The Characteristic Functions and Their Typical Values for the Nonlinear Spinors

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In this paper, we solve the eigen solutions to some nonlinear spinor equations, and compute several functions reflecting their characteristics. The numerical results show that, the nonlinear spinor equation has only finite meaningful eigen solutions, which have positive discrete mass spectra and anomalous magnetic moment. The nonlinear potential and interactions yield different contributions to the total energy, and these components of the energy lead to different energy-speed relation. The magnitude of these components can be detected by elaborate experiments. The weird properties of the nonlinear spinors might be closely related with the elementary particles and their interactions, so some deeper investigations on them are significant.

Keywords: nonlinear spinor field, anomalous magnetic moment, mass-energy relation

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I. INTRODUCTION

Since Dirac established relativistic quantum mechanics, many scientists such as H. Weyl, W. Heisenberg, have attempted to associate the elementary particles with the eigenstates of the nonlinear spinor equation\cite{1, 2, 3, 4, 5, 6}. In 1951, R. Finkelsten solved some rigorous solutions of the nonlinear spinor equation by numerical simulation, and pointed out that the corresponding particles have quantized mass spectra\cite{7, 8}. However these researches have not realized their authors’ goal due to the mathematical difficulties in analyzing nonlinear spinor equation.

In recent years, a great effort has been made along this line of research. The theoretical

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proof about the existence of solitons was investigated in [9, 10, 11, 12, 13, 14]. The symmetries and many conditional exact solutions of the nonlinear spinor equations are collected in [15]. The present work is a development of some previous works[16, 17, 18]. In this paper, we define some functions which reflect the properties of eigen solutions to the nonlinear spinor equations, and compute the typical values, then extract some important information from the data. The following are some general knowledge for the nonlinear spinors.

Denote the Minkowski metric by $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, Pauli matrices by

$$\vec{\sigma} = (\sigma^j) = \begin{cases} 
(0 \ 1) , (0 \ -i) , (1 \ 0) , \\
(1 \ 0) , (0 \ -i) , (1 \ 0) 
\end{cases}.$$  \hfill (1.1)

Define $4 \times 4$ Hermitian matrices as follows

$$\alpha^\mu = \begin{cases} 
(I \ 0) , (0 \ \vec{\sigma}) , \\
(0 \ \vec{\sigma}) , (I \ 0) 
\end{cases}, \quad \gamma = \begin{pmatrix} I & 0 \\
0 & 0 & -I 
\end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -iI \\
iI & 0 
\end{pmatrix},$$  \hfill (1.2)

where $\mu \in \{0, 1, 2, 3\}$, $x^0 = ct$ and $\alpha^\mu = \gamma^0 \gamma^\mu$. In this paper, we adopt the Hermitian matrices (1.2) instead of Dirac matrices $\gamma^\mu$ for the convenience of calculation. For Dirac’s bispinor $\phi$, the quadratic forms of $\phi$ are defined by

$$\bar{\alpha}^\mu = \phi^+ \alpha^\mu \phi, \quad \bar{\gamma} = \phi^+ \gamma \phi, \quad \bar{\beta} = \phi^+ \beta \phi,$$  \hfill (1.3)

where the superscript ‘+$’ stands for the transposed conjugation. By the definition (1.3) we have $\bar{\alpha}^\mu = \phi^+ \gamma^\mu \phi$ etc., where $\phi^+ = \phi^+ \gamma^0$ is the Dirac conjugation[19]. By transformation law of $\phi$, one can easily check that $\bar{\alpha}^\mu$ is a contra-variant 4-vector, $\bar{\gamma}$ a true scalar and $\bar{\beta}$ a pseudo-scalar. One can construct some other covariant quadratic forms, but they are not independent on (1.3) for some Pauli-Fierz identities[16, 20, 21], such as $\bar{\alpha}_\mu \bar{\alpha}^\mu = \bar{\gamma}^2 + \bar{\beta}^2$.

In general, the Lagrangian of the nonlinear bispinor $\phi$ with a vector potential $A^\mu$ and scalar $G$ is given by[22, 23]

$$L = \phi^+ \alpha^\mu (i\partial_\mu - e A_\mu) \phi - \mu \bar{\gamma} + V(\bar{\gamma}, \bar{\beta}) - s \bar{\gamma} G - \frac{1}{2} \kappa (\partial_\mu A_\nu \partial^\mu A^\nu - a^2 A_\mu A^\mu) - \frac{1}{2} \lambda (\partial_\mu G \partial^\mu G - b^2 G^2),$$  \hfill (1.4)

where $A^\mu$ and $G$ include the self and external potential, $\kappa = \pm 1$ and $\lambda = \pm 1$ are used to stand for the repulsive or attractive self interaction, e.g. $A^\mu$ stands for repulsive electromagnetic potential if ($\kappa = 1, a = 0$), but stands for attractive interactive potential similar to strong
interaction if \((\kappa = -1, a \neq 0)\). \(G\) stands for a scalar interactive potential like Higgs field, which is repulsive if \(\lambda = 1\), and attractive if \(\lambda = -1\). However, in this paper, we take \((1.4)\) as one system, and only the internal interactions are considered.

If \(\partial_\beta V \neq 0\), the eigen solution might be absent, so we only consider the case \(V = V(\dot{\gamma}) > 0\) is a concave function satisfying

\[
V'(\dot{\gamma})\dot{\gamma} > V(\dot{\gamma}), \quad \text{if } (\dot{\gamma} > 0).
\]  

The corresponding dynamical equation is given by

\[
\alpha^\mu(\hbar i \partial_\mu - e A_\mu)\phi = (\mu c + sG - V')\gamma\phi, \quad (1.6)
\]

\[
(\partial_\alpha \partial^\alpha + a^2)A^\mu = \kappa e \tilde{\alpha}^\mu, \quad (1.7)
\]

\[
(\partial_\alpha \partial^\alpha + b^2)G = \lambda s \gamma. \quad (1.8)
\]

The Hamiltonian form of \((1.6)\) is given by

\[
\hbar i \partial_\phi = \hat{H}\phi, \quad \hat{H} = c[eA_0 + \tilde{\alpha}\cdot \tilde{p} + (\mu c + sG - V')\gamma]. \quad (1.9)
\]

where \(\tilde{p} = -\hbar i \nabla + e \tilde{A}\) is the momentum operator. For \((1.9)\), by the current conservation law, we have the normalizing condition

\[
\int_{R^3} |\phi|^2 d^3x = 1. \quad (1.10)
\]

Let \(\hat{J}\) be the angular momentum operator

\[
\hat{J} = \vec{r} \times \vec{p} + \frac{1}{2} \hbar \gamma, \quad \gamma_k = \text{diag}(\sigma_k, \sigma_k), \quad (1.11)
\]

then the eigenfunctions of \(\hat{J}_3 = -\hbar i \partial_\phi + \frac{1}{2} \hbar \gamma_3\) are given by

\[
\hat{J}_3 \phi_j = j_3 \hbar \phi_j, \quad \phi_j = (u_1, u_2 e^{\phi i}, i v_1, i v_2 e^{\phi i})^T e^{j\phi i}, \quad (1.12)
\]

where the index \(T\) stands for transpose, \(j_3 = j + \frac{1}{2}, j \in \{0, \pm 1, \pm 2, \cdots\}\). For all the eigenfunctions, \(\hat{J}_3\) is commutative with the nonlinear Hamilton operator like the linear case, so under a suitable choice of coordinate system, the solutions of \((1.9)\) take the following form\([24, 25]\),

\[
\phi_j = (u_1, u_2 e^{\phi i}, i v_1, i v_2 e^{\phi i})^T \exp(j \varphi i - \frac{mc^2}{\hbar}it), \quad (1.13)
\]
where \( u_k, v_k (k = 1, 2) \) are real functions independent of \( \varphi \) and \( t \).

The system (1.9) has many symmetries such as the global gauge invariance, the spatial reversal invariance etc. The above assumptions have removed the uncertainty caused by the symmetry, and the solutions are determined except for a signature. This procedure is quite important for the numerical solving and stability analysis. If \( V(\gamma) \) take the form of polynomials, the solutions are analytic functions of \( r \), and then they can be expressed as the Taylor series of \( r \). In the cases of \( j_3 = \pm \frac{1}{2} \), which are the only cases for free particles, we have the formal solution as follows

\[
\begin{align*}
    u_1 + u_2 i &= \sum_{m=0}^{\infty} r^{2m} \left( K_m e^{-2im\theta} + \sum_{n=-m+1}^{m} A_{mn} e^{2im\theta} \right), \\
    v_1 + v_2 i &= \sum_{m=0}^{\infty} r^{2m+1} \left( J_m e^{-i(2m+1)\theta} + \sum_{n=-m}^{m} B_{mn} e^{i(2m+1)\theta} \right),
\end{align*}
\]

where \( K_m, J_m \) are real free parameters determined by boundary conditions and the normalizing condition, but \( (A_{mn}, B_{mn}) \) are real numbers determined by \( (K_n, J_n) \) with \( n \leq m \). The normalizing condition (1.10) becomes

\[
2\pi \int_0^{\infty} r^2 dr \int_0^{\pi} \sin \theta d\theta (u_1^2 + u_2^2 + v_1^2 + v_2^2) = 1. \tag{1.16}
\]

II. PROPERTIES OF THE DARK NONLINEAR SPINOR

The simplest case of (1.9) is the following dynamical equation

\[
\hbar i \partial_t \phi = \hat{H} \phi, \quad \hat{H} = c[\vec{\alpha} \cdot \hat{p} + (\mu c - w\gamma)\gamma]. \tag{2.1}
\]

Different from the linear case, the nonlinear spinor equation generally has continuous spectra if the restriction of (1.10) is absent, so the normalizing condition is a quantizing condition for nonlinear spinors, and the nonlinear coupling coefficient \( w \) is meaningful only if the solution satisfies the normalizing condition (1.10).

The eigen solutions to (2.1) with spin \( j_3 = \pm \frac{1}{2} \) can be solved rigorously as follows

\[
\begin{align*}
    \phi_{\epsilon \uparrow} &= (g, 0, if \sin \theta e^{i\phi}, -f \cos \theta, if \sin \theta e^{i\phi})^T \exp(-i \frac{mc^2}{\hbar} t), \quad \text{for } (P = 1, j_3 = \frac{1}{2}) \\
    \phi_{\epsilon \downarrow} &= (0, g, if \sin \theta e^{-i\phi}, f \cos \theta, -if \sin \theta e^{-i\phi})^T \exp(-i \frac{mc^2}{\hbar} t), \quad \text{for } (P = 1, j_3 = -\frac{1}{2}) \\
    \phi_{\epsilon \uparrow} &= (f \cos \theta, f \sin \theta e^{i\phi}, 0, ig, 0)^T \exp(-i \frac{mc^2}{\hbar} t), \quad \text{for } (P = -1, j_3 = \frac{1}{2}) \\
    \phi_{\epsilon \downarrow} &= (f \sin \theta e^{-i\phi}, -f \cos \theta, 0, ig)^T \exp(-i \frac{mc^2}{\hbar} t), \quad \text{for } (P = -1, j_3 = -\frac{1}{2})
\end{align*}
\]
where $P = 1$ corresponds to even parity, and $P = -1$ corresponds to odd parity. For the above eigenfunctions, we have

$$
\hat{\gamma} = P(g^2 - f^2), \quad 4\pi \int_0^\infty (g^2 + f^2) r^2 dr = 1. \quad (2.3)
$$

The radial equation of even parity satisfies

$$
\begin{align*}
\frac{d}{dr} g &= -\frac{1}{\hbar c}[(\mu + m)c^2 - wc(g^2 - f^2)] f, \\
\frac{d}{dr} f &= -\frac{1}{\hbar c}[(\mu - m)c^2 - wc(g^2 - f^2)] g - \frac{2}{r} f.
\end{align*} \quad (2.4)
$$

For the odd parity, we have

$$
\begin{align*}
\frac{d}{dr} g &= -\frac{1}{\hbar c}[(\mu - m)c^2 + wc(g^2 - f^2)] f, \\
\frac{d}{dr} f &= -\frac{1}{\hbar c}[(\mu + m)c^2 + wc(g^2 - f^2)] g - \frac{2}{r} f.
\end{align*} \quad (2.5)
$$

The initial data of (2.4) and (2.5) satisfy $f(0) = 0$, $g(0) > 0$. For (2.4) and (2.5), we have positive mass $0 < m < \mu$ if and only if $w > 0[17]$. Making transformation

$$
\begin{align*}
a &= \sqrt{\frac{\mu + m}{\mu - m}}, \quad r_0 = \frac{\hbar}{c\sqrt{\mu^2 - m^2}} = \frac{(a^2 + 1)\hbar}{2a\mu c}, \quad \rho = \frac{r}{r_0}, \\
u &= \sqrt{\frac{w(a^2 + 1)}{2a\mu c}} g, \quad v = -\sqrt{\frac{w(a^2 + 1)}{2a\mu c}} f.
\end{align*} \quad (2.6)
$$

where $a$ is equivalent to the spectrum, $r_0$ takes the unit of length. (2.4) and (2.5) can be rewritten in a dimensionless form. For (2.4) we have

$$
\begin{align*}
u'(a - u^2 + v^2)v, \quad u(0) = u_0 > 0, \\
v' = (\frac{1}{a} - u^2 + v^2)u - \frac{2}{\rho} v, \quad v(0) = 0,
\end{align*} \quad (2.8)
$$

where prime stands for $\frac{d}{d\rho}$. For (2.5) we have

$$
\begin{align*}
u' = (\frac{1}{a} + u^2 - v^2)v, \quad u(0) = u_0 > 0, \\
v' = (a + u^2 - v^2)u - \frac{2}{\rho} v, \quad v(0) = 0,
\end{align*} \quad (2.9)
$$

The normalizing condition (2.3) becomes

$$
(a + a^{-1})^2 \int_0^\infty (u^2 + v^2) \rho^2 d\rho = S^2 \equiv \frac{w\mu^2 c^2}{\pi \hbar^3}, \quad (2.10)
$$

where $S$ is a dimensionless constant to be determined.
The computation shows that, for any given $a > 1$, there exists a sequence of initial data $0 < u(0)_1 < u(0)_2 < \cdots$, such that (2.4) and (2.5) have eigen solutions. The theoretical analysis proves that there are infinite eigen solutions for every $a[11]$. In [17] we have shown three families of eigen solutions with even parity and the first family of eigen functions with odd parity.

To describe the properties of the eigen solutions, we define the following dimensionless functions, which are continuous functions of spectrum $a$ for the same family solutions.

1. The dimensionless norm $y(a)$

$$y \equiv \frac{1}{2} \lg \left((a + a^{-1})^2 \int_0^\infty (u^2 + v^2) \rho^2 d\rho \right). \quad (2.11)$$

For the same family of eigen solution, $y$ is a continuous function of $a$. By (2.10), the normalizing condition is equivalent to the equation $y = \lg S$.

2. The dimensionless energy $\mathcal{E}(a)$

$$\mathcal{E} \equiv \frac{1}{\mu c^2} \left( mc^2 + \frac{1}{2} wc \int_0^\infty \hat{\gamma}^2 \cdot 4\pi r^2 dr \right)$$

$$= \frac{a^2 - 1}{a^2 + 1} + \frac{a}{a^2 + 1} \frac{\int_0^\infty (u^2 - v^2)^2 \rho^2 d\rho}{\int_0^\infty (u^2 + v^2) \rho^2 d\rho}.$$ \quad (2.12)

This definition of energy is in the Nöther’s sense.

3. The mean diameter of an eigen solution $d(a)$

$$d \equiv \frac{2}{\lambda} \frac{\int r |\phi|^2 d^3 x}{\int |\phi|^2 d^3 x} = \frac{a^2 + 1}{a} \frac{\int_0^\infty (u^2 + v^2) \rho^3 d\rho}{\int_0^\infty (u^2 + v^2) \rho^2 d\rho}.$$ \quad (2.13)

where $\lambda = \frac{\hbar}{\mu c}$ is a universal Compton wave length for all solutions.

4. The total dimensionless inner pressure $P(a)$

$$P \equiv \frac{1}{3 \mu c^2} \left( mc^2 - \int_0^\infty (\mu c^2 \hat{\gamma} + \frac{1}{2} wc \hat{\gamma}^2) \cdot 4\pi r^2 dr \right)$$

$$= \frac{1}{3} \left( \frac{a^2 - 1}{a^2 + 1} - \frac{\int_0^\infty (u^2 - v^2)^2 \rho^2 d\rho}{\int_0^\infty (u^2 + v^2) \rho^2 d\rho} \right) - \frac{a}{a^2 + 1} \frac{\int_0^\infty (u^2 - v^2)^2 \rho^2 d\rho}{\int_0^\infty (u^2 + v^2) \rho^2 d\rho}. \quad (2.14)$$

The physical meanings of $y(a)$, $\mathcal{E}(a)$ and $d(a)$ are evident. Now we examine the meanings of $P(a)$. For the perfect fluid in relativity, the energy momentum tensor is given by

$$T^{\mu \nu} = (\rho_m + P) U^\mu U^\nu - Pg^{\mu \nu}, \quad T^\mu_\nu = \rho_m - 3P.$$ \quad (2.15)
For the static fluid, we have the 4-dimensional speed $U^\mu = (1, 0, 0, 0)$, and then

$$T^0_0 = \rho m, \quad P = \frac{1}{3}(T^0_0 - T^\mu_\mu). \quad (2.16)$$

For the nonlinear spinor (2.1) in curved space-time with diagonal metric, we define the corresponding concepts as follows[26, 27, 28]

$$T^{\mu\nu} = \frac{1}{2} \Re\langle \phi^+(\phi^{\mu\nu} + \phi^{\nu\mu})\phi\rangle - \mathcal{L}g^{\mu\nu} = \frac{1}{2} \Re\langle \phi^+(\phi^{\mu\nu} + \phi^{\nu\mu})\phi\rangle + (V'\gamma - V)g^{\mu\nu}. \quad (2.17)$$

For static spinor, we have

$$P = \frac{1}{3}(T^0_0 - T^\mu_\mu) = \frac{1}{3}(m|\phi|^2 - \mu\gamma - \frac{1}{2}w\gamma^2). \quad (2.18)$$

The dimensionless form of the total inner pressure of the spinor becomes (2.14).

The curves of the dimensionless functions defined above are shown in Fig.1 and Fig.2. In Fig.1, the normalizing condition $y \equiv \lg S = 0.918$ and $y \equiv \lg S = 0.647$ are derived from the anomalous magnetic moment (AMM) of an electron according to different definition of mass, as computed in the next section. A rough computation was once given in [18].

![Figure 1: The norm function $y(a)$, dimensionless energy $E = \frac{E}{\mu c^2}$ and mean diameter $d(a)$ of a spinor. Only the solutions corresponding to the intersection $y(a) = \lg S$ are meaningful in physics.](image)

For an electron, we have $\mu = m_e = 9.11 \times 10^{-21}$ kg, $\hbar = 1.055 \times 10^{-34}$ J.s, $c = 2.998 \times 10^8$ m/s. By (2.10) and $S = 8.277$, we can estimate the value

$$w = \frac{\pi \hbar^3 S^2}{\mu^2 c^2} = 4.945 \times 10^{-59} S^2 = 3.385 \times 10^{-57} (\text{Jsm}^2). \quad (2.19)$$
In this case, the nonlinear spinor equation has only two valid eigen solutions corresponding to $a = 1.95$ and $a = 45.7$. The norm function $y(a)$ of all other families of eigen solutions have no intersection points with $y = 0.918$.

The radial functions $(G, F)$ of solutions with even parity are shown in Fig.3, where

$$G(r) = \sqrt{\frac{w}{2\mu c^2}} u = \sqrt{\frac{a}{a^2 + 1}} u, \quad F(r) = -\sqrt{\frac{w}{2\mu c}} f = \sqrt{\frac{a}{a^2 + 1}} v. \quad (2.20)$$

The unit of the coordinate $r$ is the universal Compton wave length $\frac{\hbar}{\mu c}$. So the images of different solutions are visually comparable in Fig.3. More images see [17].

III. THE NONLINEAR SPINOR WITH ELECTROMAGNETIC INTERACTION

The nonlinear spinor with self electromagnetic interaction was researched by a few authors. In 1966, M. Wakano has approximately analyzed the cases of $A_0$ dominance and $\vec{A}$ dominance when $w = 0$, and reached the following conclusions[29]. In the case of $\vec{A}$ dominance, the eigen solutions or the solitons do not exist for the first order approximation. In the case of $A_0$ dominance, the eigen solutions exist but all with negative energy. In fact, the negative mass is equivalent to change the sign of $A_0$, which implies to transform the repulsive potential of $A_0$ into the absorbent one. M. Soler and A. F. Rañada calculated the eigen solutions of (1.9) by omitting $\vec{A}$. But they neglected the normalizing condition and
Figure 3: The radial distribution of the nonlinear dark spinors
did not use the true value of $e$. Their explanation for the results seems to be misguided by
some inadequate ideas\cite{30, 31}. Besides, the eigen solutions with Born-Infeld potential were
studied in \cite{32}. The detailed non-relativistic approximation of the many-spinors equations
was given in \cite{17, 33}

In general, the coordinates $r$ and $\theta$ can not be separable for nonlinear spinor with vector
potential due to the term $\vec{A}$. However $u_k$ and $v_k$ can be conveniently expressed by Fourier
series with respect to $\theta$ as (1.14) and (1.15), and the equations of the radial functions can
be derived via variation principle, because the eigen solutions are the critical points of the
following energy functional

$$J = 2\pi \int_0^\infty \int_0^\pi r^2 dr \sin \theta d\theta (\phi^+ \phi + \frac{1}{2} wc\gamma^2 - \frac{1}{2} c\nabla A_\mu \cdot \nabla A^\mu - mc^2 \bar{\alpha} + mc^2). \quad (3.1)$$

So the problem (1.9) can be changed into an ordinary differential equation system which
can be solved by numerical computation.
In this paper, we only consider the eigen solutions with $\frac{1}{2}$-spin and even parity, which is the only valid case for a free electron. In the dimensionless form, we have the magnitude for the fields

$$|\vec{A}| \sim \frac{\alpha}{a}|g|, \quad |A_0| \sim \alpha |g|, \quad |f| \sim \frac{1}{a}|g|, \quad \alpha \dot{=} \frac{1}{137},$$

(3.2)

where $a$ is the dimensionless spectrum. Since the high order terms are caused by the vector potential $|\vec{A}| \sim \alpha a|g|$, for adequately large $a$, we only keep the first order approximation for simplicity. Then we have

$$\dot{\phi} = (g, 0, i f \cos \theta, i f \sin \theta e^{i \phi})^T \exp(-i \frac{m c^2}{\hbar} t),$$

(3.3)

where $g$ and $f$ are real functions of $r$ with $g(0) > 0$. For large spectrum $a = 49.12$, the relative error of the approximation is less than $10^{-4}$, so the approximation is accurate enough to reveal the anomalous magnetic moment of a spinor with electromagnetic field. The less the value of $a$, the large the error of approximation.

The quadratic forms of $\phi$ are given by

$$\bar{\alpha}_0 = g^2 + f^2, \quad \gamma = g^2 - f^2, \quad \bar{\alpha} = 2 g f \sin(- \sin \varphi, \cos \varphi, 0).$$

(3.4)

Correspondingly we have

$$A_0 = A_0(r), \quad \vec{A} = A(r) \sin \theta(- \sin \varphi, \cos \varphi, 0).$$

(3.5)

Substituting (3.4), (3.5) into (3.1) we get the energy functional

$$J = 4 \pi e \int_0^\infty r^2 dr \left\{ \hbar ((f' + \frac{2}{r} f) g - g' f) + (\mu - m)c g^2 - (\mu + m)c f^2 - \frac{1}{2} w(g^2 - f^2)^2 + e(g^2 + f^2) A_0 - \frac{4}{3} e g f A + \frac{1}{2} A_0 (\partial_r^2 + \frac{2}{r} \partial_r) A_0 - \frac{1}{3} A (\partial_r^2 + \frac{2}{r} \partial_r - \frac{2}{r^2}) A \right\} + mc^2.$$  

(3.6)

The approximation is only caused by the vector potential $\vec{A}$. By variation, we get a closed system of ordinary differential equations

$$\begin{cases}
g' = -\frac{1}{\hbar}((\mu + m)c - e A_0 - w(g^2 - f^2)) f - \frac{2}{3\hbar} e A g,
g' = -\frac{1}{\hbar}((\mu - m)c + e A_0 - w(g^2 - f^2)) g + (\frac{2}{3\hbar} e A - \frac{2}{r}) f, \\
A''_0 + \frac{2}{r} A'_0 = -e(g^2 + f^2), \quad A'' + \frac{2}{r^2} A' = -2 e g f.
\end{cases}$$

(3.7)
Make transformation

\[ r_0 = \frac{\hbar}{\sqrt{\mu - m^2 c}}, \quad a = \sqrt{\frac{\mu + m}{\mu - m}}, \quad \alpha = \frac{e^2}{4\pi \hbar} = \frac{1}{137.035999}, \quad (3.8) \]

\[ \rho = \frac{r}{r_0}, \quad u = \sqrt{\frac{wr_0}{\hbar}} g, \quad v = -\sqrt{\frac{wr_0}{\hbar}} f, \quad P = \frac{e\alpha}{\hbar} A_0, \quad Q = \frac{2e\alpha}{3\hbar} A, \quad (3.9) \]

where \( P \) is dimensionless potential, which can not be confused with the pressure defined in (2.14). Substituting them into (3.7), we get the dimensionless form

\[
\begin{align*}
\frac{u'}{a} &= (a - P - u^2 + v^2) v - Qu, \\
\frac{v'}{a} &= \left( \frac{1}{a} + P - u^2 + v^2 \right) u + \left( Q - \frac{2}{a} \right) v, \\
\frac{P''}{a^2} + \frac{2}{a} \frac{P'}{a^2} &= -\alpha \int_0^\rho \frac{u^2 + v^2}{(u^2 + v^2)\rho^2} d\rho, \\
\frac{Q''}{a^2} + \frac{2}{a} \frac{Q'}{a^2} &= \frac{4\alpha}{3} \int_0^\rho \frac{uv}{(u^2 + v^2)\rho^2} d\rho,
\end{align*}
\]

(3.10)

In (3.10) only \( a \) is a free parameter, which acts as the spectrum similar to the dark case of \( e = 0 \). The normalizing condition is still (2.10). (3.10) is independent on the undetermined coefficient \( w \), but it becomes a global problem. The natural boundary conditions are given by

\[
\begin{align*}
u(0) &= \frac{P'(0) = Q(0) = Q'(0) = 0,}u \to u\infty e^{-\rho}, \quad v \to \frac{u\infty}{a} e^{-\rho}, \quad P \to \frac{\alpha}{4\pi \rho}, \quad Q \to \frac{Q\infty}{\rho^2}, \quad (\rho \to \infty).
\end{align*}
\]

(3.11)

The solutions of \((P, Q)\) can be expressed as

\[
\begin{align*}
P &= \frac{\alpha}{\int_0^\rho \frac{1}{\rho^2} \int_0^\rho \left[ u^2(\tau) + v^2(\tau) \right] \tau^2 d\tau d\rho,} \\
Q &= \frac{-4\alpha}{3\rho^2 \int_0^\rho (u^2 + v^2)\rho^2 d\rho} \int_0^\rho \rho^2 \int_0^\rho u(\tau)v(\tau) d\tau d\rho.
\end{align*}
\]

(3.12)

(3.13)

We have \( P > 0, Q > 0 \) for the meaningful solutions. The solution of (3.12) and (3.13) can be soundly solved by iterative algorithm.

The total energy of the system in Nöther’s sense is given by

\[
E = \int_{R^3} c \left( \sum_{\forall f} \frac{\partial L}{\partial (\partial_t f)} \partial_t f - \mathcal{L} \right) d^3 x
\]

\[= 2\pi \int_0^\infty r^2 dr \int_0^\pi \sin \theta d\theta \left( \phi^\dagger \dot{\hat{H}} \phi + \frac{1}{2} wc \gamma^2 - \frac{1}{2} c^2 A_\mu \cdot \nabla A^\mu \right), \quad (3.14)\]

Substituting (3.8), (3.9) into it, we get the dimensionless form

\[
\mathcal{E} = \frac{E}{\mu c^2} = \frac{a^2 - 1}{a^2 + 1} + \frac{a}{a^2 + 1} \int_0^\infty [(u^2 - v^2)^2 - P(u^2 + v^2) - 2Quv] \rho^2 d\rho.
\]

(3.15)
The mass of a particle is a complex classical concept, which depends on the method of measurement and the context of theory. Using different definition of mass, we will get different spectrum \( a \) and constant \( S \). Different contribution to the energy has different energy-speed relation, which can be detected by elaborate experiment\(^{23} \). Such experiment might be a key to disclose the structure of elementary particles. We give some more discussions in the next section.

In what follows, we take \( m_e \) and \( \mu \) as the classical mass for computation. To get the anomalous magnetic moment, we introduce an infinitesimal external magnetic field

\[
\vec{B}_{\text{ext}} = (0, 0, B), \quad \vec{A}_{\text{ext}} = \frac{1}{2} \vec{B}(-y, x, 0) = \frac{1}{2} Br \sin \theta (-\sin \varphi, \cos \varphi, 0).
\]

(3.16)

Adding \( \vec{A}_{\text{ext}} \) to (3.5) and substituting it into (3.6), we get the increment of the energy

\[
\Delta E = \left| \frac{8\pi}{3} ec \int_0^\infty g f r^3 dr \right| B \equiv \mu_z B,
\]

(3.17)

where \( \mu_z \) is the magnetic moment of the spinor. The dimensionless form is given by

\[
\mu_z = \frac{2(a^2 + 1)}{3a} \frac{k|\int_0^\infty uv \rho^3 d\rho|}{\int_0^\infty (u^2 + v^2)v^2 d\rho} \cdot \mu_B, \quad \mu_B \equiv \frac{e \hbar}{2m_k},
\]

(3.18)

where the constant \( \mu_B \) is the Bohr magneton,

\[
k = \begin{cases} 
1 & \text{if } m_k = \mu, \\
\mathcal{E} & \text{if } m_k = m_e.
\end{cases}
\]

(3.19)

By (3.18), we get the anomalous magnetic moment of a particle

\[
\Delta g \equiv \frac{\mu_z - \mu_B}{\mu_B} = \frac{2(a^2 + 1)k|\int_0^\infty uv \rho^3 d\rho|}{3a \int_0^\infty (u^2 + v^2)v^2 d\rho} - 1.
\]

(3.20)

The empirical value of the AMM of an electron is \( \Delta g = 0.001159652 \). The computational result suggests that (3.20) might be the truth of the AMM.

To compare with the dark spinor, we also define the dimensionless norm by (2.11). The normalizing condition (2.10) is equivalent to \( y = \lg S \). The dimensionless functions \( (\mathcal{E}, \Delta g, y) \) are all continuous functions of \( a \) for the same family of solutions. Fig.4 shows how to determine the spectrum \( a \) by the empirical AMM. Different definition of mass leads to different value of \( a \).

In Fig.4, the trends of \( \Delta g \) shows that \( \Delta g \) is a decreasing function of \( a \), and \( \Delta g \to 0 \) as \( a \to \infty \). By the empirical data of \( \Delta g \), we can compute the following undetermined
Figure 4: The anomalous magnetic moment of the system (3.10) vs. the spectra $a$, the true value for an electron is $\Delta g = 0.001159652$ or $\log(\Delta g) = -2.936$.

Figure 5: The dimensionless functions $(\mathcal{E}(a), \Delta g(a), y(a))$. The constant $S$ is determined by the intersection points A or B, which correspond to the empirical anomalous magnetic moment parameters, If taking $m_k = \mathcal{E}\mu$, we have

$$a = 49.12, \quad S = 8.277, \quad w = 3.385 \times 10^{-57} \text{Jsm}^2, \quad E_V = 1.088 \text{keV}, \quad E_A = 85 \text{eV}. \quad (3.21)$$
If taking $m_k = \mu$, we have

\[ a = 11.35, \ S = 4.434, \ w = 9.723 \times 10^{-58} \text{Js}^2, \ E_V = 15.08 \text{keV}, \ E_A = 330 \text{eV}. \] 

(3.22)

Fig. 5 shows the realistic values of some parameters such as the total energy $E$, the norm function $y(a)$. The constants $S$ or $w$ is determined by normalizing condition $y = \lg S$, and then all other parameters can be computed. By Fig. 5, we learn that, the value of $a$ is larger than that of dark spinor, namely, the electromagnetic interaction increases the rest mass $m$ of a spinor.

![Dimensionless Radial Functions](image)

Figure 6: The dimensionless radial functions, $(u, v)$ correspond to spinor fields. $(P, Q)$ correspond to the dimensionless potentials.

Since $\alpha^2 \frac{1}{137} = 1.37$ is quite small, by (3.2) we learn that, if $a > 10$, the electromagnetic field only have a little influence on the eigen solution. The numerical results also show this conclusion, Fig. 6 shows the comparison of the dimensionless fields when $a = 49.12$. 
IV. SOME INTERESTING PROPERTIES OF THE NONLINEAR SPINOR

Form the above results and some other computation and analysis, we find some special but interesting properties of the nonlinear spinor. These unusual properties might have close relationship with the nature of the elementary particles.

1. By $P \to 0$ and (2.14), for $V = \frac{1}{2}w\gamma^2$ we find

$$mc^2 \to \int_0^\infty (\mu c^2 \gamma + \frac{1}{2}wc\gamma^2) \cdot 4\pi r^2dr. \quad (4.1)$$

More calculations show that such relation also hold for other kind nonlinear potential $V(\gamma)$ satisfying $V'\gamma - V > 0$, namely we always have $|P| \ll E$. An interesting problem is whether the error is just caused by numerical approximation and $P = 0$ is a rigorous relation generally valid for nonlinear spinors?

2. All dimensionless energy $E(a)$ have a similar trend $E \to 1(a \to \infty)$. For large enough $a$, we always have $E \to \mu c^2$.

3. Taking equation system (3.10) as a developing system with respect to $\rho$, we find the initial value of $v(0), v'(0)$ etc. are determined by $u(0)$, then the eigen solutions satisfying boundary condition (3.11) only depend on $u(0)$. This is a general feature for all eigen solutions of any nonlinear spinor, which implies the eigen equation of the nonlinear spinor is over-determined. This fact might be the underlying reason of the Pauli principle[24, 25].

4. For the nonlinear spinor equation with a scalar interaction

$$\alpha^\mu h_i \partial_\mu \phi = (\mu c + sG - V')\gamma \phi, \quad (\partial_\alpha G^\alpha + b^2)G = \lambda s\gamma, \quad (4.2)$$

similar to (3.12) and (3.13), $G$ can be expressed as

$$G(r) = \frac{\lambda s}{r} \int_0^r e^{-b(r-\tau)}d\tau \int_\tau^\infty \gamma(\xi)e^{-b(\xi-\tau)}d\xi, \quad (4.3)$$

so the solution to (4.2) can be soundly solved by iteration. For the AMM $\Delta g$ defined by (3.20), computations show that we always have $\Delta g \sim 0$ similar to the above cases with electromagnetic interaction. This result implies that, it is inadequate to describe the strong interaction by a scalar field.
5. Some rough calculations show that, the AMM of a proton might be explained by the following nonlinear spinor with a strongly coupling vector interaction $G^\mu$,

\begin{align}
\alpha^\mu (\hbar \partial_\mu - eA_\mu + sG_\mu)\phi &= (Mc - V')\gamma\phi, \quad (4.4) \\
\partial_\alpha \partial^\alpha A^\mu &= e\bar{\alpha}^\mu, \quad (\partial_\alpha \partial^\alpha + b^2)G^\mu = s\bar{\alpha}^\mu. \quad (4.5)
\end{align}

If $\beta = \frac{\omega^2}{4\pi\hbar} \sim 1$, we have the following conclusions: (I). In the dimensionless form, the absolute values of $(|G_0|, |\vec{G}|)$ are comparable with $|\phi|$ near the center, so the first order approximation (3.3) is invalid. (II). The first family of the eigen solutions is absent, namely, the fields $(g(r), f(r))$ have intersections with the horizontal axis near the origin, so such spinor has complicated interior structure. (III). Although the solution has $j_3 = \frac{1}{2}$, but the solution $\phi$ includes components with orbital angular momentum, which has strong influence on the AMM, so the value of $|\Delta g|$ is much larger than that caused by the weakly coupled electromagnetic interaction.

6. The energy functional of the nonlinear spinor system (3.1) is indefinite, so the stability of the solutions is special. There are some works on this problem[34, 35, 36]. However, the usual treatment for the positive definite system might be inadequate for the nonlinear spinors.

7. From the derivations of some previous works, we find that we could almost reconstruct the physical theories based on the nonlinear spinors, so some further investigations on the nonlinear spinors are worthwhile.

V. THE TEST OF MASS-ENERGY RELATION

The Einstein’s mass-energy relation $E = mc^2$ is one of the most fundamental formulae in physics, but it has not been seriously tested by an elaborated experiment, and only some indirect evidences in nuclear reaction suggested that it holds to high precision. From the above calculation, we found the interaction potentials of a particle will yield detectable effects, which lead to different energy-speed relation, which can be used as the fingerprints of the interactive potentials of elementary particles. So the experiment may shed lights on the nature of the interaction and elementary particles.

In what follows, we give a detailed description for the experiment. The experiment only involves low energy accelerator of particles and measurement of speed. In this section, $(u, v)$
stand for the speed of a particle, which can not be confused with the fields defined in (1.12) or (2.7).

In Einstein’s original paper [37], he derived the kinetic energy of a particle $K$,

$$K = mc^2 - m_0c^2, \quad m = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \quad (5.1)$$

which implies the total energy and the speed of a particle have the following simple relation

$$E = m(v)c^2. \quad (5.2)$$

However, this relation is based on the linear classical mechanics, and it has not been directly tested by elaborated experiment. There were once some indirect evidences in the nuclear reaction. The most accurate one is provided by S. Rainville et al [38], which indicates that the mass-energy relation $E = mc^2$ holds to an error level less than 0.00004% in the process of neutron capture by nuclei of sulfur and silicon resulting in $\gamma$-radiation. As pointed out by E. Bakhoum [39], although the authors claimed it is a direct test, it is actually a test for the energy conversion $\Delta E = \Delta mc^2$ at low speed of the particles. As one of the most fundamental relation, a direct test for the original energy-speed relation (5.2) is necessary and significant.

What more important is that, for the nonlinear spinors with interactive potentials such as electromagnetic one $A^\mu$, The detailed calculation shows the interaction terms result in the fine structure of the energy-speed relation [22, 23], and the fine structure can be used as fingerprints of the interactions. (3.21) and (3.22) show that, the nonlinear potential yields energy to a magnitude of 1keV, and the electromagnetic interaction to a magnitude of 100eV, which can be easily detected by elaborated experiments.

Hereafter we take $c = 1$ as unit of speed. The general representation of the energy-speed relation is given by

$$E(u) = \frac{M_0}{\sqrt{1-u^2}} - \frac{M_1u^2}{\sqrt{1-u^2}} + \frac{M_\gamma}{\sqrt{1-u^2}} \ln \frac{1}{\sqrt{1-u^2}}, \quad (5.3)$$

where $(M_0, M_1, M_\gamma)$ are all constants of mass dimension, and $M_0$ is the total static mass of the particles, $M_1$ corresponds to interactions such as electromagnetic potential, $M_\gamma$ corresponds to the nonlinear self-interactive potential. The detailed explanation of the parameters see [23].
Because of the little value of $(M_1, M_\gamma)$ and the function $\ln \sqrt{1 - u^2}$, the nonlinear effects can be easily concealed behind $M_0$. For example, when an electron get kinetic energy $30\text{MeV}$. The corresponding speed reaches $u_1 = 0.99986c$, but
\[
E(u_1) = \frac{M_0 - M_1 + 4M_1}{\sqrt{1 - u_1^2}} \approx \frac{M_0}{\sqrt{1 - u_1^2}}.
\]
That is to say, (5.3) is a stiff equation of the coefficients $(M_0, M_1, M_\gamma)$. So we have to make some transformation to get meaningful solution.

We propose the following experimental project to measure the coefficients $(M_0, M_1, M_\gamma)$ in the energy-speed relation. The flow chart and experimental scheme are illustrated in Fig.7. The particles with unit charge are produced by the particles source, and the ones at given initial speed $u_0$ are selected by a homogeneous magnetic field. By adjusting the radius $r$, we can control the initial speed of the particles $u_0$. The series-wound accelerator is constructed by a set of uniform electrodes, which can be charged with high voltage $V$. When the selected particles pass though one pair electrodes, each particle receives an energy increment $\delta E = eV$, which converts into its kinetic energy. If $n$ pair electrodes are charged, then we get the total kinetic energy increment for each particle
\[
K = E(u_1) - E(u_0) = n\delta E = neV. \tag{5.5}
\]
(5.3) and (5.5) establish the connection between the speed $u$ and $nV$. The final speed $u = u_1$ of the particles can be measured by the position $R$ of the particle counter or film. Then we can determine the constants $(M_0, M_1, M_\gamma)$ by fitting the curve $f(u_1, V) = 0$ as defined by (5.5).

Now we make some simplification of (5.5). At first, we can solve the static mass $M_0$ at low energy $u = \tilde{u}_1 \ll c$ as follows. Assume the voltage $V = V_0$, in this nonrelativistic case, we have the approximation of (5.5) as follows
\[
neV_0 \approx \frac{1}{2}(M_0 - 2M_1 + M_\gamma)(\tilde{u}_1^2 - u_0^2). \tag{5.6}
\]
Then we get
\[
\begin{align*}
M_0 &= m_s + 2M_1 - M_\gamma, \\
E_0 &= m_s + 2M_1 - M_\gamma + \frac{1}{2}m_s u_0^2 = E(u_0), \\
m_s &\equiv 2neV_0(\tilde{u}_1^2 - u_0^2)^{-1},
\end{align*}
\]
Figure 7: The flow chart and experimental scheme to test the mass-energy relation

where $m_s$ is the non-relativistic static mass of the particle in classical sense. Therefore, we only need to determine two little coefficients $(M_1, M_\gamma)$ at high energy.

By (5.3), (5.5) and (5.7), we have the following relation

$$(2 - u_1^2)M_1 - \left(1 + \ln \sqrt{1 - u_1^2}\right)M_\gamma \left\{ = E(u_1)\sqrt{1 - u_1^2} - m_s, \quad \text{(by (5.3) and (5.7))} \right.$$  

$${\left. \right.} = \left[n eV + E(u_0)\right] \sqrt{1 - u_1^2} - m_s, \quad \text{(by (5.5) and (5.7))}$$

$${\left. \right.} \overset{\text{by (5.5) and (5.7)}}{=} (n eV + \frac{1}{2} m_s u_0^2) \sqrt{1 - u_1^2} - \frac{m_s u_1^2}{1 + \sqrt{1 - u_1^2}} + (2M_1 - M_\gamma) \sqrt{1 - u_1^2}. \quad (5.8)$$

Denoting

$$\chi = \frac{u_1^2}{1 + \sqrt{1 - u_1^2}}, \quad U = (n eV + \frac{1}{2} m_s u_0^2)c^{-2}, \quad (5.9)$$

and substituting it into (5.8), we get

$$\chi^2 M_1 - [\chi + \ln(1 - \chi)] M_\gamma = (1 - \chi)U - \chi m_s. \quad (5.10)$$

(5.10) is a linear equation of $(M_1, M_\gamma)$, which can be easily solved by the method of least squares from a sequence of measured data $(U_i, \chi_i), i = 1, 2, \cdots, N$. Define the $N$-dimensional vectors $(\vec{a}, \vec{b}, \vec{f})$ by,

$$\vec{a} = (\chi_1^2, \chi_2^2, \cdots, \chi_N^2), \quad (5.11)$$

$$\vec{b} = -[\chi_1 + \ln(1 - \chi_1), \chi_2 + \ln(1 - \chi_2), \cdots, \chi_N + \ln(1 - \chi_N)], \quad (5.12)$$

$$\vec{f} = [(1 - \chi_1)U_1 - \chi_1 m_s, (1 - \chi_2)U_2 - \chi_2 m_s, \cdots, (1 - \chi_N)U_N - \chi_N m_s]. \quad (5.13)$$
Then the solution of the least square is given by

\[ M_1 = \frac{1}{D} [\vec{b}^2 \vec{a} - (\vec{a} \cdot \vec{b}) \vec{b}] \cdot \vec{f}, \quad M_\gamma = \frac{1}{D} [\vec{a}^2 \vec{b} - (\vec{a} \cdot \vec{b}) \vec{a}] \cdot \vec{f}, \quad D = \vec{b}^2 \vec{a}^2 - (\vec{a} \cdot \vec{b})^2, \quad (5.14) \]

From the above computation, for an electron we have the typical order of magnitude for the parameters in (5.10),

\[ \chi \sim 1, \quad U \sim m_s \sim 1 \text{MeV}, \quad M_1 \sim 100 \text{eV}, \quad M_\gamma \sim 1 \text{keV}. \quad (5.15) \]

The synchrotron radiation is much less than \( M_1 \) and \( M_\gamma \), so it can be omitted. A meaningful test strongly depends on the precision of the measurement data \((\chi_i, U_i)\), which should be of relative errors less than \( 10^{-3} \). How to promote the precision of the measurement is the key for the success of a test.

Some possible solutions and its implications:

1. If \( M_\gamma = 0 \) and \( M_1 = 0 \), which means the Einstein’s mass-energy strictly holds, and the particles can not be described by the classical fields.

2. If \( M_\gamma = 0 \) and \( M_1 \neq 0 \), this kind of particles has not nonlinear self-interaction, and the balance of the particles should be explained by scalar and vector interactions.

3. If \( M_\gamma \neq 0 \), which means the standard model of particles is incomplete, and some calculations in quantum field theory should be modified.

So no matter what result the experiment provides, the implication is always important and fundamental. Although the standard model of particles has achieved a lot of progresses in explanation of the behavior of micro particles, it is essentially a phenomenology. So the test of the energy-speed relation might be a shortcut to disclose the secrets of the fundamental particles and their interactions.

VI. DISCUSSION AND CONCLUSION

we have solved the particle-like eigen solutions to some nonlinear spinor equations, and computed several functions which reflect their characteristics. The numerical results show that, the nonlinear spinor equations have positive discrete mass spectra and anomalous magnetic moment. The nonlinear potential and interactions yield different contributions
to the total energy of the system, which can be used as fingerprints of these terms. The magnitude of the components can be easily detected by elaborate experiments. The weird properties of the nonlinear spinors might be closely related with the elementary particles and their interactions, so some deeper investigations on them are significant.

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