“Entropy sum” of (A)dS Black Holes in Four and Higher Dimensions

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Abstract

We present the “entropy sum” relation of (A)dS charged black holes in higher dimensional Einstein-Maxwell gravity, \( f(R) \) gravity, Gauss-Bonnet gravity and gauged supergravity. For their “entropy sum” with the necessary effect of the un-physical “virtual” horizon included, we conclude the general results that the cosmological constant dependence and Gauss-Bonnet coupling constant dependence do hold in both the four and six dimensions, while the “entropy sum” is always vanishing in odd dimensions. Furthermore, the “entropy sum” of all horizons is related to the topology of the horizons in four and six dimensions. In these explicitly four cases, one also finds that the reversed charges \( M \) (mass), \( Q \) (charge from Maxwell field or supergravity) and \( a \) (angular momentum) play no role in the “entropy sum” relation.

1 Introduction

There has recently been some considerable ongoing interest in the “universal property” of horizon entropy for various types of multi-horizons black holes. It seems clear that these additional thermodynamic relations of entropy may provide further insight into the quantum physics for black holes. One recent interest has been focused on the “area product” or ”entropy product” rules [1–16], which is often independent of the mass of black holes [1–12] and depends only on various charges and angular momentum mostly. But the mass independence claim sometimes fails [13–16]. Another “universal property” of entropy is the “entropy sum” relation of black holes [17]. It is shown that, in the four dimensional Einstein-Maxwell-(A)dS spacetime, the “entropy sum” depends only on the cosmological constant with the necessary effect of the un-physical “virtual” horizon included. When in the spacetime with extra scalar field and in Einstein-Weyl
spacetime, they find the “entropy sum” is dependent of the constants characterizing the strength of the extra degree of freedom, the matter field. Besides, the “entropy sum” relation does not depend on the conversed charges $M$ (mass), $Q$ (charge from Maxwell field) and $J$ (from “rotation field”). Thus, it is said that the “entropy sum” is more universal and is only related to the background field properties. Furthermore, it is found that entropy product and entropy sum of some (A)dS black holes are actually equal [18], when only the effect of the physical horizons are considered, as they both can be simplified into a mass independent entropy relations of physical horizon [14,18]. That is to say, the entropy sum also somehow reveal the microscopics of black holes.

On the other hand, the entropy product of two-horizons black holes were understood well and physically via their holographic description, i.e. the thermodynamic method of black hole/CFT (BH/CFT) correspondence [10,20–25]. By using the BH/CFT correspondence, it is found that the same central charge $c_R = c_L$ is equivalent to the condition that the entropy product $S_+S_-$ being mass-independent, or equivalently the condition $T_+S_+ = T_-S_-$, where $T_\pm$, $S_\pm$ are the outer and inner horizon temperatures and entropies respectively. Therefore the entropy product $S_+S_-$ being mass-independent (equivalently thermodynamics relations $T_+S_+ = T_-S_-$) may be taken as the criterion whether there is a 2 dimensional CFT dual for the black holes in the Einstein gravity and other diffeomorphism invariant gravity theories [10,20–25]. In this sense, the equality between entropy product and entropy sum leads one to expect that the entropy sum may be understood well in the similar way. This makes it more interesting for studying the entropy sum in four and higher dimension (A)dS spacetime, especially for the cases that the mass-independence of entropy product fails.

The surprising discovery of the cosmic late stage accelerating expansion has inspired intensive research on the universe background cosmological constant, including its effects on the astrophysics and black hole physics. In this paper, we generalize the discussion about the “entropy sum” to higher dimensions cases. We will mainly test the “entropy sum” relation of (A)dS charged black holes in higher dimensional Einstein-Maxwell gravity, $f(R)$ gravity and Gauss-Bonnet gravity. For their “entropy sum” with including the necessary effect of the un-physical “virtual” horizon, we conclude the general results that the cosmological constant dependence and Gauss-Bonnet coupling constant dependence do still hold in both the four and six dimensions, while the “entropy sum” is always vanishing in odd dimensions. Furthermore, the “entropy sum” of all horizons is related to the topology of the horizons in four and six dimensions. In these clearly three cases, one also finds that the conversed charges $M$ and $Q$ play no role in the “entropy sum” relation.

This paper is organized as follows. In the next Section, we will investigate the “entropy sum” of higher dimensional (A)dS charged black holes. In Sections 3, 4 and 5 we discuss the “entropy sum” of (A)dS black hole in higher dimensional $f(R)$ gravity and Gauss-Bonnet gravity, gauged supergravity respectively. Section 6 is devoted to the conclusions.
2 “Entropy sum” of (A)dS black hole in Einstein-Maxwell gravity

The Einstein-Maxwell action in higher dimensions $d$ reads as

$$L = \frac{1}{16\pi G} \int d^d x \sqrt{-g} [R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}],$$  \hspace{1cm} (2.1)$$

where $\Lambda = \pm \frac{(d-1)(d-2)}{2\ell^2}$ is the cosmological constant associated with cosmological scale $\ell$. Here, the negative cosmological constant corresponds to AdS spacetime while positive one corresponds to dS spacetime. Varying this action with respect to the metric tensor leads to the RN-AdS solution given by [26–29]

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2_{d-2},$$  \hspace{1cm} (2.2)$$

where $d\Omega^2_{d-2}$ represents the line element of a $(d-2)$-dimensional maximal symmetric Einstein space with constant curvature $(d-2)(d-3)k$, and the $k = 1, 0$ and $-1$, corresponding to the spherical, Ricci flat and hyperbolic topology of the black hole horizon, respectively. The metric function $V(r)$ is given by

$$V(r) = k - \frac{2M}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)} r^2,$$  \hspace{1cm} (2.3)$$

where $M$ and $Q$ are the mass and the charge of the black hole, respectively. In the Einstein-Maxwell gravity, the entropy of horizon located in $r = r_i$, as usual, is given by

$$S_i = \frac{A_i}{4} = \frac{\pi^{(d-1)/2}}{2\Gamma \left(\frac{d-1}{2}\right)} r_i^{d-2},$$  \hspace{1cm} (2.4)$$

where $A_i$ is the volume of the $(d-2)$-sphere with the “radius” $r_i$$\frac{2\pi^{(d-1)/2}}{\Gamma \left(\frac{d-1}{2}\right)} r_i^{d-2}$.

Back to the horizon function Eq.(2.3), in principle this high order polynomial can have at most $2(d-2)$ roots (even number), while some of them may be equal at special parameter conditions between $M$, $Q$ and $\Lambda$. In what follows, we will only consider the case for the $2(d-2)$ roots, as we are interested in the “entropy sum” of multi-horizons black hole. That is to say, we will consider all possible roots of the horizon structure in the following paper. One needs noting that, in fact we are considering the black holes with some special black hole mass $M$ in the following discussions, in order to have multi-roots or more horizons.

However, usually the horizon function can not be solved explicitly. Thus, we will only take some specific dimensions $d$ as an example. And one will find that it is
not all necessary to list the roots explicitly out. What we emphasize is that there are four kind of roots, each of which stands for the event horizon, Cauchy horizon, cosmological horizon and the un-physical “virtual” horizon respectively. We find the “virtual” horizon can not be dropped, otherwise we can not get the “entropy sum” relation with only the background field constant dependence.

Firstly, in odd dimensions, the horizon function Eq.(2.3) is a function of \( r^2 \), which results in some pairs of root (even number) \( r_i \) and \(-r_i\). On the other hand, the entropy \( S_i \) Eq.(2.4) is an odd order function of \( r_i \), which deriving a pair of entropy vanishing, i.e. \( S(r_i) + S(-r_i) = 0 \). Hence \( \Sigma_i S_i = 0 \) with arbitrary cosmological constant, the “entropy sum” of odd dimensional (A)dS black hole in Einstein-Maxwell gravity is vanishing. Then in what follows of this section, we will focus on even dimensional black holes with the \( 2(d - 2) \) horizons. To have a further look at the “entropy sum” in other special case of the horizon function, we will present the uncharged black hole as another example. And we will test the “entropy sum” in both the four and six dimensional charged and neutral (A)dS black hole separately.

In d=4 dimensions, the horizon function Eq.(2.3) and entropy Eq.(2.4) reduce to
\[
V(r) = k - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2,
\]
\[
S_i = \pi r_i^2.
\]
(2.5)
(2.6)

We focus on this black holes with 4 horizons, including the “virtual” horizon. To obtain the “entropy sum” of all horizons, we note that there is an exact equality
\[
\sum_{i=1}^{D} r_i^2 = \left( \sum_{i=1}^{D} r_i \right)^2 - 2 \left( \sum_{1 \leq i < j \leq D} r_i r_j \right),
\]
(2.7)

where \( D \) is the number of the horizons and \( D = 4 \) for the charged black hole. We find that it is not all necessary to list the roots. We can conveniently apply the Vieta theorem on the horizon function Eq.(2.5), which shows relations between the roots and the coefficients of a polynomials as follow:
\[
\sum_{i=1}^{D} r_i = 0 \quad \sum_{1 \leq i < j \leq D} r_i r_j = \frac{-3k}{\Lambda},
\]
which results in
\[
\sum_{i=1}^{D} r_i^2 = \frac{6k}{\Lambda},
\]
(2.8)

We immediately get the “entropy sum” of all horizons
\[
\sum_{i=1}^{D} S_i = \pi \left( \sum_{i=1}^{D} r_i^2 \right) = \frac{6\pi k}{\Lambda}.
\]
(2.9)
When we consider the neutral black hole, the horizon function Eq.(2.3) reduces to

$$V(r) = k - \frac{2M}{r} - \frac{\Lambda}{3} r^2,$$

(2.10)

which shows the number of horizons we focus on is $D = 3$, including the “virtual” horizon, and the same relations between the roots and the coefficients, hence finally we get the same “entropy sum” Eq.(2.9) as the charged black hole case.

The same “entropy sum” for four dimensional charged and neutral black holes Eq.(2.9) depends on $\Lambda$ and $k$. They both are independent of $M$ and $Q$. One need note that, when $k\Lambda < 0$, there exist several “complex horizons” for certain, which is un-physical, for this reason that the sum of $r_i$ and “entropy sum” are negative. When $k = 0$, the “entropy sum” of all flat horizons is vanishing and there are also some un-physical “complex horizons”. However, one should remove the possibility of the case with no horizon, which is out of our discussion.

In d=6 dimensions, the horizon function Eq.(2.3) reduces to

$$V(r) = k - \frac{2M}{r^3} + \frac{Q^2}{r^6} - \frac{\Lambda}{10} r^2, \quad \text{for charged black hole;}$$

(2.11)

$$V(r) = k - \frac{2M}{r^3} - \frac{\Lambda}{10} r^2, \quad \text{for neutral black hole,}$$

(2.12)

and entropy Eq.(2.4) have the form

$$S_i = \frac{2}{3} \pi^2 r_i^4.$$

(2.13)

The number of horizons we focus on is $D = 8$ for charged black hole and $D = 5$ for neutral black hole, including the “virtual” horizon. For both cases, we note a similar exact equality

$$3 \sum_{i=1}^{D} r_i^4 = 4 \left( \sum_{i=1}^{D} r_i \right) \left( \sum_{i=1}^{D} r_i^3 \right) + 6 \left( \sum_{1 \leq i < j \leq D} r_i r_j \right)^2 - \left( \sum_{i=1}^{D} r_i \right)^4$$

$$- 12 \left( \sum_{1 \leq i < j < m < n \leq D} r_i r_j r_m r_n \right),$$

(2.14)

where the Vieta theorem on the horizon function Eq.(2.11) and Eq.(2.12) derive the following same relations

$$\sum_{i=1}^{D} r_i = 0, \quad \sum_{1 \leq i < j < m < n \leq D} r_i r_j r_m r_n = 0,$$

$$\sum_{1 \leq i < j \leq D} r_i r_j = -\frac{10k}{\Lambda}.$$
Thus we get
\[ \sum_{i=1}^{D} r_i^4 = \frac{200k^2}{\Lambda^2}, \]  
(2.15)
which immediately show the “entropy sum” of all horizons
\[ \sum_{i=1}^{D} S_i = \frac{2}{3} \pi^2 \left( \sum_{i=1}^{D} r_i^4 \right) = \frac{400k^2\pi^2}{3\Lambda^2}. \]  
(2.16)
The “entropy sum” for the six dimensional charged and neutral black holes Eq.(2.16) also depends on \( \Lambda \) and \( k \), and is independent of the mass \( M \) and charge \( Q \). When \( k = 0 \), the “entropy sum” of all flat horizons is vanishing and there are some un-physical “complex horizons”. When \( k = \pm 1 \), we obtain a positive “entropy sum”, which seems to describe a physical system. In addition, the “entropy sum” in AdS spacetime is the same with that in dS spacetime. Also, the spacetime with the spherical and hyperbolic horizons shares the same “entropy sum”.

To summarize, including the necessary effect of the un-physical “virtual” horizon, we find the “entropy sum” is dependent of the cosmological constant and the topology of the horizons in four and six dimensions, while it is always vanishing in odd dimensions. Besides, the conversed charges \( M \) (mass) and \( Q \) (charge from Maxwell field) play no role in the “entropy sum” relation. This agrees with the result shown in [17].

### 3 “Entropy sum” of (A)dS black hole in \( f(R) \) gravity

In four dimensions, let us first consider the action for \( f(R) \) gravity with Maxwell term,
\[ \mathcal{L} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R + f(R) - F_{\mu\nu}F^{\mu\nu}], \]  
(3.1)
where \( f(R) \) is an arbitrary function of the scalar curvature. The static, spherically symmetric constant curvature \( (R = R_0) \) solutions is given by [30–32]
\[ ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2d\Omega_2^2, \]
with the horizon function
\[ V(r) = k - \frac{2\mu}{r} + \frac{q^2}{r^2 (1 + f'(R_0))} - \frac{R_0}{12}r^2. \]  
(3.2)
Here \( d\Omega_2^2 \) represents the line element of a 2-dimensional maximal symmetric Einstein space with constant curvature \( 2k \), and \( k = 1, 0 \) and \( -1 \), corresponding to the spherical,
Ricci flat and hyperbolic topology of the black hole horizon, respectively. The cosmological constant of this theory is then have the form \( \Lambda_f = \frac{R_0}{4} \). The parameters \( \mu \) and \( q \) are related to the mass and charge of black hole. The number of horizons we focus on is \( D = 4 \) for charged black hole, including the “virtual” horizon. For the neutral black hole, the horizon function reduces to

\[
V(r) = k - \frac{2\mu}{r} - \frac{R_0}{12} r^2
\]

with the number of horizons we focus on is \( D = 3 \). For both cases, we note the entropy of all horizons are

\[
S_i = \frac{A_i}{4G} (1 + f'(R_0)).
\]

where \( f'(R_0) = \frac{\partial f(R)}{\partial R} \bigg|_{R=R_0} \) and \( A_i = 4\pi r_i^2 \). We find the equality Eq.(2.7) still holds with the different \( D \) here. Then with the help of the Vieta theorem on the horizon function Eq.(3.2) and Eq.(3.3), we obtain the following relations

\[
\sum_{i=1}^{D} r_i = 0 \quad \sum_{1 \leq i < j \leq D} r_i r_j = -\frac{12k}{R_0},
\]

which results in

\[
\sum_{i=1}^{D} r_i^2 = \frac{24k}{R_0} = \frac{6k}{\Lambda_f}.
\]

One find this “area sum” is general in both Einstein gravity and \( f(R) \) gravity in four dimensions as shown in Eq.(2.8) and Eq.(3.5). In \( f(R) \) gravity, the entropy is modified as shown in Eq.(3.4), which immediately show the “entropy sum” of all horizons

\[
\sum_{i=1}^{D} S_i = \frac{6k\pi}{\Lambda_f G} (1 + f'(R_0)).
\]

This is different from the “entropy sum” of four dimensional charged \((A)dS \) black hole, because of the modification from \( f(R) \) gravity. But one can also find “entropy sum” is depend on \( \Lambda \) and \( k \). When \( f(R) = \) constant, the theory reduces to Einstein gravity and the “entropy sum” is equal to Eq.(2.16). Again, when \( k\Lambda < 0 \) and \( k = 0 \), there are some “virtual” horizons for certain, because of the negative and vanishing “area sum”, respectively. One need note the spacetime with \( \Lambda = 0 \) is out of the discussion of this paper, for the “charge-independence” of “entropy sum” fails as shown in [17].

**In higher dimensions**, since the standard Maxwell energy-momentum tensor is not traceless, people failed to derive higher dimensional black hole/string solutions from \( f(R) \) gravity coupled to standard Maxwell field. Thus the higher dimensional charged solution are obtained only in the case of power-Maxwell field with \( d = 4p \), where \( p \) is the power of conformally invariant Maxwell lagrangian [33]. This higher solution is too
complicated to continue our discussion about the “entropy sum” of horizon. To take a further study, one can turn to the higher dimensional $f(R)$ gravity without Maxwell field, whose action behaviours as

$$\mathcal{L} = \int d^d x \sqrt{-g} [R + f(R)].$$  \hspace{1cm} (3.7)

We consider the same metric form as Eq.(2.2) with the horizon function is

$$V(r) = k - \frac{2m}{r^{d-3}} - \frac{R_0}{d(d-1)} r^2;$$  \hspace{1cm} (3.8)

which is derived from the solutions \cite{33} with vanishing charge $q$. Here $m$ is an integration constant which is related to the mass of the solution. The cosmological constant is $\Lambda_f = \frac{d-2}{2d} R_0$. The entropy has the same form as that of four dimensional charged $f(R)$ black hole as shown in Eq.(3.4) with $G = 1$ and $A_i = \frac{2^{d-1}}{1(\frac{d-1}{2})} \Gamma(i \frac{d}{2}) r_i^{-d-2}$.

**In odd dimensions**, the horizon function Eq.(3.8) is a function of $r^2$ and the entropy $S_i$ is an odd order function of $r_i$. This also result in some pairs of root $r_i$ and $-r_i$ and a pair of vanishing entropy, i.e. $S(r_i) + S(-r_i) = 0$. Hence $\Sigma_i S_i = 0$, the “entropy sum” of odd dimensional (A)dS black hole in $f(R)$ gravity are vanishing.

**In six dimensions**, the horizon function Eq.(3.8) and area of horizons reduces to

$$V(r) = k - \frac{2m}{r^3} - \frac{R_0}{30} r^2;$$  \hspace{1cm} (3.9)

and $A_i = \frac{2^{5/2}}{1(\frac{5}{2})} r_i^{-4}$. The cosmological constant is $\Lambda_f = \frac{1}{6} R_0$. The number of horizons we focus on is $D = 5$. We find the equality Eq.(2.14) still holds with the different $D$ here. Then using the Vieta theorem on the horizon function Eq.(3.9), we obtain

$$\sum_{i=1}^{D} r_i = 0, \quad \sum_{1 \leq i < j < m < n \leq D} r_i r_j r_m r_n = 0,$$

$$\sum_{1 \leq i < j \geq D} r_i r_j = \frac{30k}{R_0}.$$  \hspace{1cm}  (3.10)

Thus we get

$$\sum_{i=1}^{D} r_i^4 = \frac{1800k^2}{R_0^2} = \frac{200k^2}{\Lambda_f^2},$$  \hspace{1cm} (3.10)

which shows that the “area sum” is general in both Einstein gravity and $f(R)$ gravity in six dimensions as shown in Eq.(2.15) and Eq.(3.10). The “entropy sum” of all horizons is

$$\sum_{i=1}^{D} S_i = \frac{400k^2 \pi^2}{3\Lambda_f^2} (1 + f'(R_0)).$$  \hspace{1cm} (3.11)
Although this is different from the “entropy sum” of six dimensional charged (A)dS black hole, for the modification from $f(R)$ gravity, it has the same $\Lambda_f$-dependence and $k$-dependence, and $M$-independence. When $f(R) = $ constant, the theory reduces to Einstein gravity and the “entropy sum” is equal to Eq.(2.16). There are some unphysical “complex horizons” when $k = 0$, as the “entropy sum” of all flat horizons is vanishing. Still, the “entropy sum” in AdS and dS spacetime, and in the spacetime with the spherical and hyperbolic horizons share the same “entropy sum”, respectively.

To summarize briefly, including the necessary effect of the un-physical “virtual” horizon, we find the “area sum” in $f(R)$ gravity behaviour the same as that in Einstein gravity. Besides, the “entropy sum” depends on the cosmological constant and the topology of the horizons, does not depend on the conversed charges $M$ and $Q$, in four and six dimensions, while they are always vanishing in odd dimensions.

4 “Entropy sum” of (A)dS black hole in Gauss-Bonnet gravity

The action of the $d$-dimensional Einstein-Gauss-Bonnet-Maxwell-(A)dS theory has the form

$$L = \frac{1}{16\pi} \int d^d x \sqrt{-g} [R - 2\Lambda + \alpha (R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) - 4\pi F_{\mu\nu} F^{\mu\nu}], \quad (4.1)$$

where $\alpha$ is the Gauss-Bonnet coupling constant and the cosmological constant is $\Lambda = \pm \frac{(d-1)(d-2)}{2l^2}$ for (A)dS spacetime, $F_{\mu\nu}$ is the Maxwell field strength. The $d$-dimensional static charged Gauss-Bonne-AdS black hole solution for the above action is described by the same metric form as Eq.(2.2) with the horizon function is

$$V(r) = k + \frac{r^2}{2\tilde{\alpha}} \left( 1 - \sqrt{1 + \frac{64\pi \tilde{\alpha} M}{(d-2)r^{d-1}} - \frac{2\tilde{\alpha} Q^2}{(d-2)(d-3)r^{2d-4}} + \frac{8\tilde{\alpha} \Lambda}{(d-1)(d-2)} } \right), \quad (4.2)$$

where $\tilde{\alpha} = (d-3)(d-4)\alpha$, $M$ is the black hole mass, $Q$ is related to the charge of the black hole. The entropy has the form as

$$S = \frac{r^{d-2}}{4} \left( 1 + \frac{2(d-2)k\tilde{\alpha}}{(d-4)r_*^2} \right), \quad (4.3)$$

In odd dimensions, once again, the horizon function Eq.(4.2) is a function of $r^2$ and the entropy $S_i$ is an odd order function of $r_i$, which result in some pairs of root $r_i$ and $-r_i$ and a pair of vanishing entropy, i.e. $S(r_i) + S(-r_i) = 0$. Hence $\Sigma_i S_i = 0$, the “entropy sum” of odd dimensional (A)dS black hole in Gauss-Bonnet gravity are vanishing.
In six dimensions, the horizon function Eq. (3.8) reduces to

\[ V(r) = k + \frac{r^2}{2\tilde{\alpha}} \left( 1 - \sqrt{1 + \frac{16\pi\tilde{\alpha} M}{r^5} - \frac{\tilde{\alpha} Q^2}{6r^8} + \frac{2\tilde{\alpha} \Lambda}{5}} \right), \]

for charged black hole; \hspace{1cm} (4.4)

\[ V(r) = k + \frac{r^2}{2\tilde{\alpha}} \left( 1 - \sqrt{1 + \frac{16\pi\tilde{\alpha} M}{r^5} + \frac{2\tilde{\alpha} \Lambda}{5}} \right), \]

for neutral black hole. \hspace{1cm} (4.5)

which result in the horizons as the root of the following polynomials

\[ v(r) = 12\Lambda r^8 - 120r^6k - 120r^4k^2\tilde{\alpha} + 480\pi Mr^3 - 5Q^2, \]

for charged black hole; \hspace{1cm} (4.6)

\[ v(r) = \Lambda r^5 - 10kr^3 - 10r k^2 \tilde{\alpha} + 40\pi M, \]

for neutral black hole. \hspace{1cm} (4.7)

The number of horizons we focus on is \( D = 8 \) for charged black hole and \( D = 5 \) for charged black hole. The entropy of horizons have the form

\[ S_i = \frac{r_i^4}{4} + k\tilde{\alpha}r_i^2 \]

We find the equality Eq. (2.7) and Eq. (2.14) still hold with the different \( D \) here. Then for both charged and neutral black hole, using the Vieta theorem on the new horizon function Eq. (4.6) and Eq. (4.7), we obtain

\[ \sum_{i=1}^{D} r_i = 0, \sum_{1 \leq i < j < m < n \leq D} r_ir_jr_mr_n = -\frac{10k^2\tilde{\alpha}}{\Lambda}, \]

\[ \sum_{1 \leq i < j \leq D} r_ir_j = -\frac{10k}{\Lambda}. \]

Thus we get

\[ \sum_{i=1}^{D} r_i^4 = \frac{200k^2}{\Lambda^2} + \frac{40k^2\tilde{\alpha}}{\Lambda}, \sum_{i=1}^{D} r_i^2 = \frac{20k}{\Lambda}. \]

(4.9)

which lead to the “entropy sum” of all horizons

\[ \sum_{i=1}^{D} S_i = \frac{1}{4} \left( \sum_{i=1}^{D} r_i^4 \right) + k\tilde{\alpha} \left( \sum_{i=1}^{D} r_i^2 \right) = \frac{50k^2}{\Lambda^2} + \frac{30k^2\tilde{\alpha}}{\Lambda}. \]

(4.10)

Obviously, this is different from the “entropy sum” of six dimensional charged (A)dS black hole in Einstein gravity and \( f(R) \) gravity. But the “entropy sum” has the same \( \Lambda \)-dependence and \( k \)-dependence, and \( M \)-independence and \( Q \)-independence. Besides, It also depends on \( \alpha \). When \( \alpha = 0 \), the theory reduces to Einstein gravity and the “entropy sum” is proportional to Eq. (2.16). There are some un-physical “complex
“Entropy sum” when $k\Lambda < 0$ and $k = 0$, as the sum of $r_i^2$ of all flat horizons is negative and zero respectively. However, this “entropy sum” in AdS and dS spacetime, and in the spacetime with the spherical and hyperbolic horizons do not share the same “entropy sum”, which is different from that of Einstein gravity and $f(R)$ gravity.

Finally, to give a brief summary, including the necessary effect of the un-physical “virtual” horizon, we conclude that the “entropy sum” depends on the cosmological constant, the Gauss-Bonnet coupling constant and the topology of the horizons, does not depend on the conversed charges $M$ and $Q$, in six dimensions. The extra background field constant dependence is also appears in the spacetime with other matter source as shown in [17]. Again, the “entropy sum” are vanishing in odd dimensions.

5 “Entropy sum” of black hole in gauged supergravity theory

“Entropy sum” relation is also valid in supergravity theory. Let us check it explicitly.

In the four dimensions, a gauged pairwise equal charges solution which is constructed in [40], has the metric

$$ds^2_4 = -\Delta_r \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + W \left( \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{W} \left( adt - \frac{r_1 r_2 + a^2}{\Xi} d\phi \right)^2 ,$$

(5.1)

where

$$\Delta_r = r^2 + a^2 - 2mr + g^2 r_1 r_2(r_1 r_2 + a^2) ,$$

$$\Delta_\theta = 1 - g^2 a^2 \cos^2 \theta ,$$

$$W = r_1 r_2 + a^2 \cos^2 \theta ,$$

and

$$r_1 = r + 2m \sinh^2 \delta_1 \quad r_2 = r + 2m \sinh^2 \delta_2 .$$

As discussed in [41], when the parameter $\delta_i = 0$, the metric (5.1) should reduce to the four dimensional Kerr-AdS metric\(^1\). Solve the equation $\Delta_\theta(r) = 0$ to get the four horizons. And the entropy for each horizon is $S(r_\alpha) = \frac{A(r_\alpha)}{4}$, where the area of each horizon is

$$A(r_\alpha) = \frac{4\pi(r_1\alpha r_2\alpha + a^2)}{\Xi} .$$

(5.2)

With no difficulty we get

$$\sum_{\alpha=1}^4 A(r_\alpha) = -\frac{8\pi}{g^2} ,$$

(5.3)

\(^1\)The metric in [40] is not suitable here.
and the entropy sum
\[ \sum_{\alpha=1}^{4} S(r_{\alpha}) = \frac{2\pi}{g^{2}}. \]  
(5.4)

**In the six-dimensions**, a gauged charged rotating black hole is considered in [41]. The metric can be presented concisely in Jacobi-Carter coordinates. However, when we concern with thermodynamic quantities, we use angular velocities measured with respect to a non-rotating frame at infinity and move to Boyer–Lindquist time and azimuthal coordinates, which is discussed intensively in [41]. For simplification, we do not intend to write the details here. The horizon function is
\[ f(r) = (r^{2} + a^{2})(r^{2} + b^{2}) + g^{2}[r(r^{2} + a^{2}) + q][r(r^{2} + b^{2}) + q] - 2mr, \]
(5.5)
and the entropy for each horizon is
\[ S(r_{i}) = \frac{2\pi^{2}((r_{i}^{2} + a^{2})(r_{i}^{2} + b^{2}) + qr_{i})}{3\Xi_{a}\Xi_{b}}. \]
(5.6)
Rewrite \( f(r) = 0 \) to a polynomial function
\[
g^{2}r^{6} + (1 + g^{2}a^{2} + g^{2}b^{2})r^{4} + 2g^{2}qr^{3} + (g^{2}a^{2}b^{2} + a^{2} + b^{2})r^{2}
+ (-2m + g^{2}a^{2}q + g^{2}b^{2}q)r + a^{2}b^{2} + g^{2}q^{2} = 0.
\]
(5.7)
It has six roots at most, which correspond to six horizons, including un-physical ones. However, we do not need to solve them analytically. By using Vieta theorem, it is not difficult to compute
\[ \sum_{i=1}^{6} S(r_{i}) = \frac{4\pi^{2}}{3g^{4}}. \]
(5.8)
Both (5.4) and (5.8) only relate to gauge parameter \( g \), which can be interpreted as a cosmological constant in the position.

**In the odd dimensions**, unlike the Einstein-Maxwell, \( f(R) \) or Gauss-Bonnet theory, there is not a universal metric in arbitrary \( d \) dimensions. So we have to check explicitly. In \( d = 5 \) [42] and \( d = 7 \) [43], the “entropy sum” equals zero. It is the same as our conclusion above.

In these solutions, we note that there may exist more than one charge [40] or angular momentum [41]. However, they play no role in the entropy sum, which suggests again that the entropy sum is “universal” as we have anticipated.

## 6 Conclusions

As we have seen above, the “entropy sum” relation of (A)dS charged black holes in higher dimensional Einstein-Maxwell gravity, \( f(R) \) gravity depends only on the cosmological constant. For the case in Gauss-Bonnet gravity, there are the cosmological constant dependence and Gauss-Bonnet coupling constant dependence holding together.
We have taken the detail discussions in four, six and all odd dimensions, putting all the cases together, we conclude that the “entropy sum” shares the following properties:

1. It has got the cosmological constant dependence (and Gauss-Bonnet coupling constant dependence for Gauss-Bonnet gravity) in four and six dimensions. They are also dependent of the topology of the horizons in four and six dimensions;

2. It is always vanishing in odd dimensions;

3. It involves all possible bifurcating horizons including the “virtual” horizon, i.e., we consider all roots of the horizon function. Generically, this involves complex roots, and hence un-physical horizons. The same phenomenon has appeared in the discussion of universal entropy relation in Schwarzschild-de Sitter and Reissner-Nordström-de Sitter black holes [14].

4. The conversed charges $M$ (mass), $Q$ (charge from Maxwell field or supergravity) and $a$ (angular momentum) play no role in the “entropy sum” relation.

Generically, one may expect that the background constant dependence and topology of the horizons dependence do still hold in even dimensions. In addition, the solutions shown in this paper are all a cosmological constant included, with the reason that the asymptotical flat spacetime is expected to be a failed example (One can see [17] for that argument). Also, we have left out considering multi-rotations in higher dimensions, while the “entropy sum” does not depend on rotation in four dimensional Einstein gravity and Einstein-Weyl gravity. All these are left as future possible directions.

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