Full nonuniversality of the symmetric 16-vertex model on the square lattice

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We consider the symmetric two-state 16-vertex model on the square lattice whose vertex weights are invariant under any permutation of adjacent edge states. The vertex-weight parameters are restricted to a critical manifold which is self-dual under the gauge transformation. The critical properties of the model are studied numerically by using the Corner Transfer Matrix Renormalization Group method. The accuracy of the method is tested on two exactly solvable cases: the Ising model and a specific version of the Baxter 8-vertex model in a zero field. The numerical results imply parameter-dependent critical exponents which clearly violate weak universality hypothesis.

I. INTRODUCTION

According to the universality hypothesis [1], critical exponents of a statistical system at the second-order phase transition do not depend on details of the corresponding Hamiltonian. Equivalently, critical exponents depend only on system’s space dimensionality and symmetry of microscopic degrees of freedom (say spins). The first violation of the universality hypothesis was observed in the Baxter’s exact solution of the 8-vertex model on the square lattice in a zero electric field [2, 3] whose critical exponents are functions of the model parameters. Suzuki [5] argued that the violation of universality in the 8-vertex model is due to the identification of the deviation from the critical point with the temperature difference $|T_c - T|$. Instead, if taking the inverse correlation length $|T_c - T|^{-\nu}$ (the critical exponent $\nu$ is assumed to be the same for both limits $T \to T_c^-$ and $T \to T_c^+$) as the natural measure of the distance from the critical point, the renormalized exponents $\beta/\nu, \gamma/\nu, \delta = 1 + \frac{\gamma}{\beta}$, etc. become universal, i.e. independent of model’s parameters. This phenomenon is known as weak universality. Weak universality was observed in many two-dimensional (2D) systems, including the Ashkin-Teller model [6, 8], absorbing phase transitions [9], the spin-1 Blume-Capel model [10], frustrated spin models [11, 12], percolation models [13], etc. There are few exceptions from models with continuously varying critical exponents which violate weak universality, such as micellar solutions [14], Ising spin glasses [15], itinerant composite magnetic materials [16, 17], etc.

The partition function of the “electric” 8-vertex model on the square lattice can be mapped onto Ising spins in a field [25, 26]. The critical properties of the general 16-vertex model with an enlargement of the O(2) gauge transformation which preserves the permutation symmetry of vertex weights and its self-dual manifolds can be found relatively simply. In a certain subspace of vertex weights, the model can be mapped onto Ising spins in a field [25, 26]. The critical properties of the model were studied numerically by combining a series expansion on the lattice and the Coherent Anomaly method [27] in Ref. [28]. In spite of modest computer facilities and lack of efficient numerical methods at that time (almost 30 years ago), the numerical results indicate the full nonuniversality of the model. For a recent survey of the general 16-vertex model with an enlargement of known mappings, see Ref. [29].

The aim of this work is to revisit the study of the critical properties of the symmetric 16-vertex model on the square lattice by using the Corner Transfer Matrix Renormalization Group (CTMRG) method [30, 34]. The method is based on the renormalization of the density matrix [35, 37] and the
II. MODEL AND ITS EXACTLY SOLVABLE CASES

A. Basic facts about the model

The general two-state vertex model on the square lattice of $N$ ($N \to \infty$) sites is defined as follows. Each lattice edge can be in one of two states. These states will be denoted either by $\pm$ signs or by “dipole” arrows: the right/up oriented arrow corresponds to the $(+)$ state, while the left/down arrow to the $(-)$ state. With each vertex we associate the set of $2^4$ possible Boltzmann weights $w(s_1,s_2,s_3,s_4) = \exp \left[ -\varepsilon(s_1,s_2,s_3,s_4)/T \right]$. In units of $k_B = 1$, both energy $\varepsilon(s_1,s_2,s_3,s_4)$ and temperature $T$ are taken as dimensionless. For the symmetric version of the model, the vertex weights are invariant with respect to any permutation of state variables $(s_1,s_2,s_3,s_4)$. Let us denote by $w_i = \exp(-\varepsilon_i/T)$ ($i = 0,1,\ldots,4$) the vertex weight with $i$ incident edges in the $(-)$ state and the remaining $4-i$ incident edges in the $(+)$ state. Thus from among 16 possible configurations of vertex states there is 1 corresponding to each of the vertex weights $w_2$ and $w_4$, 4 corresponding to each of $w_1$ and $w_3$ and 6 corresponding to $w_0$, see Fig. 1.

Thermal equilibrium of the system is determined by the (dimensionless) free energy per site

$$f\{w\}/T = \lim_{N \to \infty} \frac{1}{N} \ln Z\{w\},$$

(1)

where

$$Z\{w\} = \sum_{\{s\}} \prod_{\text{vertex}} \text{(weights)},$$

(2)

is the partition function with the summation going over all possible edge configurations and the product being over all vertex weights in the lattice. The mean concentration $c_i$ of the vertices with weight $w_i$ is given by

$$c_i = -w_i \frac{\partial f\{w\}}{\partial w_i}/T \quad (i = 0,\ldots,4).$$

(3)

The mean concentrations are constrained by the obvious normalization condition $\sum_{i=0}^{4} c_i = 1$. The mean-value of the edge-state variable

$$P = \frac{1}{4} \sum_{i=0}^{4} (4-2i)c_i$$

(4)

defines the polarization. When one applies an isotropic electric field $E$ (with the same strength along either of the two axes), each arrow dipole $s = \pm 1$ gets the energy $-Es$. Since every dipole belongs to just two vertices, the vertex weights are modified to

$$w_i(E) = w_i \exp[E(2-i)/T].$$

(5)

One can trivially extend the definitions of the vertex concentrations and the polarization to $E \neq 0$, 

\[
\begin{align*}
\varepsilon_1 &= \varepsilon_2 = 0 \quad w_{0,4} = 1 \\
\varepsilon_3 &= \cdots = \varepsilon_{10} = \varepsilon \quad w_{1,3} = \exp(-\varepsilon/T) \\
\varepsilon_{11} &= \cdots = \varepsilon_{16} = 1 \quad w_2 = \exp(-1/T)
\end{align*}
\]
with the corresponding notations \( c_i(E) \) and \( P(E) \). Then the polarization susceptibility reads as

\[
\chi = \lim_{E \to 0} \frac{\partial P(E)}{\partial E} = \frac{1}{2} \sum_{i,j=0}^{4} (2 - i)(2 - j)\chi_{ij},
\]

(6)

where

\[
\chi_{ij} = -\frac{\partial}{\partial c_i} c_i(E = 0)
\]

(7)

form the tensor of generalized susceptibilities.

The partition function of the general two-state vertex model is given by [24]

\[
\text{\textup{form the tensor of generalized susceptibilities.}}
\]

(8)

Here, \( V_{ss'}(y) \) are the elements of the matrix

\[
V(y) = \frac{1}{\sqrt{1 + y^2}} \begin{pmatrix} 1 & y \\ y & -1 \end{pmatrix}
\]

(9)

with rows (columns) indexed from up to down (left to right) as +, − and a free (real) gauge parameter \( y \). For the symmetric version of the vertex model, the gauge transformation keeps the permutation symmetry of vertex weights [24], namely

\[
\tilde{w}_i = \sum_{j=0}^{4} W_{ij}(y)w_j \quad (i = 0, 1, \ldots, 4),
\]

(10)

\[
W_{ij}(y) = \frac{1}{(1 + y^2)^2} \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j-k}{k} (-1)^k y^{i+j-2k}.
\]

(11)

Points in the vertex-weight parameter space which can be mapped onto itself by gauge transformation with a nontrivial (point-dependent) value of \( y \neq 0 \), form the so-called self-dual manifold. The self-dual manifold for the symmetric 16-vertex model is given by [24]

\[
w_0^2w_3 - w_1w_4^2 - 3w_2(w_0 - w_4)(w_1 + w_3) + (w_1 - w_3)(w_0w_4 + 2(w_1 + w_3)^2) = 0.
\]

(12)

Its importance consists in the fact that critical points of second-order phase transitions are confined to this subspace of vertex weights.

In this work, we restrict ourselves to the symmetric 16-vertex model whose vertex weights are invariant with respect to the flip of all adjacent edge states \((+) \leftrightarrow (-)\). The vertex weights are parametrized as follows

\[
w_0 = w_4 = 1, \quad w_1 = w_3 = e^{-\epsilon/T}, \quad w_2 = e^{-1/T},
\]

(13)

see also Fig. 1, where the real parameter \( \epsilon \geq 0 \). It can be checked that this choice of vertex weights automatically satisfy the self-dual condition [12]. Thus for a fixed value of the energy \( \epsilon \) there should exist a critical temperature \( T_c \) at which a second-order phase transition takes place. In the disordered phase at \( T > T_c \), the state-flip symmetry ensures that the equality of concentrations \( c_i = c_{i+4} \), so that the mean polarization \( \langle P \rangle \) vanishes. In the ordered phase at \( T < T_c \), the state-flip symmetry breaking causes that \( c_i \neq c_{i+4} \) and the spontaneous polarization \( P \) becomes non-zero. Close to \( T_c \), it is the non-analytic function of \( T - T_c \):

\[
P \propto (T_c - T)^{\beta_e}, \quad T \to T_c^-
\]

(14)

with \( \beta_e \) (the subscript \( e \) means “electric”) being the critical exponent. Applying to the vertex system a small isotropic external electric field \( E \) just at the critical temperature, the polarization behaves as

\[
P(E) \propto E^{1/\delta_e}, \quad T = T_c
\]

(15)

where \( \delta_e \) is another critical exponent. Close to the critical point, the polarization susceptibility [6] exhibits the singularity of type

\[
\chi \propto \frac{1}{|T_c - T|^{\gamma_e}},
\]

(16)

where the critical exponent \( \gamma_e \) is assumed to be the same for both limits \( T \to T_c^- \) and \( T \to T_c^+ \). The pair arrow-arrow correlation function exhibits the large-distance behavior

\[
G_e(r) \propto \frac{1}{r^{\nu_e}} \exp(-r/\xi), \quad r \to \infty.
\]

(17)

Close to the critical point, the correlation length \( \xi \) behaves as

\[
\xi \propto \frac{1}{|T_c - T|^{\nu_e}}.
\]

(18)

The divergence of \( \xi \) at \( T = T_c \) reflects the fact that the short-range (exponential) decay out of the critical point changes to the long-range (inverse power-law) one at \( T = T_c \) characterized by the critical exponent \( \nu_e \).

Having at one’s disposal two critical exponents \( \beta_e \) and \( \delta_e \), the remaining ones (considered in this work) can be calculated by using scaling relations [3]:

\[
\gamma_e = \beta_e (\delta_e - 1),
\]

\[
\nu_e = \frac{1}{2} \beta_e (\delta_e + 1),
\]

(19)

\[
\eta_e = \frac{4}{\delta_e + 1}.
\]
The symmetric 16-vertex model can be mapped onto an Ising model on the square lattice if the vertex weights are constrained by \[25, 26\]

\[ w_0 w_2 w_4 - w_0 w_3^2 - w_1^2 w_4 + 2 w_1 w_2 w_3 - w_3^3 = 0. \]  

(20)

For the state-flip symmetry of vertex weights \([13]\), this equation takes the form

\[ 1 + e^{1/T} = 2e^{2(1-\varepsilon)/T}. \]  

(21)

As concerns the parameters of the Ising model for the state-flip symmetry, the external magnetic field acting on spins \(H = 0\) and the (dimensionless) coupling \(J\) between nearest-neighbor spins is given by

\[ J = \frac{1}{2} \ln \left( \frac{w_1}{w_2} \right) = \frac{1 - \varepsilon}{2T}. \]  

(22)

The known critical value of the Ising coupling reads \([3]\)

\[ J_c = \frac{1}{2} \ln \left( 1 + \sqrt{2} \right). \]  

(23)

Consequently, Eqs. (21) and (22) imply the following critical parameters of the symmetric 16-vertex model:

\[ \varepsilon^{(1)} = 1 - \frac{\ln(1 + \sqrt{2})}{\ln(5 + 4\sqrt{2})} = 0.627516 \ldots, \]  

(24)

\[ T_c^{(1)} = \frac{1}{\ln(5 + 4\sqrt{2})} = 0.422618 \ldots. \]  

(25)

In contrast to standard mappings of models on dual lattices, the mapping between the symmetric 16-vertex and Ising models is made on the same square lattice \([23, 26]\). The relation between the polarization of the symmetric 16-vertex model and the magnetization of the equivalent Ising system can be derived by using the technique presented in Ref. \([38]\). This relation is linear and therefore the critical exponents of the symmetric 16-vertex model coincide with the Ising ones. The Ising critical exponents are summarized in Table I.

![Table I. List of electric critical exponents for the symmetric 16-vertex model at the exactly solvable Ising and the Baxter 8-vertex points.](image)

**TABLE I.** List of electric critical exponents for the symmetric 16-vertex model at the exactly solvable Ising and the Baxter 8-vertex points.

| Exponent | \(\beta_c\) | \(\delta_c\) | \(\gamma_c\) | \(\nu_c\) | \(\eta_c\) |
|----------|-------------|-------------|-------------|---------|---------|
| \(\varepsilon(1)\) | \(0.6275\) | \(1/8\) | \(15/4\) | \(7/4\) | \(1/4\) |
| \(\varepsilon \to \infty\) | \(1/8\) | \(11/4\) | \(3/4\) | \(1/3\) |

When \(\varepsilon \to \infty\), the vertex configurations \(w_1\) and \(w_3\) in Fig. I with odd numbers of (+), or equivalently (−), edge states vanish. The consequent Baxter’s 8-vertex model has vertex-weight parameters \(a = w_0 = w_4 = 1\) and \(b = c = d = w_2 = \exp(-1/T)\) \([3]\). The vertex system exhibits the ferroelectric-A phase for \(a > b + c + d\). The second-order transition between the ferroelectric-A and disordered phases takes place at

\[ a_c = b_c + c_c + d_c, \quad T_c = \frac{1}{\ln 3} = 0.910239 \ldots. \]  

(26)

Introducing the auxiliary parameter

\[ \mu = 2 \arctan \left( \sqrt{\frac{a_c b_c}{c_c d_c}} \right) = \frac{2\pi}{3}, \]  

(27)

according to Ref. \([20]\) the electric critical exponents are given by

\[ \beta_e = \frac{\pi - \mu}{4\mu} = \frac{1}{8}, \]  

\[ \delta_e = \frac{3\pi + \mu}{\pi - \mu} = 11, \]  

\[ \gamma_e = \frac{\pi + \mu}{2\mu} = \frac{5}{4}, \]  

\[ \nu_e = \frac{\pi}{2\mu} = \frac{3}{4}, \]  

\[ \eta_e = 1 - \frac{\mu}{\pi} = \frac{1}{3}. \]  

These critical exponents are summarized in Table I.

**III. NUMERICAL METHOD**

The CTMRG method \([30, 32]\) is based on Baxter’s technique of corner transfer matrices \([3]\). Each quadrant of the square lattice with size \(L \times L\) is represented by the corner transfer matrix \(C\). The reduced density matrix is defined by \(\rho = \text{Tr}' C'^4\) (where the partial trace \(\text{Tr}'\) is taken), so that the partition function \(Z = \text{Tr} \rho\), see Fig. 2. The number of degrees of freedom grows exponentially with \(L\) and the density matrix is used in the process of their reduction. Namely, degrees of freedom are iteratively projected to the space generated by the eigenvectors of the reduced density matrix \(\rho\) with the largest eigenvalues. The projector on this reduced space of dimension \(m\) will be denoted by \(O\): the larger truncation parameter \(m\) is taken, the better precision of the results is attained. In each iteration the linear system size is expanded from \(2L\) to \(2L + 2\) via the inclusion of the Boltzmann weight \(W\) of the basic vertex (see Fig. I). The expansion process transforms the corner transfer matrix \(C\) to \(C'\) and the half-row transfer matrix \(H\) to \(H'\) in the way represented schematically in Fig. 2.
thick lines (lines) represent renormalized multi-arrow (arrow) variables obtained after the renormalization which consists in the summation and $O$-projection of multi-arrow (arrow) thick lines (lines) from the previous iteration. The fixed boundary conditions are used, with each arrow at the boundary line set to the value $s = -1$. This choice ensures a quicker convergence of the method in the ordered phase towards the thermodynamic limit.

IV. NUMERICAL RESULTS

According to Eq. (14), the critical temperature $T_c$ is the lowest temperature at which $P = 0$ or, equivalently, the highest temperature at which $P \neq 0$. Based on comparison with the known values of the Ising (25) and Baxter’s (26) critical temperatures, the error in estimation of $T_c(\epsilon)$ is of order $10^{-4}$ for all values of $\epsilon$. The error is even smaller (of order $10^{-5}$) when fitting data for the spontaneous polarization close to the critical point according to (14). Numerical results for the $\epsilon$-dependence of the critical temperature are shown in Fig. 3. We see that $T_c(\epsilon)$ is only weakly affected by dimension of the truncated space $m = 100$ and $m = 200$.

The inset of Fig. 3 documents the log-log plot of the small-$\epsilon$ behavior of $T_c(\epsilon)$. The power-law least-square fitting at low $\epsilon < 10^{-8}$ yields

$$T_c(\epsilon) = -6.6 \times 10^{-18} + 0.954(5)\epsilon^{0.9998(3)},$$

where the absolute term is on the accuracy border of the computer. We conclude that in the limit of small $\epsilon$ the critical temperature converges to zero linearly. On the other hand, as $\epsilon$ increases the critical temperature saturates quickly to the value 0.91024 which is close to the asymptotic $\epsilon \to \infty$ analytic result (26) of the 8-vertex model.

The critical exponent $\beta_c$ should interpolate between the same values $1/8$ at small and large $\epsilon$. It is calculated again by fitting polarization data according to formula

$$\beta_c = \left( \frac{\partial \ln P}{\partial \ln E} \right)^{-1}.$$

This formula has to be considered for a very small value

![FIG. 2. The CTMRG renormalization process. The density matrix $\rho$ is expanded as $4 \times 4$ transfer matrices $C$. The expansion process of the corner transfer matrix $C \rightarrow C' = O^T H W C O$ and the half-row transfer matrix $H \rightarrow H' = O^T W H O$ from the previous iteration RG Step, see the text.](image)

![FIG. 3. The $\epsilon$-dependence of the critical temperature $T_c$ of the symmetric 16-vertex model, for dimension of the truncated space $m = 100$ (open circles) and $m = 200$ (open circles with stars). The inset shows a linear dependence of $T_c(\epsilon)$ for small values of $\epsilon$.](image)
of field $E$, but not too small to avoid numerical errors due to the critical state of the vertex system. The obtained data for $E = 10^{-5}$ and $2.5 \times 10^{-5}$ are presented in Fig. 5 within the range of $\varepsilon \in [0, 18]$. Data for $E = 10^{-5}$, evaluated at approximation orders $m = 100$ (circles) and $m = 200$ (squares), converge well below the anticipated value $11$. On the other hand, numerical data for the optimal field $E = 2.5 \times 10^{-5}$ evaluated at approximation order $m = 100$ (triangles) are lying close to the previous data for $E = 10^{-5}$ with $m = 200$ and for large values of $\varepsilon \gtrsim 14$ tend correctly to the value $11$.

The critical exponent $\gamma_e$ is expected to interpolate between $7/4$ at small $\varepsilon$ and $5/4$ at large values of $\varepsilon$. This exponent is calculated by fitting the susceptibility data according to the formula (16). Fitting is performed in the region $T > T_c$ with the susceptibility functional values from the interval $\chi \in [10000, 50000]$. Within the range of $\varepsilon \in [0, 18]$, the obtained $m = 100$ data are represented by triangles in Fig. 6. Data tend for small and large values of $\varepsilon$ correctly to $7/4$ and $5/4$, respectively. Because fits of the singular formula (16) are accompanied by relatively large errors, we have calculated alternatively $\gamma_e$ by inserting the previous data for $\beta_e$ and $\delta_e$ into the first of scaling relations (19). Hereinafter, we adopt the convention that an exponent deduced by using scaling relations will be denoted by a tilde on its top. Numerical data for $\gamma_e$ are represented in Fig. 6 by circles. Note that the plot exhibits a monotonous decay.

The critical exponents $\bar{\nu}_c$ and $\bar{\eta}_c$, calculated by inserting the previous data for $\beta_e$ and $\delta_e$ into the second and third of scaling relations (19), respectively, are represented as functions of $\varepsilon$ in Fig. 7 by triangles and circles, respectively. Both plots exhibit non-monotonous behavior. The exponent $\bar{\nu}_c$ interpolates correctly between $1$ at small $\varepsilon$ and $3/4$ at large $\varepsilon$ and $\bar{\eta}_c$ interpolates correctly between $1/4$ at small $\varepsilon$ and $1/3$ at large $\varepsilon$.

As seen in Figs. 14 the critical exponents of the symmetric 16-vertex model on the square lattice depend on model’s parameter $\varepsilon$. To judge the validity of the hypothesis of weak universality, note that the critical points and exponents of the exactly solvable Ising $\varepsilon^{(1)} = 0.6275$ and Baxter’s $\varepsilon \to \infty$ cases lie on the same continuous curve. It is therefore sufficient to test the renormalized exponents $\beta_e/\nu_e$, $\gamma_e/\nu_e$ and the exponent $\delta_e$, which are independent of model’s parameters if weak universality works, at the two exactly solvable points. In
FIG. 7. The critical exponents \( \tilde{\nu}_e \) (triangles) and \( \tilde{\eta}_e \) (circles), calculated by inserting the previous data for \( \beta_e \) and \( \delta_e \) into the second and third of scaling relations (19), respectively, as functions of \( \varepsilon \in [0, 18] \).

V. CONCLUSION

The system under consideration in this paper was the symmetric two-state 16-vertex model on the square lattice. Its vertex weights, which are invariant under any permutation of adjacent edge states, are considered to be symmetric with respect to the flip of all adjacent edge states (+) \( \leftrightarrow \) (−) (see Fig. 1) as well. Such vertex weights automatically lie on the self-dual manifold of the gauge transformation (12), carrying critical points. The parametrization of vertex weights (13) contains two positive parameters, the temperature \( T \) and the energy \( \varepsilon \). Two exactly solvable cases, namely the Ising model and the specific version of Baxter’s 8-vertex model correspond to \( \varepsilon^{(1)} \approx 0.6275 \) and \( \varepsilon \to \infty \), respectively. To study critical properties of the model, we have applied the very accurate CTMRG method. The dependence of the critical temperature \( T_c \) on \( \varepsilon \) is pictured in Fig. 3. The fit of the plot in the region of small \( \varepsilon \) (see the inset) indicates the linear dependence with \( T_c \) going to 0 as \( \varepsilon \to 0 \). The plot of the critical exponent \( \beta_e \) versus \( \varepsilon \), calculated with optimal parameters of the temperature step \( \Delta T = 10^{-4} \) and dimension of the reduced space \( m = 200 \), is represented by square data in Fig. 4. The critical exponent \( \delta_e(\varepsilon) \) is calculated with optimal parameters of the electric field \( E = 2.5 \times 10^{-5} \) and \( m = 100 \), see triangle data in Fig. 5. The plots of the exponent \( \gamma_e \) versus \( \varepsilon \) are evaluated “from first principles” (triangles) and by using the first of scaling relations (19) (circles) in Fig. 6. The dependence of the critical exponents \( \nu_e \) and \( \eta_e \) on \( \varepsilon \), evaluated by using the second and third of scaling relations (19), are presented in Fig. 7. All critical exponents interpolate correctly between their known values at the two solvable cases \( \varepsilon^{(1)} \approx 0.6275 \) and \( \varepsilon \to \infty \). The variation of the critical exponents with model’s parameter \( \varepsilon \) is such that weak universality hypothesis is violated.

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