On the largest eigenvalue of Wishart matrices with identity covariance when \( n, p \) and \( p/n \to \infty \)

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Abstract

Let \( X \) be a \( n \times p \) matrix and \( l_1 \) the largest eigenvalue of the covariance matrix \( X^*X \). The “null case” where \( X_{i,j} \sim \mathcal{N}(0, 1) \) is of particular interest for principal component analysis. For this model, when \( n, p \to \infty \) and \( n/p \to \gamma \in \mathbb{R}_+^* \), it was shown in [Johnstone (2001)] that \( l_1 \), properly centered and scaled, converges to the Tracy-Widom law. We show that with the same centering and scaling, the result is true even when \( p/n \) or \( n/p \to \infty \), therefore extending the previous result to \( \gamma \in \mathbb{R}_+^* \). The derivation uses ideas and techniques quite similar to the ones presented in [Johnstone (2001)]. Following [Soshnikov (2002)], we also show that the same is true for the joint distribution of the \( k \) largest eigenvalues, where \( k \) is a fixed integer.

Numerical experiments illustrate the fact that the Tracy-Widom approximation is reasonable even when one of the dimension is small.

1 Introduction

Large scale principal component analysis (PCA) - concerning an \( n \times p \) matrix \( X \) where \( n \) and \( p \) are both large - is nowadays a widely used tools in many fields, such as image analysis, signal processing, functional data analysis and quantitative finance. Several examples come to mind, including Eigenfaces, subspace filtering, or [Laloux et al. (1999)] where PCA (as well as some random matrix theory) is used to try to improve on the naive solution to Markovitz’s portfolio optimization problem.

Important progress has been made recently in our understanding of the statistical properties of PCA in such settings. Emblematic of this is work of [Johnstone (2001)], which explains the properties of the square of the largest singular value of a random matrix \( X \) under the “null model” where its entries are iid \( \mathcal{N}(0, 1) \). Specifically, if we denote the sample eigenvalues of \( X'X \) by \( l_1 \geq \ldots \geq l_p \), call

\[
\begin{align*}
n_1 &= \max(n, p) - 1, \quad p_1 = \min(n, p), \\
\mu_{np} &= (\sqrt{n_1} + \sqrt{p_1})^2, \\
\sigma_{np} &= (\sqrt{n_1} + \sqrt{p_1}) \left( \frac{1}{\sqrt{n_1}} + \frac{1}{\sqrt{p_1}} \right)^{1/3},
\end{align*}
\]

and \( W_1 \) the Tracy-Widom distribution (see [A0]), it was shown in [Johnstone (2001)] that

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Theorem 1 (Johnstone) If \( n, p \to \infty \) and \( n/p \to \gamma \in (0, \infty) \),
\[
\frac{l_1 - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathcal{D}} W_1.
\]

Building on Johnstone (2001) and using properties of determinantal point processes, Soshnikov (2002) showed that the same result holds for the \( k \) largest eigenvalues, where \( k \) is a fixed integer: their joint distribution converges to their Tracy-Widom counterpart.

This is a very interesting development because the classical theory (e.g. Anderson (1984)) was developed under the assumption that \( p \) was fixed and \( n \) grew to \( \infty \), whereas in modern day applications both \( p \) and \( n \) are large. However, Johnstone’s assumption \( n/p \to \gamma \) imposes a limit on the validity of his result which one would like to remove. In an actual data analysis, with given \( p \) and \( n \), \( n = o(p) \) and \( n \approx p \) could be equally plausible. Furthermore, a specific \( X \) of size \( n \times p \) could arise in many triangular arrays settings, where we have \( X_j \) of size \( n_j \times p_j \), and the limitation \( n_j/p_j \to \gamma \) finite might only hold in some triangular situations and not in others.

Accordingly in this paper we weaken the assumption that \( n/p \to \gamma \) finite and show that

Theorem 2 If \( n, p \to \infty \) and \( n/p \to \infty \),
\[
\frac{l_1 - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathcal{D}} W_1.
\]

Moreover, with the same centering and scaling, the joint distribution of the \( k \) largest eigenvalues converges in law to its Tracy-Widom counterpart.

Dually, the same result holds if \( n/p \to 0 \).
1.1 Numerical experiments

Johnstone (2001) showed empirically that in that situation the Tracy-Widom approximation was reasonably satisfying, even for small matrices. Similarly, to try to assess its accuracy in our setup, we ran the following experiments in Matlab: we picked $n$ and $p$ and generated 10,000 $n \times p$ matrices $X$ with entries iid $\mathcal{N}(0,1)$. Then we used standard routines (normest in Matlab) to compute their spectral norms and squared them to obtain a dataset of $l_1$-s.

Following Johnstone (2003), we adjust centering and scaling to

$$
\tilde{\mu}_{np} = \sqrt{n - 1/2} + \sqrt{p - 1/2},
$$

$$
\tilde{\sigma}_{np} = (\sqrt{n - 1/2} + \sqrt{p - 1/2}) \left( \frac{1}{\sqrt{n - 1/2}} + \frac{1}{\sqrt{p - 1/2}} \right)^{1/3}.
$$

This leads to a very significant improvement in the quality of the Tracy-Widom approximation for our simulations. Simple manipulations (explained in section 2.2) show that we have some freedom in choosing the centering and scaling: if we replace $n$ by $n + a$ and $p$ by $p + b$ (where $a$ and $b$ are fixed real numbers) in the definitions of $\mu_{np}$ and $\sigma_{np}$, Theorem 1 and Theorem 2 still hold. The particular choice used here is motivated by a careful theoretical analysis of the entries of $K_N$ mentioned in section 2.2.

Table 1 summarizes the “quantile” properties of the empirical distributions we obtained and compare them to the Tracy-Widom reference. We used the same reference points as Johnstone (2001).

We picked the dimensions according to two criteria: $100 \times 4000$, $30 \times 5000$, and $50 \times 5000$ were chosen to investigate “representative” microarray situations. We chose the other to have a range of ratios and estimate how valuable the Tracy-Widom approximation would be in situations that could be considered classical, i.e. one small dimension (less than 10) and one large (several hundreds to several thousands). For the sake of completeness, we redid the simulations presented in Johnstone (2001) and present in Table 2 the results obtained with $\tilde{\mu}_{np}$ and $\tilde{\sigma}_{np}$ as centering and scaling.

We see that the fit is good to very good for the upper quantiles (0.9 and beyond) across the range of dimensions we investigated. The practical interest of this remark is clear: these are the quantiles one would naturally use in a testing problem. We note that it appears empirically that the problem gets harder when the ratio $r$ of the larger dimension to the smaller one in our notation) gets bigger: the larger $r$, the larger $p_1$ should be for the approximation to be acceptable.

1.2 Conclusions and Organization

From a technical standpoint, the method developed in Johnstone (2001) proves to be versatile, and, at least conceptually, relatively easy to adapt to the case where $n/p \to \infty$. Nevertheless, substantial technical work is needed to obtain Theorem 2. Using the elementary fact (see e.g. theorem 7.3.7 in Horn and Johnson (1990)) that the largest eigenvalue of $X^*X$ is the same as the largest eigenvalue of $XX^*$, it will be sufficient to give the proof in the case $n/p \to \infty$.

From a practical point of view, we show that the Tracy-Widom limit law does not depend on how the sequence $(n,p)$ is embedded. As long as both dimensions go to infinity, the properly re-centered and re-scaled largest eigenvalue converges weakly to this law.

We can compare this with the “classical” situation where $p$ is held fixed, in which case the limiting joint distribution is known, too (see e.g. Anderson (1984), corollary 13.3.2). In this case, the centering is done around $n$ and the scaling is $\sqrt{n}$; elementary computations show that $(l_1 - \mu_{np})/\sigma_{np}$ also has a non-degenerate limiting distribution (possibly changing with each $p$). Nevertheless, even with the classical centering, it is hard to evaluate the marginals in this context and the results are therefore difficult to use in practice.
### Table 1: Quality of the Tracy-Widom Approximation for some large matrices

| TW Quantiles | TW  | 10×1000 | 10×4000 | 10×10000 | 100×4000 | 30×5000 |
|--------------|-----|---------|---------|----------|----------|---------|
| -3.9         | .01 | 0.009   | 0.010   | 0.015    | 0.012    | 0.013   |
| -3.18        | .05 | 0.047   | 0.050   | 0.060    | 0.053    | 0.055   |
| -2.78        | .10 | 0.102   | 0.107   | 0.112    | 0.103    | 0.105   |
| -1.91        | .30 | 0.303   | 0.308   | 0.316    | 0.304    | 0.303   |
| -1.27        | .50 | 0.506   | 0.506   | 0.522    | 0.508    | 0.503   |
| -0.59        | .70 | 0.705   | 0.704   | 0.723    | 0.706    | 0.702   |
| 0.45         | .9  | 0.904   | 0.904   | 0.913    | 0.901    | 0.904   |
| 0.98         | .95 | 0.953   | 0.951   | 0.958    | 0.951    | 0.953   |
| 2.02         | .99 | 0.992   | 0.990   | 0.992    | 0.991    | 0.991   |

### Table 2: Quality of the Tracy-Widom Approximation (Continued)

| TW Quantiles | TW  | 50×5000 | 50×20000 | 50×50000 | 5×200 | 5×2000 | 5×20000 |
|--------------|-----|---------|----------|----------|-------|--------|---------|
| -3.9         | .01 | 0.010   | 0.017    | 0.021    | 0.008 | 0.014 | 0.018   |
| -3.18        | .05 | 0.053   | 0.067    | 0.079    | 0.047 | 0.057 | 0.069   |
| -2.78        | .10 | 0.104   | 0.125    | 0.139    | 0.094 | 0.110 | 0.120   |
| -1.91        | .30 | 0.309   | 0.331    | 0.345    | 0.293 | 0.314 | 0.320   |
| -1.27        | .50 | 0.502   | 0.522    | 0.538    | 0.500 | 0.506 | 0.519   |
| -0.59        | .70 | 0.705   | 0.718    | 0.727    | 0.714 | 0.712 | 0.710   |
| 0.45         | .9  | 0.899   | 0.905    | 0.911    | 0.911 | 0.906 | 0.907   |
| 0.98         | .95 | 0.949   | 0.955    | 0.957    | 0.959 | 0.951 | 0.954   |
| 2.02         | .99 | 0.991   | 0.992    | 0.992    | 0.994 | 0.992 | 0.992   |

Table 1: Quality of the Tracy-Widom Approximation for some large matrices: the leftmost columns display certain quantiles of the Tracy-Widom distribution. The second column gives the corresponding value of its cdf. Other columns give the value of the empirical distribution functions obtained from simulations at these quantiles. \( \tilde{\mu}_{np} \) and \( \tilde{\sigma}_{np} \) are the centering and scaling sequences.

Table 2: Quality of the Tracy-Widom Approximation (Continued): the columns have the same meaning as in Table 1. The ratio \( p/n \) is smaller than in Table 1 and the matrices are not as big, but the Tracy-Widom approximation is already acceptable for the upper quantiles. \( \tilde{\mu}_{np} \) and \( \tilde{\sigma}_{np} \) are the centering and scaling sequences.
Our simulation results show that the Tracy-Widom approximation is reasonably good (for the upper quantiles) even when \( p \) or \( n \) are small. As remarked by Johnstone (2001), Proposition 1.2, this implies that when doing PCA, one could develop (conservative) tests based on the Tracy-Widom distribution that could serve as alternatives to the scree plot or the Wachter plot.

The paper is organized as follows: after presenting (Section 2) the main elements of the proof of Theorem 1, we describe (Section 3) the strategy that will lead to the proof of Theorem 2. We prove the two crucial points needed in Section 4. To make the paper self-contained, we give some background information about different aspects of the problem in the appendices. Several technical issues are also treated there in order to avoid obscuring the proof of the main result.

2 Outline of Johnstone’s proof

Before describing the backbone of the proof presented in Johnstone (2001), we need to introduce a few notational conventions. In what follows, we will use \( N \) instead of \( p \) to be consistent with the literature. We also denote by \( \text{AB} \) (for “asymptotic behavior”) the situation where \( n, N, \) and \( n/N \to \infty \). We will frequently index functions that depend on both \( N \) and \( n \) with only \( N \). The reason for this is that it will allow us to keep the notations relatively light, and that we think of \( n \) as being a function of \( N \). Notations like \( E_N \) and \( P_N \) will denote expectation and probability under the measure induced by the matrices (of size \( n(N) \times N \)) we are working with. Finally, it is technically simpler to work with a matrix \( X \) whose entries are standard complex Gaussians (i.e. the real and imaginary parts are independent, and they are both \( \mathcal{N}(0,1/2) \)), rather than with entries that are \( \mathcal{N}(0,1) \). When we mention the complex case, we refer to this situation.

We now give a quick overview of the important points around which the proof of Theorem 1 was articulated.

At the core of several random matrix theory results lie the fact that the joint distribution of the eigenvalues of the random matrices of interest is known and can be represented as the Fredholm determinant of a certain operator (or a totally explicit function of it).

Building on this, if we introduce a number \( b \) that is 1 in the real case and 2 in the complex one, it turns out that one has the representation formula

\[
E_N \left( \prod_{i=1}^{N} (1 + f(l_i)) \right) = |\det(\text{Id} + S_N f)|^{b/2},
\]

where \( S_N \) is an explicit kernel, depending of course upon the kind of matrices in which one is interested. Here, \( f \) treated as an operator means multiplication by this function. It is clear that if \( \chi_t = -1\{x : x \geq t\} \), we have

\[
P_N(l_1 \leq t) = |\det(\text{Id} + S_N \chi_t)|^{b/2}.
\]

The interested reader can find background information on this in Mehta (1990), chapters 5 and 6, Tracy and Widom (1998) or Deift (2000), chapter 5, which in turn (p.109) points to Reed and Simon (1972), section 17, vol 4, for background on operator determinants. We stress the fact that all these formulas are finite dimensional.

From the last display, the strategy to show convergence in law in either Theorem 1 or 2 is clear: fix \( s_0 \), show that under the relevant assumptions, \( P(l_{1,N} \leq s_0) \to W_1(s_0) \), and use the fact that \( W_1 \) is continuous to conclude.

2.1 Complex case

We just saw that to find the asymptotic behavior of \( l_1 \) is equivalent to showing the convergence of the determinant of a certain operator. This task can be reduced to showing convergence in
trace class norm of this operator (see Reed and Simon (1972) for background on this, e.g, Lemma XIII.17.4 (p.323)). Through work from Widom (1999), Johnstone (2001) exhibits an integral representation formula for his operator, and the original problem is essentially transformed into showing that certain integrals have a predetermined limit.

In somewhat more detail, if we call $\alpha = n - N,$ and $L_k^\alpha$ the $k$-th Laguerre polynomial associated with $\alpha$ (as in Szegö (1975), p.100), let

$$\phi_k(x) = \sqrt{\frac{k!}{(k + \alpha)!}} x^{\alpha/2} e^{-x/2} L_k^\alpha(x),$$

$$\xi_k(x) = \phi_k(x)/x, \quad a_N = \sqrt{N}n, \quad \text{and finally}$$

$$\begin{cases} 
\phi(x) = (-1)^N \sqrt{\frac{\pi}{2}} (\sqrt{n} \xi_N(x) - \sqrt{N} \xi_{N-1}(x)), \\
\psi(x) = (-1)^N \sqrt{\frac{2\pi}{2}} (\sqrt{N} \xi_N(x) - \sqrt{n} \xi_{N-1}(x)).
\end{cases}$$

We note two things: first, there is a slight abuse of notation since $\phi$ and $\psi$ obviously depend on $n$ and $N,$ but as in Johnstone (2001), we choose to not carry these indices in the interest of readability. Also, $\phi$ and $\psi$ admit more “compact” representations, in terms of a single Laguerre polynomial, with a modified $\alpha,$ or another degree. These are easy to derive using Szegö (1975), p.102, for instance. Nevertheless we choose to work (except in A7) with the previous representations because of the symmetries they present.

The kernel $S_N$ mentioned in [11] has the representation (Johnstone (2001), equation (3.6))

$$S_N(x, y) = \int_0^\infty \phi(x + z)\psi(y + z) + \psi(x + z)\phi(y + z)dz.$$

Now let $\bar{S}$ be the Airy operator. Its kernel is

$$\bar{S}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y} = \int_0^\infty \text{Ai}(x + u)\text{Ai}(y + u)du,$$

where $\text{Ai}$ denotes the Airy function. It was shown in Tracy and Widom (1994) that, viewing $\bar{S}$ as an operator on $L^2[s, \infty),$ one had

$$\det(\text{Id} - \bar{S}) = W_2(s),$$

where $W_2$ is the Tracy-Widom law “emerging” in the complex case (see A0). So the complex analog of theorem [11] follows from the fact that, after defining $S_{\tau}(x, y) = \sigma_N S_N(\mu_N + \sigma_Nx, \mu_N + \sigma_Ny),$ Johnstone managed to show, for all $s,$ that

$$\det(\text{Id} - S_{\tau}) \to \det(\text{Id} - \bar{S}).$$

To do this, he introduced $\phi_{\tau}(s) = \sigma_N \phi(\mu_N + s\sigma_N),$ and similarly $\psi_{\tau}.$ Note that we have

$$S_{\tau}(x, y) = \int_0^\infty \phi_{\tau}(x + z)\psi_{\tau}(y + z) + \psi_{\tau}(x + z)\phi_{\tau}(y + z)dz.$$

Since what we are interested in is really $S_{\tau} \chi_s,$ for some fixed $s,$ we will view $S_{\tau}$ as an operator acting on $L^2[s, \infty)$ in what follows.

So the problem becomes to show that, as $n, N \to \infty$

$$\phi_{\tau}(s), \psi_{\tau}(s) \to \frac{1}{\sqrt{2}} \text{Ai}(s), \quad (2)$$

and that $\forall s_0 \in \mathbb{R},$ there exists $N_0(s_0)$ such that if $N > N_0,$ we have on $[s_0, \infty),$

$$\phi_{\tau}(s), \psi_{\tau}(s) = O(e^{-s/2}). \quad (3)$$
Once this is shown (we give more details on this later), we can show that $S_\tau \to \bar{S}$ in the trace class norm of operators on $L^2[s, \infty)$. A classical way to do it is described in the remark at the end of section 3 of Johnstone (2001), which bounds the trace class norm of the difference of $S_\tau - \bar{S}$ in terms of the Hilbert-Schmidt norm of operators whose kernels are related to $\phi_\tau, \psi_\tau$ and $Ai$. This leads to the conclusion that
$$\det(\text{Id} - S_\tau) \to \det(\text{Id} - \bar{S}),$$
since $\det$ is continuous with respect to trace class norm. Therefore, the largest eigenvalue of $X^*X$ has the behavior it was claimed it has.

### 2.2 Real Case

In the real case, using arguments from Tracy and Widom (1996) and Widom (1999), Johnstone (2001) gets a representation similar to (1), this time involving an operator with kernel a $2 \times 2$ matrix (instead of scalar in the complex case). He is then able to relate it to the complex case problem - the matrix operator determinant can be computed as the product of two scalar operator determinants - and shows that the "reduced" variable he works with ought to have the same limit as it had in the Gaussian Orthogonal Ensemble case, which was studied in depth by Tracy and Widom.

For the sake of completeness, we recall that in this situation $\alpha = n - 1 - N$ and
$$P_N(l_1 \leq t) = \sqrt{\det(\text{Id} + K_N \chi_t)}.$$
$K_N$ has the representation (in the $N$ even case)
$$K_N = \begin{pmatrix} S_N + \psi \otimes \epsilon \phi & S_N D - \psi \otimes \phi \\ \epsilon S_N - \epsilon + \epsilon \psi \otimes \epsilon \phi & S_N + \epsilon \phi \otimes \psi \end{pmatrix},$$
where $D$ is the differential operator, $\epsilon$ is convolution with the kernel $\epsilon(x - y)$, and $\epsilon(x) = \text{sgn}(x)/2$.

We note the slight change in $\alpha$ and replace $n$ by $n - 1$ when we need to use the results or formulas derived in the complex case (for instance, the $S_N$ we just mentioned is $S_{n-1,N}$, and not $S_{n,N}$). We refer the reader to Gohberg et al. (2000) for a complement of information on operator determinants and to the end of section VIII in Tracy and Widom (1996) for details on the technical problems that $K_N$ poses.

From a purely technical standpoint, one critical issue is to evaluate the large $n, N$ limit of $c_\phi = \int_0^\infty \phi(x) dx/2$. If one can show that it is $1/\sqrt{2}$ when $N \to \infty$ through even values, then Johnstone’s considerations hold true all the way and we have the same conclusion as in Theorem 1.

We note that using the interlacing properties of the singular values (as mentioned for instance in Soshnikov (2002), Remark 5; see also Horn and Johnson (1990), theorem 7.3.9), as well as the estimates of the difference (resp. ratio) between two consecutive terms of the centering (resp. scaling) sequence, the $N$ odd case follows immediately from the $N$ even case. To be more precise, we use the fact that
$$\frac{\mu_{n,N} - \mu_{n,N-1}}{\sigma_N} = O(N^{-1/3}) \to 0 \text{ as } N \to \infty$$
to check that the $N$ even terms lower and upper bounding the $N$ odd probability have the same limit. Note that the same relationship holds for $\mu_{n+a,N+b}$ and $\mu_{n,N}$, if $a$ and $b$ are fixed real numbers. Therefore, after doing the proof with centering sequence $\mu_{n+3/2,N+1/2}$ (which is technically simpler), we will be able to conclude that the theorem holds true for $\mu_{n,N}$.

Last, to be able to use Soshnikov (2002), Lemma 2, which gives the result we wish for the joint distribution of the $k$-largest eigenvalues, we will need to verify that the entries of the $2 \times 2$ operator converge pointwise, and are bounded above in an exponential way. This is what is done in the proof of Lemma 1 of Soshnikov (2002), and we will show in A8 that the arguments given there can be extended to handle our situation.
3 Further Remarks and Agenda

Most of the work in [Johnstone (2001)] is done in closed form, and in the finite dimensional case. That has two advantages from our standpoint: as the limiting behavior is only investigated in the last “step”, most of the arguments given there carry through for our problem, and the method certainly does.

Therefore, our contribution is mostly technical; it follows very closely the ideas of [Johnstone (2001)], providing solutions to technical problems appearing in the case we consider. Only at a few points could we not use the approach developed in [Johnstone (2001)]. This led us to an analysis of the complex case that is slightly different from the original one, but the core reasons for which the result holds are the same.

In what follows, we first focus on showing that (2) and (3) hold true when $n, N$ and their ratio $t$ tend to infinity. This takes care of the complex case. We then turn to the problem of the asymptotic behavior of $c_{\phi}$, and the technical points we have to verify for [Soshnikov (2002)] results to hold.

The following remarks outline the differences between the analysis we present here and the one done in [Johnstone (2001)].

3.1 Remarks on adaptation of the original proof

3.1.1 Complex case

To show that (2) and (3) held true, [Johnstone (2001)] essentially reduced his problem to studying the solution of a “perturbed” Airy equation and used tools from [Olver (1974)] to carefully study it. One point that was used repeatedly was that the turning points of the equation were bounded away from one another when $n, N$ were large. This is not true anymore in the case we consider, and we show how to get around this difficulty. So we do not work with a perturbed Airy equation anymore, but rather with Whittaker functions, which have a close relationship to Laguerre polynomials, and their expansion in terms of parabolic cylinder functions (see A9 for some background information on special functions). In [Olver (1980)], the case we are interested in was studied in detail, giving us most of the tools we need to show (3). Using [Olver (1975)], we reinterpret the parabolic cylinder functions results in terms of Airy functions and derive the elements we need to complete the proof of (2) and (3).

The reason for which we could not exactly follow the “original” method is related to the error control function called $V(\zeta)$ in [Johnstone (2001)]. This function depends upon the parameter $\omega = 2\lambda/\kappa$, which in the case $n/N \to \gamma \in \mathbb{R}$ is bounded away from 2. This essentially allows a uniform control over $V$, and it is possible to show that this error control function is bounded as a function of $N$. Since the control is actually something like $\exp(\lambda_0 V/\kappa) - 1$, it tends to zero as $N \to \infty$. This gave [Johnstone (2001)] a way to get part of (3). In our case, it seems that $V$ would tend to $\infty$, at a rate that is nevertheless $o(\kappa)$. As it seems easier and more promising to use [Olver (1980)] than to derive the growth of $V$, we choose this approach. Nevertheless, this is the only (but crucial) technicality (in the complex case) that did not carry through by the method described in [Johnstone (2001)] under AB.

3.1.2 Real Case

For the $c_{\phi}$ problem, we provide a closed form expression at given $n, N$ and show that in the limit is the “right” one as long as $n$ and $N$ tend to $\infty$. This does not use the saddlepoint method, but relies on the availability of a generating function formula for Laguerre polynomials. The proof is done in A7.

A simple modification to [Johnstone (2001)] would give the same result: in the display preceding
and expand \((1 - t^2)^{-(\alpha/2+1)}\). Multiplying by \(1 + t\) has a very simple effect on the series, and so \(c_k\) is known explicitly.

In A8, we show how to check that the conditions required for Soshnikov’s results to hold are indeed met. They are straightforward consequences of the analysis we will carry below.

Since the real case is derived from the complex one after analyzing a few technical points, we verify these in the appendices and present here the study of the complex case. We now turn to the main problem we solve in this note: showing (2) and (3) under our set of assumptions.

### 4 Complex case: study of asymptotics

In this section, we work on the problem of showing pointwise convergence and uniform boundedness, setting the problem in a way similar to section 5 of Johnstone (2001). We recall his notations, slightly modified to avoid confusions: 

\[ N_+ = N + 1/2, \quad n_+ = n + 1/2, \quad z = \mu_N + \sigma_N s, \]

with \(\mu_N = (\sqrt{(N + \alpha)^+} + \sqrt{N_+})^{1/2}\) and \(\sigma_N = (\sqrt{(N + \alpha)^+} + \sqrt{N_+})^{1/4}(1/\sqrt{N_+} + 1/\sqrt{(N + \alpha)^+}) \). For reasons that will be transparent later on, our aim is to show that

\[
F_N(z) = (-1)^N \sigma_N^{-1/2} \sqrt{N!} z^{(\alpha+1)/2} e^{-z/2} L_N^{\alpha N}(z) \rightarrow \text{Ai}(s), \quad \forall s \in \mathbb{R},
\]

and

\[
F_N(z) = O(e^{-s}) \text{ uniformly in } [s_0, \infty), s_0 \in \mathbb{R}.
\]

The scaling is slightly different from the original proof: \(N^{-1/6}\) has been replaced by \(\sigma_N^{-1/2}\). As in Johnstone (2001), we focus on \(w_N(z) = z^{(\alpha+1)/2} e^{-z/2} L_N^\alpha(z)\), which satisfies

\[
\frac{d^2 w}{dz^2} = \left( \frac{1}{4} - \frac{\kappa}{z} + \frac{\lambda^2 - 1/4}{z^2} \right) w,
\]

where \(\kappa = N(\alpha + 1)/2\) and \(\lambda = \alpha/2\). Remark that under \(\textbf{AB} \overset{\text{def}}{=} n, N, n/N \rightarrow \infty, \ k \sim \lambda\). Our strategy is to reformulate the problem in terms of so-called Whittaker functions, denoted \(W_{\kappa,\lambda}\), and to use the extensive available studies of these functions to show (2) and (3). Temme (1990), formula (3.1) p.117 shows that

\[
w_N(z) = \frac{(-1)^N}{N!} W_{\kappa,\lambda}(z).
\]

From now on, we will closely follow \(\text{Olver (1980)}\). Let us remark that

\[
F_N(z) = \sigma_N^{-1/2} \frac{1}{\sqrt{N!}} W_{\kappa,\lambda}(z).
\]

We fix \(s_0 \in \mathbb{R}\), and we work only with \(z = \mu_N + \sigma_N s\), where \(s \geq s_0\).

**Preliminaries** Following \(\text{Olver (1980)}\), we introduce \(l = \kappa/\lambda, \ \beta = \sqrt{2(l - 1)}\), and the turning points \(x_1 = 2l - 2\sqrt{l^2 - 1}, \ x_2 = 2l + 2\sqrt{l^2 - 1}\), after the rescaling \(x = z/\lambda\). We remark that the two turning points coalesce at \(2\) under the hypothesis \(\textbf{AB}\). In the new variable \(x\), we have

\[
\frac{d^2 W}{dx^2} = \left( \lambda^2 g(x) - \frac{1}{4x^2} \right) W,
\]
where \( g(x) = \frac{1}{4x^2}(x-x_1)(x-x_2) \). Using the ideas explained in Johnstone (2001), we shall be eventually interested in the asymptotics for \( z = \mu_N + \sigma_N s \), or \( x = z/\lambda = x_2 + \sigma_N s/\lambda \) of \( F_N(z) \). Let us now define an auxiliary variable \( \upsilon \) by

\[
\int_{\beta}^{\upsilon} (\tau^2 - \beta^2)^{1/2} d\tau = \int_{x_2}^{x} g^{1/2}(t) dt \quad \text{if} \quad x_2 \leq x < \infty ,
\]

\[
\int_{-\beta}^{\upsilon} (\beta^2 - \tau^2)^{1/2} d\tau = \int_{x_1}^{x} (-g)^{1/2}(t) dt \quad \text{if} \quad x_1 \leq x \leq x_2 .
\]

We limit \( x \) to this range because of the technically important following point: \( \sigma_N/\lambda \) tends to zero faster than \( x_2 - x_1 \) does, and so, when \( s \) is bounded below, \( x \) will stay in the range \((x_1, \infty)\) for all \( N \) greater than a certain \( N_0 \). This is shown in A2, along with the closely related fact that we can focus on \( \upsilon \geq 0 \). Our analysis is based on section 3 of Olver (1980), where he builds on Olver (1975), in which he expands Whittaker functions in terms of parabolic cylinder functions. The condition \( \upsilon \geq 0 \) is critical, since Olver’s expansions depend on the sign of \( \upsilon \). Therefore, A2 entitles us to focus on only one specific form of these. From (3.10) p.219 in Olver (1980), one has

\[
W_{\kappa,\lambda}(\lambda x) = (2\lambda)^{1/4} \{ (\lambda(2 + \beta^2/2)/\epsilon) \lambda^{1+\beta^2/4} \} x^{1/2} \left\{ U(-\frac{1}{2} \lambda \beta^2, \upsilon \sqrt{2\lambda}) + \epsilon_1(\lambda^2, \beta^2, \upsilon) \right\} ,
\]

where, if \( E \) and \( M \) are the weight and modulus functions associated with \( U \) in Olver (1975) (p.156), we have, according to Olver (1981) (3.11) p.219,

\[
\epsilon_1(\lambda^2, \beta^2, \upsilon) = E^{-1}(-\frac{1}{2} \lambda \beta^2, \upsilon \sqrt{2\lambda}) M(-\frac{1}{2} \lambda \beta^2, \upsilon \sqrt{2\lambda}) O(\lambda^{-2/3}) \quad (6)
\]

uniformly with respect to \( \beta \in [0, B] \) and \( \upsilon \in [0, \infty) \), \( B \) being an arbitrary positive constant. We recall that the main relationship between \( U, E \) and \( M \) : for \( b \leq 0 \) and \( x \geq 0 \), \( |U(b, x)| \leq E^{-1}(b, x)M(b, x) \).

We now show that we have uniform boundedness on \([s_0, \infty)\). The pointwise convergence result will be a straightforward consequence of the arguments we need to develop to solve this first problem.

### 4.1 Uniform Boundedness

Following up on the previous displays, if \( n, N \) are large enough so that \( \upsilon \geq 0 \), we have

\[
|W_{\kappa,\lambda}(\lambda x)| \leq (2\lambda)^{1/4} \{ (\lambda(2 + \beta^2/2)/\epsilon) \lambda^{1+\beta^2/4} \} x^{1/2} ME^{-1}(1 + O(\lambda^{-2/3})) ,
\]

where we omitted the argument \((-\frac{1}{2} \lambda \beta^2, \upsilon \sqrt{2\lambda})\) for readability purposes. Our plan is now to transform this upper bound into a somewhat similar one, involving the modulus and weight function associated with the Airy function, which have the advantage of having only one parameter and known asymptotics.

To carry out this program, we need to split the investigation into two parts: first \( s \geq 0 \) or \( \upsilon \geq \beta \). This will allow us to find a \( s_1 \geq 0 \) such that \( F_N(z) = O(e^{-\delta}) \) on \([2s_1, \infty)\). In the second part, we will just have to consider the case \( s \in [s_0, 2s_1] \), and show that \( F_N \) is merely uniformly bounded on this interval.
4.1.1 Case \( s \geq 0 \)

In order to use the results linking parabolic cylinder functions and the Airy function (proved in \cite{Olve1959} and cited in \cite{Olve1975}), let us define yet another auxiliary variable, \( \eta \), by

\[
\frac{2}{3} \eta^{3/2} \beta^2 = \int_{x_2}^{x} g^{1/2}(t)dt.
\]

Then, if we call \( \mathcal{E} \) and \( \mathcal{M} \) the weight and modulus functions associated with the Airy function, we have, as shown in A3:

\[
\mathbf{E}^{-1}(-\frac{1}{2} \lambda \beta^2, v \sqrt{2 \lambda}) \leq \mathcal{E}^{-1}(\lambda^{2/3} \beta^{4/3}) \eta (1 + \mathcal{O}(\lambda \beta^{-2})) ,
\]

\[
\mathbf{M}(-\frac{1}{2} \lambda \beta^2, v \sqrt{2 \lambda}) \leq \frac{2\pi^{1/4}}{(\lambda \beta)^{1/4}} (\Gamma((1 + \lambda \beta^2)/2))^{1/2} \beta^{1/2} \times \left( \frac{\eta}{v^2 - \beta^2} \right)^{1/4} \mathcal{M}(\lambda^{2/3} \beta^{4/3} \eta) (1 + \mathcal{O}(\lambda \beta^{-2})) .
\]

Whence, if we call \( \theta \equiv \lambda^{2/3} \beta^{4/3} \eta , \)

\[
|F_N(\lambda x)| \leq K_{n,N} x^{1/2}(\eta/(x^2 - 4lx + 4))^{1/4} \mathcal{E}^{-1}(\theta) \mathcal{M}(\theta) \left( 1 + \mathcal{O}(\lambda \beta^{-2}) \vee \lambda^{-2/3} \right) .
\]

In A4, we show that \( K_{n,N} \sim 2^{2/3}(N/n)^{1/4} \) under \( \mathbf{AB} \). From now on, \( \Delta \) will denote a generic constant; its value may change from display to display. As long as \( x \geq x_2 \), or \( s \geq 0 \), we have

\[
|F_N(\lambda x)| \leq \Delta(N/n)^{1/4} x^{1/2}(\eta/(x^2 - 4lx + 4))^{1/4} \mathcal{E}^{-1}(\theta) \mathcal{M}(\theta) .
\]

Now using the fact that (see \cite{Olve1974}, chap. 11) \( x^{1/4} \mathcal{M}(x) \leq \Delta , \mathcal{E}^{-1}(x) \leq \Delta \exp(-2x^{3/2}/3) \) for \( x \geq 0 \) and \( \lambda \beta^2 = 2N + 1 \), we get the new inequality

\[
|F_N(\lambda x)| \leq \Delta \left( \frac{N}{n} \right)^{1/4} N^{-1/6} \left( \frac{x^2}{x^2 - 4lx + 4} \right)^{1/4} \exp(-2\theta^{3/2}/3) .
\]

In A5.1, we show that there exists \( s_1 \) such that if \( s \geq 2s_1 , (2\theta^{3/2})/3 \geq s \). Also, as shown in A6.1, if \( s \geq 0, g \) is positive and increasing in \( x \) (or, equivalently, in \( s \)). Since the rational function of \( x \) appearing in the previous display is just \( (4g(x))^{-1/4} \), we can bound it by its value at \( x(2s_1) \) on \([2s_1, \infty)\). In A6.2, we show that, at \( s \) fixed, under \( \mathbf{AB} \), we have \( 4g(x) \sim \beta \sigma_N s/\lambda \), and using the equivalents mentioned in A1, we have \( \sigma_N \beta/\lambda \sim 4N^{1/3}/n \), from which we conclude that

\[
\left( \frac{N}{n} \right)^{1/4} N^{-1/6} (4g(2s_1))^{-1/4} \sim N^{1/12} n^{-1/4} (8s_1 N^{1/3}/n)^{-1/4} \sim (8s_1)^{-1/4} .
\]

Therefore, if \( N \) is large enough,

\[
\forall s \in [2s_1, +\infty) \quad |F_N(\lambda x)| \leq \Delta \exp(-s)
\]

4.1.2 Case \( s \in [s_0, 2s_1] \)

Our aim now is just to show that \( F_N \) as a function of \( s \) is bounded on this interval; from this we shall immediately have that \( F_N = \mathcal{O}(\exp(-s)) \) on this interval, and we will have a proof of (5). This part is comparatively simpler: we use equation (7), in which we have \( \mathbf{E}^{-1} \leq 1 \), by definition \cite{Olve1975}, p.156, (5.22)). Now using the display between (6.12) and (6.13) p.159 of the same article, we have for \( \lambda \beta^2 \geq 1 \) and \( v \geq 0 \),

\[
\frac{\mathbf{M}(-\lambda \beta^2/2, v \sqrt{2 \lambda})}{(\Gamma((1 + \lambda \beta^2)/2))^{1/2}} \leq \frac{\Delta^{1/2}}{(\lambda \beta^2)^{1/2}} \left( \frac{\eta}{v^2 - \beta^2} \right)^{1/4} .
\]
Hence,

\[ |F_N(\lambda x)| \leq K_{n,N} \Delta \left( \frac{\eta}{x^2 - 4lx + 4} \right)^{1/4} x^{1/2}. \]

However on this interval, \( x \to 2 \), by A5.2 \( \eta = (\lambda \beta^2)^{-2/3} s + o((\lambda \beta^2)^{-2/3}) \), and by A6.2 \((x^2 - 4lx + 4) = 4s\sigma_N \beta(1 + o(1))/\lambda \). Therefore,

\[ \frac{\eta}{x^2 - 4lx + 4} \sim 2^{-2/3} 2^{-4} n/N \]

on the whole interval, and, because of the asymptotic estimate of \( K_{n,N} \) given in A4, \( F_N \) is bounded uniformly in \( N \) on the interval \([s_0, 2s_1]\).

We can thus conclude that

\[ \forall s_0, \exists N_0(s_0) N > N_0(s_0), F_N(s) = O_s(e^{-s}) \text{ on } [s_0, \infty). \]

### 4.2 Pointwise convergence

Having studied in detail the uniform boundedness of \( F_N \) makes the pointwise convergence problem easier. First, since we bounded above \( F_N \) in terms of \( M \) and \( E^{-1} \), equation (10) shows that \( \epsilon_1 = O((\lambda^{-2/3} e^{-s}) \) on \([s_0, \infty). \) So for fixed \( s \), it tends to zero as \( N \) gets large. The pointwise limit of \( F_N \) will be the pointwise limit of the parabolic cylinder function part of the expansion. We call this part \( \varphi F_N \), for “principal part”.

Using the relationship between \( U \) and \( A_i \) that we mention in A3, we have, with \( \theta = (\lambda \beta^2)^{2/3}/\eta \),

\[ \varphi F_N(\lambda x) = K_{n,N} x^{1/2} \left( \frac{\eta}{x^2 - 4lx + 4} \right)^{1/4} (A_i(\theta) + E^{-1}(\theta) M(\theta) O((\lambda \beta^2)^{1})) \]

Since \( x \to 2 \), \( K_{n,N} \sim 2^{2/3} (N/n)^{1/4} \) and given the estimate we just mentioned for the ratio \( \eta/(x^2 - 4lx + 4) \), we have

\[ K_{n,N} x^{1/2} \left( \frac{\eta}{x^2 - 4lx + 4} \right)^{1/4} \sim 1. \]

In other respects, we show in A5.2 that \( \theta \to s \) under AB. Finally, \( E^{-1} \) and \( M \) are bounded on \( \mathbb{R} \), as shown in 11.2 (pp.394-397) of Olver [1974]. Hence \( E^{-1}(\theta) M(\theta) (\lambda \beta^2)^{-1} \to 0 \) under AB, and we can conclude that \( \varphi F_N(\lambda x) \to A_i(s) \); combining all the elements gives

\[ \forall s \in \mathbb{R}, F_N(\lambda x) \to A_i(s) \text{ under AB}. \]

### 4.3 Asymptotics for \( \phi_\tau \) and \( \psi_\tau \)

So far we have shown that \( F_N(z) = (-1)^N \sigma_N^{-1/2} \sqrt{z} \phi_N(z) \to A_i(s) \), and that \( e^s F_N \) was bounded when \( N > N_0 \) and \( s \geq s_0 \).

Our aim is to show (2) and (3). Let us write, as in Johnstone [2001],

\[ \phi_\tau = \phi_{I,N} + \phi_{II,N}, \]

where

\[ \phi_{I,N}(z) = (-1)^N \sigma_N \sqrt{a_N} \phi_N(z)/(\sqrt{2z}) = F_N(z) d_N(z/\mu_N)^{-3/2}. \]
Study of $\phi_{I,N}$ In the previous display, we have $d_N = (\sigma_N/\mu_N)^{3/2}/\sqrt{n}a_n$. As $\sigma_N \sim n^{1/2}N^{-1/6}$ and $\mu_N \sim n$, $(\sigma_N/\mu_N)^{3/2} \sim n^{-3/4}N^{-1/4}$. Since $a_n = \sqrt{n}n$, $\alpha a_n n = \frac{3/2N}{1/2}$, and therefore $d_N \to 1/\sqrt{2}$. But when $s$ is fixed, $z/\mu_N \to 1$, so it follows that

$$
\text{Under AB, } \phi_{I,N}(\mu_N + \sigma_N s) \to \frac{\text{Ai}(s)}{\sqrt{2}}.
$$

To bound $\phi_{I,N}$ for $N > N_0$ and $s \geq s_0$, we use, as in Johnstone (2001), the uniform bound for $F_N$ and $(z/\mu_N)^{-3/2} \leq \exp(-3s\sigma_N/2\mu_N))$, if $s \geq 0$. If $s \leq 0$, we have $(z/\mu_N)^{-3/2} \leq (1+s_0\sigma_N/\mu_N)^{-3/2}$, and since this converges to 1 under AB, it is bounded if $N$ is large enough. So we have shown that,

$$
\phi_{I,N}(\mu_N + s\sigma_N) \begin{cases}
\to 2^{-1/2}\text{Ai}(s), & N \to \infty, \\
\leq Me^{-s} \text{ on } [s_0, \infty) \text{ if } N \geq N_0(s_0).
\end{cases}
$$

Study of $\phi_{II,N}$ We use once again the same approach as in Johnstone (2001). We have

$$
\phi_{II,N} = u_N v_{N-1} \phi_{I,N-1},
$$

where $u_N = (\sigma_N/\sigma_{N-1})\sqrt{a_n/a_{N-1}}$ and $v_N = (N/n)^{1/2}$, and $u_{N-1}$ appearing in $\sigma_{N-1}$ is $n_{N-1}$ (for $\sigma_{N-1}$ is defined in terms of $L_{N-1}^{1/2} \mu_{N-1}$ and we should therefore have the same $\alpha = n - N = (n - 1) - (N - 1)$). Remark that under AB, $v_N \to 0$ and $u_N \to 1$. Define $s'$ by $\mu_N + s\sigma_N = \mu_{N-1} + s\sigma_{N-1}s'$. From

$$
s' = \frac{\mu_N - \mu_{N-1}}{\sigma_{N-1}} + \frac{s\sigma_N}{\sigma_{N-1}},
$$

we deduce that $s' \geq s/2$ on $[0, \infty)$, if $N$ is large enough: as a matter of fact, under AB, $\mu_N - \mu_{N-1} = O(\sqrt{n}/N)$, $\sigma_N \sim n^{1/2}N^{-1/6}$, and $\sigma_N/\sigma_{N-1} \to 1$, so it is larger than $1/2$ when $N$ is large enough. To summarize, we just showed that

$$
\phi_{II,N}(\mu_N + s\sigma_N) \leq M v_N e^{-s/2} \text{ for } s \in [0, \infty),
$$

by applying the bound we got for $\phi_{I,N}$ to $\phi_{I,N-1}$ and $s'$ as the dummy variable. Here, we are implicitly using the fact that since $n/N \to 1$, $(n-1)/(N-1)$ does too, and we can apply all the results we derived before. On the other hand, when $s \in [s_0, 0]$, we can use the fact that $(\mu_N - \mu_{N-1}) \geq 0$ and $\sigma_N/\sigma_{N-1} \leq 2$ to show that $s' \geq 2s$ and hence

$$
\phi_{II,N}(\mu_N + s\sigma_N) \leq M v_N e^{-s} \leq M' v_N e^{-s/2} \text{ for } s \in [s_0, 0].
$$

The conclusion is therefore that

$$
\phi_{II,N}(\mu_N + s\sigma_N) \begin{cases}
\to 0, & N \to \infty, \\
\leq \Delta e^{-s/2} \text{ on } [s_0, \infty), \text{ if } N > N_0(s_0).
\end{cases}
$$

Hence we have shown that (2) and (3) hold for $\phi_\tau$. The analysis for $\psi_\tau$ is similar.

5 Appendices

This section is devoted to giving background information needed to understand the problem and make the paper relatively self-contained. We also establish many of the properties needed in the course of the proofs of equations (2) and (3) here.

Before we start, let us mention a notation issue: $\alpha$ changes value depending on whether we treat the complex case or the real one. For the complex case $\alpha + N = n$, whereas for the real one $\alpha + N = n - 1$. We frequently replace $\alpha + N$ by $n$ in what follows; this is because the proof of equations (2) and (3) is done in the complex case and applies to the real one by just changing $n$ into $n - 1$ everywhere. When dealing with problems which are real case specific, we keep the notation $N + \alpha$. The definition of $\mu_N$ and $\sigma_N$ are also given in terms of $N + \alpha$ to highlight the adjustments needed when dealing with the real or the complex case.
A0: Tracy-Widom distributions

We recall here the definition of the Tracy-Widom distributions. We split the description according to whether the entries of the matrix we are considering are real or complex.

We first need to introduce the function $q$, defined as

$$
\begin{align*}
q''(x) &= xq(x) + 2q^3(x), \\
q(x) &\sim Ai(x) \quad \text{as} \quad x \to \infty.
\end{align*}
$$

- **Complex Case** The Tracy-Widom distribution appearing in the complex case, $W_2$, has cumulative distribution function $F_2$ given by

$$
F_2(s) = \exp \left( - \int_s^\infty (x-s)q^2(x)dx \right).
$$

The joint distribution is slightly more involved to define. Following Soshnikov (2002), we do it through its $k$-point correlation functions, using its determinantal point process character (see e.g. Soshnikov (2000)).

Let us first call $\bar{S}$ be the Airy operator. Its kernel is

$$
\bar{S}(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y} = \int_0^\infty Ai(x+u)Ai(y+u)du.
$$

In the complex case, the $k$-point correlation functions have the property that

$$
\rho_k(x_1, \ldots, x_k) = \det_{1 \leq i,j \leq k} \bar{S}(x_i, x_j).
$$

- **Real Case** The real counterpart of $W_2$, which is called $W_1$, has cdf $F_1$ with

$$
F_1(s) = \exp \left( - \frac{1}{2} \int_s^\infty q(x) + (x-s)q^2(x)dx \right).
$$

The $k$-point correlation functions satisfy

$$
\rho_k(x_1, \ldots, x_k) = \left( \det_{1 \leq i,j \leq k} K(x_i, x_j) \right)^{1/2},
$$

where the $2 \times 2$ matrix kernel of $K$ has entries (see Soshnikov (2002), eq (2.18) to (2.21))

$$
\begin{align*}
K_{1,1}(x, y) &= \bar{S}(x, y) + \frac{1}{2}Ai(x) \int_y^\infty Ai(u)du, \\
K_{2,2}(x, y) &= K_{1,1}(y, x), \\
K_{1,2}(x, y) &= -\frac{1}{2}Ai(x)Ai(y) - \frac{\partial}{\partial y} \bar{S}(x, y), \\
K_{2,1}(x, y) &= -\int_0^\infty dt \left( \int_{x+t}^\infty Ai(v)dv \right) Ai(y+t) - \epsilon(x-y) + \frac{1}{2} \int_y^x Ai(u)du + \frac{1}{2} \int_x^\infty Ai(u)du \int_y^\infty Ai(v)dv.
\end{align*}
$$

A1: Asymptotic behavior of some simple functions

In this appendix, we present some basic facts and identities that we used throughout the proof.

We will make repeated use of the following observations: since $\sigma_N = (\sqrt{N+\alpha} + \sqrt{N+\lambda})^{1/3}$ and $\lambda = \alpha/2$, under $AB$ we have

$$
\sigma_N \sim n^{1/2}N^{-1/6},
$$

$$
\lambda \sim n/2.
$$

We also use several times the following identities:
\textbf{Fact 1} With \( \lambda = \alpha/2, \kappa = N + (\alpha + 1)/2, \) and \( l = \kappa/\lambda, \beta = \sqrt{2(l - 1)} \), we have
\[
\begin{align*}
\lambda \beta^2 &= (2N + 1), \\
\beta &\sim 2\sqrt{N/n}.
\end{align*}
\]
The first remark is simple algebra, and the second one comes from \( \beta^2 = 2(l - 1) = 2(2N + 1)/\alpha \sim 4N/n \) under \( \text{AB} \). We have the estimates:

\textbf{Fact 2} \( x_2 - x_1 \sim 8\sqrt{N/n} \) and \( \sigma_N/\lambda \sim 2n^{-1/2}N^{-1/6} \).

The second one is obvious; the first one comes from the fact that \( x_2 - x_1 = 2l \pm 2\sqrt{l^2 - 1} \). Using Fact 1 immediately gives the claimed result. Finally, we have the following estimates

\textbf{Fact 3} \( \beta \sigma_N/\lambda \sim 4N^{1/3}/n \) and \( \sigma_N^3/(\lambda \beta^2) \sim (n/N)^{3/2}/2 \).

The result directly follows from the aforementioned estimates.

\textbf{A2: Working with} \( v \geq 0 \)

Here we assume that \( s \in [s_0, \infty) \). We also assume that \( s < 0 \), for otherwise we can work with \( v \geq \beta > 0 \). From A1, we have \( |x - x_2| = |s| \sigma_N/\lambda \leq |s_0| \sigma_N/\lambda \ll x_2 - x_1 \) by Fact 2. Now \( v = 0 \) corresponds to \( x_0 \leq \bar{x} = (x_1 + x_2)/2 \); as a matter of fact, since \( (x_2 - x)(x_1 - x) \) is symmetric around \( \bar{x} \) and \( 1/\bar{x} \) is obviously larger on \([x_1, \bar{x}]\) than it is on \([\bar{x}, x_2]\), we have
\[
\int_{-\beta}^{\bar{x}} (\beta^2 - \tau^2) d\tau = \int_{\bar{x}}^{x_1} (-g(t))^{1/2} dt \geq \int_{\bar{x}}^{x_2} (-g(t))^{1/2} dt.
\]
By symmetry, we also get
\[
\int_{-\beta}^{0} (\beta^2 - \tau^2) d\tau = \int_{0}^{\beta} (\beta^2 - \tau^2) d\tau = \frac{1}{2} \int_{\bar{x}}^{x_1} (-g(t))^{1/2} dt
\]
\[
\leq \int_{\bar{x}}^{x_1} (-g(t))^{1/2} dt = \int_{-\beta}^{\bar{x}} (\beta^2 - \tau^2) d\tau,
\]
and therefore, \( v_{\bar{x}} > 0 \).
However \( \bar{x} \) is always smaller than \( x(s_0) \) if \( N \) is large enough. So we can limit our investigations to the case \( v \geq 0 \).

\textbf{A3: Relationship between} \( E^{-1}, E^{-1}, M \) and \( M \)

We claim that if \( s \geq 0 \), and we define \( \theta = (\lambda \beta^2)^{2/3} \eta \), the following inequalities hold true:
\[
\begin{align*}
E^{-1}(-\lambda \beta^2/2, \nu \sqrt{2\lambda}) &\leq E^{-1}(\theta)(1 + O((\lambda \beta^2)^{-1})) , \\
M(-\lambda \beta^2/2, \nu \sqrt{2\lambda}) &\leq \left(\frac{4\pi}{(\lambda \beta^2)^{1/4}}\right)^{1/4} [\Gamma((1 + \lambda \beta^2)/2)]^{1/2} \beta^{1/2} \left(\frac{\eta}{\nu^2 - \beta^2}\right)^{1/4} \times M(\theta)(1 + O((\lambda \beta^2)^{-1})).
\end{align*}
\]

For the sake of simplicity we call \( \Xi \) the part that precedes the sign ‘×’ in the last inequality. According to \cite{Olver1974}, equations (5.12) and (5.13), we have
\[
\begin{align*}
U(-\lambda \beta^2/2, \sqrt{2\lambda}) &= \Xi \{ \text{Ai}(\theta) + M(\theta)E^{-1}(\theta)O((\lambda \beta^2)^{-1}) \} , \\
\bar{U}(-\lambda \beta^2/2, \sqrt{2\lambda}) &= \Xi \{ \text{Bi}(\theta) + M(\theta)E^{-1}(\theta)O((\lambda \beta^2)^{-1}) \} .
\end{align*}
\]
We have, if $s \geq 0$, $x \geq x_2$, so

$$2/3\beta^2 \eta^{3/2} = \int_{x_2}^{x} g^{1/2}(t) dt = \int_{\beta}^{\eta} (\tau^2 - \beta^2)^{1/2} d\tau .$$

(8)

For the Airy function, the weight and modulus functions had different definition depending on whether the argument was bigger than the largest root, $c$, of Ai(z) = Bi(z) or not. Likewise, the definition of $E^{-1}$ and $M$ depends on the position of the argument with respect to the largest root of the equation $\tilde{U}(b,x) = U(b,x)$, which is called $\rho(b)$ in [Olver 1975].

**Where do the auxiliary variables lie when $s \geq 0$?** We claim that the answer is that $\theta \geq 0 > c$, and $\nu \sqrt{2\lambda} \geq \rho(-\lambda \beta^2/2)$.

The first part of equation (8) implies that $\eta \geq 0$, so $\theta \geq c$, as $c < 0$. This means that we can use the definition $M^2 = 2AiBi$ and $E^{-1}M = 2^{1/2}Ai$. The second part implies that $\nu \geq \beta$; therefore, $2\nu \sqrt{2} \geq 2\lambda \beta^2 \geq \rho(-\lambda \beta^2/2)^2$, since by [Olver 1975], equation (5.21), $\rho(b) \leq 2(-b)^{1/2}$ when $b \to -\infty$. This means that we have similar relationships between $E^{-1}$, $M$, $U$, and $\tilde{U}$, to the one we had in the Airy case, $U$ playing the role of Bi, and $\tilde{U}$ playing the role of Ai.

**Consequences of their positions** The interesting consequence of the previous remarks is that we can write, if $N$ is large enough, for all $s \geq 0$

$$E^{-2}(-\lambda \beta^2/2, \nu \sqrt{2\lambda}) = \frac{U}{\bar{U}} = \frac{M(\theta)E^{-1}(\theta) 2^{-1/2} + O((\lambda \beta^2)^{-1})} {M(\theta)E(\theta) 2^{-1/2} + O((\lambda \beta^2)^{-1})} .$$

In other words, we just proved that $\exists N_0$ such that $N > N_0$ implies, $\forall s \geq 0$

$$E^{-1}(-\lambda \beta^2/2, \nu \sqrt{2\lambda}) \leq E^{-1}(\theta)(1 + O((\lambda \beta^2)^{-1}) .$$

By the same arguments, we derive that

$$M(-\lambda \beta^2/2, \nu \sqrt{2\lambda}) \leq \Xi M(\theta)(1 + O((\lambda \beta^2)^{-1}) .$$

**A4: Asymptotic behavior of $K_{n,N}$**

The aim here is to show that

$$K_{n,N} \sim 2^{2/3}(N/n)^{1/4} .$$

$K_{n,N}$ has the following expression:

$$K_{n,N} = \frac{(2\lambda)^{1/4} \{\lambda(2 + 1/2\beta^2)/e\}^{\lambda(1+\beta^2/4)} \sqrt{2\pi}^{1/4} \Gamma((1 + \lambda \beta^2)/2)]^{1/2} \beta^{1/2}}{(\lambda \beta^2)^{1/12} \sqrt{n!N!} \sigma_N} .$$

Since $\lambda \beta^2 = (2N + 1)$, $\Gamma((1 + \lambda \beta^2)/2) = \Gamma(N + 1) = N!$.

In other respects, let $A_n = \{\lambda(2 + 1/2\beta^2)/e\}^{\lambda(1+\beta^2/4)} \sqrt{2\pi}^{1/4} \sigma_N! . Note that $2\lambda + \lambda \beta^2/2 = n - N + (2N + 1)/2 = n + 1/2 = n_+$. So $A_n = (n_+/e)^{n_+/2} / \sqrt{n!} . Using Stirling’s formula, we get that $A_n \sim (n_+/n)^{n_+/2}(n+/n)^{1/4}(2\pi)^{-1/4} \sim (2\pi)^{-1/4}$.

Now rewriting

$$K_{n,N} = \frac{A_n(\lambda \beta^2)^{1/4}(8\pi)^{1/4}}{(\lambda \beta^2)^{1/12} \sqrt{\sigma_N}} ,$$

we get that $K_{n,N} \sim 2^{2/3}(N/n)^{1/4}$, from using $A_n(8\pi)^{1/4} \sim \sqrt{2}$ and the second estimates of Fact B in A1.
A5: Asymptotic properties of $\eta$

This appendix is divided into two parts. We first show that there exists $s_1$ such that, if $s \geq 2s_1$,

$$\frac{2}{3}\lambda\beta^2\eta^{3/2} \geq s. \quad \text{(P1)}$$

Then we shall show:

uniformly in $s \in [a, b]$, \quad (2N + 1)^{2/3}\eta = s + o(1). \quad \text{(P2)}$

A5.1: Proof of P1

This is the argument that was used in A8 of [Johnstone (2001)]. We repeat it for the sake of completeness.

Let us first suppose that $s$ is given. Since $g(x) = (x - x_1)(x - x_2)/(4x^2)$, we have

$$\sigma_N^2g(x) = s\frac{\sigma_N^3(x_2 - x_1) + s\sigma_N/\lambda}{4(x_2 + s\sigma_N/\lambda)^2} \sim s\frac{\sigma_N^32\sqrt{2}\beta(l + 1)^{1/2}}{16\lambda} \sim s\frac{\beta\sigma_N^3}{4\lambda},$$

the first equivalent coming from the fact that when $s$ is fixed, $x_2 - x_1 \gg s\sigma_N/\lambda$, and $x_2 \to 2$.

The second is just $l \to 1$ under $AB$. Now using the first point of Fact $3$ in A1, together with

$$\sigma_N^2 \sim nN^{-1/3},$$

we get that $(\beta\sigma_N^3)/(4\lambda) \to 1$. So at $s$ fixed,

$$\sigma_N^2g(x) \to s.$$

Having this information let us now pick $s_1 = 8$. If $N$ is large enough, we have $\sigma_N^2g(x(s_1)) \geq s_1/2 = 4$. For all (fixed) $N$ $g$ is an increasing function of $s$. Therefore for the same $N$ we will have

$$\forall s \geq s_1 \quad \sigma_N^2g(x) \geq \sigma_N^2g(x(s_1)) \geq s_1/2 = 4,$$

and hence, since $s \geq s_1 \geq 0$, $g$ is positive and we have $g^{1/2}(s) \geq 2/\sigma_N$. Therefore,

$$\frac{2}{3}\lambda\beta^2\eta^{3/2} = \lambda \int_{x_2}^x g^{1/2}(t)dt \geq \int_{x(s_1)}^x g^{1/2}(t)dt \geq \frac{2\lambda}{\sigma_N}(s - s_1) = 2(s - s_1).$$

Consequently, if $s \geq 2s_1$, we have (P1).

A5.2: Proof of P2

Without loss of generality, we can suppose that $a$ and $b$ have the same sign, and $a \geq 0$. (If it is not the case, we can split $[a, b] = [a, 0] \cup [0, b]$, apply the reasoning on each of these, and get the claimed result for the original interval.)

The idea is that on $[a, b]$, we have

$$\frac{(x - x_2)(x_2 + b\sigma_N/\lambda - x_1)}{4(x_2 + a\sigma_N/\lambda)^2} \geq g(x) \geq \frac{(x - x_2)(x_2 - x_1 + a\sigma_N/\lambda)}{4(x_2 + b\sigma_N/\lambda)^2}.$$

Now on both sides, the terms which are not $(x - x_2)$ are $(x_2 - x_1)(1 + o(1)) = 4\beta(1 + o(1))$, again because $\sigma_N/\lambda \ll \beta$. So if we integrate the square root of the previous inequality between $x_2$ and $x(s)$, we get

$$\frac{2}{3}(s\sigma_N/\lambda)^{3/2}2\sqrt{\beta}(1 + o(1))/4 \geq 2/3\eta^{3/2}\beta^2 \geq 2/3(s\sigma_N/\lambda)^{3/2}2\sqrt{\beta}(1 + o(1))/4,$$

or

$$\frac{1}{2}s^{3/2}(\sigma_N^3\beta/\lambda)^{1/2}(1 + o(1)) \geq \eta^{3/2}\lambda\beta^2 \geq \frac{1}{2}s^{3/2}(\sigma_N^3\beta/\lambda)^{1/2}(1 + o(1)).$$

The conclusion follows from A1, Fact 3 whose first point, along with the estimate of $\sigma_N$ mentioned there, shows that $\sigma_N^3\beta/\lambda \sim 4$. We note that (P2) also gives us pointwise convergence of $(\lambda\beta^2)^{2/3}\eta$ to $s$. 

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A6: Properties of $g$

We first show that $g$ is increasing - at $N$ fixed - as a function of $s$, if $s \geq 0$. Then we give an estimate of $4x^2g(x)$ as $N \to \infty$ and $s \in [a,b]$.

A6.1: $g$ is increasing on $s \geq 0$

Since $g(t) = (t-x_2)(t-x_1)/(4t^2) = (t^2 - 4lt + 4)/(4t^2)$, we have
\[ g'(t) = l/t^2 - 2/t^3 = lt - 2/t^3. \]
Now $lx_2 = 2l^2 + 2l\sqrt{l^2 - 1} \geq 2$, since $l = 1 + (2N + 1)/\alpha \geq 1$. But $lx \geq lx_2$ when $s \geq 0$, and the assertion is proved.

A6.2: On the asymptotic behavior of $4x^2g(x)$ for $s \in [a,b]$

This estimate is motivated by the fact that in the course of the proof of the main result, we have to deal with an expression of the form
\[ \eta x^2 - 4lx + 4. \]
We already studied in detail $\eta$ as a function of $s$ and $N$. We now focus on $x^2 - 4lx + 4$. Recalling that $x^2 - 4lx + 4 = (x - x_2)(x - x_1)$ and $x = x_2 + s\sigma_N/\lambda$, we have
\[ x^2 - 4lx + 4 = s\frac{\sigma_N}{\lambda}(x_2 - x_1 + s\frac{\sigma_N}{\lambda}) = s\frac{\sigma_N}{\lambda}(x_2 - x_1 + o(\beta)), \]
because the first estimate in Fact 3 shows that $\sigma_N/\lambda = o(\beta)$, and since $s \in [a,b]$, the previous statement holds true uniformly on this interval. Now $x_2 - x_1 \sim 4\beta$ under AB, and therefore, uniformly on $[a,b]$,
\[ x^2 - 4lx + 4 = s\frac{\sigma_N}{\lambda}4\beta(1 + o(1)), \]
as was claimed in 4.1.2. Also, since $x = x_2 + s\sigma_N/\lambda$, and $x_2 = 2 + (2l + 2)^{1/2}+ \beta^2$,
\[ 4g(x) = s\frac{\sigma_N}{\lambda}\beta(1 + o(1)). \]

A7: Limit of $c_\phi$

Recall that under the notation of Johnstone (2001),
\[ \sqrt{2c_\phi} = \frac{1}{2}\sqrt{\sigma_N} \left( \sqrt{N + \alpha} \int \xi_N - \sqrt{N} \int \xi_{N-1} \right) \]
where $\xi_k(x) = x^{\alpha/2-1}e^{(-x/\alpha)}L_{k}^{(\alpha)}(x)\sqrt{k!/(k+\alpha)!}$. We are interested in
\[ v_{k,\alpha} = \sqrt{k + \alpha} \int \xi_k - \sqrt{k} \int \xi_{k-1} \]
\[ = \sqrt{\frac{k!}{(k + \alpha - 1)!}} \int_0^\infty x^{\alpha/2-1}e^{(-x/\alpha)}(L_k^{(\alpha)}(x) - L_{k-1}^{(\alpha)}(x)) \, dx \]
(by Szeg"o (1975) 5.1.13 p.102) = \[ \frac{k!}{(k + \alpha - 1)!} \int_0^\infty x^{\alpha/2-1}e^{(-x/\alpha)}L_k^{(\alpha-1)}(x) \, dx \]
\[ = \sqrt{\frac{k!}{(k + \alpha - 1)!}}I_{k,\alpha}. \]
Now using Szegö (1975) 5.1.9 p.101,
\[
\sum_{k=0}^{\infty} w^k L_k^{\alpha-1}(x) = (1 - w)^{-\alpha} \exp \left( -\frac{wx}{1 - w} \right).
\]
So if \( F(\alpha) = \int_0^\infty \left( \sum_{k=0}^{\infty} w^k L_k^{\alpha-1}(x) \right) x^{\alpha/2-1} e^{-x/2} dx \), we have:
\[
F(\alpha) = (1 - w)^{-\alpha} \int_0^\infty x^{\alpha/2-1} e^{-x/2} e^{-wx/(1-w)} dx
= (1 - w)^{-\alpha} \Gamma(\alpha/2) \left( \frac{2(1 - w)}{1 + w} \right)^{\alpha/2}
= 2^{\alpha/2} (1 - w^2)^{-\alpha/2} \Gamma(\alpha/2).
\]
Now if \( x \geq 0, \) \( |L_n^{\alpha-1}(x)| \leq L_n^{\alpha-1}(-x) \), by 5.1.6 in Szegö (1975), and hence
\[
\sum_{k=0}^{\infty} w^k L_k^{\alpha-1}(x) \leq \sum_{k=0}^{\infty} |w|^k L_k^{\alpha-1}(-x) = (1 - |w|)^{-\alpha} \exp \left( \frac{x|w|}{1 - |w|} \right).
\]
Therefore, as long as \( w \in (-1/3, 1/3) \), we can switch orders of summation, and get
\[
\sum_{k=0}^{\infty} w^k I_{k,\alpha} = 2^{\alpha/2} (1 - w^2)^{-\alpha/2} \Gamma(\alpha/2).
\]
But \( (1 - w)^{-\alpha/2} \Gamma(\alpha/2) = \sum_{k=0}^{+\infty} \frac{\Gamma(\alpha/2 + k)}{k!} w^k \), since the right-hand side converges without any difficulty on \((-1/3, 1/3)\), and hence
\[
\sum_{k=0}^{\infty} w^k I_{k,\alpha} = \sum_{m=0}^{\infty} \frac{2^{\alpha/2} \Gamma(\alpha/2 + m)}{m!} w^{2m}.
\]
So we have
\[
\forall k \in 2\mathbb{N}, \quad I_{k,\alpha} = \frac{2^{\alpha/2} \Gamma((\alpha + k)/2)}{(k/2)!}.
\]
Now \( v_{k,\alpha} = \sqrt{\frac{k!}{(k + \alpha - 1)!}} I_{k,\alpha} = 2^{\alpha/2} \frac{\Gamma((\alpha + k)/2)}{\sqrt{k/\alpha + \alpha - 1}! \sqrt{k/2}!} \). Since \( \Gamma(z) \sim (z/e)^z \sqrt{2\pi/z} \), we have
\[
\frac{\Gamma((\alpha + k)/2)}{\sqrt{\Gamma(k + \alpha)}} \sim 2^{-(\alpha+k)/2} (\alpha + k)^{-1/4} (2\pi)^{1/4} \sqrt{2},
\]
\[
\frac{\sqrt{k!}}{(k/2)!} \sim 2^{k/2} (\pi k)^{-1/4} 2^{1/4} 4^{k/2} \sqrt{2},
\]
which in turn leads to
\[
v_{k,\alpha} \sim 2^{\alpha/2} (k(\alpha + k))^{-1/4} 2^{-(\alpha+k)/2} 2^{k/2} \sqrt{2} \sqrt{2}
\sim 2(k(\alpha + k))^{-1/4}
\sim 2/\sqrt{a_k}.
\]
Hence, as \( N \) is even, \( \sqrt{2} c_\phi = v_{N,\alpha} \sqrt{a_N}/2 \to 1 \).
A8: On Soshnikov (2002) Lemma 1

For Soshnikov (2002) Lemma 1 to hold true in our case, we have to check two things. First that not only does \( \sigma_N \phi(\mu_N + \sigma_N s) \to \text{Ai}(s)/\sqrt{\sigma} \), but also that this is true for the derivative:

\[
\sigma_N^2 \phi'(\mu_N + \sigma_N s) \to \frac{1}{\sqrt{2}} \text{Ai}'(s). \tag{S1}
\]

We also have to verify that \( \sigma_N^2 \phi'(\mu_N + \sigma_N s) \) is bounded above by \( \Delta(s_0) \exp(-\Delta s) \) on \([s_0, \infty)\), where \( \Delta \) is a positive constant. We need to verify this for \( \psi \) as well, but the techniques are similar, so we will verify it only for \( \phi \).

The second point that we need to check is that

\[
\int_{0}^{\infty} \left( \int_{0}^{z} \phi(u) du \right) \psi(y + z) \, dz \to 0 \text{ as } N \to \infty. \tag{S2}
\]

A8.1: Proof of (S1)

It is easy to see that all we need to work on are the properties of \( g_N(s) = F_N(\mu_N + \sigma_N s) \); if we can show that \( \sigma_N F_N'((\mu_N + \sigma_N s) \to \text{Ai}'(s) \), and that it is bounded by \( \Delta(s_0)e^{-\Delta s} \) on \([s_0, \infty)\), we will be done.

We have very easily that

\[
-\sigma_N F_N'((\mu_N + \sigma_N s_1) = \int_{s_1}^{\infty} \frac{d^2 F_N}{d s^2} \bigg|_{\mu_N + \sigma_N u} \, du.
\]

So the strategy is clear: we want to show that the integrand in the right-hand side is bounded by an integrable function and that it converges pointwise to \( \text{Ai}''(u) = u\text{Ai}(u) \). However,

\[
\frac{d^2 F_N(x)}{dx^2} \bigg|_{\mu_N + \sigma_N u} = \left[ 1 - \frac{\kappa_N}{\mu_N + \sigma_N u} + \frac{\lambda^2 - 1/4}{(\mu_N + \sigma_N u)^2} \right] F_N(\mu_N + \sigma_N u),
\]

and since we already know that \( F_N(\mu_N + \sigma_N s) \to \text{Ai}(s) \), we first need to check that, pointwise,

\[
\sigma_N^2 \left[ 1 - \frac{\kappa_N}{\mu_N + \sigma_N s} + \frac{\lambda^2 - 1/4}{(\mu_N + \sigma_N s)^2} \right] \to s.
\]

In turn, this reduces to showing that

\[
\sigma_N^2 \left[ \frac{1}{4} - \frac{\kappa_N}{\mu_N} + \frac{\lambda^2 - 1/4}{\mu_N^2} \right] \to 0, \text{ and } \sigma_N^3 \left[ \frac{\kappa_N}{\mu_N} - 2\frac{\lambda^2 - 1/4}{\mu_N^2} \right] \to 1.
\]

The first result comes from the remarkable equality \( \kappa_N/\mu_N - \lambda^2/\mu_N^2 = 1/4 \), which follows from the fact that if we call \( x = \sqrt{N_+/(N + \alpha)} \), we have \( \kappa_N/\mu_N = .5 - x/(1 + x)^2 \) and \( \lambda^2/\mu_N^2 = .25 - x/(1 + x)^2 \). Using these estimates, we see that \( \kappa_N/\mu_N - 2(\lambda/\mu_N)^2 = x/(1 + x)^2 \sim \sqrt{N_+/n_+} \), from which we conclude that the second result holds.

Note that if we changed the centering and scaling (replacing \( n \) by \( \tilde{n} = n + \alpha \) and \( N \) by \( \tilde{N} = N + \beta \)), by studying the first expression in this case as a “perturbation” of the study we just did, and using the fact that \( \mu_\tilde{N} - \mu_N = O(\sqrt{n}/N) \), one could show that the first expression is then \( O(N^{-1/3}) \), and so the result would hold. We also have corresponding results for the second expression. This
shows that we have some freedom in the centering and scaling we pick. It is also needed to show that
\[ \sigma_N^2 \phi'(\mu_N + \sigma_N s) \rightarrow \text{Ai}(s), \]
since in our splitting of \( \phi \), the second part \( \phi_{II,N} \) corresponds to parameters \((n-1,N-1)\), but is centered and scaled using \( \mu_N \) and \( \sigma_N \), defined with \((n,N)\).

To show that the sequence of functions we are interested in is bounded above by an integrable function, we split \([s_0, \infty)\) into \([s_0, \sqrt{n}]\) and \([\sqrt{n}, \infty)\). On the first interval, we can apply the previous results since \( \frac{\sigma_N s}{\mu_N} \) is small compared to 1. So in particular the whole integrand will be smaller than \( \Delta(s_0)(1 + |s|)^2 \exp(-s/2) \), after taking into account the properties of \( F_N \). On the other hand, on \([\sqrt{n}, \infty)\), \( \sigma_N^2 \leq s^2 \), and the denominators involving \( s \) are bigger than \( \mu_N \) and \( \mu_N^2 \) respectively, which gives immediately that the integrand is less than \( \Delta(s_0)s^2 \exp(-s/2) \). From this we conclude that the integrand is less than \( \Delta(s_0) \exp(-s/4) \), for instance, and that therefore the derivative we are interested in is too.

It then follows easily that (S1) is true, and we also showed that the left-hand side of (S1) is dominated on \([s_0, \infty)\) and for \( N > N_0(s_0) \) by \( \Delta(s_0)e^{-s/4} \).

A8.2: Proof of (S2)

The approach laid out in Soshnikov (2002) p.1044 works after some modifications. We first write
\[
\int_0^\infty \left( \int_0^z \phi(u) du \right) \psi(y + z) \, dz = \int_0^{n^{5/8}} \left( \int_0^z \phi(u) du \right) \psi(y + z) \, dz + \int_{n^{5/8}}^\infty \left( \int_0^z \phi(u) du \right) \psi(y + z) \, dz .
\]

Then we can check, via a third order asymptotic development in \( x \) of the right-hand side of equation (2.10) in Olver (1980), that equation (2.18) therein is still true in our case, since, with his notations, \( x_N \leq n^{-3/8} \). Therefore, the analysis carried out after equation (3.21) of the same reference applies, and after integration of the expansion following (3.22) adapted to our situation, we can show that
\[
\int_0^{n^{5/8}} \phi(u) du = O(n^{-n/16})
\]
With this estimate and this splitting of \([0, \infty)\), the rest of Soshnikov’s argument holds true and therefore (S2) can be verified.

A9: A quick look at special functions

In this note, we mentioned three types of special functions, Airy, Whittaker, and parabolic cylinder functions. We recall their definition in this appendix, as well as the main ideas behind some of the transformations Olver used. To justify their introduction, let us say that they play a special role because it is possible, in the setting we were in, to write the functions we studied as a perturbation of the differential equations these functions satisfy.

A9.1: Airy function

Let us consider the following second order differential equation:
\[
\frac{d^2 w}{dx^2} = x w .
\]
General remark: Recessive solutions  Since these functions are used to get asymptotic expansions, it makes sense to define the independent solutions with respect to their behavior at $+\infty$. Usually, independent solutions $w_1$ and $w_2$ are sought, so that $w_2 = o(w_1)$ at a particular point of the (extended) real line. In our cases, it will be $\infty$. $w_2$ is called the recessive solution. That leaves the problem underdetermined, but with this in mind, one can then give enough constraints so the problem is fully determined, and solve in terms of recessive and dominant solutions. For a more precise definition of recessivity, see Olver (1974), p.155.

In the case of the Airy function, we have for example: (from Olver (1974), 11.1, p.392)

$$ Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos(t^3/3 + xt) dt $$

$$ Bi(x) = \frac{1}{\pi} \int_0^{\infty} \{\exp(-t^3/3 + xt) + \sin(t^3/3 + xt)\} dt $$

A9.2: Whittaker functions

These are solution of the following differential equation

$$ \frac{d^2 W}{dx^2} = \left( \frac{1}{4} - \frac{\kappa}{z} + \frac{\lambda^2 - 1/4}{z^2} \right) W. \quad (9) $$

$W_{\kappa,\lambda}$, the recessive solution at $\infty$, is obtained by requiring

$$ W_{\kappa,\lambda}(x) \sim e^{-x/2} x^\kappa \text{ as } x \to \infty. $$

The other solution is $M_{\kappa,\lambda}$, which is required to satisfy

$$ M_{\kappa,\lambda}(x) \sim x^{\lambda+1/2} \text{ as } x \to 0^+. $$

For more detail on these, see Olver (1974), p.260, or Olver (1980).

A9.3: Parabolic cylinder functions

According to Olver (1959), equation (2.9) p.133, parabolic cylinder functions satisfy (in the case we are interested in)

$$ \frac{d^2 W}{dx^2} = \left( \frac{1}{4} x^2 + a \right) W. $$

$U(a, x)$ is chosen to satisfy

$$ U(a, x) \sim x^{-a-1/2} e^{-x^2/4} \text{ as } x \to +\infty. $$

On the other hand, $\bar{U}$ satisfies

$$ \bar{U}(a, x) \sim (2/\pi)^{1/2} \Gamma(1/2 - a) x^{a-1/2} e^{x^2/4} \text{ as } x \to +\infty. $$

$\bar{U}$'s definition is actually fairly complicated, and can be found in Olver (1959), equation (2.12) or in Olver (1973), section 5.1.

A9.4: On the usage of these functions

As we mentioned earlier, these functions play a central role because it is relatively easy to transform the equations in which we are interested into one of the three mentioned above, or a perturbation of it. Then a range of techniques are available to study the effect of the perturbation, and one can sometimes, and obviously in the case we examine, get asymptotic expansions in terms
of the “non-perturbed” solutions. Since these functions are quite well known, information can be
gathered about the function of original interest this way.

For example, in Johnstone (2001), section 5, after the scaling \( \xi = x/\kappa \), the Whittaker equation (9) becomes

\[
\frac{d^2 W}{d\xi^2} = \left( \frac{\kappa^2 (\xi - \xi_1)(\xi - \xi_2)}{4\xi^2} - \frac{1}{4\xi^2} \right) W.
\]

Using the Liouville-Green transformation \( \zeta (d\zeta/d\xi)^2 = (\xi - \xi_1)(\xi - \xi_2)/(4\xi^2) \), with \( w = (d\zeta/d\xi)^{-1/2} W \), one has

\[
\frac{d^2 w}{d\zeta^2} = \{ \kappa^2 \zeta + \psi(\zeta) \} w.
\]

This is a perturbation of the (scaled) Airy equation, for \( \text{Ai}(\kappa^{2/3}\zeta) \) and \( \text{Bi}(\kappa^{2/3}\zeta) \) are solutions of \( d^2 w/d\zeta^2 = \kappa^2 \zeta w \).

\( w \) is not \( W \), but it can be related to it, and it is through this mean that Johnstone did his original
analysis. As \( \psi \) is a relatively involved function of \( \xi \) and \( \zeta \), we do not explicit it, but just mention that the understanding of \( \psi \) is key to getting the uniform bound (3). For more on this, see Johnstone (2001) or Olver (1974), theorem 11.3.1 p.399.

The problem we encountered (and mentioned in 3.1.1) about the error control function is exactly here: we could not get enough information about the behavior of \( \psi \) under \( \mathbf{A}\mathbf{B} \), so we slightly changed approach and turned to other studies.

In Olver (1980), Olver starts with equation (9), where the dummy variable was \( z \). Writing \( x = z/\lambda \) and \( l = \kappa/\lambda \), he gets

\[
\frac{d^2 W}{dx^2} = \left( \lambda^2 g(x) - \frac{1}{4x^2} \right) W.
\]

As he aims to expand the solution in terms of parabolic cylinder functions, he changes variables another time, by writing

\[
W = \left( \frac{dx}{d\zeta} \right)^{1/2} w, \quad \left( \frac{d\zeta}{dx} \right)^2 = \frac{x^2 - 4lx + 4}{4x^2(\zeta^2 - \beta^2)},
\]

with \( \beta = \{2(l - 1)\}^{1/2} \). Hence, he gets

\[
\frac{d^2 w}{d\zeta^2} = \{ \kappa^2 (\zeta^2 - \beta^2) + \psi(\kappa, \beta, \zeta) \} w,
\]

with \( \psi(\kappa, \beta, \zeta) = -\dot{x}^2/(4x^2) + \dot{x}^{1/2}d^2(\dot{x}^{-1/2})/d\zeta^2 \). His Olver (1975) is a study of this type of equations, in particular of the control of the deviation of the solution of the previous equation to the corresponding parabolic cylinder function. In Olver (1980), he studies very explicitly the abstract estimate he gets in Olver (1975) in the case of Whittaker functions. We use this repeatedly in our study, as it is essential to get the crucial property (3).

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