Berry Phase and Hannay’s Angle in a Quantum-Classical Hybrid System

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Berry phase, which had been discovered for more than two decades, provides us a very deep insight on the geometric structure of quantum mechanics. Its classical counterpart—Hannay’s angle—is defined if closed curves of action variables return to the same curves in phase space after a time evolution. In this paper, we study the Berry phase and Hannay’s angle in a quantum-classical hybrid system under the Born-Oppenheimer approximation. By quantum-classical hybrid system, we denote a composite system consists of a quantum subsystem and a classical subsystem. The effects of subsystem-subsystem couplings on the Berry phase and Hannay’s angle are explored. The results show that the Berry phase has been changed sharply by the couplings, whereas the couplings have small effect on the Hannay’s angle.

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I. INTRODUCTION

The concept of Berry phase has paved a new way for understanding quantum physics, it has attracted much attention since the Berry’s discovery [18] and found potential applications in fields ranging from chemistry to condensed matter physics. Shortly after Berry’s discovery, Simon gave a mathematical interpretation to this phase that it can be regarded as the holonomy in a Hermitian line bundle since the adiabatic theorem naturally defines a connection in such a bundle [4]. After these remarkable discoveries, Hannay found that this geometrical phase not only exist in quantum system but also in classical world [5]. Analogous with the Berry phase, the angle variable of classical integrable systems [6] acquires an additional angle shift as the system slowly cycles in phase space. This angle shift is called Hannay’s angle. It was later proved by Berry that the geometric phase and Hannay’s angle possess a natural relation under semiclassical approximation [7]. As a matter of course, this quantum-classical correspondence gives rise to many impressive explorations [8–10].

It is known that a quantum Hilbert space carries the same structure of a classical phase space [11, 12]. By this virtue, a quantum system can be treated like a classical system without loss of physics [11, 12]. Particularly in Ref. [12], the authors introduced a general framework for testing nonlinear quantum mechanics. In this generalized theory, the elements of quantum mechanics, such as wave functions, observable, symmetries and time evolution, all can be treated classically. Based on this instructive work, some interesting efforts have been devoted to nonlinear quantum system and quantum-classical hybrid system [13–15]. Here come the questions: Since quantum mechanics can be put into the framework of classical mechanics, can Berry phase be presented into the form of Hannay’s angle? What is the Berry phase or Hannay’s angle of the quantum-classical hybrid system? How does the subsystem-subsystem couplings affect the Hannay’s angle and the Berry phase? We shall shed light on these questions in this paper.

The paper is organized as follows. In Sec. II, we first represent a quantum system in the framework of classical theory by Weinberg’s method [12], then we calculate the Hannay’s angle of the quantum system. Next we compare this Hannay’s angle with the Berry phase of the original quantum system and find that the Hannay’s angle and Berry phase differ only in a sign [14]. An example is given to show this result. In Sec. III, we represent a quantum-classical hybrid system in classical mechanics based on the Born-Oppenheimer approximation, a unified one-form which can deduce both of the Berry phase and the Hannay’s angle is given. By this one-form, we calculated the Hannay’s angles and Berry phase and study the effect of subsystem-subsystem couplings on the Hannay’s angle. Finally, we conclude our results in Sec. IV.

II. BERRY PHASE AND HANNAY’S ANGLE

In quantum theory, observables are represented by Hermitian matrices F or real bilinear functions ⟨ψ|F|ψ⟩. Weinberg has generalized the representation into real non-bilinear functions f(ψ, ψ∗) to include nonlinearity. Let {φn} be an orthonormal basis of Hilbert space, the Schrödinger equation can be rewritten as:

\[ i\hbar \frac{d\psi_n}{dt} = \frac{\partial h}{\partial \psi_n^*}, \]

with |ψ⟩ = \[\sum \psi_n |\phi_n⟩, \psi = (ψ₁, …, ψ_n, …, ψ_N)T \] and h(ψ, ψ∗) is the total energy of the system. If we decompose ψn into real and imaginary parts ψn = (qn + ipn)/\[2\hbar\], the schrödinger equation and its complex conjugate can be written as Hamiltonian canonical equations.
The Hamiltonian function $h(\psi, \psi^*)$ can be transformed into $h(q, p)$. It is amazing to note that the quantum hermitian structure becomes the symplectic structure of classical mechanics, $dp_n \wedge dq_n = i\hbar \psi_n^* \wedge d\psi_n$.

Now, let us turn to the first question. Consider a quantum system with $N$ levels, whose Hamiltonian function $h(q, p; \mathbf{X})$ depends on a set of slowly varying parameters $\mathbf{X} = (X_1, Y_1, \ldots)$ and its quantum state $|\psi\rangle$.

By the procedure mentioned above, i.e., by decomposing $|\psi\rangle$ with basis $\{|n\rangle\}$, $|\psi\rangle = \sum_n \psi_n(t)|n\rangle$, and setting $\psi_n(t) = (q_n(t) + ip_n(t))/\sqrt{2\hbar}$, we can write the Hamiltonian function in terms of "position variable $q$" and "momentum variable $p" as $h(p(t), q(t))$. Of course, the state vector $|\psi\rangle$ can also be expanded taking the Hamiltonian eigenstates $|E_k(\mathbf{X})\rangle$ as a basis, $|\psi\rangle = \sum \psi_k(t)|E_k(\mathbf{X})\rangle$.

By the adiabatic theorem, the occupation probability of each eigenstate $|\psi_k(t)\rangle$ remains unchanged in the adiabatic limit. One then can introduce a new pair of variables $(\theta, \mathbf{I})$ by

$$\psi_k(t) = \sqrt{\frac{I_k}{\hbar}} e^{-i\theta_k},$$

and write the Hamiltonian function as

$$\hat{h} = \hat{h}(\mathbf{I}; \mathbf{X}) = \sum_k E_k(\mathbf{X}) I_k / \hbar,$$

where $E_k(\mathbf{X})$ is the eigenvalue of the Hamiltonian $\hat{H}$ with corresponding eigenstate $|E_k(\mathbf{X})\rangle$. It has been proved that the two new variables $\theta$ and $\mathbf{I}$ satisfy the same canonical equations as the angle-action variables in classical mechanics.

$$\dot{\theta}_k = \frac{\partial \hat{h}}{\partial I_k}, \quad \dot{I}_k = 0.$$

So far, the quantum Hamiltonian $\hat{H}$ was transformed into classical Hamiltonian function $\hat{h}(\mathbf{I}; \mathbf{X})$, and the quantum unitary transformation $|E_k(\mathbf{X})\rangle = \sum_n C_{kn}(\mathbf{X})|n\rangle$ turns out to be a classical canonical transformation $(q, p) \rightarrow (\theta, \mathbf{I})$.

$$p_n = \sum_k \sqrt{2I_k} \left[ \cos \theta_k \text{Im}(C_{kn}(\mathbf{X})) - \sin \theta_k \text{Re}(C_{kn}(\mathbf{X})) \right],$$

$$q_n = \sum_k \sqrt{2I_k} \left[ \cos \theta_k \text{Re}(C_{kn}(\mathbf{X})) + \sin \theta_k \text{Im}(C_{kn}(\mathbf{X})) \right].$$

According to Berry’s theory, the Hannay’s angle of the system can be written as

$$\Delta \theta_k(\mathbf{I}; \mathbf{X}) = -\frac{\partial}{\partial I_k} \oint A_H(\mathbf{I}; \mathbf{X}),$$

where

$$A_H(\mathbf{I}; \mathbf{X}) = \langle p(\theta, \mathbf{I}; \mathbf{X}) d\mathbf{X} q(\theta, \mathbf{I}; \mathbf{X}) \rangle_\theta$$

is the angle one-form for Hannay’s angle. The angular brackets $\langle \cdots \rangle_\theta$ denote an averaging over all angles $\theta$, and $d\mathbf{X}$ is defined as $d\mathbf{X} F(\mathbf{X}) = \partial F(\mathbf{X}) / \partial \mathbf{X}$.

Substituting Eq. (6) into Eq. (8), we obtain

$$A_H = \sum_k I_k \sum_n [iC_{kn}^*(\mathbf{X}) d\mathbf{X} C_{kn}(\mathbf{X})]$$

$$= \sum_k iI_k (E_k(\mathbf{X})|d\mathbf{X} E_k(\mathbf{X})) = \sum_k iI_k A_B(k; \mathbf{X}).$$

We note that $A_B(k; \mathbf{X})$ is nothing but the one-form for Berry phase, this means that the Hannay’s angles exactly equal to the minus Berry phases of the original quantum system.

$$\Delta \theta_k(\mathbf{I}; \mathbf{X}) = -\frac{\partial}{\partial I_k} \oint A_H = -i \oint A_B(k; \mathbf{X}) = -\gamma_k(\mathbf{C}).$$

The appearance of the minus is because the angle variables of the effective classical system correspond to the opposite numbers of the phases in Eq. (3). Therefore, we can choose $A(\mathbf{I}; \mathbf{X}) = A_H$ to be a general one-form that

$$\Delta \theta_k = -\gamma_k = -\frac{\partial}{\partial I_k} \oint A(\mathbf{I}; \mathbf{X}).$$

To shed more light on the result, we consider the adiabatic evolution of a spin-half particle with magnetic moment $\mu$ in an external magnetic field $\mathbf{B}$. The Hamiltonian reads

$$\hat{H} = -\mu \mathbf{\hat{S}} \cdot \mathbf{B},$$

where $\mathbf{\hat{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ are Pauli Matrices. As aforementioned, if we choose the two spin eigenstates $|\pm\rangle$ as the basis, the Hamiltonian in Eq. (12) can be transformed into a Hamiltonian function

$$h(p, q; \mathbf{B}) = \frac{1}{\hbar} [(q_1 q_2 + p_1 p_2) B_1 + (p_1 q_2 - p_2 q_1) B_2$$

$$+ \frac{1}{2} (p_2^2 + q_2^2 - p_1^2 - q_1^2) B_3],$$

with $|\psi\rangle = \psi_1|\rangle + \psi_2|\rangle$ and $\psi_j = (q_j + ip_j)/\sqrt{2\hbar}, (j = 1, 2)$. The canonical variables ($q, p$) satisfy the normalization condition

$$\sum_{j=1}^2 (p_j^2 + q_j^2) = 2\hbar.$$

It is interesting to note that by defining a vector, $\mathbf{S} = (S_1, S_2, S_3)$, the Hamiltonian function can be written as,

$$h(\mathbf{S}; \mathbf{B}) = -\mu \mathbf{S} \cdot \mathbf{B}.$$
where
\[
\begin{aligned}
S_1 &= (q_1 q_2 + p_1 p_2)/\hbar, \\
S_2 &= (p_1 q_2 - p_2 q_1)/\hbar, \\
S_3 &= (p_2^2 + q_2^2 - p_1^2 - q_1^2)/(2\hbar).
\end{aligned}
\]

The normalization condition in terms of \( S \) is \( S^2 = S_1^2 + S_2^2 + S_3^2 = 1 \), and their Poisson bracket has a relation with the quantum commutator as \( [11] \)
\[
\{ S_i, S_j \} = 2\epsilon_{ijk} S_k/\hbar = \frac{1}{i\hbar} \langle \psi | [\hat{\sigma}_i, \hat{\sigma}_j] | \psi \rangle. 
\]

Moreover, if we choose \( |\pm\rangle \) as the basis, \( S \) is nothing but the Stokes parameters which span the Poincare sphere,
\[
\begin{aligned}
I &= |\psi_2|^2 + |\psi_1|^2 = S^2 = 1, \\
U &= 2\text{Re}(\psi_2^* \psi_1^*) = S_1 S, \\
V &= 2\text{Im}(\psi_2^* \psi_1^*) = S_2 S, \\
Q &= |\psi_2|^2 - |\psi_1|^2 = S_3 S.
\end{aligned}
\]

We now move to calculate the Hannay’s angle. Since the Hamiltonian in Eq. \([12] \) has two eigenstates
\[
|E_1\rangle = \sqrt{B + B_3/2B} |+\rangle + \frac{B_1 + iB_2}{\sqrt{2B(B + B_3)}} |-\rangle,
\]
\[
|E_2\rangle = -\sqrt{B - B_3/2B} |+\rangle + \frac{B_1 + iB_2}{\sqrt{2B(B - B_3)}} |-\rangle,
\]
with eigenenergies \(-\mu B \) and \( \mu B \), respectively, where \( B = \sqrt{B_1^2 + B_2^2 + B_3^2} \). The canonical transformation \( (q, p) \rightarrow (\theta, I) \) and the transformed Hamiltonian function can be written as
\[
q_1 = \sqrt{2I_1} \left[ \frac{B_2 \cos \theta_1}{\sqrt{2B(B + B_3)}} + \frac{B_2 \sin \theta_1}{\sqrt{2B(B + B_3)}} \right],
\]
\[
q_2 = \sqrt{2I_2} \left[ \frac{B_1 \cos \theta_2 - B_2 \sin \theta_2}{\sqrt{2B(B - B_3)}} \right],
\]
\[
p_1 = \sqrt{2I_1} \left[ \frac{B_2 \cos \theta_1 - B_1 \sin \theta_1}{\sqrt{2B(B + B_3)}} \right],
\]
\[
p_2 = \sqrt{2I_2} \left[ \frac{B_2 \cos \theta_2 - B_1 \sin \theta_2}{\sqrt{2B(B - B_3)}} \right],
\]
\[
\hbar(I; B) = \mu B (I_2 - I_1),
\]
with \( q = (q_1, q_2), p = (p_1, p_2), \theta = (\theta_1, \theta_2) \) and \( I = (I_1, I_2) \). Therefore, we obtain the angle one-form by Eq. \([3] \),
\[
A = \frac{B_2 dB_1 - B_1 dB_2}{2B(B + B_3)} I_1 + \frac{B_2 dB_1 - B_1 dB_2}{2B(B - B_3)} I_2.
\]

The Hannay’s angles can thus be obtained by Eq. \([11] \)
\[
\Delta \theta_1 = -\oint \frac{B_2 dB_1 - B_1 dB_2}{2B(B + B_3)},
\]
\[
\Delta \theta_2 = -\oint \frac{B_2 dB_1 - B_1 dB_2}{2B(B - B_3)},
\]
which differ from the Berry phases for the original quantum Hamiltonian \([1] \) only by a sign.

### III. BERRY PHASE AND HANNAY’S ANGLE IN HYBRID SYSTEM

Based on formalism in the last section, we now turn to the second question risen in the Introduction. Consider a hybrid system consisting of a classical and quantum subsystem, the Hamilton function of this quantum-classical hybrid system under Born-Oppenheimer approximation can be written as \([13, 19] \)
\[
H_{\text{hybrid}} = \langle \psi | \hat{H}_1(Q, X_1) | \psi \rangle + H_2(P, Q; X_2),
\]
where \( |\psi\rangle \) is the state of the fast quantum subsystem. The Hamilton function \( H_2 \) describes a slow classical subsystem with momentum \( P \) and coordinate \( Q \), \( X_1 = (X_1, Y_1, \ldots) \) and \( X_2 = (X_2, Y_2, \ldots) \) are slowly varying parameters of the quantum and classical subsystems, respectively. The subsystem-subsystem coupling is included in \( H_1(Q, X_1) \). Following the procedure given in the last section, we first choose a basis, then expand the state \( |\psi\rangle \) in this basis, next decompose the expansion coefficients into real parts \( q \) and imaginary parts \( p \), finally represent the hybrid system by a classical Hamiltonian function,
\[
H = H_1(p, q; Q, X_1) + H_2(P, Q; X_2).
\]
As known, the Hamiltonian of quantum subsystem \( H_1(p, q; Q, X_1) \) can be transformed into \( H_1(\theta, I; Q, X_1) \) by a canonical transformation \( (q, p) \rightarrow (\theta, I) \). But the new Hamiltonian \( \hat{H}_1 \) differs from the old one, it takes \([3, 7] \),
\[
\hat{H}_1(\theta, I; Q, X_1) = \mathcal{H}_1(I; Q, X_1) + \frac{\partial S}{\partial t} + \left( \frac{\partial \mathcal{S}}{\partial Q} - p \frac{\partial \mathcal{Q}}{\partial Q} \right)
\]
\[
+ \hat{X}_1 \cdot \left( \frac{\partial \mathcal{S}}{\partial X_1} - p \frac{\partial \mathcal{Q}}{\partial X_1} \right),
\]
where
\[
\mathcal{H}_1(I; Q, X_1) = H_1(p(\theta, I; Q, X_1), q(\theta, I; Q, X_1); Q, X_1),
\]
and \( S(q, I; Q, X_1) \) is the generating function of the transformation and the single-valued function \( \mathcal{S} \equiv S(q(\theta, I; Q, X_1), I; Q, X_1) \) is introduced to give an explicit form for \( \hat{H}_1 \). Since the variables \( Q \) can be treated
as slowly varying parameters like $X_1$, an average over $\theta$ may be taken to approximate to Eq. (25) \[6\],
\[
\langle \hat{H}_1 \rangle_\theta = \mathcal{H}(I; Q, X_1) - \hat{Q} \cdot \left( \frac{\partial \hat{Q}}{\partial \hat{Q}} \right)_\theta - \hat{X}_1 \cdot \left( \frac{\partial \hat{Q}}{\partial \hat{X}_1} \right)_\theta,
\]
where the angular brackets $\langle \cdots \rangle_\theta$ denote an averaging over all angles $\theta$, and the terms $\hat{Q} \cdot \left( \frac{\partial \hat{Q}}{\partial \hat{Q}} \right)_\theta$ and $\hat{X}_1 \cdot \left( \frac{\partial \hat{X}_1}{\partial \hat{X}_1} \right)_\theta$ are dropped for zero contribution to the equations of motion. Thus, the Hamiltonian of total system can be written as
\[
H_{av} = \mathcal{H}(I; J, X_1, X_2) + H_2(P, Q; X_2)
\]
where the first two terms is the effective Hamiltonian under Born-Oppenheimer approximation and the latter two terms are related to the Berry phase of the quantum subsystem, because the quantum one-form is defined as
\[
\langle \mathcal{Q} \rangle_\theta = \int \mathcal{Q} \cdot d\lambda,
\]
and the latter two terms are related to the Berry phases of angle-action variables via the canonical transformation. According to Berry’s theory \[7\], we introduce the canonical transformation $(q, p) \rightarrow (\phi, J)$, where $|\psi\rangle = (\cos \phi, \sin \phi, 0)$ with magnetic moment $\mu$ and coupling constant $\lambda$, and $X = (X, Y, Z)$ are the time-dependent parameters of the classical subsystems. The total state $|\psi\rangle$ is defined by the spin eigenstates $|\pm\rangle$. I.e., $|\psi\rangle = (|q_- + ip_+\rangle - (q_+ + ip_-)|\rangle)/\sqrt{2\mu}$. It is easy to get the eigenfunctions $|E_\pm\rangle$ and the corresponding eigenvalues for $\hat{H}_1$,
\[
E_\pm = \pm \mu B_{tot},\n\]
where $B_{tot} \equiv \sqrt{B^2 + \lambda^2 Q^2}$ and $\cos \Theta \equiv \lambda Q / B_{tot}$. Following the proposed procedures, we transform the quantum Hamiltonian $H_1$ into a classical form with a canonical transformation $(q_\pm, p_\pm) \rightarrow (\theta_\pm, I_\pm)$, where $|\psi\rangle = \sqrt{I_+/|e^{-i\theta_+}|E_+} + \sqrt{I_-/e^{-i\theta_-}|E_-}$. After averaging over the angles $\theta_\pm$, we obtain
\[
H_{av} = \mu B_{tot}(I_+ - I_-) + \frac{A_1(I; Q, B)}{dt} + 1/2(XQ^2 + 2YPQ + ZP^2),
\]
where
\[
A_1(I; Q, B) = -1/2 [I_+(1 - \cos \Theta) + I_- (1 + \cos \Theta)] d\varphi
\]
is the phase one-form for the quantum subsystem. So far, the Hamiltonian is fully classical and only contains variables of classical system. It is difficult to derive the action variable for this Hamiltonian, because the dependence of $B_{tot}$ on $Q$ is complicated. However, in the weak coupling limit $\lambda \ll B_{tot}$, the problem becomes easy. Expanding $B_{tot}$ to the second order in $\lambda$,
\[
B_{tot} \approx B + \frac{\lambda^2 Q^2}{2B},
\]
we obtain the Hamiltonian in the weak coupling limit,
\[
H_{av} \approx \mu B(I_+ - I_-) + \frac{\mu \lambda^2 Q^2 (I_+ - I_-)}{2B} + \frac{A_1(I; Q, B)}{dt} + 1/2(XQ^2 + 2YPQ + ZP^2).
\]
According to Berry’s theory \[7\], we introduce the canonical transformation $(Q, P) \rightarrow (\phi, J)$
\[
Q = \frac{2ZJ}{\Omega} \cos \phi,\n\]
\[
P = -\frac{2ZJ}{\Omega} \cos \phi \left( \frac{Y}{Z} \cos \phi + \frac{\Omega}{Z} \sin \phi \right),
\]
with $\Omega \equiv \{(X + \mu \lambda^2 (I_+ - I_-)/B) | Z - Y^2 | \}^{1/2}$. Substituting Eq. (38) into Eq. (33) and averaging it over $\phi$, we obtain the Hamiltonian for the hybrid system,

$$\langle H_{av} \rangle = \mu B (I_+ - I_-) + J \Omega + \frac{A(I, J; B, X)}{dt},$$

where the general one-form for the hybrid system is given by $(\cos \Theta) = 0$,

$$A(I, J; B, X) = \frac{(I_+ + I_-)}{2} d\phi - \frac{Y J}{2Z} \left( \frac{Z}{\Omega} \right).$$

Thus the Berry phases and Hannay’s angle can be given uniformly by a straightforward calculation

$$\gamma_{\pm} = \int \left[ \frac{1}{2} d\phi \pm \frac{\mu X^2 J^2}{4 \Omega} \right]$$

$$\Delta \phi = \int \frac{Y}{2Z} d\left( \frac{Z}{\Omega} \right).$$

The range of integration is determined by the common period of $X$ and $B$. It is easy to find that when $A = 0$, the Berry phases are the solid angle $\pi$ on Bloch sphere. The subsystem-subsystem couplings not only adds a correction to the Berry phases, but also change range of integration. Whereas the interaction gives the Hannay’s angle a modification $\omega \equiv (X Z - Y^2)^{1/2} \to \Omega$ through $\Omega$.

The second example is a quantum harmonic oscillator coupled with a classical one. The mean-field Hamiltonian (40) for this hybrid system is

$$\hat{H} = \langle \psi | \hat{H}_1 | \psi \rangle + \frac{1}{2} (X_2 Q^2 + 2 Y_2 P Q + Z_2 P^2),$$

where $\hat{H}_1 = \frac{1}{2} [X_1 q^2 + Y_1 (\hat{p} q + \hat{q} \hat{p}) + Z_1 \hat{p}^2] + K \hat{q} Q$ includes the free Hamiltonian of the quantum subsystem and the subsystem-subsystem coupling, $K$ is the coupling constant and $X_1 = (X_1, Y_1, Z_1)$, $X_2 = (X_2, Y_2, Z_2)$ are the time-dependent parameters of the two subsystems. Similar with Ref. [7], the eigenfunctions and eigenvalues of $\hat{H}_1$ are

$$E_n = \left( n + \frac{1}{2} \right) h \omega - \frac{Z_1 K^2 Q^2}{2 \omega^2},$$

$$\psi_n = \sqrt{\alpha} \chi_n (\alpha \left( q + \frac{K Z_1 Q}{\omega^2} \right) \exp \left( -i Y_1 q^2 / 2 Z_1 h \right),$$

with $\omega = \sqrt{X_1 Z_1 - Y_1^2}$, $\alpha = \frac{\sqrt{\omega h}}{Z_1}$ and the normalized Hermite functions $\chi_n (\xi)$. By the transformation, $(q, p) \to (\theta, I)$, we obtain the averaged Hamiltonian by averaging over the angles $\theta$,

$$H_{av} = \sum_n (n + 1/2) I_n \omega - \frac{Z_1 K^2 Q^2}{2 \omega^2} + \frac{A(I, X_1, Q)}{dt},$$

$$+ \frac{1}{2} (X_2 Q^2 + 2 Y_2 P Q + Z_2 P^2),$$

where

$$A(I, X_1, Q) = \sum_n I_n \left[ \frac{(2n + 1) Z_1}{4 \omega} + \frac{K^2 Z_1^2 Q^2}{2 \omega^4} \right] d \left( \frac{Y_1}{Z_1} \right)$$

is the phase one-form for the quantum subsystem. Note that the action variables $I$ are invariant, we can treat them as constants. After the canonical transformation $(Q, P) \to (\phi, J)$ (elliptic case)

$$Q = \left( \frac{2 Z_1 J}{\Omega} \right)^{1/2},$$

$$P = - \left( \frac{2 Z_1 J}{\Omega} \right)^{1/2} \left( \frac{Y_2}{Z_2} \cos \phi + \frac{\Omega}{Z_2} \sin \phi \right),$$

and substituting Eq. (40) into Eq. (41) as well as averaging over $\phi$, the Hamiltonian of the hybrid system becomes

$$\langle H_{av} \rangle = \sum_n \left( n + \frac{1}{2} \right) I_n \omega + J \Omega + \frac{A(I, J; X_1, X_2)}{dt},$$

with $\Omega = \left( \frac{(\omega^2 X_2 - Z_1 K^2 Z_2)}{\omega^2} - Y_2 \right)^{1/2}$, and the general one-form for the hybrid system is

$$A(I, J; X_1, X_2) = \sum_n I_n \left[ \frac{(2n + 1) Z_1}{4 \omega} + \frac{K^2 Z_1^2 Z_2 J}{2 \omega^4 \Omega} \right] d \left( \frac{Y_1}{Z_1} \right)$$

Therefore the Berry phases and Hannay’s angle can be given by

$$\gamma_n = \int \left[ \frac{(2n + 1) Z_1}{4 \omega} + \frac{K^2 Z_1^2 Z_2 J}{2 \omega^4 \Omega} \right] d \left( \frac{Y_1}{Z_1} \right),$$

$$\Delta \phi = \int \frac{Y_2}{2Z_2} d \left( \frac{Z_2}{\Omega} \right) - \frac{K^2 Z_1^2 Z_2 J}{2 \omega^4 \Omega} d \left( \frac{Y_1}{Z_1} \right).$$

The limits of the integrals are decided by the common period of $X_1$ and $X_2$.

Now we take a specific choice of the periodic parameters to see an exact result of our theory. Set

$$X_1 = A_1 \mu_1 (1 + \epsilon \cos (\omega_1 t))$$

$$Y_1 = - A_1 \epsilon \sin (\omega_1 t)$$

$$Z_1 = \frac{A_1}{\mu_1} (1 - \epsilon \cos (\omega_1 t))$$

$$X_2 = A_2 \mu_2 (1 + \epsilon \cos (\omega_2 t))$$

$$Y_2 = - A_2 \epsilon \sin (\omega_2 t)$$

$$Z_2 = \frac{A_2}{\mu_2} (1 - \epsilon \cos (\omega_2 t))$$

where $\omega_1, \omega_2$ are the frequencies of the parameters, $\epsilon$ is a dimensionless constant and the units of $A$ and $\mu$ are $s^{-1}$ and $kg/s$, respectively. If the frequency ratio $\omega_1/\omega_2$ is rational, $X_1$ and $X_2$ have a common period $T$. After
a straightforward calculation, we obtain the phases and angle in Eq. (19) as
\[
\gamma_n = \gamma_{n0} + \gamma_I, \\
\Delta \phi = \Delta \phi_0 + \Delta \phi_I.
\] (51)

The noninteracting phases \(\gamma_{n0}\) and angle \(\Delta \phi_0\) are
\[
\gamma_{n0} = \frac{(2n + 1)(1 - \sqrt{1 - \epsilon^2})T \omega_1}{4 \sqrt{1 - \epsilon^2}}, \\
\Delta \phi_0 = -\int_0^T \left[ \frac{\omega_2 \epsilon^2 \sin^2(\omega_2 t) dt}{2 \Omega (1 - \epsilon \cos(\omega_2 t))} + \frac{\epsilon A_2 \sin(\omega_2 t) d\Omega}{2 \Omega^2} \right],
\] (52)
and the coupling terms follow
\[
\gamma_I = \int_0^T -\epsilon \omega_1 A_2 D^2 J [1 - \epsilon \cos(\omega_2 t)] [1 - \epsilon \cos(\omega_1 t)] dt \\
= \frac{J}{K} \Delta \phi_I,
\] (53)
with the definition \(D \equiv K/(\sqrt{2} \mu_1 \mu_2 A_1 A_2 (1 - \epsilon^2))\) and \(\Omega = A_2 \sqrt{1 - \epsilon^2 - 2D^2}(1 - \epsilon \cos(\omega_1 t))(1 - \epsilon \cos(\omega_2 t))\). If we take the limit \(D \ll \sqrt{1 - \epsilon^2}, \Omega \approx A_2 \sqrt{1 - \epsilon^2}\) can be treated as a constant, \(\gamma_{n0}\) and \(\Delta \phi_0\) then satisfy the relation,
\[
\gamma_{n0} \approx -\left( n + \frac{1}{2} \right) \frac{\omega_1 \Delta \phi_0}{\omega_2},
\] (54)
and the coupling Berry phase and Hannay’s angle can be approximated as
\[
\gamma_I \approx \frac{\epsilon^2 A_2 J D^2 T \omega_1}{h A_1 (1 - \epsilon^2) \sqrt{1 - \epsilon^2}}, \\
\Delta \phi_I \approx -\frac{\epsilon^2 A_2 D^2 T \omega_1}{A_1 (1 - \epsilon^2) \sqrt{1 - \epsilon^2}}.
\] (55)

Considering the elliptic condition \((1 - \epsilon^2 - 2D^2(1 + \epsilon)^2 > 0)\), we perform numerical calculation for Eq. (52) and (53) with \(\epsilon = \sqrt{3}/2, A_1/A_2 = 10^8\) and \(J/h = 10^{13}\). The results are presented in Fig. (1) and Fig. (2).

Fig. (1) shows how the Berry phase of the ground state changes with \(K\). The original \(0 + \delta\) on the horizontal axis denotes a infinitely small positive number. Because when \(K = 0\), there is no interaction between the two subsystems, the range of integration of the phases and the angle are determined by the period of \(X_1\) and \(X_2\), respectively, rather than their common period. The relation between Berry phases and Hannay’s angle \(\gamma_{n0} = (2n + 1) T \omega_1 \approx -\left( n + \frac{1}{2} \right) \Delta \phi_0\) agree with the prediction in Ref. (2), \(\gamma_n = (n + \frac{1}{2}) \pi = -\left( n + \frac{1}{2} \right) \Delta \phi\). From the figure, we can find that the coupling constant \(K\) has remarkable influence on the Berry phase and, as the coupling increases, the influence becomes larger. This means that the classical subsystem play a role of driving field for the quantum system. It can also be seen from Fig. (1) that the frequency ratio of the parameters \(\omega_1/\omega_2\) can change both Berry phase \(\gamma_I\) and \(\gamma_{n0}\), since it determines the common period \(T\) of the parameters \(X_1\) and \(X_2\). In contrast, the Hannay’s angle \(\Delta \phi_I\) is much smaller than \(\Delta \phi_0 \approx -\pi T \omega_2\) as Fig. (2) shows. Therefore, we conclude that not only a classical system can drive a quantum system to obtain a extra Berry phase, but also the quantum system can react back on the classic system and induce a correction to the Hannay’s angle. This is because the quantum system possesses the same structure of the classical system, and the quantum parallel transport can be treated as a parallel transport in the effective classical system. The quantum one-form may also affect the classical subsystem due to the interaction.

It is worth addressing that this quantum-classical hybrid model is different from the full quantum coupled quantum generalized harmonic oscillator model, since they possess different structures. For full quantum
model, the Hamiltonian can be written as
\[ \hat{H} = \frac{1}{2} [X_1 \dot{q}^2 + Y_1 (\dot{p} \dot{q} + \dot{q} \dot{p}) + Z_1 \dot{p}^2] + K \dot{q} \dot{Q} \]
\[ + \frac{1}{2} [X_2 \dot{Q}^2 + Y_2 (\dot{P} \dot{Q} + \dot{Q} \dot{P}) + Z_2 \dot{P}^2]. \tag{56} \]
After a canonical transformation for \( Q \) and \( q \) by
\[ \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} q / \sqrt{Z_1} \\ Q / \sqrt{Z_2} \end{pmatrix}, \tag{57} \]
with
\[ \sin \beta = \left[ \frac{\omega_1^2 - \omega_2^2 + \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4K^2Z_1Z_2}}{2(\omega_1^2 - \omega_2^2)^2 + 4K^2Z_1Z_2} \right]^{1/2}, \]
\[ \cos \beta = \left[ \frac{\omega_1^2 - \omega_2^2 + \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4K^2Z_1Z_2}}{2(\omega_1^2 - \omega_2^2)^2 + 4K^2Z_1Z_2} \right]^{1/2}, \tag{58} \]
the frequencies of the two oscillators \( \omega_1 \equiv (X_1Z_1 - Y_1^2)^{1/2} \)
and \( \omega_2 \equiv (X_2Z_2 - Y_2^2)^{1/2} \) change into
\[ \Omega_1 = \sqrt{\frac{\omega_1^2 + \omega_2^2 + \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4K^2Z_1Z_2}}{2}}, \tag{59} \]
\[ \Omega_2 = \sqrt{\frac{\omega_1^2 + \omega_2^2 - \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4K^2Z_1Z_2}}{2}}. \]
The eigenfunctions of \( \hat{H} \) can be written as the product of the eigenfunctions of two effective harmonic oscillators,
\[ \Psi_{mn}(R_1, R_2; X_1, X_2) = \varphi_{1m}(R_1; X_1, X_2) \varphi_{2n}(R_2; X_1, X_2), \tag{60} \]
where
\[ \varphi_{kn}(R_k; X_1, X_2) = \left( \frac{\Omega_k}{\hbar} \right)^{1/4} \chi_n \left( \frac{R_k \sqrt{\Omega_k}}{\hbar} \right) \exp \left( -\frac{iY_k q^2}{2Z_k \hbar} - \frac{iY_k Q^2}{2Z_k \hbar} \right), \tag{61} \]
and \( \chi_n \) are the normalized Hermite functions. Inserting Eq. \( \text{[52]} \) into Eq. \( \text{[6]} \) and Eq. \( \text{[10]} \), we obtain the one-form and the phase
\[ A^B_{mn} = -\frac{iZ_1}{4} \left[ \frac{(m+1) \cos^2 \beta}{\Omega_1} + \frac{(n+1) \sin^2 \beta}{\Omega_2} \right] d \left( \frac{Y_1}{Z_1} \right) \]
\[ - \frac{iZ_2}{4} \left[ \frac{(m+1) \sin^2 \beta}{\Omega_1} + \frac{(n+1) \cos^2 \beta}{\Omega_2} \right] d \left( \frac{Y_2}{Z_2} \right), \tag{62} \]
\[ \gamma_{mn} = i \oint A^B_{mn} \]
\[ = \oint \left\{ \frac{Z_1}{4} \left[ \frac{(m+1) \cos^2 \beta}{\Omega_1} + \frac{(n+1) \sin^2 \beta}{\Omega_2} \right] d \left( \frac{Y_1}{Z_1} \right) \right. \]
\[ + \frac{Z_2}{4} \left[ \frac{(m+1) \sin^2 \beta}{\Omega_1} + \frac{(n+1) \cos^2 \beta}{\Omega_2} \right] d \left( \frac{Y_2}{Z_2} \right) \right\}. \tag{63} \]
It is easy to find that when there is no interaction between the two oscillators \( (K = 0) \), the Berry phase in Eq. \( \text{[53]} \) returns back to a sum of the Berry phases of two generalized harmonic oscillators \[7],
\[ \gamma_{mn} = \oint \frac{(2m+1)Z_1}{4\omega_1} d \left( \frac{Y_1}{Z_1} \right) + \oint \frac{(2n+1)Z_2}{4\omega_2} d \left( \frac{Y_2}{Z_2} \right). \tag{64} \]
If \( \hat{q} \) denotes the coordinate of a light particle and \( \hat{Q} \) the coordinate of a heavy one, we can take Born-Oppenheimer approximation, treating the heavy particle coordinate \( \hat{Q} \) as a parameter for the light particle, the Hamiltonian for the light particle then takes,
\[ \hat{H}_1 = \frac{1}{2} [X_1 \dot{q}^2 + Y_1 (\dot{p} \dot{q} + \dot{q} \dot{p}) + Z_1 \dot{p}^2] + K \dot{q} \dot{Q}. \tag{65} \]
Its eigenfunctions \( \psi_n(q; Q, X_1) \) and eigenvalues \( E_n(Q, X_1) \) are given by Eq. \( \text{[43]} \),
\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega - \frac{1}{2} K^2 Q^2, \tag{66} \]
\[ \psi_n = \sqrt{\alpha \chi_m} \left( \alpha \left( q + \frac{KZ_1Q}{\omega^2} \right) \right) \exp \left( -\frac{iY_1 q^2}{2Z_1 \hbar} \right), \]
with \( \omega = \sqrt{X_1Z_1 - Y_1^2} \) and \( \alpha = \sqrt{\frac{\omega}{Z_1 \hbar}} \). Note that \( E_n(Q) \) enters into the Hamiltonian for the heavy particle as a potential, Hamiltonian for the heavy particle then takes,
\[ H^{	ext{eff}}_{mn} = \frac{1}{2} [X_2 \dot{Q}^2 + Y_2 (\dot{P} \dot{Q} + \dot{Q} \dot{P}) + Z_2 \dot{P}^2] + E_n(Q, X_1). \tag{67} \]
The eigenvalues and eigenfunctions of the effective Hamiltonian can be calculated straightforwardly,
\[ E^{	ext{eff}}_{mn}(X_1, X_2) = \left( m + \frac{1}{2} \right) \hbar \Omega + \left( n + \frac{1}{2} \right) \hbar \omega, \tag{68} \]
\[ \psi_{m}(Q; X_1, X_2) = \sqrt{\alpha \chi_m} (\alpha' Q) \exp \left( -\frac{iY_2 Q^2}{2Z_2 \hbar} \right), \]
where the effective frequency is defined by \( \Omega = \sqrt{\frac{(\omega^2 X_2 - Z_2 K^2 Z_2 - Y_2^2)^{1/2}}{\omega^2}} \) and \( \alpha' = \sqrt{\frac{Z_2}{Z_1 \hbar}} \). \( E^{	ext{eff}}_{mn}(X_1, X_2) \)
\( (m, n = 1, 2, 3, \ldots) \) are the eigenvalues of the total Hamiltonian, the corresponding eigenvectors are
\[ \Psi_{\text{tot}}(m; Q; X_1, X_2) \approx \varphi_{m}(Q; X_1, X_2) \psi_n(q; Q, X_1). \tag{69} \]
Therefore, the Berry phase of the total system can be calculated by Eq. \( \text{[4]} \) and Eq. \( \text{[10]} \) as
\[ \gamma_{mn} = \int \int dq dQ \oint \psi^*_n(q; Q, X_1) \varphi_{m}(Q; X_1, X_2) \]
\[ \cdot d \chi [\varphi_{m}(Q; X_1, X_2) \psi_n(q; Q, X_1)] \]
\[ = \oint \left\{ \frac{(2m+1)Z_1}{4\omega} \left[ \frac{(2m+1)K^2 Z_1^2}{4\omega^2 \Omega} \right] d \left( \frac{Y_1}{Z_1} \right) \right. \]
\[ + \left[ \frac{(2m+1)Z_2}{4\Omega} \right] d \left( \frac{Y_2}{Z_2} \right) \right\}. \tag{70} \]
Interestingly, if we take the Bohr-Sommerfeld quantization rule \( J = (m + 1/2) \hbar \) into account and notice \( \oint d(\gamma_2/\Omega) = 0 \), the contribution of \( \gamma_{mn} \) to the Berry phase for the light particle (the first two term in Eq. (70)) exactly matches the Berry phase for our quantum-classical hybrid oscillator in Eq. (49), and the contribution to the Berry phase for the heavy particle (the second two term in Eq. (70)) satisfies \( \Delta \phi = -\partial \gamma_{mn}/\partial m \) (see Ref. [7]), where the Hannay angle takes Eq. (49). This is exactly the relation between Berry phase and Hannay’s angle. The description for quantum and classical system is different in physics. To treat them uniformly, we apply the Weinberg’s theory and express the quantum subsystem classically. Since the classical particle is much heavier than the quantum particle, the Born-Oppenheimer approximation turns to be a good approximation for this problem, the predictions made in this paper are reasonable.

IV. CONCLUSION

The Berry phase and Hannay’s angle in coupled quantum-classical hybrid systems have been studied in this paper. To calculate uniformly the Berry phase and Hannay’s angle, we introduced a one-form connection, by which we obtain both of the Berry phase and the Hannay’s angle for the hybrid system. In this sense, the Berry phase and Hannay’s angle in the quantum and classical subsystem can be treated uniformly. To illustrate the formalism, we give two examples. The first example is a spin-half particle coupled to a classical oscillator. In the second example, we calculated the Berry phase and the Hannay’s angle for two couple oscillators, one of which is quantum while another is classical. The effects of subsystem-subsystem coupling on the phase and angle are given and discussed. The results show that the classical subsystem provides the quantum subsystem an large correction to the Berry phase, while the quantum subsystem gives the classical Hannay’s angle a small perturbation. These predictions depend on the feature of quantum-classical hybrid system and their mutual interactions. We also found that the frequency ratio affects the phases and the angle, since it can control the evolution periods of the quantum and classical subsystems. Finally, we have calculated the Berry phase for a fully quantum version of the two coupling system.

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