KAM TORI FOR QUINTIC NONLINEAR SCHRÖDINGER EQUATIONS WITH GIVEN POTENTIAL

GUANGHUA SHI
College of Mathematics and Computer Science, Hunan Normal University
Changsha, Hunan 410081, China

DONGFENG YAN∗
School of Mathematics and Statistics, Zhengzhou University
Zhengzhou, Henan, 450001, China

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Abstract. This paper is concerned with the 1-dimensional quintic nonlinear Schrödinger equations with real valued $C^\infty$-smooth given potential

$$\sqrt{-1}u_t = u_{xx} - V(x)u - |u|^4u$$

subject to Dirichlet boundary conditions. By means of normal form theory and an infinite-dimensional Kolmogorov-Arnold-Moser (KAM, for short) theorem, it is proved that the above equation admits a family of elliptic tori where lies small amplitude quasi-periodic solutions with two frequencies of high modes.

1. Introduction. In the past few decades, KAM theory has been generalized to the infinite dimensional version and applied to construct the quasi-periodic solutions for some Hamiltonian PDEs. Among those PDEs, the nonlinear Schrödinger equations ($\sqrt{-1}u_t - u_{xx} + Vu + f(|u|^2)u = 0$) and the nonlinear wave equations ($u_{tt} - u_{xx} + Vu + f(u) = 0$) in various situations have been studied by many authors, see [4, 5, 6, 8, 10, 15, 24, 25, 28, 29] for references. As for KAM theory, it is well known that the parameters play an important role in overcoming the small denominator problem. Kuksin [14, 12, 13] and Wayne [27] firstly considered the potential $V$ as parameters, which are often referred to as the external parameters. Generally speaking, the aforementioned papers conclude that there exist many quasi-periodic solutions for “most” potentials $V$. However, for a prescribed potential $V$, there are no external parameters any more, which makes it necessary to extract parameters from the nonlinear term by means of the Birkhoff normal form technique. For the case $V(x) \equiv m$ in the nonlinear Schrödinger equations, Kuksin and Pöschel [15] obtain the parameters from the nonlinear term $|u|^2u$ while Liang and You [18] get it from the term $|u|^4u$. For the nonlinear wave equations, the same result holds true, but there are some restrictions on $m$, see [10, 25] for details. If the potential $V$ is not constant, it becomes very difficult to get a suitable normal form to extract parameter from the term $|u|^2u$ (or $u^3$). Yuan and Du [29, 9] give a positive answer by choosing

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* Corresponding author: Dongfeng Yan.
high modes as tangent frequencies. Besides, we mention that much progress has also been made on the applications of infinite KAM theorem to those kind of PDEs with nonlinearity containing spatial derivative, including KdV equations, Benjamin-Ono equations, derivative nonlinear Schrödinger equations and derivative nonlinear wave equations, the corresponding KAM theorems which are applicable to the unbounded Hamiltonian vector fields are developed to construct the quasi-periodic solutions for these PDEs, see, for instance, [3, 1, 11, 16, 17, 19, 20, 21, 22, 30] for references.

The aim of this paper is to investigate the following nonlinear Schrödinger equations with real valued smooth potential

\[ \sqrt{-1} u_t = u_{xx} - V(x)u - |u|^4u \tag{1} \]

on the finite x-interval \([0, \pi]\) subject to Dirichlet boundary conditions

\[ u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbb{R}, \tag{2} \]

and construct its quasi-periodic solutions via KAM theory. We show that this equation possesses lots of small amplitude quasi-periodic solutions lying on 2-dimensional elliptic invariant tori.

Following the idea in [12], we treat equation (1) as an infinite Hamiltonian dynamical system on some suitable phase space \(D\), one may choose the usual Sobolev space, for instance, \(D = H^1_0([0, \pi])\). The Hamiltonian for the nonlinear Schrödinger Equation (1) then reads

\[ H = \frac{1}{2} \langle Au, u \rangle + \frac{1}{6} \int_0^\pi |u|^6 dx, \]

where \(A = -\frac{d^2}{dx^2} + V(x)\), \(\langle \cdot, \cdot \rangle\) represents the usual inner-product of \(L^2([0, \pi])\), and equation (1) can be rewritten in the Hamiltonian form

\[ \dot{u} = \sqrt{-1} \nabla H(u), \]

where the gradient of the Hamiltonian \(H\) is defined with respect to \(\langle \cdot, \cdot \rangle\).

The time-quasi-periodic solutions of equation (1) we are going to construct are of small amplitude, a conventional way is to treat the high order term \(|u|^4u\) as a perturbation of the linear PDE \(\sqrt{-1} u_t = u_{xx} - V(x)u\). Let \(\phi_\mu(x)\) and \(\lambda_\mu\) \((\mu = 1, 2, \cdots)\) be the basic modes and frequencies for this linear PDE under the Dirichlet boundary conditions (2). It should be noted that when \(V(x) \equiv m\) with \(m\) a positive real number, then the basic modes and frequencies are trivial, that is, \(\phi_\mu(x) = \sqrt{\frac{2}{\pi}} \sin \mu x\) and \(\lambda_\mu = \mu^2 + m\). However, when \(V(x)\) is a given smooth potential, then the basic modes are not trivial at all. Actually, from [26] we know that \(\phi_\mu(x)\) and \(\lambda_\mu\) admit some asymptotic expressions. To be more specific, the frequencies \(\lambda_\mu\) could be expanded in the following manner

\[ \lambda_\mu = \mu^2 + V_0 + \frac{\bar{V}}{\mu^2} + O\left(\frac{1}{\mu^4}\right), \tag{3} \]

where \(V_0 = \frac{1}{\pi} \int_0^\pi V(x)dx\), \(\bar{V}\) represents some constant depending on the potential \(V(x)\) and \(O\left(\frac{1}{\mu^4}\right)\) denotes the high order term. Hence each solution is the superposition of harmonic oscillations of these modes in the following form

\[ u(t, x) = \sum_{\mu \geq 1} q_\mu(t) \phi_\mu(x), \quad q_\mu(t) = q_\mu^0 e^{\sqrt{-1} \lambda_\mu t}. \]
On the whole, these solutions lie on a rotational torus of finite or infinite dimension, which is determined by whether finite or infinite modes are excited or not. Particularly, for every choice
\[ J = \{ n_1, n_2 : 1 \leq n_1 \leq n_2 \} \subset \mathbb{N}, \]
of two basic modes there exists an invariant subspace of complex dimension 2 \( S_J \) which can be foliated into rotational tori:
\[ S_J = \{ u = q_1 \phi_{n_1} + q_2 \phi_{n_2} : q \in \mathbb{C}^2 \} = \bigcup_{I \in \mathbb{P}^2} T^J(I), \]
where \( \mathbb{P}^2 = \{ I : I_1 > 0, I_2 > 0 \} \) and
\[ T^J(I) = \{ u = q_1 \phi_{n_1} + q_2 \phi_{n_2} : |q_1|^2 = 2I_1, |q_2|^2 = 2I_2 \}. \]

In view of the perturbation term \(|u|^4 u\), the aforementioned invariant manifolds \( S_J \) will not persist in their entirety thanks to resonances among the modes and the strong perturbing effect of \(|u|^4 u\) for large amplitudes. However, if choosing the special modes \( \phi_{n_1}, \phi_{n_2} \) satisfying (4), then we shall prove that a large subfamily of rotational 2-tori with slight deformations persists in a sufficiently small neighbourhood of the origin. To be more precise, there exists a Cantor set \( C \subset \mathbb{P}^2 \), a family of 2-tori
\[ T^J[C] = \bigcup_{I \in C} T^J(I) \subset S_J \]
over \( C \), and a Lipschitz continuous embedding
\[ \Phi : T^J[C] \rightarrow D, \]
such that the restriction of \( \Phi \) to each \( T^J(I) \) is an embedding of a rotational 2-tori for equation (1). Our main results can be stated as follows.

**Theorem 1.1.** For the nonlinear Schrödinger equation (1), suppose that the potential \( V(x) \) is a periodic and real-valued \( C^\infty \)-smooth function with \( V \neq 0 \), where \( \bar{V} \) is defined by the asymptotic expression (3). Assume further that the index sets are of the form
\[ J = \{ n_1, n_2 : n_1 \geq N > 1, n_2 \geq 6n_1^2 \text{ and } (\sqrt{2}n_2 - \lfloor \sqrt{2}n_2 \rfloor) \in \left( \frac{1}{4}, \frac{3}{4} \right) \} \subset \mathbb{N}, \tag{4} \]
where \( \lfloor \cdot \rfloor \) denotes the integral part and
\[ N = \max \left\{ \sqrt{\frac{10N_\lambda}{c_1}}, \sqrt{\frac{N_\lambda}{|V|}}, 4 \sqrt{11N_\lambda + 6}, 3\pi^6(N_\lambda + N_\phi + 2) \left[ \frac{1}{2} + \frac{3}{8}N_\phi \right] \right\}^{1/3}. \]

Then there exists a set \( C^* \subset \mathbb{P}^2 \) with positive Lebesgue measure, a family of Diophantine 2-tori
\[ T^J[C^*] = \bigcup_{I \in C^*} T^J(I) \subset S_J \]
over \( C^* \), and a Lipschitz continuous embedding \( T^J[C^*] \rightarrow D \), which is a higher order perturbation of the trivial inclusion \( \Phi_0 : S_J \rightarrow D \) restricted to \( T^J[C^*] \), such that, the restriction of \( \Phi \) to each \( T^J(I) \) in the family is an embedding of invariant rotational 2-tori for the nonlinear Schrödinger equation (1). In addition, the invariant tori carry plenty of quasi-periodic solutions of high modes.

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1 See (8), (9) and (36) for the definitions of \( N_\lambda, N_\phi, c_1 \).
Remark 1. Let us make some comments on our main result with previous work. On one hand, Theorem 1.1 in the present paper extends the work of [18], where the authors deal with a special case when \( V(x) \equiv M \), \( M \) is a constant. Considering the nonlinear term is of the form \( |u|^4 u \), before applying KAM theorem, one has to ensure the combinations of six frequencies do not vanish, i.e.,
\[
\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n \neq 0.
\]
This condition is easy to verify when \( V(x) \equiv M \), since at this time the indexes of the frequencies satisfy the zero-momentum condition, i.e., \( i \pm j \pm k \pm l \pm m \pm n = 0 \).
However, for a general potential \( V(x) \) in the present paper, the zero-momentum condition does not hold true anymore, which leads to the key difficulty in this case. We manage to fix it in Lemma 4.1. On the other hand, compared with [9], where the nonlinear term is of the form \( |u|^2 u \), one only needs to check that the combinations of four frequencies do not vanish. Things turn to be more involved in the present paper, we adopt an more effective strategy to solve it, see Lemma 4.1 for details.

Remark 2. The given potential \( V(x) \) here is assumed to be real valued, which is necessary to make sure that Eq. (1) possesses the conserved quantity
\[
\int_0^{2\pi} |u|^2 dx.
\]
It is worth noting that this conservation law makes the tangential and normal frequencies to be affine function, which makes it easier to check the measure estimate part of the KAM theorem developed by Pöschel in [24].

Remark 3. The index set \( J \) is not empty. In fact, the set \( \{ \sqrt{2} n^2 - [\sqrt{2} n^2] : n^2 \geq 6n_1^2 \} \) is dense in \((0, 1)\).

2. The Sturm-Liouville problem. In this section, we shall focus on the spectra of the following Sturm-Liouville problem
\[
-\frac{d^2 y}{dx^2} + V(x)y = \lambda y,
\]
\[
y(0) = y(\pi) = 0.
\]
(5)

It is known to all that the above Sturm-Liouville problem admits infinitely many strictly increasing simple eigenvalues
\[
\lambda_1 < \lambda_2 < \cdots < \lambda_\mu < \cdots \to \infty,
\]
and normalized eigenfunction \( \phi_\mu \) corresponding to \( \lambda_\mu \). More precisely, we have the following lemma.

Lemma 2.1. The eigenvalues \( \{\lambda_\mu\} \) and eigenfunctions \( \{\phi_\mu\} \) of the Sturm-Liouville problem (5) possess the following asymptotic expressions
\[
\lambda_\mu = \mu^2 + V_0 + \frac{V}{\mu^2} + O\left(\frac{1}{\mu^4}\right),
\]
\[
\phi_\mu(x) = \kappa_\mu^{-1} \left[ \sin(\mu x) - \frac{\cos \mu x}{2 \mu} \int_0^x V(s) ds + \hat{\phi}_\mu(x) \right],
\]
(6)

where \( \kappa_\mu > 0 \) is a positive constant depending on \( \mu \) fulfilling that \( \|\phi_\mu\|_{L^2[0, \pi]} = 1 \), and
\[
\hat{\phi}_\mu(x) = O\left(\frac{1}{\mu^2}\right), \quad \hat{\phi}_\mu'(x) = O\left(\frac{1}{\mu}\right), \quad \hat{\phi}_\mu''(x) = O(1),
\]
(7)
holds true uniformly for \( x \in [0, \pi] \) and
\[
V_0 = \frac{1}{\pi} \int_0^\pi V(x)dx.
\]

**Proof.** The proof of this basic lemma can be found in [23] and [26], we omit it. \( \square \)

To specify the large number \( N \), by above Lemma 2.1, we assume that
\[
|\lambda - \mu^2 - V_0 - \bar{V}\mu^2| \leq \frac{V_1}{\mu^4}, \quad \sup_{x \in [0, \pi]} |\hat{\phi}_\mu^{(j)}(x)| \leq \frac{N_\phi}{\mu^{2-j}}.
\]

Let
\[
N_\lambda = \max\{ \sup_{x \in [0, \pi]} |V(x)|, |\bar{V}| + V_1 \},
\]
then coefficient \( \kappa_\mu \) in Lemma 2.1 admits the following asymptotic expansion.

**Lemma 2.2.** We simply have
\[
\kappa_\mu^2 = \frac{\pi^2}{2} + O\left(\frac{1}{\mu^2}\right),
\]
moreover,
\[
|\kappa_\mu^2 - \frac{\pi^2}{2}| \leq \frac{\pi^3(N_\lambda^2 + N_\phi^2)}{\mu^2}.
\]

**Proof.** Through direct computations, this lemma can be proved. See [29] for details. \( \square \)

Finally, we give an useful lemma, which can be found in the appendix in [8].

**Lemma 2.3.** Let \( \phi_j(x) = \sum_k \varphi_j^k e^{\sqrt{-1}kx} \). If \( V(x) \in C^\infty(\mathbb{T}) \), then for any \( n > 0 \), there exists a constant \( C_n \) such that
\[
|\varphi_j^k| \leq \max_{\pm} \frac{C_n}{(1 + |k \pm j|)^n}.
\]

3. **The Hamiltonian.** The Hamiltonian for the nonlinear Schrödinger Equation (1) is
\[
H = \frac{1}{2} \langle Au, u \rangle + \frac{1}{6} \int_0^\pi |u|^6 dx,
\]
where \( A = -\frac{d^2}{dx^2} + V(x) \). We rewrite \( H \) as a Hamiltonian in infinitely many coordinates by making the ansatz
\[
u = S \varrho = \sum_{j \geq 1} q_j(t) \phi_j(x).
\]
The coordinates are taken from the Hilbert space \( \ell^{a,p} \) of all complex-valued sequences \( q = (q_1, q_2, \cdots) \) with
\[
\|q\|_{a,p} = \sum_{j \geq 1} j^{2p}|q_j|^a e^{2aj} < \infty.
\]
One then gets the Hamiltonian
\[
H = \Lambda + G = \frac{1}{2} \sum_{j \geq 1} \lambda_j |q_j|^2 + \frac{1}{6} \int_0^\pi |S \varrho|^6 dx
\]
with
\[ G = \frac{1}{6} \int_0^\pi |\mathcal{S} q|^6 dx = \frac{1}{6} \sum_{i,j,k,l,m,n \geq 1} G_{ijklmn} \hat{q}_i \hat{q}_j \hat{q}_k \hat{q}_l \hat{q}_m \hat{q}_n, \quad (15) \]
where
\[ G_{ijklmn} = \int_0^\pi \phi_i \phi_j \phi_k \phi_l \phi_m \phi_n dx. \quad (16) \]
We equip the phase space \( T^{a,p} \times T^{a,p} \) with symplectic structure \( \omega = \sum_{j \geq 1} dq_j \wedge d\bar{q}_j \), then the equations of motion are
\[ \dot{q}_j = 2\sqrt{-1} \frac{\partial H}{\partial \bar{q}_j}, \quad j \geq 1. \quad (17) \]
They are the classical Hamiltonian equations of motions for the real and imaginary parts of \( q_j = x_j + \sqrt{-1} y_j \) written in complex notation. Rather than discussing the above formal validity, we shall use the following elementary observation.

**Lemma 3.1.** Let \( a > 0 \) and \( p \geq 0 \). If a curve \( I \rightarrow T^{a,p}, t \rightarrow q(t) \equiv (\{q_j(t)\}_{j \geq 1}) \) be an analytic solution of \( (17) \), then \( u(t,x) = \sum_{j \geq 1} q_j(t) \phi_j(x) \), is a solution of \( (1) \) that is \( C^\infty \)-smooth on \( I \times [0,\pi] \).

The proof can be found in [15].

Next, we study the regularity of the gradient of \( G \) in the following lemma.

**Lemma 3.2.** For \( p > 1 \) and \( a \geq 0 \), the gradient \( G_{\bar{q}} \) is real analytic as a map from some neighborhood of the origin in \( T^{a,p} \) into \( T^{a,p} \) with
\[ ||G_{\bar{q}}||_{a,p} = O(||q||_{a,p}^5). \]

**Proof.** In view of (14), it is clear that \( G_{\bar{q}_j} = \langle u^5, \phi_j \rangle \). Then, taking advantage of the Remark 3.3 in [2] and the Lemma 2.3, we have
\[ ||G_{\bar{q}}||_{a,p} \sim ||u^5||_{H^{a,p}(\mathbb{T})} \leq C ||u||_{H^{a,p}(\mathbb{T})}^5 \sim ||q||_{a,p}^5 \]
where the second inequality has using the algebraic property of the norm in Sobolev Space.

In the rest of this section, we shall calculate the coefficients \( G_{ijklkk} \), which are useful when checking the conditions of KAM theory.

**Lemma 3.3.** Fix \( 2\pi \sqrt{N_\lambda^2 + N_\phi^2} \leq N \leq n_1 \leq \frac{1}{2} n_2 \), we have
\[ G_{n_k n_k n_k n_k} = \frac{1}{4\pi^2} (6 - \delta^{2n_k} + 4\delta^n) + O(n_k^{-1}), \quad k = 1, 2, \]
\[ G_{n_1 n_2 n_1 n_2} = \frac{1}{4\pi^2} (4 - \delta^{n_1 n_2} - 2\delta_{n_1 n_2} + 2\delta_{n_1 n_2} + 2\delta_{n_2 n_1}) + O(n_1^{-1}) \]
with
\[ |O(n_k^{-1})| \leq \pi^4 (N_\lambda + N_\phi + 2) \left[ \frac{1}{2} + \pi^2 (N_\lambda^2 + N_\phi^2) \right] \frac{1}{n_1}. \quad (18) \]

**Proof.** On account of (6), one gets that
\[ \kappa^2 \kappa^2_0 \kappa^2_1 \int_0^\pi \phi^2 \phi^2_0 \phi^2_1 dx = \kappa^2 \kappa^2_0 \int_0^\pi \phi^2 \phi^2_0 (\sin lx + O(l))^2 dx \]
\[
\begin{align*}
&= \kappa_i^2 \kappa_j^2 \int_0^\pi \phi_i^2 \phi_j^2 \sin^2 l x dx + O\left(\frac{1}{l}\right) \\
&= \frac{\kappa_i^2 \kappa_j^2}{2} \int_0^\pi \phi_i^2 \phi_j^2 dx - \frac{\kappa_i^2 \kappa_j^2}{2} \int_0^\pi \phi_i^2 \phi_j^2 \cos 2l x dx + O\left(\frac{1}{l}\right) \\
&= \frac{\kappa_i^2}{4} + \frac{\pi}{16} \delta_i - \frac{\kappa_i^2 \kappa_j^2}{2} \int_0^\pi \phi_i^2 \phi_j^2 \cos 2l x dx + O\left(\frac{1}{j^2 + \frac{1}{j}}\right) 
\end{align*}
\] (20)

where in the last line we have used the Lemma 3.2 in [29]. Due to (11), the tail \(O\left(\frac{1}{j^2 + \frac{1}{j}}\right)\) satisfies

\[
|O\left(\frac{1}{j^2 + \frac{1}{j}}\right)| \leq \pi^2 (N_\lambda + N_\phi) \left[ \frac{\pi}{2} + \frac{\pi^3 (N_\lambda^2 + N_\phi^2)}{i^2} \right] \left[ \frac{\pi}{2} + \frac{\pi^3 (N_\lambda^2 + N_\phi^2)}{j^2} \right] (1 + \frac{1}{j}).
\] (22)

Then we continue to calculate the integral and obtain that

\[
\begin{align*}
\kappa_i^2 \kappa_j^2 \int_0^\pi \phi_i^2 \phi_j^2 \cos 2l x dx \\
&= \kappa_i^2 \int_0^\pi \phi_i^2 (\sin j x + O\left(\frac{1}{j}\right))^2 \cos 2l x dx \\
&= \kappa_i^2 \int_0^\pi \phi_i^2 \sin^2 j x \cos 2l x dx + O\left(\frac{1}{j}\right) \\
&= \frac{\kappa_i^2}{2} \int_0^\pi \phi_i^2 \cos 2l x dx - \frac{\kappa_i^2}{2} \int_0^\pi \phi_i^2 \cos 2j x \cos 2l x dx + O\left(\frac{1}{j}\right) \\
&:= I - II + O\left(\frac{1}{j}\right),
\end{align*}
\]

in which \(I, II\) respectively stand for the first term and the second one in the fourth line and

\[
|O\left(\frac{1}{j}\right)| \leq \pi^2 (N_\lambda + N_\phi) \left[ \frac{\pi}{2} + \frac{\pi^3 (N_\lambda^2 + N_\phi^2)}{i^2} \right] \left[ \frac{\pi}{2} + \frac{\pi^3 (N_\lambda^2 + N_\phi^2)}{j^2} \right] (1 + \frac{1}{j}).
\] (24)

Next we shall give some estimates of \(I, II\). Set \(\mathcal{V}(x) = -\frac{1}{2} \int_0^x V(s) ds\). Owing to (6), it is clear that

\[
I = \frac{1}{2} \int_0^\pi \left( \sin ix + \frac{\cos ix}{i} \mathcal{V}(x) + \tilde{\phi}_i(x) \right)^2 \cos 2l x dx \\
= \frac{1}{2} \int_0^\pi (\sin^2 ix + f_i(x)) \cos 2l x dx,
\]

where

\[
f_i(x) = \left( \sin 2ix + \frac{\cos^2 2ix}{i} \mathcal{V}(x) \right) \frac{\mathcal{V}(x)}{i} + \left( \tilde{\phi}_i(x) + 2 \sin ix + \frac{2 \cos ix}{i} \mathcal{V}(x) \right) \tilde{\phi}_i(x).
\]

In view of Lemma 2.1, one gets that \(\sup_{x \in [0, \pi]} \left| \frac{df_i(x)}{dx} \right| \leq \pi^2 (N_\lambda^2 + N_\phi^2)\). Integrating by parts one has

\[
\left| \int_0^\pi f_i(x) \cos 2l x dx \right| = \left| \frac{1}{2l} \int_0^\pi \frac{df_i(x)}{dx} \sin 2l x dx \right| \leq \frac{\pi^3 (N_\lambda^2 + N_\phi^2)}{l}.
\]

Therefore,

\[
I = \frac{1}{2} \int_0^\pi \sin^2 ix \cos 2l x dx + O\left(\frac{1}{l}\right) = \frac{\pi}{8} \delta_i + O\left(\frac{1}{l}\right)
\] (25)
with
\[ |O(\frac{1}{T})| \leq \frac{\pi^3(N_x^2 + N_y^2)}{l}. \tag{26} \]

As to II, one easily obtains that
\[ II = \frac{\kappa^2_i}{4} \int_0^\pi \phi_i^2 \left( \cos 2(j - l)x + \cos 2(j + l)x \right) dx. \tag{27} \]

If \( j = l = n_k, k = 1, 2 \), due to the fact \( \int_0^\pi \phi_i^2 dx = 1 \), then
\[ II = \frac{\kappa^2_i}{4} + \frac{\kappa^2_i}{4} \int_0^\pi \phi_i^2 \cos 4n_k x dx = \frac{\kappa^2_i}{4} - \frac{\pi}{16} \delta^{2n_k} + O(\frac{1}{n_k}) \tag{28} \]

where the technique we have used here is analogous to that of deriving (25) and
\[ |O(\frac{1}{n_k})| \leq \frac{\pi^3(N_x^2 + N_y^2)}{n_k}. \tag{29} \]

Otherwise \( j \neq l \), without loss of generality, we assume that \( j = n_1, l = n_2 \). By (27), similarly, we have that
\[ II = -\frac{\pi}{16} \delta^{2n_1-n_2} - \frac{\pi}{16} \delta^{2n_2+n_1} + O(\frac{1}{n_1}) \tag{30} \]

where the estimate of \( O(\frac{1}{n_1}) \) is the same as that in (29). Combining the relations (21), (23), (25), (28) and (30), the conclusions (18) hold true. When \( N \geq 2\pi \sqrt{N_x^2 + N_y^2} \) and \( j = n_1, l = n_2 \), the estimates (22), (24), (26), (29) and (30) lead to (19).

4. Partial Birkhoff normal form. Since the quadratic part of Hamiltonian (15), does not provide any "twist" condition required by KAM theory, we shall use the normal form technique to get the "twisted" integrable terms from the sixth order terms. To get a two dimensional KAM tori, for simplicity, we choose \((q_{n_1}, q_{n_2})\) as tangential variables. All the other variables are called normal ones. In this part, the sixth order terms with at most two normal variables will be cancelled, while the other sixth order terms are left since they have no effect on the tori. Then we define the index sets \( \triangle_*, * = 0, 1, 2 \) and \( \triangle_3 \) in the following way: \( \triangle_* \) is the set of index \((i, j, k, l, m, n)\) such that there exist right components not in \( \{n_1, n_2\} \). \( \triangle_3 \) is the set of index \((i, j, k, l, m, n)\) such that there exist at least three components not in \( \{n_1, n_2\} \). Define the resonance sets \( \mathcal{N} = \{(i, j, k, i, j, k)\} \). For our convenience, rewrite \( G = \tilde{G} + \hat{G} + \check{G} \), where
\[
\tilde{G} = \frac{1}{6} \sum_{(i, j, k, l, m, n) \in (\triangle_0 \cup \triangle_1 \cup \triangle_2) \cap \mathcal{N}} G_{ijklmn} q_i q_j q_k q_l q_m q_n,
\]
\[
\hat{G} = \frac{1}{6} \sum_{(i, j, k, l, m, n) \in (\triangle_0 \cup \triangle_1 \cup \triangle_2) \setminus \mathcal{N}} G_{ijklmn} q_i q_j q_k q_l q_m q_n
\]
and
\[
\check{G} = \frac{1}{6} \sum_{(i, j, k, l, m, n) \in \triangle_3} G_{ijklmn} q_i q_j q_k q_l q_m q_n.
\]

To remove \( \hat{G} \), we need the following lemma.
Lemma 4.1. Assume that the indexes $n_1, n_2 \in J$ defined by (4), then we have

$$|\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n| \geq \eta,$$

(31)

where $\eta$ depends on $n_1$, for any $(i,j,k,l,m,n) \in (\triangle_0 \cup \triangle_1 \cup \triangle_2) \setminus \mathcal{N}$.

Proof. For convenience, we set

$$\delta = \lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n.$$

When $\{i,j,k\} \cap \{l,m,n\} \neq \emptyset$, (31) holds true by Lemma 2.4 in [9]. So it suffices to consider the case $\{i,j,k\} \cap \{l,m,n\} = \emptyset$.

Case 1. min$\{i,j,k,l,m,n\} \geq C_1 = 1 + \max\{\sqrt{24(N_\lambda + 1)}, \sqrt{2}|V|\}$.

When $i^2 + j^2 + k^2 - l^2 - m^2 - n^2 \neq 0$, by (6)-(9), then

$$|\delta| = |i^2 + j^2 + k^2 - l^2 - m^2 - n^2| + O\left(\frac{1}{i^2} + \frac{1}{j^2} + \frac{1}{k^2} + \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}\right)$$

$$\geq |i^2 + j^2 + k^2 - l^2 - m^2 - n^2| - N_\lambda \left(\frac{1}{i^2} + \frac{1}{j^2} + \frac{1}{k^2} + \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}\right)$$

$$> 1 - \frac{1}{2} = \frac{1}{2}.$$

If $i^2 + j^2 + k^2 - l^2 - m^2 - n^2 = 0$, we will discuss the following three subcases.

Subcase 1.1. $(i,j,k,l,m,n) \in \triangle_0 \setminus \mathcal{N}$.

Due to the assumption $\{i,j,k\} \cap \{l,m,n\} = \emptyset$, we know that $i = j = k = n_2$ and $l = m = n = n_1$, or vice versa. It contradicts the condition $i^2 + j^2 + k^2 = l^2 + m^2 + n^2$.

Subcase 1.2. $(i,j,k,l,m,n) \in \triangle_1$.

Without loss of generality, we assume that $i \neq n_1, n_2$. We just consider the case $j = k = n_1, l = m = n = n_2$. Indeed, if $j = k = n_2, l = m = n = n_1$, then $i^2 + 2n_2^2 = 3n_1^2$ which is impossible to hold true by the fact that $n_1^2 \leq n_2$. Thus one obtains $i^2 + 2n_1^2 = 3n_2^2$. Again, due to the fact that $n_1^2 \leq n_2$, we know $i > n_2$. Then it is easy to get

$$|\delta| \geq \frac{|V|}{n_1^2} \left(\frac{1}{i^2} + \frac{1}{j^2} + \frac{1}{k^2} - \frac{1}{l^2} - \frac{1}{m^2} - \frac{1}{n^2}\right) - V_1 \left(\frac{1}{i^4} + \frac{1}{j^4} + \frac{1}{k^4} + \frac{1}{l^4} + \frac{1}{m^4} + \frac{1}{n^4}\right)$$

$$> 2|V| - 4|V| - 6N_\lambda \left(n_1^2 - n_2^2\right)$$

$$> \frac{|V|}{n_1^2},$$

where $n_2 \geq 2n_1^2$ and $N \geq \sqrt{\frac{7N_\lambda}{|V|}}$.

Subcase 1.3. $(i,j,k,l,m,n) \in \triangle_2 \setminus \mathcal{N}$.

In this subcase, without loss of generality, we either have

(a) $k = n_1, l = m = n = n_2$, and $i,j \neq n_1, n_2$.

or

(b) $j = k = n_1, m = n = n_2$, and $i,l \neq n_1, n_2$.

In the case (a), recall $i,j \geq C_1 \geq \sqrt{2|V|}/|V|$ and choose $N \geq \sqrt{\frac{10N_\lambda}{|V|}}$, then

$$|\delta| \geq \frac{|V|}{n_1^2} + \frac{V_1}{i^4} + \frac{V_1}{j^4} + \frac{V_1}{k^4} + \frac{V_1}{l^4} + \frac{V_1}{m^4} + \frac{|V|}{n_1^2} - \frac{3|V|}{n_2^2} - \frac{4V_1}{n_1^2}$$

$$> \frac{|V|}{2n_1^2} + \frac{|V|}{2j^2} + \frac{|V|}{2l^2} > \frac{|V|}{2n_1^2}.$$
In case (b), we have
\[ i^2 + 2n_1^2 = l^2 + 2n_2^2. \]  
(32)

Since \( n_1 \ll n_2 \), obviously, \( i > n_2 \). If \( l \geq n_1 \), let \( N \geq 4 \sqrt{\frac{n_2}{n_1}} \), then
\[ |\delta| \geq \left| \tilde{V} \right| \left( \frac{1}{l^2} + \frac{2}{n_1^2} - \frac{1}{l^2} - \frac{2}{n_2^2} \right) - V_1 \left( \frac{1}{l^2} + \frac{2}{n_1^2} + \frac{1}{l^2} \right) \]
\[ > \frac{|\tilde{V}|}{2n_1^2}. \]

When \( l < n_1 \), by (4), we know that there is no integer in \( (\sqrt{2n_2} - \frac{1}{2}, \sqrt{2n_2} + \frac{1}{2}) \).
Hence, there is no square number in \( (2n_2^2 - \frac{\sqrt{2}}{2}n_2 + \frac{1}{16}, 2n_2^2 + \frac{\sqrt{2}}{2}n_2 + \frac{1}{16}) \). Since \( n_2 \geq 6n_1^2 \), \( i^2 = l^2 + 2n_1^2 - 2n_2^2 \) belongs to \( (2n_2^2 - \frac{\sqrt{2}}{2}n_2 + \frac{1}{16}, 2n_2^2 + \frac{\sqrt{2}}{2}n_2 + \frac{1}{16}) \). This is a contradiction.

**Case 2.** \( \min \{i, j, k, l, m, n\} \leq C_1 \).

Without loss of generality, we assume that \( k = \min \{i, j, k, l, m, n\} \). In the case that \( (i, j, k, l, m, n) \in \Delta_1 \), we have \( i = j = n_1, l = m = n = n_2 \), or vice versa. Without loss of generality, we assume that \( i = j = n_1, l = m = n = n_2 \). Then ones obtain that
\[ |\delta| = |\lambda_k - V_0 + 2n_1^2 - 3n_2^2 + O \left( \frac{5}{n_1^2} \right) | \]
\[ \geq 3n_2^2 - C - 2n_1^2 - |V_0| - \frac{5N_\lambda}{n_1}, \]  
(33)

where \( C = \max \{|\lambda_k| : k \leq C_1\} \). By (8), one gets
\[ |\lambda_k| \leq k^2 + |V_0| + \frac{|\tilde{V}|}{k^2} + \frac{V_1}{k^4} \leq k^2 + 3\pi N_\lambda. \]

Therefore, \( |\delta| > 1 \), when we choose \( N \geq \sqrt{C_1^2 + 11N_\lambda} \).

In the case that \( (i, j, k, l, m, n) \in \Delta_2/N \), without loss of generality, we just consider the case: \( l \neq n_1, n_2, i = j = n_2, m = n = n_1 \). If \( l < n_2 \), we obtain
\[ |\delta| = |\lambda_k - \lambda_i + 2n_1^2 - 2n_2^2 + O \left( \frac{4}{n_1^2} \right) | \]
\[ \geq 2n_2^2 - |\lambda_k| - |\lambda_i| - 2n_1^2 - \frac{4N_\lambda}{n_1}. \]  
(34)

Similarly, \( |\delta| > 1 \) provided that \( N \geq 2\sqrt{C_1^2 + 20N_\lambda} \). On the other hand, i.e., \( l \geq n_2 \), rewrite
\[ |\delta| = |\lambda_k + 2n_2^2 - 2n_1^2 - l^2 + O \left( \frac{5}{n_1^2} \right) | \]
\[ \geq |\lambda_k + 2n_2^2 - 2n_1^2 - l^2| - \frac{5N_\lambda}{n_1^2}. \]  
(35)

Observe that
\[ c_1 := \inf_{l, n_1, n_2 \geq 1, k \leq C_1} \{ |\lambda_k + 2n_2^2 - 2n_1^2 - l^2| : |\lambda_k + 2n_2^2 - 2n_1^2 - l^2| \neq 0 \} > 0, \]  
(36)

which depends on \( C_1 \) only, independent of \( n_1 \). Thus, if \( |\lambda_k + 2n_2^2 - 2n_1^2 - l^2| \neq 0 \), then
\[ |\delta| \geq |\lambda_k + 2n_2^2 - 2n_1^2 - l^2| - \frac{5N_A}{n_1^2} \geq c_1/2, \]  
(37)

as \( N \geq \sqrt{\frac{10N_A}{c_1}} \). Now we are in position to consider \( \lambda_k + 2n_2^2 - 2n_1^2 - l^2 = 0 \). The proof is similar to that of the Subcase 1.3.

Finally, let

\[ N \geq \max \left\{ \sqrt{\frac{10N_A}{c_1}}, 4\sqrt{\frac{N_A}{|V|}} + 4\sqrt{11N_A + 6} \right\}, \quad \eta = \min \{ \frac{1}{2}, \frac{c_1}{2n_1^2} \}, \]  
(38)

we complete the proof.

Next we transform the Hamiltonian (15) into the partial Birkhoff form of order six so that the KAM Theorem can be applied.

**Proposition 1.** For any given \( n_1, n_2 \) satisfying (31), there exists a real analytic, symplectic change of coordinates \( \Gamma \) from some neighbourhood of the origin in \( \ell^{n,p} \) into \( \ell^{n,p} \) that takes it into

\[ H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K, \]  
(39)

where \( X_{\bar{G}}, X_{\hat{G}} \) and \( X_K \) are real analytic vector fields defined in a neighborhood of the origin in \( \ell^{n,p} \), and taking values in \( \ell^{n,p} \) with

\[ \bar{G} = \left( \frac{5}{12\pi^2} + O\left( \frac{1}{n_1} \right) \right)(|q_{n_1}|^6 + |q_{n_2}|^6) \]
\[ + \left( \frac{9}{4\pi^2} + O\left( \frac{1}{n_1} \right) \right)(|q_{n_1}|^4|q_{n_2}|^2 + |q_{n_1}|^2|q_{n_2}|^4) \]
\[ + \frac{3}{2} \sum_{i \neq n_1, n_2} G_{n_1n_1n_1n_1i} |q_{n_1}|^4|q_i|^2 + \sum_{i \neq n_1, n_2} G_{n_2n_2n_2n_2i}|q_{n_2}|^4|q_i|^2 \]
\[ + 6 \sum_{i \neq n_1, n_2} G_{n_1n_2n_2n_2i}|q_{n_2}|^2|q_{n_2}|^2|q_i|^2, \]
\[ \hat{G} = \frac{1}{6} \sum_{(i,j,k,l,m,n) \in \Delta_3} G_{ijklmn} q_i q_j q_k q_l q_m q_n, \]  
and \( |K| = O(\|q\|^{10}_{n,p}) \).

The proof of this proposition can be derived directly from [2], we omit it.

5. **Proof of the main theorem.** In this section, we shall prove the main theorem, our proof can be divided into the following five steps.

**Step 1.** Introduce new coordinates and rescalings. In view of (39), our Hamiltonian turns to be \( H = \Lambda + \bar{G} + \hat{G} + K \). Let us introduce the symplectic polar and complex coordinates as follows

\[ q_j = \begin{cases} \sqrt{2(\sqrt{\xi_j^2} + y_j)e^{-\sqrt{-1}x_j}}, & j = n_1, n_2 \\ \sqrt{2z_j}, & j \neq n_1, n_2 \end{cases} \]

with \( \xi = (\xi_{n_1}, \xi_{n_2}) \in \mathbb{R}^2 \). The symplectic structure arrives at

\[ \frac{\sqrt{-1}}{2} \sum_{j \geq 1} dq_j \wedge d\bar{q}_j = \sum_{j = n_1, n_2} dx_j \wedge dy_j + \sqrt{-1} \sum_{j \neq n_1, n_2} dz_j \wedge d\tilde{z}_j. \]  
(40)
According to the Lemma 3.3, the new Hamiltonian reads
\[ H^* = \langle \omega^*(\xi), y \rangle + \sum_{i\neq n_1, n_2} \Omega_i^*(\xi) z_i \bar{z}_i + P^*(y, x, z, \bar{z}; \xi), \]  
where \( \omega^*(\xi) = (\omega_{n_1}^*, \omega_{n_2}^*) \) with
\[
\omega_{n_1}^* = \lambda_{n_1} + [\frac{18}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + [-\frac{8}{\pi^2} + O(\frac{1}{n_1})] \xi_{n_1},
\]
\[
\omega_{n_2}^* = \lambda_{n_2} + [\frac{18}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + [-\frac{8}{\pi^2} + O(\frac{1}{n_1})] \xi_{n_2},
\]
\[
\Omega_{2n_1}^* = \lambda_{2n_1} + [\frac{24}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + [-\frac{6}{\pi^2} + O(\frac{1}{n_1})] \xi_{n_1},
\]
\[
\Omega_{2n_2}^* = \lambda_{2n_2} + [\frac{24}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + [-\frac{6}{\pi^2} + O(\frac{1}{n_1})] \xi_{n_2},
\]
\[
\Omega_{n_2-n_1}^* = \lambda_{n_2-n_1} + [\frac{18}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + O(\frac{1}{n_1}) \xi_{n_1} + O(\frac{1}{n_1}) \xi_{n_2},
\]
\[
\Omega_{n_2+n_1}^* = \lambda_{n_2+n_1} + [\frac{18}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + O(\frac{1}{n_1}) \xi_{n_1} + O(\frac{1}{n_1}) \xi_{n_2},
\]
\[
\Omega_{1}^* = \lambda_1 + [\frac{24}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + [-\frac{6}{\pi^2} + O(\frac{1}{n_1})] \xi_{n_1},
\]
\[
\Omega_{2}^* = \lambda_2 + [\frac{24}{\pi^2} + O(\frac{1}{n_1})](\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 + [-\frac{6}{\pi^2} + O(\frac{1}{n_1})] \xi_{n_2}, \quad (i \neq n_1, n_2, 2n_1, 2n_2, n_2 - n_1, n_1 + n_2),
\]
and
\[ P^* = \tilde{G} + K + L, \]
\[ L = O(|y|^3) + O(|\xi| \frac{1}{2} |y|^2) + O(|\xi| \frac{1}{2} |y||z|_a^2) + O(|y|^2 |z|_a^3). \]

We note that the nonlinear Schrödinger equation (1) possesses a conserved quantity
\[ \frac{2\pi}{0} \int |u|^2 dx, \]  
which indicates
\[ \sum_i |q_i|^2 = |q_{n_1}|^2 + |q_{n_2}|^2 + \sum_{i \neq n_1, n_2} |q_i|^2 = 2\chi, \]  
where \( \chi \) is a positive constant. Due to the transformations (5.1), we get
\[ y_{n_1} + \sqrt{\xi_{n_1}} + y_{n_2} + \sqrt{\xi_{n_2}} + \sum_{i \in \mathbb{Z} \setminus \{n_1, n_2\}} |z_i|^2 = \chi, \]
which implies \( \chi = O(|\xi|^{\frac{1}{2}}) \). Furthermore, by some computations, it gives
\[ (\sqrt{\xi_{n_1}} + \sqrt{\xi_{n_2}})^2 = \chi^2 + O(|\xi|^{\frac{1}{2}} |y| + |\xi|^{\frac{1}{2}} ||z||_a^2 + |y|^2 + |y||z||_a^2 + ||z||_a^4). \]  
By (42) and (45), the Hamiltonian (41) reads as
\[ H(y, x, z, \bar{z}; \xi) = \langle \omega(\xi), y \rangle + \sum_{i \neq n_1, n_2} \Omega_i(\xi) z_i \bar{z}_i + P(y, x, z, \bar{z}; \xi), \]
where $\omega(\xi) = (\omega_{n_1}(\xi), \omega_{n_2}(\xi))$ with

$$\omega_{n_1}(\xi) = \lambda_{n_1} + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_1},$$

$$\omega_{n_2}(\xi) = \lambda_{n_2} + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_2},$$

$$\Omega_{2n_1}(\xi) = \lambda_{2n_1} + \left(\frac{24}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{9}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_1},$$

$$\Omega_{2n_2}(\xi) = \lambda_{2n_2} + \left(\frac{24}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_1},$$

$$\Omega_{2n_2}(\xi) = \lambda_{2n_2} + \left(\frac{24}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_1},$$

and

$$\Omega_{n_2-n_1}(\xi) = \lambda_{n_2-n_1} + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + O\left(\frac{1}{n_1}\right) \xi_{n_1} + O\left(\frac{1}{n_1}\right) \xi_{n_2},$$

$$\Omega_{n_2+n_1}(\xi) = \lambda_{n_2+n_1} + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + O\left(\frac{1}{n_1}\right) \xi_{n_1} + O\left(\frac{1}{n_1}\right) \xi_{n_2},$$

$$\Omega_i(\xi) = \lambda_i + \left(\frac{24}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_1}$$

$$\Omega_{j}(\xi) = \lambda_{j} + \left(\frac{24}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2 + \left(-\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \xi_{n_1}, (i \neq n_1, n_2, 2n_1, 2n_2, n_2 - n_1, n_1 + n_2),$$

and

$$P(y, x, z, \bar{z}; \xi) = P^* + O(|y|^3 + |\xi|^2|y|^2 + |\xi|^2|z||z|^2 + |y|^2|z|^2 + |y||z|^2 + |y||z|^2 + |y|^2|z|^2 + |z|^2) + O(\xi^2 ||z||^2 + |y|^2|z|^2 + |y||z|^2 + |z|^2),$$

Recalling the definitions of $L$ and $\hat{G}$, we can put the last two terms in the second line in (48) into $L$ and $\hat{G}$ respectively. Thus, the perturbation $P$ in (46) can be written as

$$P = \hat{G} + K + L.$$  

(49)

On account of (47), we know that both the tangential frequency $\omega(\xi)$ and normal frequency $\Omega(\xi)$ are affine functions. For simplicity, we introduce the following denotations

$$\omega(\xi) = \alpha + F\xi, \quad \Omega(\xi) = \beta + B\xi,$$

with

$$\alpha = (\lambda_{n_1} + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2, \lambda_{n_2} + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2),$$

$$\beta = (\beta_j)_{j \neq n_1, n_2},$$

$$\beta_j = \lambda_j + \left(\frac{24}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2, j \neq n_1, n_2, n_2 + n_1, n_2 - n_1,$$

$$\beta_i = \lambda_i + \left(\frac{18}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) \chi^2, i = n_2 - n_1, n_2 + n_1.$$  

(50)
\[ F = \begin{bmatrix} -\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right) & 0 \\ 0 & -\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right) \end{bmatrix}, \] (51)

\[ B = \begin{bmatrix} -\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right) & -\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right) \\ \vdots & \vdots \\ -\frac{9}{\pi^2} + O\left(\frac{1}{n_1}\right) & -\frac{6}{\pi^2} + O\left(\frac{1}{n_1}\right) \end{bmatrix} \left(\begin{array}{c} (i) \\ \vdots \\ (2n_1) \end{array}\right) \]

where \( i \neq n_1, n_2, 2n_1, 2n_2, n_1 + n_2 - n_1 \).

**Step 2.** Check the non-degeneracy condition of frequencies.

**Claim.** We claim that the frequencies \((\omega(\xi), \Omega(\xi))\) fulfills the following Nondegeneracy conditions:

\((N1)\) \( \det F \neq 0; \)

\((N2)\) \( \langle l, \beta \rangle \neq 0, \text{ for } 1 \leq |l| \leq 2; \)

\((N3)\) \( \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \not\equiv 0, \forall (k, l) \in \mathbb{Z}^2 \times \mathbb{Z}^\infty, 1 \leq |l| \leq 2. \)

Checking the validity of nondegeneracy condition \((N1)\): In view of (51) and (19), choosing \( n_1 \geq N \geq 3\pi^6(N_{\lambda} + N_{\phi} + 2)\left[\frac{1}{2} + \pi^2(N_{\lambda}^2 + N_{\phi}^2)\right] \), we simply have

\[ \det F = \left(-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)\right)^2 \neq 0, \] (53)

which indicates \((N1)\) holds true.

Checking the condition \((N2)\): From the definition of \( \beta \), clearly we have \( \langle l, \beta \rangle \neq 0 \), for \( 1 \leq |l| \leq 2 \). In order to show the nondegeneracy condition \((N3)\), we need the following lemma.

**Lemma 5.1.** For each \( k \in \mathbb{Z}^2, l \in \mathbb{Z}^\infty \) with \( 1 \leq |l| \leq 2 \), we have

\[ \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \neq 0. \]

**Proof.** We only need to show that

\[ \langle k, \alpha \rangle + \langle l, \beta \rangle \neq 0 \quad \text{or} \quad Fk + B^T l \neq 0 \]

for all \((k, l)\) satisfying \( 1 \leq |l| \leq 2 \). Assume that \( Fk + B^T l = 0 \) for some \( k \in \mathbb{Z}^2, 1 \leq |l| \leq 2 \). Denote \( k = (k_1, k_2), l = (l_1, l_2, \ldots, l_n, \cdots) \in \mathbb{Z}^\infty \), without loss of generality, we always assume \( i \neq n_1, n_2 \). One can draw the conclusions from the following cases.

**Case I:** \(|l| = 1\). In this case, we discuss it from several possibilities.
Subcase(1): $l_i = \pm 1, i = 2n_1$,

$$Fk + B^Tl = 0 \Rightarrow \left( -\frac{8}{\pi^2} + O(\frac{1}{n_1}) \right) k_1 \pm \left( -\frac{9}{\pi^2} + O(\frac{1}{n_1}) \right) = 0$$

$$\Rightarrow k_1 = \frac{\pm 9}{\pi^2} + O(\frac{1}{n_1}) \notin \mathbb{Z},$$

which is impossible due to (52). Similar things happen when $l_{2n_2} = \pm 1$.

Subcase(2): $l_{n_2-n_1} = \pm 1$,

$$Fk + B^Tl = 0 \Rightarrow \left\{ \left( -\frac{8}{\pi^2} + O(\frac{1}{n_1}) \right) k_1 \pm O(\frac{1}{n_1}) = 0 \right\}$$

$$\Rightarrow \left\{ \begin{array}{c}
k_1 = \frac{O(\frac{1}{n_1})}{-\frac{8}{\pi^2} + O(\frac{1}{n_1})}, \\
k_2 = \frac{O(\frac{1}{n_1})}{-\frac{8}{\pi^2} + O(\frac{1}{n_1})}, \end{array} \right.$$  

From the fact that $k_1, k_2 \in \mathbb{Z}$, one gets $k_1 = k_2 = 0$. However, at this time,

$$\langle k, \alpha \rangle + \langle l, \beta \rangle = \pm \Omega_{n_2-n_1}(\xi) \neq 0,$$

which yields that

$$\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \neq 0.$$  

It is similar for the situation when $l_{n_2+n_1} = \pm 1$.

Subcase(3): $l_i = \pm 1, i \neq 2n_1, n_2-n_1, n_2+n_1, 2n_2$.

$$Fk + B^Tl = 0 \Rightarrow \left( -\frac{8}{\pi^2} + O(\frac{1}{n_1}) \right) k_1 \pm \left( -\frac{6}{\pi^2} + O(\frac{1}{n_1}) \right) = 0$$

$$\Rightarrow k_1 = \frac{\pm 6}{\pi^2} + O(\frac{1}{n_1}) \notin \mathbb{Z},$$

this cannot happen.

**Case II:** $|l| = 2$, and $l_i = \pm 2$. This Case is similar with Case I, we omit it.

**Case III:** $|l| = 2$, and $l_i = l_j = 1, i \neq j$. Without loss of generality, we suppose $i < j$.

Subcase(1): $i, j \notin \{2n_1, n_2-n_1, n_2+n_1, 2n_2\}$.

$$Fk + B^Tl = 0 \Rightarrow \left( -\frac{8}{\pi^2} + O(\frac{1}{n_1}) \right) k_1 + \left( -\frac{12}{\pi^2} + O(\frac{1}{n_1}) \right) = 0$$

$$\Rightarrow k_1 = \frac{-\frac{12}{\pi^2} + O(\frac{1}{n_1})}{-\frac{8}{\pi^2} + O(\frac{1}{n_1})} \notin \mathbb{Z},$$

which is impossible.

Subcase(2): $i \notin \{2n_1, n_2-n_1, n_2+n_1, 2n_2\}, j = 2n_1$.

$$Fk + B^Tl = 0 \Rightarrow \left( -\frac{8}{\pi^2} + O(\frac{1}{n_1}) \right) k_1 + \left( -\frac{15}{\pi^2} + O(\frac{1}{n_1}) \right) = 0$$

$$\Rightarrow k_1 = \frac{-\frac{15}{\pi^2} + O(\frac{1}{n_1})}{-\frac{8}{\pi^2} + O(\frac{1}{n_1})} \notin \mathbb{Z},$$
which cannot happen. For the cases when \( i \notin \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\} \), and \( j = n_2 - n_1 \), or \( j = n_2 + n_1 \) or \( j = 2n_2 \), one always have \( k_1 \notin \mathbb{Z} \), which contradicts with previous assumptions.

Subcase (3): \( i \in \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\}, j \notin \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\} \).

This case is similar with Subcase (2), we omit the details.

Subcase (4): \( i, j \in \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\} \). We only deal with two different situations. For the case \( i = n_2 - n_1 \), \( j = n_2 + n_1 \), one gets

\[
F_k + B^T l = 0 \Leftrightarrow \begin{cases}
-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)k_1 \pm O\left(\frac{1}{n_1}\right) = 0 \\
-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)k_2 \pm O\left(\frac{1}{n_1}\right) = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
k_1 = \frac{O\left(\frac{1}{n_1}\right)}{-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)}, \\
k_2 = \frac{O\left(\frac{1}{n_1}\right)}{-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)},
\end{cases}
\]

in view of \( k_1, k_2 \in \mathbb{Z} \), one obtains \( k_1 = k_2 = 0 \), under this circumstance, one has

\[
\langle k, \alpha \rangle + \langle l, \beta \rangle = \Omega_{n_2 - n_1}(\xi) + \Omega_{n_2 + n_1}(\xi) \neq 0.
\]

For the case \( i = 2n_1, j = n_2 - n_1 \), we have

\[
F_k + B^T l = 0 \Rightarrow \left(-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)\right)k_1 + \left(-\frac{9}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) = 0
\]

\[
\Rightarrow k_1 = \frac{9}{\pi^2} + O\left(\frac{1}{n_1}\right) \notin \mathbb{Z},
\]

this is impossible.

**Case IV:** \( |l| = 2 \), and \( l_i = 1, l_j = -1 \).

Subcase (1): \( i, j \notin \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\} \).

\[
F_k + B^T l = 0 \Leftrightarrow \begin{cases}
8 + O\left(\frac{1}{n_1}\right)k_1 \pm O\left(\frac{1}{n_1}\right) = 0 \\
8 + O\left(\frac{1}{n_1}\right)k_2 \pm O\left(\frac{1}{n_1}\right) = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
k_1 = \frac{O\left(\frac{1}{n_1}\right)}{8 + O\left(\frac{1}{n_1}\right)}, \\
k_2 = \frac{O\left(\frac{1}{n_1}\right)}{8 + O\left(\frac{1}{n_1}\right)},
\end{cases}
\]

we have \( k_1 = k_2 = 0 \) from above discussion, which yields that

\[
\langle k, \alpha \rangle + \langle l, \beta \rangle = \Omega_{i}(\xi) - \Omega_{j}(\xi) \neq 0.
\]

Subcase (2): \( i \notin \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\}, j \in \{2n_1, n_2 - n_1, n_2 + n_1, 2n_2\} \).

There are four similar subcases under this situation, we only deal with the case when \( j = 2n_1 \).

\[
F_k + B^T l = 0 \Rightarrow \left(-\frac{8}{\pi^2} + O\left(\frac{1}{n_1}\right)\right)k_1 + \left(\frac{3}{\pi^2} + O\left(\frac{1}{n_1}\right)\right) = 0
\]
Step 3: Checking the validity of spectral asymptotics:

Owing to (47), we have \( \Omega_j(\xi) = j^2 + \cdots + O\left(\frac{1}{j}\right) \), which indicates the spectral asymptotics property holds true for \( d = 2, \delta = -1 \).

Step 4: Checking the regularity property:

To this end, it requires to define the phase space and some norms. Let \( P^{a,p} = T^2 \times \mathbb{C}^2 \times l^{a,p} \times l^{a,p} \ni (x, y, z, \bar{z}) \) represent the phase space, in which \( T^2 \) denotes the complexification of the 2-torus \( T^2 \). Define

\[
D(s, r) = \{ (x, y, z, \bar{z}) \in P^{a,p} : |3x| < s, |y| < r^2, \|z\|_{a,p} + \|\bar{z}\|_{a,p} < r \},
\]

where \(|\cdot|\) represents the usual sup-norm for complex vector and \( \|\cdot\|_{a,p} \) denotes the norm defined in the Hilbert space \( l^{a,p} \). One can define the weighted norm in the following form

\[
|U|_{s,r} = |x| + \frac{1}{r^2} |y| + \frac{1}{r} \|z\|_{a,p} + \frac{1}{r} \||\bar{z}\|_{a,p}
\]

for \( U = (x, y, z, \bar{z}) \in P^{a,p} \). Let \( \Pi \) denote the parameter set, i.e., \( \xi = (\xi_1, \xi_2) \in \Pi \). For a mapping \( W : D(s, r) \times \Pi \rightarrow P^{a,p} \), set its Lipschitz semi-norm as follows:

\[
|W|_{L^{a,p}}^{Lip} = \sup_{\xi \neq \zeta} \frac{W(\cdot, \zeta) - W(\cdot, \xi)}{|\zeta - \xi|},
\]

where the supremum is taken over the parameter set \( \Pi \). Let \( X_R \) be the Hamiltonian vector field with respect to the symplectic structure \( dx \wedge dy + \sqrt{-1} dz \wedge d\bar{z} \), i.e.,

\[
X_R = (\partial_y R, -\partial_x R, \nabla_z R, \nabla_{\bar{z}} R).
\]

From Step 3, one has \( \Omega_j(\xi) = j^2 + \cdots + j^{-1} \), where the dots stand for fixed lower order term in \( j \). To be specific, there exists a fixed, parameter-dependent sequence
\( \Omega \) fulfilling \( \Omega_j = j^2 + \cdots \) such that the tails \( \hat{\Omega}_j = \Omega_j - \bar{\Omega}_j \) leads to a Lipschitz map \( \hat{\Omega} : \Pi \to l^1_\infty \), in which \( l^1_\infty \) denotes the space of all real sequences with finite form \( |v|_1 = \sup_j |v_j|j \).

Let \( M \) be the upper bound fulfilling \( |\omega|_{Lip}^{\Pi} + |\Omega|_{Lip}^{\Pi} \leq M < \infty \).

To check the regularity property, we choose \( s = s_0, r = \epsilon \) and \( \epsilon^{\frac{12}{7}} \leq \xi_1, \xi_2 \leq 2 \epsilon^{\frac{12}{7}} \). Then we have

\[
|X_G|_{D(s_0, \epsilon)} = O(|\xi|^\frac{3}{4}) = O(\epsilon^{\frac{16}{7}}),
\]

(54)

\[
|X_K|_{D(s_0, \epsilon)} = O(|\xi|^\frac{5}{2} / \epsilon^2) = O(\epsilon^{\frac{16}{7}}),
\]

(55)

therefore,

\[
|X_P|_{D(s_0, \epsilon)} \leq C \epsilon^{\frac{16}{7}}.
\]

(56)

Since \( X_P \) is analytic with respect to the parameter \( \xi \), thus one has

\[
|X_P|^{{Lip}}_{D(s_0, \epsilon)} \leq C \epsilon^{\frac{4}{7}}.
\]

(57)

**Step5:** Checking the KAM smallness condition:

Take \( \alpha = \epsilon^2 / \gamma \), then by (56) and (57) one arrives at

\[
|X_P|_{D(s_0, \epsilon)} + \frac{\alpha}{M} |X_P|^{{Lip}}_{D(s_0, \epsilon)} \leq \gamma \alpha,
\]

(58)

where \( M \) is a constant independent of \( \epsilon \) by (47), \( \gamma \) is a positive number. Using Theorem D in [24], we know that

\[
|\Pi \setminus \Pi_\alpha| \leq c \rho^{n-1} \alpha,
\]

(59)

in which \( \rho \) denotes the diameter of \( \Pi \), and \( n \) is the dimension of the tori. Note that here \( \rho = O(\epsilon^{\frac{12}{7}}) \) and \( n = 2 \). Therefore, the measure of the excluded parameter set is of \( O(\epsilon^{\frac{24}{7}}) \), which is of higher order than \( |\Pi| = O(\epsilon^{\frac{12}{7}}) \). This implies that the rotational tori persist for most of \( \xi \in \Pi \) when \( \epsilon \) is small enough. Therefore, the KAM smallness condition holds true provided \( \epsilon \) small enough.

Hence all the assumptions of KAM theorem A in [24] are verified, the proof of our main theorem 1.1 is completed by applying this KAM theorem.

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**REFERENCES**

[1] P. Baldi, M. Berti and R. Montalto, KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation, *Math. Ann.*, **359** (2014), 471–536.

[2] D. Bambusi and B. Grébert, Birkhoff normal form for partial differential equations with tame modulus, *Duke Math. J.*, **135** (2006), 507–567.

[3] M. Berti, L. Biasco and M. Procesi, KAM theory for the Hamiltonian derivative wave equations, *Arch. Ration. Mech. Anal.*, **212** (2014), 905–955.

[4] J. Bourgain, Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE, *Int. Math. Res. Not.*, **1994** (1994), 475–497.

[5] J. Bourgain, Quasi-periodic solutions of Hamiltonian perturbations for 2D linear Schrödinger equation, *Ann. Math.*, **148** (1998), 363–439.
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[6] J. Bourgain, On invariant tori of full dimension for 1D periodic NLS, *J. Funct. Anal.*, 229 (2005), 62–94.

[7] C. M. Cao and X. P. Yuan, Quasi-periodic solutions for perturbed generalized nonlinear vibrating string equation with singularities, *Discrete Contin. Dyn. Syst.*, 37 (2017), 1867–1901.

[8] L. Chierchia and J. G. You, KAM tori for 1D nonlinear wave equation with periodic boundary conditions, *Comm. Math. Phys.*, 211 (2000), 497–525.

[9] L. J. Du and X. P. Yuan, Invariant tori for nonlinear Schrödinger equations with a given potential, *Dynamics of PDE*, 3 (2006), 331–346.

[10] M. N. Gao and J. J. Liu, Quasi-periodic solutions for 1D wave equation with higher order nonlinearity, *J. Differential Equations*, 252 (2012), 1466–1493.

[11] T. Kappler and J. Pöschel, *KdV & KAM*, Springer-Verlag, Berlin, Heidelberg, 2003.

[12] S. B. Kuksin, Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum, *Funktsional. Anal. i Prilozhen.*, 21 (1987), 22–37, 95.

[13] S. B. Kuksin, Perturbation of quasiperiodic solutions of infinite-dimensional Hamiltonian systems, *Math. USSR Izv.*, 32 (1989), 39–62.

[14] S. B. Kuksin, *Nearly Integrable Infinite-Dimensional Hamiltonian Systems*, Lecture Notes in Mathematics, 1556. Springer-Verlag, Berlin, 1993.

[15] S. B. Kuksin and J. Pöschel, Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation, *Ann. Math.*, 143 (1996), 149–179.

[16] S. B. Kuksin, A KAM theorem for equations of the Korteweg-de Vries type, *Rev. Math-Math Phys.*, 10 (1998), 1–64.

[17] S. B. Kuksin, *Analysis of Hamiltonian PDEs*, Oxford Lecture Series in Mathematics and its Applications, 19. Oxford University Press, Oxford, 2000.

[18] Z. G. Liang and J. G. You, Quasi-periodic solutions for 1D Schrödinger equations with higher order nonlinearity, *SIAM J. Math. Anal.*, 36 (2005), 1965–1990.

[19] J. J. Liu and X. P. Yuan, Spectrum for quantum Duffing oscillator and small-divisor equation with large variable coefficient, *Commun. Pure Appl. Math.*, 63 (2010), 1145–1172.

[20] J. J. Liu and X. P. Yuan, A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations, *Commun. Math. Phys.*, 307 (2011), 629–673.

[21] J. J. Liu and X. P. Yuan, KAM for the derivative nonlinear Schrödinger equation with periodic boundary conditions, *J. Differential Equations*, 256 (2014), 1627–1652.

[22] L. F. Mi, Quasi-periodic solutions of derivative nonlinear Schrödinger equations with a given potential, *J. Math. Anal. Appl.*, 390 (2012), 335–354.

[23] J. Pöschel and E. Trubowitz, *Inverse Spectral Theory*, Oxford Lecture Series in Mathematics and its Applications, 19. Oxford University Press, Oxford, 2000.

[24] J. Pöschel, A KAM-theorem for some nonlinear partial differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 23 (1996), 119–148.

[25] J. Pöschel, Quasi-periodic solutions for a nonlinear wave equation, *Comment. Math. Helv.*, 71 (1996), 269–296.

[26] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I*, Second Edition, Clarendon Press, Oxford, 1962.

[27] C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Commun. Math. Phys.*, 127 (1990), 479–528.

[28] X. P. Yuan, Quasi-periodic solutions of completely resonant nonlinear wave equations, *J. Differential Equations*, 230 (2006), 213–274.

[29] X. P. Yuan, Invariant tori of nonlinear wave equations with a given potential, *Discrete Contin. Dyn. Syst.*, 16 (2006), 615–634.

[30] J. Zhang, M. N. Gao and X. P. Yuan, KAM tori for reversible partial differential equations, *Nonlinearity*, 24 (2011), 1189–1228.

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E-mail address: 12110180067@fudan.edu.cn
E-mail address: yand11@fudan.edu.cn