A THEOREM ON MEROMORPHIC DESCENT AND THE SPECIALIZATION OF THE PRO-ÉTALÉ FUNDAMENTAL GROUP

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Abstract. Given a Noetherian formal scheme \( \hat{X} \) over \( \text{Spf}(R) \), where \( R \) is a complete DVR, we first prove a theorem of meromorphic descent along a possibly infinite cover of \( \hat{X} \). Using this we construct a specialization functor from the category of continuous representations of the pro-étale fundamental group of the special fiber to the category of \( F \)-divided sheaves on the generic fiber. This specialization functor is compatible with the specialization functor of the étale fundamental groups. We also express the pro-étale fundamental group of a connected scheme \( X \) of finite type over a field as coproducts and quotients of the free group and the étale fundamental groups of the normalizations of the irreducible components of \( X \) and those of its singular loci.

Introduction

Let \( X \) be a connected locally topologically noetherian scheme. In \([3, \text{Definition 7.4.2}]\), B. Bhatt and P. Scholze introduced the pro-étale fundamental group \( \pi_1^{\text{proét}}(X) \) of \( X \), a topological group which classifies the geometric covers of \( X \) (cf. \([3, \text{Definition 7.3.1}]\)). The geometric covers include the finite étale covers. Therefore, \( \pi_1^{\text{proét}}(X) \) refines Grothendieck’s étale fundamental group \( \pi_1^{\text{ét}}(X) \), which classifies the finite étale covers. In fact, there is a natural morphism \( \pi_1^{\text{proét}}(X) \to \pi_1^{\text{ét}}(X) \) which makes \( \pi_1^{\text{ét}}(X) \) the profinite completion of \( \pi_1^{\text{proét}}(X) \) (in this introduction we ignore the important issue of base points for simplicity of exposition).

We have \( \pi_1^{\text{proét}}(X) = \pi_1^{\text{ét}}(X) \) when \( X \) is normal. If \( X \) is a degenerate curve (cf. \([12]\)) over an algebraically closed field, then \( \pi_1^{\text{proét}}(X) \) is a discrete free group (cf. Theorem IV below), a fact that naturally occurred in the theory of Mumford curves (cf. \([12]\)), but could only be phrased in an ad hoc way previously.

Let \( R \) be a complete discrete valuation ring (DVR) with \( \text{Frac}(R) = K \), and let \( X \) be a proper scheme over \( \text{Spec} R \). Let \( X_0 \) be the special fiber and \( X_K \) the generic fiber. Grothendieck’s specialization map

\[ \text{sp}_\text{ét} : \pi_1^{\text{ét}}(X_K) \longrightarrow \pi_1^{\text{ét}}(X_0) \]

has been a fundamental tool for studying the étale fundamental groups. But in general there is no analogous map \( \pi_1^{\text{proét}}(X_K) \to \pi_1^{\text{proét}}(X_0) \), as pointed out by \([10, \text{Remark 3.2}]\). For example, assume that the residue field of \( R \) is algebraically closed and \( X_K \) is a Mumford curve, then any map \( \pi_1^{\text{proét}}(X_K) \to \pi_1^{\text{proét}}(X_0) \), being a map from a profinite group to a discrete free group, is necessarily trivial.

E. Lavanda overcame this difficulty in a special case by constructing a commutative diagram of \( \bar{K} \)-group schemes:

\[
\begin{array}{ccc}
\pi_1^{\text{Fdiv}}(X_K) & \longrightarrow_{\text{sp}_{\text{proét}}} & \pi_1^{\text{proét}}(X_0)_{\bar{K}} \\
\uparrow & & \downarrow \\
\pi_1^{\text{ét}}(X_K)_{\bar{K}} & \longrightarrow_{\text{sp}_\text{ét}} & \pi_1^{\text{ét}}(X_0)_{\bar{K}},
\end{array}
\]

where \( \text{sp}_{\text{proét}} \) is a new specialization map constructed by Lavanda, \( \pi_1^{\text{Fdiv}}(X) \) is a Tannakian fundamental group classifying \( F \)-divided sheaves, and for a topological group \( \pi, \pi_{\bar{K}} \) is the algebraic hull of \( \pi \) over \( \bar{K} \) (cf. Notations and Conventions (2), (4)). For a profinite \( \pi, \pi_{\bar{K}} \) is the rather transparent
formation of the constant group scheme associated to $\pi$ (cf. Notations and Conventions (3)). Thus the above commutative diagrams means that Lavanda’s specialization map lifts Grothendieck’s. The vertical arrow on the left is a functorial surjection (cf. [16, Proposition 13, §2.4], [19, Theorem I]).

Lavanda’s hypothesis for the above theory is that $X$ is a semi-stable curve over $R$ whose generic fiber is smooth. This is quite restrictive. The first main result in this paper is to do the theory in much greater generality (another approach of generalizing the specialization map by considering the de Jong fundamental group of the rigid generic fiber can be found in [1]):

**Theorem I** (cf. 1.18). Let $R$ be a complete discrete valuation ring of equal characteristic $p > 0$, and let $X$ be a proper scheme over $R$ with connected fibers $X_K$ and $X_0$. Assume further that the residue field of $R$ is separably closed and $X(R) \neq \emptyset$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\pi^\text{Fal}(X_K) & \xrightarrow{\text{sp}^\text{proét}} & \pi^\text{proét}(X_0)_K \\
\downarrow & & \downarrow \\
\pi^\text{ét}(X_K)_K & \xrightarrow{\text{sp}^\text{ét}} & \pi^\text{ét}(X_0)_K,
\end{array}
$$

We remark that for a more general $R$-scheme $X$ the hypothesis $X(R) \neq \emptyset$ is applicable upon replacing $R$ by a finite extension (cf. 1.19). Furthermore, the above commutative diagram comes from a refinement which is a 2-commutative diagram of Tannakian categories, and this refined result is valid without assuming either $X(R) \neq \emptyset$ or that $R$ has a separably closed residue field (cf. 1.16).

There are two main ingredients in E. Lavanda’s construction of $\text{sp}^\text{proét}$:

(A) A formula for the pro-étale fundamental group of a semi-stable curve [10, Theorem 1.17];

(B) The theory of meromorphic descent along a torsor under a torsion-free discrete group on a curve, developed by D. Gieseker in [5, Lemma 1].

Her construction of $\text{sp}^\text{proét}$ works only for semi-stable curves because of both (A) and (B). In this paper, we generalize (B) considerably by the following theorem:

**Theorem II** (cf. 1.11). If $q: \tilde{Y} \to \tilde{X}$ is a $G$-torsor, where $G$ is any discrete group, and if $\tilde{X}$ is Noetherian, then the pullback along $q$ induces an equivalence between $\text{Coh}(\tilde{X}) \otimes_R K$ and the category of sheaves in $\text{Coh}(\tilde{Y}) \otimes_R K$ equipped with meromorphic descent data.

Thanks to Theorem II we can directly prove Theorem I without using anything like (A). On the other hand, we also generalize (A) to much more general schemes, even though it is not needed in Theorem I. We consider the case when $X$ is a connected Nagata Noetherian J-2 scheme (e.g. a connected scheme of finite type over a field). The idea is to express the pro-étale fundamental group of $X$ in terms of the étale fundamental groups of the normalization of its irreducible components and the pro-étale fundamental group of its singular locus. Then using Noetherian induction one may understand the pro-étale fundamental group based on one’s knowledge on the étale fundamental groups.

**Theorem III** (cf. 2.20). Let $X$ be a connected Nagata Noetherian J-2 scheme. Assume that $X$ is irreducible and the singular locus $Z$ of the reduced induced structure of $X$ is connected. Let $\tilde{X}$ be the normalization of $X$, and let $\{Z_j\}_{1 \leq j \leq n}$ be the connected components of the inverse image of $Z$ under $\tilde{X} \to X$. Then $\pi^\text{proét}_1(X)$ is the coproduct of $\pi^\text{ét}(\tilde{X})$, $\pi^\text{ét}(Z, x)$, and a free discrete group of rank $n - 1$ on generators $x_2, \ldots, x_n$ quotient by the relations

$$
\phi_j(a) = x_j \psi_j(a) x_j^{-1}, \quad a \in \pi^\text{proét}_1(Z_j), 1 \leq j \leq n,
$$

where $\phi_j : \pi^\text{proét}_1(Z_j) \to \pi^\text{proét}_1(Z)$ and $\psi_j : \pi^\text{proét}_1(Z_j) \to \pi^\text{proét}_1(\tilde{X})$ are functorial maps, and $x_1 = 1$.

We have a similar description when $X$ has more irreducible components (cf. 2.20). If $Z$ has more connected components, we also have an inductive procedure to compute $\pi^\text{proét}_1(X)$ (cf. 2.24). As an example, E. Lavanda’s formula ([10, Theorem 1.17]) is a special case of the following:
Theorem IV (cf. 2.27). If $X$ is a connected scheme of finite type over a separably closed field $k$ and if the singular locus $Z$ is 0-dimensional, then $\pi^\text{proet}_1(X)$ is the coproduct of $\pi^\text{proet}_1(X_i)$, $1 \leq i \leq n$, and a free discrete group of rank $|f^{-1}(Z)| - |Z| - n + 1$, where $X_1, \ldots, X_n$ are the irreducible components of $X$, $f : \hat{X} \to X$ is the normalization map, and $|\cdot|$ denote the number of points of the underlying set.

As another application of Theorem III we prove that a continuous representation of $\pi^\text{proet}_1(X, x)$ factors through a discrete quotient iff its restrictions to the étale fundamental groups of the normalizations of the irreducible components of $X$ do so (cf. 2.31).

Notations and Conventions

1. For a topological group $\pi$ and a field $K$, $\text{Rep}_{K}^{\text{cts}}(\pi)$ denotes the category of finite dimensional continuous $K$-representations viewing $K$ as a discrete field (even if $K$ carries a natural non-discrete topology).

2. For a topological group $\pi$ and a field $K$, we denote by $\pi_{K} := \pi^{\text{cts}}$ the algebraic hull of $\pi$ over $K$ (cf. [10, Definition 2.2]). This is the Tannaka dual of the Tannakian category $\text{Rep}_{K}^{\text{cts}}(\pi)$ equipped with the forgetful fiber functor.

3. If $\pi$ is a profinite group, say $\pi = \varprojlim_{i \in I} \pi_i$, where each $\pi_i$ is a finite discrete group and the limit is taken in the category of topological groups, then $\pi_{K} = \varprojlim_{i \in I} (\pi_i)_K$, where the projective limit is taken in the category of affine $K$-group schemes. Since the association $\pi \mapsto \pi_{K}$ induces an equivalence between profinite groups and profinite constant $K$-group schemes, we will identify $\pi$ and $\pi_{K}$ without saying.

4. Let $X$ be a scheme over a field $K$ of positive characteristic. We denote by $F_{X/K} : X \to X^{(1)}$ the relative Frobenius map. For each $i \in \mathbb{N}$, we can define, inductively, the relative Frobenius $F_{X^{(i)}/K} : X^{(i)} \to X^{(i+1)}$. An $F$-divided sheaf is a sequence $(E_i, \sigma_i)_{i \in \mathbb{N}}$ where $E_i$ is a finitely presented $O_{X^{(i)}}$-module and $\sigma_i : F_{X^{(i)}/K}E_{i+1} \to E_i$ is $O_{X^{(i)}}$-linear isomorphism. We denote by $\text{Fdiv}(X)$ the category of $F$-divided sheaves on $X/K$. It’s worth to note that if $X$ is of finite type over $K$, then $E_i$ is actually a vector bundle on $X^{(i)}$ (see [16, §2.2.1], [19, Theorem I, (3)] for more details).

1. Meromorphic descent

1.1. Noetherian meromorphic descent.

Conventions 1.1. Let $R$ be an adic Noetherian ring, and let $I \subseteq R$ be an ideal of definition. Suppose that $I = (f)$. Let $\hat{K}$ be a localization of $R$ at the element $f$. Let $\hat{X}$ be an adic formal scheme ([6, 10.4.2, p. 407]) equipped with an adic morphism $\hat{X} \to \text{Spf}(R)$ ([6, 10.12.1, p. 436]). In this case, we simply call $\hat{X}$ a formal scheme over $\text{Spf}(R)$ or over $R$. We denote by $\text{Coh}(\hat{X})$ the category of sheaves of $O_{\hat{X}}$-modules on $\hat{X}$ whose pullbacks to the scheme $X_n := (\hat{X}, O_{\hat{X}}/I^{n+1})$ are finitely presented for each $n \in \mathbb{N}$. If $\hat{X}$ is a locally Noetherian formal scheme ([6, 10.4.2, p. 407]), then $\text{Coh}(\hat{X})$ is just the category of coherent sheaves on $\hat{X}$ ([17, 01XZ]).

Definition 1.2. We denote by $\text{Coh}^m(\hat{X})$ the category whose objects are exactly the same as those in $\text{Coh}(\hat{X})$, whose set of morphisms between two objects $\mathcal{F}, \mathcal{G}$ is

$$\Gamma(\hat{X}, \text{Hom}_{O_{\hat{X}}}(\mathcal{F}, \mathcal{G}) \otimes_R K)$$

where $\text{Hom}$ denotes the internal sheaf $\text{Hom}$.

Definition 1.3. Let $\mathcal{C}$ be an $R$-linear category. We define the category $\mathcal{C} \otimes_R K$ to the category whose objects are exactly the same as those in $\mathcal{C}$, whose set of morphisms between two objects $X, Y$ is $\text{Hom}_{\mathcal{C}}(X, Y) \otimes_R K$. If $\mathcal{C}$ is an Abelian category, then so is $\mathcal{C} \otimes_R K$ ([14, Lemma 1.5, p. 7]).

Definition 1.4. Let $q : \hat{Y} \to \hat{X}$ be an adic map of locally Noetherian formal schemes over $\text{Spf}(R)$. We define the category $\text{Coh}^m(\hat{Y}/\hat{X})$ of coherent sheaves with meromorphic descent data (MDD for short) to be the category whose objects are pairs $(\mathcal{F}, \phi)$, where $\mathcal{F}$ belongs to $\text{Coh}^m(\hat{Y})$ and $\phi : p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ is a map in $\text{Coh}^m(\hat{Y} \times_{\hat{X}} \hat{Y})$ which satisfies
• the identity condition: the pullback of $\phi$ along the diagonal map $\hat{Y} \to \hat{Y} \times \hat{X} \hat{Y}$ is the identity map in $\text{Coh}^m(\hat{Y})$;

• the cocycle condition: we have an equality

$$p_{23}^* \phi \circ p_{12}^* \phi = p_{13}^* \phi$$

in $\text{Coh}^m(\hat{Y} \times \hat{X} \hat{Y})$,

whose morphisms between two objects $(\mathcal{F}, \phi)$ and $(\mathcal{G}, \varphi)$ are those morphisms in

$$\text{Hom}_{\text{Coh}^m(\hat{Y})}(\mathcal{F}, \mathcal{G})$$

which are compactifiable with $\phi$ and $\varphi$.

**Theorem 1.5.** Let $q: \hat{Y} \to \hat{X}$ be an adic map of Noetherian formal schemes over $\text{Spf}(R)$. Then there is a canonical functor

$$\text{Coh}(\hat{X}) \otimes_R K \to \text{Coh}^m(\hat{Y}/\hat{X})$$

which is an equivalence if $q$ is faithfully flat.

**Proof.** The functor is simply the formation of the canonical descent datum ([17, 023D]). A proof of this theorem, which is due to O. Gabber, can be found in [13, (1.9), p. 774] (the result was stated there when $R$ is a complete DVR and $K = \text{Frac}(R)$, but the proof works in our setting).

**Example 1.6.** Theorem 1.5 would not be true if one replaces “Noetherian” by “locally Noetherian”. Let $R$ be a complete DVR with maximal ideal $I = (\pi)$ and quotient field $K$. Let $T$ (e.g. $R[x, y]/(xy)$) be a scheme over $R$ equipped with two disjoint opens $U_x, U_y$ (e.g. $U_x = (x \neq 0)$, $U_y = (y \neq 0)$) whose special fibers are not empty and $U_x \approx U_y$ as $R$-schemes. Let $\{T_n | n \in \mathbb{Z}\}$ be $\mathbb{Z}$-copies of $T$. We glue $U_{y_1} \subseteq T_n$ with $U_{x_{n+1}} \subseteq T_{n+1}$ for all $n \in \mathbb{Z}$ in the sense of [8, Chapter II, Exercise 2.12, p. 80]. Then we get a scheme $X$ with a Zariski-covering $\{T_i\}_{i \in \mathbb{Z}}$ such that

$$T_i \cap T_j = \begin{cases} \emptyset, & \text{if } i - j \neq \pm 1; \\ T_i = T_j, & \text{if } i = j; \\ U_{y_i} = U_{x_j}, & \text{if } j = i + 1; \end{cases}$$

Set $Y = \bigsqcup_{n \in \mathbb{Z}} T_n$, and let $q: \hat{Y} \to \hat{X}$ be the formal completion of the map $Y \to X$ along the special fiber. Now if we take $\mathcal{O}_Y$ on $\hat{Y}$ and glue the restrictions on $\hat{U}_x$ by the uniformizer $\pi$. The so obtained meromorphic descent data cannot descent to $\hat{X}$.

This shows that meromorphic descent may not be effective even for a Zariski covering with by infinitely many Zariski open sets. So the next result is somewhat surprising.

### 1.2. Meromorphic descent along $G$-torsors.

**Conventions 1.7.** Let $R$ be a complete DVR, and let $k$ be the residue field of $R$. Let $K$ be the quotient field. Let $\pi$ be a uniformizer of $R$, $\mathcal{S} = \text{Spec}(R)$, $\hat{S} = \text{Spf}(R)$. Let $\hat{X} \to \hat{S}$ be an adic morphism of Noetherian formal schemes. We denote $X_n$ the scheme given by the ringed space $(\hat{X}, \mathcal{O}_X/(\pi)^{n+1})$. Let $q: \hat{Y} \to \hat{X}$ be a torsor under a discrete group $G$, i.e. $q$ is a faithfully flat adic map of formal schemes over $\hat{S}$ and there is a group homomorphism $\rho: G \to \text{Aut}_X(\hat{Y})$ such that the induced map

$$\hat{Y} \times G \to \hat{Y} \times_X \hat{Y}$$

$$(y, w) \mapsto (y, \rho(w)(y))$$

is an isomorphism, where $\hat{Y} \times G := \bigsqcup_{w \in G} \hat{Y}$. Note that $\hat{Y}$ is a locally Noetherian formal scheme, but not Noetherian when $G$ is infinite. We denote by $Y_n$ the scheme given by the ringed space $(\hat{Y}, \mathcal{O}_Y/(\pi)^{n+1})$.

By using the isomorphism $\hat{Y} \times_X \hat{Y} \simeq \hat{Y} \times G$, it is evident that $\text{Coh}^m(\hat{Y}/\hat{X})$ is equivalent to the category $\text{Coh}^m(\hat{Y}/\hat{X}, G)$, which we define now:
**Definition 1.8.** The category $\text{Coh}^m(\hat{Y}/\hat{X}, G)$ of coherent sheaves with MDD is the category whose objects are pair $(\mathcal{F}, \{h_w\}_{w \in G})$ where $\mathcal{F}$ is an object in $\text{Coh}^m(\hat{Y})$ and $\{h_w\}_{w \in W}$ is a collection of elements such that $$h_w \in \text{Hom}_{\text{Coh}^m(\hat{Y})}(\mathcal{F}, w^* \mathcal{F}) = \Gamma(\hat{Y}, \text{Hom}_{\mathcal{O}_G}(\mathcal{F}, w^* \mathcal{F}) \otimes_R K)$$ satisfying the “identity condition” and the “cocycle condition”:
- the identity condition: $h_e = \text{id}$ for the unit $e \in G$;
- the cocycle condition: we have an equality $$w^* (h_{w'}) \circ h_w = h_{ww'}$$ for all $w, w' \in G$.

A map $$(\mathcal{F}, \{h_w\}_{w \in G}) \longrightarrow (\mathcal{F}', \{h'_w\}_{w \in G})$$ between two coherent sheaves with MDD is an element in $\text{Hom}_{\text{Coh}^m(\hat{Y})}(\mathcal{F}, \mathcal{F}') = \Gamma(\hat{Y}, \text{Hom}_{\mathcal{O}_G}(\mathcal{F}, \mathcal{F}') \otimes_R K)$ which is compatible with $\{h_w\}_{w \in G}$ and $\{h'_w\}_{w \in G}$ in the obvious sense.

**Definition 1.9.** Let $(\mathcal{F}, \{h_w\}_{w \in G})$ be a coherent sheaf with MDD on $\hat{Y}$. We say that it is genuine if for all $w \in G$, the element $h_w$ lies in the subset $\text{Hom}_{\text{Coh}^m(\hat{Y})}(\mathcal{F}, w^* \mathcal{F})$ of $\text{Hom}_{\text{Coh}^m(\hat{Y})}(\mathcal{F}, w^* \mathcal{F})$.

**Lemma 1.10.** Any object in $\text{Coh}^m(\hat{Y}/\hat{X}, G)$ is isomorphic to a genuine one.

**Proof.** Suppose that $(\mathcal{F}, \{h_w\}_{w \in G})$ is an object in $\text{Coh}^m(\hat{Y}/\hat{X}, G)$. Let $\mathcal{G}$ be the image of $\mathcal{F}$ in $\mathcal{F} \otimes_R K$. Then we have $\mathcal{F} \otimes_R K \cong \mathcal{G} \otimes_R K$. Thus replacing $\mathcal{F}$ by $\mathcal{G}$ we may assume that $\mathcal{F}$ is $K$-torsion free. Now we claim that there is a coherent subsheaf $\mathcal{F}' \subseteq \mathcal{F} \otimes_R K$ such that $\mathcal{F}' \otimes_R K = \mathcal{F} \otimes_R K$ and $h_w$ sends $\mathcal{F}' \subseteq \mathcal{F} \otimes_R K$ to $w^* \mathcal{F}' \subseteq w^* \mathcal{F} \otimes_R K = w^* \mathcal{F} \otimes_R K$. Evidently, the lemma follows from this claim.

We will find $\mathcal{F}'$ using Noetherian induction on $\hat{X}$. The key is the induction step: if the claim is true when $\hat{Y}/\hat{X}$ is replaced by $q^{-1}(U)/U$ for some open $U$ of $\hat{X}$, and $U \neq \hat{X}$, then there exists an open $U'$ such that $U \subseteq U' \subseteq \hat{X}$ and the claim is true when $\hat{Y}/\hat{X}$ is replaced by $q^{-1}(U')/U'$.

Let $Z = \hat{X} \setminus U$. We regard $Z$ as a closed subscheme of $X_0$ with the reduced induced structure. Let $\eta \in Z$ be a generic point of $Z$. Since $\pi^{\text{pro-ét}}_1(\text{Spec}(\kappa(\eta))) = \pi^{\text{ét}}_1(\text{Spec}(\kappa(\eta)))$ is profinite, the $G$-torsor $\hat{Y}|_{\text{Spec}(\kappa(\eta))}$ obtained by pulling back $\hat{Y}$ through $\text{Spec}(\kappa(\eta)) \hookrightarrow Z \hookrightarrow X_0 \hookrightarrow \hat{X}$ is induced from an $H$-torsor over $\text{Spec}(\kappa(\eta))$ with $H \subseteq G$ finite. Spreading it out we find $V \subseteq Z$, an open neighborhood of $x$, and an $H$-torsor $\tilde{Z} \rightarrow V$ which induces $\hat{Y}|_V$. Thus $\tilde{Z}$ is an open in $\hat{Y}|_V = q^{-1}(V)$ and $\tilde{Y}|_V$ is the disjoint union of $w\tilde{Z}$ over $w \in G/H$.

We can find an $H$-invariant quasi-compact open subset $W$ of $Y_0$ such that $W \cap q^{-1}(Z) = \tilde{Z}$. Consider $W$ as an open formal subscheme of $\tilde{Y}$. We can convert the MDD on $\mathcal{F}$ in the sense of 1.8 to an MDD $\phi$ in the sense of 1.4, and restrict it to get an MDD for $W \rightarrow q(W)$. By Gabber’s theorem 1.5, there is a coherent sheaf $\mathcal{E}$ on $q(W)$ such that $(q|_W)^* \mathcal{E}$ is isomorphic to $(\mathcal{F}|_W, \phi)$ in $\text{Coh}^m(W/q(W))$. We may and do assume that the isomorphism is given by a morphism in $\text{Hom}_{\text{Coh}^m(W)}((q|_W)^* \mathcal{E}, \mathcal{F}|_W)$. Let $\mathcal{E}'$ be the image of this morphism.

Let $\mathcal{F}'$ be the coherent sheaf on $q^{-1}(U)$ asserted by the induction hypothesis. We now use $\mathcal{F}'$ on $q^{-1}(U)$ and $\mathcal{E}'$ on $W$ to form a sheaf $\mathcal{F}''$ on $q^{-1}(U')$ extending $\mathcal{F}'$, where $U' := U \cup q(W) = U \cup \tilde{Z}$, using a method of Gieseker and Raynaud, as follows. We first try to extend it to $q^{-1}(U) \cup W = q^{-1}(U) \cup \tilde{Z}$. On the open set $q^{-1}(U) \cap W$, the restriction of $\mathcal{F}, \mathcal{F}', \mathcal{E}'$ become the same upon tensoring with $K$. Therefore, for $n$ large enough we have $$\pi^n \mathcal{E}' \subseteq \mathcal{F}' \subseteq \pi^{-n} \mathcal{E}'$$ on $q^{-1}(U) \cap W$. 

This implies
\[ \mathcal{F}'/\pi^n\mathcal{E}' \subset \pi^n\mathcal{E}'/\pi^n\mathcal{E}' \quad \text{on } q^{-1}(U) \cap W. \]
On the other hand, the right-hand side is obviously the restriction of a coherent sheaf on \( W_{2n} \)
(by which we mean \( W \) regarded as an open subscheme of \( Y_{2n} \)). By [17, 01PD], the left-hand side
extends to a coherent sheaf \( \mathcal{G} \) on \( W \) such that \( \mathcal{G} \) is a subsheaf of \( \pi^n\mathcal{E}'/\pi^n\mathcal{E}' \). Let \( \mathcal{E}'' \) be the
subsheaf of \( \pi^n\mathcal{E}'/\pi^n\mathcal{E}' \) such that \( \mathcal{E}''/\pi^n\mathcal{E}' = \mathcal{G} \). By construction, \( \mathcal{E}'' = \mathcal{F}' \) on \( q^{-1}(U) \cap W \).
Since both \( \mathcal{E}', \mathcal{F}' \) are \( H \)-invariant and the extension in [17, 01PD] is functorial, \( \mathcal{E}'' \) is also \( H \)-invariant.

We now define a sheaf \( \mathcal{F}'' \) on
\[ q^{-1}(U') = q^{-1}(U) \cup \bigcup_{w \in G} w^{-1}W \]
as follows: it is the subsheaf of \( \mathcal{F} \otimes_R K \) generated by sections of \( \mathcal{F}' \) and sections of \( h_w^{-1}(w^*\mathcal{E}'') \) (a sheaf on \( w^{-1}W \)) for all \( w \in G \). Thus a local section of \( \mathcal{F} \otimes_R K \) is in \( \mathcal{F}'' \) iff its stalk at each point
is a sum of stalks of some local sections of \( \mathcal{F}' \) and \( h_w^{-1}(w^*\mathcal{E}'') \). We claim that \( \mathcal{F}'' \) fulfills all the
conditions required by the induction step.

From the above definition and the induction hypothesis, it is easy to see that \( \mathcal{F}'' \) = \( \mathcal{F}' \) for all \( y \in q^{-1}(U) \). For \( y \in q^{-1}(U') \setminus q^{-1}(U) \), \( \mathcal{F}'' = \sum h_w^{-1}(\mathcal{E}'') \), where the sum is over \( \{(w, z) : w \in G, z \in \tilde{Z}, w^{-1}z = y\} \), a transitive \( H \)-set. It follows that for any \( w \in G \),
\[ \mathcal{F}''|_{w^{-1}W} = \sum_{j \in H} h_{jw}^{-1}(jw)^*\mathcal{E}'') = h_w^{-1}(w^*\mathcal{E}'') \]
where the second equality follows from the fact that both \( W \) and \( \mathcal{E}'' \) are \( H \)-invariant. We conclude that \( \mathcal{F}'' \) is coherent on \( q^{-1}(U') \).

It remains to verify that \( h_u \) maps \( \mathcal{F}'' \) to \( u^*\mathcal{F}'' \) for each \( u \in G \). It suffices to check this over \( q^{-1}(U) \) and \( w^{-1}W \) for all \( w \in G \). The case of \( q^{-1}(U) \) is clear. Over \( w^{-1}W \), we have the following
commutative diagrams:

\[
\begin{array}{ccc}
w^{-1}W & \xrightarrow{w} & W \\
\uparrow{w^{-1}} & & \uparrow{w} \\
uw^{-1}W & \xrightarrow{u} & W \\
\end{array}
\begin{array}{ccc}
w^{-1}W & \xrightarrow{h_w^{-1}} & \mathcal{E}'' \\
\downarrow{w^{-1}} & & \downarrow{w} \\
h_w^{-1}(w^*\mathcal{E}'') & \xrightarrow{h_w} & w^*\mathcal{E}'' \\
\end{array}
\]
due to the cocycle condition \( h_w = (u^*h_{uw^{-1}}) \circ h_u \). Thus \( h_u \) maps \( \mathcal{F}''|_{w^{-1}W} \) to \( u^*(\mathcal{F}''|_{w^{-1}W}) \)
which is exactly \( (u^*\mathcal{F}'')|_{w^{-1}W} \). The lemma is proved completely.

**Theorem 1.11.** The pullback functor
\[ \text{Coh}(\tilde{X}) \otimes_R K \longrightarrow \text{Coh}^m(\tilde{Y}/\tilde{X}, G) \]
is an equivalence.

**Proof.** It is enough to show that the functor is fully faithful. Then the statement follows readily
from 1.10. Now suppose we are given two \( R \)-torsion free sheaves \( \mathcal{F}, \mathcal{G} \in \text{Coh}(\tilde{X}) \). Set \( \text{Coh}^m(q, G) :=
\text{Coh}^m(\tilde{Y}/\tilde{X}, G) \).

By fpqc-descent of quasi-coherent sheaves, we have an exact sequence of \( R \)-torsion free modules:
\[ 0 \longrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}_{\mathcal{O}_Y}(q^*\mathcal{F}, q^*\mathcal{G}) \longrightarrow \prod_{w \in G} \text{Hom}_{\mathcal{O}_Y}(q^*\mathcal{F}, q^*\mathcal{G}) \]
where the second map is the pullback, and the third map is \( w^* - \text{id}^* \). Note that the natural map
\[ (\prod_{w \in G} \text{Hom}_{\mathcal{O}_Y}(q^*\mathcal{F}, q^*\mathcal{G})) \otimes_R K \longrightarrow \prod_{w \in G} (\text{Hom}_{\mathcal{O}_Y}(q^*\mathcal{F}, q^*\mathcal{G}) \otimes_R K) \]
is injective, so we get an exact sequence:
\[
(1) \ 0 \longrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_R K \longrightarrow \text{Hom}_{\mathcal{O}_Y}(q^*\mathcal{F}, q^*\mathcal{G}) \otimes_R K \longrightarrow \prod_{w \in G} (\text{Hom}_{\mathcal{O}_Y}(q^*\mathcal{F}, q^*\mathcal{G}) \otimes_R K)
\]
By the very definition of the category \( \text{Coh}^m(q, G) \), we have an exact sequence

\[
0 \rightarrow \text{Hom}_{\text{Coh}^m(q, G)}(q^*F, q^*G) \rightarrow \Gamma(Y, \text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K) \\
\rightarrow \prod_{w \in G} \Gamma(Y, \text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K)
\]

(2)

where the last map is again given by \( w^* - \text{id}_* \). The inclusion

\[
\text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K \subseteq \Gamma(Y, \text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K)
\]

which identifies \( \text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K \) as the subspace of elements \( s \) of the right hand side such that \( \pi^n s \) is contained in

\[
\text{Hom}_{O_Y}(q^*F, q^*G) \subseteq \Gamma(Y, \text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K)
\]

for some \( n \in \mathbb{N} \), induces a map \((1) \Rightarrow (2)\). We have to show that the kernel part of \((1) \Rightarrow (2)\) namely, the map

\[
\text{Hom}_{O_Y}(F, G) \otimes_R K \rightarrow \text{Hom}_{\text{Coh}^m(q, G)}(q^*F, q^*G)
\]

is an isomorphism.

To show this, we just have to show that if

\[
s \in \Gamma(Y, \text{Hom}_{O_Y}(q^*F, q^*G) \otimes_R K)
\]

such that \( w^* s = s \) for all \( w \in G \), then there exist \( n \) large such that \( \pi^n s \in \text{Hom}_{O_Y}(q^*F, q^*G) \).

Let \( U \rightarrow X_0 \) be a quasi-compact étale map which trivializes the \( G \)-torsor \( Y_0 \rightarrow X_0 \). Then the image \( V \) of \( U \rightarrow Y_0 \) is a quasi-compact open whose \( G \)-translates \( \{V_w\}_{w \in G} \) cover \( Y_0 \). Since \( V \) is quasi-compact, \( \pi^n s|_V \in \text{Hom}_{O_{Y_0}}(q^*F|_V, q^*G|_V) \). Now the condition \( w^* s = s \) implies that \( \pi^n s|_{V_w} \in \text{Hom}_{O_{Y_0}}(q^*F|_{V_w}, q^*G|_{V_w}) \) for all \( w \in G \). Thus \( \pi^n s \in \text{Hom}_{O_Y}(q^*F, q^*G) \).

\[ \square \]

**Remark 1.12.** The above pullback functor is functorial in the following sense. If there is a Cartesian diagram of locally Noetherian formal schemes over \( \text{Spf}(R) \)

\[
\begin{array}{ccc}
\hat{Y}' & \rightarrow & \hat{Y} \\
\downarrow & & \downarrow \\
Y' & \rightarrow & Y
\end{array}
\]

where \( q, q' \) are \( G \)-torsors, and if \( F \) is a coherent sheaf on \( X \) and \( (q^*F, \{h_w\}_{w \in G}) \) is the corresponding pair, then the pair corresponding to \( \alpha^*F \) is \( (\beta^*q^*F, \{\beta^*h_w\}_{w \in G}) \).

1.3. The specialization functor. We are in Setting 1.7 except that here we assume that \( R \) is of characteristic \( p \). Let \( X \) be a proper scheme over \( \text{Spec}(R) \), and let \( \hat{X} \) be the formal scheme associated with the special fiber of \( X \rightarrow \text{Spec}(R) \). Suppose that the special fiber \( X_0 \) is connected and \( \xi \in X_0 \) is a geometric point. Set \( X_K = X \times_{\text{Spec}(R)} \text{Spec}(K) \).

**Theorem 1.13.** There is a natural equivalence of categories:

\[
\text{Coh}(\hat{X}) \otimes_R K \rightarrow \text{Coh}(X_K)
\]

which is additive, monoidal and exact.

**Proof.** By Grothendieck’s existence theorem we have an additive, monoidal and exact equivalence

\[
\text{Coh}(X) \otimes_R K \xrightarrow{\approx} \text{Coh}(\hat{X}) \otimes_R K
\]

One the other hand, since \( X \) is Noetherian the natural pullback functor

\[
\text{Coh}(X) \otimes_R K \rightarrow \text{Coh}(X_K)
\]

is an equivalence. This completes the proof. \( \square \)
Construction 1.14. Let $\hat{X}^{(i)}$ denote the formal scheme obtained by the following Cartesian diagram

$$
\begin{array}{ccc}
\hat{X}^{(1)} & \longrightarrow & \hat{X} \\
\downarrow & & \downarrow \\
\text{Spf}(R) & \longrightarrow & \text{Spf}(R)
\end{array}
$$

where $F_{\text{Spf}(R)}$ denotes the absolute Frobenius of $\text{Spf}(R)$. The universal property of pullback diagrams provides a map $\hat{X} \to \hat{X}^{(1)}$ over $\text{Spf}(R)$. In other words, $\hat{X} \to \hat{X}^{(1)}$ is the relative Frobenius of $\hat{X}/\text{Spf}(R)$. Then we can define the relative Frobenius maps $F_{\hat{X}^{(i)}/\text{Spf}(R)}: \hat{X}^{(i)} \to \hat{X}^{(i+1)}$ of $\hat{X}^{(i)}/\text{Spf}(R)$ inductively for each $i \in \mathbb{N}$. Similarly, we have relative Frobenius maps $F_{X_K^{(i)}/K}: X_K^{(i)} \to X_K^{(i+1)}$ of $X_K^{(i)}/K$.

We want to construct the specialization functor

$$s_{\text{proét}}^K: \text{Rep}^\text{proét}_K(X_0, \xi) \longrightarrow \text{Fdiv}(X_K)$$

(cf. Notations and Conventions (1), (4)).

Given a continuous representation

$$\rho: \pi_1^\text{proét}(X_0, \xi) \to \text{GL}(V)$$

where $V$ is a finite dimensional $K$-vector space, we take $G \subseteq \text{GL}(V)$ the image of $\rho$. By [3, Lemma 7.4.6], we get a $G$-torsor $q_0: Y_0 \to X_0$. Extending $q_0$ to higher thickenings we get a $G$-torsor $q: \hat{Y} \to \hat{X}$.

Now $V \otimes_R \mathcal{O}_Y$ equipped with the $G$-action is an object in $\text{Coh}^m(\hat{Y}/\hat{X})$, so it corresponds, via 1.11 and 1.13, to a coherent sheaf $E_0$ on $X_K$. Moreover, we have commutative diagrams

$$
\begin{array}{ccc}
\hat{Y}^{(i)} & \longrightarrow & \hat{Y}^{(i+1)} \\
\downarrow \rho^{(i)} & & \downarrow \rho^{(i+1)} \\
\hat{X}^{(i)} & \longrightarrow & \hat{X}^{(i+1)}
\end{array}
$$

which are Cartesian because $\hat{Y}^{(i)} \to \hat{X}^{(i)}$ are étale. Each $q^{(i)}$ will produce a coherent sheaf $E_i$, and 1.12 guarantees the isomorphism $\sigma_i$. In this way, we get an object $(E_i, \sigma_i)_{i \in \mathbb{N}} \in \text{Fdiv}(X_K)$.

Note that instead of taking $G$ to be the image of $\rho$ we can also take any discrete quotient

$$\pi_1^\text{proét}(X_0, \xi) \to G \to \text{GL}(V)$$

in the middle of $\rho$. The object $(E_i, \sigma_i)_{i \in \mathbb{N}} \in \text{Fdiv}(X_K)$ obtained in this way will be canonically isomorphic to the one we constructed above. Using this we see that our construction is indeed functorial, so it defines the desired functor $s_{\text{proét}}^K$.

Construction 1.15. Let’s consider the construction of the functor

$$s_{\text{ét}}^K: \text{Rep}^\text{ét}_K(\pi_1^\text{ét}(X_0, \xi)) \longrightarrow \text{Fdiv}(X_K)$$

which is a folklore.

Given a finite dimensional continuous $K$-representation $\rho: \pi_1^\text{ét}(X_0, \xi) \to \text{GL}(V)$, let $G \subseteq \text{GL}(V)$ be the image of $\rho$. Then $G$ is a finite group and the surjection

$$\pi_1^\text{ét}(X_0, \xi) \to G$$

corresponds to a pointed $G$-torsor $q_0: Y_0 \to X_0$ which extends automatically to a $G$-torsor $q: \hat{Y} \to \hat{X}$. By Grothendieck’s existence theorem we get a (unique) $G$-torsor $q: Y \to X$ whose formal completion along $X_0 \to X$ is $q$. The general fiber of $q$ is a $G$-torsor $q_K: Y_K \to X_K$ which extends to cartesian diagrams:

$$
\begin{array}{ccc}
Y_K^{(i)} & \longrightarrow & Y_K^{(i+1)} \\
\downarrow q_K^{(i)} & & \downarrow q_K^{(i+1)} \\
X_K^{(i)} & \longrightarrow & X_K^{(i+1)}
\end{array}
$$
Now we apply fpqc-descent of quasi-coherent sheaves to descent the $G$-sheaf $O_{X^{(i)}_K} \otimes_K V$ to a coherent sheaf $E_i$ over $X^{(i)}_K$ along the $G$-torsor $q^{(i)}$. We also have isomorphisms $\sigma_i: F^{et}_{X^{(i)}_K/E_{i+1}} \rightarrow E_i$ given by (3) and the functoriality of the fpqc-descent of quasi-coherent sheaves. This defines an object $(E_i, \sigma_i)_{i \in \mathbb{N}} \in \text{Fdiv}(X_K)$. Just as 1.14, the construction is functorial, so it defines the functor

$$sp^{K}_{et}: \text{Rep}^{cts}_{K}(\pi_1^{et}(X_0, \xi)) \rightarrow \text{Fdiv}(X_K).$$

**Theorem 1.16.** The following natural diagram of categories

$$\begin{array}{ccc}
\text{Rep}^{cts}_{K}(\pi_1^{et}(X_0, \xi)) & \longrightarrow & \text{Fdiv}(X_K) \\
\downarrow & & \downarrow \text{sp}^{K}_{et} \\
\text{Rep}^{cts}_{K}(\pi_1^{proet}(X_0, \xi)) & \longrightarrow & \text{Fdiv}(X_K)
\end{array}$$

is 2-commutative.

**Proof.** The category $\text{Rep}^{cts}_{K}(\pi_1^{et}(X_0, \xi))$ is the full subcategory of $\text{Rep}^{cts}_{K}(\pi_1^{proet}(X_0, \xi))$ consisting of representations whose monodromy groups are finite. Suppose in 1.14 the image $G \subseteq \text{GL}(V)$ is finite. Then the $G$-torsor $q: \tilde{Y} \rightarrow \tilde{X}$ is algebraizable, so there exists a unique $G$-torsor $q: \tilde{X} \rightarrow X$ whose formal completion is $q$. To finish the proof one just has to notice that if we identify Coh$(\tilde{X}) \otimes_R K$ (resp. Coh$(\tilde{Y}) \otimes_R K$) with Coh$(X_K)$ (resp. Coh$(Y_K)$) via 1.13, then the meromorphic descent in 1.11 is nothing but the fpqc-descent. \hfill \Box

### 1.4. The specialization map of fundamental groups

We resume the notations and conventions in the previous subsection. We assume, in addition, that $X$ and $X_K$ are connected, and that $\eta \in X_K$ is a geometric point. These additional assumptions are just to make sense of the fundamental groups of $X$ and $X_K$.

Recall that we have a specialization map of the étale fundamental groups

$$\pi_1^{et}(X, \eta) \longrightarrow \pi_1^{et}(X, \xi) \cong \pi_1^{et}(X_0, \xi) \leftarrow \pi_1^{et}(X_0, \xi)$$

which is defined up to a choice of a path from $\eta$ to $\xi$. We want to recover this map from the specialization functor $sp^K_{et}$ (cf. 1.15), which does not depend on $\eta$.

Suppose $\eta$ comes from a rational point $\eta_0$, i.e. there is a factorization

$$\eta: \text{Spec}(\bar{K}) \rightarrow \text{Spec}(K) \rightarrow X_K$$

The pro-constant quotient $\pi^G(X_K/K, \eta_0)$ of the Nori-étale fundamental group $\pi^E(X_K/K, \eta_0)$ (cf. [20], Definition 4.5) or the gerbe version [19, Definition 4.1]) is the quotient of the étale fundamental group $\pi_1^{et}(X, \eta) = \pi_1^{et}(X_K/K, \eta)$ classifying finite étale covers equipped with a $K$-rational point over $\eta_0$. We have fully faithful inclusions:

$$\text{Rep}^{cts}_{K}(\pi^G(X_K/K, \eta_0)) \subseteq \text{Rep}_{K}(\pi^E(X_K/K, \eta_0)) \subseteq \text{Fdiv}(X_K)$$

where the first inclusion corresponds to the pro-constant quotient map, and the second inclusion is obtained by taking the essentially finite objects of $\text{Fdiv}(X_K)$ (cf. [16, Corollary 12, Proposition 13],[19, Theorem 5.8, Theorem 6.23]). The category $\text{Fdiv}(X_K)$ is Tannakian (see [19, Theorem I (3)]). The rational point $\eta_0$ provides a neutral fiber functor

$$ev_{\eta_0}: \text{Fdiv}(X_K) \longrightarrow \text{Vect}_K$$

$$(E_i, \sigma_i)_{i \in \mathbb{N}} \mapsto \eta_0^* E_0$$

for $\text{Fdiv}(X_K)$. We denote $\pi^{Fdiv}_1(X_K, \eta_0)$ the affine $K$-group scheme corresponding to the neutral Tannakian category $(\text{Fdiv}(X_K), ev_{\eta_0})$. 
1.4.1. When $X$ has an $R$-rational point.

**Proposition 1.17.** Suppose that the residue field $k$ of $R$ is separably closed, and there is a $R$-rational point $(\xi, \eta_0) \in X(R)$. Then the functor $\text{sp}^K_1$ factors as

$$\text{Rep}^{\text{cts}}_K(\pi_1^{\text{et}}(X_0, \xi)) \to \text{Rep}^{\text{cts}}_K(\pi_1^G(X_K/K, \eta_0)) \subseteq \text{Fdiv}(X_K)$$

which induces maps of neutral Tannakian categories

$$\text{(5)} \quad (\text{Rep}^{\text{cts}}_K(\pi_1^{\text{et}}(X_0, \xi), F_\xi)) \to (\text{Rep}^{\text{cts}}_K(\pi_1^G(X_K/K, \eta_0), F_{\eta_0})) \to (\text{Fdiv}(X_K), \text{ev}_{\eta_0})$$

where $F_\xi$ and $F_{\eta_0}$ denote the forgetful functor to $\text{Vect}_K$. By taking Tannakian dual we get homomorphisms of $K$-group schemes

$$\pi_1^{\text{et}}(X_K, \eta) \to \pi_1^G(X_K/K, \eta_0) \to \pi_1^{\text{et}}(X_0, \xi).$$

By composing the second map with the quotient map $\pi_1^{\text{et}}(X_K, \eta) \to \pi_1^G(X_K/K, \eta_0)$ one recovers the specialization map (4)

$$\pi_1^{\text{et}}(X_K, \eta) \to \pi_1^G(X_K/K, \eta_0) \to \pi_1^{\text{et}}(X_0, \xi).$$

Moreover, there is a canonical path between the geometric points $\xi$ and $\eta$: $\text{Spec}(\overline{K}) \to \text{Spec}(K) \xrightarrow{\eta} X$.

**Proof.** Let’s revisit 1.15. Consider the pointed $G$-torsor $q_0$ corresponding to $\pi_1^{\text{et}}(X_0, \xi) \to G$

$$\text{Spec}(k) \xrightarrow{\xi} \text{Spec}(K) \xrightarrow{q_0} X_0 \xrightarrow{y_0} Y_0 \xrightarrow{q_0} Y_0$$

Since $R$ is strictly Henselian, all the finite étale covers split completely. Thus the geometric point $y_0 \in Y_0 \subseteq Y$ determines a unique $R$-rational point $(y_0, y_1)$ of $Y$.

So we get the pointed torsor

$$\text{Spec}(R) \xrightarrow{(\xi, \eta_0)} \text{Spec}(K) \xrightarrow{\eta_0} X_K \xrightarrow{\eta_0} X_K$$

Since the pointed torsor (7) is pointed by a $K$-rational point, the map $\rho_K: \pi_1^{\text{et}}(X_K, \eta) \to G$ corresponding to the it factors as

$$\pi_1^{\text{et}}(X_K, \eta) \to \pi_1^G(X_K/K, \eta_0) \to G$$

Continuing the algorithms in 1.15 we get the desired factorization of $\text{sp}^K_1$.

To check (5) one first notice that $q_1^K E_0 = \mathcal{O}_{Y_K} \otimes_K V$. Thus we have

$$F_\xi(\rho) = F_{\eta_0}(\rho_K) = V = y_1^\#(\mathcal{O}_{Y_K} \otimes_K V) = y_1^\#q_1^K E_0 = \eta_0^\# E_0 = \text{ev}_{\eta_0}((E_i, \sigma_i)_{i \in \mathbb{N}}).$$

Since all the identifications are functorial, we get (5).

Finally, the canonical path $q^{-1}(\xi) \to q^{-1}(\eta)$ is given functorially by

$$y_0 \mapsto (y_0, y_1) \mapsto (\tilde{y}_1: \text{Spec}(\overline{K}) \to \text{Spec}(K) \xrightarrow{y_1} Y_K \subseteq Y)$$

$\square$
Corollary 1.18. Suppose that the residue field \( k \) of \( R \) is separably closed, and there is an \( R \)-rational point \( (\xi, \eta_0) \in X(R) \). Then the functor \( \text{sp}_1^R \) induces a specialization map
\[
\pi_1^{\text{Fdiv}}(X_K, \eta_0) \rightarrow (\pi_1^\text{proét}(X_0, \zeta))_K
\]
which recovers (4).

Proof. From 1.14 we get a \( K \)-linear tensorial exact functor \( \text{sp}_1^R \) of Tannakian categories. The \( R \)-rational point \( (\xi, \eta_0) \) provides a \( K \)-rational point \( \eta_0 \) for \( X_K \) and therefore a neutral fiber functor \( \text{ev}_{\eta_0} \) for \( \text{Fdiv}(X_K) \). To get (8), we just have to show that \( \text{ev}_{\eta_0} \) is compatible with the forgetful functor \( F_\zeta \) of \( \text{Rep}_K^\text{proét}(X_0, \xi) \) under \( \text{sp}_1^R \).

Let’s revisit 1.14. The \( R \)-rational point \( (\xi, \eta_0) \) induces a \( \text{Spf}(R) \)-rational point \( \zeta \) of \( \hat{X} \). Consider the pointed \( G \)-torsor \( q_0 \) corresponding to \( \pi_1^\text{proét}(X_0, \xi) \rightarrow G \)
\[
\text{Spec}(k) \xrightarrow{\xi} X_0 \xrightarrow{q_0} Y_0
\]

Since \( R \) is strictly Henselian, there is a unique (hence functorial) lift \( \lambda \) of \( \zeta \) along \( q \).

\[
\text{Spf}(R) \xrightarrow{\zeta} \hat{X} \xleftarrow{q} \hat{Y}
\]

Suppose \( \mathcal{E}_0 \) is the sheaf in \( \text{Coh}(\hat{X}) \otimes_R K \) obtained by applying meromorphic descent 1.11 to \( V \otimes_R \mathcal{O}_{\hat{Y}} \). Since \( \mathcal{E}_0 \) corresponds to \( E_0 \) via 1.13, \( \xi^* \mathcal{E}_0 \in \text{Coh}(\text{Spf}(R)) \otimes_R K \) corresponds to \( \eta_0^* E_0 \in \text{Vect}_K \) by naturality of 1.13. But
\[
\xi^* \mathcal{E}_0 = \lambda^* (q^* \mathcal{E}_0) = \lambda^* (V \otimes_R \mathcal{O}_{\hat{Y}}) = V \otimes_R \mathcal{O}_{\text{Spf}(R)}
\]

corresponds to \( V \) via 1.13. Thus \( \text{ev}_{\eta_0}((E_i, \sigma_i)_{i \in \mathbb{N}}) = \eta_0^* E_0 = V \) as desired.

The last statement is nothing but a combination of 1.17 and 1.16. \( \square \)

1.4.2. When \( X \) does not have an \( R \)-rational point. We see that if \( X \) admits an \( R \)-rational point, then \( \text{sp}_1^R \) induces a map of fundamental group schemes (8) recovers the classical specialization map of fundamental groups (4). What if \( X \) has no \( R \)-rational point?

Suppose the residue field \( k \) of \( R \) is separably closed, and \( X \) is flat over \( R \). By 1.21 there is a finite ring extension \( R \rightarrow R' \) where \( R' \) is a complete DVR such that \( X(R') \neq \emptyset \). Let \( (\xi, \eta_0) \in X(R') \), and let \( k' \) (resp. \( K' \)) denotes the residue field (resp. function field) of \( R' \). Set \( X' := X \otimes_R R' \). Then \( X'_0 = X_0 \otimes_k k' \) and \( X'_K = X_K \otimes_K K' \). Since \( k'/k \) is a finite purely inseparable extension, we have
\[
\pi_1^{\text{et}}(X_0, \xi) = \pi_1^{\text{et}}(X_0 \otimes_k k', \xi) \quad \text{and} \quad \pi_1^\text{proét}(X_0, \xi) = \pi_1^\text{proét}(X_0 \otimes_k k', \xi).
\]

In this case, we have

Corollary 1.19. Suppose the residue field \( k \) of \( R \) is separably closed, and \( X \) is flat and geometrically connected over \( R \). Then a homomorphism of \( K' \)-group schemes:
\[
\pi_1^{\text{Fdiv}}(X'_K, \eta_0) \rightarrow (\pi_1^\text{proét}(X_0, \xi))_{K'}
\]
which recovers (4) for \( X'/R' \).

Proof. We apply 1.18 to the \( R' \)-scheme \( X' \). \( \square \)

Lemma 1.20. Let \( A \) be a flat finitely generated \( R \)-algebra. Suppose that \( A \) is an integral domain, and \( P \) is a maximal ideal of \( A \) containing the uniformizer \( \pi \) of \( R \). If \( AP[\frac{1}{\pi}] \) is a field, then \( A[\frac{1}{\pi}] \) is also a field.
Proof. The conditions imply that if $Q \subseteq P$ is a non-zero prime ideal of $A$, then $\pi \in Q$. Consider the minimal prime ideals of the ideal $(\pi)$ in $A_p$. If $PA_P$ was not one of them, then there would be $a \in PA_P$ such that $a$ is not contained in any of those minimal prime ideals of $(\pi)$. By Krull principal ideal theorem each minimal prime ideal $I$ of $(a)$ has height 1, but since it is non-zero, $\pi \in I$. Thus $I$ has to be also a minimal prime ideal of $(\pi)$ - a contradiction! So $PA_P$ is a minimal prime ideal of $(\pi)$. This implies that $\text{ht}(P) = 1$, so by [11, 14.C, Theorem 23, p. 84] and the fact that $R$ is universally catenary, the function field of $A[\frac{1}{g}]$ is finite over $K$, hence $A[\frac{1}{g}]$ a field. □

Lemma 1.21. If $X$ is flat and locally of finite type over a complete DVR $R$, then for any closed point $\xi \in X_0$ we can find a complete DVR $R'$ which is finite flat over $R$ and an $R$-map $\text{Spec}(R') \to X$ whose special point goes to $\xi$.

Proof. Replacing $X$ by an affine open neighborhood of $\xi$, we may assume that $X = \text{Spec}(A)$ is affine and $\xi$ corresponds to a maximal ideal $P \subseteq A$. By going down, we can find a prime ideal $I \subseteq A$ such that $I \subseteq P$ and $I \cap R = \{0\}$. Thus the ring $A_P[\frac{1}{I}]$ is non-zero. Let $J \subseteq A[\frac{1}{I}]$ be a prime ideal such that $J_P \subseteq A_P[\frac{1}{I}]$ is maximal. By 1.20, $J$ is a maximal ideal of $A[\frac{1}{I}]$. Let $\eta \in X = \text{Spec}(A[\frac{1}{I}])$ be the closed point corresponding to $J$, then the residue field $\kappa(\eta)$ is a finite extension of $K$. Note also that $\eta$ specializes to $\xi \in X$. By [17, 054F] there exists a DVR $R'$ and a map $\text{Spec}(R') \to X$, whose special point goes to $\xi$ and whose generic point goes to $\eta$. Moreover, $R'$ can be chosen in such a way that its function field is equal to $\kappa(\eta)$. Thus $R \to R'$ is a finite ring extension (cf. [17, 0335], [17, 03GH]). □

2. The Pro-étale fundamental group

In this section, we generalize E. Lavanda’s theorem [10, Theorem 1.17] to more general schemes (see Theorem 2.20, Theorem 2.24 and Theorem 2.27). The core techniques we use here are similar to those in E. Lavanda’s proof: (1) the technique of proper descent of étale morphisms [10, Proposition 1.16] due to A. Grothendieck and D. Rydh; (2) a combinatorial method which turns out to be a folklore (see e.g. [4, 8.4.1, p. 333]).

2.1. Results about Noohi groups.

Lemma 2.1. A Noohi group $\pi$ is Hausdorff and has a basis of open neighborhoods of $e \in \pi$ given by subgroups.

Proof. Let $\pi-(\text{Sets})$ be the category of discrete sets with $\pi$-actions, and let $F_\pi: \pi-(\text{Sets}) \to (\text{Sets})$ be the forgetful functor. Since $\pi$ is Noohi, the continuous map $\pi \to \text{Aut}(F_\pi)$ is an isomorphism. Let $T$ denote the set of open subgroups of $\pi$. Then for each $U \subseteq \pi$ in $T$ the coset $\pi/U$ is a discrete set equipped with a continuous $\pi$-action. Let $\text{Aut}(\pi/U)$ be the automorphism group (with the compact-open topology) of $\pi/U$. Then $\text{Aut}(F_\pi)$ is a subgroup of $\prod_{U \in T} \text{Aut}(\pi/U)$ equipped with the subspace topology. Since each $\text{Aut}(\pi/U)$ is Hausdorff and has a basis of open neighborhoods of the unit given by subgroups, so is the product $\prod_{U \in T} \text{Aut}(\pi/U)$ and its topological subgroup $\text{Aut}(F_\pi) \cong \pi$. □

Lemma 2.2. Let $f, g: G \to \pi$ be two continuous maps of topological groups, where $\pi$ is Hausdorff and has a basis of open neighborhoods of $e \in \pi$. If the induced functors $f^*, g^*: \pi-(\text{Sets}) \to G-(\text{Sets})$ coincide, then $f = g$. In other words, $\text{Hom}(G, \pi) \to \text{Fun}(\pi-(\text{Sets}), G-(\text{Sets}))$ is injective.

Proof. Since $\pi$ is Hausdorff and has a basis of open neighborhoods of $e \in \pi$, the intersection of all open subgroups of $\pi$ is $e$. Therefore, it is enough to show that $f$ is equal to $g$ after composing with $\pi \to \pi/U$ for all open subgroups $U \subseteq \pi$. But $\pi/U \in \pi-(\text{Sets})$, so its restrictions to $G-(\text{Sets})$ via $f$ and $g$ must coincide. This implies exactly that $\pi \to \pi/U$ equalizes $f$ and $g$. □

The following is a generalization of [9, Lemma 2.51, p. 18] to not necessarily Hausdorff topological groups.
Lemma 2.3. The natural inclusion

\[(\text{Noohi groups}) \subseteq (\text{Topological groups})\]

has a left adjoint, which we will denote by \((-)^{\text{Noohi}}\).

Proof. Given \(G \in (\text{Topological groups})\) the category \(G\)-\(\text{(Sets)}\) is a tame infinite Galois category. Let \(F\): \(G\)-\(\text{(Sets)} \rightarrow (\text{Sets})\) be the forgetful functor, then \(\text{Aut}(F)\) equipped with the compact-open topology is a Noohi group by [3, Theorem 7.2.5.1]. By [3, Theorem 7.2.5.3] the continuous map \(\alpha: \text{Aut}(F) \rightarrow \text{Aut}(F)\) identifies \(\text{Aut}(F)\)-\(\text{(Sets)}\) and \(G\)-\(\text{(Sets)}\). We put \(G^{\text{Noohi}} := \text{Aut}(F)\). If \(\gamma: G \rightarrow G'\) is a continuous homomorphism, where \(G'\) is Noohi, then by [3, Theorem 7.2.5.2] the functor \(G'\)-\(\text{(Sets)} \rightarrow G\)-\(\text{(Sets)}\) corresponds to a unique map \(\beta: \text{Aut}(F) \rightarrow G'\). Thus by 2.2 we have \(\beta \circ \alpha = \gamma\). This concludes the proof. \(\square\)

Definition 2.4. Let \(\pi\) be a Noohi group, and let \(H \subseteq \pi\) be a subgroup. Denote \(\langle H \rangle\) the smallest normal subgroup of \(\pi\) containing \(H\). We define the Noohi quotient of \(\pi\) by \(H\) to be \((\pi/\langle H \rangle)^{\text{Noohi}}\). Let \(\langle H \rangle^{\text{c}}\) be the closure of \(\langle H \rangle\) in \(\pi\). By [3, Proposition 7.1.5, p. 188] and 2.1, \((\pi/\langle H \rangle)^{\text{Noohi}}\) is the Raïkov completion of \(\pi/\langle H \rangle^{\text{c}}\).

If \(\{f_i,g_i\}_{i \in I}\) are elements of \(\pi\), we often form the Noohi quotient \((\pi/\langle H \rangle)^{\text{Noohi}}\) with \(H\) being the subgroup generated by \(\{f_i^{-1}g_i\}_{i \in I}\). We refer to this as the Noohi quotient of \(\pi\) by the relations \(\{f_i = g_i\}_{i \in I}\).

**Lemma 2.5.** Finite coproducts exist in the category of Noohi groups.

Proof. It suffices to show that \(\pi_1 \coprod \pi_2\) exists for any diagram \(\pi_1 \xrightarrow{\mu} \pi \xrightarrow{\eta} \pi_2\) of Noohi groups. Indeed when \(\pi\) is trivial, \(\pi_1 \coprod \pi_2\) is just the Noohi group corresponding to the tame infinite Galois category of discrete sets equipped with two independent actions from \(\pi_1\) and \(\pi_2\). See [3, Example 7.2.6].

In general, \(\pi_1 \coprod \pi_2\) is nothing but the Noohi group of \(\pi_1 \coprod \pi_2\) by the relations \(\{p(x) = q(x)\}_{x \in \pi}\). Alternatively, it can also be defined by the Noohi group corresponding to the tame infinite Galois category of discrete sets equipped with two actions from \(\pi_1\) and \(\pi_2\) which agree on \(\pi\). \(\square\)

2.2. A construction of Van Kampen. The construction we are about to discuss, in the case of discrete groups, can be found in [4, 8.4.1, p. 333], where it is attributed to Van Kampen.

Let \(s \geq 1\) and let \(\pi, \pi', \pi_1, \ldots, \pi_s\) be Noohi groups. Let \(\psi_i: \pi_i \rightarrow \pi, \phi_i: \pi_i \rightarrow \pi'\) be continuous homomorphisms. We define a (discrete) group \(F\) by generators \(\{u_{ij} : 1 \leq i, j \leq s\}\) and relations \(u_{ij} = e, u_{ij}u_{jk} = u_{ik}\). Evidently, \(F\) is free of rank \(s-1\) on the generators \(v_2, \ldots, v_s\), where \(v_i := u_{1i}\) for \(i = 2, \ldots, s\). We put \(v_1 = e\).

**Lemma 2.6.** The following Noohi groups are isomorphic:

(i) The Noohi coproduct of \(\pi'\) and \(F\).

(ii) The Noohi coproduct of \(s\) copies of \(\pi'\), and one copy of \(F\), then Noohi quotient by the relations \(u_{ij}^{-1}[y], u_{ij} = [y], 1 \leq i, j \leq s, y \in \pi'\). Here \([y]\) denote \(y\) regarded as an element in the \(i\)-th copy of \(\pi'\).

Proof. The isomorphism is characterized by \(y \leftrightarrow [y]\), for \(y \in \pi'\) and \(g \leftrightarrow g\) for \(g \in F\). The universal property of coproduct implies that there are homomorphisms in both directions matching these elements, and they are inverse to each other. \(\square\)

**Lemma 2.7.** The following Noohi groups are isomorphic:

(i) The Noohi coproduct of \(\pi\) and the group in 2.6 (i), Noohi quotient by the relations \(\psi_i(a) = v_{i}^{-1}\phi_i(a)v_{i}, a \in \pi_i, i = 1, \ldots, s\).

(ii) The Noohi coproduct of \(\pi\) and the group in 2.6 (ii), Noohi quotient by the relations \(\psi_i(a) = [\phi_i(a)], a \in \pi_i, i = 1, \ldots, s\).

(iii) The Noohi coproduct of \(F\) with \(\pi\) \(\coprod \pi',\) Noohi quotient by the relations \(\psi_i(a) = v_{i}^{-1}\phi_i(a)v_{i}, a \in \pi_i, i = 2, \ldots, s\).
(iv) The Noohi coproduct of $F$ and (the Noohi fiber coproduct of $\pi \to \pi \prod_{i_1,\ldots,i_s} \pi_i$, $\pi$), Noohi quotient by the relations $u_{ij}^{-1}(e \ast y)u_{ij} = e \ast y$ for all $y \in \pi$, $i, j = 1, \ldots, s$. Here $e \ast y$ denote the product of $e \in \pi$ and $y \in \pi'$ in $\prod_{i_1,\ldots,i_s} \pi_i'$.

Proof. The isomorphism between (i) and (ii) is induced by that of 2.6. The isomorphism between (i) and (iii) is rather obvious. The isomorphism between (iii) and (iv) is constructed in a way similar to that of 2.6. \hfill\qed

Remark 2.8. We will denote the group in 2.7 as $\text{VK}(\pi, \pi'; \pi_1''', \ldots, \pi_s'')$ and omit the homomorphisms in subscripts when there is no confusion. The description (i) is the one usually found in the literature, e.g. [4, 8.4.1, p. 333]. It relies on single out the index 1 and so does (iii). The descriptions (ii) and (iv) make it clear that the construction actually treats all indices in equal footing. This construction is very useful due to the next result.

Setting 2.9. To state the result, we work in the following situation with $s \geq 1$:

- Let $(\mathcal{C}, F), (\mathcal{D}, G), (\mathcal{E}_1, H_1), \ldots, (\mathcal{E}_s, H_s)$ be tame infinite Galois categories.
- For $j = 1, \ldots, s$, let $u_j : \mathcal{C} \to \mathcal{E}_j$ (resp. $v_j : \mathcal{D} \to \mathcal{E}_j$) be a functor such that $(\mathcal{C}, H_j \circ u_j)$ (resp. $(\mathcal{D}, H_j \circ v_j)$) is a tame infinite Galois category and $H_j \circ u_j$ is compatible with $F$ (resp. $H_j \circ v_j$ is compatible with $G$) in the sense of [9, Definition 3.14].
- Recall that an object of the 2-fibre product ([17, 003R])

$$\mathcal{C}_{\mathcal{E}_1 \times \cdots \times \mathcal{E}_s} \mathcal{D}$$

is a triple $(X, Y, \phi)$, where $X \in \mathcal{C}$, $Y \in \mathcal{D}$, and $\phi$ is an isomorphism

$$(u_j(X))_{1 \leq j \leq s} \to (v_j(Y))_{1 \leq j \leq s}$$

in $\mathcal{E}_1 \times \cdots \times \mathcal{E}_s$. We define a functor $F'$ from this 2-fibre product to (Sets) by setting $F'(X, Y, \phi) = F(X)$.
- For $j = 1, \ldots, s$, choose an isomorphism of fiber functors on $\mathcal{C} : H_j \circ u_j \cong F$ and denote the composition $\pi_1(\mathcal{E}_j, H_j) \to \pi_1(\mathcal{C}, H_j \circ u_j) \to \pi_1(\mathcal{C}, F)$ by $\psi_j$. We also choose an isomorphism of fiber functors on $\mathcal{D} : H_j \circ v_j \cong G$ and denote the composition $\pi_1(\mathcal{E}_j, H_j) \to \pi_1(\mathcal{D}, H_j \circ v_j) \to \pi_1(\mathcal{D}, G)$ by $\phi_j$.

Proposition 2.10. With the above setting, the pair

$$(\mathcal{C}_{\mathcal{E}_1 \times \cdots \times \mathcal{E}_s}, F')$$

is a tame infinite Galois category, whose fundamental group is isomorphic to

$$\text{VK}(\pi_1(\mathcal{C}, F), \pi_1(\mathcal{D}, G); \pi_1(\mathcal{E}_1, H_1), \ldots, \pi_1(\mathcal{E}_s, H_s))_{(\psi_1, \ldots, \psi_s), (\phi_1, \ldots, \phi_s)}.$$

Proof. Let us denote the pair in the proposition by $(\mathcal{C}', F')$. An object in $\mathcal{C}'$ is an $(s + 2)$-tuple $(A, B, \lambda_1, \ldots, \lambda_s)$, where $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $\lambda_j : u_j(A) \cong v_j(B)$. We will view this as an object $(A, B, \lambda_1)$ of $\mathcal{C} \times_{\mathcal{E}_1} \mathcal{D}$, together with isomorphisms $(\lambda_2, \ldots, \lambda_s)$ in $\mathcal{E}_2 \times \cdots \times \mathcal{E}_s$.

By [3, Theorem 7.2.5], we may further describe a category equivalent to $\mathcal{C}'$ using Noohi group actions. Indeed, $\mathcal{C}'$ is equivalent to the category of s-tuples $(S, \lambda'_2, \ldots, \lambda'_s)$, where $S$ is a discrete $\Pi_1$-set with $\Pi_1 := \pi_1(\mathcal{C}, F) \prod_{\pi_1(\mathcal{E}_1, H_1), \psi_1, \phi_1} \pi_1(\mathcal{D}, G)$, and for $s = 2, \ldots, s$, $\lambda'_j : S \to S$ is a bijection of $\pi_1(\mathcal{E}_j, H_j)$-sets where the source (resp. target) has the structure of a $(\pi_1(\mathcal{E}_j, H_j)$-set via $u_j(A)$ (resp. $v_j(B)$). This exactly means: $\phi_j(a)s = \lambda'_j(\psi_j(a)s)$ for all $a \in \pi_1(\mathcal{E}_j, H_j), s \in S$. Therefore, $\mathcal{C}'$ is exactly equivalently the category of discrete $\pi$-set, where $\pi$ is the Noohi group given by the proposition, by 2.7 (iii). This completes the proof by [3, Example 7.2.2]. \hfill\qed

Remark 2.11. With the notation of the proof, it is obvious that under the identification $\pi_1(\mathcal{C}', F') \simeq \pi$, the maps $\pi_1(\mathcal{C}, F) \to \pi_1(\mathcal{C}', F'), \pi_1(\mathcal{D}, G) \to \pi_1(\mathcal{C}', F')$ correspond to the canonical maps $\pi_1(\mathcal{C}, F) \to \pi, \pi_1(\mathcal{D}, G) \to \pi$. 
Remark 2.12. More sophisticated forms of this sort of result can be found in [18, Corollary 5.3, p. 19] and [9, Corollary 3.18, p. 32]. However, the form given here is more readily applicable and is enough for almost all known applications, such as [10, Theorem 1.17], which we generalize in 2.27. More applications are given in the next few subsections.

2.3. Van Kampen theorems.

Remark 2.13. From now on we assume that $X$ is a locally topologically Noetherian scheme. Let $\text{Cov}(X)$ denote the category of geometric covers of $X$ in the sense of [3, Definition 7.3.1]. This construction gives a category fibered over the category of locally topological Noetherian schemes in an obvious way.

Let $X_{\text{red}}$ denote the reduced induced scheme structure of $X$. Then by [2, Exposé VIII, Théorème 1.1, p. 247] or [7, Exposé IX, 4.10, p. 186] we can conclude that $\text{Cov}(X) \xrightarrow{\sim} \text{Cov}(X_{\text{red}})$ just as in [10, Lemma 1.15]. Thus if $X$ is a locally Noetherian connected scheme and $x \in X$ is a geometric point, then $\pi_1^{\text{proét}}(X_{\text{red}}, x) \to \pi_1^{\text{proét}}(X, x)$ is an isomorphism.

Lemma 2.14. Let $f_1 : Z_1 \to X, f_2 : Z_2 \to X$ be monomorphisms of schemes. Assume that the induced map $Z_1 \coprod Z_2 \to X$ is a morphism of effective descent for $\text{Cov}(-)$. Then we have an equivalence given by the pullback functor:

$$\text{Cov}(X) \simeq \text{Cov}(Z_1) \times_{\text{Cov}(Z)} \text{Cov}(Z_2)$$

where $Z = Z_1 \times_X Z_2$.

Proof. By the effectiveness of descent, an object of $\text{Cov}(X)$ amounts to a descent datum, which consists of $Y_i \in \text{Cov}(Z_i), i = 1, 2$ together with $\varphi_{ij} : p^*_i Y_i \to p^*_j Y_j$ for $i, j \in \{1, 2\}$ satisfying the cocycle condition. The cocycle condition says that $\varphi_{11}$ and $\varphi_{22}$ are just identities (by the monomorphism assumption) and $\varphi_{12}, \varphi_{21}$ give isomorphisms $Y_1|_Z \simeq Y_2|_Z$, inverse to each other. Thus we get an object in the $\text{Cov}(Z_1) \times_{\text{Cov}(Z)} \text{Cov}(Z_2)$. It is routine to verify that this construction gives an equivalence. \hfill \Box

Lemma 2.15. The assumption of Lemma 2.14 is satisfied when $Z_1, Z_2$ are open subschemes of $X$ such that $X = Z_1 \cup Z_2$.

Proof. This is essentially the classical van Kampen theorem and an easy case of fpqc descent. Notice that here $Z = Z_1 \cap Z_2$. \hfill \Box

Lemma 2.16. The assumption of Lemma 2.14 is satisfied when $X$ is locally Noetherian and $Z_1, Z_2$ are closed subschemes of $X$ such that $X = Z_1 \cup Z_2$ set-theoretically.

Proof. This follows from [10, Proposition 1.16] applied to the map $\phi : Z_1 \coprod Z_2 \to X$, which relies on Rydh’s work [15] and generalizes [7, IX Théorème 4.12]. \hfill \Box

Lemma 2.17. Let $X$ be a locally Noetherian scheme. Let $Z \subseteq X$ be a closed subscheme. Consider a proper surjective map $\tilde{X} \to X$

Denote $Z \times_X \tilde{X}$ by $\tilde{Z}$. Suppose that the union of the image of the two closed immersions

$$\Delta_f : \tilde{X} \to \tilde{X} \times \tilde{X}, \quad \tilde{Z} \times \tilde{Z} \to \tilde{X} \times \tilde{X}$$

is $\tilde{X} \times X \tilde{X}$ set-theoretically. Then the pullback functor induces an equivalence of categories:

$$\text{Cov}(X) \xrightarrow{\sim} \text{Cov}(\tilde{X}) \times_{\text{Cov}(\tilde{Z})} \text{Cov}(Z).$$

Proof. Let us construct a quasi-inverse of the pullback functor. So we start with a triple $(Y, W, \phi)$, where $Y \in \text{Cov}(	ilde{X}), W \in \text{Cov}(Z)$, and $\phi : Y \times \tilde{X} \tilde{Z} \xrightarrow{\sim} W \times \tilde{Z}$ is an isomorphism. The idea is to construct a descent datum of $Y$ for the morphism $f$, then we get the desired object in $\text{Cov}(X)$ by applying [10, Proposition 1.16].

To construct this descent datum, we need to work on $\tilde{X} \times \tilde{X}$ using the equivalence

$$\text{Cov}(\tilde{X} \times \tilde{X}) \equiv \text{Cov}(\tilde{X}) \times_{\text{Cov}(\tilde{Z})} \text{Cov}(\tilde{Z}).$$
which follows from Lemma 2.16 because the fiber product of the two closed immersions in the current lemma is the closed immersion \( \tilde{Z} \xrightarrow{\Delta_f} \tilde{X} \times X \tilde{X} \). Under this equivalence, \( p_i^* \) \((Y)\) corresponds to the triple \((Y, q_i^*(Y|_Z), \text{can}_i)\) for \( i = 1, 2 \). Here \( p_i : \tilde{X} \times X \tilde{X} \to \tilde{X} \), \( q_i : \tilde{Z} \to Z \) are projections, and \( \text{can}_i \) is the obvious canonical isomorphism. The desired descent datum \( \lambda : p_1^* \) \((Y)\) \( \to p_2^* \) \((Y)\) corresponds to \((Y, q_1^*(Y|_Z), \text{can}_1) \to (Y, q_2^*(Y|_Z), \text{can}_2)\) given by \( \text{id}_Y : Y \to Y \), \( \varphi : q_1^*(Y|_Z) \to q_2^*(Y|_Z)\), where \( \varphi \) is the canonical descent datum signifying the fact that \( Y|_Z \) is isomorphic to \( W \times_Z \tilde{Z} \) via \( \phi \).

We have to show that \( \lambda \) is indeed a descent datum. That is, considering the triple fibred product, the fibred product and the projections:

\[
\begin{array}{c}
\tilde{X} \times X \tilde{X} \times X \tilde{X} \\
\downarrow p_{23} \quad \downarrow p_{12} \quad \downarrow p_{13} \\
\tilde{X} \times X \tilde{X} \times X \tilde{X} \\
\downarrow p_1 \quad \downarrow p_2 \\
\tilde{X},
\end{array}
\]

we have to show the cocycle condition \( p_{23}^* \lambda \circ p_{12}^* \lambda = p_{13}^* \lambda \). Since the fibred product is covered by \( \Delta_f(\tilde{X}) \) and \( \tilde{Z} \times Z \tilde{Z} \), the triple fibred product is covered by the triple diagonal \( \Delta_f(\tilde{X}) \) and the closed subset

\[ \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \]

Thanks to [17, 0BTJ] it is enough to check the equality \( p_{23}^* \lambda \circ p_{12}^* \lambda = p_{13}^* \lambda \) on \( \Delta_f(\tilde{X}) \) and \( \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \) separately. On \( \Delta_f(\tilde{X}) \) all the pullbacks of \( \lambda \) are identities, so there is nothing to check. On \( \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \) the equality holds because \( \lambda|_{\tilde{Z} \times Z \tilde{Z}} = \varphi \) is a descent datum for \( \tilde{Z} \to Z \). ☐

2.4. Schemes with connected singularity.

Conventions 2.18. Let \( X \) be a Nagata quasi-compact scheme, and let \( \eta_1, \ldots, \eta_n \) be the generic points of \( X \). Then we denote \( f : \tilde{X} \to X \) the relative normalization of the map

\[ \prod_{1 \leq i \leq n} \text{Spec}(\kappa(\eta_i)) \to X \]

in the sense of [17, 0BAK]. According to [17, 0AVK], \( f \) is finite (and surjective). Moreover, according to [17, 0351], the map \( f : \tilde{X} \to X \) factors through the reduced induced structure \( X_{\text{red}} \subseteq X \), and the map \( \tilde{X} \to X_{\text{red}} \) is the normalization of \( X_{\text{red}} \) relative to the disjoint union of the spectra of the generic points. See [17, 0359] for examples of Nagata schemes.

Assume further that \( X \) is a J-2 scheme ([17, 0TR2]; see [17, 0TR5] for examples of J-2 schemes). Then the singular locus \( Z \subseteq X \) is closed. We give \( Z \) the reduced induced structure and put \( \tilde{Z} := Z \times_X \tilde{X} \). Now it is easy to see that all the hypotheses of Lemma 2.17 are satisfied.

By [17, 035A], [17, 0TR4] and 2.13, we may directly assume that \( X \) is reduced in the proof of the following theorems.

Theorem 2.19. Let \( X \) be a connected Nagata Noetherian J-2 scheme. Let \( X_1, \ldots, X_n \) be its irreducible components. Let \( Z \subseteq X \) be the singular locus of \( X_{\text{red}} \). Let \( \tilde{X}_i \) be the connected component of \( \tilde{X} \) corresponding to \( X_i \), and put \( \tilde{Z} := \tilde{X} \times_X Z \). For each \( 1 \leq i \leq n \), suppose \( \tilde{X}_i \cap \tilde{Z} = Z_{ij} \prod_{1 \leq i \neq j \leq n} Z_{in} \) is a decomposition. Then the pullback functor

\[ \text{Cov}(X) \xrightarrow{\sim} \prod_{\text{Cov}(Z)} \text{Cov}(\tilde{X}_i) \times_{\text{Cov}(Z_{ij})} \text{Cov}(Z) \]

is an equivalence.

Proof. By 2.17 the pullback map

\[ \text{Cov}(X) \xrightarrow{\sim} \left( \text{Cov} \left( \prod_{1 \leq i \leq n} \tilde{X}_i \right) \times_{\text{Cov}(\tilde{Z})} \text{Cov}(Z) \right) = \left[ \left( \prod_{1 \leq i \leq n} \text{Cov}(\tilde{X}_i) \right) \times_{\text{Cov}(Z_{ij})} \text{Cov}(Z) \right] \]

is an equivalence. Then one checks easily that the later is the same as the right hand side in the statement of the theorem. ☐
Now we come back to fundamental groups. Assumptions and notations are as above, and we assume in addition that $Z$ is connected. Choose geometric points $x \in Z$, $x_{ij} \in Z_{ij}$, and isomorphisms of fiber functors $F_x \simeq F_{f(x_{ij})}$, hence isomorphisms $\pi_1^{\text{proét}}(Z, x) \simeq \pi_1^{\text{proét}}(Z, f(x_{ij}))$. We denote $\phi_{ij}$ the map

$$\pi_1^{\text{proét}}(Z_{ij}, x_{ij}) \longrightarrow \pi_1^{\text{proét}}(Z, f(x_{ij})) \xrightarrow{\simeq} \pi_1^{\text{proét}}(Z, x)$$

induced by $Z \subseteq X$, and $\psi_{ij}$ the map

$$\pi_1^{\text{proét}}(Z_{ij}, x_{ij}) \longrightarrow \pi_1^{\text{proét}}(\tilde{X}_i, x_{ij}) \xrightarrow{\simeq} \pi_1^{\text{ét}}(\tilde{X}_i, x_{ij})$$

obtained by choosing isomorphisms of fiber functors $\phi_{ij}$ of $Z_{ij}$, $\psi_{ij}$ of $\tilde{X}_i$. Therefore, the canonical maps $\pi_1^{\text{proét}}(Z, x) \rightarrow \pi_1^{\text{proét}}(X, x)$ and $\pi_1^{\text{ét}}(\tilde{X}_i, x_{ij}) \rightarrow \pi_1^{\text{proét}}(\tilde{X}_i, x_{in_i})$. Thanks to the equivalence between the category of tame infinite Galois categories with fiber functors and the category Noohi groups, we get

**Theorem 2.20.** There is an isomorphism of topological groups

$$\pi_1^{\text{proét}}(X, x) \simeq \prod_{1 \leq i \leq n} \text{VK}(\pi_1^{\text{proét}}(\tilde{X}_i, x_{i1}), \pi_1^{\text{proét}}(Z, x); \pi_1^{\text{proét}}(Z_{i1}, x_{i1}), \ldots, \pi_1^{\text{proét}}(Z_{in_i}, x_{in_i}))$$

Moreover, the canonical maps $\pi_1^{\text{proét}}(Z, x) \rightarrow \pi_1^{\text{proét}}(X, x)$ and $\pi_1^{\text{ét}}(\tilde{X}_i, x_{ij}) \rightarrow \pi_1^{\text{proét}}(\tilde{X}_i, f(x_{ij})) \simeq \pi_1^{\text{proét}}(X, x)$ correspond, via this isomorphism, to the projections in the coproduct.

**Proof.** This is immediate from 2.10 and 2.19. \qed

2.5. **Disconnected singularities: a dévissage.** Next, we deal with the case when the singular locus is not connected.

**Setting 2.21.** Let $X$ be a connected Nagata Noetherian J-2 scheme. Let $X_1, \ldots, X_n$ be its irreducible components. Let $Z_j \subseteq X$ ($1 \leq j \leq m$) be the connected components of the singular locus $Z$ of $X_{\text{red}}$. For $j = 1, \ldots, m$, set

$$T_j := \left( \bigcup_{X_i \cap Z_j \neq \emptyset} X_i \right) \setminus \left( \bigcup_{t \neq j} Z_t \right).$$

We notice that if $m = 0$, then $X$ is regular and $n = 1$. From now on we assume $m \geq 1$.

**Lemma 2.22.** With the above setting,

(i) Each $T_j$ is connected and open in $X$.

(ii) $X = T_1 \cup \cdots \cup T_m$.

(iii) For each $1 \leq j \leq m$, $Z_j \subset T_j$ and $Z_j$ is disjoint from $T_t$ for $t \neq j$.

(iv) There exists $k$ such that $\bigcup_{t \neq k} T_t$ is connected, $1 \leq k \leq m$.

**Proof.** (i) Assume $X_i \cap Z_j \neq \emptyset$ and $X_s \cap Z_j = \emptyset$. Then $X_i \cap X_s$, being a subset of the singular locus, lies in $\bigcup_{t \neq j} Z_t$. Therefore, $X_s$ is disjoint from $T_j$. Thus

$$T_j = \left[ \left( \bigcup_{X_i \cap Z_j \neq \emptyset} X_i \right) \setminus \left( \bigcup_{X_i \cap Z_j = \emptyset} X_s \right) \right] \setminus \left( \bigcup_{t \neq j} Z_t \right) \setminus \left( \bigcup_{X_i \cap Z_j = \emptyset} X_t \right)$$

is open in $X$. Since $T_j$ is the union of the connected sets $Z_j$ and $X_i \setminus \left( \bigcup_{t \neq j} Z_t \right)$ (over those $i$ such that $X_i \cap Z_j \neq \emptyset$), and each of the latter intersects with $Z_j$, it is clear that $T_j$ is connected.

(ii) Let $x \in X$. Assume $x \in Z_j$; say $x \in Z_j$. Then it is clear that $x \in T_j$. Assume $x \notin Z_j$, say $x \in X_i$. Take any $j$ such that $X_i \cap Z_j \neq \emptyset$ (such $j$ exists; otherwise $X_i$ is a regular component and we have $n = 1$ and $m = 0$), then it is clear $x \in T_j$.

(iii) is obvious.

(iv) We claim that we can find a permutation $(j_1, \ldots, j_m)$ of $\{1, \ldots, m\}$ such that $T_{j_1} \cup \cdots \cup T_{j_r}$ is connected for all $1 \leq r \leq m$. Then the desired result follows by taking $k = j_m$. To prove the claim, we pick $j_1$ arbitrarily. Suppose that we have picked $j_1, \ldots, j_r$ with $r < m$, then it suffices to take $j_{r+1}$ such that $(T_{j_1} \cup \cdots \cup T_{j_r}) \cap T_{j_{r+1}} \neq \emptyset$. Indeed if such $j_{r+1}$ could not be found, we
would deduce that $T_{j_1} \cup \cdots \cup T_{j_r}$ is open and closed in the connected scheme $X$ by (ii). This proves the claim. \hfill \Box

**Theorem 2.23.** With the above setting, assume moreover that $T'_1 := T_2 \cup \cdots \cup T_m$ is connected. Each connected component of $T_1 \cap T'_1$ is a connected component of the regular scheme $X \setminus Z$. Let $D_1, \ldots, D_s$ be the connected components of $T_1 \cap T'_1$. Then the pullback functor

$$
\text{Cov}(X) \rightarrow \text{Cov}(T_1) \prod_{1 \leq i \leq r, T_i} \text{Cov}(T'_i)
$$

is an equivalence.

**Proof.** By Lemma 2.22 (iii), $T_1 \cap T'_1 = (T \setminus Z) \cap (T'_1 \setminus Z)$. Since $T \setminus Z$ and $T'_1 \setminus Z$ are both unions of some of the irreducible components of the regular scheme $X \setminus Z$, the first statement is clear. The second statement follows readily from 2.15. \hfill \Box

Again, we come back to fundamental groups. With the above setting, we may and do assume that $T'_1 := T_2 \cup \cdots \cup T_m$ is connected by Lemma 2.22 (iv). Let $d_i \in D_i$. Choose isomorphisms of fiber functors $F_{d_1} \simeq \cdots \simeq F_{d_s}$ on $T_1$, and those $F_{d_1} \simeq \cdots \simeq F_{d_s}$ on $T'_1$, so we get

$$
\pi^\text{proét}(T_1, d_1) \simeq \cdots \simeq \pi^\text{proét}(T_1, d_s)
$$

$$
\pi^\text{proét}(T'_1, d_1) \simeq \cdots \simeq \pi^\text{proét}(T'_1, d_s)
$$

**Theorem 2.24.** There is an isomorphism

$$
\pi^\text{proét}(X, d_1) \simeq VK(\pi^\text{proét}(T_1, d_1), \pi^\text{proét}(T'_1, d_1); \pi^\text{ét}(D_1, d_1), \ldots, \pi^\text{ét}(D_s, d_s)),
$$

where the relevant maps are $\phi_i : \pi^\text{ét}(D_i, d_i) \rightarrow \pi^\text{proét}(T_1, d_1) \simeq \pi^\text{proét}(T'_1, d_1)$ and $\psi_i : \pi^\text{ét}(D_i, d_i) \rightarrow \pi^\text{proét}(T'_1, d_1)$. Moreover, the canonical maps $\pi^\text{proét}(T_1, d_1) \rightarrow \pi^\text{proét}(X, d_1)$ and $\pi^\text{proét}(T'_1, d_1) \rightarrow \pi^\text{proét}(X, d_1)$ correspond, via this isomorphism, to the projections of the coproduct.

**Proof.** This follows immediately from 2.23 and 2.10. \hfill \Box

**Remark 2.25.** Observe that the singular locus of $T_1$ is $Z_1$ and that of $T'_1$ is $Z_2 \cup \cdots \cup Z_m$. Therefore, the above theorem provides an inductive way to compute pro-étale fundamental groups in terms of pro-étale fundamental groups of connected schemes with connected singular locus. The latter is already handled by 2.20. By further performing an induction on the dimension of the singular locus, we can describe the pro-étale fundamental group of an arbitrary (Nagata, J-2, and Noetherian) connected scheme in terms of etale fundamental groups of normal schemes.

**Corollary 2.26.** Let $\mathcal{C}$ be the smallest class of Noohi groups such that

- $\mathcal{C}$ contains the étale fundamental groups of connected normal schemes.
- $\mathcal{C}$ contains discrete free groups of finite rank.
- $\mathcal{C}$ is closed under fiber coproducts and quotients.

Then the pro-étale fundamental group of an arbitrary connected scheme lies in $\mathcal{C}$.

**2.6. An example.** As an example, we will compute the pro-étale fundamental group of an arbitrary connected curve over a separably closed field, generalizing [10, Theorem 1.17], which treated the stable case. In fact, we can handle more than curves. Let’s resume 2.18 and 2.21. The key assumption that we now make is the following:

**The pro-étale fundamental group of every connected component of $Z$ or $\tilde{Z}$ is trivial.**

This condition is satisfied if $Z$ is a disjoint union of spectrums of separably closed fields, e.g. when $X$ is a curve over a separably closed field. Let’s choose $x \in X$, $x_i \in \tilde{X}_i$, and paths $F_x \simeq F_{f(x_i)} \simeq \cdots \simeq F_{f(x_n)}$. Then we have

**Theorem 2.27.** With the above setting, $\pi^\text{proét}(X, x)$ is the Noohi coproduct of $\pi^\text{ét}(\tilde{X}_i, x_i)$, $i = 1, \ldots, n$, and a discrete free group of rank $\tilde{m} - m - n + 1$, where $m$ denotes the number of connected components of $Z$ and $\tilde{m}$ denotes that of $\tilde{Z}$. 

Proof. It is a nice exercise to prove the theorem by following the route provided by 2.24 and proceed by induction on the number $m$. However, it is easier to use the following argument. We will apply Theorem 2.19, and argue as in the proof of Proposition 2.10. Write the connected components of $\mathcal{Z}$ as $Z_1, \ldots, Z_m$, and those of $\mathcal{Z}$ as $\tilde{Z}_1, \ldots, \tilde{Z}_\tilde{m}$. An object of $\text{Cov}(\tilde{X}) \times_{\text{Cov}(Z)} \text{Cov}(\tilde{Z})$ is $(\{A_i\}^n_{i=1}, \{B_j\}^m_{j=1}, \{\lambda_k\}^\tilde{m}_{k=1})$, where $A_i$ is a $\pi_1(\tilde{X}_i, x_i)$-set, $B_j$ is a set, and $\lambda_k$ is a bijection from $A_i(k)$ to $B_j(k)$ if $\tilde{Z}_k \subset \tilde{X}_i(k)$ and $f(\tilde{Z}_k) = Z_j(k)$.

Again the “only if” part is obvious. Assume that $\lambda$ is a geometric point. Set $1, \ldots, \tilde{m}$ as points, and those of $\mathcal{Z}$ as $1, \ldots, m$. Then $\tilde{m} - n - m + 1$ of them. This description of $\text{Cov}(\tilde{X}) \times_{\text{Cov}(Z)} \text{Cov}(\tilde{Z})$ easily implies the theorem. □

2.7. Discrete representations. Let $\lambda: \pi \to G$ be a continuous homomorphism of topological groups. We say that $\lambda$ is discrete if it factors as $\pi \to G' \to G$ with $G'$ discrete, or equivalently, $\ker(\lambda)$ is open in $\pi$. When $\pi$ is a fundamental group, it is often interesting to know the discrete homomorphisms.

Lemma 2.28. Let $\pi_1$ and $\pi_2$ be Noohi groups and let $\pi_1 \coprod \pi_2$ be their Noohi coproduct. Let $\lambda: \pi_1 \coprod \pi_2 \to G$ be a continuous homomorphism corresponding to $\lambda_1: \pi_1 \to G$, $\lambda_2: \pi_2 \to G$. Then $\lambda$ is discrete if and only if $\lambda_1$ and $\lambda_2$ are both discrete.

Proof. The “only if” part is obvious. Assume that both $\lambda_1$ and $\lambda_2$ are discrete. Say $\lambda$ factors as $\pi_1 \to G_1' \to G$. Then clearly $\lambda$ factors as $\pi_1 \coprod \pi_2 \to G_1' \coprod G_2' \to G$, where $G_1' \coprod G_2'$ is the coproduct of $G_1'$ and $G_2'$ as topological groups. Since $G_1' \coprod G_2'$ is discrete, $\lambda$ is discrete. □

Lemma 2.29. Let $\pi$ be a Noohi group and let $\pi' := (\pi/\langle H \rangle)^{\text{Noohi}}$ be a Noohi quotient of $\pi$. Consider a commutative diagram of continuous homomorphisms with $G$ Hausdorff:

$$
\begin{array}{ccc}
\pi & \xrightarrow{\lambda} & G \\
\downarrow & & \downarrow \\
\pi' & \xrightarrow{\lambda'} & G
\end{array}
$$

Then $\lambda$ is discrete if and only if $\lambda'$ is discrete.

Proof. Again the “only if” part is obvious. Assume that $\lambda$ is discrete and factorizes as $\pi \to G' \xrightarrow{\phi} G$. We may assume that $G' \to G$ is injective. Then $\ker(\pi \to G') = \ker(\pi \to G) \supset H$. Therefore, $\pi \to G'$ factorizes as $\pi \to \pi/\langle H \rangle^c \to G'$ in the category of topological groups. Applying the functor $(\cdot)^{\text{Noohi}}$, we get $\pi \to \pi' \xrightarrow{\psi} G'$. It remains to show $\phi \circ \psi = \lambda'$.

By construction $\phi \circ \psi$ and $\lambda'$ agree on the image $M$ of $\pi$, which is a dense subset of $\pi'$. The map $(\phi \circ \psi, \lambda'): \pi' \to G \times G$ sends $M$ to the diagonal of $G \times G$, which is closed as $G$ is Hausdorff. We conclude that the whole image of $(\phi \circ \psi, \lambda')$ lies in the diagonal. This shows: $\phi \circ \psi = \lambda'$. □

Lemma 2.30. Suppose $g: Y' \to Y$ is a finite surjective map of normal integral schemes and $y' \in Y'$ is a geometric point. Set $y := g(y')$. Consider a commutative diagram of continuous homomorphisms:

$$
\begin{array}{ccc}
\pi_1^\text{proét}(Y', y') & \xrightarrow{\lambda} & G \\
\downarrow & & \downarrow \\
\pi_1^\text{proét}(Y, y) & \xrightarrow{\lambda'} & G
\end{array}
$$

Then $\lambda$ is discrete if and only if $\lambda'$ is discrete.
Proof. Indeed, the following commutative diagram:

\[
\begin{array}{ccc}
\text{Gal}(K(Y')) & \leftarrow & \text{Gal}(K(Y)) \\
\downarrow & & \downarrow \\
\pi_1^{\text{proét}}(Y', y') & \rightarrow & \pi_1^{\text{proét}}(Y, y)
\end{array}
\]

implies that $\pi_1^{\text{proét}}(Y', y') \simeq \pi_1^{\text{proét}}(Y, y') \rightarrow \pi_1^{\text{proét}}(Y, y) \simeq \pi_1^{\text{proét}}(Y, y)$ is open. Thus $\text{Ker}(\lambda)$ is open if and only if $\text{Ker}(\lambda')$ is open.

\[\square\]

**Proposition 2.31.** Let $X$ be a connected Nagata Noetherian $J$-2 scheme, and let $x \in X$ be a geometric point. Let $\{\eta_i\}_{1 \leq i \leq n}$ be the set of all generic points of $X$, and let $A := \prod_{1 \leq i \leq n} K_i$, where each $K_i$ is a finite extension of $\kappa(\eta_i)$. Let $\tilde{X}_A$ denote the normalization of $X$ inside $\text{Spec}(A)$. Let

$$
\lambda: \pi_1^{\text{proét}}(X, x) \rightarrow G
$$

be a continuous map of Hausdorff topological groups. Then $\lambda$ is discrete if and only if the induced map $\pi_1^{\text{proét}}(U, u) \rightarrow G$ is discrete for each connected component $U$ of $\tilde{X}_A$ and each geometric point $u \in U$.

Proof. We just have to show the “if” direction. By 2.30, we may assume that $K_i = \kappa(\eta_i)$ for all $i$, i.e. $\tilde{X}_A = \tilde{X}$. By 2.24, 2.28 and 2.29, we may do an induction to reduce the proposition to the case where the singular locus of $Z \subseteq X_{\text{red}}$ is connected. So we are in the situation of Lemma 2.20.

We want to show that the induced map $\pi_1^{\text{proét}}(Z, x) \rightarrow G$ is discrete. This would finish the proof by applying 2.28 and Lemma 2.29 again.

There is a ring $C := \prod_{j \in J} L_j$, where each $L_j$ is a finite extension of the residue field of a generic point of $Z$, such that the normalization $\tilde{Z}_C$ of $Z$ in $C$ lifts the inclusion $Z \hookrightarrow X$ to a map $\tilde{Z}_C \rightarrow \tilde{X}$. Thanks to the lifting, all the maps from the pro-étale fundamental groups of the connected components of $\tilde{Z}_C$ to $G$ factor through a discrete quotient. Since the dimension of $Z$ is smaller than that of $X$, we complete our proof by induction. \[\square\]

**Remark 2.32.** Proposition 2.31 can also be proved using [9, Remark 3.20]. Indeed, after we have reduced to the case $\tilde{X}_A = \tilde{X}$, we can directly apply 2.28 and 2.29 together with loc. cit. to finish the proof, because in loc. cit. $\pi_1^{\text{proét}}(X, x)$ is written as a Noohi quotient of the Noohi coproduct of the fundamental groups of the $\tilde{X}_i$s and a free group.

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**References**

[1] Piotr Achinger, Marcin Lara, and Alex Youcis. *Specialization for the pro-étale fundamental group.* 2021. arXiv: 2107.06761 [math.AG].

[2] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III.* Vol. 269, 270, 305. Lecture Notes in Mathematics. Springer, 1971.

[3] Bhargav Bhatt and Peter Scholze. “The pro-étale topology for schemes”. In: *Astérisque* 369 (2015), pp. 99–201. issn: 0303-1179.

[4] Ronald Brown. *Topology and groupoids.* Third edition of it Elements of modern topology [McGraw-Hill, New York, 1968; MR0227979], With 1 CD-ROM (Windows, Macintosh and UNIX). BookSurge, LLC, Charleston, SC, 2006, pp. xxvi+512. isbn: 1-4196-2722-8.

[5] David Gieseker. “Stable vector bundles and the Frobenius morphism”. In: *Ann. Sci. École Norm. Sup. (4) 6* (1973), pp. 95–101. issn: 0012-9593. url: http://www.numdam.org/item?id=ASENS_1973_4_6_1_95_0.
REFERENCES

[6] A. Grothendieck and J. A. Dieudonné. *Eléments de géométrie algébrique. I*. Vol. 166. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1971, pp. ix+466. ISBN: 3-540-05113-9; 0-387-05113-9.

[7] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Vol. 224. Lecture notes in mathematics. Springer-Verlag, 1971.

[8] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[9] Marcin Lara. “Homotopy Exact Sequence for the Pro-Étale Fundamental Group”. PhD thesis. 2019. URL: http://dx.doi.org/10.17169/refubium-2773.

[10] Elena Lavanda. “Specialization map between stratified bundles and pro-étale fundamental group”. In: *Adv. Math.* 335 (2018), pp. 27–59. ISSN: 0001-8708. DOI: 10.1016/j.aim.2018.06.013. URL: https://doi.org/10.1016/j.aim.2018.06.013.

[11] Hideyuki Matsumura. *Commutative algebra*. Second. Vol. 56. Mathematics Lecture Note Series. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980, pp. xv+313. ISBN: 0-8053-7026-9.

[12] David Mumford. “An analytic construction of degenerating curves over complete local rings”. In: *Compositio Math.* 24 (1972), pp. 129–174. ISSN: 0010-437X.

[13] Arthur Ogus. “F-isocrystals and de Rham cohomology. II: Convergent isocrystals.” English. In: *Duke Math. J.* 51 (1984), pp. 765–850. ISSN: 0012-7094; 1547-7398/e. DOI: 10.1215/S0012-7094-84-05136-6.

[14] Valentina Di Proietto, Fabio Tonini, and Lei Zhang. *A crystalline incarnation of Berthelot’s conjecture and Künneth formula for isocrystals*. To appear in *Journal of Algebraic Geometry*. 2021.

[15] David Rydh. “Submersions and effective descent of étale morphisms”. en. In: *Bulletin de la Société Mathématique de France* 138.2 (2010), pp. 181–230. DOI: 10.24033/bsmf.2588. URL: http://www.numdam.org/articles/10.24033/bsmf.2588/.

[16] João Pedro Pinto dos Santos. “Fundamental group schemes for stratified sheaves”. In: *J. Algebra* 317.2 (2007), pp. 691–713. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2007.03.005. URL: https://doi.org/10.1016/j.jalgebra.2007.03.005.

[17] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2019.

[18] Jakob Stix. “A general Seifert-Van Kampen theorem for algebraic fundamental groups”. In: *Publ. Res. Inst. Math. Sci.* 42.3 (2006), pp. 763–786.

[19] Fabio Tonini and Lei Zhang. “Algebraic and Nori fundamental gerbes”. In: *J. Inst. Math. Jussieu* 18.4 (2019), pp. 855–897. ISSN: 1474-7480. DOI: 10.1017/s147474801700024x. URL: https://doi.org/10.1017/s147474801700024x.

[20] Lei Zhang. “Nori’s fundamental group over a non algebraically closed field”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 18.4 (2018), pp. 1349–1394. ISSN: 0391-777X.

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