Discretization and Convergence of the EIT Optimal Control Problem in 2D and 3D Domains

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Abstract

We consider Inverse Electrical Impedance Tomography (EIT) problem on recovering electrical conductivity and potential in the body based on the measurement of the boundary voltages on the $m$ electrodes for a given electrode current. The variational formulation is pursued in the optimal control framework, where electrical conductivity and boundary voltages are control parameters, and the cost functional is the norm declinations of the boundary electrode current from the given current pattern and boundary electrode voltages from the measurements. EIT optimal control problem is fully discretized using the method of finite differences. New Sobolev-Hilbert space is introduced, and the convergence of the sequence of finite-dimensional optimal control problems to EIT coefficient optimal control problem is proved both with respect to functional and control in 2- and 3-dimensional domains.

Key words: Electrical Impedance Tomography, PDE constrained optimal control, method of finite differences, discrete optimal control problem, energy estimate, embedding theorems, convergence in functional, convergence in control

AMS subject classifications: 35R30, 35Q93, 49M25, 49M05, 65M06, 65N21

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1 Introduction

This paper analyzes inverse EIT problem of estimating an unknown conductivity inside the body based on voltage measurements on the surface of the body when electric currents are applied through a set of contact electrodes. Let $Q \in \mathbb{R}^n$ be an open and bounded set representing body, and assume $\sigma(x) : Q \rightarrow \mathbb{R}_+$ be an electrical conductivity function. Electrodes, $(E_l)_{l=1}^m$, with contact impedances vector $Z := (Z_l)_{l=1}^m \in \mathbb{R}_m^+$ are attached to the periphery of the body, $\partial Q$. Electric current vector $I := (I_l)_{l=1}^m \in \mathbb{R}^m$ is applied to the electrodes. Vector $I$ is called current pattern if it satisfies conservation of charge condition

$$\sum_{l=1}^m I_l = 0 \tag{1.1}$$

The induced constant voltage on electrodes is denoted by $U := (U_l)_{l=1}^m \in \mathbb{R}^m$. By specifying ground or zero potential it is assumed that

$$\sum_{l=1}^m U_l = 0 \tag{1.2}$$

Let $u : Q \rightarrow \mathbb{R}$ is an electrostatic potential. Mathematical model of EIT is described through the following mixed boundary-value problem for second order elliptic PDE:

$$- \nabla \cdot (\sigma(x) \nabla u(x)) = 0, \quad x \in Q \tag{1.3}$$

$$\sigma(x) \frac{\partial u(x)}{\partial \nu} = 0, \quad x \in \partial Q - \bigcup_{l=1}^m E_l \tag{1.4}$$

$$u(x) + Z_l \sigma(x) \frac{\partial u(x)}{\partial \nu} = U_l, \quad x \in E_l, \ l = 1, m \tag{1.5}$$

$$\int_{E_l} \sigma \frac{\partial u}{\partial \nu} ds = I_l, \quad l = 1, m \tag{1.6}$$

where $\nu$ is the outward normal at $x \in \partial Q$. The following is the

Forward EIT Problem: Given electrical conductivity map $\sigma$, electrode contact impedance vector $Z$, and electrode current pattern $I$ it is required to find electrostatic potential $u$ and electrode voltages $U$ satisfying (1.3)–(1.6).

The goal of the paper is to analyze the following

Inverse EIT Problem: Given electrode contact impedance vector $Z$, electrode current pattern $I$ and boundary electrode measurement $U^*$, it is required to find electrostatic potential $u$ and electrical conductivity map $\sigma$ satisfying (1.3)–(1.6) with $U = U^*$.

EIT problem has many important applications in medicine, industry, geophysics and material science [28]. We are especially motivated with medical applications on the detection of cancerous tumors from breast tissue or other parts of the body. Relevance of the inverse EIT problem for cancer detection is based on the fact that the conductivity of the cancerous tumor is higher than the conductivity of normal tissues [48]. Inverse
EIT Problem is an ill-posed problem and belongs to the class of so-called Calderon type inverse problems, due to celebrated work [20], where well-posedness of the inverse problem for the identification of the conductivity coefficient $\sigma$ of the second order elliptic PDE (1.3) through Dirichlet-to-Neumann or Neumann-to-Dirichlet boundary maps is presented. Significant development in Calderon’s inverse problem in the class of smooth conductivity function with spatial dimension $n \geq 3$, concerning questions on uniqueness, stability, reconstruction procedure, reconstruction with partial data was achieved in [41, 42, 57, 51, 12, 17, 38, 39]. Global uniqueness in spatial dimension $n = 2$ and reconstruction procedure through scattering transform and employment of the, so-called $D$-bar method was presented in a key paper [52]. Further essential development of the $D$-bar method for the reconstruction of discontinuous parameters, regularization due to inaccuracy of measurements, joint recovery of the shape of domain and conductivity are pursued in [32, 33, 34, 43].

Mathematical model (1.3)–(1.6) for the EIT, referred to as complete electrode model, was suggested in [56]. This model suggests replacement of the complete potential measurements along the boundary with measurements of constant potential along the electrodes with contact impedances. In [56] it was demonstrated that the complete electrode model is physically more relevant, and it is capable of predicting the experimentally measured voltages to within 0.1 percent. Existence and uniqueness of the solution to the problem (1.3)-(1.6) was proved in [56]. Inverse EIT Problem is more difficult than the Calderon’s problem due to the fact that the infinite-dimensional conductivity function $\sigma$ and finite-dimensional voltage vector $U$ must be identified based on the finitely many boundary electrode voltage measurements. Hence the input data is finite-dimensional current vector, while in Calderon’s problem input data is given via infinite-dimensional boundary operator ”Dirichlet-to-Neuman” or ”Neuman-to-Dirichlet”. Therefore, inverse EIT problem is highly ill-posed and powerful regularization methods are required for its solution. It is essential to note that the size of the input current vector is limited to the number of electrodes, and there is no flexibility to increase its size. It would be natural to suggest that multiple data sets - input currents can be implemented for the identification of the same conductivity function. However, note that besides unknown conductivity function, there is unknown boundary voltage vector with size directly proportional to the size of the input current vector. Accordingly, multiple experiments with ”current-to-voltage” measurements is not reducing underdeterminacy of the inverse problem. One can prove uniqueness and stability results by restricting conductivity to the finite-dimensional subset of piecewise analytic functions provided that the number of electrodes is large enough [49, 26]. Within last three decades many methods developed for numerical solution of the ill-posed inverse EIT problem. Without any ambition to present a full review we refer to some significant developments such as recovery of small inclusions from boundary measurements [14, 45]; hybrid conductivity imaging methods
multi-frequency EIT imaging methods [16, 54]; finite element and adaptive finite element method [31, 50]; imaging algorithms based on the sparsity reconstruction [16, 30]; globally convergent method for shape reconstruction in EIT [27]; D-bar method, diction reconstruction method, recovering boundary shape and imaging the anisotropic electrical conductivity [13, 21, 25, 24, 29]; globally convergent regularization method using Carleman weight function [40].

Inverse EIT problem was widely studied in the framework of Bayesian statistics [37]. In [35] inverse EIT problem is formulated as a Bayesian problem of statistical inference and Markov Chain Monte Carlo method with various prior distributions is implemented for calculation of the posterior distributions of the unknown parameters conditioned on measurement data. In [36] Bayesian model of the regularized version of the inverse EIT problem is analyzed. In [44] the Bayesian method with Whittle-Matern priors is applied to inverse EIT problem. In general the strategy of the Bayesian approach to inverse EIT problem in infinite-dimensional setting is twofold. First approach is based on discretization followed by the application of finite-dimensional Bayesian methods. All the described papers are following this approach, which is outlined in [37]. Alternative approach is based on direct application of the Bayesian methods in functional spaces before discretization [47, 22].

In this paper we introduce variational formulation of the inverse EIT problem as a PDE constrained coefficient optimal control problem in a new Hilbert space setting. The novelty of the control theoretic model is its adaptation to clinical situation when additional "voltage-to-current" measurements can increase the size of the input data from the number of electrodes $m$ up to $m!$ while keeping the size of the unknown parameters fixed. We pursue discretization of the optimal control problem with the sequence of discrete optimal control problems via finite differences. The main goal of this paper is to prove the convergence of the sequence of finite-dimensional optimal control problems to EIT optimal control problem both with respect to functional and control in 2D and 3D domains. The results on the existence of the optimal control, Fréchet differentiability in the Besov space setting, formula for the Fréchet gradient, optimality condition, and numerical solution via gradient descent method in 2D model example are addressed in another paper [9].

The organization of the paper is as follows. In Section 1.1 we introduce the notations of the functional spaces. In Section 1.2 we introduce Inverse EIT Problem as PDE constrained optimal control problem. In Section 1.3 we pursue discretization via finite differences, and introduce approximating sequence of finite dimensional discrete optimal control problems. Section 2 formulates the main result. Various key preliminary results are proved in Section 3. Proof of the main result is completed in Section 4. Finally, in Section 5 we outline the main conclusions.
1.1 Notations

Although the main results of the paper are established when number of spatial variables is 2 and 3, for technical reasons we will describe general notations in space of \( n \) independent variables. Differences for the cases \( n = 2 \) or \( n = 3 \) will be specifically mentioned.

Let \( Q \) be a bounded domain in \( \mathbb{R}^n \); \( B_r(x) = \{ y \in \mathbb{R}^n : |y - x| < r \} \); \( \mu_1, \ldots, \mu_n \)\(_d\cdot\) \( d \)-dimensional Lebesgue measure; We use the standard notation for Banach spaces \( C^k(\bar{Q}) \), \( k \in \mathbb{Z}^+ := \{0\} \cup \mathbb{Z}_+ \) of \( k \)-times continuously differentiable functions on \( \overline{Q} \), and we simply write \( C(\bar{Q}) \), if \( k = 0 \). The following standard notation will be used for Hölder spaces:

- For \( k \in \mathbb{Z}^+, 0 < \gamma \leq 1 \), Hölder space \( C^{k,\gamma}(\overline{Q}) \) is the Banach space of elements \( u \in C^k(\overline{Q}) \) with finite norm
  \[
  \|u\|_{C^{k,\gamma}(\overline{Q})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{Q})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{Q})}
  \]
  where
  \[
  [v]_{C^{0,\gamma}(\overline{Q})} := \sup_{x, x' \in Q, x \neq x'} \frac{|v(x) - v(x')|}{|x - x'|^{\gamma}}
  \]

Throughout the paper we use standard notations for \( L_p(Q), 1 \leq p \leq \infty \) spaces; the following standard notations are used for Sobolev spaces [18, 19]:

- For \( s \in \mathbb{Z}^+, 1 \leq p < \infty \), Sobolev space \( W^s_p(Q) \) is the Banach space of measurable functions on \( Q \) with finite norm
  \[
  \|u\|_{W^s_p(Q)} := \begin{cases} \left( \int_Q \left( \sum_{|\alpha| \leq s} |D^\alpha u(x)|^p dx \right)^\frac{1}{p} \right), & \text{if } 1 \leq p < \infty, \\ \sum_{|\alpha| \leq s} \|D^\alpha u(x)\|_{L^\infty(Q)}, & \text{if } p = \infty, \end{cases}
  \]
  where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n^+, |\alpha| = \alpha_1 + \ldots + \alpha_n, D_k = \frac{\partial}{\partial x_k}, D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} \). In particular if \( p = 2 \), \( H^s(Q) := W^s_2(Q) \) is a Hilbert space with inner product
  \[
  (f, g)_{H^s(Q)} = \sum_{|\alpha| \leq s} (D^\alpha f(x), D^\alpha g(x))_{L_2(Q)}
  \]
  Equivalent inner product and norm in \( H^1(Q) \) are given as
  \[
  ((f, g))_{H^1(Q)} := \int_Q Df \cdot Dg \, dx + \sum_{l=1}^m \int_{E_l} fg \, dS, \quad \|f\|_{H^1(Q)} := ((f, f))^{\frac{1}{2}}.
  \]

The following is the new Hilbert space introduced in this paper.

- \( \tilde{H}^1(Q), \ n = 2,3 \) is a linear subspace of \( H^1(Q) \), defined as
  \[
  \tilde{H}^1(Q) = \{ u \in H^1(Q) | u_{x_1x_2} \in L_2(Q) \}, \quad \text{if } Q \in \mathbb{R}^2
  \]
\[ \tilde{H}^1(Q) = \{ u \in H^1(Q) | u_{x_1 x_2}, u_{x_1 x_3}, u_{x_2 x_3}, u_{x_1 x_2 x_3} \in L_2(Q) \}, \text{ if } Q \in \mathbb{R}^3. \]

\[ \tilde{H}^1(Q) \text{ is an Hilbert space with inner product} \]

\[ (u, v)_{\tilde{H}^1(Q)} = \begin{cases} (u, v)_{H^1} + (u_{x_1 x_2}, v_{x_1 x_2})_{L_2}, & \text{if } n = 2 \\ (u, v)_{H^1} + \sum_{i,j=1}^{3} (u_{x_i x_j}, v_{x_i x_j})_{L_2} + (u_{x_1 x_2 x_3}, v_{x_1 x_2 x_3})_{L_2}, & \text{if } n = 3 \end{cases} \]

Standard notation will be employed for embedding of Banach spaces:

- \( B_1 \hookrightarrow B_2 \) means bounded embedding of \( B_1 \) into \( B_2 \), i.e. \( B_1 \subset B_2 \), and \( \|u\|_{B_2} \leq C\|u\|_{B_1}, \forall u \in B_1 \), for some constant \( C \).

- \( B_1 \Subset B_2 \) denotes compact embedding of \( B_1 \) into \( B_2 \), meaning that \( B_1 \hookrightarrow B_2 \), and every bounded subset of \( B_1 \) is precompact in \( B_2 \).

1.2 EIT Optimal Control Problem

Consider the optimal control problem on the minimization of the cost functional

\[ \mathcal{J}(v) = \sum_{l=1}^{m} \left| \int_{E_l} \frac{U_l - w(x)}{Z_l} dx - I_l \right|^2 + \beta |U - U^*|^2 \quad (1.7) \]

on the control set

\[ \mathcal{F}^R = \left\{ v = (\sigma, U) \in \tilde{H}^1(Q) \times \mathbb{R}^m \left| \sum_{l=1}^{m} U_l = 0, \right. \left. \|\sigma\|_{\tilde{H}^1}^2 + |U|^2 \leq R^2, \sigma(x) \geq \sigma_0 > 0, \forall x \in Q \right\} \quad (1.8) \]

where \( \beta > 0, R > 0, \) and \( u = u(\cdot ; v) \in H^1(Q) \) is a weak solution of the elliptic problem (1.3)-(1.5), i.e.

\[ \int_Q \sigma \nabla u \cdot \nabla \eta dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} w \eta ds = \sum_{l=1}^{m} \frac{U_l}{Z_l} \int_{E_l} \eta ds, \quad \forall \eta \in H^1(Q). \quad (1.9) \]

This optimal control problem will be called Problem \( \mathcal{E} \). Note that the first term in the cost functional \( \mathcal{J}(v) \) characterizes the mismatch of the condition (1.6) in light of the Robin condition (1.5).

1.3 Discrete Optimal Control Problem

To discretize optimal control problems \( \mathcal{E} \) we pursue finite difference method following the framework introduced in [8]. Let \( h > 0 \) and cut \( \mathbb{R}^n \) by the planes

\[ x_i = k_i h, \quad i = 1, \ldots, n \quad \forall k_i \in \mathbb{Z}. \]

into a collection of elementary cells with length \( h \) in each \( x_i \)-direction. For every \( h > 0 \) and multi-index \( \alpha = (k_1, \ldots, k_n) \) we define a cell \( C^\alpha_h \) as

\[ C^\alpha_h = \{ x \in \mathbb{R}^n \mid k_i h \leq x_i \leq (k_i + 1) h, \quad i = 1, \ldots, n \}, \quad (1.10) \]
and consider the collection of cells which have non-empty intersection with $Q$:

$$C_h^Q = \{C_h^\alpha | C_h^\alpha \cap Q \neq \emptyset\}$$  \hspace{1cm} (1.11)

We now introduce exterior approximation of $\overline{Q}$ as follows:

$$Q_h = \bigcup_{C_h^\alpha \in C_h^Q} C_h^\alpha$$  \hspace{1cm} (1.12)

Obviously, we have $\overline{Q} \subset Q_h$. Let $S_h = \partial Q_h$. The vertex of the prism $C_h^\alpha$ whose coordinates are smallest relative to the other vertices, is called its natural corner. We are going to identify each prism (cell) by its natural corner. With slight abuse of notation we denote as $C_z^\alpha h$, a cell in $\mathbb{R}^n$ of side length $h$ and with natural corner at $z$. Hence, $C_h^\alpha$ and $C_{z h}^\alpha$ are identical.

Consider a lattice

$$L = \{x \in \mathbb{R}^n | \exists \alpha \in \mathbb{Z}^n \text{ s.t. } x_i = k_i h, \ i = 1, \ldots, n\}.$$  

We will write $x_\alpha = (k_1 h, \ldots, k_n h)$. Bijection $\alpha \mapsto x_\alpha$ will henceforth be referred as natural. Given a set $X$ which is in natural bijection with a subset of the set of multi-indexes $\alpha$, we write $\mathcal{A}(X)$ as the indexing set. Moreover, if $X \subset \mathbb{R}^n$, then $\mathcal{L}(X) := \mathcal{L} \cap X$. When $X = \mathcal{L}(Y) \subset \mathbb{R}^n$, we’ll agree to write $\mathcal{A}(Y)$ instead of $\mathcal{A}(\mathcal{L}(Y))$. These indexes are also in natural bijection with the natural corners of these prisms. In particular, some of the corresponding lattice points may fall on the boundary $S_h$. We contrast this set to the set $\mathcal{A}(Q_h')$ of indexes in natural bijection to the lattice points that lie strictly in the interior of $Q_h$, and to the set $\mathcal{A}(Q_h)$, of all indexes which are in natural bijection with the lattice points that lie in $Q_h$. We will write

$$\sum_{\mathcal{A}(X)} \text{ instead of } \sum_{\alpha \in \mathcal{A}(X)},$$

and likewise for other expressions requiring subscripts. We adopt the notation

$$\alpha \pm e_i := (k_1, \ldots, k_i \pm 1, \ldots, k_n).$$

To discretize optimal control problem $\mathcal{E}$, we need to introduce some refined subsets of grid points of $Q_h$.

$$Q_h^+ = \{x_\alpha \in Q_h : C_h^\alpha \cap Q \neq \emptyset\}$$

be a subset of natural corners of the cells in $Q_h$. We denote as

$$Q_h^{(i)} = \{x_\alpha \in Q_h : x_{\alpha + e_i} \in Q_h\}$$

the subset of all grid points $x_\alpha$ in $Q_h$ such that the edge $[x_\alpha, x_{\alpha + e_i}] \subset Q_h$, and similarly

$$Q_h^{(i,j)} = \{x_\alpha \in Q_h : x_\alpha + e_i + e_j \in Q_h\}, \ i, j = 1, 2, 3 \hspace{1cm} (1.13)$$
Subset of natural corners $x_\alpha$ of cells in $Q_h$ which intersect the boundary $S$ is denoted as
\[ \dot{S}_h = \{ x_\alpha \in Q_h : C^\alpha_h \cap S \neq \emptyset \} \] (1.14)
and
\[ \dot{E}_{ih} = \{ x_\alpha \in Q_h : E_{i\alpha} := C^\alpha_h \cap E_i \neq \emptyset \}, \quad l = 1, \ldots, m \]
is a collection of grid points which are natural corners of $C^\alpha_h$ containing portion $E_{i\alpha}$ of the boundary curve $E_i$. Let $\Gamma_{i\alpha} = m_{n-1}(E_{i\alpha})$, $l = 1, \ldots, m$ is an $n - 1$-dimensional Lebesgue measure of $E_{i\alpha}$. We are going to assume that any control vector $\sigma$ is extended to a larger set $Q + B_1(0)$ as bounded measurable function with preservation of conditions in the control set (1.2). We introduce discrete grid function by discretizing $\sigma$ through Steklov average:
\[ \sigma_\alpha = \frac{1}{h^n} \int_{x_1}^{x_1+h} \cdots \int_{x_n}^{x_n+h} \sigma(y_1, \ldots, y_n) \, dy_1 \cdots \, dy_n, \quad \alpha \in \mathcal{A}(Q_h), \] (1.15)
where $x_i$ is the $i$-th coordinate of $x_\alpha$. We use standard notation for finite differences of grid functions $u_\alpha, \sigma_\alpha$:
\[
\begin{align*}
 u_{\alpha x_i} &= \frac{u_{\alpha+e_i} - u_\alpha}{h}, \quad u_{\alpha x_i} = \frac{u_\alpha - u_{\alpha-e_i}}{h}, \quad u_{\alpha x_i} = \frac{u_{\alpha+e_i} - u_\alpha}{h}, \quad i = 1, \ldots, n \\
 \sigma_{\alpha x_i x_j} &= \frac{\sigma_{(\alpha+e_j)x_i} - \sigma_{\alpha x_i}}{h} = \frac{\sigma_{\alpha+e_j+e_i} - \sigma_{\alpha+e_i}}{h^2} + \frac{\sigma_{\alpha+e_i} + \sigma_{\alpha}}{h} \\
 \sigma_{\alpha x_1 x_2 x_3} &= \frac{\sigma_{(\alpha+e_3)x_1 x_2} - \sigma_{\alpha x_1 x_2}}{h} = \frac{\sigma_{(\alpha+e_3+e_2)x_1} - \sigma_{(\alpha+e_3)x_1} - \sigma_{(\alpha+e_2)x_1} + \sigma_{\alpha x_1}}{h^2} \\
 &= \frac{\sigma_{\alpha+e_3+e_2+e_1} - \sigma_{\alpha+e_3+e_2} - \sigma_{\alpha+e_3+e_1} - \sigma_{\alpha+e_2+e_1} + \sigma_{\alpha+e_2} + \sigma_{\alpha+e_1} - \sigma_{\alpha}}{h^3}.
\end{align*}
\]
For a given discretization with step size $h$, we employ the notation
\[ [\eta]_h := \{ \eta_\alpha \in \mathbb{R} : \alpha \in \mathcal{A}(Q_h) \}, \]
for the grid function. Next, we define the discrete $\mathcal{H}^1(Q_h)$, $\mathcal{H}^1(Q_h)$ and $L_\infty(Q_h)$ norms:

$$
\| [u]_h \|_{\mathcal{H}^1(Q_h)}^2 := \sum_{\mathcal{A}(Q_h)} h^n u^2 + \sum_{i=1}^{n} \sum_{\mathcal{A}(Q_h)} h^n u^2_{pxi}
$$

$$
\| [u]_h \|_{\mathcal{H}^1(Q_h)} := \sum_{i=1}^{n} h^n \sum_{\mathcal{A}(Q_h)} u^2_{pxi} + \sum_{l=1}^{m} \sum_{\mathcal{A}(E_h)} \Gamma_{\alpha} u^2_{\alpha}
$$

$$
\| [\sigma]_h \|_{\mathcal{H}^1(Q_h)}^2 := \sum_{\mathcal{A}(Q_h)} h^2 \sigma^2_\alpha + \sum_{i=1}^{2} \sum_{\mathcal{A}(Q_h)} h^2 \sigma^2_{\alpha x i} + \sum_{\mathcal{A}(Q_h)} h^2 \sigma^2_{\alpha x 1 x 2}, \ Q \in \mathbb{R}^2
$$

$$
\| [\sigma]_h \|_{\mathcal{H}^1(Q_h)} := \sum_{\mathcal{A}(Q_h)} h^3 \sigma^2_\alpha + \sum_{i=1}^{3} \sum_{\mathcal{A}(Q_h)} h^3 \sigma^2_{\alpha x i} + \sum_{i,j=1}^{3} \sum_{\mathcal{A}(Q_h)} h^3 \sigma^2_{\alpha x i x j}
$$

$$
+ \sum_{\mathcal{A}(Q_h)} h^3 \sigma^2_{\alpha x 1 x 2 x 3}, \ Q \in \mathbb{R}^3
$$

$$
\| [\sigma]_h \|_{L_\infty(Q_h)} := \max_{\alpha \in \mathcal{A}(Q_h)} |\sigma_\alpha|
$$

For fixed $R > 0$, define the discrete control sets $\mathcal{F}_h$ as

$$
\mathcal{F}_h := \left\{ [v]_h = ([\sigma]_h, U) \left| \sum_{l=1}^{m} U_l = 0, \ |[\sigma]_h|_{\mathcal{H}^1(Q_h)}^2 + |U|_{\mathbb{R}^m}^2 \leq R^2, \ \sigma_\alpha \geq \sigma_0 > 0, \ \forall \alpha \in \mathcal{A}(Q_h) \right\}
$$

(1.16)

and the interpolating map $\mathcal{P}_h$ as

$$
\mathcal{P}_h : \bigcup_R \mathcal{F}_h \to \bigcup_R \mathcal{F}_h, \ \mathcal{P}_h([v]_h) = (\mathcal{P}_h([\sigma]_h), U) = (\sigma', U)
$$

where $\sigma'$ is a multilinear interpolation of $[\sigma]_h$, which assigns the value $\sigma_\alpha$ to each grid point of $C_h^\alpha$, and it is a piecewise linear with respect to each variable $x_i$ when the other variables are fixed. Precisely,

$$
\sigma'(x) = \sigma_\alpha + \sigma_{\alpha x_1}(x_1 - k_1 h) + \sigma_{\alpha x_2}(x_2 - k_2 h) + \sigma_{\alpha x_1 x_2}(x_1 - k_1 h)(x_2 - k_2 h), x \in C_h^\alpha, \ n = 2
$$

(1.17)

$$
\sigma'(x) = \sigma_\alpha + \sum_{i=1}^{3} \sigma_{\alpha x_i}(x_i - k_i h) + \sum_{i,j=1}^{3} \sigma_{\alpha x_i x_j}(x_i - k_i h)(x_j - k_j h) + \sigma_{\alpha x_1 x_2 x_3} \prod_{1 \leq i \leq 3} (x_i - k_i h), x \in C_h^\alpha, \ n = 3.
$$

(1.18)

We also define the discretizing map $\mathcal{Q}_h$ as

$$
\mathcal{Q}_h : \bigcup_R \mathcal{F}_h \to \bigcup_R \mathcal{F}_h, \ \mathcal{Q}_h(v) = (\mathcal{Q}_h(\sigma), U) = ([\sigma]_h, U)
$$
where $[\sigma]_h = \{\sigma_\alpha\}$, with $\sigma_\alpha$ given by (1.15) for each $\alpha \in \mathcal{A}(Q_h)$.

Next, we define a discrete state vector, which is a solution of the discretized elliptic problem (1.3)–(1.5).

**Definition 1.** Given $[v]_h$, the grid function $[u([v]_h)]_h$ is called a discrete state vector of problem $\mathcal{E}$ if it satisfies

$$h^n \sum_{\mathcal{A}(Q_h)} \sum_{i=1}^n \sigma_\alpha u_{\alpha x_i} \eta_{\alpha x_i} + \sum_{l=1}^m \frac{1}{Z_l} \sum_{\mathcal{A}(E_{ih})} \Gamma_{\alpha \eta} u_\alpha + J_h([u]_h, [\eta]_h) = \sum_{l=1}^m \frac{U_l}{Z_l} \sum_{\mathcal{A}(E_{ih})} \Gamma_{\alpha \eta} \eta_\alpha$$

for arbitrary grid function $[\eta]_h$, where

$$J_h([u]_h, [\eta]_h) = h^n \sum_{\mathcal{A}(S_h)} \sum_{i=1}^n \theta^i_\alpha u_{\alpha x_i} \eta_{\alpha x_i}, \quad \theta^i_\alpha = \begin{cases} 1 & \text{if } \alpha \in \mathcal{A}(Q^i_h) \setminus Q^+_h \\ 0 & \text{otherwise} \end{cases}$$

The necessity of adding $J_\alpha$ to (1.19) is that some $u_{\alpha x_i}$ and $\eta_{\alpha x_i}$ values on $S_h$ are not present in the first term of (1.19). Addition of these terms to (1.19) through $J_\alpha$ is essential for the proof of stability of our discrete scheme.

In Section 3, it will be proved that for a given $[v]_h \in \mathcal{F}_h^R$ there exists a unique discrete state vector of problem $\mathcal{E}$. Consider minimization of the discrete cost functional

$$J_h([v]_h) = \sum_{l=1}^m \left( \sum_{\mathcal{A}(E_{ih})} \Gamma_{\alpha \eta} \frac{U_l - u_\alpha}{Z_l} - I_l \right)^2 + \beta |U - U^*|^2$$

on a control set $\mathcal{F}_h^R$, where $u_\alpha$’s are components of the discrete state vector $[u([v]_h)]_h$ of the Problem $\mathcal{E}$. The formulated discrete optimal control problem will be called Problem $\mathcal{E}_h$.

Next, we define three interpolations of the discrete state vector $[u]_h$.

**Piecewise constant interpolation $\tilde{u}$:** $Q_h \rightarrow \mathbb{R}$ assigns to the interior of each cell in $Q_h$ the value of $u_\alpha$ at its natural corner:

$$\tilde{u}_h |_{Q^\alpha_i} = u_\alpha, \quad \forall \alpha \in \mathcal{A}(Q^+_h).$$

**Piecewise constant interpolation of the discrete $x_i$-derivative $\tilde{u}^i$:** $Q_h \rightarrow \mathbb{R}$, $i = 1, \ldots, n$ assigns to the interior of each cell in $Q_h$ the value of the forward spatial difference at the natural corner:

$$\tilde{u}^i_h |_{Q^\alpha_i} = u_{\alpha x_i}, \quad \forall \alpha \in \mathcal{A}(Q^+_h).$$

**Multilinear interpolation $u'$:** $Q_h \rightarrow \mathbb{R}$ assigns the value $u_\alpha$ to each grid point in $\mathcal{L}(Q_h)$, and it is a piecewise linear with respect to each variable $x_i$ when the rest of variables are fixed.
2 Main Result

The following is the main result of the paper.

**Theorem 2.** The sequence of discrete optimal control problems $\mathcal{E}_h$ approximates the optimal control problem $\mathcal{E}$ with respect to functional, i.e.

$$\lim_{h \to 0} \mathcal{J}_{h*} = \mathcal{J}_*,$$

where

$$\mathcal{J}_{h*} = \inf_{\mathcal{F}_h} \mathcal{J}_h([v]_h), \quad \mathcal{J}_* := \inf_{f \in \mathcal{F}} \mathcal{J}(v).$$

Furthermore, let $\{\epsilon_h\}$ be a sequence of positive real numbers with $\lim_{h \to 0} \epsilon_h = 0$.

If the sequence $[v]_{h,\epsilon} = ([\sigma]_{h,\epsilon}, U_{h,\epsilon}) \in \mathcal{F}_h$ is chosen so that

$$\mathcal{J}_{h*} \leq \mathcal{J}_h([v]_{h,\epsilon}) \leq \mathcal{J}_{h*} + \epsilon_h,$$

then we have

$$\lim_{h \to 0} \mathcal{J}(\mathcal{P}_h([v]_{h,\epsilon})) = \mathcal{J}_*,$$

the sequence $\{\mathcal{P}_h([\sigma]_{h,\epsilon}, U_{h,\epsilon})\}$

- is precompact in Tikhonov topology of $\tilde{H}^1(Q) \times \mathbb{R}^m$ formed with the product of the weak topology of $\tilde{H}^1(Q)$ and Euclidean topology of $\mathbb{R}^m$;
- is precompact in Tikhonov topology of $C^{0,\mu}(\overline{Q}) \times \mathbb{R}^m$, $0 < \mu < \frac{1}{2}$, formed with the product of the strong topology of Hölder space $C^{0,\mu}(\overline{Q})$ and Euclidean topology of $\mathbb{R}^m$;

and all the corresponding limit points $v_* = (\sigma_*, U_*)$ are optimal controls of the problem $\mathcal{E}$. Moreover, if $v_* = (\sigma_*, U_*)$ is any such limit point, then there exists a subsequence $h'$ such that the multilinear interpolations $u_{h'}$ of the discrete state vectors $[u([v]_{h',\epsilon})]_{h'}$ converge to weak solution $u = u(x; v_*)$ of the elliptic problem (1.3)-(1.5), weakly in $H^1(Q)$, strongly in $L^2(Q)$, and almost everywhere on $Q$.

3 Preliminary Results

The following lemma presents a key discrete energy estimate for the elliptic PDE problem:

**Lemma 3** (Discrete Energy Estimate). For any $[v]_h \in \mathcal{F}_h$, discrete state vector $[u([v]_h)]_h$ satisfies the energy estimate:

$$\|\|u_h\|\|_{\mathcal{H}^1(Q_h)} \leq M|U|,$$

where $M$ depends on $\sigma_0, Z$ and $Q$.

**Proof:** We set $\eta_\alpha = u_\alpha$ in (1.19) to get

$$h^n \sum_{\mathcal{A}(Q_h')} \sigma_\alpha \sum_{i=1}^n u_{\alpha x_i}^2 + \sum_{l=1}^m \frac{1}{Z_l} \sum_{\mathcal{A}(E_{ih})} \Gamma_{l\alpha} u_\alpha^2 + J_\alpha(u_\alpha, u_\alpha) = \sum_{l=1}^m \frac{U_l}{Z_l} \sum_{\mathcal{A}(E_{ih})} \Gamma_{l\alpha} u_\alpha,$$
and by recalling the definition of $J_\alpha$ and the fact that $0 < \sigma_0 \leq \sigma_\alpha$ we have

$$
\mu \|[u([v]_h)]_h\|^2_{H^1(Q_h)} \leq \sum_{l=1}^m Z_l^{-1} U_l \sum_{\mathcal{A}(E_{ih})} \Gamma_{l \alpha} u_\alpha
$$

(3.2)

where $\mu = \min\{\sigma_0, \min_l Z_l^{-1}\}$. Using Cauchy-Schwarz inequality we derive

$$
\sum_{l=1}^m Z_l^{-1} U_l \sum_{\mathcal{A}(E_{ih})} \Gamma_{l \alpha} u_\alpha \leq m^{\frac{1}{2}} Z^{-1}_l \max_l \frac{1}{2} |\partial Q| \max_l |[u([v]_h)]_h||_{H^1(Q_h)}.
$$

(3.3)

From (3.2) and (3.3), (3.1) follows with $M = \mu^{-1} m^{\frac{1}{2}} Z^{-1}_l \max_l \frac{1}{2} |\partial Q|$. □

**Corollary 4.** For any $[v]_h \in \mathcal{F}_h^R$, there exists a unique discrete state vector $[u([v]_h)]_h$.

Assertion of the corollary follows from energy estimate with similar arguments as in [46]. By replacing $u_{x_i}, \eta_{x_i}$ with respective difference quotients, from (1.19) it follows

$$
\sum_{\mathcal{A}(Q_h)} \left\{ \mathcal{L}_\alpha \cdot [u]_h - \mathcal{G}_\alpha(U) \right\} \eta_\alpha = 0,
$$

(3.4)

where $\mathcal{L}_\alpha$ is a vector of the same size as $[u]_h$ and $\mathcal{G}_\alpha : \mathbb{R}^m \to \mathbb{R}$ is a linear functional. Since the values of $\eta_\alpha$ are independent, (3.4) is equivalent to the following system of linear algebraic equations (SLAE)

$$
\mathcal{L}_\alpha \cdot [u]_h = \mathcal{G}_\alpha(U), \quad \alpha \in \mathcal{A}(Q_h).
$$

(3.5)

Note that the number of equations, and the number of unknowns $u_\alpha$ in (3.5) are both equal to number of vertices in a grid $Q_h$. Addition of the expression $J_\alpha(u_\alpha, \eta_\alpha)$ to the discrete identity (1.19) served exactly to this purpose. From the energy estimate (3.1) it easily follows that the corresponding homogeneous SLAE has only a zero solution. Therefore, claim of the corollary is a consequence of the well-known result of linear algebra.

Another crucial consequence of the energy estimate (3.1) is uniform $H^1(Q)$-bounded of the interpolations of the discrete state vector:

**Corollary 5.** Multilinear interpolation $u'_h$ of the discrete state vector is uniformly bounded in $H^1(Q)$:

$$
\sup_{[v]_h \in \mathcal{F}_h^R} \|u'_h\|_{H^1(Q)} \leq C,
$$

(3.6)

where $C$ depends on $\sigma_0, Z, Q, R, n$.

Indeed, first of all from [8] (formula (4.13)) it follows that

$$
\int_Q |Du'_h|^2 \leq 2^{n-1} \sum_{i=1}^n \sum_{\mathcal{A}(Q_h)} h^n |u_{x_i}|^2.
$$

(3.7)
Next, we establish that the sequences \( u'_h \) and \( \tilde{u}_h \) are equivalent in strong topology of \( L_2(E_i) \), as \( h \to 0 \). The proof is similar to the statement (d) of Theorem 14 in [8]. The following estimate is proved in [8] (estimate (4.23)):
\[
|\tilde{u}_h(x) - u'_h(x)| \leq (2^n - 1)n \sum_{\text{edges of } C^0_h} h^2|u_{\alpha'x_j}|^2, \quad x \in C^0_h,
\]  
(3.8)
where the summation on the right-hand side is taken over all \( \alpha' \) of \( \alpha \) of Theorem 14 in [8].

Therefore, from (3.8) and (3.1) it follows that
\[
\|\tilde{u}_h - u'_h\|_{L_2(E_i)}^2 \leq L(2^n - 1)2^{n-1}n \sum_{i=1}^n \sum_{\alpha \in A'} h^{n+2}u_{\alpha x_i}^2 \to 0,
\]  
(3.9)
as \( h \to 0 \). Assuming that \( h \leq 1 \), from (3.7) and (3.9) it follows that
\[
\|u'_h\|_{H^1(Q_h)}^2 \leq C_1\|u_h\|_{H^1(Q_h)}^2,
\]  
(3.10)
where \( C_1 = 2^{n-1}(2L(2^n - 1)nm + 1) \). Due to equivalency of the norms \( \| \cdot \| \) and \( \| \cdot \| \) in \( H^1(Q) \) (see Lemma 5.1 in [9]), from (3.10), (3.6) follows.

Discrete energy estimate implies the following interpolation

**Lemma 6.** Let \( R > 0 \) is fixed, and \( \{[v]_h\} \) is a sequence of discrete control vectors such that \([v]_h \in \mathcal{F}_h^R \) for each \( h \). Then the following statements hold:

(a) The sequences \( \{u'_h\} \) and \( \{\tilde{u}_h\} \) are uniformly bounded in \( L_2(Q_h) \).

(b) For each \( i \in \{1, \ldots, n\} \), the sequences \( \{\tilde{u}'_h\} \), \( \{\partial u'_h/\partial x_i\} \) are uniformly bounded in \( L_2(Q_h) \).

(c) the sequence \( \{\tilde{u}_h - u'_h\} \) converges strongly to 0 in \( L_2(Q) \) as \( h \to 0 \).

(d) For each \( i \in \{1, \ldots, n\} \), the sequences \( \{\partial u'_h/\partial x_i - \tilde{u}'_h\} \) converges weakly to zero in \( L_2(Q) \) as \( h \to 0 \).

(e) the sequence \( \{\tilde{u}_h - u'_h\} \) converges strongly to 0 in \( L_2(S) \) as \( h \to 0 \).

The proof of the claims (a)-(d) coincides with the proofs of similar claims in Theorem 14 of [8]. The claim (e) is proved above in (3.9).

Next, we recall the necessary and sufficient condition for the convergence of the discrete optimal control problems \( \mathcal{E}_h \), which is the suitable criteria to employ for the proof of method of finite differences for the optimal control problems with distributed parameters ([1]-[10]).

**Lemma 7.** [58] The sequence of discrete optimal control problems \( \mathcal{E}_h \) approximates the continuous optimal control problem \( \mathcal{E} \) with respect to the functional \( \mathcal{J} \) if and only if the following conditions are satisfied:

1. For arbitrary sufficiently small \( \epsilon > 0 \) there exists \( h_1 = h_1(\epsilon) \) such that \( \mathcal{D}_h(v) \in \mathcal{F}_h^R \) for all \( v \in \mathcal{F}_h^{R-\epsilon} \) and \( h \leq h_1 \); Moreover, for any fixed \( \epsilon > 0 \) and for all \( v \in \mathcal{F}_h^{R-\epsilon} \) the following inequality is satisfied:
\[
\limsup_{h \to 0} (\mathcal{J}_h(\mathcal{D}_h(v)) - \mathcal{J}(v)) \leq 0.
\]  
(3.11)
2. For arbitrary sufficiently small $\varepsilon > 0$ there exists $h = h_2(\varepsilon)$ such that $P_h([v]_h) \in \mathcal{F}^{R+\varepsilon}$ for all $[v]_h \in \mathcal{F}_h^R$ and $h \leq h_2$; moreover, for all $[v]_h \in \mathcal{F}_h^R$, the following inequality is satisfied:

$$\limsup_{h \to 0} (J(P_h([v]_h)) - J_h([v]_h)) \leq 0. \quad (3.12)$$

3. For arbitrary sufficiently small $\varepsilon > 0$, the following inequalities are satisfied:

$$\limsup_{\varepsilon \to 0} J_*(\varepsilon) \geq J_*, \quad \liminf_{\varepsilon \to 0} J_*(-\varepsilon) \leq J_*,$$

where $J_*(\pm \varepsilon) = \inf_{\mathcal{F}^{R\pm\varepsilon}} J(v)$.

Our next goal is to show that the mappings $P_h$ and $Q_h$ satisfy the conditions of Lemma 7. The following lemma plays a key role to prove this claim. The proof is similar to the proof of Proposition 11 in [8].

**Lemma 8.** Let $Q \in \mathbb{R}^2$. Then for $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{\mathcal{A}(Q^+_h)} h^2 |\sigma_{\alpha x_1 x_2}|^2 \leq (1 + \varepsilon) \left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L_2(Q_h)}^2$$

whenever $h < \delta$.

**Proof:** For each $h > 0$, define the function $\tilde{\sigma}^{12}_{h}$ as

$$\tilde{\sigma}^{12}_{h} \bigg|_{C^\alpha_h} = \sigma_{\alpha x_1 x_2}, \quad \forall \alpha \in \mathcal{A}(Q^+_h) \quad (3.13)$$

In the following we will prove that

$$\tilde{\sigma}^{12}_{h} \to \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \quad \text{strongly in } L_2(Q) \text{ as } h \to 0 \quad (3.14)$$

As an element of $\tilde{H}^1(Q)$, almost all restrictions of $\sigma$ to lines parallel to the $x_i$ direction are absolutely continuous, moreover, restrictions of $\frac{\partial \sigma}{\partial x_1}$ (or $\frac{\partial \sigma}{\partial x_2}$) to lines parallel to the $x_2$ (or $x_1$) direction are absolutely continuous. Therefore, for almost every $z = (z_1, z_2) \in Q$ we have

$$\int_{C^\alpha_h} \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} dy = \sigma(z + he_2 + he_1) - \sigma(z + he_2) - \sigma(z + he_1) + \sigma(z)$$

$$\quad (3.15)$$

In the following transformation, we write simply $\mathcal{A}$ instead of summation index set $\mathcal{A}(Q^+_h)$. Using the definition of Steklov average (1.15), (3.15)
and Cauchy-Schwartz inequality, we get

$$\left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L^2(Q_h)}^2 = \sum \int_{C_h^{\alpha + e_1 + e_2}} \left| \frac{\partial^2 \sigma(x)}{\partial x_1 \partial x_2} \right|^2 \, dx = $$

$$\sum \int_{C_h^{\alpha + e_1 + e_2}} \int \left[ \frac{1}{h^4} \left( \int \sum_{C_h^{\alpha}} \left( \sum_{C_h^{\alpha + e_1}} \left( \sum_{C_h^{\alpha + e_2}} \left( \sum_{C_h^{\alpha + e_1 + e_2}} \sigma \right) \right) \right) \right) + \frac{1}{h^4} \left( \int \sum_{C_h^{\alpha}} \left( \sum_{C_h^{\alpha + e_1}} \left( \sum_{C_h^{\alpha + e_2}} \left( \sum_{C_h^{\alpha + e_1 + e_2}} \sigma \right) \right) \right) \right) \right] \, dx \, dz \, dy \, dz \, dx \, dy \, dz \, dx \, dy \, dz \, dx \, dy \, dz \, dx  (3.16)$$

Changing integration order with respect to $y$ and $z$, we have

$$\int \int \left| \frac{\partial^2 \sigma(y)}{\partial x_1 \partial x_2} - \frac{\partial^2 \sigma(x)}{\partial x_1 \partial x_2} \right|^2 \, dy \, dz = \left( \int_{C_h^{\alpha}} \left( (y_1 - k_1 h)(y_2 - k_2 h) \right) \right) \right) \leq h^2 \times$$

$$\int \left( (k_1 + 2)h - y_1 \right) \left( (y_2 - k_2 h) \right) \right) \right) \leq h^2 \times$$

From (3.16), (3.17) it follows that

$$\left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L^2(Q_h)}^2 \leq \sum \int_{C_h^{\alpha + e_1 + e_2}} \left( \int \sum_{C_h^{\alpha}} \left( \sum_{C_h^{\alpha + e_1}} \left( \sum_{C_h^{\alpha + e_2}} \left( \sum_{C_h^{\alpha + e_1 + e_2}} \sigma \right) \right) \right) \right) \, dx \, dz \, dy \, dz \, dx \, dy \, dz \, dx  \ (3.18)$$

Let $\forall \epsilon > 0$ be fixed. Since $C^2(Q + B_1(0))$ is dense in $\tilde{H}^1(Q + B_1(0))$ we can choose $g \in C^2(Q + B_1(0))$ such that

$$\left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} - \frac{\partial^2 g}{\partial x_1 \partial x_2} \right\|_{L^2(Q + B_1(0))}^2 < \frac{\epsilon}{24(1 + m_n(Q))}.  \ (3.19)$$
From (3.18) it follows
\[
\left\| \sigma_h^{12} - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L_2(Q_h)}^2 \leq \sum_{\mathcal{A}} \frac{3}{h^2} \int_{C_h^\alpha} (I_1 + I_2 + I_3) \, dx, \tag{3.20}
\]
where
\[
I_1 = \left( \int_{C_h^{\alpha+e_1+e_2}} dz + \int_{C_h^{\alpha+e_1}} dz + \int_{C_h^{\alpha+e_2}} dz \right) \left( \frac{\partial^2 \sigma(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(z)}{\partial x_1 \partial x_2} \right)
\]
\[
I_2 = \left( \int_{C_h^{\alpha+e_1+e_2}} dz + \int_{C_h^{\alpha+e_1}} dz + \int_{C_h^{\alpha+e_2}} dz \right) \left( \frac{\partial^2 g(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(x)}{\partial x_1 \partial x_2} \right)
\]
\[
I_3 = \left( \int_{C_h^{\alpha+e_1+e_2}} dz + \int_{C_h^{\alpha+e_1}} dz + \int_{C_h^{\alpha+e_2}} dz \right) \left( \frac{\partial^2 g(x)}{\partial x_1 \partial x_2} - \frac{\partial^2 \sigma(x)}{\partial x_1 \partial x_2} \right)
\]
Since \(\frac{\partial^2 g}{\partial x_1 \partial x_2}\) is uniformly continuous on \(Q + B_1(0)\), there exists \(\delta = \delta(\epsilon) > 0\) such that
\[
\left| \frac{\partial^2 g(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(x)}{\partial x_1 \partial x_2} \right| < \frac{\epsilon}{24(1 + m_n(Q))} \tag{3.21}
\]
whenever \(|z - x| < \delta\). Let \(h_\epsilon > 0\) satisfy
\[
\sqrt{8} h_\epsilon < \delta. \tag{3.22}
\]
Then (3.21) is satisfied for each \(h < h_\epsilon\), any \(\alpha \in \mathcal{A}\), and any \(x, z \in C_h^{\alpha+e_1+e_2} \cup C_h^{\alpha+e_1} \cup C_h^{\alpha+e_2} \cup C_h^{\alpha}\). Assuming \(h\) is chosen so small that \(m_n(Q_h) \leq 2m_n(Q)\), from (3.19), (3.21) it follows
\[
\sum_{\mathcal{A}} \frac{3}{h^2} \int_{C_h^\alpha} I_1 \, dx < 12 \left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} - \frac{\partial^2 g}{\partial x_1 \partial x_2} \right\|_{L_2(Q_h)}^2 < \frac{\epsilon}{2(1 + m_n(Q))}
\]
\[
\sum_{\mathcal{A}} \frac{3}{h^2} \int_{C_h^\alpha} I_2 \, dx < \frac{\epsilon m_n(Q)}{1 + m_n(Q)}
\]
\[
\sum_{\mathcal{A}} \frac{3}{h^2} \int_{C_h^\alpha} I_3 \, dx < 12 \left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} - \frac{\partial^2 g}{\partial x_1 \partial x_2} \right\|_{L_2(Q_h)}^2 < \frac{\epsilon}{2(1 + m_n(Q))}
\]
From (3.20) we deduce
\[
\left\| \sigma_h^{12} - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L_2(Q_h)}^2 < \epsilon, \quad \forall h \leq h_\epsilon \tag{3.23}
\]
Lemma is proved. \(\square\)

The following lemma expresses similar result for 3D domains:

**Lemma 9.** Let \(Q \in \mathbb{R}^3\). Then for \(\forall \epsilon > 0\), there exists \(\delta > 0\) such that
\[
\sum_{\mathcal{A}(Q_h^3)} h^3 |\sigma_{\alpha x_1 x_2 x_3}| \leq (1 + \epsilon) \left\| \frac{\partial^3 \sigma}{\partial x_1 \partial x_2 \partial x_3} \right\|_{L_2(Q_h^3)}^2
\]
whenever \(h < \delta\).
Although it is more technical, the proof of Lemma 9 is very similar to the proof of Lemma 8. Lemmas 8 and 9 imply that the mappings $\mathcal{P}_h$ and $\mathcal{Q}_h$ satisfy the conditions of Lemma 7.

**Corollary 10.** Assume $Q \in \mathbb{R}^2$ or $\mathbb{R}^3$. For arbitrary sufficiently small $\epsilon > 0$ there exists $h_\epsilon$ such that

\begin{align*}
\mathcal{Q}_h(v) &\in \mathcal{F}_h^R \quad \text{for all } v \in \mathcal{F}^{R-\epsilon} \\
\mathcal{P}_h([v]) &\in \mathcal{F}^{R+\epsilon} \quad \text{for all } [v] \in \mathcal{F}_h^R \quad \text{and } h \leq h_\epsilon.
\end{align*}

To prove (3.24), we first choose $h'_\epsilon$ such that for $\forall h < h'_\epsilon$ we have

$$
\|\sigma\|_{\mathcal{H}^1(Q_h)}^2 \leq \left( R - \frac{\epsilon}{2} \right)^2
$$

Then we apply Lemmas 8, 9, Proposition 11 in [8] with $\epsilon_1 = \left( \frac{R}{R-\frac{\epsilon}{2}} \right)^2 - 1$, and select $h_\epsilon < h'_\epsilon$ such that for $\forall h < h_\epsilon$

$$
\|\mathcal{Q}_h(\sigma)\|_{\mathcal{H}^1(Q_h)}^2 \leq (1 + \epsilon_1)\|\sigma\|_{\mathcal{H}^1(Q_h)}^2 \leq R^2,
$$

which proves (3.24). To prove (3.25), we derive the following estimation via straightforward calculation of the respective norm of multilinear interpolation $\sigma_h'$:

$$
\|\mathcal{P}_h([\sigma])\|_{\mathcal{H}^1(Q)}^2 = \|\sigma_h'\|_{\mathcal{H}^1(Q)}^2 \leq \|\sigma_h\|_{\mathcal{H}^1(Q_h)}^2 + C h,
$$

where $C$ is independent of $h$. The latter easily imply (3.25). Final statement of this section is the following embedding result of [11]:

**Lemma 11.** [11] If $Q \subset \mathbb{R}^2$ or $\mathbb{R}^3$, then

$$
\mathcal{H}^1(Q) \hookrightarrow C^{0, \frac{1}{2}}(\mathcal{Q}); \quad \mathcal{H}^1(Q) \subset C^{0, \mu}(\mathcal{Q}), 0 < \mu < \frac{1}{2}.
$$

### 4 Approximation Theorem and Convergence of the Discrete Optimal Control Problems

The following approximation theorem establishes the convergence of the discretized PDE problem:

**Theorem 12.** Let $\{[v]_h\} = \{([\sigma]_h, U^h)\}$ be a sequence of discrete control vectors such that there exists $R > 0$ for which $[v]_h \in \mathcal{F}_h^R$ for each $h$, and such that the sequence $\{([\sigma]_h, U^h)\}$ converges to $v = (\sigma, U)$ in Tikhonov topology of $\mathcal{H}^1(Q) \times \mathbb{R}^m$ formed with the product of the weak topology of $\mathcal{H}^1(Q)$ and Euclidean topology of $\mathbb{R}^m$. Then the sequence of multilinear interpolations $\{u'_h\}$ of associated discrete state vectors $\{[u]_h([v]_h)\}$ converges to the solution $u = u(x; v) \in H^1(Q)$ of the elliptic problem (1.3)–(1.5), weakly in $H^1(Q)$, strongly in $L_2(Q)$, strongly in $L_2(S)$, and almost everywhere on $Q$. 

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Proof. By (3.6) of Corollary 5, sequence \( \{u'_h\} \) is uniformly bounded in \( H^1(Q) \). Consequently, \( \{u'_h\} \) is weakly precompact in \( H^1(Q) \). Let \( u \in H^1(Q) \) be any weak limit point. By the Rellich-Kondrachov Theorem [53], it is known that there is a subsequence of \( \{u'_h\} \) that converges to \( u \), weakly in \( H^1(Q) \), and strongly in \( L_2(Q) \) and \( L_2(S) \). By selecting further subsequence, if necessary, one can achieve that the convergence is almost everywhere on \( Q \). We proceed to show that \( u \) satisfies the integral identity (1.9). Without loss of generality, we assume that the whole sequence \( \{u'_h\} \) converges to \( u \in H^1(Q) \). Let \( Q' \subset \mathbb{R}^n \) be bounded open domain such that \( \tilde{Q} \subset Q' \) and choose arbitrary function \( \eta \in C^1(\overline{Q'}) \). We assume \( h > 0 \) is small enough that \( Q_h \subset Q' \). We choose a grid function

\[
[\eta]_h = \{\eta_\alpha : \eta_\alpha = \eta(x_\alpha), \alpha \in \mathscr{A}(Q_h)\}
\]

in (1.19). Introducing standard interpolations \( \tilde{\eta}_h \) and \( \eta_h^i \) as

\[
\tilde{\eta}_h|_{C^0_h} = \eta_\alpha, \quad \eta_h^i|_{C^0_h} = \eta_{\alpha i}, \quad \forall \alpha \in \mathscr{A}(Q_h^+) \]

we write (1.19) in an equivalent form:

\[
\sum_{i=1}^{n} \int \tilde{\sigma}_h \tilde{u}_h \tilde{\eta}_h^i \, dx + \sum_{l=1}^{m} \frac{1}{|E_l|} \int \tilde{u}_h \tilde{\eta}_h \, dS - \sum_{l=1}^{m} \frac{U_l^h}{|E_l|} \int \tilde{\eta}_h \, dS = -J_h([u]_h, [\eta]_h) - \sum_{i=1}^{n} \int_{Q_h \setminus Q} \tilde{\sigma}_h \tilde{u}_h \tilde{\eta}_h^i \, dx \quad (4.1)
\]

Since, \( \tilde{\sigma}_h \) converges to \( \sigma \) strongly in \( L_2(Q) \), \( \tilde{u}_h^i \) converges to \( \frac{\partial u}{\partial x_i} \) weakly in \( L_2(Q) \), \( \tilde{u}_h \) converges to \( u \) strongly in \( L_2(S) \), \( \tilde{\eta}_h^i \) and \( \tilde{\eta}_h \) converges to \( \frac{\partial \eta}{\partial x_i} \) and \( \eta \) uniformly on \( \overline{Q} \), the limit of three terms on the left hand side of (4.1) imply the respective terms of the integral identity (1.9) as \( h \to 0 \). Hence, it remains to prove that limit of the remaining terms in (4.1) vanishes. By applying Cauchy-Schwartz inequality we have

\[
|J_h([u]_h, [\eta]_h)| \leq \left( \sum_{i=1}^{n} \sum_{\mathscr{A}(Q_h^0)} h^n u_{\alpha i}^2 \right)^{\frac{1}{2}} \|\eta\|_{C^1(\overline{Q'})} \sqrt{n} h \left( \sum_{\mathscr{A}(S_h)} h^{n-1} \right)^{\frac{1}{2}}. \quad (4.2)
\]

Noting that every grid point on \( S_h \) belongs to cell (with \( 2^n \) vertices) which intersects \( S \), and by recalling the definition of \( \hat{S} \) we can estimate

\[
\sum_{\mathscr{A}(S_h)} h^{n-1} \leq 2^n \sum_{\mathscr{A}(S_h)} h^{n-1} \leq 2^n \sup_{h > 0} \sum_{\mathscr{A}(S_h)} h^{n-1} = 2^n \mathcal{H}^{n-1}(S), \quad (4.3)
\]

where \( \mathcal{H}^{n-1}(\cdot) \) is \( n - 1 \)-dimensional Hausdorff measure on \( \mathbb{R}^n \). Since \( S \) is Lipschitz, \( \mathcal{H}^{n-1}(S) \) coincides with the surface measure \( m_{n-1}(S) \) [23]. Therefore, from (4.2),(4.3) and discrete energy estimate (3.1) it follows that

\[
J_h([u]_h, [\eta]_h) = O(\sqrt{h}) \to 0, \text{ as } h \to 0. \quad (4.4)
\]
Using Cauchy-Schwartz inequality for the second term in the right hand side of (4.1) we have
\[
\left| \sum_{i=1}^{n} \int_{Q_h \setminus Q} \tilde{\sigma}_h \tilde{u}_h^i \tilde{n}_h^i \, dx \right| \leq \sup_{Q_h \setminus Q} |\tilde{\sigma}_h| \sum_{i=1}^{n} \| \tilde{u}_h^i \|_{L^2(\partial Q_h \setminus Q)} \| \tilde{n}_h^i \|_{L^2(\partial Q_h \setminus Q)} \\
\leq \| \sigma'_h \|_{C(\partial Q_h)} \||u|_{\tilde{H}^1(Q_h)} \|_{H^{1} (Q_h)} \| D\eta \|_{C^1(\Sigma)} (m_n(Q_h \setminus Q))^{\frac{1}{2}} \tag{4.5}
\]
From the embedding result of Lemma 11 and (3.28) it follows that for sufficiently small \( h \)
\[
\| \sigma'_h \|_{C(\partial Q_h)} \leq C \| \sigma'_h \|_{\tilde{H}^1(Q_h)} \leq C \| [\sigma]_h \|_{\tilde{H}^1(Q_h)}^2 + 1 \leq CR + 1. \tag{4.6}
\]
Since Lebesgue measure of \( Q_h \setminus Q \) converges to zero as \( h \to 0 \), from the energy estimate (3.1) and (4.6) it follows that (4.5) converges to zero as \( h \to 0 \). Hence, passing to limit as \( h \to 0 \), from (4.1) it follows that the limit function \( u \) satisfies the integral identity (1.9).

Approximation Theorem 12 imply the existence of the optimal control.

**Corollary 13.** The optimal control problem \( \mathcal{E} \) has a solution, i.e.
\[
\mathcal{F}_* := \left\{ v \in \mathcal{F}^R \mid \mathcal{J}(v) = \mathcal{J}_* \right\} \neq \emptyset
\]

The proof of the corollary is similar to the proof of existence Theorem 4.4 in [9].

In light of the approximation Theorem 12, to complete the proof of Theorem 2 it remains to prove that the conditions of Lemma 7 are satisfied.

Proof of the condition (iii) of Lemma 7 coincide with the proof of similar fact from [1, 5]. Hence, it only remains to prove that the conditions (3.11) and (3.12) of Lemma 7 are satisfied (see Corollary 10).

Let \( v \in \mathcal{F}^{(R-\epsilon)} \). By Corollary 10 we have \( \mathcal{D}_h(\sigma) = [\sigma]_h \in \mathcal{F}^R \). Applying Corollary 10 again, we deduce that \( \mathcal{P}_h([\sigma]_h) \) belong to \( \mathcal{F}^{R+\epsilon} \), and therefore it forms a weakly precompact sequence in \( \tilde{H}^1(Q) \). From compact embedding result of Lemma 11 it follows that it forms a precompact sequence in a strong topology of \( C^{0,\mu} (\overline{Q}) \), \( 0 < \mu < \frac{1}{2} \). It easily follows that the whole sequence \( \mathcal{P}_h([\sigma]_h) \) converges to \( \sigma \) weakly in \( \tilde{H}^1(Q) \), and strongly in \( C^{0,\mu} (\overline{Q}) \). From Theorem 12 it follows that the sequence of multilinear interpolations \( \{ u_h^i \} \) of associated discrete state vectors \( \{ ||u||_{[v]_h} \} \) converges to the solution \( u = u(x; v) \in H^1(Q) \) of the elliptic problem (1.3)–(1.5), weakly in \( H^1(Q) \), strongly in \( L^2(Q) \) and \( L^2(S) \), and almost everywhere on \( Q \). Claim (e) of Lemma 6 implies that the sequence \( \tilde{u}_h \) converges to \( u \) strongly in \( L^2(S) \). Therefore, we have
\[
\lim_{h \to 0} \mathcal{J}_h(\mathcal{D}_h(v)) = \lim_{h \to 0} \left( \sum_{i=1}^{m} \left( \int_{E_i} \frac{U_i - \tilde{u}_h}{Z_i} ds - I_i \right)^2 + \beta |U - U^*|^2 \right) \\
= \sum_{i=1}^{m} \left( \int_{E_i} \frac{U_i - u}{Z_i} ds - I_i \right)^2 + \beta |U - U^*|^2 = \mathcal{J}(v)
\]
which proves \((3.11)\).

Let \([v_h] = ([\sigma], U) \in \mathcal{F}_R^h\) be arbitrary sequence. From the Corollary \(10\) it follows that \((\mathcal{P}_h([\sigma]), U) \in \mathcal{F}_{R+1}\) for sufficiently small \(h\), and therefore it is a precompact sequence in Tikhonov topology of \(H^1(Q) \times \mathbb{R}^m\) formed as a product of weak topology of \(H^1(Q)\) and Euclidean topology of \(\mathbb{R}^m\). From compact embedding result of Lemma \(11\) it follows that \(\{\mathcal{P}_h([\sigma])\}\) is a precompact sequence in a strong topology of \(C^{\mu,\mu}(\overline{Q})\), \(0 < \mu < 1\). Without loss of generality assume that the whole sequence \((\mathcal{P}_h([\sigma]), U)\) converges to some limit \(\vec{v} = (\sigma, \vec{U}) \in \tilde{H}^1(Q) \times \mathbb{R}^m\). We have

\[
J(\mathcal{P}_h([v]) - J_h([v]) = J(\mathcal{P}_h([v]) - J(\vec{v}) + J(\vec{v}) - J_h([v])
\]

From Theorem \(12\) it follows that

\[
\lim_{h \to 0} (J(\mathcal{P}_h([v]) - J(\vec{v})) = 0.
\]

The proof of the limit

\[
\lim_{h \to 0} (J(\vec{v}) - J_h([v])) = 0
\]

is almost identical to the preceding proof of \((3.11)\). Hence, \((3.12)\) is proved and this completes the proof of the Theorem \(2\).

5 Conclusions

This paper is on the analysis of the Inverse Electrical Impedance Tomography (EIT) problem on recovering electrical conductivity and potential in the body based on the measurement of the boundary voltages on the \(m\) electrodes for a given electrode current. The variational formulation is pursued in the PDE constrained optimal control framework, where electrical conductivity and boundary voltages are control parameters, state vector-potential is a solution of the mixed problem for the second order elliptic PDE, and the cost functional is the norm difference of the boundary electrode current from the given current pattern and boundary electrode voltages from the measurements. The novelty of the control theoretic model is its adaptation to clinical situation when additional "voltage-to-current" measurements can increase the size of the input data from the number of electrodes \(m\) up to \(m!\) while keeping the size of the unknown parameters fixed. EIT optimal control problem is fully discretized using the method of finite differences. New Sobolev-Hilbert space is introduced, and the convergence of the sequence of finite-dimensional optimal control problems to elliptic coefficient optimal control problem is proved both with respect to functional and control in 2- and 3-dimensional domains.

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