circ dF$.

A $\tau$-bimonad with an antipode is called a $\tau$-Hopf monad.

As described in \cite{22}, if a $\tau$-bimonad $F$ has a left or right adjoint $G$, then the mates under the adjunction of the structure maps of the monad and comonad $F$, equip $G$ with a monad and a comonad structure, respectively. Moreover, the mate $\bar{\tau}$ of $\tau$ under the adjunction is a double entwining for $G$, and $G$ is a $\bar{\tau}$-bimonad. If $F$ is a $\tau$-Hopf monad, then $G$ is a $\bar{\tau}$-Hopf monad.

8.10. The bimonad $- \otimes_R B$. For an $R$-bialgebra $(B, \mu, \iota, \Delta, \varepsilon)$, the functor $- \otimes_R B : \mathcal{M}_R \to \mathcal{M}_R$ is a tw-bimonad, hence a bimonad with respect to the mixed distributive law $\Psi := m \otimes_R B \otimes_R \tau \otimes_R d \otimes_R B$.

By duality, $\text{Hom}_R(B, -)$ is a tw-bimonad, with coproduct $[\mu, -]$ and counit $[\iota, -]$ in \textbf{3.4} product $[\Delta, -]$ and unit $[\varepsilon, -]$ in \textbf{4.2}, where $\overline{\tau} : \text{Hom}_R(B, \text{Hom}_R(B, -)) \to \text{Hom}_R(B, \text{Hom}_R(B, -))$
is given by switching the arguments. Thus \( \text{Hom}_R(B, -) : \mathbb{M}_R \to \mathbb{M}_R \) is a bimonad with respect to the mixed distributive law \( \text{Hom}_R(\psi, -) \):

\[
\text{Hom}_R(B, \text{Hom}_R(B, -)) \cong \text{Hom}_R(B, \text{Hom}_R(B, -))
\]

\[
\text{Hom}_R(B \otimes_R B, -) \xrightarrow{\text{Hom}_R(\psi, -)} \text{Hom}_R(B \otimes_R B, -).
\]

A motivating example of a \((\text{tw})\)Hopf monad in [22] is the functor \(- \otimes_R \text{H} : \mathbb{M}_R \to \mathbb{M}_R\) induced by a Hopf algebra \(\text{H}\).

Summarising the preceding observations we obtain the following.

8.11. Characterisations of Hopf algebras. For an \(R\)-bialgebra \((\text{H}, \mu, \iota, \Delta, \varepsilon)\), the following assertions are equivalent.

(a) \(\text{H}\) is a Hopf algebra;
(b) the map \(\gamma : \text{H} \otimes_R \text{H} \xrightarrow{\Delta \otimes_R \mu} \text{H} \otimes_R \text{H} \xrightarrow{I \otimes_R \mu} \text{H} \otimes_R \text{H}\) is an isomorphism;
(c) \(\text{H}\) is an \(\text{H} \otimes_R \text{H}\)-Galois right (equivalently, left) comodule;
(d) \(\text{H}\) is an \(\text{H} \otimes_R \text{H}\)-Galois right (equivalently, left) comodule;
(e) \(- \otimes_R \text{H}\) is a \((\text{tw})\)-Hopf monad on \(\mathbb{M}_R\);
(f) for the \((\text{tw})\)-bimonad \([\text{H}, -] = \text{Hom}_R(\text{H}, -)\), the natural transformation

\[
[\gamma, -] : [\text{H}, [\text{H}, -]] \xrightarrow{[H, [\mu, -]]} [\text{H}, [\text{H}, -]] [\Delta, [\varepsilon, -]] [\text{H}, [\text{H}, -]]
\]

is an isomorphism;
(g) \(\text{Hom}_R(\text{H}, -)\) is a \((\text{tw})\)-Hopf monad on \(\mathbb{M}_R\);
(h) \(- \otimes_R \text{H} : \mathbb{M}_R \to \mathbb{M}_R^\text{H}\) is an equivalence;
(i) \(\text{Hom}_R(\text{H}, -) : \mathbb{M}_R \to \mathbb{M}_R^H\) is an equivalence;
(j) \(\text{H}\) is a \(\text{Hom}_{-, \text{H}}(\text{H} \otimes_R \text{H}, -, \text{H})\)-Galois left comodule (equivalently, a \(\text{Hom}_{\text{H}, \text{H}}(\text{H} \otimes_R \text{H}, \text{H}, -)\)-Galois right comodule);
(k) \(\text{H}\) is a \(\text{Hom}_{-, \text{H}}(\text{H} \otimes_R \text{H}, -, \text{H})\)-Galois left comodule (equivalently, a \(\text{Hom}_{\text{H}, \text{H}}(\text{H} \otimes_R \text{H}, \text{H}, -)\)-Galois right comodule).

Proof. (a)-(d) and (h) are standard equivalent characterisations of Hopf algebras, see e.g. [9, 15.2 and 15.5].

(a)\(\Leftrightarrow\)(c)\(\Leftrightarrow\)(f)\(\Leftrightarrow\)(g) is proven in [22].

(c)\(\Rightarrow\)(j) and (d)\(\Rightarrow\)(k) follow by 6.7.

(i)\(\Rightarrow\)(j) follows by 6.8 (a)\(\Rightarrow\)(c)(i).

(h)\(\Rightarrow\)(i) There is a sequence of equivalences,

\[
\mathbb{M}_R^\text{H} \cong \mathbb{M}_R^H \cong \mathbb{M}_R[H \otimes_R \text{H}, -] \cong \mathbb{M}_R[H \otimes_R \text{H}, -] \cong \mathbb{M}_R^H \cong \mathbb{M}_R^H.
\]

cf. 8.4, 8.8, 8.7 and 8.5 (note that (h)\(\Rightarrow\)(a)). Combining this composite with the equivalence in part (h), we obtain an equivalence functor

\[
\text{Hom}_R(\text{H} \otimes_R \text{H}, -) : \mathbb{M}_R \to \mathbb{M}_R^H.
\]

We claim that the functor in part (i) is naturally isomorphic to this equivalence, hence it is an equivalence, too.

The equivalence in part (h) gives rise to an \(R\)-module isomorphism

\[
\text{Hom}_R(\text{H} \otimes_R \text{H}, M \otimes_R \text{H}) \to \text{Hom}_R(\text{H}, M), \; \Psi \mapsto (I_M \otimes_R \varepsilon) \circ \Psi(- \otimes_R \iota),
\]

Proof. (a)-(d) and (h) are standard equivalent characterisations of Hopf algebras, see e.g. [9, 15.2 and 15.5].
for any $R$-module $M$, that is natural in $M$. Using the coring isomorphism $H \otimes_R H \cong H \otimes_R \delta H$ in $\mathcal{X}$ we can transfer it to a natural isomorphism

$$
\beta_M : \text{Hom}_{H \otimes_R H}^H(\delta H, M \otimes_R H) \to \text{Hom}_R(H, M), \Phi \mapsto (I_M \otimes_R \varepsilon) \circ \Phi \circ \Delta.
$$

An easy computation shows that $\beta_M$ is a morphism of $[H \otimes_R H, -]$-modules, what completes the proof. □

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