Non-uniform measure in
four-dimensional simplicial quantum gravity

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Abstract.

Four-dimensional euclidean quantum gravity has been studied as a discrete model based on dynamical triangulations by Ambjørn and Jurkiewicz and by Agishtein and Migdal. We discuss a particular implementation of a Monte Carlo simulation of simplicial quantum gravity. As an application we introduce a non-uniform measure and examine its effect on simple aspects of the mathematical geometry. We find that the transition region from the hot to the cold phase is shifted and that the criticality of the transition changes.
1 Introduction

To date there does not exist a single self-consistent theory of four-dimensional quantum gravity. Even leaving aside profound conceptual issues like how to understand the concept of time in quantum cosmology, already at the technical level there are problems with the quantization of the gravitational field. Perturbation theory, which has been successfully applied to gauge theories fails for gravity due to its non-renormalizability. Alternatives to the standard field theoretic treatment include string theory, but the perturbative loop expansion is not summable \[3\], and it is not clear how to recover a four dimensional spacetime. Canonical quantization methods \[4\] offer the hope that quantum gravity can be constructed as a fundamentally non-perturbative theory, but though promising this approach is far from completion.

Here we will be concerned with an attempt to regularize the formal euclidean path integral for quantum gravity by a discretization of the integration over the space of metrics. Such a regularization leads to a well defined theory, which however may be completely unrelated to the ‘correct’ quantum theory of gravity. The relevant question to ask is therefore whether the so regularized path integral leads to reasonable physical statements. A much more modest goal is to understand the statistical model resulting from such a discretization, and this is what we want to contribute to.

As was shown by Regge \[5\] and further explored by others \[6\], a classical curved spacetime can be approximated by a piecewise linear manifold composed of simplices, whose edge lengths define a unique metric. The traditional Regge calculus fixes the connectivity of the simplicial complex but allows the edge lengths to vary to obtain all possible metrics. A related approach referred to as dynamical triangulations calls for fixing the edge lengths and varying the connectivity of the simplicial complex \[7\].

The euclidean path integral can be regulated by first of all fixing the topology of the manifold, replacing the Einstein action by its simplicial equivalent and by replacing the infinite dimensional, ill-defined integration over the space of metrics by a finite summation over simplicial complexes. This regularization leaves, however, one ambiguity unresolved, which is the choice of the proper measure in the path integral. The best way to pick out a unique measure in the continuum theory is through BRST invariance \[10\], which results in a path-integral that still requires a regularization, and the integration variables are no longer the metric components so triangulations do not seem to be applicable.

It is not known what kind of effective measure the transition to triangulated spacetimes introduces into the path integral. In fact, one reason why the method of dynamical triangulations has become popular is based on results obtained in two dimensions indicating that the dynamical triangulated theory is equivalent to exact results obtained in Liouville theory if in the former the measure factor is set equal to one \[8\]. Naively interpreted, this suggests that even in four dimensions the summation over dynamical triangulations somehow takes into account the proper measure factor. It is therefore of interest to examine whether a non-uniform measure motivated by the four dimensional continuum theory leads to changes in the dynamically triangulated theory.

The discretization of the path integral opens the way for a Monte Carlo simulation. In what follows we will discuss results obtained in four dimensional simplicial quantum gravity via dynamical triangulations. For results obtained via Regge calculus see the review by Hamber \[6\]. There have been two studies in four dimensions, Ambjørn and Jurkiewicz \[1\] and Agishtein and Migdal \[2\] (both using the uniform measure). It is still open whether there is a second order phase transition, which would allow for the existence of a continuum limit, but in \[2\] a
diffusion dimension of 4.0 is found in the antigravity phase.

In this article we focus on the relation between the cosmological and gravitational constants, the net scalar curvature, and the average distance of two simplices. Since [1] and [2] do not contain directly comparable data we thought it to be useful to demonstrate that the computer code written for this article produces data in agreement with both. The main result of this paper is that the inclusion of a measure factor of the type \( \prod_x (\det g)^{n/2} \) leads to qualitatively different results, noticeably that the criticality of the transition from the hot to the cold phase depends on \( n \).

In section 2 we introduce the model in detail. In section 3 we comment on the particular implementation of the data structure representing the simplicial complex and the Monte Carlo algorithm as far as it differs from [1, 2]. In section 4 we present results obtained for uniform and non-uniform measures. Section 5 concludes with a discussion.

2 The Model

2.1 Discretized action

We consider the case where the manifold has the topology of \( S^4 \). The euclidean Einstein-Hilbert action is

\[
S_E[g] = \lambda \int d^4x \sqrt{g} - \frac{1}{G} \int d^4x \sqrt{g} R(g),
\]

where \( \lambda \) is the cosmological constant, \( G \) is the gravitational constant, \( g \) is the determinant of the metric and \( R(g) \) its scalar curvature.

As discussed in [1, 2], if one considers only manifolds which are simplicial complexes with \( S^4 \) topology and defines the metric by the condition that all edges have length 1, then the volume integral \( V \) and the net scalar curvature \( R \) can be replaced by

\[
V \equiv \int d^4x \sqrt{g} \quad \longleftrightarrow \quad N_4[T],
\]

\[
R \equiv \int d^4x \sqrt{g} R(g) \quad \longleftrightarrow \quad 2\pi N_2[T] - 10\alpha N_4[T],
\]

where \( N_i[T] \) denotes the number of \( i \)-simplices of the triangulation \( T \) and \( \alpha \) is derived from the condition that for approximately flat triangulations the curvature vanishes, \( \alpha = \arccos (1/4) \approx 1.318 \). The action is then simply [2]

\[
S_E[T] = \lambda N_4[T] + \lambda_0 R[T],
\]

\[
\lambda_0 = -\frac{1}{G},
\]

setting the relative factor between \( V \) and \( R \) equal to 1. Equivalently, the action can be written as [1]

\[
S_E[T] = k_4 N_4[T] - k_2 N_2[T],
\]

\[
k_4 = \lambda - 10\alpha \lambda_0 = \lambda + \frac{10}{G},
\]

\[
k_2 = -2\pi \lambda_0 = \frac{2\pi}{\alpha G},
\]

where \( R \) is replaced by \( R/\alpha \).
Notice that (6) is the most general action of the type $S_E = \sum_i N_i$ in four dimensions. Euler’s relation for $S^4$ and the Dehn-Sommerville relations leave only two of $N_0, \ldots, N_4$ independent. The number of vertices $N_0$ in the simplicial complex for example is

$$N_0 = \frac{1}{2}N_2 - N_4 + 2.$$  \hfill (9)

In addition, there are inequalities between the $N_i$. Denote by $o(a)$ the order of vertex $a$, i.e. the number of 4-simplices that contain $a$. For the average order of simplices we have

$$5 \leq \frac{1}{N_0} \sum_o o(a) = \frac{5N_4}{N_0} < \infty.$$ \hfill (10)

From (9) and (3),

$$2 < \frac{N_2}{N_4} < 4,$$  \hfill (11)

$$-0.614 < \frac{R}{N_4} < 12.0,$$ \hfill (12)

that is, the average curvature is asymmetrically bounded from below and above. This is a reflection of the fact that the choice of $S^4$ topology restricts the space of possible metrics.

The path integral is regulated by

$$Z = \int \mathcal{D}g e^{-S_E[g]} \longleftrightarrow \sum_{T \in \mathcal{T}} e^{-S_E[T]},$$ \hfill (13)

where the integration over all metrics is replaced by a summation over all triangulations $T$ with $S^4$ topology. Monte Carlo simulations of the sum over triangulations produce results which are consistent with a second order phase transition \[1, 2\] and therefore a continuum limit may exist. In three dimensions there is a first order phase transition \[11\], while in two dimensions the equivalent of $S_E$ does not lead to a well-defined theory since only one of the $N_i$ is independent.

The choice of measure in the path integral is tantamount to defining the theory. In four-dimensional simplicial quantum gravity no preferred choice for the measure is known, preferred in the sense that BRST-invariance singles out a measure in the continuum. Setting the measure equal to one as in (13), which is used in \[1, 2\], is natural only because it is simple, and because in two-dimensional quantum gravity this uniform measure leads to numerical results (e.g. \[9\]) which are consistent with the Liouville field theory approach. Of course, there is no reason to expect that a generalization to higher dimensions should contain the uniform measure. Rather the opposite is likely, that the non-triviality of higher dimensions involves the measure, as is suggested by the increased complexity of higher dimensional differentiable manifolds.

Since we would like to maintain the metric components $g_{\mu\nu}$ as integration variables so that a regularization by triangulations is possible (as opposed to the BRST approach), let us give two examples for measures in the continuum theory suggested by different physical arguments. These are

$$\mathcal{D}_1 g_{\mu\nu} = \prod_x g^{-5/2} \prod_{\mu \leq \nu} g_{\mu\nu},$$ \hfill (14)

$$\mathcal{D}_2 g_{\mu\nu} = \prod_x g^{00} g^{-3/2} \prod_{\mu \leq \nu} g_{\mu\nu}. \hfill (15)$$
The first one is suggested by diffeomorphism invariance of the measure and is scale-invariant. The second is the Leutwyler measure, which in addition takes into account how the path integral depends on a particular foliation used in the Hamiltonian formalism, and is also chosen to cancel certain divergences in the path integral. Both measures are unsatisfactory because they do not lead to a well-defined perturbation theory. As an aside, a scale invariant measure has been considered in the context of quantum Regge calculus, but a direct comparison with our results from dynamical triangulations does not seem possible.

Here we want to propose that factors of the type \( \prod g^{n/2} \) are of interest even though the true measure is not known and examine the change they introduce in the Monte Carlo simulations. We make the assumption that the summation over triangulations does not incorporate the full measure \( \mathcal{D}g_{\mu\nu} = \prod g_{\mu\nu} \) but only the flat part \( \prod g_{\mu\nu} \) and hence the equivalent of \( \prod g^{n/2} \) has to be included in the discrete theory. Notice that as long as the effective measure of summing over all \( T \in \mathcal{T} \) is not known — it probably is neither equivalent to \( \prod g_{\mu\nu} \) nor to \( \prod g^{n/2} g_{\mu\nu} \) — one can at best hope to get some qualitative insight into the influence of the measure. Even if the discrete model does depend on the measure, if there is a second order phase transition, one can attempt to establish universality of critical exponents under change of the measure.

The discrete version of the volume element is obtained by \( \sqrt{g(x)} \to o(a)/5 \), which is consistent with (2) using \( \int d^4x \to \sum_a \) and \( \sum_a o(a) = 5N_4 \). The family of measures including the uniform measure can be incorporated into the discretized theory by replacing \( S_E \) by

\[
S = S_E + S_M \quad (16)
\]

\[
S_M = -n \sum_a \log \frac{o(a)}{5}. \quad (17)
\]

One could also imagine different definitions of local volume, e.g., based on the number of edges sharing a vertex.

### 2.2 Monte Carlo simulation

The Monte Carlo evaluation of the path-integral is largely standard. One constructs a Markov chain \( \{T_i\} \) of triangulations with equilibrium configuration \( \exp(-S_E) \) so that

\[
\langle f \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(T_i). \quad (18)
\]

Of interest is the phase diagram obtained for different values of \( k_2 \) and \( k_4 \). As it turns out, for a given \( k_2 \) there is a unique value of \( k_4 \), \( k_4^c(k_2) \), such that if \( k_4 < k_4^c \) the volume \( N_4 \) goes to infinity and if \( k_4 > k_4^c \) then \( N_4 \) tends to zero. Hence we fix \( N_4 \) (canonical simulation) and look for critical points at some value of \( k_2 \) on the ‘critical’ line defined by \( k_4 = k_4^c(k_2) \).

As shown in [15], the following set of five elementary moves is ergodic for four dimensional simplicial complexes. Denoting a 4-simplex by its five vertices,

\[
abcd + 1abcd + 1abce + 1acde + 1bcde, \quad (19)
\]

\[
abcd + abcd2 \leftrightarrow 12abc + 12abd + 12acd + 12bcd, \quad (20)
\]

\[
abc + abc13 + abc23 \leftrightarrow 123ab + 123ac + 123bd, \quad (21)
\]

where \( a, b, \ldots \) and \( 1, 2, \ldots \) denote the vertices which are common to all 4-simplices on the left- and the right-hand side, respectively. Move \( i \) is the exchange of \( i \)-simplices for appropriate
There are two restrictions on when a move may be performed which ensure that the simplicial complex does not change topology. First of all, move $i$ is only possible when the order of the simplex which is to be removed is $5 - i$. For move 0, for example, a vertex can only be removed if it is of order 5, since otherwise its neighboring vertices do not form a 4-simplex which always has 5 vertices. And second, a move is not allowed if it creates simplices which are already present. For example, move 3 — left to right in (20) — introduces a new link 12 that, if already part of the simplicial complex, would lead to overlapping 4-volumes.

Given this set of ergodic moves, we have to define a random walk through the space of simplicial complexes and compute the weights required for detailed balance. Any algorithm to pick a move introduces a probability $P(T \rightarrow T')$ for a transition from triangulation $T$ to $T'$. This probability is not independent of the triangulations involved since the simplicial complex, and therefore for example the number of allowed moves, is changing. In general, suppose the transition probability is

$$W(T \rightarrow T') = P(T \rightarrow T')W_M(T \rightarrow T'),$$

where $W_M$ is the standard Metropolis weight. Detailed balance for $W$ implies

$$P(T \rightarrow T') = P(T' \rightarrow T),$$

since $W_M$ already satisfies detailed balance.

We choose a prescription for picking a move randomly that does not require knowledge of the number of possible moves of a given type, since this number is not readily available in our implementation. Given a triangulation $T$, first pick a 4-simplex, then any of its subsimplices with equal probability. The probability of this algorithm to suggest a particular $i$-simplex for a move of type $i$ from $T$ to $T' = M_iT$ is

$$P_A(T \rightarrow M_iT) = \frac{o(i\text{-simplex})}{N_4(T) \binom{5}{i + 1}},$$

where $o(i\text{-simplex})$ is the number of 4-simplices in which the subsimplex appears. The point is that if a move is allowed, the order $o$ of the subsimplex must be $5 - i$. Since the inverse of move $i$ is move $4 - i$ and

$$f_d(i) := \frac{d - i + 1}{\binom{d + 1}{i + 1}} = f_d(d - i),$$

we obtain for the factor $c(T \rightarrow T')$ such that $P = cP_A$ satisfies detailed balance that

$$\frac{c(T \rightarrow M_iT)}{c(M_iT \rightarrow T)} = \frac{N_4(T)}{N_4(M_iT)},$$

where $N_4(M_iT) = N_4(T) + 2(i - 2)$. The remaining freedom in the $c(T \rightarrow M_iT)$ can be adjusted, for example, to speed up thermalization.

The Metropolis weights derived from $S_E$ only depend on the type of move and not on where it acts on the triangulation, $\Delta S_E(M_i) = \Delta S_E(i)$ (see [1, 2]). To compute the change
in $S_M$ the order of the neighboring vertices enters because of the logarithm in the definition of $S_M$. For a move of type $i$ the change in the order of old and new common vertices is

$$\Delta o(\text{old}) = 2i - 5, \quad \Delta o(\text{new}) = 2i - 3.$$  \hfill (27)

For example, under move 2 (21) the order of vertex $a$, which is common among the old simplices, changes by -1.

There remain two interesting issues related to ergodicity, one technical and easily solved, the other fundamental and potentially a serious problem. First notice that we want to simulate for fixed values of $N_4$ but that the elementary moves discussed above change the value of $N_4$, and no set of ergodic moves is known which preserves $N_4$. The approach we use here is to allow $N_4$ to vary in a certain range which is a small fraction of $N_4$ but as large as feasible to approach ergodicity. For details of our implementation see section 3.

Even though the elementary moves introduced above are ergodic, they are not finitely ergodic [16]. This is directly related to the classical result by Markov that there is no algorithm to determine the type of simplicial 4-manifolds. It is not known whether this leads to a severe restriction on the space of metrics sampled by the conventional set of five elementary moves. Of course, for a given finite number $N_4$ of 4-simplices the elementary moves are finitely ergodic, but if the infinite volume limit is not just to be approximated by a finite number of simplicial complexes, then an algorithm to find all simplicial complexes for all $N_4$ is required and such an algorithm does not exist.

## 3 Computer implementation

In this section we discuss briefly computer related issues of Monte Carlo simulations of simplicial quantum gravity using dynamical triangulations. The reader only interested in the results may skip this section.

The challenge posed by dynamical triangulations is to find a data structure for the simplicial complex which allows efficient updating under the moves (19)–(21). As a starting point, suppose a 4-simplex is represented by the list of its five vertices which are labeled by integers. A simplicial complex is a list of 4-simplices.

The nature of the five moves requires frequent use of all types of $i$-simplices, and therefore one is lead to store additional information about the simplicial complex so that the search for a particular subsimplex does not become too time consuming. Storing just vertex labels is one extreme, storing pointers to and maintaining lists of all $i$-simplices, say, is another. In [4] an optimized version of the latter approach is employed.

Memory constraints of current workstations may be serious in schemes which call for storing data about all subsimplices (a 4-simplex has thirty subsimplices). If 100 bytes are used for each simplex, the system size is limited to about $N_4 = 100000$, which is not much considering that in a flat configuration $N_4/N_0 \approx 20$ and we are dealing with four dimensions. The same holds for super/parallel computers since the non-linear memory access creates a worst-case scenario for common caching schemes, so that all the data should be stored in fast memory.

Another aspect is speed. Clearly, one has to strike a balance between the speed gained by accessing a subsimplex directly through a pointer versus the speed lost for updating the data structure.

Notice that the elementary moves all act locally. Our code therefore stores for each 4-simplex its five vertex labels and pointers to its five neighboring simplices. This minimal
additional information allows already to perform each move with only local searches which are rather fast, while the data structure to be maintained is small and its updating therefore is also fast and simple. Two neighboring simplices share a 3-simplex face of 4 vertices. For the moves the label of the fifth vertex of the neighboring simplex is needed. This may be one of the reasons why in [3] for each simplex the five extra vertices of its neighbors are stored so that one does not have to search for them. The alternative we use is to store with each simplex the sum of its vertex labels, and the extra vertices are obtained simply by subtraction.

While the geometric nature of the elementary moves may be complicated, we have presented them in (14)–(21) in a regular form that suggests a straightforward implementation. Notice that always six simplices and six vertices are involved. A move is given by a choice of subsimplex which is to be exchanged. The test whether the order of the subsimplex is correct can be replaced by the observation that otherwise one cannot write the suggested move in the regular form, or equivalently, more than six vertices would be involved. Furthermore, the simplices appearing in each move are subsets of five out of possible six vertices in an obvious fashion, and the same is true for the pointers to neighbors attached to each simplex. Since this structure is the same for each move, we can use one compact routine that handles all five moves, which makes debugging simple.

The test for overlapping 4-volumes (the crash test) is performed by a local recursive search for doubly present subsimplices. Given the new common subsimplex which will be introduced by a move, all except one of its vertices are present in the 4-simplex chosen for the move, and one just has to check whether any of the 4-simplices containing these vertices also contains the additional vertex of the new common subsimplex.

To summarize, even though the geometric structure of the dynamic changes of the triangulation is complicated, there exist a fast, simple, and memory efficient implementation.

A few comments about the Monte Carlo simulations are in order. The system is initialized in the minimal $S^4$ configuration, i.e. six vertices connected in all possible ways, or the six 4-simplices forming the surface of a 5-simplex. Then moves of type 4 are performed until a previously fixed value $N_4^0$ is reached. For the reasons explained in section 2, we perform a constrained simulation such that $N_4 \approx N_4^0$. There are several approaches [17, 1, 2] that avoid defining a rigid cutoff which might introduce extraneous effects. They all amount to modifying the action by a potential-like term with a minimum at $N_4^0$, such that the volume $N_4$ is driven towards $N_4^0$. We choose

$$S'(k_2, k_4) = \begin{cases} 
S(k_2, k_4) - \Delta k_4 N_4 & \text{if } N_4 < N_4^0 - \Delta N_4 \\
S(k_2, k_4) & \text{if } N_4^0 - \Delta N_4 \leq N_4 \leq N_4^0 + \Delta N_4 \\
S(k_2, k_4) + \Delta k_4 N_4 & \text{if } N_4 < N_4^0 + \Delta N_4 
\end{cases} \tag{28}$$

where $\Delta N_4$ is the fixed width of the potential well and $\Delta k_4$ is the fixed slope of the walls of the potential. This method has also been used with $|x|$ and $x^2$ type potentials. We checked that the outcome of our simulations does not depend too sensitively on $\Delta N_4$ and $\Delta k_4$.

The actual simulations are performed for $k^c_4(k_2)$, for which the system is in an instable state at $N_4^0$. The modification of $S$ is not introduced as a regulation but serves only to approximate a canonical simulation with grand canonical elementary moves.

As noted before, if $k_4 < k^c_4$, then the geometric constraints are such that the system moves to larger and larger $N_4$, and vice versa (which explains the sign in the definition of $S'$). If a good initial guess $k^g_4$ is available for $k^c_4$, then there are analytic methods to estimate $k^c_4$ from simulations for $S'(k_2, k_4)$ [17, 1, 2]. In our simulations we were confronted with such widely varying conditions that a fully automated fine-tuning algorithm for $k^c_4$ was adopted. We use...
a modified bisection method to adjust $k_4^g$ until the probabilities for the system to move up or down near $N_4^0$ are balanced. The modifications take into account that a high level of noise is present by fixing a minimal bisection stepsize and allow for the stepsize to increase in case the value of $k_4^c$ is found to be drifting with thermalization.

4 Results

4.1 Results for uniform measure

We begin by presenting results for the uniform measure. A typical run consists of 500 to 5000 sweeps for systems with $N_4^0 = 4000, 8000$ and $16000$. A sweep is defined as $N_4^0$ elementary moves not counting those that are rejected by the Metropolis algorithm or by the geometric constraints. Error bars were obtained via coarse graining and not drawn if smaller than the symbol size. We trust the errors for large $k_2$. CPU time per run was of the order of 48 hours on a IBM/RISC 6000.

Figures 1 and 2 show the critical lines $k_4 = k_4^c(k_2)$ and $\lambda = \lambda^c(\lambda_0)$, the latter of which can be directly compared with [2], and we find good agreement. The points for $N_4^0 = 4000$ and $8000$ are taken at the same set of $k_2$ and often coincide. In these data there is no indication of a phase transition, the transition from a crumpled to a smooth phase takes place at a value away from the minimum in figure 2 (see below). Thermalization was very fast, on the order of 20 sweeps.

One may be surprised by the fact that $k_4^c(k_2)$ is almost exactly a straight line. Notice that since $S_E = k_4 N_4 - k_2 N_2$ the dominant classical configurations are obtained for $k_4/k_2 = N_2/N_4$, and $N_2/N_4$ is found to range from 2.0 to 2.5. This is close to the slopes in figure 1. The constants in the linear transformation between $k_2$ and $k_4$ and $\lambda_0$ and $\lambda$ happen to be such that the difference in slope for different $k_2$ becomes much clearer in figure 2 for $\lambda^c(\lambda_0)$.

Figure 3 and 4 show the curvature per volume, $R/\alpha V$ from (2)–(3), and the average geodesic distance $d$ of two 4-simplices, which is the average of the minimal number of steps from 4-simplex to 4-simplex. Both plots are consistent with [1]. We have noticed that an increase in $\Delta N_4$ (see section 3) leads to a small shift of the curves to smaller $k_2$, which might explain why our data is slightly shifted that way in comparison with [1] where $\Delta N_4 = 0$. Thermalization ranged from 50 up to 500 sweeps in the transition region.

In figure 3 for $R$, one may suspect that there is a ‘kink’ at

$$k_2^c \approx 1.1 \iff \lambda_0 \approx -0.18.$$  (29)

The data for $d$ displays clearly a transition from a crumpled phase for $k_2 < k_2^c$ to a smooth phase. When $k_2$ is small, i.e. $G$ is large, the average distance does not depend on the size of the system: essentially any simplex is as close as possible to any other simplex. If $k_2$ is large, i.e. $G$ is small, then the average distance increases with the system size: the configuration is smoothed out. The crumpled and the smooth phase (terminology borrowed from two-dimensional random surfaces) have also been called hot and cold phase, respectively.

As observed in [1, 2], the Hausdorff dimension in the crumpled phase tends to infinity while in the smooth phase it approaches unity, so the smoothing out happens for extended linear structures while the crumpling involves higher dimensions, which explains how more and more simplices can be added without increasing distances in the system. As an aside, the effective dimension of the simplicial complex also influences the estimate whether the volume...
$N_4$ is small or large for certain purposes. If a system is large enough in the cold, linear phase that might be far from true in the hot, crumpled phase.

The curvature per volume is somewhat smaller the larger the volume. Our data indicates that there are better estimates for the minimal and maximal curvature per volume than (12) (see also figure 11). As mentioned before, these limits depend on the topology and are not features of quantum gravity as such. For $S^4$ it is not clear whether the transition region is far enough removed from the topology imposed limits so that results can be generalized to different compact topologies. The average distance of 4-simplices is larger for larger volumes except for a possible crossover at $k_2 = 1$.

The two phases are also reflected in the acceptance rates of moves, which are each suggested with roughly the same frequency up to the factor in (24). Figure 5 shows the acceptance rates of moves as the number of moves which had to be suggested for a sweep when $N_4^0 = 4000$, and the number of moves accepted by the Metropolis weight. Typically, even though 70% of the moves passed the Metropolis test, only on the order of a few percent satisfy the geometric constraint. The test for geometric acceptance should therefore be highly optimized.

Figure 6 and 7 display for each type of move separately its rate of geometric acceptance and its percentage of all moves performed due to the Metropolis weight. Inserting a simplex is always possible (move 4) and is not shown in figure 6. For example, while the geometry allows move 3 in about 30% of all cases, move 2 drops from about 4% to 1%.

In figure 7, one recognizes the balance between each move and its inverse. While in the crumpled phase move 0 is suppressed since the order of each vertex tends to be large but has to be 5 for the move to be allowed, in the smooth phase moves 0 and 4 dominate.

### 4.2 Results for non-uniform measure

So far we have restricted ourselves to simple aspects of the mathematical geometry for uniform measure, which nevertheless clearly characterize the character of the two phases, and more measurements can be found in [1, 2]. The emphasize will now be on the influence of a non-uniform measure factor.

In numerical simulations of two-dimensional quantum gravity it was actually found that the effect of a measure of the type (17) is negligible since $S_M$ is about constant for fixed measure coupling [3]. Therefore we first measured the value of $s_M = \sum a \log o(a)$ in a simulation using the uniform measure. Figure 8 clearly shows that $s_M(k^2)$ is not constant. At first glance one might suspect that $s_M(k^2)$ behaves like $N_0$, since a $k_2-N_0$ plot looks exactly like figure 3 for $R$ because of (3). However, $R$ may actually become zero and $s_M(k^2)$ must be greater than a finite positive number, which shows in figure 9 for $s_M/N_0$.

Already from the size of $S_M(k^2)$ for different states of the geometry one can predict the changes when the non-uniform measure factor is built into the simulations. Both the very crumpled and very smooth phase will exist since in these regions $S_M(k^2)$ is constant because $o(a)$ is constant and the terms proportional to the coupling constants will dominate $S(k^2)$ assuming that they do not exactly cancel each other. The contribution from $S_M(k^2)$ is comparable to that of $S_E(k^2)$ near $k_2 = 0$. If the factor $n$ is positive, then $S_M(k^2)$ leads to an effectively larger value of $k_2$ in $S_E(k^2)$ and the transition region will be shifted to smaller values of $k_2$, and the other way round if $n$ is negative.

Let us see whether the actual simulations agree with these predictions. Figures 10, 11, and 12 show the $k_2$ dependence of $k_4^e$, $R/\alpha V$, and $d$ for $n = -5, -1, 0, +1, +5$. The curves do not intersect and show a continuous monotone dependence on $n$. For $n = -5$ in figure 11 there is
evidence for a deformation due to (12). Notice that the curves are qualitatively similar in scale and shape. This is important because adding a term like $S_M$ could have led to completely new features of the statistical model. And the curves are indeed shifted as predicted.

The important point is that the shift in origin and the fact that the curves are stretched in the $k_2$ direction correspond to genuinely different results. First of all, the supposedly critical point $k_2^c$ ranges from the gravity to the antigravity phase. It is therefore possible to choose $n$ to obtain particular critical values.

What is potentially much more relevant is that the criticality of the transition depends on the measure. If there is a second order phase transition, then physically relevant statements are obtained for critical exponents of the transition, and the critical exponents depend on the criticality of the transition. Figure 11 for $R/\alpha V$ shows that the slope decreases for both positive and negative large $n$. The same holds for $d$ in figure 12, where actually the transition seems to be sharpest for $n = +1$.

The $n$ dependence of $d$ suggests that the Hausdorff dimension depends on the choice of measure. For fixed $n$ and for large enough absolute values of $k_2$ the action should become independent of $S_M$ and $d$ should approach its $n = 0$ value.

5 Discussion

While perhaps not fundamental, simplicial quantum gravity in the dynamical triangulation approach has the advantage of being an easily accessible statistical model that may well share some of the features of full quantum gravity. The question of the effective measure introduced by the summation over simplicial complexes together with the ill-understood nature of the measure in the continuum theory render the ansatz $S = S_E + S_M$ considered in this paper of quite an arbitrary nature.

However, at least the mathematical geometry can be discussed in some detail, last but not least because the action $S_E$ is a linear combination of two global characteristics of the simplicial complex, say the number of 2- and 4-simplices. As result, the weights in a Monte Carlo simulation for large $N_4$ depend only on the type of moves. Thermalization can therefore be discussed for fixed weights but changing geometry and geometric constraints. For example, the effect of the measure term $S_M$ for $n > 0$ is to introduce a bias towards vertices with low order which amounts to increasing the average distance of 4-simplices and smoothing out the mathematical geometry.

The continuum theory singles out different values of $n$ for different physical reasons. In the discretized model there also exist preferred values of $n$, e.g. such that $G$ becomes infinite at the transition point. Or one may be able to choose $n$ such that the critical value of $\lambda$ is zero.

The main result is that the criticality of the phase transition depends on the coupling of the measure term. This we believe may be quite important for physical predictions based on critical exponents and is currently under investigation [19]. One clearly has to study the influence of finite size effects on the measure dependence.

A next step could be to determine the influence of the measure on some physical definition of the effective dimension of the simplicial complex (see [4]). If one finds that the dimension is independent of this type of measure, then a generalized type of universality may hold. Otherwise the measure could possibly be chosen to obtain the correct physical dimension.
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