Möbius transformation for left-derivative quaternion holomorphic functions

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Holomorphic quaternion functions only admit affine functions; thus, the Möbius transformation for these functions, which we call quaternionic holomorphic transformation (QHT), only comprises similarity transformations. We determine a general group $X$ which has the group $G$ of QHT as a particular case. Furthermore, we observe that the Möbius group and the Heisenberg group may be obtained by making $X$ more symmetric. We provide matrix representations for the group $X$ and for its algebra $x$. The Lie algebra is neither simple nor semi-simple, and so it is not classified among the classical Lie algebras. They prove that the group $G$ comprises $\text{SU}(2, \mathbb{C})$ rotations, dilations and translations. The only fixed point of the QHT is located at infinity, and the QHT does not admit a cross-ratio. Physical applications are addressed at the conclusion.

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I. INTRODUCTION

For complex $a, b, c, d$ and $z$, the Möbius transformation (MT) is defined by

$$M(z) = \frac{az + b}{cz + d}. \quad (1)$$

This expression may be decomposed into the symmetry operations of translation, rotation, dilation and the special conformal transformation, and a $\text{SL}(2, \mathbb{C})$ matrix group representation is naturally built as well. The Möbius transformation is also the symmetry of hyperbolic geometry [1] and the symmetry it contains has a broad application in physics through the conformal field theory [2, 3]. The generalization of MT for higher dimensional flat geometries is also a well-known subject [4–9].

On the other hand, (1) may be generalized using hyper-complex numbers. A quaternionic Möbius transformation [8, 10, 11] may be defined according to

$$\mu(q) = (aq + b)(cq + d)^{-1}, \quad \text{with} \quad \Delta = |a|^2|d|^2 + |b|^2|c|^2 - 2\Re[a\bar{c}d\bar{b}] \neq 0 \quad (2)$$

and quaternionic $a, b, c, d$, and $q$. The non-commutativity of quaternions implies that the transformation (2) is not only a Möbius group in four dimensions, and even a matrix representation is not natural, due to quaternion matrices being more restrictive than complex matrices [12]. Nevertheless, there is new research in the field, such as attempts to classify the quaternionic Möbius transformations [11, 13], their properties [14], and transformations involving alternative definitions for quaternion functions [15].

These previous studies consider quaternion Möbius transformations comprising all symmetry operations found in two-dimensional transformations. The approach used in this article is different because the quaternion transformations are defined for holomorphic quaternionic functions. This class of functions only admits affine quaternion functions, given by

$$F(q) = qa + b, \quad (3)$$

with quaternionic $a, b$ and $q$. The order of the product $qa$ is fixed because of quaternion non-commutativity. In order to investigate a Möbius transformation, we determine infinitesimal Lie operators that transform (3), and we determine a matrix group representations for the algebra and for the group as well. We call this matrix representation $X$, and its algebra is $x$. Assuming suitable constraints, the Lie group $X$ may be identified with the usual Möbius group $M$, the Heisenberg group $H$ and the group $G$ of quaternion holomorphic transformation (QHT) of (3). Accordingly, the group $X$ generalizes these groups, and this generalization permits us to unify the usual differential operators of

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the Möbius algebra, the Heisenberg algebra and the QHT algebra into a single algebra, \( \mathfrak{r} \). Additionally, we determine that the QHT group \( G \) comprises \( \text{SU}(2, \mathbb{C}) \) rotations, dilations and translations, as expected.

The article is organized as follows; in Section II, we give a brief survey of quaternion holomorphic functions and define the quaternion holomorphic transformation. In section III, we determine the Lie algebra \( \mathfrak{g} \) in terms of differential Lie operators and provide matrix representations for the algebra and for the Lie group \( \mathfrak{X} \). In section IV, we study several properties of the QHT and relate them to \( \mathfrak{X} \). Furthermore, we relate this group to the Möbius group, the Heisenberg group and the QHT group. Section V presents the Lie algebra \( g \) of \( G \) in detail. Section VI presents our conclusions and future perspectives.

II. FUNCTIONS AND TRANSFORMATIONS

A. Holomorphic quaternion functions

In this section, we give a quaternion analogue for the Cauchy-Riemann condition for complex functions. Further details on quaternion analysis may be found at [10, 16]. We begin defining the left-derivative of a quaternion valued function \( F \), so that

\[
\frac{dF}{dq} = \lim_{\Delta \to 0} \frac{1}{\Delta} [F(q + \Delta) - F(q)],
\]

where \( q \) is the quaternionic variable. Remembering quaternionic non-commutativity, the differential relation for the left derivative is defined in the product

\[
dF = dq \frac{dF}{dq}.
\]

Using the extended notation for quaternions in terms of real variables \( x_0, x_1, x_2, \) and \( x_3 \), we get

\[
q = x_0 + x_1 i + x_2 j + x_3 k,
\]

and then the differential relation (5) becomes

\[
\partial_0 F dx_0 + \partial_1 F dx_1 + \partial_2 F dx_2 + \partial_3 F dx_3 = (dx_0 + dx_1 i + dx_2 j + dx_3 k) \frac{dF}{dq}.
\]

Equating the real differential coefficients, we get

\[
\frac{dF}{dq} = \partial_0 F = -i \partial_1 F = -j \partial_2 F = -k \partial_3 F.
\]

We now define the symplectic notation, where quaternions are written as

\[
q = z + \zeta j
\]

where both \( z \) and \( \zeta \) are complex. Accordingly, a quaternionic function is written in symplectic notation as

\[
F = \mathcal{E}_0 + \mathcal{E}_1 j,
\]

with \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) complex functions. From (8) and (10), we obtain

\[
\partial_0 \mathcal{E}_0 = -i \partial_1 \mathcal{E}_0 = \partial_2 \mathcal{E}_1 = i \partial_3 \mathcal{E}_1, \quad \partial_0 \mathcal{E}_1 = i \partial_1 \mathcal{E}_0 = -\partial_2 \mathcal{E}_0 = i \partial_3 \mathcal{E}_0 \quad \text{and} \quad \partial_2 F = \partial_2 \mathcal{E}_0 = 0,
\]

where \( z = x_0 + ix_1, \zeta = x_2 + ix_3 \) and

\[
\partial_z = \partial_0 - i \partial_1 \quad \text{and} \quad \partial_\zeta = \partial_2 - i \partial_3.
\]

From (11) we obtain

\[
\partial_z \mathcal{E}_0 = \partial_\zeta \mathcal{E}_1 \quad \text{and} \quad \partial_\zeta \mathcal{E}_0 = -\partial_z \mathcal{E}_1.
\]
The generator of the translation may simply be
\[ \partial q \]  
function \( (13) \) we obtain
\[ q \]  
cannot be
\[ q \]  
a differential generator for
where
\[ \epsilon \]  
consider an infinitesimal similarity transformation and dilations and the addition of a quaternionic constant to a quaternion variable comprises translations. Let us then linear functions like \( (13) \). Multiplication of the quaternionic variable by a quaternion constant comprises rotations which generate a non-analytic quaternion function. Thus, the quaternionic transformation we consider generates quaternion functions exclude the special conformal transformation, because they contain the inversion operation, translations, namely the rotation, the dilation, and a rotation and a dilation included in \( u \), and not the inversion symmetry operation. The \( (16) \) transformation is thus classified as a similarity transformation.

### III. LIE ALGEBRA AND LIE GROUP OF THE QHT

As discussed in the previous section, the Möbius transformation in two dimensions has four symmetry operations, the translation, namely the rotation, the dilation and the special conformal transformation. Left-derivative quaternion functions exclude the special conformal transformation, because they contain the inversion operation, which generate a non-analytic quaternion function. Thus, the quaternionic transformation we consider generates linear functions like \( (13) \). Multiplication of the quaternionic variable by a quaternion constant comprises rotations and dilations and the addition of a quaternionic constant to a quaternion variable comprises translations. Let us then consider an infinitesimal similarity transformation
\[ q \rightarrow q(1 + \epsilon) + \delta \]  
where \( \epsilon \) and \( \delta \) are infinitesimal quaternionic constants. Using the aforementioned transformation on the quaternion function \( (13) \) we obtain
\[ \mathcal{F} \rightarrow [q(1 + \epsilon) + \delta]a + b = \mathcal{F} + qea + \delta a. \]  
The generator of the translation may simply be \( \partial_q \), but the differential generator for the transformation involving \( \epsilon \) cannot be \( q \partial_q \). Because of the non-commutativity of quaternions, the \( \epsilon \) may not be put on the left of \( q \) in order to get a differential generator for \( qea \) in \( (18) \). In order to get the Lie algebra generators, we then consider the quaternion generator comprising two complex parts. Thus, using symplectic notation, where
\[ e = \epsilon_0 + \epsilon_1 j \]  
for complex \( \epsilon_0 \) and \( \epsilon_1 \), we write
\[ qea = (z + \bar{z}j)(\epsilon_0 + \epsilon_1 j)a \]
\[ = [\epsilon_0 z - \bar{\epsilon}_1 \bar{z} + (\epsilon_1 z + \bar{\epsilon}_0 \bar{z})]a \]
\[ = (\epsilon_0 z - \bar{\epsilon}_1 \bar{z})\partial_\bar{z} \mathcal{F} + (\epsilon_1 z + \bar{\epsilon}_0 \bar{z})\partial_z \mathcal{F} \]  
(19)
Then, we have four generators: $z\partial_z$, $z\partial_{\zeta}$, $\zeta\partial_z$ and $\zeta\partial_{\zeta}$. Furthermore the translation represented by the third term of (18) is generated by $\partial_z$ and $\partial_{\zeta}$, and hence we have a total of six generators. On the other hand, we repeat the procedure for the complex conjugated quaternion variable $\bar{q}$ on a left-derivative holomorphic function $G$, and hence we obtain

$$G \to [\bar{q}(1+\epsilon) + \delta a + b = G + \bar{q}\epsilon a + \delta a]. \quad (20)$$

Consequently, from $\bar{q} = z - \zeta j$ we obtain

$$\bar{q}\epsilon = (z - \zeta j)(\epsilon_0 + \epsilon_1 j) = \epsilon_0 z + \bar{\epsilon}_1 \bar{\zeta} + (\epsilon_1 \bar{z} - \bar{\epsilon}_0 \zeta) j. \quad (21)$$

Consequently, $z\partial_z$, $z\partial_{\zeta}$, $\zeta\partial_z$ and $\zeta\partial_{\zeta}$ are the generators for the dilations and for the rotations, and $\partial_z$ and $\partial_{\zeta}$ are the generators for the translation. Note that the function $G$ cannot be obtained from a complex conjugation of a left-derivatives holomorphic function, because $\bar{F} = \bar{a}q + b$ is not left-derivative.

Naming the generator sets of the Lie algebras for the quaternionic transformations as $\mathfrak{r} = \{x_i\}$ for a quaternionic variable and as $\bar{\mathfrak{r}} = \{\bar{x}_i\}$ for a complex conjugated variable, we get

$$x_1 = \partial_z, \quad x_2 = z\partial_z \quad x_3 = \zeta\partial_z \quad x_4 = \partial_{\zeta}, \quad x_5 = \zeta\partial_{\zeta}, \quad x_6 = z\partial_{\zeta}, \quad \text{and}$$

$$\bar{x}_1 = \partial_{\bar{z}}, \quad \bar{x}_2 = \bar{z}\partial_{\bar{z}} \quad \bar{x}_3 = \bar{\zeta}\partial_{\bar{z}} \quad \bar{x}_4 = \partial_{\bar{\zeta}}, \quad \bar{x}_5 = \bar{\zeta}\partial_{\bar{\zeta}}, \quad \bar{x}_6 = \bar{z}\partial_{\bar{\zeta}}. \quad (22)$$

We note that the generators $\partial_{\zeta}$ and $\zeta\partial_{\zeta}$ are common for both the cases. The Lie algebra for $\mathfrak{r}$ is written through the commutation relations

$$[x_1, x_2] = x_1$$

$$[x_1, x_3] = 0 \quad [x_2, x_3] = -x_3$$

$$[x_1, x_4] = 0 \quad [x_2, x_4] = 0 \quad [x_3, x_4] = -x_1$$

$$[x_1, x_5] = 0 \quad [x_2, x_5] = 0 \quad [x_3, x_5] = -x_3 \quad [x_4, x_5] = x_4$$

$$[x_1, x_6] = x_4 \quad [x_2, x_6] = x_6 \quad [x_3, x_6] = x_5 - x_2 \quad [x_4, x_6] = 0 \quad [x_5, x_6] = -x_6$$

As we will see below, the $\mathfrak{r}$ algebra is not the QHT algebra, but we will need to adopt a constraint for recovering the QHT from $\mathfrak{r}$. The algebra $\bar{\mathfrak{r}}$ is of course isomorphic to $\mathfrak{r}$.

On the other hand, the direct sum of $\mathfrak{r}$ and $\bar{\mathfrak{r}}$ is not an algebra. Due to the commutator

$$[x_6, x_3] = [z\partial_{\zeta}, \zeta\partial_z] = z\partial_z, \quad (23)$$

$\mathfrak{r} \oplus \bar{\mathfrak{r}}$ is not closed, and then these algebras must always be considered separately. Perhaps this direct sum may be achieved using a right-derivative holomorphic function, but this hypothesis has not been considered here.

We can obtain the adjoint matrix representation for this algebra using the structure constants, so that the matrix elements could be obtained from $(x_i)_k^j = c_{ij}^k$. However, this representation comprises sixth order matrices that are cumbersome to manipulate. A simpler representation may be obtained defining a linear space represented by the vector

$$v = (z, \zeta, 1). \quad (24)$$

Let us then act on $v$ the differential generators (22) of $\mathfrak{r}$ in order to determine a matrix representation of the algebra. Collecting the results for each generator and adjusting the signals, we obtain a third order matrices representation

$$x_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 = (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_4 = (-1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_5 = (-1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

From the commutation relations of the algebra, we can see that $\mathfrak{r}$ has a non-trivial ideal $I$, so that $I = \{x_1, x_4\}$, and thus $\mathfrak{r}$ is neither a simple nor a semi-simple algebra. We may get further understanding looking at the Lie group, which may be built using the exponential mapping. If $X_i$ is an element of the Lie matrix group $X$, which is the covering Lie group of $\mathfrak{r}$, then

$$X_i = \exp(tx_i) = \begin{cases} 1 + tx_i & \text{for } i = \{1, 3, 4, 6\} \\ 1 + (1 - e^{-t})x_i & \text{for } i = \{2, 5\} \end{cases} \quad (26)$$
so that $t$ is a real parameter. This group can be represented by the matrix

$$X = \begin{bmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{bmatrix}, \quad (27)$$

where $X$ has complex entries. We will now examine several properties of the Lie group represented by the $X$ matrix in order to illustrate its relation to the QHT (16).

IV. PROPERTIES OF THE LIE GROUP $X$

First, we shall define a map $m : V \rightarrow \mathbb{H}$ between the vector space $V$ which contains $v$ defined in (24) and a quaternion number $q = z + \zeta j$, so that

$$m(v) = q. \quad (28)$$

Thence we use $m$ to relate $Xv$ to the QHT. In general,

$$Xv = \begin{bmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ \zeta \\ 1 \end{bmatrix} = \begin{bmatrix} az + b\zeta + p \\ cz + d\zeta + q \\ 1 \end{bmatrix}, \quad \text{and} \quad m(Xv) = az + b\zeta + p + (cz + d\zeta + q)j. \quad (29)$$

The $a, b, c$ and $d$ entries promote rotations and dilations on $z$ and $\zeta$, and $p$ and $q$ generate quaternionic translations. The group $X$ defines a transformation on $q$ that is more general than a QHT. In order to obtain a QHT (16) we must impose several constraints on $X$. We shall then write a general QHT as

$$G(q) = qu + v$$
$$= (z + \zeta j)(a + bj) + p + qj$$
$$= az - b\zeta + (bz + a\zeta)j + p + qj. \quad (30)$$

A QHT may then be represented by a matrix $G$, so that

$$G(q) \Leftrightarrow m(Gv), \quad \text{for} \quad G = \begin{bmatrix} a & -\bar{b} & p \\ b & \bar{a} & q \\ 0 & 0 & 1 \end{bmatrix}. \quad (31)$$

Then we have obtained the group $G$ of QHT as a restriction of the group $X$. The determinant of $X$, given by $|X| = ad - bc$ may have an arbitrary value. After the discussion of the QHT cross ratio and the fixed points in the sequel, we will study the consequences of normalizing this determinant.

A. Cross-ratio and fixed points

A general Möbius transformation (1) admits the existence of an invariant relation. This function is the cross-ratio, given by the expression

$$[z, z_1, z_2, z_3] = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}. \quad (32)$$

Using the identity

$$M(z) - M(z_1) = \frac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)}, \quad (33)$$

the cross-ratio may be easily obtained. (32) has the important property of invariance under Möbius transformations, and then

$$[z, z_1, z_2, z_3] = [M(z), M(z_1), M(z_2), M(z_3)]. \quad (34)$$
The cross-ratio implies, if an arbitrary Möbius transformation \( M \) is applied on three of any four points, \( M \) is the unique transformation that, applied on the fourth, will preserve the cross-ratio built with the other three transformed points. Accordingly, it can be proven that three arbitrary points are related to another set of three points by an unique Möbius transformation. A standard three points set comprises 0, 1, and \( \infty \), and then there is a unique Möbius transformation that relates this set to an arbitrary set of three complex numbers.

In order to determine whether there is some kind of cross ratio for the QHT, we could take the complex similarity transformation

\[
N(z) = az + b, \tag{35}
\]

and obtain a cross ratio using

\[
N(z) - N(z_1) = (z - z_1)a, \quad \text{so that} \quad [z, z_1, z_2] = \frac{z - z_1}{z - z_2}.
\]

Considering complex numbers as vectors, the similarity transformation (35) rotates both of the vectors by the same angle and dilates their norms by the same factor, and consequently the difference vector suffers the same transformation. The cross-ratio, on this side means that, for any given two points, a third point and its image are related by a unique similarity condition. On the other hand, for the QHT, the relation

\[
G(q) - G(q_1) = (q - q_1)(a + bj) \tag{36}
\]

informs us that the rotation and dilation are preserved in the difference. However, due to the anti-commutativity of quaternions, we cannot build a cross ratio to prove that there is a unique QHT relating every pair of quaternions. This is the first indication of a lack of symmetry of the QHT compared to the complex Möbius transformation. In the next section we discuss constraints that lead QHT to more symmetric cases. The difference (36), may generate a cross ratio by restricting the QHT to translations or real dilations, for example.

The QHT, as the similarity transformation for complex numbers, has only one fixed point, which obeys

\[
G(q) = q.
\]

For the complex case, a Möbius transformation with only one fixed point is classified as belonging to the parabolic type. This kind of transformation has a fixed point at infinity, and this fixed point cannot be moved to a finite coordinate complex point by a parabolic transformation. In a general Möbius transformation, due to the inverse operation, the infinity may be related to the zero point by inversion. The QHT, considered a parabolic transformation on each complex number that constitutes the quaternion in the symplectic notation, has analogously the infinity as its only fixed point, which cannot be moved to another point by any QHT, and consequently infinity is the only fixed point of the QHT.

### B. \( X \) group as a Möbius transformation

Imposing \( p = q = 0 \) on (27), we have a matricial representation of the Möbius group, represented by

\[
M = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}
\]

In terms of quaternions, this representation is simply a translationless transformation

\[
m(Mv) = az + b\zeta + (cz + d\zeta)j.
\]

In terms of differential operators, the algebra of the group is given by the set subset \( \{ x_2, x_3, x_5, x_6 \} \) of (22). On the other hand, by imposing \( |M| = 1 \), we have the \( \text{sl}(2, \mathbb{C}) \) algebra of the normalized Möbius group. This is achieved by using the operators set comprising

\[
\{ x_3, x_6, x_5 - x_2 \} = \{ \zeta \partial_{x_3}, z \partial_{x_5}, \zeta \partial_{x_6} - z \partial_{x_2} \}, \tag{39}
\]

whose elements satisfy the algebra

\[
[x_2 - x_5, x_3] = [x_2 - x_5, x_6] = 0, \quad [x_3, x_6] = x_5 - x_2. \tag{40}
\]

Then we have a representation of the \( \text{SL}(2, \mathbb{C}) \) group, as expected. The representation by the differential operators (27) is not new, but its interpretation in terms of quaternions is certainly new. In terms of physics, it is fascinating to observe that the above operators may be used to represent the quantum harmonic oscillator.
C. X group as the Heisenberg group

The Heisenberg group, also called the Weyl group, comprises upper triangular matrices of the form

\[
H = \begin{bmatrix}
1 & b & p \\
0 & 1 & q \\
0 & 0 & 1
\end{bmatrix}.
\] (41)

The group is isomorphic to the group of matrices of the type

\[
\tilde{H} = \begin{bmatrix}
1 & 0 & p \\
c & 1 & q \\
0 & 0 & 1
\end{bmatrix}.
\] (42)

Thus algebra of the group may be represented by matrices either as

\[
h = \begin{bmatrix}
0 & b & p \\
0 & 0 & q \\
0 & 0 & 0
\end{bmatrix}
\quad \text{or as} \quad \tilde{h} = \begin{bmatrix}
0 & 0 & p \\
c & 0 & q \\
0 & 0 & 0
\end{bmatrix}.
\] (43)

These algebras may be represented by sub-algebras of \(X\), either the matrix generators \(\{x_1, x_4, x_3\}\) or \(\{x_1, x_4, x_6\}\). The Heisenberg algebra is then a sub-algebra of \(\frak{f}\), but it is not a sub-algebra of the algebra of a QHT. Only for \(b = 0\) in \(h\) and for \(c = 0\) in \(\tilde{h}\) the algebras coincide. In this sense, only the translation of a QHT is identified with a sub-algebra of Heisenberg.

In terms of differential operators, the Heisenberg algebra is usually represented as

\[
X = \partial_x - \frac{1}{2}y\partial_z, \quad Y = \partial_y + \frac{1}{2}x\partial_z, \quad Z = \partial_z,
\] (44)

which obey the algebra

\[
[X, Y] = Z, \quad [Z, X] = [Z, Y] = 0.
\] (45)

The operators \(X\) and \(Y\) may be interpreted as momentum and position operators in certain quantum mechanical applications [17]. This algebra may be represented by two subsets of \(\frak{g}\), either by

\[
\{x_1, x_4, x_3\} = \{\partial_z, \partial_\xi, z\partial_\xi\} \quad \text{or by} \quad \{x_1, x_4, x_6\} = \{\partial_z, \partial_\xi, \xi\partial_\xi\}.
\] (46)

Then, in terms of the differential operators \(x_i\) of \(\frak{f}\), we have a new representation for the Heisenberg algebra.

V. THE QHT LIE ALGEBRA

The QHT group was obtained as represented by the matrix (31)

\[
G = \begin{bmatrix}
a - b & p \\
b & a + q \\
0 & 1
\end{bmatrix}.
\] (47)

The set of matrix generators for its algebra \(\frak{g}\) may be obtained from the \(x_i\) generators of \(\frak{f}\) (25) defining

\[
g_1 = x_3 + x_6, \quad g_2 = i(x_6 - x_3), \quad g_3 = x_2 - x_5, \quad g_4 = -(x_2 + x_5), \quad g_5 = x_1, \quad g_6 = -x_4,
\] (48)

so that

\[
g_1 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad g_2 = \begin{bmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad g_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
\] (49)
We see that the sub-algebra
From (48) and from (22), we obtain the representation in terms of the differential operators
\[ g_1 = \zeta \partial_z + z \partial_{\zeta}, \quad g_2 = i(z \partial_{\zeta} - \zeta \partial_z), \quad g_3 = z \partial_z - \zeta \partial_{\zeta}, \quad g_4 = -z \partial_z - \zeta \partial_{\zeta}, \quad g_5 = \partial_z, \quad g_6 = -\partial_{\zeta}, \] (50)
which obeys the algebra
\[
\begin{align*}
[g_1, g_2] &= 2ig_3 \\
[g_1, g_3] &= 2ig_2 \\
[g_2, g_3] &= 2ig_1 \\
[g_1, g_4] &= 0 \\
[g_2, g_4] &= 0 \\
[g_3, g_4] &= 0 \\
[g_1, g_5] &= g_6 \\
[g_2, g_5] &= ig_6 \\
[g_3, g_5] &= -g_5 \\
[g_4, g_5] &= g_5 \\
[g_1, g_6] &= g_5 \\
[g_2, g_6] &= -ig_5 \\
[g_3, g_6] &= g_6 \\
[g_4, g_6] &= g_6 \\
[g_5, g_6] &= 0
\end{align*}
\]
We see that the sub-algebra \{g_1, g_2, g_3\} generates the \(su(2, \mathbb{C})\) algebra, and then we can associate these operators with rotations on a two sphere. The operator \(g_4\) is responsible for dilations. If we exclude it from \(g\), this is equivalent to impose \(|G| = 1\). This phenomenon has already been observed in the discussion of \(X\) as a Möbius group in Section IVB.

VI. CONCLUSION

In this article we have presented the Lie algebra for a similarity transformation of quaternion holomorphic functions. The similarity transformation is the Möbius transformation for quaternion holomorphic functions, which does not have the inversion operation because this would generate a non-holomorphic quaternion function.

For this purpose, we studied the group of symmetry of a transformation \(X\), in which independent similarity transformations are applied to the complex components of a quaternion, and we have called this group \(X\). Compared to the complex Möbius transformation \(M\), quaternionic transformation \(X\) is less symmetric and admits further possibilities. Adopting some constraints on \(X\), we obtain the complex Möbius group \(M\), the Heisenberg group \(H\) and the QHT group \(G\). Thus, we have set very symmetric groups as particular cases of a group \(X\) with many degrees of freedom. From the mathematical standpoint, a possible future direction for research is to investigate whether the group presented here may be used for geometric studies, as has been done for conformal differential geometry [9]. The study of right-derivative holomorphic functions also seems to be an interesting direction for research.

There may be several physical applications for \(X\) and \(G\), and we have just mentioned a several few among many others we are unable to imagine. Some of them are very simple, and this makes these possibilities interesting. The quantum harmonic oscillator could be studied through the operators [39], and a new connection between complex quantum mechanics and quaternionic quantum mechanics [18] may be established. Also the study of quaternionic quantum field theory may be benefited [19]. As the conformal symmetry is present in \(G\), the study of quantum mechanics [20] and field theories [21] using conformal symmetry may also be studied using the algebra presented here, and the connection to quaternion quantum mechanics might also be tested here as well. Complex numbers have been used to build the two-dimensional conformal field theory, and quaternions may be used to build a four-dimensional conformal field theory. In fact, there are already some attempts in this direction, using quaternionic matrices, quaternionic projective spaces and quaternionic Fuller analytic functions [22,23]. Holomorphic quaternion functions seem too strong a constraint for these theories, but the construction of a field theory involving quaternion holomorphic functions might the attempted. On the other hand, Möbius transformations are metric symmetries for hyperbolic geometries, and they may be used for defining conformal field theories in such spaces. This has also already been done [24] and makes explicit use of the cross-ratio. A similar study using the restricted cross-ratio discussed here seems possible. The connection with Heisenberg algebra is also a very interesting direction, in order to understand the interpretation in terms of quaternionic quantum mechanics of the differential operators of position and momentum. However, there are many other possibilities in this subject, as can be seen from the studies that have already been done for with Heisenberg group [17].

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[1] J. Anderson “Hyperbolic Geometry” Springer (2008).
[2] P. H. Ginsparg “Applied conformal field theory” Les Houches summer school (1988) hep-th/9108028.
[3] R. Blumenhagen; E. Plauschinn “Introduction to conformal field theory” Lect. Notes Phys., 779:1–256 (2009).
[4] J. B. Wilker “The quaternion formalism for Möbius groups in four or fewer dimensions” Lin. Alg. Appl., 190:99–136 (1993).
[5] P. L. Waterman “Möbius transformations in several dimensions” Adv. Math., 101:87–113 (1993).
[6] A. F. Beardon “The Geometry of Discrete Groups” Springer (1995).
[7] I. R. Porteous “Clifford Algebras and the Classical Groups” Cambridge University Press (1995).
[8] P. Lounesto “Clifford Algebras and Spinors” Cambridge University Press (1997).
[9] U. Hertrich-Jeromin “Introduction to Möbius Differential Geometry” Cambridge University Press (2003).
[10] A. Sudbery “Quaternionic Analysis” Math. Proc. Camb. Phil. Soc., 85:199–225 (1979).
[11] R. Lavicka; A. G. O’Farrell; I. Short “Reversible maps in the group of quaternionic Möbius transformations” Math. Proc. Camb. Phil. Soc., 143:57 (2007).
[12] K. Gürlebeck; W. Sprössig “Quaternionic and Clifford Calculus for Physicists and Engineers” Wiley (1998).
[13] W. Cao; J. R. Parker; X. Wang “On the classification of quaternionic Möbius transformations” Math. Proc. Camb. Phil. Soc., 137:349 (2004).
[14] H. Huang “Discrete subgroups of Möbius transformations” J. Math. Anal. Appl., 397:233–241 (2013).
[15] C. Stoppato “Regular Moebius transformations of the space of quaternions” Ann. Glob. Anal. Geom., 190:387–401 (2011).
[16] C. A. Deavours “Quaternion Calculus” Am. Math. Monthly, 80:995–1008 (1973).
[17] E. Binz; S. Pods “The Geometry of Heisenberg Groups” American Mathematical Society (2008).
[18] S. L. Adler “Quaternionic Quantum Mechanics and Quantum Fields” Oxford University Press (1995).
[19] S. Giardino; P. Teotonio-Sobrinho “A non-associative quaternion scalar field theory” Mod. Phys. Lett., A28(35):1350163 (2013) arXiv:1211.5049[math-ph].
[20] V. de Alfaro; S. Fubini; G. Furlan “Conformal Invariance in Quantum Mechanics Nuovo Cim., A34:569 (1976).
[21] S. Fubini “A New Approach to Conformal Invariant Field Theories” Nuovo Cim., A34:521 (1976).
[22] F. Gursey; H. C. Tze “Complex and Quaternionic Analyticity in Chiral and Gauge Theories. Part 1” Annals Phys., 128:29 (1980).
[23] M. Evans; F. Gursey; V. Ogievetsky “From 2-D conformal to 4-D selfdual theories: Quaternionic analyticity” Phys. Rev., D47:3496–3508 (1993).
[24] P. Kleban; I. Vassileva “Conformal field theory and hyperbolic geometry” Phys. Rev. Lett., 72:3929–3932 (1994).