ORIENTED MATROIDS FROM TRIANGULATIONS OF PRODUCTS OF SIMPLICES

MARCEL CELAYA, GEORG LOHO, CHI HO YUEN

Abstract. We introduce a construction of oriented matroids from a triangulation of a product of two simplices. For this, we use the structure of such a triangulation in terms of polyhedral matching fields. The oriented matroid is composed of compatible chirotopes on the cells in a matroid subdivision of the hypersimplex, which might be of independent interest. In particular, we generalize this using the language of matroids over hyperfields, which gives a new approach to construct matroids over hyperfields.

Using the polyhedral structure, we derive a topological representation of the oriented matroid. This relies on a variant of Viro’s patchworking and insights on the structure of tropical oriented matroids. A recurring theme in our work is that various tropical constructions can be extended beyond tropicalization with new formulations and proof methods.

1. Introduction

1.1. Oriented Matroids and Matching Fields. An oriented matroid is a combinatorial object abstracting linear dependence over \( \mathbb{R} \), and can be thought as a matroid with sign data, i.e., with signs attached to its bases that satisfy certain exchange axioms. The standard example of an oriented matroid is given by the signs of the maximal minors of a real matrix, but not all oriented matroids are realizable as in coming from this way. Oriented matroids play an important role in discrete and computational geometry as well as optimization, ranging from the study of geometric configurations to linear programming; they also make appearances in algebraic geometry and topology [14, Chapter 1 & 2]. In particular, the Topological Representation Theorem of Folkman and Lawrence [20] states that every oriented matroid can be represented by a pseudosphere arrangement (a topological generalization of real hyperplane arrangements) and vice versa.

A matching field is a collection of matchings of the complete bipartite graph \( K_{R,\mathcal{E}} \cong K_{d,n} \), one perfect matching between \( R \) and \( \sigma \) for every subset \( \sigma \subset \mathcal{E} \) of size \( d \). The simplest construction of a matching field is by taking all weight minimal maximal matchings selected by a generic matrix on the complete bipartite graph, yielding a coherent matching field. They were introduced by Sturmfels and Zelevinsky in [55] to capture the combinatorics of the leading terms of maximal minors: a matching field is essentially given by choosing a term from each maximal minor of a generic matrix. Along with follow-up works such as [12, 38], it was demonstrated that much of the Gr"obner theory of maximal minors can be deduced from the purely combinatorial

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linkage property. This is analogous to the exchange property of matroids and leads to a generalization of coherent matching fields.

Signed tropicalization shows that coherent matching fields induce realizable oriented matroids. Sturmfels and Zelevinsky noted this in their paper, thereby illustrating the aforementioned analogy between the linkage property and the general exchange axiom. Motivated by their remarks, we study the relation between linkage matching fields and oriented matroids, and show that the linkage property is not enough to guarantee an oriented matroid in Example 3.14 Nevertheless, our first main result is that the statement is true for polyhedral matching fields. Such matching fields arise from triangulations of the product of two simplices. It is known that every coherent matching field is polyhedral while every polyhedral matching field is linkage, so our result extends the relation between coherent matching fields and realizable oriented matroids to an appropriate generality. We elaborate more on why polyhedral matching fields form an interesting and important intermediate class of matching fields in the next subsections, but first we state our main theorem.

By fixing an arbitrary ordering for $R$ and $E$, every matching in a matching field can be thought as a permutation. We define the sign of the matching as the sign of the permutation.

**Theorem A** (Theorem 3.7). Given a polyhedral matching field $(M_\sigma)$ and a full $d \times n$ sign matrix $A$, the sign map $\chi: (E_d) \rightarrow \{+,-\}$ defined by

$$\sigma \mapsto \text{sign}(M_\sigma) \prod_{e \in M_\sigma} A_e$$

is the chirotope of an oriented matroid (Definition 2.1).

Conceptually, our theorem says that instead of taking the signs of maximal minors, we can pick out only one term per determinant (carefully) and still obtain an oriented matroid. This result demonstrates that the matchings in a polyhedral matching field take the role of signed bases of an oriented matroid. We show that this correspondence goes even further by considering other special graphs associated with a polyhedral matching field, such as the linkage covectors, which are local unions of matchings, and the Chow covectors, which are the minimal transversals of the matchings. We describe how these graphs directly yield the signed circuits, signed cocircuits, and more generally covectors of the oriented matroid in Theorem A. We also develop a notion of duality for matching fields that descends to the duality of oriented matroids via Theorem A.

1.2. **Connections to Complexity Questions.** Our work adds a new piece to the connection between two major open complexity questions. On one hand, Smale’s 9th problem asks for a strongly polynomial algorithm in linear programming. Oriented matroids play an important role for this as it is the framework for the simplex method, which is still a natural candidate for such a strongly polynomial algorithm. On the other hand, determining the winning states of a mean payoff game is a problem in $\text{NP} \cap \text{co-NP}$ but no polynomial time algorithm is known. The latter problem is also equivalent to deciding feasibility of a tropical linear program and it turns out that matching fields are the combinatorial framework for describing tropical linear programming. From the viewpoint of mean payoff games, the matchings can be considered as partial strategies. While the tropicalization of the simplex method based on sign patterns already gave a connection between...
pivoting and strategy iteration, we directly derive the correspondence on the level of oriented matroids. The interplay between classical and tropical linear programming has already lead to a proof that a wide class of interior point methods can not be strongly polynomial [3], and we elaborate further in Section 7 how our work can contribute to the understanding of this interplay.

1.3. Triangulations of \( \Delta_{d-1} \times \Delta_{n-1} \) and Matroid Subdivisions, with Signs. Triangulations of a product of two simplices are fundamental in combinatorics and algebraic geometry [7, 16, 24], and via the connection explained in Section 1.2 also in computer science, just to name a few references. Matroid subdivisions are fundamental objects emerging from the context of valuated matroids [18], tropical linear spaces [51], and discrete convex analysis [42].

Drawing various motivations from the literature, we take a crucial new point of view and directly connect triangulations of products of simplices, matroid subdivisions and oriented matroids. Besides questions from complexity, we were inspired by the connection between regular subdivisions of products of simplices and tropical convexity established in [17]. This already lead to the concept of a tropical oriented matroid by Ardila and Develin [8], which is equivalent to the (not-necessarily regular) subdivisions of a product of two simplices [31, 43]. Polyhedral matching fields, or more precisely the pointed version thereof, give yet another equivalent description in the generic case of triangulations [38, 44].

In Section 5 of their seminal paper, Ardila and Develin asked for a connection between oriented matroids and tropical oriented matroids, especially for exploring questions related to non-realizability. The extraction of a realizable oriented matroid from a regular triangulations of \( \Delta_{d-1} \times \Delta_{n-1} \) with signs is implicit in [11], but with our perspective, we finally manage to formalize such speculated connection in full generality. As asymptotically almost every triangulation of the product of two simplices is non-regular [49], our work allows a vast generalization. In particular, we derive Ringel’s non-realizable oriented matroid of rank 3 on 9 elements from a non-regular triangulation of \( \Delta_2 \times \Delta_5 \) in Section 5.2 breaking new ground for the study of realizability of oriented matroids.

Now we sketch our proof of Theorem A. Every triangulation of \( \Delta_{d-1} \times \Delta_{n-1} \) gives rise to a matroid subdivision of the hypersimplex by transversal matroid polytopes [29]. Considering matroid polytopes allows us to manipulate the sign map \( \chi \) geometrically; in particular, the restriction of \( \chi \) to each subpolytope is a realizable chirotope (Lemma 3.8). To finish the proof, we establish a general local-to-global principle (Theorem 3.9) for oriented matroids to show \( \chi \) is a chirotope from local information. The interaction between oriented matroids and matroid subdivisions seems to be largely unexplored, although some very recent works on positive Dressians and positive tropical Grassmannians [11, 69, 52] share some common ideas with ours. Even more, Section 3.3 provides a starting point for understanding the interplay between sign patterns of chirotopes and compatible matroid subdivisions beyond the restrictive positivity condition for positroids.

1.4. Viro’s Patchworking. The method of patchworking goes back to Viro in the 1980s [56]. Viro’s method has numerous applications in real algebraic geometry and tropical geometry (see the survey by Viro [57]), and is related to the Gelfand–Kapranov–Zelevinsky theory [24]. The idea of (combinatorial) patchworking is that one can construct piecewise linear objects isotopic to real algebraic varieties by some
“cut and paste” procedure, starting with a *regular* subdivision of a Newton polytope with sign data. Extending the variant of patchworking by Sturmfels for constructing complete intersections [54], we prove:

**Theorem B.** *Given a fine mixed subdivision of $n\Delta_{d-1}$ and a sign matrix, we can construct a pseudosphere arrangement representing the oriented matroid in Theorem A via a patchworking procedure.*

From this patchworking procedure, we implicitly derive an abstract real phase structure in the sense of [13, 46] from the interplay of the subdivision and the sign matrix. Since most works on patchworking aim to construct real algebro geometric objects, their proofs usually use *tropicalization* of polynomials or similar techniques. In contrast, the aforementioned non-realizable example shows that we can produce non-algebro geometric objects. This suggests that patchworking could be applied for other topological problems beyond tropicalization.

Our proof uses a combination of combinatorial and topological methods, and is loosely based on Horn’s second topological representation theorem for tropical oriented matroids [31]. Roughly speaking, we show that it is possible to interpolate between the dual complex of a patchworking complex, which may be regarded as a cell decomposition of the boundary of the sphere, and a pseudosphere arrangement representing our oriented matroid. This is done by carefully “merging” cells together, ensuring at each step that the combinatorics and the topology are controlled. A similar technique was used by Hersh in her work on total positivity [30].

We note the work of Itenberg and Shustin in [33] which says that in dimension two, patchworking with arbitrary subdivisions produces real pseudoholomorphic curves. However, it seems not much work on patchworking with general subdivisions has been done in higher dimension.

1.5. **Matroids over Hyperfields.** In [10], Baker and Bowler introduced the theory of *matroids over hyperfields*, which provides a common framework for many “matroids with extra data” theories. The base of the theory is the notion of *hyperfields*, which can be thought as ordinary fields with multi-valued addition. Besides unifying parallel notions and propositions among these theories, the new language allows one to explain features in a particular matroid theory using the property of its base hyperfield, thereby finding the correct generality for those features to hold. We do the same in Theorem 6.8 where we extend Theorem A to all matroid theories whose base hyperfield has the *inflation property*, first introduced in the literature with different motivations [4, 40]. Unlike the special case of oriented matroids, Theorem 6.8 is new even for coherent matching fields. In view of the connection between hyperfields and tropicalization (see the other survey by Viro [58]), we ask if there can be tropicalization theories along this direction beyond the degree one case.

Taking maximal minors of a matrix over a field yields a Grassmann–Plücker function. Hence one can construct matroids over common hyperfields, such as the sign hyperfield or the tropical hyperfield (corresponding to oriented matroids and valuated matroids, respectively), using the observation that those hyperfields are images of fields under hyperfield morphisms. However, such an approach does not work in general: not every hyperfield comes from a field (indeed, the inflation property was introduced in [40] to provide such examples), while taking determinant over hyperfields is usually multi-valued. Hence, as a contribution to the theory, our work gives a new approach to construct matroids over hyperfields. Moreover, we use
hyperfied theory to provide a weaker statement (Corollary 6.11) for any hyperfield that our construction gives a matroid over a possibly perturbed base hyperfield.

1.6. Organization of the Paper. In Section 2 we collect essential definitions and background for the central objects in this paper. We prove Theorem A in Section 3 together with other axiomatic connections between matching fields and oriented matroids. Section 4 is devoted to stating and proving Theorem B. The study of the uniform oriented matroids arising via our construction is initiated in Section 5. We extend our work to matroids over hyperfields in Section 6, where we also give a brief introduction to the theory. Finally, we conclude with a few open problems in Section 7. Since the arguments in Section 4 involve several technical results in PL-topology and cellular complexes, an appendix is provided for the background and most of the proofs involving these techniques.

2. Background

Throughout the paper, we fix a ground set $E$ of size $n$ and a set $R$ of size $d \leq n$. We often identify $E, R$ with $[n] = \{1, 2, \ldots, n\}$ and $[d]$, hence fixing an ordering for them.

2.1. Oriented Matroids. We refer the reader to [14] for a comprehensive survey on oriented matroids. As in the case of matroids, there are multiple equivalent axiom systems for oriented matroids, but we mainly use the following definition. We often use $\{+, -, 0\}$ and $\{1, -1, 0\}$ for signs interchangeably, and we adopt the ordering $+, - > 0$ of signs.

**Definition 2.1.** A chirotope on $E$ of rank $d$ is a non-zero, alternating map $\chi : E^d \to \{+, -, 0\}$ that satisfies the Grassmann–Plücker (GP) relation: For any $x_1, \ldots, x_{d-1}, y_1, \ldots, y_{d+1} \in E$, the $d+1$ expressions

$$(-1)^k \chi(x_1, \ldots, x_{d-1}, y_k) \chi(y_1, \ldots, y_k, \ldots, y_{d+1}), \quad k = 1, \ldots, d + 1,$$

either contain both a positive and a negative term, or are all zeros.

With the ordering on $E$, we can specify a chirotope by its values over $\binom{E}{d}$ using the alternating property, here $\chi(\sigma) := \chi(\sigma_1, \ldots, \sigma_d)$ where $\sigma = (\sigma_1 < \ldots < \sigma_d)$.

**Example 2.2.** A chirotope is the generalization of the signs of maximal minors of a real matrix (oriented matroids coming from this way are said to be realizable). More precisely, let $A \in \mathbb{R}^{d \times n}$ be a rectangular matrix with rank $d$. Then

$$\chi(j_1, j_2, \ldots, j_d) = \text{sign det} \left(a^{(j_1)}, a^{(j_2)}, \ldots, a^{(j_d)}\right),$$

where $a^{(j_k)}$ denotes columns of $A$, is the chirotope of an oriented matroid of rank $d$. Note that we can normalize $A$ to the form $(I_{d,d} | B)$ for some $B \in \mathbb{R}^{d \times (n-d)}$ by multiplying with the inverse of a full-rank submatrix of $A$. This incurs only a global sign change of the chirotope.

To state an important characterization of chirotopes, we briefly recall the definition of a matroid.

**Definition 2.3.** Let $B(M)$ be a non-empty subset of $\binom{E}{d}$. Setting $e_B := \sum_{i \in B} e_i \in \mathbb{R}^E$ for each $B \in B(M)$, $M$ is a matroid with bases $B(M)$ if the convex hull of $\{e_B : B \in B(M)\}$ has only edge directions $e_i - e_j$ for unit vectors $e_i, e_j$. The polytope itself is the matroid polytope of $M$ and the rank of $M$ is $d$.  

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(Reproduced with slight alterations for legibility.)
Now if we know in advance that there is a matroid underneath, we can check whether a sign map is a chirotope locally \cite[Theorem 3.6.2]{14}.

**Proposition 2.4.** Suppose \( \chi : E^d \to \{+,-,0\} \) is an alternating map such that \( \Sigma := \{ \sigma \in \binom{E}{d} : \chi(\sigma) \neq 0 \} \) is the collection of bases of some matroid. Then the GP relation is equivalent to the 3-term GP relation:

For any \( x_1, x_2, y_1, y_2 \in E, X := \{ x_3, \ldots, x_d \} \subset E, \) the three expressions

\[
(2) \quad \chi(x_1, x_2, X)\chi(y_1, y_2, X), \chi(x_1, y_1, X)\chi(y_2, x_2, X), \chi(x_1, y_2, X)\chi(x_2, y_1, X),
\]

either contain both a positive and a negative term, or are all zeros.

Finally, we give the definition of pseudosphere arrangements in the statement of the Topological Representation Theorem mentioned in the introduction.

**Definition 2.5.** A pseudosphere arrangement of rank \( d \) is a collection \((S_e : e \in E)\) of \((d - 2)\)-spheres piecewise-linearly (PL), central symmetrically embedded on \( S^{d-1} \) together with sign data, i.e., for each \( S_e \), specify a positive and a negative side for the two connected components of \( S^{d-1} \setminus S_e \). Furthermore, we require that for any \( E' \subset E \), \( S_{E'} := \bigcap_{e \in E'} S_e \) is also a PL sphere, and that for every other \( S_e \), either \( S_{E'} \subset S_e \) or \( S_{E'} \cap S_e \) is a PL sphere of codimension 1 within \( S_e \).

The face lattice of such an arrangement is isomorphic to the covector lattice of the oriented matroid; we again refer the reader to \cite[Chapter 5]{14} for details.

### 2.2. Matching Fields.

A \((d, n)\)-matching field is a collection of perfect matchings \( M_{\sigma}'s \) on bipartite node sets \( R \cup \sigma \), one for each \( d \)-subset \( \sigma \) of \( E \).

**Example 2.6 (Diagonal matching field \cite{55}).** The diagonal matching field has exactly the edges \{\((1,j_1), \ldots, (d,j_d)\)\} in the matching on the ordered subset \( j_1 < j_2 < \cdots < j_d \) of \( E \). For \( d = 2 \) and \( n = 4 \) this is depicted in Figure 1.

**Example 2.7 (Coherent matching field).** A \((d, n)\)-matching field is coherent if there is a generic matrix \( M \in \mathbb{R}^{d \times n} \) such that the matching on \( R \cup \sigma \) is the weight maximal perfect matching induced by the entries of \( M \) as weights on \( K_{d,n} \). The genericity here means that the maximal matching is unique. A diagonal matching field is coherent as it is induced by the weight matrix \(((i - 1) \cdot (j - 1))_{(i,j) \in [d] \times [n]}\).

\[
\begin{array}{cccc}
1 & 2 & 1 & 1 \\
3 & 2 & 3 & 3 \\
4 & 4 & 4 & 4 \\
\end{array}
\]

**Figure 1.** The diagonal \((2,4)\)-matching field.

**Example 2.8 (Linkage matching field).** A matching field is linkage if it fulfills a local compatibility condition. Namely, for every \((d+1)\)-subset \( \tau \) of \( E \), the union of the matchings on \( \tau \) is a spanning tree on \( R \cup \tau \); these are the linkage covectors. Note that each node in \( R \) of a linkage covector has to have degree 2 by a counting argument. Figure 2 shows a linkage matching field which is the pointed extension (see Section 3.1) of the non-linkage matching field depicted in Figure 2. It was used in \cite[Prop. 2.3]{55} to show the existence of non-coherent linkage matching fields.
2.3. **Triangulations of $\Delta_{d-1} \times \Delta_{n-1}$ and Polyhedral Matching Fields.**

We give a brief introduction to the necessary polyhedral notions and refer the reader to [16] for more details. We denote the $(k-1)$-simplex by $\Delta_{k-1}$, that is the convex hull of $k$ affinely independent points. Even if we are mainly interested in combinatorial properties, we use the standard embedding $\Delta_{k-1} = \text{conv}\{e_i : i \in [k]\} \subset \mathbb{R}^k$. The product $\Delta_{d-1} \times \Delta_{n-1}$ of a $(d-1)$-simplex and an $(n-1)$-simplex is the convex hull $\Delta_{d-1} \times \Delta_{n-1} = \text{conv}\{(e_i, e_j) : i \in [d], j \in [n]\} \subset \mathbb{R}^d \times \mathbb{R}^n$.

A **triangulation** of $\Delta_{d-1} \times \Delta_{n-1}$ is a collection of full-dimensional simplices $\mathcal{T}$ whose vertices form a subset of the vertices of $\Delta_{d-1} \times \Delta_{n-1}$, such that

1. (Union) the union of all simplices in $\mathcal{T}$ is $\Delta_{d-1} \times \Delta_{n-1}$,
2. (Intersection) two simplices in $\mathcal{T}$ intersect in a common face.

Each simplex in $\mathcal{T}$ gives rise to a subgraph of the complete bipartite graph on the vertex set $\mathbb{R} \sqcup E$ by identifying a vertex $(e_i, e_j)$ with an edge. Rephrasing the definition of a triangulation in terms of graphs leads to the following characterization.

**Proposition 2.9** ([7] Proposition 7.2]). A set of graphs on the bipartite node set $\mathbb{R} \sqcup E$ encodes the maximal simplices of a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ if and only if:

1. Each graph is a spanning tree.
2. For each tree $G$ and each edge $e$ of $G$, either $G - e$ has an isolated node or there is another tree $H$ containing $G - e$.
3. If two trees $G$ and $H$ contain perfect matchings on $I \sqcup J$ for $I \subseteq \mathbb{R}$ and $J \subseteq E$ with $|J| = |I|$, then the matchings agree.
Starting from a triangulation of $\triangle_{d-1} \times \triangle_{n-1}$, we collect all $R$-saturating matchings (those covering all nodes in $R$) that appear as subgraphs of the trees corresponding to the simplices. By (3) of Proposition 2.9, there is at most one matching for each $d$-subset of $E$. Considering the barycentre of the subpolytope corresponding to $K_{R,\sigma}$ shows that there is indeed a matching for each $d$-subset [38, Proposition 2.5].

**Definition 2.10** (Polyhedral matching field). A polyhedral matching field is the collection of all $R$-saturating matchings appearing in the trees encoding $\triangle_{d-1} \times \triangle_{n-1}$.

**Example 2.11.** Figure 4 depicts four trees comprising a triangulation of $\triangle_1 \times \triangle_3$. The $2 \times 2$ matchings arising in these trees yield the matching field shown in Figure 1.

![Figure 4](image)

**Figure 4.** Trees corresponding to maximal simplices in a triangulation of $\triangle_1 \times \triangle_3$.

Using [38, Theorem 3.16], one can construct the collection $\mathcal{I} := \mathcal{I}(\triangle_{d-1} \times \triangle_{n-1})$ of trees such that each node in $R$ has degree at least 2 by iteratively taking linkage covectors. On the other hand, for $n > d$, all the matchings of the matching field occur in at least one of these trees as one can see from the barycentre construction. These trees have another nice property which will be useful later.

**Lemma 2.12.** Every tree $T$ in $\mathcal{I}$ contains an $R$-saturating matching.

**Proof.** Since each node in $R$ has degree bigger than 1, there is a leaf $\ell$ of $T$ in $E$. Include the incident edge $(i, \ell)$ in the matching and iterate the procedure with $T - \{i, \ell\}$. This is possible as all remaining nodes in $R$ are still of degree at least 2. \qed

Every coherent matching field is the polyhedral matching field extracted from a regular triangulation of $\triangle_{d-1} \times \triangle_{n-1}$, induced by the same weight matrix. On the other hand, all polyhedral matching fields are linkage [38, Section 3.1]. We see in Example 3.14 that the linkage matching field in Figure 3 is not polyhedral.

A special class of polyhedral matching fields are the pointed ones (following the terminology in [55]), which comprise the full information of a triangulation of $\triangle_{d-1} \times \triangle_{n-1}$. We augment the ground set $E$ by a copy $\tilde{R}$ of $R$ to obtain a ground set $\tilde{E}$ of size $n + d$, and we set all elements of $\tilde{R}$ to be smaller than all elements of $E$. To take the full information of all trees into account, for each tree in $\mathcal{T}$ we add edges between each node in $R$ to its copy $\tilde{R}$, yielding a set $\tilde{\mathcal{T}}$ of trees on $R \sqcup \tilde{E}$.

**Definition 2.13.** The pointed polyhedral matching field associated with a triangulation $\mathcal{T}$ of $\triangle_{d-1} \times \triangle_{n-1}$ is the collection of $R$-saturating matchings on $R \sqcup \tilde{E}$ in the trees in $\tilde{\mathcal{T}}$. 
The discussion after [38, Theorem 3.16] shows that the latter construction actually yields a correspondence between triangulations and matching fields.

One can see that pointed polyhedral matching fields are actually polyhedral matching fields by extending the original triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ to a triangulation of $\Delta_{d-1} \times \Delta_{n+d-1}$ by using a placing triangulation as in [10, Lemma 4.3.2]. Note that, on the other hand, each polyhedral matching field is a sub-matching field of a pointed polyhedral matching field. We use both points of view as they allow different constructions as we see in Section 3.6 and 4. It is similar to the relation between transversal matroids and fundamental transversal matroids, and it is reminiscent of the correspondence between matroids and linking systems [50].

Cutting appropriately through a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ yields a fine mixed subdivision of $n\Delta_{d-1}$. This is formalized as the Cayley trick, see [49]. A direct way to identify the polyhedral pieces in such a subdivision of $n\Delta_{d-1}$ is the following. For each tree $G$ corresponding to a simplex in the triangulation $\mathcal{T}$, we form the Minkowski sum

$$\sum_{j \in E} \text{conv}\{e_i : i \in N_G(j)\},$$

where $N_G(j)$ is the neighbourhood of an element $j \in E$ in $G$. The collection of these Minkowski sums tiles the dilated simplex $n\Delta_{d-1}$.

The dual polyhedral complexes to these mixed subdivisions were introduced as tropical pseudohyperplane arrangements in [8]. This draws from the correspondence between tropical hyperplane arrangements and regular subdivisions of products of simplices established in [17]. After starting from an axiomatic study of tropical oriented matroids in [8], it was subsequently shown that these combinatorial objects are indeed cryptomorphic to subdivisions of $\Delta_{d-1} \times \Delta_{n-1}$ and tropical pseudohyperplane arrangements, partially in [43] and finished in [31].

![Figure 5](https://via.placeholder.com/150)

**Figure 5.** A triangulation of $\Delta_1 \times \Delta_2$. The vertices are labeled by the corresponding edges in $K_{2,3}$. This picture was created with polymake [23].

## 3. Polyhedral Matching Fields Induce Oriented Matroids

### 3.1. Matroid Subdivisions from Triangulations

We first briefly recall the definition of a matroid subdivision.
Definition 3.1. A collection of matroids is a matroid subdivision of a matroid $M$ if they have the same ground set and rank as $M$, and their matroid polytopes (together with their faces) form a subdivision of the matroid polytope of $M$.

In the proof of [53, Proposition 2.2], the argument for (1) $\Rightarrow$ (2) does not depend on the actual tropical Plücker vector but only on the induced subdivision of the octahedron. Together with the equivalence with (3), this shows the following result, which also applies to non-regular subdivisions. The proof of our main theorem in the next section is in a similar spirit.

Proposition 3.2. A polyhedral subdivision of matroid polytope is a matroid subdivision if and only if the induced subdivision of each octahedral face is a matroid subdivision.

The construction of matroid subdivisions from triangulations of products of simplices goes back to [36] and was further refined in [19, 29, 47]. We take a more combinatorial point of view described in [38]. Given a subgraph $G$ of $K_{R,E}$, the transversal matroid $M(G)$ is the matroid whose bases are the subsets of $E$ which are matched by some $R$-saturating matching of $G$. Starting from the set of all trees corresponding to the full-dimensional simplices in a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$, we restrict to the subset $I$ of all trees where each node in $R$ has degree at least 2. By Lemma 2.12, the transversal matroid of each such tree has rank $d$.

Theorem 3.3 ([19, §5.2], [29, §2], [36, 47]). The matroids $M(T)$ ranging over all $T \in I$ form a matroid subdivision of $U_{d,n}$.

Example 3.4 (Matroid subdivisions of an octahedron). An octahedron is the matroid polytope of the uniform matroid $U_{2,4}$. There are exactly two types of matroid subdivisions of an octahedron. The trivial subdivision has only the octahedron as a cell. Apart from that, one can subdivide it into two pyramids as depicted in Figure 6; there are three such subdivisions. While one can also subdivide the octahedron into four tetrahedra, it is not a matroid subdivision by the edge direction criterion.

The non-trivial subdivision in Figure 6 arises through the construction of Theorem 3.3 from Example 2.10 and Figure 4. We restrict our attention to the two trees with $R$-degree vector $[2, 3]$ and $[3, 2]$. The combinatorial Stiefel map gives the two sets $\{12, 13, 14, 23, 24\}$ and $\{13, 14, 23, 24, 34\}$, which form a matroid subdivision of $U_{2,4}$.

![Figure 6](image-url)
Recall from Section 2.3 the construction of the set $\tilde{T}$ of trees on $R \cup \tilde{E}$. For each tree $T$ in $\tilde{T}$, we get an induced transversal matroid $\tilde{M}(T)$ on $\tilde{E} = R \cup E$. In particular, the trees give rise to a matroid subdivision of $U_{d,d+n}$.

**Corollary 3.5.** The matroids $\tilde{M}(T)$ ranging over all trees $T$ comprising a triangulation of $\triangle_{d-1} \times \triangle_{n-1}$ form the maximal cells for a matroid subdivision of $U_{d,d+n}$.

There is another way to think about this subdivision more geometrically. Following [47, §4], all cells in this subdivision are principal (or fundamental) transversal matroids with fundamental basis $\tilde{R}$. Indeed, one can obtain this subdivision concretely as follows: In the matroid polytope of $U_{d,d+n}$, the vertices adjacent to $e_{\tilde{R}}$ are in natural bijection with the vertices of $\triangle_{d-1} \times \triangle_{n-1}$ and their convex hull is $(-\triangle_{d-1}) \times \triangle_{n-1}$ up to translation. Triangulate the convex hull as the initial triangulation, and cone over the cells from $e_{\tilde{R}}$. The matroid subdivision in Corollary 3.5 is the intersection of these cones with the matroid polytope.

### 3.2. Proof of Theorem A

Fix a sign matrix $A \in \{-, +\}^{R \times E}$ and a polyhedral matching field $(M_\sigma)$ extracted from the special trees $I$ of a triangulation of $\triangle_{d-1} \times \triangle_{n-1}$.

**Definition 3.6 (Sign of a matching).** Let $M_\sigma$ be a perfect matching between $R$ and $\sigma \subset E$ in $K_{R,E}$. Define $\text{sign}(M_\sigma)$ to be the sign of the permutation

$$[d] \rightarrow R \rightarrow \sigma \rightarrow [d],$$

where the first and last maps are order-preserving bijections, with $\sigma$ inheriting the order from $E$, and the middle map is the bijection determined by $M_\sigma$.

With the terminology in place we restate our first main Theorem A.

**Theorem 3.7.** The sign map $\chi : (E_d) \rightarrow \{+, -\}$ given by

$$(3) \quad \sigma \mapsto \text{sign}(M_\sigma) \prod_{e \in M_\sigma} A_e,$$

is the chirotope of an oriented matroid.

Let $T$ be a tree in $I$ and let $A(T)$ be any matrix obtained by setting every entry of $A$ not in $T$ to 0 and replacing every remaining non-zero entry by an arbitrary real number of the same sign. Furthermore, let $\chi_T$ be the map $\chi$ restricted to the bases of the transversal matroid $M(T)$.

**Lemma 3.8.** The map $\chi_T$ is a chirotope of rank $d$ realized by $A(T)$.

**Proof.** By Lemma 2.12, $\chi_T$ is non-zero. If $\sigma \in \binom{E_d}{d}$ is not a basis of the transversal matroid, then there are no perfect matchings between $R$ and $\sigma$ in $T$, thus $\det(A(T)|\sigma)$ is 0. Otherwise there is a unique matching, namely $M_\sigma$, between $R$ and $\sigma$ in $T$. This matching corresponds to the unique non-zero term in the expansion of $\det(A(T)|\sigma)$, whose sign coincides with $\chi(\sigma)$ by definition.

**Theorem 3.9.** Let $M_1, \ldots, M_k$ be a matroid subdivision of some matroid $M$. Let $\chi$ be an alternating sign map supported on the bases $B(M)$. Suppose each restriction $\chi_i : B(M_i) \rightarrow \{+, -\}$ is a chirotope. Then $\chi$ itself is a chirotope.
Proof. Suppose \( \chi \) is not a chirotope. Since \( \chi \) is by assumption a matroid, by Proposition 2.4 there exist \( x_1, x_2, y_1, y_2, x_3, \ldots, x_d \in E \) such that (2) is violated. It is routine to check that all \( d + 2 \) elements must be distinct here, and the six \( d \)-element subsets involved in (2) form three “antipodal” pairs. Moreover, at least one such pair consists of bases of \( M \) as at least a term in (2) is non-zero.

Construct a vector \( \mathbf{c} \in \mathbb{R}^E \) where \( c(x_1) = c(x_2) = c(y_1) = c(y_2) = 1 \), \( c(x_i) = 0 \), \( \forall i = 3, \ldots, d \), and \( c(\ell) \)’s are sufficiently large numbers for every other \( \ell \in E \). Consider the face \( F_c \) of the matroid polytope of \( M \) that minimizes \( x \mapsto \mathbf{c}^\top x \). A vertex of the matroid polytope of \( M \) achieves the minimum if and only if it corresponds to one of the aforementioned subsets while being a basis of \( M \). \( F_c \) is a matroid polytope and the global matroid subdivision restricts to a matroid subdivision of it. We distinguish two cases depending on the shape of \( F_c \):

Case I: \( F_c \) is an octahedron, i.e., all three terms of (2) are non-zero. There are four possible matroid subdivisions of an octahedron as discussed in Ex. 3.4, the trivial one and three that subdivide it into two pyramids. In either case, there is a 3-dimensional cell \( C' \) that contains at least two pairs of antipodal vertices, pick a full-dimensional cell \( C := C_{M_i} \) of the global matroid subdivision that contains \( C' \). Then restricting \( \chi \) to \( M_i \) means that we are setting at most one term out of the three terms in (2) to zero, which still yields a violation of the 3-term GP relation in \( \chi_{M_i} \).

Case II: \( F_c \) is not an octahedron (thus either a square or a pyramid). The only possible matroid subdivision of \( F_c \) is the trivial one. Similar to Case I, we pick a cell \( C := C_{M_i} \) containing \( F_c \), and \( \chi_{M_i} \) will violate (2) exactly like \( \chi \). \( \square \)

**Proof of Theorem 3.7:** By Corollary 3.5 there is a matroid subdivision of the uniform matroid \( U_{d,n} \) consisting of transversal matroids, one for each spanning tree in \( \mathcal{T} \). By Lemma 3.8, the sign maps \( \chi_{T_i} \)’s are chirotopes. Hence \( \chi \) itself is a chirotope by Theorem 3.9. \( \square \)

**Example 3.10.** The diagonal matching field depicted in Figure 1 with the sign-matrix

\[
\begin{pmatrix}
+ & - & + & - \\
+ & - & - & +
\end{pmatrix}
\]

gives rise to the chirotope

\[
(12, -), (13, -), (14, +), (23, +), (24, -), (34, +)
\]

This is shown in Figure 7.

**Remark 3.11.** Instead of the full polytope \( \triangle_{d-1} \times \triangle_{n-1} \), one could equally start with a subpolytope of it. The whole construction then yields a chirotope for the transversal matroid of the subgraph of \( K_{d,n} \) corresponding to the subpolytope. However, it is no restriction to study only the full polytope as each triangulation can be extended to the full product of simplices. This is further explained in [16] and (also with signs) in [37].

**Example 3.12.** The assumption that \( A \) is a full sign matrix in Theorem 3.7 is important. Consider the following sign matrix together with the (3,5)-diagonal
Example 3.13. We note that the converse of Theorem 3.9 is not true in general. Consider the assignment of signs $(12, +), (13, +), (14, +), (23, +), (24, -), (34, -)$, together with the matroid subdivision as in Figure 7. It is routine to check that the assignment is a valid chirotope. However, when restricted to the upper piece (the pyramid containing the basis 12 but not 34), the assignment no longer satisfies the 3-term GP relation.

Example 3.14. As the linkage property was introduced in [55] in analogy with the basis exchange axiom for matroids, it is tempting to assume that a linkage matching field gives rise to an oriented matroid, generalizing our construction for polyhedral matching fields. However, the linkage but not polyhedral matching field depicted in Figure 3 shows that this is not correct. Take the sign matrix to be all positive. We have the following matchings (each tuple is a subset $\{i_1 < i_2 < i_3\} \in \binom{[3]}{3}$ followed by the permutation $\sigma(i_1), \sigma(i_2), \sigma(i_3)$ of $[3]$ and the sign of the permutation):

- $125$ $123$ $+$
- $135$ $132$ $-$
- $145$ $123$ $+$
- $235$ $231$ $+$
- $245$ $231$ $+$
- $345$ $231$ $+$. 

The support of the induced $\chi$ is $\{123, 125, 145, 245\}$, which is not the collection of bases of any matroid.

Figure 7. A matroid subdivision with signs induced by the trees and signs from Example 3.10.
Take \((x_1, x_2, y_1, y_2, X) = (1, 2, 3, 4, \{5\})\) in (2). Since all products
\[
\chi(1, 2, 5) \cdot \chi(3, 4, 5) = + \cdot + = + \\
\chi(1, 3, 5) \cdot \chi(4, 2, 5) = - \cdot - = + \\
\chi(1, 4, 5) \cdot \chi(2, 3, 5) = + \cdot + = +
\]
are positive, this contradicts the 3-term GP relation.

We elaborate more on this. A linkage matching field also determines a collection of spanning trees [38], but those trees might contain new matchings. For example, in [38, Fig. 11], the top tree and the left tree contain different matchings on 3456. In such a case, while each tree still yields a chirotope, their values need not agree. The values on some of the bases correspond to distinct matchings which means geometrically that these chirotopes can not be “glued” together.

3.3. Compatibility of chirotopes. It is interesting to ask when does the converse of Theorem 3.9 hold. In particular, in light of Example 3.13, this asks for a classification of those matroid subdivisions which are compatible with a prescribed chirotope. For example, in the very recent works on positive Dressians and positive tropical Grassmannians [9, 39, 52], they consider the following special case of the problem: Assign + to all (increasingly ordered) bases of some uniform matroid (it is known that such a sign map is a chirotope, cf. Example 5.5). Given a matroid subdivision of that matroid, when does the restriction of this chirotope to every piece also form a chirotope? These works mostly consider regular matroid subdivisions, but in [52] they also constructed a non-regular matroid subdivision of \(U_{3,12}\) by transversal matroids with such property.

Using a similar approach as in the proof of Theorem 3.9 we provide a criterion for deciding if, given a chirotope and a matroid subdivision of some matroid, its converse holds.

Let \(\chi\) be a chirotope supported on \(U_{2,4}\). There are exactly two terms among \(\chi(1, 2)\chi(3, 4), -\chi(1, 3)\chi(2, 4), \chi(1, 4)\chi(2, 3)\) having the same sign, hence exactly one matroid subdivision of \(U_{2,4}\) (necessarily non-trivial) does not satisfy the converse of Theorem 3.9 (see Example 3.13). We call such a subdivision forbidden with respect to \(\chi\).

**Proposition 3.15.** Let \(\chi\) be a chirotope supported on some matroid \(M\) and let \(M_1, \ldots, M_k\) be a matroid subdivision of \(M\). The restrictions \(\chi_i\)'s of \(\chi\) to \(M_i\)'s are all chirotopes if and only if the restriction of the matroid subdivision to every octahedron face is not forbidden.

**Proof.** Suppose the restriction of the matroid subdivision to an octahedron face is forbidden with (3-dimensional) cells \(C, C'\). Pick a full-dimensional cell \(M_i\) that contains \(C\), then the 3-term GP relation corresponding to the vertices of \(C\) is violated in \(M_i\).

Conversely, suppose some 3-term GP relation (2) is violated in \(M_i\); denote by \(F\) and \(F'\) the faces of (the matroid polytopes of) \(M\) and \(M_i\) that contain bases involved in (2), respectively. Since \(\chi\) is a chirotope but \(\chi_i\) is not, some bases of \(M\) involved in (2) must be non-bases of \(M_i\), so we have \(F' \subsetneq F\). Suppose \(F\) is an octahedron. \(F'\) is either a pyramid or a square, and \(F\) must have been subdivided in the matroid subdivision. If \(F'\) is a pyramid, then the subdivision of \(F\) is by definition forbidden; if \(F'\) is a square, pick any pyramid of the subdivision and the 3-term GP relation is still violated there, so the subdivision is also forbidden.
Now suppose $F$ is a pyramid. $F'$ must be a square, but in such case $F$ would have violated the 3-term GP relation as $F'$ does.

Proposition 3.15 is quite hard to check algorithmically, but if $\chi$ has a nice description, one might expect to understand the situation better by analyzing the relative position of the octahedron faces together with their forbidden subdivisions. For example, [52, Theorem 4.3] is a special case of our criterion.

3.4. Signed Circuits and Cocircuits. Knowing that $\chi$ is a chirotope, one can use the equivalence of axiom systems as described in [14, Chapter 3] to convert $\chi$ into other objects associated to the oriented matroid. However, it is also possible to construct many of these objects more directly using graphs in the theory of matching fields. We shall describe the construction of signed circuits and cocircuits here, while the next subsection will study covectors in general. In particular, linkage covectors form the analogue of signed circuits and Chow covectors form the analogue of signed cocircuits. Observe that a matching field can be recovered from linkage covectors and equally well from Chow covectors in analogy to the cryptomorphic description of an oriented matroid by signed circuits or signed cocircuits.

Recall that for every $(d + 1)$-subset $\tau \subset E$, the linkage tree $T_{\tau} := \bigcup_{s \subset \tau} M_s$ is a spanning tree of $K_{R, \tau}$ in which every node in $R$ has degree 2. One can then define an auxiliary tree $\tilde{T}_{\tau}$ whose node set is $\tau$ and two nodes are connected by an edge if they are both incident to the same $r \in R$ in $T_{\tau}$. Note that this is the same as identifying a node $j$ in $\tau$ with the matching on $R \cup (\tau \setminus \{j\})$ and connecting two matchings if they only differ by one edge. Such a tree $\tilde{T}_{\tau}$, together with an edge labeling using $R$, is a linkage tree in [55]. We equip $\tilde{T}_{\tau}$ with a sign labeling in the following way. To the edge $(u, v) \in \tilde{T}_{\tau}$ associated to the two edges $(u, r), (v, r) \in T_{\tau}$, we assign the negative of the product of the signs on $(u, r)$ and $(v, r)$.

Proposition 3.16. For any $\tau$, there exist precisely the two sign assignments of $\tau$ such that two adjacent nodes in $\tilde{T}_{\tau}$ are of the same sign if and only if the edge between them is positive. These are the two signed circuits supported on $\tau$.

Proof. Pick an arbitrary root for $\tilde{T}_{\tau}$ and choose a sign for it, then the sign of every other node is uniquely determined by its parent. We show that a signed circuit with respect to $\chi$ satisfies the constraint described in the statement.

Let $C$ be a signed circuit supported on $\tau$. Write $R = \{r_1, \ldots, r_d\}, \tau = \{e_1 < \ldots < e_{d+1}\}$ and let $x = e_i, y = e_j$ be any two nodes. By [14, Theorem 3.5.5],

$$C(e_j) = -C(e_i)\chi(e_i, e_2, \ldots, \hat{e_i}, \ldots, e_{d+1})\chi(e_j, e_2, \ldots, \hat{e_j}, \ldots, e_{d+1}).$$

By permuting elements in $R$ and $\tau$ if necessary, we may assume $i = 1, j = 2$, and $M_{R \setminus \{e_1\}}$ matches each $r_k$ to $l_{k+1}$. We first assume that $x, y$ are adjacent in $\tilde{T}_{\tau}$, in particular, $M_{R \setminus \{e_2\}} = M_{R \setminus \{e_1\}} \setminus \{r_1e_2\} \cup \{r_1e_1\}$. If the edges $r_1e_1$ and $r_1e_2$ have the same sign, then $-\chi(e_1, e_3, \ldots, e_{d+1})\chi(e_2, \ldots, e_{d+1}) = -\text{sign}(M_{R \setminus \{e_2\}})\text{sign}(M_{R \setminus \{e_1\}}) = -1$ as the two chirotopes have the same products of signs of edges. Now $-1 = -A_{r_1e_1}A_{r_1e_2}$ is what we have labeled the edge between $x, y$ in $\tilde{T}_{\tau}$. The opposite case when $r_1e_1$ and $r_1e_2$ having opposite signs is similar, and we can extend the comparison of signs to any pair of nodes by induction.

Now we consider signed cocircuits of our oriented matroid. Recall that cocircuits correspond to vertices in an arrangement encoding an oriented matroid. These appear in the framework of abstract tropical linear programming [37] as certain...
trees with prescribed degree sequence, see also Remark 3.34. The statement of [45, Theorem 12.9] identifies the correct tree for our purpose.

**Proposition 3.17.** For any \((n - d + 1)\)-subset \(\rho\), there exists a unique spanning tree \(T_\rho\), encoding a cell in the triangulation, such that every node in \(\rho\) is of degree 1 and every node in \(E \setminus \rho\) is of degree 2.

We say we flip a node in \(R\) if we negate the row of \(A\) indexed by the node.

**Proposition 3.18.** For any \(\rho\), there exist precisely two flippings of \(R\) such that every node of \(E \setminus \rho\) is incident to edges of different signs in \(T_\rho\). In each case, the signs of the edges incident to \(\rho\) together give a signed cocircuit supported on \(\rho\).

It is possible to give a combinatorial proof in the same vein of Proposition 3.16, but as cocircuits are special instances of covectors, we apply the result in Section 3.6 (with no circular argument). In particular, we give a more formal definition of flips there. We put a separate discussion here because of the extra structural properties of cocircuits.

**Proof.** We construct an auxiliary tree \(\tilde{T}_\rho\) whose node set is \(R\). Two nodes are connected by an edge if they are both incident to some node in \(E \setminus \rho\) in \(T_\rho\) and we label the edge by the said node in \(E \setminus \rho\). Pick an arbitrary root for \(\tilde{T}_\rho\) and choose a flipping decision for it, then the flipping decision of every other node is uniquely determined by its parent. Apply Corollary 3.33 to \(T_\rho\) and any of the two flipping decisions, we know that the signs of the edges incident to \(\rho\) is a covector of the oriented matroid supported on \(\rho\), i.e., it is a cocircuit. 

**Remark 3.19.** Perhaps a more intuitive way to understand Proposition 3.18 is to use the geometric picture in Section 4. The mixed cell representing \(T_\rho\), reflected to the correct orthant in the patchworking complex specified by the flipping, is dual to the intersection of \(d - 1\) pseudo-hyperplanes. Thus it recovers the intuition from classical hyperplane arrangements.

**Example 3.20.** We illustrate the construction of the signed circuits and cocircuits by continuing Example 3.10. The left graph in Figure 8 shows the linkage covector \(T_\tau\) for \(\tau = \{1, 2, 3\}\) used in Proposition 3.16. The signs on the nodes in \(R\) are chosen negative so that one can think of multiplying the signs along paths instead of forming a signed linkage tree as constructed before Proposition 3.16.

The right graph in Figure 8 depicts the spanning tree \(T_\rho\) for \(\rho = \{1, 2, 4\}\) as constructed in Proposition 3.18. The sign vector on \(R\) corresponds to the flippings. Note that the two sign vectors \((+, +)\) and \((-,-)\) on the nodes in \(R\) correspond to the two orthants in which the \((\text{pseudo-})\)spheres corresponding to the elements in \(E \setminus \rho\) intersect.

We give a further interpretation of the trees from Proposition 3.17 in terms of **Chow covectors**. These were introduced in [55, Eqn. 5.1] and the name was coined in [38].

**Proposition 3.21.** The collection of edges \(\Omega_\rho\) incident to \(\rho\) in \(T_\rho\) is the Chow covector supported on \(\rho\).

**Proof.** We claim that there is a (unique) perfect matching \(\tilde{M}_{R'}\) between any \((d - 1)\)-subset \(R' \subset \bar{R}\) and \(E \setminus \rho\) in \(T_\rho\): set the missing node of \(R'\) as the root of \(\tilde{T}_\rho\) in the
proof of Proposition \ref{prop:chow} and match every other node with (the label of) the edge towards its parent. From this, for any \( x \in \rho \) with adjacent node \( r \), \( \tilde{M}_{R \setminus \{x\}} \cup \{x\} \) is a perfect matching between \( R \) and \( \mathcal{E} \setminus \rho \cup \{x\} \). This gives rise to the matchings required in the definition \cite{Bjorner:2000} Eqn. 5.1] of a Chow covector. \( \square \)

The next corollary follows from \cite[Theorem 1]{Mathai:1999}, and is a matching field analogue of the matroidal fact that cocircuits are precisely the minimal transversals of bases.

**Corollary 3.22.** The minimal transversals of a polyhedral matching field are given by the leafs of \( T_{\rho} \) in \( \rho \) where \( \rho \) ranges over all \( (n-d+1) \)-subsets.

We also have a matching field analogue of the orthogonality of circuits and cocircuits, i.e., a circuit can not intersect a cocircuit by exactly one element. Let \( \tau, \rho \subset \mathcal{E} \) be of size \( d+1 \) and \( n-d+1 \), respectively, and let \( T_{\tau} \) and \( T_{\rho} \) be the trees constructed for Prop. \ref{prop:tree} and Prop. \ref{prop:chow} respectively.

**Proposition 3.23.** \( T_{\tau} \) contains at least two leafs of \( T_{\rho} \).

**Proof.** Pick an arbitrary \( \sigma \subset \tau \) of size \( d \). By Corollary \ref{cor:transversals}, \( M_{\sigma} \subset T_{\tau} \) contains at least an edge \( re \) of \( T_{\rho} \). Now \( M_{\tau \setminus \{e\}} \subset T_{\tau \setminus \{re\}} \) must contain at least another. \( \square \)

3.5. Duality. Another important concept for oriented matroids is duality. Recall from Example \ref{ex:chirotope} that an oriented matroid arising from a rectangular matrix can be represented in the form \((I_d \mid B)\), where we let \( B \) be a \( d \times n \) matrix for now. Following \cite[§7.2.5]{Edmonds:1970}, its dual oriented matroid is represented by \((-B^\top \mid I_n)\). We mimic this by fixing a triangulations \( \mathcal{T} \) of \( \Delta_{d-1} \times \Delta_{n-1} \) as well as a sign matrix \( A \in \{-1,+1\}^{d \times n} \), and constructing a pair of pointed polyhedral matching fields that gives a dual pair of oriented matroids. This illustrates an advantage of considering pointed matching fields: they allow a more satisfactory duality theory.

The paragraph on “Duality” in [14] §3.5] gives the following condition for two chirotopes to be dual.

**Lemma 3.24.** Let \( \mathcal{E} \) be a ground set of size \( d+n \). Let \( \chi: \mathcal{E}^d \to \{+,\,-,0\} \) and \( \chi': \mathcal{E}^n \to \{+,\,-,0\} \) be two chirotopes of a uniform oriented matroid of rank \( d \) and \( n \), respectively. They are dual if and only if for each ordering \((x_1,\ldots,x_{d+n})\) of the ground set \( \mathcal{E} \) we have

\[
\chi(x_1,\ldots,x_d) \cdot \chi'(x_{d+1},\ldots,x_{d+n}) = \text{sign}(x_1,\ldots,x_{d+n}),
\]

where the latter sign is the sign of the permutation represented by the ordering of the \( x_k \).
As mentioned for the definition of a pointed polyhedral matching field (Def. 2.13), one has to take the information of all matchings occurring in the trees of $\mathcal{T}$ into account. This corresponds to looking at all minors of $B$. Equally, one can associate two different pointed polyhedral matching fields with $\mathcal{T}$ by swapping the factors of the product of simplices. This leads to a pointed polyhedral matching field $(M_\sigma)$ of size $(d, n + d)$ and $(N_\sigma)$ of size $(n, d + n)$.

$(M_\sigma)$ and $(N_\sigma)$ are related combinatorially as follows. Let $\mu$ be a matching on $R \sqcup E$. Augment $R$ by another copy $\overline{E}$ of $E$ and $E$ by another copy $\overline{R}$ of $R$. This leads to the two node sets $U = R \cup \overline{E}$ and $V = \overline{R} \cup E$, each union in this order. Next, introduce another copy $\mu'$ between $E$ and $R$ by adding the edge $(j, i)$ for each edge $(i, j)$ in $\mu$. Then, connect each isolated node to its copy. We obtain a perfect matching $\tau$ on $U \sqcup V$, which can be thought as a permutation in $S_{d+n}$. The two restricted matchings $\tau|_R$ and $\tau|_E$ form a dual pair of matchings in $(M_\sigma)$ and $(N_\sigma)$, respectively. The terminology is justified as they have complementary supports in $V$. The notation is visualized in Figure 9.

Finally, we let $\chi_1$ be the chirotope induced by the matching field $(M_\sigma)$ on $R \sqcup V = R \cup (R \cup E)$ with the sign matrix $(I_{d,d} | A)$ and let $\chi_2$ be the chirotope induced by the matching field $(N_\sigma)$ on $E \sqcup V = E \cup (R \cup E)$ with the sign matrix $(-A^\top | I_{n,n})$. This can also be interpreted as the sign matrix

$$H = \begin{pmatrix} I_{d,d} & A \\ -A^\top & I_{n,n} \end{pmatrix}$$

on the bipartite graph $K_{d+n, d+n}$.

**Proposition 3.25.** The two chirotopes $\chi_1$ and $\chi_2$ describe a dual pair of oriented matroids.

**Proof.** Let an arbitrary partition of $V \cong [d + n]$ into two ordered sets $P_1 := \{x_1 < \cdots < x_d\}$ and $P_2 := \{x_{d+1} < \cdots < x_{d+n}\}$ be given. This defines the bijection $\pi$ in $S_{d+n}$ which maps $(x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+n})$ to $(1, 2, \ldots, d + n)$.

Let $\mu_1$ be the matching in $(M_\sigma)$ on $P_1$ and let $\mu_2$ be the matching in $(N_\sigma)$ on $P_2$. By construction, their union is a perfect matching $\tau$ on $U \sqcup V$, coming from a matching $\mu$ on $R \sqcup E$. Using Definition 3.6 and (3), we obtain

$$\chi_1(P_1) = \text{sign}(\pi \cdot \tau|_R) \prod_{e \in \tau|_R} H_e \quad \text{and} \quad \chi_2(P_2) = \text{sign}(\pi \cdot \tau|_E) \prod_{e \in \tau|_E} (-H_e).$$
As the concatenation \( \pi \cdot \tau \) of the matchings \( \tau \) and \( \pi \) is disjoint on \( \mathbb{R} \) and \( \mathbb{E} \), we obtain
\[
\chi_1(P_1) \cdot \chi_2(P_2) = \text{sign}(\pi) \text{sign}(\tau) \prod_{e \in \tau_{\mathbb{E}}} (-H_e) \prod_{e \in \tau_{\mathbb{R}}} H_e.
\]

Using the definition of \( \tau \) based on \( \mu \), we get
\[
\prod_{e \in \tau_{\mathbb{E}}} (-H_e) \prod_{e \in \tau_{\mathbb{R}}} H_e = \prod_{e \in \mu} (-H_e) \prod_{e \in \mu} H_e = (-1)^{|\mu|}.
\]

Now we compute the sign of \( \tau \) by counting inversions. Every inversion in \( \mu \) is paired up with an inversion within \( \pi \) and vice versa, while there are \(|\mu|^2 \equiv |\mu| \mod 2 \) many inversions between \( \mu \) and \( \pi \), so \( \text{sign}(\tau) = (-1)^{|\mu|} \). Since \( \pi \) is just the inverse of the permutation encoded by \( (x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+n}) \), the proposition follows from Lemma 3.24.

3.6. **Topes and Covectors.** In the development of tropical oriented matroids, the tropical analogue of *topes* and *covectors* played an important role. We adapt the notion to our terminology and provide the connection with the oriented matroid \( \mathcal{M} \) represented by the chirotope \( \chi \).

Fix a polyhedral matching field \( (M_\sigma) \) extracted from the special trees \( T \) of a triangulation \( T \) of \( \Delta_{d-1} \times \Delta_{n-1} \), and a sign matrix \( A \in \{-1,1\}^{\mathbb{R} \times \mathbb{E}} \). A *covector graph* is a subgraph of a tree in \( T \) with no isolated node in \( \mathbb{E} \). A *tope graph* is a covector graph for which each node in \( \mathbb{E} \) has degree 1; it is called *inner tope graph*, if it is actually a subgraph of a tree in \( T \).

**Definition 3.26.** Given \( S \in \{-1,0,1\}^R \) and \( F \subseteq \mathbb{R} \times \mathbb{E} \), we define the sign matrix \( SA_F \in \{-1,0,1\}^{R \times \mathbb{E}} \) by
\[
(SA_F)_{i,j} = \begin{cases} 
S_iA_{i,j}, & (i,j) \in F, \\
0, & \text{otherwise.}
\end{cases}
\]

**Definition 3.27.** Given a subgraph \( F \) of \( K_{R,E} \) and a sign vector \( S \in \{-1,0,1\}^R \), define the sign vector \( \psi_A(S,F) = Z \in \{-1,0,1\}^E \), where
\[
Z_j = \begin{cases} 
0, & \text{column } j \text{ of } SA_F \text{ contains positive and negative entries, or all zeros} \\
1, & \text{column } j \text{ of } SA_F \text{ contains only non-negative entries} \\
-1, & \text{column } j \text{ of } SA_F \text{ contains only non-positive entries.}
\end{cases}
\]

**Example 3.28.** The graphs in Figure 10 together with the sign vector \( S = (0, -1, 1) \) and sign matrix \( A = (I_{3,3}|A) \) with
\[
A = \begin{pmatrix}
1 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & -1 & -1
\end{pmatrix}
\]
give rise to the matrices
\[
F = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & -1
\end{pmatrix}
\quad \text{and} \quad
T = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1
\end{pmatrix}.
\]

Their images are
\[
X = \psi_A(S,F) = (0, -1, 1, 0, 1, -1) \quad \text{and} \quad
Y = \psi_A(S,F) = (-1, -1, 1, 1, 1, -1).
\]
These sign vectors fulfill $Y \geq X$ and highlight the construction used in the proof of Proposition 3.32.

Recall the notion of a weak map: we say there is a weak map $\mathcal{M} \hookrightarrow \mathcal{N}$ between two oriented matroids $\mathcal{M}, \mathcal{N}$ on $E$, if for all covectors $X$ of $\mathcal{N}$, there exists a covector $Y$ of $\mathcal{M}$ such that $Y \geq X$. If $\mathcal{M}$ and $\mathcal{N}$ have the same rank, then this is the same as saying that, up to a global sign change, $\chi_{\mathcal{M}} \geq \chi_{\mathcal{N}}$ by [14, Proposition 7.7.5]. In particular, if $\chi_{\mathcal{N}}$ is a chirotope obtained by restricting $\chi_{\mathcal{M}}$ to a cell in some matroid subdivision of $\mathcal{M}$, such as in Lemma 3.8, then $\mathcal{M} \hookrightarrow \mathcal{N}$ is a weak map.

**Proposition 3.29.** For each sign vector $S \in \{-1, 1\}^R$ and each inner tope graph $F$, $\psi_A(S,F)$ is a tope of the oriented matroid given by the chirotope $\chi$.

**Proof.** Pick a tree $T \in I$ that contains $F$ and denote by $\mathcal{N}$ the oriented matroid representing $\chi_T$. By Lemma 3.8, $\mathcal{N}$ is realized by any matrix of the form $A(T)$, in which we choose $\pm 1$ for entries in $T$ but not $F$, and $\pm 2d$ for entries in $F$. This ensures that the sign pattern of the row space element $S^T \cdot A(T)$ equals $\psi_A(S,F)$, hence $\psi_A(S,F)$ is a tope of $\mathcal{N}$. From the discussion above, there is a weak map $\mathcal{M} \hookrightarrow \mathcal{N}$, so $\psi_A(S,F)$ is a tope of $\mathcal{M}$ as well. \[\square\]

To extend this correspondence to more general covectors, we make use of a result by Mandel.

**Theorem 3.30 ([14, Thm. 4.2.13]).** Let $\mathcal{T}_o$ be the set of topes of the oriented matroid $\mathcal{M}$. Then $X \in \{-1, 0, 1\}^E$ is a covector of $\mathcal{M}$ exactly if $X \circ \mathcal{T}_o \subseteq \mathcal{T}_o$.

Recall that for $X, Y \in \{-1, 0, 1\}^E$, the composition $X \circ Y$ agrees with $X$ in all positions $e \in E$ with $X_e \neq 0$, and agrees with $Y$ otherwise.

**Corollary 3.31.** Let $X \in \{-1, 0, 1\}^E$ be a sign vector such that every $Y \in \{-1, 0, 1\}^E$ satisfying $Y \geq X$ is a tope of $\mathcal{M}$. Then $X$ is a covector of $\mathcal{M}$.

We now consider the pointed polyhedral matching field $\widetilde{(M_\sigma)}$ encoding the starting triangulation, as well as the sign matrix $\tilde{A} := (I_{d,d} | A)$. By Theorem 3.7, they induce an oriented matroid $\tilde{\mathcal{M}}$. Let $F$ be a subgraph of a tree in $\tilde{\mathcal{T}}$ without isolated nodes in $E \subset \tilde{E}$ and such that a node in $\tilde{R} \subset \tilde{E}$ is isolated only if the corresponding node in $R$ is isolated as well. Furthermore, let $S \in \{-1, 0, 1\}^R$ be a sign vector whose support contains the set of non-isolated nodes of $F$ in $\tilde{R}$.

**Proposition 3.32.** The sign vector $\psi_{\tilde{A}}(S,F)$ is a covector of $\tilde{\mathcal{M}}$. 
Proof. By Corollary 3.31, it suffices to show that every full sign vector $Y \geq \psi_A(S, F)$ is a tope of $\tilde{M}$. This amounts to finding a full sign vector $S'$ and an inner tope graph $T$ such that $Y = \psi_A(S', T)$ by Proposition 3.29, since the extended trees in $\tilde{T}$ have degree at least 2 everywhere in $R$. We first construct $S'$ by setting each zero coordinate $i \in R$ of $S$ to $Y_i$, and we include those missing edges between $R$ and its copy $\tilde{R}$. Now for each coordinate $j \in E$ of $\psi_A(S, F)$, the $j$-th column of $S\tilde{A}\tilde{F}$ must contain an entry whose sign equals $Y_j$, so we remove all edges incident to $j$ except the one corresponding to that entry. This results in an inner tope graph $T$, and from our construction it is easy to see that $Y = \psi_A(S', T)$ as desired. \hfill \Box

Restricting the covectors to $E$ yields the following.

Corollary 3.33. For a covector graph $F$, $\psi_A(S, F)$ is a covector of $M$ for every sign vector $S$.

We shall see in Section 4 that the map is actually surjective.

Remark 3.34. Now that we can transfer the covector graphs to covectors of the oriented matroid $M$, we give a brief overview on the connection with simplex-like algorithms for tropical linear programming.

The iteration in the framework of abstract tropical linear programming \cite{37} is described in the language of trees encoding a triangulation. The main object in each iteration is a basic covector, which represents the analogue of a basic point in the simplex method. These are formed by a subclass of those trees from Proposition 3.17 and, from our new perspective, give rise to certain cocircuits of $M$.

One can go even further and consider the tropicalization of the simplex method \cite{2}. Using their genericity assumption, one sees that the covector graph of the basic point defined in \cite[Prop.-Def. 3.8]{2} is also such a tree. Their pivoting depends on the signs of the tropical reduced costs, which are deduced from a “Cramer digraph” \cite[§5]{2}. The latter can also be considered as covector graphs and hence give rise to covectors for $\chi$.

4. Patchworking Pseudosphere Arrangements

4.1. Patchworking Pseudolines on an Example. The classical theory of patchworking states that the structure of the real zero set of a polynomial in one orthant, parameterized by $t > 0$, is captured for sufficiently small $t$ by the regular triangulation of its Newton polytope induced by the exponents of $t$. Hence, one can recover the structure of the real zero set by gluing the triangulations for all orthants. This uses an appropriate assignment of signs to the vertices of the Newton polytope. By considering coherent fine mixed subdivisions, see Section 2.3, this was extended to complete intersections in \cite{54}.

We use patchworking of not-necessarily coherent fine mixed subdivisions of $n\triangle_{d-1}$ to derive a representation theorem for the oriented matroids induced from polyhedral matching fields. This can be seen as a generalization of the linear case of \cite[Thm. 4]{54} for generic hyperplane arrangements. While a complete intersection for generic hyperplanes would only yield one specific cell, the oriented matroid captures the information of all intersections in a generic hyperplane arrangement.

Example 4.1 is a toy example that illustrates our construction, which is generalized to larger $E$ and higher rank in this section.
Figure 11. Fine mixed subdivision of $3\Delta_2$ with cells labeled by their summands and sign; filled vertices denote ‘+’, empty ones ‘−’.

Figure 12. Pseudohyperplane arrangement derived from a fine mixed subdivisions of $3\Delta_2$ with signs indicated in Figure 11.

Example 4.1. We start with the regular triangulation of $\Delta_2 \times \Delta_2$ induced by the height matrix

$$H = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}.$$
This gives rise to the following height function on the lattice points of $3\triangle_2$:

$$(300; 5), (201; 6), (210; 5), (102; 6), (111; 5), (120; 3), (003; 4), (012; 4), (021; 3), (030; 0)$$

Here, the height of the lattice point $(p_1, p_2, p_3)$ is the weight of the maximal matching on $K_{3,3}$ for which the weight function is obtained from $H$ by taking $p_\ell$ copies of the $\ell$-th row of $H$. Note that, alternatively, the latter height function of the mixed subdivision can be obtained by multiplying the max-tropical linear polynomials

$$(x_0 \oplus 3 \odot x_1 \oplus 3 \odot x_2) \odot (x_0 \oplus x_1 \oplus 2 \odot x_2) \odot (x_0 \oplus 1 \odot x_1 \oplus x_2).$$

We refer the reader further interested in this connection to [35].

Additionally, we equip the subdivision by the sign matrix

$$
\begin{pmatrix}
- & + & - \\
+ & + & - \\
- & - & -
\end{pmatrix}
$$

The fine mixed subdivision of $3\triangle_2$ induced by $H$ is shown in the upper-right quartile of Figure 11. The cells are labeled by their Minkowski summands (cf. the Cayley trick in Section 2.3) as follows. The elements of $E$ (as the three columns from left to right) are represented by the red, green, and blue simplices, respectively; the elements of $R$ (as the three rows from top to bottom) are represented by the top, lower-left, and right vertices of each simplex, respectively. Furthermore, the vertices of these simplices are labeled with signs coming from the sign matrix.

The faces of a mixed cell correspond to the subgraphs of its spanning tree without isolated nodes in $E$. In particular, a vertex of a mixed cell can be specified by choosing a vertex from each colored simplex, thus it encodes a sign vector $\{+, -\}^E$. Such a forest associated with a vertex is independent of the mixed cell containing it. Hence, we have a well-defined assignment of sign vectors to the vertices of the mixed subdivision. For example, the lower left vertex $v$ of the square in the upper-right quartile of Figure 11 is the Minkowski sum of a filled red vertex, an empty green and an empty blue vertex. Therefore, it encodes the sign vector $(+, +, -)$.

Next we reflect the dilated simplex across the coordinate hyperplanes in $\mathbb{R}^3$ so that there is a copy in every octant. We only show the upper half of that patchworking complex in Figure 11 as the construction is centrally symmetric. We keep the same subdivision in all copies and label the vertices of these copies with sign vectors similar to the above, but instead of the original sign matrix, we negate a row of it if the corresponding coordinate in the octant is negative. For the vertex $v$, e. g., this yields $(+, +, -)$ for its reflection in the upper-left quartile of Figure 11. Again, the sign vector assigned to a vertex that appears in multiple copies of the dilated simplex is independent of the copy chosen: whenever a vertex lies on a hyperplane $\{x_i = 0\}$, the $i$-th node of $R$ must be an isolated one in the forest corresponding to the vertex, thus the negation of the $i$-th row does not affect the sign vector.

As our example is of rank 3, we obtain a subdivision of the boundary of a dilated octahedron (which is PL homeomorphic to $S^2$), with vertices of the subdivision labeled by sign vectors.

Finally, we define a “zero locus” for each element $e \in E$ as a subset of the patchworking complex. This zero locus is dual to the cells which have a Minkowski summand with vertices of different sign. Given a cell of the subdivision, select the edges (one-dimensional faces) of the cell in which the sign vectors of their endpoints disagree on the $e$-th coordinate, and take the convex hull of the midpoints of them.
Take the union of all such convex hulls, it can be seen from Figure 12 that each of such “zero loci” is a pseudosphere on the patchworking complex.

Note that the boundary of $3\Delta_2$ in $\mathbb{R}^3$ can be seen as the intersection with the three hyperplanes bounding the non-negative orthant. Extending these through the reflections of $3\Delta_2$ yields three further pseudospheres. This gives rise to an interpretation of Figure 12 as an arrangement of six pseudospheres. By ‘fattening’ the latter three coordinate pseudospheres we arrive at the extended patchworking complex introduced in the next section.

4.2. From Fine Mixed Subdivisions to Pseudosphere Arrangements. We now state precisely and prove our method for constructing a pseudosphere arrangement representing an oriented matroid associated to a polyhedral matching field. For this, we fix a fine mixed subdivision $S$ of $n\Delta_{d-1}$. By the Cayley trick, this corresponds to a triangulation $T$ of $\triangle_{d-1} \times \triangle_{n-1}$. By means of Definition 2.13 it gives rise to a pointed polyhedral matching field $(\tilde{M}_\sigma)$ on $\mathbb{R} \cup \tilde{E}$ with $\tilde{E} = \mathbb{R} \cup E$. For an arbitrary matrix $A \in \{+, -\}^{\mathbb{R} \times E}$, we consider the augmented matrix $\tilde{A} = (I_{\mathbb{R}} | A)$ as sign matrix for $(\tilde{M}_\sigma)$. Let $\mathcal{M}$ denote the oriented matroid on $\tilde{E} = \mathbb{R} \cup E$ associated to the pointed polyhedral matching field $(\tilde{M}_\sigma)$ with the sign matrix $\tilde{A}$, and let $M$ be its restriction to $E$.

Recall from Section 2.3 that we may identify the maximal simplices in $T$ with spanning trees of $K_{\mathbb{R} \cup E}$. The cells $\sigma_F$ of $S$ are in 1-1 correspondence with the forests $F$ contained in a spanning tree of $T$ for which $\deg_F(j) \geq 1$ for all $j \in E$.

We denote the cube $[-1, 1]^d \subset \mathbb{R}^d$ and its polar dual, the crosspolytope, by $\Box_d$ and $\Diamond_d$, respectively. For a sign vector $S \in \{-1, 0, 1\}^d$ and a set $K$ contained in the coordinate subspace $\mathbb{R}^{\text{supp}(S)} \times \{0\}^{\text{supp}(\Box_d)}$ of $\mathbb{R}^d$, define

$$S \cdot K := \{(S_1x_1, \ldots, S_dx_d) \in \mathbb{R}^d : (x_1, \ldots, x_d) \in K\}$$

$$\Box_S := \{x \in \Box_d : x_i = S_i \text{ for all } i \in \text{supp}(S)\}.$$ 

Hence, $S \cdot K$ denotes the reflections of $K$ to the orthant indicated by $S$, and $\Box_S$ comprises the sign patterns of orthants containing the sign vector $S$. For a subgraph $F$ of $K_{\mathbb{R} \cup E}$, let $\text{supp}_R(F) := \{i \in \mathbb{R} : \deg_F(i) \geq 1\}$, encoding which face of $n\Delta_{d-1}$ the cell corresponding to $F$ lies on.

**Proposition 4.2.** The subdivision $S$ of $n\Delta_{d-1}$ gives rise to the subdivision

$$\text{supp}_R(F) \cap \{S_1x_1, \ldots, S_dx_d\} \subseteq \mathbb{R}^d : (x_1, \ldots, x_d) \in K\}$$

of the boundary of $\Box_d := \Box_d + n\Diamond_d$, where $\sigma_{(S,F)} := \Box_S + S \cdot \sigma_F$.

We call the complex arising in the latter Proposition the extended patchworking complex; we prove the statement together with more technical properties of the extended patchworking complex in Section A. It is a polyhedral complex subdividing the boundary of a polytope, and is therefore a PL sphere. Hence, we may consider its dual complex

$$\Delta := S^\vee := \{\sigma_{(S,F)} : \sigma_{(S,F)} \in S\}.$$
Figure 13. The complex $S_8$ (left) and its dual $\Delta = S^\vee_8$ (right).

The five subcomplexes of $\Delta$ that yield pseudospheres are highlighted: there are three of the form $\Delta_i$, $i \in \widetilde{R}$ (shown in green) and two of the form $\Delta_j$, $j \in \mathcal{E}$ (shown in red and yellow).

The realization of the poset as a polyhedral cell complex is further explained in Section 3.1. For $i \in \widetilde{R}$ and $j \in \mathcal{E}$, define the subcomplexes

\begin{align}
\Delta_i &:= \{ \sigma^\vee_<(S,F) \in \Delta : i \notin \text{supp}(S) \}, \\
\Delta_j &:= \{ \sigma^\vee_<(S,F) \in \Delta : \text{there exist edges } (i,j), (\ell,j) \text{ in } F \\
&\quad \text{such that } S_i A_{i,j} = -S_\ell A_{\ell,j} \neq 0 \}.
\end{align}

Recall the notion of a pseudosphere arrangement from Definition 2.5.

**Theorem 4.3.** The spaces $\|\Delta_k\|$ ranging over all $k \in \widetilde{E}$ form an arrangement of pseudospheres within $\|\Delta\|$ representing the oriented matroid $\mathcal{M}$.

Deleting the pseudospheres $\|\Delta_i\|$ for $i \in \widetilde{R}$ yields the following.

**Corollary 4.4.** The spaces $\|\Delta_j\|$ ranging over all $j \in \mathcal{E}$ form an arrangement of pseudospheres representing the oriented matroid $\mathcal{M}$.

4.3. **Proof of Theorem 4.3** Since it involves a lot of ingredients, we spread the proof of Theorem 4.3 out over this subsection, with some of the more technical proofs relegated to the appendix.

Before getting into the technical details of our proof, we explain the overall picture. The codimension one skeleton of $\Delta$ in each orthant is a tropical pseudohyperplane arrangement in the sense of [8], that is, a union of PL-homeomorphic images of tropical hyperplanes (codimension one skeleton of $\Delta^{\vee}_{d-1}$). Including the sign data, each $\Delta_k$ restricted to an orthant is either empty or is (the boundary of) a tropical (pseudo)halfspace in the sense of [34]. The latter is obtained from a tropical pseudohyperplane by removing the facets that lie between two regions of the same sign. As such, the arrangement of $\Delta_k$’s can be thought as the end product of a facet removal process for multiple tropical pseudohyperplanes across multiple orthants.

Using the results from Section 3.6, we can show that the face poset of the arrangement of $\Delta_k$’s equals the covector lattice of $\mathcal{M}$. The challenge now becomes topological: we need to make sure that the facet removal process does not create pathologies, so the topological structure reflects the combinatorial structure. Our
The approach is to formulate this process as a stepwise cell merging process, using the formalism of regular cell complexes. The removal of a facet determines an equivalence relation on the cells of the tropical pseudohyperplane arrangement: two cells are equivalent if their interiors intersect the interior of a common cell once the facet is removed. By taking the union of the cells in each equivalence class, we show that we get another regular cell complex with the same underlying topological space. Iterating this procedure, we end up at a regular cell complex. Since the face poset of a regular cell complex determines the complex up to cellular homeomorphism, this completes the proof.

We remark that the work in this section is very closely related to the work done in [31, Chapter 6]. Indeed, one approach to proving Theorem 4.3 is to directly use Horn’s second topological representation theorem for tropical oriented matroids. However, one of our goals of this section is to carefully delineate the combinatorial versus the topological content of Theorem 4.3, in part by making use of the correspondence between triangulations of products of two simplices and generic tropical oriented matroids. In particular, we show how the elimination axiom of tropical oriented matroids enables our cell merging process to work, which might lead to extensions of our method.

4.3.1. Poset and lattice quotients. The following definition is due to Hallam and Sagan [27], which proved useful in their work on factorizing characteristic polynomials of lattices. This definition turns out to be the right one for us as well.

Definition 4.5. Let $P$ be a finite poset. An equivalence relation $\sim$ on the ground set of $P$ is $P$-homogeneous provided the following condition holds: if $\tau \leq \sigma$ in $P$, then for every $u \in \tilde{\tau}$ there exists $v \in \tilde{\sigma}$ such that $u \leq v$ in $P$. We denote by either $\tilde{\sigma}$ or $\sigma/\sim$ the equivalence class of $\sigma$ in $\sim$.

Proposition 4.6 ([27, Lemma 5]). Suppose $\sim$ is $P$-homogeneous. Then we have a well-defined poset $P/\sim$ on the classes of $\sim$ defined as follows: $\tilde{\tau} \leq \tilde{\sigma}$ in $P/\sim$ if and only if there exists $u \in \tilde{\tau}$ and $v \in \tilde{\sigma}$ such that $u \leq v$ in $P$. Equivalently, for every $u \in \tilde{\tau}$ there exists $v \in \tilde{\sigma}$ such that $u \leq v$ in $P$.

We call the poset $P/\sim$ the homogeneous quotient of $P$ by $\sim$. We are particularly interested in homogeneous quotients which have nice factorizations in the following sense:

Definition 4.7. A homogeneous quotient $P/\sim$ is an elementary quotient if every equivalence class of $\sim$ is either a singleton, or consists of exactly three elements $\sigma, \tau, \gamma \in P$ such that $\sigma$ and $\tau$ both cover $\gamma$ in $P$.

Definition 4.8. We say that $P/\sim$ admits a factorization into elementary quotients if there exist posets $P = P_0, P_1, \ldots, P_k = P/\sim$ such that $P_i = P_{i-1}/\sim_i$ is an elementary quotient of $P_{i-1}$ for all $i = 1, 2, \ldots, k$.

Since the following notion appears several times in this paper, we give the definition here:

Definition 4.9. The augmented poset of a poset $P$ is the poset $L(P) := P \cup \{\hat{0}, \hat{1}\}$, where $\hat{0}$ and $\hat{1}$ are two additional elements such that $\hat{0} < \sigma < \hat{1}$ for all $\sigma \in P$. 
4.3.2. Elimination systems. To obtain cleaner arguments, we consider a generalization of the set of forests arising from a fine mixed subdivision.

For a subgraph $F \subseteq \mathbb{R} \times \mathbb{E}$ of the complete bipartite graph $K_{\mathbb{R}, \mathbb{E}}$ and $j \in \mathbb{E}$, define the neighbourhood $F_j := \{ i : (i, j) \in F \}$.

**Definition 4.10.** Let $\mathbb{S}$ be a collection of subsets of $\mathbb{R} \times \mathbb{E}$. Then $\mathbb{S}$ is an elimination system provided:

- (E1) For each $F \in \mathbb{S}$ and for each $j \in \mathbb{E}$, $F_j$ is non-empty.
- (E2) If $F \subseteq G \in \mathbb{S}$ and $F_j$ is non-empty for all $j \in \mathbb{E}$, then $F \in \mathbb{S}$.
- (E3) If $F, G \in \mathbb{S}$ and $j \in \mathbb{E}$, then there exists $H \in \mathbb{S}$ such that $H_j = F_j \cup G_j$ and $H_k \in \{ F_k, G_k, F_k \cup G_k \}$ for all $k \in \mathbb{E}$ with $k \neq j$.

Let $\mathcal{F}$ be the set of forests on $\mathbb{R} \sqcup \mathbb{E}$ corresponding to the cells in the fine mixed subdivision $\mathbb{S}$. Recall that, by the Cayley trick, they encode the simplices in $\mathcal{T}$, for which no node in $\mathbb{E}$ is isolated. This directly shows that they fulfill property (E1) and (E2) of elimination systems (Definition 4.10). Property (E3) is the elimination axiom for tropical oriented matroids; see [8, Definition 3.5]. A proof that $\mathcal{F}$ satisfies (E3) can be found in [13, Proposition 4.12], and this result has been generalized to arbitrary mixed subdivisions in [31, Theorem 7.11].

Hence, we conclude:

**Proposition 4.11.** The forests $\mathcal{F}$ form an elimination system. \hfill \Box

4.3.3. The poset associated to an elimination system. Generalizing the face poset of the polyhedral complex of Proposition 4.2 subdividing the boundary of $\triangle_d := \square_d + n \triangle_d$, we introduce a poset associated with an elimination system.

**Definition 4.12.** Given an elimination system $\mathbb{S}$, we define the following poset:

$$\mathcal{P}(\mathbb{S}) := \{ (S,F) : S \in \{-1,0,1\}^\mathbb{R}, F \in \mathbb{S}, \text{ supp}(S) \supseteq \text{ supp}_{\mathbb{R}}(F) \}.$$  

Recall from Proposition 4.2 that $\text{ supp}_{\mathbb{R}}(F)$ denotes those $i \in \mathbb{R}$ such that $(i,j) \in F$ for at least one $j \in \mathbb{E}$. The ordering of the poset $\mathcal{P}(\mathbb{S})$ is given as follows: $(S,F) \leq (T,G)$ if and only if $S \subseteq T$ and $F \subseteq G$. Here $S \subseteq T$ means that $S$ is obtained from $T$ by setting some entries to zero. For example, $0 - 0+ \leq + - +$; another way to see it is that the orthant labeled by $S$ is contained in the orthant labeled by $T$.

4.3.4. Quotients of $\mathcal{P}(\mathbb{S})$. Let $\Pi$ be a partition of a finite set $K$. We say that two sign vectors $X, Y \in \{-1,0,1\}^K$ are equivalent (with respect to $\Pi$), and write $X \sim Y$, if for all $s \in \{-, +\}$ and $\pi \in \Pi$, we have $X^s \cap \pi$ is nonempty iff $Y^s \cap \pi$ is nonempty. For example, the following two sign vectors are equivalent with respect to the indicated partition of the coordinates:

$$
\begin{align*}
X : & 0 + 0 - 0 0 + - 0 0 0 + + \\
Y : & 0 + - 0 0 - + 0 + + 0 + 0
\end{align*}
$$

This defines an equivalence relation on $\{-1,0,1\}^K$. We may think of each equivalence class $X/\sim$ of this equivalence relation as a sign vector in $\{0, +, -, \pm\}^\Pi$. For the above example, this would look like

$$
X/\sim = Y/\sim : \begin{array}{cccc}
0 & + & - & \pm \\
\end{array}
$$
Recall the construction of the sign matrix $SA_F$ associated with a sign vector $S$ and a graph $F$ on $\mathbb{R} \cup E$ from Definition 3.26. We introduce an equivalence relation $\sim_A$ based on the set of signs in each column of the sign matrix $SA_F$. It is the crucial notion for merging those cells which give rise to the same covector of the oriented matroid as in Proposition 3.32.

**Definition 4.13.** Let $\Pi := \{\mathbb{R} \times \{j\} : j \in E\}$ be a partition of the edges of $K_{\mathbb{R} \cup E}$. Define the following equivalence relation $\sim_A$ on $P(S)$: Given $(S, F)$ and $(T, G)$ in $P(S)$, we say that $(S, F) \sim_A (T, G)$ if $S = T$ and $SA_F \sim SA_G$ with respect to the partition of $\mathbb{R} \times \Pi$ given by $\Pi$.

**Example 4.14.** Depicted below are four elements from the poset $P(S)$ for Example 4.1.

We show each element $(S, F)$ as $(S, SA_F)$, noting that $F = \text{supp}(SA_F)$:

$$(S_1, S_1A_{F_1}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} + & - & + \\ + & 0 & 0 \\ + & 0 & 0 \end{pmatrix}\right), (S_2, S_2A_{F_2}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 & - & + \\ + & 0 & 0 \\ + & 0 & 0 \end{pmatrix}\right),$$

$$(S_3, S_3A_{F_3}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 & 0 & + \\ + & 0 & 0 \\ + & 0 & 0 \end{pmatrix}\right), (S_4, S_4A_{F_4}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 & 0 & - \\ + & 0 & 0 \\ + & 0 & 0 \end{pmatrix}\right).$$

Observe that these four sign vectors correspond to four full-dimensional cells in Figure 11, of which three are in the lower right orthant and the last is in the upper right orthant. They correspond to cells following the red pseudoline in Figure 12, starting from the triangle in the lower right orthant. We see right away that $(S_4, F_3) \not\sim_A (S_\ell, F_\ell)$ for $\ell = 1, 2, 3$ as they differ in the first component. To check for the equivalence of the other three pairs, we can consider the image of the columns of $S_1A_{F_1}, S_2A_{F_2}, S_3A_{F_3}$ to $\{0, +, -, \pm\}^3$ as indicated before Definition 4.13. This yields the three vectors $(\pm, - , +), (\pm, - , +), (\pm, - , \pm)$. Hence, we get $(S_1, F_1) \sim_A (S_2, F_2) \not\sim_A (S_3, F_3)$.

With this definition, we can formulate an important intermediate result on our way to the representation theorem. The proof is given in Section C.

**Theorem 4.15.** The poset $P(S)/\sim_A$ admits a factorization $P(S) = P_0, P_1, \ldots, P_k = P(S)/\sim_A$ into elementary quotients, such that the augmented poset $L(P_i)$ is a lattice for each $i = 0, 1, \ldots, k - 1$.

4.3.5. **Quotients of regular cell complexes.** To arrive at the desired representation of the oriented matroid, we use complexes which arise by merging cells of the extended patchworking complex. They are no polyhedral but only regular cell complexes. As we want to apply Theorem 4.25 for concluding our representation, this is good enough.

Recall the notion of a regular cell complex, as defined in Section B.1. Our next goal is to develop a notion of a quotient of a regular cell complex $\Delta$, in which cells are “merged together” according to a given equivalence relation on the cells of $\Delta$. We remark that this construction appears to be similar to one given by Hersh in her work on total positivity [30].

Let $\Delta$ be a regular cell complex with face poset $P$, so that $||\Delta|| \subseteq \mathbb{R}^d$. Given a homogeneous quotient $P/\sim$ of $P$, define the set

$$\Delta/\sim := \{\bar{\sigma} : \bar{\sigma} \in P/\sim\}$$
where $\bigcup \hat{\sigma}$ denotes the union $\bigcup_{x \in \hat{\sigma}} \tau$. Note that homogeneity of $\sim$ implies that $\bigcup \hat{\sigma} \subseteq \bigcup \hat{\tau}$ as sets if and only if $\bigcup \hat{\sigma} \leq \bigcup \hat{\tau}$ in $P/\sim$.

Under certain conditions, $\Delta/\sim$ is again a regular cell complex:

**Theorem 4.16.** Suppose:

1. The poset $P/\sim$ is an elementary quotient,
2. The augmented poset $L(P)$ is a lattice, and
3. Each $\sigma \in \Delta$ is a PL ball.

Then $\Delta/\sim$ is a regular cell complex with face poset $P/\sim$, such that each $\sigma/\sim \in \Delta/\sim$ is a PL ball.

The proof of this theorem is given in Section 4.2. From this theorem we immediately deduce the following corollary:

**Corollary 4.17.** Suppose $P$ admits a factorization $P = P_0, P_1, \ldots, P_k = P/\sim$ into elementary quotients, such that $L(P_i)$ is a lattice for each $i = 0, 1, 2, \ldots, k - 1$. Suppose further that each $\sigma \in \Delta$ is a PL ball. Then $\Delta/\sim$ is a regular cell complex with face poset $P/\sim$.

4.3.6. The map $\varphi : P(S)/\sim_A \to L(\tilde{M})$. As final ingredient, we need to consider the labeling of the regular cell complex by sign vectors. For this, we use the connection between the pairs $(S, F)$ denoting cells of the extended patchworking complex and covectors established in Corollary 3.33. Suppose $S$, $A$, and $M$ as in Section 4.2. Now, we look at the particular elimination system given by the fine mixed subdivision $S$. In the following proposition, let $L(M)$ denote the poset of non-zero covectors of $M$. Let $P(S)/\sim_A$ be the poset as in Section 4.3.3.

Recall from Section 4.3.4 the system of mixed signs $\{0, +, -, \pm\}$. Using this as an intermediate step, one sees that the following map extending Definition 3.27 is well-defined on its equivalence classes.

**Definition 4.18.** Define the map $\varphi : P(S)/\sim_A \to \{-1, 0, 1\}^{\tilde{E}}$ by $\varphi_A(S, F) = (S, \psi_A(S, F))$.

As we fix $A$ most of the time, we just set $\varphi(S, F) = \varphi_A(S, F)$.

**Example 4.19 (Ex. 4.14 continued).** Recall that we could identify the equivalence classes of the four sign vectors $S_\ell A_{F_\ell}$ for $\ell \in [4]$ with $(\pm, -, +)$, $(\pm, +, -)$, and $(\pm, \pm, \pm)$. This shows that the images of the three equivalence classes of $(S_\ell, F_\ell)$ (as $(S_1, A_{F_1}) \sim_A (S_2, A_{F_2})$) are the sign vectors

$(-, +, +, 0, -, +), (-, +, +, 0, -, 0), (+, +, +, 0, -, -)$.

Note that a similar map was used in [31 §6] to prove the representation theorem for tropical oriented matroids.

We show the next claim by reducing it to the results from Section 3.6. Since $\varphi$ is constant on the equivalence classes of $/\sim_A$, we just fix an element $(S, F) \in P(S)$. With this, we associate the bipartite graph $T$ on $R \sqcup \tilde{E}$ having the edges of $F$ and edges between the nodes of $R$ and its copy $\tilde{R}$ within $\tilde{E}$ for each element $R$ in the support of $S$. Now, the claim follows from Proposition 3.32.

**Corollary 4.20.** For all $(S, F) \in P(S)/\sim_A$, we have $\varphi(S, F) \in L(\tilde{M})$.

**Proposition 4.21.** The map $\varphi : P(S)/\sim_A \to L(\tilde{M})$ is a poset map.
\[ \text{Proof.} \] Suppose \((S, F) \leq (S', F')\) in \(P\). Let \((S, X) = \varphi(S, F)\), and let \((S', X') = \varphi(S', F')\). Then \(S \leq S'\), and because \(F \supseteq F'\), the passage from \(SA_F\) to \(SA_{F'}\) only decreases the number of non-zero entries in each column of \(SA_F\). However \(SA_{F'}\) still has at least one non-zero entry in each column. From this we conclude \(X \leq X'\), and therefore \(\varphi\) respects order. \(\square\)

Recall that a poset \(P\) is a sphere if its order complex is a sphere; see Definition B.4.

**Theorem 4.22** (Borsuk–Ulam). Let \(P, Q\) be posets such that both are homeomorphic to \(S^{d-1}\) and both are equipped with a fixed-point free involutive automorphism \(x \mapsto -x\). Let \(\varphi : P \to Q\) be a poset map satisfying \(\varphi(-x) = -\varphi(x)\) for all \(x \in P\). Then \(\varphi\) is surjective.

**Corollary 4.23.** Assuming \(P(S)/\sim_A\) is a \((d - 1)\)-sphere, the map \(\varphi : P(S)/\sim_A \to L(\tilde{M})\) is an isomorphism.

**Remark 4.24.** We show that \(P(S)/\sim_A\) is indeed a \((d - 1)\)-sphere in Section 4.3.8.

**Proof.** For \((S, F) \in P(S)/\sim_A\), the interpretation of \(SA_F\) as a generalized sign vector in \(\{0, -, +, \pm\}\) shows that \(\varphi\) is injective. To see that \(\varphi\) is surjective, we simply note that \((S, F) \mapsto -(S, F) := (-S, F)\) is a fixed-point free involutive automorphism of \(P(S)\) which descends to \(P(S)/\sim_A\), while \(X \mapsto -X\) is one of \(L(\tilde{M})\). Furthermore, by definition of \(\varphi\), we have \(\varphi(-S, F) = (-S, -X) = -(S, X) = -\varphi(S, F)\). As \(\tilde{M}\) has rank \(d\), the poset \(L(\tilde{M})\) is a \((d - 1)\)-sphere by Theorem 4.25 and hence the conclusion follows from the Borsuk-Ulam Theorem. \(\square\)

**4.3.7. Pseudosphere arrangements from regular cell complexes.** The following result shows how to get pseudosphere arrangements from regular cell complexes:

**Theorem 4.25** (\cite{14} Theorem 4.3.3, Proposition 4.3.6). Let \(M\) be an oriented matroid of rank \(d\) on the ground set \(E\). Let \(\Delta\) be a regular cell complex with face poset \(P\), such that there is a poset isomorphism \(P \simeq L(\tilde{M})\). Thus each cell \(\sigma_X\) of \(\Delta\) is labeled by some non-zero covector \(X\) of \(M\). For each \(k \in E\), define the subcomplex

\[ \Delta_k := \{ \sigma_X \in \Delta : X_k = 0 \} . \]

Then \(\|\Delta\|\) is a \((d - 1)\)-sphere, and the spaces \(\|\Delta_k\|\) ranging over all \(e \in E\) form an arrangement of pseudospheres within \(\|\Delta\|\) representing \(\tilde{M}\).

**Remark 4.26.** This theorem is really the uniqueness assertion of \cite{14} Theorem 4.3.3], whose proof can be traced back to \cite{14} Proposition 4.7.23.

**4.3.8. Putting it all together.** With all the pieces now in place, we are ready to prove Theorem 4.13 which asserts that our patchworking procedure yields a pseudosphere representation of \(\tilde{M}\).

**Proof of Theorem 4.13.** As shown for Proposition 4.11 a fine mixed subdivision \(S\) of \(n\Delta_{d-1}\) gives rise to an elimination system as in Definition 4.10. Abusing notation, we denote this elimination system also by \(S\). We let \(P(S)\) be the poset of \(S\) obtained by introducing signs as in Definition 4.12.

Let \(P(\Delta)\) denotes the face poset of \(\Delta = S^\vee\). By Corollary A.2, we have \(P(\Delta) \simeq P(S)\). Hence the poset quotient \(P(S)/\sim_A\) induces a quotient \(P(\Delta)/\sim_A\). By Theorem 4.15 then, \(P(\Delta)/\sim_A\) admits a factorization \(P(\Delta) = P_0, P_1, \ldots, P_k = P(\Delta)/\sim_A\) into
elementary quotients, such that the augmented poset $\mathcal{L}(P_i)$ is a lattice for each $i = 0, 1, \ldots, k - 1$.

As a polyhedral complex on the boundary of a $d$-dimensional polytope, $S_\circ$ is a PL $(d-1)$-sphere. Hence, by Proposition [3.12] $\Delta = S_\circ^\vee$ is also a PL $(d-1)$-sphere. In particular, by Proposition [3.11] each cell $\sigma^\vee$ in $\Delta$ is a PL ball. It follows, by Corollary [4.17], that $\Delta/\sim_A$ is a regular cell complex with face poset $P(\Delta)/\sim_A$. In particular, $P(\Delta)/\sim_A$ is a $(d-1)$-sphere.

By Corollary [4.23] we have isomorphisms $P(\Delta)/\sim_A \simeq P(S)/\sim_A \simeq \mathcal{L}(\tilde{M})$. For $k \in \mathcal{E}$, define the subcomplex

$$(\Delta/\sim_A)_k := \left\{ \sigma^\vee_{(S,F)} \in \Delta/\sim_A \mid \varphi(S,F)_k = 0 \right\}$$

of $\Delta/\sim_A$. Now, Theorem [4.25] implies that the spaces $\|(\Delta/\sim_A)_k\|$ ranging over all $k \in \mathcal{E}$ form an arrangement of pseudospheres within $\|\Delta/\sim_A\| = \|\Delta\|$ representing $\tilde{M}$.

It remains to show that $\| (\Delta/\sim_A)_k \| = \| \Delta_k \|$ for all $k \in \mathcal{E}$, where $\Delta_k$ is a subcomplex of $\Delta$ defined in [4] and [5]. For this note that the closed cells of $\Delta/\sim_A$, and hence all subcomplexes of $\Delta/\sim_A$, each consist of a union of members of $\Delta$. Hence, it suffices to show that for all $\sigma^\vee \in \Delta$ and $k \in \mathcal{E}$, we have $\sigma^\vee \subseteq \| (\Delta/\sim_A)_k \|$ if and only if $\sigma^\vee \in \Delta_k$.

For $\sigma^\vee \in \Delta$ and $k \in \mathcal{E}$, we have $\sigma^\vee \subseteq \| (\Delta/\sim_A)_k \|$ if and only if there exists $(S,F) \in P(S)$ such that $\varphi(S,F)_k = 0$ and

$$\sigma^\vee \subseteq \bigcup_{(S,F) \sim_A (S,G)} \sigma^\vee_{(S,G)}.$$  

As $\Delta$ is a regular cell complex, the interiors of the balls in $\Delta$ are disjoint, and so the above containment holds true if and only if $\sigma^\vee = \sigma^\vee_{(S,F)}$ for some $(S,F) \in P(S)$ such that $\varphi(S,F)_k = 0$. If $k \in \mathcal{R}$, then $\varphi(S,F)_k = 0$ if and only if $F_k = 0$. If $k \in \mathcal{E}$, we have $\varphi(S,F)_k = 0$ if and only if there exist $(i,k), (\ell,k) \in F$ such that $S_iA_{i,k} = -S_\ell A_{\ell,k} \neq 0$. In either case, we conclude $\sigma^\vee \subseteq \| (\Delta/\sim_A)_k \|$ if and only if $\sigma^\vee \in \Delta_k$. \hfill $\Box$

5. On Oriented Matroids arising from Matching Fields

5.1. Sets of Oriented Matroids. Starting from a polyhedral matching field $(M_{\sigma})$, we can associate an oriented matroid for each sign matrix $A \in \{+,-\}^{d \times n}$. More generally, we denote the sign map defined in [3] by $\chi((M_{\sigma}), A)$. This leads to the set of sign maps

$$(\mathcal{D}((M_{\sigma}))) = \{ \chi((M_{\sigma}), A) \mid A \in \{+,-\}^{d \times n} \}.$$ 

or just $\mathcal{D}$ if the matching field is clear. By the support $\text{supp} [(M_{\sigma})]$ of a matching field, we mean the union of all edges occurring in a matching. Clearly, only the signs in the entries corresponding to elements in the support of the matching field influence the resulting sign map.

$\text{Example 5.1.}$ A $(d,d)$-matching field consists of a single matching. Hence, among the $2^d$ sign matrices differing on the support of the matching field, we get $+$ for half of the matrices and $-$ for the other half.

$\text{Example 5.2.}$ A linkage $(d,d+1)$-matching field $(M_{\sigma})$, see Example [2.8] and Section [3.4] can be identified with its linkage covector $T$. Its support has cardinality $d + (d + 1) - 1 = 2d$. To determine all oriented matroids in $\mathcal{D}$, it suffices to fix the
signs on a matching. The remaining degrees of freedom correspond to the $d$ edges of the linkage tree $\tilde{T}$.

In particular, all $2^d$ assignments of + or − to the edges of $\tilde{T}$ yield different sign maps: Let $k$ be the node in $\tilde{T}$ corresponding to the fixed matching and let $A, A'$ be different assignments. Then there is a path emerging from $k$ on which the signs assigned by $A$ and $A'$ differ. Let $(u, v)$ be the edge closest to $k$ on this path, where $u$ denotes the node closer to $k$; observe that $u$ could be equal to $k$. Then the signs $\chi((M_\sigma), A)(\sigma_v)$ and $\chi((M_\sigma), A')(\sigma_v)$ differ, where $\sigma_v$ is the subset of $E$ corresponding to the node $v$ in $\tilde{T}$.

We summarize from Theorem 3.7.

**Corollary 5.3.** If $(M_\sigma)$ is a polyhedral matching field, then all sign maps in $\mathcal{O}[(M_\sigma)]$ are chirotopes.

For the subclass of coherent matching fields, see Example 2.7, we know that the matchings are given as weight maximal matchings induced by a weight matrix $M = (m_{ij})_{(i,j)\in[d] \times [n]} \in \mathbb{R}^{d \times n}$. As a commonly used trick in tropical geometry, we consider the matrix $(tM) := (tm_{ij})_{(i,j)\in[d] \times [n]} \in \mathbb{R}^{d \times n}$ for a sufficiently large parameter $t > 0$. Then the maximal term in the expansion of the determinant corresponds to the weight maximal matching of the coherent matching field.

**Corollary 5.4.** If $(M_\sigma)$ is a coherent matching field, then all sign maps in $\mathcal{O}[(M_\sigma)]$ are chirotopes of oriented matroids realizable over $\mathbb{R}$.

**Example 5.5.** For the diagonal $(d, n)$-matching field, we obtain all reorientations of the unique uniform positroid, the alternating matroid; see also [14, §8.2].

The observation of Example 5.2 can be extended to arbitrary linkage matching fields. Note that the sign map does not have to be a chirotope if the matching field is not polyhedral. We fix a linkage matching field $(M_\sigma)$. By [55, Thm. 3.2], each node of $R$ in the support of a linkage matching field, considered as a subgraph of $K_{d,n}$, has $n - d + 1$ neighbours. In particular, the support has cardinality $d \times (n - d + 1)$. We fix a $d$-subset $\sigma_0$ of $E$. Let $A$ and $A'$ be two different sign matrices supported on the support $\text{supp}[(M_\sigma)]$ which agree on $\sigma_0$.

**Proposition 5.6.** The sign maps $\chi((M_\sigma), A)$ and $\chi((M_\sigma), A')$ are distinct.

**Proof.** Consider the flip graph of the linkage matching field as introduced in [38, §3.5]; that is the graph which has the matchings as nodes and two nodes are connected if the matchings differ by exactly one edge. Let $\mu_1$ and $\mu_2$ be two matchings such that $A$ and $A'$ differ on the matchings. Since the linkage graphs cover the flip graph, there is a path from $\mu_1$ to $\mu_2$. As in Example 5.2, the subset of $E$ corresponding to the node in the flip graph, where the signs of $A$ and $A'$ differ for the first time on the path, shows that also the sign vectors differ.

Taking the consideration of Example 5.1 into account, we obtain the cardinality of $\mathcal{D}$.

**Corollary 5.7.** For a linkage matching field, the set $\mathcal{D}$ contains $2^{d(n-d)+1}$ different sign maps.

Two chirotopes for uniform matroids are isomorphic, if they differ by a reorientation or a relabeling; see [14, §3] for more details on these notions. In particular, for a fixed chirotope, there are at most $2^n \cdot n!$ isomorphic chirotopes.
Corollary 5.8. If \((M_\sigma)\) is a polyhedral matching field, then \(\mathcal{O}[(M_\sigma)]\) contains at least \(2^{d(n-d)-n+1}/n!\) non-isomorphic chirotopes. For \(2\log_2(n) < d < n/\log_2(n)\) with \(n > 8\), this shows that there are non-isomorphic chirotopes.

Proof. The first part of the claim just follows from the estimate on the number of isomorphic chirotopes.

For the second part, we use the estimate \(n! < 2^{n\log_2 n}\) and get

\[
2^{d(n-d)-n+1}/n! > 2^{d(n-d)-n+1-n\log_2(n)}.
\]

Thus, we derive a bound for the exponent, using the specified range of \(d\):

\[
d(n-d) - n + 1 - n \log_2(n) > 2\log_2(n)(n - \frac{n}{\log_2(n)}) - n + 1 - n \log_2(n)
\]

\[
= 2n \log_2(n) - 2n + n + 1 - n \log_2(n)
\]

\[
= n \log_2(n) - 3n + 1 \geq 1.
\]

□

Question 5.9. For a polyhedral matching field, how many non-isomorphic chirotopes does \(\mathcal{O}\) contain?

We can also vary the matching field and observe how the set of sign maps differs. If two matching fields do not have the same support, one can choose a sign matrix \(A \in \{+, -\}^{d \times n}\) such that \(\chi((M_\sigma), A)\) and \(\chi((M'_\sigma), A)\) are distinct; that is simply by modifying the entries which are in the difference of the supports. Though, for two (even coherent!) matching fields with different support still the set of chirotopes \(\mathcal{O}\) can agree as the next example shows.

Example 5.10. The two coherent \((3, 4)\)-matching fields depicted in Figure 14 by their linkage covectors give rise to the same set \(\mathcal{O}\). They agree on all matchings except for the one on \(\{2, 3, 4\}\); however, they differ there by a permutation with positive sign.

Conjecture 5.11. A matching field is polyhedral if and only if the sign map induced by each sign matrix is a chirotope.

Remark 5.12. The study of coherent matching fields is closely related to the structure of the tropical maximal minors of rectangular matrices; see [19]. In this case, considering subpolytopes of \(\Delta_{d-1} \times \Delta_{n-1}\) can be thought as setting some entries of the height matrix to tropical zero. Over the min-plus semiring, this means that non-bases of the transversal matroid are those submatrices whose tropical determinant is \(\infty\). It is natural to consider tropical singularity, another tropical analogue of zero
determinant. A square matrix is *tropically singular* if the minimum in its tropical determinant is achieved at least twice. The polyhedral interpretation is that we work with general subdivisions instead of triangulations, and only consider simplices in the subdivision as bases. However, the following example shows that this does not always produce a matroid. A conceptual explanation is that a system of tropical non-singularity conditions can not always be lifted back to non-singularity over a (valued) field.

**Example 5.13.** Consider the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 2 \\
0 & 0 & 1 & 1 & 2
\end{pmatrix}.
\]

The (column sets of) min-tropically non-singular submatrices are

\[123, 124, 125, 145, 234, 235, 345,\]

which do not form a collection of bases of any matroid. This can be seen from the pair of bases 123 and 145 by removing 2 from the first set.

5.2. **The non-realizable (3, 9) uniform oriented matroid.** We show that the (up to isomorphism) unique non-realizable uniform oriented matroid \(R\) of rank 3 on 9 elements can be realized by patchworking a suitable non-coherent fine mixed subdivision of \(6\Delta_2\) and choosing appropriate signs.

This oriented matroid goes back to Ringel [14, §8.3] and its corresponding pseudoline arrangement is depicted in Figure 15a. Our patchworked version is depicted in Figure 15b.

Each pseudoline is intersected by all other pseudolines. This gives rise to a (circular) sequence of elements of the oriented matroid for each pseudoline. In [15, Thm. 5.2], it was shown that the set of these sequences determines the oriented matroid. Using the starting points and directions indicated, it can be verified that the sequences for [15a] and [15b] agree. For example, the pseudoline sequence associated to \(B\) in both figures is \(AFHECGDI\).

It is not clear to the authors, if the oriented matroid \(R\) can be constructed from another non-coherent fine mixed subdivision of \(6\Delta_2\).

### 6. Extension to Matroids over Hyperfields

6.1. **Introduction to Matroids over Hyperfields.**

**Definition 6.1.** A *hyperfield* \((\mathbb{H}, \boxplus, \otimes, 0, 1)\) consists of a set \(\mathbb{H}\) with distinguished elements \(0 \neq 1\), together with a possibly multi-valued hyperoperation \(\boxplus : \mathbb{H} \times \mathbb{H} \to 2^\mathbb{H} \setminus \{\emptyset\}\) and an operation \(\otimes : \mathbb{H} \times \mathbb{H} \to \mathbb{H}\), such that:

- \(x \boxplus y = y \boxplus x, \forall x, y \in \mathbb{H}\).
- \(\bigcup_{y \in \mathbb{H}} a \boxplus y = \bigcup_{y \in \mathbb{H}} x \boxplus b =: x \boxplus y \boxplus z, \forall x, y, z \in \mathbb{H}\).
- \(0 \boxplus x = \{x\}, \forall x \in \mathbb{H}\).
- For any \(x \in \mathbb{H}\), there exists a unique \(-x \in \mathbb{H}\) such that \(0 \in x \boxplus (-x)\).
- If \(x \in \mathbb{H}\) and only if \(z \in x \boxplus (-y)\).
- \((\mathbb{H} \setminus \{0\}, \otimes, 1)\) is an abelian group, and \(0 \otimes x = x \otimes 0 = 0, \forall x \in \mathbb{H}\).
- \(a \otimes (x \boxplus y) = (a \otimes x) \boxplus (a \otimes y), \forall a, x, y \in \mathbb{H}\).
(a) Grünbaum’s drawing of the Ringel arrangement.

(b) Non-coherent fine mixed subdivision of $6\Delta_2$ patchworking the Ringel arrangement.

Figure 15. Two isomorphic pseudoline arrangements showing the claim of Section 5.2.
Given two hyperfields \((\mathbb{H}, \boxplus, \otimes, 0, 1), (\mathbb{H}', \boxplus', \otimes', 0', 1')\), a map \(\varphi : \mathbb{H} \to \mathbb{H}'\) is a \textit{hyperfield morphism} if \(\varphi(0) = 0', \varphi(1) = 1', \varphi(x \otimes y) = \varphi(x) \otimes' \varphi(y), \forall x, y \in \mathbb{H}\), and \(\varphi(x \boxplus y) \subset \varphi(x) \boxplus' \varphi(y), \forall x, y \in \mathbb{H}\).

\textbf{Definition 6.2.} Let \(\mathbb{H} = (\mathbb{H}, \boxplus, \otimes, 0, 1)\) be a hyperfield. A \textit{strong matroid over} \(\mathbb{H}\) (on \(E\) and of rank \(d\)) is a non-zero, alternating function \(\chi : E^d \to \mathbb{H}\) such that for any \(x_1, \ldots, x_{d-1}, y_1, \ldots, y_{d+1} \in E\),
\[
0 \in \mathbb{H}_{k=1}^{d+1} (-1)^k \chi(x_1, \ldots, x_{d-1}, y_k) \chi(y_1, \ldots, \hat{y}_k, \ldots, y_{d+1}).
\]
A non-zero alternating function \(\chi : E^d \to \mathbb{H}\) is a \textit{weak matroid over} \(\mathbb{H}\) if \(\chi\) is a matroid and that for any \(x_1, x_2, y_1, y_2 \in E, X := \{x_3, \ldots, x_d\} \subseteq E\),
\[
0 \in \chi(x_1, x_2, X) \chi(y_1, y_2, X) \boxplus \chi(x_1, y_1, X) \chi(y_2, x_2, X) \boxplus \chi(x_1, y_2, X) \chi(x_2, y_1, X).
\]

Note the similarity between the definitions of a strong and a weak matroid and the two \textit{equivalent} characterizations of an oriented matroid given in Section 2.1.

We state a proposition that the theory of matroids over hyperfields is functorial.

\textbf{Proposition 6.3 \([^10\text{ Lemma 3.40}]\).} Let \(\varphi : \mathbb{H} \to \mathbb{H}'\) be a hyperfield morphism, and \(\chi\) be a (strong, respectively weak) matroid over \(\mathbb{H}\). Then \(\varphi_* \chi := \varphi \circ \chi\) is a (strong, respectively weak) matroid over \(\mathbb{H}'\).

We collect a few essential examples here and refer the reader to \([10]\) for more. We include their Examples 3.37 and 3.38, which show that in general, the notions of strong and weak matroids do not coincide.

\textbf{Example 6.4.} We omit the arithmetic of hyperfields in the list that can be directly deduced from the axioms.

- Every field is a hyperfield. A (strong or weak) matroid over a field is a linear subspace over the field (represented by its Plücker coordinates).
- The \textit{Krasner hyperfield} \(K = \{0, 1\}\) with \(\boxplus 1 = \{0, 1\}\). A (strong or weak) matroid over \(K\) is a matroid in the usual sense.
- The \textit{sign hyperfield} \(S = \{0, 1, -1\}\) with \(\boxplus 1 = \{1\}, -1 \boxplus -1 = \{-1\}, 1 \boxplus -1 = \{0, 1, -1\}\). A (strong or weak) matroid over \(S\) is an oriented matroid.
- The \textit{min-tropical hyperfield} \(T = \mathbb{R} \cup \{\infty\}\), where \(0 \in \mathbb{R}\) is the multiplicative identity and \(\infty\) is the additive identity, \(a \boxplus b = \min\{a, b\}\) if \(a \neq b\), \(a \boxplus a = [a, \infty]\), and \(a \otimes b = a + b\). A (strong or weak) matroid over \(T\) is a \textit{valuated matroid} (also known as a tropical linear space).
- The \textit{phase hyperfield} \(P = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\}\), where \(w \boxplus z\) consists of the open minor arc between \(w, z\) if \(w \neq -z\) are non-zero, \(w \boxplus (-w) = \{w, -w, 0\}\), and \(w \otimes z = wz\). Matroids over \(P\), known as \textit{phase matroids}, were first considered by Anderson and Delucchi \([6]\). It is an important example because the notions of strong and weak matroids are different over \(P\).

\textbf{Remark 6.5.} The more general notion of \textit{matroids over tracts} was considered in \([10]\), and there is a straightforward generalization of our work at that generality. However, the exposition of such theory is more complicated and goes beyond the scope of this work, so we omit the details here.

6.2. \textbf{Hyperfields with the Inflation Property.} The following definition was probably first considered by Massouros \([40]\) under the name of \textit{monogene hyperfields}, but we follow the terminology of Anderson \([4]\).

\textbf{Definition 6.6.} A hyperfield \(\mathbb{H}\) has the \textit{inflation property} (IP) if \(1 \boxplus (-1) = \mathbb{H}\).
The following proposition is [4, Proposition 6.10].

**Proposition 6.7.** The statements are equivalent for a hyperfield $\mathbb{H}$:

1. $\mathbb{H}$ has the IP.
2. $a \boxplus (-a) = \mathbb{H}$ for any $a \neq 0$.
3. $a \in a \boxplus b$ for any $a \neq 0$.
4. Suppose $0 \in \oplus_{i=1}^{k} a_i$ for some $a_1, \ldots, a_k$ that are not all zero. Then $0 \in (\oplus_{i=1}^{k} a_i) \boxplus a_{k+1}$ for any $a_{k+1} \in \mathbb{H}$.

We show that Theorem 3.7 can be extended to (weak) matroids over hyperfields with the IP, this includes Theorem 3.7 as a special case as $\mathbb{S}$ has the IP.

**Theorem 6.8.** Let $(M_\sigma)$ be a polyhedral matching field. Let $\mathbb{H}$ be a hyperfield with the IP and $A$ be an $\mathbb{H}$-matrix with no zero entries. Then $\chi : E^d \to \mathbb{H}$ given by $\sigma \mapsto \text{sign}(M_\sigma) \otimes \bigotimes_{e \in M_\sigma} A_e$ is a weak matroid over $\mathbb{H}$.

**Proof.** We show that the proof in Section 3 can be adapted to this setting. First of all, Theorem 3.3 is a pure polyhedral geometric statement which requires no modification.

Following the argument of Lemma 3.8 each maximal minor of $A_T$ is either 0 or consists of one non-zero term, so they are all single valued and altogether realize $\chi_T$. Now we claim that the maximal minors of an $\mathbb{H}$-matrix whose support is a spanning tree induce a weak matroid over $\mathbb{H}$. Treating the non-zero entries as algebraically independent indeterminates and denoting such matrix as $\tilde{A}$, we still have the 3-term GP relation: $\det(A_{\mid x_1, x_2, X}) \det(A_{\mid y_1, y_2, X}) + \det(A_{\mid x_1, y_1, X}) \det(A_{\mid y_2, x_2, X}) + \det(A_{\mid x_1, y_2, X}) \det(A_{\mid y_2, y_1, X}) = 0$. Since each determinant is either 0 or a monomial, either all three terms are zeros, or one term is zero while the other two are the same monomial with coefficient 1 and $-1$, respectively. Replacing the indeterminates by values from $\mathbb{H}$ will still preserve the 3-term GP relation over $\mathbb{H}$.

Finally, the only non-polyhedral geometric argument in the proof of Theorem 3.9 is to show that the restriction of a violation of the 3-term GP relation (in the ambient matroid polytope) to a subpolytope is still a violation. The only non-trivial restriction is from an octahedron face to a pyramid/square cell. In the language of hyperfields this means that if $0 \not\in a \boxplus b \boxplus c$ with $a, b, c \neq 0$, then 0 is also not in the sum if we set one of the terms to zero. But this is Property (4) of Proposition 6.7 concerning hyperfields with the IP.

**Example 6.9.** We note that the statement of Theorem 6.8 actually characterizes hyperfields with the IP. Suppose $\mathbb{H}$ does not have the IP, pick $a \in \mathbb{H}$ such that $-a \not\in 1 \boxplus (-1)$, i.e., $0 \not\in 1 \boxplus (-1) \boxplus a$. Take $n = 4, d = 2$ and the diagonal matching field of the corresponding size depicted in Figure 1. Also take the $\mathbb{H}$-matrix to be 

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & a & 1 & 1
\end{pmatrix}.
$$

The induced $\chi$ is not a (weak) matroid over $\mathbb{H}$.

We list several examples and constructions of hyperfields that have the IP, besides $\mathbb{K}$ and $\mathbb{S}$.

- The **tropical phase hyperfield** $\Phi = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$, where $w \boxplus z$ consists of the closed minor arc between $w, z$ if they are both non-zero and not antipodal of each other, $w \boxplus (-w) = \Phi$ if $w \neq 0$, and $w \otimes z = wz$. 

• Let \((G, \otimes, 1)\) be an arbitrary abelian group. Introduce a new element 0 and define the hyperoperation \(\boxplus\) as \(a \boxplus b = \{a, b\}\) if \(a, b \in G\) and \(a \neq b\), \(a \boxplus a = G \cup \{0\}\) if \(a \in G\), and \(a \boxplus 0 = 0 \boxplus a = \{a\}\). Then \((G \cup \{0\}, \boxplus, \otimes, 0, 1)\) is a hyperfield with the IP. Such hyperfields are considered in [40].

• Again start with an arbitrary abelian group \((G, \otimes, 1)\) together with a new element 0. Define \(\boxplus'\) as \(a \boxplus' b = G\) if \(a, b \in G\) and \(a \neq b\), \(a \boxplus' a = G \cup \{0\}\), and \(a \boxplus' 0 = 0 \boxplus' a = \{a\}\). Then \((G \cup \{0\}, \boxplus', \otimes, 0, 1)\) is a hyperfield with the IP. Such hyperfields are called weak hyperfields in [10].

• Let \(\mathbb{H} = (\mathbb{H}, \boxplus, \otimes, 0, 1)\) be a hyperfield. Define a new hyperfield \(\tilde{\mathbb{H}}\) with the same ground set and multiplicative structure as \(\mathbb{H}\), but with the new hyperoperation \(\boxplus\) where \(a \boxplus b = (a \boxplus b) \cup \{a, b\}\) for \(a, b \neq 0\) and \(a \neq -b\), and \(a \boxplus (-a) = \mathbb{H}\) for non-zero \(a\). See [40] for a proof that it is indeed a hyperfield.

We further elaborate our last example above. We suggest the name canonical inflation for the construction of \(\tilde{\mathbb{H}}\) from \(\mathbb{H}\): in view of Proposition 6.7, this is the “minimum” change needed to make a hyperfield to become one that has the IP. More rigorously, the (set-theoretic) identity map \(i : \mathbb{H} \to \tilde{\mathbb{H}}\) is a hyperfield morphism and we have the following universal property:

**Proposition 6.10.** Let \(\varphi : \mathbb{H} \to \mathbb{H}'\) be a hyperfield morphism with \(\mathbb{H}'\) having the IP. Then we have the factorization \(\mathbb{H} \xrightarrow{i} \tilde{\mathbb{H}} \xrightarrow{\varphi} \mathbb{H}'\) of hyperfield morphisms.

**Proof.** The only non-trivial part is that \(\tilde{\mathbb{H}} \xrightarrow{i} \mathbb{H}'\) preserves addition of two non-zero values. Denote by \(\boxplus, \boxplus', \boxplus''\) the addition hyperoperators of \(\mathbb{H}, \tilde{\mathbb{H}}, \mathbb{H}'\), respectively. Suppose \(a \neq -b\) are non-zero. Then \(\varphi(a) \boxplus' \varphi(b)\) contains \(\varphi(a \boxplus b)\) (as \(\tilde{\mathbb{H}} \xrightarrow{i} \mathbb{H}'\) is a hyperfield morphism) as well as \(\varphi(a)\) (as \(\mathbb{H}'\) has the IP), so \(\varphi(a) \boxplus' \varphi(b)\) contains \(\varphi(a \boxplus b)\). For \(a \neq 0\), we have \(\varphi(a \boxplus (-a)) = \varphi(0) \subset \mathbb{H}' = \varphi(a) \boxplus' \varphi(-a)\). \(\square\)

**Corollary 6.11.** Let \((M_\sigma)\) be a polyhedral matching field. Let \(\mathbb{H}\) be an arbitrary hyperfield and \(A\) be an \(\mathbb{H}\)-matrix with no zero entries. Then \(\chi : E^d \to \mathbb{H}\) given by \(\sigma \mapsto \text{sign}(M_\sigma) \otimes \bigotimes_{e \in M_\sigma} A_e\) is a weak matroid over the canonical inflation \(\tilde{\mathbb{H}}\) of \(\mathbb{H}\). In general, given a hyperfield morphism \(\varphi : \mathbb{H} \to \mathbb{H}'\) with \(\mathbb{H}'\) having the IP, \(\varphi_*\chi\) is a weak matroid over \(\mathbb{H}'\).

**Proof.** Note that \(\chi\) does not involve the additive structure of \(\mathbb{H}\) (respectively \(\tilde{\mathbb{H}}\)), so we can simply interpret \(A\) as a \(\tilde{\mathbb{H}}\)-matrix and apply Theorem 6.8 to \(\chi\). The general statement follows from Proposition 6.10 and Proposition 6.3. \(\square\)

**Example 6.12.** Let \(pb : \mathbb{C} \to \mathbb{P}\) be given by \(z \mapsto z/|z|\) if \(z \neq 0\) and \(0 \mapsto 0\). Then we have the hyperfield morphisms \(\mathbb{C} \xrightarrow{pb} \mathbb{P} \xrightarrow{i} \Phi\), where the second map is the canonical inflation. A matroid over \(\mathbb{P}\) or \(\Phi\) is \(\mathbb{C}\)-realizable if it is the pushforward of some matroid over \(\mathbb{C}\). Consider the \((2,4)\)-diagonal matching field together with the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & 1 & 1
\end{pmatrix}.
\]

The induced function \(\chi : U_{2,4} \to \{z \in \mathbb{C} : |z| = 1\}\) takes the (ordered) basis 12 to \(i\) and every other basis to 1. It is easy to check that \(\chi\) is a (weak) matroid over \(\Phi\) but not over \(\mathbb{P}\), nor it is \(\mathbb{C}\)-realizable. However, with the natural Euclidean topology of \(\mathbb{C}, \mathbb{P}, \Phi\) and their corresponding Grassmannians (viewed as subsets of \(\mathbb{C} P^5\), \(\chi\) is a limit point of the Grassmannians \(\text{Gr}_P(2,4)\) as well as the subset of \(\mathbb{C}\)-realizable.
matroids: we can approximate $\chi$ by $p h^* \chi_R$ as $R \to \infty$, where $\chi_R$ is the matroid over $\mathbb{C}$ realized by the matrix

$$
\begin{pmatrix}
e^R & e^R & e^R & e^R \\
1 & i e^R & e^{2R} & e^{3R}
\end{pmatrix}.
$$

This suggests the potential application of matroids induced by matching fields in the general theory of hyperfields and their Grassmannians (we refer the reader to [5] for further discussion).

7. Conclusion

With a new point of view and machinery, we extend the connections between several prominent objects in matroid theory, discrete geometry, and tropical geometry beyond the “realizable” territory. We also initiate the study of the interaction between matroid subdivisions and signs, or more generally, hyperfields. As mentioned throughout in our paper, many of these ideas are interesting in its own right and could potentially be applied to other settings. Recall that the original motivation of matching fields comes from combinatorial commutative algebra and the corresponding algebraic geometry, it is also interesting to see if something can be said in these areas. We end with a few more open problems, focusing on aspects that were not fully discussed in the main content.

As already mentioned in the introduction, our work adds a new piece to the connection between the quest for a strongly polynomial algorithm for linear programming and a (weakly) polynomial algorithm for mean payoff games.

Many constructions of (sub)exponential instances for pivot rules for the simplex method are derived from parity or mean payoff games, respectively [21]. This was achieved by relating strategy iteration for these games with pivoting in the simplex method. On the positive side, one can solve mean payoff games using their equivalence with tropical linear programming [1] by a tropicalized version of the simplex method [2]. Our work gives the manifestation of this correspondence on the level of oriented matroids and matching fields, as further discussed in Remark 3.34.

An indicator on the hardness of mean payoff games comes from the richness of the oriented matroids arising for coherent matching fields. If only a subclass of uniform oriented matroids arises from coherent matching fields, this would exhibit a deep structural difference between linear programming and tropical linear programming.

Question 7.1. Which oriented matroids are realizable from a coherent matching field or a polyhedral matching field?

This adds well to the problems posed in Section 5 and the second part of the question goes even further in the direction to get tools for representing not-necessarily realizable oriented matroids. This is already interesting for pseudoline arrangements which occur frequently in combinatorics. In this case, the question reduces to the study of fine mixed subdivisions of dilated triangles $n \triangle_2$, which are substantially better understood than for arbitrary $d > 3$, see [7, 49]. Moreover, the two dimensional case has some extra symplectic flavor, and it might shed some light to our question (or vice versa): on one hand, the result of Itenberg–Shustin says that patchworking produces real pseudoholomorphic curves, on the other hand, the work of Ruberman and Starkton states that every pseudoline arrangement can be complexified into an arrangement of symplectic spheres [48].
The space of all matrices which give rise to a prescribed coherent matching field is a full-dimensional cone of the normal fan of the Newton polytope of the product of all maximal minors of a matrix of indeterminates \[55\]. Hence, the space of matrices which induce the same oriented matroid by the construction of Theorem \[A\] is a union \(U\) of cones. On the other hand, the realization space of an oriented matroid is in general a very complicated object \[41\]. One can consider such a realization space over Puiseux series and take its tropicalization \(V\), which is also a polyhedral complex; see \[35\] for more on tropicalization.

**Question 7.2.** What is the relation of the two sets \(U\) and \(V\)?

An answer to the latter question might give a new approach for the understanding of realization spaces. This is even interesting for the case \(d = 3\), as already uniform oriented matroids of rank 3 have arbitrarily complicated realization spaces.

Our proof of Theorem \[3.9\] as well as many results in the literature on matroid subdivisions, relies crucially on the reduction to the 3-term GP relation, which is a local condition and rather easy to check. This is only good enough to guarantee a weak matroid in Theorem \[6.8\] and the conclusion poses the obvious question of whether we can say something stronger.

**Question 7.3.** Is the function \(\chi\) in Theorem \[6.8\] always a strong matroid over \(\mathbb{H}\) as well?

If one wants to apply a similar polyhedral approach to this problem, it is likely that one has to analyze the global structure of matroid subdivisions. On the other hand, we do not rule out the possibility that counterexamples exist over imperfect hyperfields (with the IP). In any case, this problem might provide a polyhedral angle to understand the differences between strong and weak matroids.

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Then the following statements hold:

1. Define the collection of polytopes given by
   \[ S \subset \{ \sigma \in S \mid |{\sigma \cap T}| = n \} \cap \bigcap S \]
   where \( S, T \) are conformal, we have \( S \cap T \) is non-empty, then \( S \cap T \) is a face of both. If \( \sigma \cap T \) is in bijection with a nonempty subset \( I \subseteq R \).

2. For each \( \sigma \in S \), both \( S \) and \( \sigma \) can be recovered from \( \sigma S \).

3. For each \( \sigma S, \tau T \in S \), we have \( \sigma S \subseteq \tau T \) if and only if \( S \geq T \) and \( \sigma \subseteq \tau \).

### Proof

First note that (2) follows from the fact that each \( \sigma S = \square S + \sigma \cdot \sigma \) is a Minkowski sum of two affinely independent polytopes. Therefore, projection allows us to recover both \( \square S \) and \( \sigma \cdot \sigma \), and therefore the pair \((S, \sigma)\).

Recall the general fact that \( F \) is a proper face of the Minkowski sum \( K + L \) of two full-dimensional polytopes \( K \) and \( L \) if and only if there exists a non-zero objective function \( c \) such that \( F = Kc + Lc \), where \( Kc \) and \( Lc \) denote the faces of \( K \) and \( L \), respectively, maximized by \( c \). Specializing to the case \( K = \square d \) and \( L = n \hat{\square} d \), we have \( Kc = \square d \) and \( Lc = \sigma \cdot (n \triangle I) \), where \( S \) is the componentwise sign vector of \( c \), \( I \) is the set of all \( i \in R \) such that \( |c_i| = \max_{k \in R} |c_k| \), and \( \triangle I := \text{conv}(c_i : i \in I) \). It follows that the collection of proper faces of \( \square d + n \hat{\square} d \) is given by

\[ \left\{ \square S + \sigma \cdot (n \triangle I) : S \in \{-1, 0, 1\}^d \setminus 0, \emptyset \subseteq I \subseteq \text{supp}(S) \right\} . \]

Since \( \square S + \sigma \cdot (n \triangle I) \) is the union of the cells \( \{ \sigma S : \sigma \in S \} \), this shows that the cells in \( S \) cover the boundary of \( \square d + n \hat{\square} d \). The above fact about faces of Minkowski sums can also be used to show that the faces of \( \sigma S = \square S + S \cdot \sigma \) are given by \( \{\tau T : \text{face of } \sigma, T \geq S\} \). This establishes (3), and that \( S \) is closed under taking faces.

To establish (1), it remains to show that the intersection of two intersecting cells of \( \sigma S, \tau T \in S \) is a face of both. If \( \sigma S \cap \tau T \) is non-empty, then \( S \) and \( T \) are conformal sign vectors since otherwise, if \( i \) \( \in S \cap T \), then \( \sigma S \) would lie in the halfspace \( x_i \geq 1 \), while \( \tau T \) lies in the halfspace \( x_i \leq -1 \). Now, we would like to show \( \sigma S \cap \tau T = (\sigma \cap \tau)S \cap T \), where \( S \cap T \) denotes sign vector composition. As argued above, \( (\sigma \cap \tau)S \cap T \) is a face of both \( \sigma S \) and \( \tau T \). Thus, it remains to show that \( \sigma S \cap \tau T \) is contained in \( (\sigma \cap \tau)S \cap T \).

Suppose \( u + s = v + t \), where \( u \in \square S, v \in \square T, s \in S \cdot \sigma, t \in T \cdot \tau \). We are done if we can show \( u = v \) and \( s = t \). For this it suffices to show \( u_i = v_i \) for all \( i \in R \). Since \( S, T \) are conformal, we have \( u_i = v_i \) for all \( i \in \text{supp}(S) \cap \text{supp}(T) \). For
if there is a triangulation $\Delta$ of $P$ cell complex taking subsets can be realized as a geometric simplicial complex in some Euclidean space.

Given Proposition B.5. Every abstract simplicial complex (i.e. set system closed under order complex $\Delta(P)$ of a poset $P$ is the simplicial complex whose vertices are the elements of $P$ and whose simplices are the chains of $P$. We denote by $\|P\|$ the topological space $\|\Delta(P)\|$.

Proposition B.5. Every abstract simplicial complex (i.e. set system closed under taking subsets) can be realized as a geometric simplicial complex in some Euclidean space.

B.1.1. Regular cell complexes.

Definition B.1. A regular cell complex $\Delta$ is a Hausdorff space $\|\Delta\|$ together with a finite collection of balls $\Delta$ such that:

1. The interiors of the balls in $\Delta$ partition the space: $\|\Delta\| = \bigcup_{\sigma \in \Delta} \sigma^\circ$.
2. The boundary of any $\sigma \in \Delta$ is a union of members of $\Delta$: $\bd(\sigma) = \bigcup_{\tau \subset \sigma} \tau$.

Definition B.2. An important special case of the above definition is a polyhedral cell complex. This is a regular cell complex $\Delta$ such that each $\sigma \in \Delta$ is a polytope in $\mathbb{R}^d$, and for each $\sigma, \tau \in \Delta$ we have $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$. The underlying space of a polyhedral cell complex is a polyhedron. If every polytope in $\Delta$ is a simplex, we call $\Delta$ a geometric simplicial complex. A triangulation of a set $Q \subset \mathbb{R}^d$ is a geometric simplicial complex with underlying space $Q$.

Definition B.3. The face poset $\mathcal{P}(\Delta)$ of a regular cell complex $\Delta$ is the poset whose underlying set is the set of balls $\Delta$, and whose ordering is given by inclusion.

Definition B.4. The order complex $\Delta(P)$ of a poset $P$ is the simplicial complex whose vertices are the elements of $P$ and whose simplices are the chains of $P$. We denote by $\|P\|$ the topological space $\|\Delta(P)\|$.

B.1.2. PL balls and spheres.

Definition B.6. Given $P \subset \mathbb{R}^k$, $Q \subset \mathbb{R}^\ell$, a map $f : P \to Q$ is piecewise linear (PL) if there is a triangulation $\Delta$ of $P$ into simplices such that $f$ restricted to each simplex of $\Delta$ is an affine function. That is, if $\sigma = \conv(v_0, \ldots, v_k) \in \Delta$ then $f|_{\sigma}$ satisfies

$$f(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda_0 f(v_0) + \lambda_1 f(v_1) + \cdots + \lambda_k f(v_k)$$
for all convex combinations \( \sum \lambda_i v_i \) of the vertices \( v_0, \ldots, v_k \) of \( \sigma \). We call a PL map that is also a homeomorphism a PL homeomorphism.

**Definition B.7.** A polyhedron \( P \subset \mathbb{R}^k \) is a PL d-sphere (resp. PL d-ball) if there is a PL homeomorphism from \( P \) to the boundary of the standard d-simplex (resp. to the standard d-simplex).

**Proposition B.8.**

1. [14] Theorem 4.7.21(i)] The union of two PL d-balls, whose intersection is a PL \((d-1)\)-ball lying in the boundary of each, is a PL \(d\)-ball.

2. [14] Theorem 4.7.21(ii)] The union of two PL d-balls, which intersect along their entire boundaries, is a PL \(d\)-sphere.

3. [14] Theorem 4.7.21(iii)] (Newman’s Theorem) The closure of the complement of a PL d-ball embedded in a PL d-sphere is itself a PL d-ball.

**Lemma B.9.** Let \( \sigma, \tau \) be two PL d-balls, such that \( \sigma \cap \tau \) is a PL \((d-1)\)-ball contained in the boundaries of both \( \sigma \) and \( \tau \). Then the interior of \( \sigma \cup \tau \) is equal to \( \sigma^o \cup \tau^o \cup (\sigma \cap \tau)^o \).

**Proof.** By Proposition B.8(1), \( \sigma \cup \tau \) is a PL d-ball. We start by showing that \( \sigma^o \) contains \( (\sigma \cup \tau)^o \). Let \( x \in (\sigma \cup \tau)^o \). Then there is an open set \( U \subset (\sigma \cup \tau)^o \) containing \( x \) and a homeomorphism \( \varphi : U \to B^d_1 \subset \mathbb{R}^d \) sending \( x \) to \( 0 \). Here \( B^d_1 \) denotes the ball of radius 1 in \( \mathbb{R}^d \) centred at the origin. Since \( \sigma \) is closed in \( \sigma \cup \tau \), and since \( x \notin \tau \), we further have that \( \varphi(U \setminus \tau) \) is an open set which contains the origin: hence there exists \( \delta > 0 \) such that the scaled open ball \( \delta \cdot B^d_2 \) is contained in \( \varphi(U \setminus \tau) = \varphi(U \setminus \varphi(\tau)) \). It follows that \( \varphi^{-1}(\delta \cdot B^d_2) \) is an open neighbourhood of \( x \), homeomorphic to \( B^d_2 \), and entirely contained in \( \sigma \). In particular, this means that \( x \in \sigma^o \). We conclude \( \sigma^o \supseteq (\sigma \cup \tau)^o \).

From this containment we immediately get

\[
\partial \sigma \subseteq \sigma \setminus ((\sigma \cup \tau)^o \setminus \tau) = (\sigma \cap \tau) \cup (\sigma \cap \partial (\sigma \cup \tau)),
\]

and in particular

\[
U := \partial \sigma \setminus (\sigma \cap \tau) \subseteq \partial (\sigma \cup \tau).
\]

Since boundaries are closed, \( V := \partial (\sigma \cup \tau) \cap \partial \sigma \) is closed inside \( \partial \sigma \). Now \( W := \partial (\sigma \cap \tau) \) is the boundary of \( U \) in \( \partial \sigma \), thus it is contained in \( \overline{U} \subset V \subset \partial (\sigma \cup \tau) \).

Similarly, \( U' := \partial \sigma \setminus (\sigma \cap \tau) \subseteq \partial (\sigma \cup \tau) \). By Proposition B.8(3), both \( U \cup W, U' \cup W \) are PL \((d-1)\)-balls with common boundary \( W \), so by Proposition B.8(2), \( U \cup W \cup U' \) is a PL \((d-1)\)-sphere contained in \( \partial (\sigma \cup \tau) \). Invariance of Domain implies the containment is an equality, see for example [28 Corollary 2B.4]. After taking the complement with respect to \( \sigma \cup \tau \), this equality yields an expression for \((\sigma \cup \tau)^o \) which simplifies to \( \sigma^o \cup \tau^o \cup (\sigma \cap \tau)^o \). \( \Box \)

**B.1.3. Regular cell complexes that are PL spheres.**

**Definition B.10.** We say that a regular cell complex \( \Delta \) with face poset \( \mathcal{P} \) is a PL sphere if some realization of the order complex \( \Delta(\mathcal{P}) \) in some Euclidean space is a PL sphere.

**Proposition B.11 ([14] Proposition 4.7.26(iii)]).** Let \( \Delta \) be a regular cell complex that is a PL sphere. Then every \( \sigma \in \Delta \) is a PL ball.

An important fact about PL spheres is that they admit a dual cell structure:
Proposition B.12 ([14] Proposition 4.7.26(iv))). Let $\Delta$ be a regular cell complex that is a PL sphere. Then there exists a regular cell complex $\Delta^\vee$, also a PL sphere, such that $\|\Delta\| = \|\Delta^\vee\|$ and $P(\Delta^\vee) \simeq P(\Delta)^\vee$.

Here $P^\vee$ denotes the dual poset of $P$. In the special case when $\Delta$ is a polyhedral cell complex, there is a non-canonical way to construct this $\Delta^\vee$:

Definition B.13. Let $\Delta$ be a polyhedral cell complex. A first derived subdivision $\Delta^1$ is a subdivision of $\Delta$ obtained as follows: choose a point $x_\sigma$ in the relative interior of each $\sigma \in \Delta$. Then, $\Delta^1$ is given by

$$\Delta^1 := \{ \text{conv}(x_{\sigma_1}, \ldots, x_{\sigma_k}) : \sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \sigma_k, \text{ each } \sigma_i \in \Delta \}.$$ 

Theorem B.14 ([82] § 1.6]). If $\Delta$ is a polyhedral cell complex then $\Delta^\vee$ may be constructed as follows: Choose a first derived subdivision $\Delta^1$ of $\Delta$. For each cell $\sigma \in \Delta$, define

$$\sigma^\vee := \bigcap_{v \text{ vertex of } \sigma} \|\text{star}(v; \Delta^1)\|$$

where $\text{star}(\sigma; \Delta) := \{ \tau \in \Delta : \tau \text{ is contained in a cell containing } \sigma \}$. Then let

$$\Delta^\vee := \{ \sigma^\vee : \sigma \in \Delta \}.$$ 

B.2. Quotients of regular cell complexes. In this section we prove Theorem 4.16. Assume $\Delta$ is a regular cell complex with face poset $P$, and let $P/\sim$ be a homogeneous quotient. Recall that we define $\Delta/\sim := \{ \bigcup \hat{\sigma} : \hat{\sigma} \in P/\sim \}$ where $\bigcup \hat{\sigma}$ denotes the union $\bigcup_{\tau \in \hat{\sigma}} \tau$.

Lemma B.15. Suppose that each $\bigcup \hat{\sigma}$ in $\Delta/\sim$ is a ball whose interior equals the union of the interiors of the cells of $\hat{\sigma}$. Then $\Delta/\sim$ is a regular cell complex with face poset $P/\sim$.

Proof. We first show that $\Delta/\sim$ is a regular cell complex. It is clear that the underlying topological spaces of $\Delta$ and $\Delta/\sim$ are the same. To see that the interiors of the balls in $\Delta/\sim$ are disjoint, let $\bigcup \hat{\sigma}_1$ and $\bigcup \hat{\sigma}_2$ be two balls in $\Delta/\sim$ such that

$$(\bigcup \hat{\sigma}_1)^\circ \cap (\bigcup \hat{\sigma}_2)^\circ = \left( \bigcup_{\tau_1 \in \hat{\sigma}_1} \tau_1^\circ \right) \cap \left( \bigcup_{\tau_2 \in \hat{\sigma}_2} \tau_2^\circ \right) = \bigcup_{\tau_1 \in \hat{\sigma}_1} \tau_1^\circ \cap \bigcup_{\tau_2 \in \hat{\sigma}_2} \tau_2^\circ$$

is non-empty. In particular, there must exist $\tau_1 \in \hat{\sigma}_1$ and $\tau_2 \in \hat{\sigma}_2$ such that $\tau_1^\circ$ and $\tau_2^\circ$ intersect. This can only happen if $\tau_1 = \tau_2$, and hence $\hat{\sigma}_1 = \hat{\sigma}_2$. To see that the boundary of each $\bigcup \hat{\sigma}$ in $\Delta/\sim$ is a union of members of $\Delta/\sim$, let $\bigcup \hat{\sigma}$ be an element of $\Delta/\sim$. Then

$$\bigcup_{\tau < \hat{\sigma}} \left( \bigcup \hat{\tau} \right) = \bigcup_{\delta \in \hat{\sigma}} \bigcup_{\tau < \delta} \tau = \bigcup_{\delta \in \hat{\sigma}} \bigcup_{\tau < \delta} \tau^\circ. \quad (6)$$

We justify the last equality. We may write $\tau = \bigcup_{\gamma \leq \tau} \gamma^\circ$ for every $\tau \in \Delta$. Hence, the last equality holds provided we can show the following statement: whenever we have $\gamma \leq \tau < \delta \in \hat{\sigma}$ where $\tau \notin \hat{\sigma}$, we must also have $\gamma \notin \hat{\sigma}$. The condition $\gamma \leq \tau$ implies $\hat{\gamma} \leq \hat{\tau}$. The condition $\tau < \delta$ implies $\hat{\tau} < \hat{\delta} = \hat{\sigma}$. On the other hand, the condition $\tau \notin \hat{\sigma}$ implies $\hat{\tau} \notin \hat{\sigma}$, and therefore $\hat{\tau} < \hat{\sigma}$. We conclude $\hat{\gamma} \leq \hat{\tau} < \hat{\sigma}$, and in particular $\gamma \notin \hat{\sigma}$. Note that this argument uses the fact that $P/\sim$ is a poset, which
follows from homogeneity of $\sim$. Now, since the interiors of cells of $\Delta$ partition $\|\Delta\|$, we have by (6) that
\[
\bigcup_{\tilde{\tau} < \sigma} \left( \bigcup_{\delta \in \sigma} \left( \bigcup_{\tilde{\tau} \leq \delta} \gamma \right) \right) = \bigcup_{\delta \in \sigma} \left( \bigcup_{\gamma} \left( \bigcup_{\tilde{\tau} < \sigma} \gamma \right) \right) = \left( \bigcup_{\gamma} \right) \bigcup_{\tilde{\tau} < \sigma} \gamma.
\]
We therefore conclude
\[
\bd(\bigcup_{\sigma}) = (\bigcup_{\sigma}) \setminus (\bigcup_{\sigma})^o = (\bigcup_{\gamma}) \setminus \bigcup_{\tilde{\tau} < \sigma} \gamma = \bigcup_{\tilde{\tau} < \sigma} \left( \bigcup_{\sigma} \right).
\]

The proof that the face poset of $\Delta/\sim$ is $P/\sim$ is straightforward. If $\bigcup_{\tilde{\tau}} \subseteq \bigcup_{\tilde{\sigma}}$, then this means in particular that $\tilde{\tau} \subseteq \tilde{\sigma}$, hence $\tau \leq \sigma$ in $P$, hence $\tilde{\tau} \leq \tilde{\sigma}$ in $P/\sim$. Conversely, if $\tilde{\tau} \leq \tilde{\sigma}$ in $P/\sim$, then there exists a cell of $\tilde{\tau}$ contained in some cell of $\tilde{\sigma}$. By homogeneity, then, every cell of $\tilde{\tau}$ in contained in some cell of $\tilde{\sigma}$. Hence $\bigcup_{\tilde{\tau}} \subseteq \bigcup_{\tilde{\sigma}}$. $\square$

**Proposition B.16** ([12] Section 4.7, pp.204). Let $\Delta$ be a regular cell complex with face poset $P$. Then the augmented poset $L(P) = P \cup \{0, 1\}$ is a lattice if and only if $\Delta$ is closed under non-empty intersections: for all $\sigma, \tau \in \Delta$ such that $\sigma \cap \tau$ is non-empty, we have $\sigma \cap \tau \in \Delta$. $\square$

Recall the statement of Theorem 4.16, which says that if $\Delta$ is a regular cell complex with face poset $P$, $P/\sim$ is an elementary quotient such that $L(P)$ is a lattice, and each $\sigma \in \Delta$ is a PL ball, then $\Delta/\sim$ is a regular cell complex with face poset $P/\sim$ whose cells are PL balls.

**Proof of Theorem 4.16** It is clear that for any singleton class $\tilde{\sigma} = \{\sigma\}$, $\bigcup_{\tilde{\sigma}}$ satisfies the hypothesis of Lemma B.15. Now suppose $\tilde{\sigma} = \{\sigma, \tau, \gamma\}$ is a class in $\sim$. It is known that the function $\sigma \mapsto \dim(\sigma)$ is a rank function on $P$. In particular, since $\sigma$ and $\tau$ cover $\gamma$, then we must have $\dim(\sigma) = \dim(\tau) = \dim(\gamma) + 1$. Moreover, since $L(P)$ is a lattice, we must have $\gamma = \sigma \cap \tau$ by Proposition B.16, Proposition B.8 (1) and Lemma B.9 then show that $\bigcup_{\tilde{\sigma}}$ is a PL ball which satisfies the hypothesis of Lemma B.15. $\square$

**Appendix C. Elimination systems and quotients**

In this section we prove Theorem 4.15. We assume we are given an elimination system $S$ on $R \times E$, and a sign matrix $A \in \{-1, 1\}^{R \times E}$. We denote the poset $P(S)$ by $P$.

**Proposition C.1.** Suppose $(S, F)$ is covered by $(T, G)$ in $P$. Then either $F = G$ and $|S| = |T| - 1$, or $S = T$ and $|F| = |G| + 1$.

**Proof.** The fact that $(S, F) \leq (T, G)$ means that $S \leq T$ and $F \supseteq G$, and either $S \leq T$ or $F \supseteq G$. If $S \leq T$, then let $i \in \supp(T) \setminus \supp(S)$. Then $i \notin \supp(S)$, which means $i \notin \supp(F)$. Since $F \supseteq G$, this means $i \notin \supp_R(G)$. Hence, $(T \setminus i, G)$ is an element of $P$ such that
\[
(S, F) \leq (T \setminus i, G) \leq (T, G).
\]
Since $(S, F)$ is covered by $(T, G)$, we conclude the first inequality holds with equality, and hence $F = G$ and $|S| = |T| - 1$. 

Otherwise, \( F \supseteq G \). Let \((i, j) \in F \setminus G\). Then \((i, j)\) is not the only element of \( F_j \), since otherwise we would have \( G_j = \emptyset \) which is forbidden by (E1). We therefore have \((S, F \setminus (i, j)) \in P\) by (E2), and hence \((S, F) \leq (S, F \setminus (i, j)) \leq (T, G)\).

By covering, we conclude the second inequality holds with equality, and hence \( S = T \) and \(|F| = |G| - 1\). \( \square \)

**Corollary C.2.** The poset \( P \) is graded, with grading \( \rho(S, F) = n + |S| - |F| \). \( \square \)

Given two sign vectors \( S, T \in \{-1, 0, 1\}^R \), define their intersection \( S \cap T \in \{-1, 0, 1\}^R \) to be the sign vector such that \( (S \cap T)^+ = S^+ \cap T^+ \) and \( (S \cap T)^- = S^- \cap T^- \).

**Proposition C.3.** The augmented poset \( \mathcal{L}(P) := P \cup \{\hat{0}, \hat{1}\} \) is a lattice: if \((S, F), (T, G) \in P\) have a common lower bound, then a greatest lower bound for both is given by \((S \cap T, F \cup G)\).

**Proof.** Let \((S, F)\) and \((T, G)\) be elements of \( P \) with a common lower bound \((L, H)\). Then \( H \supseteq F \cup G \supseteq F, G \) which implies by (E2) that \( F \cup G \in \mathcal{S} \). Similarly, we have \( L \leq S \cap T \) and so

\[
\text{supp}_R(F \cup G) \subseteq \text{supp}_R(H) \subseteq \text{supp}(L) \subseteq \text{supp}(S \cap T).
\]

We conclude \((S \cap T, F \cup G) \in P\) and is a lower bound of \((S, F)\) and \((T, G)\). The fact that \( H \supseteq F \cup G \) and \( L \leq S \cap T \) shows that \((S \cap T, F \cup G)\) is in fact a greatest lower bound, as \((L, H)\) was chosen arbitrarily. \( \square \)

Our next task is to generalize the equivalence relation \( \sim_A \) on \( P \) from Definition 1.13 by allowing the partition \( \Pi \) of \( R \times E \) to vary. We assume fixed a partition \( \Pi \) of \( R \times E \) which refines the partition \( \{R \times \{j\} : j \in E\} \). Recall that \( X \sim Y \) means \( X^* \cap \pi \) is nonempty iff \( Y^* \cap \pi \) is nonempty, for all \( s \in \{-, +\} \) and \( \pi \in \Pi \).

**Definition C.4.** For \((S, F), (T, G) \in P\), we say \((S, F) \sim_A (T, G)\) if and only if \( S = T \) and \( SA_F \sim SA_G \).

**Proposition C.5.** The equivalence relation \( \sim_A \) on \( P \) is \( P \)-homogeneous. In particular, \( P/\sim_A \) is a poset.

**Proof.** Let \((S, F) \leq (T, G)\) be two elements of \( P \), and choose \((S, F') \sim_A (S, F)\). Our goal is to find \( G' \in \mathcal{S} \) such that \((T, G') \in P\) and \((S, F') \leq (T, G') \sim_A (T, G)\). Define

\[
G' := \{(i, j) \in F' : \text{if } \pi \in \Pi \text{ contains } (i, j), \text{ then there exists } (\ell, j) \in \pi \text{ such that } (T_{AG})_{\ell,j} = (SA_{F'})_{i,j}\}.
\]

Thus \((S, F') \leq (T, G')\). The definition of \( G' \) ensures that every sign appearing in the restricted sign vector \( T_{AG} \mid \pi \) also appears in \( T_{AG} \mid \pi \), for all \( \pi \in \Pi \). Conversely, if \( \pi \in \Pi \) and \((T_{AG})_{\ell,j}\) is nonzero for some \((\ell, j) \in \pi\), then \( SA_F \sim SA_G = T_{AG} \) implies there exists \((i, j) \in \pi\) such that \((T_{AG})_{i,j} = (SA_F)_{i,j} = (SA_G)_{i,j}\), and therefore \( T_{AG} \mid \pi \) contains the sign \((T_{AG})_{\ell,j}\). Note that we are using here the fact that \( \Pi \) refines the partition \( \{R \times \{j\} : j \in E\} \). We conclude \( T_{AG} \sim T_{AG'} \).

Observe that \( G_j \) is nonempty for every \( j \in E \) by (E1), and since \( T_{AG} \sim T_{AG'} \) we also have \( G'_j \) is nonempty for every \( j \in E \). Therefore, since \( G' \subseteq F' \), we have by (E2) that \( G' \in \mathcal{S} \). Moreover, \( \text{supp}_R(G') \subseteq \text{supp}_R(F') \subseteq \text{supp}(S) \subseteq \text{supp}(T) \), so that \((T, G') \in P\). We conclude \((T, G') \sim_A (T, G)\). \( \square \)
For a generalized sign vector $X/\sim \in \{0, +, -, \pm\}^\Pi$, let $|X/\sim|$ count the number of nonzero coordinates in $X/\sim$, with each $\pm$ counted twice. For example, if $X/\sim = (0, \pm, +, -) = (0, \pm, +, -)$ then $|X/\sim| = 7$. Note that if $\Pi$ is the singleton partition, then $X/\sim$ is an ordinary sign vector and $|X/\sim| = |X|$.

**Proposition C.6.** The poset $P/\sim_A$ is graded, with grading

$$\rho((S, F)/\sim_A) = n + |S| - |SA_F/\sim|. $$

**Proof.** Fix $(S, F) \in P$. First note that $(S, F)$ is a maximal element in the equivalence class $(S, F)/\sim_A$ if and only if $(|SA_F|^+ \cap \pi| \leq 1$ for all $s \in \{-, +\}$ and all $\pi \in \Pi$. Indeed, choose any $(S, G) \sim_A (S, F)$. Then (E2) implies that we may find $(S, H) \geq (S, G)$ inside $(S, F)/\sim_A$ such that $(|SA_H|^+ \cap \pi| \leq 1$ for all $s \in \{-, +\}$ and all $\pi \in \Pi$. In particular, this statement holds for the maximal elements of $(S, F)/\sim_A$.

Now, for every maximal element $(S, G) \sim_A (S, F)$, we have

$$\rho((S, G)/\sim_A) = n + |S| - |SA_G/\sim|$$

$$= n + |S| - \sum_{\pi \in \Pi} (|SA_G|^+ \cap \pi| + |SA_G|^\cap \pi|)$$

$$= n + |S| - |G|$$

$$= \rho(S, G).$$

It remains to show that $\rho$ respects the covering relations. Suppose that $(S, F)/\sim_A$ is covered by $(T, G)/\sim_A$ in $P/\sim_A$. By homogeneity, we may choose representatives $(S, F)$ and $(T, G)$ so that $(S, F)$ is covered by $(T, G)$ in $P$. Such an element $(S, F)$ is necessarily a maximal element of the equivalence class $(S, F)/\sim_A$, which implies $|(SA_F|^+ \cap \pi| \leq 1$ and $|(SA_F|^\cap \pi| \leq 1$ for all $\pi \in \Pi$. Since $(S, F) < (T, G)$, we have $TA_G = SA_G \leq SA_F$, and hence $|(TA_G|^+ \cap \pi| \leq 1$ and $|(TA_G|^\cap \pi| \leq 1$ for all $\pi \in \Pi$. It follows $(T, G)$ is maximal in $(T, G)/\sim_A$. We conclude

$$\rho((S, F)/\sim_A) = \rho(S, F) = \rho(T, G) - 1 = \rho((T, G)/\sim_A) - 1. \quad \Box$$

**Proposition C.7.** The augmented poset $\mathcal{L}(P/\sim_A)$ is a lattice.

**Proof.** Choose $(S, F)/\sim_A$ and $(T, G)/\sim_A$ with a common lower bound in $P/\sim_A$. By homogeneity and Proposition C.3, we may choose the representatives $(S, F)$ and $(T, G)$ so that $(S \cap T, F \cup G) \in P$. By homogeneity, then, $(S \cap T, F \cup G)/\sim_A$ is a lower bound for both $(S, F)/\sim_A$ and $(T, G)/\sim_A$.

We show this is a greatest lower bound. Given a lower bound $(L, H)/\sim_A$, we may find $(S, F') \sim_A (S, F)$ and $(T, G') \sim_A (T, G)$ such that $(L, H) \leq (S, F')$ and $(L, H) \leq (T, G')$ in $P$. Hence, by Proposition C.3, $(L, H) \leq (S \cap T, F' \cup G') \in P$. Therefore, it suffices to show

$$(S \cap T, F' \cup G') \sim_A (S \cap T, F \cup G).$$

For all $\pi \in \Pi$ and $s \in \{-, +\}$, we have

$$((S \cap T)A_{F' \cup G'})^s \cap \pi \text{ nonempty} \iff ((SA_F)^s \cup (TA_{G'})^s) \cap \pi \text{ nonempty}$$

$$\iff ((SA_F)^s \cup (TA_{G})^s) \cap \pi \text{ nonempty}$$

$$\iff ((S \cap T)A_{F \cup G})^s \cap \pi \text{ nonempty}. $$

In particular, this shows $(S \cap T, F' \cup G') \sim_A (S \cap T, F \cup G). \quad \Box$
Recall our objective: we wish to prove Theorem 4.15 which states that $\mathcal{P}/\sim_A$ admits a factorization $\mathcal{P} = P_0, P_1, \ldots, P_k = \mathcal{P}/\sim_A$ into elementary quotients, such that $\mathcal{L}(P_i)$ is a lattice for each $i$. Here $\Pi$ denotes the partition $\{R \times \{j\} : j \in E\}$.

**Proof of Theorem 4.15.** By Proposition C.3, $\mathcal{L}(\mathcal{P})$ is a lattice. Thus, let $\Pi$ be a partition of $\mathcal{P}$ which refines the partition $\Pi$ and has at least one part $\pi \in \Pi$ such that $|\pi| \geq 2$. Let $e := (i, j) \in \pi$, and let $\bar{\Pi}$ be the refinement of $\Pi$ obtained by splitting the part $\pi$ into two parts: $\{e\}$ and $\pi \setminus \{e\}$. That is,

$$\bar{\Pi} = (\Pi \setminus \{\pi\}) \cup \{\{e\}, \pi \setminus \{e\}\}.$$ 

Let $\sim$ and $\sim_A$ denote the equivalence relations on sign vectors on $R \times E$ induced by $\Pi$ and $\bar{\Pi}$, respectively. These determine $\mathcal{P}$-homogeneous equivalence relations $\sim_A$ and $\sim_A$ by Proposition C.5. Let $\bar{\Pi} = \mathcal{P}/\sim_A$. Since $\sim_A$ is $\mathcal{P}$-homogeneous, and since $\sim_A$ refines $\sim_A$, we have that $\sim_A$ is $\bar{\Pi}$-homogeneous. Moreover, there is a natural identification $\bar{\Pi}/\sim_A = \mathcal{P}/\sim_A$. Therefore, by induction, the theorem is proved if we can show that $\bar{\Pi}/\sim_A$ is an elementary quotient whose augmented poset is a lattice. In fact the lattice assertion follows from Proposition C.7.

Fix $(S, F) \in \mathcal{P}$. We would like to show that the equivalence class containing $(S, F)/\sim_A$ in $\bar{\Pi}/\sim_A$ is either a singleton, or consists of exactly three elements two of which cover a third. Note that if $SA_F|_{\pi}$ is the zero vector, then this equivalence class is indeed a singleton. This is because we would immediately know that $(SA_F)|_{\pi} = 0$ and $SA_F|_{\pi \setminus \{e\}} = 0$, hence in this case $(S, F)/\sim_A$ is completely determined by $(S, F)/\sim_A$.

Otherwise, the sign vector $SA_F|_{\pi}$ is non-zero, and in this case there are exactly three generalized sign vectors $X_1, X_2, X_3/\sim_A \in \{0, -, +, \pm\}^\Pi$, depending on $SA_F|_{\pi}$ and $(SA)_e$, such that $X_1 \sim X_2 \sim X_3 \sim SA_F$. The restrictions of these to $\pi$ are depicted below, in all of four possible cases:

\[
\begin{array}{cccc}
\pi \setminus e & e & \sim & \pi \\
X_1 & X_2 & X_3 & SA_F \\
\ | & | & | & \\
0 & + & + & \pm \\
\pm & - & \pm & \\
\end{array}
\]

\[
\begin{array}{cccc}
\pi \setminus e & e & \sim & \pi \\
X_1 & X_2 & X_3 & SA_F \\
\ | & | & | & \\
0 & + & + & \pm \\
\pm & + & \pm & \\
\end{array}
\]

\[
\begin{array}{cccc}
\pi \setminus e & e & \sim & \pi \\
X_1 & X_2 & X_3 & SA_F \\
\ | & | & | & \\
0 & + & + & + \\
+ & 0 & + & \\
\end{array}
\]

\[
\begin{array}{cccc}
\pi \setminus e & e & \sim & \pi \\
X_1 & X_2 & X_3 & SA_F \\
\ | & | & | & \\
0 & - & - & - \\
- & 0 & - & \\
\end{array}
\]

The following argument applies simultaneously to all four cases shown above. Suppose there are at least two distinct elements $(S, F)/\sim_A$ and $(S, F')/\sim_A$ in the same equivalence class of $\bar{\Pi}/\sim_A$. Then there exists a unique $i \in \{1, 2, 3\}$ such that

$$\{SA_F/\sim_A, SA_F'/\sim_A, X_i/\sim_A\} = \{X_1/\sim_A, X_2/\sim_A, X_3/\sim_A\},$$

for some $i \in \{1, 2, 3\}$. We consider the three cases separately.

- If $i = 1$ or $2$, then without loss of generality assume $SA_F \sim X_3$.
  - If $i = 1$, then by (E2) the set $F'' = F \setminus e$ is in $S$, and $SA_F'' \sim X_1$.
  - If $i = 2$, then by (E2) the set $F'' = F \setminus ((SA_F)^{\prime} \cap \pi)$ is in $S$, where $s$ is the unique sign appearing in $X_3|_{\pi \setminus e}$ but not $X_2|_{\pi \setminus e}$, and $SA_F'' \sim X_2$.
• If $i = 3$, then by (E3), we can find $F'' \in S$ such that

$$SA_{F''}|_\pi = SA_{F' \cup F''}|_\pi$$

$$SA_{F''}|_\tau \in \{SA_F|_\tau, SA_{F'}|_\tau, SA_{F' \cup F''}|_\tau\} \text{ for all } \tau \in \bar{\Pi} \setminus \pi.$$ 

This shows $SA_{F''} \sim X_3$.

In all three cases, we therefore have found $(S, F'') \sim_A (S, F)$ such that $SA_{F''} \sim X_i$. Therefore the equivalence class of $(S, F)/\sim_A$ in $\bar{P}/\sim_A$ consists of the three distinct elements $(S, F)/\sim_A$, $(S, F')/\sim_A$, and $(S, F'')/\sim_A$. Their gradings in $\bar{P} = P/\sim$ are given by, by Proposition C.6 $n - |S| - |X_i/\sim|$ for $i = 1, 2, 3$. Inspecting the above four tables, we conclude that two of these elements cover the third in $\bar{P}$. \qed

Technische Universität Berlin, Institut für Mathematik, Sekr. MA4-1, Straße des 17 Juni 136, D-10623 Berlin
E-mail address: celaya@math.tu-berlin.de

Department of Mathematics, London School of Economics and Political Science, London, WC2A 2AE, UK
E-mail address: g.loho@lse.ac.uk

Division of Applied Mathematics, Brown University, Providence, RI 02912, USA
E-mail address: ChiHoYuen@Brown.edu