The Complexity of Max-Min $k$-Partitioning

Anisse Ismaili

ABSTRACT

In this paper we study a max-min $k$-partition problem on a weighted graph, that could model a robust $k$-coalition formation. We settle the computational complexity of this problem as complete for class $\Sigma^P_2$. This hardness holds even for $k = 2$ and arbitrary weights, or $k = 3$ and non-negative weights, which matches what was known on MaxCut and Min-3-Cut one level higher in the polynomial hierarchy.

KEYWORDS

$k$-Partition; Robustness; Complexity

1 PRELIMINARIES

A max-min $k$-partition instance is defined by $(N, L, w, k, m, \theta)$. Let $N = [n]$, where $n \in \mathbb{N}$ is a set of nodes. The set of links $L \subseteq \binom{N}{2}$ consists of unordered node pairs. Link $\ell = (i, j)$ maps to weight $w_{\ell} \in \mathbb{Z}$. Equivalently, $w : L^2 \rightarrow \mathbb{Z}$ satisfies for any $(i, j) \in L^2$ that $w(i, j) = 0, w(i, j) = w(j, i)$ and $w(i, j) \neq 0 = \{i, j\} \in L$. $k$ is the size of a partition, $2 \leq k < n$. $m \in \mathbb{N}$ is the number of nodes that could be removed. $\theta \in \mathbb{Z}$ is a threshold value.

Let $\pi$ denote a $k$-partition of $N$, which is a collection of node subsets $\{S_1, \ldots, S_k\}$, such that for each $i \in [k], S_i \subseteq N$, and $\forall S_i, S_j \in \pi$, where $i \neq j, S_i \cap S_j = \emptyset$ holds. We say that a $k$-partition $\pi$ is complete when $\bigcup_{i \in [k]} S_i = N$ holds (otherwise, it is incomplete). For a complete partition $\pi$ and an incomplete partition $\pi'$, we say that $\pi$ subsumes $\pi'$ if $S_i \supseteq S_i'$ holds for all $i \in [k]$. For node $i \in N$, $\pi(i)$ is the node subset to which it belongs. For any $S \subseteq N$, define $W(S) = \sum_{i, j \in S} w(i, j)$.

Then, let $W(\pi)$ denote $\sum_{S \in \pi} W(S)$. We require that no node subset be empty; hence, if some node subset is empty, we set $W(\pi) = -\infty$.

Given a $k$-partition $\pi = \{S_1, \ldots, S_k\}$ and a set $M \subseteq N$, the remaining incomplete partition $\pi_{-M}$ after removing $M$ is defined as $\{S_1', \ldots, S_k'\}$, where $S_i' = S_i \setminus M$. Let $W_{-M}(\pi)$ denote the minimum value after removing at most $m$ nodes, i.e., it is defined as:

$$W_{-M}(\pi) = \min_{M \subseteq N, |M| \leq m} \{W(\pi_{-M})\}.$$ 

To obtain $W_{-M}(\pi) \neq -\infty$, every $S \in \pi$ needs to contain at least $m+1$ nodes, so that no node subset of $\pi_{-M}$ is empty. For partition $\pi = \{S_1, \ldots, S_k\}$, we define its deficit count $df(\pi)$ as $\sum_{i \in [k]} \max(0, m + 1 - |S_i|)$. Thus, $df(\pi) = 0$ must hold in order to obtain $W_{-M}(\pi) \neq -\infty$.

Definition 1.1. The decision version (1) of our main problem is defined below. It may also be referred to as the defender’s problem.

1. **Max-Min-$k$-Partition**: Given a max-min $k$-partition instance, is there any $k$-partition $\pi$ satisfying $W_{-M}(\pi) \geq \theta$?

2. **Max-Min-$k$-Partition/Verif**: Given an instance of a max-min $k$-partition and a partition $\pi$, does $W_{-M}(\pi) \geq \theta$ hold?

A key step is to study the natural verification problem (2), to which complement we refer as the attacker’s problem. (Does an attack $M \subseteq N, |M| \leq m$ on $\pi$ exist such that $W(\pi_{-M}) \leq \theta - 1$?)

2 COMPLEXITY OF MAX-MIN-$k$-PARTITION

In this section, we address the computational complexity of the defender’s problem. The verification (resp. attacker’s) problem itself turns out to be coNP-complete (resp. NP-complete), which incites one more level in the polynomial hierarchy (PH). We show that Max-Min-$k$-Partition is complete for class $\Sigma^P_2$, even in two cases:

(a) when $k = 2$ for arbitrary link weights $w \leq 0$, or

(b) when $k = 3$ for non-negative link weights $w \geq 0$.

These results seem to match what was known on MaxCut [3] (contained in Min-2-Cut when $w \leq 0$ and NP-complete) and Min-3-Cut [1] (NP-complete for $w \geq 0$ when one node is fixed in each node subset), but one level higher in PH.

**Observation 1.** Max-Min-$k$-Partition/Verif is coNP-complete. It holds even for $k = 1$, weights $w \in \{0, 1\}$ and threshold $\theta = 1$.

**Proof.** Decision problem Max-Min-$k$-Partition/Verif is in class coNP, since for any no-instance, a failing set $M$ such that $W(\pi_{-M}) \leq \theta - 1$ is a no-certificate verifiable in polynomial-time.

We show coNP-hardness by reduction from MinVertexCover to the (complement) attacker’s problem. Let graph $G = (V, E)$ and vertex number $m \in \mathbb{N}$ be any instance of MinVertexCover. MinVertexCover asks whether there exists a vertex subset $U \subseteq V, |U| \leq m$ such that $\forall (i, j) \in E, i \in U \lor j \in U$, i.e. every edge is covered by a vertex in $U$. We reduce it to an attacker’s instance with nodes $N \equiv V$, weights $w(i, j) \in \{0, 1\}$ equal to one if and only if $(i, j) \in E$ and threshold $\theta = 1$. The verified partition is simply $\pi = \{N\}$. The idea is that constraints $W(\pi_{-M}) \leq 0$ is equivalent to damaging every link, hence to finding a vertex-cover $U \equiv M$ with $|M| \leq m$. □

We now proceed with the computational complexity of the main defender’s problem under $w \leq 0$ and $w \geq 0$. We show $\Pi^P_2$-hardness of the ∀∃ complement by reduction from MaxMinVertexCover or ∀∃SAT. The idea is to (1) enforce that only some proper partitions are meaningful. One possible proper partition corresponds to
Theorem 2.1. Problem Max-Min-k-Partition is \(\Sigma^P_2\)-complete, even for \(k = 2\) node-subsets and \(w \in \{-n^2, 1, 2\}\).

Proof. Decision problem Max-Min-k-Partition asks whether \(\exists k\)-partition \(\pi, \forall M \subseteq N, |M| \leq m, W(\pi, M) \geq 0\). Therefore, it lies in class \(\Sigma^P_2\), since, for yes instances, such a \(k\)-partition is a certificate that can be verified by an NP-oracle on the remaining coNP problem Max-Min-k-Partition/Verif. We show \(\Sigma^P_2\)-hardness by a (complementary) reduction from \(\Pi^P_2\)-complete problem MaxMinVertexCover, defined as follows. Given graph \(G = (V, E)\), for function \(p : I \rightarrow \{0, 1\}\), we define \(V(\pi) = \bigcup_{i \in I} V_i, p(i)\) and induced subgraph \(G(\pi) = (V(\pi), E(p))\). Given \(m \in N\), it asks whether:

\[ \forall p : I \rightarrow \{0, 1\}, \quad \exists V \subseteq V(P), |U| \leq m, \quad U \text{ is a vertex cover of } G(p). \]

where \(U\) is a vertex cover of \(G^{(p)}\) means \(\forall (u, v) \in E(V^{(p)}), u \in U \) or \(v \in U\). Since edges between \(V_i, 0\) and \(V_i, 1\) are never relevant, we can remove them. By [4, Th. 10, proof], all \(V_j, i\) sets have the same size, hence \(\forall p^{(p)}\) has a constant size \(n\) for any \(p\).

The reduction is described in Figure 1. We reduce any instance of MaxMinVertexCover (as described above) to the following complementary instance of Max-Min-k-Partition. Nodes \(N \equiv V\) are identified with vertices, hence can also be partitioned by \(I \times \{0, 1\}\) into \(N = \bigcup_{i \in I} (N_{i, 0} \cup N_{i, 1})\) with \(N_{i, j} \equiv V_j, i\). We ask for \(k = 2\) node-subsets and choose a large number \(\Lambda\), e.g. \(\Lambda = n^2\). For every link \(\{i, j\} \in N\), if \(\{i, j\} \in E\), we define \(w(i, j) = 2\); otherwise \(\{i, j\} \notin E\), we define \(w(i, j) = 1\). The case \(\{i, j\} \in E\) is trivially satisfied on 2-partitions \(\pi\) where for \(\ell \in I\), two nodes \(\{i, j\} \in N_{\ell, 0} \times N_{\ell, 1}\) in the same node-subset. Indeed, even with a empty attack \(M = \emptyset\), weight \(W(\pi, M)\) incurs synergy \(w(i, j) = -\Lambda\) and \(W(\pi, M) < 0 \leq \text{fn}(m)\). Therefore, the interesting part of this condition is the other 2-partitions: the proper 2-partitions \(\pi = \{S_1, S_2\}\), which satisfy \(\forall \ell \in I, (v, i, j) \in N_{\ell, 0} \times N_{\ell, 1}, \pi(i) \neq \pi(j)\) (it’s easy to see that \(\pi\) can be characterized by a function \(p : I \rightarrow \{0, 1\}\) such that \(S_1 = \bigcup_{i \in I} N_{i, p(i)}\) and \(S_2 = \bigcup_{i \in I} N_{i, 1-p(i)}\), and \(|S_1| = |S_2| = n\). Since the remaining weights inside \(S_1\) and \(S_2\) are positive, the largest failures are the most damaging, \(|M| = 2m\) holds.

We now define function \(\text{fn}\). It maps \(x \in [0, 2m]\) to the number of in-subset pairs in a proper 2-partition \(\pi = \{S_1, S_2\}\) \(|(|S_1| = |S_2| = n)\) after \(x\) nodes fail in \(S_1\) and \(2m - x \in S_2\) (total 2m failures). One has:

\[ \text{fn}(m)(x) = 2^{\left(\begin{array}{c} n \\ 2 \end{array}\right)} - x \sum_{i=1}^{m-x} \sum_{j=1}^{n-j} g_{n, m} + x(2m), \]

where \(g_{n, m}\) is constant w.r.t. \(x\). Since \(\text{fn}(m)(x) = 2(x - m)\) and \(\text{fn}^*(m)(x) = 2\), it is a strictly convex function with minimum point at \(x = m\). Therefore, for integers \(x \in [2m]\), if \(x \neq m\), the inequality \(\text{fn}(m)(x) > \text{fn}(m)(m)\) holds. By definition, \(\text{fn}(m)(x)\) is a lower bound on \(W(\pi, M)\) (by assuming that all remaining weights in \(\pi, M\) have a value of 1, instead of 1 or 2). Therefore, the main condition can only be satisfied by balanced failures \(M = M_1 \cup M_2\) such that \(M_1 \subseteq S_1, M_2 \subseteq S_2\) and crucially: \(|M_1| = |M_2| = m\).

(Yes=Yes) Any subgraph \(G^{(p)}\) admits a vertex cover \(U \subseteq V(P)\) with \(|U| \leq m\). Let us show that any proper 2-partition \(\pi = \{S_1, S_2\}\) (characterized by a function \(p : I \rightarrow \{0, 1\}\)) can be failed down to \(\text{fn}(m)(m)\). Let \(M_1 \subseteq S_2\) correspond to the vertex cover of subgraph \(G^{(p)}\) and \(M_2 \subseteq S_2\) to the vertex cover of subgraph \(G^{(1-p)}\). Then, the failing set \(M = M_1 \cup M_2\) is a size of \(|M| \leq 2m\), is balanced, and any node pair \((i, j)\) of weight two in \(\pi\) (edge in \(E\)) has \(i \neq j\) in \(M\), by the vertex covers. All in all, \(|W(\pi, M) - \text{fn}(m)\) (Yes=Yes) Any proper 2-partition \(\pi = \{S_1, S_2\}\) (characterized by function \(p : I \rightarrow \{0, 1\}\)) admits a well balanced failing set \(M = M_1 \cup M_2\) such that \(W(\pi, M) \leq \text{fn}(m)(m)\). Then it must be the case that \(M_1\) and \(M_2\) covers all the node pairs of synergy two in \(S_1\) (resp. \(S_2\) that correspond to the edges of \(G^{(p)}\) resp. \(G^{(1-p)}\)). Then, for any subgraph \(G^{(p)}\) attack \(U \equiv M_1\) is a vertex cover.

Adding a constant to all weights does not preserve optimal solutions. Thus, we cannot modify a problem with negative weights to an equivalent non-negative weight problem. Still, a hardness result for \(k = 3\) can also be obtained from \(\forall \exists \Theta SAT\).

Theorem 2.2. Max-Min-k-Partition is \(\Sigma^P_2\)-complete, even for \(k = 3\) node-subsets and weights \(w \in \{0, \Lambda, \Lambda + 1\}\), where \(\Lambda \geq n^2\).

Proof. Let us first recall a classical reduction from 3SAT to IndependentSet, and how the later relates to VertexCover. Let any 3SAT instance be defined by formula \(F = C_1 \land \ldots \land C_a\), where \(C_1\) is a 3-clause on variables \(X\). Every clause \(C_i = \ell_{i, 1} \lor \ell_{i, 2} \lor \ell_{i, 3}\) is reduced to triangle of vertices \(V_i = \{v_{i, 1}, v_{i, 2}, v_{i, 3}\}\) representing the literals of the clause. The set of \(3\alpha\) vertices is then \(V = \bigcup_{i=1}^{3} V_i\). Between any two subsets \(V_i, V_j\), edges exist between two vertices if and only if the corresponding literals are on the same variable and are complementary (hence incompatible). It is easy to see that an independent-set \(U \subseteq V\) of size \(a\) must have exactly one vertex per triangle \(V_i\), and will exist (no edges within) if and only if there exists an instantiation of \(X\) that makes at least one literal per clause \(C_i\) true. Given a graph \(G = (V, E)\), if \(U \subseteq V\) is an independent-set, it means that \(i \in U \land j \in U \Rightarrow (i, j) \notin E\). Hence, contrapositron \((i, j) \in E \Rightarrow (i \in V \cup j \in V) \lor (j \in U \cup i \in U)\) means that \(V \cup U\) is a vertex cover. For instance, in the reduction from 3SAT, one can equivalently ask...
We extend this proof to including clauses with no negative literals. The rationale is to forbid two inconsistent scenarios on a same node-subset, that is, two vertices per triangle \( V_i \); Set \( V_i \) of three vertices shall have no edge left to cover.

Let any instance of \( \exists \mathbb{S} \text{AT} \) be defined by 3CNF formula \( F(X,Y) = \bigwedge_{i=1}^n C_i \) on variables \( X = \{ x_1, \ldots, x_N \} \) and \( Y = \{ y_1, \ldots, y_M \} \). This problem asks whether:

\[
\forall_{x_i}: x_i \rightarrow \{0, 1\}, \quad \exists_{y_j}: y_j \rightarrow \{0, 1\}, \quad F(x_i, y_j) \text{ is true.}
\]

Without loss of generality, one can assume there is at most one \( X \)-literal per clause \( C_i \). Indeed, if there are three \( X \)-literals, some \( x_i \) can make the clause false, and it is trivially a no-instance. If there are two \( X \)-literals: \( C = x \lor x' \lor y \), then adding a fresh \( Y \)-variable \( z \), one easily obtains \( C = (x \lor z \lor y) \land (x' \lor z \lor y) \). For ease of presentation, we assume exactly one \( X \)-literal and two \( Y \)-literals. We extend this proof to including clauses with no \( X \)-literal, in its final remark. Let \( X(C) \) be the \( X \)-literal in clause \( C \).

We build a \( \text{MAX-MIN-3-PARTITION} \) instance on \( n = 10a + 2 \) nodes with \( m = 2a \) failures. We first describe the nodes. To define the weights, we denote a number \( X \) or \( Y \) for a vertex cover \( \mathcal{C} \) and a total of 4 \( X \)-nodes, 3 \( Y \)-nodes, and a total of 4 \( \mu \)-variables. This construct is depicted in Figure 3.

Whether the \(\Lambda\) is a link or a node, we define the number \( \Lambda \rightarrow 1 \), and only one 3-partition can be linked with weight \( \Lambda \) or \( \Lambda \rightarrow 1 \). We call \(\Lambda\)-link any link with weight \( \Lambda \) or \( \Lambda \rightarrow 1 \). We call 1-link any link with weight \( 1 \) or \( \Lambda \rightarrow 1 \). We call 1-node any node that is the negation of the former's. We achieve this construct by defining threshold \( \theta \) as:

\[
\theta - 1 = \left| \frac{2m}{2} \right| \Lambda + 2 \left( \frac{m + 1}{2} \right) \Lambda + \mu_1 + \mu_2,
\]

and asking whether \( \exists \mathbb{S} \text{AT} \) is true.

A proper-3-partition \( \pi \) is characterized by an instantiation \( p: X \rightarrow \{0, 1\} \) of \( X \)-variables extended to literals by \( p(\neg x) = 1 - p(x) \), and which defines:

\[
S(\pi) = K \cup \bigcup_{i=1}^{\#(\pi)} N_{i:p(X(C_i))} \quad (3m \text{ nodes})
\]

\[
S^{1/2} = \{ \nu^{1/2} \} \cup \bigcup_{i=1}^{\#(\pi)} N_{i:1-p(X(C_i))} \quad (m + 1 \text{ nodes})
\]

\[
S^{2/2} = \{ \nu^{2/2} \} \cup \bigcup_{i=1}^{\#(\pi)} N_{i:1-p(X(C_i))} \quad (m + 1 \text{ nodes})
\]

Note that in \( S^{1/2} \) (resp. \( S^{2/2} \)) the number of 1-links is constant \( \mu_1 \) (resp. \( \mu_2 \)) for any \( p \), since the formula on \( Y \)-literals is the same and 1-link \( \nu^{1/2}, \nu^{2/2}_i \) (resp. \( \nu^{2/2}, \nu^{1/2}_i \)) compensates for \( \nu^{2/2}, \nu^{1/2}_i \).

We that in our construct, any 3-partition is which is not a proper-3-partition does not trivially satisfy the complement question above. First, let us reason as if all three node-subsets were cliques of \(\Lambda\)-links. Crucially, in a node-subset of \( n \)-size, the number of links (\( \nu^{1/2}_i \)) is quadratic. Therefore, the largest node-subsets will be the first attacked, and the only way \( \pi \rightarrow M \) contains as many as \( (m + 1)^2 \) links if the node-subsets of \( \pi \) had sizes \( 3m, m + 1 \) and \( m + 1 \). Second, assume \(\Lambda\)-links are missing in some node-subsets. Then, an attack would focus on more connected subsets and \( \pi \rightarrow M \) cannot contain as many as \( (m + 1)^2 \) \(\Lambda\)-links. Therefore, 3-partition \( \pi \) must consist in \(\Lambda\)-link cliques of size \( 3m, m + 1 \) and \( m + 1 \). If the largest did not follow consistently some instantiation \( p: X \rightarrow \{0, 1\} \), then some \(\Lambda\)-links would be missing (see (\#)). Also, the only way to obtain two \(\Lambda\)-linked cliques of size \( m + 1 \) on \( N \setminus S(\pi) \) is by \( S^{1/2} \) and \( S^{2/2} \). We also know that \( S^{1/2} \) and \( S^{2/2} \) contain \(\mu_1 + \mu_2 \) 1-links.

Crucially, attack \( M \) occurs where it does the largest damage w.r.t. \(\Lambda\)-links: on node-subset \( S^{(p)} \), and the number of remaining \(\Lambda\)-links is \( (2m + 1)^2 \) \(\Lambda\)-links. Given a proper-3-partition, what could make the inequality false would be a surviving 1-link in \( S^{(p)} \) \( M \). Consequently, condition \( \exists M, W(\pi \rightarrow M) \leq \theta - 1 \) amounts to

\footnote{It is the same idea as in the standard reduction from \( \text{SAT} \) to \( \text{INDEPENDENTSET} \).}
a 2α node attack M that covers every 1-link in $\bigcup_{i=1}^{j=α} N_i.p(X(C_i))$. A crucial observation is that we necessarily attack/cover exactly two nodes per tetrad $N_{i,j}$, since each tetrad contains a triangle. In negative tetrads $N_{i,0}$, because of 1-link $\{v_{i,0}^x, v_{i,0}^y\}$, one of these nodes has to be $v_{i,0}^y \in M$. In positive tetrads $N_{i,1}$, since node $v_{i,1}^x$ is not involved in other 1-links than the triangle, choosing both $v_{i,1}^y$ and $v_{i,1}^x$ in 1-link cover M is the best choice. As in 3SAT $\leq$ IndependentSet, this amounts to a 1-link-independent-set $\overline{M} = S^{(p)} \setminus (K \cup M)$ with size 2α and two nodes per tetrad $N_{i,j}$; first, node $v_{i,j}^x$, second if $j = 0$ then $v_{i,0}^y$ xor $v_{i,0}^y$ or otherwise if $j = 1$ then $v_{i,1}^x$.

(yes $\Rightarrow$ yes) Assume that for every $r_x : X \rightarrow \{0, 1\}$, there exists $r_y : Y \rightarrow \{0, 1\}$ such that in every clause $C_i$ with $r_x(X(C_i)) = 0$, a $Y$-literal is made true by instantiation $r_y$. We show that given any proper-3-partition $\{S^{(p)}, S^{1/2}, S^{2/2}\}$, in $S^{(p)} \setminus K = \bigcup_{i=1}^{j=α} N_i.p(X(C_i))$, there exists a 1-link-independent-set $\overline{M}$ of size 2α, as below. Taking $r_x \equiv p$, let $r_y : Y \rightarrow \{0, 1\}$ be as above mentioned. Then,$$
abla \overline{M} = \bigcup_{i \in [α]} \left\{ \begin{array}{ll}
if p(X(C_i)) = 0: & \{v_{i,0}^y, v_{i,0}^x \mid r_y(t_i^p) = 1\} \\
nif p(X(C_i)) = 1: & \{v_{i,1}^x, v_{i,1}^x\}
\end{array} \right. $$is a 1-link-independent-set of size 2α: node $v_{i,0}^y$ exists since instantiation $r_y$ gives at least one true literal per clause where $r_y(X(C_i)) = 0$, and nodes are not 1-linked (no literal contradiction).

(yes $\Rightarrow$ yes) Assume that for any $r_x \equiv p : X \rightarrow \{0, 1\}$, a 1-link-independent-set $\overline{M}$ with size 2α exists in node-subset $S^{(p)} \setminus K = \bigcup_{i=1}^{j=α} N_i.p(X(C_i))$. Then, nodes $v_{i,0}^y \in \overline{M}$ consistently define $r_y : Y \rightarrow \{0, 1\}$ that makes any clause $C_i$ true whenever $r_x(X(C_i)) = 0$.

Crucially, we also include clauses without any $Y$-literal in the same construct. Assume w.l.o.g. that there are less than $α/2$ such $Y$-clauses, within the first indexes in [α]. To any $Y$-clause $C = t_i^p \lor t_j^p \lor t_k^p$, one associates two tetrads $N_{i,j} = \{v_{i,j}^y, v_{i,j}^y, v_{i,j}^y, v_{i,j}^y\}$, $j \in \{0, 1\}$. For $C_i, C_j$ $Y$-clauses, between $N_{i,0}$ and $N_{j,0}$ weights are zero. Negative tetrads $N_{i,0}$ are fully $\Lambda$-linked inside, between themselves, with previous tetrads of one $X$-variable and set $K$. Positive tetrads $N_{i,1}$ are fully $\Lambda$-linked inside, between themselves and with $S^{1/2}$. Given a $Y$-clause $C$, we define $X(C) = 0$. For proper-3-partitions, we extend $p(\emptyset) = 0$; hence in $\{S^{(p)}, S^{1/2}, S^{2/2}\}$, for $C_i$ a $Y$-clause, one has $N_{i,0} \subseteq S^{(p)}$ and $N_{i,1} \subseteq S^{1/2}$. Similarly, in any $Y$-clause tetrad $N_{i,j}$, there are 1-links $\{v_{i,j}^y, v_{i,j}^y, v_{i,j}^y\}$, $(v_{i,j}^y, v_{i,j}^y)$, $(v_{i,j}^y, v_{i,j}^y)$, $(v_{i,j}^y, v_{i,j}^y)$ (optional 1-links $\{v_{i,j}^y, v_{i,j}^y\}$), and whenever two $Y$-literals are complementary. It follows that the same proof holds. \hfill $\Box$

3 RELATED WORK

Partitioning of a set into (non-empty) subsets may also be referred as coalition structure formation of a set of agents into coalitions. When a number of coalitions $k$ is required and there are synergies between vertices/agents, this problem is referred as $k$-cut, or $k$-way partition, where one minimizes the weight of edges/synergies between the coalitions, or maximizes it inside the coalitions. For positive weights and $k \geq 3$, this problem is NP-complete [1], when one vertex is fixed in each coalition. For positive weights and fixed $k$, a polynomial-time $O(n^k T(n, m))$ algorithm exists [2], when no vertex is fixed in coalitions, and where $T(n, m)$ is the time to find a minimum $(s, t)$ cut on a graph with $n$ vertices and $m$ edges. When

not too many negative synergies exist (that is, negative edges can be covered by $O(\log(n))$ vertices), an optimal $k$-partition can be computed in polynomial-time [? ].

REFERENCES

[1] Elias Dahlhaus, David Johnson, Christos H. Papadimitriou, Paul D. Seymour, and Mihalis Yannakakis. 1992. The Complexity of Multiway Cuts (Extended Abstract). In Proceedings of the 24th Annual ACM Symposium on Theory of Computing (STOC-1992) 241–251.

[2] Olivier Goldschmidt and Dorit S. Hochbaum. 1994. A Polynomial Algorithm for the k-Cut Problem for Fixed k. Mathematics of Operations Research 19, 1 (1994), 24–37. http://www.jstor.org/stable/20186673.

[3] M. Karp, Richard. 1972. Reducibility among combinatorial problems. In Complexity of computer computations. Springer, 85–103.

[4] Ker-I Ko and Chih-Lung Lin. 1995. On the Complexity of Min-Max Optimization Problems and their Approximation. Springer US, Boston, MA, 219–239. https://doi.org/10.1007/978-1-4613-3557-3_15