The Factorial-Basis Method for Finding Definite-Sum Solutions of Linear Recurrences With Polynomial Coefficients

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Abstract

The problem of finding a nonzero solution of a linear recurrence $Ly = 0$ with polynomial coefficients where $y$ has the form of a definite hypergeometric sum, related to the Inverse Creative Telescoping Problem of [15, Sec. 8], has now been open for three decades. Here we present an algorithm (implemented in a SageMath package) which, given such a recurrence and a quasi-triangular, shift-compatible factorial basis $B = \langle P_k(n) \rangle_{k=0}^{\infty}$ of the polynomial space $\mathbb{K}[n]$ over a field $\mathbb{K}$ of characteristic zero, computes a recurrence satisfied by the coefficient sequence $c = \langle c_k \rangle_{k=0}^{\infty}$ of the solution $y_n = \sum_{k=0}^{\infty} c_k P_k(n)$ (where, thanks to the quasi-triangularity of $B$, the sum on the right terminates for each $n \in \mathbb{N}$). More generally, if $B$ is $m$-sieved for some $m \in \mathbb{N}$, our algorithm computes a system of $m$ recurrences satisfied by the $m$-sections of the coefficient sequence $c$. If an explicit nonzero solution of this system can be found, we obtain an explicit nonzero solution of $Ly = 0$.

Keywords: definite hypergeometric sums; shift-compatible factorial bases; (formal) polynomial series; quasi-triangular bases; binomial-coefficient bases; solutions of linear recurrences

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1. Introduction

By definition, a P-recursive (or: holonomic) sequence over a field $\mathbb{K}$ of characteristic 0 is given by a homogeneous linear recurrence with polynomial coefficients, together with suitable initial conditions. Often one wishes to find explicit representations of P-recursive sequences, so algorithms have been devised to find solutions of such recurrences within a given class of explicitly representable sequences. Some well-known examples of this kind are the algorithms for finding
polynomial [1], rational [2, 3], hypergeometric [28, 19], d’Alembertian [5], and Liouvillian solutions [18], as well as hypergeometric solutions in the setting of $\Pi\Sigma^*$-fields [4]. These classes do not exhaust explicitly representable P-recursive sequences; for instance, every definite hypergeometric sum on which Zeilberger’s Creative Telescoping algorithm [36, 37] succeeds is a P-recursive sequence, but such sequences are typically not Liouvillian. Hence it makes sense to consider the Inverse Creative Telescoping Problem (ICTP): given a homogeneous linear recurrence with polynomial coefficients with no Liouvillian solutions, find its solutions in the form of definite sums of a given type. Problems of this kind have been posed in [27, p. 84, item 2], and again in [15, Sec. 8].

Here we make a small but important step towards solving ICTP. Let $Ly = 0$ be the equation we wish to solve, where $L$ is a linear recurrence operator with polynomial coefficients. We provide an algorithm which, given $L$ and a sequence $B = \langle P_k(n) \rangle_{k=0}^\infty$ of polynomials in $n$ which is a quasi-triangular, shift-compatible factorial basis of the polynomial space $\mathbb{K}[n]$ (see Definitions 3, 9 and 28), returns a linear recurrence operator $L'$ such that for any sequence $y \in \mathbb{K}^N$ of the form

$$y_n = \sum_{k=0}^\infty c_k P_k(n) \quad \text{for all } n \in \mathbb{N}$$

for some $c \in \mathbb{K}^N$, we have $Ly = 0$ if and only if $L'c = 0$. So, if we can solve the latter equation for the unknown sequence $c$ (by, say, one of the algorithms mentioned in the preceding paragraph, or by using our factorial-basis method (FBM) recursively), $y$ in (1) will be an explicit definite-sum solution of $Ly = 0$.

We point out that FBM can be viewed as a kind of a discrete converse of the method of generating functions (GFM) for solving recurrences of the form $Ly = 0$. As is well known, GFM produces a differential equation $L'f = 0$ satisfied by the (ordinary) generating function $f(x) = \sum_{n=0}^\infty y_n x^n$, which we then solve (if we can) and read off the coefficient sequence $\langle y_n \rangle_{n=0}^\infty$ from the obtained solution. On the other hand, FBM assumes that the unknown sequence $y$ is of the form (1) (with the basis $B$ given as part of the input, playing the role of the power basis $\langle x^n \rangle_{n=0}^\infty$ in GFM), and produces a recurrence equation $L'c = 0$ which we then solve for $c$ (if we can), and obtain the solution $y$ from (1). Like other solution methods, FBM can also be used for factoring linear recurrence operators (cf. Example 44), as well as for deriving summation identities (cf. Examples 25, 44) when some other form of the solution is already known.

There is also some remote similarity between our problem where, given a recurrence operator $L$ and a basis $\langle P_k(n) \rangle_{k=0}^\infty$ of the space of polynomials $\mathbb{K}[n]$, we seek sequences $c \in \mathbb{K}^N$ such that $y$ in (1) satisfies $Ly = 0$, and the classical Fredholm or Volterra integral equations of the first kind, as well as the Stieltjes
moment problem (cf. [26]) having the form
\[
\begin{align*}
g(n) &= \int_0^\infty K(n,t)f(t)dt \quad \text{(Fredholm)} \\
g(n) &= \int_0^n K(n,t)f(t)dt \quad \text{(Volterra)} \\
m_n &= \int_0^\infty t^n f(t)dt \quad \text{(Stieltjes)}
\end{align*}
\]
where the left-hand sides \(g(n)\) resp. \(\langle m_n \rangle_{n=0}^\infty\), as well as the kernels \(K(n,t)\) resp. \(t^n\) are given, and one seeks the unknown function \(f(t)\). Apart from our problem being “discrete” while the above three are “continuous”, the main difference between them lies in the fact that our sequence \(y\) is given recursively, and our goal is to find its explicit representation in terms of the unknown \(c\), while in the other three problems the left-hand sides are presumably given explicitly, and finding \(f(t)\) (corresponding to our \(c\)) is the final goal. Nevertheless, we will occasionally write our polynomial basis element \(P_k(n)\) as \(K(n,k)\), and call it the kernel of \(K\).

Note that recently, Imamoglu and van Hoeij [20] have solved the important related problem of finding definite-sum solutions of second-order linear differential equations with rational-function coefficients. Their algorithms (very effective in practice, but called “heuristic” by the authors as they haven’t been fully proven yet) find solutions in the form \(A \cdot 2F_1(a_1,a_2;b_1;f)\) or in the form \(A \cdot (r_0 \cdot 2F_1(a_1,a_2;b_1;f) + r_1 \cdot 2F_1(a_1,a_2;b_1;f))\) where \(A\) has algebraic logarithmic derivative, and \(f, r_0, r_1\) are algebraic.

The contents of the rest of the paper are as follows: In Section 2, we define factorial bases of the polynomial algebra \(K[x]\), and the notion of their compatibility with endomorphisms of \(K[x]\). Following [6], to each factorial basis \(B\) we assign the algebra \(K[[B]]\) of formal polynomial series as a generalization of the algebra \(K[[x]]\) of formal power series, with the basis element \(P_k(x) \in B\) in the former algebra playing the role of \(x^k\) in the latter. We extend the action of an endomorphism \(L\) of \(K[x]\) to \(K[[B]]\) in a natural way, then assign to \(L\) its associated operator \(L' = RBL\) acting on sequences in such a way that \(L(\sum_{k=0}^\infty c_kP_k(x)) = 0\) iff \(\sum_{k=0}^\infty (L'c_k)P_k(x) = 0\). This enables us to solve the equation \(Ly = 0\) for \(y \in K[[B]]\) by solving the (perhaps simpler) equation \(L'c = 0\) for \(c \in K^N\).

In order for FBM to be useful, we need our formal polynomial series \(y\) to have a definite value \(y_n \in K\) for every \(n \in N\). For example, when \(P_k(n) = \binom{n}{k}\), this is true since the series \(y\) in \([1]\) is terminating (in fact, \(y\) is the classical binomial transform of \(c\)). With this example in mind, in Section 3 we define quasi-triangular bases which are factorial shift-compatible bases with the properties that for a fixed \(n\), we have \(P_k(n) = 0\) for all \(k\) large enough compared with \(n\), and that for each \(a \in K^N\) there is some \(b \in K^N\) such that \(a_n = \sum_{k=0}^\infty b_kP_k(n)\).

Section 4 defines generalized binomial-coefficient bases, and provides a mechanism for creating many new compatible bases by taking products of the already
constructed ones. In bases that are products of \( m > 1 \) factors, the coefficients \( \alpha_{k,i} \) expressing the actions of the operators on the basis elements are quite complicated conditional expressions depending on the residue class of \( k \mod m \). Therefore in Section 5 we extend our approach to the so-called sieved polynomial bases where the definition of the \( k \)-th basis element \( P_k(x) \) depends on the residue class of \( k \mod m \). To facilitate the computations, we do not attempt to compute the associated operator \( L' = RB \) directly but instead represent it by a matrix of operators \( [RB] = [L_{r,j}]_{r,j=0}^{m-1} \) where the operator \( L_{r,j} \) expresses the contribution of the \( j \)-th \( m \)-section of the coefficient sequence of \( y \) to the \( r \)-th \( m \)-section of the coefficient sequence of \( Ly \). Section 6 presents several nontrivial applications of the developed theory and algorithms, such as the explicit solution of a recurrence equation of order 7 (which leads to complete factorization of the corresponding operator – see Example 44), and construction of explicit solutions of the so-called Apéry recurrences for \( \zeta(2) \) and \( \zeta(3) \) (45, and 46).

In Section 7 we introduce a particular instance of sieved polynomial bases called shuffled polynomial bases, which are defined as some specific interlacing or shuffling of basic sieved polynomial bases. This can be seen as a straightforward generalization of the product bases defined in Section 4. This section includes a fully constructive way of extending the compatibilities of different operators.

**Notation 1.** \( \mathbb{N} = \{0, 1, 2, \ldots \} \) denotes the set of nonnegative integers, \( \mathbb{K} \) a field of characteristic zero, \( \mathbb{K}^\mathbb{N} \) the set of all sequences with terms from \( \mathbb{K} \), \( \mathbb{K}[x] \) the \( \mathbb{K} \)-algebra of univariate polynomials over \( \mathbb{K} \), and \( \mathcal{L}_{\mathbb{K}[x]} \) the \( \mathbb{K} \)-algebra of linear operators \( L : \mathbb{K}[x] \to \mathbb{K}[x] \).

**Definition 2.** Let \( m \in \mathbb{N} \setminus \{0\} \) and \( j \in \{0, 1, \ldots, m-1\} \).

- A sequence \( c \in \mathbb{K}^\mathbb{N} \) is called the \( j \)-th \( m \)-section of a sequence \( a \in \mathbb{K}^\mathbb{N} \) if \( c_k = a_{mk+j} \) for all \( k \in \mathbb{N} \). We say that \( c \) is obtained from \( a \) by multisection, and denote it by \( s_j^m a \).

- A sequence \( c = \Lambda(a^{(0)}, a^{(1)}, \ldots, a^{(m-1)}) \in \mathbb{K}^\mathbb{N} \) is called the interlacing of sequences \( a^{(0)}, a^{(1)}, \ldots, a^{(m-1)} \in \mathbb{K}^\mathbb{N} \) if \( c_k = a_k^{(r)} \) where \( k = qm + r \) with \( q \in \mathbb{N} \) and \( r \in \{0,1,\ldots,m-1\} \) for all \( k \in \mathbb{N} \).

2. Formal polynomial series

The power-series method is a time-honored approach to solving differential equations by reducing them to recurrences satisfied by the coefficient sequences of their power series solutions. In [6] it was shown how, by generalizing the notion of formal power series to formal polynomial series, one can use this method to find solutions of other linear operator equations such as \( q \)-difference equations, and recurrence equations themselves, which interest us here. In this section we summarize some relevant definitions, examples and results from [6].

**Definition 3.** We call a sequence of polynomials \( B = \langle P_k(x) \rangle_{k=0}^\infty \) from \( \mathbb{K}[x] \) a factorial basis of \( \mathbb{K}[x] \), if for all \( k \in \mathbb{N} \):
\textbf{P1.} \deg P_k(x) = k, \\
\textbf{P2.} P_k(x) \mathbin{|} P_{k+1}(x) \text{ in } \mathbb{K}[x].

Note that due to property \textbf{P1}, any factorial basis of \mathbb{K}[x] is a basis of \mathbb{K}[x] as a vector space over \mathbb{K}.

\textbf{Notation 4.} Denote by \mathcal{P} = \langle x^k \rangle_{k=0}^\infty \text{ the power basis}, and by \mathcal{C} = \langle (\xi^k) \rangle_{k=0}^\infty \text{ the binomial-coefficient basis of } \mathbb{K}[x], \text{ respectively.}

\textbf{Example 5.} Clearly, both \mathcal{P} and \mathcal{C} are factorial bases of \mathbb{K}[x].

\textbf{Proposition 6.} \mathcal{B} = \langle P_k(x) \rangle_{k=0}^\infty \text{ is a factorial basis if there are a root sequence } \rho = \langle \rho_1, \rho_2, \rho_3, \ldots \rangle \in \mathbb{K}^{\mathbb{N}\setminus\{0\}} \text{ and a sequence } \langle c_0, c_1, c_2, \ldots \rangle \in (\mathbb{K}^*)^\mathbb{N} \text{ such that}

\begin{equation}
P_k(x) = c_k(x - \rho_1)(x - \rho_2) \cdots (x - \rho_k) \text{ for all } k \in \mathbb{N}.
\end{equation}

\textit{Proof.} If \textbf{[2]} holds then \langle P_k(x) \rangle_{k=0}^\infty \text{ clearly satisfies } \textbf{P1} \text{ and } \textbf{P2}.

Conversely, if \langle P_k(x) \rangle_{k=0}^\infty \text{ satisfies } \textbf{P1} \text{ and } \textbf{P2} \text{ then for each } k \in \mathbb{N} \text{ there are } u_k \in \mathbb{K}^* \text{ and } v_k \in \mathbb{K} \text{ such that } P_{k+1}(x) = (u_k x - v_k) P_k(x) = u_k P_k(x) (x - v_k / u_k).

Let \( c_0 := P_0(x) \in K^* \). By induction on \( k \), we see that each \( P_k(x) \) \text{ is of the form \textbf{[2]} with } c_k = c_0 \prod_{j=0}^{k-1} u_j \text{ and } \rho_k = v_k / u_k \text{ for all } k \in \mathbb{N} \setminus \{0\}. \quad \blacksquare

\textbf{Example 7.} The root sequence of the power basis \( \mathcal{P} \) is \( \rho = \langle 0, 0, 0, 0, \ldots \rangle \), and \( c_k = 1 \) for all \( k \in \mathbb{N} \). The root sequence of the binomial-coefficient basis \( \mathcal{C} \) is \( \rho = \langle 0, 1, 2, 3, \ldots \rangle \), and \( c_k = 1 \) for all \( k \in \mathbb{N} \).

Note that in the umbral calculus, factorial bases with \( c_k = 1 \) for all \( k \in \mathbb{N} \) are known as \textit{sequences of polynomials with persistent roots} (cf. [17]).

\textbf{Notation 8.} Denote by \( D, E, Q, X \in \mathcal{L}_{\mathbb{K}[x]} \) the \textit{differentiation}, \textit{shift}, \textit{q-shift}, and \textit{multiplication-by-the-independent-variable operators}, respectively, acting on polynomials \( p \in \mathbb{K}[x] \) by

\begin{align*}
Dp(x) &= p'(x), \\
Ep(x) &= p(x + 1), \\
Qp(x) &= p(qx), \\
Xp(x) &= xp(x)
\end{align*}

where \( q \in \mathbb{K}^* \) is not a root of unity, so that the operators \( \{1, Q, Q^2, \ldots\} \) are linearly independent.

\textbf{Definition 9.} A factorial basis \( \mathcal{B} \) of \( \mathbb{K}[x] \) and an operator \( L \in \mathcal{L}_{\mathbb{K}[x]} \) are \textit{compatible} with each other if there are \( A, B \in \mathbb{N} \) such that, for all \( k \in \mathbb{N} \), there are \( \alpha_{k,i} \in \mathbb{K} \) with \( -A \leq i \leq B \), such that

\begin{equation}
LP_k(x) = \sum_{i=-A}^{B} \alpha_{k,i} P_{k+i}(x) \text{ for all } k \in \mathbb{N},
\end{equation}

with \( P_j(x) = 0 \) when \( j < 0 \). To assert that \textbf{[3]} holds for specific \( A, B \in \mathbb{N} \), we will say that \( \mathcal{B} \) is \((A, B)\)-\textit{compatible} with \( L \). 

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Example 10.

- $(x^k)' = kx^{k-1}$, so $P$ is $(1,0)$-compatible with $D$ (simply, take $\alpha_{k,-1} = k$, $\alpha_{k,0} = 0$),
- $(x+1)^k = \binom{x}{k-1} + \binom{x}{k}$, so $C$ is $(1,0)$-compatible with $E$ (take $\alpha_{k,-1} = 1$, $\alpha_{k,0} = 1$),
- $(qx)^k = q^k x^k$, so $P$ is $(0,0)$-compatible with $Q$ (take $\alpha_{k,0} = q^k$),
- the basis $B = \langle (x+k)^k \rangle_{k=0}^\infty$ is not compatible with $E$, since the well-known identity $(x+k+1)^k = \sum_{j=0}^{k} \binom{x+j}{j}$ implies that $EP_k(x) = \sum_{j=0}^{k} P_j(x)$, where the upper bound depends on $k$.

Proposition 11. A factorial basis $B$ of $K[x]$ is $(A,B)$-compatible with $L \in L_\mathbb{K}[x]$ if and only if

C1. $\deg LP_k(x) \leq k + B$ for all $k \geq 0$,
C2. $P_k-A(x) \mid LP_k(x)$ for all $k \geq A$.

Proof. Necessity of these two conditions is obvious. For sufficiency, let

$$LP_k(x) = \sum_{j=0}^{\deg LP_k(x)} \lambda_{j,k} P_j(x)$$

be the expansion of $LP_k(x)$ w.r.t. $B$. By C1, we can replace the upper summation bound by $k + B$. Rewriting the resulting equation as

$$LP_k(x) - \sum_{j=k-A}^{k+B} \lambda_{j,k} P_j(x) = \sum_{j=0}^{k-A-1} \lambda_{j,k} P_j(x),$$

we see by C2 and P2 that $P_{k-A}(x)$ divides the left side, while the right side is of degree less than $k-A = \deg P_{k-A}(x)$. Hence both sides vanish, and so

$$LP_k(x) = \sum_{j=k-A}^{k+B} \lambda_{j,k} P_j(x) = \sum_{i=-A}^{B} \lambda_{k+i,k} P_{k+i}(x) = \sum_{i=-A}^{B} \alpha_{k,i} P_{k+i}(x)$$

where $\alpha_{k,i} := \lambda_{k+i,k}$. This proves $(A,B)$-compatibility of $B$ with $L$.

Corollary 12. Every factorial basis is $(0,1)$-compatible with $X$.

Proof. Since $\deg xP_k(x) = k + 1$ and $P_k(x) \mid xP_k(x)$ for all $k \geq 0$, this follows from Proposition 11.

Proposition 13. If a factorial basis $B$ is $(A,B)$-compatible with $E$, then $B$ is also $(A,0)$-compatible with $E$.

Proof. Since the shift operator $E$ preserves polynomial degrees, the coefficients of $P_j(x)$ with $j > k$ in the expansion of $EP_k(x)$ w.r.t. $B$ all vanish.
Proposition 14. A factorial basis $\mathcal{B}$ of $\mathbb{K}[x]$ having the root sequence $\rho$ is $(A,0)$-compatible with the shift operator $E$ if and only if for all $k \in \mathbb{N}$ the following inclusion of multisets is valid:

$$[\rho_1 + 1, \rho_2 + 1, \ldots, \rho_k + 1] \subseteq [\rho_1, \rho_2, \ldots, \rho_{k+A}].$$

(4)

Proof. We use Proposition [11] with $L = E$ and $B = 0$. Since for every factorial basis we have $\deg EP_k(x) = \deg P_k(x) = k$, condition $\text{C1}$ is always satisfied. Hence $(A,0)$-compatibility of $\mathcal{B}$ with $E$ is equivalent to condition $\text{C2}$ which requires that $P_{k-A}(x)$ divides $P_k(x + 1)$ for all $k \geq A$. In terms of $\rho$ this is equivalent to $[\rho_1, \rho_2, \ldots, \rho_{k-A}] \subseteq [\rho_1 - 1, \rho_2 - 1, \ldots, \rho_k - 1]$ for all $k \geq A$, or

$$[\rho_1, \rho_2, \ldots, \rho_k] \subseteq [\rho_1 - 1, \rho_2 - 1, \ldots, \rho_{k+A} - 1]$$

for all $k \geq 0$. By adding 1 to all the terms on both sides, this turns into (4). $\square$

Example 15. Since $[1, 1, 1, \ldots, 1] \nsubseteq [0, 0, \ldots, 0]$ for all $k \geq 1$ and $A \geq 0$, Proposition [14] implies that $\mathcal{P}$ is not compatible with $E$.

- Since $[1, 2, 3, \ldots, k] \subseteq [0, 1, 2, \ldots, k - 1, k]$ for all $k \geq 0$, Proposition [14] implies that $\mathcal{C}$ is $(1,0)$-compatible with $E$.

Compatibility of a factorial basis with the differentiation operator $D$ can also be characterized in terms of its root sequence as shown in the next result.

Proposition 16. Let $\mathcal{B}$ be a factorial basis with the root sequence $\rho$. Then $\mathcal{B}$ is $(p,0)$-compatible with $D$ if and only if, for all $n \in \mathbb{N}$ we have

$$\{\rho_1, \ldots, \rho_n\} \subset \{\rho_{n+1}, \ldots, \rho_{n+p}\}.$$  

Proof. Let $g_n(x) = \gcd(P_n(x), P_n'(x))$. It is well known that, if we write $P_n(x) = A_n(x)g_n(x)$ and $P_n'(x) = B_n(x)g_n(x)$, we have that $A_n(x)$ is a polynomial with simple roots and such that $A_n(\rho_m) = 0$ for all $m \leq n$ (i.e., it contains all the different roots up to and including $\rho_n$).

By Proposition [11], condition $\text{C2}$, we have that $D$ is $(p,0)$-compatible with $\mathcal{B}$ if and only if $P_n(x)$ divides $P_{n+p}'(x)$ for all $n \geq 0$. Write

$$P_{n+p}(x) = P_n(x)(x - \rho_{n+1}) \cdots (x - \rho_{n+p}).$$

Using the product rule, we have that $P_n(x)$ divides $P_{n+p}'(x)$ if and only if

$$P_n(x) \mid P_n'(x)(x - \rho_{n+1}) \cdots (x - \rho_{n+p}).$$

(5)

Using the definition of $A_n(x)$ and $B_n(x)$ described above, we have that (5) holds if and only if:

$$A_n(x) \mid B_n(x)(x - \rho_{n+1}) \cdots (x - \rho_{n+p}),$$

and using the fact that $\gcd(A_n(x), B_n(x)) = 1$, this is equivalent to:

$$A_n(x) \mid (x - \rho_{n+1}) \cdots (x - \rho_{n+p}).$$

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By the definition of $A_n(x)$, this is equivalent to
\[
\{\rho_1, \ldots, \rho_n\} \subset \{\rho_{n+1}, \ldots, \rho_{n+p}\}.
\]

\[\square\]

**Corollary 17.** Let $B$ be a factorial basis with the root sequence $\rho$. If $B$ is $(p,0)$-compatible with $D$, then the number of distinct roots in $\rho$ is at most $p$.

*Proof.* Using Proposition 16 if $B$ is $(p,0)$-compatible, then the inclusion of roots implies that the distinct roots of $P_n(x)$ are at most $p$. \[\square\]

**Corollary 18.** Let $B$ be a factorial basis whose root sequence $\rho$ is periodic with period length $p$. Then $B$ is $(p,0)$-compatible with $D$. Conversely, if $B$ is $(p,0)$-compatible with $D$ and $B$ has $p$ different roots, then there is $n_0 \in \mathbb{N}$ such that $\rho_{n+p} = \rho_n$ for all $n \geq n_0$.

*Proof.* If the root sequence of $B$ is periodic with period length $p$, then $\rho_{n+p} = \rho_n$, so in particular,
\[
\{\rho_1, \ldots, \rho_n\} \leq \{\rho_1, \ldots, \rho_p\} = \{\rho_{n+1}, \ldots, \rho_{n+p}\},
\]
hence, by Proposition 16 $B$ is $(p,0)$-compatible with $D$.

On the other hand, assume that $\rho$ has $p$ different elements and let $n_0$ be the least positive integer such that $|\{\rho_1, \ldots, \rho_{n_0}\}| = p$. Since $B$ is $(p,0)$-compatible with $D$, we have that, for $n \geq n_0$, all the elements $\rho_{n+1}, \ldots, \rho_{n+p}$ are different. We can see that from this point on $\rho_n = \rho_{n+p}$ for all $n \geq n_0$. \[\square\]

**Example 19.**

- Since $P$ has as root sequence $\langle 0, 0, 0, \ldots \rangle$, which is periodic with period length 1, it is $(1,0)$-compatible with $D$ by Corollary 18.

- Since the root sequence of $C$ contains infinitely many different elements, by Corollary 17 $C$ is not compatible with $D$.

Let $B = \langle P_k(x) \rangle_{k=0}^{\infty}$ be a factorial basis, and let $\ell_k : K[x] \to K$ for $k \in \mathbb{N}$ be linear functionals such that $\ell_k(P_m(x)) = \delta_{k,m}$ for all $k,m \in \mathbb{N}$ (i.e., $\ell_k(p(x))$ is the coefficient of $P_k(x)$ in the expansion of $p(x) \in K[x]$ w.r.t. $B$). Property P2 implies that $\ell_k(P_j(x)P_m(x)) = 0$ when $k < \max\{j,m\}$, hence $K[x]$ naturally embeds into the algebra $K[[B]]$ of formal polynomial series of the form
\[
y(x) = \sum_{k=0}^{\infty} c_k P_k(x) \quad (c_k \in K),
\]
with multiplication defined by
\[
\left(\sum_{k=0}^{\infty} c_k P_k(x)\right) \cdot \left(\sum_{k=0}^{\infty} d_k P_k(x)\right) = \sum_{k=0}^{\infty} c_k P_k(x),
\]
\[
e_k = \sum_{\max\{i,j\} \leq k \leq i+j} c_i d_j \ell_k(P_i(x)P_j(x)).
\]
Notation 20.  • For any factorial basis \( \mathcal{B} \) of \( \mathbb{K}[x] \), let \( \mathcal{L}_B \) denote the set of all operators \( L \in \mathcal{L}_{\mathbb{K}[x]} \) such that \( \mathcal{B} \) is compatible with \( L \).

• Let \( \mathcal{E} \) denote the \( \mathbb{K} \)-algebra of recurrence operators of the form \( L' = \sum_{i=-s}^{r} c_i(k) S^i \) with \( r, s \in \mathbb{N} \) and \( c_i : \mathbb{Z} \to \mathbb{K} \) for \( -s \leq i \leq r \), acting on the \( \mathbb{K} \)-algebra of all two-way infinite sequences \( c \in \mathbb{K}^\mathbb{Z} \) by \( S^i c(k) = c(k+i) \) for all \( k, i \in \mathbb{Z} \).

• For any factorial basis \( \mathcal{B} \) of \( \mathbb{K}[x] \), let \( \sigma_{\mathcal{B}} \) denote the map \( \mathbb{K}[[\mathcal{B}]] \to \mathbb{K}^\mathbb{Z} \) assigning to \( y(x) = \sum_{k=0}^{\infty} c_k P_k(x) \in \mathbb{K}[[\mathcal{B}]] \) its coefficient sequence \( c = \langle c_k \rangle_{k \in \mathbb{N}} \) extended to \( \langle c_k \rangle_{k \in \mathbb{Z}} \) by taking \( c_k = 0 \) whenever \( k < 0 \). We will omit the subscript \( \mathcal{B} \) when it is clear from the context.

Definition 21. Let \( \mathcal{B} = \{P_k(x)\}_{k=0}^{\infty} \) be a factorial basis of \( \mathbb{K}[x] \), \( (A, B) \)-compatible with \( L \in \mathcal{L}_{\mathbb{K}[x]} \). Extend \( L \) to an operator acting on \( \mathbb{K}[[\mathcal{B}]] \) by setting

\[
L \sum_{k=0}^{\infty} c_k P_k(x) := \sum_{k=0}^{\infty} c_k L_P(x) = \sum_{k=0}^{\infty} c_k \sum_{i=-A}^{B} \alpha_{k,i} P_{k+i}(x) = \sum_{k=0}^{\infty} \left( \sum_{i=-B}^{A} \alpha_{k+i,-i} S^i \right) P_k(x) = \sum_{k=0}^{\infty} (\mathcal{R}_B L) c_k P_k(x) \tag{7}
\]

where

\[
\mathcal{R}_BL := \sum_{i=-B}^{A} \alpha_{k+i,-i} S^i \in \mathcal{E} \tag{8}
\]

is the operator, associated to \( L \) in basis \( \mathcal{B} \), with \( A, B, \alpha_{k,i} \) as in (3), and \( P_k(x) = c_{k-1} = 0 \) whenever \( k < 0 \) or \( i > k \).

Theorem 22. Let \( \mathcal{B}, L \) and \( \mathcal{R}_B L \) be as in Definition 21 and let \( y, p \in \mathbb{K}[[\mathcal{B}]] \).

1. \( \sigma_{\mathcal{B}}(Ly) = (\mathcal{R}_B L) \sigma_{\mathcal{B}}y, \)
2. \( Ly = p \iff (\mathcal{R}_B L) \sigma_{\mathcal{B}}y = \sigma_{\mathcal{B}}p. \)

Proof. Write \( y(x) = \sum_{k=0}^{\infty} c_k P_k(x) \). Then \( \sigma_{\mathcal{B}}y = c \) and (7) imply that \( \sigma_{\mathcal{B}}(Ly) = (\mathcal{R}_B L)c, \) hence \( \sigma_{\mathcal{B}}(Ly) = (\mathcal{R}_B L) \sigma_{\mathcal{B}}y, \) proving item 1. Furthermore,

\[
Ly = p \iff \sigma_{\mathcal{B}}(Ly) = \sigma_{\mathcal{B}}p \iff (\mathcal{R}_B L) \sigma_{\mathcal{B}}y = \sigma_{\mathcal{B}}p,
\]

proving item 2. \( \square \)

Example 23. Using (8) we read off from the \( \alpha_{k,i} \) given in Example 10 that

\[
\begin{align*}
\mathcal{R}_P D &= (k+1) S, \\
\mathcal{R}_C E &= S+1, \\
\mathcal{R}_P Q &= q^k,
\end{align*}
\]

while \( xx^k = x^{k+1} \) and \( x \binom{x}{k} = (k+1) \binom{x}{k+1} + k \binom{x}{k} \) imply by (3) and (8) that

\[
\begin{align*}
\mathcal{R}_P X &= S^{-1}, \\
\mathcal{R}_C X &= k(1+S^{-1}).
\end{align*}
\]
Theorem 24. [6, Prop. 2 & Thm. 1] \( \mathcal{L}_B \) is a \( \mathbb{K} \)-algebra, and the transformation \( \mathcal{R}_B : \mathcal{L}_B \to \mathcal{E} \), defined in (8), is an isomorphism of \( \mathbb{K} \)-algebras.

Let \( L \in \mathcal{L}_B \) and \( p \in \mathbb{K}[\![B]\!] \) where \( \mathcal{B} = \langle P_k(x) \rangle_{k=0}^\infty \) is a factorial basis of \( \mathbb{K}[x] \). Note that Theorem 22 opens the way to finding solutions \( y \in \mathbb{K}[\![\mathcal{B}]\!] \) of the equation \( Ly = p \) by the following three-step procedure:

**Procedure** \( \text{DEFINITESUMSOLS} \)

1. Compute \( L' = \mathcal{R}_B L \in \mathcal{E} \).
2. Solve \( L'c = \sigma_B p \) for the unknown \( c \in \mathbb{K}^Z \) with \( c_k = 0 \) for \( k < 0 \).
3. Return \( y(x) = \sum_{k=0}^\infty c_k P_k(x) \).

As our goal is finding definite-sum solutions of linear recurrence equations, we henceforth limit our attention to linear recurrence operators \( L \in \mathbb{K}[x]\langle E \rangle \) and their associated operators \( L' = \mathcal{R}_B L \in \mathcal{E} \) with respect to various factorial bases \( \mathcal{B} \) compatible with the shift operator \( E \) (shift-compatible bases, for short), and polynomial right-hand sides \( p \in \mathbb{K}[x] \). In this case, we can use Definitions 9 and 21, Corollary 12 and Proposition 13 to elaborate step 1 of procedure **DEFINITESUMSOLS** as follows:

**Procedure** \( \text{ASSOCIATEDOP} \)

**INPUT:** \( L \in \mathbb{K}[x]\langle E \rangle; \)
\hspace{1cm} a factorial basis \( \mathcal{B} = \langle P_k(x) \rangle_{k=0}^\infty, (A,0) \)-compatible with \( E \)

**OUTPUT:** \( L' = \mathcal{R}_B L \in \mathcal{E} \)

1. Using linear algebra, compute \( \alpha_{k,-A}, \alpha_{k,-A+1}, \ldots, \alpha_{k,0} \in \mathbb{K} \) such that
   \[
P_k(x + 1) = \sum_{i=-A}^0 \alpha_{k,i} P_{k+i}(x) \quad \text{for all } k \in \mathbb{N}.
   \]
   Let \( E' = \sum_{i=0}^A \alpha_{k+i,-i} S^i \in \mathcal{E} \).
2. Using linear algebra, compute \( \beta_{k,0}, \beta_{k,1} \in \mathbb{K} \) such that
   \[
x P_k(x) = \beta_{k,0} P_k(x) + \beta_{k,1} P_{k+1}(x).
   \]
   Let \( X' = \beta_{k-1,1} S^{-1} + \beta_{k,0} \in \mathcal{E} \).
3. Return the operator \( L' \in \mathcal{E} \), obtained from \( L \) by substituting \( E' \) for \( E \) and \( X' \) for \( x \).

---

1by a procedure we mean a high-level algorithm where not all steps are fully specified yet
3. The binomial transform and quasi-triangular bases

In the rest of the paper we occasionally use \( n \) instead of \( x \) to denote the independent variable of basis polynomials as well as of recurrence equations resp. operators. In particular, the shift operator \( E \) acts both by \( Ex = x + 1 \) and by \( En = n + 1 \).

In order for a formal-series solution \( y(n) = \sum_{k=0}^{\infty} c_k P_k(n) \) of an equation \( Ly = p \) obtained by procedure DefiniteSumSols given at the end of Section 2 to be a definite-sum solution of our original equation where \( L \in \mathbb{K}[x]\langle E \rangle \) and \( y, p \in \mathbb{K}^\mathbb{N} \), we need to impose some additional requirements on the basis \( B \). One obvious such requirement (satisfied, e.g., by the binomial-coefficient basis \( C = \langle (x)_k \rangle_{k=0}^{\infty} \)) is that it is locally finite, meaning that for each \( n \in \mathbb{N} \), there is an \( f(n) \in \mathbb{N} \) such that \( P_k(n) = 0 \) for all \( k > f(n) \). If this is the case, we have \( y(n) = \sum_{k=0}^{f(n)} c_k P_k(n) \in \mathbb{K} \), hence \( y \in \mathbb{K}^\mathbb{N} \). Another desirable property of \( C \) is its invertibility, meaning that for each \( a \in \mathbb{K}^\mathbb{N} \) there exists \( b \in \mathbb{K}^\mathbb{N} \) such that \( a_n = \sum_{k=0}^{n} b_k \binom{n}{k} \). In this section, we give some examples of computing \( R_B L \) when \( B = C \), and define the class of quasi-triangular bases which are locally finite and invertible.

It follows from Example 23 and Theorem 24 that every linear recurrence operator \( L \in \mathbb{K}[x]\langle E \rangle \) is compatible with the binomial-coefficient basis \( C = \langle (x)_k \rangle_{k=0}^{\infty} \). To compute the associated operator \( L' = R_C L \in \mathcal{E} \), we apply the substitution

\[
E \mapsto S + 1, \quad x \mapsto k(1 + S^{-1})
\]

to all terms of \( L \in \mathbb{K}[x]\langle E \rangle \). Clearly, \( L' \in \mathbb{K}[k]\langle S, S^{-1} \rangle \), and every \( h \in \ker L' \) gives rise to a solution \( y_n = \sum_{k=0}^{\infty} \binom{n}{k} c_k \) of \( Ly = 0 \).

Example 25. Here we list some operators \( L \in \mathbb{K}[n]\langle E \rangle \), their associated operators \( L' = R_C L \in \mathcal{E} \) with respect to the binomial-coefficient basis \( C \), and some of the elements of their kernels.

1. \( L = E - c \) where \( c \in \mathbb{K}^* \): Here \( L' = S - (c - 1) \), and

\[
y_n = \sum_{k=0}^{\infty} \binom{n}{k} (c - 1)^k = c^n
\]

is indeed a solution of \( Ly = 0 \).

2. \( L = E^2 - 2E + 1 \): Here \( L' = S^2 \), and by Theorem 22 any \( c \in \ker L' \) satisfies \( c_{n+2} = 0 \) for all \( n \geq 0 \), or equivalently, \( c_n = 0 \) for all \( n \geq 2 \). Hence

\[
y_n = \sum_{k=0}^{\infty} \binom{n}{k} c_k = c_0 + c_1 n
\]

is indeed a solution of \( Ly = 0 \).
3. \( L = E^2 - E - 1 \): Here \( L' = S^2 + S - 1 \), and any \( h \in \ker L' \) is of the form \( c_n = (-1)^n (C_1 F_n + C_2 F_{n+1}) \) where \( C_1, C_2 \in \mathbb{K} \) and \( F = \langle 0, 1, 1, 2, \ldots \rangle \) is the sequence of Fibonacci numbers. Hence every \( y \in \ker L \) is of the form

\[
y_n = \sum_{k=0}^{\infty} \binom{n}{k} (-1)^k (C_1 F_k + C_2 F_{k+1}).
\]

In particular, by setting \( y_0 = F_0 \) and \( y_1 = F_1 \), we find the identity

\[
F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} F_k.
\]

4. \( L = E - (n + 1) \): Here \( L' = S - n - nS^{-1} \), and the equation to solve is

\[
c_{n+1} - nc_n - nc_{n-1} = 0 \quad \text{for } n \geq 0,
\]

which yields \( c_1 = 0 \) as well as

\[
c_n = (n - 1)(c_{n-1} + c_{n-2}) \quad \text{for } n \geq 2.
\]

The general solution of the latter equation is of the form

\[
c_n = n! \left( C_1 \sum_{k=0}^{n} \frac{(-1)^k}{k!} + C_2 \right)
\]

where \( C_1, C_2 \in \mathbb{K} \) (cf. [30, Example 8.6.1]). Now \( c_1 = 0 \) implies \( C_2 = 0 \), hence every \( y \in \ker L \) is of the form

\[
y_n = C \sum_{k=0}^{\infty} \binom{n}{k} k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}
\]

for some \( C \in \mathbb{K} \). In particular, by setting \( y_0 = 0! = 1 = C \), we find the identity

\[
n! = \sum_{k=0}^{n} \binom{n}{k} k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}
\]

or equivalently,

\[
\sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} = 1.
\]

5. \( L = E^3 - (n^2 + 6n + 10)E^2 + (n + 2)(2n + 5)E - (n + 1)(n + 2) \): Unlike in the preceding four cases, the equation \( Ly = 0 \) has no nonzero Liouvillian solutions (i.e., solutions which are interlacings of d’Alembertian sequences: cf. [32, Corollary 15.2] or [31, Theorem 12]). Here \( L' = S^3 - (n^2 + 6n + 7)S^2 - (2n^2 + 8n + 7)S - (n + 1)^2 \), and equation \( L'h = 0 \) has a hypergeometric solution \( h_n = n^2 \). So

\[
y_n = \sum_{k=0}^{\infty} \binom{n}{k} k!^2 = \sum_{k=0}^{n} \binom{n}{k} k!^2
\]

is a non-Liouvillian definite-sum solution of equation \( Ly = 0 \).
Remark 26. Since $R_C(n^i) = k^i \sum_{j=0}^{i} \binom{i}{j} S^{-j}$ (as can be easily seen by induction on $i$), every negative power $S^{-j}$ in $L'$ is multiplied by $k^i$ for some $i \geq j$. So all terms of $L'$ containing $S^{-j}$ vanish for $k = 0, 1, \ldots, j - 1$ (cf. the term $kS^{-1}$, renamed as $nS^{-1}$ in $L'$ of item 4 in Example [25]).

Note that any sequence $a \in \mathbb{K}^N$ can be represented in the form

$$a_n = \sum_{k=0}^{\infty} \binom{n}{k} b_k \quad \text{for all } n \in \mathbb{N} \quad (9)$$

where $b \in \mathbb{K}^N$ satisfies

$$b_n = \sum_{k=0}^{\infty} (-1)^{n-k} \binom{n}{k} a_k \quad \text{for all } n \in \mathbb{N}.$$ 

This is because the infinite matrix $B = \left[ \binom{n}{k} \right]_{n,k=0}^{\infty}$ of the system of linear equations [9] for the unknown $b_0, b_1, b_2, \ldots$ is lower triangular with unit diagonal, so it is invertible, and it is easy to see that its inverse is $B^{-1} = \left[ (-1)^{n-k} \binom{n}{k} \right]_{n,k=0}^{\infty}$. Some authors call the sequence $a$ the binomial transform of $b$, and $b$ the inverse binomial transform of $a$ (cf. [24] seq. A007317). Others define the binomial transform as an involution: $a_n = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} b_k$, and $b_n = \sum_{k=0}^{\infty} (-1)^{k} \binom{n}{k} a_k$ (cf. [25] p. 137, Exercise 36). For our purposes, the actual values of leading coefficients of the basis elements are not important.

Proposition 27. Let $L = \sum_{i=0}^{r} p_i(n) E^i \in \mathbb{K}[n]\langle E \rangle$, and let its associated operator w.r.t. the binomial-coefficient basis $C$ be $L' = R_C(L) = \sum_{i=-r}^{s} q_i(k) S^i \in \mathbb{K}[k]\langle S, S^{-1} \rangle$. Then

1. $s = r$, and
2. $q_\ast(k) S^s = p_\ast(k) S^r$.

In other words, ord $L'$ = ord $L$, and the leading term of $L'$ (after renaming $k \to n$, $S \to E$) agrees with that of $L$.

Proof. Clearly

$$R_C(n^i) = (k + kS^{-1})^i = k^i + O(S^{-1}),$$
$$R_C(E^i) = (S + 1)^i = S^i + O(S^{-i}),$$

where $O(S^\rho)$ denotes an operator from $\mathbb{K}[k]\langle E, E^{-1} \rangle$ of order at most $\rho$. Hence for $p_i(n) = \sum_{j=0}^{d_i} c_{i,j} n^j$ we have

$$R_C(p_i(n)) = \sum_{j=0}^{d_i} c_{i,j} R_C(n^j) = \sum_{j=0}^{d_i} c_{i,j} (k^j + O(S^{-1})) = p_i(k) + O(S^{-1}),$$
$$R_C(p_i(n)E^i) = (p_i(k) + O(S^{-1})) (S^i + O(S^{-i})) = p_i(k) S^i + O(S^{-i})$$
if there is a strictly increasing function \( f \).

Proposition 29. Call a shift-compatible basis \( B \) quasi-triangular with the property that any sequence \( a \in \mathbb{K}^N \) can be represented in the form

\[
a_n = \sum_{k=0}^{\infty} b_k P_k(n)
\]

for some \( b \in \mathbb{K}^N \).

Definition 28. Call a shift-compatible basis \( B = \langle P_k(n) \rangle_{k=0}^{\infty} \) quasi-triangular if there is a strictly increasing function \( f : \mathbb{N} \to \mathbb{N} \) such that

1. \( \forall k, n \in \mathbb{N}: (k > f(n) \implies P_k(n) = 0) \),
2. \( \forall n \in \mathbb{N}: P_{f(n)}(n) \neq 0 \).

Clearly, the basis \( C = \langle \binom{n}{k} \rangle_{k=0}^{\infty} \) is quasi-triangular with \( f(n) = n \).

Proposition 29. A basis \( B = \langle P_k(n) \rangle_{k=0}^{\infty} \) is quasi-triangular if and only if its root sequence \( \rho = \langle \rho_1, \rho_2, \rho_3, \ldots \rangle \) satisfies

1. \( \langle 0, 1, 2, 3, \ldots \rangle \) is a subsequence of \( \rho \),
2. for every \( n \in \mathbb{N} \), the first appearance of \( n \) in \( \rho \) precedes the first appearance of \( n + 1 \) in \( \rho \).

Proof. Assume first that \( B \) is quasi-triangular with \( f : \mathbb{N} \to \mathbb{N} \) as in Definition 28 and let \( n \in \mathbb{N} \). Then \( P_{f(n)}(n) \neq 0 \) and \( P_{f(n)+1}(n) = 0 \), hence \( \rho_1, \rho_2, \ldots, \rho_{f(n)} \neq n \) and \( P_{f(n)+1} = n \). Since \( f \) is strictly increasing, \( \langle \rho_{f(0)+1}, \rho_{f(1)+1}, \rho_{f(2)+1}, \ldots \rangle = \langle 0, 1, 2, \ldots \rangle \) is a subsequence of \( \rho \), proving item 1. Since \( P_{f(n)}(n) \neq 0 \) and \( B \) is factorial, \( P_k(n) \neq 0 \) for all \( k \leq f(n) \), so \( \rho_1, \rho_2, \ldots, \rho_{f(n)} \neq n \), hence the first term of \( \rho \) equal to \( n \) is \( \rho_{f(n)+1} \). As \( f \) is strictly increasing, this proves item 2.

Assume now that the root sequence \( \rho \) of \( B \) satisfies items 1 and 2, and let \( f(n) := \min \{ k \in \mathbb{N} \setminus \{ 0 \} : \rho_k = n \} - 1 \). Then \( f : \mathbb{N} \to \mathbb{N} \) is strictly increasing, \( \rho_1, \rho_2, \ldots, \rho_{f(n)} \neq n \), while \( \rho_{f(n)+1} = n \), so \( P_k(n) = 0 \) for all \( k > f(n) \), and \( P_{f(n)}(n) \neq 0 \), hence \( B \) is quasi-triangular.

Theorem 30. Let \( B = \langle P_k(n) \rangle_{k=0}^{\infty} \) be a quasi-triangular basis with \( f : \mathbb{N} \to \mathbb{N} \) as in Definition 28. Then for every \( a \in \mathbb{K}^N \) there exists \( b \in \mathbb{K}^N \) such that \( E_a = \sum_{k=0}^{\infty} b_k P_k(n) \) holds.
Proof. Since \( f \) is strictly increasing, it is injective, and we can define \( b_k \in \mathbb{K} \) for \( k = 0, 1, 2, \ldots \) recursively as follows:

1. If \( k = f(n) \) for some \( n \in \mathbb{N} \) then let \( b_k = \frac{1}{P_{f(n)}(n)} \left( a_n - \sum_{i=0}^{k-1} b_i P_i(n) \right) \).
2. If \( k \notin f(\mathbb{N}) \) then let \( b_k \in \mathbb{K} \) be arbitrary.

Then for every \( n \in \mathbb{N} \) we have \( b_{f(n)} = \frac{1}{P_{f(n)}(n)} \left( a_n - \sum_{i=0}^{f(n)-1} b_i P_i(n) \right) \), hence

\[
a_n = b_{f(n)} P_{f(n)}(n) + \sum_{i=0}^{f(n)-1} b_i P_i(n) = \sum_{k=0}^{\infty} b_k P_k(n),
\]

proving equality \([10]\). \(\square\)

4. Products of compatible bases

To be able to use formal polynomial series to find other definite-sum solutions of linear recurrence equations, we need a rich supply of shift-compatible bases.

**Definition 31.** For \( a \in \mathbb{N} \setminus \{0\}, b \in \mathbb{K}, \) and for all \( k \in \mathbb{N}, \) let \( P^{(a,b)}_k(n) := \binom{an+b}{k}. \)

We denote the polynomial basis \( \left\{ P^{(a,b)}_k(n) \right\}_{k=0}^{\infty} \) by \( \mathcal{C}_{a,b}, \) and call it a generalized binomial-coefficient basis of \( \mathbb{K}[[n]]. \)

**Proposition 32.** Any generalized binomial-coefficient basis \( \mathcal{C}_{a,b} \) is a factorial basis of \( \mathbb{K}[[n]], \) which is \((a,0)\)-compatible with the shift operator \( E. \) If \( b \in \mathbb{N} \) then \( \mathcal{C}_{a,b} \) is quasi-triangular.

**Proof.** Clearly \( \deg_n P^{(a,b)}_k(n) = k \) and

\[
P^{(a,b)}_{k+1}(n) = \frac{an+b-k}{k+1} P^{(a,b)}_k(n) \quad \text{for all } k \in \mathbb{N},
\]

so \( P^{(a,b)}_k(n) \mid P^{(a,b)}_{k+1}(n), \) and \( \mathcal{C}_{a,b} \) is factorial. By Chu-Vandermonde’s identity,

\[
EP^{(a,b)}_k(n) = P^{(a,b)}_k(n+1) = \binom{an+a+b}{k} = \sum_{i=0}^{a} \binom{a}{i} \binom{an+b}{k-i} = \sum_{i=-a}^{0} \binom{a}{i} P^{(a,b)}_{k+i}(n),
\]

so \( \mathcal{C}_{a,b} \) is \((a,0)\)-compatible with \( E, (\alpha_{k,i} = \binom{a}{i}) \) for \( i = -a, -a+1, \ldots, 0). \)

Finally, if \( b \in \mathbb{N}, \) let \( f(n) = an + b. \) Then \( f : \mathbb{N} \to \mathbb{N} \) is strictly increasing, \( P^{(a,b)}_k(n) = \binom{an+b}{k} = 0 \) for \( k > an+b = f(n), \) and \( P^{(a,b)}_{f(n)}(n) = \binom{an+b}{an+b} = 1, \) so \( \mathcal{C}_{a,b} \) is quasi-triangular by Definition \([28]\). \(\square\)
Definition 33. Let \( m \in \mathbb{N} \setminus \{0\} \), and for \( i = 1, 2, \ldots, m \), let \( \mathcal{B}_i = \langle P_k^{(i)}(n) \rangle_{k=0}^\infty \) be a basis of \( \mathbb{K}[n] \). For all \( k \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, m-1\} \), let

\[
P_{mk+j}^{(\pi)}(n) := \prod_{i=1}^j P_{k+1}^{(i)}(n) \cdot \prod_{i=j+1}^m P_k^{(i)}(n).
\]

Then the sequence \( \prod_{i=1}^m \mathcal{B}_i := \langle P_n^{(\pi)}(n) \rangle_{n=0}^\infty \) is the product of \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m \).

Theorem 34. Let \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m \) be factorial bases of \( \mathbb{K}[n] \), and \( L \in \mathcal{L}_{\mathbb{K}[n]} \).

1. \( \prod_{i=1}^m \mathcal{B}_i \) is a factorial basis of \( \mathbb{K}[n] \).

2. Let \( L \) be a ring endomorphism of \( \mathbb{K}[n] \), and let all \( \mathcal{B}_i \) be \((A_i, B_i)\)-compatible with \( L \). Write \( A = \max_{1 \leq i \leq m} A_i \) and \( B = \min_{1 \leq i \leq m} B_i \). Then \( \prod_{i=1}^m \mathcal{B}_i \) is \((mA, B)\)-compatible with \( L \).

Proof. 1. Clearly \( \deg P_{mk+j}^{(\pi)}(n) = j(k+1) + (m-j)k = mk+j \).

If \( \ell = mk+j \) with \( 0 \leq j \leq m-2 \), then \( \ell+1 = mk+(j+1) \) and

\[
\frac{P_{\ell+1}^{(\pi)}(n)}{P_{\ell}^{(\pi)}(n)} = \frac{\prod_{i=1}^{j+1} P_{k+1}^{(i)}(n)}{\prod_{i=1}^j P_{k+1}^{(i)}(n)} \cdot \frac{\prod_{i=j+2}^m P_k^{(i)}(n)}{\prod_{i=j+1}^m P_k^{(i)}(n)} = \frac{P_{k+1}^{(j+1)}(n)}{P_k^{(j+1)}(n)} \in \mathbb{K}[n]
\]
as \( \mathcal{B}_{j+1} \) is factorial. If \( \ell = mk+(m-1) \), then \( \ell+1 = m(k+1) + 0 \) and

\[
\frac{P_{\ell+1}^{(\pi)}(n)}{P_{\ell}^{(\pi)}(n)} = \frac{1}{\prod_{i=1}^{m-1} P_{k+1}^{(i)}(n)} \cdot \frac{\prod_{i=1}^m P_k^{(i)}(n)}{\prod_{i=1}^m P_k^{(i)}(n)} = \frac{P_{k+1}^{(m)}(n)}{P_k^{(m)}(n)} \in \mathbb{K}[n]
\]
because \( \mathcal{B}_m \) is factorial. Hence \( \prod_{i=1}^m \mathcal{B}_i \) is factorial as well.

2. Let \( p \in \mathbb{K}[n] \) be arbitrary. For \( i = 1, 2, \ldots, m \), let \( p = \sum_{k=0}^{\deg p} c_k^{(i)} P_k^{(i)} \) be the expansion of \( p \) w.r.t. \( \mathcal{B}_i \). Then \( Lp = \sum_{k=0}^{\deg p} c_k^{(i)} LP_k^{(i)} \), and by condition \textbf{C1} of Proposition 11

\[
\deg Lp \leq \max_{0 \leq k \leq \deg p} \deg LP_k^{(i)} \leq \max_{0 \leq k \leq \deg p} (k+B_i) = \deg p + B_i.
\]

Since this holds for all \( i \), we have \( \deg Lp \leq \deg p + B \) for all \( p \in \mathbb{K}[n] \). In particular, \( \deg LP_k^{(\pi)} \leq k + B \), so \( \prod_{i=1}^m \mathcal{B}_i \) satisfies \textbf{C1}.

Condition \textbf{C2} of Proposition 11 and our definition of \( A \) imply that \( P_{k+1}^{(i)} \mid LP_k^{(i)} \) and \( P_k^{(i)} \mid LP_k^{(i)} \) for all \( k \geq A \) and \( i \in \{1, 2, \ldots, m\} \), so

\[
P_{m(k-A)+j}^{(\pi)} = \prod_{i=1}^j P_{k+1-A}^{(i)} \cdot \prod_{i=j+1}^m P_k^{(i)} \mid \prod_{i=1}^j LP_{k+1}^{(i)} \cdot \prod_{i=j+1}^m LP_k^{(i)}
\]
or equivalently, since \( L \) is an endomorphism of the ring \( \mathbb{K}[n] \),

\[
P_{m(k-A)+j}^{(\pi)} \mid L \left( \prod_{i=1}^j P_{k+1}^{(i)} \cdot \prod_{i=j+1}^m P_k^{(i)} \right) = LP_{m(k+A)}^{(\pi)}.
\]
Dividing both sides of

For \( \ell = mk + j \geq mA \), this turns into \( P^{(\pi)}_{\ell - mA} \big| LP^{(\pi)}_\ell \), so \( \prod_{i=1}^{m} K_i \) satisfies C2 as well. By Proposition 34 this proves the claim.

\[ \square \]

**Definition 35.** Let \( m \in \mathbb{N} \setminus \{0\} \), and let \( a = (a_1, a_2, \ldots, a_m) \), \( b = (b_1, b_2, \ldots, b_m) \) where \( a_i \in \mathbb{N} \setminus \{0\}, b_i \in \mathbb{K} \) for \( i = 1, 2, \ldots, m \). We denote the product of generalized binomial-coefficient bases \( \prod_{i=1}^{m} C_{a_i, b_i} \) by \( C_{a, b} \), and call it a **product binomial-coefficient basis** of \( \mathbb{K}[x] \) having length \( m \).

**Corollary 36.** Any product binomial-coefficient basis \( C_{a, b} \) is a factorial basis of \( \mathbb{K}[x] \) which is \((ma, 0)\)-compatible with \( E \), where \( a = \max_{1 \leq i \leq m} a_i \).

**Proof.** Use Theorem 34 and Proposition 32.

By Corollary 36 we now have at our disposal a family of factorial bases \( C_{a, b} \) which, given an operator \( L \in \mathbb{K}[n](E) \) and a kernel \( K(n, k) \) of the form

\[
K(n, k) = \prod_{i=1}^{m} \binom{a_i + b_i}{k}
\]

where \( a_i, b_i \in \mathbb{N} \setminus \{0\} \) and \( b_i \in \mathbb{K} \), can be used to find solutions of \( Ly = 0 \) having the form \( y_n = \sum_{k=0}^{\infty} K(n, k) c_k \). To this end, we need to compute expansions of \( E P_n(x) \) and \( X P_n(x) \) in the basis \( C_{a, b} = (P_n(x))_{n=0}^{\infty} \).

**Example 37.** Take \( K(n, k) = \binom{n}{k}^2 \). The polynomial basis to be used here is \( C_{(1,1),(0,0)} = (P_n(x))_{n=0}^{\infty} \) where for all \( k \in \mathbb{N} \),

\[
P_{2k}(x) = \binom{x}{k}^2, \quad P_{2k+1}(x) = \binom{x}{k+1} \binom{x}{k}.
\]

According to Corollary 36, \( C_{(1,1),(0,0)} \) is a factorial basis of \( \mathbb{K}[x] \) with \( m = 2 \) and \( a = \max\{1, 1\} = 1 \), so it is \((2, 0)\)-compatible with \( E \). In particular, this means that \( P_{2k}(x + 1) \) can be expressed as a linear combination of \( P_{2k}(x) \), \( P_{2k-1}(x) \), \( P_{2k-2}(x) \), and \( P_{2k+1}(x + 1) \) as a linear combination of \( P_{2k+1}(x) \), \( P_{2k}(x) \), \( P_{2k-1}(x) \), with coefficients depending on \( k \). In the case of \( P_{2k}(x + 1) \), this is just an application of Pascal’s rule:

\[
P_{2k}(x + 1) = \binom{x + 1}{k}^2 = \left[ \binom{x}{k} + \binom{x}{k-1} \right]^2
\]

\[
= \binom{x}{k}^2 + 2\binom{x}{k} \binom{x}{k-1} + \binom{x}{k-1}^2
\]

\[
= P_{2k}(x) + 2P_{2k-1}(x) + P_{2k-2}(x).
\]

In the case of \( P_{2k+1}(x + 1) \), we can use the method of undetermined coefficients. Dividing both sides of

\[
P_{2k+1}(x + 1) = u(k)P_{2k+1}(x) + v(k)P_{2k}(x) + w(k)P_{2k-1}(x), \quad \text{or}
\]

\[
\binom{x + 1}{k + 1} \binom{x + 1}{k} = u(k) \binom{x}{k + 1} \binom{x}{k} + v(k) \binom{x}{k}^2 + w(k) \binom{x}{k} \binom{x}{k-1}
\]

\[
\binom{x + 1}{k + 1} \binom{x + 1}{k} = u(k) \binom{x}{k + 1} \binom{x}{k} + v(k) \binom{x}{k}^2 + w(k) \binom{x}{k} \binom{x}{k-1}
\]

\[
\binom{x + 1}{k + 1} \binom{x + 1}{k} = u(k) \binom{x}{k + 1} \binom{x}{k} + v(k) \binom{x}{k}^2 + w(k) \binom{x}{k} \binom{x}{k-1}
\]
where \( u(k), v(k), w(k) \) are undetermined functions of \( k \), by \((\genfrac{[}{]}{0pt}{}{m}{k})_{k+1}^{\infty}\), yields
\[
\frac{(x + 1)^2}{k(k + 1)} = u(k)\frac{(x - k + 1)(x - k)}{k(k + 1)} + v(k)\frac{x - k + 1}{k} + w(k),
\]
which is an equality of two quadratic polynomials from \( K(k)[x] \). Plugging in the values \( x = -1, k, k - 1 \), we obtain a triangular system of linear equations
\[
\begin{align*}
u(k) - v(k) + w(k) &= 0 \\
\frac{1}{k}v(k) + w(k) &= \frac{k + 1}{k + 1} \\
w(k) &= \frac{k}{k + 1}
\end{align*}
\]
whose solution is \( u(k) = 1 \) (as expected), \( v(k) = \frac{2k + 1}{k + 1} \), \( w(k) = \frac{k}{k + 1} \), and so
\[
P_{2k+1}(x + 1) = P_{2k+1}(x) + \frac{2k + 1}{k + 1} P_{2k}(x) + \frac{k}{k + 1} P_{2k-1}(x).
\]
Alternatively, we could obtain a system of linear equations for \( u(k), v(k), w(k) \) by equating the coefficients of \( x^j \) on both sides of (14) for \( j = 0, 1, 2 \).

For the expansion of \( XP_n(x) \), recall that by Corollary 12 every factorial basis is \((0,1)\)-compatible with \( X \). Indeed, as \( x(\genfrac{[}{]}{0pt}{}{k}{a}) \) is \((0,1)\)-compatible with \( X \), we have
\[
x P_{2k}(x) = \left(\genfrac{[}{]}{0pt}{}{x}{k}\right)^2 = (k + 1)\left(\genfrac{[}{]}{0pt}{}{x}{k+1}\right)\left(\genfrac{[}{]}{0pt}{}{x}{k}\right) + k\left(\genfrac{[}{]}{0pt}{}{x}{k}\right)^2
\]
\[
= (k + 1)P_{2k+1}(x) + kP_{2k}(x),
\]
\[
x P_{2k+1}(x) = \left(\genfrac{[}{]}{0pt}{}{x}{k+1}\right)\left(\genfrac{[}{]}{0pt}{}{x}{k}\right) = (k + 1)\left(\genfrac{[}{]}{0pt}{}{x}{k+1}\right)^2 + k\left(\genfrac{[}{]}{0pt}{}{x}{k+1}\right)\left(\genfrac{[}{]}{0pt}{}{x}{k}\right)
\]
\[
= (k + 1)P_{2k+2}(x) + kP_{2k+1}(x).
\]
It is easy to see that \( C_{(1,1),(0,0)} \) is quasi-triangular with \( f(n) = 2n \).

For additional examples of expansions of shifted basis elements in the basis \( C_{a,b} \), see [29].

If our kernel is as in [29], we can use the product binomial-coefficient basis \( C_{a,b} = \langle P_n(x) \rangle_{n=0}^{\infty} \) which, by Corollary 36, is \((ma,0)\)-compatible with \( E \) where \( a = \max_{1 \leq i \leq m} a_i \). In order to compute \( \alpha_{k,j,i} \in K(k) \) such that
\[
P^{(\pi)}_{mk+j}(x + 1) = \sum_{i=-ma}^{0} \alpha_{k,j,i} P^{(\pi)}_{mk+j+i}(x)
\]
for all \( k \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, m - 1\} \), we divide both sides of this equation by \( P^{(\pi)}_{mk+j+ma}(x) \) which turns it into an equality of two polynomials of degree \( ma \) from \( K(k)[x] \). From this equality a system of \( ma + 1 \) linear algebraic equations for the \( ma + 1 \) undetermined coefficients \( \alpha_{k,j,i}, i = 0, 1, \ldots, ma \), can be obtained by equating the coefficients of like powers of \( x \) on both sides, or (as in Example
by substituting $ma + 1$ distinct values from $\mathbb{K}(k)$ for $x$ in this equality. Note that for each $j \in \{0, 1, \ldots, m-1\}$, this system is uniquely solvable since $C_{a,b}$ is a basis of $\mathbb{K}[x]$, that the $\alpha_{k,j,i}$ will be rational functions of $k$, and that, as the shift operator preserves leading coefficients and degrees of polynomials, $\alpha_{k,j,0} = 1$.

To compute the coefficients of the expansion of $xP_n^{(\pi)}(x)$ w.r.t. $C_{a,b}$, we use the fact that by Corollary 12, $C_{a,b}$ is $(0,1)$-compatible with $X$:

**Proposition 38.** For $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, m-1\}$, let

$$P_n^{(\pi)}_{mk+j}(x) := \prod_{i=1}^{m} \left( \frac{a_i x + b_i}{k+1} \right) \cdot \prod_{i=j+1}^{m} \left( \frac{a_i x + b_i}{k} \right).$$

Then

$$xF_n^{(\pi)}_{mk+j}(x) = \frac{k+1}{a_j+1} P_n^{(\pi)}_{mk+j+1}(x) + \frac{k-b_j+1}{a_j+1} P_n^{(\pi)}_{mk+j}(x).$$

**Proof.**

$$\frac{k+1}{a_j+1} P_n^{(\pi)}_{mk+j+1}(x) + \frac{k-b_j+1}{a_j+1} P_n^{(\pi)}_{mk+j}(x) = P_n^{(\pi)}_{mk+j}(x) \cdot f(x)$$

where

$$f(x) = \frac{k+1}{a_j+1} \cdot \frac{P_n^{(\pi)}_{mk+j+1}(x)}{P_n^{(\pi)}_{mk+j}(x)} + \frac{k-b_j+1}{a_j+1}$$

$$= \frac{k+1}{a_j+1} \cdot \frac{(a_{j+1}x+b_{j+1})_{k+1}}{(a_{j+1}x+b_{j+1})_{k}} + \frac{k-b_j+1}{a_j+1}$$

$$= \frac{k+1}{a_j+1} \cdot \frac{a_{j+1}x+b_{j+1}-k}{k+1} + \frac{k-b_j+1}{a_j+1} = x.$$

Examples of factorial shift-compatible quasi-triangular polynomial bases that are not of the type $C_{a,b}$ are given in Section 6.

5. Sieved polynomial bases

Now we can use Procedure ASSOCIATEDOP on p. 10 to find the associated operator $R_{B,L}$ where $B = C_{a,b}$. Notice however that for $m > 1$, the coefficients $\alpha_{k,i}$ expressing the actions of $E$ resp. $X$ on $B$ are not rational functions of $k$ anymore, but conditional expressions evaluating to $m$ generally distinct rational functions, depending on the residue class of $k \mod m$ (cf. Example 37 with $m = 2$, and Proposition 38). So the coefficients of $R_{B,L}$, obtained by composing and adding the operators $R_{B,E}$ and $R_{B,X}$ repeatedly, will contain quite complicated conditional expressions. In addition, $\text{ord} R_{B,L}$ may exceed $\text{ord} L$ by a factor of $ma$ which can be exponential in input size.
To overcome these inconveniences, we note that product bases represent a special case of sieved polynomial bases where the definition of the $k$-th basis element $P_k(x)$ depends on the residue of $k$ modulo some $m \in \mathbb{N}$, $m \geq 1$ (for similar phenomena in the theory of orthogonal polynomials satisfying three-term recurrences, cf. [7] and the series of papers [8]–[14], [21]–[24]). For a sieved basis $\mathcal{B}$ with modulus $m$ (an $m$-sieved basis, for short) we do not attempt to compute $R_{\mathcal{B}}L$ directly but represent it by a matrix $[R_{\mathcal{B}}L] = \left[ L_{r,j} \right]_{r,j=0}^{m-1}$ of operators where $L_{r,j} \in \mathcal{E}$ expresses the contribution of the $j$-th $m$-section $s^m_j \sigma_{\mathcal{B}}y$ of the coefficient sequence of $y$ to the $r$-th $m$-section $s^m_r \sigma_{\mathcal{B}}(Ly)$ of the coefficient sequence of $Ly$ (see Definition 2 and Notation 20 for the definitions of $s^m_j$ and $\sigma_{\mathcal{B}}$, $\mathcal{E}$, respectively). Note that being $m$-sieved is not an intrinsic property of a polynomial basis, but rather describes its presentation.

**Proposition 39.** Let $L \in \mathcal{L}_{\mathbb{K}[x]}$, $\mathcal{B} = \langle P_n(x) \rangle_{n=0}^\infty$ (a factorial basis of $\mathbb{K}[x]$), $m \in \mathbb{N} \setminus \{0\}$, $A, B \in \mathbb{N}$ and $\alpha_{k,j,i} \in \mathbb{K}$ be such that for all $k \in \mathbb{N}$ and $j \in \{0,1,\ldots,m-1\}$,

$$LP_{mk+j}(x) = \sum_{i=-A}^B \alpha_{k,j,i} P_{mk+j+i}(x).$$

(18)

Furthermore, for all $r, j \in \{0,1,\ldots,m-1\}$ define

$$L_{r,j} := \sum_{A \leq i \leq B \atop i+j \equiv r \pmod{m}} \alpha_{k+i+j,i,j} S^{r-j-i} \in \mathcal{E}$$

(19)

(to keep notation simple, we do not make the dependence of $L_{r,j}$ on $m$ explicit).

Then for every $y \in \mathbb{K}[\mathcal{B}]$ and $r \in \{0,1,\ldots,m-1\}$,

$$s^m_r \sigma_{\mathcal{B}}(Ly) = \sum_{j=0}^{m-1} L_{r,j} s^m_j \sigma_{\mathcal{B}}y.$$  

(20)

**Proof.** Write $y(x) = \sum_{n=0}^\infty c_n P_n(x)$ as the sum of its $m$-sections

$$y(x) = \sum_{j=0}^{m-1} \sum_{k=0}^\infty c_{mk+j} P_{mk+j}(x) = \sum_{j=0}^{m-1} \sum_{k=0}^\infty (s^m_j c)_{k} P_{mk+j}(x).$$

(21)
Then

\[ L_y(x) = \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} (s_j^m c)_k L P_{mk+j}(x) = \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} (s_j^m c)_k \sum_{i=-A}^{B} \alpha_{k,j,i} P_{mk+j+i}(x) \]

(22)

\[ = \sum_{j=0}^{m-1} \sum_{r=0}^{m-1} \sum_{\frac{r-j}{m} \in \mathbb{Z}} \sum_{k=0}^{\infty} \alpha_{k,j,i} (s_j^m c)_k P_{mk+r+j}(x) \]

(23)

\[ = \sum_{r,j=0}^{m-1} \sum_{\frac{r-j}{m} \in \mathbb{Z}} \sum_{k=0}^{\infty} \alpha_{k,j,i} \left( s_j^m c \right)_{k+\frac{r-i-j}{m}} P_{mk+r}(x) \]

(24)

\[ = \sum_{r,j=0}^{m-1} \sum_{\frac{r-j}{m} \in \mathbb{Z}} \sum_{k=0}^{\infty} \alpha_{k,j,i} \left( s_j^m c \right)_{k+\frac{r-i-j}{m}} P_{mk+r}(x) \]

(25)

\[ = \sum_{r=0}^{m-1} \sum_{j=0}^{m-1} \left( \sum_{k=0}^{\infty} L_{r,j} s_j^m c \right) P_{mk+r}(x) \]

(26)

where in (22) we used (18), in (23) we reordered summation on \( i \) with respect to the residue class of \( i+j \mod m \), (24) was obtained by replacing \( k \) with \( k+\frac{r-i-j}{m} \), (25) by noting that

\[ k < 0 \implies mk + r < 0 \implies P_{mk+r} = 0, \]

\[ k < \frac{i+j-r}{m} \implies (s_j^m c)_{k+\frac{r-i-j}{m}} = (s_j^m c)_{k-\frac{i+j-r}{m}} = 0, \]

and (26) by using (19). Now the equality of \( L_y(x) \) and the series in (26) can be restated as (20). □

**Corollary 40.** Let \( L, B, m, \) and \( L_{r,j} \) for \( r \in \{0, 1, \ldots, m-1\} \) be as in Proposition 39 and let \( p \in \mathbb{k}[[B]] \). Then

\[ L y = p \iff \forall r \in \{0, 1, \ldots, m-1\}: \sum_{j=0}^{m-1} L_{r,j} s_j^m \sigma_B y = s_r^m \sigma_B p. \]

**Proof.** By Proposition 39

\[ L y = p \iff \sigma_B (L y) = \sigma_B p \]

\[ \iff \forall r \in \{0, 1, \ldots, m-1\}: s_r^m \sigma_B (L y) = s_r^m \sigma_B p \]

\[ \iff \forall r \in \{0, 1, \ldots, m-1\}: \sum_{j=0}^{m-1} L_{r,j} s_j^m \sigma_B y = s_r^m \sigma_B p. \] □
Note that for \( m = 1 \), Proposition 39 and Corollary 40 turn into Theorem 22.1 and Theorem 22.2, respectively (with \( \alpha_{k,i} = \alpha_{k,0,i} \) and \( R_B L = L_{0,0} \)).

**Notation 41.** \( [R_B L] := [L_{r,j}]_{r,j=0}^{m-1} \in M_m(\mathcal{E}) \) where \( L_{r,j} \) is as given in (19).

**Proposition 42.** Let \( L^{(1)}, L^{(2)} \in L_B \). Then

\[
[R_B \left( L^{(1)} L^{(2)} \right)] = [R_B L^{(1)}] [R_B L^{(2)}].
\]

**Proof.** Write \( L = L^{(1)} L^{(2)} \) and \( \sigma = \sigma_B \). By (20),

\[
s^m \sigma(Ly) = \sum_{j=0}^{m-1} L_{t,j} s^m_j \sigma y = \sum_{j=0}^{m-1} [R_B L]_{t,j} s^m_j \sigma y.
\]

On the other hand, by (20) applied to \( L^{(1)} \) and \( L^{(2)} \),

\[
s^m \sigma(L^{(1)} L^{(2)} y) = \sum_{r=0}^{m-1} L^{(1)}_{t,r} s^m_r \sigma(L^{(2)} y)
= \sum_{r=0}^{m-1} L^{(1)}_{t,r} \sum_{j=0}^{m-1} L^{(2)}_{r,j} s^m_j \sigma y
= \sum_{j=0}^{m-1} \left( \left[ R_B L^{(1)} \right] [R_B L^{(2)}] \right)_{t,j} s^m_j \sigma y.
\]

Hence

\[
\sum_{j=0}^{m-1} [R_B L]_{t,j} s^m_j \sigma y = \sum_{j=0}^{m-1} \left( \left[ R_B L^{(1)} \right] [R_B L^{(2)}] \right)_{t,j} s^m_j \sigma y
\]

for any \( y \in \mathbb{K}[[B]] \), which implies the claim. \( \square \)

It follows that to compute \([R_B L]\) for an arbitrary operator \( L \in \mathbb{K}[x]\langle E \rangle\), it suffices to apply the substitution

\[
E \mapsto [R_B E],
\]

\[
x \mapsto [R_B X],
\]

\[
1 \mapsto I_m
\]

where \( I_m \) is the \( m \times m \) identity matrix, to all terms of \( L \). We adapt procedure \texttt{AssociatedOp} from p. 10 to compute the associated matrix of operators \([R_B L]\) for an \( m \)-sieved basis \( B \) in the following way:

**Procedure** \texttt{AssociatedOpSieved}
INPUT: \( L \in \mathbb{K}[x] \langle E \rangle; \ A \in \mathbb{N}; \) 
an \(m\)-sieved factorial basis \( \mathcal{B} = \langle P_{mk+j}(x) \rangle_{m \in \mathbb{N}, j \in \{0,1,\ldots,m-1\}}; \)
\((A,0)\)-compatible with \( E \)

OUTPUT: \( [\mathcal{R}_B L] = [L_{r,j}]_{r,j=0}^{m-1} \in M_m(\mathcal{E}) \)

1. Using linear algebra, compute \( \alpha_{k,j,i} \in \mathbb{K} \) for \( j \in \{0,1,\ldots,m-1\} \) and \( i \in \{-A,-A+1,\ldots,0\} \) such that

\[
P_{mk+j}(x+1) = \sum_{i=-A}^{0} \alpha_{k,j,i} P_{mk+j+i}(x) \quad \text{for all } k \in \mathbb{N}.
\]

For \( r,j \in \{0,1,\ldots,m-1\} \) let \( E_{r,j} = \sum_{-A \leq i \leq j \mod m} \alpha_{k+r-i-j,m,i} S_{r-i}^{j} \in \mathcal{E}. \)

Let \( [\mathcal{R}_B E] = \{E_{r,j}\}_{r,j=0}^{m-1}. \)

2. Using linear algebra, compute \( \beta_{k,j,0}, \beta_{k,j,1} \in \mathbb{K} \) for \( j \in \{0,1,\ldots,m-1\} \) such that

\[
x P_{mk+j}(x) = \beta_{k,j,0} P_{mk+j}(x) + \beta_{k,j,1} P_{mk+j+1}(x) \quad \text{for all } k \in \mathbb{N}.
\]

For \( r,j \in \{0,1,\ldots,m-1\} \) let \( X_{r,j} = \sum_{-A \leq i \leq 1 \mod m} \beta_{k+r-i-j,m,i} S_{r-i}^{j} \in \mathcal{E}. \)

Let \( [\mathcal{R}_B X] = \{X_{r,j}\}_{r,j=0}^{m-1}. \)

3. Return the matrix of operators \( [\mathcal{R}_B L] = [L_{r,j}]_{r,j=0}^{m-1} \), obtained by applying substitution \( \phi \) to \( L. \)

For a product binomial-coefficient basis \( \mathcal{C}_{a,b}, \) we can make the above procedure more specific, as already explained in part on p. 18 ff.

**Proposition 43.** Let \( \mathcal{B} = \mathcal{C}_{a,b}. \) Then for all \( r,j \in \{0,1,\ldots,m-1\}, \)

\[
X_{r,j} = \left\lfloor r = j \right\rfloor \frac{k-b_{j+1}}{a_{j+1}} + \left\lfloor r = 0 \land j = m-1 \right\rfloor \frac{k}{a_{j+1}} S_{-1}^{j-1} + \left\lfloor r = j+1 \right\rfloor \frac{k+1}{a_{j+1}}
\]

where

\[
[\varphi] = \begin{cases} 
1, & \text{if } \varphi \text{ is true}, \\
0, & \text{otherwise}
\end{cases}
\]
is the Iverson bracket.

**Proof.** From Proposition 38 we read off that in this case

\[
\beta_{k,j,0} = \frac{k-b_{j+1}}{a_{j+1}}, \quad (28)
\]
\[
\beta_{k,j,1} = \frac{k+1}{a_{j+1}}. \quad (29)
\]
From (19) with \( A = 0, B = 1 \) it follows that
\[
X_{r,j} = \begin{cases} 
[j \equiv r \pmod{m}] \beta_{k + \frac{r-i-j}{m},0} S_{\frac{r-i-j}{m}} \\
[j \equiv r-1 \pmod{m}] \beta_{k + \frac{r-i-j}{m},1} S_{\frac{r-i-j}{m}}.
\end{cases}
\tag{30}
\]
Combining (28) – (30) with \( 0 \leq r, j \leq m - 1 \) yields the assertion.

\[\square\]

**Algorithm** \textsc{AssociatedOpBC}

\textbf{Input:} \( L \in \mathbb{K}[x] \langle E \rangle, m \in \mathbb{N} \setminus \{0\}, a = (a_1, a_2, \ldots, a_m) \in (\mathbb{N} \setminus \{0\})^m, b = (b_1, b_2, \ldots, b_m) \in \mathbb{Z}^m \)

\textbf{Output:} \([\mathcal{R}_B L] = [L_{r,j}]_{r,j=0}^{m-1}\) where \( \mathcal{B} = C_{a,b} \)

1. \( A := m \max_{1 \leq i \leq m} a_i \).
2. For \( j = 0, 1, \ldots, m - 1 \) let
   \[
P_{m(k+j)}(x) := \prod_{i=1}^{j} \left( \frac{a_i x + b_i}{k+1} \right) \cdot \prod_{i=j+1}^{m} \left( \frac{a_i x + b_i}{k} \right).
\]
3. For \( j = 0, 1, \ldots, m - 1 \) do
   for \( i = -A, -A+1, \ldots, 0 \) compute by simplification
   \[
   Q_{m(k+j+i)}(x) := \frac{P_{m(k+j+i)}(x)}{P_{m(k+j-A)}(x)} \in \mathbb{K}(k)[x],
   \]
   \[
   R_{m(k+j)}(x) := \frac{P_{m(k+j)}(x+1)}{P_{m(k+j-A)}(x)} \in \mathbb{K}(k)[x].
   \]
   Equate coefficients of \( 1, x, x^2, \ldots, x^A \) on both sides of
   \[
   R_{m(k+j)}(x) = \sum_{i=-A}^{0} \alpha_{k,j,i} Q_{m(k+j+i)}(x)
   \]
   and solve the resulting system of \( A + 1 \) linear algebraic equations
   for the \( A + 1 \) unknowns \( \alpha_{k,j,i} \in \mathbb{K}(k), i = -A, -A+1, \ldots, 0 \).
4. For \( r, j = 0, 1, \ldots, m - 1 \) let
   \[
   E_{r,j} := \sum_{i+j \equiv \pi(r \pmod m) \leq 0} \alpha_{k + \frac{r-i-j}{m},i} S_{\frac{r-i-j}{m}},
   \]
   \[
   X_{r,j} := \begin{cases} 
[r = j] \frac{k - b_{j+1}}{a_{j+1}} + [r = 0 \land j = m-1] \frac{k}{a_{j+1}} S^{-1} + [r = j+1] \frac{k+1}{a_{j+1}}.
\end{cases}
\]
   Let \([\mathcal{R}_B E] = [E_{r,j}]_{r,j=0}^{m-1}, [\mathcal{R}_B X] = [X_{r,j}]_{r,j=0}^{m-1}\).
5. Return the matrix of operators \([\mathcal{R}_B L] = [L_{r,j}]_{r,j=0}^{m-1}\), obtained by applying substitution (27) to \( L \).
6. Main examples

To find definite-sum solutions \(y \in \mathbb{K}^N\) of a recurrence equation of the form \(Ly = p\) where \(L \in \mathbb{K}[n](E)\) and \(p \in \mathbb{K}[n]\), we select a quasi-triangular, shift-compatible, \(m\)-sieved factorial basis \(\mathcal{B}\), and use Procedure DEFINITESUMSOLS on p. 10. First, we follow Procedure ASSOCIATEDOPSIEVED on p. 23 (or, if \(\mathcal{B}\) is a product-binomial coefficient basis, Algorithm ASSOCIATEDOPBC on p. 24) to compute the matrix of operators \([R_{\mathcal{B}}L] = [L_{r,j}]_{r,j=0}^{m-1}\). Then we set up the system of linear recurrence equations

\[
L_{0,0}s_0^n c + L_{0,1}s_1^m c + \cdots + L_{0,m-1}s_{m-1}^m c = s_0^n d
\]

\[
L_{1,0}s_0^n c + L_{1,1}s_1^m c + \cdots + L_{1,m-1}s_{m-1}^m c = s_1^n d
\]

\[
\vdots
\]

\[
L_{m-1,0}s_0^n c + L_{m-1,1}s_1^m c + \cdots + L_{m-1,m-1}s_{m-1}^m c = s_{m-1}^m d
\]

for the unknown sequence \(c\) where \(d = (d_0, d_1, \ldots, d_{\deg p}, 0, 0, \ldots)\) is the sequence of coefficients of polynomial \(p(n)\). From Corollary 40 it follows that \(y \in \mathbb{K}^N\) with \(y_n = \sum_{k=0}^{\deg p} c_k P_k(n)\) and \(f(n)\) as in Definition 28 satisfies \(Ly = p\) if and only if the \(m\)-sections \(s^m_j c\) of the coefficient sequence \(c\) of \(y\) satisfy (31). So finally we solve this system for the unknown sequence \(c\) in any way we can.

Example 44. We illustrate the process just described on the linear recurrence equation \(Ly = 0\) where \(L \in \mathbb{Q}[n](E)\) is the 7\textsuperscript{th}-order operator

\[
L = (n + 8)(27034107689 n + 247037440535) E^7
\]

\[
- 2(n + 7)(27034107689 n^2 + 707256640479 n + 3519513987204) E^6
\]

\[
+ (27034107689 n^4 + 1763504948043 n^3 + 29534526868562 n^2
\]

\[
+ 187161930754966 n + 404930820118700) E^5
\]

\[
- 4(121973169216 n^4 + 3928755304511 n^3 + 43197821249228 n^2
\]

\[
+ 198945697078905 n + 329021406797184) E^4
\]

\[
+ (2167208392754 n^4 + 45326791213914 n^3 + 347739537911929 n^2
\]

\[
+ 1165212776491303 n + 143993706115596) E^3
\]

\[
- 2(613023852648 n^4 + 8954947813901 n^3 + 52565810509778 n^2
\]

\[
+ 141274453841469 n + 142893654078876) E^2
\]

\[
- (n + 2)^2(1109455476579 n^2 + 3624719391913 n - 357803625948) E
\]

\[
+ 24(n + 1)^2(n + 2)(8996538731 n + 29816968829),
\]

using the basis \(\mathcal{B} = C_{(1,1),(0,0)}\) from Example 37 with

\[
P_{2k}(n) = \binom{n}{k}^2, \quad P_{2k+1}(n) = \binom{n}{k+1} \binom{n}{k}.
\]
To compute $[\mathcal{R}_BE]$, comparing (18) with (13) and (15) yields

$$
\alpha_{k,0,0} = 1, \quad \alpha_{k,1,0} = 1, \\
\alpha_{k,0,-1} = 2, \quad \alpha_{k,1,-1} = \frac{2k+1}{k+1}, \\
\alpha_{k,0,-2} = 1, \quad \alpha_{k,1,-2} = \frac{k}{k+1},
$$

hence by (19)

$$
[\mathcal{R}_BE] = \begin{bmatrix}
E_{0,0} & E_{0,1} \\
E_{1,0} & E_{1,1}
\end{bmatrix} = \begin{bmatrix}
S + 1 & \frac{2k+1}{k+1} \frac{2k+1}{k+1} S + 1 \\
2S & k+1
\end{bmatrix}, \tag{32}
$$

while from Proposition 13 it follows that

$$
[\mathcal{R}_BX] = \begin{bmatrix}
X_{0,0} & X_{0,1} \\
X_{1,0} & X_{1,1}
\end{bmatrix} = \begin{bmatrix}
k & kS^{-1} \\
k+1 & k
\end{bmatrix}. \tag{33}
$$

To obtain $[\mathcal{R}_BL] = [L_{r,j}]_{r,j=0}^1 \in M_2(\mathcal{E})$, we take $L$ and substitute $[\mathcal{R}_BE]$ for $E$, $[\mathcal{R}_BX]$ for $n$ and the $2 \times 2$ identity matrix, multiplied by $z$, for any constant $z \in \mathbb{K}$. This yields the operators $L_{0,0}, L_{0,1}, L_{1,0}, L_{1,1}$ given in Appendix C.

The next step is to find a non-zero solution of the system of recurrences (31), which in the case $m = 2$ turns into

$$
L_{0,0}s_0^2c + L_{0,1}s_1^2c = 0, \\
L_{1,0}s_0^2c + L_{1,1}s_1^2c = 0. \tag{34}
$$

By means of any algorithm for finding hypergeometric solutions of recurrence equations, we discover that the sequence $\langle k! \rangle_{k=0}^\infty$ is annihilated both by $L_{0,0}$ and $L_{1,0}$, while the sequence $\langle 2^k \rangle_{k=0}^\infty$ is annihilated both by $L_{0,1}$ and $L_{1,1}$. Hence the pairs $\langle k! \rangle$ and $\langle 0, 2^k \rangle$ are two linearly independent solutions of (34), and the interlacing

$$
c^{(1)} = \Lambda(k!, 0), \quad c^{(2)} = \Lambda(0, 2^k)
$$

give rise to two linearly independent definite-sum solutions of $Ly = 0$:

$$
\begin{align*}
y^{(1)}_n &= \sum_{k=0}^{2n} c_k^{(1)} P_k(n) = \sum_{j=0}^{n} j!P_{2j}(n) = \sum_{j=0}^{n} j! \binom{n}{j}^2, \\
y^{(2)}_n &= \sum_{k=1}^{2n-1} c_k^{(2)} P_k(n) = \sum_{j=0}^{n-1} 2^j P_{2j+1}(n) = \sum_{j=0}^{n-1} 2^j \binom{n}{j+1} \binom{n}{j}.
\end{align*}
$$

These solutions can be used to factor the operator $L$ in $\mathcal{Q}(n)(E)$. Zeilberger’s algorithm 36 37 computes operators

$$
L_1 = E^2 - 2(n+2)E + (n+1)^2, \\
L_2 = (n+1)(n+3)E^2 - 3(n+2)(2n+3)E + (n+1)(n+2)
$$

\footnote{see Definition 2}
such that $L_1y^{(1)} = 0$ and $L_2y^{(2)} = 0$. Algorithm Hyper [28] shows that $L_1$ and $L_2$ are minimal annihilators of $y^{(1)}$ resp. $y^{(2)}$, hence they divide $L$ from the right, and so does their least common left multiple

$$L_4 = (5 + n)(-2 - 37n - 138n^2 - 123n^3 - 33n^4 + n^5 + n^6)E^4$$

$$- (4 + n)(-276 - 1434n - 2946n^2 - 2342n^3 - 718n^4 - 35n^5 + 18n^6 + 2n^7)E^3$$

$$+ (-6896 - 32704n - 60998n^2 - 55528n^3 - 26184n^4 - 5888n^5 - 239n^6$$

$$+ 144n^7 + 24n^8 + n^9)E^2$$

$$- (2 + n)^2(-1686 - 6102n - 8388n^2 - 5286n^3 - 1430n^4 - 44n^5 + 47n^6$$

$$+ 6n^7)E$$

$$+ (1 + n)^2(2 + n)(-331 - 803n - 680n^2 - 225n^3 - 13n^4 + 7n^5 + n^6).$$

Indeed, in $\mathbb{Q}(n)\langle E \rangle$, $L = L_3L_4$ where

$$L_3 = \frac{1}{p(n)} \cdot 24(29816968829 + 8996538731 n)$$

$$+ \frac{1}{p(n)q(n)} \cdot (-416468942295148 - 1125518881632823 n$$

$$- 1053315627513055n^2 - 421766464222932n^3 - 59182885147037n^4$$

$$+ 6062805507491n^5 + 2492093169923n^6 + 186046100685n^7)E$$

$$- \frac{1}{q(n)r(n)} \cdot (-9439737938111061 - 12584340359048430 n$$

$$- 5882475183305081n^2 - 948888850906974n^3 + 102412128495705n^4$$

$$+ 57283235096898n^5 + 7386765251173n^6 + 325688030730n^7)E^2$$

$$+ \frac{1}{r(n)} \cdot (270347440535 + 27034107689 n)E^3$$

with

$$p(n) = -331 - 803n - 680n^2 - 225n^3 - 13n^4 + 7n^5 + n^6,$$  \hspace{1cm} (35)

$$q(n) = -2044 - 2849n - 1348n^2 - 187n^3 + 37n^4 + 13n^5 + n^6,$$

$$r(n) = -6377 - 5887n - 1542n^2 + 111n^3 + 117n^4 + 19n^5 + n^6.$$

This means that any solution $y$ of $L_1y = 0$ or $L_2y = 0$ also satisfies our original equation $Ly = 0$. By using the well-known method of variation of constants (a.k.a. reduction of order) we find out that whenever $c_2(n)a_{n+2} + c_1(n)a_{n+1} + c_0(n)a_n = 0$ for some $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and for all $n \in \mathbb{N}$, then also $c_2(n)b_{n+2} + c_1(n)b_{n+1} + c_0(n)b_n = 0$ where

$$b_n = a_n \sum_{k=0}^{n-1} (-1)^k \prod_{i=1}^{k} \left(1 + \frac{c_1(i-1) \cdot a_i}{c_2(i-1) \cdot a_{i+1}}\right),$$  \hspace{1cm} (36)
provided that it is well defined. Thus from $L_1 y^{(1)} = 0$ we obtain another solution

$$y^{(3)}_n = y^{(1)}_n \sum_{k=0}^{n-1} (-1)^k \prod_{i=1}^{k} \left(1 - 2(1 + i) \frac{y^{(1)}_i}{y^{(1)}_{i+1}}\right)$$

satisfying $L_1 y^{(3)} = 0$ and linearly independent from $y^{(1)}$. If we now use formula \[36\] to construct a sequence $y^{(4)}$, linearly independent from $y^{(2)}$ and satisfying $L_2 y^{(4)} = 0$, we unfortunately obtain the same solution $y^{(2)}$ again (but, in compensation, discover a summation identity). However, increasing the lower bound in the product within \[36\] by 1, we do obtain a solution

$$y^{(4)}_n = y^{(2)}_n \sum_{k=0}^{n-1} (-1)^k \prod_{i=2}^{k} \left(1 - 3 \frac{(1 + i)(1 + 2i) y^{(2)}_i}{y^{(2)}_{i+1}}\right)$$

satisfying $L_2 y^{(4)} = 0$ at all $n \in \mathbb{N}$ except at $n = 0$. It is easy to check that $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ are four linearly independent sequences, represented explicitly and satisfying $L y_n = 0$ for all $n \geq 1$.

Since $L_4 = L_1 L_1 = \tilde{L}_2 L_2$ where

$$\tilde{L}_1 = (5+n)(-2 - 37n - 138n^2 - 123n^3 - 33n^4 + n^5 + n^6)E^2$$

$$- (4+n)(-256 - 1060n - 1492n^2 - 836n^3 - 142n^4 + 21n^5 + 6n^6)E$$

$$+ (2+n) p(n)$$

$$\tilde{L}_2 = \frac{1}{3+n}(-2 - 37n - 138n^2 - 123n^3 - 33n^4 + n^5 + n^6)E^2$$

$$- \frac{1}{3+n}(-393 - 1698n - 2727n^2 - 1917n^3 - 574n^4 - 36n^5 + 14n^6 + 2n^7)E$$

$$+ (1+n) p(n)$$

with $p(n)$ as given in \[35\], it follows that $L$ factorizes as

$$L = L_3 \tilde{L}_1 L_1 = L_3 \tilde{L}_2 L_2. \quad (37)$$

Since ord $L_3 = 3$, ord $L_1 = ord L_2 = ord \tilde{L}_1 = ord \tilde{L}_2 = 2$, and the operators $L_3, L_1, L_2, \tilde{L}_1, \tilde{L}_2$ have neither right nor left first-order factors in $\mathbb{Q}(n)(E)$, equation \[37\] gives two distinct factorizations of $L$ into irreducible factors in $\mathbb{Q}(n)(E)$.

When we are interested in finding $y \in \ker L$ of the form

$$y(x) = \sum_{k=0}^{\infty} c_k P_{mk}(x)$$

for some $m$-sieved basis $B = \langle P_k(x) \rangle_{k=0}^{\infty}$, we have $s^m_0 \sigma_{BY} = c$ and $s^m_j \sigma_{BY} = 0$ for all $j \in \{1, 2, \ldots, m-1\}$, hence Corollary \[40\] implies

$$L y = 0 \iff \forall r \in \{0, 1, \ldots, m-1\}: L_{r,0} s^m_0 \sigma_{BY} = 0$$

$$\iff \forall r \in \{0, 1, \ldots, m-1\}: L_{r,0} c = 0$$

$$\iff \gcd(L_{0,0}, L_{1,0}, \ldots, L_{m-1,0}) c = 0.$$
This means that any nonzero element of the first column of $[R_B L] = [L_{r,j}]_{r,j=0}^{m-1}$ may serve as a nontrivial annihilator $L'$ of $h$, and taking their greatest common right divisor might yield $L'$ of lower order. The fact that we only need the first column $[R_B L]e^{(1)}$ of $[R_B L]$ – where $e^{(1)} = (1, 0, \ldots, 0)^T \in \mathbb{K}^m$ – can be used to advantage in the following way:

In step 3 of Procedure ASSOCIATEDOPSIEVED from p. 23, as well as in step 5 of Algorithm ASSOCIATEDOPBC from p. 24, instead of computing the entire matrix $[R_B L]$ we only compute the vector $[R_B L]e^{(1)}$. To do so, we start with $e^{(1)}$, and proceed from right to left through the expression obtained from $L$ by substitution (27), multiplying a matrix by a vector at each point. Finally, we return $L' = \gcd(L_0, 0, L_1, 0, \ldots, L_{m-1}, 0)$.

**Example 45** (Apéry’s $\zeta(2)$-recurrence [9, 33]). Let

$$L := (n + 2)^2 E^2 - (11n^2 + 33n + 25)E - (n + 1)^2$$

and

$$K(n, k) = \binom{n}{k} \binom{n + k}{2k}.$$ 

Take

$$P_{3k}(x) = K(x, k) = \binom{x}{k} \left( \frac{x + k}{2k} \right),$$

$$P_{3k+1}(x) = \binom{x}{k} \left( \frac{x + k}{2k + 1} \right),$$

$$P_{3k+2}(x) = \binom{x}{k+1} \left( \frac{x + k}{2k + 1} \right),$$

or, more concisely,

$$P_k(x) = \binom{x}{\left\lfloor \frac{k+1}{3} \right\rfloor} \left( \frac{x + \left\lfloor \frac{k}{3} \right\rfloor}{\left\lfloor \frac{2k+1}{3} \right\rfloor} \right).$$

Clearly $B = \langle P_k(x) \rangle_{k=0}^\infty$ is factorial, so $X$ is $(0, 1)$-compatible with $B$:

$$xP_{3k}(x) = (2k + 1)P_{3k+1}(x) + kP_{3k}(x),$$

$$xP_{3k+1}(x) = (k + 1)P_{3k+2}(x) + kP_{3k+1}(x),$$

$$xP_{3k+2}(x) = 2(k + 1)P_{3k+3}(x) - (k + 1)P_{3k+2}(x).$$

It is not hard to see that $B$ is quasi-triangular with $f(n) = 3n$. Furthermore, the root sequence of $B$

$$\rho = \langle 0, 0, -1, 1, 1, -2, 2, 2, -3, 3, 3, -4, 4, 4, -5, 5, 5, -6, \ldots \rangle$$

satisfies

$$[\rho_1 + 1, \rho_2 + 1, \ldots, \rho_k + 1] \subseteq [\rho_1, \rho_2, \ldots, \rho_k, \rho_{k+1}, \ldots, \rho_{k+3}]$$

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for all \( k \geq 0 \), hence by Proposition 14 \( E \) is \((3,0)\)-compatible with \( B \):

\[
P_{3k}(x + 1) = P_{3k}(x) + \frac{3}{2} P_{3k-1}(x) + \frac{8k - 3}{2k} P_{3k-2}(x) + P_{3k-3}(x),
\]

\[
P_{3k+1}(x + 1) = P_{3k+1}(x) + \frac{3k + 1}{2k + 1} P_{3k}(x) + \frac{k}{2k + 1} P_{3k-1}(x) + \frac{2k - 1}{2k + 1} P_{3k-2}(x),
\]

\[
P_{3k+2}(x + 1) = P_{3k+2}(x) + \frac{3k + 2}{k + 1} P_{3k+1}(x) + P_{3k}(x).
\]

The associated operator matrices are:

\[
[R_B X] = \begin{bmatrix}
    k & 0 & 2k S^{-1} \\
    2k + 1 & k & 0 \\
    0 & k + 1 & -(k + 1)
\end{bmatrix},
\]

\[
[R_B E] = \begin{bmatrix}
    S + 1 & \frac{3k + 1}{2k + 1} S & 1 \\
    \frac{8k + 5}{2(k + 1)} S & \frac{2k + 1}{2k + 3} S + 1 & \frac{3k + 2}{k + 1} S \\
    \frac{S}{2} & \frac{k + 1}{2k + 3} S & 1
\end{bmatrix}.
\]

For \( L \) as defined in (38) we obtain:

\[
L_{0,0} = (k + 2)^2 S^2 + \frac{29k^3 + 46k^2 + 14k - 1}{2k + 1} S
\]

\[- 2(37k^2 + 41k + 11),
\]

\[
L_{1,0} = \frac{(k + 2)(4k + 5)(12k^2 + 26k + 11)}{2(k + 1)(2k + 3)} S^2
\]

\[- \frac{79 + 237k + 199k^2 + 47k^3}{2(1 + k)} S - (2k + 1)(49k + 31),
\]

\[
L_{2,0} = \frac{(k + 2)(22k^2 + 62k + 43)}{2(2k + 3)} S^2
\]

\[- \frac{3}{2}(11k^2 + 34k + 25) S - 11(k + 1)(2k + 1),
\]

and

\[
gcrd(L_{0,0}, L_{1,0}, L_{2,0}) = S - 2 \frac{2k + 1}{k + 1}. \tag{39}
\]

So \( c_k = \binom{2k}{k} \) satisfies \( L_{0,0} c_k = L_{1,0} c_k = L_{2,0} c_k = 0 \). Since

\[
c_k P_{3k}(n) = \binom{2k}{k} \binom{n}{k} \binom{n + k}{2k} = \binom{n^2}{k} \binom{n + k}{k},
\]

we have found that Apéry’s \( \zeta(2) \)-sequence

\[
y_n^{(1)} = \sum_{k=0}^{\infty} c_k P_{3k}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n + k}{k}
\]

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is a solution of $Ly = 0$ with $L$ as in (38). Using formula (36) as in Example 44, we obtain another, linearly independent solution $y^{(2)}$ of $Ly = 0$

$$y^{(2)}_n = y^{(1)}_n \sum_{k=0}^{n-1} (-1)^k \prod_{i=1}^{k} \left( 1 - \frac{111^2 + 11i + 3}{(i+1)^3} \frac{y^{(1)}_i}{y^{(1)}_{i+1}} \right).$$

Example 46 (Apéry’s $\zeta(3)$-recurrence [9, 33]). Let

$$L := (n + 2)^3E^2 - (2n + 3)(17n^2 + 51n + 39)E + (n + 1)^3$$

and

$$K(n,k) = \left( \frac{n+k}{2k} \right)^2.$$ 

Take

$$P_{4k}(x) = K(x,k) = \left( \frac{x+k}{2k} \right)^2,$$

$$P_{4k+1}(x) = \left( \frac{x+k}{2k} \right) \left( \frac{x+k}{2k+1} \right),$$

$$P_{4k+2}(x) = \left( \frac{x+k}{2k+1} \right)^2,$$

$$P_{4k+3}(x) = \left( \frac{x+k}{2k+1} \right) \left( \frac{x+k+1}{2k+2} \right).$$

Clearly $B = \langle P_k(x) \rangle_{k=0}^{\infty}$ is factorial, so $X$ is $(0,1)$-compatible with $B$:

$$xP_{4k}(x) = (2k+1)P_{4k+1}(x) + kP_{4k}(x),$$

$$xP_{4k+1}(x) = (2k+1)P_{4k+2}(x) + kP_{4k+1}(x),$$

$$xP_{4k+2}(x) = 2(k+1)P_{4k+3}(x) - (k+1)P_{4k+2}(x),$$

$$xP_{4k+3}(x) = 2(k+1)P_{4k+4}(x) - (k+1)P_{4k+3}(x).$$

It is not hard to see that $B$ is quasi-triangular with $f(n) = 4n$. Furthermore, the root sequence of $B$

$$\rho = \langle 0, 0, -1, -1, 1, 1, -2, -2, 2, -3, -3, 3, -4, -4, 4, 4, \ldots \rangle$$

satisfies

$$[\rho_1 + 1, \rho_2 + 1, \ldots, \rho_k + 1] \subseteq [\rho_1, \rho_2, \ldots, \rho_k, \rho_{k+1}, \ldots, \rho_{k+4}]$$
for all $k \geq 0$, hence by Proposition 14, $E$ is $(4, 0)$-compatible with $\mathcal{B}$:

$$
P_{4k}(x + 1) = P_{4k}(x) + 2P_{4k-1}(x) + \frac{3k - 1}{k} P_{4k-2}(x) + \frac{4k - 1}{k} P_{4k-3}(x)
+ P_{4k-4}(x),
$$

$$
P_{4k+1}(x + 1) = P_{4k+1}(x) + \frac{4k + 1}{2k + 1} P_{4k}(x) + \frac{2k}{2k + 1} P_{4k-1}(x)
+ \frac{2k - 1}{2k + 1} (P_{4k-2}(x) + P_{4k-3}(x)),
$$

$$
P_{4k+2}(x + 1) = P_{4k+2}(x) + 2P_{4k+1}(x) + P_{4k}(x),
$$

$$
P_{4k+3}(x + 1) = P_{4k+3}(x) + \frac{4k + 3}{2(k + 1)} P_{4k+2}(x) + \frac{6k + 5}{2(k + 1)} P_{4k+1}(x) + P_{4k}(x).
$$

The associated operator matrices are:

$$
[R \mathcal{B} X] = \begin{bmatrix}
  k & 0 & 0 & 2k S^{-1} \\
  2k + 1 & 0 & 0 & 0 \\
  0 & 2k + 1 & -(k + 1) & 0 \\
  0 & 0 & 2(k + 1) & -(k + 1)
\end{bmatrix},
$$

$$
[R \mathcal{B} E] = \begin{bmatrix}
  S + 1 & \frac{4k + 1}{k + 1} S + 1 & 1 & 1 & \frac{6k + 5}{k + 1} S + 1 & 2 & \frac{6k + 5}{k + 1} S + 1 & 2 \\
  \frac{2k + 3}{k + 1} S & \frac{2k + 3}{k + 1} S & 1 & 1 & \frac{2k + 3}{k + 1} S & 2 & \frac{2k + 3}{k + 1} S & 2
\end{bmatrix}.
$$

For $L$ as defined in (40) we obtain:

$$
L_{0,0} = (k + 2)^3 S^2 + \left(\frac{58k^4 + 105k^3 - 25k^2 - 121k - 45}{2k + 1}\right) S
- 4(2k + 1) \left(90k^2 + 101k + 27\right),
$$

$$
L_{1,0} = \left(\frac{(k + 2)^2}{k + 1}(2k + 3)\right)^2 \left(28k^3 + 96k^2 + 103k + 34\right) S^2
- \frac{2 \left(75k^4 + 414k^3 + 796k^2 + 636k + 177\right) S}{k + 1}
- 8(37k + 27)(2k + 1)^2,
$$

$$
L_{2,0} = \left(\frac{(k + 2)^2}{k + 1}(2k + 3)\right)^2 \left(26k^3 + 87k^2 + 90k + 28\right) S^2
- \frac{4 \left(42k^4 + 215k^3 + 390k^2 + 295k + 77\right) S}{k + 1}
- 16(10k + 7)(2k + 1)^2,
$$

$$
L_{3,0} = \left(\frac{(k + 2)^2}{2k + 3}\right)^2 \left(12k^2 + 33k + 22\right) S^2
- \frac{8 \left(22k^3 + 96k^2 + 137k + 64\right) S}{k + 1}
- 64(k + 1)(2k + 1)^2,
$$

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and
\[ \text{gcd}(L_{0,0}, L_{1,0}, L_{2,0}, L_{3,0}) = S - \frac{4(2k + 1)^2}{(k + 1)^2}. \]

(41)

So \( c_k = \binom{2k}{k}^2 \) satisfies \( L_{0,0} c_k = L_{1,0} c_k = L_{2,0} c_k = L_{3,0} c_k = 0 \). Since
\[ c_k P_{4k}(n) = \binom{2k}{k}^2 \left( \frac{n+k}{2k} \right)^2 = \binom{n}{k}^2 \left( \frac{n+k}{k} \right)^2, \]
we have found that Apéry’s \( \zeta(3) \)-sequence
\[ y_n^{(1)} = \sum_{k=0}^{\infty} c_k P_{4k}(n) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \left( \frac{n+k}{k} \right)^2 \]
is a solution of \( Ly = 0 \) with \( L \) as in (40). Using formula (36) as in Example 44, we obtain another, linearly independent solution \( y^{(2)} \) of \( Ly = 0 \)
\[ y_n^{(2)} = y_n^{(1)} \sum_{k=0}^{n-1} (-1)^k \prod_{i=1}^{k} \left( 1 - \frac{(2i+1)(17i^2 + 17i + 5)}{(i+1)^3} \frac{y_i^{(1)}}{y_{i+1}^{(1)}} \right). \]

7. Shuffled polynomial bases

In Examples 45 and 46, we built two quasi-triangular bases that involved some binomial coefficients. These bases, however, are sieved polynomial bases that cannot be written as a product basis of those binomial coefficients.

In the case of product bases (see Definition 33) the root sequences of the factors are interlaced in a balanced way. However, as illustrated in the previous examples, sometimes we need to interlace the root sequences in an unbalanced way. The root sequence of the basis for Example 45 was:
\[ \rho = \langle 0, 0, -1, 1, 1, -2, 2, 2, -3, 3, 3, \ldots \rangle, \]
while the root sequences of the two factors \( \binom{x}{k} \) and \( \left\{ \binom{x+k}{2k}, \binom{x+k}{2k+1} \right\} \) are, respectively:
\[ \rho^{(1)} = \langle 0, 1, 2, 3, \ldots \rangle, \quad \rho^{(2)} = \langle 0, -1, 1, -2, 2, -3, 3, \ldots \rangle. \]

The usual interlacing of these two root sequences is different from the sequence \( \rho \). However, it is possible to build up the sequence \( \rho \) from \( \rho_1 \) and \( \rho_2 \):
\[ \rho_{3k+1} = \rho_{2k+1}^{(2)}, \quad \rho_{3k+2} = \rho_{k+1}^{(1)}, \quad \rho_{3k+3} = \rho_{2k+2}^{(2)} \]

It is this idea on which we base the concept of a shuffled polynomial basis.
Definition 47. For $F, m \in \mathbb{N} \setminus \{0\}$, let $B_i = \langle P_k^{(i)} \rangle_{k=0}^\infty$ for $i = 1, 2, \ldots, F$ be polynomial bases, and let $c = (c_0, \ldots, c_{m-1}) \in \{1, \ldots, F\}^m$. For all $k \in \mathbb{N}$ and index $j \in \{0, \ldots, m - 1\}$, let

$$Q_{mk+j}(x) = \prod_{i=1}^{F} P_{c_i(mk+j)}^{(i)}(x),$$

where $e_i(mk+j) = ks_i(m) + s_i(j)$, and $s_i(t) = |\{r \in \{0, \ldots, t-1\}; c_r = i\}|$ for all $t \in \{0, \ldots, m\}$. Then the sequence $\langle Q_n(x) \rangle_{n=0}^\infty$ is the $c$-shuffled basis of $B_1, \ldots, B_F$.

Intuitively, $F$ is the number of bases we are shuffling, $m$ is the length of the period of this shuffling, and the vector $c$ indicates which basis is used in each step for each period. Next, the function $e_i(n)$ gives the index of the element of the $i$-th basis that is used in the $n$-th element of the shuffled basis. Finally, the function $s_i : \{0, \ldots, m\} \to \mathbb{N}$ counts how many times we have used the $i$-th basis in a particular point of a period.

Example 48. Let $B_1, \ldots, B_m$ be factorial bases. Then $\prod_{i=1}^{m} B_i$ is the $(1, \ldots, m)$-shuffled basis of $B_1, \ldots, B_m$. Namely, the number of factors coincides with the length of the period (i.e., $F = m$), and the vector $c = (1, 2, \ldots, m)$ indicates in what order we interlace the bases.

In this particular example, the functions $s_i(j)$ are the step functions:

$$s_i(j) = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{otherwise} \end{cases}$$

which leads to the following formulas for $e_i(n)$:

$$e_i(mk+j) = k \cdot 1 + s_i(j) = \begin{cases} k & \text{if } j < i, \\ k+1 & \text{otherwise} \end{cases}$$

turning (42) into (11).

Example 49. Let us show that the basis of Example 45 is a shuffled basis using the following two bases:

$$B_1 = \langle \left( \frac{x}{n} \right)_n \rangle_n, \quad B_2 = \langle \left( \frac{x+n}{2n}, \frac{x+n}{2n+1} \right)_n \rangle_n.$$

Take $F = 2$, $m = 3$, and $c = (2, 1, 2)$. Then we have that

$$s_1(0) = 0, s_1(1) = 0, \quad s_1(2) = 1, \quad s_1(3) = 1,$$

$$s_2(0) = 0, s_2(1) = 1, \quad s_2(2) = 1, \quad s_2(3) = 2.$$

This implies that the index functions $e_1(n)$ and $e_2(n)$ have the following values:

$$e_1(3k+j) = k \cdot 1 + s_1(j) = \begin{cases} k & \text{if } j = 0, \\ k & \text{if } j = 1, \\ k+1 & \text{if } j = 2, \end{cases}$$
\[
e_2(3k + j) = k \cdot 2 + s_2(j) = \begin{cases} 2k & \text{if } j = 0, \\ 2k + 1 & \text{if } j = 1, \\ 2k + 1 & \text{if } j = 2, \end{cases}
\]

Hence, we have that the \((2, 1, 2)\)-shuffled basis of \(B_1\) and \(B_2\) is:

\[
Q_{3k}(x) = P^{(1)}_{k}(x)P^{(2)}_{2k}(x) = \left( x \right) \frac{x + k}{2k},
\]

\[
Q_{3k+1}(x) = P^{(1)}_{k}(x)P^{(2)}_{2k+1}(x) = \left( x \right) \frac{x + k}{2k + 1},
\]

\[
Q_{3k+2}(x) = P^{(1)}_{k+1}(x)P^{(2)}_{2k+1}(x) = \left( x \right) \frac{x + k}{2k + 1},
\]

which are exactly the formulas displayed in Example 45.

Shuffled bases are a particular case of sieved polynomial bases, since the elements are defined modulo \(m\).

**Definition 50.** Let \(L \in \mathcal{L}\{x\}, m \in \mathbb{N} \setminus \{0\}\), and let \(B = \langle P_n(x) \rangle_{n=0}^{\infty}\) be a factorial basis of \(K[x]\). We say that \(L\) is \((A, B)\)-compatible in \(m\) sections with \(B\) if there are \(\alpha_{k,j,i} \in \mathbb{K}(k)\) such that for all \(k \in \mathbb{N}\) and \(j \in \{0, \ldots, m-1\}\),

\[
LP_{mk+j}(x) = \sum_{i=-A}^{B} \alpha_{k,j,i}P_{mk+j+i}(x).
\]

Observe that in this definition we have used formula (18), but restricting ourselves to \(\alpha_{k,j,i}\) being rational expressions in \(k\). This means that being \((A, B)\)-compatible with \(B\) in the usual sense (see Definition 9) does not imply that the operator is \((A, B)\)-compatible with \(B\) in one section. We include this restriction for computational reasons. In general, the coefficients \(\alpha_{k,j,i}\) could belong to any computable ring of sequences closed under shift and dilation.

**Example 51.** Let \(B = \langle P_n(x) \rangle_{n=0}^{\infty}\) be the factorial basis from Example 37, defined by the formulas:

\[
P_{2n}(x) = \left( x \right)^2, \quad P_{2n+1}(x) = \left( x \right) (\frac{x}{n+1}).
\]

Equations (13) and (15) show that \(E\) is \((2, 0)\)-compatible in two sections with \(B\). However, \(E\) is not \((2, 0)\)-compatible in one section with \(B\). To see that, it is enough to write a unified compatibility formula that holds for all \(n \in \mathbb{N}\):

\[
P_n(x + 1) = P_n(x) + \frac{n}{\lfloor n/2 \rfloor} P_{n-1}(x) + \frac{\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} P_{n-2}(x).
\]

This formula fits Definition 9 but the coefficients of \(P_{n-1}(x)\) and \(P_{n-2}(x)\) are not rational in \(n\), so \(E\) is not \((2, 0)\)-compatible in one section with \(B\).
Let \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_F \) be factorial bases, compatible in sections with an endomorphism \( L \) or with \( X \), and let \( \mathcal{B} \) be their \( c \)-shuffled basis. The following lemmas show that then \( \mathcal{B} \) is also compatible in sections with \( L \) (resp. with \( X \)). Moreover, the proofs of these lemmas show how to construct the coefficients \( \alpha_{k,j,i} \in \mathbb{K}(k) \) for \( cB \), given the corresponding coefficients for each of \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_F \).

First, we show in Lemma \([52]\) how to expand the number of sections of compatibility. This result holds for any polynomial basis. In Proposition \([53]\) we show that a shuffled basis of quasi-triangular bases is again quasi-triangular. Then we proceed to extend the desired compatibilities to shuffled bases (Theorems \([54]\) and \([55]\)). These extensions make use of Proposition \([11]\) so they hold for factorial bases.

**Lemma 52.** Let \( L \in \mathcal{L}_{\mathbb{K}[x]} \) and \( \mathcal{B} = (P_n(x))_{n=0}^{\infty} \) be a polynomial basis. If \( L \) is \( (A, B) \)-compatible in \( m \) sections with \( \mathcal{B} \) then it is \( (A, B) \)-compatible in \( tm \) sections with \( \mathcal{B} \) for all \( t \in \mathbb{N} \setminus \{0\} \).

**Proof.** Let \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, tm - 1\} \). Write \( j = jm + j_1 \) with \( j_1 \in \{0, \ldots, m - 1\} \). Then we have:

\[
LP_{(tm)k+j} = LP_{(tk+j_0)m+j_1} = \sum_{i=-A}^{B} \alpha_{tk+j_0,j_1,i}P_{m(tk+j_0)+j_1+i}(x) =
\]

\[
= \sum_{i=-A}^{B} \tilde{\alpha}_{k,j,i}P_{(tm)k+j+i}(x),
\]

proving the compatibility in \( tm \) sections, with \( \tilde{\alpha}_{k,j,i} = \alpha_{tk+j_0,j_1,i} \in \mathbb{K}(k). \)

**Proposition 53.** Let \( \mathcal{B} = (Q_n(x))_{n=0}^{\infty} \) be the \( c \)-shuffled basis of the factorial bases \( \mathcal{B}_i = (P^{(i)}_n(x))_{n=0}^{\infty} \) for \( i = 1, \ldots, F \). Then \( \mathcal{B} \) is also factorial. Moreover, if all \( \mathcal{B}_i \) are quasi-triangular, so is \( \mathcal{B} \).

**Proof.** Let \( n \in \mathbb{N} \) be written as \( n = km + j \) with \( j \in \{0, \ldots, m - 1\} \). Using Definition \([47]\) we have that

\[
Q_n(x) = \prod_{i=1}^{F} P^{(i)}_{k_s_i(m)+s_i(j)}(x)
\]

where

\[
s_i(t) = |\{r \in \{0, 1, \ldots, t-1\}; c_r = i\}|
\]

for all \( t \in \{0, 1, \ldots, m\} \). We wish to show that the quotient \( Q_{n+1}(x)/Q_n(x) \) is a polynomial of degree 1. To this end, we distinguish two cases.

a) \( j \in \{0, 1, \ldots, m-2\} \)

Here \( n + 1 = km + (j + 1) \) where \( j + 1 \in \{1, 2, \ldots, m - 1\} \). Obviously

\[
s_i(j) = |\{r \in \{0, 1, \ldots, j-1\}; c_r = i\}|, \quad \text{and}
\]

\[
s_i(j + 1) = |\{r \in \{0, 1, \ldots, j-1, j\}; c_r = i\}|
\]

\[
= \begin{cases} s_i(j), & c_j \neq i \\ s_i(j) + 1, & c_j = i, \end{cases}
\]
hence the quotient

\[
\frac{P_{k_s(m)+s_i(j+1)}(x)}{P_{k_s(m)+s_i(j)}(x)} = \begin{cases} 
1, & c_j \neq i \\
\frac{P_{k_s(m)+s_i(j+1)}(x)}{P_{k_s(m)+s_i(j)}(x)}, & c_j = i
\end{cases}
\]

is a polynomial in \( x \) of degree at most 1, and so, since the basis \( \mathcal{B}_i \) is factorial, the quotient

\[
\frac{Q_{n+1}(x)}{Q_n(x)} = \prod_{i=1}^{r} \frac{P_{k_s(m)+s_i(j+1)}(x)}{P_{k_s(m)+s_i(j)}(x)} = \frac{P_{c_i}(k)_{s_i(m)+s_i(j)+1}(x)}{P_{k_s(m)+s_i(j)}(x)}
\]

is a polynomial in \( x \) of degree 1.

\[b)\] \( j = m - 1 \) (hence \( m = j + 1 \))

Here \( n = k m + (m - 1) \) and \( n + 1 = (k + 1)m + 0 \). Obviously

\[
\begin{align*}
    s_i(0) &= |\{r \in \emptyset; c_r = \hat{i}\}| = 0, \\
    s_i(m - 1) &= |\{r \in \{0, 1, \ldots, m - 2\}; c_r = \hat{i}\}|, \text{ and} \\
    s_i(m) &= |\{r \in \{0, 1, \ldots, m - 2, m - 1\}; c_r = \hat{i}\}|
\end{align*}
\]

\[
s_i(m - 1), \quad c_{m-1} \neq \hat{i} \\
    s_i(m - 1) + 1, \quad c_{m-1} = \hat{i},
\]

hence the quotient

\[
\frac{P_{(k+1)s_i(m)+s_i(0)}(x)}{P_{k_s(m)+s_i(m-1)}(x)} = \begin{cases} 
\frac{P_{(k+1)s_i(m)-1}(x)}{P_{k_s(m)-1}(x)}, & c_{m-1} \neq \hat{i} \\
\frac{P_{(k+1)s_i(m)-1+1}(x)}{P_{k_s(m)-1+1}(x)}, & c_{m-1} = \hat{i}
\end{cases}
\]

is a polynomial in \( x \) of degree at most 1, and the quotient

\[
\frac{Q_{n+1}(x)}{Q_n(x)} = \prod_{i=1}^{r} \frac{P_{(k+1)s_i(m)+s_i(0)}(x)}{P_{k_s(m)+s_i(m-1)}(x)} = \frac{P_{(c_{m-1})}(k+1)s_{m-1}(m-1) + k+1(x)}{P_{(k+1)s_{m-1}(m-1) + k}(x)}
\]

is again a polynomial in \( x \) of degree 1, so we conclude that the \( c \)-shuffled basis \( \mathcal{B} \) is indeed factorial as claimed.

Note that in the case \( b) \) it follows from (44) that \( s_{c_{m-1}}(m - 1) + 1 = \)
\(s_{cm-1}(m)\), and since \(m - 1 = j\), we have

\[
\frac{Q_{n+1}(x)}{Q_n(x)} = \frac{P_{k(s_{cm-1}(m-1)+s_{cm-1}(m-1)+1)}^{(cm-1)}}{P_{k(s_{cm-1}(m-1)+s_{cm-1}(m-1)+1)}^{(cm-1)}} = \frac{P_{k(s_{cm-1}(m-1)+s_{cm-1}(m-1)+1)}^{(cm-1)}}{P_{k(s_{cm-1}(m-1)+s_{cm-1}(m-1)+1)}^{(cm-1)}}
\]

so we conclude that equation (43) in fact holds for all \(j \in \{0, 1, \ldots, m-1\}\).

Now, if all the bases \(B_i\) are quasi-triangular, we can use Proposition 29 to prove that \(B\) is also quasi-triangular. First, it is clear from Definition 29 that the root sequence \(\rho\) of \(B\) contains the root sequences \(\rho^{(i)}\) of \(B_i\) for all \(i \in \{1, 2, \ldots, F\}\), so \(\rho\) is certainly a subsequence of \(\rho\).

Next, it is also clear that \(\rho\) is a kind of interlacing of \(\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(F)}\), in which the relative order of terms originating from \(\rho^{(i)}\) is preserved within \(\rho\) for each \(i \in \{1, 2, \ldots, F\}\). Let \(n \in \mathbb{N}\) be arbitrary, and let \(\mu\) be the minimal index such that \(\rho_\mu = n + 1\). Then \(n + 1 = \rho_\mu = \rho_\nu^{(i)}\) for some \(i \in \{1, 2, \ldots, F\}\) and \(\nu \in \mathbb{N} \setminus \{0\}\). Since \(B_i\) is quasi-triangular, there is \(\nu' < \nu\) such that \(\rho_\nu^{(i')} = n\). Hence there is also \(\mu' < \mu\) such that \(\rho_\mu' = n\), so the first appearance of \(n\) in \(\rho\) (which occurs at some \(\mu'' \leq \mu'\)) precedes the first appearance of \(n + 1\) in \(\rho\). As \(n \in \mathbb{N}\) was arbitrary, \(B\) is quasi-triangular.

**Theorem 54.** Let \(m \in \mathbb{N} \setminus \{0\}\) and \(B = (Q_n(x))_{n=0}^\infty\) be the c-shuffled basis of \(B_i = (P_n(x))_{n=0}^\infty\) for \(i = 1, \ldots, F\). If each \(B_i\) is \((0, 1)\)-compatible in \(t_i\) sections with \(X\) (the multiplication-by-x operator), then \(B\) is \((0, 1)\)-compatible in \(mt\) sections with \(X\) for any \(t \in \mathbb{N} \setminus \{0\}\) such that \(t_i\) divides \(t\) for each \(i = 1, \ldots, F\).

**Proof.** By hypothesis, for each basis \(B_i, k \in \mathbb{N}\) and \(j \in \{0, \ldots, t_i - 1\}\) we have:

\[
xP_{k, t_i+j+1}^{(i)}(x) = \alpha_{k, t_i+j+1}^{(i)}(x) + \alpha_{k, t_i+j+1}^{(i)}(x).
\]

Let \(t \in \mathbb{N}\) be such that \(t_i\) divides \(t\) for each \(i = 1, \ldots, F\). Let \(k \in \mathbb{N}\) and \(j \in \{0, \ldots, mt - 1\}\). Let us see how we can express \(xQ_{kmt+j}(x)\) in terms of \(Q_{kmt+1}(x)\) and \(Q_{kmt+j+1}(x)\).

Let \(j = ja + j_1\) with \(j_1 \in \{0, \ldots, m - 1\}\). We then have \(kmt + j = (kt + ja)m + j_1\), meaning that the only difference between the element \(Q_{kmt+j}(x)\) and \(Q_{kmt+j+1}(x)\) is in the \(c_{j_1}\)-th factor of the shuffle basis. Let \(c = c_{j_1}\). In the polynomial \(Q_{kmt+j}(x)\), the index of the factor from \(B_c\) is

\[(kt + ja)s_c(m) + s_c(j_1)\]

Let \(a_c = ts_c(m)/t\). We know that \(a_c \in \mathbb{N}\) by definition of \(t\). Hence:

\[(kt + ja)s_c(m) + s_c(j_1) = (a_c k + j_2)t_c + j_3,\]
where \( j_0 s_e(m) + s_e(j_1) = j_2 t_e + j_3 \) with \( j_4 \in \{0, \ldots, t_e - 1\} \).

In other terms, the element from the basis \( B_c \) that appears in \( Q_{kmt+j}(x) \) is the element \( P^{(c)}_{(a_i,k+j_2)t_e+j_3}(x) \) and we can use formula \(46\):

\[
x \frac{Q_{kmt+j}(x)}{Q_{kmt+j+1}(x)} = x \frac{P^{(c)}_{(a_i,k+j_2)t_e+j_3}(x)}{P^{(c)}_{(a_i,k+j_2)t_e+j_3+1}(x)} = \frac{xP^{(c)}_{(a_i,k+j_2)t_e+j_3}(x)}{xP^{(c)}_{(a_i,k+j_2)t_e+j_3+1}(x)}
\]

\[
= \frac{\alpha^{(c)}_{a_i,k+j_2,j_3,0} P^{(c)}_{(a_i,k+j_2)t_e+j_3}(x) + \alpha^{(c)}_{a_i,k+j_2,j_3,1} P^{(c)}_{(a_i,k+j_2)t_e+j_3+1}(x)}{\alpha^{(c)}_{a_i,k+j_2,j_3,0} Q_{kmt+j+1}(x) + \alpha^{(c)}_{a_i,k+j_2,j_3,1} Q_{kmt+j+1}(x)}
\]

Multiplying the last equation by \( Q_{kmt+j+1}(x) \) we get that

\[
x Q_{kmt+j}(x) = \alpha^{(c)}_{a_i,k+j_2,j_3,0} Q_{kmt+j}(x) + \alpha^{(c)}_{a_i,k+j_2,j_3,1} Q_{kmt+j+1}(x),
\]

which proves that \( B \) is \((0, 1)\)-compatible in \(tm\) sections with \(X\) and provides a direct formula for the compatibility coefficients in each section. \(\square\)

It is interesting to remark that the condition on \( t \) guarantees that the new compatibility coefficients are rational functions of \(k\). If we pick \( t \) to be minimal with such property, then we have a minimal number of sections for the compatibility of \(X\). It could be that \(X\) is compatible with \(B\) in fewer sections, but that is not the general case and should be taken care of individually.

**Theorem 55.** Let \( L \in \mathcal{L}_{K[x]} \) be an endomorphism, \( m \in \mathbb{N} \setminus \{0\} \), and \( B = \langle Q_n(x) \rangle_{n=0}^{\infty} \) be the \(c\)-shuffled basis of \( B_i = \langle P^{(i)}_n(x) \rangle_{n=0}^{\infty} \) with \(i = 1, \ldots, F\). If each \(B_i\) is \((A_i, B_i)\)-compatible in \(t_i\) sections with \(L\) then \(B\) is \((mA, B)\)-compatible in \(mt\) sections with \(L\) where:

- \( B = \min\{B_i; i \in \{1, \ldots, F\}\} \),
- \( A = \max\{A_i/s_i(m); i \in \{1, \ldots, F\}\} \),
- \( t \) is a natural number such that \( t_i \) divides \( ts_i(m) \) for each \( i = 1, \ldots, F \).

**Proof.** By the equivalence of Proposition 11, it is enough to check conditions C1 and C2 in each section. For C1, we can repeat the same proof as in Theorem 54 to show that

\[
\deg LQ_n(x) \leq n + \min\{B_i; i \in \{1, \ldots, F\}\},
\]

showing that \(B\) was chosen correctly.
Consider \( t \in \mathbb{N} \) as defined in this theorem. Let us see that, for any \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, mt - 1\} \), the polynomial \( Q_{kmt+j-mA}(x) \) divides \( Q_{kmt+j}(x) \). Since \( L \) is an endomorphism, we have:

\[
LQ_{kmt+j}(x) = L \left( \prod_{i=1}^{F} P_{kt,ts_i(m) + s_i(j)}^{(i)}(x) \right) = \prod_{i=1}^{F} \left( LP_{kt,ts_i(m) + s_i(j)}^{(i)}(x) \right). \tag{47}
\]

On the other hand,

\[
Q_{kmt+j-mA}(x) = Q_{(kt-A)m+j}(x) = \prod_{i=1}^{F} P_{(kt-A)s_i(m) + s_i(j)}^{(i)}(x). \tag{48}
\]

At this point, we only need to show that, for all \( i = 1, \ldots, F \), the polynomial \( P_{(kt-A)s_i(m) + s_i(j)}^{(i)}(x) \) divides \( LP_{kt,ts_i(m) + s_i(j)}^{(i)}(x) \). Let \( a_i = \frac{(ts_i(m))}{t_i} \), which is a natural number by the construction of \( t \). We can also write \( s_i(j) = j_i a_i t_i + j_{i,1} \) with \( j_{i,1} \in \{0, \ldots, t_i - 1\} \). Then

\[
kts_i(m) + s_i(j) = (ka_i + j_{i,0})t_i + j_{i,1}.
\]

Using the \((A_i, B_i)-compatibility\) of \( \mathcal{B}_i \) with \( L \), we have

\[
LP_{kt,ts_i(m) + s_i(j)}^{(i)}(x) = LP_{(ka_i + j_{i,0})t_i + j_{i,1}}^{(i)}(x) = \sum_{l = -A_i}^{B_i} a_{(ka_i + j_{i,0})j_{i,1},l}^{(i)} P_{kt,ts_i(m) + s_i(j) + l}^{(i)}(x).
\]

If we now show that \((kt-A)s_i(m) + s_i(j) \leq kts_i(m) + s_i(j) - A_i \) for all \( i \), then using the fact that \( \mathcal{B}_i \) is a factorial basis, we get that

\[
P_{(kt-A)s_i(m) + s_i(j)}^{(i)}(x) \text{ divides } P_{kt,ts_i(m) + s_i(j)}^{(i)}(x) \text{ for all } l = -A_i, \ldots, B_i,
\]

and, in particular, that \( P_{(kt-A)s_i(m) + s_i(j)}^{(i)}(x) \) divides \( LP_{kt,ts_i(m) + s_i(j)}^{(i)}(x) \).

But this is simple to prove using the construction of \( A \). Since \( A \geq A_i/s_i(m) \) for all \( i = 1, \ldots, F \), then we have \( A_i s_i(m) \geq A_i \). Hence,

\[
(kt-A)s_i(m) + s_i(j) = kts_i(m) + s_i(j) - A_i s_i(m) \leq kts_i(m) + s_i(j) - A_i.
\]

In the rest of the proof, we analyze the quotient between (47) and (48) to show that \( L \) is \((mA, B)-compatible\) with \( \mathcal{B} \) in \( mt \) sections, and that all compatibility coefficients for \( \mathcal{B} \) are rational functions in \( k \) for all the sections.

We first need to define a set of polynomials for each of the bases \( \mathcal{B}_i \). Let \( i = 1, \ldots, F \), \( j \in \{0, \ldots, t_i - 1\} \), \( s \in \mathbb{N} \) and \( k \in \mathbb{N} \). Consider the following quotient:

\[
D_{j,s,k}^{(i)}(x) = \frac{LP_{km+j}^{(i)}(x)}{LP_{km+j-(A_i+s)}^{(i)}(x)}.
\]

This is always a polynomial by Proposition 11. Moreover, the coefficients of these polynomials are rational functions in \( k \), since the compatibility conditions for each basis \( \mathcal{B}_i \) are given by rational functions in \( k \).
Also, for \( i \in \{1, \ldots, F \} \), consider the integers \( b_i = \As_i(m) - A_i \). These are always non-negative integers since we have taken \( A \) such that \( \As_i(m) \geq A_i \). Now, we can analyze the quotient between (47) and (48):

\[
\frac{LQ_{kmt+j}(x)}{Q_{kmt+j-mA}(x)} = \prod_{i=1}^{F} \frac{LP^{(i)}_{k\alpha_i(m)+s_i(j)}(x)}{P^{(i)}_{(kt-A)s_i(m)+s_i(j)}}(x) = \prod_{i=1}^{F} \frac{LP^{(i)}_{(k\alpha_i+j_i,0)t_i+j_{i,1}}(x)}{P^{(i)}_{(k\alpha_i+j_i,0)t_i+j_{i,1}-(A_i+b_i)}}(x)
\]

(49)

We can now follow the proof of Proposition 11 to convert the coefficients of the quotient (49) to the compatibility coefficients for \( B \). Let \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, mt - 1\} \). Consider, for \( l \in \mathbb{N} \), the following family of polynomials

\[ I_{k,j,l} = \frac{Q_{kmt+j-mA+l}(x)}{Q_{kmt+j-mA}(x)} \in \mathbb{K}[x]. \]

Since \( B \) is a factorial basis, it is clear that, for any fixed values of \( k \) and \( j \), the set \( \{I_{k,j,l}; l \in \mathbb{N}\} \) is a factorial basis of \( \mathbb{K}[x] \). Let \( \beta_{k,j,l} \) be the coefficients of the quotient (49) in terms of this basis. Then we get

\[
\frac{LQ_{kmt+j}(x)}{Q_{kmt+j-mA}(x)} = \sum_{l=0}^{mA+B} \beta_{k,j,l} I_{k,j,l} = \sum_{l=0}^{mA+B} \beta_{k,j,l} I_{k,j,l} = \frac{\sum_{l=0}^{mA+B} \beta_{k,j,l} LQ_{kmt+j-mA+l}(x)}{Q_{kmt+j-mA}(x)} .
\]

(50)

The coefficients \( \beta_{k,j,l} \) can be computed using linear algebra, performing a change of basis from the standard power basis \( \{x^n; n \in \mathbb{N}\} \) to the basis created by \( \{I_{k,j,l}; l \in \mathbb{N}\} \). Moreover, the upper bound for the sum in (50) is at most \( mA + B \) since we already know that \( L \) is \((mA, B)\)-compatible with \( B \).

If we multiply both sides of equation (50) by \( Q_{kmt+j-mA}(x) \) we obtain:

\[
LQ_{kmt+j}(x) = \sum_{l=0}^{mA+B} \beta_{k,j,l} Q_{kmt+j-mA+l}(x) = \sum_{i=-mA}^{B} \beta_{k,j,i+mA} Q_{kmt+j+i}(x).
\]

This formula provides the compatibility coefficients for the \((mA, B)\)-compatibility of \( L \) with \( B \) in \( mt \) sections, by taking \( \alpha_{k,j,i} = \beta_{k,j,i+mA} \). These coefficients are always rational functions in \( k \) since \( B \) is a shuffled basis of length \( m \).

This theorem is a direct generalization of Theorem 34 since for a product basis we have \( s_i(m) = 1 \) for each factor. Hence, in this case the definition of \( A \) in Theorem 34 coincides with the definition of \( A \) in this theorem.
8. Concluding remarks

The first author has implemented the results of this paper in a SageMath package which allows for a fully automatic computation of the examples throughout this document.

The software, still under active development at the time of writing, is distributed under the GNU General Public License\(^3\) and is available at:

https://github.com/Antonio-JP/pseries_basis

We conclude by listing some possible extensions of the results of this paper.

- **Better analysis of sections:** in Examples 45 and 46 we obtained solutions analyzing only the first column of the matrix of operators \(R_B L\). This is helpful to study solutions with a fixed kernel \(K(n, k)\). However, it may happen that this approach yields no nonzero solutions. As shown in Example 44 we can extract more information if we analyze the solutions of all the columns of \(R_B L\). It could be interesting to study how we can solve these systems in an automatic fashion, and what kind of information they can provide.

- **Compatibilities of derivations:** although compatibility with the derivation operator \(D\) seems limited by Proposition 16 there are factorial bases that are compatible with derivation operators. All the results extending compatibilities for product bases (Definition 33), sieved bases, and shuffled bases (Definition 47) can be extended for arbitrary derivation operators in a similar way as for endomorphisms (Theorem 34 and Theorem 55). These results are already implemented in the package \texttt{pseries\_basis}.

- **Other polynomial bases:** this paper has focused on factorial bases. However, we can prove Theorem 22 for any polynomial basis \(\langle P_n(x) \rangle_{n=0}^{\infty}\) with \(\deg P_n(x) = n\), hence we can study similar compatibility problems for orthogonal polynomial bases.

  A basic implementation of these properties and bases is included in our package (see documentation for the class \texttt{OrthoBasis}).

- **Other series bases:** similarly, we can study other types of bases for \(\mathbb{K}[[x]]\). Instead of having a basis consisting of polynomials, we can consider a formal power series basis \(\langle f_n(x) \rangle_{n=0}^{\infty}\) where \(f_n(x) = \sum_{k=n}^{\infty} c_k x^k\) with \(c_n \neq 0\). In this setting, the same definition of compatibility carries over, and a corresponding version of Theorem 22 can be proven.

  In particular, let \(f(x) \in \mathbb{K}[[x]]\) have order 1. Then if an operator \(L\) is compatible with the basis \(\langle f(x)^n \rangle_{n=0}^{\infty}\), all the solutions for \(L\) can be written as a composition of a holonomic function with \(f(x)\).

  A simple implementation for this type of bases is included in our software (check documentation for \texttt{OrderBasis} and \texttt{FunctionalBasis}).

---

\(^3\)See https://www.gnu.org/licenses/gpl-3.0.txt
Appendix A. Implementation

All the results in this paper are included in the SageMath package named \texttt{pseries\_basis}, which allows a fully automated computation of all the examples throughout this document.

At the time of writing, this software is still under development (the current version is v0.3) and has not been added to the official Sage distribution. Readers are invited to test the functionalities included in the package and report any desired features, errors or comments.

The software is distributed under the GNU General Public License\footnote{See https://www.gnu.org/licenses/gpl-3.0.txt} on the GitHub repository:

\url{https://github.com/Antonio-JP/pseries_basis}

Any Sage user can install it locally using the PyPi system included in Sage by running the command

\begin{verbatim}sage --pip install git+https://github.com/Antonio-JP/pseries_basis\end{verbatim}

or by cloning the repository and running \texttt{make install} in the repository folder. This process will install all required dependencies for a proper functionality. Once installed, the package is available in Sage and can be imported with the code:

\begin{verbatim}sage: from pseries_basis import *\end{verbatim}

In case the user does not want or could not install Sage locally, we offer the possibility of using it via Binder. A complete demo of the package with explanations of its implementation can be found at:

\url{https://mybinder.org/v2/gh/Antonio-JP/pseries_basis/master?labpath=notebooks\%2Fpaper_examples.ipynb}

All the documentation of the code can be also found at:

\url{https://antonio-jp.github.io/pseries_basis/}

Appendix A.1. Data structures

The package \texttt{pseries\_basis} provides a class \texttt{FactorialBasis} to represent the factorial bases $\mathcal{B}$ described throughout this paper (see Definition 3). These bases have a main method \texttt{element} that, given an index $n \in \mathbb{N}$, returns their $n$th element.

Then, several general functionalities are included to manage the compatibilities with linear operators:

- \texttt{set\_compatibility}: given an operator $L$ and some coefficients $\alpha_{k,j,i}$, it sets the compatibility of the operator $L$ with the provided coefficients (see Definitions 9 and 50).
• **compatibility**: given an operator $L$, it returns the compatibility coefficients associated with it if it is compatible with $\mathcal{B}$.

• **recurrence**: returns the recurrence equation (or system in case of sieved bases) associated with an operator $L$. This method is based on Theorem 24.

### Appendix A.2. Building factorial bases

The package includes several built-in bases that can be easily obtained in the code:

- **Power basis**: let $\mathcal{P}_{a,b} = \langle (ax + b)^n \rangle$. This basis can be built using `PowerBasis(a,b)` and includes automatically the compatibility with $X$ and $D$. In particular, we can build the power basis $\mathcal{P} = \langle x^n \rangle$ that has been used in the previous sections.

- **Falling factorial basis**: let $\mathcal{B} = \langle \prod_{k=0}^{n-1} (ax + b - kc) \rangle$. This basis can be built using `FallingBasis(a,b,c)` and includes automatically the compatibility with $X$ and $E_{c/a}$: $x \mapsto x + \left(\frac{c}{a}\right)$. This includes the falling factorial basis (when $a = 1$, $b = 0$ and $c = 1$) and the rising factorial basis (when $a = 1$, $b = 0$ and $c = -1$).

- **Binomial basis**: recall $\mathcal{C}_{a,b} = \langle \binom{ax + b}{n} \rangle$. These bases can be obtained using `BinomialBasis(a,b)` and include automatically their compatibility with $X$ and $E$.

From these basic pieces, the user can build even further bases with the following functionality:

- **Scalar product**: given a basis $\mathcal{B} = \langle P_n(x) \rangle$ and a hypergeometric sequence $(a_n)_n$, the new basis $a\mathcal{B} = \langle a_n P_n(x) \rangle$ can be computed with usual multiplication in Sage. Moreover, the compatibilities of $\mathcal{B}$ are automatically extended to $a\mathcal{B}$.

- **Product basis**: the product basis of $\mathcal{B}_1,...,\mathcal{B}_m$ can be built in the code with the command `ProductBasis([\mathcal{B}_1,...,\mathcal{B}_m], \text{ends}=[])`, where the content of \text{ends} is a list of the names of endomorphisms that the resulting product basis will be compatible with.

- **Shuffled basis**: more generally, the user can build a shuffled basis with the command `SievedBasis([\mathcal{B}_1,...,\mathcal{B}_m], c, \text{ends}=[])`, where $c$ is the cycle determining how the roots of the factor bases are shuffled (see Definition 47).
Appendix A.3. Building generalized binomial bases

In Examples 45 and 46 we have used a basis whose even elements were the binomial coefficients \( \binom{x+n}{2n} \). In this subsection we illustrate how we can use our package to build this basis automatically.

By shifting \( n \rightarrow (n+1) \) in \( \binom{x+n}{2n} \) and taking the quotient, we obtain:

\[
\frac{\binom{x+n+1}{2n+2}}{\binom{x+n}{2n}} = \frac{(x+n+1)(x-n)}{(2n+1)(2n+2)},
\]

so at every two steps we added the roots \( n \) and \((-n-1)\) to the root sequence of the basis and a factor of \( \frac{1}{(2n+1)(2n+2)} \) to the leading coefficients.

The falling and rising factorial basis have root sequences

\[ \rho_f = (0, 1, 2, \ldots), \quad \rho_r = (-1, -2, -3, \ldots), \]

respectively. Hence, we can build the product of these two bases to obtain a basis with the desired root sequence:

\[
sage: pos_roots = FallingBasis(1,0,1)
sage: neg_roots = FallingBasis(1,1,-1)
sage: almost = ProductBasis([pos_roots, neg_roots])
\]

In this piece of code, the object \( \texttt{almost} \) contains a product basis that guarantees the desired root sequence. This is not yet the basis we want because the leading coefficient sequence is all 1:

\[
sage: [almost.cn(i) for i in range(10)]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
\]

And if we look at the bases used in Examples 45 and 46, the leading coefficients are precisely \( \frac{1}{n!} \). We can build this also in the code since \( \frac{1}{n!} \) is a hypergeometric sequence:

\[
sage: basis = (1/factorial(n))*almost
\]

This process can be generalized to any desired binomial coefficient of the shape

\[
\binom{ax+bn+c}{mn+r}
\]

This is automatized in the method \( \texttt{GeneralizedBinomial} \) that receives the constant parameters \( a, b, c, m \) and \( r \) and returns a basis whose \((mn)\)-th elements are precisely the binomial coefficients shown above. Moreover, the compatibilities with \( X \) and \( E \) are automatically computed whenever they are possible (i.e, whenever \( r = 0 \)).

Appendix A.4. Revisiting Example 45

Now, we are going to show how to use the package to reproduce Example 45. We are interested in studying compatibility of the linear operator

\[
L = (n+2)^2E^2 - (11n^2 + 33n + 25)E - (n+1)^2,
\]
with respect to the kernel:

\[ K(n, k) = \binom{n}{k} \binom{n+k}{2k}. \]

This kernel is built as a product of two simpler binomial coefficients that can be created with the code:

```sage
sage: b1 = BinomialBasis(1,0)
sage: b2 = GeneralizedBinomial(1,1,0,2,0)
```

As we saw in Example 45, we are interested in a basis that has \( K(n, k) \) as some of its elements. This can be achieved with a shuffled basis:

```sage
sage: A2 = SievedBasis([b1,b2],[1,0,1], ends=['E'])
```

Now that we have built the basis of \( K[x] \), we need to build the linear operator associated to \( L \). For doing so, we are going to use the package `ore_algebra` developed by M. Kauers and M. Mezzarobba that provides functionality to represent such linear operators:

```sage
sage: OE.<E> = OreAlgebra(QQ['x'], ('E',
    lambda p : p(x=x+1),
    lambda p : 0))
sage: L = (x +2)^2*E^2 -
    (11* x^2 + 33* x + 25)*E -
    (x +1)^2
```

And now we follow the process described in Example 45: we build the recurrence matrix, take the first column and compute a greatest common right divisor of its elements:

```sage
sage: recurrence_matrix = A2.recurrence(L)
sage: first_column =
    [A2.remove_Sni(recurrence_matrix[j,0])
    for j in range(recurrence_matrix.nrows())]
sage: gcrd = first_column[0].gcrd(*first_column[1:])
```

Which is exactly the recurrence we obtained in (39).

### Appendix A.5. Revisiting Example 46

In Example 46 we studied compatibility of the linear operator

\[ L = (n + 2)^3E^2 - (2n + 3)(17n^2 + 51n + 39)E + (n + 1)^3, \]

with respect to the kernel:

\[ K(n, k) = \left( \frac{n + k}{2k} \right)^2. \]

This kernel is the square of a simple binomial coefficient that can be created with the code:

```sage
https://github.com/mkauers/ore_algebra
```
This basis contains \( \binom{x+n}{2n} \) as its even positions. Hence, to obtain the basis described in Example 40, we only need to build the `ProductBasis` of `b2` with itself:

```
sage: A3 = ProductBasis([b2,b2], ends=['E'])
```

In a similar way as we did in Example 45, we now build the linear operator using `ore_algebra`, then obtain the recurrence matrix and consider the greatest common right divisor of the elements of its first column:

```
sage: OE.<E> = OreAlgebra(QQ[x], ('E',
lambda p : p(x=x+1),
lambda p : 0))
sage: L = (x +2)^3* E^2 -
(2*x + 3)*(17*x^2 + 51*x + 39)*E +
(x +1)^3
sage: recurrence_matrix = A3.recurrence(L)
sage: first_column =
[A3.remove_Sni(recurrence_matrix[j,0])
for j in range(recurrence_matrix .nrows())]
sage: gcrd = first_column[0].gcrd(*first_column[1:])
sage: gcrd
(n^2 + 2*n + 1)*Sn - 16* n^2 - 16* n - 4
```

Which is exactly the recurrence we obtained in (41).

### Appendix B. More examples

In this section we include additional examples that are not in the original paper. These examples showcase how to use the results on this paper, and illustrate (see Examples 58 and 59) how we can obtain definite-sum solutions containing not just one, but several nested definite sums.

All these examples (and more) can be found and tested in the repository.

We also offer a binder notebook to try these examples out without installing Sage or the package:

```
https://mybinder.org/v2/gh/Antonio-JP/pseries_basis/master?labpath=notebooks/paper_examples_appB.ipynb
```

**Example 56** (Binomial transform of the Catalan numbers). Let us start with the sequence \((e_n)\) defined by \(e_0 = 1\), \(e_1 = 2\) and \(Le = 0\) where:

\[
L = (x +3) E^2 - 2 (3x + 5) E + 5 (x + 1).
\]

According to the OEIS database, this sequence\(^7\) is the binomial transform of the Catalan numbers, i.e.,

\[
e_n = \sum_{k=0}^{n} c_k \binom{n}{k},
\]
where \((c_n)_n\) is the sequence of Catalan numbers\(^8\). The methods of this paper are a great tool for proving automatically this type of identities. For doing so, we compute a new sequence \((b_n)_n\) such that we know \(e_n = \sum_{k=0}^n b_k \binom{n}{k}\). This sequence will be annihilated by \(\mathcal{R}_C(L)\) and will have as initial conditions \(b_0 = 1, b_1 = 1, b_2 = 2, b_3 = 5\).

Computing \(\mathcal{R}_C(L)\) yields the operator:

\[
\mathcal{R}_C(L) = (n + 3) S_n^2 - (3n + 4) S_n - 2(2n + 1).
\]

By closure properties of P-recursive sequences it is easy to show that \(b_n = c_n\) for all \(n \in \mathbb{N}\).

**Example 57** (Franel numbers). Franel numbers \((f_n)_n\) satisfy \(f_0 = 1, f_1 = 2\) and \(L f = 0\) where \(L = (n + 2)^2 E^2 - (7n^2 + 21n + 16) E - 8(n + 1)^2\).

It is known that Franel numbers are the sum of the cubes of binomial coefficients: \(f(n) = \sum_{k=0}^n \binom{n}{k}^3\). We can check this identity using our methods with the product basis \(C((1,1),(0,0,0))\). In this case, the compatibility of \(C((1,1),(0,0,0))\) with \(E\) and \(X\) can be written with \(3 \times 3\) matrices:

\[
\mathcal{R}(X) = \begin{pmatrix}
 n & 0 & n S_n^{-1} \\
 (n + 1) & n & 0 \\
 0 & (n + 1) & n
\end{pmatrix},
\]

\[
\mathcal{R}(E) = \begin{pmatrix}
 S_n + 1 & \frac{3n+1}{n+1} & \frac{3n^2+3n+1}{(n+1)^2} \\
 3S_n & \frac{n+1}{n+2} S_n + 1 & \frac{3n+2}{n+1} \\
 3S_n & \frac{3n+3}{n+2} S_n & \frac{(n+1)^2}{(n+2)^2} S_n + 1
\end{pmatrix}.
\]

The associated matrix for the operator \(L\) that defines the Franel numbers is then a \(3 \times 3\) matrix. We have included in the GitHub repository a folder with the description of each of its elements\(^10\). But since we want to see solutions of the shape

\[
f(x) = \sum_n c_n \left(\frac{x}{n}\right)^3,
\]

we have to do as we did for Examples 45 and 46 and consider the greatest common right divisor of the elements of the first column of the matrix which

\(^8\)https://oeis.org/A000108
\(^9\)See https://oeis.org/A000172
\(^10\)https://github.com/Antonio-JP/pseries_basis/tree/master/notebooks/example57
are:

\[ L_{0,0} = (n + 3)^2 S_n^3 + \frac{158n^4 + 686n^3 + 1088n^2 + 756n + 199}{(n + 1)^2} S_n^2 + \]
\[ \frac{62n^4 - 8n^3 - 334n^2 - 402n - 141}{(n + 1)^2} S_n - (221n^2 + 244n + 67) \]
\[ L_{1,0} = \frac{(n + 3)(11n^2 + 42n + 37)}{n + 2} S_n^3 + \]
\[ \frac{274n^4 + 1598n^3 + 3414n^2 + 3166n + 1073}{(n + 1)(n + 2)} S_n^2 - \]
\[ \frac{200n^3 + 883n^2 + 1217n + 531}{(n + 1)} S_n - (85n + 61)(n + 1) \]
\[ L_{2,0} = \frac{55n^4 + 420n^3 + 1168n^2 + 1398n + 607}{(n + 2)^2} S_n^3 + \]
\[ \frac{266n^4 + 1806n^3 + 4568n^2 + 5106n + 2129}{(n + 2)^2} S_n^2 - \]
\[ (307n^2 + 914n + 670) S_n - 14(n + 1)^2 \]

Computing the greatest common right divisor for these operators yields the operator \( S_n - 1 \), which is only satisfied by constant sequences. A simple computation guarantees that if \( f(x) = \sum_n c_n \binom{x}{n} \) for \( f(x) \) the solution yielding the Franel numbers, then \( c_0 = 1 \) meaning that \( c_n = 1 \) for all \( n \) proving the desired identity:

\[ f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad \text{for all } n \in \mathbb{N}. \]

**Example 58** (First double binomial sum). Let us consider now the sequence \((d_n)_n\) where \( d_n \) is the sum over all Dyck paths of semilength \( n \) of the arithmetic mean of the \( x \) and \( y \) coordinates.\footnote{https://oeis.org/A258431} It is known that this sequence satisfies the following recurrence

\[ (n - 1)d_n = (8n - 10)d_{n-1} - (16n - 24)d_{n-2} \quad \text{for } n > 2, \]

and has as its first terms \( d_0 = 0, d_1 = 1 \) and \( d_2 = 5 \). Let \( d(x) \) be a function such that \( d(n) = d_n \) for all \( n \in \mathbb{N} \). Then it is annihilated by the following linear operator:

\[ L = (x + 2) E^3 - (8x + 14) E^2 + (16x + 24) E. \quad \text{(B.1)} \]

We want to find an explicit formula for this sequence, so we try to compute a sum with respect to the binomial basis \( C \), i.e., we write \( d(x) = \sum_k c_k \binom{x}{k} \). Computing the operator \( R_C(L) \) yields:

\[ (n + 3) S_n^4 - (4n + 12) S_n^3 - 2n S_n^2 + 12(n + 2) S_n + 9(n + 1). \]
Hence, the sequence \((c_n)_n\) defined by \(R_C(L)\) with initial terms 0, 1, 3, 11, 36, \ldots allows us to write:

\[
d_n = \sum_{k=0}^{n} c_k \binom{n}{k}.
\]

If we look at OEIS for this sequence \((c_n)_n\), we do not find anything. So we compute now a new sequence \((b_n)_n\) for which we can write \(c_n = \sum_{l=0}^{n} b_l \binom{n}{l}\).

To this end, we compute the operator:

\[
R_C(R_C(L)) = (n + 3) S_n^4 + n S_n^3 - 2 (4n + 9) S_n^2 - 8 n S_n + 16 n S_n^{-1} + 8(2n + 3).
\]

Using the results of this paper, we know that the sequence \((b_n)_n\) with initial terms 0, 1, 1, 5, 6 is annihilated by \(R_C(R_C(L))\). This sequence still does not show up in OEIS. However, we observe that the sequence \((b_n)_n\) is the interleaving of two simpler sequences. More specifically, the odd terms look like the original sequence \((d_n)_n\). Hence, if we put everything together, we can prove by using closure properties that, if \((d_n)_n\) are defined as above and \((b_n)_n\) are defined by the formula:

\[
d_n = \sum_{l=0}^{n} \sum_{k=0}^{l} b_l \binom{k}{l} \binom{l}{n},
\]

then for all \(n \in \mathbb{N}\), we have \(b_{2n-1} = d_n\).

**Example 59** (Second double binomial sum). For this example we are going to consider the sequence:

\[
a_n = \sum_{k=0}^{n} \binom{2n}{k}.
\]

This sequence is half of the sum of the binomial coefficients having even upper argument. Let \(a(x)\) be a function such that \(a(n) = a_n\). We can check that this function is annihilated by the following recurrence operator:

\[
L = (x + 2) E^2 - 2 (4x + 5) E + 8(2x + 1).
\]

We proceed now similarly to Example 58. Let \((c_n)_n\) be a sequence with \(a_n = \sum_{k=0}^{n} c_k \binom{n}{k}\). Then \((c_n)_n\) is annihilated by \(R_C(L)\), which we can compute:

\[
R_C(L) = (n + 2) S_n^2 - (5n + 6) S_n + 3n + 9 n S_n^{-1}.
\]

This recurrence operator involves the inverse shift \(S_n^{-1}\). In order to apply again the recurrence compatibility with the binomial basis \(\mathcal{C}\), we need to remove this inverse shift. For doing so we simply multiply \(R_C(L)\) by \(S_n\) from the left. Our sequence \((c_n)_n\) is still annihilated by \(S_n R_C(L)\).

The sequence \((c_n)_n\) can be found in OEIS as A027914, defined as the sum of the first half of trinomial coefficients. Let us now consider the sequence:

\[\text{https://oeis.org/A032443}\]
\((b_n)_n\) defined again as \(c_n = \sum_{k=0}^{n} b_k \binom{n}{k}\). The sequence \((b_n)_n\) is annihilated by \(\mathcal{R}_C(S_n\mathcal{R}_C(L))\), which we can compute:

\[
\mathcal{R}_C(S_n\mathcal{R}_C(L)) = (n + 3) S_n^3 - (n + 2) S_n^2 - 2(3n + 5) S_n + 4(n + 1) + 8n S_n^{-1}.
\]

The sequence \((b_n)_n\), defined by \(\mathcal{R}_C(\mathcal{R}_C(L))\) and the initial terms

\[
b_0 = 1, \quad b_1 = 1, \quad b_2 = 3, \quad b_3 = 4, \quad b_4 = 11,
\]

appears in OEIS as the sequence A027306 (which can be checked automatically using closure properties of D-finite sequences). This sequence has a closed form formula:

\[
b_n = 2^{n-1} + \left(\frac{1 + (-1)^n}{4}\right) \binom{n}{n/2}.
\]

Putting everything together, this process has proved the following identity:

\[
\sum_{k=0}^{n} \binom{2n}{k} = \sum_{k=0}^{n} \sum_{l=0}^{k} \left(2^{l-1} + \left(\frac{1 + (-1)^l}{4}\right) \binom{l}{l/2} \binom{k}{l} \binom{n}{l}\right).
\]

**Example 60** (Third double binomial sum). We can illustrate the recursive use of our methods on the linear recurrence equation \(Ly = 0\) where \(L \in \mathbb{Q}[x]\langle E\rangle\) is the recurrence operator of order 5 defined by

\[
L = -64(1 + x)(2 + x)(3 + x)(-151 - 39x + 8x^2 + 2x^3)
+ 16(2 + x)(3 + x)(-3867 - 2400x - 182x^2 + 108x^3 + 16x^4)E
- 4(3 + x)(-48214 - 42707x - 10472x^2 + 530x^3 + 500x^4 + 48x^5)E^2
+ 2(-163088 - 179069x - 66637x^2 - 6360x^3 + 1822x^4 + 480x^5 + 32x^6)E^3
- (-61566 - 62939x - 21344x^2 - 1644x^3 + 580x^4 + 132x^5 + 8x^6)E^4
+ (5 + x)(-106 - 49x + 2x^2 + 2x^3)E^5.
\]

If we look for solutions \(y(x)\) such that \(Ly = 0\) we will not find any hypergeometric solutions. Then we can try to use the methods of this paper to find a definite-sum solution for this recurrence. We start by taking the binomial basis \(C\) that we have used throughout the paper. If we write the solutions \(y(x)\) in the binomial basis \(y(x) = \sum_{n=0}^{\infty} z_n \binom{\alpha}{n}\), then we know that the sequence \((z_n)_n\) is annihilated by the recurrence \(\mathcal{R}_C(L)\):

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\[ R_C(L) = -8(n - 5)(n - 4)(n - 3)(n - 2)(n - 1)nS^{-6} \]
\[ - 4(n - 4)(n - 3)(n - 2)(n - 1)n(15 + 4n)S^{-5} \]
\[ + 2(n - 3)(n - 2)(n - 1)n(17 - 114n + 12n^2)S^{-4} \]
\[ + 2(n - 2)(n - 1)n(451 - 493n + 32n^2 + 32n^3)S^{-3} \]
\[ - (n - 1)n(-941 + 2804n + 692n^2 - 576n^3 + 16n^4)S^{-2} \]
\[ - n(4128 + 8413n + 1816n^2 - 1772n^3 + 152n^4 + 96n^5)S^{-1} \]
\[ + 180 + 10988n + 19519n^2 + 8440n^3 - 752n^4 - 680n^5 - 16n^6 \]
\[ + (50990 + 66145n + 22205n^2 - 1808n^3 - 1000n^4 + 288n^5 + 64n^6)S \]
\[ + (-35864 - 77301n - 45009n^2 - 6616n^3 + 1596n^4 + 472n^5 + 24n^6)S^2 \]
\[ - (85212 + 67646n + 12843n^2 - 1308n^3 - 236n^4 + 108n^5 + 16n^6)S^3 \]
\[ - (-58916 - 60882n - 21043n^2 - 1728n^3 + 562n^4 + 132n^5 + 8n^6)S^4 \]
\[ + (5 + n)(-106 - 49n + 2n^2 + 2n^3)S^5. \]

In particular, the sequence \((z_n)_n\) will be annihilated by \(S^6R_C(L)\). This operator only has forward shifts and can be considered as an element of \(\mathbb{Q}[x]\langle E \rangle\) (as the operator \(L\)), by mapping \(S \mapsto E\) and \(n \mapsto x\). Now, we consider a function \(z(x)\) with \(z(n) = z_n\) that is annihilated by the following operator:

\[ M = -8(x + 1)(x + 2)(x + 3)(x + 4)(x + 5)(x + 6) \]
\[ - 4(x + 2)(x + 3)(x + 4)(x + 5)(x + 6)(39 + 4x)E \]
\[ + 2(x + 3)(x + 4)(x + 5)(x + 6)(-235 + 30x + 12x^2)E^2 \]
\[ + 2(x + 4)(x + 5)(x + 6)(5557 + 3347x + 608x^2 + 32x^3)E^3 \]
\[ - (x + 5)(x + 6)(-62885 - 37276x - 6220x^2 - 192x^3 + 16x^4)E^4 \]
\[ - (x + 6)(680718 + 592237x + 210112x^2 + 36436x^3 + 3032x^4 + 96x^5)E^5 \]
\[ - (4416936 + 4645888x + 1770833x^2 + 323528x^3 + 29792x^4 + 1256x^5 + 16x^6)E^6 \]
\[ + (4786184 + 4125565x + 1639901x^2 + 354352x^3 + 42200x^4 + 2592x^5 + 64x^6)E^7 \]
\[ + (3309382 + 4225311x + 1666719x^2 + 305288x^3 + 28716x^4 + 1336x^5 + 24x^6)E^8 \]
\[ - (1951356 + 1322930x + 482643x^2 + 101028x^3 + 11644x^4 + 684x^5 + 16x^6)E^9 \]
\[ - (573028 + 1214154x + 509885x^2 + 93840x^3 + 8842x^4 + 420x^5 + 8x^6)E^{10} \]
\[ + (11 + x)(104 + 191x + 38x^2 + 2x^3)E^{11}, \]

which is the same operator as \(S^6R_C(L)\) but written as an element of \(\mathbb{Q}[x]\langle E \rangle\). Similarly to what we did with \(y(x)\), we can write the function \(z(x) = \sum_{k=0}^{\infty} w_k(x)\) and then the sequence \((w_n)_n\) is annihilated by the operator \(R_C(M)\):
\[ R_C(M) = 256(n - 2)^2(n - 1)n(2n - 3)S^{-3} \]

\[-128(n - 1)n \left(4n^4 - 66n^3 + 124n^2 - 77n + 14\right) S^{-2} \]

\[-32n \left(144n^5 - 1036n^4 - 954n^3 + 52n^2 + n + 32\right) S^{-1} \]

\[-16 \left(1176n^6 - 260n^5 - 18480n^4 - 42580n^3 - 37988n^2 - 12369n - 252\right) \]

\[-16 \left(2880n^6 + 16930n^5 + 9973n^4 - 120405n^3 - 307141n^2 - 257421n - 54488\right) S \]

\[-8 \left(9420n^6 + 109214n^5 + 441622n^4 + 610729n^3 - 368468n^2 - 1447147n - 656803\right) S^2 \]

\[-2 \left(43344n^6 + 740620n^5 + 4880834n^4 + 15304796n^3 + 22173471n^2 + 11018808n - 25704\right) S^3 \]

\[-\left(71912n^6 + 1612772n^5 + 14331384n^4 + 63562068n^3 + 144253438n^2 + 150787607n + 51445066\right) S^4 \]

\[-\left(43344n^6 + 1198400n^5 + 13254962n^4 + 74139652n^3 + 216230077n^2 + 297858761n + 134496052\right) S^5 \]

\[-\left(18840n^6 + 617420n^5 + 8122956n^4 + 54243220n^3 + 189442880n^2 + 311777199n + 161972208\right) S^6 \]

\[-\left(5760n^6 + 217784n^5 + 3308558n^4 + 25513428n^3 + 102653179n^2 + 192498561n + 107373416\right) S^7 \]

\[-\left(1176n^6 + 50300n^5 + 863968n^4 + 7519420n^3 + 33976774n^2 + 70410893n + 39850322\right) S^8 \]

\[-\left(144n^6 + 6864n^5 + 131182n^4 + 1266684n^3 + 6307771n^2 + 14115123n + 7618716\right) S^9 \]

\[-\left(8n^6 + 420n^5 + 8812n^4 + 93012n^3 + 502140n^2 + 1188245n + 560444\right) S^{10} \]

\[+ (n + 11) \left(2n^3 + 38n^2 + 191n + 104\right) S^{11}. \]

If we now clear the inverse shifts as we did for \( R_C(L) \) then we obtain the operator of order 14 defined by \( N = S^3R_C(M) \). We again transform the operator \( N \) to the operator ring \( \mathbb{Q}[x] \langle E \rangle \) by mapping \( S \mapsto E \) and \( n \mapsto x \):
$N = 256(x + 1)^2(x + 2)(x + 3)(2x + 3)$
- $128(x + 2)(x + 3) \left( 4x^4 - 18x^3 - 254x^2 - 683x - 559 \right) E$
- $32(x + 3) \left( 144x^3 + 1124x^4 - 426x^3 - 25598x^2 - 79013x - 74179 \right) E^2$
- $16 \left( 1176x^6 + 20908x^5 + 136380x^4 + 347300x^3 - 60488x^2 - 1776489x - 2231667 \right) E^3$
- $16 \left( 2880x^6 + 68770x^5 + 652723x^4 + 3078171x^3 + 7218056x^2 + 6781572x + 179368 \right) E^4$
- $8 \left( 9420x^6 + 278774x^5 + 3351532x^4 + 20826253x^3 + 69908761x^2 + 118492934x + 77352791 \right) E^5$
- $2 \left( 43344x^6 + 1520812x^5 + 21841574x^4 + 213028306x^3 + 5038293899x^2 + 512343213 \right) E^6$
- $\left( 71912x^6 + 2907188x^5 + 48231084x^4 + 419520636x^3 + 2013028306x^2 + 118492934x + 77352791 \right) E^7$
- $\left( 43344x^6 + 1978592x^5 + 37082402x^4 + 364460956x^3 + 1975485853x^2 + 577093275x + 6372374530 \right) E^8$
- $\left( 18840x^6 + 956540x^5 + 19927656x^4 + 217460092x^3 + 1305865484x^2 + 4067804487x + 5088538521 \right) E^9$
- $\left( 5760x^6 + 321464x^5 + 7352918x^4 + 87927084x^3 + 576736243x^2 + 1951205055x + 2622724016 \right) E^{10}$
- $\left( 1176x^6 + 71468x^5 + 1777228x^4 + 23049076x^3 + 163315666x^2 + 592690529x + 842959919 \right) E^{11}$
- $\left( 144x^6 + 9456x^5 + 253582x^4 + 353648x^3 + 26819995x^2 + 103319745x + 153333162 \right) E^{12}$
- $\left( 8x^6 + 564x^5 + 16192x^4 + 240876x^3 + 7845869x + 11977427 \right) E^{13}$
- $+ (x + 14) \left( 2x^3 + 56x^2 + 473x + 1073 \right) E^{14}$

Now we can use any software to find explicit solutions to $Nv = 0$. For example, we find two linearly independent hypergeometric solutions:

$v^{(1)}(n) = (2n + 1)!,$
$v^{(2)}(n) = \frac{1}{n!}.$

Hence, we can write two linearly independent solutions of the original operator $L$ by unrolling the identities we have found:

$y^{(1)}(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (2j + 1)!$,
$y^{(2)}(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!}{k!} \frac{1}{j!},$

both satisfy $Ly = 0$.

The operators $L$, $M$ and $N$ can be found in the repository together with the code.\textsuperscript{13}

\textsuperscript{13} https://github.com/Antonio-JP/pseries_basis/tree/master/notebooks/example60
Appendix C. The matrix elements of $[R_S L]$ from Example 44

Here we present explicit formulas for the operators $L_{0,0}, L_{0,1}, L_{1,0}$ and $L_{1,1}$ from Example 44. The computations for obtaining these operators are explained in detail in the aforementioned Example, but can also be automatically computed using the software `pseries_basis`. They can also be found (ready to be used in SageMath) in the repository:

[1] https://github.com/Antonio-JP/pseries_basis/tree/master/notebooks/example44
\[ L_{0,0} = (k + 8)(27034107689k + 247037440535) S^7 \\
- \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} (54068215378k^9 \\
- 315669611138k^8 - 45148617745347k^7 - 78269684213291k^6 \\
- 645424039244505k^5 - 3005053417965383k^4 - 8221511641457480k^3 \\
- 129113197043173300k^2 - 10575731420946896k - 34247146225582080) S^6 \\
+ \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} (27034107689k^9 - 3508146051312k^8 \\
- 127964289486598k^7 - 174117484722631k^6 - 1252449880356964k^5 \\
- 53047967564919031k^4 - 136503346354387959k^3 - 20904577928727562k^2 \\
- 173958661328760224k - 59682736706956320) S^5 \\
+ \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} (1972211122835k^9 + 62134267567378k^7 \\
+ 61671804410852k^6 + 261914159085683k^5 + 4315508250526315k^4 \\
- 1167669149632785k^3 - 9620009176334670k^2 - 4014526382135216k \\
+ 3400385599899936) S^4 \\
- \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} (2972566483581k^7 - 1482758853584245k^6 \\
- 2589937042152480k^5 - 16499978058431541k^4 - 52671318555658357k^3 \\
- 88255097542772662k^2 - 71905587088529204k - 21524025761438520) S^3 \\
- \frac{1}{(k+1)(k+2)} (64025119688979k^6 + 916316298831859k^5 \\
+ 5515823411381379k^4 + 17451451407071553k^3 + 30016470047039710k^2 \\
+ 26136807998134436k + 885455058834008) S^2 \\
+ \frac{4}{k+1} (12511390805301k^5 + 48661327183573k^4 - 74830042870409k^3 \\
- 512087325174801k^2 - 633098967293677k - 198345093160056) S \\
+ 4(34604693659372k^4 + 175550020109206k^3 + 291507102636319k^2 \\
+ 199874021738859k + 49640119659704) \\
+ 8k^2 (2263487310112k^2 + 6642551248868k + 2276852470297) S^{-1} \\
- 413236428752(k - 1)^2 k^2 S^{-2}, \]
\[ L_{0,1} = (432545723024k^2 + 5219471638609k + 13834096669960) S^6 \\
- \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} (81102320670k^8 \\
+ 19046230918120k^7 + 165247796584595k^6 + 627022054492313k^5 \\
+ 729851398238169k^4 + 1509733892578497k^3 - 430873559384614k^2 \\
- 221225807013576k + 614777382717120) S^5 \\
+ \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} (378477507646k^9 \\
+ 262294950120k^6 - 178977301246832k^7 - 3536993458109041k^6 \\
- 28948803267991653k^5 - 128904824467220745k^4 - 33282449676929377k^3 \\
- 491118612333672390k^2 - 377639831328665120k - 114882404752528800) S^4 \\
+ \frac{1}{(k+1)(k+2)(k+3)(k+4)} (3985703076428k^7 + 19378717050920k^7 \\
+ 2713882670024520k^6 + 18619321752485251k^5 + 73349659346823626k^4 \\
+ 174605063422565737k^3 + 248149371724462126k^2 + 193178386633211432k \\
+ 62908960035990144) S^3 \\
- \frac{2}{(k+1)(k+2)(k+3)} (15771010694581k^7 + 181528774532964k^6 \\
+ 745276353701544k^5 + 977855992237626k^4 + 1343180715641631k^3 \\
- 4587331633883826k^2 - 2904023672777446k + 307621359311628) S^2 \\
- \frac{4}{(k+1)(k+2)} (10843329249882k^6 + 197606913503225k^5 \\
+ 1312326808327958k^4 + 4181515523826039k^3 + 6805531587905072k^2 \\
+ 5342072311504845k + 1547863353158842) S \\
+ \frac{8}{k+1} (17316549266881k^5 + 10808747519889k^4 + 241637933180241k^3 \\
+ 250891662715092k^2 + 128154251184293k + 28521786904807) \\
+ 16k (4536557506826k^3 + 16485306406473k^2 + 1475049943633k \\
+ 4048952022402) S^{-1} \\
+ 384(k-1)k^2 (1547923253k + 13567625316) S^{-2}, \]
\[
L_{1,0} = \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} (432545723024k^8
+ 17547024744793k^7 + 30072935395324k^6 + 2834950240712954k^5
+ 16000429195865408k^4 + 55071479995635089k^3 + 11209951018863348k^2
+ 12224718229835548k + 53987648288898960) S^7
- \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)} (81102320670k^8
+ 2472339332810k^7 + 292833072628835k^6 + 162676275049999k^5
+ 3369314689609239k^4 - 6528259273082053k^3 - 46958468605880528k^2
- 8544980480836572k - 52432761655513872) S^6
+ \frac{1}{(k+2)(k+3)(k+4)(k+5)} (378477507646k^8 + 3190665711589k^7
- 176165113042889k^6 - 3680820024183060k^5 - 30578022192058416k^4
- 133391475526561039k^3 - 319542866474066205k^2 - 394131699873504978k
- 193003564034906648) S^5
+ \frac{1}{(k+2)(k+3)(k+4)} (3985703076428k^7 + 19976727165562k^6
+ 292676432119304k^5 + 20553495786751855k^4 + 79525367763859646k^3
+ 173129687209637083k^2 + 197180830938857338k + 9004378927857560) S^4
- \frac{1}{(k+2)(k+3)} (31542021389162k^6 + 410370581149671k^5
+ 1889632865572464k^4 + 2742859006263721k^3 - 4455809171785822k^2
- 1697330513822344k - 1316272750386721) S^3
- \frac{4}{k+2} (10843329249882k^5 + 213871907378048k^4 + 1533963953805732k^3
+ 5087511194624529k^2 + 785466084698885k + 4507808585185441) S^2
+ 4 (3463309533762k^4 + 268124598840421k^3 + 719049847857749k^2
+ 7871885237468017k + 289840947961864) S
+ 8(k+1)(9073115013652k^3 + 46580285333424k^2 + 68719863652441k
+ 31063488457919) + 192k^2(k+1)(3095846506k + 28683173885) S^{-1},
\]
\[ L_{1,1} = (k + 1)(27034107689k + 247037440535) S^7 \]
\[ + \frac{1}{(k + 2)(k + 3)(k + 4)(k + 5)(k + 6)} (54068215378k^8 - 694147118784k^7 - 55428688219699k^6 - 878006017531531k^5 - 6681392431254415k^4 - 28097754885306673k^3 - 66148844332414088k^2 - 80650521283582156k - 3895899854945400) S^6 \]
\[ - \frac{1}{(k + 2)(k + 3)(k + 4)(k + 5)(k + 6)} (27034107689k^9 - 3508146051312k^8 - 152565327483588k^7 - 2467626608265457k^6 - 21224638658568224k^5 - 10797246647419135k^4 - 333950933455101617k^3 - 612421038700564968k^2 - 605231342224899340k - 243934827298760208) S^5 \]
\[ + \frac{1}{(k + 2)(k + 3)(k + 4)(k + 5)} (1972211122835k^8 + 71974682766174k^7 + 87577306381164k^6 + 4554430750938027k^5 + 6786812575418620k^4 - 31583019684887547k^3 - 1571619705098751k^2 - 25788832376543354k - 1491050984687384) S^4 \]
\[ - \frac{1}{(k + 2)(k + 3)} (2972566483581k^7 - 14918406365101k^6 - 347324917413492k^5 - 291196390635209k^4 - 124841735272806629k^3 - 292268791433903686k^2 - 353135255931287712k - 170560133319385632) S^3 \]
\[ - \frac{1}{(k + 2)} (64025119688979k^6 + 1150730119088011k^5 + 8609373148451587k^4 + 33951954000293401k^3 + 73784589829185334k^2 + 83087956017304548k + 37513154389125452) S^2 \]
\[ + \frac{4}{k + 2} (12511390805301k^5 + 68878950332595k^4 - 682226097908060k^3 - 1049675078236094k^2 - 214978935833503k - 1346833666634436) S \]
\[ + 4 (34604693659372k^4 + 23568629244298k^3 + 553354051695523k^2 + 545211853981501k + 192577819165746) \]
\[ + 8k(k + 1)(2263487310112k^2 + 8868888400908k + 581054296121) S^{-1}. \]

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