Gaussian Quantum Marginal Problem

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Abstract: The quantum marginal problem asks what local spectra are consistent with a given spectrum of a joint state of a composite quantum system. This setting, also referred to as the question of the compatibility of local spectra, has several applications in quantum information theory. Here, we introduce the analogue of this statement for Gaussian states for any number of modes, and solve it in generality, for pure and mixed states, both concerning necessary and sufficient conditions. Formally, our result can be viewed as an analogue of the Sing-Thompson Theorem (respectively Horn’s Lemma), characterizing the relationship between main diagonal elements and singular values of a complex matrix: We find necessary and sufficient conditions for vectors \((d_1,\ldots,d_n)\) and \((c_1,\ldots,c_n)\) to be the symplectic eigenvalues and symplectic main diagonal elements of a strictly positive real matrix, respectively. More physically speaking, this result determines what local temperatures or entropies are consistent with a pure or mixed Gaussian state of several modes. We find that this result implies a solution to the problem of sharing of entanglement in pure Gaussian states and allows for estimating the global entropy of non-Gaussian states based on local measurements. Implications to the actual preparation of multi-mode continuous-variable entangled states are discussed. We compare the findings with the marginal problem for qubits, the solution of which for pure states has a strikingly similar and in fact simple form.

1. Introduction

What reduced states are compatible with some quantum state of a composite system having a certain spectrum? The study of this question has in fact a long tradition – as the natural quantum analogue of the marginal problem in classical probability theory. Very recently, this problem, now coined the quantum marginal problem, has seen a revival of interest, motivated by applications in the context of quantum information theory [1–6]. In fact, in the quantum information setting, notably in quantum channel capacity expressions, in assessments of quantum communication protocols, or in the
separability problem, one often encounters questions of compatibility of reductions with global quantum states [7–11].

Since it is only natural to look at the full orbit under local unitary operations, the quantum marginal problem immediately translates to a question of the compatibility of spectra of quantum states. The **mixed quantum marginal problem** then amounts to the following question: Is there a state $\rho$ of a quantum system with $n$ subsystems, each with a reduction $\rho_k$, that is consistent with

$$\text{spec}(\rho) = r,$$

$$\text{spec}(\rho_k) = r_k$$

for $k = 1, \ldots, n$, $r$ and $r_k$ denoting the respective vectors of spectra?. In the **pure marginal problem**, one assumes $\rho = |\psi\rangle \langle \psi|$ to be pure. In the condensed-matter context [12,13], related questions are also of interest: For example, once one had classified all possible two-qubit reductions of translationally invariant quantum states, then one would be able to obtain the ground state energy of any nearest-neighbor Hamiltonian of a spin chain. The quantum marginal problem was solved in several steps: Higuchi et al. [1] solved the pure quantum marginal problem for qubits. Subsequently, Bravyi was able to solve the mixed state case for two qubits, followed by Franz [6] and Higuchi [2] for a three qutrit system. The general solution of the quantum marginal problem for finite-dimensional systems was found in the celebrated work of Klyachko [5], see also Refs. [14,15]. This is indeed a closed-form solution. Yet the number of constraints grows extremely rapidly with the system size, rendering the explicit check whether the conditions are satisfied unfeasible even for relatively small systems.

In this work, we introduce the Gaussian version of the quantum marginal problem. Gaussian states play a key role in a number of contexts, specifically whenever bosonic modes and quadratic Hamiltonians become relevant, which are ubiquitous in quantum optical systems, free fields, and condensed matter lattice systems. For general infinite dimensional systems the marginals problem may well be intractable. However, given that in turn these Gaussian states can be described by merely their first and second moments [16,17], one could reasonably hope that it could be possible to give a full account of the **Gaussian quantum marginal problem**. This gives rise, naturally, not to a condition to spectra of quantum states, but to symplectic spectra, as explained below. For the specific case of three modes, the result is known [18], see also Ref. [19]. In this work we will show that this program of characterizing the reductions of Gaussian states can be achieved in generality, even concerning both necessary and sufficient conditions. This means that one can give a complete answer to what reductions entangled Gaussian states can possible have.$^1$

Equivalently, we can describe this Gaussian marginal problem as a problem of compatibility of temperatures of standard harmonic systems: Given a state $\rho$, what **local temperatures** – or equivalently for single modes, what **local entropies** – are compatible with this joint state? Of course, one can always take the temperatures to be equal. But if they are different, they constrain each other in a fairly subtle way, as we will see. In a sense, the result gives rise to the interesting situation that by looking at local temperatures, one can assess whether these reductions may possibly originate from a joint system in a pure state. Finally, it is important to note, since sufficiency of the conditions

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$^1$ We refer here to the marginal problem for Gaussian states, which are quantum states fully defined by their first and second moments of canonical coordinates. However, clearly, our result equally applies to general and hence non-Gaussian states, in that it fully answers the question what local second moments are consistent with global second moments of quantum states of several modes.
is always proven by an explicit construction, the result also implies a recipe for preparing multi-mode continuous-variable entangled states.

2. Main Result

We consider states on \( n \) modes, and consider reductions to single modes. Gaussian states are represented by the matrix of second moments, the \( 2n \times 2n \) covariance matrix \( \gamma \) of the system, together with the vector \( \mu \) of first moments. For a definition and a survey of properties, see Refs. [16,17]. In this language, the vacuum state of a standard oscillator becomes \( \gamma = \frac{1}{2} a \sigma a^T \), as the \( 2 \times 2 \) identity matrix. The canonical commutation relations are embodied in the symplectic matrix

\[
\sigma = \bigoplus_{k=1}^{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

for \( n \) modes. The covariance matrices of \( n \) modes are exactly those real matrices satisfying

\[
\gamma + i \sigma \geq 0,
\]

which is simply a statement of the Heisenberg uncertainty principle. The first moments can always be made zero locally, and are hence not interesting for our purposes here. Note also that the set of Gaussian states is closed under reductions, so reduced states of Gaussian states are always Gaussian as well.

Real matrices that leave the symplectic form invariant, \( S \sigma S^T = \sigma \), form the real symplectic group \( \text{Sp}(2n, \mathbb{R}) \). In the same way as symmetric matrices \( M \) can be diagonalized with orthogonal matrices to a diagonal matrix \( O M O^T = D \), one can diagonalize strictly positive matrices using such \( S \in \text{Sp}(2n, \mathbb{R}) \), according to

\[
S M S^T = D.
\]

The simply counted main diagonal elements of \( D \) form then the symplectic spectrum of \( M \), and the collection of symplectic eigenvalues can be abbreviated as \( \text{sspec}(M) = (d_1, \ldots, d_n) \),

\[
D = \text{diag}(d_1, d_1, \ldots, d_n, d_n).
\]

This procedure is nothing but the familiar normal mode decomposition. In turn, by definition, the symplectic eigenvalues are given by the square roots of the eigenvalues of the matrix \( -M \sigma M^T \). Again, for the vacuum, the symplectic eigenvalues are all given by unity. In a mild abuse of notation, we will refer to the symplectic spectrum of a Gaussian state as the symplectic spectrum of the respective covariance matrix.

Finally, for a given covariance matrix \( \gamma \), and in fact any strictly positive real matrix, we refer to the symplectic main diagonal elements \( (c_1, \ldots, c_n) \) as the symplectic eigenvalues of the \( 2 \times 2 \) main diagonal blocks. This is the natural analogue of main diagonal elements. Equivalently, the symplectic main diagonal elements are the main diagonal elements after the main diagonal \( 2 \times 2 \) blocks have been brought into the form

\[
\gamma_k = \begin{bmatrix} c_k & 0 \\ 0 & c_k \end{bmatrix}.
\]

We are now in the position to state our main result. It relates the symplectic spectrum of composite systems to the ones of the reductions. We will first state it as a mere matrix constraint, then as the actual Gaussian marginal problem, and finally for the important special case of having a pure joint state.
Theorem 1 (Necessary and sufficient conditions). Let \((d_1, \ldots, d_n)\) and \((c_1, \ldots, c_n)\) be two vectors of positive numbers in non-decreasing order. Then there exists a strictly positive real \(2n \times 2n\)-matrix \(\gamma\) such that \((d_1, \ldots, d_n)\) are its symplectic eigenvalues and \((c_1, \ldots, c_n)\) the symplectic main diagonal elements if and only if the \(n + 1\) conditions

\[
\sum_{j=1}^{k} c_j \geq \sum_{j=1}^{k} d_j, \quad k = 1, \ldots, n \tag{8}
\]

\[
c_n - \sum_{j=1}^{n-1} c_j \leq d_n - \sum_{j=1}^{n-1} d_j \tag{9}
\]

are satisfied.

This set of inequalities may be conceived as a general analogue of the Sing-Thompson theorem [20–22], see below. More physically speaking, this means the following:

Corollary 1 (Gaussian marginal problem). Assume that \(\rho\) is a Gaussian state of \(n\) modes satisfying \(\text{sspec}(\rho) = (d_1, \ldots, d_n)\). Then the possible reduced states \(\rho_k\) to each of the individual modes \(k = 1, \ldots, n\) are exactly those Gaussian states with

\[
\text{sspec}(\rho_k) = c_k \tag{10}
\]

satisfying Eq. (8) and (9).

These conditions hence fully characterize the possible reduced marginal states. For two modes, \(n = 2\), for example, the given conditions read

\[
c_1 + c_2 \geq d_1 + d_2, \tag{11}
\]

\[
c_2 - c_1 \leq d_2 - d_1, \tag{12}
\]

for \(c_2 \geq c_1\) and \(d_2 \geq d_1\). The constraint \(c_1 \geq d_1\) is then automatically satisfied. For pure Gaussian states, the above conditions take a specifically simple form. Quite strikingly, we will see that the resulting conditions very much resemble the situation of the marginal problem for qubits.

Corollary 2 (Pure Gaussian marginal problem). Let \(\rho = |\psi\rangle\langle\psi|\) be a pure Gaussian state of \(n\) modes. Then the set \((b_1 + 1, \ldots, b_n + 1)\) of symplectic eigenvalues

\[
\text{sspec}(\rho_k) = b_k + 1, \tag{13}
\]

\(k = 1, \ldots, n\), of the reduced states \(\rho_k\) of each of the \(n\) modes is given by the set defined by

\[
b_j \leq \sum_{k \neq j} b_k \tag{14}
\]

for all \(j\), for \(b_j \geq 0\).
To reiterate, these conditions are necessary and sufficient for the local symplectic spectra being consistent with the global state being a pure Gaussian state.

Equivalently, this can be put as follows: If $\gamma$ is the covariance matrix of a pure Gaussian state with reductions

$$
\gamma_k = \begin{bmatrix}
  b_k + 1 & 0 \\
  0 & b_k + 1
\end{bmatrix},
$$

$k = 1, \ldots, n$. Then, Eq. (14) defines the local temperatures $T_k$ per mode consistent with the whole system being in a pure Gaussian state, according to

$$
b_k = 2(\exp(1/T_k) - 1)^{-1},
$$

for the standard harmonic oscillator (an oscillator with unit mass and frequency). The above condition hence determines the temperatures that modes can have, given that a composite system is in a pure Gaussian state. The form of Eq. (15) can always be achieved by means of local rotations and squeezings in phase space. One can hence equally think in terms of local symplectic spectra or local temperatures.

It is instructive to compare the results for the pure Gaussian marginal problem with the one for qubits as solved in Ref. [1]. There, it has been found that for a system consisting of $n$ qubits, one has

$$
\lambda_j \leq \sum_{k \neq j} \lambda_k
$$

for the spectral values $r_k = (\lambda_k, 1 - \lambda_k), \lambda_k \in [0, 1]$. Moreover, these conditions are both necessary and sufficient. It is remarkable that this form is identical with the result for $n$ single modes

$$
b_j \leq \sum_{k \neq j} b_k,
$$

$b_k \geq 0$, as necessary and sufficient conditions. Again, the admissible symplectic eigenvalues are defined by a cone the base of which is formed by a simplex. Note that the methods used in Ref. [1] to arrive at the above result are entirely different. Once again, a striking formal similarity between the case of qubit systems and Gaussian states is encountered.

Finally, from the perspective of matrix analysis, the above result can be seen as a general analogue of the Sing-Thompson Theorem [20–22] (or Horn’s Lemma [23] in case of Hermitian matrices), first posed in Ref. [24], where the role of singular values is taken by the symplectic eigenvalues.

**Sing-Thompson Theorem ([20–22]).** Let $(x_1, \ldots, x_n)$ be complex numbers such that $|x_k|$ are non-increasingly ordered and let $(y_1, \ldots, y_n)$ be non-increasingly ordered positive numbers. Then an $n \times n$ matrix exists with $x_1, \ldots, x_n$ as its main diagonal and $y_1, \ldots, y_n$ as its singular values if and only if

$$
\sum_{j=1}^{k} |x_j| \leq \sum_{j=1}^{k} y_j, \ k = 1, \ldots, n,
$$

$$
\sum_{j=1}^{n-1} |x_j| - |x_n| \leq \sum_{j=1}^{n-1} y_j - y_n.
$$
It is interesting to see that – although the symplectic group \( \text{Sp}(2n, \mathbb{R}) \) is not a compact group – there is so much formal similarity concerning the implications on main diagonal elements of matrices. Note, however, that the ordering of singular values and symplectic eigenvalues, respectively, is different in Theorem 1 and in the Sing-Thompson theorem.

### 3. Proof

As a preparation of the proof, we will identify a simple set of necessary conditions that constrains the possible reductions that are consistent with the assumption that the state is pure and Gaussian. These simple conditions derive from a connection between the symplectic trace and the trace of the covariance matrix. Quite surprisingly, we will see that they already define the full set of possible marginals consistent with a Gaussian state of \( n \) modes. We shall start by stating the condition to the reductions.

**Lemma 1 (Symplectic trace).** Let \( \gamma \) be a strictly positive real \( 2n \times 2n \)-matrix such that its main diagonal \( 2 \times 2 \) blocks are given by Eq. (7) for \( c_k \in [1, \infty) \). Then the symplectic eigenvalues \( (d_1, \ldots, d_n) \) of the matrix \( \gamma \) satisfy

\[
\sum_{k=1}^{n} d_k \leq \sum_{k=1}^{n} c_k. \tag{21}
\]

**Proof.** Note that the right-hand side of Eq. (21) is nothing but half the trace of the covariance matrix \( \gamma \), whereas the left-hand side is the symplectic trace \( \text{str}(\gamma) \) of \( \gamma \), so

\[
\text{str}(\gamma) = \sum_{j=1}^{n} d_j \tag{22}
\]

if \( \text{sspec}(\gamma) = (d_1, \ldots, d_n) \), see, e.g., Ref. [25]. We arrive at this relationship by making use of a property of the trace-norm. The symplectic eigenvalues \( d_1, \ldots, d_n \) of \( \gamma \) are given by the square roots of the simply counted eigenvalues of the matrix \( (i\sigma)\gamma(i\sigma)\gamma \) [16,17]. Hence, the symplectic spectrum is just given by the spectrum of the matrix

\[
M = |\gamma^{1/2}(i\sigma)\gamma^{1/2}|, \tag{23}
\]

where \( |\cdot| \) denotes the matrix absolute value\(^2\). So we have that

\[
2 \sum_{k=1}^{n} d_k = \text{tr}(M) = \|\gamma^{1/2}(i\sigma)\gamma^{1/2}\|_1, \tag{24}
\]

where \( \|\cdot\|_1 \) is the trace norm. The property we wish to prove then immediately follows from the fact that the trace-norm is a unitarily invariant norm: this implies that

\[
2 \sum_{k=1}^{n} d_k = \|\gamma^{1/2}(i\sigma)\gamma^{1/2}\|_1 \leq \|(i\sigma)\gamma\|_1, \tag{25}
\]

as \( \|AB\|_1 \leq \|BA\|_1 \) for any matrices \( A, B \) for which \( AB \) is Hermitian. This inequality holds for any unitarily invariant norm whenever \( AB \) is a normal operator [26]. Now,

\(^2 |A| = (A^2)^{1/2} \) for Hermitian matrices \( A \).
since any covariance matrix is positive, $\gamma \geq 0$, and the largest singular value of $i\sigma$ is clearly given by unity, we can finally conclude that

$$2 \sum_{k=1}^{n} d_k \leq \|\gamma\|_1 = \text{tr}(\gamma) = 2 \sum_{k=1}^{n} c_k,$$

(26)

which is the statement that we intended to show. □

This observation implies as an immediate consequence a necessary condition for the possible reductions, given a Gaussian state of an $n$-mode system: Let $\gamma$ be the covariance matrix of a Gaussian pure state of $n$ modes, with reductions as above. We can think of the state as a bi-partite state between a distinguished mode labeled $k$, without loss of generality being the last mode $k = n$, and the rest of the system. We can in fact Schmidt decompose this pure state with respect to this split using Gaussian unitary operations [29,28,34]. This means that we can find symplectic transformations $S_A \in \text{Sp}(2(n-1), \mathbb{R})$ and $S_B \in \text{Sp}(2, \mathbb{R})$ such that

$$(S_A \oplus S_B)\gamma (S_A \oplus S_B)^T = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},$$

(27)

where

$$A = \text{diag}(1, \ldots, 1, a_n, a_n),$$

(28)

$$B = \text{diag}(a_n, a_n),$$

(29)

with some $2(n-1) \times 2$-matrix $C$. The symplectic eigenvalues of modes $1, \ldots, n-1$ are hence given by $1, \ldots, 1, a_n$. The above statement therefore implies the inequality

$$n - 2 + a_n \leq a_1 + \cdots + a_{n-1},$$

(30)

or, by substituting $b_k = a_k - 1$ for all $k = 1, \ldots, n$,

$$b_n \leq b_1 + b_2 + \cdots + b_{n-1}.$$

(31)

This must obviously hold for all distinguished modes and not only the last one, and hence, we arrive at the following simple necessary conditions:

**Corollary 3 (Necessary conditions for pure states).** Let $\gamma$ be the covariance matrix of a pure Gaussian state with thermal reductions

$$\gamma_k = \begin{bmatrix} b_k + 1 & 0 \\ 0 & b_k + 1 \end{bmatrix},$$

(32)

$k = 1, \ldots, n$. Then, for all $j$,

$$b_j \leq \sum_{k \neq j} b_k.$$

(33)
That is, the largest value of $b_j$ cannot exceed the sum of all the other ones.

So far, we have assumed the global state $\rho$ to be a pure state. In the full problem, however, we may of course allow $\rho$ to be any Gaussian state, and hence a mixed one, with symplectic spectrum

$$\text{sspec}(\rho) = (d_1, \ldots, d_n) \geq (1, \ldots, 1),$$

(34)

instead of being $(1, \ldots, 1)$. This is the Gaussian analogue of the mixed marginal problem. For this mixed state case, we provide necessary conditions for the main reductions, in form of $n$ inequalities on partial sums, and one where the largest symplectic eigenvalue of a reduction plays an important role. The first set of $n$ conditions is up to the different ordering a weak majorization relation for symplectic eigenvalues, which is in fact essentially a corollary of a result from Ref. [27] due to Hiroshima. The second statement, the $n+1$th condition, as well as showing sufficiency of the general conditions, will turn out to be significantly more involved.

**Lemma 2 (Necessity of the first $n$ conditions).** Let $(d_1, \ldots, d_n)$, and $(c_1, \ldots, c_n)$ be defined as in Theorem 1. For any given $(d_1, \ldots, d_n)$, the admissible $(c_1, \ldots, c_n)$ satisfy

$$\sum_{j=1}^{k} c_j \geq \sum_{j=1}^{k} d_j$$

(35)

for all $k = 1, \ldots, n$.

**Proof.** Let $S \in \text{Sp}(2n, \mathbb{R})$ be the matrix from the symplectic group that brings $\gamma$ into diagonal form, so

$$S \gamma S^T = \text{diag}(d_1, d_1, \ldots, d_n, d_n).$$

(36)

The main diagonal elements of $\gamma$, in turn, again without loss of generality in non-decreasing order, are given by $(c_1, \ldots, c_n)$. Now according to Ref. [27], we have that

$$\min \text{tr}(T \gamma T^T) = 2 \sum_{j=1}^{k} d_j$$

(37)

for $k = 1, \ldots, n$, where the minimum is taken over all real $2k \times 2n$-matrices $T$ for which

$$T \sigma_k T^T = \sigma_k.$$  

(38)

Here, $\sigma_k$ denotes the symplectic matrix on $k$ modes as defined in Eq. (3). Now we can actually take $S \in \text{Sp}(2n, \mathbb{R})$ according to $S = I$, we see that $T$, consisting of the first $2k$ rows of $S$, satisfies Eq. (38). Since this submatrix does not necessarily correspond to a minimum in Eq. (37), we find

$$2 \sum_{j=1}^{k} c_j = \text{tr}(T \gamma T^T) \geq 2 \sum_{j=1}^{k} d_j,$$

(39)

for any $k = 1, \ldots, n$. $\Box$

We will now prove the necessity of the $n+1$th inequality constraint in Theorem 1.
Lemma 3 (Necessity of the last condition). Let \((d_1, \ldots, d_n)\) and \((c_1, \ldots, c_n)\) be defined as in Theorem 1. For any given vector of symplectic eigenvalues \((d_1, \ldots, d_n)\), the admissible \((c_1, \ldots, c_n)\) satisfy
\[
c_n - \sum_{j=1}^{n-1} c_j \leq d_n - \sum_{j=1}^{n-1} d_j.
\] (40)

Proof. We will define the function \(f : S_n \to \mathbb{R}\), where \(S_n\) is the set of strictly positive real \(2n \times 2n\)-matrices, as follows: We define the vector \(c = (c_1, \ldots, c_n)\) as
\[
c_j = (\gamma_{2j-1,2j-1}\gamma_{2j,2j} - \gamma_{2j-1,2j})^{1/2},
\] (41)
j = 1, \ldots, n, as the usual vector of symplectic spectra of each of the \(n\) modes, and then set
\[
f(\gamma) := 2 \max(c) - \sum_{j=1}^{n} c_j.
\] (42)
For a diagonal matrix \(D = \text{diag}(d_1, d_1, \ldots, d_n, d_n)\) with entries in non-decreasing order, we have
\[
f(D) = d_n - \sum_{j=1}^{n-1} d_j.
\] (43)

We will now investigate the orbit of this function \(f\) under the symplectic group,
\[
\tilde{f} = \sup \left\{ x \in \mathbb{R} : x = f(SDS^T), S \in \text{Sp}(2n, \mathbb{R}) \right\},
\] (44)
and will see that the supremum is actually attained as a maximum for \(S = 1\). Each of the matrices \(\gamma = SDS^T\) have by construction the same symplectic spectrum as \(D\). This is a variation over \(2n^2 + n\) real parameters, as any \(S \in \text{Sp}(2n, \mathbb{R})\) can be decomposed according to the Euler decomposition as
\[
S = O Q V,
\] (45)
where \(O, V \in K(n) := \text{Sp}(2n, \mathbb{R}) \cap O(2n)\) and
\[
Q \in \{(z_1, 1/z_1, \ldots, z_n, 1/z_n) : z_k \in \mathbb{R}\setminus\{0\}\}.
\] (46)
That is, \(O, V\) reflect passive operations, whereas \(Q\) stands for a squeezing operation.

We will now see that the maximum of this function \(\tilde{f}\) – which exists, albeit the group being non-compact – is actually attained when the matrix is already diagonal. This means that in general, we have that
\[
\tilde{f} = 2 \max \text{sspec}(\gamma) - \text{str}(\gamma).
\] (47)
For any global maximum, any local variation will not increase this function further. Let us start from some \(\gamma = SDS^T\). For any such covariance matrix \(\gamma\) we can find a \(T \in \text{Sp}(2(n - 1), \mathbb{R})\) such that
\[
(T \oplus \mathbb{1}_2)\gamma(T \oplus \mathbb{1}_2)^T = \begin{bmatrix} E & F \\ FT & G \end{bmatrix} =: \gamma',
\] (48)
where

\[ E = \text{diag}(c'_1, c'_1, \ldots, c'_{n-1}, c'_{n-1}) \quad (49) \]

is a \((2n - 2) \times (2n - 2)\) matrix and \(G\) is a \(2 \times 2\) matrix. Using Lemma 1 again, we find that

\[ \sum_{j=1}^{n-1} c'_j \leq \sum_{j=1}^{n-1} c_j, \quad (50) \]

so

\[ f(\gamma') \geq f(\gamma). \quad (51) \]

In other words, it does not restrict generality to assume the final covariance matrix to be of the form as in the right hand side of Eq. (48), and we will use the notation

\[ \gamma = SDS^T = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix} \quad (52) \]

with \(E = \text{diag}(c_1, c_1, \ldots, c_{n-1}, c_{n-1})\) and \(G = \text{diag}(c_n, c_n)\).

We can now investigate submatrices of \(\gamma\) associated with modes \(m\) and \(n\), \(1 \leq m < n\),

\[ M_{m,n} = \begin{bmatrix} c_m 1_2 \ & C_{n,m} \\ C_{n,m}^T \ & c_n 1_2 \end{bmatrix}. \quad (53) \]

This we can always bring to a diagonal form, using symplectic diagonalization, only affecting the main diagonal elements of modes \(n\) and \(m\), and leaving the other main diagonal elements invariant. This brings this submatrix into the form

\[ M'_{m,n} = \begin{bmatrix} c'_m 1_2 \ & 0 \\ 0 \ & c'_n 1_2 \end{bmatrix}, \quad (54) \]

with \(c'_n \geq c'_m\). From Lemma 5 we know that

\[ c'_n - c'_m \leq c_n - c_m, \quad (55) \]

so we have increased the function \(f\), whenever \(C_{n,m} \neq 0\). Hence, for global and hence local optimality, we have to have \(C_{n,m} = 0\). However, each of the matrices \(C_{n,m} = 0\) for all \(m = 1, \ldots, n - 1\) exactly if the matrix \(\gamma\) is already diagonal.

What remains to be shown is that the function \(f\) is bounded from above, to exclude the case that the maximum does not even exist. One way to show this is to make use of the upper bound in Lemma 4 to have for every covariance matrix \(\gamma\) with symplectic spectrum \((d_1, \ldots, d_n)\) in non-decreasing order

\[ f(\gamma) \leq \sum_{j=2}^{n} d_j + (3 - 2n)d_1, \quad (56) \]

which shows that \(f\) is always bounded from above. If \(\gamma\) is merely a strictly positive real matrix, but no covariance matrix, an upper bound follows from a rescaling with a positive number. \(\square\)

We now prove the upper bound required for the proof of Lemma 3.
Lemma 4 (Upper bound). Let \((d_1, \ldots, d_n), (c_1, \ldots, c_n)\) be defined as in Theorem 1, and \(\gamma\) be additionally a \(2n \times 2n\) covariance matrix. For any given \((d_1, \ldots, d_n)\), the admissible \((c_1, \ldots, c_n)\) satisfy

\[
c_n - \sum_{j=1}^{n-1} c_j \leq \sum_{j=2}^{n} d_j + (3 - 2n)d_1. \tag{57}
\]

Proof. We start from a \(4n \times 4n\)-covariance matrix

\[
\gamma = \begin{bmatrix} A & C \\ C^T & A \end{bmatrix},
\]

corresponding to a pure Gaussian state, where

\[
A = \bigoplus_{k=1}^{n} \begin{bmatrix} d_k & 0 \\ 0 & d_k \end{bmatrix}, \tag{59}
\]

\[
C = \bigoplus_{k=1}^{n} \begin{bmatrix} (d_k^2 - 1)^{1/2} & 0 \\ 0 & -(d_k^2 - 1)^{1/2} \end{bmatrix} \tag{60}
\]

are real \(2n \times 2n\)-matrices. Physically, this means that we start from a collection of \(n\) two mode squeezed states, with the property that the reduction to the first \(n\) modes is just a diagonal covariance matrix with symplectic eigenvalues \((d_1, \ldots, d_n)\), again in non-decreasing order. Let us first assume that \(d_1 = 1\); this assumption will be relaxed later. Let us now consider

\[
\begin{bmatrix} S_1 & 0 \\ 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} S_1^T & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} S_1AS_1^T & S_1C \\ C^T S_1 & S_1 \end{bmatrix}, \tag{61}
\]

for \(S_1 \in \text{Sp}(2n, \mathbb{R})\). Obviously, the set we seek to characterize is the set \(B\) of main diagonals of the upper left block

\[
U = S_1AS_1^T \tag{62}
\]

of this matrix. We can always start from a diagonal matrix having the symplectic eigenvalues on the main diagonal, and consider the orbit under all symplectic transformations \(S \in \text{Sp}(4n, \mathbb{R})\).

We will now relax the problem by allowing all \(S \in \text{Sp}(4n, \mathbb{R})\) instead of symplectic transformations of the form \(S = S_1 \oplus 1, S_1 \in \text{Sp}(2n, \mathbb{R})\). We hence consider the full orbit under all symplectic transformations. This set \(C \supset B\) is characterized by the reductions to single modes of

\[
\gamma' = S\gamma S^T = \begin{bmatrix} A' & C' \\ C'^T & A \end{bmatrix} \tag{63}
\]

for some \(S \in \text{Sp}(4n, \mathbb{R})\), such that again \(A = \text{diag}(d_1, d_1, \ldots, d_n, d_n)\) This includes the case (61).

We are now in the position to make use of the statement that we have established before: From exploiting the Schmidt decomposition on the level of second moments, and using Lemma 1 relating the trace to the symplectic trace, we find

\[
c_n - \sum_{j=1}^{n-1} c_j \leq \sum_{j=2}^{n} d_j + 3 - 2n, \tag{64}
\]
as \( d_1 = 1 \) was assumed.

Let us now consider the case of \( d_1 > 1 \). We will apply the previous result, after appropriately rescaling the covariance matrix. Indeed, we can construct a covariance matrix \( \tilde{\gamma} \) as in Eq. (58) for \((\tilde{d}_1, \ldots, \tilde{d}_n) = (1, d_2/d_1, \ldots, d_n/d_1)\). (65)

We then investigate the orbit of \( \tilde{\gamma} \) under the symplectic group, and look at the main diagonal elements of \( S\tilde{\gamma}S^T \). By construction, we have that \( \tilde{\gamma} + i\sigma \geq 0 \). We can hence apply Eq. (64) to this case. Multiplying both sides of Eq. (63) by \( d_1 \) gives rise to the condition in Eq. (57).

Lemma 5 (Solution to two-mode problem). There exists a strictly positive real \( 4 \times 4 \)-matrix \( \gamma \) with main diagonal blocks \( \text{diag}(c_1, c_1), \text{diag}(c_2, c_2) \) and symplectic eigenvalues \((d_1, d_2)\) if and only if

\[
\begin{align*}
    c_1 + c_2 &\geq d_1 + d_2, \\
    c_2 - c_1 &\leq d_2 - d_1,
\end{align*}
\]

assuming \( c_2 \geq c_1 \) and \( d_2 \geq d_1 \). Moreover, \( c_1 - c_2 = d_1 - d_2 \) if and only if the \( 2 \times 2 \) off diagonal block of \( \gamma \) vanishes.

Proof. The necessary conditions that \(|c_1 - c_2| \leq |d_1 - d_2|\) are a consequence of Lemma 4. The necessary conditions \( c_1 + c_2 \geq d_1 + d_2 \) and \( c_1 \geq d_1 \) have been previously shown in Lemma 2. Hence, we have to show that these conditions can in fact be achieved. This can be done by considering a

\[
\gamma = \begin{bmatrix}
    c_1 & 0 & e & 0 \\
    0 & c_1 & 0 & f \\
    e & 0 & c_2 & 0 \\
    0 & f & 0 & c_2
\end{bmatrix} = S\text{diag}(d_1, d_1, d_2, d_2)S^T.
\]

(68)

The relationship between \( c_1, c_2, e, f \) and \( d_1, d_2 \) is given by

\[
d_{1/2}^2 = (c_1^2 + c_2^2 + 2ef \\
\quad \pm (c_1^4 + c_2^4 + 4ef c_2^2 - 2c_1^2 c_2^2 - 2ef) + 4c_1 c_2 (e^2 + f^2))^{1/2}/2.
\]

(69)
as \( d_1, d_2 \) are the square roots of the eigenvalues of \(-\sigma \gamma \sigma \gamma \) [16], compare also Ref. [32]. An elementary analysis shows that the above inequalities can always be achieved. Also, the extremal values are achieved if and only if \( e = f = 0 \). \( \Box \)

What we finally need to show is that the conditions that we have derived are in fact sufficient. This will be the most involved statement.

Lemma 6 (Sufficiency of the conditions). For any vectors \((c_1, \ldots, c_n)\) and \((d_1, \ldots, d_n)\) satisfying the conditions (8) and (9) there exists a \( 2n \times 2n \) strictly positive real matrix with \( \text{diag}(c_1, \ldots, c_n) \) as its symplectic main diagonal elements and \((d_1, \ldots, d_n)\) as its symplectic eigenvalues.
Proof. The argument will essentially be an argument by induction, in several ways resembling the argument put forth in Refs. [20–22]. The underlying idea of the proof is essentially as follows: On using the given constraints, one constructs an appropriate $2(n - 1) \times 2(n - 1)$-matrix, in a way that it can be combined to the desired $2n \times 2n$-matrix by means of an appropriate $S \in \text{Sp}(4, \mathbb{R})$ acting on a $4 \times 4$ submatrix only. Note, however, that compared to Ref. [22], we look at variations over the non-compact symplectic group $\text{Sp}(2n, \mathbb{R})$, and not the compact $U(2n)$.

For a single mode, $n = 1$, there is nothing to be shown. For two modes, Lemma 5 provides the sufficiency of the conditions. Let us hence assume that we are given vectors $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_n)$ as above, and that we have already shown that for $2(n - 1) \times 2(n - 1)$-matrices, the conditions (8) and (9) are indeed sufficient. We complete the proof by explicitly constructing an $2n \times 2n$-matrix with the stated property.

We have that $c_1 \geq d_1$ by assumption. We could also have $c_1 \geq d_j$ for some $2 \leq j \leq n$, so let $k \in \{1, \ldots, n\}$ be the largest index such that

$$c_1 \geq d_k. \quad (70)$$

Let us first consider the case that $k \leq n - 2$, and we will consider the cases $k = n - 1$ and $k = n$ later. Then we can set $x := d_k + d_{k+1} - c_1$, which means that $x \geq 0$, and that all conditions

$$c_1 + x \geq d_k + d_{k+1}, \quad (71)$$
$$c_1 - x \geq d_k - d_{k+1}, \quad (72)$$
$$-c_1 + x \geq d_k - d_{k+1} \quad (73)$$

are satisfied: (71) by definition, (72) because $c_1 \geq d_k$ and (73) as $d_{k+1} \geq c_1$. This means that we can find a matrix of the form

$$\gamma' := \begin{bmatrix} c_1 & 1/2 & C \\ C^T & x & 1/2 \end{bmatrix}, \quad (74)$$

for some $2 \times 2$-matrix $C$, with symplectic eigenvalues $(d_k, d_{k+1})$, using Lemma 5. Therefore, the matrix

$$\gamma'' = \gamma' \oplus \text{diag}(d_1, d_1, d_2, d_2, \ldots, d_{k-1}, d_{k-1}, d_{k+2}, d_{k+2}, \ldots, d_n, d_n) \quad (75)$$

has the symplectic spectrum $(d_1, \ldots, d_n)$.

We will now show that we can construct a $2(n - 1) \times 2(n - 1)$ matrix $\gamma'''$ with symplectic eigenvalues $(d_1, \ldots, d_{k-1}, x, d_{k+2}, \ldots, d_n)$ and main diagonal elements $(c_2, c_2, \ldots, c_n, c_n)$, by invoking the induction assumption. This matrix $\gamma'''$ we can indeed construct, as we have

$$c_2 + \cdots + c_l \geq d_1 + \cdots + d_{l-1}, \quad l = 2, \ldots, k, \quad (76)$$
$$c_2 + \cdots + c_{k+1} \geq d_1 + \cdots + d_{k-1} + x, \quad (77)$$
$$c_2 + \cdots + c_s \geq d_1 + \cdots + d_{k-1} + x + d_{k+2} + \cdots + d_s, \quad s = k + 2, \ldots, n, \quad (78)$$

as one can show using $d_k \leq c_1 \leq d_{k+1}$ and $x = d_k + d_{k+1} - c_1$. Also, we have

$$c_n - c_2 - \cdots - c_{n-1} \leq d_n - d_1 - \cdots - d_{k-1} - x - d_{k+2} - \cdots - d_n. \quad (79)$$
fulfilling all of the conditions that we need invoking the induction assumption to construct $\gamma''''$. This matrix has the same symplectic eigenvalues as the right lower $2(n-1) \times 2(n-1)$ submatrix $\gamma''''$ of $\gamma''$. Therefore, there exists an $S \in \text{Sp}(2(n-1), \mathbb{R})$ such that

$$\gamma'''' = S\gamma''''S^T.$$  \hfill (80)

So the matrix

$$\gamma := (\mathbb{1}_2 \oplus S)\gamma''(\mathbb{1}_2 \oplus S)^T$$  \hfill (81)

has the symplectic spectrum $(d_1, \ldots, d_n)$ and symplectic main diagonal elements $(c_1, \ldots, c_n)$. Hence, by invoking the induction assumption, we have been able to construct the desired matrix with the appropriate symplectic spectrum and main diagonal elements. Note that only two-mode operations have been needed in order to achieve this goal.

We now turn to the two remaining cases, $k = n$ and $k = n - 1$. In both cases this means that we have $c_1 \geq d_{n-1}$, as $d_n \geq d_{n-1}$, and both cases can be treated in actually exactly the same manner. Obviously, this implies that also $c_n \geq c_1 \geq d_{n-1}$. We can now define again an $x$, by means of a set of inequalities. This construction is very similar to the one in Ref. [22]. We can require on the one hand

$$x \geq \max\{d_{n-1}, d_{n-1} + d_n - c_n, d_{n-1} - d_n + c_n, d_1 + \cdots + d_{n-2} + c_{n-1} - c_1 - \cdots - c_{n-2}\}. \hfill (82)$$

On the other hand, we can require

$$x \leq \min\{d_n - d_{n-1} + c_n, c_1 + \cdots + c_{n-1} - d_1 - \cdots - d_{n-2}, 0\}. \hfill (83)$$

Both these conditions can be simultaneously satisfied, making use of $c_n \geq c_{n-1}$ and $c_n \geq d_{n-1}$. This in turn means that we have

$$c_n + x \geq d_{n-1} + d_n,$$  \hfill (84)

$$c_n - x \geq d_{n-1} - d_n,$$  \hfill (85)

$$x - c_n \geq d_{n-1} - d_n,$$  \hfill (86)

where the latter two inequalities mean that $|x - c_n| \leq |d_{n-1} - d_n|$. Moreover, we satisfy all the inequalities

$$c_1 + \cdots + c_l \geq c_1 + \cdots + d_l, \quad l = 1, \ldots, n - 2,$$  \hfill (87)

$$c_1 + \cdots + c_{n-1} \geq d_1 + \cdots + d_{n-2} + x,$$  \hfill (88)

and

$$c_{n-1} - c_1 - \cdots - c_{n-2} \leq x - d_1 - \cdots - d_{n-2}.$$  \hfill (89)

Again, we can hence invoke the induction assumption, and construct in the same way as before the desired covariance matrix with symplectic spectrum $(d_1, \ldots, d_n)$ and symplectic main diagonal elements $(c_1, \ldots, c_n)$. This ends the proof of sufficiency of the given conditions. \qed
4. Physical Implications of the Result and Outlook

The results found in this work can also be read as a full specification of what multipartite Gaussian states may be prepared. Since the argument is constructive it readily provides a recipe of how to construct multi-mode Gaussian entangled states with all possible local entropies. For pure states, starting from squeezed modes, all is needed is a network of passive operations. Applied to optical systems of several modes, notably, this gives rise to a protocol to prepare multi-mode pure-state entangled light of all possible entanglement structures from squeezed light, using passive linear optical networks, via

$$\gamma = OPO^T,$$  \hspace{1cm} (90)

with $P = (z_1, 1/z_1, \ldots, z_n, 1/z_n)$, $z_k \in \mathbb{R}\setminus\{0\}$, and $O \in K(n)$. $P$ is the covariance matrix of squeezed single modes, whereas $O$ represents the passive optical network. The latter can readily be broken down to a network of beam splitters and phase shifters, according to Ref. [33]. Hence, our result also generalizes the preparation of Ref. [18] from the case of three modes to any number of modes. Similarly, for mixed states, the given result readily defines a preparation procedure, but now using also squeezers in general.

The above statement also settles the question of the sharing of entanglement of single modes versus the rest of the system in a multi-mode system: For a pure Gaussian state with $d_1 = \cdots = d_n = 1$, the entanglement entropy $E_j|\{1,\ldots,n\}\{j\}$ of a mode labeled $j$ with respect to the rest of the system is given by

$$E_j|\{1,\ldots,n\}\{j\} := S(\rho_j) = s(c_j) := \frac{c_j + 1}{2} \log_2 \frac{c_j + 1}{2} - \frac{c_j - 1}{2} \log_2 \frac{c_j - 1}{2},$$  \hspace{1cm} (91)

where $s : [1, \infty) \to [0, \infty)$ is a monotone increasing, concave function.

**Corollary 4 (Entanglement sharing in pure Gaussian states).** For pure Gaussian states, the set of all possible entanglement values of a single mode with respect to the system is given by

$$\left(E_1|\{2,\ldots,n\}, \ldots, E_n|\{1,\ldots,n-1\}\right) \in \left\{ (s(c_1), \ldots, s(c_n)) : c_j - 1 \leq \sum_{k \neq j} (c_k - 1), \quad c_j \geq 1 \right\}.$$  \hspace{1cm} (92)

This result is an immediate consequence of the above pure marginal problem, Corollary 2. In fact, this is for pure Gaussian states more than a monogamy inequality: it constitutes a full characterization of the complete set of consistent degrees of entanglement.

A further practically useful application of our result is the following: It tells us how pure a state must have been, based on the information available from measuring local properties like local photon numbers. This is expected to be a very desirable tool in an experimental context: In optical systems, such measurements are readily available with homodyne or photon counting measurements.

**Corollary 5 (Locally measuring global purity in non-Gaussian states).** Let us assume that one has acquired knowledge about the local symplectic eigenvalues $c_1, \ldots, c_n$ of
a global state $\rho$. Then one can infer that the global von-Neumann entropy $S(\rho)$ of $\rho$ satisfies

$$S(\rho) \leq s\left(\sum_{k=1}^{n} c_k\right).$$

This estimate is true regardless whether the state $\rho$ is a Gaussian state or not.

**Proof.** Let us denote with $\omega$ the Gaussian state with the same covariance matrix $\gamma \geq 0$ as the (unknown) state $\rho$. The vectors $(d_1, \ldots, d_n)$ and $(c_1, \ldots, c_n)$ are the symplectic eigenvalues and symplectic main diagonal elements of $\omega$, respectively. From the fact that $\text{diag}(d_1, d_1, \ldots, d_n, d_n)$ reflects a tensor product of Gaussian states, we can conclude that

$$S(\omega) = \sum_{j=1}^{n} s(d_j).$$

In turn, from Lemma 1 we find that

$$\sum_{j=1}^{n} c_j \geq \sum_{j=1}^{n} d_j.$$  \hfill (95)

By means of an extremality property of the von-Neumann entropy (see, e.g., Ref. [34,35]) that a Gaussian state has the largest von-Neumann entropy for fixed second moments, we find that $S(\rho) \leq S(\omega)$. Since the function $s : [1, \infty) \to [0, \infty)$ defined in Eq. (91) is concave and monotone increasing, we have that

$$S(\rho) \leq \sum_{j=1}^{n} s(d_j) \leq s(d_1 + \cdots + d_n) \leq s(c_1 + \cdots + c_n).$$

This is the statement that we intended to prove. Clearly, this bound is tight, as is obvious when applying the inequality to the Gaussian state with covariance matrix $\text{diag}(d_1, d_1, \ldots, d_n, d_n)$ itself. \hfill \qed

For example, if one obtains $c_1 = 3/2 = c_2 = 3/2$ and $c_3 = 2$ in local measurements on the local photon number, then one finds that the global state necessarily satisfies $S(\rho) \leq s(5)$. This is a powerful tool when local measurements in optical systems are more accessible than global ones, for example, when no phase reference is available, or bringing modes together is a difficult task.

To finally turn to the role of Gaussian operations in this work: Our result highlights an observation that has been encountered already a number of times in the literature: That global Gaussian operations applied to many modes at once are often hardly more powerful than when applied to pairs of modes. This resembles to some extent the situation in the distillation of entangled Gaussian states by means of Gaussian operations [16,36–38].

In this work, we have given a complete characterization of reductions of pure or mixed Gaussian states. In this way, we have also given a general picture of the possibility of sharing quantum correlations in a continuous-variable setting. Since our proof is constructive, it also gives rise to a general recipe to generate multi-mode entangled states with all possible reductions. Formally, we established a connection to a compatibility argument of symplectic spectra, by means of new matrix inequalities fully characterizing the set in question. These matrix inequalities formally resemble the well-known
Sing-Thompson Theorem relating singular values to main diagonal elements. It is the hope that this work can provide a significant insight into the achievable correlations in composite quantum systems of many modes.

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