INVERSE PROBLEM FOR A TIME-DEPENDENT CONVECTION-DIFFUSION EQUATION IN ADMISSIBLE GEOMETRIES

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Abstract. We consider a partial data inverse problem for a time-dependent convection-diffusion equation on an admissible manifold. We prove that the time-dependent convection term and time-dependent density can be recovered uniquely modulo a known gauge invariance. There have been several works on inverse problems related to the steady state convection-diffusion operator in Euclidean as well as in Riemannian geometry settings; however, inverse problems related to time-dependent convection-diffusion equation on a manifold are not studied in the prior works, which is the main aim of this paper. In fact, to the best of our knowledge, the problem studied here is the first work related to a partial data inverse problem for recovering both first and zeroth-order time-dependent perturbations of evolution equations in the Riemannian geometry setting.

Keywords: Inverse problems, time-dependent coefficients, convection-diffusion equation, partial boundary data, admissible manifold, Carleman estimates, geometric optics solutions.

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1. Introduction and statement of main result

The paper deals with a partial data inverse problem related to a convection-diffusion equation on $M_T := (0, T) \times M$ where $0 < T < \infty$ and $(M, g)$ is a smooth $n$–dimensional ($n \geq 2$) Riemannian manifold having smooth boundary $\partial M$. We denote by $\Sigma := (0, T) \times \partial M$ as the lateral boundary of $M_T$ and $\partial M_T := \Sigma \cup \{(0) \times M\} \cup \{(T) \times M\}$ the topological boundary of $M_T$. We also denote by $TM$ and $T^*M$ the tangent and cotangent bundle of $M$. For a convection term $A \in W^{1, \infty}(M_T; T^*M)$ given by $A(t, x) := \sum_{j=1}^{n} A_j(t, x)dx^j$ in local coordinates $x_1, x_2, \ldots, x_n$ of manifold $M$ and density $q \in L^\infty(M_T)$, the initial boundary value problem (IBVP) for the convection-diffusion equation on $M_T$ is modeled by the following IBVP for second order linear parabolic partial differential equation (PDE)

$$\begin{cases}
\left[\partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}} \left( \partial_{x_j} + A_j(t, x) \right) \left( \sqrt{|g|} g^{jk} \left( \partial_{x_k} + A_k(t, x) \right) \right) + q(t, x) \right] u(t, x) = 0, & (t, x) \in M_T \\
\sigma(0, x) = \phi(x), & x \in M \\
\sigma(t, x) = f(t, x), & (t, x) \in \Sigma
\end{cases}
$$

(1.1)

where $g^{-1} := \left((g^{ij})\right)_{1 \leq i, j \leq n}$ denote the inverse of metric tensor $g := \left((g_{ij})\right)_{1 \leq i, j \leq n}$, $|g| = \det(g)$ and the initial value $\phi$ and the Dirichlet data $f$ are assumed to be non-zero. Throughout this article, we denote by $L_{A,q}$ the following operator

$$L_{A,q} := \partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}} \left( \partial_{x_j} + A_j(t, x) \right) \left( \sqrt{|g|} g^{jk} \left( \partial_{x_k} + A_k(t, x) \right) \right) + q(t, x).$$

(1.2)

In this paper, we are interested in determining the convection term $A$ and density coefficient $q$ from the boundary measurements of the solution. To define the boundary operators, we need to have the existence and uniqueness of a solution to the forward problem for IBVP given by (1.1).

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Motivated by [15] [18], we define the following spaces
\[ \mathcal{K}_0 := \{(f|_{t=0}, f|_{\Sigma}) : f \in H^1(0, T, H^{-1}(M)) \cap L^2(0, T; H^1(M))\} \] and
\[ \mathcal{K}_T := \{(f|_{t=T}, f|_{\Sigma}) : f \in H^1(M_T)\}
\]
where we refer to [25] for the definition of function spaces \( H^m(0, T; H^k(M)) \), for \( k, m \in \mathbb{R} \). Now for \((\phi, f) \in \mathcal{K}_0\), it can be shown by following arguments from [15] [25] [42] that there exists a unique solution \( u \in H^1(0, T, H^{-1}(M)) \cap L^2(0, T; H^1(M)) \) of IBVP (1.1). Based on the existence and uniqueness of solution and following [15] [39] [48], we observe that for any solution \( u \in H^1(0, T, H^{-1}(M)) \cap L^2(0, T; H^1(M)) \) the operator \( \mathcal{N}_{A,q}u \) given by
\[
\langle \mathcal{N}_{A,q}u, w|_{\partial M_T^*} \rangle := \int_{M_T^*} \left( -u \partial_t \overline{w} + \langle \nabla_g u, \nabla_g \overline{w} \rangle_g + 2u(A, \nabla_g \overline{w})_g + (\delta_g A)u \overline{w} - |A|^2 u \overline{w} + qu \overline{w} \right) dV_g dt
- \int_M u(0,x) \overline{w}(0,x) dV_g
\]
is well-defined for all \( w \in H^1(M_T^*) \) where \( \mathcal{K}_T^* := (\{T\} \times M) \cup \Sigma \). Now if we assume the sufficient regularity on the coefficients \( A, q \) and Dirichlet data \( f \), then as shown in [15] the operator \( \mathcal{N}_{A,q}u \) is given by
\[
\Lambda_{A,q}u := \left( u|_{t=T}, \left[ \partial_\nu u(t,x) + 2\nu(t) \cdot A(t,x) u(t,x) \right] \right|_{\Sigma})
\]
where \( \nu \) stands for the outward unit normal vector to \( \partial M \) and \( u \) solves the IBVP given by (1.1). This motivates us to define our input-output operator \( \Lambda_{A,q} : \mathcal{K}_0 \rightarrow \mathcal{K}_T^* \) by
\[
\Lambda_{A,q}(\phi, f) := \mathcal{N}_{A,q}u
\]
where \( \mathcal{K}_T^* \) stands for dual of \( \mathcal{K}_T \) and \( u \) is solution to the IBVP (1.1) when the initial data is \( \phi \) and Dirichlet boundary data equal to \( f \).

This work is concerned with the determination of time-dependent coefficients \( A \) and \( q \) appearing in (1.1) using the measurements of the input-output operator \( \Lambda_{A,q} \) on a proper subset of \( \Sigma \) for the case when \( (M, g) \) is an admissible manifold whereby an admissible manifold, we mean the following.

**Definition 1.1.** (Admissible manifold [11] [29]) We say that a compact Riemannian manifold \((M, g)\) of dimension \( n \geq 2 \) with boundary \( \partial M \), is admissible if \( M \) is orientable and \((M, g)\) is a submanifold of \( \mathbb{R} \times (\text{int}(M_0), g_0) \) where \((M_0, g_0)\) is a compact, simply connected Riemannian manifold with boundary \( \partial M_0 \) which is strictly convex in the sense of the second fundamental form and \( M_0 \) has no conjugate points.

In order to state the main result of this article, we first need to specify the subset of \( \partial M \) where the measurements are given. Now if we write \( x \in M \), as \( x := (x_1, x') \in \mathbb{R} \times M_0 \) and \( \varphi(x) := x_1 \) then \( \partial M \) can be decomposed into the two parts given by
\[ \partial M_+ := \{ x \in \partial M : \partial_\nu \varphi(x) > 0 \} \] and \( \partial M_- := \{ x \in \partial M : \partial_\nu \varphi(x) \leq 0 \} \)
where \( \nu(x) \) stands for outward unit normal to \( \partial M \) at \( x \in \partial M \) and \( \partial_\nu \varphi \) denote the normal derivative of \( \varphi \) with respect to the metric \( g \). In this paper, we will be assuming that our boundary measurements are given on slightly bigger than half of \( \partial M \). To specify this portion of \( \partial M \), we take \( \epsilon > 0 \) small enough and define \( \partial M_{\pm, \epsilon/2} \) by
\[
\partial M_{\epsilon/2, \pm} := \left\{ x \in \partial M : \partial_\nu \varphi(x) \geq \frac{\epsilon}{2} \right\} \] and \( \partial M_{-\epsilon/2, \pm} := \left\{ x \in \partial M : \partial_\nu \varphi(x) < \frac{-\epsilon}{2} \right\}
\]
as \( \partial M_{-\epsilon/2, \pm} \) is the small enough open neighborhood of \( \partial M_- \). We denote the corresponding lateral part of \( \Sigma \) by \( \Sigma_+ := (0, T) \times \partial M_+ \), \( \Sigma_{\pm, \epsilon/2} := (0, T) \times \partial M_{\pm, \epsilon/2} \), \( \Sigma_- := (0, T) \times \partial M_- \) and \( \Sigma_{-, \epsilon/2} := (0, T) \times \partial M_{-, \epsilon/2} \). We also denote \( M_{\pm, \epsilon/2}^T := (\{T\} \times M) \cup \Sigma_\pm \) and \( M_{\pm, \epsilon/2}^T := (\{T\} \times M) \cup \Sigma_{\pm, \epsilon/2} \). Now, using these notations, we define the partial input-output operator by
\[
\Lambda_{A,q}^{\text{partial}}(\phi, f) := \mathcal{N}_{A,q}u|_{\partial M_{\pm, \epsilon/2}^T}.
\]
Our aim in this article is to recover $A$ and $q$ uniquely from the knowledge of $\Lambda_{A,q}^{\text{partial}}$; however, due to gauge invariance, it is impossible to recover these coefficients fully. Since this is the first work related to the time-dependent convection-diffusion equation on manifolds therefore before stating the main result of this paper, we first provide quick proof of the gauge invariance associated with our problem. In the Euclidean setting, this has been well observed in prior works; see, for example, [15], [38] and references therein.

**Definition 1.2.** (Gauge Invariance) Let $A^{(i)} \in W^{1,\infty}(M_T)$ and $q_i \in L^{\infty}(M_T)$ for $i = 1, 2$. We say $(A^{(1)}, q_1)$ and $(A^{(2)}, q_2)$ are gauge equivalent if there exists $\Psi \in W^{2,\infty}_0(M_T)$ such that

$$A^{(2)}(t,x) = A^{(1)}(t,x) - \nabla_y \Psi(t,x)$$

and $q_2(t,x) = q_1(t,x) - \partial_t \Psi(t,x)$, for $(t,x) \in M_T$.

**Proposition 1.3.** Suppose $u_1(t,x)$ is a solution to the following IBVP

$$\begin{cases}
\partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}} \left( \partial_{x_j} + A^{(1)}_j \right) \left( \sqrt{|g|} g^{jk} \left( \partial_{x_k} + A^{(1)}_k \right) \right) + q_1 u_1(t,x) = 0, \quad (t,x) \in M_T \\
u_1(0,x) = \phi(x), \quad x \in M \\
u_1(t,x) = f(t,x), \quad (t,x) \in \Sigma
\end{cases} \tag{1.6}$$

and $\Psi \in W^{2,\infty}_0(M_T)$, then $u_2(t,x) = e^{\Psi(t,x)} u_1(t,x)$ satisfies the following IBVP

$$\begin{cases}
\partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}} \left( \partial_{x_j} + A^{(2)}_j \right) \left( \sqrt{|g|} g^{jk} \left( \partial_{x_k} + A^{(2)}_k \right) \right) + q_2 u_2(t,x) = 0, \quad (t,x) \in M_T \\
u_2(0,x) = \phi(x), \quad x \in M \\
u_2(t,x) = f(t,x), \quad (t,x) \in \Sigma
\end{cases} \tag{1.7}$$

where $A^{(2)}(t,x) = A^{(1)}(t,x) - \nabla_y \Psi(t,x)$ and $q_2(t,x) = q_1(t,x) - \partial_t \Psi(t,x)$. Now if $\Lambda_{A^{(i)},q_i}$ for $i = 1, 2$, are the input-output operators associated with $u_i$ and defined by (1.3) then

$$\Lambda_{A^{(1)},q_1}(\phi,f) = \Lambda_{A^{(2)},q_2}(\phi,f), \quad \text{for all } (\phi,f) \in K_0.$$ 

**Proof.** Substituting $u_1(t,x) = e^{-\Psi(t,x)} u_2(t,x)$ in Equation (1.6), from simple computations, we get

$$0 = L_{A^{(1)},q_1}(e^{-\Psi(t,x)} u_2(t,x)) = L_{A^{(1)},-\nabla_y \Psi,q_1-\partial_t \Psi}(u_2(t,x)) = L_{A^{(2)},q_2}(u_2(t,x)), \quad (t,x) \in M_T$$

and

$$u_2(0,x) = e^{\Psi(0,x)} u_1(0,x) = \phi(x), \quad x \in M,$n_2(t,x) = e^{\Psi(t,x)} u_1(t,x) = f(t,x), \quad (t,x) \in \Sigma.$$ 

Hence $u_2$ solves (1.7). Also, we have that

$$u_2(T,x) = e^{\Psi(T,x)} u_1(T,x) = u_1(T,x), \quad x \in M, \quad \partial_{\nu} u_2 \bigg|_{\Sigma} = \left( e^{\Psi} \partial_{\nu} \Psi \right) u_1 \bigg|_{\Sigma} = \partial_{\nu} u_1 \bigg|_{\Sigma},$$

and

$$\left. \left( \nu \cdot A^{(2)} u_2 \right) \right|_{\Sigma} = \left. \left( \nu \cdot (A^{(1)} - \nabla_y \Psi \Psi) e^{\Psi} u_1 \right) \right|_{\Sigma} = \left. \left( \nu \cdot A^{(1)} u_1 \right) \right|_{\Sigma}$$

where in the above equations, we have used the fact that $\Psi \in W^{2,\infty}_0(M_T)$. Thus combining the above equations together with (1.3) we get

$$\Lambda_{A^{(1)},q_1}(\phi,f) = \Lambda_{A^{(2)},q_2}(\phi,f), \quad \text{for all } (\phi,f) \in K_0.$$ 

With this preparation, we are ready to state the main result of this paper as follows.

**Theorem 1.4.** Let $(M,g)$ be an admissible manifold. Let $A^{(i)} \in W^{1,\infty}(M_T; T^*(M))$ for $i = 1, 2$ given by $A^{(i)}(t,x) = \sum_{j=1}^{n} A^{(i)}_j(t,x) dx^j$, in local coordinates on $(M,g)$ and $q_i \in L^{\infty}(M_T)$ for $i = 1, 2$. Suppose $u_i$ for
Let \( i = 1, 2 \), be solution to (1.1) when \((A, q) = (A^{(i)}, q_i)\) for \( i = 1, 2 \) and \( \Lambda^{\text{partial}}_{A^{(i)}, q_i} \) are input-output operator given by (1.5) corresponding to \( u_i \) for \( i = 1, 2 \). Now for \( \epsilon > 0 \) small enough if

\[
\Lambda^{\text{partial}}_{A^{(1)}, q_1}(\phi, f) = \Lambda^{\text{partial}}_{A^{(2)}, q_2}(\phi, f), \quad \text{for all } (\phi, f) \in \mathcal{K}_0
\]

then there exists a function \( \Psi \in W^{2, \infty}_0(M_T) \) such that

\[
A^{(1)}(t, x) - A^{(2)}(t, x) = \nabla \psi(t, x) \quad \text{and} \quad q_1(t, x) - q_2(t, x) = \partial_t \psi(t, x), \quad \text{for } (t, x) \in M_T
\]

provided \( A^{(1)}(t, x) = A^{(2)}(t, x), \) for \((t, x) \in \Sigma \).

**Remark 1.5.**

1. Observe that the measurement data used in Theorem 1.4 is an input-output map, unlike the usual Dirichlet to Neumann (DN) map used in the Euclidean setting. This is due to the fact that in our boundary Carleman estimate stated in Theorem 2.1 we do not have the estimate on weighted \( L^2 \) norm of solution \( u \) at \( t = T \) which is because of the choice of weight function \( \varphi(t, x) := \lambda^2 \beta^2 t + \lambda x_1 \) while in Euclidean setting one can actually take \( \beta = 1 \) which helped one to the Carleman estimate with a bound on the weighted \( L^2 \) norm of the solution \( u \) at \( t = T \). We refer to [15, 50] for details about it. However, if we assume the coefficients are small enough then we can obtain the boundary Carleman estimate (see Theorem 3.1 in [48]) with a bound on the weighted \( L^2 \) norm of the solution \( u \) at \( t = T \) and can determine the coefficients from the knowledge of DN map measured on a suitable subset of \( \Sigma \).

2. We observe that because of gauge invariance proved in Proposition 1.3 it is impossible to prove that \( A^{(1)} = A^{(2)} \) and \( q_1 = q_2 \) in \( M_T \) from the given hypothesis of Theorem 1.4. As far as the uniqueness issue is concerned, gauge invariance guarantees that the result obtained in Theorem 1.4 is optimal.

3. Unique recovery of \( A \) and \( q \) is also possible with some extra conditions on the unknown vector field \( A \). For instance, if \( A \) is divergence-free, that is, \( \delta_q A = 0 \) in \( M_T \), then one can recover both \( A \) and \( q \) uniquely in \( M_T \). This divergence-free condition has been exploited in earlier works to get the full recovery, please refer [48, 50].

4. On the other side, if we assume that vector field \( A \) is time-independent, then recovery of \( A \) is possible up to a potential of the form \( \nabla \psi(t, x) \) and \( q \) can be fully recovered in \( M_T \). The proof follows similarly as we have done for Theorem 1.4.

The problem considered in this article can be put under the umbrella of Calderón type inverse problems for parabolic partial differential equations (PDEs), which was initially proposed by Calderón in [14] for elliptic PDEs and studied by Nachman [13] in two dimensions and by Sylvester-Uhlmann in [53] in dimension three and higher. Analogous problems for parabolic and hyperbolic PDEs have been studied in [1, 30, 44, 47]. Choulli-Kian [21] derived a stability estimate for recovering the time-dependent coefficient, which is a product of functions depending only on time and only space variables, from the boundary measurements. We also refer to [20], where an abstract inverse problem for parabolic pde is studied. All these works are concerned with the recovery of zeroth order perturbation of elliptic, parabolic, and hyperbolic PDEs from full boundary data. In [22], the recovery of general time-dependent zeroth order perturbation of heat operator from partial boundary measurements is considered. Inverse problems of recovering the coefficients appearing in the steady state convection-diffusion from full and partial boundary measurements in Euclidean geometry have been studied in [12, 17, 19, 24, 28, 37, 35, 46, 51]. Recovery of first-order perturbation of a parabolic pde from final and single measurement has been studied in [16] and [18] respectively. In [8], stable recovery of time-dependent coefficients appearing in a convection-diffusion from full boundary data has been studied. Choulli-Kian in [22] proved a stability estimate for recovering a time-dependent potential from partial boundary data, and motivated by their work, authors of [48] proved the unique recovery of time-dependent coefficients appearing in a convection-diffusion equation from partial boundary data. In [35], a uniqueness result is proved with a smallness assumption on the convection term, which is later on removed in a recent work of [50] where stability estimates for recovering
We denote by \( L \) the space of all functions \( \partial M \) and its formal adjoint. For a compact Riemannian manifold \((M, g)\) with boundary denoted by \( \partial M \), we denote by \( dV_g \) the volume form on \((M, g)\) and by \( dS_g \) the induced volume form on \( \partial M \). Then the \( L^2 \)-norm of a function \( u \) on \( M \) and \( f \) on \( \partial M \) are given by

\[
\|u\|_{L^2(M)} := \left( \int_M |u(x)|^2 \, dV_g \right)^{1/2}
\]

and

\[
\|f\|_{L^2(\partial M)} := \left( \int_{\partial M} |f(x)|^2 \, dS_g \right)^{1/2},
\]

respectively.

We denote by \( L^2(M) \) as the space of all functions \( u \) defined on \( M \) for which \( \|u\|_{L^2(M)} < \infty \) and \( L^2(\partial M) \) as the space of all functions \( f \) defined on \( \partial M \) for which \( \|f\|_{L^2(\partial M)} < \infty \). Then \( (L^2(M), \|\cdot\|_{L^2(M)}) \) and \( (L^2(\partial M), \|\cdot\|_{L^2(\partial M)}) \) are Hilbert spaces with respect to the inner-products defined by

\[
\langle f, g \rangle_{L^2(M)} := \int_M f(x) \overline{g(x)} \, dV_g
\]

and

\[
\langle f, g \rangle_{L^2(\partial M)} := \int_{\partial M} f(x) \overline{g(x)} \, dS_g
\]

respectively.

We state the boundary Carleman estimate as follows.

**Theorem 2.1.** Let \((M, g)\) be an admissible manifold. For \( \beta \in \left( \frac{1}{\sqrt{3}}, 1 \right) \), let \( \varphi(t, x) := \lambda^2 \beta^2 t + \lambda x_1 \), \( A \in (W^{1,\infty}(MT))^n \) and \( q \in L^\infty(M_T) \). Then there exists a constant \( C > 0 \) depending only on \( M, T, A \) and...
then one can check that

\[ \lambda^2 \| e^{-\varphi} u \|_{L^2(M_T)}^2 + \| e^{-\varphi} \nabla g u \|_{L^2(M_T)}^2 + \| e^{-\varphi(T,\cdot)} \nabla_g u(T,\cdot) \|_{L^2(M)}^2 + \lambda \int_{\Sigma_t} e^{-2\varphi} |\partial_t u(t,x)|^2 \langle \nu, e_1 \rangle_g \, dS_g \, dt \]

\[ \leq C \left( \| e^{-\varphi} \mathcal{L}_{A,q} u \|_{L^2(M_T)}^2 + \lambda^2 \| e^{-\varphi(T,\cdot)} u(T,\cdot) \|_{L^2(M)}^2 + \lambda \int_{\Sigma_t} e^{-2\varphi} |\partial_t u(t,x)|^2 \langle \nu, e_1 \rangle_g \, dS_g \, dt \right) \]

(2.1)

hold for \( \lambda \) large enough and for all \( u \in C^2(M_T) \) satisfying the following

\[ u(0,0) = 0, \text{ for } x \in M \text{ and } u(t,x) = 0, \text{ for } (t,x) \in \Sigma. \]

**Proof.** In order to prove the weighted \( H^1 - L^2 \) estimate given by (2.1), we need to convexify the Carleman weight \( \varphi \). This convexification will help us to absorb the first-order perturbation \( A \) appearing in \( \mathcal{L}_{A,q} \), which has been used in [15, 50] for Euclidean case and [29] for anisotropic magnetic Schrödinger operator. Now for \( s > 0 \), we denote the convexified weight function by \( \varphi_s \) and define by

\[ \varphi_s(t,x) := \varphi(t,x) - \frac{s(x_1 + 2\ell)^2}{2} = \lambda^2 \beta^2 t + \lambda x_1 - \frac{s(x_1 + 2\ell)^2}{2}. \]

(2.2)

A direct computation gives

\[ \partial_t \varphi_s = \lambda^2 \beta^2, \quad \partial_{x_1} \varphi_s = \lambda - s(x_1 + 2\ell), \quad \partial_{x_1}^2 \varphi_s = -s, \text{ and } |\partial_{x_1} \varphi_s|^2 = (\lambda - s(x_1 + 2\ell))^2. \]

(2.3)

Before we proceed further, let us observe that

\[ \mathcal{L}_{A,q} v(t,x) = \partial_t v(t,x) - \partial^2_{x_1} v(t,x) - \Delta g_0 v(t,x) - 2(A(t,x), \nabla_g v(t,x))_g + \hat{q}(t,x)v(t,x) \]

where \( \langle \cdot, \cdot \rangle_g \) and \( \nabla_g \) denote the inner-product and gradient operator w.r.t. metric \( g \) respectively and

\[ \hat{q}(t,x) := q(t,x) - \delta_g A(t,x) - |A(t,x)|_g^2, \]

here in expression of \( \hat{q}, \delta_g A \) given by

\[ \delta_g A = \frac{1}{\sqrt{|g|}} \sum_{j,k=1}^n \partial_j \left( g^{jk} \sqrt{|g|} A_k \right) \]

is known as the divergence operator w.r.t. to metric \( g \) and \( |A|^2_g = \sum_{j,k=1}^n g^{jk} A_j A_k \). With this, we define the conjugated operator \( P_s \) with a convexified weight function \( \varphi_s \) by

\[ P_s v := e^{-\varphi_s} \mathcal{L}_{A,q}(e^{\varphi_s} v) = e^{-\varphi_s} \left( \partial_t - \partial_{x_1}^2 - \Delta g_0 - 2(A, \nabla_g A)_g + \hat{q} \right) \left( e^{\varphi_s} v \right). \]

(2.4)

Upon expanding the above expression, \( P_s \) will take the following form

\[ P_s v(t,x) = (\partial_t + (\partial_t \varphi_s)) v(t,x) - (\partial_{x_1}^2 + 2\partial_{x_1} \varphi_s \partial_{x_1} + (\partial_{x_1} \varphi_s)^2 + \partial_{x_1}^2 \varphi_s) v - \Delta g_0 v(t,x) - 2(A(t,x), \nabla_g v(t,x))_g - 2(A(t,x), \nabla_g \varphi_s(t,x))_g v(t,x) + \hat{q}(t,x)v(t,x). \]

Using (2.3) and the fact that \( (M,g) \) is admissible manifold, we get

\[ P_s v(t,x) = \partial_t v(t,x) + \lambda^2 \beta^2 v(t,x) - (\partial_{x_1}^2 + 2(\lambda - s(x_1 + 2\ell)) \partial_{x_1} + (\lambda - s(x_1 + 2\ell))^2 - s) v(t,x) - \Delta g_0 v(t,x) - 2(A(t,x), \nabla_g v(t,x))_g - 2(\lambda - s(x_1 + 2\ell)) g^{1k} A_k(t,x)v(t,x) + \hat{q}(t,x)v(t,x). \]

Now if we define \( P_1, P_2 \) and \( P_3 \) by

\[ P_1 v(t,x) := (\partial_t v - (2\lambda - s(x_1 + 2\ell)) \partial_{x_1} v + 4sv)(t,x), \]

\[ P_2 v(t,x) := -(\partial_{x_1}^2 v - \Delta g_0 v - \lambda^2 (1 - \beta^2) v + 2\lambda s(x_1 + 2\ell)v - s^2(x_1 + 2\ell)^2 v - 3sv)(t,x) = -(\partial_{x_1}^2 - \Delta g_0 + \mathcal{K}(x_1)) v(t,x), \text{ where } \mathcal{K}(x_1) := 2\lambda s(x_1 + 2\ell) - \lambda^2 (1 - \beta^2) - s^2(x_1 + 2\ell)^2 - 3s \]

\[ P_3 v(t,x) := -2(A(t,x), \nabla_g v(t,x))_g - 2(\lambda - s(x_1 + 2\ell)) g^{1k} A_k(t,x)v(t,x) + \hat{q}(t,x)v(t,x) \]

then one can check that \( P_s v(t,x) \) has the following compact form

\[ P_s v(t,x) = P_1 v(t,x) + P_2 v(t,x) + P_3 v(t,x). \]

(2.5)
Our first aim is to estimate the $L^2$ norm of $P_s v$ on $M_T$, therefore we define $I_s$ by

$$ I_s := \int_{M_T} |P_s v(t, x)|^2 \, dV_g(x) \, dt = \int_{M_T} |P_1 v(t, x) + P_2 v(t, x) + P_3 v(t, x)|^2 \, dV_g dt $$

$$ \geq \frac{1}{2} \int_{M_T} (P_1 v(t, x) + P_2 v(t, x))^2 \, dV_g dt - \int_{M_T} |P_3 v(t, x)|^2 \, dV_g dt $$

$$ \geq \int_{M_T} P_1 v(t, x) P_2 v(t, x) \, dV_g dt - \int_{M_T} |P_3 v(t, x)|^2 \, dV_g dt. $$

This gives us

$$ I_s \geq \frac{1}{2} \int_{M_T} P_1 v(t, x) P_2 v(t, x) \, dV_g dt - \int_{M_T} I_{s,2} \, dV_g dt. \tag{2.6} $$

We aim to estimate the right-hand side of (2.6). To do that, we start with the first term in the above inequality and, therefore consider

$$ P_1 v(t, x) P_2 v(t, x) = -\partial_t v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) + K(x_1) v(t, x) \partial_t v(t, x) - 4 s v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) v(t, x) $$

$$ + 4 s K(x_1) |v(t, x)|^2 + 2 \left( \lambda - s(x_1 + 2 \ell) \right) \partial_{x_1} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) $$

$$ - 2 K(x_1) v(t, x) \partial_{x_1} v(t, x). $$

Now consider $I_{s,1}$ from (2.6)

$$ I_{s,1} = \int_{M_T} P_1 v(t, x) P_2 v(t, x) \, dV_g dt = -\int_{M_T} \partial_t v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt $$

$$ + 4 s \int_{M_T} K(x_1) |v(t, x)|^2 \, dV_g dt + \frac{1}{2} \int_{M_T} K(x_1) \partial_t |v(t, x)|^2 \, dV_g dt $$

$$ - 4 s \int_{M_T} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt $$

$$ + 2 \int_{M_T} (\lambda - s(x_1 + 2 \ell)) \partial_{x_1} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt $$

$$ - \int_{M_T} K(x_1) (\lambda - s(x_1 + 2 \ell)) \partial_{x_1} |v(t, x)|^2 \, dV_g dt $$

$$ := I_1 + I_2 + I_3 + I_4 + I_5 + I_6 $$

where

$$ I_1 := -\int_{M_T} \partial_t v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt; \quad I_2 := 4 s \int_{M_T} K(x_1) |v(t, x)|^2 \, dV_g dt $$

$$ I_3 := \frac{1}{2} \int_{M_T} K(x_1) \partial_t |v(t, x)|^2 \, dV_g dt; \quad I_4 := -4 s \int_{M_T} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt $$

$$ I_5 := 2 \int_{M_T} (\lambda - s(x_1 + 2 \ell)) \partial_{x_1} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt $$

$$ I_6 := -\int_{M_T} K(x_1) (\lambda - s(x_1 + 2 \ell)) \partial_{x_1} |v(t, x)|^2 \, dV_g dt. $$

In order to estimate $I_{s,1}$ in (2.6), we need to estimate each $I_j$ for $1 \leq j \leq 6$. To estimate these $I_j$s, we use integration by parts repeatedly along with initial and boundary conditions on $v$. Consider

$$ I_1 = -\int_{M_T} \partial_t v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt = \frac{1}{2} \int_M |\nabla_g v(T, x)|^2 \, dV_g. \tag{2.7} $$
\[ I_2 = 4s \int_{M_T} \mathcal{K}(x_1)|v(t, x)|^2 \, dV_g dt \]
\[ = -4s \int_{M_T} (\lambda^2(1 - \beta^2) - 2\lambda s(x_1 + 2\ell) + s^2(x_1 + 2\ell)^2 + 3s) |v(t, x)|^2 \, dV_g dt. \quad (2.8) \]

\[ I_3 = \frac{1}{2} \int_{M_T} \mathcal{K}(x_1)\partial_t|v(t, x)|^2 \, dV_g dt = \frac{1}{2} \int_{M} \mathcal{K}(x_1)|v(T, x)|^2 dV_g \]
\[ = \frac{1}{2} \int_{M} (-\lambda^2(1 - \beta^2) + 2\lambda s(x_1 + 2\ell) - s^2(x_1 + 2\ell)^2 - 3s) |v(T, x)|^2 \, dV_g. \]

Recall \( \ell \leq (x_1 + 2\ell) \leq 3\ell \), therefore choosing \( \lambda \) large enough, we obtain
\[ I_3 \geq -C\lambda^2\|v(T, \cdot)\|_{L^2(M)}^2, \text{ for some constant } C > 0 \text{ independent of } \lambda. \quad (2.9) \]

Next, consider
\[ I_4 = -4s \int_{M_T} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt = 4s \int_{M_T} |\nabla_g v(t, x)|^2_g \, dV_g dt. \quad (2.10) \]

The next integral in the line is
\[ I_5 = 2 \int_{M_T} (\lambda - s(x_1 + 2\ell))\partial_{x_1} v(t, x) \left( \partial^2_{x_1} v + \Delta_{g_0} v \right) (t, x) \, dV_g dt \]
\[ = 2 \int_{M_T} (\lambda - s(x_1 + 2\ell))\partial_{x_1} v(t, x) \Delta_{g} v(t, x) \, dV_g dt. \]

Using the integration by parts, we have
\[ I_5 = -2 \int_{M_T} \partial_{x_1} v(t, x) \left\langle \nabla_g v, \nabla_g (\lambda - s(x_1 + 2\ell)) \right\rangle_g \, dV_g dt \]
\[ - 2 \int_{M_T} (\lambda - s(x_1 + 2\ell)) \left\langle \nabla_g v, \partial_{x_1} \nabla_g v \right\rangle_g \, dV_g dt \]
\[ + 2 \int_{\Sigma} (\lambda - s(x_1 + 2\ell))\partial_{x_1} v(t, x)\partial_{\nu} v(t, x) \, dS_g dt \]

where \( \nu(x) \) is outward unit normal vector to \( \partial M \) at \( x \in \partial M \), \( \partial_{\nu} v(t, x) \) stands for the normal derivative w.r.t. \( x \) of \( v \) at \( (t, x) \in (0, T) \times \partial M \) and \( dS_g \) denote the surface measure on \( \partial M \). Again using the integration by parts, we have that
\[ I_5 = 2s \int_{M_T} |\partial_{x_1} v(t, x)|^2 \, dV_g dt - \int_{\Sigma} (\lambda - s(x_1 + 2\ell)) \langle \nu, e_1 \rangle_g |\nabla_g v(t, x)|^2_g \, dS_g dt \]
\[ - s \int_{M_T} |\nabla_g v(t, x)|^2_g \, dV_g dt + 2 \int_{\Sigma} (\lambda - s(x_1 + 2\ell))\partial_{x_1} v(t, x)\partial_{\nu} v(t, x) \, dS_g dt. \]

Now \( v|_{(0, T) \times \partial M} = 0 \), implies that \( \nabla_g v|_{\Sigma} = (\partial_{\nu} v) \nu \) and \( \partial_{x_1} v|_{\Sigma} = \langle \nabla_g v, e_1 \rangle_g \Sigma = \partial_{\nu} v(\nu, e_1) \). Using these, we get
\[ I_5 = s \int_{M_T} (|\partial_{x_1} v(t, x)|^2 - |\nabla_{g_0} v(t, x)|^2_{g_0}) \, dV_g dt + \int_{\Sigma} (\lambda - s(x_1 + 2\ell))\partial_{\nu} v(t, x) \langle \nu, e_1 \rangle_g \, dS_g dt. \]
Combining $I_5$ and $I_4$, we have

\[
I_4 + I_5 = 5s \int_{M_T} |\partial_{x_1} v(t,x)|^2 dV_g dt + 3s \int_{M_T} |\nabla_{g_0} v(t,x)|^2_{g_0} dV_g dt \\
+ \int_{\Sigma} (\lambda - s(x_1 + 2\ell)) |\partial_v v(t,x)|^2_{g_0} |\nu, e_1 g| dS_g dt \\
\geq 3s \|\nabla_g v\|_{L^2(M_T)}^2 + \int_{\Sigma} (\lambda - s(x_1 + 2\ell)) |\partial_v v(t,x)|^2_{g_0} |\nu, e_1 g| dS_g dt. \tag{2.11}
\]

Next, we consider the last term of $I_{s,1}$

\[
I_6 = - \int_{M_T} K(x_1)(\lambda - s(x_1 + 2\ell)) |\partial_{x_1} v(t,x)|^2 dV_g dt \\
= \int_{M_T} (\lambda - s(x_1 + 2\ell)) |v(t,x)|^2 |\partial_{x_1} K(x_1)| dV_g dt - s \int_{M_T} K(x_1)|v(t,x)|^2 dV_g dt \\
= 2s \int_{M_T} (\lambda - s(x_1 + 2\ell))^2 |v(t,x)|^2 dV_g dt \\
+ s \int_{M_T} (\lambda^2(1 - \beta^2) - 2\lambda s(x_1 + 2\ell) + s^2(x_1 + 2\ell)^2 + 3s) |v(t,x)|^2 dV_g dt.
\]

After simplifying, we get

\[
I_6 = s\lambda^2(3 - \beta^2) \int_{M_T} |v(t,x)|^2_g dV_g dt - 6s^2 \int_{M_T} (x_1 + 2\ell)|v(t,x)|^2_g dV_g dt \\
+ 3s^2 \int_{M_T} (s(x_1 + 2\ell)^2 + 1) |v(t,x)|^2_g dV_g dt. \tag{2.12}
\]

Next, we estimate $I_{s,2}$ in the following way:

\[
I_{s,2} = \int_{M_T} |P_{3} v(t,x)|^2 dV_g dt \\
= \int_{M_T} \left(-2(A(t,x), \nabla_g v(t,x))_g - 2(\lambda - s(x_1 + 2\ell)) g^{kk} A_k(t,x)v(t,x) + \tilde{q}(t,x)v(t,x)\right)^2 dV_g dt \\
\leq 8\|A\|_{L^\infty(M_T)}^2 \|\nabla_g v\|_{L^2(M_T)}^2 + 8\lambda^2\|A\|_{L^\infty(M_T)}^2 \|v\|_{L^2(M_T)}^2 \|\nabla g\|_{L^2(M_T)}^2 + 2\|\tilde{q}\|_{L^\infty(M_T)} \|v\|_{L^2(M_T)}^2 \tag{2.13}
\]

Combining $I_2$, $I_4$, $I_5$, $I_6$, and $I_{s,2}$ in the following way:

\[
I_2 + I_4 + I_5 + I_6 - I_{s,2} = \lambda^2 \left(s(3\beta^2 - 1) - 8\|A\|_{L^\infty(M_T)}^2 - \frac{2}{\lambda^2} \|\tilde{q}\|_{L^\infty(M_T)}^2 \right) \|v\|_{L^2(M_T)}^2 \\
+ (\text{terms having lower power of } \lambda) \|v\|_{L^2(M_T)}^2 + \left(3s - 8\|A\|_{L^\infty(M_T)}^2 \right) \|\nabla g\|_{L^2(M_T)}^2 \\
+ \int_{\Sigma} (\lambda - s(x_1 + 2\ell)) |\partial_v v(t,x)|^2_{g_0} |\nu, e_1 g| dS_g dt.
\]

After choosing $s$ and $\lambda$ large enough together with using the fact that $\beta \in (1/\sqrt{3}, 1)$, and a combination of all estimates obtained above, will amount to have the following estimate on $\|P_s v\|_{L^2(M_T)}^2$:

\[
\|P_s v\|_{L^2(M_T)}^2 \geq C \left(s\lambda^2 \|v\|_{L^2(M_T)}^2 + \|\nabla g v(T, \cdot)\|_{L^2(M)}^2 - \lambda^2 \|v(T, \cdot)\|_{L^2(M)}^2 - s \|\nabla g\|_{L^2(M_T)}^2 \right) \\
+ \lambda \int_{\Sigma} |\partial_v v(t,x)|^2_{g_0} |\nu, e_1 g| dS_g dt \tag{2.14}
\]

where constant $C$ depends only on $A$, $q$, $T$ and $M$. 
This provides the estimate for the operator \( P_s = e^{-\phi_s(t,x)} \mathcal{L}_{A,q} e^{\phi_s(t,x)} \). To obtain the required Carleman estimate, we put \( v(t, x) = e^{-\phi_s(t,x)} u(t, x) \)
\[
\|e^{-\phi_s} \mathcal{L}_{A,q} u\|_{L^2(M_T)}^2 + \lambda^2 C \|e^{-\phi_s(T, \cdot)} u(T, \cdot)\|_{L^2(M)}^2 + \lambda \int_{\Sigma_+} |e^{-\phi_s} \partial_\nu u(t, x)|^2 \langle \nu, e_1 \rangle_g \ dS_g dt
\]
\[
\geq s\lambda^2 \|e^{-\phi_s} u\|_{L^2(M_T)}^2 + \|e^{-\phi_s(T, \cdot)} \nabla_g u(T, \cdot)\|_{L^2(M)}^2 + s \|e^{-\phi_s} \nabla_g u\|_{L^2(M_T)}^2 + \lambda \int_{\Sigma_+} |e^{-\phi_s} \partial_\nu u(t, x)|^2 \langle \nu, e_1 \rangle_g \ dS_g dt.
\]
Finally, using the expression for \( \phi(t, x) \) and the fact that \( e^{-\frac{s(\epsilon_1 + 2\theta)}{2}} \) has a strictly positive lower and upper bound, we get the following required estimate
\[
\lambda^2 \|e^{-\phi} u\|_{L^2(M_T)}^2 + \|e^{-\phi} \nabla_g u\|_{L^2(M_T)}^2 + \|e^{-\phi(T, \cdot)} \nabla_g u(T, \cdot)\|_{L^2(M)}^2 + \lambda \int_{\Sigma_+} e^{-2\phi} |\partial_\nu u(t, x)|^2 \langle \nu, e_1 \rangle_g \ dS_g dt
\]
\[
\leq C \left( \|e^{-\phi} \mathcal{L}_{A,q} u\|_{L^2(M_T)}^2 + \lambda^2 \|e^{-\phi(T, \cdot)} u(T, \cdot)\|_{L^2(M)}^2 + \lambda \int_{\Sigma_+} e^{-2\phi} |\partial_\nu u(t, x)|^2 \langle \nu, e_1 \rangle_g \ dS_g dt \right)
\]
for some constant \( C > 0 \) independent of \( \lambda \). This completes the proof of the theorem. \( \square \)

Our next aim of this section is to derive the interior Carleman estimates in a Sobolev space of negative order for \( \mathcal{L}_{A,q} \) and its formal \( L^2 \)-adjoint \( \mathcal{L}^*_{A,q} \). Before going to state and prove the interior Carleman estimates, we first give some definitions and notations for semi-classical Sobolev spaces of arbitrary order. This will help us to represent the Carleman estimates in a nice form. Let us begin by assuming that \((M, g)\) is embedded in a compact Riemannian manifold \((N, g)\) without boundary and denote by \(N_T := (0, T) \times N\). Following [29], we denote by \( J^s \) for \( s \in \mathbb{R} \), the semi-classical pseudo-differential operator of order \( s \) on \((N, g)\) and it is defined by \( J^s := (\lambda^2 - \Delta_g)^{s/2} \). Using this we define the semi-classical Sobolev space \( H^s_\lambda(N) \) for \( s \in \mathbb{R} \), as the completion of \( C^\infty(N) \) with respect to the following norm
\[
\|u\|_{H^s_\lambda(N)} := \|J^s u\|_{L^2(N)}.
\]
Since \((N, g)\) is a Riemannian manifold without boundary therefore the dual of \( H^s_\lambda(N) \), for any \( s \in \mathbb{R} \) can be identified with \( H^{-s}_\lambda(N) \). Also note that for \( s = 1 \), we have that
\[
\|u\|_{H^1_\lambda(N)}^2 := \lambda^2 \|u\|_{L^2(N)}^2 + \|\nabla_g u\|_{L^2(N)}^2.
\]
Now following [25] the time-dependent Sobolev space \( L^2(0, T; H^s_\lambda(N)) \) is defined as the set of all strongly measurable functions \( u : [0, T] \to H^s_\lambda(N) \) such that
\[
\|u\|_{L^2(0, T; H^s_\lambda(N))} := \left( \int_0^T \|u(t, \cdot)\|_{H^s_\lambda(N)}^2 \ dt \right)^{1/2} < \infty.
\]
Then \( L^2(0, T; H^s_\lambda(N)) \) is a Banach space with respect to the norm \( \| \cdot \|_{L^2(0, T; H^s_\lambda(N))} \) defined by (2.15) and the dual of \( L^2(0, T; H^s_\lambda(N)) \) can be identified with \( L^2(0, T; H^{-s}_\lambda(N)) \). If we take \( v \in C^\infty_c(M_T) \) in (2.14) then we have the following estimate
\[
\|v\|_{L^2(0, T; H^s_\lambda(N))}^2 \leq C \|L^-\phi v\|_{L^2(0, T; L^2(N))}^2
\]
where \( \phi \) is same as in Theorem [2.1] and \( L^-\phi := e^{-\phi} \mathcal{L}_{A,q} e^{-\phi} \). Now if we denote by \( L^* = e^\phi \mathcal{L}^*_{A,q} e^{-\phi} \) where \( \mathcal{L}^*_{A,q} \) stands for a formal \( L^2 \)-adjoint of \( \mathcal{L}_{A,q} \) then using the arguments similar to the one used in deriving (2.14), the following estimate
\[
\|u\|_{L^2(0, T; H^s_\lambda(N))}^2 \leq C \|L^*_{\phi} u\|_{L^2(0, T; L^2(N))}^2
\]
holds for all \( u \in C^\infty_c(M_T) \) where \( \phi \) is same as in Theorem [2.1] and constant \( C > 0 \) is independent of \( \lambda \) and \( u \).
In order to construct the suitable solutions to $\mathcal{L}_{A,q}^* u = 0$ and $\mathcal{L}_{A,q} v = 0$, we need to shift the index by $-1$ for spatial variable in (2.16) and (2.17) respectively, which we will do in the following lemma.

Lemma 2.2. Let $\mathcal{L}_{\phi}^* := e^{\phi} \mathcal{L}_{A,q}^* e^{-\phi}$, and $\mathcal{L}_{\phi} := e^{-\phi} \mathcal{L}_{A,q} e^{\phi}$, where $\mathcal{L}_{A,q}^*$ denote the formal $L^2$-adjoint of $\mathcal{L}_{A,q}$ and $\phi$, $A$ and $q$ be as in Theorem 2.1. Then there exists a constant $C > 0$ independent of $\lambda$ and $v$ such that

$$\|v\|_{L^2((0,T];L^2(N))} \leq C \|\mathcal{L}_{\phi}^* v\|_{L^2(0,T;H^{-1}_\lambda(N))}$$

(2.18)

holds for all $\lambda$ large enough and for all $v \in C_0^\infty(M_T)$ and

$$\|v\|_{L^2((0,T];L^2(N))} \leq C \|\mathcal{L}_{\phi} v\|_{L^2(0,T;H^{-1}_\lambda(N))}$$

(2.19)

holds for all $\lambda$ large enough and for all $v \in C_0^\infty(M_T)$.

Proof. First, we establish (2.18), and the proof for (2.19) can be carried out in a similar manner. We begin with the inequality:

$$\|v\|_{L^2((0,T];H^1_\lambda(N))} \leq C \|\mathcal{L}_{\phi}^* v\|_{L^2(0,T;L^2(N))}$$

holds for all $v \in C_0^\infty(M_T)$. Next, we shift the index by $-1$ in the above estimate. Let $w \in C_0^\infty(M_T)$ and consider the adjoint operator defined as:

$$\mathcal{L}_{A,q}^* := \left(-\partial_t - \sum_{j,k=1}^n \frac{1}{\sqrt{|g|}} (\partial_{x_j} - A_j)(\sqrt{|g|} g^{jk}(\partial_{x_k} - A_k)) + \bar{q}\right).$$

For $s > 0$, define the convexified weight function $\varphi_s$ as follows:

$$\varphi_s(t,x) := \varphi(t,x) + \frac{s(x_1 + 2\ell)^2}{2} = \lambda^2 \beta^2 t + \lambda x_1 + \frac{s(x_1 + 2\ell)^2}{2}.$$  

Let $P_s^* := e^{\varphi_s} \mathcal{L}_{A,q}^* e^{-\varphi_s}$, we have

$$P_s^* w = e^{\varphi_s} \left(-\partial_t - \partial_{x_1}^2 - \Delta_{g_0} + 2\langle A, \nabla g \rangle + \bar{q}_s\right) e^{-\varphi_s} w$$

where $\bar{q}_s(t,x) := \bar{q}(t,x) + \delta_q A(t,x) - |A(t,x)|^2$.

Expressing $P_s^* w$ as a sum of three components:

$$P_s^* w := P_1^* w(t,x) + P_2^* w(t,x) + P_3^* w(t,x)$$

where

$$P_1^* w(t,x) := (-\partial_t + 2(\lambda + s(x_1 + 2\ell))(\partial_{x_1} + 4sw)) (t,x),$$

$$P_2^* w(t,x) := (-\partial_{x_1}^2 - \Delta_{g_0} - \lambda^2 (1 - \beta^2) w - 2\lambda s(x_1 + 2\ell) w - s^2 (x_1 + 2\ell)^2 w - 3sw) (t,x),$$

$$P_3^* w(t,x) := 2\langle A(t,x), \nabla g w(t,x) \rangle g^{1k} A_k(t,x) w(t,x) + \bar{q}_s(t,x) w(t,x).$$

Furthermore, we have $J^{-1}(P_1^* + P_2^*) J^1 w = (P_1^* + P_2^*) w$, from this we get

$$\|(P_1^* + P_2^*) J^1 w\|_{L^2(0,T;H^{-1}_\lambda(N))} = \|J^{-1}(P_1^* + P_2^*) J^1 w\|_{L^2(0,T;L^2(N))}$$

$$= \|(P_1^* + P_2^*) w\|_{L^2(0,T;L^2(N))}.$$  

From the same calculation as done for the Carleman estimate (2.1), we obtain

$$\|(P_1^* + P_2^*) J^1 w\|_{L^2(0,T;H^{-1}_\lambda(N))} \geq \sqrt{s} \|\nabla g w\|_{L^2(0,T;L^2(N))} + \sqrt{s} \lambda \|w\|_{L^2(0,T;L^2(N))}.$$  

Now consider

$$\|P_3^* J^1 w\|_{L^2(0,T;H^{-1}_\lambda(N))} \leq C(\|A\| \|\nabla g w\|_{L^2(0,T;L^2(N))} + \lambda \|A\| \|w\|_{L^2(0,T;L^2(N))} + \|\bar{q}_s\| \|w\|_{L^2(0,T;L^2(N))}).$$

Hence using the inequality and choosing $s$ and $\lambda$ large enough, we get

$$\|P_s^* J^1 w\|_{L^2(0,T;H^{-1}_\lambda(N))} \geq C \|w\|_{L^2(0,T;H^{-1}_\lambda(N))}.$$
Now, consider $\chi \in C_c^\infty(\bar{M})$ such that $\chi = 1$ in $\bar{M}_1$ where $\bar{M} \subset M_1 \subset \hat{M}$. By taking $w = \chi J^{-1}v$ in the above estimate and using
\[
\|(1 - \chi)J^{-1}v\|_{L^2(0,T;H^1_\lambda(N))} \leq \frac{C}{\lambda^2} \|v\|_{L^2(0,T;L^2(N))}
\]
and
\[
\|v\|_{L^2(0,T;L^2(N))} = \|J^{-1}v\|_{L^2(0,T;H^1_\lambda(N))} \leq \|v\|_{L^2(0,T;H^1_\lambda(N))} + \frac{C}{\lambda^2} \|v\|_{L^2(0,T;L^2(N))}.
\]
We get
\[
\|P_s^*v\|_{L^2(0,T;H^{-1}_\lambda(N))} \geq \|P_s^*J^1w\|_{L^2(0,T;H^{-1}_\lambda(N))} - \frac{C}{\lambda^2} \|v\|_{L^2(0,T;L^2(N))}
\]
\[
\geq \|w\|_{L^2(0,T;H^1_\lambda(N))} - \frac{C}{\lambda^2} \|v\|_{L^2(0,T;L^2(N))}
\]
\[
\geq C\|v\|_{L^2(0,T;L^2(N))}
\]
hold for $\lambda$ large. Therefore, we have
\[
\|v\|_{L^2(0,T;L^2(N))} \leq C\|P_s^*v\|_{L^2(0,T;H^{-1}_\lambda(N))}.
\]
Now, using the expression for $\varphi(t,x)$ and the fact that $e^{\frac{\|z\|^4}{2(1+2\beta^2)}}$ has a strictly positive lower and upper bound, we conclude
\[
\|v\|_{L^2(0,T;L^2(N))} \leq C\|L^*_\varphi v\|_{L^2(0,T;H^{-1}_\lambda(N))}
\]
holds for all $\lambda$ large enough and for all $v \in C_c^\infty(M_T)$. \hfill $\Box$

The above estimates, together with the Hahn-Banach theorem and the Riesz representation theorem, give the following solvability result, proof of which follows from [18, 50].

**Lemma 2.3.** Let $\varphi$, $A$ and $q$ be as before and $\lambda > 0$ be large enough. Then for $F \in L^2(M_T)$ there exists a solution $u \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$ of
\[
\mathcal{L}_\varphi w(t,x) = F(t,x), \ (t,x) \in M_T
\]
satisfying the following estimate
\[
\|u\|_{L^2(0,T;H^1(M))} \leq C\|F\|_{L^2(M_T)} \tag{2.20}
\]
for some constant $C > 0$ independent of $\lambda$ and $u$ and there exists a solution $v \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$ of
\[
\mathcal{L}_\varphi^* w(t,x) = F(t,x), \ (t,x) \in M_T
\]
satisfying the following estimate
\[
\|v\|_{L^2(0,T;H^1(M))} \leq C\|F\|_{L^2(M_T)} \tag{2.21}
\]
for some constant $C > 0$ independent of $\lambda$ and $v$.

**Proof.** The proof for $\mathcal{L}_\varphi$ is presented below, and the proof for $\mathcal{L}_\varphi^*$ can be established using analogous arguments.

Consider the subspace $S$ of $L^2(0,T;H^{-1}_\lambda(N))$ defined as
\[
S := \{ \mathcal{L}_\varphi^* w(t,x) : w \in C_c^\infty(M_T) \}.
\]
Define the linear operator $T$ on $S$ by
\[
T(\mathcal{L}_\varphi^* z) = \int_{M_T} z(t,x) F(t,x) \, dV_g dt, \text{ for } F \in L^2(M_T).
\]
For any $\mathcal{L}_\varphi^* z \in S$, we have
\[
|T(\mathcal{L}_\varphi^* z)| \leq \int_{M_T} |z(t,x)||F(t,x)| \, dV_g \, dt \leq \|z\|_{L^2(M_T)}\|F\|_{L^2(M_T)}.
\]
Using the Carleman estimate \cite{2,15}, we obtain
\[
|T(\mathcal{L}_\varphi^* z)| \leq C\|F\|_{L^2(M_T)}\|\mathcal{L}_\varphi^* z\|_{L^2(0,T;M_0^1(N))}.
\]
This inequality holds for $z \in C_c^\infty(M_T)$. By the Hahn-Banach theorem, extend the linear operator $T$ to $L^2(0,T;H_1^1(N))$. Denote the extended map as $T$ and note that it satisfies the inequality
\[
\|T\| \leq C\|F\|_{L^2(M_T)}.
\]
By the Riesz representation theorem, as $T$ is a bounded linear functional on $L^2(0,T;H_1^1(N))$, there exists a unique $u \in L^2(0,T;H_1^1(N))$ such that
\[
T(f) = \langle f, u \rangle_{L^2(0,T;H_1^1(N)),L^2(0,T;H_1^1(N))}
\]
for $f \in L^2(0,T;H_1^1(N))$, with
\[
\|u\|_{L^2(0,T;H_1^1(N))} \leq C\|F\|_{L^2(M_T)}.
\]
Now, for $z \in C_c^\infty(M_T)$, choosing $f = \mathcal{L}_\varphi^* z$ in the above equation, we get $\mathcal{L}_\varphi u = F$.
Using the expression for $\mathcal{L}_\varphi$ and the fact that $u \in L^2(0,T;H^1(M))$ and $F \in L^2(M_T)$, we conclude that $\partial_t u \in L^2(0,T;H^{-1}(M))$. Hence, we have $u \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$.

\[\square\]

3. Construction of geometric optics solutions

In this section, we aim to construct the exponential growing and decaying solutions to the convection-diffusion operator $\mathcal{L}_{A,g}$ and its $L^2$-adjoint $\mathcal{L}_{A,g}^*$, respectively. Construction of these solutions will be proved with the help of the interior Carleman estimates in negative order Sobolev spaces stated in Lemma \cite{2,2}.

3.1. Construction of exponentially growing solutions. In this subsection, we will construct the exponential growing solutions to $\mathcal{L}_{A,g} u(t,x) = 0$, in $M_T$ which takes the following form
\[
u(t,x) = e^{(\varphi+i\psi)(t,x)} \left( T_g(t,x) + R_{g,\lambda}(t,x) \right)
\]
where $\varphi$ is same as in Theorem \cite{2,1} and $\psi$, $T_g$ will be constructed using the WKB construction in such a way that the correction term $R_{g,\lambda}$ satisfies the following
\[
e^{-(\varphi+i\psi)}\mathcal{L}_{A,g} \left( e^{(\varphi+i\psi)} R_{g,\lambda}(t,x) \right) = F_{\lambda}(t,x), (t,x) \in M_T
\]
for $F_{\lambda} \in L^2(M_T)$ such that $\|F_{\lambda}\|_{L^2(M_T)} \leq C$, for some constant $C > 0$ independent of $\lambda$ and $R_{g,\lambda}$ satisfies $\|R_{g,\lambda}\|_{L^2(0,T;H_1^1(M))} \leq C\|F_{\lambda}\|_{L^2(M_T)}$, for some constant $C > 0$, not depending on $\lambda$. More precisely, we prove the following theorem.

**Theorem 3.1.** Let $M_T$, $\mathcal{L}_{A,g}$ and $\varphi$ be as before. Suppose $(D,g_0)$ be a simple manifold which is extension of $(M_0,g_0)$ in the sense that $M_0 \subset D$ and $y_0 \in D$ is such that $(x_1,y_0) \notin M$ for all $x_1$. Now if $(r,\theta)$ denote the polar normal coordinates on $(D,g_0)$, $(x_1,r,\theta)$ denote the points in $M$ and $A_1$ and $A_r$ are components of $A$ in $x_1$ and $r$ coordinates respectively, then for $\lambda$ large enough the following equation
\[
\mathcal{L}_{A,g} u(t,x) = 0, (t,x) \in M_T
\]
has a solution taking the following form
\[
u(t,x) = e^{\varphi+i\psi} \left( T_g(t,x_1,r,\theta) + R_{g,\lambda}(t,x_1,r,\theta) \right)
\]
\[\text{(3.2)}\]
where
\[ \psi = \lambda(\sqrt{1 - \beta^2})r, \] and \( T_g(t, x_1, r, \theta) = \phi(t)e^{i\eta\left(\sqrt{1 - \beta^2}\right)x_1}e^{-\nu r}e^{i\Phi_1(t, x_1, r, \theta)b(r, \theta) - 1/4}h(\theta) \]
here \( \phi \in C_c^{\infty}(0, T) \), \( \Phi_1 \) is solution to
\[ \partial_t \Phi_1 + i(\sqrt{1 - \beta^2})\partial_r \Phi_1 + \left(-iA_1 + (\sqrt{1 - \beta^2})A_r\right) = 0 \]
and \( R_{g, \lambda} \) satisfies the following
\[ \mathcal{L}_x \left(e^{i\psi}R_{g, \lambda}\right)(t, x) = -e^{i\psi}\mathcal{L}_{A, q}T_g(t, x), \quad (t, x) \in M_T \]
and \( \|R_{g, \lambda}\|_{L^2(0, T; H^1(M))} \leq C \) for some constant \( C > 0 \) independent of \( \lambda \).

Proof. Following [29], if we denote \( \rho := \varphi + i\psi \), then simple calculations show that the conjugated operator \( \mathcal{L}_\rho := e^{-\rho}\mathcal{L}_{A, q}e^{\rho} \) will have the following expression
\[ \mathcal{L}_\rho = \mathcal{L}_{A, q} + \left(\partial_t \rho - \Delta_g \rho - g^{jk}\partial_j \rho \partial_k \rho\right) - 2 \left(g^{jk}\partial_j \rho \partial_k \psi + g^{jk}\partial_j \rho A_k\right). \]
Using \( \rho = \varphi + i\psi \), and \( \varphi = \lambda^2\beta^2 t + \lambda x_1 \), we get
\[ \mathcal{L}_\rho = \mathcal{L}_{A, q} + \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j \psi \partial_k \psi\right) - 2 \left(2\lambda \partial_t + 2ig^{jk}\partial_j \psi \partial_k + 2\lambda A_1 + 2ig^{jk}\partial_j \psi A_k + i\Delta_g \psi + 2i\lambda \partial_t \psi - i\partial_t \psi\right). \]
Now \( u \) given by (3.1) solves \( \mathcal{L}_{A, q}v = 0 \) if and only if \( \mathcal{L}_\rho \left(e^{-\rho}u\right) = 0 \). This will give us
\[ \mathcal{L}_\rho R_{g, \lambda}(t, x) = -\mathcal{L}_{A, q}T_g(t, x) - \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j \psi \partial_k \psi\right)T_g(t, x) \]
\[ + \left(2\lambda \partial_t + 2ig^{jk}\partial_j \psi \partial_k + 2\lambda A_1 + 2ig^{jk}\partial_j \psi A_k + i\Delta_g \psi + 2i\lambda \partial_t \psi - i\partial_t \psi\right)T_g(t, x), \quad (t, x) \in M_T. \]
In order to have \( \|R_{g, \lambda}\|_{L^2(0, T; H^1(M))} \leq C \), we choose \( \psi \) and \( T_g \) such that
\[ \partial_t \psi = 0, \quad g^{jk}\partial_j \psi \partial_k \psi = \lambda^2(1 - \beta^2) \]
and
\[ \left(2\lambda \partial_t + 2ig^{jk}\partial_j \psi \partial_k + 2\lambda A_1 + 2ig^{jk}\partial_j \psi A_k + i\Delta_g \psi + 2i\lambda \partial_t \psi\right)T_g(t, x) = 0, \quad (t, x) \in M_T. \]
To solve equations (3.5) and (3.6) for \( \psi \) and \( T_g \), we use the polar normal coordinates \((r, \theta)\) on \((D, g_0)\) centered at \( y_0 \in D \) as mentioned in statement of theorem. We consider the polar normal coordinates on \( D \) which are denoted by \((r, \theta)\) and given by \( x_0 = \exp_{y_0}(r \theta) \), where \( r > 0 \) and \( \theta \in S_{y_0}(D) := \{v \in T_{y_0}D : |v|_g = 1\} \), here \( T_{y_0}D \) denote the tangent space to \( D \) at \( y_0 \in D \). Then using the Gauss lemma (see Lemma 15 in Chapter 9 of [52]) there exists a smooth positive definite matrix \( P(r, \theta) \) with \( \det(P) := b(r, \theta) \) such that the metric \( g_0 \) in the polar normal coordinates \((r, \theta)\), takes the following form
\[ g_0(r, \theta) = \begin{bmatrix} 1 & 0 \\ 0 & P(r, \theta) \end{bmatrix}. \]
Now since the points in \( M \) are denoted by \((x_1, r, \theta)\) where \((r, \theta)\) are polar normal coordinates in \((D, g_0)\), therefore after using the previous form of \( g_0 \), the metric \( g \) has the following form
\[ g(x_1, r, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P(r, \theta) \end{bmatrix}. \]
Using (3.8), we see that
\[ \psi(x) = (\lambda \sqrt{1 - \beta^2})\text{d} \text{ist}_g(y_0, x) = (\lambda \sqrt{1 - \beta^2})r, \]
solves equation (3.5) on $M$. Using this choice of $\psi$ and form of $g$ given by (3.8) in equation (3.6), we have
\[
\left( \partial_t + i(\sqrt{1-\beta^2})\partial_r + A_1 + i(\sqrt{1-\beta^2})A_r + i(\sqrt{1-\beta^2}) \frac{\partial b(r, \theta)}{4b(r, \theta)} \right) T_g(t, x_1, r, \theta) = 0.
\]
Now, one can check that the solution of the above equation can be given by
\[
T_g(t, x_1, r, \theta) = \phi(t) e^{i\mu(\sqrt{1-\beta^2} - 1/4)t} e^{-\mu t} e^{i\Phi_1(t, x_1, r, \theta)} b(r, \theta)^{-1/4} h(\theta)
\]
where $\phi \in C^\infty_c(0, T)$, $\mu \in \mathbb{R}$, $h \in C^\infty(S_{g_0}(D))$ are arbitrary but fixed and $\Phi_1(t, x_1, r, \theta)$ satisfies the following
\[
\left( \partial_t \Phi_1 + i \left( \sqrt{1-\beta^2} \right) \partial_r \Phi_1 \right) + \left( -i A_1 + (\sqrt{1-\beta^2}) A_r \right) = 0.
\]
Now using (3.9) and (3.10) in (3.4), we get
\[
\mathcal{L}_\rho R_{g, \lambda}(t, x) = -\mathcal{L}_{A,q} T_g(t, x), \quad (t, x) \in M_T
\]
But $\mathcal{L}_\rho R_{g, \lambda} = e^{-i\psi} \mathcal{L}_\phi (e^{i\psi} R_{g, \lambda})$ therefore if we denote $\tilde{R}_{g, \lambda} = e^{i\psi} R_{g, \lambda}$ then $\tilde{R}_{g, \lambda}$ satisfies the following equation
\[
\mathcal{L}_\rho \tilde{R}_{g, \lambda}(t, x) = -e^{i\psi} \mathcal{L}_{A,q} T_g(t, x), \quad (t, x) \in M_T
\]
Now using the expressions for $\psi$ and $T_g$ from (3.9) and (3.10) respectively and assumptions on $A$ and $q$, we have that $-e^{i\psi} \mathcal{L}_{A,q} T_g \in L^2(M_T)$ and $\|e^{i\psi} \mathcal{L}_{A,q} T_g\|_{L^2(M_T)} \leq C$, for some constant $C > 0$ independent of $\lambda$. Hence using Lemma 2.3 together with above estimate for right hand side of (3.12), we conclude that there exists $\tilde{R}_{g, \lambda} \in H^1(0, T; H^{-1}(M)) \cap L^2(0, T; H^1(M))$ solving (3.12) and it satisfies the following estimate $\|\tilde{R}_{g, \lambda}\|_{L^2(0, T; H^1(M))} \leq C$, for some constant $C > 0$, independent of $\lambda$. Hence, we conclude that $R_{g, \lambda}$ solves the required equation and satisfies the desired estimate. This completes the proof of the Theorem. □

3.2. Construction of exponentially decaying solutions. The aim of this subsection is to construct the exponential decaying solutions to
\[
\mathcal{L}_{A,q}^* u := \left( -\partial_t - \sum_{j,k=1}^n \frac{1}{\sqrt{|g|}} (\partial_{x_j} - A_j)(\sqrt{|g|} g^{jk}(\partial_{x_k} - A_k)) + \bar{q} \right) u = 0, \quad (t, x) \in M_T
\]
Taking the following form
\[
u(t, x) = e^{-(\varphi - i\psi)(t, x)} \left( T_d(t, x) + R_{d, \lambda}(t, x) \right)
\]
where $\varphi$ is the same as in Theorem 2.1 and $\psi, T_d$ will be constructed using the WKB construction in such a way that the correction term $R_{d, \lambda}$ satisfies the following
\[
e^{(\varphi - i\psi)} \mathcal{L}_{A,q}^* e^{-(\varphi - i\psi)} R_{d, \lambda}(t, x) = F_{\lambda}(t, x), \quad (t, x) \in M_T
\]
for some $F_{\lambda} \in L^2(M_T)$ such that $\|F_{\lambda}\|_{L^2(M_T)} \leq C$, for some constant $C > 0$ independent of $\lambda$ and $R_{d, \lambda}$ satisfies $\|R_{d, \lambda}\|_{L^2(0, T; H^1(M))} \leq C \|F_{\lambda}\|_{L^2(M_T)}$, for some constant $C > 0$ not depending on $\lambda$. To construct these solutions, we first start with the construction of $\psi$ and $T_d$ following the arguments used in Theorem 3.1 Denote by $\rho := \varphi - i\psi$, then one can check that the conjugated operator
\[
\mathcal{L}_\rho^* := e^\rho \mathcal{L}_{A,q}^* e^{-\rho} = e^\rho \left( -\partial_t - \sum_{j,k=1}^n \frac{1}{\sqrt{|g|}} (\partial_{x_j} - A_j)(\sqrt{|g|} g^{jk}(\partial_{x_k} - A_k)) + \bar{q} \right) e^{-\rho}
\]
is given by
\[
\mathcal{L}_\rho^* = \mathcal{L}_{A,q}^* + \left( \partial_t \rho - g^{jk} \partial_j \rho \partial_k \rho \right) + \left( 2g^{jk} \partial_j \rho \partial_k \rho - 2g^{jk} \partial_j \rho \partial_k A_k + \Delta_q \rho \right).
\]
Using $\rho = \varphi - i\psi$ and $\varphi = \lambda^2\beta^2t + \lambda x_1$, we have
\[
\mathcal{L}_\rho^* = \mathcal{L}_{A,q}^* + \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi\right) + \left(2\lambda\partial_1 - 2ig^{jk}\partial_j\psi\partial_k - 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k - i\Delta_g\psi + 2i\lambda\partial_t\psi - i\partial_t\psi\right).
\]
Now we observe that $u$ given by (3.13) solves $\mathcal{L}_{A,q}^* v = 0$ in $M_T$ if and only if $\mathcal{L}_\rho^*(e^{\rho}u) = 0$ in $M_T$. Using this, we see that $R_{d,\lambda}$ satisfies the following equation
\[
\mathcal{L}_\rho^* R_{d,\lambda}(t, x) = -\mathcal{L}_{A,q}^* T_d(t, x) - \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi\right) T_d(t, x)
- \left(2\lambda\partial_1 - 2ig^{jk}\partial_j\psi\partial_k - 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k - i\Delta_g\psi + 2i\lambda\partial_t\psi - i\partial_t\psi\right) T_d(t, x).
\]
(3.14)
To get the estimate $\|R_{d,\lambda}\|_{L^2(0, T; H^1(M))} \leq C$, for some constant $C > 0$ independent of $\lambda$, we choose $\psi$ and $T_d$ satisfying the following equations
\[
\partial_t\psi = 0, \quad \lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi = 0
\]
(3.15)
and
\[
\left(2\lambda\partial_1 - 2ig^{jk}\partial_j\psi\partial_k - 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k - i\Delta_g\psi + 2i\lambda\partial_t\psi - i\partial_t\psi\right) T_d(t, x) = 0, \quad (t, x) \in M_T
\]
(3.16)
respectively. To solve equations (3.15) and (3.16) for $\psi$ and $T_d$, we again use the polar normal coordinates $(r, \theta)$ on $(D, g_0)$ centered at $y_0 \in D$ as used in the proof of Theorem 3.1. For a fixed $y_0 \in D$, we consider the polar normal coordinates on $D$ which are denoted by $(r, \theta)$ and given by $x_0 = \exp_{y_0}(r\theta)$, where $r > 0$ and $\theta \in S_{y_0}(D) : \{v \in T_{y_0}D : |v|_g = 1\}$, here $T_{y_0}D$ denote the tangent space to $D$ at $y_0 \in D$. Then using the Gauss lemma (see Lemma 15 in Chapter 9 of [22]) there exists a smooth positive definite matrix $P(r, \theta)$ with $\det P(r, \theta) = b(r, \theta)$ such that the metric $g_0$ in the polar normal coordinates $(r, \theta)$, takes form given by (3.8). Now since the points in $M$ are denoted by $(x_1, r, \theta)$ where $(r, \theta)$ are polar normal coordinates in $(D, g_0)$, therefore after using the form of $g_0$ given by (3.8), the metric $g$ takes the form given by equation (3.13) and using this, we observe that
\[
\psi(x) = \left(\lambda\sqrt{1 - \beta^2}\right) \text{dist}_g(y_0, x) = \left(\lambda\sqrt{1 - \beta^2}\right) r,
\]
(3.17)
solves equation (3.15) and
\[
T_d(t, x_1, r, \theta) = \phi(t) e^{i\Phi_2(t, x_1, r, \theta)} b(r, \theta)^{-1/4} h(t)
\]
(3.18)
solves equation (3.16) where $\phi \in C_c^\infty(0, T)$, $h \in C^\infty(S_{y_0}(D))$ are arbitrary but fixed and $\Phi_2(t, x_1, r, \theta)$ satisfies the following
\[
\left(\partial_t\Phi_2 - i \left(\sqrt{1 - \beta^2}\right) \partial_r\Phi_2\right) + \left(iA_1 + \sqrt{1 - \beta^2} A_r\right) = 0,
\]
(3.19)
$A_1$ and $A_r$ are components of $A$ in $x_1$ and $r$ coordinates respectively. Now if we use (3.17) and (3.18) in equation (3.14) and repeating the arguments used in showing the estimate for $R_{d,\lambda}$ in Theorem 3.1, then we get that there exists $R_{d,\lambda} \in H^1(0, T; H^{-1}(M)) \cap L^2(0, T; H^1(M))$ solving
\[
\mathcal{L}_\varphi^* \left(e^{i\psi} R_{d,\lambda}\right)(t, x) = -e^{i\psi} \mathcal{L}_{A,q} T_d(t, x), \quad (t, x) \in M_T
\]
(3.20)
and $R_{d,\lambda}$ satisfies the following estimate
\[
\|R_{d,\lambda}\|_{L^2(0, T; H^1(M))} \leq C
\]
(3.21)
for some constant $C > 0$ independent of $\lambda$. Combining all these, we end up with proving the following theorem.

**Theorem 3.2.** Let $M_T$, $\mathcal{L}_{A,q}$ and $\varphi$ be as before. Suppose $(D, g_0)$ be a simple manifold which is extension of $(M_0, g_0)$ in the sense that $M_0 \subset D$ and $y_0 \in D$ is such that $(x_1, y_0) \notin M$ for all $x_1$. Now if $(r, \theta)$ denote
the polar normal coordinates on \((D,g_0)\), \((x_1,r,\theta)\) denote the points in \(M\) and \(A_1\) and \(A_r\) are components of \(A\) in \(x_1\) and \(r\) coordinates respectively, then for \(\lambda\) large enough the following equation

\[ L^{\ast}_{A,q}v(t,x) = 0, \quad (t,x) \in M_T \]

has a solution taking the following form

\[ v(t,x) = e^{-(\rho-\psi)(t,x)}(T_d(t,x_1,r,\theta) + R_d,\lambda(t,x_1,r,\theta)) \]

where \(\psi, T_d\) are given by (3.17), (3.18) and \(R_d,\lambda\) satisfies (3.20) and (3.21).

4. Derivation of Integral Identity and Proof of Main Theorem

We use this section to derive an integral identity, which will be required to prove our main result. Later, using the geometric optics solutions constructed in Section 3 we conclude the proof of Theorem 1.4. We start by recalling

\[ L_{A,q} = \partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}} \left( \partial_{x_j} + A_j(t,x) \right) \left( g^{jk} \sqrt{|g|} (\partial_{x_k} + A_k(t,x)) \right) + q(t,x) \]

and

\[ L^{\ast}_{A,q} = -\partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}} \left( \partial_{x_j} - A_j(t,x) \right) \left( \sqrt{|g|} g^{jk} (\partial_{x_k} - A_k(t,x)) \right) + \mathcal{Q}(t,x). \]

For \(l = 1,2\), let \(A^{(l)}\) and \(q_l\) be as in Theorem 1.4. Further assume that \(u_l\) is solution to the corresponding IBVP for \(L_{A^{(l)},q_l}\) given by (1.1) when \((A,q) = (A^{(l)},q_l)\) for \(l = 1,2\), that is, for \(l = 1,2\), we have

\[
\begin{cases}
L_{A^{(l)},q_l} u_l(t,x) = 0, & (t,x) \in M_T \\
u_l(0,x) = \phi(x), & x \in M \\
u_l(t,x) = f(t,x), & (t,x) \in \Sigma.
\end{cases}
\]

Then \(u := u_1 - u_2\), satisfies the following IBVP with zero initial and boundary conditions

\[
\begin{cases}
L_{A^{(1)},q_1} u(t,x) = \mathcal{Q} u_2(t,x), & (t,x) \in M_T \\
u(0,x) = 0, & x \in M \\
u(t,x) = 0, & (t,x) \in \Sigma,
\end{cases}
\]

where \(\mathcal{Q} u_2(t,x) := \left( |A^{(1)}|^2_g - |A^{(2)}|^2_g \right) u_2 + 2 \left( A^{(1)} - A^{(2)} \right) \nabla_g u_2 + \delta_g \left( A^{(1)} - A^{(2)} \right) u_2 + (q_2 - q_1) u_2\). To simplify the notation, let us denote by \(\tilde{q}(t,x) := (\tilde{q}_1 - \tilde{q}_2)(t,x)\) and \(\tilde{A}(t,x) := (\tilde{A})_{1 \leq j \leq n} := (A^{(1)} - A^{(2)})(t,x)\). The \(\tilde{A}_i := |A^{(i)}|^2_g + \delta_g A^{(i)} - q_i\), for \(i = 1,2\), then with these notations \(\mathcal{Q} u_2\) becomes

\[ \mathcal{Q} u_2(t,x) = 2(\tilde{A}(t,x), \nabla_g u_2(t,x))_g + \tilde{q}(t,x) u_2(t,x). \]

Now since \(\mathcal{Q} u_2 \in L^2(M_T)\) therefore using Theorem 1.43 in [23] we have that there exists a unique solution \(u \in L^2(0,T;H^2(M) \cap H^1(0,T;L^2(M)))\) to (4.1) with \(\partial_x u \in L^2(0,T;H^{1/2}(\Sigma))\). Now if \(v(t,x)\) is a solution to the adjoint operator of \(L_{A^{(1)},q_1}\), given by

\[ L^{\ast}_{A^{(1)},q_1} v(t,x) = 0, \quad (t,x) \in M_T, \quad (4.3) \]
then we observe that

\[
(\langle \Lambda_{A(1),q_1} - \Lambda_{A(2),q_2}\rangle (\phi, f), v|_{\partial M_T^*}) = \langle \mathcal{N}_{A(1),q_1} u_1 - \mathcal{N}_{A(2),q_2} u_2, v|_{\partial M_T^*} \rangle
\]

\[
= \int_{M_T} \left( -u_1 \partial_t \mathcal{V} + \langle \nabla_g u_1, \nabla_g \mathcal{V} \rangle_g + 2u_1 \langle A^{(1)}, \nabla_g \mathcal{V} \rangle_g + (\delta_g A^{(1)}) u_1 \mathcal{V} - |A^{(1)}|^2 u_1 \mathcal{V} + q_1 u_1 \mathcal{V} \right) dV_g dt
\]

\[- \int_M u_1 (0, x) \hat{v}(0, x) dV_g
\]

\[- \int_{M_T} \left( -u_2 \partial_t \mathcal{V} + \langle \nabla_g u_2, \nabla_g \mathcal{V} \rangle_g + 2u_2 \langle A^{(2)}, \nabla_g \mathcal{V} \rangle_g + (\delta_g A^{(2)}) u_2 \mathcal{V} - |A^{(2)}|^2 u_2 \mathcal{V} + q_2 u_2 \mathcal{V} \right) dV_g dt
\]

\[+ \int_M u_2 (0, x) \hat{v}(0, x) dV_g.\]

Using integration by parts with \(u|_{\Sigma} = 0, u|_{t=0} = 0\) and \(v\) is solution to (4.3), we get

\[
(\langle \Lambda_{A(1),q_1} - \Lambda_{A(2),q_2}\rangle (\phi, f), v|_{\partial M_T^*}) = -2 \int_{M_T} \langle \hat{A}(t, x), \nabla_g u_2(t, x) \rangle \bar{v}(t, x) dV_g dt - \int_{M_T} \bar{q}(t, x) u_2(t, x) \bar{v}(t, x) dV_g dt.
\]

(4.4)

Multiplying equation \((4.2)\) by \(\bar{v}(t, x)\) and integrate it over \(M_T\), we get

\[
\int_{M_T} \mathcal{L}_{A(1),q_1} u(t, x) \bar{v}(t, x) dV_g dt = 2 \int_{M_T} \langle \hat{A}(t, x), \nabla_g u_2(t, x) \rangle \bar{v}(t, x) dV_g dt + \int_{M_T} \bar{q}(t, x) u_2(t, x) \bar{v}(t, x) dV_g dt.
\]

Now use the integration by parts together with \(u|_{\Sigma} = 0, u|_{t=0} = 0, \hat{A}|_{\Sigma} = 0\) and the fact that \(v\) is a solution to (4.3), to obtain the following identity

\[
2 \int_{M_T} \langle \hat{A}(t, x), \nabla_g u_2(t, x) \rangle \bar{v}(t, x) dV_g dt + \int_{M_T} \bar{q}(t, x) u_2(t, x) \bar{v}(t, x) dV_g dt
\]

\[= - \int_{M} \delta_{ij} \nu_j \partial_{x_i} u(t, x) \bar{v}(t, x) dS_g dt + \int_{M} u(T, x) \bar{v}(T, x) dV_g.
\]

(4.5)

From Equations (4.4) and (4.5), we have

\[
(\langle \Lambda_{A(1),q_1} - \Lambda_{A(2),q_2}\rangle (\phi, f), v|_{\partial M_T^*}) = \int_{\Sigma} g^{ik} \nu_j \partial_{x_k} u(t, x) \bar{v}(t, x) dS_g dt - \int_{M} u(T, x) \bar{v}(T, x) dV_g.
\]

Using (1.8), we get \(\partial_t u|_{\Sigma - \epsilon/2} = 0\) and \(u|_{t=T} = 0\). Therefore, Equation (4.5) becomes

\[
2 \int_{M_T} \langle \hat{A}(t, x), \nabla_g u_2(t, x) \rangle \bar{v}(t, x) dV_g dt + \int_{M_T} \bar{q}(t, x) u_2(t, x) \bar{v}(t, x) dV_g dt
\]

\[= - \int_{\Sigma \setminus \Sigma_{-\epsilon/2}} g^{ik} \nu_j \partial_{x_k} u(t, x) \bar{v}(t, x) dS_g dt.
\]

(4.6)

Let us define \(J_1, J_2\) and \(J_3\) by

\[
J_1 := 2 \int_{M_T} \langle \hat{A}(t, x), \nabla_g u_2(t, x) \rangle \bar{v}(t, x) dV_g dt, \quad J_2 := \int_{M_T} \bar{q}(t, x) u_2(t, x) \bar{v}(t, x) dV_g dt \quad \text{and}
\]

\[J_3 := - \int_{\Sigma \setminus \Sigma_{-\epsilon/2}} \partial_t u(t, x) \bar{v}(t, x) dS_g dt.
\]

With these notations (4.6) becomes

\[
J_1 + J_2 = J_3.
\]

(4.7)

Our next aim is to substitute the exponentially growing and decaying solutions constructed in section 3 for \(u_2\) and \(v\) respectively, in each term of equation (4.7). Recall that \(u_2\) satisfies

\[
\mathcal{L}_{A(2),q_2} u_2 = 0, \quad \text{in } M_T
\]
and $v$ satisfies

$$\mathcal{L}^{*\dagger} A_0 v = 0, \text{ in } M_T$$

therefore we choose the expressions for solutions $u_2$ and $v$ from (3.2) and (3.22) respectively, substitute in each term of (1.7). We start with the following calculations

$$\left\langle \tilde{A}(t,x), \nabla_g u_2(t,x) \right\rangle_g = \left( \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle_g T_g(t,x) + \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle_g R_{g,\lambda}(t,x) \right)$$

$$= \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle_g T_g(t,x)\tilde{T}_d(t,x) + \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle_g T_g(t,x)\tilde{R}_{d,\lambda}(t,x)$$

$$+ \left\langle \tilde{A}, \nabla_g T_g(t,x) \right\rangle \tilde{T}_d(t,x) + \left\langle \tilde{A}, \nabla_g T_g(t,x) \right\rangle \tilde{R}_{d,\lambda}(t,x)$$

$$+ \left\langle \tilde{A}, \nabla_g R_{g,\lambda}(t,x) \right\rangle \tilde{T}_d(t,x) + \left\langle \tilde{A}, \nabla_g R_{g,\lambda}(t,x) \right\rangle \tilde{R}_{d,\lambda}(t,x)$$

$$:= \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle g T_g(t,x)\tilde{T}_d(t,x) + \tilde{Z}_1(t,x).$$

Similarly, we see that

$$\tilde{q}(t,x)u_2(t,x)\tilde{v}(t,x) = \tilde{q}(t,x)(T_d T_g(t,x) + T_d R_{g,\lambda}(t,x) + T_g \tilde{R}_{d,\lambda}(t,x) + \tilde{R}_{d,\lambda}(t,x) R_{g,\lambda}(t,x)) := \tilde{Z}_2(t,x).$$

Using the above expressions in definitions of $J_1$ and $J_2$, we get

$$J_1 + J_2 = 2 \int_{M_T} \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle_g T_g(t,x)\tilde{T}_d(t,x) \, dV_g dt$$

$$+ 2 \int_{M_T} \tilde{Z}_1(t,x) \, dV_g dt + \int_{M_T} \tilde{Z}_2(t,x) \, dV_g dt.$$  \hspace{1cm} (4.8)

Now using the expression for $v$ from (3.22) in the expression of $J_3$, we obtain

$$J_3 = - \int_{\Sigma \setminus \Sigma_{-\epsilon/2}} e^{-\varphi + i\psi} \partial_\nu u(t,x) T_d(t,x) \, dS_g dt - \int_{\Sigma \setminus \Sigma_{-\epsilon/2}} e^{-\varphi + i\psi} \partial_\nu u(t,x) \tilde{R}_{d,\lambda}(t,x) \, dS_g dt.$$

We use the boundary Carleman estimate given in Theorem 2.1 and follow the arguments used in deriving Lemma 5.1 in [48] to get the following estimate for $J_3$

$$|J_3| \leq C\lambda^{1/2}, \text{ for some constant } C > 0, \text{ independent of } \lambda.$$  \hspace{1cm} (4.9)

Using (4.8) together with the estimate on $J_3$ given by (4.9) in (1.7), we get

$$2 \int_{M_T} \left\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi) \right\rangle_g T_g(t,x)\tilde{T}_d(t,x) \, dV_g dt \leq 2 \int_{M_T} \tilde{Z}_1(t,x) \, dV_g dt + \int_{M_T} \tilde{Z}_2(t,x) \, dV_g dt + |J_3|$$

$$\leq C (\|\tilde{Z}_1\|_{L^2(M_T)} + \|\tilde{Z}_2\|_{L^2(M_T)} + |J_3|).$$

Let $(x_1, r, \theta)$ be the polar normal coordinate on $(M, g)$ and $\tilde{A}_1$ and $\tilde{A}_r$ be components of $\tilde{A}$ in $x_1$ and $r$ coordinates respectively as in Theorem 3.1. Then, the above estimate can be rewritten as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{S_g M_0} \int_{r_1}^{r_2} (\tilde{A}_1 + i\sqrt{1 - \beta^2} \tilde{A}_r) T_g(t, x_1, r, \theta)\tilde{T}_d(t, x_1, r, \theta) \, dV_g dt \, dr \, d\theta dt \leq \frac{C}{\lambda} (\|\tilde{Z}_1\|_{L^2(M_T)} + \|\tilde{Z}_2\|_{L^2(M_T)} + |J_3|).$$  \hspace{1cm} (4.10)
where we used the fact \( dV_g = b(r, \theta)^{1/2} dx_1 dr d\theta \) in polar normal coordinates on \((M, g)\) and \(\tau_+(y_0, \theta)\) is length of the geodesic in \(M\), starting at \(y_0\) in the direction of \(\theta\). After using the estimates on \(R_{d, \lambda}\) and \(R_{g, \lambda}\) together with the expressions for \(T_d\) and \(T_g\) from Theorems 3.1 and 3.2, we get that

\[
\|Z_i\|_{L^2(M_T)} \leq C, \text{ for } i = 1, 2 \text{ and constant } C > 0 \text{ independent of } \lambda.
\]

Using this estimate along with equation (4.9) in equation (4.10) and taking \(\lambda \to \infty\), we get

\[
\int_{\mathbb{R}} \int_{S_y M_0} \int_0^{\tau_+(y_0, \theta)} \left( 1 + i \sqrt{1 - \beta^2} \hat{A}_r \right) (\phi(t))^{2} e^{i\mu \left( \sqrt{1 - \beta^2} x_1 \right)} e^{-\mu r} e^{i\Phi(t, x_1, r, \theta)} (h(\theta))^2 dV_1 dr d\theta dt = 0,
\]

where \(\Phi(t, x_1, r, \theta) := (\Phi_1 - \Phi_2) (t, x_1, r, \theta)\) with \(\Phi_1\) and \(\Phi_2\) satisfying equations (3.1) and (3.19) respectively. As this relation is true for all cutoff functions \(\phi \in C^\infty_c(0, T)\) therefore we get

\[
\int_{\mathbb{R}} \int_{S_y M_0} \int_0^{\tau_+(y_0, \theta)} \left( 1 + i \sqrt{1 - \beta^2} \hat{A}_r \right) (t, x_1, r, \theta) e^{i\mu \left( \sqrt{1 - \beta^2} x_1 \right)} e^{-\mu r} e^{i\Phi(t, x_1, r, \theta)} (h(\theta))^2 dV_1 dr d\theta dt = 0,
\]

for all \(t \in (0, T)\) and \(h \in C^\infty(S_{y_0}(D))\). Now following the arguments from [29, see Section 6] we obtain that there exists a \(\Psi \in W^{2, \infty}_0(M_T)\) such that \(\tilde{A}(t, x) = \nabla_g \Psi(t, x)\) for \((t, x) \in M_T\). This proves the required uniqueness for the convection term.

Next, we prove the uniqueness of density coefficient \(q\). To prove this, we replace the pair \((A^{(1)}, q_1)\) by \((A^{(3)}, q_3)\), by taking \(A^{(3)} = A^{(2)}\) in \(M_T\), where \(A^{(3)}(t, x) = A^{(1)}(t, x) - \nabla_g \Psi(t, x)\) and \(q_3(t, x) = q_1(t, x) - \partial_r \Psi(t, x)\). From Proposition 3.3 and Equation (1.3), we get \(\Lambda_{A^{(3)}, q_3} = \Lambda_{A^{(2)}, q_2}\). Using this in Equation (4.11), we get

\[
\int_{M_T} (q_2 - q_3)(t, x) u_2(t, x) \tilde{v}(t, x) dV_g dt = - \int_{\Sigma \setminus \Sigma_{-\epsilon/2}} g^{jk} \nu_j \partial_x k u(t, x) \tilde{v}(t, x) dS_g dt,
\]

Again, we use the explicit expressions of \(u_2\) and \(v\) from Theorems 3.1 and 3.2 and take \(\lambda \to \infty\) together with the estimate \(\|Z_2\|_{L^2(M_T)} \leq C/\lambda\), for some constant \(C > 0\), independent of \(\lambda\), to end up with getting

\[
\int_{\mathbb{R}} \int_{S_y M_0} \int_0^{\tau_+(y_0, \theta)} q(t, x_1, r, \theta) (\phi(t))^{2} e^{i\mu \left( \sqrt{1 - \beta^2} x_1 \right)} e^{-\mu r} e^{i\Phi(t, x_1, r, \theta)} (h(\theta))^2 dt dx_1 dr d\theta dt = 0,
\]

where \(q(t, x_1, r, \theta) := (q_2 - q_3)(t, x_1, r, \theta)\) is assumed to be zero outside \(M_T\). Finally by varying \(\phi \in C^\infty_c(0, T), h \in C^\infty(S_{y_0}(D))\) and taking \(\Phi \equiv 0\) which is possible since \(A^{(3)} = A^{(2)}\) in \(M_T\) and if \(\Phi_1\) solves (3.11) then we can choose \(\Phi_2 = \Phi_1\) which solves (3.19), we get that

\[
\int_{\mathbb{R}} \int_0^{\tau_+(y_0, \theta)} q(t, x_1, r, \theta) e^{i\mu \left( \sqrt{1 - \beta^2} x_1 + i r \right)} dx_1 dr = 0, \text{ for all } \theta \in S_{y_0}(D), \mu > 0, \beta \in \left( \frac{1}{\sqrt{3}}, 1 \right) \text{ and } t \in (0, T).
\]

Now following [33], we vary \(y_0 \in D\) such that \((x_1, y_0) \notin M\) for all \(x_1\) to get \(q \equiv 0\) in \(M_T\) which gives \(q_1(t, x) - q_2(t, x) = \partial_r \Psi(t, x)\) for \((t, x) \in M_T\). This completes the proof of the main theorem.

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PARTIAL DATA INVERSE PROBLEM

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