Moments of Central $L$-values for Maass Forms over Imaginary Quadratic Fields

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Abstract. In this paper, over imaginary quadratic fields, we consider the family of $L$-functions $L(s, f)$ for an orthonormal basis of spherical Hecke–Maass forms $f$ with Archimedean parameter $t_f$. We establish asymptotic formulae for the twisted first and second moments of the central values $L(\frac{1}{2}, f)$, which can be applied to prove that at least 33% of $L(\frac{1}{2}, f)$ with $t_f \leq T$ are non-vanishing as $T \to \infty$. Our main tools are the spherical Kuznetsov trace formula and the Voronoï summation formula over imaginary quadratic fields.

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1. Introduction

A recurring theme in analytic number theory is the study of central value of a family of \( L \)-functions. In this paper, we prove asymptotic formulae for the twisted first and second moments of central \( L \)-values for the family of Hecke–Maass cusp forms over the classical modular group \( \text{PGL}_2(\mathbb{Z}) \) or a Bianchi modular group \( \text{PGL}_2(O) \) (here \( O \) is the ring of integers of an imaginary quadratic field \( F \)). As a standard application, we obtain non-vanishing results for the central value of such Maass form \( L \)-functions in the Archimedean aspect.

There are abundant non-vanishing results for holomorphic modular forms over \( \mathbb{Q} \) in [Duk, IS, KMI, KM2, Van, KMV, Dja, Rou1, Rou2, BF1, LT, Luo, BF2, Liu2, Job] and also over a totally real field in [Tro].

Recently there are two papers [Liu1] and [BHS] on the non-vanishing of central \( L \)-value for the family of Maass forms for \( \text{PGL}_2(\mathbb{Z}) \) in the aspect of spectral parameter \( t_f \); in the former, the existence of a positive proportion of non-vanishing is proven for eigenvalues in short intervals, while in the latter, a lower bound for the proportion is obtained effectively. In both works, a formula of Motohashi\(^1\) (see [Mot1, Lemma] or [Mot2, Lemma 3.8]) is used for the twisted second moment, but the authors of [BHS] are able to obtain an asymptotic formula so that their effective non-vanishing result becomes possible.

In this paper, we use the formula of Kuznetsov instead of Motohashi. The reader might wonder: “What is new here? The Kuznetsov formula has already been used for a lot of problems.” To illustrate the novelty of this work, we need to answer two questions:

1. Why do the previous authors abandon the Kuznetsov formula?
2. Why do we abandon the Motohashi formula?

There are two Kuznetsov formulae for \( \text{PSL}_2(\mathbb{Z}) \) (see [Kuz1, Kuz2] or [Mot2, Theorem 2.2, 2.4]): weighted either with or without root number \( \epsilon_f \) and containing either the \( J \)- or the \( K \)-Bessel function. The Kuznetsov formula for \( \text{PGL}_2(\mathbb{Z}) \) is deduced from summing up these two formulae (Maass forms for \( \text{PGL}_2(\mathbb{Z}) \) are termed even forms in the literature).

In [Mot2, §3.3], Motohashi derives his formula from the Kuznetsov formula for \( \text{PSL}_2(\mathbb{Z}) \) that is weighted by \( \epsilon_f \) and contains \( K_{2it}(x) \). A simple but crucial observation is that \( L(\frac{1}{2}, f) = 0 \) if \( \epsilon_f = -1 \). He avoids using the other Kuznetsov formula with \( J_{2it}(x) \), as “the relevant transformation is difficult to handle because its integrand is not of rapid decay”.

In [Liu], Shenhui Liu follows Wenzhi Luo’s mollification analysis for the holomorphic case in [Luo]. In his Introduction, he lists several advantages of Motohashi’s formula for the mollified second moment and explains that there is a “deeper reason for using Motohashi’s formula”: If one were to use the Kuznetsov formula, it would not be easy “to extract information from the off-diagonal terms by using properties of Estermann zeta-functions”, “since the Mellin–Barnes representation of \( J_{2it}(x) \) gives very narrow room for contour shifting”.

A direct approach by the Kuznetsov formula is certainly of its own interests and merits. A more general goal on our mind is to obtain a positive proportion for the non-vanishing problem over an imaginary quadratic field \( F \). However, there is currently no Motohashi formula over a field other than \( \mathbb{Q} \). Indeed, the root number \( \epsilon_f \) is always +1 for any spherical Maass form \( f \) over \( \text{PSL}_2(\mathbb{O}) \), so Motohashi’s idea

\(^1\) As indicated in [Mot2, §3.6], this formula was claimed by Kuznetsov [Kuz2] with no rigorous proof. It should therefore be called the Kuznetsov–Motohashi formula. Nevertheless, to avoid confusion, we shall still name it after Motohashi.
does not work here directly. At any rate, one must reconsider the problem over \( \mathbb{Q} \) and find a way to solve it without recourse to the Motohashi formula.

Our idea is to bypass the difficulties encountered in [Mot2 §3.3] and [Liu1] by

1. applying the Voronoï summation as a substitute of the functional equation for Estermann zeta-functions, and
2. applying the Fourier-type representation instead of the Mellin–Barnes representation for Bessel functions.

The Voronoï summation has occurred in the case of holomorphic modular forms in [Hou, BF2], but their analyses are quite different from ours. A key feature of our analysis is a uniform treatment of the integrals involving \( J_{2\ell}(x) \) and \( K_{2\ell}(x) \) which can be extended to the complex setting. More explicitly, we shall have Fourier integrals in the off-diagonal terms, and the problem will be reduced to estimating the area of certain regions defined via hyperbolic or trigonometric-hyperbolic functions.

**Statement of Results.** Let \( F = \mathbb{Q} \) or an imaginary quadratic field \( \mathbb{Q}(\sqrt{d_F}) \) of discriminant \( d_F \) and class number \( h_F = 1 \). Let \( \mathcal{O} \) be its ring of integers. Let \( N = 1 \) or \( 2 \) be the degree of \( F \). For a nonzero integral ideal \( n \subset \mathcal{O} \) let \( N(n) = |\mathcal{O}/n| \) be its norm and \( \tau(n) = \sum_{\mathcal{O}/n} 1 \).

Let \( \gamma_0 \) and \( \gamma_1 \) respectively be the constant term and the residue of Dedekind’s \( \zeta_F(s) \) at \( s = 1 \). Define \( c_1 = 1/8\pi^2 \) or \( \sqrt{|d_F|}/8\pi^3 \) according as \( F = \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{d_F}) \). Moreover, let \( \theta = \frac{13}{84} \) and \( \theta' = \frac{9}{56} \) (see (2.3)), and for \( 0 < \beta \leq 1 \) define

\[
\alpha_1 = \begin{cases} 0, & \text{if } \frac{1273}{4053} + \varepsilon \leq \beta \leq 1, \\ 2\theta, & \text{if } 0 < \beta < \frac{1273}{4053} + \varepsilon, \end{cases} \quad \alpha_2 = \begin{cases} 0, & \text{if } \frac{7}{3} + \varepsilon \leq \beta \leq 1, \\ 2\theta, & \text{if } \frac{1273}{4053} + \varepsilon \leq \beta < \frac{7}{3} + \varepsilon, \\ 4\theta, & \text{if } 0 < \beta < \frac{1273}{4053} + \varepsilon, \end{cases}
\]

if \( F = \mathbb{Q} \), and

\[
\alpha_1 = \begin{cases} 0, & \text{if } \frac{7}{8} + \varepsilon \leq \beta \leq 1, \\ 2\theta', & \text{if } 0 < \beta < \frac{7}{8} + \varepsilon, \end{cases} \quad \alpha_2 = \begin{cases} 2\theta', & \text{if } \frac{7}{8} + \varepsilon \leq \beta \leq 1, \\ 4\theta', & \text{if } 0 < \beta < \frac{7}{8} + \varepsilon, \end{cases}
\]

if \( F = \mathbb{Q}(\sqrt{d_F}) \).

Let \( \mathcal{B} \) be an orthonormal basis consisting of Hecke–Maass cusp forms for the spherical cuspidal spectrum for \( \text{PGL}_2(\mathcal{O}) \). For \( f \in \mathcal{B} \), let \( t_f \in [0, \infty) \cup i[0, \frac{1}{2}) \) be its Archimedean parameter, \( \lambda_f(n) \) be its Hecke eigenvalues, \( L(s, f) \) and \( L(s, \text{Sym}^2 f) \) respectively be its standard and symmetric square \( L \)-functions. For any sequence of complex numbers \( a_f \) we introduce the harmonic summation

\[
\sum_{f \in \mathcal{B}} h_f a_f = \sum_{f \in \mathcal{B}} \omega_f a_f, \quad \omega_f = \frac{1}{2L(1, \text{Sym}^2 f)}.
\]

For large \( T > M \) we define

\[
k(t) = e^{-(t-T)^2/M^2} + e^{-(t+T)^2/M^2},
\]

and for \( q = 1 \) or \( 2 \) we introduce the smoothly weighted twisted moments:

\[
\mathcal{M}_q(m) = \sum_{f \in \mathcal{B}} k(t_f) \lambda_f(m) L\left(\frac{1}{2}, f\right)^q.
\]
THEOREM 1.1. Define $\gamma_0' = \gamma_0 - \gamma_1 \log \left( (2\pi)^N / |d_F| \right)$. Let $M = T^\beta$ with $\varepsilon \leq \beta \leq 1 - \varepsilon$. Then

\begin{equation}
\mathcal{M}_1(m) = 4\sqrt{\pi} c_1 \frac{MT^N}{\sqrt{N(m)}} \left( 1 + O_\varepsilon \left( (M/T)^2 \right) \right) + O_\varepsilon \left( MT^{N\alpha_1 + \varepsilon} \right),
\end{equation}

for $N(m) \leq T^{N-\varepsilon}$, and

\begin{equation}
\mathcal{M}_2(m) = 8\sqrt{\pi} c_1 \tau(m)MT^N \left( \gamma_1 \log \frac{T^N}{\sqrt{N(m)}} + \gamma_0' + O_\varepsilon \left( MT^{\varepsilon/T} \right) \right) + O_\varepsilon \left( MT^{N\alpha_2 + \varepsilon} + \frac{\sqrt{N(m)T^{N/2 + \varepsilon}}}{M^{1-N/2}} \right),
\end{equation}

for $N(m) \leq T^{2N-\varepsilon}$. Moreover, if $F = \mathbb{Q}$, then the error term $O_\varepsilon \left( MT^{\varepsilon/T} \right)$ in the first line of (1.13) may be improved into $O_\varepsilon \left( (M/T)^2 \log T \right)$ and the second error term in the second line may be removed if $N(m) \leq M^{2-\varepsilon}$.

The error terms are always inferior to the main term in (1.10) as long as $N(m) \leq T^{N-\varepsilon}$, while this holds for (1.11) as long as

\begin{equation}
N(m) \leq \min \left\{ M^{2-N/2}T^{N-\varepsilon}, T^{2N(1-\alpha_2)-\varepsilon} \right\}.
\end{equation}

When $F = \mathbb{Q}$, with cleaner error terms, (1.10) and (1.11) are essentially Theorem 4.1 and 5.1 in [BHS] prior to the averaging process for the $T$-parameter.

For $T^\varepsilon \leq 3H \leq T$ define

\begin{equation}
\mathcal{N}_q(T, H) = \sum_{|\gamma_1| \leq H} L(\frac{1}{2}, f)^q + \frac{1}{2\pi} \gamma_1 \int_{T-H}^{T+H} \frac{\zeta_F \left( \frac{1}{2} + it \right)^2}{\zeta_F \left( \frac{1}{2} + 2it \right)} dt.
\end{equation}

Then we have simpler asymptotic formulae for $\mathcal{N}_q(T, H)$ as follows.

COROLLARY 1.2. For $F = \mathbb{Q}$ we have

\begin{equation}
\mathcal{N}_1(T, H) = \frac{1}{\pi^2} HT + O_\varepsilon \left( T^{1+\varepsilon} \right),
\end{equation}

and

\begin{equation}
\mathcal{N}_2(T, H) = \frac{1}{\pi^2} \int_{T-H}^{T+H} K(\log K + \gamma_0') dK + O_\varepsilon \left( T^{1+\varepsilon} \right).
\end{equation}

For $F = \mathbb{Q}(\sqrt{d_F})$ we have

\begin{equation}
\mathcal{N}_1(T, H) = \frac{\sqrt{|d_F|}}{3\pi^3} (3HT^2 + H^3) + O_\varepsilon \left( T^{2+\varepsilon} \right),
\end{equation}

and

\begin{equation}
\mathcal{N}_2(T, H) = \frac{\sqrt{|d_F|}}{\pi^3} \int_{T-H}^{T+H} K^2(2\gamma_1 \log K + \gamma_0') dK + O_\varepsilon \left( T^{2+\varepsilon} \right).
\end{equation}

In the case $H = T/3$, the formulae (1.10) and (1.11) should be compared with [LI] Theorem 1 and [Mot1] Theorem 2.

As a consequence of the mollification technique as in [IS], [KMV], one may derive from the asymptotic formulae in Theorem 1.1 the following effective lower bound for the proportion of non-vanishing $L(\frac{1}{2}, f)$.
THEOREM 1.3. For any $\varepsilon > 0$ and sufficiently large $T$, we have

$$\sum_{|t_j - T| \leq H} h \quad \geq \quad \left( \frac{\Delta}{1 + \Delta} - \varepsilon \right) \sum_{|t_j - T| \leq H} 1,$$

where $3H = T^\beta$ with $\varepsilon \leq \beta \leq 1$ and

$$\Delta \leq \min \left\{ 1 - \alpha_2, \frac{1}{4} + \left( \frac{1}{N} - \frac{1}{4} \right) \beta \right\}.$$  

For $F = \mathbb{Q}$ this is essentially Theorem 1.2 in [BHS]. For $F = \mathbb{Q}(\sqrt{d_F})$ it follows from almost the same arguments in [BF2, §8] and [BHS, §7]. As such, we omit the details of proof and only remark that the limitation (1.15) comes from the inequality (1.8). To avoid extra work on $L(1,\text{Sym}^2 f)$, we allow the harmonic weight to be present (see the paragraph below (2.9) in [IS]).

The following results follow if we choose $H = T/3$ in Theorem 1.3 and use a dyadic partition.

COROLLARY 1.4. We have

$$\sum_{|t_j - T| \leq T} h \quad \geq \quad \left( \frac{1}{2} - \varepsilon \right) \sum_{|t_j - T| \leq T} 1$$

if $F = \mathbb{Q}$, and

$$\sum_{|t_j - T| \leq T} h \quad \geq \quad \left( \frac{1}{3} - \varepsilon \right) \sum_{|t_j - T| \leq T} 1$$

if $F = \mathbb{Q}(\sqrt{d_F})$.

Finally, we remark that, with some efforts, our results can be extended to an arbitrary imaginary quadratic field (see [Qi3]).

**Notation.** By $X \ll Y$ or $X = O(Y)$ we mean that $|X| \leq cY$ for some constant $c > 0$, and by $X \asymp Y$ we mean that $X \ll Y$ and $Y \ll X$. We write $X \ll p, q, \ldots$ or $X = O_p, q, \ldots (Y)$ if the implied constant $c$ depends on $P, Q, \ldots$. We say that $X$ is negligibly small if $X = O_A(T^{-A})$ for arbitrarily large but fixed $A > 0$.

We adopt the usual $\varepsilon$-convention of analytic number theory; the value of $\varepsilon$ may differ from one occurrence to another.

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character $\psi_F(x) = e(-x)$ if $F_x = \mathbb{R}$ and $\psi_F(z) = e(-(z + \bar{z}))$ if $F_x = \mathbb{C}$. We choose the Haar measure $dx$ of $F_x$, self-dual with respect to $\psi_F$: the Haar measure is the ordinary Lebesgue measure on the real line if $F_x = \mathbb{R}$, and twice the ordinary Lebesgue measure on the complex plane if $F_x = \mathbb{C}$.

In general, we use Gothic letters $a, b, m, n, \ldots$ to denote nonzero integral ideals of $F$. Let $p$ always stand for a prime ideal. Let $N(a)$ denote the norm of $a$.

### 2.2. Arithmetic Functions

Let $\tau(n)$ and $\mu(n)$ be the divisor function and the Möbius function. For $s \in \mathbb{C}$ define

\begin{equation}
\tau_s(n) = \tau_{-s}(n) = \sum_{ab=n} N(ab^{-1})^s.
\end{equation}

### 2.3. Kloosterman and Ramanujan Sums

For $m, n \in \mathcal{O}$ and $c \in \mathcal{O}$ we define

\begin{equation}
S(m, n; c) = \sum_{a \in (\mathcal{O}/c)^*} \psi_F\left(\frac{ma + n\bar{a}}{c}\right),
\end{equation}

where $\bar{a}$ is the multiplicative inverse of $a$ modulo $c$. The sum $S(m, 0; c)$ is usually named after Ramanujan. We have

\begin{equation}
S(m, 0; c) = \sum_{d | (m, 2c)} N(d)\mu(d)c^{-1}.
\end{equation}

### 2.4. The Dedekind Zeta Function

Let $\zeta_F(s)$ be the Dedekind $\zeta$ function for $F$:

\begin{equation}
\zeta_F(s) = \sum_{n \in \mathcal{O}} \frac{1}{N(n)^s}, \quad \text{Re}(s) > 1.
\end{equation}

It is well-known that $\zeta_F(s)$ is a meromorphic function on the complex plane with a simple pole at $s = 1$. Define the constants $\gamma_0$ and $\gamma_1$ by

\begin{equation}
\zeta_F(s) = \frac{\gamma_1}{s - 1} + \gamma_0 + O(|s - 1|), \quad s \to 1.
\end{equation}

For $F = \mathbb{Q}(\sqrt{d_F})$ we have $\zeta_F(s) = \zeta(s)L(s, \chi_{d_F})$ with $\chi_{d_F}$ the primitive quadratic character associated to $F$.

Let $\theta > 0$ be a sub-convex exponent for $\zeta_F(s)$; namely,

\begin{equation}
\zeta_F\left(\frac{1}{2} + it\right) \ll_{\varepsilon} (1 + |t|)^{N\theta + \varepsilon}
\end{equation}

for any $\varepsilon > 0$. For the Riemann $\zeta(s)$ the best sub-convex exponent to date $\theta = \frac{13}{84}$ is due to Bourgain [Bou]. This together with the Weyl sub-convex bound for $L(s, \chi_{d_F})$ (see for example [HB1]) yields $\theta = \frac{2}{30}$ in the case $F = \mathbb{Q}(\sqrt{d_F})$. It should be remarked that for arbitrary $F$ the Weyl exponent $\theta = \frac{1}{6}$ is always admissible [HB2].

### 3. Automorphic Forms on $GL_2$

In this section, we briefly compile some results and introduce the relevant notation from the theory of spherical automorphic forms on $\text{PGL}_2(\mathcal{O})/\text{PGL}_2(F_x)/K_x$, where $K_x = O_2(\mathbb{R})/\{\pm I_2\}$ or $U_2(\mathbb{C})/\{\pm I_2\}$ according as $F_x = \mathbb{R}$ or $\mathbb{C}$, especially the Kuznetsov trace formula and the Voronoï summation formula (for Eisenstein series). The reader is referred to [Q13, Q15, Ven] for further details.
3.1. Archimedean Representations. In this paper, we shall be concerned only with spherical representations of $\text{PGL}_2(F_\infty)$. By definition, an irreducible representation of $\text{PGL}_2(F_\infty)$ is spherical if it contains a nonzero $K_\infty$-invariant vector.

Let $Y = (-\infty, \infty) \cup i(-\frac{1}{2}, \frac{1}{2})$. We associate to $t \in Y$ a unique spherical unitary irreducible representation $\pi(it)$ of $\text{PGL}_2(F_\infty)$. Namely, the parameter $t$ determines a character of the diagonal torus via

$$
\left(\begin{array}{c}x \\
y\end{array}\right) \to \|x/y\|^it, \quad x, y \in F_\infty^\times,
$$

and we let $\pi(it)$ be the irreducible spherical constituent of the representation unitarily induced from this character. For $t$ real, the spherical $\pi(it)$ is tempered, and the Plancherel measure $d\mu(t)$ is defined by

$$
d\mu(t) = \begin{cases} 
t \tanh(\pi|t|) \, dt, & \text{if } F_\infty \text{ is real}, \\
t^2 \, dt, & \text{if } F_\infty \text{ is complex.} \end{cases}
$$

Moreover, we define

$$
\text{Pl}(t) = \begin{cases} 
4 \cosh(\pi t), & \text{if } F_\infty \text{ is real}, \\
8\pi \sinh(2\pi t)/t, & \text{if } F_\infty \text{ is complex.} 
\end{cases}
$$

Compared with [Q3] (3.2)], we have normalized $\text{Pl}(t)$ here by the factors 4 and $8\pi$. Let $W_{it}$ be the spherical ($K_\infty$-invariant) Whittaker vector so that

$$
W_{it}(\left(\begin{array}{c}x \\
1\end{array}\right)) = \begin{cases} 
\|x\|^{\frac{1}{2}} K_{it}(2\pi|x|), & \text{if } F_\infty \text{ is real,} \\
\|x\|^{\frac{1}{2}} K_{2it}(4\pi|x|), & \text{if } F_\infty \text{ is complex.} 
\end{cases}
$$

3.2. Hecke–Maass Cusp Forms. Fix an orthonormal basis $\mathcal{B}$ for the cuspidal subspace of $L^2(\text{PGL}_2(\mathbb{O})/\text{PGL}_2(F_\infty)/K_\infty)$ that consists of eigenforms for the Hecke algebra as well as the Laplacian operator (Hecke–Maass cusp forms). Each $f \in \mathcal{B}$ transforms under a certain representation $\pi(it)$ of $\text{PGL}_2(F_\infty)$, for some $t_f \in Y$. In general, we have the Kim–Sarnak bound in [BB]:

$$
|\text{Im}(t_f)| \leq \frac{7}{64},
$$

but it is known that $t_f$ is real for $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{d_F})$ with $d_F = -3, -4, -7, -8, -11$ (see [EGM] §7.6). Accordingly, define $Y_{KS} = (-\infty, \infty) \cup i\left[-\frac{7}{64}, \frac{7}{64}\right]$. The Fourier expansion of $f$ is of the form:

$$
f(g_\infty) = \sum_{n \in \mathcal{O} \setminus \{0\}} \frac{a_f(n\mathcal{O})}{\sqrt{N(n\mathcal{O})}} W_{it_f}(\left(\begin{array}{c}n \\
1\end{array}\right) g_\infty), \quad g_\infty \in \text{GL}_2(F_\infty).
$$

As indicated by the notation, the Fourier coefficient $a_f(n) = a_f(n\mathcal{O})$ only depends on the ideal $n = n\mathcal{O}$. Let $\lambda_f(n)$ denote the $n$-th Hecke eigenvalue of $f$. It is known that $\lambda_f(n)$ are real. We have the Hecke relation:

$$
\lambda_f(n_1)\lambda_f(n_2) = \sum_{\mathfrak{d} | (n_1, n_2)} \lambda_f(n_1n_2/\mathfrak{d}^2).
$$

As usual, there is a constant $C_f$ so that

$$
a_f(n) = C_f \lambda_f(n)
$$
for any nonzero integral ideal \( n \). By the Rankin–Selberg method, we have

\[
|C_f|^2 = \frac{\text{Pl}(t_f)}{2L(1, \text{Sym}^2 f)}.
\]

3.3. Kuznetsov Trace Formula.

**Definition 3.1** (Space of test functions). Let \( S > \frac{1}{2} \). We set \( \mathcal{H}(S) \) to be the space of functions \( h(t) \) which extends to an even holomorphic function on the strip \( \{ t + i\sigma : |\sigma| \leq S \} \) such that

\[
h(t + i\sigma) \ll e^{-\pi|t|(1 + |t|)^{-N}},
\]

holds uniformly for some \( N > 6 \).

**Definition 3.2** (Bessel kernel). Let \( s \in \mathbb{C} \).

1. When \( F_\infty = \mathbb{R} \), for \( x \in \mathbb{R}_+ \) we define

\[
B_s(x) = \frac{\pi}{\sin(\pi s)} \left( J_{2s}(4\pi \sqrt{x}) - J_{2s}(4\pi \sqrt{x}) \right),
\]

\[
B_s(-x) = 4\cos(\pi s)K_{2s}(4\pi \sqrt{x}).
\]

2. When \( F_\infty = \mathbb{C} \), for \( z \in \mathbb{C}^\times \) we define

\[
B_s(z) = \frac{2\pi^2}{\sin(2\pi s)} \left( J_{2s}(4\pi \sqrt{z})J_{-2s}(4\pi \sqrt{z}) - J_{2s}(4\pi \sqrt{z})J_{-2s}(4\pi \sqrt{z}) \right).
\]

The Kuznetsov trace formula of Bruggeman and Miatello in the spherical case is as follows. See [Qi3] Proposition 3.5 or [Ven] Proposition 1.

**Proposition 3.3** (Kuznetsov trace formula). Let \( h(t) \) be a test function in \( \mathcal{H}(S) \) and define

\[
\mathcal{H} = \int_{-\infty}^{\infty} h(t) d\mu(t), \quad \mathcal{H}(x) = \int_{-\infty}^{\infty} h(t) B_{it}(x) d\mu(t), \quad x \in F_\infty^\times.
\]

For nonzero integral ideals \( m = m\mathfrak{D} \) and \( n = n\mathfrak{D} \) we have

\[
\sum_{f \in \mathfrak{B}} \omega_f h(t_f) \lambda_f(m) \lambda_f(n) + \frac{1}{4\pi c_0} \int_{-\infty}^{\infty} \omega(t) h(t) \tau_\mathfrak{D}(m) \tau_\mathfrak{D}(n) dt
\]

\[
= c_1 \delta_{m,n} \mathcal{H} + c_2 \sum_{\epsilon \in \mathfrak{D}^\times} \sum_{\epsilon^2 c \in \mathfrak{D}^\times \setminus \{0\}} \frac{S(m; \epsilon n; c)}{|\mathfrak{N}(c)| \mathcal{H}\left( \frac{\epsilon mn}{c^2} \right)},
\]

where

\[
\omega_f = \frac{|C_f|^2}{\text{Pl}(t_f)} = \frac{1}{2L(1, \text{Sym}^2 f)} \quad \omega(t) = \frac{1}{|\zeta_f(1 + 2it)|^2},
\]

\( \delta_{m,n} \) is the Kronecker \( \delta \) that detects \( m = n \), and \( c_0, c_1, \) and \( c_2 \) are given by

\[
c_0 = 1, \quad c_1 = \frac{1}{8\pi^2}, \quad c_2 = \frac{1}{16\pi^2},
\]

if \( F = \mathbb{Q} \), and

\[
c_0 = \frac{2\pi}{w_F \sqrt{|d_F|}} \quad c_1 = \frac{\sqrt{|d_F|}}{8\pi^3}, \quad c_2 = \frac{1}{16\pi^3},
\]

if \( F = \mathbb{Q}(\sqrt{d_F}) \).
For our normalized $P_l(t)$ the constants in [Q13] (3.18), (3.19) have been modified here accordingly. Note that $c_0 = \gamma_1$ and $c_2 = c_1/2\sqrt{|d_F|}$.

By the discussions below [LC] Lemma 2.2, it is known that the lower bound $|\zeta_p(1 + 2it)| \gg \log(3 + |t|)$ holds, and hence

$$\omega(t) \ll \log(3 + |t|).$$

### 3.4. Voronoï Summation Formula

The Voronoï summation formula for the divisor function $\tau(n) = \tau_0(n)$ is as follows. Compare [IK] (4.49).

**Proposition 3.4** (Voronoï summation formula). Let $a, \tilde{a}, c \in \mathbb{C}$ with $c \neq 0$ be such that $(a, c) = 1$ and $\tilde{a}a \equiv 1 \pmod{c}$. For $w(x) \in C_c^\infty(F^\infty)$ we define its Hankel transform $\tilde{w}_0(y)$ with Bessel kernel $B_0$ (as in Definition 3.2).

$$\tilde{w}_0(y) = \int_{F^\infty} w(x)B_0(xy)dx, \quad y \in F^\infty,$$

and define its associated Mellin integrals:

$$\tilde{w}_0(0) = \int_{F^\infty} w(x)dx, \quad \hat{\tilde{w}}_0(0) = \int_{F^\infty} w(x)\log \|x\|_\infty dx.$$

Then we have the identity

$$\frac{|N(c)|}{\sqrt{|d_F|}} \sum_{n \in \mathbb{Z}} \psi_{\chi_n}(\frac{an}{c}) \tau(n\mathcal{D})w(n) = \gamma_1\tilde{w}_0(0) + 2\left(\gamma_0 - \gamma_1 \log \frac{|N(c)|}{\sqrt{|d_F|}}\right)\tilde{w}_0(0)$$

$$+ \frac{1}{\sqrt{|d_F|}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \psi_{\chi_n}(\frac{-\bar{a}n}{c}) \tau(n\mathcal{D})\tilde{w}_0\left(\frac{n}{c^2}\right),$$

where the constants $\gamma_0$ and $\gamma_1$ are defined as in (2.3).

**Proof.** Apply [Q15] Corollary 1.4 with $\zeta = a/c$, $\alpha = (1)$, $S = \{p : p|\langle c \rangle\}$, and $b = (c^2)$. Note that every $\zeta \in F$ may be expressed as a fraction $\zeta = a/c$ with $(a, c) = 1$ since the class number $h_F = 1$.

It will be more convenient to interpret the zero frequency as the limit:

$$\gamma_1\tilde{w}_0(0) + 2\left(\gamma_0 - \gamma_1 \log \frac{|N(c)|}{\sqrt{|d_F|}}\right)\tilde{w}_0(0) = \lim_{s \to 0} \sum_{n \in \mathbb{Z}} \frac{\zeta_p(1 \pm 2s)\tilde{w}_s(0)}{|N(c)|^{2/d_F}|x|^s},$$

where $\tilde{w}_s(0)$ is the Mellin transform

$$\tilde{w}_s(0) = \int_{F^\infty} w(x)\|x\|^s x dx.$$

See [Q15] Theorem 1.3.

### 4. Approximate Functional Equations

Let $f \in \mathcal{B}$ be a (spherical) Hecke–Maass cusp form for $\mathrm{PGL}_2(\mathbb{R})$ with Hecke eigenvalues $\lambda_f(n)$ and Archimedean parameter $t_f \in Y$. The $L$-function attached to $f$ is defined by

$$L(s, f) = \sum_{n \in \mathbb{Z}} \frac{\lambda_f(n)}{N(n)^s}.$$

The completed $L$-function for $f$ is $\Lambda(s, f) = N(\mathcal{D})^s \gamma(s, t_f)L(s, f)$, where

$$\gamma(s, t) = (N\pi)^{-s} \Gamma \left( \frac{N(s - it)}{2} \right) \Gamma \left( \frac{N(s + it)}{2} \right).$$
Recall that $N = 1$ or $2$ according as $F$ is rational or imaginary quadratic. It is known that $\Lambda(s, f)$ is entire and has the functional equation

$$L(s, f) = \Lambda(1 - s, f).$$

For $\text{Re}(s) > 1$, it follows from the Hecke relation that

$$L(s, f)^2 = \sum_{n_1, n_2 \in \mathbb{O}, (n_1, n_2)} \frac{\lambda_f(n_1 n_2 / \mathfrak{d}^2)}{N(n_1 n_2)^s} = \sum_{\mathfrak{d} \in \mathbb{O}} \frac{1}{N(\mathfrak{d})^{2s}} \sum_{n_1, n_2 \in \mathbb{O}} \frac{\lambda_f(n_1 n_2)}{N(n_1 n_2)^s},$$

and hence

$$L(s, f)^2 = \zeta_F(2s) \sum_{n \in \mathbb{O}} \frac{\lambda_f(n) \tau(n)}{N(n)^s},$$

where $\tau(n)$ is the divisor function.

Similarly, if $\lambda_f(n)$ are replaced by $\tau_{it}(n)$, then

$$\zeta_F(s + it)\zeta_F(s - it) = \sum_{n \in \mathbb{O}} \frac{\tau_{it}(n) \tau(n)}{N(n)^s},$$

and

$$\zeta_F(s + it)^2\zeta_F(s - it)^2 = \zeta_F(2s) \sum_{n \in \mathbb{O}} \frac{\tau_{it}(n) \tau(n)}{N(n)^s}.$$

We have the Approximate Functional Equations for $L(s, f)$ and $L(s, f)^2$ (see [IK, Theorem 5.3]):

$$L\left(\frac{1}{2}, f\right) = 2 \sum_{n \in \mathbb{O}} \frac{\lambda_f(n)}{\sqrt{N(n)}} V_1(N(n\mathfrak{D}^{-1}); t_f),$$

(4.6)

$$L\left(\frac{1}{2}, f\right)^2 = 2 \sum_{n \in \mathbb{O}} \frac{\lambda_f(n) \tau(n)}{\sqrt{N(n)}} V_2(N(n\mathfrak{D}^{-2}); t_f),$$

(4.7)

with

$$V_1(y; t) = \frac{1}{2\pi i} \int_{(3)} G(v, t) y^{-v} \frac{dv}{v},$$

(4.8)

$$V_2(y; t) = \frac{1}{2\pi i} \int_{(3)} G(v, t)^2 \zeta_F(1 + 2v) y^{-v} \frac{dv}{v},$$

(4.9)

for $y > 0$, where

$$G(v, t) = \frac{\gamma(\frac{1}{2} + v, t)}{\gamma(\frac{1}{2}, t)} \cdot e^{it^2}.$$}

In parallel, we have

$$\left|\zeta_F\left(\frac{1}{2} + it\right)^2\right| = 2 \sum_{n \in \mathbb{O}} \frac{\tau_{it}(n)}{\sqrt{N(n)}} V_1(N(n\mathfrak{D}^{-1}); t) + O(e^{-\epsilon t^2/2}),$$

(4.11)

$$\left|\zeta_F\left(\frac{1}{2} + it\right)^4\right| = 2 \sum_{n \in \mathbb{O}} \frac{\tau_{it}(n) \tau(n)}{\sqrt{N(n)}} V_2(N(n\mathfrak{D}^{-2}); t) + O(e^{-\epsilon t^2}),$$

(4.12)

in which the errors arise from the polar terms.
Lemma 4.1. For \( t \in Y_{KS} \) define
\[
C(t) = \sqrt{\frac{1}{t} + t^2}.
\]

(1) Let \( U > 1, A > 0, \) and \( \epsilon > 0. \) We have
\[
V_1(y; t) \ll_A \left(1 + \frac{y}{C(t)^N}\right)^{-A}, \quad V_2(y; t) \ll_A \left(1 + \frac{y}{C(t)^{2N}}\right)^{-A},
\]
and
\[
V_1(y; t) = \frac{1}{2\pi i} \int_{\epsilon-iU}^{\epsilon+iU} G(v, t) y^{-v} \frac{dv}{v} + O_\epsilon \left(\frac{C(t)^\epsilon}{y^{\epsilon e^{U/2}}}\right),
\]
\[
V_2(y; t) = \frac{1}{2\pi i} \int_{\epsilon-iU}^{\epsilon+iU} G(v, t)^2 \zeta_F(1 + 2v) y^{-v} \frac{dv}{v} + O_\epsilon \left(\frac{C(t)^\epsilon}{y^{\epsilon e^{U/2}}}\right).
\]

(2) Define
\[
\psi(t) = \frac{N}{2} \left(\frac{\Gamma'}{\Gamma} \left(\frac{N(1+2it)}{4}\right) + \frac{\Gamma'}{\Gamma} \left(\frac{N(1-2it)}{4}\right) - \log(N\pi)\right).
\]
(This \( \psi \) should not be confused with the additive character as it stands here for the digamma function.) We have
\[
V_1(y; t) = 1 + O_A \left(\frac{y}{C(t)^N}\right)^A,
\]
for \( 1 \ll y < C(t)^N, \) and
\[
V_2(y; t) = \gamma_0 + \gamma_1 (\psi(t) - \log \sqrt{y}) + O_A \left(\frac{y}{C(t)^{2N}}\right)^A,
\]
for \( 1 \ll y < C(t)^{2N}, \) where \( \gamma_0 \) and \( \gamma_1 \) are defined as in \([2.5],\) and
\[
\psi(t) = N \log(C(t)/2\pi) + O(1/C(t)^2).
\]

Proof. The asymptotics in (1) are analogous to those in [QI3, Lemma 5.1 (1)]. See also [IK, Proposition 5.4], [Blo, Lemma 1], and [QI1, Lemma 3.7]. To derive (4.17) and (4.18), we choose \( U = \sqrt{C(t)}, \) say, and shift the integral contour in (4.13) and (4.15) from \( \Re(v) = \epsilon \) further down to \( \Re(v) = -A; \) the main term is the residue from the pole at \( v = 0 \) while the error term is from the Stirling formula. Note that the integrand in (4.13) has a double pole at \( v = 0, \) and its residue may be computed using
\[
\zeta_F(1 + 2v) = \frac{\gamma_1}{2v} + \gamma_0 + O(|v|), \quad v \to 0,
\]
by (2.5), and
\[
G(v, t) = 1 + \psi(t)v + O\left(|v|^2\right), \quad v \to 0,
\]
by (4.14) and (4.10). Moreover, (4.19) follows readily from
\[
\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right),
\]
for \(|s| \to \infty\) and \(|\arg(s)| \leq \pi - \delta < \pi. \) Q.E.D.
5. Choice of Weight Function

**Definition 5.1.** Let \( 1 \leq T^\varepsilon \leq M \leq T^{1-\varepsilon} \). Define the function
\[
k(t) = k_{T,M}(t) = e^{-(t-T)^2/M^2} + e^{-(t+T)^2/M^2}.
\]

Next we introduce an unsmoothing process as in [LI §3] by an average of the weight \( k_{T,M} \) in the \( T \)-parameter.

**Definition 5.2.** Let \( 3H \leq T \) and \( T^\varepsilon \leq M \leq H^{1-\varepsilon} \). Define
\[
w(t) = w_{T,M,H}(t) = \frac{1}{\sqrt{T}M} \int_{T-H}^{T+H} k_{K,M}(t)dK.
\]

Let \( \chi(t) = \chi_{T,H}(t) \) denote the characteristic function for \( ||t| - T| \leq H \) (\( t \) real). By adapting the arguments in [LI §3] (see also [BHS §3]), it is easy to prove that \( w(t) - 1 \) is exponentially small if \( ||t| - T| \leq H - M^{1+\varepsilon} \),
\[
w(t) - \chi(t) = O \left( \frac{M^3}{(M + \min \{|||t| - T| - H|\})^{3/2}} \right),
\]
if \( ||t| - T \pm H| \leq M^{1+\varepsilon} \), and \( w(t) \) is exponentially small if otherwise. From these, along with (2.6) and (3.11), one may prove the following lemmas (compare [LI (3.6), (3.7)]).

**Lemma 5.3.** Let \( \lambda \) be a real constant. Suppose that \( a_f \geq 0 \) and that
\[
\sum_{f \in \mathbb{A}}^h k(t_f)a_f = O_{\lambda,\epsilon}(MT^\lambda)
\]
for any \( M \) with \( T^\varepsilon \leq M \leq T^{1-\varepsilon} \). Then for \( M^{1+\varepsilon} \leq 3H \leq T \) we have
\[
\sum_{f \in \mathbb{A}}^h a_f = \sum_{f \in \mathbb{A}}^h w(t_f)a_f + O_{\lambda,\epsilon}(MT^\lambda).
\]

Note that we obtain [BHS Lemma 3.1] by applying Lemma 5.3 with \( M = H^{1-\varepsilon} \) and \( a_f = \delta_{L(\mathbb{A},f),\neq0} \) (the Kronecker \( \delta \) that detects \( L(\mathbb{A},f) \neq 0 \)).

**Lemma 5.4.** For \( M^{1+\varepsilon} \leq 3H \leq T \) we have
\[
2 \int_{T-H}^{T+H} \frac{\zeta_F(\frac{1}{2} + it)^{2q}}{\zeta_F(1 + 2it)^2} dt = \int_{-\infty}^{\infty} \frac{\zeta_F(\frac{1}{2} + it)^{2q}}{\zeta_F(1 + 2it)^2} w(t)dt + O_{\epsilon}(MT^{2q,N\theta + \varepsilon}),
\]
where \( \theta \) is a sub-convex exponent for \( \zeta_F(s) \) as in (2.6).

Part 2. Analysis of Integrals

In the subsequent sections, we shall analyze the Bessel integrals, their Hankel and Mellin integral transforms over \( F_x = \mathbb{R} \) or \( \mathbb{C} \). Henceforth, \( x, y \) will always stand for real variables, while \( z, u \) for complex variables.

6. Asymptotics for Bessel Kernels

Let \( B_s(x) \) and \( B_s(z) \) be the real and complex Bessel kernels as in Definition 3.2 respectively.

By the works in [LQ][Q13], the Bessel integrals \( \mathcal{H}(x) \) and \( \mathcal{H}(z) \) in the Kuznetsov trace formula are well understood, and their results will be recollected in the next

\[2\text{Note that the 1 in [BHS (3.4)] should be the characteristic function.}\]
section. In this section, we are mainly concerned with the Bessel kernel \( B_0(x) \) or \( B_0(z) \) for the Hankel transform arising in the Voronoi summation formula.

In view of Definition \ref{def:BesselKernel} the connection formulae in \cite{Wat} 3.61 (1), (2)] may be applied to deduce

\begin{equation}
B_0(x) = \pi i (H_0^{(1)}(4\pi \sqrt{x}) - H_0^{(2)}(4\pi \sqrt{x})), \quad B_0(-x) = 4K_0(4\pi \sqrt{x}),
\end{equation}

and

\begin{equation}
B_0(z) = \pi^2 i (H_0^{(1)}(4\pi \sqrt{z})H_0^{(1)}(4\pi \sqrt{z}) - H_0^{(2)}(4\pi \sqrt{z})H_0^{(2)}(4\pi \sqrt{z})).
\end{equation}

By the asymptotic expansions in \cite{Wat} 7.2 (1, 2), 7.23 (1)], for any non-negative integer \( K \), there are smooth functions \( W_0(x) \) and \( W_0(z) \) (depending on \( K \)) with

\begin{equation}
x^j \frac{d^j W_0(x)}{dx^j} \ll_{j,K} 1,
\end{equation}

\begin{equation}
x^j z^k \frac{d^{j+k} W_0(z)}{dz^j \partial \bar{z}^k} \ll_{j,k,K} 1,
\end{equation}

such that

\begin{equation}
B_0(x) = \sum \pm \frac{e(\pm (2\sqrt{x} + 1/8))}{\sqrt{x}} W_0(\pm \sqrt{x}) + O_K \left( \frac{1}{x^{(2K+1)/4}} \right),
\end{equation}

\begin{equation}
B_0(-x) = O \left( \frac{\exp(-4\pi \sqrt{x})}{\sqrt{x}} \right),
\end{equation}

\begin{equation}
B_0(z) = \sum \pm \frac{e(\pm 4\text{Re} \sqrt{z})}{\sqrt{|z|}} W_0(\pm \sqrt{z}) + O_K \left( \frac{1}{|z|^{(K+1)/2}} \right),
\end{equation}

for \( x > 1 \), and

\begin{equation}
B_0(z) = \sum \pm \frac{e(\pm 4\text{Re} \sqrt{z})}{\sqrt{|z|}} W_0(\pm \sqrt{z}) + O_K \left( \frac{1}{|z|^{(K+1)/2}} \right),
\end{equation}

for \( |z| > 1 \).

\textbf{Remark 6.1.} For the real case, \ref{eq:asymp1} has a cleaner form without the error term. For the complex case, however, the error term must be included in \ref{eq:asymp2}, for the two product functions in \ref{eq:asymp3} are not individually well defined on \( \mathbb{C} \setminus \{0\} \).

\section{Properties of Bessel Integrals}

For \( 1 \leq T^\varepsilon \leq M \leq T^{1-\varepsilon} \), let \( k(t) = k_{T,M}(t) \) be the weight function as defined in \S 5. Define

\begin{equation}
h^q(t; v) = h_{T,M}^q(t; v) = k_{T,M}(t)G(v, t)^q,
\end{equation}

with \( \text{Re}(v) = \varepsilon \) and \( |\text{Im}(v)| \leq \log T \). Note that \( h^q(t; v) \) lies in the space \( \mathcal{H} \left( \frac{1}{2} + \varepsilon \right) \) as in Definition \ref{def:SpaceH}. Since \( q \) and \( v \) are inessential to our analysis, we shall simply write \( h(t) = h^q(t; v) \). Let \( \mathcal{H}(x) \) or \( \mathcal{H}(z) \) be its associated Bessel integral (see \ref{eq:BesselIntegral}) defined by

\begin{equation}
\mathcal{H}(x) = \int_{-\infty}^{\infty} h(t) B_{d,t}(x) t \tanh(\pi t) dt, \quad \mathcal{H}(z) = \int_{-\infty}^{\infty} h(t) B_{d,t}(z) t^2 dt.
\end{equation}

The following results for \( \mathcal{H}(x) \) and \( \mathcal{H}(z) \) are essentially established in \cite{LQ} and \cite{Qi} \S 8 in different settings (see also \cite{LJ} \Lp \You for the real case). For the complex case, however, it will be more convenient to work here in the Cartesian coordinates.

The estimates above may be derived from shifting the integral contour to \( \text{Im}(t) = \frac{1}{2} + \varepsilon \). See \cite{You} and \cite{LQ}.
First of all, let us assume if for any $A \geq 0$, and such that

1. if $F_{\infty}$ is real, then $\mathcal{H}(x) = \mathcal{H}_+(x) + \mathcal{H}_-(x) + O(T^{-A})$ for $|x| > 1$, with

   \begin{equation}
   \mathcal{H}_{\pm}(x^2) = MT^{1+\epsilon} \int_{-M'/M}^{M'/M} g(Mr)e(Tr/\pi \mp 2x \cosh r)dr,
   \end{equation}

   and

2. if $F_{\infty}$ is complex, then $\mathcal{H}(z) = \mathcal{H}_+(z) + \mathcal{H}_-(z) + O(T^{-A})$ for $|z| > 1$, with

   \begin{equation}
   \mathcal{H}_{\pm}(z^2) = MT^{2+\epsilon} \int_{-M'/M}^{M'/M} g(Mr)e(2Tr/\pi \mp 4\text{Re}(\text{trh}(r, \omega)))dr\omega,
   \end{equation}

   for $\text{arg}(z) \in [0, \pi)$, where trh($r, \omega$) is the “trigonometric-hyperbolic” function defined by

   \begin{equation}
   \text{trh}(r, \omega) = \cosh r \cos \omega + i \sinh r \sin \omega.
   \end{equation}

   Furthermore,

   3. for real $x$ with $1 < |x| \ll T^2$, we have $\mathcal{H}(x) = O(T^{-A})$;

   4. for complex $z$ with $1 < |z| \ll T^2$, we have $\mathcal{H}(z) = O(T^{-A})$;

   5. for real $x$ with $|x| \leq 1$, we have

   \begin{equation}
   \mathcal{H}(x) \ll_{A, \epsilon} MT^{1-2A} \sqrt{|x|};
   \end{equation}

6. for complex $z$ with $|z| \leq 1$, we have

   \begin{equation}
   \mathcal{H}(z) \ll_{A, \epsilon} MT^{2-4A}|z|.
   \end{equation}

Remark 7.2. In [Q13], for the proof in the case $|x| \leq 1$ or $|z| \leq 1$ a certain polynomial is introduced to annihilate the poles of the gamma factor, but it is redundant because the residues of the integrand in (7.2) at these poles are actually exponentially small in view of $|\text{Im}(v)| \leq \log T$.

In the real case, Lemma 7.1 (3) may be strengthened for $x > 1$ as follows.

Lemma 7.3. We have $\mathcal{H}_+(x) = O(T^{-A})$ for $1 < x \leq M^{2-\epsilon}T^3$.

8. Analysis of Hankel Transforms

Let $w(x) \in C_c^\infty[1, 2]$ satisfy $w^{(j)}(x) \ll_j (\log T)^j$ for all $j \geq 0$. For $|A| \gg T^2$, define

\begin{equation}
w(x, A) = w(|x|)\mathcal{H}(Ax),
\end{equation}

if $F_{\infty}$ is real, and

\begin{equation}
w(z, A) = w(|z|)\mathcal{H}(Az),
\end{equation}

if $F_{\infty}$ is complex. Let $\tilde{w}_0(y, A)$ and $\tilde{w}_0(u, A)$ be their Hankel transform defined by

\begin{equation}
\tilde{w}_0(y, A) = \int w(x, A)B_0(xy)dx, \quad \tilde{w}_0(u, A) = \int w(z, A)B_0(zu)dz.
\end{equation}

First of all, let us assume $\Lambda > 0$ with no loss of generality, as

\begin{equation}
\tilde{w}_0(y, A) = \tilde{w}_0(\epsilon y, \epsilon A), \quad \tilde{w}_0(u, A) = \tilde{w}_0(\epsilon u, \epsilon A),
\end{equation}

for any $\epsilon \in F_{\infty}$ with $|\epsilon| = 1$. 
Lemma 8.1. Suppose that $A \gg T^2$.

(1) When $F_{\infty}$ is real, for $y \geq T^\epsilon$ we have

\[
\hat{w}_0(\pm y, A) = \frac{MT^{1+\epsilon}}{\sqrt{y}} \psi^\pm(\sqrt{y}/A, \sqrt{A}) + O(T^{-A}),
\]

with

\[
\psi^+(x, \Delta) = \int_{-M^*/M}^{M^*/M} e(Tr/\pi)g(Mr)\hat{V}(\Delta(x - \cosh r))dr,
\]
or $\psi^+(x, \Delta) = 0$ according as $\Delta > M^{1-\epsilon}T$ or not, and

\[
\psi^-(x, \Delta) = \int_{-M^*/M}^{M^*/M} e(Tr/\pi)g(Mr)\left(\hat{V}(\Delta(x + \sinh r)) - \hat{V}(\Delta(x - \sinh r))\right)dr,
\]

where $\hat{V}(x)$ is a Schwartz function satisfying

\[
\frac{d^j\hat{V}(x)}{dx^j} \ll_{j, A} \left(1 + \frac{|x|}{\log T}\right)^{-A}
\]

for any $j, A \geq 0$.

(2) When $F_{\infty}$ is complex, for $|u| \geq T^\epsilon$ we have

\[
\hat{w}_0(u, A) = \frac{MT^{2+\epsilon}}{\sqrt{|u|}} \psi(\sqrt{|u|}/A, \sqrt{A})d\omega + O(T^{-A}),
\]

with

\[
\psi(z, \Delta) = \int_0^{2\pi} \int_{-M^*/M}^{M^*/M} e(2Tr/\pi)g(Mr)\hat{V}(\Delta(z - \text{trh}(r, \omega)))drd\omega,
\]

where $\hat{V}(z)$ is a Schwartz function satisfying

\[
\frac{\partial^{j+k}\hat{V}(z)}{\partial z^j \partial \bar{z}^k} \ll_{j, k, A} \left(1 + \frac{|z|}{\log T}\right)^{-A}
\]

for any $j, k, A \geq 0$.

Proof. First, let $F_{\infty}$ be real. By (6.4) (with $K$ large in terms of $\epsilon$ and $A$), (8.2), and (7.4) in Lemma (1), along with the substitution $\pm 2\sqrt{T} \to x$, it follows that, up to a negligible error, $\hat{w}_0(y, A)$ or $\hat{w}_0(-y, A)$ becomes the sum of

\[
MT^{1+\epsilon} \frac{1}{\sqrt{y}} \int_{-M^*/M}^{M^*/M} e(Tr/\pi)g(Mr)\left(\int_{-\infty}^{\infty} V(x)e(-x(\sqrt{y} \pm \sqrt{A}\cosh r))dx\right)dr,
\]
or

\[
MT^{1+\epsilon} \frac{1}{\sqrt{y}} \int_{-M^*/M}^{M^*/M} e(Tr/\pi)g(Mr)\left(\int_{-\infty}^{\infty} V(x)e(-x(\sqrt{y} \pm \sqrt{A}\sinh r))dx\right)dr,
\]

respectively, where $V(x)$ is a certain smooth weight function supported in $|x| \in [1/2, 1/\sqrt{2}]$ with

\[
V^{(j)}(x) \ll_{j, A} (\log T)^j.
\]

(To be explicit, $V(\pm 2x) = (1 \mp i)\sqrt{2}w(x^2)W_0(\mp \sqrt{\bar{x}})$. ) By Lemma (7.3) the first integral is negligibly small unless $\sqrt{A} > M^{1-\epsilon}T$. Observe that the inner integral is a Fourier integral, and that $\sqrt{y} + \sqrt{A}\cosh r \gg T$ is large, so the results follow immediately.
Second, let $F_\alpha$ be complex. Similar to the real case, one may prove (8.9) on applying (6.6) and (7.5), along with the substitution $\pm 2\sqrt{z} \to z$. Q.E.D.

8.1. Analysis for the Hyperbolic Functions.

**Lemma 8.2.** Let $\delta < \rho \leq 1$. For $0 \leq x < 1$ define the region $\Gamma^{-}(\delta, \rho; x)$ by

$$|r| \leq \rho, \quad |\sinh r \pm x| \leq \delta.$$  

(1) $\Gamma^{-}(\delta, \rho; x)$ is non-empty unless $x \leq \rho$.

(2) $\Gamma^{-}(\delta, \rho; x)$ has length $O(\delta)$.

**Proof.** The first assertion is obvious in view of $\sinh r = O(\rho)$. By the mean value theorem, the second inequality in (8.12) implies $|r \pm \text{arcsinh} x| \leq \delta$, and hence the length of $\Gamma^{-}(\delta, \rho; x)$ is bounded by $O(\delta)$. Q.E.D.

**Lemma 8.3.** Let $\sqrt{3} < \rho \leq 1$. For $0 < x < \sqrt{2}$ define the region $\Gamma^{+}(\delta, \rho; x)$ by

$$|r| \leq \rho, \quad |\cosh r - x| \leq \delta.$$  

(1) $\Gamma^{+}(\delta, \rho; x)$ is non-empty unless $|x - 1| \leq \rho^2$.

(2) $\Gamma^{+}(\delta, \rho; x)$ has length $O(\delta/\sqrt{|x - 1|})$.

(3) We have $\sinh r \leq \sqrt{3}$ on the region $\Gamma^{+}(\delta, \rho; 1)$.

**Proof.** By $\sinh^2 r = \cosh^2 r - 1$, the second inequality in (8.13) implies

$$|\sinh^2 r - (x^2 - 1)| \leq \delta.$$  

Then (1) and (3) are obvious. As for (2), (8.13) yields $|r| \leq \sqrt{3}$ if $|x - 1| \leq \delta$, the empty set if $1 - x \not\geq \delta$, and $|r \pm \text{arcsinh}\sqrt{x^2 - 1}| \leq \delta/\sqrt{x - 1}$ if $x - 1 \geq \delta$ (again, by the mean value theorem), and hence the length of $\Gamma^{+}(\delta, \rho; x)$ is bounded by $O(\delta/\sqrt{|x - 1|})$ in every case. Q.E.D.

8.2. Analysis for the Trigonometric-Hyperbolic Function.

**Lemma 8.4.** Let $\delta < \rho \leq 1$. For $|x| < \sqrt{2}$ and $|y| < 1$ define $I(\delta, \rho; x + iy)$ to be the set of $(r, \omega)$ such that

$$|r| \leq \rho, \quad |\cos \omega \cosh r - x| \leq \delta, \quad |\sin \omega \sinh r - y| \leq \delta.$$  

(1) $I(\delta, \rho; x + iy)$ is non-empty unless $|x| < 1 + 2\rho$ and $|y| \leq \rho$.

(2) The area of $I(\delta, \rho; x + iy)$ has bound as follows,

$$\text{Area} I(\delta, \rho; x + iy) \leq \frac{\delta^2}{\sqrt{(|x| - 1)^2 + y^2}}.$$  

(3) We have $\sinh r, \sin \omega \leq \sqrt{3}$ on the region $I(\delta, \rho; \pm 1)$.

**Proof.** We shall focus on (2), since (1) is obvious while (3) will be transparent in the last case of its proof.

By symmetry, we only need to work in the setting with $(r, \omega) \in [0, \rho] \times [0, \pi/2]$ and $(x, y) \in [0, \sqrt{2}) \times [0, 1)$.

Consider the mapping

$$f : (r, \omega) \to (\cos \omega \cosh r, \sin \omega \sinh r),$$

so that $I(\delta, \rho; x + iy)$ is contained in the preimage under $f$ of the square with center $(x, y)$ and area $4\delta^2$. The Jacobian matrix

$$J_f(r, \omega) = \begin{pmatrix} \cos \omega \sinh r & \sin \omega \cosh r \\ -\sin \omega \cosh r & \cos \omega \sinh r \end{pmatrix}.$$
On the semi-closed rectangle \((0, \rho] \times (0, \pi/2)\), since all the principal minors of \(J_f(r, \omega)\) are positive, by the Univalence Theorem of Gale and Nikaitō ([GN] §§4.2, 4.3), \(f\) is a univalent mapping. Note that the Jacobian determinant is equal to \(\sinh^2 r + \sin^2 \omega\). Therefore \(f\) may be used as a coordinate transform, and if we are able to prove the lower bound
\[
\sinh^2 r + \sin^2 \omega \gtrsim \sqrt{(x-1)^2 + y^2}
\]
on \(I(\delta, \rho; x + iy)\) for either \(|x-1| \geq \delta\) or \(y \geq \delta\), then (8.16) follows immediately in this case.

Now we prove (8.18). For \(x \leq 1/2\), say, the second inequality in (8.15) implies \(\cos \omega \leq 1/\sqrt{2}\) (provided that \(\rho \leq 1\), so that \(\cosh r\) is near 1 and \(\delta < \rho\) is small), and hence (8.18) is clear. For \(x > 1/2\), observe that the second inequality in (8.15) implies
\[
|\sinh^2 r - \sin^2 \omega - \sin^2 \omega \sinh^2 r - (x^2 - 1)| \leq \delta,
\]
due to \(\cos^2 \omega \cosh^2 r = 1 + \sinh^2 r - \sin^2 \omega - \sin^2 r\). In the case when \(|x-1| \geq \delta\) and \(y \geq \delta\), the last inequality in (8.15) and (8.19) together yield
\[
\sinh^2 r - \sin^2 \omega = x^2 - 1, \quad \sin \omega \sinh \tau = y,
\]
and hence (8.18) by \(\sinh^2 r + \sin^2 \omega = \sqrt{(\sinh^2 r - \sin^2 \omega)^2 + 4 \sin^2 \omega \sin^2 r}\). The proof is similar for the remaining two cases when \(|x-1| \leq \delta\) or \(y \leq \delta\).

Finally, in the case when \(|x-1| \leq \delta\) and \(y \leq \delta\), we have \(|\cos \omega \cosh r - 1| \leq \delta\) and \(|\sin \omega \sinh \tau| \leq \delta\) (so the Jacobian of \(f\) could be very small or vanish). Since \((\cosh r - \cos \omega)^2 = (\cos \omega \cosh r - 1)^2 + (\sin \omega \sinh r)^2\) and \(\cosh^2 r - \cos^2 \omega = \sin^2 \omega + \sin^2 r\), it follows that the area of \(I(\delta, \rho; x + iy)\) is bounded by \(O(\delta)\), and hence (8.16). Moreover, (3) is also clear from these arguments.

Q.E.D.

In practice \(z = \sqrt{n/m} (m, n \in \mathcal{O} \setminus \{0\})\). The simple lemma below will help us take care of the square root in the complex case, with (1)–(4) corresponding to (12.22)–(12.29) in [12.4].

**Lemma 8.5.** Write \(z = x + iy\) and \(z^2 = x_2 + iy_2\). Let \(y \leq \rho\).

1. If \(|x| \leq \rho\), then \(|z^2| \leq \rho^2\).
2. If \(|x| \geq \rho\), then \(x_2 = x^2\) and \(y_2 \leq \rho|x|\).
3. If \(|x-1| \leq \rho\), then \(|z^2 - 1| \leq \rho\) and \(|z^2 - 1|^2 = (|x| - 1)^2 + y^2\).
4. If \(|1 - |x|| \geq \rho\), then \(|x_2 - 1| \leq 1 - |x|\) and \(y_2 \leq \rho\).

**8.3. Estimates for the \(\Psi\)-integrals.** Let
\[
\rho = M^e/M, \quad \delta = T^e/\Delta.
\]
It is then clear that the \(\Psi\)-integrals \(\Psi^\pm(x, \Delta)\) and \(\Psi(z, \Delta)\) defined in Lemma 8.1 are trivially bounded by the area of \(I^\pm(\delta, \rho; x)\) and \(I(\delta, \rho; z)\) respectively. A direct consequence of Lemma 8.2, 8.3 and 8.4 is the following proposition. For brevity, we shall allow \(M^e\) to absorb absolute constants—for example, the factor 2 in \(|x| < 1 + 2\rho\) and the implied constant in \(|y| \leq \rho\) \((\rho = M^e/M)\).

**Proposition 8.6.** Let \(\Psi^\pm(x, \Delta)\) and \(\Psi(z, \Delta)\) be as in (8.6), (8.7) and (8.10).

1. \(\Psi^-(x, \Delta)\) or \(\Psi^+(x, \Delta)\) is negligibly small unless \(x < M^e/M\) or \(|x-1| < M^e/M^2\) respectively, in which case
\[
\Psi^-(x, \Delta) \leq \frac{T^e}{\Delta}, \quad \Psi^+(x, \Delta) \leq \frac{T^e}{\Delta \sqrt{|x-1|}}.
\]
\( (2) \) \( \Psi(x + iy, \Delta) \) is negligibly small unless \( |x| < 1 + M^\varepsilon/M \) and \( |y| < M^\varepsilon/M \), in which case
\[
\Psi(x + iy, \Delta) \ll \frac{T^\varepsilon}{\Delta^2 \sqrt{(|x|-1)^2 + y^2}}.
\]

Finally, by recourse to partial integration for the \( r \)-integral, we prove that \( \Psi^+(1, \Delta) \) and \( \Psi(\pm 1, \Delta) \) are negligibly small for \( \Delta \leq T^{2-\varepsilon} \).

**Proposition 8.7.** Let \( \Psi^+(x, \Delta) \) and \( \Psi(z, \Delta) \) be defined as in (8.6) and (8.10).

1. We have \( \Psi^+(1, \Delta) = O_{A, \varepsilon}(T^{-A}) \) if \( \Delta \leq T^{2-\varepsilon} \).
2. We have \( \Psi(\pm 1, \Delta) = O_{A, \varepsilon}(T^{-A}) \) if \( \Delta \leq T^{2-\varepsilon} \).

**Proof.** There are three steps. First, smoothly truncate the \( r \)-integral to the range \( |r| \leq \rho \). Second, repeat partial integration. Faà di Bruno’s formula (\cite{Joh}) and its extension are required to calculate the higher \( r \)-derivatives of \( \tilde{V}(\Delta(x - \cosh r)) \) and \( \tilde{V}(\Delta(z - \text{trh}(r, \omega))) \). Third, confine the integration to the region \( \Gamma^+(\delta, \rho; 1) \) or \( \Gamma(\delta, \rho; \pm 1) \), and use the bounds for \( \sinh r \) or \( \sin \omega \) in Lemma 8.3 (3) or Lemma 8.4 (3), respectively. In this way, one obtains high powers of \( \Delta \sqrt{\delta'/T} = \Delta T^{1-\varepsilon} \). The details are left to the readers. Q.E.D.

**8.4. Remarks on the Complex Case.** The results in the complex case may be improved when \( x \) is close to \( \pm 1 \), in correspondence to the case of \( \Psi^+(x, \Delta) \). However, the improvements will not be useful, since the worst case scenario is when \( x \) stays away from 0 and \( \pm 1 \), say around 1/2. See \cite{12.4}

### 9. Mellin Transform of Bessel Kernels

In this section, we derive explicit formulae for the Mellin transform of the Bessel kernel \( B_t(x) \) and \( B_t(z) \). To be precise, define
\[
(9.1) \quad \tilde{B}_t(s) = \int B_t(x)|x|^{s-1}dx,
\]
or
\[
(9.2) \quad \tilde{B}_t(s) = \int B_t(z)|z|^{2s-2}dz,
\]
according as \( F_x \) is real or complex.

**Lemma 9.1.** For \( |\text{Im}(t)| < \text{Re}(s) < \frac{1}{2} \) the Mellin integral \( \tilde{B}_t(s) \) in (9.1) or (9.2) is absolutely convergent, and
\[
(9.3) \quad \tilde{B}_t(s) = \frac{\gamma(s, t)}{\gamma(1-s, t)},
\]
with \( \gamma(s, t) \) defined in (9.2).

**Proof.** For \( |\text{Im}(t)| < \frac{1}{2} \) we have crude estimates:
\[
B_t(x) \ll_{t, \varepsilon} \min \left\{ \frac{1}{|x||\text{Im}(t)|+\varepsilon}, \frac{1}{\sqrt{|x|}} \right\}, \quad B_t(z) \ll_{t, \varepsilon} \min \left\{ \frac{1}{|z||\text{Im}(2z)|+\varepsilon}, \frac{1}{\sqrt{|z|}} \right\},
\]
so the convergence of integrals is clear.

For the real case, by \cite{EMOT} §7.7.3 (19), (27), along with Euler’s reflection formula, we have
\[
\int_0^\infty J_\mu(4\pi x)x^{\rho-1}dx = \frac{1}{(2\pi)^{\rho+1}} \sin \left( \frac{\pi(\rho - \mu)}{2} \right) \Gamma\left( \frac{\rho + \mu}{2} \right) \Gamma\left( \frac{\rho - \mu}{2} \right),
\]
for $-\operatorname{Re}(\mu) < \operatorname{Re}(\rho) < \frac{1}{2}$, and
\[
\int_0^\infty K_\mu(4\pi x) x^{\rho - 1} dx = \frac{1}{4(2\pi)^\rho} \Gamma\left(\frac{\rho + \mu}{2}\right) \Gamma\left(\frac{\rho - \mu}{2}\right),
\]
for $|\operatorname{Re}(\mu)| < \operatorname{Re}(\rho)$.

For the complex case, we have
\[
\int_0^{2\pi} \int_0^\infty J_\mu(x e^{i\phi}) x^{2\rho - 1} dx d\phi = \frac{\cos(\pi\mu) - \cos(\pi\rho)}{(2\pi)^{2\rho + 2}} \Gamma\left(\frac{\rho + \mu}{2}\right)^2 \Gamma\left(\frac{\rho - \mu}{2}\right)^2.
\]
for $|\operatorname{Re}(\mu)| < \operatorname{Re}(\rho) < \frac{1}{2}$, with
\[
J_\mu(z) = \frac{1}{\sin(\pi\mu)} (J_{-\mu}(4\pi z) J_{-\mu}(4\pi \bar{z}) - J_\mu(4\pi z) J_\mu(4\pi \bar{z})).
\]
This is a simple consequence of Theorem 1.1 and Proposition 3.2 in [Qi4], specialized to the case $d = 0$ and $y = 0$. Note that Gauss’ hypergeometric function is equal 1 at the origin.

In view of Definition 4.2 one derives
\[
\int B_{it}(x) |x|^{s-1} dx = \frac{2}{(2\pi)^{2s}} \Gamma(s + it) \Gamma(s - it),
\]
\[
\iint B_{it}(z) |z|^{2s-2} dz = \frac{2}{(2\pi)^{4s}} \Gamma(s + it)^2 \Gamma(s - it)^2.
\]
Then (9.3) readily follows from Euler’s reflection formula and Legendre’s duplication formula (the latter is needed only for the real case). Q.E.D.

**Remark 9.2.** The formula (9.3) can also be interpreted from the viewpoint of representation theory for local functional equations. See [Qi2] §17.

**Part 3. The Twisted First and Second Moments**

**10. Setup: Application of the Kuznetsov Formula**

Now we turn to the investigation of the twisted first and second moments:

\[
M_q(m) = \sum_{f \in \mathcal{H}} k(t_f) \lambda_f(m) L\left(\frac{1}{2}, f\right)^q
\]

for $q = 1$ or 2, and weight function $k(t)$ defined as in (13.4) or (13.1). In the sequel, we shall always let $m = n\mathcal{D}$.

By the Approximate Functional Equations (4.6) and (4.7), we infer that

\[
M_q(m) = 2 \sum_{n \in \mathcal{O}} \frac{\tau(n)^{q-1}}{\sqrt{N(n)}} \sum_{f \in \mathcal{H}} k(t_f) \lambda_f(m) \lambda_f(n) V_q(N(n\mathcal{D}^{-q}); t_f).
\]

In view of (4.13) in Lemma 4.1 (1), at the cost of a negligible error term, we may truncate the summations over $n$ to the range $N(n) \leq T e^{N + \varepsilon}$.

Next, we use the expressions of $V_q(N(n\mathcal{D}^{-q}); t)$ as in (4.14) and (4.15) in Lemma 4.1 (1) with $U = \log T$ (so that the errors therein are negligible), and then apply the Kuznetsov trace formula in Proposition 3.3 inside the $v$-integral with test function:

\[
h^q(t; v) = k(t) G(v, t)^q;
\]

see (7.1). Moreover, for the diagonal and the Eisenstein contributions, with the loss of negligible errors, we revert the $v$-integral to $V_q(N(n\mathcal{D}^{-q}); t)$, and for the
latter convert the $n$-sum to $|\zeta_F(\frac{1}{2} + it)|^{2q}$ by the Approximate Functional Equations (4.11) and (4.12). It follows that

$\mathcal{M}_q(m) = \mathcal{D}_q(m) - \mathcal{E}_q(m) + \mathcal{O}_q(m) + O(T^{-A})$,

where $\mathcal{D}_q(m)$ is the diagonal term (it exists when $N(m) \leq T^{qN + \varepsilon}$)

$\mathcal{D}_q(m) = 2c_1 \frac{\tau(m)^{q-1}}{\sqrt{N(m)}} \mathcal{H}_q(m),$

with

$\mathcal{H}_q(m) = \int_{-\infty}^{\infty} k(t) V_q(N(m \mathcal{D}^{-q}); t) d\mu(t),$

$\mathcal{E}_q(m)$ is the Eisenstein (continuous spectrum) term

$\mathcal{E}_q(m) = \frac{1}{4\pi} \int_{-\infty}^{\infty} k(t) \tau_d(m) \omega(t)|\zeta_F(\frac{1}{2} + it)|^{2q} dt,$

and $\mathcal{O}_q(m)$ is the off-diagonal term

$\mathcal{O}_1(m) = \frac{2}{\pi i} \frac{c_2}{|d_F|^2} \int_{\lambda - i \log T}^{\lambda + \log T} \mathcal{O}_1(m; \nu) \frac{d\nu}{\nu},$

$\mathcal{O}_2(m) = \frac{2}{\pi i} \frac{c_2}{|d_F|^2} \int_{\lambda - i \log T}^{\lambda + \log T} \mathcal{O}_2(m; \nu) \zeta(1 + 2\nu) |d_F| \frac{d\nu}{\nu},$

with

$\mathcal{O}_q(m; \nu) = \sum_{(c) \in \mathfrak{O}} \frac{1}{|N(c)|} \sum_{n \in \mathfrak{D}'} \frac{\tau(n \mathcal{D})^{q-1}}{|N(n)|^{1/2 + \nu}} S(m, n; c) \mathcal{H}_q \left( \frac{mn}{c^2}; \nu \right),$

and

$\mathcal{H}_q(x; \nu) = \int_{-\infty}^{\infty} h^q(t; \nu) B_{\nu}(x) d\mu(t).$

Note that the factor 2 arises in (10.8) and (10.9) when we combine the $\epsilon$- and $n$-sums into an $n$-sum, and fold the $\epsilon$-sum into a $(\epsilon)$-sum over ideals.

11. The Twisted First Moment

Let us first treat the diagonal term $\mathcal{D}_1(m)$ as defined by (10.5) and (10.6). It contains the main term for $\mathcal{M}_1(m)$.

Recall the definitions of $d\mu(t)$ and $k(t)$ given by (5.1) and (5.1). Now we apply (4.17) in Lemma 4.1 (1) to analyze $\mathcal{H}_1(m)$. The main term yields

$\int_{-\infty}^{\infty} k(t) d\mu(t) = 2\sqrt{\pi} M T^N (1 + O((M/T)^2)),$

which can be easily seen by truncation near $t = \pm T$ and the change of variable $t \to Mt \pm T$. The error-term contribution is bounded by $(N(m)/T^N)^A$ and hence negligibly small if $N(m) \leq T^{N - \varepsilon}$ and $A$ is large in terms of $\varepsilon$. We conclude that

$\mathcal{D}_1(m) = 4\sqrt{\pi c_1} \frac{MT^N}{\sqrt{N(m)}} (1 + O((M/T)^2)).$
For the Eisenstein term $\mathcal{E}_1(m)$, on inserting (2.6) and (6.14) into (10.7) and estimating the integral trivially, we obtain
\begin{equation}
\mathcal{E}_1(m) = O_{\varepsilon}(MT^{2N\theta + \varepsilon}).
\end{equation}

However, (11.2) may be improved into
\begin{equation}
\mathcal{E}_1(m) = O_{\varepsilon}(MT^{\varepsilon}),
\end{equation}
if $M \geq T^{4\theta + \varepsilon}$ for $F = \mathbb{Q}$ or $M \geq T^{2 + \varepsilon}$ for $F = \mathbb{Q}(\sqrt{d_F})$. For this use the estimate for the second moment of $\zeta\left(\frac{1}{2} + it\right)$ on short intervals in [BW, Theorem 3] or the asymptotic formula for the second moment of $\zeta_F\left(\frac{1}{2} + it\right)$ in [MW].

Finally, we consider the off-diagonal term $\mathcal{O}_1(m)$ given by (10.8), (10.10), and (10.11). Since $|N(m)| \leq T^{N-\varepsilon}$ and $|N(n)| \leq T^{N+\varepsilon}$, one may adjust $\varepsilon$ so that $|mn/c^2| \ll T^2$, and Lemma 7.1 (3)-(6) implies that $\mathcal{H}_1(mn/c^2; \nu$, $\mathcal{O}_1(m; \nu$, and hence $\mathcal{O}_1(m)$ are negligibly small. A remark is that Weil’s bound for $S(m, n; c)$ is needed (one could use $O\left(\sqrt{N(m)c}\tau(c)\right)$ to ensure that the $(c)$-sum is convergent.

In conclusion, the asymptotic formula (1.6) in Theorem 1.1 is established on the foregoing arguments.

### 12. The Twisted Second Moment

This section is devoted to the proof of the asymptotic formula (1.7) for $\mathcal{M}_2(m)$ in Theorem 1.1.

The analysis of $\mathcal{D}_2(m)$, albeit slightly more involved, is similar to that of $\mathcal{D}_1(m)$. By (4.18) and (4.19) in Lemma 4.1 (2), $\mathcal{H}_2(m)$ is equal to
\begin{equation}
\int_{-\infty}^{\infty} k(t)\left(\gamma_1 \left(N \log \sqrt{\frac{1}{4} + t^2} - \log \sqrt{N(m)}\right) + \gamma_0^1\right) \frac{d\mu(t)}{\log \frac{MT^{N-\varepsilon}}{\sqrt{N(m)}}} + O_{\varepsilon}(MT^{N-\varepsilon}),
\end{equation}
with $\gamma_0^1$ defined as in Theorem 1.1. Consequently,
\begin{equation}
\mathcal{D}_2(m) = 4\sqrt{\pi} \tau(m)MT^N \left(\gamma_1 \frac{T^N}{\sqrt{N(m)}} + \gamma_0^1 + O_{\varepsilon}\left((M/T)^2 \log T\right)\right).
\end{equation}

It should be stressed that $\mathcal{D}_2(m)$ only contributes half the main term for $\mathcal{M}_2(m)$.

The trivial estimate for $\mathcal{E}_2(m)$ obtained from (2.6) and (8.14) is as follows:
\begin{equation}
\mathcal{E}_2(m) = O_{\varepsilon}(MT^{4\theta + \varepsilon}).
\end{equation}

By (2.6) and (11.3), we improve (12.2) into
\begin{equation}
\mathcal{E}_2(m) = O_{\varepsilon}(MT^{2N\theta + \varepsilon}),
\end{equation}
for $M \geq T^{4\theta + \varepsilon}$ or $M \geq T^{2 + \varepsilon}$ according as $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{d_F})$. Further, if $F = \mathbb{Q}$, then (12.3) may be improved into
\begin{equation}
\mathcal{E}_2(m) = O_{\varepsilon}(MT^{\varepsilon})
\end{equation}
for $M \geq T^{2 + \varepsilon}$, by the estimate for the fourth moment of $\zeta\left(\frac{1}{2} + it\right)$ on short intervals in [IVI, §6] (see also [LM]). As for the fourth moment of $\zeta_F\left(\frac{1}{2} + it\right)$ for $F = \mathbb{Q}(\sqrt{d_F})$, an explicit spectral formula is known over the Gaussian field in [BM] but currently we do not know how it can be used to obtain non-trivial estimate (asymptotic is beyond our reach as $|\zeta_F(s)|^4$ is of degree 8).

Now we turn to the study of the off-diagonal term $\mathcal{O}_2(m)$ (see 10.9)–10.11).

First of all, by Lemma 7.1 (3)-(6), one may impose the condition $|mn/c^2| \gg T^2$ to the summations, with the cost of a negligible error. Let $\sum_{n} \nu(|x|/R)$ be a dyadic
partition of unity for $F^c_x$, with $R = 2^{j/2}$ and $v(r) \in C^2_c[1, 2]$. It may be exploited to partition the sum in (10.10) into $O(\log T)$ many sums of the form

$$
\Theta_2(m; R; v) = \frac{1}{R^{N/2+\varepsilon}v} \sum_{|c| \leq \sqrt{|m|/R^{N/2+\varepsilon}}} \frac{1}{|N(c)|} \sum_{n \in 0 \setminus \{0\}} \tau(n) S(m, n; c) w \left( \frac{n}{R} \cdot \frac{mR}{c^2}; v \right),
$$

(12.5)

for $R \leq T^{2+\varepsilon}$, where

$$
w(x, A; v) = w(|x|; v) \mathcal{H}_2(Ax; v), \quad w(r; v) = \frac{v(r)}{r^{N/2+\varepsilon}v}.
$$

Clearly, the weight function $w(x, A; v)$ is of the form in (5.1) or (5.2). Note that $w^{(j)}(r; v) \ll_s (\log T)^j$ holds uniformly for $v \in [\varepsilon - i \log T, \varepsilon + i \log T]$.

12.1. Application of the Voronoï Summation. Next, in (12.5) we open the Kloosterman sum $S(m, n; c)$ (as in (2.2)) and apply the Voronoï summation formula (see (3.17) and (3.18)) to the $n$-sum. It is clear that the exponential sum over $(\mathbb{Z}/c\mathbb{Z})^\times$ turns into the Ramanujan sum $S(m - n, 0; c)$.

For the entire zero-frequency contribution, we reverse the procedures above—truncation and partition of unity—and shift the integral contour for $v$ to $\text{Re}(v) = \frac{1}{4}$, costing only negligible errors. We obtain

$$
\mathcal{E}(m) = \frac{2}{\pi^2} c_2 \int_{-\infty}^{\infty} k(t) \int_{(\frac{1}{4})} G(v, t)^2 \zeta(1 + 2v) \tilde{Z}(m; v, t) \frac{dv}{v} d\mu(t),
$$

(12.6)

where

$$
\tilde{Z}(m; v, t) = \lim_{\delta \to 0} \sum_{n \perp t} \zeta_F(1 \pm 2\delta) d_F |v^{\pm \delta} \sum_{(c) \subset 0}^{(c) \subset 0} \frac{S(m, 0; c)}{|N(c)|^{\frac{2\pm 2\delta}{2}} \hat{B}_{it}(m/c^2; \frac{1}{2} - v \pm \delta)},
$$

(12.7)

and

$$
\hat{B}_{it}(y; s) = \int_{(b_2)} B_{it}(xy) x^{s-1} dx.
$$

(12.8)

Note that we can effectively truncate the $t$-integral near $\pm T$ and the $v$-integral at height $\log T$, that the $(c)$-sum and the $x$-integral are absolutely convergent (see the proof of Lemma 9.1), and that the expression in the limit is analytic in the $\delta$-variable. At any rate, it is legitimate to arrange the order of sums and integrals in the above manner.

The next lemma manifests that $\mathcal{E}(m)$ contributes the other half of the main term for $\mathcal{M}_2(m)$. Compare (12.1).

**Lemma 12.1.** We have

$$
\mathcal{E}(m) = 4\sqrt{\pi c_1} \frac{\tau(m) MT^N}{\sqrt{N(m)}} \left( \gamma_1 \log \frac{T^N}{\sqrt{N(m)}} + \gamma_0' + O_\varepsilon ((M/T)^2 \log T) \right).
$$

(12.9)

For the dual sum, it remains to prove the following estimates. For brevity, we have suppressed $v$ from our notation.
Lemma 12.2. Let $R \leq T^{2+\varepsilon}$. Let $w(r) \in C_c^\infty[1, 2]$ satisfy $w^{(j)}(r) \leq (\log T)^j$. Define

$$\tilde{\sigma}_2(m; R) = \sum_{(c) \in \mathbb{Z}} \frac{1}{|N(c)|^2} \sum_{n \neq 0, x \neq 0} \tau(n) S(m - n, 0; c) \tilde{w}_0 \left( \frac{nR}{c^2}, \frac{mR}{c^2} \right),$$

with

$$w(x, A) = w(|x|) \mathcal{H}_2(Ax), \quad \tilde{w}_0(y, A) = \int_{F\mathbb{R}} w(x, A) B_0(xy) dx.$$ Then

$$\sqrt{R} \tilde{\sigma}_2(m; R) \leq \begin{cases} T^{-A}, & \text{if } 0 < m \leq M^{2-\varepsilon}, \\ \frac{\sqrt{mT^{1/2+\varepsilon}}}{\sqrt{M}}, & \text{if } M^{2-\varepsilon} < m \leq T^{2-\varepsilon}, \end{cases}$$

for $F = \mathbb{Q}$, and

$$R \tilde{\sigma}_2(m; R) \leq \sqrt{N(m)} T^{1+\varepsilon} + \frac{M^2 T^{1+\varepsilon}}{\sqrt{N(m)}},$$

for $F = \mathbb{Q}(\sqrt{d_F})$.

The asymptotic formula in (12.7) now follows by combining (12.1)–(12.4), (12.9), (12.12), and (12.13).

12.2. Proof of Lemma 12.1 We start with cleaning up the expression of $\mathcal{E}(m)$ in (12.6)–(12.8). By the change of variable $x \to x/y$ in (12.8),

$$\tilde{B}_{it}(y; s) = \|y\|^{-s} \tilde{B}_{it}(s),$$

where $\tilde{B}_{it}(s) = \tilde{B}_{it}(1; s)$. Then the factor $|N(c)|^{1-2\varepsilon \pm \delta} / |N(m)|^{1/2 - \varepsilon \pm \delta}$ is extracted from $\tilde{B}_{it}(m/c^2; 1/2 - \varepsilon \pm \delta)$. The resulting $(c)$-sum may be evaluated by the Ramanujan identity:

$$\sum_{(c) \in \mathbb{Z}} S(m, 0; c) |N(c)|^{1+2\varepsilon} = \frac{\tau_0(m)}{N(m)^{1/2}} \frac{1}{\zeta(1 + 2\varepsilon)},$$

due to (2.3) and $m = m\mathcal{D}$ (so that $N(m) = |d_F N(m)|$). The two $\zeta(1 + 2\varepsilon)$ in (12.6) and (12.11) cancel, so there is now only a simple pole at $\nu = 0$. By Lemma 9.1, the Mellin integral

$$\tilde{B}_{it}(1/2 - \varepsilon \pm \delta) = \frac{\gamma(1/2 - \varepsilon \pm \delta, t)}{\gamma(1/2 \pm \delta, t)}.$$ Moreover, $c_2 = c_1/2\sqrt{|d_F|}$ (see (3.12) and (3.13)). Thus $\mathcal{E}(m)$ is simplified into

$$\mathcal{E}(m) = c_1 \gamma(m) \int_{-\infty}^\infty k(t) \frac{1}{\pi i} \int_{(\varepsilon)} G(v, t) Z(m; v, t) \frac{dv}{v} dv dt,$$

where

$$Z(m; v, t) = \lim_{\delta \to 0} \sum_{\pm} \zeta_{d_F}^\delta(1 \pm 2\delta) |d_F|^{\pm 2\delta} \frac{\gamma(1/2 - \varepsilon \pm \delta, t)}{\gamma(1/2 \pm \delta, t)},$$

and

$$\gamma(m) = \lim_{\delta \to 0} \sum_{\pm} \zeta_{d_F}^\delta(1 \pm 2\delta) |d_F|^{\pm 2\delta} \frac{\gamma(1/2 - \varepsilon \pm \delta, t)}{\gamma(1/2 \pm \delta, t)}.$$
In view of (4.10) and (12.17), it is clear that $G(v, t)^2 Z(m; v, t)$ is even in the $v$-variable, and therefore the $v$-integral in (12.16) is equal to exactly its value at $v = 0$ (to see this, apply $v \to -v$ to half of the integral). Consequently,

(12.18) \[ \mathcal{E}(m) = c_1 \frac{\tau(m)}{N(m)} \int_{-\infty}^{\infty} k(t) Z(m; 0, t) d\mu(t). \]

We have

(12.19) \[ Z(m; 0, t) = 2\gamma_0 + \gamma_1 \left( \log \frac{|dF|^2}{N(m)} + 2\psi(t) \right), \]

for $\gamma_0, \gamma_1, \text{and} \psi(t)$ as in (2.9) and (4.10). By (12.19), (12.18), and (12.19), we can conclude the proof with the same arguments for the diagonal term $D_2(m)$.

### 12.3. Proof of Lemma [12.2] for $F = \mathbb{Q}$

In this subsection, let $c, d, m,$ and $n$ be positive integers.

It follows from $m \leq T^{2-\varepsilon}$ and $c^2 \leq mR/T^2$ that $nR/c^2 \geq T^\varepsilon$, so Lemma 8.1 (1) yields

\[
\tilde{\omega}_0 \left( \pm \frac{nR}{c^2}, \frac{mR}{c^2} \right) = \sqrt{cMT^{1+\varepsilon}} \sum \frac{\tau(n)}{\sqrt{n}} \sum_{d|m+n} \delta_{\epsilon} \sum_{cd \leq \sqrt{mR}/c} \frac{|\mu(c)|}{\sqrt{c}},
\]

Recall that we defined $\Psi^+(x, \Delta) = 0$ unless $\Delta > M^{1-\varepsilon}/T$ (due to Lemma 7.3). Moreover, by $m \leq T^{2-\varepsilon}$ and $R \leq T^{2+\varepsilon}$, one may adjust $\varepsilon$ so that $\sqrt{mR}/c \leq T^{2-\varepsilon}$.

By the formula for the Ramanujan sum $S(m \pm n, 0; c)$ in (2.3) and the estimates for the $\Psi^\pm$-integrals in (8.3) in particular Proposition 8.6 (1) and 8.7 (1), we infer that, up to a negligibly small error, $\sqrt{R} \tilde{D}_2(m; R)$ is bounded by the sum of

(12.20) \[ \tilde{D}^-(m) = \frac{MT^{1+\varepsilon}}{\sqrt{mR}} \sum_{0<n<m/M^{2-\varepsilon}} \frac{\tau(n)}{\sqrt{n}} \sum_{d|m+n} \delta_{\epsilon} \sum_{cd \leq \sqrt{mR}/M^{1-\varepsilon}T} \frac{|\mu(c)|}{\sqrt{c}}, \]

and

(12.21) \[ \tilde{D}^+(m) = \frac{MT^{1+\varepsilon}}{\sqrt{mR}} \sum_{0<l|m/M^{2-\varepsilon}} \frac{\tau(m+l)}{\sqrt{|l|}} \sum_{d|m} \delta_{\epsilon} \sum_{cd \leq \sqrt{mR}/M^{1-\varepsilon}T} \frac{|\mu(c)|}{\sqrt{c}}, \]

with $l = n - m$. A critical point is that the diagonal term with $n = m$ ($l = 0$) is removed from the second sum $\tilde{D}^+(m)$ because it is negligibly small by Proposition 8.7 (1). Finally, if $m \leq M^{2-\varepsilon}$ then $\tilde{D}^-(m)$ and $\tilde{D}^+(m)$ vanish since the $n$-sum and $l$-sum have no terms, and if otherwise we have estimates

\[
\tilde{D}^-(m) \ll \frac{MT^{1/2+\varepsilon}}{\sqrt{m}} \sum_{0<n<m/M^{2-\varepsilon}} \frac{\tau(n)\tau(m+n)}{\sqrt{n}} \ll \frac{\sqrt{mT^{1/2+\varepsilon}}}{\sqrt{M}},
\]

\[
\tilde{D}^+(m) \ll \sqrt{mT^{1/2+\varepsilon}} \sum_{0<l|m/M^{2-\varepsilon}} \frac{\tau(l)\tau(m+l)}{\sqrt{|l|}} \ll \frac{\sqrt{mT^{1/2+\varepsilon}}}{\sqrt{M}},
\]

as desired.
12.4. Proof of Lemma 12.2 for \( F = \mathbb{Q}(\sqrt{d_F}) \). For the case \( F = \mathbb{Q}(\sqrt{d_F}) \) we use Lemma 8.1 (2), Proposition 8.6 (2) and 8.7 (2). Let \( z = x + iy = \sqrt{n/m} \) (\( m, n \in \mathcal{O}' \setminus \{0\} \)). We partition the region \( |x| < 1 + \rho \) and \( |y| < \rho \) in Proposition 8.6 (2) (\( \rho = M^2/M \)) according to the \( x \)-coordinate as follows:

\[
|x| < \rho, \quad \delta < |x| \leq 2\delta, \quad |x| - 1 < \rho, \quad \delta < 1 - |x| \leq 2\delta,
\]

for dyadic \( \delta \) of the form \( 2^{-j} (j = 2, 3, \ldots) \) with \( \rho \ll \delta < 1/2 \). In view of Lemma 8.3 the problem is reduced to proving that the following four sums have bounds as in (12.13):

\[
\tilde{\mathcal{O}}^{-}(m) = \frac{MT^{2+\varepsilon}}{|m|\sqrt{R}} \sum_{0 < |n| < |m|/M^{2-\varepsilon}} \frac{\tau(n \mathcal{D})}{\sqrt{|n|}} \mathcal{R}(m - n, m),
\]

\[
\tilde{\mathcal{O}}_{\delta}^-(m) = \frac{MT^{2+\varepsilon}}{|m|\delta \sqrt{|m|\mathcal{R}}} \sum_{\mathbb{R}(n/m) = \delta^2, \mathbb{I}(n/m) < \delta/M^{1-\varepsilon}} \tau(n \mathcal{D})\mathcal{R}(m - n, m),
\]

\[
\tilde{\mathcal{O}}^{+}(m) = \frac{MT^{2+\varepsilon}}{\sqrt{|m|\delta}} \sum_{0 < |l| < |m|/M^{1-\varepsilon}} \frac{\tau((m + l) \mathcal{D})}{|l|} \mathcal{R}(l, m),
\]

\[
\tilde{\mathcal{O}}_{\delta}^{+}(m) = \frac{MT^{2+\varepsilon}}{|m|\delta \sqrt{|m|\mathcal{R}}} \sum_{\mathbb{R}(l/m) = \delta, \mathbb{I}(l/m) < M^2/M} \tau((m + l) \mathcal{D})\mathcal{R}(l, m),
\]

where

\[
\mathcal{R}(l, m) = \sum_{\delta | \mathcal{D}} \sqrt{N(\mathcal{D})} \sum_{\mathbb{N}(\mathcal{D}) | |m|/T^2} \frac{|\mu(\mathcal{D})|}{\sqrt{N(\mathcal{D})}}
\]

It is clear that

\[
\mathcal{R}(l, m) = O\left(\frac{\tau(\mathcal{D})\sqrt{|m|\mathcal{R}}}{T}\right),
\]

therefore

\[
\tilde{\mathcal{O}}^{-}(m) \ll \frac{MT^{1+\varepsilon}}{|m|} \sum_{0 < |n| < |m|/M^{2-\varepsilon}} \frac{\tau(n \mathcal{D})\tau((m - n) \mathcal{D})}{|n|} \ll \frac{|m|T^{1+\varepsilon}}{M^2},
\]

\[
\tilde{\mathcal{O}}_{\delta}^{-}(m) \ll \frac{MT^{1+\varepsilon}}{|m|\delta} \sum_{\mathbb{R}(n/m) = \delta^2, \mathbb{I}(n/m) < \delta/M^{1-\varepsilon}} \tau(n \mathcal{D})\tau((m - n) \mathcal{D})
\]

\[
\ll \frac{MT^{1+\varepsilon}}{|m|\delta} \sum_{\mathbb{R}(n/m) = \delta^2, \mathbb{I}(n/m) < \delta/M^{1-\varepsilon}} 1,
\]

and similarly

\[
\tilde{\mathcal{O}}^{+}(m) \ll \frac{MT^{1+\varepsilon}}{|m|} \sum_{0 < |l| < |m|/M^{1-\varepsilon}} \frac{\tau(l \mathcal{D})\tau((m + l) \mathcal{D})}{|l|} \ll |m|T^{1+\varepsilon},
\]

\[
\tilde{\mathcal{O}}_{\delta}^{+}(m) \ll \frac{MT^{1+\varepsilon}}{|m|\delta} \sum_{\mathbb{R}(l/m) = \delta, \mathbb{I}(l/m) < M^2/M} \tau(l \mathcal{D})\tau((m + l) \mathcal{D})
\]

\[
\ll \frac{MT^{1+\varepsilon}}{|m|\delta} \sum_{\mathbb{R}(l/m) = \delta, \mathbb{I}(l/m) < M^2/M} 1.
\]
we infer that

\[ \lambda \leq 5.4 \text{ (with } a \text{ and } b \text{)} \]

\[ \text{(13.1)} \]

The final estimation for \( \tilde{\mathcal{O}}_\delta (m) \) can be done by the next lemma.

**Lemma 12.3.** Let \( m \in \mathcal{O}' \). For \( Q \preceq P \) define the rectangle \( R(P, Q) = \{ x + iy : |x| < P, |y| < Q \} \). The number of points in \( m^{-1} \mathcal{O}' \cap R(P, Q) \) has bound \( O((|m|P + 1)(|m|Q + 1)) \).

**Proof.** Firstly, it is clear that \( m \cdot R(P, Q) \) is contained in a parallelogram of the form \( R_a(|m|P, |m|Q) = \{ x + iy : |x| \leq |m|P, |y| \leq |m|Q \} \). Exchanging \( x \leftrightarrow y \) if necessary, one may assume that \( |a| \leq 1 \). Secondly, \( \mathcal{O}' \) is contained in a certain rectangular lattice spanned by a real scalar and an imaginary scalar. By rescaling, it is reduced to counting the integral lattice points in \( R_a(|m|P, |m|Q) \), which can be done very easily.

It follows from Lemma 12.3 along with \( M^\varepsilon /M \leq \delta < 1/2 \), that

\[ \tilde{\mathcal{O}}_\delta (m) \leq |m|T^{1+\varepsilon} + \frac{M^2T^{1+\varepsilon}}{|m|}. \]

### 13. Moments without Twist and Smooth Weight

In this section, we use the unsmoothing technique in §5 to prove Corollary 1.2.

By the proof of Theorem 1.1 in the previous sections, for \( T^\varepsilon \leq M \leq T^{1-\varepsilon} \) we have

\[ M_1(1) + \mathcal{E}(1) = 4\sqrt{\pi}c_1MTN + O_\varepsilon\left(M^3/T^{2-N}\right), \]

and

\[ M_2(1) + \mathcal{E}(1) = 8\sqrt{\pi}c_1MT\left(\log T + \gamma_0\right) + O_\varepsilon\left(M^3\log T/T\right), \]

if \( F = \mathbb{Q} \), and

\[ M_2(1) + \mathcal{E}(1) = 8\sqrt{\pi}c_1MT^2(2\gamma_1\log T + \gamma_0') + O_\varepsilon\left(M^2T^{1+\varepsilon}\right), \]

if \( F = \mathbb{Q}(\sqrt{d_F}) \). It follows that

\[ M_q(1) + \mathcal{E}_q(1) = O_\varepsilon\left(M^{N+\varepsilon}\right) \]

for any \( T^\varepsilon \leq M \leq T^{1-\varepsilon} \).

It is known that \( L(\tfrac{1}{2}, f) \) is non-negative by [Guo]. Applying Lemma 5.3 and 5.4 (with \( \lambda = N + \varepsilon \) and \( a_f = L(\tfrac{1}{2}, f) \)) and the averaging process to (13.1)–(13.3), we infer that

\[ N_1(T, H) = 4c_1 \int_{T-H}^{T+H} K^N dK + O_\varepsilon\left(M^{N+\varepsilon}\right), \]

and

\[ N_2(T, H) = 8c_1 \int_{T-H}^{T+H} K(\log K + \gamma_0')dK + O_\varepsilon\left(M^{1+\varepsilon}\right), \]

if \( F = \mathbb{Q} \), and

\[ N_2(T, H) = 8c_1 \int_{T-H}^{T+H} K^2(2\gamma_1\log K + \gamma_0')dK + O_\varepsilon\left(M^{2+\varepsilon}\right), \]

\[ \text{[For the case } F = \mathbb{Q}, \text{ the reader may compare our formulae with those in [Liu1] Proposition 1].} \]
if $F = \mathbb{Q}(\sqrt{d_F})$. Then Corollary 12 follows on choosing $M = T^\varepsilon$.

Finally, we remark that the arguments for $\mathcal{E}_q(m)$ in 111 and 112 may be easily employed here to show that, except when $T^{\varepsilon} < H < T^{\varepsilon + \varepsilon}$ for $q = 2$ and $F = \mathbb{Q}(\sqrt{d_F})$, the Eisenstein contribution in $\mathcal{N}_q(T, H)$ is $O(T^{N + \varepsilon})$ so that it may be removed from the asymptotic formulae in Corollary 12.

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