Uniformly resolvable \((C_4, K_{1,3})\)-designs of order \(v\) and index 2

Mario Gionfriddo *
Dipartimento di Matematica e Informatica
Università di Catania
Catania
Italia
gionfriddo@dmi.unict.it

Selda Küçükçifçi †‡
Department of Mathematics
Koç University
Istanbul
Turkey
skucukcifci@ku.edu.tr

Salvatore Milici §
Dipartimento di Matematica e Informatica
Università di Catania
Catania
Italia
milici@dmi.unict.it

E. Şule Yazıcı
Department of Mathematics
Koç University
Istanbul
Turkey

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Abstract

In this paper we consider the uniformly resolvable decompositions of the complete graph \(2K_v\) into subgraphs where each resolution class contains only blocks isomorphic to the same graph. We completely

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†Corresponding author, phone: +90 212 338 1523, fax: +90 212 338 1559
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determine the spectrum for the cases in which all the resolution classes are either $C_4$ or $K_{1,3}$.

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1 Introduction

Given a collection of graphs $\mathcal{H}$, an $\mathcal{H}$-decomposition of a graph $G$ is a decomposition of the edges of $G$ into isomorphic copies of graphs in $\mathcal{H}$. The copies of $H \in \mathcal{H}$ in the decomposition are called blocks. Such a decomposition is called resolvable if it is possible to partition the blocks into classes $P_i$ such that every point of $G$ appears exactly once in some block of each $P_i$.

A resolvable $\mathcal{H}$-decomposition of $G$ is sometimes also referred to as an $\mathcal{H}$-factorization of $G$ and a resolution class is called an $\mathcal{H}$-factor of $G$. The case where $\mathcal{H}$ is a single edge ($K_2$) is known as a 1-factorization of $G$ and it is well known to exist for $G = K_v$ if and only if $v$ is even. A single class of a 1-factorization, a pairing of all points, is also known as a 1-factor or a perfect matching.

In many cases we wish to impose further constraints on the classes of an $\mathcal{H}$-decomposition. For example, a class is called uniform if every block of the class is isomorphic to the same graph in $\mathcal{H}$. Uniformly resolvable decompositions of $K_v$ have also been studied in [2], [3], [5]–[7], [9]–[16].

In this paper we study the existence of a uniformly resolvable decomposition of $2K_v$ having the following type:

$r$ classes containing only copies of 4-cycles and $s$ classes containing only copies of 3-stars.

We will use the notation $(C_4, K_{1,3})$-URD$(v, 2; r, s)$ for such a uniformly resolvable decomposition of $2K_v$. Let now

$$J(2K_v; C_4, K_{1,3}) = \{(r, s) : \text{there exists a uniformly resolvable decomposition of } 2K_v \text{ into } r \text{ classes containing only copies of } C_4 \text{ and } s \text{ classes containing only copies of } K_{1,3}\}.$$ 

For $v \geq 4$, divisible by 4, define $I(v)$ according to the following table:

| $v$      | $I(v)$                                      |
|----------|---------------------------------------------|
| $0 \pmod{12}$ | $\{(v - 1 - 3x, 4x), x = 0, 1, \ldots, \frac{v - 3}{3}\}$ |
| $4 \pmod{12}$ | $\{(v - 1 - 3x, 4x), x = 0, 1, \ldots, \frac{v - 1}{3}\}$ |
| $8 \pmod{12}$ | $\{(v - 1 - 3x, 4x), x = 0, 1, \ldots, \frac{v - 2}{3}\}$ |

Table 1: The set $I(v)$.
In this paper we completely solve the spectrum problem for such systems; that is, characterize the existence of uniformly resolvable decompositions of $2K_v$ into $r$ classes of 4-cycles and $s$ classes of 3-stars, by proving the following result:

**Main Theorem.** For every integer $v \geq 4$, divisible by 4, the set $J(2K_v; C_4, K_{1,3})$ is identical to the set $I(v)$ given in Table 1.

Now let us introduce some useful definitions, notations, results and discuss constructions we will use in proving the main theorem. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [1] and its online updates. For some results below, we also cite this handbook instead of the original papers.

For any four vertices $a_1, a_2, a_3, a_4$, let the 3-star, $K_{1,3}$, be the simple graph with the vertex set $\{a_1, a_2, a_3, a_4\}$ and the edge set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$ and the 4-cycle $C_4$ be the simple graph with the vertex set $\{a_1, a_2, a_3, a_4\}$ and the edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\}$. In what follows, we will denote the 3-star by $(a_1, a_2, a_3, a_4)$ and the 4-cycle by $(a_1, a_2, a_3, a_4)$, $(a_4, a_3, a_2, a_1)$ or any cyclic shift of these.

A resolvable $H$-decomposition of the complete multipartite graph with $u$ parts each of size $g$ is known as a resolvable group divisible design $H$-RGDD of type $g^u$, the parts of size $g$ are called the groups of the design. When $H = K_n$ we will call it an $n$-RGDD.

A $(C_4, K_{1,3})$-URGDD (2; $r$, $s$) of type $g^u$ is a uniformly resolvable decomposition of the complete multipartite graph of index 2 with $u$ parts each of size $g$ into $r$ classes containing only copies of 4-cycles and $s$ classes containing only copies of 3-stars.

If the blocks of an $H$-GDD of type $g^u$ can be partitioned into partial parallel classes, each of which contain all points except those of one group, we refer to the decomposition as a frame. When $H = K_n$ we will call it an $n$-frame and it can be deduced that the number of partial parallel classes missing a specified group $G$ is $\lceil \frac{[G]}{n-1} \rceil$.

An incomplete resolvable $(C_4, K_{1,3})$-decomposition of $2K_v$ with a hole of size $h$ is a $(C_4, K_{1,3})$-decomposition of $2K_{v+h} - 2K_h$ in which there are two types of classes, full classes and partial classes which cover every point except those in the hole (the points of $2K_h$ are referred to as the hole). Specifically a $(C_4, K_{1,3})$-IURD($2K_{v+h} - 2K_h; [r, s], [\bar{r}, \bar{s}]$) is a uniformly resolvable $(C_4, K_{1,3})$-decomposition of $2K_{v+h} - 2K_h$ with $r$ partial classes of 4-cycles which cover only the points not in the hole, $s$ partial classes of 3-stars which cover only the points not in the hole, $\bar{r}$ full classes of 4-cycles which cover every point of $2K_{v+h}$ and $\bar{s}$ full classes of 3-stars which cover every point of $2K_{v+h}$.

We also need the following definitions. Let $(s_1, t_1)$ and $(s_2, t_2)$ be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If $X$
and $Y$ are two sets of pairs of non-negative integers, then $X + Y$ denotes the set \{$(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y$\}. If $X$ is a set of pairs of non-negative integers and $h$ is a positive integer, then $h \ast X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any $h$ elements of $X$ together (repetitions of elements of $X$ are allowed).

Similarly to what was done in [S], the following results can be proven.

**Lemma 1.1.** If there exists a $(C_4, K_{1,3})$-URD $(2K_v; r, s)$ of $2K_v$ with $r > 0$ and $s > 0$ then $v \equiv 0 \pmod{4}$ and $(r, s) \in I(v)$.

The following lemma will be very useful in proving the main results of this paper.

**Lemma 1.2.** Let $v$, $g$, and $u$ be non-negative integers such that $v = gu$. If there exists

1. a $4$-RGDD of type $g^u$;
2. a $(C_4, K_{1,3})$-URD $(4K_v; r_1, s_1)$ with $(r_1, s_1) \in J_1 = \{(3, 0), (0, 4)\}$;
3. a $(C_4, K_{1,3})$-URD $(2K_g; r_2, s_2)$, with $(r_2, s_2) \in J_2$, where $(r_2, s_2) \in J_2$ and $J_2 = \{(r_2, s_2) :$ there exists a $(C_4, K_{1,3}) - URD(2K_g; r_2, s_2)\}$;

then there exists a $(C_4, K_{1,3})$-URD $(2K_v; r, s)$ for each $(r, s) \in J_2 + t \ast J_1$, where $t = \frac{u(a-1)}{3}$ is the number of parallel classes of the $4$-RGDD of type $g^u$.

## 2 Small cases

**Lemma 2.1.** $J(2K_4; C_4, K_{1,3}) = \{(3, 0), (4, 0)\}$.

*Proof.* Let $V(K_4) = \mathbb{Z}_4$.

- $(3,0)$
  
  3 classes of 4-cycles are: $\{(0,1,2,3)\}, \{(0,2,3,1)\}, \{(0,2,1,3)\}$.

- $(0,4)$
  
  4 classes of 3-stars can be obtained from the base block $\{\{0;1,2,3\}\}$.

\[\square\]

**Lemma 2.2.** $J(2K_8; C_4, K_{1,3}) = \{(7,0), (4,4), (1,8)\}$.

*Proof.* Let $V(K_8) = \mathbb{Z}_8$. 

4
Proof. Let the groups to be \( (C_4, K_{13}) \)-URGDD(2; 4, 0) of type 4\(^2\) and replace each group of size 4 with the same \( (C_4, K_{13}) \)-URD(2\(K_4\); r, s), with (r, s) \( \in \{ (3, 0), (0, 4) \} \) which exists by Lemma 2.3.

\( \Box \)

Lemma 2.3. \( J(2 K_{12}; C_4, K_{13}) = \{ (11, 0), (8, 4), (5, 8), (2, 12) \} \).

Proof. Let \( V(K_{12}) = \mathbb{Z}_{12} \).

- (11, 0), (8, 4)
  Take a \( (C_4, K_{13}) \)-URGDD(2; 8, 0) of type 4\(^3\) and replace each group of size 4 with the same \( (C_4, K_{13}) \)-URD(2\(K_4\); r, s), with (r, s) \( \in \{ (3, 0), (0, 4) \} \), which exists by Lemma 2.3.

- (5, 8)
  5 classes of 4-cycles and 8 classes of 3-stars are:
  \{ (0, 1, 4, 7), (2, 3, 6, 5), (8, 11, 9, 10) \}, \{ (0, 11, 10, 3), (1, 2, 9, 8), (4, 6, 7, 5) \},
  \{ (3, 1, 4, 8), (2, 0, 6, 10), (7, 11, 9, 5) \}, \{ (3, 11, 5, 0), (1, 2, 9, 7), (4, 6, 8, 10) \},
  \{ (1, 3, 2, 0), (4, 8, 11, 7), (6, 10, 9, 5) \},
  \{ (0, 4, 5, 6), (7, 8, 9, 10), (11, 1, 2, 3) \}, \{ (1, 5, 6, 7), (4, 9, 10, 11), (8, 0, 2, 3) \},
  \{ (2, 4, 6, 7), (5, 8, 10, 11), (9, 0, 1, 3) \}, \{ (3, 4, 5, 7), (6, 8, 9, 11), (10, 0, 1, 2) \},
  \{ (3, 4, 10, 6), (8, 7, 9, 5), (11, 1, 2, 0) \}, \{ (1, 10, 6, 8), (4, 9, 5, 11), (7, 0, 2, 3) \},
  \{ (2, 4, 6, 8), (10, 7, 5, 11), (9, 0, 1, 3) \}, \{ (0, 4, 10, 8), (6, 7, 9, 11), (5, 3, 1, 2) \}.

- (2, 12)
  2 classes of 4-cycles are:
  \{ (0, 5, 6, 1), (2, 7, 8, 3), (4, 9, 10, 11) \}, \{ (0, 7, 6, 11), (1, 8, 9, 2), (3, 4, 5, 10) \}.
  12 classes of 3-stars can be obtained from the base blocks:
  \{ (4, 10, 1, 6), (9, 2, 5, 7), (11, 3, 8, 0) \}.

\( \Box \)

Lemma 2.4. There exists a \( (C_4, K_{13}) \)-URGDD(2; r, s) of type 12\(^2\) with (r, s) \( \in \{ (12, 0), (6, 8), (0, 16) \} \).

Proof. Take the groups to be \( \{ a_1, a_2, \ldots , a_{12} \} \) and \( \{ b_1, b_2, \ldots , b_{12} \} \).
• The case $(12, 0)$ follows by [2].

• $(0, 16)$

12 parallel classes of 4-cycles are obtained by considering $i = 1, 4, 7, 10$ as listed below:

\[
\{(a_i; b_i, b_{i+1}, b_{i+2}), (a_{i+1}; b_{i+3}, b_{i+4}, b_{i+5}), (a_{i+2}; b_{i+6}, b_{i+7}, b_{i+8}), \\
(b_{i+9}; a_{i+3}, a_{i+4}, a_{i+5}), (b_{i+10}; a_{i+6}, a_{i+7}, a_{i+8}), (b_{i+11}; a_{i+9}, a_{i+10}, a_{i+11})\},
\]

\[
\{(a_i; b_{i+3}, b_{i+4}, b_{i+5}), (a_{i+1}; b_{i+6}, b_{i+7}, b_{i+8}), (a_{i+2}; b_{i+9}, b_{i+10}, b_{i+11}), \\
(b_i; a_{i+3}, a_{i+4}, a_{i+5}), (b_{i+1}; a_{i+6}, a_{i+7}, a_{i+8}), (b_{i+2}; a_{i+9}, a_{i+10}, a_{i+11})\},
\]

\[
\{(a_{i}; b_{i+6}, b_{i+7}, b_{i+8}), (a_{i+1}; b_{i+9}, b_{i+10}, b_{i+11}), (a_{i+2}; b_i, b_{i+1}, b_{i+2}), \\
(b_{i+3}; a_{i+3}, a_{i+4}, a_{i+5}), (b_{i+4}; a_{i+6}, a_{i+7}, a_{i+8}), (b_{i+5}; a_{i+9}, a_{i+10}, a_{i+11})\},
\]

\[
\{(a_{i}; b_{i+9}, b_{i+10}, b_{i+11}), (a_{i+1}; b_i, b_{i+1}, b_{i+2}), (a_{i+2}; b_{i+3}, b_{i+4}, b_{i+5}), \\
(b_{i+6}; a_{i+3}, a_{i+4}, a_{i+5}), (b_{i+7}; a_{i+6}, a_{i+7}, a_{i+8}), (b_{i+8}; a_{i+9}, a_{i+10}, a_{i+11})\}.
\]

• $(6, 8)$

6 parallel classes of 4-cycles are:

\[
\{(a_{12}; b_{12}, a_5, b_1), (a_{1}; b_2, a_7, b_3), (a_{2}; b_4, a_8, b_5), (a_{3}; b_7, a_4, b_8), \\
(a_6; b_9, a_{10}, b_{10})\},
\]

\[
\{(a_{12}; b_2, a_8, b_3), (a_{1}; b_4, a_3, b_5), (a_{2}; b_6, a_{11}, b_7), (a_{4}; b_8, a_6, b_{11}), \\
(a_{5}; b_9, a_7, b_{10})\},
\]

\[
\{(a_{12}; b_2, a_2, b_7), (a_{1}; b_4, a_9, b_6), (a_{3}; b_5, a_4, b_9), (a_{5}; b_{12}, a_7, b_{10}), \\
(a_6; b_8, a_{18}, b_1), (a_{10}; b_1, a_{11}, b_3)\}
\]

\[
\{(a_{12}; b_4, a_{10}, b_5), (a_{1}; b_6, a_6, b_7), (a_{2}; b_9, a_4, b_{10}), (a_{3}; b_3, a_{11}, b_8), \\
(a_{5}; b_2, a_9, b_{11}), (a_{7}; b_1, a_8, b_1)\},
\]

\[
\{(a_{12}; b_4, a_{11}, b_5), (a_{1}; b_7, a_5, b_6), (a_{2}; b_8, a_7, b_{11}), (a_{3}; b_{12}, a_4, b_6), \\
(a_6; b_1, a_8, b_{10}), (a_{9}; b_2, a_{10}, b_3)\}
\]

\[
\{(a_{12}; b_6, a_2, b_9), (a_{1}; b_5, a_5, b_8), (a_{3}; b_7, a_4, b_{10}), (a_6; b_{12}, a_8, b_3), \\
(a_7; b_1, a_9, b_11), (a_{10}; b_2, a_{11}, b_4)\}, \text{ and 8 parallel classes of 3-stars are the last 8 parallel classes of the solution for } (0, 16)
\]

\[\square\]

**Lemma 2.5.** $J(24, 2; C_4, K_{1,3}) = I(24)$.

**Proof.** Take a $(C_4, K_{1,3})$-URGDD$(2; r, s)$ of type $12^2$ and index 2 with $(r, s) \in \{(12, 0), (0, 16)\}$ which exists by Lemma [2.4]. Replace each group of size 12 with the same $(C_4, K_{1,3})$-URD$(2K_{12}; r, s)$, where $(r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\}$ which exists by Lemma [2.3]

\[\square\]

**Lemma 2.6.** $J(36, 2; C_4, K_{1,3}) = I(36)$. 

6
Proof. Take a \((C_4, K_{1,3})\)-UGDD\((2; r, s)\) of type 12\(^3\) and index 2 with \((r, s) \in \{(24, 0), (0, 32)\}\) which exists by Lemma 2.4. Replace each group of size 12 with the same \((C_4, K_{1,3})\)-URD\((2K_{12}; r, s)\), where \((r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\}\) which exists by Lemma 2.3.

\[\square\]

Lemma 2.7. There exists a \((C_4, K_{1,3})\)-IURD\((2K_{20} - 2K_8; [r, s], [\bar{r}, \bar{s}])\) with \((r, s) \in \{(7, 0), (4, 4), (1, 8)\}\) and \((\bar{r}, \bar{s}) \in \{(12, 0), (9, 4), (6, 8), (3, 12), (0, 16)\}\).

Proof. Let the point set of \(K_{20}\) be \(\mathbb{Z}_{20}\) and the point set \(\{0, 1, \ldots, 7\}\) be the hole. The following resolution classes \((7, 0), (4, 4), (1, 8)\) cover the same edges on the point set \(K_{20} - K_8\).

- The parallel classes of resolution \((7, 0)\):
  \[
  \{(8, 9, 11, 10), (12, 13, 15, 14), (16, 17, 19, 18)\}, \\
  \{(8, 11, 15, 12), (9, 10, 19, 16), (13, 14, 18, 17)\}, \\
  \{(8, 13, 10, 15), (9, 18, 11, 19), (12, 16, 14, 17)\}, \\
  \{(8, 14, 16, 10), (12, 18, 15, 9), (11, 17, 19, 13)\}, \\
  \{(8, 16, 15, 12), (14, 10, 19, 11), (18, 9, 13, 17)\}, \\
  \{(8, 18, 10, 15), (14, 13, 16, 19), (12, 11, 9, 17)\}, \\
  \{(8, 9, 15, 14), (10, 11, 17, 16), (12, 13, 19, 18)\}.
  \]

- The parallel classes of resolution \((4, 4)\) are the last 4 parallel classes in resolution \((7, 0)\) above and the following 4 parallel classes of 3-stars:
  \[
  \{(8, 9, 10, 11), (14, 12, 13, 15), (19, 16, 17, 18)\}, \\
  \{(9, 10, 11, 19), (15, 8, 12, 13), (17, 14, 16, 18)\}, \\
  \{(10, 11, 15, 19), (12, 8, 13, 17), (18, 9, 14, 16)\}, \\
  \{(11, 15, 18, 19), (13, 8, 10, 17), (16, 9, 12, 14)\}.
  \]

- The parallel classes of resolution \((1, 8)\) are the last parallel class in resolution \((7, 0)\) above and the following 8 parallel classes of 3-stars:
  \[
  \{(8, 9, 10, 11), (12, 13, 14, 15), (16, 17, 18, 19)\}, \\
  \{(8, 10, 12, 13), (9, 11, 15, 16), (14, 17, 18, 19)\}, \\
  \{(9, 10, 11, 12), (13, 14, 17, 19), (15, 8, 16, 18)\}, \\
  \{(10, 11, 13, 19), (14, 8, 15, 16), (17, 9, 12, 18)\}, \\
  \{(10, 14, 15, 16), (17, 11, 12, 13), (18, 8, 9, 19)\}, \\
  \{(11, 12, 14, 18), (15, 8, 10, 13), (19, 9, 16, 17)\}, \\
  \{(11, 13, 15, 19), (16, 8, 12, 14), (18, 9, 10, 17)\}, \\
  \{(12, 8, 15, 18), (13, 9, 14, 16), (19, 10, 11, 17)\}.
  \]

Now all the edges that are not covered in the above resolution classes on the point set \(K_{20} - K_8\) will be covered by the following full parallel classes on the point set \(K_{20}\).

\[\square\]
• The parallel classes of resolution (12, 0):

\[
\{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 14, 18, 16), (7, 17, 15, 19)\},
\{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 16, 7, 18), (14, 17, 15, 19)\},
\{(0, 10, 1, 11), (2, 8, 3, 9), (4, 14, 12, 16), (5, 15, 13, 18), (6, 17, 7, 19)\},
\{(0, 10, 111), (2, 8, 3, 9), (4, 17, 5, 19), (6, 12, 7, 14), (13, 16, 15, 18)\},
\{(0, 12, 1, 13), (2, 14, 3, 15), (4, 8, 16, 9), (5, 17, 6, 19), (7, 10, 18, 11)\},
\{(0, 12, 1, 13), (2, 14, 11, 16), (3, 15, 4, 17), (5, 8, 19, 9), (6, 10, 7, 18)\},
\{(0, 14, 1, 15), (2, 18, 3, 19), (4, 9, 7, 16), (5, 8, 17, 10), (6, 12, 11, 13)\},
\{(0, 14, 1, 15), (2, 13, 8, 17), (3, 16, 5, 18), (4, 10, 12, 19), (6, 9, 7, 11)\},
\{(0, 16, 1, 17), (2, 15, 4, 18), (3, 13, 10, 14), (5, 9, 6, 11), (7, 8, 19, 12)\},
\{(0, 16, 1, 17), (2, 12, 3, 19), (4, 11, 8, 18), (5, 10, 9, 14), (6, 13, 7, 15)\},
\{(0, 18, 1, 19), (2, 13, 3, 16), (4, 11, 5, 14), (6, 8, 7, 15), (9, 12, 10, 17)\},
\{(0, 18, 1, 19), (2, 12, 3, 17), (4, 8, 6, 10), (5, 15, 11, 16), (7, 13, 9, 14)\}.
\]

• The parallel classes of resolution (9, 4) are the following 9 parallel classes of 4-cycles:

\[
C_1 = \{(0, 10, 1, 11), (2, 8, 3, 9), (4, 17, 5, 19), (6, 12, 7, 14), (13, 16, 15, 18)\},
C_2 = \{(0, 12, 1, 13), (2, 14, 3, 15), (4, 8, 16, 9), (5, 17, 6, 19), (7, 10, 18, 11)\},
C_3 = \{(0, 14, 1, 15), (2, 12, 11, 16), (3, 13, 6, 18), (4, 17, 8, 19), (5, 9, 7, 10)\},
C_4 = \{(0, 10, 1, 11), (2, 8, 3, 9), (4, 14, 12, 16), (5, 15, 13, 18), (6, 17, 7, 19)\},
C_5 = \{(0, 18, 1, 19), (2, 13, 3, 17), (4, 8, 11, 16), (5, 10, 6, 15), (7, 12, 9, 14)\},
C_6 = \{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 14, 18, 16), (7, 17, 15, 19)\},
C_7 = \{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 16, 7, 18), (14, 17, 15, 19)\},
C_8 = \{(0, 12, 1, 13), (2, 14, 3, 15), (4, 10, 6, 11), (5, 16, 7, 18), (8, 17, 9, 19)\},
C_9 = \{(0, 14, 4, 18), (1, 16, 2, 19), (3, 12, 10, 17), (5, 8, 13, 11), (6, 9, 7, 15)\},
\]

and the following 4 parallel classes of 3-stars:

\[
S_1 = \{(0; 15, 16, 17), (7; 8, 11, 13), (12; 6, 10, 19), (14; 1, 5, 9), (18; 2, 3, 4)\},
S_2 = \{(1; 16, 17, 18), (4; 9, 10, 15), (5; 8, 11, 14), (13; 2, 6, 7), (19; 0, 312)\},
S_3 = \{(2; 12, 18, 19), (6; 8, 9, 11), (10; 13, 14, 17), (15; 1, 4, 7), (16; 0, 3, 5)\},
S_4 = \{(3; 12, 16, 19), (8; 6, 7, 18), (9; 5, 10, 13), (11; 4, 14, 15), (17; 0, 1, 2)\}.
\]

• The 6 parallel classes of 4-cycles of resolution (6, 8) are \(C_1, C_2, C_3, C_4, C_5\) in the resolution (9, 4) above together with the following parallel class of 4-cycles:

\[
\{(0, 8, 17, 9), (1, 12, 4, 13), (2, 10, 6, 11), (3, 14, 19, 15), (5, 16, 7, 18)\} \text{ and the 8 parallel classes of 3-stars are } S_1, S_2, S_3, S_4 \text{ in the resolution (9, 4) above together with the following 4 parallel classes of 3-stars:}
\]

\[
S_5 = \{(0; 9, 14, 18), (5; 11, 12, 13), (10; 3, 4, 17), (16; 1, 2, 6), (19; 7, 8, 15)\},
S_6 = \{(1; 8, 9, 19), (7; 15, 16, 18), (11; 2, 3, 13), (12; 0, 5, 10), (14; 4, 6, 17)\},
S_7 = \{(2; 10, 14, 19), (4; 11, 12, 13), (6; 9, 16, 18), (8; 0, 1, 5), (17; 3, 7, 15)\},
S_8 = \{(3; 10, 11, 12), (9; 1, 7, 19), (13; 0, 5, 8), (15; 2, 6, 17), (18; 4, 14, 16)\}.
\]

• The 3 parallel classes of 4-cycles of resolution (3, 12) are \(C_1, C_2, C_3\) in the
resolution \((9, 4)\) and the 12 parallel classes of 3-stars are \(S_1, S_2, S_3, S_4\) in the resolution \((9, 4)\), \(S_5, S_6, S_7, S_8\) in the resolution \((6, 8)\) above together with the following 4 parallel classes of 3-stars:

\[
S_9 = \{(0; 9, 10, 18), (1; 11, 12, 13), (4; 8, 14, 16), (15; 3, 5, 19), (17; 2, 6, 7)\},
\]

\[
S_{10} = \{(2; 9, 10, 11), (3; 8, 13, 17), (5; 15, 16, 18), (12; 4, 7, 14), (19; 0, 1, 6)\},
\]

\[
S_{11} = \{(6; 10, 11, 15), (8; 0, 2, 17), (14; 3, 9, 19), (16; 4, 7, 12), (18; 1, 5, 13)\},
\]

\[
S_{12} = \{(7; 14, 18, 19), (9; 3, 12, 17), (10; 1, 5, 6), (11; 0, 8, 16), (13; 2, 4, 15)\}.
\]

- The 16 parallel classes of 3-stars of resolution \((0, 16)\) are \(S_i, i = 2, 3, \ldots, 12\) above together with the following 5 parallel classes of 3-stars:

\[
\{(0; 10, 11, 17), (2; 8, 12, 15), (13; 7, 16, 18), (14; 1, 3, 9), (19; 4, 5, 6)\},
\]

\[
\{(0; 13, 15, 16), (5; 14, 17, 19), (7; 8, 9, 10), (12; 1, 6, 11), (18; 2, 3, 4)\},
\]

\[
\{(1; 10, 13, 14), (12; 0, 7, 19), (16; 2, 8, 9), (17; 4, 5, 6), (18; 3, 11, 15)\},
\]

\[
\{(3; 8, 13, 15), (4; 9, 17, 19), (10; 5, 12, 18), (11; 1, 7, 16), (14; 0, 2, 6)\},
\]

\[
\{(6; 12, 13, 18), (7; 10, 11, 14), (8; 4, 17, 19), (9; 2, 3, 5), (15; 0, 1, 16)\}.
\]

\[\square\]

Lemma 2.8. \(J(2K_{20}; C_4, K_{1,3}) = I(20)\).

Proof. Replace the hole of size 8 in Lemma 2.7 by a \((C_4, K_{1,3})\)-URD\((2K_8; r, s)\), with \((r, s) \in \{(7, 0), (4, 4), (1, 8)\}\) which exists by Lemma 2.2 \[\square\]

3 Main results

Lemma 3.1. For every \(v \equiv 0 \pmod{12}\) \(I(v) \subseteq J(2K_v; C_4, K_{1,3})\).

Proof. For \(v = 12, 24, 36\) the conclusion follows from Lemmas 2.3, 2.5 and 2.6. For \(v \geq 48\) start with a 4-RGDD \(G\) of type \(12\frac{v}{3}\) \[1\] and apply Lemma 1.2 with \(g = 12, u = \frac{v}{12}\) and \(t = \frac{(v - 12)}{4}\) (The input designs are a \((C_4, K_{1,3})\)-URD\((2K_4; r, s)\), with \((r, s) \in \{(3, 0), (0, 4)\}\), which exists by Lemma 2.1 and a \((C_4, K_{1,3})\)-URD\((2K_{12}; r, s)\), with \((r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\}\), which exists by Lemma 2.3. This implies

\[
J(2K_v; C_4, K_{1,3}) \supseteq \{(11, 0), (8, 4), (5, 8), (2, 12)\} + \frac{(v - 12)}{3} * \{(3, 0), (0, 4)\}.
\]

Since \(\frac{(v - 12)}{3} * \{(3, 0), (0, 4)\} = \{(v - 12 - 3x, 4x) \mid x = 0, \ldots, \frac{v - 12}{3}\}\), it is easy to see that \(\{(11, 0), (8, 4), (5, 8), (2, 12)\} + \frac{(v - 12)}{6} * \{(3, 0), (0, 4)\}\) \(= I(v)\). This completes the proof. \[\square\]

Lemma 3.2. For every \(v \equiv 8 \pmod{24}\) \(I(v) \subseteq J(2K_v; C_4, K_{1,3})\).
Proof. For \( v = 8 \) the result follows by Lemma 2.2. For \( v \geq 32 \), start with a 4-RGDD of type \( 8\bar{\bar{8}} \) and apply Lemma 1.2 with \( g = 8, u = \frac{v}{2} \) and \( t = \frac{v-8}{3} \) (The input designs are a \((C_r, K_{1,3})\)-URD\((2K_4; r, s)\), with \((r, s) \in \{(3,0), (0,4)\}\), which exists by Lemma 2.1) and a \((C_r, K_{1,3})\)-URD\((2K_5; r, s)\), with \((r, s) \in \{(7,0), (4,4), (1,8)\}\), which exists by Lemma 2.2. Proceeding as in Lemma 3.1 the result follows.

**Lemma 3.3.** For every \( v \equiv 4 \pmod{12} \), \( I(v) \subseteq J(2K_v; C_4, K_{1,3}) \).

**Proof.** For \( v = 4 \) the result follows by Lemma 2.4. For \( v \geq 16 \), start with a 4-RGDD of type \( 4\bar{\bar{4}} \) and apply Lemma 1.2 with \( g = 4, u = \frac{v}{4} \) and \( t = \frac{v-4}{3} \) (The input design is a \((C_r, K_{1,3})\)-URD\((2K_4; r, s)\), with \((r, s) \in \{(3,0), (0,4)\}\), which exists by Lemma 2.1). Proceeding as in Lemma 3.1 the result follows.

**Lemma 3.4.** For every \( v \equiv 20 \pmod{24} \), \( I(v) \subseteq J(2K_v; C_4, K_{1,3}) \).

**Proof.** The case \( v = 20 \) follows by Lemma 2.8. For \( v \geq 44 \) start from a 2-frame \( \mathcal{F} \) of type \( 1\bar{\bar{2}} \) with groups \( G_i, i = 1, 2, \ldots, \frac{v-8}{12} \). Then expand each point by 12 points and add a set \( H = \{a_1, a_2, \ldots, a_8\} \). For \( i = 1, 2, \ldots, \frac{v-8}{12}, \) let \( P_i \) be the partial factor which miss the group \( G_i \).

Replace each block \( b \in P_i \) by a \((C_r, K_{1,3})\)-URGDD\((2; r_1, s_1)\) of type \( 12^2 \) and index 2, say \( D^b_i \) on the vertex set of \( b \times \{1, 2, \ldots, 12\} \) with \((r_1, s_1) \in \{(12,0), (6,8), (0,16)\}\), which exists by Lemma 2.4.

For \( i = 1, 2, \ldots, \frac{v-8}{12} \) place on \( H \cup \{G_i \times \{1, 2, \ldots, 12\}\} \) a copy of a \((C_r, K_{1,3})\)-URD\((2K_{20} - 2K_8; [x_1, y_1], [x, y])\), say \( D_i \) with \((x_1, y_1) \in \{(7,0), (4,4), (1,8)\}\) and \((x, y) \in \{(12,0), (6,8), (0,16)\}\), which exists by Lemma 2.7. Combine the parallel classes of \( D^b_i \) with the full classes of \( D_i \) so to obtain \( r_2 \) classes of \( C_4 \) and \( s_2 \) classes of \( K_{1,3} \) with \((r_2, s_2) \in \{(\frac{v-8}{12}) \ast \{(12,0), (6,8), (0,16)\}\} \).

Fill the hole \( H \) with a copy of a \((C_r, K_{1,3})\)-URD\((2K_8; r, s)\) say \( D \) with \((r_4, s_4) \in \{(7,0), (4,4), (1,8)\}\), which exists by Lemma 2.2. Combine the classes of \( D \) with the partial of \( D_i \) so to obtain \( r_4 \) classes of \( C_4 \) and \( s_4 \) classes of \( K_{1,3} \) with \((r_4, s_4) \in \{(7,0), (4,4), (1,8)\}\).

This gives a \((C_r, K_{1,3})\)-URD\((2K_4; r, s)\), with \((r, s) \in \{(7,0), (4,4), (1,8)\}\) and \( J(2K_v; C_4, K_{1,3}) = I(v) \). Proceeding as in Lemma 3.1 we obtain the result.

Combining Lemmas 3.1, 3.2, 3.3 and 3.4 we obtain the main theorem of this article.

**Theorem 3.5.** For each \( v \equiv 0 \pmod{4} \), we have \( J(2K_v; C_4, K_{1,3})=I(v) \).
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