GOLDBACH VERSUS DE POLIGNAC NUMBERS

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Abstract. In this article we prove a second moment estimate for the Maynard-Tao sieve and give an application to Goldbach and de Polignac numbers. We show that at least one of two nice properties holds. Either consecutive Goldbach numbers lie within a finite distance from one another or else the set of de Polignac numbers has full density in $2\mathbb{N}$.

1. Introduction

Let $\mathcal{P}$ denote the set of prime numbers and write $p_n$ for its $n$-th member. Given an admissible tuple of integers $\mathcal{H} = \{h_1, ..., h_k\}$ the Hardy-Littlewood $k$-tuple prime conjecture is the assertion that

$$\{n + h_1, ..., n + h_k\} \subset \mathcal{P}$$

for infinitely many integers $n$. The problem has seen a number of breakthroughs over the past decade and these efforts spawned an international collaboration known as the Polymath8 project ([2]). Assuming the generalised Elliott-Halberstam conjecture, it was demonstrated that any admissible configuration $\{n + h_1, n + h_2, n + h_3\}$ contains at least two primes for infinitely many values of $n$. These ideas can be applied to Goldbach numbers, that is to say, positive integers which can be expressed as the sum of two primes. Fixing some large natural number $N$ one considers the collection $\{n, n + 2, N - n\}$ and in this manner it can be shown, under suitable hypotheses, that at least one of the following statements must hold

(i) There are infinitely many twin primes.
(ii) One has $g_{n+1} - g_n \leq 4$ for all sufficiently large $n$.

Here $g_n$ denotes the $n$-th Goldbach number. In this paper we prove a result of the same nature. To state the theorem, we say $m$ is a de Polignac number if there exist infinitely many pairs of primes $(p, p')$ such that $p - p' = m$. Let $\mathcal{D}$ denote the set of de Polignac numbers.

Theorem 1.1. At least one of the following statements must hold

(i) There exists an absolute constant $C > 0$ so that $g_{n+1} - g_n \leq C$ for all sufficiently large $n$.
(ii) The set $\mathcal{D}$ has full asymptotic density in the even numbers and more precisely

$$|\mathcal{D}^c \cap [0, N]| \leq N^\kappa$$

for all large $N$ and some $\kappa < 1$.

We note that this result is unconditional while the Polymath theorem relies on the powerful Elliott-Halberstam Conjecture. It will, however, be necessary to push just beyond the reach of the Bombieri-Vinogradov Conjecture. The proof of Theorem 1.1 employs a second moment estimate for the Maynard-Tao sieve weight (see [5]) together with a Cauchy-Schwarz argument. We will show that a similar argument can be applied to the sequence of normalised prime gaps. Letting $\mathcal{L}$ denote the set of limit points for the
sequence \((p_{n+1} - p_n) / \log p_n\), Banks, Freiberg and Maynard \[1\] established the lower bounds
\[
\liminf_{T \to \infty} \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{8} \quad \text{and} \quad \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{22} \quad \forall T > 0,
\]
where \(m\) denotes the Lebesgue measure on \(\mathbb{R}\). The asymptotic density estimate is ineffective in \(T\).

We give a simple extension of this result.

**Proposition 1.2.** The limit set \(\mathcal{L}\) obeys the estimates
\[
\liminf_{T \to \infty} \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{4} \quad \text{and} \quad \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{3}{25} \quad \forall T > 0,
\]
with the first estimate being ineffective in \(T\).

**Remark.** Assuming a variation on the Elliott-Halberstam conjecture we will prove in Section 5 that the constant \(1/4\) may be replaced with \(1/2\).

**Notation** We introduce some standard notation that will be used throughout the paper. For functions \(f\) and \(g\) we will use the symbols \(f \ll g\) and \(f = O(g)\) interchangeably to express Landau’s big O symbol. A subscript of the form \(\ll \eta\) means the implied constant may depend on the quantity \(\eta\). The statement \(f \sim g\) means \(f\) and \(g\) are asymptotically equivalent, i.e., \(\lim_{x \to \infty} f(x)/g(x) = 1\) and we will write \(r(N, k) = o_k(1)\) when \(\lim_{k \to \infty} r(N, k) = 0\), independently of \(N\). Given a natural number \(m\), we write \(P^+(m)\) for its largest prime divisor. We reserve the letter \(\mu\) for the Möbius function and write \([-N] = \{1, 2, \ldots, N\}\) for any natural number \(N\).

2. Setting up the sieve

2.1. The general framework. In order to obtain clusters of primes in bounded intervals one considers sums of the form
\[
S = \sum_{n \leq N} \left( \sum_{i=1}^{k} 1_p(n + h_i) - (m - 1) \right) w(n)^2.
\]
When \(S > 0\) we necessarily have some \(m\)-tuple \((n + h_1, \ldots, n + h_m)\) consisting entirely of primes.

The weight function \(w(n)\) takes the shape
\[
w(n) = \sum_{\substack{d_1, \ldots, d_k \in \mathbb{R}^+ \mid \sum_{i=1}^{k} d_i \leq \tau \atop d_i \mid n + h_i \forall i}} \lambda_d
\]
where \(d = (d_1, \ldots, d_k)\) denotes a \(k\)-tuple of positive integers, \(d = \prod_{i=1}^{k} d_i\) and
\[
\lambda_d = \left( \prod_{i=1}^{k} \mu(d_i) \right) f \left( \frac{\log d_1}{\log R}, \ldots, \frac{\log d_k}{\log R} \right)
\]
for some smooth function \(f : [0, \infty)^k \to \mathbb{R}\) supported on \(\Delta_k := \left\{ t_1, \ldots, t_k \geq 0 \right\} \setminus \left\{ \sum_{i=1}^{k} t_i \leq 1 \right\}\).

Due to a technical restriction, which will be pointed out in the appendix, it is in fact necessary to reduce the size of the simplex. We define for any pair of real numbers \(0 \leq \eta < \tau < 1\), the region
\[
\Delta_k(\eta, \tau) = \left\{ t_1, \ldots, t_k \geq \eta \left| \sum_{i=1}^{k} t_i \leq \tau \right. \right\}. \quad \text{The truncation parameter } R = N^\delta, \quad \text{with } 0 < \delta < 1, \quad \text{depends on the level of distribution of the primes. We also let } w := \log \log \log N, \text{ set } W = \prod_{p \leq w} p.
and choose a residue class $b_0 \mod W$ with $(b_0, W) = 1$. With regards to the partial derivatives of $f$, define for each $1 \leq j \leq k$,

$$Df = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} f$$

and

$$D_j f = \frac{\partial^{k-1}}{\partial t_1 \cdots \partial t_{j-1} \partial t_{j+1} \cdots \partial t_k} f.$$

We record here the asymptotic estimates required to compute the sums appearing in $S$. They are essentially proven in [2, Section 5] but we will give a short sketch of these results in the appendix.

**Proposition 2.1.** Let $k \in \mathbb{N}$ be sufficiently large. Under the assumptions and notation introduced above, there exist constants $\delta > 1/4$ and $\sigma > 0$ with the following property. For any smooth function $f : [0, \infty)^k \to \mathbb{R}$ supported on $\Delta_k(0, \sigma)$ and for each choice of index $1 \leq i_0 \leq k$ one has the estimates

$$\sum_{n \leq N} 1_P(n + h_{i_0}) w(n)^2 \sim \delta N \beta(N) J_{i_0}^{(i_0)}(f) \quad \text{and}$$

$$\sum_{n \leq N} w(n)^2 \sim N \beta(N) I_k(f).$$

The superscript $'$ indicates that $n$ is made to run through natural numbers in the residue class $b_0 \mod W$. $J$ and $I$ are integrals given by

$$J_{i_0}^{(i_0)}(f) = \int D_{i_0} f(t_1, \ldots, t_{i_0-1}, 0, t_{i_0+1}, \ldots, t_k)^2 \, dt_1 \cdots dt_{i_0-1} dt_{i_0+1} \cdots dt_k,$$

$$I_k(f) = \int D f(t_1, \ldots, t_k)^2 \, dt_1 \cdots dt_k$$

and

$$\beta(N) = \beta(N, W) = \frac{W^k}{\varphi(W)^k} (\log N)^k.$$

The main ingredient in the proof of Theorem 1.1 will be a second moment estimate for the weight $\sum_{i=1}^k 1_P(n + h_i) w(n)$. Using this bound we will finish the argument in Section 4.

**Proposition 2.2** (Second moment estimate). Let $\psi : \mathbb{N} \to \mathbb{R}$ be a positive sequence tending to zero and suppose that $\psi(k) \log k \to \infty$. Under the assumptions outlined above, there exists a smooth function $f : [0, \infty)^k \to \mathbb{R}$ supported on $\Delta_k(0, \sigma)$ satisfying the estimates

$$\sum_{n \leq N} 1_P(n + h_{i_0}) w(n)^2 \sim \psi(k) \frac{\log k}{k} \delta N \beta(N) I_k(f)(1 + o_k(1)),$$

$$\sum_{n \leq N} 1_P(n + h_i) 1_P(n + h_j) w(n)^2 \leq \delta \psi(k)^2 \frac{(\log k)^2}{k^2} N \beta(N) I_k(f)(1 + o_k(1))$$

for all $h_{i_0}$ and all pairs $h_i \neq h_j$ in $\mathcal{H}$.

\footnote{Recall the notation $r(N, k) = o_k(1)$ when $\lim_{k \to \infty} r(N, k) = 0$, independently of $N$.}
A VARIATIONAL PROBLEM

We begin the proof of Proposition 2.2 with a nice observation made by T. Tao in the blog post [7]. Given \( h_i \neq h_j \), one has that

\[
1_p(n + h_j)1_p(n + h_i) \left( \sum_{d_i, \ldots, d_k} \lambda_d \right)^2 \leq 1_p(n + h_i) \left( \sum_{d_i, \ldots, d_k} \lambda_d \right)^2
\]

provided that \( \tilde{\lambda} \) satisfies \( \tilde{\lambda}_d = \lambda_d \) whenever \( d_i = d_j = 1 \). Since the values of \( i \) and \( j \) will have no bearing on the argument, let us assume for notational convenience that \( i = 1 \) and \( j = k \). Defining \( \lambda \) as in (2.3), we have thus reduced our problem to that of finding a function \( \tilde{f} \) which minimises

\[
\tilde{M}_k := \frac{\sum_{n \leq N} 1_p(n + h_1) \tilde{w}(n)^2}{\sum_{n \leq N} w(n)^2}
\]

subject to the condition

(3.1) \[ \tilde{f}(0, t_2, \ldots, t_{k-1}, 0) = f(0, t_2, \ldots, t_{k-1}, 0). \]

An application of Proposition 2.1 gives the asymptotic

\[
\tilde{M}_k \sim \delta \frac{\int_{\Delta_k^{-1}} D_k \tilde{f}(t_1, t_2, \ldots, t_{k-1}, 0)^2 dt_1 \ldots dt_{k-1}}{\int_{\Delta_k} D_f(t_1, t_2, \ldots, t_k)^2 dt_1 \ldots dt_k}.
\]

Using the techniques developed in [2] it can be shown that for some specific choice of \( f \) one has the bound

(3.2) \[ \frac{\int_{\Delta_k^{-1}} D_k f(t_1, t_2, \ldots, t_{k-1}, 0)^2 dt_1 \ldots dt_{k-1}}{\int_{\Delta_k} D_f(t_1, t_2, \ldots, t_k)^2 dt_1 \ldots dt_k} \sim \psi(k) \frac{\log k}{k}(1 + o_k(1)). \]

We will revisit this estimate in the next section but let us assume for the time being that (3.2) holds. Then it remains to minimise \( \int D_k \tilde{f}(t_1, t_2, \ldots, t_{k-1}, 0) \) under the constraint (3.1). By the Euler-Lagrange equations, the extremiser \( \tilde{f} \) must satisfy \( \frac{\partial}{\partial t_1} D_k \tilde{f} = 0 \). Applying the boundary conditions \( f(\partial \Delta) = 0 \) together with (3.1) one finds the minimiser \( \tilde{f} \) for which

\[
D_k \tilde{f}(t_1, t_2, \ldots, t_{k-1}, 0) = - \left[ \frac{\partial^{k-2}}{\partial t_2 \ldots \partial t_{k-1}} f(0, t_2, \ldots, t_{k-1}, 0) \right] / (1 - t_2 - \ldots - t_{k-1}).
\]

To avoid issues on the boundary \( \partial \Delta_k \) recall that we defined \( \Delta_k(\eta, \tau) = \left\{ t_1, \ldots, t_k \geq \eta \mid \sum_{i=1}^k t_i \leq \tau \right\} \) for any pair of real numbers \( 0 \leq \eta < \tau < 1 \). Introducing the notation \( F(t_1, \ldots, t_k) = \frac{\partial f}{\partial t_1 \ldots t_k} \) and applying the fundamental theorem of calculus, it remains to choose a \( 0 < \tau < 1 \) and bound the
ratio
\[
\frac{\left( \int F(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^2}{(1 - \sum_{i=2}^{k-1} t_i)^2} / \int F(t_1, \ldots, t_k) dt_1 \ldots dt_k \\
\leq \frac{\int_{\Delta_k(0, \tau)} \left( \int F(t_1, \ldots, t_k) dt_1 \right)^2}{(1 - \sum_{i=2}^{k-1} t_i)^2} dt_2 \ldots dt_k \\
\int \left( \int F(t_1, \ldots, t_k) dt_1 \right)^2 dt_2 \ldots dt_k \\
\leq \int_{\Delta_k(0, \tau)} \left( \int F(t_1, \ldots, t_k) dt_1 \right)^2 / (1 - \sum_{i=2}^{k-1} t_i)^2 dt_2 \ldots dt_k + R \\
\leq \int_{\Delta_k(0, \tau)} \int F(t_1, \ldots, t_k) dt_1 \ldots dt_k \\
\leq I' + R.
\]

In the second line we removed the integration over the region \( \Gamma := \text{supp}(F) \setminus \Delta_k(0, \tau) \) at the cost of an error term
\[
R \leq \text{vol}(\Gamma) \sup_{x \in \Gamma} F(x)^2.
\]

Now let \( \eta > 0 \) be a small constant (to be chosen later) and introduce the function
\[
F(t_1, \ldots, t_k) := \begin{cases} 
\prod_{i=1}^k g(kt_i) & \text{for } (t_1, \ldots, t_k) \in \Delta_k(2\eta, \tau) \\
\phi(t_1, \ldots, t_k) & \text{for } (t_1, \ldots, t_k) \in \Delta_k(\eta, \tau + \eta) \setminus \Delta_k(2\eta, \tau) \\
0 & \text{otherwise}
\end{cases}
\]
with \( g \) taking the form
\[
g(t) = 1_{[0, T]}(t) / (l + At).
\]

Here \( l > 1 \) and \( \tau = l^{-1} \). This is a smooth modification of the function given in [3, Section 7]. Our intention is to choose a bump function \( \phi \) which makes \( F \) smooth and satisfies the bound
\[
\max_{x \in \Delta_k(\eta, \tau + \eta) \setminus \Delta_k(2\eta, \tau)} \phi(x) \leq \max_{x \in (2\eta, \tau)} F(x).
\]

For the construction of such a bump function it suffices to use a \( C^\infty \) version of Urysohn’s lemma (see [3, 8.18]). We then gain control over \( R \) by choosing \( \eta(k) > 0 \) to be sufficiently small.

Before proceeding with the evaluation of \( I' \) and \( I \), we note the estimates
\[
\left( \int g(x) \, dx \right)^2 = \frac{\log(1 + 4\tau)}{A^2} \quad \text{and} \quad \nu := \int g(x)^2 \, dx = \frac{1}{l} \left( 1 - \frac{1}{1 + \frac{AT}{l}} \right)^{-1}.
\]

Due to the presence of the factor \( (1 - \sum_{i=2}^{k-1} t_i)^{-2} \) in the integral \( I' \) it will be convenient to assume that \( g \)'s center of mass is much smaller than 1. We impose the condition
\[
m_c := \int x g(x)^2 \, dx / \int g(x)^2 \, dx \leq \frac{T}{l} \left( 1 - \frac{T}{k} \right).
\]

3.1. Estimates for \( \tilde{M}_k \), \( I' \) and \( I \). Let us first prove the estimate (3.2) for \( \tilde{M}_k \).

Lemma 3.1. For \( g \) defined as above and \( \epsilon := (1 - T/k)/l \), one has the estimates
\[
\left( \sum_{i=1}^k t_i \geq \epsilon \prod_{i=1}^k g(kt_i) \right)^2 dt_1 \ldots dt_k \leq k^{-k} \nu^k \frac{T}{kl} \left( 1 - \frac{T}{k} \right) \left( 1 - \frac{T}{l} \right)^{-1}.
\]
and

\[ (3.7) \]
\[
\int_{\sum_{i=1}^{k} t_i \geq \varepsilon} \left( \prod_{i=1}^{k} g(kt_i) \right)^2 dt_1 \cdots dt_k \leq k^{-(k+1)} \nu^{k-1} \frac{T}{kl} \left( \frac{1 - \frac{T}{l}}{1 - \frac{T}{l} - m_c} \right)^{-2} \left( \int g(x) \, dx \right)^2.
\]

**Proof.** Let \( \rho := (k - T)/l(k - 1) - m_c > 0 \) and write \( x_i = kt_i \) for \( i = 1, \ldots, k \). We proceed as in [5 Section 7] and observe that the condition \( \sum_{i=1}^{k} t_i \geq \varepsilon \) implies \( \sum_{i=1}^{k} x_i > (k - 1)m_c \). This gives the inequality \( 1 \leq \rho^{-2} (\sum_{i=1}^{k} x_i - m_c) \) from which we gather that

\[
\int_{\sum_{i=1}^{k} t_i \geq \varepsilon} \left( \prod_{i=1}^{k} g(kt_i) \right)^2 dt_1 \cdots dt_k \leq \rho^{-2} k^{-k} \int \cdots \int \left( \prod_{i=1}^{k} x_i - m_c \right)^2 \left( \prod_{i=1}^{k} g(x_i) \right)^2 \, dx_1 \cdots dx_k.
\]

After expanding the square, a straightforward computation shows that the RHS is no greater than \( (\rho^{-2} k^{-k} m_c T \nu^k)/(k - 1) \). The inequality (3.6) now follows since \( (k - 1) \rho^2 \geq k((1 - T)/l - m_c)^2 \) and \( m_c \leq 1/l \). A small modification of this argument gives (3.7). \( \square \)

For the remainder of this section we impose the restrictions

\[ (3.8) \]
\[
\frac{T}{kl} \left( \frac{1 - \frac{T}{l}}{1 - \frac{T}{l} - m_c} \right)^{-2} = o_k(1) \quad \text{and} \quad \log \left( 1 + \frac{AT}{l^2} \right) \sim \log k.
\]

To prove (3.2) we first observe that \( \int_{\sum_{i=1}^{k} t_i \geq \tau} F(t_1, \ldots, t_k)^2 = \int_{\sum_{i=1}^{k} t_i \geq \tau} \phi(t_1, \ldots, t_k)^2 \) can be controlled, as in the previous section, by taking \( \eta \) sufficiently small with respect to \( k \). Combining (3.6) with the first estimate in (3.8) we now get

\[
\int_{\sum_{i=1}^{k} t_i \leq \tau} F(t_1, \ldots, t_k)^2 \, dt_1 \cdots dt_k = \int_{\sum_{i=1}^{k} t_i \leq \tau} \left( \prod_{i=1}^{k} g(kt_i) \right)^2 \, dt_1 \cdots dt_k = k^{-k} \nu^k (1 + o_k(1)).
\]

On the other hand (3.7), together with the estimates in (3.8), yields

\[
\int \left( \int F(t_1, \ldots, t_k) \, dt_1 \right)^2 \, dt_2 \cdots dt_k \sim \int_{\sum_{i=2}^{k} t_i \leq \tau} \left( \prod_{i=1}^{k} g(kt_i) \right)^2 \, dt_2 \cdots dt_k
\]
\[
\sim \int_{\sum_{i=2}^{k} t_i \leq \tau} \left( \int g(kt_i) \, dt_1 \right)^2 \, dt_2 \cdots dt_k
\]
\[
= k^{-(k+1)} \nu^{k-1} \left( \int g(x) \, dx \right)^2 (1 + o_k(1)).
\]

Combining all of the preceding estimates, we find that

\[
\frac{J_k}{l_k} \sim \frac{l \log k)^2}{kA} \left( 1 - \frac{T}{kl} \left( \frac{1 - \frac{T}{l}}{1 - \frac{T}{l} - m_c} \right)^{-2} \right).
\]

In order to find an appropriate choice of parameters \( A, T, l \), we set \( 1 + AT/l = e^\alpha \) with \( \alpha = \log k - c \log \log k \) for some constant \( c > 0 \). After a simple calculation one arrives at the expression
Let \( H \) be a tuple. Assume furthermore that \( N \in \mathbb{N} \) is sufficiently large with respect to \( k \). Associated to \( H \) there is another tuple \( H' = \{ h'_1, ..., h'_k \} \), where \( h'_j = N - h_j \) for each \( j \). Now write \( \overline{H} = H \cup H' \) and define

\[
\tilde{S}(\overline{H}) = \sum_{n \in [N/2,N]} \left( \sum_{h \in \overline{H}} a_h(n) \right) w(n)^2.
\]

where \( a_h(n) = 1_{P}(n + h) \) when \( h \in H \) and \( a_h(n) = 1_{P}(h - n) \) when \( h \in H' \). An application of Cauchy-Schwarz gives

\[
\tilde{S}(\overline{H}) \leq \left( \sum_{n \in [N/2,N]} 1_{\{X > 0\}}(n)w(n)^2 \right)^{1/2} \left( \sum_{n \in [N/2,N]} \sum_{h \in \overline{H}} a_h(n)a_{h'}(n)w(n)^2 \right)^{1/2}
\]

(4.1)
where $X(n) = X_{\mathcal{H}}(n) = \sum_{h \in \mathcal{H}} \alpha_h(n)$. From (4.1) and Proposition 2.2 we get the lower bound
\[
\sum_{n \in [N/2, N]} w(n)^2 \geq \sum_{n \in [N/2, N]} 1_{(X > a)}(n) w(n)^2
\]
\[
\geq \left( \sum_{n \in [N/2, N]} \sum_{h \in \mathcal{H}, h \neq h'} a_h(n) w(n)^2 \right)^2 \left( \sum_{n \in [N/2, N]} \sum_{h, h' \in \mathcal{H}, h \neq h'} a_h(n) a_{h'}(n) w(n)^2 \right)^{-1}
\]
\[
= \left[ \frac{\delta N}{2} \beta(N) \psi(2k) \log(2k) I_{2k}(f) \right]^2 \left( \sum_{n \in [N/2, N]} \sum_{h, h' \in \mathcal{H}, h \neq h'} a_h(n) a_{h'}(n) w(n)^2 \right)^{-1} (1 + o_k(1)).
\]

Now let $\mathcal{M} := (\mathcal{H} \times \mathcal{H}) \cup (\mathcal{H}' \times \mathcal{H}')$ and write $\mathcal{M}'$ for the complement of $\mathcal{M}$ in $\mathcal{H} \times \mathcal{H}$. We gather that
\[
(4.2) \quad \sum_{n \in [N/2, N]} \sum_{(h, h') \in \mathcal{M}'} a_h(n) a_{h'}(n) w(n)^2 \geq \frac{(\delta(N/2) \beta(N) \psi(2k) \log(2k) I_{2k}(f))^2}{(N/2) \beta(N) I_{2k}(f)} (1 + o_k(1))
\]
\[
- \sum_{n \in [N/2, N]} \sum_{(h, h') \in \mathcal{M}} a_h(n) a_{h'}(n) w(n)^2.
\]

To prove Theorem 1.1 we set $\delta = 1/4 + \epsilon$ and consider two mutually exclusive assumptions.

**Hypothesis A** We say hypothesis A holds if there exists an increasing sequence of natural numbers $k$ satisfying the following condition. For each admissible $k$-tuple $\mathcal{H}$ at least $1/2 - \epsilon$ of all pairs $1 \leq i < j \leq k$ produce a difference $h_j - h_i$ which is not a de Polignac number.

Suppose hypothesis A is true and let $n \in [N/2, N]$. It follows that $a_h(n) a_{h'}(n) = 0$ for at least $1/2 - \epsilon$ of all pairs $(h, h') \in \mathcal{M}$. Plugging this information back into (4.2) and applying Proposition 2.2 we find that
\[
\sum_{n \in [N/2, N]} \sum_{(h, h') \in \mathcal{M}'} a_h(n) a_{h'}(n) w(n)^2 \geq \frac{\delta^2 N}{2} \beta(N) (\psi(2k))^2 (\log 2k)^2 I_{2k}(f) (1 + o_k(1))
\]
\[
- 2(1/2 + \epsilon) \frac{\delta N}{2} \beta(N) \psi(2k) \left( \frac{(\log 2k)^2}{4k^2} (k^2 - k) I_{2k}(f) (1 + o_k(1)) \right).
\]

A simple calculation shows that the RHS is a positive quantity for $N$ and $k$ sufficiently large. From this we deduce the existence of some $n \in [N/2, N]$ and a pair $h_i, h_j \in \mathcal{H}$ for which $n + h_i$ and $N - n - h_j$ are both prime. This implies that all sufficiently large $N$ lie within a bounded distance from a Goldbach number.

Now consider the case where hypothesis A fails and write $\mathcal{D}$ for the set of de Polignac numbers. Let $k$ be any sufficiently large number and $\mathcal{H}$ an admissible $k$-tuple. Then at least $1/2 + \epsilon$ of all pairs $1 \leq i < j \leq k$ give a difference $h_j - h_i$ which is a de Polignac number. As an immediate consequence we get the following useful property. Let $U := \{u_1, ..., u_k\} \subset 2N$ and $V := \{v_1, ..., v_k\} \subset 2N$ be a
pair of sets for which $U \cap V = \emptyset$ and $U \cup V$ is admissible. Then there exists a $(u, v) \in U \times V$ with $|u - v| \in \mathcal{D}$. We will say $(\mathcal{D}, k)$ satisfies the cross product property. To finish the proof of Theorem 4.1 we need the following lemma which was proven in a private communication with S. Miner and S. Das.

**Lemma 4.1.** Let $k \in \mathbb{N}$ be arbitrary and suppose $(\mathcal{D}, k)$ satisfies the cross product property. Then $\mathcal{D}$ has full asymptotic density in $2\mathbb{N}$. Moreover, we have the power saving

$$|\mathcal{D}^c \cap [N]| \ll N^\kappa$$

for some $\kappa < 1$ depending on $k$.

**Remark** There is an expedient way of establishing the full density of $\mathcal{D}$ without the power saving result. Indeed, suppose for a contradiction that the set $A := \mathcal{D}^c \cap 2\mathbb{N}$ has positive upper density and let $\mathcal{P}(y) = \prod_{p \leq y} p$. An application of Szemerédi’s Theorem gives a $(2k - 2)$-term arithmetic progression $P = (b + ra)_{r \leq 2k - 2} \subset A$. We may assume without loss of generality that $a \equiv 0 \mod \mathcal{P}(2k)$. Now consider the pair $U = \{a, 2a, \ldots, ka\}$ and $V = \{b + ka, b + (k + 1)a, \ldots, b + (2k - 1)a\}$. Clearly $U$ and $V$ do not intersect and their union is admissible. Since the difference set $|U - V| = P \subset A$, the cross product property gives the desired contradiction.

We now turn to the estimate for $\mathcal{D}^c \cap [N]$. The result will follow from two simple lemmas. Call a pair $\{x, y\}$ an $A$-pair if $|y - x| \in A$.

**Lemma 4.2.** For every $k$ there is an $\ell = \ell(k)$ such that if $U$ and $T$ are two disjoint subsets of $2[N]$ of size $\ell$, then there are $X \subset U$ and $Y \subset T$ such that $|X| = |Y| = k$ and $X \cup Y$ is admissible.

Given the above lemma, we shall use the classic result of Kövári–Sós–Turán on the Turán number of complete bipartite graphs to resolve the problem.

**Theorem 4.3** (Kővári–Sós–Turán, 1954). If $G$ is a graph on $n$ vertices that does not contain $K_{\ell, \ell}$ as a subgraph, then $G$ has at most $cn^{1/\ell}n^{2-1/\ell} + O(n)$ edges.

**Proof of Lemma 4.2** Let $H$ be a graph with vertices $V = 2[N]$, and edges

$$E = \{\{x, y\} : \{x, y\} \text{ is not an A-pair}\}.$$

Let $(\mathcal{D}, k)$ satisfy the cross product property and let $\ell = \ell(k)$ be as in Lemma 4.2. We claim that $H$ is $K_{\ell, \ell}$-free. Indeed, suppose for contradiction $K_{\ell, \ell} \subset H$, and let $U$ and $T$ be the two vertex sets on which this copy of $K_{\ell, \ell}$ is realised. In particular, we must have $U \times T \subset E(H)$, and so there are no $A$-pairs in $U \times T$.

However, by Lemma 4.2 we can find two $k$-sets $X \subset U$ and $Y \subset T$ such that $X \cup Y$ is admissible. By assumption there must be some $A$-pair in $X \times Y \subset U \times T$, giving the necessary contradiction. Thus $H$ is indeed $K_{\ell, \ell}$-free, and by Theorem 4.3 has $O(N^{2-1/\ell})$ edges. However, since there are $N - d$ edges corresponding to a difference of $2d$, we need at least $(t + 1)$ edges to cover $t$ differences. Since $H$ has only $O(N^{2-1/\ell})$ edges, it can cover at most $O(N^{1-1/(2\ell)})$ differences, and thus we must have $|A| \geq N - O(N^{1-1/(2\ell)})$.

It remains to prove Lemma 4.2.

**Proof of Lemma 4.2** Since we wish to find sets $X$ and $Y$ of size $k$, we need only consider primes of size at most $2k$. Since $U, T \subset 2[N]$ consist solely of even integers, we need only take into account odd primes $p_2 < \ldots < p_m \leq 2k$, where $m = \pi(2k)$. To begin, set $X_1 = U$ and $Y_1 = T$ and $\ell_0 = \ell = 3^m k$. 

Now suppose for $2 \leq i \leq m - 1$ we are given subsets $X_i \subset U$ and $Y_i \subset T$, both of size $\ell_i$, such that $X_i \cup Y_i$ does not occupy all residue classes modulo $p_j$, for any $2 \leq j \leq i$. By the pigeonhole principle, there is some residue class $C$ modulo $p_{i+1}$ such that $|\{X_i \cup Y_i\} \cap C| \leq 2\ell_i/p_{i+1}$. Let $X'_{i+1} = X_i \setminus C$ and $Y'_{i+1} = Y_i \setminus C$. Let $\ell_{i+1} = \ell_i(1 - 2/p_{i+1})$, and observe that this gives a lower bound on the sizes of $X'_{i+1}$ and $Y'_{i+1}$. Finally, take $X_{i+1}$ and $Y_{i+1}$ to be arbitrary subsets of $X'_{i+1}$ and $Y'_{i+1}$ of size $\ell_{i+1}$, and note that these sets do not occupy the residue class $C$ modulo $p_{i+1}$.

Repeating these process, we arrive at sets $X_m \subset U$ and $Y_m \subset T$ of size $\ell_m$ that do not occupy all residue classes modulo $p_j$ for any $1 \leq j \leq m$, and hence are admissible. If $\ell_m \geq k$, we can take $X$ and $Y$ to be arbitrary $k$-subsets of $X_m$ and $Y_m$.

We have $\ell_m = \ell_{m-1}(1 - 2/p_m) = \ldots = \ell \prod_{j=2}^{m}(1 - 2/p_j) \geq \ell/3^m = k$, completing the proof. \qed

5. A note on the sequence of normalised prime gaps

In this section we give a proof of Proposition 1.2 and discuss a conditional improvement of the result which relies on a conjectural form of [1, Theorem 4.2] combined with a simple Cauchy Schwarz estimate. We first require some notation and background results. Recall [1, Lemma 4.1].

**Lemma 5.1.** Let $T \geq 3$ and assume $P \geq T^{\log_2 T}$. Then there exists an absolute constant with the following property. Ranging over all moduli $q$ satisfying $q \leq T$ and $P^+(q) \leq P$ there is at most one primitive character $\chi$ mod $q$ for which $L(s, \chi)$ has a zero in the region

$$Re(s) \geq 1 - \frac{c}{\log P}, \quad |Im(s)| \leq \exp\left[\log P/(\log T)^{1/2}\right].$$

In this case, one has the bounds

$$P^+(q) \gg \log q \gg \log_2 T.$$

Following the notation of Lemma 5.1 we introduce the quantities

$$Z(NT) = P^+(q), \quad w = \epsilon \log N, \quad W = \prod_{p \leq w} p,$$

and consider a modified form of the Bombieri-Vinogradov theorem. For squarefree $q_0$ satisfying $P^+(q_0) \leq N^{\log_2 N}$ we require an estimate of the following form. There exists a constant $0 < \theta < 1$ so that for any small $\delta > 0$

$$\sum_{q \leq N^{\theta - \delta}} \max_{(a, q) = 1} \left| \psi(N; q, a) - \frac{N}{\varphi(q)} \right| \ll_{A, \delta} \frac{N}{\varphi(q_0)(\log N)^A}.$$

In [1, Theorem 4.2] it was demonstrated that (5.1) holds with $\theta = 1/2$.

Now assume $k \in \mathbb{N}$ is large and let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be an admissible $k$-tuple for which each member is bounded in size by $N$. Assume also that each prime dividing $\prod_{1 \leq i < j \leq k} (h_i - h_j)$ is smaller than $w$.

In our current setting we require a modified version of the weight (2.2). For a $k$-tuple $d = (d_1, \ldots, d_k)$, define

$$\lambda_d = \left( \prod_{i=1}^k \mu(d_i) \right) \sum_{j=1}^k \prod_{l=1}^k F_{j,l} \left( \frac{\log d_l}{\log N} \right).$$
with $J$ a fixed number, $F_{j,t} : [0, \infty] \to \mathbb{R}$ smooth and compactly supported. We also assume $\lambda_d$ is supported on $k$-tuples for which $(\prod_{i=1}^k d_i, Z(N^{4\varepsilon})) = 1$ and $(\prod_{i=1}^k d_i) \leq N^\delta$. We let $\nu$ denote the associated weight function given in (2.2).

**Proposition 5.2.** Let $a = [2/\theta]$, $m \in \mathbb{N}$ and $k$ a large natural number with $am+1|k$. Suppose, in addition to all of the above assumptions, that the $k$-tuple $\mathcal{H}$ is partitioned into two parts

$$\mathcal{H} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{am+1}$$

with $|\mathcal{H}_i^{(1)}| = k/(am+1)$ for each $i$. Then there exists an $n_1 \in [N, 2N]$ for which $n_1 \equiv b \mod W$ together with a set of $m+1$ distinct indices $\{i_1^{(1)}, \ldots, i_{m+1}^{(1)}\} \subset \{1, \ldots, am+1\}$ satisfying

$$|\mathcal{H}_i^{(1)} \cap \mathcal{P}| = 1 \text{ for all } i \in \{i_1, \ldots, i_{m+1}\}$$

**Proof.** Define $Y_j(n) = \sum_{h \in \mathcal{H}_j} 1_P(n+h)$ for each $1 \leq j \leq am+1$ and consider the sum

$$A = \sum_{n \leq N} \left( \sum_{j=1}^{am+1} 1_{Y_j > 0}(n) - m - \sum_{j=1}^{am+1} \sum_{h,h' \in \mathcal{H}_j} 1_P(n+h)1_P(n+h') \right) \nu(n)^2.$$ 

Observe that the result will follow if we are able to demonstrate that $A > 0$. By choosing the functions $F_{j,t}$ appropriately the following bounds were proven in [1] Lemma 4.5 parts (i), (ii), (iii). For any $0 < \rho < 1$ and any small $\delta > 0$ one has

$$\sum_{n \leq N} \nu(n)^2 \sim N \beta(N)I_k(F)(1 + o_k(1))$$

$$\sum_{n \leq N} 1_P(n+h)\nu(n)^2 \sim N \beta(N) \frac{\log k}{k}(\rho \delta)I_k(F)(1 + o_k(1))$$

$$\sum_{n \leq N} 1_P(n+h)1_P(n+h')\nu(n)^2 \leq N \beta(N) \left( \frac{2}{\theta} + O(\delta) \right) \frac{(\log k)^2}{k^2}(\rho \delta)^2I_k(F)(1 + o_k(1)),$$

where $F(y_1, \ldots, y_k) := \sum_{j=1}^J \prod_{i=1}^k F_{j,t}^i(y_i)$. We address the first summation in $A$ with a Cauchy-Schwarz argument. Writing $\rho \delta \log k = cm$ for some small constants $\delta, c > 0$ it follows that

$$\sum_{n \leq N} 1_{Y_j > 0}(n)\nu(n)^2 \geq \left( \sum_{n \leq N} \sum_{h \in \mathcal{H}_j} 1_P(n+h)\nu(n)^2 \right)^2 \left( \sum_{n \leq N} \sum_{h,h' \in \mathcal{H}_j} 1_P(n+h)1_P(n+h')\nu(n)^2 \right)^{-1}$$

$$\geq \left[ (\rho \delta)N \beta(N) \frac{\log k}{k} \frac{k}{am+1} I_k(F) \right]^2 (1 + o_k(1))$$

$$\times \left[ \frac{2}{\theta} (1 + O(\delta))(\rho \delta)^2N \beta(N) \frac{(\log k)^2}{k^2} \left( \frac{k}{am+1} \right)^2 \frac{k}{am+1} I_k(F) \right]^{-1}$$

$$\geq N \beta(N)I_k(F) \frac{\theta}{2} (1 + o_k(1))(1 + O(\delta)).$$
Applying the bounds given in \[5.2\] we get
\[A \geq N^2 \beta(N)I_k(F)(1 + o_k(1))(1 + O(\delta)) \left( \frac{\theta}{2}(am + 1) - m - (am + 1) \left( \frac{k}{am + 1} - \frac{k}{am + 1} \right) \right) \left( \frac{2}{k} \right) \cdot \left( \frac{c^2m^2}{k^2} \right) .\]
Since \( a \geq (2/\theta) \), the result follows after taking \( c \) to be sufficiently small and \( k \) sufficiently large. \( \square \)

From this point onwards the demonstration of Proposition 1.2 is carried out as in \[1, Section 6\].

Let \( m \geq 1 \) and suppose \( k \) is a large positive integer with \((am + 1) | k\). Given \( \beta_{am+1} \geq ... \geq \beta_1 \geq 0 \) one obtains a \( k \)-tuple \( \mathcal{H} = \mathcal{H}_1 \cup ... \cup \mathcal{H}_{am+1} \) for which each set in the partition is of size \( k/(am + 1) \) and
\[ h_j = (\beta_j + \epsilon + o(1)) \log N \text{ for all } h_j \in \mathcal{H}_j .\]
Furthermore, one finds an integer \( n > y \) and \( z > 0 \) so that \([n, n + z] \cap \mathcal{P} = \mathcal{H}(n) \cap \mathcal{P} \). We gather that the primes in \( \mathcal{H}(n) \) are consecutive. By Proposition 5.2 there are at least \( m + 1 \) primes, each coming from a distinct member of the partition. In this manner we obtain a string of indices \( 1 \leq i_1 < i_2 < ... < i_l \leq am + 1 \), coming from distinct cells in the partition, with \( l \geq m + 1 \) and associated representations
\[ \frac{p_{r+1} - p_r}{\log p_r} = \beta_{i_j} - \beta_{i_{j-1}} + o(1) \]
for some value \( r = r(i_j) \). From this we easily deduce the following property.

(5.4)
Any sequence \( 0 \leq \beta_1 < ... < \beta_{am+1} \) contains a subset \( \beta_{i_1} < \beta_{i_2} < ... < \beta_{i_l} \) of length \( l \geq m + 1 \) satisfying \( \beta_{i_{j+1}} - \beta_{i_j} \in \mathcal{L} \forall i_j \).

Lemma 5.3. Property (5.4) implies Proposition 1.2. Assuming a level of distribution \( \theta = 1 \) in (5.1), one gets the improvement
\[ \liminf_{T \to \infty} \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{2} .\]

Proof. The unconditional claim is an application of \[1, Corollary 1.2\] for the case \( m = 1, \theta = 1/2 \). \( \square \)

APPENDIX

In this final section we will discuss Proposition 2.1. Since the proof follows that of \[2, Lemma 4.1\] very closely, we will limit ourselves to a sketch of the argument, pointing out important differences when necessary. Expanding the expression \( S \) in (2.1) we get two sums. First consider

(A-1)
\[ \sum_{n \leq N} w(n)^2 = \sum_{d \leq x} \left( \prod_{i=1}^{k} \mu(d_i)\mu(e_i) \right) f \left( \frac{\log d_1}{\log x}, ..., \frac{\log d_k}{\log x} \right) f \left( \frac{\log e_1}{\log x}, ..., \frac{\log e_k}{\log x} \right) \sum_{n \leq N \atop n \equiv b \mod W} \frac{1}{\prod_{j=1}^{k} [d_j, e_j]} . \]
\[ = \sum_{d \leq x} \left( \prod_{i=1}^{k} \mu(d_i)\mu(e_i) \right) f \left( \frac{\log d_1}{\log x}, ..., \frac{\log d_k}{\log x} \right) f \left( \frac{\log e_1}{\log x}, ..., \frac{\log e_k}{\log x} \right) \left( \frac{N}{W \prod_{j=1}^{k} [d_j, e_j]} + O(1) \right) . \]
Since $f$ is a compactly supported, smooth function, we may apply Fourier inversion to write

$$\exp \left( \sum_{i=1}^{k} t_i \right) f(\xi) = \int_{\mathbb{R}^k} \exp \left( -i \sum_{i=1}^{k} t_i \xi_i \right) g(\xi) \, d\xi \quad \forall \xi = (t_1, \ldots, t_k) \in \mathbb{R}^k.$$  \hspace{1cm} (A-2)

for some smooth function $g : \mathbb{R}^k \to \mathbb{R}$ obeying decay estimates of the form $g(\xi) \ll_A (1 + \|\xi\|)^{-A}$ for any $A > 0$. This leads to the expression

$$f \left( \frac{\log d_1}{\log x}, \ldots, \frac{\log d_k}{\log x} \right) = \int_{\mathbb{R}^k} \frac{g(\xi)}{\prod_{i=1}^{k} d_i^{\xi_i}} \, d\xi,$$

Inserting this integral representation into the main term of (A-1) we find the sum

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} K(\xi, \xi') g(\xi) g(\xi') \, d\xi d\xi',$$

where

$$K(\xi, \xi') = \sum_{\mathcal{D}(\xi, \xi')}^\ast \prod_{i=1}^{k} \frac{\mu(d_i)\mu(e_i)}{d_i^{\xi_i} e_i^{\xi_i}}.$$

The superscript $\ast$ means the summation takes place over squarefree integers for which $[d_1, e_1], \ldots, [d_k, e_k], W, Q$ are pairwise coprime. In [2, Equation 41] the asymptotic for $K$ was shown to be

$$K(\xi, \xi') = (1 + o(1)) \beta(N) \prod_{j=1}^{k} \frac{(1 + i\xi_j)(1 + i\xi'_j)}{2 + \xi_j + i\xi'_j}.$$  

To prove the identity

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \prod_{j=1}^{k} \frac{(1 + i\xi_j)(1 + i\xi'_j)}{2 + i\xi_j + i\xi'_j} g(\xi) g(\xi') \, d\xi d\xi' = \int_{\mathbb{R}^k} f(t)^2 \, dt,$$

divide the RHS of (A-2) by $\exp(\sum_{i=1}^{k} t_i)$ and differentiate the integrand with respect to each variable $t_i$. This gives

$$f(t) = \int_{\mathbb{R}^k} \prod_{j=1}^{k} \left( 1 + i\xi_j \right) \exp \left( - \sum_{r=1}^{k} t_r (1 + i\xi_r) \right) g(\xi) \, d\xi$$

which is then squared and integrated to get the desired representation. It remains to evaluate the summation over primes appearing in (2.1). For any index $1 \leq r \leq k$ one expands the sum

$$\sum_{n \leq N} 1_p(n + h_r)w(n)^2.$$
to find the expression

\[(A-3)\]

\[
\sum_{d,e} \left( \prod_{i=1}^{k} \mu(d_i) \mu(e_i) \right) f \left( \frac{\log d_1}{\log x}, \ldots, \frac{\log d_k}{\log x} \right) f \left( \frac{\log e_1}{\log x}, \ldots, \frac{\log e_k}{\log x} \right) \sum_{n \leq N} \phi(n + h_r) \]

\[
= \sum_{d,e} \left( \prod_{i=1}^{k} \mu(d_i) \mu(e_i) \right) f \left( \frac{\log d_1}{\log x}, \ldots, \frac{\log d_k}{\log x} \right) f \left( \frac{\log e_1}{\log x}, \ldots, \frac{\log e_k}{\log x} \right) \times \left[ \frac{N}{\phi(q(W,d,e))} \log N + \Delta \left( 1_{|N+h_r,N+2h_r|} \theta, a(W,d,e), q(W,d,e) \right) \right].
\]

Here we have used the notation \( q(W,d,e) = W \prod_{j=1}^{k} [d_j, e_j] \) and \( a(W,d,e) \) is the unique residue class mod \( q(W,d,e) \) satisfying all the conditions on \( n \). The main term in \( A-3 \) is treated as before except that the factors \([d_i, e_i]\) in the denominator become \( \phi([d_i, e_i]) \), which will have no effect on the argument.

For the remainder term \( \Delta \), we refer the reader to [2, Section 4.3], where it is demonstrated that a bound of the form

\[
\sum_{q \leq x^{1/2+\varepsilon}} \Delta \left( 1_{|N+h_r,N+2h_r|} \theta, a(q), q \right) \ll_A \frac{N}{(\log N)^A}
\]

holds for some pair of constants \( \varepsilon, \sigma > 0 \). We observe that the smoothness parameter \( \sigma \), which plays an important role in the above equidistribution estimate, forces the support of \( f \) to lie within \( \Delta_k(0, \sigma) \). This accounts for the occurrence of \( \sigma \) in Propositions 2.1 and 2.2.

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