Real time dynamics quantum simulation of (1+1)-D lattice QED with Rydberg atoms

Simone Notarnicola,1 Mario Collura,1,2 and Simone Montangero1

1Dipartimento di Fisica e Astronomia “G. Galilei” & INFN, via Marzolo 8, I-35131, Padova, Italy
2Theoretische Physik, Universität des Saarlandes, D-66123 Saarbrücken, Germany.

We show how to implement a Rydberg-atom quantum simulator to study the non-equilibrium dynamics of an Abelian (1+1)-D lattice gauge theory. The implementation locally codifies the degrees of freedom of a $\mathbb{Z}_3$ gauge field, once the matter field is integrated out by means of the Gauss’ local symmetries. The quantum simulator scheme is based on current available technology and scalable to considerable lattice sizes. It allows, within experimentally reachable regimes, to explore different string dynamics and to infer information about the Schwinger $U(1)$ model.

Rydberg-atom systems are nowadays one of the most promising platform in the field of quantum simulation for the achievement of results unaccessible via classical numerical simulations [1–4]. Single neutral atoms with a Rabi frequency coupling their internal ground state to an highly excited Rydberg state constitute a benchmarked qubit prototype [5–8]. Optical tweezer arrays allow to arrange a large number of atoms geometries, from one dimensional lattices to three-dimensional structures [2, 9]. The van der Waals interactions between excited atoms result in the Rydberg blockade mechanism [10], which recently led to the observation of phase transitions in 1D lattices and to the realization of Schrodinger’s cat states [3, 11, 12]. In the last decade, a lot of effort has been spent to investigate the properties of lattice gauge theories by exploiting suitable mappings towards universal quantum simulators [13, 14]. First proposals made use of ultracold atoms trapped in optical lattices to engineer the degrees of freedom of Abelian and non-Abelian lattice gauge theories [13–21]. In recent experiments, trapped ions have been used to explore the mechanism of pairs of particle-antiparticle production as well as the equilibrium properties of the Schwinger model [22, 23]. Lately, combined quantum and classical computations have been performed by using available quantum computers [24]. Finally, a connection between a Rydberg atom experiment and 1D gauge quantum link models has been unveiled [25]. However, to date, there is no versatile quantum simulation proposal available to study gauge theory real-time dynamics on scalable systems.

Here we introduce a quantum simulator scheme to study the dynamics of an Abelian lattice gauge theory in which an initial electric field string can persist in time or break, in analogy with predictions of QCD [26–28]. We consider the Schwinger model with spinless fermions coupled to the electric field defined on links. We integrate out the matter field degrees of freedom and derive the Hamiltonian relative to the dynamics of the gauge field only. We approximate then the Schwinger model by replacing the continuous symmetry group $U(1)$ with $\mathbb{Z}_n$ in which the electric field spectrum is discretized and truncated. We choose $n = 3$ so that the electric field locally takes three values $(0, \pm 1)$ – Fig. 1 (a). As a consequence, its energy contribution is non trivial (see Eq. (3)), and we observe different string behaviors depending on the values of the couplings. By using three Rydberg atoms to represent each link we map the eigenstates of the local electric field into the configurations with only one excited atom. The atomic lattice string is represented by the separable state $|s(0)\rangle$ at $\tau = 0$: By setting the parameters in the Hamiltonian $\hat{H}_c$ (see Eq. (4)) the system evolves to the state $|s(\tau_f)\rangle$ in which the initial string is broken. Numerical parameters of Hamiltonian in Eq. (3): $m = 0$, $t = 0.682$ MHz, $g^2/t = 0.5$.

![FIG. 1. (a) Dynamics of the $\mathbb{Z}_3$ gauge and fermion field relative to the Hamiltonian $\hat{H}$ of Eq. (3): A rotation of the local gauge field state is accompanied by a fermion hopping process which preserves gauge invariance. (b) The ground state of the atoms (empty dot) is coupled to a Rydberg state (full dot) by a Rabi frequency with a blue-shifted detuning; given a set of three equidistant atoms, the electric field eigenstates $|E\rangle_j$ are mapped into those configurations with only one excited atom. (c) Chain of atomic triangular sets. An electric field string is represented by the separable state $|s(0)\rangle$ at $\tau = 0$: By setting the parameters in the Hamiltonian $\hat{H}_c$ (see Eq. (4)) the system evolves to the state $|s(\tau_f)\rangle$ in which the initial string is broken. Numerical parameters of Hamiltonian in Eq. (3): $m = 0$, $t = 0.682$ MHz, $g^2/t = 0.5$.](image-url)
The Hamiltonian for the (1+1)-D lattice model is [31, 32]. Hereafter, we set $\hat{\Delta} = 0$ and we consider the bare-vacuum dynamics of the Hamiltonian $\hat{H}_3$ and of the $U(1)$ models: Our quantum simulator remarkably allows to study properties of both the continuous and the discrete gauge model.

The Hamiltonian for the (1+1)-D lattice QED in the Schwinger formulation is [29, 30]

$$\hat{H}_S = \sum_j \left[ -t \hat{\psi}_j^{\dagger} \hat{U}_j \hat{\psi}_{j+1} + \text{h.c.} + m \hat{p}_j \hat{n}_j + g^2 \hat{E}_j^2 \right],$$

where a staggered, fermionic, spinless matter field with mass $m$ is defined on the lattice sites. It satisfies $\{ \hat{\psi}_j, \hat{\psi}_k \} = \delta_{jk}$ and $\{ \hat{\psi}_j, \hat{\psi}_k \} = \{ \hat{\psi}_j^{\dagger}, \hat{\psi}_k^{\dagger} \} = 0$, while $\hat{n}_j = \hat{\psi}_j^{\dagger} \hat{\psi}_j$ and $p_x = (-1)^j$. The gauge field propagator $\hat{U}_j$ and the electric field $\hat{E}_j$ are defined on each link between the nearest-neighbor sites $j, j + 1$: They commute according to $[\hat{E}_j, \hat{U}_k] = \delta_{jk} \hat{U}_j$ and $g^2$ is the electric field energy coupling.

Due to the staggering, the electrons (positrons) are represented by filled (empty) even (odd) sites and, therefore, the gauge-matter interaction term proportional to $t$ is responsible for electron-positron pair creation and annihilation. During these processes, the electric field is incremented or decremented in order to satisfy the Gauss’ law on each site: Equivalently, given the set of gauge operators $\hat{Q}_j = \hat{\Delta} \hat{E}_j - \frac{1}{2} \hat{p}_j + \hat{n}_j$, with $\hat{\Delta} \hat{E}_j = \hat{E}_j - \hat{E}_{j-1}$ and $e = -1$, we have $[\hat{H}_S, \hat{Q}_j] = 0 \forall j$. It follows that the Hamiltonian has a block-diagonal form in the basis of the gauge operators’ eigenstates: Each block is identified by a set of static charges $g_j$ and the relative states satisfy $\hat{Q}_j | \psi \rangle = g_j | \psi \rangle \forall j$.

Once the boundary conditions and the set of static charges $g_j$ are fixed, the gauge operators $\hat{Q}_j$ fix a one-to-one correspondence between the eigenstates of the matter and the electric fields operators $\hat{\psi}_j^{\dagger} \hat{\psi}_j$ and $\hat{E}_j$, indicated as $\{|m, \mathcal{E}\}$ and $\{|\mathcal{E}\rangle \}$ respectively: Therefore, the basis of the gauge sector characterized by $\{ g_j \}$ is in the form $|m, \mathcal{E}; \{ g_j \} \rangle$. It follows that $\hat{H}_S$ can be recast in each sector as a function of the matter or the gauge field operators [31, 32]. Hereafter, we set $g_j = 0 \forall j$ and write the Hamiltonian as a function of the gauge field operators: The Hamiltonian is still local, namely (see the Supplementary Material (SM))

$$\hat{H}_S^\theta = -t \sum_j \hat{U}_j \hat{P}_{\Delta E_j}(0) \hat{P}_{\Delta E_{j+1}}(1) + \text{h.c.}$$

$$+ m \sum_j \left( \frac{1}{2} \hat{p}_j - \hat{E}_j \right) + g^2 \sum_j \hat{E}_j^2.$$


\[ \langle \theta \rangle = 0, 1 \mid -1 \rangle \mid -1 \rangle \mid 0 \rangle \mid 1 \rangle \mid 0 \rangle \mid 1 \rangle \quad d \kappa \gamma = 2 \gamma = 0 \gamma = 1 \]

(b) A label \( \vartheta \) is assigned to the atoms, uniformly for each link, to map each \( |E_j\rangle \) into the states \( |\vartheta j\rangle \).

**Z\(_3\)** model dynamics.— Although the ground state of the Hamiltonian \( \hat{H} \) is always confined, the unitary dynamics induced by quenching the ratio \( g^2 / \ell \equiv \Gamma \) allows us to observe different regimes [28]. We take as initial state \( |s(0)\rangle \) an electric field string of length \( s \) originated by a positron-electron pair: The state evolves as \( |s(\tau)\rangle = \exp(-i\hat{H}\tau) |s(0)\rangle \) (\( \hbar = 1 \)).

For \( \Gamma \lesssim 1 \) fluctuations of local charge density lead to the creation of positron-electron pairs which annihilate the electric field between them and break the string: We show an example of this dynamics in Fig. 2 (a), where a chain of \( L = 21 \) links has been prepared with a string originated from an electron on the site \( j = 8 \) and a positron on the site \( j = 14 \). By quenching the coupling from \( g = 0 \) to \( \Gamma = 0.5 \) the string breaks during its evolution. A different scenario emerges by choosing instead \( \Gamma > 1 \): In Fig. 2 (c) we take the same initial state and evolve it with \( \Gamma = 4.0 \). Due to the large gap between the zero and nonzero electric field states the string breaking process is strongly off-resonant and does not occur in accessible times. It is worth noting the oscillations in the middle of the string, which are a peculiar of the \( \mathbb{Z}_3 \) model: The electric field is oscillating between the two local degenerate states \( |E = \pm 1\rangle \) and the transient in which \( \langle E_j\rangle = 0 \) is due to their superposition during the population inversion process.

In the limit \( \Gamma > 1 \) the \( \mathbb{Z}_3 \) model better approximates the Schrödinger one. By taking as initial state the bare vacuum, its dynamics is naturally constrained in the low energy sector due to the large electric field coupling and it is not affected by the truncation of the electric field spectrum: As an example, in Fig. 4 (upper panel) we show the local dynamics of the electric field for \( \Gamma = 4 \): The \( \mathbb{Z}_3 \) dynamics (red curve) and the Schrödinger one (squares) coincide. On the opposite regime, when tuning \( \Gamma = 0.5 \), after a transient in which the dynamics of the two models coincides, it eventually relaxes to different values (see bottom panel of Fig. 4).

**Rydberg quantum simulator.**— Our quantum simulator consists of a quasi-1D lattice of neutral atoms coupled to a Rydberg state \( nS \), with \( n \gg 1 \), by an effective Rabi frequency \( \Omega \). The atoms are initially trapped into a tweezers array [9, 35, 36] and then released. By locally modulating the laser detuning, a configuration in which some atoms are excited to a Rydberg state and the others are in their internal ground state is created.

A non-trivial dynamics is then induced by remodulating the laser detuning: The atoms move from ground to Rydberg states and thus interact among each other. In the following we show that this process effectively reproduces the gauge invariant dynamics of the electric field in the \( \mathbb{Z}_3 \) model. We start by showing that the gauge invariant electric field eigenstates are mapped into a set of atomic configurations in which the atoms are in their ground or Rydberg states.

In general, two atoms at distance \( r \) can be simultaneously excited only if \( \Omega > V_\text{r} = c_0 / r^6 \), where \( V_\text{r} \) is the Van der Waals interaction energy, due to the so-called Rydberg blockade mechanism. \( c_0 \) depends on the atomic species and on the specific excited state [10]. We impose gauge invariance by mapping gauge-breaking states into atomic configurations forbidden by the Rydberg blockade.

The lattice is shaped as a prism with an equilateral triangular section, as shown in Fig. 3 (a). Sets of three atoms, called links, lay in planes perpendicular to the main prism axes. The links are placed at a fixed distance \( d \) among each others while the distance between atoms in the same set is \( \ell \). Their dynamics is described by the Hamiltonian [2, 3, 37]:

\[ \hat{H}_r = \sum_{j, \vartheta} \left[ \Omega \hat{\sigma}_j^x \delta_{\vartheta}^j - \Delta_j \delta_{\vartheta}^j n_{j, \vartheta}^2 + \frac{1}{2} \sum_{\vartheta, k} V_{j, k}^\vartheta \delta_{\vartheta}^j n_{j, \vartheta}^2 \right], \quad (4) \]

where each atom is labeled by the link-index \( j \) and \( \vartheta \) which indicates the position inside the link (Fig. 3 (b)).

The ground state of each atom \( |g\rangle_{j, \vartheta} \) is coupled to the excited state \( |r\rangle_{j, \vartheta} \) by the operator \( \hat{\sigma}_j^x \delta_{\vartheta}^j \) with Rabi frequency \( \Omega \). The projector \( \hat{\pi}_{j, \vartheta} = |r\rangle_{j, \vartheta} \langle r|_{j, \vartheta} \) is multiplied by a detuning term with \( \Delta_j = - \delta_{j\vartheta} > 0 \) and \( \delta_{j\vartheta} \ll \Delta \).

The interaction depends only on the distance between the atoms. We call \( V_{l} = V_{j, j+1}^\vartheta \), \( V = V_{j, j+1}^\vartheta \) with \( \vartheta \neq \vartheta' \) and \( V_d = V_{j, j+1}^{\vartheta, \vartheta'} \). We neglect \( V_{j, k}^{\vartheta, \vartheta'} \) with \( |k - j| > 1 \) since they are much smaller than the other energies involved.

We first map the gauge invariant states into a set of atomic configurations and then we map the Hamiltonian \( \hat{H} \) into \( \hat{H}_r \). We choose \( \Delta \gg \Omega \) and \( V_l \gg \Omega \): for each single link, Rydberg excitations are enhanced by the large detuning but simultaneous excitations are prevented by the Rydberg blockade. By applying second-order perturbation theory we restrict the dynamics of the \( j \)-th link to the subspace spanned by set of states \( \Sigma_j = \{ |\vartheta j\rangle \}_{0 \leq \vartheta \leq 2} \) in which the atom in position \( \vartheta \) is excited (Fig. 1 (a)).
We now consider a chain of L links, arranged as shown in Fig. 3, with the distance d between consecutive links such that \( V_d \gg \Omega \) and \( V \sim \Omega \). Simultaneous excitations of aligned atoms (red filled circles) are forbidden, while non-aligned excitations (blue filled circles) are allowed. We map in a staggered fashion the electric field eigenstates \( |E_j \rangle \) into the states \( |\vartheta_j \rangle \) according to \( \vartheta_j = (E_j + 4 + (-1)^j) \mod 3 \). The set of allowed atomic configurations \( \Phi \subset \bigotimes_{j=1}^L \Sigma_j \) corresponds to the set of the \( \mathbb{Z}_3 \) gauge invariant states. In Fig. 3 (a) we show two neighboring links and the site in between with charge \( q_j = 0, +1 \). Since the electric field cannot decrease from the link \( j - 1 \) to the link \( j \), electric field states such as \( |0 \rangle_{j-1} |1 \rangle_j \) are mapped into configurations of excited atoms prevented by Rydberg blockade. On the other side, the configurations corresponding to the states \( |0 \rangle_{j-1} |0 \rangle_j \) or \( |0 \rangle_{j-1} |1 \rangle_j \) are allowed.

In order to map the Hamiltonian \( \hat{H} \) (Eq. (3)) into \( \hat{H}_r \) defined in Eq. (4) we must confine the dynamics into the subspace spanned by \( \Phi \). By applying second order perturbation theory we obtain the effective Hamiltonian \( \hat{H}_r^\delta = -t \sum'_{j, \vartheta \neq \vartheta'} |\vartheta \rangle \langle \vartheta' |_{j} + \sum_{j, \vartheta} \delta_{j, \vartheta} |\vartheta \rangle \langle \vartheta |_{j} \) where \( t = \Omega^2 (1/(\Delta - 2V) + 1/(V + 2V - \Delta)) \) and the energy shift \( \Delta\Sigma - (L - 1)V \) has been applied (see the SM). The primed sum is restricted to the transitions between states in \( \text{span}(\Phi) \) and is equivalent to the hopping term of the Hamiltonian \( \hat{H}_r \) by construction. The electric field energy coupling \( g^2 \) is obtained by modulating the local detuning \( \delta_{j, \vartheta} \) such that \( g^2 = \delta_{j, \vartheta} - \delta_{j, \vartheta'} \), with \( |\vartheta' \rangle \equiv |E = \pm 1 \rangle \) and \( |\vartheta \rangle \equiv |E = 0 \rangle \).

The mass term involves an interaction between nearest-neighbor links in the Hamiltonian \( \hat{H} \): Its implementation should be encoded in the inter-links interaction term of the Hamiltonian \( \hat{H}_r \). The Hamiltonian \( \hat{H}_r^\delta \) implements the case with \( m = 0 \): Indeed, two-link states with or without charge are both represented by configurations whose interaction energy is \( V \). In the SM we show that the case \( m \neq 0 \) can be implemented by modifying the geometry of the lattice.

**Results.** We benchmark the dynamics of the quantum simulator via a numerical analysis. The experimental parameters we use refer to \(^{87}\text{Rb} \) atoms excited to the state \( |68S; m = 1/2 \rangle \), with \( c_0 = 612 \text{ GHz} \). We set \( \Omega = 3 \text{ MHz} \), \( \Delta = 27 \text{ MHz} \), \( \ell = 4 \mu \text{m} \), \( d = 5.5 \mu \text{m} \) so we have \( t = 0.682 \text{ MHz} \). We set \( \delta_{j, \vartheta} = 0 \) for \( \vartheta, j \) such that \( E_j = 0 \) and \( \delta_{j, \vartheta} = g^2 \) for \( \vartheta, j \) such that \( E_j = \pm 1 \). By calling \( V^{(2)} \) the amplitude of next-nearest neighbor interactions, we have \( V^{(2)} \ll V, V_r, V_d \) by construction. Nevertheless, next-nearest neighbors interactions add a gauge invariant interaction term into the Hamiltonian \( \hat{H}_r^\delta \). The values we choose for the parameters of the Hamiltonian \( \hat{H}_r \) guarantee that \( V^{(2)} < t, g^2 \). We have numerically checked that, under these conditions, the dynamics of the \( \mathbb{Z}_3 \) model is not significantly affected by next-nearest neighbor interactions and therefore we neglect them.

**FIG. 4.** Electric field bulk local dynamics. The dynamics of the quantum simulator (light green curve) and the \( \mathbb{Z}_3 \) model (red curve) exhibit a fair agreement both in for \( \Gamma = 4.0 \) (upper curves) and \( \Gamma = 0.5 \) (lower curves). In the first case they coincide also with the dynamics of the Schwinger model (blue squares), as expected in the limit \( \Gamma \gg 1 \). In the latter they deviate due to the truncation of the electric field spectrum in the \( \mathbb{Z}_3 \) model. (\( L = 21, t = 0.682 \text{ MHz}, m = 0 \))

We use a TEBD algorithm \(^{38, 39}\) to simulate the dynamics of the Rydberg Hamiltonian \( \hat{H}_r \) (see SM) and compare it with the exact diagonalization of the \( \mathbb{Z}_3 \) model for a chain of \( L = 21 \) links. The implementation thus requires 60 atoms and is achievable on the basis of a recent experiment \(^3\). In Fig. 4 we show the evolution of the local electric field after a quench from the bare vacuum \( |E = 0 \rangle \) for both \( \mathbb{Z}_3 \) model (red line) and to the quantum simulator (light blue line) are in a fair agreement both in both the cases with \( \Gamma > 1 \) (upper panel) and \( \Gamma < 1 \) (lower panel). The high frequency oscillations in the curve relative to the Rydberg dynamics are reminiscent of the second order processes in \( \Omega \) which generate in the transitions between different states \( |\vartheta \rangle \).

This Rydberg-atom quantum simulator is able to catch the string dynamics: As we show in Fig. 2 (b), (d), by taking \( |s \rangle \) as initial state, it is possible to distinguish the string breaking and persisting regimes predicted by the \( \mathbb{Z}_3 \) model.

**Conclusions and outlook.** In this work we have introduced a quantum simulator for the study of the real time dynamics of an Abelian quantum lattice gauge theory with scalable sizes of the lattice up to 20 sites. Thanks to the Rydberg blockade mechanism, which is able to guarantee an high reliability between the original model and the experimental realization in the case of local interactions, we explored the dynamics of the \( \mathbb{Z}_3 \) gauge model in different parameters regimes and with different initial states. Remarkably, we compared the dynamics of the \( \mathbb{Z}_3 \) model with that of the \( U(1) \) gauge theory, finding qualitative similarities in the string dynamics. Our quantum simulator is therefore a versatile and reliable experimental setup useful for investigating exotic properties of discrete and continuous Abelian lattice.
gauge theories [40, 41]. Possible outlook for this work is the extension to two-dimensional theories, in continuity with a recent proposal about the study of 2D pure gauge systems [42], as well as the application of this protocol for simulating clock variables to different models such as time crystals [43, 44].

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**Supplementary Material**

**Integrating the matter field**

In this section we derive the Hamiltonian $\hat{H}^g_S$ of Eq. (2) starting from $\hat{H}_S$ defined in Eq. (1) of the main text. The subspace we are considering contains those states $|\Phi\rangle$ such that $Q_j |\Phi\rangle = 0 \quad \forall j$. The basis vectors of this subspace, \{|m, \hat{E}\}\, are the eigenstates of a projector operator $\hat{P}$ which can be written as a product of local terms $\hat{P} = \prod_{j=1}^N \hat{P}_j$: We express $\hat{P}_j$ in terms of the local computational basis $\prod_{j=1}^N \{|m\rangle_j \otimes \{\hat{E}\}_j\}$. In particular, since the Gauss’ law establishes a relation between the expectation value of the charge density operator $\hat{n}_j$ and the related electric field flux $\Delta \hat{E}_j = \hat{E}_j - \hat{E}_{j-1}$, we define the following projectors:

$$\hat{P}_{\Delta \hat{E}_j} (n_j) = \sum_\mathcal{E} |\mathcal{E}\rangle\langle \mathcal{E}|_{j-1} \otimes |\mathcal{E} + \Delta \mathcal{E}^{n_j}\rangle\langle \mathcal{E} + \Delta \mathcal{E}^{n_j}|_j,$$

with $\Delta \mathcal{E}^{n_j} \equiv \mathcal{E}_j - \mathcal{E}_{j-1} = \frac{1 - P_{\hat{S}^r_j}}{2} - n_j$. With this representation we have that

$$\hat{P}_j = |0\rangle_0 \otimes \hat{P}_{\Delta \hat{E}_j} (0) + |1\rangle_1 \otimes \hat{P}_{\Delta \hat{E}_j} (1), \quad \text{Eq. (A.2)}$$

namely $\hat{P}_j$ selects those configurations such that the difference between the electric field values in neighboring links and the occupation of the site between them satisfy the Gauss’ law.

Now we focus on the aforementioned Hamiltonian $\hat{H}^g_S$. We show that, if we consider two basis vectors $|m, \mathcal{E}\rangle$ and $|m', \mathcal{E}'\rangle$, it exists an operator $\hat{H}^g_S$ such that

$$\langle m, \mathcal{E} | \hat{H}^g_S | m', \mathcal{E}' \rangle = \langle \mathcal{E} | \hat{H}^g_S | \mathcal{E}' \rangle.$$ \quad \text{Eq. (A.3)}$$

In the following we explicitly computes the operator $\hat{H}^g_S$. We start by considering the mass term at the site $j$:

$$\langle m, \mathcal{E} | \hat{\psi}_j^\dagger \hat{\psi}_j \otimes 1^\mathcal{E} | m', \mathcal{E}' \rangle = \langle \mathcal{E} | \left(1 - \frac{\hat{P}_j}{2} - \Delta \hat{E}_j\right) | \mathcal{E}' \rangle,$$

where $1^\mathcal{E}$ is identity operator acting on the electric field space, and we used the fact that all the basis vectors \{|m, \mathcal{E}\}\ satisfy the Gauss’ law. The computation of the electric field energy is straightforward since it is diagonal in the electric field local computational basis.

The hopping term is composed by the unitary propagator, which provides the evolution of the gauge field, and the fermionic operators, which constraint the allowed transitions. In order to integrate out the matter field, we need to recast the fermionic constraints in terms of the gauge field operators only. To this purpose we observe that

$$\langle m, \mathcal{E} | \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \otimes 1^\mathcal{E} | m', \mathcal{E}' \rangle$$

$$= \langle m, \mathcal{E} | \hat{P}_j \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P} | m', \mathcal{E}' \rangle$$

$$= \langle m, \mathcal{E} | \hat{P}_j \hat{P}_{\Delta \hat{E}_j} (0) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P}_{\Delta \hat{E}_j} (0) \hat{P} | m', \mathcal{E}' \rangle,$$

where we have used the definition of $\hat{P}$ as well as the fact that $\hat{P}_j = \hat{P} \forall j$. Discarding the overall projector $\hat{P}$ and using the definition (A.2) we obtain

$$\langle m, \mathcal{E} | \left[1|1\rangle_j \otimes \hat{P}_{\Delta \hat{E}_j} (1) \hat{\psi}_j^\dagger \hat{\psi}_{j+1}\right]$$

$$\otimes \left[0\rangle_0 \otimes \hat{P}_{\Delta \hat{E}_j} (0) + \hat{\psi}_j^\dagger \hat{\psi}_{j+1}\right]$$

$$\langle 1\rangle_0 \otimes \hat{P}_{\Delta \hat{E}_j} (1) | m', \mathcal{E}' \rangle$$

$$= \langle \mathcal{E} | \left[\hat{P}_{\Delta \hat{E}_j} (1) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1) \right] | \mathcal{E}' \rangle$$

$$= \langle \mathcal{E} | \hat{U}^j \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1) | \mathcal{E}' \rangle.$$

The first passage is justified by observing that $\hat{\psi}_j^\dagger \hat{\psi}_{j+1} = |1\rangle_0 \otimes |0\rangle_1 \otimes |1\rangle_{j+1}$. The second passage is a consequence of the definition (A.1) which allows to write

$$\langle m | \hat{1}_1 \otimes |0\rangle_0 | 1\rangle_{j+1} | m' \rangle = 1$$

$$\langle \mathcal{E} | \left[\hat{P}_{\Delta \hat{E}_j} (1) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1) \right] | \mathcal{E}' \rangle = 1.$$ \quad \text{Eq. (A.7)}$$

In the last passage of Eq. (A.6), since $\hat{P}_{\Delta \hat{E}_j} (0) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1) \hat{U}_j$, it follows

$$\hat{P}_{\Delta \hat{E}_j} (0) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P}_{\Delta \hat{E}_j} (1)$$

$$\hat{P}_{\Delta \hat{E}_j} (0) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \hat{P}_{\Delta \hat{E}_j} (1) \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1)$$

$$\hat{U}^j \left(\hat{P}_{\Delta \hat{E}_j} (0) \hat{\psi}_j^\dagger \hat{\psi}_{j+1} \right)^2 \hat{U}^j \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1).$$

As a conclusion, we have derived the Hamiltonian

$$\hat{H}^g_S = - t \sum_j \hat{U}_j^d \hat{P}_{\Delta \hat{E}_j} (0) \hat{P}_{\Delta \hat{E}_j} (1) + h.c.$$ \quad \text{Eq. (A.9)}$$

$$+ m \sum_j \left(\frac{1 - P_j}{2} - \Delta \hat{E}_j\right) + g^2 \sum_j \hat{E}_j^2.$$

**Second-order derivation of the Hamiltonian $\hat{H}^g_S$**

In this section we show how to map the Rydberg Hamiltonian to the $\mathbb{Z}_3$ Hamiltonian (3) by using second-order perturbation theory. We start by considering
the single link case and then we extend the result to
the entire chain. The Hamiltonian \( \hat{H}_r \) (Eq. 4) of
the main text), in terms of single link 3-atom states 
\( \{|rrg\}, |grg\}, |ggg\}, |ggr\}, |rrg\}, |rrr\} \) reduces to the following \( 8 \times 8 \) matrix

\[
\begin{pmatrix}
\hat{H}_{r,s}^{PP} & \hat{H}_{r,s}^{PN} \\
\hat{H}_{r,s}^{PN} & \hat{H}_{r,s}^{NN}
\end{pmatrix}
\]

The first three states contain only one excited atom and
they form the set \( \Sigma \) \( (|\vartheta\rangle)_{0<\vartheta<2} \) defined in the main
text (links indexes are omitted here). We have applied a
shift of the energy equal to \( \Delta \) so the energy of the states in
\( \Sigma \) is \( E_s = \delta_s \). We call \( \mathcal{H}_s \) the subspace spanned by \( \Sigma \) and
\( \mathcal{H}_N \) the complementary one such that \( \mathcal{H}_s = \mathcal{H}_d \oplus \mathcal{H}_N \).
We defined the projectors \( \hat{P}_s \) and \( \hat{N}_s = 1 - \hat{P}_s \) such that

\[
\begin{align*}
\hat{H}_{r,s}^{PP} &= \hat{P}_s \hat{H}_{r,s} \hat{P}_s \quad \text{(A.11)} \\
\hat{H}_{r,s}^{NN} &= \hat{N}_s \hat{H}_{r,s} \hat{N}_s \\
\hat{H}_{r,s}^{PN} &= \hat{P}_s \hat{H}_{r,s} \hat{N}_s = (\hat{H}_{r,s}^{NP})^\dagger.
\end{align*}
\]

They correspond respectively to the top left, bottom
right and off-diagonal parts of \( \hat{H}_{r,s} \) delimited by
double lines in Eq. (A.10). The effective Hamiltonian relative
to the subspace \( \mathcal{H}_p \) can be derived by assuming
that there exists a set of eigenstates of \( \hat{H}_{r,s} \) whose
energies are perturbations of the spectrum of \( \hat{H}_{r,s}^{PP} \).
Let us consider an eigenstate \( |\Psi\rangle \) whose energy satis-
ifies \( E_s = \delta_s \ll V_s - \Delta, \Delta \) and define \( |\Psi_P\rangle = \hat{P}_s |\Psi\rangle \) and
\( |\Psi_N\rangle = \hat{N}_s |\Psi\rangle \); the eigenvalue equation can be written as

\[
\begin{align*}
E_s \begin{pmatrix} |\Psi_P\rangle \\ |\Psi_N\rangle \end{pmatrix} &= \begin{pmatrix} \hat{H}_{r,s}^{PP} & \hat{H}_{r,s}^{PN} \\
\hat{H}_{r,s}^{PN} & \hat{H}_{r,s}^{NN} \end{pmatrix} \begin{pmatrix} |\Psi_P\rangle \\ |\Psi_N\rangle \end{pmatrix},
\end{align*}
\]

from which it emerges that \( |\Psi_P\rangle \) obeys the equation

\[
\begin{align*}
E_s |\Psi_P\rangle &= \hat{H}_{r,s}^{PP} + \hat{H}_{r,s}^{PN} \frac{1}{E_s - \hat{H}_{r,s}^{NN}} \hat{H}_{r,s}^{NP} |\Psi_P\rangle \\
&= \hat{\tilde{H}}_{r,s}^{PP} |\Psi_P\rangle.
\end{align*}
\]

We compute the matrix elements of \( \hat{\tilde{H}}_{r,s}^{PP} \) in the basis
of the states \( \{|rrg\}, |grg\}, |ggg\} \). We consider its action
on the state \( |rrg\rangle \):

\[
\begin{align*}
\hat{\tilde{H}}_{r,s}^{PP} |rrg\rangle &= \delta_0 |rrg\rangle + \Omega \hat{\tilde{H}}_{r,s}^{PN} \frac{1}{E_s - \hat{H}_{r,s}^{NN}} (|ggg\rangle + |rrg\rangle + |ggr\rangle),
\end{align*}
\]

with \( t_s = \Omega^2 \left( \frac{1}{V_s - \Delta} + \frac{1}{\Delta} \right) \), where we are neglecting the
overall constant \( -\Omega^2 \left( \frac{1}{V_s - \Delta} + \frac{1}{\Delta} \right) \).

When we consider a chain of \( L \) links we must take
into accounts the inter-link energies \( V_i \) and \( V_d \). In fact,
the interactions between nearest-neighbour modify the
local Hamiltonian \( \hat{H}_{r,s}^{NN} \). Transitions between to different
gauge invariant states \( |\Psi\rangle \) and \( |\Psi'\rangle \) are mediated by local
link changes \( |\vartheta\rangle \to |\vartheta'\rangle \),

For example, let us suppose the transition \( |\Psi\rangle \to |\Psi'\rangle \) corresponds to only changing the local state \( |rrg\rangle \) to
the the state \( |ggr\rangle \). This transition is now mediated by
the doubly excited local state \( |rrg\rangle \), and by the local
state \( |ggg\rangle \), with no excitations. Due to the inter-
link interaction, the transition \( |rrg\rangle \to |ggr\rangle \) induces a
local change in the energy \( V_i + 2V - \Delta \). Analogously,
the transition \( |rrg\rangle \to |ggg\rangle \) leads to the an energy
change \( \Delta - 2V \). As a consequence, the allowed transi-
tion $|rgr\rangle_j \rightarrow |grg\rangle_j$ is a second-order process with rate 
\[ t = \Omega^2 \left( \frac{1}{V_l + 2V - \Delta} + \frac{1}{\Delta - 2V} \right). \]

So far, we only consider transition between gauge invariant many-body states. However, the Hamiltonian $\hat{H}_r$ may allow local transitions such as $|rgr\rangle_j |grg\rangle_{j+1} \rightarrow |grg\rangle_j |ggr\rangle_{j+1}$ with two excited atoms at distance $d$. Such transition is suppressed by an energy penalty $V_d$ and breaks the gauge invariance. As far as $V_d \gg V$, $\Omega$, these second-order processes can be neglected. In conclusion, we have shown how $\hat{H}_r$ reduces to $\hat{H}_r^6$ up to second-order corrections in $\Omega$.

**Mass-term implementation**

In this section we describe how to implement a non-zero mass term in the Hamiltonian $\hat{H}_r$. Let us recall that, in the $\mathbf{Z}_3$ model, even matter sites can contain a zero or negative charge corresponding respectively to a zero mass energy. If we consider the electric-field Hamiltonian $\hat{H}$, given an even site $j$, the energies of the two local states $|E\rangle_{j-1,0} \otimes |(E-1) \text{ mod } 3 \rangle_j$ and $|E\rangle_{j-1} \otimes |E\rangle_j$ differ by $m$. Similar considerations apply for odd sites.

In order to achieve this condition, we tilt the triangular structures relative to the even links by a small angle $\phi$ in an anticlockwise way, as shown in Fig. 1. A rotation in the plane perpendicular to the lattice axis makes the distances between the site corresponding to $E_{j-1} = -1$ and the sites $E_j = -1$ and $E_j = 0$ to be different: In this way, different interaction strengths are engineered and thus the energy difference between the vacuum and the charged configuration can be implemented. We define the following characteristic inter-links distances:

\[ R_{r_{yd}} = \left[ d^2 + 4 \ell^2 \sin^2 (\phi/2) \right]^{1/2}, \quad (A.17) \]

\[ R_+ = \left[ d^2 + 4/3 \ell^2 \sin^2 (\pi/3 + \phi/2) \right]^{1/2}, \quad (A.18) \]

\[ R_- = \left[ d^2 + 4/3 \ell^2 \sin^2 (\pi/3 - \phi/2) \right]^{1/2}. \quad (A.19) \]

$R_{r_{yd}}$ is the smallest distance, corresponding to forbidden configurations, while $R_\leq$ correspond to the allowed ones.

The energies corresponding to charged and vacuum configurations are $V_v^c = c_6/R_6^c$ and $V_v^c = c_6/R_6^v$, respectively. Analogously, we define the same energies for odd matter sites, namely $V_o^v = c_6/R_6^o$ and $V_o^c = c_6/R_6^c$.

Staggering is implemented by a further lattice deformation: Indeed, the energy of the vacuum configuration for an even matter site must be equal to the energy of the charged configuration for an odd matter site, i.e. $V_v^c = V_v^o$. In order to achieve the above statement we change the inter-links distance of the chain by a small amount $\varepsilon$, such that $d^o = d + \varepsilon$ and $d^c = d - \varepsilon$ relative to odd and even sites respectively. We choose a value of $\phi$ and thereafter choose a value of $\varepsilon$ to satisfy the condition

\[ V_v^c = V_v^c. \]

As a result we get $m = V_v^c - V_v^c \approx V_o^o - V_o^o$. For example, with the parameters used in the main text, by applying a rotation $\phi = 0.05 \text{ rad}$ we obtain $t = 0.667 \text{ MHz}, m_{-} = V_v^c - V_v^c = 0.385 \text{ MHz}$ and $m_{+} = V_v^o - V_v^o = 0.356 \text{ MHz}$. We can simulate therefore a mass $m = (m_{+} + m_{-})/2$ with a relative error $\sim 4\%$.

**TEBD-MPS numerical simulation details**

The dynamics of the Rydberg atoms quantum simulator has been numerically simulated by using the Hamiltonian $\hat{H}_r$ defined in Eq. (4) of the main text. We considered only interactions between nearest-neighbour links. We used the MPS representation of the many-body state. We took the link as local subspace: Since it is composed by three atoms which can be in a Rydberg or in the internal ground state, its Hilbert space dimension is $2^3 = 8$. The auxiliary dimension was set to 128 and we checked the convergence of the dynamics by repeating the same simulations with larger bond dimension equal to 256. Note that the dynamics of each link is mostly constrained in the three-dimensional subspace spanned by $\Sigma_j$, allowing accurate results with a relatively small bond dimension.

The dynamics has been computed by using the TEBD algorithm with second-order Suzuki-Trotter decomposition of the evolution operator, with time-step $dt = 0.01 (2\pi)^{-1} \mu s$. Local and interaction parameters have been chose such that $\{ \Delta = 27 \text{ MHz}, V_I = 149.414 \text{ MHz}, \Omega = 3 \text{ MHz}, \delta_\theta = g^2 \}$ and $\{ V = 6.186 \text{ MHz}, V_d = 22.109 \text{ MHz} \}$.

![FIG. 1. Mass implementation: In the case with $m \neq 0$ the positions of the atoms belonging to even links are rotated anticlockwise by an angle $\phi$. The correspondent value of the mass is $m = V_v^c - V_v^c$ is implemented.](image-url)