About a mathematical model of market

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Abstract. In the paper a famous mathematical model of macroeconomics, which is called "market model" was considered. Traditional versions of this model have no periodic solutions and, therefore, they cannot describe a cyclic recurrence of the market economy. In the paper for the corresponding equation a delay was added. It allows obtaining sufficient conditions for existence of the stable cycles.

1. Introduction
The presence of cyclic recurrence in a market economy was mentioned in the works of economists of 19th century (see, for example, [1]). From the middle of the 20th century for the study of economic dynamics, the mathematical methods were widely used. Particularly, in macroeconomics a number of mathematical models: the aggregate demand-aggregate supply model (market model), the Solow model, the multiplier-accelerator model and so on were proposed (see [2-4]).

Let \( p(t) \) is the price of the product at the moment \( t \). Then the dynamics of the price are described by the following equation [2-4]

\[
\dot{p} = D(p) - S(p),
\]

where \( D(p) \) - the demand function, \( S(p) \) - the supply function. Usually, these functions are assumed with the following properties (see [4], p.146):

1. non-negative and smooth functions \( D(p), S(p) \) are defined if \( p \in (0, \infty) \);
2. if \( p \in (0, \infty) \) then the inequalities \( D'(p) < 0, S'(p) > 0 \) are hold;
3. \( \lim_{p \to 0} D(p) = D_0 \), where \( D_0 >> 1 \) or \( D_0 = \infty \), \( \lim_{p \to 0} S(p) = S_0 \), where \( S_0 << 1 \) or \( S_0 = 0 \);
4. \( \lim_{p \to \infty} D(p) = D_\infty \), where \( D_\infty << 1 \), \( \lim_{p \to \infty} S(p) = S_\infty \), where \( S_\infty \in R_+ \) or \( S_\infty = \infty \).

The inequality \( D'(p) < 0 \) means that demand for goods falls if the price of the product is rising. On the contrary, a supply is rising if the market price of the product is rising. Condition \( \lim S(p) = 0 \) is natural since for a zero price the supply of the product is absent.

Equation (1.1) has only one positive equilibrium state \( (p(t) = p_0 > 0) \). Indeed, if we consider the function \( F(p) = D(p) - S(p) \) for which \( F'(p) = D'(p) - S'(p) < 0 \), i.e. these functions are monotonically decreasing. Moreover, \( \lim_{p \to 0} F(p) = \infty \) or it equals the sufficiently large constant and \( \lim_{p \to \infty} F(p) < 0 \). Therefore, the equation \( F(p) = 0 \) has only one root \( p(t) = p_0 > 0 \). The
inequality $F'(p) < 0$ means that equilibrium state $p(t) = p_0$ is asymptotically stable. Hence, the equation (1.1) cannot have any cycles.

In the papers [5,6] the modification of the basic equation (1.1) was suggested. Instead of this equation we consider the delay-differential equation

$$\dot{p} = D(p) - S(p_h), \quad (1.2)$$

where $p = p(t)$, $p_h = p(t - h)$, i.e. we add a delay in the supply function $S(p)$. Now this function depends not on the price of trading day $p(t)$, but on the price in the previous period, i.e. the price of yesterdays trading day. This latest statement is only natural, because the manufacturer depends not on the price of trading day $p$.

At the same time the characteristic equation (2.4) has a countable number of roots

$$\lambda + b_\alpha \left( \alpha + \exp(-\lambda h) \right) = 0. \quad (2.4)$$

Here, as the points we denote terms having at zero an order of smallness greater than three.

2. Analysis of the first model

Here, we consider the following equation

$$\dot{p} = a_\alpha p^{-\alpha} - a_\beta p(t - h). \quad (2.1)$$

This equation after substitution $p = \mu p_1$, $\mu = (a_\alpha / b_\alpha)^{\alpha_1}$, $\alpha_1 = (1 + \alpha)^{-1}$ has the form

$$\dot{p}_1(t) = b_\alpha \left[ p_1^{-\alpha}(t) - p_1(t - h) \right].$$

Last differential equation has an equilibrium state $p_1(t) = p_0 = 1$.

Let $p_1(t) = 1 + x(t)$, $p_1(t - h) = 1 + y(t)$, $y(t) = x(t - h)$. As the result for the deviation $x(t)$ we obtain the equation

$$\dot{x}(t) = b_\alpha [-\alpha x - y] + a_2 x^2 + a_3 x^3 + \ldots, \quad a_2 = b_\alpha \frac{\alpha(\alpha + 1)}{2}, \quad a_3 = -b_\alpha \frac{\alpha(\alpha + 1)(\alpha + 2)}{6}. \quad (2.2)$$

Here, as the points we denote terms having at zero an order of smallness greater than three.

2.1. Stability of an equilibrium state of the equation (2.2)

This equation has a zero equilibrium state. For the analysis of its stability we consider the linearized variant of the equation

$$\dot{x}(t) = -b_\alpha [\alpha x + y], \quad x = x(t), \quad y = x(t - h). \quad (2.3)$$

It is well known [7] that a question about stability of solutions of the equation (2.3) comes down to the analysis of location of the roots of the characteristic equation

$$\lambda + b_\alpha \left( \alpha + \exp(-\lambda h) \right) = 0. \quad (2.4)$$

At the same time the characteristic equation (2.4) has a countable number of roots $\lambda_j \in C$. If for them the inequalities $Re\lambda_j \leq -\gamma_0 < 0$ are hold, then the solutions of the equation (2.3)
are asymptotically stable and unstable if at least for one eigenvalue $\lambda_k$ the inequality $\text{Re}\lambda_k > 0$ is hold. Note that the equation (2.4) has no zero root. Only critical case of a pair of pure imaginary roots of characteristic equation $\lambda_{1,2} = \pm i\sigma$, $\sigma > 0$, $\text{Re}\lambda_j \leq -\gamma_0 < 0$ ($j = 3, 4, 5, \ldots$) can take a place. For this $\sigma$ from the following equation we have

$$\cos\sigma h = -\alpha, \quad (2.5)$$

and $b_\alpha = b_{cr} = \min\{b_{ak}\} > 0$. In turn, $b_{ak} = \sigma_k/(\sin\sigma_k h)$, where $\sigma_k$ - one of the positive roots of the equation (2.5) for which $\sin\sigma_k h > 0$. It is elementary to verify that the appropriate pair $(\sigma, b_{cr})$ is the numbers $\sigma = \omega_\alpha/h$, $b_{cr} = \omega_\alpha/(h\sqrt{1-\alpha^2})$, where $\omega_\alpha = \arccos(-\alpha) \in (\pi/2, \pi)$. There exist the corresponding solutions if $\alpha \in (0, 1)$. If $\alpha \geq 1$ for the differential equation (2.2), a critical case in the problem about stability cannot be realized.

2.2. Bifurcation problem

In the equation (2.2) we set $b_\alpha = b_{cr}(1+\varepsilon)$, $0 < \varepsilon << 1$. As the result, we obtain the following equation which depends on $\varepsilon$

$$\dot{x} = A(\varepsilon)x + F_2(x, \varepsilon) + F_3(x, \varepsilon) + F_0(x, \varepsilon), \quad (2.6)$$

where

$$A(\varepsilon)x = -b_{cr}(1+\varepsilon)[\alpha x + y], \quad F_2(x, \varepsilon) = b_{cr}(1+\varepsilon)\frac{\alpha(\alpha+1)}{2}x^2,$$

$$F_3(x, \varepsilon) = -b_{cr}(1+\varepsilon)\frac{\alpha(\alpha+1)(\alpha+2)}{6}x^3.$$

We denote $F_0(x, \varepsilon)$ as a function for which the inequality $|F_0(x, \varepsilon)| \leq M_0|x|^4$, $M_0 > 0$ is hold. For the equation (2.6) we add the initial condition

$$x(t, \varepsilon) = g(t), \quad t \in [-h, 0], \quad (2.7)$$

where the given function is $g(t) \in C[-h, 0]$. The solution of the Cauchy problem (2.6), (2.7) with sufficiently small $g(t)(\|g(t)\|_{C[-h,0]} \leq r_0)$ generates a local semiflow. This semiflow has a two-dimensional local central manifold $M_2(\varepsilon)(\dim M_2(\varepsilon) = 2)$. On this manifold the dynamics of the solutions of the equation (2.6) is reduced to the analysis of the system of two ordinary differential equations - the Poincare-Dulac normal form (NF) (see [8,9]). We write this equation in a complex form

$$z' = \left[(\beta_1 + i\beta_2) + (l_1 + il_2)|z|^2\right]z + O(\varepsilon), \quad (2.8)$$

where $\beta_1, \beta_2, l_1, l_2, \in R$ and $z = z(s)$, $s = \varepsilon t$ - “slow” time. All the rest solutions of the Cauchy problem (2.6), (2.7) with sufficiently small initial conditions tend to $M_2(\varepsilon)$ with a speed of exponent. Solutions on the $M_2(\varepsilon)$ may be reconstructed by using the solutions of the normal form (2.8). The link between solutions of the problem (2.6), (2.7) and NF (2.8) is established by using the algorithm which has the origin in a famous Krylov-Bogolyubov algorithm.

We will find the solutions of the equation (2.6) belonging to $M_2(\varepsilon)$ in the following form

$$x(t, \varepsilon) = \varepsilon^{1/2}x_1(t, z, \bar{z}) + \varepsilon x_2(t, z, \bar{z}) + \varepsilon^{3/2}x_3(t, z, \bar{z}) + o(\varepsilon^{3/2}), \quad (2.9)$$

where $z(s)$ - solutions of the equation (2.8), $x_j(t, z, \bar{z})$, $j = 1, 2, 3, \ldots$ which sufficiently smoothly depend on arguments. They have a period of $2\pi/\sigma$ of variable $t$. Also $x_1(t, \varepsilon) = z(s)\exp(i\sigma t) + \bar{z}(s)\exp(-i\sigma t)$ and for $x_2(t, z, \bar{z})$, $x_3(t, z, \bar{z})$ the identities $M_{\pm}(x_j) =$
\[ \sigma/(2\pi) \int_{0}^{2\pi/\sigma} x_j \exp(\pm i\sigma t) dt = 0, \quad j = 2, 3, \ldots \text{ are hold.}\]

Functions \( x_2(t, z, \bar{z}), \ x_3(t, z, \bar{z}) \) are determined as the solutions of the nonhomogeneous differential equations. For their formation we substitute the sum (2.9) in the equation (2.6) and equate the terms of \( \varepsilon^{1/2}, \ \varepsilon, \ \varepsilon^{3/2} \) and so on. Equating the terms of \( \varepsilon \) we obtain the equation for \( x_2(t) \)

\[ \dot{x}_2 + b_{cr}(\alpha x_2 + y_2) = b_{cr} \frac{\alpha(\alpha + 1)}{2} x_1^2. \]  

(2.10)

For \( x_3(t) \) we obtain the following nonhomogeneous equation

\[ \dot{x}_3 + b_{cr}(\alpha x_3 + y_3) = -b_{cr} \frac{\alpha(\alpha + 1)(\alpha + 2)}{6} x_1^3 + b_{cr} \alpha(\alpha + 1) x_1 x_2 + b_{cr} [-\alpha x_1 - y_1] + [b_{cr} h \exp(-i\sigma h) - 1] \psi(z, \bar{z}) \exp(i\sigma t) \]

(2.11)

\[ \psi(z, \bar{z}) = \beta z + l|z|^2 \bar{z}, \ y_j = y_j(t, z, \bar{z}) = x_j(t - \bar{h}, z, \bar{z}), \ z = z(s), \ j = 1, 2, 3, \ldots \]

**Remark:** nonhomogeneous differential equation \( \dot{u} + b_{cr}[\alpha u + v] = \varphi(t), \ v = u(t - \bar{h}) \), where \( \varphi(t) - 2\pi/\sigma \) - periodic function has \( 2\pi/\sigma \) - periodic solution if

\[ M_{\pm}(\varphi) = \frac{\sigma}{2\pi} \int_{0}^{2\pi/\sigma} \varphi(t) \exp(\pm i\sigma t) dt = 0. \]

Equality \( M_{\pm}(u) = 0 \) determines one such solution.

Nonhomogeneous equation (2.10) has the following appropriate solution

\[ x_2(t, z, \bar{z}) = \eta z^2 \exp(2i\sigma t) + \xi + \bar{\eta}z^2 \exp(-2i\sigma t), \]

where \( \xi = \alpha, \ \eta = \alpha/(2(\alpha - 1) + 4i\sqrt{1 - \alpha^2}) \). The solvability conditions of the nonhomogeneous differential equation (2.11) in the class of periodic functions let us to determine the coefficients of NF \( \beta = \beta_1 + i\beta_2, \ l = l_1 + i l_2 \). After the corresponding calculations, we have

\[ \beta_1 = b_{cr} \frac{(1 - \alpha^2)^{3/2} w_\alpha}{L_0} > 0, \ \beta_2 = b_{cr} \frac{(1 - \alpha^2)[\sqrt{1 - \alpha^2} + \alpha w_\alpha]}{L_0}, \]

\[ L_0 = 1 - \alpha^2 + \omega_\alpha^2 + 2\alpha w_\alpha \sqrt{1 - \alpha^2}, \]

\[ l_1 = -b_{cr} \frac{(1 - \alpha^2)^{3/2}}{(5 - 4\alpha)L_0} \sqrt{5 - \alpha \sqrt{1 - \alpha^2} + 6\alpha \omega_\alpha} < 0, \]

\[ l_2 = -b_{cr} \frac{\alpha(\alpha + 1)(1 - \alpha^2)}{(5 - 4\alpha)L_0} \left[ \alpha \sqrt{1 - \alpha^2} + (6\alpha - 5) \omega_\alpha \right]. \]

Consider a truncated form of the equation (2.8)

\[ z' = (\beta_1 + i\beta_2)z + (l_1 + il_2)z^2. \]  

(2.12)

**Lemma 1.** Differential equation of the form (2.12) has a periodic solution

\[ z(s) = \rho_0 \exp(i\omega_0 s), \quad \rho_0 = \sqrt{-\frac{\beta_1}{l_1}}, \quad \omega_0 = \beta_2 - \frac{l_2 \beta_1}{l_1} \]

if \( \beta_1 l_1 < 0 \). This solution is stable if \( \beta_1 > 0 \ (l_1 < 0) \).

A verifying of the statement is standard.

We note that in our case \( \beta_1 > 0, \ l_1 < 0 \), i.e. there exists a stable periodic solution.

**Theorem 1.** There exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) the differential equation (2.1) has a stable cycle. This cycle generates the periodic solutions

\[ p(t, \varepsilon) = \left( \frac{a_\varepsilon}{b_\varepsilon} \right)^{1/(1 + \alpha)} \left[ 1 + \varepsilon^{1/2} \rho_0 \exp(i\sigma t + i\varphi_0) + \exp(-i\sigma t - i\varphi_0) \right] + \varepsilon \rho_0^2 \left[ \eta \exp(2i\sigma t + 2i\varphi_0) + \xi + \bar{\eta} \exp(-2i\sigma t - 2i\varphi_0) \right] + o(\varepsilon), \ \sigma_\varepsilon = \sigma + \varepsilon \omega_0 + o(\varepsilon). \]

The validity of the statement follows from the works [8,9].
3. Supply function with "saturation"
In this section we consider the second variant of model "aggregate demand-aggregate supply model" for which the function $S(p_t)$ is chosen with saturation ($\lim_{p \to \infty} S(p) = b < \infty$). The corresponding equation after renormalization $p = cp_2$ has the form

$$
\dot{p}_2(t) = b_c \left[ \frac{\alpha}{1 + p_2(t)} - \left( 1 - \frac{1}{1 + p_2(t-h)} \right) \right], \quad \alpha = \frac{a}{b_c}, \quad b_c = \frac{b}{c}.
$$

(3.1)

The equation (3.1) has only one positive equilibrium state $p_2(t) = \alpha$. Let

$$
p_2(t) = \alpha + x_1(t), \quad p_2(t-h) = \alpha + y_1(t), \quad y_1(t) = x_1(t-h), \quad x_1(t) = (1+\alpha)x(t)(y_1(t) = (1+\alpha)y(t)).
$$

As the result of these two substitutions, we obtain the equation for $x(t)(y(t) = x(t-h))$

$$
\dot{x} = b_\alpha \left[ (-\alpha x - y) + (\alpha x^2 + y^2) - (\alpha x^3 + y^3) \right] + R_0(x,y),
$$

(3.2)

where $b_\alpha = b_c/(1 + \alpha)^2$, $|R_0| \leq M_0[x^4 + y^4]$, $M_0 = const > 0$, in particular, $R_0(0,0) = 0$.

3.1. Stability of a zero equilibrium state of the equation (3.2)
For the investigation of a zero equilibrium state we consider a linearized at zero form of the equation (3.2)

$$
\dot{x} = -b_\alpha [\alpha x + y],
$$

(3.3)

which is the same like for the equation (2.3) from the previous section. In particular, for $b_\alpha = b_{cr}$ a critical case is realized in the problem about stability of a zero solution. The spectrum of stability has a pair of pure imaginary eigenvalues $\pm i\sigma$, where $\sigma = \omega_\alpha/h$, $\omega_\alpha = \arccos(-\alpha)$, $b_{cr} = \omega_\alpha/(h\sqrt{1-\alpha^2})$.

3.2. About bifurcation problem in the second variant of the supply function
In the equation (3.2) let $b_\alpha = b_{cr}(1 + \varepsilon)$ and as the result we obtain the differential equation which is very close to the equation (2.6)

$$
\dot{x} = A(\varepsilon)x + F_2(x,\varepsilon) + F_3(x,\varepsilon) + F_0(x,\varepsilon),
$$

(3.4)

$$
A(\varepsilon)x = b_{cr}(1 + \varepsilon)[\alpha x - y], \quad F_2(x,\varepsilon) = b_{cr}(1 + \varepsilon)(\alpha x^2 + y^2),
$$

$$
F_3(x,\varepsilon) = -b_{cr}(1 + \varepsilon)(\alpha x^3 + y^3), \quad y(t) = x(t-h).
$$

In the equations (3.4) and (2.6) a quadratic and cubic terms differ. As in the second section, the question about existence of cycles, their stability is reduced to the analysis of NF

$$
z' = [(\beta_1 + i\beta_2) + (l_1 + il_2)|z|^2]z + O(\varepsilon), \quad z = z(s), \quad s = \varepsilon t,
$$

(3.5)

$\beta_1, \beta_2$ are computed by using the formulas from the previous section. Here, in (3.5)

$$
l_1 = -b_{cr} \frac{(1 - \alpha)\sqrt{1-\alpha^2}}{L_0(5 - 4\alpha)} \left[ (8\alpha - 4)\sqrt{1-\alpha^2} + \omega_\alpha(7 - \alpha + 4\alpha^2) \right],
$$

$$
l_2 = -b_{cr} \frac{(1 - \alpha^2)}{L_0(5 - 4\alpha)} \left[ (7 - 4\alpha)\sqrt{1-\alpha^2} + \omega_\alpha(4 - 5\alpha + 4\alpha^2) \right],
$$
where $L_0 = (1 - \alpha^2) + 2\alpha \omega_a \sqrt{1 - \alpha^2} + \omega_a^2$, $\alpha \in (0, 1)$. It is not difficult to see that if $l_t < 0$ for all values of the parameter $\alpha \in (0, 1)$. Indeed, for the given $\alpha$ the expression has the form

$$(8\alpha - 4) \sqrt{1 - \alpha^2} + \omega_a (7 - \alpha + 4\alpha^2) \geq 6.89 > 0.$$  

Therefore, the shortened NF $z' = (\beta_1 + i\beta_2)z + (l_1 + il_2)\rho z|z|^2$ has the stable periodic solution $z(s) = \rho_0 \exp(i\omega s)$, where $\rho_0 = (-\beta_1/l_1)^{1/2} (-\beta_1 l_1 > 0), \omega = \beta_2 - l_2 \beta_1 / l_1$.

**Theorem 2.** There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the equation (3.1) has an attracting limit cycle which is generated by the periodic solutions

$$p(t, \varepsilon) = \varepsilon \left[ \alpha + (1 + \alpha) \varepsilon^{1/2} \rho_0 [\exp(i\sigma \varepsilon t + i\varphi_0) + \exp(-i\sigma \varepsilon t - i\varphi_0)] \right] + o(\varepsilon^{1/2}), \quad \sigma = \varepsilon + \varepsilon \omega + o(\varepsilon).$$

The validity of the statement follows from the works [7-9].

4. Conclusion

In the work we consider two variants of a mathematical market model which is more famous as "aggregate demand-aggregate supply model". As in the works [5,6] a classic form of the model was changed, and in the supply function the delay was added. It significantly changes a dynamics of the solutions. Without taking into account a delay, the corresponding equations have an equilibrium state and they cannot have the periodic solutions which describe the cyclic recurrence of a market economy. Taking into account a delay, the cycles can exist with a corresponding choice of parameters of the model (coefficients of the equation). The period of the cycles is close to $T(h) = 2\pi h / \omega_0$, $\omega_0 = \arccos(-\alpha)$, where $\alpha$ - one of the main parameters for both of the models. The period is close to zero if a constant $h$ is sufficiently small and, in opposite, sufficiently large with $h \to \infty$. In the second case when $h \to \infty$ the long cycles of the economic which were opened in the 1930s of the past century by a famous economist N.D. Kondratyev are realized. We will once again emphasize that the periodic solutions in the both variants of the choice of the demand function and the supply function exist, and they are stable for all realizable choices of the parameters of the problem. In the Solow-Swan model it was suggested to add a delay in the work [10], but in this work the analysis which is correct from a mathematical point of view is absent.

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