Perturbative Gravity and Twistor Space

N. E. J. Bjerrum-Bohr\textsuperscript{a, b}, David C. Dunbar\textsuperscript{a} and Harald Ita\textsuperscript{a} SWAT-06/464

\textsuperscript{a}Department of Physics, University of Wales Swansea

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1. Introduction

Tree amplitudes in gravity theories can be computed from those of gauge theory using the KLT relations \cite{1}. Inspired by this observation, investigations \cite{2,3,4,5} have been carried out in the hope of discovering a deeper connection at the perturbative level between $\mathcal{N} = 8$ supergravity \cite{6} and $\mathcal{N} = 4$ super Yang-Mills. A link between these theories, beyond structural similarities like the non-abelian gauge symmetry and maximal supersymmetry, is not obvious and could have surprising implications on the UV behaviour of $\mathcal{N} = 8$ supergravity. After all, $\mathcal{N} = 4$ super Yang-Mills can be proven to be a finite theory, to all orders in perturbation theory \cite{7}. Explicit computations seem to be the way forward in order to derive concrete statements about $\mathcal{N} = 8$ supergravity.

The “weak-weak” duality, between $\mathcal{N} = 4$ super Yang-Mills and a topological string theory propagating in twistor space \cite{8} implies the existence of a single perturbative $S$-matrix for these two theories. Within the $S$-matrix of gauge theory the duality lead to the discovery of surprising structures as the MHV-vertex construction \cite{9,10} and the BCFW recursion relations \cite{11}. Although the duality applies more readily to $\mathcal{N} = 4$ SYM \cite{12} the ideas motived by the twistor space duality have been applied successfully to a much wider range of theories and in particular to gravity. Among other applications are theories both with less supersymmetry \cite{13}, massive particles \cite{14,15} and computations of QCD one-loop amplitudes \cite{16}.

In this talk we discuss how the ideas inspired by the “weak-weak” duality can be applied to calculate one-loop $\mathcal{N} = 8$ supergravity amplitudes. We interprete the result to give evidence for the ”no-triangle” hypothesis, which states that $\mathcal{N} = 8$ supergravity contains neither triangle nor bubble scalar integral functions. The absence of these integral functions implies the de-facto power counting to be similar to that of $\mathcal{N} = 4$ SYM and thus stronger than needed for finiteness at one-loop. Such simplifications require, if they extend beyond one-loop, a rethinking of the ultra-violet structure of maximal supergravity.

2. Tree Amplitudes in Gravity Theories

Tree amplitudes in gravity theories are linked to those of gauge theory \cite{1} via the heuristic relation in string theory,

$$ (\text{closed string)} \sim (\text{open string)}_{\text{Left}} \times (\text{open string)}_{\text{Right}} $$

(2.1)

The concrete realisation of the relationship up to six points at $\alpha' = 0$ is,

$$ M_{[1,2,3]}^{\text{tree}} = -i A_{[1,2,3]}^{\text{tree}} \times A_{[1,2,3]}^{\text{tree}}, $$

$$ M_{[1,2,3,4]}^{\text{tree}} = -i s_{12} A_{[1,2,3,4]}^{\text{tree}} \times A_{[1,2,3,4]}^{\text{tree}}, $$

$$ M_{[1,2,3,4,5]}^{\text{tree}} = i s_{12} s_{34} A_{[1,2,3,4,5]}^{\text{tree}} \times A_{[1,2,3,4,5]}^{\text{tree}} + i s_{13} s_{24} A_{[1,2,3,4,5]}^{\text{tree}} \times A_{[1,3,2,4,5]}^{\text{tree}}, $$

$$ M_{[1,2,3,4,5,6]}^{\text{tree}} = -i s_{12} s_{34} s_{45} A_{[1,2,3,4,5,6]}^{\text{tree}} \times \left[ s_{35} A_{[1,2,3,4,5,6]}^{\text{tree}} + (s_{34} + s_{35}) A_{[2,1,5,3,4,6]}^{\text{tree}} \right] + \mathcal{P}(2,3,4), $$

where $s_{ij} = (k_i + k_j)^2$, $\mathcal{P}(2,3,4)$ represents the sum over permutations of legs 2, 3, 4 and the $A_{ij}^{\text{tree}}$ are tree-level colour-ordered gauge theory partial amplitudes. These relations are the Kawai, Lewellen and Tye (KLT) relations \cite{1}. Even in low energy effective field theories for gravity \cite{17,18} the KLT-relations can be seen to be valid.
For Yang-Mills amplitudes, the twistor space duality has motivated the development of a MHV-vertex reformulation for tree amplitudes. In this the Parke-Taylor expressions for MHV amplitudes \[ A^{\text{tree}}_{[1^+,...,j^-,...,k^-,...,n^+]} = i \frac{(j \ k)^4}{(1 \ 2) \ (2 \ 3) \cdots (n \ 1)}, \tag{2.3} \]

are promoted to vertices for a diagrammatic expansion \[9\]. The twistor variables for the intermediate momentum between vertices are calculated through the relation \( \lambda_a(q) = q_{ab} \eta^b \) with \( \eta \) being a reference spinor. Since the Parke-Taylor expression only involves holomorphic variables \( \lambda_a \) and not anti-holomorphic variables \( \lambda_a \), this is sufficient to define the expansion. Gravity MHV amplitudes \[20\] involve both \( \eta^a \) and \( \overline{\eta}^a \) shifts \[23\] in the anti-holomorphic variables \( \overline{\eta}^a \). It has proven difficult to find the correct continuation of \( \lambda_a \) \[21\]. Despite this in ref. \[4\], it was demonstrated that gravity amplitudes satisfy the same type of localisation properties in twistor space as Yang-Mills amplitudes. More recently it was shown in ref. \[22\] that a MHV construction was possible provided that one carries out special shifts \[23\] in the anti-holomorphic variables \( \lambda_a \).

Gravity amplitudes are also amenable to a BCFW-like shift as demonstrated in refs. \[23\]. Thus, although more complicated, the new techniques may be applied to amplitudes of theories with gravity.

3. One-Loop in \( \mathcal{N} = 8 \) Supergravity

At one-loop level, string theory would suggest that the KLT relations extends within the loop momentum integrals. After integration, however, such relations would not be expected to persist in the amplitude. To illustrate this we examine the one-loop amplitudes in maximal supergravity/Yang-Mills. In evaluating loop amplitudes one performs integrals over the loop momenta, \( \ell^u \), with polynomial numerator \( P(\ell^u) \). In a Yang-Mills theory, the loop momentum polynomial will generically be of degree \( \leq n \) for a \( n \)-point loop. \( \mathcal{N} = 4 \) one-loop amplitudes exhibits considerable simplification and the loop momentum integral will be of degree \( n - 4 \) \[24\]. Consequently, the amplitudes can be expressed as a sum of scalar box integrals with rational coefficients, as follows from a Passarino-Veltman reduction \[27\].

\[ A^{1\text{-loop}} = \sum_a c_a I_a^4. \tag{3.1} \]

Considerable progress has recently been made in determining such coefficients, \( c_a \), using a variety of methods based on unitarity \[20\].

For maximal \( \mathcal{N} = 8 \) supergravity \[6\] the equivalent power counting arguments \[29\] give a loop momentum polynomial of degree

\[ 2(n - 4), \tag{3.2} \]

which is consistent with eq. \[20\]. Reduction for \( n > 4 \) leads to a sum of tensor box integrals with integrands of degree \( n - 4 \) which would then reduce to scalar boxes and triangle, bubble and rational functions,

\[ M^{1\text{-loop}} = \sum_a c_a I_a^4 + \sum_a d_a I_a^3 + \sum_a e_a I_a^2 + R \tag{3.3} \]

where the \( I_3 \) are present for \( n \geq 5 \), \( I_2 \) for \( n \geq 6 \) and the rational terms for \( n \geq 7 \).

The first calculation of an one-loop \( \mathcal{N} = 4 \) amplitude was of the four point \[30\].

\[ A^{1\text{-loop}}_{[1,2,3,4]} = st \times A^{\text{tree}}_{[1,2,3,4]} \times I_{4(s,t)} . \tag{3.4} \]

Here \( I_{4(s,t)} \) denotes the scalar box integral with attached legs in the order 1234 and \( s, t \) and \( u \) are the usual Mandelstam variables. The \( \mathcal{N} = 8 \) amplitude was also given,

\[ M^{1\text{-loop}}_{[1,2,3,4]} = stu M^{\text{tree}}_{[1,2,3,4]} \left[ I_{4(s,t)} + I_{4(s,u)} + I_{4(t,u)} \right], \tag{3.5} \]

so that, like the \( \mathcal{N} = 4 \) Yang-Mills amplitude, the \( \mathcal{N} = 8 \) amplitude can be expressed in terms of scalar box-functions. For \( n = 4 \) this similarity between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \) is consistent with the previous power counting arguments.

4. The no-triangle hypothesis

Despite the power counting argument, there is evidence \[4\] that one-loop amplitudes of \( \mathcal{N} = 8 \) can be expressed simply as a sum over scalar
box integrals analogous to the $\mathcal{N} = 4$ amplitudes \[4\]. We label this as the “no-triangle hypothesis.” We emphasis that this is a hypothesis: we will present evidence in its favour but not a proof.

Firstly, in the few definite computations at one-loop level, triangle or bubble functions do not appear. In ref. \[3\] the five and six point “Maximally Helicity Violating” amplitudes were computed and contrary to expectations, consisted entirely of scalar box-functions.

Secondly, factorisation properties of the physical amplitudes do not seem to demand the presence of these functions. Since the four and five point amplitudes do not seem to demand the presence of these functions. Since the four and five point amplitudes are triangle-free then in any soft, collinear, or multi-particle pole limit of a higher point function the triangles would by necessity drop out or be absent in the first place. For further support we refer to ref. \[3\], where an ansatz for the $n$-point MHV amplitude was constructed, entirely of box functions consistent in all soft limits. The simplification is peculiar to $\mathcal{N} = 8$ and does not apply for $\mathcal{N} < 8$ supergravities \[29,31\].

Thirdly, one can calculate the box-coefficients for the amplitude using unitarity and examine whether the amplitude has the correct soft behaviour. At one-loop, the expected soft divergence in a $n$ graviton amplitude is \[32\],

\[
M_{[1,2,...,n]}^{\text{one-loop}} = i\epsilon \kappa^2 \sum_{i<j} s_{ij} \ln \left( -s_{ij} \right) / 2\epsilon \times M_{[1,2,...,n]}^{\text{tree}}. \tag{4.1}
\]

In the expansion of the one-loop amplitude \[33\] both box and triangle integral functions contain divergences of the form $\ln\left( -P^2 / \epsilon \right)$ and the bubble integrals contains $1/\epsilon$ divergences. Thus if the boxes contain all the correct IR divergences we can conclude that the remaining parts cannot contribute any IR divergences. The triangle integrals can be organised according to the number of “massive” legs. The one and two mass triangles are not all independent but choosing an appropriate subset as a basis we can immediately deduce that these must be absent from the amplitude when the boxes correctly contain the IR singularities. (A caveat is that the three-mass triangle contains no IR singularity and so cannot be excluded by an IR argument.) In ref. \[3\] the box coefficients were explicitly computed for the six-point NMHV amplitudes. The result consisted of a sum of one and two adjacent mass boxes:

\[
M_{[a,b,c,d,e,f]}^{1\text{-loop}} = c\Gamma \left( \sum_{(abcdef) \in P_6} \hat{c}_{(abc)def} I_{4}^{(abc)def} + \sum_{(abcdef) \in P_6} \hat{c}_{a(bc)(de)f} I_{4}^{(bc)(de)f} \right). \tag{4.2}
\]

The six-point amplitude thus calculated gives exactly the entire expected IR structure of the one-loop amplitude confirming the absence of triangles (with at least one massless leg).

The “no-triangle” hypothesis applies to one-loop amplitudes. However, by factorisation it conceivably extends beyond one-loop. We wish to comment that the hypothesis, if true, implies a significantly softer UV behaviour of gravity than expected from power counting.

### 5. Multi-loop amplitudes

The two-loop four-point amplitudes are \[33\],

\[
A_4^{2\text{-loop}} = g^6 st A_4^{\text{tree}} \left[ s I_{4}^{2\text{-loop},P} + s I_{4}^{2\text{-loop},NP} + \text{perms} \right]. \tag{5.1}
\]

and \[2\],

\[
M_4^{2\text{-loop}} = \left( \frac{\kappa}{2} \right)^6 stu M_4^{\text{tree}} \times \left[ s^2 I_{4}^{2\text{-loop},P} + s^2 I_{4}^{2\text{-loop},NP} + \text{perms} \right], \tag{5.2}
\]

where $P$ and $NP$ indicate the planar and non-planar two-loop scalar box functions. The amplitudes are both UV finite for $D \leq 6$. A thorough discussion of the power counting of amplitudes with $L > 2$ for $D \geq 4$ was presented in ref. \[2\].

By examining the cuts of higher point functions the leading UV behaviour of $\mathcal{N} = 4$ SYM is expected to be,

\[
\int (d^D p)^L \frac{(p^2)^{(L-2)}}{(p^2)^{3L+1}}, \tag{5.3}
\]

implying amplitudes to be finite when

\[
D < \frac{6}{L} + 4.
\]
The expectation for three-loop Yang-Mills has recently been confirmed in [34]. For \( D = 4 \) this is restating the known finiteness of \( \mathcal{N} = 4 \) SYM [7].

For supergravity, the expectation is that the integrands will be the square of Yang-Mills integrands and thus the divergence will go as,

\[
\int (d^D p)^L \frac{(p^2)^{2(L-2)}}{(p^2)^{2L+1}},
\]

implying finiteness when \( D < \frac{10}{L} + 2 \).

This suggest that \( \mathcal{N} = 8 \) supergravity is infinite at five loops in \( D = 4 \). Explicitly, this behaviour was checked for parts of the amplitude which are “two-particle cut-constructible”. Although we do not have a candidate mechanism for the no-triangle hypothesis, it does suggest a cancellation between diagrams. Such cancellations, if they persist beyond one-loop would suggest the above power counting is too conservative.

6. Conclusions

The recent progress in computing gauge theory amplitudes can be extended, in many cases, to theories incorporating gravity. This has improved our understanding of the perturbative expansion of, in particular, maximal \( \mathcal{N} = 8 \) supergravity.

The current status on the perturbative expansion is that \( \mathcal{N} = 8 \) is two loop finite, but expected to diverge from power counting arguments at five loops [35]. However it is surprising that in concrete calculations the large momentum structure in \( \mathcal{N} = 8 \) supergravity appears to be much simpler than power counting suggests. These simplifications are completely unexpected from the currently known symmetries of \( \mathcal{N} = 8 \) supergravity. One might suspect, this implies the existence of further symmetries and additional constraints on the scattering amplitudes. It seems promising, although challenging, to utilise the simplification as well as new techniques to determine the ultra-violet behaviour of higher loop scattering amplitudes in \( \mathcal{N} = 8 \) supergravity.

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