How to Construct Curves Over Finite Fields With Many Points

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Introduction

In 1940 A. Weil proved the Riemann hypothesis for curves over finite fields. As an immediate corollary he obtained an upper bound for the number of points on an irreducible curve $C$ of genus $g$ over a finite field of cardinality $q$, namely

$$\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}.$$ 

This bound was proved for elliptic curves by Hasse in 1933. Ever since, the question of the maximum number $N_q(g)$ of points on an irreducible curve of genus $g$ over a finite field of cardinality $q$ could have been investigated. But for a long time it attracted no attention and it was only after Goppa introduced geometric codes in 1980 that this question aroused systematic attention, cf. [G], [M], [S 1,2,3].

For $g \leq (q - \sqrt{q})/2$ the Hasse-Weil bound is in general best possible; for other $g$ there is a better bound. In general the so-called ‘formules explicites’ can provide improvements of the Hasse-Weil bound. Therefore, the question arises of the actual value of the maximum number $N_q(g)$ for given $g$ and $q$.

Serre gave tables in [S 1,2,3] listing the value of $N_q(g)$ for small values of $q$ and $g$, or if this value was not known, a small interval in which the value of $N_q(g)$ lies. Wirtz extended in [Wi] these tables for small values of $q$ (powers of 2 or 3) mainly by a computer search on certain families. However, in many instances the entries are large intervals of the form $[a, b]$ in which $N_q(g)$ lies; here $b$ is the best upper bound known to him (from Hasse-Weil-Serre or Ihara), while $a$ indicates that he knows that there exists a curve of genus $g$ with $a$ points over $\mathbb{F}_q$. For completeness’ sake we reproduce here his table. In order to improve on his table one needs ways to construct curves with many points.

The notion of generalized weight distribution in coding theory applied to so-called trace codes suggested to us a construction of curves with many points. This led us to considerable improvements and extensions of the table of Wirtz. The benefit is here both ways: coding theory leads to curves with many points and to curves with other interesting properties like supersingular curves. On the other hand results in algebraic geometry lead to the determination of generalized Hamming weights in coding theory.

In this paper we concentrate on constructing curves with many points. We will give the relation between curves and codes, explain our method and illustrate this with examples of the results. We will also recall some other methods to find curves with
many points. At the end we give in addition to the table by Wirtz two tables listing
some relatively good intervals or actual values for $N_q(g)$.

It should be clear to the reader that the story does not end here. As the tables and
the paper show there is ample room for improvement, both theoretically and practically.

1. The function $N_q(g)$

We begin with some notation. Let $q = p^m$ be a power of a prime and let $\mathbb{F}_q$ be a finite
field of cardinality $q$. Let $C$ be an irreducible (complete non-singular) curve defined
over $\mathbb{F}_q$. We set $c(r) = \#C(\mathbb{F}_q^r)$ and consider the zeta function

$$Z(t) = \exp\left(\sum_{r=1}^{\infty} c(r) \frac{t^r}{r}\right).$$

According to Weil we can write

$$Z(t) = \frac{P_1(t)}{(1-t)(1-qt)},$$

where $P_1(t) \in \mathbb{Z}[t]$ is of the form $P_1(t) = \prod_{i=1}^{2g}(1-\alpha_it)$ with $\alpha_i \in \mathbb{C}$ algebraic integers
of absolute value $\sqrt{q}$. Thus we have

$$c(r) = q^r + 1 - \sum_{i=1}^{2g} \alpha_i^r$$

and this gives the famous Hasse-Weil bound

$$|\#C(\mathbb{F}_q) - (q + 1)| \leq 2g\sqrt{q},$$

or more precisely

$$|\#C(\mathbb{F}_q) - (q + 1)| \leq \lfloor 2g\sqrt{q} \rfloor.$$  

(Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.) Using some algebraic number theory
Serre improved this in 1983 to:

$$|\#C(\mathbb{F}_q) - (q + 1)| \leq g\lfloor 2\sqrt{q} \rfloor.$$  

We denote by $N_q(g)$ the maximum number of rational points on a curve of genus $g$ over $\mathbb{F}_q$. Then by the above we have immediately

$$N_q(g) \leq q + 1 + g\lfloor 2\sqrt{q} \rfloor.$$  

For $g$ large with respect to $q$ there is a better bound. The basic idea for this is due
to Ihara [I]. Suppose that $\#C(\mathbb{F}_q)$ is large, then the $\alpha_i$ occurring above are near to $-\sqrt{q}$
in $\mathbb{C}$. Hence the $\alpha_i^2$ are near to $q$, so $\#C(\mathbb{F}_q^2)$ is small, but we have $\#C(\mathbb{F}_q) \leq \#C(\mathbb{F}_q^2)$. Working this out with Cauchy-Schwarz gives a better bound. This bound (the Ihara bound) is $N_q(g) \leq q + 1 + [(\sqrt{8q + 1})g^2 + 4(q^2 - q)g - g]/2$. Combined with Serre's bound we find

$$N_q(g) \leq \min\left\{q + 1 + g\lfloor 2\sqrt{q} \rfloor, q + 1 + [(\sqrt{8q + 1})g^2 + 4(q^2 - q)g - g]/2\right\}.$$
which is better than the Hasse-Weil bound for $g > (q - \sqrt{q})/2$.

Ihara’s method was generalized by Drinfeld and Vladut in [D-V] and by Serre. Serre uses the idea of a ‘formule explicite’. Take a trigonometric polynomial

$$f = 1 + 2 \sum_{n \geq 1} u_n \cos n\theta$$

which is even with real coefficients $u_n$ such that $f$ satisfies i) $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$, ii) $u_n \geq 0$ for all $n \geq 1$. We set

$$\psi_1 = \sum_{n \geq 1} u_n t^n.$$  

Then we have the estimate:

$$N \leq a_f g + b_f,$$

with

$$a_f = \frac{1}{\psi_1(1/\sqrt{q})} \quad \text{and} \quad b_f = 1 + \frac{\psi_1(\sqrt{q})}{\psi_1(1/\sqrt{q})}.$$  

The game is to find the appropriate function $f$, cf. [S 1]. Oesterlés managed to find optimal choices for such $f$; we refer to [Sch] for an exposition of the Oesterlé algorithm.

Consider now the case where $g \leq (q - \sqrt{q})/2$. It was conjectured by Stichtenoth and Xing that if $q$ is a square and if $C$ is a maximal curve of genus $g$ over $\mathbb{F}_q$, i.e. with $\#C(\mathbb{F}_q) = q + 1 + 2g\sqrt{q}$, then one should have

$$g \leq (\sqrt{q} - 1)^2/4 \quad \text{or} \quad g = (q - \sqrt{q})/2.$$  

After considerable progress by Stichtenoth and Xing [S-X] this was recently proved by Fuhrmann and Torres [F-T]. This excludes certain values for $N_q(g)$ and improves the Hasse-Weil bound in some places (slightly).

Although the present paper deals with curves of small genus over a small field it is natural to study the asymptotics of $N_q(g)$. Define

$$A(q) = \limsup N_q(g)/g \quad \text{as} \quad g \to \infty.$$  

The Drinfeld-Vladut bound gives

$$A(q) \leq \sqrt{q} - 1$$

and then equality for $q = p^{2m}$ follows since in that case Ihara proves $A(q) \geq \sqrt{q} - 1$ using modular curves. Tsfasman, Vladut and Zink used this to construct geometric Goppa codes which are better than the Gilbert-Varshamov bound in coding theory, see [T-V-Z]. Recently, for $q$ a square Garcia and Stichtenoth gave an explicit construction with repeated Artin-Schreier extensions of a sequence of curves for which $N_q(g)/g$ goes to $\sqrt{q} - 1$, cf. [G-S1].

Finally, we want to mention a recent survey paper [G-S2] in this field of a different nature.
2. Trace Codes, Higher Weights and Curves

We consider a (linear) code of length $n$: an $\mathbb{F}_q$-linear subspace $C$ of $\mathbb{F}_q^n$. The elements of $C$ are called code words. A good deal of information of a code is stored in the weight distribution, i.e. in the polynomial

$$A = \sum_{c \in C} X^{w(c)} = \sum_{i=0}^{n} A_i X^i \quad (\in \mathbb{Z}[X])$$

where $w(c) = \#\{i : c_i \neq 0\}$ is the weight of the word $c = (c_1, \ldots, c_n)$ and $A_i = \#\{c \in C : w(c) = i\}$ is the frequency of weight $i$.

Although this is a very simple situation, for some naturally defined codes it turns out to be very difficult to determine the weight distribution and determining the weight distribution is one of the central problems of coding theory.

To convince the algebraic geometer of the difficulty of this, we consider the classical Reed-Muller codes $R(r,m)$ of order $r$ over $\mathbb{F}_p$. For the vector space $P_r = \{f \in \mathbb{F}_p[X_1, \ldots, X_m] : \deg(f) \leq r\}$ we have an evaluation map $\beta : P_r \rightarrow \mathbb{F}_p^n$ with $n = p^m$ given by $f \mapsto (f(v)_{v \in \mathbb{F}_p^m})$. The image $\beta(P_r)$ is the code $R(r,m)$. The weight distribution of these codes is unknown for $r \geq 3$. Indeed, e.g. for $r = 3$ we need to know the distribution of the number of points on all cubic hypersurfaces (of a fixed space) and this seems out of reach.

For reasons of simplicity and practice one is often first interested in codes over a prime field, especially over $\mathbb{F}_2$ and $\mathbb{F}_3$. In order to get a code over the prime field from a code over an extension field we have two methods: applying the trace map or the restriction map. If $\tilde{C} \subset \mathbb{F}_q^n$ is a (linear) code of length $n$ over $\mathbb{F}_q$, then the trace code $\text{Tr}(\tilde{C})$ is obtained by applying the usual trace map $\text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ to all coordinates of all codewords in $\tilde{C}$. The trace map is the most important for us. Cyclic codes and many other classical codes or their duals are trace codes.

A basic example of trace codes that we shall exploit in the sequel is obtained as follows.

Fix $h \geq 0$ and let

$$\mathcal{R}_h = \{ R = \sum_{i=0}^{h} a_i X^{p^i} : a_i \in \mathbb{F}_q \}.$$

This is a $\mathbb{F}_q$-vector space of additive polynomials. We define the code $C_h$ as

$$C_h = \{ c_R = (\text{Tr}(xR(x)))_{x \in \mathbb{F}_q} : R \in \mathcal{R}_h \}.$$

Since additive polynomials behave as $\mathbb{F}_p$-linear functions the expression $\text{Tr}(xR(x))$ defines a quadratic form on the $\mathbb{F}_p$-vector space $\mathbb{F}_q$. Thus the code $C_h$ is a subcode of the classical binary Reed-Muller code $R(2,m)$.

Instead of the code $C_h$ above we can also consider the modified code $C_h^*$ obtained from $C_h$ by puncturing at $x = 0$, that is, deleting the coordinate corresponding to $x = 0$. It has the same weights as $C_h$. For $h = 1, 2$ the code $C_h^*$ is the dual of the classical 2- and 3- error correcting BCH-codes BCH(2) and BCH(3).
Other examples of classical codes which are trace codes and useful for our purposes include the dual Melas codes $M(q) \perp$ of length $q - 1$, the words of which are

$$c_{a,b} = (\text{Tr}(ax + b/x))_{x \in \mathbb{F}_q^*} \quad \text{for} \quad a, b \in \mathbb{F}_q.$$ 

A simple observation links trace codes with the theory of algebraic curves. The word $c_R = (\text{Tr}(xR(x)))_{x \in \mathbb{F}_q} \in C_h$ has weight

$$w(c_R) = \#\{x \in \mathbb{F}_q : \text{Tr}(xR(x)) \neq 0\} = q - \frac{1}{p}(\#C_R(\mathbb{F}_q) - 1),$$ 

where $C_R$ is the (complete smooth) algebraic curve over $\mathbb{F}_q$ given by $y^p - y = xR(x)$. The relation (1) is a simple consequence of the observation that

$$\text{Tr}(a) = 0 \iff \text{there exists } b \in \mathbb{F}_q \text{ with } b^p - b = a.$$ 

We find a correspondence $c_R \leftrightarrow C_R$ between non-zero words of $C_h$ and algebraic curves. Thus codes lead to families of curves and the weight distribution is equivalent to the distribution of the number of points on the curves in our family. It explains why the weight distributions of such codes are hard to get by. Note that words with low weight correspond to curves with many points.

The dual Melas codes mentioned above lead to families of curves $C_{a,b}$ given by $y^p - y = ax + b/x$ with

$$w(c_{a,b}) = q - 1 - \frac{1}{p}(\#C_{a,b}(\mathbb{F}_q) - 2).$$ 

For $p = 2$ this gives a (universal) family of elliptic curves, while for $p = 3$ we find a family of genus 2 curves which also leads to a universal family of elliptic curves, see [G-V 1]. The codes $C_h$ lead to supersingular curves and these were studied in [G-V 2].

We now come to the higher weight distribution of a code. Although the concept of the higher weights could have been introduced long ago, interest for it arose only recently and was motivated by applications, see [W]. Define for an $r$-dimensional linear subcode (subspace) $D \subseteq C$ of a code $C$ of length $n$ over $\mathbb{F}_p$ the weight $w(D)$ by

$$w(D) = \frac{1}{p^r - p^{r-1}} \sum_{c \in D} w(c).$$ 

An alternative definition is

$$w(D) = n - \#\{i : 1 \leq i \leq n, c_i = 0 \text{ for all } c \in D\}.$$ 

The $r$-th generalized Hamming weight $d_r(C)$ of $C$ is

$$d_r(C) = \min\{w(D) : D \subseteq C, \dim(D) = r\}.$$
So $d_1(C)$ is the usual minimum distance of $C$. The weight hierarchy of $C$ is defined as
\[ \{d_r : 1 \leq r \leq \dim(C)\}. \]

For the trace codes above we saw that the words in the code are related to curves in a family of curves. Similarly, the subcodes are related to fibre products of such curves. To be more precise, we start with a trace code $C$ with words of the form
\[ c_f = (\text{Tr}(f(x)))_{x \in \mathbb{P}^1(\mathbb{F}_q) - P}. \]
Here $f$ runs over a finite-dimensional $\mathbb{F}_q$-subspace $\mathcal{L}$ of the function field $\mathbb{F}_q(x)$ of $\mathbb{P}^1$. We require that the non-zero elements of $\mathcal{L}$ are not of the form $g^p - g + a$ with $g \in \mathbb{F}_q$ and $a \in \mathbb{F}_q$. Furthermore, $P$ is a fixed subset of $\mathbb{P}^1(\mathbb{F}_q)$ such that the elements of $\mathcal{L} - \{0\}$ have poles only in $P$. Note that for the codes $C_h$ we have $P = \{\infty\}$ and for $M(q)^\perp$ the pole set $P = \{0, \infty\}$.

Take an $r$-dimensional subcode $D$ of $C$ with basis $c_{f_1}, \ldots, c_{f_r}$. To these words correspond functions $f_1, \ldots, f_r$ from $\mathcal{L}$ and they span an $r$-dimensional $\mathbb{F}_p$-subspace $\mathcal{L}_D$ of $\mathcal{L}$. Let $C_{f_i}$ be the curve defined by $y^p - y = f_i(x)$. Each of these is an Artin-Schreier extension of $\mathbb{P}^1$ and thus admits a map $\phi_i : C_{f_i} \to \mathbb{P}^1$. Then we can associate to the subcode $D$ the curve
\[ C_D = \text{Normalization of } C_{f_1} \times_{\mathbb{P}^1} \ldots \times_{\mathbb{P}^1} C_{f_r}. \]
Up to isomorphism this does not depend on the chosen basis.

We then have the following generalization of the relation between the weight of the subcode $D$ and the number of points of $C_D$, see [G-V 3]:

(2.1) **Proposition.** The weight of the $r$-dimensional subcode $D$ of a trace code $C$ of length $n$ is related to the number of points of $C_D$ by
\[ w(D) = n - (\#C_D(\mathbb{F}_q) - \sum_{Q \in P} p^{\varepsilon(Q)})/p^r, \] (3)
where $\varepsilon(Q) = \dim_{\mathbb{F}_p}\{f \in \mathcal{L}_D : f \text{ is regular at } Q\}$.

Of course, we need to know the trace of Frobenius for $C_D$. If $\tau_f$ denotes the trace of Frobenius on $C_f$ (i.e. $\#C_f(\mathbb{F}_q) = q + 1 - \tau_f$) then we have
\[ (p - 1)\tau_D = \sum_{f \in \mathcal{L}_D - \{0\}} \tau_f. \] (4)
This can be proved by analyzing the number of points on $C_D$ in relation to the number of points on the $C_f$, as we did in [G-V 6]. It implies the existence of an isogeny
\[ \text{Jac}(C_D) \sim \prod_{f \in \mathbb{F}^{*}(\mathcal{L}_D)} \text{Jac}(C_f), \] (5)
where the summation is over a complete set of representatives for the canonical action of $\mathbb{F}_p^*$ on $\mathcal{L}_D - \{0\}$. Note that we have isomorphisms $C_f \cong C_{\lambda f}$ for any $\lambda \in \mathbb{F}_p^*$, so
that it makes sense to sum over $f$ in the projective space of the $\mathbb{F}_p$-vector space $\mathcal{L}_D$.

Alternatively, using some Galois theory one can prove (5) and then deduce (4) from (5).

For the genera of the curves we have

$$(p - 1)g(C_D) = \sum_{f \in \mathcal{L}_D - \{0\}} g(C_f). \quad (6)$$

We see that if we minimize the weight of subcodes $D$ of a fixed dimension we are maximizing the number of $\mathbb{F}_q$-rational points on curves of the form $C_D$. Of course, we need to control the genera of the curves $C_f$ with $f \in \mathcal{L}_D - \{0\}$.

3. Curves with Many Points

In this section we survey several methods to obtain linear spaces of curves with many points. These methods are closely related to coding theory. By taking the fibre product corresponding to these linear spaces we obtain new curves with many points which are distinguished by the fact that they are given quite explicitly. We illustrate each method with some examples of the results. By combining the methods we find results that go beyond those of [G-V 3-8]; there the reader can find a more detailed description of the methods and of some of the results.

**Method I.** For $q = p^m$ with $m$ even the kernel of the canonical $\mathbb{F}_p$-linear map

$$\phi : \mathcal{R}_{m/2} \rightarrow C_{m/2} \quad \text{defined by} \quad \phi(R) = (\text{Tr}(xR(x))_{x \in \mathbb{F}_q})$$

has dimension $m/2$ and consists of the additive polynomials $R = ax\sqrt{q}$ with $a \in \mathbb{F}_q^*$ satisfying $a\sqrt{q} + a = 0$. For $a \neq 0$ the corresponding curves, given by an affine equation

$$y^p - y = ax\sqrt{q} + 1,$$

have genus $g = (p - 1)\sqrt{q}/2$ and all have $pq + 1$ points over $\mathbb{F}_q$.

Observe that these curves attain the Hasse-Weil upper bound. Applying the fibre product construction and using the relations (4) and (6) yields immediately the following result.

**(3.1) Theorem.** For $q = p^m$ with $m$ even and $1 \leq r \leq m/2$ there is an $r$-dimensional subspace $L$ of curves over $\mathbb{F}_q$ with $pq + 1$ rational points. The corresponding fibre product yields a maximal curve $C_L$ over $\mathbb{F}_q$ with

$$g(C_L) = (p^r - 1)\sqrt{q}/2 \quad \text{and} \quad #C_L(\mathbb{F}_q) = p^rq + 1.$$  

For $q = p^m$ with $m$ odd the kernel of the canonical map

$$\phi : \mathcal{R}_{(m+1)/2} \rightarrow C_{(m+1)/2} \quad \text{defined by} \quad \phi(R) = (\text{Tr}(xR(x))_{x \in \mathbb{F}_q})$$

has dimension $m$ and consists of the additive polynomials

$$R = ax^{p^{(m-1)/2}} - (ax)^{p^{(m-1)/2}} \quad \text{with} \quad a \in \mathbb{F}_q.$$  

For $a \neq 0$ these curves have genus $(p - 1)\sqrt{pq}/2$ and $pq + 1$ points over $\mathbb{F}_q$. In the same way as above we find:
(3.2) Theorem. For $q = p^m$ with $m$ odd and $1 \leq r \leq m$ there is an $r$-dimensional subspace $L$ of curves over $\mathbb{F}_q$ with $pq + 1$ rational points. The corresponding fibre product yields a curve $C_L$ over $\mathbb{F}_q$ with

$$g(C_L) = (p^r - 1)\sqrt{pq}/2$$

and

$$\#C_L(\mathbb{F}_q) = p^r q + 1.$$ 

This gives some immediate improvements of Wirtz’s table:

(3.3) Example. $p = 2$

| Field | dim | genus | $\#C_r(\mathbb{F}_q)$ | Wirtz | upper bound |
|-------|-----|-------|-----------------------|-------|-------------|
| $\mathbb{F}_8$ | 1   | 2     | 17                    | 18    |             |
|       | 2   | 6     | 33                    | 25 - 36 |             |
|       | 3   | 14    | 65                    | 65    |             |
| $\mathbb{F}_{32}$ | 1   | 4     | 65                    | 65 - 77 |             |
|       | 2   | 12    | 129                   |       | 165         |
|       | 3   | 28    | 257                   | 137 - 298 |         |
|       | 4   | 60    | 513                   |       | 542         |
|       | 5   | 124   | 1025                  |       | 1025        |
| $\mathbb{F}_{128}$ | 1   | 8     | 257                   | 257 - 305 |         |
|       | 2   | 24    | 513                   |       | 657         |
|       | 3   | 56    | 1025                  |       | 1361        |

(3.4) Example. $p = 3$

| Field | dim | genus | $\#C_r(\mathbb{F}_q)$ | Wirtz | upper bound |
|-------|-----|-------|-----------------------|-------|-------------|
| $\mathbb{F}_3$ | 1   | 3     | 10                    | 10    |             |
| $\mathbb{F}_{27}$ | 1   | 9     | 82                    | 82 - 110 |             |
|       | 2   | 36    | 244                   | 184 - 319 |         |
|       | 3   | 117   | 730                   |       | 877         |

Variations on this theme in which one avails oneself of other trace forms which vanish on $\mathbb{F}_q$ also yield quite a good harvest. We mention a few fruitful ones:

- $\text{Tr}(x^t(ax^q - ax))$ for $a \in \mathbb{F}_{q=p^m}$ and $\gcd(t, p) = 1$.
- $\text{Tr}(ax^r - a^{p^t}x^s)$ for $a \in \mathbb{F}_{q=p^m}$ and $s \equiv rp^t \pmod{p^m - 1}$.
- $\text{Tr}(x(ax^{p^m-1} + b^{p^{m/2}} - a^p x^p))$ for $a \in \mathbb{F}_{q=p^m}$ with $m \equiv 0 \pmod{2}$ and $b \in \mathbb{F}_q$ satisfying $b\sqrt{q} + b = 0$.

(3.5) Example. Using for $\mathbb{F}_4$ the building blocks $\text{Tr}(x^3)$, $\text{Tr}(ax^3 + ax)$ with $a \in \mathbb{F}_4 - \mathbb{F}_2$, $\text{Tr}(x^5 + x^2)$, and $\text{Tr}(x^7 + x^4)$ we find the following curves.
The upper bounds shown here are obtained with Oesterlé’s optimal choice algorithm. In the sequel we call them Oesterlé bounds.

Method II. This method consists of studying for \( R \in \mathcal{R}_h \) (with \( R \neq 0 \)) the quadratic form \( Q = \text{Tr}(xR(x)) \) on the \( \mathbb{F}_p \)-vector space \( \mathbb{F}_q \) associated to words in \( C_h \). The bilinear form

\[
B(x, y) = \text{Tr}(xR(y) + yR(x))
\]

is symmetric with radical

\[
W = \{ x \in \mathbb{F}_q : B(x, y) = 0 \text{ for all } y \in \mathbb{F}_q \}.
\]

We denote the dimension of the \( \mathbb{F}_p \)-space \( W \) by \( w \). For \( p \neq 2 \) the quadratic form \( Q \) has rank \( m - w \). For \( p = 2 \) the form \( B(x, y) \) is symplectic and this implies \( w \equiv m \pmod{2} \). Moreover, if \( a_h \neq 0 \) then \( 0 \leq w \leq 2h \).

From now on we shall restrict to the case \( p = 2 \). In characteristic 2 we also consider the space

\[
W_0 = \{ x \in W : Q(x) = 0 \}.
\]

We have \( W = W_0 \) or \( \dim(W_0) = \dim(W) - 1 \). From the theory of quadratic forms we deduce that the rank of \( Q \) satisfies

\[
\text{rank}(Q) = \begin{cases} 
m - w & \text{if } W = W_0, \\
m - w + 1 & \text{if } W \neq W_0. 
\end{cases}
\]

If \( W \neq W_0 \) the number of points on \( Q \) is \( q/2 \), while if \( W = W_0 \) it is either \( (q + \sqrt{2wq})/2 \) or \( (q - \sqrt{2wq})/2 \). The sign depends on the Witt-index of the quadratic form.

To construct suitable quadrics we start with elements \( a_i, b_i \) for \( i = 1, \ldots, (m - w)/2 \). We require that these \( m - w \) elements are \( \mathbb{F}_2 \)-linearly independent in \( \mathbb{F}_q \). Here \( w \) is an integer with \( w \equiv m \pmod{2} \). Then we claim that

\[
Q_{a, b} = \sum_{i=1}^{(m-w)/2} \text{Tr}(a_i x) \text{Tr}(b_i x)
\]
is a quadratic form with \((q + \sqrt{q^2w})/2\) zeros in \(\mathbb{F}_q\). Note that \(\text{Tr}(ax)\) is a linear form so that the expression defines indeed a quadratic form. By simple linear algebra one transforms the quadratic form into the form \(X_1X_2 + \ldots X_{m-w-1}X_{m-w}\) with Witt-index \((m - w)/2\) and this implies the result. So we can associate to

\[\{a_1, \ldots, a_{(m-w)/2}, b_1, \ldots, b_{(m-w)/2}\}\]

a quadratic form \(Q_{(a,b)}\). We now try to arrange this so that \(Q_{(a,b)}\) lies in the code \(C_h\) or rather \(C_h^\ast\), see Section 2. That means that the coefficients in \(\sum_{i=1}^{(m-w)/2} \text{Tr}(a_ix)\text{Tr}(b_ix) = \sum_{i=1}^{(m-w)/2} \text{Tr}(\text{Tr}(a_ix)b_ix)\) of terms \(x^{2j+1}\) with \(j > h\) must disappear. The condition is:

\[(3.6) \text{Proposition. If the elements } a_i, b_i \in \mathbb{F}_q \text{ with } 1 \leq i \leq (m - w)/2 \text{ satisfy the system of equations}
\]

\[\sum_{i=1}^{(m-w)/2} (a_i^{2j}b_i + a_i^{-1}b_i^{2j}) = 0\]  \(\text{for } j = h + 1, \ldots, (m - 1)/2\) then the word of length \(q - 1\) obtained by evaluating \(\sum_{i=1}^{(m-w)/2} \text{Tr}(a_ix)\text{Tr}(b_ix)\) on \(\mathbb{F}_q^\ast\) is a codeword in \(C_h^\ast\).

Note that these equations define so-called Deligne-Lusztig varieties. For fixed \(a_1, a_2, \ldots, a_{(m-w)/2}\) the words induced by the solutions \(b_1, \ldots, b_{(m-w)/2}\) of (7) form a linear space of minimum weight words in \(C_h^\ast\). In [G-V 8] we analyzed the dimension of this space. We find for example:

\[(3.7) \text{Theorem. For } 1 \leq r \leq m-3 \text{ with } m \geq 5 \text{ and } m \text{ odd there exist curves } C_r \text{ defined over } \mathbb{F}_{2^m} \text{ of genus } g(C_r) = (2^r - 1)2^{(m-5)/2} \text{ and with } \#C_r(\mathbb{F}_{2^m}) = 2^m + 1 + (2^r - 1)2^{m-2}.
\]

\[(3.8) \text{Example. For } m = 7 \text{ Theorem (3.7) yields:}
\]

- There exist curves \(C_r\) defined over \(\mathbb{F}_{128}\) with:

| \(r\) | \(g(C_r)\) | \#\(C_r(\mathbb{F}_{128})\) | Wirtz |
|------|-------------|----------------|------|
| 1    | 2           | 161            | 172  |
| 2    | 6           | 225            | 225 - 261 |
| 3    | 14          | 353            | 289 - 437 |
| 4    | 30          | 609            | 369 - 789 |

\[(3.9) \text{Theorem. For } 1 \leq r \leq (m - 2)/2 \text{ with } m \text{ even and } m \geq 4 \text{ there exist curves } C_r \text{ defined over } \mathbb{F}_{2^m} \text{ of genus } g(C_r) = (2^r - 1)2^{(m-4)/2} \text{ and with } \#C_r(\mathbb{F}_{2^m}) = 2^m + 1 + (2^r - 1)2^{m-1}; \text{ these curves attain the Hasse-Weil bound.}
\]

\[(3.10) \text{Example. For } m = 6 \text{ (resp. } m = 8\) this Theorem yields:
\]

- There exist curves \(C_r\) defined over \(\mathbb{F}_{64}\) (resp. over \(\mathbb{F}_{256}\)) as indicated:
Method III. The third approach also arose from quadratic forms associated to the codes \( C_h \). In studying the weight distribution of \( C_h \) in [G-V 2] it was convenient first to determine for fixed \( R \in \mathcal{R}_h \) with \( R \neq 0 \) the weight distribution in the 1-dimensional family \( \mathcal{F}_R \) of words
\[
(\text{Tr}(xR(x) + bx))_{x \in \mathbb{F}_q} \quad \text{with} \quad b \in \mathbb{F}_q.
\]
The weight distribution of \( \mathcal{F}_R \) follows from the theory of quadratic forms over finite fields. Only few weights occur and the values of \( b \) which provide minimum weight words or curves with many points happen to lie in quadratic spaces. Using the Witt-index, which determines the dimension of a maximal totally singular subspace in the quadratic space, we find linear or affine spaces of curves with many points. Again we have a situation in which we can profitably apply the fibre product construction. From [G-V 7] we recall a result for fields \( \mathbb{F}_q = p^m \) with \( p \) an odd prime.

(3.11) Theorem. i) If \( m \) is odd there exists for \( 1 \leq r \leq (m - 1)/2 \) a curve \( C_r \) of genus \( g = p^r(p - 1)/2 \) with \( \#C_r(\mathbb{F}_q) = q + 1 + p^r\sqrt{pq} \). ii) If \( m \) is even there exists for \( 1 \leq r \leq m/2 \) a curve \( C_r \) of genus \( g = p^r(p - 1)/2 \) which attains the Hasse-Weil upper bound: \( \#C_r(\mathbb{F}_q) = q + 1 + p^r(p - 1)\sqrt{q} \).

If we specialize to characteristic 3 and use some specific properties of the occurring situations we derive the following results.

(3.12) Theorem. i) For \( q = 3^m \) with \( m \) odd, \( m \geq 3 \) there exists a curve of genus 4 over \( \mathbb{F}_q \) with \( q + 1 + 4\sqrt{3q} \) points over \( \mathbb{F}_q \). ii) For \( q = 3^m \) with \( m \equiv 0(\mod 4) \) there exists a curve of genus 12 over \( \mathbb{F}_q \) which realizes the Hasse-Weil upper bound.

(3.13) Example.

| Field | genus | \#C_r(\mathbb{F}_q) | Wirtz | upper bound |
|-------|-------|---------------------|-------|-------------|
| \( \mathbb{F}_{27} \) | 3 | 55 | 55 - 58 | |
| | 4 | 64 | 55 - 68 | |
| \( \mathbb{F}_{81} \) | 12 | 298 | 226 - 298 | |
| \( \mathbb{F}_{243} \) | 3 | 325 | | 337 |
| | 4 | 352 | | 368 |
| | 9 | 487 | | 523 |
Method IV. This method uses results from coding theory, especially results about weight hierarchies of codes. From formula (3) which describes the relation between the weight of a subcode $D$ and $\#C_D(\mathbb{F}_q)$ we derive immediately:

(3.14) Proposition. i) For an $r$-dimensional subcode $D$ of the punctured code $C_h^*$ of weight $w(D)$ the corresponding fibre product curve $C_D$ satisfies:

$$g(C_D) = \frac{1}{p-1} \sum_{f\in L_D-\{0\}} g(C_f) \leq (p^r - 1)p^h/2$$

and

$$\#C_D(\mathbb{F}_q) = p^r(q-w(D)) + 1.$$  

ii) For an $r$-dimensional subcode $D$ of the dual Melas code $M(q)\perp$ of weight $w(D)$ the corresponding curve $C_D$ satisfies:

$$g(C_D) = \frac{1}{p-1} \sum_{f\in L_D-\{0\}} g(C_f) \leq p^r - 1$$

and

$$\#C_D(\mathbb{F}_q) \geq p^r(q-1-w(D)) + 2.$$  

Equality holds in (8) if $D$ contains no code words corresponding to rational curves.

An interesting situation appears if $D$ is a subcode of which the weight equals the $r$-th generalized Hamming weight of the code. To determine the value of the genus $g(C_D)$ one has to investigate carefully the words occurring in $D$. In the following examples we apply Proposition (3.14).

(3.15) Example. For $p = 2$ and $q = 2^m$ with $m \geq 5$ we proved in [G-V 5] that the second generalized Hamming weight of $BCH(2)\perp$ of length $q - 1$ satisfies:

$$d_2(BCH(2)\perp) = d_2(C_1^*) = \frac{3}{2}d_1(BCH(2)\perp) = \begin{cases} 3(q - \sqrt{2}q)/4 & \text{for } m \text{ odd,} \\ 3(q - 2\sqrt{q})/4 & \text{for } m \text{ even.} \end{cases}$$

The curves corresponding to the 2-dimensional subcode in which the non-zero words all have minimum weight, have genus 1. We deduce from Proposition (3.14):

(3.16) Theorem. For $m \geq 5$ there exists a curve $C$ over $\mathbb{F}_{2^m}$ of genus 3 with $\#C(\mathbb{F}_q) = q + 1 + 3\sqrt{q}$ for $m$ odd, while $C$ attains the Hasse-Weil upper bound for $m$ even.

(3.17) Example. Since the code $BCH(2)$ of length 7 is the very simple repetition code $\mathbb{F}_2 \cdot \mathbf{1} \in \mathbb{F}_2^7$ the weight hierarchy of $BCH(2)\perp$ of length 7 is:

$$\{2, 3, 4, 5, 6, 7\}.$$  

From Proposition (3.14) and an analysis of the genera of the curves belonging to the subcodes we get
**Example.** For \( p = 2 \) and \( h = \lceil m/2 \rceil \) one derives by equating dimensions that the code \( \mathbb{F}_2 \cdot 1 + C_h \) is the binary second order Reed-Muller code \( R(2, m) \) of length \( 2^m \). Elementary properties of the minimum weight words in \( R(2, m) \) imply that

\[
d_r = (2^r - 1)d_1 = (2^r - 1)2^{m-2}/2^{r-1} \quad \text{for} \quad 1 \leq r < m.
\]

For odd \( m \) the curves associated to minimum weight words have genus \( 2^{h-1} \). If we apply Proposition (3.14) to an \( r \)-dimensional subcode of minimum weight words we find:

**Theorem.** For odd \( m \) with \( m \geq 3 \) and \( r \) with \( 1 \leq r < m \) there exists a curve \( C_r \) over \( \mathbb{F}_{q=2^m} \) of genus \( g = (2^r - 1)\sqrt{q}/8 \) with \( \#C_r(\mathbb{F}_q) = q + 1 + (2^r - 1)q/2 \).

Theorem (3.19) yields for instance:

| Field | \( r \) | \( d_r(BCH(2)^{\perp}) \) | genus | \( \#C_r(\mathbb{F}_q) \) | Wirtz |
|-------|--------|----------------|-------|-------------------|-------|
| \( \mathbb{F}_8 \) | 1 | 2 | 1 | 13 | 14 |
| | 2 | 3 | 3 | 21 | 24 |
| | 3 | 4 | 6 | 33 | 25 – 36 |
| | 3 | 4 | 7 | 33 | 25 – 39 |

We conclude with an example in which Melas codes and dual Melas codes are central (i.e. involving non-supersingular elliptic curves).

**Example.** From [G-V 4] one can derive the weight hierarchy of the dual Melas code \( M(16)^{\perp} \) of length 15:

\[
\{4, 6, 8, 9, 11, 12, 14, 15\}.
\]

For \( 1 \leq r \leq 4 \) one can find an \( r \)-dimensional subcode realizing the first four generalized Hamming weights in which all the non-zero words belong to elliptic curves. Application of Proposition (3.14) provides us with the following table.

| Field | \( r \) | genus | \( \#C_r(\mathbb{F}_q) \) | Wirtz |
|-------|--------|-------|-------------------|-------|
| \( \mathbb{F}_{16} \) | 1 | 4 | 1 | 24 | 25 |
| | 2 | 6 | 3 | 38 | 38 |
| | 3 | 8 | 7 | 58 | 49 – 70 |
| | 4 | 9 | 15 | 98 | 49 – 113 |

For other results where we use Melas codes and dual Melas codes we refer to [G-V 3] and [G-V 7].
We finish this section by observing that combining the methods, especially I and IV, can be rewarding. We mention some results for $\mathbb{F}_8$.

(3.21) Example.

| genus | $\#C(\mathbb{F}_8)$ | upper bound |
|-------|----------------------|-------------|
| 5     | 29                   | 32          |
| 11    | 41                   | 54          |
| 13    | 49                   | 61          |
| 18    | 57                   | 77          |
| 22    | 65                   | 89          |
| 23    | 65                   | 92          |
| 27    | 81                   | 103         |
| 29    | 97                   | 109         |
| 38    | 113                  | 135         |

The upper bound in the table is Oesterlé’s.

4. Final Remarks

There are of course other ways of producing curves with many points. Besides a direct computer search on certain families as performed by Wirtz, one could use specific curves like modular curves. In our tables we have incorporated some values obtained from modular curves. But more in line with the spirit of this paper, one can also use coverings of curves of genus $>0$ with many points to produce new curves with many points. Note that we used so far only Artin-Schreier covers of $\mathbb{P}^1$, but there is no reason to stick to this restriction. For example, let $C$ be a complete non-singular curve over $\mathbb{F}_q$ with genus $g$ whose $\mathbb{F}_q$-rational points are $\{P_1, \ldots, P_m\}$. Furthermore, let $Q = Q_n$ be a divisor of degree $n$ defined over $\mathbb{F}_q$ whose support contains none of the $P_i$. Let $L \subset \mathbb{F}_q(C)$ be an $r$-dimensional $\mathbb{F}_p$-vector space of functions contained in

$$\{f \in \mathbb{F}_q(C)^* : \text{div}(f) \geq -Q\} \cup \{0\}.$$ 

We shall assume that $L - \{0\}$ does not contain elements of the form $g^p - g + a$ with $g \in \mathbb{F}_q(C)$ and $a \in \mathbb{F}_q$. For $f \in L$ we can form the covering

$$C_f \to C, \quad \text{given by} \quad y^p - y = f.$$ 

This is a $p$-fold cover of $C$ ramified only in the poles of $f$. As before we can also define $C_L$ using the fibre product $C_{f_1} \times_C \ldots \times_C C_{f_r}$ for a basis $f_1, \ldots, f_r$ of $L$. The number of points can be expressed as

$$\#C_L(\mathbb{F}_q) = p^r \cdot \#\{P_i : \text{Tr}(f(P_i)) = 0 \text{ for all } f \in L\}.$$ 

The genus and trace of Frobenius can be computed from the isogeny

$$\text{Jac}(C_L) \sim \text{Jac}(C) \times \prod_{f \in \mathbb{F}(L)} P(C_f/C),$$
where the product is over a complete set of representatives in \( L - \{0\} \) for the natural action of \( \mathbb{F}_p^* \) on \( L - \{0\} \). Moreover, \( P(C_f/C) \) stands for the Prym variety of \( C_f \to C \), which is the connected component of the kernel of the norm map \( \text{Nm} : \text{Jac}(C_f) \to \text{Jac}(C) \). We find
\[
\tau_{C_L} = \tau_C + \sum_{f \in \mathcal{P}(L)} (\tau_{C_f} - \tau_C),
\]
a relation that can also be obtained by counting points on the curves \( C_L \) and \( C_f \) as in [G-V 6], and
\[
g(C_L) = g(C) + \sum_{f \in \mathcal{P}(L)} (g(C_f) - g(C)).
\]
We need the genera of the \( C_f \). If we assume that \( f \) has \( n_f \) simple poles we find
\[
g(C_f) = g(C) + (p-1)(g(C) + n_f - 1).
\]

In order to construct curves with many points one starts with a curve \( C \) with many rational points over \( \mathbb{F}_q \) and one considers \( C_L \) such that \( \text{Tr}(f(P)) = 0 \) for all (or at least for many) rational points of \( C \). So we are considering the trace codes associated to geometric Goppa codes.

The simplest case is of course when \( q = p \) so that \( \text{Tr}(f(P)) = 0 \) simplifies to \( f(P) = 0 \).

**Example.** Let \( E \) be an elliptic curve over \( \mathbb{F}_2 \) with 5 rational points \( P_1, \ldots, P_5 \). Let \( Q_n \) be a divisor of degree \( n \geq 5 \) over \( \mathbb{F}_2 \). Then \( \dim_{\mathbb{F}_2}(L(Q_n - (P_1 + \ldots + P_5))) = n-5 \) for \( n \geq 6 \) and 0 or 1 for \( n = 5 \). We find for the choice \( L = L(Q_n - (P_1 + \ldots + P_5)) \) the formulas
\[
g(C_L) = 1 + (2^{n-5} - 1)n, \quad \#C_L(\mathbb{F}_2) = 5 \cdot 2^{n-5} \quad \text{for} \quad n \geq 6.
\]
If we take for \( n = 5 \) a divisor \( Q_5 \) which is linearly equivalent to \( \sum_{i=1}^5 P_i \) then we get a curve \( C_L \) with \( g = 7 \) and \( \#C(\mathbb{F}_2) = 10 \).

One should compare this with the example that Serre gives in [S 1] using class field theory. He defines a series of curves \( C_n \) over \( \mathbb{F}_2 \) with
\[
g(C_n) = 1 + (2^{n-4} - 1)n, \quad \#C_n(\mathbb{F}_2) = 5 \cdot 2^{n-4} \quad \text{for} \quad n \geq 5.
\]
and thus finds a curve of genus 6 with 10 points. The reason for the non-optimality of our example comes from the fact that we require all functions \( f \in L \) to have simple poles in \( Q_n \), which excludes using functions that have multiple poles there but which are Artin-Schreier equivalent to a function \( \tilde{f} \) with simple poles at \( Q_n \).

This phenomenon also explains why often with the methods of Section 3 we find curves with a large number of points, but not the maximum possible. On the other hand, our curves are given explicitly.

**Example.** Variations of Serre’s method provide other good curves. Take a curve of genus 2 over \( \mathbb{F}_2 \) with 6 points. By taking a 4-fold cover ramified in a point of degree 7 we find a curve \( C_{26} \) of genus \( g = 26 \) with \( \#C_{26}(\mathbb{F}_2) = 24 \).

In this context we wish to mention a result which was communicated to us by René Schoof. It follows from class field theory and yields good curves over \( \mathbb{F}_2 \).
(4.3) **Theorem.** Let $C$ be a curve of genus $g$ over $\mathbb{F}_2$ which has $n$ rational points and a point of degree $m$ with $m \geq n$. For $1 \leq l \leq m - n + 1$ there exists a fibre product $\tilde{C}$ over $C$ of degree $2^l$ with

$$\tilde{g} \leq (2^l - 1)(g + m - 1) + g$$

and with

$$\#\tilde{C}(\mathbb{F}_2) = 2^l n.$$

If the inequality for the genus is strict then there exists an Artin-Schreier cover $C_f$ of $C$ with $g(C_f) = 2g - 1$ and $\#C_f(\mathbb{F}_2) = 2n$.

Finally, Drinfeld modules provide a theory of explicit class fields of function fields. Very recently Niederreiter and Xing obtained along these lines some good curves over $\mathbb{F}_3$ and $\mathbb{F}_4$ which we include here in our tables, cf. [N-X] and [X].

**Tables**

We reproduce here the table by Wirtz (in which we corrected a few misprints) and give two other tables. In his table Wirtz mentions not only the intervals for $N_q(g)$, but also indicates for many values where they can be found in the literature or from which curves or families they are obtained. We refrain from giving this information. In many places Wirtz mentions instead of an interval $a - b$ only an upper bound $-b$, for which he takes the Ihara-Serre bound. Note that often the Oesterlé bound is better.

In the other two tables, one for characteristic 2 and one for characteristic 3, we give for $g \leq 50$ an interval $[a, b]$ in which $N_q(g)$ lies. For $b$ we take either the Ihara-Serre bound (taking into account [F-T]) or the Oesterlé bound if that one is better. For $g \leq 50$ these bounds coincide if $q \geq 27$.

We only enter new values in the table if we consider them reasonable. We have taken this to mean that $a \geq \lceil \frac{b}{\sqrt{2}} \rceil$. This ratio is inspired by the fact that the Ihara bound implies

$$A(q) \leq \frac{1}{2}(\sqrt{8q + 1} - 1) < \sqrt{2q}$$

while we know that $A(q) \leq \sqrt{q} - 1 < \sqrt{q}$. We would like to stress here that in no way we aimed at completeness and just entered values or intervals that we happen to know. A systematic search using our methods should lead to improvements and extensions of the tables.
The Table of Wirtz.

| $g \backslash q$ | 2    | 3    | 4    | 8    | 9    | 16   | 27   | 32   | 64   | 81   | 128  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|
| 1               | 5    | 7    | 9    | 14   | 16   | 25   | 38   | 44   | 81   | 100  | 150  |
| 2               | 6    | 8    | 10   | 18   | 20   | 33   | 48   | 53   | 97   | 118  | 172  |
| 3               | 7    | 10   | 14   | 24   | 28   | 38   | 55   | 58   | 62   | 66   | 113  |
| 4               | 8    | 12   | 15   | 29   | 30   | 38   | 41   | 49   | 55   | 62   | 113  |
| 5               | 9    | 15   | 18   | 24   | 28   | 38   | 55   | 58   | 62   | 66   | 113  |
| 6               | 10   | 17   | 21   | 25   | 36   | 39   | 49   | 57   | 68   | 81   | 100  |
| 7               | 10   | 19   | 23   | 25   | 39   | 49   | 64   | 98   | 110  | 154  | 200  |
| 8               | 11   | 21   | 25   | 43   | 47   | 76   | 108  | 121  | 193  | 226  | 257  |
| 9               | 12   | 23   | 28   | 33   | 47   | 51   | 81   | 118  | 132  | 173  | 209  |
| 10              | 12–13| 25   | 30   | 50   | 55   | 61   | 87   | 128  | 143  | 199  | –349 |
| 11              | 13–14| 37   | 57   | 63   | 61   | 91   | 148  | 257  | 226  | 193  | 393  |
| 12              | 14–15| 37   | 57   | 63   | 61   | 91   | 148  | 257  | 226  | 193  | 393  |
| 13              | 14–15| 37   | 57   | 63   | 61   | 91   | 148  | 257  | 226  | 193  | 393  |
| 14              | 15–16| 65   | 43   | 70   | 81   | 108  | 113  | 187  | 193  | 289  | 437  |
| 15              | 17   | 43   | 68   | 49   | 113  | 97   | 196  | 197  | 305  | 369  | 459  |
| 16              | 16–18| 46   | 78   | 100  | 178  | 199  | 321  | 370  |      |      |      |
| 17              | 17–18|      |      |      |      |      |      |      |      |      |      |
| 18              | 18–19|      |      |      |      |      |      |      |      |      |      |
| 19              | 20   |      |      |      |      |      |      |      |      |      |      |
| 20              | 19–21|      |      |      |      |      |      |      |      |      |      |
| 21              | 21   | 57   | 89   | 94   | 145  | 121  | 244  | 257  | 401  | 250  | 460  |
| 24              |      | 64   | 108  | 136  | 235  | 157  | 283  | 313  | 568  | 297  | 591  |
| 26              |      |      |      |      |      |      |      |      |      |      |      |
| 27              |      |      |      |      |      |      |      |      |      |      |      |
| 28              |      | 65   | 114  | 73   | 123  | 97   | 181  | 136  | 263  | 137  | 298  |
| 30              |      | 81   | 192  | 105  | 218  | 157  | 277  | 129  | 313  | 257  | 536  |
| 35              |      | 113  | 254  |      |      | 157  | 360  | 193  | 428  | 401  | 706  |
| 39              |      | 33   | 129  | 129  | 291  |      |      |      |      |      |      |
| 42              |      |      |      |      |      |      |      |      |      |      |      |
| 45              |      |      |      |      |      |      |      |      |      |      |      |
| 48              |      |      |      |      |      |      |      |      |      |      |      |
| 49              |      |      |      |      |      |      |      |      |      |      |      |
| 50              |      |      |      |      |      |      |      |      |      |      |      |
Table p=2.

| $g \setminus q$ | 2   | 4   | 8   | 16  | 32  | 64  | 128 |
|-----------------|-----|-----|-----|-----|-----|-----|-----|
| 1               | 5   | 9   | 14  | 25  | 44  | 81  | 150 |
| 2               | 6   | 10  | 18  | 33  | 53  | 97  | 172 |
| 3               | 7   | 14  | 24  | 38  | 63–66 | 113 | 190–195 |
| 4               | 8   | 15  | 25–29 | 45–49 | 70–77 | 129 | 200–217 |
| 5               | 9   | 17–18 | 29–32 | 49–56 | 73–88 | 130–145 | 227–239 |
| 6               | 10  | 20  | 33–36 | 65  | 81–99 | 161 | 225–261 |
| 7               | 10  | 21–22 | 33–39 | 58–70 | 89–110 | 177 | 241–283 |
| 8               | 11  | 21–24 | 31–43 | 61–76 | 173–209 | 241–327 |
| 9               | 12  | 26  | 36–47 | 62–81 | 193–225 | 289–349 |
| 10              | 13  | 27–28 | 65–87 | 199–225 | 257–305 |
| 11              | 14  | 25–30 | 41–54 | 75–92 | 201–241 |
| 12              | 14–15 | 28–31 | 47–57 | 129–165 | 257 | 321–393 |
| 13              | 15  | 33  | 49–61 | 97–103 | 199–273 |
| 14              | 15–16 | 29–35 | 65  | 97–108 | 145–187 | 353–437 |
| 15              | 17  | 33–37 | 98–113 | 257–304 | 369–459 |
| 16              | 16–18 | 36–38 | 56–71 | 93–118 |
| 17              | 17–18 | 40  | 61–74 |
| 18              | 18–19 | 33–42 | 57–77 | 99–129 |
| 19              | 20  | 36–43 |
| 20              | 19–21 | 61–83 | 121–140 |
| 21              | 21  | 40–47 | 125–145 |
| 22              | 21–22 | 33–48 | 66–89 | 129–150 |
| 23              | 22–23 | 36–50 | 68–92 | 123–155 |
| 24              | 20–23 | 513–657 |
| 25              | 24  |
| 26              | 24–25 | 55  | 129–171 |
| 27              | 22–25 | 49–56 | 81–103 | 401–497 |
| 28              | 24–26 | 91–106 | 129–181 | 257–298 | 513 | 577–745 |
| 29              | 25–27 | 97–109 |
| 30              | 24–27 | 273–313 | 609–789 |
| 31              | 27–28 |
| 32              | 26–29 |
| 33              | 28–29 | 65–66 |
| 34              | 27–30 | 57–68 |
| 35              | 28–31 | 49–69 | 253–352 |
| 36              | 30–31 | 105–130 |
| 37              | 28–32 | 65–72 | 121–132 |
| 38              | 28–33 | 113–135 |
| 39              | 33  | 117–138 |
| 40              | 32–34 |
| 41              | 30–35 | 65–78 |
| 42              | 30–35 | 65–80 |
| 43              | 33–36 |
| 44              | 32–37 | 68–83 | 121–153 |
| 45              | 32–37 | 80–84 | 144–156 | 242–268 |
| 46              | 34–38 | 129–158 |
| 47              | 36–38 | 116–161 |
| 48              | 34–39 | 117–164 |
| 49              | 36–40 | 66–90 |
| 50              | 40  | 91–92 |
Table p=3.

| $g \setminus q$ | 3  | 9  | 27 | 81  |
|-----------------|----|----|----|-----|
| 1               | 7  | 16 | 38 | 100 |
| 2               | 8  | 20 | 48 | 118 |
| 3               | 10 | 28 | 55–58 | 136 |
| 4               | 12 | 28–30 | 64–68 | 154 |
| 5               | 12–14 | 32–36 | 55–78 | 152–172 |
| 6               | 14–15 | 29–40 | 76–88 | 190 |
| 7               | 16–17 |     | 160–208 |   |
| 8               | 15–18 | 38–47 | 218–226 |   |
| 9               | 19 | 40–51 | 82–118 | 244 |
| 10              | 19–21 | 46–55 |     | 226–262 |
| 11              | 20–22 | 55–59 |     |   |
| 12              | 22–24 | 55–63 | 109–148 | 298 |
| 13              | 24–25 | 60–66 | 136–156 |   |
| 14              | 24–26 | 56–70 |     |   |
| 15              | 28 | 58–74 |     |   |
| 16              | 27–29 | 74–78 |     | 370 |
| 17              |     |     |     |   |
| 18              | 24–31 | 74–85 |     |   |
| 19              |     | 76–88 |     |   |
| 20              | 30–34 | 74–91 |     |   |
| 21              | 28–35 | 82–95 |     |   |
| 22              | 28–36 | 76–98 |     |   |
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| 24              | 28–38 | 82–104 |     |   |
| 25              | 30–40 |     | 110–111 |   |
| 26              |     |     |     |   |
| 27              |     |     |     |   |
| 28              | 36–43 |     |     |   |
| 29              |     |     |     |   |
| 30              |     |     |     |   |
| 31              |     |     |     |   |
| 32              | 36–48 |     |     |   |
| 33              |     |     |     |   |
| 34              |     |     |     |   |
| 35              |     |     |     |   |
| 36              |     |     |     | 244–319 |
| 37              |     |     |     | 730 |
| 38              |     |     | 109–145 |   |
| 39              |     |     | 109–152 | 244–339 |
| 40              |     |     | 244–346 |   |
| 41              |     |     |     |   |
| 42              |     |     |     |   |
| 43              |     |     |     |   |
| 44              |     |     |     |   |
| 45              |     |     |     |   |
| 46              | 55–63 |     |     |   |
| 47              |     |     |     |   |
| 48              | 55–66 |     |     |   |
| 49              |     |     |     |   |
| 50              |     | 182–186 |     |   |
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