Linear Quadratic Mean Field Games – Part I: The Asymptotic Solvability Problem

Minyi Huang  Mengjie Zhou

Abstract—This paper investigates the so-called asymptotic solvability problem in linear quadratic (LQ) mean field games. The model has asymptotic solvability if for all sufficiently large population sizes, the corresponding game has a set of feedback Nash strategies subject to a mild regularity requirement. We provide a necessary and sufficient condition and show that in this case the solution converges to a mean field limit. This is accomplished by developing a re-scaling method to derive a low dimensional ordinary differential equation (ODE) system, where a non-symmetric Riccati ODE has a central role.

Index Terms—Asymptotic solvability, mean field game, re-scaling, Riccati equation.

I. INTRODUCTION

Mean field game theory has undergone a phenomenal growth. It provides a powerful methodology for handling complexity in noncooperative mean field decision problems. The readers are referred to [2], [3] for an overview of the theory and applications. The analysis in the LQ setting has attracted substantial interest due to its appealing analytical structure [8], [14], [16]. Specifically, the strategy of an individual player can be determined in a feedback form using its own state.

Two important methodologies called the top-down approach and bottom-up approaches [3], respectively, have been widely used in the analysis of mean field games. By the top-down approach [8], [9], one determines the best response of a representative agent to a mean field of an infinite population, and next all the agents’ best responses should regenerate that mean field. This procedure formalizes a fixed point problem which can be solved and further used to design decentralized strategies. The bottom-up approach (also called the direct approach [11]) starts by formally solving an $N$-player game to obtain a large coupled solution equation system. The next step is to derive a simple liming equation system by taking $N \to \infty$ [13]; also see [12] for a probabilistic framework.

This paper considers the LQ mean field game and addresses the so-called asymptotic solvability. We start with an entirely conventional solution of the game by dynamic programming and derive a set of coupled Riccati ODEs. This method may be viewed as an instance of the bottom-up approach. Our objective is to find a necessary and sufficient condition for the sequence of games to be asymptotically solvable. It turns out that such a condition is completely determined by a single low dimensional non-symmetric Riccati ODE. The derivation of this condition involves a novel re-scaling method for large-scale coupled equations with two-scales of interactions. We further determine the mean field limit of the individual strategies. Our approach has connection with an early model of mean field social optimization, which studies a high dimensional algebraic Riccati equation and uses symmetry for dimension reduction [6, Sec. 6.3]. The methodology of extracting a low dimensional structure, here as a non-symmetric Riccati ODE, to capture essential information of the large scale decision model shares similarity to identifying low dimensional dynamics of coupled oscillators in the physics literature [15], [17], [19]. Other related works include [5], [18], [21]. An optimal control problem for a set of agents with mean field coupling is solved in [5] by a large-scale Riccati ODE, where a mean field limit is derived for the Riccati equation using the scalar state and symmetry. An LQ Nash game of infinite horizon is analyzed in [18] where the number of players increases to infinity. The method is to postulate the strategies of all players and examine the control problem of a fixed player subject to the mean field dynamics. Then a family of low dimensional control problems and the parameterized algebraic Riccati equations can be solved by an implicit function theorem. Sufficient conditions are obtained for solvability when the population size is large. The solvability of LQ games with increasing population sizes in the set-up of [13] is studied in [21] analyzing $2N$-coupled steady-state Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck-Kolmogorov (FPK) equations, where each player’s control is restricted to be local state feedback from the beginning. Some algebraic conditions are obtained. However, it requires some restrictions on the model parameters, including symmetric state coefficients in dynamics.

Note that the top-down approach can also be used to solve the LQ mean field game [3]. In part II [11] of this paper, we investigate the relation between the top-down approach and the bottom-up approach as developed in this paper. A surprising finding is that they are not always equivalent.

The organization of the paper is as follows. Section II describes the LQ Nash game together with its solution via dynamic programming and Riccati equations. Section III presents the necessary and sufficient condition for asymptotic solvability, for which we give the proof in Section IV. Section V presents further mean field limits related to the dynamic programming equation and derives decentralized strategies. An illustrative example is provided in Section VI.

This work was supported by Natural Sciences and Engineering Research Council (NSERC) of Canada. In Proc. 23rd MTNS, 2018, pp. 489-495.

M. Huang is with the School of Mathematics and Statistics, Carleton University, Ottawa, K1S 5B6 ON, Canada (mhuang@math.carleton.ca). M. Zhou is with the School of Mathematics and Statistics, Carleton University, Ottawa, K1S 5B6 ON, Canada (mengjiezhou@cmail.carleton.ca).
Section VII concludes the paper.

II. THE LQ NASH GAME

Consider a population of $N$ players (or agents) denoted by $\omega_i$, $1 \leq i \leq N$. The state process $X_i(t)$ satisfies the following stochastic differential equation (SDE)

$$dX_i(t) = (AX_i(t) + Bu_i(t) + Gx_i(t))dt + dW_i(t),$$  
where the state $X_i \in \mathbb{R}^n$, control $u_i \in \mathbb{R}^n$, and $X_i(0) = \frac{1}{N} \sum_{k=1}^{N} X_k(0)$. The initial states $\{X_i(0), 1 \leq i \leq N\}$ are independent with $E(X_i(0) = x_i(0))$ and finite second moment. The $N$ standard $n_2$-dimensional Brownian motions $\{W_i, 1 \leq i \leq N\}$ are independent and also independent of the initial states. For symmetric matrix $S \geq 0$, we may write $x^T S x = \|x\|_S^2$. The cost of player $i$ is given by

$$J_i = E \int_0^T \left( x_i(t)^T - \Gamma x_i(t) - \eta_i^2 \right) dt + E^{\omega_i} \left[ -T \eta_i^2 \right].$$

The constant matrices (or vectors) $A, B, G, D, \Gamma, Q, R, \eta, \eta_i$ above have compatible dimensions, and $Q \geq 0, R > 0, \eta, \eta_i > 0$. For notational simplicity, we only consider constant parameters for the model. Our analysis and results can be easily extended to the case of time-dependent parameters.

Define

$$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{Nn}, \quad W(t) = \begin{bmatrix} W_1(t) \\ \vdots \\ W_N(t) \end{bmatrix} \in \mathbb{R}^{Nn},$$

$$\bar{A} = \text{diag}[A, \ldots, A] + \frac{G}{N} \in \mathbb{R}^{Nn \times Nn},$$

$$\bar{D} = \text{diag}[\eta, \ldots, \eta] \in \mathbb{R}^{Nn \times Nn},$$

$$B_k = \delta_k^N \otimes B \in \mathbb{R}^{N \times N}, \quad 1 \leq k \leq N.$$ We denote by $\delta_k$ a $k \times 1$ matrix with all entries equal to 1, by $\otimes$ the Kronecker product, and by the column vectors $\{e_1^T, \ldots, e_k^T\}$ the canonical basis of $\mathbb{R}^k$. We may use a subscript $n$ to indicate the identity matrix $I_n$ to be $n \times n$.

Now we write (1) in the form

$$dX(t) = (\bar{A}X(t) + \sum_{k=1}^{N} B_k u_k(t))dt + \bar{D}dW(t).$$

Under closed-loop state information, we denote the value function of $\omega_i$ by $V_i(t, x)$, $1 \leq i \leq N$, which corresponds to the initial condition $X(t) = x = (x_1^T, \ldots, x_N^T)^T$. The set of value functions is determined by the system of Hamilton-Jacobi-Bellman (HJB) equations

$$0 = \frac{\partial V_i}{\partial t} + \min_{u_i \in \mathbb{R}^n} \left( \frac{\partial^T V_i}{\partial x} (\bar{A}x + \sum_{k=1}^{N} B_k u_k) + u_i^T R u_i \\
+ |x_i - \Gamma x_i(t) - \eta_i^2|_Q^2 + \frac{1}{2} \text{Tr}\left( \bar{D}^T (V_i)_{xx} \bar{D} \right). \right),$$  

$$V_i(t, x) = |x_i - \Gamma x_i(t) - \eta_i^2|_Q^2, \quad 1 \leq i \leq N,$$

where $x_i^{(N)} = (1/N) \sum_{k=1}^{N} x_k$ and the minimizer is

$$u_i = -\frac{1}{2} R^{-1} B_i^T \frac{\partial V_i}{\partial x}.$$  

Next we substitute $u_i$ into (3).

$$0 = \frac{\partial V_i}{\partial t} + \frac{\partial^T V_i}{\partial x} (\bar{A}x + \sum_{k=1}^{N} B_k R^{-1} B_i^T \frac{\partial V_i}{\partial x})$$

$$+ |x_i - \Gamma x_i(t) - \eta_i^2|_Q^2 + \frac{1}{4} \text{Tr}\left( \bar{D}^T (V_i)_{xx} \bar{D} \right).$$

Denote

$$K_i = [0, \ldots, 0, \eta_i, 0, \ldots, 0] - \frac{1}{N} [\Gamma_i, \Gamma_i, \ldots, \Gamma_i],$$

$$K_i^f = [0, \ldots, 0, \eta_i, 0, \ldots, 0] - \frac{1}{N} [\Gamma_i, \Gamma_i, \ldots, \Gamma_i],$$

$$Q_i = K_i^T Q K_i, \quad Q_i^f = K_i^f Q K_i,$$

where $I_n$ is the $n \times n$ identity matrix. We write

$$|x_i - \Gamma x_i(t) - \eta_i^2|_Q^2 = x_i^T Q_i x - 2x_i^T K_i^T Q_i \eta_i + \eta_i^T Q_i \eta_i,$$

and we can write $|x_i - \Gamma x_i(t) - \eta_i^2|_Q^f$ in a similar form.

Suppose $V_i(t, x)$ has the following form

$$V_i(t, x) = x^T P_i(t) x + 2 S_i^T(t) x + r_i(t),$$

where $P_i$ is symmetric. Then

$$\frac{\partial V_i}{\partial x} = 2 P_i(t) x + 2 S_i(t), \quad \frac{\partial^2 V_i}{\partial x^2} = 2 P_i(t).$$

We substitute (6) and (7) into (5) and derive the equation systems:

$$\begin{cases} P_i(t) = - \left( P_i(t) \bar{A} + \bar{A}^T P_i(t) \right) + \\
\left( P_i(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T P_i(t) \right) + \\
\left( P_i(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T P_i(t) \right)
\right),$$

$$P_i(T) = Q_i,$$

$$\begin{cases} S_i(t) = - \bar{A}^T S_i(t) - P_i(t) B_k R^{-1} B_k^T S_i(t) + \\
\left( P_i(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T S_i(t) \right) + \\
\left( P_i(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T S_i(t) \right)
\right),$$

$$S_i(T) = - K_i^T Q_i \eta_i,$$

$$\begin{cases} r_i(t) = 2 S_i^T(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T S_i(t) + \\
- S_i^T(t) B_k R^{-1} B_k^T S_i(t) + \\
- \eta_i^T Q_i \eta_i - \text{Tr}(D^T (V_i)_{xx} \bar{D})
\right),$$

$$r_i(T) = \eta_i^T Q_i \eta_i.$$  

Remark 1: If (3) has a solution $(P_1, \ldots, P_N)$ on $[\tau, T] \subseteq [0, T]$, such a solution is unique due to the local Lipschitz continuity of the vector field [4]. Taking transpose on both sides of (3) gives an ODE system for $P_i^T$, $1 \leq i \leq N$, which
shows that \((P_1^T, \cdots, P_N^T)\) still satisfies \((\mathcal{S})\). So the ODE system \((\mathcal{S})\) guarantees each \(P_i\) to be symmetric.

**Remark 2:** If \((\mathcal{S})\) has a unique solution \((P_1, \cdots, P_N)\) on \([0, T]\), then we can uniquely solve \((S_1, \cdots, S_N)\) and \((r_1, \cdots, r_N)\) by using linear ODEs.

For the \(N\)-player game, we consider closed-loop perfect state information, so that the state vector \(X(t)\) is available to each player.

**Theorem 1:** Suppose that \((\mathcal{S})\) has a unique solution \((P_1, \cdots, P_N)\) on \([0, T]\). Then we can uniquely solve \((S_1, \cdots, S_N)\) and \((r_1, \cdots, r_N)\) by using linear ODEs.

We present some continuous dependence result of parameterized ODEs. This will play a key role in establishing Theorem 5 below. Consider
\[
\dot{z} = f(t, x), \quad x(0) = z \in \mathbb{R}^K, \\
\dot{y} = g(t, e, t, y),
\]
where \(y(0) = z_e \in \mathbb{R}^K, 0 < \varepsilon \leq 1\).

Let \(\phi(t, x) = f(t, x), \) or \(f(t, x) + g(t, e, t, x).\)

A1) \(\sup_{0 \leq t \leq T} |f(t, 0)| + |g(t, e, 0)| \leq C_1.\)

A2) \(\phi(t, x)\) is Lebesgue measurable for each fixed \(x \in \mathbb{R}^K.\)

A3) For each \(t \in [0, T], \) \(\phi(t, x) : \mathbb{R}^K \to \mathbb{R}^K\) is locally Lipschitz continuous in \(x,\) uniformly with respect to \((t, e, t, \cdot).\) i.e., for any fixed \(r > 0,\) and \(x, y \in B_r(0)\) which is the open ball of radius \(r\) centered 0,
\[
|\phi(t, x) - \phi(t, y)| \leq \text{Lip}(r)|x - y|,
\]
where \(\text{Lip}(r)\) depends only on \(r,\) not on \(e, t \in [0, T].\)

A4) For each fixed \(r > 0,\)
\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T, y \in B_r(0)} |g(t, e, t, y)| = 0, \quad \lim_{\varepsilon \to 0} |z_e - z| = 0.
\]

If the solutions to \((14)\) and \((15),\) denoted by \(x^\varepsilon(t)\) and \(y^\varepsilon(t),\) exist on \([0, T]\), they are unique by the local Lipschitz condition; for \((14)\) in this case denote \(\bar{\delta}_\varepsilon = \int_0^T |g(t, \varepsilon, t, x^\varepsilon(t))| dt,\) which converges to 0 as \(\varepsilon \to 0\) due to A4).

**Theorem 4:** i) If \((14)\) has a solution \(x^\varepsilon(\cdot)\) on \([0, T],\) then there exists \(0 < \bar{\varepsilon} \leq 1\) such that for all \(0 < \varepsilon \leq \bar{\varepsilon}, (15)\) has a solution \(y^\varepsilon(\cdot)\) on \([0, T]\) and
\[
\sup_{0 \leq t \leq T} |y^\varepsilon(t) - x^\varepsilon(t)| = O(|z_e - z| + \bar{\delta}_\varepsilon). \tag{16}
\]

ii) Suppose there exists a sequence \(\{e_i, i \geq 1\}\) where \(0 < e_i \leq 1\) and \(\lim_{i \to \infty} e_i = 0\) such that \((15)\) with \(\varepsilon = e_i\) has a solution \(y^\varepsilon(\cdot)\) on \([0, T]\) and \(\sup_{0 \leq t \leq T} |y^\varepsilon(t)| \leq C_2\) for some constant \(C_2.\) Then \((14)\) has a solution on \([0, T].\)

**Proof:** See Appendix B.

**Remark 3:** If \((14)\) and \((15),\) are replaced by matrix ODEs and (or) a terminal condition at \(T\) is used in each equation, the results in Theorem 4 still hold.

Let
\[
M = BR^{-1}B^T.
\]

Before presenting further results, we introduce two Riccati ODEs:
\[
\begin{aligned}
\dot{A}_1 &= A_1 M A_1 - (A_1 A + A^T A_1) - Q, \\
A_1(T) &= Q_f,
\end{aligned}
\tag{17}
\]
and
\[
\begin{cases}
\dot{A}_2 = A_1 M A_2 + A_2 M A_1 + A_2 M A_2 \\
-(A_1 G + A_2 (A + G) + A^T A_2) + Q G, \\
A_2(T) = -Q_f I_f.
\end{cases}
\] (18)

Note that (17) is the standard Riccati ODE in LQ optimal control and has a unique solution $A_1$ on $[0, T]$. Equation (18) is a non-symmetric Riccati ODE where $A_1$ is now treated as a known function. We state the main theorem on asymptotic solvability. The proof is postponed to Section IV.

**Theorem 5:** The sequence of games in (1)-(2) has asymptotic solvability if and only if (18) has a unique solution on $[0, T]$.

Our method of proving Theorem 5 is to re-scale by definition
\[
A_1^N = \Pi_1(t), \quad A_2^N = N \Pi_2(t), \quad A_3^N = N^2 \Pi_3(t),
\] (19)
and examine their ODE system. We introduce the additional equation
\[
\begin{cases}
\dot{A}_3 = A_1^N M A_2 + A_3 M A_1 + A_1 M A_3 + A_1^N M A_2 + A_2^N M A_3 \\
-(A_1^N G + G^T A_2 + A_3 (A + G) + (A^T + G^T) A_3) \\
-
\end{cases} \quad \Gamma^T \dot{Q} G, \\
A_3(T) = \Gamma_f^T Q_f I_f.
\] (20)

Note that after (17) and (18) are solved on $[0, T]$ or otherwise on a maximal existence interval for the latter, (20) becomes a linear ODE.

**Theorem 6:** Suppose (18) has a solution on $[0, T]$. Then we have
\[
\sup_{0 \leq t \leq T} \left| |\Pi_1 - A_1| + |N \Pi_2 - A_2| + |N^2 \Pi_3 - A_3| \right| = O(1/N).
\]

**Proof:** The bound follows from Theorem 4) by use of $g_1, g_2, g_3$ and the initial conditions which appear in the equations of $A_1^N, A_2^N, A_3^N$.

IV. PROOF OF THEOREM 5

Note that $\Pi_3 = \Pi_4$. We rewrite the system of (A.3), (A.4) and (A.5) by use of a set of new variables
\[
A_1^N = \Pi_1(t), \quad A_2^N = N \Pi_2(t), \quad A_3^N = N^2 \Pi_3(t).
\]

Here and hereafter $N$ is used as a superscript in various places. This should be clear from the context. We can determine functions $g_k, 1 \leq k \leq 3,$ and obtain
\[
\begin{align*}
A_1^N &= A_1^N M A_1^N - (A_1^N A + A^T A_1^N) - Q \\
&\quad + g_1(1/N, A_1^N, A_2^N), \\
A_1^N(T) &= (I - \Gamma^T N) Q_f (I - \Gamma_f N),
\end{align*}
\] (21)
\[
\begin{align*}
A_2^N &= A_1^N M A_2^N + A_2^N M A_1^N + A_2^N M A_2^N \\
&\quad - (A_1^N G + A_2^N (A + G) + A^T A_2^N) + Q G \\
&\quad + g_2(1/N, A_1^N, A_2^N), \\
A_2^N(T) &= -(I - \Gamma^T N) Q_f I_f,
\end{align*}
\] (22)
\[
\begin{align*}
A_3^N &= (A_2^N)^T M A_2^N + A_3^N M A_1^N + A_3^N M A_2^N \\
&\quad + A_1^N M A_2^N + (A_2^N)^T M A_4^N \\
&\quad - (A_2^N G + A_2^N (G + A) + A^T A_2^N) + Q G \\
&\quad - \Gamma^T Q G \\
&\quad + g_3(1/N, A_2^N, A_3^N), \\
A_3^N(T) &= \Gamma_f^T Q_f I_f.
\end{align*}
\] (23)

In particular, we can determine
\[
g_1 = \frac{1}{N}(1 - \frac{1}{N})(A_1^N M A_2^N + (A_2^N)^T M (A_2^N)^T) \\
- \frac{1}{N}(A_1^N G + G^T A_2^N) \\
- \frac{1}{N}(1 - \frac{1}{N})(A_2^N G + G^T (A_2^N)^T) \\
+ \frac{1}{N}(\Gamma^T Q + Q G) - \frac{1}{N^2} \Gamma^T Q G.
\]

The expressions of $g_2, g_3$ can be determined in a similar way and the detail is omitted here.

Note that if (18) has a unique solution on $[0, T]$, we can uniquely solve $A_3$. In view of $g_1, g_2, g_3$ and the terminal conditions in (21)-(23), by Theorem 4 and Remark 5 we obtain the desired result.

V. DECENTRALIZED CONTROL

**Proposition 7:** Suppose that (8) has a solution $(P_1, \cdots, P_N)$ on $[0, T]$. Then $S_i(t)$ in (9) has the form
\[
S_i(t) = [\theta_2^i(t), \cdots, \theta_2^i(t), \cdots, \theta_2^i(t)]^T,
\] (24)
in which the $i$th sub-vector is $\theta_i(t) \in \mathbb{R}^n$ and the remaining sub-vectors are $\theta_j(t) \in \mathbb{R}^n$. Moreover, $r_1 = \cdots = r_N$.

**Proof:** We can show that
\[
(J^T S_i) = J^T S_i,
\]
which satisfies (9) as $(S_1, \cdots, S_N)$ does. Hence $S_1 = J^T S_i$. We can further show $S_1 = J^T S_i$. Following the argument in the proof of Lemma A.1, we obtain the representation (24).

By (24), we further obtain
\[
r_i(t) = \theta_i^T M \theta_i + 2(N - 1) \theta_i^T M \theta_i - Tr(D^T \Pi_1 D) \\
- (N - 1) Tr(D^T \Pi_1 D) - \eta^T Q \eta
\] (25)

for all $i$, so that $r_1 = \cdots = r_N$. 

\[
\]
Recalling $M = BR^{-1}B^T$, we derive

$$\dot{\theta}_1(t) = \Pi_1 M \theta_1 + (N - 1) (\Pi_2 M \theta_1 + \Pi_2^T M \theta_2) - (A^T + G^T) \theta_1 - \frac{N - 1}{N} G^T \theta_2$$

$$+ (I - \frac{\Gamma^T}{N}) Q \eta,$$

$$\theta_1(T) = -(I - \frac{\Gamma^T}{N}) Q \eta_f,$$  \hspace{1cm} (26)

$$\dot{\theta}_2(t) = (\Pi_2^T + (N - 1) \Pi_3) M \theta_1$$

$$+ (\Pi_1 + (N - 2) \Pi_2^T) M \theta_2 - \frac{1}{N} G^T \theta_1$$

$$- (A^T + \frac{N - 1}{N} G^T) \theta_2 - \frac{1}{N} \Gamma^T Q \eta,$$  \hspace{1cm} (27)

$$\theta_2(T) = \frac{1}{N} \Gamma^T Q \eta_f.$$

Let

$$\chi^N_1(t) = \theta_1(t), \quad \chi^N_2(t) = N \theta_2(t).$$

We may write the ODEs of $\chi^N_1(t)$ and $\chi^N_2(t)$, which have the limiting form:

$$\dot{\chi}_1(t) = (A_1 M + A_2 M - A^T) \chi_1 + Q \eta,$$  \hspace{1cm} (28)

and

$$\dot{\chi}_2(t) = ((A_2 + A_4) M - G^T) \chi_1$$

$$+ ((A_1 + A_2) M - (A^T + G^T)) \chi_2 - \Gamma^T Q \eta,$$  \hspace{1cm} (29)

$$\chi_2(T) = \Gamma^T Q \eta_f.$$

For (25), we have the limiting form

$$\dot{r}(t) = \chi^T M \chi_1 + 2 \chi^T M \chi_1 - Tr(D^T \tilde{A} D) - \eta^T Q \eta,$$

$$r(T) = \eta^T Q \eta_f.$$

**Proposition 8:** If asymptotic solvability holds,

$$\sup_{0 \leq t \leq T} \left( |\theta_1(t) - \chi_1(t)| + |N \theta_2(t) - \chi_2(t)| \right) = O(1/N).$$

**Proof:** Under asymptotic solvability, we uniquely solve $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \chi_1, \chi_2)$ on $[0, T]$. We obtain (30) by writing the ODE system of $(A_1^N, A_2^N, A_3^N, \chi^N_1, \chi^N_2)$ and next applying Theorem 4.

**A. Decentralized control and mean field dynamics**

By Theorem 11 the strategy of player $i$ is

$$u_i = -R^{-1}B^T \left( \Pi_i(t) X_i + \Pi_2(t) \sum_{j \neq i} X_j + \theta_1(t) \right).$$

(31)

The closed-loop equation of $X_i$ is now given by

$$dX_i(t) = \left( AX_i - M \left( \Pi_1(t) X_i + \Pi_2(t) \sum_{j \neq i} X_j + \theta_1(t) \right) + \Gamma^T Q \eta \right) dt + DdW_i.$$  \hspace{1cm} (32)

We introduce

$$\frac{d\tilde{X}}{dt} = (A - M(\tilde{A}_1 + \tilde{A}_2) + G) \tilde{X} - M\chi_1(t),$$

where $\tilde{X}(0) = x_0$, and further approximate $X^N(t) \rightarrow \tilde{X}$. When $N \rightarrow \infty$, we obtain the decentralized control law

$$u^N_i = -R^{-1}B^T (\tilde{A}_1(t) X_i(0) + \tilde{A}_2(t) \tilde{X} + \chi_1(t)).$$

The next lemma provides an error estimate for the mean field approximation.

**Proposition 9:** Suppose $E \sup_{0 \leq t \leq T} |X_i(t)|^2 \leq C$ for some fixed $C$ and $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N} E X_i(0) = x_0$. Then

$$\sup_{0 \leq t \leq T} E |X^N(t) - \tilde{X}(t)|^2 = O\left( \frac{1}{N} \sum_{i=1}^{N} E X_i(0) - x_0 \right)^2 + 1/N).$$

**Proof:** We first write the SDE for $X^N(t)$ and find the explicit expression of $X^N(t) - \tilde{X}(t)$. The proposition follows from elementary estimates by use of Theorem 5 and Proposition 8.

**VI. An Example**

**Example** Take $A = 0.2$, $B = G = Q = R = 1, \Gamma = 1.2, \Gamma_f = 0, \eta_f = 0, T = 3$. Consider the equation system (17) and (18):

$$\tilde{A}_1 = \tilde{A}_2 = 0.4 A_1 - 1, \quad \tilde{A}_1(T) = 0,$$

$$\tilde{A}_2 = 2A_1 A_2 + A_2 - (A_1 + 1.4 A_2) + 1.2, \quad \tilde{A}_2(T) = 0.$$  \hspace{1cm} (30)

$A_1$ can be solved explicitly. It is numerically illustrated in Fig. 1 that $A_2$ does not have a solution on the whole interval $[0, T]$, implying no asymptotic solvability.

**VII. Conclusion**

This paper studies the asymptotic solvability problem for LQ mean field games and obtains a necessary and sufficient condition via a non-asymmetric Riccati ODE. The re-scaling technique used in this paper can be extended to more general models in terms of dynamics and interaction patterns [7], [10]. This will be reported in our future work.

**Appendix A: Proof of Theorem 3**

We prove the following lemma first.

**Lemma A.1:** We assume that (8) has a solution $(P_1(t), \cdots, P_N(t))$ on $[0, T]$. Then the following holds.
i) \( P_1(t) \) has the representation

\[
P_1(t) = \begin{bmatrix}
\Pi_1(t) & \Pi_2(t) & \Pi_3(t) & \cdots & \Pi_N(t)
\end{bmatrix}
\]

where \( \Pi_1, \Pi_2 \) and \( \Pi_N \) are \( n \times n \) symmetric matrices.

ii) For \( i > 1 \), \( P_i(t) = J^T_{i} P_i(t) J_{i} \).

Proof: Step 1. For \( 1 \leq i \leq N \), denote

\[
P_i = (P^{jk})_{1 \leq j,k \leq N},
\]

where each \( P^{jk} \) is an \( n \times n \) matrix. Define the new functions

\[
J^T_{23} P_i J_{23}, \quad i = 1, \ldots, N.
\]

By elementary calculations, we see that

\[
(J^T_{23} P_i J_{23}, J^T_{23} P_3 J_{23}, J^T_{23} P_2 J_{23}, J^T_{23} P_4 J_{23}, \ldots, J^T_{23} P_N J_{23})
\]
satisfies (8) together with its terminal condition as \((P_i(t), \ldots, P_N(t))\) does. Hence \( P_1 = J^T_{23} P_i J_{23} \), which implies

\[
P_{12}^{12} = P_{13}^{13}, \quad P_{12}^{22} = P_{13}^{33}, \quad P_{12}^{33} = P_{13}^{12}.
\]

(A.2)

Repeating the above by using \( J_{2k} \), \( k \geq 4 \), in place of \( J_{23} \), we obtain

\[
P_{12}^{12} = P_{13}^{13} = \cdots = P_{1N}^{1N}, \quad P_{12}^{22} = P_{13}^{33} = \cdots = P_{1N}^{N1},
\]

We similarly obtain \( P_1 = J^T_{24} P_1 J_{24} \), and this gives

\[
P_{12}^{12} = P_{13}^{24}.
\]

(A.2)

Repeating the similar argument, we can check all other remaining off-diagonal submatrices. Since \( P_1 \) is symmetric (also see Remark [1]), \( (P_1^{jk})^T = P_1^{jk} \). \( P_1^{jk} \) is symmetric by \( (A.2) \). By the above method we can show that all off-diagonal submatrices on neither the first row nor the first column are identical and symmetric. Therefore we obtain the representation of \( P_1 \).

Step 2. We can verify that both

\[
(J^T_{12} P_2 J_{12}, J^T_{12} P_3 J_{12}, J^T_{12} P_3 J_{12}, \ldots, J^T_{12} P_N J_{12})
\]

and \( (P_1(t), \ldots, P_N(t)) \) satisfy (8). Hence \( P_2 = J^T_{12} P_1 J_{12} \). All other cases can be similarly checked.

Proof of Theorem [4] By Lemma [A.1] we have

\[
\Pi_1(t) = \Pi_1 M \Pi_1 + (N-1)(\Pi_2 M \Pi_2 + \Pi_2 M \Pi_2^T) + (N-2)(\Pi_3 M \Pi_3 + \Pi_3 M \Pi_3^T)
\]

\[
- \left( \Pi_1 (A + \frac{G}{N}) + (A^T + \frac{G^T}{N}) \Pi_1 \right)
\]

\[
- \left( 1 - \frac{1}{N} \right) (\Pi_2 G + G^T \Pi_2^T)
\]

\[
- \left( I - \frac{\Gamma^T}{N} \right) (\Gamma - \frac{\Gamma}{N}),
\]

(A.3)

\[
\Pi_1(T) = \left( I - \frac{\Gamma^T}{N} \right) \Gamma (I - \frac{\Gamma}{N}),
\]

and

\[
\Pi_2(t) = \Pi_1 M \Pi_2 + \Pi_2 M \Pi_1 + \Pi_2^T M \Pi_3
\]

\[
+ (N-2)(\Pi_3 M \Pi_2 + \Pi_3^T M \Pi_3)
\]

\[
+ \left( I - \frac{\Gamma^T}{N} \right) \Gamma \left( I - \frac{\Gamma}{N} \right),
\]

(A.4)

\[
\Pi_3(t) = \Pi_1^2 \Pi_1 + \Pi_3 M \Pi_1 + \Pi_3 M \Pi_3
\]

\[
+ (N-2)(\Pi_3 M \Pi_2 + \Pi_3^T M \Pi_3)
\]

\[
- \left( I - \frac{\Gamma^T}{N} \right) \Gamma \left( I - \frac{\Gamma}{N} \right),
\]

(A.5)

\[
\Pi_4(T) = \left( I - \frac{\Gamma^T}{N} \right) \Gamma (I - \frac{\Gamma}{N}),
\]

\[
\Pi_3(T) = \Pi_1 \Pi_2 + \Pi_4 M \Pi_1 + \Pi_4 M \Pi_4
\]

\[
\Pi_5(T) = (\Pi_1^2 + \Pi_4 M \Pi_1 + \Pi_4 M \Pi_4)
\]

Then we can further show that \( \Pi_3 - \Pi_4 \) satisfies a linear ODE when \( \Pi_1 \) and \( \Pi_2 \) are fixed and that \( \Pi_3(T) - \Pi_4(T) = 0 \). This gives \( \Pi_3 = \Pi_4 \) on \( [0,T] \).
exists $0 < \epsilon < \epsilon_0$ such that (b) occurs with the corresponding $0 < t_{\text{max}} < T$. Since $[0,t_{\text{max}}]$ is the maximal existence interval, we have $\lim_{t \to t_{\text{max}}} \left| y^f(t) \right| = \infty$ [4]. Therefore for some $0 < t_m < t_{\text{max}}$,
\[ y^f(t_m) \in \partial B_{2C_z}(0), \quad \text{(B.1)} \]
and
\[ y^f(t) \in B_{2C_z}(0), \quad \forall 0 \leq t < t_m. \quad \text{(B.2)} \]
For $t < t_{\text{max}}$, we have
\[
y^f(t) - x^i(t) = \epsilon_z - z + \int_0^t \left[ f(\tau, y^f(\tau)) + g(\epsilon, \tau, y^f(\tau)) - f(\tau, x^i(\tau)) \right] d\tau.
\]
Denote $\zeta(\tau) = f(\tau, y^f(\tau)) + g(\epsilon, \tau, y^f(\tau)) - f(\tau, x^i(\tau))$ and it follows that
\[
\left| \zeta(\tau) \right| = \left| \zeta(\tau) - g(\epsilon, \tau, x^i(\tau)) + g(\epsilon, \tau, x^i(\tau)) \right| 
\leq \text{Lip}(2C_z) \left| y^f(\tau) - x^i(\tau) \right| + \left| g(\epsilon, \tau, x^i(\tau)) \right|.
\]
Now for $0 \leq t < t_m$, we find $\epsilon > 0$ such that for all $\epsilon \leq \epsilon$, 
\[
|y^f(t) - x^i(t)| \leq \left( \epsilon_z - z \right) + \delta_\epsilon + \int_0^t \text{Lip}(2C_z) |y^f(\tau) - x^i(\tau)| d\tau.
\]
Note that $\delta_\epsilon = \int_0^t |g(\epsilon, \tau, x^i(\tau))| d\tau \to 0$ as $\epsilon \to 0$. By Gronwall’s lemma,
\[
|y^f(t) - x^i(t)| \leq (\delta_\epsilon + |\epsilon_z - z|) e^{\text{Lip}(2C_z) t},
\]
for all $t \leq t_m$. We can find $\delta > 0$ such that for all $\epsilon \leq \delta$, 
\[
(\delta_\epsilon + |\epsilon_z - z|) e^{\text{Lip}(2C_z) t} < \frac{C_3}{3}.
\]
Then for all $0 \leq t < t_m$, $y^f(t) \in B_{3C_z/2}(0)$, which is a contradiction to (B.1). We conclude for all $0 < \epsilon \leq \delta$, $y^f$ is defined on $[0,T]$. Next, (16) follows readily.

ii) We have
\[ y^E(t) = \epsilon_\ell + \int_0^t \left[ f(\tau, y^E(\tau)) + g(\epsilon, \tau, y^E(\tau)) \right] d\tau, \quad \text{(B.3)} \]
and
\[
\left| f(\tau, y^E(\tau)) + g(\epsilon, \tau, y^E(\tau)) \right| 
\leq \text{Lip}(C_2) \left| y^E(\tau) \right| + \left| f(\tau, 0) + g(\epsilon, \tau, 0) \right| 
\leq \text{Lip}(C_2) \left| y^E(\tau) \right| + C_1 
\leq \text{Lip}(C_2) C_2 + C_1, \quad \text{(B.4)}
\]
where $C_1$ is given in A1).

By (B.3) - (B.4), the functions $\{y^E(\cdot), i \geq 1\}$ are uniformly bounded and equicontinuous. By Ascoli’s lemma, there exists a subsequence $\{y^{E_j}(\cdot), j = 1, 2, 3, \ldots\}$ such that $y^{E_j}$ converges to $y^E \in C([0,T], \mathbb{R}^K)$ uniformly on $[0,T]$. Hence,
\[ y^E(t) = z + \int_0^t f(\tau, y^E(\tau)) + g(\epsilon, \tau, y^E(\tau)) d\tau \]
for all $t \in [0,T]$. So (13) has a solution.

The proof in part i) follows the method in [20, sec. 2.4] and [22, pp. 486].

References
[1] T. Basar and G.J. Olsder. Dynamic Noncooperative Game Theory. SIAM, Philadelphia, 1999.
[2] A. Bensoussan, J. Frehse, and P. Yam. Mean Field Games and Mean Field Type Control Theory. Springer, New York, 2013.
[3] P.E. Caines, M. Huang, and R.P. Malhame. Mean Field Games, In Handbook of Dynamic Game Theory, T. Basar and G. Zaccour Eds., Springer, Berlin, 2017.
[4] J. K. Hale. Ordinary Differential Equations, New York: Wiley, 1969.
[5] M. Herty, L. Pareschi, and S. Steffen. Mean-field control and Riccati equations. Networks and Heterogeneous Media, vol. 10, no. 3, pp. 699-715, 2015.
[6] M. Huang. Stochastic control for distributed systems with applications to wireless communications, Ph.D. dissertation, Dep. Elect. Comp. Eng., McGill Univ., Montreal, Canada, 2003.
[7] M. Huang. Large-population LQG games involving a major player: the Nash certainty equivalence principle. SIAM J. Control Optim., vol. 48, no. 5, pp. 3318-3353, 2010.
[8] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized $\epsilon$-Nash equilibria. IEEE Trans Autom. Contr., vol. 52, pp. 1560-1571, Sep. 2007.
[9] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Commun. Inform. Systems, vol. 6, no. 3, pp. 221-252, 2006.
[10] M. Huang and S.L. Nguyen. Stochastic control of mean field models with mixed players. Proc. 18th IFAC World Congress, Milan, Italy, Aug. 2011.
[11] M. Huang and M. Zhou. Linear quadratic mean field games – Part II: from asymptotic solvability to the fixed point approach, under review for the 57th IEEE CDC, submitted Mar 2018.
[12] D. Lacker. A general characterization of the mean field limit for stochastic differential games. Probab. Theory Relat. Fields, 165:581-648, 2016. DOI: 10.1007/s00440-015-0641-9.
[13] J.-M. Lasry and P.-L. Lions. Mean field games. Japan. J. Math., vol. 2, no. 1, pp. 229-260, 2007.
[14] T. Li and J.-F. Zhang. Asymptotically optimal decentralized control for large population stochastic multiagent systems. IEEE Trans. Auto. Control, vol. 53, no. 7, pp. 1643-1660, 2008.
[15] E. A. Martens, E. Barreto, S. H. Strogatz, E. Ott, P. So, and T. M. Antonsen. Exact results for the Kuramoto model with a bimodal frequency distribution. Physical Review E., vol. 79, 026204, 2009. DOI: 10.1103/PhysRevE.79.026204.
[16] J. M. Moon and T. Basar. Linear-quadratic stochastic differential stackelberg games with a high population of followers. Proc. 54th IEEE CDC, Osaka, Japan, pp. 2270-2275, Dec. 2015.
[17] E. Ott and T. M. Antonsen. Low dimensional behavior of large systems of globally coupled oscillators. Chaos, vol. 18, 037113, 2008.
[18] G. P. Papavassilopoulos. Limiting behavior of LQ deterministic infinite horizon Nash games with symmetric players as the number of players goes to infinity. arXiv:1403.3300, 2014.
[19] D. Pazó and E. Montbrió. Low-dimensional dynamics of populations of pulse-coupled oscillators. Physical Review X, DOI: 10.1103/PhysRevX.4.011009, 2014.
[20] L. Perko. Differential Equations and Dynamical Systems, 2nd ed., New York: Springer, 1996.
[21] F. S. Priuli. Linear-quadratic $N$-person and mean-field games: infinite horizon games with discounted cost and singular limits. Dyn. Games Appl. vol.5, pp. 397-419, 2015.
[22] E.D. Sontag. Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd ed., New York: Springer, 1998.