Linear Spinor Field Equations for Arbitrary Spins

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Abstract

When utilizing a cluster decomposable relativistic scattering formalism, it is most convenient that the covariant field equations take on a linear form with respect to the energy and momentum dispersion on the fields in the manner given by the Dirac form for spin $\frac{1}{2}$ systems. A general spinor formulation is given for arbitrary spins by minimally extending the Lorentz algebra to include operators whose matrix representation give general Dirac matrices. The forms of these matrices are explicitly demonstrated for spin $\frac{1}{2}$ and spin 1 fields.

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1 Introduction

The Dirac equation utilizes a matrix algebra to construct a linear operator relationship between the energy and the momentum. When constructing a scattering formalism for relativistic quantum systems, the off-diagonal (off-shell) nature of the intermediate states considerably complicates the implementation of the separation of the purely kinematic variables of the non-interacting components of a multi-particle system from the dynamical variables of the interacting components, due to the generally non-linear dispersion relationships between energies and momenta. One is able to demonstrate the general cluster decomposibility of a multiparticle system if the off-shell behavior is parametric, and the resolvants satisfy linear dispersions[1][2][3], as is the case using the resolvants resulting from the Dirac equation. This makes the development of linear dispersion equations for systems of arbitrary spin crucial for the implementation of physical cluster decomposibility in general, non-perturbative, formal scattering theory.

It is therefore advantageous to extend the Lorentz group to include operators whose matrix elements reduce to the Dirac matrices for spin $\frac{1}{2}$ systems, but generally require that the form $\hat{\Gamma}^\mu \hat{P}_\mu$ be a scalar operation. The form $\hat{\Gamma}^\mu$ will be seen to be associated with the particle type current, where $\hat{P}_\mu$ is associated with the mass-energy current (space-time translations). For the present purposes, we will assume that the group is internal, commuting with global space-time translations. In a subsequent paper we will present the extended Poincare group behavior for systems that transform under this group structure. The finite dimensional representations of the extended Poincare group will there be constructed in a manner analogous to the use of the Little group on the standard state vectors in the construction of finite dimensional representations for spin and helicity in the Poincare algebra.

$$\hat{U}(\lambda) |\Psi_\lambda \bar{a} \rangle = \sum_{\lambda'} |\Psi'_{\lambda'} \bar{z}(\bar{a}) \rangle D_{\lambda'}^{\lambda}(\lambda'; \bar{a})$$  \hspace{1cm} (1.1)

A single-particle wave equation can be developed for configuration space eigenstates of the operator $\hat{\Gamma}^\mu \hat{P}_\mu$,

$$\Gamma^\mu i\partial_\mu \psi(x) = \lambda \psi(x)$$ \hspace{1cm} (1.2)

which implies that

$$\partial_\mu \left[ \psi(x) \Gamma^\mu \psi(x) \right] = 0.$$ \hspace{1cm} (1.3)

The conserved current defined in Equation 1.3 need not be the probability current, since the spinor metric need not be related to the $\Gamma$ matrices in general. However, scattering equations can be developed to express the evolution of whatever physical parameter is represented by this operator.

In what follows, the group structure of an extended Lorentz algebra will be developed. Finite dimensional representations of this group will be expressed in terms of spinors, and matrix representations will be developed for two systems of interest.
2 Extended Lorentz Group

The finite dimensional representations of this extended group will be constructed by developing a spinor representation of the algebra. The group elements will for the present include 3 parameters representing angles, 3 boost parameters, and 4 group parameters associated with the operators \( \Gamma \). Later, it can be expanded to include at least the fourteen parameters including the space-time translations \( \vec{b} \). For the present, the extended Lorentz group sub-algebra will be developed.

2.1 Extended Lorentz Group Commutation Relations

The extended group commutation relations will be chosen to be consistent with the Dirac matrices as follows:

\[
\begin{align*}
[J_j, J_k] &= i \epsilon_{jkm} J_m \\
[J_j, K_k] &= i \epsilon_{jkm} K_m \\
[K_j, K_k] &= -i \epsilon_{jkm} J_m \\
[J^0, \Gamma^k] &= i K_k \\
[J^0, J_k] &= 0 \\
[J^0, K_k] &= -i \Gamma^k \\
[\Gamma^j, \Gamma^k] &= -i \epsilon_{jkm} J_m \\
[\Gamma^j, J_k] &= i \epsilon_{jkm} \Gamma^m \\
[\Gamma^j, K_k] &= -i \delta_{jk} \Gamma^0 \\

For convenience, define \( \Delta^{(\pm)}_k \) as follows:

\[
\Delta^{(\pm)}_k = \Gamma^k (\pm) i K_k. \\
\]

Then the algebra can be expressed in terms of raising and lowering operators

\[
\begin{align*}
[\Gamma^0, J_k] &= 0 \\
[\Gamma^0, \Delta^{(\pm)}_k] &= (\pm) \Delta^{(\pm)}_k \\
[J_z, J_{\pm}] &= \pm J_{\pm} \\
[J_+, J_-] &= 2 J_z \\
[J_z, \Delta^{(\pm)}_z] &= 0 \\
[J_z, \Delta^{(\pm)}_{\pm}] &= \pm \Delta^{(\pm)}_{\pm} \\
[J_{\pm}, \Delta^{(\pm)}_{\pm}] &= 0
\end{align*}
\]
\[
\begin{align*}
\left[ J_\pm, \Delta^{(\pm)}_\mp \right] &= \pm 2 \Delta^{(\pm)}_\pm \\
\left[ J_\pm, \Delta^{(\pm)}_\pm \right] &= \mp \Delta^{(\pm)}_\pm \\
\left[ \Delta^{(+)}_\pm, \Delta^{(-)}_\pm \right] &= -2 \Gamma^0 \\
\left[ \Delta^{(\pm)}_\pm, \Delta^{(\pm)}_\pm \right] &= 0 \\
\left[ \Delta^{(+)}_\pm, \Delta^{(-)}_\pm \right] &= \mp 2 J_\pm \\
\left[ \Delta^{(-)}_\pm, \Delta^{(+)}_\pm \right] &= \mp 2 J_\pm \\
\left[ \Delta^{(\pm)}_\pm, \Delta^{(\pm)}_\mp \right] &= 0 \\
\left[ \Delta^{(\pm)}_\pm, \Delta^{(\mp)}_\mp \right] &= 0 \\
\left[ \Delta^{(\pm)}_\mp, \Delta^{(\pm)}_\mp \right] &= -4 \left[ J_\pm (\pm) \Gamma^0 \right] 
\end{align*}
\]

which will be a convenient form from which to construct the spinor representation.

### 2.2 Group metric

In general, a group metric can be developed from the adjoint representation in terms of the structure constants. For the algebra represented by

\[
\left[ \hat{G}_r, \hat{G}_s \right] = -i (g_s)_r^m \hat{G}_m
\]

the group metric can be defined by

\[
\eta_{ab} \equiv (g_a)^r_s (g_b)_r^s. 
\]

The non-vanishing components of the extended Lorentz group metric are given by

\[
\eta^{(EL)}_{J_m J_n} = -6 \delta_{mn}, \quad \eta^{(EL)}_{K_s K_n} = +6 \delta_{mn}, \quad \eta^{(EL)}_{\Gamma^\mu \Gamma^\nu} = +6 \eta_{\mu\nu}
\]

It is interesting to note that the group structure of the extended Lorentz group generates the Minkowski metric. Neither the group structure of the usual Lorentz group nor that of the Poincare group can generate the Minkowski metric due to the abelian nature of the generators for infinitesimal space-time translations.

### 3 Spinor Equations

A Casimir operator can be constructed for the extended Lorentz group, given by

\[
C = \hat{J} \cdot \hat{J} - \hat{K} \cdot \hat{K} + \Gamma^0 \Gamma^0 - \hat{\Gamma} \cdot \hat{\Gamma}.
\]

This operator can directly be verified to commute with all generators of the group. We will construct eigenstates of this Casimir operator, along with the commuting operators \( J_2 \) and \( \Gamma^0 \). To develop a basis
of states, it is convenient to construct an operator which raises and lowers eigenvalues of the operator \( \Gamma^0 \) analogous to the angular momentum raising and lowering operators. This operator is given by

\[
\Delta^{(\pm)}_J = J \cdot \Delta^{(\pm)}
\]  

(3.2)

The relevant commutation relations needed to construct the basis are given by

\[
[J_z, J_\pm] = \pm J_\pm
\]  

(3.3)

\[
[\Gamma^0, \Delta^{(\pm)}] = (\pm) \Delta^{(\pm)}_J
\]  

(3.4)

\[
[J_z, \Delta^{(\pm)}] = 0
\]  

(3.5)

\[
[\Gamma^0, J_\pm] = 0
\]  

(3.6)

\[
[J_z, \Gamma^0] = 0
\]  

(3.7)

\[
[J^2, J] = 0
\]  

(3.8)

\[
[J^2, \Gamma^0] = 0
\]  

(3.9)

\[
[J^2, \Delta^{(\pm)}] = 0
\]  

(3.10)

### 3.1 Conjugate Spinor Forms

We can construct conjugate spinor forms for the operators that satisfy Equations 2.11-2.26.

\[
J_z = \frac{1}{2} \left[ \chi^{(+)}_+ \partial^{(+)}_+ + \chi^{(-)}_- \partial^{(-)}_- - \chi^{(+)}_- \partial^{(-)}_+ - \chi^{(-)}_+ \partial^{(+)}_- \right]
\]  

(3.11)

\[
J_\pm = \chi^{(+)}_+ \partial^{(+)}_+ + \chi^{(-)}_- \partial^{(-)}_+ + \chi^{(+)}_- \partial^{(-)}_- + \chi^{(-)}_+ \partial^{(+)}_-
\]  

(3.12)

\[
\Gamma^0 = \frac{1}{2} \left[ \chi^{(+)}_+ \partial^{(+)}_+ - \chi^{(-)}_- \partial^{(-)}_- + \chi^{(+)}_- \partial^{(+)}_- - \chi^{(-)}_+ \partial^{(-)}_+ \right]
\]  

(3.13)

\[
\Delta^{(\pm)}_z = (\pm) \left[ \chi^{(\pm)}_+ \partial^{(\mp)}_+ - \chi^{(\pm)}_- \partial^{(\mp)}_- \right]
\]  

(3.14)

\[
\Delta^{(\pm)}_\pm = (\pm) 2 \chi^{(\pm)}_\pm \partial^{(\mp)}_\mp
\]  

(3.15)

This representation provides a convenient mechanism to construct spinor and matrix representations.

### 3.2 Symmetry Behavior of Spinor Forms

Examine the behavior of the operators under the transformation given by

\[
\chi^{(\pm)}_{\pm} \leftrightarrow \bar{\chi}^{(\mp)}_{\pm}
\]  

(3.16)

The angular momentum and \( \Gamma^0 \) operators can be seen to transform as

\[
J \leftrightarrow \bar{J}
\]  

(3.17)

\[
\Gamma^0 \leftrightarrow -\bar{\Gamma}^0
\]  

(3.18)
The commutation relations are preserved for the various generators in the bar representation. For the Dirac case, this will be seen to represent a "particle-antiparticle" symmetry of the system, and it will represent a general symmetry under negation of the eigenvalues of the operator $\Gamma_0$.

### 3.3 General Construction of States

We can next construct general spinor states by operation of the raising operators $\Delta_+^{(+)j}$ and $J_+$. This gives a general form for $\psi_{\gamma,M}^{\Lambda,j}$ given by

$$\psi_{\gamma,M}^{\Lambda,j} = A^{\Lambda,j} \sqrt{\frac{(J-M)!}{(J+M)!}} [x-y]^{\Lambda-J} \chi_+^{(+)} M+\gamma \chi_-^{(-)} M-\gamma \times$$

$$\left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right]^{J+M} x^{J-\gamma} y^{J+\gamma} \bigg| x = \chi_+^{(+)} \chi_-^{(-)}$$

$$y = \chi_+^{(-)} \chi_-^{(+)}$$

The action of the spinor forms of the operators given in Equations (3.11-3.15) results in the following set of equations:

$$\hat{J}^2 \psi_{\gamma,M}^{\Lambda,j} = J(J+1) \psi_{\gamma,M}^{\Lambda,j}, \quad \hat{C} \psi_{\gamma,M}^{\Lambda,j} = \gamma \psi_{\gamma,M}^{\Lambda,j}$$

$$\hat{J}_+ \psi_{\gamma,M}^{\Lambda,j} = M \psi_{\gamma,M}^{\Lambda,j}, \quad \hat{\Gamma}_0 \psi_{\gamma,M}^{\Lambda,j} = \gamma \psi_{\gamma,M}^{\Lambda,j}$$

$$\hat{J}_- \psi_{\gamma,M}^{\Lambda,j} = \sqrt{(J+M+1)(J-M+1)} \psi_{\gamma,M}^{\Lambda,j} \pm 1, \quad \hat{\Delta}^{(\pm)} \psi_{\gamma,M}^{\Lambda,j} = (\pm) (J+1) [J(\mp)\gamma] \psi_{\gamma,M}^{\Lambda,j}$$

Equation (3.22) clearly allows the construction of finite dimensional representation if $J$ is included in the eigenvalue spectrum of $\Gamma_0$.

### 3.4 Number of States

The order of the spinor polynomial of the finite dimensional state with $\Lambda = J_{max}$ can be determined by examining the minimal state from which other states can be constructed using the raising operators and orthonormality:

$$\psi_{\gamma,-\Lambda}^{\Lambda,A} = A^{\Lambda,j} \chi_-^{(2A)}.$$  (3.23)

The general state involves spinor products of the order

$$\chi_+^{(a)} \chi_-^{(b)} \chi_+^{(-c)} \chi_-^{(-d)}.$$  (3.24)

A complete basis of states requires then that $a + b + c + d = 2A$. By direct counting this yield the number of states for a complete basis:

$$N_\Lambda = \frac{1}{3}(A+1)(2A+1)(2A+3).$$  (3.25)

For instance, $N_0 = 1, N_1 = 4, N_2 = 10, N_3 = 20$, and so on.
A single J basis with \((2J + 1)^2\) states does not cover this space of spinors. However, one can directly verify that

\[
N_{J_{\text{max}}} = \sum_{J=J_{\text{min}}}^{J_{\text{max}}} (2J + 1)^2,
\]

(3.26)

where \(J_{\text{min}}\) is zero for integral systems and \(\frac{1}{2}\) for half integral systems. Thus we see that \(\Lambda\) represents the maximal angular momentum state of the system:

\[
J \leq \Lambda = J_{\text{max}}.
\]

(3.27)

### 3.5 Spinor metrics

Invariant amplitudes are defined using dual spinors so that under transformations the inner product is a scalar

\[
< \bar{\psi}\phi > = < \bar{\psi}'\phi' > \\
\psi_a^\dagger g_{ab}\phi_b = (D_{ca}\psi_a)^\dagger g_{cd} (D_{db}\psi_b)
\]

(3.28)

This means the metric should satisfy

\[
g = D^\dagger g D
\]

(3.29)

The eigenvalues of the angular momentum operator and \(\Gamma^0\) will be given by real numbers

\[
J^\dagger = J , \quad \Gamma^{0\dagger} = \Gamma^0
\]

(3.30)

Since the spinor metric is likewise hermitian, it satisfies

\[
g \Gamma^0 = \Gamma^0 g , \quad g J = J g
\]

(3.31)

From equation (3.22) it can be seen that the representation is finite dimensional if \(\Lambda = J\). Within the \(\Lambda = J\) subspace, we can assume the metric is proportional to the identity matrix for angular momentum. We can then generally show that

\[
g_{M'\gamma';M\gamma}^{\Lambda;J';J} = g^{\Lambda J} \delta_{J';J} \delta_{M'M} \delta_{\gamma'\gamma}
\]

\[
\Delta^{(\pm)}_{M'\gamma';M\gamma} = \Delta^{(\pm)}_{J';J M'M} \delta_{\gamma'\gamma} (\pm 1)
\]

(3.32)

If the condition \(g \Delta_{J}^{(\pm)} = -\Delta_{J}^{(\pm)} g\) is satisfied, then the metric can be chosen such that

\[
g^{\Lambda J} = (-)^{\Lambda - \gamma}
\]

(3.33)

By expanding the anticommutator \(\{g, \Delta_J^{\pm}\}\), this then implies that the vector components of the boost generator and \(\Gamma\) matrices must satisfy

\[
g \Gamma = -\Gamma g , \quad g K = -K g
\]

(3.34)
3.6 Construction of $\Lambda = \frac{1}{2}$ Systems

The forms of the matrices corresponding to $\Lambda = \frac{1}{2}$ are expected to have dimensionality $N_{\frac{1}{2}} = 4$, and can be expressed in terms of the Pauli spin matrices as shown below:

$$
\Gamma^0 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{i}{2} \gamma_0 \\
\mathbb{I} = \frac{i}{2} \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \\
\mathbf{K} = \frac{i}{2} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \\
\Delta^{(+)}_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \\
\Delta^{(-)}_k = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma_k \end{pmatrix}
$$

The $\Gamma^\mu$ matrices can directly be seen to be proportional to a representation of the Dirac matrices\(^4\)\(^5\).

3.7 Construction of $\Lambda = 1$ Spinor States

The forms of the matrices corresponding to $\Lambda = 1$ are expected to have dimensionality $N_1 = 10$. We choose the normalization of the spinors to satisfy

$$
\left( \chi^{(A)}_a \chi^{(B)}_b \right) \left( \chi^{(A')}_a \chi^{(B')}_{b'} \right) = \frac{1}{2} \left( \delta^{AA'} \delta_{aa'} \delta^{BB'} \delta_{bb'} + \delta^{AB} \delta_{ab} \delta^{A'B'} \delta_{ba'} \right)
$$

which results in a state normalization of the form

$$
\psi^{A,J}_{\gamma,M} \psi^{A',J'}_{\gamma',M'} = \delta^{JJ'} \delta_{\gamma\gamma'} \delta_{MM'}.
$$

The spinor states satisfying this normalization are given by

$$
\begin{align*}
\psi^{1,0}_{0,0} &= \chi^{(+)}_+ \chi^{(-)}_- - \chi^{(+)}_- \chi^{(-)}_+ \\
\psi^{1,1}_{1,1} &= \chi^{(+)}_+ \\
\psi^{1,1}_{1,0} &= \sqrt{2} \chi^{(+)}_+ \chi^{(-)}_- \\
\psi^{1,1}_{1,-1} &= \chi^{(+)}_- \\
\psi^{1,1}_{0,1} &= -\sqrt{2} \chi^{(+)}_- \chi^{(-)}_+ \\
\psi^{1,1}_{1,-1} &= -\chi^{(+)}_+ \chi^{(-)}_- - \chi^{(+)}_- \chi^{(-)}_+ \\
\psi^{1,1}_{0,0} &= -\chi^{(+)}_+ \chi^{(-)}_- - \chi^{(+)}_- \chi^{(-)}_+ \\
\psi^{1,1}_{0,-1} &= -\sqrt{2} \chi^{(+)}_- \chi^{(-)}_- \\
\psi^{1,1}_{-1,1} &= \chi^{(-)}_+ \\
\psi^{1,1}_{-1,0} &= \sqrt{2} \chi^{(-)}_+ \chi^{(-)}_- \\
\psi^{1,1}_{-1,-1} &= \chi^{(-)}_- 
\end{align*}
$$

\(\psi^{1,1}_{-1,-1}\)
3.8 Matrix Representation of $\Lambda = 1$ Systems

We finally construct matrix elements of the operators given by Equations 3.11-3.15 using the states given in Equation 3.38 to obtain the matrix representation for $\Lambda = 1$ systems:

\[
\Gamma^0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
J_z = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad J^{\pm} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Delta^{(+)} = \begin{pmatrix}
0 & 0 & 0 & 2v_z^T \\
0 & J_z & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_z
\end{pmatrix}, \quad \Delta^{(-)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2J_z \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_z
\end{pmatrix}
\]

\[
\Delta^{(+)} = \begin{pmatrix}
0 & 0 & 0 & 2v_+^T \\
0 & J_+ & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_+
\end{pmatrix}, \quad \Delta^{(-)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2J_+ \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_+
\end{pmatrix}
\]

where

\[
v_z = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad v_+ = \begin{pmatrix}
-\sqrt{2} \\
0 \\
0
\end{pmatrix}, \quad v_- = \begin{pmatrix}
0 \\
0 \\
\sqrt{2}
\end{pmatrix}
\]

\[
J_z = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad J_+ = \begin{pmatrix}
0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2} \\
0 & 0 & 0
\end{pmatrix}, \quad J_- = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \sqrt{2} \\
0 & \sqrt{2} & 0
\end{pmatrix}
\]

Since not all of the generators are Hermitian in this representation, the representation is seen to be finite dimensional, but not unitary, (the same as for the $\Lambda = \frac{1}{2}$ system).

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