ALCOVE PATH MODEL FOR $B(\infty)$

ARTHUR LUBOVSKY AND TRAVIS SCRIMSHAW

ABSTRACT. We construct a model for $B(\infty)$ using the alcove path model of Lenart and Postnikov. We show that the continuous limit of our model recovers the Littelmann path model for $B(\infty)$ given by Li and Zhang. Furthermore, we consider the dual version of the alcove path model and obtain analogous results for the dual model.

1. INTRODUCTION

The theory of Kashiwara’s crystal bases [Kas90, Kas91] has been shown to have deep connections with numerous areas of geometry and combinatorics, well-beyond its origin in representation theory and mathematical physics. A crystal basis is a particularly nice basis for a representation of a quantum group $U_q(\mathfrak{g})$ that is preserved in the $q \to 0$, or crystal limit. In particular, for a symmetrizable Kac–Moody algebra $\mathfrak{g}$, the integrable highest weight modules $V(\lambda)$, so $\lambda$ is a dominant integral weight, were shown by Kashiwara to admit crystal bases $B(\lambda)$. Moreover, Kashiwara has shown that the lower half of the quantum group $U_q^{-1}(\mathfrak{g})$ admits a crystal basis $B(\infty)$.

Roughly speaking, the algebraic action of $U_q(\mathfrak{g})$ gets transformed into a combinatorial action on the bases in the $q \to 0$ limit. While Kashiwara’s grand loop argument showed the existence of the crystal bases $B(\lambda)$, it did not give an explicit (combinatorial) description. Thus the problem was to determine a combinatorial model for $B(\lambda)$. This was first done for $\mathfrak{g}$ of type $A_n$, $B_n$, $C_n$, and $D_n$ in [KN94] and $G_2$ in [KM94] by using tableaux. A uniform model (for all symmetrizable types) for crystals using piecewise-linear paths in the weight space was constructed in [Lit95a, Lit95b], which is now known as the Littelmann path model. A special case of the Littelmann path model includes Lakshmibai–Seshadri (LS) paths, where the combinatorial definition was given by Stembridge [Ste02].

Both of these models arose from examining a particular aspect of the representation theory of $\mathfrak{g}$ and the related combinatorics or geometry. There are numerous (but not necessarily uniform) models for $B(\lambda)$ that have been constructed from geometric objects such as quiver varieties [KS97, Sai02, Sav05] and MV polytopes [BKT14, Kam07, MT14, TW12]. Another uniform model for crystals came from the study of $(t$-analogs of) $q$-characters [Kas03, Nak03a, Nak03b, Nak04], which is now known as Nakajima monomials. Additionally, some models for crystals have also arose from mathematical physics, in particular, solvable lattice models [KKM+92a, KKM+92b] (the Kyoto path model) and Kirillov–Reshetikhin modules [Sch06, SS15a, SS15b, SS15c, SS16a, SS16b] (the rigged configuration model).

Many of these models are known to have extensions to $B(\infty)$. Some authors have used the direct limit construction of Kashiwara [Kas02] to extend a particular crystal model for $B(\lambda)$ to $B(\infty)$. Examples include the tableau model [Cli98, HL08, HL12] and rigged configurations [SS15a, SS15b, SS16a, SS16b], where the model reflects the naturality of the inclusion of $B(\lambda) \to B(\mu)$ for $\lambda \leq \mu$. In contrast, other authors have used other characterizations of $B(\infty)$ to construct their extensions, such as the polyhedral realization [NZ97] (which has a $B(\lambda)$ version [HN05, Hos05, Hos13, Nak99]), Nakajima monomials [KKS07], and Littelmann paths [LZ11].

The model we will be focusing on is a discrete version of the Littelmann path model known as the alcove path model that was given for $B(\lambda)$ in [LP07, LP08]. The alcove path model in finite types is related to LS galleries and Mirković–Vilonen (MV) cycles [GL05] and the equivariant $K$-theory of the generalized flag variety [LP07]. Moreover, the alcove path model can be described in terms of certain saturated chains in the (strong) Bruhat poset. While the Littelmann path model came first, it is perhaps more proper to consider

2010 Mathematics Subject Classification. 17B37, 05E10.
Key words and phrases. crystal, alcove path, quantum group.
TS was partially supported by RTG grant NSF/DMS-1148634.
the Littelmann path model as the continuous limit of the alcove path model. Moreover, the alcove path model carries with it more information, specifically the order in which the hyperplanes are crossed, allowing a non-recursive description of the elements in full generality.

The primary goal of this paper is to construct a model for $B(\infty)$ using the alcove path model. Our approach is to use the direct limit construction of Kashiwara restricted to $\{B(k\rho)\}_{k \geq 0}$, where the inclusions $i_{k\rho,k'\rho}: B(k\rho) \to B(k'\rho)$, for $k' > k$, are easy to compute. We then complete our proof by using the fact that for every $b \in B(\infty)$, there exists a $k \gg 1$ such that $b$ and $f_i(b)$, for all $i$, is not in the kernel of the natural projection onto $B(k\rho)$. Next, the continuous limit of the alcove path model for $B(\lambda)$ to the Littelmann path model for $B(-\lambda)$ is given explicitly by [LP08, Thm. 9.4] as a “dual” crystal isomorphism $\varpi_\lambda$. We extend $\varpi_\lambda$ to an explicit crystal isomorphism between the alcove path model and Littelmann path model for $B(\infty)$. A strength of the alcove path model for $B(\infty)$ over the Littelmann path model is that we can non-recursive describe the elements in $B(\infty)$. This is analogous to the statement about highest weight crystals as we still retain the notion of an admissible sequence.

As an intermediary step, we need to construct a Littelmann path model for the contragredient dual of $B(\infty)$. We note that we can construct the contragredient dual crystal explicitly in terms of (finite length) Littelmann paths by reversing a path and changing the starting point. Using this as a base, we construct a new model that, unlike the usual Littelmann path model (or the natural model for Littelmann paths by reversing a path and changing the starting point. Using this as a base, we construct a new model that, unlike the usual Littelmann path model (or the natural model for $B(-\infty)$ as the direct limit of $\{B(-k\rho)\}_{k \geq 1}$), it no longer starts at the origin, but, roughly speaking, “at infinity.” However, in an effort to avoid this, we are led to use a dual alcove path model that is essentially given by reversing the alcove path, mimicking the contragredient dual construction on Littelmann paths. We then show that dual alcove path model is dual isomorphic to the usual Littelmann path model.

This paper is organized as follows. In Section 2, we give the necessary background on crystals and the alcove path model. In Section 3, we describe our alcove path model for $B(\infty)$. In Section 4, we prove our main results. In Section 5, we construct an isomorphism between our model and the Littelmann path model.

2. Background

In this section, we give a background on general crystals, the crystal $B(\infty)$, and the alcove path model.

2.1. Crystals. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with index set $I$, generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$, weight lattice $P$, root lattice $Q$, fundamental weights $\{\Lambda_i \mid i \in I\}$, simple roots $\{\alpha_i \mid i \in I\}$, and simple coroots $\{\alpha_i^\vee \mid i \in I\}$. We also denote the canonical pairing $\langle \cdot , \cdot \rangle: P^\vee \times P \to \mathbb{Z}$ given by $\langle \alpha_i^\vee , \alpha_j \rangle = A_{ij}$, where $P^\vee$ is the coweight lattice. Let $\Phi^+$ denote positive roots, and let $P^+$ denote the dominant weights.

An abstract $U_q(\mathfrak{g})$-crystal is a nonempty set $B$ together with maps

$$e_i, f_i: B \to B \sqcup \{0\},$$
$$\varepsilon_i, \varphi_i: B \to \mathbb{Z} \sqcup \{-\infty\},$$
$$\text{wt}: B \to P,$$

which satisfy the properties

1. $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee , \text{wt}(b) \rangle$ for all $i \in I$,
2. if $b \in B$ satisfies $e_ib \neq 0$, then
   (a) $\varepsilon_i(e_ib) = \varepsilon_i(b) - 1$,
   (b) $\varphi_i(e_ib) = \varphi_i(b) + 1$,
   (c) $\text{wt}(e_ib) = \text{wt}(b) + \alpha_i$,
3. if $b \in B$ satisfies $f_ib \neq 0$, then
   (a) $\varepsilon_i(f_ib) = \varepsilon_i(b) + 1$,
   (b) $\varphi_i(f_ib) = \varphi_i(b) - 1$,
   (c) $\text{wt}(f_ib) = \text{wt}(b) - \alpha_i$,
4. $f_ib = b'$ if and only if $b = e_ib'$ for $b, b' \in B$ and $i \in I$,
5. if $\varphi_i(b) = -\infty$ for $b \in B$, then $e_ib = f_ib = 0$.

The maps $e_i$ and $f_i$, for $i \in I$, are called the crystal operators or Kashiwara operators. We refer the reader to [HK02, Kas91] for details.
We call an abstract $U_q(\mathfrak{g})$-crystal upper regular if

$$
\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k b \neq 0\}
$$

for all $b \in B$. Likewise, an abstract $U_q(\mathfrak{g})$-crystal is lower regular if

$$
\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k b \neq 0\}
$$

for all $b \in B$. When $B$ is both upper regular and lower regular, then we say $B$ is regular. For a regular crystal, we can express an entire $i$-string through an element $b \in B$ diagrammatically by

$$
e_i^{\varepsilon_i(b)}(b) \rightarrow \cdots \rightarrow e_i^{2\varepsilon_i(b)}(b) \rightarrow e_i b \rightarrow f_i b \rightarrow f_i^2 b \rightarrow \cdots \rightarrow f_i^{\varphi_i(b)}(b).
$$

An abstract $U_q(\mathfrak{g})$-crystal is called highest weight if there exists an element $u \in B$ such that $e_i u = 0$ for all $i \in I$ and there exists a finite sequence $(i_1, i_2, \ldots, i_s)$ such that $b = f_{i_1} f_{i_2} \cdots f_{i_s} u$ for all $b \in B$. The element $u$ is called the highest weight element.

Let $B_1$ and $B_2$ be two abstract $U_q(\mathfrak{g})$-crystals. A crystal morphism $\psi : B_1 \rightarrow B_2$ is a map $B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ such that

1. $\psi(0) = 0$;
2. if $b \in B_1$ and $\psi(b) \in B_2$, then $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$;
3. for $b \in B_1$, we have $\psi(e_i b) = e_i \psi(b)$ Provided $\psi(e_i b) \neq 0$ and $e_i \psi(b) = 0$;
4. for $b \in B_1$, we have $\psi(f_i b) = f_i \psi(b)$ Provided $\psi(f_i b) \neq 0$ and $f_i \psi(b) = 0$.

A morphism $\psi$ is called strict if $\psi$ commutes with $e_i$ and $f_i$ for all $i \in I$. Moreover, a morphism $\psi : B_1 \rightarrow B_2$ is called an embedding or isomorphism if the induced map $B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ is injective or bijective, respectively. If there exists an isomorphism between $B_1$ and $B_2$, say they are isomorphic and write $B_1 \cong B_2$.

The tensor product $B_2 \otimes B_1$ is the crystal whose set is the Cartesian product $B_2 \times B_1$ and the crystal structure given by

$$
e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\
b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases}$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\
b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases}$$

$$\varphi_i(b_2 \otimes b_1) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \alpha_i^\vee, \text{wt}(b_1) \rangle\},$$

$$\psi_i(b_2 \otimes b_1) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle \alpha_i^\vee, \text{wt}(b_2) \rangle\},$$

$$\text{wt}(b_2 \otimes b_1) = \text{wt}(b_2) + \text{wt}(b_1).$$

**Remark 2.1.** Our convention for tensor products is opposite the convention given by Kashiwara in [Kas91].

We say an abstract $U_q(\mathfrak{g})$-crystal is simply a $U_q(\mathfrak{g})$-crystal if it is crystal isomorphic to the crystal basis of a $U_q(\mathfrak{g})$-module.

The highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ for $\lambda \in \mathbb{P}^+$ has a crystal basis [Kas90, Kas91]. The corresponding (abstract) $U_q(\mathfrak{g})$-crystal is denoted by $B(\lambda)$, and we denote the highest weight element by $u_{\lambda}$. Moreover, the negative half of the quantum group $U_q^-(\mathfrak{g})$ admits a crystal basis denoted by $B(\infty)$, and we denote the highest weight element by $u_{\infty}$. Note that $B(\lambda)$ is a regular $U_q(\mathfrak{g})$-crystal, but $B(\infty)$ is only upper regular.

Consider a directed system of abstract $U_q(\mathfrak{g})$-crystals $\{B_j\}_{j \in J}$ with crystal morphisms $\psi_{k,j} : B_j \rightarrow B_k$ for $j \leq k$ (with $\psi_{j,j}$ being the identity map on $B_j$) such that $\psi_{k,j} \psi_{j,i} = \psi_{k,i}$ for $i \leq j \leq k$. Let $\bar{B} = \lim_{\rightarrow} B_j$ be the direct limit of this system, and let $\bar{\psi}_{j,j} : B_j \rightarrow \bar{B}$. Then Kashiwara showed in [Kas02] that $\bar{B}$ has a crystal structure induced from the crystals $\{B_j\}_{j \in J}$; in other words, direct limits exist in the category of abstract $U_q(\mathfrak{g})$-crystals. Specifically, for $\bar{b} \in \bar{B}$ and $i \in I$, define $e_i \bar{b}$ to be $\psi_{(j)}(e_i b_j)$ if there exists $b_j \in B_j$ such that $\bar{\psi}_{j,j}(b_j) = \bar{b}$ and $e_i(b_j) \neq 0$, otherwise set $e_i \bar{b} = 0$. Note that this definition does not depend on the choice of $b_j$. The definition of $f_i \bar{b}$ is similar. Moreover, the functions $\text{wt}$, $\varepsilon_i$, and $\varphi_i$ on $B_j$ extend to functions on $\bar{B}$.
**Definition 2.2.** For a weight $\lambda$, let $T_\lambda = \{t_\lambda\}$ be the abstract $U_q(g)$-crystal with operations defined by
\[
\begin{align*}
e_i t_\lambda &= f_i t_\lambda = 0, \\
\varepsilon_i(t_\lambda) &= \varphi_i(t_\lambda) = -\infty, \\
\text{wt}(t_\lambda) &= \lambda,
\end{align*}
\]
for any $i \in I$.

Consider an abstract $U_q(g)$-crystal $B$, then the tensor product $T_\lambda \otimes B$ has the same crystal graph as $B$ (but the weight, $\varepsilon_i$, and $\varphi_i$ have changed). Next, we recall from [Kas02] that the map
\[
\psi_{\lambda+\mu}: T_{-\lambda} \otimes B(\lambda) \longrightarrow T_{-\lambda-\mu} \otimes B(\lambda+\mu)
\]
which sends $t_{-\lambda} \otimes u_{\lambda} \mapsto t_{-\lambda-\mu} \otimes u_{\lambda+\mu}$ is a crystal embedding, and this morphism commutes with $e_i$ for each $i \in I$. Moreover, for any $\lambda, \mu, \xi \in P^+$, the diagram
\[
\begin{array}{ccc}
T_{-\lambda} \otimes B(\lambda) & \xrightarrow{\psi_{\lambda+\mu}} & T_{-\lambda-\mu} \otimes B(\lambda+\mu) \\
\downarrow{\psi_{\lambda+\mu+\xi,\lambda}} & & \downarrow{\psi_{\lambda+\mu+\xi,\lambda+\mu}} \\
T_{-\lambda-\mu-\xi} \otimes B(\lambda+\mu+\xi) & & \\
\end{array}
\]
commutes. Furthermore, if we order $P^+$ by $\mu \leq \lambda$ if and only if $\langle \alpha_\mu, \lambda - \mu \rangle \geq 0$ for all $i \in I$, the set \( \{T_{-\lambda} \otimes B(\lambda)\}_{\lambda \in P^+} \) is a directed system.

**Theorem 2.3 ([Kas02]).** We have
\[
B(\infty) = \lim_{\lambda \in P^+} T_{-\lambda} \otimes B(\lambda).
\]

From Theorem 2.3, we have that for any $\lambda \in P^+$, there exists a natural projection $p_\lambda: B(\infty) \rightarrow T_{-\lambda} \otimes B(\lambda)$ and inclusion $i_\lambda: T_{-\lambda} \otimes B(\lambda) \rightarrow B(\infty)$ such that $p_\lambda \circ i_\lambda$ is the identity on $T_{-\lambda} \otimes B(\lambda)$.

We can also form the **contragredient dual crystal** $B^\vee$ of $B$ as follows. Let $B^\vee = \{b^\vee | b \in B\}$, and define the crystal structure on $B^\vee$ by
\[
\begin{align*}
f_i(b^\vee) &= (e_i b)^\vee, & e_i(b^\vee) &= (f_i b)^\vee, \\
\varphi_i(b^\vee) &= \varepsilon_i(b), & \varepsilon_i(b^\vee) &= \varphi_i(b), \\
\text{wt}(b^\vee) &= -\text{wt}(b).
\end{align*}
\]
Note that $(B^\vee)^\vee$ is canonically isomorphic to $B$. We say the $B$ is **dual isomorphic** to $C$ if there exists a crystal isomorphism $\Psi: B \rightarrow C^\vee$ and the canonically induced bijection $\bar{\Psi}: B \rightarrow C$ is a **dual crystal isomorphism**. Explicitly, a dual crystal isomorphism satisfies
\[
\begin{align*}
f_i(\bar{\Psi}(b)) &= \bar{\Psi}(e_i b), & e_i(\bar{\Psi}(b)) &= \bar{\Psi}(f_i b), \\
\varphi_i(\bar{\Psi}(b)) &= \varepsilon_i(b), & \varepsilon_i(\bar{\Psi}(b)) &= \varphi_i(b), \\
\text{wt}(\bar{\Psi}(b)) &= -\text{wt}(b).
\end{align*}
\]

### 2.2. Alcove Path Model

Let $A_0$ denote the fundamental alcove, and let $A_\lambda = A_0 + \lambda$ be the translation of $A_0$ by $\lambda$. Consider some $\lambda \in P^+$. For a pair of adjacent alcoves $A$ and $B$, we write $A \xrightarrow{\alpha} B$ if the common wall of $A$ and $B$ is orthogonal to the root $\alpha \in \Phi$ and $\alpha$ points in the direction from $A$ to $B$. A sequence of positive roots $(\beta_1, \beta_2, \ldots, \beta_m)$ (where possibly $m = \infty$) is called a **$\lambda$-chain** if
\[
A_0 = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}
\]
is an alcove path of shortest length from $A_0$ to $A_{-\lambda}$. Let $\Gamma = (\beta_1, \ldots, \beta_m)$ and $\Gamma' = (\beta'_1, \ldots, \beta'_m)$ be a $\lambda$-chain and a $\mu$-chain respectively. Let $\Gamma \ast \Gamma'$ denote the concatenated $(\lambda + \mu)$-chain $(\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_m)$. We denote $\Gamma^k$ as $\Gamma$ concatenated with itself $k$ times.
Next, we recall the definition of the \textit{lex \( \lambda \)-chain} \( \Gamma_\lambda = (\beta_1, \beta_2, \ldots, \beta_m) \) from [LP08, Prop. 4.2]. We start by defining a total ordering on 
\[ R := \{ (\beta, k) \mid \alpha \in \Phi^+, 0 \leq k < \langle \lambda, \alpha \rangle \}. \]
First, fix a total order on the set of simple roots \( \alpha_1 < \alpha_2 < \cdots < \alpha_r \). For each \( (\beta, k) \in R \), let \( \beta^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee \), and define the vector \( v_{\beta,k} := \frac{1}{\langle \lambda, \beta^\vee \rangle}(k, c_1, \ldots, c_r) \) in \( \mathbb{Q}^{r+1} \). Then \( (\beta, k) < (\beta', k') \) if and only if \( v_{\beta,k} < v_{\beta',k'} \) in the lexicographic order on \( \mathbb{Q}^{r+1} \), which defines a total order on \( R \). If the \( i \)-th entry in \( R \) is \( (\beta, k) \), we set \( \beta_i = \beta \), we also set \( \ell_i = k \).

Let \( r_j = s_{\beta_j} \), and \( \hat{r}_j = s_{\beta_j} - \epsilon_j \). We consider a set of \textit{folding positions} \( J = \{ j_1 < j_2 < \cdots < j_p \} \subseteq \{ 1, 2, \ldots, m \} \), and call \( J \) \textit{admissible} if we have
\[ \iota(J) = 1 < r_{j_1} < r_{j_2} < \cdots < r_{j_1}r_{j_2} \cdots r_{j_p} = \tau(J). \]
In other words, \( J \) is admissible if it corresponds to a path in the Bruhat graph of \( W \). Let \( \mathcal{A}(\Gamma) \) denote the set of all \( J \subseteq \{ 1, 2, \ldots, m \} \) such that \( J \) is admissible. We also write \( \mathcal{A}(\lambda) := \mathcal{A}(\Gamma_\lambda) \). We will identify the integers \( j_a \) of the admissible set with the corresponding \( j_a \)-th element in the \( \lambda \)-chain; in other words, we identify \( \{ j_1 < \cdots < j_p \} = \{ (\beta_{j_1}, \ell_{j_1}) < \cdots < (\beta_{j_p}, \ell_{j_p}) \} \).

First, the weight function \( \text{wt} : \mathcal{A}(\lambda) \rightarrow \mathcal{P} \), following [LP08], is defined as
\[ \text{wt}(J) = -\hat{r}_{j_1} \cdots \hat{r}_{j_p}(-\lambda). \]
Next consider some \( J \in \mathcal{A}(\lambda) \), and define \( \Gamma(J) = (\gamma_1, \gamma_2, \ldots, \gamma_m) \), where
\[ \gamma_k = r_{j_1}r_{j_2} \cdots r_{j_q}(\beta_k) \]
with \( q = \max \{ a \mid j_a < k \} \). Next, we describe the crystal operators. Our description is in terms of \( \Gamma(J) \), and is equivalent to the one in [LP08]. We show this connection in Appendix B. Fix some \( i \in I \), and we define the sets
\[ I_{\alpha_i} = \{ k \mid \gamma_k = \pm \alpha_i \}, \]
\[ I_{\alpha_i} \setminus J = \{ k_1 < k_2 < \cdots < k_q \}. \]
Consider the word on the alphabet \{\text{+}, \text{-}\} given by
\[ \text{sgn}(\gamma_{k_1}) \text{sgn}(\gamma_{k_2}) \cdots \text{sgn}(\gamma_{k_q}). \]
Cancel \text{-} \text{+} pairs in this word until none remain, and we call this the \textit{reduced \( i \)-signature}. If there is no \text{+} in the reduced \( i \)-signature, then define
\[ f_i(J) = \begin{cases} J \setminus \{ \text{min}(J \cap I_{\alpha_i}) \} & \text{if } \langle \iota(J)(\rho), \alpha_i^\vee \rangle < 0, \\ 0 & \text{otherwise.} \end{cases} \]
Otherwise, let \( a \) be the index corresponding to the rightmost \text{+} in the reduced \( i \)-signature. Let \( A = \{ j \in J \cap I_{\alpha_i} \mid j > a \} \), and define
\[ f_i(J) = \begin{cases} J \cup \{ a \} & \text{if } A = \emptyset, \\ (J \setminus \{ \text{min} A \}) \cup \{ a \} & \text{otherwise.} \end{cases} \]

\textbf{Remark 2.4.} Since \( \iota(J) = 1 \) in Equation (2.7), we have \( \langle \iota(J)(\rho), \alpha_i^\vee \rangle > 0 \), and hence, \( f_i(J) \) will always be \( 0 \) in this case. The reason for defining \( f_i \) this way is to simplify construction of crystal operators in the dual model in Section 2.5.

The definition for \( e_i \) is similar. If \text{no} \text{-} exists in the reduced \( i \)-signature, then define
\[ e_i(J) = \begin{cases} J \setminus \{ \text{max}(J \cap I_{\alpha_i}) \} & \text{if } \langle \iota(J)(\rho), \alpha_i^\vee \rangle < 0, \\ 0 & \text{otherwise.} \end{cases} \]
Otherwise, let $a$ be the index corresponding to the leftmost $-$ in the reduced $i$-signature. Let $A = \{ j \in J \cap I_\alpha \mid j < a \}$, and define

$$e_i(J) = \begin{cases} J \cup \{a\} & \text{if } A = \emptyset, \\ (J \setminus \{\max A\}) \cup \{a\} & \text{otherwise} . \end{cases}$$

For any $\lambda$-chain $\Gamma$, we define $\varepsilon_i$ and $\varphi_i$ by requiring that $A(\Gamma)$ is a regular crystal.

**Theorem 2.5 ([LP08]).** Fix some $\lambda \in P^+$. Then

$$A(\lambda) \cong B(\lambda).$$

2.3. Littelmann path model. Let $\pi_1, \pi_2 : [0, 1] \to \mathfrak{h}_R^+$, and define an equivalence relation $\sim$ by saying $\pi_1 \sim \pi_2$ if there exists a piecewise-linear, nondecreasing, surjective, continuous function $\phi : [0, 1] \to [0, 1]$ such that $\pi_1 = \pi_2 \circ \phi$. A path is an equivalence class $[\pi]$ such that $\pi(0) = 0$. For ease of notation, we will simply write a path by $\pi$.

Let $\pi_1$ and $\pi_2$ be paths. Define the concatenation $\pi = \pi_1 \star \pi_2$ by

$$\pi(t) := \begin{cases} \pi_1(2t) & 0 \leq t \leq 1/2, \\ \pi_1(1) + \pi_2(2t - 1) & 1/2 < t \leq 1. \end{cases}$$

Next, consider a path $\pi$. Define $s_i \pi$ as the path given by $(s_i \pi)(t) = s_i(\pi(t))$.

We will now define a crystal structure on the set of all paths. Fix some $i \in I$ and path $\pi$. Define functions $H_{i, \pi} : [0, 1] \to \mathbb{R}$ by

$$\pi(t) = \sum_{i \in I} H_{i, \pi}(t) \Lambda_i,$$

and so $H_{i, \pi}(t) = \langle \alpha_i^\vee, \pi(t) \rangle$. Let $m_{i, \pi} := \min\{H_{i, \pi}(t) \mid t \in [0, 1]\}$ denote the minimal value of $H_{i, \pi}$.

If $-m_{i, \pi} < 1$, then define $e_i \pi = 0$, otherwise define $e_i \pi$ as the path given by

$$(e_i \pi)(t) = \begin{cases} \pi(t) & \text{if } t \leq t_0, \\ \pi(t_0) + s_i(\pi(t) - \pi(t_0)) & \text{if } t_0 < t \leq t_1, \\ \pi(t) + \alpha_i & \text{if } t_1 \leq t, \end{cases}$$

where

$$t_1 := \min\{t \in [0, 1] \mid H_{i, \pi}(t) = m_{i, \pi}\},$$

$$t_0 := \max\{t \in [0, t_1] \mid H_{i, \pi}(t') \geq m_{i, \pi} + 1 \text{ for all } t' \in [0, t]\}.$$

Next, if $H_{i, \pi}(1) - m_{i, \pi} < 1$, then define $f_i \pi = 0$, otherwise define $f_i \pi$ as the path given by

$$(f_i \pi)(t) = \begin{cases} \pi(t) & \text{if } t \leq t_0, \\ \pi(t_0) + s_i(\pi(t) - \pi(t_0)) & \text{if } t_0 < t \leq \overline{t}_1, \\ \pi(t) - \alpha_i & \text{if } \overline{t}_1 \leq t, \end{cases}$$

where

$$\overline{t}_0 := \max\{t \in [0, 1] \mid H_{i, \pi}(t) = m_{i, \pi}\},$$

$$\overline{t}_1 := \min\{t \in [\overline{t}_0, 1] \mid H_{i, \pi}(t') \geq m_{i, \pi} + 1 \text{ for all } t' \in [t, 1]\}.$$

For the remaining crystal structure, we define

$$\varepsilon_i(\pi) = -m_{i, \pi},$$

$$\varphi_i(\pi) = H_{i, \pi}(1) - m_{i, \pi},$$

$$\text{wt}(\pi) = \pi(1).$$

Let $\Pi(\lambda)$ denote the closure under the crystal operators of the path $\pi_\lambda(t) = t \lambda$.

**Theorem 2.6 ([Kas96, Lit95a, Lit95b]).** Let $g$ be of symmetrizable type and $\lambda \in P^+$. Then

$$\Pi(\lambda) \cong B(\lambda).$$

Furthermore, $\Pi(\lambda)$ is the set of all Lakshmibai–Seshadri (LS) paths of shape $\lambda$. Moreover, elements $\pi \otimes \xi \in \Pi(\lambda) \otimes \Pi(\mu)$ are given by $\xi \ast \pi$. 
Remark 2.7. The reversal of the concatenation is due to our order of the tensor product. See Remark 2.1.

Furthermore, we note that the contragredient dual path \(\pi'^\vee\) is given explicitly by

\[
\pi'^\vee(t) = \pi(1-t) - \pi(1).
\] (2.9)

Moreover, we have \((f_i\pi')^\vee = e_i(\pi'^\vee)\). This gives the following proposition.

Proposition 2.8. We have \(\Pi(-\lambda) \cong \Pi(\lambda)^\vee\) given by \(\pi \mapsto \pi'^\vee\).

For \(\lambda \in P^+\) and \(\mathfrak{g}\) of finite type, the lowest weight vector of \(\Pi(\lambda)\) is precisely \(\pi_{-\lambda}\), and hence we have \(\Pi(\lambda) = \Pi(\lambda)^\vee\) as sets [Lit95a, Lit95b].

Now we recall the construction of \(B(\infty)\) using the (modified) Littelmann paths from [LZ11]. An extended path is an equivalence class \(\pi: [0, \infty) \to \mathfrak{h}^*_\mathfrak{g}\), with the same equivalence relation \(\sim\) above, that eventually result in the direction \(\rho\): There exists a \(T\) such that for all \(t > T\), we have \(\pi'(t) = \rho\), where \(\pi' = \frac{d\pi}{dt}\).

Define \(\Pi(\infty)\) as the closure under the crystal operators of \(\pi_\infty(t) = tp\). For \(\Pi(\infty)\), we need to modify the definition of weight and \(\varphi_i\) to be

\[
\varphi_i(\pi) = \xi_i(\pi) + \langle \alpha_i^\vee, \operatorname{wt}(\pi) \rangle = -m_{i,\pi} + H_{i,\pi}(T) - T,
\]

where \(T = \min\{t \mid \pi'(\tilde{t}) = \rho, \tilde{t} \geq t\}\), whereas \(\xi_i(\pi) = -m_{i,\pi}\) as for \(\Pi(\lambda)\). For the definition of the crystal operators, we replace the intervals \([0, 1]\) with \([0, \infty)\) and drop the condition for \(f_i\pi = 0\) (alternatively, it is never satisfied because \(\lim_{t \to \infty} H_{i,\pi}(t) - m_{i,\pi} = \infty\).

Theorem 2.9 ([LZ11]). Let \(\mathfrak{g}\) be of symmetrizable type. Then

\[
\Pi(\infty) \cong B(\infty).
\]

2.4. Continuous limit. We recall the dual crystal isomorphism \(\varpi_\lambda: \mathcal{A}(\lambda) \to \Pi(-\lambda)\) from [LP08, Thm. 9.4].

Consider the admissible set \(J = \{(\zeta_1, \ell_1) < \cdots < (\zeta_p, \ell_p)\} \in \mathcal{A}(\lambda)\). Let \(R_j = s_{\zeta_j}\) and let \(t_k = \ell_k / \langle \lambda, \zeta_k^\vee \rangle\), and note that \(t_1 \leq t_2 \leq \cdots \leq t_p\). Next define the set

\[
\{0 = a_0 < a_1 < a_2 < \cdots < a_q\} := \{0\} \cup \{t_1, \ldots, t_p\},
\]

which may be of smaller size due to repetition. For \(0 \leq k \leq q\), define \(\mu_k := -R_1 \cdots R_{n_k}(\lambda)\), where \(n_k = \max\{1 \leq i \leq p \mid a_k = t_i\}\) and we consider \(\mu_0 = -\lambda\) if there is no \(i\) such that \(t_i = a_k\). Now, we define \(\varpi_\lambda(J)\) as the Littelmann path \(\pi: [0, 1] \to \mathfrak{h}_{\mathfrak{g}}\) given by

\[
\pi(t) = (t - a_k)\mu_k + \sum_{m=0}^{k-1} (a_{m+1} - a_m)\mu_m,
\] (2.10)

for \(a_k \leq t \leq a_{k+1}\) and all \(0 \leq k \leq q\) with \(a_{q+1} = 1\).

Theorem 2.10 ([LP08]). Let \(\mathfrak{g}\) be of symmetrizable type. The map \(\varpi_\lambda: \mathcal{A}(\lambda) \to \Pi(-\lambda)\) is a dual crystal isomorphism.

Indeed, the map \(\varpi_\lambda\) is dual in the sense that the map \(\varpi_\lambda^\vee: \mathcal{A}(\lambda) \to \Pi(-\lambda)^\vee\) given by \(\varpi_\lambda^\vee(J) := \varpi_\lambda(J)^\vee\) is a crystal isomorphism. From Proposition 2.8, we can consider \(\varpi_\lambda^\vee\) as a crystal isomorphism \(\mathcal{A}(\lambda) \cong \Pi(\lambda)\).

We can also roughly describe the map \(\varpi_\lambda\) geometrically as follows. Define \(F\) to be the set of alcoves that contain the origin, and we note that we can tile by \(Q\) translates of \(F\) (i.e., \(F\) is a fundamental domain with respect to translation by elements in \(Q\)). For example, in type \(A_2\), these are the 6 chambers that form a hexagon and are in bijection with elements of the Weyl group \(S_3\). We then construct the LS path as a slight perturbation of the path corresponding to a folded alcove path and contracting each translate of \(F\) to its corresponding element in \(Q\).
2.5. **Contragredient dual alcove paths.** We recall an equivalent formulation of the alcove path model from [Len12]. Fix a $\lambda$-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$ with alcove walk

$$A_o = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_\lambda.$$  

We can construct the dual path from $A_o$ to $A_\lambda$ by

$$A_o = A'_0 \xrightarrow{\beta_m} A'_1 \xrightarrow{\beta_{m-1}} \cdots \xrightarrow{\beta_1} A'_m = A_\lambda,$$

where $A'_i = A_{m-i} + \lambda$, with the **dual $\lambda$-chain** $\Gamma^\vee = (\beta_m, \beta_{m-1}, \ldots, \beta_1)$. Note that the dual path is simply given by the “$(-\lambda)$-chain” $(-\beta_m, -\beta_{m-1}, \ldots, -\beta_1)$.

We reindex the dual $\lambda$-chain by $i \mapsto m + 1 - i$ to be $\Gamma^\vee = (\beta_1, \beta_2, \ldots, \beta_m)$. A subset $J = \{ j_1 < j_2 < \cdots < j_p \} \subset \{ 1, 2, \ldots, m \}$ is **dual admissible** if there exists some $w \in W$ with

$$w \succ w_{j_1} \succ w_{j_1}r_{j_2} \succ \cdots \succ w_{j_1}r_{j_2} \cdots r_{j_p} = 1,$$

cf. Equation (2.2). We set

$$\tau(J) = r_{j_1}r_{j_2} \cdots r_{j_p}, \quad \iota(J) = w = \tau(J)^{-1}.$$

As before we have $\Gamma^\vee(J) = (\gamma_1, \gamma_2, \ldots, \gamma_q)$, where

$$\gamma_k = w_{j_1}r_{j_2} \cdots r_{j_q}(\beta_k) = r_{j_1}r_{j_2} \cdots r_{j_q}(\beta_k)$$

with $q = \max \{ a \mid j_a \leq k \}$.

**Remark 2.11.** The sequence $\Gamma^\vee(J)$ in this section can be obtained by reversing the sequence $\Gamma(J)$ from Section 2.2. This construction is analogous to taking the contragredient dual of the Littelmann path $\pi^\vee$, cf. Equation (2.9).

Let $\tilde{\ell}_i = (\lambda, \beta_i^\vee) - \ell_i$, and let $\tilde{\gamma}_i = s_{\beta_i, \tilde{\ell}_i}$ then

$$\text{wt}(J) = \iota(J)\tilde{\gamma}_{j_1} \cdots \tilde{\gamma}_{j_p}(\lambda).$$

Let $\mathcal{A}^\vee(\Gamma)$ be defined as the set $J \subset \{ 1, \ldots, m \}$ such that $J$ is dual admissible with respect to $\Gamma^\vee$. For brevity, we denote $\mathcal{A}^\vee(\lambda) := \mathcal{A}^\vee(\Gamma_\lambda)$. In this case, we define crystal operators $e_i$ and $f_i$ by Equations (2.8) and (2.7) respectively, using the sets $I_{\alpha_i}$ and $I_{\alpha_i} \setminus J$ as defined in Equation (2.5), and the word

$$\text{sgn}(-\gamma_{k_1})\text{sgn}(-\gamma_{k_2}) \cdots \text{sgn}(-\gamma_{k_q}),$$

instead of Equation (2.6). Thus, we have the following.

**Proposition 2.12.** We have 

$$\mathcal{A}^\vee(\Gamma) \cong \mathcal{A}(\Gamma)^\vee,$$

where the crystal isomorphism is given by $J \mapsto J$.

Proposition 2.12 and Theorem 2.10 gives us the following.

**Corollary 2.13.** Let $\mathfrak{g}$ be of symmetrizable type. Then there exists a dual crystal isomorphism $\varpi_\lambda^\vee : \mathcal{A}^\vee(\lambda) \cong \Pi(\lambda)$.

3. **Infinite alcove paths**

In this section, we describe the alcove path model for $B(\infty)$ using the limit of $\mathcal{A}(\lambda)$ and $\mathcal{A}^\vee(\lambda)$ as $\lambda \to \infty$. We denote these by $\mathcal{A}(\infty)$ and $\mathcal{A}^\vee(\infty)$, respectively.
3.1. The crystal $A(\infty)$. We first give a combinatorial interpretation for $A(\infty)$ and then a geometric one. Fix some $\rho$-chain $\Gamma = (\beta_1, \ldots, \beta_m)$. We define the $\infty$-chain of $\Gamma$ as $\cdots * \Gamma * \Gamma$, which in terms of alcove walks is

$$\cdots \xrightarrow{-\beta_{m-1}} A_{-m-1} \xrightarrow{-\beta_m} A_{-m} \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_{m-1}} A_{-1} \xrightarrow{-\beta_m} A_0 = A_0.$$ 

Then $A(\infty)$ is the set of all admissible folding positions. Note that if we write the folding positions as $\{(\zeta_1, \ell_1), \ldots, (\zeta_p, \ell_p)\}$, then we have $\ell_k < 0$ for all $k$.

Geometrically, we start with $\emptyset$ denoting the infinite alcove walk ending at the dominant alcove $A_0$ and indefinitely repeating backwards along the $\rho$-chain. All subsequent elements in $A(\infty)$ are foldings of this alcove walk. In particular, it will not necessarily end in the dominant alcove.

We define $f_i$ and $e_i$ by Equations (2.7) and (2.8), respectively, $e_i$ by specifying $A(\infty)$ is an upper regular crystal, and wt by Equation (2.3) with $\lambda = 0$. Thus, we can define $\varphi_i$ by Condition (1) of an abstract $U_q(\mathfrak{g})$-crystal.

**Lemma 3.1.** The set $A(\infty)$ is an abstract $U_q(\mathfrak{g})$-crystal with the crystal structure given above.

**Proof.** First note that any reduced $i$-signature is of the form $\cdots + - + \cdots$, where there are at most $\varepsilon_i(J)$ number of $-$. Thus, the crystal operators $e_i$ and $f_i$ are well-defined. Next, note that $\emptyset$ is the highest weight element of $A(\infty)$, and we have

$$\varepsilon_i(\emptyset) = \varphi_i(\emptyset) = \langle \alpha_i^\vee, \text{wt}(\emptyset) \rangle = 0$$

for all $i \in I$. Thus it is sufficient to show that Conditions (3) and (4) hold as it is clear $\varphi_i(J) > -\infty$ for all $J \in A(\infty)$. However, these follow from similar arguments as given in [LP08, Sec. 7]. \qed

3.2. The dual crystal $A^\vee(\infty)$. The construction of $A^\vee(\infty)$ will be similar to the construction done in Section 2.5.

For a fixed dual $\rho$-chain $\Gamma^\vee = (\beta_1, \ldots, \beta_m)$, we define the dual $\infty$-chain of $\Gamma$ as $\Gamma^\vee * \Gamma^\vee * \cdots$, which in terms of alcove walks is

$$A_0 = A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} A_m \xrightarrow{\beta_1} A_{m+1} \xrightarrow{\beta_2} \cdots.$$ 

We can also define crystal operators on $A^\vee(\infty)$ as in Section 2.5.

**Proposition 3.2.** We have a crystal isomorphism

$$A^\vee(\infty) \cong A(\infty)^\vee$$

given by $J \mapsto J$.

**Proof.** Similar to the proof of Proposition 2.12. \qed

Unlike for the model $A(\infty)$, the alcove paths for $A^\vee(\infty)$ will always end in an (closed) alcove that contains the origin (i.e., it will start in an alcove of the fundamental domain with respect to the action of $Q$).

4. Main results

Let $\lambda, \mu \in P^+$. We embed $A(\lambda)$ into $A(\mu + \lambda)$ as follows. Note that the $(\mu + \lambda)$-chain $\Delta = \Gamma_{\mu} * \Gamma_{\lambda}$ gives rise to admissible sequences $A(\Delta) \subseteq A(\mu) \otimes A(\lambda)$. Therefore, we have a canonical isomorphism $A(\Delta) \cong A(\mu + \lambda)$.

We can embed $A(\lambda)$ into $A(\Delta)$ by a $\mu$-shift. More precisely, let $\{(\zeta_1, \ell_1), \ldots, (\zeta_p, \ell_p)\} = J \in A(\lambda)$, and define $S_{\mu} : A(\lambda) \rightarrow T_{-\mu} \otimes A(\Delta)$ by

$$S_{\mu}(J) := t_{-\mu} \otimes \{(\zeta_1, \langle \zeta_1^\vee, \mu \rangle + \ell_1), \ldots, (\zeta_p, \langle \zeta_p^\vee, \mu \rangle + \ell_p)\}.$$ (4.1)

Observe that $S_{\mu}$ is a crystal embedding since $S_{\mu}(\emptyset) = 0$ and if $f_i(J) \neq 0$, then $f_i$ either adds a folding position to $J$ or moves a folding position. This operation depends entirely on the folded $\lambda$ chain, and is not affected by the shift. In other words, we have $S_k(f_i J) = f_i S_k(J)$. Similar statements hold for $e_i$.

**Lemma 4.1.** Fix some $\lambda \in P^+$. Suppose $\mu \in P$ is such that $\mu + \lambda = k \lambda$ for some $k \in \mathbb{Q}_{\geq 0}$. The map $S_{\mu} : A(\lambda) \rightarrow T_{-\mu} \otimes A(\lambda + \mu)$ given by Equation (4.1), where $S_{\mu}(J) = 0$ if the result is not admissible, is a crystal morphism. Moreover, if $k \geq 1$, then $S_{\mu}$ is a crystal embedding, and if $k \leq 1$, then $S_{\mu}$ is a surjection.
Proof. From our assumptions, there exists some $\nu \in P^+$ such that $\Gamma_{\mu+\nu} = \Gamma_{\nu+\lambda}$ and $\Gamma_\lambda = \Gamma_{\nu+\lambda}$. Note that $k = m_{\mu+\nu}/m_\lambda$. If $m_\lambda \leq m_{\mu+\nu}$, then similar to the discussion above, the crystal operators act only on the $\lambda$-chain part of the $(\mu + \lambda)$-chain. Furthermore, it is straightforward to see that every admissible sequence in $A(\lambda)$ is admissible in $A(\lambda+\mu)$. Likewise, if $m_\lambda \geq m_{\mu+\nu}$, then the crystal operators act only on the $(\mu + \lambda)$-chain part of the $\lambda$-chain and $A(\lambda+\mu) \subseteq A(\lambda)$. Thus, the claim follows. □

Next, for $k \geq 0$, define $S_{\infty,k}^p : T_{-k} \otimes A(k)p \rightarrow A(\infty)$ by

$$S_{\infty,k}^p(t_{-k} \otimes J) = \{ (\zeta_1, 1 - k (\zeta_1', \rho), \ldots, (\zeta_p, h_p - k (\zeta_p', \rho)) \}$$

and $S_{k}^p, A(\infty) \rightarrow T_{-k} \otimes A(k)p$ by

$$S_{k}^p(J) = t_{-k} \otimes \{ (\zeta_1, h_1 + k (\zeta_1', \rho), \ldots, (\zeta_p, h_p + k (\zeta_p', \rho)) \}$$

if the result is admissible and $S_{k}^p(J) = 0$ otherwise.

Lemma 4.2. The maps $S_{k}^p$ and $S_{\infty,k}^p$ are crystal surjections and embeddings, respectively.

Proof. This is similar argument the proof of Lemma 4.1. □

Lemma 4.3. The family $\{ A(k)p \}_{k=0}^{\infty}$ forms a directed system with inclusion maps $\psi_{k',k} = S_{(k'-k)p}$, for all $k' > k$. Moreover, the map

$$S : \lim_{k \in \mathbb{Z}_{\geq 0}} A(k)p \rightarrow A(\infty)$$

is given by $S_{\infty,k}^p \circ \psi_{(k)}$.

Proof. First, $\{ A(k)p \}_{k=0}^{\infty}$ is a directed system by Lemma 4.1. Next, we note that Lemma 4.2 implies that $S$ is well-defined. It is straightforward to see that there exists a $K$ such that $S_{k}^p(J) = T_{-k} \otimes A(k)p$ for all $k \geq K$ and $J \in A(\infty)$. Hence, $S$ is invertible, and the claim follows. □

Theorem 4.4. Let $\mathfrak{g}$ be of symmetrizable type. Then we have

$$A(\infty) \cong B(\infty).$$

Proof. We will define a map $\Psi : A(\infty) \rightarrow B(\infty)$ as follows. Fix some $b \in B(\infty)$. Let $k$ be such that $p_{k}(b) \neq 0$. From Theorem 2.5, we have a (canonical) isomorphism $\Phi : A(k)p \rightarrow B(k)p$. Thus, we define $\Psi(b)$ by the composition

$$A(\infty) \xrightarrow{S_{k}^p} T_{-k} \otimes A(k)p \xrightarrow{\Phi} T_{-k} \otimes B(k)p \xrightarrow{i_{kp}} B(\infty).$$

Note that Lemma 4.2 and Lemma 4.3 states that this is independent of the choice of $k$ and well-defined. Additionally, the (local) inverse of $\Psi$ is given by the composition

$$B(\infty) \xrightarrow{p_{k}^{-1}} T_{-k} \otimes B(k)p \xrightarrow{\Phi^{-1}} T_{-k} \otimes A(k)p \xrightarrow{S_{k}^p_{\infty}} A(\infty).$$

Therefore, the map $\Psi$ is an isomorphism as desired. □

Our construction, geometrically speaking, is to extend the alcove walk in the anti-dominant chamber to infinity, but to shift the origin so that it is at the end of the path. Note that this differs from the construction of $A^\vee(\lambda)$ in Section 2.5, where the direction of the path is also reversed.

Remark 4.5. If $A_{ij}A_{ji} < 4$ for all $i \neq j \in I$ (i.e., the restriction to any rank 2 Levi subalgebra is of finite type), then we could use the Yang–Baxter moves of [Len07] to construct the directed system $\{ T_{-\lambda} \otimes A(\lambda) \}_{\lambda \in P^+}$. However, it would be interesting to construct this for general symmetrizable types as it could allow one to determine the subset of $A(\infty)$ that corresponds to $A(\lambda)$ and generalize the model for any $\infty$-chain of the lex $\rho$-chain.

We also have the following for the dual alcove path model.

Corollary 4.6. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Then we have

$$A^\vee(\infty) \cong B(\infty)^\vee.$$

Proof. This follows from Theorem 4.4 and Proposition 3.2. □
5. Continuous limit of infinite alcove walks

We will show that we can extend the dual crystal isomorphism \( \varpi_\lambda : A(\lambda) \rightarrow \Pi(\lambda)^\vee \) to a dual crystal isomorphism \( \varpi_\infty : A(\infty) \rightarrow \Pi(\infty)^\vee \). We first need to construct a model \( \Pi'(\infty) \) using somewhat different paths such that \( \Pi'(\infty) \cong \Pi(\infty)^\vee \).

From Theorem 2.3 and the tensor product rule, for any sequence \( (a_j \in I)_{j=1}^N \), there exists a \( K \) such that

\[
   f_{a_1} \cdots f_{a_N} u_\infty \mapsto t_{-k_0} \otimes u_\infty \otimes (f_{a_1} \cdots f_{a_N} u_{k_0}) \in T_{-k_0} \otimes B(\infty) \otimes B(k_0)
\]

for all \( k > K \). In terms of the Littelmann path model, there is some \( k \) such that

\[
   f_{a_1} \cdots f_{a_N} \pi_\infty = (f_{a_1} \cdots f_{a_N} \pi_{k_0}) * \pi_\infty.
\]

Define \( \Pi'(\infty) \) be the set of paths (up to \( \sim \)) \( \xi : (-\infty, 0] \rightarrow h_\mathfrak{g}^* \) in the closure of \( \xi_\infty(t) = t\rho \) under the crystal operators given in Section 2.3 except with \( m_{i,\pi} = \max\{H_{i,\pi}(t) \mid t \in (-\infty, 0]\} \), interchanging \( e_i \) and \( f_i \), and \( \text{wt}(\xi) = -\xi(0) \). We can also make this construction geometrically by considering the paths as in the one-point compactification of \( h_\mathfrak{g} \) and performing the usual path reversal and shifting the endpoint. Indeed, \( \Pi'(\infty) \) is a subset of all paths \( \xi : (-\infty, 0] \rightarrow h_\mathfrak{g}^* \) such that there exists a \( T \) where \( \xi'(t) = \rho \) for all \( t \leq T \).

However, unlike for paths with finite length and \( \Pi(\infty) \), we have \( \xi(0) = 0 \) if and only if \( \xi = \xi_\infty \). We also have the following analog of Proposition 2.8.

**Proposition 5.1.** We have \( \Pi'(\infty) \cong \Pi(\infty)^\vee \), where the dual crystal isomorphism is given by

\[
   \xi'(t) = \xi(-t) - \xi(0).
\]

**Proof.** This follows immediately from the definition of \( e_i \) and \( f_i \) and that \( \xi_\infty = \pi_\infty \). \( \square \)

**Remark 5.2.** The set \( \Pi'(\infty) \) should not be considered as \( \Pi(-\infty) = \lim_{k \rightarrow \infty} \Pi(-k\rho) \) as the latter consists of paths \( \pi : [0, \infty) \rightarrow h_\mathfrak{g}^* \) and must start at the origin. Additionally, note that \( \Pi(-\infty) \) is isomorphic to \( \Pi(\infty)^\vee \) by Proposition 2.8 applied to the direct limit (or by restricting to \( [0, T] \), where \( T \) is minimal such that \( \pi'(t) = \rho \) for all \( t > T \) and then appending \( \pi_{-\infty}(t) = -\rho t \)). However, in order to obtain the continuous limit of \( A(\infty) \), we require \( \Pi'(\infty) \).

Therefore, we define our desired dual crystal isomorphism \( \varpi_\infty \) as the following composition

\[
   A(\infty) \rightarrow T_{-k_0} \otimes A(\infty) \otimes A(k_0) \rightarrow \Pi(-k_0) \otimes \Pi'(\infty) \otimes T_{k_0} \rightarrow \Pi'(\infty)
\]

\[
   J \mapsto t_{-k_0} \otimes 0 \otimes S_{k_0}(J) \mapsto \varpi_{-k_0}(S_{k_0}(J)) \otimes \xi_\infty \otimes t_{k_0} \mapsto \xi_\infty * \varpi_{-k_0}(S_{k_0}(J))
\]

for some \( k \gg 1 \) depending on the element \( J \). Hence, by Theorem 2.10 we have the following.

**Theorem 5.3.** Let \( \mathfrak{g} \) be of symmetrizable type. Then the map

\[
   \varpi_\infty : A(\infty) \rightarrow \Pi'(\infty)
\]

defined above is a dual crystal isomorphism. Moreover, the dual crystal isomorphism is given explicitly by the same description as \( \varpi_\lambda \) given in Section 2.4.

We can also directly describe an isomorphism \( A(\infty) \cong \Pi(\infty) \) by combining the results of Theorem 5.3 and Proposition 5.1. We also have a dual version of Theorem 5.3.

**Theorem 5.4.** Let \( \mathfrak{g} \) be of symmetrizable type. Then the map

\[
   A'(\infty) \rightarrow T_{-k_0} \otimes A'(\infty) \otimes A'(k_0) \rightarrow T_{-k_0} \otimes \Pi(\infty) \otimes \Pi(k_0) \rightarrow \Pi(\infty),
\]

\[
   J \mapsto t_{-k_0} \otimes 0 \otimes S_{k_0}(J) \mapsto t_{-k_0} \otimes \pi_\infty \otimes \varpi_{k_0}(S_{k_0}(J)) \mapsto \varpi_{k_0}(S_{k_0}(J)) * \pi_\infty,
\]

where \( k \gg 1 \) depends on the element \( J \), is a dual crystal isomorphism.

**Proof.** The proof is similar to Theorem 5.3, but using Proposition 2.8 in conjunction with Theorem 2.10. \( \square \)

Alternatively this follows from taking the contragredient dual at each step of \( \varpi_\infty \).

11
APPENDIX A. CALCULATIONS USING SAGE

The crystal $\mathcal{A}(\lambda)$ (resp. $\mathcal{A}(\infty)$) has been implemented by the first (resp. second) author in Sage [Sag16, SCc08]. We conclude with examples.

We construct $\mathcal{A}(\infty)$ in type $A_3$ and compute the element $b = f_{2} f_{1} f_{2} f_{3} f_{1} f_{2} 0$:

```
sage: A = crystals.infinity.AlcovePaths([['A', 3]])
sage: mg = A.highest_weight_vector()
sage: b = mg.f_string([2, 1, 3, 2, 2, 1, 3, 2])
sage: b
((alpha[2], -2), (alpha[2] + alpha[3], -2),
 (alpha[1] + alpha[2], -2), (alpha[1] + alpha[2] + alpha[3], -2))
sage: b.weight()
(-4, -4, 0, 0)
```

Next, we construct the projection onto $\mathcal{A}(2\rho)$ by computing $S^p_{2\rho}(b)$:

```
sage: b.projection()
((alpha[2], 0), (alpha[2] + alpha[3], 2),
 (alpha[1] + alpha[2], 2), (alpha[1] + alpha[2] + alpha[3], 4))
sage: b.to_highest_weight()
[(), [2, 1, 2, 1, 3, 2, 3, 2]]
```

Note that $((k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3)^\vee, \rho) = k_1 + k_2 + k_3$. Therefore, compare the result to the corresponding elements in $\mathcal{A}(3\rho)$ and $\mathcal{A}(4\rho)$:

```
sage: A = crystals.AlcovePaths([['A', 3]], [3, 3, 3])
sage: mg = A.highest_weight_vector()
sage: mg.f_string([2, 1, 3, 2, 2, 1, 3, 2])
((alpha[2], 1), (alpha[2] + alpha[3], 4),
 (alpha[1] + alpha[2], 4), (alpha[1] + alpha[2] + alpha[3], 7))
sage: A = crystals.AlcovePaths([['A', 3]], [4, 4, 4])
sage: mg = A.highest_weight_vector()
sage: mg.f_string([2, 1, 3, 2, 2, 1, 3, 2])
((alpha[2], 2), (alpha[2] + alpha[3], 6),
 (alpha[1] + alpha[2], 6), (alpha[1] + alpha[2] + alpha[3], 10))
```

APPENDIX B. ALCOVE MODEL: CRYSTAL OPERATORS

In this section we show that our description of crystal operators in Section 2.2 is equivalent to the one given in [LP08].

Let $J = \{j_1 < j_2 < \cdots < j_p\}$. Recall $\gamma_k$ from Equation (2.4) and the set $I_{\alpha_i}$ from Equation (2.5a). Let $I_{\alpha_\infty} = \{i_1 < i_2 < \cdots < i_N\}$ and $\tilde{I}_{\alpha_i} = I_{\alpha_\infty} \cup \{\infty\}$.

Let $\gamma_{\infty} = r_{j_1} r_{j_2} \cdots r_{j_p}(\rho)$, and define

$$
\varsigma_i := \begin{cases} 
1 & \text{if } i \not\in J, \\
-1 & \text{if } i \in J.
\end{cases}
$$

Crystal operators are defined in terms of the piecewise linear function $g_{\alpha_i} : [0, N + \frac{1}{2}] \to \mathbb{R}$ given by

$$
g_{\alpha_i}(0) = -\frac{1}{2}, \quad \frac{dg_{\alpha_i}}{dx}(x) = \begin{cases} 
\text{sgn}(\gamma_{i_j}) & \text{if } x \in (j - 1, j - \frac{1}{2}), \ j = 1, \ldots, N, \\
\varsigma_i \text{sgn}(\gamma_{i_j}) & \text{if } x \in (j - \frac{1}{2}, j), \ j = 1, \ldots, N, \\
\text{sgn}(\gamma_{\infty}, \alpha_i^\vee) & \text{if } x \in (N, N + \frac{1}{2}).
\end{cases}
$$

The graph $g_{\alpha_i}$ is used to define crystal operators in the alcove model. Let

$$
\sigma_j := (\text{sgn}(\gamma_{i_j}), \varsigma_i \text{sgn}(\gamma_{i_j})),
\sigma_{N+1} := \text{sgn}(\gamma_{\infty}, \alpha_i^\vee)),
$$

where $1 \leq j \leq N$. We note the following two conditions from [LP08]:

...
(C1) $\sigma_j \in (1,1), (1,-1), (-1,-1) \text{ for } 1 \leq j \leq N$.
(C2) $\sigma_j = (1,1)$ implies $\sigma_{j+1} \in \{(1,1), (1,-1), 1\}$.

In the language of Section 2.2, we identify $(1,1)$ with the symbol $+$ and $(-1,-1)$ with the symbol $-$.

We identify $(1,-1)$ with the symbol $\pm$ and note that if $\sigma_j = (1,-1)$, then $i_j \in J$. Finally identify $\sigma_{N+1} = 1$ with $+$ and $\sigma_{N+1} = -1$ with $-$. Condition (C1) says that we can describe $g_{\alpha_i}$ as a word in the alphabet $\{+, -\}$. Condition (C2) says that the transition from $+$ to $-$ must pass through $\pm$.

We now by recall the definition of $f_i$. Let $M$ be the maximum of $g_{\alpha_i}$. Let $h^J_{1,i} = g_{\alpha_i}((j - \frac{1}{2})$ and
\[ h^J_{\infty} = g_{\alpha_i}(N + \frac{1}{2}). \]
Let $\mu$ be the minimum index in $I_{\alpha_i}$ for which we have $h^J_{1,i} = M$. Then $\mu \in J$ or $\mu = \infty$. If $M > 0$, then $\mu$ has a predecessor $k$ in $I_{\alpha_i}$, with $k \not\in J$. Define
\[ f_i(J) := \begin{cases} (J \setminus \{\mu\}) \cup \{k\} & \text{if } M > 0, \\ 0 & \text{otherwise}. \end{cases} \]

We use the convention that $J \setminus \{\infty\} = J$ and $J \cup \{\infty\} = J$. Observe that after canceling out $+-$ terms as in Section 2.2 the rightmost remaining $+$ corresponds to $k$ and the $\pm$ ($+\text{ if } \mu = \infty$) term immediately following corresponds to $\mu$. This follows from conditions (C1) and (C2).

We now recall the definition of $e_i$. If $M > h^J_{\infty}$, let $k$ be the maximum index in $I_{\alpha_i}$, for which we have $h^J_{1,k} = M$, then $k \in J$ and $k$ has a successor $\mu$ in $I_{\alpha_i}$ with $\mu \notin J$. Define
\[ e_i(J) := \begin{cases} (J \setminus \{k\}) \cup \{\mu\} & \text{if } M > h^J_{\infty}, \\ 0 & \text{otherwise}. \end{cases} \]

Here it is also the case by (C1) and (C2) that the leftmost $-$, which exists if $M > h^J_{\infty}$, corresponds to $\mu$ and the immediately preceding $\pm$ corresponds to $k$.

ACKNOWLEDGEMENTS

The authors thank Cristian Lenart for many helpful discussions. This work benefited from computations using SAGEMath [Sag16, SCe08].

REFERENCES

[BKT14] Pierre Baumann, Joel Kamnitzer, and Peter Tingley. Affine Mirković-Vilonen polytopes. Publ. Math. Inst. Hautes Études Sci., 120:113–205, 2014.
[Cli98] Gerald Cliff. Crystal bases and Young tableaux. J. Algebra, 202(1):10–35, 1998.
[GL05] S. Gaussent and P. Littelmann. LS galleries, the path model, and MV cycles. Duke Math. J., 127(1):35–88, 2005.
[HK02] Jin Hong and Seok-Jin Kang. Introduction to quantum groups and crystal bases, volume 42 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002.
[HL08] Jin Hong and Hyeonmi Lee. Young tableaux and crystal $B(\infty)$ for finite simple Lie algebras. J. Algebra, 320(10):3680–3693, 2008.
[HL12] Jin Hong and Hyeonmi Lee. Young tableaux and crystal $B(\infty)$ for the exceptional Lie algebra types. J. Combin. Theory Ser. A, 119(2):397–419, 2012.
[HN05] Ayumu Hoshino and Toshiki Nakashima. Polyhedral realizations of crystal bases for modified quantum algebras of type $A$. Comm. Algebra, 33(7):2167–2191, 2005.
[Hos05] Ayumu Hoshino. Polyhedral realizations of crystal bases for quantum algebras of finite types. J. Math. Phys., 46(11):113514, 2005.
[Hos13] A. Hoshino. Polyhedral realizations of crystal bases for quantum algebras of classical affine types. J. Math. Phys., 54(5):053511, 28, 2013.
[Kam07] Joel Kamnitzer. The crystal structure on the set of Mirković-Vilonen polytopes. Adv. Math., 215(1):66–93, 2007.
[Kas90] Masaki Kashiwara. Crystalizing the $q$-analogue of universal enveloping algebras. Comm. Math. Phys., 133(2):249–260, 1990.
[Kas91] Masaki Kashiwara. On crystal bases of the $q$-analogue of universal enveloping algebras. Duke Math. J., 63(2):465–516, 1991.
[Kas96] Masaki Kashiwara. Similarity of crystal bases. In Lie algebras and their representations (Seoul, 1995), volume 194 of Contemp. Math., pages 177–186. Amer. Math. Soc., Providence, RI, 1996.
[Kas02] Masaki Kashiwara. Bases cristallines des groupes quantiques, volume 9 of Cours Spécialisés [Specialized Courses]. Société Mathématique de France, Paris, 2002. Edited by Charles Cochet.
[Kas03] Masaki Kashiwara. Realizations of crystals. In Combinatorial and geometric representation theory (Seoul, 2001), volume 325 of Contemp. Math., pages 133–139. Amer. Math. Soc., Providence, RI, 2003.
