Spline finite element updating of a reinforced concrete beam

A. Carminelli, G. Catania
DIEM, Dept. of Mechanical Design, University of Bologna, viale Risorgimento 2, 40136, Bologna, Italy
antonio.carminelli@mail.ing.unibo.it

Abstract. An updating procedure based on measured Frequency Response Function (FRF) data is proposed to correct the numerical parameters of the continuous B-spline model of a reinforced concrete beam. The beam model coefficients (e.g. flexural stiffness, mass density, damping ratio, joint stiffness) and displacement field are described as continuous parametric functions by means of B-spline shape functions. The Euler-Bernoulli kinematic assumption is considered but the rotatory inertia effect is taken into account as well. Moreover, the B-spline beam model employs only transversal degrees of freedom (dofs). The updating algorithm is based on the minimization of an objective function, expressed as a nonlinear least squares problem considering the difference between the model-based and the experimentally measured response, at the same frequency. The minimization is solved by using a truncated linear Taylor series of the objective functions and an iterative formulation. The use of FRF data measurements as input provides a large amount of input data in the frequency domain. Nevertheless, this problem can generally be ill conditioned, especially when the number of updating parameters is increased. Any singularity of the problem is addressed by the singular value decomposing the system matrix resulting at each solution iteration. An example is reported dealing with a reinforced concrete beam considered within the Bri.Vi.Di. research project, supported by the Italian Ministero dell'Università e della Ricerca (MIUR) under the "Progetti di Interesse Nazionale" (PRIN07) framework. The updating procedure was tested by adopting measured data as input. Results are illustrated and discussed.

1. Introduction

The study of the bending vibration of beams is very important in many engineering civil and industrial applications, such as simulation of the dynamical behaviour of bridges or identification of the model of some machine components. Mathematical models can be used to predict the vibration response of structures, for design purposes or for diagnosing machine faults. The finite element (FE) method [1] is a widely used method for accurate modelling of structures. It is well known that FE models can be inaccurate because of incorrect modelling of boundary conditions, joint properties [2-3], and damping effects [4] among others.

Estimated data from measurements on a real system, such as frequency response functions (FRFs) or modal parameters, can be used to reduce the inaccuracies of the FE model by means of model updating techniques. There are many papers in the literature dealing with different FE updating methods [5], and much attention has recently been addressed towards iterative techniques [5-7], making it possible to update a large number of parameters and to obtain parameter corrections usually having physical meanings. The most evident disadvantage of iterative methods is their computational cost, which increases with the number of degrees of freedom (dofs) of the model. Several papers on FE models based on B-spline shape functions have been published in recent years [8-9]. Some papers
showed the superior accuracy of B-spline FE models compared with classic polynomial FE models, especially when dealing with vibration problems [10]. This result may be useful in applications such as FE updating to reduce the number of dofs.

In this paper an updating procedure of a B-spline based FE beam model is proposed. The FE beam model is based on the Euler-Bernoulli kinematic assumption but the rotation inertia effect is considered as well. The FE beam model employs only transversal displacement dofs. The updating approach is based on the least squares minimization of an objective function dealing with residues, defined as the difference between the model based response and the experimental measured response, at the same frequency and same excitation. The parameters to be identified during the updating procedure are the coefficients of a distributed constraint stiffness model and the damping ratios, both modelled by means of B-spline functions. A numerical application dealing with the updating of the model parameters of a reinforced concrete beam tested within the Bri.Vi.Di. project is reported. Results are critically discussed and the conclusion follows.

2. Beam model

In this section, the equations modelling the plane dynamic response of a linear elastic beam element in bending are derived based on the principle of total potential energy. The equations are based on the Euler-Bernoulli theory but modelling of rotatory inertia is included.

Consider a straight beam with length L in the x–y plane, with x as the longitudinal beam axis and y as the transverse direction of bending deflection, as shown in Figure 1. The position vector of the centroid axis is expressed by means of n B-spline functions [11]:

\[ x \xi = \sum_{i=1}^{n} B_{i}^{p} \xi \cdot x_{i}, \]

\[ y = 0 \]

where:

- ξi is the parameter abscissa variable spanning the centroid axis 0 ≤ ξ ≤ 1;
- \( B_{i}^{p} \xi \) is the i-th univariate normalized B-spline function of degree p defined by means of the knot vector:

\[ U = \xi_{1}, \ldots, \xi_{m} = \{ 0, \ldots, 0, \xi_{p+1}, \ldots, \xi_{m-p}, 1, \ldots, 1 \} ; \]

- xi is the i-th geometric control coefficient.

The kinematic conditions of the Euler-Bernoulli assumption are:

\[ \begin{align*}
    u & = -\frac{dv}{dx} \\
    v & = \frac{dy}{dx}
\end{align*} \]

The longitudinal linear strain is given by:

\[ \varepsilon_{x} = \frac{du}{dx} = -\frac{d^{2}v}{dx^{2}}, \]

and the longitudinal stress is:

\[ \sigma_{x} = E \cdot \varepsilon_{x}, \]

where E is the Young modulus.

The transverse displacement is expressed by means of a linear combination of B-spline shape functions in natural coordinates:

\[ v_{0} \xi = \sum_{i=1}^{n} B_{i}^{p} \xi \cdot v_{i}, \]
or in matrix format:

$$v_0, \xi = \begin{bmatrix} B_1^T & \cdots & B_n^T \end{bmatrix} \cdot v_1, \cdots, v_n^T = B^T \cdot \xi \cdot V.$$  \hfill (7)

In order to evaluate the differential operator shown in equation(3), the following relations can be written:

$$\frac{dv}{dx} = \frac{dv_0}{d\xi} \cdot \frac{d\xi}{dx} = \frac{dv_0}{d\xi} \cdot J = \frac{1}{J} B^T \cdot V,$$  \hfill (8)

where the Jacobian is:

$$J = \sum_{i=1}^{n} B_i \cdot x_i = B^T \cdot X,$$  \hfill (9)

and:

$$\frac{d^2 v}{dx^2} = \frac{1}{J^2} \left( \frac{d^2 v}{d\xi^2} - \frac{dv}{d\xi} \cdot J \right)$$  \hfill (10)

or in matrix format:

$$\frac{d^2 v}{dx^2} = \frac{1}{J^2} \left( B^T - \frac{J}{J} B^T \right) \cdot V = L \cdot V$$  \hfill (11)

The displacement field can be rewritten in matrix format:

$$d = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{y}{J} \cdot B^T \\ B^T \end{bmatrix} \cdot V$$  \hfill (12)

The expressions of the elasticity, inertia matrices and of the force vector can be obtained by means of the principle of minimum total potential energy:

$$\Pi = U + W \rightarrow \min$$  \hfill (13)

where $U$ is the potential of the strain energy of the system:

$$U = \frac{1}{2} \int_{0}^{l} E \cdot \sigma_s \cdot dS \cdot dx = \frac{1}{2} \int_{0}^{l} E \cdot y^2 \left( \frac{d^2 V}{dx^2} \right)^2 \cdot dS \cdot dx = \frac{1}{2} \int_{0}^{l} E I \left( \frac{d^2 V}{dx^2} \right)^2 \cdot dx$$  \hfill (14)

$$= \frac{1}{2} \cdot V^T \cdot \left( \int_{0}^{l} E I / L^T L \cdot dx \right) \cdot V$$

and $I$ is the second area moment of inertia of the beam cross section:

$$I = \int_S y^2 \cdot dS$$  \hfill (15)

and $S$ is the geometry of the beam section under analysis.
W is the potential of the distributed force per unit length $f$ and of transverse force $T$ and bending moment $Q$ concentrated on a generic point corresponding to the abscissa $\xi$, and includes the potential $W_i$ of the inertial forces:

$$ W = -\int_{\xi_0}^{\xi} f(x) v(x) dx - T v(\xi) - Q \frac{dv(\xi)}{dx} + W_i $$

where:

$$ W_i = \int_{\xi_0}^{\xi} \int_{S} \rho d^T d S dx = J \int_{\xi} \left( \frac{\rho}{J^2} B^T T + \frac{\rho}{J^2} B^T B T \right) dx d\hat{V} $$

(16)

where $\rho$ is the mass density, and $A$ is the beam cross section area. In the second member of equation (17) the term concerning the inertia of rotation of the beam cross section is clearly outlined.

The stationarity of the potential energy (equation (13)) with respect to the degrees of freedom in the vector $V$ yields the equation of motion:

$$ M \ddot{V} + K_i V = F $$

(18)

where the unconstrained stiffness matrix is:

$$ K_i = \int_{0}^{L} E I L J d\xi $$

(19)

the mass matrix is:

$$ M = \int_{0}^{L} \rho \left( \frac{1}{J^2} B^T T + \frac{1}{J^2} B^T B T \right) J d\xi $$

(20)

and the force vector is:

$$ F = \int_{0}^{L} f(x) B J d\xi + T B(\xi) + Q \frac{1}{J(\xi)} B'(\xi) $$

(21)

2.1. Constraint modelling

Distributed elastic constraints are taken into account by including an additional term $\Delta W$ in the functional of the total potential energy. The additional term $\Delta W$ takes into account the potential energy of the constraint force per unit length $f_c$, assumed as being applied along the vertical direction $y$:

$$ f_c(\xi) = -r \frac{d}{d\xi} v(\xi) $$

(22)

where $r$ is the stiffness coefficient of a distributed elastic constraint, modelled by means of B-spline functions:

$$ r \frac{d}{d\xi} v(\xi) = \sum_{i=1}^{c} B_i(\xi) r_i = B^T r $$

(23)

and the B-spline functions $B_i(\xi)$ are defined by means of the knot vector $U^C$:

$$ \Delta W = -\int_{0}^{L} f_c(\xi) v(\xi) J d\xi $$

(24)

The stiffness matrix due to the constraint force is:

$$ \Delta K = \int_{0}^{L} r \frac{d}{d\xi} B B^T J d\xi $$

(25)
2.2. Damping modelling
For lightly damped structures, appropriate results may be obtained by imposing the real damping assumption (real modeshapes). The real damping assumption is imposed by adding a viscous term in the equation of motion:

\[ M\ddot{\mathbf{V}} + C\dot{\mathbf{V}} + K\mathbf{V} + \Delta K \mathbf{V} = \mathbf{F}. \]  

The damping matrix \( C \) is calculated as explained in [12], where the modal damping ratios are evaluated from:

\[ \zeta_i = \zeta_i f_i = \zeta_i 2\pi \Omega_i, \]  

and the damping \( \zeta_i f_i \) is defined by means of control coefficients \( \gamma_i \) and B-spline functions \( B_{\xi i} \) defined on the knot vector \( U^i \):

\[ \zeta_i f_i = \zeta_i f_i \eta_i = \sum_{i=1}^{\xi_i} B_{\xi_i} \eta_i \gamma_i. \]  

Experimental FRF data are usually available up to a maximum frequency value \( f_X \): the parameterization \( f \eta \) and, consequently, the knot vector \( U^i \) can be chosen in order to take into account the contribution of the damping of each eigen-mode, beyond the \( f_X \) value, to the FRFs shape:

\[ f \eta = \eta f_X; \quad \eta \in \left[ 0, \frac{f_{\text{MAX}}}{f_X} \right] \]  

where \( f_{\text{MAX}} \) is the maximum frequency for which the damping model is defined, and:

\[ U^i = \begin{bmatrix} \eta_1 = 0 & \cdots & \eta_{\xi_i + \rho_i + 1} = \frac{f_{\text{MAX}}}{f_X} \end{bmatrix}. \]  

Given the fact that the FRF data are only available up to the \( f_X \) frequency value, the solution of the system of equations resulting from the updating procedure can be an ill-conditioned problem: the expected singularities will be dealt with by means of the singular value decomposition technique.

2.3. Verification of the B-spline beam model
A numerical test is reported to show the correctness of the beam model. In the example the free vibration analysis of a simply supported beam is considered: the resulting eigen-frequencies are compared both with the results of the analytical solution of the Euler-Bernoulli homogeneous, uniform, simply supported beam and those of an FEM model.

The free vibrations analysis of a simply supported beam with the following geometric and material properties is considered:

\[ L = 3m; \quad A = 7.2 \cdot 10^{-3} m^2; \quad I = 4.303 \cdot 10^{-4} m^4; \]
\[ E = 53.703 GPa; \quad \rho = 2500 Kg m^{-3}. \]  

The previous properties can be representative of the beam analyzed during an experimental campaign organized within the Bri.Vi.Di. research project (figure 2).

The infinite set of natural frequencies obtained by the analytical solution of the Euler-Bernoulli beam differential equations can be evaluated by [13]:

\[ f_i = \frac{\lambda_i^2}{2\pi} \sqrt{\frac{EI}{\rho A}}. \]  

**Figure 2.** Beam analyzed during an experimental campaign organized within the Bri.Vi.Di. research project.

Table 1. First ten natural frequencies $f_i$ and normalized parameters $f_i/f_1$ by employing three different beam models.

| $i$ | Exact Analytical$^a$ | standard FEM$^b$ | B-spline$^c$ |
|-----|----------------------|-----------------|--------------|
| 1   | 19.8145              | 19.808          | 19.808       |
| 2   | 79.2578              | 79.154          | 79.155       |
| 3   | 178.3301             | 177.80          | 177.8112     |
| 4   | 317.0314             | 315.38          | 315.3976     |
| 5   | 495.3615             | 491.34          | 491.3929     |
| 6   | 713.3206             | 705.02          | 705.1416     |
| 7   | 970.9086             | 955.63          | 955.8649     |
| 8   | 1268.1255            | 1242.2          | 1242.673     |
| 9   | 1604.9713            | 1563.8          | 1564.5783    |
| 10  | 1981.4461            | 1919.3          | 1920.5106    |

$^a$ Exact analytical results from the Euler-Bernoulli beam model
$^b$ Numerical results from standard FE software (202 dofs) with rotary inertia but without considering shear
$^c$ Numerical results from the proposed B-spline model (50 dofs)

where the $\lambda_i$ are:

$$\lambda_i = \frac{i\pi}{L} \quad i = 1, 2, 3, \ldots$$  \hspace{1cm} (33)

The B-spline beam model is set by adopting the following parameters:
- $p=4$
- $n=50$
- $\mathbf{U}$ is an equally spaced knot vector with repeated values for the first and last ($p+1$) knots (equation (2)).

The B-spline beam model is constrained by adopting the following parameters:
- $p^C=1$
- $n^C=4$ and $\mathbf{r} = r_1 \quad r_2 \quad r_3 \quad r_4 = 10^{10} \cdot 1 \quad 0 \quad 0 \quad 1 \quad Nm^{-2}$
- $\mathbf{U}^C = 0 \quad 0 \quad 0.001 \quad 0.999 \quad 1 \quad 1 \quad$. 

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The damping matrix is not considered. Table 1 reports the comparison of the first ten natural frequencies solution by employing both the classical analytical solution of the Euler-Bernoulli beam and the proposed B-spline beam model with 50 dofs. Table 1 also reports the solution provided by standard FEM software by adopting a beam model with 101 nodes and two dofs per node (the translation $u$, dofs has been constrained to exclude axial modes). Table 1 also compares the normalized parameters $\varphi = f_i / f_i$ by employing the three different beam models: the Euler-Bernoulli beam model, without considering the rotatory inertia effect, does not seem completely adequate to effectively represent the eigen-solution of the high frequency modes of a beam with the geometric and material properties listed in equation (31).

3. Updating procedure
The parametrization adopted for the elastic constraints and for the damping model is employed in an updating procedure based on Frequency Response Functions (FRFs) experimental measurements.

The $\ell$ measured FRFs $H^X_b \omega$, with $b=1,\ldots, \ell$, are collected in a vector $h_\chi \omega$:

$$ h_\chi \omega = \begin{bmatrix} H^X_1 \omega \\ \vdots \\ H^X_\ell \omega \end{bmatrix} $$

(34)

The dynamic equilibrium equation in the frequency domain, for the spline-based finite element model, can be defined by Fourier transforming equation (26), where $F(\cdot) =$:

$$ -\omega^2 M + j\omega C + K + \Delta K \ \tilde{V} = Z \ \omega \ \tilde{V} = H^{-1} \omega \ \tilde{V} = \tilde{F} $$

where $Z \omega$ is the dynamic impedance matrix and $H \omega = Z \omega^{-1}$ is the receptance matrix.

Since the vector $\tilde{V}$ contains non-physical displacements, the elements of the matrix $H \omega$ cannot be directly compared with the measured FRFs $H^X_b \omega$. The analytical FRFs related to physical dofs of the model can be obtained by means of the FE shape functions:

$$ H^{i,s} \omega = B^T(\xi_i) H \omega B(\xi_j) $$

(36)

The sensitivity of the FRF $H^{i,s}$ with respect to a generic parameter $p_k$ is:

$$ \frac{\partial H^{i,s} \omega,\mathbf{p}}{\partial p_k} = B^T(\xi_i) \frac{\partial H \omega,\mathbf{p}}{\partial p_k} B(\xi_j) = $$

$$ = -B^T(\xi_i) H \omega,\mathbf{p} \frac{\partial Z \omega,\mathbf{p}}{\partial p_k} H \omega,\mathbf{p} B(\xi_j) $$

(37)

where $\mathbf{p} = p_1 \cdots p_n$ is the vector containing the updating parameters $p_k$.

Since each measured FRF $H^X_b \omega$ corresponds to a well-defined set $i,r$, it is possible to collect, with respect to each measured FRF, the corresponding analytical FRFs in the vector:

$$ h_{\omega} \omega,\mathbf{p} = \begin{bmatrix} H^{i,s} \omega,\mathbf{p} \\ \vdots \\ H^{i,j} \omega,\mathbf{p} \end{bmatrix} $$

(38)

The elements of $h_{\omega,\mathbf{p}}$ are generally nonlinear functions of $\mathbf{p}$. The problem can be linearized, for a given angular frequency $\omega_i$, by expanding $h_{\omega,\mathbf{p}}$ in a truncated Taylor series at $\mathbf{p}=\mathbf{p}_0$:

$$ h_{\omega,\mathbf{p}} + \sum_{k=1}^{n'} \frac{\partial h_{\omega,\mathbf{p}}}{\partial p_k} \Delta p_k = h_{\omega,\mathbf{p}} $$

(39)
and in matrix form:

\[
\begin{bmatrix}
\frac{\partial h_a}{\partial p_1} \omega_i \cdot p_a, \ldots, \frac{\partial h_{an}}{\partial p_1} \omega_i \cdot p_a, \ldots, \frac{\partial h_{an}}{\partial p_{np}} \omega_i \cdot p_a
\end{bmatrix}
\begin{bmatrix}
\Delta p_1 \\
\vdots \\
\Delta p_k \\
\vdots \\
\Delta p_{np}
\end{bmatrix}
= h_a \omega_i - h_a \omega_i \cdot p_a
\] (40)

or:

\[
S_i \cdot \Delta p = \Delta h_i
\] (41)

where \( S_i \) is the sensitivity matrix for the \( i \)-th angular frequency value \( \omega_i \).

It is possible to obtain a least squares estimation of the \( n^p \) parameters \( p_k \), by defining the error function \( e \):

\[
e = \sum_{i=1}^{n'} S_i \cdot \Delta p - \Delta h_i, \quad n' \gg n^p
\] (42)

and by minimizing the objective function \( g \):  

\[
g = e^T \cdot e \rightarrow \min
\] (43)

Since FRF data available from measurement are usually large in quantity, a least squares estimation of the parameters can be obtained by adopting various FRF data at different frequencies. The proposed technique is iterative because a first order approximation was made during derivation of equation (39). In order to prevent updating parameters assuming non-physical values during the iterative procedure, the variable transformation proposed in [12] was adopted to constrain the parameters in a compact domain without using additional variables.

![Figure 3. Comparison of (input in dof \( l=3 \); output in dof \( l=3 \)) FRF before updating: the input data (blue dotted line) and the model (black continuous line).](image)

![Table 2. values of the updating parameters, identified by means of the proposed approach.](table)

| Parameter | \( r_2 \) | \( r_3 \) | \( \gamma_1 \) | \( \gamma_2 \) | \( \gamma_3 \) | \( \gamma_4 \) | \( \gamma_5 \) | \( \gamma_6 \) | \( \gamma_7 \) | \( \gamma_8 \) |
|-----------|----------|----------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| Value     | \( 1.21 \cdot 10^8 \) | \( 4.25 \cdot 10^9 \) | 0.030         | 0.005         | 0.309         | 0.096         | 0.169         | 0.223         | 0.022         | 0.0006         |
4. Application

The updating procedure is tested on a real problem dealing with a reinforced concrete beam analyzed within the Bri.Vi.Di. research project. Experimental estimated data are used as input data in the proposed updating procedure of the B-spline beam model with the geometric and material properties in equation (31) and the same geometric parameters \( p, n \) and \( U \) adopted in section 2.3.

The beam model is constrained by adopting the following parameters:

- \( p^C = 2 \)
- \( n^C = 9 \) and \( r = r_1, r_2, \ldots, r_9 = 5 \cdot 10^8 \) 0 1 0 0 0 0 0 1 0 \( Nm^{-2} \)
- \( U^C = 0 \quad 0 \quad 0 \quad 0.005 \quad 0.005 \quad 0.005 \quad 0.995 \quad 0.995 \quad 0.995 \quad 1 \quad 1 \quad 1 \).

The damping is considered by adopting the following parameters:

- \( p^C = 1 \)
- \( n^C = 8 \)
- \( \gamma^C = 0.03 \quad z = 1, \ldots, 12 \)
- \( f_X = 600Hz \)
- \( f_{MAX} = 900Hz \)
- \( U^C = 0 \quad 0 \quad 0.03 \quad 0.1 \quad 0.2 \quad 0.4 \quad 0.55 \quad 0.85 \quad 1 \quad 1.5 \).

The coefficients \( r_1, r_5 \) and all the \( \gamma^C \) are assumed as updating identification variables.

The estimated experimental data are the inertances, measured along the vertical direction \( y \), on 3 points of the beam whose \( x \) coordinates are:

\[
x^X_i; l = 1, 2, 3 = 0 \quad 0.46 \quad 1.6 \quad m,
\]

the reference dofs being in the middle of the beam.

A comparison of the resulting FRFs before the updating process is reported in figure 3. The values of the updating parameters, identified by means of the proposed approach, are reported in table 2. The comparison of the resulting FRFs after the updating process is reported in figure 4. The values of the updated FRFs are slightly different from measured FRFs near resonances but overall matching between identified and measured FRFs is good.
5. Conclusion
An updating procedure of a B-spline FE model of a concrete beam using measured FRFs was proposed, the updating parameters being the coefficients of a distributed constraint stiffness model and the damping ratios, both modelled by means of B-spline functions. The beam model was based on B-spline shape functions and on Euler-Bernoulli kinematic assumption. The rotatory inertia effect was included in the formulation. An application dealing with the updating of the B-spline FE beam model of a reinforced concrete beam considered within the Bri.Vi.Di. research project was reported. Measured FRF data were employed as input and good results were obtained. Future applications will be addressed towards the optimal choice of experimental dofs and frequency data.

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