Nonequilibrium phase transition in the coevolution of networks and opinions

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Models of the convergence of opinion in social systems have been the subject of a considerable amount of recent attention in the physics literature. These models divide into two classes, those in which individuals form their beliefs based on the opinions of their neighbors in a social network of personal acquaintances, and those in which, conversely, network connections form between individuals of similar beliefs. While both of these processes can give rise to realistic levels of agreement between acquaintances, practical experience suggests that opinion formation in the real world is not a result of one process or the other, but a combination of the two. Here we present a simple model of this combination, with a single parameter controlling the balance of the two processes.

We find that the model undergoes a continuous phase transition as this parameter is varied, from a regime in which opinions are arbitrarily diverse to one in which most individuals hold the same opinion. We characterize the static and dynamic properties of this transition.

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I. INTRODUCTION

Simple mathematical models describing emergent phenomena in human populations\(^{15}\), such as voter models and market models, have a long history of study in the social sciences. It is only relatively recently, however, that physicists have noted the close conceptual and mathematical connections between these models and traditional models in statistical physics such as spin models. Building on this observation, there have been a number of important advances in the understanding of these models in the last decade or so, most notably in the study of social networks\(^{1,7,14}\). While the physics community has been concerned primarily with studies of network structure, there has also been a substantial line of investigation focusing on dynamical processes on networks. One example, which has a long history in sociology but is also well suited to study using physics methods, is the dynamics of opinion formation. This problem highlights one of the fundamental questions in network dynamics, namely whether dynamics controls the structure of a network or the structure controls the dynamics.

It is observed that real social networks tend to divide into groups or communities of like-minded individuals. An obvious question to ask is whether individuals become like-minded because they are connected via the network\(^{6,10,18,19}\), or whether they form network connections because they are like-minded\(^{11}\). Both situations have been studied with physics-style models, the first using opinion formation models\(^{6,10,13}\) and the second using models of “assortative mixing” or “homophily”\(^{6,9,17}\). Common sense, however, tells us that the distinction between the two scenarios is not clear-cut. Rather, the real world self-organizes by a combination of the two, the network changing in response to opinion and opinion changing in response to the network. In this paper we study a simple model—perhaps the simplest—that combines opinion dynamics with assortative network formation, revealing an apparent phase transition between regimes in which one process or the other dominates the dynamics.

II. MODEL DEFINITION

Consider a network of \(N\) vertices, representing individuals, joined in pairs by \(M\) edges, representing acquaintance between individuals\(^{1}\). Each individual is assumed to hold one of \(G\) possible opinions on some topic of interest. The opinion of individual \(i\) is denoted \(g_i\). In the past, researchers have considered both cases where \(G\) is a fixed small number, such as a choice between candidates in an election\(^{6,13}\), and cases in which the number of possible opinions is essentially unlimited\(^{6}\), so that \(G\) can be arbitrarily large. An example of the latter might be religious belief (or lack of it)—the number of subtly different religious beliefs appears to be limited only by the number of people available to hold them.

The case of fixed small \(G\) has relatively simple behavior compared to the case of arbitrarily large \(G\), and so it is on the latter that we focus here. We will assume that the number of possible opinions scales in proportion to the number of individuals, and parameterize this proportionality by the ratio

\(^{1}\) Although acquaintance networks are typically simple graphs, with multiedges and self-edges disallowed, we have in the interest of simplicity, allowed multiedges and self-edges in our calculation. Since these form only a small fraction of all edges, we expect that our results would change little if we were to remove them.
we do the following (see Fig. 1). To be specific, on each step

we either move an edge to lie between two individuals whose

opinions agree or, we change the opinion of an individual to agree with one of their neighbors. Step 1 represents the influence of people of similar opinions. Step 2 involves the random rearrangement of edges within components of a random graph. Assuming we are in the regime \( k > 1 \) in which a giant component exists in the random graph, we will then have one giant (extensive) community and an exponential distribution of small communities. Thus, in varying \( \phi \) we go from a situation in which we have only small communities with constant average size \( \gamma \) to one in which we have a giant community plus a set of small ones.

This is the classic behavior seen in a system undergoing a continuous phase transition and it leads us to conjecture that our model displays a phase transition with decreasing \( \phi \) at which a giant community of like-minded individuals forms. In other words, there is a transition between a regime in which the population holds a broad variety of views and one in which most people believe the same thing. We now offer a variety of further evidence to support this conjecture. (Phase transition behavior is also seen in some models of opinion formation on static networks, such as the model of Ref. [10], although the mechanisms at work appear to be different from those considered here.)

In Fig. 2 we show plots of \( P(s) \) from simulations of our model for \( k = 4 \) and \( \gamma = 10 \). As the figure shows, we do indeed see a qualitative change from a regime with no giant community to one with a giant community. At an intermediate value of \( \phi \) around 0.458 we find a distribution of community sizes that appears to follow a power law \( P(s) \sim s^{-\alpha} \) over a significant part of its range, another typical signature of criticality. The exponent \( \alpha \) of the power law is measured to be \( 3.5 \pm 0.3 \), which is incompatible with the value 2.5 of the corresponding exponent for the phase transition at which a gi-

The primary interest in our model therefore is in the number and sizes of the communities that form and in the dynamics of the model as it comes to consensus. Let us consider the distribution \( P(s) \) of the sizes \( s \) of the consensus communities. In the limit \( \phi \to 1 \), only updates that move edges are allowed and hence the consensus state is one in which the communities consist of the sets of initial holders of the individual opinions. Since the initial assignment of opinions is random, the sizes of these sets follow the multinomial distribution, or the Poisson distribution with mean \( \gamma \) in the limit of large \( N \). Conversely, in the limit \( \phi \to 0 \), only changes of opinion are allowed and not edge moves, which means that the communities correspond to the initial components in the graph, which are simply the components of a random graph. Assuming we are in the regime \( \phi > 0 \) in which a giant component exists in the random graph, we will then have one giant (extensive) community and an exponential distribution of small communities. Thus, in varying \( \phi \) we go from a situation in which we have only small communities with constant average size \( \gamma \) to one in which we have a giant community plus a set of small ones.

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ant component forms in a random graph (a transition which belongs to the mean-field percolation universality class).

To further investigate our transition, we perform a finite size scaling analysis in the critical region. To do this, we need first to choose an order parameter for the model. The obvious choice is the size $S$ of the largest community in the consensus state as a fraction of system size. The arguments above suggest that this quantity should be of size $O(N^{-1})$ for values of $\phi$ above the phase transition (and hence zero in the thermodynamic limit) and $O(1)$ below it. We assume a scaling relation of the form

$$S = N^{-\alpha} F(N^b(\phi - \phi_c)).$$

where $\phi_c$ is the critical value of $\phi$ (which is presumably a function of $\bar{k}$ and $\gamma$), $F$ is a universal scaling function (bounded as its argument tends to $\pm\infty$), and $\alpha$ and $b$ are critical exponents. To estimate $\phi_c$, we plot $N^bS$ against $\phi$ and tune $a$ such that the results for simulations at different $N$ but fixed $\bar{k}$ and $\gamma$ cross at a single point, which is the critical point. Such a plot for $\bar{k} = 4$ and $\gamma = 10$ is shown in Fig. 3a. With $a = 0.61 \pm 0.05$ we obtain a unique crossing point at $\phi_c = 0.458 \pm 0.008$, which agrees well with the previous rough estimate of $\phi_c$ from Fig. 4.

Using this value we can now determine the exponent $b$ by plotting $N^bS$ against $N^a(\phi - \phi_c)$. Since $F(x)$ is a universal function, we should, for the correct choice of $b$, find a data collapse in the critical region. In Fig. 3b we show that such a data collapse does indeed occur for $b = 0.7 \pm 0.1$.

We have performed similar finite size scaling analyses for a variety of other points $(\bar{k}, \gamma)$ in the parameter space and, as we would expect, we find that the position $\phi_c$ of the phase transition varies—see Fig. 4—but that good scaling collapses exist at all parameter values for values of the critical exponents consistent with the values $a = 0.61$ and $b = 0.7$ found above.

Despite the qualitative similarities between the present phase transition and the percolation transition, our exponent values for $a$ and $b$ show that the two transitions are in different universality classes: the corresponding exponents for random graph percolation are $a = b = \frac{1}{4}$, which are incompatible with the values measured above.

IV. DYNAMICAL CRITICAL BEHAVIOR

Our model differs from percolation in another important respect also: percolation is a static, geometric phase transition,
whereas the present model is fundamentally dynamic, the consensus arising as the limiting fixed point of a converging nonequilibrium dynamics. It is interesting therefore to explore the way in which our model approaches consensus. In previous studies of opinion formation models of this type on fixed networks a key quantity of interest is the average convergence time $\tau$, which is the number of updates per vertex needed to reach consensus. If $\phi = 0$ then $\tau$ is known to scale as $\tau \sim N$ as system size becomes large \[18\]. In the opposite limit ($\phi = 1$), opinions are fixed and convergence to consensus involves moving edges one by one to fall between like-minded pairs of individuals. This is a standard sampling-with-replacement process in which the number $U$ of unsatisfied edges is expected to decay as $U \sim M e^{-t/\tau}$ for large times $t$. Thus the time to reach a configuration in which $U = O(1)$ is $t \sim M \log M$, and the convergence time is this quantity divided by the system size $N$. For fixed average degree $k = 2M/N$, this then implies that $\tau \sim \log N$. This result is confirmed numerically in Fig. 3(a).

For $\phi$ close to $\phi_c$, experience with other phase transitions leads us to expect critical fluctuations and critical slowing down in $\tau$. Figure 3(b) shows that indeed there are large fluctuations in the convergence time in the critical region. The figure shows the value of the coefficient of variation $V_\tau$ of the consensus time (i.e., the ratio of the standard deviation of $\tau$ to its mean) as a function of $\phi$ and a clear peak is visible around $\phi_c = 0.46$. To characterize the critical slowing down we assume that $\tau$ takes the traditional scaling form $\tau \sim N^{\phi_c}$ at the critical point, where $z$ is a dynamical exponent \[12\]. Figure 3(c) shows a plot of $\ln \tau N^{-\phi_c}$ as a function of $\phi_c$. If the system follows the expected scaling at $\phi_c$, then the resulting curves should cross at the critical point. Although good numerical results are considerably harder to obtain in this case than for the community sizes presented earlier, we find that the curves cross at a single point if $z = 0.61 \pm 0.15$ and $\phi = 0.44 \pm 0.03$, the latter being consistent with our previous value of $\phi_c = 0.46$ for the position of the phase transition.

V. SUMMARY AND CONCLUSIONS

To summarize, we have proposed a simple model for the simultaneous formation of opinions and social networks in a situation in which both adapt to the other. Our model contrasts with earlier models of opinion formation in which social structure is regarded as static and opinions are an outcome of that pre-existing structure \[11\]. Our model is a dynamic, nonequilibrium model that reaches a consensus state in finite time on a finite network. The structure of the consensus state displays clear signatures of a continuous phase transition as the balance between the two processes of opinion change and network rewiring is varied. We have demonstrated a finite size scaling data collapse in the critical region around this phase transition, characterized by universal critical exponents independent of model parameters. The approach to the consensus state displays critical fluctuations in the time to reach consensus and critical slowing down associated with an additional dynamical exponent. The phase transition in the model is of particular interest in that it provides an example of a simple process in which a fundamental change in the social structure of the community can be produced by only a small change in the parameters of the system.

Finally, we note that for the specific example of opinion formation mentioned in the introduction—that of choice of religion—it is known that the sizes of the communities of adherents of religious sects are in fact distributed, roughly speaking, according to a power law \[23\]. This may be a signature of critical behavior in opinion formation, as displayed by the model described here, although other explanations, such as the Yule process \[16\], are also possible.

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