A Sequence Convergence of 1 – Dimensional Subspace in a Normed Space

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Abstract. In this paper, the researchers will introduce the concept of a sequence convergence of 1 –dimensional subspaces (lines) in a normed space and shall discuss some properties of it. Furthermore, it will be proved a continuity property of angles among subspaces in inner product spaces. Finally, the notion of limit of a sequence of 2 –dimensional subspaces (planes) in a normed space is studied. The researchers also obtain a result which describe how the convergent of a sequence of lines is associated to the convergent of a sequence of planes in a normed space.

1. Introduction

A sequence in the metric space and its properties has been widely used in investigating functional continuity [1], discussing fixed point theorems [2], discussing the central limit theorem in statistics [3], and discussing the numerical regression convergence [4].

The sequence in the metric space can be vector sequence, function sequence, matrix sequence, real numbers sequence, and so on. This has been discussed in some research conducted by many researchers:[5-13] discussing the convergence of vector sequences and their properties.

The application of vector sequences’ convergence are widely studied by [14] [15]. The two research have discussed the angle between vectors in the inner product space and the concept of convergence of vector sequences in the inner product space are used to prove the continuous properties of the angular function between two vectors in the inner product space. In addition to both research, there are still many mathematicians who discuss the angle between subspace and its properties in the inner product space such as [3] [16] and [17]. However, neither of which shows the classification of the continuous properties of the angular function between the two subspaces in the inner product space.

The question arises: “Can we show the continuity of the angular function between two vector subspaces in the inner product space?” This will be answered if we have the convergence concept of subspace sequences from vector space. Readers may also agree if the concept is researched. It will also be obtained many applied to be produced.

In this paper, the researchers will define the convergence of sequence in one dimensional subspaces. This is a new concept introduced in the discussion of the convergence of sequence. The researchers also will discuss the basic properties of it. As a consequence of this definition, the property of continuity from the angle between the two subspaces will be shown. At the end, we will generalize the concept of convergence these subspaces in two-dimensional subspaces.

Before discussing the convergence concept of vector subspaces sequences, we first recall the definition of convergence of vector sequences as follows:
“a vector sequence \((u_n)\) in normed space is said convergent to a vector \(u\) if \(||u_n - u||\to 0\).

The definition of vector sequences convergence cannot be derived directly to define the convergence of subspace sequences. For example, a one-dimensional subspace can not only be replaced by \(u_n\) and \(u\) respectively as the basis of its subspace sequence, because when the definition is used, the convergence concept of subspaces sequences dependent on basis selection. Thus, by using the definition, there are subspaces sequences in normed space are convergent and divergent depending on the basis selection. For example \(X = \mathbb{R}^2, U_n = span\{(1,0)\} = U \subset X\). If we take the basis of \(U_n\) where \(\{u_n = (1,0)\}\) for every \(n\) and the basis of \(U\) is \(\{(1,0)\}\), then the sequence \((U_n)\) converges to \(U\) because \(||u_n - u|| = 0\to 0\). But if we take the basis of \(U\) \(\{u_n = (2,0)\}\) and \(\{(1,0)\}\) is the basis of \(U\), then the sequence \((U_n)\) is divergent to \(U\) because \(||u_n - u|| = ||(2,0) - (1,0)|| = \|1\|\to 0\).

To overcome these weaknesses, it is necessary to construct the definition of convergent in finite dimensional subspace in normed space that independent on basis selection. According to the survey conducted by the author (journals, magazines, proceedings, national and international seminars), the concept of convergence in finite dimensional subspace in normed space has been researched by none.

If we choose the basis of one dimensional subspace \(U = span\{u\}\), then for each \(\alpha \in R, \alpha u\) is also the basis of \(U\). Therefore, we can make the basis become the unit basis which the norm is 1 by multiplying the basis vector by \(\frac{1}{||u||}\) or \(\frac{u}{||u||}\). However, the one dimension subspace has two unit basis, that are \(\frac{u}{||u||}\) and \(\frac{-u}{||u||}\). Geometrically, it can be presented in Figure 1.

**FIGURE 1.** Representation of unit basis of a line.

By considering that the definition of \(||u_n - u|| \to 0\ for \( n \to \infty\), it can be modified by

\[
\left\|\frac{u_n}{||u_n||} - \frac{u}{||u||}\right\| \to 0
\]

or

\[
\left\|\frac{u_n}{||u_n||} + \frac{u}{||u||}\right\| \to 0
\]

for \( n \to \infty\).

However, if the result of modification is used as a definition, then the concept of convergence of line sequence is still dependent on the basis selection.

For example, let \(x = \mathbb{R}^2, U_n = span\{(1,0)\} = U \subset X\). If we take the base \( U_n \) for every \(n\) where \(u_n = (1,0)\) and the basis of \(U\) is \(\{(1,0)\}\), then sequences \((HU_n)\) converges to \(U\) because \(||u_n - \frac{u}{||u||}|| = 0\to 0\) for \(n \to 0\). But if we take the basis of \(U_n\) is

\[
u_n = \begin{cases} (1,0), & \text{if } n \text{ even} \\ (-1,0), & \text{if } n \text{ odd} \end{cases}
\]

and the basis of \(U\) is \(\{(1,0)\}\), then sequence \((U_n)\) divergent to \(U\), since both \(||u_n - \frac{u}{||u||}||\) and \(||u_n + \frac{u}{||u||}||\) divergent to \(0\) for \(n \to 0\). So to resolve it will be modified again to be

\[
\min\left\{||u_n - \frac{u}{||u||}||, ||u_n + \frac{u}{||u||}||\right\} \to 0
\]

for \( n \to \infty\).
Next, Let $X$ is a normed space. A 1-dimensional subspace of a normed space $X$ is said a line in $X$, and 2-dimensional subspace of $X$ is said a plane in $X$.

2. A Sequence of Lines

Let $X$ is a normed space. Base on the introduction, we get the notion of limit of a sequence of lines in $X$ (1-dimentional subspace of a normed space $X$) and it will be the focus on this section.

Definition 2.1 Let $(U_n)$ is a sequence of lines in $X$ with $U_n = \text{span}\{u_n\}$ and $U = \text{span}\{u\}$ is a line in $X$. A sequence $(U_n)$ is said to converge to $U$, or $U$ is said to be a limit of $(U_n)$, if and only if
\[
\min\left\{\frac{u_n}{\|u\|} - \frac{u}{\|u\|}, \frac{u_n}{\|u\|} + \frac{u}{\|u\|}\right\} \to 0
\]
as $n \to \infty$.

If a sequence has a limit, we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent. We will sometimes use the symbolism $U_n \to U$, which indicates the intuitive idea that the lines $U_n$ "approach" the line $U$ as $n \to \infty$.

The following theorem convinces us that our definition make sense.

Theorem 2.2. The definition 2.1 satisfies homogeneity property. It means, The definition is dependent of the choice of bases for $U$ and $U_n$ for every $n \in \mathbb{N}$.

Proof.
For every $n \in \mathbb{N}$, if $u_n$ change with another basis of $U_n$, that is $\alpha_n u_n$ with $\alpha_n \in \mathbb{R}$ and $u$ were replaced by $\beta u$ with $\beta \in \mathbb{R}$, then there were two cases. If $\alpha_n \beta \geq 0$ then we find that
\[
\frac{\alpha_n u_n - \beta u}{\|\alpha_n u_n\|} = \frac{u_n - u}{\|u\|} \quad \text{and} \quad \frac{\alpha_n u_n + \beta u}{\|\alpha_n u_n\|} = \frac{u_n + u}{\|u\|}.
\]
On the other hand, if $\alpha_n \beta < 0$ then we get
\[
\frac{\alpha_n u_n - \beta u}{\|\alpha_n u_n\|} = \frac{u_n + u}{\|u\|} \quad \text{and} \quad \frac{\alpha_n u_n + \beta u}{\|\alpha_n u_n\|} = \frac{u_n - u}{\|u\|}.
\]
Consequently,
\[
\min\left\{\frac{u_n}{\|u\|} - \frac{u}{\|u\|}, \frac{u_n}{\|u\|} + \frac{u}{\|u\|}\right\} = \min\left\{\frac{\alpha_n u_n}{\|\alpha_n u_n\|} - \frac{\beta u}{\|\beta u\|}, \frac{\alpha_n u_n}{\|\alpha_n u_n\|} + \frac{\beta u}{\|\beta u\|}\right\}
\]

Now, it will begin by establishing some basic properties of convergent sequences of lines since it will be needed in this section.

Fact 2.3. If $U_n = \text{span}\{u\} = U$ then $U_n \to U$.

Proof.
It is known that
\[
\left\|\frac{u_n}{\|u\|} - \frac{u}{\|u\|}\right\| = \left\|\frac{u_n}{\|u\|} - \frac{u}{\|u\|}\right\| = \|0\| = 0 \to 0 \text{ as } n \to \infty. \quad \text{Hence, } U_n \to U.
\]

Theorem 2.4. Let $U_n = \text{span}\{u_n\}$ and $U = \text{span}\{u\}$. If $U_n \to U$ then
\[
\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\| \to 0
\]
for $n \to \infty$.

Proof.
If $U_n \to U$ then, by Definition 2.1, it was obtained
Finally, \( \lim_{n \to \infty} \) there exists.

2.6. (Uniqueness of Limit) Let \( U_n = \text{span}\{u_n\} \) and \( U = \text{span}\{u\} \). If \( U_n \to U \) and \( U_n \to V \) then \( U = V \).

Proof.

Let \( U_n \to U \). By applying Theorem 2.5, then we can find \( a_n \in \mathbb{R}, a_n \neq 0 \) for every \( n \in \mathbb{N} \) such that

\[
\left\| \frac{u_n}{\|u_n\|} \right\| = \frac{a_n u}{\|a_n u\|} \to 0
\]
as \( n \to \infty \). Since \( \frac{u_n}{\|u_n\|} + \frac{u}{\|u\|} \leq \frac{u_n}{\|u_n\|} + \frac{u}{\|u\|} = 2 \) (Triangle Inequality) and also \( \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \leq 2 \), then

\[
\left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\| \left\| \frac{u_n}{\|u_n\|} + \frac{u}{\|u\|} \right\| \leq 2 \min \left\{ \left( \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2, \left\| \frac{u_n}{\|u_n\|} + \frac{u}{\|u\|} \right\|^2 \right) \right\} \to 0
\]
as \( n \to \infty \).
\[
\left\| \frac{a_n u_n}{\|a_n u_n\|} - \frac{b_n v}{\|b_n v\|} \right\| = \left\| \frac{a_n u_n}{\|a_n u_n\|} - \frac{b_n v}{\|b_n v\|} \right\| = \left\| \frac{u_n}{\|u_n\|} - \frac{a_n |b_n v|}{\|b_n a_n v\|} \right\| = \left\| u - (a_n b_n) v \right\| \leq \left\| u \left( u_n \right) - (a_n b_n) v \right\| \leq 0.
\]

Therefore, \( \left\| \frac{u_n}{\|u_n\|} - \frac{v}{\|v\|} \right\| \to 0 \) or \( \left\| \frac{u_n}{\|u_n\|} + \frac{v}{\|v\|} \right\| \to 0 \). Hence, \( \left\| \frac{u_n}{\|u_n\|} - \frac{v}{\|v\|} \right\| = 0 \) or \( \left\| \frac{u_n}{\|u_n\|} + \frac{v}{\|v\|} \right\| = 0 \). It means that \( u = cv \) for some \( c \in \mathbb{R}, c \neq 0 \). So, \( U = \text{span}\{u\} = \text{span}\{cv\} = \text{span}\{v\} = V \). \( \blacksquare \)

**Theorem 2.7.** Let \( U_n = \text{span}\{u_n\} \) and \( U = \text{span}\{u\} \). \( U_n \to U \) if and only if for every \( a \in \mathbb{R}, a_n \neq 0 \), there exists \( a \in \mathbb{R}, a_n \neq 0 \) such that

\[
\left\| \frac{a_n u_n}{\|a_n u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| \to 0
\]

as \( n \to \infty \).

**Proof.**

(\( \Rightarrow \)) If \( U_n \to U \) then by applying Theorem 2.5, there were existed \( a_n \in \mathbb{R}, a_n \neq 0 \), such that

\[
\left\| \frac{u_n}{\|u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| \to 0
\]

as \( n \to \infty \). It meant that for all \( a \in \mathbb{R}, a \neq 0 \), if \( a > 0 \) then

\[
\left\| \frac{a u_n}{\|a u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| = \left\| \frac{u_n}{\|u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| \to 0,
\]

If \( a < 0 \), then there were exists \( b_n \in \mathbb{R}, b_n \neq 0 \), that is \( b_n = -a_n \), such that

\[
\left\| \frac{a u_n}{\|a u_n\|} - \frac{b_n u}{\|b_n u\|} \right\| = \left\| \frac{u_n}{\|u_n\|} - \frac{-a_n u}{\|a_n u\|} \right\| \to 0
\]

as \( n \to \infty \).

(\( \Leftarrow \)) Let for every \( a \in \mathbb{R}, a \neq 0 \) there were exists \( a_n \in \mathbb{R}, a_n \neq 0 \) such that

\[
\left\| \frac{a u_n}{\|a u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| \to 0
\]

as \( n \to \infty \). There were two cases, if \( a > 0 \) then it was obtained

\[
\left\| \frac{u_n}{\|u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| = \left\| \frac{a u_n}{\|a u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| \to 0,
\]

or if \( a < 0 \) then it was obtained that

\[
\left\| \frac{u_n}{\|u_n\|} - \frac{-a_n u}{\|a_n u\|} \right\| = \left\| \frac{a u_n}{\|a u_n\|} - \frac{-a_n u}{\|a_n u\|} \right\| \to 0,
\]

as \( n \to \infty \). So, by applying Theorem 2.5, \( U_n \to U \). \( \blacksquare \)

### 3. Relationship to Angle in an Inner Product Space

Let \( (X, \langle \cdot, \cdot \rangle) \) is an inner product spaces. Inspired by the notion of limit of a sequence of lines in \( X \), it could be described the continuity property of angle between two lines.

**Theorem 3.1. (Continuity Property)** Let \( U_n, V_n, U \) and \( V \) are 1-dimensional subspaces of a normed space \( X \) and \( \theta(U, V) \) is an angle between \( U \) and \( V \). If \( U_n \to U \) and \( V_n \to V \) then \( \theta(U_n, V_n) \to \theta(U, V) \).

**Proof.**

Since \( U_n \to U \) and \( V_n \to V \) then by applying Theorem 2.5, it was obtained and found \( a_n, b_n \in \mathbb{R}, a_n \neq 0 \), and \( b_n \neq 0 \) for every \( n \in \mathbb{N} \) such that

\[
\left\| \frac{u_n}{\|u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| \to 0 \text{ and } \left\| \frac{b_n v}{\|b_n v\|} - \frac{v_n}{\|v_n\|} \right\| \to 0 \text{ as } n \to \infty.
\]

It was known that

\[
\left\| \frac{u_n}{\|u_n\|} - \frac{v_n}{\|v_n\|} \right\| \leq \left\| \frac{a_n u}{\|a_n u\|} - \frac{b_n v}{\|b_n v\|} \right\| \leq \left\| \frac{u_n}{\|u_n\|} - \frac{a_n u}{\|a_n u\|} + \frac{b_n v}{\|b_n v\|} - \frac{v_n}{\|v_n\|} \right\| \leq \left\| \frac{u_n}{\|u_n\|} - \frac{a_n u}{\|a_n u\|} \right\| + \left\| \frac{b_n v}{\|b_n v\|} - \frac{v_n}{\|v_n\|} \right\| \to 0.
\]
as \( n \to \infty \). Therefore, \( \frac{\|u_n\|}{\|v_n\|} \to \frac{\|a_nu\|}{\|b_nv\|} = \frac{\|a_nu - b_nv\|}{\|a_nu\|} \) as \( n \to \infty \). Hence, by the definition of angle between two subspaces of a normed space ([17]), it was obtained that

\[
\theta(U_n, V_n) = \arccos \left[ \frac{\langle u_n, v_n \rangle}{\|u_n\| \|v_n\|} \right] = \arccos \left[ \frac{1}{2} \left( 2 - \left( \frac{\|a_nu\|}{\|b_nv\|} \right)^2 \right) \right] \\
\to \arccos \left[ \frac{1}{2} \left( 2 - \left( \frac{\|a_nu\|}{\|b_nv\|} \right)^2 \right) \right] = \arccos \left[ \frac{\langle a_nu, b_nv \rangle}{\|a_nu\| \|b_nv\|} \right] = \theta(U, V)
\]

as \( n \to \infty \). □

Relationship between the notion of limit of a sequence of lines in \( X \) and angle between vectors in \( X \) ([18] and [19]) is described in the following theorem.

**Theorem 3.2.** Let \( U_n = \text{span} \{ u_n \} \) and \( U = \text{span} \{ u \} \) are 1-dimensional subspaces of a normed space \( X \) and \( \theta_{THy}(u, v) \) is an angle between two vector \( u \) and \( v \). \( U_n \to U \) if and only if \( \left| \cos \theta_{THy}(u_n, u) \right| \to 1 \) as \( n \to \infty \).

**Proof.**

In [18] and [19], angle between \( u_n \) and \( u \), \( \theta_{THy}(u_n, u) \), is defined as

\[
\cos \theta_{THy}(u_n, u) = \frac{1}{4} \left( \left\| u_n \right\| + u \right)^2 - \left( \left\| u_n \right\| - u \right)^2.
\]

In inner product spaces \( X \), for \( n \to \infty \), if \( \left\| u_n \right\| + u \to 0 \) then \( \left\| u_n \right\| - u \to 0 \), and if \( \left\| u_n \right\| - u \to 0 \) then \( \left\| u_n \right\| + u \to 0 \). Thus,

\[
\min \left\{ \left\| u_n \right\| - u , \left\| u_n \right\| + u \right\} \to 0 \iff \frac{1}{4} \left( \left\| u_n \right\| + u \right)^2 - \left( \left\| u_n \right\| - u \right)^2 \to 1. \quad \square
\]

**4. A Sequence of Planes**

Theorem 2.7 was used to get the notion of limit of a sequence of planes in \( X \) (2-dimentional subspace of a normed space \( X \)).

**Definition 4.1.** Let \( (U_n) \) is a sequence of planes in \( X \) with \( U_n = \text{span}\{u_{1n}, u_{2n}\} \) and \( V = \text{span}\{v_1, v_2\} \) is a plane in \( X \). A sequence \( (U_n) \) is said to be convergent to \( V \) if and only if for every \( b \in \mathbb{R} \), there exists \( a_n, b_n \in \mathbb{R} \) such that

\[
\left\| \frac{(a_nu + b_nv)}{\|a_nu + b_nv\|} - \frac{(a_nu + b_nv)}{\|a_nu + b_nv\|} \right\| \to 0
\]

As \( n \to \infty \).

Note that \( a, b \in \mathbb{R} \) in Definition 4.1 are arbitrary. It means that our definition is independent of the choice of basis for \( U_n \) and \( V \). So, our definition make sense. Also, if \( U_n \) and \( V \) in Definition 4.1 are a 1-dimensional subspace of \( X \), then the definition is equivalent with Theorem 2.7.

The following theorem describe a relation between the convergent of a sequence of lines in \( X \) and the convergent of a sequence of planes in \( X \).

**Theorem 4.2.** Let \( U_n = \text{span}\{u_n\} \), \( V_n = \text{span}\{v_n\} \), \( U = \text{span}\{u\} \), \( V = \text{span}\{v\} \) with \( u \) and \( v \) are linearly independent, and \( B_n = \text{span}\{u_n, v_n\} \) and \( B = \text{span}\{u, v\} \). If \( U_n \to U \) and \( V_n \to V \) then \( B_n \to B \).

**Proof:**

It was easy to prove this theorem by using Theorem 2.7, Definition 4.1, and Triangle Inequality. □
**Remark 4.3.** As the reader, it might have realized that the Definition 2.9 may also be generalized to define the limit of a sequence of \( m \)-dimensional subspaces of a normed space. Let \( X \) is a normed space with \( \dim X > m \). Let \( (V_n) \) is a sequence of \( m \)-dimensional subspaces of \( X \) with \( V_n = \text{span}\{v_1, v_2, ..., v_m\} \) and \( V = \text{span}\{v_1, v_2, ..., v_m\} \) is an \( m \)-dimensional subspace of \( X \) with \( v_1, v_2, ..., v_m \in \mathbb{N} \). A sequence \( (V_n) \) is said to converge to \( V \) if and only if for every \( a_1, a_2, ..., a_m \in \mathbb{R} \) there exists \( b_1, b_2, ..., b_m \in \mathbb{R} \) such that

\[
\left\| \sum_{i=1}^{m} a_i v_i \right\| - \left\| \sum_{j=1}^{m} b_j v_j \right\| \to 0
\]

as \( n \to \infty \).

Note that \( a_1, a_2, ..., a_m \) in Definition 4.3 were arbitrary. It meant that the definition was also invariant under any change of basis for \( V_n \) and \( V \). So, the definition made sense.

5. References

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