Discrete phase-space mappings, tomographic condition and permutation invariance

C Muñoz and A B Klimov

Departamento de Física, Universidad de Guadalajara, 44420, Revolucion 1500, Guadalajara, Mexico

E-mail: klimov.andrei@gmail.com

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Abstract
We analyze various families of discrete maps in $N$-qubit systems in the context of permutation invariance. We prove that the tomographic condition imposed on the self-dual (Wigner) map is incompatible with the requirement of the invariance under particle permutations (except for the two-qubit case), which makes it impossible to project the Wootters-like Wigner function into the space of symmetric measurements. We also provide several explicit forms of the self-dual mappings: (a) tomographic and (b) permutation invariant, and analyze the symmetric projection in the latter case.

Keywords: phase-space, discrete, Wigner function, symmetric

(Some figures may appear in colour only in the online journal)

1. Introduction

Phase-space methods [1–3] have been widely applied in quantum mechanics both for state visualization and analysis of quantum-classical transitions from kinematical and dynamic perspectives. According to this approach quantum states are mapped into distributions on some manifold, which is associated with a ‘classical’ phase-space. The structure of the mappings as well as their principal characteristics depend on the symmetry of a given quantum system. The basic requirement for any meaningful phase-space representation is covariance under an appropriate group of transformations (which is one of the fundamental Stratonovich–Weyl conditions [4, 5, 6]). In the case of continuous symmetries, and when the operators from the group representation act irreducibly in the Hilbert space of the quantum system, the phase-space manifold can be constructed as a certain quotient space, and has an intimate relationship with the set of coherent states [7–14]. In such cases, a systematic procedure for the so-called
s-parametrized phase-space mapping can be suggested at least for some types of dynamic symmetry group [6, 13, 15].

The situation is essentially more involved in the case of discrete systems. Although several approaches for representation of states of a generic d-dimensional system in a discrete lattice have been considered [16], an explicit self-consistent map possessing all the required properties can be constructed only when \( d = p^n \), where \( p \) is a prime number [18]. Then, a discrete \( p^N \times p^N \) grid, playing the role of the phase-space \( \mathcal{M}_{p^N} \), possesses the same basic geometric properties as an ordinary plane and allows a direct association of states with specific geometrical structures [18, 19] related to the notion of mutually unbiased bases [20]. A family of maps from the Hilbert space \( \mathcal{H}_{p^N} \) to \( \mathcal{M}_{p^N} \) of the form

\[
\hat{\rho} \mapsto W_p^{(s)}(\Omega), \quad \Omega = (a_1, \ldots, a_N) \in \mathcal{M}_{p^N}, \quad a_j \in \mathbb{Z}_p,
\]

with \(-1 \leq s \leq 1\) a parameter labelling elements in the family, can be introduced in a similar way as in the continuous case [21–23]. It is worth noting that in this approach only the ‘boundary’ maps (\( s = \pm 1 \)), corresponding to the familiar \( P \) and \( Q \) functions [5–9], are uniquely defined once a fiducial state for generation of the so-called discrete coherent states [23, 25] is chosen. All the other maps can still be ‘refined’ by imposing additional conditions to the standard Stratonovich–Weyl requirements (invertibility, Hermiticity, normalization and covariance under group transformations). One possible condition could be the marginal reduction i.e. that, summing the image of the density matrix in \( \mathcal{M}_{p^N} \) (a quasidistribution function) along a set of points associated with a given state, one obtains the probability distribution in this state. This requirement gives a clear geometric interpretation of the discrete map and is also known as the tomographic condition.

Regrettably, none of the discrete phase-space representations is efficient for the visualization of states of large compound (many-particle) quantum systems [25]. The phase-space images of even relatively simple states become especially involved in the macroscopic limit, when the number of particles is sufficiently large. In this case the corresponding distributions have the form of almost randomly distributed peaks. This is mainly related to the ordering problem of elements labelling points of \( \mathcal{M}_{p^N} \), but is also connected to the classical indistinguishability between irreducible degenerated subspaces (subspaces of the same dimensions that appear in the decomposition of \( \mathcal{H}_{p^N}^{\otimes \mathcal{N}} \)).

On the other hand, precise information (say, all multipartite correlation functions) about an \( N \)-particle state when \( N \gg 1 \) is not only quite difficult to process and represent in a reasonable way, but it is also extremely complicated to measure. In addition, to grasp general properties of a macroscopic quantum system, which frequently consists in indistinguishable elements, a detailed description is not really needed. Instead, collective observables can be successfully used for a global characterization and statistical description of large quantum systems [26]. In the \( N \)-qubit case such observables can be expressed as functions of collective spin operators

\[
S_{x,y,z} = \sum_{i=1}^{N} \alpha_{x,y,z}^{(i)}
\]

One should stress that (experimentally assessed) mean values of such symmetric—i.e. invariant under particle permutation—observables, contain partial information about all \( SU(2) \) irreducible subspaces appearing in the decomposition of an \( N \) qubit state. For instance, in a 3-qubit case, average values of correlation functions of operators (2) in each two-dimensional \( SU(2) \) invariant subspace are non-trivial but cannot be used to distinguish such subspaces. In other words, by measuring symmetric operators we collect averaged (over irreps of the same dimension) information about an arbitrary multipartite state.
In order to represent and analyze global properties of \( N \)-qubit systems the so-called projected \( Q \)-function was recently introduced in [27]. Such a function is defined in a three-dimensional discrete \( \times \times N \times N \) space and contains full and non-redundant information about results of measurements of any invariant-under-particle-permutation observable in an arbitrary \( N \)-qubit state. Unfortunately, being a positive image of the density matrix, the projected \( Q \)-function is not convenient for graphical representation of the interference pattern and it is desirable to find a symmetric map allowing distinction between coherent and incoherent superpositions. In the continuous case the so-called Wigner, self-dual map is commonly used for picturing quantum interference in the phase-space [9, 28].

In this paper we analyze the \( s \)-parametrized family of discrete maps for \( N \)-qubit systems from the perspective of projection into the space of symmetric measurements and discuss several possibilities for constructing discrete maps according to established invariance properties. We will show that while the dual maps, \( s = \pm 1 \), are symmetrizable, i.e. under an appropriate choice of the fiducial state \( Q \) and \( P \) of symbols of permutationally invariant operators are symmetric functions of their arguments, the symmetrization of the self-dual map, corresponding to \( s = 0 \), contradicts the appealing geometrical idea of marginal reduction [18, 21]. In other words, it is not possible to introduce a Wigner-like discrete mapping that satisfies the tomographic condition and is symmetric under particle permutations. In addition, it results that even fixing one of the above conditions a self-dual discrete map is still not uniquely defined. We discuss several possible Wigner-like maps and their projections into the space of symmetric measurements.

In section 2 we briefly recall the concept of \( s \)-parametrized discrete mappings. In section 3 we provide explicit forms for those mappings under different symmetry conditions and prove the inconsistency between the tomographic and permutation-invariance requirements. We also discuss the projected form of self-dual mappings invariant under particle permutations. In section 4 we briefly discuss the obtained results.

2. Discrete phase-space and discrete mappings

Let us consider a system of \( N \) identical qubits, living in the Hilbert space \( \mathcal{H}_{2^N} = \bigotimes_{i=1}^{N} \mathbb{C}^2 \). In order to label both states and operators acting in \( \mathcal{H}_{2^N} \) it is advantageous to use elements of the finite field \( \mathbb{F}_{2^N} \), which can be considered as a linear space spanned by an abstract basis \( \{ \theta_1, \ldots, \theta_N \} \), so that any element \( \alpha \in \mathbb{F}_{2^N} \) is represented as a linear combination

\[
\alpha = \sum_{i=1}^{N} a_i \theta_i, \quad a_i \in \mathbb{Z}_2, \tag{3}
\]

allowing the establishment of an association \( \alpha \leftrightarrow (a_1, \ldots, a_N) \). Choosing the so-called self-dual basis, orthonormal with respect to the trace operation, i.e. \( \text{tr}(\theta_i \theta_j) = \delta_{ij} \), \( \text{tr}(\alpha) = \alpha + \alpha^2 + \ldots + \alpha^{2^{N-1}} \), so that \( a_i = \text{tr}(a_i \theta_i) \), we can associate each qubit with a particular element of the basis: qubit \( \theta_i \leftrightarrow \theta_i \). The components of an orthonormal basis \( \{ |k_1, \ldots, k_N \rangle, k_i \in \mathbb{Z}_2 \} \) in \( \mathcal{H}_{2^N} \), where \( k_i \) are expansion coefficients of \( \kappa \) in the self-dual basis, are then labelled by elements of \( \mathbb{F}_{2^N} \) as \( \{ |\kappa \rangle, \kappa \in \mathbb{F}_{2^N} \} \), \( |\kappa \rangle = |\kappa \rangle = \delta_{\kappa \kappa} \). In the same way \( N \)-particle monomial operators

\[
Z_\alpha = \sigma_1^{a_1} \otimes \ldots \otimes \sigma_N^{a_N}, \quad X_\beta = \sigma_1^{b_1} \otimes \ldots \otimes \sigma_N^{b_N}, \tag{4}
\]

\[
\alpha = (a_1, \ldots, a_N), \quad \beta = (b_1, \ldots, b_N), \tag{5}
\]

where \( \sigma_i = |0\rangle\langle 0| - |1\rangle\langle 1|, \sigma_i = |0\rangle\langle 1| + |1\rangle\langle 0| \), acting in \( \mathcal{H}_{2^N} \) can be represented as
\[
Z_\alpha = \sum_\kappa \chi(\alpha \kappa) |\kappa\rangle \langle \kappa|, \quad X_\beta = \sum_\kappa |\kappa + \beta\rangle \langle \kappa|,
\]
where \(\chi(\alpha) = (-1)^{\mu(\alpha)}\). The monomials (6) satisfy the commutation relation \(Z_\alpha X_\beta = \chi(\alpha \beta) X_\beta Z_\alpha\) and generate the generalized Pauli group \(\mathcal{P}^N = \mathcal{P}^1 \otimes \ldots \otimes \mathcal{P}^1\) [29, 30].

The discrete phase-space (DPS) is introduced [18] as a \(2^N \times 2^N\) grid, which points \((\alpha, \beta)\) label elements of a monomial operational basis \(Z_\alpha X_\beta\). This DPS (isomorphic to a product of two-dimensional discrete torus \(T^2 \otimes T^2 \otimes \ldots\)) is endowed with a finite geometry [17, 18] (where e.g. two non-parallel lines have only one intersection) and admits a set of discrete symplectic transformations [21, 22] (such as rotations and squeezing). In complete similarity with the continuous case, the axes of the discrete phase-space are associated with the complementary observables \(Z_\alpha\) and \(X_\beta\) in the sense that any eigenstate of either one of them is a state of maximum uncertainty with respect to the other.

The finite field-based construction of the DPS, in contradistinction to some other previously proposed schemes [16], allows to make a direct connection with the concept of Mutually Unbiased Bases [20]. In particular, one can assign quantum states to lines in such a way that a whole (orthonormal) basis is associated with a set of parallel lines (striation) and bases corresponding to different striations are mutually unbiased [21].

The discrete phase-space can be used for representation of \(N\)-qubit states in form of distributions through a one-to-one map (1). An \(s\)-parametrized set of quasidistribution functions satisfying the standard Stratanovich–Weyl conditions is defined as [21, 23]

\[
W_f^s(\alpha, \beta) = \text{Tr} \left[ \hat{f} \Delta^s(\alpha, \beta) \right],
\]

\[
\hat{f} = \frac{1}{2^N} \sum_{\alpha, \beta} W_f^s(\alpha, \beta) \Delta^{-s}(\alpha, \beta),
\]

where the mapping kernel has the form

\[
\Delta^s(\alpha, \beta) = \frac{1}{2^N} \sum_{\gamma, \delta} \chi(\alpha \delta + \beta \gamma) |\{D(\gamma, \delta)|\zeta\}^{-s} D(\gamma, \delta),
\]

here

\[
D(\gamma, \delta) = \phi(\gamma, \delta) Z_\gamma X_\delta,
\]

\[
\phi(\gamma, \delta) \phi^*(\gamma, \delta) = 1, \quad \phi(0, \delta) = \phi(\gamma, 0) = 1,
\]

are the (unitary) displacement operators and the fiducial state \(|\zeta\rangle\) is chosen in such a way that \(|\{D(\gamma, \delta)|\zeta\} \neq 0\).

In general, the choice of a fiducial state is a subtle question and several proposals for fixing such a state have been considered [23–25]. For multipartite systems the fiducial state is usually chosen in factorized and permutationally invariant form, which in the \(N\)-qubit case is just a spin coherent state \(|\zeta\rangle \propto |\delta, \varphi_1 \otimes \ldots \otimes |\delta, \varphi_N\rangle\). Taking into account that the monomials \(Z_\gamma X_\delta\) are factorizable into single qubit operators and that \(|\zeta|\rangle = 1 - |\zeta|\), \(|\zeta|\rangle \sim \text{Re} \zeta, \langle \zeta|\rangle \sim \text{Im} \zeta\), one can easily observe that the map (9) with \(s = 1\) is singular for real \(\zeta\), imaginary \(\zeta\), and when \(||\zeta|\rangle = 1\), i.e. on the equator of the Bloch sphere. Thus, some reference states, appearing in the odd-dimensional case—such as e.g. eigenstates of the finite Fourier transform operator—cannot be used for discrete mapping in the qubit case. An appropriate choice of the fiducial state ensures that the \(s\)-parametrized kernels (9) form a non-orthogonal
informationally complete operational basis. One such possibility, corresponding to the most symmetric distribution of vectors on the Bloch sphere in the single-qubit case, will be considered below. The kernel (9) is normalized,

\[ \sum_{\alpha,\beta} \Delta^{(\alpha, \beta)} = 2^N, \, \text{Tr} \Delta^{(\alpha, \beta)} = 1, \]  

(12)
covariant under action of the discrete displacements (10)

\[ D(\kappa, \lambda) \Delta^{(\alpha, \beta)} D^\dagger(\kappa, \lambda) = \Delta^{(\alpha + \kappa, \beta + \lambda)}, \]  

(13)
and Hermitian, \( \Delta^{(\alpha, \beta)} = \Delta^{(\beta, \alpha)} \), if the phase (11) satisfies the condition,

\[ \phi^2(\gamma, \delta) = \chi(\gamma \delta), \]  

(14)
which also leads to the hermiticity of the displacement operator, \( D^\dagger(\gamma, \delta) = D(\gamma, \delta) \).

The overlap relation

\[ \text{Tr}(\Delta^{(\alpha, \beta)} \Delta^{(\alpha', \beta')}) = 2^N \delta_{\alpha \alpha'} \delta_{\beta \beta'}, \]  

(15)
is automatically fulfilled and allows the evaluation of the trace of a product in the form of a convolution,

\[ \text{Tr}(\hat{f} \hat{g}) = 2^{-N} \sum_{\alpha,\beta} W^{\alpha}_f(\alpha, \beta) W^{\beta}_g(\alpha, \beta), \]  

(16)
thus, mean values of are computed as

\[ \langle \hat{f} \rangle = 2^{-N} \sum_{\alpha,\beta} W^{\alpha}_f(\alpha, \beta) W^{\beta}_g(\alpha, \beta), \]  

(17)
where \( W^{\alpha, \beta}_f(\alpha, \beta) \) is the symbol of the density matrix.

3. Symmetries of DPS mapping

The representation of an \( N \)-qubit state \( \rho \) in DPS by any of \( W^{\alpha}_f(\alpha, \beta) \) has an important drawback: while for a small number of particles the plot of quasidistributions is representative, it becomes extremely involved and is practically useless for analysis of quantum states when \( N \gg 1 \) [25]. In part it is a consequence of the absence of a natural ordering of elements of \( \mathbb{F}_2^N \). In addition, the central limit theorem is not directly applicable to distributions labeled by \( N \)-tuples (representations of \( \mathbb{F}_2^N \)) \( \{(a_1, ..., a_N), a_i \in \mathbb{Z}_2\} \), so that inclusively ‘semiclassical’ states (as e.g. spin coherent states) are not represented as functions tending to smooth Gaussians in the limit \( N \to \infty \). This explains an essentially smaller number (with respect to the continuous case) of applications of discrete phase-space representations in many-body quantum mechanics.

3.1. Permutation-invariant \( s = \pm 1 \) mapping

Nevertheless, this problem can be fixed if the available set of measurements is restricted only to symmetric observables \( \{\hat{S}\} \), i.e. invariant under particle permutations, \( \hat{S} = \hat{P}_{ij} \hat{S} \hat{P}_{ij} \), \( i, j = 1, ..., N \), where \( \hat{P}_{ij} \) is the permutation operator [27]. It results that if the fiducial state \( |\zeta\rangle \) in (9) is permutation-invariant (i.e. it is a spin coherent state), and \( \langle \zeta | D(\gamma, \delta) | \zeta \rangle \neq 0 \), the image
$W^{(+1)}_S(\alpha, \beta)$ of any symmetric operator $\hat{S}$ is a function only of the (permutation) invariants constructed on the phase-space coordinates ($\alpha$, $\beta$)

\[
h(\alpha) = \sum_{i=1}^{N} a_i, \quad h(\beta) = \sum_{i=1}^{N} b_i, \quad h(\alpha + \beta) = \sum_{i=1}^{N} (a_i + b_i),
\]

(18)

where $\{a_i + b_i\}$ means sum mod 2, and $0 \leq h(\kappa) \leq N$. In other words,

\[
W^{(+1)}_S(\alpha, \beta) = W^{(+1)}_S(h(\alpha), h(\beta), h(\alpha + \beta)),
\]

(19)

are permutation-invariant functions of the phase-space coordinates, where $\alpha$ and $\beta$ are considered as $N$-tuples, $\alpha = (a_1, \ldots, a_N)$, according to (3).

Thus, according to (16), the full information about the results of measurements of any symmetric observable in an (arbitrary) $N$-qubit state $\rho$ is contained in the projections of $W^{(+1)}_S(\alpha, \beta)$ into the three-dimensional measurement space $\mathcal{M}$ spanned by $h(\alpha)$, $h(\beta)$, $h(\alpha + \beta)$ (space of symmetric measurements),

\[
\tilde{W}^{(+1)}_\rho(h) = \sum_{\alpha, \beta} W^{(+1)}_\rho(\alpha, \beta)\delta_{m,\alpha,0}\delta_{n,\beta,0}\delta_{k,\alpha+\beta,0},
\]

(20)

where $h = (m, n, k)$, and the average value of any symmetric operator as it follows from (17) is computed according to

\[
\langle \hat{S} \rangle = 2^{-N} \sum_h \tilde{W}^{(+1)}_\rho(h)\tilde{W}^{(+1)}_S(h).
\]

(21)

While the distribution $\tilde{W}^{(+1)}_\rho(h)$ (corresponding to the $P$-function, $P(h)$) becomes quite singular for large number of qubits, $\tilde{W}^{(-1)}_\rho(h)$ (the $Q$-function, $Q(h)$) tends to a smooth distribution when $N \gg 1$ and it is very convenient for analysis of quantum states in the macroscopic limit [27].

It is worth noting here that the mapping kernel

\[
\Delta^{(-1)}(\alpha, \beta) = |\alpha, \beta\rangle\langle\alpha, \beta|,
\]

(22)

where $|\alpha, \beta\rangle = D(\alpha, \beta)\zeta$ are the so-called discrete coherent states, that form an informationally complete set of Positive Operator Valued Measures when $\langle\zeta|D(\gamma, \delta)\zeta\rangle \neq 0$ and acquires the most symmetric form for the fiducial spin coherent state with $\zeta = \frac{\sqrt{2} - 1}{\sqrt{2}}e^{i\pi/4}$, corresponding to the direction $n_0 = (1, 1, 1)/\sqrt{3}$ on the Bloch sphere. In the single-qubit case the set of coherent states $\{|\alpha, \beta\rangle : |\zeta\rangle, \sigma_x|\zeta\rangle, \sigma_y|\zeta\rangle, \sigma_z|\zeta\rangle\}$ form a symmetric tetrahedron inscribed in the Bloch sphere (for generic coherent fiducial states see e.g. [24]). For this choice of the fiducial state one has $W^{(-1)}_{S,n}(h) \sim N/2 - h \cdot n$, being $S = (S_x, S_y, S_z)$ the collective spin operators and $n$ is a unit three-dimensional vector, directions in the measurement space $\mathcal{M}$ are associated with vectors in the Bloch sphere: $m \leftrightarrow x, k \leftrightarrow y, n \leftrightarrow z$.

3.2. Covariant (Wigner) mapping

Unfortunately, the projection $\tilde{W}^{(-1)}_\rho$ does not distinguish very well between coherent and incoherent superpositions, due to the typical—for the $Q$-function—form of the completely
positive mapping kernel (22). For instance, the magnitude of the interference term in $\tilde{W}_{\rho}^{-1}(h)$ for the GHZ state $|\text{GHZ}\rangle \sim |0...0\rangle + |1...1\rangle$ [27],

$$W_{\text{GHZ}}^{-1}(h) \sim |\zeta|^{N-2n} + |\zeta|^{-N+2n} + 2(-1)^m \cos\left(\frac{\pi}{4}(N - 2n)\right),$$

(23)
is negligible compared with the principal maxima. On figure 1 we plot $W_{\text{GHZ}}^{-1}(Q(h))$ and $W_{\text{GHZ}}^{+1}(P(h))$ functions as a set of spheres and diamonds in the three-dimensional $M$—space. The size of the spheres/diamonds and their colors indicate the density of the distribution. The largest and darkest spheres/diamonds represent points of highest density. The spheres and diamonds represent points with positive and negative value of distribution respectively.

In the continuous case it is known that the appropriate representation, that ‘sees’ the interference pattern is provided by the Wigner function, defined as a self-dual image of the density matrix (when the same type of mapping is used both for the density operator and for the observables, in order to compute average values by convoluting corresponding symbols). In addition, the continuous analog of the kernel (9) for $s = 0$ possesses another important property: integration of the Wigner function along a strip in phase-space gives the marginal probability related to the corresponding area. This important feature allows construction of the Wigner function of a quantum state in terms of probability distribution of some particular observables (‘rotated’ position operator in the $x - p$ plane). A similar condition can

Figure 1. $W_{\text{GHZ}}^{-1}(h)$ (left) and $W_{\text{GHZ}}^{+1}(h)$ (right) functions as a set of spheres and diamonds in the measurement space. The size of the spheres/diamonds and their colors indicate the density of the distribution. The largest and darkest spheres/diamonds represent points of highest density. The spheres and diamonds represent points with positive and negative value of distribution respectively.
in principle be imposed in the discrete case, where the integration along certain directions is substituted by a summation over straight lines [18], defined according to [17]

\[ \beta = \xi \alpha + \kappa. \]  

(24)

In the DPS the lines (24) can be associated with (appropriately ordered [22]) eigenstates \( \{ | \psi_{\xi}^{\kappa} \rangle \} \) of sets of commuting monomials \( \{ X_\alpha Z_\kappa \} \). A convenient way to arrange a set \( \{ | \psi_{\xi}^{\kappa} \rangle \} \) is by assigning the state \( | \psi_{\xi}^{\kappa} \rangle = 0 \rangle \) (with all positive eigenvalues) to the ray \( \beta = 0 \), while the parallel lines (24) correspond to the shifted states \( | \psi_{\xi}^{\kappa} \rangle = 0 \rangle \). In particular, elements of the logical basis \( | \kappa \rangle = | \beta \rangle = 0 \rangle \) (eigenstates of \( \alpha \) operator) are associated with the ‘horizontal’ axis \( \beta = 0 \) and all the parallel lines \( \beta = \kappa \) while the dual basis \( | \bar{\kappa} \rangle = | \bar{\beta} \rangle = 0 \rangle \), being \( | \bar{\kappa} \rangle \) eigenstates of \( \beta \) operators, correspond to the ‘vertical’ axis \( \alpha = 0 \) and the parallel lines \( \alpha = \kappa \).

3.2.1. Tomographic condition and permutation invariance. The discrete self-dual map (7), \( s = 0 \), only guarantees that summing the Wigner function along axes \( \alpha = 0 \) and \( \beta = 0 \) leads to the correct projections on the bases \( | \kappa \rangle \) and \( | \bar{\kappa} \rangle \) respectively. The requirement that a summation along any line (24) gives the marginal probability distribution for the observable associated with that line,

\[ 2^{-N} \sum_{\alpha, \beta} W^{(0)}_{\rho}(\alpha, \beta) \delta_{\lambda, \xi \alpha + \kappa} = \langle \psi_{\xi}^{\kappa} | \rho | \psi_{\lambda}^{\kappa} \rangle = P(\xi, \kappa), \]  

(25)

\[ 2^{-N} \sum_{\beta} W^{(0)}_{\rho}(\alpha, \beta) = \langle \alpha | \rho | \bar{\alpha} \rangle = \bar{P}(\alpha), \]  

(26)

is an additional condition. This so-called tomographic condition allows representation of the kernel \( \Delta^{(0)}(\alpha, \beta) \) in the form of the sum of projectors corresponding to the lines crossing at the phase-space point \( (\alpha, \beta) \) [18],

\[ \Delta^{(0)}(\alpha, \beta) = | \bar{\alpha} \rangle \langle \bar{\alpha} | + \sum_{\xi, \kappa} \delta_{\lambda, \xi \alpha + \kappa} | \psi_{\xi}^{\kappa} \rangle \langle | \psi_{\lambda}^{\kappa} | - I. \]  

(27)

The set of eigenstates \( \{ | \psi_{\xi}^{\kappa} \rangle \} \) are constructed in form of expansion

\[ | \psi_{\xi}^{\kappa} \rangle = 2^{-N/2} \sum_{\nu} \chi(\nu) c_{\nu, \xi} | \nu \rangle, \]  

(28)

where \( c_{\nu, \xi} \) satisfy the following non-linear recurrence equation [22]

\[ c_{\nu + \alpha, \xi} = \chi(\nu \alpha) c_{\nu, \xi}, \]  

(29)

so that the symbol of the state \( | \psi_{\xi}^{\kappa} \rangle \) is just a straight line (24),

\[ W^{(0)}_{\psi_{\xi}^{\kappa}}(\alpha, \beta) = \delta_{\lambda, \xi \alpha + \kappa}, \]  

(30)

and the states associated to the lines (24) with different slopes satisfy the unbiasedness relation:

\[ | \langle \psi_{\xi}^{\kappa} | \psi_{\lambda}^{\kappa} \rangle |^2 = 2^{-N} (1 - \delta_{\xi, \lambda} + \delta_{\xi, \lambda} \delta_{\xi, \lambda}). \]  

The tomographic condition (25) leads to a ‘diagonal’ representation of the density matrix, expanded on the unbiased projectors

\[ \rho = \sum_{\xi, \kappa} P(\xi, \kappa) | \psi_{\xi}^{\kappa} \rangle \langle | \psi_{\lambda}^{\kappa} | - I, \]  

(31)

and, conversely, allows the expression of the Wigner function in terms of measured probabilities,
Such representation of a quantum state using data obtained directly from projective measurements is very useful in applications to discrete quantum tomography and quantum state estimation [16, 18].

3.2.2. Tomographic phases. The condition (25) restricts the possible choices of the phase (11) of the displacement operator (10), which become tied to the coefficients $c_{\xi,\zeta}$ through

$$
\phi(\gamma, \delta) = c_{\gamma, \gamma^{-1}\delta}.
$$

One of the possible family of solutions of (29) is

$$
c_{\alpha, \xi} = (-i)^{\delta_{\alpha, \xi}}, \quad p = 1, 2, 4, 8, \ldots, 2^{N-1},
$$

which can be verified by direct substitution, leading to the phase

$$
\phi(\alpha, \beta) = c_{\alpha, \alpha^{-1}\beta} = (-i)^{\delta_{\alpha, \alpha^{-1}\beta}}, \quad p = 1, 2, \ldots, 2^{N-1}.
$$

The solution with $p = 1$,

$$
c_{\alpha, \xi} = (-i)^{\tilde{\phi}(\alpha, \sqrt{\xi})},
$$

where $\sqrt{\xi}$ is the square root of $\xi$ uniquely defined on $F_2^N$, possesses an extra symmetry: in this case the states $\psi|_{\frac{\xi}{\sqrt{\xi}}} = 1$ are factorized and permutation invariant:

$$
|\psi|_{\frac{\xi}{\sqrt{\xi}}} = 2^{-N/2} \otimes \sum_{k_i \in F_2^N} (-1)^{k_m}(-i)^{k_i} |k_i\rangle.
$$

where $n_i$ are components of $\nu$ in the self-dual basis $\{\theta_i\}$.

Examples. In the simplest case, $p = 1$, when

$$
\phi(\alpha, \beta) = (-i)^{\tilde{\phi}(\alpha, \sqrt{\beta})},
$$

the Wigner function $W_{\rho}^{(0)}(\alpha, \beta)$ has the following form for

(a) GHZ-state

$$
W_{\rho}^{(0)}(\alpha, \beta) = \left( \sum_{\xi} P(\xi, \beta + \xi \alpha) + \hat{P}(\alpha) - 1 \right).
$$

(b) W-state, which in terms of $F_2^N$ elements can be conveniently represented as

$$
W = N^{-1/2} \sum_{i=1}^N |\theta_i\rangle \langle \theta_i|,
$$

where the last term clearly represents the interference;

(b) W-state, which in terms of $F_2^N$ elements can be conveniently represented as

$$
W = N^{-1/2} \sum_{i=1}^N |\theta_i\rangle \langle \theta_i|,
$$

where the last term clearly represents the interference;
The Wigner function of SU(2) coherent states $|\zeta\rangle$ is in general fairly complicated, except for the equatorial states, when $\zeta = 1$ and $|\zeta = 1\rangle = 2^{-N/2} \sum |\kappa\rangle$, here $|\kappa\rangle$ is the logical basis,

$$W_{|\kappa\rangle}(\alpha, \beta) = \delta_{\alpha,0}. \quad (41)$$

Another possible solution of equation (29), which is closely connected to the so-called graph-state formalism [31], has the form

$$c_{\alpha,\xi} = (\pm i)^{\alpha^T \xi}, \quad (42)$$

where $\alpha^T = [\alpha_1, ..., \alpha_N]$ and $\Gamma_{pq} = [\text{tr}(\xi^T \beta^T \rho)]$ is the adjacency matrix of the graph (with loops) corresponding to the ray $\beta = \xi \alpha$. The Wigner functions of even simple states have quite cumbersome form for this choice of the phase. Equation (42) is a particular case of a more general solution for $c_{\alpha,\xi}$ found in [22].

### 3.2.3. Symmetric Wigner mapping.

In order to construct a self-contained projection of $W^0_{\alpha,\beta}$ on the space of symmetric measurements similar to (20), the map $\Delta^0(\alpha, \beta)$ should satisfy the basic condition that the symbol $W^0_{\alpha,\beta}$ of any symmetric operator $\hat{S}$ is a permutation-invariant function of the phase-space coordinates. This condition requires invariance of $\Delta^0(\alpha, \beta)$ under particle permutations,

$$\hat{\Pi}_j \Delta^0(\alpha, \beta) \hat{\Pi}_j = \Delta^0(\alpha, \beta). \quad (43)$$

In view of (9) the above condition can be fulfilled only when the phase (11) is an invariant function under the same permutations of $\alpha$ and $\beta$.

**Theorem.** There does not exist a complete set of permutation-invariant phases $\phi(\alpha, \beta)$ satisfying the condition (33).

**Proof.** The relation (33) leads to the following recurrence equation for the phase $\phi(\alpha, \beta)$

$$\phi(\alpha + \beta, \alpha \xi + \beta \xi) = \chi(\alpha, \beta)(\alpha, \alpha \xi)(\beta, \beta \xi). \quad (44)$$

Thus, it is sufficient to show that in an arbitrary set $\{\phi(\alpha, \beta), \alpha, \beta \in \mathbb{F}_p^N\}$ of solutions of (44) there exists at least one non-permutational-invariant phase. Let us suppose that any $\phi(x, y)$ satisfying (44) is invariant under a permutation of $x$ and $y$, then the character $\chi(\alpha, \beta)(\alpha \xi, \beta \xi)$ (here we have used the property $\text{tr}(\alpha) = \text{tr}(\alpha^T)$) must be also invariant under the same permutation of $\alpha, \beta, \beta \xi$ and $\alpha \xi$. Nevertheless, a permutation of $r$th and $s$th qubits does not leave invariant $\chi(\alpha, \beta)(\alpha \xi, \beta \xi)$ when $\alpha = \theta p^r, \beta = \theta p^s, \xi = (\theta_r + \theta_s)^{-1}$ such that $\text{tr}(\theta_q \theta_p) = \text{tr}(\theta_q \theta_p)$ for any $q$ satisfying $q = p = r = s$ (here $\theta_j$ are elements of a self-dual basis). In fact, taking into account that under permutation of $r$th and $s$th qubits the $N$-tuple $\alpha$ is transformed into $\alpha' = \alpha + \varepsilon \text{tr}(\alpha\varepsilon), \varepsilon = \theta_r + \theta_s$, we observe that

$$\chi(\alpha', \beta')(\alpha \varepsilon^{-1} \xi^{-1}, \beta \varepsilon^{-1} \xi^{-1}) = -\chi(\alpha, \beta)(\alpha \xi, \beta \xi) \quad (45)$$

which means that for these values $\alpha, \beta$ and $\xi$ the phase $\chi(\alpha, \beta)(\alpha \xi, \beta \xi)$ is not invariant under the permutation $\varepsilon$.

This means that one cannot construct a symmetric Wigner map satisfying the tomographic condition (25) and thus, the map (27) cannot be projected into the space of symmetric measurements in a consistent way, since such a projection (according to (20)) would not contain information about an arbitrary symmetric observable. \qed
The above proof is obviously not applicable in the two-qubit case (since it is required to have at least three different qubit labels). Moreover, in this case a tomographic and permutation-invariant phase \( \phi(\alpha, \beta) \) can be found. The particular case of two-qubit systems is analyzed below.

3.2.4. Permutation-invariant phases. As we have proved, the tomographic condition is incompatible with the permutation invariance of the phase (11). Nevertheless, withdrawing this requirement and demanding only the Hermiticity of the map (9) one can find multiple permutation-invariant solutions of equation (14).

(a) The simplest form of \( \phi(\alpha, \beta) \) is an arbitrary distributed (on \( \pm \) signs) set of square roots,

\[
\phi(\alpha, \beta) = \pm \sqrt{\chi(\alpha, \beta)},
\]

which can be also represented in the form \( \phi(\alpha, \beta) = i^{\text{Tr}(\alpha, \beta)}(-1)^{g(\alpha, \beta)} \), where \( g(\alpha, \beta) \) is an invariant under particle permutations function.

(b) The phase \( \phi(\alpha, \beta) \) can be also represented directly as a function of the invariants (18):

\[
\phi(\alpha, \beta) = \phi(h(\alpha), h(\beta), h(\alpha + \beta)).
\]

The simplest form of such a phase is

\[
\phi(\alpha, \beta) = (-1)^{f(\alpha, \beta)}(h(\alpha) + h(\beta) - h(\alpha + \beta))/2,
\]

(observe that \( h(\alpha) + h(\beta) - h(\alpha + \beta) = 2 \sum a_i b_i \)) where \( f(\alpha, \beta) \) is an arbitrary permutation invariant function. If the function \( f(\alpha, \beta) \) is in addition factorizable,

\[
f(\alpha, \beta) = \sum_i f_i(a_i, b_i),
\]

the kernel \( \Delta^0(\alpha, \beta) \) has a product form,

\[
\Delta^0(\alpha, \beta) = \otimes_i \Delta^0_{(a_i, b_i)},
\]

\[
\Delta^0_{(a_i, b_i)} = \frac{1}{2} \sum_{c, d \in \mathbb{Z}_2} (-1)^{\sigma_i c d + b_i c + f(c, d)} \sigma_i c d \sigma_i d,
\]

(49)

in contrast to the standard construction (27) (see e.g. [34]).

Although the property (30) is not true for the permutation invariant map (49) in general, it still holds for the factorized bases (eigenstates of the sets \( \{Z_i\}, \{X_i\}, \{Z_\alpha X_\beta\} \)),

\[
W_\delta(\alpha, \beta) = \delta_{\beta, \alpha},
\]

(50)

\[
W_\delta(\alpha, \beta) = \delta_{\alpha, \kappa},
\]

(51)

\[
W_\delta(\alpha, \beta) = \delta_{\alpha + \beta, \kappa},
\]

(52)

Correspondingly, the symbols of symmetric operators are permutation invariant functions, in particular, for the image of the \( SU(2) \) group element

\[
g = \exp(i \varphi \Sigma_x) \exp(i \theta \Sigma_z) \exp(i \psi \Sigma_y),
\]

where \( \varphi, \theta, \psi \) are \( \sum_{j=1}^N \sigma_{x,y,z}^{(j)} \), being \( \sigma_{x,y,z}^{(j)} \) Pauli matrices,

\[
W_g(\alpha, \beta) = \cos^N \theta
\]

(54)
\[
[e^{i\phi} + i \sqrt{2} \tan \theta \cos(\phi - \psi - \pi/4)]^N - h(\alpha) + h(\beta) + h(\alpha + \beta) \]
(55)

\[
[e^{-i\phi} + i \sqrt{2} \tan \theta \cos(\phi - \psi + \pi/4)]^N - h(\alpha) + h(\beta) + h(\alpha + \beta) \]
(56)

\[
[e^{i\phi} - i \sqrt{2} \tan \theta \cos(\phi - \psi - \pi/4)]^N - h(\alpha) - h(\beta) + h(\alpha + \beta) \]
(57)

\[
[e^{-i\phi} - i \sqrt{2} \tan \theta \cos(\phi - \psi + \pi/4)]^N - h(\alpha) + h(\beta) - h(\alpha + \beta) \]
(58)

held for \( f(\alpha, \beta) = 0 \).

The Wigner function defined with permutation-invariant phases can be faithfully mapped into the measurement space according to (20) so that average values of an arbitrary symmetric operator \( \hat{S} \) are computed as a convolution

\[
\langle \hat{S} \rangle = 2^{-N} \sum_{m,n=0}^{N} \sum_{k} \hat{W}_{\rho}(m, n, k) \hat{W}_{S}(m, n, k),
\]
(59)

where \( \hat{W}_{\rho}(m, n, k) \) is the Wigner symbol of \( \hat{S} \) and \( k = |m - n|, |m - n + 2|, \ldots, \min(m + n, N, 2N - m - n) \).

For instance, the Wigner function under the choice (47) with \( f(\alpha, \beta) = 0 \), for the GHZ state has the form

\[
\hat{W}_{\rho}(GHZ, 0) = \frac{1}{2} \delta_{\rho,0} \delta_{m,k} C_k^N + \frac{1}{2} \delta_{\rho,N} \delta_{m,N-k} C_m^N
\]

\[
+ (-1)^{m+n} C_{m+n+k}^{2N} \sum_{n=0}^{N} \binom{m+n+k}{2} \binom{m+n-k}{2} [1 + i]^{N/2}
\]

where \( C_k^N \) are the Binomial coefficients. One can observe a large interference term (the last term in the above equation) centered at \((N/2, N/2, N/2)\) while the maxima are located at \((N/2, 0, N/2)\) and \((N/2, N/N/2)\) respectively. This picture is apparently quite similar to the representation of the interference of Schrödinger cat-like states in the flat phase-space. Nevertheless, in the macroscopic limit, \( N \gg 1 \), the interference term is \( 2N^2/N \) times larger than the ‘principal’ terms, representing states \([0 \ldots 0]\) and \([1 \ldots 1]\). In figures 2 and 3 we plot the projected Wigner functions \((N = 8)\) for the symmetric phase (47) with \( f(\alpha, \beta) = 0 \) and the tomographic phase (38) respectively. In the insert plots we plot the corresponding Wigner functions in the discrete phase space. One can see that the discrete distributions look practically the same for both phases, while the projected ones have essentially different form.

One can observe on figure 2 that in case of the symmetric phase the interference, centered at \((N/2, N/2, N/2)\), is clearly dominant, although the appropriate marginal reduction \( \sum_{m,k} \hat{W}_{\rho}(GHZ, 0) \) shows only two maxima in the \( n \) (\( z \)) direction, as it is expected (the interference terms are mutually canceled). On the other hand, the projected Wigner function for the tomographic phase plotted in figure 3, being not a faithful representation of the GHZ-state (it cannot be used for computing average values of arbitrary symmetric operators), has quite an intuitive form: two clear maxima with an interference pattern. The corresponding marginal reduction \( \sum_{m,k} \hat{W}_{\rho}(m,k) \) has the same form as in the symmetric case.
Figure 2. The projected Wigner function, $\hat{W}_{\text{GHZ}}^{(0)}(\hbar)$, of the GHZ state for the symmetric phase (47) with $f(\alpha, \beta) = 0$ ($N = 8$). The upper insert plot: the Wigner function in the discrete phase-space, $W^{(0)}(\alpha, \beta)$; the lower insert plot: the marginal reduction $\sum_{m,k} \hat{W}_{\text{GHZ}}^{(0)}(\hbar)$.

Figure 3. The projected Wigner function, $\hat{W}_{\text{GHZ}}^{(0)}(\hbar)$, of the GHZ state for the tomographic phase (38) ($N = 8$). The upper insert plot: the Wigner function in the discrete phase-space, $W^{(0)}(\alpha, \beta)$; the lower insert plot: the marginal reduction $\sum_{m,k} \hat{W}_{\text{GHZ}}^{(0)}(\hbar)$.
We have performed an extensive numerical search for a symmetric function $f(\alpha, \beta)$ in (47) but could not find any appropriate function such that the projected symmetric Wigner function for typical states (GHZ, W, spin coherent) acquired intuitively appealing forms.

### 3.2.5. Two-qubit case

As it was mentioned above, in the two-qubit case there exists a permutationally symmetric phase satisfying (44). This phase can be chosen e.g. in an explicitly factorizable form

$$\phi(\alpha, \beta) = (-i)^{\alpha \beta} \alpha \beta.$$  \hspace{1cm} (60)

The corresponding symmetric Wigner mapping kernel satisfying the tomographic condition (25) is also factorized

$$\Delta^{(0)}(\alpha, \beta) = \frac{1}{2} \left[ |0\rangle\langle 0| + (-1)^{\alpha} |1\rangle\langle 1| + i(-1)^{\alpha + \beta} |1\rangle\langle 1| \right] \otimes \frac{1}{2} \left[ |0\rangle\langle 0| + (-1)^{\beta} |1\rangle\langle 1| + i(-1)^{\alpha + \beta} |1\rangle\langle 1| \right].$$

As a simple example of such map we plot in figure 4 the Wigner function of X-type states (see e.g. [35] and references therein for recent results)

$$\rho = \begin{bmatrix} \frac{\varepsilon}{2} & 0 & 0 & \frac{\varepsilon}{2} \\ 0 & (1 - \varepsilon)m & 0 & 0 \\ 0 & 0 & (1 - \varepsilon)(1 - m) & 0 \\ \frac{\varepsilon}{2} & 0 & 0 & \frac{\varepsilon}{2} \end{bmatrix}$$  \hspace{1cm} (61)

for values of the parameters $\varepsilon$ and $m$ corresponding to varying values of quantum discord [36]. One can observe that the corresponding Wigner functions have essentially different form, allowing the distinction of regions with different physical characteristics in the parameter space $(\varepsilon, m)$ (more detailed analysis of X-type states in terms of quasidistribution functions is beyond the scope of the present paper).

### 4. Conclusions

Visualization of quantum states as distributions in a phase-space may considerably improve our appreciation of state structures. In the continuous case, different types of maps from the Hilbert space into phase-space reveal distinct facets of a state under consideration. In order to analyze the interference pattern (proper to a given state) the so-called Wigner (self-dual) map...
is likely the most appropriate. This map has an important and useful property: the possibility of reconstructing the form of the distribution directly from the measurement data, obtained by assessing mean values of a certain observables. This feature is commonly called ‘the tomographic condition’ and, in algebraic language, means a connection between the integral of a distribution along a line in phase-space and a corresponding marginal distribution.

When we try to translate such a scheme to the $N$-qubit case, we are faced with the problem of extremely ‘noisy’ pictures in the discrete $2^N \times 2^N$ grid, with no smooth limit in the macroscopic regime, $N \gg 1$, where the phase-space methods should be especially useful. One possible solution consists in visualizing only those parts of the complete system that contribute to the detection of observables symmetric under particle permutations. This type of reduction leads to the concept of distributions in the measurement space [27]. It results that the representation of states in the measurement space is faithful, i.e. allows the description (and computation of mean values) of any symmetric operator, but only in the case where the map from the Hilbert space to the discrete $2^N \times 2^N$ phase-space is also symmetric. Unfortunately, the symmetry property is incompatible with the requirement to satisfy the tomographic condition (25) (except for the two-qubit case). In other words, the discrete Wigner map corresponding to (7)–(9) at $s = 0$ is either (a) permutation symmetric or (b) reconstructible through projective measurements on the elements of mutually unbiased bases according to (32), i.e. in terms of marginal probabilities (25).

In both cases there exist multiple constructions of Wigner-like mappings. We have provided some explicit forms of maps satisfying either the tomographic or permutation-invariance conditions. The permutation invariant maps allow the projection of the Wigner function into the space of symmetric measurements and efficiently detect the interference patterns, separated from the contribution of the incoherent terms. A shortcoming of using the symmetric map consists in losing a clear geometrical interpretation of the Wigner function—and, thus, its relation to the (measurable) marginal probabilities. On the other hand, there is no appropriate symmetric projection of the Wootters-like Wigner function defined by the map (27); this may limit the applicability of this map for the analysis of large $N$-qubit systems.

During the preparation of this paper we found an article by Zhu (2016 Phys. Rev. Lett. 116 040501) where the relation of the discrete Wigner function with permutation symmetry was analyzed in a different context.

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