Counting the Identities of a Quantum State

Ivan Horváth
University of Kentucky, Lexington, KY, USA
ORCiD: 0000-0001-8810-9737

Robert Mendris
Shawnee State University, Portsmouth, OH, USA
ORCiD: 0000-0003-3267-3552

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Quantum physics frequently involves a need to count the states, subspaces, measurement outcomes and other elements of quantum dynamics. However, with quantum mechanics assigning probabilities to such objects, it is often desirable to work with the notion of a “total” that takes into account their varied relevance so acquired. For example, such effective count of position states available to lattice electron could characterize its localization properties. Similarly, the effective total of outcomes in the measurement step of quantum computation relates to the efficiency of quantum algorithm. Despite a broad need for effective counting, well-founded prescription has not been formulated. Instead, the assignments that do not respect the measure-like nature of the concept, such as versions of participation number or exponentiated entropies, are used in some areas. Here we develop and solve the theory of effective number functions (ENFs), namely functions assigning consistent totals to collections of objects endowed with probability weights. Our analysis reveals the existence of a minimal total, realized by the unique ENF, which leads to effective counting with absolute meaning. Touching upon the nature of measure, our results may find applications not only in quantum physics but also in other quantitative sciences.

Keywords: quantum identities, quantum uncertainty, localization, quantum computing, effective number, effective measure, diversity measure, effective choices

ihorv2@u.uky.edu
rmendris@shawnee.edu

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I. MOTIVATION AND OVERVIEW

Among distinctive features of quantum mechanics is that, in some regards, quantum system acts as being simultaneously in multiple states of a given type. Thus, Schrödinger particle with sharp momentum is often said to reside at all positions, having an equal chance of being detected anywhere. But how many of such “position identities” of which will be referred to as the quantum identity problem here. The answer would, among other things, provide a new angle raising a deeper issue whether the effective number of quantum identities can be elevated to a meaningful notion at all.

In this paper, we develop a theoretical framework (effective number theory) which gives rationale to the following answer:}

[A] Let \( P = (p_1, p_2, \ldots, p_N) \), \( p_i = \langle \psi | \psi \rangle \), be the probability vector assigned to quantum state \( | \psi \rangle \) and basis \( \{ | i \rangle \} \). The system described by \( | \psi \rangle \) is effectively in \( N_\ast \{ | \psi \rangle, \{ | i \rangle \} \} = N_\ast[C] \) states from \( \{ | i \rangle \} \), where

\[
N_\ast[C] = \sum_{i=1}^{N} n_\ast(c_i), \quad n_\ast(c) = \operatorname{min} \{c, 1\}
\]  

To arrive at [A], we start with the axiomatic definition of effective number function (ENF) \( N[| \psi \rangle, \{ | i \rangle \}] = N[C] \), namely a function consistently assigning effective totals. To solve [Q] then amounts to finding such \( N \) and using it to specify the effective number of quantum identities in all situations. The subsequent analysis shows, however, that there exists an entire continuum spanned by the orthonormal basis \( \{|i\rangle\} \). Consequently, the additivity property we impose is

\[
\sum_{i=1}^{N} c_i = N \leq N[C] \leq N[| \psi \rangle, \{ | i \rangle \}] = N[C]
\]

Interestingly, this is not the case because it turns out that \( N_\ast \) is an ENF with absolute meaning. Indeed, we will prove that \( N_\ast[C] \leq N[C] \) for all \( C \) and all \( N \), making \( N_\ast \) the unique minimum (least element) on the set of all ENFs. Having revealed that the system in state \( | \psi \rangle \) has to be characterized as being simultaneously in at least \( N_\ast[| \psi \rangle, \{ | i \rangle \}] \) states from \( \{ | i \rangle \} \), this result is used in [A] as a basis for the meaningful canonical choice of ENF. It should be noted in this regard that the maximal ENF, \( N[C] \), whose interpretation would otherwise be on equal footing with \( N_\ast \), doesn’t exist (see Theorem 2).

A crucial novelty in our approach is the inclusion of additivity as a requirement for ENFs. This step is necessary since the effective number of states is an additive concept. However, a proper formulation requires some care. To that end, as well as to start invoking parallels with localization, consider the simple setting of a spinless Schrödinger particle on a finite lattice. In the position basis, its state \( | \psi \rangle \) is represented by \( N \)-tuple \( \langle \psi(x_1), \ldots, \psi(x_N) \rangle \), with \( p_i = \langle \psi(x_i) \rangle \) the probability of detection at location \( x_i \). Denoting by \( C \) the set of all counting weight vectors \( C = (c_1, \ldots, c_N) \), \( c_i = Np_i \), namely

\[
C = \bigcup_{i=1}^{N} C_i, \quad C_i = \{ (c_1, c_2, \ldots, c_N) \mid c_i \geq 0, \sum_{i=1}^{N} c_i = N \}
\]

the additivity property for \( N \) arises as follows. Assume that the particle is restricted to the lattice of \( N_1 \) sites in a state generating the weight vector \( C_1 \in C_{N_1} \). Separately, let it be restricted to a non-overlapping adjacent lattice of \( N_2 \) sites and characterized by \( C_2 \in C_{N_2} \). With symbol \( \boxplus \) representing the concatenation operation, since

\[
C = C_1 \boxplus C_2 \in \bigcup_{N=N_1+N_2} C_{N_1+N_2}
\]

there exists a state of the particle on the combined lattice, producing this composite \( C \). Given the additivity of numbers, the sum rule for number of available states \( N = N_1 + N_2 \) has to hold for its effective counterpart as well (\( N[C] = N[C_1] + N[C_2] \)). Consequently, the additivity property we impose is

\[
N[C_1 \boxplus C_2, N_1 + N_2] = N[C_1, N_1] + N[C_2, N_2], \quad \forall C_1, C_2
\]
where the dimensions of vector arguments are indicated explicitly to emphasize that \( N[C] \) represents \( N \) modified by distribution \( C \). Notice that \( N \) is evidently additive, and that the above reasoning doesn’t depend on the system, state or basis in question.

Several decades ago, Bell and Dean \[6\] dealt with the problem analogous to \[Q\] while analyzing localization properties of vibrations in glassy silica. In particular, they asked how many atoms do these vibrations effectively spread over. Their quantifier, the participation number \( N_p \), is given by

\[
\frac{1}{N_p[C]} = \frac{1}{N^2} \sum_{i=1}^{N} c_i^2
\]

and is still widely used in the analysis of localization. In other areas, it is common to exponentiate a suitable entropy, such as Shannon entropy \[7\], and use it for analogous purposes. However, none of these quantifiers is \((A)\)-additive. Their interpretation as effective totals is thus vague and they tend to be too arbitrary. In contrast, incorporating additivity into the definition of ENFs leads to the resolution of the quantum identity problem and suggests new possibilities both in physics (quantum uncertainty \[2\], localization \[4\], effective description of states \[5\]) and measure-related aspects of mathematics. The effective number theory, which we develop here as a tool to solve \[Q\], provides a theoretical starting point for such developments.

In the rest of this section, we describe the construction of ENFs and discuss the key results of the effective number theory. The goal here is to provide a concise but rigorous overview, including the motivations for axiomatic properties, as well as the ramifications of deduced features. The fully mathematical treatment in the technically convenient dual form of effective complementary numbers (co-numbers) is then given in Sec. \[II\]. Generalizations of the quantum identity problem are discussed in Sec. \[III\]. A broadly construed application of effective numbers in quantum theory, namely as a tool to characterize quantum states, is described in Sec. \[IV\]. Concluding remarks are given in Sec. \[V\].

### A. Effective Numbers

We now develop the notion of ENF as a function \( N = N[C] \), assigning effective total to each distribution of weights \( C \in C \) over the elements of a basis. Such construction clearly doesn’t depend on the fact that counted objects are quantum states, and we will thus use generic terms in that regard from now on. The underlying goal is to extend the “counting measure” for a collection of distinct but otherwise equivalent objects (natural number \( N \in \mathbb{N} \)) to the situation when these objects acquire varied importance expressed by their counting weights (effective number \( N[C] \in \mathbb{R} \)). The additivity property \((A)\) is thus a basic consistency requirement for acceptable ENFs.

Like in ordinary counting, no specific relation among individual objects is assumed. Thus, in the same way the number of balls in a bag does not change upon their reshuffle, the effective number will not change upon the permutation of counting weights. In other words, ENFs are required to be totally symmetric in their arguments, namely\[7\]

\[
N(\ldots c_i \ldots c_j \ldots) = N(\ldots c_j \ldots c_i \ldots) , \quad \forall i \neq j
\]

(S)

Extensions \( N \to N[C] \) are by definition such that ordinary counting corresponds to all objects being equally important, and thus to a uniform distribution. More precisely,

\[
N(1,1,\ldots,1) = N , \quad (1,1,\ldots,1) \in C_N , \quad \forall N
\]

(B1)

On the other hand, whenever all weight is given to a single object, all others being irrelevant, the effective number is required to be one, namely

\[
N(N,0,\ldots,0) = N(0,N,0,\ldots,0) = \ldots = N(0,\ldots,0,N) = 1
\]

(B2)

within each \( C_N \). Note that \((1,1,\ldots,1) \in C_N\) and \((\ldots,0,N,0,\ldots) \in C_N\) are the opposite extremes in cumulation of weight. Hence, the effective number of objects with arbitrary weights has to fall between the corresponding extremal values, namely

\[
1 \leq N[C] \leq N , \quad \forall C \in C_N , \quad \forall N
\]

(B)

The degree of weight cumulation plays more detailed role in effective numbers than just determining the boundary properties. Indeed, the concept has to respect that increasing the cumulation in the distribution cannot increase the effective number. To formulate such monotonicity, consider two objects weighted by \( C = (c_1, c_2) \in C_2 \) with \( c_1 \leq c_2 \). The deformation \( C \to C_\varepsilon = \)

\[7\] We write \( N(c_1,\ldots,c_N) \) when weights need to be distinguished, but use the functional notation \( N[C] \) otherwise.
Once an ENF is fixed and used to assign totals, the conventional “number of objects” is replaced by the “effective number of objects”. While we use the term effective number only in this restricted sense here, the underlying algebraic structure makes the concept similar to standard types of numbers.

(c₁ − ϵ, c₂ + ϵ) leads to further cumulation in favor of the second object, and thus \( N[C_1] \leq N[C] \) is imposed for all \( 0 \leq \epsilon \leq c_1 \). In situation with arbitrary \( N \), we require the same for each ordered pair \( c_i \leq c_j \) and deformation \( 0 \leq \epsilon \leq c_i \), namely:

\[
N(\ldots c_i - \epsilon \ldots c_j + \epsilon \ldots) \leq N(\ldots c_i \ldots c_j \ldots) \quad (M^-)
\]

It is easy to check that \( (M^-) \)-monotonic \( N \) attains its maximal value over \( C_N \) at \((1,1,\ldots,1)\), while the minimum is at one or multiple fully cumulated vectors \((\ldots N, \ldots)\). Conditions \((B1), (B2), (B)\) are thus compatible with \( (M^-) \)\(^8\). Note that, although not an ENF, the participation number \((A)\) satisfies \( (M^-) \) monotonicity.

The final requirement in definition of ENFs is continuity. The nature of problems with admitting discontinuities can be illustrated by

\[
N_+(C) = \sum_{i=1}^{N} n_+(c_i) \quad , \quad n_+(c) = \begin{cases} 0, & c = 0 \\ 1, & c > 0 \end{cases} \quad (5)
\]

which counts the number of non-zero weights in \( C \), and will be relevant later in our analysis. Consider again two objects with \( C = (c, 2 - c) \). When \( c \) approaches zero, thus marginalizing the first object to arbitrary degree, the effective number should approach one. However, this does not materialize in \( N_+ \) due to its discontinuity. In general, we require that ENF cannot jump upon arbitrarily small change of weights, namely

\[
N = N[C] \text{ is continuous on } C_N \quad , \quad \forall N \quad (C)
\]

The properties discussed above define the set \( \mathcal{N} \) of all effective number functions. However, there are dependencies among these requirements. In particular, it can be easily checked that the boundary condition \((B1)\) is a consequence of \((B2)\) and additivity. Similarly, \((B)\) follows from \((B1), (B2)\), symmetry and monotonicity. This leaves us with

**Definition 0.** A real-valued function \( N = N[C] \) on \( C \) is called an effective number function (belongs to set \( \mathcal{N} \)) if it is simultaneously additive \((A)\), symmetric \((S)\), continuous \((C)\), monotonic \((M^-)\) and satisfies the boundary condition \((B2)\).

Some of the features imprinted on the corresponding notion of effective numbers are visualized in Fig. 1\(^9\). On the left, natural numbers are shown as a theoretical model for expressing and manipulating the quantities of like objects (bags of balls) or of varied objects treated as equivalent. The bags containing differing amounts are assigned different discrete points on the real axis (natural numbers), with the operation of “merging the bags” \((\sqcup)\) realized by ordinary addition. Extension to objects distinguished by counting weights is shown on the right. Here the bags assigned equal amounts \( N \) by ordinary counting, may be assigned different effective numbers \( N \), depending on the cumulation of their weight distributions. With maximal cumulation \((\delta\)-function) producing \( N=1 \), the effective number continuously and monotonically increases as cumulation decreases, reaching \( N=N \) when cumulation is absent (uniform distribution). The operation of merging bags is represented by the additivity property \((A)\). Each element of \( \mathcal{N} \), if any, implements a specific version of this scheme. Thus, to assess the conceptual value and practical impact of effective numbers, it is necessary to decipher the structure of \( \mathcal{N} \).

\(^8\) To visualize how the elementary deformation in \( (M^-) \) increases cumulation, one may picture each object as a cylindrical column of incompressible liquid in the amount of its counting weight. Arranging the columns by increasing height from the left to the right produces a half-peak profile with cumulation on the right. Consider the segment of this profile delimited by columns \( c_i \) and \( c_j \) entering \( (M^-) \). The monotonicity operation is represented by the transverse flow of liquid from the left to the right endpoint through columns between them. It is understood that the columns are ordered at every moment of the flow and thus, as the amount of liquid at the endpoints changes, the length of the segment may increase. Since the liquid flows toward the center of cumulation at every point of the process, the resulting distribution is more cumulated than the original one.

\(^9\) Monotonicity \((M^-)\) is closely related to Schur concavity. The latter is equivalent to imposing \((M^-)\) and symmetry \((S)\) simultaneously (see e.g. \([8]\)).

\(^10\) Once an ENF is fixed and used to assign totals, the conventional “number of objects” is replaced by the “effective number of objects”. While we use the term effective number only in this restricted sense here, the underlying algebraic structure makes the concept similar to standard types of numbers.
B. Effective Counting

It is not difficult to establish that ENFs do exist. For example, one can verify that one parameter family of functions

\[ n(\alpha) = \sum_{i=1}^{\infty} n_i(\alpha) \text{, } n(\alpha) = \min \{ e^\alpha, 1 \} , \quad 0 < \alpha \leq 1 \]  

belongs to \( \mathfrak{M} \), with \( N(1) = N \). However, it is rather remarkable that all \( N \in \mathfrak{M} \) have the additively separable structure of (6). Indeed, Theorem 14 (Sec. II) implies the following central result specifying \( \mathfrak{M} \) explicitly.

Theorem 1. Function \( N \) on \( C \) belongs to \( \mathfrak{M} \) if and only if there exists a real-valued function \( n = n(c) \) on \( [0, \infty) \) that is concave, continuous, \( n(0) = 0 \), \( n(c) = 1 \) for \( c \geq 1 \), and

\[ N[C] = \sum_{i=1}^{N} n(c_i) , \quad \forall C \in C_N , \quad \forall N \]  

Such function \( n \) associated with \( N \in \mathfrak{M} \) is unique.

Thus, there is one to one correspondence between ENFs and functions of single variable specified by Theorem 1. Such \( n \) associated with given \( N \) will be referred to as its counting function.

The necessity of additively separable form (7) for ENFs is interesting conceptually. Indeed, it is common and familiar to represent the ordinary total (natural number) by a sequential process of adding a unit amount for each object in the collection. According to Theorem 1, this applies to every consistent extension to the effective total (effective number), albeit with objects contributing weight-dependent amounts specified by the counting function. It thus turns out that the construction of ENFs generalizes the process of ordinary counting to the process of effective counting.

C. Minimal Effective Number

A key insight into the nature of effective counting is provided by the following results concerning the structure of set \( \mathfrak{M} \). They follow directly from Theorem 18 in Sec. II.

Theorem 2. Let \( N_+ \in \mathfrak{M} \) and \( N_+ \notin \mathfrak{M} \) be functions on \( C \) defined by (1) and (5) respectively. Then

\( (a) \) \( N_+[C] \leq N[C] \leq N_+|C| \) , \( \forall N \in \mathfrak{M} \) , \( \forall C \in C \)

\( (b) \) \( \{ N[C] \mid N \in \mathfrak{M} \} = [\alpha, \beta] \) , \( \alpha = N_+[C] \) , \( \beta = N_+|C| \) , \( \forall C \in C \)

To elaborate, first note that (a) is the refinement of defining condition (B). While the upper bound is intuitive (\( N_+[C] \) counts the number of non-zero weights in \( C \)), the lower one is unexpected and consequential. In particular, the effective number of objects weighted by \( C \) cannot be smaller than \( N_+[C] \). Since \( N_+ \) is an ENF, this feature is inherent to the concept itself: there is a meaningful notion of minimal effective number. In technical terms, \( N_+ \) is the least element of function set \( \mathfrak{M} \) with respect to partial order \( (N_1 \leq N_2) \iff (N_1[C] \leq N_2[C], \forall C \in C) \), and thus a unique ENF with this property.

Part (b) conveys that, for any fixed \( C \in C \), effective counting can be adjusted so that \( N[C] \) assumes any desired value from the allowed range specified by (a). While reflecting a certain degree of arbitrariness built into the concept of effective numbers, the associated freedom of choice is in fact quite natural. To illustrate this, consider \( N \) objects with non-zero weights of very disparate magnitudes so that the collection is usefully characterized by an effective number. The insistence on ordinary count in this situation constitutes a “large extrapolation” since it forces each object to contribute equally despite the disparity. According to (b), such extrapolation can be realized by a sequence of ENFs that gradually bring the effective total to \( N \). This ability to accommodate the needed continuum of consistent schemes can be considered a feature in a framework designed to deal with generalized aspects of counting.

Note that (b) also confirms an intuitive expectation that there is no maximal ENF since, although specifying a supremal value for each \( C \), function \( N_+ \) doesn’t belong to \( \mathfrak{M} \). Taken together, the results of Theorem 2 form the basis for our canonical solution [A] of quantum identity problem [Q]. The existence of minimal total \( N_+ \) is particularly consequential in applications of effective numbers. One notable example is that it facilitates the notion of minimal (intrinsic) quantum uncertainty [2].

\[ \text{Note that it suffices to require continuity at } c = 0 \text{ since concavity guarantees it elsewhere.} \]
II. EFFECTIVE NUMBER THEORY

In this section, we will present the theory of effective numbers in a requisite mathematical detail. The aim is to do this in a self-contained manner using elementary mathematics so that it is readily accessible. Certain generalizations and abstract aspects of the underlying structure will be elaborated upon in a dedicated mathematical account.

A. Effective Complementary Numbers

Due to its technical convenience, the discussion will be carried out in terms of effective complementary numbers (effective co-numbers) realized by function\(^{12}\)

\[ M[C] = N - N[C] \quad , \quad C \in C_N \quad , \quad N \in \mathfrak{M} \]

where \( N = N[C] \) are the ENFs introduced in Sec. I. We start by defining the effective co-number functions (co-ENFs), as implied by the above relationship.

**Definition 1** \( \mathfrak{M} \) is the set of effective co-number functions \( M \), where \( M : C \to \mathbb{R} \) have the following properties. For all \( N, M, \) for all \( 1 \leq i, j \leq N, i \neq j, \) for all \( C = (c_1, ..., c_N) \in C_N, \) and for all \( B \in C_M \)

(A) additivity: \( M[C \oplus B] = M[C] + M[B] \)

(co-B2) boundary values: \( M(N, 0, ..., 0) = N - 1, \) where \( (N, 0, ..., 0) \in C_N \)

(C) continuity of \( M \) restricted to \( C_N \) whose topology is inherited from the standard topology on \( \mathbb{R}^N \)

\( M^+ \) monotonicity: \( 0 < \varepsilon \leq \min\{c_i, N - c_j\}, \ c_i \leq c_j \Rightarrow M(..., c_i, ..., c_j, ...) \leq M(..., c_i - \varepsilon, ..., c_j + \varepsilon, ...) \)

(S) symmetry: \( M(..., c_i, ..., c_j, ...) = M(..., c_j, ..., c_i, ...) \)

The following examples will be useful in the context of our analysis.

**Example 2** The function \( M_{(\alpha)}[C] = \sum_i m_{(\alpha)}(c_i) = \sum_{i=0}^N \sum_{c_i} 1 + \sum_{c_i, \in (0,1)} (1 - c_i^\alpha), \) where

\[ m_{(\alpha)}(c) = \begin{cases} 1, & c = 0 \\ 1 - c^\alpha, & 0 < c < 1 \\ 0, & 1 \leq c \end{cases} \]

belongs to \( \mathfrak{M} \) for \( \alpha \in (0, 1]. \) This example is complementary to \( N_{(\alpha)} \) introduced in (6).

**Example 3** The function \( M_+[C] = M_{(0)}[C] = \sum_{i=1}^N m_+(c_i), \) where

\[ m_+(c) = m_{(0)}(c) = \begin{cases} 1, & c = 0 \\ 0, & 0 < c \end{cases} \]

satisfies Definition except for continuity. Thus, \( M_+ \notin \mathfrak{M} \) being complementary to \( N_+ \) in (5).

**Example 4** The function \( M_* = M_{(1)}[C] = \sum_{i=1}^N m_*(c_i), \) where

\[ m_*(c) = m_{(1)}(c) = \begin{cases} 1 - c, & 0 \leq c < 1 \\ 0, & 1 \leq c \end{cases} \]

satisfies all the properties from Definition so \( M_* \in \mathfrak{M}. \) This co-ENF is complementary to \( N_* \) in (1).

Due to its repeated use in what follows, it is useful to formalize the following obvious Lemma.

**Lemma 5** If \( M \) satisfies (A) and (S) then for all \( N \)

(i) \( M[C] = M(c_1, ..., c_{i-1}, 1, c_{i+1}, ..., c_N) = M(c_1, ..., c_{i-1}, c_{i+1}, ..., c_N) + M(1), \) where \( C \in C_N. \)

(ii) \( M(1, ..., 1) = N M(1), \) where \( (1, ..., 1) \in C_N. \)

\(^{12}\) They could also be called effective dual numbers but the duality is not a central property here.
B. Separability

We now focus on demonstrating the results that will ultimately clarify the structure of effective co-numbers, and thus of effective numbers. The main conclusion will be that all co-ENFs are of additively separable form, such as the one exhibited by the family of functions $M_{(\alpha)}$. The property of additive separability is defined as follows:\(^{13}\)

**Definition 6** Additively separable function $G$ on $C$ is one that can be expressed as

$$G[C] = \sum_{i=1}^{N} g(c_i), \quad C \in C_N, \quad N = 1, 2, \ldots$$

where $g(c)$ is some function defined on $[0, \infty)$. Function $g(c)$ is called a generating function of $G[C]$.

Additively separable $G$ is generated by infinitely many distinct functions. However, for co-ENFs, a canonical representative can be singled out that is continuous and bounded on $[0, \infty)$ (see Proposition\(^{12}\) and Corollary\(^{15}\)).

The relevant insight into additive separability is provided by Lemma 7 below. Before formulating it, let us associate with $C \in C_N$ the vector $C^\uparrow \in C_N$ obtained by permuting the components of $C$ into ascending order. Also, $C^\uparrow_c$ will denote a vector obtained from $C^\uparrow$ by keeping only components less than one and removing the rest. Note that for any symmetric function $M$ on $C$, we have $M[C] = M[C^\uparrow]$, and also that $C^\uparrow_c \notin C$. In Proposition\(^{8}\) we will work with $C^\uparrow_c$ with an analogous meaning.

**Lemma 7 (Separability)** Let $G$ be a function on $C$ satisfying $(A)$, $(S)$ and the property

$$\forall N, \quad \forall C, B \in C_N : \quad C^\uparrow_c = B^\uparrow_c \implies G[C] = G[B]$$

Then following statements hold:

(a) $G$ is additively separable.

(b) If, in addition, $G$ is continuous on $C_2$, then there exists a continuous function generating it.

Proof:

(a) Assuming $C \neq (1, 1, \ldots, 1)$, let $C^\uparrow = (c_1, \ldots, c_N) \in C_N$ and $C^\downarrow_c = (c_1, \ldots, c_m)$. We will distinguish two cases, namely $2m \leq N$ and $2m > N$, for which we respectively get by using \(^{10}\)

$$G[C] = G(c_1, \ldots, c_m, c_m+1, \ldots, c_N) = G(c_1, \ldots, c_m, 2 - c_1, \ldots, 2 - c_m, 1, \ldots, 1),$$

$$G[C \uplus (1, \ldots, 1)] = G(c_1, \ldots, c_m, c_m+1, \ldots, c_N, 1, \ldots, 1) = G(c_1, \ldots, c_m, 2 - c_1, \ldots, 2 - c_m),$$

with $2 - c_\ell > 1$ for $\ell = 1, \ldots, m$. The vectors on the top line (case $2m \leq N$) are from $C_N$, while those on the bottom line (case $2m > N$) are from $C_{2m}$. In both cases, Lemma\(^{5}\) symmetry $(S)$ and additivity $(A)$ lead to

$$G[C] = G(c_1, 2 - c_1, \ldots, c_m, 2 - c_m) + (N - 2m) G(1)$$

$$= G([c_1, 2 - c_1] \uplus \ldots \uplus [c_m, 2 - c_m]) + (N - 2m) G(1)$$

$$= \sum_{\ell=1}^{m} (G(c_\ell, 2 - c_\ell) - G(1)) + (N - m) G(1).$$

Consequently, introducing the generating function

$$g(x) = \begin{cases} G(x, 2 - x) - G(1), & x \in [0, 1] \\ G(1), & x \in (1, \infty) \end{cases}$$

facilitates the claimed separability $G[C] = \sum_{i=1}^{N} g(c_i)$. Note that for $C = (1, 1, \ldots, 1)$, which was initially excluded, the separability holds in the same form.

\(^{13}\) Unless stated otherwise, referencing “function” *in this section* applies to both real and complex-valued function, and referencing “number” applies to both real and complex options.

\(^{14}\) Continuity on $C_2$ may appear weaker than $(C)$ continuity, but this lemma shows that they are equivalent.
Given the proof of (a), it is sufficient to show that continuity of $g$ on $C_k$ implies the continuity of $g$ in $\Omega$. For that, one only needs to ascertain the continuity at the gluing point $x = 1$, which holds since we have two continuous functions with the same value at the gluing point: $g(1,2 - 1) = g(1)$. \hfill \Box

We will now demonstrate that all co-ENFs satisfy $\text{(10)}$ and hence they are additively separable.

**Proposition 8** All functions $M \in \mathcal{M}$ are additively separable.

Proof:

Because of symmetry (S), we will without loss of generality assume $C = C^\uparrow$, i.e. $C$ is in ascending order. For $C = (1,1,\ldots,1)$ the implication in $\text{(10)}$ is vacuously true\(^{15}\) so $C \neq (1,1,\ldots,1)$ is assumed in what follows. We will use index $\ell$ to label the elements of $C_{\leq}$ and index $j$ for the rest of entries in $C$. Hence, $c_\ell \leq 1 < c_j$ for $\ell = 1,\ldots,m$, $j = m + 1,\ldots,m + n = N$, and $n = \sum_j 1$. Then by monotonicity $(M^+)$, Lemma 5 and $M(1) = 0$, which is the $N = 1$ case of (co-B2), we have:

$$\begin{align*}
M(\ldots c_\ell, \ldots c_j, \ldots) &\leq M(\ldots c_\ell, \ldots c_m + 1 - \varepsilon, c_m + 2, \ldots, c_m + n - 1 + (c_{m + n} - 1 + (c_m + 1 - 1)) \\
&\leq M(\ldots c_\ell, \ldots, c_m + 2 - \varepsilon, c_m + 3, \ldots, c_m + n - 1, 1 + (c_{m + n} - 1) + (c_m + 1 - 1) + \varepsilon) \\
&\leq M(\ldots c_\ell, \ldots, 1, 1 + \sum (c_j - 1)) \\
&= M(\ldots c_\ell, \ldots, 1 - n + \sum c_j).
\end{align*}$$

The opposite inequality follows from additivity (A), (co-B2), and $(M^+)$. We start with $\sum ([c_j] - 1)$ zeroes:

$$\begin{align*}
M(\ldots c_\ell, \ldots c_j, \ldots) &= M \left(\ldots c_\ell, \ldots c_j, \ldots, 0, 0, 0, 1 + \sum_{j=m+1}^{m+n} ([c_j] - 1) \right) - \sum_{j=m+1}^{m+n} ([c_j] - 1) \\
&\geq M \left(\ldots c_\ell, \ldots, c_m + 1, \varepsilon, c_m + 2, \ldots, c_m + n, 0, 0, 0, 1 + \sum (\lceil c_j \rceil - 1) - \varepsilon \right) - \sum (\lfloor c_j \rfloor - 1) \\
&\geq M \left(\ldots c_\ell, \ldots, [c_m + 1], \ldots, [c_m + n], 0, 0, 0, 1 + \sum ([c_j] - 1) - \sum ([c_j] - c_j) \right) - \sum ([c_j] - 1) \\
&= M \left(\ldots c_\ell, \ldots, 1 + \sum_j (-1) - \sum (-c_j) \right) + \sum M(0, \ldots, 0, [c_j] - 1) - \sum ([c_j] - 1) \\
&= M(\ldots c_\ell, \ldots, 1 - n + \sum c_j).
\end{align*}$$

The resulting equality implies\(^{16}\) the property $\text{(10)}$ upon recalling that the value of $M$ doesn’t change when removing the entries $c_\ell = 1$ from $C$, so that $C_\ell^\uparrow$ becomes $C_k^\downarrow$. Moreover, since (A) and (S) are among the defining properties of co-ENFs, $M$ is additively separable by Lemma 7(a). \hfill \Box

**Corollary 9** All functions $M \in \mathcal{M}$ satisfy $\text{(10)}$.

Next we investigate non-uniqueness of the generating function which requires some groundwork to begin with.

**Lemma 10** In real numbers, the following two statements are true:

\begin{enumerate}
  \item[(i)] $(\forall a)(\forall b > a)(\forall x \neq 1) (\exists m, n \in \mathbb{Z}) (\exists z \in [a, b]) (mx + nz = m + n)$
  \item[(ii)] $(\forall a > 1)(\forall b > a) (\exists B) (\forall x \in [0, \frac{1}{2}]) (\exists m, n \in \mathbb{Z}) (\exists z \in [a, b]) (mx + nz = m + n, \text{ and } \frac{n}{m} \leq B)$
\end{enumerate}

Proof:

(i) Fit $\frac{m}{n}$ between $\frac{a - 1}{1 - x}$ and $\frac{b - 1}{1 - x}$ using density of rationals in $\mathbb{R}$. Then choose $z = 1 + (1 - x)\frac{m}{n}$ to get the equality.

(ii) Choose $B = \frac{1}{a - 1}$ and $n = \lceil \frac{1}{a - x} \rceil$, then

$$\frac{1}{n} \leq b - a \leq \frac{b - a}{1 - x} = \frac{1}{1 - x} - \frac{a - 1}{1 - x}$$

\(^{15}\) $C_\ell^\downarrow$ is undefined here.

\(^{16}\) Notice that $\sum c_j = m + n - \sum c_\ell$. 
since \(0 \leq x < 1\). Hence there exists \(m\) to fit \(\frac{m}{n}\) between \(\frac{a - 1}{1 - x}\) and \(\frac{b - 1}{1 - x}\). Then again choose \(z = 1 + (1 - x)\frac{m}{n}\) to get the equality. Moreover,

\[
(a - 1) \leq \frac{a - 1}{1 - x} \leq \frac{m}{n} \quad \text{and so} \quad \frac{n}{m} \leq \frac{1}{a - 1} = B.
\]

That concludes the proof. \(\square\)

**Lemma 11** Let \(G\) be an additively separable function on \(C\) and \(g, g_1, g_2\) its generating functions. Then

(i) \(g(c) + (1 - c)K\) is also a generating function of \(G\) for every number \(K\).
(ii) if \(g_1(c) - g_2(c)\) is bounded on some interval \([a, b]\), \(0 \leq a < b\), then \(g_1(c) - g_2(c) = (1 - c)K_0\) for some number \(K_0\) and all \(c \in [0, \infty)\).

Proof:

(i) This follows from \(\sum_{i=1}^{N} (1 - c_i) = 0\) for all \(C = (\ldots c_i \ldots) \in C_N\).

(ii) Setting \(\tilde{g} = g_1 - g_2\) gives the following equation:

\[
\sum \tilde{g}(c_i) = 0 \quad \text{for all} \quad N \quad \text{and all} \quad C = (c_1, \ldots, c_N).
\]

We will show that

\[
\tilde{g}(c) = (1 - c)\tilde{g}(0) \quad \text{for all} \quad c \in [0, \infty).
\]

Case \(c = 1\). We get \(\tilde{g}(1) = 0\) from (12) for \(N = 1\) and then (13) is satisfied for \(c = 1\).

Case \(c > 1\). Without loss of generality we can assume that \(1 \notin [a, b]\). Moreover, if \(0 \leq a < b < 1\) then \(\tilde{g}\) is bounded also on \([2 - b, 2 - a]\) because \(\tilde{g}(2 - c) = -\tilde{g}(c)\), which follows from (12) when \(N = 2\). As a result, we can assume without loss of generality even that \(1 < a < b\). Under this assumption, we will first show that \(\tilde{g}\), bounded on \([a, b]\), must be bounded also on \([0, 1]\). According to Lemma 10(iii) we have

\[
(\forall a > 1)(\forall b > a) \quad (\exists B) \quad (\forall x \in [0, \frac{1}{2}]) \quad (\exists m, n \in \mathbb{Z}^+)(\exists z \in [a, b]) \quad (mx + nz = m + n, \quad \text{and} \quad \frac{n}{m} \leq B)
\]

Now we choose \(C = (x, \ldots, x, z, \ldots, z) \in C_{m+n}\) in equation (12), where \(x\) repeats \(m\) times and \(z\) repeats \(n\) times. Then

\[
m\tilde{g}(x) + n\tilde{g}(z) = 0 \quad \text{and so} \quad |\tilde{g}(x)| = \left|\frac{-n}{m}\tilde{g}(z)\right| \leq BA, \quad \text{where}
\]

\(A\) is a bound for \(|\tilde{g}|\) on \([a, b]\). Thus we have \(\tilde{g}\) bounded on \([0, \frac{1}{2}]\) by \(BA\).

Now suppose by contradiction that there is \(c > 1\) such that \(\tilde{g}(c) \neq (1 - c)\tilde{g}(0)\). Let \(k\) be an integer large enough that

\[
N - k - 2 = [kc] - k - 2 \geq 0 \quad \text{and} \quad \frac{4BA}{k} < |\tilde{g}(c) - (1 - c)\tilde{g}(0)|
\]

and set \(N = [kc], \quad x = \frac{1}{2}(N - kc)\). Then \(kc + 2x = N \quad \text{and} \quad x \in [0, \frac{1}{2}]\). Choosing \(C = (c, \ldots, c, 0, \ldots, 0, x, x) \in C_N\) in (12), where \(c\) repeats \(k\) times, will produce

\[
k\tilde{g}(c) + (N - k - 2)\tilde{g}(0) + 2\tilde{g}(x) = 0
\]

and the first condition in (14) insures that \(C\) will not have a negative number of zeroes. Given that \(N = kc + 2x\), we get

\[
\tilde{g}(c) + \left(c - 1 + \frac{2x}{k} - \frac{2}{k}\right)\tilde{g}(0) + \frac{2\tilde{g}(x)}{k} = 0
\]

\[
\tilde{g}(c) = (1 - c)\tilde{g}(0) + \varepsilon,
\]

where we set \(\varepsilon = -\frac{2}{k}(x - 1)\tilde{g}(0) + \tilde{g}(x)\). Since \(k\varepsilon\) is bounded by \(4BA\) and from the second condition in (14) we have

\[
0 < |\varepsilon| \leq \frac{4BA}{k} < |\tilde{g}(c) - (1 - c)\tilde{g}(0)|,
\]
which is a contradiction with (15). Thus the original assumption \( \tilde{g}(c) \neq (1 - c) \tilde{g}(0) \) failed and (13) holds for \( c > 1 \).

Case \( c < 1 \). We can use the previous case for \( 2 - c > 1 \) and get \( \tilde{g}(2 - c) = (1 - (2 - c)) \tilde{g}(0) \). This equation transforms into (13) since we already know that \( \tilde{g}(2 - c) = -\tilde{g}(c) \) and thus (13) holds for \( c < 1 \) too.

We have shown that if \( \tilde{g} \) satisfies (12) then it satisfies (13). Finally, setting \( K_0 = \tilde{g}(0) \) completes the proof. \( \square \)

**Proposition 12**

Let \( M \in \mathcal{M} \). Then for each number \( t \), there is a unique generating function \( m \) of \( M \) that is continuous and \( m(0) = t \).

Proof:

The existence of one continuous generating function, not necessarily satisfying \( m(0) = t \), follows from Corollary 9 and Lemma 7(b). Part (i) of Lemma 11 implies that there is at least one continuous generating function for arbitrary value of \( t = m(0) \). To show the uniqueness of such function for every \( t \), assume that there are two continuous generating functions \( m_1 \) and \( m_2 \), such that \( m_1(0) = m_2(0) \). Since \( m_1(c) - m_2(c) \) is then bounded on any finite interval due to continuity, we can use (ii) of Lemma 11 to infer that \( 0 = m_1(0) - m_2(0) = K_0 \). Using (ii) of Lemma 11 again, we finally conclude \( m_1(c) = m_2(c) \) as claimed.

\( \square \)

### C. Description and Structure of Co-ENFs

Separability results of the previous section give us access to the content and the structure of set \( \mathcal{M} \), ultimately providing a key insight into the concept of effective (co-)numbers. We start with the following proposition.  

**Proposition 13**

(i) Let \( \mathcal{G} \) be a real additively separable function defined on \( C \). \( \mathcal{G} \) is continuous (C) and monotone (\( M^+ \)) if and only if it can be generated by a function \( g(c) \) that is continuous at \( c = 0 \) and convex.

(ii) If \( M \in \mathcal{M} \) then all its continuous generating functions \( m(c) \) are convex.

Proof:

(i) (\( \Rightarrow \)) Convexity and continuity of \( g \) at \( c = 0 \) imply its continuity on \([0, \infty)\), which guarantees continuity (C) of \( \mathcal{G} \). In the presence of additive separability, conditions entailed by (\( M^+ \)) take the form

\[
g(c_i) + g(c_j) \leq g(c_i - \varepsilon) + g(c_j + \varepsilon), \quad c_i \leq c_j
\]

(16)

To show that this also follows from stated properties of \( g \), consider function \( \tilde{g} \) that equals to \( g \) everywhere except on interval \([c_i, c_j]\) where it is replaced by a linear segment with boundary values \( \tilde{g}(c_i) \) and \( \tilde{g}(c_j) \). Such function \( \tilde{g} \) is still convex, which implies

\[
\tilde{g} \left( \frac{(c_i - \varepsilon) + (c_j + \varepsilon)}{2} \right) \leq \frac{\tilde{g}(c_i - \varepsilon) + \tilde{g}(c_j + \varepsilon)}{2}.
\]

Then by linearity the left-hand side is

\[
\tilde{g} \left( \frac{c_i + c_j}{2} \right) = \frac{\tilde{g}(c_i) + \tilde{g}(c_j)}{2} = \frac{g(c_i) + g(c_j)}{2}.
\]

And the inequality turns into

\[
g(c_i) + g(c_j) \leq \tilde{g}(c_i - \varepsilon) + \tilde{g}(c_j + \varepsilon) = g(c_i - \varepsilon) + g(c_j + \varepsilon)
\]

as needed.

(\( \Rightarrow \)) Consider the (\( M^+ \)) condition \( \mathcal{G}(...c_i...c_j...) \leq \mathcal{G}(...c_i - \varepsilon...c_j + \varepsilon...) \) for additively separable \( \mathcal{G} \). Setting \( c_i = c_j = c \), and

\[\text{17} \quad \text{The claim (i) of Proposition 13 is likely to be known in the context of majorization but we haven’t found a suitable reference.}\]
subsequently \( a = c - \epsilon, \ b = c + \epsilon \), we obtain in turn

\[
\begin{align*}
g(c) + g(c) & \leq g(c - \epsilon) + g(c + \epsilon) \\
g\left(\frac{a + b}{2}\right) & \leq \frac{g(a) + g(b)}{2}.
\end{align*}
\]

Hence any \( g \), a generating function of \( G \), is midpoint convex on \([0, N]\) for all \( N \). It is well-known that every such function is convex if it is continuous. We thus select a continuous generating function \( g \), whose existence is guaranteed by Proposition 12. Such resulting \( g \) is then both continuous at \( c = 0 \) and convex. Note that \( g \) is an arbitrary continuous generating function, so all continuous generating functions \( g \) are convex. This is needed in the proof of (ii) that follows.

(ii) Proposition 8 implies additive separability of \( M \in \mathcal{M} \) and the rest of the demonstration is contained in the proof of (i)(\( \Rightarrow \)) above.

We are now in a position to describe the set \( \mathcal{M} \), specified by Definition 11 explicitly.

**Theorem 14 (Set of co-ENFs)**

\( M \in \mathcal{M} \) if and only if it is generated by a convex and continuous function \( m \), which is zero on \([1, \infty)\) and \( m(0) = 1 \). Such a generating function \( m \) of \( M \) is unique.

**Proof:**

(\( \Rightarrow \)) \( M \in \mathcal{M} \) is additively separable by Proposition 8. Then, as a consequence of additivity (A) and boundary conditions (co-B2), we have \( M(0, \ldots, 0, N) = (N - 1) \cdot m(0) + m(N) = N - 1 \), so that

\[
m(N) = (N - 1) \cdot (1 - m(0))
\]

(17) for all \( N \). Given the continuity of \( M \in \mathcal{M} \), Proposition 12 guaranties the existence of its unique continuous generating function with \( m(0) = 1 \). In conjunction with Eq. (17), this implies that \( m(N) = 0 \) for all \( N \). Furthermore, this continuous generating function is convex by (ii) of Proposition 13 and, consequently, it is zero on the entire \([1, \infty)\). This demonstrates the existence of unique \( m \) with all required properties.

(\( \Leftarrow \)) For the opposite direction, let \( M[C] = \sum m(c_i) \), where \( m \) is continuous, convex, \( m(0) = 1 \), and \( m(c) = 0 \) on \([1, \infty)\). Then (A), (C), (S) and (co-B2) follow immediately, while (M') is a consequence of Proposition 13(i). \( \square \)

Note that the unique choice of continuous generating function for co-ENF has been facilitated by a natural choice \( m(0) = 1 \), expressing the fact that the object assigned zero probability should not contribute to the effective number total (\( n(0) = 0 \)). However, it is worth pointing out that, as shown below, the same unique choice of a generating function is selected by the requirement of boundedness on entire \([0, \infty)\).

**Corollary 15** Let \( m \) be the generating function of \( M \in \mathcal{M} \), specified in Theorem 14. Then

(i) \( 0 \leq m(c) \leq 1 \) for all \( c \)

(ii) \( m \) is the only generating function of \( M \) that is bounded on its whole domain \([0, \infty)\).

**Proof:**

(i) This immediately follows from \( m(0) = 1, m(c) = 0 \) on \([1, \infty)\), and convexity.

(ii) Boundedness of \( m \) follows from (i). To demonstrate uniqueness, assume there is another bounded generating function \( m_1 \) of \( M \). Thus, \( m_1 - m \) satisfies the assumptions of Lemma 11(ii), implying the existence of non-zero \( K_0 \) such that \( m_1(c) = m(c) + K_0(1 - c) \) for \( c \in [0, \infty) \). However, this contradicts the boundedness of \( m_1 \) which demonstrates the claimed uniqueness. \( \square \)

Below we will make use of the following obvious lemma and a simple corollary.

**Lemma 16** Let \( G_1 \) and \( G_2 \) be real additively separable functions on \( C \). If there exist respective generating functions such that \( g_1(c) \leq g_2(c) \), for all \( c \), then \( G_1(C) \leq G_2(C) \), for all \( C \in C \).
**Corollary 17** If $M \in \mathfrak{M}$, then $0 \leq M[C] \leq N - 1$, for all $C \in \mathcal{C}$.

Proof:

Let $m$ be the generating function specified in Theorem 14. From (i) of Corollary 15 we have $\sum_i 0 \leq \sum c_i \leq \sum_i 1$, which translates into $0 \leq M[C] \leq N$ by Lemma 16. To put the second inequality into the claimed form, note that there is always at least one $c_j \geq 1$. For this $c_j$, we have $m(c_j) = 0$ by Theorem 14. This lowers the upper bound for $M[C]$ by unity and proves the second inequality. \hfill \Box

Using the above preparation, we will now demonstrate several structural properties of $\mathfrak{M}$.

**Theorem 18 (Maximality)**

If $M \in \mathfrak{M}$ then the following holds for all $C = (\ldots, c_i, \ldots) \in \mathcal{C}$

(i) $M(0)[C] = M_{+}[C] \leq M[C] \leq M_{*}[C] = M(1)[C]$

(ii) $M_{+}[C] = M[C] = M_{*}[C] \iff c_i \notin (0, 1)$, $i = 1, 2, \ldots, N$

(iii) $\beta_0 = M_{+}[C] < M_{*}[C] = \beta_1 \Rightarrow \{M[C] : M \in \mathfrak{M}\} = [\beta_0, \beta_1]$

Proof:

(i) To show both inequalities, let $m$ be the continuous generating function of $M$ guaranteed by Theorem 14. We will show that $m$ satisfies $m_{+}(x) \leq m(x) \leq m_{*}(x)$ on $[0, \infty)$ and then Lemma 16 will complete the proof of this part. The first inequality $m_{+}(x) \leq m(x)$ follows directly from Theorem 14 Corollary 15(i), and the definition of $m_{+}$. The second inequality holds as equality on $[1, \infty)$ by Theorem 14 and the definition of $m_{*}$. To show the second inequality on $[0, 1)$, note that the graphs of both $m$ and $m_{*}$ pass through the points $(0, 1)$ and $(1, 0)$, $m$ is convex by Proposition 13(ii), and $m_{*}$ is linear between those points. So $m_{+}(x) \leq m(x) \leq m_{*}(x)$ on $[0, \infty)$ as promised.

(ii) $(\Leftarrow)$ If $c_i \notin (0, 1)$ for $i = 1, 2, \ldots, N$, then $M_{+}[C] = M_{*}[C]$ and the equality for $M[C]$ follows from (i).

$(\Rightarrow)$ Let $M_{+}[C] = M_{*}[C]$. Then

$$\sum_{c_i = 0} 1 = \sum_{c_i = 0} 1 + \sum_{c_i \in (0, 1)} (1 - c_i).$$

Hence $0 = \sum_{c_i \in (0, 1)} (1 - c_i)$ and so $c_i \notin (0, 1)$ for $i = 1, 2, \ldots, N$.

(iii) From $M_{+}[C] < M_{*}[C]$ we have the existence of at least one $c_i \in (0, 1)$ by (ii). Then for a fixed $C$ define

$$g(\alpha) = \sum_{c_i = 0} 1 + \sum_{c_i \in (0, 1)} (1 - c_i^\alpha) = M(\alpha)[C].$$

The function $g$ is continuous, increasing, and maps interval $[0, 1]$ onto interval $[\beta_0, \beta_1]$. Then

$$\{M[C] : M \in \mathfrak{M}\} \supseteq \{M(\alpha)[C] : \alpha \in [0, 1]\} = [M_{+}[C], M_{*}[C]] = [\beta_0, \beta_1].$$

The opposite inclusion follows from (i). \hfill \Box

**III. COUNTING THE GENERAL QUANTUM IDENTITIES**

With the effective number theory in place, we now return to the topic that motivated its construction, namely the quantum identity problem. In particular, we will make explicit some of the straightforward but useful and relevant generalizations of $[Q]$. This serves, in part, as a stepping stone toward the most generic application of effective numbers in quantum theory, namely as a tool to characterize quantum states (Sec. IV).

Conceptually important extension of $[Q]$ and $[A]$ is made possible by additive separability of ENFs. Indeed, instead of an orthonormal basis, consider any collection $\{|j\rangle\}$ of $n$ orthonormal states from $N$-dimensional Hilbert space ($1 \leq n \leq N$). How many states from $\{|j\rangle\}$ is a system described by $|\psi\rangle$ effectively in? Let $N \in \mathfrak{N}$ be an ENF and $n$ the counting function uniquely
associated with it by virtue of Theorem 1. We define

\[
N[|\psi\rangle, \{|i\}\}] = \sum_{j=1}^{n} n(c_j) \quad , \quad c_j = N(|\langle j|\psi\rangle|^2)
\]

(18)

for each $|\psi\rangle$ and $\{|i\}\}$. This assignment is meaningful in the following sense. Given a fixed $\{|i\}\}$, let $\{|i\}$ be its arbitrary completion into a basis of the Hilbert space. Then, owing to additive separability of ENFs,

\[
N[|\psi\rangle, \{|i\}\}] = N[|\psi\rangle, \{|j\}\}] + N[|\psi\rangle, \{|i\}\} \setminus \{|j\}\}]
\]

(19)

where “\setminus” denotes the set subtraction. In other words, the contribution of $\{|j\}$ to $N[|\psi\rangle, \{|i\}\}]$, defined by (18), is independent of the completion $\{|i\}$, and is thus uniquely associated with $\{|j\}$ for fixed $N \in \mathbb{N}$. It is the effective number of states from $\{|j\}$ contained in $|\psi\rangle$ according to $N$. Moreover, it is straightforward to check that $N[|\psi\rangle, \{|j\}\}]$ minimizes the effective number so assigned, and we have

[A'] Let $|\psi\rangle$ be a state vector from $N$-dimensional Hilbert space and $\{|j\}\} \equiv \{|j\} \mid j = 1, 2, \ldots, n \leq N \}$ the set of $n$ orthonormal states in this space. The physical system described by $|\psi\rangle$ is effectively in $N[|\psi\rangle, \{|j\}\}]$ states from $\{|j\}\}$, specified by (18) with $n = n_\ast$.

Few simple points regarding the above are worth emphasizing.

(i) The extension (18) and the ensuing generalization of [A] to [A'] arises because the abundance of quantum identities is determined “locally”, namely without reference to basis elements orthogonal to the subspace in question. Apart from being natural for a measure-like characteristic, this feature has practical consequences in many-body applications where the dimension of Hilbert space grows exponentially with the size of the system. Indeed, the above avoids such complexity in certain calculations, thus providing a computational benefit.

(ii) None of the above applies to the abundance of quantum identities determined by the participation number $N_p$, of Eq. (4), since this value doesn’t split into contributions from orthogonal subspaces generated by the partitioned basis. This is of course due to the lack of additivity and hence of additive separability.

(iii) The above considerations are clearly not limited to counting quantum identities. In a generic situation, the inquiry is concerned with the contribution to effective number from a subset of weighted objects. To formalize such assignments directly in the effective number theory, one simply extends $N \in \mathbb{N}$ from domain $\mathcal{C}$ of counting vectors to the domain of general weights

\[
\mathcal{W} = \bigcup_n \mathcal{W}_n \quad , \quad \mathcal{W}_n \equiv \{ W = (w_1, w_2, \ldots, w_n) \mid w_j \geq 0 \}
\]

(20)

by virtue of its counting function $n$, namely $N[W] = \sum_j n(w_j)$.

While the effective number $N[|\psi\rangle, \{|i\}\}]$ specifies how many identities from basis $\{|i\}$ can the state $|\psi\rangle$ effectively take, it is frequently useful to inquire about a coarse-grained version of such effective count. To formalize the corresponding generalization, consider the decomposition of Hilbert space $\mathcal{H}$ into $M$ mutually orthogonal subspaces $\{\mathcal{H}_m\}$

\[
\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_M \quad , \quad \{\mathcal{H}_m\} \equiv \{ \mathcal{H}_m \mid m = 1, 2, \ldots, M \}
\]

(21)

Subspaces $\mathcal{H}_m$ are implicitly treated as equivalent entities. How many subspaces from $\{\mathcal{H}_m\}$ is the system described by $|\psi\rangle$ effectively in?

Given a state of the system, the rules of quantum mechanics assign probability to each subspace of the associated Hilbert space. The effective number theory therefore provides an immediate answer to the above question. More specifically, let $|\chi_m\rangle$ be the (unnormalized) projection of $|\psi\rangle$ onto $\mathcal{H}_m$. Then the probability vector is $P = (p_1, p_2, \ldots, p_M)$, where $p_m = \langle \chi_m |\chi_m\rangle$, and the corresponding counting vector is $C = MP$. Hence, given an ENF, we assign $N[|\psi\rangle, \{\mathcal{H}_m\}] \equiv N[C]$, which leads to the following generalization $[A_\ast]_g$ of [A].

[A_\ast] Let $C$ be the counting vector assigned by quantum mechanics to state $|\psi\rangle$ and the orthogonal decomposition $\{\mathcal{H}_m\}$ of the Hilbert space. Then the system described by $|\psi\rangle$ is effectively contained in $N[|\psi\rangle, \{\mathcal{H}_m\}] = N[|\psi\rangle, \{\mathcal{H}_m\}]$ subspaces from $\{\mathcal{H}_m\}$.

Applying the logic identical to one producing $[A']$, it is straightforward to generalize $[A_\ast]$ into $[A_\ast']$ for counting the identities from arbitrary sets (not necessarily full decompositions) of mutually orthogonal subspaces.

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18 What is considered “equivalent” is dictated by the physics involved, rather than the concept of effective number itself. For example, it is possible to encounter a situation where dimensions of “usefully equivalent” subspaces are not the same.
IV. APPLICATION: THE STRUCTURE OF QUANTUM STATES

While our discussion was carried out in the context of quantum identity problem, the constructed effective number framework clearly offers a much larger scope of uses. Apart from physics and mathematics, these also appear in the areas of applied science simply due to the very basic role of measure and probability in quantitative analysis. Examples of such applications will be discussed in the follow-up works. Here we describe a broader outlook on the utility of effective numbers in quantum theory.

With the quantum state encoding all “options” for the system, the physical content of \(|\psi\rangle\) closely relates to all probability vectors \(P\) induced in this manner. Retracing the steps leading to \([A]\), it is then meaningful to associate the effective number \(N_*\) with any complete set of mutually exclusive possibilities. We propose the collection of all such reductions

\[
|\psi\rangle \rightarrow P \rightarrow N_*
\]

as a systematic and physically relevant characterization of a quantum state. The general quantum identities of Sec. [III], which can also be thought of as measurement outcomes, are prime examples of objects/possibilities in question. However, any meaningful \(|\psi\rangle \rightarrow P\) can be considered. The resulting variety offers a wide range of options for targeted insight into the structure of \(|\psi\rangle\).

We wish to highlight few points in this regard.

(i) Each effective number characteristic can be refined by totals assigned to subsets of the associated sample space or coarse-grained by counting its partitions. Note that for quantum identities in \([Q]\) this corresponds to refining by totals involving parts of the basis and coarse-graining by totals involving orthogonal decompositions.

(ii) The proposed approach is universal with respect to the nature of the quantum system being described. Indeed, \(|\psi\rangle\) may be as simple as a state of harmonic oscillator, but as complex as a many-body state of quantum spins or the vacuum of non-Abelian gauge theory.

(iii) Our present focus on the finite discrete case is not restrictive either. Even in situations involving the space-time continuum, the intermediate steps of ultraviolet (lattice) and infrared (volume) regularizations yield such descriptions. Removing the cutoffs then involves a suitable \(N \rightarrow \infty\) limit. For this purpose, it is more convenient to work with effective fraction rather than effective number. In terms of probabilities, it is defined as

\[
\mathcal{F}[P] \equiv \frac{1}{N} N[P] \quad , \quad N \in \mathcal{N} \quad , \quad P \in \mathcal{P}_N
\]

Function \(N_*\) produces the minimal effective fraction, namely

\[
\mathcal{F}_*[P] = \sum_{i=1}^{N} f_*(p_i, N) \quad , \quad f_*(p, N) \equiv \min \{ p, 1/N \}
\]

Restricting ourselves to quantum identities of \([Q]\), the cutoff removal schematically proceeds as follows. Assume that \(|\psi\rangle\) is the target state (from infinite-dimensional Hilbert space) to be assigned an effective fraction in basis \(\{|i\}\). Let \(|\psi^{(k)}\rangle\) represents \(|\psi\rangle\) at \(k\)-th step of the regularization process, involving the state space of increasing dimension \(N_k\). If the latter is spanned by \(\{|i^{(k)}\}\) targeting \(\{|i\}\) then

\[
\mathcal{F}_*[|\psi\rangle, \{ |i\} \}] = \lim_{k \rightarrow \infty} \mathcal{F}_*[|\psi^{(k)}\rangle, \{ |i^{(k)}\}\}] = \lim_{k \rightarrow \infty} \mathcal{F}_*[P_k]
\]

where \(P_k = (p_1^{(k)}, \ldots, p_{N_k}^{(k)}), p_i^{(k)} = \langle i^{(k)} | \psi^{(k)} \rangle)^2\). Note that \(\mathcal{F}_*\) inversely reflects “localization” of \(|\psi\rangle\) in \(\{|i\}\).

(iv) Example of an application where the assignment (22) is carried out without direct reference to the underlying Hilbert space is provided by the problem of vacuum structure in Quantum Chromodynamics (QCD). This is often studied in the Euclidean path integral formalism, with regularized vacuum represented by the statistical ensemble of lattice gauge configurations \(U = \{U_{x,\mu}\}\). Given a composite field \(O = O(x, U)\) and the induced space-time probability distribution \(P(x, U) \propto |O(x, U)|^2\), effective number framework can be used to determine the effective fraction of space-time occupied by \(O\) for each \(U\). The corresponding quantum averages are of vital interest in this context. Yet more indirect vacuum characteristics of such type reflect the space-time properties of Dirac eigenmodes. Their main utility is in probing the features of quark dynamics.

\[19\] The meaning of \(\{|i^{(k)}\}\) is \(\{|i^{(k)}\} \equiv \{|i\} \mid i = 1, 2, \ldots, N_k\} \) targeting \(\{|i\}\) depends on the context but is usually clear on physics grounds.
V. CONCLUDING REMARKS

Emergence of a quantum system in one of many possible “identities” upon (strong) probing is among the key features of quantum behavior. Indeed, it underlies the notion of quantum uncertainty and is closely connected to a fruitful concept of localization. A well-founded prescription for the corresponding abundances is thus desirable. As a contemporary example, one may use it in the analysis of a quantum algorithm which produces the state \( |\psi\rangle_o \) as an output of quantum computation, and follows up with a measurement involving the basis \( \{ |i\rangle \} \). The effective abundance of distinct collapsed states \( |i\rangle \) obtained upon repetition of these steps is relevant for the assessment of algorithm efficiency.

In this work, we showed that requiring the desired characteristics to be measure-like (additive) produces meaningful answers. It results in the theoretical structure (effective number theory) revealing that a consistent assignment of totals to collections of objects with probability weights requires the existence of an inherent (minimal) amount \( N_* \). The appearance of such qualitative feature in basic measure considerations suggests the utility of the constructed framework already in contexts much less abstract than quantum mechanics. For example, the effective number theory can be viewed as an extension of the ordinary counting measure to what can be referred to as a diversity measure \( N_* \), with its wide range of contemporary applications (social sciences, ecosystems; see e.g. \[9\]). Other viewpoints can cast it as a choice measure, facilitating a probabilistic notion of effective choices, or as a support measure, conveying the effective size of a function support (effective domain) \[5\]. Given this universality, \( N_* \) may find uses in multiple areas of quantitative science.

Finally, we wish to point out that the existence of minimal amount is rooted in the simultaneous requirement of monotonicity (Schur concavity) and additivity for ENFs. This combination is rather unusual from the mathematics standpoint. Indeed, while monotonicity is important for the theory of majorization \[10\], it has no role in the standard formalization of measure. Conversely, additivity is crucial for the latter but not native to the former. In fact, relaxing either (M) or (A) in Definition 0 leaves the respective effective pseudo-number assignments too arbitrary. But their combination, which is necessary on conceptual grounds, leads to \( N_* \) and the associated insight.

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