A SPINORIAL ANALOGUE OF AUBIN’S INEQUALITY

B. AMMANN, J.-F. GROSJEAN, E. HUMBERT AND B. MOREL

Abstract. Let \((M, g, \sigma)\) be a compact Riemannian spin manifold of dimension \(\geq 2\). For any metric \(\tilde{g}\) conformal to \(g\), we denote by \(\tilde{\lambda}\) the first positive eigenvalue of the Dirac operator on \((M, \tilde{g}, \sigma)\). We show that

\[
\inf_{\bar{g} \in [g]} \tilde{\lambda} \Vol(M, \tilde{g})^{1/n} \leq \left(\frac{n}{2}\right) \Vol(S^n)^{1/n}.
\]

This inequality is a spinorial analogue of Aubin’s inequality, an important inequality in the solution of the Yamabe problem. The inequality is already known in the case \(n \geq 3\) and in the case \(n = 2, \ker D = \{0\}\). Our proof also works in the remaining case \(n = 2, \ker D \neq \{0\}\). With the same method we also prove that any conformal class on a Riemann surface contains a metric with \(2\tilde{\lambda}^2 \leq \tilde{\mu}\), where \(\tilde{\mu}\) denotes the first positive eigenvalue of the Laplace operator.

MSC 2000: 53A30, 53C27 (Primary) 58J50, 58C40 (Secondary)

Contents

1. Introduction 2
2. A variational formulation for the spin conformal invariant 3
3. The Bourguignon-Gauduchon-trivialization 3
4. Development of the metric at the point \(p\) 6
5. The test spinor 8
6. The proof of Theorem 1.1 for \(n \geq 3\) 9
7. The case \(n = 2\) 17
References 21

1. INTRODUCTION

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\). We assume that \(M\) is spin, and we fix a spin structure \(\sigma\) on \(M\). For any metric \(\tilde{g}\) in the conformal class \([g]\) of \(g\), we write \(\tilde{\lambda}^+ (\tilde{g})\) for the smallest positive eigenvalue of the Dirac operator with respect to \((M, \tilde{g}, \sigma)\). We define

\[
\lambda_{\min}^+ (M, g, \sigma) = \inf_{\bar{g} \in [g]} \lambda^+_1 (\tilde{g})\Vol(M, \tilde{g})^{1/n}.
\]

If \((M, g)\) is the round sphere \(S^n\) equipped with the unique spin structure on \(S^n\), we simply write \(\lambda_{\min}^+ (S^n)\). It was proven in [Lot86] (\(\ker D = \{0\}\)) and [Am03b] (\(\ker D \neq \{0\}\)) that

\[
\lambda_{\min}^+ (M, g, \sigma) > 0.
\]

Several articles have been devoted to the study of this spin-conformal invariant. A non-exhaustive list is [Hij86, Lot86, B"ar92, Am03a]. In this article we will prove the following.

**Theorem 1.1.** Let \((M, g, \sigma)\) be a compact spin manifolds of dimension \(n \geq 2\). Then,

\[
\lambda_{\min}^+ (M, g, \sigma) \leq \lambda_{\min}^+ (S^n) = \frac{n}{2} \frac{\omega_n}{\omega_{n+2}}.
\]
where $\omega_n$ stands for the volume of the standard sphere $S^n$.

The proof relies on constructing a suitable test spinor. The main idea of this construction is to start with a Killing spinor on the round sphere. Under stereographic projection this spinor then yields a solution to the equation $D\psi = c|\psi|^{2/(n-1)}\psi$ on flat $\mathbb{R}^n$. This solution will be rescaled, cut off and finally transplanted to a neighborhood of a given point $p$ of the manifold $M$. For this transplantation we carry out several calculations in a well-adapted trivialization of the spinor bundle.

The first steps in our proof are common in all dimensions. However, in some final estimates one has to distinguish between the cases $n \geq 3$ and $n = 2$.

In dimension $n \geq 3$ two other proofs for the theorem have already been published: a geometric construction [Am03b, Theorem 3.1] and a proof using an invariant for non-compact spin manifolds [Gro06]. In these dimensions, it is mostly the method of proof that is interesting and helpful: the trivialization presented here has less terms in the Taylor expansion than the trivialization by using parallel transport along radial geodesics. Some formulae of our article also enter in [Gro06]. The calculations of our article also provide helpful formulae used in [AAM03], [AH03] and [Rau06].

The main interest of the theorem however lies in the case $n = 2$. The easier subcase $n = 2$, $\ker D = \{0\}$ could be dealt with by a modification of the geometric proof [Am03b, Theorem 3.2], but the subcase $n = 2$ and $\ker D \neq \{0\}$ remained open for longtimes. Große’s method fails as well for $n = 2$ as the contribution of a cut-off function in [Gro06, Lemma 2.1(ii)] is too large. We assume that her method can be adapted by using a logarithmic cut-off function, but the details have not been worked out yet.

Our method of proof in dimension 2 actually admits applications to other problems as well. For example, one obtains the following proposition that provides a negative answer to a question raised in [AAF99].

**Proposition 1.2** (See Corollary 7.2). Let $(M, g)$ be a Riemann surface with fixed spin structure $\sigma$. For any metric $\bar{g}$ in the conformal class $[g]$, let $\mu_1(\bar{g})$ be the first positive eigenvalue of the Laplacian, and let $\lambda_1^+(\bar{g})$ be the first positive eigenvalue of the Dirac operator on $(M, \bar{g}, \sigma)$. Then

$$\inf_{\bar{g} \in [g]} \frac{\lambda_1^+(\bar{g})^2}{\mu_1(\bar{g})} \leq \frac{1}{2}.$$ 

The article is organized as follows: in Section 2, we recall that $\lambda_{\text{min}}^+(M, g, \sigma)$ has a variational characterization. Then, in Section 3 we introduce a well-adapted local trivialization of the spinor bundle, called the Bourguignon-Gauduchon-trivialization. In Section 4 we calculate the first terms of the Taylor development of the Dirac operator in this trivialization. In the following, i.e. in Section 5, we construct a good test spinor using a Killing spinor on $S^n$, and then in Section 6, we set this spinor in the functional to get Theorem 1.1 in dimension $n \geq 3$. In the last section, i.e. in Section 7, we describe the modifications for the case $n = 2$ and prove the proposition.

**Acknowledgments.** The authors want to thank Oussama Hijazi (Nancy) for his support and encouragement for working at this article. B. Ammann wants to thank C. Bär for some discussions about related subjects. We thank the referee for drawing our attention to the article [Tak02]. B. Ammann thanks the Max-Planck institute for gravitational physics, at Potsdam-Golm for its hospitality.

2. A variational formulation fo the spin conformal invariant

For a section $\psi \in \Gamma(\Sigma M)$ we define

$$J(\psi) = \left( \frac{\int_M |D\psi|^2_{g} \, v_g}{\int_M (D\psi, \psi)_{g} \, v_g} \right)^{\frac{n-1}{2}}.$$
At some places we will write $J_g$ instead of $J$ inorder to indicate, that the functional is defined with respect to $g$. Based on some idea from [Lot86], Ammann proved in [Am03a] that
\[
\lambda_{\min}^+(M, g, \sigma) = \inf_{J} J(\psi)
\]
where the infimum is taken over the set of smooth spinor fields for which
\[
\left( \int_M \langle D\psi, \psi \rangle v_g \right) > 0.
\]
Hence, to prove Theorem 1.1, we are reduced to find a smooth spinor field $\psi$ satisfying the condition below and such that $J(\psi) \leq \lambda_{\min}^+(\mathbb{S}^n) + \varepsilon$ where $\varepsilon > 0$ is arbitrary small.

3. The Bourguignon-Gauduchon-trivialization

As already explained before, the proof of our main theorem is based on a the construction of a suitable test spinor. We first construct a “good” spinor field of $\mathbb{R}^n$ and then transpose it on the manifold. In order to carry this out, we need to locally identify spinor fields on $(\mathbb{R}^n, g_{\text{eucl}})$ and spinor fields on $(M, g)$. Such an identification will be provided by a well-adapted local trivialization of the spinor bundle of $\Sigma(M, g)$.

If a spin manifold $N$ carries two metrics $g_1$ and $g_2$, then it is a priori unclear how to identify spinors on $(N, g_1)$ and spinors on $(N, g_2)$. Bourguignon and Gauduchon [BG92] constructed a convenient map from the spinor bundle of $(N, g_1)$ to the spinor bundle of $(N, g_2)$ that allows us to identify spinors, and it is this identification that will provide the necessary identification to us. The trivialization will be called Bourguignon-Gauduchon-trivialization.

This trivialization is more efficient than the commonly used “trivialization by parallel transport along radial geodesics”: with respect to the Bourguignon-Gauduchon-trivialization less terms appear in the Taylor expansion in Section 4.

Let $(M, g)$ be a Riemannian manifold with a spin structure $\sigma : \text{Spin}(M, g) \to \text{SO}(M, g)$. Let $(x_1, \ldots, x_n)$ be the Riemannian normal coordinates given by the exponential map at $p \in M$:
\[
\exp_p : U \subset T_p M \cong \mathbb{R}^n \longrightarrow V \subset M \quad (x_1, \ldots, x_n) \longmapsto m
\]

Let
\[
G : V \longrightarrow S^2_+(n, \mathbb{R}) \quad m \longmapsto G_m := (g_{ij}(m))_{ij}
\]
denote the smooth map which associates to any point $m \in V$, the matrix of the coefficients of the metric $g$ at this point, expressed in the basis $(\partial_i := \frac{\partial}{\partial x_i})_{1 \leq i \leq n}$. Since $G_m$ is symmetric and positive definite, there is a unique symmetric and positive definite matrix $B_m$ such that
\[
B_m^2 = G_m^{-1}.
\]
Since
\[
^t(B_m X) G_m (B_m Y) = g_{\text{eucl}}(X, Y), \quad \forall X, Y \in \mathbb{R}^n,
\]
where $g_{\text{eucl}}$ stands for the Euclidean scalar product, we get the following isometry defined by
\[
B_m : (T_{\exp^{-1}_p(m)} U \cong \mathbb{R}^n, g_{\text{eucl}}) \longrightarrow (T_m V, g_m)
\]
\[
(a^1, \ldots, a^n) \longmapsto \sum_{i,j} b_i^j(m) a^i \partial_j(m)
\]
for each point $m \in V$, where $b_i^j(m)$ are the coefficients of the matrix $B_m$ (from now on, we use Einstein’s summation convention). As the matrix $B_m$ depends smoothly on $m$, we can identify the following $\text{SO}_n$-principal bundles:
\[ \text{SO}(U, g_{\text{eucl}}) \xrightarrow{\eta} \text{SO}(V, g) \]
\[ U \subset T_p M \xrightarrow{\exp_p} V \subset M \]
where \( \eta \) is given by the action of \( B \) on each component vector of a frame in \( \text{SO}(U, g_{\text{eucl}}) \). The map \( \eta \) commutes with the right action of \( \text{SO}_n \), therefore the map \( \eta \) can be lifted to the spin structures

\[ \text{Spin}(n) \times U = \text{Spin}(U, g_{\text{eucl}}) \xrightarrow{\eta} \text{Spin}(V, g) \subset \text{Spin}(M, g) \]
\[ U \subset T_p M \xrightarrow{\exp_p} V \subset M \]

Hence, we obtain a map between the spinor bundles \( \Sigma U \) and \( \Sigma V \) in the following way:

\[ \Sigma U = \text{Spin}(U, g_{\text{eucl}}) \times \rho \Sigma_n \rightarrow \Sigma V = \text{Spin}(V, g) \times \rho \Sigma_n \]
\[ \psi = [s, \varphi] \rightarrow \bar{\psi} = [\bar{\eta}(s), \varphi] \]

where \( \rho \) is the complex spinor representation, and where \([s, \varphi] \) denotes the equivalence class of \((s, \varphi)\) under the diagonal action of \( \text{Spin}(n) \).

We now define

\[ e_i := b_i^j \partial_j, \]

so that \((e_1, \ldots, e_n)\) is an orthonormal frame of \((TV, g)\). Denote by \( \nabla \) (resp. \( \tilde{\nabla} \)) the Levi-Civita connection on \((TU, g_{\text{eucl}})\) (resp. \((TM, g)\)) as well as its lift to the spinor bundle \( \Sigma U \) (resp. \( \Sigma V \)). The Christoffel symbols of the second kind \( \tilde{\Gamma}_{ij}^k \) are defined by

\[ \tilde{\Gamma}_{ij}^k := \langle \tilde{\nabla}_{e_i} e_j, e_k \rangle, \]

where, for any point \( m \in V \), and the coefficients of the inverse matrix of \( B_m \) are denoted by \((b^{-1})^j_i(m)\).

**Proposition 3.1.** If \( D \) and \( \bar{D} \) denote the Dirac operators acting respectively on \( \Gamma(\Sigma U) \) and \( \Gamma(\Sigma V) \), then we have

\[ \bar{D}\bar{\psi} = D\psi + \mathbf{W} \cdot \bar{\psi} + V \cdot \bar{\psi} + \sum_{ij} (b_i^j - \delta_i^j) \partial_i \cdot \tilde{\nabla}_{\partial_j} \psi, \]

where \( \mathbf{W} \in \Gamma(\text{Cl} TV) \) and \( V \in \Gamma(TV) \) are defined by

\[ \mathbf{W} = \frac{1}{4} \sum_{i,j,k} b_i^j (\partial_i b_j^k) (b^{-1})^k_i e_i \cdot e_j \cdot e_k \]

and

\[ V = \frac{1}{4} \sum_{i,k} \left( \tilde{\Gamma}_{ik} - \tilde{\Gamma}_{ki} \right) e_k = \frac{1}{2} \sum_{i,k} \tilde{\Gamma}_{ik} e_k \]

where, for any point \( m \in V \), and the coefficients of the inverse matrix of \( B_m \) are denoted by \((b^{-1})^j_i(m)\).

**Proof.** We denote Clifford multiplication on \( \Sigma V \) by \(" \cdot "\). For all spinor field \( \psi \in \Gamma(\Sigma U) \), since \( \bar{\psi} \in \Gamma(\Sigma V) \) and by definition of \( \tilde{\nabla} \) (see e.g. [LM89, Theorem 4.14], [Bär91, I Lemma 4.1]), we have

\[ \tilde{\nabla}_{e_i} \bar{\psi} = \frac{\tilde{\nabla}_{e_i} (\psi)}{4} + \sum_{j,k} \tilde{\Gamma}_{ij}^k e_j \cdot e_k \cdot \bar{\psi}. \]

Taking Clifford multiplication by \( e_i \) on each member of (7) and summing over \( i \) yields

\[ \bar{D}\bar{\psi} = \sum_i e_i \cdot \tilde{\nabla}_{e_i} \psi + \frac{1}{4} \sum_{i,j,k} \tilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \bar{\psi}. \]
Now, using that \( e_i = \sum_j b^j_i \partial_j \) and that \( e_i \cdot \overline{\nabla_{e_i} \psi} = \partial_i \cdot \overline{\nabla_{e_i} \psi} \), we obtain that

\[
\hat{D} \tilde{\psi} = \sum_{i,j} b^j_i \partial_i \cdot \overline{\nabla_{\partial_j} \psi} + \frac{1}{4} \sum_{i,j,k} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k \cdot \tilde{\psi}
\]

and hence,

\[
\hat{D} \tilde{\psi} = \overline{D \psi} + \sum_{i,j} (b^j_i - \delta^j_i) \partial_i \cdot \overline{\nabla_{\partial_j} \psi} + \frac{1}{4} \sum_{i,j,k} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k \cdot \tilde{\psi}.
\]

See also [Pfa02] for a similar formula, worked out in more detail.

Note that by the definition of \( e_k \), we have

\[
\tilde{\Gamma}^{k}_{ij} e_k = \tilde{\Gamma}^{k}_{ij} b^k_i \partial_i.
\]

On the other hand, we compute the Christoffel symbols of the second kind

\[
\tilde{\Gamma}^{k}_{ij} e_k = \nabla_{e_i} e_j = b^l_i \nabla_{\partial_l} (b^j_s \partial_s) = b^l_i (\partial_s \partial_j^s - \delta^j_i) \partial_s + b^l_i b^l_j \Gamma^l_{rs} \partial_r,
\]

where as usually the Christoffel symbols of the first kind \( \Gamma^l_{rs} \) are defined by

\[
\Gamma^l_{rs} \partial_l = \nabla_{\partial_l} \partial_r.
\]

Therefore we have

\[
\tilde{\Gamma}^{k}_{ij} b^l_i = b^l_i (\partial_s \partial_j^s - \delta^j_i) + b^l_i b^l_j \Gamma^l_{rs} \partial_r,
\]

and hence

\[
\tilde{\Gamma}^{k}_{ij} = \left( b^l_i (\partial_s \partial_j^s - \delta^j_i) + b^l_i b^l_j \Gamma^l_{rs} \right) (b^{-1})^k_i.
\]

Now, we can write

\[
\frac{1}{4} \sum_{i,j,k} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k = W + V
\]

where \( W \in \Gamma(\Lambda^3 TV) \) and \( V \in \Gamma(TV) \) are defined by

\[
W = \frac{1}{4} \sum_{i,j,k} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k
\]

and

\[
V = \frac{1}{4} \left( \sum_{i \neq j \neq k \neq i} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k + \sum_{i \neq j \neq k \neq i} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k + \sum_{j \neq i \neq k \neq j} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k + \sum_{i \neq j = k} \tilde{\Gamma}^{k}_{ij} e_i \cdot e_j \cdot e_k \right)
\]

which is (6).

First note that by (8) we have

\[
W = \frac{1}{4} \sum_{i,j,k} \left( b^l_i (\partial_s \partial_j^s - \delta^j_i) (b^{-1})^k_i + b^l_i b^l_j \Gamma^l_{rs} (b^{-1})^k_i \right) e_i \cdot e_j \cdot e_k.
\]

However,

\[
\sum_{i,j,k} b^l_i b^l_j \Gamma^l_{rs} (b^{-1})^k_i e \cdot e_j \cdot e_k = 0
\]

since \( \Gamma^l_{rs} = \Gamma^l_{sr} \) and \( e_i \cdot e_j = -e_j' \cdot e_i \). Therefore we obtain (5). \( \square \)
4. Development of the metric at the point \( p \)

In this section we give the development of the coefficients \( \tilde{\Gamma}^k_{ij} \) in the coordinates \((x_1, \ldots, x_n)\) at the fixed point \( p \in M \).

For any point \( m \in M \), \( r \) denotes the distance from \( p \) to \( m \). Recall that in the neighborhood of \( p \), we have the following development of the metric (see for example [LP87]):

\[
\begin{align*}
g_{ij} &= \delta_{ij} + \frac{1}{3} R_{i\alpha\beta j}(p) x^\alpha x^\beta + \frac{1}{6} R_{i\alpha\beta j;\gamma}(p) x^\alpha x^\beta x^\gamma \\
&\quad + \left( \frac{1}{20} R_{i\alpha\beta j;\gamma\lambda}(p) + \frac{2}{45} \sum_m R_{i\alpha\beta m}(p) R_{j\gamma\lambda m}(p) \right) x^\alpha x^\beta x^\gamma x^\lambda + O(r^5)
\end{align*}
\]

where

\[
R_{ijkl} = \langle \nabla_{e_j} \nabla_{e_k} e_i, e_l \rangle - \langle \nabla_{e_i} \nabla_{e_k} e_j, e_l \rangle - \langle \nabla_{[e_j, e_i]} e_k, e_l \rangle
\]

and where

\[
R_{ijkl;m} = (\nabla R)_{mijkl} \quad R_{ijkl;mn} = (\nabla^2 R)_{nmijkl}
\]

are the covariant derivatives of \( R_{ijkl} \) in direction of \( e_m \) (and \( e_p \)). Therefore we write

\[
G_m = \text{Id} + G_2 + G_3 + O(r^4)
\]

with

\[
\begin{align*}
\left( G_2 \right)_{ij} &= \frac{1}{3} R_{i\alpha\beta j}(p) x^\alpha x^\beta \\
\left( G_3 \right)_{ij} &= \frac{1}{6} R_{i\alpha\beta j;\gamma}(p) x^\alpha x^\beta x^\gamma
\end{align*}
\]

Writing

\[
B_m = \text{Id} + B_1 + B_2 + B_3 + O(r^4)
\]

with

\[
\begin{align*}
\left( B_1 \right)_{ij} &= B_{ij\alpha} x^\alpha \\
\left( B_2 \right)_{ij} &= B_{ij\alpha\beta} x^\alpha x^\beta
\end{align*}
\]

and

\[
\left( B_3 \right)_{ij} = B_{ij\alpha\beta\gamma} x^\alpha x^\beta x^\gamma,
\]

the relation \( B_m^2 G_m = \text{Id} \) yields \( B_1 = 0 \) and

\[
0 = \left( 2B_2 + G_2 \right) + \left( 2B_3 + G_3 \right),
\]

hence

\[
\begin{align*}
b^i_j &= \delta^i_j - \frac{1}{6} R_{i\alpha\beta j} x^\alpha x^\beta - \frac{1}{12} R_{i\alpha\beta j;\gamma} x^\alpha x^\beta x^\gamma + O(r^4) \\
\left( b^{-1} \right)_j^i &= \delta^i_j + \frac{1}{6} R_{i\alpha\beta j} x^\alpha x^\beta + \frac{1}{12} R_{i\alpha\beta j;\gamma} x^\alpha x^\beta x^\gamma + O(r^4)
\end{align*}
\]

Writing

\[
\partial b^i_j = -\frac{1}{6} \left( R_{i\alpha\beta j} + R_{i\alpha j\beta} \right) x^\alpha - \frac{1}{12} \left( R_{i\alpha\beta j;\gamma} + R_{i\alpha j\beta;\gamma} + R_{i\alpha j\beta j;\gamma} \right) x^\alpha x^\beta + O(r^3)
\]
4.1. Development of $\Gamma^k_{ij}$, $V$ and $W$.

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \]

\[ = \frac{1}{2} \left( \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) + O(r^2) \]

\[ = \frac{1}{6} \left( R_{jik} + R_{jki} + R_{ijk} - R_{ikj} - R_{kij} \right) x^\alpha + O(r^2) \]

Using the relations $R_{jik} + R_{jki} = 0$, $R_{jki} - R_{ikj} = -2R_{ikj}$ and $R_{ikj} - R_{kij} = -2R_{ikj}$ we then have

\[ \Gamma^k_{ij} = -\frac{1}{3} \left( R_{ikj} + R_{ikj} \right) x^\alpha + O(r^2) \quad (12) \]

On the other hand, since $\partial_r b^i_j$ and $\Gamma^i_{rs}$ have no constant term, Formula (8) yields

\[ \tilde{\Gamma}^k_{ij} = \left( \delta^k_i \partial_r b^r_j + \delta^k_j \partial_r b^r_i \right) \delta^r_s + O(r^2) , \]

and hence

\[ \tilde{\Gamma}^k_{ij} = \partial_r b^k_j + \Gamma^k_{ij} + O(r^2) . \]

We have

\[ V = \frac{1}{4} \sum_{i,k} \left( \tilde{\Gamma}^i_{ik} - \tilde{\Gamma}^k_{ii} \right)e_k \]

\[ = \frac{1}{4} \sum_{i,k} \left( \partial_r b^k_j + \Gamma^i_{ik} - \partial_r b^k_i - \Gamma^k_{ii} \right)e_k \]

\[ = \frac{1}{4} \sum_{i,k} \left( \Gamma^i_{ik} - \Gamma^k_{ii} \right)e_k \]

since $\partial_r b^i_j = \partial_r b^k_j$.

Moreover, we have

\[ \sum_i \left( \Gamma^i_{ik} - \Gamma^k_{ii} \right) = -\frac{1}{3} \sum_i \left( R_{iak} + R_{iak} \right) x^\alpha + \frac{1}{3} \sum_i \left( R_{ikai} + R_{ikai} \right) x^\alpha + O(r^2) \]

\[ = -(\text{Ric})_{ak} + O(r^2) \]

Therefore we proved that

\[ V = \left( -\frac{1}{4}(\text{Ric})_{ak} x^\alpha + O(r^2) \right)e_k , \quad (13) \]

The aim now is to show that

\[ W = \frac{1}{4} \sum_{i,j,k} b^k_j (\partial_r b^i_j)(b^{-1})_k^i e_i \cdot e_j \cdot e_k \]

is $O(r^3)$. First note that by Equations (10) and (11) $b^i_k$ has no term of order 1 and $\partial_r b^i_j$ has no term of order 0. Hence, any term in $W$ of order $< 3$ is a product of the 0-order term of $b^i_k$ and of a term of order 1 or 2 of $\partial_r b^i_j$. 
Therefore $W$ has no term of order 0. To compute the terms of order 1 and 2, we write

$$ W = \frac{1}{4} \sum_{i,j,k} \left( \delta^i_j (\partial_ib_j^{k}) \delta^k_i + O(r^3) \right) e_i \cdot e_j \cdot e_k. $$

We have

$$ \sum_{i,j,k} \partial_i b_j^k e_i \cdot e_j \cdot e_k = 0 $$

since

$$ \partial_i b_j^k = \partial_j b_k^i \quad \text{and} \quad e_j \cdot e_k = -e_k \cdot e_j. $$

Therefore $W$ has no term of order 1 and 2. We proved that

$$ W = O(r^3) \quad (14) $$

Remark 4.1. Similar calculations yield

$$ V = -\left( \frac{1}{4} \left( \text{Ric} \right)_{\alpha k} x^\alpha + \frac{1}{6} \left( \text{Ric} \right)_{\alpha k, \beta} x^\alpha x^\beta + O(r^3) \right) e_k. $$

$$ W = -\frac{1}{144} \sum_{i,j,k} R_{i\beta j\gamma k} \left( R_{j\alpha i} + R_{j\alpha l} \right) x^\alpha x^\beta x^\gamma e_i \cdot e_j \cdot e_k + O(r^4). $$

We do not give details here because we do not need explicit computations of terms of order 2 for $V$ and terms of order 3 for $W$ in the proof of Theorem 1.1.

5. The test spinor

5.1. The explicit spinor. In this section we construct a good test spinor on $\mathbb{R}^n$. The spinor bundle on $\mathbb{R}^n$ is trivial, so we can identify the fibers. Let $\psi_0 \in \Sigma_0 \mathbb{R}^n$. We set $f(x) := \frac{2}{1+r^2}$, where $r := |x|$, hence

$$ \partial_i f = -x_i f^2. $$

Then define

$$ \psi(x) = f^{\tilde{\psi}}(x) (1-x) \cdot \psi_0. \quad (15) $$

One calculates

$$ \nabla_{\partial_i} \psi = -f^{\tilde{\psi}} \partial_i \psi_0 - \frac{n}{2} f^{\tilde{\psi} + 1} x_i (1-x) \cdot \psi_0, \quad (16) $$

and hence

$$ D\psi = \frac{n}{2} f^\psi \quad (17) $$

$$ |\psi| = \frac{n}{2} f^{\frac{n-1}{2}} \quad (18) $$

$$ |D\psi| = \frac{n}{2} f^{\frac{n+1}{2}}. \quad (19) $$

5.2. Conformal change of metrics. In order to explain a geometric interpretation of this spinor, we have to recall the behavior of spinors and the Dirac operators under conformal changes. See e.g. [Hit74, Hij01] for proofs.

Let $(N, h)$ be a spin manifold of dimension $n$. Consider a conformal change of metric $\tilde{h} = F^{-2} h$ for any positive real function $F$ on $(N, h)$. The map $TN \to TN$, $X \mapsto \tilde{X} = FX$ induces an isomorphism of principal bundles from $\text{SO}(N, h)$ to $\text{SO}(N, \tilde{h})$. It lifts to a bundle isomorphism between the Spin($n$)-principal bundles $\text{Spin}(N, h)$ and $\text{Spin}(N, \tilde{h})$, and passing to the associated bundles one obtains a map

$$ \Sigma_h N = \text{Spin}(N, h) \times_{\rho} \Sigma \quad \rightarrow \quad \Sigma_{\tilde{h}} N = \text{Spin}(N, \tilde{h}) \times_{\rho} \Sigma $$

$$ \varphi \quad \mapsto \quad \tilde{\varphi} $$
between the spinor bundles, which is a fiberwise isometry and we have
\[ \tilde{X} \tilde{\varphi} = \tilde{X} \varphi \]
(see [Hij01] for more details on this construction).
By conformal covariance of the Dirac operator, we have, for \( \varphi \in \Gamma(\Sigma N) \),
\[ \tilde{D} \left( F^{\frac{n-1}{2}} \tilde{\varphi} \right) = F^{\frac{n+1}{2}} \tilde{D} \varphi, \tag{20} \]

5.3. **Geometric interpretation.** We apply this formula to a particular case: let \( p \) be any point of the round sphere \( S^n \). Then \( S^n \setminus \{p\} \) is isometric to \( \mathbb{R}^n \) with the metric
\[ g_S = f^2 g_{eucl}, \tag{21} \]
with
\[ f(x) = \frac{2}{1 + r^2}. \]
Hence we set \( N := \mathbb{R}^n, h = g_{eucl}, F = f^{-1} \). One calculates with (17) and (20) that \( \Phi := F^{\frac{n-1}{2}} \tilde{\psi} \) satisfies \( D \Phi = \frac{n}{2} \Phi \) on \( S^n \setminus \{p\} \), and \( |\Phi| = 1 \). Hence, the possible singularity at \( p \) can be removed (see e.g. the Removal of singularity theorem [Am03c, Theorem 5.1]), and one sees that \( \Phi \) is an eigenspinor to the eigenvalue \( n/2 \) on the round sphere \( S^n \). The equality discussion in Friedrich’s inequality [Fri80] implies that \( \Phi \) is a Killing spinor to the constant \(-1/2\), i.e. it satisfies
\[ \nabla_X \Phi = -\frac{1}{2} X \cdot \Phi. \]
Hence we have seen that our spinor \( \psi \) is the “conformal image” of a Killing spinor on \( S^n \).

6. **The proof of Theorem 1.1 for \( n \geq 3 \)**

We begin with the following Proposition.

**Proposition 6.1.** The metric \( g \) on \( M \) can be chosen such that
\[ \text{Ric}_g(p) = 0 \quad \text{and} \quad \Delta_g(\text{Scal}_g)(p) = 0. \]

**Proof.** Consider a conformal change of the metric \( \tilde{g} = e^{2u} g \) for any real function \( u \) on \((M, g)\). Then it is well known that the Ricci curvature \((2,0)\)-tensor \( \text{Ric}_g \), the scalar curvature \( \text{Scal}_g \) and the Laplacian \( \Delta_g \) corresponding to the metric \( \tilde{g} \) satisfy (see for example Hebey [Heb97] or Aubin [Aub76])
\[ \text{Ric}_{\tilde{g}} = \text{Ric}_g - (n-2) \nabla^2 u + (n-2) \nabla u \otimes \nabla u + (\Delta_g u - (n-2)|\nabla u|_g^2) g, \]
\[ \text{Scal}_{\tilde{g}} = e^{-2u} \left( \text{Scal}_g + 2(n-1)\Delta_g u - (n-1)(n-2)|\nabla u|_g^2 \right), \tag{22} \]
As a first step, we can assume that \( \text{Scal}_g(p) = 0 \). Then, let us choose \( u \) such that
\[ u(x) = \frac{1}{2(n-2)} \left( \frac{\text{Ric}_g(p)_{ij}}{n} - \frac{\text{Scal}_g(p)}{n} g_{ij}(p) \right) x^i x^j - \frac{\Delta_g(\text{Scal}_g)(p)}{48(n-1)} (x^1)^4 \]
in a neighborhood of the point \( p \). Since \( u(p) = 0 \) and \( (\nabla u)(p) = 0 \), it is straightforward to see that \( \text{Ric}_{\tilde{g}}(p) = 0 \). Moreover, taking the Laplacian of both members of Equation (22), a simple computation shows that \( \Delta_g \text{Scal}_g(p) = 0 \). \( \square \)

Let \( \tilde{\varphi} \in \Sigma_U M \) where \( U \) is the open neighborhood of a point \( p \in M \) as defined in the previous sections. With the help of formulas (13) and (14), we have the following
Corollary 6.2. For any metric $g$ on $M$ chosen as in Proposition 6.1, we have
\[ \bar{D}\bar{\varphi} = D\bar{\varphi} + \sum_{i,j,k} A_{i j k} x^\alpha x^\beta x^\gamma e_i \cdot e_j \cdot e_k \cdot \bar{\varphi} + W' \cdot \bar{\varphi} + V \cdot \bar{\varphi} + \sum_{ij} (b_i^j - \delta_i^j) \partial_i \cdot \nabla_{\partial_j \psi} \] (23)
where $A_{i j k} \in \mathbb{R}$ and where $W' \in \Gamma(\Lambda^3 TV)$, $V \in \Gamma(TV)$, $|W'| \leq C r^4$ and $|V| \leq C' r^2$ ($C$ and $C'$ being positive constants independent of $\varphi$).

Remark 6.3. Using the formulae in Remark 4.1, we obtain the formula
\[ A_{i j k} \gamma = -\frac{1}{144} R_{\beta \gamma k} \left( R_{j i a l} + R_{j l o i} \right), \]
Assume now that $\psi$ is the test spinor constructed in Section 5. Let $\varepsilon > 0$ be a small positive number. We set
\[ \varphi(x) := \eta \psi \left( \frac{x}{\varepsilon} \right) =: \psi_{\varepsilon}(x) \]
where $\eta = 0$ on $\mathbb{R}^n \setminus B_\rho(2\delta)$ and $\eta = 1$ on $B_\rho(\delta)$, and that $\psi$, defined as in (15) satisfies the following relations (16), (17), (18) and (19) where $f$ is again defined by
\[ f(x) = \frac{2}{1 + r^2}. \]
We now prove some lemmas which will be useful in the proof of Theorem 1.1.

Lemma 6.4. We have
\[ \left| \sum_{ij} (b_i^j - \delta_i^j) \partial_i \cdot \nabla_{\partial_j} \psi \left( \frac{x}{\varepsilon} \right) \right| \leq C \frac{\varepsilon}{\varepsilon} f \left( \frac{x}{\varepsilon} \right) \] (25)
where $f = \frac{x}{1 + r^2}$.

Proof. At first, we prove that:
\[ \sum_{ij} R_{i j a} x^\alpha x^\beta \partial_i \cdot \left( \nabla_{\partial_j} \psi \left( \frac{x}{\varepsilon} \right) \right) = 0. \] (26)
Indeed, using (16), we compute that
\[ (\nabla_{\partial_j} \psi) \left( \frac{x}{\varepsilon} \right) = -\frac{f \bar{\psi} \left( \frac{x}{\varepsilon} \right)}{\varepsilon} \partial_j \cdot \psi_0 - \frac{nf \bar{\psi} \left( \frac{x}{\varepsilon} \right)}{2\varepsilon} x^j (1 - \frac{x}{\varepsilon}) \cdot \psi_0. \]
and obtain
\[ \sum_{i j a} R_{i j a} x^\alpha x^\beta \partial_i \cdot (\nabla_{\partial_j} \psi) \left( \frac{x}{\varepsilon} \right) = -\frac{f \bar{\psi} \left( \frac{x}{\varepsilon} \right)}{\varepsilon} \sum_{i j a} R_{i j a} x^\alpha x^\beta \partial_i \cdot \partial_j \cdot \psi_0 \frac{nf \bar{\psi} \left( \frac{x}{\varepsilon} \right)}{2\varepsilon} \sum_{i j a} R_{i j a} x^\alpha x^\beta x^j \partial_i \cdot (1 - \frac{x}{\varepsilon}) \cdot \psi_0. \]
Now, since if $i \neq j$, $\partial_i \cdot \partial_j = -\partial_j \cdot \partial_i$ and since
\[ \sum_{i j a} R_{i j a} x^\alpha x^\beta = \sum_{i j a} R_{i j a} x^\alpha x^\beta = \sum_{i j a} R_{i j a} x^\alpha x^\beta, \]
(we have used that $R_{i j a} = R_{i j a} = R_{i j a}$), we get that
\[ \sum_{i j a} R_{i j a} x^\alpha x^\beta \partial_i \cdot \partial_j \cdot \psi_0 = -\sum_{i j a} R_{i j a} x^\alpha x^\beta \psi_0 = 0 \]
since $\text{Ric}(p) = 0$. The first summand vanishes.
The second summand vanishes as $\sum_{\beta j} R_{i j a} x^\alpha x^j = 0$. 
This proves (26). Now, by the development of $b^j_i$ (10), we easily obtain that

$$
\sum_{ij} (b^j_i - \delta^j_i) \partial_i \cdot \nabla \partial_j (\psi(x)) \leq C r^3 \varepsilon |\nabla \psi(x)|.
$$

Differentiating expression (15), we see that

$$
|\nabla \psi| \leq C (f^{\#} + rf^{\#})
$$

Together with $rf(r) = \frac{2r}{1+r} = 1 - \frac{(1-r)^2}{1+r} \leq 1$ we obtain the lemma. \(\square\)

Now, we can start the proof of Theorem 1.1. We have, with the notations of Corollary 6.2:

$$
\tilde{D} \tilde{\psi}_\varepsilon(x) = \tilde{\nabla} \eta \cdot \tilde{\psi}(\frac{x}{\varepsilon}) + \eta \tilde{D}(\tilde{\psi}(\frac{x}{\varepsilon})) = \tilde{\nabla} \eta \cdot \tilde{\psi}(\frac{x}{\varepsilon}) + \eta \tilde{D}(\tilde{\psi}(\frac{x}{\varepsilon})) + \eta \sum_{i} A^{ijk\alpha\beta\gamma} \tilde{\psi}(\frac{x}{\varepsilon}) + \eta \sum_{ij} (b^j_i - \delta^j_i) \partial_i \cdot \nabla \partial_j (\psi(\frac{x}{\varepsilon})).
$$

Therefore we have

$$
\tilde{D} \tilde{\psi}_\varepsilon(x) = \tilde{\nabla} \eta \cdot \tilde{\psi}(\frac{x}{\varepsilon}) + \eta \sum_{i} A^{ijk\alpha\beta\gamma} \tilde{\psi}(\frac{x}{\varepsilon}) + \eta \sum_{ij} (b^j_i - \delta^j_i) \partial_i \cdot \nabla \partial_j (\psi(\frac{x}{\varepsilon})).
$$

We write that

$$
|\tilde{D} \tilde{\psi}_\varepsilon|^2(x) = I + II + III + IV + V + VI + VII + VIII + IX + X + XI + XII + XIII + XIV + XV + XVI + XVII + XVIII + XIX + XX + XXI
$$
where

\[ I = |\vec{\nabla}\eta|^2 |\tilde{\psi}|^2 \left( \frac{x}{\varepsilon} \right) \]
\[ II = \frac{\eta^2}{\varepsilon^2} \frac{n^2}{4} |\tilde{\psi}|^2 \left( \frac{x}{\varepsilon} \right) f^2 \left( \frac{x}{\varepsilon} \right) \]
\[ III = \eta^2 |\sum_{ij,\alpha\beta\gamma} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma e_i \cdot e_j \cdot e_k \cdot \tilde{\psi}^2 \left( \frac{x}{\varepsilon} \right) \]
\[ IV = \eta^2 |W|^2 |\tilde{\psi} \left( \frac{x}{\varepsilon} \right)|^2 \]
\[ V = \eta^2 |V|^2 |\tilde{\psi} \left( \frac{x}{\varepsilon} \right)|^2 \]
\[ VI = 2 \text{Re} < \vec{\nabla}\eta \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \eta \frac{n}{2} f \left( \frac{x}{\varepsilon} \right) \psi \left( \frac{x}{\varepsilon} \right) > \]
\[ VII = 2 \text{Re} < \vec{\nabla}\eta \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \eta \sum_{ij,\alpha\beta\gamma} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma e_i \cdot e_j \cdot e_k \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ VIII = 2 \text{Re} < \vec{\nabla}\eta \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \eta W' \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ IX = 2 \text{Re} < \vec{\nabla}\eta \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \eta V \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ X = \frac{\eta^2}{\varepsilon} n f \left( \frac{x}{\varepsilon} \right) \eta \sum_{ij,\alpha\beta\gamma} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \text{Re} < e_i \cdot e_j \cdot e_k \cdot \tilde{\psi}, \tilde{\psi} > \left( \frac{x}{\varepsilon} \right) \]
\[ XI = \frac{\eta^2}{\varepsilon} n f \left( \frac{x}{\varepsilon} \right) \text{Re} < \tilde{\psi} \left( \frac{x}{\varepsilon} \right), W' \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ XII = \frac{\eta^2}{\varepsilon} n f \left( \frac{x}{\varepsilon} \right) \text{Re} < \tilde{\psi} \left( \frac{x}{\varepsilon} \right), V \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ XIII = 2 \eta^2 \sum_{ij,\alpha\beta\gamma} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \text{Re} < e_i \cdot e_j \cdot e_k \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), W' \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ XIV = 2 \eta^2 \sum_{ij,\alpha\beta\gamma} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \text{Re} < e_i \cdot e_j \cdot e_k \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), V \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ XV = 2 \eta^2 \text{Re} < W' \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), V \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right) > \]
\[ XVI = 2 \text{Re} < \vec{\nabla}\eta \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \eta \sum_{ij} (\delta_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j \left( \psi \left( \frac{x}{\varepsilon} \right) \right) > \]
\[ XVII = \frac{n\eta^2}{\varepsilon} f \left( \frac{x}{\varepsilon} \right) \text{Re} < \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \sum_{ij} (\delta_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j \left( \psi \left( \frac{x}{\varepsilon} \right) \right) > \]
\[ XVIII = 2 \eta^2 \text{Re} < \sum_{ij,\alpha\beta\gamma} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma e_i \cdot e_j \cdot e_k \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \sum_{ij} (\delta_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j \left( \psi \left( \frac{x}{\varepsilon} \right) \right) > \]
\[ XIX = 2 \eta^2 \text{Re} < W' \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \sum_{ij} (\delta_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j \left( \psi \left( \frac{x}{\varepsilon} \right) \right) > \]
\[ XX = 2 \eta^2 \text{Re} < V \cdot \tilde{\psi} \left( \frac{x}{\varepsilon} \right), \sum_{ij} (\delta_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j \left( \psi \left( \frac{x}{\varepsilon} \right) \right) > \]
\[ XXI = \eta^2 \left| \sum_{ij} (\delta_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j \left( \psi \left( \frac{x}{\varepsilon} \right) \right) \right|^2 . \]
Since \( \mathbf{V} \) is a vector field, we have
\[
\text{XII} = 0
\]
Assume now that \( x \in B_p(2\delta) \). Using the fact that \( |\nabla \eta| \leq C r^4 \) (\( C \) being a constant independent of \( \varepsilon \)) and since \( r \leq \delta \leq 1 \), we have:
\[
\text{I} + \text{III} + \text{IV} + \text{V} + \text{VIII} + \text{IX} + \text{XIII} + \text{XIV} + \text{XV} \leq C r^4 f^{n-1}(\frac{x}{\varepsilon})
\]
and
\[
\text{VI} + \text{XI} \leq \frac{C}{\varepsilon} r^4 f^{n/2}(\frac{x}{\varepsilon}).
\]
Since \( f \leq 2 \) and since \( r^2 \leq C \) on \( B_p(2\delta) \), we obtain that
\[
\text{VI} + \text{XI} \leq \frac{C}{\varepsilon} r^2 f^{n-\frac{3}{2}}(\frac{x}{\varepsilon}).
\]
In the same way, using relation (25) and the fact that for all \( \varepsilon, \frac{r}{\varepsilon} f(\frac{x}{\varepsilon}) \leq 1 \), we have also
\[
\text{X} + \text{XVI} + \text{XVII} + \text{XVIII} + \text{XIX} + \text{XX} + \text{XXI} \leq C r^2 f^{n-\frac{1}{2}}(\frac{x}{\varepsilon}).
\]
Therefore we obtain that
\[
|\overline{D} \overline{\psi}_\varepsilon|^2(x) \leq \frac{n^2}{4\varepsilon^2} f^{n+1}(\frac{x}{\varepsilon}) + C r^4 f^{n-1}(\frac{x}{\varepsilon}) + \frac{C}{\varepsilon} r^2 f^{n-\frac{3}{2}}(\frac{x}{\varepsilon}) \leq \frac{n^2}{4\varepsilon^2} f^{n+1}(\frac{x}{\varepsilon}) [1 + \Delta]
\]
where
\[
\Delta = C \varepsilon^2 r^4 f^{-2}(\frac{x}{\varepsilon}) + C \varepsilon r^2 f^{-\frac{3}{2}}(\frac{x}{\varepsilon}).
\]
Since \( |\overline{D} \overline{\psi}_\varepsilon|^2 \geq 0 \) we have \( \Delta \geq -1 \). Moreover, if we define
\[
g(x) = 1 + \frac{n}{n+1} x - (1 + x)^{\frac{n}{n+1}}, \quad \forall x \geq -1,
\]
then
\[
g'(x) = \frac{n}{n+1} \left(1 - (1 + x)^{\frac{1}{n+1}}\right), \quad \forall x > -1.
\]
Therefore \( g \) admits a minimum at 0 on the interval \([-1, +\infty[\]. This yields that, \( \forall x \geq -1, \)
\[
(1 + x)^{\frac{n}{n+1}} \leq 1 + \frac{n}{n+1} x.
\]

We then have
\[
|\overline{D} \overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}(x) \leq \left(\frac{n}{2\varepsilon}\right)^{\frac{2n}{n+1}} f^n(\frac{x}{\varepsilon}) [1 + \Delta]^{\frac{n}{n+1}} \leq \frac{n}{2\varepsilon}^{\frac{2n}{n+1}} f^n(\frac{x}{\varepsilon}) + \frac{n}{n+1} \left(\frac{n}{2\varepsilon}\right)^{\frac{2n}{n+1}} f^n(\frac{x}{\varepsilon}) \Delta.
\]
Taking into account the definition of \( \Delta \) and integrating over \( M \) leads to
\[
\int_M |\overline{D} \overline{\psi}_\varepsilon|^{\frac{2n}{n+1}} dv_g \leq \varepsilon^{\frac{2n}{n+1}} [A + B + C], \tag{27}
\]
where
\[
A = \int_{B_p(2\delta)} \left(\frac{n}{2}\right)^{\frac{2n}{n+1}} f^n(\frac{x}{\varepsilon}) dv_g,
\]
\[
B = C \int_{B_p(2\delta)} \varepsilon^{2} r^4 f^{n-2}(\frac{x}{\varepsilon}) dv_g,
\]
\[
C = C \int_{B_p(2\delta)} \varepsilon^{2} r^2 f^{n-\frac{3}{2}}(\frac{x}{\varepsilon}) dv_g.
\]
Since the function $f$ is radially symmetric, we can compute $A$ with the help of spherical coordinates:
\[
A = \int_{B_r(2\delta)} \left( \frac{n}{2} \right) ^{\frac{2n}{n+1}} f^n \left( \frac{x}{\varepsilon} \right) \omega_{n-1} G(r) r^{n-1} dr,
\]
where $\omega_{n-1}$ stands for the volume of the unit sphere $S^{n-1}$ and
\[
G(r) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |g| dx \quad |g|_y := \det g_{ij}(y).
\]
From Proposition 6.1, Hebey [Heb97] or Lee-Parker [LP87], we know that
\[
G(r) \leq 1 + O(r^4).
\]
Therefore, we can estimate $A$ in the following way:
\[
A \leq \left( \frac{n}{2} \right) ^{\frac{2n}{n+1}} \omega_{n-1} \varepsilon^n \left\{ \int_0^{2\delta} f^n \left( \frac{x}{\varepsilon} \right) r^{n-1} dr + C \int_0^{2\delta} f^n \left( \frac{x}{\varepsilon} \right) r^{n+3} dr \right\}.
\]
Since
\[
\int_0^{2\delta} r^{n+3} \frac{1}{(1 + r^2)^n} dr \leq O \left( \int_1^{2\delta} r^{3-n} dr \right)
\]
we get that
\[
A \leq \left( \frac{n}{2} \right) ^{\frac{2n}{n+1}} \omega_{n-1} \varepsilon^n \left\{ \int_0^{2\delta} 2^n r^{n-1} \frac{1}{(1 + r^2)^n} dr + o(1) \right\},
\]
and hence
\[
A \leq \left( \frac{n}{2} \right) ^{\frac{2n}{n+1}} \omega_{n-1} \varepsilon^n \left\{ \int_0^{2\delta} 2^n r^{n-1} \frac{1}{(1 + r^2)^n} dr + o(1) \right\}.
\]
Let us show that
\[
B = o(\varepsilon^n).
\]
Since $dv_y \leq C dx$, setting $y = \frac{x}{\varepsilon}$, we have
\[
\int_{B_r(2\delta)} r^4 f^{n-2} \left( \frac{x}{\varepsilon} \right) dv_y \leq C \varepsilon^{n+4} \int_{B_r(2\delta)} r^4 f^{n-2} \left( \frac{y}{\varepsilon} \right) dy \leq C \varepsilon^{n+4} \int_0^{2\delta} 2^n \frac{r^{n+3}}{(1 + r^2)^n} dr \leq C \varepsilon^{n+4} O \left( \int_1^{2\delta} r^{7-n} dr \right).
\]
It is easy to check that relation (29) follows if $n \geq 3$. In the same way, we can prove that $C = o(\varepsilon^n)$.
Together with Equation (28), we can conclude that
\[
\int_M |\tilde{D} \tilde{\psi}_x|^{\frac{2n}{n+1}} dv_y \leq \varepsilon^{-\frac{2n}{n+1} + n} \left[ \left( \frac{n}{2} \right) ^{\frac{2n}{n+1}} \omega_{n-1} \int_0^{+\infty} r^{n-1} f^n(r) dr + o(1) \right],
\]
which yields
\[
\left( \int_M |\tilde{D} \tilde{\psi}_x|^{\frac{2n}{n+1}} dv_y \right)^{\frac{n}{n-1}} \leq \varepsilon^{-1} \left[ \left( \frac{n^2}{4} \right) ^{\frac{2n}{n+1}} \omega_{n-1} I \right]^{\frac{n}{n-1}} (1 + o(1)),
\]
where
\[
I = \int_0^{+\infty} 2^n r^{n-1} \frac{1}{(1 + r^2)^n} dr.
\]
We are now going to estimate \( \left| \int_M \Re e < \tilde{D}\psi_\varepsilon, \psi_\varepsilon > dv_g \right| \). We start by computing

\[
\left| \int_M \Re e < \tilde{D}\psi_\varepsilon, \psi_\varepsilon > dv_g \right| \geq A' - B' - C' - D' - E',
\]

where

\[
A' = \int_{B_\varepsilon(\delta)} \frac{n}{2\varepsilon} f^n(\frac{x}{\varepsilon}) dv_g,
\]

\[
B' = \left| \int_M \Re e < \tilde{\nabla}\eta \cdot \psi(\frac{x}{\varepsilon}), \eta \psi(\frac{x}{\varepsilon}) > dv_g \right| ,
\]

\[
C' = \left| \int_M \eta^2 \sum_{i,j,k} A_{ijk\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \Re e < e_i \cdot e_j \cdot e_k \cdot \psi, \psi > (\frac{x}{\varepsilon}) dv_g \right| ,
\]

\[
D' = \left| \int_M \eta^3 \Re e < W', \psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon}) > dv_g \right| ,
\]

\[
E' = \left| \int_M \eta^3 \Re e < \sum_{ij} (b_i^j - \delta_i^j) \partial_i \cdot \nabla \partial_j (\psi(\frac{x}{\varepsilon})), \psi(\frac{x}{\varepsilon}) > dv_g \right| .
\]

(The term in \( V \) is zero). Note that \( A' = \frac{1}{\varepsilon^n} \frac{\delta(\frac{n}{\varepsilon})}{\varepsilon^n} A \) where \( \eta \) has been replaced by \( 2\eta \). As to obtain (29), we get that

\[
B' + C' + D' \leq C \int_{B_\varepsilon(2\delta)} f^{n-1}(\frac{x}{\varepsilon}) dv_g \leq 0(\varepsilon^n) = o(\varepsilon^{-1})
\]

and

\[
E' \leq C \int_{B_\varepsilon(2\delta)} f^{n-\frac{1}{2}}(\frac{x}{\varepsilon}) dv_g \leq o(\varepsilon^{n-1}).
\]

Moreover, with the same method which was used to obtain (28), we get

\[
A' \geq \frac{n}{2} \omega_{n-1} \varepsilon^{n-1} I [1 + o(1)].
\]

This proves that

\[
\left| \int_M \Re e < \tilde{D}\psi_\varepsilon, \psi_\varepsilon > dv_g \right| \geq \frac{n}{2} \omega_{n-1} \varepsilon^{n-1} I [1 + o(1)].
\]

Finally, Equations (30) and (31) allow to estimate \( J(\tilde{\psi}_\varepsilon) \) in the following way:

\[
J(\tilde{\psi}_\varepsilon) = \left( \frac{\int_M |\tilde{D}\tilde{\psi}_\varepsilon|^{\frac{2n}{n+1}} dv_g}{\int_M \Re e < \tilde{D}\psi_\varepsilon, \psi_\varepsilon > dv_g} \right)^{\frac{n+1}{2n}} \leq \frac{n}{2} \omega_{n-1} \frac{\varepsilon}{I^{\frac{n+1}{2}}} [1 + o(1)].
\]

By (21), we have

\[
w_{n-1} I = \int_{\mathbb{S}^n} f^n dx = \omega_n
\]

Therefore, we proved that for the test spinor \( \varphi \), we have

\[
J(\tilde{\psi}_\varepsilon) \leq \lambda_{\text{min}}(S^n) [1 + o(1)].
\]

Hence Theorem 1.1 is proven.
Remark 6.5. There is a variant of this proof which needs less calculations. As a first step, one proves that for any \( \varepsilon > 0 \) there is a test spinor \( \varphi_\varepsilon \) on \( \mathbb{R}^n \) with support in \( B_0(1) \) such that \( J_{g_{eucl}}(\varphi_\varepsilon) \leq \lambda_\min^+(S^n) + \varepsilon \) where \( \varepsilon > 0 \). The argument for this coincides with the above proof, but the terms IV to XXI vanish, as \( \mathbb{R}^n \) is flat.

In a second step, one transplants this compactly supported spinor \( \varphi_\varepsilon \) to the arbitrary compact spin manifold \( (M, \Lambda^2 g) \), where \( \Lambda > 0 \) is constant, and one obtains a spinor \( \overline{\varphi_\varepsilon} \) on \( (M, \Lambda^2 g) \). The terms IV to XXI reappear. However, from our Taylor expansion worked before, it is easy to see that for \( \Lambda \to \infty \) these terms disappear.

One concludes that there for any \( \varepsilon > 0 \) there is a \( \Lambda_\varepsilon > 0 \) and a spinor \( \varphi_\varepsilon \) on \( (M, \Lambda_\varepsilon^2 g) \) such that

\[
J_{\Lambda_\varepsilon^2 g}(\varphi_\varepsilon) < \lambda_\min^+(S^n) + 2\varepsilon.
\]

Together with

\[
\lambda_\min(M, g, \sigma) = \lambda_\min(M, \Lambda_\varepsilon^2 g, \sigma) \leq J_{\Lambda_\varepsilon^2 g}(\overline{\varphi_\varepsilon})
\]

the theorem follows.

This proof is simpler. We chose the way presented above because of various reasons. One the other hand, as indicated in the introduction, in the case \( n \geq 3 \) it is not the result, but the method of proof which is interesting. The above formulae enter at several places in the literature, e.g. [AHM03], [AH03] and [Rau06]. Secondly, the simpler proof is close to Große’s proof [Gro06] and we refer to her article for the probably most elegant proof in dimension \( n \geq 3 \). Also in her proof some Taylor expansions from the present article are used.

7. The case \( n = 2 \)

The 2-dimensional case is simpler since \( g \) is locally conformally flat. On the other hand, some estimates of the last section are no longer valid in dimension 2, hence some parts have to be modified. These modifications will be carefully carried out in this section.

Let \( (M, g) \) be a compact Riemannian surface equipped with a spin structure. If \( \bar{g} \) is conformal to \( g \) we denote by \( \mu_1(\bar{g}) \) the smallest positive eigenvalue of \( \Delta_{\bar{g}} \). We prove the theorem.

**Theorem 7.1.** There exists a family of metrics \( (g_\varepsilon) \) conformal to \( g \) for which

\[
\limsup_{\varepsilon \to 0} \lambda_1^+(g_\varepsilon)^2 \text{Vol}_{g_\varepsilon}(M) \leq 4\pi
\]

and

\[
\liminf_{\varepsilon \to 0} \mu_1(g_\varepsilon) \text{Vol}_{g_\varepsilon}(M) \geq 8\pi.
\]

Theorem 7.1 clearly implies Theorem 1.1.

Roughly, these metrics can be described as follows. At first we choose a metric in the conformal class which is flat in a neighborhood of a point \( p \). We remove a small ball around it and glue in a large truncated sphere. This removal and gluing can be done in such a way that we stay within a conformal class. \( \varepsilon \to 0 \). In the limit this truncated sphere is getting larger and larger compared to the original part of \( M \).

Agricola, Ammann and Friedrich asked the following question [AAF99]:

**Let** \( M \) **be a two-dimensional torus equipped with a trivial spin structure, can we find on** \( M \) **a Riemannian metric** \( \bar{g} \)** **for which** \( \lambda_1^+(\bar{g})^2 < \mu_1(\bar{g}) \)?

To understand this question, recall that the two-dimensional torus carries 4 spin structures. Three of them (the non-trivial ones) are spin boundaries: for these spin structures it is easy to find flat examples with \( \lambda_1^+(\bar{g})^2 = \frac{1}{2}\mu_1(\bar{g}) \). For the trivial spin structure, one has \( \lambda_1^+(\bar{g})^2 = \mu_1(\bar{g}) \) for all flat metrics and \( \lambda_1^+(\bar{g})^2 > \mu_1(\bar{g}) \) for many \( S^1 \)-equivariant one’s.
Clearly, Theorem 7.1 answers this question but says much more: firstly, the result is true on any compact Riemannian surface equipped with a spin structure and not only when $M$ is a two-dimensional torus. In addition, the metric $\tilde{g}$ can be chosen in a given conformal class. Finally, this metric $\tilde{g}$ can be chosen such that $(2 - \delta)\lambda_1^+(\tilde{g})^2 < \mu_1(g)$ where $\delta > 0$ is arbitrarily small. More precisely Theorem 1.1 shows the corollary

**Corollary 7.2** (Proposition 1.2 of the Introduction). On any compact Riemannian surface $(M, g)$, we have

$$\inf \frac{\lambda_1^+(\tilde{g})^2}{\mu_1(g)} \leq \frac{1}{2}$$

where the infimum is taken over all metrics $\tilde{g}$ conformal to $g$.

### 7.1. $C^0$-metrics

Let $f$ be a smooth positive function and set $\tilde{g} = f^2g$. Let also for $u \in C^\infty(M)$

$$I_{\tilde{g}}(u) = \int_M |\nabla u|_{\tilde{g}}^2 dv_{\tilde{g}} \int_M u^2 dv_{\tilde{g}}$$

It is well known that $\mu_1(\tilde{g}) = \inf I_{\tilde{g}}(u)$ where the infimum is taken over the smooth non-zero functions $u$ for which $\int_M udv_{\tilde{g}} = 0$. Another way to express $\mu_1(\tilde{g})$ is

$$\mu_1(\tilde{g}) := \inf_V \sup_{u \in V \setminus \{0\}} I_{\tilde{g}}(u) \quad (33)$$

where the infimum runs over all 2-dimensional subspaces $V$ of $C^\infty(M)$. We now can write all these expressions in the metric $g$. We then see that for $u \in C^\infty(M)$, we have

$$I_{\tilde{g}}(u) = \int_M |\nabla u|_{g}^2 dv_{\tilde{g}}$$

and $\mu_1(\tilde{g})$ is characterized in a way analogous to (33). Now if $f$ is only continuous, we can define $\tilde{g} = f^2g$. The symmetric 2-tensor $\tilde{g}$ is not really a metric since $f$ is not smooth. We then say that $g$ is a $C^0$-metric.

We can define the first eigenvalue $\mu_1(\tilde{g})$ of $\Delta_{\tilde{g}}$ using the definition above. Suppose that

$$(1 + \rho)^{-1} f \leq \tilde{f} \leq (1 + \rho)f. \quad (34)$$

Then

$$(1 + \rho)^{-2} I_{f^2g}(u) \leq I_{\tilde{f}^2g}(u) \leq (1 + \rho)^2 I_{f^2g}(u).$$

From the variational characterization (33) it follows that

$$(1 + \rho)^{-2} \mu_1(f^2g) \leq \mu_1(\tilde{f}^2g) \leq (1 + \rho)^2 \mu_1(f^2g),$$

which is a special case of a result by Dodziuk [Dod82, Proposition 3.3]. In particular, we get

**Lemma 7.3.** If $(f_n)$ is a sequence of smooth positive functions that converges uniformly to $f$, then $\mu_1(f_n^2g)$ tends to $\mu_1(f^2g)$.

In the same way, if $\tilde{g} = f^2g$ is a metric conformal to $g$ where $f$ is positive and smooth, we define

$$\mathcal{J}_g(\psi) = \frac{\int_M |D_g\psi|_{\tilde{g}}^2 dv_{\tilde{g}}}{\int_M (D_g\psi, \psi)_{\tilde{g}} dv_{\tilde{g}}}.$$  

The first eigenvalue of the Dirac operator $D_\tilde{g}$ is then given by $\lambda_1^+(\tilde{g}) = \inf \mathcal{J}_\tilde{g}(\psi)$ where the infimum is taken over the smooth spinor fields $\psi$ for which $\int_M (D_\tilde{g}\psi, \psi)_{\tilde{g}} dv_{\tilde{g}} > 0$. Now, as explained in paragraph 5.2 we can identify spinors for the metric $g$ and spinors for the metric $\tilde{g}$ by a fiberwise isometry. Moreover, using this identification, we have for all smooth spinor field:

$$D_\tilde{g}(f^{-\frac{1}{2}} \varphi) = f^{-\frac{1}{2}} D_g \varphi.$$
This implies that if we set $\varphi = f^2 \psi$, we have
\[
J_\varphi^f(\varphi) := \frac{\int_M |D_g \varphi|^2 f^{-1} dv_g}{\int_M \langle D_g \varphi, \varphi \rangle dv_g} = J_\varphi(\psi)
\]
and the first eigenvalue of the Dirac operator $D_g$ is given by
\[
\lambda_1^+(\bar{g}) = \inf \, J_\varphi^f(\varphi)
\]
where the infimum is taken over the smooth spinor fields $\varphi$ for which $\int_M \langle D_g \varphi, \varphi \rangle dv_g > 0$. Now, when $\bar{g} = f^2 g$ is no longer smooth, but a $C^0$-metric, we can use (35) to define $\lambda_1^+(\bar{g})$.

Under the assumption (34), we get
\[
(1 + \rho)^{-1} J_{f^2 g}^\varphi(\varphi) \leq J_{f^2 g}^\varphi(\varphi) \leq (1 + \rho) J_{f^2 g}^\varphi(\varphi),
\]
and hence
\[
(1 + \rho)^{-1} \lambda_1^+(f^2 g) \leq \lambda_1^+(\bar{g}).
\]

We have proven a result similar to Lemma 7.3:

**Lemma 7.4.** If $(f_n)$ is a sequence of smooth positive functions that converges uniformly to $f$, then $\lambda_1^+(f_n^2 g)$ tends to $\lambda_1^+(\bar{g})$.

### 7.2. The metrics $(g_{\alpha, \varepsilon})_{\alpha, \varepsilon}$

In this paragraph, we construct the metrics $(g_{\alpha, \varepsilon})_{\alpha, \varepsilon}$ conformal to $g$ which will satisfy:

\[
\limsup_{\varepsilon \to 0} \lambda_1^+(g_{\alpha, \varepsilon})^2 \operatorname{Vol}_{g_{\alpha, \varepsilon}}(M) \leq 4 \pi
\]

and

\[
\liminf_{\alpha \to 0} \liminf_{\varepsilon \to 0} \mu_1(g_{\alpha, \varepsilon}) \operatorname{Vol}_{g_{\alpha, \varepsilon}}(M) \geq 8 \pi.
\]

Clearly this implies Theorem 1.1. By Lemmas 7.3 and 7.4, it suffices to construct $C^0$-metrics $(g_{\alpha, \varepsilon})_{\alpha, \varepsilon}$. Recall that the volume of $M$ for a $C^0$-metric is defined by $\operatorname{Vol}_{f^2 g}(M) = \int_M f^2 dv_g$. At first, without loss of generality, we can assume that $g$ is flat near a point $p \in M$. Let $\alpha > 0$ be a small number to be fixed later such that $g$ is flat on $B_p(\alpha)$. We set for all $x \in M$ and $\varepsilon > 0$,

\[
f_{\alpha, \varepsilon}(x) = \begin{cases} \frac{\varepsilon^2}{r^2} & \text{if } r \leq \alpha \\ \frac{\varepsilon^2}{r^2 + \alpha^2} & \text{if } r > \alpha \end{cases}
\]

where $r = d_g(., p)$. The function $f_{\alpha, \varepsilon}$ is continuous and positive on $M$. We then define for all $\varepsilon > 0$, $g_{\alpha, \varepsilon} = f_{\alpha, \varepsilon}^2 g$. The symmetric 2-tensors $(g_{\alpha, \varepsilon})_{\alpha, \varepsilon}$ will be the desired $C^0$-metrics. For these metrics, we have

\[
\operatorname{Vol}_{g_{\alpha, \varepsilon}}(M) = \int_M f_{\alpha, \varepsilon}^2 dv_g = \int_{B_p(\alpha)} f_{\alpha, \varepsilon}^2 dv_g + \int_{M \setminus B_p(\alpha)} f_{\alpha, \varepsilon}^2 dv_g.
\]

Since $g$ is flat on $B_p(\alpha)$, we have

\[
\int_{B_p(\alpha)} f_{\alpha, \varepsilon}^2 dv_g = \int_0^{2\pi} \int_0^\alpha \frac{\varepsilon^4 r}{(\varepsilon^2 + r^2)^2} d\theta dr.
\]

Substituting $\rho = r^2 / \varepsilon^2$ we obtain

\[
\int_{B_p(\alpha)} f_{\alpha, \varepsilon}^2 dv_g = \pi \varepsilon^2 \int_0^{\alpha^2} \frac{1}{(1 + \rho)^2} d\rho = \pi \varepsilon^2 + o(\varepsilon^2).
\]
Since \( f_{\alpha,\varepsilon}^2 \leq \frac{4}{\alpha^2} \) on \( M \setminus B_p(\alpha) \), we have \( \int_{M \setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = o(\varepsilon^2) \). We obtain
\[
\text{Vol}_{\varepsilon}(M) = \pi \varepsilon^2 + o(\varepsilon^2).
\] (38)

7.3. **Proof of relation (36).** We define on \( \mathbb{R}^2 \) as in subsection 5.1 the spinor field
\[
\psi(x) = f(x)(1-x) \cdot \psi_0
\]
where \( f(x) = \frac{2}{1 + |x|^2} \), \( |\psi_0| = 1 \). We have
\[
D\psi = f \psi \quad \text{and} \quad |\psi| = f^\frac{1}{2}.
\] (39)

Now, we fix a small number \( \alpha > 0 \) such that \( g \) is flat on \( B_p(2\alpha) \). Then, let \( \delta \) be a small number such that we take \( 0 \leq \delta \leq \alpha \). Assume that \( \varepsilon \) tends to 0. Furthermore let \( \eta \) be a smooth cut-off function defined on \( M \) by
\[
\eta(x) = \begin{cases} 1 & \text{if } r \leq \delta^2 \\ \frac{\log(r)}{\log(\delta^2)} - 1 & \text{if } r \in [\delta^2, \delta] \\ 0 & \text{if } r \geq \delta 
\end{cases}
\]
The function \( \eta \) is such that \( 0 \leq \eta \leq 1 \), \( \eta(B_p(\delta)) = \{1\} \), \( \eta(\mathbb{R}^n \setminus B_p(2\delta)) = \{0\} \) and
\[
\kappa_\delta := \int_M |\nabla \eta|^2 dv_g \to 0 \quad \text{for } \delta \to 0.
\] (40)

Identifying \( B_p(2\delta) \) in \( M \) with \( B_0(2\delta) \) in \( \mathbb{R}^2 \), we can define a smooth spinor field on \( M \) by \( \psi_\varepsilon \equiv \eta(x) \psi(\frac{x}{\varepsilon}) \). Using (39), we have
\[
D_g(\psi_\varepsilon) = \nabla \eta \cdot \psi(\frac{x}{\varepsilon}) + \frac{\eta}{\varepsilon} f \left( \frac{x}{\varepsilon} \right) \psi(\frac{x}{\varepsilon}).
\] (41)

Since \( \langle \nabla \eta \cdot \psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon}) \rangle \in i\mathbb{R} \) and since \( |D_g \psi_\varepsilon|^2 \in \mathbb{R} \), we have
\[
\int_M |D_g \psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g = I_1 + I_2
\] (42)
where
\[
I_1 = \int_M |\nabla \eta|^2 \left| \psi \left( \frac{x}{\varepsilon} \right) \right|^2 dx \quad \text{and} \quad I_2 = \int_M \frac{\eta^2}{\varepsilon^2} f^2 \left( \frac{x}{\varepsilon} \right) \left| \psi \left( \frac{x}{\varepsilon} \right) \right|^2 f_{\alpha,\varepsilon}^{-1} dx.
\]
By (39), \( \left| \psi \left( \frac{x}{\varepsilon} \right) \right|^2 \leq 2 \) and hence
\[
I_1 \leq 2 \int_M |\nabla \eta|^2 dv_g = 2 \kappa_\delta \to 0
\] (43)
for \( \delta \to 0 \). Now, by (39),
\[
I_2 \leq \frac{2}{\varepsilon^2} \int_{B_\varepsilon(2\delta)} f^3 \left( \frac{x}{\varepsilon} \right) f_{\alpha,\varepsilon}^{-1} dx.
\]
Since \( f_{\alpha,\varepsilon} = \frac{1}{2} f(\frac{x}{\varepsilon}) \) on the support of \( \eta \), we have
\[
I_2 \leq \frac{2}{\varepsilon^2} \int_{B_p(2\delta)} f^2 (\frac{x}{\varepsilon}) dx.
\]
Mimicking what we did to get (38), we obtain that
\[
I_2 \leq 8\pi + o_\varepsilon(1)
\]
where \( o_\varepsilon(1) \) denotes a term tending to 0 for \( \varepsilon \to 0 \). Together with (42) and (43), we obtain
\[
\int_M |D_g \psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g \leq 8\pi + 2\kappa_\delta + o_\varepsilon(1).
\] (44)

In the same way, by (41), since \( \int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g \in \mathbb{R} \) and since \( \langle \nabla \eta \cdot \psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon}) \rangle \in i\mathbb{R} \), we have
\[
\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f \left( \frac{x}{\varepsilon} \right) \left| \psi \left( \frac{x}{\varepsilon} \right) \right|^2 dv_g.
\]
By (39), this gives
\[
\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \frac{\eta^2}{\varepsilon^2} f^2 \left( \frac{x}{\varepsilon} \right) dv_g.
\]
With the computations made above, it follows that
\[
\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = 4\pi \varepsilon + o(\varepsilon).
\]
Together with (44) and (38), we obtain
\[
\lambda_1^+(g_{\alpha,\varepsilon})^2 \text{Vol}_{g_{\alpha,\varepsilon}}(M) \leq \left( J_{g_{\alpha,\varepsilon}}'(\psi_\varepsilon) \right)^2 \text{Vol}_{g_{\alpha,\varepsilon}}(M) \leq \left( \frac{8\pi + 2\kappa_\delta + o_\varepsilon(1)}{4\pi \varepsilon + o(\varepsilon)} \right)^2 \left( \pi \varepsilon^2 + o(\varepsilon^2) \right)
= 4\pi + 2\kappa_\delta + \frac{1}{4\pi} \kappa_\delta^2 + o_\varepsilon(1).
\]
Letting $\varepsilon$ then $\delta$ go to 0, we get Relation (36).

7.4. Proof of relation (37). As pointed out by the referee the metrics $g_{\alpha,\varepsilon}$ coincide with metrics constructed in [Tak02], and relation (37) is proven in this article.

References

[AAF99] I. Agricola, B. Ammann and T. Friedrich, A comparison of the eigenvalues of the Dirac and Laplace operators on a two-dimensional torus, Manuscripta Math. 100 (1999), No 2, 231–258.

[Am03a] B. Ammann, A variational problem in conformal spin Geometry, Habilitationsschrift, Universität Hamburg, May 2003, downloadable on http://www.berndammann.de/publications.

[Am03b] B. Ammann, A spin-conformal lower bound of the first positive Dirac eigenvalue, Diff. Geom. Appl. 18 (2003), 21–32.

[Am03c] B. Ammann, The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions, Preprint, 2003.

[AH03] B. Ammann, E. Humbert, Positive mass theorem for the Yamabe problem on spin manifolds, Geom. Funct. Anal. 15 (2005), 567–576.

[AHM03] B. Ammann, E. Humbert, B. Morel, Mass endomorphism and spinorial Yamabe type problems on conformally flat manifolds, Comm. Anal. Geom. 14 (2006), 163–182.

[Aub76] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl., IX. Sér. 55 (1976), 269–296.

[Bär91] C. Bär, Das Spektrum des Dirac-Operators, PhD thesis, Bonner Math. Schr. 217, Bonn 1991.

[Bär92] C. Bär, Lower eigenvalue estimates for Dirac operators, Math. Ann. 293 (1992), 39–46.

[BG92] J.-P. Bourguignon and P. Gauduchon, Spineurs, opérateurs de Dirac et variations de métriques, Commun. Math. Phys. 144 (1992), 581–599.

[CES03] B. Colbois and A. El Soufi, Extremal eigenvalues of the Laplacian in a conformal class of metrics: the ‘conformal spectrum’ Ann. Global Anal. Geom. 24 (2003), 337–349.

[Dod82] J. Dodziuk, Eigenvalues of the Laplacian on forms, Proc. Amer. Math. Soc. 85 (1982), 437–443.

[Fri80] T. Friedrich, Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nicht-negativer Krümmung, Math. Nachr., 97 (1980), 117–146.

[Gro06] N. Große, On a conformal invariant of the Dirac operator on noncompact manifolds, Ann. Glob. Anal. Geom. 30 (2006), 407–416.

[Heb97] E. Hebey, Introduction à l’analyse non-linéaire sur les variétés, Diderot Éditeur, Arts et sciences., 1997.

[Hij86] O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing Spinors, Commun. Math. Phys. 104 (1986), 151–162.

[Hij91] O. Hijazi, Première valeur propre de l’opérateur de Dirac et nombre de Yamabe, C. R. Acad. Sci. Paris, Série I 313, (1991), 865–868.

[Hij01] O. Hijazi, Spectral properties of the Dirac operator and geometrical structures, Ocampo, Hernan (ed.) et al., Geometric methods for quantum field theory, Proceedings of the summer school, Villa de Leyva, Colombia, July 12-30, 1999, Singapore: World Scientifi. 116-169, 2001.

[Hit74] N. Hitchin, Harmonic spinors, Adv. Math. 14 (1974), 1–55.

[LM89] H.-B. Lawson and M.-L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, 1989.

[LPr87] J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Am. Math. Soc., New Ser. 17 (1987), 37–91.

[Lott86] J. Lott, Eigenvalue bounds for the Dirac operator, Pacific J. Math. 125 (1986), 117–126.

[Pfa02] F. Pfäflle, Eigenwertkonvergenz für Dirac-Operatoren, PhD thesis, University of Hamburg, Germany, 2002, Shaker Verlag Aachen 2003, ISBN 3-8322-1294-9.

[Rau06] S. Raulot, Aspect conforme de l’opérateur de Dirac sur une variété à bord, PhD thesis, Université Nancy I, 2006.
[Sch84] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Diff. Geom.* 20 (1984), 479–495.

[Tru68] N.S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.* 22 (1968), 265–274.

[Tak02] J. Takahashi, Collapsing of connected sums and the eigenvalues of the Laplacian, *J. Geom. Phys.* 40 (2002), 201–208.

[Yam60] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* 12 (1960), 21–37.

Authors’ addresses:

Bernd Ammann, Jean-François Grosjean, and Emmanuel Humbert, Institut Élie Cartan BP 239_ Université de Nancy 1_ 54506 Vandoeuvre-lès-Nancy Cedex France

Bertrand Morel  Départements QLIO et GIM IUT Epinal - Hubert Curien 7, rue des Fusillés de la Résistance BP 392 88010 Epinal Cedex France

E-Mail: ammann@iecn.u-nancy.fr, grosjean@iecn.u-nancy.fr, humbert@iecn.u-nancy.fr, and bertrand.morel@univ-nancy2.fr