AN ALTERNATIVE CONSTRUCTION OF ZIP PERIOD MAPS FOR SHIMURA VARIETIES

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ABSTRACT. Let $S$ be the special fibre of a Shimura variety of Hodge type, with good reduction at a place above $p$. We give an alternative construction of the zip period map for $S$, that is used to define the Ekedahl-Oort strata of $S$. The method employed is local, $p$-adic, and group-theoretic in nature.

1. INTRODUCTION

1.1. History of zip period map. The zip period map in the title arises in the development of Ekedahl-Oort (EO, for simplicity) stratification theory for Shimura varieties. Initially, the EO stratification was defined by Ekedahl and Oort [Oor01] for the moduli space of principally polarized abelian varieties $A_g \otimes \mathbb{F}_p$ of dimension $g$ in characteristic $p > 0$ (can be viewed as the Siegel type Shimura variety), by declaring that two points $(A, \lambda)$ and $(A', \lambda')$ over $\overline{\mathbb{F}}_p$ lie in the same stratum if their $p$-kernels are isomorphic.

Later on, this stratification was extended to PEL type Shimura varieties in [GO00], [Moo01], [Moo04], [MW04], [VW13], and to Hodge type Shimura varieties in [Vas10], [Zha18]. The underlying idea is the same as in the Siegel case, that is, considering isomorphism classes of $p$-kernels of abelian varieties with additional structures. The way of defining and studying these strata evolves over time. Let $S$ be the special fibre of a PEL type Shimura variety of good reduction at $p$; it is defined over a finite field, say $\kappa$. In order to give the dimension formula for the EO strata of $S$, Wedhorn [Wed01] constructed a sequence of morphisms of stacks over $\kappa$ (later viewed as a period map in characteristic $p$)

$$S \rightarrow \text{BT}_1 \rightarrow \text{DS}_1,$$

where $\text{BT}_1$ is the stack of BT 1’s (i.e., the $p$-kernel of $p$-divisible groups) with PEL structure and $\text{DS}_1$ is the stack of Dieudonné spaces with PEL structures (i.e., Dieudonné modules associated with BT 1’s with extra structure). He shows that the map $S \rightarrow \text{BT}_1$ is smooth and the natural map $\text{BT}_1 \rightarrow \text{DS}_1$ given by the crystalline Dieudonné functor is a homeomorphism.

Soon, Moonen and Wedhorn [MW04] established the theory of $F$-zips with the underlying idea that an $F$-zip structure on a vector bundle in characteristic $p$ is like a Hodge structure in characteristic 0. Moreover, they constructed a morphism of $\kappa$-stacks

$$S \rightarrow [G/X_\mu],$$

where $X_\mu$ is the moduli of trivialized $F$-zips with PEL structures (of certain type $\mu$ determined by $S$). Here the map is given by taking the $F$-zip associated with the universal BT 1 over $S$, which by definition is the de Rham cohomology $H^1_{\text{dR}}(A/S)$ of the universal abelian scheme $A$, equipped with its $F$-zip structure. In fact, an $F$-zip associated with a BT 1 is equivalent to the corresponding Dieudonné space defined in [Wed01] and hence the map (1.1.2) is essentially the same as the map (1.1.1). Thanks to the analogy of $F$-zip structures to Hodge structures, the map in (1.1.2) is considered as a period map in characteristic $p$. 
Based on the theory of $F$-zips, Pink, Wedhorn and Ziegler [PWZ15] defined the notion of $G$-zips (as $F$-zips with $G$-structures) and the stack of $G$-zips of type $\mu$, denoted by $G$-$\text{Zip}^\mu$; we refer to §4.1 for its precise definition. In the meanwhile, they show that the stack $G$-$\text{Zip}^\mu$ can be realized as the quotient stack of $G$ by some zip group $E_\mu$ (a notion defined in [PWZ11]), i.e., we have an isomorphism of $\kappa$-$T$-schemes

\[
G$-$\text{Zip}^\mu \cong [G_\kappa/E_\mu].
\]  

(1.1.3)

Suppose now that $\mathcal{S}$ is of Hodge type and $p \geq 3$. In order to extend EO stratification to Shimura varieties of Hodge type, Zhang [Zha18] (see also [Wor13]) constructed a map of algebraic stacks (see §4.2 for a review of construction)

\[
\zeta: \mathcal{S} \rightarrow G$-$\text{Zip}^\mu,
\]

(1.1.4)

and showed that $\zeta$ is smooth. The EO strata of $\mathcal{S}$ are defined as geometric fibres of $\zeta$. The strata thus defined are automatically smooth and many properties on these strata are obtained by translating the information of the target stack into that of $\mathcal{S}$ via $\zeta$; see loc. cit. for details.

We call map $\zeta$ the zip period map for $\mathcal{S}$. There are also some other variants of this map: for example, the perfectly smooth map $\text{Sh}_\mu \rightarrow \text{Sh}_\mu^{\text{loc}(2.1)}$ in [XZ17, Rem. 7.2.5] (see also [SYZ20] for its generalization) and the map $\eta: \mathcal{S} \rightarrow \mathcal{D}_1/\mathcal{X}_\infty$ in [Yan18, Thm. 8.5.2]. The aim in this paper is to give an alternative construction of $\zeta$ (more precisely, the composition of $\zeta$ with the isomorphism in (1.1.3)) avoiding the language of $G$-zips, which (we hope) provides a different perspective of understanding the already existing zip period map.

1.2. Main results and the strategy of proof. Let $(\mathcal{G}, \mathcal{X})$ be a Shimura datum of Hodge type and denote by $S_K$ the Kisin-Vasiu integral model of the associated Shimura variety $\text{Sh}_K(\mathcal{G}, \mathcal{X})$ of level $K$ which is hyperspecial at $p$. This hyperspecial condition on $K$ implies that $\mathcal{G}_\mathbb{Q}_p$ admits a reductive $\mathbb{Z}_p$-model $\mathfrak{G}$, whose special fibre we denote by $G$. Recall that the integral model $S_K$ is a quasi-projective and smooth scheme over $\mathcal{O}$, the localization at some place above $p$ of the ring of integers of the reflex field of $(\mathcal{G}, \mathcal{X})$. Thanks to the hyperspecial assumption $\mathcal{O}$ is unramified at $p$. Write $\kappa$ for the residue field of $\mathcal{O}$ and $S := S_K \otimes \mathcal{O} \kappa$. Let $\mu: \mathbb{G}_{m,k} \rightarrow G_\kappa$ be a representative for the reduction over $\kappa$ of the $\mathbb{G}(\mathbb{C})$-conjugacy class $[\mu]_C$ of the inverses of Hodge cocharacters $\mathbb{G}_{m,C} \rightarrow G_\mathbb{C}$ determined by $(\mathcal{G}, \mathcal{X})$.

Denote by $P_\pm \subseteq G_\kappa$ the opposite parabolic subgroups of $G_\kappa$ defined by $\mu$ and $U_\pm \subseteq P_\pm$ the corresponding unipotent radicals, and $U^\sigma$ the base change of $U_-$ along the $p$-power Frobenius $\sigma: \kappa \rightarrow \kappa$; the same convention applies to other notations of the form $(\cdot)^\sigma$. The stack $G$-$\text{Zip}^\mu$ is in fact isomorphic to some quotient stack $[G_\kappa/E_\mu]_\kappa$ of $G_\kappa$ by the smooth algebraic group $E_\mu = P_+ \times U^\sigma$ (see §4.1 for the action), but such an isomorphism is not quite formal. The following nearly trivial observation turns out to be important to this work: since $U^\sigma$, as a normal subgroup of $E_\mu$, acts freely on $G_\kappa$ by right multiplication, by passing to quotient we obtain a canonical isomorphism of algebraic stacks over $\kappa$ (§5.6):

\[
[(G_\kappa/U^\sigma)/P_+] \cong [G_\kappa/E_\mu].
\]

where $G_\kappa/U^\sigma$ is represented by a smooth $\kappa$-scheme. Hence to give the zip period map $\zeta$ above is equivalent to a give a $P_+$-torsor, say $T$, over $S$ and a $P_+$-equivariant map of $\kappa$-schemes $T \rightarrow G_\kappa/U^\sigma$. The natural candidate for $T$ is the scheme $I_+$ of trivializations of the Hodge filtration $H^1_{dR}(A/S) \supseteq \omega_A/S$, respecting certain tensors that we do not specify in this introduction. The torsor $I_+$ is part of the datum for the universal $G$-zip which defines $\zeta$. 

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Theorem 1.2.1 (Thm. [5.5.1] Thm. [5.6.1]). There exists an (explicitly constructed) morphism of $\kappa$-schemes
\[ \gamma: I_+ \to G_\kappa/U^\sigma. \]

(2). The map $\gamma$ is $P_\kappa$-equivariant, and hence induces a morphism of algebraic $\kappa$-stacks
\[ \eta: S \to [(G_\kappa/U^\sigma)/P_+] \cong [G_\kappa/E_I] \cong G\text{-Zip}^\mu. \]

Theorem 1.2.2 (Thm. [5.6.1]). We have a natural 2-isomorphism $\cong \zeta$. Consequently, we give an alternative construction of the zip period map for $S$.

We describe now the construction of $\gamma$ on geometric points. Take $k = \overline{F}_p$. From now on we fix a cocharacter $\tilde{\mu}: \mathbb{G}_{m,k(\kappa)} \to \mathcal{G}_{W(\kappa)}$ of $\mathcal{G}_{W(\kappa)}$ which lifts $\mu: \mathbb{G}_{m,\kappa} \to G_\kappa$. A point $\tilde{x} = (\tilde{x}, \beta_x) \in I_+(k)$ consists of a point $\tilde{x} \in S(k)$ and a trivialization
\[ \beta_x: [\Lambda^1_{\mathcal{A}_x(\kappa)} \supseteq \Lambda^1_{\mathcal{W}(\kappa)}] \cong [H^1_{\text{dR}}(A_x/\kappa) \supseteq \mathcal{O}_{A_x/\mathcal{W}(\kappa)}] \cong [\mathcal{M} \supseteq \mathcal{M}_1], \]
respecting tensors on both sides, where $\mathcal{M}$ and $\mathcal{M}_1$ denote the reduction modulo $p$ of the contravariant Dieudonné module $\mathcal{M}$ of the $p$-divisible group $A_x[p^\infty]$ over $k$ and its Hodge filtration respectively $\mathcal{M}_1$ (§5.3.). Here $\Lambda^1_{\mathcal{A}_x}$ is the weight 1 subspace of $\Lambda^1_{\mathcal{A}_x}$ induced by $\mu_k: \mathbb{G}_{m,k} \to G_k$. Let $\tilde{I}_+$ be the integral model over $\overline{S}$ of $I_+$. The first step of the construction of $\gamma$ on $k$-points is to choose a lift $\tilde{x}^1 = (x, \beta_{x^1}) \in \tilde{I}_+(W(k))$ of $\tilde{x}$ which provides a lift of $\beta_x$,
\[ \beta_{x^1}: [\Lambda^1_{W(\kappa)} \supseteq \Lambda^1_{\mathcal{W}(\kappa)}] \cong [H^1_{\text{dR}}(A_x/W(\kappa)) \supseteq \mathcal{O}_{A_x/\mathcal{W}(\kappa)}], \]
and hence a trivialization of $\mathcal{M}$ via the canonical isomorphism $H^1_{\text{dR}}(A_x/W(\kappa)) \cong \mathcal{M}$, and then show that via the trivialization $\beta_{x^1}$ the Frobenius of $\mathcal{M}$ admits a uniform decomposition
\[ \tilde{I}_+ \tilde{\mu}_{W(\kappa)}(p) \text{ with } \tilde{I}_+ \tilde{\mu} \in \mathcal{G}(W(k)) \cong \text{GL}(\Lambda^1_{\mathcal{W}(\kappa)}), \]
where $\tilde{\mu}_{W(\kappa)}: \mathbb{G}_{m,k(\kappa)} \to \mathcal{G}_{W(\kappa)}$ is the base change along $\sigma: W(k) \to W(k)$ of $\tilde{\mu}_{W(\kappa)}$. The key point here is that the element $\tilde{I}_+$ is integral and hence we can take its reduction modulo $p$, denoted by $\tilde{I}_+$ in $G_\kappa$. Then one proceeds by showing that the image of $\tilde{I}_+$ in $G_\kappa/U^\sigma(k)$ is independent of lifts $\tilde{x}^1$ (Lem. [5.4.2]; we denote it by $\gamma_{\tilde{x}}^0$). To summarize, the map $\gamma$ on $k$-points is given by performing the following operations (§5).

(1.2.1) $\tilde{x}^0 \in I_+(k) \xrightarrow{\text{choose } \tilde{x}^1} \tilde{I}_+ \subset \mathcal{G}(W(k)) \xrightarrow{\text{mod } p} \tilde{I}_+ \subset G(k) \xrightarrow{\text{projection}} \gamma_{\tilde{x}}^0 \in G/U^\sigma(k).

The technical heart of the construction of $\gamma$ in Thm. [1.2.1] is to justify the operations in (1.2.1) and to show that these operations can be performed in a relative sense: for every smooth $\kappa$-algebra $\bar{R}$ which (automatically) admits a simple frame (equivalently, a crystalline prism if one prefers) and every point $\tilde{x} \in I_+(\bar{R})$, we can construct a point $\gamma_{\tilde{x}} \in G_\kappa/U^\sigma(\bar{R})$ whose specialization at geometric points coincides with (1.2.1); see Prop. [5.4.1]. This relative construction relies on relative classifications of $p$-divisible groups as in [4195] and is more complicated in the sense that in the relative setting we need to compare not only different lifts $\tilde{x}^1$ as aforementioned, but also different choices of simple frames for $\bar{R}$. The independence of these two different types of choices are proved via matrices calculations; see §5.4. Finally the global map $\gamma: I_+ \to G_\kappa/U^\sigma$ is obtained by first constructing it on Zariski opens of $I_+$ and then gluing the local maps together.

We now give some comments on the comparison of $\zeta$ and $\eta$. Zhang’s construction of $\zeta$ uses the global geometry over characteristic $p$, namely the language of $G$-zips (which is somewhat complicated: for example, a $G$-torsor involves three torsors plus some delicate zip relations) but follows in spirit the original intuitive definition of EO stratification for $A_g$ since the stack $G\text{-Zip}^\mu$ can be viewed as the moduli space of BT 1’s while the universal
$G$-zip for $\zeta$ corresponds to $A[p]$, the universal BT 1 over $S$. In particular $\zeta$ is determined by $A[p]$. In contrast, the construction of $\eta$ is local and group-theoretic; it avoids the fancy language of $G$-zips and uses only one torsor, $I_x$. But since it does not start with $A[p]$, in the end the dependence of $\gamma$ (hence of $\eta$) on $A[p]$ is obscured. From our local construction one sees better the role that the zip group $E_\mu = P_+ \ltimes U^\sigma$ plays in the business of zip period map. For example, given a $k$-point $x^p$ of the $P_+$-torsor $I_x$, different lifts $x^p$ produce the same $U^\sigma$-coset in $G(k)$; this coincides with Faltings deformation theory which says that the deformation of the $p$-divisible group $A[p]$ is controlled by the integral model of $U_\sigma$. The proof of Thm. [1.2.2] is not formal, partly because the canonical isomorphism $[G_k/E_\mu] \cong G$-Zip$^\mu$ is not formal.

This work has a certain amount of overlap (not on main results) with my PhD thesis [Yan18]. The connection between these two works will be made in a subsequent paper.

1.3. Notational convention. Throughout the paper we fix a prime number $p \geq 3$. The Dieudonné crystals (resp. modules) used in this paper are contravariant. Let $R$ be a ring and $M$ an $R$-module. If $\sigma : R \to R$ is a ring endomorphism we write $M^\sigma = \sigma^* M$ for the base change $M \otimes_{R, \sigma} R$. If $M$ is finite locally free, we denote by $M^\ast$ its dual $R$-module. Then we have the canonical identification $M^\otimes \cong M^{\ast\otimes}$ of $R$-modules, where $M^\otimes$ is the direct sum of all $R$-modules obtained from $M$ by applying the operations of taking duals, tensor products, symmetric powers and exterior powers. Here, as a general convention, the notation “$\cong$” means canonical isomorphism between mathematical objects. For any $R$-automorphism $f : M \cong M$, we have an induced isomorphism $(f^{-1})^* : M^\ast \to M^\ast, a \mapsto f^{-1} \circ a$, and hence a canonical isomorphism of $R$-group schemes $(\cdot)^\vee : \text{GL}(M) \cong \text{GL}(M^\ast), g \mapsto g^\vee := (g^{-1})^*$. We also use the letter $M$ to denote a Levi subgroup (of some algebraic group) but it shall be always clear from the context whether $M$ is a module or an algebraic group. The decoration $(\cdot)$ usually indicates that the object in question is over the characteristic $p$ world or is the reduction modulo $p$ of $(\cdot)$; it shall be clear if it has some other meaning.

For an $F_p$-algebra $\bar{R}$, we use $\sigma : \bar{R} \to \bar{R}$ for the absolute (i.e., $p$-power) Frobenius of $\bar{R}$. If $X$ is a scheme over $\bar{R}$, we write $X^\sigma$ for its pull back along $\sigma$ and $\sigma : X \to X^\sigma$ the relative Frobenius over $\bar{R}$. In particular, when $X$ is defined over $F_p$, sometimes we also write $\sigma : X \to X$ for the composition of the relative Frobenius of $X$ with the canonical isomorphism $\sigma : X^\sigma \cong X$. Similarly, if $f : X \to Y$ is a map between objects over $\bar{R}$, we write $f^\sigma$ for its base change along $\sigma : \bar{R} \to \bar{R}$. Now let $k$ be a perfect field and $\mathcal{G}$ a group scheme over $W(k)$, which is defined over $\mathbb{Z}_p$. For a $W(k)$-algebra $R$ with a Frobenius lift $\sigma = \sigma_R : R \to R$ over $W(k)$, we often denote by

$$\sigma : \mathcal{G}(R) \to \mathcal{G}(R)$$

(1.3.1)

the homomorphism induced by $\sigma : R \to R$ (note that $\sigma : R \to R$ is only a $\mathbb{Z}_p$-endomorphism, but not a $W(k)$-endomorphism in general). We abuse language and call also (1.3.1) ”Frobenius” of $\mathcal{G}$. In case $\mathcal{G}$ is defined over $F_p$, this Frobenius coincides with relative Frobenius mentioned above.

In this paper, for quotient stacks we systematically use right actions instead of left or mixed actions; for example, the stack $[G_k/E_\mu]$ in this paper corresponds to $[E_\mu\backslash G_k]$ in the literature.

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2. Classification of $p$-divisible groups (recollection)

Throughout this section we let $k$ be a perfect field of characteristic $p$ and denote by
$\sigma : W(k) \to W(k)$ its unique ring automorphism inducing the absolute Frobenius of $k$.

2.1. Existence of simple frames.

\textbf{Lemma 2.1.1.} Let $\bar{R}$ be a $k$ algebra which Zariski locally admits a finite $p$-basis ([dJ95, 
Def. 1.1.1], or [BM90, Def. 1.1.1]). The following holds:

1. \textbf{There exists a} $\sigma : W(k) \to W(k)$ \textbf{its unique ring automorphism inducing the absolute Frobenius of $k$}.

2. \textbf{There is a ring endomorphism} $\sigma : R \to R$ \textbf{lifting the absolute Frobenius of $\bar{R}$,}

3. \textbf{Let $R, \bar{R}$ be as above and $\bar{A}$ an étale $\bar{R}$ algebra. Then there exists a formally étale $R$-}

4. \textbf{Let $(R, \sigma)$ be as above. If $m$ is a maximal ideal of $R$, then $\sigma$ extends uniquely to a Frobenius lift of the $m$-adic completion $\bar{R}$, which is a lift of the $m$-adic completion of $\bar{R}$.}

\textit{Proof.} (1) and (2) is a special case of [Kim15, Lem. 2.1] (take $I = (p)$) and (3) is a special case
of the first part of [Kim15, Lem. 2.5]. For (4), note first that $\sigma(m) \subseteq m$. This follows from
the fact that $m$ contains $p$, and the fact that the morphism $\text{Spec} \bar{R} \to \text{Spec} R$ induced
by the absolute Frobenius of $\bar{R}$ is identity on topological spaces. Hence we can define
$\sigma_{\bar{R}} : \bar{R} \to \bar{R}$ by sending an element $(r_i) \in \lim_{i} R/m^i = \bar{R}$ to $(\sigma(r_i)) \in \bar{R}$. \hfill $\Box$

\textbf{Example 2.1.2 ([BM90, 1.1.2]).} The main examples of $\bar{R}$ in our later applications are:

1. $\bar{R}$ is a perfect $k$-algebra (the empty $p$-basis case). In this case, the unique simple frame

2. $\bar{R}$ is a smooth $k$-algebra of finite type. Here Zariski locally $\bar{R}$ indeed admits a finite $p$-

basis: Zariski locally $\bar{R}$ is étale over some polynomial algebra $A = k[x_1, \cdots, x_n]$ which

3. the image of this $p$-basis in $\bar{R}$ form a $p$-basis of $\bar{R}$, as the relative Frobenius map $A \otimes_{\sigma A} \bar{R} \to \bar{R}$, $a \otimes r \mapsto ar^p$ is an isomorphism

4. hence the Frobenius $\sigma : \bar{R} \to \bar{R}$ can be identified with the canonical ring map $\bar{R} \to \bar{A} \otimes_{\sigma A} \bar{R}, r \mapsto 1 \otimes r$. 

Definition 2.1.3. Let \( \hat{R} \) be as in Lem. 2.1.1. A simple frame of \( \hat{R} \), relative to \( W(k) \), is a pair \( \hat{R} = (R, \sigma) \), where \( R \) is a lift of \( \hat{R} \) and \( \sigma : R \to R \) is a Frobenius lift of \( R \).

Remark 2.1.4. A simple frame \((R, \sigma)\) over \( W(k) \) of \( \hat{R} \) is the same thing as a crystalline prismatic over the base prism \((W(k), \sigma)\) in the sense of Bhatt-Scholze \([BS21]\). Hence in fancier language, simple frames of \( \hat{R} \) should perhaps be termed as (crystalline) prismatic charts of \( \hat{R} \).

2.2. Classification of \( p \)-divisible groups over \( \hat{R} \). Let \( \hat{R} \) be as in Lem. 2.1.1 and \( \hat{R} = (R, \sigma) \) a simple frame of \( \hat{R} \). Till the end of this section, we assume further that \( \hat{R} \) is as in Exam. 2.1.2. As a preparation for later sections, we review in this subsection results on classification of \( p \)-divisible groups over \( \hat{R} \) (and over \( R \) in the next subsection \( \S 2.3 \)), in terms of linear data over the simple frame \((R, \sigma)\).

We denote by \( \hat{\Omega}_R \) the module of \( p \)-adically continuous differentials of \( R \), i.e.,

\[
\hat{\Omega}_R := \lim_{\to} \Omega^1_{[R/p^n R]/W(k)}.
\]

It is a finite projective \( R \)-module due to the finite \( p \)-basis assumption on \( \hat{R} \). We denote by \( \text{DM}(\hat{R}, \nabla) \) the category of Dieudonné modules with connections. Here a Dieudonné module with connection (or simply a Dieudonné module) over \( R \) (or simply over \( R \) when \( \sigma \) is chosen) is a tuple \((M, F, \nabla, \nabla_M)\), where \( M \) is a finite locally free \( R \)-module and \( F : M^\sigma \to M, \nabla : M \to M^\sigma \) are maps between \( R \)-modules such that

\[
F \circ \nabla = p \cdot \text{id}_{M^\sigma}; \quad \nabla \circ F = p \cdot \text{id}_M,
\]

and where \( \nabla_M : M \to M \otimes_R \hat{\Omega}_R \) is an integrable topologically quasi-nilpotent connection over the \( p \)-adically continuous derivation \( d_R : R \to \hat{\Omega}_R \) of \( R \), with respect to which \( F \) is horizontal, i.e., \( \nabla_M \circ F = (F \otimes \text{id}_{\hat{\Omega}_R}) \circ \sigma^*(\nabla_M) \).

For a \( p \)-divisible group \( \hat{H} \) over \( \hat{R} \), we denote by \( \mathbb{D}^\ast(\hat{H}) \) the Dieudonné crystal\(^1\) of \( \hat{H} \) as in \([BBM82]\), which coincides with the construction in \( \text{[Mes72]} \) up to duality: to be precise, our \( \mathbb{D}^\ast(\hat{H}) \) here corresponds to the Dieudonné crystal \( \mathbb{D}(\hat{H}^c) \) in \( \text{[Mes72]} \), with \( \hat{H}^c \) the dual \( p \)-divisible group of \( \hat{H} \). Following usual convention, we write \( \mathbb{D}^\ast(\hat{H})(R) \) for the evaluation of \( \mathbb{D}^\ast(\hat{H}) \) at the canonical PD-thickening \( R \to \hat{R} \). By functoriality of the formation of Dieudonné crystals, we have \( \mathbb{D}^\ast(\hat{H}) = \mathbb{D}^\ast(\hat{H}^c) \), where \( \mathbb{D}^\ast(\hat{H}) \) is the pullback along \( \sigma : \hat{R} \to \hat{R} \) of \( \mathbb{D}^\ast(\hat{H}) \). Consequently we have canonical isomorphism \( \mathbb{D}^\ast(\hat{H})(R) \cong \mathbb{D}^\ast(\hat{H}^c) \) of \( R \)-modules. The Frobenius \( \hat{H} \to \hat{H}^c \) and Verschiebung \( \hat{H} \to \hat{H}^\sigma \) induces morphism of crystals,

\[
F : \mathbb{D}^\ast(\hat{H}^c) \to \mathbb{D}^\ast(\hat{H}), \quad \nabla : \mathbb{D}^\ast(\hat{H}) \to \mathbb{D}^\ast(\hat{H}^c),
\]

such that \( F \circ \nabla = p \cdot \text{id}_{\mathbb{D}^\ast(\hat{H})} \) and \( \nabla \circ F = p \cdot \text{id}_{\mathbb{D}^\ast(\hat{H}^c)} \). Evaluating at the thickening \( R \to \hat{R} \) we obtain \( R \)-linear maps \( F, \nabla \) for \( \mathbb{D}^\ast(\hat{H})(R) \), just like an object in \( \text{DM}(\hat{R}, \nabla) \) satisfying \( \text{(2.2.1)} \). Denote by \( (BT/R) \) the category of \( p \)-divisible groups over \( \hat{R} \). The following classification result is known.

Remark 2.2.1. If \( H = \hat{A}[p^\infty] \) for some abelian scheme \( \hat{A} \) over \( \hat{R} \) (the case we mostly concern for later applications), we have canonical isomorphism of Dieudonné crystals (\([BBM83 \text{ 3.3.7, 2.5.6}]\)),

\[
\mathbb{D}^\ast(\hat{H}) \cong \mathbb{D}^\ast(\hat{A}) \cong \mathbb{R}^1\pi_{\text{CRIS},*}\mathcal{O}^\text{cris}_{\hat{A}},
\]

\(^1\)The superscript \( * \) in \( \mathbb{D}^\ast(\hat{H}) \) is used to indicate that our Dieudonné crystal here is contravariant.
where \( \pi : \mathcal{A} \to \text{Spec} \mathcal{R} \) is the structure morphism. It follows then that we have the following canonical isomorphism of \( R \)-modules, which is Frobenius equivariant

\[
H^1_{\text{cris}}(\mathcal{A}/R) \cong \mathbb{D}^+(\mathcal{H})(R).
\]

**Theorem 2.2.2.** For any \( p \)-divisible group \( \mathcal{H} \) over \( \mathcal{R} \), there exists a natural connection \( \nabla_M : M \to M \otimes_R \hat{\Omega}_R \) for \( M = \mathbb{D}^+(\mathcal{H})(R) \) such that the tuple

\[
(2.2.3) \quad M = (M, F, \nabla_M)
\]
is an object in \( \text{DM}(\mathcal{R}, \nabla) \). Moreover, such an assignment gives an equivalence of categories between \( (\text{BT}/\mathcal{R}) \) and \( \text{DM}(\mathcal{R}, \nabla) \).

**Proof.** If \( \mathcal{R} \) a perfect \( k \)-algebra, this is a (unpublished) result of Gabber, relying on a result of Berthelot [Ber80] where the case of a perfect discrete valuation ring is dealt; see also [Lau13] and [SW13] for different proofs. In this case, the connection can even be suppressed in the definition of a Dieudonné module. If \( \mathcal{R} \) and \( \mathcal{R} \) for different proofs. In this case, the connection can even be suppressed in the definition of a Dieudonné module. If \( \mathcal{R} \) is a smooth \( \mathcal{R} \)-algebra of finite type, this follows from [dJ95, 4.1.1, 2.3.4, 2.4.8]; indeed, since \( \mathcal{R} \) satisfies [dJ95, 1.3.1.1] by (1.3.2.1) in loc. cit. and \( \mathcal{X} = \text{Spec} \mathcal{R} \) satisfies the hypothesis of [dJ95, 4.1.1] by (2.4.7.2) in loc. cit.

\( \square \)

### 2.3. Classification of \( p \)-divisible groups over \( R \)

The same setting as in the previous subsection §2.2. Now we start with a \( p \)-divisible group \( \mathcal{H} \) over \( R \) and write \( \mathcal{H} = H \otimes_R \mathcal{R} \). For the \( p \)-divisible group \( H^* \) there is constructed in [Mes72, IV. 1.14] a universal extension \( 0 \to \omega_H \to \text{E}(H^*) \to H^* \to 0 \); taking Lie (following notation in loc. cit.), we get an exact sequence of locally free \( R \)-modules

\[
0 \to \omega_H \to \text{Lie}(\text{E}(H^*)) \to \text{Lie}(H^*) \to 0
\]

where \( \omega_H \) is the sheaf of invariant differential of \( H \). Moreover, it follows from the construction of \( \mathbb{D}^+(\mathcal{H}) \) that we have canonical isomorphism \( \text{Lie}(\text{E}(H^*)) \cong \mathbb{D}^+(\mathcal{H})(R) \) of \( R \)-modules ([Mes72, IV. 2.5.4], see also [BBM82, 3.3.5]) and thus we can identify them. Similarly we have an exact sequence for \( \mathcal{H} \),

\[
0 \to \omega_\mathcal{H} \to \mathbb{D}^+(\mathcal{H})(R) \to \text{Lie}(\mathcal{H}^*) \to 0
\]

Here we stress that the submodule \( \omega_\mathcal{H} \subseteq \mathbb{D}^+(\mathcal{H})(R) \) is a locally direct summand of \( \mathbb{D}^+(\mathcal{H})(R) \) which lifts the locally direct summand \( \omega_\mathcal{H} \subseteq \mathbb{D}^+(\mathcal{H})(R) \) of \( \mathbb{D}^+(\mathcal{H})(R) \). Let \( \mathcal{M} \subseteq (\text{BT}/\mathcal{R}) \) be the Dieudonné module corresponding to \( \mathcal{H} \). Write \( \mathcal{F} : \mathcal{M}^\sigma \to \mathcal{M} \) for the reduction modulo \( p \) of \( \mathcal{F} \). Set \( \mathcal{M}^1 := \omega_\mathcal{H} \); it is called the Hodge filtration of \( \mathcal{M} \). Then we have the relation,

\[
(2.3.1) \quad \mathcal{M}^{1, \sigma} = \text{Ker}(\mathcal{F}) \subseteq \mathcal{M}^\sigma.
\]

Denote by \( (\text{BT}/\mathcal{R}) \) the common category of \( p \)-divisible groups over \( \text{Spec} \mathcal{R} \) and over \( \text{Spf} \mathcal{R} \) (justified by [dJ95, 2.4.4]) and by \( \text{AFDM}(\mathcal{R}, \nabla) \) the category of tuples \( (\mathcal{M}, M^1, F, \nabla_M) \), where \( (\mathcal{M}, F, \nabla_M) \) is an object in \( \text{DM}(\mathcal{R}, \nabla) \), and where \( M^1 \subseteq \mathcal{M} \) is a locally direct summand, lifting the locally direct summand \( \mathcal{M}^1 \subset \mathcal{M} \). Morphisms are obvious ones. We call an object in \( \text{AFDM}(\mathcal{R}, \nabla) \) an admissibly filtered Dieudonné module over \( R \) over \( \mathcal{R} \) (or simply over \( R \); when \( \sigma \) is chosen); cf. [Mes72, V. 1.4].

**Remark 2.3.1.** For the purpose of future reference, we recall the following well-known comparison result, that underlines the crystalline Dieudonné theory. If \( \mathcal{A} \) is an abelian scheme over \( R \), with \( \mathcal{A} \) its pullback to \( \mathcal{R} \), we have canonical isomorphism of \( R \)-modules ([Ber74, V. 2.3.7], also cf. [BO78, 7.26.3])

\[
(2.3.2) \quad H^1_{\text{dR}}(\mathcal{A}/R) \cong H^1_{\text{cris}}(\mathcal{A}/R).
\]
Moreover, we have the following a canonical isomorphism of filtered $R$-modules

$\left(\mathbb{D}^+(\hat{H})(R) \supset \omega_H\right) \cong \left(\mathbb{H}^1_{dR}(A/R) \supset \omega_A\right)$.

**Theorem 2.3.2.** The assignment $G \mapsto \left(\mathbb{D}^+(\hat{H})(R), \omega_H, \mathcal{F}, \mathcal{V}, \nabla_M\right)$ gives a category equivalence between $(\mathcal{B}T/R)$ and $\text{AFDM}(\mathcal{R}, \mathcal{V})$.

**Proof.** To lift a $p$-divisible group $\hat{H}$ over $\bar{R}$ to $R$ is the same thing as lifting its dual $\hat{H}^*$ to $R$, the assertion follows from the combination of Thm. 2.2.2 and Grothendieck-Messing deformation theory ([Mes72, V, 1.6]) which in our setting says that lifting $\hat{H}^*$ to $R$ is equivalent to lifting the locally direct summand $\omega_H \subseteq \mathbb{D}^+(\hat{H})(\bar{R})$ to a locally direct summand of $\mathbb{D}^+(\hat{H})(R)$. \hfill $\square$

### 2.4. Base change along simple frames.

Let $\mathcal{R}$ be as in Exam. 2.1.2 and $\mathcal{R} = (\mathcal{R}', \sigma')$ a simple frame of $\mathcal{R}$ over $W(k)$. Let $f : \bar{\mathcal{R}} \to \mathcal{R}'$ be a morphism of simple frames over $W(k)$ (i.e., a map $f : R \to \mathcal{R}'$ of $W(k)$-algebras, compatible with Frobenius lifts). Then we have commutative diagrams as below induced by base change along $f$ in obvious senses.

\[
\begin{array}{ccc}
\mathcal{B}T/\bar{R} & \overset{\cong}{\longrightarrow} & \mathcal{D}M(\bar{R}, \mathcal{V}) \\
\downarrow & & \downarrow \\
\mathcal{B}T/R' & \overset{\cong}{\longrightarrow} & \mathcal{D}M(\mathcal{R}', \mathcal{V})
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{B}T/\bar{R} & \overset{\cong}{\longrightarrow} & \mathcal{A}FDM(\mathcal{R}, \mathcal{V}) \\
\downarrow & & \downarrow \\
\mathcal{B}T/R' & \overset{\cong}{\longrightarrow} & \mathcal{A}FDM(\mathcal{R}', \mathcal{V})
\end{array}
\]

### 2.5. Partially divided Frobenius.

The setting is the same as in the previous two subsections §2.2, §2.3. Let $(\mathcal{M}, \mathcal{M}^1, \mathcal{F}, \mathcal{V}, \nabla_M)$ be an object in $\mathcal{A}FDM(\mathcal{R}, \mathcal{V})$. Assume now that the submodule $\mathcal{M}^1 \subset \mathcal{M}$ is a (not just locally) direct summand of $\mathcal{M}$. Let $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^0$ be a decomposition of $\mathcal{M}$ into $R$-submodules; such a decomposition is called a normal decomposition of $\mathcal{M}$ (or simply of $\mathcal{M}$). Define the following maps,

\[
\Gamma := \frac{1}{p} F\mathcal{M}^1, \sigma \oplus F\mathcal{M}^0, \sigma, \quad f := p \text{id}_{\mathcal{M}^1, \sigma} \oplus \text{id}_{\mathcal{M}^0, \sigma},
\]

so that we have $F = \Gamma \circ f$. We shall call $\Gamma$ the partially divided Frobenius of $\mathcal{M}$ w.r.t. the normal decomposition $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^0$. The next lemma describes the most important property of $\Gamma$, with the point being that a normal decomposition of $\mathcal{M}$ enables us to decompose $F$ as the composition of an integral part $\Gamma$ with a rational part $f$. Such a decomposition is important for later applications.

**Lemma 2.5.1.** The map $\Gamma$ defined in (2.5.1) is an isomorphism of $R$-modules.

**Proof.** Let us first note that $\Gamma$ is surjective: indeed from (2.3.1) we obtain the equality displayed below, which implies $\text{Im}(\Gamma) = \mathcal{M}$:

\[
\mathcal{F}^{-1}(p\mathcal{M}) = \pi^{-1}(\bar{\mathcal{M}}^1, \sigma) = \mathcal{M}^1, \sigma + p\mathcal{M}^0 = \mathcal{M}^1, \sigma + p\mathcal{M}^0, \sigma,
\]

where $\pi : \mathcal{M}^\sigma \to \bar{\mathcal{M}}^\sigma$ is the canonical reduction modulo $p$ map.

It is enough to show that for every maximal ideal $m$ of $R$, the pull back to $\bar{R}_m$ of $\Gamma$ is an isomorphism. To ease notation, write $A = \bar{R}_m$ and let $(A, \sigma)$ be the unique simple frame of the $m$-adic completion of $\bar{R}$, induced by $(R, \sigma)$ as in Lem. 2.1.1. By functoriality as discussed above (2.4.1), the base change along $(R, \sigma) \to (A, \sigma)$ of $\mathcal{M}$ is equal to the admissibly filtered Dieudonné module of $H \otimes_R A$ if $H$ is the $p$-divisible group over $R$ corresponding to $\mathcal{M}$. So we are reduced to show the $\Gamma$ map over $(A, \sigma)$ corresponding to $H \otimes_R A$ and the decomposition $\mathcal{M}_A = \mathcal{M}_A^1 \oplus \mathcal{M}_A^0$ is an isomorphism. But as $A$ is local, the source and target
of $\Gamma : M^0 \to M$ are free $A$-modules of the same rank, by Nakayama’s lemma the assertion follows from the surjectivity of $\Gamma$. \qed

We remark that in later sections we will not use the full power of the classification results, Thm. 2.2.2 and Thm. 2.3.2. We only need the fact that given a $p$-divisible group over $\mathcal{R}$ (resp. over $\mathcal{R}$), one can associate with it an object in $\text{DM}(\mathcal{R}, \mathcal{V})$ (resp. in $\text{AFDM}(\mathcal{R}, \mathcal{V})$) and such an association is compatible with base change of simple frames.

3. Good reduction of Shimura varieties of Hodge type

3.1. Shimura varieties of Hodge type. Let $G$ be a (connected) reductive group over $\mathbb{Q}$ and $X$ a $G(\mathbb{R})$-conjugacy class of homomorphisms

$$h : S := \text{Res}_{\mathbb{C}/\mathbb{R}} G_m \to G_{\mathbb{R}}$$

of algebraic groups over $\mathbb{R}$, such that $(G, X)$ is a Shimura datum in the sense that they satisfy axioms (2.1.1.1)-(2.1.1.3) of [Del79, 2.1.1]. Suppose that $V$ is a finite-dimensional $\mathbb{Q}$-vector space with a perfect alternating pairing $\psi$ and write $\text{GSp} = \text{GSp}(V, \psi)$ for the corresponding group of symplectic similitudes. Then we get the most important example of Shimura datum $(\text{GSp}, S^\pm)$ with $S^\pm$ the Siegel double space, which is defined to be the set of homomorphisms $\mathbb{C} \to \text{GSp}_{\mathbb{R}}$ such that: (1) The $\mathbb{C}^\times$ action on $V_{\mathbb{R}}$ gives rise to a Hodge structure of type $(-1, 0)$ and $(-1, 0)$; (2) $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

In this paper we consider a Shimura datum $(G, X)$ of Hodge type; i.e., there exists an embedding of Shimura data $(G, X) \hookrightarrow (\text{GSp}, S^\pm)$ for some $(\text{GSp}, S^\pm)$. Let $K = K_p \mathbb{A}_f \subseteq G(\mathbb{A}_f)$ be an open compact subgroup such that $K_p \subseteq G(\mathbb{Q}_p)$ is a hyperspecial subgroup and that $K^p \subseteq G(\mathbb{A}_f^p)$ is sufficiently small (hence is neat). The condition that $K_p$ is hyperspecial means that there is a reductive group $\mathcal{S}$ over $\mathbb{Z}_{(p)}$, which we fix from now on, such that $K_p = \mathcal{S}(\mathbb{Z}_{(p)})$. The condition that $K^p$ is sufficiently small guarantees that the double quotient

$$\text{Sh}_K(G, X)_C := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$$

has the structure of a smooth quasi-projective complex variety by a theorem of Baily-Borel. Results of Shimura, Deligne, Milne and others imply that, up to isomorphism, $\text{Sh}_K(G, X)_C$ has a unique quasi-projective smooth model $\text{Sh}_K(G, X)$ over the reflex field $E$ of $(G, X)$. The reflex field $E$ only depends on the Shimura datum $(G, X)$. For $(\text{GSp}, S^\pm)$, the reflex field is $\mathbb{Q}$.

3.2. Integral canonical models. As explained in [Kis10, 2.3.1, 2.3.2], for a given Shimura datum $(G, X)$ with embedding $(G, X) \hookrightarrow (\text{GSp}, S^\pm)$, using Zarhin’s trick we may modify $(V, \psi)$ so that there exists a $\mathbb{Z}_{(p)}$-lattice $\Lambda$ of $V$ with the following property: (1), the pairing $\psi$ induces a perfect $\mathbb{Z}_{(p)}$-pairing on $\Lambda$, still denoted by $\psi$; (2) the embedding $G \to \text{GSp}$ is induced by an embedding $\mathcal{S} \hookrightarrow \text{GSp}(\Lambda, \psi)$ of reductive group schemes over $\mathbb{Z}_{(p)}$. From now on, we fix such an embedding and accordingly the modified embedding of Shimura data $(G, X) \hookrightarrow (\text{GSp}, S^\pm)$. Set $K_p = \text{GSp}(\mathbb{Z}_{(p)})$. By [Kis10, 2.1.2] there exists an open compact subgroup $\mathcal{K}^p \subseteq \text{GSp}(\mathbb{A}_f)$ containing $K^p$ such that $\mathcal{S}$ induces an embedding of Shimura varieties over $E$,

$$\text{Sh}_K \hookrightarrow \text{Sh}_E \otimes_{\mathbb{Q}} E.$$ 

Moreover, if $\mathcal{K}^p$ is sufficiently small, $\text{Sh}_E$ has a quasi-projective smooth model over $\mathbb{Z}_{(p)}$, denoted by $\hat{S} = \hat{S}_E$, which has an explicit moduli interpretation as described in [Kis17, 1.3.4]). In what follows we always assume that $K^p$ and $\mathcal{K}^p$ are sufficiently small, and we will also fix a $\mathbb{Z}$-lattice $\Lambda_S$ of the $\mathbb{Z}_{(p)}$-module $\Lambda$ such that $\Lambda_S \otimes \mathbb{Z}$ is $\mathcal{K}$-stable. The choice
of such a \( \mathbb{Z} \)-lattice allows one to describe the scheme \( \tilde{S} \) as moduli space of polarized abelian varieties (not just up to prime to \( p \)-isogeny). In particular, it comes with a universal abelian scheme, denoted by \( \mathcal{A} \).

Fix a place \( v \) of \( E \) above \( p \). Denote by \( \mathcal{O}_{E,(v)} \) the localization at \( v \) of the ring of integers \( \mathcal{O}_E \) of \( E \). Denote by \( S = S_K(G, X) \) the normalization of the schematic closure of \( \mathcal{S}_K \) in \( \tilde{S} \otimes \mathbb{Z}_p, \mathcal{O}_{E,(v)} \). Recall that we have the assumption \( p \geq 3 \). The following theorem is now well-known and is due to Vasiu and Kisin, independently.

**Theorem 3.2.1.** The scheme \( S \) is smooth over \( \mathcal{O}_{E,(v)} \) and is the integral canonical model over \( \mathcal{O}_{E,(v)} \) of \( \mathcal{S}_K \).

Strictly speaking, **integral canonical model** refers to a tower of models \( \{S_K\}_{K'} \) over \( \mathcal{O}_{E,(v)} \) for the tower \( \{\mathcal{S}_K\}_{K'} \), with \( K = K''K' \) and \( K' \) varying; see [Mil92, §2] for its precise meaning. Here we abuse language since soon \( K \) will be fixed till the end of this paper.

In particular, we obtain a finite morphism \( \varepsilon : S \to \tilde{S} \) of schemes over \( \mathcal{O}_{E,(v)} \). We call the pull-back to \( S \) of \( \mathcal{A} \) the **universal abelian scheme** of \( S \), still denoted by \( \mathcal{A} \). Write \( \kappa \) for the residue field of \( \mathcal{O}_{E,(v)} \) and \( S = S_K \) for the special fibre of \( S \). In particular, \( S \) is a quasi-projective smooth scheme over \( \kappa \), coming with a universal abelian scheme \( \mathcal{A} = \mathcal{A}_\kappa \).

In fact, the existence of the hyperspecial subgroup \( K_p \) implies that \( E \) is unramified at \( p \) (Mil94, 4.7), and hence we have \( \mathcal{O}_{E,v} = W(\kappa) \), where \( \mathcal{O}_{E,v} \) is the completion of \( \mathcal{O}_{E,(v)} \) with respect to its maximal ideal. In what follows, we will mainly work over \( W(\kappa) \) or over \( \kappa \). We will use the same notations for the base change to \( W(\kappa) \) of those objects defined over \( \mathcal{O}_{E,(v)} \) (e.g., the integral model \( S \)).

### 3.3. Reduction of Hodge cocharacters and their Frobenius twists.

As shown in [Kis10, 1.3.2], the \( \mathbb{Z}_p \)-reductive group scheme \( \mathfrak{g} \) can be realized as the schematic stabilizer of a finite set of tensors \( \{s_{\alpha}\}_{\alpha} \subseteq \Lambda^\otimes = (\Lambda^*)^\otimes \); i.e., for any \( \mathbb{Z}_p \)-algebra \( R \),

\[
\mathfrak{g}(R) = \{ g \in \text{GL}(\Lambda_k) \mid g(s_{\alpha}) = s_{\alpha} \text{, } \forall \alpha \},
\]

where \( s_{\alpha \cdot} \in (\Lambda_k^*)^\otimes \) denotes the tensor induced by \( s_{\alpha} \). Here for the functorial consideration later, we view \( \mathfrak{g} \) as a reductive \( \mathbb{Z}_p \)-subgroup scheme of \( GL(\Lambda^*) \) via the dual representation \( GL(\Lambda) \to GL(\Lambda^*) \),

\[
(3.3.1) \quad \iota : \mathfrak{g} \to \text{GSp}(\Lambda, \psi) \to GL(\Lambda) \cong GL(\Lambda^*).
\]

Write \( G \) for the special fibre of \( \mathfrak{g} \). It is a (connected) reductive group over \( \overline{\mathbb{F}}_p \).

For any \( h \in \mathbf{X} \), there is an associated Hodge cocharacter \( v_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}} \) which can be described as follows. For any \( \mathbb{C} \)-algebra \( R \), we have \( R \otimes_\mathbb{R} \mathbb{C} = R \times c^\ast(\mathbb{R}) \) where \( c \) denotes complex conjugation. Then on \( R \)-points \( v_h \) is given by

\[
R^x \to R^x \times c^\ast(\mathbb{R})^x = (R \otimes_\mathbb{R} \mathbb{C})^x \to \mathbb{S}(R) \to G_{\mathbb{C}}(R),
\]

where the first inclusion is given by \( x \in R^x \to (x, 1) \). Denote by [\( \mu \)] the unique \( G(\mathbb{C}) \)-conjugacy class in \( \text{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}}) \) which contains the inverses of all the \( v_h \)'s. Let \( Z = \text{Hom}_{\mathbb{Z}_p}(\mathbb{G}_{m,\mathbb{Z}_p}, \mathfrak{g}) \) be the fpqc sheaf of cocharacters, and \( \text{Ch} = \mathfrak{g} \setminus \mathbb{Z} \) the fpqc quotient sheaf of \( Z \) by the adjoint action of \( \mathfrak{g} \). By [DG11, Chap. XI, Cor. 4.2], the sheaf \( Z \) is represented by a smooth separated scheme over \( \mathbb{Z}_p \), and it is shown in [Zha18, 2.2.2] that \( \text{Ch} \) is represented by a disjoint union of connected finite étale schemes over \( \mathbb{Z}_p \). Moreover, it is shown in loc. cit. that the \( \mathbb{C} \)-point of \( \text{Ch} \) corresponding to the conjugacy class [\( \mu \)]
where the condition \( \lim_{t \to 0} \tilde{\mu} (t) g(t)^{-1} \) exists. Then we have
\[
\tilde{\mu} (p) \mathcal{P}_\mu (A) \tilde{\mu} (p)^{-1} \subseteq \mathcal{S}(A), \quad \tilde{\mu} (p) \mathcal{U}_\mu (A) \tilde{\mu} (p)^{-1} \subseteq K_1(\mathcal{S}(A)),
\]
with \( K_1(\mathcal{S}(A)) := \{ g \in \mathcal{S}(A) | \bar{g} = 1 \in G(\bar{A}) \} \), where \( \bar{g} \) denotes the image of \( g \) in \( G(\bar{A}) \) under the canonical reduction map \( \mathcal{S}(A) \to G(\bar{A}) \).

**Proof.** Recall the dynamic descriptions of \( \mathcal{P}_\mu \) and \( \mathcal{U}_\mu \) (see for example [CGP15, 2.1]):
\[
\mathcal{P}_\mu (A) = \{ g \in \mathcal{S}(A) | t \lim_{t \to 0} \tilde{\mu} (t) g(t)^{-1} \text{ exists} \}, \quad \mathcal{U}_\mu (A) = \{ g \in \mathcal{P}_\mu (A) | t \lim_{t \to 0} \tilde{\mu} (t) g(t)^{-1} = 1 \},
\]
where the condition \( \lim_{t \to 0} \tilde{\mu} (t) g(t)^{-1} \) exists means that the homomorphism of \( A \)-group schemes \( f_{\tilde{\mu}, g} : G_m \to \mathcal{S}_A, t \mapsto \tilde{\mu} (t) g(t)^{-1} \), extends to a morphism of \( A \)-schemes \( F_{\tilde{\mu}, g} : G_{a, A} \to \mathcal{S}_A, t \mapsto \tilde{\mu} (t) g(t)^{-1} \), while the condition \( \lim_{t \to 0} \tilde{\mu} (t) g(t)^{-1} = 1 \) requires \( F_{\tilde{\mu}, g}(0) = 1 \in \mathcal{S}(A) \).
Now let $g \in \mathcal{P}_+(A)$. Since $p \in \mathbb{G}_m(A)[\frac{1}{p}] \cap \mathbb{G}_a(A)$ ($A$ is $p$-torsion free), one finds that
\[
\overline{\mu}(p)g\overline{\mu}(p)^{-1} = f_{g,\overline{\mu}}(p) = F_{\overline{\mu},g}(p) \in G(A).
\]
If moreover, $g \in \mathcal{U}_+(A)$, the functoriality of $F_{\overline{\mu},g}$ for the canonical projection $A \to \bar{A}$, viewed as a map between $W(\kappa)$-algebras, implies
\[
\overline{\mu}(p)g\overline{\mu}(p)^{-1} = F_{\overline{\mu},g}(p) = 1.
\]
\[
\square
\]

For later applications, we fix an embedding of $\mathcal{G}_{W(\kappa)}$ into $\text{GL}_{2g, W(\kappa)}$ as follows. Choose a $W(\kappa)$-basis $v_1, \cdots, v_\kappa, v_{\kappa+1}, \cdots, v_{2g} \in \Lambda^*_W(\kappa)$ such that the first $g$ elements above lies in $\Lambda^*_W(\kappa)$ and the remaining ones lie in $\Lambda^*_W(\kappa)$. Then by sending an element $h \in \text{GL}(\Lambda^*_W(\kappa))$ to the matrix $X_h \in \text{GL}_{2g, W(\kappa)}$ such that
\[
h(v_1, \cdots, v_\kappa) = (v_1, \cdots, v_{2g})X_h,
\]
we obtain an isomorphism of $W(\kappa)$-group schemes between $\text{GL}(\Lambda^*_W(\kappa))$ and $\text{GL}_{2g, W(\kappa)}$. Hence from (3.3.1) we obtain an embedding of $\mathcal{G}_{W(\kappa)}$ into $\text{GL}_{2g, W(\kappa)}$, as $W(\kappa)$-reductive group schemes,
\[
(3.4.2)\quad t : \mathcal{G}_{W(\kappa)} \hookrightarrow \text{GS}_{\text{Sp}(\Lambda, \psi)_{W(\kappa)}} \hookrightarrow \text{GL}(\Lambda^*_W(\kappa)) \cong \text{GL}_{2g, W(\kappa)},
\]
and accordingly a cocharacter $\overline{\mu}' := t \circ \overline{\mu}$ of $\text{GL}_{2g, W(\kappa)}$. For every $W(\kappa)$-algebra $R$ such that $p \in R^\times$, we have
\[
(3.4.3)\quad \overline{\mu}'(p) = \left( \begin{array}{cc} pI_g & \ast \\ \ast & I_g \end{array} \right) \in \text{GL}_{2g}(R).
\]
We denote by $\mathcal{P}_\pm', \mathcal{U}_\pm', \mathcal{M}'$ the counterparts of $\mathcal{P}_\pm, \mathcal{U}_\pm, \mathcal{M}$ respectively for the cocharacter $\overline{\mu}'$ of $\text{GL}_{2g, W(\kappa)}$. Clearly these subgroups can be described explicitly in term of matrices; for example $\mathcal{U}_+'' \subseteq \mathcal{G}_{W(\kappa)}$ consists of matrices of the form $\left( \begin{array}{cc} I_g & \ast \\ \ast & \ast \end{array} \right)$, where $\ast$ denotes a $g$ by $g$-matrix. It is a general fact that we have (see [Con14, 4.1.10] for example)
\[
(3.4.4)\quad \mathcal{P}_\pm = \mathcal{P}_\pm' \cap \mathcal{G}, \quad \mathcal{U}_\pm = \mathcal{U}_\pm' \cap \mathcal{G}, \quad \mathcal{M} = \mathcal{M}' \cap \mathcal{G}.
\]
We shall see that the embedding $t$ in (3.4.2) will enable us to reduce some group-theoretic arguments in later sections to much easier problems like multiplying $2$ by $2$ block matrices.

3.5. **Tensors on $H_{\text{dR}}(A/S)$.** For all $i \geq 0$, write $H_{\text{dR}}^i(A/S) := \textbf{R}^i\pi_*(\Omega^*_{A/S})$ for the $i$-th relative de Rham cohomology of $A$ over $S$, where $\pi : A \to S$ is the structure morphism. As is shown in [BBM82, 2.5.2] (generalizing the well-known case where the base is a field to the case where the base is an arbitrary scheme), for all $i \geq 0$ (resp. all $r, s \geq 0$), the $\mathcal{O}_S$-module $H_{\text{dR}}^i(A/S)$ (resp. $\textbf{R}^i\pi_*(\Omega^r_{A/S})$) are locally free and their formations commute with arbitrary base change. Moreover, the Hodge-de Rham spectral sequence
\[
\mu \mathcal{E}^r = \textbf{R}^r\pi_*(\Omega^*_{A/S}) \Longrightarrow H_{\text{dR}}^{r+s}(A/S),
\]
deregenerates at $E_1$-page. In particular, we have an exact sequence of locally free $\mathcal{O}_S$-modules
\[
(3.5.1)\quad 0 \to \omega_{A/S} \to H_{\text{dR}}^1(A/S) \to \textbf{R}^1\pi_* \mathcal{O}_S \to 0,
\]
where the Hodge filtration $\omega_{A/S} = \pi_*\Omega^1_{A/S}$ is of rank $g$ and $H_{\text{dR}}^1(A/S)$ is of rank $2g$. 

Q. YAN
For typographical reason, in this and the next subsections we write \(V_{\text{dR}}\) for \(H^1_{\text{dR}}(A/S)\). Below we explain the so-called “(integral) de Rham tensors” on \(V_{\text{dR}}\). We will need these tensors to define interesting torsors over \(S\) in §3.7.

The \(\mathbb{Q}\)-representation \(V\) of \(G\) coming from the embedding \(G \hookrightarrow \text{GSp}(V, \psi)\) gives rise to a \(\mathbb{Q}\)-local system \(V_{B,\mathbb{Q}} = R^1\pi_{\mathbb{Q}}^\ast\mathbb{Q}\) on \(\text{Sh}_{K,\mathbb{C}}\). Below we first explain how the tensors \((s_\alpha)_\alpha \subseteq V^\otimes\) that cut out \(G\) inside \(\text{GL}(V^\ast)\) induce global sections on \(V_B^\otimes\); cf. \cite[2.2]{Kis10} and \cite[2.3]{CS17}. Write:

\[
\text{Sh}_K = X \times G(\mathbb{A}^f)/K, \quad \text{Sh}_{K'} = S^\perp \times \text{GSp}(\mathbb{A}^f)/K'.
\]

Then the canonical projection \(\text{Sh}_K \rightarrow \text{Sh}_{K',\mathbb{C}}\) (resp. \(\text{Sh}_K' \rightarrow \text{Sh}_{K',\mathbb{C},\text{an}}\)) makes \(\text{Sh}_K\) a \(G(\mathbb{Q})\)-torsor over \(\text{Sh}_{K,\mathbb{C}}\) (resp. \(\text{Sh}_K'\) a \(G(\mathbb{Q})\)-torsor over \(\text{Sh}_{K',\mathbb{C}}\)). To make distinctions, we write \(A'\) for the universal (analytic) abelian variety over \(\text{Sh}_{K',\mathbb{C}}\) with \(\pi' : A' \rightarrow \text{Sh}_{K',\mathbb{C}}\) the structure map. Then we know that the isogeny class of \(A'\) corresponds to the dual of the \(\mathbb{Q}\)-local system \(R^1\pi'_\mathbb{Q}\) (viewed as a variation of Hodge structure over \(\text{Sh}_{K',\mathbb{C}}\)), which in turn corresponds to the constant \(\mathbb{Q}\)-local system \(V\) over the cover \(\text{Sh}_K\), together with the structure morphism \(G(\mathbb{Q}) \rightarrow \text{GL}(V)\). Clearly we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Sh}_K & \xrightarrow{\quad} & \text{Sh}_{K'} \\
\downarrow & & \downarrow \\
\text{Sh}_{K,\mathbb{C}} & \longrightarrow & \text{Sh}_{K',\mathbb{C}}
\end{array}
\]

where the top horizontal map is equivariant w.r.t. the group homomorphism \(G(\mathbb{Q}) \rightarrow \text{GSp}(\mathbb{Q})\). Hence the variation of Hodge structure \(V_{B,\mathbb{Q}}\) over \(\text{Sh}_{K,\mathbb{C}}\), that corresponds to the isogeny class of \(A\), also corresponds to the constant \(\mathbb{Q}\)-local system \(V^\ast\) over the cover \(\text{Sh}_K\), together with the representation \(G(\mathbb{Q}) \hookrightarrow \text{GSp}(\mathbb{Q}) \rightarrow \text{GL}(V) \cong \text{GL}(V^\ast)\). Now it is clear that the set of tensors \((s_\alpha)_\alpha \subseteq V^\otimes\) gives rise to a set of global sections (we simply call them Betti-tensors),

\[
(s_{\alpha, B})_\alpha \subseteq \Gamma(\text{Sh}_{K,\mathbb{C}}^\text{an}, V_{B,\mathbb{Q}}^\otimes).
\]

By the Riemann-Hilbert correspondence \cite{Del70} we have the following equivalences of tensor categories,

\[
\text{Loc}_C(\text{Sh}_{K,\mathbb{C}}^\text{an}) \xrightarrow{(\cdot)^\otimes \mathcal{O}_{\text{Sh}_{K,\mathbb{C}}}^\text{an}} \text{VBIC}(\text{Sh}_{K,\mathbb{C}}^\text{an}) \xrightarrow{(\cdot)^\text{an}} \text{VBIC}(\text{Sh}_{K,\mathbb{C}})\text{reg},
\]

where \(\text{Loc}_C(\text{Sh}_{K,\mathbb{C}}^\text{an})\) denotes the tensor category of \(C\)-local systems over \(\text{Sh}_{K,\mathbb{C}}\), \(\text{VBIC}(\text{Sh}_{K,\mathbb{C}}^\text{an})\) (resp. \(\text{VBIC}(\text{Sh}_{K,\mathbb{C}})\text{reg}\)) denotes the tensor category of holomorphic (resp. algebraic) vector bundles with integrable connections (resp. with integrable connections, with regular singularities at infinity). Under these category equivalences, the \(C\)-local system \(V_{B,C} := V_{B,\mathbb{Q}} \otimes \mathbb{C}\) corresponds to the vector bundle \(\mathcal{V}_{\text{dR}, C} := H^1_{\text{dR}}(A/\text{Sh}_{K,C})\) over \(\text{Sh}_{K,C}\), and we have a parallel isomorphism of analytic vector bundles over \(\text{Sh}_{K}^\text{an}\),

\[
e : V_{B,C} \otimes \mathcal{O}_{\text{Sh}_{K,C}}^\text{an} \cong V_{\text{dR}, C}^\text{an},
\]

where the left hand side is equipped with trivial connections. Hence by transport of structure, we obtain from the Betti tensors \((3.5.2)\) our desired horizontal global sections (call them de Rham tensors),

\[
(s_{\alpha, dR})_\alpha \subseteq \Gamma(\text{Sh}_{K,\mathbb{C}}, V_{\text{dR}, C}^\otimes),
\]
such that $e(s_{a,B}) = s_{a,dr}^{an}$, with $s_{a,dr}^{an}$ understood. Here note that although $V_{B,C}^\otimes$ does not live inside Loc$_C$(Sh$_{k,\mathbb{C}}^{an}$), each Betti tensor $s_{a,B}$ lies in some direct summand of $V_{B,C}^\otimes$ which does live inside Loc$_C$(Sh$_{k,\mathbb{C}}^{an}$).

**Proposition 3.5.1** ([Kis10], 2.2.1, 2.3.9). Each of the de Rham tensors $s_{a,dr}$ in (3.5.3) descends to $\mathcal{O}_{E,\{\psi\}}$; i.e., there exist (necessarily unique) horizontal global sections

$$(s_{a,dr})_\alpha \subseteq \Gamma(S, V_{dr}^\otimes),$$

whose restriction on Sh$_{k,\mathbb{C}}$ are the tensors in (3.5.3).

We call the global sections $s_{a,dr}$ obtained here (integral) de Rham tensors.

### 3.6. Crystalline nature of integral de Rham tensors

In this subsection we make clear of a (perhaps well-known) consequence of Prop. 3.5.1 concerning the property of integral tensors $s_{a,dr}$ being horizontal, as it will be needed later.

Let $R$ be a $p$-complete flat $W(k)$-algebra and $x, y : \text{Spec} R \to S$ morphisms of $W(k)$-schemes which are congruent modulo $p$. Write $V_{dr,x}, s_{a,dr,x}$ the pull backs to $R$ along $x$ of $V_{dr}, s_{a,dr}$ respectively; similarly for $V_{dr,y}, s_{a,dr,y}$. It is a well-known fact that the vector bundle $V_{dr}$ on $S$ has an $F$-crystal structure in the sense of [Kat73] and the Gauss-Manin connection on $V_{dr}$ provides a canonical isomorphism of $R$-modules (see for example (1.2) of loc. cit.), $\varepsilon(x,y) : V_{dr,x} \cong V_{dr,y}$.

Since $s_{a,dr}$ is horizontal, we have

$$\varepsilon(x,y)(s_{a,dr,x}) = s_{a,dr,y}.$$ 

### 3.7. Torsors over Shimura varieties

For simplicity, from now on we write $s$ instead of $(s_{a})_\alpha$; similarly we simply write $s_{dr}$ instead of $(s_{a,dr})_\alpha$. For a $W(k)$-morphism $x : \text{Spec} R \to S$, we also write $s_{dr,R}$ for $s_{dr,x}$ (i.e., the pull-back of $s_{dr}$ along $x$), if the structure morphism $x$ is understood. However, in order to keep notations suggestive, we still write $H^1_{cr}(A/S)$ instead of $V_{dr}$. Now we are ready to define two $S$-schemes below, which will play important roles later.

$$I := \text{Isom}_{\mathcal{O}_S} \left( [A^\otimes_{W(k)}, s_{W(k)}] \otimes \mathcal{O}_S, \ [H^1_{dr}(A/S), s_{dr}] \right),$$

$$I^+ := \text{Isom}_{\mathcal{O}_S} \left( [A^\otimes_{W(k)} \supseteq \Lambda^+_1 W(k), s_{W(k)}] \otimes \mathcal{O}_S, \ [H^1_{dr}(A/S) \supseteq \omega_{A/S}, s_{dr}] \right).$$

Unwinding definition: for every $W(k)$-algebra $R$, a point $x^p \in I^+(R)$ consists of a pair $(x, \beta_x)$, where $x \in S(R)$ corresponds to a morphism $x : \text{Spec} R \to S$ of $W(k)$-schemes, and

$$\beta_x : (A^+_R \supseteq \Lambda^+_1) \cong (H^1_{dr}(A_x/R) \supseteq \omega_x)$$

is an isomorphism of $R$-modules, which maps $s_R$ to $s_{dr,R}$ termwise. Here following our notational convention we denote $A_x, \omega_x, s_{dr,R}$ the pull back to $R$ along $x$ of $A, \omega_{A/S}, s_{dr}$ respectively. We have similar descriptions for points $x \in I(R)$ by omitting filtrations in $\beta_x$.

Clearly $S$ resp. $\mathcal{P}_+$ naturally acts on $I$, resp. $I^+$ on the right, freely and transitively. To be precise, the action of a section $h \in \mathcal{G}(R)$ (resp. $h \in \mathcal{P}_+(R)$) on $I(R)$ (resp. on $I^+(R)$) is given by

$$x^p \cdot h = (x, \beta_x) \cdot h = (x, \beta_x h).$$

**Lemma 3.7.1.** The scheme $I^+$ (resp. $I$) is a $\mathcal{P}_+$-torsor (resp. $\mathcal{G}$-torsor) over $S$.

**Proof.** We only show here the assertion for $I^+$ as the assertion for $I$ can be shown in the same way; or maybe better, it follows from the fact that $I$ is the push-forward of $I^+$ along the homomorphism $\mathcal{P}_+ \to \mathcal{G}$.

Since $I^+$ is an $S$-scheme of finite presentation and the action of $\mathcal{P}_+$ on $I^+$ is free and transitive, it suffices to show that $I^+$ is faithfully flat over $S$. In other words, we need to
show that for each closed point $s$ of $S$, the pullback of $I_+$ to Spec$O_{S,x}$ along the natural map \( \text{Spec}O_{S,x} \to I_+ \), denoted by $I_{+,x}$, is a $P_+$-torsor over Spec$O_{S,x}$. Since we know already that when restricted to the generic fibre $\text{Sh}_K$ of $S$, $I_+$ is a $P_+$-torsor over $S$ (as it is so after further base change to $\text{Sh}_{K,\mathbb{C}}$), we may assume that $s$ lies in the special fibre of $S$. But as stated in [Zha18, Lem. 2.3.2, 2)], it is essentially shown in [Kis10] that $I_{+,x}$ is a trivial $P_+$-torsor over $O_{S,x}$.

\[ \square \]

4. The Zip Period Map $\zeta$ for $S$

In this section we review the zip period map $\zeta: S \to G\text{-Zip}^\mu$, that Zhang constructs in [Zha18], following loc. cit. and [Wor13].

4.1. The stack of $G$-zips. Let $\mu: \mathbb{G}_{m,\kappa} \to G_{\kappa}$ be as in (3.3.2), and

\[ M, U_+ \subseteq P_+, \]

the special fibres of the algebraic groups $M, U_+ \subseteq P_+$ defined in (3.3.4). Recall our notational convention in §1.3: for a subgroup $H \subseteq G_{\kappa}$, we write $H^\sigma \subseteq G^\sigma_{\kappa} \cong G_{\kappa}$ for its base change along $\sigma: \kappa \to \kappa$.

**Definition 4.1.1 ([PWZ16, 3.1]).** Let $T$ be a scheme over $\kappa$. A $G$-zip of type $\mu$ over $T$ is a quadruple $I = (I, I_+, I_-, I_0)$ consisting of a right $G$-torsor $I$ over $T$, a $P_+$-torsor $I_+ \subseteq I$, and $P^\sigma_-$-torsor $I_- \subseteq I$, and an isomorphism of $M^\sigma_0$-torsors:

\[ \iota: I^0_+ / U^\sigma_+ \cong I_- / U^\sigma_. \]

A morphism $I \to I' = (I', I'_+, I'_-, I'_0)$ of $G$-zips of type $\mu$ over $T$ consists of a $G$-equivariant morphism $I \to I'$ which sends $I_+$ to $I'_+$ and $I_-$ to $I'_-$, and which is compatible with the isomorphisms $\iota$ and $\iota'$. The category of $G$-zips over all $\kappa$-schemes form an algebraic stack over $\kappa$.

For the cocharacter $\mu$ there is an associated group scheme $E_\mu \subseteq P_+ \times P^\sigma_-$, called the **zip group** of $\mu$, which is given on points of a $\kappa$-scheme $T$ by

\[
E_\mu(T) = \{ (u_+, m, u_-(m)) \mid m \in M(T), u_+ \in U_+(T), u_- \in U^\sigma_-(T) \}.
\]

Here we use the decomposition of $\kappa$-groups $P_+ = U_+ \times M$, $P^- = U_- \times M$. Clearly we have an isomorphism of $\kappa$-group schemes

\[ U_+ \times M \times U_- \cong E_\mu, \quad (u_+, m, u_-) \mapsto u_+mu_-, \]

where we omit describing the group law of the LHS. In particular, $E_\mu$ is a smooth connected linear algebraic group over $\kappa$. Consider its right action on $G_{\kappa}$ by

\[
g \cdot (p_+, p_-) = p_-^{-1}gp_- = m^{-1}u_+^{-1}gu_- \sigma(m).
\]

With respect to this action one can form the quotient stack $[G_{\kappa}/E_\mu]$ over $\kappa$. Here we use the right action while in [PWZ11] and [PWZ15], as well as in [Zha18], the left action is used, but apparently the resulting stacks $[E_\mu/G_{\kappa}]$ and $[G_{\kappa}/E_\mu]$ are canonically isomorphic.

**Theorem 4.1.2 ([PWZ15, 3.11, 3.12]).** The stacks $G\text{-Zip}^\mu$ and $[G_{\kappa}/E_\mu]$ are naturally isomorphic. They are smooth algebraic stacks of dimension 0 over $\kappa$. 
4.2. The universal G-zip over S. In this subsection we give definitions of those torsors appearing in the universal G-zip constructed in [Zha18] and refer to loc. cit. for more details.

For the relative de Rham cohomology $H^1_{\text{dR}}(A/S)$, apart from the well-known Hodge filtration $\omega_{A/S} \subseteq H^1_{\text{dR}}(A/S)$, there is another filtration

$$\mathcal{O}_{A/S} := R^1\pi_*\mathcal{F}^0(\Omega^\bullet_{A/S}),$$

called the conjugate filtration of $H^1_{\text{dR}}(A/S)$, fitting into the short exact sequence

$$(4.2.1) 0 \to \mathcal{O}_{A/S} \to H^1_{\text{dR}}(A/S) \to \pi_*\mathcal{F}^1(\Omega^\bullet_{A/S}) \to 0,$$

of locally free $O_S$-modules. This short exact sequence is a particular consequence of the degeneration at $E_2$-page of the conjugate spectral sequence of the filtration. As discussed in [MW04, 9.1-9.5], Cartier isomorphisms (9.4 in loc. cit.) induces the following direct-summand-wise isomorphism of $O_S$-modules,

$$(4.2.2) \delta : \omega_{A/S}^\sigma \oplus (H^1_{\text{dR}}(A/S)/\omega_{A/S})^\sigma \cong (H^1_{\text{dR}}(A/S)/\mathcal{O}_{A/S}) \oplus \mathcal{O}_{A/S}.$$

We call the direct-summand-wise isomorphism $\delta$ the zip isomorphism associated with the universal abelian scheme $A$ over $S$. The tuple $(H^1_{\text{dR}}(A/S), \omega_{A/S}, \mathcal{O}_{A/S}, \delta)$ is an “F-zip” in the terminology of [MW04]. We call it the universal F-zip over $S$. The universal $G$-zip over $S$, to be defined below, should be viewed as the universal F-zip over $S$ with a $G$-structure.

Remark 4.2.1. The zip isomorphism $\delta$ above can also be constructed using crystalline Dieudonné theory, without explicit reference to Cartier isomorphisms, as is done in [Zha18]. Indeed, there are canonical isomorphism of $O_S$-modules $H^1_{\text{dR}}(A/S) \cong D^+(A[p^\infty])_S \cong D^+(A)_S$, where $D^+(A)_S$ is the restriction on $S$ zar (namely, the Zariski site of $S$) of the Dieudonné crystal $D^+(A)$ associated to $A$ ([BBM82, 2.5.7]). Under this canonical isomorphism, the Hodge filtration on both sides coincide ([BBM82, 2.5.8]) and the conjugate filtration $\mathcal{O}_{A/S}$ is equal to $\text{Ker}(V : D^+(A)_S \to D^+(A)[1])$. Then one can proceed to construct $\delta$ in the same way as described in (4.2.2) below.

Write $\Lambda^\sigma_k = \Lambda^\sigma \oplus \Lambda^{\sigma, -1}$ for the weight decomposition of $\Lambda^\sigma_k$ induced by the inverse of $\mu^\sigma$. Due to the canonical isomorphism $\Lambda^\sigma_k \cong \Lambda^{\lambda^\sigma}_k$, such a decomposition can be described in a different way: if $\Lambda^\sigma_k = \Lambda^{\sigma, 0} \oplus \Lambda^{\sigma, -1}$ is the weight decomposition of $\Lambda^\sigma_k$ induced by $\mu$ as in (3.3.5), we have

$$(4.2.3) \Lambda^0_k = \text{can}^{-1}(\Lambda^{\sigma, 0}), \quad \Lambda^{-1}_k = \text{can}^{-1}(\Lambda^{\sigma, -1}).$$

Here $\Lambda^{\sigma, 0} : = (\Lambda^{\sigma, 0})^\sigma$; similarly for $\Lambda^{\sigma, 1}$.$^\sigma$. Then $P^\sigma$ is the schematic stabilizer in $G_k$ of the filtration $\Lambda^\sigma_k \subseteq \Lambda^\sigma_k$. Now we come to the definitions of the following $k$-schemes,

$$(4.2.4) I := \text{Isom}_{O_S}([\Lambda^\sigma_k, s_k] \otimes O_S, [H^1_{\text{dR}}(A/S), s_{\text{dR}}]),$$

$$(4.2.4) I_+ := \text{Isom}_{O_S}([\Lambda^{\lambda^\sigma}_k \supseteq \Lambda^{\lambda^\sigma, 1}_k, s_k] \otimes O_S, [H^1_{\text{dR}}(A/S) \supseteq \omega_{A/S}, s_{\text{dR}}]),$$

Clearly $I$ and $I_+$ are special fibres of $I$ and $I_+$ respectively (§3.7). The group $G_k$ (resp. $P^\sigma_+$, resp. $P^{-\sigma}$) acts on $I$ (resp. $I_+$, resp. $I_-$) on the right, as in (4.1.2).

---

2The degeneration of the conjugate spectral sequence at E2-page follows from that of the Hodge-de Rham spectral sequence at E1-page; see for example [Kar74, 2.3.2].
**Theorem** 4.2.2 ([Zha18, 2.4.1, 3.1.2]). (1) The scheme \( I (\text{resp. } I_+\text{, resp. } I_-) \) is a \( G_\kappa \)-torsor (resp. \( P_+\)-torsor, resp. \( P_-\)-torsor) over \( S \).

(2) The direct-summand-wise isomorphism \( \delta \) in (4.2.2) induces an isomorphism

\[
t : I^\delta_+/U^\sigma_+ \cong I_-/U^\sigma_-
\]

Hence, the tuple \( \underline{I} : = (I, I_+, I_-, t) \) is a \( G \)-zip of type \( \mu \) over \( S \), inducing a morphism of algebraic stacks over \( \kappa \)

\[
\zeta : S \longrightarrow G\text{-Zip}^\mu \cong [G_\kappa/E_\mu].
\]

(3) The map \( \zeta \) is a smooth map of \( \kappa \)-stacks.

We shall call the tuple \( \underline{I} : = (I, I_+, I_-, t) \) the **universal \( G \)**-**zip** over \( S \) and \( \zeta \), the **zip period map** for \( S \). As indicated at the beginning of this section, our focus in this paper is the map \( \zeta \) itself. But for the reader’s curiosity, we end this section by giving the definition of EO strata for \( S_{G_\mu} \).

**Definition** 4.2.3 ([Zha18, 3.1.1]). Set \( k = \tilde{F}_p \). For a geometric point \( w : \text{Spec}k \rightarrow G\text{-Zip}^\mu \), the **EO stratum** of \( S_k \) associated to \( w \); denoted by \( S_k^w \), is defined to be the fibre of \( w \) under the zip period map \( \zeta : S_k \rightarrow G\text{-Zip}^\mu_k \).

Merely by definition of \( \zeta \), being a morphism of algebraic stacks, and the property of \( [G_k/E_\mu] \) being a \( 0 \)-dimensional stack, one learns that each \( S_k^w \) is a locally closed subscheme of \( S_k \). Moreover, the smoothness of \( \zeta \) implies that each \( S_k^w \) is automatically a smooth \( \kappa \)-scheme. See [Zha18] for more properties of these EO strata.

5. **Construction of \( \gamma : I_+ \rightarrow G_\kappa/U^\sigma_+ \)**

The main goal of this section is to construct a morphism of \( \kappa \)-schemes \( \gamma : I_+ \rightarrow G_\kappa/U^\sigma_+ \) and to deduce from it a morphism of \( \kappa \)-stacks, \( \eta : S \rightarrow [G_\kappa/E_\mu] \). The comparison of \( \eta \) with \( \zeta \) will be given in \S 6. Here \( G_\kappa/U^\sigma_+ \) is the quotient fpqc sheaf of \( G_\kappa \) by the \( U^\sigma_+ \)-action via right multiplication. It is represented by a scheme, smooth separated of finite type over \( \kappa \), and the canonical projection \( G_\kappa \rightarrow G_\kappa/U^\sigma_+ \) is smooth; see for example [Mil17, 7.15, 7.17].

5.1. **Trivialized Frobenius.** Let \( \bar{R} \) be a \( \kappa \)-algebra which Zariski locally admits a finite \( p \)-basis, and \( \bar{x} = (\bar{x}, \beta_\bar{A}) \in I_+(\bar{R}) \) an \( \bar{R} \)-point of \( I_+ \); (cf. \S 3.7). Let \( \bar{R} = (R, \sigma) \) be a simple frame of \( \bar{R} \) (which exists by Lem. 2.1.1) and \( x^\sigma = (x, \beta_x) \in \mathbb{I}_+(R) \) a lift of \( \bar{x} \); here \( x^\sigma \) exists since \( \mathbb{I}_+ \) is smooth over \( W(\kappa) \).

By Thm. 2.3.2, the \( p \)-divisible group \( A_+[p^\infty] \) corresponds to an object in \( \text{AFDM}(\bar{R}, \nabla) \), namely an admissibly filtered Dieudonné module over \( \bar{R} = (R, \sigma) \),

\[
\mathcal{M} = (M, F, V, \nabla_M, M^1_+) \quad \text{with} \quad M = \text{D}^+(A_+[p^\infty])(R),
\]

where, with the simple frame \( \bar{R} \) fixed, the Dieudonné module \( (M, F, V, \nabla_M) \) is determined by \( A_+[p^\infty] \), and hence by \( \bar{x} \), while the admissible filtration \( M^1_+ \subseteq M \) depends on the lift \( x \) of \( \bar{x} \). In particular, depending on the objects \( (x \text{ or } \bar{x}) \) we want to emphasize, we can write

\[
F = F_x = F_{\bar{x}} : M^\sigma \rightarrow M,
\]

By Rem. 2.3.1 that we have canonical isomorphism of filtered \( R \)-modules

\[
(M \supseteq M^1_+) \cong (H^1_{\text{DR}}(A_+/R) \supseteq \omega_x).
\]
For this reason we identify them and this identification equips $M$ with a set of tensors $s_{IR,R} \in M^{\otimes}$. With this identification we view $\beta_x$ as a trivialization of the filtered module $(M \supseteq M^1)$. Let $\beta_x : \left(\Lambda^+_k \supseteq \Lambda^+_{IR,R} \right) \cong (M \supseteq M^1)$.

Note that since $\Lambda^*$ is a free $\mathbb{Z}_{(p)}$-module, we have canonical isomorphisms $\varepsilon : (\Lambda^+_R, s_R) \cong (\sigma^* \Lambda^+_R, \sigma^* s_R)$. By transport of structure we obtain the \textbf{trivialized Frobenius} $(5.1.2)$

\[ F_{\phi} = \beta^{-1}_x \circ \phi(\beta_x) : \Lambda^+_R \to \Lambda^+_R, \]

where we set $\phi(\beta_x) : \Lambda^+_R \to M^\sigma$ to be $\beta^{\sigma}_x \varepsilon$. For an element $h \in \mathcal{P}_+(R)$, we have $(5.1.3)$

\[ F_{\phi^h} = h^{-1} F_{\phi} \circ \sigma(h), \]

by definition of the action of $\mathcal{P}_+(R)$ on $\mathbb{I}_+(R)$ ($\S3.7$). Here, $\sigma(h)$ is defined as,

\[ \sigma(h) := \varepsilon^{-1} h^{\sigma} \varepsilon; \]

this coincides with our notational convention $(1.3.1)$. Sometimes, we simply identify $\sigma(h)$ and $h^{\sigma}$ by suppressing the canonical isomorphism $\varepsilon$ above. Clearly, for an element $x^\phi \in \mathbb{I}(R)$, we can define $F_{\phi^h}$ in the same way.

\section{Frobenius invariance of tensors.} The setting in this subsection is the same as in the previous subsection $\S5.1$

\textbf{Lemma 5.2.1.} For an element $x^\phi = (x, \beta_x) \in \mathbb{I}(R)$, the Frobenius $F_{\phi^h}$ defined above preserves tensors $s_R$ termwise. In particular, we have

\[ F_{\phi^h} \in \mathcal{S}(R[1/p]). \]

\textit{Proof.} This follows from the next lemma and the definition of $F_{\phi^h}$. \hfill $\square$

\textbf{Lemma 5.2.2.} The Frobenius $F : M^\sigma \to M$, after inverting $p$, sends $\sigma^*_R s_{DR,R}$ to $s_{DR,R}$ termwise.

\textit{Proof.} For any maximal ideal $m$ of $R$, by Lem. 2.1.1 (4) the Frobenius lift $\sigma : R \to R$ induces a simple frame $(\hat{R}_m, \sigma)$ of $\hat{R}_m$, compatible with $(R, s_R)$. Note that $\hat{R}_m$ is necessarily $p$-complete (since $m$ contains $p$). Hence it suffices to show the lemma after base change to $\hat{R}_m$ for all $m$. In particular, we may assume that $R$ is a local ring.

Let $s \in \mathcal{S}$ be the image of the closed point of $\text{Spec} R$, which necessarily lies in the special fibre $S \subseteq \mathcal{S}$. Let $s \in S$ be a closed point which is a specialization of $s'$. Then the morphism $x : \text{Spec} R \to \mathcal{S}$ facts through the canonical embedding $s : \text{Spec} A \to \mathcal{S}$, where $A := \hat{O}_{S, \sigma}$ is the complete local ring of $\mathcal{S}$ at $x$. Choose a $W(k)$-isomorphism $A \cong W(k)[X_1, \cdots, X_r]$ and consider the Frobenius lift $\sigma_A : A \to A$ of $A$ given by sending each $X_i$ to its $p$-th power. Write $N := (N, \mathcal{F}_N, \nabla_N, \nabla_N)$ for the Dieudonné module over $(A, \sigma_A)$ of $A[p]\epsilon$. Then the induced de Rham tensor $s_{DR,A} \in N^\otimes$ is horizontal. We claim that $s_{DR,A}$ is also Frobenius invariant. Prior to showing the claim, let us note that the claim implies our lemma. Indeed, if we let $f : A \to \hat{R}_m$ to denote the structure morphism, then $M$ is canonically isomorphic to the pull back $f^* N = N \otimes_{A,f} R$. If we identify this canonical isomorphism, then the Frobenius $F_M$ is equal to

\[ M^\sigma \cong \sigma^*_R f^* N \cong f^* \sigma_A^* N \cong \sigma^* f\circ \sigma_A^* N \cong \sigma^* f^* N \cong \sigma^* M, \]

where the isomorphism $\sigma^*_R f^* N \cong \sigma^* f\circ \sigma_A^* N$ is provided by the integrable connection $\nabla_N$ (note that $\sigma_R \circ f$ and $f \circ \sigma_A$ become the same after modulo $p$); in fact, due to our choice of free
coordinates $X_s$, it is possible to give an explicit formula for $e$ (see, for example [Kis10, 1.5]). Then since $s_{dR,A}$ is horizontal, one sees that $e$ sends $\sigma_R f^* s_{dR,A}$ to $f^* \sigma_R s_{dR,A}$ (this can also be seen from the explicit expression of $e$, cf. [Kis10, 1.5.4]). Now it is clear that we are reduced to show the claim.

The proof of the claim is already in the proof of [Kis10, Prop. 2.3.5]. We now write $B$ for the adapted deformation ring $R_{G_{cris}}$ in loc. cit. of the $p$-divisible group $A[p^\infty]$ over $k := k(s)$, where we use $\breve{s} : Speck \to S$ to denote the special fibre of $s$. Again the $W(k)$-algebra $B$ is isomorphic to some power series ring over $W(k)$ and one can equip it with a Frobenius lift $\sigma_B : B \to B$ by sending free coordinates to their $p$-th powers. Write $\underline{L} := (L, F_L, V_L, \nabla_L)$ for the Dieudonné module over $(B, \sigma_B)$ that corresponds to the universal $p$-divisible group over $B$; the construction of $\underline{L}$ is explained in [Kis10, 1.5.4]. By construction the Dieudonné module $L$ comes with Frobenius-invariant tensors which we denote by $s_{crys}$; we know from the proof of [Kis10, 1.5.4] (see also [Lov17, Thm. 3.3.12] for more details) that there exists a $W(k)$-algebra homomorphism (in fact an isomorphism) $g : B \to A$ such that the tuple $(N, s_{dR,A})$ is obtained as the pull back along $g$ of the tuple $(\underline{L}, s_{crys})$, expect that one has to use the integrable connection $\nabla_L$, as we did for $\nabla_N$, to deal with the possible incompatibility of Frobenius lifts between $\sigma_A$ and $\sigma_B$. Again using the fact that $s_{crys}$ is horizontal, we conclude that $s_{dR,A}$ is also Frobenius invariant. 

5.3. Trivialized partially divided Frobenius. Let $R, \tilde{R}, x^\phi$ and $x^\mu$ be as in §5.1 For the next lemma, which plays important roles for our construction of $\gamma : I_+ \to G_\kappa/U^\sigma_\kappa$ below, one may recall the characters $\tilde{\mu}$ and $\mu^\sigma = \sigma(\tilde{\mu})$ in §3.3.

Lemma 5.3.1. For every $x^\phi \in \mathcal{I}_+(R)$ we have $F_{x^\phi} \in \mathcal{S}(R)\mu^\sigma(p) \subseteq \mathcal{S}(R[1/p])$; i.e., we have $F_{x^\phi} = f_{x^\phi}\tilde{\mu}^\sigma(p)$ for some (necessarily unique) element $f_{x^\phi} \in \mathcal{S}(R)$.

Proof. The weight decomposition $A^+_R = A_R^{+1} \oplus A_R^{+0}$ given by $\tilde{\mu}$ (see §3.3), induces via $\beta_x$ a normal decomposition $M = M^1 \oplus M^0$ of $M$. Then by §2.5 we have the decomposition $F_x = \Gamma_{x^\phi} \circ f_{x^\phi}$, with $\Gamma_{x^\phi}$ and $f_{x^\phi}$ defined as in (2.5.1). Note that by Lem. 2.5.1 the partially divided Frobenius $\Gamma_{x^\phi}$ is an isomorphism of $R$-modules. Now we have

\begin{equation}
F_{x^\phi} = \beta_x^{-1} F_{x^\phi}\sigma(\beta_x) = \beta_x^{-1}(\Gamma_{x^\phi} f_{x^\phi}) \sigma(\beta_x) = f_{x^\phi}(\sigma(\beta_x)^{-1} f_{x^\phi} \sigma(\beta_x)),
\end{equation}

where $f_{x^\phi}$ is defined as

\begin{equation}
f_{x^\phi} := \beta_x^{-1} \Gamma_{x^\phi} \sigma(\beta_x).
\end{equation}

Clearly we have $f_{x^\phi} \in \text{GL}(A^+_R)$. Unwinding the definition of $\tilde{\mu}^\sigma$ in §3.3 we see that $\sigma(\beta_x)^{-1} f_{x^\phi} \sigma(\beta_x) = \tilde{\mu}(p)$.

Now the equality (5.3.1) becomes $F_{x^\phi} = f_{x^\phi} \tilde{\mu}^\sigma(p)$, and hence by Lem. 5.2.1 we have $f_{x^\phi} \in \mathcal{S}(R[1/p]) \cap \text{GL}(A^+_R) = \mathcal{S}(R)$. 

\hfill \Box

We will call $f_{x^\phi} \in \mathcal{S}(R)$ the trivialized partially divided Frobenius attached to $x^\phi$, w.r.t. the simple frame $R = (R, \sigma)$. 


5.4. **Local version of \( \gamma \).** We continue our discussion in the setting of the previous subsection.

With the simple frame \( R \) fixed, for an element \( x^\phi \in \mathbb{I}_+(R) \), as usual we write \( \overline{f}_\phi \in G(\overline{R}) \) for the reduction modulo \( p \) of the trivialized partially divided Frobenius \( f_\phi \). Denote by \( \gamma_{\overline{f}} \in G_\kappa/U_\alpha(\overline{R}) \) for the image of \( \overline{f}_\phi \) along the canonical projection \( G(\overline{R}) \to G_\kappa/U_\alpha(\overline{R}) \). The notation \( \gamma_{\overline{f}} \) is justified by the following proposition.

**Proposition 5.4.1.** The element \( \gamma_{\overline{f}} \in G_\kappa/U_\alpha(\overline{R}) \) is determined by \( \overline{f} \), i.e., it is independent of the choice of lifts \( x^\phi \) and the choice of simple frames \( R \) of \( \overline{R} \). In particular, it induces a morphism of \( \kappa \)-schemes,

\[
\gamma_{\overline{f}} : \text{Spec} \overline{R} \to G_\kappa/U_\alpha.
\]

**Proof.** Let \( R' = (R', \sigma') \) be another simple frame of \( \overline{R} \) and \( y^\phi = (y, \beta_y) \in \mathbb{I}_+(R') \) another lift of \( \overline{f} \). Denote by \( f_{\phi, \sigma'} \in \mathbb{S}(R') \) be the trivialized partially divided Frobenius attached to \( y^\phi \), w.r.t. \( R' \). We will compare below the two elements \( \overline{f}_{\phi, \sigma'} \in G(\overline{R}) \).

Note first that we may assume \( R = R' \). Indeed, by Lem. \[2.1.1\] we can choose an isomorphism of \( W(\kappa) \)-algebras \( \varepsilon : R' \cong R \) whose reduction modulo \( p \) is id\( \overline{R} \). This isomorphism induces another frame structure,

\[
(R, \sigma') := \varepsilon^+(R', \sigma'),
\]
on \( R \). Denote by \( f_{\phi, \sigma'} \) the trivialized partially divided Frobenius of \( \varepsilon^+(y^\phi) \in \mathbb{I}_+(R) \), w.r.t. simple frames \( (R, \sigma') \). Then clearly we have \( \overline{f}_{\phi, \sigma'} = f_{\phi, \sigma'} \).

From now on, we assume \( R = R' \) and denote by \( \overline{f}_\phi \in \mathbb{S}(R) \) be the trivialized partially divided Frobenius attached to \( y^\phi \), w.r.t. \( R \). Now the proposition follows from the combination of Lem. \[5.4.2\] below which compares \( f_{\phi, \sigma} \) and \( f_{\phi, \sigma'} \), and Lem. \[5.4.3\] below which compares \( \overline{f}_{\phi, \sigma} \) and \( \overline{f}_{\phi, \sigma'} \). \hfill \Box

**Lemma 5.4.2.** With the simple frame \( R \) fixed, if \( y^\phi = (y, \beta_y) \in \mathbb{I}_+(R) \) is another lift of \( x^\phi \). Then there exists \( u_- \in U_-(\mathbb{R}) \), such that following holds in \( G(\mathbb{R}) \)

\[
(5.4.1) \quad \overline{f}_\phi \cdot \sigma(u_-).
\]

**Proof.** Recall that by \[3.6\] we have the canonical parallel isomorphism

\[
\varepsilon(x, y) : H^1_{\text{dR}}(A_x/R) \cong \mathbb{D}_*^{\text{can}}(A_x) \cong H^1_{\text{dR}}(A_y/R),
\]

which carries \( s_{\text{dR}, x} \) to \( s_{\text{dR}, y} \). Hence for our purpose we may assume \( x = y \); note however that this does NOT mean\[3\] that \( \beta_x = \beta_y \). Denote by \( \mathbb{I}_x \) the trivial \( \mathbb{G}_m \)-torsor over \( \text{Spec} R \), obtained as the pullback to \( \text{Spec} R \) of \( \mathbb{I} \) along \( x : \text{Spec} R \to \mathbb{S} \) and view \( \beta_x, \beta_y \) as elements in \( \mathbb{I}_x(R) \).

Write \( h := \beta_y^{-1} \circ \beta_x \in \mathbb{I}_x(\mathbb{R}) \). By \[5.1.3\] we have

\[
\overline{f}_\phi = h^{-1} f_{\phi, \sigma}(\mu(p)h\mu(p)^{-1})).
\]

Hence it suffices to show the following

\[
(5.4.2) \quad \mu(p)h\mu(p)^{-1} \in \mathbb{I}(\mathbb{R}), \quad \mu(p)h\mu(p)^{-1} \in U_-(\mathbb{R}).
\]

To show these we use the embedding \( \mathbb{I} : S_{W(\kappa)} \hookrightarrow \text{GL}_{2g, W(\kappa)} \) introduced in \[3.4\]. Since we have \( \mu(p)h\mu(p)^{-1} \in \mathbb{I}(R', \mathbb{F}_p) \), in order to show that it lies in \( \mathbb{I}(\mathbb{R}) \), it suffices to show that it lies in \( \text{GL}_{2g, W(\kappa)}(R') \). Moreover, by \[3.4.4\], in order to show \( \mu(p)h\mu(p)^{-1} \in U_-(\mathbb{R}) \),

\[3\]One may instead write \( \beta'_y \) for \( \beta_y \), in the discussion below, but for typographical reason we choose not to do so.
we may replace $\bar{\mu}$ by the induced cocharacter $\bar{\mu}' = t \circ \bar{\mu}$ of $GL_{2g, W(\kappa)}$. Inside $GL_{2g}(R[1/p])$, $\bar{\mu}'(p)$ and $h$ (note that $\bar{h} = 1$) are represented by matrices of the following forms respectively

\[(5.4.3) \quad \begin{pmatrix} pI_g \\ I_g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_g + pA \\ pC \\ I_g + pD \end{pmatrix},\]

where $A, B, C, D$ are $g$ by $g$ matrices with entries in $R$. Now the problems become trivial due to our discussion at the end of §3.4.

\[(5.4.4) \quad \begin{pmatrix} pI_g \\ I_g \end{pmatrix}(I_g + pA pB)^{-1} = \begin{pmatrix} I_g + pA \\ C \\ I_g + pD \end{pmatrix}. \]

\[\square\]

**Lemma 5.4.3.** Fix a lift $x' \in I_+(R)$ of $x$, and let $R' = (R, \sigma')$ be another simple frame of $R$. Then there exists an element $u_- \in U_-^\sigma(R)$, such that

\[(5.4.5) \quad \int_{\gamma} = \int_{x, \sigma'}(u_-).\]

**Proof.** By basic properties of the Dieudonné crystal $D^\sigma(A_T)$ (cf. also §3.6), we have canonical parallel isomorphism $t : \sigma' M \to \sigma^\sigma M$, such that $F = F' \circ t$. By direct computation one sees that

$$f_{\sigma} = f_{\sigma'}(\bar{\mu}^\sigma(p)h\bar{\mu}^\sigma(p)^{-1}),$$

where $h := \sigma'(\beta_s)^{-1}t \sigma(\beta_s) \in G(R)$, and the superscript “$\sigma'$” in $\bar{\mu}^\sigma$ certainly refers to the Frobenius lift $\sigma : W(\kappa) \to W(\kappa)$. We use again the embedding $t : GL_{2g, W(\kappa)} \to GL_{2g, W(\kappa)}$ in §3.4 but in a twisted manner. To be precise, the pull back of $t$ along $\sigma : W(\kappa) \to W(\kappa)$ induces another embedding

$$\sigma(t) : G \cong G_\kappa^\sigma \to GL_{2g, W(\kappa)} \cong GL_{2g, W(\kappa)}.$$ 

Exactly as in the proof of Lem. 5.4.3 it suffices to show $\bar{\mu}(p)h\bar{\mu}^\sigma(p)^{-1} \in U_-^\sigma(R)$, with $U_-^\sigma \subseteq GL_{2g, \kappa}$, the counterpart of $U_-^\sigma \subseteq G_\kappa$ for the cocharacter

$$G_{m, \kappa} \cong G_{m, \kappa}^\sigma \to GL_{2g, \kappa} \cong GL_{2g, \kappa}.$$ 

Via the embedding $\sigma(t)$, $\bar{\mu}^\sigma(p)$ and $h$ are represented inside $GL_{2g}(R[1/p])$ by matrices of the same forms as in (5.4.3) respectively, and $U_-^\sigma(R)$ consists of matrices of the form

$$\begin{pmatrix} I_g \\ * \\ I_g \end{pmatrix}. \quad \text{Now we finish by the same calculation as in (5.4.3).} \quad \square$$

**5.5. The global map $\gamma : I_+ \to G_\kappa/U_-^\sigma$ via gluing.** In this subsection we apply Prop. 5.4 to construct the global map $\gamma : I_+ \to G_\kappa/U_-^\sigma$. For this we take a Zariski affine open covering $\{ \tilde{X}_i : Spec\tilde{R}_i \to I_+ \}$ of $I_+$. As each $\tilde{R}_i$ is smooth over $\kappa$, by Exam. 2.1.2 Zariski locally it admits a finite $p$-basis, and hence we can apply Prop. 5.4.1 and obtain morphisms of $\kappa$-schemes,

\[(5.5.1) \quad \gamma_i = \gamma_i^{\tilde{X}_i} : Spec\tilde{R}_i \to G_\kappa/U_-^\sigma.\]

**Theorem 5.5.1.** The maps $\gamma_i$ defined in (5.5.1) glue to a map of $\kappa$-schemes,

$$\gamma : I_+ \to G_\kappa/U_-^\sigma.$$
Proof. Since $L_\ast$ is quasi-projective (hence separated), the intersection of $\text{Spec} R_i$ and $\text{Spec} R_j$ is again affine. Denote it by $\text{Spec} R_{ij}$. We need to show $\gamma_i$ and $\gamma_j$ restrict to the same map on $\text{Spec} R_{ij}$, for all $i,j$. But since $R_{ij}$ is again a smooth $\kappa$-algebra (hence Zariski locally admits finite $p$-basis), this follows from the next lemma, Lem. 5.5.2. \qed

Lemma 5.5.2. Given two $\kappa$-algebras $R, R'$ which Zariski locally admit finite $p$-basis, and a morphism of $\kappa$-schemes $\tilde{x} : \text{Spec} R \to \text{Spec} R'$, then for any morphism of $\kappa$-schemes $\xi : \text{Spec} R' \to \text{R}$, we have

$$\gamma_{\tilde{x} \circ \pi} = \gamma_{\tilde{x} \circ \xi}.$$  

Proof. This is immediate from our construction in Prop. 5.4 if there exists a homomorphism of simple frames $f : (R', \sigma) \to (R, \sigma)$ which lifts the structure map $\tilde{R} \to \tilde{R}'$. In general we do not know such an $f$ always exists; below we proceed by reducing a general case to cases where $f$ exists by “passing to perfection”.

Note first that for each $\mathbb{F}_p$-algebra $\tilde{A}$ which Zariski locally admits a $p$-basis, the absolute Frobenius map $\sigma : \tilde{A} \to \tilde{A}$ is faithfully flat (the $p$-basis assumption implies that $\tilde{A}$ as an $\tilde{A}$-module via $\sigma$, is locally free). Consequently the canonical ring map

$$\tilde{A} \to \tilde{A}_{\text{perf}} := \lim_{\longrightarrow\ a \rightsquigarrow a^p} \tilde{A}$$

is faithfully flat. In particular, horizontal arrows in the following commutative diagram,

$$
\begin{array}{ccc}
G_\kappa/\mathcal{U}_\sigma(\tilde{R}) & \longrightarrow & G_\kappa/\mathcal{U}_\sigma(\tilde{R}_{\text{perf}}) \\
\downarrow & & \downarrow \\
G_\kappa/\mathcal{U}_\sigma(\tilde{R}') & \longrightarrow & G_\kappa/\mathcal{U}_\sigma(\tilde{R}'_{\text{perf}}),
\end{array}
$$

are injective. Now since the formation of $W(\cdot)$ is functorial, we are reduced to show $\gamma_{\tilde{x} \circ \pi} = \gamma_{\tilde{x} \circ \xi}$ with $\pi : \text{Spec} \tilde{R}_{\text{perf}} \to \tilde{R}$ the canonical morphism. This follows from the following fact: there is a sequence of homomorphisms of simple frames over $W(\kappa)$,

$$(R, \sigma) \to (R, \sigma)_{\text{perf}} := (\tilde{R}_{\text{perf}}, \sigma) \cong (W(\tilde{R}_{\text{perf}}), \sigma),$$

which lifts the structure map $\tilde{R} \to \tilde{R}_{\text{perf}}$ (see also [Yan18 Lem. 6.12] for another construction). We need to explain this fact: $R_{\text{perf}}$ is defined as the colimit perfection,

$$R_{\text{perf}} := \lim_{\longrightarrow\ \sigma : R \rightarrow R} R,$$

and $\tilde{R}_{\text{perf}}$ is the $p$-completion of $R_{\text{perf}}$. Clearly we have $R_{\text{perf}}/pR_{\text{perf}} = \tilde{R}_{\text{perf}}$. The Frobenius lift $\sigma_R : R \to R$ induces a Frobenius lift $\sigma$ on $R_{\text{perf}}$ (hence on $R_{\text{perf}}$) compatible with $\sigma_R$, and hence we get the homomorphism $(R, \sigma) \to (R, \sigma)_{\text{perf}}$ of simple frames displayed above. In fact $R_{\text{perf}}$ is $p$-torsion free and the simple frame $(R_{\text{perf}}, \sigma)$, viewed as a crystalline prism (see [2,1.4]), is nothing but the perfection of the prism $(R, \sigma)$ in the sense of Bhatt-Scholze [BS21, Lem. 3.9] and hence we justified the isomorphism of simple frames $(R, \sigma)_{\text{perf}} \cong (W(\tilde{R}_{\text{perf}}), \sigma)$; see [BS21, Cor. 2.31]. \qed

5.6. The zip period map $\eta$. The natural embedding $U^\sigma \hookrightarrow E^\kappa$ realizes $U^\sigma$ as a normal subgroup of $E^\kappa$. Via this embedding $U^\sigma$ acts on $G_\kappa$ by right multiplication. Passing to quotient, we obtain an action of $P_+ = E^\kappa/U^\sigma$ on $G_\kappa/U^\sigma$ given on local sections by $g \cdot p_+ = p_+^{-1} g \sigma(m)$, where $p_+ = u_+ m$, with $u_+ \in U_+$ and $m \in M$. Denote by $[(G_\kappa/U^\sigma)/P_+]$ the resulting quotient algebraic stack over $\kappa$. Since the action of $U^\sigma$ on $G_\kappa$ is free, the
canonical projection $G_\kappa \to G_\kappa/U_\kappa^\sigma$ induces a canonical isomorphism of algebraic stacks over $\kappa$,

$$[G_\kappa/E_\kappa] \cong [(G_\kappa/U_\kappa^\sigma)/P_+].$$

**Theorem 5.6.1.** The map $\gamma$ is equivariant w.r.t. the actions of $P_+$ on $I_+$ and on $G_\kappa/U_\kappa^\sigma$, and hence induces a morphism of algebraic stacks over $\kappa$,

$$\eta : S \cong I_+ / P_+ \to [(G_\kappa/U_\kappa^\sigma)/P_+] \cong [G_\kappa/E_\kappa] \cong G \cdot \text{Zip}^\mu.$$

**Proof.** We need to show the commutativity of the following diagram of $\kappa$-schemes

$$\begin{array}{ccc}
I_+ \times_\kappa P_+ & \xrightarrow{\gamma \times \text{id}_{P_+}} & (G_\kappa/U_\kappa^\sigma) \times_\kappa P_+ \\
\downarrow \quad & \quad & \downarrow \\
I_+ & \xrightarrow{\gamma} & G_\kappa/U_\kappa^\sigma,
\end{array}$$

where vertical arrows are given by $P_+$-actions. Since $I_+ \times_\kappa P_+$ is geometrically reduced, it suffices to check the commutativity on $k$-points for an algebraically closed field extension $k$ of $\kappa$. Note first that for any $\tilde{x}^\circ \in I_+(k)$, by Lem. 5.5.2 we have $\gamma(\tilde{x}^\circ) = \gamma_{\tilde{x}^\circ}$. For any $k$-point $(\tilde{x}^\circ, \tilde{p})$ of $I_+ \times_\kappa P_+$, take a $W(k)$-point $(x^\circ, p_+)$ of $I_+ \times W(k) P_+$, which lifts $(\tilde{x}^\circ, \tilde{p})$. Then $x^\circ \cdot p_+$ is a lift of $\tilde{x}^\circ \cdot \tilde{p}_+$. Applying the construction in §5.4, we obtain an element $\int_{x^\circ \cdot p_+} \in \mathcal{S}(W(k))$. A direct calculation using the relation (5.1.3) gives the following

$$\int_{x^\circ \cdot p_+} = \int_{x^\circ} (\tilde{\mu}^\sigma(p) \sigma(p)) = p_+ \int_{x^\circ} \sigma(\tilde{\mu}(p) u_+ \tilde{\mu}(p)^{-1}) \sigma(m),$$

where $p_+ = u_+ m$, with $u_+ \in \mathcal{U}_+(W(k))$ and $m \in \mathcal{M}(W(k))$, and where for the second “=” one uses the fact that $m$ commutes with $\tilde{\mu}(p)$ and that $\tilde{\mu}^\sigma(p) = \sigma(\tilde{\mu}(p))$. But by Lem. 3.4.1 the element $\tilde{\mu}(p) u_+ \tilde{\mu}(p)^{-1} \in \mathcal{S}(W(k)(\frac{1}{p}))$ actually lies in $\mathcal{S}(W(k))$ and we have $\tilde{\mu}(p) u_+ \tilde{\mu}(p)^{-1} = 1 \in G(\tilde{R})$. \qed

6. **COMPARISON OF $\eta$ WITH $\zeta$**

In this section we show that the map $\eta : S \to G \cdot \text{Zip}^\mu$ constructed in Thm. 5.6.1 coincides with the map $\zeta : S \to G \cdot \text{Zip}^\mu$ in [Zha18], in the sense that there are naturally 2-isomorphic. The strategy is to show that there is a natural isomorphism between their corresponding objects in the groupoid $[G_\kappa/E_\kappa](S)$.

6.1. **Zip isomorphisms associated with Dieudonné modules.** As a preparation for the next subsection, as in §5.1 we let $\bar{R}$ be a $\kappa$-algebra which Zariski locally admits a finite $p$-basis, and choose a simple frame $\bar{R} = (R, \sigma)$ for $\bar{R}$. Take a point $\bar{r} \in S(\bar{R})$ and denote by $\bar{M} = (M, F, V, \nabla)$ the Dieudonné module over $\bar{R}$ that is associated with the $p$-divisible group $A_{\bar{r}}[p^n]$. Write $\bar{F} : \bar{M}^\sigma \to \bar{M}, \bar{V} : \bar{M} \to \bar{M}^\sigma$ for the reduction modulo $p$ of $F, V$ respectively; note however that $\bar{M}, \bar{F}, \bar{V}$ are independent of the choice of $\bar{R}$, as they can be obtained by taking evaluation at the trivial PD thickening $\bar{R} \leftarrow \tilde{R}$ of the Dieudonné crystal $\mathcal{D}^+(A_{\bar{r}})[p^n]$; see §2.2. Then the relations $\bar{F} \circ \bar{V} = p \cdot \text{id}_M$ and $\bar{V} \circ \bar{F} = p \cdot \text{id}_{M^\sigma}$ give rise to an exact sequence of $\bar{R}$-modules

$$\bar{M}^\sigma \xrightarrow{\bar{F}} \bar{M} \xrightarrow{\bar{V}} \bar{M} \xrightarrow{\bar{V}} \bar{M}.$$ And hence canonical isomorphisms $\bar{F} : \bar{M}^\sigma / \text{Ker}(\bar{F}) \xrightarrow{\sim} \text{Ker}(\bar{V}), [\bar{V}] : \bar{M}/\text{Ker}(\bar{V}) \xrightarrow{\sim} \text{Ker}(\bar{F})$; combining them, we obtain a canonical direct-summand-wise isomorphism of $\bar{R}$-modules

$$(6.1.1) \quad \delta : \text{Ker}(\bar{F}) \oplus \bar{M}^\sigma / \text{Ker}(\bar{F}) \xrightarrow{[\bar{V}]^{-1} \oplus \bar{F}} \bar{M}/\text{Ker}(\bar{V}) \oplus \text{Ker}(\bar{V}).$$
We call $\delta$ above the zip isomorphism associated with the Dieudonné module $M$. Now we make connection to the zip isomorphism we defined in $\text{4.2.2}$ (cf. Rem. $\text{4.2.1}$). Let $\bar{M}^1 \subseteq \bar{M}$ be the Hodge filtration of $\bar{M}$ as introduced in $\text{2.3.3}$. As recalled in $\text{2.3.1}$, we have $\bar{M}^1 - \sigma = \ker(\bar{F}) \subseteq \bar{M}^\sigma$. We identify the following canonical isomorphism,

$$
(\bar{M} \supseteq \bar{M}^1) \cong (H^1_{\text{dR}}(A_\mathbf{f}/\bar{R}) \supseteq \omega_\mathbf{f}).
$$

Write $\bar{M}_0 := \ker(\bar{V}) = \text{Im}(\bar{F}) \subseteq \bar{M}$. Under the canonical isomorphism $\bar{M} \cong H^1_{\text{dR}}(A_\mathbf{f}/\bar{R})$, $\bar{M}_0$ corresponds to the conjugate filtration $\mathfrak{f} \bar{\mathfrak{x}}$ of $H^1_{\text{dR}}(A_\mathbf{f}/\bar{R})$. We also identify the canonical isomorphism

$$
(\bar{M}_0 \subseteq \bar{M}) \cong (\mathfrak{f} \bar{\mathfrak{x}} \subseteq H^1_{\text{dR}}(A_\mathbf{f}/\bar{R})).
$$

With these identifications, the zip isomorphism (6.1.1) is nothing but the pull back to $\bar{\mathfrak{x}}$ along $\bar{\mathfrak{x}}$ of the zip isomorphism (4.2.2). In what follows we write $\delta$ in this form:

(6.1.2) \hspace{1cm} \delta : \bar{M}^1, \sigma \oplus \bar{M}^\sigma / \bar{M}^1, \sigma \xrightarrow{\mathfrak{f} \bar{\mathfrak{x}} \to \omega_\mathbf{f}} \bar{M} / \bar{M}_0 \oplus \bar{M}_0,

6.2. Comparison of $\eta$ and $\zeta$. Under the isomorphism $G$-Zip$^\mu \cong [G_\kappa/E_\mathbf{m}]$, the universal $G$-zip in $\text{4.2.1}$ corresponds to an $E_\mathbf{m}$-torsor $\mathcal{Z} = \mathcal{Z}_\kappa$ over $S$, together with an $E_\mathbf{m}$-equivariant map $\zeta : \mathcal{Z} \to G_\kappa$. The $E_\mathbf{m}$-torsor $\mathcal{Z}$ is given by the pull-back of the canonical projection $I_+ \to \mathbf{L}_-/U^\sigma$ of $S$-morphism along the $S$-morphism

$$
I_+ \xrightarrow{\sigma} I^\sigma_+ \xrightarrow{1} I_+ / U^\sigma \
$$

The map $\tilde{\zeta} : \mathcal{Z} \to G_\kappa$ is given by sending a local section $(\bar{x}^+, \bar{x})$ of $\mathcal{Z} \subseteq I_+ \times S I_-$ to

$$
\tilde{\zeta}(\bar{x}^+, \bar{x}) := \beta_{\bar{x}}^{-1} \circ \theta_{\bar{x}},
$$

which is a local section of $G \subseteq \text{GL}(\Lambda^+)$. Here $\bar{x} = (\bar{x}, \bar{x})$ is a local section of $I_-$ with the same underling point $\bar{x}$ as that of $\bar{x}^+$. On the other hand, under the isomorphism $[(G_\kappa/U^\sigma) / P_+] \cong [G_\kappa/E_\mathbf{m}]$, the $P_+$-equivariant map corresponds to an $E_\mathbf{m}$-torsor $\mathcal{Z}_\eta$ over $S$, given by the pull back of $\gamma : I_+ \to G_\kappa / U^\sigma$ along the canonical projection $G_\kappa \to G_\kappa / U^\sigma$, together with an $E_\mathbf{m}$-equivariant map $\eta : \mathcal{Z}_\eta \to G_\kappa$ given by the canonical projection from $\mathcal{Z}_\eta$ to $G_\kappa$. The right $E_\mathbf{m}$-action on $\mathcal{Z}_\eta$ is given by

$$
(\bar{x}^+, \gamma) \cdot (p_+, p_-) = (\bar{x}^+, \beta_{\bar{x}}^{-1} \circ \theta_{\bar{x}}).
$$

Theorem 6.2.1. There is a natural isomorphism $\mathcal{Z} \cong \mathcal{Z}_\eta$ of $E_\mathbf{m}$-torsors over $S$. In other words, the two morphisms of $\kappa$-algebraic stacks $\zeta$ and $\eta$ are 2-isomorphic.

Proof. Note that it is enough to show the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\zeta} & G_\kappa \\
p_{\mathbf{m}} & \xrightarrow{\gamma} & G_\kappa / U^\sigma. \\
\end{array}
$$

This is because, once it is shown, one sees readily that the induced morphism $\mathcal{Z} \to \mathcal{Z}_\eta$ of $S$-schemes, given on local sections by

$$
(\bar{x}^+, \beta_{\bar{x}}^{-1} \circ \theta_{\bar{x}}),
$$

is $E_\mathbf{m}$-equivariant, and hence is a morphism between $E_\mathbf{m}$-torsors over $S$, and hence is automatically an isomorphism.
Clearly the problem is local on $Z$. Let $\tilde{z} : \text{Spec} \overline{R} \to Z$ be an affine open of $Z$. In the discussion below, the underlying point $\tilde{x} \in S(\overline{R})$ is fixed. Hence to ease notation, we may write $\beta_\tilde{x} \in I_+(\overline{R})$ instead of $(\tilde{x}, \beta_\tilde{x}) \in I_+(\overline{R})$; similarly for points in $I_-(\overline{R})$. We need to show the image of $\tilde{\zeta}(\beta_\tilde{x}, \theta_\tilde{x}) = \beta_\tilde{x}^{-1} \circ \theta_\tilde{x} \in G(\overline{R})$ in $G_K/U^\sigma(\overline{R})$ coincides with $\gamma(\beta_\tilde{x})$. Again since $Z$ is a smooth $\kappa$-scheme, by Exam. 5.2.1 Zariski locally $\overline{R}$ admits a finite $p$-basis, and hence we may choose a simple frame $(R, \sigma)$ for $\overline{R}$ and a lift $x^\flat \in \mathbb{I}_+(R)$ for $\tilde{x}$, we can apply the discussion in §5 Then by Lem. 5.5.2 $\gamma(\beta_\tilde{x})$ is equal to the image in $G_K/U^\sigma(\overline{R})$ of

$$\int_{x^\flat} = \beta_\tilde{x}^{-1} \Gamma_{x^\flat} \sigma(\beta_\tilde{x}) \in G(\overline{R}).$$

**Lemma 6.2.2.** We have $(\beta_{\tilde{x}}, \theta_{\tilde{x}}^\flat) \in Z(\overline{R})$.

Before showing Lem. 6.2.2 let us note the following: it implies Thm. 6.2.1 Indeed, if Lem. 6.2.2 is shown, then by definition of a $G$-zip, $\theta_{\tilde{x}}^\flat$ and $\theta_{\tilde{x}}$ have the same image in $I_+/U^\sigma(\overline{R})$, as they both correspond to the image of $\beta_\tilde{x}$ under the isomorphism $i : I_+^\sigma/U^\sigma_+(\overline{R}) \cong I_+/U^\sigma(\overline{R})$. Hence we have $\theta_{\tilde{x}}^\flat = \theta_{\tilde{x}} \cdot u_-$ for some $u_- \in U^\sigma(\overline{R})$, and hence the following equality holds

$$\int_{x^\flat} = \tilde{\zeta}(\beta_{\tilde{x}}, \theta_{\tilde{x}}^\flat) = \tilde{\zeta}(\beta_{\tilde{x}}, \theta_{\tilde{x}})u_- \in G(\overline{R}),$$

which implies that the image of $\tilde{\zeta}(\beta_{\tilde{x}}, \theta_{\tilde{x}})$ in $G_K/U^\sigma(\overline{R})$ is equal to $\gamma(\beta_{\tilde{x}})$, as desired.

**Proof of Lem. 6.2.2.** We first show $\theta_{\tilde{x}}^\flat \in I_-(\overline{R})$. Note that by our discussion in §6.1 the subset $I_-(\overline{R}) \subseteq I(\overline{R})$ consists of elements $\theta_{\tilde{x}} \in I(\overline{R})$ which carries the direct summand $\Lambda^*_0 \overline{R}$ of $\Lambda^*_R$ isomorphically onto the conjugate filtration $\tilde{M}_0$ of $\tilde{M}$.

Using notations in Lem. 5.3.1 the normal decomposition $M = M^1 \oplus M^0$ induces a decomposition $\tilde{M} = \tilde{M}^1 \oplus \tilde{M}^0$ of $\tilde{M}$, and hence a decomposition $\tilde{M}^\sigma = \tilde{M}^{1,\sigma} \oplus \tilde{M}^{0,\sigma}$ of $\tilde{M}^\sigma$. With this decomposition, we have $\tilde{M}_0 = \tilde{F}(\sigma^*(\tilde{M}^0))$. From this equality we see that the direct summand of $M$,

$$M_0 := \theta_{\tilde{x}}^\flat(\Lambda^{*,0}) = \Gamma_{x^\flat}(M^{0,\sigma}) = F(\sigma^*(\tilde{M}^0)),$$

is a lift of the conjugate filtration $\tilde{M}_0$ of $\tilde{M}$ and we have $\theta_{\tilde{x}}^\flat(\Lambda^*_0, \overline{R}) = F(\sigma^*(\tilde{M}^0))$. In other words, $\theta_{\tilde{x}}^\flat \in I_-(\overline{R})$.

To finish the proof, we still need to show that the image of $\beta_{\tilde{x}}$ in $I^\sigma_+/U^\sigma_+\overline{R}$ coincides with the image of $\theta_{\tilde{x}}^\flat$ in $I_-U^\sigma(\overline{R})$, via the isomorphism $i : \Gamma_{x^\flat}(M^{0,\sigma}) \cong I_-U^\sigma(\overline{R})$. Denote by $\mu' : \mathbb{G}_{m, \kappa} \xrightarrow{\mu} G_{\kappa} \hookrightarrow \text{GL}(\Lambda^*_\kappa)$ the cocharacter of $\text{GL}(\Lambda^*_\kappa)$ induced by $\mu$, as in §3.4 Then we can form the $\kappa$-stack $\text{GL}(\Lambda^*_\kappa)$-Zip$^{\mu'}$. By forgetting tensors everywhere in $\mathbb{I}$, we obtain a $\text{GL}(\Lambda^*_\kappa)$-Zip $\mathbb{I} = (\mathbb{I}', \mathbb{I}_+, \mathbb{I}_{-}, \mathbb{I}_+)$. Then by functoriality of the formation of $G$-zips, we have the following commutative diagram

$$\begin{array}{ccc}
I^\sigma_+/U^\sigma_+\overline{R} & \overset{i}{\longrightarrow} & I_-U^\sigma(\overline{R}) \\
\downarrow \downarrow & & \downarrow \\
I^\sigma_+/U^\sigma_+\overline{R} & \overset{i'}{\longrightarrow} & I_-U^\sigma(\overline{R}),
\end{array}$$

where the vertical arrows are injective: this can be seen by working fppf locally and using the fact (see (3.4.4))

$$\text{Cent}_{\text{GL}(\Lambda_\kappa)}(\mu') \cap G_{\kappa} = \text{Cent}_{G_{\kappa}}(\mu).$$
Hence we are reduced to show that the image of $\beta_2$ and $\theta_\ell$; that is, we are reduced to the case $G_\kappa = \text{GL}(\Lambda^*_\kappa)$.

Let us now unwind the definition of $t$ for $G_\kappa = \text{GL}(\Lambda^*_\kappa)$. In this special case the set $I^0_\kappa/U^\alpha_\kappa(\bar{R})$ can be realized as the set of equivalence classes in $I_\ell(\bar{R})$ with equivalence relations given by declaring $\beta_1, \beta_2 \in I^0_\kappa(\bar{R})$ equivalent if

$$\text{gr}(\beta_1) = \text{gr}(\beta_2) : (\Lambda^*_\kappa/\Lambda^*_0)^\kappa \oplus \Lambda^*_0\kappa \cong (\bar{M}/\bar{M}^1)^\kappa \oplus \bar{M}^1.$$

Similarly, the set $I_-/U^\alpha(\bar{R})$ can be realized as the set of equivalence classes in $I_- (\bar{R})$ with equivalence relations given by declaring $\theta_1, \theta_2 \in I_- (\bar{R})$ equivalent if

$$\text{gr}(\theta_1) = \text{gr}(\theta_2) : \Lambda^*_0 \oplus \Lambda^*_0/\Lambda^*_0 \cong \bar{M}_0 \oplus \bar{M}/\bar{M}_0.$$

The map $t$ is given by sending the equivalence class of $\beta \in I^0_\kappa(\bar{R})$ to the unique equivalence class of $\theta \in I_- (\bar{R})$ with $\text{gr}(\theta)$ equal to composition of

$$\Lambda^*_0 \oplus \Lambda^*_0 \cong \Lambda^*_0 \oplus \Lambda^*_0 \cong (\Lambda^*_\kappa/\Lambda^*_0)^\kappa \oplus \Lambda^*_0 \cong (\bar{M}/\bar{M}^1)^\kappa \oplus \bar{M}^1.$$

with the zip isomorphism $\bar{M}/\bar{M}^1 \cong \bar{M}^1$ defined in (6.1.2). Here the isomorphism $\cong$ is induced by (4.2.3). Up to all kinds of identifications described above, $t$ is simply given by

$$\text{gr}(\theta) \mapsto \delta \circ \text{gr}(\sigma(\theta)).$$

Now we are reduced to verify the equality $\delta = \text{gr}(\Gamma_\varphi^\ell)$, which amounts to verifying the commutativity of the diagrams below, with vertical arrows canonical projections,

$$\begin{align*}
M^1,\kappa & \oplus M^0,\kappa \xrightarrow{\Gamma_\varphi^\ell} M_{-1} \oplus M_0 \\
\delta & \downarrow \\
\bar{M}^1,\kappa \oplus (\bar{M}/\bar{M}^1)^\kappa & \xrightarrow{\delta} \bar{M}/\bar{M}_0 \oplus \bar{M}_0,
\end{align*}$$

where $M_{-1} := \Gamma_\varphi^\ell(M^1,\kappa)$. For the commutativity of (6.2.1), we only need to check that for every element $m \in M^1,\kappa$, we have $[\bar{V}]^{-1}(\bar{m}) = \Gamma_\varphi^\ell(m)$. Note that the image $[\bar{V}]^{-1}(\bar{m})$ is the unique element $\bar{n} \in \bar{M}/\bar{M}_0$ such that $\bar{V}(\bar{n}) = \bar{m}$. But $\bar{V}(\Gamma_\varphi^\ell(m)) = (V \circ \Gamma_\varphi^\ell)(\bar{m}) = \bar{m}$. This finishes the proof of Lem. 6.2.2 and hence that of Thm. 6.2.1.

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