Fibration theorem for Waldhausen $K$-theory

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Abstract

The goal of this note is to give a variant of the generic fibration theorem for Waldhausen $K$-theory without assuming the factorization axiom.

Introduction

The main purpose of this short note is giving a variant of the generic fibration theorem for Waldhausen $K$-theory. The theorem is first proven in [Wal85] and improved in [Sch06]. To give a more precise information, let $C$ be a small category with cofibrations in the sense of [Wal85] and $v \subset w$ sets of weak equivalences in $C$ such that $w$ is extensional. Then the full subcategory $C^w$ of $C$ spaned by those of objects $x$ such that the canonical morphism $0 \to x$ is in $w$ is a subcategory with cofibrations in $C$ and the inclusion functor $C^w \hookrightarrow C$ and the identity functor of $C$ induces a sequence of simplicial categories:

$$vS \cdot C^w \to vS \cdot C \to wS \cdot C.$$  

The original theorem in [Wal85] or [Sch06] says that if $w$ is saturated and the pair $(C, w)$ satisfies the factorization axiom, then the sequence above is a fibration sequence up to homotopy. But in practice, the factorization axiom is a strong condition for applications. In this paper we give an another sufficient applicable condition which makes the sequence above a fibration sequence up to homotopy. (See Theorem 2.2.)

Conventions. We mainly follows the notations in [Wal85]. For a pair of small categories $X$ and $Y$, we write $Y^X$ for the category whose objects are functors from $X$ to $Y$ and whose morphisms are natural transformations.

Acknowledgements. The author wishes to express his deep gratitude to Marco Schlichting for stimulating discussions.

1 Sets of morphisms in a small category

In this section, let $C$ and $D$ be small categories and we write $\text{Mor} C$ for the set of all morphisms in $C$. We mainly study heritability of properties for sets of morphisms in $C$ by taking a right cofinal subset in $\mathbb{1.3}$, a pull-back by a functor in $\mathbb{1.5}$ and simplicial constructions in $\mathbb{1.7}$ We start by giving a glossary about properties for sets of morphisms.

Definition 1.1. Let $S$ and $T$ be sets of morphisms in $C$.

(1) We set

$$T \circ S := \{fg \in \text{Mor} C; \bullet \xrightarrow{g} \bullet \quad \text{where } g \in S \text{ and } f \in T\}.$$  

(2) We say that $S$ is a multiplicative set (resp. strictly multiplicative set, saturated set or satisfies the saturated axiom) if $S$ is closed under finite compositions (resp. closed under isomorphisms and is a multiplicative set, resp. satisfies the two out of three property.) Namely it satisfies the following condition(s) (a) and (c) (resp. (b) and (c), resp. (d)):  

(a) For any object $x$ in $C$, the identity morphism $\text{id}_x$ of $x$ is in $w$.  
(b) All isomorphisms in $C$ are in $w$.  
(c) For any pair of composable morphisms $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ in $S$, $gf$ is also in $S$.  

(d) For any pair of composable morphisms \( f \circ g \circ \bullet \circ \bullet \circ \bullet \in C \), if two of \( f, g \) and \( g f \) are in \( S \), then the other one is also in \( S \).

For a multiplicative system \( S \), we regard it as a subcategory of \( C \).

(3) We say that \( S \) is right permutative (resp. right reversible, resp. right Ore) with respect to \( T \) if it satisfies the following condition(s) (a) (resp. (b), resp. (a) and (b)):

(a) For any morphisms \( a : x \to z \) in \( S \) and \( b : y \to z \) in \( T \), there are an object \( u \) in \( C \) and morphisms \( a' : u \to y \) in \( S \) and \( b' : u \to x \) in \( T \) such that \( ba' = ba \).

(b) For any morphisms \( a, a' : x \to y \) in \( T \), if there exists a morphism \( b : y \to z \) in \( S \) such that \( ba = ba' \), then there exists a morphism \( c : u \to x \) in \( S \) such that \( ac = a'c \).

(4) We say that \( S \) is right localizing (in \( C \)) if \( S \) is a multiplicative and right Ore set with respect to \( \text{Mor} C \).

(5) We say that \( T \) is right cofinal in \( S \) if \( T \subset S \) and for any morphism \( x \to y \) in \( S \), there is a morphism \( z \to x \) in \( T \) such that the composition \( z \to y \) is also in \( T \).

**Example 1.2 (Set of all isomorphisms).** We write \( i_C \) or shorty \( i \) for the class of all isomorphisms in \( C \). Then \( i_C \) is a saturated, strictly multiplicative, right localizing set in \( C \).

**Lemma 1.3.** Let \( T \subset S \) and \( \mathcal{U} \) be sets of morphisms in \( C \). Assume that \( T \) is right cofinal in \( S \).

(1) If \( S \) is right permutative with respect to \( \mathcal{U} \) and \( \mathcal{U} \circ T \subset \mathcal{U} \), then \( T \) is also right permutative with respect to \( \mathcal{U} \).

(2) If \( S \) is right reversible with respect to \( \mathcal{U} \), then \( T \) is also right reversible with respect to \( \mathcal{U} \).

(3) If \( T \) is right permutative with respect to \( \mathcal{U} \) and \( T \circ \mathcal{U} \subset \mathcal{U} \), then \( S \) is also right permutative with respect to \( \mathcal{U} \).

(4) Assume that \( S \) is saturated and right localizing in \( C \) and \( T \) is a multiplicative set. Then the inclusion functor \( i : T \to S \) is a homotopy equivalence.

**Proof.** (1) For any morphisms \( f : a \to c \) and in \( \mathcal{U} \) and \( s : b \to c \) in \( T \), there are morphisms \( f' : d \to b \) in \( \mathcal{U} \) and \( s' : d \to a \) in \( S \) such that \( fs' = s'f' \). Since \( T \) is right cofinal in \( S \), there is a morphism \( s'' : e \to d \) in \( T \) such that the composition \( s's'' : e \to a \) is in \( T \). By the condition \( \mathcal{U} \circ T \subset \mathcal{U} \), the morphism \( f's'' \) is in \( \mathcal{U} \).

(2) For any morphisms \( f, g : a \to b \) in \( \mathcal{U} \), assume that there is a morphism \( s : b \to c \) in \( T \) such that \( sf = s'g \). Then there is a morphism \( s' : d \to a \) in \( S \) such that \( fs' = g s' \). Since \( T \) is right cofinal in \( S \), there is a morphism \( s'' : e \to d \) such that the composition \( s's'' : e \to a \) is in \( T \).

(3) Let \( f : a \to c \) and \( s : b \to c \) be morphisms in \( \mathcal{U} \) and \( S \) respectively. Then there is a morphism \( s' : b' \to b \) in \( T \) such that the composition \( s' \) : \( b' \to c \) in \( T \). Then there are morphisms \( s'' : d \to a \) in \( T \) and \( f' : d \to b' \) in \( \mathcal{U} \) such that \( ss'f' = fs'' \). By assumption the composition \( s'f' : d \to b' \) is in \( \mathcal{U} \).

(4) Let \( x \) be an object in \( S \). We write \( i/x \) for the category whose object is a pair \((y, a)\) of an object \( y \) in \( T \) and a morphism \( a : y \to x \) in \( S \) and whose morphism \( (y, a) \to (z, b) \) is a morphism \( \alpha : y \to z \) in \( T \) such that \( a = ba \). Since the object \((x, i_d)\) is in \( i/x \), the category \( i/x \) is a non-empty category.

**Claim.** \( i/x \) is a cofiltering category. Namely

(a) For any objects \((y, a)\) and \((z, b)\) in \( i/x \), there are morphisms \( \alpha : (w, c) \to (y, a) \) and \( \beta : (w, c) \to (z, b) \) in \( i/x \).

(b) For any morphisms \( \alpha, \beta : (y, a) \to (z, b) \) in \( i/x \), there is a morphism \( \gamma : (w, c) \to (y, a) \) such that \( \alpha \gamma = \beta \gamma \).

**Proof of Claim.**

(a) Since \( S \) is right permutative with respect to \( \text{Mor} C \), there are morphisms \( b' : w' \to y \) and \( a' : w' \to z \) such that \( a' \) is in \( S \). Then by the saturated axiom for \( S \), \( ba' = ab' \) and \( b' \) are also in \( S \). By right cofinality of \( T \) in \( S \), there is a morphism \( \alpha : \gamma' \to \gamma \) such that the composition \( a' \) is in \( T \). By right cofinality of \( T \) in \( S \), there is a morphism \( \alpha : \gamma' \to \gamma \) such that the composition \( b' \gamma' \) is in \( T \). Then we set \( \alpha : \gamma' \to \gamma \) and \( c : = ac = ba \).

(b) Since \( S \) is right reversible with respect to \( \text{Mor} C \), the equalities \( ba = a = c \) implies that there is a morphism \( \gamma : \gamma' \to \gamma \) in \( S \) such that \( \alpha \gamma = \beta \gamma \). By right cofinality of \( T \) in \( T \), there is a morphism \( \gamma : \gamma' \to \gamma \) in \( T \) such that the composition \( \gamma \) is in \( T \). We set \( \gamma : = \gamma' \) and \( e : = a \gamma \).

By Corollary 2 and Theorem A in \([Qui73\, \S1]\), we obtain the desired result.
Definition 1.4. Let \( \phi : C \to D \) be a functor and \( S \) a non-empty set of morphisms in \( D \). We define the set of morphisms \( \phi^{-1} S \) in \( C \) the pull-back of \( S \) by \( \phi \) by the formula.

\[
\phi^{-1} S : = \{ f \in \text{Mor}_C; \phi(f) \in S \}.
\]

Lemma 1.5. Let \( \phi : C \to D \) be a functor and \( S \) a non-empty set of morphisms in \( D \). Then

1. If \( S \) is a multiplicative, (resp. strictly multiplicative, saturated) set in \( C \). Then \( \phi^{-1} S \) is also.
2. If \( \phi \) is full and essentially surjective and if \( T \) is right cofinal in \( S \), then \( \phi^{-1} T \) is right cofinal in \( \phi^{-1} S \).
3. If \( \phi \) is an equivalence of categories, \( S \) and \( T \) are strictly multiplicative sets (resp. \( S \) is a strictly multiplicative set) and \( S \) is right permutative (resp. reversible) with respect to \( T \), then \( \phi^{-1} S \) is right permutative (resp. reversible) with respect to \( \phi^{-1} T \).

Proof. (1) Since a functor sends an identity morphism (resp. isomorphism) to an identity morphism (resp. isomorphism), if \( S \) is closed under identity morphisms (resp. isomorphisms), then \( \phi^{-1} S \) is also. Let \( x \xrightarrow{a} y \xrightarrow{b} z \) be a pair of composable morphisms in \( C \). If two of \( ba \), \( b \) and \( a \) are in \( \phi^{-1} S \), then two of \( \phi(b)\phi(a) \), \( \phi(b) \) and \( \phi(a) \) are in \( S \). Therefore if \( S \) is closed under compositions (resp. a saturated set), then \( \phi^{-1} S \) is also.

(2) For any morphism \( s : x \to y \) in \( \phi^{-1} S \), there is a morphism \( t : z' \to \phi(x) \) in \( T \) such that the composition \( \phi(s)t \) is in \( T \). By essential surjectivity of \( \phi \), there are an object \( z \in C \) and an isomorphism \( t' : \phi(z) \cong z' \). By fullness of \( \phi \), there is a morphism \( t'' : z \to x \) in \( C \) such that \( \phi(t'') = t \). Since \( T \) is a strictly multiplicative set, the composition \( t'' \) is in \( T \) and therefore \( t'' \) is in \( \phi^{-1} T \).

(3) First assume that \( S \) is a strictly multiplicative, right permutative set with respect to \( T \) and \( T \) is a strictly multiplicative set. Let \( s : x \to z \) and \( t : y \to z \) be morphisms in \( \phi^{-1} S \) and \( \phi^{-1} T \) respectively. Then there are morphisms \( s' : w \to \phi(y) \) and \( t' : w' \to \phi(x) \) in \( S \) and \( T \) respectively such that \( \phi(s')t' = \phi(t)s' \). By essential surjectivity of \( \phi \), there are an object \( w \) in \( C \) and an isomorphism \( u : \phi(w) \cong w' \) in \( D \). Since \( S \) and \( T \) are strictly multiplicative sets, \( s'u \) and \( t'u \) are in \( S \) and \( T \) respectively. By fullness of \( \phi \), there are morphisms \( s'' : w \to y \) and \( t'' : w \to x \) in \( C \) such that \( s''u = \phi(s'') \) and \( t''u = \phi(t'') \). Notice that \( s''u \) and \( t''u \) are in \( \phi^{-1} S \) and \( \phi^{-1} T \) respectively. The equality \( \phi(st'') = \phi(s)t''u = \phi(t)s'u = \phi(ts'') \) and faithfulness of \( \phi \) imply the equality \( st'' = ts'' \).

Next assume that \( S \) is a strictly multiplicative, right permutative set with respect to \( T \). Let \( a, b : x \to y \) be morphisms in \( \phi^{-1} T \) and \( s : y \to z \) a morphism in \( \phi^{-1} S \) such that \( sa = sb \). Then there is a morphism \( t : w' \to \phi(x) \) in \( S \) such that \( at = bt \). By essential surjectivity of \( \phi \), there are an object \( u \) in \( C \) and an isomorphism \( u : \phi(w) \cong w' \) in \( D \). Since \( S \) is a strictly multiplicative set, \( tu \) is in \( S \). By fullness of \( \phi \), there is a morphism \( t' : w \to x \) in \( C \) such that \( \phi(t') = tu \). Notice that \( t' \) is in \( \phi^{-1} S \). The equalities \( \phi(at') = \phi(a)tu = \phi(b)tu = \phi(bt') \) and faithfulness of \( \phi \) imply the equality \( at' = bt' \).

Definition 1.6. For a multiplicative set \( S \) of \( C \), we define the simplicial subcategory \( C(-, S) \) in \( C^{[-]} \)

\[
[m] \to C(m, S)
\]

where \( C(m, S) \) is the full subcategory of \( C^{[m]} \) consisting of those functors which take values in \( S \). For each \( m \), we denote an object \( x \) in \( C(m, S) \) by

\[
x : x_0 \xrightarrow{i_0^1} x_1 \xrightarrow{i_1^2} x_2 \xrightarrow{i_2^3} \cdots \xrightarrow{i_{m-1}^m} x_m.
\]

For a set \( T \) of morphisms in \( C \), we define \( TC(m, S) \) to be the set of morphisms in \( C(m; S) \) by the formula

\[
TC(m, S) : = \{ f \in \text{Mor}_C(m; S); f_i \text{ is in } T \text{ for any } 0 \leq i \leq m \}.
\]

Lemma 1.7. Let \( S, T, U \) and \( V \) be non-empty sets of morphisms in \( C \) and \( n \) a non-negative integer. Then

1. Assume that \( T \) is a multiplicative set and right cofinal in \( S \), \( S \) is right permutative with respect to \( U \) and \( U \circ T \subset T \circ U \). Then \( TC(n; U) \) is right cofinal in \( SC(n; U) \).
2. Assume that \( U \circ S \subset T \) and \( U \circ S \subset U \circ S \subset T \circ U \), \( S \) is right permutative with respect to both \( T \) and \( U \) and all morphisms in \( S \) are monomorphisms. Then \( SC(n; U) \) is right permutative with respect to
\( T\mathcal{C}(n;\mathcal{U}) \).

(3) Assume that \( \mathcal{V} \) is a multiplicative, right cofinal set in \( S, \mathcal{U} \circ \mathcal{V} \subset \mathcal{V} \circ \mathcal{U} \) and \( S \) is a multiplicative, right permutative and reversible set with respect to both \( \mathcal{U} \) and \( T \) respectively. Then \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) is right reversible with respect to \( T\mathcal{C}(n;\mathcal{U}) \).

Proof. We proceed by induction on \( n \). If \( n = 0 \), then the assertion is hypothesis. We assume that \( n \geq 1 \). Let \( i: [n-1] \to [n] \) be the inclusion functor. We write \( i^*: \mathcal{C}(n;\mathcal{U}) \to \mathcal{C}(n-1;\mathcal{C}) \) for the induced functor by the composition with \( i \).

(1) Let \( s: a \to b \) a morphism in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \). Then there is a morphism \( t': c' \to a \) in \( T \) such that \( st' : c' \to b \) is in \( T \). Then since \( S \) is right permutative with respect to \( \mathcal{U} \), there is a morphism \( t'' : c'' \to a \) in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) such that \( d'' = c' \) and \( t'' = t' \). Applying the inductive hypothesis to \( i^*(st'')t'' \), there is a morphism \( i^*(st'')t'' \) in \( \mathcal{S}\mathcal{C}(n-1;\mathcal{U}) \) such that the composition \( i^*(st'')t'' \) is in \( \mathcal{S}\mathcal{C}(n-1;\mathcal{U}) \). Then by the condition \( \mathcal{U} \circ T \subset T \circ \mathcal{U} \), there are morphisms \( j: c_{n-1} \to c_{n} \) and \( t'' \) in \( \mathcal{U} \) and \( c_{n} \) in \( T \) such that \( j \). We define \( t : c \to b \) to be a morphism in \( \mathcal{T}\mathcal{C}(n;\mathcal{U}) \) by setting

\[
\begin{align*}
t_k: &= \frac{c_{n-1}^k}{c_k}, \quad t_k: = \frac{c_{n-1}^k}{c_k} \quad \text{for } 0 \leq k \leq n-2, \\
t_{n-1}: &= \frac{c_{n-1}}{c}, \quad t_n: = \frac{c_{n-1}}{c}
\end{align*}
\]

Then we can easily check that \( st : c \to b \) is in \( \mathcal{T}\mathcal{C}(n;\mathcal{U}) \).

(2) Let \( t: a \to b \) and \( s: c \to b \) be morphisms in \( \mathcal{T}\mathcal{S}(n;\mathcal{U}) \) and \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) respectively. Then there are morphisms \( p': d' \to a \) and \( q': d' \to c \) in \( S \) and \( T \) respectively such that \( s, q' \). By applying the inductive hypothesis to \( i^*(tq')q'' \), there is a morphism \( i^*(tq')q'' \) in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) such that \( i^*(tq')q'' \) is in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) respectively such that \( i^*(tq')q'' \). By assumption \( U \circ S \subset S \circ U \), there are morphisms \( j: d_{n+1}' \to d' \) and \( q'' \) in \( \mathcal{U} \) and \( p'' \) in \( \mathcal{U} \) such that \( p''j = i^*(p')p'' \) and \( q'' \). Notice that \( s_n \) is a monomorphism by assumption and we have equalities

\[
\begin{align*}
s_nq_{n+1}^d &= s_nq_{n+1}^d, \\
n_{n+1}^t &= t_n^d
\end{align*}
\]

Therefore we have the equality \( q_{n+1}^d = i_{n-1}^d \). Namely \( q : d \to c \) is actually a morphism in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \).

(3) Let \( f, g : a \to b \) be morphisms in \( \mathcal{T}\mathcal{C}(n;\mathcal{U}) \) and \( x: b \to c \) a morphism in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) such that \( sf = sq \). Then there is a morphism \( t': d' \to a \) in \( S \) such that \( fnt' = gn't' \). Since \( S \) is right permutative with respect to \( \mathcal{U} \), there is a morphism \( t'' : d'' \to a \) in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) such that \( d'' = d' \) and \( t'' = t' \). By applying the inductive hypothesis to \( i^*(ft'')i^*(gt'') \), there is a morphism \( i^*(ft'')i^*(gt'') \) in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) such that \( i^*(ft'')i^*(gt'') \) is in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) respectively such that \( i^*(ft'')i^*(gt'') \). Since \( \mathcal{V}(n-1;\mathcal{U}) \) is a right cofinal set in \( \mathcal{S}\mathcal{C}(n-1;\mathcal{U}) \) by (1), there is a morphism \( t'' \) in \( \mathcal{U} \mathcal{C}(n-1;\mathcal{U}) \) such that the composition \( t'' \) is in \( \mathcal{S}\mathcal{C}(n-1;\mathcal{U}) \). Then by the condition \( \mathcal{U} \circ T \subset T \circ \mathcal{U} \), there are morphisms \( j: d_{n-1} \to d'' \) and \( t'' \) in \( \mathcal{U} \) such that \( i^*(ft'')i^*(gt'') \). We define \( t : d \to a \) to be a morphism in \( \mathcal{S}\mathcal{C}(n;\mathcal{U}) \) by setting \( d_k = k_{n-1}^d \) and \( t_k = t_k^{d'^{(j)}}k_{n-1}^d \) for \( 0 \leq k \leq n-1 \) and \( d_{n-1}^d = k_{n-1}^d \) for \( 0 \leq k \leq n-2 \) and \( d_n = k_n^d \). Then \( f \) is strictly multiplicative. For any pair of sets of weak equivalences \( w \subset w \) in \( \mathcal{C} \) such that \( w \) is extensional, the inclusion functor \( \mathcal{C} \to \mathcal{C} \) and the identity functor of \( \mathcal{C} \) induce the sequence

\[
\mathcal{V}\mathcal{S}\mathcal{C} \to \mathcal{V}\mathcal{S}\mathcal{C} \to \mathcal{V}\mathcal{S}\mathcal{C}.
\]

\[\text{(1)}\]

2 Fibration theorem revisited

In this section, let \( \mathcal{C} \) be a small category with cofibrations and we write \( 0 \) and \( \text{Cof} \mathcal{C} \) for the specific zero object and the set of all cofibrations in \( \mathcal{C} \) respectively. For any set of morphisms \( \mathcal{U} \) in \( \mathcal{C} \), we write \( \mathcal{C}^{\mathcal{U}} \) for the full subcategory of those objects \( x \) such that the canonical morphism \( 0 \to x \) is in \( \mathcal{U} \). If a set of weak equivalences \( w \) in \( \mathcal{C} \) is extensional, then \( \mathcal{C}^{\mathcal{U}} \) is subcategory with cofibrations in \( \mathcal{C} \). For any set of weak equivalences \( w \subset w \) in \( \mathcal{C} \) such that \( w \) is extensional, the inclusion functor \( \mathcal{C}^{\mathcal{U}} \to \mathcal{C} \) and the identity functor of \( \mathcal{C} \) induce the sequence

\[
v\mathcal{S}\mathcal{C} \to v\mathcal{S}\mathcal{C} \to v\mathcal{S}\mathcal{C}.
\]

\[\text{(1)}\]
The main objective of this section, we will give a sufficient condition that the sequence (1) is a fibration up to homotopy. We start by looking into the proof of the generic fibration theorem in [Wal85].

**Proposition 2.1.** Let \( v \subseteq w \) be sets of weak equivalences in \( C \). Assume that \( w \) is extensional. Then the sequence (1) is a fibration up to homotopy if and only if the inclusion functor \( \bar{w}.C(\_, v) \to w.S.C(\_, v) \) of bisimplicial categories is a homotopy equivalence.

**Proof.** Since \( w \) is extensional, we have an isomorphism of bisimplicial categories

\[
\begin{bmatrix}
vS.Cw \\
\downarrow
\end{bmatrix} \simeq \begin{bmatrix}
\bar{w}.C(\_, v) \\
\downarrow
\end{bmatrix}.
\]

(See [Wal85, p.352].) Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
vS.Cw & \to & vS.C \\
\downarrow & & \downarrow \\
\bar{w}.C(\_, v) & \to & w.S.C(\_, v)
\end{array}
\]

Here the top line is a fibration sequence, up to homotopy and the map II is a homotopy equivalence by [Wal85, 1.5.7., 1.6.5.]. Hence the bottom line is a fibration sequence up to homotopy if and only if the map I is a homotopy equivalence.

**Theorem 2.2 (Fibration theorem).** Let \( v \subseteq w \) be sets of weak equivalences in \( C \) such that \( w \) is saturated, extensional and right localizing in \( C \) and right permutative with respect to both \( v \) and \( Cof \) \( C \) and \( v \circ \bar{w} \subseteq \bar{w} \circ v \) and assume that all cofibrations in \( C \) are monomorphisms. Then the sequence (1) is a fibration up to homotopy.

**Remark 2.3.** Let \( w \) be a set of weak equivalences \( w \) in \( C \) and \( v = i_C \) the set of all isomorphisms in \( C \). Then we have \( v \circ \bar{w} \subseteq \bar{w} \circ v \). Namely \( v = i_C \) always satisfies the assumption in Theorem 2.2.

**Example 2.4.** Let \( E \) be a small exact category, \( A \) a right \( s \)-filtering subcategory of \( E \) and \( w \) a set of all weak isomorphisms associated to \( A \) in \( E \) in the sense of [Sch04]. Then the sets \( w \) and \( v = i_E \) satisfy the assumptions in Theorem 2.2 by [Sch04]. Therefore we have the fibration sequence

\[
K(A) \to K(E) \to K(E; w).
\]

**Proof of Theorem 2.2.** Let \( n \) and \( m \) be a non-negative integer and a positive integer respectively. We set \( B : = C(n, v) \) and \( A : = B(m - 1; Cof B) \). We enumerate assumptions in Theorem 2.2:

(A) \( w \) is saturated and extensional.
(B) \( w \) is right permutative with respect to \( Mor C \).
(C) \( w \) is right permutative with respect to \( v \).
(D) \( w \) is right reversible with respect to \( Mor C \).
(E) \( \bar{w} \) is right cofinal in \( w \).
(F) \( \bar{w} \) is right permutative with respect to \( Cof C \).
(G) \( \bar{w} \) is right permutative with respect to \( v \).
(H) \( v \circ \bar{w} \subseteq \bar{w} \circ v \).
(I) All cofibrations in \( C \) are monomorphims.
**Claim.** We have the following:

(a) \( \bar{w} \) is right permutative with respect to \( \text{Mor} \mathcal{C} \).

(b) \( \bar{w} \mathcal{B} \) is right cofinal in \( w \mathcal{B} \).

(c) \( \bar{w} \mathcal{B} \) is right permutative with respect to \( \text{Mor} \mathcal{B} \).

(d) \( \bar{w} \mathcal{B} \) is right permutative with respect to \( \text{Cof} \mathcal{B} \).

(e) \( \bar{w} \mathcal{B} \) is right reversible with respect to \( \text{Mor} \mathcal{B} \).

(f) \( \bar{w} \mathcal{A} \) is right cofinal in \( w \mathcal{A} \).

(g) \( \bar{w} \mathcal{A} \) is extensional and saturated in \( \mathcal{A} \).

(h) \( \bar{w} \mathcal{A} \) is right permutative with respect to \( \text{Mor} \mathcal{A} \).

(i) \( \bar{w} \mathcal{A} \) is right permutative with respect to \( \text{Mor} \mathcal{A} \).

(j) \( \bar{w} \mathcal{A} \) is right reversable with respect to \( \text{Mor} \mathcal{A} \).

(k) \( \bar{w} \mathcal{A} \) is right reversable with respect to \( \text{Mor} \mathcal{A} \).

**Proof of claim.** (a) We apply Lemma 1.7 (1) to \( S = w \), \( T = \bar{w} \) and \( \mathcal{U} = \text{Mor} \mathcal{C} \). Assumptions in Lemma 1.3 (1) follow from (B), (E) and \( \text{Mor} \mathcal{C} \circ \bar{w} \subset \text{Mor} \mathcal{C} \).

(b) We apply Lemma 1.7 (1) to \( S = w \), \( T = \bar{w} \) and \( \mathcal{U} = v \). Assumptions in Lemma 1.7 (1) follow from assumptions (E) and (H). Hence we get the result.

(c) We apply Lemma 1.7 (2) to \( S = \bar{w} \), \( T = \text{Mor} \mathcal{C} \) and \( \mathcal{U} = v \). Assumptions in Lemma 1.7 (2) follow from assumptions (a), (C), (H), (I) and \( \text{Mor} \mathcal{C} \circ \bar{w} \subset \text{Mor} \mathcal{C} \). Hence we get the result.

(d) We apply Lemma 1.7 (2) to \( S = \bar{w} \), \( T = \text{Cof} \mathcal{C} \) and \( \mathcal{U} = v \). Assumptions in Lemma 1.7 (2) follow from assumptions (F), (G), (H), (I) and \( \text{Cof} \mathcal{C} \circ \bar{w} \subset \text{Cof} \mathcal{C} \). Hence we get the result.

(e) We apply Lemma 1.3 (3) to \( S = w \mathcal{B} \), \( T = \bar{w} \mathcal{B} \) and \( \mathcal{U} = \text{Cof} \mathcal{B} \). Assumptions in Lemma 1.3 (3) follow from (b), (d) and \( \bar{w} \mathcal{B} \circ \text{Cof} \mathcal{B} \subset \text{Cof} \mathcal{B} \). Hence we obtain the result.

(f) We apply Lemma 1.7 (3) to \( S = \bar{v} = w \), \( T = \text{Mor} \mathcal{C} \) and \( \mathcal{U} = v \). Assumptions in Lemma 1.7 (3) follow from assumptions (A), (C), (D) and \( w \circ v \subset w \circ v \). The last condition follows from assumption \( v \subset w \). Hence we get the result.

(g) We apply Lemma 1.7 (1) to \( S = w \mathcal{B} \), \( T = \bar{w} \mathcal{B} \) and \( \mathcal{U} = \text{Cof} \mathcal{B} \). Assumptions in Lemma 1.7 (1) follow from (e) and \( \bar{w} \mathcal{B} \circ \text{Cof} \mathcal{B} \subset \text{Cof} \mathcal{B} \circ \text{Cof} \mathcal{B} \). The last condition follows from \( \bar{w} \mathcal{B} \circ \text{Cof} \mathcal{B} \subset \text{Cof} \mathcal{B} \circ \text{Cof} \mathcal{B} \). Hence we obtain the result.

Assertion (h) follows from (A) as in [Wal85].

(i) We apply Lemma 1.7 (2) to \( S = \bar{w} \mathcal{B} \), \( T = \text{Mor} \mathcal{B} \) and \( \mathcal{U} = \text{Cof} \mathcal{B} \). Assumptions in Lemma 1.7 (2) follow from assumptions (e) and \( \text{Cof} \mathcal{C} \circ \bar{w} \mathcal{B} \subset \bar{w} \mathcal{B} \circ \text{Cof} \mathcal{C} \) and \( \text{Mor} \mathcal{C} \circ \bar{w} \mathcal{B} \subset \text{Mor} \mathcal{B} \). Hence we get the result.

(j) We apply Lemma 1.3 (3) to \( S = w \mathcal{A} \), \( T = \bar{w} \mathcal{A} \) and \( \mathcal{U} = \text{Mor} \mathcal{A} \). Assumptions in Lemma 1.7 (3) follow from assumptions (g), (i) and \( \bar{w} \mathcal{A} \circ \text{Mor} \mathcal{A} \subset \text{Mor} \mathcal{A} \). Hence we get the result.

(k) We apply Lemma 1.7 (3) to \( S = w \mathcal{B} \), \( T = \text{Mor} \mathcal{B} \) and \( \mathcal{U} = \text{Cof} \mathcal{B} \) and \( V = \bar{w} \mathcal{B} \). Assumptions in Lemma 1.7 (2) follow from assumptions (b) and (e) and (f) and the facts that \( \text{Cof} \mathcal{B} \circ \bar{w} \mathcal{B} \subset \bar{w} \mathcal{B} \circ \text{Cof} \mathcal{B} \) and \( \bar{w} \mathcal{B} \) and \( w \mathcal{B} \) are multiplicative sets in \( B \). Hence we get the result.

Since the forgetful functor gives an equivalence of categories with cofibrations

\[
S_m \mathcal{A} \overset{\sim}{\to} B(m - 1; \text{Cof} \mathcal{B}),
\]

it turns out that \( \bar{w}S_m \mathcal{C}(n; v) \) is right cofinal in \( wS_m \mathcal{C}(n; v) \) and \( wS_m \mathcal{C}(n; v) \) is saturated, extensional and right localizing in \( S_m \mathcal{C}(n; v) \) by Lemma 1.5. Therefore the inclusion functor

\[
\bar{w}S_m \mathcal{C}(n; v) \hookrightarrow wS_m \mathcal{C}(n; v)
\]

is a homotopy equivalence by Lemma 1.3 (4). Hence by the realization lemma in [Seg74 Appendix A] or [Wal78, 5.1], the inclusion functor between bisimplicial categories

\[
\bar{w}S.C(\cdot; v) \hookrightarrow wS.C(\cdot; v)
\]

is a homotopy equivalence. Then the sequence (1) is a fibration sequence up to homotopy by Proposition 2.1. 

\[\square\]
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