An error bound in the Sudakov-Fernique inequality

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Abstract

We obtain an asymptotically sharp error bound in the classical Sudakov-Fernique comparison inequality for finite collections of gaussian random variables. Our proof is short and self-contained, and gives an easy alternative argument for the classical inequality, extended to the case of non-centered processes.

1 Statement of the result

Gaussian comparison inequalities are among the most important tools in the theory of gaussian processes, and the Sudakov-Fernique inequality (named after Sudakov [11, 12] and Fernique [3]) is perhaps the most widely used member of that class.

We will concentrate on the Sudakov-Fernique inequality in this article; general discussions about comparison inequalities can be found in Adler [1], Fernique [4], Ledoux & Talagrand [9], and Lifshits [10].

The classical Sudakov-Fernique inequality goes as follows:

**Theorem 1.1.** [Sudakov-Fernique inequality] Let \( \{X_i, i \in I\} \) and \( \{Y_i, i \in I\} \) be two centered gaussian processes indexed by the same indexing set \( I \). Suppose that both the processes are almost surely bounded. For each \( i, j \in I \), let \( \gamma_{ij}^X = \mathbb{E}(X_i - X_j)^2 \) and \( \gamma_{ij}^Y = \mathbb{E}(Y_i - Y_j)^2 \). If \( \gamma_{ij}^X \leq \gamma_{ij}^Y \) for all \( i, j \), then \( \mathbb{E}(\sup_{i \in I} X_i) \leq \mathbb{E}(\sup_{i \in I} Y_i) \).

As mentioned before, this inequality is attributed to Sudakov [11, 12] and Fernique [3]. Later proofs were given in Alexander [2] and an unpublished work of S. Chevet. Important variants were proved by Gordon [5, 6, 7] and Kahane [8]. More recently, Vitale [14] has shown, through a clever argument, that we only need \( \mathbb{E}(X_i) = \mathbb{E}(Y_i) \) instead of \( \mathbb{E}(X_i) = \mathbb{E}(Y_i) = 0 \) in the hypothesis of Theorem 1.1. We will prove the following result, which gives an sharp error bound when the indexing set is finite, and also contains Vitale’s extension of the Sudakov-Fernique inequality.

**Theorem 1.2.** Let \( (X_1, \ldots, X_n) \) and \( (Y_1, \ldots, Y_n) \) be gaussian random vectors with \( \mathbb{E}(X_i) = \mathbb{E}(Y_i) \) for each \( i \). For \( 1 \leq i, j \leq n \), let \( \gamma_{ij}^X = \mathbb{E}(X_i - X_j)^2 \) and \( \gamma_{ij}^Y = \mathbb{E}(Y_i - Y_j)^2 \), and let \( \gamma = \max_{1 \leq i, j \leq n} |\gamma_{ij}^X - \gamma_{ij}^Y| \). Then

\[
|\mathbb{E}(\max_{1 \leq i \leq n} X_i) - \mathbb{E}(\max_{1 \leq i \leq n} Y_i)| \leq \sqrt{\gamma \log n}.
\]

Moreover, if \( \gamma_{ij}^X \leq \gamma_{ij}^Y \) for all \( i, j \), then \( \mathbb{E}(\max_i X_i) \leq \mathbb{E}(\max_i Y_i) \).
The asymptotic sharpness of the error bound is easy to see from the case where all the \(X_i\)'s are independent standard normals and all the \(Y_i\)'s are zero.

## 2 Proof

We first need to state the following well-known “integration by parts” lemma:

**Lemma 2.1.** If \(F : \mathbb{R}^n \to \mathbb{R}\) is a \(C^1\) function of moderate growth at infinity, and \(X = (X_1, \ldots, X_n)\) is a centered Gaussian random vector, then for any \(1 \leq i \leq n\),

\[
\mathbb{E}(X_i F(X)) = \sum_{j=1}^{n} \mathbb{E}(X_i X_j) \mathbb{E} \left( \frac{\partial F}{\partial x_i}(X) \right).
\]

A proof of this lemma can be found in the appendix of [13], for example.

**Proof of Theorem 1.2.** Let \(X = (X_1, \ldots, X_n)\) and \(Y = (Y_1, \ldots, Y_n)\). Without loss of generality, we may assume that \(X\) and \(Y\) are defined on the same probability space and are independent. Fix \(\beta > 0\), and define \(F_\beta : \mathbb{R}^n \to \mathbb{R}\) as:

\[
F_\beta(x) := \beta^{-1} \log \left( \sum_{i=1}^{n} e^{\beta x_i} \right).
\]

(Note that \(x\) denotes the vector \((x_1, \ldots, x_n)\), a convention that we shall follow throughout.) Now, for each \(i\), let \(\mu_i = \mathbb{E}(X_i) = \mathbb{E}(Y_i)\), \(\tilde{X}_i = X_i - \mu_i\), and \(\tilde{Y}_i = Y_i - \mu_i\). For \(1 \leq i, j \leq n\), let \(\sigma_{ij}^X = \mathbb{E}(\tilde{X}_i \tilde{X}_j)\) and \(\sigma_{ij}^Y = \mathbb{E}(\tilde{Y}_i \tilde{Y}_j)\). For \(0 \leq t \leq 1\) define the random vector \(Z_t = (Z_{t,1}, \ldots, Z_{t,n})\) as

\[
Z_{t,i} = \sqrt{1-t} \tilde{X}_i + \sqrt{t} \tilde{Y}_i + \mu_i.
\]

For all \(t \in [0, 1]\), let \(\varphi(t) = \mathbb{E}(F_\beta(Z_t))\). Then \(\varphi\) is differentiable, and

\[
\varphi'(t) = \mathbb{E} \left[ \sum_{i=1}^{n} \frac{\partial F_\beta}{\partial x_i}(Z_t) \left( \frac{\tilde{Y}_i}{2\sqrt{t}} - \frac{\tilde{X}_i}{2\sqrt{1-t}} \right) \right].
\]

Again, for any \(i\), Lemma 2.1 gives us

\[
\mathbb{E} \left( \frac{\partial F_\beta}{\partial x_i}(Z_t) \tilde{X}_i \right) = \sqrt{1-t} \sum_{j=1}^{n} \sigma_{ij}^X \mathbb{E} \left( \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(Z_t) \right)
\]

and

\[
\mathbb{E} \left( \frac{\partial F_\beta}{\partial x_i}(Z_t) \tilde{Y}_i \right) = \sqrt{t} \sum_{j=1}^{n} \sigma_{ij}^Y \mathbb{E} \left( \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(Z_t) \right).
\]

Combining, we have

\[
\varphi'(t) = \frac{1}{2} \sum_{1 \leq i, j \leq n} \mathbb{E} \left( \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(Z_t) \right) (\sigma_{ij}^Y - \sigma_{ij}^X).
\]
Now
\[ \frac{\partial F_\beta}{\partial x_i}(x) = p_i(x) := \sum_{j=1}^n e^{\beta x_j}. \]

Note that for each \( x \in \mathbb{R}^n \), the numbers \( p_1(x), \ldots, p_n(x) \) as defined above are nonnegative and sum to 1. In other words, they induce a probability measure on \( \{1, 2, \ldots, n\} \). It is straightforward to verify that
\[ \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(x) = \begin{cases} \beta (p_i(x) - p_i(x)^2) & \text{if } i = j, \\ -\beta p_i(x)p_j(x) & \text{if } i \neq j. \end{cases} \]

Thus,
\[ \sum_{1 \leq i, j \leq n} \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(x)(\sigma_{ij}^Y - \sigma_{ij}^X) = \beta \sum_{i=1}^n p_i(x)(\sigma_{ii}^Y - \sigma_{ii}^X) - \beta \sum_{1 \leq i, j \leq n} p_i(x)p_j(x)(\sigma_{ij}^Y - \sigma_{ij}^X). \]

Now observe that since \( \sum_{i=1}^n p_i(x) = 1 \), therefore
\[ \sum_{i=1}^n p_i(x)(\sigma_{ii}^Y - \sigma_{ii}^X) = \frac{1}{2} \sum_{1 \leq i, j \leq n} p_i(x)p_j(x)(\sigma_{ii}^Y - \sigma_{ii}^X + \sigma_{jj}^Y - \sigma_{jj}^X). \]

Combining, we have
\[ \sum_{1 \leq i, j \leq n} \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(x)(\sigma_{ij}^Y - \sigma_{ij}^X) = \beta \sum_{1 \leq i, j \leq n} p_i(x)p_j(x)(\gamma_{ij}^Y - \gamma_{ij}^X). \]

Now note that
\[ \sigma_{ii}^X + \sigma_{jj}^X - 2\sigma_{ij}^X = \mathbb{E}(\tilde{X}_i - \tilde{X}_j)^2 = \mathbb{E}(X_i - X_j)^2 - (\mu_i - \mu_j)^2 \]
and similarly
\[ \sigma_{ii}^Y + \sigma_{jj}^Y - 2\sigma_{ij}^Y = \mathbb{E}(\tilde{Y}_i - \tilde{Y}_j)^2 = \mathbb{E}(Y_i - Y_j)^2 - (\mu_i - \mu_j)^2. \]

Therefore,
\[ \sum_{1 \leq i, j \leq n} \frac{\partial^2 F_\beta}{\partial x_j \partial x_i}(x)(\sigma_{ij}^Y - \sigma_{ij}^X) = \beta \sum_{1 \leq i, j \leq n} p_i(x)p_j(x)(\gamma_{ij}^Y - \gamma_{ij}^X). \]

Thus, if \( \gamma_{ij}^X \leq \gamma_{ij}^Y \) for all \( i, j \), then \( \varphi'(t) \geq 0 \) for each \( t \), which implies
\[ \mathbb{E}(F_\beta(Y)) = \varphi(1) \geq \varphi(0) = \mathbb{E}(F_\beta(X)). \]
Now observe that
\[
\max_i x_i = \beta^{-1} \log e^{\beta \max_i x_i} \\
\leq \beta^{-1} \log \left( \sum_i e^{\beta x_i} \right) \\
\leq \beta^{-1} \log (ne^{\beta \max_i x_i}) \\
= \beta^{-1} \log n + \max_i x_i.
\]

(2)

In other words, \( \max x_i \leq F_\beta(x) \leq \beta^{-1} \log n + \max x_i \).

Thus, taking \( \beta \to \infty \) in (1), we get the second assertion of the theorem. For the first, note that with \( \gamma = \max_{1 \leq i,j \leq n} |\gamma_{ij}^Y - \gamma_{ij}^X| \), we have
\[
\left| \sum_{1 \leq i,j \leq n} \frac{\partial^2 F_\beta(x)}{\partial x_j \partial x_i} (\sigma_{ij}^Y - \sigma_{ij}^X) \right| \leq \frac{\beta \gamma}{2} \sum_{1 \leq i,j \leq n} p_i(x)p_j(x) = \frac{\beta \gamma}{2}.
\]

This shows that
\[
|\mathbb{E}(F_\beta(Y)) - \mathbb{E}(F_\beta(X))| \leq \frac{\beta \gamma}{4}.
\]

Combined with (2), this gives
\[
|\mathbb{E}(\max_i Y_i) - \mathbb{E}(\max_i X_i)| \leq \frac{\beta \gamma}{4} + \frac{\log n}{\beta}.
\]

Choosing \( \beta = 2 \sqrt{\frac{\log n}{\gamma}} \) gives the desired result.

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