Scattering in one dimension: The coupled Schrödinger equation, threshold behaviour and Levinson’s theorem

K.A. Kiers
Department of Physics, University of British Columbia,
Vancouver, British Columbia, Canada V6T 1Z1

W. van Dijk
Redeemer College, Ancaster, Ontario, Canada L9G 3N6
and Department of Physics and Astronomy, McMaster University,
Hamilton, Ontario, Canada L8S 4M1
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We formulate scattering in one dimension due to the coupled Schrödinger equation in terms of the S matrix, the unitarity of which leads to constraints on the scattering amplitudes. Levinson’s theorem is seen to have the form

$$\eta(0) = \pi \left( n_b + \frac{1}{2} n - \frac{1}{2} N \right),$$

where $\eta(0)$ is the phase of the S matrix at zero energy, $n_b$ the number of bound states with nonzero binding energy, $n$ the number of half-bound states, and $N$ the number of coupled equations. In view of the effects due to the half-bound states, the threshold behaviour of the scattering amplitudes is investigated in general, and is also illustrated by means of particular potential models.

I. INTRODUCTION

The quantum mechanics of one-dimensional scattering describes many actual physical phenomena to a good approximation. (For example, see Ref. [1] for a review of tunneling times.) One-dimensional models are furthermore often employed to gain deeper insight into the approximations used in order to make the more complex three-dimensional systems tractable. It is therefore not surprising that there have been many articles, also in this journal, dealing with various aspects of such scattering. In particular a number of papers have appeared in recent years on the threshold behaviour of one-dimensional scattering and Levinson’s theorem [2–7]. These studies have been limited to systems without coupling.

In this paper we wish to investigate scattering described by a system of coupled differential equations with a particular interest in developing a formulation for Levinson’s theorem and in gaining insight into the threshold properties of the scattering amplitudes. This work can be seen as a special case of multichannel scattering for which the threshold energies are equal. In subsequent work, we intend to generalize to the case of differing threshold energies. Although in previous work on one-dimensional scattering one has at times employed a “partial wave” analysis or a parity-eigenstate representation, we have chosen to use the traditional, more “physical”, approach involving states with incident waves coming from a single direction.

In Sec. II we express the scattering properties in terms of the S matrix, the unitarity of which leads to specifiable constraints on the scattering amplitudes. For the proof of the generalized Levinson’s theorem we make use of the complete set of orthonormal states of the Hamiltonian; this is an alternative to the approach involving the analyticity of the scattering amplitudes. The proof of Levinson’s theorem depends on the threshold properties of the scattering amplitudes. These properties are of interest in their own right, especially in connection with scattering time delay and advance [6], and therefore we discuss the zero-energy behaviour of the amplitudes at some length.

The factorization of the S matrix is generalized to the coupled system in Sec. III. We also indicate that there is a class of finitely periodic matrix potentials for which the scattering amplitudes can be found in a way analogous to the case with no coupling.

In Sec. IV we discuss a number of specific potential models to elucidate and amplify general results. We conclude with a brief discussion of our results in Sec. V.

II. S-MATRIX FORMULATION

The one-dimensional scattering problem has been studied in terms of the S matrix by a number of authors. (See, for example, Refs. [8, 9].) We extend the formalism to include a matrix potential function. The Schrödinger equation for a stationary state of such a system is

$$-\frac{d^2 \Psi}{dx^2} + V(x)\Psi = k^2 \Psi, \quad (2.1)$$

where $V(x)$ is a real, symmetric $N \times N$ matrix, $k^2$ the energy of the system, and $\Psi(k, x)$ the wave function, which is an $N$-dimensional column vector. For large values of $|x|$ the potential matrix $V(x)$ approaches zero sufficiently fast, so that in the asymptotic region $\Psi(k, x)$ represents a free-particle wave function. To ensure this we will take
$V(x) = 0$ for $|x| > R$, $R$ being the range of the potential. Furthermore, we assume that $|V_{ij}(x)|$ is integrable for $i, j = 1, \ldots, N$.

The physical scattering solutions of Eq. (2.1) at a given energy $k^2$ can be written as the columns of $N \times N$ matrices $\psi(k, x)$ and $\tilde{\psi}(k, x)$ which are uniquely determined by the boundary conditions,

$$
\psi(k, x) \sim \begin{cases} 
1e^{ikx} + \rho(k)e^{-ikx}, & x \to -\infty \\
\tau(k)e^{ikx}, & x \to \infty 
\end{cases} \quad (2.2)
$$

and

$$
\tilde{\psi}(k, x) \sim \begin{cases} 
\bar{\tau}(k)e^{-ikx}, & x \to -\infty \\
1e^{-ikx} + \bar{\rho}(k)e^{ikx}, & x \to \infty. 
\end{cases} \quad (2.3)
$$

We will refer to $\psi$ and $\tilde{\psi}$ as the solution matrices. Note that the columns of $\psi$ contain the wave functions with an incident wave from the left, whereas the columns of $\tilde{\psi}$ includes those with an incident wave from the right. The $N \times N$ matrices $\rho, \bar{\rho}, \tau, \bar{\tau}$ are generalizations of the usual reflection and transmission amplitudes (13-14). The set of $N$-dimensional column vectors of matrices $\psi$ and $\tilde{\psi}$ represent solutions of Eq. (2.1). The linear independency of these solutions can be shown by considering the $2N \times 2N$ matrix,

$$
W(\psi, \tilde{\psi}) = \begin{pmatrix} \psi & \tilde{\psi} \\ \psi' & \tilde{\psi}' \end{pmatrix},
$$

in which the prime indicates the derivative with respect to $x$. In Appendix A we show that the $\det W(\psi, \tilde{\psi})$ is a constant, which is nonzero if and only if the solutions are linearly independent. Its value, determined from the asymptotic forms of $\psi$ and $\tilde{\psi}$, is

$$
\det W(\psi, \tilde{\psi}) = \det(\tau(k))(-2ik)^N. \quad (2.5)
$$

Thus when both $k$ and $\det \tau(k)$ are nonzero, the columns of $\psi$ and $\tilde{\psi}$ give $2N$ linearly independent solutions.

In order to define the $S$ matrix, we consider the general solution matrix in the asymptotic region,

$$
\Phi(k, x) \sim \begin{cases} 
Ae^{ikx} + B'e^{-ikx}, & x \to -\infty \\
A'e^{ikx} + Be^{-ikx}, & x \to \infty, 
\end{cases} \quad (2.6)
$$

where $A, A', B, B'$ are $N \times N$ matrices. Since the unprimed matrices are associated with incoming waves and the primed matrices with outgoing waves, the $S$ matrix can be defined as the matrix that transforms the coefficients of the incoming waves into those of the outgoing waves $\Phi(1)$, so that

$$
\begin{pmatrix} A' \\ B' \end{pmatrix} = S \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \quad (2.7)
$$

Clearly, $S$ is a $2N \times 2N$ matrix. We can write it in terms of the transmission and reflection amplitudes by making use of the special cases for which $(A, B) = (1, 0)$ and $(A, B) = (0, 1)$. The result is

$$
S = \begin{pmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{pmatrix} = \begin{pmatrix} \tau & \bar{\rho} \\ \bar{\tau} & \rho \end{pmatrix}. \quad (2.8)
$$

The $S$ matrix contains $4N^2$ complex elements or $8N^2$ real parameters. As we will see below there are a number of relations between the transmission and reflection amplitudes, which will reduce the number of independent real parameters.

A. Relation between reflection and transmission amplitudes

Constraints on the transmission and reflection amplitudes follow from the Schrödinger equation (2.1). Consider two solution matrices $\psi_1(k, x)$ and $\psi_2(k, x)$, then

$$
\psi_2^* = (k^2 - V)\psi_2 \quad \text{and} \quad \psi_2^{1''} = \psi_2^1(k^2 - V), \quad (2.9)
$$

so that

$$
\psi_2^{1''} \psi_1 = \psi_2^1(k^2 - V)\psi_1. \quad (2.10)
$$

The Schrödinger equation, Eq. (2.1), for $\psi_1$ premultiplied by $\psi_2^1$ yields

$$
\psi_2^1 \psi_1^* = \psi_2^1(k^2 - V)\psi_1. \quad (2.11)
$$

Subtracting Eqs. (2.10) and (2.11), we obtain

$$
\psi_2^{1''} \psi_1 - \psi_2^{1'} \psi_1^* = \frac{d}{dx} \left[ \psi_2^1 \psi_1 - \psi_2^1 \psi_1^* \right] = 0, \quad (2.12)
$$

which leads to

$$
\psi_2^1 \psi_1 - \psi_2^{1'} \psi_1^* = \text{constant matrix}. \quad (2.13)
$$

If we now insert the asymptotic forms of $\psi$ or $\tilde{\psi}$ for $\psi_1$ or $\psi_2$ into Eq. (2.13) and equate the expression at $-\infty$ to that at $+\infty$, we obtain the following relations.

$$
\tau^* \tau + \rho^* \rho = \bar{\tau}^* \bar{\tau} + \bar{\rho}^* \bar{\rho} = 1, \quad (2.14)
$$

$$
\rho^* \bar{\tau} + \bar{\rho}^* \tau = 0. \quad (2.15)
$$

Eqs. (2.14) and (2.13) are equivalent to the statement that $S' S = I$, where $I$ is the $2N \times 2N$ identity matrix.

Further relations are found by using the time-reversal symmetry of the system. Since $V(x)$ is real, the complex conjugate solution matrices $\psi^*$ and $\tilde{\psi}^*$ are also solutions of the Schrödinger equation. By complex conjugating Eq. (2.1), we see that the roles of the incoming and outgoing asymptotic waves are reversed, and indeed

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1 The boldface 0 and 1 refer to the zero and identity $N \times N$ matrices, respectively.
Eq. (2.7) is valid when $A \rightarrow B^*$, $B \rightarrow A^*$, $A' \rightarrow B^*$, and $B' \rightarrow A^*$. Thus Eq. (2.7) may be written as

$$
\left( \begin{array}{c} B^* \\ A^* \end{array} \right) = S \left( \begin{array}{c} B^* \\ A^* \end{array} \right),
$$

(2.16)

Multiplying on the left by $S^\dagger$ and using $S^\dagger S = I$, we find that

$$
\left( \begin{array}{c} B^* \\ A^* \end{array} \right) = S^\dagger \left( \begin{array}{c} B^* \\ A^* \end{array} \right),
$$

(2.17)

which leads to

$$
\left( \begin{array}{c} A' \\ B' \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) S^\dagger \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \left( \begin{array}{c} A \\ B \end{array} \right),
$$

(2.18)

where $S^T$ is the transpose of $S$. Thus

$$
S = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) S^T \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
$$

(2.19)

from which it follows that

$$
\bar{\tau} = \tau^T, \quad \rho = \rho^T, \quad \text{and} \quad \bar{\rho} = \bar{\rho}^T.
$$

(2.20)

Inserting these expressions into Eqs. (2.14) and (2.15), we obtain

$$
\bar{\tau} \bar{\bar{\tau}}^\dagger + \rho \rho^\dagger = \tau \tau^\dagger + \bar{\rho} \bar{\rho}^\dagger = 1,
$$

(2.21)

$$
\tau \rho^\dagger + \bar{\tau} \bar{\rho}^\dagger = \rho \tau^\dagger + \bar{\bar{\tau}} \bar{\bar{\rho}}^\dagger = 0.
$$

(2.22)

These equations yield the other half of the unitarity condition of the $S$ matrix, so that

$$
S^\dagger S = SS^\dagger = I.
$$

(2.23)

For a parity-invariant potential function, i.e., $V(-x) = V(x)$, there are further constraints on the transmission and reflection amplitudes. In that case the amplitudes are symmetric matrices and the two types of amplitudes are the same, i.e.,

$$
\rho = \bar{\rho} = \rho^T = \bar{\rho}^T, \quad \text{and} \quad \tau = \bar{\tau} = \tau^T = \bar{\tau}^T.
$$

(2.24)

There is also a useful relation between the scattering amplitudes at $k$ and at $-k$, which is easily obtained by generalizing the result for uncoupled potentials [4]. Since $\psi^*(k, x)$ and $\tilde{\psi}^*(-k, x)$ are solution matrices of the Schrödinger equation with the same boundary conditions as $\psi(k, x)$ and $\tilde{\psi}(k, x)$ respectively, we find that

$$
\tau^*(-k) = \tau(k), \quad \rho^*(-k) = \rho(k),
$$

$$
\bar{\tau}^*(-k) = \bar{\tau}(k), \quad \bar{\rho}^*(-k) = \bar{\rho}(k).
$$

(2.25)

From these relations it follows immediately that the reflection and transmission amplitudes at threshold (i.e., $k = 0$) are real.

### B. Levinson’s Theorem

Levinson’s theorem in its most common formulation for a spherically symmetric potential gives a relationship of the scattering phase shifts at zero and infinite energy. The theorem has also been studied for one-dimensional systems without coupling [3]. We generalize the theorem to the matrix-potential case. Levinson’s theorem is a consequence of the orthogonality and completeness relation of the eigenstates of the total Hamiltonian.

The scattering states of the Schrödinger equation (2.1) are defined by Eqs. (2.2) and (2.3) along with the bound states. We label the state vectors with subscript $E_j$, but allow the possibility of different subscripts $j$ for the same energy in order to include all independent bound-state vectors. For example, one could have $E_j = E_i$ where $i \neq j$ when $E_i$ is a degenerate energy eigenvalue. The orthonormality relations are

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, \psi_{ij}^\dagger(k, x) \psi_{ij}(k', x)
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, \tilde{\psi}_{ij}^\dagger(k, x) \tilde{\psi}_{ij}(k', x) = \delta_{ij} \delta(k' - k),
$$

(2.26)

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, \psi_{ij}^\dagger(k, x) \psi_{ij}(x)
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, \tilde{\psi}_{ij}^\dagger(k, x) \tilde{\psi}_{ij}(x) = \delta_{ij},
$$

(2.27)

where $\psi_i$ and $\tilde{\psi}_i$ are the $i$th columns of the $\psi$ and $\tilde{\psi}$ matrices respectively. Thus the completeness relation is

$$
\sum_{j=1}^{n_b} \psi_{E_j}(x) \psi_{E_j}(x') + \sum_{i=1}^{N} \frac{1}{2\pi} \int_{0}^{\infty} dk \, \psi_i(k, x) \psi_i^\dagger(k, x')
$$

$$
+ \sum_{i=1}^{N} \frac{1}{2\pi} \int_{0}^{\infty} dk \, \tilde{\psi}_i(k, x) \tilde{\psi}_i^\dagger(k, x') = 1 \delta(x - x').
$$

(2.29)

The completeness relation may be written in a more compact form in terms of the matrices themselves rather than the column vectors, i.e.,

$$
\sum_{j=1}^{n_b} \psi_{E_j}(x) \psi_{E_j}^\dagger(x') + \frac{1}{2\pi} \int_{0}^{\infty} dk \, \psi(k, x) \psi(k', x')
$$

$$
+ \tilde{\psi}(k, x) \tilde{\psi}(k', x') = 1 \delta(x - x').
$$

(2.30)
For the free-particle case when $V(x) = 0$, there are no bound states and the completeness relation is
\[
\frac{1}{2\pi} \int_0^\infty dk \left[ \psi^0(k,x) \psi^0(k,x') + \bar{\psi}(k,x) \bar{\psi}(k,x') \right] = 1 \delta(x-x').
\] (2.31)

We now subtract Eq. (2.31) from Eq. (2.30), set $x' = x$, and integrate over $x$ from $-R$ to $R$. The resulting equation may be written as
\[
\int_0^\infty dk \int_{-R}^R dx \left[ \psi(k,x) \psi^\dagger(k,x) + \bar{\psi}(k,x) \bar{\psi}^\dagger(k,x) \right] - \bar{\psi}(k,x) \psi^\dagger(k,x) - \bar{\psi}^0(k,x) \psi^0(k,x) = -2\pi \eta_b.
\] (2.32)

The trace of Eq. (2.32) in the limit as $R$ approaches infinity gives
\[
\lim_{R \to \infty} \int_0^\infty dk \int_{-R}^R dx \left\{ \text{Tr} \left[ \psi(k,x) \psi^\dagger(k,x) + \bar{\psi}(k,x) \bar{\psi}^\dagger(k,x) \right] - \bar{\psi}(k,x) \psi^\dagger(k,x) - \bar{\psi}^0(k,x) \psi^0(k,x) \right\} = -2\pi \eta_b.
\] (2.33)

To perform the integration over $x$ in the right side of Eq. (2.33), we use the identity
\[
\text{Tr} \left[ \psi \psi^\dagger \right] = \frac{1}{2k} \partial \partial^\dagger \left[ \text{Tr} \left[ \partial \partial^\dagger \right] - \partial^2 \psi \partial \partial^\dagger \right],
\] (2.34)

which may be obtained by taking the derivative with respect to $k$ of the Schrödinger equation (2.1). Since in the limit $R$ exceeds the range of the potential, we can insert the asymptotic forms of the wave functions in Eq. (2.33) to obtain
\[
\lim_{R \to \infty} \int_0^\infty \frac{dk}{2k} \text{Tr} \left[ -2ik \left( \partial \partial^\dagger + \partial \partial^\dagger \right) - i(\rho + \bar{\rho}) e^{2ikR} + i(\rho^\dagger + \bar{\rho}^\dagger) e^{-2ikR} \right] = -2\pi \eta_b.
\] (2.35)

Following Newton and Jost [19] we define the phase as
\[
\eta(k) = \frac{1}{2i} \ln \det S(k),
\] (2.36)

where we require $\eta(k)$ to be continuous for $k \in (0, \infty)$. Since the $S$ matrix is unitary, we may write $S = U S_D U$ where $U$ is a real orthogonal matrix and $S_D$ is the diagonal matrix $S_D = \text{diag}(e^{i\delta_1}, \ldots, e^{i\delta_{2N}})$ where the $\delta_j$’s are real phases [17]. Let us write therefore $S = e^{i\Delta}$ where $\Delta = U \Delta_D U^\dagger$ for $\Delta_D = \text{diag}(\delta_1, \ldots, \delta_{2N})$. Then
\[
2i\eta(k) = \ln \det S(k) = \ln \det S_D(k) = \sum_{j=1}^{2N} i\delta_j(k) = \text{Tr}[i\Delta_D(k)]
\] (2.37)

and
\[
2i \frac{\partial \eta}{\partial k} = \text{Tr} \left[ \frac{\partial \Delta_D}{\partial k} \right] = \text{Tr} \left[ S_D^\dagger \frac{\partial S_D}{\partial k} \right] = \text{Tr} \left[ S^\dagger \frac{\partial S}{\partial k} \right].
\] (2.38)

Thus Eq. (2.33) may be written as
\[
\eta(0) - \eta(\infty) = \pi \eta_b - \lim_{R \to \infty} X(R),
\] (2.39)

where
\[
X(R) = \frac{1}{2} \int_0^\infty \frac{dk}{2k} \text{Tr} \left[ i(\rho(k) + \bar{\rho}(k)) e^{2ikR} - i(\rho^\dagger(k) + \bar{\rho}^\dagger(k)) e^{-2ikR} \right].
\] (2.40)

In the next section we show that $\rho(k) \sim O(1/k)$ for large $|k|$, so that the integration in Eq. (2.41) converges for large $|k|$. We now take the limit as $R$ approaches $\infty$, using the relation
\[
\lim_{R \to \infty} \frac{e^{2ikR}}{k} = i\pi \delta(k),
\] (2.42)

where $\delta(k)$ is the Dirac delta function. In that limit $X(R)$ goes to $-\pi/4 \text{Tr}[\rho(0) + \bar{\rho}(0)]$, so that the statement of Levinson’s theorem now is
\[
\eta(0) = \pi \eta_b + \pi/4 \text{Tr}[\rho(0) + \bar{\rho}(0)],
\] (2.43)

where we have set $\eta(\infty)$ equal to zero. In the next section we also show that
\[
\text{Tr}[\rho(0) + \bar{\rho}(0)] = -2(N-n),
\] (2.44)

where $n$ is the number of “half-bound states” [16,18]. Thus in its final form Levinson’s theorem states,
\[
\eta(0) = \pi n_b + \frac{1}{2} N - \frac{1}{2} N.
\] (2.45)

This expression of the theorem is consistent with that for the uncoupled case given in Ref. [1].

C. Threshold behaviour of $\rho$ and $\tau$

The threshold behaviour of reflection and transmission amplitudes and coefficients has been discussed recently in several articles [4,5,6,7]. In order to study the behaviour of the $\rho$ and $\tau$ matrices at $k = 0$, we introduce a different set of solutions of the Schrödinger equation, according to Eq. (2.23) the columns of $\psi$ and $\bar{\psi}$ fail to be linearly independent at $k = 0$. Let $\phi(k,x)$ and $\chi(k,x)$ be solution matrices of Eq. (2.1) which satisfy the boundary conditions,
\[ \phi(k, -R) = \chi'(k, -R) = 1 \]
\[ \phi'(k, -R) = \chi(k, -R) = 0, \quad (2.46) \]

where \( R \) is the range of the potential as defined at the beginning of Sec. II. By evaluating \( \det W(\phi, \chi) \) at \( x = -R \) and using the results of Appendix A, we readily show that for all \( x \) and \( k \) the \( \det W(\phi, \chi) = 1 \), where the matrix \( W \) is defined as in Eq. (2.4). Thus unlike the column vectors of \( \psi \) and \( \bar{\psi} \), the column vectors of \( \phi \) and \( \chi \) are linearly independent at zero energy.

In order to obtain the scattering amplitudes, we expand \( \psi \) in terms of \( \phi \) and \( \chi \) so that
\[ \psi(k, x) = \phi(k, x)B(k) + \chi(k, x)C(k), \quad (2.47) \]
where \( B(k) \) and \( C(k) \) are matrices of expansion coefficients. Evaluating \( \psi \) and \( \psi' \) at \( \pm R \) using Eqs. (2.2) and (2.47), we obtain four equations involving \( B(k), C(k), \rho(k) \) and \( \tau(k) \). By eliminating three of these we find that \( \rho \) can expressed in terms of \( \phi \) and \( \chi \) and their derivatives evaluated at \( R \). Thus
\[ \rho(k) = \{k^2\chi(k, R) + ik[\chi'(k, R) + \phi(k, R)] - \phi'(k, R)\}^{-1} \]
\[ \{k^2\chi(k, R) + ik[\chi'(k, R) - \phi(k, R)] + \phi'(k, R)e^{-2ikR}\}. \quad (2.48) \]

Similarly by expanding \( \bar{\psi} \) in terms of \( \phi \) and \( \chi \) we obtain
\[ \bar{\rho}(k) = \{k^2\chi(k, R) - ik[\chi'(k, R) - \phi(k, R)]\}
\[ \{k^2\chi(k, R) + ik[\chi'(k, R) + \phi(k, R)] - \phi'(k, R)e^{-2ikR}\}. \quad (2.49) \]

and, since \( \tau(k) = \bar{\tau}^T(k) \),
\[ \tau(k) = 2ik\{k^2\chi^T(k, R) + ik[\chi'^T(k, R) + \phi^T(k, R)]
\[ - \phi'^T(k, R)e^{-2ikR}\}. \quad (2.50) \]

Thus all four scattering amplitudes, and consequently the \( S \) matrix, are determined by \( \phi \) and \( \chi \) and their derivatives evaluated at \( R \). In Appendix B it is shown that \( \phi(k, x) \) and \( \chi(k, x) \) are entire functions of complex \( k \) for all \( x \in [-R, R] \) so that the analytic properties of the scattering amplitudes can be determined using these wave functions. By inserting the expressions for the large real \( k \) behaviour of the wave functions, Eqs. (B14) and (B17), into the expressions for the scattering amplitudes, we obtain
\[ \tau(k) \sim 1 + O(1/k) \text{ and } \rho(k) \sim 0 + O(1/k) \text{ for } k \to \infty, \quad (2.52) \]

and similar expressions for \( \bar{\tau} \) and \( \bar{\rho} \).

If \( \det[\phi'(0, R)] \neq 0 \), the reflection and transmission amplitudes at zero energy are
\[ \rho(0) = \bar{\rho}(0) = -1 \quad \text{and} \quad \tau(0) = \bar{\tau}(0) = 0. \quad (2.53) \]

The case for which \( \det[\phi'(0, R)] = 0 \) needs special attention. In order to understand the significance of this condition, we look at the bound states of the system. In the Schrödinger equation (2.2) for bound states, we denote the bound-state energy as \( \alpha^2 = -k^2 \) with \( \alpha > 0 \). The bound-state wave functions can be expressed as column vectors of a matrix \( \psi_b(\alpha, x) \) with the asymptotic boundary conditions,
\[ \psi_b(\alpha, x) \sim \begin{cases} e^{\alpha x}Q, & x \leq -R \\ e^{-\alpha x}T, & x \geq R \end{cases} \quad (2.54) \]

where \( Q \) and \( T \) are matrices of constants. The number of independent bound states at a given energy will depend on the rank of the matrix \( \psi_b \), and consequently cannot exceed \( N \). Proceeding as we did for the scattering states, we expand the bound-state wave functions in terms of the functions \( \phi_b(\alpha, x) \) and \( \chi_b(\alpha, x) \) which are solution matrices of the Schrödinger equation with energy \(-\alpha^2\) and satisfy the boundary conditions
\[ \phi_b(\alpha, -R) = \chi_b(\alpha, -R) = 1 \]
\[ \chi_b(\alpha, -R) = \phi_b(\alpha, -R) = 0. \quad (2.55) \]

Thus
\[ \psi_b(\alpha, x) = \phi_b(\alpha, x)\beta(\alpha) + \chi_b(\alpha, x)\gamma(\alpha), \quad (2.56) \]

where \( \beta \) and \( \gamma \) are matrices of expansion coefficients. At \( R \) and \( -R \) we match the asymptotic form of the wave function, Eq. (2.54), and its derivative to the expanded form, Eq. (2.54), and its derivative. Eliminating \( T, Q \) and \( \gamma \) from the four equations so obtained, we are left with the equation
\[ \{\alpha^2\chi_b(\alpha, R) + \alpha[\chi_b(\alpha, R) + \phi_b(\alpha, R)]
\[ + \phi_b(\alpha, R)\beta(\alpha) = 0 \quad (2.57) \]

Since one of the four matching equations is \( Q = \beta(\alpha)e^{\alpha R} \), there will be bound states only if the matrix \( \beta(\alpha) \) contains nonzero entries. Such a nontrivial matrix exists only when
\[ \det[\alpha^2\chi_b(\alpha, R) + \alpha[\chi_b(\alpha, R) + \phi_b(\alpha, R)]
\[ + \phi_b(\alpha, R)] = 0. \quad (2.58) \]

This is the bound-state eigenvalue equation for energy \(-\alpha^2\). In contrast to the case with no coupling, the bound-state eigenenergies can be degenerate.

Let us consider the eigenstates when \( \alpha = 0 \). These will occur only if \( \det[\phi_b'(0, R)] = 0 \). In general the solutions represented by the column vectors of \( \psi_b(0, x) \) are bounded but not square integrable; hence they are referred to as half-bound states [10]. The restriction on the potential function that it vanishes for \( |x| \geq R \), precludes the possibility of having normalizable state functions at zero energy. For this to be the case a linear combination
of the columns of $\psi_b$ would yield $\Psi(0, x) = 0$ for $|x| \geq R$. Such a boundary condition would lead to the trivial solution of Eq. (2.1). Normalizable zero-energy bound states can exist for potentials which are less restrictive than those of this paper [13].

Since $\phi(0, x)$ and $\phi_b(0, x)$ are solutions of the same system of differential equations and both have the same boundary conditions, $\phi_b(0, x) = \phi(0, x)$. Thus the condition that

$$\det[\phi'(0, R)] = 0 \quad (2.59)$$

is equivalent to the condition for the existence of half-bound states.

Consider the matrix eigenvalue equation,

$$\phi'(0, R) \tilde{\beta} = \lambda \tilde{\beta}, \quad (2.60)$$

where $\tilde{\beta}$ is a column vector and $\lambda$ is its eigenvalue. There will be a nontrivial solution only if

$$\det[\phi'(0, R) - \lambda I] = 0. \quad (2.61)$$

Suppose the eigenvalues obtained are $\lambda_1, \ldots, \lambda_N$. At least one of these must be zero if there is a half-bound state. Actually there may be $n(\leq N)$ zero eigenvalues. We can order these in the following way: $0, \ldots, 0, \lambda_{n+1}, \ldots, \lambda_N$. These $n$ zero eigenvalues will have $n$ linearly independent eigenvectors associated with them, which represent $n$ distinct half-bound states.

We now return to the discussion of reflection and transmission amplitudes. The inverse of $\tau$ of Eq. (2.50) can be written as

$$2i k \tau^{-1}(k) = \{k^2 \chi(k, R) + ik[\chi'(k, R) + \phi(k, R)] \}
- \phi'(k, R)) e^{2ikR}. \quad (2.62)$$

If this equation is combined with Eq. (2.49), we obtain

$$2i k \tilde{\rho}(k) \tau^{-1}(k) = \{k^2 \chi(k, R) - ik[\chi'(k, R) - \phi(k, R)]
+ \phi'(k, R)). \quad (2.63)$$

The factor in the curly brackets of Eq. (2.62) is precisely that in the determinant of Eq. (2.58) (with $k = i\alpha$), i.e., it is the factor which determines the bound states of the system. In Appendix B we show that matrices $\phi$ and $\chi$ are elementwise entire functions of $k$. From Eq. (2.62) and the fact that $\tau^T = \tau$ we see that $2i k \tau^{-1}(k)$ and $(2ik)^N \det \tau^{-1}(k)$ are also entire functions of $k$. According to Eqs. (2.58) and (2.62), $\det \tau^{-1}(k)$ has a zero at $k = i\alpha$ when $-\alpha$ is the energy of a bound state. Thus $\det \tau(k)$ has poles (possibly multiple) on the positive imaginary axis of the complex $k$ plane. In the absence of a half-bound state, $\det \tau^{-1}(k)$ has an $N^{th}$ order pole at $k = 0$ and $\det \tau$ has an $N^{th}$ order zero at $k = 0$. Since there can be no more than $N$ half-bound states $\det \tau(k)$ and $\tau(k)$ are analytic in a neighborhood of $k = 0$.

In the following we consider real $k$. Taking the limits as $k$ goes to zero of Eqs. (2.62) and (2.63) we obtain

$$\lim_{k \to 0} 2ik \tau^{-1}(k) = -\phi'(0, R) \quad \text{and} \quad \lim_{k \to 0} 2ik \tilde{\rho}(k) \tau^{-1}(k) = \phi'(0, R). \quad (2.64)$$

Similarly using Eq. (2.51) and the transpose of Eq. (2.48), we get

$$\lim_{k \to 0} 2ik \tau^{-1}(k) = -\phi'^T(0, R) \quad \text{and} \quad \lim_{k \to 0} 2ik \tilde{\rho}(k) \tau^{-1}(k) = \phi'^T(0, R). \quad (2.65)$$

We introduce unitary matrices $U(k)$ and $V(k)$ which diagonalize $\tau$ [21, page 192], so that

$$\tau_D(k) = U^\dagger(k) \tau(k) V(k) \quad \text{and} \quad 2ik \tau^{-1}_D(k) = V^\dagger(k) 2ik \tau^{-1}(k) U(k). \quad (2.66)$$

In the limit as $k$ approaches zero $U(0)$ and $V(0)$ also diagonalize $\phi'^T(0, R)$, i.e., $\phi'^T_D(0, R) = V^\dagger(0) \phi'^T(0, R) U(0)$, so that

$$\lim_{k \to 0} 2ik \tau^{-1}_D(k) = -\phi'^T_D(0, R). \quad (2.67)$$

We define the matrix

$$r(k) \equiv V^\dagger(k) \rho(k) V(k), \quad (2.68)$$

so that

$$\lim_{k \to 0} 2ik r(k) \tau^{-1}_D(k) = \phi'^T_D(0, R). \quad (2.69)$$

Combining Eqs. (2.67) and (2.69) gives

$$r(0) \phi'^T_D(0, R) = -\phi'^T_D(0, R). \quad (2.70)$$

The matrices $\phi'^T(0, R)$ and $\phi'^T_D(0, R)$ have the same rank [21, page 55]. Thus $\phi'^T_D(0, R)$ will have the same number of nonzero diagonal elements as there are nonzero eigenvalues of $\phi'^T(0, R)$ or $\phi'(0, R)$. Writing $\phi'^T_D(0, R) = \text{diag}(0, \ldots, 0, s_{n+1}, \ldots, s_N)$ and using Eq. (2.70) we see that the matrix $r(0)$ must have the form

$$r(0) = \begin{pmatrix} R_{11} & 0 \\ R_{21} & -1 \end{pmatrix}, \quad (2.71)$$

where the matrices $R_{11}, R_{21}, 0$, and $1$ have dimensions $n \times n, (N - n) \times n, n \times (N - n)$ and $(N - n) \times (N - n)$ respectively.

To study the behaviour of $\tau_D(k)$ near $k = 0$, we consider the Hermitian positive semi-definite matrix $T(k) = \tau^T(k) \tau(k)$, whose real nonnegative eigenvalues we denote by $t_1(k), \ldots, t_N(k)$. The singular values of $\tau(k)$ are defined as the nonnegative square root of these, i.e., $t_1(k), \ldots, t_N(k)$, and they form the diagonal elements of $\tau_D(k)$ in Eq. (2.67), so that $\tau_D(k) = \text{diag}(t_1(k), \ldots, t_N(k))$ [21, page 192]. Since $\tau(k)$ is analytic in the neighborhood of $k = 0$, so is $T(k)$. For small real $k > 0$, we invoke a theorem of Rellich [21, page
In order to simplify Eq. (2.43) we write \( \tilde{t}_1(\kappa) = t_1^2(\kappa) \). Hence the \( t_1^2(\kappa) \)'s are power series of \( k^2 \) and the \( t_i(\kappa) \)'s are power series of \( k \).

The same \( \tilde{t}_1(\kappa)\)'s are power series of \( k^2 \) and the \( t_i(\kappa) \)'s are power series of \( k \).

By applying the diagonalization transformation of \( \tau \), we find that

\[
\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11} & 0 \\ R_{21} & -1 \end{pmatrix} + \begin{pmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = 0,
\]

where \( A_1 = \text{diag}(t_1, \ldots, t_n) \) and the block matrices have the appropriate dimensions. From Eq. (2.81) it follows that

\[
A_1\tilde{R}_{11} + \tilde{R}_{11}A_1 = 0^{n \times n}.
\]

Consequently, the second term in the right side of Eq. (2.78) is zero, and we have the simple relation

\[
\text{Tr}[\rho(0) + \tilde{\rho}(0)] = -2(N - n)
\]

In light of the discussion of the \( N = 1 \) normal and anomalous threshold behavior [7,13], our result seems surprising since the right side of Eq. (2.83) is an integer, whereas the anomalous threshold effect for parity-noninvariant potentials gives values for \( \rho(0) \) which lie between 0 and 1. However, if one considers the sum of \( \rho(0) \) and \( \tilde{\rho}(0) \) in the uncoupled case, one obtains an even integer. For a parity-invariant potential, i.e., \( V(-x) = V(x) \), \( \rho \) and \( \tilde{\rho} \) are equal and the zero-energy value of \( \rho \) will always be an integer.

### III. SEGMENTED POTENTIALS AND FACTORIZATION OF THE S MATRIX

For the Schrödinger equation without coupling it is well known that the reflection and transmission amplitudes satisfy a factorization formula. That is, if the potential is subdivided into a number of sections, then the total transmission and reflection amplitudes for the system can be expressed in terms of the amplitudes for each of the truncated pieces of the potential. The recent proof of Aktosun [22] may be generalized immediately to the case of \( N \) coupled equations.

Following Aktosun, then, we subdivide the real line into \( J \) pieces. The boundaries of the segments are denoted by \( x_i, \ i = 0, 1, \ldots, J \), with \( -R = x_0 < x_1 < \ldots < x_{J-1} < x_J = R \). The potential may then be written as a sum of truncated potentials as follows

\[
V(x) = \sum_{j=0}^{J-1} V_j(x),
\]

where

\[
V_j(x) = \begin{cases} V(x), & x_j < x < x_{j+1} \\ 0, & \text{otherwise} \end{cases}
\]

The single indices on the truncated potentials should not be confused with the implicit double indices which label the various elements of the potential matrix.

For a given \( j \), then, let us define the matrix
\[ \Lambda_j(k) = \begin{pmatrix} \tau_j^{-1}(k) & -\tau_j^{-1}(k)\tilde{\rho}_j(k) \\ \rho_j(k)\tau_j^{-1}(k) & (\tilde{\tau}_j(k))^{-1} \end{pmatrix}, \] (3.3)

where \( \tau_j(k), \tilde{\tau}_j(k), \rho_j(k), \tilde{\rho}_j(k) \) represent the various amplitude matrices for the truncated potentials with the usual boundary conditions. The amplitudes for the original potential are similarly arranged into a matrix,

\[ \Lambda(k) = \begin{pmatrix} \tau^{-1}(k) & -\tau^{-1}(k)\tilde{\rho}(k) \\ \rho(k)\tau^{-1}(k) & (\tilde{\tau}(k))^{-1} \end{pmatrix}, \] (3.4)

and the factorization formula is then simply given by

\[ \Lambda(k) = \prod_{j=0}^{J-1} \Lambda_j(k), \] (3.5)

where the factors on the right side of the equation are ordered so that factors with lower subscripts occur to the left of the ones with higher subscripts. The proof of Eq. (3.3) is completely analogous to that advanced by Aktosun in the \( N = 1 \) case and so we shall not review it here. The only added complication is the fact that the various amplitude matrices do not generally commute, but this has been properly accounted for in the definitions of the \( \Lambda \) matrices. The utility of Eq. (3.5) will become apparent below when we use it to derive the amplitudes for scattering from two different delta-function matrix potentials in terms of the amplitudes for scattering from each of them separately.

This approach effectively factorizes the \( S \) matrix in the sense that if the \( S \) matrix of the \( j \)th potential segment, i.e.,

\[ S_j(k) = \begin{pmatrix} \tau_j(k) & \tilde{\rho}_j(k) \\ \rho_j(k) & \tilde{\tau}_j(k) \end{pmatrix}, \] (3.6)

is known, then the scattering amplitudes of the potential segment are determined, and from them \( \Lambda_j(k) \). Using Eq. (3.3) we can obtain \( \Lambda(k) \) for the whole potential, and this allows us to solve for the scattering amplitudes and the \( S \) matrix of the whole potential.

Another generalization of the uncoupled to the coupled problem involves the finitely periodic potentials, recently discussed by a number of authors \[3,23–25\]. In the derivation of the factorization formula, Eq. (3.3), we have to be careful in the ordering of the products such as \( \rho \tau^{-1} \). This non-commutativity of the amplitude matrices would typically prevent us from generalizing the closed-form solutions of the finitely periodic potentials. There are however classes of potentials for which the various amplitude matrices do commute with each other and for which the results for no coupling can be generalized.

An example of such a class of potentials consists of those potentials which can be expressed as

\[ V(x) = U \text{diag}[v_1(x), v_2(x), \ldots, v_N(x)]U^T, \] (3.7)

where \( U \) is a constant (real) orthogonal matrix. Note that the potentials of Eq. (3.7) include as a subclass those of the form \( V(x) = v(x)M \), where \( v(x) \) is a real-valued function of \( x \) and \( M \) is a constant symmetric matrix. Such potentials have been used previously in various applications (see, for example, Ref. \[26\]). It is easy to prove that when the potential is diagonalizable by a constant orthogonal matrix, then all of the amplitude matrices (as well as their inverses and hermitian conjugates) commute with each other.

If we use potentials of this type to construct finitely periodic potentials with nonoverlapping subpotentials, the analysis of Rozman et al. \[23,24\] follows in the same way for the matrix potential problem and the expressions for the amplitude matrices are straightforward generalizations of their results.

\section*{IV. POTENTIAL MODELS}

Below we consider some potential models which lend themselves to solutions in closed form. These models help to elucidate some of the results obtained in the previous sections.

\subsection*{A. Constant potential matrix}

An example of a potential for which solutions can be obtained in closed form is the square-well or square-barrier potential matrix for which

\[ V(x) = \begin{cases} V_0 & \text{for } a \leq x \leq b \text{ where } a \geq -R \text{ and } b \leq R \\ 0 & \text{otherwise}, \end{cases} \] (4.1)

where \( V_0 \) is a real symmetric \( N \times N \) matrix. The Schrödinger equation (2.1) is equivalent to a first-order differential equation of the matrix function \( W(x) = W(\phi, \chi) \) of Eq. (4.2), i.e.,

\[ W'(x) = F(x)W(x), \] (4.2)

where

\[ F(x) = \begin{pmatrix} 0 & 1 \\ V(x) - k^2 & 0 \end{pmatrix}, \] (4.3)

with the boundary condition \( W(-R) = I \). As we saw earlier the function \( W(x) \) provides the advantage of giving solutions that are linearly independent at \( k = 0 \). In addition, Eq. (4.2) gives us an initial value problem, rather than the two-point boundary condition problem of the original Schrödinger equation. Solving for the scattering amplitudes numerically is simpler for the initial value problem. An alternative to this approach is the variable amplitude formulation which also casts the problem into a system of first order differential equations with an initial value condition \[13,14\]. In principle, Eq. (4.2) can be
used to solve the Schrödinger equation for any arbitrary potential matrix.

For the constant potential matrix the differential equation Eq. (1.2) can be integrated starting at \( x = -R \) over the three regions \((-R, a), (a, b)\) and \((b, R)\) in turn [27]. The result is

\[
W(R) = \begin{pmatrix}
\cos k(R - b) & k^{-1} \sin k(R - b) \\
-k \sin k(R - b) & \cos k(R - b)
\end{pmatrix}
\begin{pmatrix}
\cosh K(b - a) & K^{-1} \sinh K(b - a) \\
K \sinh K(b - a) & \cosh K(b - a)
\end{pmatrix}
\begin{pmatrix}
\cos k(a + R) & k^{-1} \sin k(a + R) \\
-k \sin k(a + R) & \cos k(a + R)
\end{pmatrix},
\tag{4.4}
\]

where \( K^2 = V_0 - k^2 \). To simplify matters, but still to allow us to study a model with a potential function without definite parity, we set \( a = -R \), so that we obtain explicit forms for the wave functions at \( x = R \).

\[
\begin{align*}
\phi(k, R) &= \cos k(R - b) \cosh K(R + b) + k^{-1} \sin k(R - b) \sinh K(R + b) \\
\phi'(k, R) &= -k \sin k(R - b) \cosh K(R + b) + \cos k(R - b) \sinh K(R + b) \\
\chi(k, R) &= \cos k(R - b) K^{-1} \sinh K(R + b) + k^{-1} \sin k(R - b) \cosh K(R + b) \\
\chi'(k, R) &= -k \sin k(R - b) \sinh K(R + b) + \cos k(R - b) \cosh K(R + b).
\end{align*}
\tag{4.5}
\]

These expressions can be inserted in the equations for \( \rho, \tilde{\rho}, \tau, \tilde{\tau} \), Eqs. (2.43) to (2.51), to obtain the scattering amplitudes.

Note that \( K^2 = V_0 - k^2 \) is a real symmetric matrix and may therefore be diagonalized by an orthogonal transformation \( U \), giving \( K_D^2 = U K^2 U^T \). The diagonal matrix \( K_D \) has the square root of the diagonal elements of \( K_D^2 \) along its diagonal. Thus \( K = U^T K_D U \), which is a symmetric matrix. Consequently, \( \phi, \phi', \chi, \chi' \) are symmetric matrices, which leads to \( \tau = \tilde{\tau} \). Furthermore, since each of the wave-function matrices or their derivatives at \( R \) is a power series (or polynomial) of the matrix \( K \), the wave-function matrices commute. It is not difficult to show that in the case of \( b = R \), i.e., when the potential function has even parity, \( \rho(k) = \tilde{\rho}(k) \).

The threshold behaviour of the transition amplitudes for a potential lacking specific parity can be studied explicitly with this model. Since Levinson’s theorem involves the trace of the amplitudes at zero energy, we need to consider the diagonalized forms of the amplitudes only. We diagonalize each of the wave-function matrices of Eq. (4.5) using the same orthogonal matrix \( U \) for each, and we denote the diagonal matrices at zero energy as

\[
\begin{align*}
\phi_D(0, R) &= \text{diag}(p_1, \ldots, p_N) \\
\phi_D'(0, R) &= \text{diag}(\lambda_1, \ldots, \lambda_N) \\
\chi_D(0, R) &= \text{diag}(x_1, \ldots, x_N) \\
\chi_D'(0, R) &= \text{diag}(x_1, \ldots, x_N).
\end{align*}
\tag{4.6}
\]

When there is no half-bound state, we obtain, by inserting these expressions into Eqs. (2.43) and (2.51),

\[
\rho(0) = \rho_D(0) = -1 \quad \text{and} \quad \tau(0) = \tau_D(0) = 0. \tag{4.7}
\]

When there are \( n \) half-bound states and \( \det \phi'(0, R) = 0 \), we write \( \phi_D'(0, R) = \text{diag}(0, \ldots, 0, \lambda_{n+1}, \ldots, \lambda_N) \). Using Eq. (2.13) we find that

\[
\phi'(k, x) \chi(k, x) - \phi(k, x) \chi'(k, x) = -1. \tag{4.8}
\]

In general the matrices \( \phi \) and \( \chi \) are real and for the constant potential matrix they are symmetric as well. Thus

\[
\phi'(0, R) \chi(0, R) - \phi(0, R) \chi'(0, R) = \phi_D'(0, R) \chi_D(0, R) - \phi_D(0, R) \chi_D'(0, R) = -1. \tag{4.9}
\]

From this relation it follows that \( x'_i = 1/p_i \) for \( i = 1, \ldots, n \), and furthermore

\[
\rho_D(0) = \text{diag} \left( -\frac{p_1^2}{1 + p_1^2}, \ldots, -\frac{p_n^2}{1 + p_n^2}, -1, \ldots, -1 \right), \tag{4.10}
\]

\[
\tilde{\rho}_D(0) = \text{diag} \left( 1 - \frac{p_1^2}{1 + p_1^2}, \ldots, 1 - \frac{p_n^2}{1 + p_n^2}, -1, \ldots, -1 \right), \tag{4.11}
\]

\[
\tau_D(0) = \tilde{\tau}_D(0) \equiv \text{diag} \left( \frac{2 p_1}{1 + p_1^2}, \ldots, \frac{2 p_n}{1 + p_n^2}, 0, \ldots, 0 \right). \tag{4.12}
\]

Clearly the relation \( \phi^2 D + \tilde{\phi}^2 D = \text{diag} \) is satisfied by Eqs. (4.10) and (4.11). For the parity invariant potential obtained by setting \( b = R \) in Eq. (1.3), \( \phi(0, R) = \chi'(0, R) \). Hence \( p_i = x'_i \) for \( i = 1, \ldots, N \), and it follows that \( p_i^2 = 1 \). Such a potential therefore yields transition amplitudes of the form

\[
\rho_D(0) = \tilde{\rho}_D(0) = \text{diag}(0, \ldots, 0, -1, \ldots, -1) \quad \text{and} \quad \tau_D(0) = \text{diag}(\pm 1, \ldots, \pm 1, 0, \ldots, 0). \tag{4.13}
\]

When the \( i \)-th diagonal element of \( \phi_D(0, x) \) is an even (odd) function, then the \( i \)-th diagonal element of \( \tau_D(0) \) will have a plus (minus) sign with the one. The converse is not necessarily true.

Consider the special case of \( N = 1 \). For the parity invariant potential with a half-bound state, one has \( \rho(0) = 0 \) and \( \tau(0) = \pm 1 \). The plus sign corresponds to \( \phi(0, x) \) being an even solution and the negative sign to it being an odd solution. When there is no half-bound state, then \( \rho(0) = -1 \) and \( \tau(0) = 0 \). In the case of a potential without definite parity, \( \rho(0) = \tilde{\rho}(0) = -1 \) and \( \tau(0) = \tilde{\tau}(0) = 0 \) when the potential does not support a half-bound state. When there is a half-bound state, \( \tau(0) \) and \( \tilde{\tau}(0) \) are not equal to zero, nor are the \( \rho \)'s equal to \(-1\). However, the sum of the \( \rho \)'s is an integer, i.e., \( \rho(0) + \tilde{\rho}(0) = 0 \). These results, which are clearly valid for the square-well potential, are actually valid for any \( N = 1 \) potential function.
B. Models involving delta-function potentials

We now consider two examples involving delta functions for which results can be obtained in closed form. The results exhibit qualitative features which are also found in much more complicated examples. First we examine the case of a single delta-function matrix potential positioned at the origin. Then we will use the factorization formula derived earlier to look at the case for which there are two delta-function matrices symmetrically positioned on both sides of the origin.

1. Delta function at the origin

We write the Schrödinger equation for this case as

\[ \left( -\frac{d^2}{dx^2} + \delta(x)\lambda \right) \psi = k^2 \psi, \]  
(4.14)

where \( \lambda \) is an \( N \times N \) symmetric matrix, and \( \psi \) can be taken to be either a column vector solution or a solution matrix. The former approach will be used when we consider bound states and the latter when we examine scattering solutions.

First consider the scattering solutions. Since the potential has even parity, we immediately have the result that \( \tilde{\rho} = \rho \) and \( \tilde{\tau} = \tau \). Thus we need only consider the solution with the incident wave from the left,

\[ \psi(k, x) = \begin{cases} 1 e^{ikx} + \rho e^{-ikx}, & x < 0 \\ \tau e^{ikx}, & x \geq 0. \end{cases} \]  
(4.15)

Here we see the utility of working directly with a matrix of column eigenvectors (as opposed to working with individual column vectors); \( \rho \) and \( \tau \) may be solved for directly in terms of matrix operations. The scattering amplitudes are

\[ \rho(k) = (2ik - \lambda)^{-1} \lambda, \]  
(4.16)
\[ \tau(k) = 1 + \rho(k) = 2ik(2ik - \lambda)^{-1}. \]  
(4.17)

Of particular interest to us, due to its connection with the version of Levinson’s theorem given in Eq. (2.43), is the quantity \( \text{Tr}[\rho(0) + \tilde{\rho}(0)] \). If \( \lambda^{-1} \) exists, then \( \rho(0) = -1 \) and \( \text{Tr}[\rho(0) + \tilde{\rho}(0)] = 2\text{Tr}[\rho(0) = -2N \). As expected. If \( \lambda \) is not invertible, however, we must be a bit more careful.

In order to determine the significance of the noninvertibility of \( \lambda \), consider the bound-state solutions of Eq. (4.14). Setting \( \alpha^2 = -k^2 \) and insisting that \( \alpha \geq 0 \), we find that the column eigenvector for the bound state is

\[ \Psi_\beta(\alpha, x) = \begin{cases} A e^{\alpha x}, & x \leq 0 \\ A e^{-\alpha x}, & x \geq 0. \end{cases} \]  
(4.18)

where \( A \) is a normalized column matrix. By integrating Eq. (1.14) over an infinitesimal interval including the origin, we obtain an expression between the derivatives of \( \Psi_\beta(\alpha, x) \) on both sides of the origin, which leads to the relation

\[ (2\alpha + \lambda) A = 0. \]  
(4.19)

In order to avoid the trivial solution, we demand that

\[ \text{det}(2\alpha + \lambda) = 0. \]  
(4.20)

The non-negative values of \( \alpha \) which solve the above equation define the bound-state energies. Clearly there is at least one half-bound state if \( \det \lambda = 0 \).

Returning to the scattering problem, we find that the easiest way to proceed is to first diagonalize the matrix \( \lambda \). Since \( \lambda \) is real and symmetric, the diagonalization can be accomplished by using an orthogonal matrix \( U \), so that

\[ \lambda_D = U\lambda U^{-1}, \]  
(4.21)

where \( \lambda_D \) is diagonal and \( U^T = U^{-1} \). If we now define

\[ \psi_D \equiv U\psi U^{-1}, \]  
(4.22)

we see that Eq. (4.14) may be rewritten as

\[ \left( -\frac{d^2}{dx^2} + \delta(x)\lambda_D \right) \psi_D = \tilde{k}^2 \psi_D. \]  
(4.23)

The orthogonal transformation similarly transforms the boundary conditions, Eq. (4.15), to

\[ \psi_D(k, x) = \begin{cases} 1 e^{ikx} + \rho_D(k)e^{-ikx}, & x \leq 0 \\ \tau_D(k)e^{ikx}, & x \geq 0. \end{cases} \]  
(4.24)

We see here another advantage of working directly with square matrices. If we had been working with column vector wave functions, the transformed wave functions would have been given by \( U\Psi \), so that the normalization of the incoming wave would in general have been changed. Working with \( N \times N \) wave-function matrices gives the above result that the form of the boundary conditions is unchanged under the transformation. In fact the transformed wave function is itself a diagonal matrix, and we essentially have \( N \) decoupled copies of the problem with no coupling, with (possibly) different potential strengths.

Suppose now that \( \det \lambda = 0 \). Then it follows that \( \lambda \) has at least one zero eigenvalue. Let us again suppose that there are in fact \( n \) zero eigenvalues, so that

\[ \lambda_D = \text{diag}(0, \ldots, 0, \lambda_{n+1}, \ldots, \lambda_N), \]  
(4.25)

\[ \text{Note that this same trick can be employed any time the potential is of the form } V(x) = v(x)M, \text{ where } M \text{ is a real symmetric matrix. Diagonalizing } M \text{ gives } N \text{ decoupled systems with potentials } V_i(x) = m_i v(x), i = 1, \ldots, N, \text{ where the } m_i \text{ are the eigenvalues of the matrix } M. \]
where the $\lambda_i$, $i = n + 1, \ldots, N$, are the remaining (nonzero) eigenvalues. Then the diagonalized reflection and transmission amplitude matrices are given by

$$
\rho_D(k) = \text{diag} \left( 0, \ldots, 0, \frac{\lambda_{n+1}}{2ik - \lambda_{n+1}}, \ldots, \frac{\lambda_N}{2ik - \lambda_N} \right)
$$

(4.26)

and

$$
\tau_D(k) = \text{diag} \left( 1, \ldots, 1, \frac{2ik}{2ik - \lambda_{n+1}}, \ldots, \frac{2ik}{2ik - \lambda_N} \right),
$$

(4.27)

so that

$$
\text{Tr}[\rho(0) + \bar{\rho}(0)] = 2\text{Tr}[\rho_D(0)] = -2(N - n).
$$

(4.28)

Thus we see in this example how the trace of $\rho(0) + \bar{\rho}(0)$ keeps track of the number of half-bound states in the system. In fact it is easy to verify that Levinson’s theorem holds for the coupled system, since it holds separately for each decoupled equation of the diagonalized problem.

It is instructive to consider the relation

$$
2ik\tau_D^{-1}(k) = \text{diag}(2ik, \ldots, 2ik, 2ik - \lambda_{n+1}, \ldots, 2ik - \lambda_N),
$$

(4.29)

which follows from Eq. (4.27). It demonstrates for this model that in the limit as $k \to 0$ the expression Eq. (2.63) is real, as expected.

2. Potential with two delta functions

We now turn to a slightly more complicated example, in which there are two delta-function matrix potentials, one at $x = a$ and the other at $x = -a$. The $N = 1$ version of this model was studied by Senn [4]. The Schrödinger equation for this case is given by

$$
\left( -\frac{d^2}{dx^2} + \delta(x + a)\lambda + \delta(x - a)\bar{\lambda} \right) \psi = k^2\psi,
$$

(4.30)

with boundary conditions

$$
\psi(k, x) = \begin{cases} 1e^{ikx} + \rho(k)e^{-ikx}, & x \leq -a \\ \tau(k)e^{ikx}, & x \geq a, \end{cases}
$$

(4.31)

for the wave incident from the left and

$$
\bar{\psi}(k, x) = \begin{cases} \bar{\tau}(k)e^{-ikx}, & x \leq -a \\ 1e^{-ikx} + \bar{\rho}(k)e^{ikx}, & x \geq a, \end{cases}
$$

(4.32)

for the wave incident from the right. Rather than solve the Schrödinger equation again, we may now simply substitute the results of the previous section into the factorization formula, Eq. (3.5).

An evaluation of the resulting expressions yields

$$
\rho(k) = \left( \lambda e^{-4ika} + (2ik + \lambda)(2ik - \lambda)^{-1}\bar{\lambda} \right) \Gamma^{-1}(k; a; \lambda, \bar{\lambda})^{-1},
$$

(4.33)

$$
\tau(k) = -4k^2e^{-2ika}(2ik - \lambda)^{-1}\Gamma^{-1}(k; a; \lambda, \bar{\lambda})^{-1},
$$

(4.34)

$$
\bar{\rho}(k) = (2ik - \lambda)^{-1}\Gamma^{-1}(k; a; \lambda, \bar{\lambda})^{-1}(2ik - \lambda)^{-1}\lambda e^{-4ika} + 2ik + \bar{\lambda},
$$

(4.35)

$$
\bar{\tau}(k) = \tau^T(k),
$$

(4.36)

where

$$
\Gamma(k; a; \lambda, \bar{\lambda}) = (2ik - \lambda)^{-1}e^{-2ika} - (2ik - \bar{\lambda})^{-1}\lambda e^{2ika}.
$$

(4.37)

Let us assume for the moment that both $\lambda^{-1}$ and $\bar{\lambda}^{-1}$ exist so that the above expressions are well defined. (For $k > 0$, it is actually sufficient that only one or the other exists – it is possible to rewrite the expressions so that they contain only $\lambda^{-1}$ and not $\lambda^{-1}$.) Performing a Taylor expansion of $\rho$ and $\bar{\rho}$ for small $k$, we find that in the typical case $\rho(0) = \bar{\rho}(0) = -1$, so that $\text{Tr}[\rho(0) + \bar{\rho}(0)] = -2N$, as expected. The atypical case is defined by the condition $\det(\lambda^{-1} + \bar{\lambda}^{-1} + 2a) = 0$, which, as we shall see, is also the condition for a half-bound state.

Let us then work out the bound-state condition. This may be done in a manner similar to that for the single delta-function case to obtain

$$
\det \Gamma(ia; a; \lambda, \bar{\lambda}) = 0.
$$

(4.38)

Solutions of Eq. (4.38) with $\alpha > 0$ correspond to bound states. As $\alpha \to 0$, Eq. (4.38) yields the half-bound-state condition,

$$
\det(\lambda^{-1} + \bar{\lambda}^{-1} + 2a) = 0.
$$

(4.39)

Alternatively, if we employ the wave functions $\phi$ and $\chi$ of Sec. II C for the model potential and use Eq. (2.53) as the condition for the bound state, we obtain the equation

$$
\det \left( \left( \lambda + 2\alpha (\lambda + 2\alpha)(\lambda + 2\alpha) - \lambda \bar{\lambda} e^{-4\alpha a} \right) / 2\alpha \right) = 0,
$$

(4.40)

which in the limit as $\alpha$ approaches zero reduces to

\[\text{Some care must be taken with the reflection amplitude matrices, for they acquire a phase when the potential is translated. The transmission amplitude matrices are, however, unchanged.}\]
\[ \det(\lambda + \tilde{\lambda} + 2a\lambda\tilde{\lambda}) = 0. \quad (4.41) \]

This equation is preferred over Eq. (4.39) since it is not artificially singular when one of the inverse matrices does not exist.

Let us now consider an explicit example with \( N = 2 \). Since one of the two matrices \( \lambda \) or \( \tilde{\lambda} \) may always be diagonalized by an orthogonal transformation, we will let \( \lambda \) be diagonal right from the start. An example which gives a half-bound state for \( a = 1 \) is one for which

\[ \lambda = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\lambda} = \begin{pmatrix} -6 & -2 \\ -2 & -1 \end{pmatrix}. \quad (4.42) \]

Fig. 1 shows a parametric plot of \( \rho_{11}(k) \) as a function of \( k \) in the complex plane for the cases \( a = 0.95, a = 1.00, \) and \( a = 1.05 \). In the two typical cases \( (a = 0.95, 1.05), \rho_{11}(0) = -1 \), while for the atypical case \( (a = 1), \rho_{11}(0) = 0.777\ldots \). This is then the analog of Senn’s “threshold anomaly” for the generalized matrix version of his model [4]. Examination of the other diagonal reflection amplitudes yields the expected result that \( \text{Tr}[\rho(0) + \tilde{\rho}(0)] \) is equal to \(-4\) in the typical case and \(-2\) in the case where one half-bound state exists.

The behaviour of \( \rho_{11}(k) \) as a function of \( k \) may strike the reader as being somewhat peculiar: for sufficiently large \( k \) as \( k \) increases, \( \rho_{11}(k) \) traces out a never-ending counter-clockwise spiral towards the origin. A plot of the argument of \( \rho_{11}(0) \) for the three cases would show that the phase shifts are not bounded – they keep on increasing to infinity. This peculiar feature does not exist when there is no coupling (the phase shifts are bounded due to Levinson’s theorem), but is a rather generic feature of \( N > 1 \) models. The important thing to bear in mind when \( N > 1 \) is that the phase shift which obeys Levinson’s theorem is defined as being proportional to the logarithm of the determinant of the \( S \) matrix. This phase shift can in general be a nontrivial function of the “physical” phase shifts associated with the scattering amplitudes.

**FIG. 1.** Plot of the \( \text{Im}(\rho_{11}(k)) \) versus the \( \text{Re}(\rho_{11}(k)) \) for \( k = 0 \) to \( k = 5 \) for double delta-function matrix potential. For the \( a = 1.05 \) case the curve reverses the direction of travel around the origin when \( k \approx 1.1 \).

The determinant function for the double delta-function potential graphed as a function of \( \alpha \).

**FIG. 2.** The determinant function for the double delta-function potential graphed as a function of \( \alpha \).

\[ f(\alpha, a; \lambda, \tilde{\lambda}) = \det \left( \left( \lambda + 2\alpha \right) \left( \tilde{\lambda} + 2\alpha \right) - \lambda\tilde{\lambda}e^{-4\alpha a} \right) / 2\alpha \], \quad (4.43)\]

as a function of \( \alpha \) for \( a \approx 1 \). The inset in this figure shows an expanded view of the function near the origin for the three cases \( a = 0.95, 1.00, \) and \( 1.05 \). In addition to the two regular bound states that all three cases possess (near \( \alpha = 0.5164 \) and \( \alpha = 3.3508 \)), the \( a = 1.05 \) case has an extra bound state near \( \alpha = 0.0259 \), and the \( a = 1.00 \) case has a new bound state just emerging at \( \alpha = 0 \).

Finally, Fig. 3 shows a plot of the “Levinson’s theorem” phase shift as a function of \( k \) for the three cases. Clearly this phase shift is well-behaved and is bounded. As the potential “strength” is adjusted so that the system goes through a half-bound state the phase shift at \( k = 0 \) jumps by \( \pi \) in two increments of \( \pi/2 \).
FIG. 3. The phase of the $S$ matrix (divided by $\pi$) of the double delta-function potential as a function of $k$. For $k$ larger than shown on the graph the three curves remain close to one another and approach zero as $k \to \infty$.

V. DISCUSSION

In this section we make a few observations. The problem of one-dimensional coupled-equation scattering using a representation of wave functions which have incoming waves from the left or the right is readily solvable. Despite the advantages of a “partial wave” representation [7,8] or a parity-eigenstate representation [2,6] for parity-invariant potential functions, our analysis (which is valid for any potential) is quite manageable.

The use of wave function matrices (see Refs. [16,28] for three-dimensional scattering and also [13,14] for one-dimensional scattering) rather than column-vector wave functions, leads to simplified notation for a number of relations, e.g., the closure relation, Eq. (2.30). One also finds that performing a unitary transformation on the scattering wave function matrix does not alter the normalization of the incoming waves, whereas it does for column wave functions. The introduction of the real matrix wave function solutions $\phi$ and $\chi$ has two distinct advantages. In the first place the Schrödinger equation for the scattering problem can be reduced to a system of first-order differential equations with one-point boundary conditions, Eq. (2.1). The scattering amplitudes (and the $S$ matrix) are algebraic expressions of these functions evaluated at $R$. Furthermore, unlike the solution matrices $\psi$ and $\tilde{\psi}$, matrices $\phi$ and $\chi$ have linearly independent columns at threshold and consequently are convenient for investigating threshold behaviour.

Our starting point with the wave functions $\psi$ and $\tilde{\psi}$, which gives the definition of the reflection amplitudes $\rho$ and $\tilde{\rho}$, yields a generalized and simplified understanding of threshold behaviour. Whereas previous work [2,4] indicates that for parity-noninvariant potentials the reflection amplitude at threshold can have noninteger values, unlike that for parity-invariant potentials, we find that $\text{Tr}[\rho(k) + \tilde{\rho}(k)]$ at threshold is always an integer (see Eq. (2.83)). The results of de Bianchi [2], however, already imply such a relation, as well as noninteger reflection amplitudes at threshold, for some parity-noninvariant potentials with no coupling.

Finally, the phases of the reflection and transmission amplitudes of the coupled system are not simple functions of $k$, as is the case for uncoupled scattering, for which these phases satisfy the appropriate form of Levinson’s theorem. The phase $\eta$ of the $S$ matrix, which appears in Levinson’s theorem, is in general a nontrivial function of the phases of the scattering amplitudes. Furthermore, the function $\eta(k)$ is bounded, unlike the phases of the scattering amplitudes which are not necessarily bounded.

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APPENDIX A: SOME PROPERTIES OF SOLUTIONS OF THE SCHRÖDINGER EQUATION

In this appendix we determine the condition for linear independence of the solutions of the Schrödinger equation. Consider the matrix Schrödinger equation

$$-\frac{d^2 \psi}{dx^2} + V(x)\psi = k^2 \psi,$$  \hspace{1cm} (A1)

where $\psi$ is the $N \times N$ solution matrix whose columns are solutions to Eq. (2.1). Suppose we have two such solution matrices, $f$ and $\tilde{f}$. We define a $2N \times 2N$ matrix functional

$$W(f, \tilde{f}) = \begin{pmatrix} f & \tilde{f} \\ f' & \tilde{f}' \end{pmatrix},$$  \hspace{1cm} (A2)

in which the prime indicates the derivative with respect to $x$. Since $f$ and $\tilde{f}$ satisfy Eq. (A1) the matrix $W$ is a solution of the matrix equation,

$$W' = FW,$$  \hspace{1cm} (A3)
where
\[
F = \begin{pmatrix}
0 & 1 \\
V - k^2 & 0
\end{pmatrix}.
\] (A4)

**Lemma 1** (det \(W\)' = 0 for all \(x \in (-\infty, \infty)\)).

**Proof:** Let us write \(f\) and \(\tilde{f}\) in terms of their \(N\)-component row vectors:
\[
f = \begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_N
\end{pmatrix} \quad \text{and} \quad \tilde{f} = \begin{pmatrix}
\tilde{f}_1 \\
\tilde{f}_2 \\
\vdots \\
\tilde{f}_N
\end{pmatrix}.
\] (A5)

We then obtain [page 96]
\[
(\text{det } W)' = \text{det } \begin{pmatrix}
f'_{\tilde{f}_1} \\
f_{\tilde{f}_1} \\
\vdots \\
f_{\tilde{f}_N}
\end{pmatrix} + \cdots + \text{det } \begin{pmatrix}
f'_{\tilde{f}_1} \\
f_{\tilde{f}_1} \\
\vdots \\
f_{\tilde{f}_N}
\end{pmatrix} = \begin{pmatrix}
f_1 \tilde{f}_1 \\
f_2 \tilde{f}_1 \\
\vdots \\
f_N \tilde{f}_1
\end{pmatrix} + \cdots + \begin{pmatrix}
f_1 \tilde{f}_1 \\
f_2 \tilde{f}_1 \\
\vdots \\
f_N \tilde{f}_1
\end{pmatrix}.
\] (A6)

The determinant of the matrix on the right side of Eq. (A9) is zero, and consequently the (det \(W\)' = 0. The \(W\) is a constant function of \(x\).

**Lemma 2** The solutions contained in the columns of \(f\) and \(\tilde{f}\) are linearly independent if and only if det \(W\) ≠ 0 for all \(x \in (-\infty, \infty)\).

**Proof:** First suppose that det \(W\) ≠ 0. We consider a linear combination of solutions which is equal to the trivial solution,
\[
f h + \tilde{f} \tilde{h} = 0 \quad \text{for all } x,
\] (A10)
where \(h\) and \(\tilde{h}\) are \(N\)-component column vectors of constants and \(o\) is the \(N\)-component zero column vector. A similar relation holds for the derivatives of \(f\) and \(\tilde{f}\), so that
\[
Wc = o,
\] (A11)
where
\[
c = \begin{pmatrix}
h \\
\tilde{h}
\end{pmatrix},
\] (A12)
and \(c\) and \(o\) are now 2\(N\)-component column vectors. If det \(W\) ≠ 0 for some \(x\), which according to the previous result means it is nonzero for all \(x\), \(c = o\) and the column solutions contained in \(f\) and \(\tilde{f}\) are linearly independent.

Suppose now that det \(W\) = 0. If det \(W\) = 0 for some \(x = x_0\), then the system of linear equations \(W(x_0)c = o\) has a nontrivial solution \(c\). We form a column solution of the system of differential equations, Eq. (A13), \(w(x) = W(x)c\) which vanishes at \(x_0\). This is the trivial solution of Eq. (A13); \(w(x) = o\) for all \(x\). It follows that the column solutions contained in \(f\) and \(\tilde{f}\) are linearly dependent.

**APPENDIX B: ANALYTIC PROPERTIES OF THE SOLUTION MATRICES \(\phi\) AND \(\chi\)**

We consider solution matrices \(\phi(k, x)\) and \(\chi(k, x)\) of Eq. (A1) with boundary conditions \(\phi(k, -R) = \)
\( \chi'(k, -R) = 1 \) and \( \phi'(k, -R) = \chi(k, -R) = 0 \), where \( R \) is the range of the potential. According to a theorem of Poincaré an ordinary differential equation containing an entire function of some parameter has solutions which are entire functions of the parameter provided these solutions have boundary conditions which are independent of the parameter. We will show that \( \phi \) and \( \chi \) are entire functions of \( k \), following a similar derivation for partial-wave solutions in three-dimensional scattering [28,29].

It is straightforward to verify that the matrix functions \( \phi \) and \( \chi \) with the given boundary conditions are solutions of integral equations of the Volterra type,

\[
\phi(k, x) = 1 \cos k(x + R) + \int_{-R}^{x} dx' \frac{\sin k(x - x')}{k} V(x') \phi(k, x') \quad (B1)
\]

\[
\chi(k, x) = \frac{1}{2} \sin k(x + R) + \int_{-R}^{x} dx' \frac{\sin k(x - x')}{k} V(x') \chi(k, x') \quad (B2)
\]

In order to show that each element of the solution matrices is an entire function of \( k \), we rewrite Eq. \( \text{(B1)} \) in the form

\[
\phi(k, x) = 1 \cos k(x + R) + \int_{-R}^{x} dx' \int_{0}^{x-x'} dt \cos kt V(x') \phi(k, x') \quad (B3)
\]

We solve Eq. \( \text{(B3)} \) by successive approximations of the form

\[
\phi(k, x) = \sum_{s=0}^{\infty} \phi^{(s)}(k, x), \quad (B4)
\]

where

\[
\phi^{(s)}(k, x) = \int_{-R}^{x} dx' \int_{0}^{x-x'} dt \cos kt V(x') \phi^{(s-1)}(k, x'), \quad \text{for } s \geq 1 \text{ and } \phi^{(0)}(k, x) = 1 \cos k(x + R). \quad (B5)
\]

Thus

\[
|\phi_{ij}^{(s)}(k, x)| \leq \int_{-R}^{x} dx' \int_{0}^{x-x'} dt \cos kt |V_{ij}(x')||\phi_{ij}^{(s-1)}(k, x')|, \quad s \geq 1 \quad (B6)
\]

and

\[
|\phi_{ij}^{(0)}(k, x)| = \delta_{ij} |\cos k(x + R)|. \quad (B7)
\]

Denoting \( 3k \) for the imaginary part of \( k \) and using the relation |cos \( kt \)| \( \leq \cosh 3kt \) for \( t \) real, we obtain upon iteration

\[
|\phi_{ij}^{(s)}(k, x)| \leq \sum_{i'j'} |\phi_{ij}^{(s)}(k, x)|
\]

\[
\leq \left( \frac{\sinh(23kR)}{3k} \right)^s \cosh(23kR) \int_{-R}^{x} dx_1 \int_{-R}^{x_1} dx_2 \cdots \int_{-R}^{x_{s-1}} dx_s \sum_{i',j',l_1,\ldots,l_s} |V_{i'j'_l_1}(x_1)||V_{i'j'_l_2}(x_2)| \cdots |V_{i'j'_l_s}(x_s)|. \quad (B8)
\]

Since the integrand is a symmetric function under the interchange of any pair \((x_i, x_j)\),

\[
|\phi_{ij}^{(s)}(k, x)| \leq \left( \frac{\sinh(23kR)}{3k} \right)^s \cosh(23kR) \frac{1}{s!} \sum_{i',j',l_1,\ldots,l_s} \int_{-R}^{x} dx_1 \int_{-R}^{x} dx_s |V_{i'j'_l_s}(x_s)|. \quad (B9)
\]

Let

\[
M_0 = \max_{i,j} \int_{-R}^{R} dx' |V_{ij}(x')| < \infty. \quad (B10)
\]

Then

\[
|\phi_{ij}^{(s)}(k, x)| \leq \left( \frac{\sinh(23kR)}{3k} \right)^s \cosh(23kR) \frac{N^{s+2}}{s!} M_0^s. \quad (B11)
\]

Thus the series \( \sum_s \phi_{ij}^{(s)} \) converges absolutely and uniformly for \( x \in [-R, R] \) and for every region in the complex \( k \) plane. To determine the existence of \( \frac{\partial \phi}{\partial k}(k, x) \), we differentiate Eq. \( \text{(B3)} \) with respect to \( k \),

\[
\frac{\partial \phi}{\partial k}(k, x) = -(x + R)1 \sin k(x + R)
\]

\[
- \int_{R}^{x} dx' \int_{0}^{x-x'} dt \sin kt V(x') \phi(k, x')
\]

\[
+ \int_{R}^{x} dx' \int_{0}^{x-x'} dt \cos kt V(x') \frac{\partial \phi}{\partial k}(k, x'). \quad (B12)
\]

When Eq. \( \text{(B4)} \) is differentiated with respect to \( k \) it yields

\[
\frac{\partial \phi}{\partial k}(k, x) = \sum_{s=0}^{\infty} \frac{\partial \phi^{(s)}(k, x)}{\partial k}, \quad (B13)
\]

where now we have

\[
\frac{\partial \phi^{(s)}(k, x)}{\partial k} = \int_{R}^{x} dx' \int_{0}^{x-x'} dt \cos kt V(x') \frac{\partial \phi^{(s-1)}(k, x')}{\partial k}. \quad (B14)
\]

with

\[
\frac{\partial \phi^{(0)}(k, x)}{\partial k} = -(x + R)1 \sin k(x + R)
\]

\[
- \int_{R}^{x} dx' \int_{0}^{x-x'} dt \sin kt V(x') \phi(k, x'). \quad (B15)
\]
It is not difficult to show that \( \frac{\partial \phi}{\partial k}(k, x) \) is bounded, and the convergence of series (B13) follows in the same manner as that of \( \phi(k, x) \). Since \( \phi(k, x) \) and its derivative with respect to \( k \) exist for all \( k \), \( \phi(k, x) \) is an entire function of \( k \). Similarly \( \chi(k, x) \) can be shown to be an entire function of \( k \).

For real \( k \) the behaviour of \( \phi(k, x) \) and \( \chi(k, x) \) as \( k \) becomes very large can be determined by iterating Eqs. (B1) and (B2). Thus

\[
\phi(k, x) \sim_{k \to \infty} \frac{1}{k} \cos k(x + R) + \frac{1}{k} \int_{-R}^{x} dx' \sin k(x - x')V(x') \cos k(x' + R) + O(1/k^2)
\]

(B16)

and

\[
\chi(k, x) \sim_{k \to \infty} \frac{1}{k^2} \sin k(x + R) \frac{1}{k} + \frac{1}{k^2} \int_{-R}^{x} dx' \sin k(x - x')V(x') \sin k(x' + R) + O(1/k^3).
\]

(B17)

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