On Coverings of Tori with Cubes

I. I. Bogdanov*, O. R. Grigoryanb, and M. E. Zhukovskii∗,∗∗

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Abstract—We obtain new bounds and exact values of the minimum number of cubes with side length $\varepsilon \in (0, 1)$ covering the torus $[\mathbb{R}/\mathbb{Z}]^3$.

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1. HISTORY AND FORMULATION OF THE PROBLEM

For an arbitrary $d \in \mathbb{N}$ and $\varepsilon \in (0, 1)$, the problem is to find the minimum number $\mu(d; \varepsilon)$ of cubes $A_1, \ldots, A_n$ with side length $\varepsilon$ that cover the torus $T^d := [\mathbb{R}/\mathbb{Z}]^d$. As usual, by a cube of side length $\varepsilon$, we mean a set of the form $\{(x_1, \ldots, x_d) : x_i \in [x_i^0, x_i^0 + \varepsilon]\}$, and, by a covering, we mean a collection of sets $A_1, \ldots, A_n$ such that $A_1 \cup \ldots \cup A_n = T^d$. It is known [1] that, for all $d$ and $\varepsilon$,

$$\mu(d; \varepsilon) \geq \lceil 1/\varepsilon \rceil^d,$$  \hspace{2cm} (1)

where $\lceil t \rceil^d = \lceil t \rceil \cdot \lceil t \rceil \cdot \ldots \cdot \lceil t \rceil$ and $\lceil x \rceil^d = \lceil x \rceil$. The case of growing dimension $d$ has been rather well studied. Note that (1) implies $\mu(d; \varepsilon) \geq (1/\varepsilon + o(1))^d$. The general result of Erdős and Rogers on coverings [2] implies that $1/\varepsilon$ is the correct base of the exponential function, i.e., $\mu(d; \varepsilon) = (1/\varepsilon + o(1))^d$. More precisely, Erdős and Rogers proved that $\mu(d; \varepsilon) = O(d(1/\varepsilon)^d)$ (for both rational and irrational $\varepsilon$). Note that the lower bound (1) implies only $\mu(d; \varepsilon) = \Omega((1/\varepsilon)^d)$.

For small values of $d$, we know only the work [1], where it is proved that, for $d = 2$, the lower bound (1) is sharp, i.e., $\mu(2; \varepsilon) = \lceil 1/\varepsilon \rceil^d$.

Consider the case $d = 3$. Since $\mu(1; \varepsilon) = \lceil 1/\varepsilon \rceil$ and $\mu(2; \varepsilon) = \lceil 1/\varepsilon \rceil^d$, it follows that

$$\lceil 1/\varepsilon \rceil^d / \lceil 1/\varepsilon \rceil \leq \mu(3; \varepsilon) \leq \lceil 1/\varepsilon \rceil^d / \lceil 1/\varepsilon \rceil.$$  \hspace{2cm} (2)

Note that, for $d = 3$, the lower bound is no longer sharp. For example, it was noted in [1] that $\mu(3; 3/7) > [7/3]^{[7/3]}$, and we were able to find the exact value of $\mu(3; \varepsilon)$ for all $\varepsilon \geq 7/15$ and for $\varepsilon$ close to $1/r$.

Finally, we note that, for $\varepsilon = 2/r$, the corresponding packing problem (i.e., the problem of finding the minimum number $\nu(d; \varepsilon)$ of disjoint cubes of side $\varepsilon$ in $T^d$) is related to finding the Shannon capacity $c(C_r)$ of a simple cycle on $r$ vertices [4]; specifically, $c(C_r) = \sup_{x \in \mathbb{Z}} \nu(d; 2r)^{1/d}$ (a similar relationship holds for all other rational $\varepsilon$, but the corresponding graphs are more difficult to describe). For even $r$, obviously, $c(C_r) = r/2$. If $r$ is odd, then the Shannon capacity value is known only for $r = 3$, namely, $c(C_3) = \sqrt{3}$ [5].

2. NEW RESULTS

Assume first that $\varepsilon \geq 1/2$.

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*a Moscow Institute of Physics and Technology (National Research University), Dolgoprudny, Moscow oblast, Russia

**National Research University Higher School of Economics, Moscow, Russia

*e-mail: zhukmax@gmail.com
Theorem 1. It is true that
\[ \mu(3; \varepsilon) = 8, \quad \varepsilon \in [1/2, 2/3), \quad \mu(3; \varepsilon) = 5, \quad \varepsilon \in [2/3, 3/4), \quad \mu(3; \varepsilon) = 4, \quad \varepsilon \in [3/4, 1). \]

Note that, for \( \varepsilon \in [1/2, 1) \), the quantity \( \mu(3; \varepsilon) \) is equal to its lower bound in (2) if and only if \( \varepsilon \in [1/2, 4/7) \cup [2/3, 3/4). \)

Since, for integer \( r \geq 2 \) and \( \varepsilon \in \left[ \frac{1}{r + 1/(r^2 + r + 1)}, \frac{1}{r} \right] \), the lower and upper bounds in (2) coincide, we obviously have \( \mu(3; \varepsilon) = r^3 \). Additionally, we found a left neighborhood of \( 1/r \) such that the lower bound is sharp for all \( \varepsilon \) from this neighborhood. Note that, for such \( \varepsilon \), the difference between the upper and lower bounds is, on the contrary, large.

Theorem 2. Suppose that \( r \in \mathbb{N} \) and \( \varepsilon \in \left[ \frac{1}{r + 1/(r^2 + r + 1)}, \frac{1}{r} \right] \). Then \( \mu(3; \varepsilon) = r^3 + r^2 + r + 1 \).

Additionally, the right neighborhood was expanded.

Theorem 3. Suppose that \( r \geq 2 \) is an integer and \( \varepsilon \in \left[ \frac{1}{r^2 - 1}, \frac{r^2 - 1}{r^3 - r - 1} \right] \). Then \( \mu(3; \varepsilon) = r^3 \).

In certain cases, we were able to strengthen the lower bound in (2).

Theorem 4. Suppose that \( r \geq 2 \) is an integer and \( \xi \in \{1, \ldots, r\} \) is such that
\[ \xi^2 \leq \xi + (r + 1) \left( \frac{\xi}{r + 1} \right) . \]

Additionally, assume that
(i) \( s = r^2 + r + \xi \),
(ii) \( t = r^3 + r^2 + 2\xi r + \left( \frac{\xi}{r + 1} \right) \) is coprime to \( s \).

Then \( \mu \left( \frac{3; \xi}{t} \right) \geq t + 1 \).

Since the condition \( \xi^2 \leq \xi + (r + 1) \left( \frac{\xi}{r + 1} \right) \) implies either \( \xi \geq \sqrt{r + 1} \) or \( \xi = 1 \), there are only two such values in the interval \( [1/3, 1/2) \): \( \xi \in \left\{ \frac{7}{16}, \frac{8}{21} \right\} \).

Finally, in certain cases, we managed to strengthen the upper bound in (2).

Theorem 5. Suppose that \( r \geq 2 \) is an integer and \( \xi \in \{1, \ldots, r\} \). Additionally, assume that
(i) \( s = r^2 + r + \xi \),
(ii) \( t = r^3 + r^2 + \xi(r + 1) \).

Then \( \mu \left( \frac{3; \xi}{t} \right) \leq t \).

To prove the above results, we established that the problem of finding \( \mu(d; \varepsilon) \) is reduced to its discrete analogue, namely, to the problem of finding the minimum number of cubes with side length \( s \) that cover \( [\mathbb{Z}]/[\mathbb{Z}]^d \) for some \( s, t \) such that \( s/t \) is close to \( \varepsilon \). Thus, these auxiliary assertions imply that the problem is reduced to considering a countable set of values \( \varepsilon \), and the number of such values in each interval of the form \( [1/r, 1/(r - 1)] \) is finite.

3. TRANSITION TO THE DISCRETE CASE

Lemma 1. There exists an infinite sequence of rational numbers \( 1 > \frac{s_1}{t_1} > \frac{s_2}{t_2} > \ldots > 0 \) such that, for each \( i \in \mathbb{N} \), it is true that \( t_i \leq \mu \left( d; \frac{s_i}{t_i} \right) \) and \( \mu(d; \varepsilon) = \mu \left( d; \frac{s_i}{t_i} \right) \) for all \( \varepsilon \in \left[ \frac{s_i}{t_i}, \frac{s_{i+1}}{t_{i+1}} \right) \), where \( s_0 = t_0 = 1 \).

It follows from (2) that, for each \( r \geq 2 \), to solve the problem for all \( \varepsilon \) in the interval \( \left[ \frac{1}{r}, \frac{1}{r^2 - 1} \right] \), it suffices to consider at most \( \frac{r^2 (r^3 + 1)}{2(r - 1)} + r^3 \) distinct values of \( \varepsilon \).

Let \( s \) and \( t \) be positive integers such that \( s \leq t \). Let \( \mu_0(d; s, t) \) be the smallest number of cubes with an side consisting of \( s \) points that cover the torus \( [\mathbb{Z}]/[\mathbb{Z}]^d \).

Lemma 2. Suppose that \( r \geq 2 \) is an integer and \( \varepsilon \in \left[ \frac{1}{r}, \frac{1}{r^2 - 1} \right] \). Additionally, suppose that \( \frac{s}{t} \in \mathbb{N} \) is the rational number nearest to \( \varepsilon \) such that \( t \leq r^d \). Then \( \mu(d; \varepsilon) = \mu_0(d; s, t) \).

4. SOME SPECIAL CASES

For \( \varepsilon \in \left[ \frac{1}{2}, 1 \right] \), the exact values of \( \mu(3; \varepsilon) \) are given in Theorem 2. Below are the exact results and some estimates following from Theorems 2–5 and Lemma 1 for \( \varepsilon \in \left[ \frac{1}{3}, \frac{1}{2} \right] \):

- for \( \varepsilon \in \left[ \frac{1}{3}, \frac{1}{5} \right] \), \( \mu(3; \varepsilon) = 27 \);
- for \( \varepsilon \in \left[ \frac{8}{21}, \frac{5}{13} \right] \), \( \mu(3; \varepsilon) \in [22, 24] \).
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• for $\varepsilon \in \left[\frac{7}{16}, \frac{4}{9}\right]$, $\mu(3; \varepsilon) \in [17, 21]$;
• for $\varepsilon \in \left[\frac{4}{9}, \frac{7}{15}\right]$, $\mu(3; \varepsilon) \in [16, 18]$;
• for $\varepsilon \in \left[\frac{7}{15}, \frac{1}{2}\right]$, $\mu(3; \varepsilon) = 15$.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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