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On fractal-fractional Covid-19 mathematical model

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\textbf{A B S T R A C T}

In this article, we are studying a Covid-19 mathematical model in the fractal-fractional sense of operators for the existence of solution, Hyers-Ulam (HU) stability and computational results. For the qualitative analysis, we convert the model to an equivalent integral form and investigate its qualitative analysis with the help of iterative convergent sequence and fixed point approach. For the computational aspect, we take help from the Lagrange’s interpolation and produce a numerical scheme for the fractal-fractional waterborne model. The scheme is then tested for a case study and we obtain interesting results.

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1. Introduction

As we know the Covid-19 is violent acute aspiration syndrome, and it is also a pandemic [1]. After the end of 2019, he Covid-19 has caused significant economic loss and destruction, and a few million people also died from this virus.

Due to the enormous public health problems and the need to direct health measures, many researchers have focused their efforts on the Covid-19 modeling and its spread in the population [2–4,6,8,12,14–19]. Both mathematical models [20–22] and statistical approaches [32] were used.

During the recent years, numerous mathematical models of fractal and fractional order to the Covid-19 have also been constructed by researchers [2–32]. Now, we would like to summarize some works on the Covid-19 briefly. Also, for some very interesting and recent works on Covid-19, we refere to the readers to the papers of [33–35].

In Kolabje et al. [4], the authors investigated the time-series evolution of the cumulative number of confirmed cases of Covid-19, the novel coronavirus disease for some African countries.

In Ullah et al. [6], the transmission dynamics of a Covid-19 pandemic model with vertical transmission has been improved for nonsingular kernel type of fractional differentiation. Here, numerical simulations have also been given depending upon based on real data of Covid-19 in Indonesia to show the plots of the impacts of the fractional order derivative with the expectation. The constructed model gives better than classical models.

Das and Samanta [7] discussed transmission dynamics of the Covid-19 in Italy 2020. Here, taking into account the uncertainty due to the limited information about the Covid-19, the authors have taken the modified susceptible-asymptomatic-infectious-recovered compartmental model under fractional order framework. The validity of the Covid-19 model is justified by comparing real data with the results obtained from simulations.

In Baba and Nasidi [9] presented a fractional order SIR model incorporating individual with mild cases as a compartment to become SMIR model. Here, it was shown that when the rate of infection of the mild cases increases, there is equivalent increase in the overall population of infected individuals. Hence, it is noticed that to curtail the spread of the disease, there is need to take care of the mild cases as well.

Oname et al. [25] considered and analyzed a fractional order model for Covid-19 and tuberculosis co-infection, using the Atangana-Baleanu derivative. The model was simulated using data relevant to both diseases in New Delhi, India. Simulations of the fractional order model revealed that reducing the risk of Covid-19 infection by latently-infected TB individuals will not only bring down the burden of Covid-19, but will also reduce the co-infection of both diseases in the population. Rezapour et al. [30] provided a SEIR epidemic model for the spread of Covid-19 using the Caputo fractional derivative. Using the fractional Euler method, they have got an approximate solution to the model. To predict the transmis-
sion of ovid-19 in Iran and in the world, they provided a numerical simulation based on real data.

Tuan et al. [31] gave a mathematical model for the transmission of Covid-19 by the Caputo fractional-order derivative. Using the generalized Adams-Bashforth-Moulton method, they solved the system and obtain the approximate solutions. They also presented a numerical simulation for the transmission of Covid-19 in the world. Here, the reproduction number was also obtained as which shows that the epidemic continues.

Using the fractal-fractional sense of differential and integral operators we get the following the Covid-19 model:

\[
\begin{align*}
\mathcal{F}_{D}^{\alpha} e^t &\frac{d}{dt} \phi(t) = -\frac{\alpha_1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \phi(s) \, ds,
\end{align*}
\]

where \( \phi(t) \) is the Mittag-Leffler type is given by

\[
\begin{align*}
\mathcal{F}_{D}^{\alpha} e^t &\frac{d}{dt} \phi(t) = -\frac{\alpha_1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \phi(s) \, ds,
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\end{align*}
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where, \( \phi(t) \) is the Mittag-Leffler type is given by

\[
\begin{align*}
\mathcal{F}_{D}^{\alpha} e^t &\frac{d}{dt} \phi(t) = -\frac{\alpha_1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \phi(s) \, ds,
\end{align*}
\]
Theorem 2.1. The kernels $Q_1, Q_2, Q_3, Q_4$ are satisfying the Lipschitz condition if the assumption $C'$ holds and satisfies $\phi_i < 1$ for $i \in \mathbb{N}_1^4$ and are contractions provided that $\psi_i < 1$ for every $i \in \mathbb{N}_1^4$.

Proof. First, we prove that $Q_1(t, S)$ satisfies Lipschitz condition. Using $S(t)$ and $S^*(t)$, we have

$$
\|Q_1(t, S) - Q_1(t, S^*)\| = \left\| \Lambda_1 - (\mu + \theta_1)S(t) - \beta_1 \frac{S(t)E(t)}{N} - \beta_2 \frac{S(t)I(t)}{N} \right\|
\leq \left[ \mu + \theta_1 + \beta_1 \frac{1}{N} \|S(t)\| + \beta_2 \frac{1}{N} \|I(t)\| \right] \|S - S^*\|
\leq \psi_1 \|S - S^*\|
$$

where $\phi_1 = \mu + \theta_1 + \beta_1 c_1 + \beta_2 c_2 < 1$. Hence, $Q_1$ satisfies Lipschitz condition and $\phi_1 < 1$. Next, we prove that $Q_2(t, E)$ satisfies Lipschitz condition for this $E, E^*$ we have

$$
\|Q_2(t, E) - Q_2(t, E^*)\| = \left\| \beta_1 \frac{S(t)E(t)}{N} + \beta_2 \frac{S(t)I(t)}{N} - (\mu + \alpha_1 + \theta_2)E(t) \right\|
\leq \left[ \beta_1 \frac{1}{N} \|S(t)\| + \lambda_1 + \theta_1 + \mu \right] \|E - E^*\|
\leq \psi_2 \|E - E^*\|
$$

where, $\psi_2 = \beta_1 c_3 + \lambda_1 + \theta_1 + \mu < 1$. Hence, $Q_2$ satisfies Lipschitz condition and $\psi_2 < 1$.

Next, we prove that $Q_3(t, S_2)$ satisfies Lipschitz condition for this using $I, I^*$ we have

$$
\|Q_3(t, I) - Q_3(t, I^*)\| = \left\| \alpha_1 E(t) - (\alpha_2 + \theta_3 + \mu + \delta_1)I(t) \right\|
\leq \left[ \alpha_2 + \theta_3 + \mu + \delta_1 \right] \|I - I^*\|
\leq \psi_3 \|I - I^*\|
$$

where, $\psi_3 = \alpha_2 + \theta_3 + \mu + \delta_1 < 1$. Hence, $Q_3$ satisfies Lipschitz condition and $\psi_3 < 1$. Next, we prove that $Q_4(t, R)$ satisfies Lipschitz condition. For this we have

$$
\|Q_4(t, R) - Q_4(t, R^*)\| = \left\| \alpha_2 I(t) - \mu R(t) \right\|
\leq \mu \|R - R^*\|
\leq \psi_4 \|R - R^*\|
$$

where, $\psi_4 = \mu < 1$.

Hence, $Q_4$ satisfies Lipschitz condition and $\psi_4 < 1$.

Next, we prove that $Q_5(t, Q_2)$ satisfies Lipschitz condition. For this we have

$$
\|Q_5(t, Q_2) - Q_5(t, Q_2^*)\| = \left\| \theta_1 S(t) + \theta_2 E(t) + \theta_3 I(t) - \mu Q_2(t) \right\|
\leq \mu \|Q_2 - Q_2^*\|
\leq \psi_4 \|Q_2 - Q_2^*\|
$$

Hence, $Q_5$ satisfies Lipschitz condition and $\psi_4 < 1$.

Ultimately all the functions satisfies Lipschitz conditions with $\psi_i < 1$ for $i \in \mathbb{N}_1^4$ complete the proof. \hfill \Box
\[ I_3(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_2-1}S^{k_2-1}Q_3(s, I(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_3(t, I(t)) \\
R(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_1-1}Q_4(s, R(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_4(t, R(t)) \\
Q(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_2-1}Q_5(s, Q(t))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_5(t, R(t)). \]

Now, we define the following recursive formulas:
\[ S_n(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_2-1}Q_4(s, S_{n-1}(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_4(t, S_{n-1}(t)). \]
\[ E_n(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_2-1}Q_4(s, E_{n-1}(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_4(t, E_{n-1}(t)). \]
\[ I_n(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_2-1}Q_5(s, I_{n-1}(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_5(t, I_{n-1}(t)) \]
\[ R_n(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_2-1}Q_4(s, R_{n-1}(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_4(t, R_{n-1}(t)). \]
\[ Q_n(t) = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1}S^{k_2-1}Q_5(s, Q_{n-1}(s))\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}Q_5(t, Q_{n-1}(t)). \]

Next, we consider the differences as follow:
\[ \Delta S_{n+1}(t) = S_{n+1}(t) - S_n(t) \]
\[ = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1} \times S^{k_2-1}[Q_4(s, S_n(s)) - Q_4(s, S_{n-1}(s))]\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}[Q_4(t, S_n(t)) - Q_4(t, S_{n-1}(t))]. \]
\[ \Delta E_{n+1}(t) = E_{n+1}(t) - E_n(t) \]
\[ = \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1} \times S^{k_2-1}[Q_4(s, E_n(s)) - Q_4(s, E_{n-1}(s))]\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}[Q_4(t, E_n(t)) - Q_4(t, E_{n-1}(t))]. \]
\[ \Delta I_{n+1}(t) = I_{n+1}(t) - I_n(t) \\
+ \frac{k_1 k_2}{AB(k_1) \Gamma(k_1)} \int_0^t (t-s)^{k_1-1} \times S^{k_2-1}[Q_5(s, I_n(s)) - Q_5(s, I_{n-1}(s))]\,ds \\
+ \frac{k_2(1-k_1)}{AB(k_1)} t^{k_2-1}[Q_5(t, I_n(t)) - Q_5(t, I_{n-1}(t))]. \]
\[ \Delta R_{n+1}(t) = R_{n+1}(t) - R_n(t) \]
\[ \]
\[ \Delta Q_{Tn+1}(t) = Q_{Tn+1}(t) - Q_{Tn}(t) \]

\[ = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[Q_5(s, Q_{Tn}(s) - Q_5(s, Q_{Tn-1}(s))]ds \]
\[ + \frac{K_2(1-\kappa_1)}{\Gamma(\kappa_1)} t^{\kappa_2-1}[Q_5(t, Q_{Tn}(t) - Q_5(t, Q_{Tn-1}(t))]. \]

Now, taking norm of the above system on both sides,

\[ \| \Delta S_{n+1}(t) \| = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[\|Q_1(s, S_n(s) - Q_1(s, S_{n-1}(s))\|]ds \]
\[ + \frac{K_2(1-\kappa_1)}{\Gamma(\kappa_1)} t^{\kappa_2-1}\|Q_1(t, S_n(t) - Q_1(t, S_{n-1}(t))\|. \]

\[ \| \Delta E_{n+1}(t) \| = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[\|Q_2(s, E_n(s) - Q_2(s, E_{n-1}(s))\|]ds \]
\[ + \frac{K_2(1-\kappa_1)}{\Gamma(\kappa_1)} t^{\kappa_2-1}\|Q_2(t, E_n(t) - Q_2(t, E_{n-1}(t))\|. \]

\[ \| \Delta I_{n+1}(t) \| = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[\|Q_3(s, I_n(s) - Q_3(s, I_{n-1}(s))\|]ds \]
\[ + \frac{K_2(1-\kappa_1)}{\Gamma(\kappa_1)} t^{\kappa_2-1}\|Q_3(t, I_n(t) - Q_3(t, I_{n-1}(t))\|. \]

\[ \| \Delta R_{n+1}(t) \| = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[\|Q_4(s, R_n(s) - Q_4(s, R_{n-1}(s))\|]ds \]
\[ + \frac{K_2(1-\kappa_1)}{\Gamma(\kappa_1)} t^{\kappa_2-1}\|Q_4(t, R_n(t) - Q_4(t, R_{n-1}(t))\|. \]

\[ \| \Delta Q_{Tn+1}(t) \| = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[\|Q_5(s, Q_{Tn}(s) - Q_5(s, Q_{Tn-1}(s))\|]ds \]
\[ + \frac{K_2(1-\kappa_1)}{\Gamma(\kappa_1)} t^{\kappa_2-1}\|Q_5(t, Q_{Tn}(t) - Q_5(t, Q_{Tn-1}(t))\|. \]

**Theorem 2.2.** The fractal fractional order COVID-19 model (1) has a solution if the following holds true,

\[ \Delta = \max \{|\psi_1, \psi_2, \psi_3, \psi_4, \psi_5| < 1. \]

**Proof.** We define the function

\[ H_{1n}(t) = S_{n+1}(t) - S(t) \]
\[ H_{2n}(t) = E_{n+1}(t) - E(t) \]
\[ H_{3n}(t) = I_{n+1}(t) - I(t) \]
\[ H_{4n}(t) = R_{n+1}(t) - R(t) \]
\[ H_{5n}(t) = Q_{Tn+1}(t) - Q_T(t) \]

Taking norm of the above system we have,

\[ \|H_{1n}(t)\| = \|S_{n+1}(t) - S(t)\| \]

\[ = \frac{K_1K_2}{\Gamma(\kappa_1)\Gamma(\kappa_1)} \int_0^t (t-s)^{\kappa_1-1} \]
\[ \times S^{\kappa_2-1}[\|Q_1(s, S_n(s) - Q_1(s, S(s)))\|]ds \]
Theorem 3.1. The fractal fractional (1) has unique solution if the following holds true:

\[
\left[ \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} \right] \phi_i \leq 1, \ i \in N^2.
\]

Proof. Let us consider contradiction that there exists another solution of fractal fractional model (1) such that \( S(t), \hat{E}(t), \hat{I}(t), \hat{R}(t), \hat{Q}(t) \) such that

\[
\begin{align*}
\dot{S}(t) &= \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} Q_1(s, \hat{S}(s)) \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} t^{k_2 - 1} Q_3(t, \hat{S}(t)) \\
\dot{E}(t) &= \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} Q_2(s, \hat{E}(s)) \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} t^{k_2 - 1} Q_4(t, \hat{E}(t)) \\
\dot{I}(t) &= \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} Q_3(s, \hat{I}(s)) \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} t^{k_2 - 1} Q_5(t, \hat{I}(t)) \\
\dot{R}(t) &= \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} Q_4(s, \hat{R}(s)) \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} t^{k_2 - 1} Q_6(t, \hat{R}(t)) \\
\dot{Q}(t) &= \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} Q_3(s, \hat{Q}(s)) \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} t^{k_2 - 1} Q_5(t, \hat{Q}(t)).
\end{align*}
\]

Now taking differences of \( S(t), \hat{S}(t) \) and then take norm, we have

\[
\begin{align*}
\| S(t) - \hat{S}(t) \| &= \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} \| Q_1(S, S(t)) - Q_1(S, \hat{S}(t)) \| \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} \| Q_3(S, \hat{S}(s)) - Q_3(S, \hat{S}(s)) \| \, ds \\
&\leq \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} \| Q_1(s, \hat{S}(s)) \| \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} \| Q_3(s, \hat{S}(s)) \| \, ds \\
&\leq \frac{\kappa_1 \kappa_2}{AB(k_1)} \Gamma(k_1 + k_2) \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} \| Q_1(s, \hat{S}(s)) \| \, ds \\
&\quad + \frac{\kappa_2 (1 - \kappa_1)}{AB(k_1)} \int_0^t (t - s)^{k_1 - 1} s^{k_2 - 1} \| Q_3(s, \hat{S}(s)) \| \, ds \leq 0.
\end{align*}
\]
The above inequality is true if
\[ \|S - \hat{S}\| = 0 \]
\[ \Rightarrow S = \hat{S}. \]

\[ \|E - \hat{E}\| \leq \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_2 \|E - \hat{E}\| \]
\[ \left[ 1 - \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_2 \right] \|E - \hat{E}\| \leq 0. \]

The above inequality is true if
\[ \|E - \hat{E}\| = 0 \]
\[ \Rightarrow E = \hat{E}. \]

\[ \|I - \hat{I}\| \leq \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_3 \|I - \hat{I}\| \]
\[ \left[ 1 - \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_3 \right] \|I - \hat{I}\| \leq 0. \]

The above inequality is true if
\[ \|I - \hat{I}\| = 0. \quad \text{this implies } I = \hat{I}. \]

\[ \|R - \hat{R}\| \leq \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_4 \|R - \hat{R}\| \]
\[ \left[ 1 - \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_4 \right] \|R - \hat{R}\| \leq 0. \]

The above inequality true if
\[ \|R - \hat{R}\| = 0 \]
\[ \Rightarrow R = \hat{R}. \]

Similarly,
\[ \|Q_r - \hat{Q}_r\| \leq \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_5 \|Q_r - \hat{Q}_r\| \]
\[ \left[ 1 - \left[ \frac{k_1 k_2 \Gamma(k_2)}{AB(k_1) \Gamma'(k_1 + k_2)} + \frac{k_2 (1 - k_1)}{AB(k_1)} \right] \phi_5 \right] \|Q_r - \hat{Q}_r\| \leq 0. \]

The above inequality true if
\[ \|Q_r - \hat{Q}_r\| = 0 \]
\[ \Rightarrow Q_r = \hat{Q}_r. \]

Thus the (1) has unique solution. □

4. Hyers-Ulams stability

Definition 4.1. The fractal fractional integral system (1) is to be Hyers-Ulam stability if there exist a constant \( \varphi_1 > 0, i \in \mathbb{N}_1^2 \) satisfying for every \( \beta_i > 0, i \in \mathbb{N}_1^2 \)

Definition 4.2.

\[ \|S(t) - \int_0^t (t-s)^{k_1-1} s^{k_2-1} Q_1(s, S(s)) ds \| \leq \beta_1 \]
\[ \|E(t) - \int_0^t (t-s)^{k_1-1} s^{k_2-1} Q_2(s, E(s)) ds \| \leq \beta_2 \]
\[ \|I(t) - \int_0^t (t-s)^{k_1-1} s^{k_2-1} Q_3(s, I(s)) ds \| \leq \beta_3 \]
\[ \| R(t) - \frac{\kappa_1 \kappa_2}{A \Gamma(k_1 + k_2)} \int_0^t (t-s)^{k_2-1} \| Q_2(s, E(s)) ds \]
\[ - \frac{\kappa_1 (1-k_1)}{A \Gamma(k_1)} t^{k_1-1} Q_4(t, R(t)) \leq \beta_4 \]

\[ \| Q_2(t) - \frac{\kappa_1 \kappa_2}{A \Gamma(k_1 + k_2)} \int_0^t (t-s)^{k_2-1} \| Q_2(s, Q_2(t)) ds \]
\[ - \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} t^{k_2-1} Q_5(t, Q_2(t)) \leq \beta_5. \]

There exist approximate solution of the model (1) \( S^*(t), E^*(t), I^*(t), R^*(t), Q_2^*(t) \) that satisfies the given model, such that

\[ \| S(t) - S^*(t) \| = \frac{\kappa_1 \kappa_2}{A \Gamma(k_1) \Gamma(k_2)} \int_0^t (t-s)^{k_2-1} \| Q_1(t, S(t)) - Q_1(t, S^*(t)) \|
\]
\[ \times \| Q_1(t, S(t)) - Q_1(t, S^*(t)) \|
\]
\[ + \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} t^{k_2-1} \| Q_2(t, S(t)) - Q_2(t, S^*(t)) \|
\]
\[ \leq \left[ \frac{\kappa_1 \kappa_2 \Gamma(k_2)}{A \Gamma(k_1) \Gamma(k_1 + k_2)} + \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} \right] \phi_1 \| S - S^* \|. \]

Let
\[ \xi_1 = \left[ \frac{\kappa_1 \kappa_2 \Gamma(k_2)}{A \Gamma(k_1) \Gamma(k_1 + k_2)} + \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} \right] \phi_1 \| S - S^* \|. \]

\( \eta_1 = \phi_1 \) the above an equalities become

\[ \| S(t) - S^*(t) \| \leq \eta_1 \xi_1. \]

Similarly we have

\[ \| E(t) - E^*(t) \| \leq \eta_2 \xi_2 \]

\[ \| I(t) - I^*(t) \| \leq \eta_3 \xi_3 \]

\[ \| R(t) - R^*(t) \| \leq \eta_4 \xi_4 \]

\[ \| Q_2(t) - Q_2^*(t) \| \leq \eta_5 \xi_5. \]

**Theorem 4.1.** *If the above assumptions hold, then the fractal fractional COVID-19 model (1) is HU stable*

**Proof.** We know that the fractal fractional COVID-19 model (1) has unique solution let \( S^*(t), E^*(t), I^*(t), R^*(t), Q_2^*(t) \) be approximate solution of model (1) which satisfy the model then we have

\[ \| S(t) - S^*(t) \| = \frac{\kappa_1 \kappa_2}{A \Gamma(k_1) \Gamma(k_2)} \int_0^t (t-s)^{k_2-1} \| Q_1(t, S(t)) - Q_1(t, S^*(t)) \|
\]
\[ \times \| Q_1(t, S(t)) - Q_1(t, S^*(t)) \|
\]
\[ + \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} t^{k_2-1} \| Q_2(t, S(t)) - Q_2(t, S^*(t)) \|
\]
\[ \leq \left[ \frac{\kappa_1 \kappa_2 \Gamma(k_2)}{A \Gamma(k_1) \Gamma(k_1 + k_2)} + \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} \right] \phi_1 \| S - S^* \|. \]

Let
\[ \alpha_1 = \left[ \frac{\kappa_1 \kappa_2 \Gamma(k_2)}{A \Gamma(k_1) \Gamma(k_1 + k_2)} + \frac{\kappa_2 (1-k_1)}{A \Gamma(k_1)} \right] \phi_1 \| S - S^* \|
\]
and
\[ \beta_1 = \phi_1. \]
so the above inequality become

\[ \| S - S^* \| \leq \alpha_1 \beta_1. \]
similarly

\[ \| E - E^* \| \leq \alpha_2 \beta_2 \]
\[ \| I - I^* \| \leq \alpha_3 \beta_3 \]
\[ \| R - R^* \| \leq \alpha_4 \beta_4 \]
\[ \| Q_2 - Q_2^* \| \leq \alpha_5 \beta_5. \]
consequently by definition the COVID-19 model (1) is hyers-ulans stable which is complete the proof. \( \Box \)
5. Numerical scheme

Let us consider
\[
F_{0}^{\text{FM}}D_{t}^{\alpha}x_{2}(t) = V(t, \varphi(t)), \quad \text{where} \; \varphi(0) = \varphi_{0}.
\]

The above equation can be written in Antangana-Baleanu fractional derivative as following
\[
F_{0}^{\text{AB}}D_{t}^{\alpha}x_{2}(t) = \kappa_{2}t^{\varepsilon_{2}-1}L(t, \varphi(t)) = V(t, \varphi(t)).
\]
Taking Antangana-Baleanu integral, we get
\[
\varphi(t) = \varphi(0) + \frac{\kappa_{1}}{\Gamma(\kappa_{1})} \int_{0}^{t} (t - \tau)^{\varepsilon_{1}-1}V(\tau, \varphi(\tau)) d\tau .
\]
Replacing (t) by \( t + 1 \) we have
\[
\varphi^{n+1} = \varphi(0) + \frac{\kappa_{1}}{\Gamma(\kappa_{1})} \int_{0}^{t+1} (t + 1 - \tau)^{\varepsilon_{1}-1}V(\tau, \varphi(\tau)) d\tau .
\]

By applying two step Lagrange Polynomial we obtain
\[
V(x, \varphi(t)) = \frac{V(t_{k}, \varphi(t_{k}))}{t_{k} - t_{k-1}} (x - t_{k-1}) - \frac{V(t_{k-1}, \varphi(t_{k-1}))}{t_{k} - t_{k-1}} (x - t_{k}) = \frac{V(t_{k}, \varphi(t_{k}))}{t_{k} - t_{k-1}} (x - t_{k-1}) - \frac{V(t_{k-1}, \varphi(t_{k-1}))}{t_{k} - t_{k-1}} (x - t_{k}).
\]

Proceeding to consider the equation, we get
\[
\varphi^{n+1} = \varphi(0) + \frac{\kappa_{1}}{\Gamma(\kappa_{1})} \int_{0}^{t+1} \frac{V(t_{k}, \varphi(t_{k}))}{t_{k} - t_{k-1}} (x - t_{k-1}) - \frac{V(t_{k-1}, \varphi(t_{k-1}))}{t_{k} - t_{k-1}} (x - t_{k}) d\tau .
\]

Now solving the integral we get
\[
\varphi^{n+1} = \varphi(0) + \frac{\kappa_{1}}{\Gamma(\kappa_{1})} \int_{0}^{t+1} \frac{V(t_{k}, \varphi(t_{k}))}{t_{k} - t_{k-1}} (x - t_{k-1}) - \frac{V(t_{k-1}, \varphi(t_{k-1}))}{t_{k} - t_{k-1}} (x - t_{k}) d\tau = \varphi(0) + \frac{\kappa_{1}}{\Gamma(\kappa_{1})} \int_{0}^{t+1} \frac{V(t_{k}, \varphi(t_{k}))}{t_{k} - t_{k-1}} (x - t_{k-1}) - \frac{V(t_{k-1}, \varphi(t_{k-1}))}{t_{k} - t_{k-1}} (x - t_{k}) d\tau .
\]

5.1. Numerical results

In this portion, we present the numerical description of the Covid-19 model 1. The numerical values are taken from the article 4, where \( \mu = 0.02, \beta_{2} = 0.3, \rho_{1} = 0.9, \delta_{1} = 0.4, \beta_{1} = 0.4, \delta_{2} = 0.2, \gamma = 0.1, \delta_{3} = 0.1, \sigma = 0.1, \lambda_{1} = 8 \times 10^{5}, \lambda_{2} = 5 \times 10^{7}, \text{and the initial values are: } S(0) = 3 \times 10^{3}, \; E(0) = 2.5 \times 10^{5}, \; I(0) = 6 \times 10^{7}, \; R(0) = 5000. \; Q_{T}(0) = 10^{6} \).

From the numerical results which are explained via graphs we have observed that the population of the susceptible people and exposed people are suddenly reduced and are transferred in to the infected class. As they are quarantined, the infection is control. After few days, the recovery begins and the infection is then reduced. This shows that in the controlling of the infection, the quarantine has an important role.

The numerical description of the model is presented with the help of eight figures. The first Fig. 1 is the numerical data for the susceptible class of the model, the Fig. 2 is the exposed people to the Covid-19 patients, the Fig. 3 is the Covid-19 infected class, Fig. 4 are the simulations for the recovered class from the Covid-19, Fig. 5 represents the graphical data about quarantined people.
6. Conclusion

In this work, a fractal-fractional order Covid-19 model based on five classes of a population was taken into consideration. The classes of the considered populations are the susceptible, exposed, infected, quarantined and recovered. Here, the existence of solutions, stability of the considered system and numerical simulation based on an iterative numerical scheme were investigated for the considered model. The scheme is supported by the Lagrange’s interpolation polynomial. We also applied the numerical scheme to the available data in literature and got very much interesting results for different fractional orders. We suggest the readers for reconsideration of the fractal-fractional Covid-19 model (1) for other fractional derivatives and stability results. They can also generate numerical schemes with the help of other polynomials. The researchers can also consider the model for variable order and develop more general results as a continuation of this study.

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Authors contributions

The first author (H.K) worked in the conceptualization, formal analysis and methodology. The second author (FA) investigated the suggested model and helped in the writing. The third author (O.T) worked in the methodology, software and validation. The fourth author (M.J) worked in the project administration and supervision.

Declaration of Competing Interest

Authors declare that they have no conflict of interest.

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