Testing GR with Galactic-centre Stars

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Abstract. The Galactic Centre S-stars orbiting the central supermassive black hole reach velocities of a few percent of the speed of light. The GR-induced perturbations to the redshift enter the dynamics via two distinct channels. The post-Newtonian regime perturbs the orbit from the Keplerian (Zucker et al., 2006, Kannan & Saha 2009), and the photons from the Minkowski (Angélil & Saha 2010). The inclusion of gravitational time dilation at $O(v^2)$ marks the first departure of the redshift from the line-of-sight velocities. The leading-order Schwarzschild terms curve space, and enter at $O(v^3)$. The classical Keplerian phenomenology dominates the total redshift. Spectral measurements of sufficient resolution will allow for the detection of these post-Newtonian effects. We estimate the spectral resolution required to detect each of these effects by fitting the redshift curve via the five keplerian elements plus black hole mass to mock data. We play with an exaggerated S2 orbit - one with a semi-major axis a fraction of that of the real S2. This amplifies the relativistic effects, and allows clear visual distinctions between the relativistic terms. We argue that spectral data of S2 with a dispersion $\sim 10\text{ km s}^{-1}$ would allow for a clear detection of gravitational redshift, and $\sim 1\text{ km s}^{-1}$ would suffice for leading-order space curvature detection.

1. Introduction

The stars orbiting the supermassive black hole ($M \approx 4.4 \cdot 10^6 M_\odot$) within the central arcsecond are on highly relativistic orbits. In comparison, the velocity of a geosynchronous Earth satellite is $v_{\text{satell}} \approx 0.00005c$. Mercury, whose orbital Schwarzschild precession has been measured has $v_{\text{merc}} \approx 0.00016c$. Binary pulsar systems manage to reach $v_{\text{binary pulsar}} \sim 0.003c$, while the galactic centre S-Stars boast $v_{S2} \sim 0.03c$. This, due to their proximity to the black hole during pericenter passage (down to $\sim 3000$ of the gravitational radius) make this class of stars the fastest resolvable ballistic objects known, and allow for the prospect of detecting post-Newtonian effects.

The dynamics of the orbit and light trajectories provide an opportunity to test the form of the metric, and in doing so, General Relativity. The Kerr metric is the external solution to the Einstein Field Equations for a rotating body. The geodesics of such a space-time exhibit some well-known features: gravitational time dilation, prograde precession, lensing, and frame-dragging. Gravitational time dilation and precession are orbital effects. The former due to a temporal stretching, and the latter due to space curvature. Such a curvature also affects photon trajectories — veering the trajectories away from those of straight lines.

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Each of these effects contributes to the redshift of the Galactic-centre stars in a distinct manner.

These features, although markedly distinct, are all due to the same metric. The post-Newtonian formalism, valid provided \( r \gg 0 \), allows us to cleanly disentangle these effects, and investigate the detection of each separately. Not only are the stars on post-Keplerian orbits, but the photons must travel through spacetime on nontrivial paths before arriving at Earth. This nontrivial path through spacetime affects the time of arrival, and therefore the redshift.

2. Model

The star’s orbit can be described by the Hamiltonian (Angéil & Saha 2010)

\[
\mathcal{H}_{\text{star}} = -\frac{p_t^2}{2} + \frac{p_r^2}{2} + \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2r^2} \sin^2 \theta - \frac{p_t^2}{r} \propto v^1 \quad \text{No gravity}
\]

\[
+ \frac{p_r^2}{2} + \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2r^2} \sin^2 \theta - \frac{p_t^2}{r} \propto v^1, v^2 \quad \text{Kepler + Time-dilation}
\]

\[
-\frac{p_t^2}{r} - \frac{p_r^2}{r} \propto v^3 \quad \text{space curvature}
\]

\[
+ \text{frame dragging, torquing, ...}
\]

At leading order, the system is spatially invariant, and the star feels no acceleration. At next-to-leading order, gravity debuts. Classically, the potential term is \( 1/r \). GR however demands the modification to \( p_t^2/r \), which results in gravitational time-dilation \( \propto v^2 \), a consequence of the Einstein Equivalence Principle. Spatially the problem has remained unchanged. However, because the time-dilation term affects the photon arrival times, the redshift is affected. Space curvature enters one order higher. This is the leading-order Schwarzschild term, and causes the orbit to precess. Higher order effects, such as frame-dragging and torquing, we choose not to delve into here.

The Hamiltonian governing photon paths, being null, contains a different selection of pre-truncation terms.

\[
\mathcal{H}^{\text{null}} = -\frac{p_t^2}{2} + \frac{p_r^2}{2} + \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2r^2} \sin^2 \theta \propto v^0 \quad \text{Minkowski}
\]

\[
-\frac{p_t^2}{r} - \frac{p_r^2}{r} \propto v^3 \quad \text{space curvature}
\]

\[
+ \text{frame dragging, torquing, ...}
\]

At leading order, the trajectories are straight lines. Lensing occurs at \( \mathcal{O}(v^3) \) via the leading order Schwarzschild contribution. There is no contribution at \( \mathcal{O}(v^4) \). For photons, the frame-dragging term debuts at \( \mathcal{O}(v^5) \) along with higher-order Schwarzschild terms, and spin-induced torquing terms. In this work, we consider effects only up to \( \mathcal{O}(v^3) \) for both the null and timelike cases.

*For a maximally spinning black hole, the frame-dragging photon signal on S2's redshift at pericenter is \( \sim 10 \text{ m s}^{-1} \) — two orders of magnitude weaker than the Schwarzschild photon
3. Calculating the redshift

To calculate the redshift curve of the star, we integrate the star’s orbit using timelike solutions to (1), and then, on chosen points along the star’s orbit, we find the paths of those particular photons emitted by the star which hit the observer (Figure 1). To do this, the initial angular momentum of the trajectories (corresponding to null solutions of (2)) is varied until the termination position of the photon converges on the observer position. Once the trajectories of these photons are known, the redshift may then be calculated directly from the definition:

\[
z = \frac{t_{a2} - t_{a1}}{\tau_{e2} - \tau_{e1}} - 1, \tag{3}
\]

where \(\tau_{e1}\) and \(\tau_{e2}\) are the proper times of a pair of photons emitted at neighbouring points on the star’s orbit, and \(t_{a1}\) and \(t_{a2}\) are their respective arrival times.

4. Post-Newtonian detection

The redshift curve of a galactic centre star is dependent on a handful of parameters. These include the Keplerian elements, the black hole mass, as well as discrete parameters which toggle the post-Newtonian terms. We proceed as follows. In order to put upper bounds on the spectral resolution required to detect the post-Newtonian effects, we generate mock spectral data consisting of 200 data points with a chosen dispersion, using relativistic terms up to \(O(v^3)\) for both the null and timelike metrics, and determine whether or not we are able recover the parameter values by fitting with these effects turned off. For illustrative purposes, we consider an exaggerated S2 orbit. For the semi-major axis, we take \(a = a_{S2}/100\). In doing so, \(v = 10v_{S2}\) — the classical contribution to the redshift is raised 10-fold. The redshift due to gravitational time dilation, which enters the dynamics at \(O(v^2)\) is increased 100-fold. The space curvature redshift contribution, entering at \(O(v^3)\), is enhanced 1000-fold. Figure 2 shows the results of the fitting procedure. The classical fit manages a \(\chi^2_{\text{red}} = 4.88\), the time dilation fit \(\chi^2_{\text{red}} = 2.68\), and the space curvature fit \(\chi^2_{\text{red}} = 1.07\). Hence, a spectral dispersion of \(10^3\) km s\(^{-1}\) suffices for clear visual and numerical distinction of these relativistic effects. In undoing our exaggeration of the orbit, we argue that a spectral dispersion of \(\sim 10\) km s\(^{-1}\) would allow for a clear detection of gravitational redshift for the real S2, and \(\sim 1\) km s\(^{-1}\) would yield a test for space curvature.

References

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Figure 1. Schematic illustration of the redshift calculation method. Each photon shot by the star hits the observer. Each of the above cases yields four points on the redshift curve. The first panel shows Minkowski photons, the second space-curved photons, and the third frame-dragged photons.

Figure 2. Our S2-like star has $a = a_{S2}/100$. The mock redshift data in the above examples is generated all in the same way: 200 data points are distributed along the complete orbit with a dispersion of $10^3$ km s$^{-1}$, gravitational time dilation and space curvature are all turned on. In the first panel, we fit with gravitational time dilation turned on, and space curvature turned off. For the second, we turn gravitational time dilation on, and for the third, we further turn space curvature on. The mismatch between the fit and the simulated data is discernable in the first two panels. Only in the last panel, when all the effects are included in the fit, is a satisfactory fit with $\chi^2_{\text{red}} \approx 1$ obtained.

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