A numerical method for the solution of the two-phase fractional Lamé-Clapeyron-Stefan problem

M. Blasik

Institute of Mathematics, Czestochowa University of Technology
Armii Krajowej 21, 42-201 Czestochowa, Poland
marek.blasik@gmail.com

Abstract

In this paper we present a numerical solution of a two-phase fractional Stefan problem with time derivative described in the Caputo sense. In the proposed algorithm, we use a special case of front-fixing method supplemented by the iterative procedure, which allows us to determine the position of the moving boundary. In the final part, we also present some examples illustrating the comparison of the analytical solution with the results received by the proposed numerical method.

Keywords— moving boundary problems, fractional derivatives and integrals, Stefan problems, phase changes, finite difference methods

1 Introduction

Moving boundary problems are one of the most important area within partial differential equations. This particular kind of boundary-value problem was originally intended to describe the solid-liquid phase change process, but also refers to such phenomena as solute transport, molecular diffusion or controlled drug release [1,2,3]. Moving boundary problems, as the name implies, are characterized by having a moving boundary of the domain, which has to be determined as a part of the solution. These kinds of problems are also called Stefan problems, in connection with the early work of Slovene scientist Joseph Stefan, who investigated ice formation in the polar Arctic seas [4]. In recent years, the classical Stefan problem has been very well studied and described in many papers and monographs (compare [5,6,7,8,9] and the references therein).

The fractional Stefan problem is a natural generalization of the classical Stefan problem. The first paper devoted to this issue was published in 2004 and concerned the mathematical modeling of the controlled release of a drug from slab matrices [10]. Recently, there has been an increase in the number of scientific publications concerning the moving boundary problems modeled by the anomalous diffusion equation. Basically, published papers deal with three classes of phenomena. As mentioned earlier, the first concerned the controlled release of a drug from slab matrices [11,12]. A second class of problems relates to mathematical modeling of the thermal conductivity with phase transitions [13,14,15,16] and the last class refers to mathematical modeling a movement of the shoreline in a sedimentary ocean basin [17,18].

Most of the analytical solutions of fractional Stefan problem were obtained for the one-phase case. Liu Junyi and Xu Mingyu [10] studied the one-phase Stefan problem with fractional anomalous diffusion (Riemann-Liouville derivative with respect to time variable was used) and got an exact solution (concentration of the drug in the matrix)
in terms of the Wright’s function. They also showed that position of the penetration of solvent at time $t$ moving as $\sim t^{\frac{\alpha}{\beta} + 1}$, $0 < \alpha \leq 1$. Xicheng Li et al. [19] considered one-phase fractional Stefan problem with Caputo derivative with respect to time and two types of space-fractional derivatives (Caputo and Riemann-Liouville). They got the solution in terms of the generalized Wright’s function. It should be noted that in both cases deliberated by the authors, function describing the moving boundary is $\sim t^{\frac{\alpha}{\beta}}, 0 < \alpha \leq 1, 1 < \beta \leq 2$, where $\alpha, \beta$ denotes the orders of the fractional derivatives with respect to time and spatial variable respectively. In the paper [14] Voller analytically solved a limit case fractional Stefan problem describing the melting process. For the governing equation with a Caputo derivative with respect to time of order $0 < \beta \leq 1$ and for the same fractional derivative with respect to space for the flux of order $0 < \alpha \leq 1$, he showed that a melting front is described by power function $t^{\frac{\beta}{\alpha} + 1}$. More analytical results were published in the papers [20, 21, 22].

An important result with respect to this paper is the closed analytical solution of the two-phase fractional Lamé-Clapeyron-Stefan problem in a semi-infinite region obtained by Roscani and Tarzia [23], which will be recalled in Section 3. The problem involves determination of three functions, namely, $u_1, u_2$ and $S$ fulfilling two subdiffusion equations [13, 14] and additional differential equation (interface energy balance condition) [15] governing function $S$. The authors showed that $u_1$ and $u_2$ can be expressed in terms of the Wright’s function, moreover location of the phase-change interface is described by power function $t^{\frac{\beta}{\alpha} + 1}$, where parameter $p$ can be evaluated form transcendental equation (22). This solution will be especially useful for validation of the numerical method proposed in Section 4.

An alternative to the closed analytical solutions are those received by numerical methods. There are many techniques of numerical solving of classical Stefan problem, some of them have been generalized to the case of the fractional order. Xiaolong Gao et al. [24] generalized the boundary immobilisation technique (also known as front-fixing method [5]) to the fractional Stefan problem with a space-fractional derivative. Another variant of the front-fixing method was proposed in paper [25] and applied to the fractional Stefan problem with time-fractional derivative.

Our aim is to develop the numerical method of solving the two-phase, one-dimensional fractional Stefan problem. The proposed numerical scheme is an extension of the front-fixing method [25] to the two-phase problem. Our new approach is based on the suitable selection of the new space coordinates. The original coordinate system $(x, \tau)$ is transformed into a two new orthogonal systems $(v_1, \tau)$ and $(v_2, \tau)$ using transformation (27) and (43), respectively. Both new spatial variables $v_1, v_2$ depend on the parameter $p$ which is unknown and chosen a priori. The proposed numerical method uses integro-differential equations equivalent to the corresponding governing differential equations of the problem. The solutions of the integro-differential equations are obtained separately for each phase and fulfill the interface energy balance condition (the moving boundary is fixed ) only when the value of parameter $p$ is correct. Selection of the appropriate value of parameter $p$ is implemented by iterative algorithm on the basis of the fractional Stefan condition.

The paper is organized as follows. In the next Section, we introduce definitions of the fractional integrals and derivatives together with some of their properties. In Section 3, we formulate a mathematical model describing the melting process and recall closed analytical similarity solution for two-phase fractional Stefan problem. Section 4 is devoted to the new numerical method of solving of two-phase fractional Stefan problem, which is an extension of the method developed in the paper [25] to the two-phase case. Section 5 contains the analytical and numerical results. The last Section includes a summary of the paper and conclusions.
2 Preliminaries

Let us recall the basic definitions and properties from fractional calculus [26, 27] which will be further applied to formulate and solve the two-phase fractional Stefan problem modeled by two linear equations with Caputo time derivative of order $\alpha \in (0, 1]$. First, we define the left-sided Riemann-Liouville integral.

Definition 1 The left-sided Riemann-Liouville integral of order $\alpha$, denoted as $I_{0+}^{\alpha}$, is given by the following formula for $\Re(\alpha) > 0$:

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{0}^{t} f(u) (t-u)^{1-\alpha} du,$$  \hspace{1cm} (1)

where $\Gamma$ is the Euler gamma function.

The left-sided Caputo derivative of order $\alpha \in (0, 1)$ denoted by $^{c}D_{0+}^{\alpha}$ is defined via the above left-sided Riemann-Liouville integral $^{c}D_{0+}^{\alpha} f(t) := I_{0+}^{1-\alpha} f'(t)$.

Definition 2 Let $\Re(\alpha) \in (0, 1]$. The left-sided Caputo derivative of order $\alpha$ is given by the formula:

$$^{c}D_{0+}^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} f'(u) (t-u)^{\alpha-1} du, & 0 < \alpha < 1, \\ \frac{df(t)}{dt}, & \alpha = 1. \end{cases}$$  \hspace{1cm} (2)

An important property of fractional integral operators given in Definition 1 is called a semigroup property, which allows composition of two left-sided fractional integrals.

Property 3 (cf. Lemma 2.3 [26]) If $\Re(\alpha) > 0$, and $\Re(\beta) > 0$, then the equation

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t)$$  \hspace{1cm} (3)

is satisfied at almost every point $t \in [0, b]$ for $f(t) \in L_{p}(0, b)$ where $1 \leq p \leq \infty$. If $\alpha + \beta > 1$, then the above relation holds at any point of $[0, b]$.

The composition rule of the left-sided Riemann-Liouville integral with the left-sided Caputo derivative is a consequence of the semigroup property for fractional integrals.

Property 4 (cf. Lemma 2.22, [26]) Let function $f \in C^{1}(0, b)$. Then, the composition rule for the left-sided Riemann-Liouville integral and the left-sided Caputo derivative is given as follows:

$$I_{0+}^{\alpha} {^{c}D_{0+}^{\beta}} f(t) = f(t) - f(0).$$  \hspace{1cm} (4)

The two-parameter Wright function determined in a complex plane plays important role in the theory of partial differential equations of fractional order. It is a generalization of the complementary error function.

Definition 5 Let $\gamma > -1$, $\delta \in \mathbb{C}$, $z \in \mathbb{C}$. The two-parameter Wright function is given as the following series:

$$W(z; \gamma, \delta) := \sum_{k=0}^{\infty} \frac{z^{k}}{k! \Gamma(\gamma k + \delta)}.$$  \hspace{1cm} (5)

We note that for $\gamma = -\frac{1}{2}$, $\delta = 1$ the above two-parameter Wright function becomes complementary error function [28]:

$$W\left(-z; -\frac{1}{2}, 1\right) = \text{erfc}\left(\frac{z}{2}\right).$$  \hspace{1cm} (6)

For $\gamma = -\frac{1}{2}$, $\delta = -\frac{1}{2}$ two-parameter Wright function can be expressed by the formula [29]:

$$W\left(-z; -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^{2}}{4}\right).$$  \hspace{1cm} (7)
3 Mathematical formulation of the problem

The two-phase fractional Stefan problem is a mathematical model describing the solidification and melting process. The mathematical formulation of the problem involves an anomalous diffusion equation for the liquid and solid phases and a condition at the liquid-solid interface, called the fractional Stefan condition, which describes the position of the phase change front. At the moving boundary \( X = s(t) \), the temperature \( U \) is constant and equal to melting point \( U_s \). We are considering the melting of a semi-infinite, one dimensional slab occupying \( X \geq 0 \), where the phase change interface moves from the heat source at a temperature \( U = U_0 \) at the boundary \( X = 0 \). A simple scheme of the model is shown in Figure 1.

Consider the following governing equations of the model

\[ c_1 \rho_1 c D_{0+}^\alpha U_1(X, t) = K_1 \frac{\partial^2 U_1(X, t)}{\partial X^2}, \quad 0 < X < s(t) \leq l_1, \quad t > 0, \tag{8} \]
\[ c_2 \rho_2 c D_{0+}^\alpha U_2(X, t) = K_2 \frac{\partial^2 U_2(X, t)}{\partial X^2}, \quad s(t) < X < \infty, \quad t > 0, \tag{9} \]

where \( U_1 \) denotes the temperature in the liquid phase, \( U_2 \) the temperature in the solid phase, \( s(t) \) represents the position of the moving boundary, \( K \) the modified thermal conductivity (has SI unit \([J \cdot s^{-\alpha} \cdot m^{-1} \cdot K^{-1}]\)), \( \rho \) the density, \( c \) the specific heat and subscripts 1 and 2 indicate liquid and solid phase, respectively. Equations (8,9) should be supplemented by Dirichlet conditions:

\[ U_1(0, t) = U_0, \quad U_1(s(t), t) = U_2(s(t), t) = U_s, \quad \lim_{X \to \infty} U_2(X, t) = U_\infty, \quad t > 0. \tag{10} \]

At \( t = 0 \) the liquid phase does not exist, so we use two initial conditions:

\[ U_2(X, 0) = U_\infty, \quad s(0) = 0. \tag{11} \]

The position of the moving boundary \( s(t) \) is determined by the fractional Stefan condition, which expresses the heat balance in the melting layer:

\[ L \rho_1 c D_{0+}^\alpha s(t) = K_2 \frac{\partial U_2(X, t)}{\partial X}\bigg|_{X=s(t)} - K_1 \frac{\partial U_1(X, t)}{\partial X}\bigg|_{X=s(t)}, \tag{12} \]
where $L$ is the latent heat. Let us note that the above mathematical model depends on eight parameters. To simplify of the studied problem, we introduce the following dimensionless variables

$$x = \frac{X}{l_1}, \quad \tau = t \left( \frac{K_0}{\epsilon_0 \rho_1 l_1^2} \right)^{\frac{1}{\alpha}}, \quad u_1 = \frac{U_1 - U_s}{U_0 - U_s}, \quad u_2 = \frac{U_2 - U_s}{U_0 - U_s}, \quad S = \frac{s}{l_1},$$

which reduces the number of free parameters to five and makes it possible study their impact on the solution.

We can rewrite governing equations (13) in a non-dimensional form

$$^cD_{0+}^{\alpha+} u_1(x, \tau) = \kappa_1 \frac{\partial^2 u_1(x, \tau)}{\partial x^2}, \quad 0 < x < S(\tau) \leq 1, \quad \tau > 0, \quad (13)$$

$$^cD_{0+}^{\alpha+} u_2(x, \tau) = \kappa_2 \frac{\partial^2 u_2(x, \tau)}{\partial x^2}, \quad S(\tau) < x < \infty, \quad \tau > 0, \quad (14)$$

supplemented with the boundary conditions

$$u_1(0, \tau) = 1, \quad u_1(S(\tau), \tau) = u_2(S(\tau), \tau) = 0, \quad \lim_{x \to \infty} u_2(x, \tau) = \frac{U_\infty - U_s}{U_0 - U_s}, \quad \tau > 0, \quad (15)$$

initial conditions

$$u_2(x, 0) = \frac{U_\infty - U_s}{U_0 - U_s}, \quad (16)$$

$$S(0) = 0, \quad (17)$$

and the fractional Stefan condition

$$^cD_{0+}^{\alpha+} S(\tau) = \lambda_2 \frac{\partial u_2(x, \tau)}{\partial x} \bigg|_{x=S(\tau)} - \lambda_1 \frac{\partial u_1(x, \tau)}{\partial x} \bigg|_{x=S(\tau)}, \quad (18)$$

where $\kappa_1 = (K_1/K_0)(c_0/c_1)$, $\kappa_2 = (K_2/K_0)(c_0/c_2)$, $\lambda_1 = (U_0 - U_s)K_1c_0/LK_0$, $\lambda_2 = (U_0 - U_s)K_2c_0/LK_0$, are standard values of the respective variables.

According to results obtained by Roscani and Tarzia [23] the closed analytical solution of the two-phase fractional Stefan problem (13)-(18) is given by the functions

$$u_1(x, \tau) = 1 - \frac{W \left( \frac{-x}{\sqrt{\kappa_1} \tau ^{\alpha/2}} : -\frac{\alpha}{2}, 1 \right)}{W \left( \frac{-p}{\sqrt{\kappa_1}} : -\frac{\alpha}{2}, 1 \right)} - 1, \quad (19)$$

$$u_2(x, \tau) = \frac{U_\infty - U_s}{U_0 - U_s} \frac{W \left( \frac{-p}{\sqrt{\kappa_2}} : -\frac{\alpha}{2}, 1 \right) - W \left( \frac{-x}{\sqrt{\kappa_2} \tau ^{\alpha/2}} : -\frac{\alpha}{2}, 1 \right)}{W \left( \frac{-p}{\sqrt{\kappa_2}} : -\frac{\alpha}{2}, 1 \right)}, \quad (20)$$

$$S(\tau) = p \tau ^{\alpha/2}, \quad (21)$$

$$\frac{\Gamma(1 + \frac{\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2})} = \lambda_2 \frac{U_\infty - U_s}{U_0 - U_s} \frac{W \left( \frac{-p}{\sqrt{\kappa_2}} : -\frac{\alpha}{2}, 1 - \frac{\alpha}{2} \right)}{W \left( \frac{-p}{\sqrt{\kappa_2}} : -\frac{\alpha}{2}, 1 \right)} - \lambda_1 \frac{W \left( \frac{-p}{\sqrt{\kappa_1}} : -\frac{\alpha}{2}, 1 - \frac{\alpha}{2} \right)}{W \left( \frac{-p}{\sqrt{\kappa_1}} : -\frac{\alpha}{2}, 1 \right)} - 1. \quad (22)$$

It should be noted that the above solution reduces to the fractional one-phase problem [20, 21] for $u_2 \to 0$.

When $\alpha \to 1$, then from (19), (22) the following results of the classical two-phase Stefan problem are recovered:

$$u_1(x, \tau) = 1 - \frac{\text{erfc} \left( \frac{x}{2\sqrt{\kappa_1} \tau} \right) - 1}{\text{erfc} \left( \frac{p}{2\sqrt{\kappa_1}} \right) - 1}, \quad (23)$$
\[
  u_2(x, \tau) = \frac{U_\infty - U_s}{U_0 - U_s} \text{erfc} \left( \frac{p}{2\sqrt{\kappa_2}} \right) - \text{erfc} \left( \frac{x}{2\sqrt{\kappa_2}} \right), \\
  S(\tau) = p\sqrt{\tau},
\]

\[
  p = \lambda_2 \frac{U_\infty - U_s}{U_0 - U_s} \sqrt{\pi \kappa_2} \text{erfc} \left( \frac{p}{2\sqrt{\kappa_2}} \right) - \lambda_1 \frac{\exp \left( -\frac{p^2}{4\kappa_1} \right)}{\sqrt{\pi \kappa_1}} \left( \text{erfc} \left( \frac{p}{2\sqrt{\kappa_1}} \right) - 1 \right).
\]

4 Numerical solution

The numerical method proposed in the paper is an extension of the technique developed in [25] to the two-phase problem. Applying the finite difference method to the fractional Stefan problem, we encounter a number of difficulties related to the discretization of the fractional derivative with respect to the time variable. As we know, the Caputo derivative is a non-local operator, which is defined on an interval. Suppose the positions, that the moving boundary will reach at different times are known and are at uniform spaced intervals in the same time layer, see at the left side of Figure 2. For such a grid, the discretization of the Caputo derivative at some node with respect to time is very difficult because it requires values of function \(u_1\) (or \(u_2\)) for all previous times (where the spatial variable is fixed) in points, which do not overlap with the mesh nodes. Let us also notice one important fact regarding discretization of equation (13) leading to some ambiguity. The Caputo derivative requires for each point of the liquid phase their history from time \(\tau = 0\), but they do not exist before they reach the melting point. It seems that the only way to solve a mesh problem is a suitable choice of new space coordinates for equations (13) and (14).

![Figure 2: Meshes for \(\alpha = 1, p = 1, m_1 = m_2 = 5, n = 5, l_2/l_1 = 2\).](image)

Let us first consider the region occupied by the first phase bounded by \(x = 0\) and \(x = S(\tau)\), marked in blue in Figure 2. For the equation (13), it is convenient to replace the spatial variable with the following similarity variable [30]:

\[
v_1 = \frac{x}{S(\tau)} = \frac{x}{p\tau^{\alpha/2}},
\]
which has an important property, namely fixes the moving boundary at \( v_1 = 1 \) for all \( \tau \). We transform the first order derivative with respect to the spatial variable:

\[
\frac{\partial u_1(x, \tau)}{\partial x} = \frac{\partial v_1}{\partial x} \frac{\partial u_1(v_1, \tau)}{\partial v_1} = \frac{1}{p^2 \tau^\alpha} \frac{\partial u_1(v_1, \tau)}{\partial v_1}.
\] (28)

Subsequently, for the second order spatial derivative we have:

\[
\frac{\partial^2 u_1(x, \tau)}{\partial x^2} = \frac{1}{p^2 \tau^\alpha} \frac{\partial^2 u_1(v_1, \tau)}{\partial v_1^2}.
\] (29)

Respectively, for the first order partial derivative in time:

\[
\frac{\partial u_1(x, \tau)}{\partial \tau} = \frac{\partial u_1(v_1, \tau)}{\partial \tau} = \frac{\alpha v_1}{2 \tau} \frac{\partial u_1(v_1, \tau)}{\partial v_1}.
\] (30)

Caputo derivative of function \( u_1(v_1, \tau) \) with respect to the time variable can be expressed as follows:

\[
^{c}D_{0+, \tau}^\alpha u_1(v_1, \tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau - \xi)^{-\alpha} \frac{\partial}{\partial \xi} u_1(v_1, \xi) d\xi - \frac{\alpha v_1}{2 \Gamma(1-\alpha)} \int_0^\tau (\tau - \xi)^{-\alpha} \frac{1}{\xi} \frac{\partial}{\partial \xi} u_1(v_1, \xi) d\xi.
\] (31)

Using formulas (29) and (31) we write equation (13) in coordinate system \((v_1, \tau)\):

\[
\frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau - \xi)^{-\alpha} \frac{\partial}{\partial \xi} u_1(v_1, \xi) d\xi - \frac{\alpha v_1}{2 \Gamma(1-\alpha)} \int_0^\tau (\tau - \xi)^{-\alpha} \frac{1}{\xi} \frac{\partial}{\partial \xi} u_1(v_1, \xi) d\xi = \frac{\kappa_1}{p^2 \tau^\alpha} \frac{\partial^2 u_1(v_1, \tau)}{\partial v_1^2}.
\] (32)

We integrate the previous equation applying the left-sided Riemann-Liouville integral of order \( \alpha \in (0,1) \) and we get the following equation:

\[
I_{0+, \tau}^\alpha \frac{\partial}{\partial \tau} u_1(v_1, \tau) = \frac{\alpha v_1}{2} I_{0+, \tau}^\alpha (\frac{\partial}{\partial v_1} u_1(v_1, \tau)) = \frac{\kappa_1}{p^2 \Gamma(\alpha)} I_{0+, \tau}^\alpha (\frac{1}{\tau^\alpha} \frac{\partial^2 u_1(v_1, \tau)}{\partial v_1^2})
\] (33)

Finally, using Properties 3 and 4 to equation (33), we obtain an integro-differential equation in the form of:

\[
u_1(v_1, \tau) = u_1(v_1, 0) + \frac{\alpha v_1}{2} \int_0^\tau \frac{\partial}{\partial v_1} u_1(v_1, \xi) d\xi + \frac{\kappa_1}{p^2 \Gamma(\alpha)} \int_0^\tau \frac{1}{(\tau - \xi)^{1-\alpha}} \frac{\partial^2 u_1(v_1, \xi)}{\partial v_1^2} d\xi
\] (34)

The kernel of the second integral on the right-hand side of formula (34) causes some difficulties in deriving a numerical scheme. For this reason we are introducing an auxiliary function:

\[
\tilde{u}_1(v_1, \tau) = u_1(v_1, \tau)^{-\alpha},
\] (35)

which leads to the integro-differential equation:

\[
\tilde{u}_1(v_1, \tau)^\alpha = \tilde{u}_1(v_1, 0) \tau_0^\alpha + \frac{\alpha v_1}{2} \int_0^\tau \frac{\partial}{\partial v_1} \tilde{u}_1(v_1, \xi) d\xi + \frac{\kappa_1}{p^2 \Gamma(\alpha)} \int_0^\tau \frac{1}{(\tau - \xi)^{1-\alpha}} \frac{\partial^2 \tilde{u}_1(v_1, \xi)}{\partial v_1^2} d\xi,
\] (36)
supplemented with the boundary conditions

\[ \bar{u}_1(0, \tau) = \tau^{-\alpha}, \quad \bar{u}_1(1, \tau) = 0, \]

and initial condition

\[ \bar{u}_1(v_1, 0) = 0. \]

The region marked in blue in Figure 2 in the \((x, \tau)\) plane is transformed into the unit square (in the general case to the rectangle) using the transformation (27). Rectangular mesh created in this way consists of horizontal lines spaced \(\Delta x = 1/(np^2/\alpha)\) units apart and vertical lines spaced \(\Delta x = 1/m_1\) units apart. The points \(((v_1), \tau_j) = (i\Delta v_1, j\Delta \tau)\) where \(i = 0, \ldots, m_1\) and \(j = 0, \ldots, n\), of intersection of the horizontal and vertical lines are grid nodes. We denote the value of function \(\bar{u}_1\) at point \((i\Delta v_1, j\Delta \tau)\) by \((\bar{u}_1)_{i,j}\). At this stage, the value of parameter \(\tau_0\) is not known and chosen a priori. Construction of similarity variable (27) requires an additional assumption for variable \(\tau\). Let \(\tau_0\) be a very small positive number.

The method of discretization of the integro-differential equation (36) is described in detail in the paper [25] and leads to an implicit scheme:

\[
(\bar{u}_1)_{i+1,j} = (\bar{u}_1)_{i,j-1} - 2(\bar{u}_1)_{i,j} + (\bar{u}_1)_{i-1,j} + \sum_{j=1}^{k} q_{i,j}((\bar{u}_1)_{i+1,j} - (\bar{u}_1)_{i-1,j}),
\]

where

\[
q_{i,j} = \frac{\alpha i^{\alpha-1} \Delta \tau}{4},
\]

\[
\frac{(\Delta \tau)^{\alpha}}{m_1 \alpha^{(\alpha+1)}}\left(\frac{\alpha}{m_1 \alpha + 1}\right)(k^{\alpha+1} - (k - \alpha)(k + 1)^{\alpha}) \quad \text{for } j = 0
\]

\[
\frac{\alpha^{(\alpha+1)}}{m_1 \alpha + 1}\left((k - j + 2)^{\alpha+1} + (k - j)^{\alpha+1}ight) \quad \text{for } 1 \leq j \leq k
\]

\[
\frac{-2(k - j + 1)^{\alpha+1}}{m_1 \alpha + 1} \quad \text{for } j = k + 1
\]

Weights (39) are the result of application of the trapezoidal rule [31] to the last integral term in formula (36). Obtained numerical scheme can be written in the matrix form

\[ A(\bar{U}_1)_{k+1} = B, \]

where \(\bar{U}_1\) is a vector of unknown values of function \(\bar{u}_1\) at instant \(\tau_{k+1}\).
The elements of matrices $A$ and $B$ are defined as follows:

$$B = \begin{bmatrix}
  b_1 - a_{1,k+1}^1(\bar{u}_1)_{0,k+1} & b_2 & b_3 \\
  \vdots & \vdots & \vdots \\
  b_{m_1-2} - a_{m_1-1,k+1}^3(\bar{u}_1)_{m_1,k+1}
\end{bmatrix}_{(m_1-1) \times 1}$$

where

$$a_{i,k+1}^1 := -r_{k+1,i,k+1} + q_{i,k+1}^1,$$

$$a_{k+1}^2 := \tau_{k+1}^\alpha + 2r_{k+1,k+1}^1,$$

$$a_{i,k+1}^3 := -r_{k+1,i,k+1} - q_{i,k+1}^1,$$

$$b_i := (\bar{u}_1)_{0,0}^\alpha + \sum_{j=0}^k r_{j,k+1}^1((\bar{u}_1)_{i-1,j} - 2(\bar{u}_1)_{i,j} + (\bar{u}_1)_{i+1,j}) + \sum_{j=1}^k q_{i,j}^1((\bar{u}_1)_{i+1,j} - (\bar{u}_1)_{i-1,j}).$$

We recover the values of function $u_1$ in coordinate system $(x, \tau)$ by applying formulas

$$u_{1,i,j} = (\bar{u}_1)_{i,j}^\tau_0^\alpha, \quad (41)$$

$$x_{i,j} = (v_1)_i^{p\tau_0^\alpha/2}. \quad (42)$$

The region marked in red in Figure 2 in the $(x, \tau)$ plane is transformed into the unit square using the following transformation:

$$v_2 = \frac{x - p\tau_0^\alpha/2}{l_2/l_1 - p\tau_0^\alpha/2}, \quad (43)$$

which fixes the moving boundary at $v_2 = 0$ for all $\tau$. Let us note, that the mesh for the semi-infinite region contains an infinite number of nodes, which makes it impossible to perform any numerical calculations. Therefore, from a practical point of view the Dirichlet boundary condition in infinity (15) can be replaced by the following condition:

$$u_2(l_2/l_1, \tau) = \frac{U_\infty - U_s}{U_0 - U_s}, \quad (44)$$

where $l_2$ is a sufficiently large positive real number.

We transform the first order derivative with respect to the spatial variable:

$$\frac{\partial u_2(x, \tau)}{\partial x} = \frac{\partial v_2}{\partial x} \frac{\partial u_2(v_2, \tau)}{\partial v_2} = \frac{1}{l_2/l_1 - p\tau_0^\alpha/2} \frac{\partial u_2(v_2, \tau)}{\partial v_2}. \quad (45)$$

Consequently, for the second-order spatial derivative

$$\frac{\partial^2 u_2(x, \tau)}{\partial x^2} = \frac{1}{(l_2/l_1 - p\tau_0^\alpha/2)^2} \frac{\partial^2 u_2(v_2, \tau)}{\partial v_2^2}. \quad (46)$$

Respectively, for the first order partial derivative with respect to the time variable:

$$\frac{\partial u_2(x, \tau)}{\partial \tau} = \frac{\partial u_2(v_2, \tau)}{\partial \tau} - \frac{\alpha p(v_2 - 1)}{2\tau(p - l_2/l_1\tau_0^{-\alpha/2})} \frac{\partial u_2(v_2, \tau)}{\partial v_2}. \quad (47)$$
Caputo derivative of function \( u_1(v_1, \tau) \) with respect to the time variable can be expressed as follows:

\[
^cD_{0+}^{\alpha, \tau} u_2(v_2, \tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau - \xi)^{-\alpha} \frac{\partial u_2(v_2, \xi)}{\partial \xi} d\xi - \frac{\alpha p(v_2 - 1)}{2\Gamma(1-\alpha)} \int_0^\tau \frac{(\tau - \xi)^{-\alpha}}{\xi (p - l_2/l_1 \xi^{-\alpha/2})} \frac{\partial u_2(v_2, \xi)}{\partial v_2} d\xi.
\]

(48)

Using formulas (46) and (48) we can write equation (14) in the new coordinate system \((v_2, \tau)\):

\[
\frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{1}{(\tau - \xi)^{\alpha/2}} \frac{\partial u_2(v_2, \xi)}{\partial \xi} d\xi = \frac{\alpha p(v_2 - 1)}{2\Gamma(1-\alpha)} \int_0^\tau \frac{(\tau - \xi)^{-\alpha}}{\xi (p - l_2/l_1 \xi^{-\alpha/2})} \frac{\partial u_2(v_2, \xi)}{\partial v_2} d\xi = \frac{\kappa_2}{(l_2/l_1 - pr^{-\alpha/2})^2} \frac{\partial^2 u_2(v_2, \tau)}{\partial v_2^2}.
\]

(49)

We integrate both sites of equation (49) applying the left-sided Riemann-Liouville integral of order \( \alpha \in (0, 1) \):

\[
I_{0+}^\alpha I_{0+}^{\alpha, \tau} u_2(v_2, \tau) = \frac{\alpha p(v_2 - 1)}{2} I_{0+}^{\alpha, \alpha} I_{0+}^{\alpha, \alpha} \left( \frac{\partial}{\partial v_2} \frac{u_2(v_2, \tau)}{\tau (p - l_2/l_1 \tau^{-\alpha/2})} \right) = I_{0+}^\alpha \left( \frac{\kappa_2}{(l_2/l_1 - pr^{-\alpha/2})^2} \frac{\partial^2 u_2(v_2, \tau)}{\partial v_2^2} \right).
\]

(50)

Finally, using Properties 8 and 4 to equation (50), we obtain an integro-differential equation in the form of:

\[
u_2(v_2, \tau) = u_2(v_2, 0) + \frac{\alpha p(v_2 - 1)}{2} \int_0^\tau \frac{\partial}{\partial v_2} \frac{u_2(v_2, \xi)}{\xi (p - l_2/l_1 \xi^{-\alpha/2})} d\xi + \frac{\kappa_2}{\Gamma(\alpha)} \int_0^\tau \frac{(\tau - \xi)^{1-\alpha}}{(l_2/l_1 - p\xi^{-\alpha/2})^2} \frac{\partial^2 u_2(v_2, \xi)}{\partial v_2^2} d\xi.
\]

(51)

We introduce the second auxiliary function

\[
\tilde{u}_2(v_2, \tau) = \frac{u_2(v_2, \tau)}{(l_2/l_1 - pr^{-\alpha/2})^2},
\]

(52)

which leads to the integro-differential equation:

\[
\tilde{u}_2(v_2, \tau)(l_2/l_1 - pr^{-\alpha/2})^2 = \tilde{u}_2(v_2, 0)(l_2/l_1 - pr_0^{-\alpha/2})^2 + \frac{\alpha p(v_2 - 1)}{2} \int_0^\tau \frac{\partial}{\partial v_2} \left( \frac{p\xi^{\alpha-1} - l_2^{\alpha-1} - l_1^{\alpha-1}}{l_2^{\alpha-1} - l_1^{\alpha-1}} \right) \tilde{u}_2(v_2, \xi) d\xi + \frac{\kappa_2}{\Gamma(\alpha)} \int_0^\tau \frac{1}{\xi^{1-\alpha}} \frac{\partial^2 \tilde{u}_2(v_2, \xi)}{\partial v_2^2} d\xi,
\]

(53)

supplemented with the boundary conditions

\[
\tilde{u}_2(0, \tau) = 0, \quad \tilde{u}_2(1, \tau) = \frac{1}{(l_2/l_1 - pr_0^{-\alpha/2})^2} \frac{U_\infty - U_s}{U_0 - U_s},
\]

(54)

and initial condition

\[
\tilde{u}_2(v_2, 0) = \frac{1}{(l_2/l_1 - pr_0^{-\alpha/2})^2} \frac{U_\infty - U_s}{U_0 - U_s}.
\]

(55)
The integro-differential equation (53) is discretized in analogy to equation (36). For the second phase we also operate on a rectangular uniform grid consist of horizontal lines spaced $\Delta \tau = 1/(np^{2/\alpha})$ units apart and vertical lines spaced $\Delta v_2 = 1/m_2$ units apart. The points $((v_2)i, \tau_j) = (i\Delta v_2,j\Delta \tau)$ where $i = 0,...,m_2$ and $j = 0,...,n$, of intersection of the horizontal and vertical lines are grid nodes. At this stage of the calculation, the value (for both phases the same) of parameter $p$ is not known and chosen a priori.

The implicit numerical scheme for second phase is in the form of:

\[
\begin{align*}
(\bar{u}_2)_{i-1,k+1} - r^2_{k+1,k+1} + q^2_{i,k+1} + (\bar{u}_2)_{i,k+1}(l_2/l_1 - \alpha p^{2/\alpha}2) & = \\
+ 2r^2_{k+1,k+1} + (\bar{u}_2)_{i+1,k+1} - q^2_{i,k+1} & = (\bar{u}_1)_{i,0}(l_2/l_1 - \alpha p^{2/\alpha}2) \sum_{j=0}^{k} (\bar{u}_2)_{i-1,j}-2(\bar{u}_2)_{i,j} + \\
+ (\bar{u}_2)_{i+1,j} & \sum_{j=1}^{k} q^2_{i,j}((\bar{u}_2)_{i-1,j} - (\bar{u}_2)_{i-1,j}),
\end{align*}
\]

\[ r^2_{j,k+1} = \frac{c_{j,k+1}\kappa_2}{\Gamma(\alpha)(\Delta v_2)^2}, \quad q^2_{i,j} = \frac{\alpha p(i\Delta v_2 - 1)(\alpha p^{-1} - l_2/l_1 \alpha^{2/\alpha} - 1)\Delta \tau}{4\Delta v_2} \]  

The obtained numerical scheme can be written in the matrix form

\[ D(\bar{U}_2)_{k+1} = E, \]  

where $\bar{U}_2$ is a vector of unknown values of function $\bar{u}_2$ at instant $\tau_{k+1}$.

\[
D = \begin{bmatrix}
  & & & & & & & \\
- & & & & & & & \\
- & & & & & & & \\
- & & & & & & & \\
- & & & & & & & \\
- & & & & & & & \\
- & & & & & & & \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{bmatrix}_{(m_2-1) \times (m_2-1)}
\]

\[
E = \begin{bmatrix}
e_1 - d^1_{1,k+1}\bar{u}_2)_{0,k+1} \\
e_2 \\
e_3 \\
\vdots \\
e_i \\
\vdots \\
e_{m_2-1} - d^2_{m_2-1,k+1}\bar{u}_2)_{m_2,k+1} \\
\end{bmatrix}_{(m_2-1) \times 1}
\]
The elements of matrices $D$ and $E$ are defined as follows:

\[
\begin{align*}
d_{i,k+1}^1 & := -r_{k+1,k+1}^2 + q_{i,k+1}^2 \\
d_{k+1}^0 & := (l_2/l_1 - pr_k^{\alpha/2})^2 + 2r_{k+1,k+1}^2 \\
d_{i,k+1}^3 & := -r_{k+1,k+1}^2 + q_{i,k+1}^2 \\
e_i & := (\bar{u}_2)_{i,0}((l_2/l_1 - pr_k^{\alpha/2})^2 + \sum_{j=0}^{k} r_j^{2}(\bar{u}_2)_{i-1,j} - 2(\bar{u}_2)_{i,j} + (\bar{u}_2)_{i+1,j}) \\
& + \sum_{j=1}^{k} q_{i,j}((\bar{u}_2)_{i+1,j} - (\bar{u}_2)_{i-1,j}),
\end{align*}
\]

Applying the following formulas:

\[
\begin{align*}
(u_2)_{i,j} & = (\bar{u}_2)_{i,j}((l_2/l_1 - pr_j^{\alpha/2})^2, \\
x_{i,j} & = (v_2)_{i,0}((l_2/l_1 - pr_j^{\alpha/2}) + pr_j^{\alpha/2},
\end{align*}
\]

we recover the values of function $u_2$ in coordinate system $(x, \tau)$.

The presented numerical scheme allows us solve the governing equations [13], but only when the correct value of the parameter $p$ is known. We can determine it using the fractional Stefan condition. First, we integrate both sides of equation [18] applying the left-sided Riemann-Liouville integral of order $\alpha \in (0,1)$. Next, using Property 4 we obtain:

\[
S(\tau) - S(0) = \lambda_2 I_{0+}^\alpha \frac{\partial u_2(x, \tau)}{\partial x} \bigg|_{x=S(\tau)} - \lambda_1 I_{0+}^\alpha \frac{\partial u_1(x, \tau)}{\partial x} \bigg|_{x=S(\tau)} .
\]

Applying the difference quotient approximation for the first order derivative with respect to space variable, weights (59) and condition (17), we get:

\[
S(\tau_n) = \frac{\lambda_2}{\Gamma(\alpha)} \sum_{j=0}^{n} c_{j,n} \frac{(u_2)_{i,j} - (u_2)_{i,0,j}}{x_{i,j} - x_{i,0,j}} - \frac{\lambda_1}{\Gamma(\alpha)} \sum_{j=0}^{n} c_{j,n} \frac{(u_1)_{m,j} - (u_1)_{m-1,j}}{x_{m,j} - x_{m-1,j}}, j = 0, ..., n.
\]

Let us note that the value of function $S$ at the final time instant $\tau_n$ for the correct value of parameter $p$ should be equal to 1. From this relation the following criterion of convergence results in:

\[
|1 - S(\tau_n, p)| < \epsilon,
\]

where $\epsilon > 0$ is a certain arbitrarily small real number. We denote as $S(\tau_n, p)$ the value of $S(\tau_n)$ for a fixed value of parameter $p$. Below we present an iterative algorithm for determining the value of parameter $p$, based on the bisection method [25].

1. choose interval $[p_a, p_b]$ for parameter $p$, define the value of parameter $\epsilon$
2. calculate numerically the solution of equations (59) for $p_a$ and $p_b$
3. if one of the conditions:
   \[
   |1 - S(\tau_n, p_a)| < \epsilon \text{ or } |1 - S(\tau_n, p_b)| < \epsilon
   \]
   is fulfilled, then end the algorithm otherwise go to the next step
4. if the condition:
   \[
   (1 - S(\tau_n, p_a)) \cdot (1 - S(\tau_n, p_b)) < 0
   \]
   is fulfilled, then go to the next step, otherwise go to step 1
5. calculate $p_c := \frac{p_a + p_b}{2}$
6. calculate numerically the solution of equations (59) for $p_c$
7. if $|1 - S(\tau_n, p_c)| < \epsilon$
   is fulfilled, then end the algorithm, otherwise go to the next step.

8. if $(1 - S(\tau_n, p_a)) \cdot (1 - S(\tau_n, p_c)) > 0$
   then substitute $p_a := p_c$ and go to step 5 or
   if $(1 - S(\tau_n, p_b)) \cdot (1 - S(\tau_n, p_c)) > 0$
   then substitute $p_b := p_c$ and go to the step 5.

The discussed algorithm was used to determine the value of the parameter $p$, whose
values are presented in Table 2.

5 Numerical examples

In order to validate the results obtained by the proposed numerical method with the
analytical solution one, twelve computer simulations were performed. We assumed
four values of order of the Caputo derivative $\alpha \in \{0.25, 0.5, 0.75, 1\}$ and three sets
of physical parameter values. Due to the fact that the proposed method is iterative
(involves multiple solving of governing equations for different values of parameter $p$),
the following mesh parameters were adopted: $l_2/l_1 = 10$, $m_1 = 100$, $m_2 = 500$, $n = 400$.

Table 1 shows the values of coefficient $p$ obtained from transcendental equation (22)
depending on order of the Caputo derivative and three sets of physical parameters. The
results collected in Table 1 indicates, that for a fixed set of physical parameters, $p$ is an
increasing function of order $\alpha$. This means that for $\alpha$ values less than one the melting
process is slower than in the case of the classical Stefan problem.

Let us now analyze the impact of the physical parameters on the solution for the
fixed value of $\alpha$. Comparing the first and second row of the Table 1 we notice that $p$ is a
decreasing function of $\lambda_2$. This observation results directly from equation (18). For a
growing heat a flux in solid zone ($\lambda_2$ is increasing), heat balance in the melting layer is
preserved only when value of the Caputo derivative of function $S$ decreases. For $\alpha = 1$
fractional derivative of function $S$ can be interpreted as a velocity of movement of the
melting front. The opposite behavior of the melting process is observed by analyzing
the first and third row of Table 1 for the fixed value of $\alpha$ - coefficient $p$ is an increasing
function of parameter $\kappa_1$.

Results given in Table 2 were obtained by applying the criterion of convergence
formulated in (63) and the iterative algorithm discussed in the previous section.

Table 3 contains the values of the variable $\tau$ for which the liquid phase propagates
to the region bounded by $S(\tau) = 1$. The collected data leads to the following conclusion
- changing parameter $\lambda_2$ has a greater effect on process dynamics, than the changing
parameter $\kappa_1$ for the fixed value of $\alpha$.

In Figure 3, 5, 7 and 9 we present dimensionless temperature profiles. On each
graph we show the solutions obtained numerically (blue, yellow and green line) and
the corresponding analytical solutions (black lines) plotted for $x \in [0, 10]$. In two cases
Table 2: Value of parameter $p$ from numerical solution.

| $\lambda_1$ | $\lambda_2$ | $\kappa_1$ | $\kappa_2$ | $\alpha$ |
|--------------|--------------|-------------|-------------|----------|
| 1            | 1            | 1           | 1           | 0.25     |
|              |              |             |             | 0.7053   |
| 1            | 2            | 1           | 1           | 0.5      |
|              |              |             |             | 0.7358   |
| 1            | 1            | 2           | 1           | 0.75     |
|              |              |             |             | 0.8041   |
|              |              |             |             | 0.9311   |

Table 3: Value of variable $\tau$ fulfilling equation $S(\tau) = 1$ obtained from numerical solution.

| $\lambda_1$ | $\lambda_2$ | $\kappa_1$ | $\kappa_2$ | $\alpha$ |
|--------------|--------------|-------------|-------------|----------|
| 1            | 1            | 1           | 1           | 0.25     |
|              |              |             |             | 16.329   |
| 1            | 2            | 1           | 1           | 0.5      |
|              |              |             |             | 90.885   |
| 1            | 1            | 2           | 1           | 0.75     |
|              |              |             |             | 11.241   |

for $\alpha = 1$ and $\alpha = 0.75$ we have adopted $x \in [0, 2]$ and $x \in [0, 6.5]$ respectively, because analytical solution (20) is difficult to calculate for large values of the spatial variable $x$. This is related to the definition of the Wright function, see formula (5). Therefore, due to time constraints of calculations, we used 500 terms of the series. However, this is not sufficient for large values of the similarity variable $\tau_{\alpha/2}$.

In Figure 4, 6, 8 and 10, we present graphs of the function $S$ which describe the position of the phase change front. The black and red line represents the analytical and numerical solution respectively.

![Figure 3](image)

Figure 3: Numerical solution of $u_1$, $u_2$ for $\alpha = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$. 
Figure 4: Numerical solution of $S$ for $\alpha = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$.

Figure 5: Numerical solution of $u_1$, $u_2$ for $\alpha = 0.75$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$. 
Figure 6: Numerical solution of $S$ for $\alpha = 0.75$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$.

Figure 7: Numerical solution of $u_1, u_2$ for $\alpha = 0.5$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$. 
Figure 8: Numerical solution of $S$ for $\alpha = 0.5$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$. 

Figure 9: Numerical solution of $u_1$, $u_2$ for $\alpha = 0.25$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$. 
Figure 10: Numerical solution of $S$ for $\alpha = 0.25$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 1$, $\kappa_2 = 1$.

6 Conclusions

In this paper, we have developed a numerical method for solving the two-phase fractional Stefan problem, which can be used as an effective tool for determining a solution alternative to the closed analytical one. The applied special case of front-fixing technique allows us to use the mesh with constant spatial and time step, which provides discretization of the Caputo derivative with respect to time. Let us point out that the proposed method has some limitations directly derived from the construction of the new spatial variables requiring the analytical explicit form of the function describing the moving boundary depending on the unknown parameter $p$.

Based on the numerical results presented in Section 5, we can draw the following conclusions:

1. The error generated by the numerical solution of the governing subdiffusion equations is a decreasing function of the order of the Caputo derivative. In the special case for $\alpha = 1$, we observe very good matching of the numerical solution to the analytical one.

2. The relationship between the accuracy of the numerical solution of function $S$ and the order of the Caputo derivative is not as obvious as it was listed in the previous point. This is confirmed by the graphs in the Figures 6 and 8. It should be emphasized that the accuracy of the numerical solution of the function $S$ is the result of many factors, which explains formula (62). First, the numerical solutions of the functions $u_1$ and $u_2$ are burdened by different errors. Second, the approximation of the heat flux calculated in the liquid phase is usually more accurate than that calculated for the solid phase, because for the example considered in the paper, there is the following relationship $x_{m_1,j} - x_{m_1-1,j} < x_{1,j} - x_{0,j}$ (see formula (62) and Figure 4). In summary, the accuracy of the numerical solution of the function $S$ is not only dependent on the order of Caputo derivative $\alpha$, but also depends on mesh parameters and the model’s physical parameters.

3. The proposed numerical method is particularly useful for large values of $\alpha$, i.e. when the closed analytical solution is difficult to calculate (for large values of the similarity variable $\frac{x}{\tau^{\alpha/2}}$).
Acknowledgements

This research was supported by the Czestochowa University of Technology Grant Number BS/MN-1-105-301/17/P. I would like to express my warm thanks to Vaughan R. Voller, who gave scientific guidance that greatly assisted the research.

References

[1] T. Higuchi. Rate of release of medicaments from ointment bases containing drugs in suspension. *Journal of Pharmaceutical Sciences*, 50:874–875, 1961.

[2] T. Higuchi. Mechanism of sustained-action medication: Theoretical analysis of solid drugs dispersed institutionsolid matrices. *Journal of Pharmaceutical Sciences*, 52:1145–1149, 1963.

[3] D.S. Cohen and T. Erneux. Free boundary problems in controlled release pharmaceuticals. ii: Swelling-controlled release. *SIAM J Appl Math*, 48:1466–1474, 1988.

[4] J. Stefan. Uber die theorie der eisbildung, insbesondere uber die eisbildung im polarmeere. *Annalen der Physik und Chemie*, 278(2):269–286, 1891.

[5] J. Crank. *Free and Moving Boundary Problems*. Clarendon Press, Oxford, 1984.

[6] S.C. Gupta. *The Classical Stefan Problem. Basic Concepts, Modeling and Analysis*. Elsevier, Amsterdam, 2003.

[7] J.M. Hill. *One-Dimensional Stefan Problems: An Introduction*. Longman Scientific and Technical, New York, 1987.

[8] L.I. Rubinstein. *The Stefan Problem*. American Mathematical Society, 1971.

[9] M. N. Ozisik. *Heat Conduction*. Wiley, 2 edition, 1993.

[10] J. Liu and M. Xu. An exact solution to the moving boundary problem with fractional anomalous diffusion in drug release devices. *Zeitschrift fur Angewandte Mathematik und Mechanik*, 84:22–28, 2004.

[11] Li Xi-cheng. *Fractional Moving Boundary Problems and Some of Its Applications to Controlled Release System of Drug*. PhD thesis, Shandong University, 2009.

[12] Chen Yin and Xicheng Li. Anomalous diffusion of drug release from slab matrix: Fractional diffusion models. *International Journal of Pharmaceutics*, 418:78–87, 2011.

[13] J. Singh, P.K. Gupta, and K.N. Rai. Homotopy perturbation method to space?time fractional solidification in a finite slab. *Applied Mathematical Modelling*, 35:1937–1945, 2011.

[14] V.R. Voller. An exact solution of a limit case stefan problem governed by a fractional diffusion equation. *International Journal of Heat and Mass Transfer*, 53:5622–5625, 2010.

[15] V.R. Voller and F. Falcini. Two exact solutions of a stefan problem with varying diffusivity. *International Journal of Heat and Mass Transfer*, 58:80–85, 2013.

[16] V.R. Voller. Fractional stefan problems. *International Journal of Heat and Mass Transfer*, 74:269–277, 2014.

[17] Rajeev and M.S. Kushwaha. Homotopy perturbation method for a limit case stefan problem governed by fractional diffusion equation. *Applied Mathematical Modelling*, 37:3589–3599, 2013.
[18] Rajeev, M.S. Kushwaha, and A. Kumar. An approximate solution to a moving boundary problem with space-time fractional derivative in fluvio-deltaic sedimentation process. *Ain Shams Engineering Journal*, 4(4):889–895, 2013.

[19] Xicheng Li, Mingyu Xu, and Shaowei Wang. Scale-invariant solutions to partial differential equations of fractional order with a moving boundary condition. *Journal of Physics A: Mathematical and Theoretical*, 41:155202, 2008.

[20] Liu Junyi and Xu Mingyu. Some exact solutions to Stefan problems with fractional differential equations. *Journal of Mathematical Analysis and Applications*, 351:536–542, 2009.

[21] S. Roscani and E. Marcus. Two equivalent Stefan’s problems for the time fractional diffusion equation. *Fractional Calculus and Applied Analysis*, 16(4):802–815, 2013.

[22] S.D. Roscani. Hopf lemma for the fractional diffusion operator and its application to a fractional free-boundary problem. *Journal of Mathematical Analysis and Applications*, 434(1):125–135, 2015.

[23] S.D. Roscani and Tarzia D.A. A generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem. 24(2):237–244.

[24] X. Gao, X. Jiang, and S. Chen. The numerical method for the moving boundary problem with space-fractional derivative in drug release devices. *Applied Mathematical Modelling*, 39:2385–2391, 2015.

[25] M. Blasik and M. Klimek. Numerical solution of the one phase 1d fractional Stefan problem using the front fixing method. *Mathematical Methods in the Applied Sciences*, 38(15):3214–3228, 2015.

[26] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[27] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives*. Gordon Breach, Amsterdam, 1993.

[28] M. El-Shahed and A. Salem. An extension of Wright function and its properties. *Journal of Mathematics*, pages 1–11, 2015.

[29] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[30] H.G. Landau. Heat conduction in a melting solid. *Quarterly of Applied Mathematics*, 8:81–94, 1950.

[31] K. Diethelm. *The Analysis of Fractional Differential Equations*. Springer-Verlag, Berlin, 2010.