integral points of bounded height
on a log fano threefold

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abstract. we determine an asymptotic formula for the number of integral
points of bounded height on a blow-up of \( \mathbb{P}^3 \) outside certain planes using
universal torsors.

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1. introduction

manin’s conjecture [fmt89, bm90] is concerned with the number of rational
points on fano varieties \( X \) (that is, smooth, projective varieties with ample anti-
canonical bundle \( \omega_X^\vee \)) over a number field \( K \) with zariski dense \( K \)-rational points.
we may associate height functions \( H : X(K) \to \mathbb{R}_{>0} \) with the anticanonical bundle.
manin’s conjecture gives a prediction for the number of rational points of bounded
anticanonical height that lie in the complement \( V \) of all accumulating subvarieties,
whose rational points would dominate the total number. more precisely, it predicts
that the number of rational points of bounded height
\[
\# \{ x \in V(K) \mid H(x) \leq B \}
\]
grows asymptotically as \( c B (\log B)^{r-1} \), where \( r \) is the picard number of \( X \).

peyre [pey95, pey03] gave a conjectural interpretation of the constant \( c \) as a
product \( \alpha \beta \tau \), where \( \alpha \) depends on the geometry of the effective cone, \( \beta \) is a coho-
mological constant connected to the brauer group and \( \tau \) is an adelic volume that
can be interpreted as a product of local densities. such asymptotics are in par-
ticular known for generalized flag varieties [fmt89], toric varieties [bt98], equi-
variant compactifications of vector groups [clt02], and some smooth del pezzo
surfaces [bre02, bf04, bb11].

fano threefolds were classified by iskovskih, mori and mukai [isk77, mm82].
for these, manin proved a lower bound for the number of rational points after a
finite extension of the base field [man93]. those fano threefolds that are toric or
additive and for which manin’s conjecture is thus known have been classified by
batyrev [bat81] and huang–montero [hm18], respectively. besides such results for
general classes of varieties, manin’s conjecture for fano threefolds remains open.

date: january 19, 2021.
2010 mathematics subject classification. primary 11d45; secondary 11g35, 14g05.
On proper varieties, integral points on an integral model and rational points coincide as a consequence of the valuative criterion for properness. A set-up concerning integral points on a non-proper variety analogous to Manin’s conjecture is the following: Consider a smooth log Fano variety over a number field $K$, by which we shall mean a smooth, projective variety $X$ together with a reduced, effective divisor $D$ with strict normal crossings over an algebraic closure such that the log-anticanonical bundle $\omega_X(D)^\vee$ is ample. Let $H$ be a log-anticanonical height function, let $U$ be a flat integral model of $X-D$ and consider the complement $V \subset X$ of those subvarieties whose points would dominate the number of integral points on $U$. How does the number of integral points of bounded height
\[ \# \{ x \in U(\mathcal{O}_K) \cap V(K) \mid H(x) \leq B \} \]
behave asymptotically?

Results in this direction include complete intersections of large dimension compared to their degree [Bir62], algebraic groups and homogeneous spaces [DRS93, EMS96, EM93, BR95, Mau07, GOS09, WX16], and partial equivariant compactifications [CLT10b, CLT12, TBT13], that is, equivariant compactifications $X$ together with an invariant divisor $D$. The first case is an application of the circle method; for the latter cases, the group structure is exploited by means of harmonic analysis or similar methods.

In [CLT10a], Chambert-Loir and Tschinkel describe a framework allowing a geometric interpretation of such asymptotic formulas. These results suggest that the asymptotic for a split variety $X$ (i.e., such that $\text{Pic}(X) \to \text{Pic}(X_{\overline{\mathbb{Q}}})$ is an isomorphism) over the field $K = \mathbb{Q}$ of rational numbers, with a geometrically integral divisor $D$, has the form
\[ \alpha \tau_\infty \tau_\text{fin} B(\log B)^{-1}(1 + o(1)), \]
where $\alpha$ depends on the geometry of the effective cone, $\tau_\infty$ is a Tamagawa volume of the boundary $D(R)$ and $\tau_\text{fin}$ is product of local volumes of integral points $U(\mathbb{Z}_p)$.

Our main result is such an asymptotic formula for a log Fano threefold that does not belong to any of the above classes. To this end, we parametrize the integral points using universal torsors. Universal torsors have been defined and studied by Colliot-Thélène and Sansuc [CTS87]; their usage to count rational points goes back to Salberger [Sal98], who used them to reprove Manin’s conjecture for toric varieties. Since then, the technique has been used to count rational points on many other varieties. This is the first application of the torsor method to integral points.

We will count integral points on a smooth log Fano threefold $(X, D)$, where $X$ is in particular Fano, has Picard number 2 and is of type 30 in the classification of Fano threefolds [MM82]. Let $\pi: X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3 = \text{Proj} \mathbb{Q}[a, b, c, d]$ along the smooth conic $C = V(a^2 + bc, d)$. We will provide asymptotic formulas for the number of integral points on $X-D_i$, where $D_i$ is the preimage $\pi^{-1}(V(b))$ of a plane intersecting $C$ twice in one rational point and $D_2$ is the preimage $\pi^{-1}(V(a))$ of a plane intersecting $C$ in two rational points. Up to $\mathbb{Q}$-automorphism, these are precisely the planes intersecting $C$ in rational points. To construct integral models $\mathcal{U}_i$ of $U_i = X-D_i$, we consider the blow-up $\mathcal{X}$ of $\mathbb{P}^3$ along $V(a^2 + bc, d)$ and define $\mathcal{U}_1 = \mathcal{X} - D_1, \mathcal{U}_2 = \mathcal{X} - D_2$. Manin’s conjecture for rational points on this variety is known by [CLT02], since it is a compactification of $G^1_2$, so it provides a natural starting point for the investigation of integral points on threefolds by new methods. Note that even though the complete variety $X$ is an equivariant compactification, the open subvarieties $U_i$ whose integral points we are counting are not partial equivariant compactifications (Lemma 2.4 and Remark 2.5), so our result is not a special case of [CLT12].
We describe the sets of integral points explicitly by a universal torsor in Section 2. In Section 3, we construct a log-anticanonical height function \( H : X(\mathbb{Q}) \to \mathbb{R}_{>0} \), measures \( \tau_{(X,D_i),p} \) on \( X(\mathbb{Q}_p) \) and \( \tau_{D_i,\infty} \) on \( D_i(\mathbb{R}) \) together with convergence factors associated with the Artin \( L \)-function of the virtual Galois module 
\[
\text{EP}(U_i) = [H^0(U_i,\mathbb{Q},\mathbb{G}_m)/\mathbb{Q}] - [H^1(U_i,\mathbb{Q},\mathbb{G}_m) \otimes \mathbb{Q}],
\]
defined in [CLT10a], and a renormalization factor \( c_R \). We continue with a description of a constant \( \alpha \) and the exponent of log \( B \) in the expected asymptotic.

In Sections 4 and 5, we prove an asymptotic formula for the number of integral points of bounded height on \( U_1 \) and \( U_2 \). A comparison of these formulas with the computations in the preceding section results in the following:

**Theorem 1.1.** For \( i \in \{1, 2\} \), let \( X, X', D_i, U_i, \) and \( H \) be as above. There exists an open subvariety \( V_i \subset X \) such that the number of integral points of bounded height \( N_i(B) = \#\{x \in U_i(\mathbb{Z}) \cap V_i(\mathbb{Q}) \mid H(x) \leq B\} \) satisfies the asymptotic formula
\[
N_i(B) = \alpha_i \tau_{i,\text{fin}} \tau_{i,\infty} B \log B(1 + o(1)),
\]
where
\[
\alpha_i = \frac{1}{(\text{rk Pic}(X) - 1)!} \chi_{\text{H}^0(X)}(\omega^i_{X(D_i)})^\vee,
\]
\[
\tau_{i,\text{fin}} = L(1; \text{EP}(U_i)) \prod_p L_p(1; \text{EP}(U_i)) \tau_{(X,D_i),p}(U_p(\mathbb{Z}_p)), \text{ and}
\]
\[
\tau_{i,\infty} = c_R \tau_{D_i,\infty}(D_i(\mathbb{R})).
\]
More explicitly, we have
\[
N_1(B) = \frac{20}{3\zeta(2)} B \log B + O(B) \quad \text{and}
\]
\[
N_2(B) = \frac{20}{3} \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right) B \log B + O(B(\log \log B)^2).
\]

**Acknowledgments.** Parts of this article were prepared at the Institut de Mathématiques de Jussieu – Paris Rive Gauche, supported by DAAD. I wish to thank Antoine Chambert-Loir for his remarks and the institute for its hospitality.

## 2. A Universal Torsor

The Cox ring of \( X \) over \( \mathbb{Q} \) is
\[
R(X) = \bigoplus_{d \in \text{Pic}(X)} H^0(X, \mathcal{L}_d),
\]
where \( (\mathcal{L}_d)_d \) is a suitable system of representatives of every class in the geometric Picard group; its ring structure is induced by the sum and tensor product of sections. By [DHH+15, Theorem 4.5, Case 30] (which contains a typo in the degrees of \( x \) and \( y \)), it is
\[
R(X) = \mathbb{Q}[a, b, c, x, y, z]/(a^2 + bc - yz),
\]
and its grading by \( \text{Pic}(X) \cong \mathbb{Z}^2 \) is given by
\[
\begin{array}{ccccccc}
a & b & c & x & y & z \\
1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & -1 & -1 & 1
\end{array}
\]
The pullbacks of planes along \( \pi \) correspond to degree \([1, 0]\), the exceptional divisor \( E \) to degree \([0, 1]\), and the anticanonical bundle thus to degree \([4, -1]\).
Lemma 2.1. The variety
\[ T_{\mathcal{X}} = \text{Spec } R(\mathcal{X}_{\mathbb{Q}}) - V(I_{\text{irr}}), \]
where \( I_{\text{irr}} = (a, b, c, z)(x, y) \), is a universal torsor over \( \mathcal{X}_{\mathbb{Q}} \).

Proof. In addition to the ring itself, we argue using the bunch of cones \( \Phi \) associated with \( X \) \cite[3.2]{ADHL15}. It consists of all cones \( \text{Cone}(\{\deg(t) \mid t \in M\}) \) generated by the degrees \( \deg(t) \in \text{Pic}(X)_{\mathbb{Q}} \) of a subset \( M \subset \{a, b, c, x, y, z\} \) of the generators satisfying the following: We have \( \prod_{t \in M} t \notin \sqrt{\{t \mid t \notin M\}} \), that is, the equation \( a^2 + bc - yz \) has a solution with \( t = 0 \) for \( t \notin M \) and \( t \neq 0 \) for \( t \in M \); and we have \( \omega^* \in \text{Cone}(\{\deg(t) \mid t \in M\}) \). The bunch of cones is thus 
\[ \Phi = \{\text{Cone}([0], [1]), \text{Cone}([1], [1]), \text{Cone}([0], [1]), \text{Cone}([0], [2]), \text{Cone}([1], [2])\}, \]
given by, for example, the generators \( \{b, x\}, \{z, x\}, \{a, y\}, \{a, y, z\} \), respectively (these are all possible cones containing the anticanonical bundle); the condition is seen to hold by considering the solution \((0, 1, 0, 1, 0, 0), (0, 0, 0, 1, 0, 1), (0, 1, 0, 0, 1, 0), (1, 0, 0, 0, 1, 1)\), respectively. Indeed, \( X \) is defined by a bunched ring with a maximal bunch by \cite[Theorem 3.2.1.9 (ii)]{ADHL15}, which can only be the bunch \( \Phi \) just defined.

The irrelevant ideal \( I_{\text{irr}} \) is generated by all elements of the form \( \prod_{t \in M} t \) such that \( M \) is a subset of the generators satisfying \( \text{Cone}(\deg(t) \mid t \in M) \in \Phi \). This yields
\[ I_{\text{irr}} = (ax, bx, cx, zx, ay, by, cy, zy) = (a, b, c, z)(x, y), \]
since the minimal subsets \( M \) suffice. \( \square \)

Denote by \( p: T \to X \) a morphism rendering \( T \) a universal torsor. We note that the composition of morphisms \( T \to X \to \mathbb{P}^3 \) maps \((a, b, c, x, y, z) \to (a : b : c : xz)\), that \( V(x) \subset T \) is the preimage of the strict transform of \( V(c) \subset \mathbb{P}^3 \), that \( V(y) \subset T \) is the preimage of the strict transform of \( V(a^2 + bc) \), and that \( V(z) \subset T \) is the preimage of the exceptional divisor \( E \subset X \). Next, we construct an integral model of this torsor. Consider the ring
\[ R_Z = \mathbb{Z}[a, b, c, x, y, z]/(a^2 + bc - yz) \]
and the ideal \( I_{\text{irr}, Z} = (a, b, c, z)(x, y) \subset R_Z \).

Lemma 2.2. The scheme \( T = \text{Spec } R_Z - V(I_{\text{irr}, Z}) \) is a \( \mathbb{G}^2_{m, Z} \)-torsor over \( X \).

Proof. We note that removing \( yz \) from the generators of \( I_{\text{irr}} \) does not change the radical of the ideal and that the degrees of the two factors of any of the remaining generators \( f_i \in \{ax, bx, cx, zx, ay, by, cy\} \) form a basis of the Picard group. Thus, \cite[Theorem 3.3]{FP16} shows that \( T = \text{Spec } R_Z - V(I_{\text{irr}, Z}) \) is a \( \mathbb{G}^2_{m, Z} \)-torsor over the \( Z \)-scheme \( X' \) obtained by gluing the spectra of the degree-0-parts \( R_Z[f_i^{-1}(0)] \) of the localizations in the generators \( f_i \) of the irrelevant ideal.

This integral model \( X' \) of \( \mathcal{X}_{\mathbb{Q}} \) coincides with the blow-up \( X \). Indeed, we can embed both the Cox ring \( R_Z \) and the Rees algebra
\[ A = \bigoplus_{n \geq 0} I^n = \mathbb{Z}[a, b, c, d]/[(a^2 + bc)\xi, d\xi] \]
for \( I = (a^2 + bc, d) \) into the field \( \mathbb{Q}(a, b, c, d, \xi) = \text{Frac}(A) \), where the first embedding maps \( z \mapsto \xi^{-1}, x \mapsto d\xi \), and \( y \mapsto (a^2 + bc)\xi \). The blow-up is then given by gluing the spectra of the seven rings \( A_s, t \subset \text{Frac}(A) \) arising the following way: First take the degree-0-part (with respect to the usual grading of \( \mathbb{Z}[a, b, c, d] \), not considering the natural grading of the Rees algebra) of the localizations of \( A \) in \( s \in \{a, b, c, d\} \), then further localize in one of the generators \( t \in \{\frac{a^2 + bc}{x^2} \xi, \frac{a^2 + bc}{z^2} \xi\} \) \((t = \xi \text{ suffices for } s = d)\) of the Rees algebra and take the degree-0-part with respect to the grading.
induced by the natural grading of the Rees algebra. The rings $R_{\mathbb{Z}}[f^{-1}]^{(0)}$ for $f$ in $ax, bx, cx, zy, ay, by, cy$ coincide with the rings $A_{s,t}$ for $(s,t)$ in 

\[(a, d\xi/a), (b, d\xi/b), (c, d\xi/c), (d, \xi), (a, (a^2 + bc)\xi), (b, (a^2 + bc)\xi), (c, (a^2 + bc)\xi),\]

so the two schemes defined by the blow-up and [FP16, Construction 3.1] coincide. 

\[\square\]

**Lemma 2.3.** The morphism $p$ induces a 4-to-1-correspondence between integral points on $X$ and 

\[(1) \quad T(\mathbb{Z}) = \{(a, b, c, x, y, z) \in \mathbb{Z}^6 \mid a^2 + bc - yz = 0, \gcd(a, b, c, z) = \gcd(x, y) = 1\} ,\]

between integral points on $U_1$ and 

\[(2) \quad T_1(\mathbb{Z}) = \{(a, b, c, x, y, z) \in \mathbb{Z}^6 \mid a^2 + bc - yz = 0, b = \pm 1, \gcd(x, y) = 1\} ,\]

and between integral points on $U_2$ and 

\[(3) \quad T_2(\mathbb{Z}) = \{(a, b, c, x, y, z) \in \mathbb{Z}^6 \mid a^2 + bc - yz = 0, a = \pm 1, \gcd(x, y) = 1\} .\]

**Proof.** The fiber $f^{-1}(P)$ of any point $P \in \mathcal{X}(\mathbb{Z})$ is a $\mathbb{G}_a^2$-$\mathbb{Z}$-torsor. Since such torsors are parameterized by $H^2_{\text{fppf}}(\text{Spec}\, \mathbb{Z}, \mathbb{G}_a^2) = \text{Cl}(\mathbb{Z})^2 = 1$, all fibers are isomorphic to $\mathbb{G}_m^2$, and we get a 4-to-1-correspondence between integral points on the torsor $T$ and those on $X$.

Since $T$ is quasi-affine, its integral points have a description as lattice points satisfying the equation of the Cox ring and coprimality conditions given by the irrelevant ideal. Points on the preimages of $U_1$ and $U_2$ under the morphism $p : T \to X$ are defined by the additional condition $(b) = 1$ and $(a) = 1$, respectively. 

\[\square\]

We conclude this section with some observations on the geometry of $X$.

**Lemma 2.4.** There is no action of $\mathbb{G}_a^3$ on $X$ with an open orbit under which $D_1$ or $D_2$ are invariant, neither is $X$ toric.

**Proof.** Since $\mathbb{G}_a^3$ has to act continuously on $\text{Pic}(X)$, the exceptional divisor has to be invariant and we thus get an action on $\mathbb{P}^3 - C$. If one of the planes not containing $C$ is invariant, the action further restricts to the complement $\mathbb{A}^3 - C$ of a conic in $\mathbb{A}^3$. Since the action needs to have an open orbit, we would get an open immersion $\mathbb{A}^3 \to \mathbb{A}^3 - C$, an impossibility by Ax–Grothendieck.

Since its Cox ring is not polynomial, $X$ cannot be toric, cf. [HK00]. 

\[\square\]

**Remark 2.5.** The total variety $X$ is a compactification of $\mathbb{G}_a^3$, as classified by Huang and Montero [HM18] (induced by the action of $\mathbb{G}_a^3$ on $\mathbb{P}^3$, where the group acts trivially on the plane $V(d)$ and by addition on the complement). Manin’s conjecture for rational points [CLT02] and asymptotics for integral points on some open subvarieties [CLT12] are known due to Chambert-Loir and Tschinkel: The admissible divisors $D$ are the exceptional divisor, the strict transform of $V(d)$, and their sum. Even though $X$ is an equivariant compactification of $\mathbb{G}_a^3$, the pairs $(X,D_i)$ are neither partial equivariant compactifications of $\mathbb{G}_a^3$ nor toric by the previous lemma. Our result is thus not a special case of [CLT10b] or [CLT12].

Lastly, we can describe the geometric Picard group with the information we gathered in the proof of Lemma 2.1: The pseudo-effective cone is generated by the degrees of the generators of the Cox ring, so $\underline{\text{Eff}}(X) = \text{Cone}(E, H - E)$. The semi-ample cone is the intersection of all cones in $\Phi$ and thus $\text{SAmp}(X) = \text{Cone}(H, 2H - E)$. In particular, the log-anticanonical bundles 

\[\omega(D_1)^{\vee} \cong \omega(D_2)^{\vee} \cong \mathcal{O}_X(3H - E)\]

are in its interior, hence ample.
3. Metrics, heights, Tamagawa measures, and predictions

3.1. Adelic metrics. To fix notation, we start by recalling the definition of adelic metrics and methods to construct them, as found for example in [Pey03]. An adelic metric on a line bundle \( L \) on our smooth, projective variety \( X \) is a collection of norms \( \| \cdot \|_v : L(x_v) \to \mathbb{R}_{\geq 0} \) for any completion \( K_v \) of \( K \) and any \( K_v \)-point \( x_v \in V(K_v) \) that satisfy the following conditions:

(1) For every local section \( s \in \Gamma(U, L) \), the map \( U(K_v) \to \mathbb{R}_{\geq 0} : x_v \mapsto \| s(x_v) \|_v \) is continuous with respect to the analytic topology.

(2) For almost all finite places \( v \), the norm is defined by an integral model \( V \) of \( V \) and \( \mathcal{L} \) of \( L \) over \( V \) in the following way: Since \( V \) is proper, any point \( x_v \in V(K_v) \) lifts uniquely to a point \( \tilde{x}_v \in \mathcal{V}(\mathcal{O}) \). Then \( \tilde{x}_v^* \mathcal{L} = \mathcal{L}(\tilde{x}_v) \) is a lattice in \( x^* \mathcal{L} = \mathcal{L}(x_v) \), and we take the unique norm \( \| \cdot \|_v \) on \( \mathcal{O}(x) \) that assigns to any generator of \( \mathcal{L}(\tilde{x}_v) \) the norm 1. Since any two flat models are isomorphic over almost all finite places \( v \), this is independent of the choice of a model.

There are several methods to construct adelic metrics:

- **Pull-backs.** Let \( f : V \to V' \) be a morphism between smooth, projective \( K \)-varieties and consider an adelic metric on a line bundle \( \mathcal{L} \) on \( V' \). Then we get an adelic metric on \( f^* \mathcal{L} \) in the following way: Locally, any section of \( f^* \mathcal{L} \) has the form \( s' = h \cdot f^* s \) for local sections \( s \) of \( \mathcal{L} \) and \( h \) of \( V \). We set \( \| s'(x_v) \|_v = \| h(x_v) \|_v \| f^* s(x_v) \|_v \).

- **Tensor products and inverses.** If \( \mathcal{L} \) and \( \mathcal{L}' \) are metrized line bundles, there is an induced metric on \( \mathcal{L} \otimes \mathcal{L}' \) defined by \( \| s \otimes s' \|_v = \| s \|_v \| s' \|_v \) and an induced metric on \( \mathcal{L}'^{-1} \) defined by \( \| h(x) \|_v = \| (h(s))(x) \|_v \| s(x) \|_v^{-1} \), independent of the choice of a local section \( s \) of \( \mathcal{L} \) that does not vanish in \( x \).

- **Base point free bundles.** There is a canonical adelic metric on \( \mathcal{O}_{\mathbb{P}^n}(1) \): Any section \( s \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n}(1)) \) is a homogeneous rational function in \( x_0, \ldots, x_n \) of degree 1 that is defined everywhere on \( U \). Thus, for any point \( x = (x_0 : \cdots : x_n) \in U(K_v) \), the norm \( \| s(x) \|_v \cdot \max\{ |x_i|_v \}^{-1} \) is well-defined. This norm is defined by the integral model \( \mathbb{P}^n_{K_v} \) at all finite places. Using this, we can associate a metric with any base point free line bundle \( \mathcal{L} \) together with a set of global sections \( s_0, \ldots, s_n \) that do not vanish simultaneously: We have a morphism

\[
    f : V \to \mathbb{P}^n, x \mapsto (s_0(x) : \cdots : s_n(x))
\]

with \( \mathcal{L} \cong f^* \mathcal{O}_{\mathbb{P}^n}(1) \). Then the pull-back construction gives a metric induced by

\[
    \| f^* s(x) \|_v = \frac{|s(f(x))|}{\max\{|s_0(x)|_v, \ldots, |s_n(x)|_v\}}
\]

for rational functions \( s \) as above.

Since every line bundle on a smooth, projective variety is a quotient of very ample bundles, this allows the construction of metrics on any bundle.

- **Metrics on \( X \).** Returning to our variety \( X \), we endow certain line bundles with adelic metrics. For fixed \( d \in \text{Pic}(X) \), the elements of degree \( d \) in the Cox rings are the global sections of a line bundle \( \mathcal{L}_d \) with isomorphism class \( d \) (such that \( \mathcal{L}_d \otimes \mathcal{L}_e = \mathcal{L}_{d+e} \) by the construction of the Cox ring). We consider the bundles \( \mathcal{L}_{[3,-1]} \) and \( \mathcal{L}_{[1,0]} \) that are isomorphic to the log-anticanonical bundles \( \omega_X(D_1)^\vee \cong \omega_X(D_2)^\vee \) and the pullback of the tautological bundle \( \pi^* \mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \). Neither of the sets

\[
    \{a^2x, b^2x, c^2x, z^2x^3, ay, by, cy\} \quad \text{and} \quad \{a, b, c, xz\}
\]
of sections of these bundles vanish simultaneously, so (4) gives us the metrics
\[(5) \quad \langle s, (a:b:c:x:y:z) \rangle \mapsto \max\{|a|^2x_1, |b|^2x_1, |c|^2x_1, |z|^2x_1, |ay|, |by|, |cz|\}
\]
on $\mathcal{L}_{[3,-1]}$ and
\[(6) \quad \langle t, (a:b:c:x:y:z) \rangle \mapsto \max\{|a|^2, |b|^2, |c|^2, |x|^2\}
\]
on $\mathcal{L}_{[1,0]}$, where $(a:b:c:x:y:z)$ is the image of $(a,b,c,x,y,z) \in T(\mathbb{Q}_v)$ (i.e., a point in Cox coordinates) in $X(\mathbb{Q}_v)$, $s \in R(X)$ has degree $[3,-1]$, and $t \in R(X)$ has degree $[1,0]$. 

3.2. Heights. Any line bundle $\mathcal{L}$ on a smooth, projective variety $V$ over a number field $K$ together with an adelic metric determines a height function
\[H: V(K) \to \mathbb{R}_{\geq 0}, \quad x \mapsto \prod_v \|s(x)\|_v^{-1},\]
where $s$ is a section that does not vanish in $x$. Since $\prod_v |a|_v = 1$ for all $a \in K$, this does not depend on the choice of $s$. The number of rational points of bounded height $\#\{x \in V(K) \mid H(x) \leq B\}$ is finite if the line bundle $\mathcal{L}$ is ample.

The chosen metric on $\mathcal{L}_{[3,-1]}$ defines a log-anticanonical height function on $X$, which we can easily describe in Cox coordinates: Since $X$ is proper, every rational point in $X(\mathbb{Q})$ lifts to a unique integral point in $X(\mathbb{Z})$, which in turn corresponds to four integral points $(a, b, c, x, y, z) \in T(\mathbb{Z})$ by Lemma 2.3. By the coprimality condition and the equation, no prime can divide all of the monomials in the denominator of (5). Thus we get
\[(7) \quad H(a:b:c:x:y:z) = \max\{|a|^2x_1, |b|^2x_1, |c|^2x_1, |z|^2x_1, |ay|, |by|, |cz|\}
\]
for the image $(a:b:c:x:y:z) \in X(\mathbb{Q})$ of $(a,b,c,x,y,z) \in T(\mathbb{Z})$ (with the usual real absolute value).

3.3. Tamagawa measures. An adelic metric on the canonical bundle $\omega_V$ of a smooth, projective variety $V$ over a number field $K$ determines a Borel measure on the $K_v$-points $V(K_v)$ for all places $v$, called a Tamagawa measure. In local coordinates $x_1, \ldots, x_n$, it is given by $d\tau_{V,v} = \|dx_1 \wedge \cdots \wedge dx_n\|^{-1}_v d\mu_v$, where $\mu_v$ is the Haar measure with $\mu_v(s_v) = 1$ for finite places, $d\mu_v = dx$ the usual Lebesgue measure for real places and $d\mu_v = 2dx dy$ for complex places.

Let $D$ be a divisor on $V$ with strict normal crossings (over $\overline{K}$), and let $U = V - D$. We consider the following measures defined by Chambert-Loir and Tschinkel in [CLT10a]: A metric on $\mathcal{O}_V(D)$ induces another measure $\tau_{V,D,v}$ on $V(K_v)$ via $d\tau_{V,D,v} = \|1_D\|^{-1}_v d\tau_{V,v}$, where $1_D$ is the canonical section of $\mathcal{O}_V(D)$. For a component $D'$ of $D$, metrics on $\omega_V$ and $\mathcal{O}_V(D')$ determine an adelic metric on the bundle $\omega_V(D')$ and, using the adjunction isomorphism $\omega_D' \cong \omega_V(D'|D)$, on the canonical bundle $\omega_D'$ of $D'$. This metric defines a residue measure $\tau_{D',v}$ on $D'(K_v)$, and the process can be repeated to define measures on intersections of components of $D$.

These measures are renormalized with factors associated with the virtual Galois module
\[\text{EP}(U) = [H^0(U_{\overline{K}}, \mathbb{G}_m)/\overline{K} \otimes \mathbb{Q}] - [H^1(U_{\overline{K}}, \mathbb{G}_m) \otimes \mathbb{Q}].\]
The Artin $L$-function of a continuous representation $M$ of the absolute Galois group $\Gamma_K$ is the Euler product
\[L(s; M) = \prod_v L_v(s; M)\]
with local factors

\[ L_v(s; M) = \det((\text{id} - q_{\overline{v}}^{-s} \text{Fr}_v)|_{M^s})^{-1}, \]

where Fr_v is a geometric Frobenius element, I_v is an inertia subgroup at v, and q_v is the cardinality of the residue field k_v at v. The L-function of this virtual module is the quotient of the two L-functions. In particular, the Artin L-function of a trivial representation is simply \( \prod_v (1 - q_{\overline{v}}^{-s})^{-\dim M} \).

The residue measures at the infinite places are renormalized by factors depending on the fields of definition of the divisor components. For a geometrically integral divisor \( D \), this is simply a factor of \( c_1 \) at any real place and \( c_\mathbb{C} = 2\pi \) at any complex place; see [CLT10a, 3.1.1, 4.1] for details.

To explicitly calculate volumes with respect to such measures on our variety \( X \), we need metrics on the bundles \( \omega_X \), \( \mathcal{O}(D) \) and \( \mathcal{O}(D^2) \), not just on bundles isomorphic to them. To this end, we choose isomorphisms between those bundles and the bundles \( \mathcal{L}_{[4,-1]} \) and \( \mathcal{L}_{[1,0]} \) and identify sections corresponding under those isomorphisms. Up to scalar, the canonical section \( 1_{D} \) (resp. \( 1_{D^2} \)) is the unique section of \( \mathcal{O}(D) \) (resp. \( \mathcal{O}(D^2) \)) corresponding to \( D \) (resp. \( D^2 \)). This also holds for the elements \( b \) (resp. \( a \)) of the degree-[1,0]-part of the Cox ring (regarded as the global sections of the bundle \( \mathcal{L}_{[1,0]} \), so there are isomorphisms with \( 1_D \mapsto b \) and \( 1_{D^2} \mapsto a \), and we will use these. For the (anti-)canonical bundle, we consider the chart

\[ f : V \to \mathbb{A}^3, \quad (a : b : c : x : y : z) \mapsto \left( \frac{a}{xz}, \frac{b}{xz}, \frac{c}{xz} \right) \]

and its inverse

\[ g : \mathbb{A}^3 \to V, \quad (a, b, c) \mapsto (a : b : c : 1 : a^2 + bc : 1), \]

where \( V = X - V(xz) = \pi^{-1}(V(d)) \cong \mathbb{A}^3 \). The sections \( da \wedge db \wedge dc \) and \( \frac{da}{a} \wedge \frac{db}{b} \wedge \frac{dc}{c} \) of the canonical and anticanonical bundle have neither zeroes or poles on \( \mathbb{A}^3 \cong V \), and their tensor product is 1. Up to scalar, they are the only sections with this property. Since the analogous property holds for \( x^{-4}z^{-3} \) and \( x^4z^3 \), we can fix isomorphisms identifying \( da \wedge db \wedge dc \) with \( x^{-4}z^{-3} \) and \( \frac{da}{a} \wedge \frac{db}{b} \wedge \frac{dc}{c} \) with \( x^4z^3 \).

Finite places. For any prime \( p \), we equip \( \mathbb{Q}_p \) with the Haar measure \( \mu \), normalized such that \( \mu(\mathbb{Z}_p) = 1 \).

**Lemma 3.1.** For any prime \( p \), we have

\[ \tau_{(X,D),p}(\mathcal{U}_1(\mathbb{Z}_p)) = 1 + \frac{1}{p} = \frac{\#\mathcal{U}_1(\mathbb{F}_p)}{p^3} \quad \text{and} \]

\[ \tau_{(X,D),p}(\mathcal{U}_2(\mathbb{Z}_p)) = 1 + \frac{1}{p^2} = \frac{\#\mathcal{U}_2(\mathbb{F}_p)}{p^3}. \]

**Proof.** Under the above chart, the set of integral points \( \mathcal{U}_1(\mathbb{Z}_p) \cap V(\mathbb{Q}_p) \) corresponds to the set

\[ \left\{ \left( \frac{a}{d}, \frac{1}{d}, \frac{c}{d} \right) \mid a, c, d \in \mathbb{Z}_p \right\} = \{(a, b, c) \in \mathbb{Q}_p^3 \mid |b| \geq 1, |a|, |c| \leq |b| \}, \]

and, analogously, \( \mathcal{U}_2(\mathbb{Z}_p) \cap V(\mathbb{Q}_p) \) corresponds to the set

\[ \{(a, b, c) \in \mathbb{Q}_p^3 \mid |a| \geq 1, |b|, |c| \leq |a| \}. \]

On \( \mathcal{U}_1(\mathbb{Z}_p) \cap V(\mathbb{Q}_p) \), we have \( \| f_{D_1} \|_{\mathcal{O}(D_1)} = |b|\|_{\mathcal{O}(D_1)} = \max\{|a|, |b|, |c|, |xz|\} = 1 \), while \( \| da \wedge db \wedge dc \|_{\omega_X} \) evaluates to

\[
\max\{|a^2x|, |b^2x|, |c^2x|, |z^2x^3|, |ay|, |by|, |cy|\} \cdot \max\{|a|, |b|, |c|, |xz|\} \cdot \max\{|a|, |b^2 + bc|\} \cdot \|b|.
\]
This means that
\[ df_\tau(X,D_1),g = (\max\{|b^2|, 1, |b(a^2 + bc)|\})^{-1} |b| d\mu \]
on \( g^{-1}(U_1(Z_p)) \) and, by an analogous argument, that
\[ df_\tau(X,D_2),g = (\max\{|a^2|, 1, |a(a^2 + bc)|\})^{-1} |a| d\mu \]
on \( g^{-1}(U_2(Z_p)) \).

With these descriptions we can explicitly calculate the volumes. In the first case, we get
\[
\tau(X,D_1),p(U_1(Z_p)) = \int_{|b| \geq 1} \frac{1}{|b|} \max\{|b^2|, |b(a^2 + bc)|\} \, da \, db \, dc
\]
\[ = \int_{a^2 + bc \leq \frac{1}{|b|}} \frac{1}{|b|} \, da \, db \, dc + \int_{a^2 + bc > \frac{1}{|b|}} \frac{1}{|b|^2} \, da \, db \, dc.
\]
The first of these terms is
\[
\int_{1 \leq \frac{a^2 + c}{|b|}} \frac{1}{|b|^3} \, da \, db \, dc = \int_{|b| > 1, |a| \leq |b|} \frac{1}{|b|^3} \, da \, db = \int_{|b| > 1} \frac{1}{|b|^2} \, db = \sum_{k \geq 0} \frac{1}{p^k} \left( 1 - \frac{1}{p} \right) p^k = 1,
\]
while the second is
\[
\int_{1 \leq \frac{a^2 + c}{|b|}} \frac{1}{|b|^3} \, da \, db \, dc = \int_{|b| > 1, |a| \leq |b|} \frac{1}{|b|^3} \sum_{k=1}^{\lfloor v(b) \rfloor} \frac{1}{p^k} \left( 1 - \frac{1}{p} \right) p^k \, da \, db = \left( 1 - \frac{1}{p} \right) \int_{|a| \leq |b|} \frac{1}{|b|^2} \, db = \left( 1 - \frac{1}{p} \right) \sum_{k \geq 0} \frac{k}{p^{2k}} \left( 1 - \frac{1}{p} \right) p^k = \frac{1}{p},
\]
so \( \tau(X,D_1),p(U_1(Z_p)) = 1 + \frac{1}{p} = \frac{\#U_1(Z_p)}{p}. \) The volume \( \tau(X,D_2),p(U_2(Z_p)) \) is calculated similarly. \qed

**Archimedean place.** Next, we calculate the volumes of the divisors \( D_1 \) and \( D_2 \) with respect to the residue measures explicitly described in [CLT10a, 2.1.12].

**Lemma 3.2.** We have \( \tau_{D_1,\infty}(D_1(\mathbb{R})) = \tau_{D_2,\infty}(D_2(\mathbb{R})) = 20. \)

**Proof.** The adjunction isomorphism induces a metric on \( \omega_{D_1} \) via
\[ \|da \wedge dc\|_{\omega_{D_1}} = \|da \wedge db \wedge dc\|_{\omega_X} \|b\|^4 \cdot \frac{1}{\|c\|^4} \cdot \chi_{(-D_1)} \cdot \chi_{(-D_1)}. \]
Since \( da \wedge db \wedge dc \) corresponds to \( x^{-4}z^{-3} \in R(X) \), the first factor of 8 is
\[ \max\{|a^2x|, |b^2y|, |c^2x|, |z^2x^3|, |ay|, |by|, |cy|, |e|, |x|, |z|, xz|\}
\[ = \max\{|a^2|, |c^2|, 1, |a^3|, |a^2c|\} \max\{|a|, |c|, 1, |x|, |z|, xz|\}, \]
when evaluated in \( (a : 0 : c : 1 : a^2 : 1) \in V \cap D_1. \) On the affine variety \( V \), regarding \( b \) as an element of \( \Gamma(V, \mathcal{O}_V(-D_1)) \subset \mathcal{O}_V(V) \) and using the canonical trivialization
of $\mathcal{O}(-D_1)$ outside $D_1$ with the fact that $1_{D_1}$ corresponds to $b \in R(X)$ under our chosen isomorphism gives us

$$\lim_{b \to 0} \left( \frac{|b|}{\|1_{D_1} \|_{\mathcal{O}(D_1)}} \right)^{-1} = \lim_{b \to 0} \left( \frac{|b| \max\{|a|, |b|, |c|, |xz|\}}{|b|} \right)^{-1} = \max\{|a|, |c|, 1\}^{-1}$$

for the second factor. We thus have explicit descriptions

$$df \tau_{D_1, \infty} = \|da \wedge dc\|_{\omega_{D_1}}^{-1} \frac{1}{\max\{|a^2|, |c^2|, 1, |a^2|, |a^2c|\}} \, da \, dc$$

and, by an analogous argument,

$$df \tau_{D_2, \infty} \, db \, dc = \frac{1}{\max\{|b^2|, |c^2|, 1, |b^2c|, |bc^2|\}} \, db \, dc$$

of the Tamagawa measures $\tau_{D_1, \infty}$ and $\tau_{D_2, \infty}$ with respect to the Lebesgue measure.

For the volume of the first divisor, we now get

$$\tau_{D_1, \infty}(D_1(\mathbb{R})) = \int_{|a|, |a^2c| \leq 1} \frac{1}{\max\{|c^2|, 1\}} \, da \, dc + \int_{|a| > |c|} \frac{1}{|a^2|} \, da \, dc$$

$$+ \int_{|c| > |a|} \frac{1}{\max\{|c^2|, |a^2c|\}} \, dc \, dc.$$

The first term of this expression is $\frac{82}{3}$ by (11) below, the second is $\int_{|a|} \frac{2}{|a^2|} = 4$ and the third is

$$\int_{|c| > |a|} \frac{1}{|a^2c|} \, da \, dc + \int_{|c| > |a|} \frac{1}{|a^2c|} \, dc \, dc.$$

In (9), the first term is

$$\int_{|c| > |a|} \frac{1}{|a^2c|} \, da \, dc = \int_{|c| \geq 1} \frac{2}{|c|^2} \, dc = 4$$

and the second is

$$\int_{a \in \mathbb{R}} \frac{2}{\max\{|a^2|, |a^2c|\}} \, da = \int_{|a| \leq 1} \frac{2}{|a^2|} \, da + \int_{|a| > 1} \frac{2}{|a^2|} \, da = 16 \frac{3}{3}.$$

Thus, (9) is $\frac{22}{3}$ and $\tau_{D_1, \infty}(D_1(\mathbb{R})) = \frac{20}{3} + 4 + \frac{28}{3} = 20$.

For the other divisor, we get $\tau_{D_2, \infty}(D_2(\mathbb{R})) = 20$ by similar arguments. □

**Convergence Factors.** Since both $U_1$ and $U_2$ have only constant nowhere vanishing global sections over any algebraically closed field and the Galois group acts trivially on the Picard group $H^1(U_1, \mathbb{Q}/\mathbb{Z}, \mathbb{G}_m) = \mathbb{Z} \cdot |E|$ (where $E$ is the exceptional divisor), the Euler factors of the Artin $L$-function of the virtual Galois module

$$\text{EP}(U_i) = [H^0(U_1, \mathbb{Q}/\mathbb{Z}, \mathbb{G}_m) \otimes \mathbb{Q}] \otimes [-H^1(U_1, \mathbb{Q}/\mathbb{Z}, \mathbb{G}_m)] \otimes [-|Q|]$$

are simply $L_p(s; \text{EP}(U_i)) = 1 - \frac{1}{p}$. In particular, the Artin $L$-function is

$$L(s; \text{EP}(U_1)) = \prod_p L_p(s; \text{EP}(U_i)) = \zeta(s)^{-1},$$

it has a simple zero at $s = 1$ and its principal value at 1 is $L_*(1; \text{EP}(U_i)) = \lim_{s \to 1} L(s; \text{EP}(U_i))(s - 1)^{-1} = 1$.

At the infinite place, we get a renormalization factor $c_R = 2$, since the divisor $D$ is geometrically irreducible.
3.4. The constant $\alpha$. Previous results such as [CLT12] suggest that in the case of a number field with only one infinite place and a geometrically irreducible divisor $D$, we get a factor $\alpha$ of the leading constant in the following way: Consider the pseudo-effective cone $\text{Eff}(V) \subset \text{Pic}(V)_\mathbb{R}$ and its characteristic function $\omega_{\text{Eff}(V)}(L) = \int_{\text{Eff}(V)} e^{-\langle L, t \rangle} \, dt$ (with respect to the Haar measure on $\text{Pic}(V)_\mathbb{R}$ normalized by $\text{Pic}(V)^\vee$). Then
\[
\alpha = \frac{1}{(\text{rk Pic}(V) - 1)!} \omega_{\text{Eff}(V)}(D)^\vee.
\]
should be a factor of the leading constant $c$.

In our case, we have $\text{Eff}(X) = \text{Cone}(E, H - E)$ and $\omega_X(D_i)^\vee \cong O_X(3(H - E) + 2E)$, hence $\alpha_i = 1/6$ for both $i = 1, 2$.

3.5. The exponent of $\log B$. In the case of a geometrically irreducible divisor and a number field with one infinite place, the same previous results suggest that the exponent of $\log B$ in the asymptotic formula should be $\text{rk}(\text{Pic}(X)) - 1 = 1$.

4. Integral points on $X - D_1$

We study the number
\[
N_1(B) = \# \{ x \in U_1(Z) \cap V_1(\mathbb{Q}) \mid H(x) \leq B \}
\]
of integral points of bounded height on $U_1 = X - V(b)$ that, as rational points, are in the complement $V_1$ of the subvariety $V(abx) = \pi^{-1}(V(abd))$.

Using the 4-to-1-correspondence (2) with integral points on the universal torsor $T_1$ and noticing the symmetry in the two values $\pm 1$ of $b$ in (2), this description of integral points on the universal torsor yields the formula
\[
N_1(B) = \frac{1}{2} \# \{ (a, c, x, y, z) \in \mathbb{Z}^5 \mid a^2 + c - yz = 0, \text{gcd}(x, y, z) = 1, \text{gcd}(x, y, z) \leq B, a, x, z \neq 0 \},
\]
where
\[
H(a, b, c, x, y, z) = \max \{ |a^2x|, |b^2x|, |c^2x|, |z^2x^3|, |ay|, |by|, |cz| \}
\]
by (7). Solving the equation, we can simplify this to
\[
\frac{1}{2} \# \{ (a, x, y, z) \in \mathbb{Z}^4 \mid \text{gcd}(x,y) = 1, H(a, x, y, z) \leq B, a, x, z \neq 0 \},
\]
where
\[
\tilde{H}(a, x, y, z) = H(a, 1, yz - a^2, x, y, z)
\]
\[= \max \{ |a^2x|, |x|, |(yz - a^2)^2 x|, |z^2 x^3|, |ay|, |by|, |(yz - a^2)y| \}.
\]

Lemma 4.1. We have
\[
N_1(B) = \frac{1}{2} \sum_{\alpha > 0} \mu(\alpha) \sum_{a \in \mathbb{Z}_{\geq 0}} \int_{a^2 x^2 \leq B} \frac{1}{|z|} \frac{1}{\alpha^2 + c} \frac{1}{c^2 + c} \frac{1}{|y(z + 1)|} \, da \, dc + O(B).
\]

Proof. A M"obius inversion yields
\[
N_1(B) = \sum_{\alpha > 0} \mu(\alpha) \sum_{a, x', z \in \mathbb{Z}_{\geq 0}} \# \{ y' \in \mathbb{Z} \mid \tilde{H}(a, ax', ay', z) \leq B \}.
\]
We get
\[
\# \{ y' \in \mathbb{Z} \mid \tilde{H}(a, ax', ay', z) \leq B \} = V_1(a, a', z; B) + O(1),
\]
where
\[
V_1(a, a', x', z; B) = \int_{\tilde{H}(a, ax', ay', z) \leq B} dy',
\]
and similar estimates with an error term of the form $O(\sup_{x} f(x))$ in the following steps when replacing the sum of a function $f$ whose derivative changes a bounded number of times by an integral, cf. [DF14, Lemma 3.6]. We can bound the sum over the error term by

$$
\ll \sum_{\alpha > 0, a, x', z \in \mathbb{Z}_{\geq 0}, |\alpha a^{2} x'|, |\alpha^{3} z x'| < B} 1 \ll \sum_{\alpha > 0} \frac{B}{|\alpha x'|^{2}} \ll B
$$

to get $N_1(B) = \sum_{a} \mu(a) \sum_{a, x', z} \int_{H_{1}(a, \alpha z', \alpha y, z) \leq B} dy' + O(B)$.

Turning to the variable $a$ next we estimate the sum $\sum_{a \in \mathbb{Z}_{\geq 0}} V_1(a, a, x', z; B)$ by the integral $V_2(\alpha, \alpha, x', z; B) = \int V_1(\alpha, a, x', z; B) da$, introducing an error bounded by

$$
\ll \sum_{\alpha > 0, a, x', z \in \mathbb{Z}_{\geq 0}, |\alpha a^{2} x'|, |\alpha^{3} z x'| \leq B} \frac{B^{1/2}}{\alpha |x'|^{1/2} |z|} \ll \sum_{\alpha > 0, \alpha, z \in \mathbb{Z}_{\geq 0}, |\alpha a^{3} z^{2}| \leq B} \frac{B^{2/3}}{\alpha^{4/3} |z|^{4/3}} \ll B^{2/3},
$$

where we use the condition $|\alpha y' z - a^{2}| \leq B$ to estimate the integral $V_1$. A change of variables $c = \alpha y' z - a^{2}$ now results in the description

$$
V_2(\alpha, \alpha, x', z; B) = \int_{|a| \geq 1} V_1(\alpha, a, x', z; B) da = \int_{|a| \geq 1} \frac{1}{|\alpha|} \int_{\alpha z^{2}, z \in \mathbb{Z}_{\geq 0}, |\alpha z| \leq |a|} \frac{1}{|a|} \frac{1}{|\alpha z|} da dc
$$

of the main term.

\begin{lemma}
\textbf{Lemma 4.2.} We have

$$
N_1(B) = \frac{1}{2} \sum_{\alpha > 0} \frac{\mu(a)}{\alpha^{2}} \int_{|a^{2} x|, |a^{3} z| \leq B, |c^{2} x|, |c^{3} z| \leq B, |a|, |z| \geq 1, |c| \geq 1} \frac{1}{|a z|} da dc dx dz + O(B).
$$
\end{lemma}

\begin{proof}
We first want to replace the two instances of $a^{2} + c$ by $a^{2}$ in the inequalities defining the region for the volume function $V_2$ of the previous lemma, to get a new volume function $V'_2(\alpha, \alpha, x', z; B)$. The error we introduce when replacing $|a(a^{2} + c) z^{-1}|$ by $|a^{2} z^{-1}|$ is bounded by the integral over the region

$$
B = \left|\frac{ac}{z}\right| \leq \left|\frac{a^{3}}{z}\right| \leq B + \left|\frac{ac}{z}\right|, \quad \text{i.e.,} \quad \left|a^{2} - \frac{B |z|}{|a|}\right| \leq |c|.
$$

With a change of variables $a' = a^{2} - B |z| |a|^{-1}$, where

$$
\left|\frac{da'}{da}\right| = 2 |a| + \frac{B |z|}{|a|} \geq \sqrt{|a'|},
$$

we have

$$
\frac{1}{|a|} \frac{1}{|c|} \frac{1}{|a z|} \leq \frac{1}{|a|} \frac{1}{|c|} \frac{1}{|a^{2} z|}.
$$

and

$$
\frac{1}{|a|} \frac{1}{|c|} \frac{1}{|a^{2} z|} \leq \frac{1}{|a|} \frac{1}{|c|} \frac{1}{|a^{2} z|}.
$$

We continue with the estimation

$$
\ll \sum_{\alpha > 0, a, x', z \in \mathbb{Z}_{\geq 0}, |\alpha a^{2} x'|, |\alpha^{3} z x'| \leq B} \frac{B^{1/2}}{\alpha |x'|^{1/2} |z|} \ll \sum_{\alpha > 0, \alpha, z \in \mathbb{Z}_{\geq 0}, |\alpha a^{3} z^{2}| \leq B} \frac{B^{2/3}}{\alpha^{4/3} |z|^{4/3}} \ll B^{2/3},
$$

where we use the condition $|\alpha y' z - a^{2}| \leq B$ to estimate the integral $V_1$. A change of variables $c = \alpha y' z - a^{2}$ now results in the description

$$
V_2(\alpha, \alpha, x', z; B) = \int_{|a| \geq 1} V_1(\alpha, a, x', z; B) da = \int_{|a| \geq 1} \frac{1}{|\alpha|} \int_{\alpha z^{2}, z \in \mathbb{Z}_{\geq 0}, |\alpha z| \leq |a|} \frac{1}{|a z|} da dc
$$

of the main term.

\end{proof}
we can bound the total error by

\[ \ll \sum_{\alpha > 0} \sum_{x', z \in \mathbb{Z}} \int_{|\alpha^2 x' z|^2 \leq B} \frac{1}{\sqrt{|\alpha^2 x' z|^2}} \, da' dc \]
\[ \ll \sum_{\alpha > 0} \sum_{x', z \in \mathbb{Z}} \int_{|\alpha^2 x' z|^2 \leq B} \frac{1}{\sqrt{|\alpha^2 x' z|^2}} \, dc \ll \sum_{\alpha > 0} \sum_{x', z \in \mathbb{Z}} \frac{B^{3/4}}{\alpha^{3/4} |x|^3 / 4 |z|} \]
\[ \ll \sum_{\alpha > 0} \frac{B^{5/6}}{\alpha^{7/4} |z|^{7/6}} \ll B^{5/6}. \]

When modifying the other inequality, the error we introduce is bounded by an integral over a similar region, and, after an analogous change of variables, we get the same bound.

Next, we estimate the summation over \( z \). Using the height conditions \(|a| \leq B^{1/3} |z|^{1/3}\) and \(|c| \leq B^{1/2} |\alpha x|^{1/2}\), we can bound the volume

\[ V_2'(\alpha, x', z; B) \ll B^{5/6} |\alpha x|^{-1/2} |z|^{-2/3}. \]

Replacing the sum over \( z \) by an integral, we introduce an error

\[ \ll \sum_{\alpha > 0} \frac{1}{\alpha^{3/2}} \sum_{1 \leq |x'| \leq B^{1/3}} \frac{B^{5/6}}{|x'|^{1/2} / 2 |z|^{7/6}} \ll B. \]

For \( V_3(\alpha, x'; B) = \int_{|z| \geq 1} V_2'(a, x', z; B) \, dz \), we get an upper bound

\[ V_3(\alpha, x'; B) \ll \int_{|\alpha^2 x'^2 z|^2 \leq B} \frac{B^{5/6}}{\alpha^{1/2} |x'|^{1/2} / 2 |z|^{7/6}} \, dz \ll \frac{B}{\alpha |x'|}. \]

Finally, replacing the sum over \( x' \) by an integral \( \int_{|x'| \geq 1} V_3(\alpha, x'; B) \) introduces an error term

\[ \ll \sum_{\alpha > 0} \frac{B}{\alpha^2} \ll B, \]

and a change of variables \( x = \alpha x' \) completes the proof. \( \square \)

**Proposition 4.3.** The number of integral points of bounded height on \( U_2 \) satisfies the asymptotic formula

\[ N_1(B) = \frac{20}{3\zeta(2)} B \log B + O(B). \]

**Proof.** We first remove the condition \(|a| \geq 1\) in (10) and get an error term

\[ \ll \sum_{\alpha} \frac{1}{\alpha^2} \int_{|x|^2 |x|^2 \leq B,} \, dx \, dz \ll \sum_{\alpha} \frac{1}{\alpha^2} \int_{|x| \geq \alpha} \frac{B}{|x|} \, dx \ll \sum_{\alpha} \frac{B}{\alpha^5} \ll B. \]
By a change of variables \( a \mapsto az^{1/3}B^{-1/3} \), \( c \mapsto cz^{1/3}B^{-1/3} \), \( x \mapsto az^{2/3}B^{-1/3} \), we now have

\[
N_1(B) = \frac{1}{2} \sum_{\alpha > 0} \frac{\mu(\alpha)}{\alpha^2} \int_{\alpha^2 \leq |x| \leq 1} |B| dx \, dc \, dz + O(B)
\]

\[
= \frac{1}{2} \sum_{\alpha > 0} \frac{\mu(\alpha)}{\alpha^2} \int_{\alpha^2 \leq |x| \leq 1} (B \log |x| \alpha^{-3}) dx \, dc \, dz + R_1(B) + O(B)
\]

\[
= \frac{1}{2} \sum_{\alpha > 0} \frac{\mu(\alpha)B \log B}{\alpha^2} \int_{\alpha^2 \leq |x| \leq 1} |B| dx \, dc \, dz + R_2(B) + O(B)
\]

\[
= \frac{B \log B}{\zeta(2)} \int \frac{1}{\max\{1, |a^2|, \alpha^2\}} \, da \, dc \, + O(B),
\]

since the error terms satisfy

\[
|R_1(B)| \ll \sum_{\alpha \geq 1} \frac{B}{\alpha^2} \int_{\alpha^2 \leq |x| \leq 1} \log \left( B |x| \alpha^{-3} \right) \, da \, dc \, dx
\]

\[
\ll \sum_{\alpha \geq 1} \frac{B}{\alpha^2} \int_{|x| \leq \alpha B^{-1/3}} 3 \log \left( B^{1/3} |x| \alpha^{-1} \right) \, dx
\]

\[
\ll \sum_{\alpha \geq 1} \frac{B \alpha^{1/2}}{B^{1/6}} \left( 2 \log \left( B^{1/3} \alpha^{-1} B^{-1/3} \alpha \right) \right) \ll B^{5/6}
\]

and

\[
|R_2(B)| \ll \sum_{\alpha \geq 1} \frac{B}{\alpha^2} \int_{|x| \leq 1} \log \left( \frac{|x|}{\alpha} \right) \, dx \ll \sum_{\alpha \geq 1} \frac{B}{\alpha^2} \left( 2 + \log(\alpha) \right) \ll B.
\]

Finally, we note that the integral evaluates to

\[
\left(1\right) \int_{\alpha^2 \leq |x| \leq 1} \frac{1}{\max\{1, |x| \}} \, da \, dc = \int_{|c| \leq 1} 2 dc + \int_{|c| > 1} \frac{2}{|c|^{5/2}} dc = \frac{20}{3},
\]

and arrive at the asymptotic expression.

\[
\square
\]

5. Integral Points on \( X - D_2 \)

We count the number

\[
N_2(B) = \# \{ x \in U_1(\mathbb{Z}) \cap V_2(\mathbb{Q}) \mid H(x) \leq B \}
\]

of integral points of bounded height on \( U_2 = X - V(a) \), that, as rational points, are in the complement \( V_2 \) of \( V(abcxyz) = \pi^{-1}(V(abc)) \). With the 4-to-1-correspondence to integral points on the torsor, and noticing the symmetry in the two possible values \( a = \pm 1 \) of \( a \) in (3), we get

\[
\left(11\right) \int_{\alpha^2 \leq |x| \leq 1} \frac{1}{\max\{1, |x| \}} \, da \, dc = \int_{|c| \leq 1} 2 dc + \int_{|c| > 1} \frac{2}{|c|^{5/2}} dc = \frac{20}{3},
\]

and arrive at the asymptotic expression.

\[\square\]

**Lemma 5.1.** We have

\[
N_2(B) = \sum_{b,x,z \in \mathbb{Z}_{\geq 0}} \theta_1(b, x, z)V_1(b, x, z; B) + O(B),
\]

where

\[
\theta_1(b, x, z) = \sum_{y \in \mathbb{Z}} \chi(y) \mathbf{1}_{\text{gcd}(y, b, x, z) = 1}.
\]
where
\[ V_1(b, x, z; B) = \frac{1}{2} \int_{|H_2(b, c, x, z) \leq B} \frac{1}{\alpha} \, dc \]

with
\[ \tilde{H}_2(b, c, x, z) = H(1, b, c, x, (1 + bc)z^{-1}, z) = \max \left\{ |x|, |b^2x|, |c^2x|, |z^2x^3|, \left| \frac{1 + bc}{z} \right|, \left| \frac{b(1 + bc)}{z} \right|, \left| \frac{c(1 + bc)}{z} \right| \right\}, \]

and \( \theta_1(b, x, z) = \prod_p \theta_1^{(p)}(b, x, z) \) with
\[ \theta_1^{(p)}(b, c, z) = \begin{cases} 0, & p \mid b, p \mid z, \\ 1 - \frac{1}{p}, & p \nmid b, p \mid x, \\ 1, & \text{else.} \end{cases} \]

Proof. Using a Möbius inversion to remove the condition gcd\((x, y) = 1\) in (12), and setting \( y' = \frac{y}{\alpha} \), we get
\[ N_2(B) = \frac{1}{2} \sum_{b, x, z \in \mathbb{Z}_{>0}} \sum_{\alpha \mid x} \mu(\alpha) \tilde{N}_2(\alpha, b, x, z; B), \]

where
\[ \tilde{N}_2(\alpha, b, x, z; B) = \# \left\{ (c, y') \in \mathbb{Z}^2 \mid \frac{c \neq 0, 1 + bc - y'z = 0}{H(1, b, c, x, y', z) \leq B} \right\}. \]

To estimate \( \tilde{N}_2 \), we first note that \( \tilde{N}_2(\alpha, b, x, z; B) = 0 \) whenever \( \alpha z \) and \( b \) are not coprime. If they are coprime, we estimate
\[ \tilde{N}_2(\alpha, b, x, z; B) = \# \left\{ c \in \mathbb{Z}_{>0} \mid b^e b^{-1} \equiv c (\mod \alpha z), \tilde{H}_2(b, c, x, z) \leq B \right\} = \int_{|c| \geq 1} \frac{1}{\alpha z} \, dc + O(1), \]

analogously to the first case. This inequality together with the height conditions \( |b^2x| \leq B \) and \( |z^2x^3| \leq B \) allows us to bound the summation over the error terms by
\[ \ll \sum_{b, x, z \in \mathbb{Z}_{>0}} \sum_{\alpha \mid x} |\mu(\alpha)| \ll \sum_{x \in \mathbb{Z}_{>0}} \frac{2e(x) B}{|x|^2} \ll B. \]

We arrive at
\[ N_2(B) = \sum_{b, x, z} \sum_{\alpha \mid x} \frac{\mu(\alpha)}{\alpha} V_1(b, x, z; B) + O(B), \]

where
\[ V_1(b, x, z; B) = \frac{1}{2} \int_{|H_2(b, c, x, z) \leq B} \frac{1}{\alpha} \, dc. \]

Using the multiplicativity of \( \mu \) and gcd, we can factor the sum over \( \alpha \)
\[ \sum_{\alpha \mid x} \frac{\mu(\alpha)}{\alpha} = \prod_p \begin{cases} 0, & p \mid b, p \mid z, \\ 1 - \frac{1}{p}, & p \nmid b, p \mid x, \\ 1, & \text{else} \end{cases} \]

to get a description of the arithmetic term \( \theta_1 \). \( \square \)
Lemma 5.2. We have
\[ N_2(B) = \sum_{b, z} \theta_2(x, z)V_2(x, z; B) + O(B \log \log B)^2, \]
where
\[ V_2(x, z; B) = \frac{1}{2} \int_{H_2(b, c, x, z) \leq B} \frac{1}{|x|} \, db \, dc \]
and \( \theta_2(x, z) = \prod_p \theta_2^{(p)} \)
with
\[
\theta_2^{(p)} = \begin{cases} 
\left( 1 - \frac{1}{p} \right)^2, & p \mid x, z, \\
1 - \frac{1}{p} + \frac{1}{p^2}, & p \mid x, p \nmid z, \\
1 - \frac{1}{p}, & p \nmid x, p \mid z, \\
1, & p \nmid x, z.
\end{cases}
\]

Proof. Using the height conditions \(|c^2x|, |b(1 + bc)z^{-1}| \leq B\) to estimate the integral, we can bound the volume function by the geometric average
\[
V_1(b, x, z; B) \ll \frac{1}{|x|} \left( \frac{B^{1/2}}{|x|^{1/2}} \right)^{2/3} \left( \frac{B |z|}{|b|^2} \right)^{1/3} \ll \frac{B}{|xz|} \left( \frac{B}{|b^2x|} \right)^{-1/6} \left( \frac{B}{|z^2x^3|} \right)^{-1/6}.
\]
Since the integral is zero whenever \(|b^2x| \geq B\) or \(|z^2x^3| \geq B\), the assertion follows by [Der09, Proposition 3.9] with \(r = 0, s = 2\). (In the notation of loc. cit. we consider the ordering \(n_0 = b, n_1 = x, n_2 = z\) of the variables, take \(a_1 = a_2 = 1/6\), and \(k_{i,j}\) to be the exponents in these two height conditions. Note that \(\theta_1\) satisfies [Der09, Definition 7.8], and hence the requirements of the proposition.) \(\square\)

Lemma 5.3. We have
\[ N_2(B) = \frac{1}{2} \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right) \int_{H_2(b, c, x, z) \leq B} \frac{1}{|x|} \, db \, dx \, dz + O(B \log \log B)^2. \]

Proof. Using the same estimate for the integral over \(c\) as in the previous lemma and estimating the integral over \(b\) using the height condition \(|b^2x| \leq B\), we get the bound
\[
V_2(x, z; B) \ll \int_{1 \leq |b| \leq B^{1/2}|x|^{-1/2}} B^{2/3} |b|^{-2/3} |x|^{1/3} |z|^{2/3} \ll B \left( \frac{B}{|xz|} \right)^{-1/6} \left( \frac{B}{|b^2x|} \right)^{-1/3} \left( \frac{B}{|z^2x^3|} \right)^{-1/3},
\]
for the volume function \(V_2\). Since \(V_2(b, z; B) = 0\) whenever \(|z^2x^3| > B\), we get an asymptotic formula by [Der09, Proposition 4.3] (with \(r = s = 1\)). We are only left to see that the constant is indeed
\[
\prod_p \left( \frac{1}{p^2} \left( 1 - \frac{1}{p} \right)^2 + \frac{1}{p} \left( 1 - \frac{1}{p} \right) \left( 2 - \frac{2}{p} + \frac{1}{p^2} \right) + \left( 1 - \frac{1}{p} \right)^2 \right) = \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right).
\]
\(\square\)

Proposition 5.4. We have
\[ N_2(B) = cB \log(B) + O(B \log \log B)^2, \]
where
\[ c = \frac{20}{3} \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right). \]

Proof. We have to estimate the integral in the previous lemma. We first want to replace \((1 + bc)/z\) by \(bc\) in the height conditions. In the case of the condition \(b(1 + bc)/z\), this leaves us with an error term that can be bounded by the integral over the region defined by \(B - \frac{1}{z} \leq \frac{|Bz|}{z} \leq B + \frac{1}{z}\), i.e., \(\frac{Bz}{z} - \frac{1}{z} \leq |c| \leq \frac{Bz}{z} + \frac{1}{z}\), and the remaining height conditions, hence is at most

\[ \ll \int_{|b|,|z| \geq 1} \frac{1}{|b^2z|} \, db \, dx \, dz \ll \int \frac{B^{1/2}}{|b^2z|^{3/2}} \, db \, dz \ll B^{1/2}. \]

The condition \(c(1 + bc)/z\) can be dealt with analogously. Next, we remove the condition \(|b| \geq 1\), where we get an error term

\[ \ll \int_{|b|,|z| \geq 1} \frac{1}{|b|} \, db \, dx \, dz \ll \int_{|x| \geq \frac{1}{|b|}} \frac{B^{1/2}}{|b^2z|^{3/2}} \, dx \, dz \]

\[ \ll \int_{|x| \geq \frac{1}{|b|}} \frac{B^{2/3}}{|b|^{4/3}} \, dz \ll B^{2/3}, \]

and subsequently remove \(|c| \geq 1\) analogously. Thus, we can estimate the integral in the previous lemma as \(V_3(B) + O(B^{2/3})\), where

\[ V_3(B) = \int_{|b^2x|,|c^2x|,|x^2z^2|, |b^2z|, |bcz^2| \leq B, \quad |x|, |z| \geq 1} \frac{1}{|z|} \, db \, dc \, dx \, dz. \]

By a change of variables \(b \mapsto B^{-1/3}bz^{-1/3}, c \mapsto B^{-1/3}cz^{-1/3}, x \mapsto B^{-1/3}xz^{-2/3}\), we get

\[ V_3(B) = B \int_{|b^2x|, |c^2x|, |x|, |b^2z|, |bcz^2| \leq 1, \quad 1 \leq |x| \leq B^{1/2}/|b^2|^{3/2}} \frac{1}{|z|} \, db \, dc \, dx \, dz \]

\[ = 2B \int_{|b^2x|, |c^2x|, |x|, |b^2z|, |bcz^2| \leq 1} \log \left( B^{1/2} |x|^{3/2} \right) \, db \, dc \, dx + R_3(B) \]

\[ = B \log B \int_{|b^2z|, |bcz^2| \leq 1} \max\{ |b^2|, |c^2|, 1\}^{-1} \, db \, dc + R_3(B) + R_4(B) \]

\[ = 2B \log B \int_{|b^2z|, |bcz^2| \leq 1} \max\{ |b^2|, |c^2|, 1\}^{-1} \, db \, dc + R_3(B) + R_4(B). \]

The error terms are

\[ |R_3(B)| \ll B \int_{|x| \leq B^{-1/3}} \log(B^{1/2} |x|^{3/2}) \, db \, dx \]

\[ \ll B \int_{|b^2z|, |bcz^2| \leq 1} \frac{3}{2} B^{-1/3} db \, dc \ll B^{2/3} \left( \int_{|c| \leq 1} \frac{1}{|c|} \, dc + \int_{|c| > 1} \frac{1}{c^2} \, dc \right) \]

\[ \ll B^{2/3} \]
and
\[
|R_4(B)| \ll B \int_{|x| \leq 1} \left| \log \left( \frac{|x|}{|x^3/2|} \right) \right| \, db \, dc \ll \int_{|b^2|, |bc^2| \leq 1} \frac{1}{\max\{|b^2|, |c^2|, 1\}} \, db \, dc.
\]
The integral at the end of (13) then further evaluates to
\[
\int_{|b^2|, |bc^2| \leq 1} \frac{1}{\max\{|b^2|, |c^2|, 1\}} \, db \, dc = 2 \int_{|b| \geq 1} \frac{1}{\max\{|b^2|, 1\}} \, db \, dc = 2 \int_{|b| \leq 1} 4 |b| \, db + \int_{|b| > 1} \frac{4}{b^4} \, db = \frac{20}{3},
\]
and we get the desired asymptotic. \qed

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