Gravity coupled to a scalar field in extra dimensions

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Abstract. In \(d+1\) dimensions we solve the equations of motion for the case of gravity minimally or conformally coupled to a scalar field. For the minimally coupled system the equations can either be solved directly or by transforming vacuum solutions, as shown before in \(3+1\) dimensions by Buchdahl. In \(d+1\) dimensions the solutions have been previously found directly by Xanthopoulos and Zannias. Here we first rederive these earlier results, and then extend Buchdahl’s method of transforming vacuum solutions to \(d+1\) dimensions. We also review the conformal coupling case, in which \(d+1\) dimensional solutions can be found by extending Bekenstein’s method of conformal transformation of the minimal coupling solution. Combining the extended versions of Buchdahl transformations and Bekenstein transformations we can in arbitrary dimensions always generate solutions of both the minimal and the conformal equations from known vacuum solutions.

1. Introduction
Theories in which gravity couples to a scalar field are common in extra-dimensional problems. A few examples are Kaluza-Klein, Jordan and Brans-Dicke theories, as well as string theory in general. Consequently, the corresponding Einstein equations have been studied extensively for the last sixty years, with particular emphasis on searching for black hole solutions.

Minimally coupled scalar fields in \(3+1\) dimensions were first studied by Fisher [1] in 1948, and later by Bergmann and Leipnik [2] and by Janis, Newman and Winicour [3]. These were all solving the equations directly. However, in 1959 Buchdahl showed that it is always possible to generate a solution for the minimal coupling case by means of a particular transformation [4] of a vacuum solution metric. The problem was later revisited by Janis, Robinson and Winicour [5], who also included electromagnetism in their solutions.

The extension from \(3+1\) dimensions to \(d+1\) dimensions was first done by Xanthopoulos and Zannias [6], using a direct solution technique. In the present paper we generalize Buchdahl’s transformation method to arbitrary dimensions to solve the same problem.

Solutions of the conformally coupled equations in \(3+1\) dimensions were first found by Bocharova, Bronnikov and Melnikov in 1970 [7], using a direct approach. However, these results were published in a Russian journal, and did therefore not get the attention of physicists in the western world. As a result, the solutions were re-discovered independently by Bekenstein [8] in 1974, using a conformal transformation method. These solutions included a black hole-like solution [9], known as the BBMB (Bocharova–Bronnikov–Melnikov–Bekenstein) black hole. The first direct solution in the West was found by Frøyland [10] in 1982, who demonstrated that the

1 This is a more complete version of a talk given at the XXIX Spanish Relativity Meeting Einstein’s legacy: from the theoretical paradise to astrophysical observation (ERE2006), Palma de Mallorca, Spain, Sept. 4-8, 2006.
metric coincides with the extremal Reissner-Nordström solution. In a later work, Xanthopoulos and Zannias [11] further showed that the BBMB solution is unique.

Again, the extension to $d + 1$ dimensions was also done by Xanthopoulos, this time together with Dialynas [12]. They found solutions for the conformally coupled case by extending Bekenstein’s method of transforming minimal solutions. They also demonstrated that the BBMB black hole only exists in $3 + 1$ dimensions. Klmick [13] immediately after found these solutions by a direct integration technique.

These issues have been investigated in collaboration with T angen [14]. One goal of this investigation was to extend Buchdahl’s and Bekenstein’s methods of generating solutions by conformal transformations to spacetimes with extra dimensions. However, during the course of the project, we found that most results had already been obtained by others. We will here give a pedagogical introduction to the field based on the more formal approach of Tangen. In particular, we will show how the Buchdahl transformation can be extended to arbitrary dimensions.

### 2. Gravity coupled to a scalar field

In $D = d + 1$ dimensions the theory of gravity coupled to a free scalar field is described by the action

$$S = \int d^Dx \sqrt{-g} \left( R - \xi R \phi^2 - g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right)$$

We work in natural units where $c = 1 = M_D$, $M_D$ being the $D$-dimensional reduced Planck mass. For instance, in four spacetime dimensions we have $M_D^{-2} = 8\pi G_4 = 1$.

Putting the parameter $\xi$ to zero gives us the common case where the scalar field is minimally coupled to gravity, while keeping $\xi$ non-zero allows for more general couplings. In this paper we will also be interested in the conformal coupling case defined by $\xi = \frac{d-1}{4d}$. Varying the action with respect to $\phi$ gives the equation of motion for the scalar field

$$\Box^2 \phi - \xi R \phi = 0$$

while varying with respect to the metric $g^{\mu\nu}$ gives the Einstein equations

$$(1 - \xi \phi^2) E_{\mu\nu} = T_{\mu\nu}^\phi + \Delta T_{\mu\nu}^\phi$$

Here $T_{\mu\nu}^\phi$ is the ordinary scalar field energy-momentum tensor for the minimal coupling ($\xi = 0$) case

$$T_{\mu\nu}^\phi = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha}$$

and $\Delta T_{\mu\nu}^\phi$ is the Huggins term [15] coming from the extra term $\xi R \phi^2$ in the Lagrangian [16]

$$\Delta T_{\mu\nu}^\phi = \xi \left[ g_{\mu\nu} \Box^2 (\phi^2) - (\phi^2)_{,\mu\nu} \right]$$

To summarize, the total energy-momentum tensor is given by

$$T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} + \xi \left[ g_{\mu\nu} \Box^2 (\phi^2) - (\phi^2)_{,\mu\nu} \right] + \xi \phi^2 E_{\mu\nu}$$

Only for $\xi = \frac{d-1}{4d}$ will $T_{\mu\nu}$ be traceless and the theory will be conformally invariant. By contracting equation (3) we see that in this case also the Ricci scalar vanishes, $R = 0$. Taking the trace of equation (4), we notice that only in $1 + 1$ dimensions is a minimally coupled scalar field conformally invariant.
2.1. Statical, spherical symmetric solutions

We are searching for black hole-like solutions and are only interested in static and spherical symmetrical solutions. The most general static and spherical symmetric metric in $d + 1$ dimensions can be written

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2d\Omega_d^2$$

where $\alpha$, $\beta$ and $\gamma$ are unknown functions of the radial coordinate $r$, and $d\Omega_d^2$ is the solid angle element in $d - 1$ dimensions. Putting $\gamma = 1$ gives us Schwarzschild coordinates in which the equations of motion often take the simplest form. In our case the solutions can often not be explicitly written down in Schwarzschild coordinates and it is better to work in isotropic coordinates, defined by $\beta(r) = \gamma(r)$. In Schwarzschild coordinates the Einstein tensor has the independent components

$$E_{tt} = \frac{(d - 1)\beta' + (d - 2)(d - 1)(e^{2\beta} - 1)}{2r^2}$$

$$E_{rr} = \frac{(d - 1)\alpha' - (d - 2)(d - 1)(e^{2\beta} - 1)}{2r^2}$$

The scalar field configuration must also be static and spherically symmetric, so $\phi_{,\mu}$ can only have one non-zero component, $\phi_{,r} \equiv \phi'$. Using this when calculating the energy-momentum tensor we find for the minimal case

$$e^{2\beta - 2\alpha} T_{tt} = T_{rr} = \frac{1}{2} \phi'^2$$

and for the Huggins term

$$e^{2\beta - 2\alpha} \Delta T_{tt} = -2\xi \left[ \phi \phi'' + \phi \phi' \left( \frac{d - 1}{r} - \beta' \right) + \phi'^2 \right]$$

$$\Delta T_{rr} = 2\xi \phi \phi' \left( \frac{d - 1}{r} + \alpha' \right)$$

We also find the following expression for the D’Alambertian operator

$$\Box^2 = e^{-2\beta} \left[ \partial_r^2 + \left( \alpha' - \beta' + \frac{d - 1}{r} \right) \partial_r \right]$$

Using this we get the following simple expression for the first component of the Ricci tensor

$$R_{tt} = e^{-2\alpha} \Box^2 \alpha$$

while the Ricci scalar may be written

$$R = -2\Box^2 \alpha + \frac{d - 1}{r^{d - 1}} \left[ r^{d - 2} \left( 1 - e^{-2\beta} \right) \right]'$$

3. Minimal coupling

3.1. Fundamental equations

Putting $\xi = 0$ gives us the minimal coupling case. Equation (2) then reduces to $\Box^2 \phi = 0$, and using equation (13) we obtain the equation of motion for $\phi$,

$$\phi'' = -(\alpha' - \beta' + \frac{d - 1}{r})\phi'$$

For a non-constant scalar field this can easily be integrated to give
\[ \phi' = C e^{-\alpha + \beta} r^{-(d-1)} \]  
(17)
where \( C \) is a constant of integration. We further notice from (14) that \( R_{tt} = 0 \) gives us \( \Box^2 \alpha = 0 \) so for both \( \alpha \) and \( \phi \) non-zero we have
\[ \alpha' = K \phi' \]  
(18)
for some constant \( K \). When using (8-9) and (10) the Einstein equations can be simplified to
\[ e^{2\beta} - 1 = \frac{r}{d-2} (\alpha' - \beta') \]  
(19)
\[ \phi'^2 = \frac{d-1}{r} (\alpha' + \beta') \]  
(20)
Using equation (18) to substitute for \( \phi' \), and then eliminating \( \beta \) and \( \beta' \) from equations (17), (19) and (20) we are left with a first-order differential equation for \( \alpha \). But this can not be explicitly solved to find \( \alpha \), indicating that the general solution of the minimally coupled equations can not be explicitly written i Schwarzschild coordinates. We can however look at two special cases.

First we notice that for constant \( \phi \) equation (20) imply \( \alpha' + \beta' = 0 \). Then equation (19) can be rewritten as
\[ \left[ r^{d-2} \left( 1 - e^{-2\beta} \right) \right]' = 0 \]  
(21)
which may be integrated explicitly, and we end up with the trivial Schwarzschild vacuum solution for the metric (17) in \( d + 1 \) dimensions (18)
\[ ds^2 = -\left( 1 - \frac{B_s}{r^{d-2}} \right) dt^2 + \left( 1 - \frac{B_s}{r^{d-2}} \right)^{-1} dr^2 + r^2 d\Omega_d^2 \]  
(22)
were the integration constant \( B_s \) is canonically normalized to give Newtonian gravity in the large \( r \) limit, \( B_s = \frac{G d M}{2(d-2)} \) with \( G_d \) being the \( d \)-dimensional Newtonian gravitational constant, and \( M \) is the mass of the black hole.

Second, for the special case \( \alpha = 0 \), equation (19) can be rewritten as
\[ \left[ r^{2(d-2)} \left( 1 - e^{-2\beta} \right) \right]' = 0 \]  
(23)
giving
\[ e^{-2\beta} = 1 - \frac{C'}{r^{2(d-2)}} \]  
(24)
Inserting this into (20) we see that for \( \phi'^2 \) to be positive, the integration constant \( C' \) has to be negative. We put \( C' = -A^2 \) and integrate \( \phi' \) to get
\[ \phi = \pm \sqrt{\frac{d-1}{d-2} \ln \left( \sqrt{1 + \frac{A^2}{r^{2(d-2)}}} - \frac{A}{r^{d-2}} \right) + C''} \]  
(25)
Since we want \( \phi \) to go to zero for large \( r \) when the metric approaches flat Minkowski space we choose the integration constant \( C'' = 0 \). Our final solution thus reads
\[ ds^2 = -dt^2 + \left( 1 + \frac{A^2}{r^{2(d-2)}} \right)^{-1} dr^2 + r^2 d\Omega_d^2 \]  
(26)
\[ \phi = \pm \sqrt{\frac{d-1}{d-2} \ln \left( \sqrt{1 + \frac{A^2}{r^{2(d-2)}}} - \frac{A}{r^{d-2}} \right) + C''} \]  
(27)
Xanthopoulos and Zannias [6] found a general two-parameter solution in arbitrary dimensions by solving the equations of motion in isotropic coordinates. This solution may be written as

$$ ds^2 = - \left[ \frac{r^{d-2} - r_0^{d-2}}{r^{d-2} + r_0^{d-2}} \right]^{2\alpha} dt^2 + \left[ 1 - \frac{r_0^{2(d-2)}}{r^{2(d-2)}} \right]^{-\frac{2\alpha}{2}} \left[ \frac{r^{d-2} - r_0^{d-2}}{r^{d-2} + r_0^{d-2}} \right]^{-\frac{2\alpha}{2}} \left( dr^2 + r^2 d\Omega_d^2 \right) $$

where \( r_0 \) and \( a \) are arbitrary constants. The parameter \( a \) can run between 0 and 1, and \( a = 1 \) corresponds to the Schwarzschild metric [22] plus a constant scalar field solution. \( a = 0 \) gives the upper sign version of (26-27). As showed by Xanthopoulos and Zannias this latter solution is a naked singularity, and no value of \( a \) yields a black hole solution [6].

3.2. Buchdahl transformations

We will now show that for a given solution of the \( d+1 \) dimensional Einstein equations in vacuum, one can always generate a solution of the same equations minimally coupled to a scalar field. In \( 3+1 \) dimensions this was first shown by Buchdahl [4] and later by Janis, Robinson and Winicour [5]. For the general \( d+1 \) dimensional case see Tangen [14].

For a metric of the form

$$ ds^2 = -e^{2V(x^i)} dt^2 + e^{-2V(x^i)} \delta_{ij} x^i x^j $$

\( \delta_{ij} \) being a \( d \)-dimensional spatial metric and \( V \) is a function of spatial coordinates only, the Einstein tensor reads

$$ R_{00} = e^{4V} \left[ \nabla^2 V - (d - 3) V_i V^i \right] = e^{2V} \nabla^2 V $$

$$ R_{ij} = \hat{R}_{ij} + (d - 3) V_i V^i + (d - 5) V_i V_j + \hat{h}_{ij} \left[ \nabla^2 V - (d - 3) V_i V^i \right] $$

The hat denotes quantities derived using the metric \( \hat{h}_{ij} \) which also is used to raise indexes. If the metric (30) is a solution of the Einstein equations in vacuum we must have \( R_{\mu\nu} = 0 \), which implies

$$ \hat{\nabla}^2 V - (d - 3) V_i V^i = 0 $$

$$ \hat{R}_{ij} + (d - 3) V_{ij} + (d - 5) V_i V_j = 0 $$

We now introduce a new metric

$$ ds^2 = -e^{2U(x^i)} dt^2 + e^{-2U(x^i)} \delta_{ij} x^i x^j $$

where again \( U \) is a function of spatial coordinates only, and the spatial metric \( \delta_{ij} \) is conformal to the spatial vacuum metric \( \hat{h}_{ij} \). We want our metric (35) to be a solution of the minimally coupled equations. For this metric the Einstein equations for gravity minimally coupled to a static scalar field, \( \tilde{R}_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} \), reads

$$ \hat{\nabla}^2 U - (d - 3) U, U^i = 0 $$

$$ \hat{R}_{ij} + (d - 3) U_{ij} + (d - 5) U_i U_j = \phi, \phi_{,i} $$
while we have the equation of motion for the scalar field $\phi$

$$\Box^2 \phi = 0 = e^{2U} \left[ \tilde{\Box}^2 \phi - (d-3)\phi, U^{-1} \right]$$  \hspace{1cm} (38)$$

Since the Ricci tensor in $d$ dimensions transforms as

$$\tilde{R}_{\mu \nu} = \tilde{R}_{\mu \nu} - (d-2)\Omega^{-2} \Omega,_{\mu \nu} - \Omega^{-1} \tilde{h}_{\mu \nu} \Box^2 \Omega$$

\hspace{1cm} + 2(d-2)\Omega^{-2} \Omega,_{\mu} \Omega,_{\nu} - (d-3)\Omega^{-2} \tilde{h}_{\mu \nu} \Omega^\alpha \Omega,_{\alpha}$$  \hspace{1cm} (39)$$

under a Weyl transformation $\hat{h}_{ij} = \Omega^2 \tilde{h}_{ij}$ of the metric, we find that when setting

$$U = aV$$  \hspace{1cm} (40)$$

$$\hat{h}_{ij} = \Omega^2 \tilde{h}_{ij} = e^{2d-3 \frac{1-a^2}{d-2}} U \tilde{h}_{ij}$$  \hspace{1cm} (41)$$

$$\phi = \pm \sqrt{\frac{d-1}{d-2}} \frac{1-a^2}{a^2} U$$  \hspace{1cm} (42)$$

the equations (33-34) are transformed into (36-37). The equation of motion for $\phi$ (38) is also fulfilled. We see from (42) that a necessary constraint is $a^2 \leq 1$. In conclusion, given a vacuum solution of the Einstein equations on the form (30), we can always find a solution of the Einstein equations for a minimally coupled scalar field given by

$$ds^2 = -e^{2V} dt^2 + e^{-2V} \tilde{h}_{ij} dx^i dx^j$$  \hspace{1cm} (43)$$

$$\phi = \pm \sqrt{\frac{d-1}{d-2}} (1-a^2) V$$  \hspace{1cm} (44)$$

where $a$ is an arbitrary constant and $b = \frac{a+d-3}{d-2}$.

We adopt the higher-dimensional Schwarzschild solution (22) written in the form

$$ds^2 = -e^{2V} dt^2 + e^{-2V} \left[ dr^2 + e^{2V} r^2 d\Omega_d^2 \right]$$  \hspace{1cm} (45)$$

where

$$e^V = \sqrt{1 - \frac{B}{r^{d-2}}}$$  \hspace{1cm} (46)$$

as our vacuum solution. Using the above transformation, we end up with the following two-parameter set of solutions

$$ds^2 = -e^{2V} dt^2 + e^{-2V} \left[ \frac{d-1}{d-2} \left( 1 - a^2 \right) \ln \left( 1 - \frac{B}{r^{d-2}} \right) \right]$$  \hspace{1cm} \left(47\right)$$

$$\phi = \pm \frac{1}{2} \sqrt{\frac{d-1}{d-2}} \left( 1 - a^2 \right) \ln \left( 1 - \frac{B}{r^{d-2}} \right)$$  \hspace{1cm} \left(48\right)$$

Renaming the constant $B = 4r_0^{d-2}$ and making the coordinate transformation to isotropic coordinates

$$r \rightarrow r \left( r^{d-2} - r_0^{d-2} \right)^{\frac{1}{d-2}}$$  \hspace{1cm} \left(49\right)$$

we see when choosing the upper sign in (48) that these are exactly the same solutions as those found by Xanthopoulos and Zannias [6].
4. Conformal coupling

4.1. Fundamental equations

We now consider the case \( \xi = \frac{d-1}{4d} \) for which the system is conformally invariant. Since \( T_{\mu \nu} \), \( E_{\mu \nu} \) and \( R_{\mu \nu} \) are traceless in a conformal theory, equation (2) still reduces to \( \Box^2 \phi = 0 \) and the Einstein equations in Schwarzschild coordinates can be written

\[
\frac{\phi''}{(d-1)} + \frac{\phi' \phi''}{d} = \left(1 - \frac{d-1}{4d} \phi^2 \right) \left[ \frac{2}{r} \phi' - \frac{d-2}{r^2} (\phi^2 - 1) \right]
\]

(50)

Further, since \( \Box^2 \phi = 0 \), equations (16) and (17) are still valid.

When trying to solve the above equations, it is tempting to choose \( \alpha' + \beta' = 0 \). Then we get from adding (50) and (51)

\[
\phi'' = \frac{d}{d-1} (\phi')^2
\]

(52)

with solution

\[
\phi = \frac{A}{(r - B)^{\frac{d+1}{2}}}
\]

(53)

where \( A \) and \( B \) are constants. Combining with the \( \phi \)-equation (17) which for the case \( \alpha' + \beta' = 0 \) reads

\[
\phi' = C e^{-2\alpha} r^{-(d-1)}
\]

(54)

we can solve for \( e^{2\alpha} \) and find

\[
e^{2\alpha} = e^{-2\beta} = \left(1 - \frac{B}{r}\right)^{\frac{d+1}{2}} r^{-\frac{d-1}{2}}
\]

(55)

Here we have chosen the integration constant \( C \) in (54) such that the metric approaches Minkowski for large \( r \). But only for \( d = 3 \) dimensions is this a solution of the Einstein equations (50) and (51). In \( 3+1 \) dimensions we find \( A^2 = \frac{B^2}{\xi} = 6B^2 \) and the metric given by (55) reduces to the \( d = 3 \) version of the extremal Reissner-Nordström metric\cite{19,20,18}

\[
ds^2 = -\left(1 - \frac{B}{r^{d-2}}\right)^2 dt^2 + \left(1 - \frac{B}{r^{d-2}}\right)^{-2} dr^2 + r^2 d\Omega^2_\perp
\]

(56)

The corresponding scalar field solution is

\[
\phi = \sqrt{\frac{B}{r - B}}
\]

(57)

To get canonical normalization we must put \( B = B_s/2 \). For \( d \neq 3 \) the \( d+1 \)-dimensional extremal Reissner-Nordström metric is not a solution of the combined gravity and scalar field equations. In this case there are no solutions having \( \alpha' + \beta' = 0 \).

4.2. Bekenstein transformations

Introducing a new metric \( \tilde{g}_{\mu \nu} \) in the form of the following conformal transformation of the old metric \( g_{\mu \nu} \)

\[
\tilde{g}_{\mu \nu} = \cosh^4 (\sqrt{\xi} \phi) \, g_{\mu \nu}
\]

(58)
together with a redefinition of the scalar field
\[ \psi = \frac{1}{\sqrt{\xi}} \tanh(\sqrt{\xi} \phi) \] (59)

brings us from the minimal coupling case of the Einstein plus scalar field equations in the old metric \( g_{\mu\nu} \)

\[ E_{\mu\nu}(g) = T^{\phi}_{\mu\nu}(g) \] (60)

\[ \Box^2 \phi = 0 \] (61)

to the conformal coupling case of the same equations for the new metric \( \tilde{g}_{\mu\nu} \)

\[ (1 - \xi \psi^2)E_{\mu\nu}(\tilde{g}) = T^{\psi}_{\mu\nu}(\tilde{g}) + \Delta T^{\psi}_{\mu\nu}(\tilde{g}) \] (62)

\[ \Box^2 \psi = 0 = R \] (63)

Here we use the fact that the minimal scalar field energy-momentum tensor [4] does not change under a conformal transformation like (58)

\[ T^{\phi}_{\mu\nu}(\tilde{g}) = T^{\phi}_{\mu\nu}(g) \] (64)

and that the Einstein tensor under a conformal transformation \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) in \( D \) dimensions change like

\[ \tilde{E}_{\mu\nu} = E_{\mu\nu} + (D - 2)\Omega^{-1} \left[ g_{\mu\nu} \Box^2 \Omega - \Omega_{\mu\nu} \right] + 2(D - 2)\Omega^{-2} \Omega_{,\mu} \Omega_{,\nu} + \frac{1}{2}(D - 2)(D - 5) \Omega^{-2} g_{\mu\nu} \Omega_{,\alpha} \Omega_{,\alpha} \] (65)

The result is the same if we instead of transformations \( [58, 59] \) use

\[ \tilde{g}_{\mu\nu} = \sinh \frac{1}{\sqrt{\xi}}(\sqrt{\xi} \phi) \ g_{\mu\nu} \] (66)

\[ \psi = \frac{1}{\sqrt{\xi}} \coth(\sqrt{\xi} \phi) \] (67)

The latter correspond to \( 1 - \xi \psi^2 \) being negative while the first transformations \( [58, 59] \) is used for positive \( 1 - \xi \psi^2 \). The transformations \( [58, 59] \) and \( [60, 61] \) were first found in 3+1 dimensions by Bekenstein [8]. Maeda [21] showed very generally that Lagrangians with arbitrary couplings between \( \phi \) and \( R \) in arbitrary dimensions can always be transformed to a minimally coupled Einstein Frame theory by means of a conformal transformation. The specific extension of equations \( [68, 69] \) and \( [66, 67] \) to arbitrary dimensions was done by Xanthopoulos and Dialynas [12].

We now take as our minimal solution the solution \( [46, 48] \) found in section 3.2 which we write like

\[ ds^2 = -e^{2V_a} dt^2 + e^{-2V_b} dr^2 + e^{2V(1-b)} d\Omega_d^2 \] (68)

\[ \phi = \pm \sqrt{\frac{d-1}{d-2}} (1-a^2) V \] (69)

Performing the transformations \( [58, 59] \) we arrive at a two-parameter solution of the conformal equations

\[ ds^2 = \left( \frac{e^{2V_c} + e^{-2V_c}}{2} \right)^{-1} \left[ -e^{2V_a} dt^2 + e^{-2V_b} dr^2 + e^{2V(1-b)} r^2 d\Omega^2 \right] \] (70)

\[ \sqrt{\xi} \psi = \tanh(\pm 2V_c) = \pm \frac{e^{4V_c} - 1}{e^{4V_c} + 1} \] (71)
where the constant \( c \) is given by
\[
c = \frac{d - 1}{4} \sqrt{\frac{1 - a^2}{d(d - 2)}}
\]
(72)

The conformal solution (70-71) has the same two parameters \( a \) and \( B \) as the minimal solution (46-48). \( V \) is still given by (46) and we still have \( b = \frac{a + d - 3}{d - 1} \).

Choosing now the particular solution given by \( c = \frac{1}{4} \), corresponding to \( a = \frac{1}{d - 1} \) and \( b = \frac{d - 2}{d - 1} \), the metric (70) simplifies to
\[
ds^2 = \left( \frac{e^V + 1}{2} \right)^{\frac{d-1}{2}} [-dt^2 + e^{-2V} dr^2 + r^2 d\Omega^2]
\]
(73)

In order to write this particular solution in Schwarzschild coordinates, we introduce a new radial coordinate \( R \) given by
\[
R = \left( \frac{1 + \sqrt{1 - B/r}}{2} \right)^{\frac{d-1}{2}} r
\]
(74)

Then the metric (73) can be written as
\[
ds^2 = -\left( \frac{R}{r(R)} \right)^2 dt^2 + \left( \frac{2}{d - 1} \left( \frac{R}{r(R)} \right)^{\frac{d-1}{2}} + \frac{d - 3}{d - 1} \right)^{-2} dR^2 + R^2 d\Omega^2
\]
(75)

where \( r(R) \) is given implicitly from (74). When choosing the lower sign in (71) the scalar field \( \phi \) now takes the form
\[
\sqrt{\xi} \psi = \left( \frac{r(R)}{R} \right)^{\frac{d-1}{2}} - 1
\]
(76)

Only in \( d = 3 \) dimensions can (74) be solved explicitly to give
\[
\frac{1}{r} = \frac{1}{R} \left( 1 - \frac{B}{4R} \right)
\]
(77)

so that we can write (75) and (70) in Schwarzschild coordinates
\[
ds^2 = -\left( 1 - \frac{B}{4R} \right)^2 dt^2 + \left( 1 - \frac{B}{4R} \right)^{-2} dR^2 + R^2 d\Omega^2
\]
\[
\psi = \sqrt{6} \frac{B/4R}{R - B/4}
\]
(78)

which is the same BBMB solution with the extremal Reissner-Nordström metric as we found in last section (60-67). As demonstrated by Xanthopoulos and Zannias [10] and Xanthopoulos and Dialynas [12], this is the only known black hole solution for gravity coupled to scalar fields except for the trivial solutions where \( \phi \) is constant and the metric is a vacuum black hole. The BBMB solution has been extensively studied by for instance Zannias [22] who has shown it not to have a continuous new parameter, and therefore not to contradict the no scalar hair-theorem [23] [24] [25] [26]. For an overview see [27].
5. Conclusions
Given a vacuum solution of the Einstein equations, solutions of the equations for gravity coupled either minimally or conformally to a massless scalar field can be generated in arbitrary spacetime dimensions. To obtain a minimal solution we perform a generalized Buchdahl transformation on our vacuum metric. To obtain a conformal solution we perform a generalized Bekenstein transformation on this minimal solution.

In the search for static and spherical symmetric black hole-like solutions we choose the Schwarzschild black hole as our seeding metric. This gives us both Xanthopoulos and Zannias’ minimal solutions and Xanthopoulos and Dialynas’ conformal solutions. It is known that only in 3 + 1 dimensions do these solutions include a black hole, namely the BBMB black hole where the metric is the extremal Reissner-Nordström metric. This makes us wonder what is special with the four-dimensional spacetime we normally call home.

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