Equality of Secure Domination and Inverse Secure Domination Numbers

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Abstract

Let $G = (V, E)$ be a graph. Let $D$ be a minimum secure dominating set of $G$. If $V - D$ contains a secure total dominating set $D'$ of $G$, then $D'$ is called an inverse secure dominating set with respect to $D$. The smallest cardinality of inverse secure dominating set of $G$ is the secure domination number $\gamma_s^{-1}(G)$ of $G$. In this paper, we obtain some graphs for which $\gamma_s(G) = \gamma_s^{-1}(G)$ and establish some results on this respect. Also we obtain some graphs for which $\gamma_s(G) = \gamma_s^{-1}(G) = \frac{p}{2}$, where $p$ is the number of vertices of $G$.

Key words: dominating set, secure dominating set, inverse secure dominating set, inverse secure domination number.

Mathematics Subject Classification: 05C69, 05C78

1. Introduction

All graphs considered here are finite, undirected without isolated vertices, loops and multiple edges. For all further notation and terminology we refer the reader to $^1$. Let $G = (V, E)$ be a graph. A set $D$ of vertices in a graph $G$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. Recently several domination parameters are given in the books by Kulli in $^2, 3, 4$. Let $D$ be a minimum dominating set of $G$. If $V - D$ contains a dominating set $D'$ of $G$, then $D'$ is called an inverse dominating set of $G$ with respect to $D$. The inverse domination number $\gamma^{-1}(G)$ of $G$ is the minimum cardinality of an inverse dominating set of $G$. This concept was introduced by Kulli and Sigarkanti in $^5$. Many other inverse domination parameters were studied, for example, in $^6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24$.

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A secure dominating set of a graph $G$ is a dominating set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $v \in D$ adjacent to $u$ such that $(D - \{v\}) \cup \{u\}$ is a dominating set. The smallest cardinality of a secure dominating set is the secure domination number $\gamma_s(G)$ of $G$. This concept was studied, for example, in \cite{21,22}. Let $D$ be a minimum secure dominating set of $G$. If $V - D$ contains a secure dominating set $D'$ of $G$, then $D'$ is called an inverse secure dominating set with respect to $D$. The inverse secure domination number $\gamma_{s}^{-1}(G)$ of $G$ is the minimum cardinality of an inverse secure dominating set of $G$. This concept was introduced by Enriquez in \cite{22} and was studied by Kulli in \cite{23}.

A $\gamma_{s}^{-1}$-set is a minimum inverse secure dominating set. Similarly other sets can be expected. If $D = \{u\}$ is a secure dominating set of $G$, then $u$ is called a secure dominating vertex of $G$. A vertex $u$ of $G$ is said to be a $\gamma_{s}$-required vertex of $G$ if $u$ lies in every $\gamma_{s}$-set of $G$.

An application of inverse secure domination is found in Computer Science. In the event that there is a need for all nodes in a system to have direct access to needed resources (for example, large database) a secure dominating set furnishes such a configuration. If a second important resource is needed, then a separated disjoint secure dominating set provides duplication in case the first is corrupted in some way. We have $\gamma_{s}(G) \leq \gamma_{s}^{-1}(G)$.

From the point of above, one may demand $\gamma_{s}(G) = \gamma_{s}^{-1}(G)$, whereas many graphs do not enjoy such a property.

For Example, we consider the graph $G$ in Figure 1. Then $\gamma_{s}(G) = 2$ and $\gamma_{s}^{-1}(G) = p - 2$. In this case, if $p$ is large, then $\gamma_{s}^{-1}(G)$ is sufficiently large compared to $\gamma_{s}(G)$.

\begin{figure}[h]
\centering
\includegraphics{figure1.png}
\caption{Figure 1}
\end{figure}

2. Graphs with $\gamma_{s}(G) = \gamma_{s}^{-1}(G)$

Proposition 1. If $K_p$ is a complete graph with $p \geq 2$ vertices, then
\[ \gamma_{s}(K_p) = \gamma_{s}^{-1}(K_p) = 1. \]

Proposition 2. If $K_{m,n}$ is a complete bipartite graph with $4 \leq m \leq n$, then
\[ \gamma_{s}(K_{m,n}) = \gamma_{s}^{-1}(K_{m,n}) = 4. \]

Proposition 3. If $K_{m,n}$ is a complete bipartite graph with $4 \leq m \leq n$, then
\[ \gamma_{s}(K_{m,n}) = \gamma_{s}^{-1}(K_{m,n}) = 2. \]

Proof: Clearly $K_{m,n} = K_m \cup K_n$. Therefore
\[ \gamma_{s}(K_{m,n}) = \gamma_{s}(K_m) + \gamma_{s}(K_n) = 2. \]
\[ \gamma_{s}^{-1}(K_{m,n}) = \gamma_{s}^{-1}(K_m) + \gamma_{s}^{-1}(K_n) = 2. \]

Hence the result follows.

Theorem 4: Let $G$ be a graph with $\gamma_{s}(G) = \gamma_{s}^{-1}(G)$. Then $G$ has no $\gamma_{s}$-required vertex.

Proof: Let $G$ be a graph with $\gamma_{s}(G) = \gamma_{s}^{-1}(G)$. Let $D$ be a $\gamma_{s}$-set and $D'$ be a $\gamma_{s}^{-1}$-set of $G$. On the contrary, assume $G$ contains a $\gamma_{s}$-required vertex $u$. Then $u$ lies in every $\gamma_{s}$-set of $G$. Hence $u \in D$ and $u \in D'$, which is a contradiction to $D \subseteq V - D$. Thus the result follows.

Theorem 5. Let $u$ be a secure dominating vertex of a graph $G$. Then
\[ \gamma_{s}^{-1}(G) = \gamma_{s}(G - u). \]
Proof: Let $u$ be a secure dominating vertex of $G$. Then $\{u\}$ is a $\gamma_s$-set of $G$. Thus any $\gamma_s^{-1}$-set of $G$ lies in $G - \{u\}$ and is a minimum dominating set of $G - \{u\}$. Hence $\gamma_s^{-1}(G) = \gamma_s(G - u)$.

Construct the graph $G$ as follows:

Let $H_i = K_{m_i}, i = 1, 2, ..., r$ and $2 \leq m_1 \leq m_2 \leq \ldots \leq m_r$. Let $v_i \leq H_i, i = 1, 2, \ldots, r$. Consider the graph $G$ obtained from joining the vertices $v_i, v_{i+1}, i = 1, 2, \ldots, r-1$, see Figure 2. Consider the vertices $u_i \in H_i$ such that $u_i \neq v_i, i = 1, 2, \ldots, r$. 

![Graph illustration](image.png)

**Figure 2**

**Proposition 6.** Let $G$ be a graph as shown in Figure 2. Then $\gamma_s(G) = \gamma_s^{-1}(G) = r$.

**Proof:** The set $D = \{v_1, v_2, \ldots, v_r\}$ is a $\gamma_s$-set in $G$. Then the set $D' = \{u_1, u_2, \ldots, u_r\}$ is a $\gamma_s^{-1}$-set in $G$ for $u_i, u_{i+1}, i = 1, 2, \ldots, r$. Thus $\gamma_s(G) = \gamma_s^{-1}(G) = r$.

**Corollary 7.** Let $G$ be a graph as shown in Figure 2 such that $m_1 = m_2 = \ldots = m_r = 2$. Then $\gamma_s(G) = \gamma_s^{-1}(G) = \frac{p}{2}$, where $p = 2r$ is the number of vertices of $G$.

**Proposition A:** Let $G$ be a connected non-complete graph with $p \geq 4$ vertices. If $\gamma_s^{-1}(G) = 2$, then $\gamma_s(G) = 2$.

**Proposition 8.** If $G$ be a connected graph with $p \geq 4$ vertices such that $G \neq K_p$ and $\gamma_s^{-1}(G) = 2$, then $\gamma_s(G) = \gamma_s^{-1}(G) = 2$.

**Proof:** This follows from Proposition A.

**Proposition 9.** Let $G$ and $H$ be complete graphs. Then $\gamma_s(G + H) = \gamma_s^{-1}(G + H) = 1$.

**Proof:** If $G$ and $H$ are complete graphs, then $G + H$ is complete. Thus $\gamma_s(G + H) = \gamma_s^{-1}(G + H) = 1$.

3. Graphs with $\gamma(G) = \gamma_s^{-1}(G) = \frac{p}{2}$
In this section, we establish some results for which \( \gamma(G) = \gamma_s^{-1}(G) = \frac{p}{2} \).

**Theorem 10.** If \( G = K_{s^2}, K_s \) or \( K_s - e \), then \( \gamma(G) = \gamma_s^{-1}(G) = \frac{p}{2} \), where \( p \) is the number of vertices of \( G \).

**Proof:** If \( G = K_{s^2} \), then by Proposition 2, \( \gamma(G) = \gamma_s^{-1}(G) = \frac{p}{2} \). If \( G = K_s \), then by Proposition 1, \( \gamma(G) = \gamma_s^{-1}(G) = \frac{p}{2} \). If \( G = K_s - e \), then we have \( \gamma(G) = \gamma_s^{-1}(G) = \frac{p}{2} \), where \( p \) is the number of vertices of \( G \).

Construct the graph \( G \) as follows: Let \( e_i = u_i v_i \), \( 1 \leq i \leq m \) and \( e_{i+1} = v_{i+1} u_{i+1} \) be the edges of a cycle \( C_{2m} \). For each \( e_i = u_i v_i \), join the vertices \( u_i, v_i \) to new vertices \( x_i, y_i \) to form the graph \( G \), see Figure 3.

**Theorem 11:** Let \( G \) be a graph with \( 4m \) vertices as shown in Figure 3. Then

\[
\gamma(G) = \gamma_s^{-1}(G) = 2m.
\]

**Proof:** In the graph \( G \) of Figure 3, \( V(G) = \{ u_1, ..., u_m, v_1, ..., v_m, x_1, ..., x_m, y_1, ..., y_m \} \). Then the set \( D = \{ u_1, ..., u_m, v_1, ..., v_m \} \) is a \( \gamma \)-set with \( 2m \) vertices and \( D' = \{ x_1, ..., x_m, y_1, ..., y_m \} \) is a \( \gamma_s^{-1} \)-set with \( 2m \) vertices. Thus \( \gamma(G) = \gamma_s^{-1}(G) = 2m \).

**Remark 10.** Let \( G_1, G_2, ..., G_m \) be the \( m \) connected components of a graph \( G \). Let \( D_i \) be a \( \gamma \)-set of \( G_i \), and \( D'_i \) be a \( \gamma_s^{-1} \)-set of \( G_i \), for \( i = 1, 2, ..., m \). Then \( D_1 \cup D_2 \cup ... \cup D_m \) is a \( \gamma \)-set of \( G \) and \( D'_1 \cup D'_2 \cup ... \cup D'_m \) is a \( \gamma_s^{-1} \)-set of \( G \). Thus \( \gamma(G) = \sum_{i=1}^{m} \gamma(G_i) \) and

**Theorem 13.** Let \( G_1, G_2, ..., G_m \) be the \( m \) components of a graph \( G \). Then \( \gamma(G) = \gamma_s^{-1}(G) \) if and only if \( \gamma(G_i) = \gamma_s^{-1}(G_i) \), for \( i = 1, 2, ..., m \).

**Proof:** Let \( G_1, G_2, ..., G_m \) be the \( m \) connected components of graph \( G \).

By Remark 12, \( \gamma(G) = \sum_{i=1}^{m} \gamma(G_i) \) and \( \gamma_s^{-1}(G) = \sum_{i=1}^{m} \gamma_s^{-1}(G_i) \). Therefore, \( \gamma(G) = \gamma_s^{-1}(G) \) if \( \gamma(G) = \gamma_s^{-1}(G) \) for \( i = 1, 2, ..., m \).

Conversely suppose \( \gamma(G) = \gamma_s^{-1}(G) \). We have \( \gamma(G_i) \leq \gamma_s^{-1}(G_i) \), for \( i = 1, 2, ..., m \). We now prove that \( \gamma(G_i) = \gamma_s^{-1}(G_i) \), for \( i = 1, 2, ..., m \). On the contrary, assume \( \gamma(G_i) < \gamma_s^{-1}(G_i) \) for some \( i \). Then \( \gamma(G_i) > \gamma_s^{-1}(G_i) \), for some \( j \neq i \), which is a contradiction. Thus \( \gamma(G_i) = \gamma_s^{-1}(G_i) \) for \( i = 1, 2, ..., m \).

**Corollary 14.** If the connected components \( G_i \) of \( G \) are either \( K_2 \) or \( K_{s^2} - e \) or \( K_{s^4} \) or \( G \) as shown in Figure 3, then \( \gamma(G) = \gamma_s^{-1}(G) = p \), where \( p \) is the number of vertices of \( G \).

**Proof:** This follows from Theorems 10, 11, 13.
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