LINEAR SUPERPOSITION
AS A CORE THEOREM OF QUANTUM EMPIRICISM

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Abstract. Clarifying the nature of the quantum state $|\Psi\rangle$ is at the root of the problems with insight into (counterintuitive) quantum postulates. We provide a direct—and math axiom-free—empirical derivation of this object as an element of a vector space. Establishing the linearity of this structure—quantum superposition—is based on a set-theoretic creation of ensemble formations and invokes the following three principia: (I) quantum statics, (II) the notion of a number in a physical theory, and (III) mathematization of matching the two observations with each other; quantum invariance. All of the constructs rest upon a formalization of the minimal experimental entity: micro-event, detector click. This is sufficient for producing the $\mathbb{C}$-numbers, axioms of linear vector space (superposition principle), statistical mixtures of states, eigenstates and their spectra, and a non-commutativity of observables. No use is required of the concept of time. As a result, the foundations of theory are liberated to a significant extent from the issues associated with physical interpretations, philosophical exegeses, and mathematical reconstruction of the entire quantum edifice.

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Contents

1 Introduction ........................................................................................................... 3
   1.1 Modern status of quantum theory ............................................................ 3
   1.2 Formula of superposition ............................................................................ 4
   1.3 Physics = mathematics ................................................................................. 5
2 Points of departure .............................................................................................. 6
   2.1 Variations as micro-level transitions ............................................................ 6
   2.2 Observation .................................................................................................... 9
   2.3 Numeric realizations ..................................................................................... 10
   2.4 Macro and micro .......................................................................................... 11
   2.5 Quantum ensembles and statistics .............................................................. 13
   2.6 Distinguishability and numbers ................................................................... 14
3 Ensemble formations ............................................................................................ 15
   3.1 Mixtures of ensembles .................................................................................. 15
   3.2 Ensemble brace ............................................................................................ 17
4 Why domain C arises? ......................................................................................... 17
   4.1 Continuum of quantum phases .................................................................... 18
   4.2 Statistic + phases ........................................................................................ 20
5 Empiricism and mathematics ............................................................................. 22
   5.1 Union of ensembles ..................................................................................... 22
   5.2 Semigroup .................................................................................................... 24
   5.3 Measurement ................................................................................................ 25
   5.4 Invariance with respect to observations ...................................................... 26
6 Quantum superposition ....................................................................................... 29
   6.1 Representations of states ............................................................................. 29
   6.2 Representations of devices. Spectra ............................................................ 30
   6.3 Physical properties ....................................................................................... 31
   6.4 Superposition of states ................................................................................ 32
   6.5 Interference .................................................................................................. 35
7 Numbers ................................................................................................................ 37
   7.1 Replications of ensembles ......................................................................... 37
   7.2 Number as an operator ............................................................................... 38
   7.3 QM and the nature of arithmetic ................................................................. 39
   7.4 2-dimensional numbers .............................................................................. 41
   7.5 Involutions and C*-algebra ......................................................................... 43
   7.6 Naturalness of C-numbers .......................................................................... 45
8 State space ............................................................................................................. 46
   8.1 Linear vector space ...................................................................................... 47
   8.2 Bases, infinities, and countability ................................................................. 49
   8.3 The theorem .................................................................................................. 49
9 Numbers, minus, and equality, revisited ............................................................ 52
   9.1 Separation of the number matters ............................................................... 52
   9.2 Operations on numbers ............................................................................... 53
   9.3 Naturalness of abstracta .............................................................................. 54
10 About interpretations ......................................................................................... 55
11 Closing remarks ................................................................................................. 57
   11.1 ‘Math-assembler’ and philosophy .............................................................. 57
   11.2 Well, where’s probability? .......................................................................... 59
Acknowledgments ................................................................................................... 60
References .............................................................................................................. 60
1. Introduction

... somewhat curious that, even after nearly a full century, physicists still do not quite agree on what the theory tells us...

G. ’t Hooft [64, p. 5]

It is almost a crying shame that we are nowhere close to that with quantum mechanics, given that it is over 70 years old now

C. Fuchs [50, p. 32]

The contradiction between fundamental nature of quantum theory and phenomenological feature of its mathematics [96] is likely to never cease instigating the attempts to overcome it. The subject-matter and leitmotif of what follows is the fact that the linear superposition and theory’s axioms are entirely empirical in origin, and the challenges that accompany their interpretations are a nonexistent problem; “semantic confusion” [123, p. 10].

1.1. Modern status of quantum theory. The debates concerning the foundations of quantum mechanics (QM) hitherto “show no sign of abating” [120, p. 222], [112, 78], and despite widespread scepticism [79, 41, 37, 53, 101, 82] it is generally acknowledged that the problem is a real one [118, 78, 98, 24]—it is not something made up or “just a dispute over words” [2, p. 5]—and sometimes “has been regarded as a very serious one” [82, p. 418], [144]. It is worth noting that in recent decades the discussions have even worsened [50, 51], and current research has intensified, due to the tremendously increased and formerly inconceivable technological possibilities of operating with individual micro-objects and the urge to implement the idea of quantum computing and computer [30, 2].

The reason for this state of affairs remains the same as it was before. Unlike the classical theories—e.g., thermodynamics or relativity theory, the QM-axiomatics seems wholly divorced from human language [102, 32, 3, 55, 14, 38, 63, 78, 103, 112, 123, 124, 82]. Quantum postulates are not merely formal. They express themselves in terms of linear operators on a complex Hilbert space \( H \) [7, 11, 29, 53, 66, 100, 102, 74, 79] and there is not literally one word here that can be brought into conjunction with reality by means that have at least some kind of relationship with classical description. What is more, it is very well known that the abstract character of these terms is required by the essence of the point (invariance) and, at the same time, that the attempt to link them with physical images leads to the famous paradoxes associated with such concepts as causality, (non)locality, and realism [57, 145, 4, 5, 6, 14, 52, 54, 78, 83, 55]. All of that causes a problem with interpretations of QM.

It is well known that the theory has steadfastly resisted reconciliation between interpretations, which is reflected not only in the voluminoseness of the literature. The differences in viewpoint are often based on points of principle [12, 117, 119, 58, 112, 82, 101, 49, 96, 126], and even highly qualified publications face criticism [40, 89, 108, 94]. Among other things, we encounter appeals [68, 37, 52, 3, 96, 1, 88, 118] (there is even a manifest [137]), striking titles such as “scandal of quantum mechanics” [69], “the Oxford Questions . . . to two clouds” [21, p. 6], “quantum mechanics for the Soviet naval officers” [154, p. 161], “Church of the Larger Hilbert Space” (J. Smolin) [50], and also April Fools’ jokes [17], political parallels with “Marxism” and “the Cold War” [48], and many more [10, 22, 78, 96, 127, 130, 55]. An interesting fact. Cambridge University Press has published a 500-page-long book [50] containing the remarkable electronic correspondence between C. Fuchs and modern researchers in the field.
of quantum foundations. This correspondence has continued [51, over 2200 pages], and now covers the past two decades. It characterizes the state of affairs in the field ([49])!, and does not merely add to one’s impression of the unending discussions about quantum matters (see also introductory section in [154]), it also, due to the lack of formality, represents a source of ideas and of valuable thoughts. Schlosshauer’s very informative ‘quantum interviews’ [118] pursue the same goals.

The lack of transparent motivations for mathematics—a pressing requirement of physics—means that QM-formalism is hard to distinguish from a “cook book of procedures and rituals” (J. Nash) or “user-manual” [124, p. 1690], [152], [55, p. xiii]. Therefore the “dissatisfaction regarding comprehension” and the “need for interpretation that is alien to an exact science” [151, pp. 7–8] lead to the fact that “we admit, be it willingly or not, that quantum mechanics is not a physical theory but a mathematical model” [124, p. 1701] or that “nature imitates a mathematical scheme” (Heisenberg [65, p. 347]). In fact, “we have essentially no grasp on why the theory takes the precise structure that it does” [50, p. 32], which raises the suspicion that “this quantum skyscraper is built on very shaky ground” [154, p. 8].

At the same time, well-founded opinions have long been known to the effect that “quantum theory needs no ‘interpretation’” [52], [37, 69, 96] or that “only consequences of the basic tenets of quantum mechanics can be verified by experiment, and not its basic laws” [41, p. 16]. In other words, the discrepancies between opinions are significant, and often radical: from epithets such as “schizoid”, situation is desperate” [82, p. 420], [105, p. 424] to substantiation of quantum computations [30] and whole books written on the subject [112]. In any case, the contradictions in the views [7, 63], [133, sect. 5.5] cannot be considered an acceptable state of affairs (see also sect. 11.1), for the simple reason that the ‘quantum philosophy’ issues turn into an ‘industry’ of interpretations, while at the same time the very same philosophers call for its denial: “interpretation of QM emerged as a growth industry” [91, p. 92].

1.2. Formula of superposition. In contrast, the “dominant role of mathematics in constructing quantum mechanics” has lead to that mathematical “assumptions are usually considered to be physical” [124, p. 1691]. That is to say, “there has been a substitution of concepts” [152, p. 295] and “one of the consequences of the quantum revolution was the replacement of explanations of physical phenomena by their mathematical description” [152, p. 296]. These characteristics convey, in the best possible way, the dissatisfaction with the fact that quantum physics “was actually reduced to a physical interpretation of the Hilbert space theory” [124, p. 1690]. The $\mathbb{H}$-space in itself is a fairly cumbersome mathematical structure, and even determines a crucial principle—superposition of states [32]. It is thus not surprising that this principle becomes “one of the vague points . . . the [Dirac] argument is difficult to consider as rational . . . the physical principle simply fits underneath it” (excerpt from preface to the Russian edition of [121]).

Mathematics of the $\mathbb{H}$-space contains three constituents: a vector space, the inner-product add-on over it, and topology. The two latter ones invoke the first one which is completely independent (algebra) and begins with the formula

$$|\psi\rangle = a \cdot |\varphi\rangle + b \cdot |\chi\rangle.$$  \hspace{1cm} (1)

*The case in point is the many-world conception by Everett–DeWitt. See also pages 158, 161, 176–179 in [31] regarding the “state of schizophrenia” and ‘explanations’ as to why “schizophrenia cannot be blamed on quantum mechanics” [31, p. 182].
This is the core expression of quantum theory. Understanding its genesis is tantamount to understanding the nature of the linearity of QM. In (1), there appear symbols of complex numbers \(a, b \in \mathbb{C}\), of operations \(\cdot\) and \(+\), and also of vectors \(|\psi\rangle, |\varphi\rangle, |\chi\rangle \in \mathbb{H}\). It is clear that until an empirical basis for all these devices is found, the interpretation of the abstraction (1) and questions of the kind “Quantum States: What the Hell Are They?” (55 times in [51]) will remain a problem. To all appearances, the problem is considered so difficult—“quantum states . . . cannot be ‘found out’” [112, p. 428]—that the (non-axiomatic) meaning of these symbols was not even discussed in the literature. In the meantime, not only is the situation far from hopeless, but it also admits a solution. The present work is devoted to gradual progress towards an understanding of formula (1). Stated differently, equality (1) becomes an empirical theorem (p. 50).

- The main part of the challenge is not only to ascertain what is being added/multiplied in (1), but also to realize what ‘to add/multiply’ is.

Moreover, besides the symbols \(\{a, b, |\psi\rangle, |\varphi\rangle, |\chi\rangle, \cdot, +\}\) the expression (1) contains an equals sign =, and, surprising as it may seem, it conceals one of the key points: the 3-rd principium of QM (III, p. 27).

The guiding observation is based on the fact that the only thing that we have access to are the microscopic events, and therefore “we have little to begin with other than what an experimental physicist would call experiments with a single microsystem” [84, p. 5]. Consequently, we must begin from them and from collecting them into ensemble formations. It is precisely in this context that we will use the word empiricism—quantum empiricism of micro-acts—and it is in this respect that quantum theory has a statistical nature; as predicted by Einstein* [63, Ch.7–8] and von Neumann [102, 127], long emphasized by Ballentine [9] and justified in detail by Ludwig [84, 85, 86, 87]. A. Leggett proposes in this regard the “extreme statistical interpretation” [80], [118, p. 79] and remarks clearly that “to seek any further “meaning” in the formalism is pointless and can only generate pseudoquestions”. With that, he overtly adds such characteristics as “complete gibberish” [80, p. 70] and “verbal window dressing” [118, p. 79]. The difficulty is, of course, in creating the object \(|\psi\rangle\) itself. A step-by-step characterization of this procedure (sects. 3–8) and key words to what follows are reflected in the very (sub)section titles listed in the Contents.

1.3. **Physics = mathematics.** Thus the situation appears to be one whereby the mathematical add-ons are difficult to reconcile with physical motivations (physical principles), while attempts to axiomatize an interface between them [59] only conceal a deeper insight [144]. T. Maudlin: “... physicists have been misled by the mathematical language they use to represent the physical world”.

The approach proposed below is an implementation of the idea that such a view must be abandoned and replaced by a negation of the prior existence of both physical (pre)conceptions and mathematical structures. Physics and mathematics should be created ‘from scratch’. Due to the initial absence of mathematics, the introduction of mathematical structures is almost ruled out, proofs must yield to empirical inference, and the physic’s language—the language of physical reasoning—is initially under a linguistic ban whatsoever; it cannot exist a priori. In other words, even the natural language conjunction of notions with physical adjectives (and verbs [61]) becomes far from being free, as in the case of the classical description’s

* A. Einstein: “It may be a correct theory of statistical laws, but an inadequate conception of individual elementary processes” [63, p. 156]; see also [35, pp. 38–40].
language; see sects. 2.1, 5.4, 6.3, 6.5. The reasoning should be subordinated only to the low-level microscopic empiricism. This immediately touches on its closest creation—the notion of a number, since numbers do not come ‘from the sky’, and the theory will have to be a quantitative one. Numbers then turn into a kind of ‘problem of numbers’ (principium II), and a substantial part of what follows is devoted to this.

On the whole, the aforesaid ideology is supported by the common belief—often certainty even—that QM is not perturbative, its linearity is not associated with linear approximation of something else, and that in general, it is not extensible (ultimate) [28] and must be free of interpretations [37]. All of these issues, in one way or another, are directly related to the derivation of formula (1).

2. Points of departure

In the Beginning was the Word
A. Zeilinger [145, 01:06’05’’]

Most of the time the apparatus
is empty and sometimes you have
a photon is coming through
A. Zeilinger [145, 12’43’’]

Since empiricism is in essence supra-mathematical [23], i.e., it is concerned with meta-mathematics [73, 109], its mathematization (theory construction) should begin not with postulates and definitions, but with the formation of an object language and of terminology [86, 88]. Therefore, relying on the established understanding of the fundamental reasons for the quantum view of physical observation [102, 7, 53, 55], up until the end of this section we will adopt the natural-language meaning of the words observation, state, numbers, physical influence, large/small, micro/macro, etc. Their contents will later be defined more precisely or entirely changed. For instance, the meaning of the word ‘state’ will be radically transformed, to which we are drawing attention in advance. Accordingly, a degree of informality, clarified on page 27 in Remark 10, is inevitable here.

2.1. Variations as micro-level transitions. We will (and ‘must’ [73, Ch. 3]) first view the concept of a system at an intuitive level [34, sect. 1.1]: there is what is referred to as ‘system \( S \).

Let us tentatively (a priori) relate the concept of a state to the associated context described by the words ‘the system \( S \) can be different, or in different states’. That is to say, system \( S \) is always in a certain state \( \Psi \) belonging to the set \( \mathcal{T} = \{ \Psi, \Phi, \ldots \} \), each element of which is admissible for \( S \), and all of them are different from each other: \( \Psi \neq \Phi \).

The statement ‘states are different’ does not require a consideration when \( \Psi \) and \( \Phi \), referred to as state, are the abstract elements of an abstract set \( \mathcal{T} = \{ \Psi, \Phi, \ldots \} \). However, in order to tie its elements to reality, we have to introduce the criteria of coincidence/distinguishability of one from the other. Criteria may come exclusively from observation procedures, without which it is impossible to either detect states, or claim that they differ, coincide, exist or that they actually occur.

On the other hand, the nature of micro-phenomena shows that observations are always associated with irreducible intervention into the system, manifesting in what is known as transition \( \Psi \leadsto \Psi' \) (or destruction\(^*\)). Due to a lack of criteria, there is no sense in attributing

\(^*\) As an example, observations at colliders are literally the destructions, and mass at that.
adjectives small/large, (in)significant/partial, or collocations like ‘comparison of destructions at instants $t_1, t_2$’ to this concept. Let us proceed from the idea that initially there is nothing but the transition; transitions may actually occur without destructions $\Psi \leadsto \Psi$, however.

Two different $\Psi, \Phi$ may be destroyed into new $\Psi', \Phi'$, as well as into the combinations of old/new. Thus, strictly speaking, the meaning of words ‘different, new, . . . ’ eludes us in this case, which is why even the identification of $\Psi$-elements and the $\mathcal{T}$ itself, as a set, become questionable. Therefore, besides the formal writings $\Psi = \Phi$ and $\Psi \neq \Phi$ for $\Psi, \Phi \in \mathcal{T}$, the physical distinguishability/equivalence (recognizability $\not\approx/\approx$) needs to be established. As to the identity in this regard, see von Neumann’s reasoning “One might object against II . . . ” on page 302 of his book [102]. The sole thing that distinguishability may rely on are the transition acts. In turn, variation is a key element in transitions, that is why we will begin constructing with distinguishability.

Let us take the still virtually unlimited way $\mathcal{A}$ of intervening $\leadsto$ in $\mathcal{S}$, and attempt to introduce distinguishability $\Psi \not\approx \Phi$ as $\mathcal{A}$-distinguishability. Due to the fact that micro-transition $\Psi \leadsto \Psi'$ is not pre-determined, initial states $\Psi$ undergo arbitrarily free changes. Next time, the results will be different and absolutely arbitrary (the term ‘different’ is understood to be $\neq$), and each act is indiscernible from a case in which it contains ones similar to itself within itself. It would be natural to associate such a case to the absurd, which is unrelated to the meaning of the word (physical) observation, and to discard the given $\mathcal{A}$.

Non-meaninglessness arises only if we impose the negation of random combinations of $\neq$ and $=$ in transitions, at least for a part of $\mathcal{T}$, i.e., introduce the preservation acts $\Psi \not\leadsto \Psi$. The ‘preservation’ should be read here as indestructibility of state, i.e., as a ($=$)-coincidence under the secondary act $\Psi \leadsto \Psi \leadsto \Psi$. Otherwise, the vanishing difference between ‘preservation’ and ‘variation’ leads to a linguistic chaos [46, p. 232]. This means that the destruction $\Psi \leadsto \Psi'$ may not be considered as a 1-fold one. State $\Psi'$ on the right should be examined for changeability and transform into the left part of the subsequent transition: $\Psi \leadsto \Psi' \leadsto \Psi''$. Thereby, the structure $\Psi' \leadsto \Psi''$ with the binate entity ‘before/after’ or ‘on the left/right’ becomes the key one, and we consider it an initial object in subsequent constructs. Preserved states are, by definition, those that pass the reproducibility test.

Thus logic requires to begin with the transition compositions

$$\Psi \not\leadsto \Psi' \not\leadsto \Psi'' \not\leadsto \ldots,$$

wherein the cases like

$$\Psi' \not\leadsto \Psi' \not\leadsto \Psi''$$

are ruled out (a ban on changing of what has been unchanged), and the never-ending sequence

$$\Psi \not\leadsto \Psi' \not\leadsto \ldots \not\leadsto \Psi'' \not\leadsto \ldots$$

(non-recognisability of states) must be terminated

$$\Psi \not\leadsto \Psi' \not\leadsto \ldots \not\leadsto \Psi'' \not\leadsto \alpha \not\leadsto \alpha,$$

yielding a ‘finiteness’ ($=$ the realistic) and the concept of conserved/distinctive $\alpha$-states. The terminology $\alpha$-event [37] could be used instead.
Freedom of elements in sequence (4), including the choice of $\alpha$-states, is not limited by anything besides the ban on (2). Therefore this arbitrariness, which is physically never recognizable, reduces the generic chain (4) into the shortened one

$$\Psi \rightarrow \alpha \rightarrow \alpha,$$

which is identical to the scheme

$$\cdots \Psi \cdots \rightarrow \alpha \rightarrow \alpha$$

with certain $\alpha \in \Xi$.

Discussions on the fact that “how …” and “what happens” [102, p. 217] at the very microscopic level are extremely widespread in the literature [54, 5, 68, 83, 89, 98, 151] (see [4, 7, 114, 24] for the exhaustive references), although it is not difficult to predict the fact that the attempts to understand the inner structure of box (6) will only lead back to an identical box.

Actually, the uncontrollability of micro-changes is universally known, yet describing them as a process in time $t \mapsto t + \varepsilon$ will start employing the language terminology (functions, arithmetic operations, etc) that has not yet been created even for the fixed instants $t_1$, $t_2$. However, what may be associated with fixed time are only non-temporal entities, for which we have nothing but transitions (5). The attempt to manage them, i.e., to control intervention into $S$, leads to looping, or ‘measuring the measurement’, not to mention the ambiguity of the term itself. G. Ludwig: “it is not meaningful to speak of a measurement “at time $\tilde{t}$.” … the real physical meaning of the time parameter … has nothing to do with the notion “time of measurement.” [84, p. 288]. See also [85, p. 365], [87, p. 150], [88, p. 100]. Just as before, the physical assessments like ‘abrupt’, ‘(ir)reversible’, ‘(non)simultaneous’, “immediately following …,” [102, p. 231, 410], etc are unacceptable here. No temporal process may be present in the foundations of the theory [84, Ch. VII.4, 6], [85, Chs. III, XVII], [86], since it is not immediately clear ‘And what exactly are we having at instants $t_1$ or $t_2$?’.

Remark 1. All said above means that attempts to deduce QM dynamically [118, 10 · Reconstructions] are beforehand doomed to vicious circles ‘round the boxes’ and time $t$ like attempts to ‘substantiate’ dynamically Lorenz’s contraction instead of kinematic postulates of the relativity theory [22]. Consistent theory must rest either upon ‘irreducible’ elements (6) or upon ‘boxes’ of a different kind.

In the latter case, the theory becomes a particular model with interpretation; e.g., the Lindblad equations [141, 142], decoherence [67, 148, 114, 116], stochastic dynamics, and other statistic-dynamical models [4, 151]. Anyway, simulation and comprehension are not the same thing, and this point, with regard to quantum theory, was repeatedly emphasized in the literature [124], [118].

Now, if theory is built as a fundamental one, rather than as a model [118, p. 144], with primary entity changeability $\cdots \Psi \cdots \rightarrow \alpha$, a box (6) may only be involved in it as the initial starting point and as an indescribable object. Elements of reality, in any understanding, may not exist inside/outside of the box; it can be only structureless. Accordingly, the notions of preparation, of measurement, and of a physical process are meaningless without construction (6). These statements are clearly in agreement with the fact that any reasoning must not contradict the formal logical rules [73], hence, there must exist [109, 122] the empirically undefinable logical atoms. A. Peres: “While quantum theory can in principle describe anything, a quantum description cannot include everything. In every physical situation something must remain unanalyzed” [104, p. 173; emphases original]. Or, expressed by Pauli’s words, “Like the ultimate fact without any cause, the individual outcome of a measurement is … not
comprehended by laws”. In particular, the set $\mathcal{T}$ and transitions-arrows $\Leftrightarrow$ are also the atoms. From the aforesaid, we may formulate the following observation.

I Quantum statics should precede quantum dynamics (the 1-st principium of QM).

The rationales do not end here, and will be later reinforced once we begin to exploit the terminology that is usually taken for granted from the outset; namely, the quantitative descriptions (see [50, p. 178]). If they arise not as numeric interpretations of something, but out of an experiment, then observation should be the beginning, and the ‘production of numbers’—the end. In other words, the model theory ‘with boxes’ other than (5)–(6) implicitly implies the logical sequence \{model of process $\rightarrow$ numeric interpretation\}, where empiricism holds a role other than primary. It is clear that, regardless of the model, such a situation will always remain unsatisfactory in the physical respect.

2.2. Observation. The sequences addressed above lead to the following outcome.

- Any physically meaningful micro-observation $\Leftrightarrow$ either saves a state ($\alpha$ $\Leftrightarrow$ $\alpha$) or turns it into a conserved one ($\Psi$ $\Leftrightarrow$ $\alpha$).

The two extremes do not contradict this fact. The first—maximally rough observations—is when all states are destroyed into a certain one: $\Psi \leadsto \Psi_0$ (‘whatever and however we watch, all we see is one and the same’). In this, the state $\Psi_0$ is not destroyed: $\Psi_0 \leadsto \Psi_0$. Another extreme is when none of the states are destroyed: $\Psi \leadsto \Psi$. This is the case of ideal (quantum) observation, but, due to the absence of any changes, it is indistinguishable from the case of if observations are entirely absent.

A case with two distinctive states

$$\alpha_1 \Leftrightarrow \alpha_1, \quad \alpha_2 \Leftrightarrow \alpha_2$$

is the simplest among these extremes. Of course, the $\alpha$’s are prohibited from transitioning into each other. Because there is still the free admissibility of transitions $\Psi \Leftrightarrow \alpha_1$, $\Psi \Leftrightarrow \alpha_2$, we can turn the semantic sequence

\{arbitrariness $\rightarrow$ preservation $\rightarrow$ distinctive $\alpha$’s\}

into the more rigorous scheme

$$\left( \mathcal{T} = \{\Psi, \Phi, \ldots\} + \mathcal{A}\text{-observations} \right) \rightarrow \{\alpha_1, \alpha_2, \ldots\} \equiv \mathcal{T}_{\mathcal{A}} \subset \mathcal{T},$$

which gives rise to the concept of a physical distinguishability (‘distinguo’), albeit partially. It is formally defined only on the subset $\mathcal{T}_{\mathcal{A}}$: the statement $\alpha_1 \not\approx \alpha_2$ is equivalent to (7). To avoid overloading the further notation, we do not use symbols like $\approx_{\mathcal{A}}$ and $\not\approx_{\mathcal{A}}$; the context is always obvious.

O By a physical observation $\mathcal{A}$ or, in short, observation we will mean such interventions* $\Leftrightarrow$, in which the ‘never-ending’ chaos (3) is replaced by a chaos with the notion of a preservation, i.e., ‘chaos with rule (6)’:

$$\Psi \Leftrightarrow \alpha, \quad \text{where} \quad \alpha \Leftrightarrow \alpha.$$
The set of $\alpha$-objects $\mathcal{I}_{\alpha}$ with the property

$$\begin{align*}
\alpha_1 \rightarrow A \alpha_1, \quad \alpha_2 \rightarrow A \alpha_2, \quad \ldots
\end{align*}$$

(10)

is discrete, and the $\alpha_s$ themselves are termed the **eigen** (proper) for observation $A$. They define $A$ and do not depend on $S$. No logical connection between $\Psi$ (the left of (9)), family $\mathcal{I}_{\alpha}$, and system $S$ exists.

Expressed another way, introduction of the conception ‘the eigen’ is equivalent to the following informal, yet minimal motivation: at least some certainty instead of total arbitrariness.

Two instruments $A$ and $B$ may have arbitrarily different eigen-states $\{\alpha_1, \ldots, \alpha_n\} \neq \{\beta_1, \ldots, \beta_m\}$. Accordingly, as regards observation $B$, the (distinctive) states $\{\alpha_s\}$ do not differ, in general, from the ‘regular’ $\Psi$, i.e., from those chaotically destroyable into the $B$-eigen states $\beta_k$. All kinds of instruments $A, B, \ldots$ are thus defined by aggregates $\mathcal{I}_{\alpha}$, $\mathcal{I}_{\alpha}$, $\ldots$. The number $|\mathcal{I}_{\alpha}|$ of corresponding $\alpha$-objects therein may be an arbitrary integer.

There are also no (logical) grounds for restricting/prescribing the composition of $\mathcal{I}_{\alpha}$; any element of $\mathcal{I}$ may be the conserved one for a certain instrument. Parenthetically, the notion of eigen-state—in different forms—is sometimes present in axiomatics of QM [106, 15, 98].

In a generic case, the chaos present in (9) leaves open the problem of correlating the recognizability $\Psi \not\approx \Phi$ (or $\Psi \approx \Phi$) with physics. Clearly, the issue is linked to the ambiguity of the term $\Psi$-state itself, which is used in pt. $S$—an important point—due to the need to start with something, since building the mathematical description without some sort of a set is impossible.

**Remark** 2. Informally, metalinguistic aspects (semantics) are in general as follows. Inasmuch as we are receiving different $\alpha$-responses to each micro-act $\rightarrow A$, let us say that ‘on the other side from us there is something that can also be different, and all of that ought to be described’. This reflects our intuitive perception of reality, which, both at micro- and macro-level, boils down to an ineradicable pair: ‘something outside’ + ‘that which can be different for us’ (pt. $S$). If we give up any of these conceptual premises—‘something outside’ ($\Psi$) and ‘can be different’ ($\alpha_j$), then, as above, we face a linguistic dead end, as the basis for reasoning disappears. Because of this the arrow $\rightarrow A$ must be accompanied by ‘some thing’ to the left and the right of it; the low-level set $\mathcal{I} = \{\Psi, \Phi, \ldots, \alpha_1, \alpha_2, \ldots\}$ does arise. Then the arbitrary elements $\Psi \in \mathcal{I}$ (unrestricted chaos) are assigned to the left of this act instead of ‘some thing’, and the micro $\not\approx$-distinguishable $\alpha$-objects ($\alpha_j \not\approx \alpha_s$)—to the right. Put another way,

- what’s being abstracted is not “concrete things” [53, p. 27] but a primitive element of perception—micro-event $\alpha$-click*. The initial mathematical premises of QM-theory should contain nothing but the $\not\approx$-distinguishability and formalization (9)–(10).

It may be added that the micro-observation, as such, is terminated at the eigen elements; one and the same $\alpha_j$ has always remained on the right.

As a result, the minimal experimental entity $\Psi \rightarrow A \alpha$ constitutes, mathematically, an ordered pair $(\Psi, \alpha_j)$ of elements of the set $\mathcal{I}$, which are labeled by the symbol $A$ that is equivalent to the $\mathcal{I}_{\alpha}$-family (8).

### 2.3. Numeric realizations.

Is there a possibility to rely exclusively on the inflexibility of the eigen-type elements (10)? Or to define the sought-for ultimate distinguishability $\not\approx$ through the $A$-(micro)distinguishabilities $\alpha_j \not\approx A \alpha_s$? Let us formulate a thesis.

* As to these ‘qualia’, see answers of Č. Brukner on pp. 41–43 in [118] and page 635 in [144].
There is no (linguistic) means of recognizing the system $S$ to be different (pt. $S$), other than through the results of its destructions into the $\{\alpha_1, \alpha_2, \ldots\}$-objects of observational instruments $\mathcal{A}$.

Certainly, the stringency of this linguistic taboo ($T$) must be accompanied by something constructive, and we will adopt the following programme that reflects the fact that the unequivocal description may take only the form of a quantitative mathematical theory.

$R$  (●) Of primary elements $\{\Psi, \alpha_1, \ldots\} \in \mathfrak{X}$ one constructs a new set $\mathbb{H}$, of which the elements

$$|\Xi\rangle \equiv \bigoplus(a_1, |\alpha_1\rangle; a_2, |\alpha_2\rangle; \ldots) \in \mathbb{H} \quad (11)$$

are said to be (numeric) representations in the ‘reference frame for instrument $\mathcal{A}$’, and $a_s$ are the numeric objects. The distinguishability relation $\not\approx_{\mathcal{A}}$ is carried over to $\mathbb{H}$ and admits an $a$-coordinate realization there; symbol $\not\approx$.

(●●) No preferential or preordained observational reference frame $\mathcal{A}\{\alpha_1, \alpha_2, \ldots\}$ (‘instrument the absolute’) exists.

Identification (11) is always tied to a certain family $\mathfrak{X}_{\mathcal{A}}$. Accordingly, images of $\alpha_s$—symbols $|\alpha_s\rangle$—will be present in (11), and character $\bigoplus$ is also no more than a symbol here. Even though coordinates $a_s$ are declared to be numbers or aggregates of numbers, there is no arithmetic stipulated for them yet. Distinguishability $|\Psi\rangle \not\approx |\tilde{\Psi}\rangle$ of two representatives

$$\bigoplus(a_1, |\alpha_1\rangle; a_2, |\alpha_2\rangle; \ldots) \equiv |\Psi\rangle, \quad \bigoplus(\tilde{a}_1, |\alpha_1\rangle; \tilde{a}_2, |\alpha_2\rangle; \ldots) \equiv |\tilde{\Psi}\rangle$$

by means of numbers $a_k \neq \tilde{a}_k$ and mathematical implementation of (11) and of the $\mathbb{H}$-space, i.e., a ‘coordinatization’ scheme, have yet to be established. This will comprise the meaning of the word ‘constructs’ (sects. 7–8), which may not be even linked to the mathematical term mapping yet, since no math of QM exists at the moment. It immediately follows that the question about numeric entities—in particular, about (11)—is nontrivial in physics.

II To speak of an exact correspondence between empiricism and mathematics makes no sense until the mechanism of emergence of that which is understood by number* has been detailed (the 2-nd principium of QM).

From pts. $T$, $R$, and $II$ it also follows that the search for a description through hidden variables, over which something is averaged, is indistinguishable from the utopian attempts to find out an intrinsic content of boxes (5).

2.4. Macro and micro. The task becomes more precise at this point. Instead of nonphysical identity/noncoincidence ($\Psi = \Phi$ or $\Psi \neq \Phi$) of two abstract elements $\Psi, \Phi$ of the abstract set $\mathfrak{X}$, the concept of a physical $\not\approx$-equivalence ($\not\approx$-distinguishability) of $\mathbb{H}$-representatives $\{|\Psi\rangle, |\Phi\rangle, \ldots\}$ is needed. That is, there must hold either relation $|\Psi\rangle \approx |\Phi\rangle$ or its negation $|\Psi\rangle \not\approx |\Phi\rangle$ for all $|\Psi\rangle, |\Phi\rangle \in \mathbb{H}$. The primitive set $\mathfrak{X}$, initially required by point $S$, must disappear from the ultimate mathematics in terms of $|\Psi\rangle \in \mathbb{H}$. Therefore elements $\Psi \in \mathfrak{X}$ are henceforth named primitives. In summary,

- There is no a priori way to endow the term (quantum) state of system $S$ with any meaning [82, p. 419]; this concept should be created. Meanwhile, one cannot get

* What an empiric/observer understands by the word number. The underlying message here implies that the reliance upon the all-too-familiar arithmetic explains nothing. The empirical nature of its emergence must be scrutinized.
around the concept of a (micro)observation $\mathcal{A}$. Essentially, no one thing, including $\Psi$, $\alpha$, or the $\mathcal{I}$-set itself, can be the primary bearer of data on $\mathcal{S}$.

The notions of a physical observable and of its observable values [84] are also ambiguous at this point. Their ambiguity is even greater than that of state, due to questions like ‘what is being measured?’ and even ‘what is a measurement?’. Nonetheless, we will not discard the habitual (pt. S) term state up until the end of sect. 2.

Irreproducibility of outcomes, i.e., ‘turnability of $\Psi$-primitives into the various’ leaves only one option: ‘to take a look at $\mathcal{S}$ again, once again, . . . ’. In other words, to seek the source of description in repeatability. It is necessary, then, to move to the subject of macro-, rather than micro-observation. This intention fits perfectly with the undefined verb ‘constructs’ in pt. R, and the following paradigm should be understood as the macro.

$\mathcal{M}$ The only way of handling the uncontrollable micro-level changes is the consideration of the results of repeated destructions, accompanied by what is called the common physical macro-environment (experimental context):

$$
\begin{align*}
\Psi & \quad \ldots \quad \Psi \\
\alpha_1 & \quad \ldots \quad \alpha_1 \\
\vdots & \\
\alpha_2 & \quad \ldots \quad \alpha_2
\end{align*} + \{\text{common macro-environment } \mathcal{M}\}. \quad (12)
$$

An importance of repetitions and distinguishability has long been noted (Bohr, von Neumann et al [65]) and recently it was particularly emphasized in the work [149]. The words ‘copy/repeat.../distin...’ occur 90 times therein.

Thus empiricism of quantum statics forces us to operate exclusively with such formations of copies $\alpha$, . . . , $\Psi$, and this is the maximum amount of data provided by the supra-mathematical problem setup. All further mathematical structures may come only from constructions like (12) and from nothing else. Getting ahead of ourselves, let us turn our attention to the fact that implementation of this idea is not brief—“the mathematization process (cor) is not simple” [88, p. 24] (see also a footnote on page 54), and sects. 3–9 are devoted specifically to this idea.

One can once again repeat (sect. 1.3) that much of what follows does not and cannot contain the mathematical proofs as they are usually present in the literature on quantum foundations. Instead, there appears the step-by-step inference of objects as they result themselves: numbers, operations, groups, algebras, etc. The only instrument that may be applicable here is empirical inference.

The common macro-environment $\mathcal{M}$ in (12) is also considered as a supra-mathematical notion [109], the mathematical implementation of which is yet to be created. The same considerations regarding qualitative adjectives are applicable as to the physical convention $\mathcal{M}$ as well as the transition acts in sect. 2.1. Representations (11) are the formalization of the meaning \{observation + data on system $\mathcal{S}$\}, but now with no references to the elementary acts in (12). The physical distinguishability criteria $|\Psi\rangle \not\sim |\Phi\rangle$ may not be formulated yet (there are no physical attributes yet), but $|\alpha_s\rangle$-elements have already appeared in (11) as prototypes of explicitly distinguishable $\alpha_s$.

Remark 3. Dual form of the typical quantum statements like ‘$\mathcal{S}$ is a micro-system and $\mathcal{A}$-instrument is a macro-object’ (Bohr) is identical to the initial premise ‘observation does always destroy a system’. It follows that there is actually little need for that terminology. Indeed, QM-micro has no internal structure; hence, an often discussed issue about boundary (and limits*) between micro and macro

\* A. Zeilinger: “... no limit. The limit is only a question of ... money and of experience” [145, 13’09’’].
[112, 63, 9, 7] is devoid of sense. This may be a matter only of ‘different macros’—the ‘smaller/bigger’; i.e., when they describe certain models.

As a (partially philosophical) aside, we note that what is understood by observational indeterminism does, in fact, boil down to distinguishability, and more specifically, to postulating the micro-chaos (9). In considering the denial of (9) as an impossible proposition, we arrive at the M-paradigm and conclude that the only way to deal with that which is contemplated for the subject-matter of a physical description must be the treatment of micro-acts as ensembles [97, Lect. 6]. In other words, and in accordance with the outline of the analysis of clicks set out below, the determinism of micro-processes is meaningless as a concept, since they are not processes but structure-less acts that have not even any relationship to each other. Since there are as yet no any physical phenomena as such, the claim that ‘phenomenon-1 appears to be the cause that precedes phenomenon-2 as the effect’ is no more than a collection of words. To attribute to them physical content and mathematical formulation at the micro-level is impossible in principle; the ‘problem of boxes’ noted above. Accordingly, the cause of (classical) macro-indeterminism is the absurdity of the notion of its twin concept—micro-determinism—and the unavoidable repetition of the arrows \( \rightarrow \) (M); “there can be no question of causality in the ordinary sense of the word” [65, p. 351]. See also [97, p. 223].

2.5. **Quantum ensembles and statistics.** Let us call the upper row in (12), as a \( \Psi \)-copy set, (quantum) homogeneous ensemble (von Mises Kollektiv [97]). We will designate it, simplifying when needed, by any way

\[
\{\Psi \Psi \cdots \Psi\} \equiv \{\Psi \cdots \Psi\}_N \equiv \{\Psi\}_N,
\]

where \( N \) is understood to be an arbitrary large number. Scheme (12) dictates also to consider the generic ensembles

\[
\{\{\alpha_1 \cdots \alpha_1\}_n_1 \{\alpha_2 \cdots \alpha_2\}_n_2 \cdots \cdots \}, \quad \{\cdots \Psi \cdots \Psi \Phi \cdots \Phi \Theta \cdots \Theta \cdots\}
\]

(13)
as collections of homogeneous sub-ensembles. Ensembles are symbolized in the same manner as sets, but, for the convenience of perception, without the numerous commas and internal parentheses \{\} in (13); for example,

\[
\{ab \cdots b\{bca\} \cdots\} = \{a, b, \ldots, b, b, c, a, \ldots\} = \cdots \equiv: \{ab \cdots bbca \cdots\}.
\]

Scheme (12) is the first place where numbers emerge in a theory, and conversion

\[
\{\alpha\text{-ensemble (13)}\} \rightarrow (n_1, n_2, \ldots)
\]

into the integer collection precedes a numerical \( \mathfrak{A} \)-measurement of \( S \). Quantities \( n_s \in \mathbb{Z}^+ \), however, should not be associated with such, as they are potentially infinite. The minimal way of creating the knowingly finite numbers out of independent and potential infinities \( n_s \) (without loss of their independence) is to divide each of them by a greater infinity, which is a constant \( \Sigma \) for the entire ensemble. It is clear that one should put

\[
\Sigma := n_1 + n_2 + \cdots \quad \text{and} \quad \nu_1 := \frac{n_1}{\Sigma}, \quad \nu_2 := \frac{n_2}{\Sigma}, \quad \cdots \quad (\Sigma \rightarrow \infty), \quad (14)
\]

and that ensemble (13) does not provide any numerical data besides the relative frequencies (14); all the other data are functions of \( \nu_s \). An independence of the theory from the ensemble \( \Sigma \)-constant, i.e., the limit \( \Sigma \rightarrow \infty \), is also implied to be a principle—the \( \Sigma \)-postulate.

Thus the M-paradigm (12) does not only give birth to a concept of ‘numerical data in the theory’ per se, but also converts their \( \mathbb{Z}^+ \)-discreteness into the \( \mathbb{R} \)-continuum of real measurements. Namely, numbers \( \nu_s \in \mathbb{R} \) are the statistic \((\nu_1,\nu_2,\ldots)\) of destructions \( \rightarrow \mathfrak{a} \) into the ensemble of primitives \( \{\{\alpha_1\}_n_1, \{\alpha_2\}_n_2, \ldots\} \).
2.6. Distinguishability and numbers. Distinguishability of the two ensembles now turns out to be the $\mathbb{R}$-numerical; it is determined by the difference between $\nu$-numbers. As a result, and according to pt. R, the two elements $|\Psi\rangle \not\approx |\tilde{\Psi}\rangle$ of $\mathbb{H}$ will differ in the numbers $a_s$ and $\tilde{a}_s$, if the latter turn out to be the bearers of different statistics

$$\nu_j(a_1, a_2, \ldots) \neq \tilde{\nu}_j(\tilde{a}_1, \tilde{a}_2, \ldots).$$

Hence, distinguishability $\not\approx$ is carried over to $\mathbb{H}$ with an extension to the non-eigen objects, but it is inherently incomplete, since it does not take into consideration the most significant fact—arbitrariness of transitions (6).

Actually, the collection $(\nu_1, \ldots)$, as a final result of transitions $\{\Psi \rightarrow \alpha_s\}$, ‘knows nothing’ about their left hand side, much less about its uniqueness $\Psi$. If, for instance, we would require $\not\approx = \Psi$ under equal $\alpha$-statistics $\{\nu_s\}$ for the two families $\{\not\rightarrow \alpha_s\}$ and $\{\Psi \rightarrow \alpha_s\}$ (collectivity of $\not$’s), that would mean a mass control over transitions (9). Instead of a ‘black box’ above, we find that prior to acts $\not\rightarrow$ all the $\not$’s were equal to $\Psi$. This, however, is the declaration of a property: ‘prior to observation system $S$ was/existed in . . . ’. With any continuation of this sentence, it is pointless and prohibited, if it is theoretically accepted that prior to observation nothing exists and there are no properties (sect. 2.1). The indeterminacy of the ingoing $\not$’s is therefore mandatory, and numbers $(\nu_1, \nu_2, \ldots)$ required for recognition are apparently insufficient. Since the micro-changeability of single primitives $\Psi$ means nothing [5], [7, p. 493], [82, p. 419(!), left column], only a generic ensemble

$$\{\not \cdots \not\} \mapsto \{\cdots \Psi \cdots \tilde{\Psi} \cdots \tilde{\Theta} \cdots \Theta \cdots \} \equiv: \mathcal{A}$$

(16)
can be the intermediary in the sought-for translation of $\Psi \in \mathcal{T}$ onto representations $|\Xi\rangle \in \mathbb{H}$, when constructing (11).

In the accustomed physics terminology, the above is expressed in the sequence

state $\not\rightarrow$ quant $\rightarrow$ state’ $\rightarrow$ measurement. (17)

The removal of the intermediate component here, i.e., switch to the sequence

state $\not\rightarrow$ class $\rightarrow$ · · · $\rightarrow$ measurement, (18)
is amount to the rejection of micro-destructibility and of unpredictability. Even with the classical consideration, this supposition is questionable, since the notion of ‘change when observed’ disappears. The relationships between the dual concepts—(micro/macro)-scopicity, big/middle/small, etc—are also lost. That is the reason why, strictly speaking, all observations, regardless of (envisioned physical) macro/meso/micro characteristics, have the structure (17), i.e., are the quantum observations; no non-quantal observations exist. With their idealized ‘roughening’, the classical description appends numeric $\nu$-statistic to (18), which is when the left/right parts of (18) become indistinguishable with respect to the arrow symbols; the arrows may then be replaced with the equivalence

state $\not\rightarrow$ class $\{\nu$-statistic numbers$\}$. (19)

Supplementing the right hand side here with the concept of spectral values $\{\alpha_s\}$ for all of the observables $\mathcal{A} = A(q, p)$ (or for phase observables $\{q_1, q_2, \ldots p_1, p_2, \ldots\}$), this side will turn into an exhaustive numerical realization of the left hand side. Criterion $\approx$, then, turn into the $\mathbb{R}$-numeric equality $=$ of all the $\mathcal{A}$-statistics or into an equality of phase distributions.
\( q(1, \ldots; p_1, \ldots) \). This is a situation of the classical (statistical) physics (ClassPhys). In both these cases, distinguishability \( \not\approx \) depends on the concept of \( \alpha \)-states.

**Remark 4.** From this point onward, by state we will strictly mean representations (11). So it makes no sense to speak of transitions between states in this case. The writing \( |\Xi\rangle \to |\alpha\rangle \) and its typical the wave function collapse interpretation are not correct. In fact, treating transition \( |\Xi\rangle \to |\alpha_1\rangle \) as a state-to-state destruction, its left hand side cannot carry any information about \( \nu_{(\alpha)} \)-frequencies for other events \( |\Xi\rangle \to |\alpha_s\rangle \), much less about amount of destruction from envisioned \( \mathcal{B} \)-observations \( |\Xi\rangle \to |\beta\rangle \). Such \( \nu_{(\beta)} \)-amounts’ are always present at standard comprehension of the \( |\Xi\rangle \)-symbol. By this reason, the concept of a state should not be used as a correct term at all [88]; the terminology, however, has been settled.

The motivation given above—S (system, primitives), O (observation) and R (representations), T (taboo), principia I (QM-statics) and II (numbers)—is sufficient for further creating the basis of mathematical formalism of QM. These principles and motivation can hardly be considered as postulates, at least in the common meaning of the phrase ‘postulates of a physical theory’, since they are of a naturally language nature, and are, as we believe, the points of departure for reasonings, whatever the approach to the micro-world. As it was emphasized above, these principles do not stipulate for pre-determined mathematics and physics, with the exception of an already established metamathematical (and linguistic) understanding [73, 109] of how to approach the mathematical axioms, structures, theories, and their interpretations altogether; see also Remarks 7 and 10, and sects. 5, 10.

### 3. Ensemble formations

*Your acquaintance with reality grows literally by buds or drops of perceptions. . . . they come totally or not at all*

W. James

The key corollary of the macro-paradigm (12) is not merely the appearance of numeric data in the theory, but also the fact that the further construct must rely not on isolated primitives, but on their aggregates being considered as an integrated whole, i.e., as a set. This causes a choice for the ensemble notation.

#### 3.1. Mixtures of ensembles.

Returning to the analysis of transitions \( \Xi \to \alpha_s \), one gets that the lower row in (12) comes actually from indeterminacy

\[
\{\alpha_1\}_{n_1} \{\alpha_2\}_{n_2} \cdots \cdots
\]

\[
\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow
\]

\[
\cdots \Xi \cdots \Xi \cdots \Xi \cdots
\]

and, thus, (12) should be replaced with the scheme

\[
\{\cdots \Psi \cdots \Phi \cdots \Theta \cdots\}
\]

\[
\mathcal{A} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \mathcal{A}
\]

\[
\{\cdots \alpha_1 \cdots \alpha_2 \cdots \}
\]

wherein the composition of the upper ingoing row may not be predetermined. Fundamentally, according to (17) it may not be withdrawn from (20), and, at the same time, the meaning of the row can in no way be aligned with the adjective ‘observable’ via typical empirical/physical
words: properties, readings, quantities/amounts, and other ‘observable’ characteristics. Such a non-detectability is the equivalent of that a box may be prepended to the scheme (6):

\[ \cdots \cdots \longrightarrow \Psi \longrightarrow \alpha. \]  

(21)

If \( \beta \)'s serve as \( \Psi \) in (21), then we have the schemes of precedence and of continuation:

\[ \cdots \cdots \longrightarrow \alpha \rightarrow \beta \rightarrow \beta \text{ or } \cdots \cdots \rightarrow \beta \rightarrow \alpha. \]

Let an observer capture the fact of any distinguishability in the intermediate \( \mathcal{A} \). Section 2.1 tells us that this may be only the distinguishability of objects \( \{\alpha_1, \alpha_2, \ldots\} \); hence, this very \( \mathcal{A} \) turns into an observation (pt. O). The \( \mathcal{M} \)-paradigm then gives rise to the numbers of \( \alpha \)-events \( (n_1, n_2, \ldots) \) and their relative frequencies \( (w_1, w_2, \ldots) \) by the rule (14). If subsequently micro-observations \( \mathcal{B} \) are to follow, then a composite macro-observation \( \mathcal{B} \circ \mathcal{A} \) has been formed, and frequencies \( \{w_j\} \) cannot but impact on statistical characteristics of these later \( \mathcal{B} \)-observation’s micro-events. However, being an ingoing ensemble for \( \mathcal{B} \), each homogeneous \( \{\alpha_s \cdots \alpha_s\}_n \) is indistinguishable from an indefinite ensemble \( \{\cdots \Psi \cdots \Phi \cdots\}_n \), since the concept of ‘\( \approx_s \)-sameness’ is unknown for \( \mathcal{B} \). Instrument \( \mathcal{B} \) is ‘aware of only its own \( \approx_s \) and cannot know what it destroys\(^*\), or that the source-object consists of one and the same \( \alpha_s \). According to pts. O and M, an instrument produces nothing more than its own ‘destruction list’; in this case \( \{\beta_1\}_m, \{\beta_2\}_m, \ldots \). This list, in general, is completely independent of the other ones, since, according to pt. R(●●), there can not be restrictions on \( \Sigma_\alpha \) and \( \Sigma_\beta \). In case the set \( \{\alpha_s \cdots \alpha_s\}_n \) transits into collection \( \{\beta_k \cdots \beta_k\}_n \), this means that \( \alpha_s \) has always transited into one and the same \( \beta_k \) every time (under the convention \( \Sigma \rightarrow \infty \)), and there takes place merely a coincidence \( \alpha_s = \beta_k \) of eigen-primitives in the lists \( \Sigma_\alpha \) and \( \Sigma_\beta \).

If \( \mathcal{B} \circ \mathcal{A} \) is proceeded with a third observation \( \mathcal{C} \), the preceding analysis is repeated recursively with the same result; only the values \( \{w_j\} \) will be changed. As a result, only the following two ingoing types for macro-scheme (20) are conceivable:

\[ \{\cdots \Psi \cdots \Phi \cdots \Theta \cdots\} \]  

indefinite ensemble  

(no statistic),

(22)

\[ \{\cdots \Psi \cdots \Phi \cdots\}^{(w_1)} \{\cdots \Psi \cdots \Phi \cdots\}^{(w_2)} \cdots \]  

ensemble mixture  

(with statistic \( (w_1, w_2, \ldots) \)).

(23)

It is reasonable to regard case (23) as the preparation of systems \( \{S_1^{(w_1)}, S_2^{(w_2)}, \ldots\} \), which are non-isolated from each other, and to each of which we assign the number \( 0 < w_s < 1 \), referred to as its statistical weight. These weights are all that is inherited from the preparation \( \mathcal{A} \), and subsequent micro-observation acts \( \mathcal{B} \) are performed on indefinite ensembles (22) again. It is clear that in the view of transitions \( \longrightarrow \) in scheme (20), this situation is a derivative of (22), and this very type (22) is crucial [11, p. 53]. In other words, if preparation is considered a concept as essential as observation, we still remain within the framework of the primary essence of a transition:

\[ \Psi \longrightarrow \beta, \quad \alpha \longrightarrow \beta. \]

Its left hand side should always be seen as an undetermined primitive, even though we treat/call it preparatory (micro)observation; see also “preparation-measurement reciprocity” in [13].

\(^*\) Rejection of this point brings us once again (p. 8) to attempts at penetrating into the ‘black box’ of transitions (5), i.e., to attempts at creating the physics of a more primary level.
3.2. Ensemble brace. According to pts. R and M, representations (11) must reflect all information about physics of the problem: primitives/incomes, transitions (‘arrows’ \( \rightarrow \)), and outgoing statistics. All that data are contained in scheme (20), which is why the maximum that the model of a future mathematical object, which characterizes everything we get while watching the \( \mathcal{S} \), can rely on is the ensemble brace:

\[
\{\Xi\} \equiv \begin{cases} 
\{\cdots \Psi \cdots \Phi \cdots \Theta \cdots \} \\
\{\cdots \alpha_1 \cdots \alpha_2 \cdots \}
\end{cases}
\]

(or a couple of ensemble bunches). It is immediately seen that (24) carries the radical difference between situation (17) and its ‘roughening’ (19); because of the upper row. The enormous arbitrariness within the brace and arrows \( \rightarrow \) must inherently give birth to different processing rules of statistics and to effects typical for QM. Thanks to maximality of (24), it has been just this row which encloses all the sought-for cases of distinguishability \( \not\approx \). In particular, by varying the upper row, while the lower one remains unchanged, we get into a situation, when \( \alpha \)-statistic \( (\nu_1, \nu_2, \ldots) \) turns out to be the same for \( \{\Xi\} \) and \( \{\tilde{\Xi}\} \), meanwhile, \( \{\Xi\} \not\approx \{\tilde{\Xi}\} \).

The further problem is thus as follows. With the indefinite \( \mathfrak{H}\)-ensemble (16) in hand, i.e., with the upper row of (24), is it possible, based on the principles described above, to bring the still incomplete relation \( \not\approx \) to the maximal quantum-physical distinguishability of states?

4. Why domain \( \mathbb{C} \) arises?

\[
\ldots \text{quod ideo sint imaginariae,} \ldots \]

\[
\text{quod ideo sint} \ldots \text{tum certe forent reales ideoque non imaginariae*}
\]

L. Euler (1736)

The first priority, in the \( \not\approx \)-distinguishability of objects (24), is to separate the closest and unconditional criterion—the outgoing \( \alpha \)-statistic. To do this, let us split the lower row into families \( \{\{\alpha_1\}_\infty \{\alpha_2\}_\infty \cdots\} \), where

\[
\infty_1 + \infty_2 + \cdots = \infty, \quad (25)
\]

and, subsequently (rather than the reverse, otherwise (23)), taking into account the arbitrariness of ‘arrows’, we also split the upper row:

\[
\{\Xi\} = \begin{cases} 
\{\cdots \Psi \cdots \Phi \cdots \}_\infty \\
\{\cdots \alpha_1 \cdots \alpha_1 \}_\infty 
\end{cases}
\]

(26)

\[
\begin{cases} 
\{\cdots \Psi \cdots \Phi \cdots \}_\infty \\
\{\cdots \alpha_1 \cdots \alpha_1 \}_\infty
\end{cases}
\]

(27)

* ... this is why they are imaginary. Were they ... they would certainly be real and therefore not imaginary.
that differ from each other in the upper row composition.

4.1. **Continuum of quantum phases.** Cardinality of the $\mathcal{T}$-set cannot be finite; this would entail finitely many $\alpha$-primitives for all kinds of instruments. But finiteness of this number $N$ would mean an exclusivity of its value, that does not follows from anywhere. At the same time, all the $\mathcal{A}$-ensembles (16) are subsets of the set $\mathcal{T}$ (boolean $2^\mathcal{T}$); any finite portion of it is ruled out. Hence, the infinite variety of upper rows in (27) is uncountable.

Aside from numeric $\nu$-statistic, the program $R$ does also require an association of the numeric objects with each row

$$\mathcal{A} = \{\ldots \Psi \ldots \Phi \ldots \Theta \ldots \} \infty \iff \ldots,$$

because primitive’s symbols must disappear in the ultimate description. In order to avoid introducing the structures ad hoc, we will produce numbers here in the same manner, in which statistics were producing in sect. 2.5. Indeed, the logical genesis of the concept of a number must be single in theory. That is, taking into account the presence of copies of primitives, one may write

$$\ldots \iff \{\Psi'\}_{\infty'}\{\Psi''\}_{\infty''} \ldots,$$

and numbers per se will come into being by the $\Sigma$-convention like (14). Thus, on the one side, the discreteness of micro-transition acts is embodied in (28) with the sequence $(\Psi', \Psi'', \ldots)$, and on the other, the uncountability of micro-arbitrariness is inherited by attaching the symbolic ‘quantities’—characters $(\infty', \infty'', \ldots)$—to elements of this sequence. The global discreteness says that there are no grounds to assume a more than countable infinity $\aleph_0$ for the set $\mathcal{T}$, i.e., $|\mathcal{T}| = \aleph_0$. The infinity of the family (28), hence, has the type

$$2^{\aleph_0} = \aleph,$$

i.e., it is continual [76]. Which possibilities exist for the form of row (28)?

The trivial case $\{\{\Psi'\}_{\infty'}\}$ for (28) drops out at once, since element $\Psi'$ would always go into the one and the same primitive:

$$\{\Psi' \ldots \Psi'\}_1 \iff \{\Psi'\}_1.$$

But this is tantamount to the identity $\Psi' = \alpha_1$, which robs of any meaning the concept of transition $\rightarrow$ whatsoever. Hence, for each of the formations (28), the following options are admissible:

$$\{\{\Psi'\}_{\infty'}\{\Psi''\}_{\infty''}\} \ldots, \quad \{\{\Psi'\}_{\infty'}\{\Psi''\}_{\infty''}\{\Psi'''\}_{\infty'''}\ldots\} \ldots$$

$$\ldots, \quad \{\{\Psi'\}_{\infty'}\{\Psi''\}_{\infty''}\{\Psi'''\}_{\infty'''}\ldots\}.$$
it is unclear why one (unrecognizable upper) row should differ from another in the number (what?) of defining primitives \( \{\Psi, \Psi', \ldots\} \) (which ones?). By analogy with (14), the only way to implement the said above is the introducing the cardinals as

\[
\kappa' \equiv \infty', \quad \kappa'' \equiv \infty'', \quad \kappa''' \equiv \infty''', \quad \ldots. \tag{30}
\]

**Remark 5.** A few remarks may be made in connection with case \( N = \infty \). It is related to a conglomerate of infinites, which has the form of discretely infinite family of continual infinities \( \{\kappa', \kappa'', \ldots\} \), and things would have been even ‘worse’, if staticity of one of the schemes (29) would be changed to variability. Such formations would need to be equipped with topology and with associated concepts of convergence and of limit. But all this touches on principally unobservable numeric entities, for which it is not clear how to motivate the further reductions to a ‘finite mathematics’ as required: dimensions, finite approximations, finite numbers (which ones?), etc. Moreover, all of that would pertain only to the global structure parameters of the theory prior to constructing it per se, not to mention the physical models. To put it plainly, such an assumption would result not to a theory but to a theory of theories and so on ad infinitum. For these reasons, we leave the case \( N = \infty \) aside, though it might be worth scrutinizing it. However, in sect. 7.6, we will give a further justification of that the numeric domain of the theory is just what it has already been known in QM.

By examining the options \( N = 3, N = 4, \ldots \), we conclude that they should be dismissed, since there immediately arises an issue associated with the questionable empirical exclusivity of a certain ‘world integer’ \( N \), which characterizes the number of ‘physically inaccessible’ \( \Psi \)-objects. That is, these options would be topologically related to a certain nontrivial dimension \( N \geq 3 \) with unmotivated origin.

For the remaining case \( N = 2 \) one establishes that, in the following writing

\[
\{\Psi' \cdots \Psi'\}_{\infty_1} \cup \{\Psi'' \cdots \Psi''\}_{\infty_2} \equiv \{\alpha_1 \cdots \alpha_1\}_{\infty_1}
\]

of the scheme (27), none of the primitives \( \Psi', \Psi'' \) may coincide with \( \alpha_1 \). Otherwise, the unrestricted adjunction of identical transitions \( \alpha_1 \rightarrow \alpha_1 \) to (27) would mean indeterminacy of both the number \( \kappa_1 \), and the actual statistic \((\nu_1, \nu_2, \ldots)\).

Not counting the ‘extremely complex’ case \( N = \infty \), the only option \( N = 2 \) remains. It should have been adopted even before, on the ground that the most minimalistic construction, which set-theoretically gives rise, as a minimum, to the minimal numerical object—a single number, corresponds to the minimal \( N = 2 \) in (29). The maximal case is problematic, while mid-ones are ruled out. Hence, all possible assumptions regarding the upper row structure in (27) are indistinguishable from a case just as if the row contained two primitives only. Functionality of the symbol \( \cup \), with regard to inclusion of copies \( \Psi' \) and \( \Psi'' \), is unchanged as is; see further sect. 5.1 below.

We take into account that each of the numbers (30) is mathematically generated by the standard scheme \{\text{(ordered) integers}\} \rightarrow \{\text{(ordered) rationals}\} \rightarrow \{\text{(ordered) continuum}\}. The ordering \( < \) is always present here and, as is well known [131], is a natural equivalent of the set-theoretic inclusion \( \subset \). The latter, in turn, is intimately associated with the semantic of sect. 2—gathering ensembles in sets (see sect. 5.1).

We now conclude that all kinds of schemes (27) form an \( \aleph \)-continuum, for which there is no reasonable rationale for equipping it with a topology other than the standard order topology of the 1-dimensional real \( \mathbb{R} \)-axis or its equivalents. Call the quantity \( \kappa \in \mathbb{R} \) quantum phase.

It should be added, that in considering the two upper rows (27) as infinite sets

\[
\{\Psi \cdots \Psi\}_{\infty'} \cup \{\Phi \cdots \Phi\}_{\infty''} \quad \text{and} \quad \{\Psi \cdots \Psi\}_{\infty'} \cup \{\Phi \cdots \Phi\}_{\infty''},
\]

linearity of quantum superposition
one can always establish their formal identity. However, physics requires to distinguish the rows, which is what the numeric part of pt. \( R \) and comparison of cardinals \((\infty', \infty'')\) ‘serve’.

4.2. **Statistic + phases.** Thus the closest reconciliation of scheme (26) with the \( R(\bullet) \)-postulate is an ensemble brace of the form

\[
(\Xi) = \left\{ \begin{array}{l}
\{ \Psi \} \infty \{ \Phi \} \infty' \\
\{ \alpha_1, \ldots, \alpha_1 \} \infty_1 \\
\{ \alpha_2, \ldots, \alpha_2 \} \infty_2 \\
\end{array} \right\} \quad \ldots \ldots \quad (31)
\]

followed by the (upper) continual numeration through \( \mathbb{R} \)-numbers

\[
x_\nu \equiv \frac{\infty'}{\infty} \quad (\nu \equiv \infty' + \infty''). \quad (32)
\]

In other words, quantitative description in the theory is created on a basis of the minimal building bricks

\[
\left\{ \begin{array}{l}
\{ \Psi \} \infty \{ \Phi \} \infty' \\
\{ \alpha \cdots \alpha \} \infty \\
\end{array} \right\} \quad (unitary brace) \quad (33)
\]

with two abstract ingoing primitives.

Now, we have cardinals connected by relation (25) and the structure (31)–(32). In the above-described context, parentheses \{\} and symbols \( \Psi, \Phi \) \(\rightarrow\) no longer carry meaning at this point. Therefore, we may omit them as ‘extraneous’ and write (31) as

\[
(\Xi) \iff \begin{array}{c}
x_1 \\
\infty_1 \\
\end{array} \begin{array}{c}
x_2 \\
\infty_2 \\
\end{array} \begin{array}{cc}
\ddots \\
\end{array} = \ldots ,
\]

where \( \alpha \)‘s are well represented by a subscripted numeration; observation \( \mathcal{A} \) has been fixed so far. Let us now introduce a statistic from the ‘embracing infinity’ (25):

\[
\cdots = \begin{array}{c}
x_1 \\
\nu_1 \cdot \infty \\
\end{array} \begin{array}{c}
x_2 \\
\nu_2 \cdot \infty \\
\end{array} \begin{array}{cc}
\ddots \\
\end{array} = \begin{array}{c}
x_1 \\
\nu_1 \\
\end{array} \begin{array}{c}
x_2 \\
\nu_2 \\
\end{array} \begin{array}{cc}
\ddots \\
\end{array} \cdot \infty , \quad \nu_s \equiv \frac{\infty_s}{\infty}.
\]

Then, by \( \Sigma \)-postulate, one arrives at a continually numerical labeling of objects (31):

\[
(\Xi) \iff \left\{ \left( \frac{x_1}{\nu_1} \right), \left( \frac{x_2}{\nu_2} \right), \ldots \right\}.
\]

Recall that the arithmetical operations on the emergent pairs \( (\nu, x) \) are still out of the question, and \( \Sigma \)-limit is indifferent to ‘innards’ of \( (\Xi) \). Only one of all the potentially infinite quantities tends to infinity: the total cardinality (25) of brace (31). What is remained ‘non-extraneous’ in (31) is \( \alpha \)‘s, and we return them to their place. Hence, from the viewpoint of observation \( \mathcal{A} \), the aggregate of all the possible brace (24) is indistinguishable from an order-indifferent 2-parametric family of data

\[
(\Xi) = \left\{ \left( \frac{x_1}{\nu_1} \right) \alpha_1, \left( \frac{x_2}{\nu_2} \right) \alpha_2, \ldots \right\}. \quad (34)
\]

We drop a lower bar in the symbolic designation \( (\Xi) \), highlighting the fact that the meaning of the \( (\Xi) \)-object becomes increasingly divorced from primitives in pt. \( S \) and gets into the numeric domain to match the programme \( R \).
As an outcome, despite the freedom of ingoing collection in (26) and quantum micro-arbitrariness, the distinguishability \( (\Xi) \not\approx (\tilde{\Xi}) \) is indeed determinable, it is determinable not only by statistic and is the \((\mathbb{R} \times \mathbb{R})\)-numerical:

\[
(\Xi) \not\approx (\tilde{\Xi}), \quad \text{if} \quad (\nu_s, \kappa_s) \neq (\tilde{\nu}_s, \tilde{\kappa}_s).
\]  

(35)

What is more, the preliminary (classical) \( \not\approx \)-criterion (15) fits in (34)–(35) as a particular case by omitting the \( \kappa \)-numbers and middle link from (17). That is to say, the ignoring of quantum ‘\( \kappa \)-effects’ is only possible via the \((3 \mapsto 2)\)-reduction (17) into (18), with an automatic imposition of the ClassPhys description. A simplified and hypothetical version of QM over \( \mathbb{R}^1 \) is also ruled out. It would mean a reduction of the two numbers \((\nu, \kappa)\) to a single one. But they have fundamentally different origin. The construct and reasoning in sect. 2.1 tell also us that the attempt at a greater ‘quantum specification’ to (5) and (17) is impossible by virtue of the 2-row structure—ingoing/outgoing—of the object \((\Xi)\), and distinguishability by numeric pairs (35) is highest possible.

The \((\Xi)\)-objects (34) remain and they, as a family, exhaustively inherit the problem’s physics. The quantities \( \nu_s \) are the really observable (unitless) numbers—the percentage quantity of events, which an instrument/observer declares to be distinguishable \( \alpha \)-objects. The quantities \( \kappa_s \) are the internal and unremovable degrees of freedom. Loosely, the \( \kappa \)'s may be speculatively referred to as phases, but they may not be associated with an actual quantity of something. Not only is any ‘material’ treatment of these ‘amounts’ impossible, but it is fundamentally prohibited, since the converse would have meant endowing the ‘nonexistent boxes’ (5)–(6) with notional content or asserting the nature of their origin. Substantiation is only allowed here for the fact of their existence, which is reflected by presence of the left hand side in the concept of transition \( \Psi \not\Rightarrow \alpha \) (Remark 2).

In view of numerous and ongoing discussions of the meaning of the quantum state, note that any its (even merely similar) classical/ontological ‘visualization mechanisms’ [123, p. 137] as wave functions of a ‘certain matter’ or of a hypothetical observable are pointless for the same reason. This is why it is impossible to create, observe, transmit or measure a state, or endow it with the property of being known/unknown, or ‘physically recognize/distinguish’ it from another. Cf. the works [89, 108] and the “methods to directly measure general quantum states . . . by weak measurements” in [90]. The state will itself, when created as a mathematical object, determine the meaning of such words (see sect. 5.3) with an appropriate concept of the physical distinguishability (sect. 2.4).

All the \( \kappa_s \) and \( \nu_s \) are independent of each other, except for relation \( \nu_1 + \nu_2 + \cdots = 1 \). Taking into account the admissible renormalization of both \( \mathbb{R} \)-numbers, the pair \((\nu, \kappa)\) can be topologically identified with a point on the complex plane:

\[
\left( \frac{\kappa}{\nu} \right) \cong (\lambda, \mu) \in \mathbb{R}^2 \equiv \mathbb{C}.
\]

The issue of the numerical domain over which the quantum description is being done—the real \( \mathbb{R} \), complex \( \mathbb{C} \), the quaternions \( \mathbb{Q} \) or whatever—is non-trivial and continues to be the subject of study [86, 1, 8]. The complexity \( \mathbb{C} \) is often motivated by a quantum dynamics (Schrödinger’s equation) [66, p. 132, Stueckelberg], [132], however, such a motivation is inconsistent, and as we have seen, there is no need for it. The rigidity of the \( \mathbb{C} \)-domain points to the fact that, in particular, the quaternion QM also has no place to originate from [7, sect. 10.1], although it was the object of theoretical constructs in the 1960–70’s [44].

Note that even the most comprehensive works [86, p. 217], [66, p. 131], [15, p. 234] observe a difficulty in the full substantiation of the \( \mathbb{C} \)-domain.
5. Empiricism and mathematics

Set theory does not seem today to have . . . organic interrelationship with physics
P. Cohen & R. Hersh [27, p. 116]

. . . physics has something to say about the foundations of mathematics . . .
“if we believe in ZF there is nothing for physics to say” is not right
P. Benioff [50, p. 31]

Up to this point we have dealt, roughly speaking, with a single abstract aggregate \( \Xi \) isolated from the others. However, the constructional nature of the ensemble brace (31) entails the following closedness relation between them. Every brace \( \Xi \) is composed of some others in infinitely many ways (for remote analogies, see [4, sect. 11.2]), i.e., it is a union

\[
(\Xi) = (\Xi') \cup (\Xi''),
\]

and, to put it in reverse, any union of two brace is a third object-brace. In collections (36), the operation \( \cup \), which generates them, is commutative and associative:

\[
A \cup B = B \cup A, \quad A \cup (B \cup C) = (A \cup B) \cup C,
\]

and these 2- and 3-term relations not only are not a formal supplement, but should be read as the structural properties in general.

5.1. Union of ensembles. Consider the lower \( \alpha \)-rows of brace (26) and experimental forming the new real \( \alpha \)-ensembles from them. Let the procedure of such forming be denoted by \( U(A, B, \ldots) \), where \( (A, B, \ldots) \) are the ensembles per se. Its essence is such that it is comprehensively determined by the following minimum: a rule that involves the minimal (i.e., two) number of arguments \( U(A, B) = \langle ? \rangle \) and a rule of the repeated applying \( U \) to itself: \( U(U(\ldots), \ldots) = \langle ? \rangle \). Obviously, we should write

\[
U(A, B) = U(B, A), \quad U(A, U(B, C)) = U(U(A, B), C),
\]

which is of course merely the empirical rephrasing the standard properties (37) of operation \( \cup \). However, the converse is logically preferable: empiricism (38) is formalized into the abstract properties (37). If we now attach the upper ‘quantum’ primitives to the low \( \alpha \)-rows—a requirement of sect. 2.1—then the operationality of actions with the resulting \( \Xi \)-braces would be just like that of \( U \), i.e., (38). In other words, we carry over properties (38) (and use them everywhere) to the general operation on \( \Xi \)-brace, without distinguishing between the essences of symbols \( \cup \) and \( U \). ‘Micro-operationality’ of empiricism and its formalization are confined, at most, by the rules (37) and (38).

Let us temporarily discontinue using the numerical terminology as applied to \( \Xi \)-objects. They differ from each other due to relationships between their ‘innards’, rather than because of our assignment of symbols \( [\lambda, \mu] \) to them. The brace are comprised of elements that are combined into sets and are added to them. In the language of an abstract logic, we are dealing with the fact that transitions \( x \) form the brace \( A, B, \ldots \), i.e., they are in the membership relationships \( x \in A, x \in B, \ldots \) or, when accumulated as micro-acts, ‘get belonged to them’. That is to say, the brace themselves and their formation (accumulation of statistic for the \( \Sigma \)-limit) are equivalent to a huge number of propositional ‘micro-sentences \( x \in A \) or \( x \in B \) or . . . ’. However, again, this is nothing but a logically formal equivalent of the union
It is preferable, however, to adhere to the sequence order in ideology more stringently—{observation \(\rightarrow\) mathematics}, {empiricism \(\rightarrow\) numeric representation}, without substitution it for the opposite. At least, if we rely upon the following comprehension of the empiricism: our elementary perceptions are formalized only into sets and set-theoretic \(\cup\)-abstraction (39). See also [88, Ch. 3, [50, p. 178], [65, p. 323], [23], [26, pp. 12, 86, Ch. 4], sect. 11.1, and [16].

Summing up, we detect a kind of a junction point: the physical and mathematical fundamentality of operation \(\cup\) for describing the elementary acts. That is to say, the mathematics

\[ A \cup B = \{ x \mid (x \in A) \lor (x \in B) \}, \]

which is already being constantly exploited above.

Remark 6. As is well known [131, 76], due to properties of logical atoms \(\in\) (membership) and \(\lor\) (or), the properties of sentences like (39) are determined precisely by rules (37) for \(\cup\). Technically, we should also take an idempotence \(A \cup A = A\) into account, however. At the same time, the need to have a number requires that the duplicates in ensembles have to be taken into consideration. Nevertheless, this situation is easily simulated by the set theory itself. Indeed, consider first the lower row in (24) as a strictly abstract set \(\{ \alpha', \alpha'', \ldots \} \subseteq \Xi\). Then, instrument \(\mathcal{A}'\) ‘asserts’ the distinguishable elements \(\{ \alpha_1, \alpha_2, \ldots \}\) and those that should be thought of as their equivalents:

\[ \alpha_1' \approx \alpha_1'' \approx \cdots : \alpha_1, \quad \alpha_2' \approx \alpha_2'' \approx \cdots : \alpha_2, \quad \ldots \]

This equivalence can be characterized, say, by words ‘a detector click at one and the same place \(\alpha_1\)’. Upon such a formalization, one obtains the formation \(\{ \alpha_1', \alpha_1'', \ldots \} \{ \alpha_2', \alpha_2'', \ldots \} \cdots \approx \{ \alpha_1, \ldots \} \{ \alpha_2, \ldots \} \cdots\), i.e., the very lower row in (26). It is within this context that we think of the union operation without falling into contradiction. Accordingly, \(\Xi \cup \Xi \neq (\Xi)\) but the standard symbol \(\cup\) continues to be used for simplicity.

Therefore if we get back to the numerical labels (34), but ignore the ‘inner composition’ of \(\Xi\), i.e., the \(\mathbf{M}\)-paradigm, thus excluding \(\cup\) and (37) from the reasoning, then all kinds of \(\Xi\)-objects would turn into the semantically ‘segregated ideograms’. Micro-transitions, their mass nature, arbitrariness, \(\not\approx\)-distinguishability, and the quantumness of the task at all simply disappear. To take an illustration, the obvious statement

the brace \(\{ \Psi \rightarrow \alpha \} \equiv : \Xi\) has an empirical relationship

with its duplication \(\{ \Psi \rightarrow \alpha, \Psi \rightarrow \alpha \} \equiv : (\Xi')\)

becomes pointless, because the property \(\Xi \cup \Xi = (\Xi')\) is missing. This is despite the fact that the creation of the transition copies in \((\Xi')\) is a primary operation for generating the objects and reasoning at all. Construction of the theory would then become possible only with the interpretative introduction of the vanished concepts ‘from scratch’. Therefore macro-empiricism necessitates that the relationships (37) be operative rules, and with that the quantumness or classicality of consideration is of no significance.

Remark 7. Let us look at the situation on the opposite—mathematical—side. The union of sets \(\cup\) is a fundamental operation already at the level of the set-theoretic formalization; e.g., the Zermelo–Fraenkel \(\{ZF\}\) axioms [76]. This is one of the first ways on how to create sets—the axiom of union. Therefore if we believe in the set-theoretic mode of explaining/creating the quantum foundations then quantitative description will inevitably invoke operationality of the mathematical primitive \(\cup\) through rules (37). This would be suffice to declare,

- inasmuch as we have nothing but \(\cup\) and \(\Xi\) (taboo \(\mathbf{T}\)), commutativity/associativity of theory is then postulated at the outset, with the subsequent carrying these structures over to numeric representations, i.e., to \(\mathbb{R}\) or \(\mathbb{C}\).

It is preferable, however, to adhere to the sequence order in ideology more stringently—{observation \(\rightarrow\) mathematics}, {empiricism \(\rightarrow\) numeric representation}, without substitution it for the opposite. At least, if we rely upon the following comprehension of the empiricism: our elementary perceptions are formalized only into sets and set-theoretic \(\cup\)-abstraction (39). See also [88, Ch. 3, [50, p. 178], [65, p. 323], [23], [26, pp. 12, 86, Ch. 4], sect. 11.1, and [16].
of \((\Xi)\)-brace (31) and of objects (34) may not inherently be exhausted by them as ‘bare’ sets without structures. Recalling now pr. 11, we draw a conclusion regarding the very construction of the theory.

- Reconciliation of the \(R\)-paradigm with empiricism must transform itself into \textit{rewriting} the primary ensemble \(\cup\)-constructions (26), (31), (33), and relationships between them into the language of numeric symbols.

More formally, we have the following continuation of pt. \(R(\bullet)\).

\(R^+\) \textit{Homomorphism of the ensemble brace properties ‘onto numbers’:} mutual \(\cup\)-relationships (37) between the \((\Xi)\)-brace should be carried over to relations between their numeric \((\Xi)\)-representations (34).

Thereby, we once again fix the maximum that is available for the construction of quantum mathematics. One may handle only the \(\cup\)-aggregates of transitions—constructions (31), (34)—and the minimal modules (33).

5.2. \textbf{Semigroup}. In line with (36), let us split the unitary brace (33) into two ones or combine two brace into one. Remove also the symbols of primitives \(\Psi\) and \(\Phi\) from there; as was pointed out above, they are not necessary at this stage. Replacing the notation of upper cardinals (33) with pairs \((\infty', \infty_2')\) and \((\infty'', \infty_2'')\), upon the union one obtains

\[
(\infty', \infty_2') \cup (\infty'', \infty_2'') = (\infty' + \infty', \infty_2' + \infty_2'').
\]  

(40)

Here, addition \(+\) obviously satisfies the properties (37). If the cardinal ‘\(\infty\)-coordinates’ are replaced with the ‘finite percentages’ \((\nu, \mathcal{S})\) introduced above, i.e., if one puts

\[
\begin{align*}
\nu &= \frac{\infty_1}{\infty_1 + \infty_2}, \quad \mathcal{S} = \infty_1 + \infty_2, \\
\{ \nu = \frac{\infty_1}{\infty_1 + \infty_2}, \quad \mathcal{S} = \infty_1 + \infty_2 \} &\text{,} \\
\infty_1 &= \nu \mathcal{S}, \quad \infty_2 = (1 - \nu) \mathcal{S}
\end{align*}
\]

(41)

as in (32), then rule (40) acquires the form of a number composition:

\[
(\nu', \mathcal{S}') \circ (\nu'', \mathcal{S}'') = \left( \frac{\nu' \mathcal{S}' + \nu'' \mathcal{S}''}{\mathcal{S}' + \mathcal{S}'}, \mathcal{S}' + \mathcal{S}'' \right).
\]  

(42)

The commutativity/associativity properties of operation \(\circ\) hold here due to birationality of (41). Then, the formal application of \(\Sigma\)-postulate, i.e., \(\mathcal{S}' + \mathcal{S}'' \to \infty\), breaks, however, the symmetry \(' \leftrightarrow ''\) and associativity of \(\circ\), since

\[
(\nu', \mathcal{S}') \circ (\nu'', \mathcal{S}'') \quad \rightarrow \quad \nu' \circ \nu'' = s \cdot \nu' + (1 - s) \cdot \nu'', \quad s \equiv \frac{\mathcal{S}'}{\mathcal{S}' + \mathcal{S}''}, \quad (43)
\]

where \(s\) is an undefined parameter. Consequence of the same kind holds also true for the \(\nu\)-components of pairs (34), for which a convex \(w\)-combination of statistics does arise:

\[
(\nu'_1, \nu'_2, \ldots) \circ (\nu''_1, \nu''_2, \ldots) \equiv (\nu' \circ \nu'') = w \cdot \nu' + (1 - w) \cdot \nu'', \quad w \equiv \frac{\Sigma'}{\Sigma' + \Sigma''}. \quad (44)
\]

At the same time, the splitting (40) is no more than an ‘intrinsic reshuffle’ of one and the same \((\Xi)\)-brace that ‘knows nothing’ about concept of a number (numbers \(s, w\)), much less about the concept of observation or its numeric form. Therefore mathematics of the ensemble structures should be independent of any representation for (36) by operations like (42). Composition \((\Xi') \circ (\Xi'') = (\Xi)\) should be determined solely by its constituents \((\nu', \nu')\) and \((\nu'', \nu'')\).
Remark 8. In classical statistics, the foregoing has an analog as insensitivity to data on events to the way of their gathering and of layout; for example, \((2, 3) + (1, 4) \equiv (0, 6) + (3, 1) \equiv \cdots \equiv \text{data}\). Then, the observation proper is being created by the scheme \(\text{data} \mapsto (3, 7) \mapsto \left(\frac{3}{3+7}, \frac{7}{3+7}\right) = (0.3, 0.7) = (\nu_1, \nu_2) \equiv \text{observ}\). Parameters like \(w\) can appear in \((\Xi)\) only if, prior to any of the \(\cup\)-unions \((36)\), a construction like \((23)\) has been fixed. That is, the invariantly number-free brace \((36)\) has been supplemented by an external number \(w\) and ratio \(w : (1-w)\). The correction \((\Xi) \mapsto \{(\Xi')^w, (\Xi'')^{1-w}\}\) of the theory, related to this number and to arrays \((23)\), is very well known as a \(w\)-statistical mixture \(\{(w; \psi'), (1-w; \psi'')\}\) of wave functions, accompanied by the formalization in terms of statistical operator \(w \cdot |\psi'\rangle \langle \psi'| + (1-w) \cdot |\psi''\rangle \langle \psi''|\).

Now, in order that numeric \((\nu, \varkappa)\)-realization \((34)\) of ensemble brace \((31)\) inherits quantum empiricism \((O, M)\) and structure properties \((36)-(37)\) correctly, we reassign the quantities \((\nu, \varkappa)\) with a ‘percentage meaning’ and replace them with different numbers \([\lambda, \mu]\):

\[
(\Xi) = \left\{\left[\lambda_1^1\right]_\lambda \alpha_1, \left[\lambda_2^2\right]_\lambda \alpha_2, \ldots \right\}.
\]

In so doing, each pair \([\nu^\prime]_\lambda, [\nu''\nu]_\lambda\) behaves as a whole, and, under coinciding \(\alpha_\nu\), the pairs are endowed with a composition \([\nu^\prime]_\lambda \oplus [\nu''\nu]_\lambda\) that is to be commutative. Along with this, if symbol \(\cup\) denotes a composition of objects \((45)\) then it should obviously copy properties \((46)\):

\[
(\Xi) \cup (\Psi) = (\Psi) \cup (\Xi), \quad (\Xi) \cup ((\Psi) \cup (\Phi)) = ((\Xi) \cup (\Psi)) \cup (\Phi).
\]

The finite ensembles are vanishingly small in their contribution into infinite ones \((\Sigma\)-postulate), i.e., elements of the \((\Xi)\)-family, as infinite sets, are considered modulo finite ensembles. Let us designate their image as \((0)\), and, due to property \((\Xi) \cup (0) = (\Xi)\), it is naturally referred to as zero. The collection \((45)\) itself also has been formed by the \(\cup\)-combining the ingredients

\[
\left\{\left[\lambda_1^1\right]_\lambda \alpha_1, \left[\lambda_2^2\right]_\lambda \alpha_2, \ldots \right\} \equiv \left\{\left[\lambda_1^1\right]_\lambda \alpha_1\right\} \cup \left\{\left[\lambda_2^2\right]_\lambda \alpha_2\right\} \cup \cdots = \cdots,
\]

and which is why the same symbol \(\cup\) may be freely used between objects with different \(\alpha_\nu\):

\[
\cdots = \left\{\left[\lambda_1^1\right]_\lambda \alpha_1\right\} \cup \left\{\left[\lambda_2^2\right]_\lambda \alpha_2\right\} \cup \cdots.
\]

For the sake of brevity, we omit the redundant curly brackets further, redefining

\[
(\Xi)_\alpha := \left[\lambda_1^1\right]_\lambda \alpha_1 \cup \left[\lambda_2^2\right]_\lambda \alpha_2 \cup \cdots.
\]

As a result, we have had that the set-theoretic prototypes \((26)-(27)\), \((31)\) of states \((11)\) do invariantly exist in form of all kinds of \(\cup\)-decompositions. Thus, in dealing with the only instrument \(\alpha\), one reveals the following property.

- For each observation \(\alpha\), the set of \((\Xi)_\alpha\)-objects forms an infinite commutative semigroup \(\mathfrak{G}\) with respect to operation \(\cup\).

An internal (beyond the observation) nature of \((\Xi)_\alpha\)-objects \((46)\) is characterized by their commutative superpositions \((\Xi')_\alpha \cup (\Xi'')_\alpha\), which are independent of the classical composition of observational \(\nu\)-statistics.

5.3. Measurement. The described above numerical \((\Xi)\)-version of the \((\Xi)\)-brace \(\cup\)-phenomenology now allows one to preliminarily formalize the concept, the absence of which deprives the theory of its logical foundation. Namely, measuring statistic by observation \(\alpha\) over \(S\):

\[
\text{QM-measurement : } \quad \left\{\left[\lambda_1^1, \mu_1\right], \left[\lambda_2^2, \mu_2\right], \ldots \right\} \mapsto (\nu_1, \nu_2, \ldots).
\]

That is, the \([\lambda, \mu]\)-collection gets mathematically mapped into the \(\nu\)-statistic. This is a maximum of information provided by observation \(\alpha\). The mapping \((47)\) annihilates the pairs
Therefore the inheritance/homomorphism of operations $\cup$ and $\sqcup$ onto anything at all is eliminated. Upon operation (47), both $(\kappa, \nu)$-sets and their $\cup$-unions, $\sqcup$-operations, and the semigroup $\mathcal{G}$ proper disappear; as a result, the distinctive feature subsequently referred to as superposition will also disappear. The new numbers $\{\nu_s\}$ may be ‘added up’ only as required by the different, i.e., classical, rule: forming the convex combinations (44). We note that the formalization of measurement does not now depend on how the mathematical projection $[\lambda, \mu] \mapsto (\nu)$ would be further implemented—it is a separate job [18]—or how the $t$-dynamics would be introduced.

Remark 9. Incorporation of $t$-dynamics into the theory is still impossible due to the absence of mathematics to be applied to instants $t_1, t_2$. Accordingly, no physical $t$-process may correspond to the mathematical mapping (47), while the known ‘conceptual’ problems with collapses [53, 141, 64, 83, 98] are, in fact, non-existent [82, 84, 86]. More precisely, they stem from the blurring of meaning which we typically give to the words ‘states’ (what is that?), ‘ensembles’ (what are they comprised of?), and ‘collapse’ (of what?). In regard to the latter, the authors of the book [88] speak out in the most definitive manner—“fairy tales”.

In sect. 2.1, the fundamental premise of the $\alpha$-symbol-based distinguishability $\neq$ was the foundation of the entire subsequent language. One then observes that measurement will essentially remain a senseless term—“For microsystems nothing can be directly measured” [85, p. 304]—up until it invokes the concept of a quantum state, i.e., the $(\Xi)$- and $\alpha$-objects.

5.4. Invariance with respect to observations. Up to this point we had had no need for the matching of observation $\mathcal{A}$ with observation $\mathcal{B}$, although it is clear that a description based on a certain specified $\mathcal{A}$ will inevitably be non-invariant with respect to the tool ${\mathcal{A}, \mathcal{B}, \ldots}$ (‘observation space’) and unacceptable (pt. R) due to the impermissible exclusivity of the set $\{\alpha_1, \ldots\}$. At the same time, we do not have anything but ${\mathcal{A}, \mathcal{B}, \ldots}$ and micro-acts (12) (pts. T and M). In the brace, this fact has already been reflected: transitions $\not\rightarrow$ are combined into integrities (24). Logically, however, the $(\Xi)_{\mathcal{A}}$, $(\Xi)_{\mathcal{B}}$-objects are incomparable and isolated from each other, as carriers of statistics of different origin. On the other hand, ‘the same’ is observed by instrument $\mathcal{B}$, as by instrument $\mathcal{A}$. Although this context has not yet been invoked, without it the application of set-theoretic constructs to physics is devoid of meaning*, just like the union of the speeds of an electron and of the Moon into a

\[ \frac{1}{2}(v_e + v_M). \]

Thus the global structuredness is required in the set of various $(\Xi)$-data according to the context ‘the same’ and its negation. Apparently, this addition implies entities like ‘the same particle’, ‘in the same preparation/state’, ‘under the same temperature, $M$-environment (12)’, ‘the same closed system $S$’, ‘in the same external field’, ‘in the same interferometer’ with ‘the same detectors’, etc [4, 7, 54]; short and generalized notation $\langle S, M, \ldots \rangle$. All the concepts here, including state, are physical conventions, yet their formalization and modeling are required for the creation of a theory (sect. 2.4).

The notion ‘with the same initial data’ falls under the same category, if the intention is to use the term time $t$. Again, the very creating the $(\Xi)$-brace as a set ‘by the piece’ is from the outset thought (sect. 2.4) of as a creation on the assumption of common $\langle S, M, \ldots \rangle$. For instance, the $\mathcal{A}$-statistic $(\Xi)_{\mathcal{A}}$ is gathered with ‘the same’ $\langle S, M, \ldots \rangle$ as the $\mathcal{B}$-statistic $(\Xi)_{\mathcal{B}}$. On its part, any variation is sufficient to obtain ‘not the same’, even if we ‘envision it as null’

* “The statements of quantum mechanics are meaningful and can be logically combined only if one can imagine a unique experimental context” [4, p. 115; emphasis original].
in the spirit of the widely known “without in any way disturbing a system” [113, p. 234]. To take an illustration, equipment of interferometer (sect. 6.5) with additional ‘which-slit’ detectors is already in conflict with the notion of ‘the same S’. In similar cases, we end up in situations of type (23), since the detectors cause a distinguishability.

It follows from the above that in order to match \( A \) and \( B \), the supra-mathematical categories \( \langle S, M, \ldots \rangle \) are required, however, we are only in possession of the ensemble brace \((\Xi)_A\) and \((\Xi)_B\) (pt. T). On the other hand, without joint consideration of the two instruments, i.e., without introducing a mechanism of the mathematical matching \((\Xi')_A \leftrightarrow (\Xi')_B\), \((\Xi'')_A \leftrightarrow (\Xi'')_B\), ..., the segregation of the \((\Xi)\)-objects is absolute. (It is clear that the matching of micro-observations \( \Psi \rightarrow \alpha_s \) and \( \Psi \rightarrow \beta_j \) is also futile.) It is impossible to associate physics with the abstractly segregated \((\Xi)_A\)-brace. Otherwise, the solitary object \((\Xi)_A\), generating nothing more than a statistic provided by the single instrument \( A \), would yield a description of everything, which is absurd by pt. R(●●). The physical contents arise precisely through the above-mentioned matching; see sect. 6.3 further below. As a result, we adopt a kind of the relativity principle analogue: a tenet on the quantum observational invariance (cf. [144, p. 632]).

### III Theory should introduce a means of equating the macro-observations (pts. O, M)

by differing instruments \{\( \alpha_1, \ldots \}_A \not\equiv \{\beta_1, \ldots \}_B\} under a common experimental environment \( \langle S, M, \ldots \rangle \) (**the 3-rd principium of QM**).

We are currently returning once again to sect. 2.1, falling into a situation when the case in hand does not just entail fundamental theory in the form of \{math\} + \{physical ‘bla-bla-bla’\}, while, continuing on an informal note, the mathematics of physics—quantum mathematics—is being created ‘from scratch’. During its construction, it is impossible to foresee physical conventions \( \langle S, M, \ldots \rangle \), meanwhile, any preliminary and the formal characterization for \( \langle S, M, \ldots \rangle \) is ruled out. In fact, the attempts to mathematically formalize the physical context of observation, rather than observation itself, will not logically manage without another observation, in this case, of the experimental environment. The semantic cycling, or the retrogression of definitions into infinity, are apparent here, which is why, once again, the ‘box (6) method’ prohibitions are required. (See also a paragraph containing the capitalized emphasize “CANNOT IN PRINCIPLE” on p. 418 of the work [82].) Sooner or later, it will have to be declared that mathematics will be created for the conventions \( \langle S, M, \ldots \rangle \), and that this mathematics will be a mathematical model for \( \langle S, M, \ldots \rangle \). The analogous argument—“mathematics is there to serve physics, and not the other way round” (L. Hardy [118, p. 242])—has already long been met in the literature [4, 7, 51]. In connection with the “general contextual models”, see the book [70] (the Växjö-model, “quantum contextuality”) and bibliography therein.

**Remark** 10. In order to avoid the just mentioned semantic/linguistic closedness—(mathematically ‘pathological’) characteristic of the physical and natural languages, a description that lays claim to the role of an unambiguous/rigorous theory requires a careful separation of the object- and meta-languages. For more detail, see [73, §§14–16], [109, sect. V.1], [131, sect. 3.9]. For this reason, the constructs should track the blending of the object QM-domain (syntax) and the meta-domain (semantics) and, more generally, the penetration of extra-linguistic elements of thinking [26] into QM. The notion of ‘the same \( \langle S, M, \ldots \rangle \)’, which is intuitive in a natural language, should explicitly be indicated as an external and fundamental category (pr. III), and its circular re-interpretations within the theory should be banned. The physics terminology per se (sects. 6.3, 6.5, and 9.1) will become accessible when physical concepts are introduced via the originating (and obligatorily very primitive) quantum mechanics language. See also a selected thesis in sect. 11.1 on p. 58.
It is crucial to immediately note that, in the same manner, the classical description contains the
cited arguments in their entirety. It is easy to convince that such a description also implies
implicitly that which is designated above as \( \{ \mathcal{S}, \mathcal{M}, \ldots \} \); otherwise, the physical reasoning
would be entirely impossible*.

**Remark** 11. Here, the situation is similar to the role of the axiom of choice in the ZF-system
[27, 76]. It has been well known for a long time that the axiom is often subconsciously implied [47, Chs. II, IV]; it can also not be either circumvented or ignored. Another counterexample to ‘infinite
retrogression and circularity’ in logic comes from the very same system. This is a ban on infinite chain
of set memberships \( \in \) on the left (regularity axiom: \( (\forall x \in X, x \cap X \neq \emptyset) \Rightarrow X = \emptyset \))
under the permissibility of infinite (\( \in \))-continuability to the right (the infinity axiom) [122, 76]:
\[
X_0 \in X_1 \in X_2 \in \ldots \in X_n \in \ldots \in \ldots .
\]
The obvious parallels here are the famous Russell paradox [47] or a chaos in the computer file system
when the ‘hard links’ from a folder to the parent folder are allowed. Thus the relations \( \in \) ‘downwards’
to the left and necessarily terminates in something, i.e., in a set which contains nothing: \( \emptyset = X_0 \in \ldots \in X_n \in \ldots \). Therefore one needs to give ‘meaning’ to the only set—the empty one \( \emptyset \). Incidentally,
it is these axioms guarantee the existence of infinitely many ordinal numbers (103) and *uniqueness* of
this structure; the ordinals, and numbers at all, have yet to be dealt with further below.

All that remains is to add that no theory in physics is feasible without recalculations of
physical units, of vectors/tensors, without transformations in the fibre add-ons over mani-
folds, etc. Accordingly, the considerations on invariance and on transformations should be
present in the quantum case as well, but it—which is its principal difference from the classical
case—still lacks the concepts of physical quantities/properties (see sect. 6.3). Therefore such
argumentation may only be applied to those objects that we have at our disposal, i.e., to the
(\( \in \))-brace. In fact, the renunciation of pr. **11** would be tantamount to the inability to make
the theory a physical one.

Now, pr. **11** and the ‘quantum diversity of the reference frames’ \( \mathcal{A} \) and \( \mathcal{B} \) require a kind of
factorization of the entire family \( \{ (\Xi)_{\mathcal{A}}, (\Xi)_{\mathcal{B}}, \ldots, (\Xi)_{\mathcal{A}'}(\Xi)_{\mathcal{B}'}, (\Xi)_{\mathcal{A}''}, (\Xi)_{\mathcal{B}''}, \ldots \} \) with respect to \( \{ \mathcal{S}, \mathcal{M}, \ldots \} \),
i.e., the introduction of an operation of equating the results \( (\Xi)_{\mathcal{A}}, (\Xi)_{\mathcal{B}} \) that came from \( \mathcal{S} \).
The immediate setting \( (\Xi)_{\mathcal{A}} \not\approx (\Xi)_{\mathcal{B}} \) should not be made, since these brace are simply different
sets. That is why, with isolated semigroups
\[
\{ \uparrow; (\Xi)_{\mathcal{A}}, (\Xi)_{\mathcal{A}'}, \ldots \}, \quad \{ \uparrow; (\Xi)_{\mathcal{B}}, (\Xi)_{\mathcal{B}'}, \ldots \}, \quad \ldots
\]
at our disposal, we have to conceive of them as elements of a new set \( \mathcal{H} \) of objects of a single
nature, i.e., 1) to carry out the mapping \( \{ \mathcal{G}_{\mathcal{A}}, \mathcal{G}_{\mathcal{B}}, \ldots \} \mapsto \mathcal{H} \), assigning new representatives
\( |\Xi_{\mathcal{A}}| \in \mathcal{H} \) to the (\( \in \))-brace, and 2) to equip \( \mathcal{H} \) with an equivalence relation \( |\Xi_{\mathcal{A}}| \approx |\Xi_{\mathcal{B}}| \) (the
concept ‘the same’ above). Let us implement all of that by the scheme
\[
(\Xi)_{\mathcal{A}} := (\mu_1') \downarrow_{\lambda_1'} \uparrow (\mu_2') \downarrow_{\lambda_2'} \uparrow \cdots \Rightarrow (\mu_1')_{\lambda_1'} \perp (\mu_2')_{\lambda_2'} \perp \cdots \equiv: |\Xi_{\mathcal{A}}| \in \mathcal{H},
\]
\[
(\Xi)_{\mathcal{B}} := (\mu_1') \downarrow_{\lambda_1'} \uparrow (\mu_2') \downarrow_{\lambda_2'} \uparrow \cdots \Rightarrow (\mu_1')_{\lambda_1'} \perp (\mu_2')_{\lambda_2'} \perp \cdots \equiv: |\Xi_{\mathcal{B}}| \in \mathcal{H},
\]
(48)

* Foulis–Randall: “... we often prefer to regard a number of outcomes of distinct physical operations
as registering the same property, ... representing the same measurement. ... permitting an unrestricted
identification of outcomes would lead to "grammatical" chaos” [46, p. 232].
In this, the new addition + must of course homomorphically inherit operations $\otimes$, $\oplus$, $\cdots$, and extension of this definition throughout $H$ is then made with the aid of the very equivalence $\approx$:

$$|\Xi_\alpha\rangle \pm |\Xi_\beta\rangle = |\Xi_\alpha\rangle \approx |\Xi_\beta\rangle \Rightarrow |\Xi_\alpha\rangle \pm |\Xi_\beta\rangle = |\Xi_\alpha\rangle \pm |\Xi_\beta\rangle.$$  

Negation $\not\approx$ of the relation $\approx$, e.g., $|\Xi_\alpha\rangle \not\approx |\Xi_\beta\rangle$, is exactly the very same distinguishability that was discussed in sects. 2–3.

For the sake of convenience, we adopt the regular sign $=$ for $\approx$, in order not to introduce yet another homomorphism, which are already numerous, with more underway. In other words, the physics $\langle S, M, \ldots \rangle$ is ‘concentrated’ in the sign $=$, turning the empirical structures (48) into the $\mathcal{A}$-, $\mathcal{B}$-implementations of the object $|\Xi\rangle \equiv |\Xi_\alpha\rangle = |\Xi_\beta\rangle$ that is being constructed. The adequate term for it—DataSource—corresponds to the preliminary prototype of the concept of a state, but we will adhere to the standard term, disregarding its variance.

6. Quantum superposition

**Why the quantum?**

J. Wheeler

… postulation of something as a Primary Observable is itself a sort of theoretical act and may turn out to be wrong

T. Maudlin [93, p. 142]

6.1. Representations of states. Let us simplify notation according to the rule $^{[\mu]} \equiv a$. The sought-for relationships between $\mathcal{A}$, $\mathcal{B}$, $\ldots$ then turn into the equalities

$$\text{representations of } |\Xi\rangle\text{-state } \left\{ \begin{array}{c}
a_1|\alpha_1\rangle \pm a_2|\alpha_2\rangle \pm \cdots = b_1|\beta_1\rangle \pm b_2|\beta_2\rangle \pm \cdots = \cdots.
\end{array} \right. \quad (49)$$

They furnish representations $|\Xi_\alpha\rangle$, $|\Xi_\beta\rangle$, $\ldots$ of quantum state $|\Xi\rangle$ of system $S$. By design, the DataSource object $|\Xi\rangle$ carries data $(\Xi)_\mathcal{A}$, $(\Xi)_\mathcal{B}$ and, more generally, $(\Xi)$-data (46) from the arrays of any observations, including the imaginary ones. That is what eliminates the initial need for the $(\Xi)_\mathcal{A}$-brace (24) to be came from the observation $\mathcal{A}$, which is reflected in the shortening of the term ‘representation of state’ to simply ‘state’ $|\Xi\rangle$. It should be added that the straightforward storing of objects $\{(|\Xi_\alpha\rangle, |\Xi_\beta\rangle, \ldots \}$ in a certain set $H$, but with the independence of operations $\{\pm(\alpha\beta), \pm(\beta\beta), \ldots \}$ preserved, would not differ from the tautological substitution of symbols. Accordingly, the semantic autonomy of $(\Xi)$-brace would be inherited, whereas invariance III requires the elimination of precisely this autonomy. What is more, the set-theoretic original copy for operations $\{\mathcal{A}, \mathcal{B}, \ldots \}$ and $\pm$ is one and the same—the union $\cup$.

The symbols $|\alpha_s\rangle$ and $|\beta_s\rangle$ in (49) are no more than symbols. Hence the property of objects (49) of being identical must be reflected in terms of their coordinate $a, b$-components; pt. $R(\bullet)$. This means that any aggregate $(a_1, a_2, \ldots)$ is unambiguously calculated by means of a certain transformation $\hat{U}$ into any other $(b_1, b_2, \ldots)$, if the two aggregates represent a common $|\Xi\rangle$:

$$(a_1, a_2, \ldots) = \hat{U}(b_1, b_2, \ldots).$$

The $\hat{U}$ then becomes an isomorphism between these aggregates and, accordingly, their lengths must coincide. This length—a certain single constant—will be symbolized as $D$. 


6.2. Representations of devices. Spectra. Naturally, instrument is converted to the \( H \)-structure language along with \((\Sigma)\)-objects. It is a set of symbols \( \{|\gamma_1\}, \{|\gamma_2\}, \ldots \} \) in place of the previous \( \{\gamma_1, \gamma_2, \ldots\} \). As has just been shown, their number for any \( \mathcal{C}\)-instrument should be equal to \( D \). However, generally speaking, \(|\Sigma_{\mathcal{C}}| \neq |\Sigma_{\mathcal{D}}|\), since \( \Sigma_{\mathcal{C}} \) and \( \Sigma_{\mathcal{D}} \) are assigned in an arbitrary way (pt. \( O \)). Therefore if we take an illustration \( \mathcal{A}\{\alpha_1, \alpha_2\} \) and \( \mathcal{B}\{\beta_1, \beta_2, \beta_3\} \), then \( H \)-representation of instrument \( \mathcal{A} \) should appear at least as \( \{|\alpha_1\}, \{|\alpha_2\}, \{|\alpha_3\}\} \). Clearly, the already present distinguishability \( \alpha_1 \not\equiv \alpha_2 \) (sect. 2.2) is automatically converted into an abstract distinguishability of new symbols \(|\alpha_1\) \neq |\alpha_2\), and empirical \( \mathcal{A}\)-distinguishability is confined exclusively by these two symbols. In that case, for the purpose of noncontradiction, it is not difficult to see that if \(|\alpha_3\) \simeq |\alpha_1\), then either \(|\alpha_3\) \simeq |\alpha_1\) or \(|\alpha_3\) \simeq |\alpha_1\). By an extension of this argument one gets that every \( \mathcal{A}\)-instrument should be endowed with the (non)equivalence relation \( \simeq/\not\equiv \) in terms of the \( H \)-structure by its formal \( \{|\alpha_1\}, \ldots\} \)-representations.

Let us proceed further from a self-suggested extension of pt. \( R \). Let us declare—and it is more than natural—that the numeric representations \( \alpha_s \) are linked not only to observations, but to instruments as well. Each \( \alpha_s \) is the new object of a numeric type: a number or a collection of numbers. Then, indiscernibility, say \(|\alpha_3\) \simeq |\alpha_1\), is recorded by coincidence of the numeric labels \( \alpha_3 = \alpha_1 \) attached to the symbols \(|\alpha_3\) and \(|\alpha_1\) respectively. The abstract distinguishability \(|\alpha_3\) \neq |\alpha_1\), meanwhile, remains as it is. From these we have the following formalization of relationship between \( \simeq \) and \( = \) by means of dropping/adding the brackets \(|\) :}

\[
\begin{align*}
|\alpha_s\) \neq |\alpha_k\) & \iff \alpha_s \neq \alpha_k \quad \text{under } |\alpha_s\) \neq |\alpha_k\). \tag{50}
|\alpha_s\) \simeq |\alpha_k\) & \iff \alpha_s = \alpha_k
\end{align*}
\]

Call the quantity \( \alpha_s \) (numeric) \textit{spectral label/marker} of eigen-element \(|\alpha_s\). Then, by \( H \)-representation \([\mathcal{A}]\) of instrument \( \mathcal{A} \) we will mean the set of objects \( \{|\alpha_1\}, \ldots, |\alpha_D\}\) supplemented with the spectral structure (50):

\[
[\mathcal{A}] := \{|\alpha_1|_{\alpha_1}, |\alpha_2|_{\alpha_2}, \ldots\}. \tag{51}
\]

It is not difficult to see that if \(|\alpha_1\) \neq |\alpha_2\) then either \(|\alpha_3\) \simeq |\alpha_1\) or \(|\alpha_3\) \simeq |\alpha_2\). Otherwise, spectral markers \( \alpha_1 = \alpha_2 \) should coincide, and primary primitives \( \alpha_1 \not\equiv \alpha_2 \) lose their empirical distinguishability in contrast with (7). Multiple coincidence of \( \alpha_s\)-markers is acceptable.

In the presence of relations (50), it is natural to state that instrument \( \mathcal{A} \) is coarser (more symmetrical) than \( \mathcal{B} \), and, terminologically, to declare that the degeneration of the spectral-label values takes place. In cases of embeddability like \( \mathcal{A}_2\{\alpha_1, \alpha_2\} \subset \mathcal{A}_3\{\alpha_1, \alpha_2, \alpha_3\} \), instrument \( \mathcal{A}_2 \) can even be called the same as (coinciding with) \( \mathcal{A}_3 \), but with a more rough scale. Or conversely, \( \mathcal{A}_3 \) is a more precise extension of \( \mathcal{A}_2 \). In particular, the natural notion of a device resolution fits here. All instruments may then be mathematically imagined as having the same resolution, but, perhaps, with degeneration of spectra. The non-coinciding instruments may be interpreted as non-equivalent reference frames \( \mathcal{A} \not\equiv \mathcal{B} \) in an observation space; according to pts. \( R(\bullet\bullet) \) and \( III \), they are mandatorily present in the description. The spectral degenerations are also present at all times, since element \( \alpha_1 \) can always be removed from \( \Sigma_{\mathcal{A}} \), and there are no logical foundations to prohibit an observational instrument with family \( \Sigma_{\mathcal{A}} - \{\alpha_1\} \). Hence it follows that introducing the spectra—instrumental readings—is required even formally; without physics. It is, of course implied here that spectral (in)distinguishability is realized in the same manner as its statistical counterpart in sects. 2.6 and 4, i.e., by numbers.
Incidentally, such a property of $\alpha_s$—i.e., of being a numeric type object—is not necessary at the moment. Spectrum \( \{\alpha_1, \alpha_2, \ldots\} \) may be thought of as an abstract set of labels attached to the eigen-elements. As numbers, it is introduced for the subsequent creation of models to classical/macroscopic dynamic, and they are numeric.

Returning to \( D \), we note that, in any case, the toolkit \( \{\mathcal{A}, \mathcal{B}, \ldots\} \equiv \mathcal{O} \) in real use has always been defined, fixed, and is finite. Hence the constant

\[
D \geq 2
\]

has also been defined and fixed, and it becomes the globally static observable characteristic; an empirically external parameter. Meanwhile, the entire scheme internally contains the natural method of its own extension \( D \mapsto D + 1 \), and the potentially all-encompassing choice \( D = \infty \) may be considered the universally preferable one in quantum theory. By freezing the different \( D < \infty \), the theory makes it possible to create models, and they are not only admissible but also well-known. Their efficiency is examined in experiments. Once again, the \( D \)-constant, notion of spectra, and their degenerations are created by the \( \mathcal{OBp} \)-invariance requirement, i.e., by pr. III.

As a result, the structure of \( H \)-representations of states and of instruments are liberated from the arbitrariness in assigning the subsets \( \Xi_{\mathcal{A}} \) in (8). The statistical unitary pre-images (33) and \( H \)-elements of the form \( \zeta |\gamma_s\rangle \) can be associated with any ‘eigen symbol’ \( |\gamma_s\rangle \). They are always available because all kinds of brace (31) are known to contain subfamilies when ingoing \( \Psi, \Phi \)-primitives get to a single one, e.g., to \( \gamma_1 \). Therefore every representation \( a_1|\alpha_1\rangle \pm \cdots \) is always equivalent to a \( \Xi_{\mathcal{A}} \)-brace for some observation \( \mathcal{C} \) with a homogeneous outgoing ensemble \( \{\gamma_1 \cdots \gamma_1\} \). That is, one may always write

\[
a_1|\alpha_1\rangle \pm a_2|\alpha_2\rangle \pm \cdots = c_1|\gamma_1\rangle \pm 0|\gamma_2\rangle \pm \cdots \equiv: c_1|\gamma_1\rangle,
\]

while naturally referring to \( c_1|\gamma_1\rangle \) as one of the eigen-states of instrument \( \mathcal{C} \), with an appropriate correction of the similar definition in pt. O.

6.3. Physical properties. Now, the ‘general physics’ \( \{S, M, \ldots\} \) is mathematized into representations (49) of states \( |\Xi\rangle \) of system \( S \). There is, however, an ambiguity, the source of which is the fact that the natural/classical language also lays claims to a similar formulation. This refers to the belief in the existence of mathematics (‘bad habit’ [96]; see also [57], [88, p. 122], and [95]) that describes \( S \) as an individual object with properties regardless of observation; an observation that is not a functioning attribute of the mathematics itself. In classical description, it is specified by definitions: point \( P \) of a phase space, \((q,p)\)-coordinatization of the point (manifold), and statistical distribution \( \varrho(q,p) \).

On the other hand, quantum empiricism provides nothing more to us besides the ensemble brace and \( |\Xi\rangle \)-states (pt. T). Preordained definienda with physical contents are unacceptable, i.e., \( S \) should not be conceived as ‘something with physical properties’ or as an ‘individual system’ [86, 87]. However, since the observational data (in the broadest sense of the word) may not originate from anywhere but a certain \( |\Xi\rangle \)-object, 1) the very concept of physical phenomena/properties [140, pp. 211–230] and 2) their numeric values should be subsequently created. This is habitually referred to as elements/images of reality [55, pp. 194], [4, sect. 10.2], [87, §XIII.4.8], or what we have been calling attributes of a physical system.

Indeed, the primary ideology of sects. 1.3 and 2.1 tells us that an invasion of physically self-apparent images into the theory should be avoided, because “quantum theory not only does not use—it does not even dare to mention—the notion of a “real physical situation” (E. Jaynes [55, p. 198]). Therefore everything, with no exceptions, should be (mathematically) created:
coordinates/momenta, energies, optical spectra, device readings, sizes, number/numeration of particles, their (in)determinacy/boundary (bosons/fermions), the notion of a subsystem of system $S$ (see (23)), etc. Degrees of freedom and the numeric forms of what is known as the classical reference frames—manifold coordinates—need also to be created; a fact required for careful posing the questions regarding the problem on quantum gravity. The concept of a (non-elementary) particle, which is conceptually close to the notion of a subsystem/part, is also a physical convention, and can only arise from the $|\Xi\rangle$ or its models: Bose-condensates, quantum theories of various fields (relativistic or non), deconfinement excitations in crystal lattices, quasi-particles in a superfluid phase, and more. (QFTs is a subclass of QM-theory, not its extension.) In this regard, the well-known dualism problem is eliminated because both the particles and waves are the classical notions, and in quantum language, they turn into the derivatives of the notions of state/mixture (23) and of spectrum. Note that the $\nu$-statistic also falls under observable quantities, and constant $D^*$, if declared finite, is an example of an already created characteristic.

In other words, the logic of the above constructs prohibits not only the endowing the terms ‘internal state of an individual object $S^{***}$ and ‘the system is in a state’ [86]–[88], [100] with meaning, but also the indirect using their numerical forms. That would work in circumvention of empiricism, assuming the a priori availability of mathematical structures that do not rely on the state space. The construction of the latter is far from being complete, since it is still a ‘bare’ semigroup $H$.

6.4. Superposition of states. Since writings (49) exist for any ensemble $|\Xi\rangle$-brace, let us consider the following two representations:

$$a_1|\alpha_1\rangle \pm a_2|\alpha_2\rangle = b_1|\beta_1\rangle \pm b_2|\beta_2\rangle \pm \cdots,$$

$$a_2|\alpha_2\rangle = b'_1|\beta_1\rangle \pm b'_2|\beta_2\rangle \pm \cdots.$$  (54)

Comparison of these equalities tells us that the second one is a solution of the first one with respect to $a_2|\alpha_2\rangle$. Hence, the semigroup operation $\pm$ admits a cancellation of element $a_1|\alpha_1\rangle$. This means that there exists an element $\tilde{a}_1|\alpha_1\rangle$ such that

$$\{\tilde{a}_1|\alpha_1\rangle \pm a_1|\alpha_1\rangle\} \pm a_2|\alpha_2\rangle = \tilde{a}_1|\alpha_1\rangle \pm \{b_1|\beta_1\rangle \pm b_2|\beta_2\rangle \pm \cdots\} \downarrow$$

$$0|\alpha_1\rangle \pm a_2|\alpha_2\rangle = \tilde{a}_1|\alpha_1\rangle \pm b_1|\beta_1\rangle \pm b_2|\beta_2\rangle \pm \cdots \downarrow \quad \text{(due to (53))}$$

$$a_2|\alpha_2\rangle = b'_1|\beta_1\rangle \pm b'_2|\beta_2\rangle \pm \cdots \downarrow \quad \downarrow$$

$$|0\rangle \equiv 0|\alpha_1\rangle = \tilde{a}_1|\alpha_1\rangle \pm a_1|\alpha_1\rangle, \quad \tilde{a}_1|\alpha_1\rangle \pm b_1|\beta_1\rangle \pm \cdots = b'_1|\beta_1\rangle \pm \cdots,$$

where $|0\rangle$ stands for a zero in semigroup $H$ (image $|0\rangle$ of the finite length brace $|\Xi\rangle$) and 0 in $|0|\alpha_1\rangle$ is a symbol of its $[\lambda, \mu]$-coordinates. By canceling out $a_s|\alpha_s\rangle$, one by one, if necessary, one deduces that any element of $H$ does have an inversion. That is, $H$ is actually a group.

---

* Dimension of a state space to come. A tensorial structure of this space pertains directly to the physical properties but we do not touch upon this point here. As an aside, the same structure will provide the means of distinguishing the aforementioned models under $D = \infty$.

** L. Ballentine: “The habit of considering an individual particle to have its own wave function is hard to break” [11, p. 238]; cf. “To speak of a single possible initial apparatus state is pure fantasy” (N. Graham [31, pp. 241–242]).
We re-denote inverse elements $a_s|\alpha_s$ by $(-a_s)|\alpha_s$ and inversions of sums are formed from their $(\pm)$-sums. Moreover, all the $[\lambda, \mu]$-pairs turn into a set $\{a, b, \ldots\}$ equipped with the above mentioned composition $\oplus$, which follows from an obvious property of unitary brace:

$$a|\alpha_1 \pm b|\alpha_1 = (a \oplus b)|\alpha_1$$  \hspace{1cm} (55)

(inheritance of closedness under the $\cup$-operation). This composition is also a $\oplus$-operation of a group and of a commutative one:

$$a \oplus b = b \oplus a, \quad (a \oplus b) \oplus c = a \oplus (b \oplus c), \quad a \oplus 0 = a, \quad a \oplus (-a) = 0.$$  \hspace{1cm} (56)

Therefore the group nature of semigroup $H$ and the group (56) come from the scheme

$$\begin{align*}
\text{single observation } &\quad \Rightarrow \quad \{\text{semigroup } \mathcal{G}\} \quad \Rightarrow \quad \{(\mathcal{A}, \mathcal{B})\text{-invariance, } \langle \mathcal{S}, \mathcal{M}, \ldots \rangle \text{ and } \mathbb{1}\}\quad \Rightarrow \quad \{\text{group } H\}
\end{align*}$$

and, technically, from equatings/identifyings (49).

Thus handling the $|\Xi\rangle$-objects breaks free from its ties to the notion of observation, and the objects admit the formal writings $a|\Psi \pm b|\Phi \pm \cdots$. Call them superpositions. However, as soon as they or the state are associated in meaning with the word ‘readings’ (i.e. with the subject of sect. 6.3), this term should be replaced with a non-truncated one: representation of the state with respect to a certain observation. In particular, the statistical data $\nu_j$ are obtained from such expressions only after their conversion into a sum over eigen-states of the form (49); a task of the subsequent mathematical tool. No superposition $a|\Psi \pm b|\Phi \pm \cdots$ has any physical sense in and of itself [123, p. 137], nor is it preferable to any other one. It merely reflects the closedness of states with respect to operation $\pm$, since any $|\Xi\rangle$ is recorded as a sum of various $\{a|\Psi, b|\Phi, \ldots\}$ in a countless number of ways and is linked to any other such sum. Without a system of $|\alpha_s\rangle$-symbols for instrument $\mathcal{A}$, nothing observable out of the aggregate of coefficients $\{\pm a, \pm b, \ldots\}$ (and, of course, of the $|\Psi\rangle$-letters themselves) is extractable in any imaginable way. Accordingly, it is incorrect to speak of—and that is a widespread misconception—the destruction of the superposition or of the “relative-phase information” [116, p. 253], associating the word destruction with physical/observational meanings or processes.

As a result, even without having a numeric theory yet and without resorting to the concept of a physical quantity, we arrive at the paramount property, which characterizes the most general type of macroscopic observations (17).

- **Superposition principle.**

  A $(\pm)$-composition of quantum states $a|\Psi$ and $b|\Phi$, which are admissible for system $\mathcal{S}$, is an admissible state

  $$a|\Psi \pm b|\Phi = c|\Xi,$$  \hspace{1cm} (57)

  and with that the set $\{a|\Psi, b|\Phi, c|\Xi, \ldots\}$ $\subseteq H$ forms a commutative group with respect to operation $\pm$. The family $\{a, b, \ldots\}$ of coordinate $\mathbb{R}^2$-representatives of states (49) is also equipped with the same group structure under the $\oplus$-operation (56) and with rule of carrying the operation $\pm$ over to $\oplus$:

  $$a|\Psi \pm b|\Psi = (a \oplus b)|\Psi.$$  \hspace{1cm} (58)

Let us clarify the transferring (55) to (58). The union of the state prototypes $a|\Psi, b|\Phi \in H$ is known to belong to $\mathcal{G}$. So the composition $a|\Psi \pm b|\Psi$ should be identical to a certain
element $c|\Psi\rangle \in H$. It is clear that $c$ depends on $a$, $b$ and, hence, $a|\Psi\rangle \pm b|\Psi\rangle = c(a, b)|\Psi\rangle$. The exhaustive properties of dependence $c(a, b)$ are given by formulas (56) and (58) under notation $c(a, b) \equiv (a \oplus b)$.

Besides the essential non-physical nature of the (±)-addition—“superpositions . . . we cannot recognize them” [37, p. 13], the primary and typical property of quantum superposition is in the fact that, due to subtraction, it is possible the experimental obtainment of a ‘quantum zero’ in statistic from ‘non-zeroes’. With that, the ‘non-zeroes seem’ to be positive, but there are ‘negative non-zeroes’: negative numbers (see also sect. 9). Subtraction manifests by the typical obscurations in interference pictures. S. Aaronson: “We’ve got minus signs, and so we’ve got interference” [2, p. 220]. No classical composition

$$w\varrho_1 + (1 - w)\varrho_2$$

of non-zero statistics $\varrho_1$, $\varrho_2$ can provide a zero value, since the zero will never be obtained via the $\cup$-unions. The same is true for the pre-superposition in isolated brace ($\Xi$)$_w$, i.e., when one instrument is in question.

Remark 12. One cannot but mention yet another counterexample to the superposition’s ‘physicality’—‘the quantum cat’. Any combination of ‘dead and living animal’ is meaningless as a statement about new/nonclassical entity like a ‘(half-)dead/alive cat’ or about thinking of a ‘particle as being both here and not here at the same time’. It makes absolutely no sense to add up one absent nature’s phenomena to the other. **What’s being added is states, not their names or verbal descriptions of envisioned (‘fantasized’) physical properties; cf.** [93, pp. 134(!), 135]. Accordingly, the word combination ‘system is in a superposition’ is, at most, an interpretative allegory (sect. 10) without physical content, much less a mathematical one. Meaning of the word ‘add’ is still being created and implemented at objects to be thought of as ‘atomic irreducible’ entities—numbers (see sect. 7). T. Maudlin notes on p. 133 of [93]: “Our job . . . is to invent mathematical representations . . . , rather than merely linguistic terms such as “z-up.” . . . we are in some danger of confusing physical items with mathematical items” (italics supplied).

A statement about QM-superposition (without $\mathbb{C}$-numbers) as a non-independent axiom can be found in the book [66, p. 108] but arguments given there are circular: Hilbert space $\rightarrow$ quantum logic of propositions $\rightarrow$ superposition principle. Similarly, in the works [106] and [15, p. 164], all of that is ‘derived’ from modular lattices [19]. However, it is known that the lattices come into QM from the Hilbert space structure and, on the other hand, the purging QM-foundations of such a space’ axiomatics constitutes Birkhoff’s 110-th problem [19, p. 286]. Note also that, in connection with formal logic approaches to theory construction [15, 46, 66, 78, 106, 138, 146], the issue of substantiating the matters that this logic deals with [88, 86, 125] should not be neglected. A. Stairs: “If by “logic” we mean something like “correct reasoning,” then it would make no sense to think of logic as “just another theory.” [128, p. 258]. See also [45, p. 29].

Yet another fact that results from the above constructs is that the availability of a superposition math structure (57) reflects the presence of at least two $\mathcal{A}$, $\mathcal{B}$ with non-coinciding families of eigen-primitives $\{\alpha_s\}$, $\{\beta_k\}$; a consequence of pt. $\mathbb{R}(\bigotimes)$. This point should be particularly emphasized, since in the future it will manifest in the non-commutativity of operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. Even though this work does not get to operators as a mathematical structure, it is clear that the emergent eigen-states and spectra are directly related to them. In this context, the ‘commuting instruments’ $\{|\alpha_1\rangle, |\alpha_2\rangle, \ldots\} = \{|\beta_1\rangle, |\beta_2\rangle, \ldots\}$ can be treated, roughly speaking, as coinciding, because this fact is independent of the specific spectrums $\{\alpha_1, \alpha_2, \ldots\}$, $\{\beta_1, \beta_2, \ldots\}$ assigned to them. If they differ, this is merely a different (numeric) graduation of the spectrum scale; it is the same for all instruments, and its length
is the parameter $D$. Notice that definition of an $A$-observation is not different from the formal assignment of the family $\mathcal{T}_A$ (pt. O and (8)), which is why the non-coinciding sets $\mathcal{T}_A$, $\mathcal{T}_B$ do always exist. This provides a kind of abstractly deductive existence’s proof for the QM-interference and for the ‘utmost low-level finality’ of QM altogether \[86, 53, 7\]. The whys and wherefores of theory do not require invoking the physical conceptions; independence of the classical physics, which is yet to be created from the quantum one. In particular, no use is required of the notion of a certain pretty small—again the classical term—quantity, i. e., the Plank constant $\hbar$.

6.5. **Interference.** Let us supplement once again sect. 6.3 with comments as to the involving the *physics-related* argumentation in the explication of quantal behavior. We have already mentioned above that for this purpose there is simply no language of physics (see sects. 2.1, 6.3) and of mathematics yet (see sects. 2.3, 5). That is why analogies of this sort are not only deceptive, but must be prohibited for exactly the same reasons that accompanied boxes (5). The typical examples in this connection are the notion of a simultaneous measurability and the 2-slit interference \[123, 3\].

First and foremost, the two cases—whether one or two slits are open—are entirely different experimental situations; one has here $\langle S, M, \ldots \rangle' \neq \langle S, M, \ldots \rangle''$. There is nowhere to seek a means of their comparison or the transference of one into another. Nonetheless, the classical approach, when opening another slit $\langle S, M, \ldots \rangle_2$ together with the first one $\langle S, M, \ldots \rangle_1$, in a literal sense, envisions characteristics for $\langle S, M, \ldots \rangle''$ (see sect. 6.3). In doing so, the transference method itself—‘addition of two physics’ $\langle S, M, \ldots \rangle'$ by the rule of arithmetic addition of statistics (59)—is meanwhile considered self-apparent. Thus the natural questions, such as ‘why/where are the zeroes coming from, they shouldn’t be there’. In accordance with the above exposition, everything here is erroneous, including the ‘natural’ questions. There are no rules at the outset—(non)classical and even quantum, just as there is no addition per se. A priori assuming them to stem from the obvious images is, in fact, the declaration of physical properties for $\langle S, M, \ldots \rangle''$ that, however, do not follow from anywhere \[154, p. 55\], \[134\], \[3\]. The (illegal) assumption of the ‘negligible effect of the which-slit detectors’ mentioned on p. 27 is identical with a declaration of a physical property.

Taken alone, the $\rho$-distributions—separate for $\langle S, M, \ldots \rangle_1$ and $\langle S, M, \ldots \rangle_2$—are entirely correct observational pictures, but introducing the rule (59) is indistinguishable from ‘invention’ of physics*; a logically prohibited operation. In other words, the mere fact of non-adherence to this rule means that the grammatical conjunction of the verbs ‘to understand/deduce’ with the noun ‘micro-phenomena’ is unacceptable even linguistically. It is the point $T$ that prohibits predefined (classical) semantics; a fact that was faithfully summarized by C. Fuchs: “badly calibrated linguistics is the predominant reason for quantum foundations continuing to exist as a field of research” \[50, p. xxxix\]. To understand or deduce (from mathematics) quantal phenomena is unfeasible \[123, p. 111\] and is “absolutely impossible, to explain in any classical way” (quote and italics by Feynman). Just as with the elucidation of the nature of the quantum state on p. 21, any classical justification is guaranteed to fail here, since it is based on significant implications.

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*D. Slavnov: “... to invent the physical exegesis of a ... mathematical scheme” \[152, p. 304\]. W. de Muynck: “Our custom of seeing classical mechanics as a no-nonsense description of 'reality as it is' does not seem to be justified. This custom is actually based on a confusion of categories ...” \[100, p. 89\].
The classical theory is a theory of observational objects with observational properties expressed by observational numbers. We possess none of the three items required to create a quantum (= correct) description (sect. 2). Accordingly, the description can only be changed into ‘to describe in newly created terms’. A. Leggett notes [80] that which is understood as common-sense should also be replaced. The reason is clear. The common-sense operates—and that’s perfectly normal—in observational categories rather than in structureless ‘microscopy’ (9) and ∪-abstractions of sect. 5.1.

For similar reasons, we cannot think (or imagine) that a particle in an interferometer ‘flies through the slit’, ‘has (not) arrived’, ‘is here/there, now/later’, that a “photon . . . interferes . . . with itself” [32, p. 9], and that, generally, ‘something is flying along a trajectory’, and ‘something’ is a particle at an intuitive understanding. Cf. Dirac’s description of “the translational states of a photon” in §3 of [32]. “Photons are just clicks in photon detectors; nothing real is traveling from the source to the detector” (ascribed to A. Zeilinger), and this point is supported by all the known varieties and types of interferometers. An interferometer—the entire installation—should be perceived as nothing more than a black box ⟨⟨S, M, . . .⟩⟩—the box (5)—‘outside of space and time’, i.e., as a kind of irreducible element producing the only entity—distinguishable α-events, and no other. The inspection of the screen clicks observed within it—‘is zero statistic possible in any spot?’—lacks meaning until the theory’s numeric apparatus is presented. The criticism of the typical (a common event-space) analysis of the 2-slit experiment [42] is already abundant in the literature. See, for example, the works [134] (!), [154, pp. 55–58], [71, Ch. 2], [43, 3], [104, p. 93].

Summing up, it is not the quantum interference that requires interpretative comprehension, but its classic ‘roughening’. In other words, a scheme that latently presumes the rule (59) of extrapolation of what is observed in macro and micro [140, (!) last sentence on p. 101]. It is this scheme—ascribing the ontological status [81, 80] to everything, not the quantum approach, that contradicts the logic and experience. Moreover, the paramount component of constructs—{observation ⇒ state'}—is cast out and replaced with (19) under such a transformation. The DataSource object (p. 29) begins to be identified with the observational and numeric characteristics, while the logic of the micro-world requires the distancing of precisely these two concepts, with no need for the characteristics themselves.

Thus we should not be deriving the physics of one phenomenon from another and making (super)generalizations, as soon as the incorrectness of the previous derivation method was established. Hence,

- quantum mathematics constitutes not a physical theory*—and that is its distinguishing feature—but, rather, a single (syntactic) principle of forming the mathematical models being subsequently turned into (the physical) theories. This principle is not subject to any (physical) treatment.

To create the models, we already have a good deal of latitude: the toolkit $O = \{\mathcal{A}, \mathcal{B}, \ldots\}$, the parameter $D$, the families $\{\mathcal{T}_{\mathcal{A}}, \mathcal{T}_{\mathcal{B}}, \ldots\}$, numbers $\{w_s\}$ of mixtures (23), and—thanks to the notion of invariance III—spectra, a structure of a group, and the concept of (different) representations of the one mathematical structure. This freedom will be subsequently

*S. Aaronson: “… it’s not a physical theory in the same sense as electromagnetism or general relativity . . . quantum mechanics sits at a level between math and physics . . . is the operating system . . .” [2, p. 110; emphasis in the original].
augmented with the notions of a composite system and of time \( t \), with models of the classical Lagrangians, and also with their symmetries and phenomenological constants. All that remains is to examine the numeric constituent of quantum mathematics.

7. Numbers

By number we understand not so much a multitude of unities, as the abstracted ratio of any quantity to another quantity of the same kind, which we take for unity.

I. Newton (1707)

\[ \ldots \text{where do units come from?} \]

S. Gryb & F. Mercati [56, p. 91]

7.1. Replications of ensembles. In connection with the emergence of a group, the numeric representation of brace also undergoes change, since the ‘doubling’ of the semigroup into a group* through adjoining the inversions deprives coordinate \( a \) of its distinction in comparison with the inversion \(-a\). Given the involution

\[ -(a) = a, \]

it makes no difference what to call an element, and what to call its inversion in the pair \( \{a, -a\} \).

The aforesaid is best demonstrated by another way of ‘digitalizing’ the empiricism, which is realized as the infinite replication of finite ensembles

\[ \{\{\Psi\}_n, \{\Psi\}_n, \ldots \} = \{\{\Psi\}_n\}_\infty \equiv: \{\Psi\}_{n\infty}. \]

That is, empirically, any infinite ensemble is thought of as created by repetitions (copies) of the finite objects \( \{\Psi\}_n \). They, in turn, are replications of the atomic primitive \( \{\Psi\}_1 \). Replication is thus an operation of the same significance as \( \cup \) and \( \sqcup \). With it, the \( (\Xi) \)-brace is characterized by the ‘numeric’ combination

\[ \{[n_1\infty, m_1\infty], [n_2\infty, m_2\infty], \ldots \} \Rightarrow (\Xi) \]

(indices label the \( \alpha_s \)-primitives), which has been created from the unitary brace by the scheme

\[
\begin{align*}
\{\{\Psi\}_\infty, & \{\Phi\}_\infty\} \\
\downarrow & \downarrow \\
\{\alpha \cdot \ldots \cdot \alpha\}_\infty
\end{align*}
\Rightarrow \begin{align*}
\{\{\Psi\}_\infty, & \{\Phi\}_\infty\}_m \\
\downarrow & \downarrow \\
\{\alpha \cdot \ldots \cdot \alpha\}_{(n+m)\infty}
\end{align*} \Rightarrow [n\infty, m\infty].
\]

The semigroup union \((\Xi') \sqcup (\Xi'')\) is then conformed with the writing

\[
\{[n_1\infty, m_1\infty], [n_2\infty, m_2\infty], \ldots \} \sqcup \{[n'_1\infty, m'_1\infty], [n'_2\infty, m'_2\infty], \ldots \} =
= \{[(n'_1 + n'_2)\infty, (m'_1 + m'_2)\infty], \ldots \}. \]

Moreover, the \( n \)-, \( m \)-quantities may be freely thought of as real ones due to the \( \mathbb{R}^2 \)-continual infinity of ensembles proven above (sect. 4). The empirical rationale of this is apparent; namely, fractions of arbitrarily large ensembles \( \{\Psi\Psi \ldots \} \).

*Formally known as a symmetrization of the commutative associative law (monoid) [20]. Curiously, under commutativity and associativity [25, sect. 1.10], solution to the problem of embedding is unique [20, pp. 15–17], and otherwise it is, generally speaking, absent. There exist the classes (Mal’cev (1936)), which are not axiomatized by finitely many \( \forall \)-formulas [92, pp. 216–217].
This way of matching an infinity with the Σ-postulate automatically inherits the translation of associativity/commutativity, because ‘percentages’ like \( s \) and \( w \), just as the rules \((43)–(44)\) themselves, do not even emerge. There, these numbers were originating from Σ-postulate, and it, in turn, was *demolishing the pair* \((x, \mathfrak{G})\) *itself* in \((42): \mathfrak{G} \rightarrow \infty\). It is clear that, according to \((63)\), the semigroup structure \(G\) is also inherited, turning into the addition of the numeric pairs

\[
(n', m') \oplus (n'', m'') = (n' + n'', m' + m'').
\]  

(64)

Returning to the group, we observe that the ‘negative symbols’ \((-n, -m)\) might be initially taken as the semigroup \(G\) under duplication, with equal success and with the same arithmetical addition \(\oplus\), while positive \((n, m)\) could be considered as their inversions. Summing up, let us specify the rules of passing to the numeric representations

\[
(33) \iff \{\pm \hat{p}, \pm \hat{q}\}\alpha, \quad (p, q) \in \mathbb{R}^2,
\]  

(65)

and, to avoid ambiguity, let us replace the binary composition symbol \(\cup\) with a new symbol \(\mp\) for objects \((65)\):

\[
\{\hat{p}, \hat{q}\}\alpha \mp \{\hat{n}, \hat{m}\}\alpha.
\]

The previously dropped primitives \(\Psi, \Phi\) have been restored here, since they will be further needed for theory’s invariance (sects. 7.4, 7.5), although they are still unnecessary at the moment.

It is not accidental that we spoke of ‘numerical labeling’ the brace (p. 20), since the question of arithmetic on them has not yet arisen. Although \(\nu\)-statistics—the real \(\mathbb{R}\)-numbers—are already involved, their use was based on the accustomed perception of a number. In accordance with pr. 11, the number formalization of ensemble empiricism should be considered in greater detail.

**7.2. Number as an operator.** Let us begin with the classical simplification

\[
\mathfrak{A} = \{\{\Psi\}, \{\Psi\Psi\}, \{\Psi\Psi\Psi\}, \{\Psi\Psi\Psi\Psi\}, \ldots\},
\]  

(66)

and the notion of the number does not yet appear in any form; it should be created (11).

The mathematical abstracting the observation micro-acts is an employment of the operation \(\cup\) and of its closedness (see sect. 5.1); for example, \(\{\Psi\} \cup \{\Psi\Psi\Psi\} = \{\Psi\Psi\Psi\Psi\}\). All the symbols in \((66)\), as well as the character \(\cup\), is of course merely a convention, and they may be changed. By writing \((66)\) in symbols like \(+\) and \(\{a, b, c, d, \ldots\}\), this set should be supplemented with identities as \(a + b = c, b + b = d, \ldots\), i.e., with a binary construction \(+\). Then (semi)group and commutative superpositions arise. At the same time we notice that the introduction of numbers at this point—even if only as symbols—is not necessary. More precisely, it would reduce to re-notation of the set’s elements. But the empirical description requires their unification, as manifested in the numeric notation like \(\{\Psi\} \equiv \mathbb{1}\{\Psi\}, \{\Psi\Psi\} \equiv 2\{\Psi\}, \ldots\). It is precisely this pattern that was implicitly kept in mind in procedures \((33)–(34)\) and \((61)–(62)\), i.e., when introducing the numbers \(n\) by means of replication of finite or infinite ensembles:

\[
\{\Psi \cdots \Psi\}_n \iff n\{\Psi\}, \quad \{\Psi\}_\infty \cdots \{\Psi\}_\infty\}_n \iff n\{\Psi\}.
\]

The symbol \(\iff\) should read here as ‘the same thing as’. Clearly, the very idea of conjunction of the two entities—empirical brace \((33)\) and the notion of a number (sects. 5.1, 5.2)—is not
otherwise implementable. That is to say, we have no any means of translating the macro-observations into the numerical language, other than through the notion of ‘quantity of something’:

\[
\begin{align*}
\downarrow & \ \mathcal{A}\text{-transitions} \ \downarrow \\
\downarrow & \ \{\text{‘quantity of’}\} \ \{\text{‘something’}\} \ (\text{replication}) \ \downarrow \\
\downarrow & \ \{\text{numbers}\} \ \{\Psi\text{-primitives, ensembles}\} \\
\ & \ n\{\Psi\}
\end{align*}
\]

(67)

Otherwise, the quantitative theory would have nowhere to originate even at the level of the natural \(\mathbb{N}\)-number characters.

On the other hand, the numerical tokens are ‘affixed’ not only to the ‘atom’ \(\{\Psi\}\), but also to other objects, any ones at that. Therein lies the primary meaning of this, still supra-mathematical concept; one might even say, a definition according to which this notion has been conceived and is being used universally. Here are examples:

\[
3\{\Psi\} \equiv \{\Psi\Psi\Psi\}, \ 2\{\Theta\Phi\} \equiv \{\Theta\Theta\Phi\Phi\}, \ \{\Psi\Phi\} \overset{2}{\rightarrow} \{\Psi\Psi\Psi\}, \ \ a^{3} \overset{1}{\rightarrow} 3a, \ \ c^{1} \overset{}{\rightarrow} 1c. \quad (68)
\]

Accordingly, there emerge identities like \(2b \equiv 4a\), \(3a \equiv c\), \(1c \equiv c\). In other words, as we complete simplification \((66)\), while abstracting the empirical contents of numeric symbols, they should be defined as unary operations \(\{\hat{1}, \hat{2}, \widehat{\pi}_4, \hat{\pi}, \ldots\}\) that operate at the \(\mathcal{A}\)-set as automorphisms.

Now, replication is formalized as an operator \(\hat{n}\) with its numeric symbol \(n\):

\[
\psi \mapsto n\psi, \quad \psi, n\psi \in \mathcal{A}, \quad n \in \mathbb{R},
\]

(69)

where \(\psi\) is understood to be any (sub)ensemble/(sub)set*. We will refer to this fact as implementation of a replication operator by numbers.

### 7.3. QM and the nature of arithmetic
We immediately observe the following properties.

The operators are applicable to each other, i.e., being a family \(\{\hat{n}, \hat{m}, \hat{p}, \ldots\}\), they are closed with respect to their composition \(\hat{n}(\hat{m}\psi) = (\hat{n} \cdot \hat{m})\psi = \hat{p}\psi\), and among them, there is an identical operator \(\hat{1}\psi = \psi\). The empirical meaning of the concept a fractional portion of the infinite ensemble (see \((61)\)) tells us that for each \(\hat{n}\) there exists its inversion \(\hat{n}^{-1}\). Hence, the composition of replications \(\hat{n} \cdot \hat{n}^{-1}\) must return the former ‘quantity’: \((\hat{n} \cdot \hat{n}^{-1})\psi = \hat{1}\psi\). As the family \(\{\hat{n}, \hat{m}, \hat{p}, \ldots\}\) provides automorphisms of the \(\mathcal{A}\)-set, these operators entail the associative identities \((\hat{n} \cdot \hat{m}) \cdot \hat{p}\psi = (\hat{n} \cdot (\hat{m} \cdot \hat{p}))\psi\). The common nature of the replication and of the \(\cup\)-union also signifies that there are relations in place that mix the actions of the unary \(\hat{n}\)'s and the binary union of ensembles. At a minimum, one suffices to define the action of replicator on a ‘\(\cup\)-sum’ of replications. Clearly, the case in point is a distributive coordination of \(\cdot\) and \(\cup\):

\[
\hat{p}(\hat{n}\psi \cup \hat{m}\psi) = (\hat{p} \cdot \hat{n})\psi \cup (\hat{p} \cdot \hat{m})\psi.
\]

We now observe that the indication of \(\psi\) everywhere in ensemble identities loses both necessity and significance, and the \(\psi\)-label becomes a semblance of a dummy index or the unit symbol \([kg]\), which can be changed. As we omit it, the theory is freed of \(\psi\) as of a ‘numeration

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*Formally, in the language of the ZF-theory, \(n\psi\) would be organized as an ordered pair \((n, \psi) : \equiv \{(n), \{n, \psi\}\}\) [76], where \(n\) is a cardinality of a set consisting of copies of the object/set \(\psi\).
unit’. Then the last relation, as an example, acquires the form of a property between the operator \( n \)-symbols \((69)\), if \( \{\cup, \circ\} \) are replaced with the symbols of binary operations \( \{+, \times\} \):

\[
p \times (n + m) = (p \times n) + (p \times m).
\]

Supplementing this relation with other empirically determining properties, one infers that the unary operationality of \( \hat{n} \)-replications is indistinguishable from the binary operationality on their \( n \)-symbols. The latter, in turn, acquires the multiplicative structure of a commutative group

\[
n \times m = m \times n, \quad (n \times m) \times p = n \times (m \times p), \quad n \times 1 = n, \quad n \times n^{-1} = 1,
\]

and, as for the addition +, it is already binary and commutative due to properties of \( \cup \):

\[
n + m = m + n, \quad (n + m) + p = n + (m + p), \quad n + 0 = n.
\]

After acquiring properties \((70)–(72)\)—call them arithmetics, symbols \( \{n, m, \ldots\} \) turn into abstract numbers, although their operator genesis does not go away and is yet to be involved. This is where a full list of requirements for the concept of a real number should be added, and which have to do with ordering \( < \), completeness/continuity, and their relations with algebraic rules \((70)–(72)\). We will take that this is done axiomatically [147, pp. 35–38], although the algebraic part of this ‘axiomatics’, as we have seen, is not axiomatic but deducible from empiricism.

As an outcome, we reveal an essential asymmetry in genesis of the standard binary structures + and \( \times \), and thereby a greater primacy of QM-consideration even over the (seemingly natural) arithmetic. Indeed, binarity may come only from operation \( \cup \), which is primordially unique and, thereby, is inherited only to the one natural prototype—addition.

- Multiplication is not featured in superposition principle \((57)–(58)\), nor does it arise directly as a binary structure.

It originates in the closedness of replications \( \hat{n} \circ \hat{m} \), and they are required according to the \( \mathbf{M} \)-paradigm \((12)\). In effect, the non-operatorial way of introducing the \( n \)-numbers is not a self-evidence for empiricism. However, the pure axiomatic declaration of arithmetic \((70)–(72)\) will, in one way or another, require a (reciprocal to \((67)\)) treatment of a number in the context of ‘the quantity of what?’, while its empirical pre-image always appears in the pair ‘the quantity’ + ‘of something’. Another way to put it is that,

- in the foundations of theory, the predecessor/analog of the notion of a physical unit arises,

though the ultimate description is a description in terms of binary structures \((70)–(72)\). It is carried out by dropping/attaching the symbols like \( \psi \), which is a quantum generalization of the physical theory’s independence of measurement units. Certainly, upon formalization, the \( \hat{n} \)-replication and its binary \( n \)-twin become universally abstract. For example, the \( \hat{n} \)-operator may be applied to the quantum case in which the object \( \psi \) has already an internal structure associated with the presence of \( \Psi, \Phi \)-primitives; this does not change the essence of the matter. Another example is when numbers \( n \) give birth to the really observable quantities. See also sect. 5.3, Remark 15, and additional discussion in sect. 9.
Let us now proceed from the fact that the comprehension/relation of the number and its operator has been formalized as described above, and ‘axioms’ (70)–(72) have been complemented with the negative numbers

\[ n + (-n) = 0, \]

for they have been fully justified in the superposition principle.

7.4. 2-dimensional numbers. A number in and of itself, as a replication operator, may be applied to any ensemble or, in fact, to anything at all. However, in quantum case, the ‘upper’ primitives are attached to every ‘lower’ \( \alpha \)-event. These primitives, as was noted above, have to be got rid of. At the same time, the minimal structure, associated with the statistical array \( \{ \alpha_s, \ldots, \alpha_s \} \) as a whole, is a unitary brace \( \{ \hat{n}, \hat{m} \} \alpha_s \), containing two ‘upper’ primitives \( \Psi, \Phi \). Their order, however, is arbitrary there. That is to say, given \((n, m) \alpha \) there are two quite equal objects \( \{ \hat{n}, \hat{m} \} \alpha \) and \( \{ \hat{n}, \hat{m} \} \alpha \) that are subjected to a replication. Each of them should be in a relation (see sect. 5.1) to any other brace (62), which is already apparent in the example of ‘1-dimensional’ versions \((n, 0) \alpha \) and \((n', 0) \alpha \).

As in the classical case (68), the sought-for generalizations of replicators, which are the transitive automorphisms on unitary \( \alpha \)-brace, are not abstract and not arbitrary. They are strictly bound to the declared meaning of a number: \( \hat{N} \)-operation of creating the copies. Therefore, by virtue of the equal rights of \( \Psi \) and \( \Phi \), it is imperative to bring the two 1-fold copying acts \( \hat{N} \{ \hat{n}, \hat{m} \} \alpha \) and \( \hat{M} \{ \hat{n}, \hat{m} \} \alpha \) into play, which differ in the permutation of primitives \( \Psi \equiv \Phi \). This point will determine a quantum extension of the replication.

As a result, since we have nothing but the ‘copying’ \( \hat{N} \) and ‘union’ \( \hat{M} \), the most general transformation of the brace \( \{ \hat{n}, \hat{m} \} \alpha \) into (any) brace \( \{ \hat{n}', \hat{m}' \} \), which is in a quantum replication relation with it, is determined by the rule

\[
\{ \hat{n}, \hat{m} \} \alpha \xrightarrow{(N,M)} \{ \hat{n}', \hat{m}' \} \alpha, \quad \{ \hat{n}', \hat{m}' \} \alpha = \hat{N} \{ \hat{n}, \hat{m} \} \alpha \hat{+} \hat{M} \{ \hat{n}, \hat{m} \} \alpha. \tag{73}
\]

This is the quantum version of operators (68)–(69), and the foregoing ideology of \( \hat{N} \)-operators and of liberation from the \( \Psi \)-symbols remains in force and entails the following. The numeric implementation of replicating the unitary brace (65), along with the \((n, m)\)-representation of itself, is also determined by a certain pair \((N, M) \in \mathbb{R}^2 \), i.e., by an operator symbol \((\hat{N}, \hat{M})\).

The aforesaid means that the numeric form \((n, m) \xrightarrow{(N,M)} (n', m')\) of transformation (73) is indistinguishable from a composition of pairs

\[(N, M) \odot (n, m) = (n', m'),\]

where \( \odot \) is a designation for the new binary operation. Its resultant structure is derived from the arithmetic nature (71) of the 1-dimensional replication (68) described above, i.e., from the rules

\[
\hat{N} \{ \hat{n}, \hat{m} \} \alpha = \{ N \times n, N \times m \} \alpha, \quad \hat{M} \{ \hat{n}, \hat{m} \} \alpha = \{ M \times n, M \times m \} \alpha. \tag{74}
\]

*As concerns the philosophical literature, the issue of numbers was, likely, discussed, and it would be appropriate to quote: “... numbers: they can be added to one another, perhaps multiplied by one another, .... But it is typically obscure what sort of physical relation these mathematical operations could possibly represent” (T. Maudlin [93, p. 138]; first emphasis ours, second in original). Cf. Einstein’s remarks concerning “concepts and propositions” and “the series of integers” on p. 287 in [36].
Here, a positivity/negativity of symbols \((n, m)\) in (65) should also be taken into account. Having regard to the foregoing, rules (73)--(74) generate the Ansatz

\[(N, M) \odot (n, m) = (\pm Nn \pm Mm, \pm Nm \pm Mn), \tag{75}\]

wherein all the four signs \(\pm\) are independent of each other, and the \((\times)\)-multiplication of 1-dimensional numbers in (71) and (74) have been re-denoted by the habitual standard \(Nm \equiv N \times m\).

As was the case previously, just emerged binarity for \(\odot\) should inherit associativity, existence of unity \(\mathbb{I}\), and of inversions. Namely, if the \((n, m)\)-pairs are identified with the notation (56) according to the convention

\[(n, m) \equiv a, \tag{76}\]

then the following properties should be declared:

\[(a \odot b) \odot c = a \odot (b \odot c), \quad a \odot \mathbb{I} = a, \quad a \odot a^{-1} = \mathbb{I}. \tag{77}\]

From (73)--(74) it is not difficult to see that the combining (77) with (56) leads to a distributive coordination of operations \(\oplus\) and \(\odot\):

\[c \odot (a \oplus b) = (c \odot a) \oplus (c \odot b). \tag{78}\]

However, the direct examination of this property shows that Ansatz (75) satisfies it automatically, and examination of an associativity in (77) particularizes (75) into the expression

\[(N, M) \odot (n, m) = \pm (Nn \pm Mm, Nm + Mn); \tag{79}\]

now, with two independent signs \(\pm\). Moreover, in passing we reveal the commutativity

\[a \odot b = b \odot a, \tag{80}\]

though it was not presumed prior to that. The search for unity \(\mathbb{I}\) and subsequent finding an inversion of the element \((n, m)\) yield:

\[\mathbb{I} = (\pm 1, 0), \quad (n, m)^{-1} = \left(\frac{n}{\Delta}, -\frac{m}{\Delta}\right), \quad \Delta \equiv n^2 \pm m^2. \tag{81}\]

Both the \((\pm)\)-symbols continue to be independent here. The choice \(\Delta = n^2 - m^2\) leads to the absence of inversions \((n, n)^{-1}\). This is in conflict with the group property (77) and also gives rise to the unmotivated exclusivity of the unitary brace \(\{\hat{n}, \hat{n}\}\). The case \(\Delta = n^2 + m^2\) remains, and it reduces the scheme to the form

\[\mathbb{I} = \pm (1, 0), \quad (N, M) \odot (n, m) = \pm (Nn - Mm, Nm + Mn) \tag{82}\]

with a single symbol \(\pm\). It is not difficult to see that the choice of sign \(+\) or \(-\) leads to models that are isomorphic in regard to which of representatives \((+1, 0)\) or \((-1, 0)\) should be assigned for the identical replication \(\mathbb{I}\). By virtue of (60) it does not matter, and we declare

\[\mathbb{I} \equiv (1, 0), \quad (N, M) \odot (n, m) = (Nn - Mm, Nm + Mn). \tag{83}\]

This is nothing more nor less than the canonical multiplication of complex numbers \(n + i \cdot m = a \in \mathbb{C}\), if the following identifications are performed:

\[(1, 0) \equiv \mathbb{I}, \quad (0, 1) \equiv i, \quad \{\oplus, \odot\} \equiv \{+, \cdot\}, \quad (n, m) \equiv (n + i \cdot m). \tag{84}\]

In view of the paramount importance of the \(\mathbb{C}\)-number field in QM, let us provide additional substantiations to the rigidity of emergence of this specific numeric structure, i.e., the axiom
collection (56), (77)–(80). Among other things, the transpositions \( \Psi \rightleftharpoons \Phi \) used above fit more general reasoning.

7.5. **Involutions and \( \hat{C} \)-algebra.** Apart from a freedom in ordering the primitives \( \Psi \rightleftharpoons \Phi \) in brace \( \{\hat{n}, \hat{m}\}_\alpha \), there is one more arbitrariness: reappointing them \( (\Psi \mapsto \Theta, \ldots) \) as elements of the set \( \mathcal{I} \). However, no physics predetermines any of these degrees of freedom. Say, if other ingoing \( \mathcal{I} \)-elements \( \Theta, \Omega \) were present in (31) instead of \( \Psi, \Phi \), then the \( \mathcal{G} \)-semigroup theory, strictly speaking, should be declared the segregated theories \( \mathcal{G}_{\Psi \Phi}, \mathcal{G}_{\Theta \Omega} \), etc. It is clear that the marking the theories, or they as a family, is a manifest absurdity, and they should be thus factorized with respect to all kinds of ways to label them by \( \mathcal{I} \)-primitives. The liberation from the \( \Psi, \Phi \)-icons and reconciliation of the result with pt. \( R^+ \) (p. 24) are then performed by the scheme \{primitive has changed \( \mapsto \) a number character is changing\}.

Inasmuch as declaring the \( \{\Psi, \Phi, \Theta, \ldots\} \) to be ingoing primitives in (31) is a replacement of one to another, any such an appointment boils down to permutations of no more than pairs, with two types (inner/outer):

\[
\begin{align*}
\mathcal{J}_{\Psi \Phi} : & \quad (\Psi, \Phi) \leftrightarrow (\Phi, \Psi), \\
\mathcal{R}_{\Phi \Theta} : & \quad (\Psi, \Phi) \leftrightarrow (\Psi, \Theta).
\end{align*}
\]  

(82)

However, it is immediately obvious that these reappointments change nothing in the  \( \cup \)-relationships between (31) and are defined by the structure relations \( \mathcal{J}_{\Psi \Phi}^2 = \mathcal{I}, \mathcal{R}_{\Phi \Theta}^2 = \mathcal{I} \). Then the need to indicate the primitives themselves, as required, is eliminated, and their symbols may be thrown away, if semigroup \( \mathcal{G} \) is correctly furnished with the two abstract involutions \( \mathcal{J} \) and \( \mathcal{R} \). The \( \mathcal{G} \) itself, of course, possesses also involution (60) that turns it into the group \( H \), but this involution has already had a numeric representation (65) by signs \( \pm \). Let us specify. Here, it suffices to identify the term ‘numeric’ with the group arithmetic of the \( \oplus \)-addition (56) coming from the superposition principle realized on pairs (64)–(65). Therefore the operators’ actions (82) should be carried over onto objects defined in precisely this manner; nothing more needs to be assumed.

Operator \( \mathcal{J}_{\Psi \Phi} \) is immediately translated into a numeric form independently of the property that the objects \( \{\hat{n}, \hat{m}\}_\alpha \) form a (semi)group. In fact, since the swap \( \Psi \rightleftharpoons \Phi \) in the unordered pair

\[
\mathcal{J}_{\Psi \Phi} : \quad \{\hat{n}, \hat{m}\} \mapsto \{\hat{m}, \hat{n}\} = \{\hat{m}, \hat{n}\}
\]

(\( \alpha \)-label is dropped here as superfluous) is indistinguishable from permutation of numbers \( n \rightleftharpoons m \), the symbols \( \Psi \) and \( \Phi \) may be thrown away, organizing the numbers themselves into the ordered pairs

\[
\cdots \Rightarrow (n, m) \mapsto (m, n).
\]

When required, the \( \alpha \)-symbol returns hereinafter.

Let us now proceed to the outer involution \( \Phi \rightleftharpoons \Theta \) in (82):

\[
\mathcal{R}_{\Phi \Theta} : \quad \{\hat{n}, \hat{m}\} \mapsto \{\hat{n}, \hat{m}\}.
\]

It is indifferent to the (first) \( \Psi \)-element of the pair, and, extracting it by the rule

\[
\{\hat{n}, \hat{m}\} = \{\hat{n}, \hat{0}\} \oplus \{\hat{0}, \hat{m}\},
\]

the question boils down to finding a representation to the transformations

\[
(\{\hat{n}, \hat{0}\} \oplus \{\hat{0}, \hat{m}\}) \mapsto (\{\hat{n'}, \hat{0}\} \oplus \{\hat{0}, \hat{m'}\}) \quad (n', m').
\]
The component \( \{ \overset{\circ}{n}, \overset{\circ}{m} \} \) must go into itself, since the symbol \( \Psi \) attached to it has not changed. It means that \( n' = n \), and one is left with the task

\[
\{ \overset{\circ}{0}, \overset{\circ}{m} \} \overset{\overset{\circ}{R}}{\rightarrow} \{ \overset{\circ}{0}, \overset{\circ}{m} \}.
\]

However, operation \( \overset{\circ}{R}_{\circ} \) recognizes only the primitive symbols rather than their numbers. That is, replications \( \overset{\circ}{m} \{ \overset{\circ}{0}, \pm \overset{\circ}{1} \} = \{ \overset{\circ}{0}, \pm \overset{\circ}{m} \} \) do formally commute with \( \overset{\circ}{R}_{\circ} \). Hence, omitting the letters \( \{ \Psi, \Phi, \Theta \} \), it will suffice to look for the representation of \( \overset{\circ}{R} \) by numeric pairs \((0, \pm m)\) factorized with respect to replications \( \overset{\circ}{m} \), i.e., by the set \( \{(0, 1), (0, -1)\} \). It, for its part, remains to be transformed into itself, and the replication operators \( \overset{\circ}{n}, \overset{\circ}{m} \) will recreate the generic case. The identical transformation \((0, \pm 1) \rightarrow (0, \pm 1)\) is ruled out due to \( \overset{\circ}{R}_{\circ} \neq \overset{\circ}{I} \); therefore, \((0, \pm 1) \overset{\overset{\circ}{R}}{\rightarrow} (0, \mp 1) \). Restoring all the symbols that were dropped, effect of \( \overset{\circ}{R} \) reduces to the sign change for the 2-nd element of the coordinate pair:

\[
(n, m) \overset{\overset{\circ}{R}}{\rightarrow} (n, -m). \tag{83}
\]

There is no need to change sign for the 1-st element, as this change is the operator \(-\overset{\circ}{I} \circ \overset{\circ}{R}\). Furthermore, one observes that the already existing group inversion \(-\overset{\circ}{I}\) coincides with composition

\[
(\overset{\circ}{R} \circ \overset{\circ}{I})^2 = -\overset{\circ}{I}, \tag{84}
\]

and we may even ‘forget’ about it, leaving the equipment

\[
\{ \oplus, \overset{\circ}{I}, \overset{\circ}{m}, \overset{\circ}{R}, \overset{\circ}{I} \} \tag{85}
\]

of semigroup \( \mathcal{G} \) as an irreducible set of mathematical structures over it.

In this connection yet another—more formal—motivation of the passage \( \{\text{semigroup} \rightarrow \text{group}\} \), and thus of the superposition principle, does arise. In fact, the derivation of \( \overset{\circ}{R} \) above engaged the inversion \((60)\), but reappointment of primitives \( \overset{\circ}{\Phi} \overset{\circ}{\rightarrow} \overset{\circ}{\Theta} \) in \((82)\) is a fully independent act. Therefore if we forget about ‘(-)-copies of the positive pairs’ \((0, m)\), then the involutory nature of automorphism \( \overset{\circ}{R}_{\circ} \) would still reproduce the semigroup \( \mathcal{G} \) in numbers by ‘duplication’ \( m \mapsto \pm m \), i.e., create the negative pairs \((0, -m)\), thus turning \( \mathcal{G} \) into a group \( H \). An analogous reasoning on the symbol ‘-’ could be cited even earlier, when the \( \mathbb{C} \)-field was being derived.

Now, remembering the above-described passage to the binarity of \( \circ \)-multiplication on the \((n, m)\)-pairs, we arrive at the problem of matching it with structures \((85)\). Clearly, one needs only to ascertain the functionality of operators \( \overset{\circ}{I} \) and \( \overset{\circ}{R} \) that were not available yet. Relation \((84)\) immediately gives us the correspondence \( \overset{\circ}{R} \circ \overset{\circ}{I} \overset{\circ}{\rightarrow} i \), since \( i^2 = -1 \). Hence, one of these operators, say \( \overset{\circ}{I} \), manifests itself in the imaginary unit \( i \). Origin of this operator—permutation \( \overset{\circ}{I}_{\circ} \) in \((82)\)—is the very same permutation \( \overset{\circ}{\Psi} \overset{\circ}{\rightarrow} \overset{\circ}{\Phi} \) that generated the \( i \)-object in algebra \((80)\)–\((81)\). The second operator, i.e., \((83)\), as is directly seen, is also not related to the binary \( \circ \) and \( \circ \) but determines the change \( i \mapsto -i \). This means that the QM-consideration does not just lead to the field \( \mathbb{C} \) but to a division \( \hat{\mathbb{C}}^* \)-algebra, which is equipped with the two non-binary operations

\[
\overset{\circ}{R} a \rightarrow a^*, \quad a \overset{\circ}{\rightarrow} \overset{\circ}{a}.
\]

Informally, it defines all the basic actions on ‘complex quantities’ and thereby determines a QM-extension/generalization of the intuitive (arithmetical) manipulations with the habitual
real quantities (67)–(72). Hence, the four binary arithmetical operations—addition/subtraction and multiplication/division—should be supplemented with the two unary ones: conjugation $\hat{\mathcal{R}}$ and swap $\hat{\mathcal{J}}$.

**Remark 13.** A curious observation for the complex number ‘mathematics’ is in place. None of these operations boils down (algebraically) to involution $-\hat{1}I$. That is to say, each of the pairs $(\hat{\mathcal{R}}, -\hat{1})$ or $(\hat{\mathcal{J}}, -\hat{1})$ is expressible through $(\hat{\mathcal{J}}, \hat{\mathcal{R}})$, and not the reverse. To put it plainly, the self-suggested going from the natural sign change (i.e., $-\hat{1}$ over $\mathbb{R}$) to the inversion of the $\oplus$-addition (i.e., $-\hat{1}$ over $\mathbb{C}$) deprives the involution $-\hat{1}I$ of its primary character in the domain $\mathbb{R}$. Furthermore, the 2-nd operation $\hat{\mathcal{J}}$ is more primary even than the complex conjugation $\hat{\mathcal{R}}$ in that it has to do with a formal pair $(n, m)$—merely transposes it—and does not invoke the arithmetic, as does $\hat{\mathcal{R}}$ when changing the sign $m \mapsto -m$ in (83).

We note—and this is important [18]—that the observational statistic are unchanged upon both operations $\hat{\mathcal{J}}, \hat{\mathcal{R}}$.

### 7.6. Naturalness of $\mathbb{C}$-numbers.

Thus the $\mathcal{T}$-set primitives have been entirely banished from the theory, with the exception of the eigen-state $\alpha$-markers, which are needed only for distinguishability (sect. 2.1) of $\mathcal{A}$-observations. These markers may be interchanged, but permutability $\alpha_s \leftrightarrow \alpha_n$ is already reflected by the superposition’s commutativity. Taking now into account the fact that reassigning the $\alpha$-labels does not touch on the concept of the number, one infers: the invariance attained above is exhaustive. As a result, we draw the following conclusion.

- The coordinate representatives $\{a, b, c, \ldots\}$ of states and of their superpositions (57) form the complex number field $\tilde{\mathbb{C}}^*$ equipped with the structures of conjugation and of swap:
  
  $$(n + im) \mapsto (n - im), \quad (n + im) \sim (m + in). \tag{86}$$

  Statistical weights $\nu_s$ in object (34) are invariant with respect to both the involutions $\nu_j(a^*) = \nu_j(a) = \nu_j(\tilde{\alpha})$ for each component $a_s$ independently.

What is more, the commentary on the primacy of QM over the abstract arithmetic (see p. 40) has a logical continuation.

- Quantum-theoretic description invokes no $\mathbb{C}$-numbers; it creates them together with the $\tilde{\mathbb{C}}^*$-algebra.

This fact is remarkable in its own right because the ‘2-dimensional’ numbers arise at the lowest empirical level, not from the need for solving any mathematical problems; mathematics is still lacking. Therefore pt. $\mathbb{R}^+$ (p. 24) could have even been weakened by replacing ‘homomorphism onto numbers’, roughly speaking, with the ‘homomorphism onto continuum’. Our minimal points of departure are replications and the ‘ingoing/ongoing’ structure of brace (31). Neither does theory depend on the meanings that will be later attached to the physical concepts—observables, measurement, spectra, means, etc—to their interpretations or rigorous definitions. At the same time, the interferential ‘effects of subtraction and of zeroes’ are intrinsically present in the very foundation of the construct.

Let us add, in conclusion, two more formal vindications of rigidity of emerging the $\mathbb{C}$-structure. In doing so, one assumes that we have already had the $\mathbb{R}$-numbers.

Unitary brace contain pairs of the form $\{\hat{n}, 0\}$. The binary operations $\{\oplus, \odot\}$ on their numeric representatives $(n, 0)$ are closed and, as easily seen, form a commutative field, which is isomorphic to $\mathbb{R}$. It is a subset of the generic pair set $(n, m)$. From the operator nature of $\odot$
it follows that these pairs form a certain distributive ring with general-group properties (77)–(78). The presence of the field \( \mathbb{R} \) contained in it tells us that these pairs can be realized by the elements \( n + mx \) of, at most, associative algebra \( A \) over \( \mathbb{R} \). Here, \( n, m \in \mathbb{R}, x \) is a generator of any ring’s element beyond \( \mathbb{R} \), and the habitual \(+\) replaces the sign \( \oplus \). Multiplication of two such elements
\[
(n + mx) \odot (n' + m'x) = nn' + (mn' + nm')x + mm'x^2 = \cdots
\]
immediately shows that result does not depend on order of factors, i.e.,
\[
\cdots = (n' + m'x) \odot (n + mx),
\]
due to permutability of \( \{n, m, n', m'\} \) between each other and of any \( x \) with itself. This is a direct consequence of 2-dimensionality of the algebra \( A \); it must be commutative. Invoking now the well-known Frobenius theorem on associative and commutative structures containing the field \( \mathbb{R} \) [139], we arrive once again at a multiplication of the form (80).

Yet another reasoning about exclusivity of \( \mathbb{C} \)-numbers follows from matching the topological and algebraic properties of the general numeric systems [107, sect. 27]. The case in hand is the uniqueness and non-arbitrariness in emergence of the topological field \( \mathbb{C} \); Pontryagin (1932). In our case, we have two continuums—numeric symbols \( n \) and \( m \), each of which, by the very method of constructing the \( \Xi \)-objects (33), is equipped only with the natural ordering \( < \). Since we have no any more math-structures yet, the topology, continuity, and limits on each of the continuums can already be introduced with respect to this relation. For one example, there is no need to introduce a topology by a priori creating the arithmetic operation of multiplication/divisibility of rationals, as it is done in the \( p \)-adic approaches to QM [153, 150, 71]. The ‘non-naturalness’ of multiplication as compared with addition was already noted above. Besides, in the \( p \)-adic versions for a numeric domain, the topologically and physically required matching between the natural ordering, connection, and continuity [107, Ch. 4] is destroyed, and the approaches themselves stipulate the existence of the observations numbers with a comprehensive arithmetic. At the same time, questions about ‘structure’ of the \( x \)-space at Planck’s scale\(^*\) and about measurements by rationals (see motivation in [150, 153]) are not arising yet, because we are not relying on physical conceptions and are not yet introducing these notions as numerical. From the low-level empiricism standpoint, any objects, apart from the \( \mathbb{R}^2 \)-continuality and frequencies \( \nu \), require call for independent axioms. In turn, the primary nature of the \( \mathbb{R} \)-continuality itself follows directly from the boolean \( 2^\Xi \) (p. 18) and \( \Sigma \)-postulate of infinity.

8. State space

Quantum states … cannot be ‘found out’
[112, p. 428]

They want to know “what is really going on”
B.-G. Englert [37, p. 12]

\(^*\)D. Mermin: “when I hear that spacetime becomes a foam at the Planck scale, I don’t reach for my gun” [96].
8.1. **Linear vector space.** Once replication \((\hat{N}, \hat{M})\) of brace \(\{\hat{n}, \hat{m}\}_\alpha\) has acquired a binary character
\[
(\hat{N}, \hat{M})(\{\hat{n}, \hat{m}\}_\alpha) \iff ((N, M) \circ (n, m))|\alpha\rangle = a|\alpha\rangle,
\]
(87)
the difference between ‘what is replicated’ and ‘how many times’ disappears. A symbol \(|\alpha\rangle\) of the eigen-state has been attached to the abstract \(\mathbb{C}\)-number \(a\). Construing this point as a quantum analog of re-choosing (liberation of) the measurement units (p. 40), we obtain that the two formal states \(a|\alpha\rangle\) and \(b|\alpha\rangle\) are always connected by a certain number operator \(\hat{p}\):
\[
b|\alpha\rangle = \hat{p}(a|\alpha\rangle), \quad \hat{p} \leftrightarrow b \circ a^{-1}.
\]
Manipulating the numbers becomes independent of symbols \(|\alpha\rangle\). The way to formalize this is to think of generic states \(a|\Psi\rangle \in \mathcal{H}\) as the ‘solid characters’
\[
a|\Psi\rangle \mapsto |\Xi\rangle \in \mathbb{H},
\]
i.e., as the \(|\Xi\rangle\)-elements of a new set \(\mathbb{H}\), which is equipped with the \(\hat{p}\)-replication images represented by the \(p\)-family \((p \in \mathbb{C})\) of maps
\[
\mathbb{C} \times \mathbb{H} \mapsto \mathbb{H} : \quad p \cdot |\Xi\rangle = |\Phi\rangle \in \mathbb{H},
\]
(89)
and which is obliged to inherit the structure (87). This inheritance tells us that coordination of \(\circ\)-multiplication in (87) with the replication’s \(p\)-realization is performed by a new operation \(\cdot\) of the unary kind on \(\mathbb{H}\), i.e., (89), which should be subordinated to the rule
\[
p \cdot (a \cdot |\Psi\rangle) = (p \circ a) \cdot |\Psi\rangle \quad (p, a \in \mathbb{C}, \quad |\Psi\rangle \in \mathbb{H}).
\]
(90)
Due to this connection between operations \(\circ\) and \(\cdot\), it would be logical to refer to the latter as multiplication as well. An analogous rule had already occurred in relationship (58) between the \(\oplus\)-structure and the (\(\oplus\))-group superposition, i.e., when the multiplicative structures \(\{\circ, \cdot\}\) were not available yet.

Among replication operators \(\hat{p}\), there exists an identical transformation
\[
\hat{p} = \hat{I} : \quad a|\Psi\rangle \xrightarrow{\hat{I}} a|\Psi\rangle,
\]
to which a symbol of the numeric unity \(p = 1\) corresponds. Hence, in accordance with (88)–(89), there follows the rule
\[
1 \cdot |\Xi\rangle = |\Xi\rangle, \quad \forall |\Xi\rangle \in \mathbb{H}.
\]

It is clear that the \((\cdot)\)-multiplication needs to be agreed with the \(\cup\)-union. Let us make use of the fact that the object of (quantum) replication may be not only the unitary brace \(\{\hat{n}, \hat{m}\}_\alpha\), which is equivalent to the eigen-element \(a|\alpha\rangle\), but a \((\oplus)\)-sum of the like objects and, in general, any constituents of quantum ensembles (see p. 39). Therefore the \(\hat{p}\)-replication
\[
\hat{p}(a|\alpha\rangle \oplus b|\beta\rangle) = \cdots
\]
(91)
is known to have its twin-sum
\[
\cdots = a'|\alpha\rangle \oplus b'|\beta\rangle = \cdots
\]
(92)
with certain coefficients \(a', b'\). Let us, for the moment, give back (91) to the initial language of operators/brace according to the scheme
\[
(\hat{N}, \hat{M}), \quad \{\hat{n}', \hat{m}'\}_\alpha + \{\hat{n}'', \hat{m}''\}_\beta.
\]
(93)
Take also into account a pre-image of operation $\dagger$ on objects (91), i.e., $\dagger$. Then, (93) and content of sects. 7.4–7.5 shows certainly that expression (92) must be of the form

$$\cdots = (p \circ a)|\alpha\rangle \dagger (p \circ b)|\beta\rangle = \cdots.$$  

Reconverting, by (90), expressions like $(p \circ a)|\alpha\rangle$ into the operatorial $\hat{p}(a|\alpha\rangle)$, we complete the ellipsis

$$\cdots = \hat{p}(a|\alpha\rangle) \dagger \hat{p}(b|\beta\rangle).$$

Passing now to the $p$-number and to the $|\Xi\rangle$-objects (88), i.e., replacing $a|\alpha\rangle \mapsto |\Psi\rangle$, $b|\beta\rangle \mapsto |\Phi\rangle$, one derives a linearity of operation $\cdot$ when acting on a sum:

$$p \cdot (|\Psi\rangle \dagger |\Phi\rangle) = p \cdot |\Psi\rangle \dagger p \cdot |\Phi\rangle.$$  

Here, the $H$-addition $\dagger$ has been carried over to the group $\mathbb{H}$ as a new symbol $\hat{+}$. This is nothing but a distributive coordination of the ($\cdot$)-multiplication with the group addition $\dagger$.

In a similar way—through a number operator, one establishes yet another relation $a \cdot |\Xi\rangle \dagger b \cdot |\Xi\rangle = (a \oplus b) \cdot |\Xi\rangle$ between $\cdot$ and $\dagger$; its origin is equivalent to (58). From the constructs above, it is not difficult to see that we have examined all the possibilities of replicating the superpositions (57) or their constituents by arbitrary $\mathbb{C}$-numbers, which is why we have exhausted all the compatibility rules that stem from the two fundamental operations—replication and union ($\dagger$).

Thus, having considered the passage (88)–(89) as a final homomorphism of the $H$-group elements $a|\Psi\rangle$ onto the objects $|\Xi\rangle \in \mathbb{H}$, i.e., adjusting the previous concept of a state—dataSource (p. 28), we infer the following.

- The minimal and mathematically invariant bearer of the observation’s empiricism is an abstract space $\mathbb{H}$ of states $|\Psi\rangle$ of the system $\mathcal{S}$. The structure properties

  $$\mathbb{H} \equiv \{|\Psi\rangle, |\Phi\rangle, \ldots\} \quad \text{(commutative group under operation $\dagger$)},$$  
  $$\mathbb{C} \equiv \{a, b, \ldots\} \quad \text{(field of complex numbers: (56), (76)–(80))},$$  
  $$a \cdot |\Psi\rangle \in \mathbb{H} \quad \text{(closedness under external operation $\cdot$)},$$  
  $$a \cdot (b \cdot |\Psi\rangle) = (a \circ b) \cdot |\Psi\rangle, \quad a \cdot |\Psi\rangle \dagger b \cdot |\Psi\rangle = (a \oplus b) \cdot |\Psi\rangle,$$

  $$\mathbb{I} \cdot |\Psi\rangle = |\Psi\rangle, \quad a \cdot (|\Psi\rangle \dagger |\Phi\rangle) = a \cdot |\Psi\rangle \dagger a \cdot |\Phi\rangle$$

  (96)

  of the space coincide with axioms of a linear vector space (LVS) over the field $\mathbb{C}$.

This ‘axiom list’ should be complemented with a declaration of the global number $\mathbb{D}$ value (52) established above. Stated in a nutshell, the nature of the quantum state space is twofold: group superposition (57) and operator nature of the ‘$a$-numbering’ its elements. Relations (96) describe rules of their ‘interactions’. It is known that such formations, while being implemented by a binary algebra of numbers, turn into the vector spaces and modules [77, Ch. 5], [139, sect. I(7.1–2), II(13.4)]. Concerning consistency of these rules—say, of numeric distributivity (78)—with relations (96), see the work [110].

**Remark** 14. A certain oddity is in place. QM-empiricism is such that the standard definition of LVS by the all-too-familiar axioms (94)–(96) is more nonphysical by its nature than the ‘generalistic’ abstraction of a group with operator automorphisms [77]. Anyway, a point like this might be expected because, as noted in sect. 1.2, meaning of all of tokens in (1) and their origin are entirely unknown, and linearity of QM is radically different from other ‘linearities’ in physics.
8.2. Bases, infinities, and countability. From a ban on transitions $\alpha_s \rightarrow \alpha_n$ under $s \neq n$ it follows that unitary $\alpha_s$-brace (33) correspond to vectors $a_s \cdot |\alpha_s\rangle$ that are linearly inexpressible through each other. Aside from the general ensemble brace (31), no other elements exist, and all of them are in one-to-one correspondence with the vector representations $a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots$. Each such vector has a statistical pre-image (31), and vice versa; there are no gaps. This means that the system of vectors $\{|\alpha_1\rangle, |\alpha_2\rangle, \ldots\}$ forms a basis of $\mathbb{H}$—basis of LVS, and the number of symbols $|\alpha_s\rangle$ is its dimension: $\dim \mathbb{H} = D$. The $D = \infty$ case, just like anything associated with infinity, can not be formalized without topology, and its presence is presumed, but discussion is dropped. We just remark that even earlier, when arising the 2-dimensional continuum, we have silently assumed the $\mathbb{R} \times \mathbb{R}$-product topology on it. This assumption is natural, inasmuch as it does not involve additional constructions/requirements. What is more, if we bring into play the concept of measurement (47)—i.e., projection $\mathbb{R}^2 \mapsto \mathbb{R}$, then such a topology becomes a weakest one, when the projection is continuous (Tikhonov (1930)). Thus if properties (94)–(96) are directly accepted to be empirical, then the mathematical rigors augment them axiomatically on the outside; one constructs the mathematical theory.

The micro-transition $\rightarrow^{\Psi}$ in sect. 2.1—the click—is a solitary entity. This means that the number of the eigen $\alpha_s$-primitives for an actual instrument may be either finite or discretely unbounded. We base upon the fact that continual formations is a product of mathematics rather than empiricism; see also [88, p. 35]. The $\Sigma$-set, as an example, is also non-continual, but that premise may even be given up, because only a discrete portion of this set (transitions $\rightarrow^{\Psi}$) is present at arguments. Notice incidentally that continuum does not feature in the ZF-axioms [76] but is also created, just as “an infinity is actually not given to us at all, but is ... extrapolated through an intellectual process.” (Hilbert–Bernays [73, p. 55]). One obtains a countability of the set of vectors $\{|\alpha_s\rangle\}$. Hence follows a completeness of $\mathbb{H}$ and countability of dimension (52), as of numeric LVS-invariant:

$$D = 2, 3, \ldots, \aleph_0 \quad (= \dim \mathbb{H}).$$

8.3. The theorem. The states $|\Psi\rangle$ and their sums, at the moment, form a formal family of different elements. Recall that symbols $\{\approx, \neq\}$ in pt. R, as from the end of sect. 5.4, have been replaced with the standard ones $\{=, \neq\}$. The physical aspects $\{S, M, \ldots\}$ were being left aside so far, and, for example, $|\Psi\rangle$ and $c \cdot |\Psi\rangle$ were the different vectors of the $\mathbb{H}$-space. However,

- empiricism (deals with and) yields not states and their superpositions but $|\alpha\rangle$-representations.

It is these representations that carry information about statistic $(\nu_1, \nu_2, \ldots)$ through coefficients $a_j$. The replicative character of $c$-multipliers and $\Sigma$-postulate entail however that the two vectors $1 \cdot |\Psi\rangle$ and $c \cdot |\Psi\rangle$ should correspond to the one and the same statistic $\nu_s = (1, 0, \ldots) = \tilde{\nu}_s$ under an observation $\mathcal{D}$ with the eigen collection $\{|\Psi\rangle, \ldots\}$. Let us write the equalities

$$\nu_s \iff 1 \cdot |\Psi\rangle = a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots \quad \Rightarrow \quad \nu_s$$

$$\tilde{\nu}_s \iff c \cdot |\Psi\rangle = c \cdot (a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots) \quad \Rightarrow \quad \tilde{\nu}_s$$

(98)
and look at them in the following order: the first line—from right to left, and the second—in the reverse direction. Their right hand sides are the carriers of some statistics $\nu_{\sigma}$ and $\tilde{\nu}_{\sigma}$. The frequencies $\nu_{\sigma} = (\nu_1, \nu_2, \ldots)$ come from the numeric set $(a_1, a_2, \ldots)$ under the same environments $\langle S, M, \ldots \rangle$ that give rise to the statistic $\nu_{\sigma}$. But it is also generated by the representative $c \cdot |\Psi\rangle$, which is associated with the same $\langle S, M, \ldots \rangle$; hence, $\tilde{\nu}_{\sigma} = \nu_{\sigma}$. By virtue of the second equal sign in (98), the same $\langle S, M, \ldots \rangle$ are associated with the second $\mathcal{A}$-collection $(c \circ a_1, c \circ a_2, \ldots)$. Therefore the frequencies $\tilde{\nu}_{\sigma}$ that emanate from it have to be identical to those emanating from the first collection $(a_1, a_2, \ldots)$. That is to say $\tilde{\nu}_{\sigma} = \nu_{\sigma}$, and the scale dilatations $|\Psi\rangle \mapsto c \cdot |\Psi\rangle$ are not recognized by any $\mathcal{A}$-instruments. A more concise reasoning is that the quantum replication $c = \eta + \imath \eta$ may be viewed as the 1-dimensional replications $\tilde{n}(a|\alpha\rangle)$, $\tilde{\lambda}(a|\alpha\rangle)$ of all the brace $a_s|\alpha_s\rangle$-images and of sums like $\tilde{n}(a|\alpha\rangle) \pm (\imath \cdot \tilde{m})(a|\alpha\rangle)$. These replications do not change the superposition statistic as a whole.

The aforesaid gives birth to a universal—stronger than $\approx$ and irrespective to instruments—observational equivalence relation

$$|\Psi\rangle \cong \text{const} \cdot |\Psi\rangle$$

on the space $\mathbb{H}$; the ‘physical’ indistinguishability (sect. 2.4).

The basis vectors $|\alpha_s\rangle$ and their ($\cong$)-equivalents will be referred to as eigen vectors/states of instrument $\mathcal{A}$; clearly, the concepts of instrument and of (macro)-observation (O) should now be distinguished. Accordingly, the spectral construction (50)–(51) should be corrected. Call the data set

$$\{|\alpha_1\rangle|\alpha_1\rangle, |\alpha_2\rangle|\alpha_2\rangle, \ldots \} \equiv: [\mathcal{A}]$$

the $[\mathcal{A}]$-representative of instrument $\mathcal{A}$ in $\mathbb{H}$. The add-on (99) does not touch on $\mathbb{H}$-space, since the spectral labels $|\alpha_j\rangle$ are the self-contained objects independently of vectors $|\alpha_k\rangle$. These labels and corresponding degenerations determine ‘internal properties’ of the formalized notion of an instrument (99). Conversely, any state vector $|\Psi\rangle$ or $c \cdot |\Psi\rangle$ may be treated as one element of the $[\mathcal{C}]$-representative for imagined/actual instrument $\mathcal{C}$ (spectrum is arbitrary) and is a ($\hat{+}$)-sum of the eigen elements for any other $[\mathcal{A}]$-representative. Remembering (23), we arrive at the final conclusion that determines the (pre-dynamical) theory of macroscopic data on micro-transitions.

- The core (1-st) theorem of quantum empiricism.
  1. The mathematical representatives of physical observations and of preparations are the quantum states $|\Psi\rangle$ and statistical mixtures of eigen $|\alpha\rangle$-states

$$\{|\alpha_1\rangle|^{w_1\rangle}, |\alpha_2\rangle|^{w_2\rangle}, \ldots \}, \quad w_1 + w_2 + \cdots = 1.$$  

2. Properties (94)–(97) define objects $|\Psi\rangle$ as elements of a (complete separable) linear vector space $\mathbb{H}$ over the algebra of complex numbers $\mathbb{C}^*$.

3. Dimension $\dim \mathbb{H} = D \geq 2$, representing an observable quantity ($D \neq \infty$), is set to the value $\max\{|\Sigma_{\sigma}|, |\Sigma_{\sigma}|, \ldots \} = D$ as required by an accuracy of the toolkit $\mathcal{O} = \{\mathcal{A}, \mathcal{B}, \ldots \}$. The eigen $|\alpha\rangle$-vectors for each $[\mathcal{A}]$-representative provide a basis of $\mathbb{H}$ independently of spectra (99).

4. Rules (94)–(97), for a fixed $D \neq \infty$, are categorical as an axiomatic system; they admit no non-isomorphic models. States $|\Psi\rangle$ and $c \cdot |\Psi\rangle$ are statistically indistinguishable, and statistics $\nu_{\sigma}(a)$ are invariant to involutions (86).
The words ‘complete separable’ have been supplemented here for mathematical reasons. Indeed, the algebra constructed above needs some amendments of a topological nature because all the construction contains three infinities: continuum $\mathbb{C}$, continuum $\mathbb{H}$, and dimension $D$.

In this connection, see the work [107].

To summarize: having considered the micro-destruction arrays with empirical rather than a formal eye on arithmetic, ideology of creating the quantitative theory leads to the key feature of quantum states—their addition, and the ‘quantities under addition are measured’ by complex numbers.

Remark 15. Again, as in sect. 7.3, we draw attention to a hidden and unremovable extension of the physical units’ concept.

• “... units. Despite the rudimentary nature of units, they are probably the most inconsistently understood concept in all of physics ... where do units come from?” [56, p. 91].

Surprisingly, the naive and straightforward conjunction of this concept with an abstract number seems to contravene the multiplication arithmetic; but not the addition one. The typical example illustrates the point: $(2\, \text{kg}) \times (3\, \text{kg}) \neq 6\, \text{kg}$. On the other hand, $2 \times (3\, \text{kg}) = 6\, \text{kg}$ and $(2\, \text{kg}) + (3\, \text{kg}) = 5\, \text{kg}$, and the ‘$\text{kg}$’ may be replaced here with any other entity: the classical metres, the abstract ‘Quanten Stücken $\psi$', etc. They have no any operational meaning, but one cannot get by without them.

The digital characters—say, 5—acquire their usual (abstract) numerical meaning (matematization) only when we throw the ‘units’ \{Stück, °C, sheep, $\psi$, ...\} out of data like \{5Stück, 5°C, 5sheep, 5ψ, ...\}. The symbol ‘five 5’ in 5°C is the very same ‘five 5’ as in 5ψ. It is pointless without such a matching/abstracting. At the same time, an inversion of this abstracting—attaching \{Stück, °C, sheep, $\psi$, ...\} to the character 5—is always an interpretation of the abstraction: interpretation ‘the Stück’, ‘C-interpretation the Celsius’, etc. See also comments by D. Darling concerning “sheep, numbers, add, things”, etc on p. 178 in the book [50]. Incidentally, within this (physical and quantum) context, even the very LVS should be regarded as no less a primary math structure than the numbers themselves. Empiricism generates both these structures simultaneously. However, the structure ‘LVS’, in contrast to arithmetic, simply does not ‘forget’ an operator nature of the structure ‘number’ and its empirical inseparability from the notion of the unit.

As we have seen, nothing above and beyond what was used in constructing the mathematics (94)–(97) is required to explain the meaning of the quantum state. Besides, we have obtained not merely a completion of construction (11):

$$\oplus(a_1, |\alpha_1\rangle; a_2, |\alpha_2\rangle) = a_1 \cdot |\alpha_1\rangle \hat{+} a_2 \cdot |\alpha_2\rangle \hat{+} \cdots .$$

In the first place, one establishes the nature of the quantal discreteness—distinguishability of discrete $\alpha$-events (sect. 2.2). Accordingly, “indivisibility, or “individuality”, characterizing the elementary processes” (N. Bohr [113, p. 203]) must be formalized into the ‘elemental’ click. We also clarify the formalization of measurement/preparation and of genesis of the $\mathbb{C}$-numbers. The well-known (*)-conjugation operation also finds its origin. Moreover, it is supplemented with a transposition $\Re(a) \rightleftharpoons \Im(a)$ of the real/imaginary part of the $\mathbb{C}$-number, and this transposition should be regarded just as natural operation as conjugation. The emergent concepts of spectra, of their degenerations and eigen-states provide near comprehensive mathematical image of physical observables. The state becomes devoid its mysteriousness [81, 103], since it is explicitly built in terms of the unique model of the ‘statistical’ $|\alpha\rangle$-representatives supplemented with microscopic mixtures (100).
9. Numbers, minus, and equality, revisited

... quas decet numeris negativis exprimantur, additio et subtractio consueto more peracta nullis premitur difficultatibus*

L. Euler (1735)

9.1. Separation of the number matters. The empirical adequacy of quantum theory can be based only on phenomenological ensembles (sects. 2.5, 4). Creation of their mathematics tells us, then, that the ‘quantity of something’ (67)–(69) turns into a formal operational algebra through labeling the operator replications (sects. 7.1–7.2) and yields the numbers proper. At first, they emerge merely as

\[ \text{n-symbols of abstractions (67)-(68) with ordering } < \]

\[ \downarrow \downarrow \]

\[ \ldots \ldots \]

and then as internal objects of theory:

\[ \downarrow \downarrow \]

\[ \text{numbers } n \text{ as elements of arithmetic (70)-(72) \quad \downarrow \downarrow } \]

\[ (m,n)\text{-numbers } a \text{ and their } \tilde{C}^*\text{-algebra (56), (76)-(80), (86) \quad \downarrow \downarrow } \]

\[ \ldots \ldots \]

These steps are necessary and mean that not only are the complex numbers far from self-evident, but even the negative ones are; a key place (49), (54) wherein a group arises. All the other structural points, first and foremost the observational quantities, may be further produced (even as conceptions) only through certain mathematical mappings:

\[ \downarrow \downarrow \]

\[ \text{observations numbers } \nu_s \text{ and } \alpha_s: \]

\[ a_s \mapsto (\text{statistic } \nu_s), \quad |\alpha_s\rangle \mapsto (\text{spectra } \alpha_s) \quad . \]

(101)

In other words, if a concept is a numeric one already in empiricism—frequencies, spectra, etc.—then its meaningful formalization by means of a mathematical definitio can resort only to a mathematics that we have at our disposal; LVS and algebra of numbers.

Thus numerical quantities in the constructions are initially divided up by their emergence mechanism (II): the intrinsic abstracta and reifications (101). Without such a division, the circular logic is inevitable, and the above-mentioned ‘unit’ exposition of numbers would still be supplemented with the task of their observational treatment, complicated by ‘2-dimensionality’. This task would be present in formalism not merely as a problem, but as an inherently intractable challenge. Actually, any entity can be identified with numbers, and

*... if we represent the notions, which are necessary, by negative numbers, then addition and subtraction ... are executed without any difficulty.
this is why, the quantum empiricism and principium II—paradigm of the very number in a physical theory—insist on the need to pay the closest possible attention to all these things.

**Remark 16.** In this regard, the situation has a parallel with the known history of electrodynamics of moving bodies; as was pointed out just before wording the principium III. Lorentz’s theory is inconsistent, if the space-time tags to events are not linked up to the precise empirical definitions in different reference frames: clocks/simultaneity and rigid rods. In quantum case, the chief subject of empirical definition is a concept of the number. Otherwise, the meaning of the very notion of a quantitative theory has been blurring.

We have seen now that there is no way of founding the theory only on the observable categories (see Remark 2). The attempts to use statistics at the very beginning of the theory are known [99, 60, 9, 11, 38, 84, 86], and rightly so; they were initiated by H. Margenau (1936) [7, Ch. 15]. However, the scheme just given is rigid. In order to obviate the premature appearance of the very need for an interpretation, the scheme must not be varied. Being a sequence of steps, it provides in essence an answer to the principium II.

9.2. Operations on numbers. The last step in this scheme contains, in particular, the map \( a \mapsto \nu \), i.e., measurement (47). Its form should be established in its own right; Born’s rule [18]. To illustrate, the naive transformation of negative numbers by a ‘seemingly natural’ rule like \( |\pm p| \) is not correct and does not follow from anywhere. For the built algebra (94)-(96), the operation \( \pm p \mapsto |\pm p| \) is extrinsic and illegal; it is just not there. According to ideology of sect. 1.3, not only objects—numbers, vectors, quantities, characteristics, etc—but also all the math operations should be created; one without the other is meaningless. What is more, the numeric object of the theory—the complex pair \( (\pm p, \pm q) \)—is as yet single, it contains a principally ‘non-materializable’ ingredient (sect. 7.4) and behaves as a whole. As regards empiricism, the negative and the \( \mathbb{C} \)-numbers are equally ‘nonexistent, imaginary’, since the state algebra (94)-(96) has not been supplemented with notions of the ‘empirically perceived’ quantities (101). In fact, the step-by-step transformation of the binary operation \( \cup \) to symbols \( \oplus, \pm, \mp \) and, finally, to operations \( \{\oplus, \cdot, \oplus, \otimes\} \) does not terminate at states. Algebra (94)-(96) will be further required to create now the mathematically correct calculation rules of the proper observational quantities.

The foregoing is reinforced by the fact that pr. II has been involved in the classical description and in vindication/refutation of, say, the hidden variable theories. Here, numbers are identified with the reified quantities, and subtraction is taken for granted from the outset. However, the negative quantities are also being created here, and they are constructed in the same manner as in the case of ‘quantum zero’ for the \( \mathbb{H} \)-group in sect. 5.4.

Indeed, the readings and physical quantities are no more than notches, and ‘negative notches’ are introduced prior to mathematics of symbols according to the following (subconsciously) intended scheme. The self-apparent physical conventionality, which have been calling ‘an addition’ of two such notches, must give, in accordance with the supra-mathematical requirements of physics, what is named ‘nought, zero’. Two waves at a point, for instance, compensate each other. The result is asserted to be identical with a zero the mathematical; and that is the subtraction. Stated differently, the minus is a fairly abstract construction in its own right. In this respect one might state that the very classical physics needs an interpretation; in terms of strictly positive ‘the number of Stücke’. Mathematization of empiricism into numbers is not a distinctive feature of quantum description. However, comprehension of ‘abstracting the minus sign’ is not confined even to this. A word of explanation is necessary with regard to the situation.
The positive/negative $\pm p$ are formalized [76] into the class pairs $(m, n)$ being equivalent with respect to an ‘adding’ of the class $(\ell, \ell)$:

$$
(+) \equiv (0, 0) \approx (m + \ell, 0 + \ell), \quad (-n) \equiv (0, n) \approx (0 + \ell, n + \ell),
$$

$$
\pm p \iff (m - n) \equiv (m, n) \approx (m + \ell, n + \ell),
$$

(102)

where $m, n, \ell$ are to be seen as ‘something strictly positive’. This ‘adding’ is yet another tacitly assumed and much more abstract action: addition of objects of some other kind—‘positive couples’ $(n, m)$. Technically, at an appropriate place of sect. 7, we would have to introduce such classes and to assign their own algebraic operations for them. The result might be called the ‘genuine’ arithmetic and could be enlarged to the ‘complete’ arithmetic with multiplication and division. Hence, the single-token object $(+m)$ or $(-n)$, which we perceive as self-evident (cf. pr. II), is a rather unobvious construction—the generic class of two-token $(m, n)$-abstractions (102). Essence of the symbol of a negative number $(-n)$ is revealed only when contrasting the two positive ones; exactly the same situation has occurred in (48)–(49) and (54). It is clear that once all the $\pm p$-numbers, and the ‘normal’ positive $+p$’s among, have been formalized into the equivalences (102), the fact that they possess any ‘natural meanings’—like ‘operation of the quantity $p \mapsto |p|$’ invented above—becomes more than unnatural; the abstract class operations that appear out of nowhere. It is the same with $\mathbb{Q}$-numbers and their $\mathbb{R}$-extension: classes of equivalent pairs $(n/m) \equiv (n, m) \approx (n\ell, m\ell)$.

9.3. Naturalness of abstracta. We thus infer that rejection or disregard of the similar ‘naturally abstract’ set-theoretic models would be tantamount to rejection of the minus sign even in the elementary physics. This is an absurd, but its root is a need for abstracting. On the other hand, the motivated deduction of these models cannot be replaced with (hidden) axiomatic assumptions. Such an ambivalence is, in our view, one reason why the problem with ‘decrypting’ quantum postulates is so difficult, because it touches upon the metamathematical (and metaphysical) aspects of the very thinking [73, 26, 75]. The stream of subconscious homomorphisms∗ is considerable and is always larger than it seems. In sects. 3–9, we have described not all of them. ‘Difficulties’ with complex numbers, strictly speaking, should already be attributed to the level of the ‘usual’ negative ones. Bearing in mind that the ‘minus’ comes from the equals sign $=$, and the equality comes from the scheme $\triangleright$ (48) $\leqslant$ (49), both the principia—II and III—are very important (and functioning) also in the classical case. In quantum case, they are just fundamentally unavoidable for the very deduction of the theory. The nature of $\text{QM}$ theory, of arithmetic, of complex numbers, and of their algebras is one and the same.

Transferring the reasoning above to the natural numbers $\mathbb{N}$, the degrees of classical and quantum abstraction become even indistinguishable. Empirical motivation leads, in one way or another, to the standard von Neumann’s representation for ordinals

$$
0 \equiv \emptyset, \quad 1 \equiv \{\emptyset\}, \quad 2 \equiv \{\emptyset, \emptyset\}, \quad 3 \equiv \{\emptyset, \{\emptyset\}, \emptyset, \{\emptyset\}\}, \quad \ldots,
$$

(103)
i.e., to using the ZF-axiom of union: $n + 1 \equiv n \cup \{n\}$ [76]. Therefore less obvious is the $\mathbb{N}$-numbers themselves, to be followed by the ordering, topologies, extensions, generalizations,

∗For a philosophical discussion to these representations and the origin of models, see [140, pp. 1–230]. M. Wartoński: “The very notion of ‘phenomenon’ or of ‘the appearance of things,’ . . . is a cognitive and perceptual act of abstraction” [140, p. 220]. As one more comment concerning the abstracting/realism we refer to the first half of a letter from A. Einstein to H. Samuel in [111, pp. 157–160]; see also [36].
etc. The formal characterization of all the experimental reduces then to the successive creating from the set-theoretic atoms—unions of sets—their direct products and mappings into other constructions of the same kind. Hence, both the physical images ‘being under a ban above’ and auxiliary structures—dimensionalities, orders, etc—should equally well become homomorphisms onto certain formal constructions regardless of the description’s classicality/quantumness. Presence of, say, non-binary operation $\mathbb{C}^*$-conjugation does not stand out, because its nature does not differ from the one of (habitual) subtraction and of division. All of these are involutory structures that have been mathematically inherited from the empirical meta-requirements: repetitions ($M$-paradigm), physics $\{S, M, \ldots\}$, and invariance $\text{III}$ (sect. 5.4).

To close the section we add that the distancing the concepts of state/DataSource and of physical properties is the continuation of a more primary idea—detaching the macro-observation themselves from their theory [84, 86] (Σ-limit and number). The atomic constituent of observation is a physical event [86, 59], and it begins and terminates, in effect, in $(\not\equiv_\alpha)$-distinguishability of $\alpha$-clicks (sect. 2.1). All the further matters—numbers, arithmetics, time (cause/effect), (non)inertial reference frames, the notions of an observer/event/coordinates in the relativity theory (a quantum view on the equivalence principle), measurement readings/records, etc, no matter how self-evident, are the math add-ons, which could originate only in the ‘∪-theory’ of sect. 5. Following von Weizsäcker, it might be called ‘Ur-theory’. There are no contradictions in observations themselves, whether we call them macroscopic or microscopic. Contradictions do arise in the ‘mathêmaticae being constructed’.

10. About interpretations

It is therefore not . . . a question of a re-interpretation . . . quantum mechanics would have to be objectively false, in order that another description . . . than the statistical one be possible J. von Neumann [102, p. 325]

The source of the numerous treatments of QM [63, Ch. 10], [78, Ch. 10], [7, 114, 118, 112, 120] is the fact that the $\alpha$-event and subconscious comprehension of $\Psi$ (pt. S) are a priori endowed with physical properties, with observational characteristics of the DataSource, and with operationality of the canonical QM-concept $|\Psi\rangle^*$. However, none of these initially exist. There are only abstractions $\Psi \xrightarrow{\sim} \alpha$ of the primitive $\alpha$-events. An important point is that even the ‘eigen $\alpha$-click’ (of a photon/electron) should not be identified with an $|\alpha\rangle$-state, and, broadly speaking, it should not be an element of the language in which $|\Psi\rangle$-terminology and numbers have been employed; see also the second epigraph to sect. 2. Then something subsequently referred to as a state (the abstract) and a measurement (the concrete) is created. However, the process of abstracting—the set-theoretic homomorphisms $\{\Psi, \cup\} \Rightarrow \cdots \Rightarrow \{|\Psi\rangle, +, \cdot\}$—is a rather multistage one (sects. 3–9), and reduction of this sequence ‘for physical reasons’ always contains phenomenological axioms a priori. Clearly, in the reverse direction we confront hard-to-disentangle assumptions.

* An illustrative example in this regard is one of the first sentences from Everett’s PhD: “The state function $\psi$ is thought of as objectively characterizing the physical system . . . at all times . . . independently of our state of knowledge of it” [39, p. 3]; and also, on p. 8, “The general validity of pure wave mechanics, without any statistical assertions, is assumed for all physical systems, including observers and measuring apparata” (emphasis original). And, again Everett’s: “The physical ‘reality’ is assumed to be the wave function of the whole universe itself” [48, p. 100].
In order to avoid paradoxes like ‘quantum cat’ ("state vector does not describe ... a single cat" [10, p. 37]), ‘the presence of a particle here and there’, or like ‘quantum bomb-testing’ (Elitzur–Vaidman) [7, 55], a stringent notional differentiation between states and adjectives with physical images is required*. It seems preferable to radicalize the non-identity of these categories, i.e., to proclaim it a postulate; for instance, a boldface italics in Remark 12. At least, the differentiation between them should not be neglected in reasoning, inasmuch as it seems impossible to change the deeply ingrained terminology such as ‘state of system $S$’ [137, p. 7], ‘an observable has/acquires a (numeric) value when being measured’, or ‘system is in superposition’ [86, 87], etc. With this mixing, circular logic (see Remark 10) will be present at all times. See also pages 29–30 in the work [45] and, in particular, an emphasized warning by D. Foulis about “a mistake, and a serious one!” on p. 29 therein.

The most illustrative example is the (in)famous problem of measurement, which is the subject of a vast literature containing opposing opinions [68, 63, 78, 83, 98, 24], and which is the source of questions around locality in QM. The point here, put very briefly, is that the measuring ‘problem’ is one of principle, not of practice; a pseudoproblem [34]. As we have seen, in measurements nothing either propagates (at superluminal speeds), nor is anything collapsed [10, 88], nullified, localized, there are no such things as quantum jumps, and no pieces of the wave function are cut out. G. Ludwig: “... there is no collapse of wave packets in reality. Do not believe in fairy tales!” [88, p. 104]. Again (see p. 21 and sect. 6.3), direct or indirect attempts to physically characterize this function are hopeless, since it is the very DataSource around which all corresponding objects—readings, frequencies and other quantities—are only slated to be created. It is impossible to ‘reconcile’ the (nonclassical) notion of a quantum state with (any) its observational prototypes. However, the appeals regarding the irrelevance of the projective postulate are often encountered in the literature [124, 82, 74, 53, 86, 87, 88].

Another example of circular logic is the critiqued [63, 129, 130] meaning of the phrase “an ensemble of similarly prepared systems” [9, 63, 112]. The complete revision of this idea, as was set forth above, does, in fact, demonstrate that, like in the ensemble approaches, “QM is a usual statistical theory” [9, 88, p. 123], [97, p. 223], [11, 154, 4, 86, 88, 123, 99, 100] with a frequency content of the indeterminism (and with the classical logic [88]), but with a different mathematics of statistic’s calculus. It is different due to the fact that the theory is not tied, as in classical description, to the notion of an observable quantity, and the $\nu$’s are calculated through the ‘other/abstract’ numbers.

Although we have not yet touch on other significant attributes—the means over statistic, operators, and products of $\mathbb{H}$-spaces will be considered in their own rights, it is clear that the need to search for a description in terms of hidden variables is also eliminated. Even from a formalistic perspective, the proof of presence/absence [102, 33, 55] of these ‘physical’ quantities should be attributed to the semantic conclusions of meta-theory (＝physics) [109], i.e., to theorems about formal theory rather than to theorems of its calculus. In our case, and more generally, the formal theory is the syntactical axioms of QM. It is known that corrolaries of such axioms are inherently unable to lead to statements about interpretations [109], since theorems about object-theory itself is not provable by means of its object-language [122, 109].

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* T. Maudlin: “... we need to keep the distinction between mathematical and physical entities sharp. Unfortunately, the usual terminology makes this difficult” [93, p. 129]. Even indirect usage of the terminology that came from the classical description can be a source of confusing. For example, so-called exchange interaction as a cause of ‘correlation’ between identical parts of system.
A similar line of reasoning has accompanied quantum theory for quite a while: “claim that the formalism by itself can generate an interpretation is unfounded and misleading” (L. Ballentine [10]). That stands in stark contrast with the known statement of DeWitt that “mathematical formalism of the quantum theory is capable of yielding its own interpretation” [31, p. 160, 165, 168] or that “conventional statistical interpretation of quantum mechanics thus emerges from the formalism itself” [31, p. 185]. Especially if we take into account the fact that it is not the theory itself, but only its (formal) interpretation that determines the very semantic terms truth/falsity of sentences (Gödel). In turn, there may be more than one such interpretation. See also [65, p. 310], [16], and especially Ch. III in [73].

In any case, the fact that we were initially constructing a model simply eliminates the problem, or, at most, transfers it into the domain of questions of micro-transitions \( \rightarrow \) and of \( \exists \)-family as employed entities (see Remark 2); i.e., the domain of questions, which invoke the set theory at all. Be it as it may, logic (formalized or not) does not allow us to make statements about statements, much less a statement that refers to itself. The self-referentiality is one of the main troubles encountered in quantum foundations [98, 100]. All of this, of course, does not depend on whether the interpretation is built over in a strictly formalized [122] or in a physically natural form. In effect, the issue of interpretations—in the rigorous definition sense [109, Ch. 2], [122], [131]—is simply nonexistent. Accordingly, the demystification of the known and the search for ontological interpretations to \( \alpha \)-coordinates of the \( |\Psi\rangle \)-vector [7, 43, 60]—the wave function—is no longer a problem, and with it disappears the Feynman question of “the only consistent interpretation of this quantity” [42, p. 22; italics original]. See also the review [81] wherein the exhaustive literature references are given.

11. Closing remarks

quantum mechanics has been a rich source for the invention of fairy tales
G. Ludwig & G. Thurler [88, p. 122]

I simply do not know how to change
quantum mechanics by a small amount
without wrecking it altogether
S. Weinberg

11.1. ‘Math-assembler’ and philosophy. Remembering and continuing sect. 1.3, it is generally tempting to infer that when creating the theory, we may not rest upon any meanings that are tacitly associated with the typical terminology; no matter, physical or mathematical. To illustrate, even the very natural wordings—‘here/there’, ‘bigger/smaller’, ‘let a two-particle \( \mathcal{S}' \), ‘electron for Alice/Bob’, ‘consisting of . . . ’, ‘subsystem \( \mathcal{S}_1 \)’—have already comprised an equivalent of a measurement/preparation (sect. 3.1), of physical images, and of some mathematics; including mixtures (23). As in sect. 6.5, this may well be non-correct [93]. A source of contradictions is in implicit implying, i.e., in the illogical confounding of observations, clicks, numbers, physics, maths, and imagination. From this, there result the sense confusions, well-known no-go theorems* [14, 54], and paradoxes like EPR or the jocular Bell question “Was the world wave function waiting to jump for thousands of millions of years . . . for some more highly qualified measurer — with a Ph.D.?" [14, p. 117], [78, p. 18],

* “... combined with some peculiar terminology, has led to confusion ... A woefully common feature, ... each protagonist had some interpretation of the quantum state in mind, but never stated clearly what it was” (L. Ballentine [13, pp. 2, 6]).
Clearly, quantum effects of observations do not depend on whether the personified (homosapiens) or biological observer (Heisenberg–Zeilinger dog [72, pp. 171–174], [37, 17]) perceives them. Without numbers, solely a quantitative theory is not possible (sect. 6.5), because the entire terminology becomes indefinite.

Thus, once a mathematics and unambiguous language—spectra, means, and macroscopic dynamical models—have been created, not only is there no longer a need to call on the ‘otherworldly’ explanation ways, but the very presence of a certain share of (circum-)philosophy and subjectivity in quantum foundations becomes extremely questionable. There is no longer any freedom to invent exegeses (of ‘the quantum postulate Bible’). Moreover, the freedom to ask questions is no longer there, since the created object-language of states, of spectra, and of frequencies narrows down the entire admissible lexicon. It may generate questions that are not only ill-posed but should also be qualified as “meaningless” [101, p. 422], as in sect. 6.5; those that are based on an intuitive perception of the term observation by human being.

In the classical paradigm, the language sentences are always interrelated, since, one way or another, all of them refer to the observational concepts. This fact—thinking (even if partly/implicitly) in a classical manner with ‘quantum conclusions’—is the very source of paradoxes, since the human intuition is rather problematic and personal category. (A. Stairs: “Don’t trust intuition” [128, p. 256]) Because of this, in order to avoid collisions between theory and meta-language*, the subconscious striving of the natural language to include one in the other has to be limited; see Remark 10. A. Leggett’s comments on “pseudoquestions” at the end of sect. 1.2 may then be strengthened so that the meaninglessness should become an element of language, including the language of ‘philosophy of quanta’.

- Quantum (meta)mathematics/physics creates the notion of a prohibited statement/question, one that is devoid of meaning. These sentences, which involve classical analogies in circumvention of 1) the physically non-interpretable abstraction \( |\Psi\rangle \), 2) its \( |\alpha\rangle \)-representations, and 3) the numerical quantities’ nature (sect. 9.1).

See also pages 234–235 in [113] with Bohr’s appeals regarding the “necessity of a radical revision of basic principles for physical explanation . . . revision of the foundation for the unambiguous use of elementary concepts” and his comment on words “phenomena”, “observations”, “attributes”, and “measurements” on p. 237.

The literature on this subject, even taking only the qualified sources into account, is vast [1, 5, 30, 38, 43, 50, 54, 64, 93, 96, 95, 103, 112, 116, 116, 118, 55, 151, 154, 78, 63, 130, 7] and abounds with terminology—“... words, ostensibly English” (A. Leggett [78, p. 300]), that completely defies translation into the language of events or of concretization: observer’s consciousness, parallel/branching universes/worlds, free will, world branch, mental information, and also the word combinations like rational agent, information has been recorded/transmitted/(not)reached an observer (Wigner’s friend), teleporting a state, many-minds/worlds/words, quantum psychology, psycho-physical parallelism (in this connection, see [26, p. 86(!)]), and more.

Of course, “without philosophy, science would lose its critical spirit and would eventually become a technical device” [7, p. 800] but, on the other hand, “the concept of the free will cannot be defined by indications on devices” [87, p. 151] and “one must not confuse physics with philosophy” [37, p.12]. As concerns the attitudes toward QM—at the suggestion of M. Tegmark in the 1990’s, there even carried out polls and statistical analysis of their

* And also situations when “er führt dazu, überhaupt alle sprachlich ausdrückbaren Sätze als sinnleer zu erklären” (A. Einstein [35, p. 33]).
There are also known attempts to involve here the biology of consciousness/brain [130, 136], [115, sect. 6], [116, Ch. 9]. Regarding them, however, there have been not merely sceptical but quite opposite opinions [143, sects. 5–6], [87, §XII.5(!)]. Of special note is Ballentine’s remark “to stop talking about “consciousness” or “free will” on the last page of the preprint [13]; see also [87, §XII.5].

As a result, we gain “a contribution to philosophy, but not to physics” [91, p. 86]. At the same time, the proposed math ‘∪-assembler’

\[ \{\Psi \rightarrow \alpha, (\Xi)\text{-brace (31), } (\forall, \in, \cup)\text{-logic (36)}(39)\} \]

is quite sufficient for creating the object-language. Giving a natural form to it would be acceptable, however, it is clear that the set-theoretic ∪-base of the language cannot be avoided. Nevertheless, the syntactically more formal description of transitions/brace/numbers is surely of interest and value. This would turn, however, all the above material into a mathematical logic text; we avoid this in the present work. It is probably for this reason that the very important and extremely thorough works* by G. Ludwig [84, 85, 86, 87] and by his school often get omitted from the literature on quantum foundations. Among other things, in spite of explicit pointing out a “solution in principle of the measuring problem” in [86, p. V] (and subtitle “Derivation of Hilbert Space Structure” of [86]), name of this author has not been mentioned in the detailed reviews [81, 114, 148] and even in the books [123, 50, 116, 98, 112, 103].

11.2. Well, where’s probability? An answer to this question in quantum theory is brief enough—nowhere, because relationship of this concept with empiricism is unique—the frequencies \{\nu_j\}. Cf. the famous De Finetti (1970) remark “probability does not exist” and A. Khrennikov’s comment “It seems that the machinery of randomness has no applications in quantum physics. Experimenters are only interested in . . . frequencies” [70, p. 36]. Or, expressed differently by von Mises’ words, “If we base the concept of probability, not on the notion of relative frequency, . . . at the end of the calculations, the meaning of the word ‘probability’ is silently changed from that adopted at the start to a definition based on the concept of frequency” [97, p. 134; emphasis added]. Otherwise, QM-theory would require an interpretation of Kolmogorov’s axioms, and they, in turn, require interpreting the concept of the number—an axiomatic add-on over the ZF theory [76]. Bearing in mind the primary nature of numbers and nontriviality of their emergence in a physical theory (sect. 7.2), it is not just impossible to avoid the phenomenological frequencies. Logic forbids them from being subsidiary with reference to probability in any definition.

The ensemble empiricism, for its part, is self-sufficient**, and the only conventionality within it is an infinite number of repetitions. Its formalization is an appropriate axiom in the ZF-theory [76, 62]. Expressed another way, any non-frequency framework for QM-probability ‘attracts vague justifications’ in terms of: potentiality, tendency, propensity (K. Popper), amount of ignorance, subjective likelihood, degree of belief, etc [43]. But even from philosophical point of view “probability is a deeply troublesome notion” (L. Hardy [118, p. 78]), that is supported by the vast literature on this subject [135] (!), [133, pp. 41–43], [7, 3, 30, 71, 72].

* Pretheories, 76 axioms [86, p. 241], ordered sets, morphisms, absence of the word superposition in [86, 84], (valid) criticism of “theories of states” [88], etc.

** In this connection we cannot agree with statement of theorem III in van Kampen’s work [68] and with further comment as to “a single system” and “calculation of spectra”.

\[ \{\Psi \rightarrow \alpha, (\Xi)\text{-brace (31), } (\forall, \in, \cup)\text{-logic (36)}(39)\} \]
The ultimate conclusion completes Remark 7. If we accept the set-theoretic viewpoint of things then sect. 5.1, by all appearances, provides a positive answer to the question of the rigidity of QM-theory [28]; see also the 2-nd epigraph to this section. At least, it is hard to imagine what any other (axiom-free) way of turning empiricism into quantum mathematics would look like, as soon as we abandon the minimalistic entity $\Psi \rightarrow \alpha$.

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REFERENCES

[1] AARONSON S. Is Quantum Mechanics An Island In Theoryspace? http://arXiv.org/abs/0401062
[2] AARONSON S. Quantum Computing since Democritus. Cambridge University Press (2013).
[3] ACCARDI L. Urne e camaleonti. Dialogo sulla realtà, le leggi del caso e l’interpretazione della teoria quantistica. il Saggiatore (1997).
[4] ALLAHVERDYAN A. E., BALIAN R. & NIEUWENHUIZEN T. M. Understanding quantum measurement from the solution of dynamical models. Physics Reports (2013) 525(1), 1–166.
[5] ALTER O. & YAMAMOTO Y. Can we measure the wave function of a single wave packet of light? In: [54], pp. 103–109.
[6] ANSMANN M., WANG H., BIALCZAK R. C., HOFHEINZ M., LUCERO E., NEELEY M., O’CONNELL A. D., SANK D., WEIDES M., WENNER J., CLELAND A. N. & MARTINIS J. M. Violation of Bell’s inequality in Josephson phase qubits. Nature (2009) 461, 504–506.
[7] AULETTA G. Foundations and Interpretation of Quantum Mechanics. World Scientific (2001).
[8] BAEZ J. C. Division Algebras and Quantum Theory. Found. Phys. (2012) 42, 819–855.
[9] BALLENTINE L. E. The Statistical Interpretation of Quantum Mechanics. Rev. Mod. Phys. (1970) 42(4), 358–381.
[10] BALLENTINE L. E. The formalism is not the interpretation. Physics Today (1971) April, 36–38.
[11] BALLENTINE L. E. Quantum Mechanics. A Modern Development. World Scientific (2000).
[12] BALLENTINE L. E. Classicality without Decoherence: A Reply to Schlosshauer. Found. Phys. (2008) 38, 916–922.
[13] BALLENTINE L. E. Ontological Models in Quantum Mechanics: What do they tell us? http://arXiv.org/abs/1402.5689
[14] BELL J. S. Speakable and unspeakable in quantum mechanics. Cambridge University Press (1993).
[15] BELTRAMETTI E. G. & CASSINELLI G. The Logic of Quantum Mechanics. Encyclopedia of mathematics and its applications 15. Addison–Wesley (1981).
[16] BENIOFF P. A. Models of Zermelo Frankel set theory as carriers for the mathematics of physics. I, II. Journ. Math. Phys. (1976) 17(5), 618–628, 629–640.
[17] BERGOU J. A. & ENGLERT B.-G. Heisenberg’s dog and quantum computing. Journ. Mod. Optics (1998) 45(4), 701–711.
[18] BREZHNEV, YU. V. The Born rule. Why Hilbert space? In preparation.
[19] BIRKHOFF G. Lattice theory. Providence (1967).
[20] BOURBAKI N. Elements of mathematics. Algebra I: Chapters 1–3. Springer (1974).
[21] BRIGGS G. A. D., BUTTERFIELD J. N. & ZEILINGER A. The Oxford Questions on the foundations of quantum physics. Proc. Royal Soc. A (2013) 469, 0299(8).
[22] BUB J. & PITOWSKY I. Two Dogmas About Quantum Mechanics. In: [112], pp. 433–459.
[23] BUENO O., FRENCH S. & LADYMAN J. On Representing the Relationship between the Mathematical and the Empirical. Philosophy of Science (2002) 69(3), 497–518.
[24] BUSCH P., LAHTI P. J. & MITTELSTAEDT P. The Quantum Theory of Measurement. Springer (1996).
[13] SUDBERRY A. Quantum Mechanics and the Particles of Nature: An Outline for Mathematicians. Cambridge University Press (1986).
[134] SABÓ L. Quantum Structures Do Not Exist in Reality. Int. Journ. Theor. Phys. (1998) 37(1), 449–456.
[135] SABÓ L. Objective probability-like things with and without objective indeterminism. Studies Hist. Phil. Mod. Phys. (2007) 38, 628–634.
[136] TEGMARK M. The importance of quantum decoherence in brain processes. Phys. Rev. E (2000) 61, 4194–4206.
[137] DE TOUZALIN A., MARCUS C., HEIJMAN F., CIRAC I., MURRAY R. & CARLACO T. Quantum Manifesto. A New Era of Technology. (2016) May. http://qurope.eu/system/files/u7/93056_Quantum%20Manifesto_WEB.pdf
[138] VARADARAJAN V. S. Geometry of Quantum Theory. Springer (2007).
[139] VAN DER WAERDEN B. L. Algebra I, II. Springer (1970).
[140] WARTOFSKY M. W. Models. Representation and the Scientific Understanding. D. Reidel Publishing Co. (1979).
[141] WEINBERG S. Collapse of the State Vector. Phys. Rev. A (2012) 85, 062116(6).
[142] WEINBERG S. What happens in a measurement? Phys. Rev. A (2016) 93, 032124(5).
[143] WISEMAN H. M. & EISERT J. Nontrivial quantum effects in biology: A skeptical physicist’ view. In: Abbott D., Davies P. C. W. & Pati A. K. (eds) Quantum aspects of life. Imperial College Press (2008), pp. 381–401.
[144] ZEILINGER A. A Foundational Principle for Quantum Mechanics. Found. Phys. (1999) 29(4), 631–643.
[145] ZEILINGER A. Quantum Information and the Foundations of Quantum Mechanics. Newton Lecture in Institute of Physics. London (2008) June 17th. https://www.youtube.com/watch?v=7DiEl7msEZc
[146] ZIERLER N. Axioms for non-relativistic quantum mechanics. Pacific Journ. Math. (1961) 11(3), 1151–1169.
[147] ZORICH V. A. Mathematical analysis I. Springer (2004).
[148] ZUREK W. H. Decoherence, einselection, and the quantum origins of the classical. Rev. Mod. Phys. (2003) 75(3), 715–775.
[149] ZUREK W. H. Wave-packet collapse and the core quantum postulates: Discreteness of quantum jumps from unitarity, repeatability, and actionable information. Phys. Rev. A (2013) 87, 052111(6).
[150] ВЛАДИМИРОВ В. С., ВОЛОВИЧ И. В., ЗЕЛЕНОВ Е. И. p-адический анализ и математическая физика. ФизМатЛит (1994).
[151] КАДОМЦЕВ Б. Б. Динамика и информация. Редакция журнала УФН (1997).
[152] СЛАВНОВ Д. А. Измерения и математический аппарат квантовой физики. Физика элемент. частиц и атомн. ядра (2007) 38(2), 295–359.
[153] ХРЕННИКОВ А. Ю. Ненаргымедов анализ и его приложения. ФизМатЛит (2003).
[154] ХРЕННИКОВ А. Ю. Введение в квантовую теорию информации. ФизМатЛит (2008).

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