Invited Comment

Chiral EFT based nuclear forces: achievements and challenges

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Abstract

During the past two decades, chiral effective field theory has become a popular tool to derive nuclear forces from first principles. Two-nucleon interactions have been worked out up to sixth order of chiral perturbation theory and three-nucleon forces up to fifth order. Applications of some of these forces have been conducted in nuclear few- and many-body systems—with a certain degree of success. But in spite of these achievements, we are still faced with great challenges. Among them is the issue of a proper uncertainty quantification of predictions obtained when applying these forces in ab initio calculations of nuclear structure and reactions. A related problem is the order by order convergence of the chiral expansion. We start this review with a pedagogical introduction and then present the current status of the field of chiral nuclear forces. This is followed by a discussion of representative examples for the application of chiral two- and three-body forces in the nuclear many-body system including convergence issues.

Keywords: low-energy QCD, effective field theory, chiral perturbation theory, nucleon–nucleon potentials, three-nucleon forces, nuclear matter, microscopic nuclear structure

(Some figures may appear in colour only in the online journal)

1. Historical perspective

In 1975, when Bohr, Mottelson, and Rainwater were honored with the Nobel Prize, the Reid potential [1] was the most popular nucleon–nucleon (NN) potential within the international nuclear physics community. It was applied in most microscopic nuclear structure calculations produced in the 1970s. The Reid potential is a phenomenological potential that was considered very quantitative by the standards of the time and easy to use, which explains its popularity. However, attempts to derive the NN interaction on fundamental grounds had been around for quite a while. Since the mid 1960s, one-boson exchange potentials were being developed [2], which by the mid 1970s assumed a quantitative character comparable to the Reid potential [3–5]. Moreover, research that went beyond the simple one-boson-exchange assumption (which always includes a ‘fictitious’ σ-boson) was also under way. The most notable work of this kind became known as the Paris [6] and the Bonn potentials [7].

Since the more sophisticated meson models seemed to have a sound theoretical foundation and, in addition, were quantitatively very successful, it appeared that they were the solution of the nuclear force problem. However, with the discovery (in the 1970s) that the fundamental theory of strong interactions is quantum chromodynamics (QCD)2 and not meson theory, all ‘meson theories’ had to be viewed as models, and the attempts to derive the nuclear force from first principals had to start all over again.

The problem with a derivation of nuclear forces from QCD is two-fold. First, each nucleon consists of three valence quarks, quark–antiquark pairs, and gluons such that the system of two nucleons is a complicated many-body problem. Second, the force between quarks, which is created by the exchange of gluons, has the feature of being very

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2 For the abbreviations and acronyms used in the article, see table 1.
strong at the low energy-scale that is characteristic of nuclear physics. This extraordinary strength makes it difficult to find converging expansions. Therefore, during the first round of new attempts, QCD-inspired quark models became popular. The positive aspect of these models is that they try to explain nucleon structure (which consists of three quarks) and \( NN \) interactions (six-quark systems) on an equal footing. Some of the gross features of the two-nucleon force, like the 'hard core', are explained successfully in such models. However, from a critical point of view, it must be noted that these quark-based approaches are yet another set of models and not a theory. Alternatively, one may try to solve the six-quark problem with brute computing power, by putting the six-quark system on a four-dimensional lattice of discrete points which represents the three-dimensions of space and one-dimension of time. This method has become known as lattice QCD and is making progress. However, such calculations are computationally very expensive and cannot be used as a standard nuclear physics tool.

Around 1980, a major breakthrough occurred when the nobel laureate Steven Weinberg applied the concept of an effective field theory (EFT) to low-energy QCD \([8, 9]\). He simply wrote down the most general Lagrangian that is consistent with all the properties of low-energy QCD, since that would make this theory equivalent to low-energy QCD. A particularly important property is \( SU(2)_L \times SU(2)_R \) symmetry, the so-called chiral symmetry, which is 'spontaneously' broken. Massless spin-\( \frac{1}{2} \) fermions possess chirality, which means that their spin and momentum are either parallel to each other ('right-handed') or anti-parallel ('left-handed') and remain so forever. Since the quarks, which nucleons are made of ('up' and 'down' quarks), are almost massless, approximate chiral symmetry is a given. Naively, this symmetry should have the consequence that one finds in nature mesons of the same mass, but with positive and negative parity. However, this is not the case and such failure is termed a 'spontaneous' breaking of the symmetry. According to a theorem first proven by Goldstone, the spontaneous breaking of a symmetry implies the existence of a particle, here, the pseudoscalar pion. Thus, the pion becomes the main player in the interactions of pions with nucleons. However, this EFT can be expanded in powers of momentum over 'scale', where scale denotes the 'chiral symmetry breaking scale' which is about 1 GeV. This scheme is also known as chiral perturbation theory (ChPT) and allows to calculate the various terms that make up the nuclear potential systematically power by power, or order by order. Another advantage of the chiral EFT approach is its ability to generate not only the force between two nucleons, but also many-nucleon forces, on the same footing \([10]\). In modern theoretical nuclear physics, the chiral EFT approach is becoming increasingly popular and is applied with great success \([11–15]\).

This article is organized as follows. In section 2, we will present a pedagogical introduction into the EFT approach to low-energy QCD, including the development of effective Lagrangians. Section 3 provides a broad overview of the hierarchy of nuclear forces as they emerge from EFT. Sections 4–6 then spell out in detail the development of the two-nucleon forces from long-range to short-range and the construction of quantitative \( NN \) potentials. Chiral many-body forces are presented in section 7. Applications of chiral forces in the many-body problem and convergence issues are discussed in sections 8 and 9 concludes the article.

### 2. EFT for low-energy QCD

QCD provides the theoretical framework to describe strong interactions, namely interactions involving quarks and gluons. According to QCD, objects which carry color interact weakly at short distances and strongly at large distances, where the separation between the two regimes is about 1 fm. Naturally, short distances and long distances can be associated with high and low energies, respectively, causing the quarks to be confined into hadrons, which carry no color. At the same time, the weak nature of the force at high energies results into what is known as 'asymptotic
freedom’. (We note that these behaviors originate from the fact that QCD is a non-abelian gauge field theory with color $SU(3)$ the underlying gauge group.) Therefore, QCD is perturbative at high energy, but strongly coupled at low-energy. The energies typical for nuclear physics are low and, thus, nucleons are appropriate degrees of freedom. The nuclear force can then be regarded as a residual color interaction acting between nucleons in a way similar to how the van der Waals forces bind neutral molecules. If described in terms of quark and gluon degrees of freedom, the interaction between nucleons is an extremely complex problem, which can be confronted with the computational methods known as lattice QCD. In a recent paper [16], the $NN$ system is investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV. Over the range of energies that are studied, the scattering phase shifts are investigated at a pion mass of about 450 MeV.

The central component of this potential exhibits repulsion at the core as well as intermediate-range attraction. This is encouraging, but one must keep in mind that the pion masses employed in this study are still quite large. In summary, although calculations within lattice QCD are being performed and improved, they are computationally very costly, and thus they are useful, in practice, only to explore a few cases. Clearly, a different approach is necessary to address a full variety of nuclear structure problems.

The concept of an EFT shows an alternative and realistic way to proceed. The first step towards the development of an EFT is the identification of appropriate scales. The large difference between the masses of the pions and the masses of the vector mesons, like $\rho(770)$ and $\omega(782)$, provides a clue. From that observation, one is prompted to take the pion mass as the identifier of the soft scale, $Q \sim m_\pi$, while the rho mass sets the hard scale, $\Lambda_q \sim m_\rho$, often referred to as the chiral-symmetry breaking scale. It is then natural to consider an expansion in terms of $Q/\Lambda_q$. With regard to the choice of degrees of freedom, we observed earlier that, as far as conventional nuclear physics is concerned, quarks and gluons are ineffective and thus nucleons and pions should be taken as the appropriate degrees of freedom. Of course, we do not wish to construct yet one more phenomenological model and, therefore, our EFT must be firmly linked with QCD. This strong link is present if we require the EFT to be consistent with the symmetries of QCD. The meaning and relevance of such statement is expressed in the so-called ‘folk theorem’ by Weinberg [8]:

If one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible $S$-matrix consistent with analyticity, perturbative unitarity, cluster decomposition, and the assumed symmetry principles.

To summarize, the development of a proper EFT must proceed as follows:

(i) Identify the low- and high-energy scales, and the degrees of freedom suitable for (low-energy) nuclear physics.

(ii) Recognize the symmetries of low-energy QCD and explore the mechanisms responsible of their breakings.

(iii) Build the most general Lagrangian which respects those (broken) symmetries.

(iv) Formulate a scheme to organize contributions in order of their importance. Clearly, this amounts to performing an expansion in terms of (low) momenta.

(v) Using the expansion mentioned above, evaluate Feynman diagrams to any desired accuracy.

In what follows, we will discuss each of the steps above. Note that the first one has already been addressed, so we will move directly to the second one.

2.1. Symmetries of low-energy QCD

Our purpose here is to provide a compact introduction into (low-energy) QCD, with particular attention to the symmetries and their breakings. For more details the reader is referred to [11, 18].

2.1.1. Chiral symmetry. We begin with the QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \bar{q}(i\gamma^\mu D_\mu - \mathcal{M})q - \frac{1}{4} G_{\mu\nu\sigma\tau} G^{\mu\nu\sigma\tau}$$

with the gauge-covariant derivative

$$D_\mu = \partial_\mu - i g \lambda_\mu A_\mu,$$

and the gluon field strength tensor

$$G_{\mu
\nu\sigma\tau} = \partial_\mu A_{\nu\sigma\tau} - \partial_\nu A_{\mu\sigma\tau} + g f_{\mu\nu\sigma\tau} A_{\mu\sigma\tau}.$$

In the above, $q$ denotes the quark fields and $\mathcal{M}$ the quark mass matrix. Further, $g$ is the strong coupling constant and $A_{\mu\sigma\tau}$ are the gluon fields. Moreover, $\lambda_\mu$ are the Gell-Mann matrices and $f_{\mu\nu\sigma\tau}$ the structure constants of the SU(3) color Lie algebra $(a, b, c = 1, \ldots, 8)$; summation over repeated indices is always implied. The gluon–gluon term in the last equation arises from the non-abelian nature of the gauge theory and is the reason for the peculiar features of the color force.

The current masses of the up ($u$), down ($d$), and strange ($s$) quarks are in a $\overline{MS}$ scheme at a scale of $\mu \approx 2$ GeV [19]:

$$m_u = 2.3 \pm 0.7 \text{ MeV},$$

$^3$ For $SU(N)$ group indices, we use Latin letters,..., $a, b, c,..., i, j, k,...$, and, in general, do not distinguish between subscripts and superscripts.
where

\[ m_d = 4.8 \pm 0.5 \text{ MeV}, \]

\[ m_u = 95 \pm 5 \text{ MeV}. \]

These masses are small as compared to a typical hadronic scale such as the mass of a light hadron other than a Goldstone bosons, e.g., \( m_p = 0.78 \text{ GeV} \approx 1 \text{ GeV}. \)

Thus it is relevant to discuss the QCD Lagrangian in the case when the quark masses vanish:

\[ \mathcal{L}^0_{\text{QCD}} = \bar{q}i\gamma^\mu D_\mu q - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}. \]

Right- and left-handed quark fields are defined as

\[ q_R = P_R q, \quad q_L = P_L q, \]

with

\[ P_R = \frac{1}{2}(1 + \gamma_5), \quad P_L = \frac{1}{2}(1 - \gamma_5). \]

Then the Lagrangian can be rewritten as

\[ \mathcal{L}^0_{\text{QCD}} = \bar{q}_R i\gamma^\mu D_\mu q_R + \bar{q}_L i\gamma^\mu D_\mu q_L - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}. \]

This equation reveals that the right- and left-handed components of massless quarks do not mix in the QCD Lagrangian. For the two-flavor case, this is \( SU(2)_R \times SU(2)_L \) symmetry, also known as chiral symmetry. However, this symmetry is broken in two ways: explicitly and spontaneously.

**2.1.2. Explicit symmetry breaking.** The mass term \(-\bar{q}Mq\) in the QCD Lagrangian equation (1) breaks chiral symmetry explicitly. To better see this, let us rewrite \( \mathcal{M} \) for the two-flavor case

\[ \mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \]

\[ = \frac{1}{2}(m_u + m_d) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(m_u - m_d) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ = \frac{1}{2}(m_u + m_d) I + \frac{1}{2}(m_u - m_d) \gamma_5. \]

The first term in the last equation is invariant under \( SU(2)_L \) (isospin symmetry) and the second term vanishes for \( m_u = m_d \). Therefore, isospin is an exact symmetry if \( m_u = m_d \). However, both terms in equation (11) break chiral symmetry. Since the up and down quark masses (equations (4) and (5)) are small as compared to the typical hadronic mass scale of \( \sim 1 \text{ GeV} \), the explicit chiral symmetry breaking due to non-vanishing quark masses is very small.

**2.1.3. Spontaneous symmetry breaking.** A (continuous) symmetry is said to be spontaneously broken if a symmetry of the Lagrangian is not realized in the ground state of the system. There is evidence that the (approximate) chiral symmetry of the QCD Lagrangian is spontaneously broken—

for dynamical reasons of non-perturbative origin which are not fully understood at this time. The most plausible evidence comes from the hadron spectrum.

From chiral symmetry, one naively expects the existence of degenerate hadron multiplets of opposite parity, i.e., for any hadron of positive parity one would expect a degenerate hadron state of negative parity and vice versa. However, these ‘parity doublets’ are not observed in nature. For example, take the \( \rho \)-meson which is a vector meson of negative parity (\( J^P = 1^- \)) and mass 776 MeV. There does exist a \( 1^+ \) meson, the \( a_1 \), but it has a mass of 1230 MeV and, therefore, cannot be perceived as degenerate with the \( \rho \). On the other hand, the \( \rho \) meson comes in three charge states (equivalent to three isospin states), the \( \rho^0 \) and the \( \rho^\pm \), with masses that differ by at most a few MeV. Thus, in the hadron spectrum, \( SU(2)_L \) (isospin) symmetry is well observed, while axial symmetry is broken: \( SU(2)_R \times SU(2)_L \) is broken down to \( SU(2)_L \).

A spontaneously broken global symmetry implies the existence of (massless) Goldstone bosons. The Goldstone bosons are identified with the isospin triplet of the (pseudoscalar) pions, which explains why pions are so light. The pion masses are not exactly zero because the up and down quark masses are not exactly zero either (explicit symmetry breaking). Thus, pions are a truly remarkable species: they reflect spontaneous as well as explicit symmetry breaking. Goldstone bosons interact weakly at low energy. They are degenerate with the vacuum and, therefore, interactions between them must vanish at zero momentum and in the chiral limit (\( m_\pi \to 0 \)).

**2.2. Chiral effective Lagrangians**

The next step in our EFT program is to build the most general Lagrangian consistent with the (broken) symmetries discussed above. An elegant formalism for the construction of such Lagrangians was developed by Callan, Coleman, Wess, and Zumino (CCWZ) [20] who developed the foundations of non-linear realizations of chiral symmetry from the point of view of group theory.\(^4\) The Lagrangians we give below are built upon the CCWZ formalism.

We already addressed the fact that the appropriate degrees of freedom are pions (Goldstone bosons) and nucleons. Because pion interactions must vanish at zero momentum transfer and in the limit of \( m_\pi \to 0 \), namely the chiral limit, the Lagrangian is expanded in powers of derivatives and pion masses. More precisely, the Lagrangian is expanded in powers of \( Q/\Lambda_c \) where \( Q \) stands for a (small) momentum or pion mass and \( \Lambda_c \approx 1 \text{ GeV} \) is identified with the hard scale. These are the basic steps behind the chiral perturbative expansion.

Schematically, we can write the effective Lagrangian as

\[ \mathcal{L} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \mathcal{L}_{NN} + \ldots, \]

where \( \mathcal{L}_{\pi\pi} \) deals with the dynamics among pions, \( \mathcal{L}_{\pi N} \) describes the interaction between pions and a nucleon, and

\(^4\) An accessible introduction into the rather involved CCWZ formalism can be found in [18].
\( \mathcal{L}_{NN} \) contains two-nucleon contact interactions which consist of four nucleon-fields (four nucleon legs) and no meson fields. The ellipsis stands for terms that involve two nucleons plus pions and three or more nucleons with or without pions, relevant for nuclear many-body forces (an example for this in lowest order are the last two terms of equation (18), below). The individual Lagrangians are organized order by order:

\[
\mathcal{L}_{\pi\pi} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi\pi}^{(4)} + \ldots,
\]

\[
\mathcal{L}_{\pi N} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(3)} + \mathcal{L}_{\pi N}^{(5)} + \mathcal{L}_{\pi N}^{(7)} + \ldots,
\]

and

\[
\mathcal{L}_{NN} = \mathcal{L}_{NN}^{(0)} + \mathcal{L}_{NN}^{(2)} + \mathcal{L}_{NN}^{(4)} + \mathcal{L}_{NN}^{(6)} + \ldots,
\]

where the superscript refers to the number of derivatives or pion mass insertions (chiral dimension) and the ellipsis stands for terms of higher dimensions.

Above, we have organized the Lagrangians by the number of derivatives or pion-masses. This is the standard way, appropriate particularly for considerations of \( \pi - \pi \) and \( \pi - N \) scattering. As it turns out (see section 3.1), for interactions among nucleons, sometimes one makes use of the so-called index of the interaction

\[
\Delta \equiv d + \frac{n}{2} - 2,
\]

where \( d \) is the number of derivatives or pion-mass insertions and \( n \) the number of nucleon field operators (nucleon legs). We will now write down the Lagrangian in terms of increasing values of the parameter \( \Delta \) and we will do so using the so-called heavy-baryon formalism which we indicate by a ‘hat’ [21].

The leading-order Lagrangian reads,

\[
\mathcal{L}_{\Delta=0} = \frac{1}{2} \partial_{\mu} \pi \cdot \partial^{\mu} \pi - \frac{1}{3} m_{\pi}^{2} \pi^{2}
\]

\[
+ \frac{1 - 4\alpha}{2f_{\pi}^{2}} (\pi \cdot \partial_{\mu} \pi)(\pi \cdot \partial^{\mu} \pi)
\]

\[
- \frac{\alpha}{f_{\pi}^{2}} m_{\pi}^{2} \pi \cdot \partial^{\mu} \pi + \frac{8\alpha}{1} - \frac{8}{f_{\pi}^{2}} m_{\pi}^{2} \pi^{4}
\]

\[
+ \tilde{N} \left\{ i\partial_{\mu} - \frac{g_{A}}{2f_{\pi}} \tau \cdot (\partial_{\mu} \vec{\pi}) \pi
\]

\[
+ \frac{1}{4f_{\pi}^{2}} \tau \cdot (\partial_{\mu} \partial_{\nu} \pi) \right\} \tilde{N}
\]

\[
+ \tilde{N} \left\{ \frac{g_{A}(4\alpha - 1)}{4f_{\pi}^{2}} (\tau \cdot \pi)[\pi \cdot (\partial_{\mu} \vec{\pi}) \pi]
\]

\[
+ \frac{g_{A}\alpha}{2f_{\pi}^{2}} \pi^{2}[\tau \cdot (\partial_{\mu} \vec{\pi}) \pi]
\]

\[
- \frac{1}{2} C_{s} \tilde{N} \tilde{N} \tilde{N} \tilde{N} = \frac{1}{2} C_{f} (\tilde{N} \tilde{\sigma} N) \cdot (\tilde{N} \tilde{\sigma} N) + \ldots,
\]

and subleading Lagrangians are

\[
\mathcal{L}_{\Delta=1} = \mathcal{N} \left\{ \frac{\nabla^{2} \pi}{2M_{N}} - \frac{i g_{A}}{4M_{N} f_{\pi}} \tau \cdot [\vec{\partial} \cdot (\vec{\nabla} \partial_{\mu} \pi - \partial_{\mu} \vec{\pi})]
\]

\[
- \frac{i}{8M_{N} f_{\pi}^{2}} \tau \cdot \left[ \vec{\nabla} \cdot (\pi \times \vec{\nabla} \pi)
\right.
\]

\[
- (\pi \times \vec{\nabla} \pi) \cdot \vec{\nabla} N
\]

\[
+ \tilde{N} \left[ 4\alpha m_{\pi}^{2} - \frac{2\alpha}{f_{\pi}^{2}} m_{\pi}^{2} \pi^{2}
\right.
\]

\[
+ \frac{c_{1}}{f_{\pi}^{2}} \left( \partial_{\mu} \pi \cdot \partial^{\mu} \pi \right) + \left( c_{4} + \frac{1}{4M_{N}} \right)
\]

\[
\times \frac{1}{2f_{\pi}^{2}} e^{i(\vec{\rho} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B})(\vec{\tau} \cdot \vec{C})} \bar{N} \left[ \tau \cdot (\partial_{\mu} \vec{\pi}) \pi \right] \right\} N
\]

\[
- \frac{D}{4f_{\pi}^{2}} \left[ (\bar{N} \pi N) \bar{N} \pi N \right] \cdot (\bar{N} \pi N) + \ldots,
\]

\[
\mathcal{L}_{\Delta=2} = \mathcal{L}_{\Delta=2}^{(4)} + \mathcal{L}_{\Delta=2}^{(6)} + \ldots,
\]

\[
\mathcal{L}_{\Delta=3} = \mathcal{L}_{\Delta=3}^{(4)} + \ldots,
\]

\[
\mathcal{L}_{\Delta=4} = \mathcal{L}_{\Delta=4}^{(5)} + \mathcal{L}_{\Delta=4}^{(6)} + \ldots,
\]

\[
\mathcal{L}_{\Delta=6} = \mathcal{L}_{\Delta=6}^{(6)} + \ldots,
\]

where we included terms relevant for a calculation of the two-nucleon force up to sixth order. The Lagrangians \( \mathcal{L}_{\pi\pi}^{(3)} \) and \( \mathcal{L}_{\pi\pi}^{(4)} \) can be found in [22] and \( NN \) contact Lagrangians are given below. The pion fields are denoted by \( \pi \) and the heavy baryon nucleon field by \( \bar{N} \) (\( \bar{N} = N' \)). Furthermore, \( g_{A}, f_{\pi}, m_{\pi}, \) and \( M_{N} \) are the axial-vector coupling constant, pion decay constant, pion mass, and nucleon mass, respectively. Numerical values for these quantities will be given later. The \( c_{i} \) are low-energy constants (LECs) from the dimension two \( \pi N \) Lagrangian and \( \alpha \) is a parameter that appears in the expansion of a SU(2) matrix \( U \) in powers of the pion fields, see [11] for more details. Results are independent of \( \alpha \).

The lowest order (or leading order) \( NN \) Lagrangian has no derivatives and reads [9]

\[
\mathcal{L}_{NN}^{(0)} = -\frac{1}{2} C_{s} \bar{N} N \bar{N} - \frac{1}{2} C_{f} \bar{N} \tilde{\sigma} N \cdot (\bar{N} \tilde{\sigma} N),
\]

where \( C_{s} \) and \( C_{f} \) are free parameters to be determined by fitting to the NN data.
The second order $NN$ Lagrangian can be stated as follows:

\[ \mathcal{L}_{NN}^{(2)} = - C_1' [(\bar{\mathcal{N}} \mathcal{N})^2 + (\mathcal{N} \bar{\mathcal{N}})^2] - C_2' (\bar{\mathcal{N}} \mathcal{N}) \cdot (\mathcal{N} \bar{\mathcal{N}}) \]

\[ - C_3' \mathcal{N}[\mathcal{N} \bar{\mathcal{N}}^2 + \mathcal{N} \bar{\mathcal{N}}] \]

\[ - i C_4' [\bar{\mathcal{N}} \mathcal{N} \cdot (\mathcal{N} \bar{\mathcal{N}} \times \partial \mathcal{N})] \]

\[ + (\mathcal{N} \bar{\mathcal{N}}) \cdot (\bar{\mathcal{N}} \sigma \times \bar{\mathcal{N}}) \]

\[ - i C_5' \mathcal{N} \bar{\mathcal{N}} (\mathcal{N} \sigma \times \bar{\mathcal{N}}) \]

\[ - i C_6' (\bar{\mathcal{N}} \sigma \mathcal{N}) \cdot (\bar{\mathcal{N}} \sigma \mathcal{N} \times \partial \mathcal{N}) \]

\[ - (C_7' \delta_{\alpha \beta} \delta_{\gamma \delta} + C_8' \delta_{\alpha \beta} \delta_{\gamma \delta} + C_9' \delta_{\alpha \beta} \delta_{\gamma \delta}) \]

\[ \times [\bar{\mathcal{N}} \sigma_\alpha \bar{\mathcal{N}} \sigma_\beta \mathcal{N} \sigma_\gamma \partial \mathcal{N} + \bar{\mathcal{N}} \sigma_\alpha \mathcal{N} \sigma_\beta \bar{\mathcal{N}} \sigma_\gamma \partial \mathcal{N}] \]

\[ - (C_{10}' \delta_{\alpha \beta} \delta_{\gamma \delta} + C_{11}' \delta_{\alpha \beta} \delta_{\gamma \delta} + C_{12}' \delta_{\alpha \beta} \delta_{\gamma \delta} + C_1 \delta_{\alpha \beta} \delta_{\gamma \delta}) \]

\[ \times [\bar{\mathcal{N}} \sigma_\alpha \mathcal{N} \sigma_\beta N + \bar{\mathcal{N}} \sigma_\alpha \mathcal{N} \sigma_\beta N] \bar{\mathcal{N}} \sigma_\gamma \partial \mathcal{N}. \]

(24)

Similar to $C_5$ and $C_7$ of equation (23), the $C_i'$ of equation (24) are free parameters which are determined in a fit to the NN data. Clearly, the contact Lagrangians grow considerably in size and complexity as the order increases. Therefore we do not provide here explicit expressions for $L_{NN}^{(3)}$ and $L_{NN}^{(4)}$. The $NN$ contact potentials derived from some of the NN Lagrangians are given in section 6.1.

3. Nuclear forces from EFT: overview

We proceed here with discussing the various steps towards a derivation of nuclear forces from EFT. In this section, we will discuss the expansion we are using in more details as well as the various Feynman diagrams as they emerge at each order.

3.1. ChPT and power counting

An infinite number of Feynman diagrams can be evaluated from the effective Langrangians and so one needs to be able to organize these diagrams in order of their importance. ChPT provides such an organizational scheme.

In ChPT, graphs are analyzed in terms of powers of small external momenta over the large scale: $(Q/\Lambda_{\chi})^\nu$, where $Q$ is generic for a momentum (nucleon three-momentum or pion four-momentum) or a pion mass and $\Lambda_{\chi} \sim 1$ GeV is the chiral symmetry breaking scale (hadronic scale, hard scale). Determining the power $\nu$ has become known as power counting.

For the moment, we will consider only so-called irreducible graphs. By definition, an irreducible graph is a diagram that cannot be separated into two by cutting only nucleon lines. Following the Feynman rules of covariant perturbation theory, a nucleon propagator carries the dimension $Q^{-1}$, a pion propagator $Q^{-2}$, each derivative in any interaction is $Q$, and each four-momentum integration $Q^4$. This is also known as naive dimensional analysis. Applying then some topological identities, one obtains for the power of an irreducible diagram involving $A$ nucleons [11]

\[ \nu = -2 + 2A - 2C + 2L + \sum_i \Delta_i, \quad (25) \]

with

\[ \Delta_i \equiv d_i + \frac{n_i}{2} - 2. \quad (26) \]

In the two equations above, for each vertex $i$, $C$ represents the number of individually connected parts of the diagram while $L$ is the number of loops; $d_i$ indicates how many derivatives or pion masses are present and $n_i$ the number of nucleon fields. The summation extends over all vertices present in that particular diagram. Notice also that chiral symmetry implies $\Delta_i \geq 0$. Interactions among pions have at least two derivatives ($d_i \geq 2, n_i = 0$), while interactions between pions and a nucleon have one or more derivatives ($d_i \geq 1, n_i = 2$). Finally, pure contact interactions among nucleons ($n_i = 4$) have $d_i \geq 0$. In this way, a low-momentum expansion based on chiral symmetry can be constructed.

Naturally, the powers must be bounded from below for the expansion to converge. This is in fact the case, with $\nu \geq 0$.

Furthermore, the power formula equation (25) allows to predict the leading orders of connected multi-nucleon forces. Consider a $m$-nucleon irreducibly connected diagram (multi-nucleon force) in an $A$-nucleon system ($m \leq A$). The number of separately connected pieces is $C = A - m + 1$. Inserting this into equation (25) together with $L = 0$ and $\sum \Delta_i = 0$ yields $\nu = 2m - 4$. Thus, two-nucleon forces ($m = 2$) appear at $\nu = 0$, three-nucleon forces ($m = 3$) at $\nu = 2$ (but they happen to cancel at that order), and four-nucleon forces (4NFs) at $\nu = 4$ (they do not cancel). More about this in the next subsection.

For later purposes, we note that for an irreducible $NN$ diagram ($A = 2, C = 1$), the power formula collapses to the very simple expression

\[ \nu = 2L + \sum_i \Delta_i. \quad (27) \]

To summarize, at each order $\nu$ we only have a well defined number of diagrams, which renders the theory feasible from a practical standpoint. The magnitude of what has been left out at order $\nu$ can be estimated (in a very simple way) from $(Q/\Lambda_{\chi})^{\nu+1}$. The ability to calculate observables (in principle) to any degree of accuracy gives the theory its predictive power.

3.2. The ranking of nuclear forces

As shown in figure 1, nuclear forces appear in ranked orders in accordance with the power counting scheme.

The lowest power is $\nu = 0$, also known as the leading order (LO). At LO we have only two contact contributions with no momentum dependence ($\sim Q^0$). They are signified by the four-nucleon-leg diagram with a small-dot vertex shown in the first row of figure 1. Besides this, we have the static
In spite of its simplicity, this rough description contains some of the main attributes of the $NN$ force. First, through the 1PE it generates the tensor component of the force known to be crucial for the two-nucleon bound state. Second, it predicts correctly $NN$ phase parameters for high partial waves. At LO, the two terms which result from a partial-wave expansion of the contact term impact states of zero orbital angular momentum and produce attraction at short- and intermediate-range.

Notice that no terms with power $n = 1$, as they would violate parity conservation and time-reversal invariance. The next order is then $\nu = 2$, next-to-leading order, or NLO.

The next order is then $\nu = 2$, next-to-leading order, or NLO.

Note that the two-pion exchange (2PE) makes its first appearance at this order, and thus it is referred to as the ‘leading 2PE’. As is well known from decades of nuclear physics, this contribution is essential for a realistic account of the intermediate-range attraction. However, the leading 2PE has insufficient strength, for the following reason: the loops present in the diagrams which involve pions carry the power $\nu = 2$ (see equation (27)), and so only $\pi NN$ and $\pi \pi NN$ vertices with $\Delta_i = 0$ are allowed at this order. These vertices are known to be weak. Moreover, seven new contacts appear at this order which impact $L = 0$ and $L = 1$ states. (As always, two-nucleon contact terms are indicated by four-nucleon-leg diagrams and a vertex of appropriate shape, in this case a solid square.) At this power, the appropriate operators include spin–orbit, central, spin–spin, and tensor terms, namely all the spin and isospin operator structures needed for a realistic description of the 2NF, although the medium-range attraction still lacks sufficient strength.

At the next order, $\nu = 3$ or next-to-next-to-leading order (NNLO), the 2PE contains the so-called $\pi \pi NN$ seagull vertices with two derivatives. These vertices (proportional to the $c_j$ LECs and denoted by a large solid dot in figure 1), bring in

| LO  | 2N Force | 3N Force | 4N Force | 5N Force |
|-----|----------|----------|----------|----------|
| $(Q/\Lambda)^0$ | ![LO 2N Force](image) | ![LO 3N Force](image) | ![LO 4N Force](image) | ![LO 5N Force](image) |
| NLO | ![NLO 2N Force](image) | ![NLO 3N Force](image) | ![NLO 4N Force](image) | ![NLO 5N Force](image) |
| NNLO | ![NNLO 2N Force](image) | ![NNLO 3N Force](image) | ![NNLO 4N Force](image) | ![NNLO 5N Force](image) |
| N^3LO | ![N^3LO 2N Force](image) | ![N^3LO 3N Force](image) | ![N^3LO 4N Force](image) | ![N^3LO 5N Force](image) |
| N^4LO | ![N^4LO 2N Force](image) | ![N^4LO 3N Force](image) | ![N^4LO 4N Force](image) | ![N^4LO 5N Force](image) |
| N^5LO | ![N^5LO 2N Force](image) | ![N^5LO 3N Force](image) | ![N^5LO 4N Force](image) | ![N^5LO 5N Force](image) |

Figure 1. Hierarchy of nuclear forces in ChPT. Solid lines represent nucleons and dashed lines pions. Small dots, large solid dots, solid squares, triangles, diamonds, and stars denote vertices of index $\Delta = 0$, 1, 2, 3, 4, and 6, respectively. Further explanations are given in the text.
correlated 2PE and intermediate \( \Delta(1232) \)-isobar contributions. Consistent with what meson theory of the nuclear force [6, 7] has shown since a long time concerning the importance of these effects, at this order the 2PE finally provides medium-range attraction of realistic strength, bringing the description of the NN force to an almost quantitative level. New new contacts become available at NNLO.

The discussion above reveals how two- and many-nucleon forces are generated and increase in number as we move to higher orders. Three-nucleon forces appear at NLO, but their net contribution vanishes at this order [10]. The first non-zero 3NF contribution is found at NNLO [24, 25]. It is therefore easy to understand why 3NF are very weak as compared to the 2NF which contributes already at \((Q/\Lambda_{\rho})^6\).

For \( \nu = 4 \), or next-to-next-to-next-to-leading order (N\(^5\)LO), we display some representative diagrams in figure 1. There is a large attractive one-loop 2PE contribution (the bubble diagram with two large solid dots \( \sim Q^4 \)), which slightly over-estimates the 2NF attraction at medium range. Two-pion-exchange graphs with two loops are seen at this order, together with three-pion exchange (3PE), which was determined to be very weak at N\(^3\)LO [26, 27]. The most important feature at this order is the presence of 15 additional contacts \( \sim Q^5 \), signified by the four-nucleon-leg diagram in the figure with the diamond-shaped vertex. These contact impacts states with orbital angular momentum up to \( L = 2 \), and are the reason for the quantitative description of the two-nucleon force (up to approximately 300 MeV in terms of laboratory energy) at this order [11, 28]. More 3NF diagrams show up at N\(^3\)LO, as well as the first contributions to 4NFs. We then see that forces involving more and more nucleons appear for the first time at higher and higher orders, which gives theoretical support to the fact that 2NF \( \gg 3NF \gg 4NF \ldots \).

Further, 2PE and 3PE occur at N\(^3\)LO (fifth order). The contribution to the 2NF at this order has been first calculated by Entem et al [29]. It turns out to be moderately repulsive, thus compensating for the attractive surplus generated at N\(^3\)LO by the bubble diagram with two solid dots. The long- and intermediate-range 3NF contributions at this order have been evaluated [22, 30], but not yet applied in nuclear structure calculations. They are expected to be sizeable. Moreover, a new set of 3NF contact terms appears [31]. The N\(^3\)LO 4NF has not been derived yet. Due to the subleading \( \pi\piNN \) seagull vertex (large solid dot \( \sim Q^5 \)), this 4NF could be sizeable.

Finally turning to N\(^4\)LO (sixth order): the dominant 2PE and 3PE contributions to the 2NF have been derived by Entem et al in [32], which represents the most sophisticated investigation ever conducted in chiral EFT for the NN system. The effects are small indicating the desired trend towards convergence of the chiral expansion for the 2NF. Moreover, a new set of 26 NN contact terms \( \sim Q^6 \) occurs that contributes up to F-waves (represented by the NN diagram with a star in figure 1) bringing the total number of NN contacts to 50 [33]. The three-, four-, and five-nucleon forces of this order have not yet been derived.

This section has provided an overview. In the following sections, we will present more details.

4. Pion-exchange contributions to the NN interaction

The various pion-exchange contributions to the NN potential may be analyzed according to the number of pions being exchanged between the two nucleons:

\[
V = V_{1\pi} + V_{2\pi} + V_{3\pi} + V_{4\pi} + \ldots, \tag{28}
\]

where the meaning of the subscripts is obvious and the ellipsis represents \( 5\pi \) and higher pion exchanges. For each of the above terms, we assume a low-momentum expansion:

\[
\begin{align*}
V_{1\pi} &= V_{1\pi}^{(0)} + V_{1\pi}^{(2)} + V_{1\pi}^{(4)} + V_{1\pi}^{(6)} + \ldots, \\
V_{2\pi} &= V_{2\pi}^{(1)} + V_{2\pi}^{(3)} + V_{2\pi}^{(5)} + V_{2\pi}^{(7)} + \ldots, \\
V_{3\pi} &= V_{3\pi}^{(2)} + V_{3\pi}^{(4)} + V_{3\pi}^{(6)} + V_{3\pi}^{(8)} + \ldots, \\
V_{4\pi} &= V_{4\pi}^{(3)} + V_{4\pi}^{(5)} + V_{4\pi}^{(7)} + \ldots. \tag{29}
\end{align*}
\]

where the superscript denotes the order \( \nu \) of the expansion and the ellipses stand for contributions of seventh and higher orders. Due to parity and time-reversal, there are no first order contributions. Moreover, since \( n \) pions create \( L = n - 1 \) loops, the leading order for \( n \)-pion exchange occurs at \( \nu = 2n - 2 \) (see equation (27)).

Order by order, the pion-exchange part of the NN potential builds up as follows:

\[
\begin{align*}
V_{LO} &\equiv V^{(0)}_{1\pi} = V_{1\pi}^{(0)}, \tag{33} \\
V_{NLO} &\equiv V^{(2)}_{1\pi} = V_{1\pi}^{(2)} + V_{1\pi}^{(4)} + V_{1\pi}^{(6)}, \tag{34} \\
V_{NNLO} &\equiv V^{(3)}_{1\pi} = V_{1\pi}^{(3)} + V_{1\pi}^{(5)} + V_{1\pi}^{(7)}, \tag{35} \\
V_{NNNLO} &\equiv V^{(4)}_{1\pi} = V_{1\pi}^{(4)} + V_{1\pi}^{(6)} + V_{1\pi}^{(8)}, \tag{36} \\
V_{NNNLO} &\equiv V^{(5)}_{1\pi} = V_{1\pi}^{(5)} + V_{1\pi}^{(7)} + V_{1\pi}^{(9)}, \tag{37} \\
V_{NNNLO} &\equiv V^{(6)}_{1\pi} = V_{1\pi}^{(6)} + V_{1\pi}^{(8)} + V_{1\pi}^{(10)} + \ldots, \tag{38}
\end{align*}
\]

where LO stands for leading order, NLO for next-to-leading order, etc.

The explicit expressions for the potentials will be stated in terms of contributions to the momentum-space NN amplitudes in the center-of-mass system (CMS), which arise from the following general decomposition:

\[
V(\vec{p}', \vec{p}) = V_C + \tau_1 \cdot \tau_2 W_C + \left[ V_S + \tau_1 \cdot \tau_2 W_S \right] \hat{d}_1 \cdot \hat{d}_2 + \left[ V_L + \tau_1 \cdot \tau_2 W_L \right] \left( -i\sigma \times \left( \vec{q} \times \vec{k} \right) \right) + \left[ V_T + \tau_1 \cdot \tau_2 W_T \right] \hat{d}_1 \cdot \vec{q} \hat{d}_2 \cdot \vec{q} + \left[ V_D + \tau_1 \cdot \tau_2 W_D \right] \times \left( \hat{d}_1 \cdot \vec{q} \times \vec{k} \right) \hat{d}_2 \cdot \left( \vec{q} \times \vec{k} \right), \tag{39}
\]

where \( \vec{p}' \) and \( \vec{p} \) denote the final and initial nucleon momenta in the CMS, respectively. Moreover, \( \vec{q} = \vec{p}' - \vec{p} \) is the momentum transfer, \( \vec{k} = (\vec{p}' + \vec{p})/2 \) the average momentum, and \( \vec{S} = (\hat{d}_1 + \hat{d}_2)/2 \) the total spin, with \( \hat{d}_1, \hat{d}_2, \tau_1, \tau_2 \) the spin and isospin operators of nucleon 1 and 2, respectively.

For on-shell scattering, \( V_C \) and \( W_C(\alpha = C, S, LS, T, \sigma L) \) can be expressed as functions of \( q = |\vec{q}| \) and \( p = |\vec{p}'| = |\vec{p}| \), only.

We will now discuss the contributions order by order.
4.1. Leading order

At leading order, there is only the $1\pi$-exchange contribution, see figure 1. The charge-independent $1\pi$-exchange is given by
\[
V_{1\pi}^{(CI)}(\hat{p'}, \hat{p}) = -\frac{g_A^2}{4f_\pi^2} \tau_1 \cdot \tau_2 \frac{\hat{\sigma}_1 \cdot \hat{q} \hat{\sigma}_2 \cdot \hat{q}}{q^2 + m_\pi^2}.
\] (40)

Higher order corrections to the $1\pi$-exchange are taken care of by mass and coupling constant renormalizations which, in turn, are accounted for by working with the physical values. Note also that, on shell, there are no relativistic corrections. Thus, we apply $1\pi$-exchange in the form equation (40) through all orders.

We use $g_A = 1.290$ (instead of $g_A = 1.276$ [34]) to account for the so-called Goldberger–Treiman discrepancy. Via the Goldberger–Treiman relation, $g_{\pi NN}^2 = g_A^2 M_N f_\pi$, our value for $g_A$ together with $f_\pi = 92.4$ MeV and $M_N = 938.918$ MeV implies $g_{\pi NN}^2/4\pi = 13.67$ which is consistent with the empirical values obtained from $\pi N$ and $NN$ data analysis [35, 36].

For results presented below, we will be specifically calculating neutron–proton (np) scattering and take the charge-dependence of the $1\pi$-exchange into account. Thus, the $1\pi$-exchange potential that we actually apply reads
\[
V_{1\pi}^{(np)}(\hat{p'}, \hat{p}) = -V_{1\pi}(m_\pi) + (-1)^{I+1} 2 V_{1\pi}(m_\pi),
\] (41)
where $I = 0, 1$ denotes the total isospin of the two-nucleon system and
\[
V_{1\pi}(m_\pi) \equiv -\frac{g_A^2}{4f_\pi^2} \frac{\hat{\sigma}_1 \cdot \hat{q} \hat{\sigma}_2 \cdot \hat{q}}{q^2 + m_\pi^2}.
\] (42)

We use $m_\pi = 134.9766$ MeV and $m_p = 139.5702$ MeV. Formally speaking, the charge-dependence of the 1PE exchange is of order NLO [11], but we include it already at leading order to make the comparison with the np phase shifts more meaningful.

4.2. Next-to-leading order (NLO)

The NN diagrams that occur at NLO (see figure 1) contribute in the following way [37]:
\[
W_C = \frac{L(\Lambda; \; q)}{384\pi^2f_\pi^2} \left[ 14m_\pi^2 (1 + 4g_A^2 - 5g_A^4) + q^2 (1 + 10g_A^2 - 23g_A^4) - \frac{48g_A^4 m_\pi^4}{w^2} \right],
\] (43)
\[
V_T = -\frac{1}{q^2} V_S = -\frac{3g_A^4}{64\pi^2f_\pi^2} L(\Lambda; \; q),
\] (44)

where the (regularized) logarithmic loop function is given by:
\[
L(\Lambda; \; q) = \frac{w}{2q} \times \left( \frac{\Lambda^2 + q^2}{2m_\pi^2} \right) - \frac{\Lambda^2 - 4m_\pi^2 q w}{2m_\pi^2(\Lambda^2 + q^2)}
\] (45)

with $w = \sqrt{4m_\pi^2 + q^2}$. $\Lambda$ denotes the cutoff of the spectral-function regularization (SFR) [38]. Note that
\[
\lim_{\Lambda \to \infty} L(\Lambda; \; q) = \frac{w}{2m_\pi} \ln \frac{w + q}{2m_\pi},
\] (46)

is the logarithmic loop function of dimensional regularization.

4.3. Next-to-next-to-leading order (NNLO)

The NNLO contribution (see the 2NF diagrams of the NNLO row in figure 1) is given by [37]:
\[
V_C = \frac{3g_A^4}{16\pi^2f_\pi^2} \left[ 2m_\pi^2 (c_3 - 2c_1) + c_3 q^2 \right] (2m_\pi^2 + q^2) A(\Lambda; \; q),
\] (47)
\[ W_T = -\frac{1}{q^2} W_S = -\frac{g_s^4}{32\pi^2} e_s w^2 A(\tilde{\Lambda}; q). \]  

48

The loop function that appears in the above expressions, regularized by spectral-function cut-off \( \tilde{\Lambda} \), is

\[ A(\tilde{\Lambda}; q) = \frac{1}{2q} \arctan \frac{q(\tilde{\Lambda} - 2m_\pi)}{q^2 + 2\tilde{\Lambda}m_\pi}, \]  

and

\[ \lim_{\tilde{\Lambda} \to \infty} A(\tilde{\Lambda}; q) = \frac{1}{2q} \arctan \frac{q}{2m_\pi} \]  

yields the loop function used in dimensional regularization.

4.4. Next-to-next-to-next-to-leading order (N^3\text{LO})

The number of diagrams involved is now dramatically increasing. Therefore, we will provide additional figures showing the full complexity of the diagrams representing the nuclear forces at higher orders.

The 2PE contributions at N^3\text{LO} are shown in figure 2. They consist of three parts, which we will discuss one by one.

4.4.1. Football diagram at N^3\text{LO}.

The football diagram at N^3\text{LO}, figure 2(a), generates

\[ V_C = \frac{3}{16\pi f_\pi^2} \left[ \left( \frac{c_2}{6} w^2 + c_3(2m_\pi^2 + q^2) \right) 
- 4c_3 m_\pi^2 + \frac{c_2^2}{45} \right] L(\tilde{\Lambda}; q), \]  

\[ W_T = -\frac{1}{q^2} W_S = \frac{c_2^2}{96\pi^2 f_\pi^4} w^2 L(\tilde{\Lambda}; q). \]  

4.4.2. Leading two-loop contributions.

The leading order 2\pi-exchange two-loop diagrams are shown in figure 2(b). In terms of spectral functions, the results are [39]:

\[ \text{Im } V_C(i\mu) = \frac{2\kappa}{3\mu(8\pi f_\pi^2)} \int_0^1 dx [g_s^2(\mu^2 - 2m_\pi^2) \]
\[ + 2(1 - g_s^2) \kappa^2 x^2] \]
\[ \times [ 96\pi^2 f_\pi^2 (2m_\pi^2 - \mu^2)(d_1 + d_2) \]
\[ - 2\kappa^2 x^2 d_3 + 4\mu^2 d_4 \]
\[ + 4m_\pi^2 (1 + 2g_s^2)] \]
\[ \times \frac{\mu}{2m_\pi} \ln \left( \frac{\mu^2}{2m_\pi} \right) \]
\[ \times \frac{\mu^2}{12} (5 + 13g_s^2) \]
\[ + 2m_\pi^2 (1 + 2g_s^2)] \]
\[ \times \frac{\kappa}{m_\pi} \ln \left( \frac{\kappa x + \sqrt{m_\pi^2 + \kappa^2 x^2}}{m_\pi} \right) \]
\[ \times \left[ \frac{5}{6} + \frac{m_\pi^2}{\kappa^2 x^2} - \left( 1 + \frac{m_\pi^2}{\kappa^2 x^2} \right)^{3/2} \right]. \]  

49

\[ \text{Im } V_T(i\mu) = \mu^2 \]  

\[ \text{Im } V_S(i\mu) = \frac{g_s^2 \mu^3}{8\pi f_\pi^4} (d_{15} - d_{14}) \]
\[ + \frac{2g_s^6 \mu^3}{(8\pi f_\pi^2)} \int_0^1 dx (1 - x^2) \]
\[ \times \left[ \frac{1}{6} - \frac{m_\pi^2}{\kappa^2 x^2} + \left( 1 + \frac{m_\pi^2}{\kappa^2 x^2} \right)^{3/2} \right] \]
\[ \times \ln \left( \frac{\kappa x + \sqrt{m_\pi^2 + \kappa^2 x^2}}{m_\pi} \right). \]  

50

Figure 3. N^3\text{LO} three-pion exchange contributions. Notation as in figure 1. (Figure reproduced with permission from [11]. Copyright Elsevier 2011)
Two-pion-exchange contributions that occur at N4LO are displayed proportional to corrections of NNLO diagrams. Notation as in \( \text{Figures 1 and 2.} \) The leading one-loop \( \pi N \) amplitude is folded with the chiral \( \pi N N \) vertices proportional to \( c_s \). The one-loop \( \pi N \) amplitude proportional to \( c_t \) is folded with the leading order chiral \( \pi N \) amplitude. (c) Relativistic corrections of NNLO diagrams. Notation as in \( \text{Figures 1 and 2.} \)

\[
\text{Im } W_5(i\mu) = \mu^2 \text{ Im } W_T(i\mu) = \frac{g^4_A}{\pi^4} \frac{(4m^2_\pi - \mu^2)}{\mu^2} \times \left[ \frac{m^2_\pi - \mu^2}{4} \right] \ln \frac{\mu + 2m_\pi}{\mu - 2m_\pi} + (1 + 2g^2_A) \mu m_\pi, \tag{56}
\]

where \( \kappa = \sqrt{\mu^2/4 - m^2_\pi} \).

The momentum space amplitudes \( V_a(q) \) and \( W_a(q) \) are obtained from the above expressions by means of subtracted dispersion integrals:

\[
V_{C.S}(q) = -\frac{2g^{m+3}}{\pi^2} \int_{m_\pi}^{\Lambda} \text{d}\mu \, \frac{\text{Im } V_{C.S}(i\mu)}{\mu^{m+3} + q^2}, \tag{57}
\]

\[
V_T(q) = -\frac{2g^{m+1}}{\pi^2} \int_{m_\pi}^{\Lambda} \text{d}\mu \, \frac{\text{Im } V_T(i\mu)}{\mu^{m+2} + q^2}, \tag{58}
\]

and similarly for \( W_{C,S,T} \). We use \( m = 3 \) for the dispersion integrals that contribute at N3LO and N4LO, and \( m = 5 \) at N5LO. Moreover, \( n = 2 \) is applied for 2PE and \( n = 3 \) for 3PE. For \( \Lambda \to \infty \) the above dispersion integrals yield the results of dimensional regularization, while for finite \( \Lambda \geq m_\pi \) we have SFR [38]. The purpose of the finite scale \( \Lambda \) is to constrain the imaginary parts to the low-momentum region where chiral EFT is applicable.

4.4.4. Leading relativistic corrections. Counting \( Q/M_\pi \sim Q^2/\Lambda^2 \), the relativistic corrections of the NLO diagrams, which are shown in \( \text{Figures 1 and 2.} \) They are of order N3LO and are given by [11]:

\[
V_C = \frac{3g^4_A}{128\pi^2 f^2_\pi M_N} \left[ \frac{m^5_\pi}{2w^2} + (2m^2_\pi + q^2)(q^2 - m^2_\pi)A(\tilde{\Lambda}; q) \right], \tag{59}
\]

\[
W_C = \frac{g^2_A}{64\pi^2 f^2_\pi M_N} \left[ \frac{3g^4_A m_\pi^5}{2w^2} \right. &
\left. + [g^2_A (3m^2_\pi + q^2) - 2m^2_\pi - q^2] \times (2m^2_\pi + q^2)A(\tilde{\Lambda}; q), \right. \tag{60}
\]

\[
V_T = \frac{1}{q^2} V_C = \frac{3g^4_A}{256\pi^2 f^2_\pi M_N} (5m^2_\pi + 2q^2)A(\tilde{\Lambda}; q), \tag{61}
\]

\[
W_T = -\frac{1}{q^2} W_C = \frac{g^2_A}{128\pi^2 f^2_\pi M_N} \left[ g^2_A (3m^2_\pi + q^2) - w^2 \right] A(\tilde{\Lambda}; q), \tag{62}
\]

\[
V_{LS} = \frac{3g^4_A}{32\pi^2 f^2_\pi M_N} (2m^2_\pi + q^2)A(\tilde{\Lambda}; q), \tag{63}
\]

\[
W_{LS} = \frac{g^2_A (1 - g^2_A)}{32\pi^2 f^2_\pi M_N} w^2 A(\tilde{\Lambda}; q). \tag{64}
\]

4.4.4. Leading 3PE contributions. The leading 3\( \pi \)-exchange contributions that occur at N3LO are shown in \( \text{Figures 3.} \) They have been calculated in \([26, 27]\) and are found to be negligible. Therefore, we omit them.

4.5. Next-to-next-to-next-to-leading order (N4LO)

At this order, we have two- and 3PE contributions, which we will now discuss one by one.

4.5.1. Two-pion exchange contributions at N4LO. The 2\( \pi \)-exchange contributions that occur at N4LO are displayed

\[
\text{Figure 4. } N^3\text{LO two-pion-exchange contributions. (a) The leading one-loop } \pi N \text{ amplitude is folded with the chiral } \pi N N \text{ vertices proportional to } c_s. \text{ (b) The one-loop } \pi N \text{ amplitude proportional to } c_t \text{ is folded with the leading order chiral } \pi N \text{ amplitude. (c) Relativistic corrections of NNLO diagrams. Notation as in Figures 1 and 2.}
\]
graphically in figure 4. We can distinguish between three groups of diagrams.

First, there are the N^4LO 2π-exchange two-loop contributions of class (a), figure 4(a). For this class the spectral functions are obtained by integrating the product of the leading one-loop πN amplitude and the chiral ππNN vertex proportional to c_i over the Lorentz-invariant 2π-phase space.

Second, we have the N^4LO 2π-exchange two-loop contributions of class (b), figure 4(b). Here, the product of the one-loop πN amplitude proportional to c_i (see [22] for details) and the leading order chiral πN amplitude is integrated over the 2π-phase space.

The analytic expressions for the spectral functions of class (a) and (b) are very involved, which is why we do not reprint them here. The interested reader is referred to [29].

Finally, there also some relativistic corrections. This group consists of diagrams with one vertex proportional to c_i and one 1/M_N correction. A few representative graphs are shown in figure 4(c). Since in this investigation we count Q/M_N ~ (Q/A)^2, these relativistic corrections are formally of order N^3LO. The result for this group of diagrams is [39]:

\[
V_C = \frac{g^2_\alpha}{32\pi^2M_N f_\pi^2} [(6c_3 - c_2)q^4 + 4(3c_3 - c_2 - 6c_1)q^2m^2_\pi + 6(2c_3 - c_2)m^4_\pi - 24(2c_1 + c_3)m^2_\pi w^2],
\]

\[
W_C = -\frac{c_4}{48\pi^2M_N f_\pi^4} \left[ g^2_\alpha (8m^2_\pi + 5q^2) + w^2 \right]q^2 L(\bar{\Lambda}; q).
\]

4.5.2. 3PE contributions at N^4LO. The 3π-exchange of order N^4LO is shown in figure 5. The spectral functions for these diagrams have been calculated in [40]. We use here the classification scheme introduced in that reference and note that class XI vanishes. Moreover, we find that the class X and part of class XIV make only negligible contributions. Thus, we include in our calculations only class XII and XIII, and the V_S contribution of class XIV. For the very involved expressions, we refer the interested reader to [29].

4.6. Next-to-next-to-next-to-next-to-leading order (N^5LO)

At N^5LO, we are faced with two-, three-, and four-pion exchange contributions.

4.6.1. 2PE contributions at N^5LO. The 2π-exchange contributions that occur at N^5LO are displayed graphically in figure 6. We will now discuss each class separately.

The N^5LO 2π-exchange two-loop contributions, denoted by class (a), are shown in figure 6(a). For this class the spectral functions are obtained by integrating the product of the subleading one-loop πN-amplitude (see [22] for details).
and the chiral $\pi NN$-vertex proportional to $c_i$ over the Lorentz-invariant $2\pi$-phase space [32].

A first set of $2\pi$-exchange contributions at three-loop order, denoted by class (b), is displayed in figure 6(b). Here, the leading one-loop $\pi N$-scattering amplitude is multiplied with itself and integrated over the $2\pi$-phase space [32].

Further $2\pi$-exchange three-loop contributions at $N^3$LO, denoted by class (c), are shown in figure 6(c). For these, the two-loop $\pi N$-scattering amplitude (which is of order five) would have to be folded with the tree-level $\pi N$-amplitude. To our knowledge, the two-loop elastic $\pi N$-scattering amplitude has never been evaluated in some decent analytical form. Note that the loops involved in the class (c) contributions include only leading order chiral $\pi N$-vertices. According to our experience such contributions are typically small. For these reasons, class (c) is neglected.

Besides the above, there are also some relativistic $1/M_\pi^2$-corrections. This group consists of the $1/M_\pi^2$-corrections to the leading chiral $2\pi$-exchange diagrams. Since we count $Q/M_\pi \sim (Q/A)^2$, these relativistic corrections are formally of sixth order ($N^3$LO). The expressions for the corresponding $NN$-amplitudes can be found in [41].

4.6.2. 3PE contributions at $N^3$LO. The $3\pi$-exchange contributions of order $N^3$LO are shown in figure 7. We can distinguish between two classes.

Class (a) consists of the diagrams displayed in figure 7(a). They are characterized by the presence of one subleading $\pi\pi NN$-vertex in each nucleon line. Using a notation introduced in [29, 40], we distinguish between the various sub-classes of diagrams by roman numerals.

Class (b) is shown in figure 7(b). Each $3\pi$-exchange diagram of this class includes the one-loop $\pi N$-amplitude (completed by the LECs $d_j$). Only those parts of the $\pi N$-scattering amplitude, which are either independent of the pion CMS-energy or depend on it linearly could be treated with the techniques available. The contributions are, in general, small. Results presented below include only the larger portions within this class. The omitted pieces are about one order of magnitude smaller. To facilitate a better understanding, we have subdivided this class into sub-classes labeled by roman numerals, following [29, 40].

The very involved analytic expressions for the spectral function can be found in [32].

4.6.3. Four-pion exchange at $N^3$LO. The exchange of four pions between two nucleons occurs for the first time at $N^3$LO. The pertinent diagrams involve three loops and only leading order vertices, which explains the sixth power in small momenta. 3PE with just leading order vertices turned out to be negligibly small [26, 27], and so we expect four-pion exchange with leading order vertices to be even smaller. Therefore, we can safely neglect this contribution.

5. Perturbative $NN$ scattering in peripheral partial waves

We will now discuss $NN$ scattering involving states of high orbital angular momentum. We recall that peripheral scattering is the best tool to explore the $NN$ force beyond short distances. Due to the high angular momentum ‘barrier’, the contribution from short-range terms is marginal. In fact, since the contact terms at $N^3$LO do not contribute for $L \geq 3$, there exists the unique opportunity to study the $NN$ force when it is controlled entirely by pion exchanges, which carry the signature of chiral symmetry. In short, states with $L \geq 3$ are a suitable ground to test the predictive power of chiral EFT. The LECs can be taken from $\pi N$ analysis, leaving no free parameters. Furthermore, the scattering phases in high angular momentum states are small, suggesting that a perturbative treatment would be appropriate. On the other hand, the latter is not suitable for the central partial waves, which require a non-perturbative approach to the solution of the Lippmann–Schwinger (LS) equation, with all its model (cutoff) dependence.

The perturbative $K$-matrix for $np$ scattering is calculated as follows:

$$K(\vec{p}, \vec{p}) = V_{12}^{(np)}(\vec{p}, \vec{p}) + V_{22}^{(np)}(\vec{p}, \vec{p})$$

with $V_{12}^{(np)}(\vec{p}, \vec{p})$ as in equation (41), and $V_{22}^{(np)}(\vec{p}, \vec{p})$ representing the once iterated 1PE given by

$$V_{22}^{(np)}(\vec{p}, \vec{p}) = \mathcal{P} \int \frac{d^3p'}{(2\pi^3)} \frac{1}{E_{p'}} |V_{12}^{(np)}(\vec{p}, \vec{p}) V_{22}^{(np)}(\vec{p}, \vec{p})|,$$

where $\mathcal{P}$ denotes the principal value integral and $E_{p'} = \sqrt{M_0^2 + p'^2}$. A calculation at LO includes only the first term on the right-hand side of equation (70), $V_{12}^{(np)}(\vec{p}, \vec{p})$, while calculations at NLO or higher order also include the second term on the right-hand side, $V_{22}^{(np)}(\vec{p}, \vec{p})$. At NNLO, the twice iterated 1PE should be included as well; and at higher orders further iterations should be accounted for. However, we found that the difference between the once iterated 1PE and the infinitely iterated 1PE is so small that it could not be identified on the scale of our phase shift figures. For that reason, we omit iterations of 1PE beyond what is contained in $V_{22}^{(np)}(\vec{p}, \vec{p})$. Furthermore, $V_{12}^{(np)}(\vec{p}, \vec{p})$ stands for terms where irreducible 2PE is iterated with 1PE.

Finally, the fourth term on the rhs of equation (70), $V_{22}^{(np)}(\vec{p}, \vec{p})$, stands for the irreducible multi-pion exchange contributions that occur at the order at which the calculation is conducted. In multi-pion exchanges, we use the average pion mass $m_\pi = 138.039$ MeV and, thus, neglect the charge-dependence due to pion-mass splitting in irreducible multi-pion diagrams.
We start with the individual N^4LO (fifth-order) contributions. For this purpose, we display in figure 8 phase shifts for six important peripheral partial waves, namely, $F_3$, $F_2$, $F_2$, $F_3$, $F_3$, $F_3$, $F_3$, $F_3$, $F_3$, $F_3$, and $G_5$. In each frame, the following curves are shown:

1. N^3LO.
2. The previous curve plus the $c_i/M_N$ corrections (denoted by ‘c/M’), figure 4(c).
3. The previous curve plus the N^4LO 2π-exchange two-loop contributions of class (a), figure 4(a).
4. The previous curve plus the N^4LO 2π two-loop contributions of class (b), figure 4(b).
5. The previous curve plus the N^4LO 3π-exchange contributions, figure 5.

In summary, the various curves add up successively the individual N^4LO contributions in the order indicated in the curve labels. The last curve in this series, curve (5), is the full N^4LO result. In these calculations, a SFR cutoff $\hat{\Lambda} = 1.5$ GeV is applied (see equations (57) and (58)) and the KH LECs (see table 2) are used.

From figure 8, we make the following observations. In triplet $F$-waves, the $c_i/M_N$ corrections as well as the 2PE two-loops, class (a) and (b), are all repulsive and of about the same strength. As a consequence, the problem of the excessive attraction, that N^3LO is beset with, is overcome. A similar trend is seen in $G_5$. An exception is $F_3$, where the class (b) contribution is attractive leading to phase shifts above the data for energies higher than 150 MeV.

Now turning to the N^4LO 3PE contributions (curve (5) in figure 8): they are substantially smaller than the 2PE two-loop ones, in all peripheral partial waves. This can be interpreted as an indication of convergence with regard to the number of pions being exchanged between two nucleons—a trend that is very welcome. Further, note that the total 3PE contribution is a very comprehensive one, see figure 5. It is the sum of ten terms which, individually, can be fairly large. However, destructive interference between them leads to the small net result.

For all $F$ and $G$ waves (except $F_3$), the final N^4LO result is close to the empirical phase shifts. Notice that this includes also $G_5$, which posed persistent problems at N^4LO [49].

It is also of interest to know how predictions change with variations of $\hat{\Lambda}$ within a reasonable range. We have, therefore, varied $\hat{\Lambda}$ between 0.7 and 1.5 GeV and show the predictions for all $F$ and $G$ waves in figures 9 and 10, respectively, in terms of colored bands. It is seen that, at N^3LO, the variations of the predictions are very large and always too attractive while, at N^4LO, the variations are small and the predictions are close to the data or right on the data. Figures 9 and 10 also include the lower orders (LO, NLO, and NNLO) such that a comparison of the relative size of the order-by-order contributions is possible. We observe that there is not much of a convergence, since obviously the magnitudes of the NNLO, N^3LO, and N^4LO contributions are about the same.

To obtain more insight into the convergence issue, we need to proceed to the next order, which is N^5LO. As shown in figures 6 and 7, the sixth-order corrections consist of

### Table 2. Low-energy constants as determined in [22]. The sets ‘GW’ and ‘KH’ are based upon relativistic kinematics, the CMS on-shell masses, respectively. The $c_i$ appear in equation (18) and are in units of GeV^{-1}. The $d_i$ and $\hat{c}_i$ belong to $L_{2N}$ and $L_{3N}$ (see equations (19) and (20)) and are in units of GeV^{-2} and GeV^{-3}, respectively.

|        | GW | KH |
|--------|----|----|
| $c_1$  | -1.13 | -0.75 |
| $c_2$  | 3.69 | 3.49 |
| $c_3$  | -5.51 | -4.77 |
| $c_4$  | 3.71 | 3.34 |
| $\delta_1 + \delta_2$ | 5.57 | 6.21 |
| $\delta_3$ | -5.35 | -6.83 |
| $d_5$ | 0.02 | 0.78 |
| $d_6 - d_{15}$ | -10.26 | -12.02 |
| $\hat{c}_4$ | 1.75 | 1.52 |
| $\hat{c}_6$ | 0.88 | -10.41 |
| $\hat{c}_8$ | -1.74 | -6.08 |
| $\hat{c}_{10}$ | 0.58 | 0.37 |
| $\hat{c}_{12}$ | 0.96 | 3.26 |

Throughout this paper, we use

$$M_N = \frac{2M_p M_n}{M_p + M_n} = 938.9183 \text{ MeV}. \quad (72)$$

Based upon relativistic kinematics, the CMS on-shell momentum $p$ is related to the kinetic energy of the incident neutron in the laboratory system (‘Lab. Energy’), $T_{lab}$, by

$$p^2 = \frac{M_p^2 T_{lab} (T_{lab} + 2M_p)}{(M_p + M_n)^2 + 2T_{lab}M_p}, \quad (73)$$

with $M_p = 938.2720$ MeV and $M_n = 939.5654$ MeV the proton and neutron masses, respectively.

The $K$-matrix, equation (70), is decomposed into partial waves following [42] and phase shifts are then calculated via

$$\tan \delta_i (T_{lab}) = -\frac{M_p^2 p}{16\pi^2 E_p} K_i (p, p). \quad (74)$$

For more details concerning the evaluation of phase shifts, including the case of coupled partial waves, see [43] or the appendix of [44].

Chiral symmetry establishes a link between the dynamics in the $\pi N$-system and the $NN$-system (through common LECs). In order to check the consistency, we use the LECs for subleading $\pi N$-couplings as determined in analyses of low-energy elastic $\pi N$-scattering. Appropriate analyses for our purposes are contained in [22], where $\pi N$-scattering has been calculated at fourth order using the same power-counting of relativistic $1/M_N$-corrections as in the present work. Reference [22] performed two fits, one to the GW [45] and one to the KH [46] partial wave analysis resulting in the two sets of LECs listed in table 2.

The contributions up to N^3LO and their impact on peripheral $NN$ scattering have been discussed and demonstrated in detail in [11] and, therefore, we will not repeat that demonstration here. But we will discuss the recent progress that has been made in the calculation of orders beyond N^3LO.
several contributions. As in the case of $N^4$LO, we will first show how the individual $N^5$LO contributions impact $NN$-phase-shifts in peripheral waves. In figure 11, we display phase-shifts for two peripheral partial waves, namely, $1^1G_4$, and $3^3G_5$. The following curves are shown:

(1) $N^4$LO.
(2) The previous curve plus the $N^4$LO $2\pi$-exchange contributions of class (a), figure 6(a).
(3) The previous curve plus the $N^4$LO $2\pi$-exchange contributions of class (b), figure 6(b).

(4) The previous curve plus the $N^4$LO $3\pi$-exchange contributions of class (a), figure 7(a).
(5) The previous curve plus the $N^4$LO $3\pi$-exchange contributions of class (b), figure 7(b).
(6) The previous curve plus the $1/M^2_N$-corrections (denoted by ‘$1/M^2_N’$’ [41].

The last curve in this series, curve (6), includes all $N^5$LO contributions calculated in [32]. For all curves of this figure, a SFR cutoff $\hat{\Lambda} = 800$ MeV (see equations (57) and (58)) is employed and the GW (see table 2) LECs are used.
Figure 9. Phase-shifts of neutron–proton scattering at various orders up to N^4LO. The colored bands show the variation of the predictions when the SFR cutoff $\tilde{\Lambda}$ is changed over the range 0.7–1.5 GeV. The KH LECs are applied. Empirical phase shifts as in figure 8. (Figure reproduced with permission from [29]. Copyright the American Physical Society 2015.)
From figure 11, we see that the two-loop $2\pi$-exchange class (a), figure 6(a), generates a strong repulsive central force, while the spin–spin and tensor forces provided by this class are negligible. The fact that this class produces a relatively large contribution is not unexpected, since it is proportional to $c^2$. The $2\pi$-exchange contribution class (b), figure 6(b), creates a moderately repulsive central force as seen by its effect on $^3G_4$ and a noticeable tensor force as the

Figure 10. Same as figure 9, but for $G$-waves. (Figure reproduced with permission from [29]. Copyright the American Physical Society 2015.)
impact on $^3G_5$ demonstrates. The $3\pi$-exchange class (a), figure 7(a), is negligible in $^1G_4$, but noticeable in $^3G_5$ and, therefore, it should not be neglected. This contribution is proportional to $c_7^2$, which suggests a non-negligible size but it is typically smaller than the corresponding $2\pi$-exchange contribution class (a). The $3\pi$-exchange class (b) contribution, figure 7(b), turns out to be negligible (see the difference between curve (4) and (5) in figure 11). This may not be
unexpected since it is a three-loop contribution with only leading-order vertices. Finally the relativistic $1/M^2$-corrections to the leading $2\pi$-exchange [41] have a small but non-negligible impact, particularly in $^3G_5$.

The $^5NLO$ predictions for all $G$ and $H$ waves are displayed in figure 12 in terms of colored bands that are generated by varying the SFR cutoff $\tilde{\Lambda}$ (see equations (57) and (58)) between 700 and 900 MeV. The figure clearly reveals again that, at $^3NLO$, the predictions are, in general, too attractive. As discussed, the $^4NLO$ contribution, essentially, compensates this attractive surplus. $^5NLO$ then adds additional repulsion bringing the final prediction right onto the data (i.e. empirical phase-shifts). Moreover, the $^6NLO$ contribution is, in general, substantially smaller than the one at $^5NLO$, thus, showing a signature of convergence of the chiral expansion.

To summarize, we present in figure 13 a comparison between all orders from LO to $^5NLO$. Note that the difference between the LO prediction (1PE, dotted line) and the data (filled and open circles) is to be provided by 2PEs and 3PEs, i.e. the intermediate-range part of the nuclear force. How well that is accomplished is a crucial test for any theory of nuclear forces. NLO produces only a small contribution, but $^2NLO$ creates substantial intermediate-range attraction (most clearly seen in $^1G_4$, $^3G_5$, and $^3H_6$). In fact, $^3NLO$ is the largest contribution among all orders. This is due to the one-loop $2\pi$-exchange triangle diagram which involves one $\pi\piNN$-contact vertex proportional to $c_1$. This vertex represents correlated 2PE as well as intermediate $\Delta(1232)$-isobar excitation. It is well-known from the traditional meson theory of nuclear forces [6, 7] that these two features are crucial for a realistic and quantitative 2PE model. Consequently, the one-loop $2\pi$-exchange at $^2NLO$ is attractive and assumes a realistic size describing the intermediate-range attraction of the nuclear force almost correctly. At $^3NLO$, more one-loop 2PE is added by the bubble diagram with two $c_1$-vertices, a contribution that seems to overestimate the attraction. This attractive surplus is then compensated by the prevailing repulsive two-loop $2\pi$- and $3\pi$-exchanges that occur at $^5NLO$ and $^6NLO$.

In this context, it is worth noting that also in conventional meson theory [7] the one-loop models for the 2PE contribution always show some excess of attraction (see figures 7–9 of [49]). The same is true for the dispersion theoretic approach pursued by the Paris group [6]. In conventional meson theory, the surplus attraction is reduced by heavy-meson exchange ($\rho$- and $\omega$-exchange) which, however, has no place in chiral EFT (as a finite-range contribution). Instead, in the latter approach, two-loop $2\pi$- and $3\pi$-exchanges provide the corrective action.

6. Constructing complete chiral $NN$ potentials

Previously, we addressed the long- and medium-range parts of the nuclear interaction, which involve pion-exchange contributions. Because of their long-range nature, these terms control partial waves with high values of $L$ and are governed by chiral symmetry. Of course, to obtain quantitative predictions of low-energy $NN$ scattering observables or nuclear properties, all partial waves must be described realistically, in particular the most central ones ($L \lesssim 2$), which carry information on the dynamics at short range. The latter will be our next concern.
6.1. NN contact terms

It has been known for a long time that the bulk of the short-distance behavior of the nuclear force can be explained with the introduction of heavy bosons, e.g. the $\omega$(782). Applying Fourier transformation to the propagator of the meson

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{-i\vec{q}\cdot\vec{r}}}{m^2 + q^2} = \frac{1}{4\pi} \frac{e^{-m_{\omega}r}}{r},$$ (75)

provides a qualitative description of the NN force at short range.

Since ChPT is an expansion valid for small values of the momentum, mesons such as the $\omega$(782) are outside its resolution power (notice that $\Lambda_\chi \approx m_{\omega,\omega}$). However, the propagator of the heavy boson under consideration can be handled with an expansion

$$\frac{1}{m^2 + Q^2} \approx \frac{1}{m^2} \left(1 - \frac{Q^2}{m^2} + \frac{Q^4}{m^4} - \ldots\right).$$ (76)

One may then approach the short-range NN interaction guided by the expansion above, namely as a power series in $Q/m_{\omega}$. This is the origin of the contributions referred to as contact terms.

Contact terms play an important role in renormalization. Contributions involving the exchange of more than one pion entail loop integrals, which produce polynomial terms whose coefficients can be divergent or scale dependent (see appendix B of [11]). Contact terms are then crucial to remove those divergences or scale dependencies and so they act as ‘counter terms’.

Our procedure will involve partial-wave expansion of terms polynomial in $Q$, where $Q$ stands for the momentum transfer between the two nucleons, $q$, or their average momentum $k$ (see below equation (39) for their definitions). In any case, for even $\nu$

$$Q' = f_\nu(\cos \theta),$$ (77)

where $f_\nu$ stands for a polynomial of degree $m$ and $\theta$ is the scattering angle in the CMS. When expanding $Q'$ in partial waves, we encounter the integral

$$I_{L}^{(\nu)} = \int_{-1}^{1} Q' P_L(\cos \theta) d\cos \theta = \int_{-1}^{1} f_\nu(\cos \theta) P_L(\cos \theta) d\cos \theta,$$ (78)

where $L$ is the orbital angular momentum and $P_L$ is a Legendre polynomial. Since Legendre polynomials are orthogonal

$$I_{L}^{(\nu)} = 0 \quad \text{for} \quad L > \frac{\nu}{2},$$ (79)

Therefore, we can see that contact terms of a particular order can only contribute up to some partial wave.

Parity conservation forbids the presence of odd powers of $Q$. Therefore, the contact interaction can be formally written as

$$V_\alpha = V_\alpha^{(0)} + V_\alpha^{(2)} + V_\alpha^{(4)} + V_\alpha^{(6)} + \ldots,$$ (80)

with the superscript indicating a given order.

Next, we display the contact NN potentials at each order as they emerge from the Lagrangians shown at the end of section 2.2.

6.1.1. Zeroth order (LO). From the Lagrangian $\mathcal{L}_{NN}^{(0)}$, equation (23) (which is part of $\mathcal{L}_{\Delta=0}$, equation (17)), we can generate the contact potential

$$V_\alpha^{(0)}(\vec{p}', \vec{p}) = C_5 + C_7 \vec{\sigma}_1 \cdot \vec{\sigma}_2,$$ (81)

whose partial-wave contributions are

$$V_\alpha^{(0)}(S_0) = \tilde{C}_5 S_0 = 4\pi (C_5 - 3 C_7)$$ (82a)

$$V_\alpha^{(0)}(S_1) = \tilde{C}_7 S_1 = 4\pi (C_5 + C_7).$$ (82b)

6.1.2. Second order (NLO). For this, we refer to $\mathcal{L}_{NN}^{(2)}$, equation (24) (part of $\mathcal{L}_{\Delta=2}$, equation (19)). We now have

$$V_\alpha^{(2)}(p', p) = C_1 q^2 + C_2 k^2 + (C_4 q^2 + C_6 k^2) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + C_8 (-\vec{q} \cdot \vec{k}) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + C_6 (\vec{\sigma}_1 \cdot \vec{q}) (\vec{\sigma}_2 \cdot \vec{q}) + C_7 (\vec{\sigma}_1 \cdot \vec{k}) (\vec{\sigma}_2 \cdot \vec{k}).$$ (83)

Notice that the constants $C_i$ which appear in these expressions are related to the coefficients $C_i'$ present in the Lagrangian $\mathcal{L}_{NN}^{(2)}$, equation (24), see [23, 51] for details (not relevant for us at this point).

One way to partial-wave decompose the potential above is the method proposed by Erkelenz, Alzetta, and Holinde [42]. One obtains

$$V_\alpha^{(2)}(S_0) = C_{S_0} (p^2 + p'^2)$$ (84a)

$$V_\alpha^{(2)}(P_0) = C_{P_0} pp'$$ (84b)

$$V_\alpha^{(2)}(P_1) = C_{P_1} pp'$$ (84c)

$$V_\alpha^{(2)}(P_2) = C_{P_2} pp'$$ (84d)

$$V_\alpha^{(2)}(S_1) = C_{S_1} (p^2 + p'^2)$$ (84e)

$$V_\alpha^{(2)}(S_2) = C_{S_2} - D_1 p^2$$ (84f)

$$V_\alpha^{(2)}(D_2) = C_{S_2} - D_1 p^2$$ (84g)

$$V_\alpha^{(2)}(P_2) = C_{P_2} pp'$$ (84h)

with

$$C_{S_0} = 4\pi \left(C_1 + \frac{1}{4} C_2 - 3 C_3 - \frac{3}{4} C_4 - C_6 - \frac{1}{4} C_7\right)$$ (85a)

$$C_{P_0} = 4\pi \left(-\frac{2}{3} C_1 + \frac{1}{6} C_2 - \frac{2}{3} C_3 + \frac{1}{6} C_4 - \frac{2}{3} C_6 - \frac{1}{2} C_7\right)$$ (85b)
\[ C_{V_1} = 4\pi \left( -\frac{2}{3} C_1 + \frac{1}{6} C_2 + 2 C_3 - \frac{1}{2} C_4 + \frac{2}{3} C_5 - \frac{1}{6} C_6 \right) \]  
(85c)

\[ C_{V_2} = 4\pi \left( -\frac{2}{3} C_1 + \frac{1}{6} C_2 - \frac{2}{3} C_3 + \frac{1}{6} C_4 - \frac{4}{3} C_5 + \frac{1}{3} C_6 \right) \]  
(85d)

\[ C_{S_1} = 4\pi \left( C_1 + \frac{1}{4} C_2 + C_3 + \frac{1}{4} C_4 + \frac{1}{3} C_6 + \frac{1}{12} C_7 \right) \]  
(85e)

\[ C_{S_1^{\prime}} = 4\pi \left( \frac{2\sqrt{2}}{3} C_6 - \frac{\sqrt{2}}{6} C_7 \right) \]  
(85f)

\[ C_{V_2} = 4\pi \left( -\frac{2}{3} C_1 + \frac{1}{6} C_2 - \frac{2}{3} C_3 + \frac{1}{6} C_4 + \frac{1}{3} C_5 \right) \]  
(85g)

6.1.3. Fourth order (\(N^2\)LO). The contact potential of order four reads

\[ V_{\text{LO}}^{(4)}(\vec{p}', \vec{p}) = D_1 \, q^4 + D_2 \, k^4 + D_3 \, q^2 \, k^2 + D_4 \, (\vec{q} \times \vec{k})^2 + \frac{1}{2} D_5 \, q^4 + D_6 \, k^4 + D_7 \, q^2 \, k^2 + D_8 \, (\vec{q} \times \vec{k})^2 \delta_1 \cdot \delta_2 + \frac{1}{2} D_9 \, q^4 + D_{10} \, k^2 \cdot (\vec{q} \times \vec{k}) + (D_{11} \, q^2 + D_{12} \, k^2) (\delta_1 \cdot \vec{q}) (\delta_2 \cdot \vec{k}) + (D_{13} \, q^2 + D_{14} \, k^2) (\delta_2 \cdot \vec{k}) (\delta_1 \cdot \vec{q}) \]  
(86)

The corresponding partial-wave expressions at this order can be found in Appendix E of [11].

6.1.4. Sixth order (\(N^3\)LO). At sixth order, 26 new contact terms appear, bringing the total number to 50. These terms as well as their partial-wave decomposition have been worked out in [33]. So far, these terms have not been used in the construction of NN potentials.

6.2. Definition of NN potential

At this point, we have all the ‘ingredients’ required to describe the well-known phenomenology of the nuclear force at long, medium, and short distances. When approaching the most central waves, though, we are faced with one more hurdle. As is known from the most elementary nuclear physics, the NN system at \(L = 0\) admits a bound state, the weakly bound deuteron, and large scattering lengths, which do not allow for a perturbative treatment. Moreover, unlike what happens with \(\pi - \pi\) and \(\pi - N\) in the chiral limit, the interaction of nucleons does not vanish when \(Q \to 0\). As argued by Weinberg [9], intermediate states with only nucleons are responsible for the large increase of the scattering amplitude commonly referred to as ‘infrared enhancement’. A way to circumvent this problem, as suggested by Weinberg, is to calculate the NN potential perturbatively and then to apply it in a scattering equation to obtain the NN amplitude. This is the strategy we will adopt.

The pion-exchange parts of the NN potential were spelled out in equations (33)–(38). To obtain the complete potential, one just has to add to this the contact terms listed in equation (80). Thus, one has to do the following extensions to some of the equations (33)–(38):

\[ V_{\text{LO}} \mapsto V_{\text{LO}} + V_{\text{ct}}^{(0)}, \]  
(87)

\[ V_{\text{NLO}} \mapsto V_{\text{NLO}} + V_{\text{ct}}^{(2)}, \]  
(88)

\[ V_{\text{NNLO}} \mapsto V_{\text{NNLO}} + V_{\text{ct}}^{(4)}, \]  
(89)

\[ V_{\text{NNNLO}} \mapsto V_{\text{NNNLO}} + V_{\text{ct}}^{(6)}, \]  
(90)

and no changes to \(V_{\text{NNLO}}\) and \(V_{\text{NNNLO}}\).

The potential \(V\) as derived in previous sections is, in principal, an invariant amplitude and, thus, satisfies a relativistic scattering equation, for which we choose the BhS equation [11], which reads explicitly

\[ T(p', \vec{p}) = V(p', \vec{p}) + \int \frac{d^3p''}{(2\pi)^3} \, V(p', \vec{p}'') \times \frac{M_N}{E_{p'}} \frac{1}{p^2 - p''^2 + i\epsilon} \, T(p'', \vec{p}) \]  
(91)

with \(E_{p''} \equiv \sqrt{M_N^2 + p''^2}\). The use of a relativistic equation implies that relativistic corrections are already included to all orders (no additional corrections are needed when increasing the EFT order).

If we define

\[ \overline{V}(p', \vec{p}) \equiv \frac{1}{(2\pi)^3} \, \sqrt{\frac{M_N}{E_{p'}}} \, V(p', \vec{p}) \, \sqrt{\frac{M_N}{E_p}}, \]  
(92)

and

\[ \overline{T}(p', \vec{p}) \equiv \frac{1}{(2\pi)^3} \, \sqrt{\frac{M_N}{E_{p'}}} \, T(p', \vec{p}) \, \sqrt{\frac{M_N}{E_p}}, \]  
(93)

where the factor \(1/(2\pi)^3\) is simply a convenient choice, the BhS equation assumes the form of the non-relativistic LS equation

\[ \overline{T}(p', \vec{p}) = \overline{V}(p', \vec{p}) + \int d^3p'' \, \overline{V}(p', \vec{p}'') \times \frac{M_N}{p^2 - p''^2 + i\epsilon} \, \overline{T}(p'', \vec{p}). \]  
(94)

Since \(\overline{V}\) satisfies equation (94), it may be regarded as a non-relativistic potential. By the same arguments, \(\overline{T}\) may be regarded as the non-relativistic \(T\)-matrix. All technical aspects associated with the solution of the LS equation can be found in appendix A of [44], including specific formulas for the \(np\) and \(pp\) phase shifts. Additional details concerning the relevant operators and their decompositions are given in section 4 of [42]. Finally, computational methods to solve the LS equation are found in [43].

6.3. Regularization and non-perturbative renormalization

Iteration of \(\overline{V}\) in the LS equation, equation (94), requires cutting \(\overline{V}\) off for high momenta to avoid infinities. This is
consistent with the fact that ChPT is a low-momentum expansion which is valid only for momenta $Q \ll \Lambda_c \approx 1$ GeV. Therefore, the potential $\tilde{V}$ is multiplied with the regulator function $f(p', p)$,

$$\tilde{V}(\vec{p}', \vec{p}) \rightarrow \tilde{V}(\vec{p}', \vec{p}) f(p', p)$$

(95)

with

$$f(p', p) = \exp\left[-(p'/\Lambda)^{2n} - (p/\Lambda)^{2n}\right],$$

(96)

such that

$$\tilde{V}(\vec{p}', \vec{p}) f(p', p) \approx \tilde{V}(\vec{p}', \vec{p}) \left\{1 - \left[p'/\Lambda\right]^{2n} + \left[p/\Lambda\right]^{2n} + \ldots\right\}.$$  

(97)

Typical choices for the cutoff parameter $\Lambda$ that appears in the regulator are $\Lambda \approx 0.5$ GeV $< \Lambda_c \approx 1$ GeV. At $N^3$LO and $N^4$LO, an appropriate choice for $n$ is three.

We display equation (97) to demonstrate that the exponential cutoff may not impact the order at which we are working. If $n$ is sufficiently large, the regulating function generates terms beyond the given order. Under the assumption of a reasonable convergence of the chiral expansion, these terms are sufficiently small not to impact the accuracy at the present order. But note that the form as given in equation (96), and not its expansion equation (97), is used in actual calculations. We also mention in this context that the square-root factors in equations (92) and (93) are not expanded, as their full structure ensures consistency with relativistic elastic unitarity.

It is pretty obvious that results for the $T$-matrix may depend sensitively on the regulator and its cutoff parameter. The removal of such regulator dependence is known as renormalization. Proper renormalization of the chiral $NN$ interaction is a controversial issue, see section 4.5 of [11] for a more comprehensive discussion.

For a successful EFT (in its domain of validity), one must be able to claim independence of the predictions on the regulator. Also, truncation error must decrease as we go to higher and higher orders. These are precisely the goals of renormalization.

Lepage [52] has stressed that the cutoff independence should be examined for cutoffs below the hard scale and not beyond. Ranges of cutoff independence within the theoretical error are to be identified using Lepage plots [52]. A systematic investigation of this kind has been conducted in [53]. In that work, the error of the predictions was quantified by calculating the $\chi^2$/datum for the reproduction of the $np$ elastic scattering data as a function of the cutoff parameter $\Lambda$ of the regulator function equation (96). Predictions by chiral $np$ potentials at order NLO and NNLO were investigated applying Weinberg counting for the counter terms ($NN$ contact terms). The results from this study for the energy range 35–125 MeV are shown in the upper frame of figure 14 and for 125–183 MeV in the lower frame. It is seen that the reproduction of the $np$ data at these energies is generally poor at NLO, while at NNLO the $\chi^2$/datum assumes acceptable values (a clear demonstration of order-by-order improvement). Moreover, at NNLO one observes ‘plateaus’ of constant low $\chi^2$ for cutoff parameters ranging from about 450–850 MeV. This may be perceived as cutoff independence (and, thus, successful renormalization) for the relevant range of cutoff parameters.

6.4. NN potentials order by order

As discussed, $NN$ potentials can be calculated at various orders, see equations (33)–(38) and equations (87)–(90), with better accuracy at higher orders. The convergence properties of the chiral expansion in the most central partial waves can be seen in figure 15. There, we display the $J \leq 2$ phase parameters for potentials constructed at order NLO, NNLO, and $N^3$LO with cutoffs ranging between 450 and 600 MeV. (The NLO and NNLO potentials are from [53] and the $N^3$LO ones from [11, 62].) There is noticeable improvement in the agreement between the predictions and the empirical phase shifts as the order increases.

For a more direct comparison of theory and experiment, we can calculate observables, rather than phase shifts. The quality of the agreement with experimental data is typically expressed in terms of the $\chi^2$/datum, with a value close to unity indicating a nearly perfect agreement.

In table 3, we report the $\chi^2$/datum for the comparison between the world $np$ data below 290 MeV and the predictions of $np$ potentials at NLO and NNLO by the Bochum group [38]. The NLO potentials generate a very large $\chi^2$/datum (between 67 and 105), while the NNLO potentials give values between 12 and 27, consistent with the findings of [53] shown in figure 14. It is promising to see that there is order-by-order
improvement, but the np data at NLO and NNLO are not reproduced with sufficient quality.\(^5\)

The most natural strategy is then to proceed to the next order, as suggested already in 2002\(^{[49, 54]}\). The first N\(^3\)LO potential followed shortly after\(^{[28]}\).

### Table 3.

| Energy bin       | \(T_{\text{lab}}\) (MeV) | \(\chi^2/\text{datum}\) | Bochum np potentials |
|------------------|--------------------------|--------------------------|----------------------|
|                  |                          |                          | NLO (550/700–400/500) | NNLO (600/700–500/500) |
| 0–100            | 1058                     | 4–5                      | 1.4–1.9              |
| 100–190          | 501                      | 77–121                   | 12–32                |
| 190–290          | 843                      | 140–220                  | 25–69                |
| 0–290            | 2402                     | 67–105                   | 12–27                |

\(^5\) For an optimized NNLO potential see\(^{[63]}\) and for local NLO and NNLO potentials see\(^{[64]}\).

At N\(^3\)LO \((Q^3)\), 24 contact terms bring in a total of 24 parameters which impact partial waves with \(L \leq 2\), while at NLO and NNLO there are only 9 contacts with \(L \leq 1\) (see section 6.1 and table 4). These LECs are free constants employed to parametrize the short-range phenomenology. Table 4 shows how many terms with a certain power of \(Q\) participate in a given NN state. One can see from the table that contacts appear for the first time in D-waves at N\(^3\)LO. This is one important mechanism behind the considerable improvement in the reproduction of the NN data at this order. Because the D-states are somewhat in between central and peripheral waves, contact terms, in addition to the 1PEs and 2PEs, are important to describe the D-phases correctly. Moreover, at N\(^3\)LO, every P-wave also benefits from an additional contact term, leading to further improvement, especially in \(^3\)P\(_0\) and \(^3\)P\(_1\) at incident laboratory energies greater than 100 MeV (see figure 15). Table 4 also displays the number of free parameters used in the Nijmegen partial wave analysis (PWA93)\(^{[47]}\) and in the high-precision CD-Bonn potential\(^{[44]}\). For S and P waves, that number is approximately equal to the one required by EFT at N\(^3\)LO \((Q^3)\). Interestingly, we find in EFT a retroactive motivation for the phenomenology which became popular in the 1990s to construct high-precision potentials.

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**Figure 15.** Phase shifts of neutron–proton scattering for the lower partial waves with \(J \leq 2\). The yellow, red, and blue bands show the variations of the predictions with changing cutoffs between 450 and 600 at NLO, N\(^2\)LO, and N\(^3\)LO, respectively. The predictions by N\(^3\)LO potentials lie within the dark blue band and are, therefore, not explicitly shown. The solid dots and open circles are the results from the Nijmegen multi-energy np phase shift analysis\(^{[47]}\) and the VPI/GW single-energy np analysis SM99\(^{[48]}\), respectively. (Figure reproduced with permission from\(^{[66]}\). Copyright the American Physical Society 2015.)

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For an optimized NNLO potential see\(^{[63]}\) and for local NLO and NNLO potentials see\(^{[64]}\).
Table 4. Number of parameters needed for fitting the \( np \) data in the Nijmegen phase-shift analysis and by the high-precision CD-Bonn potential versus the total number of \( NN \) contact terms of EFT based potentials to different orders.

|                  | Nijmegen PWA93 | CD-Bonn pot. | EFT contact potentials [33] |
|------------------|----------------|--------------|----------------------------|
|                  | [47]           | [44]         | \( Q^0 \) \( Q^2 \) \( Q^4 \) \( Q^6 \) |
| \( ^3S_0 \)      | 3              | 4            | 1 2 4 6  |
| \( ^3P_1 \)      | 3              | 4            | 1 2 4 6  |
| \( ^3S_1−D_1 \)  | 2              | 2            | 0 1 3 6  |
| \( ^3P_0 \)      | 3              | 2            | 0 1 2 4  |
| \( ^3P_1 \)      | 3              | 2            | 0 1 2 4  |
| \( ^3P_2 \)      | 3              | 3            | 0 1 2 4  |
| \( ^3P_3 \)      | 3              | 3            | 0 1 2 4  |
| \( ^3P_2−F_2 \)  | 2              | 1            | 0 0 1 3  |
| \( ^3D_2 \)      | 2              | 3            | 0 0 1 2  |
| \( ^3D_1 \)      | 2              | 2            | 0 0 1 2  |
| \( ^3D_2 \)      | 2              | 2            | 0 0 1 2  |
| \( ^3D_3 \)      | 1              | 2            | 0 0 1 2  |
| \( ^3D_2−G_2 \)  | 1              | 0            | 0 0 0 1  |
| \( ^3F_1 \)      | 1              | 1            | 0 0 0 0  |
| \( ^3F_2 \)      | 1              | 2            | 0 0 0 1  |
| \( ^3F_3 \)      | 1              | 2            | 0 0 0 1  |
| \( ^3F_4 \)      | 2              | 1            | 0 0 0 1  |
| \( ^3F_5−H_3 \)  | 0              | 0            | 0 0 0 0  |
| \( ^4G_4 \)      | 1              | 0            | 0 0 0 0  |
| \( ^4G_3 \)      | 0              | 1            | 0 0 0 0  |
| \( ^4G_4 \)      | 0              | 1            | 0 0 0 0  |
| \( ^4G_5 \)      | 0              | 1            | 0 0 0 0  |
| Total            | 35             | 38           | 2 9 24 50 |

Thanks to the larger number of parameters, \( N^3LO \) potentials can be constructed which are of about the same quality as the high-precision \( NN \) potentials of the 1990s [44, 55, 56]. This fact is clearly revealed in the \( \chi^2 \)/datum for the fit of the \( np \) and \( pp \) data below 290 MeV shown in table 5 and 6, respectively. Table 5, which is pretty self-explanatory, displays the \( \chi^2 \)/datum for various chiral potentials as well as the Argonne potential, compared with the world \( np \) data below 290 MeV.

As we turn now to proton-proton (\( pp \)), note first that the \( \chi^2 \) for \( pp \) data are typically larger than for \( np \) because of the higher precision of \( pp \) data (table 6). Thus, the Argonne \( V_{18} \) produces a \( \chi^2 \)/datum = 1.4 for the world \( pp \) data below 290 MeV and the best Idaho \( N^3LO \) \( pp \) potential obtains 1.5. The fit by the best Bochum \( N^3LO \) \( pp \) potential results in a \( \chi^2 \)/datum = 2.9 and the worst produces 22.3. In view of these poor \( \chi^2 \), the Bochum group has recently launched an attempt towards improving their chiral potentials [58, 59]. However, as in their previous work [37], they have fitted their new potentials only to \( NN \) phase shifts and not to the \( NN \) data. The \( \chi^2 \) for the reproduction of the \( NN \) data by the new Bochum potentials are not available and, thus, no reliable statement about the quality of the new potentials can be made. In the 1990s, the Nijmegen group has pointed out repeatedly that for high quality potentials it is insufficient to fit phase shifts only. A seemingly ‘good’ fit of phase shifts can be misleading and can result in a poor \( \chi^2 \) for the reproduction of the data.

Concerning alternative \( N^1LO \) potentials, we note that a minimally non-local \( NN \) potential of this kind has been constructed in [65] which produces a \( \chi^2 \)/datum of about 1.3 for the \( pp \) plus \( np \) data.

Now turning to \( N^4LO \): based upon the derivation of the 2PE and 3PE contributions to the \( NN \) interaction at \( N^4LO \) by Entem et al [29] presented in section 4.5 and applied in peripheral scattering in section 5, \( NN \) potential at \( N^4LO \) have recently been developed [29, 59]. Note that the lower partial waves, which are crucial for a quantitative reproduction of the \( NN \) data, are ruled by the contact terms The number of contacts at \( N^4LO \) (\( Q^6 \)) is the same as at \( N^3LO \) (\( Q^5 \)). Thus, the \( N^4LO \) potentials are not very different from the \( N^3LO \) ones. Note also that the high quality of some of the \( N^3LO \) potentials [11, 28, 65] leaves little room for improvements.

A further increase in accuracy (if needed) could be achieved at \( N^6LO \) (\( Q^8 \)), where the number of contact terms advances to 50 (table 4) [33]. As discussed in section 4.6, the dominant 2PE and 3PE contributions at \( N^6LO \) have been derived [32]. Thus, all the mathematical material for the construction of \( N^6LO \) potentials is available. However, it is debatable if there is a need for them.

7. Nuclear many-body forces

Two-nucleon forces derived from chiral EFT as described above have been applied, often successfully, in the many-body system. On the other hand, over the past several years we have learnt that, for some few-nucleon reactions and nuclear structure issues, 3NFs cannot be neglected. The most well-known cases are the so-called A, puzzle of \( N−d \) scattering [60], the ground state of \( ^{10}B \) [61], and the saturation of nuclear matter [62, 66]. As we observed previously, the EFT approach generates consistent two- and many-nucleon forces in a natural way (see the overview given in figure 1). We now shift our focus to chiral 3NFs- and 4NFs.

7.1. Three-nucleon forces

Weinberg [10] was the first to discuss nuclear three-body forces. Not long after that, the first 3NF at NNLO was derived by van Kolck [24].

For a 3NF, we have \( A = 3 \) and \( C = 1 \) and, thus, equation (25) implies

\[
\nu = 2 + 2L + \sum_i \Delta_i. \tag{98}
\]

We will use this equation to analyze 3NF contributions order by order.

7.1.1. Next-to-leading order. The lowest possible power is obviously \( \nu = 2 \) (NLO), which is obtained for no loops (\( L = 0 \)) and only leading vertices (\( \sum_i \Delta_i = 0 \)). As discussed by Weinberg [10] and van Kolck [24], the contributions from these diagrams vanish at NLO. So, the bottom line is that
there is no genuine 3NF contribution at NLO. The first non-vanishing 3NF appears at NNLO.

7.1.2. Next-to-next-to-leading order. The power $\nu = 3$ (NNLO) is obtained when there are no loops ($L = 0$) and $\sum_1 \Delta_i = 1$, i.e., $\Delta_i = 1$ for one vertex while $\Delta_i = 0$ for all other vertices. There are three topologies which fulfill this condition, known as the 2PE, 1PE, and contact graphs [24, 25] (figure 16).

The 2PE 3N-potential is derived to be

$$V^{3NF}_{2PE} = -\frac{g_A}{2f_\pi} \sum_{i\neq j\neq k} \frac{1}{q_i^2 + m_i^2} \frac{1}{q_j^2 + m_j^2} \frac{1}{q_k^2 + m_k^2} \left[ \bar{q}_i \cdot \bar{q}_j \right] \left( \tau_i \cdot \tau_j \right)$$

(99)

with $\bar{q}_i \equiv \bar{p}_i^t - \bar{p}_i$, where $\bar{p}_i$ and $\bar{p}_i^t$ are the initial and final momenta of nucleon $i$, respectively, and

$$F_{ab}^{ij} = \delta^{ab} \left[ \frac{2c_3 m_\pi^2}{f_\pi^2} \bar{q}_i \cdot \bar{q}_j + \frac{2c_3}{f_\pi^2} \bar{q}_i \cdot \bar{q}_j \right]$$

$$+ \frac{c_4}{f_\pi^2} \sum_c a^{abc} \bar{q}_k \cdot (\bar{q}_i \times \bar{q}_j)$$

(100)

It is interesting to observe that there are clear analogies between this force and earlier 2PE 3NFs already proposed decades ago, particularly the Fujita–Miyazawa [67] and the Tucson–Melbourne [68] forces.

The 2PE 3NF does not introduce additional fitting constants, since the LECs $c_1$, $c_2$, and $c_4$ are already present in the 2PE 2NF. These LECs are constrained by $NN$ and $\pi N$ data.

The other two 3NF contributions shown in figure 16 are easily derived by taking the last two terms of the $\Delta = 1$ Langrangian, equation (18), into account. The 1PE contribution is

$$V^{3NF}_{1PE} = -D \frac{g_A}{q_i^2} \sum_{i\neq j\neq k} \left( \tau_i \cdot \tau_j \right)$$

(101)

and the 3N contact potential is given by

$$V^{3NF}_{ct} = E \frac{1}{2} \sum_{i\neq j\neq k} \left( \tau_i \cdot \tau_j \right)$$

(102)

These 3NF potentials introduce two additional constants, $D$ and $E$, which can be constrained in more than one way. One may use the triton binding energy and the $nd$ doublet scattering length $^2a_{nd}$ as done in [25]. Alternative choices include the binding energies of $^3$H and $^4$He [69] or an optimal global fit of the properties of light nuclei [70]. Another method makes use of the triton binding energy and the Gamow–Teller matrix element of tritium $\beta$-decay [71]. When the values of $D$ and $E$ are determined, the results for other observables involving three or more nucleons are true theoretical predictions.

Applications of the leading 3NF include few-nucleon reactions, spectra of light- and medium-mass nuclei [72, 73], and nuclear and neutron matter (NM) [62, 66], often with satisfactory results. Some problems, though, remain unresolved, such as the well-known ‘$A_0$ puzzle’ in nucleon-
The leading 3NFs underestimate the analyzing power in $p$-$d$ scattering to a larger degree than in $p$-$d$. Although the $p$-$^3\text{He}$ $A_1$ improves considerably (more than in the $p$-$d$ case) when the leading 3NF is included [74], the disagreement with the data is not fully removed. Also, predictions for light nuclei are not quite satisfactory [70].

In summary, the leading 3NF of ChPT is an outstanding contribution. It gives validation to, and provides a better framework for, 3NFs which were proposed already 5 decades ago; it alleviates existing problems in few-nucleon reactions and the spectra of light nuclei. Nevertheless, we still face several challenges. With regard to the 2NF, we have discussed earlier that it is necessary to go to order 4 for high-quality predictions. Thus, the 3NF at N3LO must be considered simply as a matter of consistency with the 2NF sector. At the same time, one hopes that its inclusion may result in further improvements with the aforementioned unresolved problems.

### 7.1.3. Next-to-next-to-next-to-leading order

At N$^3$LO, there are loop and tree diagrams. For the loops (figure 17), we have $L = 1$ and, therefore, all $\Delta_i$ have to be zero to ensure $\nu = 4$. Thus, these one-loop 3NF diagrams can include only leading order vertices, the parameters of which are fixed from $nN$ and $NN$ analysis. One sub-group of these diagrams (the 2PE graphs, see figure 17) has been calculated by Ishikawa and Robilotta [75], and the other topologies have been evaluated by the Bochum–Bonn group [76]. The N$^3$LO 2PE 3NF has been applied in the calculation of nucleon-deuteron observables in [75] causing little impact. Very recently, the long-range part of the chiral N$^3$LO 3NF has been tested in the triton [77] and in three-nucleon scattering [78] yielding only moderate effects. The long- and short-range parts of this force have been used in NM calculations (together with the N$^3$LO 4NF) producing relatively large contributions from the 3NF [79]. Thus, the ultimate assessment of the N$^3$LO 3NF is still outstanding and will require more few- and many-body applications.

#### 7.1.4. The 3NF at N$^4$LO

In the meantime, one may go ahead and look at the next order of 3NFs, which is N$^4$LO or $\nu = 5$. The loop contributions that occur at this order are obtained by replacing in the N$^3$LO loops one vertex by a $\Delta_i = 1$ vertex (with LEC $c_i$), figure 18, which is why these loops may be more sizable than the N$^3$LO loops. The 2PE, 1PE-2PE, and ring topologies have been evaluated [22, 30] so far. In addition, we have three ‘tree’ topologies (figure 19), which include a new set of 3N contact interactions that has recently been derived by the Pisa group [31]. Contact terms are typically simple (as compared to loop diagrams) and their coefficients are essentially free. Therefore, it would be an attractive project to test some terms (in particular, the spin–orbit terms) of the N$^4$LO contact 3NF [31] in calculations of few-body reactions (specifically, the $p$–$d$ and $p$–$^3\text{He}$) and spectra of light nuclei.

#### 7.2. Four-nucleon forces

For connected ($C = 1$) $A = 4$ diagrams, equation (25) yields

$$\nu = 4 + 2L + \sum_i \Delta_i.$$  \hfill (103)
We then see that the first (connected) non-vanishing 4NF is generated at \( \nu = 4 \) (N^4LO), with all vertices of leading type, figure 20. This 4NF contribution has no loops and introduces no novel parameters [80]. (See [11] for a more detailed discussion on these diagrams.)

For a reasonably convergent series, terms of order \((Q/\Lambda)^2\) must be small, and therefore chiral 4NF are predicted to be very weak. This expectation was confirmed in a recent calculation of the \(^4\text{He}\) binding energy including the leading 4NF (figure 20). Its effect was found to be a few 100\,keV [81], to be compared with the actual size of the binding energy, 28.3\,MeV. Although obtained with the help of several approximations, this preliminary predictions supports the notion that 4NF may indeed be negligible.

The effects of the leading chiral 4NF in symmetric nuclear matter (SNM) and pure NM have been worked out by Kaiser et al [82, 83].

8. Applications in the nuclear many-body problem

In this section, we will address some recent applications of the few-nucleon forces which were derived and discussed in previous sections. In particular, we will concentrate on applications where the analysis is conducted in the spirit of exploring order-by-order convergence of the predictions.

It should be clear from the historical perspectives presented at the opening of this article that our present knowledge of nuclear forces in free space and in the few-nucleon system is the result of decades of struggle. The nature of the nuclear force in a dense medium is an even more complex problem, as it involves aspects of the forces that cannot be constrained through free-space NN scattering or the properties of the (relatively ‘simple’) few-nucleon system.

Although predictions for finite nuclei are the ultimate test for many-body predictions, infinite nuclear matter is an alternative and convenient testing ground for many-body theories. By ‘nuclear matter’ we mean an infinite system of nucleons acted on by their mutual strong forces and no electromagnetic interactions. Nuclear matter is characterized by its energy per particle as a function of density and other quantities as appropriate (e.g. temperature, isospin asymmetry, spin asymmetry). Such relation is known as the nuclear matter equation of state (EoS). The translational invariance of the system facilitates theoretical calculations. At the same time, adopting what is known as ‘local density approximation’, one may use the EoS directly in calculations of finite systems, as we will discuss below.

When proton and neutron densities are different (that is, in the presence of isospin asymmetry), the energy per particle becomes a function of both the total density and the relative concentrations of neutrons and protons. The EoS of isospin-asymmetric matter naturally introduces the symmetry energy, similarly to the appearance of the symmetry term in the well-known Bethe–Weizäcker formula. As will be discussed later in more details, the symmetry energy is typically approximated as the difference between the energy per particle of SNM and pure NM as a function of density. Of particular contemporary interest is the EoS of highly neutron-rich matter, all the way to nearly pure NM. This quantity is important for understanding wide ranging questions in modern nuclear physics, from the properties of rare isotopes to those of neutron stars. On the one hand, the density dependence of the symmetry energy is known to correlate strongly with the neutron skin thickness of a heavy nucleus. On the other hand, the pressure in neutron-rich matter is the main input for the structure equations of compact stars. Therefore, microscopic predictions together with empirical constraints from observables that are sensitive to the EoS are an ideal combination to learn about the in-medium behavior of nuclear forces, particularly in isospin-asymmetric medium. We also recall that the EoS is an important part of the input of transport models describing heavy-ion (HI) collisions and thus can be constrained through analyses of carefully selected observables in ion–ion scattering. Concerning non-terrestrial observations, partnership between nuclear physics and
astrophysics is increasingly important as better constraints on the high-density part of the EoS become available through more accurate measurements of neutron star masses. In summary, studies of nucleonic matter are especially timely and important, as they support rich on-going and future experimental effort, both in terrestrial laboratories and the cosmos.

8.1. Order-by-order predictions of the energy per nucleon in nuclear and NM

The problem shared by all non-EFT based approaches is that it is essentially impossible to estimate reliably the uncertainty associated with a particular prediction. On the other hand, EFT provides a well-defined framework to calculate observables where the truncation error decreases systematically as higher orders are included. Earlier in this article, we have seen that such task can be accomplished quite successfully at the level of NN phase shifts.

In this section, we will review and discuss recent calculations of the energy per particle in infinite matter at different orders of chiral EFT [66]. The discussion will also emphasize the importance of error quantification and how it should be addressed in chiral EFT.

Estimates of theoretical uncertainties [84] for calculations of the EoS have mostly focused on varying the LECs and resolution scale at which nuclear dynamics are probed [62, 85–89]. In a recent work [66] we layed the foundation for order-by-order calculations of nuclear many-body systems by presenting consistent NLO and N^2LO chiral nuclear forces whose relevant short-range three-body forces are fit to \( \Lambda = 3 \) binding energies and the lifetime of the triton. We then assessed the accuracy with which infinite nuclear matter properties and the isospin asymmetry energy can be predicted from order-by-order calculations in chiral EFT.

Uncertainty originates from:

- The choice of the many-body method (a source of error not inherent to EFT).
- Error in the determination of the LECs. Short-range LECs (NN) and long-range LECs (\( \pi N \)) must be considered separately.
- Regulator dependence.
- Truncation error.

In the following, we will address those items briefly but systematically.

A variety of many-body methods are available and have been used extensively in nuclear matter predictions. They include: the coupled-cluster method, many-body perturbation theory, variational Monte Carlo or Greens function Monte Carlo methods. In computing the EoS, we employ the non-perturbative particle-particle ladder approximation. In the traditional hole-line expansion, it represents the leading-order contribution. To quantify the uncertainty carried by this choice, it is insightful to compare with [90, 91]. In [90], the authors report on coupled-cluster calculations in SNM including particle-particle and hole–hole diagrams (as well as an exact treatment of the Pauli operator). The overall effect, as seen from comparing the first and last entries in table II of [90], is very small around saturation density, consistent with table II in [62], and grows to 1.5 MeV at the highest Fermi momentum included in the study. Note that these calculations adopt the N^3LO potential [28] (with \( \Lambda = 500 \text{ MeV} \)) and no three-nucleon forces. On the other hand, in [91] coupled-cluster calculations in nucleonic matter were performed at N^2LO with two- and three-body forces and with the inclusion of selected triples clusters, namely correlations beyond particle–particle and hole-hole ladders. The effect of these contributions is found to be negligible in NM and about 1 MeV per nucleon in symmetric matter in the density range under consideration [91]. In the light of the above considerations, we conclude that a realistic estimate of the impact of using a non-perturbative approach beyond particle-particle correlations is about 1 MeV in nuclear matter around saturation density and much smaller in NM. As we show below, such uncertainties are significantly smaller than those associated with variations in the cutoff scale.

In order to quantify the error associated with possible variations of the (short-range) NN LECs, we refer to recent findings from the Granada group [92]. They applied 205 samples of smooth local potentials, all with \( \chi^2/\text{datum} \) of approximately 1, and found a variation of 15 keV in the triton binding energy. From our part, we performed Brueckner–Hartree–Fock calculations in nuclear matter using local high-precision potentials from the Nijmegen group [55] and observed an uncertainty of 0.6 MeV in the energy per particle at normal density. In summary, we conclude that the uncertainty arising from the error in the NN data has negligible impact on the many-body system. Concerning the (long-range) \( \pi N \) LECs, they are likely to impact mostly peripheral partial waves (namely, those high partial waves where no contact terms are present). At NLO and N^2LO, that means D-waves and higher, whereas at N^3LO no contacts exist in F-waves and higher. Therefore, we expect variations of the \( \pi N \) LECs (within the range allowed by \( \pi N \) scattering data) to have only minor impact in nuclear matter, since its sensitivity is limited to peripheral partial waves. Nevertheless, we stress that a systematic investigation with consideration of \( \pi N \) LECs uncertainty consistently in the 2NF and the 3NF, has not yet been done and is part of our future plans.

Keeping in mind the uncertainty considerations made above, we now move to nuclear and NM predictions. Our results for the energy per particle as a function of the nuclear density are shown in figure 21 for SNM. We note that the \( pp \) ladder approximation employed in the present work is in good agreement with the perturbative results available at N^3LO from [62] including up to third-order particle-particle diagrams. In figure 21, the shaded bands in yellow and red represent the spread of our complete calculations conducted at NLO and N^2LO, respectively. The blue band is the result of a calculation that employs N^3LO \( \pi N \) potentials together with N^2LO 3NFs. In all cases shown, the cutoff is varied over the range 450–600 MeV. As noted before, the N^3LO 3NFs and 4NFs are at present omitted, and the resulting convergence pattern gives an estimate on the theoretical uncertainty of the calculation (and not necessarily of the chiral EFT expansion.
We note that at NLO the potentials constructed at lower cutoff scales do not exhibit saturation until very high densities. On the other hand, for the 600 MeV cutoff potential the $^3S_0$ partial wave (together with the $^3S_1$ partial wave) is sufficiently repulsive to enable saturation at a relatively smaller density. We observe that the convergence pattern for the low-cutoff ($\Lambda = 450–500$ MeV) potentials is significantly better than for the 600 MeV potential. Overall there is a large spread from cutoff variations both at NLO and N2LO beyond nuclear matter saturation density. Moreover, the bands at these two orders do not overlap, suggesting that their width is not a suitable representation of the uncertainty. Although the (not yet complete) N3LO calculation reveals a strong reduction of the cutoff dependence, it is important to notice that an uncertainty of about 8 MeV remains at saturation density. While we do not expect much of a change in nuclear matter predictions from 4NFs [80, 81, 86], it is quite possible that the inclusion of N3LO 3NFs might reduce either the cutoff dependence or improve the convergence pattern. This will be an interesting subject for future investigations.

The results for NM are presented in figure 22, where the bands have the same meaning as in figure 21. Note that the range of densities under consideration is smaller for NM in order to keep the Fermi momentum below the cutoff in all cases. We see a large spread at NLO for the largest densities considered, whereas the band has only moderate size at the next order and remains small for our N3LO calculation. Similar to what was observed in SNM, the bands at NLO and N2LO do not overlap in NM. In addition, the N3LO band does not generally overlap with the N2LO band. Therefore, the variation obtained by changing the cutoff does not seem to provide a reliable representation of the uncertainty at the given order. A better way to estimate such uncertainty is to consider the difference between the predictions at two consecutive orders.

In figure 23 we present the results for the symmetry energy, which is defined as the strength of the quadratic term in an expansion of the energy per particle in asymmetric matter with respect to the asymmetry parameter $\alpha$:

$$ E(\rho, \alpha) \approx E(\rho, \alpha = 0) + E_{\text{sym}} \alpha^2 + O(\alpha^4), $$

where $E = E/A$ is the energy per particle and $\alpha = (\rho_n - \rho_p)/(\rho_n + \rho_p)$. The nearly linear behavior of $E(\rho, \alpha)$ with $\alpha^2$ has been confirmed by many microscopic calculations (see for instance [93, 94], but see also [95]). It is a common approximation to neglect powers beyond $\alpha^2$ in the expansion above and thus defining the symmetry energy as the difference between the energy per particle in NM and SNM.

As mentioned at the beginning of this section, systematic efforts are ongoing to set better empirical constraints on the
symmetry energy, through both laboratory and astrophysical measurements. It is therefore important to have an understanding of the theoretical uncertainty affecting calculations of this quantity. The spread due to the change of the cutoff values in our NLO, N2LO, and N3LO calculations is represented by the three bands as before. As observed previously for symmetric matter, the spread due to cutoff variations remains large at N3LO, with some minimal overlap with the NLO band. The N3LO band reflects the large cutoff sensitivity previously observed in symmetric matter. Again, we conclude that the spread generated by changing the cutoff does not in general provide a reliable estimate of the theoretical uncertainty.

8.2. Spin-polarized NM

Polarized NM is an interesting system for various reasons. Among them is the impact that spin instabilities in the interior of stellar matter would have on neutrino interactions and thus the star cooling mechanism.

Spin polarization is also of interest in symmetric or nearly SNM. For instance, for the purpose of scattering from polarized nuclei, one may define a spin dependent optical potential which, for the spin degree of freedom, plays the same role as the Lane potential [97] for the isospin degree of freedom. Such spin symmetry potential can be obtained from the difference between the single-particle potentials for spin-up and spin-down nucleons in polarized SNM.

To address the most general case, one must include both spin and isospin polarizations. From the astrophysics point of view, stellar matter contains a small, but not insignificant proton fraction. With regard to experiments in terrestrial laboratories, the spin dependence of the nuclear interaction in nuclear matter can be explored through collective excitations, such as giant resonances. Most typically, a nucleus with non-zero spin is also isospin asymmetric, making it necessary to include both spin and isospin polarizations. For those reasons, in previous work [96, 98, 99] we explored matter with different densities of neutrons and protons where each type of nucleon can have arbitrary degree of spin polarization. We obtained predictions employing the Dirac–Brueckner–Hartree–Fock approach to nuclear matter and a relativistic one-boson-exchange NN potential and did not see any indications of a phase transition to a spin-polarized state. We note that all models which start from the bare NN force and apply it in the medium (see, for instance, [100]) end up with similar conclusions. In contrast, approaches based on parametrizations of Skyrme forces, or other phenomenological forces, report different findings. For instance, with the SLy4 and SLy5 forces and the Fermi liquid formalism a phase transition to the antiferromagnetic state is predicted in asymmetric matter at a critical density equal to about 2–3 times normal density [101]. Qualitative disagreement is also encountered with other approaches such as relativistic Hartree–Fock models based on effective meson–nucleon Lagrangians. For instance, in [102] it was reported that the onset of a ferromagnetic transition in NM, and its critical density, are crucially determined by the inclusion of isovector mesons and the nature of their couplings.

The brief review given above summarizes the findings of many useful and valid calculations. However, the problem common to all of them is that it is essentially impossible to estimate, in a statistically meaningful way, the uncertainties associated with a particular prediction, or to quantify the error related to the approximations applied in a particular model. Therefore, in this section, we apply the same philosophy as in section 8.1 to study the EoS of polarized NM at different orders of ChPT.

Based on the literature mentioned above, a phase transition to a polarized phase (at least up to normal densities) seems unlikely, although the validity of such conclusion must be assessed in the context of EFT errors. Furthermore, polarized NM is a very interesting system for several reasons. Because of the large neutron–neutron scattering length, NM displays behaviors similar to those of a unitary Fermi gas. In fact, up to nearly normal density, unpolarized NM is found to display the behavior of an S-wave superfluid [103, 104]. The possibility of simulating low-density NM with ultracold atoms near a Feshbach resonance [105] has also been discussed. When the system is totally polarized, it has been observed to behave like a weakly interacting Fermi gas [106]. Here, we wish to explore to which extent and up to which densities we are in agreement with such conclusions, and how this and other observations depend on the chiral order and the resolution scale.

In contrast with previous calculations, our recent work summarized here contains the following novelties:

- We consider both cutoff dependence and truncation error for the purpose of uncertainty quantification of chiral EFT. Although incomplete in the 3NF at N3LO, our calculations are a substantial step in that direction. We note, further, that the contribution from the 3NF at N3LO
was found to be very small in NM for the potentials in our perview [86], about $-0.5$ MeV at normal density. Here, we consider NM or highly neutron-rich matter.

- For the first time, we present results for both spin and isospin asymmetries within the framework of chiral forces. These tools are necessary to assess, for instance, the sensitivity of the results (particularly, the potential onset of a phase transition) to the presence of a non-zero proton fraction.

For a detailed description of the formalism, the reader is referred to [109]. Here, we will just summarize some definitions which are necessary for the discussion which follows.

In a spin-polarized and isospin asymmetric system with fixed total density, $\rho$, the partial densities of each species are

$$\rho_n = \rho_{na} + \rho_{nd}, \quad \rho_p = \rho_{pu} + \rho_{pd}, \quad \rho = \rho_n + \rho_p.$$  \hfill (105)

where $a$ and $d$ refer to up and down spin-polarizations, respectively, of protons ($p$) or neutrons ($n$). The isospin and spin asymmetries, $\alpha$, $\beta_n$, and $\beta_p$, are defined in a natural way:

$$\alpha = \frac{\rho_n - \rho_p}{\rho}, \quad \beta_n = \frac{\rho_{na} - \rho_{nd}}{\rho_n}, \quad \beta_p = \frac{\rho_{pu} - \rho_{pd}}{\rho_p}.$$  \hfill (106)

The density of each individual component can be related to the total density by

$$\rho_{na} = (1 + \beta_n)(1 + \alpha)\frac{\rho}{4},$$  \hfill (107a)

$$\rho_{nd} = (1 - \beta_n)(1 + \alpha)\frac{\rho}{4},$$  \hfill (107b)

$$\rho_{pu} = (1 + \beta_p)(1 - \alpha)\frac{\rho}{4},$$  \hfill (107c)

$$\rho_{pd} = (1 - \beta_p)(1 - \alpha)\frac{\rho}{4},$$  \hfill (107d)

where each partial density is related to the corresponding Fermi momentum through $\rho_{\sigma \pi} = (k_{F\pi}^2)/(6\pi^2)$. The average Fermi momentum and the total density are related in the usual way as $\rho = (2k_{F\pi}^2)/(3\pi^2)$.

We show in figure 24 the energy per particle in fully polarized NM as a function of density. The yellow and red bands represent the uncertainties due to cutoff variations obtained in the complete calculations at NLO and N$^3$LO, respectively. The blue band is the result of the same cutoff variations applied to our exploratory N$^3$LO calculation, see text for details. The dotted curve shows the energy of the free Fermi gas. (Figure reproduced with permission from [109]. Copyright the American Physical Society 2015.)

![Figure 24.](image)

**Table 7.** Ratio of the energy per particle of a free Fermi gas to the energy per particle of polarized neutron matter around saturation density at N$^3$LO (as described in the text) and for different values of the cutoff.

| Density (fm$^{-3}$) | $\Lambda$ (MeV) | $E_{\text{free}}/E$ |
|----------------------|-----------------|---------------------|
| 0.15                 | 450             | 0.95                |
|                      | 500             | 0.92                |
|                      | 600             | 0.95                |
| 0.17                 | 450             | 0.95                |
|                      | 500             | 0.91                |
|                      | 600             | 0.93                |

Clearly, the variations associated with changing the cutoff are not a good indicator of the uncertainty at a given order of chiral EFT, as the results from one order to the other do not overlap. Furthermore, the predictions do not show a good convergence pattern, although some indication of slow convergence can be seen when moving from N$^2$LO to our N$^3$LO calculation.

As can be concluded from table 7, the predictions from the N$^3$LO calculation are close to the free Fermi gas energy, at least up to saturation densities. Our results with the N$^3$LO [28] ($\Lambda = 500$ MeV) potential are in good agreement with those from [106] using the same potential as well as three- and 4NFs at N$^3$LO.

In figure 25, for our N$^3$LO calculation, we compare predictions (along with their cutoff variations) of the energy...
per neutron in: unpolarized NM (green band), partially polarized NM (pink band), and fully polarized NM (blue band). For the partially polarized case, the value of $\beta_0$ is equal to 0.5, corresponding to 75% of the neutrons being polarized in one direction and 25% in the opposite direction, see equations (106). Clearly, a lesser degree of spin asymmetry (as compared to the ferromagnetic case) yields considerably less repulsion. There is definitely no sign of a phase transition, particularly to a ferromagnetic state, nor an indication that such transition may occur at higher densities. This is consistent with what we observed earlier [98] with meson-theoretic interactions.

As a baseline comparison, we also include, for the unpolarized case, predictions based on a different approach, shown by the black dotted line in figure 25. These are taken from [107] and are based on the Argonne $v_{18}$ two-nucleon interaction plus the Urbana IX three body-force, using variational methods. The predictions are overall in reasonable agreement with our green band, although those from [107] show more repulsion as compared to the softer chiral interactions.

Most typically, models which do predict spin instability of NM find the phase transition to occur at densities a few times normal density. Such high densities are outside the domain of ChPT. With some effective forces, though, it was found [108] that a small fraction of protons can significantly reduce the onset of the threshold density for a phase transition to a spin-polarized state of neutron-rich matter. We explored this scenario by adding a small fraction of protons to fully polarized or unpolarized neutrons. From equations (105)–(107), a proton fraction of 10% is obtained with $\alpha = 0.8$. The results are displayed in figure 26, where a crossing of the bands labeled with "0.8, 1.0" and "0.8, 0.0", respectively, would indicate a phase transition. Thus we conclude that such transition is not predicted with chiral forces. By extrapolation, a transition to a polarized state would also appear very unlikely at higher densities.

To summarize this section, we have calculated the EoS of (fully and partially) polarized neutron-rich matter. We performed complete calculations at second and third order of chiral EFT and calculations employing the N$^3$LO 2NF plus the leading 3NF. Results with both spin and isospin asymmetries have been presented for the first time with chiral forces in [109].

In all calculations, the cutoff dependence is moderate and definitely underestimates the uncertainty of each order. Concerning the latter, we do not see a satisfactory convergence pattern. The missing 3NFs are most likely not the main cause of uncertainty at N$^3$LO, since [106] has demonstrated that large cancelations take place between the $2\pi$-exchange 3NF and the $\pi$-ring 3NF at N$^3$LO, while other 3NF contributions are very small (about 0.1–0.2 MeV). Clearly a calculation at N$^4$LO is necessary to get a realistic indication of the EFT error at N$^3$LO. Such effort is in progress. If such calculation displays a reasonable convergence pattern, it will be strong evidence that polarized NM, indeed, behaves nearly like a free Fermi gas, at least up to normal densities.
In our N^3LO calculation, the energies of the unpolarized system at normal density are close to 16 MeV for all cutoffs, whereas those in the polarized case are approximately 60 MeV. Thus, even in the presence of the large uncertainties discussed above, a phase transition to a ferromagnetic state can be excluded. This conclusion remains valid in the presence of a small proton fraction.

8.3. Uncertainty analysis for predictions of the neutron skin in ^208Pb at different orders of chiral EFT

As mentioned at the beginning of section 8, intense effort is ongoing to obtain reliable empirical information for the less known aspects of the EoS. HI reactions are a popular way to seek constraints on the symmetry energy, through analyses of observables that are sensitive to the difference between the pressure in nuclear and NM. Isospin diffusion data in HI collisions together with analyses based on isospin-dependent transport models, provide information on the slope of the symmetry energy. For a recent review on available constraints from a broad spectrum of experiments, see [110].

Concerning the lower densities, isospin-sensitive observables can also be identified among the properties of normal nuclei. The neutron skin of neutron-rich nuclei is a powerful observable, being sensitive to the slope of the symmetry energy, which determines to which extent neutrons are pushed outwards to form the skin [111]. Parity-violating electron scattering experiments are now a realistic option to determine neutron distributions with unprecedented accuracy. These experiments at low momentum transfer are especially suitable to probe neutron densities, because the Z boson couples primarily to neutrons [112]. From the first electro-weak observation of the neutron skin in a neutron-rich heavy nucleus, a value of 0.33±0.16 for the neutron skin of ^208Pb was determined [113], but the next PREX experiment aims to measure the skin within an uncertainty smaller by a factor of 3 (see [113] and references therein).

From the theoretical point of view, we stress once again that microscopic calculations with statistically meaningful uncertainties are essential to guide experiments. Therefore, following the spirit of [66], it is the purpose of this section to systematically examine and discuss predictions of the neutron skin in ^208Pb at different orders of chiral EFT and changing resolution scale.

It is well established that the neutron skin thickness correlates with the derivative of the symmetry energy. The latter is often represented through the L parameter

\[ L = 3\rho_0 \left( \frac{\partial E_{\text{sym}}(\rho)}{\partial \rho} \right)_{\rho_0} \approx 3\rho_0 \left( \frac{\partial e_{\text{nu}}(\rho)}{\partial \rho} \right)_{\rho_0}, \]

which originates from an expansion of the symmetry energy around the saturation point, \( \rho_0 \). The second (approximate) equality is due to the vanishing of the first derivative of the energy per particle in SNM at \( \rho_0 \), leaving a term proportional to the pressure in NM. Nevertheless, L depends sensitively on the saturation density, which can be quite different from model to model, particularly when considering different chiral orders and regulators. In other words, theoretical predictions of L carry larger EFT uncertainties than the ones of just NM pressure at some fixed density. To explore this point further, we will also compare predictions and uncertainties with those obtained using a phenomenological EoS for SNM consistent with the empirical saturation point.

8.3.1. Predictions with microscopic EoS for NM and SNM.

We calculate proton and neutron density distributions with a method described in an earlier work [114]. The method is based on an energy functional derived from the semi-empirical mass formula, where the volume and symmetry terms are contained in the isospin-asymmetric EoS. Thus, we write the energy of a (spherical) nucleus as

\[ E(Z, A) = \int d^3r \ e(\rho, \alpha)\rho(r) + \int d^3r_0 (|\nabla \rho|^2) + \beta |\nabla \rho|^2 + I_C, \]

where \( I_C \) stands for the Coulomb term. In the above equation, \( \rho \) and \( \rho_0 \) are the usual isoscalar and isovector densities, given by \( \rho_0 + \rho_p \) and \( (\rho_0 - \rho_p) \), respectively, \( \alpha \) is the neutron asymmetry parameter, \( \alpha = \rho_n/\rho \), and \( e(\rho, \alpha) \) is the energy per particle in isospin-asymmetric nuclear matter. The constant \( f_0 \) in equation (109) is approximately 70 MeV fm^3, whereas the magnitude of \( \beta \) is about 1/4 [115]. (Even with variations of \( \beta \) between −1 and +1, we found that the contribution from this term was negligibly small, so we disregarded its contribution.)

The symmetry energy, \( E_{\text{sym}} \), has been defined in equation (104). As discussed earlier, it is customary to retain only the term quadratic in \( \alpha \) in equation (104).

The proton and neutron density functions are obtained by minimizing the value of the energy, equation (109), with respect to the parameters of Thomas–Fermi distributions for proton and neutron densities. Although simple, this method has the advantage of allowing a very direct connection between the EoS and the properties of finite nuclei. Furthermore, microscopic structure calculations for \( A = 208 \) are presently not possible. In [114], our method was shown to yield realistic predictions for ^40Ca, ^90Zr, and ^208Pb with some of the Bonn meson-exchange potentials [5].

In the figures which follow, the size of each band is obtained from variations of the cutoff between 450 and 600 MeV in the regulator applied to the 2NF and the 3NF. In figure 27, the pressure in NM is shown. The yellow and red bands represent the uncertainties in the predictions due to cutoff variations as obtained in complete calculations at NLO and N^3LO, respectively. The blue band is the result of a calculation employing N^3LO NN potentials together with 3NFs at N^3LO. The pressure is proportional to the slope of the various curves which make up the corresponding bands shown in figure 24. We observe moderate cutoff dependence except at NLO and a slow convergence tendency with increasing order.

As already pointed out, the L parameter, defined as in equation (108), is sensitive to the characteristics of the EoS of symmetric matter through \( \rho_0 \). The latter changes dramatically from order to order as well as with changing cutoff, which can be clearly seen from figure 21. In figure 28, we show the L.
parameter as a function of density, i.e.

\[ L(\rho) = 3\rho \left( \frac{\partial E_{\text{sym}}(\rho')}{\partial \rho'} \right)_{\rho' = \rho} , \tag{110} \]

which reflects the difference between the pressures in NM and in SNM at each density. The derivative of the EoS of SNM comes in through the symmetry energy and determines larger uncertainties than those seen in figure 27.

The predictions for the skin thickness of $^{208}$Pb are summarized in table 8, along with the corresponding values of the $L$ parameter and the saturation density are given in the last two columns.

Table 8. Neutron skin thickness, $S$, in $^{208}$Pb at the specified order of chiral EFT as explained in the text. The corresponding values of the $L$ parameter and the saturation density are given in the last two columns.

| Order  | $S$(fm)   | $L(\rho_0)$(MeV) | $\rho_0$(fm$^{-3}$) |
|--------|-----------|------------------|---------------------|
| N$^2$LO | 0.21$^{+0.04}_{-0.02}$ | 77.4$^{+11.2}_{-16.2}$ | 0.167$^{+0.043}_{-0.022}$ |
| N$^3$LO | 0.17$^{+0.02}_{-0.01}$ | 39.9$^{+12.3}_{-17.3}$ | 0.144$^{+0.032}_{-0.003}$ |

Table 9. As table 8, but employing a phenomenological model for the EoS of SNM. See text for details.

| Order  | $S$(fm)   | $L(\rho_0)$(MeV) |
|--------|-----------|------------------|
| NLO    | 0.126$^{+0.004}_{-0.003}$ | 20.4$^{+8.8}_{-6.3}$ |
| N$^2$LO | 0.20$^{+0.01}_{-0.01}$ | 70.6$^{+14.1}_{-8.0}$ |
| N$^3$LO | 0.172$^{+0.002}_{-0.003}$ | 44.9$^{+3.8}_{-5.4}$ |

8.3.2. Using a phenomenological EoS for SNM. The nearly linear correlation between skins and NM pressure typically observed in phenomenological investigations of skins [116, 117] refers to a family of models with the same, or very similar, SNM properties which differ mostly in the slope of NM. This scenario can be simulated, for instance, by combining an empirical SNM EoS together with different (microscopic) NM EoS, thus separating out the role of NM pressure and removing any model dependence originating from the details of the saturation point.

We repeated the calculations adopting, this time, the empirical EoS from [118] for SNM. The latter is obtained...
from a Skyrme-type energy density functional and has a realistic saturation point at $p_0 = 0.16$ fm$^{-3}$ with energy per particle equal to $-16.0$ MeV. The corresponding findings are displayed in table 9. For this test, we also show the results at NLO, since the saturation point can be defined for all cases. Although the midvalues are reasonably consistent with those in table 8, the uncertainties are much smaller, particularly for the $L$ parameter, as to be expected based on the previous observations. The much smaller uncertainty at N$^3$LO reflects the negligible cutoff dependence of NM pressure at that order, see figure 27.

With similar considerations as above with respect to the truncation error, we define the uncertainty at N$^3$LO as the difference between the prediction at this order and the one at the next order, which gives approximately 0.03. Assuming a similar uncertainty at N$^1$LO, we estimate the skin thickness at N$^3$LO, when adopting an empirical parametrization for the EoS of SNM, to be $0.17 \pm 0.03$. We note, again, that this reflects the uncertainty in pure NM at the low densities probed by the skin. Such uncertainty is small, consistent with the low-density behavior seen in figure 27.

We observe that our final estimate is consistent with the value reported in [119], where the skin is obtained through correlations from [117], and including a study based on the liquid drop model. This strengthens our confidence in the method we adopt to obtain the skin.

To summarize, the neutron skin is an important isospin-sensitive ‘observable’, essentially determined by the difference in pressure between symmetric and NM. We calculated the neutron skin of $^{208}$Pb with two- and three-body chiral interactions. The neutron and proton density functions are obtained in a simple approach based on the semi-empirical mass formula. We observed that, in fully microscopic calculations, model dependence from the details of SNM at the saturation point does impact predictions of the symmetry pressure and, to a lesser extent, the neutron skin.

At the low densities typically probed by studies of the skin, EFT theoretical uncertainties for the skin are small on a scale set by a realistic experimental uncertainty, particularly at the higher orders of chiral EFT.

Calculations at N$^3$LO are needed for a better quantification of the truncation error at N$^1$LO, and thus a reliable comparison of the EFT error with the target uncertainty set by future PREX experiments. Concerning the latter, from [113] we learn that the target uncertainty of PREX II is a factor of 3 smaller than the one from the first PREX experiment, thus approximately $\pm 0.05$. If accomplished, this will allow to discriminate between theoretical predictions, along with the measured central value. For instance, the present EFT predictions would not be consistent with a measurement such as 0.33 (the current central value) $\pm 0.05$.

### 9. Conclusions

The past 20 years have seen great progress in our understanding of nuclear forces in terms of low-energy QCD. Key to this development was the realization that low-energy QCD is equivalent to an EFT which has become known as ChPT. In this framework, two- and many-body forces emerge on an equal footing and the empirical fact that nuclear many-body forces are substantially weaker than the two-nucleon force is explained naturally.

We presented the current status of the development of chiral nuclear forces and discussed open questions and future challenges. We also reviewed some representative examples for typical applications of chiral forces in many-body systems. For this we chose, specifically, nuclear and neutron-rich matter, including isospin and spin asymmetries, as well as an analysis of neutron skin thickness predictions in a neutron-rich nucleus.

Chiral forces have also been applied in ab initio calculations of finite nuclei (structure and reactions). Because of lack of space, we could not discuss this topic in this review and, therefore, we like to refer the interested reader to the comprehensive literature [64, 73, 120–127].

The importance of error quantification has finally been recognized in theoretical nuclear physics. We explored various sources of uncertainty systematically and noticed that the largest uncertainty comes from the truncation error of the chiral expansion (as given by the difference between the predictions at two consecutive orders). We also found that the predictions up to N$^3$LO (fourth order) for many-body observables carry a truncation error that is, in general, substantially larger than the error of the empirical information, rendering the predictions inconclusive. Thus, in many applications of chiral EFT it may be necessary to proceed beyond fourth order. In any case, the convergence of the chiral expansion is one of the most important issues to which more work needs to be devoted in the near future.

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