Adaptive tracking control for a class of stochastic nonlinearly parameterized systems with time-varying input delay using fuzzy logic systems

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Abstract
This paper addresses the problem of adaptive tracking control for a class of stochastic nonlinear systems with time-varying input delays, and the nonlinear functions of the systems are with not only the unknown parameters but also the unknown state time-varying delays, which is different from the previous work. In this paper, through a state transformation, the system can be easily transformed into a system without the time-varying input delay; the appropriate Lyapunov–Krasovskii functionals are used to compensate the unknown time-varying delay terms, and the quadratic functions instead of the quartic functions often utilized in the existing results are used as Lyapunov functions to analyze the stability of systems and the hyperbolic tangent functions are introduced to deal with the Hessian terms. Fuzzy logic systems (FLSs) in Mamdani type are used to approximate the unknown nonlinear functions. Then, based on the backstepping technique, the adaptive fuzzy controller is designed. The three main advantages of the developed scheme are that (i) unlike the existing results which deal with the nonlinearly parameterized functions by using the separation principle, the nonlinearly parameterized functions are lumped into the continuous functions which can be approximated by using the FLS; (ii) the number of the adjusted parameters only depend on the order of the investigated systems, which can reduce the computational burden greatly; and (iii) the existence of the time-varying input delay such that the controller design becomes much more difficult, and in this paper, it can be dealt with by using an appropriate state transformation. It is proven that all the signals of the closed-loop system are semi-globally uniformly ultimately bounded in probability, whereas the tracking error converges to a small neighborhood of the origin. Finally, simulation results are provided to show the effectiveness of the proposed approach.

Keywords
Fuzzy logic system, state time-varying delays, nonlinearly parameterized functions, hyperbolic tangent functions, quadratic functions, time-varying input delay

Introduction
In the past decades, the adaptive fuzzy or neural network (NN) control design for uncertain stochastic nonlinear systems has received increasing attention, such as1–4,39 many fuzzy problems can be effectively solved by either the homotopy perturbation method5 or the variational iteration method,6 both methods were proposed by Chinese mathematician, Dr. Ji-Huan He.7 Based on Itô stochastic differential equation and backstepping design technique, many adaptive fuzzy control design results obtained for deterministic nonlinear systems were extended to those stochastic nonlinear systems, for example, references.8–12 and the references therein. However, in consequence of the appearance of higher order Hessian matrix term in the infinitesimal generator, the quartic functions and the Young inequality are combined to design the...
controller and analyze the stability of the closed-loop systems in the most of the existing results, such as the references 13–15, and so on, and according to these references, it can be seen that this kind of controller design scheme is very complex. In order to optimize the control theory of the stochastic nonlinear systems, in references 11 and 12, the classical quadratic functions instead of the quartic functions are used as Lyapunov functions to analyze and synthesize of stochastic nonlinearly parameterized systems with distributed input delay and stochastic nonlinear systems with unknown time-varying delays, respectively; during the controller design procedure, the hyperbolic tangent functions and the FLS are combined to deal with the Hessian terms. Thus, the long-standing obstacle for stochastic systems control by the quadratic Lyapunov function is overcome. Nevertheless, up to now, to the author’s knowledge, the second moment approach is not extended to the other kinds of stochastic nonlinear systems.

In recent years, many results were concerned with the systems with linear parameterizations, such as references 16–18, and so on. However, it is well known that nonlinear parameterizations are inevitable in most realistic practical problems. Actually, designing the algorithms for nonlinearly parameterized systems is an interesting and meaningful problem. Many valuable results have been achieved for the nonlinearly parameterized systems with the nonlinearly parameterized functions \( f(x(t), \rho) \).11,19–22 However, it should be emphasized that in the existing results, the separation principle was introduced to deal with the functions which can be separated into two parts: the one part is unknown parameters and the other part is function without the nonlinearity parameterization problem. This design method is much complicated and the separated out parameters should be designed parameter adaptive laws, which can increase the computational burden greatly. Therefore, for the nonlinearly parameterized systems, if not using the separation principle, how to handle with the nonlinearly parameterized functions is worth studying. Furthermore, if the function \( f(x(t-d(t)), \rho) \) is with not only the nonlinearly parameterization problem but also the unknown state time-varying delay, how to handle with the function is a challenging problem.

Time-delay phenomena exist in many practical systems such as physical, biological, and economical systems. The existence of time delay is often a significant cause of instability and deteriorative performance, so the design and analysis of controllers for time-delay systems has received much attention. State delay and input delay are two kinds of the time-delay phenomena, and it is worth noting that the investigation for the nonlinear systems with state delay by using the FLS and NN has been received a lot of results, for example, references 10, 12, 23–26 and the references therein. The adaptive fuzzy tracking controllers were proposed for a class of stochastic nonlinear systems with state time-varying delays in references 10 and 12. In references 23 and 24, the authors studied the problem of adaptive fuzzy tracking control for a class of perturbed nonlinear systems with state time-varying delay and unknown dead zone and the problem of adaptive output tracking control for a class of nonlinear systems subject to unknown state time-varying delay and input saturation, respectively. For the systems with input delay, the investigation has received some results.11,27–31 In references 27–31, for the deterministic systems with input delay, the effective control schemes have been proposed, and in reference 11, an adaptive fuzzy tracking controller was developed for a class of stochastic nonlinearly parameterized systems with distributed constant input delay. However, the above literature are all about the time-invariant input delays, and for the systems with time-varying input delays, it is very difficult to design the controllers and analyze the stability of the closed-loop systems because of the existence of the derivatives of the time-varying input delays. Therefore, for the deterministic nonlinear systems or the stochastic nonlinear systems with the time-varying input delay \( u(t-t(\tau)) \), how to design the controller is a worth studying subject.

Based on the above observations, in this paper, the problem of output tracking is revisited for stochastic strict feedback nonlinear systems with nonlinearly parameterized function \( f(x(t-d(t)), \rho) \) and time-varying input delay \( u(t-t(\tau)) \) by using fuzzy control. The system will be transformed into a system without the time-varying input delay by using the state transformation with the integrator. Then, the appropriate Lyapunov–Krasovskii functionals are used to compensate the unknown time-varying delays terms, and the quadratic functions are used as Lyapunov functions to analyze the stability of systems and the hyperbolic tangent functions are introduced to deal with the Hessian terms. The nonlinearly parameterized functions will be lumped into the unknown continuous functions which can be approximated by using the FLS. Finally, the adaptive backstepping approach is utilized to construct the fuzzy controller. The three main advantages of the scheme are as follows:

1. In this paper, the control theory of the stochastic nonlinear system is optimized. That is because the classical quadratic functions instead of the fourth moment approach often utilized in the most existing results are used as Lyapunov functions to analyze the stability of the stochastic nonlinear systems. The computational burden of the algorithm is reduced. Therefore, it is extremely convenient to the practical applications in engineering. The hyperbolic tangent functions are introduced to deal with the higher order Hessian terms.
2. This is the attempt to deal with the control problem of stochastic nonlinear systems with the function \( f(x(t - d(t)), \rho) \). The appropriate Lyapunov–Krasovskii functionals are used to deal with the time-varying delays terms. The FLS are applied to approximate the unknown nonlinearly parameterized functions. This is unlike the results since the condition of function separation principle is reduced. In this paper, the number of the adjusted parameters only depends on the order of the investigated systems, which can reduce the computational burden greatly.

3. Compared with the references, the existence of the time-varying input delay \( u(t - \tau(t)) \) makes the controller design to become much more difficult. During the controller design procedure, the derivatives of the time-varying input delays are dealt with by using the appropriate assumptions, a state transformation through which the system with time-varying input delay is transformed into a system without input delay.

It can be proven that all the signals in the closed-loop system are bounded in probability and the tracking error can converge to a small residual set around the origin in the mean square sense. Simulation results are provided to show the effectiveness of the proposed approach.

**Problem formulation and preliminaries**

**Problem formulation**

Consider the following stochastic nonlinearly parameterized systems

\[
\begin{align*}
\dot{x}_i &= (x_{i+1} + f_i(\bar{x}_i(t - d_i(t)), \bar{y}_i)) dt + h_i^T(\bar{x}_i) d\omega, \quad i = 1, 2, \ldots, n - 1, \\
\dot{x}_n &= (m(t)u(t - \tau(t)) + f_n(\bar{x}_n(t - d_n(t)), \bar{y}_n)) dt + h_n^T(\bar{x}_n) d\omega,
\end{align*}
\]

(1)

where \( \bar{x}_n = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the state vector of the system with \( x_i = [x_1, \ldots, x_i]^T \in \mathbb{R}^i \) and \( \bar{x}_i(t - d_i(t)) = [x_1(t - d_i(t)), \ldots, x_i(t - d_i(t))]^T \in \mathbb{R}^i \), \( \tau(t) \), \( d_i(t) \), and \( y \in \mathbb{R} \) are the time-varying input delay, state time delay, and output signal, respectively. \( \omega \) is an \( r \)-dimensional standard Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, P) \) with \( \Omega \) being a sample space, \( \mathcal{F} \) being \( \sigma \)-field, and \( P \) being a probability measure. \( \bar{y}_i = [\rho_1, \rho_2, \ldots, \rho_i]^T \) is an unknown parameter vector with \( \rho_i \) being the unknown parameter. \( f_i(\cdot) \) and \( h_i(\cdot) \) are unknown smooth nonlinear functions, and \( m(t) \) is a continuous function. The control objective is to design an adaptive fuzzy state-feedback tracking controller for the system (1) such that (i) all the signals in the closed-loop system are bounded in probability and (ii) the output \( y(t) \) follows the desired trajectory \( \bar{y}_d \) as expected in the mean square sense. To the end, define a vector function as \( \bar{y}_d_T = [y_{d1}^{(1)}, \ldots, y_{dn}^{(n)}], i = 1, \ldots, n \), where \( y_{di}^{(i)} \) is the \( i \)th time derivative of \( y_d \).

Consider an \( n \)-dimensional stochastic system

\[
\dot{x}(t) = f(x)dt + g(x)d\omega, \quad \forall t \geq 0
\]

where \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( g: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r} \) are locally Lipschitz and \( x \) and \( \omega \) are the same ones defined in (1).

Define a differential operator \( L \) as follows

\[
LV(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{Tr} \left\{ g^T \frac{\partial^2 V}{\partial x^2} g \right\}
\]

(2)

where \( V(x) \in \mathbb{C}_2 \).

In addition, the following assumptions and lemma are needed in this paper.

**Assumption 1.** The desired trajectory vectors \( \bar{y}_{di} \) are known and continuous, and \( \bar{y}_{di} \in \Omega_{di} \in \mathbb{R}^{i+1} \) with \( \Omega_{di} \) being known compact sets, \( i = 1, 2, \ldots, n \).

**Assumption 2.** \( 0 \leq d_i(t) \leq d \) and their derivatives satisfy \( \dot{d}_i(t) \leq d^* \) with \( d \) and \( d^* \) being positive unknown constants.

**Assumption 3.** The time-varying input delay \( \tau(t) \) should satisfy \( 0 < \tau_1(t) \leq \tau_2(t) \leq \tau \) and \( \dot{\tau}(t) < 1 \) with \( \tau_1 \) and \( \tau_2 \) being unknown constants.
Assumption 4. The function \( m(t) \) satisfies \( m(t) = 1 - \tau(t) \).

Lemma 1. For \( 1 \leq j \leq n \) and \( \theta_j > 0 \), consider the set \( \Theta_{\theta_j} \) given by \( \Theta_{\theta_j} = \{ z_j | |z_j| \leq 0.2554 \theta_j \} \). Then, for \( z_j \not\in \Theta_{\theta_j} \), the inequalities \( 1 - 16\tanh^2(z_j/\theta_j) < 0 \) are satisfied.

Fuzzy logic systems

In this paper, the following rules are used to develop the adaptive fuzzy controller \( Fl \):

\[ \text{(5)} \]

where \( \alpha_i \) and \( \theta_i \) are designed parameters. It has been proven that when the membership functions are chosen as Gaussian functions, the above fuzzy logic system is capable of uniformly approximating any continuous function over a compact set with any degree of accuracy. This property is shown by the following lemma.

Lemma 2. Let \( f(x) \) be a continuous function defined on compact set \( \Omega \). Then, for any constant \( t > 0 \), there exists an FLS (5) such that

\[ \sup_{x \in \Omega} |f(x) - \phi^T \zeta(x)| < t \] (6)

Controller design

In this section, we will use the recursive backstepping technique to develop the adaptive fuzzy tracking control laws as follows

\[ \alpha_i = -k_i z_i - \frac{\hat{\theta}_i \zeta_i^T(Z_i) \zeta_i(Z_i) z_i}{2\eta^2} \] (7)

\[ \hat{\theta}_i = \zeta_i^T(Z_i) \zeta_i(Z_i) z_i - a_i \theta_i \] (8)

where \( \alpha_i, i = 1, \ldots, n - 1 \) is called the intermediate control function, and when \( i = n, \alpha_n \) is the true control input \( u(t) \). \( \eta > 0 \) are designed parameters. \( \theta_i = \| \phi_i \|^2 \), where \( \| \phi_i \|^2 \) are unknown parameters and will be specified later. \( \theta_i \) are the estimations of \( \theta \) and the estimation errors are defined as \( \tilde{\theta}_i = \theta_i - \hat{\theta}_i \). The control gain \( k_i \) satisfies \( k_i > 1/\lambda, i = 1, 2, \ldots, n - 1 \), with \( \lambda \) being a positive design parameter, and especially, \( k_{n-1} \) satisfies \( k_{n-1} > (1/\lambda) + (\sigma/2) \) with \( \sigma \) being a positive design parameter, and \( \zeta_i(Z_i) \) is a fuzzy basis function vector with \( Z_i \) being the input vector. Note that \( \alpha_0 \) is equal to \( y_d \).

The \( n \)-step adaptive fuzzy backstepping tracking control design is based on the following change of coordinates
will be transformed into a system without the input delay by using the state transformation with an integrator (9).

Therefore, it cannot be approximated by the FLS. Similar to reference 32, we introduce the hyperbolic tangent function

\[ \text{tanh}(z_1) = \frac{e^{z_1} - e^{-z_1}}{e^{z_1} + e^{-z_1}} \]

Now, we propose the following backstepping-based design procedure.

Step 1. From (1) and (9), we yield that

\[ dz_1 = dy - dy_d = \left( x_2 + f_i\left( x_1(t - d_i(t)), \gamma_1 \right) \right) dt + h_i^j(x_1)dc \]

Choose the Lyapunov function candidate as

\[ V_1 = \frac{1}{2} z_1^2 + \frac{e^{-\gamma(t-d)}}{2(1 - d^s)} \int_{t-d_i(t)}^t e^{\gamma s} f_i^2(x_1(s), \gamma_1) ds \]

then, we can get

\[ LV_1 = z_1 \left( x_2 + f_i\left( x_1(t - d_i(t)), \gamma_1 \right) \right) - \dot{y}_d + \frac{1}{2} h_i^j(x_1)h_i(x_1) + \frac{e^{\epsilon d}}{2(1 - d^s)} f_i^2(x_1(t), \gamma_1) \]

\[ - \frac{ye^{-\gamma(t-d)}}{2(1 - d^s)} \int_{t-d_i(t)}^t e^{\gamma s} f_i^2(x_1(s), \gamma_1) ds - \frac{e^{-\gamma(d_i(t)-d)}}{2(1 - d^s)} f_i^2(x_1(t - d_i(t)), \gamma_1) \]

Using the triangular inequality and Assumption 2 gives that

\[ zifi\left( z_1(t - d_i(t)), \gamma_1 \right) \leq \frac{z_1^2}{2} + \frac{f_i^2(x_1(t - d_i(t)), \gamma_1)}{2(1 - d^s)} \frac{e^{-\gamma(d_i(t)-d)}}{2(1 - d^s)} \geq \frac{1}{2} \]

Combining (12) with (13) yields that

\[ LV_1 \leq z_1 \left( x_2 - \dot{y}_d \right) + \frac{1}{2} \frac{z_1^2}{2} + \frac{1}{2} h_i^j(x_1)h_i(x_1) + \frac{e^{\gamma d}}{2(1 - d^s)} f_i^2(x_1(t), \gamma_1) - \frac{ye^{-\gamma(t-d)}}{2(1 - d^s)} \int_{t-d_i(t)}^t e^{\gamma s} f_i^2(x_1(s), \gamma_1) ds \]

\[ = z_1 \left( x_2 + H_i \right) - \frac{ye^{-\gamma(t-d)}}{2(1 - d^s)} \int_{t-d_i(t)}^t e^{\gamma s} f_i^2(x_1(s), \gamma_1) ds \]

where \( \phi(Z_1) = (1/2)x_1 + (H_1/z_1) - \dot{y}_d \), \( H_1 = (1/2)h_i^j(x_1)h_i(x_1) + (e^{\gamma d}/2(1 - d^s))f_i^2(x_1(t), \gamma_1) \) with \( Z_1 = [x_1, \gamma_d, \gamma_d]^T \in \Omega_{Z_1} \subset R^3 \) and \( \Omega_{Z_1} \) being some known compact set in \( R^3 \). Notice that in (14), the term \( H_1/z_1 \) is discontinuous at \( z_1 = 0 \). Therefore, it cannot be approximated by the FLS. Similar to reference 32, we introduce the hyperbolic tangent function \( \text{tanh}(z_1/\theta_1) \) to deal with the term. Define

\[ \gamma_1(Z_1) = \phi_1(Z_1) - \frac{H_i}{z_1} + \frac{16}{z_1} \tanh^2 \left( \frac{z_1}{\theta_1} \right) \]

(15)
where $\theta_1$ is a positive design parameter. Note that $\lim (16/z_1)\tanh^2(z_1/\theta_1)H$ exists; thus, the nonlinear function $\varphi(Z_1)$ can be approximated by an FLS $\theta_1^T \xi(Z_1)$ such that $z_1 \to 0$

$$\varphi(Z_1) = \theta_1^T \xi(Z_1) + \delta_1(Z_1) \quad \text{(16)}$$

furthermore, using the triangular inequality yields that

$$z_1(\theta_1^T \xi(Z_1) + \delta_1(Z_1)) \leq \frac{\theta_1}{2\eta} \xi^T \xi z_1^2 + \frac{z_1}{2\lambda} + \frac{\eta^2}{2} + \frac{\lambda \xi^2}{2} \quad \text{(17)}$$

from (14)-(17), we can get

$$L \leq z_1 \left( z_2 + \alpha_1 + \frac{\theta_1}{2\eta} \xi^T \xi z_1 + \frac{z_1}{2\lambda} \right) + \frac{\eta^2}{2} + \frac{\lambda \xi^2}{2} + \frac{\dot{\theta}_1 }{2\eta^2} \left( 1 - 16\tanh^2 \left( \frac{z_1}{\theta_1} \right) \right) H_1 \quad \text{(18)}$$

where $\lambda$ and $\eta$ are the positive design parameters, and $\epsilon_1$ is the upper bound of $\delta(Z_1)$.

**Remark 2.** The unknown continuous function $f_1(x_1, \varphi_1)$ is nonlinearly parameterized. In many existing results, the separation principle was used to deal with the nonlinearly parameterized function $f_1(x_1, \varphi_1)$ such that it can be expressed as linear parameterized function. Nevertheless, in this paper, whether the unknown continuous function $f_1(x_1, \varphi_1)$ is linearly parameterized or not, since $\varphi_1$ is constant vector other than variate, that is to say there is only one variable $x_1$ in function $f_1(x_1, \varphi_1)$; thus, the function $f_1(x_1, \varphi_1)$ can be lumped into the continuous function $\varphi_1(Z_1)$ such that it can be approximated by using the FLS.

Then, choose the following Lyapunov candidate as

$$V_1 = V_1 + \frac{\theta_1^2}{4\eta^2} \quad \text{(19)}$$

then, we have

$$L \leq z_1 \left( z_2 + \alpha_1 + \frac{\theta_1}{2\eta} \xi^T \xi z_1 + \frac{z_1}{2\lambda} \right) + \frac{\eta^2}{2} + \frac{\lambda \xi^2}{2} + \frac{\dot{\theta}_1 }{2\eta^2} \left( 1 - 16\tanh^2 \left( \frac{z_1}{\theta_1} \right) \right) H_1 \quad \text{(20)}$$

choosing the virtual controller $\alpha_1$ and the parameter adaptive laws defined in (7) and (8) gives that

$$L \leq z_1 z_2 - \left( k_1 \frac{1}{2\lambda} \right) z_2^2 + \frac{\lambda \xi^2}{2} + \frac{\eta^2}{2} - \frac{\sigma_1 }{4\eta^2} - \frac{\sigma_1 }{4\eta^2} \quad \text{(21)}$$

moreover, in (21), the residual term $z_1 z_2$ will be dealt with in the next step.

Step i: Considering $z_1 = x_i - a_{i-1}$, where $a_{i-1}$ defined in (8), then

$$dz_1 = dx_i - da_{i-1} = \left[ x_{i+1} + f_1 \left( \varphi(t - d_i(t), \varphi_1) \right) - W_i - \sum_{j=i}^{i-1} \frac{\partial^2 a_{i-1} }{\partial x_j } f_1 \left( \varphi(t - d_i(t), \varphi_1) \right) \right] dt \quad \text{(22)}$$

where
\[
W_i = \frac{1}{2} \sum_{j=1}^{i-1} \frac{\partial^2 a_{i-1}}{\partial x_j \partial x_j} h_j^T(x_i) h_j^T(x_i) + \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} \partial x_j + \sum_{j=i-1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} \partial x_j + \frac{\partial a_{i-1}}{\partial x_i} \partial x_i = \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} \partial x_j + \frac{\partial a_{i-1}}{\partial x_i} \partial x_i
\]

Choose the Lyapunov function candidate as
\[
V_i = \frac{1}{2 \alpha} + \frac{e^{-\gamma(t-d)}}{2(1-d^2)} \sum_{j=1}^{i} \int_{t-d_j(t)}^{t} e^{\alpha f_j^2} \left( \bar{x}_j(s), \bar{\xi}_j \right) ds
\]

then, we have
\[
LV_i = z_i \left( x_{i+1} + f_i \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right) - W_i - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} f_j \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right) + z_{i-1} \right) - z_{i-1} z_i
\]
\[
+ \frac{1}{2} \left( h_j^T(x_i) - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} h_j^T(x_i) \right) \left( h_j(x_i) - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} h_j(x_i) \right) - \frac{\gamma e^{-\gamma(t-d)}}{2(1-d^2)} \sum_{j=1}^{i} \int_{t-d_j(t)}^{t} e^{\alpha f_j^2} \left( \bar{x}_j(s), \bar{\xi}_j \right) ds
\]
\[
- \sum_{j=1}^{i} \frac{e^{-\gamma(t-d_i(t))}}{2(1-d^2)} \left( 1 - d_j(t) \right) f_j^2 \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right) + \frac{e^{\alpha d_i(t)}}{2(1-d^2)} \sum_{j=1}^{i} f_j^2 \left( \bar{x}_j(t), \bar{\xi}_j \right)
\]

using the triangular inequality and Assumption 2 gives that
\[
-z_i \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} f_j \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right) \leq \frac{1}{2 \alpha} \sum_{j=1}^{i-1} \left( \frac{\partial a_{i-1}}{\partial x_j} \right)^2 + \frac{1}{2} \sum_{j=1}^{i-1} f_j^2 \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right),
\]
\[
xz_i f_i \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right) \leq \frac{1}{2 \alpha} + f_i^2 \left( \bar{x}_i(t - d_j(t)), \bar{\xi}_j \right) \frac{e^{-\gamma(t-d_i(t))}}{2(1-d^2)} \left( 1 - d_j(t) \right) \geq \frac{1}{2}
\]

From (23) and (24), we get
\[
LV_i \leq z_i \left( x_{i+1} - W_i + z_{i-1} + \frac{1}{2} \sum_{j=1}^{i-1} \left( \frac{\partial a_{i-1}}{\partial x_j} \right)^2 + \frac{1}{2} \frac{H_i}{z_i} \right) - z_{i-1} z_i + \frac{e^{\alpha d_i(t)}}{2(1-d^2)} \sum_{j=1}^{i} f_j^2 \left( \bar{x}_j(t), \bar{\xi}_j \right)
\]
\[
+ \frac{1}{2} \left( h_j^T(x_i) - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} h_j^T(x_i) \right) \left( h_j(x_i) - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} h_j(x_i) \right) - \frac{\gamma e^{-\gamma(t-d)}}{2(1-d^2)} \sum_{j=1}^{i} \int_{t-d_j(t)}^{t} e^{\alpha f_j^2} \left( \bar{x}_j(s), \bar{\xi}_j \right) ds
\]
\[
= z_i \left( x_{i+1} - W_i + z_{i-1} + \frac{1}{2} \sum_{j=1}^{i-1} \left( \frac{\partial a_{i-1}}{\partial x_j} \right)^2 + \frac{1}{2} \frac{H_i}{z_i} \right) - z_{i-1} z_i + \frac{\gamma e^{-\gamma(t-d)}}{2(1-d^2)} \sum_{j=1}^{i} \int_{t-d_j(t)}^{t} e^{\alpha f_j^2} \left( \bar{x}_j(s), \bar{\xi}_j \right) ds
\]

where
\[
H_i = \frac{1}{2} \left( h_j^T(x_i) - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} h_j^T(x_i) \right) \left( h_j(x_i) - \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} h_j(x_i) \right) + \frac{e^{\alpha d_i(t)}}{2(1-d^2)} \sum_{j=1}^{i} f_j^2 \left( \bar{x}_j(t), \bar{\xi}_j \right)
\]

Define
\[
\varphi_i(Z_i) = -W_i + z_{i-1} + \frac{1}{2} \sum_{j=1}^{i-1} \left( \frac{\partial a_{i-1}}{\partial x_j} \right)^2 + \frac{1}{2} \frac{H_i}{z_i}
\]

with \( Z_i = [x_1, \ldots, x_i, \bar{\theta}_i, \bar{\theta}_i, \ldots, \bar{\theta}_i]^T \in \Omega_{Z_i} \subset \mathbb{R}^3 \) and \( \Omega_{Z_i} \) being some known compact set in \( \mathbb{R}^3 \). Similar to step 1, define
\[
\varphi_i(Z_i) = \varphi_i(Z_i) - \frac{H_i}{z_i} + \frac{16}{z_i} \tan h^2 \left( \frac{z_i}{\bar{\theta}_i} \right) H_i
\]
where $\theta_i$ is a positive design parameter. Thus, the nonlinear function $\varphi(Z_i)$ can be approximated by an FLS $\phi_i^T \xi_i(Z_i)$ such that

$$\varphi(Z_i) = \phi_i^T \xi_i(Z_i) + \delta_i(Z_i)$$  \hspace{1cm} (28)$$

then, using

$$z_i(\phi_i^T \xi_i(Z_i) + \delta_i(Z_i)) \leq \frac{\theta_i}{2\eta} z_i^2 \xi_i^2 + \frac{\eta^2}{2} + z_i^2 + \frac{\lambda \xi_i^2}{2} \hspace{1cm} (29)$$

yields that

$$LV \leq z_i(z_{i+1} + \alpha_i) - z_{i-1} z_i + \frac{\theta_i}{2\eta} z_i^2 \xi_i^2 + \frac{\eta^2}{2} + \frac{\lambda \xi_i^2}{2} + \left(1 + 16\tanh^2\left(z_i^{1/\eta}\right)\right) H_i$$

$$- \frac{\eta^2}{2(1-d^2)} \int_{b(t)}^t e^{\eta^2 f^2_j(s)} (\xi_j(s), \bar{\xi}_j) ds$$

where $\epsilon_i$ is the upper bound of $\delta_i(Z_i)$ with $|\delta_i(Z_i)| \leq \epsilon_i$. Choose the Lyapunov function as

$$\mathcal{V}_i = V_i + \frac{\delta_i^2}{4\eta^2}$$

(31)

from (8) and (9), we get

$$LV \leq z_i z_{i+1} - z_{i-1} z_i - \left(k_i - \frac{1}{2\lambda} \right) z_i^2 + \frac{\lambda \xi_i^2}{2} + \frac{\eta^2}{2} + \frac{\lambda \xi_i^2}{2} + \left(1 - 16\tanh^2\left(z_i^{1/\eta}\right)\right) H_i$$

$$- \frac{\eta^2}{2(1-d^2)} \int_{b(t)}^t e^{\eta^2 f^2_j(s)} (\xi_j(s), \bar{\xi}_j) ds$$

Step $n - 1$: Considering $z_{n-1} = x_{n-1} - a_{n-2}$, where $a_{n-2}$ defined in (8), then

$$dz_{n-1} = dx_{n-1} - da_{n-2}$$

$$= \left[ x_n + f_{n-1}\left( x_{n-1}(t - d_{n-1}(t)), \bar{\xi}_{n-1}, \xi_{n-1} \right) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j} f_j(t - d_{j}(t), \bar{\xi}_{j}) - W_{n-1} \right] dt$$

$$+ \left( h_{n-1}(x_{n-1}) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j} h_j(x_{n-1}) \right) d\omega$$

(33)

where

$$W_{n-1} = \frac{1}{2} \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j} h_j^2(x_{n-1}) + \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j} h_j^2(\xi_{n-1})$$

$$+ \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j} h_j^2(\xi_{n-1})$$

$$+ \frac{1}{2} \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j} h_j^2(\xi_{n-1}) + \frac{\partial a_{n-2}}{\partial x_j} h_j^2(\xi_{n-1})$$

(34)

Choose the Lyapunov function candidate as

$$V_{n-1} = \frac{1}{2} z_{n-1}^2 + \frac{e^{-\eta^2 f^2_j(s)}}{2(1-d^2)} \int_{b(t)}^t e^{\eta^2 f^2_j(s)} (\xi_j(s), \bar{\xi}_j) ds$$

(35)

then, according to $z_n = x_n - a_{n-1} + \int_{t-d_{n-1}(t)}^{t} u(s) ds$, we yield
\[ LV_{n-1} = z_{n-1} \left( z_n + a_{n-1} - \int_{t-r(t)}^{t} u(s)ds + f_{n-1} \left( x_{n-1} (t - d_{n-1}(t)) \overline{\varphi}_{n-1} \right) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j^2} f_j \left( x_j (t - d_j(t)) \overline{\varphi}_j \right) - W_{n-1} + z_{n-2} \right) \]

\[-z_{n-2} z_{n-1} + \frac{1}{2} \left( h_{n-1}^T (x_{n-1}) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j^2} h_j^T (x_j) \right) \left( h_{n-1} (x_{n-1}) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j^2} h_j (x_j) \right)\]

\[-\frac{\gamma e^{-\gamma (t-d)}}{2(1-d')} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\omega t} f_j^2 (x_j (s), \overline{\varphi}_j) ds + \frac{\omega d}{2(1-d')} \sum_{j=1}^{n-1} f_j^2 (x_j (t), \overline{\varphi}_j)\]

\[-\sum_{j=1}^{n-1} e^{-\gamma (t-d_j(t)-d)} \left( 1 - \frac{1}{2} \right) \]

Using the triangular inequality and Assumption 2 gives that

\[-z_{n-1} \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j^2} f_j \left( x_j (t - d_j(t)) \overline{\varphi}_j \right) \leq \frac{1}{2} z_{n-1} + \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{\partial^2 a_{n-2}}{\partial x_j^2} \right)^2 f_j^2 (x_j (t - d_j(t)) \overline{\varphi}_j),\]

\[z_{n-1} f_{n-1} (x_{n-1} (t - d_{n-1}(t)) \overline{\varphi}_{n-1}) \leq \frac{1}{2} z_{n-1} + f_{n-1}^2 \left( x_{n-1} (t - d_{n-1}(t)) \overline{\varphi}_{n-1} \right),\]

\[\times \frac{e^{-\gamma (d_{n-1}(t)-d)}}{2(1-d')} \left( 1 - \frac{1}{2} \right).\]

Combining (9), (36), and (37) yields that

\[LV_{n-1} \leq z_{n-1} \left( z_n + a_{n-1} - \int_{t-r(t)}^{t} u(s)ds + \frac{1}{2} z_{n-1} + \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{\partial^2 a_{n-2}}{\partial x_j^2} \right)^2 W_{n-1} + z_{n-2} \right)\]

\[-z_{n-2} z_{n-1} + \frac{1}{2} \left( h_{n-1}^T (x_{n-1}) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j^2} h_j^T (x_j) \right) \left( h_{n-1} (x_{n-1}) - \sum_{j=1}^{n-2} \frac{\partial^2 a_{n-2}}{\partial x_j^2} h_j (x_j) \right)\]

\[-\frac{\gamma e^{-\gamma (t-d)}}{2(1-d')} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\omega t} f_j^2 (x_j (s), \overline{\varphi}_j) ds + \frac{\omega d}{2(1-d')} \sum_{j=1}^{n-1} f_j^2 (x_j (t), \overline{\varphi}_j)\]

Moreover, if \( u(t) \) is a continuous function on the closed interval \( [t - r(t), t] \), by using the integral mean value theorem, the triangular inequality, and Assumption 3, we can obtain

\[-z_{n-1} \int_{t-r(t)}^{t} u(s)ds \leq \frac{\omega}{2} z_{n-1} + \frac{1}{2} \left( \int_{t-r(t)}^{t} u(s)ds \right)^2\]

\[= \frac{\omega}{2} z_{n-1} + \frac{1}{2} \left( u(\chi) (t - r(t))^2 \leq \frac{\omega}{2} z_{n-1} + \frac{\omega}{2} u(\chi), \chi \in [t - r(t), t],\right.\]

\[z_{n-2} z_{n-1} \leq \frac{1}{2} z_{n-1} + \frac{1}{2} z_{n-1}^2\]

where \( \omega > 0 \) is a design parameter. From (38)-(39), one gets
\[LV_{n-1} \leq z_{n-1}(a_{n-1} + \varphi(Z_{n-1})) + \frac{1}{2} \sigma^2 + \frac{r^2}{2 \sigma} u^2(\chi) - z_{n-2}z_{n-1}\]
\[+ \frac{1}{2} \left( h_{n-1}^T \left( \tilde{x}_{n-1} \right) - \sum_{j=1}^{n-2} \frac{\partial a_{n-2} \partial h_j^T(x_j)}{\partial x_j} \right) \left( h_{n-1} \left( \tilde{x}_{n-1} \right) - \sum_{j=1}^{n-2} \frac{\partial a_{n-2} \partial h_j^T(x_j)}{\partial x_j} \right) + \frac{e^{\eta d}}{2(1 - d^2)} \sum_{j=1}^{n-1} f_j^2(\tilde{x}_j(t) \tilde{\gamma}_j) - \frac{\gamma e^{-\gamma (t-d)}}{2(1 - d^2)} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\eta f_j^2(\tilde{x}_j(s) \tilde{\gamma}_j)} ds\]  

where
\[\varphi(Z_{n-1}) = \frac{\sigma}{2} z_{n-1} + z_{n-1} + \frac{1}{2} z_{n-1} \sum_{j=1}^{n-1} \left( \frac{\partial a_{n-2} \partial h_j^T(x_j)}{\partial x_j} \right)^2 - W_{n-1} + z_{n-2} + H_{n-1},\]
\[H_{n-1} = \frac{1}{2} \left( h_{n-1}^T \left( \tilde{x}_{n-1} \right) - \sum_{j=1}^{n-2} \frac{\partial a_{n-2} \partial h_j^T(x_j)}{\partial x_j} \right) \left( h_{n-1} \left( \tilde{x}_{n-1} \right) - \sum_{j=1}^{n-2} \frac{\partial a_{n-2} \partial h_j^T(x_j)}{\partial x_j} \right) \]  

with \(Z_{n-1} = [x_1, ..., x_{n-1}; \tilde{\gamma}_{d_1}, ..., \tilde{\gamma}_{d_{n-2}}]^T \in \Omega_{Z_{n-1}} \subset \mathbb{R}^{3n-3}\) and \(\Omega_{Z_{n-1}}\) being some known compact set in \(\mathbb{R}^{3n-3}\). Similar to step 1, define
\[\varphi_n(Z_{n-1}) = \varphi_n(Z_{n-1}) + \frac{H_{n-1}}{z_{n-1}} + \frac{16}{z_{n-1}} \tanh^2 \left( \frac{z_{n-1}}{\varphi_n(Z_{n-1})} \right) H_{n-1}\]
where \(\varphi_{n-1}\) is a positive design parameter. Thus, the nonlinear function \(\varphi(Z_{n-1})\) can be approximated by an FLS \(\phi_{n-1, n-1}(Z_{n-1})\) such that
\[\varphi_n(Z_{n-1}) = \phi_{n-1, n-1}(Z_{n-1}) + \delta_{n-1}(Z_{n-1})\]  

then, using
\[z_{n-1} \left( \frac{\partial}{\partial x_{n-1}^T} \tilde{x}_{n-1}(Z_{n-1}) + \delta_{n-1}(Z_{n-1}) \right) \leq \frac{\theta_{n-1}^2}{2\eta^2} \tilde{x}_{n-1}(Z_{n-1})^2 + \frac{\eta^2}{2} + \frac{\lambda \sigma_{n-1}^2}{2}\]
yields that
\[LV_{n-1} \leq z_{n-1}(a_{n-1} + \frac{1}{2} \sigma^2 + \frac{r^2}{2 \sigma} u^2(\chi) - z_{n-2}z_{n-1} + \frac{\theta_{n-1}^2}{2\eta^2} \tilde{x}_{n-1}(Z_{n-1})^2 + \frac{\eta^2}{2} + \frac{\lambda \sigma_{n-1}^2}{2}\]
\[+ \frac{\sigma_{n-1}^2}{2\eta^2} + \frac{\eta^2}{2} + \frac{\lambda \sigma_{n-1}^2}{2\eta^2} - \frac{\gamma e^{-\gamma (t-d)}}{2(1 - d^2)} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\eta f_j^2(\tilde{x}_j(s) \tilde{\gamma}_j)} ds\]  

where \(\eta_{n-1}\) is the upper bound of \(\delta_i(Z_{n-1})\) with \(|\delta_{n-1}(Z_{n-1})| \leq \eta_{n-1}\). Choose the Lyapunov function as
\[V_{n-1} = V_{n-1} + \frac{\varphi_{n-1}^2}{4\eta^2}\]  

from (8)–(10), we yield
\[LV_{n-1} \leq z_{n-1}^2 - \left( k_{n-1} - \frac{1}{2\lambda} - \frac{\sigma}{2} \right) z_{n-1}^2 + \frac{\sigma_{n-1}^2}{4\eta^2} - \frac{\lambda \sigma_{n-1}^2}{4\eta^2} + \frac{\sigma_{n-1}^2}{4\eta^2} + \frac{\eta^2}{2} + \left( 1 - 16 \tanh^2 \left( \frac{z_{n-1}}{\varphi_{n-1}(Z_{n-1})} \right) \right) H_{n-1} - \frac{\gamma e^{-\gamma (t-d)}}{2(1 - d^2)} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\eta f_j^2(\tilde{x}_j(s) \tilde{\gamma}_j)} ds\]  

Step n: Considering \(z_n = x_n - a_{n-1} + \int_{t-d_{n-1}(t)}^{t} u(s) ds\) and using Assumption 4 yields that
\[
\begin{align*}
\dot{z}_n &= d x_n - d a_{n-1} + \left( u(t) - \left( 1 - \hat{\tau}(t) \right) u(t - \tau(t)) \right) dt \\
&= \left[ m(t)u(t - \tau(t)) + f_h \left( x_n(t - d_n(t)) \bar{\gamma}_n \right) + u(t) - \left( 1 - \hat{\tau}(t) \right) u(t - \tau(t)) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} f_j \left( \bar{\gamma}_j(t - d_j(t)) \right) - W_n \right] dt \\
&+ \left( h^T_n \left( \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} h^T_j \left( \bar{\gamma}_j \right) \right) d \omega \\
&= \left[ u(t) + f_h \left( x_n(t - d_n(t)) \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} f_j \left( \bar{\gamma}_j(t - d_j(t)) \right) - W_n \right] dt \\
&+ \left( h^T_n \left( \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} h^T_j \left( \bar{\gamma}_j \right) \right) d \omega \\
&\quad \text{where} \\
W_n &= \frac{1}{2} \sum_{j=1}^{n-1} e \frac{\partial^2 a_{n-1}}{\partial x_j^2} h^T_j \left( \bar{\gamma}_j \right) h^T_j \left( \bar{\gamma}_j \right) + \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} \bar{\gamma}_j + j = 1 \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} \bar{\gamma}_j + e \frac{\partial a_{n-1}}{\partial \bar{\gamma}_{d(n-1)}} \bar{\gamma}_{d(n-1)} \\
\end{align*}
\]

(46)

Similar to Step \( i \), choose the Lyapunov function candidate as

\[
V_n = \frac{1}{2} \sum_{i=1}^{n-1} \frac{e^{-\gamma(t-d)}}{2(1 - d^2)} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\omega f^2_j \left( \bar{\gamma}_j(s) \bar{\gamma}_j \right)} ds
\]

(48)

then, we have

\[
LV_n = z_n \left( u(t) + f_h \left( x_n(t - d_n(t)) \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} f_j \left( \bar{\gamma}_j(t - d_j(t)) \bar{\gamma}_j \right) - W_n \right) \\
+ \frac{1}{2} \left( h^T_n \left( \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} h^T_j \left( \bar{\gamma}_j \right) \right)^T \left( h^T_n \left( \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} h^T_j \left( \bar{\gamma}_j \right) \right) - \frac{e^{-\gamma(t-d)}}{2(1 - d^2)} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\omega f^2_j \left( \bar{\gamma}_j(s) \bar{\gamma}_j \right)} ds \\
+ \frac{e^{\omega d}}{2(1 - d^2)} \sum_{j=1}^{n-1} f^2_j \left( \bar{\gamma}_j(s) \bar{\gamma}_j \right) - \sum_{j=1}^{n-1} \frac{e^{-\gamma(\hat{\tau}(t)-d)}}{2(1 - d^2)} \left( 1 - \hat{\tau}(t) \right) f^2_j \left( \bar{\gamma}_j(t - d_j(t)) \bar{\gamma}_j \right)
\]

(49)

Similar to Step \( i \), we can have

\[
LV_n \leq z_n \left( u(t) - W_n + \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{\partial a_{n-1}}{\partial x_j} \right)^2 + \frac{1}{2} \sum_{j=1}^{n-1} H_n \right) - \frac{e^{-\gamma(t-d)}}{2(1 - d^2)} \sum_{j=1}^{n-1} \int_{t-d_j(t)}^{t} e^{\omega f^2_j \left( \bar{\gamma}_j(s) \bar{\gamma}_j \right)} ds
\]

(50)

where

\[
H_n = \frac{1}{2} \left( h^T_n \left( \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} h^T_j \left( \bar{\gamma}_j \right) \right) \left( h_n \left( \bar{\gamma}_n \right) - \sum_{j=1}^{n-1} e \frac{\partial a_{n-1}}{\partial x_j} h_j \left( \bar{\gamma}_j \right) \right) + \frac{e^{\omega d}}{2(1 - d^2)} \sum_{j=1}^{n-1} f^2_j \left( \bar{\gamma}_j(s) \bar{\gamma}_j \right)
\]

(51)

Define

\[
\varphi_n(\bar{\gamma}_n) = \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{\partial a_{n-1}}{\partial x_j} \right)^2 - W_n + \frac{1}{2} \sum_{j=1}^{n-1} \frac{16 \tan^2 \left( \frac{z_n}{\theta_n} \right)}{z_n^{\theta_n} H_n}
\]

(52)
where $Z_n = [x_1, ..., x_n, \bar{\theta}_1, ..., \bar{\theta}_n]^{T} \in \Omega_{Z} \subset R^{3n}$ and $\Omega_{Z}$ is some known compact set in $R^{3n}$. $\theta_n$ is a positive design parameter. Thus, the nonlinear function $\bar{\psi}(Z_n)$ can be approximated by an FLS $\phi_n^{T} \xi_n(Z_n)$ such that

$$
\bar{\psi}_n(Z_n) = \phi_n^{T} \xi_n(Z_n) + \delta_n(Z_n)
$$

then, using

$$
z_n(\phi_n^{T} \xi_n(Z_n) + \delta_n(Z_n)) \leq \frac{\theta_n}{2} \xi_n^{T} \xi_n + \frac{\eta^2}{2} + \frac{\lambda \bar{\lambda}^2}{2}
$$
yields that

$$
LV_n \leq z_n u(t) + \frac{\theta_n}{2} \xi_n^{T} \xi_n + \frac{\eta^2}{2} + \frac{\lambda \bar{\lambda}^2}{2} + \left(1 - 16 \text{tanh}^2 \left(\frac{\eta_n}{\bar{\lambda}_n}\right)\right) H_n - \frac{\lambda \bar{\lambda}^2}{2(1 - d^\tau)} \sum_{j=1}^{n} \int_{t-\delta(t)}^{t} e^{\sigma^2_2 \left(\xi_j(s), \bar{\psi}_j\right)} ds
$$

where $\epsilon_j$ is the upper bound of $\delta_j(Z_n)$ with $|\delta_j(Z_n)| \leq \epsilon_i$. Choose the Lyapunov function as

$$
\bar{\psi}_n = V_n + \frac{\theta_n^2}{4\eta^2}
$$

(8) and (9) yield that

$$
LV_n \leq - \left( k_n - \frac{1}{2\lambda} \right) \xi_n^{T} \xi_n + \frac{\lambda \bar{\lambda}^2}{2} + \frac{\eta^2}{2} - \frac{\sigma^2_2}{4\eta^2} + \frac{\sigma^2_2}{4\eta^2} + \left(1 - 16 \text{tanh}^2 \left(\frac{\eta_n}{\bar{\lambda}_n}\right)\right) H_n
$$

Remark 3. The main technical obstacle in the design for stochastic systems is that the Ito stochastic differentiation involves not only the gradients but also the higher order Hessian terms $(1/2) \text{Tr} \{g^{T} (\partial^2 V / \partial x^2) g\}$. In order to handle the Hessian terms conveniently, in the most existing results, the authors used the quartic functions $(1/4) z^4$ to analyze the stability of the systems. In addition, for the stochastic tracking problem, in references 34 and 35, the controllers are designed by using the quadratic Lyapunov function under a risk-sensitive cost criterion. Unlike them in this paper, according to (11), (22), (35), and (48), we can see that without using a risk-sensitive cost criterion, the classical quadratic functions are chosen as Lyapunov functions to investigate the stochastic tracking problem, and for the higher order Hessian terms, the hyperbolic tangent functions $\text{tan} h(z_1/\nu_1)$ are introduced to deal with them so that the obstacle for stochastic systems control by the quadratic Lyapunov function is overcome. It should be pointed out that by combining the classical quadratic Lyapunov functions and the hyperbolic tangent functions, a new scientific and feasible control method for the stochastic nonlinear systems is provided.

Stability analysis

Theorem 1. For stochastic nonlinear system (1), under Assumptions 1–4, the controller (8) and the parameter adaptive law (9) guarantee that all the signals in the closed-loop system is semi-globally uniformly ultimately bounded in probability. Moreover, the tracking error in probability may be made arbitrarily small by appropriately adjusting the design parameters.

Remark 4: From (45), it can be seen that boundedness of the residual term $(e^2/2\sigma^2) \xi^2(\xi)$ cannot be sure; thus, the boundedness of the whole closed-loop system cannot be proven directly. Then, in the following, we will firstly prove that the true controller signal $u(t)$ is bounded.

Proof. Now, we assume that for $j = 1, ..., n$, $\theta_j = \bar{\theta}$, where $\bar{\theta} > 0$ is an arbitrary small constant. Then, in Lemma 1, the mentioned set $\Theta_n$ can be rewritten as $\Theta_n$.

If $|z_n| \leq 0.2554 \bar{\theta}$, according to (9), it is obvious that $\bar{\theta}_n$ is bounded. Further, $\bar{\theta}_n$ is bounded as $\theta_n$ is a constant. Define $\theta_n = \max \{ \theta_n(0), (0.2554 \bar{\theta})^2/\sigma_n \}$; then, we have $\theta_n \leq \theta_n$. Combining (9) gives that
If $|z_n| > 0.2554 \theta$, using Lemma 1 gives that $(1 - 16 \tanh^2(z_n/\theta))H_n < 0$. Then, from (56), we yield that

$$L \bar{V}_n \leq - \left( k_n - \frac{1}{2} \lambda \right)^2 + \frac{\rho^2}{2} + \frac{\sigma^2}{4} + \frac{\theta^2}{4} - \frac{\gamma^2}{2} \sum_{j=1}^{n-1} \int_{t_j}^{t} e^{\sigma f_j^2(\tilde{z}_j)} ds$$

where $k_n$ satisfies $k_n > (1/\lambda)$, so we can deduce $k_n - (1/2\lambda) > 0$. Now, denote $\psi = \min(2k_n - (1/\lambda), \sigma_n)$; one gets

$$L \bar{V}_n \leq - \psi \bar{V}_n + C_n$$

where $C_n = (\sigma^2/\psi) + (\eta^2/2)$. Note that $C_n$ is bounded; therefore, based on the conclusion of reference 16 and (59), it follows that $z_n$ and $\theta_n$ are bounded in probability. Therefore, we can get $z_n \leq N$ in probability with $N$ being the upper bound of $z_n$. Further, $\theta_n$ is bounded as $\theta_n$ is a constant. Then, according to (9), we can get that $u(t)$ is bounded and $u(t) \leq M$ with $M$ being the upper bound of $u(t)$.

Finally, let

$$\bar{u} = \max \left\{ \left( k_n + \frac{\rho}{2\theta^2} \right) 0.2554 \theta M \right\}, \quad \bar{z}_\eta = \max \{ 0.2554 \theta, \bar{N} \}$$

then, we have

$$|u(t)| \leq \bar{u}, \quad |z_n| \leq \bar{z}_\eta$$

In the following, we will prove that all the other signals in the closed-loop system are bounded, and the proof process is divided into the following three cases.

**Case 1.** For $1 \leq j \leq n - 1$, $z_j \notin \Theta_\theta$. In this case, $|z_j| > 0.2554 \theta$. Let $\bar{V}_{n-1} = \sum_{j=1}^{n-1} \bar{V}_j$; then, according to (21), (32) and (45) yields that

$$L \bar{V}_{n-1} \leq - \sum_{j=1}^{n-1} \tilde{k}_j \bar{z}_j^2 + \sum_{j=1}^{n-1} \tilde{\sigma}_j \bar{\theta}_j^2 - \sum_{j=1}^{n-1} \tilde{\sigma}_j \bar{\theta}_j^2 - \sum_{j=1}^{n-1} \gamma^2 \sum_{j=1}^{n-1} \int_{t_j}^{t} e^{\sigma f_j^2(\tilde{z}_j)} ds$$

where for $j = 1, 2, \ldots, n - 2$, $\tilde{k}_j = k_j - (1/2\lambda)$ and $\tilde{k}_{n-1} = k_{n-1} - (1/2\lambda) - (\sigma/2)$. It follows from the definitions of $H_j$ and Lemma 1 that $\sum_{j=1}^{n} (1 - 16 \tanh^2(z_j/\theta))H_j < 0$; then, from (61) and (62), we have

$$L \bar{V}_{n-1} \leq - \sum_{j=1}^{n-1} \tilde{k}_j \bar{z}_j^2 - \sum_{j=1}^{n-1} \tilde{\sigma}_j \bar{\theta}_j^2 - \sum_{j=1}^{n-1} \gamma^2 \sum_{j=1}^{n-1} \int_{t_j}^{t} e^{\sigma f_j^2(\tilde{z}_j)} ds + C$$

where $C = (1/2)^2 + (\tau^2/2\rho^2) + (\sigma^2/4\eta^2) + (\theta^2/4\eta^2) + (\gamma^2/2), k_j$ satisfies $k_j > (1/\lambda)$ for $j = 1, 2, \ldots, n - 2$ and $k_{n-1}$ satisfies $k_{n-1} > (1/\lambda) + (\sigma/2)$. Thus, we can deduce $\tilde{k}_j > 0$ and $\tilde{k}_{n-1} > 0$. Now, denote $\psi_{n-1} = \min(2\tilde{k}_1, \ldots, 2\tilde{k}_{n-1}, \tilde{\sigma}_1)$; one gets

$$L \bar{V}_{n-1} \leq - \psi_{n-1} \bar{V}_{n-1} + C$$

Note that $C$ is bounded; therefore, based on the conclusion of reference 16 and (64), we can get that $z_j$ and $\theta_j$ are bounded in probability. Further, $\bar{\theta}_j$ are bounded as $\theta_j$ are constants. Since $z_1$ and $\gamma_1$ are bounded, $y$ is also bounded. Using (8) with $i = 1$, and noting that $z_1, \theta_1$, and $z_1(\bar{z}_1)$ are all bounded, we can conclude that $\alpha_i$ is bounded.
Consequently, it follows from $x_2 = z_2 + \alpha_1$ that $x_2$ is bounded. Following the same way, $\alpha_{j-1}$ and $x_{j, j = 3, \ldots, n}$ can be proven to be bounded. Thus, in Case 1, all the signals in the closed-loop system are bounded in probability.

Furthermore, based on (64) and Dynkin’s formula,38 we have

$$E\left[\dot{V}_{n-1}\right] \leq e^{-\gamma_{n-1} t}E\dot{V}_{n-1}(0) + \psi^{-1}_{n-1}C, \forall t \geq 0$$

(65)

this (65) together with $E(||Z_{n-1}||^2) \leq 2E(\dot{V}_{n-1})$ indicates that

$$E(||Z_{n-1}||^2) \leq 2E\left[\dot{V}_{n-1}\right] \leq 2e^{-\gamma_{n-1} t}E\dot{V}_{n-1}(0) + 2\psi^{-1}C, \forall t \geq 0$$

(66)

where $Z_{n-1} = [z_1, z_2, \ldots, z_{n-1}]^T$.

**Case 2.** For $1 \leq j \leq n - 1, |z_j| \leq 0.2554\theta$. The boundedness of $\hat{\theta}_j$ for $1 \leq j \leq n - 1$ is obtained according to $|z_j| \leq 0.2554\theta$ and (9). Moreover, for $1 \leq j \leq n - 1$, $\hat{\theta}_j$ are bounded as $\theta_j$ are all constants. According to Assumption 1, $y_d, \hat{y}_d, \ldots, \hat{y}_{(n)}_d$ are all bounded; thus, it follows from $z_1 = x_1 - y_d$ that $x_1$ is also bounded. In addition, using (8) and noting that $z_j, \hat{\theta}_j$ and $\hat{z}_j$ are all bounded, one gets that $\alpha_j, j = 1, \ldots, n - 1$ are also bounded. Furthermore, in the light of $z_2 = x_2 - \alpha_1$, the boundedness of $x_2$ is ensured. Similarly, we can conclude that the state variables $x_j, j = 3, \ldots, n - 1$ in the closed-loop system are bounded in probability.

Furthermore, we have

$$E(||Z_{n-1}||^2) \leq (0.2554)^2 \|\theta\|^2$$

(67)

where $\theta$ is a $n - 1$ dimensional vector with $\theta = [\theta, \ldots, \theta]^T$.

**Case 3.** Some $z_m \in \Theta_\theta$, while some $z_j \notin \Theta_\theta$ with $m = 1, 2, \ldots, n - 1$. Define $\Sigma_M$ and $\Sigma_J$ as the index sets of subsystems consisting of $z_m \in \Theta_\theta$ and $z_j \notin \Theta_\theta$, respectively. Then, for $j \in \Sigma_J$, choose the Lyapunov function candidate as

$$\dot{V}_{j} = \sum_{i \in \Sigma_J} \psi_i$$

by the controller design process, we have the infinitesimal generator of $V_{j}$, as follows

$$L\dot{V}_{j} \leq -\sum_{i \in \Sigma_J} \frac{e_j^2}{Z_j} + \sum_{i \in \Sigma_J} \frac{\lambda_j^2}{2} + \sum_{i \in \Sigma_J} \frac{\sigma_j^2}{4\eta_j^2} - \sum_{i \in \Sigma_J} \frac{\alpha_j^2}{4\eta_j^2} + \sum_{i \in \Sigma_J} \frac{\eta_j^2}{2} + q(z_{n-1}) \left(1 - \text{tanh}^2 \left(\frac{z_j}{\delta}\right)\right) H_j < 0$$

where for $j = 1, 2, \ldots, n - 2$, $\delta_j = k_j - (1/2\lambda)$, $K_{n-1} = k_{n-1} - (1/2\lambda) - (\alpha/2)$, and $q(z_{n-1}) = \begin{cases} 1, & \text{if } n - 1 \in \Sigma_J, \\
0, & \text{if } n - 1 \in \Sigma_M \end{cases}$

Then, from Lemma 1, we can have $\sum_{i \in \Sigma_J} (1 - 16\text{tanh}^2(z_j/\delta)) H_j < 0$. Further, similar to the proof of Theorem 1 in reference 32, the last term of (68) can be expressed as

$$\sum_{j \in \Sigma_J} \frac{e_j^2}{Z_j} - \sum_{j \in \Sigma_J} \frac{\lambda_j^2}{2} \sum_{j \in \Sigma_J} \left(\frac{e_j^2}{4\lambda_j^2} + \lambda_j^2 \sum_{j \in \Sigma_J} \frac{e_j^2}{4\lambda_j^2} \right) + \sum_{j \in \Sigma_J} \left(\frac{e_j^2}{4\lambda_j^2} + \lambda_j^2 \right)$$

(69)

Based on the above discussion, (68) can be rewritten as
where
\[
C_{\Sigma_j} = \frac{1}{2} s^2_n + \frac{r^2}{2\sigma^2} + \sum_{j \in \Sigma_j} \frac{\lambda_l^2}{2} + \sum_{i \in \Sigma_j} \frac{\sigma_l^2}{4\theta^2} + \sum_{j+1 \in \Sigma_{M}, j \in \Sigma_i} 2\lambda (0.2554 \theta)^2 + \sum_{j \in \Sigma} \frac{\eta^2}{\theta^2}
\]

Further, since for \(j = 1, 2, \ldots, n-2\), \(k_j > (1/\lambda)\), and \(k_{n-1} > (1/\lambda) + (\sigma/2)\), we can get that for \(j = 1, \ldots, n-1\), \(k_j - (1/\lambda) > 0\). Now, letting \(v_j = \min(2k_j - 2(1/\lambda), \sigma_j)\) gives that
\[
LV_{\Sigma_j} \leq -v_j \hat{V}_{\Sigma_j} + C_{\Sigma_j}
\]
(71)

Similar to Case 1, it can be shown that \(z_{\Sigma}, \hat{\theta}_{\Sigma} \) and \(\hat{\hat{\theta}}_{\Sigma} \) are, in probability, bounded for \(j \in \Sigma\).

For \(m \in \Sigma_M\), we know that \(z_m\) are bounded. Similar to Case 2, by combining with the conclusions of \(j \in \Sigma_M\), we can obtain that the variables \(x_m, \hat{\theta}_m, \) and \(\hat{\hat{\theta}}_m\) are bounded in probability.

Furthermore, for \(m \in \Sigma_M\), we can get
\[
E\left(\left\|Z_{\Sigma_m}\right\|^2\right) \leq (0.2554)^2 \left\|\theta_{\Sigma_m}\right\|^2
\]
(72)
where \(\left\|Z_{\Sigma_m}\right\|^2 = \sum_{m \in \Sigma_M} z^2_m\) and \(\left\|\theta_{\Sigma_m}\right\|^2 = \sum_{m \in \Sigma_M} \theta^2_m\). Then, similar to Case 1, using (71) yields that
\[
E\left(\left\|Z_{\Sigma_m}\right\|^2\right) \leq 2E\left[\hat{V}_{\Sigma_m}\right] \leq 2e^{-v_1} E\hat{V}_{\Sigma_m}(0) + 2v_1 C_{\Sigma_m}
\]
(73)

Finally, from Cases 1–3, we can conclude that
\[
E\left(\left\|Z_{n-1}\right\|^2\right) \leq \max\left(2e^{-v_1} E\hat{V}_{n-1}(0) + 2v_1 C_{\Sigma_1}, (0.2554)^2 \left\|\theta\right\|^2, (0.2554)^2 \left\|\theta_{\Sigma_0}\right\|^2 + 2e^{-v_1} E\hat{V}_{\Sigma_0}(0) + 2v_1 C_{\Sigma_0}\right)
\]
(74)
which means that \(Z_{n-1}\) eventually in probability converges to the following set
\[
\Xi_{\Sigma} := \left\{E\left(\left\|Z_{n-1}\right\|^2\right) \leq \Pi_{\Sigma}\right\}
\]
where
\[
\Pi_{\Sigma} = \max\left(2v_1 C_{\Sigma_1}, (0.2554)^2 \left\|\theta\right\|^2, (0.2554)^2 \left\|\theta_{\Sigma_0}\right\|^2 + 2v_1 C_{\Sigma_0}\right)
\]
(75)

This concludes the proof.

**Remark 5.** From (75), one yields that the error signals \(z_1, z_2, \ldots, z_n\) are all bounded in probability. Moreover, according to (75), the tracking error in probability may be made arbitrarily small by appropriately adjusting the design parameters \(\eta, \sigma, \hat{\theta}, \) and \(k_0\). Theoretically, \(\sigma_j\) should be chosen smaller while the positive constants \(\eta, \hat{\theta}, \) and \(k_0\) should be chosen appropriately. However, how to choose the optimal parameters to get the optimal tracking performance is still an open problem. In the simulation, the design parameters are set using a trial-and-error method.

**Remark 6.** In references 36 and 37, the authors studied the fuzzy adaptive practical tracking problem for a class of nonlinear pure feedback systems with quantized input signal and addressed the adaptive neural tracking control problem for a class of uncertain stochastic nonlinear systems with non-strict feedback form and pre-specified tracking accuracy, respectively. In references 36 and 37, to analyze the convergence of the tracking error using the Barbalat’s Lemma via some nonnegative functions rather than the positive-definite Lyapunov functions, hence, the issue has been overcome that the tracking error cannot converge to an accuracy assigned a priori. For more detailed proofs, refer in the literature 37 and 38.
Simulation

Example 1: To show the feasibility of the developed scheme, the simulation example is given in the following. The developed adaptive fuzzy controllers are applied to the following stochastic nonlinear system with time-varying input delay

\[
\begin{align*}
    dx_1 &= \left( x_1 + \rho_1 e^{x_2(t-d_1(t))} \right) dt + 0.2 x_1 d\omega,
    \\
    dx_2 &= \left( m(t) u(t - \tau(t)) + \ln \left( 1 + \left( \rho_2 \rho_1 x_2(t - d_2(t)) \right)^2 \right) \right) dt + x_2^2 d\omega,
    \\
    y &= x_1
\end{align*}
\] (76)

in the simulation, \(d_1(t) = 1 + 0.5 \cos(0.5t), d_2(t) = 1 + 0.2 \cos(2t), \tau(t) = 0.1 + 0.01 \cos(-5t), m(t) = 1 - 0.05 \sin(-5t), \rho_1 = 0.5, \rho_2 = 1.6, \) and the initial states are chosen as \(x_1(0) = 1.8, \theta_1(0) = \theta_2(0) = 0, \) and \(x_2(0) = 0.\) The simulation objective is to apply the developed adaptive fuzzy controller such that the boundedness of all the signals in the closed-loop system is guaranteed and the system output \(y\) follows the reference signal \(y_d\) to a small neighborhood of zero with \(y_d = \sin t + \sin(0.5t).\) The virtual controller, the true controller, and the parameter adaptive laws are chosen as

\[
\begin{align*}
    a_1 &= -k_1 z_1 - \frac{\theta_1^T(Z_1) \xi_1(Z_1) z_1}{2\eta^2}, \\
    u(t) &= -k_2 z_2 - \frac{\theta_2^T(Z_2) \xi_2(Z_2) z_2}{2\eta^2}, \\
    \dot{\theta}_1 &= \xi_1^T(Z_1) \xi_1(Z_1) z_1^2 - \sigma_1 \theta_1, \\
    \dot{\theta}_2 &= \xi_2^T(Z_2) \xi_2(Z_2) z_2^2 - \sigma_2 \theta_2
\end{align*}
\] (77)

(78)

select the design parameters as \(\eta = 0.7, \sigma_1 = \sigma_2 = 0.1, k_1 = 10, \) and \(k_2 = 28.\)

For the stochastic nonlinearly parameterized system (76), authors make many simulation results by choosing different random seeds \(\omega\); all of them show that the signals in the closed-loop system are bounded and the system output \(y\) follows the reference signal \(y_d\) to a small neighborhood of zero. In this paper, one case of them is listed and shown in Figures 1–6. From

![Figure 1](image1.png)

**Figure 1.** \(x_1\) (solid line) and \(y_d\) (dashed line) (Example 1).

![Figure 2](image2.png)

**Figure 2.** The tracking error \(z_1\) (Example 1).
Figure 1 and Figure 2, it can be seen that good tracking performance is obtained. The boundedness of $u$ and $x_2$ are illustrated in Figures 3 and 4, respectively. The adaptive parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ are also bounded by Figures 5 and 6.

**Example 2**: Consider a one-link manipulator with the dynamic equation as follows

\[
\begin{align*}
D\ddot{q} + B\dot{q} + N\sin(q) &= \tau + \tau_d \\
M\dddot{\tau} + H\tau &= u - K_m\dot{q} + u_d
\end{align*}
\]  

(79)

**Figure 3.** The trajectory of $u(t)$ (Example 2).

**Figure 4.** The state $x_2$ (Example 1).

**Figure 5.** The trajectory of $\hat{\theta}_1$ (dashed line) (Example 1).
where \( q, \dot{q}, \) and \( \ddot{q} \) denote the link position, velocity, and acceleration, respectively. \( \tau \) is the torque produced by the direct-current motor. \( D \) is the mechanical inertia, while \( B \) denotes the viscous friction coefficient. \( N \) is the gravity coefficient along with mass of load. \( M \) and \( H \) are the armature inductance and resistance, respectively. \( K_m \) is the back electromotive force coefficient. \( \tau_d \) denotes the uncertainties with model errors with \( \tau_d = q(t - d_2) + q \sin(q) \dot{\omega} \) and with the unknown time-varying delay \( d_2 \), while the stochastic disturbance is chosen as \( u_d = \dot{q} \cos(q(t - d)) + 2q^2 \dot{\omega} \) with the unknown time-varying delay \( d_3 \), where \( \omega \) denotes the torque stochastic disturbance defined as (1). \( u \) is the control input representing the electromechanical torque. Let \( x_1 = q, x_2 = \dot{q}, \) and \( x_3 = \tau \); system (79) can be rewritten in the following form
Figure 9. The trajectory of $u(t)$ (Example 2).

Figure 10. The state $x_2$ (solid line) and $x_3$ (dashed line) (Example 2).

Figure 11. The trajectory of $\hat{\theta}_1$ (dashed line) (Example 2).

Figure 12. The trajectory of $\hat{\theta}_2$ (dashed line) (Example 2).
\[
\begin{align*}
    dx_1 &= x_2 dt \\
    dx_2 &= \left( \frac{1}{D} x_3 - \frac{B}{D} x_2 - \frac{N}{D} \sin(x_1) + \frac{1}{D} \sin(x_1(t - d_2)) \right) dt + \frac{1}{D} x_1 \sin(x_1) d\omega \\
    dx_3 &= \left( \frac{1}{M} u(t - \tau(t)) - \frac{H}{M} x_3 - \frac{K_m}{M} x_3 + \frac{1}{M} \sqrt{2} \cos(x_1(t - d_3)) \right) dt + \frac{2}{M} \theta_2 d\omega
\end{align*}
\]

It is noted that the first subsystem is a linear differential equation. Then, the virtual control \( a_2 \) is described as \( a_2 = -k_1 z_1 \). In the simulation, the system parameters are chosen as \( D = 1 \text{ kg} \cdot \text{m}^2, M = 1 \text{ kg}, B = 1 \text{ Nm} \cdot \text{s} \cdot \text{rad}, K_m = 10 \text{ Nm/A}, H = 0.5 \Omega, \) and \( N = 0.5, d_2(t) = 1 + 0.5 \cos(0.5t), d_3(t) = 1 + 0.2 \cos(2t), \tau(t) = 0.02, m(t) = 1, \) and the initial states are chosen as \( x_1(0) = x_2(0) = x_3(0) = 0, \theta_1(0) = \theta_2(0) = 0 \).

The simulation objective is to apply the developed adaptive fuzzy controller such that the boundedness of all the signals in the closed-loop system is guaranteed and the system output \( y \) follows the reference signal \( y_d \) to a small neighborhood of zero with \( y_d = 0.3 \left( 1 - \cos(t) \right) \). The controller parameters are \( \eta = 0.2, \sigma_1 = \sigma_2 = 0.9, k_1 = 15, k_2 = 28, \) and \( k_3 = 60 \). The simulation results are presented by Figures 7–12. From Figures 7 and 8, it can be seen that good tracking performance is obtained. The boundedness of \( u, x_2, \) and \( x_3 \) are illustrated in Figures 9 and 10, respectively. The adaptive parameters \( \theta_2 \) and \( \theta_3 \) are also bounded by Figures 11 and 12.

**Remark 7.** Figures 1–12 shows that for the proposed control design scheme is effective, the output can track the desired signal successfully and all the other signals in the closed-loop system are bounded in probability. Moreover, unlike the most existing results13, 14,15 in which the fourth moment approach is used as Lyapunov functions to analyze the stability of the stochastic nonlinear systems, in this paper, the use of the classical quadratic functions optimizes the control theory of the stochastic nonlinear systems.

**Conclusion**

In this paper, the problem of adaptive fuzzy tracking control has been addressed for a class of stochastic non-linearly parameterized systems with time-varying input delay. Without using the parameter separation principle, the adaptive fuzzy controller has been constructed by combining the backstepping approach. The proposed controller ensures that all the signals of the resulting closed-loop system are bounded in probability, and the tracking error converges to a small neighborhood of the origin in the mean square sense. However, under the proposed controller, the number of adjustable parameter depends on the order of the system; thus, by combining the classical quadratic Lyapunov functions, how to design the controller containing less adoption parameters to reduce the computation burden is very significant. In the future, the stochastic nonlinear with distributed time-varying input delays will be considered, of course, how to design the effective tracking controller is a challenging problem.

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