Discrete Riesz transforms and sharp metric $X_p$ inequalities

By Assaf Naor

Abstract

For $p \in [2, \infty)$, the metric $X_p$ inequality with sharp scaling parameter is proven here to hold true in $L_p$. The geometric consequences of this result include the following sharp statements about embeddings of $L_q$ into $L_p$ when $2 < q < p < \infty$: the maximal $\theta \in (0, 1]$ for which $L_q$ admits a bi-$\theta$-Hölder embedding into $L_p$ equals $q/p$, and for $m, n \in \mathbb{N}$, the smallest possible bi-Lipschitz distortion of any embedding into $L_p$ of the grid $\{1, \ldots, m\}^n \subseteq \ell_q^n$ is bounded above and below by constant multiples (depending only on $p, q$) of the quantity $\min\{n^{(p-q)(q-2)/(q^2(p-2))}, m^{(q-2)/q}\}$.

1. Introduction

The purpose of the present article is to positively resolve three conjectures that were posed by the author in collaboration with G. Schechtman in [NS16]. Specifically, we shall prove here that Conjectures 1.5, 1.8 and 1.12 of [NS16] all have a positive answer. As we shall explain below, of these three conjectures, Conjecture 1.8 was a longstanding open problem in embedding theory, while Conjecture 1.12 asserts the validity of a quite subtle and perhaps unexpected phase transition phenomenon that was first formulated as conceivably holding true in [NS16]. Conjecture 1.5 relates to a bi-Lipschitz invariant that was introduced in [NS16], asking about finer properties of this invariant in terms of a certain auxiliary parameter.

It was proven in [NS16] that Conjectures 1.8 and 1.12 follow from Conjecture 1.5. Thus Conjecture 1.5 is the heart of the matter and the main focus of the present article, but we shall first describe all of the above conjectures since, by proving their validity, we establish delicate geometric phenomena related to the metric structure of $L_p$ spaces. In addition to these applications, a key contribution of the present article is the use of a deep result of Lust-Piquard [LP98] for geometric purposes. While [NS16] proposed an approach to resolve the above conjectures, formulated as Question 6.1 in [NS16] and discussed at

Supported in part by the BSF, the Packard Foundation and the Simons Foundation.
© 2016 Department of Mathematics, Princeton University.
length in [NS16, §6], where it was shown to imply the above conjectures, we
do not pursue this approach here, and indeed Question 6.1 of [NS16] remains
open. Below we take a different route, yielding a novel connection between
purely geometric questions and investigations in modern harmonic analysis

and operator algebras.

1.1. Geometric statements. Following standard notation in Banach space
theory and embedding theory (as in, say, [LT77], [Ost13]), for \( n \in \mathbb{N} \) and
\( p \in [1, \infty) \), we let \( \ell_p^n \) denote the space \( \mathbb{R}^n \) equipped with the \( \ell_p \) norm.
When referring to the space \( L_p \), we mean for concreteness the Lebesgue space \( L_p(\mathbb{R}) \),
though all of our new geometric results apply equally well to any infinite di-
mensional \( L_p(\mu) \) space. The \( L_p \) distortion of a metric space \((X, d_X)\), denoted
\( c_p(X) \in [0, \infty] \), is the infimum over those \( D \in [0, \infty] \) for which there exists a
mapping \( f : X \to L_p \) that satisfies

\[ \forall x, y \in X, \quad d_X(x, y) \leq \|f(x) - f(y)\|_{L_p} \leq D d_X(x, y). \]

\((X, d_X)\) is said to admit a bi-Lipschitz embedding into \( L_p \) if \( c_p(X) < \infty \).

Given \( m, n \in \mathbb{N} \) and \( q \in [1, \infty) \), the metric space whose underlying
set is \( \{1, \ldots, m\}^n \) (the \( m \)-grid in \( \mathbb{R}^n \)), equipped with the metric inherited
from \( \ell_q^m \), will be denoted below by \( [m]^n \). It follows from the classical work
[Pal36] of Paley, in combination with general principles related to differen-
tiation of Lipschitz functions (see [BL00, Ch. 7]), that if \( 2 < q < p < \infty \),
then \( \lim_{m \to \infty} c_p(\ell_q^m) = \infty \). Since \( [m]^n \) becomes “closer” to \( \ell_q^m \) as \( m \to \infty \),
one can apply an ultrapower argument (see [Hei80]) to deduce from this that
\( \lim_{m,n \to \infty} c_p([m]^n) = \infty \), but such reasoning does not yield information on the
rate of growth of \( c_p([m]^n) \). Effective estimates here follow from an alternative
approach of Bourgain [Bou87] (with an improvement in [GNS12]), as well as
the approach of [NS16], but the resulting bounds are far from being sharp.

Resolving Conjecture 1.12 of [NS16], Theorem 1 below computes the quantity
\( c_p([m]^n) \) up to constant factors that may depend on \( p, q \) but not on \( m, n \).

**Theorem 1** (Sharp evaluation of the \( L_p \) distortion of \( \ell_q^m \) grids). Suppose
that \( p, q \in [2, \infty) \) satisfy \( q < p \). Then for every \( m, n \in \mathbb{N} \), we have

\[
(1) \quad c_p([m]^n) \asymp_{p,q} \min \left\{ \frac{(p-q)(q-2)}{q(p-2)}, m^{1 - \frac{2}{q}} \right\} = \left( \min \left\{ \frac{n^{\frac{p-q}{q(p-2)}}, m} \right\} \right)^{1 - \frac{2}{q}}.
\]

In the statement of Theorem 1, as well as in what follows, we use stan-
dard asymptotic notation. Namely, the notation \( a \lesssim b \) (respectively \( a \gtrsim b \))
stands for \( a \leq cb \) (respectively \( a \geq cb \)) for some universal constant \( c \in (0, \infty) \).
The notation \( a \asymp b \) stands for \( (a \lesssim b) \land (b \lesssim a) \). When we allow for im-
plicit constants to depend on parameters, we indicate this by subscripts. Thus
\( a \lesssim_{p,q} b \) (respectively \( a \gtrsim_{p,q} b \)) means that there exists \( c(p, q) \in (0, \infty) \) that
may depend only on \( p, q \) such that \( a \leq c(p, q)b \) (respectively \( a \geq c(p, q)b \)). The notation \( a \asymp_{p,q} b \) stands for \((a \lesssim_{p,q} b) \land (b \lesssim_{p,q} a)\).

Very few results at the level of precision of Theorem 1 are known, and analogous questions are open even for some values of \( p, q \) that are not covered by Theorem 1; see [NS16, Rem. 1.13] for more on this interesting topic. The asymptotic formula (1) expresses the statement that there exist two specific embeddings of \([m]^n_q\) into \( L_p \) such that one of them is always the best possible embedding of \([m]^n_q\) into \( L_p \), up to constant factors that do not depend on \( m, n \).

One of these embeddings arises from the work of Rosenthal [Ros70] (relying also on computations in [GPP80], [FJS88]), and the other is due to Mendel and the author [MN06] (relying also on a construction from [Sch38]). These issues, including precise descriptions of the above two embeddings, are explained in detail in [NS16].

The following immediate corollary of Theorem 1 asserts that if \( 2 < q < p < \infty \) and \( m, n \in \mathbb{N} \), then the \( L_p \) distortion of \([m]^n_q\) exhibits a phase transition at \( m \asymp n^{(p-q)/(q(p-2))} \).

**Corollary 2** (Sharp phase transition of the \( L_p \) distortion of \( \ell^n_q \) grids).

Suppose that \( m, n \in \mathbb{N} \) and \( p, q \in (2, \infty) \) satisfy \( q < p \). Then

\[
m \gtrsim n^{\frac{p}{q(p-2)}} \implies c_p([m]^n_q) \asymp_{p,q} c_p(\ell^n_q),
\]

while as \( n \to \infty \), we have

\[
m = o\left(n^{\frac{p-q}{q(p-2)}}\right) \implies c_p([m]^n_q) = o\left(c_p(\ell^n_q)\right).
\]

It would be illustrative to state a concrete example of the situation whose validity is established in Corollary 2. So, consider the (arbitrary) choices \( q = 3 \) and \( p = 4 \). When one tries to embed the grid \([m]^n_3\) into \( L_4 \), one sees that there is a phase transition at \( m \asymp \sqrt[3]{n} \). If \( m \gtrsim \sqrt[3]{n} \), then any embedding of \([m]^n_3\) into \( L_4 \) incurs the same distortion (up to universal constant factors) as the distortion required to embed all of \( \ell^n_3 \) into \( L_4 \), which grows like \( \sqrt[3]{n} \). However, if \( m = o(\sqrt[3]{n}) \), then one can embed \([m]^n_3\) into \( L_4 \) with distortion \( o(\sqrt[3]{n}) \), and in this case the \( L_4 \) distortion of \([m]^n_3\) is \( \sqrt[3]{m} \), up to universal constant factors.

Our second geometric result is Theorem 3 below, which resolves Conjecture 1.8 of [NS16].

**Theorem 3** (Evaluation of the critical \( L_p \) snowflake exponent of \( L_q \)).

Suppose that \( p, q \in (2, \infty) \) satisfy \( q < p \). Then the maximal \( \theta \in (0, 1] \) for which the metric space \((L_q, \|x - y\|_L^q)\) admits a bi-Lipschitz embedding into \( L_p \) equals \( q/p \).

In the setting of Theorem 3, the fact that \((L_q, \|x - y\|_{L_q}^{q/p})\) does indeed admit a bi-Lipschitz (even isometric) embedding into \( L_p \) was established by
Mendel and the author in [MN04]. Since then, it has been a well-known conjecture that in this context the Hölder exponent \( q/p \) cannot be increased, but before [NS16] it was not even known that if \( (L^q, \|x-y\|_{L^q}) \) admits a bi-Lipschitz embedding into \( L_p \), then necessarily \( \theta < 1 - \delta \) for some \( \delta = \delta(p,q) > 0 \). Note that the endpoint case \( q = 2 \) must be removed from Theorem 3 since \( L_2 \) embeds isometrically into \( L_p \).

1.2. Optimal scaling in the \( L_p \)-valued metric \( X_p \) inequality. In what follows, given \( n \in \mathbb{N} \) we shall denote the set \( \{1, \ldots, n\} \) by \( [n] \). The coordinate basis of \( \mathbb{R}^n \) will be denoted by \( e_1, \ldots, e_n \), and for a sign vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n \) and a subset \( S \subseteq [n] \), we shall use the notation

\[
\varepsilon_S \overset{\text{def}}{=} \sum_{j \in S} \varepsilon_j e_j.
\]

Fix \( p \in (0, \infty) \). Following [NS16], a metric space \( (X,d_X) \) is said to be an \( X_p \) metric space if there exists \( X \in (0, \infty) \) such that for every \( n \in \mathbb{N} \) and \( k \in [n] \), there exists \( m \in \mathbb{N} \) such that, denoting as usual \( \mathbb{Z}_{2m} = \mathbb{Z}/(2m\mathbb{Z}) \), every function \( f : \mathbb{Z}_{2m}^n \to X \) satisfies the following distance inequality:

\[
(3) \quad \left( \frac{1}{k} \sum_{S \subseteq [n]} \mathbb{E}[d_X(f(x + m\varepsilon_S), f(x))^p] \right)^{1/p} \leq X_m \left( \frac{k^n}{n} \sum_{j=1}^n \mathbb{E}[d_X(f(x + e_j), f(x))^p] + \left( \frac{k}{n} \right)^{\frac{p}{2}} \mathbb{E}[d_X(f(x + \varepsilon), f(x))^p] \right)^{1/p}.
\]

The expectations in (3) are with respect to \( (x, \varepsilon) \in \mathbb{Z}_{2m}^n \times \{-1,1\}^n \) chosen uniformly at random. We refer to [NS16] for a detailed discussion of the meaning of (3); see also Sections 1.2.1 and 1.3 below.

The above definition of \( X_p \) metric spaces introduces the auxiliary integer \( m \in \mathbb{N} \), which we call the scaling parameter corresponding to \( n \) and \( k \). For some purposes \( m \) can be allowed to be arbitrary, but for other purposes one needs to obtain good bounds on \( m \) (as a function of \( n,k \)). It can, however, be quite difficult to obtain sharp bounds on scaling parameters in metric inequalities. (For example, an analogous question in the context of metric cotype [MN08] is longstanding and important.) In [NS16] it was proven that if \( p \in [2, \infty) \), then \( L_p \) is an \( X_p \) metric space. The proof in [NS16] yields the validity of (3) when \( X = L_p \) whenever \( m \geq_p n^{\frac{3}{2}}/\sqrt{k} \). It was also shown in [NS16, Prop. 1.4] that if \( p \in (2, \infty) \) and \( k \) is sufficiently large (as a function of \( p \)), then for (3) to hold true in \( L_p \) one must necessarily have \( m \geq_p \sqrt{n}/k \). Conjecture 1.5 of [NS16] asks whether for every \( p \in (2, \infty) \), this lower bound on \( m \) actually expresses the asymptotic behavior of the best possible scaling parameter, i.e., whether
the metric $X_p$ inequality (3) holds true in $L_p$ for every $m \gtrsim_p \sqrt{n/k}$. Theorem 4 below resolves this conjecture positively.

**Theorem 4** ($L_p$ is an $X_p$ metric space with sharp scaling parameter). Suppose that $k, m, n \in \mathbb{N}$ satisfy $k \in [n]$ and $m \gtrsim \sqrt{n/k}$. Suppose also that $p \in [2, \infty)$. Then every $f : \mathbb{Z}_{8m}^n \to L_p$ satisfies

\[
(4) \quad \left( \frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\mid |S| = k}} \mathbb{E} \left[ \|f(x + 4m\varepsilon_S) - f(x)\|_{L_p}^p \right] \right)^{\frac{1}{p}} \lesssim_p m \left( \frac{k}{n} \sum_{j=1}^n \mathbb{E} \left[ \|f(x + e_j) - f(x)\|_{L_p}^p \right] + \left( \frac{k}{n} \right)^{\frac{p}{2}} \mathbb{E} \left[ \|f(x + \varepsilon) - f(x)\|_{L_p}^p \right] \right)^{\frac{1}{p}},
\]

where the expectations are taken with respect to $(x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1, 1\}^n$ chosen uniformly at random.

**Remark 5.** Our proof of Theorem 4 shows that the implicit constant in (4) is $O(p^4/\log p)$. As explained in [NS16], this constant must be at least a (universal) constant multiple of $p/\log p$. While it is conceivable that a more careful implementation of our approach could somewhat decrease the dependence on $p$ that we obtain, it seems that a new idea is required in order to establish the sharp dependence of $O(p/\log p)$ in (4) (if true). We leave the question of determining the correct asymptotic dependence on $p$ in (4) as an interesting (and perhaps quite challenging) open question.

1.2.1. **Applications of Theorem 4.** The usefulness of the metric $X_p$ inequality for $L_p$ stems in part from the fact that it allows one to rule out the existence of metric embeddings in situations where the classical differentiation techniques fail. Examples of such situations include the treatment of discrete sets as in Theorem 1, where it is not clear how to interpret the notion of derivative, as well as the treatment of Hölder mappings as in Theorem 3, where, unlike the Lipschitz case, mappings need not have any point of differentiability. In fact, by [NS16, Th. 1.14] both Theorems 1 and 3 follow from Theorem 4. For completeness, we shall now briefly sketch why this is so.

Suppose that $2 \leq q < p < \infty$ and $m, n \in \mathbb{N}$. It is simple to check, as done in [NS16, Lemma 3.1], that there exists $h : \mathbb{Z}_{8m}^n \to [32m]^n_q$ such that for $(x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1, 1\}^n$, $S \subseteq [n]$ and $j \in [n]$,

\[
\|h(x + 4m\varepsilon_S) - h(x)\|_{q^*} \approx m|S|^{\frac{1}{q}}, \quad \|h(x + e_j) - h(x)\|_{q^*} \approx 1
\]

and

\[
\|h(x + \varepsilon) - h(x)\|_{q^*} \approx n^{\frac{1}{q^*}}.
\]
Fix $D \in [1, \infty)$, and suppose that $\phi : [32m]_q^n \to L_p$ satisfies $\|x - y\|_{L_q^p} \leq \|\phi(x) - \phi(y)\|_{L_p^q} \leq D\|x - y\|_{L_q^p}$ for every $x, y \in [32m]_q^n$. An application of Theorem 4 to $f = \phi \circ h$ (with $m$ replaced by $4m$), which we are allowed to do only when $k \in [n]$ is such that $4m \geq \sqrt{n/k}$, yields the bound

$$\tag{5} D \geq_p \max_{k \in [n]} \frac{\frac{1}{k^q}}{k^{\frac{p}{n}} (k + \frac{k^{\frac{q}{n}}}{n^{\frac{q}{p}}})^{\frac{1}{p}}}.$$

By evaluating the maximum in (5), one arrives at the asymptotic lower bound on $c_p([32m]_q^n)$ that appears in (1). As we explained earlier, the matching upper bound in (1) corresponds to the better of two explicit embeddings that are described in equations (11) and (27) of [NS16]. This completes the deduction of Theorem 1. Next, fix $L \in [1, \infty)$ and $\theta \in (0, 1]$. Suppose that $\psi : L_q \to L_p$ satisfies $\|x - y\|_{L_q^p} \leq \|\psi(x) - \psi(y)\|_{L_p^q} \leq L\|x - y\|_{L_q^p}$ for every $x, y \in L_q$. For $k, n \in \mathbb{N}$ with $k \in [n]$, fix $m = \lceil \sqrt{n/(2k)} \rceil$ and apply Theorem 4 to $f = \psi \circ h$. The estimate thus obtained is

$$\left( \frac{n}{k} \right)^{\frac{q}{p}} k^{\frac{q}{p}} \leq_p L \sqrt{\frac{n}{k}} \left( k + \frac{k^{\frac{q}{n}}}{n^{\frac{q}{p}}} \right)^{\frac{1}{p}}.$$

Hence, for every $n \in \mathbb{N}$, we have

$$\tag{6} 1 \leq_p L n^{\frac{1-q}{2}} \min_{k \in [n]} \left( k + k^{\frac{q}{n}} n^{\frac{q}{p}(\frac{1}{2} - \frac{1}{2})} \right)^{\frac{1}{p}} k^{\theta(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}}.$$

Theorem 3 now follows by choosing the optimal $k$ in (6) and letting $n \to \infty$; complete details of this computation appear in the proof of Theorem 1.14 in [NS16].

1.3. Hypercube Riesz transforms and an $X_p$ inequality for Rademacher chaos. Fixing $n \in \mathbb{N}$, for every $h : \{-1, 1\}^n \to \mathbb{R}$ and $j \in [n]$, let $\partial_j h : \{-1, 1\}^n \to \mathbb{R}$ be given by

$$\tag{7} \forall \varepsilon \in \{-1, 1\}^n, \quad \partial_j h(\varepsilon) \overset{\text{def}}{=} h(\varepsilon) - h(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n).$$

Also, given $S \subseteq [n]$ we shall denote by $E_S h : \{-1, 1\}^n \to \mathbb{R}$ the function that is obtained from $h$ by averaging over the coordinates in $S$, i.e., recalling the notation (2), we define

$$\tag{8} \forall \varepsilon \in \{-1, 1\}^n, \quad E_S h(\varepsilon) \overset{\text{def}}{=} \frac{1}{2^n} \sum_{\delta \in \{-1, 1\}^n} h(\delta_S + \varepsilon_{[n] \setminus S}).$$

In particular, $E_S h$ depends only on those entries of $\varepsilon \in \{-1, 1\}^n$ that belong to $[n] \setminus S$. Given $p \in [1, \infty)$, from now on we shall reserve the notation
\[ \|h\|_p \text{ exclusively for the } L_p \text{ norm of } h \text{ with respect to the normalized counting measure on the discrete hypercube } \{-1,1\}^n, \text{ i.e.,} \]

\[ \|h\|_p \overset{\text{def}}{=} \left( \frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} |h(\varepsilon)|^p \right)^{\frac{1}{p}} = \left( \mathbb{E}_{[n]} |h|^p \right)^{\frac{1}{p}}. \]

In what follows, \( L^0_p(\{-1,1\}^n) \) denotes the subspace of all those \( h \in L^p_p(\{-1,1\}^n) \) with \( \mathbb{E}_{[n]} h = 0 \).

We shall work with the usual Fourier–Walsh expansion of a function \( h : \{-1,1\}^n \to \mathbb{R} \). Thus, for every \( A \subseteq [n] \), consider the corresponding Walsh function \( W_A : \{-1,1\}^n \to \mathbb{R} \) given by

\[ \forall \varepsilon \in \{-1,1\}^n, \quad W_A(\varepsilon) \overset{\text{def}}{=} \prod_{j \in A} \varepsilon_j, \]

and denote

\[ \hat{h}(A) \overset{\text{def}}{=} \frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} h(\varepsilon) W_A. \]

Then we have

\[ \forall \varepsilon \in \{-1,1\}^n, \quad h(\varepsilon) = \frac{1}{2^n} \sum_{A \subseteq [n]} \hat{h}(A) W_A(\varepsilon). \]

In probabilistic terminology, the above representation of \( h \) as a multilinear polynomial in the variables \( \varepsilon_1, \ldots, \varepsilon_n \) expresses it as Rademacher chaos. A useful inequality for Rademacher chaos of the first degree, i.e., for weighted sums of independent and identically distributed Bernoulli random variables, served as the inspiration for the metric \( X_p \) inequality (3). Specifically, (3) is a nonlinear extension of the following inequality, which holds true for every \( p \in [2, \infty) \), \( k, n \in \mathbb{N} \) with \( k \in [n] \), and every \( a_1, \ldots, a_n \in \mathbb{R} \):

\[ \left( \frac{1}{2^n (k^n)} \sum_{S \subseteq [n]} \sum_{\varepsilon \in \{-1,1\}^n} \left| \sum_{j \in S} \varepsilon_j a_j \right|^p \right)^{\frac{1}{p}} \overset{\text{def}}{=} \left( \frac{1}{\log p} \left( \frac{k}{n} \sum_{j=1}^n |a_j|^p + \frac{(k/n)^{\frac{p}{2}}}{2^n} \right) \right)^{\frac{1}{p}}. \]

(9)

This inequality is due to Johnson, Maurey, Schechtman and Tzafriri, who proved it in [JMST79] with a constant factor that grows to \( \infty \) with \( p \) faster than the \( p/\log p \) factor that appears in (9). The factor \( p/\log p \) that is stated in (9) is best possible; in the above sharp form, (9) is due to Johnson, Schechtman and Zinn [JSZ85]. As a step towards Theorem 4, we shall prove the following theorem in Section 3 below, thus extending (9) to Rademacher chaos of arbitrary degree.
Theorem 6 (Xp inequality for Rademacher chaos). Suppose that \( p \in [2, \infty) \), \( n \in \mathbb{N} \) and \( k \in [n] \). Then every \( h \in L^0_p(\{-1,1\}^n) \) satisfies
\[
\frac{1}{n!} \sum_{S \subseteq [n]} \left\| E_{[n] \setminus S} h \right\|_p^p \lesssim_p \left( \frac{k}{n} \sum_{j=1}^n \| \partial_j h \|_p^p + \left( \frac{k}{n} \right)^\frac{p}{2} \| h \|_p^p \right)^\frac{1}{p}.
\]

The deduction of Theorem 4 from Theorem 6 appears in Section 2 below.

Remark 7. As in (4), the implicit constant that we obtain in (10) is \( O(p^4/\log p) \). In fact, our proof yields the following slightly more refined estimate in the setting of Theorem 6:
\[
\frac{1}{n!} \sum_{S \subseteq [n]} \left\| E_{[n] \setminus S} h \right\|_p^p \lesssim_p \left( \frac{k}{n} \sum_{j=1}^n \| \partial_j h \|_p^p \right)^\frac{1}{p} + \frac{p^2}{\log p} \sqrt{\frac{k}{n}} \cdot \| h \|_p.
\]

It remains open to determine the growth rate as \( p \to \infty \) of the implicit constant in (10).

1.3.1. Lust-Piquard’s work. Our proof of Theorem 6 uses deep work [LP98] of Lust-Piquard on dimension-free bounds for discrete Riesz transforms. Specifically, for every \( h : \{-1,1\}^n \to \mathbb{R} \) and \( j \in [n] \), the \( j \)th (hypercube) Riesz transform of \( h \), denoted \( R_j h : \{-1,1\}^n \to \mathbb{R} \), is defined as follows:
\[
\forall \varepsilon \in \{-1,1\}^n, \quad R_j h(\varepsilon) \overset{\text{def}}{=} \sum_{A \subseteq [n]} \frac{\hat{h}(A)}{\sqrt{|A|}} W_A(\varepsilon).
\]

Lust-Piquard proved the following inequalities, which hold true for \( p \in [2, \infty) \) and \( h \in L^0_p(\{-1,1\}^n) \):
\[
\frac{1}{p^{3/2}} \| h \|_p \lesssim_p \left( \sum_{j=1}^n (R_j h)^2 \right)^\frac{1}{2} \lesssim_p p \| h \|_p.
\]

The inequalities in (13) were proved by Lust-Piquard in [LP98], though with a dependence on \( p \) that is worse than what we stated above. The dependence on \( p \) that appears in (13) follows from [ELP08] (specifically, the left-hand inequality in (13) follows from [ELP08, Th. 1.1] and the right-hand inequality in (13) follows from [ELP08, Th. 5.1]. Actually, the right-hand inequality in (13) with the stated dependence on \( p \) follows implicitly from ingredients that are already present in [LP98]). Note that these estimates are stated in [ELP08] in terms of the strong \((p, p)\) norm of the Hilbert transform with values in the Schatten–von Neumann trace class \( S_p \), but this norm was shown to be \( O(p) \) by Bourgain in [Bou86], and the bounds that we stated in (13) result from a direct substitution of Bourgain’s bound into the statements in [ELP08].
The availability of dimension independent bounds for Riesz transforms is a well-known paradigm in other (nondiscrete) settings, originating from important classical work of Stein [Ste83] in the case of $\mathbb{R}^n$ equipped with Lebesgue measure (see also [GV79], [DRdF85], [Bañ86]). Most pertinent to the present context is the classical theorem of P. A. Meyer [Mey84] (see also [Gun86]) that obtained dimension independent bounds for the Riesz transforms that are associated to $\mathbb{R}^n$ equipped with the Gaussian measure (and the Ornstein–Uhlenbeck operator). Pisier discovered in [Pis88] an influential alternative proof of P. A. Meyer’s theorem, based on a transference argument (see [CW76]) that allows one to reduce the question to the boundedness of the (one dimensional) Hilbert transform.

Lust-Piquard’s work generally follows Pisier’s strategy, but it also uncovers a phenomenon that is genuinely present in the hypercube setting and not in the Gaussian setting. Specifically, Lust-Piquard reduces the task of bounding the hypercube Riesz transforms to that of bounding the $S_p$ norm of certain operators in a noncommutative $*$ algebra of $(2^n \times 2^n)$ matrices, and proceeds to do so using operator-theoretic methods, including her noncommutative Khinchine inequalities [LP86].

This indicates why the $S_p$-valued Hilbert transform makes its appearance in Lust-Piquard’s inequality (recall the paragraph above, immediately following (13)), despite the fact that (13) deals with real-valued functions on the (commutative) hypercube. Significantly, while the classical results on Riesz transforms (with respect to either Lebesgue measure or the Gaussian measure) yield dimension independent bounds for every $p \in (1, \infty)$, it turns out that (13) actually fails to hold true when $p \in (1, 2)$, as explained in [LP98] (where this observation is attributed to unpublished work of Lamberton); see also [ELP08, §5.5]. The reason for this disparity between the ranges $p \in (1, 2)$ and $p \in [2, \infty)$ becomes clear when one transfers the question to the noncommutative setting, and this suggests a more complicated (but still dimension-free) replacement for (13) in the range $p \in (1, 2)$, which Lust-Piquard also proved in [LP98]. So, while it is conceivable that a proof of (13) could be found that does not proceed along Lust-Piquard’s noncommutative route, such a proof has not been found to date, and the qualitative divergence between the discrete situation and its continuous counterparts indicates that there may be an inherently different phenomenon at play here. Since its initial publication, Lust-Piquard’s work influenced developments by herself and others that focused on proving related inequalities in other situations; we do not have anything new to add to this interesting body of work other than showing here that in addition to their intrinsic interest, such results can have a decisive role in understanding geometric embedding questions.
2. Deduction of Theorem 4 from Theorem 6

Assuming the validity of Theorem 6 for the moment, we shall now proceed to show how it implies Theorem 4. Note that since (4) involves only the $p$th powers of distances in $L_p$, by integration it suffices to prove Theorem 4 for real valued functions. So, from now on we shall assume that $m, n \in \mathbb{N}$ and we are given a function $f : \mathbb{Z}_m^n \to \mathbb{R}$, the goal being to prove the validity of (4) for every $k \in [n]$ provided that $m \geq \sqrt{n/k}$, with the $L_p$ norms replaced by absolute values in $\mathbb{R}$.

In what follows, given $S \subseteq [n]$ and $f : \mathbb{Z}_m^n \to \mathbb{R}$, define a function $T_S f : \mathbb{Z}_m^n \to \mathbb{R}$ by
\[
\forall x \in \mathbb{Z}_m^n, \quad T_S f(x) \overset{\text{def}}{=} \frac{1}{2^n} \sum_{\delta \in \{-1, 1\}^n} f(x + 2\delta S).
\]

For future use we record the following simple lemma.

**Lemma 8.** For every $p \in [1, \infty)$, $m, n \in \mathbb{N}$, $S \subseteq [n]$ and $f : \mathbb{Z}_m^n \to \mathbb{R}$, we have
\[
\left( \frac{1}{(8m)^n} \sum_{x \in \mathbb{Z}_m^n} |f(x) - T_S f(x)|^p \right)^{\frac{1}{p}} \leq 2 \left( \frac{1}{(16m)^n} \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_m^n} |f(x + \varepsilon) - f(x)|^p \right)^{\frac{1}{p}}.
\]

**Proof.** By convexity, for every $x \in \mathbb{Z}_m^n$, we have
\[
|f(x) - T_S f(x)|^p \leq \frac{1}{2^n} \sum_{\delta \in \{-1, 1\}^n} |f(x) - f(x + 2\delta S)|^p
\]
\[
\leq \frac{2^{p-1}}{2^n} \sum_{\delta \in \{-1, 1\}^n} \left( |f(x) - f(x + \delta S + \delta_{[n]}S)|^p + |f(x + \delta S + \delta_{[n]}S) - f(x + 2\delta S)|^p \right).
\]

The desired estimate (15) follows by averaging (16) over $x \in \mathbb{Z}_m^n$ while using the translation invariance of the uniform measure on $\mathbb{Z}_m^n$ and the fact that if $\delta$ is uniformly distributed over $\{-1, 1\}^n$, then the sign vectors $\delta_S + \delta_{[n]\setminus S}$ and $-\delta_S + \delta_{[n]\setminus S}$ are both also uniformly distributed over $\{-1, 1\}^n$. ∎

**Lemma 9.** Suppose that $m, n \in \mathbb{N}$ and $k \in [n]$. If $p \in [2, \infty)$, then every $f : \mathbb{Z}_m^n \to \mathbb{R}$ satisfies
It therefore follows from (19) that
\[ S \subseteq x, \varepsilon \]

Recalling (8) and (14), observe that for every (21)
\[ \forall y \]

where for (19), make the change of variable (18)
\[ \frac{1}{m} \]

Proof. For every fixed \( S \subseteq [n], \) we have
\[ \left( \frac{1}{(16m)^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_m^n} \left| T_{[n]} \cdot s f(x + 4m\varepsilon S) - T_{[n]} \cdot s f(x) \right|^p \right)^{\frac{1}{p}} \]

(17)
\[ \leq \frac{k/n}{(8m)^n} \sum_{j=1}^n \sum_{x \in \mathbb{Z}_m^n} |f(x + e_j) - f(x)|^p \]

\[ + \frac{(k/n)^{\frac{3}{2}}}{(16m)^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_m^n} |f(x + \varepsilon) - f(x)|^p. \]

For every fixed \( S \subseteq [n], \) we have
\[ \left( \frac{1}{(16m)^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_m^n} \left| T_{[n]} \cdot s f(x + 4m\varepsilon S) - T_{[n]} \cdot s f(x + 4(k - 1)\varepsilon S) \right|^p \right)^{\frac{1}{p}} \]

(18)
\[ \leq \sum_{k=1}^m \left( \frac{1}{(16m)^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_m^n} \left| T_{[n]} \cdot s f(x + 4k\varepsilon S) - T_{[n]} \cdot s f(x + 4(k - 1)\varepsilon S) \right|^p \right)^{\frac{1}{p}} \]

(19)
\[ = m \left( \frac{1}{(16m)^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{y \in \mathbb{Z}_m^n} \left| T_{[n]} \cdot s f(y + 2\varepsilon S) - T_{[n]} \cdot s f(y - 2\varepsilon S) \right|^p \right)^{\frac{1}{p}}, \]

where for (19), make the change of variable \( y = x + 2(2k + 1)\varepsilon S \) in each of the summands of (18).

For every \( x \in \mathbb{Z}_m^n, \) define \( h_x : \{-1,1\}^n \to \mathbb{R} \) by (20)
\[ \forall \varepsilon \in \{-1,1\}^n, \quad h_x(\varepsilon) \overset{\text{def}}{=} f(x + 2\varepsilon) - f(x - 2\varepsilon). \]

Recalling (8) and (14), observe that for every \( (x, \varepsilon) \in \mathbb{Z}_m^n \times \{-1,1\}^n \) and \( S \subseteq [n], \) we have
\[ T_{[n]} \cdot s f(x + 2\varepsilon S) - T_{[n]} \cdot s f(x - 2\varepsilon S) = E_{[n]} \cdot s h_x(\varepsilon). \]

It therefore follows from (19) that
\[ \left( \frac{1}{(16m)^n(n)} \sum_{S \subseteq [n]} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_m^n} \left| T_{[n]} \cdot s f(x + 4m\varepsilon S) - T_{[n]} \cdot s f(x) \right|^p \right)^{\frac{1}{p}} \]

(21)
\[ \leq \left( \frac{1}{(8m)^n(n)} \sum_{S \subseteq [n]} \sum_{x \in \mathbb{Z}_m^n} \left\| E_{[n]} \cdot s h_x \right\|_p^p \]

\[ \leq \sum_{j=1}^n \frac{k/n}{(8m)^n} \sum_{x \in \mathbb{Z}_m^n} \| \partial_j h_x \|_p + \frac{(k/n)^{\frac{3}{2}}}{(8m)^n} \sum_{x \in \mathbb{Z}_m^n} \| h_x \|_p^p, \]
where in the last step of (21) we applied Theorem 6 with \( h \) replaced by \( h_x \), separately for each \( x \in \mathbb{Z}_{8m}^n \), which we are allowed to do because the function \( h_x \) is odd, so \( h_x \in L_p^0(\{-1,1\}^n) \).

Next, observe that for every \( (x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1,1\}^n \) and \( j \in [n] \), we have

\[
|\partial_j h_x(\varepsilon)|^p \overset{(20)}{=} |f(x + 2\varepsilon) - f(x - 2\varepsilon) - f(x + 2\varepsilon - 4\varepsilon_j e_j) + f(x - 2\varepsilon + 4\varepsilon_j e_j)|^p \\
\leq 2^{p-1}|f(x + 2\varepsilon) - f(x + 2\varepsilon - 4\varepsilon_j e_j)|^p + 2^{p-1}|f(x - 2\varepsilon) - f(x - 2\varepsilon + 4\varepsilon_j e_j)|^p.
\]

By summing (22) over \( (x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1,1\}^n \), we therefore see that

\[
\forall j \in [n], \quad \frac{1}{(8m)^n} \sum_{x \in \mathbb{Z}_{8m}^n} \|\partial_j h_x\|_p^p \leq \frac{2^p}{(8m)^n} \sum_{y \in \mathbb{Z}_{8m}^n} |f(y + 4e_j) - f(y)|^p.
\]

Since for every \( y \in \mathbb{Z}_{8m}^n \) we have

\[
|f(y + 4e_j) - f(y)|^p \leq 4^{p-1} \sum_{k=1}^4 |f(y + ke_j) - f(y + (k-1)e_j)|^p,
\]

it follows from (23) that

\[
\frac{1}{(8m)^n} \sum_{j=1}^n \sum_{x \in \mathbb{Z}_{8m}^n} \|\partial_j h_x\|_p^p \leq \frac{8^p}{(8m)^n} \sum_{j=1}^n \sum_{z \in \mathbb{Z}_{8m}^n} |f(z + e_j) - f(z)|^p.
\]

In the same vein to the above reasoning, for every \( (x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1,1\}^n \), we have

\[
|h_x(\varepsilon)|^p \overset{(20)}{=} 4^{p-1} \sum_{k=-1}^2 |f(x + k\varepsilon) - f(x + (k-1)\varepsilon)|^p.
\]

Consequently,

\[
\frac{1}{(8m)^n} \sum_{x \in \mathbb{Z}_{8m}^n} \|h_x\|_p^p \leq \frac{4^p}{(16m)^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{z \in \mathbb{Z}_{8m}^n} |f(z + \varepsilon) - f(z)|^p.
\]

The desired estimate (17) now follows from a substitution of (24) and (25) into (21).

\[\square\]

\textit{Proof of Theorem 4.} Fixing \( (x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1,1\}^n \) and \( S \subseteq [n] \), observe that

\[
|f(x + 4m\varepsilon_S) - f(x)|^p \leq 3^{p-1} \left( |T_{[n] \setminus S} f(x + 4m\varepsilon_S) - T_{[n] \setminus S} f(x)|^p + |f(x) - T_{[n] \setminus S} f(x)|^p + |f(x + 4m\varepsilon_S) - T_{[n] \setminus S} f(x + 4m\varepsilon_S)|^p \right).
\]
By averaging (26) over \((x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \{-1, 1\}^n\) and all those \(S \subseteq [n]\) with \(|S| = k\), while using translation invariance in the variable \(x\), we see that

\[
\frac{1}{(16m)^n \binom{n}{k}} \sum_{S \subseteq [n]} \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{8m}^n} \frac{|f(x + 4m\varepsilon S) - f(x)|^p}{m^p}
\]

(27) \[
\lesssim_p \frac{1}{(16m)^n \binom{n}{k}} \sum_{S \subseteq [n]} \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{8m}^n} \frac{|T_{[n]} \cdot Sf(x + 4m\varepsilon) - T_{[n]} \cdot Sf(x)|^p}{m^p}
\]

(28) \[
+ \frac{1}{m^p (8m)^n \binom{n}{k}} \sum_{S \subseteq [n]} \sum_{x \in \mathbb{Z}_{8m}^n} |f(x) - T_{[n]} \cdot Sf(x)|^p.
\]

The quantity that appears in (27) can be bounded from above using Lemma 9, and the quantity that appears in (28) can be bounded from above using Lemma 8. The resulting estimate is

\[
\frac{1}{(16m)^n \binom{n}{k}} \sum_{S \subseteq [n]} \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{8m}^n} \frac{|f(x + 4m\varepsilon S) - f(x)|^p}{m^p}
\]

\[
\lesssim_p \frac{k/n}{(8m)^n} \sum_{j=1}^{n} \sum_{x \in \mathbb{Z}_{8m}^n} |f(x + e_j) - f(x)|^p
\]

\[
+ \frac{k/n^2 + 1/m^p}{(16m)^n} \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{8m}^n} |f(x + \varepsilon) - f(x)|^p.
\]

This implies the desired estimate (4), since we are assuming that \(m \geq \sqrt{n/k}\).

\[\square\]

3. Proof of Theorem 6

Suppose that \(n \in \mathbb{N}\) and \(h : \{-1, 1\}^n \to \mathbb{R}\). For every \(k \in \{0, \ldots, n\}\), the \(k\)th Rademacher projection of \(h\) is the function \(\text{Rad}_k h : \{-1, 1\}^n \to \mathbb{R}\) that is given by

\[\text{Rad}_k h(\varepsilon) \overset{\text{def}}{=} \sum_{A \subseteq [n] \atop |A| = k} \hat{h}(A) W_A(\varepsilon).\]

We also have the common notation \(\text{Rad}_1 h = \text{Rad} h\). Note that \(\text{Rad}_0\) is the mean of \(h\); i.e., recalling the notation (8), \(\text{Rad}_0 h = \mathbb{E}_{[n]} h\). By a classical theorem of Bonami [Bon68], if \(\eta : \{-1, 1\}^n \to \mathbb{R}\) is a Rademacher chaos of order at most \(k\), i.e., \(\hat{\eta}(A) = 0\) whenever \(A \subseteq [n]\) is such that \(|A| > k\), then for every \(p \in [2, \infty)\), we have \(\|\eta\|_p \leq (p - 1)^{k/2} \|\eta\|_2 \leq p^{k/2} \|\eta\|_2\). Consequently,

\[\|\text{Rad}_k h\|_p \leq p^{\frac{k}{2}} \|\text{Rad}_k h\|_2 \leq p^{\frac{k}{2}} \|h\|_2 \leq p^{\frac{k}{2}} \|h\|_p,\]
where we used the fact that (by Parseval’s identity) \(\|\text{Rad}_k h\|_2 \leq \|h\|_2\), and that \(\|h\|_2 \leq \|h\|_p\) since \(p \geq 2\). This was a quick (and standard) derivation of the following well-known operator norm bound for \(\text{Rad}_k\), which we state here for ease of future reference:

\[
\|\text{Rad}_k\|_{p \to p} \leq p^{\frac{1}{2}}.
\]

Given \(S \subseteq [n]\) and \(\alpha \in \mathbb{R}\), for every \(h : \{-1, 1\}^n \to \mathbb{R}\), define a function \(\Delta^\frac{1}{2} h : \{-1, 1\}^n \to \mathbb{R}\) by

\[
\forall \varepsilon \in \{-1, 1\}^n, \quad \Delta^\frac{1}{2} h(\varepsilon) \equiv \sum_{\substack{A \subseteq [n] \atop \alpha \in S}} |A \cap S| \delta h(A) W_A(\varepsilon).
\]

Thus, recalling the notation (7) for the hypercube partial derivatives \(\partial_1, \ldots, \partial_n\), as well the notation (12) for the hypercube Riesz transforms \(R_1, \ldots, R_n\), we have the following standard identities:

\[
\Delta_S \equiv \Delta^\frac{1}{2} = \frac{1}{2} \sum_{j \in S} \partial_j = \frac{1}{4} \sum_{j \in S} \partial_j^2 \partial_j = \sum_{j \in S} \partial_j^2
\]

(using that \(\{\partial_j\}_{j=1}^n\) are self-adjoint operators with \(\partial_j^2 = 2 \partial_j\) for all \(j \in [n]\)) and

\[
\forall j \in [n], \quad R_j = \frac{1}{2} \partial_j \Delta^{-\frac{1}{2}}.
\]

This means that Lust-Piquard’s inequality (13) can we rewritten as follows:

\[
\frac{1}{p^{3/2}} \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p \leq \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p.
\]

By Khinchine’s inequality (with asymptotically sharp constant; see [PZ30, Lemma 2]), we have

\[
\left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p \leq \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p.
\]

In combination with (30), this implies that

\[
\frac{1}{p^{3/2}} \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p \leq \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p.
\]

For ease of future reference, we also record here the following formal consequence of (31):

\[
\forall S \subseteq [n], \quad \frac{1}{p^{3/2}} \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p \leq \left( \sum_{j=1}^n (\partial_j h)^2 \right)^\frac{1}{2} \|p \| \Delta^\frac{1}{2} h \|_p.
\]

Indeed, by symmetry it suffices to establish the validity of (32) when \(S = \{1, \ldots, k\}\) for some \(k \in [n]\), in which case (32) follows by first fixing the values
of \( \varepsilon_{k+1}, \ldots, \varepsilon_n \in \{-1, 1\} \), applying (31) (with \( n \) replaced by \( k \)) to the mapping \( (\varepsilon_1, \ldots, \varepsilon_k) \mapsto h(\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n) \), raising the resulting estimates to the power \( p \), and then averaging over the remaining variables \( (\varepsilon_{k+1}, \ldots, \varepsilon_n) \in \{-1, 1\}^{n-k} \).

Lemma 10 below contains bounds on negative powers of the hypercube Laplacian \( \Delta_{[n]} \) that will be used later, but are more general and precise than what we actually need for the proof of Theorem 6; we will only use the following operator norm estimate corresponding to the case \( \alpha = 1/2 \) of Lemma 10, and a worse dependence on \( p \) would have sufficed for our purposes as well:

\[
\forall p \in [2, \infty), \quad \sup_{n \in \mathbb{N}} \left\| \Delta_{[n]}^{-\frac{1}{2}} \right\|_{p \to p} \lesssim \sqrt{\log p}.
\]

We include here the sharp estimates of Lemma 10 because they are interesting in their own right and our proof yields them without additional effort. The boundedness of negative powers of the hypercube Laplacian were studied in [NS02, §3] in the context of vector valued mappings. By specializing the bounds that are stated in [NS02] to the case of real valued mappings, one obtains a variant of (33), but with a much worse dependence on \( p \). (The resulting bound grows exponentially with \( p \).) The (simple) proof below of Lemma 10 follows the strategy of [NS02] while using additional favorable properties of real valued mappings and taking care to obtain asymptotically sharp bounds.

**Lemma 10.** Suppose that \( p \in [2, \infty) \) and \( \alpha \in (0, \infty) \) satisfy

\[
\alpha \leq \frac{5 + \log p}{4}.
\]

Then

\[
\sup_{n \in \mathbb{N}} \left\| \Delta_{[n]}^{-\alpha} \right\|_{p \to p} \asymp \frac{(\log p)^\alpha}{2^{\alpha} \Gamma(1 + \alpha)}.
\]

**Remark 11.** Some restriction on \( \alpha \) in the spirit of (34) is needed for (35) to hold true, since \( \lim_{\alpha \to \infty} \Delta_{[n]}^{-\alpha} = \text{Rad} \) and it is known that \( \|\text{Rad}\|_{p \to p} \gtrsim \sqrt{p} \) for \( n \) large enough (as a function of \( p \)).

**Proof of Lemma 10.** The lower estimate

\[
\sup_{n \in \mathbb{N}} \left\| \Delta_{[n]}^{-\alpha} \right\|_{p \to p} \gtrsim \frac{(\log p)^\alpha}{2^{\alpha} \Gamma(1 + \alpha)}
\]

holds true for every \( \alpha \in (0, \infty) \), without the restriction (34). Indeed, denote \( p^* \overset{\text{def}}{=} p/(p-1) \) and observe that since \( \Delta_{[n]}^{-\alpha} \) is self-adjoint, it follows by duality that (36) is equivalent to the estimate

\[
\sup_{n \in \mathbb{N}} \left\| \Delta_{[n]}^{-\alpha} \right\|_{p^* \to p^*} \gtrsim \frac{(\log p)^\alpha}{2^{\alpha} \Gamma(1 + \alpha)}.
\]
Fix an integer \( n \geq 2 \), and consider the following \( f_p^n \in L_{p^*}(\{-1,1\}^n) \), for which \( \|f_p^n\|_{p^*} = 1 \):

\[
\forall \varepsilon \in \{-1,1\}^n, \quad f_p^n(\varepsilon) \overset{\text{def}}{=} 2^{\frac{n}{p^*}} \delta(1,\ldots,1)(\varepsilon) = 2^{-\frac{n}{p}} \prod_{j=1}^n (1 + \varepsilon_j) = 2^{-\frac{n}{p}} \sum_{A \subseteq [n]} W_A.
\]

For every \( u \in (0,\infty) \) and \( \alpha \in (0,\infty) \), the following identity holds true:

\[
\frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-su} ds.
\]

Consequently,

\[
\Delta_{[n]}^- = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-s \Delta_{[n]}(I - \text{Rad}_0)} ds.
\]

Note that for every \( s \in (0,\infty) \) and \( \varepsilon \in \{-1,1\}^n \), we have

\[
e^{-s \Delta_{[n]}(I - \text{Rad}_0)} f_p^n(\varepsilon) = 2^{-\frac{n}{p}} \sum_{A \subseteq [n]} e^{-s|A|} W_A
\]

\[
= 2^{-\frac{n}{p}} \left( \prod_{j=1}^n (1 + e^{-s} \varepsilon_j) - 1 \right) = -2^{-\frac{n}{p}} \left( 1 - (1 + e^{-s})^{\kappa(\varepsilon)}(1 - e^{-s})^{n - \kappa(\varepsilon)} \right),
\]

where we use the notation

\[
\forall \varepsilon \in \{-1,1\}^n, \quad \kappa(\varepsilon) \overset{\text{def}}{=} \left| \{ j \in [n] : \varepsilon_j = 1 \} \right|.
\]

Since the function \( k \mapsto (1 + e^{-s})^k (1 - e^{-s})^{n-k} \) is increasing on \( \{0,\ldots,n\} \), it follows from (39) that

\[
\forall \varepsilon \in \{-1,1\}^n, \quad \kappa(\varepsilon) \leq \frac{n}{2} \implies 2^{-\frac{n}{p}} e^{-s \Delta_{[n]}(I - \text{Rad}_0)} f_p^n(\varepsilon)
\]

\[
\leq - \left( 1 - (1 - e^{-2s})^{\frac{n}{2}} \right).
\]

Recalling (38), it therefore follows that if \( \varepsilon \in \{-1,1\}^n \) satisfies \( \kappa(\varepsilon) \leq n/2 \), then

\[
2^{\frac{n}{p}} \left| \Delta_{[n]}^- f_p^n(\varepsilon) \right| \geq \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} \left( 1 - (1 - e^{-2s})^{\frac{n}{2}} \right) ds
\]

\[
\geq \frac{1}{\Gamma(\alpha)} \int_0^{\log n/2} s^{\alpha-1} \frac{ds}{4} = \frac{(\log n)^\alpha}{2^{2+\alpha} \Gamma(1 + \alpha)}.
\]

Hence, since \( \left| \{ \varepsilon \in \{-1,1\}^n : \kappa(\varepsilon) \leq \frac{n}{2} \} \right| \geq 2^{n-1} \), we have

\[
\|\Delta_{[n]}^- f_p^n\|_{p^*} \overset{\sim}{\geq} \frac{2^{-\frac{n}{p}} (\log n)^\alpha}{2^{2+\alpha} \Gamma(1 + \alpha)}.
\]

The desired estimate (37) now follows by choosing \( n = \lceil p \rceil \) in (40).
Having proven (36), it remains to show that under the assumption (34) we have

\[ \sup_{n \in \mathbb{N}} \left\| \Delta^{-\alpha}_{[n]} \right\|_{p \to p} \lesssim \frac{(\log p)^\alpha}{2^\alpha \Gamma(1 + \alpha)}. \]

To this end, observe first that the identity

\[ e^{-s\Delta_{[n]}}(I - \text{Rad}_0) = \sum_{k=1}^{n} \frac{1}{e^{sk}\text{Rad}_k} \]

implies that

\[ \left\| e^{-s\Delta_{[n]}}(I - \text{Rad}_0) \right\|_{p \to p} \lesssim \sum_{k=1}^{n} \frac{1}{e^{sk}} \left\| \text{Rad}_k \right\|_{p \to p} \lesssim \sum_{k=1}^{n} \left( \frac{\sqrt{p}}{e^s} \right)^k. \]

Hence,

\[ e^{s} > \sqrt{p} \implies \left\| e^{-s\Delta_{[n]}}(I - \text{Rad}_0) \right\|_{p \to p} \lesssim \frac{\sqrt{p}}{e^s - \sqrt{p}}. \]

Suppose that \( M \in (0, \infty) \) satisfies

\[ e^{M} > \sqrt{p} \iff M \geq \frac{1 + \log p}{2}. \]

Then, by (42) we have

\[ \frac{1}{\Gamma(\alpha)} \int_{M}^{\infty} s^{\alpha - 1} \left\| e^{-s\Delta_{[n]}}(I - \text{Rad}_0) \right\|_{p \to p} ds \lesssim \frac{\sqrt{p}}{\Gamma(\alpha)} \int_{M}^{\infty} \frac{s^{\alpha - 1}}{e^s} ds. \]

Due to (34) and (43), we have \( M \geq 2(\alpha - 1) \). Since the function \( s \mapsto s^{\alpha - 1}e^{-s/2} \) is decreasing on \( [2(\alpha - 1), \infty) \supseteq [M, \infty) \), it follows that for every \( s \geq M \), we have \( s^{\alpha - 1}e^{-s} \leq M^{\alpha - 1}e^{-M/2}e^{-s/2} \). So,

\[ \int_{M}^{\infty} \frac{s^{\alpha - 1}}{e^s} ds \leq \frac{M^{\alpha - 1}}{e^{M/2}} \int_{M}^{\infty} \frac{ds}{e^{s/2}} \lesssim \frac{M^{\alpha - 1}}{e^M}. \]

A substitution of (45) into (44) yields the estimate

\[ \frac{1}{\Gamma(\alpha)} \int_{M}^{\infty} s^{\alpha - 1} \left\| e^{-s\Delta_{[n]}}(I - \text{Rad}_0) \right\|_{p \to p} ds \lesssim \frac{M^{\alpha - 1} \sqrt{p}}{e^M \Gamma(\alpha)}. \]

At the same time, since for every \( s \in [0, \infty) \) we have \( \left\| e^{-s\Delta_{[n]}} \right\|_{p \to p} \leq 1 \) (because \( e^{-s\Delta_{[n]}} \) is an averaging operator) and \( \left\| I - \text{Rad}_0 \right\|_{p \to p} \leq 2 \), we have

\[ \frac{1}{\Gamma(\alpha)} \int_{0}^{M} s^{\alpha - 1} \left\| e^{-s\Delta_{[n]}}(I - \text{Rad}_0) \right\|_{p \to p} ds \lesssim \frac{1}{\Gamma(\alpha)} \int_{0}^{M} s^{\alpha - 1} ds = \frac{M^{\alpha}}{\Gamma(1 + \alpha)}. \]

Making the choice

\[ M \overset{\text{def}}{=} \frac{1 + \log p}{2}, \]
we see that
\[
\|\Delta_{[n]}^{-\alpha}h\|_p \leq \left( \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha-1} \|e^{-s\Delta_{[n]}(I - \text{Rad}_0)}\|_p ds \right)^{\alpha} \approx \frac{M^\alpha}{\Gamma(1 + \alpha)} + \frac{M^\alpha \sqrt{p}}{e^{M\Gamma(\alpha)}}
\]
\[
\approx \frac{(1 + \log p)^\alpha}{2^\alpha \Gamma(1 + \alpha)} + \frac{(1 + \log p)\alpha^{-1}}{2^{\alpha-1} \Gamma(\alpha) \sqrt{e}}
\]
\[
\leq \frac{(\log p)^\alpha}{2^\alpha \Gamma(1 + \alpha)} \left( \frac{1}{\alpha} + \frac{1}{\log p} \right) \left( 1 + \frac{1}{\log p} \right)^\alpha
\]
\[
\leq \frac{(\log p)^\alpha}{2^\alpha \Gamma(1 + \alpha)}.
\]

This is precisely the desired estimate (41), completing the proof of Lemma 10.

\[\square\]

**Lemma 12.** Fix \( n \in \mathbb{N} \), \( p \in [1, \infty) \) and \( \alpha \in \mathbb{R} \). Then for \( h \in L^0_p([-1,1]^n) \) and \( S \subseteq [n] \), we have
\[
\|E_{[n]} \setminus S h\|_p \leq \|\Delta_{[n]} \Delta_{[n]}^{-\alpha}h\|_p.
\]

**Proof.** We have the following identity of operators on \( L^0_p([-1,1]^n) \):
\[
E_{[n]} \setminus S \Delta_{[n]} \Delta_{[n]}^{-\alpha} = E_{[n]} \setminus S.
\]

Indeed, if \( \emptyset \neq A \subseteq [n] \), then \( E_{[n]} \setminus S W_A = 1_{\{A \subseteq S\}} W_A \), and at the same time we have
\[
E_{[n]} \setminus S \Delta_{[n]} \Delta_{[n]}^{-\alpha} W_A = \frac{|A \cap S|^\alpha}{|A|^\alpha} E_{[n]} \setminus S W_A = \frac{|A \cap S|^\alpha}{|A|^\alpha} 1_{\{A \subseteq S\}} W_A = 1_{\{A \subseteq S\}} W_A.
\]

Consequently,
\[
\|\Delta_{[n]} \Delta_{[n]}^{-\alpha}h\|_p = E_S \|E_{[n]} \setminus S |\Delta_{[n]} \Delta_{[n]}^{-\alpha}h|_p \geq E_S \|E_{[n]} \setminus S \Delta_{[n]} \Delta_{[n]}^{-\alpha}h\|_p
\]
\[
= \left( \frac{1}{(\log p)^\alpha} \sum_{S \subseteq [n]} \|E_{[n]} \setminus S h\|_p \right)^\frac{1}{p} \leq \left( \frac{1}{(\log p)^\alpha} \sum_{S \subseteq [n]} \|\Delta_{[n]} \Delta_{[n]}^{-\alpha}h\|_p \right)^\frac{1}{p}.
\]

**Proof of Theorem 6.** By Lemma 12 we have
\[
\left( \frac{1}{(\log p)^\alpha} \sum_{S \subseteq [n]} \|E_{[n]} \setminus S h\|_p \right)^\frac{1}{p} \leq \left( \frac{1}{(\log p)^\alpha} \sum_{S \subseteq [n]} \|\Delta_{[n]} \Delta_{[n]}^{-\alpha}h\|_p \right)^\frac{1}{p}.
\]
By Lust-Piquard’s discrete Riesz transform inequality (32), for every fixed \( S \subseteq [n] \), we have

\[
(52) \quad \left\| \Delta_S^\frac{1}{2} \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p \lesssim p^{\frac{3}{2}} \left( \frac{1}{2^n} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j \in S} \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p^p \right)^{\frac{1}{p}}.
\]

A substitution of (52) into (51) yields

\[
(53) \quad \left( \frac{1}{(n)} \sum_{\frac{|S|}{|S|} \subseteq [n]} \left\| E_{[n]-S} h \right\|_p^p \right)^{\frac{1}{p}} \lesssim p^{\frac{3}{2}} \left( \frac{1}{2^n} \sum_{\frac{|S|}{|S|} \subseteq [n]} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j \in S} \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p^p \right)^{\frac{1}{p}}.
\]

For fixed \( \varepsilon \in \{-1,1\}^n \), the linear \( X_p \) inequality (9) with \( \{a_j = \partial_j \Delta_{[n]}^{-\frac{1}{2}} h(\varepsilon)\}_{j=1}^n \) yields the estimate

\[
(54) \quad \left( \frac{1}{2^n} \sum_{\frac{|S|}{|S|} \subseteq [n]} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j \in S} \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h(\varepsilon) \right\|_p^p \right)^{\frac{1}{p}} \lesssim p^{\frac{3}{2}} \left( \frac{k}{n} \sum_{j=1}^n |\partial_j \Delta_{[n]}^{-\frac{1}{2}} h(\varepsilon)|^p + \frac{(k/n)^{\frac{p}{2}}}{2^n} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j \in S} \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h(\varepsilon) \right\|_p^p \right)^{\frac{1}{p}}.
\]

By taking \( L_p \) norms with respect to \( \varepsilon \in \{-1,1\}^n \), it follows from (54) that

\[
(55) \quad \left( \frac{1}{2^n} \sum_{\frac{|S|}{|S|} \subseteq [n]} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j \in S} \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p^p \right)^{\frac{1}{p}} \lesssim p^{\frac{3}{2}} \left( \frac{k}{n} \sum_{j=1}^n |\partial_j \Delta_{[n]}^{-\frac{1}{2}} h|_p^p + \frac{(k/n)^{\frac{p}{2}}}{2^n} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j \in S} \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p^p \right)^{\frac{1}{p}}.
\]

By Lemma 10 we have

\[
(56) \quad \left( \sum_{j=1}^n \left\| \partial_j \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p^p \right)^{\frac{1}{p}} \lesssim \sqrt{\log p} \left( \sum_{j=1}^n \left\| \partial_j h \right\|_p^p \right)^{\frac{1}{p}},
\]

and another application of Lust-Piquard’s discrete Riesz transform inequality (31) shows that

\[
(57) \quad \left( \frac{1}{2^n} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j=1}^n \delta_j \partial_j \Delta_{[n]}^{-\frac{1}{2}} h \right\|_p^p \right)^{\frac{1}{p}} \lesssim p^{\frac{3}{2}} \|h\|_p.
\]

The desired estimate (10) (in its slightly more refined form (11)) now follows by substituting (56) and (57) into (55) and then substituting the resulting inequality into (53). \( \square \)
4. Beyond $L_p$

Fix $p \in [2, \infty)$. Following [NS16], a Banach space $(X, \| \cdot \|_X)$ is said to be an $X_p$ Banach space if for every $k, n \in \mathbb{N}$ with $k \in [n]$, for every $v_1, \ldots, v_n \in X$ we have

$$\frac{1}{2^n k!} \sum_{S \subseteq [n]} \sum_{|S| = k} \left\| \sum_{j \in S} \varepsilon_j v_j \right\|_X^p \lesssim \frac{1}{n} \sum_{j=1}^n \|v_j\|_X^p + \frac{(k/n)^{2}\varepsilon}{2^2 n} \sum_{\varepsilon \in \{-1, 1\}^n} \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_X^p.$$

Inequality (9) implies that $L_p$ is an $X_p$ Banach space when $p \in [2, \infty)$, and in [NS16] it was proven that for $p$ in this range, also the Schatten–von Neumann trace class $S_p$ is an $X_p$ Banach space. Thus, due to [McC67], there exists an $X_p$ Banach space that is not isomorphic to a subspace of $L_p$.

By [NS16] we know that any $X_p$ Banach space is also an $X_p$ metric space. (See [Nao12], [Bal13] for the significance of such results in the context of the Ribe program.) Our proof of Theorem 4 does not imply this general statement, since it relies on additional properties of the target Banach space $X$, which in our case is $L_p$. An inspection of our proof reveals that it uses only two nontrivial properties of the target space Banach $X$. Firstly, we need the following operator norm bounds:

$$(58) \sup_{n \in \mathbb{N}} \left\| \Delta^{-\frac{1}{2}}_{[n]} \otimes I_X \right\|_{L_p((-1,1)^n, X) \to L_p((-1,1)^n, X)} < \infty.$$ 

By [NS02, Th. 5], the requirement (58) is equivalent to $X$ being a $K$-convex Banach space. (For background on $K$-convexity, see the survey [Mau03].) Secondly, we need $X$ to satisfy the following vector valued version of Lust-Piquard’s inequality (31) for every $n \in \mathbb{N}$ and $h : \{-1, 1\}^n \to X$:

$$(59) \frac{1}{2^n} \sum_{\delta \in \{-1, 1\}^n} \left\| \sum_{j=1}^n \delta_j (\partial_j \otimes I_X) h \right\|_{L_p((-1,1)^n, X)}^p \lesssim_X \left( \left\| \Delta^{-\frac{1}{2}}_{[n]} \otimes I_X \right\|_{L_p((-1,1)^n, X)} \right)^p.$$

So, the argument of the present article actually shows that any $K$-convex $X_p$ Banach space $X$ that satisfies (59) is also an $X_p$ metric space, with the same scaling parameter as in the statement of Theorem 4. However, as we shall explain in Corollary 15 below, the validity of (59) already implies that $X$ is $K$-convex. This means that (58) is a consequence of (59) and there is no need to stipulate the validity of (58) as a separate assumption. We therefore have the following theorem.

**Theorem 13.** Suppose that $p \in [2, \infty)$ and that $(X, \| \cdot \|_X)$ is an $X_p$ Banach space that satisfies (59). Suppose also that $k, m, n \in \mathbb{N}$ satisfy $k \in [n]$. 

...
and \( m \geq \sqrt{n/k} \). Then every \( f : \mathbb{Z}_m^n \to X \) satisfies

\[
(60) \quad \left( \frac{1}{k} \sum_{S \subseteq [n] \atop |S| = k} E \left[ \| f(x + 4m\varepsilon_S) - f(x) \|_X^p \right] \right)^{\frac{1}{p}} \leq_X \left( \frac{k}{n} \sum_{j=1}^{n} E \left[ \| f(x + e_j) - f(x) \|_X^p \right] + \left( \frac{k}{n} \right)^{\frac{p}{2}} E \left[ \| f(x + \varepsilon) - f(x) \|_X^p \right] \right)^{\frac{1}{p}},
\]

where the expectations are taken with respect to \((x, \varepsilon) \in \mathbb{Z}_m^n \times \{-1, 1\}^n\) chosen uniformly at random.

It seems to be quite challenging to obtain a clean and useful characterization of the class of Banach spaces that satisfy the dimension-independent vector valued discrete Riesz transform inequality (59). We did verify, in collaboration with A. Eskenazis, that the Schatten–von Neumann trace class \( S_p \) satisfies (59) when \( p \in [2, \infty) \), but in order to see this one needs to reexamine Lust-Piquard’s proof in [LP98] while checking in several instances that her argument could be adjusted so as to apply to \( S_p \)-valued functions as well. Since including such an argument here would be quite lengthy (and mostly a repetition of Lust-Piquard’s work), we postpone the justification of (59) when \( X = S_p \) to forthcoming work that is devoted to vector valued Riesz transforms. Due to the fact that \( S_p \) was shown to be an \( X_p \) Banach space in [NS16], Theorem 13 holds true when \( X = S_p \), with our current proof showing that the implicit constant in (60) (with \( X = S_p \)) is \( O(p^4/\sqrt{\log p}) \).

It remains to prove that if a Banach space \( X \) satisfies (59), then \( X \) is \( K \)-convex. In fact, the following stronger statement holds true. (See [Mau03] for background on type of Banach spaces.)

**Proposition 14.** Suppose that \( p \in [1, \infty) \) and \( \alpha \in (0, 1) \), and suppose that \( (X, \| \cdot \|_X) \) is a Banach space such that for every \( n \in \mathbb{N} \) and every \( h : \{-1, 1\}^n \to X \), we have

\[
(61) \quad \frac{1}{2^n} \sum_{\delta \in (-1, 1)^n} \left\| \sum_{j=1}^{n} \delta_j (\partial_j \otimes I_X) h \right\|_{L_p((-1,1)^n, X)}^p \lesssim_X \left\| (\Delta_{[n]}^\alpha \otimes I_X) h \right\|_{L_p((-1,1)^n, X)}^p.
\]

Then \( X \) has type \( \frac{1}{\alpha} - \tau \) for every \( \tau \in (0, 1) \). In particular, if (59) holds true, then \( X \) has type \( 2 - \tau \) for every \( \tau \in (0, 1) \).

By Pisier’s \( K \)-convexity theorem [Pis82], a Banach space \( X \) has type strictly larger than 1 if and only if \( X \) is \( K \)-convex. We therefore have the following corollary of Proposition 14.
Corollary 15. If $p \in [1, \infty)$ and $(X, \| \cdot \|_X)$ is a Banach space that satisfies (61), then $X$ is $K$-convex.

Proof of Proposition 14. Let $r_X \in [1, 2]$ be the supremum over those $r \in [1, 2]$ such that $X$ has type $r$. Our goal is to show that $r_X \geq 1/\alpha$. By the Maurey–Pisier theorem [MP76], for every $n \in \mathbb{N}$, there exists a linear operator $J_n : L_{r_X}([-1, 1]^n) \to X$ such that

$$\forall g \in L_{r_X}([-1, 1]^n), \quad \|g\|_{r_X} \leq \|J_n g\|_X \leq 2\|g\|_{r_X}.$$  

(62)

Fixing $n \in \mathbb{N}$ and $g \in L_{r_X}([-1, 1]^n)$, for every $\omega \in \{-1, 1\}^n$, define $g_\omega \in L_{r_X}([-1, 1]^n)$ by

$$\forall \varepsilon \in \{-1, 1\}, \quad g_\omega(\varepsilon) \overset{\text{def}}{=} g(\omega \varepsilon) = g(\omega_1 \varepsilon_1, \ldots, \omega_n \varepsilon_n).$$

Next, define $h_g : \{-1, 1\}^n \to X$ by setting

$$\forall \omega \in \{-1, 1\}^n, \quad h_g(\omega) \overset{\text{def}}{=} J_n g_\omega \in X.$$  

(64)

It follows from (63) and (64) that

$$\forall \omega \in \{-1, 1\}^n, \quad h_g(\omega) = J_n \left( \sum_{A \subseteq [n]} \hat{g}(A) W_A(\omega) W_A \right) = \sum_{A \subseteq [n]} \hat{g}(A) W_A(\omega) J_n(W_A).$$

(65)

By (65), for every $\omega \in \{-1, 1\}^n$, we have

$$\left( \Delta_{[n]}^\alpha \otimes I_X \right) h_g(\omega) = \sum_{A \subseteq [n]} |A|^\alpha \hat{g}(A) W_A(\omega) J_n(W_A) = J_n(\Delta_{[n]}^\alpha g_\omega) = J_n(\left( \Delta_{[n]}^\alpha g \right)_\omega).$$

Consequently,

$$\forall \omega \in \{-1, 1\}^n, \quad \left\| \left( \Delta_{[n]}^\alpha \otimes I_X \right) h_g(\omega) \right\|_X \overset{(66) \wedge (62)}{\leq} 2 \left\| \left( \Delta_{[n]}^\alpha g \right)_\omega \right\|_{r_X} = 2 \left\| \Delta_{[n]}^\alpha g \right\|_{r_X}.$$  

(67)

In a similar vein, it follows from (65) that for every $\omega, \delta \in \{-1, 1\}^n$, we have

$$\sum_{j=1}^n \delta_j \left( \partial_j \otimes I_X \right) h_g(\omega) = J_n \left( \left( \sum_{j=1}^n \delta_j \partial_j g \right)_\omega \right).$$

Hence,

$$\forall \omega, \delta \in \{-1, 1\}^n, \quad \left\| \sum_{j=1}^n \delta_j \left( \partial_j \otimes I_X \right) h_g(\omega) \right\|_X \overset{(68) \wedge (62)}{\geq} \left\| \sum_{j=1}^n \delta_j \partial_j g \right\|_{r_X}.$$  

(69)
By combining (67) and (69) with an application of (61) to $h = h_g$, it follows that

$$\left(\frac{1}{2^n}\sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \delta_j \partial_j g \right\|_{r_X} \right)^{\frac{1}{r_X}} \lesssim_p \left(\frac{1}{2^n}\sum_{\delta \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \delta_j \partial_j g \right\|_{r_X}^p \right)^{\frac{1}{p}} \lesssim_p X \left\| \Delta^\alpha_n g \right\|_{r_X},$$

(70)

where the first step of (70) uses Kahane’s inequality [Kah64]. When $\alpha = 1/2$, by a result of Lamberton [LP98, p. 283], and for general $\alpha \in (0,1)$ by a result of the author and Schechtman [ELP08, §5.5], the validity of (70) for every $n \in \mathbb{N}$ and $g \in L_{r_X}(\{-1,1\}^n)$ implies that $r_X \geq 1/\alpha$. □

Acknowledgements. I am very grateful to Alexandros Eskenazis for many helpful discussions.

References

[Bal13] K. Ball, The Ribe programme, in Séminaire Bourbaki. Vol. 2011/2012, Astérisque 352, Math. Soc. France, Paris, 2013, pp. Exp. No. 1047, viii, 147–159. MR 3087345. Zbl 1303.46019.

[Bań86] R. Bańuelos, Martingale transforms and related singular integrals, Trans. Amer. Math. Soc. 293 (1986), 547–563. MR 0816309. Zbl 0591.60045. http://dx.doi.org/10.2307/2000021.

[BL00] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis. Vol. 1, Amer. Math. Soc. Colloq. Publ. 48, Amer. Math. Soc., Providence, RI, 2000. MR 1727673. Zbl 0946.46002.

[Bon68] A. Bonami, Construction d’opérateurs de convolution sur le groupe $D^\infty$, C. R. Acad. Sci. Paris Sér. A-B 266 (1968), A864–A866. MR 0231131. Zbl 0159.42302.

[Bou87] J. Bourgain, Remarks on the extension of Lipschitz maps defined on discrete sets and uniform homeomorphisms, in Geometrical Aspects of Functional Analysis (1985/86), Lecture Notes in Math. 1267, Springer-Verlag, New York, 1987, pp. 157–167. MR 0907692. Zbl 0633.46018. http://dx.doi.org/10.1007/BFb0078143.

[Bou86] J. Bourgain, Vector-valued singular integrals and the $H^1$-BMO duality, in Probability Theory and Harmonic Analysis (Cleveland, Ohio, 1983), Monogr. Textbooks Pure Appl. Math. 98, Dekker, New York, 1986, pp. 1–19. MR 0830227. Zbl 0602.42015.

[CoW76] R. R. Coifman and G. Weiss, Transference Methods in Analysis, Conf. Board Math. Sci. Reg. Conf. Ser. Math. 31, Amer. Math. Soc., Providence, RI, 1976. MR 0431928. Zbl 0377.43001.

[DuRdF85] J. Duoandikoetxea and J. L. Rubio de Francia, Estimations indépendantes de la dimension pour les transformées de Riesz, C. R. Acad. Sci. Paris Sér. I Math. 300 (1985), 193–196. MR 0780616. Zbl 0577.42015.
[ELP08] L. B. EFRAIM and F. LUST-Piquard, Poincaré type inequalities on the
discrete cube and in the CAR algebra, Probab. Theory Related Fields
141 (2008), 569–602. MR 2391165. Zbl 1141.60005. http://dx.doi.org/10.1007/s00440-007-0094-x.

[FJS88] T. Figiel, W. B. Johnson, and G. Schechtman, Random sign embeddings from $l_p^r$, $2 < r < \infty$, Proc. Amer. Math. Soc. 102 (1988), 102–106. MR 0915724. Zbl 0655.46016. http://dx.doi.org/10.2307/2046039.

[GNS12] O. Giladi, A. Naor, and G. Schechtman, Bourgain’s discretization
theorem, Ann. Fac. Sci. Toulouse Math. 21 (2012), 817–837. MR 3052032. Zbl 1283.46015. http://dx.doi.org/10.5802/afst.1352.

[GPP80] E. D. Gluskin, A. Pietsch, and J. Puhl, A generalization of Khint-
chine’s inequality and its application in the theory of operator ideals, Studia
Math. 67 (1980), 149–155. MR 0583295. Zbl 0441.47032.

[Gun86] R. F. Gundy, Sur les transformations de Riesz pour le semi-groupe
d’Ornstein-Uhlenbeck, C. R. Acad. Sci. Paris Sér. I Math. 303 (1986),
967–970. MR 0877182. Zbl 0606.60063.

[GV79] R. F. Gundy and N. T. Varopoulos, Les transformations de Riesz et
les intégrales stochastiques, C. R. Acad. Sci. Paris Sér. A-B 289 (1979),
A13–A16. MR 0545671. Zbl 0413.60003.

[Hei80] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew.
Math. 313 (1980), 72–104. MR 0552464. Zbl 0412.46017. http://dx.doi.org/10.1515/crll.1980.313.72.

[JMST79] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, Sym-
metric structures in Banach spaces, Mem. Amer. Math. Soc. 19 (1979),
v+298. MR 0527010. Zbl 0421.46023. http://dx.doi.org/10.1090/memo/0217.

[JSZ85] W. B. Johnson, G. Schechtman, and J. Zinn, Best constants in mo-
ment inequalities for linear combinations of independent and exchange-
able random variables, Ann. Probab. 13 (1985), 234–253. MR 0770640. Zbl 0564.60020. Available at http://www.jstor.org/stable/2243635.

[Kah64] J.-P. Kahane, Sur les sommes vectorielles $\sum \pm u_n$, C. R. Acad. Sci. Paris
259 (1964), 2577–2580. MR 0169279. Zbl 0127.35504.

[LT77] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. I, Ergeb.
Math. Grenzgeb. 92, Springer-Verlag, New York, 1977, Sequence spaces.
MR 0500056. Zbl 0362.46013.

[LP86] F. Lust-Piquard, Inégalités de Khintchine dans $C_p (1 < p < \infty)$, C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), 289–292. MR 0859804. Zbl 0592.47038.

[LP98] F. Lust-Piquard, Riesz transforms associated with the number operator
on the Walsh system and the fermions, J. Funct. Anal. 155 (1998), 263–
285. MR 1623158. Zbl 0908.47027. http://dx.doi.org/10.1006/jfan.1997.3217.

[Mau03] B. Maurey, Type, cotype and $K$-convexity, in Handbook of the Ge-
ometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003,
B. Maurey and G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, *Studia Math.* **58** (1976), 45–90. MR 0443015. Zbl 0344.47014.

C. A. McCarthy, *c*ₚ, *Israel J. Math.* **5** (1967), 249–271. MR 0225140. Zbl 0156.37902. http://dx.doi.org/10.1016/S1874-5849(03)80037-2.

B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, *Studia Math.* **58** (1976), 45–90. MR 0443015. Zbl 0344.47014.

C. A. McCarthy, *c*ₚ, *Israel J. Math.* **5** (1967), 249–271. MR 0225140. Zbl 0156.37902. http://dx.doi.org/10.1016/S1874-5849(03)80037-2.

M. Mendel and A. Naor, Euclidean quotients of finite metric spaces, *Adv. Math.* **189** (2004), 451–494. MR 2101227. MR 2101227. Zbl 1088.46007. http://dx.doi.org/10.1016/j.aim.2003.12.001.

M. Mendel and A. Naor, Some applications of Ball’s extension theorem, *Proc. Amer. Math. Soc.* **134** (2006), 2577–2584. MR 2213735. Zbl 1108.46052. http://dx.doi.org/10.1090/S0002-9939-06-08298-0.

M. Mendel and A. Naor, Metric cotype, *Ann. of Math.* **168** (2008), 247–298. MR 2415403. Zbl 1187.46014. http://dx.doi.org/10.4007/annals.2008.168.247.

P.-A. Meyer, Transformations de Riesz pour les lois gaussiennes, in *Seminar on Probability, XVIII*, *Lecture Notes in Math.* **1059**, Springer-Verlag, New York, 1984, pp. 179–193. MR 0770960. Zbl 0543.60078. http://dx.doi.org/10.1007/BFb0100043.

A. Naor, An introduction to the Ribe program, *Jpn. J. Math.* **7** (2012), 167–233. MR 2995229. Zbl 1261.46013. http://dx.doi.org/10.1007/s11537-012-1222-7.

A. Naor and G. Schechtman, Remarks on non linear type and Pisier’s inequality, *J. Reine Angew. Math.* **552** (2002), 213–236. MR 1940437. Zbl 1033.46013. http://dx.doi.org/10.1515/crll.2002.092.

A. Naor and G. Schechtman, Metric $X_p$ inequalities, *Forum Math.* **Pi** **4** (2016), e3, 81. MR 3456183. Zbl 06539068. http://dx.doi.org/10.1017/fmp.2016.1.

M. I. Ostrovskii, *Metric Embeddings*, De Gruyter Stud. Math. no. 49, De Gruyter, Berlin, 2013, Bilipschitz and coarse embeddings into Banach spaces. MR 3114782. Zbl 1279.46001. http://dx.doi.org/10.1515/9783110264012.

R. E. A. C. Paley, Some theorems on abstract spaces, *Bull. Amer. Math. Soc.* **42** (1936), 235–240. MR 1563277. Zbl 62.0464.03. http://dx.doi.org/10.1090/S0002-9904-1936-06277-4.

R. E. A. C. Paley and A. Zygmund, On some series of functions, (1), *Math. Proc. Cambridge Philos. Soc.* **26** (1930), 337–357. Zbl 56.0254.01. http://dx.doi.org/10.1017/S0305004100016078.

G. Pisier, Holomorphic semigroups and the geometry of Banach spaces, *Ann. of Math.* **115** (1982), 375–392. MR 0647811. Zbl 0487.46008. http://dx.doi.org/10.2307/1971396.

G. Pisier, Riesz transforms: a simpler analytic proof of P.-A. Meyer’s inequality, in *Séminaire de Probabilités, XXII*, *Lecture Notes in Math.* **1321**, pp. 1299–1332. MR 1999197. Zbl 1074.46006. http://dx.doi.org/10.1016/S1874-5849(03)80037-2.
Springer-Verlag, New York, 1988, pp. 485–501. MR 960544. Zbl 0645.60061. http://dx.doi.org/10.1007/BFb0084154.

[Ros70] H. P. Rosenthal, On the subspaces of $L^p$ ($p > 2$) spanned by sequences of independent random variables, Israel J. Math. 8 (1970), 273–303. MR 0271721. Zbl 0213.19303. http://dx.doi.org/10.1007/BF02771562.

[Sch38] I. J. Schoenberg, Metric spaces and positive definite functions, Trans. Amer. Math. Soc. 44 (1938), 522–536. MR 1501980. Zbl 064.0617.02. http://dx.doi.org/10.2307/1989894.

[Ste83] E. M. Stein, Some results in harmonic analysis in $\mathbf{R}^n$, for $n \to \infty$, Bull. Amer. Math. Soc. 9 (1983), 71–73. MR 0699317. Zbl 0515.42018. http://dx.doi.org/10.1090/S0273-0979-1983-15157-1.

(Received: January 21, 2016)

Princeton University, Princeton, NJ
E-mail: naor@math.princeton.edu