“Moduli Space” of Asymptotically Anti-de Sitter Spacetimes in (2+1)-Dimensions

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Abstract

Setting an ansats that the metric is expressible by a power series of the inverse radius and taking a particular gauge choice, we construct a “general solution” of (2+1)-dimensional Einstein’s equations with a negative cosmological constant in the case where the spacetime is asymptotically anti-de Sitter. Our general solution turns out to be parametrized by two centrally extended quadratic differentials on $S^1$. In order to include 3-dimensional Black Holes naturally into our general solution, it is necessary to exclude the region inside the horizon. We also discuss the relation of our general solution to the moduli space of flat $\tilde{SL}(2, \mathbb{R}) \times \tilde{SL}(2, \mathbb{R})$ connections.

classification numbers (PACS nos.): 04.20, 04.60

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1 Introduction

When we try to quantize General Relativity (GR), we usually run into serious obstructions, which include the “issue of time in quantum gravity” and the “problem of finding (local) observables” [1]. These two obstructions are both closely related to the general covariance of GR.

To see from the viewpoint of the canonical formalism, in the case of pure gravity on a compact spatial manifold, the Hamiltonian of GR is expressed by a linear combination of the first class constraints. The time evolution of the canonical variables via the Hamiltonian is nothing but a gauge transformation which cannot be observed physically. These first class constraints generates general coordinate transformations of the spacetime, which prevents us from making a distinction between a point on spacetime and another point.

One of the ways to circumvent these difficulties is to consider spacetimes which are asymptotically isometric to some well-behaved space (such as Minkowski, de Sitter, or anti-de Sitter space). The Hamiltonian defined on these space has a nontrivial contribution from the boundary, according to which we have a possibility to define a meaningful time evolution [2]. Moreover, we can construct physical observables at spatial infinity because the asymptotic condition restricts the types of diffeomorphisms allowed at the infinity. It would therefore be important to investigate GR on a spacetime which is asymptotically flat (or (anti-) de Sitter).

In this paper we investigate the asymptotically anti-de Sitter spacetimes in (2+1)-dimensions. In (2+1)-dimensions, Einstein’s equations with a negative cosmological constant tell us that the spacetimes be locally anti-de Sitter ($ADS^3$) [3].

If we consider naively from this fact, solutions of Einstein’s equations which are asymptotically $ADS^3$ seem to be exhausted by 3-dimensional black holes (3DBH) [4] possibly with a negative mass. The main purpose of this paper is to investigate whether this is indeed the case.

In §2 while reviewing the canonical formalism of asymptotically $ADS^3$ spacetimes [3] in terms of Chern-Simons formulation of GR [3][7], we show that the diffeomorphism equivalence classes of asymptotically $ADS^3$ spacetimes are characterized by two centrally extended quadratic differentials on $S^1$. In §3 we solve Einstein’s equations...
explicitly by imposing a particular gauge-fixing condition and by making an ansatz
that the metric should be expressible by a power series of inverse radial coordinate.
3DBH’s turn out to be naturally involved in our general solution if we neglect the
region inside the outer horizon. To obtain some intuition about our general solution,
we investigate some simple cases in §4. While we find new solutions which do not
belong to 3DBH, these solutions appear to be physically irrelevant because they
involve closed timelike curves. §5 is devoted to the analysis of topological structure
of the moduli space. Its relation to the moduli space of flat $\tilde{SL}(2,\mathbb{R}) \times \tilde{SL}(2,\mathbb{R})$
connections is also suggested. In §6, after summarizing the main results, we discuss
the remaining issues on the asymptotically $ADS^3$ spacetimes.

2 Effective Theory of Asymptotically Anti-de Sitter Spacetimes

We work in the first-order Einstein gravity in (2+1)-dimensions with a negative
cosmological constant $\Lambda = -\frac{1}{l^2}$, which is shown to be
equivalent to the SO(2, 2) Chern-Simons gauge theory [3, 4]. We use as fundamental variables the triad $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$. [5] If we assume that the spacetime manifold $M$ has a boundary $\partial M$, the action is

$$I = \int_M \epsilon_{abc} e^a \wedge [d\omega^{bc} + \omega^b_d \wedge \omega^{dc} - \frac{1}{3} \Lambda e^b \wedge e^c] + B'(\partial M)$$

$$= \int_M \text{Tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A] + B'(\partial M),$$

(2.1)

where $B'(\partial M)$ is the boundary term which is necessary for the variational principle
to give local equations of motion. We have also introduced SO(2, 2) connection
$A \equiv P_a e^a + \frac{1}{7} \epsilon_{abc} J^a \omega^{bc}$ with $(J_a, P_a)$ being the generators of SO(2, 2) Lie algebra:

$$[J_a, J_b] = \epsilon_{abc} J^c, [J_a, P_b] = \epsilon_{abc} P^c, [P_a, P_b] = \frac{1}{l^2} \epsilon_{abc} J^c.$$ 

Tr in eq.(2.1) denotes an invariant bilinear form on SO(2, 2):

$$\text{Tr}(J_a P_b) = \eta_{ab}, \quad \text{Tr}(J_a J_b) = \text{Tr}(J_a J_b) = 0.$$ 

[1]Our convention for the indices and the signatures of the metrics is the following: $\mu, \nu, \rho, \cdots (= t, r, \phi)$ denote 2+1 dimensional spacetime indices and the metric $g_{\mu \nu}$ has the signature $(-,+,+)$; $i, j, k, \cdots$ are used for spatial indices; $a, b, c, \cdots$ represent indices of the local Lorentz group, with the metric $\eta_{ab} = \text{diag}(-,+,+)$; $\epsilon_{abc}$ is the totally antisymmetric pseudo-tensor with $\epsilon_{012} = -\epsilon_{012} = 1$. 
The action in the canonical formalism is obtained by performing the 2+1 decomposition $M \approx \mathbb{R} \times \Sigma$, $A = A_t dt + \tilde{A}$ and $d = dt \partial_t + \tilde{d}$. 

\[
I = \int dt \left[ \int_{\Sigma} \text{Tr}(-\tilde{A} \wedge \partial_t \tilde{A}) - H \right], \\
H \equiv -2 \int_{\Sigma} \text{Tr}[A_t(\tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A})] + B(\partial \Sigma). 
\tag{2.2}
\]

We should notice that the boundary term $B(\partial \Sigma)$ of the Hamiltonian $H$ is introduced in order to make Hamilton’s principle well-defined. This $B(\partial \Sigma)$ is determined by the following functional differential equation

\[
\int dt \delta B(\partial \Sigma) = -2 \int_{\partial M} \text{Tr}(A_t dt \wedge \delta \tilde{A}) = 2 \int dt \oint_{\partial \Sigma} d\phi \text{Tr}(A_t \delta A \phi). 
\tag{2.3}
\]

As is well known this system is a first class constraint system. Assume we take the gauge-fixing method in which we explicitly solve the constraints by imposing particular gauge-fixing conditions as many as the number of the first class constraints. The spatial part $\tilde{A}$ of the connection, which are the dynamical degrees of freedom in the unconstrained system, are thus determined by solving the constraint $\tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A} = 0$. While the temporal part $A_t$ remains as gauge degrees of freedom, we can represent it in terms of $\tilde{A}$ by solving the equations of motion $F_{ti} = 0$. The problem thus reduces to that of solving the equations of motion

\[
F \equiv dA + A \wedge A = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu = 0 
\tag{2.4}
\]

under a particular gauge choice. We mention that this equations of motion involve Einstein’s equations $d\omega^{ab} + \omega^a_c \wedge \omega^c_b = 0$ and torsion-free conditions $de^a + \omega^a_b \wedge e^b = 0$.

Next we determine the asymptotic form of $A$. The condition that the metric should be asymptotically $ADS^3$ is given by [3]:

\[
ds^2 = -(\frac{r^2}{l^2} + O(1)) dt^2 + (\frac{l^2}{r^2} + O(\frac{1}{r^4})) dr^2 + (r^2 + O(1)) d\phi^2 \\
+ O(1) dt d\phi + O(\frac{1}{r^3}) dr dt + O(\frac{1}{r^3}) dr d\phi. 
\tag{2.5}
\]

Due to the gauge degrees of freedom associated with the local Lorentz transformations, there are an infinitely many number of SO(2, 2) connections $A$ which give the

\footnote{$\partial_t$ denotes a partial differentiation with respect to $t$. Tilde refers to the quantities or the operations defined on $\Sigma$.}
above metric. By loosely fixing the local Lorentz gauge degrees of freedom, we put the following asymptotic condition on $A$:

$$
A = P_0 l\{(\frac{l}{r} - \frac{M}{2r^2}) + O(\frac{1}{r^2})\} \, dt + O(\frac{1}{r}) \, dr + O(\frac{1}{r^2}) \, d\phi \\
+ P_1 l\{(\frac{l}{r} + \frac{M}{2r^2}) + O(\frac{1}{r^2})\} \, dt + O(\frac{1}{r}) \, dr + O(\frac{1}{r^2}) \, d\phi \\
+ P_2 l\{(\frac{l}{r} + (\frac{M - M'}{2r^2}) + O(\frac{1}{r^2})\} \, dt + O(\frac{1}{r}) \, dr + O(\frac{1}{r^2}) \, d\phi \\
+ J_0 \{(\frac{l}{r} - \frac{M}{2r^2}) + O(\frac{1}{r^2})\} \, dt + O(\frac{1}{r}) \, dr + O(\frac{1}{r^2}) \, d\phi \\
+ J_1 \{(\frac{l}{r} - \frac{M}{2r^2}) + O(\frac{1}{r})\} \, dt + O(\frac{1}{r}) \, dr + O(\frac{1}{r^2}) \, d\phi \\
+ J_2 \{(\frac{l}{r} + (\frac{M - M'}{2r^2}) + O(\frac{1}{r})\} \, dt + O(\frac{1}{r}) \, dr + O(\frac{1}{r^2}) \, d\phi \}
$$

(2.6)

In setting the above asymptotic form we have solved the equations of motion (2.4) asymptotically. $M$, $M'$ and $J$ in eq. (2.6) correspond to the $O(1)$-part of $g_{tt}$, $O(\frac{1}{r})$-part of $g_{rr}$, and $O(1)$-part of $g_{t\phi}$, respectively.

According to ref. [5], the group of the asymptotic symmetries which preserve boundary condition (2.5) is isomorphic to the pseudo-conformal group in 2 dimensions, which in turn is isomorphic to the direct product group of two Virasoro groups. In our formulation, these transformations are generated by the SO(2,2) gauge transformation $\delta \xi A = d\xi + [A, \xi]$. The gauge parameter $\xi$ is of the following form

$$
\xi = P_0 (B^0 r + C^0 + O(\frac{1}{r})) + J_0 (B^0 r + C^0 + O(\frac{1}{r})) \\
+ P_1 (\beta^1 + O(\frac{1}{r})) + J_1 (\gamma^1 + O(\frac{1}{r})) \\
+ P_2 (B^2 r + C^2 + O(\frac{1}{r})) + J_2 (B^2 r + C^2 + O(\frac{1}{r}))
$$

(2.7)

with the coefficients subject to the relations:

$$
\begin{align*}
\partial_\beta B^2 &= l \partial_\beta B^0 = -\beta^1 / l \\
C^2 + lD^0 &= -\frac{M^2}{2} B^2 - \frac{M}{2} B^0 \\
C^0 + lD^2 &= -\frac{M^2}{2} B^0 - \frac{M}{2} B^0 \\
C^2 - lD^0 &= l^2 (\frac{2M - M'}{2} - \partial_\beta^2) B^2 - \frac{M}{2} B^0 \\
1D^2 - C^0 &= l^2 (\frac{2M - M'}{2} - \partial_\phi^2) B^0 - \frac{M}{2} B^2.
\end{align*}
$$

(2.8)

The $O(\frac{1}{r^2})$ terms depend on more detailed information on the connection $A$ which are not explicitly written in eq. (2.6).

In the canonical formalism, the generator of the asymptotic gauge transformation (2.7) is expressed by a linear combination of the first class constraints plus a surface
\[ G[\xi] = -2 \int \sum \text{Tr}[\xi(\tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A})] + \int \sum d^2 x \text{Tr}[(\partial_t \xi + [A_t, \xi])\Pi] + Q[\xi], \]
\[ \delta Q[\xi] = 2 \oint_{\partial \Sigma} \text{Tr}(\xi \delta \tilde{A}), \] (2.9)

where \( \Pi \) denotes the conjugate momentum of \( A_t \) and \( \Pi \approx 0 \) also gives first class constraints.

From now on we will consider that the boundary \( \partial \Sigma \) of the spatial manifold consists only of the spatial infinity at \( r \rightarrow \infty \). The charge \( Q[\xi] \) in this case is given by
\[ Q[\xi] = 2 \oint \text{Tr}[\xi(\tilde{A} - (\tilde{A})_0)], \] (2.10)

where \((A)_0\) is a fiducial connection. If we use as \((A)_0\) the “vacuum configuration” of 3DBH
\[ (A)_0 = P_0 \frac{r}{l} dt + P_1 \frac{l}{r} dr + P_2 r d\phi + J_0 \frac{r}{l} d\phi + J_2 \frac{r}{l^2} dt, \] (2.11)
the charge is explicitly given by
\[ Q[\xi] = \oint d\phi [B^0(2M - M')l - B^2 J] = -\frac{l}{2} \oint d\phi [\xi^- M^+ + \xi^+ M^-], \] (2.12)

where \( M^\pm \equiv 2M - M' \pm \frac{\tilde{t}}{l} \) and
\[ \xi^\pm \equiv -(B^0 \pm B^2) = \xi^\pm(t^\pm) \text{ with } t^\pm \equiv \frac{\tilde{t}}{l} \pm \phi. \]

Owing to this charge, we can compute the Poisson bracket of two generators following the usual definition (see, e.g., ref.\[3\]). The result is
\[ \{G[\xi], G[\eta]\}_{P.B.} = -G[[\xi, \eta] + \delta \xi \eta - \delta \eta \xi + (\cdots)] + 2 \oint_{\partial \Sigma} \text{Tr}[\xi \delta \eta (\tilde{A})_0], \] (2.13)

where \( \delta \xi = \{\xi, G[\eta]\}_{P.B.}, \) and \((\cdots)\) denotes a linear combination of the constraints. The last term in the R.H.S. gives the central term. If we solve the constraints and formally take the Dirac bracket, we find that the charges form a pseudo-conformal algebra with a central term\[3\]:
\[ \{Q[\xi], Q[\eta]\}_{D.B.} = -Q[[\xi, \eta]] + l \oint d\phi (-\xi^+ \partial^3_\phi \eta^+ + \xi^- \partial^3_\phi \eta^-), \] (2.14)

\[ ^3 \text{In practice these generators generates under the Poisson bracket the transformation:} \]
\[ \{A, G[\xi]\}_{P.B.} = \delta \xi A + \text{(terms linear in the constraints with coefficients \( \{A, \xi\}_{P.B.} \)).} \]

The second term in the R.H.S., however, vanishes because we consider the constraints to be solved.
where we have given the central term explicitly. \( \delta \xi \eta \) and \( \delta \eta \xi \) do not contribute to the R.H.S., because only the \( O(r) \) terms in eq.(2.7) contribute to the expression of the charge (2.12) and because \( \delta \xi \eta \) is of \( O(\frac{1}{r}) \).

By substituting
\[
([\xi, \eta])^\pm = \pm (\xi^\pm \partial_\phi \eta^\pm - \eta^\pm \partial_\phi \xi^\pm)
\]
into (2.14), we can extract the Dirac bracket of \( M^\pm \),
\[
\{M^\pm(\phi), M^\pm(\phi')\}_{D.B.} = \mp 2 \frac{2}{l^3} \partial_\phi \delta(\phi, \phi') M^\pm(\phi') - \partial_\phi' \delta(\phi, \phi') M^\pm(\phi) + 2 \partial^3_\phi \delta(\phi, \phi'),
\]
\[
\{M^+(\phi), M^-(\phi')\}_{D.B.} = 0.
\]

While in principle transformation of \( M^\pm \) under the asymptotic gauge transformation (2.7) can be computed by examining the asymptotic form of \( \delta \xi A \) up to \( O(\frac{1}{r}) \), it is much easier to use the charge (2.12) and the Dirac bracket (2.15). We will give the result only:
\[
\delta \xi M^\pm = \{M^\pm, Q[\xi]\}_{D.B.} = \pm (\xi^\pm \partial_\phi M^\pm + 2 \partial_\phi \xi^\pm M^\pm - 2 \partial^3_\phi \xi^\pm).
\] (2.16)

Because the Hamiltonian \( B(\partial \Sigma) \) given by eq.(2.3) is a particular charge \( Q[\xi] \) with \( \xi^+ = \xi^- = -1/l \), we find the time evolution of \( M^\pm \) as
\[
\partial_t M^\pm = \{M^\pm, B(\partial \Sigma)\}_{D.B.} = \mp \frac{1}{l} \partial_\phi M^\pm,
\]
After solving the equations of motion (2.4), \( M^\pm \) thus reduces to the function defined on \( S^1 \) (coordinatized by \( t^\mp \)),
\[
M^\pm(t, \phi) = M^\pm(t^\mp).
\] (2.17)

By noticing that \( \xi^+ \) and \( \xi^- \) are respectively functions of \( t^+ \) and of \( t^- \) only, transformation (2.16) turns out to be the centrally extended transformation of a quadratic differential on \( S^1 \). The integrated version of (2.16) is as follows \footnote{\( \{, \} \) denotes the Schwarzian derivative which is defined to be \( \{\zeta, z\} = \frac{d^3 \zeta/dz^3}{d\zeta/dz} - \frac{3}{2} \left( \frac{d^2 \zeta/dz^2}{d\zeta/dz} \right)^2 \).}
\[
M^\pm(t^\mp) \rightarrow M^\pm(t'^\mp) = \left( \frac{dt'^\mp}{dt^\mp} \right)^2 M^\pm(t'^\mp) - 2 \{t'^\mp, t^\mp\},
\] (2.18)
where \( t^{\pm'}(t^{\pm}) \) is a “new coordinate on \( S^1 \)” which is subject to the periodic condition:

\[
t^{\pm'}(t^{\pm} + 2\pi) = t^{\pm'}(t^{\pm}) + 2\pi.
\]

The effective theory of the asymptotic \( ADS^3 \) spacetimes thus reduces to a pseudo-conformal field theory defined on a cylinder which is coordinatized by \((t, \phi)\) and whose metric is conformal to

\[
ds^2_{(2)} = -\frac{dt^2}{l^2} + d\phi^2.
\]

\( M^{\pm} \) plays the role of “stress-energy tensor” in this effective theory.

Because the asymptotic gauge transformation \( (2.7) \) induces the diffeomorphism of the metric (in the case where the metric is nondegenerate \( [7] \)), we naively expect that the two sets of parameters \((M^+, M^-)\) and \((M'^+, M'^-)\) give diffeomorphism equivalent spacetimes when they are related with each other by eq.\( (2.18) \). To see whether this is indeed the case requires a detailed investigation on the global structure such as singularity, horizon, etc.. This statement is, however, true as long as the asymptotic behavior of the spacetime is concerned. The moduli space of asymptotically \( ADS^3 \) spacetimes is therefore given by

\[
Q^+ \times Q^-,
\]

where \( Q^{\pm} \) is the space of centrally extended quadratic differentials on \( S^1 \) with periodic coordinates \( t^{\pm} \).

(Remark:) In order to check that the asymptotic gauge transformation \( (2.7) \) preserves the asymptotic form \( (2.6) \), we have to give next-to-leading order terms in \( (2.6) \) so that the curvature should vanish asymptotically up to eq.\( (3.7) \), and we have to give adequate \( O\left( \frac{1}{r^2} \right) \) terms in eq.\( (2.7) \). The next-to-leading order terms are determined uniquely by six arbitrary functions of \((t, \phi)\), which gives (leading order terms of) the gauge degrees of freedom irrelevant to the asymptotic physics. Invariance of the action \( (2.1) \) under asymptotic transformation \( (2.7) \) is established only after we give the next-to-leading order terms in \( (2.6) \).
3 Generic Solution

Now we explicitly solve equations of motion (2.4) to obtain the spacetime which is parametrized by arbitrary “stress energy tensor” $M^{\pm}(t^\mp)$. To eliminate the gauge degrees of freedom which are at most of $O(1/r^2)$ and which are irrelevant to the asymptotic physics, we have to impose appropriate gauge-fixing conditions which have nonvanishing Poisson brackets with the first class constraints. First we fix the local Lorentz gauge degrees of freedom. This can be done by fixing the triad parts.

We impose

$$e_r^0 = e_\phi^0 = e_r^2 = 0.$$  

(3.1)

In order to fix the remaining gauge, we have to set further three gauge-fixing conditions, two of which we will take

$$\omega_r^{01} = \omega_r^{12} = 0.$$  

(3.2)

To simplify the analysis we will make the ansats that the metric components can be expressed by power series $\sum_{n \geq n_0} a_n r^{-n}$ with $a_n$ being in general some function of $(t, \phi)$. Owing to this ansats, we can assert that:

if a quantity $q$ of $O(1/r^n)$ with $n > 0$ satisfies  

$$\partial_r q = 0,$$  

then $q = 0$. (3.3)

By properly combining eqs.(2.4), (3.1-3.3), we find

$$e^1 = l d\rho \quad \rho = \ln \frac{r}{l} - \frac{M[l]}{4r^2} + O(\frac{1}{r^3}),$$  

$$\omega^{20} = d\sigma \quad \sigma = -\frac{J}{4r^2} + O(\frac{1}{r^3}).$$  

(3.4)

Let us now consider the gauge transformed connection:

$$A' \equiv g A g^{-1} - dgg^{-1}, \quad g \equiv \exp(P_1 l \rho + J_1 \sigma).$$  

(3.5)

5The remaining condition is given by $e_\phi^2 = r$ or by $e_r^1 = \frac{l}{r}$. Our gauge seems to be good as least in the sense that the matrix of Poisson brackets between the constraints and the gauge-fixing conditions does not degenerate weekly almost everywhere sufficiently inside the infinity. At infinity, however, the matrix asymptotically degenerates. This seems to be the source of the appearance of the asymptotic gauge degrees of freedom.
The result of substituting eqs. (2.6)(3.1-3.4) is considerably simplified as follows:

\[
A' = P_0 l[(1 - \frac{2M - M'}{4} + O(\frac{1}{r})) \frac{dt}{t} + (\frac{1}{4} + O(\frac{1}{r})) d\phi] \\
+ P_2 l[(1 + \frac{2M - M'}{4} + O(\frac{1}{r})) d\phi + (-\frac{1}{4} + O(\frac{1}{r})) dt] \\
+ J_0 [(1 - \frac{2M - M'}{4} + O(\frac{1}{r})) d\phi + (\frac{1}{4} + O(\frac{1}{r})) dt] \\
+ J_2 [(1 + \frac{2M - M'}{4} + O(\frac{1}{r})) dt + (-\frac{I}{4} + O(\frac{1}{r})) d\phi].
\]

(3.6)

By solving the equations of motion \(dA' + A' \wedge A' = 0\) and by using (3.3), we see that all the \(O(1/r)\)-terms in \(A'\) vanish and that the following equations hold:

\[
\partial_\phi (2M - M') + \partial_t J = 0, \quad t^2 \partial_t (2M - M') + \partial_\phi J = 0.
\]

(3.7)

We can easily solve these equations and find

\[
M^\pm \equiv 2M - M' \pm \frac{J}{l} = M^\pm (t^\pm),
\]

(3.8)

where \(t^\pm \equiv \frac{t}{l} \pm \phi\). This is the very equation (2.17) which has been obtained by the general analysis.

To obtain the explicit form of the SO(2, 2) connection \(A\), we have only to perform the inverse transformation

\[
A = g^{-1} A' g - dg^{-1} g.
\]

The result is:

\[
A = P_0 l B \frac{dt}{l} + P_2 l (C d\phi + D \frac{dt}{l}) + P_1 l d\rho
\]

\[
+ J_0 B d\phi + J_2 (C \frac{dt}{l} + D d\phi) + J_1 d\sigma,
\]

(3.9)

with

\[
\begin{align*}
B &= C^{-1} [e^{2\rho} - \frac{M^+ M^-}{16} e^{-2\rho}] \\
C &= [e^{2\rho} + \frac{M^+ M^-}{4} + \frac{M^+ M^-}{16} e^{-2\rho}]^{1/2} \\
D &= -\frac{J}{2lC} \\
e^{2\sigma} &= [e^{2\rho} + \frac{M^+}{4}]^{-1} [e^{2\rho} + \frac{M^-}{4}].
\end{align*}
\]

(3.10)

The last equation is necessary for \(A\) to satisfy the gauge-fixing condition \(e_0^0 = 0\). This connection gives the spacetime which is specified by the numerical values of the “stress-energy tensor” \(M^\pm\).

To look for the residual gauge degrees of freedom, it is convenient to use (anti-)chiral SL(2, \(\mathbb{R}\)) connections \(A^{(\pm)} \equiv J_a (\frac{1}{2} \epsilon^{ac}_{\ b} \omega^b_{\ c} \pm e^a/l)\):

\[
A'(\pm) = \pm J_0 (1 - \frac{M^\pm}{4}) dt^\pm + J_2 (1 + \frac{M^\pm}{4}) dt^\pm.
\]

(3.11)
The gauge transformations which keeps this form of $A^{(\pm)}$ are uniquely determined up to one arbitrary function $\xi^\pm(t^\pm)$:

$$
\xi^{(\pm)} = \pm J_0 \{ \xi^\pm - \frac{1}{2} \left( \frac{M^\mp}{2} - \partial_\pm^2 \right) \xi^\pm \} \pm J_1 \partial_\pm \xi^\pm - J_2 \{ \xi^\pm + \frac{1}{2} \left( \frac{M^\mp}{2} - \partial_\pm^2 \right) \xi^\pm \}. \quad (3.12)
$$

The induced transformation of $M^\pm$ is given by

$$
\delta_\xi M^\pm = -\xi^\mp \partial_\mp M^\pm - 2 \partial_\mp \xi^\mp M^\pm + 2 \partial_\mp^3 \xi^\mp. \quad (3.13)
$$

This is nothing but the transformation (2.16). So we realize that, in our space of solutions, there are no gauge degrees of freedom other than the transformation of centrally extended quadratic differentials on $S^1$, i.e., transformation (2.16).

In passing, the asymptotic gauge transformation (2.7) in our gauge is recovered by first setting $\xi^\pm = -(B^0 \pm B^2)$ and then substituting eq. (3.10) into the following expression of $\xi^\pm$

$$
\xi = g^{-1} \xi^0 g + J_1 \delta_\xi \sigma + P_1 \delta_\xi \rho. \quad (3.14)
$$

### 4 Simple Examples

To investigate spacetimes given by (3.9), it is necessary to choose an adequate radial coordinate. In order to see that our solution involves 3DBH (possibly with a negative mass), we first fix the gauge by

$$
c_\phi^2 = \tilde{r}. \quad (4.1)
$$

The metric constructed from (3.9) is then expressed as

$$
\text{ds}^2 = -(\frac{\tilde{r}^2}{\tilde{T}^2} - M + \frac{\tilde{r}^2}{4\tilde{T}^2}) \text{dt}^2 + (\tilde{r}d\phi - \frac{\tilde{J}}{2\tilde{T}} \text{dt})^2 + \frac{1}{(\frac{\tilde{r}^2}{\tilde{T}^2} - M + \frac{\tilde{r}^2}{4\tilde{T}^2})} \left\{ \frac{\tilde{r} \text{df} - \frac{\tilde{J}}{2\tilde{T}} \text{dt}}{\tilde{T}} - \frac{\tilde{J}dM + M^-}{4M + M^-} (\frac{\tilde{r}^2}{\tilde{T}^2} - \frac{M}{2} - \sqrt{(\frac{\tilde{r}^2}{\tilde{T}^2})^2 - M \frac{\tilde{r}^2}{\tilde{T}^2} + \frac{J^2}{4\tilde{T}^2}})^2 \right\}. \quad (4.2)
$$

---

5 This form of $\xi$ is derived by considering the following sequence of transformations:

$$
A \to A' \to A' + \delta_\xi A' \to A + \delta_\xi A.
$$

6 In the last two sections we have seen that the combination $2M - M'$ plays a crucial role. We will henceforth rename $2M - M'$ as $M$ because of the notational convenience.

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When $M^+$ and $M^−$ are both constant, this gives 3DBH possibly with a negative mass. When not both of $M^+$ and $M^−$ are constant and $M^+M^− > 0$, however, this metric becomes essentially complex in the region $r_− < \tilde{r} < r_+$ with $\tilde{r}^2 \equiv \frac{M^±\sqrt{M^+M^−}}{2}$. It would be natural to consider that such an eccentric metric is physically meaningless unless we can bypass the region of complex metric e.g. by a coordinate redefinition.

Thus we take the following radial gauge

$$e^{2\rho} = \frac{r^2}{l^2} \equiv \zeta \ (i.e., \ e^1_r = \frac{l}{r}).$$

The metric in this gauge is

$$ds^2 = -\frac{(\zeta^2 - \frac{M^+M^−}{16})^2}{(\zeta + \frac{M^+}{4})(\zeta + \frac{M^−}{4})}\,dt^2 + \frac{l^2}{4\zeta^2}\,d\zeta^2$$

$$+ \left(\sqrt{\frac{(\zeta + \frac{M^+}{4})(\zeta + \frac{M^−}{4})}{\zeta}}\,l\,d\phi - \frac{J}{2l}\left(\frac{\zeta}{(\zeta + \frac{M^+}{4})(\zeta + \frac{M^−}{4})}\,dt\right)^2\right)^2$$

$$= -\left(\frac{r^2}{l^2} - \frac{M^+M^−l^2}{16r^2}\right)dt^2 + \left(r^2 + \frac{M^2}{2} + \frac{M^+M^−l^4}{16r^2}\right)d\phi^2 - J\,dt\,d\phi + \frac{l^2}{r^2}\,dr^2.$$
In this case the spacetime is a three dimensional black hole. The parametrization in the (2+2)-dimensional Minkowskii space is given by ref.[4]:

\[ (T, X) = \frac{r^2 + \sqrt{M^+ M^-} l^2}{(M^+ M^-)^{1/4} r} (\cosh \tilde{\phi}, \sinh \tilde{\phi}) \quad \tilde{\phi} = \frac{-\sqrt{M^+} + \sqrt{M^-}}{2}, \]
\[ (Y, Z) = \frac{r^2 - \sqrt{M^+ M^-} l^2}{(M^+ M^-)^{1/4} r} (\cosh \tilde{t}, \sinh \tilde{t}) \quad \tilde{t} = \frac{\sqrt{M^+} - \sqrt{M^-}}{2}. \quad (4.5) \]

At \( r = r_0 = \frac{(M^+ M^-)^{1/4}}{2} \) there exists a horizon of the Rindler space type which splits the spacetime into two causally-independent regions \( r > r_0 (Y > |Z|) \) and \( r < r_0 (Y < -|Z|) \). This horizon \( r = r_0 \) is the remnant of the outer horizon of 3DBH. \( r = \infty \) and \( r = 0 \) correspond respectively to spatial infinities of the two regions.

ii) \( M^+ < 0, M^- < 0 \). The spacetime is a “negative-mass black hole”. The conical and helical singularity appears at \( r = r_0 = \frac{(M^+ M^-)^{1/4}}{2} \). This spacetime involves “obvious” closed timelike curves (CTC) with \( t \) and \( r \) being constant, in the region \( r_0 < r < \max\{\frac{(-M^+)^{1/2}}{2}, \frac{(-M^+)^{1/2}}{2}\} \). Hence this case is usually ruled out from the physical spectrum[3] (except the case with \( M^+ = M^- \), where the CTC’s necessarily pass through the conical singularity at \( r = r_0 \)).

iii) \( M^+ M^- < 0 \). This spacetime does not have either conical singularity or horizon and so we can naturally take the domain of \( r \) to be \((0, \infty)\). This spacetime, however, necessarily involves the obvious CTC’s in the region \( r < \max\{\frac{(-M^+)^{1/2}}{2}, \frac{(-M^+)^{1/2}}{2}\} \) and so we usually exclude this from the physical spectrum.

Now, in order to look for nontrivial spacetimes, let us investigate the case with \( M^\pm = m_0^\pm + \delta M^\pm \), where \( m_0^\pm \) is a constant and \( \delta M^\pm \) is a small fluctuation. For \( M^\pm \) not to be gauge equivalent to \( m_0^\pm \), we must have \( \delta M^\pm \) which cannot be absorbed into the gauge transformations, i.e.

\[ \delta M^\pm \neq -\xi^\mp \partial_\mp m_0^\pm - 2 \partial_\mp \xi^\mp m^\pm + 2 \partial^3_\mp \xi^\mp \quad (4.6) \]

We consider that

\[ T^2 - X^2 - Y^2 + Z^2 = L^2 \]

holds. The metric is obtained by substituting the parametrization into the pseudo-Minkowski metric:

\[ ds^2 = -dT^2 + dX^2 + dY^2 - dZ^2. \]
for any well-defined function $\xi^\pm(t^\mp)$ on $S^1$. By substituting the fourier decomposition $\xi^\pm = \sum_n \xi^\pm_n e^{int^\pm}$, we find the two cases.

a) When $m^\pm_0 \neq -n^2$ for $\forall n \in \mathbb{Z}\{0\}$, only $\delta M^\pm = const.$ survives. Thus we have only to consider the constant $M^\pm$ which gives 3DBH.

b) When $m^\pm_0 = -n^2$ with $\exists n \in \mathbb{Z}\{0\}$, $\sin nt^\mp$ and $\cos nt^\mp$ also survive as non-trivial $\delta M^\pm$. We therefore expect the appearance of nontrivial spacetimes in this case.

We will only consider the case with $m^+_0 = m^-_0 = -n^2$.

Otherwise case b) corresponds to the fluctuation about physically irrelevant spacetimes in which naked CTC’s appear. To see the behavior of new solutions, it is sufficient to investigate the case $m^+_0 = m^-_0 = -1$. We will only consider the contribution of oscillating fluctuation. By a proper constant shift of $(t, \phi)$, we can take the form of $M^\pm$ as

$$M^\pm = -1 + 4\epsilon^\pm \cos t^\mp,$$

where $\epsilon^\pm$ is a positive infinitesimal constant. Substituting this into (4.4), we find the metric of the new spacetime:

$$ds^2 = -\left(\frac{\zeta + \frac{1}{4} - \epsilon^- \cos t^-}{\zeta + \frac{1}{4} - \epsilon^+ \cos t^+}\right)dt^2 + \frac{l^2}{\zeta}\left(\frac{\zeta + \frac{1}{4} + \epsilon^- \cos t^-}{\zeta + \frac{1}{4} - \epsilon^+ \cos t^+}\right)d\phi^2 + 2\frac{d\zeta}{4\epsilon^\mp}\frac{l^2 d\phi^2}{4\epsilon^\mp}.$$  (4.8)

This spacetime would not probably have any curvature singularity because it is a solution of Einstein’s equations in (2+1)-dimensions, which give at most conical singularities. Because of the fluctuation it is difficult to see whether there exist conical singularities in this spacetime.

We will only investigate whether CTC’s exist or not. Because CTC’s pass through the region $g_{\phi\phi} < 0$ at least twice, it suffices to consider CTC’s of the following form

$$x(\phi) = (t, \phi, \zeta) = (t_0 + \delta t(\phi), \phi, \frac{1}{4} + \delta \zeta(\phi)),$$  (4.9)

where $t_0$ is a constant and $\delta t$ and $\delta \zeta$ are small fluctuations of $O(\epsilon)$ which are periodic in $\phi$. The condition for the $x(\phi)$ to be a CTC is for all $\phi$ the following inequality holds:

$$0 \geq \left(\frac{ds}{d\phi}\right)^2 = -(\frac{d(\delta t)}{d\phi} + \epsilon^+ \cos t_0^- - \epsilon^- \cos t_0^+)^2 + l^2(2\delta \zeta + \epsilon^- \cos t_0^- + \epsilon^+ \cos t_0^+)^2 + 4l^2(\frac{d\delta \zeta}{d\phi})^2 + O(\epsilon^3),$$

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where we have set \( t_0^\pm = \frac{t_0}{T} \pm \phi \). We may restrict the analysis to the case where

\[
\delta t = A \cos \phi + B \sin \phi, \quad \delta \zeta = C \cos \phi + D \sin \phi.
\]

Substituting this into the above inequality and making an elementary but tedious analysis, we find that CTC’s of the form (4.9) appear in the region

\[
\frac{1}{4} - \frac{1}{2} \sqrt{(\epsilon^+ + \epsilon^-)^2 \cos^2 \frac{t_0}{T} + (\epsilon^+ - \epsilon^-)^2 \sin^2 \frac{t_0}{T}} \leq \zeta \leq \frac{1}{4} + \frac{1}{2} \sqrt{(\epsilon^+ + \epsilon^-)^2 \cos^2 \frac{t_0}{T} + (\epsilon^+ - \epsilon^-)^2 \sin^2 \frac{t_0}{T}}.
\]

It would therefore be probable to exclude the new spacetime (4.8) from the physical spectrum, unless the CTC’s can be shielded by some singular structure.

5 Several Aspects of the Moduli Space

In this section we investigate some properties of the moduli space, which is a direct product of two copies of the space \( Q \) of centrally extended quadratic differentials \( T(\phi) \) on \( S^1 \). The transformation of \( T \) generated by a vector field \( v(\phi) \frac{\partial}{\partial \phi} \) is already given by (2.16):

\[
\delta_v T = -v T' - 2v' T + 2v'' , \tag{5.1}
\]

where \( v' \equiv \frac{dv}{d\phi} \). Thus our problem can be translated into that of finding all the functions \( T(\phi) \) which do not mutually transform by (5.1).

Instead of dealing with \( T \) itself, it is convenient to consider the “stabilizer” of a given \( T \). A stabilizer of \( T \) is a vector field \( f(\phi) \frac{\partial}{\partial \phi} \) which leaves \( T \) unchanged:

\[
0 = \delta_f T = -f T' - 2f' T + 2f'' . \tag{5.2}
\]

According to ref.[10], we can say the following.

1) For a fixed \( T(\phi) \), the stabilizers form a vector space whose dimension is either 1 or 3.

2) If a stabilizer \( f \) is given, \( T \) can be expressed by using \( f \):

\[
T = \frac{d - (f')^2}{f^2} + 2\frac{f''}{f} , \tag{5.3}
\]

where \( d \) is a constant adjusted by requiring the regularity of \( T \). 9

3) The stabilizers necessarily belong to one of the following three types: O) \( f \) has

\[
\text{Under the vector transformation: } f(\phi) \to \tilde{f}(\phi) = \frac{d\phi}{d\tilde{\phi}} f(\phi), \text{ the } T \text{ given by (5.3) is subject to the desired transformation:}
\]

\[
T(\phi) \to \tilde{T}(\phi) = \left( \frac{d\tilde{\phi}}{d\phi} \right)^2 T(\tilde{\phi}) - 2\{\tilde{\phi}, \phi\}.
\]
no zeros. Then $d$ is an arbitrary parameter. I) $f$ has only (even number of) single zeros. At each zero $(f')^2$ has the same value which should equal to $d$. II) $f$ has only double zeros where $f'''$ must vanish. $d$ is zero in this case.

4) The type O)-stabilizers are diffeomorphic to $f = \text{const}$. Because $\frac{d\phi}{f(\phi)}$ is then nonsingular and, by taking an appropriate coordinate $\tilde{\phi}$, we can set:

$$\frac{d\phi}{f(\phi)} = \frac{d\tilde{\phi}}{a}, \quad \left(\frac{2\pi}{a} \equiv \oint \frac{d\phi}{f(\phi)}\right). \tag{5.4}$$

By substituting $f(\phi) = a\frac{d\phi}{d\phi}$ into (5.3), we can see that the $T(\phi)$ with a type O)-stabilizer is equivalent to a constant $T$.

5) The type I)-stabilizers are splitted into the non-diffeomorphic classes which are characterized by the number of zeros $2n$ and the magnitude of $\Delta$ which is defined by

$$\Delta = \lim_{\epsilon \to 0} \int_{|\phi - \phi_k| > \epsilon} d\phi \left| \frac{f'(\phi_k)}{f(\phi)} \right|, \tag{5.5}$$

where $\phi_k (k = 1, \ldots, 2n)$ are the zeros of $f$.

6) Two type II)-stabilizers are diffeomorphic with each other if they have the same number of double zeros and the same signature of $U(f)$. $^{10}$

We can construct concrete realization of generic type I)- and type II)-stabilizers.

The type I)-stabilizers are represented by

$$f_{n,a}(\phi) = \sin n\phi + a \cos 2n\phi, \quad -1 < a < 1. \tag{5.6}$$

This $f_{n,a}$ has $2n$ zeros and $\Delta = n\pi \frac{4a^2 - 1 + \sqrt{1 + 8a^2}}{2a\sqrt{1 - a^2}}$. The $T(\phi)$ which is stabilized by $f_{n,a}$ is

$$T(\phi) = -n^2 \frac{16a^2(\sin n\phi - \alpha)^2 + 4a(\sin n\phi - \alpha) - 2 + \sqrt{1 + 8a^2}}{4a^2(\sin n\phi - \alpha)^2}, \tag{5.7}$$

where $\alpha = \frac{1 + \sqrt{1 + 8a^2}}{4a}$. From eq.(5.6) we can see that $f_n(\phi) = \sin n\phi$ corresponds to the limit $a \to 0$ in which $\Delta \to 0$ and $T(\phi) \to -n^2$.

$^{10}$ $U(f)$ is defined as follows. In the interval $\phi_k < \phi < \phi_{k+1}$, we consider $\tau_k$ which is defined by

$$d\tau_k = \frac{d\phi}{f(\phi)},$$

and which is normalized by: $\tau_k \sim \frac{1}{\phi - \phi_k}$ as $\phi \to \phi_k + 0$. Then $\tau_k$ always behaves as $\tau_k \sim \frac{-1}{\phi - \phi_{k+1}} + a_k$, for $\phi \to \phi_{k+1} - 0$. The definition of $U(f)$ is: $U(f) = \sum_k a_k$.
The type II-stabilizers should be diffeomorphic to
\[ \tilde{f}_{n,b}(\phi) = 1 - (1 - b) \cos n\phi - b \cos 2n\phi, \quad -\frac{1}{3} < a < 1, \] (5.8)
which has \( n \) double zeros and \( U(\tilde{f}_{n,b}) = \frac{2\pi n^2 b}{\sqrt{(1-b)(1+3b)}} \). This \( \tilde{f}_{n,b} \) stabilizes
\[ T(\phi) = -n^2 \frac{1 + 3b^2 + (6b + 2b^2)(1 + 2 \cos n\phi) + 4b^2(1 + 2 \cos n\phi)^2}{(1 + b(1 + 2 \cos n\phi))^2}. \] (5.9)
If we set \( b = 0 \), eq.(5.8) reduces to \( \tilde{f}_n = 1 - \cos n\phi \) which has \( U(\tilde{f}_n) = 0 \) and which stabilizes \( T(\phi) = -n^2 \). On account of statement 6),
the space of the type II-stabilizers splits into three equivalence classes under diffeomorphisms on \( S^1 \), whose representatives are \( \tilde{f}_{n,b} \) with \( b \) being positive, zero, and negative respectively.

Let us now consider the “tangent space” \( TQ \) of \( Q \), i.e. the space of small fluctuations \( \delta T \) which cannot be absorbed into the transformation given by (5.1). By looking at eq.(5.3) and its associated footnote, we see that, at least when the space of the stabilizers is one-dimensional, we have only to look for the fluctuation \( \delta f \) of the stabilizer which cannot be expressed by the vector transformation:
\[ \delta f = \delta_v f = -v f' + v' f. \] (5.10)
This equation has a formal solution
\[ v(\phi) = f \int d\phi \frac{\delta f}{f'^2}. \] (5.11)
Our problem reduces to that of finding \( \delta f \) which is well-defined on \( S^1 \) and which gives non-periodic \( v \). The desired \( \delta f \) is given by \( f'^2 \), which gives \( v = f \phi \). Taking the fluctuation of eq.(5.3) and substituting \( \delta f = -\frac{1}{2} f^2 \), we can compute \( \delta T \) which cannot be represented by (5.1). The result is:
\[ \delta T = fT - 3f'' . \] (5.12)

\[ ^{11} \] Multiplication by a regular function \( C(\phi) \) with
\[ \oint d\phi C(\phi) = 2\pi C_0 \neq 0 \] does not influence essential results. The reasoning is as follows. We can decompose \( C(\phi) \) into \( C_0 + c(\phi) \), where \( C_0 \) is a constant and \( \oint d\phi c(\phi) = 0 \). The constant gives the same result and \( c(\phi) \) part gives the portion of \( \delta f \) which is well-defined on \( S^1 \). While \( \delta f = 1 \), \( \delta f = f \), etc. give \( v \) which is singular, such fluctuations also make \( \delta T \) singular. We can therefore rule out these fluctuations.
Because we can find one $\delta T$ in the above form per one stabilizer $f$, we conclude that the dimension of $TQ$ equals one when there is only one linearly-independent stabilizer.

What about the case where the space of stabilizers being three dimensions? By using eq. (5.2) we can show that the space of the stabilizers of a given $T$ forms a Lie algebra under the Lie derivative $[f, g] = fg' - gf'$. Because this Lie algebra generates a three dimensional subgroup of $DiffS^1$ (the group of diffeomorphisms of a circle, i.e., the Virasoro group), it would be natural to identify this group with $SL(n)(2, \mathbb{R})$ (the n-fold covering of $SL(2, \mathbb{R})$). By performing an appropriate diffeomorphism we can choose the generators of $SL(n)(2, \mathbb{R})$ to be $(1, \cos n\phi, \sin n\phi)$, which are the stabilizers of $T(\phi) = -n^2$. Thus $T(\phi)$ which has three linearly independent stabilizers turns out to be equivalent to $T = -n^2$. Under the action of $SL(n)(2, \mathbb{R}) \subset DiffS^1$ the stabilizers

$$f = z + x \cos n\phi + y \sin n\phi$$

of $T = -n^2$ split into the equivalence classes whose representatives are $f = z$ with $z \in \mathbb{R}$, $f = y\hat{f}_n$ with $y \in [0, \infty)$, and $f = \pm f_n$ respectively. The net tangent space of $Q$ at a "triple stabilizer point" is thus given by a T-shaped space plus two points (fig.2a).

By gluing the adjacent tangent spaces into together we find the topology of the whole space $Q$ of the centrally extended quadratic differentials: $Q$ is an almost one-dimensional pectinated space which is constructed by gluing infinitely-many half-lines ($f_{n,a}$ with fixed $n$) to one line of $\mathbb{R}^1$ ($f = const.$) at the points $-n^2$, and then by associating two points ($\hat{f}_{n,\pm \epsilon}$ with $n$ fixed) to the points $-n^2$ (fig.2b).

Finally we mention the relation between $Q$ and the moduli space $C$ of flat $\tilde{SL}(2, \mathbb{R})$ connections on a cylinder, where $\tilde{SL}(2, \mathbb{R})$ is the universal covering group of $SL(2, \mathbb{R})$. It is well known that the moduli space of flat connections is parametrized by the conjugation classes of the holonomies around noncontractible loops[12]. Using this as in the case of the flat $\tilde{SL}(2, \mathbb{R})$ connections on a torus[11], we find that the moduli space $C$ is represented by the sum of infinitely many sectors:

$$C = C_T \cup \left( \bigcup_{n \in \mathbb{Z}} C^n_S \right) \cup \left( \bigcup_{n \in \mathbb{Z}, \sigma = \pm} C^n_{N, \sigma} \right).$$

(5.13)
$SL(2, \mathbb{R})$ connections which parametrize each sector are given, for example, by:

\[
\begin{align*}
C_T & : A = J_0 \tilde{\beta} d\phi, \\
C^n_S & : A = J_0 n d\phi + (J_2 \cos n\phi + J_1 \sin n\phi) \tilde{\beta} d\phi, \\
C^n_{N,\pm} & : A = J_0 n d\phi \pm e^{\lambda} [J_0 + (J_2 \cos n\phi + J_1 \sin n\phi)] d\phi,
\end{align*}
\]

where $\beta \in \mathbb{R}$ and $\tilde{\beta} \in [0, \infty)$. $\lambda$ is an arbitrary parameter which can be absorbed by a boost. We can easily see that the point $\tilde{\beta} = 0 \in C^n_S$ coincides with $\beta = n \in C_T$, and that $C^n_{N,\pm}$ are “very close to” (but not connected to) the point $\beta = n \in C_T$ (fig.3). From this we find that $Q$ is homeomorphic to the following subspace of $C$:

\[
Q \approx (C_T \setminus \{0\}) / \mathbb{Z}_2 \cup C^0_{N,+} \cup C^0_{S} \cup \left( \bigcup_{n \in \mathbb{N}} C^n_S \right) \cup \left( \bigcup_{n \in \mathbb{N}, \sigma = \pm} C^n_{N,\sigma} \right),
\]

where $\mathbb{Z}_2$ is generated by the inversion: $A \rightarrow -A$.

This can partially be expected from eq.(3.11). When $M^\pm$ is constant, $A'^{\pm}(z)$ in (3.11) can be conjugated into the form of $A \in (C_T \setminus \{0\}) / \mathbb{Z}_2 \cup C^0_{N,+} \cup C^0_{S}$ given above. The spaces of $M^\pm$ each of which is left invariant by a vector field $f_{n,a}(t^\pm)$ and by $\tilde{f}_{n,b}(t^\pm)$ stretch from the points $M^\pm = -n^2$ (with $n \in \mathbb{N}$) which correspond to $A^{(\mp)} = \mp J_0 n dt^{\mp}$. These spaces should therefore be related to $C^n_S$ and $C^n_{N,\pm}$. While it seems to be difficult to establish the gauge equivalence between $A \in C^n_S$ (or $A \in C^n_{N,\pm}$) and eq.(3.11) with $M^\pm$ given, for example, by (5.7) (or by (5.9)), we conjecture that relation (5.15) is in fact the gauge equivalence relation.

Thus we see that the moduli space of asymptotically $ADS^3$ spacetimes is a subspace of the moduli space of flat $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ connections.\[^{12}\] This result is expected by naively investigating Chern-Simons formulation of (2+1)-anti-de Sitter gravity on a cylinder. It is nontrivial, however, that the argument of the chiral connection $A^{(\pm)}$ is not simply $\phi$ but $\pm t^\pm = \pm t + \phi$. This cannot be extracted by a naive analysis of Chern-Simons formulation.

\[^{12}\text{Roughly speaking, the rest of flat } SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \text{ connections can be interpreted as follows. Because simultaneous changes of signs in } A^{(\pm)} \text{ does not affect the metric, the relevant connections give spacetimes with opposite orientation. A change of the relative sign between } A^{(+) \text{ and } A^{(-)} \text{ corresponds to exchanging } t \text{ and } \phi. These connections are therefore expected to give spacetime with CTC’s at spatial infinity, which cannot be considered as our universe.}\]
6 Discussion

We have seen that, in (2+1)-dimensions, the moduli space of asymptotically anti-de Sitter spacetimes modulo diffeomorphisms is parametrized by two centrally extended quadratic differentials. We found two new families of solutions corresponding to type I)- and type II)-stabilizers. From the analysis of small fluctuations about \( M^\pm = -n^2 \) made in §4, however, these solutions turn out to have CTC’s and thus seem to be ruled out as physically irrelevant. If we take the equivalence class under all asymptotic transformations, only 3DBH’s and anti-de Sitter space (possibly with a conical singularity) seem to survive as physically relevant configurations.

If we consider only the region with sufficiently large radius, however, these CTC’s do not appear. It is possible that we can get rid of these CTC’s by an appropriate surgery of the spacetime. Nor we know whether we can really take the equivalence under all the asymptotic transformations. Behavior of the metric (1.4) would be extremely complicated when, for example, \( M^{(\pm)} \) has zeros. While we have investigated the structure of the moduli space by a somewhat topological consideration, it has not been shown directly that all the positive-definite \( M^\pm \) reduce to a positive constant by an adequate transformation (2.18). For the time being we only mention that the asymptotic behavior of the asymptotically \( AD{S^3} \) spacetimes is described by two centrally extended quadratic differentials. There seem to be numerous issues which require further investigation.

Finally we comment on the effective action. In the analysis made in §2 we have considered the spatial infinity as the only boundary. Unless we include a source term, however, we have to add the contribution from the inner boundary into eq.(2.2). The result of substituting (3.9) into (2.2) turns out to be constant even if we do not impose eq.(3.7), which should be derived from the variational principle. To look for the origin of this inconsistency, we consider the variation of the action:

\[
\delta I = 2 \int_M \text{Tr}[\delta A \wedge (dA + A \wedge A)].
\]

From (3.9) we see that \( F = dA + A \wedge A \) is proportional to \( dt \wedge d\phi \) with nonvanishing terms being the coefficients of \( J_0, J_2, P_0 \) and of \( P_2 \). To obtain the equations of motion (3.7), therefore, we need nontrivial values of \( e^0_r, e^2_r, \omega^{12}_r \) or of \( \omega^0_0 \), which vanish in
our gauge. This result is closely related to the fact that, in pure Chern-Simons
gauge theories on a cylinder, the moduli space of flat connections does not have
any nontrivial symplectic structure. To obtain a nontrivial dynamics, we have to
take account of the gauge degrees of freedom, whose values on the boundary become
dynamical degrees of freedom described by Chiral Wess-Zumino-Witten action \[13\].

In our formulation we should first introduce the “small” gauge degrees of freedom.
The relevant connection is:
\[ g_s A g_s^{-1} - d g_s g_s^{-1}, \quad g_s = \exp(P_a \zeta^a + J_0 \eta^0 + J_2 \eta^2), \]

where \( A \) is given by \( (3.9) \), and \( \zeta^a, \eta^0, \eta^2 \sim O(\frac{1}{r^3}) \) are the small gauge degrees of freedom which are subject to some boundary condition imposed on an appropriately
chosen inner boundary. While it is desirable to obtain the action which reproduce the
equations of motion \( (3.7) \) and which is invariant under the transformation \( (2.16) \), it is
probable that the boundary condition on the inner boundary violates the symmetry
under \( (2.16) \), leaving only a subgroup as the symmetry of the system. \[13\]

Acknowledgments

I would like to thank Prof. K. Kikkawa, Prof. H. Itoyama and H. Kunitomo
for helpful discussions and careful readings of the manuscript. I am also grateful
to Prof. T. Kubota for letting me aware of some useful literature. The author is
supported by the Japan Society for the Promotion of Science.

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Figure Captions

Fig.1 Behavior of the conventional radial coordinate $\tilde{r}$ in the complex $\zeta$-plane (in the case $M^+, M^- > 0$).

Fig.2 (a) Tangent space of $Q$ at the “triple stabilizer point”. (b) Topology of the space $Q$ of centrally extended quadratic differentials. The solid line, dotted lines, and dots denote the spaces of $T$ stabilized by type O)-, type I)- and type II)-stabilizers respectively.

Fig.3 Topology of the moduli space $\mathcal{C}$ of flat $\tilde{SL}(2, \mathbb{R})$ connections on a cylinder. The part drawn by bold lines is homeomorphic to $Q$. 