Almost quasi-Sasakian manifolds equipped with skew-symmetric connection

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Abstract—On a sub-Riemannian manifold, a connection with skew-symmetric torsion is defined as the unique connection from the class of $N$-connections that has this property. Two cases are considered separately: sub-Riemannian structure of even rank, and sub-Riemannian structure of odd rank. The resulting connection, called the canonical connection, is not a metric connection in the case when the sub-Riemannian structure is of even rank. The structure of an almost quasi-Sasakian manifold is defined as an almost contact metric structure of odd rank that satisfies additional requirements. Namely, it is required that the canonical connection is a metric connection and that the transversal structure is a Kähler structure. Both the quasi-Sasakian structure and the more general almost contact metric structure, called an almost quasi-Sasakian structure, satisfy these requirements. Sufficient conditions are found for an almost quasi-Sasakian manifold to be an Einstein manifold.

Keywords and phrases: Sub-Riemannian manifold of contact type, almost contact metric manifold, inner connection, almost quasi-Sasakian manifold, skew-symmetric connection

1. INTRODUCTION

By an almost quasi-Sasakian manifold (AQS-manifold) we mean an almost normal almost contact metric manifold with a closed fundamental form for which the condition
\[ d\eta(\hat{\xi}, \cdot) = 0 \]
holds true. An almost contact metric manifold is called by the author of this paper almost normal if the equality
\[ \tilde{N}\varphi = N\varphi + 2\varphi^* d\eta \otimes \hat{\xi} = 0 \]
holds [5]. The „almost normality” condition is equivalent to the integrability of the transversal structure. An AQS-manifold is a generalization of a quasi-Sasakian manifold and reduces to the latter if
\[ d\eta = \varphi^* d\eta. \]

An AQS-structures naturally arises on the distribution of an almost contact metric manifold [3]. In this paper, an AQS-structure is equipped with a connection with skew-symmetric torsion. The study of such connections (called canonical in this work) is motivated by the demands of theoretical physics [2,4]. Initially, an $N$-connection $\nabla^N$ is defined on a sub-Riemannian manifold of contact type endowed with an endomorphism $N : TM \to TM$ of the tangent bundle of the manifold $M$ ($N\hat{\xi} = \mathbf{0}$, $N(D) \subset D$). The connection $\nabla^N$ expressed as follows in terms of the Levi-Civita connection $\nabla$:
\[
\nabla_X^N Y = \nabla_X Y + (\nabla_X \eta)(Y)\hat{\xi} - \eta(Y)\nabla_X \xi - \eta(X)(\nabla_{\hat{\xi}} \eta)(Y)\hat{\xi} - \eta(X)(C + \psi - N)Y.
\]
We divide sub-Riemannian structures into two classes: structures of even rank and structures of odd rank. For structures of odd rank, the equality
\[ \tilde{\nabla}_\xi \eta = 0 \]
is satisfied, in this case the expression for the \( N \)-connection becomes simpler:
\[ \nabla^N_X Y = \tilde{\nabla}_X Y + (\tilde{\nabla}_X \eta) (Y) \tilde{\xi} - \eta(Y) \tilde{\nabla}_X \tilde{\xi} - \eta(X)(C + \psi - N) Y. \]

The case \( \tilde{\nabla}_\xi \eta = 0 \) is considered in detail in [1]. For the corresponding manifolds, conditions are found under which the \( N \)-connection has skew-symmetric torsion. This connection is uniquely defined and corresponds to the endomorphism \( N = 2\psi \), where the endomorphism \( \psi \) is given by the equality
\[ \omega(X,Y) = g(\psi X,Y) \]
and is called in the paper the second structural endomorphism of an almost contact metric manifold.

In the same paper, the concept of an \( \nabla^N \)-Einstein almost contact metric manifold is introduced. For the case \( N = 2\psi \), conditions are found under which an almost contact metric manifold is a \( \nabla^N \)-Einstein manifold.

This work consists of two main parts. In the first part, conditions are found under which an \( N \)-connection on a sub-Riemannian manifold of contact type has skew-symmetric torsion. In the second part, \( N \)-connections with skew-symmetric torsion are considered applied to AQSM-manifolds. Conditions are found under which AQSM-manifolds are Einstein manifolds with respect to the canonical connection. Examples of such manifolds are given.

2. SUB-RIEMANNIAN MANIFOLDS EQUIPPED WITH A CANONICAL
SKEW-SYMMETRIC CONNECTION

Let \( M \) be a smooth manifold of odd dimension \( n = 2m + 1 \) with a sub-Riemannian structure \( (M, \xi, \eta, g, D) \) of a contact type, where \( \eta \) and \( \xi \) is a 1-form and a unit vector field generating, respectively, mutually orthogonal distributions \( D \) and \( D^\perp \).

Throughout the work we use adapted coordinates. A chart \( k(x^i), i, j, k = 1, \ldots, n \), of the manifold \( M \) will be call adapted to the distribution \( D \) if
\[ D^\perp = \text{Span}(\partial_n), \quad \partial_n = \xi, \]
see [3]. Let \( P : TM \to D \) be the projector defined by the decomposition
\[ TM = D \oplus D^\perp, \]
and \( k(x^i) \) be an adapted chart. The vector fields
\[ P(\partial_a) = \tilde{\varepsilon}_a = \partial_a - \Gamma^a_{kn} \partial_n, \quad a, b, c = 1, \ldots, n - 1, \]
are linearly independent and linearly generate the distribution \( D = \text{Span}(\tilde{\varepsilon}_a) \) in the definition domain of the corresponding chart.

For the adapted charts \( k(x^i) \) and \( k'(x'^i) \) the following coordinate transformation formulas are fulfilled:
\[ x'^a = x^a(x'^i), \quad x'^n = x'^{n'} + x^n(x'^i). \]

A tensor field \( t \) of type \( (p, q) \) defined on an almost contact metric manifold is called admissible (to the distribution \( D \)) or transversal if \( t \) vanishes whenever one of its arguments is \( \tilde{\xi} \) or \( \eta \). The coordinate representation of an admissible tensor field with respect to an adapted chart is as follows:
\[ t = t^{a_1 \cdots a_p}_{b_{q'}} \tilde{\varepsilon}_{a_1} \otimes \cdots \otimes \tilde{\varepsilon}_{a_p} \otimes dx^{b_{q'}} \otimes \cdots \otimes dx^{b_q}. \]

The transformation of the components of an admissible tensor field in adapted coordinates obeys the following law:
\[ t'^a_b = A^a_{a'} A^b_{b'} t^{a'}_{b'}. \]
where \( A_a^a = \frac{\partial x^a}{\partial x^a} \).

Thus adapted coordinates play the role of "holonomic" coordinates for the non-involutive distribution. Adapted coordinates are essentially used in foliation geometry [7].

Let \( \omega = d\eta \). The equality
\[
[\tilde{e}_a, \tilde{e}_b] = 2\omega_{ba}\partial_n
\]
holds true. This, in particular, implies an important statement for what follows: the condition \( d\eta(\xi, X) = 0 \) is equivalent to the equality \( \partial_n \Gamma_a^a = 0 \).

Let \( \tilde{\nabla} \) be the Levi-Civita connection and \( \tilde{\Gamma}_{jk}^i \) its Christoffel symbols. The following proposition may be obtained by direct computations based on the use of the equality
\[
2\tilde{\Gamma}_{ij}^m = g^{km}(\tilde{\varepsilon}_j g_{ik} + \tilde{\varepsilon}_k g_{ij} - \tilde{\varepsilon}_i g_{jk} + \Omega^l_{kj} g_{li} + \Omega^l_{ki} g_{lj}) + \Omega^m_{ij}.
\]

**Proposition 1.** The Christoffel symbols \( \tilde{\Gamma}_{ij}^k \) of the Levi-Civita connection of a sub-Riemannian manifold in adapted coordinates have the form:
\[
\tilde{\Gamma}_{ab}^c = \Gamma_{ab}^c, \quad \tilde{\Gamma}_{ab}^n = \omega_{ba} - C_{ab}, \quad \tilde{\Gamma}_{bn}^a = \tilde{\Gamma}_{na}^b = C_a^b + \psi_a^b,
\]
\[
\tilde{\Gamma}_{an}^n = -\partial_n \Gamma_a^n, \quad \tilde{\Gamma}_{nm}^a = g^{ab}\partial_n \Gamma_b^n,
\]
where
\[
\Gamma_{bc} = \frac{1}{2}g^{ad}(\tilde{\varepsilon}_b g_{cd} + \tilde{\varepsilon}_c g_{bd} - \tilde{\varepsilon}_d g_{bc}), \quad \psi^b_a = g^{bc}\omega_{ac}, \quad C_{ab} = \frac{1}{2}\partial_n g_{ab}, \quad C_a^b = g^{bc}C_{ac}.
\]

Here the endomorphism \( \psi : TM \to TM \) is determined by the equality
\[
\omega(X, Y) = g(\psi(X, Y),
\]
and we set
\[
C(X, Y) = \frac{1}{2}(L_{\xi}g)(X, Y).
\]

An \( N \)-connection \( \nabla^N \) is defined on a sub-Riemannian manifold endowed with the endomorphism \( N : TM \to TM \) of the tangent bundle of \( M \) (\( N\xi = 0 \), \( N(D) \subset D \)). The connection \( \nabla^N \) may be expressed in terms of the Levi-Civita connection \( \tilde{\nabla} \),
\[
\nabla^N_X Y = \tilde{\nabla}_X Y + (\tilde{\nabla}_X \eta)(Y)\xi - \eta(Y)\tilde{\nabla}_X \xi - \eta(X)\tilde{\nabla}_\xi \eta)(Y)\xi - \eta(X)(C + \psi - N)Y.
\]

**Proposition 2.** A linear connection \( \nabla^N \) defined on a sub-Riemannian manifold is skew-symmetric if and only if \( N = 2\xi \).

**Proof.** It may be directly checked that with respect to adapted coordinates the nonzero Christoffel symbols \( G_{jk}^i \) of the connection \( \nabla^N_X \) have the form
\[
G_{bc}^a = \frac{1}{2}g^{ad}(\tilde{\varepsilon}_b g_{cd} + \tilde{\varepsilon}_c g_{bd} - \tilde{\varepsilon}_d g_{bc}), \quad G^b_{na} = N^b_a, \quad G^n_{ma} = -\partial_n \Gamma^n_a.
\]

The rank of a sub-Riemannian structure is equal to \( 2p \) if \( (d\eta)^p \neq 0 \), \( \eta \wedge (d\eta)^p = 0 \), and equal to \( 2p + 1 \) if \( \eta \wedge (d\eta)^p \neq 0 \), \( (d\eta)^{p+1} = 0 \). It is easy to check that the rank of a sub-Riemannian structure is \( 2p + 1 \) if and only if \( \partial_n \Gamma^n_a = 0 \).

Put \( \tilde{S}(X, Y, Z) = g(S(X, Y), Z), \ X, Y, Z \in \Gamma(TM) \). With respect to adapted coordinates the nonzero components of the tensor \( \tilde{S}(X, Y, Z) \) have the following form:
\[
\tilde{S}(\tilde{e}_a, \tilde{\partial}_n) = 2\omega_{ab},
\]
\[
\tilde{S}(\tilde{e}_a, \tilde{\partial}_n, \tilde{e}_b) = -g(N\tilde{e}_a, \tilde{e}_b),
\]
The tensor $\tilde{S}(X,Y,Z)$ is skew-symmetric if and only if $2\omega_{ab} = g(N\tilde{e}_a, \tilde{e}_b)$. This proves Proposition 2.

If $N = 2\psi$, the connection $\nabla^N$ will be called the canonical connection. Note that the canonical connection in the case of a sub-Riemannian structure of even rank is not a metric connection. Indeed,

$$\nabla^N g_{na} = -G^a_{na} = \partial_n \Gamma^a_n.$$ 

3. BASIC INFORMATION FROM THE GEOMETRY OF ALMOST QUASI-SASAKIAN MANIFOLDS

Consider an almost contact metric manifold $M$ of odd dimension $n = 2m + 1$. Let $(M, \xi, \eta, \varphi, g, D)$ be an almost contact metric structure on a manifold $M$, where $\varphi$ is a tensor of type $(1,1)$, called a structural endomorphism, $\xi$ and $\eta$ are a vector and a covector, called, respectively, a structure vector and a contact form, $g$ is a (pseudo-)Riemannian metric. In this case, the following equalities hold true:

1) $\varphi^2 = -I + \eta \otimes \tilde{\xi}$,
2) $\eta(\tilde{\xi}) = 1$,
3) $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, $X,Y \in \Gamma(TM)$.

The smooth distribution $D = \ker(\eta)$ is called the distribution of an almost contact structure. As a consequence of conditions 1) - 3) we obtain:

5) $\varphi \tilde{\xi} = \tilde{0}$,
6) $\eta \circ \varphi = 0$,
7) $\eta(X) = g(X, \tilde{\xi})$, $X \in \Gamma(TM)$.

The skew-symmetric tensor $\Omega(X,Y) = g(X,\varphi Y)$ is called the fundamental form of the structure. An almost contact metric structure is called a contact metric structure if the equality $\Omega = d\eta$ holds.

The smooth distribution $D^\perp = \text{Span}(\tilde{\xi})$, orthogonal to the distribution $D$, is called the framing of the distribution $D$. There is the decomposition $TM = D \oplus D^\perp$.

A Sasakian manifold is a contact metric space satisfying the additional condition

$$N^{(1)}_\varphi = N_\varphi + 2d\eta \otimes \tilde{\xi} = 0,$$

where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]$$

is the Nijenhuis tensor of the endomorphism $\varphi$. The condition $N^{(1)}_\varphi = N_\varphi + 2d\eta \otimes \tilde{\xi} = 0$ means that the Sasakian space is a normal space.

An almost contact metric manifold is called an almost contact Kähler manifold [5] if the following conditions hold:

$$d\Omega = 0, \quad \tilde{N}_\varphi = N_\varphi + 2\varphi^*d\eta \otimes \tilde{\xi} = 0.$$ 

A manifold for which the condition $\tilde{N}_\varphi = N_\varphi + 2\varphi^*d\eta \otimes \tilde{\xi} = 0$ is satisfied is called by us an almost normal manifold. It is easy to check that an almost normal almost contact metric manifold is a normal manifold if and only if $d\eta = \varphi^*d\eta$.

Let $P : TM \to D$ be the projector defined by the decomposition $TM = D \oplus D^\perp$. Then the following proposition holds.
Proposition 3. For any almost contact metric manifold, the following equality holds: $PN^{(1)}_{\varphi} = \tilde{N}_{\varphi}$.

Proof. 

$$PN^{(1)}_{\varphi}(X, Y) = P(\varphi X, \varphi Y) + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + 2d\eta(X, Y)\xi$$

$$= P[\varphi X, \varphi Y] - P[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

$$= [\varphi X, \varphi Y] - \eta(\varphi X, \varphi Y)\xi - [X, Y] + \eta([X, Y])\xi - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

$$= [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + 2d\eta(\varphi X, \varphi Y)\xi = \tilde{N}_{\varphi}(X, Y).$$

The proposition is proved. □

Note that the just proved proposition implies the relation:

$$N^{(1)}_{\varphi}(X, Y) = \tilde{N}_{\varphi}(X, Y) + 2(d\eta(X, Y) - d\eta(\varphi X, \varphi Y))\xi. \quad (1)$$

An internal linear connection $\nabla$ [3] on a manifold with an almost contact metric structure is a map

$$\nabla : \Gamma(D) \times \Gamma(D) \to \Gamma(D),$$

satisfying the following conditions:

1) $\nabla_{f_1\tilde{x} + f_2\tilde{y}} = f_1\nabla\tilde{x} + f_2\nabla\tilde{y},$

2) $\nabla_{\tilde{x}} f\tilde{y} = (\tilde{x}f)\tilde{y} + f\nabla\tilde{x}\tilde{y},$

3) $\nabla_{\tilde{x}}(\tilde{y} + \tilde{z}) = \nabla\tilde{x}\tilde{y} + \nabla\tilde{x}\tilde{z},$

where $\Gamma(D)$ is the module of admissible vector fields (vector fields at each point belonging to the distribution $D$).

The Christoffel symbols of $\nabla$ are determined from the relation $\nabla_{\tilde{a}}\tilde{c} = \Gamma^c_{ab}\tilde{e}_b$. From the equality $\tilde{e}_a = A^a_{\tilde{a}}\tilde{e}_{\tilde{a}}$, where $A^a_{\tilde{a}} = \frac{\partial x^a}{\partial x^{\tilde{a}}}$, the transformation formula follows

$$\Gamma^c_{ab} = A^a_{\tilde{a}} A^b_{\tilde{b}} A^c_{\tilde{c}} \Gamma^\tilde{c}_{\tilde{a} \tilde{b}} + A^a_{\tilde{a}} \tilde{e}_a A^c_{\tilde{c}}.$$

Hence, in particular, it follows that the derivatives $\partial_a\Gamma^{d}_{ac}$ are components of an admissible tensor field.

We give two examples of almost contact Kähler manifolds.

Example 1. Let $M = \{(x, y, z, u, v) \in R^5 : y \neq 0\}$ be a smooth manifold of dimension 5 equipped with an almost contact metric structure $(M, \tilde{\xi}, \eta, \varphi, g, D)$, where

1) $D = \text{Span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, here $\tilde{e}_1 = \partial_1 - y\partial_5$, $\tilde{e}_2 = \partial_2$, $\tilde{e}_3 = \partial_3$, $\tilde{e}_4 = \partial_4$, and $(\partial_1, \ldots, \partial_5)$ is the basis of vector fields corresponding to the coordinates $(x, y, z, u, v)$ on $R^5$,

2) $\tilde{\xi} = \partial_5$,

3) $\eta = dz + ydx$,

4) $\varphi\tilde{e}_1 = \tilde{e}_3$, $\varphi\tilde{e}_2 = \tilde{e}_4$, $\varphi\tilde{e}_3 = -\tilde{e}_1$, $\varphi\tilde{e}_4 = -\tilde{e}_2$, $\varphi\tilde{\xi} = 0$,

5) the basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{\xi})$ consists of orthonormal vectors.
It may be directly checked that the almost contact metric manifold $M$ is not normal, but almost normal. Indeed,

$$N_{\varphi}^{(1)}(\vec{e}_1, \vec{e}_2) = \varphi^2[\vec{e}_1, \vec{e}_2] + [\vec{e}_3, \vec{e}_4] - \varphi[\vec{e}_3, \vec{e}_2] - \varphi[\vec{e}_1, \vec{e}_4] + 2d\eta(\vec{e}_1, \vec{e}_2)\vec{\xi} = \varphi^2\vec{\xi} - \eta(\vec{\xi})\vec{\xi} = -\vec{\xi}. $$

On the other hand,

$$\tilde{N}_{\varphi}(\vec{e}_1, \vec{e}_2) = 2d\eta(\vec{e}_3, \vec{e}_4)\vec{\xi} = 0. $$

For the structure under consideration, the equality $d\eta(\vec{\xi}, X) = 0$, $X \in \Gamma(TM)$, holds. Thus $\omega = d\eta$ is an admissible tensor field, to which the internal connection $\nabla$ may be applied [3]. Moreover, $\nabla \omega = 0$. Let, further, $\psi$ be the endomorphism defined by the equality $\omega(X, Y) = g(\psi X, Y)$. The coordinate representation of the endomorphism $\psi$ is of the form $\psi^b_a = g^{bc}\omega_{ac}$. Thus the trace of the square of the endomorphism $\psi$ is constant, $\text{tr}(\psi^2) = \text{const}$. 

**Example 2.** Considers the same manifold $M$ as in the previous example with the only difference that

$$\vec{e}_1 = \partial_1 - yz\partial_5, \quad \eta = dz + yzd\bar{x}. $$

Unlike the previous case, the condition $d\eta(\vec{\xi}, \cdot) = 0$ is not satisfied. Indeed,

$$2d\eta(\vec{\xi}, \vec{e}_1) = -\eta(\vec{\xi}, \vec{e}_1) = y \neq 0. $$

An almost contact metric manifold is called an almost quasi-Sasakian manifold (AQS-manifold) if the following conditions are satisfied:

$$d\Omega = 0, \quad \tilde{N}_{\varphi} = N_{\varphi} + 2\varphi^*d\eta \otimes \vec{\xi} = 0, \quad d\eta(\vec{\xi}, \cdot) = 0. \quad (2) $$

Note that Example 1 implies that there exist an almost quasi-Sasakian manifolds satisfying the conditions

$$\nabla \omega = 0, \quad \text{tr}(\psi^2) = \text{const}. $$

Moreover the equality $\nabla \omega = 0$ is equivalent to the equality $\nabla \psi = 0$.

The following theorem holds.

**Theorem 1.** An almost contact metric structure is an almost quasi-Sasakian structure if and only if the following equality holds:

$$(\nabla_X \varphi)Y = g((\varphi \circ \varphi)Y, X)\vec{\xi} - \eta(Y)(\varphi \circ \varphi)(X) - \eta(X)(\varphi \circ \psi - \psi \circ \varphi)Y. \quad (3) $$

**Proof.** Let $M$ be a AQS-manifold. Let us show that the condition (3) is satisfied. The equality

$$2g((\nabla_X \varphi)Y, Z) = 3(d\Omega(X, \varphi Y, \varphi Z) - d\Omega(X, Y, Z)) + g(N_{\varphi}^{(1)}(Y, Z), \varphi X)$$

$$+ 2N_{\varphi}^{(2)}(Y, Z)\eta(Y) + 2(d\eta(\varphi Y, X)\eta(Z) - d\eta(\varphi Z, X)\eta(Y)), $$

where the operator $N_{\varphi}^{(2)}$ is similar to $N_{\varphi}^{(1)}$, see, e.g., [1]. holds for any almost contact metric manifold. Using (2), we get

$$g((\nabla_X \varphi)Y, Z) = \eta(X)(g((\varphi \circ \varphi)Y, Z) + d\eta(Y, \varphi Z)) + g(d\eta(\varphi Y, X)\vec{\xi}, Z) + d\eta(X, \varphi Z)\eta(Y). $$

This proves the equality (3). The inverse statement may be easily proved using adapted coordinates. \hfill \Box

The following propositions are direct consequences of Theorem 1.

**Proposition 4.** An almost contact metric structure is a quasi-Sasakian structure if and only if the equality holds:

$$(\nabla_X \varphi)Y = g(AY, X)\vec{\xi} - \eta(Y)AX, A = \varphi \circ \psi. $$

**Proposition 5.** An almost quasi-Sasakian manifold is a quasi-Sasakian manifold if and only if one of the following conditions holds
1) \( d\eta = \varphi^*d\eta \),
2) \( \varphi \circ \psi - \psi \circ \varphi = 0 \),
3) \( g(X, AY) = g(AX, Y), A = \varphi \circ \psi \).

4. ALMOST QUASI-SASAKIAN MANIFOLDS WITH CANONICAL SKewed-SYMMETRIC CONNECTION

Let \((M, \vec{\xi}, \eta, \varphi, g, D)\) be an almost quasi-Sasakian structure given on a manifold \(M\), and let \(\nabla^N\) be the canonical connection. Proposition 5 implies the following

Proposition 6. An almost quasi-Sasakian manifold is a quasi-Sasakian manifold if and only if \(\nabla^N \varphi = 0\).

Further, we restrict our attention to the case when the torsion \(\tilde{S}(X, Y, Z)\) of the connection \(\nabla^N\) is parallel. It is known that this condition holds for Sasakian manifolds. At the same time, it follows from the above example that there exist such almost quasi-Sasakian manifolds that are neither quasi-Sasakian, nor Sasakian and for which the torsion is skew-symmetric and parallel.

Let \(K\) be the curvature tensor of the canonical connection \(\nabla^N\). For nonzero components of the tensor \(K\) it holds

\[
K^d_{abc} = R^d_{abc} + 4\omega_{ab}\psi^d_c,
\]

\[
K^d_{anc} = 2\nabla^a\psi^d_c.
\]

Here \(R^d_{abc} = 2\varepsilon^d_{[a|b|c]} + 2\Gamma^d_{[a|c]} \Gamma^c_{b|c}\) are components of the Schouten curvature tensor \([3]\) defined by the equality

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{P[X, Y]} Z - P[Q[X, Y], Z], \quad Q = 1 - P.
\]

The tensor \(K\) may be written in the following way:

\[
K(X, Y)Z = R(X, Y)Z + \eta(Y)(\nabla_X N)Z - \eta(X)(\nabla_Y N)Z + 4\omega(X, Y)\psi(Z), \quad X, Y, Z \in \Gamma(TM).
\]

Tensor field \(r(X, Z) = \text{tr}(Y \mapsto R(X, Y)Z), X, Z \in \Gamma(D)\), will be called the Ricci-Wagner tensor. Using adapted coordinates, we write down the components of the Ricci tensor \(k\) of the connection \(\nabla^N\):

\[
k_{ab} = r_{ab} + 4\omega_{ad}\psi^d_b,
\]

\[
k_{an} = k_{nn} = 0,
\]

\[
k_{na} = -\nabla^a\psi^d_c.
\]

From the definition of the endomorphism \(\psi\) it follows that the equality \(\nabla\omega = 0\) holds if and only if \(\nabla\psi = 0\). Hence the components of the Ricci tensor \(k\) of the connection \(\nabla^N\) with parallel torsion take the form:

\[
k_{ab} = r_{ab} + 4\omega_{ad}\psi^d_b,
\]

\[
k_{an} = k_{na} = k_{nn} = 0.
\]

Thus the following theorem turns out to be true.

Theorem 2. An almost quasi-Sasakian manifold is an Einstein manifold with respect to the canonical connection with parallel torsion if and only if

\[
r_{ab} = 4\omega_{da}\psi^d_b.
\]
As follows from Theorem 2, the existence of Einstein metric on an almost quasi-Sasakian manifold with respect to the canonical connection essentially depends on the structure of the Ricci-Wagner tensor.

We complete the work with an example of an almost quasi-Sasakian Einstein manifold.

**Example 3.** Let us introduce a quasi-Sasakian structure on the manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5 : y \neq 0\}$ by setting:

1) $D = \text{Span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, where $\tilde{e}_1 = \partial_1 - y\partial_5$, $\tilde{e}_2 = \partial_2$, $\tilde{e}_3 = \partial_3$, $\tilde{e}_4 = \partial_4$, and $(\partial_1, \ldots, \partial_5)$ is the basis of vector fields corresponding to the coordinates $(x, y, z, u, v)$ on $\mathbb{R}^5$,

2) $\tilde{\xi} = \partial_5$,

3) $\eta = dv + ydx$,

4) $\varphi \tilde{e}_1 = \tilde{e}_2$, $\varphi \tilde{e}_2 = -\tilde{e}_1$, $\varphi \tilde{e}_3 = \tilde{e}_4$, $\varphi \tilde{e}_4 = -\tilde{e}_3$, $\varphi \tilde{\xi} = 0$,

5) the metric tensor is given by the equality

$$g = \frac{1}{(1 + x^2 + y^2)^2}((dx)^2 + (dy)^2) + (dz)^2 + (du)^2 + \eta^2.$$

The condition $r_{ab} = 4\omega_{da}^d \psi_b^d$ reduces to the equality $r_{\alpha\beta} = -4g_{\alpha\beta}$, $\alpha, \beta = 1, 2$. It is easy to check that the above metric satisfies the condition $r_{\alpha\beta} = -4g_{\alpha\beta}$. Thus, we have obtained an example of an Einstein quasi-Sasakian manifold. If we redefine the first structural endomorphism by setting

$$\varphi \tilde{e}_1 = \tilde{e}_3$$
$$\varphi \tilde{e}_2 = \tilde{e}_4$$
$$\varphi \tilde{e}_3 = -\tilde{e}_1$$
$$\varphi \tilde{e}_4 = -\tilde{e}_2$$
$$\varphi \tilde{\xi} = 0,$$

then the quasi-Sasakian manifold reduces to an almost quasi-Sasakian manifold. Thus we have obtained an example of an Einstein AQS-manifold with respect to the canonical connection.

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