LEVEL SETS OF ASYMPTOTIC MEAN OF DIGITS FUNCTION FOR 4-ADIC REPRESENTATION OF REAL NUMBER

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Abstract. We study topological, metric and fractal properties of the level sets

\[ S_\theta = \{ x : r(x) = \theta \} \]

of the function \( r \) of asymptotic mean of digits of a number \( x \in [0; 1] \) in its 4-adic representation,

\[ r(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i(x) \]

if the asymptotic frequency \( \nu_j(x) \) of at least one digit does not exist, were

\[ \nu_j(x) = \lim_{n \to \infty} \frac{n-1}{n} \# \{ k : \alpha_k(x) = j, k \leq n \}, \quad j = 0, 1, 2, 3. \]

1. Introduction

Let \( 2 \leq s \in \mathbb{N} \) and \( A_s = \{ 0, 1, \ldots, s-1 \} \) be an alphabet of \( s \)-adic number system. By \( \Delta_s^{a_1(x)a_2(x)\ldots a_k(x)\ldots} \) denote the \( s \)-adic representation of a number \( x \in [0; 1] \), i.e.,

\[ x = \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_n}{s^n} + \cdots \equiv \Delta_s^{a_1a_2\ldots}, \]

where \( A_s \ni \alpha_i(x) \) are digits of the \( s \)-adic representation of the number \( x \in [0; 1] \). The value \( n^{-1} \sum_{i=1}^{n} \alpha_i(x) \equiv r_n(x) \) is called relative mean of digits in the \( s \)-adic representation of \( x \).

In this paper we study properties of the function \( r \) of asymptotic mean of digits, in particular, topological, metric, and fractal properties of number sets with a preassigned

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asymptotic mean of digits. Namely, we investigate sets

$$S_\theta \equiv \left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i(x) = \theta \in [0; s - 1] \right\},$$

that are level sets of the function \( r \) (indeed, \( r^{-1}(\theta) = S_\theta \)). If \( \theta \notin [0; s - 1] \) then it is easily proved that the set \( S_\theta \) is empty.

Asymptotic mean of digits of a number \( x \) is closely related to the concept of digit frequency of the number.

Let \( N_i(x, k) \) be the number the digits \( "i" \in A_s \) appears in the \( s \)-adic representation \( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^s \) of the real number \( x \in [0; 1] \) to \( k \)th place including, i.e.

$$N_i(x, k) = \# \{ j : \alpha_j(x) = i, j \leq k \}.$$

The frequency (asymptotic frequency) of a digit \( "i" \) in the \( s \)-adic representation of a number \( x \in [0; 1] \) is the limit (if it exists) such that

$$\nu_i(x) = \lim_{k \to \infty} v_i^{(k)},$$

where \( v_i^{(k)} = k^{-1}N_i(x, k) \) is called relative frequency of the digit \( "i" \) in the \( s \)-adic representation of a number \( x \).

The frequency function \( \nu_i(x) \) of a digit \( "i" \) in the \( s \)-adic representation of a number \( x \in [0; 1] \) is well defined for \( s \)-adic–irrational numbers, and, for \( s \)-adic–rational numbers, it is well defined after agreement to use representation with period (0) only.

Different mathematical objects with fractal properties were defined and studied in terms of frequencies. First of all it is Besicovitch–Eggleston’s sets [3, 6]

$$E_{\tau_0, \tau_1, \ldots, \tau_{s-1}} = \{ x : x = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^s, \nu_i(x) = \tau_i \geq 0, i = 0, s - 1 \},$$

the Hausdorff–Besicovitch dimension of the sets is equal to [4] to

$$\alpha_0(E_{\tau_0, \tau_1, \ldots, \tau_{s-1}}) = -\frac{\ln \tau_0^{\tau_0} \tau_1^{\tau_1} \ldots \tau_{s-1}^{\tau_{s-1}}}{\ln s}.$$

The number \( x \) is called normal for basis \( s \) if the value \( \nu_i(x) \) exists for all \( i \in A_s \) and equals to \( s^{-1} \). The set of all normal for basis \( s \) numbers is the only Besicovitch–Eggleston’s set of positive and even full Lebesgue measure.

Normal for every natural basis \( s \geq 2 \) number \( x \) is called normal. According to the famous Borel’s theorem [5] we see that Lebesgue measure of the set of normal numbers is equal to 1.

In the papers [1, 11] it was proved that the Hausdorff–Besicovitch dimension of abnormal and essentially abnormal number sets (i.e. number sets having not frequency of at least one digit or having not frequencies of all digits respectively) is equal to 1.

If a number \( x \) has all digits frequencies then the relationship between asymptotic mean of digits and digits frequencies of the number \( x \) is the following:

$$r(x) = \nu_1(x) + 2\nu_2(x) + \cdots + (s - 1)\nu_{s-1}(x).$$

When \( s = 2 \), it is obvious that the asymptotic mean of digits is equal to the frequency of digit “1”. So we do not examine this case. The case \( s = 3 \) was studied in papers [8, 10]. It is unique since it is the only case where the set \( S_\theta \) is a union of two disjoint sets \( \Theta_1 \) and \( \Theta_2 \) such that

$$\Theta_1 \equiv \{ x : \text{frequencies of all digits exist} \},$$
$$\Theta_2 \equiv \{ x : \text{frequency of any digit does not exist} \}.$$
In cases \( s > 3 \) the set \( S_0 \) is a union of three disjoint sets such that
\[
\Theta_1 \equiv \{ x : \text{frequencies of all digits exist} \},
\Theta_2 \equiv \{ x : \text{frequency of at least one digit exists and of at least one digit does not exist} \},
\Theta_3 \equiv \{ x : \text{frequency of any digit does not exist} \}.
\]

In this paper we study the case \( s = 4 \) since it is the easiest and modeling in the last class. Our previous paper \([9]\) was devoted to studying properties of the set \( \Theta_1 \), and this one deals with the sets \( \Theta_2 \) and \( \Theta_3 \).

### 2. The Object of Study

**Lemma 1.** If in the 4-adic representation of a real number \( x \in [0; 1] \) the frequency of one digit does not exist, then the frequency of at least one more digit does not exist.

**Proof.** Suppose the frequency \( \nu_k(x) \) does not exist, i.e., \( \lim_{n \to \infty} \frac{N_k(x_0, n)}{n} \) does not exist. Since
\[
\frac{N_k(x_0, n)}{n} = 1 - \frac{N_j(x_0, n)}{n} - \frac{N_m(x_0, n)}{n} - \frac{N_l(x_0, n)}{n},
\]
we see that
\[
\lim_{n \to \infty} \left( \frac{N_j(x_0, n)}{n} + \frac{N_m(x_0, n)}{n} + \frac{N_l(x_0, n)}{n} \right)
\]
does not exist. It means that at least one of the limits \( \lim_{n \to \infty} \frac{N_i(x_0, n)}{n} \), \( \lim_{n \to \infty} \frac{N_m(x_0, n)}{n} \) or \( \lim_{n \to \infty} \frac{N_l(x_0, n)}{n} \), where \( \{ j, k, l, m \} = \{ 0, 1, 2, 3 \} \) do not exist. \( \square \)

**Lemma 2.** If in the 4-adic representation of a real number \( x \in [0; 1] \) the asymptotic mean of digits, \( r(x) \), and at least two 4-adic digits frequencies \( \nu_i(x) \), \( \nu_j(x) \), where \( i, j \in \{ 0, 1, 2, 3 \} \), exist, then the remaining two 4-adic digits frequencies of the number \( x \) exist.

**Proof.** Consider the system
\[
\begin{align*}
\nu_0^{(n)} + \nu_1^{(n)} + \nu_2^{(n)} + \nu_3^{(n)} &= 1, \\
\nu_1^{(n)} + 2\nu_2^{(n)} + 3\nu_3^{(n)} &= r_n,
\end{align*}
\]

Let \( i, j \in \{ 1, 2, 3 \} \). Since \( \lim_{n \to \infty} \nu_i^{(n)} = \nu_i(x) \), \( \lim r_n = \theta \), we see that from the second equation of system \((\text{I})\) it follows that \( \lim_{n \to \infty} \nu_k^{(n)} \), \( k \in \{ 1, 2, 3 \} \setminus \{ i, j \} \), exists, i.e. the frequency \( \nu_k(x) \) exists. Then from the first equation of system \((\text{I})\) it follows that \( \nu_0(x) \) exists.

Let \( i = 0, j = 1 \). Then from system \((\text{I})\) we have
\[
\begin{align*}
\nu_0^{(n)} &= r_n + \nu_1^{(n)} - 2\nu_0^{(n)} - 2, \\
\nu_2^{(n)} &= 1 - \nu_0^{(n)} - \nu_1^{(n)} - \nu_3^{(n)} = 3 + \nu_3^{(n)} - 2\nu_1^{(n)} - r_n,
\end{align*}
\]
which implies existence of the frequencies \( \nu_2(x) \) and \( \nu_3(x) \).

Let \( i = 0, j = 2 \). Then from system \((\text{I})\) we obtain
\[
\begin{align*}
\nu_3^{(n)} &= \frac{1}{2}(r_n - \nu_2^{(n)} + \nu_0^{(n)} - 1), \\
\nu_1^{(n)} &= 1 - \nu_0^{(n)} - \nu_2^{(n)} - \nu_3^{(n)} = \frac{3}{2} - \frac{3}{2}\nu_0^{(n)} - \frac{3}{2}\nu_2^{(n)} - \frac{r_n}{2},
\end{align*}
\]
which implies existence of the frequencies \( \nu_1(x) \) and \( \nu_3(x) \).
Let \( i = 0, j = 3 \). Then from system (11) we have
\[
\begin{align*}
\nu_2^{(n)} &= r_n - 2\nu_3^{(n)} + \nu_0^{(n)} - 1, \\
\nu_1^{(n)} &= 1 - \nu_0^{(n)} - \nu_2^{(n)} - \nu_3^{(n)} = 2 - r_n - 2\nu_0^{(n)} + 2\nu_3^{(n)},
\end{align*}
\]
which implies existence of the frequencies \( \nu_1(x) \) and \( \nu_2(x) \). \( \square \)

**Corollary 1.** (from Lemmas 1 and 2). A number \( x \in S_0 \) cannot have frequencies of only two or of only three 4-adic digits.

From Lemma 1 it follows that if a number does not have frequency of at least one 4-adic digit then it does not have frequency of one more digit, therefore, the number \( x \in S_0 \) cannot have frequencies of only three digits. According to Lemma 2 if a number \( x \in S_0 \) has frequencies of at least two digits then it has frequencies of all digits, therefore, the number \( x \) cannot have frequencies of only two 4-adic digits.

Hence, the set \( S_0 \) can be represented as a union of three disjoint sets \( \Theta_1, \Theta_2 \) and \( \Theta_3 \) such that
\[
\begin{align*}
\Theta_1 & \equiv \{ x : \nu_i(x) \text{ exist, } \forall i \in \mathcal{A}_1 \}, \\
\Theta_2 & \equiv \{ x : \text{exist frequency of only one } \text{4-adic digit } \nu_i(x), i \in \mathcal{A}_4 \}, \\
\Theta_3 & \equiv \{ x : \nu_i(x) \text{ do not exist, } \forall i \in \mathcal{A}_4 \}.
\end{align*}
\]

In the following sections we study properties of the sets \( \Theta_2 \) and \( \Theta_3 \).

### 3. Abnormal Numbers That Have Asymptotic Mean of Digits

**Theorem 1.** If \( \theta = 0 \) or \( \theta = 3 \), then \( \Theta_2 = \Theta_3 = \emptyset \).

**Proof.** Let \( \theta = 0 \). If \( \lim_{n \to \infty} r_n(x) = 0 \), then for any \( i \in \{1, 2, 3\} \) the following inequality holds: \( 0 \leq \nu_i^{(n)}(x) \leq v_i^{(n)}(x) + 2v_2^{(n)}(x) + 3v_3^{(n)}(x) = r_n(x) \to 0 \), as \( n \to \infty \). Therefore, \( \nu_i(x) = \lim_{n \to \infty} \nu_i^{(n)}(x) = 0 \) and \( \nu_0(x) = 1 \). Hence, \( \Theta_2 = \Theta_3 = \emptyset \).

Let \( \theta = 3 \). If \( \lim_{n \to \infty} r_n(x) = 3 \), then multiplying the first equation of system (11) by \( 3 \) and subtracting the second equation of the system, we obtain that \( 3\nu_3^{(n)} + 2 \nu_1^{(n)} + \nu_2^{(n)} = 3 - r_n \).

Hence \( 0 \leq \nu_i^{(n)}(x) \leq 3\nu_3^{(n)}(x) + 2\nu_1^{(n)}(x) + \nu_2^{(n)}(x) = 3 - r_n(x) \to 0 \) as \( n \to \infty \). Therefore, \( \nu_i(x) = 0 \), for all \( i \in \{0, 1, 2\} \). Hence \( \nu_3(x) = 1 \) and \( \Theta_2 = \Theta_3 = \emptyset \). \( \square \)

**Lemma 3.** Let \( (s_k) \) be a sequence of positive integers and the following conditions hold:
\[
\lim_{k \to \infty} s_k = \infty, \quad \lim_{k \to \infty} \frac{1}{\sum_{i=1}^{k} s_i} = 0, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0, \quad \alpha_1 \neq \alpha_2, \quad \beta_1 \neq \beta_2.
\]
Then there exist sequences \( a_n(\alpha_1, \alpha_2) \) and \( b_n(\beta_1, \beta_2) \) such that \( a_n(\alpha_1, \alpha_2) \in \{\alpha_1, \alpha_2\} \) and \( b_n(\beta_1, \beta_2) \in \{\beta_1, \beta_2\} \) for all \( n \in \mathbb{N} \) and the limits
\[
\lim_{k \to \infty} \frac{\sum_{i=1}^{k} [a_i(\alpha_1, \alpha_2) \cdot s_i]}{\sum_{i=1}^{k} s_i} \quad \text{and} \quad \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [b_i(\beta_1, \beta_2) \cdot s_i]}{\sum_{i=1}^{k} s_i}
\]
do not exist.

**Proof.** Without loss of generality let \( \alpha_2 > \alpha_1, \beta_2 > \beta_1 \),
\[
\frac{k}{\sum_{i=1}^{k} s_i} \leq \frac{k}{\sum_{i=1}^{k} \lambda s_i} = \lambda \quad \text{and} \quad \frac{k}{\sum_{i=1}^{k} s_i} > \frac{k}{\sum_{i=1}^{k} \lambda (s_i - 1)} = \lambda - \frac{k}{\sum_{i=1}^{k} s_i} \to \lambda \quad \text{as} \quad k \to \infty.
\]
Hence, \( \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [\lambda s_i]}{\sum_{i=1}^{k} s_i} = \lambda \).

Suppose that \( \varepsilon > 0 \) satisfies \( \alpha_2 - \varepsilon > \alpha_1 + \varepsilon \) and \( \beta_2 - \varepsilon > \beta_1 + \varepsilon \). Let \( r_1, l_1 \) be smallest positive integers such that for any \( n > r_1 \) and \( m > l_1 \) the following inequalities hold:

\[
\frac{\sum_{i=1}^{n} [\alpha_2 s_i]}{\sum_{i=1}^{n} s_i} > \alpha_2 - \varepsilon \quad \text{and} \quad \frac{\sum_{i=1}^{m} [\beta_2 s_i]}{\sum_{i=1}^{m} s_i} > \beta_2 - \varepsilon.
\]

Denote \( n_1 = \max(r_1, l_1) \).

Let \( r_2, l_2 \) be smallest positive integers such that for all \( r_2 > n_1, l_2 > n_1 \) and \( n > r_2, m > l_2 \),

\[
\frac{\sum_{i=1}^{n_1} [\alpha_2 s_i] + \sum_{i=n_1+1}^{n} [\alpha_1 s_i]}{\sum_{i=1}^{n} s_i} < \alpha_1 + \varepsilon \quad \text{and} \quad \frac{\sum_{i=1}^{n_1} [\beta_2 s_i] + \sum_{i=n_1+1}^{m} [\beta_1 s_i]}{\sum_{i=1}^{m} s_i} < \beta_1 + \varepsilon.
\]

Denote \( n_2 = \max(r_2, l_2) \).

Let \( r_3, l_3 \) be smallest positive integers such that for any \( r_3 > n_2, l_3 > n_2 \) and \( n > r_3, m > l_3 \), the following inequalities hold:

\[
\frac{\sum_{i=1}^{n_1} [\alpha_2 s_i] + \sum_{i=n_1+1}^{n_2} [\alpha_1 s_i] + \sum_{i=n_2+1}^{n} [\alpha_2 s_i]}{\sum_{i=1}^{n} s_i} > \alpha_2 - \varepsilon
\]

and

\[
\frac{\sum_{i=1}^{n_1} [\beta_2 s_i] + \sum_{i=n_1+1}^{n_2} [\beta_1 s_i] + \sum_{i=n_2+1}^{m} [\beta_2 s_i]}{\sum_{i=1}^{m} s_i} > \beta_2 - \varepsilon.
\]

Denote \( n_3 = \max(r_3, l_3) \). And so on.

Let \( a_n(\alpha_1, \alpha_2) = \alpha_1 \) if \( n \in \{1, \ldots, n-1\} \), \( a_n(\alpha_1, \alpha_2) = \alpha_2 \) if \( n \in \{n_k, \ldots, n_{k+1} - 1\} \) and \( k \) is not an even integer; \( a_n(\alpha_1, \alpha_2) = \alpha_1 \) if \( n \in \{n_k, \ldots, n_{k+1} - 1\} \) and \( k \) is an even integer.

Let \( b_n(\beta_1, \beta_2) = \beta_1 \) if \( n \in \{1, \ldots, n_1 - 1\} \), \( b_n(\beta_1, \beta_2) = \beta_2 \) if \( n \in \{n_k, \ldots, n_{k+1} - 1\} \) and \( k \) is not an even integer; \( b_n(\beta_1, \beta_2) = \beta_1 \) if \( n \in \{n_k, \ldots, n_{k+1} - 1\} \) and \( k \) is an even integer. This is possible since for fixed \( p \) following relations hold:

\[
\lim_{k \to \infty} \frac{\sum_{i=1}^{k} [\lambda s_i]}{\sum_{i=1}^{k} s_i} = \lim_{k \to \infty} \left( \frac{\sum_{i=1}^{k} [\lambda s_i]}{\sum_{i=1}^{k} s_i} - \sum_{i=1}^{p-1} [\lambda s_i] \right) = \lambda - 0 = \lambda.
\]

Denote \( x_n = \sum_{i=1}^{n} [a_i(\alpha_1, \alpha_2) s_i] \), \( y_n = \sum_{i=1}^{n} [b_i(\beta_1, \beta_2) s_i] \). Suppose the limits \( \lim_{k \to \infty} x_n \) and \( \lim_{k \to \infty} y_n \) exist. Let \( \delta < \min(\alpha_2 - \alpha_1 - 2\varepsilon, \beta_2 - \beta_1 - 2\varepsilon) \). From the Cauchy criterion, it follows that there are \( N_1, N_2 \in \mathbb{N} \) such that for any \( a, b > N_1 \) and \( c, d > N_2 \) the inequalities
Theorem 5.

Proof. Let $\Theta_1$ be a set of all elements such that for all elements $x \in [0; 1]$, we have $\|\tau_n\|$ be a matrix of dimension $(4 \times \infty)$. Consider the following form of a real number $x \in [0; 1]$: 

$$\hat{x} = \Delta^{(4)}_{0,1,2,3} 01112333001112333...$$

In paper [9] we proved the following three theorems.

**Theorem 2.** If $\|\tau_n\|$ is a matrix of dimension $(4 \times \infty)$ such that for all $n \in \mathbb{N}$ the following conditions hold: $\tau_{3n} + \tau_{2n} + \tau_{n} = 1$, $\tau_{n} + 2\tau_{2n} + 3\tau_{3n} = \theta$, then

$$\lim_{n \to \infty} \tau_{n} = \theta.$$ 

**Theorem 3.** If $\|\tau_n\|$ is a stochastic matrix of dimension $(4 \times \infty)$ and for any fixed $j \in \{0, 1, 2, 3\}$, then

$$\nu_j(\hat{x}) = \lambda.$$ 

**Theorem 4.** Let $(s_k^{(1)}), (s_k^{(2)})$ be sequences of positive numbers such that $\lim_{k \to \infty} s_k^{(r)} = \infty$, $r \in \{1, 2\}$ and $\|p^{(1)}\| = \|p_n^{(1)}\|, \|p^{(2)}\| = \|p_n^{(2)}\|$ be stochastic matrices of dimension $(4 \times \infty)$. Let

$$x(\|p^{(r)}\|; \|s_k^{(j)}\|) = \Delta^{(4)}_{0,1,2,3} 01112333001112333...$$

If $\lim_{k \to \infty} |s_k^{(1)} - s_k^{(2)}| = \infty$, then $x(\|p^{(1)}\|; \|s_k^{(1)}\|) \neq x(\|p^{(2)}\|; \|s_k^{(2)}\|).$ 

If $\lim_{n \to \infty} \sum_{i=0}^{3} |p_i^{(1)} - p_i^{(2)}| > 0$, then $x(\|p^{(1)}\|; \|s_k^{(1)}\|) \neq x(\|p^{(2)}\|; \|s_k^{(2)}\|).$

**Theorem 5.** If $\theta \in (0; 3)$, then the set $\Theta_2$ is an everywhere dense, continuum set of zero Lebesgue measure.

Proof. The well-known Borel’s theorem states that $\nu_0 = \nu_1 = \nu_2 = \nu_3 = \frac{1}{4}$ for almost all in the sense of Lebesgue measure numbers of $[0; 1]$. From this fact it follows that Lebesgue measure of the set $\Theta_2$ is equal to zero.

Continuity. Construct a continuum subset of $\Theta_2$ such that frequency of the digit 0 exists for all elements (similarly we can construct a continuum subset of $\Theta_2$ such that the frequency of a fixed digit “i”, $i \in \{1, 2, 3\}$, exists for all elements). Let $s_k = k$ and $p = (p_0, p_1, p_2, p_3), \ q = (q_0, q_1, q_2, q_3)$ be stochastic vectors such that $p_0 = q_0,
\[ p_1 + 2p_2 + 3p_3 = q_1 + 2q_2 + 3q_3 = \theta, \quad p_1 \neq q_1, \] then \( \lim_{k \to \infty} s_k = \infty \), \( \lim_{k \to \infty} \frac{k}{\sum_{i=1}^{k} s_i} = 0 \). From Lemma 3, where \( \alpha_1 = \beta_1 = p_1, \alpha_2 = \beta_2 = q_1 \), it follows that there exists a sequence \( a_n(p_1, q_1) \) such that \( a_n(p_1, q_1) = p_1 \) or \( a_n(p_1, q_1) = q_1 \) for any \( n \in N \) and \( \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [a, (p_1, q_1), s_i]}{\sum_{i=1}^{k} s_i} \) does not exist. Denote \( \tau_{0k} = p_0, \tau_{1k} = a_k(p_1, q_1) \). Using the system

\[
\begin{align*}
\tau_{2k} + \tau_{3k} &= 1 - p_0 - a_k(p_1, q_1), \\
2\tau_{2k} + 3\tau_{3k}^{(n)} &= \theta - a_k(p_1, q_1)
\end{align*}
\]

we calculate \( \tau_{2k}, \tau_{3k} \). Namely, \( \tau_{3k} = \theta + a_k(p_1, q_1) - 2 + 2p_0 \tau_{2k} = 3 - 3p_0 - \theta - 2a_k(p_1, q_1) \). It is evident that \( \tau_{ki} \), where \( i \in \{2, 3\} \) is equal to \( p_1 \) or \( q_1 \) if \( a_k(p_1, q_1) \) is equal to \( p_1 \) or \( q_1 \), respectively. From Theorem 3 it follows that \( \nu_0(x) = p_0 \). Since \( \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [a, (p_1, q_1), s_i]}{\sum_{i=1}^{k} s_i} \) does not exist we obtain that the frequency \( \nu_1(x) \) does not exist either. From Theorem 4 it follows that different numbers \( x \) constructed as specified above correspond to different pairs of vectors \( \overline{p} \) and \( \overline{q} \) with relevant properties. Since the set of such pairs is a continuum, we see that the set \( \Theta_2 \) is a continuum.

**Everywhere density.** Since the condition \( \lim_{k \to \infty} r_k(x) = \theta \) does not depend on an arbitrary finite group of first symbols and for any interval \([a, b]\) there exists a cylinder \( [\Delta_{\gamma_1 \gamma_2 \ldots \gamma_r}(0); \Delta_{\gamma_1 \gamma_2 \ldots \gamma_r}(3)] \) completely contained in it, we see that \( \Theta_2 \) is an everywhere dense set.

**Theorem 6.** If \( \theta \in (0; 3) \), then the Hausdorff–Besicovitch dimension \( \alpha_0(\Theta_2) \) of the set \( \Theta_2 \) is positive, i.e., \( \alpha_0(\Theta_2) > 0 \).

**Proof.** Let \( (\varepsilon_i) \) be an arbitrary sequence of zeros and ones, vectors \((p_0, p_1, p_2, p_3)\) and \((q_0, q_1, q_2, q_3)\) be stochastic vectors such that \( p_0 > 0 \), \( p_1 \neq q_1 \), \( p_1 + 2p_2 + 3p_3 = \theta = q_1 + 2q_2 + 3q_3 \), \( x_i = [p_0 k(i+1)] - [p_0 k i] - r_i, r_i = \begin{cases} 0, & \text{if } \varepsilon_i = 1 \\ 1, & \text{if } \varepsilon_i = 0 \end{cases} \), \( t_i = [p_3 k(i+1)] - [p_3 k i] \).

Consider the system

\[
\begin{align*}
x_i + y_i + z_i + t_i &= k - 1, \\
y_i + 2z_i + 3t_i &= [\theta k(i+1)] - [\theta k i],
\end{align*}
\]

whence \( z_i = [\theta k(i+1)] - [\theta k i] - k + 1 + x_i - 2y_i, y_i = 2(k-1) - ([\theta k(i+1)] - [\theta k i]) - 2x_i + t_i \), We obtain that

\[
\begin{align*}
z_i &= \frac{[\theta k i] + \theta k}{k} - 1 + \frac{1}{k} + \frac{[p_0 k i] + p_0 k}{k} - \frac{2([p_3 k i] + p_3 k)}{k} \\
&= \theta - 1 + p_0 - 2p_3 + \left[\frac{[\theta k i] + \theta k}{k}\right] + \frac{1}{k} + \left[\frac{[p_0 k i] + p_0 k}{k}\right] - \frac{2([p_3 k i] + p_3 k)}{k}.
\end{align*}
\]
Since \( \theta - 1 + p_0 - 2p_3 = p_2 \) and for a sufficiently large \( k \) we have \( z_i \in N \) for all \( i \in N \). In the same way,

\[
\frac{y_i}{k} = 2 - \frac{2}{k} - \frac{\left\{ \theta k \right\} + \theta k}{k} - \frac{2\left\{p_0k + \theta k\right\}}{k} + \frac{\left\{p_0k + p_3k\right\}}{k} + \frac{2\{p_0k\} + \{p_0k\}}{k} + \frac{\{p_3k\} + \{p_3k\}}{k}
\]

\[
= 2 - \theta - 2p_0 + 3p_3 - \frac{2}{k} \left\{ \left\{ \theta k \right\} + \left\{ \theta k \right\} \right\} + \frac{2\{p_0k\} + \{p_0k\}}{k} + \frac{\{p_3k\} + \{p_3k\}}{k}
\]

Since \( 2 - \theta - 2p_0 + 3p_3 = p_1 \) and for a sufficiently large \( k \) we obtain \( y_i \in N \) for all \( i \in N \). Similarly we prove that for a sufficiently large \( k \in N \) all solutions of the system

\[
\begin{align*}
x_i + y_i + z_i + t_i &= k - 1, \\
y_i + 2z_i + 3t_i &= \left\{ \theta k(i + 1) \right\} - \theta k, \\
x_i &= \left\lfloor p_0k(i + 1) \right\rfloor - \left\lfloor p_0k \right\rfloor - r_i, \\
t_i &= \left\lfloor q_3k(i + 1) \right\rfloor - \left\lfloor q_3k \right\rfloor,
\end{align*}
\]

(3)

are positive integers for all \( i \in N \).

Let \( k \) be a sufficiently large positive integer. Let all solutions of systems (2) and (3) be positive integers for arbitrary sequence of zeros and ones \( (\varepsilon_i) \), \( i \in N \). Construct the number \( x(\varepsilon_i) \) as follows:

\[
x(\varepsilon_i) = \Delta \varepsilon_1 \ldots \varepsilon_j 0 \ldots 0 1 \ldots 1 2 \ldots 3 \ldots 3 \ldots 3 \ldots 3 x_1 y_1 z_1 t_1 x_j y_j z_j t_j
\]

Without loss of generality put \( p_3 > q_3 \), let \( \delta > 0 \) be such that \( p_3 - \delta > q_3 - \delta \). Let \( r_1 \) be a positive integer such that \( (x_i, y_i, z_i, t_i) \) is a solution of system (2) for any \( j \in \{1, 2, \ldots, r_1\} \) and

\[
\frac{N_3(x, kr_1)}{kr_1} = \frac{\sum_{i=1}^{r_1} t_i}{kr_1} = \frac{\lfloor p_3k(r_1 + 1) \rfloor}{kr_1} > p_3 - \delta,
\]

this is possible since the last value tends to \( p_3 \) as \( r_1 \to \infty \).

Let \( r_1 < r_2 \) be a positive integer such that \( (x_j, y_j, z_j, t_j) \) is a solution of system (3) for any \( j \in \{r_1 + 1, \ldots, r_2\} \) and

\[
\frac{N_3(x, kr_2)}{kr_2} = \frac{\sum_{i=1}^{r_2} t_i}{kr_2} = \frac{\lfloor p_3k(r_1 + 1) \rfloor - \lfloor q_3k(r_1 + 1) \rfloor + \lfloor q_3k(r_2 + 1) \rfloor}{kr_2} < q_3 + \delta,
\]

this is possible since the last value tends to \( q_3 \) as \( r_2 \to \infty \).

Let \( r_2 < r_3 \) be a positive integer such that \( (x_j, y_j, z_j, t_j) \) is a solution of system (2) for any \( j \in \{r_2 + 1, \ldots, r_3\} \) and

\[
\frac{N_3(x, kr_3)}{kr_3} = \frac{\sum_{i=1}^{r_3} t_i}{kr_3} = \frac{\lfloor p_3k(r_1 + 1) \rfloor - \lfloor q_3k(r_1 + 1) \rfloor + \lfloor q_3k(r_2 + 1) \rfloor - \lfloor q_3k(r_2 + 1) \rfloor + \lfloor q_3k(r_3 + 1) \rfloor}{kr_3}
\]

\[
> p_3 - \delta,
\]

this is possible since the last value tends to \( p_3 \) as \( r_k \to \infty \). And so on.

We obtain that

\[
\frac{N_3(x, kr_i)}{kr_i} - \frac{N_3(x, kr_{i+1})}{kr_{i+1}} > p_3 - q_3 - 2\delta \text{ for all } i \in N. \text{ Assume that}
\]

\[
\lim_{i \to \infty} \frac{N_3(x, kr_i)}{kr_i} \text{ exists. Hence, we have a contradiction with Cauchy’s criterion. Thus,}
\]
It is clear that \( \lim_{i \to \infty} \frac{N_\delta(x, kr_i)}{kr_j} \) does not exist, i.e., the frequency \( \nu_3(x(\varepsilon_i)) \) does not exist. On the other hand, if \( kj \leq n \leq k(j + 1) \) then

\[
\begin{align*}
N_0(x(\varepsilon_i), n) & \geq \frac{\sum_{i=1}^{j} [p_0k(i + 1)] - [p_0ki]}{k(j + 1)} = \frac{[p_0k(j + 1)] - [p_0k]}{k(j + 1)} \\
& = p_0 - \frac{\{p_0k(j + 1)] - [p_0k]}{k(j + 1)} \to p_0, \quad \text{as} \quad j \to \infty,
\end{align*}
\]

\[
\begin{align*}
N_0(x(\varepsilon_i), n) & \leq \frac{\sum_{i=1}^{j+1} [p_0k(i + 1)] - [p_0ki]}{kj} = \frac{[p_0k(j + 2)] - [p_0k]}{kj} \\
& = p_0 - \frac{\{p_0k(j + 2)] - [p_0k]}{kj} \to p_0, \quad \text{as} \quad j \to \infty,
\end{align*}
\]

hence, \( \nu_0(x(\varepsilon_i)) = p_0 \). Also

\[
\begin{align*}
\begin{align*}
\sum_{i=1}^{j} [\theta k(i + 1)] - [\theta ki] & = \frac{[\theta k(j + 1)] - [\theta k]}{k(j + 1)} \\
& \to \theta, \quad \text{as} \quad j \to \infty,
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{j+1} [\theta k(i + 1)] - [\theta ki] & = \frac{[\theta k(j + 2)] - [\theta k]}{kj} \\
& \to \theta, \quad \text{as} \quad j \to \infty,
\end{align*}
\]

hence, \( \lim_{n \to \infty} r_n(x(\varepsilon_i)) = \theta \), and from Theorem \( \Theta \) the frequencies \( \nu_1(x(\varepsilon_i)) \) and \( \nu_2(x(\varepsilon_i)) \) do not exist.

Thus, \( x(\varepsilon_i) \in \Theta_2 \).

Selecting an arbitrary quantity of (not necessarily consecutive) blocks of number \( x(\varepsilon_i) \) and changing the order of digits (except for \( \varepsilon_i \)) inside each block we get either the “old” number \( x(\varepsilon_i) \), or a new number \( \tilde{x}(\varepsilon_i) \). These numbers belong to \( \Theta_2 \) since \( N_i(x(\varepsilon_i)), kr) = N_i(\tilde{x}(\varepsilon_i), kr) \) for any \( r \in N \) and \( \ell \in \{0, 1, 2, 3\} \). Denote by \( C(x(\varepsilon_i)) \) the set of numbers \( \tilde{x}(\varepsilon_i) \) obtained from \( x(\varepsilon_i) \) by choosing an arbitrary number of blocks and changing the digit order inside them. It is obvious that the set is a continuum. Denote by \( C_1 \) a union of the sets \( C(x(\varepsilon_i)) \) with respect to all possible sequences \( (\varepsilon_i) \) and show that \( \alpha_0(C_1) = \frac{1}{2\theta} \).

To calculate the Hausdorff–Besicovitch dimension it is sufficient to use covering of 4-adic cylinders. Consider a covering of the set \( C_1 \) by cylinders of the same rank \( m \). The \( \alpha \)-volume of the covering is equal to

\[
R_\alpha^m = \begin{cases} 
2^{-(k - j)}(4^{-(kt - j)})^\alpha, & \text{if} \quad m = kt - j, \; j \in \{1, \ldots, k - 1\}, \\
2^{k\alpha}(4^{-(kt)})^\alpha, & \text{if} \quad m = kt.
\end{cases}
\]

It is clear that \( R_\alpha^m < R_\alpha^{kt-j}, \; j \in \{2, \ldots, k - 1\} \) hence, consider an \( \alpha \)-covering of the set \( C_1 \) with cylinders of rank \( n = kt - 1 \).

The Hausdorff’s box–counting \( \alpha \)-measure of the set \( C_1 \) is equal to

\[
\begin{align*}
\hat{H}_\alpha(C_1) & = \lim_{l \to \infty} \frac{2^{-(l-1)}}{4^{(kt-1)\alpha}} = 2^{2\alpha - 1} \lim_{l \to \infty} 2^{(1-2\alpha)l}.
\end{align*}
\]
Whence,
\[ \hat{H}_\alpha(C_1) = \begin{cases} 
0, & \text{if } \alpha > \frac{1}{2k}, \\
\infty, & \text{if } \alpha < \frac{1}{2k}.
\end{cases} \]

Therefore, box-counting dimension of the set \( C_1 \) is equal to \( \alpha = \frac{1}{2k} \). Let us show that \( \alpha_0(C_1) = \frac{1}{2k} \). Consider an arbitrary finite covering of the set \( C_1 \) by 4-adic cylinders \( \{v_j\}, j \in \{1, \ldots, l\} \), and prove that if \( \alpha = \frac{1}{2k} \) then the preceding rank covering is not improvable. Let \( u_i \) be one of cylinders of the covering. Then \( |u_i| = 3^{-n} \) for some \( n \in \mathbb{N} \). Let \( n = kp - r, r \in \{0, \ldots, k - 1\} \), then \( \alpha \)-volume of covering of the set \( C_1 \cap \Delta_j \) by cylinders of rank \( kp - r + m \) is equal to
\[
R_{kp-r+m}^\alpha(C_1 \cap \Delta_j) = \begin{cases} 
2^{(1-r)\alpha} \frac{4^{(1-r)\alpha}}{2}, & \text{if } m = kl + r - j, j \in \{1, \ldots, k - 1\}, \\
2^{(k-\alpha)(k+p+k+l)} \alpha, & \text{if } m = kl + r, l \in \mathbb{N}.
\end{cases}
\]

Let us show that \( R_{k(p+l)-1}^\alpha(C_1 \cap \Delta_j) \leq v_n = (4^{-(kp-r)})^\alpha \). It is obvious that \( R_{k(p+l)-1}^\alpha(C_1 \cap \Delta_j) < R_{k(p+l)-j}^\alpha(C_1 \cap \Delta_j) \), if \( j \in \{2, \ldots, k\} \). Consider an \( \alpha \)-covering \( K \cap \Delta_j \) by cylinders of rank \( k(p + l) - 1 \). Its volume is equal to
\[
2^{l-1}(4^{-(kp+3l-1)})^\alpha.
\]

Since \( \left( \frac{2}{4^k} \right)^l = 1 \) and \( \frac{4^{(1-r)\alpha}}{2} < 1 \) we obtain if \( \alpha = \frac{1}{2k} \) then
\[
2^{l-1}(4^{-(kp+3l-1)})^\alpha = (4^{-(kp-r)})^\alpha \frac{4^{(1-r)\alpha}}{2} \left( \frac{2}{4^k} \right)^l \leq (4^{-(kp-r)})^\alpha.
\]

Hence if \( \alpha = \frac{1}{2k} \) we have \( \hat{H}_\alpha(C_1) = H_\alpha(C_1) = \frac{1}{2k} \) and we see that the Hausdorff–Besicovitch dimension of the set \( C_1 \) is equal to the box-counting dimension of the set \( C_1 \subset \Theta_2 \), thus \( \alpha_0(\Theta_2) \geq \alpha_0(C_1) = \frac{1}{2k} > 0. \)

\[ \Box \]

5. The set \( \Theta_3 \)

**Theorem 7.** If \( \theta \in (0; 3) \), then the set \( \Theta_3 \) is an everywhere dense, continuum set of zero Lebesgue measure.

**Proof.** Lebesgue measure. Since almost all (in the sense of Lebesgue measure) numbers of the interval \([0; 1]\) are normal, i.e., \( \nu_0 = \nu_1 = \nu_2 = \nu_3 = \frac{1}{4} \) we see that Lebesgue measure of the set \( \Theta_3 \) is equal to zero.

**Continuity.** Let \( s_k = k, p_0 > q_0 > 0, p_1 > q_1 > 0 \). Suppose that all solutions of the system
\[
\begin{align*}
x + y + z + t & = 1, \\
y + 2z + 3t & = \theta, \\
x & = p_0 \lor q_0, \\
y & = p_1 \lor q_1 \end{align*}
\]

are positive.

It is obvious that \( \lim_{k \to \infty} s_k = \infty, \lim_{k \to \infty} \frac{k}{\sum_{i=1}^{k} s_i} = 0, \lim_{k \to \infty} \frac{\sum_{i=1}^{k+1} s_i}{\sum_{i=1}^{k} s_i} = 0. \) From Lemma \( 3 \) where
\[
\alpha_1 = p_0, \alpha_2 = q_0, \beta_1 = p_1, \beta_2 = q_1 \] it follows that there exist sequences \( a_n(p_0, q_0) = p_0 \)
or \( a_n(p_0, q_0) = q_0 \) and \( b_n(p_1, q_1) = p_1 \) or \( b_n(p_1, q_1) = q_1 \) such that for all \( n \in \mathbb{N} \) the limits

\[
\lim_{k \to \infty} \frac{\sum_{i=1}^{k} [a_i(p_0, q_0)s_i]}{\sum_{i=1}^{k} s_i} \quad \text{and} \quad \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [b_i(p_1, q_1)s_i]}{\sum_{i=1}^{k} s_i}
\]

do not exist.

Denote \( \tau_{0k} = a_k(p_0, q_0) \), \( \tau_{1k} = b_k(p_1, q_1) \). From following system

\[
\begin{align*}
\tau_{0k} + \tau_{1k} + \tau_{2k} + \tau_{3k} &= 1, \\
\tau_{1k} + 2\tau_{2k} + 3\tau_{3k} &= \theta
\end{align*}
\]

we obtain \( \tau_{2k}, \tau_{3k}, \) i.e., \( \tau_{3k} = \theta - 2 + 2\tau_{0k} + \tau_{1k} \), \( \tau_{2k} = 3 - \theta - 3\tau_{0k} - \tau_{1k} \).

Since the limits

\[
\lim_{k \to \infty} \frac{N_0(x, \sum_{i=1}^{k} s_i)}{\sum_{i=1}^{k} s_i} = \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [a_i(p_0, q_0)s_i]}{\sum_{i=1}^{k} s_i}
\]

and

\[
\lim_{k \to \infty} \frac{N_1(x, \sum_{i=1}^{k} s_i)}{\sum_{i=1}^{k} s_i} = \lim_{k \to \infty} \frac{\sum_{i=1}^{k} [b_i(p_1, q_1)s_i]}{\sum_{i=1}^{k} s_i}
\]

do not exist, the frequencies \( \nu_0(x) \) and \( \nu_1(x) \) do not exist either. Then from Theorem \( \text{2} \) and from Theorem \( \text{3} \) it follows that \( \lim_{n \to \infty} r_n(x) = \theta \) and \( \nu_2(x), \nu_3(x) \) do not exist.

From Theorem \( \text{4} \) it follows that different numbers constructed as indicated above correspond to different pairs \( (p_0, q_0) \) and \( (p_1, q_1) \). Since the set of such pairs is a continuum, we obtain that set \( \Theta_3 \) is a continuum.

**Everywhere density.** Since the condition \( \lim_{k \to \infty} r_k(x) = \theta \) does not depend on an arbitrary finite group of first symbols, and for any interval \( [a; b] \subset [0; 1] \) there exists a cylinder \( [\Delta^1_{\gamma_1\gamma_2\ldots\gamma_1(0)}; \Delta^1_{\gamma_1\gamma_2\ldots\gamma_1(3)}] \) completely contained in it, we see that \( \Theta_3 \) is an everywhere dense set. \( \square \)

**Theorem 8.** If \( \theta \in (0; 3) \), then the Hausdorff–Besicovitch dimension \( \alpha_0(\Theta_3) \) of the set \( \Theta_3 \) is positive, i.e., \( \alpha_0(\Theta_3) > 0 \).

**Proof.** Let \( (\varepsilon_i) \) be an arbitrary sequence of zeros and ones, let \( p_0^{(1)} > p_0^{(2)} > 0 \) and \( p_1^{(1)} > p_1^{(2)} > 0 \), let solutions of the system

\[
\begin{align*}
x + y + z + t &= 1, \\
y + 2z + 3t &= \theta, \\
x &= p_0^{(1)} \lor p_0^{(2)}, \\
y &= p_1^{(1)} \lor p_1^{(2)}
\end{align*}
\]

be positive.

Denote

\[
\begin{align*}
r_i &= \begin{cases} 0, & \text{if } \varepsilon_i = 1, \\
1, & \text{if } \varepsilon_i = 0 \end{cases}, \\
\bar{r}_i &= \begin{cases} 0, & \text{if } \varepsilon_i = 0, \\
1, & \text{if } \varepsilon_i = 1. \end{cases}
\end{align*}
\]
Similarly to the proof of Theorem 6 we show existence of a sufficiently large positive integer $k$ such that all solutions of the systems

$$
\begin{align*}
  x_i + y_i + z_i + t_i &= k + 1, \\
  y_i + 2z_i + 3t_i &= \lfloor \theta k(i+1) \rfloor - \lfloor \theta ki \rfloor, \\
  x_i &= [p_0^{(1)} k(i+1)] - [p_0^{(1)} ki] - r_i, \\
  t_i &= [p_1^{(1)} k(i+1)] - [p_1^{(1)} ki] - \tilde{r}_i,
\end{align*}
$$

(4)

$$
\begin{align*}
  x_i + y_i + z_i + t_i &= k + 1, \\
  y_i + 2z_i + 3t_i &= \lfloor \theta k(i+1) \rfloor - \lfloor \theta ki \rfloor, \\
  x_i &= [p_0^{(2)} k(i+1)] - [p_0^{(2)} ki] - r_i, \\
  t_i &= [p_1^{(2)} k(i+1)] - [p_1^{(2)} ki] - \tilde{r}_i,
\end{align*}
$$

(5)

are positive for all $i \in \mathbb{N}$.

Let $(\varepsilon_i)$ be a fixed sequence of zeros and ones. Construct a number $x(\varepsilon_i)$ as follows:

$$
x(\varepsilon_i) = \Delta^4_{\varepsilon_i} 0 \ldots 01 \ldots 12 \ldots 23 \ldots 3 \varepsilon_j 0 \ldots 01 \ldots 12 \ldots 23 \ldots 3 \ldots
$$

Let $\delta > 0$ be such that $p_0^{(1)} - \delta > p_0^{(2)} + \delta$, $p_1^{(1)} - \delta > p_1^{(2)} + \delta$.

Let $g_1$ be a positive integer such that $(x_j, y_j, z_j, t_j)$ is a solution of system (4) for any $j \in \{1, 2, \ldots, g_1\}$ and $\frac{N_a(x, kg_1)}{kg_1} > p_0^{(1)} - \delta$, $\frac{N_1(x, kg_1)}{kg_1} > p_1^{(1)} - \delta$.

Let $g_2$ be a positive integer such that $(x_j, y_j, z_j, t_j)$ is a solution of system (5) for all $j \in \{g_1 + 1, g_1 + 2, \ldots, g_2\}$ and $\frac{N_a(x, kg_2)}{kg_2} < p_0^{(2)} + \delta$, $\frac{N_1(x, kg_2)}{kg_2} < p_1^{(2)} + \delta$.

Let $g_3$ be a positive integer such that $(x_j, y_j, z_j, t_j)$ is a solution of system (4) for any $j \in \{g_2 + 1, g_2 + 2, \ldots, g_3\}$ and $\frac{N_a(x, kg_3)}{kg_3} > p_0^{(1)} - \delta$, $\frac{N_1(x, kg_3)}{kg_3} > p_1^{(1)} - \delta$. And so on.

Since

$$
\left| \frac{N_a(x, kg_{j+1})}{kg_{j+1}} - \frac{N_a(x, kg_j)}{kg_j} \right| > p_a^{(1)} - p_a^{(2)} - 2\delta
$$

for all $j \in \mathbb{N}$, the limits $\lim_{j \to \infty} \frac{N_a(x, kg_j)}{kg_j}$, $a \in \{0, 1\}$ do not exist (assuming the converse, we obtain a contradiction to the Cauchy criterion). Thus, $\nu_0(x(\varepsilon_i))$ and $\nu_1(x(\varepsilon_i))$ do not exist.

Let

$$
kj \leq n < k(j+1),
$$

$$
r_n \geq \frac{\lfloor \theta k(j+1) \rfloor - \lfloor \theta k \rfloor}{k(j+1)} = \theta - \frac{\{\theta k(j+1)\} + \lfloor \theta k \rfloor}{k(j+1)} \to \theta,
$$

$$
r_n \leq \frac{\lfloor \theta k(j+2) \rfloor - \lfloor \theta k \rfloor}{kj} = \theta \cdot \frac{j+2}{j} - \frac{\lfloor \theta k(j+2) \rfloor - \lfloor \theta k \rfloor}{kj} \to \theta \ (j \to \infty).
$$

Hence, $r_n \to \theta$ as $n \to \infty$ and from Theorem 3 it follows that the frequencies $\nu_2(x(\varepsilon_i))$ and $\nu_3(x(\varepsilon_i))$ do not exist, i.e., $x(\varepsilon_i) \in \Theta_3$.

Selecting an arbitrary quantity of (not necessarily consecutive) blocks of number $x(\varepsilon_i)$ and changing the order of digits inside each block (except for $\varepsilon_i$) we obtain either the “old” number $x(\varepsilon_i)$, or a new number $\tilde{x}(\varepsilon_i)$. These numbers are contained in $\Theta_3$ since $N_l(x(\varepsilon_i), kr) = N_l(\tilde{x}(\varepsilon_i), kr)$, for any $r \in \mathbb{N}$ and $l \in \{0, 1, 2, 3\}$. Denote by $C(x(\varepsilon_i))$ the
set of numbers \( \tilde{x}(\varepsilon_i) \) obtained from \( x(\varepsilon_i) \) by choosing an arbitrary number of blocks and changing digit order inside them. It is evident that the set is a continuum. Denote by \( C_1 \) a union of the sets \( C(x(\varepsilon_i)) \) of all possible sequences \( (\varepsilon_i) \) and show that \( \alpha_0(C_1) = \frac{1}{2^k} \).

Similarly to the proof of Theorem 6, we show that \( \alpha_0(C_1) = \frac{1}{2^k} \).

Thus, \( \alpha_0(\Theta_3) \geq \alpha_0(C_1) > 0 \). \( \square \)

References

1. S. Albeverio, M. Pratsiovytyi, G. Torbin, Singular probability distributions and fractal properties of sets of real numbers defined by the asymptotic frequencies of their \( s \)-adic digits, Ukrain. Mat. Zh. 57 (2005), no. 9, 1163–1170; English transl. Ukrainian Math. J. 57 (2005), no. 9, 1361–1370.

2. S. Albeverio, M. Pratsiovytyi, G. Torbin, Topological and fractal properties of real numbers which are not normal, Bull. Sci. Math. 129 (2005), no. 8, 615–630.

3. A. S. Besicovitch, Sets of fractional dimension. 2: On the sum of digits of real numbers represented in the dyadic system, Math. Ann. 110 (1934), no. 3, 321–330.

4. P. Billingsley, Ergodic Theory and Information, John Wiley and Sons Inc., New York—London—Sydney, 1965.

5. É. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Mat. 27 (1909), no. 1, 247–271.

6. H. G. Eggleston, The fractional dimension of a set defined by decimal properties, Quart. J. Math. 20 (1949), no. 4, 31–36.

7. L. Olsen, Normal and non-normal points of self-similar sets and divergence points of self-similar measures, J. London Math. Soc. 2(67) (2003), no. 1, 103–122.

8. M. V. Pratsiovytyi, S. O. Klymchuk, Linear fractals of Besicovitch–Eggleston type, Scientific journal of Dragomanov NPU. Series 1. Physics and mathematics (2012), no. 13(2), 80–92.

9. M. V. Pratsiovytyi, S. O. Klymchuk, Topological, metric and fractal properties of sets of real numbers with preassigned mean of digits of \( 4 \)-adic representation when their frequencies exist, Scientific journal of Dragomanov NPU. Series 1. Physics and mathematics (2013), no. 14, 217–226.

10. Pratsiovytyi M.V., Klymchuk S.O., Makarchuk O.P., Frequency of ternary digit of number and its asymptotic mean of digits, Ukrain. Mat. Zh. 66 (2014), no. 3, 302–310. (Ukrainian); English transl. Ukrainian Math. J. 66 (2014), no. 3, 336–346.

11. M. V. Pratsiovytyi, G. M. Torbin, Superfractality of the set of numbers having no frequency of \( n \)-adic digits, and fractal probability distributions, Ukrain. Mat. Zh. 47 (1995), no. 7, 971–975. (Ukrainian); English transl. Ukrainian Math. J. 47 (1995), no. 7, 1113–1118.

12. G. M. Torbin, Frequency characteristics of normal numbers in different number systems, Fractal Analysis and Related Problems, Kyiv: Institute of Mathematics, National Academy of Sciences of Ukraine – National Pedagogical Dragomanov University, 1998, no. 1, pp. 55–55. (Ukrainian)

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