On the complex structure of symplectic quotients

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Abstract Let $K$ be a compact group. For a symplectic quotient $M_\lambda$ of a compact Hamiltonian Kähler $K$-manifold, we show that the induced complex structure on $M_\lambda$ is locally invariant when the parameter $\lambda$ varies in $\text{Lie}(K)^*$. To prove such a result, we take two different approaches: (i) use the complex geometry properties of the symplectic implosion construction; (ii) investigate the variation of geometric invariant theory (GIT) quotients.

Keywords complex structure, symplectic reduction, symplectic implosion, geometric invariant theory

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1 Introduction

1.1 Backgrounds

For symplectic manifolds with Hamiltonian actions, the symplectic reduction has always been a powerful tool to investigate properties on such manifolds since Marsden and Weinstein [19] introduced it several decades ago. By using this technique, we can obtain a new symplectic manifold called the reduction manifold or the symplectic quotient. The symplectic reduction establishes a natural correspondence between the symplectic quotient and the original manifold. As a consequence, if some quantities can be defined on both manifolds, people can often find some interesting relations by comparing them. As an illustration of this general idea, we would like to recall two classical examples.

Riemann-Roch numbers. Since each symplectic manifold can also be viewed as an almost complex manifold, we can define Riemann-Roch (RR) numbers for the original manifold as well as the symplectic quotient. If we try to compare these two RR numbers, we will encounter the famous geometric quantization conjecture (see [9]). Because RR numbers can contain some representation theoretic information, the merit of such a comparison is that the representation theoretic information of the original manifold can be recovered from that of the symplectic quotient.

Kähler metrics. If the manifold is a Kähler manifold, the symplectic quotient can inherit a Kähler structure automatically. In this case, Kähler metrics are natural quantities that can be utilized for comparison. To obtain specific results, we usually need some assumptions on the metric of the original manifold. For example, in [5, Chapter 7], Futaki obtained a formula for the metric on the symplectic

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quotient supposing that the metric on the original manifold has positive Ricci curvature. More recent development along the same line appears in [18], where La Nave and Tian used a special equation to restrict the metric on the original Kähler manifold. Then the metrics on symplectic quotients must satisfy the famous Kähler-Ricci flow equation. Moreover, in such a situation, they discovered that the symplectic reduction construction can be reversed in a certain sense, i.e., the solution to the Kähler-Ricci flow equation can be used to construct the special metric on the original manifold.

1.2 Main results

The symplectic reduction has a nice feature, i.e., we can usually obtain a family of symplectic quotients rather than a single one. Acute readers may have noticed that the above two examples have used this family of symplectic quotients. As observed from these two examples, if we want to recover information on the original manifold from the symplectic reduction, only the input from a single symplectic quotient is insufficient, or in other words, we should study the family of symplectic quotients. For this purpose, it is beneficial to find some methods to compare the different symplectic quotients in the family. In this paper, we are going to deal with one such comparison problem. For a precise statement, we need some notations.

Let $K$ be a compact group and $\mathfrak{k}$ be its Lie algebra. Choose a maximal torus $T \subseteq K$ (resp. a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$) and a closed positive Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}^*$. By the root space decomposition, $\mathfrak{t}^*$ is naturally included in $\mathfrak{k}^*$. Let $M$ be a Kähler manifold with a holomorphic Hamiltonian $K$-action. We denote the symplectic quotient at $\lambda \in \mathfrak{k}^*$ by $M = M//_\lambda K$, where $\mathfrak{k}^*$ is the dual of the Lie algebra of $K$. It is well known that one can also give a complex structure on $M//_\lambda K$. In the following result, we compare the symplectic quotients by their complex structures. For the sake of simplicity, in this paper, we always assume that if $\lambda$ is a regular value of the moment map $\mu$ of $M$, $K$ acts on $\mu^{-1}(\lambda)$ freely. In other words, $M_\lambda$ is smooth.

**Theorem 1.1.** Let $M$ be a compact Hamiltonian Kähler $K$-manifold. Suppose that $\lambda \in \mathfrak{t}^*_+ \subseteq \mathfrak{t}^*$ is a regular value of the moment map. Let $\sigma$ be the face of the Weyl chamber where $\lambda$ lies in. Then for any $\lambda' \in \sigma$ lying in a sufficiently small neighborhood of $\lambda$, $M_{\lambda'}$ is biholomorphic to $M_\lambda$.

Roughly speaking, the above result asserts that the complex structures on $M_\lambda$ do not change when $\lambda$ varies in a small neighborhood. In the following, we also refer to this property of symplectic quotients as the **local invariance of complex structures** (with respect to the parameter $\lambda$). For abelian group actions, it is known that symplectic quotients satisfy such a property. But for the non-abelian case, it seems that the result similar to the above theorem is not documented in the literature explicitly as far as we are concerned. One aim of this paper is to fill this gap.

As we have explained at the beginning of this subsection, we will treat the above theorem as a premise to study the relation between some quantities defined both on the original manifold and symplectic quotients. In this specific situation, one possible direction along this line is trying to generalize La Nave-Tian’s correspondence of Kähler metrics mentioned before to non-abelian group actions based on the above result.

As to the proof of Theorem 1.1, we will take two different approaches: one is of more symplectic geometry flavor while the other is of more algebraic geometry flavor. Although looked different ostensibly, we find that both of them are linked to a simple idea: reducing a non-abelian problem to an abelian one.

For the symplectic geometry method for Theorem 1.1, we use a construction called the **symplectic implosion** due to Guillemin et al. [8]. Roughly speaking, such a construction enables us to construct a new symplectic space with an abelian group action to substitute the original one. We can hint the usage of this construction by the following result in [8] (see Theorem 2.2): the symplectic quotient of a manifold, which is generally obtained from a non-abelian reduction, is symplectically isomorphic to the abelian symplectic quotient of its symplectic implosion.

To prove Theorem 1.1, we need to find a Kähler manifold refinement of the result of Guillemin et al. mentioned above. More precisely, we have the following result which compares the complex structures
(in fact, the Kähler structures) on a symplectic quotient inherited from the original manifold and its symplectic implosion, respectively.

**Theorem 1.2.** Let $M$ be a Hamiltonian Kähler $K$-manifold and let $M_{\text{impl}}$ be the symplectic implosion of $M$. For any $\lambda \in \mathfrak{t}_s^* \subseteq \mathfrak{t}^*$ being a regular value of the moment map of $M$, the two symplectic quotients, $M \sslash \lambda K$ and $M_{\text{impl}} \sslash \lambda T$, are naturally isomorphic as Kähler manifolds.

As a side comment, we notice that, in [25, Theorem 3.8], Safronov proved a result similar to Theorem 1.2 for algebraic varieties with a complex symplectic structure. Compared with the method therein, i.e., the derived symplectic geometry, we follow a rather elementary calculation approach for the proof of Theorem 1.2.

With Theorem 1.2, we can complete the first proof of Theorem 1.1 by using the local invariance of complex structures for abelian symplectic quotients.

Symplectic quotients have an intimate relation with geometric invariant theory (GIT) quotients (see [16, 23, 27] for smooth manifolds and [11, 12] for general complex spaces). Considering this, we would like to give some GIT explanations of Theorem 1.1 before turning to the second proof of it. More concretely, we use the symplectic implosion again to show the following result, referred to as Proposition 4.2 for a more precise statement and a discussion about the stability condition used here.

**Proposition 1.3.** Let $\lambda$ and $\lambda'$ be in the same positions as in Theorem 1.1. Denote the coadjoint orbit going through $\lambda$ ($\lambda'$) by $O_{\lambda}$ ($O_{\lambda'}$). Then the semi-stable point sets of $M \times O_{\lambda}^{1,1}$ and $M \times O_{\lambda'}^{1,1}$ coincide.

Our second approach to Theorem 1.1 is based on the Kirwan-Ness theorem, namely, the coincidence of GIT quotients and symplectic quotients. Due to this theorem, if we are able to show Proposition 1.3 without using the symplectic implosion, we can find another proof of Theorem 1.1. Therefore, what we have actually done is using some complex algebraic geometry techniques to reprove Proposition 1.3.

The main ideas and tools behind this method come from the variation of GIT (vGIT) quotients theory as developed by [4, 30]. In the analytic language, vGIT discusses the variation of the symplectic quotients when the symplectic form (or the moment map) on the original manifold changes. From this viewpoint, it is quite reasonable to expect that Proposition 1.3 will follow from general vGIT results. In fact, Proposition 1.3 can be seen as a local and weaker version of the following result (see Theorems 5.4 and 5.5).

**Theorem 1.4.** Let $K_\sigma \subseteq K$ be the isotropic subgroup associated with the face $\sigma$. For any $\lambda \in \sigma$, let $(M \times K/K_\sigma)_\lambda^{ss} \subseteq M \times K/K_\sigma$ be the semi-stable point set associated with $\lambda$. One concludes that there is a finite partition of $\sigma = \bigsqcup_{i=1}^N \sigma_i$, such that for any $\sigma_i$ which is not a “wall”, and $\lambda$ and $\lambda'$ lying in the interior of $\sigma_i$, $(M \times K/K_\sigma)^{ss} = (M \times K/K_{\sigma'})^{ss}$.

In fact, in the above theorem, for the case where $\sigma = \mathfrak{t}_s^*\{1\}$, by using Sard’s theorem, we can see that $\sigma_i$ must not be a wall. To show the theorem, we need to generalize some vGIT results in [4, 30] from the projective algebraic varieties to the general Kähler manifolds. Especially, following [4], we discuss the relation between the stability condition and a special numerical function $M^*(x)$. The properties of $M^*(x)$ play a major role in the proof of the above theorem.

In the final part of this paper, we discuss an interesting relation between these two approaches of Theorem 1.1. Besides the similar philosophy behind them, one can even find a precise correspondence in a certain way. We observe that to reduce the vGIT problem to the abelian group action case, in [30], Thaddeus used a construction very similar to the symplectic implosion. In fact, in the setting we concerned in this paper, i.e., on the manifold $M \times O_{\lambda}^{1,1}$, the output of Thaddeus’s construction is a partial desingularization of the symplectic implosion (see Propositions 6.1 and 6.3).

1.3 An outline of this paper

The rest of this paper is organized as follows. In Section 2, we review some materials about the symplectic implosion used in this paper. After that, we prove Theorem 1.2 and give the first proof of Theorem 1.1 in Section 3, using the symplectic implosion. Section 4 is a transitional section, where we recall the stability

\[1\] The asterisk here means that we use the negative canonical symplectic form on a coadjoint orbit.
condition used in this paper and prove Proposition 1.3. In Section 5, we discuss the properties of the
numerical function $M^*(x)$ and prove two vGIT type results Theorems 5.4 and 5.5. As a corollary, we
obtain another proof of Theorem 1.1. The last section, Section 6, is devoted to the comparison of these
two approaches.

2 The symplectic implosion

In this section, we will review some backgrounds about the symplectic implosion. Basically, all the
materials in this section follow closely from [8]. Along the way, we also set up the assumptions and
notations used in the whole paper. We start with the symplectic geometry features of the symplectic
implosion, and then turn to its complex geometry properties.

2.1 Symplectic aspects

Let $(M, \omega)$ be a connected symplectic manifold with a Hamiltonian group action of a compact group $K$.
Recall that an action is called Hamiltonian if there is a moment map $\mu$ for this action, which by definition
is an equivariant map from $M$ to $\mathfrak{t}^*$, the dual of the Lie algebra of $K$, and satisfies the following equation:

$$d(\mu, X) = \iota_X \omega,$$  \hspace{1cm} (2.1)

where $X \in \mathfrak{t}$ and $X^M$ is the vector field induced by $X$ on $M$ by the infinitesimal group action. We should
remind readers that sometimes the moment map is defined to be $-\mu$ in the literature.

In this paper, we fix once and for all a maximal torus $T$ in $K$, denoting the corresponding Cartan
subalgebra by $\mathfrak{t}$. Also, we fix a closed positive Weyl chamber $\Sigma$ in the dual of $\mathfrak{t}$. By using the root space
decomposition of $\mathfrak{t}$, $\mathfrak{t}^*$ is identified as a subspace of $\mathfrak{t}^*$, and all the points lying on $\sigma$ have the same isotropic group $K_\sigma$ under the coadjoint action $\text{Ad}^*(K)$ on $\mathfrak{t}^*$. An equivalence relation $\sim$ is introduced for the points in $\mu^{-1}(\sigma)$ as follows: for $x, y \in \mu^{-1}(\sigma)$, $x \sim y$ if and only if $x = ky$ for some $k \in [K_\sigma, K_\sigma]$. The imploled cross-section\(^2\) of $M$, $[8$, Definition 2.1$]$, is defined to be the quotient space: $M_{\text{impl}} := \mu^{-1}(\mathfrak{t}_*^+)/\sim$ with the quotient map denoted by $\pi : \mu^{-1}(\mathfrak{t}_*^+) \to M_{\text{impl}}$. Set-theoretically, the imploled cross-section can be written as the following disjoint union:

$$M_{\text{impl}} = \coprod_{\sigma \in \Sigma} \mu^{-1}(\sigma)/[K_\sigma, K_\sigma],$$  \hspace{1cm} (2.2)

where $\Sigma$ denotes the index set of the faces of $\mathfrak{t}_*^+$. Note that on $\Sigma$, there is a natural partial order: $\sigma \leq_\tau$ if and only if $\sigma \subseteq \tau$.

Remark 2.1. About the imploled cross-section $M_{\text{impl}}$, the following properties hold:

(i) In general, $M_{\text{impl}}$ is not a smooth manifold, but only can be a stratified symplectic space in the
weak sense of [28]. The quotient map $\pi$ is always proper. If we assume that $M$ is compact, so is $M_{\text{impl}}$.

(ii) By [8, Corollary 2.7], every component appearing in the decomposition of (2.2) is a symplectic
quotient. More precisely, one has

$$M_{\text{impl}} = \coprod_{\sigma \in \Sigma} \mu^{-1}(\sigma)/[K_\sigma, K_\sigma] = \coprod_{\sigma \in \Sigma} \mu^{-1} \left( K_\sigma \left( \bigcup_{\tau \supseteq \sigma} \tau \right) \right) / [K_\sigma, K_\sigma].$$  \hspace{1cm} (2.3)

Moreover, by the symplectic cross-section theorem, [8, Theorem 2.5], for any $\sigma \in \Sigma$, $\mu^{-1}(K_\sigma(\bigcup_{\tau \supseteq \sigma} \tau))$ is a smooth submanifold of $M$, which implies the singularity of $M_{\text{impl}}$ is not too bad.

\(^2\) $\sigma$ is also called a wall in the literature. But in this paper, the terminology “wall” is reserved for another concept used later.

\(^3\) This is the terminology used by Guillemin et al. [8]. In this paper, the imploled cross-section and the symplectic
implosion will be used interchangeably.

\(^4\) When the symplectic quotient is taken with respect to 0, the subscript of $// \pi$ is omitted.
(iii) The minimal face \( \sigma \) satisfying \( \mu(M) \subseteq \sigma \) is called the \textit{principal face} for \( M \), denoted by \( \sigma_{\text{prin}} \). The group action \( [K_{\sigma_{\text{prin}}}, K_{\sigma_{\text{prin}}}] \) on \( \mu^{-1}(\sigma_{\text{prin}}) \) is trivial actually, which means that the stratum

\[
\mu^{-1}(\sigma_{\text{prin}})/[K_{\sigma_{\text{prin}}}, K_{\sigma_{\text{prin}}}]) = \mu^{-1}(\sigma_{\text{prin}})
\]

(called the \textit{principal cross-section} of \( M \)) in the decomposition (2.2) must be smooth. In many cases, the principal face of \( M \) is the interior of the positive Weyl chamber \((t_{+}^*)^0\). Especially, since in this paper we are mainly concerned with the case where the regular value set of \( \mu \) is non-empty, we will always assume that \( \sigma_{\text{prin}} = (t_{+}^*)^0 \).

By definition, \( M_{\text{impl}} \) inherits a \( T \)-action from the \( K \)-action of \( M \). Besides, the moment map \( \mu \) on \( M \) also induces a continuous map \( \mu_{\text{impl}} : M_{\text{impl}} \to t_{+}^* \) on the symplectic implosion. Although \( M_{\text{impl}} \) is not a manifold in general, \( (M_{\text{impl}}, \omega_{\text{impl}}, \mu_{\text{impl}}) \) can be seen as a Hamiltonian \( T \)-space. This means that when restricted to a smooth stratum, \( \mu_{\text{impl}} \) is just a moment map for the smooth \( T \)-action in the usual sense. Under such identification, the Liouville form \( \beta \) on \( T^*K \) can be written as follows:

\[
\beta_{(k,\lambda)}(X, \xi) = \langle \lambda, X \rangle \quad \text{for} \quad (k, \lambda) \in K \times t^* \simeq T^*K, \quad (X, \xi) \in t \times t^* \simeq T(k,\lambda)(K \times t^*). \tag{2.4}
\]

Let \( \omega = d\beta \) be the usual symplectic form on \( T^*K \). Note that both \( \mathcal{L} \) and \( \mathcal{R} \) can be lifted to actions on \( T^*K \). We denote the lifted actions by the same notations:

\[
\mathcal{L}_{g}(k, \lambda) = (gk, \lambda), \quad \mathcal{R}_{g}(k, \lambda) = (kg^{-1}, g\lambda),
\]

where the group action on \( t^* \) is the coadjoint action \( \text{Ad}^* \). Both of \( \mathcal{L} \) and \( \mathcal{R} \) are Hamiltonian with respect to \( \omega \), by (2.4), whose moment maps are

\[
\Phi_{\mathcal{L}}(k, \lambda) = -k\lambda, \quad \Phi_{\mathcal{R}}(k, \lambda) = \lambda. \tag{2.5}
\]

One can carry out the symplectic implosion construction for \( T^*K \) using either \( \mathcal{L} \) or \( \mathcal{R} \). Following the convention of [8], in this paper, the imploled cross-section of \( T^*K, (T^*K)_{\text{impl}} \), is always constructed out of \( \mathcal{R} \) unless otherwise declared. The decomposition of (2.3) here has a more explicit form

\[
(T^*K)_{\text{impl}} = \prod_{\sigma \in \Sigma} \left( K \times \left( K_{\sigma} \bigcup_{\tau > \sigma} \tau \right) \right) \parallel [K_{\sigma}, K_{\sigma}] = \prod_{\sigma \in \Sigma} \frac{K}{[K_{\sigma}, K_{\sigma}]} \times \sigma. \tag{2.6}
\]

Note that the principal cross-section here is \( K \times (t_{+}^*)^0 \) and all the components in the decomposition (2.6) are smooth submanifolds. As the symplectic implosion of \( T^*K \), \( (T^*K)_{\text{impl}} \) inherits a \( K \)-action from \( \mathcal{L} \) and a \( T \)-action from \( \mathcal{R} \). By (2.5), the moment map for the \( T \)-action on \( (T^*K)_{\text{impl}} \) is \( \Phi_{\text{impl,R}}([k], \lambda) = \lambda \).
for $[k] \in K/[K_\sigma,K_\sigma]$. Moreover, it is easy to check that the induced $K$-action is also Hamiltonian whose moment map is $\Phi_{impl,\mathcal{L}}([k],\lambda) = -k\lambda$. By Theorem 2.2, one can calculate the symplectic quotient of $(T^*K)_{impl}$ with respect to the $T$ action explicitly. Recall that a well-known property of $T^*K$ is that its symplectic quotients are coadjoint orbits (see [19, Subsection 4.2]). Therefore, the symplectic quotient $(T^*K)_{impl} \sslash T$ as a Hamiltonian $K$-manifold, is naturally isomorphic to the coadjoint orbit $\mathcal{O}_\tau^*$. A special feature of $(T^*K)_{impl}$ is the following universal property. By [8, Theorem 4.9], for any Hamiltonian $K$-manifold $(M,\omega,\mu)$, $M_{impl}$ as a Hamiltonian $K$-space can be constructed as follows:

$$M_{impl} = (M \times (T^*K)_{impl}) \sslash K,$$ (2.7)

where the $K$-action on the product manifold is the diagonal action. For this reason, $(T^*K)_{impl}$ is called the universal imploded cross-section.

**Remark 2.3.** Some comments about the assumption on the groups used in this paper. As we have said, we always assume that $K$ is a compact group. Let $K'$ be a finite cover of $K$. The $K$-action on $M$ also induces a Hamiltonian $K'$-action. In addition, symplectic quotients obtained with respect to the $K$-action and the $K'$-action are the same. Since in this paper, we are only interested in the properties of symplectic quotients, from now on, we will assume that $K$ is isomorphic to a product of a torus and a semi-simple simply connected group.

Moreover, in the remaining subsection of this section discussing the compact geometry properties of a symplectic implosion, we will further assume that the group $K$ is a semi-simple simply connected group to omit some technicalities involved in the general case [8, the bottom of p.174]. We remark that such an assumption causes no loss of generalities. Since, by [8, Lemma 2.4], the symplectic implosions of $M$ with respect to $K$ and $[K,K]$ are equal, one can use the complex structure of $M_{impl,[K,K]}$ to define the complex structure of $M_{impl,K}$.

### 2.3 Complex aspects

In the rest of this paper, we assume that the Hamiltonian $K$-manifold $(M,\omega,\mu)$ is endowed with a compatible integrable $K$-invariant complex structure $J$, which means that $J$ preserves the symplectic structure and $g(-,-):=\omega(-,J-)$ is a Riemannian metric on $M$. In other words, $(M,g,J)$ is a Kähler manifold and $\omega$ is the Kähler form. Unlike the symplectic structure on $M_{impl}$, the Kähler structure on $M_{impl}$ does not inherit from $M$ directly. To see this, consider the following fact: although the principal cross-section $\mu^{-1}(\sigma_{prim})$ of $M_{impl}$ can be seen as a smooth submanifold of $M$, in general, $\mu^{-1}(\sigma_{prim})$ is not a complex submanifold of $M$. To overcome this problem, one first defines the complex structure on $(T^*K)_{impl}$ and the complex structure of $M_{impl}$ is defined by using the equality (2.7).

Following [8, Section 6], to define the complex structure on the universal imploded cross-section $(T^*K)_{impl}$, we need to embed it into a $K$-representation space as an affine subvariety. Let $\Lambda = \ker(\exp |_t)$ be the exponential lattice in $t$ and $\Lambda^* = \text{Hom}_\mathbb{Z}(\Lambda,\mathbb{Z})$ be the weight lattice in $t^*$. Then $\Lambda^*_\mathbb{R} = \Lambda \cap t^*_\mathbb{R}$ is the monoid of dominant weights. Choose a set of fundamental weights $\Pi = \{\varpi_1,\ldots,\varpi_r\}$, which spans $\mathbb{Z}$-basis. Let $V_{\varpi_i}$ be the irreducible representation of $K$ with the highest weight $\varpi_i$ and $v_i$ be a fixed highest weight vector of $V_{\varpi_i}$. We will show that $(T^*K)_{impl}$ can be embedded into $E = \bigoplus_{\varpi \in \Pi} V_{\varpi}$.

Before describing the embedding, we recall some facts about the Hamiltonian action on the vector space $E$. Take a Hermitian metric $(-,-)_E$ on $E$ such that one can decompose $E$ into the direct sum of unitary $K$-subrepresentation $V_{\varpi}$ and $\|v_i\| = 1$. The symplectic form and the moment map of $E$ are given by

$$\omega_E(v,w) = -\text{Im} \langle v, w \rangle_E \quad \text{and} \quad \langle \mu_E(v), X \rangle = \frac{1}{2} \omega_E(X,v,v),$$ (2.8)

respectively, where $v,w \in E$ and $X \in t$. For the later usage, one also defines a $T$-action on $E$ by requiring $T$ acting on $V_{\varpi}$ with the weight $-\varpi$. Clearly, this $T$-action commutes with the $K$-action.

Now let $\{\alpha_1,\ldots,\alpha_r\} \subseteq t^*$ be simple roots of $t$, and $\{\alpha_1^\vee,\ldots,\alpha_r^\vee\} \subseteq t$ be the corresponding coroots, i.e., $\alpha_i^\vee = 2\alpha_i^*/(\alpha_i,\alpha_i) \in t$, where $(-,-)$ is a Weyl group invariant inner product on $t^*$ and $\alpha_i^*$ is the dual
element of $\alpha_i$ with respect to the inner product. It is well known that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in \{1, \ldots, r\}$. Since $\lambda(\alpha_j^\vee) \geq 0$ holds for any $\lambda \in \mathfrak{T}_+^*$, we can define a continuous map from $K \times \mathfrak{T}_+^*$ to $E$ as follows:

$$F(k, \lambda) = \frac{1}{\sqrt{\pi}} \sum_{p=1}^{r} \sqrt{\lambda(\alpha_p^\vee)} k.e_p. \quad (2.9)$$

By [8, Lemma 6.2], $F$ can descend to a map on $(T^* K)_{\text{impl}}$ and we will denote the descended map by the same symbol $F$. In fact, $F : (T^* K)_{\text{impl}} \to E$ gives the claimed embedding. Let $G = K^C$ be the complexification of $K$ and $N$ be a maximal unipotent group. The image of $F$ in $E$ is

$$G_N := G \left( \sum_{\varpi \in \Pi} v_{\varpi} \right)$$

with respect to the Zariski or standard topology. As an affine subvariety of $E$, $G_N$ has an induced Hamiltonian $K \times T$-space structure, which is isomorphic to $(T^* K)_{\text{impl}}$ via $F$ by [8, Propositions 6 and 8]. Therefore, by identified with $G_N$, $(T^* K)_{\text{impl}}$ is given a complex structure, which is compatible with its symplectic structure. Using such a description of $(T^* K)_{\text{impl}}$, one can find the principal cross-section is the unique open and dense $G$ orbit $G(\sum_{\varpi \in \Pi} v_{\varpi}) \simeq G/N$. Moreover, each stratum in (2.6) corresponds to a $G$ orbit, i.e.,

$$F(K/[K_\sigma, K_\sigma] \times \sigma) = G/[P_\sigma, P_\sigma],$$

where $P_\sigma$ is the parabolic subgroup associated with face $\sigma$.

When $(T^* K)_{\text{impl}}$ is equipped with a complex structure, one can use (2.7) to define the complex structure on $M_{\text{impl}}$ as a symplectic quotient of a Kähler manifold in the usual way which we review in the next section. A small inaccuracy here is that since $M \times (T^* K)_{\text{impl}}$ is a Kähler space, not smooth necessarily, the symplectic reduction needs some extension, i.e., one should use the reduction of complex spaces developed in [11,12]. However, for our purpose, we actually do not need such a general theory. The reason is that the space we are interested in can always be obtained via the symplectic reduction construction involving only a suitable smooth stratum of $(T^* K)_{\text{impl}}$.

As we have recalled, the $T$-action on $M_{\text{impl}}$ is Hamiltonian. In the Kähler case, one can further assert that the $T$-action is also holomorphic by the result in [7]. We give a proof for the simpler manifold case in the next section.

### 3 The first proof of Theorem 1.1: The symplectic implosion approach

In this section, we use the Kähler geometry properties of the symplectic implosion to prove Theorem 1.1. To this aim, we first give a proof for Theorem 1.2 by using Proposition 3.2, which is a “reduction in stages” result for Kähler manifolds. Then, we show Theorem 1.1 by combining Theorem 1.2 and the property of the abelian reduction.

#### 3.1 The Kähler structure on symplectic quotients

For readers’ convenience, we briefly recall the definition of the Kähler structure on a symplectic quotient. Let $(X, \omega_X, J_X, \mu_X)$ be a Hamiltonian Kähler $K$-manifold. Suppose that 0 is a regular value of the moment map $\mu_X$. At any point $x \in \mu^{-1}(0)$, one has the following orthogonal decomposition of the tangent space:

$$T_x \mu^{-1}(0) = Q_x \oplus \mathfrak{t} \cdot x \quad \text{and} \quad T_x X = Q_x \oplus \mathfrak{t} \cdot x \oplus J_X (\mathfrak{t} \cdot x), \quad (3.1)$$

where $\mathfrak{t} \cdot x$ is the tangent subspace induced by the infinitesimal $K$-action at $x$. By the definition of a Kähler metric, $Q_x$ is a $J$-invariant subspace, which implies that $Q_x$ is also a symplectic subspace of $T_x X$. Hence,

$$d\pi_x : Q_x \to T_{\pi(x)} X_0$$

$^{5)}$ In general, any regular value lying in $\mathfrak{g}^*$ works, where $\mathfrak{g}$ is the center of $\mathfrak{t}$. 
is an isomorphism, where \( \pi \) is the quotient map from \( \mu^{-1}(0) \) to the symplectic quotient \( X_0 = \mu^{-1}(0)/K \). Via \( d\pi(x) \), the (Riemannian) metric and the almost complex structure on \( T_{\pi(x)}X_0 \) are induced from those of \( Q_x \). One can show that (see [5, Lemma 7.2.7] or [18, Lemma 2.2]) the almost complex structure on \( X_0 \) obtained in this way is integrable and compatible with the reduced symplectic form \( \omega_0 \) on \( X_0 \).

As an additional remark, to define a complex structure on a symplectic quotient, besides the method described as above, there is another possible way, i.e., by selecting a semi-stable point set, \( X^{ss} \), of \( X \) and defining \( X_0 \) to be the holomorphic quotient of \( X^{ss} \) by \( K^{C} \) (see [16, Section 7]). Readers may find that a short review of this method in Section 4 is helpful.

Now we show that the \( T \)-action on a symplectic implosion is holomorphic. Since the complex structure on a symplectic implosion is defined by using the symplectic reduction, this result follows from the following general property about group actions on a symplectic quotient.

**Proposition 3.1.** Assume that there is another compact, Hamiltonian and holomorphic group action \( L \) on \( X \). Moreover, the \( L \)-action commutes with the \( K \)-action and preserves the moment map \( \mu \) of the \( K \)-action. Then \( L \) induces an action on the symplectic quotient \( X_0 \), which preserves the Kähler structure on \( X_0 \).

**Proof.** It is a classical result that the induced \( L \)-action preserves the reduced symplectic structure (see [19, Theorem 2]). We are going to show the induced \( L \)-action also preserves the complex structure on \( X_0 \). Let \( W \in \text{Lie}(L) \) and \( W^{X} \) be the induced vector field of \( W \) on \( X^{6} \). First, we show that the Lie derivative of \( W^{X} \) preserves the sections lying in the subbundle \( Q \subseteq T\mu^{-1}(0) \) over \( \mu^{-1}(0) \). For any \( w \in \Gamma(Q) \) being a smooth section of \( Q \), if we view \( w \) as a vector field of the submanifold \( \mu^{-1}(0) \),

\[
\mathcal{L}_{W^{X}}w = [W^{X}, w] \in \Gamma(T\mu^{-1}(0))
\]

is well defined. For any \( Y \in \mathfrak{t} \), since \( L \) preserves the restricted metric on \( \mu^{-1}(0) \), one has

\[
(\mathcal{L}_{W^{X}}w, Y^{X}) = W^{X}(w, Y^{X}) - (w, [W^{X}, Y^{X}]) = 0,
\]

which implies that \( \mathcal{L}_{W^{X}}w \in \Gamma(Q) \).

Now choose any vector field \( v \) on \( X_0 \), denoting the lift-up of \( v \) in \( \Gamma(Q) \) by \( v^{\sharp} \). Let \( J_0 \) be the reduced complex structure on \( M_0 \). By the definition of \( J_0 \) and the induced \( L \)-action, one has

\[
J_0 v = d\pi(J v^{\sharp}), \quad W^{X_0} = d\pi(W^{X}). \tag{3.2}
\]

Using (3.2), we have

\[
(\mathcal{L}_{W^{X_0}}J_0)(v) = [W^{X_0}, J_0(v)] - J_0 [W^{X_0}, v]
= [d\pi(W^{X}), d\pi(J v^{\sharp})] - J_0 [d\pi(W^{X}), d\pi(v^{\sharp})]
= d\pi([W^{X}, J v^{\sharp}]) - J_0 d\pi([W^{X}, v^{\sharp}])
= d\pi([W^{X}, J v^{\sharp}]) - d\pi J([W^{X}, v^{\sharp}])
= d\pi((\mathcal{L}_{W^{X}}J)(v^{\sharp})) = 0. \tag{3.3}
\]

Note that in the fourth equality, we have used \( [W^{X}, v^{\sharp}] = \mathcal{L}_{W^{X}}v^{\sharp} \in \Gamma(Q) \) and \( [W^{X}, v^{\sharp}] \) is a \( K \)-invariant vector field. Since \( v \) is arbitrary, one concludes that \( \mathcal{L}_{W^{X_0}}J_0 = 0 \).

With Proposition 3.1, we can state the following “reduction in stages” result for Kähler manifolds, which is a preparation for the proof of Theorem 1.2.

---

\[ \text{The restriction of } W^{X} \text{ on } \mu^{-1}(0) \text{ is denoted by the same symbol.} \]
Proposition 3.2. Let \((X, \omega_X, J_X, \mu_1 \times \mu_2)\) be a Hamiltonian Kähler \(K_1 \times T_2\)-manifold, where \(K_1\) is a compact semi-simple group and \(T_2\) is a compact torus. Suppose that \((0, \lambda)\) is a regular value of \(\mu_1 \times \mu_2\). Then as Kähler manifolds, \(X \parallel_{(0, \lambda)} (K_1 \times T_2)\) is naturally isomorphic to \((X \parallel_0 K_1) \parallel_\lambda T_2\).

The two symplectic quotients in the proposition are canonically diffeomorphic to each other and we will identify the two quotients in the following. By using reduction in stages for symplectic manifolds (or spaces) (see [19, Theorem 2] and [28, Section 4]), the reduced symplectic forms given by the two methods in the proposition are the same. Therefore, to show the two reduction procedures leads to the same complex structure on the symplectic quotient, and one only needs to show that the two reduced metrics on the symplectic quotient coincide. For this purpose, the following elementary fact about projections of linear spaces is useful.

Lemma 3.3. Let \(E\) be a complex vector space with a Hermitian inner product. \(V\) and \(W\) are two subspaces of \(E\). For the following three orthogonal projections \(P : E \to (V + W)\perp, P_1 : E \to W\perp\) and \(P_2 : W\perp \to (P_1(V))\perp \cap W\perp,\) one has \(P = P_2 P_1\).

Proof. If \(u \in V + W,\) one has \(u = v + w\) with \(v \in V, w \in W\). Then by definition, \(P_2 P_1 (u) = P_2 P_1 (v) = 0\) and \(P(u) = 0\). If \(u \in (V + W)\perp,\) one has \(w \in W\perp \cap V\perp\). Therefore, \(P_2 P_1 (u) = P_2 (u)\). On the other hand, if \(u \in (V + W)\perp,\) since any \(v' \in P_1 (V)\) can be written as \(v' = v - v''\) with \(v, v'' \in W,\) one has \(u \perp v',\) i.e., \(u \in (P_1(V))\perp\). Hence, \(P_2 (u) = u = P(u)\).

Proof of Proposition 3.2. Firstly, we assume \(X\) to be a manifold. Let \(x \in X\) lying in the level set \((\mu_1 \times \mu_2)\perp (0, \lambda).\) Denote \(W\) to be the subspace of \(T_x X\) generated by the infinitesimal \(K_1\)-action and \(V \subseteq T_x X\) to be the subspace generated by the infinitesimal \(T_2\)-action. Choose

\[v \in T_x ((\mu_1 \times \mu_2)\perp (0, \lambda)) \subseteq T_x X\]

transversal to \(W + V.\) Introduce the following two quotient maps:

\[\pi : (\mu_1 \times \mu_2)\perp (0, \lambda) \to ((\mu_1 \times \mu_2)\perp (0, \lambda) / (K_1 \times T_2) = X \parallel_{(0, \lambda)} (K_1 \times T_2),\]

\[\bar{\pi} : (\mu_1 \times \mu_2)\perp (0, \lambda) / ((\mu_1 \times \mu_2)\perp (0, \lambda) / K_1) / T_2 = (X \parallel_0 K_1) \parallel_\lambda T_2.\]

Since \(((\mu_1 \times \mu_2)\perp (0, \lambda) / (K_1 \times T_2)\) is naturally isomorphic to \(((\mu_1 \times \mu_2)\perp (0, \lambda) / K_1) / T_2,\) \(\pi\) and \(\bar{\pi}\) are essentially the same map. We decide to use different notations to remind us that they originate from different reduction procedures. We are going to show that the norm on \(d\pi (v),\) coming from reduction of \(K_1 \times T_2,\) is equal to the norm on \(d\bar{\pi} (v),\) coming from reduction first by \(K_1\) and then by \(T_2.\) As a result, the two metrics on the common quotient coincide. Suppose \(P\) to be the orthogonal projection from \(T_x ((\mu_1 \times \mu_2)\perp (0, \lambda))\) onto \((V + W)\perp,\) \(P_1\) to be the orthogonal projection from \(T_x ((\mu_1 \times \mu_2)\perp (0, \lambda))\) onto \(W\perp\) and \(P_2\) to be the orthogonal projection from \(W\perp\) onto \((P_1(V))\perp \cap W\perp.\) By the definition of the metric on a symplectic quotient, the norm of \(d\pi (v)\) is equal to \(\|P(v)\|\) and the norm of \(d\bar{\pi} (v)\) is equal to \(\|P_2 P_1 (v)\|.\) Therefore, the equality of the metrics is a result of Lemma 3.3.

Remark 3.4. We make some comments about implications of Proposition 3.2.

(i) In Proposition 3.2, the \(K_1\)-action and the \(T_2\)-action play similar roles, which means that it makes no difference which action comes first when we perform reduction in stages. As a corollary, one can see that

\[(X \parallel_0 K_1) \parallel_\lambda T_2 \simeq (X \parallel_\lambda T_2) \parallel_0 K_1.\]

(ii) The same proof of Proposition 3.2 also works for a more general case: \(K_1 \times T_2\) can be replaced by \(K_1 \times K_2,\) where \(K_1\) and \(K_2\) are arbitrary compact groups; \((0, \lambda)\) can be replaced by \((\lambda_1, \lambda_2),\) where \(\lambda_i\) lies in the dual of the center of \(K_i, i = 1, 2.\)

3.2 Proofs of Theorems 1.1 and 1.2

Let \((M, \omega, J, \mu)\) be a Hamiltonian Kähler \(K\)-manifold. As an application of Proposition 3.2, we first give a proof of Theorem 1.2.
Proof of Theorem 1.2. We first prove the case where \( K \) is a semi-simple simply connected group. We begin with an analysis of the metric of \((T^* K)^{\text{impl}}\). Here, we use notational conventions as in Subsection 2.3. Let \((R, R_+)\) be a root system for \( g = \mathfrak{f}^C \) containing simple roots \( S = \{\alpha_1, \ldots, \alpha_r\} \). One has the following decomposition of \( \mathfrak{t}^* \):

\[
\mathfrak{t} = \mathfrak{t} \oplus \sum_{\alpha \in R_+} \mathfrak{t}_\alpha, \quad \mathfrak{t} = \sum_{i=1}^r \mathbb{R} \cdot \alpha_i^\vee,
\]

where \( \mathfrak{t}_\alpha = (\langle g \rangle_\alpha \oplus \langle g \rangle_{-\alpha}) \cap \mathfrak{t} \) is a real two-dimensional vector space. Let \( R(\sigma) \subseteq R \) be a subset of roots such that \( \langle \xi, \alpha^\vee \rangle = 0 \) for any \( \xi \in \sigma, \alpha \in R(\sigma) \) and let \( S(\sigma) := S \cap R(\sigma) \). Recall that \( \mathfrak{t}_\sigma \subseteq \mathfrak{t} \) is the Lie algebra of the isotropic group of any point lying in \( \sigma \). The following decomposition also holds:

\[
[\mathfrak{t}_\sigma, \mathfrak{t}_\sigma] = \sum_{\alpha \in S(\sigma)} \mathbb{R} \cdot \alpha_i^\vee \oplus \sum_{\alpha \in R(\sigma) \cap R_+} \mathfrak{t}_\alpha,
\]

\[
\mathfrak{t}/[\mathfrak{t}_\sigma, \mathfrak{t}_\sigma] \simeq \sum_{\alpha \in S(\sigma)} \mathbb{R} \cdot \alpha_i^\vee \oplus \sum_{\alpha \in R_+ \setminus R(\sigma)} \mathfrak{t}_\alpha.
\]

One notices that for the smooth stratum \( K/\{K_\sigma, K_\tau\} \times \sigma \) of \((T^* K)^{\text{impl}}\), the tangent space at \( (\xi, \lambda) \) is \( \mathfrak{t}/[\mathfrak{t}_\sigma, \mathfrak{t}_\sigma] \oplus \sigma \). By (2.5), for any \( (\xi, \lambda) \in (T^* K)^{\text{impl}} \), one has \( \Phi^{\text{impl}, \mathcal{R}}(\xi, \lambda) = \lambda \), which means that

\[
T_{(\xi, \lambda)}\Phi^{-1, \mathcal{R}} \mathcal{F}^{-1, \mathcal{R}}(\lambda) = \mathfrak{t}/[\mathfrak{t}_\sigma, \mathfrak{t}_\sigma].
\]

Recall that in Subsection 2.3, we define the complex structure, or the Kähler structure equivalently, on \((T^* K)^{\text{impl}}\) using an embedding \( \mathcal{F} : (T^* K)^{\text{impl}} \to E \). By the definition of the symplectic form on \( E \), (2.8), for any \( H \in \sum_{\alpha \in S(\sigma)} \mathbb{R} \cdot \alpha_i^\vee \) or \( X \in \mathfrak{t}_\sigma, \alpha \in R_+ \setminus R(\sigma) \), one has

\[
d\mathcal{F}_{(\xi, \lambda)}(V) = \frac{1}{\sqrt{\pi}} \sum_{p=1}^r \sqrt{\lambda(\alpha_p^\vee)} V_{\alpha_p} \text{ for } V \text{ being equal to } H \text{ or } X.
\]

Since \( \{v_p\} \) are the highest weight vectors, the above equality implies that \( (H, v_p, X, v_q) = 0 \) for any \( p, q \in \{1, \ldots, r\} \). Therefore, \( \sum_{\alpha \in S(\sigma)} \mathbb{R} \cdot \alpha_i^\vee \) and \( \sum_{\alpha \in R_+ \setminus R(\sigma)} \mathfrak{t}_\alpha \) as subspaces at \( T_{(\xi, \lambda)}\Phi^{-1, \mathcal{R}}(\lambda) \) are always orthogonal to each other under the pullback metric. Clearly, the former subspace is the subspace generated by the right \( T \)-action at this point. To calculate the metric on the symplectic quotient of the \( T \)-action, we only need to calculate the metric on \( \sum_{\alpha \in R_+ \setminus R(\sigma)} \mathfrak{t}_\alpha \subseteq \mathfrak{t}/[\mathfrak{t}_\sigma, \mathfrak{t}_\sigma] \).

Choose a vector \( X_\alpha \) in \( g_\alpha \) for \( \alpha \in R_+ \setminus R(\sigma) \). Let \( \theta \) be the Cartan involution corresponding to \( \mathfrak{t} \subseteq \mathfrak{g} \). One has \( X_\alpha := \theta(X_\alpha) \in g_{-\alpha} \). Using a suitable normalization of \( X_\alpha \), we can assume that \( [X_\alpha, X_\alpha] = -i \alpha^\vee \).

By the definition of the Cartan involution, \( U_\alpha := X_\alpha + X_{-\alpha} \) and \( V_\alpha := iX_\alpha - iX_{-\alpha} \) are the vectors in \( \mathfrak{t}_\alpha \).

For any \( \alpha, \alpha' \in R_+ \setminus R(\sigma) \), using (2.8) and (3.6), we can calculate the inner product of \( U_\alpha \) and \( V_{\alpha'} \) as the vectors in \( T_{(\xi, \lambda)}\Phi^{-1, \mathcal{R}}(\lambda) \) under the pullback metric of \( E \) in the following way:

\[
(U_\alpha, V_{\alpha'})(\xi, \lambda) = \frac{-1}{\pi} \sum_{p=1}^r \lambda(\alpha_p^\vee) \text{Im} (U_\alpha, v_p, iV_{\alpha'}, v_p)_E
\]

\[
= \frac{-1}{\pi} \sum_{p=1}^r \lambda(\alpha_p^\vee) \text{Im} (X_{-\alpha}, v_p, X_{-\alpha'}, v_p)_E
\]

\[
= \frac{-1}{\pi} \sum_{p=1}^r \lambda(\alpha_p^\vee) \text{Im} (-X_{\alpha'}, X_{-\alpha}, v_p, v_p)_E
\]

\[
= \frac{-1}{\pi} \sum_{p=1}^r \lambda(\alpha_p^\vee) \text{Im} ([X_{\alpha'}, X_{-\alpha}], v_p, v_p)_E.
\]

In the above equalities, the first one uses the relation between the real inner product and the Hermitian inner product on \( E \). The second and fourth equalities use the fact that \( v_p \) is the highest weight vector. As for the third equality, recall that the adjoint operator of \( X_{-\alpha} \) is \(-X_{\alpha} \).
Now if $\alpha \neq \alpha'$, then $[X_{\alpha'}, X_{-\alpha}] \notin \mathfrak{t}$, and consequently $([X_{\alpha'}, X_{-\alpha}], v_p, v_p)_E = 0$. Otherwise, for $\alpha = \alpha'$, one has
\[
\text{Im} \left( ([X_{\alpha'}, X_{-\alpha}], v_p, v_p)_E \right) = \text{Im} \left( (i\omega, v_p, v_p)_E \right) = -\text{Im} \left( 2\pi \omega_p(\lambda) \right) = 0.
\]
All in all, the inner product of $U_\alpha$ and $V_{\alpha'}$ always vanishes. In the same way, one can show that
\[
(U_\alpha, U_{\alpha'})(\lambda, \lambda) = \begin{cases} 
2 \sum_{p=1}^r \lambda(p) \omega_p(\lambda') = 2\langle \lambda, \lambda' \rangle, & \alpha = \alpha', \\
0, & \alpha \neq \alpha',
\end{cases} \tag{3.7}
\]
where we have used $\langle \omega_p, \lambda' \rangle = \delta_{pq}$. For the inner product between $V_\alpha$ and $V_{\alpha'}$, the result is similar.

Recall that the Kirillov-Kostant-Souriau symplectic form on the coadjoint orbit $O_\lambda := K \cdot \lambda \subseteq \mathfrak{t}^*$ is defined as follows:
\[
\omega_{O_\lambda}(X \cdot [\lambda], Y \cdot [\lambda]) = \langle \lambda, [X, Y] \rangle, \tag{3.8}
\]
where $X, Y \in \mathfrak{t}$. As we have said before, the symplectic reduction of $(T^*K)_{\text{impl}}$ at $\lambda$ for the $T$-action is $O^*_\lambda$, i.e., $O_\lambda$ with the symplectic form $-\omega_{O_\lambda}$. Now if we define the almost complex structure on $T_\lambda O^*_\lambda \simeq \sum_{\alpha \in R_+, \lambda} r_\alpha$ as follows: for $\alpha \in R_+ \setminus R(\sigma)$,
\[
J(U_\alpha \cdot [\lambda]) = -V_\alpha \cdot [\lambda], \quad J(V_\alpha \cdot [\lambda]) = U_\alpha \cdot [\lambda], \tag{3.9}
\]
it is well known that such an almost complex structure is integrable on the coadjoint orbit. By (3.8) and (3.9), the Kähler metric on $O^*_\lambda$ is
\[
(U_\alpha \cdot [\lambda], U_{\alpha'} \cdot [\lambda])_{O^*_\lambda} = -\omega_{O_\lambda}(U_\alpha \cdot [\lambda], J(U_{\alpha'} \cdot [\lambda])) = \langle \lambda, [U_\alpha, V_{\alpha'}] \rangle = \begin{cases} 
2\langle \lambda, \lambda' \rangle, & \alpha = \alpha', \\
0, & \alpha \neq \alpha'.
\end{cases} \tag{3.10}
\]
The terms involving $V_\alpha$ can be calculated in the same way. Comparing (3.7) and (3.10), one can conclude
\[
(T^*K)_{\text{impl}} \big|_\lambda T = (K/[K_{\sigma}, K_{\sigma}] \times \sigma) \big|_\lambda T \simeq O^*_\lambda
\]
as the Kähler manifold. Now, by the definition of the Kähler reduction, Proposition 3.2 and Remark 3.4(i), we can show the conclusion of Theorem 1.2 using the following argument:
\[
M \big|_\lambda K \simeq (M \times O^*_\lambda) \big|_\lambda K \simeq ((M \times K/[K_{\sigma}, K_{\sigma}] \times \sigma) \big|_\lambda T) \big|_\lambda K \\
\simeq ((M \times K/[K_{\sigma}, K_{\sigma}] \times \sigma) \big|_\lambda K) \big|_\lambda T = M_{\text{impl}} \big|_\lambda T, \tag{3.11}
\]
where in the last equality we use the fact that
\[
\Phi^{-1}_{\text{impl}, \mathcal{R}}(\lambda) \subseteq K/[K_{\sigma}, K_{\sigma}] \times \sigma.
\]

For a general compact group $K$, let $K = K_{\text{ss}}Z$, where $K_{\text{ss}} = [K, K]$ is the semi-simple part of $K$ and $Z$ is the center of $K$. Choose a Cartan subgroup $T_{\text{ss}}$ of $K_{\text{ss}}$. Then $T = T_{\text{ss}}Z$ is a Cartan subgroup of $K$. Take $\lambda = \lambda_1 + \lambda_2 \in t_{\text{ss}}^* \oplus \mathfrak{z}^* = \mathfrak{t}^*$, where $\lambda_1$ lies in the face $\sigma$ of $t_{\text{ss}}^*$. Then $\lambda$ lies in the face $\bar{\sigma} := \sigma \oplus \mathfrak{z}^*$ of the positive Weyl chamber of $K$. Recall that in Remark 2.3, for a general $K$, the complex structure on $M_{\text{impl}, K}$ is defined by $M_{\text{impl}, K}$. Then, by Proposition 3.2 and (3.11),
\[
M \big|_\lambda K \simeq (M \times O^*_{\lambda_1}) \big|_{(0, \lambda_2)} (K_{\text{ss}} \times Z) \simeq ((M \times O^*_{\lambda_1}) \big|_{K_{\text{ss}}}) \big|_{\lambda_2} Z \\
\simeq ((M \times K_{\text{ss}}/[K_{\text{ss}}, K_{\text{ss}}] \times \sigma) \big|_{K_{\text{ss}}}) \big|_{\lambda_2} T_{\text{ss}} \big|_{\lambda_2} Z \\
\simeq ((M \times (T^*K_{\text{ss}})_{\text{impl}}) \big|_{K_{\text{ss}}}) \big|_{\lambda} T = M_{\text{impl}, K} \big|_{\lambda} T,
\]
where $O_{\lambda_1}$ is the coadjoint orbit of $\lambda_1$ in $\mathfrak{k}_{\text{ss}}$. \qed
As an application of Theorem 1.2, we can show the local invariance of complex structures on a symplectic quotient, i.e., Theorem 1.1. Firstly, we recall the corresponding result for a torus action. For simplicity, we only state the result for an $S^1$-action. The higher-dimensional case is similar.

**Lemma 3.5.** Let $K = \mathbb{S}^1$. Instead of assuming the compactness of $M$, here we only require the moment map $\mu : M \to \mathfrak{t}^* \simeq \mathbb{R}^1$ of the $K$-action is proper. If $c$ is a regular value of $\mu$ and $S^1$ acts on $\mu^{-1}(c)$ freely, then for any $c' \in \mathbb{R}$ close to $c$ enough, $M_{c'}$ is biholomorphic to $M_c$.

For readers’ convenience, we reproduce the proof of [5, Lemma 7.4.1]. One can also see [18, Proposition 2.4].

**Proof of Lemma 3.5.** Choosing a non-zero vector $X$ in $\mathfrak{k}$, since $c$ is a regular value of $\mu$ and $c$ and $c'$ are close enough, the flow generated by $JX^M$ induces a diffeomorphism between $M_{c'}$ and $M_c$. We will show that this diffeomorphism is also holomorphic.

Since the almost complex structure on $M$ is integrable, the corresponding Nijenhuis tensor vanishes. Therefore, for any vector field $v$,

$$[JX^M, Jv] - J[JX^M, v] = J[X^M, Jv] + [X^M, v].$$

(3.12)

Since $X^M$ induces a holomorphic isometry, one has

$$0 = (\mathcal{L}_{X^M} J)(v) = \mathcal{L}_{X^M}(Jv) - J(\mathcal{L}_{X^M}v) = [X^M, Jv] - J[X^M, v].$$

(3.13)

By combining (3.12) and (3.13),

$$0 = (\mathcal{L}_{X^M} J)(v) = \mathcal{L}_{X^M}(Jv) - J(\mathcal{L}_{X^M}v) = [X^M, Jv] - J[X^M, v] = 0.$$ (3.14)

Since $v$ is an arbitrary vector field, (3.14) implies that the flow generated by $JX^M$ preserves the complex structure of $M$. Note that this flow also preserves the decomposition of (3.1). Therefore, the complex structure of the subspace $Q$ is invariant under the action of the flow, which implies that the diffeomorphism between $M_{c'}$ and $M_c$ is holomorphic.

Due to Theorem 1.2, Theorem 1.1 is an easy corollary of Lemma 3.5. More precisely, we use a minor generalization of Lemma 3.5: instead of assuming the properness of $\mu$, we only need that $\mu$ is proper in a neighborhood of $\mu^{-1}(c)$, under which condition the proof of the lemma still works.

**Proof of Theorem 1.1.** Using Proposition 3.2 and Lemma 3.5, we can see the only case needed to show is that $K$ is semi-simple and simply-connected. We choose a small neighborhood $U$ of $\lambda$ in $\mathfrak{g}$ and denote $S := (M \times K/[K_{G}, K_{\sigma}] \times U) \sslash K$. Since $\lambda$ is regular, $S$ is a smooth manifold. For $\lambda, \lambda' \in U$, by the proof of Theorem 1.2, we have $M_{\lambda'} = S \sslash \lambda T$, $M_{\lambda} = S \sslash \lambda' T$. Now, by Lemma 3.5 and the remark in the previous paragraph, $S \sslash \lambda T$ is biholomorphic to $S \sslash \lambda' T$, which implies the result of Theorem 1.1.

**Remark 3.6.** If we only want to show Theorem 1.1, from the above proof, the introduction of the symplectic implosion is not so necessary. In fact, we only use a suitable smooth stratum of the symplectic implosion. This mainly because our problem is local in essence. However, if we remove all the materials about the symplectic implosion, the whole proof would be in an ad hoc flavor more or less. In our humble opinion, the symplectic implosion gives the right framework to understand the problem.

## 4 A GIT explanation of Theorem 1.1

It is a seminal result of [16, 23] that a symplectic quotient is naturally isomorphic to a GIT quotient. Inspired by this fact, we discuss a result in this section, Proposition 4.2, which can be seen as a reinterpretation of Theorem 1.1 in the GIT language.

### 4.1 Stability conditions

Firstly, let us recall the definition of a key concept: semi-stable (or stable) points. As in Subsection 2.3, let $G = K^\mathbb{C}$ be the complexification of the compact group $K$. For a compact Hamiltonian Kähler $K$-manifold $(M, \omega, \mu, J)$, there is a holomorphic $G$-action on $M$ induced by the $K$-action. Any point $m \in M$
is called a semi-stable point, if \( G \cdot m \cap \mu^{-1}(0) \neq \emptyset \). Furthermore, if \( m \) also satisfies that \( G \cdot m \cap \mu^{-1}(0) \neq \emptyset \) and the isotropic group \( G_m \) at \( m \) is finite, \( m \) is called a stable point. We denote the sets of semi-stable and stable points of \( M \) by \( M^{ss} \) and \( M^s \), respectively. Sometimes we also use the notations \( M^{ss}(\mu) \) and \( M^s(\mu) \) to emphasize the dependence of semi-stable and stable point sets on the moment map.

Remark 4.1. A variety of equivalent properties has been used to define the stability condition. Especially for the analytic stability condition on a Kähler manifold, which we use here, the definition of semi-stable points in Subsection 4.1 coincides with the definition of semi-stable points appearing in [11,12]. It turns out that the set \( M^s \) has appeared in [16] under the name the minimal stratum\(^8\), which is defined by using the gradient flow of \( \| \mu \|^2 \). For a proof for the coincidence of \( M^s \) and the minimal stratum, one can see [16, Theorem 7.4] and [27, Proposition 2.4].

We also remind readers that the terminology of semi-stable points also appears in [16]. But it seems that the author reserves this concept for algebraic manifolds exclusively therein and the definition conforms to the standard GIT one (see [20]).

4.2 Symplectic quotients and semi-stable point sets

Since we always assume 0 to be a regular value of the moment map and \( K \) to act on \( \mu^{-1}(0) \) freely, the semi-stable or stable point set behaves particularly well. In fact, by [16, Theorems 7.4 and 8.10], \( M^{ss} \) and \( M^s \) coincide in this case. In particular, \( M^{ss} \) is a \( G \) principal bundle and one can define the complex structure on the symplectic quotient by the formula: \( M^{ss}/G = \mu^{-1}(0)/K = M_0 \). As we have said, Kirwan and Ness’s theorem ensures that such a definition of the complex structure on a symplectic quotient is the same as the definition given in Subsection 3.1.

Recall that \( M_\lambda \) and \( M_{\lambda'} \) can be viewed as two symplectic quotients of a common manifold \( M \times K/K_\sigma \) with respect to two different symplectic structures. In view of the above equivalent definition of the complex structure on a symplectic quotient, we can explain Theorem 1.1 as follows. The two different symplectic structures on \( M \times K/K_\sigma \) give the same semi-stable point set. More precisely, one has the following result.

Proposition 4.2. Let \( \lambda \) and \( \lambda' \) be two points lying in a face \( \sigma \subseteq \mathbb{C}_+^* \). Identifying \( K/K_\sigma \) with \( O_\lambda^* \) using the map \( [k] \mapsto k \cdot \lambda \), one obtains a Hamiltonian Kähler structure on \( M \times K/K_\sigma \), whose moment map is denoted by \( \Xi_\lambda \). Similarly, by substituting \( O_\lambda^* \) with \( O_{\lambda'}^* \), one can define another moment map \( \Xi_{\lambda'} \) on \( M \times K/K_\sigma \). If \( \lambda \) and \( \lambda' \) are close enough, the following two semi-stable point sets are equal:

\[
(M \times K/K_\sigma)^{ss}(\Xi_\lambda) = (M \times K/K_\sigma)^{ss}(\Xi_{\lambda'}).
\]

It is possible to prove Proposition 4.2 by using techniques from GIT directly, which, as a consequence, leads to an algebraic geometry proof of Theorem 1.1. We will pursue this approach in the next section. Here, we are content to prove the proposition using the symplectic implosion again, as another example of the power of this construction.

Proof of Proposition 4.2. The idea of the proof is to use the fact that \( M \times K/K_\sigma \) can be seen as the symplectic quotient of \( M \times (T^*K)_{\text{impl}} \) at any point in \( U \subseteq \sigma \) with respect to the \( T \)-action, where \( U \) is a neighborhood of \( \lambda \) in \( \sigma \). However, if we recall the definition of \( (T^*K)_{\text{impl}} \), we will find that a smooth stratum of \( M \times (T^*K)_{\text{impl}} \) suffices to give the same symplectic quotient. Therefore, instead of using \( M \times (T^*K)_{\text{impl}} \) directly, we will use a smooth stratum \( M \times K/[K_\sigma, K_\sigma] \times \sigma \) in the following argument without changing the result.

Let \( \Phi_K \) (resp. \( \Phi_T \)) be the moment map of \( M \times K/[K_\sigma, K_\sigma] \times \sigma \) of the \( K \)-action (resp. \( T \)-action). Since \( \lambda \) is a regular value of \( \Phi_T \), due to [16, Theorems 7.4 and 8.10], \( T^\perp \Phi_T^{-1}(\lambda) \) is the semi-stable point set of the \( T^\perp \)-action and

\[
M \times K/K_\sigma \simeq M \times O_\lambda^\perp = T^\perp \Phi_T^{-1}(\lambda)/T^\perp.
\]

\(^8\) Note that in [27], the points in this set are called analytic semi-stable.
Using Lemma 4.3 proved later, one knows that $T^C\Phi^{-1}_T(\lambda)$ is $G$-invariant, which implies that
\[
G^C(\Phi^{-1}_K(0) \cap \Phi^{-1}_T(\lambda)) \subseteq G^C\Phi^{-1}_T(\lambda) = T^C\Phi^{-1}_T(\lambda).
\]
By (4.1) and (4.2), the definition of semi-stable points yields
\[
(M \times K/K_e)^{ss}(\Xi,\lambda) = (G^C(\Phi^{-1}_K(0) \cap \Phi^{-1}_T(\lambda)))/T^C.
\]
Similarly, one has
\[
(M \times K/K_e)^{ss}(\Xi,\lambda') = (G^C(\Phi^{-1}_K(0) \cap \Phi^{-1}_T(\lambda')))/T^C.
\]
On the other hand, using the same argument, one can show that $G\Phi^{-1}_K(0)$ is $T^C$-invariant and $(G^C(\Phi^{-1}_K(0) \cap \Phi^{-1}_T(\lambda')))/G$ gives the semi-stable point set of the $T^C$-action on $(M \times K/[K_e,K_e] \times \sigma)/K$ with respect to the moment map $\mu_{impl} - \lambda'$. Since $\lambda$ and $\lambda'$ are close enough, by the result of GIT quotients for a torus action, the semi-stable point sets of $\mu_{impl} - \lambda$ and $\mu_{impl} - \lambda'$ coincide, which implies that
\[
G^C(\Phi^{-1}_K(0) \cap \Phi^{-1}_T(\lambda)) = G^C(\Phi^{-1}_K(0) \cap \Phi^{-1}_T(\lambda')).
\]
As a result, $(M \times K/K_e)^{ss}(\Xi,\lambda) = (M \times K/K_e)^{ss}(\Xi,\lambda')$.

**Lemma 4.3.** Suppose $K_1$ and $K_2$ to be two compact groups. Let $X$ be a Hamiltonian Kähler $K_1 \times K_2$-manifold (not necessarily compact) and $\mu_1$ be the moment map of the $K_1$-action. If $\mu_1^{-1}(0)$ is a compact subset, the set of semi-stable points with respect to $\mu_1$, $X^{ss}(\mu_1)$, is $K_1^C \times K_2^C$-invariant.

**Proof.** By definition, $X^{ss}(\mu_1)$ is $K_1^C$-invariant. So we only need to show that it is also $K_2^C$-invariant. Since $\mu_1$ is $K_2$-invariant, the definition of semi-stable points implies that $X^{ss}(\mu_1)$ is $K_2$-invariant. Choose a pre-compact open neighborhood $Z \subseteq X^{ss}(\mu_1)$ of $\mu_1^{-1}(0)$. Due to the $K_2$-invariance of $X^{ss}(\mu_1)$, one can find a neighborhood $U$ of the identity element in $K_1^C$ such that $U \cdot Z \subseteq X^{ss}(\mu_1)$. Using the definition of $X^{ss}(\mu_1)$ again, one has $K_1^C \cdot Z = X^{ss}(\mu_1)$. Therefore,
\[
U \cdot X^{ss}(\mu_1) = UK_1^C Z = K_1^C U \cdot Z = X^{ss}(\mu_1),
\]
which implies the $K_2^C$-invariance of $X^{ss}(\mu_1)$ due to the Cartan decomposition $K_2^C = \exp\mathfrak{t}_2 \cdot K_2$.

**Remark 4.4.** A similar result for the stable point set also holds by using a similar argument.

## 5 The second proof of Theorem 1.1: The vGIT approach

The authors in [4,30] studied the variation of GIT quotients of an algebraic variety when the linearization of the group action changes. As promised before, in this section, we will use their results to give another proof of Proposition 4.2 and recover Theorem 1.1 consequently. More concretely, we first prove Theorem 1.4, which is merely a restatement of the results contained in Theorems 5.4 and 5.5. As a corollary, we can show Proposition 4.2. We begin with a discussion about a numerical function related to the stability condition.

### 5.1 A numerical function

We recall some useful definitions from [4] in our settings. Let $(X,\omega_X,J_X,\mu_X)$ be a compact Hamiltonian Kähler $K$-manifold. As before, we extend the $K$-action on $X$ to a $G$-action holomorphically. A group homomorphism from $\mathbb{C}^*$ to $G$ is called a one-parameter subgroup, if it is the complexification of a group homomorphism from $S^1$ to $K$. Naturally, we can identify such a one-parameter subgroup with an element in $\mathfrak{t}$. Let $\mathcal{X}_+(G) \subseteq \mathfrak{t}$ be the set of one-parameter subgroups of $G$. For any $x \in X$ and $\rho \in \mathcal{X}_+(G)$, following [4], one defines a numerical function,
\[
M(x) := \sup_{\rho} d_\rho(0,\mu_X(\rho(\mathbb{C}^*) \cdot x)),
\]

---

9) We use the same symbol, $\mu_{impl}$, to denote the moment map of $M_{impl}$ and its restriction on $(M \times K/[K_e,K_e] \times \sigma)/K$.

10) Actually, this result can be showed by using the flow appearing in the proof of Lemma 3.5.
in which $d_p(0, A)$ denotes the signed distance from the origin to the boundary of the set $A_p$. Note that $A_p \subseteq \mathbb{R}_+$ is the projection of $A$ onto the positive ray spanned by $\rho^*$, the dual of $\rho$ under an invariant metric of $t$. We remind readers that the definition of $M(x)$ here is a little different from the form given in [4, Subsection 2.5.2]. A typo seems to be spotted therein. Anyway, since our manifold $X$ is only a compact Kähler manifold, not necessarily projective, we would like to provide a proof for the following result, which is well known for the algebraic case\textsuperscript{11}.

**Proposition 5.1 (A numerical criterion for the stability).** Let $X^s(\mu_X)$ (resp. $X^s(\mu_X)$) be the semi-stable (resp. stable) point set defined by using $\mu_X$ as in Subsection 4.1. With the function $M(x)$, one can give the following numerical description of semi-stable (resp. stable) points (see [4, Subsection 2.5.2]):

\[
X^{ss}(\mu_X) = \{ x \in X \mid M(x) \leq 0 \}, \quad (5.2)
\]

\[
X^s(\mu_X) = \{ x \in X \mid M(x) < 0 \}. \quad (5.3)
\]

**Proof.** Using Atiyah's convexity theorem, which discusses the image of the closure of an abelian group action orbit under the moment map (see [1, Theorem 2]), we can reformulate the function $M(x)$ as follows:

\[
M(x) = - \inf_{V \in \mathcal{X}_*(G) \Rightarrow \rho \cdot t \to +\infty} \lim_{t \to +\infty} (\mu_X(\exp(iVt) \cdot x), V/\|V\|). \quad (5.4)
\]

The existence of the limit appearing in (5.4) is well known (see [21, Subsection 3.2]). By using (5.4), or using Atiyah's convexity theorem directly, we can check a special case of Proposition 5.1, i.e., the case where the group action is one-dimensional. In particular, for any $\rho \in \mathcal{X}_*(G)$, Proposition 5.1 holds for the $\rho(C^\ast)$-action.

To prove Proposition 5.1 for the general case, we deal with (5.2) and (5.3) separately. For (5.2), we can use [16, Lemma 8.9], which asserts that $x \in X$ is semi-stable for the $G$ action if and only if $x$ is semi-stable for every one-parameter subgroup of $G$. This result, combined with the one-dimensional case where we have known, yields (5.2) immediately.

To show (5.3) for general $G$, we need a function $\Lambda_x$\textsuperscript{12} used in [21] (see also [15,31]). Recall that as a non-positively curved space, the symmetric space $K\backslash G$ has a natural compactification by adding a boundary at infinity $\partial_\infty(K\backslash G)$. By definition, every point of $z \in \partial_\infty(K\backslash G)$ is an equivalent class of geodesics rays on $K\backslash G$. Since the right $G$-action on $K\backslash G$ preserves the metric, it induces a right $G$-action on $\partial_\infty(K\backslash G)$. $\Lambda_z(z)$ is a Lipschitz continuous function on $\partial_\infty(K\backslash G)$ with respect to the Tits metric on $\partial_\infty(K\backslash G)$. On $\partial_\infty(K\backslash G)$, we can define another topology called sphere topology, which, in general, is different from the topology induced by the Tits metric. Let $W \in \mathfrak{t}$ be a vector of unit norm. The map $z : W \mapsto \exp(iWt)$, $t \in [0, \infty)$ leads to a homeomorphism between the unit sphere of $\mathfrak{t}$ and $\partial_\infty(K\backslash G)$ with sphere topology. Using such a homeomorphism, by definition, $\Lambda_x$ can be calculated in the following way:

\[
\Lambda_x(z(W)) = \lim_{t \to +\infty} (\mu_X(\exp(iWt) \cdot x), W). \quad (5.5)
\]

In the following, we need an equivariant property of $\Lambda_x$ (see [21, Lemma 3.6]), i.e., for any $z \in \partial_\infty(K\backslash G)$ and $g \in G$, we have

\[
\Lambda_{g \cdot x}(z) = \Lambda_x(z \cdot g).
\]

We first show the inclusion in one direction for (5.3), i.e.,

\[
X^s(\mu_X) \subseteq \{ x \in X \mid M(x) < 0 \}. \quad (5.6)
\]

Let $x \in X^s(\mu_X)$. By definition, we can find $y \in \mu_X^{-1}(0)$ such that $y = g \cdot x$ and $G_y$ is finite. Then, (5.5) and the equivariance of $\Lambda_x$ tell us that for any $V \in \mathcal{X}_*(G)$,

\[
\lim_{t \to +\infty} (\mu_X(\exp(iVt) \cdot x), V/\|V\|) = \Lambda_x(V/\|V\|) = \Lambda_y((V/\|V\| \cdot g^{-1})).
\]

\textsuperscript{11} After this paper had been submitted, we noticed the paper [6]. Proposition 5.1 can be deduced from [6, Theorem 14.1] almost. We will discuss this a little in Remark 5.3.

\textsuperscript{12} In [21, Subsection 3.3], the same function is denoted by $\lambda_x$, and we change the notation a little to avoid the symbol ambiguity.
Therefore, by using (5.5) again, to show (5.6), we only need to find $\epsilon_0 > 0$ such that
\[
\lim_{t \to +\infty} \langle \mu_X(\exp(iWt) \cdot y), W \rangle \geq \epsilon_0
\]
holds for any unit norm vector $W$ in $\mathfrak{f}$. However, by a direct calculation, we have the following equality (see [21, (3.5)]):
\[
\lim_{t \to +\infty} \langle \mu_X(\exp(iWt) \cdot y), W \rangle = \int_0^\infty |W^X(\exp iW^\tau \cdot y)|^2 d\tau.
\]
Since $G_y$ is finite, $W^X(y) \neq 0$. Therefore, the existence of $\epsilon_0$ is a simple result of the above equality.

To show the inclusion of (5.3) in the other direction, we use Lemma 5.2, which is a parallel result of Kirwan’s lemma for the stable points. As before, such a lemma enables us to reduce the general case to the $C^*$ case, which we already know. As a result, the proof of (5.3) is completed.

Lemma 5.2. \quad $x \in X$ is a stable point for the $G$-action if and only if for any $\rho \in \mathcal{X}_+(G)$, $x$ is a stable point for the $\rho(C^*)$-action with respect to the restricted moment map.

Proof. \quad Note that we have proved the inclusion (5.6). Then the “only if” part of the lemma is a consequence of this inclusion and Proposition 5.1 for the $C^*$-action that we have known.

To show the “if” part of the lemma, we first show the following claim: if $x$ is a stable point for any one-parameter subgroup action, then $g \cdot x$ satisfies the same property for any $g \in G$. To prove it, we need the following result about the group action on $\partial_\infty(K \setminus G)$ (see [21, Sections 2 and 5]). For any $V \in \mathcal{X}_+(G)$, there exists $Y \in \mathcal{X}_+(G)$ such that
\[
z(Y/ \|Y\|) = z(V/ \|V\|) \cdot g.
\]
In other words, the $G$-action on $\partial_\infty(K \setminus G)$ preserves the “rational” points. With this property, the equivariance of $\Lambda_x$ and Proposition 5.1 for the $C^*$ action, the claim follows.

Now, we can argue by reductio ad absurdum to show the “if” part of the lemma. Assume that $x$ is not stable. By Kirwan’s lemma, or (5.2), one knows that $x$ is semi-stable at least. Moreover, since $x$ is stable with respect to any one-parameter subgroup action, the isotropic subgroup of $x$ is finite. Therefore, suppose $y \in \mu_X^{-1}(0)$ lying in the closure of $G \cdot x$. Then $y \notin G \cdot x$. On the other hand, by [31, Corollary 5.5.4], one can find a one-parameter group $\rho_0$ and a point $w \in G \cdot x$ such that $y$ lies in the closure of $\rho_0(C^*) \cdot w$. Note the claim in the former paragraph implies that $w$ is stable with any one-parameter subgroup action due to $w \in G \cdot x$. Especially, $w$ is stable with respect to the $\rho_0(C^*)$-action. Meanwhile, since $y \in \mu_X^{-1}(0)$, it entails that $y$ also lies in the zero level set of the moment map associated with the $\rho_0(C^*)$-action. Hence, one can conclude that $y \in \rho_0(C^*) \cdot w \subseteq G \cdot x$, which leads to a contradiction. Consequently, $x$ must be a stable point.

Remark 5.3. \quad Some comments about the proof and consequences of Proposition 5.1.

(1) As readers may have noted from the proof of Proposition 5.1, the stability can be characterized by using the function $\Lambda_x$, which has been done in [21]. A similar thing is also true for the semi-stability. By using $\Lambda_x$, [29, Theorem 4.3] asserts that $x$ is semi-stable if and only if $\Lambda_x(z(V)) \geq 0$ for any $V \in \mathfrak{f}$. By using an argument like [31, Remark 5.5.4], this result is equivalent to (5.2). Hence, one can obtain a proof of (5.2) without using Kirwan’s lemma.

(2) During the proof of Proposition 5.1, one has actually shown the following results:
\[
x \text{ is stable } \iff \lim_{t \to +\infty} \langle \mu_X(\exp(iVt) \cdot x), V \rangle \geq 0 \quad \text{for any } V \in \mathcal{X}_+(G), \tag{5.7}
\]
\[
x \text{ is semi-stable } \iff \lim_{t \to +\infty} \langle \mu_X(\exp(iVt) \cdot x), V \rangle > 0 \quad \text{for any } V \in \mathcal{X}_+(G), \tag{5.8}
\]
which is an analog of the classical Hilbert-Mumford numerical criterion [20, Theorem 2.1]. Such type of results has been obtained by [6]. Moreover, they also prove a similar result for the poly-stable points.

\footnote{In [31], a one-parameter subgroup does not necessarily come from the complexification of an $S^1$ subgroup of $K$. But for the one-parameter subgroup $\rho_0$ needed here, by checking the proof of [31, Corollary 5.5.4], we find that $\rho_0 \in \mathcal{X}_+(G)$ indeed.}
(3) If $X$ is projective, Ness [22, Lemma 3.5] has proved a stronger result for the stable point set, i.e.,
the infimum in (5.4) can be achieved at certain $V \in \mathcal{X}(G)$. But for the Kähler case, the same result
seems not to be true, even for a torus action. However, if we try to take the infimum in $\mathfrak{k}$, it is possible
that a similar result still holds.

(4) As we have said, in [6, Theorem 14.1], the authors proved the Hilbert-Mumford numerical criterion
in a different way. As we have seen, up to the infimum in (5.4), Proposition 5.1 is almost equivalent to
that a similar result still holds.

Theorem 5.4

Let $M$ be the moment map on $M \times K/K_\sigma$ induced by the identification between $K/K_\sigma$ and $\mathcal{O}^*_\lambda$. To emphasize
its dependence on $\Xi_\lambda$ (or $\lambda$), in the following, we denote the numerical function $M(x)$ on $M \times K/K_\sigma$
by $M^\lambda(x)$. But we should note that for any given Kähler form and the corresponding moment map on
$M \times K/K_\sigma$, we can always define $M^\star(x)$ for them. That is to say, $M^\star(x)$ is a function on the space of
moment maps on $M \times K/K_\sigma$. Via the moment map $\Xi_\lambda$, $\sigma$ can be seen as a convex subset of this space
of moment maps. In this paper, we usually only consider the restriction of $M^\star(x)$ to $\sigma$.

Following [4, Subsection 3.3], we use $M^\star(x)$ to give a partition of $\sigma \subseteq t^*_\sigma$. A subset $H$ of $\sigma$ is called a
wall if there exists
\[ x \in (M \times K/K_\sigma)_{>0} := \{ x \mid \dim G_x > 0 \} \]
such that
\[ H = H(x) := \{ \lambda \in \sigma \mid M^\lambda(x) = 0 \}. \]
For a chamber, we mean that it is a non-empty connected component of the complement of the union
of walls in $\sigma$. About the relation between chambers and the sets of semi-stable (or stable) point, one
has the following result. Note that this result, as well as the next several results in this section, is a
restatement of the corresponding result in [4] for projective manifolds in our analytic settings. For their
proofs, since we have dropped the algebraicriety condition, we will provide details about the necessary
modification compared with their original proofs.

Theorem 5.5. There are only finitely many walls in $\sigma$. 

Proof. We notice that in our analytic settings, by (5.4), $M^\star(x)$, viewed as a function on the space of
moment maps, is also a sub-additive and positively homogeneous function just like its algebraic counterpart in [4, Lemma 3.2.5]. It implies that $M^\star(x)$ is a convex function. Especially, $M^\star(x)$, viewed as a function on $\sigma$, is also convex. Given this fact, combined with Proposition 5.1 and [27, Proposition 2.4],
we can repeat all the arguments in the proof of [4, Theorem 3.3.2] verbatim.

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in a different way. As we have seen, up to the infimum in (5.4), Proposition 5.1 is almost equivalent to
that a similar result still holds.
To prove [4, Theorem 3.3.3], Dolgachev and Hu used a key fact, [4, Theorem 2.4.5]\(^\text{14}\), which asserts that there are finitely many points, \(\lambda_1, \ldots, \lambda_N\) in \(\sigma\) such that for any \(\lambda \in \sigma\), the set \((M \times K/K_\lambda)^{ss}(\Xi_{\lambda})\) equals one of the sets \((M \times K/K_\lambda)^{ss}(\Xi_{\lambda})\). So if one could show such a result holds for a Hamiltonian Kähler manifold, one can repeat Dolgachev and Hu’s proof to show Theorem 5.5. We notice that although [4, Theorem 2.4.5] is stated not only for the algebraic case, but also for the general Kähler case, the proof of which, however, uses a lemma [4, Lemma 1.3.6], that is of algebraic nature.

Here, we try to give a proof of the finiteness theorem of Dolgachev-Hu for Kähler manifolds by modifying the argument of [2, Example 5.1], which gives another proof for the finiteness theorem in the algebraic case. We begin with a precise statement of the theorem to be proved.

**Theorem 5.6** (See [4, Theorem 2.4.5(ii)]). Suppose that a compact group \(K\) acts on a compact complex manifold \(X\) holomorphically. There exist finitely many open subsets of \(X\), \(\{U_1, \ldots, U_M\}\), such that for any Hamiltonian Kähler structure on \(X\) compatible with the \(K\)-action, the corresponding semi-stable point set must be one of \(\{U_1, \ldots, U_M\}\).

**Proof.** In spirit, Białynicki’s method is similar to the method that we have used in the proof of Proposition 5.1, i.e., reducing a general compact group action to a torus action, and checking the result for the torus action using Atiyah’s convexity theorem.

**Step 1.** Reducing to the torus action case. As usual, let \(G = K^C\) be the complexification of \(K\). Choose a maximal torus \(T\) in \(K\), and then \(T^C\) is the maximal torus of \(G\). For any \(K\)-invariant Hamiltonian Kähler structure \((\omega, \Psi)\) on \(X\), denote the induced moment map for the \(T\)-action by \(\Psi_T\). One has the following relation between the semi-stable point sets for the \(K\) and \(T\)-actions:

\[
X^{ss}(\Psi) = \bigcap_{k \in K} (k \cdot X^{ss}(\Psi_T)). \tag{5.9}
\]

As in [26], (5.9) is a direct result of the Hilbert-Mumford criterion, i.e., (5.7) and the equivariance property of the moment map. By (5.9), it is clear that if Theorem 5.6 holds for the \(T\)-action, the same theorem also holds for the \(K\)-action.

**Step 2.** Verifying the theorem for the \(T\)-action. Denote \(\{F_1, \ldots, F_N\}\) to be the set of connected components of the \(T\)-action (or \(T^C\)-action equivalently) fixed points. For any \(x \in X\), following Białynicki-Birula, we introduce two sets

\[
\text{sh}(x) := \{F_i \mid T^C \cdot x \cap F_i \neq \emptyset\}, \quad \text{c}(x) := \{y \in X \mid \text{sh}(x) = \text{sh}(y)\}. \tag{5.10}
\]

By definition, there are only finitely many different sets lying in \(\{\text{c}(x) \mid x \in X\}\). For any \(x, y \in X\), one has either \(\text{c}(x) = \text{c}(y)\) or \(\text{c}(x) \cap \text{c}(y) = \emptyset\).

For any Kähler form \(\omega_X\) and the moment map \(\mu_X\) on \(X\), by [1, Theorem 2], i.e., Atiyah’s convexity theorem, the semi-stable point set has the following representation:

\[
X^{ss}(\mu_X) = \{x \in X \mid 0 \in \mu_X(T^C \cdot x)\} = \{x \in X \mid 0 \in \text{conv}(\mu_X(\text{sh}(x)))\}. \tag{5.11}
\]

By (5.10) and (5.11), there exist \(x_1, \ldots, x_k \in X\) such that \(X^{ss}(\mu_X) = \bigcup_{i=1}^k \text{c}(x_i)\), which implies that there are only finite possibilities for the shape of \(X^{ss}(\mu_X)\).

**Remark 5.7.** Some comments about Theorem 5.6.

(i) For the algebraic case, the theorem holds not only for smooth manifolds but also for projective varieties (see [2, 4]). In fact, the theorem is even true for the positive characteristic (see [24, 26]). Taking this fact into consideration, it seems reasonable to expect a similar result to hold for compact Hamiltonian Kähler spaces. As before, to obtain such a generalization, we need two main results for singular spaces. One of them, i.e., Atiyah’s convexity theorem, does have a singular space generalization, [10, p.80, Theorem]. So, the only thing left is to find a Kähler space version of the Hilbert-Mumford criterion.

\(^{14}\) This result is sometimes called Dolgachev-Hu’s finiteness theorem in the literature (see, e.g., [2, 26]).
(ii) We notice that an argument similar to that of Theorem 5.6 also appears in [13, p. 174, Proposition]. Moreover, by the result of that paper, on a projective manifold, any open set \( U \), appearing in Theorem 5.6 comes actually from the corresponding set constructed by using the algebraic method.

(iii) In [14, Theorem 3.5], the author used a method very close to the method used in Step 2 in the above proof. In particular, this means that although [14] discusses only about projective manifolds, this beautiful result is also true for general compact Kähler manifolds.

With Theorem 5.6 in hand, one can use the same argument as in [4, Theorem 3.3.3] to prove Theorem 5.5. For readers’ convenience, we incorporate Dolgachev and Hu’s proof here.

**Proof of Theorem 5.5.** For any wall \( H \), let

\[
(M \times K/K_\sigma)_H := \{ x \in (M \times K/K_\sigma)_{(>0)} \mid H \subseteq H(x) \},
\]

where \((M \times K/K_\sigma)_{(>0)}\) is the set of points in \(M \times K/K_\sigma\) with a positive-dimensional isotropic group and \(H(x)\) is the set defined in Subsection 5.2. We will show that \((M \times K/K_\sigma)_H\) can be written in the following way:

\[
(M \times K/K_\sigma)_H = \bigcap_{\lambda \in H} ((M \times K/K_\sigma)^{ss}(\Xi_\lambda) \cap (M \times K/K_\sigma)_{(>0)}).
\]

(5.12)

On one hand, if \( x \in (M \times K/K_\sigma)_H \), one has \( H \subseteq H(x) \) by definition, which implies that, due to the definition of \( H(x) \), \( M^*(x) \) vanishes on \( H \). Then by Proposition 5.1, for any \( \lambda \in H \), \( x \in (M \times K/K_\sigma)^{ss}(\Xi_\lambda) \), i.e.,

\[
x \in \bigcap_{\lambda \in H} ((M \times K/K_\sigma)^{ss}(\Xi_\lambda) \cap (M \times K/K_\sigma)_{(>0)}).
\]

On the other hand, if \( x \in \bigcap_{\lambda \in H} ((M \times K/K_\sigma)^{ss}(\Xi_\lambda) \cap (M \times K/K_\sigma)_{(>0)}) \), by Proposition 5.1 again, for any \( \lambda \in H \), one has \( M^\lambda(x) = 0 \), or equivalently \( \lambda \in H(x) \). Hence, \( H \subseteq H(x) \), i.e., \( x \in (M \times K/K_\sigma)_H \). As a result, (5.12) is true.

By Theorem 5.6, one can find a finite set of points \( \{\lambda_1, \ldots, \lambda_M\} \subseteq \sigma \) such that for any \( \lambda \in \sigma \), the set \((M \times K/K_\sigma)^{ss}(\Xi_\lambda)\) equals one of the sets \((M \times K/K_\sigma)^{ss}(\Xi_{\lambda_i})\). By (5.12), we know that there are only finitely many subsets of \(M \times K/K_\sigma\) which are of the form \((M \times K/K_\sigma)_H\) for some wall \( H \). However, by the definition of walls, two walls \( H \) and \( H' \) coincide if and only if \((M \times K/K_\sigma)_H = (M \times K/K_\sigma)_{H'}\). As a consequence, only finitely many walls exist.

\[ \square \]

6 A relation between two approaches

After two proofs of Theorem 1.1 are given, one by the symplectic implosion, one by the vGIT, it is appropriate to have a comparison between two approaches. As we have declared in Section 1, the general guideline behind two approaches is a reflection of the same plain idea: reducing a non-abelian reduction problem to an abelian one, which looms in the statement and proof of Theorems 1.2 and 5.6 for example. Besides, one can go beyond such a general discussion and work out a more concrete relation between these two approaches. As it turns out, to carry out such a comparison, it is Thaddeus’s proof of vGIT [30] that fits better for this purpose, although we have used Dolgachev and Hu’s proof of the same theory extensively in the previous section. More precisely, an interesting construction used by Thaddeus [30, Subsection 3.1] has a natural correspondence with the symplectic implosion construction. We summarize the results in Propositions 6.1 and 6.3.

6.1 Thaddeus’s master space

As before, we assume that \((X, \omega_X, J_X, \mu_X)\) is a compact Hamiltonian Kähler manifold. But here, we further require that there is a holomorphic prequantum line bundle \( L \) over \( X \). In other words, the \( K \)-action is lifted to a group action on \( L \) and there is a \( K \)-invariant Hermitian metric \( h \) on \( L \). Denoting the
Recall that $\Pi = \{\pi_1, \ldots, \pi_r\}$ is the set of fundamental weights of $\mathbb{K}$. We denote $\mathbb{C}_i$, $i = 1, \ldots, r$, to be the 1-dimensional representation of $T$ with weight $\pi_i$. Also, recall that $V_i$ is the irreducible representation of $K$ with highest weight $\pi_i$, and $v_i$ is a highest weight vector of $V_i$. Then we can and will identify $\mathbb{C}_i$ with $\mathbb{C}_i \subseteq V$. Moreover, let $\pi_0$ be the zero weight and $\mathbb{C}_0$ be the trivial representation of $T$. As usual, by using the Borel subgroup $T^C \subseteq B \subseteq G$ corresponding to the positive roots we have chosen, one constructs a line bundle $L_i = G \times B \mathbb{C}_i$, $i = 0, \ldots, r$, over $G/B$, associated with the $T^C$ (or $B$) representation $\mathbb{C}_i$.

By choosing a parameter $t \in \Delta = \{(t_i) \in \mathbb{Q}_{>0}^r \mid \sum_{i=0}^r t_i = 1\}$, one has a family of $G$-linearizations, $L \otimes \prod_{i=0}^r L_i^{-t_i}$, over $X \times G/B$.

Roughly speaking, Thaddeus [30] constructed a "master space" $X_{ms}$, that transforms GIT quotients of $X \times G/B$ defined by the $G$-action on $L \otimes \prod_{i=0}^r L_i^{-t_i}$ to GIT quotients of $X_{ms}$ defined by a family of $T^C$-actions on a fixed line bundle. In our situations, one can even construct a certain universal master space, i.e., a space independent of $X$, as follows. For $E^N = \bigoplus_{i=1}^r \mathbb{C}_i \subseteq E = \bigoplus_{i=1}^r V_i$,

$$G_{ms} := G \times_B \mathbb{P}(C_0 \oplus E^N) = K \times_T \mathbb{P}(C_0 \oplus E^N).$$

The associated (relative) hyperplane line bundle of $G_{ms}$ is denoted by $\mathcal{O}_{G_{ms}}(1)$. $G_{ms}$ has the following embedding:

$$(p, \pi) : G_{ms} \rightarrow \mathbb{P}(C_0 \oplus E) \oplus G/B,$$

where $g \in G$ and $u \in \mathbb{C}_0 \oplus E^N \subseteq C_0 \oplus E$. One has $\mathcal{O}_{G_{ms}}(1) = p^* \mathcal{O}_{\mathbb{P}(C_0 \oplus E)}(1)$.

**A notation warning.** $\mathbb{P}$ used in this paper is the projectivization of a space, i.e., taking all one-dimensional subspaces of the original space, which is different from the algebraic usage of $\mathbb{P}$ as in [30]. In fact, the description of the construction given here is equivalent to Thaddeus’s original construction except that we choose to perform the manipulation on the dual bundle. As a useful notation, we also denote $\mathbb{P}(C_0 \oplus E)$ (resp. $\mathbb{P}(C_0 \oplus E^N)$) by $E$ (resp. $E^N$) in the following.

Up to now, we still miss an important assumption in Thaddeus’s construction, i.e., the line bundles over $X \times G/B$ used in the construction should be ample. In our description of the construction, it is equivalent to the negativity of the line bundles $\{L_i\}$. However, by some direct calculations, one knows that $L_i$ is only semi-negative for any $i \in \{0, \ldots, r\}$. One can remedey this problem as follows. Let $\epsilon$ be a small fractional weight lying in $(\pi_i^*)^\mathbb{N}$. Instead of using a set of weights $\{\pi_0, \pi_1, \ldots, \pi_r\}$ to construct $G_{ms}$, we use $\{\pi_0 + \epsilon, \pi_1 + \epsilon, \ldots, \pi_r + \epsilon\}$ to do the real job\(^{16}\). The resulting space is denoted by $G'_{ms}$. It turns out that $G'_{ms}$ is holomorphically isomorphic to $G_{ms}$. But the (relative) hyperplane line bundle is changed to an ample line bundle $\mathcal{O}_{G'_{ms}}(1) = \mathcal{O}_{G_{ms}}(1) \oplus \pi^* L_1^{-1}$, where $L_1$ is the line bundle over $G/B$ associated with $\epsilon$. The first Chern form of $\mathcal{O}_{G'_{ms}}(1)$, i.e., the symplectic form on $G'_{ms}$, is $\pi^* \omega_E + \pi^* \mathcal{O}_E^*$, where $\omega_E$ is the Fubini-Study form on $E$ and we use the identification $G/B \simeq K/T \simeq \mathcal{O}_E^*$. The master space of $X^{16}$ is defined to be $X_{ms} := (X \times G'_{ms})/K$.

By the definition of $G_{ms}$, there exists a natural $G$ (or $K$)-action on it and this action can be lifted to an action on $\mathcal{O}_{G_{ms}}(1)$. About the $T^C$ (or $T$)-action on $G'_{ms}$, one can proceed to resemble the symplectic implosion construction. Namely, we define a $T^C$-action on $E^N$ by requiring that the $T^C$-action is diagonalized and the weight of the action on $\mathbb{C}_i$ is given by $-\pi_i$, $i = 0, 1, \ldots, r$, which induces the $T^C$-action on $G_{ms} = G_{ms}$ and $\mathcal{O}_{G_{ms}}(1)$. As for the $T^C$-action on $\mathcal{O}_{G_{ms}}(1)$, one should also take the natural $T^C$-action on $L_1$ into consideration. Denote the moment map of the $T$-action on $\mathcal{O}_{G_{ms}}(1)$ by $\theta_T$.

For concrete calculations, it is convenient to identify $T^C$ with $(\mathbb{C}^*)^r$ by using the chosen set of fundamental weights $\Pi$. Especially, under such an identification, for $z = (\zeta_1, \ldots, \zeta_r) \in (\mathbb{C}^*)^r \simeq T^C$ and $u = (u_1, \ldots, u_r) \in E^N$, one has $z \cdot u = (z_1^{-1}u_1, \ldots, z_r^{-1}u_r)$.

\(^{15}\) Without ambiguity, we do not distinguish a bundle on $X$ or $G/B$ and its pull-back on $X \times G/B$.

\(^{16}\) Admittedly, there are no representations associated with such fractional weights. One can either use scaling arguments to correct this little flaw or take the discussion of $G_{ms}$ and $\mathcal{O}_{G_{ms}}(1)$ as definitions directly.

\(^{17}\) If we stick to Thaddeus’s terminology, it may be more proper to call $(X \times G_{ms})/K$ the master space of $X \times G_{ms}$. 
6.2 The master space and the symplectic implosion

In [8, Proposition 7.7], the authors constructed a smooth manifold closely related to $G_{ms}^e$, i.e., $G \times_B E^N$ (the symplectic form on it depends on $e$), as a desingularization of the universal implosion section $(T^*K)_{impl} = G_N$. Let $\text{in} : E^N \rightarrow E^N$ be the inclusion map, $\text{in}(u) := [1 : u]$, which induces a $K \times T$-equivariant inclusion from $G \times_B E^N$ to $G_{ms}$, also denoted by $\text{in}$. Therefore, $G_{ms}^e$ is a smooth compactification of $G \times_B E^N$. On the other hand, one notices that the map $\text{in}$ is the restriction of the map $IN : E \rightarrow \bar{E}$, where $\bar{G}_N$ be the closure of $IN(G_N)$ in $\bar{E}$ (with respect to either standard or Zariski topology). $G_{ms}^e$ can also be seen as a desingularization of $\bar{G}_N$. In fact, the following diagram commutes, where $q : G \times_B E^N \rightarrow E$ is the multiplication:

$$
\begin{array}{ccc}
G \times_B E^N & \xrightarrow{\text{in}} & G_{ms}^e \\
q \downarrow & & \downarrow p \\
G & \xrightarrow{IN} & \bar{G}_N
\end{array}
$$

Now by using some scaling, we assume that $\mu_X(X) \cap \mathcal{U}_e$ is contained in the simplex spanned by vertices $\{0, \varpi_1/\sqrt{2\pi}, \ldots, \varpi_r/\sqrt{2\pi}\}$ and $e \in (\mathcal{U}_e) \cap \mu_X(X)$ is a small regular value of $\mu_X$. One can check that, under such an assumption, $X_{ms}$ is an orbifold. Meanwhile, due to [8, Corollary 7.13], $(X \times G \times_B E^N) / K$ is also an orbifold, which in fact is a partial desingularization of $X_{impl}$, i.e., there exists a proper surjective bimeromorphic map $(X \times G \times_B E^N) / K \rightarrow X_{impl}$. Hence, the following proposition unveils the close relation between the master space and the symplectic implosion, i.e., $X_{ms}$ is a partial desingularization of $X_{impl}$. 

**Proposition 6.1.** $(X \times G \times_B E^N) / K$ is holomorphically isomorphic to $X_{ms}$.

**Proof.** We denote the moment maps on $X \times G \times_B E^N$ and $X \times G_{ms}^e$ by $\Phi$ and $\Psi$, respectively. We are going to show that $(X \times G \times_B E^N)_{ss}(\Phi)$ is isomorphic to $(X \times G_{ms}^e)_{ss}(\Psi)$ under the map $id_X \times \text{in}$.

Let $(x, [e, u]) \in X \times G \times_B E^N$ satisfy $\Phi(x, [e, u]) = 0$. We claim that there exists $z \in (\mathcal{C}^*)^r \simeq T^c$ such that $\Psi(x, [e, i \cdot u]) = 0$. By (2.8) and [16, Subsection 2.7], for $u = (u_1, \ldots, u_r) \in \bigoplus_{i=1}^r \mathcal{C}_i = E^N \subseteq E$, one can calculate the moment maps of $E$ and $\bar{E}$ as follows:

$$
\begin{align*}
\mu_{E,K}(u) &= -\pi \sum_{i=1}^r |u_i|^2 \varpi_i, \\
\mu_{E,K}(\text{in}(u)) &= -\frac{1}{1 + \|u\|^2} \sum_{i=1}^r |u_i|^2 \varpi_i.
\end{align*}
$$

Choose

$$
z = \sqrt{\frac{1 - \pi \sum_{i=1}^r |u_i|^2}{\pi}} (1, \ldots, 1),
$$

which is well defined due to the assumption on the image of $\mu_X$. By (6.1), one can check that $\mu_{E,K}(u) = \mu_{E,K}(i \cdot u)$. Therefore, by the definition of the moment maps of $G \times_B E^N$ and $G_{ms}^e$ in Subsection 6.1 and [8, Proposition 7.7], one has

$$
\Psi(x, [e, i \cdot u]) = \mu_X(x) + \mu_{E,K}(i \cdot u) - \epsilon = \mu_X(x) + \mu_{E,K}(u) - \epsilon = \Phi(x, [e, u]) = 0.
$$

Hence,

$$
z \cdot (x, [e, \text{in}(u)]) = (x, [e, \text{in}(z \cdot u)]) \in (X \times G_{ms}^e)_{ss}(\Psi),
$$

which implies $(x, \text{in}([e, u])) \in (X \times G_{ms}^e)_{ss}(\Psi)$ due to Lemma 4.3. As a result, using the equivariant property of the map $\text{in}$, we have the inclusion for one direction,

$$
id \times \text{in}((X \times G \times_B E^N)_{ss}(\Phi)) \subseteq (X \times G_{ms}^e)_{ss}(\Psi).
$$

\footnote{A minus sign appears in $\mu_{E,K}$ compared with Kirwan’s formula, since Kirwan’s sign convention of a moment map is different from ours.}
To show the inclusion of the other direction, one only needs to notice the following fact: for any point \( p \in \Psi^{-1}(0) \), there exists \( k \in K \) such that \( k \cdot p \) is of the form \( (x, [e, [0 : u]] \} and lies in the image of \( \text{id} \times \) in consequently. To verify this fact, we note that for any point \( p \in \Psi^{-1}(0) \) that does not have the asserted property, \( p \) must be of the form \( (x, [e, [0 : u]] \} up to a \( K \)-action on \( p \). But due to the assumption on the image of \( \mu_X \),

\[
\Psi(x, [e, [0 : u]]) = \mu_X(x) + \epsilon = \frac{1}{\|u\|^2} \sum_{i=1}^r |u_i|^2 \omega_i
\]
can never vanish, which leads to a contradiction. With such a result, we are now essentially in the same situation as in the previous paragraph, which enables us to use the same argument to show that \( \text{id} \times \text{in}(X \times G \times_B \mathbb{E}^N)_{\text{ss}}(\Phi) \geq (X \times G^{s}_{\text{ms}})_{\text{ss}}(\Psi) \).

### Remark 6.2.

In a sense, the above proposition compares two kinds of symplectic quotients, coming from an affine manifold and its embedding into a projective space, respectively. One notices that similar results have been discussed in [17, pp. 224 and 242].

#### 6.3 Torus actions on the master space

In [30], Thaddeus introduced a family of torus actions on \( O_{G^{s}_{\text{ms}}} \) (1) and calculated the corresponding GIT quotients. Not unexpected, such quotients can also be obtained as symplectic quotients with respect to different values of a moment map. To make this comparison more transparent, we first recall the definition of the family of torus actions.

Let \( T = \{ \xi \in \mathbb{C}^{r+1} \mid \prod_{i=0}^r \xi_i = 1 \} \). Then \( T \) acts on

\[
(u_0', \ldots, u_r') \in (C_0 \oplus E^N) \otimes \mathbb{C}_r = \bigoplus_{i=0}^r \mathbb{C}_r \otimes \mathbb{C}_r
\]

by

\[
\xi \cdot (u_0', \ldots, u_r') := (\xi_0 u_0', \ldots, \xi_r u_r'),
\]

which induces a \( T \)-action on \( G^{s}_{\text{ms}} \) and \( O_{G^{s}_{\text{ms}}} \) (1). Now, recall that one has a parameter \( t \in \Delta \), which determines a fractional character of \( T \) by \( t(\xi) := \prod_{i=0}^r \xi_i^t \). By using this, the claimed family of \( T(t) \)-actions on \( O_{G^{s}_{\text{ms}}} \) (1) is induced from the following family of actions depending on \( t \):

\[
\xi \cdot (u_0', \ldots, u_r') := t^{-1}(\xi)(\xi_0 u_0', \ldots, \xi_r u_r').
\]

Using the identification between \( T^C \) and \( (\mathbb{C}^*)^r \), one defines a map from \( T^C \) to \( T \) as follows: let \( \eta = (\prod_{i=1}^r z_i)^{-t^{-1}} \). We have

\[
\varphi : T^C \simeq (\mathbb{C}^*)^r \to T \to \mathbb{C},
\]

\[
(\zeta_1, \ldots, \zeta_r) \mapsto (\eta, \eta z_1^{-1}, \ldots, \eta z_r^{-1}).
\]

Strictly speaking, \( \varphi \) is not a map, which is just a tuple of fractional characters of \( (\mathbb{C}^*)^r \). But for our following usage, such a “map” \( \varphi \) is enough. Therefore, we will ignore this little inaccuracy in the definition of \( \varphi \).

By composing \( \varphi \) and the \( T \)-action, one has a new \( T^C \)-action on \( O_{G^{s}_{\text{ms}}} \) (1), whose \( T \)-moment map is

\[
\vartheta'_{T} = \vartheta_{T} - \frac{1}{r+1} \sum_{i=0}^r \omega_i = \epsilon.
\]

In the same way, by composing \( \varphi \) and the \( T(t) \)-action, one has a family of \( T^C(t) \)-actions depending on \( t \in \Delta \) on \( O_{G^{s}_{\text{ms}}} \) (1), whose \( T(t) \)-moment maps are

\[
\vartheta'_{T}(t) = \vartheta_{T} - \sum_{i=0}^r t_i (\omega_i + \epsilon).
\]

Therefore, using the Kirwan-Ness theorem, we come to the following interpretation of the torus action quotients on a master space.
Proposition 6.3. Under the map $\varphi$, for any $t \in \Delta$, the GIT quotient $X_{ms} // T(t)$ is holomorphically isomorphic to the symplectic quotient $X_{ms} // \sum_{i=0}^{r} t_i (\varpi_i + \epsilon) T$.

In fact, one can also calculate and compare the two quotients explicitly as follows. On the algebraic side, by the results in [30, Subsection 3.1], $X_{ms} // T(t)$ is isomorphic to $(X \times G/B) // G(t)$, which is the GIT quotient with respect to the $G$-linearization $L \otimes \prod_{i=0}^{r} (L_i \otimes L_i)^{-t_i}$. On the symplectic side, by Proposition 6.1,

$$X_{ms} // \sum_{i=0}^{r} t_i (\varpi_i + \epsilon) T = ((X \times G \times E^N) // K) // \sum_{i=0}^{r} t_i (\varpi_i + \epsilon) T = (X \times O(\sum_{i=0}^{r} t_i (\varpi_i + \epsilon)) // K,$$

which is isomorphic to $(X \times G/B) // G(t)$ by the Kirwan-Ness theorem again.

Propositions 6.1 and 6.3 complete our interpretation of Thaddeus’s construction in terms of the symplectic implosion techniques.

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