On a local geometric property of the generalized elastic transmission eigenfunctions and application

Huaian Diao\textsuperscript{1,}, Hongyu Liu\textsuperscript{2} and Baiyi Sun\textsuperscript{3}

\textsuperscript{1} School of Mathematics, Jilin University, Changchun, Jilin 130012, People’s Republic of China
\textsuperscript{2} Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong SAR, People’s Republic of China
\textsuperscript{3} School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, People’s Republic of China

E-mail: hadiao@gmail.com, hongyu.liuip@gmail.com and 1412102726@qq.com

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Abstract

Consider the nonlinear and completely continuous scattering map $\mathcal{S}(\Omega; \lambda, \mu, V, u_i) = u_\infty(\hat{x})$, $\hat{x} \in \mathbb{S}^{n-1}$, which sends an inhomogeneous elastic scatterer $(\Omega; \lambda, \mu, V)$ to its far-field pattern $u_\infty$ due to an incident wave field $u_i$ via the Lamé system. Here, $(\lambda, \mu, V)$ signifies the medium configuration of an elastic scatterer that is compactly supported in $\Omega$. In this paper, we are concerned with the intrinsic geometric structure of the kernel space of $\mathcal{S}$, which is of fundamental importance to the theory of inverse scattering and invisibility cloaking for elastic waves and has received considerable attention recently. It turns out that the study is contained in analysing the geometric properties of a certain non-selfadjoint and non-elliptic transmission eigenvalue problem. We propose a generalized elastic transmission eigenvalue problem and prove that the transmission eigenfunctions vanish locally around a corner of $\partial \Omega$ under generic regularity criteria. The regularity criteria are characterized by the Hölder continuity or a certain Fourier extension property (in terms of the Herglotz wave approximation) of the transmission eigenfunctions. As an interesting and significant application, we apply the local geometric property to derive several novel unique identifiability results for a longstanding inverse elastic problem by a single far-field measurement.
Keywords: elastic scattering, non-scattering and invisibility, transmission eigenfunctions, geometric structure, inverse obstacle problem, unique identifiability, single far-field pattern

1. Introduction

1.1. Background and motivation

We first introduce the elastic scattering due to an embedded inhomogeneous medium and an incident wave field, which is the physical origin of our study. Let $\lambda$, $\mu$ be real constants satisfying the following strong convexity condition

$$\mu > 0, \quad n\lambda + 2\mu > 0 \quad \text{for } n = 2, 3.$$ 

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain with a connected complement $\mathbb{R}^n \setminus \overline{\Omega}$. Suppose that $V \in L^\infty(\mathbb{R}^n)$ is a real-valued function with $\text{supp}(V) \subset \Omega$. The parameters $\lambda$, $\mu$ and $V$ characterize the elastic medium configuration of the space $\mathbb{R}^n$, with $\lambda$, $\mu$ and $1 + V$ respectively denoting the bulk moduli and density. Throughout, we assume that $V$ is nontrivial, which is also referred to as a scattering potential. $(\Omega; \lambda, \mu, V)$ signifies an inhomogeneous scatterer embedded in the uniformly homogeneous space $\mathbb{R}^n$. Let $u^i$ be an incident field which is a $\mathbb{C}^n$-valued entire solution to the following Lamé system:

$$Lu^i + \omega^2 u^i = 0 \quad \text{in } \mathbb{R}^n, \quad Lu^i := \lambda \Delta u^i + (\lambda + \mu) \nabla \nabla \cdot u^i,$$

where $\omega \in \mathbb{R}_+$ signifies the angular frequency of the time-harmonic wave propagation. The interaction between the incident field $u^i$ and the scatterer $(\Omega; \lambda, \mu, V)$ generates the elastic scattering, which is governed by the following Lamé system:

$$Lu + \omega^2 (1 + V)u = 0 \quad \text{in } \mathbb{R}^n; \quad u = u^i + u^sc; \quad u^sc$$

is radiating.

Here by radiating, we mean that $u^sc$ satisfies the following Kupradze radiation condition

$$\lim_{r \to \infty} r^{\frac{n-1}{2}} \left( \frac{\partial u^sc}{\partial r} - i k_\beta u^sc \right) = 0, \quad r := |x|, \quad \beta = p, s,$$

where

$$u^sc = u^sc_p + u^sc_s, \quad u^sc_p := -\frac{1}{k_p^2} \nabla \cdot (\nabla \cdot u^sc), \quad u^sc_s := \begin{cases} \frac{1}{k_p^2} \nabla \times \nabla \times u^sc & (3D) \\ \frac{1}{k_s} \text{curl} \text{curl} u^sc & (2D) \end{cases},$$

and

$$k_p := \frac{\omega}{\sqrt{2\mu + \lambda}}, \quad k_s := \frac{\omega}{\sqrt{\mu}}.$$ 

(1.5)

In (1.4), the two-dimensional operators $\text{curl}$ and $\text{curl}$ are defined respectively by

$$\text{curl} u = \partial_1 u_2 - \partial_2 u_1, \quad \text{curl} u = (\partial_3 u, -\partial_1 u)^T,$$

with $u = (u_1, u_2)$ and $u$ being vector-valued and scalar functions, respectively. It is noted that (1.4) is the Helmholtz decomposition of the vector field $u^sc$, where $u^sc_p$ and $u^sc_s$ are respectively referred to as the compressional and shear parts of $u^sc$. This decomposition shall also play
an important role in our subsequent analysis. The Kupradze radiation condition characterizes the outward propagating nature of the scattered field $u^\infty$. The well-posedness of the scattering system (1.2) is known [32], and in particular there exists a unique solution $u \in \mathcal{H}_1^1(\mathbb{R}^n)$ which admits the following asymptotic expansions:

\[
\begin{align*}
  u^\infty_\beta(x) &= \frac{e^{i\beta r}}{r^{\frac{n-1}{2}}} \left( u^\infty_p(x)\hat{x} + O\left(\frac{1}{r}\right)\right), \quad \hat{x} := x/|x|, \quad \beta = p, s, \\
  u^\infty(x) &= \frac{e^{i\beta r}}{r^{\frac{n-1}{2}}} u^\infty_p(\hat{x})\hat{x} + \frac{e^{i\beta r}}{r^{\frac{n-1}{2}}} u^\infty_s(\hat{x})\hat{x} + O\left(\frac{1}{r^{\frac{n+1}{2}}}\right),
\end{align*}
\]

as $r = |x| \to \infty$, where $u^\infty_p$ and $u^\infty_s$ are both scalar functions defined on the unit sphere $S^{n-1} := \{x \in \mathbb{R}^n | |x| = 1\}$. Define the far-field pattern $u^\infty$ of $u^\infty_\beta$ as $u^\infty_\beta(\hat{x}) := u^\infty_p(\hat{x})\hat{x} + u^\infty_s(\hat{x})\hat{x}$, where $\hat{x}$ clearly has the unique correspondences: $u^\infty_\beta(\hat{x}) = u^\infty_p(\hat{x})\cdot \hat{x}$ and $u^\infty_\beta(\hat{x}) = u^\infty_s(\hat{x})\cdot \hat{x}$, moreover, due to the Rellich theorem [18], the correspondence between $u^\infty_p$ and $u^\infty_s$ is also one-to-one.

Define the scattering operator $\mathcal{S}$ as

\[
\mathcal{S} \left( (\Omega; \lambda, \mu, V), u^\prime \right) = u^\infty_p(\hat{x}), \quad \hat{x} \in S^{n-1},
\]

which is implicitly defined by the scattering system (1.2). It can be directly verified that $\mathcal{S}$ is nonlinear with respect to the argument $(\Omega; \lambda, \mu, V)$, whereas it is linear with respect to the argument $u^\prime$. An inverse problem of industrial importance in geophysical exploration and medical imaging is to recover $(\Omega; \lambda, \mu, V)$ by knowledge of $u^\infty_p$. In this paper, we are mainly concerned with the kernel space of $\mathcal{S}$, namely

\[
\mathcal{S} \left( (\Omega; \lambda, \mu, V), u^\prime \right) = 0,
\]

which corresponds to the physical scenario that no scattering, a.k.a. invisibility occurs. In particular, we consider the geometric structures of the kernel space $\text{Ker}(\mathcal{S})$, namely, the quantitative relationships between the configurations in $\text{Ker}(\mathcal{S})$ and the intrinsic geometries of $\Omega$. Intuitively, if non-scattering/invisibility occurs, one has nil scattering information from the far-field measurement and hence the inverse problem described above fails. However, it is much surprising that the geometric understanding of $\text{Ker}(\mathcal{S})$ cannot only provide quantitative understanding of how the waves behave inside the scattering object when invisibility occurs with respect to exterior observations, but can also provide new insights of the inverse problem (1.7) for both the derivation of new theoretical uniqueness results and the development of novel numerical reconstruction algorithms. Before discussing more about these aspects, we present the so-called elastic transmission eigenvalue problem, which shall provide a broader spectral perspective of studying the geometric structure of $\text{Ker}(\mathcal{S})$.

Consider the following PDE system for $v = (v_1)_{i=1}^N \in H^1(\Omega)^n$ and $w = (w_i)_{i=1}^N \in H^1(\Omega)^n$:

\[
\begin{align*}
  &\lambda \Delta v + (\lambda + \mu) \nabla \cdot v + \omega^2 v = \mathbf{0} & \text{in } \Omega, \quad \\
  &\lambda \Delta w + (\lambda + \mu) \nabla \cdot w + \omega^2 (1 + V) w = \mathbf{0} & \text{in } \Omega, \\
  &w = v, \quad T_n v = T_n w & \text{on } \partial \Omega,
\end{align*}
\]

\[
(1.9)
\]
with $\nu$ signifying the outward unit normal to $\partial \Omega$, and the boundary traction operator $T_\nu \mathbf{v}$ defined as

$$T_\nu \mathbf{v} = \begin{cases} 2\mu \partial_\nu \mathbf{v} + \lambda \nu \cdot (\nabla \cdot \mathbf{v}) + \mu (\partial_2 v_1 - \partial_1 v_2) \nu^\perp, & \text{for } n = 2, \\ 2\mu \partial_\nu \mathbf{v} + \lambda \nu \cdot (\nabla \cdot \mathbf{v}) + \mu \nu \times (\nabla \times \mathbf{v}), & \text{for } n = 3, \end{cases} \quad (1.10)$$

where for $n = 2$, $\nu^\perp \in \mathbb{R}^2$ denotes the unit vector obtained via rotating $\nu$ anti-clockwise by $\pi/2$. It is clear that $\mathbf{v} = \mathbf{w} \equiv 0$ are a pair of trivial solutions to (1.9). If for a certain $\omega \in \mathbb{R}^+$, there exists a pair of nontrivial solutions $(\mathbf{v}, \mathbf{w}) \in H^1(\Omega)^n \times H^1(\Omega)^n$ to (1.9), then $\omega$ is called an elastic transmission eigenvalue and $(\mathbf{v}, \mathbf{w})$ is referred to as the corresponding pair of elastic transmission eigenfunctions. The connection between Ker($S$) and the transmission eigenvalue problem (1.9) can be described as follows. If $u^c \equiv 0$, one clearly has $u^c = 0$ in $\mathbb{R}^n \setminus \Omega$. This in turn yields that $\mathbf{u} = \mathbf{u'}|_{\partial \Omega}$ for the scattering system (1.2). In such a case, one can show that $\mathbf{v} = (\mathbf{v'}|_{\partial \Omega})$ and $\mathbf{w} = (\mathbf{u'}|_{\partial \Omega})$ are a pair of transmission eigenfunctions. However, if $(\mathbf{v}, \mathbf{w})$ are a pair of transmission eigenfunctions, one cannot directly have the non-scattering/invisibility unless $\mathbf{v}$ can be (analytically) extended to an entire solution to (1.1) to generate an incident field. Nevertheless, $(\mathbf{v}, \mathbf{w})$ is located in any small neighborhood of Ker($S$) in the sense that $\mathbf{v}$ and $\mathbf{w}$ can respectively approximate $\mathbf{u'}$ and $\mathbf{u}$ in $\Omega$ within $\varepsilon$-accuracy for any $\varepsilon > 0$ such that the corresponding scattering amplitude is also of order $\varepsilon$. This viewpoint has been verified in [8, 11, 34] for the acoustic scattering governed by the Helmholtz equation, and one should be able to show similar results for the elastic scattering by following a similar spirit. However, this is beyond the scope the current study, and we choose to present it somewhere else in our future paper.

Due to its physical significance, the transmission eigenvalue problems have been extensively and extensively investigated in the literature, especially associated with the acoustic and electromagnetic scattering. It turns out that the transmission eigenvalue problems are non-elliptic and non-selfadjoint, and the corresponding mathematical study is highly challenging and intriguing, which is of significant interest in its own sake for the spectral theory of partial differential operators. We refer to [14, 19, 36] for historical accounts and surveys on the existing developments of the transmission eigenvalue problems. It is pointed out that the spectral study is less touched for the elastic transmission eigenvalue problems than that for the acoustic and electromagnetic transmission eigenvalue problems. Moreover, the existing results are concerned more about the spectral properties of the transmission eigenvalues and there are much fewer results on the intrinsic properties of the transmission eigenfunctions. Recently, there are considerable efforts in the literature in unveiling the distinct geometric structures of the transmission eigenfunctions. In [11], a local geometric structure was discovered for the acoustic transmission eigenfunctions, showing that they are generically vanishing around a corner on $\partial \Omega$. The discovery is motivated by the relevant study [8, 13, 42] which verified that if a scattering potential possesses a corner, it scatters every incident field non-trivially and stably. The vanishing property strongly depends on the regularity of the transmission eigenfunctions, and has been established under two regularity conditions. The first one is the Hölder continuity of the transmission eigenfunctions locally around the corner. This means that the transmission eigenfunctions should be more regular than $H^1$, say by the standard Sobolev embedding, $W^{1,\infty} \subset \subset W^{\alpha,\infty}$ with $\alpha \in (0, 1)$. On the other hand, as evidenced by the numerical study [6], there are indeed cases where the transmission eigenfunctions are not vanishing and instead they are localizing, especially for certain concave corners. In order to gain more insightful understanding of the regularity effect, another criterion was introduced in terms of the growth rate of the Herglotz densities which arise from the Fourier extension of the transmission eigenfunctions [11, 23, 25]. There are several further studies on the locally vanishing property of...
the transmission eigenfunctions in different physical scenarios [5, 7, 10, 12, 15]. In [17, 24], a global rigidity property is discovered, showing that the transmission eigenfunctions tend to localize on $\partial \Omega$. Both the local and global geometric structures of transmission eigenfunctions can produce interesting and significant applications. First, they are of fundamental importance to the invisibility cloaking which is a Frontier technology [29, 39]. Second, they have been used to establish novel unique identifiability results for the inverse scattering problems by a single far-field measurement [5, 7–10, 15, 16, 25, 37, 38], which constitutes a longstanding problem in the inverse scattering theory [18, 40]. Furthermore, in [17], a super-resolution wave imaging scheme was developed by making use the geometric properties of the transmission eigenfunctions.

However, most of the existing studies discussed above are concerned with the transmission eigenfunctions associated with the acoustic or electromagnetic scattering. There is very limited study on the elastic transmission eigenfunctions due to their more complicated physical and mathematical nature. In [7], the authors proved that the elastic transmission eigenfunctions locally vanish around a corner under the condition that the eigenfunctions are Hölder continuous in the domain. The result was applied to deriving a novel uniqueness result in determining an unknown elastic source from its far-field pattern. In this paper, we shall provide a much more comprehensive study of this important geometric property for the elastic transmission eigenfunctions. First, we shall consider a more general formulation of the elastic transmission eigenvalue problem which includes (1.9) as a special case. Second, we establish the local vanishing property under both the Hölder continuity and the Fourier extension property of the generalised transmission eigenfunctions. Third, we apply the newly established results to deriving two novel unique identifiability results for the inverse elastic problem in determining the polygonal/polyhedral support of an inhomogeneous medium independent of its content. Here, we would like to mention in passing some related practical background on the inverse determinations in elasticity [27, 28]. Finally, we would like to briefly discuss the mathematical strategies that we develop to derive the results. In order to establish the geometric property, one needs to track the singularity of the transmission eigenfunctions (with respect to their smoothness) induced by the geometric singularity of the domain, namely the corner. To achieve that end, we develop a microlocal argument that can be localized around the corner. An integral identity involving the difference of the two transmission eigenfunctions as well as a special type of CGO (complex geometric optics) solutions is a critical ingredient in our study. Compared to most of the existing studies mentioned earlier, there is a boundary integral terms due to the generalized transmission conditions in our study, which together with the more complicated nature of the Lamé system, makes the related analysis and estimates highly technical and subtle. In tracking the order of the asymptotic parameter in the phase of the CGO solutions, we manage to derive the desired results.

1.2. Mathematical setup and summary of the main findings

Let $\eta \in L^\infty(\partial \Omega)$ be a real-valued function. Instead of (1.9), we consider the following generalized elastic transmission eigenvalue problem for $v = (v_i)_{i=1}^n, w = (w_i)_{i=1}^n \in H^1(\Omega)^n$,

$$
\begin{aligned}
\lambda \Delta v + (\lambda + \mu) \nabla \cdot v + \omega^2 v &= 0 \quad \text{in } \Omega, \\
\lambda \Delta w + (\lambda + \mu) \nabla \cdot w + \omega^2 (1 + V) w &= 0 \quad \text{in } \Omega, \\
w = v, \quad T_v v + \eta v = T_v w \quad \text{on } \Gamma,
\end{aligned}
$$

(1.11)

where $\Gamma \subset \partial \Omega$ is an open subset. We note that if $\eta \equiv 0$ and $\Gamma = \partial \Omega$, the transmission eigenvalue problem (1.11) is reduced to (1.9). Hence, we refer to (1.11) as the generalised
transmission eigenvalue problem. It is particular to note that the transmission condition
\( T_\nu v + \eta v = T_\nu w \) not only brings mathematical generalisation but also is physically meaningful. In fact, it is referred to as the conductive transmission condition in the context of electromagnetic scattering, which arises in effectively describing a thin layer of highly conducting coating [1, 16]. In (1.11), the generalised transmission condition can also be used to effectively describe a thin layer of highly lossy elastic coating. However, we shall not explore more about this point since it is not the focus of the current article. If \( \eta \equiv 0 \), the existence and properties of the transmission eigenvalues to (1.11) have been studied in [3, 4]. In this paper, we shall study the intrinsic geometric properties of the transmission eigenfunctions assuming their existence in the general case.

The major geometric finding can be sketched as follows. Let \( \Gamma = \Gamma^- \cup \Gamma^+ \), where \( \Gamma^\pm \) are two non-collinear/non-coplanar line segments or planes in 2D and 3D, respectively. That is, \( \Gamma^- \) and \( \Gamma^+ \) form a (non-degenerate) corner on \( \partial \Omega \). Under mild conditions on \( V \) and \( \eta \) as well as the necessary regularity requirements on \( v \) and \( w \) as discussed above, it is shown that \( v \) and \( w \) are vanishing around the corner. The 2D results are contained in theorems 2.1 and 2.3 and corollary 2.2 for \( \eta \neq 0 \) and \( \eta \equiv 0 \), respectively, whereas the corresponding 3D results are contained in theorem 3.1 and corollaries 3.2 and 3.3. According to our earlier discussion, the geometric results imply that when non-scattering or nearly non-scattering occurs, the incident and the total wave fields propagate in a peculiar manner that avoids the corner places of the inhomogeneous medium. Moreover, the results can be used to establish novel unique identifiability results the geometrical inverse elastic problem, which are contained in theorems 4.1 and 4.2.

Before proceeding further to prove our main result in \( \mathbb{R}^2 \) in section 2, we would like to summarize our main methodologies to provide the readers a global picture of our study. Consider the elastic transmission eigenfunctions \((v, w)\) fulfilling (1.11). We use the elastic Herglotz wave function to approximate \( v \) with a certain accuracy and kernel increasing property, which serves as a certain regularity characterization for \( v \in H^1(\Omega)^2 \). By virtue of the complex geometric optics (CGO) solution introduced in [7], we establish the integral equality via the Green formula. The asymptotic decay of all integrals in the underlying integral equality with respect to the asymptotic parameter in the CGO is carefully studied, where we extract the leading order terms in the aforementioned asymptotic analysis. With the above preparations, we prove the vanishing property of the elastic transmission eigenfunction near a planar corner. For the 3D case, by using the dimensional reduction technique and similar to the 2D result, we can establish the local geometrical characterization of the elastic transmission eigenfunction near an edge corner in \( \mathbb{R}^3 \) under generic conditions, which shall be clearer from our subsequent analysis in section 3.

The rest of the paper is organized as follows. In sections 2 and 3, we present the studies in two and three dimensions respectively. Section 4 is devoted to the study of the inverse elastic problem.

2. Vanishing near corners of generalized elastic transmission eigenfunctions: two-dimensional case

In this section, we consider the geometric property of the generalized elastic transmission eigenfunction to (1.11) in two dimensions. First, we introduce the geometric setup of our study. For \( x = (x_1, x_2)^\top \in \mathbb{R}^2 \), the polar coordinate of \( x \) is given by \( x = (r \cos \theta, r \sin \theta)^\top \). Denote an open sector \( W \subseteq \mathbb{R}^2 \) and its boundary \( \Gamma^\pm \) as follows:
\[ W = \{ x \in \mathbb{R}^2 | x \neq 0, \quad \theta_m < \arg(x_1 + ix_2) < \theta_M \}, \]
\[ \Gamma_- = \{ x \in \mathbb{R}^2 | x \neq 0, \quad \arg(x_1 + ix_2) = \theta_m \}, \]
\[ \Gamma_+ = \{ x \in \mathbb{R}^2 | x \neq 0, \quad \arg(x_1 + ix_2) = \theta_M \}, \]  
(2.1)

where \(-\pi < \theta_m < \theta_M < \pi\). Let \(B_h\) and \(B_e\) denote open disks centered at \(0\) of radii \(h \in \mathbb{R}_+\) and \(\varepsilon \in \mathbb{R}_+\) with \(\varepsilon < h\), respectively. In the sequel, we set

\[ S_h = W \cap B_h, \quad \Gamma^\pm_h = \Gamma^\pm \cap B_h, \quad S_h^\pm = \mathbb{W} \cap \overline{B_h}, \quad \text{and} \quad \Lambda_h = \partial S_h \setminus (\Gamma^+_h \cup \Gamma^-_h). \]  
(2.2)

The elastic Herglotz wave function \(\mathbf{v}_g\) in \(\mathbb{R}^2\) is defined by

\[ \mathbf{v}_g = e^{-i \mathbf{z} \cdot \mathbf{d}} \int_{\mathcal{B}_k} \left\{ \sqrt{k_p} \frac{e^{i k_p \mathbf{d} \cdot \mathbf{x}} g_p(d) d + \sqrt{k_i} \frac{e^{i k_i \mathbf{d} \cdot \mathbf{x}} g_i(d) d^{-1}}{\omega} \right\} d\sigma(d), \]  
(2.3)

where the kernel \(g = (g_p, g_i)^T\) satisfies \(g_p, g_i \in L^2(S^1)\), \(d, d^{-1} \in S^1\), and \(d \perp d^{-1}\).

**Lemma 2.1** [2, theorem 3.4]. Let \(D \subset \mathbb{R}^2\) be a bounded Lipschitz domain with a connected complement. Then the set of elastic Herglotz wave functions is dense with respect to the \(H^1(D)\)-norm in the set of solutions to the Lamé system

\[ \mathcal{L}v + \omega^2 v = 0, \quad \mathcal{L} := \lambda \Delta + (\mu + \mu) \nabla (\nabla \cdot) \quad \text{in} \quad D. \]  
(2.4)

By virtue of lemma 2.1, for any generalized elastic transmission eigenfunctions \((v, w) \in H^1(\Omega)^2 \times H^1(\Omega)^2\) to (1.11), there exists a sequence of elastic Herglotz wave functions \(v_j\) given by

\[ v_j(\mathbf{x}) = e^{-i \mathbf{z} \cdot \mathbf{d}} \int_{\mathcal{B}_k} \left\{ \sqrt{k_p} \frac{e^{i k_p \mathbf{d} \cdot \mathbf{x}} g_p(d) d + \sqrt{k_i} \frac{e^{i k_i \mathbf{d} \cdot \mathbf{x}} g_i(d) d^{-1}}{\omega} \right\} d\sigma(d), \]  
(2.5)

which can approximate \(v\) to an arbitrary accuracy in \(H^1(\Omega)^2\). It is clear that \(v_j\) can be regarded as the Fourier extension of \(v\). Let

\[ v_j(\mathbf{x}) = v^R_j(\mathbf{x}) + iv^I_j(\mathbf{x}), \]  
(2.6)

where \(v^R_j(\mathbf{x})\) and \(v^I_j(\mathbf{x})\) are real valued functions.

In what follows, we shall split the real and imaginary parts of the elastic transmission eigenfunctions \((v, w)\) to (1.11) as

\[ v = v_R + iv_I, \quad w = w_R + iw_I. \]  
(2.7)

It is straightforward to verify that both \((v_R, w_R)\) and \((v_I, w_I)\) satisfy (1.11). In a similar manner, we let the real and imaginary part of the kernel functions \(g_{\beta}(d)\) \((\beta = p, s)\) of (2.5) be given by

\[ g_{\beta}(d) = g^R_{\beta}(d) + ig^I_{\beta}(d). \]  
(2.8)
Proposition 2.1. Let the elastic Herglotz wave function \( v_j \) be defined by (2.5). Denote

\[
\begin{align*}
    v^{R}_{j\mu}(0) &= \frac{1}{\sqrt{2}} \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^R_{j\mu}(d) \, d\sigma(d), \\
v^{I}_{j\mu}(0) &= \frac{1}{\sqrt{2}} \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^I_{j\mu}(d) \, d\sigma(d),
\end{align*}
\]

Then

\[
v_j(0) = v^{R}_{j\mu}(0) + iv^{I}_{j\mu}(0) := (v^{R}_{j\mu}(0) + v^{I}_{j\mu}(0)) + i(v^{R}_{j\mu}(0) - v^{I}_{j\mu}(0)),
\]

where \( g^R_{j\mu}(d) \) and \( g^I_{j\mu}(d) \) are defined in (2.8). Let \( J_{\ell}(t) \) be the \( \ell \)th Bessel function of the first kind for \( \ell \in \mathbb{N} \cup \{0\} \). Denote \( J_{\ell,\beta} = J_{\ell}(|k_\beta| |x|) \), \( \beta = p, s \). Furthermore, we have

\[
v^{R}_{j\mu}(x) = v^{R}_{j\mu}(0)J_{0,\mu} + v^{I}_{j\mu}(0)J_{0,0} + v^{R}_{j\mu}(0)J_{0,0} + v^{I}_{j\mu}(0)J_{0,0},
\]

and

\[
v^{I}_{j\mu}(x) = v^{I}_{j\mu}(0)J_{0,0} - v^{R}_{j\mu}(0)J_{0,0} - v^{I}_{j\mu}(0)J_{0,0},
\]

where \( p_\mu = d \), and \( p_\sigma = d_\perp \), and

\[
\begin{align*}
    A^{(0)}_{\beta,\lambda} &= \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^R_{j\mu}(d) \cos(2\ell \theta) p_\mu \, d\sigma(d) + \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^I_{j\mu}(d) \cos(2\ell \theta) p_\lambda \, d\sigma(d), \\
    A^{(0)}_{\beta,2} &= \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^R_{j\mu}(d) \cos((2\ell - 1) \theta) p_\mu \, d\sigma(d) - \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^I_{j\mu}(d) \cos((2\ell - 1) \theta) p_\mu \, d\sigma(d), \\
    A^{(0)}_{\beta,3} &= \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^R_{j\mu}(d) \cos(2\ell \theta) p_\mu \, d\sigma(d) - \int_{\Omega} \sqrt{\frac{k_\beta}{\omega}} g^I_{j\mu}(d) \cos(2\ell \theta) p_\mu \, d\sigma(d), \quad \beta = p \text{ or } s,
\end{align*}
\]

and \( \theta \) is the angle between \( x \) and \( d \) in (2.5). It holds that
\[ |A_{jp,i}^{(l)}| \leq 2 \sqrt{\frac{k_p \pi}{\omega}} \|g_jp\|_{L^2(S^1)}, \quad |A_{jp,i}^{(l)}| \leq 2 \sqrt{\frac{k_p \pi}{\omega}} \|g_jp\|_{L^2(S^1)}, \quad i = 1, \ldots, 3. \]

(2.13)

We shall make use of the complex geometrical optics solution (CGO) \( u(x) \) introduced in [7]. We next review some quantitative properties of \( u(x) \), which shall be used in our subsequent analysis.

**Lemma 2.2 [7, proposition 3.1].** Let \( \Omega \subset \mathbb{R}^2 \) such that \( \Omega \cap (\mathbb{R}^2 \cup \{0\}) = \emptyset \). Denote

\[ u(x) = \begin{pmatrix} \exp(-s \sqrt{z}) \\ i \exp(-s \sqrt{z}) \end{pmatrix} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad x = (x_1, x_2) \top, \quad (2.14) \]

where \( z = x_1 + ix_2 \) and \( s \in \mathbb{R}_+ \). The complex square root is defined as

\[ \sqrt{z} = \sqrt{|z|} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right), \]

where \(-\pi < \theta \leq \pi\) is the argument of \( z \). Then \( u \) satisfies

\[ \mathcal{L}u = 0 \quad \text{in } \Omega. \]

Let the open sector \( W \) be defined in (2.1). Then

\[ \int_W u_1(x)dx = 6i(e^{-2\theta M} - e^{-2\theta m})s^{-4}. \]

(2.15)

In addition for \( \alpha, h > 0 \) and \( j \in \{1, 2\} \) we have the upper bounds

\[ \int_W |u_j(x)||x|^{\alpha} dx \leq \frac{2(\theta M - \theta m)\Gamma(2\alpha + 4)}{\delta_w^{\alpha+4}} s^{-2\alpha - 4}, \]

(2.16)

and

\[ \int_{W \backslash \delta_w} |u_j(x)| dx \leq \frac{6(\theta M - \theta m)}{\delta_w} s^{-4} e^{-2\theta \sqrt{\Theta} \delta_w}, \]

(2.17)

where \( \delta_w = \min_{0 < \theta < \theta M} \cos \left( \frac{\theta}{2} \right) \) is a positive constant.

The following lemma states the regularity of the CGO solution \( u(x) \) defined in (2.14), where the proof follows from a similar argument to that of [25, lemma 2.3] and we skip the details.

**Lemma 2.3.** Let \( S_h \) be defined in (2.2) and \( u(x) \) be given in (2.14). Then \( u(x) \in H^1(S_h)^2 \) and \( \mathcal{L}u(x) = 0 \) in \( S_h \). Furthermore, it holds that

\[ \|u(x)\|_{L^2(S_h)^2} \leq \sqrt{\theta M - \theta m} e^{-2\sqrt{\Theta} \delta_w \Theta} h, \]

(2.18)

and

\[ \|x|^\alpha |u(x)|\|_{L^2(S_h)^2} \leq s^{-2(\alpha+1)} \frac{2(\theta M - \theta m)\Gamma(4\alpha + 4)}{(2\delta_w)^{2\alpha+2}}, \]

(2.19)

where \( \Theta \in [0, h] \) and \( \delta_w \) is defined in (2.17).
By directly calculations, we can prove lemma 2.4. One can find the detailed proof of this lemma in the arXiv version of this paper [26].

**Lemma 2.4.** Suppose that $Λ_h$ and $\mathbf{u}(\mathbf{x})$ are defined by (2.2) and (2.14) respectively. Recall that $\delta_W > 0$ is given in (2.17). We have

$$
\|\mathbf{u}(\mathbf{x})\|_{H^1(\Lambda_h)} \leq \sqrt{\frac{h}{2}} \sqrt{\theta_M - \theta_m} e^{-s\sqrt{\theta_M}}.
$$

both of which decay exponentially as $s \to +\infty$.

We proceed to derive several key lemmas in order to establish the main geometric result of this section. It is first recalled the following Green formula for the Lamé operator; see [20, lemma 3.4] and [41, theorem 4.4].

**Lemma 2.5.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Let $\mathbf{u}_1 \in H^1(\Omega)^\nu$ and $\mathbf{v}_1 \in H^1(\Omega)$ satisfying $\mathcal{L}\mathbf{u}_1 \in L^2(\Omega)^\nu$ and $\mathcal{L}\mathbf{v}_1 \in L^2(\Omega)^\nu$. The following Green identity holds

$$
\int_{\Omega} (\mathcal{L}\mathbf{u}_1 \cdot \mathbf{v}_1 - \mathcal{L}\mathbf{v}_1 \cdot \mathbf{u}_1) \, d\mathbf{x} = \int_{\partial\Omega} (\mathcal{T}_\nu \mathbf{u}_1 \cdot \mathbf{v}_1 - \mathcal{T}_\nu \mathbf{v}_1 \cdot \mathbf{u}_1) \, d\mathbf{u}.
$$

Recall the splitting (2.7). In what follows, we shall mainly focus on establishing the relevant results for $(\mathbf{v}_R, \mathbf{w}_R)$. Due to the symmetric role of $(\mathbf{v}_R, \mathbf{w}_R)$ and $(\mathbf{v}_i, \mathbf{w}_i)$, those results hold equally for $(\mathbf{v}_i, \mathbf{w}_i)$, and hence $(\mathbf{v}, \mathbf{w})$.

According to (1.11), using (2.21), one can establish the key integral equality (2.22) in lemma 2.6, where the corresponding detailed proof can be referred to the arXiv version [26] of this paper.

**Lemma 2.6.** Let $\mathbf{v}_R \in H^1(\Omega)^\nu$ and $\mathbf{w}_R \in H^1(\Omega)^\nu$ be a pair of generalized elastic transmission eigenfunctions to (1.11). Let the CGO solution $\mathbf{u}$ and the elastic Herglotz wave function $\mathbf{v}_R^\ell$ be defined in (2.14) and (2.11) respectively. Assume that the Lipschitz domain $\Omega \subset \mathbb{R}^3$ contains a corner $S_h \subset \Omega \cap W$, where $S_h$ is defined in (2.2) and $W$ is a sector defined in (2.1). Denote $q = 1 + V$, where $V$ is defined in (1.11). Then the following integral equality holds

$$
I_1 + I_2 = I_\Lambda - I_\pm - I_\pm^\ell,
$$

where

$$
I_1 = \omega^2 \int_{S_h} (q \mathbf{w}_R - \mathbf{v}_R^\ell) \cdot \mathbf{u}(\mathbf{x}) \, d\mathbf{x}, \quad I_2 = -\omega^2 \int_{S_h} (\mathbf{v}_R - \mathbf{v}_R^\ell) \cdot \mathbf{u}(\mathbf{x}) \, d\mathbf{x},
$$

$$
I_\Lambda = \int_{\Lambda_h} (\mathcal{T}_\nu (\mathbf{v}_R - \mathbf{w}_R)) \cdot \mathbf{u} - (\mathcal{T}_\nu \mathbf{u}) \cdot (\mathbf{v}_R - \mathbf{w}_R) \, d\mathbf{u},
$$

$$
I_\pm = \int_{\Gamma_\pm^W} \eta \mathbf{u} \cdot \mathbf{v}_R^\ell \, d\mathbf{u}, \quad I_\pm^\ell = \int_{\Gamma_\pm^W} \eta \mathbf{u} \cdot (\mathbf{v}_R - \mathbf{v}_R^\ell) \, d\mathbf{u}.
$$

Here $\Lambda_h$ and $\Gamma_\pm^W$ are defined in (2.2).
By using the Hölder inequality, lemma 2.4, and the trace theorem, one has

**Lemma 2.7.** Let $I_{A}$ be defined in (2.22). Under the same setup as that in lemma 2.6, we have the following estimate

\[
|I_{A}| \leq C \frac{\sqrt{2h + s^2} + \mu s}{\sqrt{2}} \sqrt{\theta_M - \theta_m} e^{-\gamma \sqrt{c_w}} \|v_R - w_R\|_{H^1(S_h^2)},
\]

where $C$ is a positive constant coming from the trace theorem, $S$ and $\delta_w > 0$ are defined in (2.2) and (2.17), respectively.

The proof of lemma 2.8 is based on the Cauchy–Schwarz inequality, the trace theorem together with lemma 2.3. We choose to skip the details and refer to the arXiv version [26] for the interested readers.

**Lemma 2.8.** Under the same setup as that in lemma 2.6, we further suppose that the boundary parameter $\eta$ of (1.11) satisfies $\eta \in C^4(\Gamma_\eta^R)$ for $0 < \alpha < 1$. For any given constants $\beta_1, \beta_2$ and $\gamma$ satisfying $\gamma > \max\{\beta_1, \beta_2\} > 0$, assume that there exits a sequence of $\{v_j^R\}_{j=1}^\infty$ defined by (2.6) with kernels $g_{j\cdot}$ and $g_{\cdot\cdot}$ can approximate $v_R$ in $H^1(S_h)$ fulfilling

\[
\|v_R - v_j^R\|_{H^1(S_h^2)} \leq \tilde{f}^{-\gamma}, \quad \|s_{j\cdot}\|_{L^2(S_h^2)} \leq \tilde{f}^{\beta_1}, \quad \|s_{\cdot\cdot}\|_{L^2(S_h^2)} \leq \tilde{f}^{\beta_2}. \tag{2.24}
\]

Recall that $I_2$ and $I_{\pm}$ are defined in (2.22). Then the following integral estimates hold:

\[
|I_2| \leq \omega^2 h \sqrt{\theta_M - \theta_m} e^{-\gamma \sqrt{c_w}} f^{-\gamma}, \tag{2.25}
\]

and

\[
|I_{\pm}| \leq \left( |\eta(0)|h \sqrt{\theta_M - \theta_m} e^{-\gamma \sqrt{c_w}} + \|\eta\|_{C^4(\Gamma_\eta^R)} \sqrt{\frac{2}{(\delta_w)^{2\alpha+2} s^{-2\alpha-1}} f^{-\gamma}} \right)f^{-\gamma}, \tag{2.26}
\]

where $\Theta \in [0, h]$, $\delta_w$ is defined in (2.17), $\theta_m$ and $\theta_M$ are defined in (1.11).

Under the Hölder continuous assumption on $q w_R(x)$ near the corner $S_h$, by lemma 2.2, we can obtain the following lemma. The detailed proof of lemma 2.9 can be found in the arXiv version [26] of this paper.

**Lemma 2.9.** Under the same setup as that in lemma 2.8, we further suppose that $q w_R \in C^\alpha(S_h^2)$ $(0 < \alpha < 1)$ and hence

\[
f_R(x) := q w_R(x) = f_{R_1}(x) + \delta f_R(x), \quad |\delta f_R| \leq \|f_R(x)\|_{C^\alpha(S_h^2)} |x|^\alpha. \tag{2.27}
\]

Then the following integral estimate holds

\[
|I_1| \leq \omega^2 \left( 4 \left( \sqrt{\frac{\mu k_p}{\omega}} (1 + k_p) \|g_{j\cdot}\|_{L^2(S_h^2)} + \sqrt{\frac{\mu k_p}{\omega}} (1 + k_p) \|g_{\cdot\cdot}\|_{L^2(S_h^2)} \right) \times \frac{(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_w^{2\alpha+4} s^{-2\alpha-4}} + \|f_R(x)\|_{C^\alpha(\Omega_\eta^R)} \frac{2\sqrt{2}(\theta_M - \theta_m)\Gamma(4\alpha + 4)}{\delta_w^{2\alpha+4} s^{-2\alpha-4}} \right.
\]

\[
+ 6 \sqrt{2} \left( \|f_{R_1}(0)\| + |v_j^R(0)| \right) e^{-2\delta_w^2} - e^{-2\delta_w x} |x|^{-4} \right), \tag{2.28}
\]
Lemma 2.10. Consider the same setup as that in lemma 2.6 and suppose that \( \eta \in C^\infty(\Gamma_h^\pm) \) and has expansion at the origin as

\[
\eta(x) = \eta(0) + \delta \eta(x), \quad |\delta \eta(x)| \leq ||\eta||_{C^\infty}|x|^\alpha.
\]

Recall that \( I_h \) is defined (2.22) and denote

\[
I_h = I_1^+ + \eta(0)I_2^+,
\]

where \( I_1^+ = \int_{\Gamma_h^+} \delta \eta \cdot v_R^j \, d\sigma \) and \( I_2^+ = \int_{\Gamma_h^+} \eta \cdot v_R^j \, d\sigma \). Then the following estimate holds

\[
|I_1^+| \leq O(s^{-2\alpha+2}) + (\|g_{p'}\|_{L^2(\Omega^j)} + \|g_{p''}\|_{L^2(\Omega^j)}) \times O(s^{-2\alpha+4}) \quad \text{as} \ s \to +\infty.
\]

Lemma 2.11. Consider the same setup as that in lemma 2.6 and recall that \( I_2^\pm \) is defined (2.30). When \( s \to +\infty \), the following results hold

\[
I_2^{\pm (p)} + I_2^{\pm (s)} = \eta(0) \left( v_{jj}'(0) + v_j'(0) + v_j''(0) \right) \cdot \left( \frac{1}{\sqrt{s}} \right)
\]

\[
\times 2s^{-2} \left( \mu(\theta_M)^{-2} - \mu(\theta_M)^{-2}e^{-s\sqrt{\theta_M}(\theta_M)} - \mu(\theta_M)^{-1}s\sqrt{\theta_M}e^{-s\sqrt{\theta_M}(\theta_M)} \right),
\]

\[
|I_2^{\pm (p)}| \leq O(s^{-6}), \quad |I_2^{\pm (s)}| \leq O(s^{-6}),
\]

where

\[
I_2^{\pm (p)} = \int_{\Gamma_h^\pm} \eta \cdot v_{jj}^p(0) \, d\sigma, \quad I_2^{\pm (s)} = \int_{\Gamma_h^\pm} \eta \cdot v_j^s(0) \, d\sigma,
\]

Denote

\[
I_2^{\pm (j)} = \int_{\Gamma_h^\pm} \eta \cdot v_j^j(0) \, d\sigma, \quad I_2^{\pm (i)} = \int_{\Gamma_h^\pm} \eta \cdot v_i^j(0) \, d\sigma,
\]

\[
I_2^{\pm (j)} = \sqrt{2\sum_{\ell=1}^{+\infty} (-1)^{\ell} \int_{\Gamma_h^\pm} \eta \cdot A^\ell_j(\ell) J_{2\ell}(k_\ell ||x||) \, d\sigma},
\]

\[
I_2^{\pm (j)} = \sqrt{2\sum_{\ell=1}^{+\infty} (-1)^{\ell} \int_{\Gamma_h^\pm} \eta \cdot A^\ell_j(\ell) J_{2\ell-1}(k_\ell ||x||) \, d\sigma}.
\]
It holds that
\[
T_{214}^{\pm(p)} \leq O(s^{-\delta}), \quad T_{214}^{\pm(t)} \leq O(s^{-\delta}), \\
T_{22}^{\pm(p)} \leq \|g_{jp}\|_{L^2(S^1)} \times O(s^{-4}), \quad T_{22}^{\pm(t)} \leq \|g_{jp}\|_{L^2(S^1)} \times O(s^{-4}), \\
T_{23}^{\pm(p)} \leq \|g_{jp}\|_{L^2(S^1)} \times O(s^{-4}), \quad T_{23}^{\pm(t)} \leq \|g_{jp}\|_{L^2(S^1)} \times O(s^{-4}),
\]
as \(s \to +\infty\).

**Theorem 2.1.** Let \((\mathbf{v}, \mathbf{w}) \in H^1(\Omega)^2 \times H^1(\Omega)^2\) be a pair of eigenfunctions to (1.11) associated with \(\omega \in \mathbb{R}_+\). Assume that the domain \(\Omega \subset \mathbb{R}^2\) contains a corner \(\Omega \cap B_\delta \subset \Omega \cap W\) with \(h \ll 1\). By rigid motions if necessary, we can assume that the vertex of the corner is \(0 \in \partial \Omega\). Let \(W\) be the sector defined in (2.1) and \(S_h = \Omega \cap B_\delta \subset \Omega \cap W\) in \(\Omega\). Suppose that \(qw \in \mathcal{C}_0^\infty(S_h)^2\) with \(q := 1 + V\) and \(\eta \in \mathcal{C}_0^\infty(\Gamma_h^0)\) for \(0 < \alpha < 1\). If the following conditions are fulfilled:

(a) For any given constants \(\gamma > \max\{\beta_1, \beta_2\} > 0\), assume that there exits a sequence Herglotz functions \(\{v_j\}_{j=1}^\infty\), where \(v_j\) is defined in (2.5), can approximate \(v\) in \(H^1(S_h)^2\) fulfilling
\[
\|v - v_j\|_{H^1(S_h)^2} \leq j^{-\gamma}, \quad \|g_{jp}\|_{L^2(S^1)} \leq j^\beta, \quad \|g_{jp}\|_{L^2(S^1)} \leq j^\beta,
\]

(2.33)

(b) The function \(\eta(x)\) does not vanish at the corner point, i.e.
\[
\eta(0) \neq 0,
\]

(2.34)

(c) The corner is non-degenerate, namely the angles \(\theta_m\) and \(\theta_M\) of the sector \(W\) satisfy
\[
-\pi < \theta_m < \theta_M < \pi \quad \text{and} \quad \theta_M - \theta_m \neq \pi;
\]
then we have
\[
\lim_{\rho \to +0} \frac{1}{m(B(0, \rho) \cap \Omega)} \int_{B(0, \rho) \cap \Omega} |v(x)| dx = 0,
\]

(2.36)
where \(m(B(0, \rho) \cap \Omega)\) is the measure of \(B(0, \rho) \cap \Omega\).

**Proof.** As remarked earlier, we shall make use of the splitting (2.7) and it is sufficient for us to show that \(v_{\rho}\) satisfies the geometric property (2.36). First, it is easy to see that \(qw_{\rho} \in \mathcal{C}_0^\infty(S_h)^2\) and \(v_{\rho} \in H^1(S_h)^2\) can be approximated by \(\{v_j\}_{j=1}^\infty\) defined in (2.6) satisfying (2.24). Therefore the assumptions in lemmas 2.6–2.11 are fulfilled.

Substituting (2.30) and (2.32) into (2.22) and rearranging the terms, we have
\[
\eta(0) \left( v_{\rho}^p(0) + v_{\rho}^i(0) + v_j^p(0) + v_j^i(0) \right) \cdot \left( 1 \right)
\]

\[
\times 2s^{-2} \left( \mu(\theta_m)^{-2} - \mu(\theta_M)^{-2} - e^{-\sqrt{\pi} \theta_m} - e^{-\sqrt{\pi} \theta_M} \right) + \mu(\theta_m)^{-2} - \mu(\theta_M)^{-2} - e^{-\sqrt{\pi} \theta_m} - e^{-\sqrt{\pi} \theta_M}
\]

\[
= -\eta(0)\left( T_{22}^{\pm(p)} + T_{22}^{\pm(i)} + T_{23}^{\pm(p)} + T_{23}^{\pm(i)} + T_{211}^{\pm(p)} + T_{211}^{\pm(i)} + T_{214}^{\pm(p)} + T_{214}^{\pm(i)} \right)
\]

\[
- I_1 + I_2 - I_3 + I_4.
\]

(2.37)
Multiplying $s^2$ on the both side of (2.37), by virtue of (2.23), (2.25), (2.26), (2.28), (2.30), (2.31), (2.32) and (2.24), and letting $s = j^{\beta_1/2} \max \{\beta_1, \beta_2\} < \theta < \gamma$ with $j \to +\infty$, we have

$$
\lim_{j \to \infty} \eta(0) \left( v_R^R(0) + v_R^I(0) + v_I^R(0) + v_I^I(0) \right) \cdot \left( \frac{1}{i} \right) \left( \mu^{-2}(\theta_m) + \mu^{-2}(\theta_M) \right) = 0,
$$

(2.38)

which further implies that

$$
\lim_{j \to \infty} \eta(0) v_R^R(0) \left( \mu^{-2}(\theta_m) + \mu^{-2}(\theta_M) \right) = 0.
$$

(2.39)

Here we use the fact that

$$
v_R^R(0) = v_R^R(0) + v_R^I(0) + v_I^R(0) + v_I^I(0),
$$

according to proposition 2.1.

Under the condition (2.35), from [25, lemma 2.10], we know that

$$
\mu^{-2}(\theta_m) + \mu^{-2}(\theta_M) \neq 0.
$$

(2.40)

Since $\eta$ is a real valued function, by virtue of (2.34) and (2.40), from (2.39), one has

$$
\lim_{j \to \infty} v_R^R(0) \cdot \left( \frac{1}{i} \right) = 0
$$

(2.41)

which implies that

$$
\lim_{j \to \infty} v_R^R(0) = 0.
$$

(2.42)

Finally, it can be directly deduced that

$$
\lim_{\rho \to 0} \frac{1}{m(B(0, \rho) \cap \Omega)} \int_{B(0, \rho) \cap \Omega} |v_R(x)| \, dx
\leq \lim_{j \to \infty} \lim_{\rho \to 0} \frac{1}{m(B(0, \rho) \cap \Omega)} \int_{B(0, \rho) \cap \Omega} |v_R(x) - v_R^R(x)| \, dx
\quad + \lim_{\rho \to 0} \frac{1}{m(B(0, \rho) \cap \Omega)} \int_{B(0, \rho) \cap \Omega} |v_R^R(x)| \, dx.
$$

(2.43)

Combining (2.24) and (2.42) with (2.43), one can prove (2.36).

The proof is complete.

□

Remark 2.1. We would like to point out that the Fourier extension property (2.33) can be generalized as follows

$$
\|v - v_j\|_{L^2(\Omega)} \leq \phi_1(j), \quad \|g_{\rho,0}\|_{L^2(\Omega)} \leq \frac{1}{\phi_2(j)}, \quad \|g_\mu\|_{L^2(\Omega)} \leq \frac{1}{\phi_3(j)}.
$$

(2.44)
where $\phi_{\ell}(j) \in \mathbb{R}_+$ are strict decreasing functions with respect to $j$ and $\lim_{j \to +\infty} \phi_{\ell}(j) = 0$, $\ell = 1, 2, 3$, satisfying

$$
(\phi_1(j))^\gamma \leq \phi_2(j) = \min\{\phi_2(j), \phi_3(j)\}, \quad 0 < \gamma < 1, \quad \forall j \in \mathbb{N}.
$$

In fact, by letting $s = \frac{1}{(\phi_1(j))^{\rho/2}}$, $\gamma < \rho < 1$, (2.45) one can show that $s^{-2} \|v - v_j\|_{H^1(S_h^2)} \leq (\phi_1(j))^{1-\rho}$, $s^{-2} \|g_{xp}\|_{L^2(S_1^1)} \leq (\phi_1(j))^{\rho - \gamma}$.

By virtue of (2.44), under the same setup of theorem 2.1, and using a similar argument as that in lemma 2.8, one can prove that

$$
\lim_{j \to +\infty} v_j(0) = 0
$$

by letting $j \to +\infty$. Therefore, we can prove (2.36) under the same setup of theorem 2.1, where the condition (2.24) is replaced by (2.44) and $q w \in C^\gamma(S_h^2)$.

We next consider the degenerate case of (1.11) with $\eta \equiv 0$ in (1.11). The proof of corollary 2.2 follows from the one of theorem 2.1 with some necessary modifications. One can establish the integral identity by noting $\eta \equiv 0$ in (2.22). Hence the asymptotic analysis with respect to the parameter $s$ in the CGO for each term in the integral equality can be derived. We skip the detailed proof of corollary 2.2, where one can find it in the arXiv version of this paper [26].

**Corollary 2.2.** Under the same setup as that in theorem 2.1 but with $\eta \equiv 0$, we have

$$
\lim_{\rho \to 0} \frac{1}{m(B(0, \rho) \cap \Omega)} \int_{B(0, \rho) \cap \Omega} |V(x)w(x)| \, dx = 0.
$$

**Remark 2.2.** The difference between theorem 2.1 and corollary 2.2 lies in their conclusions (2.36) and (2.48). If one further assumes in corollary 2.2 that $V(x) \in C(S_h)$ and $V(0) \neq 0$, it is easy to show that

$$
\lim_{\rho \to 0} \frac{1}{m(B(0, \rho) \cap \Omega)} \int_{B(0, \rho) \cap \Omega} |w(x)| \, dx = 0,
$$

which together with the transmission condition $v = w$ on $\Gamma$ further implies that (2.36) holds as well.
Finally, we establish the vanishing property under the Hölder regularity of the transmission eigenfunctions, which shall be useful for our study of the inverse elastic problem in section 4. Substituting the Hölder expansion of $v$ in (2.22), following a similar idea in the proof of theorem 2.1 to characterize the asymptotic analysis with respect to the parameter $s$ in the CGO for each term in the integral identity, we are able to show that $v$ must vanish near the planar corner. The detailed proof of theorem 2.3 is skipped, which can be found in the arXiv version of this paper [26].

**Theorem 2.3.** Let $(v, w) \in H^1(\Omega)^2 \times H^1(\Omega)^2$ be a pair of eigenfunctions to (1.11) associated with $\omega \in \mathbb{R}_+$. Let $W, S_h$ and $q$ be those described in theorem 2.1. Suppose that $v \in C^2(S_h^0)^2$, $qw \in C^2(S_h^0)^2$, $\eta \in C^\alpha(\Gamma_0^+)$ with $\eta(0) \neq 0$, for $\alpha \in (0, 1)$ and the corner $W$ is non-degenerate. Then we have

$$v(0) = 0. \quad (2.49)$$

**Remark 2.3.** Similar to corollary 2.2, under the same setup of theorem 2.3, for the degenerate case of (1.11) with $\eta \equiv 0$ in (1.11), if $V$ is Hölder-regular near the corner and $V(0) \neq 0$, one can prove that $v(0) = 0$, where $v \in H^1(\Omega)^2 \cap C^\infty(S_h^0)$ and $w \in H^1(\Omega)^2$ are a pair of the generalized elastic transmission eigenfunction to (1.11) associated with $\omega \in \mathbb{R}_+$ such that $qw \in C^\infty(S_h^0)$. We choose not to discuss the details in this paper. In [7, theorem 1.5], the corresponding vanishing property was established when $v - w \in H^2(\Omega)^2$, $V$ is Hölder continuous around the corner such that $V(0) \neq 0$, and either $v$ or $w$ is Hölder continuous around the corner. Compared with [7, theorem 1.5], the assumption $v - w \in H^2(\Omega)^2$ can be removed in our setting.

3. **Vanishing near corners of generalized elastic transmission eigenfunctions: three-dimensional case**

In this section, we establish the vanishing property of the generalized elastic transmission eigenfunctions for the 3D case. In principle, we could also consider a generic corner in the usual sense as the one for the 2D case. However, we consider a 3D edge corner described by $S_h \times (-M, M)$, where $S_h$ is defined in (2.2) and $M \in \mathbb{R}_+$. It is readily seen that $S_h \times (-M, M)$ actually describes an edge singularity. In what follows, we suppose that the Lipschitz domain $\Omega \subset \mathbb{R}^3$ with $0 \in \partial \Omega$ possesses a 3D edge corner. Let $0 \in \mathbb{R}^2$ be the vertex of $S_h$ and $x_3 \in (-M, M)$. Then $(0, x_3)$ is defined as an edge point of $S_h \times (-M, M)$.

In order to make use of the CGO solution $u(x)$ introduced in lemma 2.2 to study the vanishing property of $(v, w)$ to (1.11) at a 3D edge corner, we define the following dimension reduction operator.

**Definition 3.1.** Let $S_h \subset \mathbb{R}^2$ be defined in (2.2), $M > 0$. For a given function $g$ in the domain $S_h \times (-M, M)$. Pick up any point $x_3 \in (-M, M)$. Suppose that $\phi \in C^\infty_0((x_3 - L, x_3 + L))$ is a nonnegative function and $\phi \not\equiv 0$, where $L$ is sufficiently small such that $(x_3 - L, x_3 + L) \subset (-M, M)$, and write $x = (x', x_3) \in \mathbb{R}^3$, $x' \in \mathbb{R}^2$. The dimension reduction operator $\mathcal{R}$ is defined by
\[ R(g)(x') = \int_{x_3 - L}^{x_3 + L} \phi(x_3) g(x', x_3) dx_3, \quad (3.1) \]

where \( x' \in S_h \).

Before presenting the main results of this section, we first analyze the regularity of the functions after applying the dimension reduction operator. Using a similar argument of [5, lemma 3.4], we can prove the following lemma, whose detailed proof is omitted.

**Lemma 3.1.** Let \( g \in H^m(S_h \times (-M, M))^3 \), \( m = 1, 2 \). Then

\[ R(g)(x') \in H^m(S_h)^3. \]

Similarly, if \( g \in C^\alpha(\overline{S_h} \times [-M, M])^3 \), \( 0 < \alpha < 1 \), then

\[ R(g)(x') \in C^\alpha(\overline{S_h})^3. \]

The elastic Herglotz wave function \( v_g \) in \( \mathbb{R}^3 \) is defined by

\[ v_g(x) = \int_{\mathbb{S}^2} \{ e^{i\beta x \cdot d} g_{\rho}(d) + e^{i\beta x \cdot d} g_{\mu}(d) \} d\sigma(d), \quad (3.2) \]

where the kernel \( g = g_{\rho} + g_{\mu} \) with \( g_{\rho} \in L^2(\mathbb{S}^2)^3 \) and \( g_{\mu} \in L^2(\mathbb{S}^2)^3 \). Lemma 2.1 holds equally in the three dimensions. In view of this lemma, for any pair of the generalized elastic transmission eigenfunction \((v, w)\) to (1.11), there exists a sequence Herglotz wave function \( \{v_j\}_{j=1}^\infty \) defined by

\[ v_j(x) = \int_{\mathbb{S}^2} \{ e^{i\beta x \cdot d} g_{\rho}(d) + e^{i\beta x \cdot d} g_{\mu}(d) \} d\sigma(d), \quad (3.3) \]

where \( g = g_{\rho} + g_{\mu} \) with \( g_{\rho} \in L^2(\mathbb{S}^2)^3 \) and \( g_{\mu} \in L^2(\mathbb{S}^2)^3 \), can approximate \( v \) to an arbitrary accuracy in \( H^1(\Omega)^3 \). Henceforth, we let the real and imaginary parts of the kernel functions \( g_{\rho}(d) (\beta = \rho, \mu) \) in (3.3) be defined by

\[ g_{\beta}(d) = g_{\rho\beta}(d) + ig_{\mu\beta}(d), \quad (3.4) \]

where

\[ g_{\rho\beta}(d) = \begin{pmatrix} g_{\rho,\beta}^{(1,2)}(d) \\ g_{\rho,\beta}^{(3)}(d) \end{pmatrix} \in \mathbb{R}^3, \quad g_{\mu\beta}(d) = \begin{pmatrix} g_{\mu,\beta}^{(1,2)}(d) \\ g_{\mu,\beta}^{(3)}(d) \end{pmatrix} \in \mathbb{R}^3, \quad (3.5) \]

with \( g_{\rho,\beta}^{(1,2)}(d) \in \mathbb{R}^2 \) and \( g_{\mu,\beta}^{(1,2)}(d) \in \mathbb{R}^2 \).

Similar to proposition 2.1, using Jacobi–Anger expansion we have the expansion of the real and imaginary part of \( v_j(x) \) defined in (3.3) as follows.
Proposition 3.1. Let the elastic Herglotz wave function \( v_j \) be defined by (3.3). Denote
\[
\begin{align*}
v_{jp}^{R}(0) &= \int_{S^2} g_{jp}^{R}(d) \, d\sigma(d), \\
v_{jp}^{I}(0) &= \int_{S^2} g_{jp}^{I}(d) \, d\sigma(d),
\end{align*}
\]
Then
\[
v_j(0) = v_{jp}^{R}(0) + iv_{jp}^{I}(0) = (v_{jp}^{R}(0) + v_{jp}^{I}(0)) + i(v_{jp}^{I}(0) + v_{jp}^{I}(0)), \tag{3.6}
\]
where \( g_{jp}^{R}(d) \) and \( g_{jp}^{I}(d) \) (\( \beta = p, s \)) are defined in (3.4). Let \( j_{\ell}(t) \) be the \( \ell \)th Bessel function for \( \ell \in \mathbb{N} \cup \{0\} \).

In view of the explicit expression \( v_j(x) \) defined in (3.3), using Euler formula, one can handily compute that

Proposition 3.2. Let the elastic Herglotz wave function \( v_j \) be defined in (3.3), where \( v_j^{R}(x) \) and \( v_j^{I}(x) \) are the real and imaginary parts of \( v_j \) respectively. Then it holds that
\[
v_j^{R}(x) = \int_{S^2} (\cos(k_d \cdot x)g_{jp}^{R}(d) - \sin(k_d \cdot x)g_{jp}^{I}(d) + \cos(k_d \cdot x)g_{jp}^{R}(d)) \, d\sigma(d), \tag{3.7}
\]
and
\[
v_j^{I}(x) = \int_{S^2} (\cos(k_d \cdot x)g_{jp}^{I}(d) + \sin(k_d \cdot x)g_{jp}^{R}(d)) \, d\sigma(d).
\]

Let \( S_h \subset \mathbb{R}^2 \) be defined in (2.2) and \( M > 0 \). For any fixed \( x_3 \in (-M, M) \) and \( L > 0 \) defined in definition 3.1, we suppose that \( L \) is sufficiently small such that \( (x_3 - L, x_3 + L) \subset (-M, M) \). Write \( x = (x', x_3) \in \mathbb{R}^3 \), \( x' \in \mathbb{R}^2 \). In what follows, we consider the transmission eigenvalue problem for \( v, w \in H^1(S_h \times (-M, M))^2 \):
\[
\begin{align*}
\lambda \Delta v + (\lambda + \mu) \nabla \nabla \cdot v + \omega^2 v &= 0, & x' \in S_h, & -M < x_3 < M, \\
\lambda \Delta w + (\lambda + \mu) \nabla \nabla \cdot w + \omega^2 (1 + \text{V})w &= 0, & x' \in S_h, & -M < x_3 < M, \tag{3.8} \\
w &= v, & T_v v + \eta v &= T_w w & x' \in \Gamma_h^\pm, & -M < x_3 < M,
\end{align*}
\]
where \( \Gamma_h^\pm \) are defined in (2.2), \( T_v \) is boundary traction operator to \( \Gamma_h^\pm \times (-M, M), q \in L^\infty(S_h \times (-M, M)) \) defined in (1.10) and \( \eta \in L^\infty(\Gamma_h^\pm \times (-M, M)) \) is independent of \( x_3 \). Similar to the 2D case, we let \( (v_R, w_R) \) and \( (v_I, w_I) \) respectively signify the real and imaginary parts of \( (v, w) \), and both of them satisfy the Lamé system (3.8). We shall mainly focus on dealing with \( (v_R, w_R) \) and all the results hold equally for \( (v_I, w_I) \), and hence \( (v, w) \).

Noting that \( \eta \) in (3.8) is independent of the \( x_3 \) variable, and by applying the reduction operator \( \mathcal{R} \) defined in definition 3.1, one can show by direct verifications the following lemma.
Lemma 3.2. Denote

\[
G_1(x') = - \omega^2 R(v_R(x')) - \int_{-L}^{L} \phi''(x_3) \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ (2 \lambda + \mu) v_3 \end{bmatrix} (x', x_3) \, dx_3 \\
+ (\lambda + \mu) \int_{-L}^{L} \phi''(x_3) \begin{bmatrix} \partial_1 v_3 \\ \partial_2 v_3 \\ \partial_1 w_1 + \partial_2 w_2 \end{bmatrix} (x', x_3) \, dx_3 = \begin{bmatrix} G_1^{(1,2)}(x') \\ G_1^{(2)}(x') \end{bmatrix},
\]

\[
G_2(x') = - \omega^2 R(f_R(x')) - \int_{-L}^{L} \phi''(x_3) \begin{bmatrix} \lambda w_1 \\ \lambda w_2 \\ (2 \lambda + \mu) w_3 \end{bmatrix} (x', x_3) \, dx_3 \\
+ (\lambda + \mu) \int_{-L}^{L} \phi''(x_3) \begin{bmatrix} \partial_1 w_3 \\ \partial_2 w_3 \\ \partial_1 w_1 + \partial_2 w_2 \end{bmatrix} (x', x_3) \, dx_3 = \begin{bmatrix} G_2^{(1,2)}(x') \\ G_2^{(2)}(x') \end{bmatrix},
\]

where \( G_1^{(1,2)}(x') \in \mathbb{R}^2 \) (\( \ell = 1, 2 \)), \( f_R := (1 + V)w_R = q w_R \) and \( R \) is the dimension reduction operator associated with \( \phi \) defined in definition 3.1. Denote

\[
v_R = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3, \quad w_R = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3,
\]

where \( v^{(2)} \in \mathbb{R}^2 \) and \( w^{(2)} \in \mathbb{R}^2 \). Assume that \( \eta \in L^\infty(\Gamma_k^+ \times (-M, M)) \) in (3.8) is independent of \( x_3 \). Then it holds that

\[
\begin{align*}
\tilde{\mathcal{L}}(v_R(x')) & = G_1(x') & \text{in } S_h, \\
\tilde{\mathcal{L}}(w_R(x')) & = G_2(x') & \text{in } S_h, \\
\mathcal{R}(w_R(x')) & = \mathcal{R}(v_R(x')), & \text{on } \Gamma_k^+, \\
\begin{bmatrix} T_\nu \mathcal{R}(v^{(1,2)}) + \lambda \mathcal{R}(\partial_3 v_3) \nu \\ \mu \partial_3 \mathcal{R}(v_3) + \mu \mathcal{R}(\partial_3 v_3) \nu \end{bmatrix} + \eta \mathcal{R}(v_R) & = \begin{bmatrix} T_\nu \mathcal{R}(w^{(1,2)}) + \lambda \mathcal{R}(\partial_3 w_3) \nu \\ \mu \partial_3 \mathcal{R}(w_3) + \mu \mathcal{R}(\partial_3 w_3) \nu \end{bmatrix} & \text{on } \Gamma_k^+.
\end{align*}
\]

in the distributional sense, where \( \nu \) signifies the exterior unit normal vector to \( \Gamma_k^+ \), \( T_\nu \) is the two dimensional boundary traction operator defined in (1.10) and

\[
\mathcal{L} = \begin{bmatrix} \lambda \Delta' + (\lambda + \mu) \partial_1^2 & (\lambda + \mu) \partial_1 \partial_2 & 0 \\ (\lambda + \mu) \partial_1 \partial_2 & \lambda \Delta' + (\lambda + \mu) \partial_2^2 & 0 \\ 0 & 0 & \lambda \Delta' \end{bmatrix} = \begin{bmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} \end{bmatrix}
\]

(3.12)

with \( \Delta' := \partial_1^2 + \partial_2^2 \) being the Laplace operator with respect to the \( x' \)-variables. Here \( \mathcal{L} \) is the two dimensional Lamé operator with respect to the \( x' \)-variable.
Lemma 3.3 is a direct result of lemma 3.2. The corresponding proof can be found in the arXiv version of this paper [26].

**Lemma 3.3.** Under the same setup in lemma 3.2, the PDE system (3.11) is equivalent to

\[
\begin{aligned}
&\mathcal{L}R(v_{1}^{(1,2)})(x') = G_{1}^{(1,2)}(x') \quad \text{in } S_h, \\
&\mathcal{L}R(w^{(1,2)})(x') = G_{2}^{(1,2)}(x') \quad \text{in } S_h, \\
&R(w^{(1,2)})(x') = \mathcal{R}(v_{1}^{(1,2)})(x'), \\
&T_{\nu}R(v_{1}^{(1,2)})(x') + \eta R(v_{1}^{(1,2)})(x') = T_{\nu}R(w^{(1,2)})(x') \quad \text{on } \Gamma_{h}^\pm.
\end{aligned}
\]  

(3.13)

\[
\begin{aligned}
&\lambda \Delta' R(v_{2})(x') = G_{1}^{(2)}(x') \quad \text{in } S_h, \\
&\lambda \Delta' R(w_{2})(x') = G_{2}^{(2)}(x') \quad \text{in } S_h, \\
&R(w_{3})(x') = R(v_{2})(x'), \partial_{\nu}R(v_{3})(x') + \mu^{-1} \eta R(v_{3})(x') = \partial_{\nu}R(w_{3})(x') \quad \text{on } \Gamma_{h}^\pm.
\end{aligned}
\]  

(3.14)

where \( G_{1}^{(1,2)} \) and \( G_{2}^{(2)} \) are defined in (3.9), \( \ell = 1, 2 \).

Next we mainly study the system (3.13). Using Green’s formula (2.21) on the domain \( S_h \) and the boundary condition in (3.13), one can obtain a similar integral equality as in lemma 2.6. The proof of lemma 3.4 is skipped, which can be found in the arXiv version of this paper [26].

**Lemma 3.4.** Let \( S_h, \Lambda_h \) and \( \Gamma_{h}^\pm \) be defined in (2.2). Suppose that \( v^{(1,2)}, w^{(1,2)} \in H^{1}(S_h \times (-M, M)) \) fulfill (3.13). Recall that the CGO solution \( u(x) \) is defined in (2.14) and \( P_{\delta}(t) \) is the Legendre polynomial. Let \( \beta = p \) or \( s, j_{p,\beta} = j_{p}(k_{\beta}|x|) \) and

\[
\begin{aligned}
v_{p}^{(1,2)}(0) &= \int_{S^2} g_{p,\beta}^{(1,2)}(d) d\sigma(d), \\
v_{p}^{(1,2)}(0) &= \int_{S^2} g_{p,\beta}^{(1,2)}(d) d\sigma(d), \\
B_{\beta,1}^{(1,2)} &= \int_{S^2} g_{p,\beta}^{(1,2)}(d) P_{2\ell}(\cos \varphi) d\sigma(d), \\
B_{\beta,2}^{(1,2)} &= \int_{S^2} g_{p,\beta}^{(1,2)}(d) P_{2\ell-1}(\cos \varphi) d\sigma(d).
\end{aligned}
\]

Denote

\[
v_{j}^{(1,2)}(x) = \left( \begin{array}{c} v_{j}^{(1,2)}(x) \\ v_{j}^{(2)}(x) \end{array} \right) \in \mathbb{R}^{3}, \quad v_{j}^{(1,2)}(x) \in \mathbb{R}^{2},
\]

where

\[
v_{j}^{(1,2)}(x) = v_{p}^{(1,2)}(0) j_{p}(k_{\beta}|x|) + v_{p}^{(1,2)}(0) j_{p}(k_{\beta}|x|) + \sum_{\beta = p, \ell = 1}^{+\infty} (-1)^{\ell} (4\ell + 1) j_{2\ell+1} B_{\beta,1}^{(1,2)}
\]

\[
+ 2 \sum_{\beta = p, \ell = 1}^{+\infty} (-1)^{\ell} (4\ell - 1) j_{2\ell-1} B_{\beta,2}^{(1,2)},
\]

(3.15)
and

\[ v_j^{(3)}(x) = j_0(k_\beta |x|) \int_{S^2} g_{R,j}^{(3)}(d) \sigma(d) + j_0(k_\beta |x|) \int_{S^2} g_{R,j}^{(3)}(d) \sigma(d) \]

\[ + \sum_{\beta = \rho, \ell = 1}^{+\infty} (-1)^{4(4\ell + 1)} j_{2\ell}(k_\beta |x|) \int_{S^2} g_{R,j}^{(3)}(d) P_{2\ell}(\cos \varphi) \sigma(d) \]

\[ + \sum_{\beta = \rho, \ell = 1}^{+\infty} (-1)^{4(4\ell - 1)} j_{2\ell-1}(k_\beta |x|) \int_{S^2} g_{R,j}^{(3)}(d) P_{2\ell-1}(\cos \varphi) \sigma(d) \]

with \( g_{R,j}^{(1,2)} \), \( g_{R,j}^{(1,2)} \), \( g_{R,j}^{(3)} \) and \( g_{R,j}^{(3)} \) defined in (3.5). Then the following integral equality holds

\[ \tilde{I}_1 + \tilde{I}_2 = \tilde{I}_{\Lambda_\beta} - \tilde{I}_\pm = \tilde{I}_\pm^\alpha, \quad (3.16) \]

where

\[ \tilde{I}_1 = \int_{S^2} (f_{ij} + f_1 + f_3 \cdot u(sx') dx', \quad \tilde{I}_2 = \int_{S^2} (f_1 - f_{ij}) \cdot u(sx') dx', \]

\[ \tilde{I}_{\Lambda_\beta} = \int_{S^2} \left(T_{\rho} \mathcal{R}(v^{(1,2)} - w^{(1,2)}) \right) \cdot u - (T_{\rho} u) \cdot \mathcal{R}(v^{(1,2)} - w^{(1,2)}) \sigma, \]

\[ \tilde{I}_\pm = \int_{S^2} \eta(x') \mathcal{R}(v^{(1,2)} - v_j^{(1,2)}) \cdot u(sx') \sigma, \quad \tilde{I}_\pm^\alpha = \int_{S^2} \eta(x') \mathcal{R}(v^{(1,2)} - v_j^{(1,2)}) \cdot u(sx') \sigma, \]

\[ f_1 = -\omega^2 \mathcal{R}(v^{(1,2)}), \quad f_2 = \omega^2 \mathcal{R}(q w^{(1,2)}), \quad f_3 = -\int_{-L}^L \phi'/(x_3) \lambda(v_1 - w_1) \lambda(v_2 - w_2) \,(x', x_3) dx_3, \]

\[ f_{ij} = -\omega^2 \mathcal{R}(v_j^{(1,2)}), \quad f_4 = (\lambda + \mu) \int_{-L}^L \phi'/(x_3) \left( \frac{\partial_1(v_1 - w_1)}{\partial_2(v_3 - w_3)} \right) \,(x', x_3) dx_3. \]

(3.17)

Similar to lemma 2.7, for the integral \( \tilde{I}_{\Lambda_\beta} \) defined in (3.16) one has

Lemma 3.5. Recall that \( \tilde{I}_{\Lambda_\beta} \) is defined in (3.17). Under the same setup in lemma 3.4, the following integral estimate holds

\[ |\tilde{I}_{\Lambda_\beta}| \leq C \left( \frac{2h + s^2}{\sqrt{s}} + \frac{\mu s}{\sqrt{2}} \sqrt{\theta_{M} - \theta_{\omega}} e^{-i\sqrt{\omega} w} \|v^{(1,2)} - w^{(1,2)}\|_{H^1(S_\delta)^2}, \quad (3.18) \right. \]

where \( C \) is a positive constant coming from the trace theorem, \( S_\delta \) and \( \delta_w > 0 \) are defined in (2.2) and (2.17), respectively.

We choose to skip the corresponding proof of lemma 3.6, which can be found in the arXiv version of this paper [26].

Lemma 3.6. Under the same setup in lemma 3.4, for any given positive constants \( \gamma, \beta_1 \) and \( \beta_2 \), we assume that there exists a sequence of the Herglotz wave functions \( \{\mathbf{v}_j\}_{j=1}^{+\infty} \). where \( \mathbf{v}_j \) is defined by (3.3), can approximate \( \mathbf{v} \) in \( H^1(S_\delta \times (-M,M))^3 \) satisfying
the Hölder continuity of $f$ is detailed in the arXiv version of this paper [26].

Furthermore, suppose that $\eta \in C^\alpha(\overline{\Gamma_h} \times [-M, M])$, $0 < \alpha < 1$, and hence it holds that

$$
\eta(x') = \eta(0) + \delta \eta(x'), \quad |\delta \eta(x')| \leq \|\eta(x')\|_{C^\alpha} |x'|^\alpha
$$

for $0 < \alpha < 1$. Recall that $\tilde{I}_2$ and $\tilde{I}_m^\delta$ are defined in (3.17). Then we have the following estimates

$$
\tilde{I}_2 \leq C\|\phi\| \left( |\eta(0)| h \sqrt{\theta_M - \theta_m} e^{-\sqrt{\delta_M}} \right) \|\phi\|_{L^\infty} \tilde{j}^{-\gamma},
$$

where $C$ is a positive constant depending on $L$ and $h$.

By virtue of lemma 3.1, the regularity results on $f_1$ defined in $C^\alpha(\overline{S_h} \times [-M, M])$, then $f_2(x') \in C^\alpha(\overline{S_h})$. Furthermore we assume that

$$
v_R - w_R \in C^{1,\alpha}(\overline{S_h} \times [-M, M]),
$$

where $\alpha \in (0, 1)$. Then one has $f_1(x') \in C^\alpha(\overline{S_h})$ and $f_3(x') \in C^\alpha(\overline{S_h})$.

The integral estimate for $\tilde{I}_1$ defined in (3.17) is established in the following lemma. Using the Hölder continuity of $f_i$, the definition of the dimensional reduction operator and (2.16), we can analyze the integral asymptotic behavior with the parameter $s$ in the CGO u. One may find its detailed proof in the arXiv version of this paper [26].

**Lemma 3.7.** Let $f_2(x')$, $f_3(x')$ and $f_4(x')$ be defined in (3.17). If

$$
q w_R \in C^\alpha(\overline{S_h} \times [-M, M]),
$$

then $f_2(x') \in C^\alpha(\overline{S_h})$. Furthermore we assume that

$$
v_R - w_R \in C^{1,\alpha}(\overline{S_h} \times [-M, M]),
$$

where $\alpha \in (0, 1)$. Then one has $f_1(x') \in C^\alpha(\overline{S_h})$ and $f_3(x') \in C^\alpha(\overline{S_h})$.

The integral estimate for $\tilde{I}_1$ defined in (3.17) is established in the following lemma. Using the Hölder continuity of $f_i$, the definition of the dimensional reduction operator and (2.16), we can analyze the integral asymptotic behavior with the parameter $s$ in the CGO u. One may find its detailed proof in the arXiv version of this paper [26].

**Lemma 3.8.** Recall that $f_2(x')$, $f_3(x')$ and $f_4(x')$ are defined in (3.17). Suppose that $f_1(x') \in C^\alpha(\overline{S_h}) (\ell = 2, 3, 4)$, $0 < \alpha < 1$. Recall that $\tilde{I}_1$ is defined in (3.17), then the following integral estimates hold

$$
|\tilde{I}_1| \leq 6|f_1(0)| + |f_2(0)| + |f_3(0)| + |f_4(0)| \left( e^{-20\mu_1} - e^{-2\delta_M} \right) \|\phi\|_{L^\infty} \tilde{j}^{-\gamma} + \sum_{\ell=2}^4 \|f_\ell\|_{C^\alpha(\overline{S_h})} 2\sqrt{2(\theta_M - \theta_m)} \Gamma(2\alpha + 4) \delta_M^{2\alpha + 4} \|\phi\|_{L^\infty} \tilde{j}^{-\gamma} + 4L^2 \text{diam}(S_h) \|\phi\|_{L^\infty} \sum_{\beta=p,s} (1 + k_\beta) \left( \|g_{1,2}^{(1,2)}\|_{L^2(\partial S^\gamma)} + \|g_{1,2}^{(1,2)}\|_{L^2(\partial S^\gamma)} \right),
$$

where $g_{1,2}^{(1,2)}$ and $g_{1,2}^{(1,2)}$ are defined in (3.5) ($\beta = p, s$), $\delta_M$ is defined in (2.17), $\theta_m$ and $\theta_M$ are defined in (1.11).
Lemma 3.9. Let $j_\ell(t)$ be the $\ell$th spherical Bessel function with the form
\[
j_\ell(t) = \frac{t^\ell}{(2\ell + 1)!!} \left(1 - \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2l!(2l+1)!!} t^{2l}\right),
\]
where $N_{i,j} = (2\ell + 3)\ldots(2\ell + 2l + 1)$ and $R$ be the dimension reduction operator defined in definition 3.1. Then
\[
\mathcal{R}(j_\ell(k_\ell|x|)(x')) = C(\phi) + C_1(\phi) \sum_{l=1}^{\infty} \frac{(-1)^{l+1}(|x|^2 + a_{i,j}^2)^{l-\frac{1}{2}}}{2l!N_{i,j}} |x|^2,
\]
(3.25a)

\[
\mathcal{R}(j_{2\ell}(k_\ell|x|)(x')) = C_1(\phi) |x|^2 \frac{k_{2\ell}^2(|x|^2 + a_{i,j}^2)^{l-\frac{1}{2}}}{(4\ell + 1)!!} \left[1 - \sum_{l=1}^{\infty} \frac{(-1)^{l+1}(|x|^2 + a_{i,j}^2)^l}{2l!N_{i,j,1}} \right],
\]
(3.25b)

\[
\mathcal{R}(j_{2\ell+1}(k_\ell|x|)(x')) = C_1(\phi) |x|^2 \frac{k_{2\ell+1}^2(|x|^2 + a_{i,j}^2)^{l-\frac{1}{2}}}{(4\ell + 1)!!} \left[1 - \sum_{l=1}^{\infty} \frac{(-1)^{l+1}(|x|^2 + a_{i,j}^2)^l}{2l!N_{i,j,2}} \right],
\]
(3.25c)

where $\beta = p, s$, $\ell \in \mathbb{N}$, $a_{i,j}, \xi, a_{i,l}, a_{i,l,1} \in (-L, L)$, $N_{i,j,1} = (4\ell + 3)\ldots(4\ell + 2l + 1)$, $N_{i,j,2} = (4\ell + 1)\ldots(4\ell + 2l - 1)$ and
\[
C(\phi) = \int_{-L}^{L} \phi(x_3) dx_3 > 0, \quad C_1(\phi) = \int_{-\arctan L/|x'|}^{\arctan L/|x'|} \phi(|x'| \tan \omega) \sec^3 \omega d\omega.
\]
(3.26)

Furthermore, it holds that
\[
0 < C_1(\phi) \leq \sec^3 \omega C(\phi),
\]
(3.27)

where $\omega \in (-\arctan L/|x'|, \arctan L/|x'|)$.

Remark 3.1. We should emphasize that $C_1(\phi)$ is uniformly bounded with respect to $|x'|$ since $\omega \in (-\arctan \frac{L}{|x'|}, \arctan \frac{L}{|x'|})$ is fixed, which shall be used in the proof of theorem 3.1 in what follows.

By virtue of (3.15), the reduction operator $\mathcal{R}$ defined in (3.1), the integral mean value theorem and Cauchy–Schwarz inequality, one can establish the integral equality (3.28) and obtain the asymptotic analysis (3.29) with respect to $s$ in the CGO $u$. The corresponding proof of lemma 3.10 can be found in the arXiv version of this paper [26].

Lemma 3.10. Consider the same setup in lemma 3.4 and recall that $I_\pm$ is defined (3.17).
Suppose that $\eta \in C^1(\overline{V_h} \times [-M, M])$ is independent of $x_3$ and has the expansion (3.20), then it holds that
\[
I_\pm = I_1^\pm + \eta(0) I_2^\pm,
\]
(3.28)
Lemma 3.11. Consider the same setup in lemma 3.4 and let $\tilde{T}_2^+$ be defined in (3.28). Let

$$
\tilde{T}_{21,\beta}^+ = \int_{\Gamma_h^+} \left( \frac{1}{i} \cdot \nu_p^{(1,2)}(0) \mathcal{R}(j_0(k_\beta|x|))v_1 \right) d\sigma,
$$

$$
\tilde{T}_{23,\beta}^+ = \int_{\Gamma_h^+} \left( \frac{1}{i} \cdot \nu_p^{(1,2)}(0) \mathcal{R}(j_0(k_\beta|x|))v_1 \right) d\sigma,
$$

$$
\tilde{T}_{25,\beta}^+ = \int_{\Gamma_h^+} \left( \frac{1}{i} \cdot \nu_p^{(1,2)}(0) \mathcal{R}(j_0(k_\beta|x|))v_1 \right) d\sigma,
$$

where $\nu_p^{(1,2)}$ and $\nu_p^{(1,2)}$ are defined in (3.5).

**Lemma 3.12.** Consider the same setup in lemma 3.4 and recall that $\tilde{T}_2^+$ is defined (3.28). Denote

$$
\tilde{T}_{211} = C(\phi) \left( \frac{1}{i} \cdot \nu_p^{(1,2)}(0) \right) \int_{\Gamma_h^+} v_1 u_1 d\sigma,
$$

$$
\tilde{T}_{212} = \left( \frac{1}{i} \cdot \nu_p^{(1,2)}(0) \right) \int_{\Gamma_h^+} C_1(\phi) \sum_{l=1}^{\infty} \frac{(-1)^l k^2 l(l+1)!!}{2^l l(2l+1)!!} |x|^l \frac{1}{2} u_1 d\sigma,
$$

$$
\tilde{T}_{222} = \left( \frac{1}{i} \cdot \nu_p^{(1,2)}(0) \right) \int_{\Gamma_h^+} C_1(\phi) \sum_{l=1}^{\infty} \frac{(-1)^l k^2 l(l+1)!!}{2^l l(2l+1)!!} |x|^l \frac{1}{2} u_1 d\sigma,
$$

where $C(\phi), C_1(\phi)$ are defined in (3.25), and $a_{0j} \in (-L, L)$. Then it holds that
Lemma 3.13. Consider the same setup in lemma 3.2 and suppose that $v_R$ can be approximated by a sequence of the functions $\{v_j^R\}_{j=1}^{+\infty}$ defined by (3.3) in $H^1(S_h \times (-M, M))^3$ satisfying (3.19). If $\lambda \neq 0$, $\eta(0) \neq 0$ and $-\pi < \theta_m < \theta_M < \pi$ satisfying $\theta_M - \theta_m \neq \pi$, then we have
\[
\lim_{j \to +\infty} v_j(m)(0) = 0.
\]

Theorem 3.1. Let $\Omega, S_h \times (-M, M)$ be described above and $\alpha \in (0, 1)$. For any fixed $x_3 \in (-M, M)$ and $L > 0$ defined in definition 3.1, we suppose that $L$ is sufficiently small such that $(x_3 - L, x_3 + L) \subset (-M, M)$ and
\[
(B_h \times (-M, M)) \cap \Omega = S_h \times (-M, M),
\]
where $B_h \subset \mathbb{R}^2$ is the central ball of radius $h \in \mathbb{R}_+$. Assume that $v, w \in H^1(\Omega)^3$ are the generalized elastic transmission eigenfunctions to (1.11), where the Lamé constant $\lambda \neq 0$. Assume further that $q w \in C^0(S_h \times (-M, M))^3$ and $v - w \in C^1(S_h \times [-M, M])^3$, where $\theta$ is the vertex of $S_h$ defined in (2.1). Write $x = (x', x_3) \in \mathbb{R}^3$, $x' \in \mathbb{R}^2$. If the following conditions are fulfilled:

(a) $v$ can be approximated in $H^1(S_h \times (-M, M))^3$ by the Herglotz functions $v_j, j = 1, 2, \ldots$, with kernels $g_j$ and $g_j$, satisfying (3.19):

(b) The function $\eta = \eta(x')$ is independent of $x_3$ and
\[
\eta(0) \neq 0.
\]

(c) The angles $\theta_m$ and $\theta_M$ of the sector $S_h$ satisfy
\[
-\pi < \theta_m < \theta_M < \pi \quad \text{and} \quad \theta_M - \theta_m \neq \pi.
\]
then for every edge points \((0, x_j^*) \in \mathbb{R}^3\) of \(S_h \times (-M,M)\), namely \(x_j^* \in (-M,M)\), one has

\[
\lim_{\rho \to +0} \frac{1}{m(B((0, x_j^*), \rho) \cap \Omega)} \int_{m(B((0, x_j^*), \rho) \cap \Omega)} |v(x)| \, dx = 0, \tag{3.37}
\]

where \(m(B((0, x_j^*), \rho) \cap \Omega)\) is the measure of \(B((0, x_j^*), \rho) \cap \Omega\).

**Remark 3.2.** Similar to remark 2.1 in the 2D case, a more general Fourier extension property (cf (2.44)) can be proposed study the vanishing property in theorem 3.1. However, this will involve rather lengthy and complicate analysis, and we choose not to explore more along that direction in this paper.

**Proof of theorem 3.1.** As remarked earlier, it is sufficient for us to consider \(v_R\) and \(w_R\), and prove that (3.37) holds for \(v_R\). Since \(qw \in C^0(\overline{S_h} \times [-M,M])^3\), it is easy to see that \(qw_R \in C^0(\overline{S_h} \times [-M,M])^3\). Similarly, we know that \(v_R - w_R \in C^{1,\alpha}(\overline{S_h} \times [-M,M])^3\) under the assumption \(v - w \in C^{1,\alpha}(\overline{S_h} \times [-M,M])^3\).

Since the Herglotz functions \(v, j = 1,2, \ldots\), defined in (3.3) with kernels \(g_{jp}\) and \(g_{j\mu}\) can approximate \(v\) under the condition (3.19), we see that \(v_R \in H^1(S_h \times (-M,M))^3\) can be approximated by \(\{v_R^j\}_{j=1}^{+\infty}\) defined in (3.3) satisfying (3.19). Therefore the assumptions in lemmas 3.4–3.12 are fulfilled.

We divide the proof into two parts.

**Part I.** First we shall prove that

\[
\lim_{j \to +\infty} v_{(1,2)}^j(0) + v_{(1,2)}^\mu(0) = 0, \tag{3.38}
\]

where \(v_{(1,2)}^j(0)\) and \(v_{(1,2)}^\mu(0)\) are defined in (3.15).

By lemmas 3.2 and 3.3, we consider the PDE system (3.13). Recall that \(C(\phi)\) and \(\mu(\theta)\) are defined in (3.26) and (3.32) respectively. Substituting (3.28) and (3.30) and into (3.16), after arranging terms, we have

\[
\eta(0)2s^{-2} \left( C(\phi) \left( \frac{1}{i} \cdot v_{(1,2)}^j(0) + C(\phi) \left( \frac{1}{i} \cdot v_{(1,2)}^\mu(0) \right) \right) 
\times \left( -s \sqrt{h} e^{-s\sqrt{h}(\theta_m)} \mu^{-1}(\theta_m) - e^{-s\sqrt{h}(\theta_m)} \mu^{-2}(\theta_m) + \mu^{-2}(\theta_m) \right) 
- s \sqrt{h} e^{-s\sqrt{h}(\theta_m)} \mu^{-1}(\theta_m) - e^{-s\sqrt{h}(\theta_m)} \mu^{-2}(\theta_m) + \mu^{-2}(\theta_m) \right),
\]

\[
= -\eta(0) (I_{23,p}^+ + \bar{I}_{23,s}^+ + \bar{I}_{25,p}^+ + \bar{I}_{25,s}^+ + I_{211}^+ + \bar{I}_{212}^+ + \bar{I}_{221}^+ + \bar{I}_{222}^+ - I_{2}^+ + \bar{I}_{A}^+ - \bar{I}_{12} - \bar{I}_{2}^+),
\]

(3.39)

where \(I_{2\ell,\beta}^\pm\) \(\ell = 3,5, \beta = p,s\), \(I_{211,\ell}^\pm\) \((\ell = 1,2)\), \(I_{25,\ell}^\pm\) \((\ell = 1,2)\), \(I_{1}^+, \bar{I}_{A}^+, \bar{I}_{1}^+ (\ell = 1,2), \bar{I}_{2}^+\) are defined in (3.31), (3.30), (3.28) and (3.17) respectively.

Multiplying \(s^2\) on the both side of (3.39), by virtue of (3.18), (3.21), (3.24), (3.29) and (3.31), and letting \(s = j^{p/2}\) \((\max \{\beta_1, \beta_2\} < q < \gamma)\) with \(j \to +\infty\), we have

\[
\lim_{j \to +\infty} C(\phi)\eta(0) \left( v_{(1,2)}^j(0) + v_{(1,2)}^\mu(0) \right) \left( \frac{1}{i} \right) \left( \mu^{-2}(\theta_m) + \mu^{-2}(\theta_m) \right) = 0.
\]
Recall that the definition of $\phi$ in definition 3.1, we know that $C(\phi) > 0$. Under the condition (3.36), from [25, lemma 2.10], we know that

$$\mu^{-2}(\theta_m) + \mu^{-2}(\theta_M) \neq 0.$$  

Under the condition (3.35), since $\eta$ is real valued function, we have

$$\lim_{j \to \infty} \left( v_{jp}(0,1,2) + v_{js}(0,1,2) \right) \cdot (1,i) = 0,$$

which readily implies (3.38) by noting that $v_{jp}(0,1,2) + v_{js}(0,1,2)$ is a real vector.

**Part II.** By lemma 3.13, we have (3.34). Combining (3.38) with (3.34), by using the definition of $v_{j}(0)$ in (3.6), we can prove that

$$\lim_{j \to \infty} v_{j}(0) = 0.$$  

Using a similar argument to that for (2.43), we can finish the proof of this theorem. $\square$

We next consider the degenerate case of theorem 3.1 with $\eta \equiv 0$. The corresponding detailed proof of the following corollary is skipped and one may find it in the arXiv version of this paper [26]. Indeed, the proof follows from the one of theorem 3.1 with some necessary modifications by noting $\eta \equiv 0$ in the integral equality (3.16).

**Corollary 3.2.** Consider the same setup in theorem 3.1 but with $\eta \equiv 0$. If the following conditions are fulfilled:

(a) For any given constants $\gamma > \max\{\beta_1, \beta_2\} > 0$, there exits a sequence of Herglotz functions $v_j, j = 1, 2, \ldots$, with kernels $g_{jp}$ and $g_{js}$ can approximate $v$ in $H^1(S_h \times (-M,M))$ satisfying

$$\|v - v_j\|_{H^1(S_h \times (-M,M))} \leq j^{-\gamma}, \quad \|g_{jp}\|_{L^2(S_2)} \leq j^h \quad \text{and} \quad \|g_{js}\|_{L^2(S_2)} \leq j^h.$$  

(b) The angles $\theta_m$ and $\theta_M$ of the sector $W$ satisfy

$$-\pi < \theta_m < \theta_M < \pi \quad \text{and} \quad \theta_M - \theta_m \neq \pi;$$  

then for every edge points $(0, x_3^c) \in \mathbb{R}^3$ of $S_h \times (-M,M)$, namely $x_3^c \in (-M,M)$, one has

$$\lim_{\rho \to 0+} \frac{1}{m(B((0, x_3^c), \rho) \cap P(\Omega))} \int_{B((0, x_3^c), \rho) \cap P(\Omega)} R(Vw)(x')dx' = 0,$$

where $P(\Omega)$ is the projection set of $\Omega$ on $\mathbb{R}^2$.

**Remark 3.3.** According to corollary 3.2, the average value of each component of the function $Vw$ over the cylinder centered at the edge point $(0, x_3^c)$ with the height $L$ vanishes in the distribution sense. Moreover, if we assume that $V(x', x_3)$ is continuous near the edge point $(0, x_3^c)$ where $x_3^c \in (-M,M)$ and $V(0, x_3^c) \neq 0$, by the dominant convergence theorem and the definition of $R$, one can show that
the assumption that

\[ \eta \in H^1(\Omega)^3, \]

then we can prove that

Consider the same setup in theorem 3.4, we can prove that

Corollary 3.5.

The following corollary is omitted.

if we further require that \( v \) is Hölder continuous at the edge corner, similar to theorem 3.1, we have the following theorem. Indeed, by exploring the Hölder continuity of \( f \) (\( \ell = 2, 3, 4 \)) near the corner, adopting a similar argument in theorem 3.1, we can show that \( v \) must vanish at the edge corner point. We choose to skip the detailed proof of this theorem, which can be found in the arXiv version of this paper [26].

**Theorem 3.4.** Consider the same setup in theorem 3.1, but assume that \( v \in H^1(\Omega)^3 \cap C^\alpha(\bar{\Omega}_h \times [-M,M])^3, \) and \( qw \in C^\alpha(\bar{\Omega}_h \times [-M,M])^3 \) for \( 0 < \alpha < 1. \) If \( q \) is independent of \( x_3 \) and \( \eta(0) \neq 0, \) and the corner is non-degenerate, then \( v \) vanishes at the edge point \((0, x^*_3) \in \mathbb{R}^3 \) of \( \bar{\Omega}_h \times (-M, M), \) where \( x^*_3 \in (-M, M). \)

In corollary 3.5, if global Hölder continuous regularities for \( v \) and \( w \) are fulfilled, similar to theorem 3.4, we can prove that \( v \) and \( w \) must vanish at the edge corner point by removing the assumption that \( v - w \) is \( C^{1,\alpha} \)-continuous at the edge corner. The detailed proof of the following corollary is omitted.

**Corollary 3.5.** Consider the same setup in theorem 3.4. Assume that \( v, w \in H^1(\Omega)^3 \cap C^\alpha(\Omega)^3 (0 < \alpha < 1) \) are the generalized elastic transmission eigenfunctions to (1.11). Suppose further that \( \eta \in C^\alpha(\bar{T}_h^+ \times [-M, M]) \) with \( \eta(0) \neq 0, \) and \( qw \in C^\alpha(\Omega)^3 \) for \( 0 < \alpha < 1. \) If the corner is non-degenerate, then \( v \) and \( w \) vanish at the edge point \((0, x^*_3) \in \mathbb{R}^3 \) of \( \bar{\Omega}_h \times (-M, M), \) where \( x^*_3 \in (-M, M). \)
4. Unique recovery results for the inverse elastic problem

In this section, we apply the geometric property of generalized elastic transmission eigenfunctions established in the previous sections to the study of the unique recovery for the inverse elastic problem (1.8).

We first introduce a more general formulation of the inverse elastic problem. Let $\Omega$, $u^i$, $V$ and $q = 1 + V$ be those introduced in section 1.1. Let $\eta \in L^\infty(\partial\Omega)$ with $\Im \eta \geq 0$. Consider the following elastic transmission system

\[
\begin{aligned}
\mathcal{L} u^- + \omega^2 u^- &= 0 & & \text{in } \Omega, \\
\mathcal{L} u^+ + \omega^2 u^+ &= 0 & & \text{in } \mathbb{R}^n \setminus \Omega, \\
u^+ &= u^-, & & T_r u^+ + \eta u^+ = T_r u^- & & \text{on } \partial\Omega, \\
u^+ &= u^i + u^e & & \text{in } \mathbb{R}^n \setminus \Omega, \\
limit_{r \to \infty} i k \left( \frac{\partial u^-}{\partial R} - ik \beta u^e \right) &= 0, & & \beta = p, s.
\end{aligned}
\]  

(4.1)

If $\eta \equiv 0$, (4.1) is reduced to be (1.2). The well-posedness of (4.1) for the case $\eta \equiv 0$ was investigated in [31, 32]. By following a standard variational argument in [31, 32], one can show the unique existence of a solution $u = u^- \chi_\Omega + u^+ \chi_{\mathbb{R}^n \setminus \Omega} \in H^1_\text{loc}(\mathbb{R}^n)$ to (4.1). However, it is not the focus of this article and in what follows, we always assume the well-posedness of the system (4.1). We write $u^e(\mathbf{x}; u^i)$, $\beta = t, p, s$ to signify the far-field patterns associated with (4.1) and consider the following inverse problem

\[\mathcal{F}(\Omega; \eta) = u^e(\mathbf{x}; u^i), \quad \beta = t, p, s,\]  

(4.2)

where $\mathcal{F}$ is implicitly defined by the scattering system (4.1). We are particularly interested in the geometrical inverse problem of recovering $\Omega$ independent of $q$ and $\eta$. On the other hand, it is pointed out that if $\Omega$ can be recovered, the boundary parameter $\eta$ can be recovered as well by a standard argument. To the best of our knowledge, the inverse problem (4.2) is new to the literature, in particular for the case $\eta \equiv 0$. If the far-field pattern is given associated with a single incident wave, then it is referred to as a single far-field measurement for the inverse problem (4.2), otherwise it is referred as multiple measurements. It can be directly verified that with a single far-field measurement, the inverse problem (4.2) is formally determined since both $s^{n-1}$ (on which the far-field pattern is given) and $\partial\Omega$ (it completely determines the shape of $\Omega$) are $(n - 1)$-dimensional manifolds. The inverse shape problem associated with a single far-field measurement constitutes a longstanding challenging problem in the inverse scattering theory [19], a fortiori the one described above for the elastic scattering. In the rest of the paper, we shall apply the geometric results derived in the previous sections to establish several novel unique identifiability results to the inverse shape problem (4.2) associated with a single far-field measurement within a certain generic (though still specific) scenario. Before that, we would like to mention in passing some related results in the literature [31, 33] on the unique identifiability for the inverse problem (4.2) associated with multiple far-field measurements and $\eta \equiv 0$.

First, we introduce the admissible class of elastic scatterers in our study. Let $W_\mathbf{x}_c(\theta_W)$ be an open sector in $\mathbb{R}^2$ with the vertex $\mathbf{x}_c$ and an open angle $\theta_W$. Denote

\[\Gamma^\pm_\theta(\mathbf{x}_c) := \partial W_\mathbf{x}_c(\theta_W) \cap B_\theta(\mathbf{x}_c), \quad S_\theta(\mathbf{x}_c) := B_\theta(\mathbf{x}_c) \cap W_\mathbf{x}_c(\theta_W),\]  

(4.3)
where \( B_h(x_\ell) \) is an open disk centered at \( x_\ell \) with the radius \( h \in \mathbb{R}_+ \).

**Definition 4.1.** Let \((\Omega; q, \eta)\) be an elastic scatterer. Consider the scattering problem (4.1) and \( u^i \) is the incident wave field therein. The scattering configuration is said to be admissible if the following conditions are fulfilled:

(a) \( \Omega \) is a bounded simply connected Lipschitz domain in \( \mathbb{R}^n \) with a connected complement, and \( q \in L^\infty(\Omega) \), \( \eta \in L^\infty(\partial \Omega) \) are real valued functions.

(b) Following the notations in theorem 2.3, if \( \Omega \subseteq \mathbb{R}^2 \) possesses a planar corner \( B_h(x_\ell) \cap \Omega = \Omega \cap W_{h, \ell}(\theta_w) \) where \( x_\ell \) is the vertex of the sector \( W_{h, \ell}(\theta_w) \) and the open angle \( \theta_w \) of \( W_{h, \ell}(\theta_w) \) satisfies \( \theta_w \in (0, \pi) \), then \( q \in C^\alpha(S_h(\xi_\ell)) \) and \( \eta \in C^\alpha(\Gamma^\pm_h(\xi_\ell)) \) for \( \alpha \in (0, 1) \) with \( q(x_\ell) \neq 1 \) and \( \eta(x_\ell) \neq 0 \), where \( S_h(\xi_\ell) \) and \( \Gamma^\pm_h(\xi_\ell) \) are defined in (4.3). Similarly, following the notations in theorem 3.1, if \( \Omega \subseteq \mathbb{R}^3 \) possesses a 3D edge corner \( (B_h(x_\ell) \times (-M, M)) \cap \Omega = S_h(\xi_\ell) \times (-M, M) \), where \( x_\ell \) is the vertex of \( S_h(\xi_\ell) \) contained in the sector \( W_{h, \ell}(\theta_w) \) and the open angle \( \theta_w \) of \( W_{h, \ell}(\theta_w) \) satisfies \( \theta_w \in (0, 2\pi) \), then \( q \in C^\alpha(S_h(\xi_\ell) \times [-M, M]) \), \( \eta \equiv 0 \) on \( \partial \Omega \).

(c) The total wave field \( u \) is non-vanishing everywhere in the sense that for any \( x \in \mathbb{R}^n \),

\[
\lim_{\rho \to +0 \atop \rho \neq 0} \frac{1}{m(B(x, \rho))} \int_{B(x, \rho)} |u(x)| \, dx \neq 0. \tag{4.4}
\]

**Remark 4.1.** The assumption (4.4) can be fulfilled in certain generic scenario. For an illustration, let us consider a specific case by requiring the angular frequency \( \omega \in \mathbb{R}_+ \) sufficiently small. In the physical scenario, this is also equivalent to requiring that the size of the scatterer, namely \( \text{diam}(\Omega) \), is sufficiently small (compared to the operating wavelength). In such a case, from a physical point of view, the interruption of the incident field due to the scatterer should be small, i.e. \( u^w \) should be small compared to \( u^i \). Hence, if \( u^i \) is non-vanishing everywhere (say, \( u^i \) is a plane wave, namely, the Herglotz wave (2.3) or (3.2) with the densities being delta-distributions), then \( u = u^i + u^w \) should be non-vanishing everywhere. However, a rigorous justification of such a physical intuition will cost lengthy arguments and we choose not to explore more about this point.

Next, we present a technical lemma concerning the regularity of the solution to the Lamé system around a corner (cf [43, 44]) in \( \mathbb{R}^2 \). We would also like to refer interested readers to [21, 22, 30] on classical results of decomposing solutions to elliptic PDEs in corner domains.

**Lemma 4.1.** [43, theorem 2.3]. Let \( \Omega \) be a bounded open connected subset of \( \mathbb{R}^2 \), where the boundary \( \partial \Omega \) of \( \Omega \) is the union of a finite number of line segments \( \Gamma_\ell, \ell \in \Xi \). Fix a partition of \( \Xi \) into \( \mathcal{D} \cup \mathcal{N} \), where \( \mathcal{D} \) and \( \mathcal{N} \) correspond to Dirichlet and Neumann boundary conditions respectively. Given a vector field \( f \in L^2(\Omega)^2 \) and \( g^{(\ell)} \in H^{1/2}(\Gamma_\ell)^2 \) for all \( \ell \in \mathcal{N} \), consider the weak solution \( u \in H^1(\Omega)^2 \) of the Lamé system

\[
Lu = f \quad \text{in} \quad \Omega \tag{4.5}
\]

with mixed boundary conditions

\[
\begin{cases}
  u = 0 \quad &\text{on} \quad \Gamma_\ell, \quad \ell \in \mathcal{D}, \\
  T_\nu u = g^{(\ell)} \quad &\text{on} \quad \Gamma_\ell, \quad \ell \in \mathcal{N}.
\end{cases}
\]

If \( \Omega \) satisfies the assumption: \( \forall \ell_1, \ell_2 \in \Xi \) such that \( \Gamma_{\ell_1} \cap \Gamma_{\ell_2} \neq \emptyset \), the interior angle \( \angle(\Gamma_{\ell_1}, \Gamma_{\ell_2}) \) fulfills \( \angle(\Gamma_{\ell_1}, \Gamma_{\ell_2}) < 2\pi \) and moreover, if \( \ell_1 \in \mathcal{D} \) and \( \ell_2 \in \mathcal{N} \), \( \angle(\Gamma_{\ell_1}, \Gamma_{\ell_2}) < \pi \), then a solution
\( u \) of (4.5) with the data \( f \in L^2(\Omega)^2 \) and \( g^{(i)} \in H^{1/2}(\Gamma)^2 \), \( \forall \ell \in \mathcal{N} \) satisfies \( u \in H^{3/2+\varepsilon}(\Omega)^2 \) for some \( \varepsilon > 0 \).

**Lemma 4.2.** Let \( S_{2h} = W \cap B_{2h} \) and \( \Gamma_{2h}^\pm = \partial S_{2h} \setminus \partial B_{2h} \), where \( W \) is the infinite sector defined in (2.1) with the opening angle \( \theta_W \in (0, \pi) \). Suppose that \( u \in H^1(B_{2h})^2 \) satisfies

\[
\begin{align*}
\mathcal{L}u^+ + \omega^2 q_- u^- &= 0 && \text{in } S_{2h}, \\
\mathcal{L}u^+ + \omega^2 u^+ &= 0 && \text{in } B_{2h} \setminus S_{2h}, \\
u^+ &= u^- && \text{on } \Gamma_{2h}^\pm,
\end{align*}
\tag{4.6}
\]

where \( u^+ = u|_{B_{2h} \setminus S_{2h}} \), \( u^- = u|_{S_{2h}} \), \( \omega \) is a positive constant and \( q_- \in L^\infty(S_{2h}) \). Assume that \( u^+ \) is real analytic in \( B_{2h} \setminus S_{2h} \). There exists \( \alpha \in (0, 1) \) such that \( u^- \in C^{\alpha}(S_{2h}) \).

**Proof.** Since \( u^+ \) is real analytic in \( B_{2h} \setminus S_{2h} \), we let \( w \) be the analytic extension of \( u^+|_{B_{2h} \setminus \overline{S_{2h}}} \) in \( B_h \). By using the transmission condition on \( \Gamma_h^\pm \), one clearly has that \( u^- = u^+ = w \) on \( \Gamma_h^\pm \). Set \( v = u^- - w \). Let \( I_0 \) denote the line segment with the starting point \( h(\cos \theta_m, \sin \theta_m) \in \Gamma_h^- \) and the ending point \( h(\cos \theta_m, \sin \theta_m) \in \Gamma_h^+ \). It can be directly verified that

\[ \mathcal{L}v = f \quad \text{in } \mathcal{T}_h; \quad \mathcal{T}_h v = g \quad \text{on } I_0; \quad v = 0 \quad \text{on } \Gamma_h^\pm, \]

where \( \mathcal{T}_h \) is the open triangle formed by \( \Gamma_h^\pm \) and \( I_0 \), \( f = -\omega^2(w + q_- u^-) \in L^2(S_{2h})^2 \) and \( g \in C^\infty(\overline{\mathcal{T}_h})^2 \). By virtue of lemma 4.1, one has that \( v \in H^{3/2+\varepsilon}(\mathcal{T}_h)^2 \) with \( \varepsilon \in \mathbb{R}_+ \). Therefore by the Sobolev embedding theorem, it is clear that there exists \( \alpha \in (0, 1) \) such that \( v \in C^{\alpha}(S_{2h})^2 \).

Hence, we readily have that \( u^- \in C^{\alpha}(S_{2h})^2 \).

The proof is complete. \( \square \)

**Remark 4.2.** We would like to point out that the regularity result in lemma 4.2 in general does not hold for the three dimensional case. This is mainly due to the fact that the corresponding regularity result in lemma 4.1 is generically not true around a general polyhedral corner in \( \mathbb{R}^3 \). Hence, we exclude the generalized transmission condition in (1.11) for an admissible elastic scatterer in definition 4.1. That is, we only consider the case \( \eta \equiv 0 \) on the boundary of an admissible elastic scatterer in \( \mathbb{R}^3 \), which is different from the two-dimensional case.

**Theorem 4.1.** Consider the elastic scattering problem (4.1) associated with the incident elastic wave field \( u' \) and two elastic scatterers \( (\Omega; q_j, \eta_j) \) \( (j = 1, 2) \) in \( \mathbb{R}^n \) \((n = 2, 3)\) being admissible scattering configuration. Let \( u^{1,\infty}_j(\hat{x}; u') \) be the far-field pattern associated with the scatterer \( (\Omega_j; q_j, \eta_j) \) and the incident field \( u' \), \( \beta = t, p \) or \( s \). If

\[
u^{1,\infty}_j(\hat{x}; u') = u^{2,\infty}_j(\hat{x}; u'), \quad \hat{x} \in S^{n-1},
\tag{4.7}
\]

for a fixed incident wave \( u' \), then one has that

\[
\Omega_1 \Delta \Omega_2 := (\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)
\tag{4.8}
\]

cannot possess a corner. Hence, if \( \Omega_1 \) and \( \Omega_2 \) are convex polygons in \( \mathbb{R}^2 \) or convex polyhedra in \( \mathbb{R}^3 \), one must have

\[
\Omega_1 = \Omega_2.
\tag{4.9}
\]

**Proof.** We prove (4.8) by contradiction. Suppose that there is a corner contained in \( \Omega_1 \Delta \Omega_2 \). Without loss of generality, we may assume that the vertex \( O \) of the corner \( \Omega_2 \cap W_\mathcal{N}(\theta_W) \) is
such that \( O \in \partial \Omega_2 \) and \( O \notin \overline{\Omega}_1 \). Furthermore, one may assume that in two dimensions, \( O \) is the origin, whereas in three dimensions, the edge corner point \( O = (x'_1, x'_3) \) of the 3D edge corner \( (B_h(x'_1) \times (M, M)) \cap \Omega_2 = \overline{S_h(x'_1)} \times (M, M) \) fulfills that \( x'_1 \) is the origin of \( \mathbb{R}^2 \).

Due to (4.7), applying Rellich’s theorem (see [32, 35]), we know that \( u^+_1 = u^+_2 \) in \( \mathbb{R}^n \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2) \). Thus

\[
u_1(\mathbf{x}) = u_2(\mathbf{x}) \tag{4.10} \]

for all \( \mathbf{x} \in \mathbb{R}^n \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2) \). In what follows, we consider two separate cases.

**Case 1 \((n = 2)\):** following the notations in (2.2) and the setup of theorem 2.3, we have from (4.10) that

\[
u_2^- = u_2^+ = u_1^+, \quad T_h \nu_2^- = T_h u_2^+ + \eta_2 u_2^+ = T_h u_1^+ + \eta_2 u_1^+ \quad \text{on } \Gamma_h^\pm, \tag{4.11} \]

where the superscripts \((-)^-, (\cdot)^+\) stand for the limits taken from \( \Omega_2 \) and \( \mathbb{R}^2 \setminus \overline{\Omega}_2 \) respectively. Moreover, we take \( h \in \mathbb{R}_+ \) sufficiently small such that

\[
\mathcal{L} u_1^+ + \omega^2 u_1^+ = 0 \quad \text{in } B_h, \quad \mathcal{L} u_2^+ + \omega^2 q_2 u_2^+ = 0 \quad \text{in } \Sigma_h. \tag{4.12} \]

Since \((\Omega_j, q_j, \eta_j), \ j = 1, 2\), are admissible, we know that \( q_2 \in C^4(\overline{S_h}) \) and \( \eta_j \in C^4(\overline{\Gamma_h}) \). Clearly \( u_2^- \in H^1(S_h^2) \) and \( u_1^+ \) is real analytic in \( B_2 \setminus \overline{B_2} \). According to lemma 4.2, we know that \( u_2^- \in C^a(S_h^2) \), which implies that \( q_2 u_2^- \in C^a(\overline{S_h}) \). Using the admissibility condition \((b)\) in definition 4.1, by (4.14) and applying theorem 2.3, and also utilizing the fact that \( u_1 \) is continuous at the vertex \( 0 \), we have

\[
u_1(0) = 0, \]

which contradicts to the admissibility condition \((c)\) in definition 4.1.

**Case 2 \((n = 3)\):** since \((\Omega_j, q_j, \eta_j), \ j = 1, 2\), are admissible, we know that \( \eta_2 \equiv 0 \) on \( \partial \Omega_2 \). Therefore, from (4.10), following the setup of theorem 3.1, it yields that

\[
u_2^- = u_2^+ = u_1^+, \quad T_h \nu_2^- = T_h u_2^+ = T_h u_1^+ \quad \text{on } \Gamma_h^+ \times (-M, M). \tag{4.13} \]

Moreover, we take \( h \in \mathbb{R}_+ \) sufficiently small such that

\[
\mathcal{L} u_1^+ + \omega^2 u_1^+ = 0 \quad \text{in } B_h, \quad \mathcal{L} u_2^+ + \omega^2 q_2 u_2^+ = 0 \quad \text{in } \Sigma_h \times (-M, M). \tag{4.14} \]

By the well-posedness of the direct problem (4.1) with \( \eta \equiv 0 \), we know that \( u_2 \in H^1(B_R) \) where \( B_R \) is a ball centered at the origin with the radius \( R \in \mathbb{R}_+ \) such that \( \Sigma_h \times [-M, M] \subseteq B_R \) and \( B_R \in \mathbb{R}^3 \setminus \overline{\Omega}_1 \). Let \( \tilde{q}_2 = q_2 \chi_{\Omega_1} + 1_{\chi_{B_R \setminus \overline{\Omega}_1}} \). Then

\[
\mathcal{L} u_1 + \omega^2 u_1 = 0 \quad \text{in } B_R, \quad \mathcal{L} u_2 + \omega^2 \tilde{q}_2 u_2 = 0 \quad \text{in } B_R. \]

Since \( \tilde{q}_2 \in L^\infty(B_R) \), by the interior elliptic regularity estimate [41], we have \( u_2^- \in H^2(B_R) \), where \( B_R \in B_R \setminus \overline{S_h} \times [-M, M] \). Again using the interior elliptic regularity estimate, we have \( u_2^- \in W^{2,4}(\overline{S_h} \times [-M, M]) \). Using the Sobolev embedding theorem, we have \( u_2^- \in C^{1.1/4}(\overline{S_h} \times [-M, M]) \). Clearly, \( u_2^- \) is real analytic in \( \overline{S_h} \times [-M, M] \). Therefore one has \( u_1^+ - u_1^- \in C^{1.1/4}(\overline{S_h} \times [-M, M]) \). Since \((\Omega_j, q_j, \eta_j), \ j = 1, 2\), are admissible, we know that \( q_2 \in C^4(\overline{S_h} \times [-M, M]) \). Using the admissibility condition \((b)\) in definition 4.1, by (4.14) and applying corollary 3.3, we have

\[
u_1(0) = 0, \]

32
which contradicts to the admissibility condition (c) in definition 4.1.

The conclusion (4.9) can be immediately obtained by using the contradiction argument and (4.8).

The proof is complete. □

Based on definition 4.1, if we further assume that the surface parameter η is constant, we can recover η simultaneously in \( \mathbb{R}^2 \) once the shape of the scatterer, namely \( \Omega \) is determined. However, in determining the surface conductive parameter, we need to assume that \( q_1 = q_2 := q \) are known.

**Theorem 4.2.** Consider the elastic scattering problem (4.1) in \( \mathbb{R}^2 \) associated with the incident elastic wave field \( u' \) and two elastic scatterers \( (\Omega_j; q, \eta_j) \) being admissible scattering configuration, where \( \Omega_j = \Omega \) for \( j = 1, 2 \) and \( \eta_j \neq 0 \), \( j = 1, 2 \), are two constants. Let \( u_{j}^{\omega, \infty}(x; u') \) be the far-field pattern associated with the scatterer \( (\Omega_j; q, \eta_j) \) and the incident field \( u' \), \( \beta = t, p \) or \( s \). Suppose that

\[
u_{j}^{\omega, \infty}(x; u') = u_{j}^{\omega, \infty}(x; u'), \quad x \in \mathbb{S}^1, \quad (4.15)\]

for a fixed incident wave \( u' \). Then if \( \omega \) is not an eigenvalue of the partial differential operator \( \mathcal{L} + \omega^2 q \) in \( H^1_0(\Omega) \), we have \( \eta_1 = \eta_2 \).

**Proof.** Due to (4.15), we have \( u_{j}^{\omega, \infty} = u_{j}^{\omega} \) for all \( x \in \mathbb{R}^2 \setminus \Omega \) and thus \( T_{\omega} u_{j}^{\omega} = T_{\omega} u_{j} \) on \( \partial \Omega \). Combining with the transmission condition in the scattering problem (4.1), we deduce that

\[
u_{j} = u_{j}^{\omega} + u_{j}^{\omega} = u_{j}^{\omega} \quad \text{on} \quad \partial \Omega.\]

Thus, we have

\[
T_{\omega}(u_{j}^{\omega} - u_{j}^{\omega}) = T_{\omega}(u_{j}^{\omega} - u_{j}^{\omega}) + \eta_1 u_{j}^{\omega} - \eta_2 u_{j}^{\omega} = (\eta_1 - \eta_2) u_{j}^{\omega} \quad \text{on} \quad \partial \Omega.\]

Set \( v := u_{j}^{\omega} - u_{j}^{\omega} \). Then \( v \) fulfills

\[
\begin{cases}
(\mathcal{L} + \omega^2 q)v = 0 & \text{in} \quad \Omega, \\
v = 0 & \text{on} \quad \partial \Omega, \\
T_{\omega}v = (\eta_1 - \eta_2) u_{j}^{\omega} & \text{on} \quad \partial \Omega.
\end{cases} \quad (4.16)
\]

Since \( \omega \) is not an eigenvalue of the operator \( \mathcal{L} + \omega^2 q \) in \( H^1_0(\Omega) \), one must have \( v = 0 \) to (4.16).

Substituting this into the Neumann boundary condition of (4.16), we know that \( (\eta_1 - \eta_2) u_{j} = T_{\omega} v = 0 \) on \( \partial \Omega \).

Next, we prove the uniqueness of \( \eta \) by contradiction. Assume that \( \eta_1 \neq \eta_2 \). Since \( (\eta_1 - \eta_2) u_{j} = 0 \) on \( \partial \Omega \) and \( \eta_j \), \( j = 1, 2 \) are constants, we can deduce that \( u_{j} = 0 \) on \( \partial \Omega \). Then \( u_{j} \) satisfies

\[
\begin{cases}
(\mathcal{L} + \omega^2 q)u_{j} = 0 & \text{in} \quad \Omega, \\
u_{j} = 0 & \text{on} \quad \partial \Omega.
\end{cases} \quad (4.16)
\]

Similar to (4.16), this Dirichlet problem also only has a trivial solution \( u_{j} = 0 \) in \( \Omega \) due to that \( \omega \) is not an eigenvalue of \( \mathcal{L} + \omega^2 q \) in \( H^1_0(\Omega) \). Hence, we can derive \( u_{j} = u_{j} = 0 \) and

\[
T_{\omega} u_{j}^{\omega} = T_{\omega} u_{j}^{\omega} + \eta_1 u_{j}^{\omega} = T_{\omega} u_{j}^{\omega} = 0 \quad \text{on} \quad \partial \Omega,
\]

which contradicts to the admissibility condition (c) in definition 4.1.
which implies that \( u_1 \equiv 0 \) in \( \mathbb{R}^n \) and thus \( u_1^c = -u_1 \). This contradicts the fact that \( u_1^c \) satisfies the Kupradze radiation condition.

The proof is complete. \( \square \)

**Remark 4.3.** In theorem 4.2, it is required that \( \omega \) is not an eigenvalue of \( L + \omega^2 q \) in \( H^1_0(\Omega) \). Clearly, if \( q \) is negative-valued in \( \Omega \), this condition is obviously fulfilled. On the other hand, if \( q \) is positive-valued in \( \Omega \), then this condition can be readily fulfilled when \( \omega \in \mathbb{R}_+ \) is sufficiently small.

### 5. Concluding remarks

In this paper we have established a local geometrical characterization of a generalized elastic transmission eigenfunctions near a planar sector or a three dimensional edge corner, where the corresponding eigenfunctions fulfill certain regularity criteria. In fact, if the corresponding generalized elastic transmission eigenfunctions can be approximated by the elastic Herglotz wave functions with the associated kernel fulfilling certain increasing property or they are Hölder continuous near the corner, we showed that they must vanish near the underlying corner. The aforementioned findings generalize the corresponding one in [7] by introducing a more general transmission boundary condition of (1.11) and relaxing the regularity requirement in [7], where only Hölder continuity of the transmission eigenfunction \( (v, w) \) and \( v - w \in H^2 \) was considered. As an important physical application, the spectral result in this paper can be used to establish a novel unique identifiability for determining the shape (as well as the corresponding boundary parameter in \( \mathbb{R}^2 \)) of a polygonal or polyhedral elastic medium scatterer by a single far-field measurement.

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### Data availability statement

No new data were created or analysed in this study.

### ORCID iDs

Hongyu Liu [https://orcid.org/0000-0002-2930-3510](https://orcid.org/0000-0002-2930-3510)

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