HIGHER REGULARITY FOR SOLUTIONS TO EQUATIONS ARISING FROM COMPOSITE MATERIALS

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Abstract. We consider parabolic systems in divergence form with piecewise $C^{(s+\delta)/2\times s+1}$ coefficients and data in a bounded domain consisting of a finite number of cylindrical subdomains with interfacial boundaries in $C^{s+1+p}$, where $s \in \mathbb{N}$, $\delta \in (1/2, 1)$, and $\mu \in (0, 1]$. We establish piecewise $C^{(s+1+\mu')/2}$ estimates for weak solutions to such parabolic systems, where $\mu' = \min \{1/2, \mu\}$, and the estimates are independent of the distance between the interfaces. In the elliptic setting, our results answer an open problem (c) in Li and Vogelius (Arch. Rational Mech. Anal. 153 (2000), 91–151).

1. Introduction and main results

1.1. Introduction. In this paper, we study higher derivative estimates of solutions to divergence form parabolic systems

\begin{equation}
- u_t + D_\alpha (A^{\alpha\beta} D_\beta u) = D_\alpha f^\alpha =: \text{div } f \quad \text{in } Q := (-T, 0) \times D,
\end{equation}

where $T \in (0, \infty)$, $D := \bigcup_{j=1}^M D_j \setminus \partial D \subset \mathbb{R}^d$ is a bounded domain containing a finite number of subdomains $D_j$, $j = 1, \ldots, M$, and $d \geq 2$. Here

$u = (u_1, \ldots, u_n)^T$, $f^\alpha = (f_1^\alpha, \ldots, f_n^\alpha)^T$

are (column) vector-valued functions, $A^{\alpha\beta}$ are $n \times n$ matrices which satisfy the strong ellipticity condition: there exists a number $\nu > 0$ such that for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{n \times d}$,

$\nu |\xi|^2 \leq A^{\alpha\beta} \xi_i^\alpha \xi_j^\beta$, $|A^{\alpha\beta}| \leq \nu^{-1}$,

and are assumed to be piecewise smooth in each subdomain. Throughout this paper, we use the Einstein summation convention over repeated indices. We will denote $A^{\alpha\beta}$ by $A$ for abbreviation.

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Even though we assume the strong ellipticity condition in the proofs below, our main result still holds under a weaker ellipticity condition: $\int_D A^{\alpha\beta} D_\alpha D_\beta \phi^2 \geq \nu \int_D |D\phi|^2$ for any $\phi \in H^1_0(D)$, and in particular for the linear systems of elasticity. See [17] pp. (1.4)–(1.6) for details.
In the elliptic setting (i.e., the time-independent case), this problem arises from the stress analysis in composite materials consisting of inclusions (subdomains) embedded in the background medium (the matrix), where the inclusions have material properties different from that of the matrix. The mathematical problem is formulated in terms of a finite number of disjoint bounded subdomains $D_j, j = 1, \ldots, M$. The subdomains are assumed to be appropriately smooth. With these conditions, the physical properties of the composite media are described by a divergence form partial differential equations (PDEs) with coefficients that are smooth in each subdomain but have discontinuities across the interface separating the subdomains. From an engineering point of view, the most important quantity is the stress represented by the gradient of the solution to PDEs. There is massive literature in this direction. We will only mention the most relevant works in this paper.

A special case when $D_1$ and $D_2$ are two touching disks in a bounded domain $D \subset \mathbb{R}^2$ was studied by Bonnetier and Vogelius [2]. They considered the scalar equation

\begin{equation}
D_\alpha (a(x)D_\alpha u) = 0 \quad \text{in } D,
\end{equation}

where the coefficient $a(x)$ is given by

\[ a(x) = a_0 1_{D\cap D_1} + 1_{D\setminus (D_1 \cup D_2)}, \]

with $0 < a_0 < \infty$ and $1_{\cdot}$ is the indicator function. They showed that $|Du|$ is bounded by using a Möbius transformation and the maximum principle. The numerical analysis also indicates such result holds for certain elliptic systems; see the work [1] for the equations of elasticity. The elliptic equation with nonhomogeneous terms

\begin{equation}
D_\alpha (A^{\alpha\beta}D_\beta u) = \text{div } f \quad \text{in } D
\end{equation}

was first studied by Li and Vogelius in [18]. They showed that any weak solution $u$ to (1.3) is piecewise $C^{1,\delta}$ with $\delta \in \left(0, \frac{\mu}{2(1+\mu)}\right]$, under the assumption that the interface is in $C^{1,\mu}$, the coefficients $A^{\alpha\beta}$ and the data $f$ are piecewise $C^{\delta}$, where $\mu \in (0, 1]$. Such Schauder estimates were improved to $C^{1,\delta}$ with $\delta \in \left(0, \frac{\mu}{2(1+\mu)}\right]$ in [17], and to $C^{1,\delta}$ with $\delta \in \left(0, \frac{\mu}{1+\mu}\right]$ in [10], where second-order elliptic system in divergence form was considered. The main feature of [18] [17] [10] is that more than two components are allowed to touch and, interestingly, these estimates are independent of the distance between the subdomains. See also [5] [6] [7] [20] [9] and the references therein. The corresponding results for parabolic equations and systems were studied in [13] [16], where the subdomains are assumed to be cylindrical and coefficients satisfy some smooth regularity assumptions, and in [11], where the subdomains are allowed to be non-cylindrical, and the interfacial boundaries are assumed to be $C^{1,\text{Dini}}$ in the spatial variables and $C^{\gamma_0}$ in the time variable with $\gamma_0 > 1/2$. 
In [18], Li and Vogelius also studied higher regularity of solutions to (1.2) in the special case when \( D_1 \) and \( D_2 \) are two touching unit disks centered at \((0, -1)\) and \((0, 1)\) in \( \mathbb{R}^2 \), and \( D \) is a disk \( B_{R_0} := \{ x : |x| < R_0 \} \) with \( R_0 > 2 \). By using known results, it is easily seen that away from the origin \( u \) is smooth in each subdomain up to the interfacial boundaries. The regularity issue near the origin is subtle because of the geometry of the subdomains. In [18], by using conformal mappings it was proved that in the 2D case, for sufficiently large \( R_0 \), \( u \) is piecewise smooth up to interfacial boundaries near the origin. Among other thought-provoking questions raised in [18], the authors asked (1) whether the condition that \( R_0 \) being sufficiently large can be dropped, (2) whether a uniform estimate holds when the inclusions are close to each other but do not touch, and ultimately (3) whether a similar result holds true for general subdomains in any dimensions.

The first question was answered affirmatively by the first named author and H. Zhang in [12] using an explicit construction of Green’s function, which is represented as an infinite series of logarithmic functions composed with conformal mappings. The second question was answered a bit later by the first named author and H. Li in [9] by also using a Green function method. In both papers, the authors considered more general nonhomogeneous equations and obtained Schauder type estimates as well as optimal derivative estimates by showing the explicit dependence of the coefficients and the distance between interfacial boundaries of inclusions. In particular, when \( a_0 = 0 \) or \( \infty \), it was shown that for any positive integer \( j \), \( |D^j u| \) blows up at the rate \( \varepsilon^{-j/2} \) with \( \varepsilon \) being the distance between two disks, which agrees with the known results in [3, 19] when \( j = 1 \). See also a recent interesting paper [15] for related results about higher derivative estimates in dimension two with circular inclusions.

In this paper, we address the third question mentioned above. We consider more general divergence form parabolic systems with piecewise \( C^{(s+\delta)/2, s+\delta} \) coefficients and data in a bounded domain consisting of a finite number of cylindrical subdomains with \( C^{s+1+\mu} \) interfacial boundaries, where \( s \in \mathbb{N}, \delta \in (1/2, 1), \) and \( \mu \in (0, 1] \). We establish piecewise \( C^{(s+1+\mu')/2, s+1+\mu'} \) estimates for weak solutions to such parabolic systems, where \( \mu' = \min \{ \frac{1}{2}, \mu \} \), and the estimates are independent of the distance between the interfaces. In the time-independent case, the corresponding result for elliptic systems follows.

It is worth mentioning that in the case of two subdomains, the problem is also closely related to the transmission problem. We refer the reader to the work [20, 21] for results about sharp regularity in various spaces and interior higher-order Schauder estimates for weak solutions to the transmission problem.

1.2. Main results. For \( \varepsilon > 0 \) small, we set

\[
D_\varepsilon := \{ x \in D : \text{dist}(x, \partial D) > \varepsilon \}.
\]
Assumption 1.1. The domain, coefficients, and data satisfy the following conditions, respectively:

(a) The domain \( \mathcal{D} \) contains \( M \) disjoint subdomains \( \mathcal{D}_j, j = 1, \ldots, M \), and the interfacial boundaries are \( C^{s+1+\mu} \), where \( s \in \mathbb{N} \) and \( \mu \in (0, 1] \). We also assume that any point \( x \in \mathcal{D} \) belongs to the boundaries of at most two of the \( \mathcal{D}_j \)’s.

(b) The coefficients \( A^{\alpha\beta} \) and the data \( f^a \) are of class \( C^{(s+\delta)/2, s+\delta}((-T + \varepsilon, 0) \times (\mathcal{D}_\varepsilon \cap \overline{\mathcal{D}}_j)), j = 1, \ldots, M \), where \( \delta \in \left( \frac{1}{2}, 1 \right) \).

Here is the main result of the paper.

Theorem 1.2. Let \( \mathcal{Q} = (-T, 0) \times \mathcal{D}, \varepsilon \in (0, 1), p \in (1, \infty), A^{\alpha\beta} \) and \( f^a \) satisfy Assumption 1.1. Let \( u \in \mathcal{H}^1_p(\mathcal{Q}) \) be a weak solution to (1.1) in \( \mathcal{Q} \). Then \( u \in C^{(s+\delta)/2, s+\delta}((-T + \varepsilon, 0) \times (\mathcal{D}_\varepsilon \cap \overline{\mathcal{D}}_j)) \) and satisfies

\[
|u|_{(s+\delta)/2, s+\delta}(\cdot) \leq N \left( \|Du\|_{L^1((-T, 0) \times \mathcal{D})} + \sum_{j=1}^{M} \|f^a_{(s+\delta)/2, s+\delta}(\cdot)\|_{\mathcal{D}_j} \right)
\]

for any \( j_0 = 1, \ldots, M \), where \( \mu' = \min\left\{ \frac{1}{2}, \mu \right\} \), \( N \) depends on \( n, d, M, p, \nu, \varepsilon, \|A\|_{(s+\delta)/2, s+\delta}, \) and the \( C^{s+1+\mu} \) characteristic of \( \mathcal{D}_j \).

Remark 1.3. The proof of Theorem 1.2 actually gives the following better estimate in the time variable:

\[
[D^\alpha u]_{(1+\delta)/2, (1+\delta)} \leq N \left( \|Du\|_{L^1((-T, 0) \times \mathcal{D})} + \sum_{j=1}^{M} \|f^a_{(s+\delta)/2, s+\delta}(\cdot)\|_{\mathcal{D}_j} \right),
\]

where \( j_0 = 1, \ldots, M \). See Lemmas 3.7 and 4.5 below.

Remark 1.4. When \( s = 0 \), piecewise Hölder-regularity of \( Du \) was proved in [11]. When \( s = 1 \), to obtain piecewise regularity of \( u \), we first prove the regularity of \( u_t \) since we need to use the equation (1.1) to solve for \( D^2 u \). For this, we employ finite difference quotient argument to get

\[
u_t \in C^{1/4,1/2}((-T + \varepsilon, 0) \times (\mathcal{D}_\varepsilon \cap \overline{\mathcal{D}}_j)).
\]

See the proof of Lemma 3.7 for the details. When \( s \geq 2 \), we can differentiate the equation (1.1) with respect to \( t \) and obtain

\[
u_t \in C^{(s-1+\mu')/2, s-1+\mu'}((-T + \varepsilon, 0) \times (\mathcal{D}_\varepsilon \cap \overline{\mathcal{D}}_j)), \mu' = \min\left\{ \frac{1}{2}, \mu \right\}.
\]

See the proof of Lemma 4.5 for more details.

Remark 1.5. As explained in [11] Corollary 2.2, any \( \mathcal{H}_p^1(\mathcal{Q}) \) weak solution to (1.1) is in \( \mathcal{H}_{p, \text{loc}}^1(\mathcal{Q}) \), thus the results in Theorem 1.2 hold under the assumption that \( u \in \mathcal{H}_1^1(\mathcal{Q}) \).
Now let us end this section with the organization of the paper and an outline of the proof. In Section 2, we first present the basic notation and definitions. For simplicity, we assume that $\mathcal{D}$ is a unit ball $B_1$ and take $Q$ to be a unit cylinder $Q_1 := (-1,0) \times B_1$. Then we represent the interfacial boundaries in $B_1$ by $x^d = h_j(x')$, where $j = 1, \ldots, m + 1$, and $h_j \in C^{s+1/2}(B'_1)$.

Using the functions $h_j$, we define a vector field $\ell_{k,0} = (0,0,1,0,\ldots,\ell_{d,0})$ near 0 which is a tangential direction on each interfacial boundary, where $k = 1, \ldots, d - 1$, the $k$-th component is 1, and $\ell_{d,0}$ is defined in (2.3). From the vector field $\ell_{k,0}$, we get a unit vector field $\ell$ which is orthogonal to each other; see Figure 1. For the properties of $\ell$, we refer the reader to Lemma 2.1 below.

We give a complete proof of Theorem 1.2 with $s = 1$ in Section 3. The main idea is to consider the directional derivatives of $u$ along the vector field $\ell$ defined in Section 2. We first derive a new parabolic system in divergence form

$$-\bar{u}_t + D_\alpha(A^{\alpha\beta}D_\beta \bar{u}) = D_\alpha f^\alpha + g,$$

where $f^\alpha$ and $g$ are defined in (3.17) and (3.2), respectively, $\bar{u} := u_\ell - u$, $u$ is defined in (3.14) which is piecewise smooth,

$$u_\ell = D_\ell u - \sum_{j=1}^{m+1} \ell_{ij}D_j u(t_0, P_j x_0),$$

$P_j x_0$ is defined in (3.6), $\ell_{ij}$ is a smooth extension of $\ell|_{\mathcal{D}}$ to $\cup_{k=1, k \neq j}^{m+1} D_k$, and $(t_0, x_0) \in Q_1$ is a fixed point. Then piecewise Hölder regularity of $\bar{u}$ is derived by adapting Campanato’s approach in [4,14], and further developed in [7,8,11] and the references therein. The key point in employing Campanato’s method is to show the mean oscillation of $D\bar{u}$ in cylinders vanishes in a certain order as the radii of the cylinders go to zero. However, we cannot apply this method to $D\bar{u}$ directly since $D\bar{u}$ is discontinuous across the interfaces. To overcome this difficulty, we first use the weak type-(1, 1) estimate for solutions to parabolic systems with coefficients depending only on one direction and a certain decomposition of $\bar{u}$, to establish a decay estimate Proposition 5.3 of

(1.4) \[ \inf_{Q^{\epsilon, \tau} \subseteq \mathbb{R}^n} \left( \int_{Q^{\tau}(\Lambda x_0)} \left( |D_{\phi^\epsilon} \bar{u} - Q|^\tau + |\mathcal{A}^{\alpha\beta}D_{\phi} \bar{u} - R_{\phi}|^\tau \right) dt \, dy \right)^2, \]

where $\tau \in (0, 1/4)$, $y = \Lambda x$, $\Lambda = (\Lambda^{\alpha\beta})_{\alpha, \beta=1}^d$ is a $d \times d$ orthogonal matrix representing the linear transformation from the coordinate system associated with 0 to the coordinate system associated with $x_0$, and

$$\bar{u}(t, y) = \tilde{u}(t, x), \quad \mathcal{A}^{\alpha\beta}(t, y) = \Lambda^{\alpha k} \Lambda^{ks}(t, x) \Lambda^{\beta \gamma}, \quad \tilde{\gamma}^{\alpha}(t, y) = \Lambda^{ak} \tilde{\gamma}(t, x).$$

Then with the help of the decay estimate of the functional in (1.4) together with the estimates of $|D_{\phi^\epsilon} \bar{u} - D_{\phi} \bar{u}|$ and $|\tilde{u} - \mathcal{A}^{\alpha\beta}D_{\phi} \bar{u} + \tilde{\gamma}|$ in (3.40) below,
in Lemma 3.4 we obtain the decay estimate of
\[
\inf_{q^\alpha, Q \in \mathbb{R}^d} \left( \int_{Q \cap (z_0)} \left( |D_{t^\alpha} \tilde{u} - q^\beta| \frac{1}{2} + |	ilde{U} - Q| \right) dz \right)^2,
\]
where \( \tilde{U} := n^a(A^{a\beta}D_{\beta}\tilde{u} - \tilde{f}^a) \) and \( n^a \) is defined in (2.5) below. The desired result is then proved by utilizing the decay estimate.

In Section 4 we deal with the case when \( s \geq 2 \) by using a similar scheme. To this end, we consider the following parabolic system in divergence form
\[
-\tilde{u}_t + D_a(A^{a\beta}D_{\beta}\tilde{u}) = D_\alpha f^\alpha + g,
\]
where \( \tilde{u} := u^\ell - u \), \( u \) is defined in (4.15), which is piecewise smooth,
\[ u^\ell = D_\ell D_\ell \cdots D_\ell u - u_0, \]
u_0 is defined in (4.10), \( f^\alpha \) and \( g \) are defined in (4.19) and (4.7), respectively. Then via a similar argument used in Section 3 we establish a decay estimate of
\[
\inf_{q^\alpha, Q \in \mathbb{R}^d} \left( \int_{Q \cap (z_0)} \left( |D_{t^\alpha} \tilde{u} - q^\beta| \frac{1}{2} + |	ilde{U} - Q| \right) dz \right)^2,
\]
where \( \tilde{U} := n^a(A^{a\beta}D_{\beta}\tilde{u} - \tilde{f}^a) \). With this estimate, Theorem 1.2 follows.

In the Appendix A we present the local boundedness and piecewise Hölder-regularity of \( Du \) for the solution of (1.1), and some auxiliary estimates proved in [11], which are used frequently in the current paper.

2. Preliminaries

In this section, we first present the basic notation and the function spaces. Then we provide the assumptions of the subdomains by assuming \( \mathcal{D} \) is a unit ball \( B_1 \) for simplicity and introduce vector fields near the origin 0 together with some properties of the vector fields.

2.1. Notation and definitions. For \( r > 0 \) and \( z = (t, x) \in \mathbb{R}^{d+1}, d \geq 2 \), we denote
\[
Q_r(z) := (t - r^2, t) \times B_r(x),
\]
where \( B_r(x) := \{ y \in \mathbb{R}^d : |y - x| < r \} \). For \( x \in \mathbb{R}^d \), we denote \( x' = (x^1, \ldots, x^{d-1}) \in \mathbb{R}^{d-1} \) and \( B'_r(x') := \{ y' \in \mathbb{R}^{d-1} : |y' - x'| < r \} \). We often write \( B_r \) and \( B'_r \) for \( B_r(0) \) and \( B'_r(0') \), respectively. The parabolic distance between two points \( z_1 = (t_1, x_1) \) and \( z_2 = (t_2, x_2) \) is defined by
\[
|z_1 - z_2|_p := \max \left\{ |t_1 - t_2|^{1/2}, |x_1 - x_2| \right\}.
\]

We denote the parabolic boundary of a cylinder \( Q = (a, b) \times \mathcal{D} \) by
\[
\partial_p Q = ((a, b) \times \partial \mathcal{D}) \cup ([a] \times \mathcal{D}).
\]

For a function \( f \) defined in \( \mathbb{R}^{d+1} \), we set
\[
(f)_Q = \frac{1}{|Q|} \int_Q f(t, x) \, dx \, dt = \int_Q f(t, x) \, dx \, dt,
\]
where $|Q|$ is the $d + 1$-dimensional Lebesgue measure of $Q$.

Next, for $\gamma, \gamma' \in (0, 1)$, we denote the $C^{\gamma/2, \gamma'}$ semi-norm by

$$[u]_{\gamma/2, \gamma'; Q} := \sup_{(t, x), (s, y) \in Q, t \neq s} \frac{|u(t, x) - u(s, y)|}{|t - s|^\gamma + |x - y|^{\gamma'}},$$

and the $C^{\gamma/2, \gamma'}$ norm by

$$|u|_{\gamma/2, \gamma'; Q} := [u]_{\gamma/2, \gamma'; Q} + |u|_0, \quad \text{where } |u|_0 = \sup_Q |u|.$$

Define

$$[u]_{\gamma, Q} := \sup_{(t, x), (s, y) \in Q} \frac{|u(t, x) - u(s, x)|}{|t - s|^\gamma},$$

$$[u]_{(1+\gamma)/2, 1+\gamma; Q} := [Du]_{\gamma/2, \gamma; Q} + [u]_{(1+\gamma)/2; Q},$$

and

$$|u|_{(1+\gamma)/2, 1+\gamma; Q} := [u]_{(1+\gamma)/2, 1+\gamma; Q} + |u|_0.$$

We denote $C^{(1+\gamma)/2, 1+\gamma}$ to be the set of all bounded measurable functions $u$ for which $Du$ are bounded and continuous in $Q$ and $|u|_{(1+\gamma)/2, 1+\gamma; Q} < \infty$. The function spaces $C^{(1+\gamma)/2, l+\gamma}$, $l = 2, 3, \ldots$, are defined accordingly.

For $p \in (1, \infty)$, we denote

$$W_p^{1, 2}(Q) := \{u : u, u_t, Du, D^2 u \in L^p(Q)\}.$$

Set

$$\mathcal{H}^{-1}_p(Q) := \left\{ u : u = \sum_{|\alpha| \leq 1} D^\alpha u_\alpha, u_\alpha \in L^p(Q) \right\},$$

and

$$\|u\|_{\text{H}^{-1}_p(Q)} := \inf \left\{ \sum_{|\alpha| \leq 1} \|u_\alpha\|_{L^p(Q)} : u = \sum_{|\alpha| \leq 1} D^\alpha u_\alpha \right\}.$$
2.2. Assumptions and auxiliary results. Our objective is to prove the interior regularity of solutions by establishing the local estimates. We will slightly abuse the notation to localize the problem by taking $Q$ to be a unit cylinder $Q_1^{-} := (-1,0) \times B_1$. By suitable rotation and scaling, we may suppose that a finite number of subdomains lie in $B_1$ and that they can be represented by

$$x^d = h_j(x'), \quad \forall \; x' \in B_1', \; j = 1, \ldots, m < M,$$

where

$$-1 < h_1(x') < \cdots < h_m(x') < 1,$$

$h_j(x') \in C^{s+1+\mu}(B_1')$ with $s \in \mathbb{N}$. Set $h_0(x') = -1$ and $h_{m+1}(x') = 1$. Then we have $m + 1$ regions:

$$D_j := \{x \in \mathcal{D} : h_{j-1}(x') < x^d < h_j(x')\}, \quad 1 \leq j \leq m + 1.$$

For $j = 1, \ldots, m$, the normal direction of the interfacial boundary $\Gamma_j = \{x^d = h_j(x')\}$ is given by

$$n_j := (n_1^j, \ldots, n_d^j) = (1 + |D_{x'} h_j(x')|^2)^{-1/2}(-D_{x'} h_j(x'), 1)^T \in \mathbb{R}^d.$$

For each $k = 1, \ldots, d - 1$, we define a vector field $\ell^{k,0}_j : \mathbb{R}^d \rightarrow \mathbb{R}^d$ near 0 as follows:

$$\ell^{k,0}_k = 1, \quad \ell^{k,0}_i = 0, \quad i \neq k, d,$$

$$\ell^{k,0}_d = \begin{cases} D_k h_m(x'), & x^d \geq h_m, \\ \frac{x^d - h_{j-1}}{h_j - h_{j-1}} D_k h_j(x') + \frac{h_j - x^d}{h_j - h_{j-1}} D_k h_{j-1}(x'), & h_{j-1} \leq x^d < h_j, \\ D_k h_1(x'), & x^d < h_1. \end{cases}$$

It is easily seen that on $\Gamma_j$, $\ell^{k,0}_d = D_k h_j(x')$ so that $\ell^{k,0}_j$ is in a tangential direction. Moreover, it follows from $h_j \in C^{s+1+\mu}$ that $\ell^{k,0}_j$ is $C^{s+\mu}$ on $\Gamma_j$. Define the projection operator by

$$\text{proj}_b = \frac{\langle a, b \rangle}{\langle a, a \rangle} a,$$
where $\langle a, b \rangle$ denotes the inner product of the vectors $a$ and $b$, and $\langle a, a \rangle = |a|^2$. Then we make the vector field orthogonal to each other by using the Gram-Schmidt process:

$$
\ell^1 = \ell_1^{1,0}, \quad \ell^1 = \frac{\ell^1}{|\ell^1|},
$$

$$
\ell^2 = \ell_2^{2,0} - \text{proj}_{\ell^1} \ell_2^{2,0}, \quad \ell^2 = \frac{\ell^2}{|\ell^2|},
\quad \vdots
$$

where

$$
\ell^d = \ell_d^{d-1,0} - \sum_{j=1}^{d-2} \text{proj}_{\ell^j} \ell_d^{d-1,0}, \quad \ell^d = \frac{\ell^d}{|\ell^d|}.
\quad (2.4)
$$

From the definition of $\ell_k^{k,0}$, we define the corresponding unit normal direction which is orthogonal to $\ell_k^{k,0}$, $k = 1, \ldots, d - 1$, (and thus also $\ell^k$):

$$
n(x) = (n^1, \ldots, n^d)^T = \left(1 + \sum_{k=1}^{d-1} (\ell_k^{k,0})^2\right)^{-1/2} (-\ell_d^{1,0}, \ldots, -\ell_d^{d-1,0}, 1)^T. \quad (2.5)
$$

Obviously, $n(x) = n_j$ on $\Gamma_j$.

**Lemma 2.1.** Let $\ell^k$ be defined in (2.4), $k = 1, \ldots, d - 1$. Then the following assertions hold.

(i) We have $\ell^k \in C^{1/2}(D)$ and the $C^{1/2}$ norms are independent of the distance between the subdomains.

(ii) It holds that $D\ell^1 \ell^2 \cdots \ell^k \in C^0(D)$, where $k_1 = 1, \ldots, d - 1$, $\tau = 1, \ldots, s$, and $s \in \mathbb{N}$ with $s \geq 2$. Moreover, the $C^0$ norms are independent of the distance between the subdomains.

(iii) We have $|D\ell^k| \leq N|h_j - h_{j-1}|^{-1/2}$ and $|DD\ell^1 \ell^2 \cdots \ell^k| \leq N|h_j - h_{j-1}|^{-1}$, where $N$ depends on the $C^{s+1}$ norms of $h_j$.

**Proof.** (i) We start with proving that $\ell_k^{k,0}$ is $C^{1/2}$ in the vertical direction $x^d$.

For any two points $(x', x'^1_i), (x', x'^1_2)$ satisfying $h_{j-1}(x') \leq x'^1_i < h_j(x')$, $i = 1, 2$, we have

$$
\ell_d^{k,0}(x', x'^1_i) - \ell_d^{k,0}(x', x'^1_2) = \frac{x'^1_i - x'^1_2}{h_j - h_{j-1}}D_k(h_j - h_{j-1}).
$$

It follows from $h_j \in C^{s+1+\mu}$ and $h_j > h_{j-1}$ that

$$
|D_k h_j(x') - D_k h_{j-1}(x')| \leq Nh_j(x') - h_{j-1}(x')|^{1/2}. \quad (2.6)
$$

See, for instance, [18, (50)]. This together with $|x'^1_i - x'^1_2| \leq h_j - h_{j-1}$ gives

$$
|\ell_d^{k,0}(x', x'^1_i) - \ell_d^{k,0}(x', x'^1_2)| \leq Nh_j(x'^1_i) - h_{j-1}(x'^1_2)^{1/2}.
$$
We continue to prove that \( \ell_{d}^{k,0} \) is \( C^{1/2} \) in \( x' \). For any two points \((x'_1, x')\) and \((x'_2, x')\) with \( h_{j-1}(x'_i) \leq x' < h_j(x'_i) \), \( i = 1, 2 \), we have

\[
\ell_{d}^{k,0}(x'_1, x') - \ell_{d}^{k,0}(x'_2, x') = \frac{x' - h_{j-1}(x'_1)}{(h_j - h_{j-1})(x'_1)} D_k(h_{j-1}(x'_1)) + D_k h_{j-1}(x'_1) - \frac{x' - h_{j-1}(x'_2)}{(h_j - h_{j-1})(x'_2)} D_k(h_{j-1}(x'_2)) + D_k h_{j-1}(x'_2) + \frac{x' - h_{j-1}(x'_1)}{(h_j - h_{j-1})(x'_1)} D_k(h_{j-1}(x'_1)) - D_k h_{j-1}(x'_1)(x'_2)
\]

(2.7)

Now we estimate the last term in (2.7). Without loss of generality, we assume that

\[
|(h_j - h_{j-1})(x'_i)| = \sup_{r=0,1} |(h_j - h_{j-1})(rx'_i + (1 - r)x'_2)|.
\]

If \(|x'_1 - x'_2| > |(h_j - h_{j-1})(x'_i)|\), then by using \( h_{j-1}(x'_i) \leq x' < h_j(x'_i) \) and (2.6), \( i = 1, 2 \), we have

\[
|D_k h_{j-1}(x'_i)| \leq 2|D_k h_{j-1}(x'_2)| \leq N|h_{j-1}(x'_2)|^{1/2} \leq N|x'_1 - x'_2|^{1/2}.
\]

If \(|x'_1 - x'_2| \leq |(h_j - h_{j-1})(x'_i)|\), then by (2.6) and (2.8), we derive

\[
|D_k(h_j - h_{j-1})(x'_i)| \leq N|(h_j - h_{j-1})(x'_2)|^{1/2} \leq N|(h_j - h_{j-1})(x'_1)|^{1/2}.
\]

Using \( h_j \in C^{1+\mu} \), we obtain

\[
\left| \frac{x' - h_{j-1}(x'_1)}{(h_j - h_{j-1})(x'_1)} - \frac{x' - h_{j-1}(x'_2)}{(h_j - h_{j-1})(x'_2)} \right| = \left| \frac{h_{j-1}(x'_i) - h_{j-1}(x'_2)}{(h_j - h_{j-1})(x'_1)} + \frac{(x' - h_{j-1}(x'_2))(h_j - h_{j-1})(x'_2) - (h_j - h_{j-1})(x'_1))}{(h_j - h_{j-1})(x'_1) \cdot (h_j - h_{j-1})(x'_2)} \right|
\]

(2.10)

\[
\leq \frac{N|x'_1 - x'_2|}{(h_j - h_{j-1})(x'_i)}.
\]

Combining (2.9) and (2.10), we deduce

\[
\left| D_k(h_j - h_{j-1})(x'_i) \left( \frac{x' - h_{j-1}(x'_1)}{(h_j - h_{j-1})(x'_1)} - \frac{x' - h_{j-1}(x'_2)}{(h_j - h_{j-1})(x'_2)} \right) \right|
\]
\[ \frac{N|x'_1 - x'_2|}{|h_j - h_{j-1}|(x'_1)|^{1/2}} \leq \frac{N|x'_1 - x'_2|}{|x'|^{1/2}}. \]

Therefore, coming back to (2.7), we have

\[ |\ell^{k,0}_d(x'_1, x^d) - \ell^{k,0}_d(x'_2, x^d)| \leq \frac{N|x'_1 - x'_2|}{|x'|^{1/2}}. \]

We hence conclude that \( \ell^{k,0}_d \) is \( C^{1/2} \) in \( D \). Combining with the definition of \( \ell^k \) in (2.4), we derive the \( C^{1/2} \)-regularity of \( \ell^k \).

(ii) In view of (2.3), a direct calculation gives

\[ D_{k_1} \ell^{k_2,0}_d = \frac{D_{k_1} h_{j-1} D_{k_2} (h_j - h_{j-1})}{h_j - h_{j-1}} - \frac{(x^d - h_{j-1})D_{k_2} (h_j - h_{j-1})D_{k_1} (h_j - h_{j-1})}{(h_j - h_{j-1})^2} \]

\[ + \frac{x^d - h_{j-1}}{h_j - h_{j-1}}D_{k_1} D_{k_2} h_j + \frac{h_j - x^d}{h_j - h_{j-1}}D_{k_1} D_{k_2} h_{j-1}, \]

and

\[ D_d \ell^{k_2,0}_d = \frac{D_{k_2} (h_j - h_{j-1})}{h_j - h_{j-1}}, \]

where \( h_{j-1} \leq x^d < h_j \). Since

\[ \ell^{k,i}_d = \frac{(x^d - h_{j-1})D_{k_1} (h_j - h_{j-1})}{h_j - h_{j-1}} + D_{k_1} h_{j-1}, \quad h_{j-1} \leq x^d < h_j, \]

we have

\[ D_{k_1} \ell^{k_2,0}_d + D_d \ell^{k_1,0}_d D_d \ell^{k_2,0}_d = \frac{x^d - h_{j-1}}{h_j - h_{j-1}}D_{k_1} D_{k_2} h_j + \frac{h_j - x^d}{h_j - h_{j-1}}D_{k_1} D_{k_2} h_{j-1}, \]

when \( h_{j-1} \leq x^d < h_j \). Therefore, we obtain

\[ D_{\ell^{k,0}} D_{\ell^{k_2,0}_d} = D_{k_1} \ell^{k_2,0}_d + D_d \ell^{k_1,0}_d D_d \ell^{k_2,0}_d \]

\[ = \begin{cases} D_{k_1} D_{k_2} h_m(x'), & x^d \geq h_m, \\ \frac{x^d - h_{j-1}}{h_j - h_{j-1}}D_{k_1} D_{k_2} h_j(x') + \frac{h_j - x^d}{h_j - h_{j-1}}D_{k_1} D_{k_2} h_{j-1}(x'), & h_{j-1} \leq x^d < h_j, \\ D_{k_1} D_{k_2} h_1(x'), & x^d < h_1. \end{cases} \]

This together with (2.1) implies that \( D_{\ell^{k,0}} D_{\ell^{k_2,0}_d} \in C^0(D) \) and the \( C^0 \) norm is independent of the distance between the subdomains. From this, and using the definition of \( \ell^k \) in (2.4) and the fact that \( \ell^k \) is a linear combination of \( \ell^{j,0} \), \( j = 1, \ldots, k \), we also deduce that

\[ D_{\ell^{k,0}} \ell^{k_2} \in C^0(D). \]

By an induction argument, we conclude

\[ D_{\ell^{k,0}} D_{\ell^{k_2,0}} \cdots D_{\ell^{k_{n-1},0}} \ell^{k_n,0}_d \]
\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{ll}
D_{k_1} D_{k_2} \cdots D_{k_n}(x') ,
\end{array} \right. \\
&\quad \left. \begin{array}{l}
x^d \geq h_m, \\
1 - \frac{h_j - x^d}{h_j - h_{j-1}}, \quad h_{j-1} \leq x^d < h_j, \\
D_{k_1} D_{k_2} \cdots D_{k_n}h_1(x'), \quad x^d < h_1,
\end{array} \right.
\end{aligned}
\end{equation}

and thus, similarly, \( D_{\ell_{k_1}} D_{\ell_{k_2}} \cdots D_{\ell_{k_n}} \ell_k \in C^0(\mathcal{D}) \).

(iii) By virtue of (2.11), (2.12), and (2.6), we have in \( B_r(x_0) \cap \mathcal{D}_j \),

\[ |D\ell_k| \leq N|h_j - h_{j-1}|^{-1/2}. \]

Then recalling the definition of \( \ell_k \) in (2.4), this also holds for \( \ell_k \). Similarly, from (2.13), we have

\[ |DD_{\ell_{k_1}} D_{\ell_{k_2}} \cdots D_{\ell_{k_n}} \ell_k| \leq N|h_j - h_{j-1}|^{-1}, \]

and thus

\[ |DD_{\ell_{k_1}} D_{\ell_{k_2}} \cdots D_{\ell_{k_n}} \ell_k| \leq N|h_j - h_{j-1}|^{-1}. \]

The lemma is proved.

\end{proof}

2.3. \textbf{Coordinate systems.} In the proofs, we will use different coordinate systems associated with different points defined as follows.

We fix a coordinate system such that \( 0 \in \mathcal{D}_{j_0} \) for some \( j_0 \in \{1, \ldots, m+1\} \) and the closest point on \( \partial \mathcal{D}_{j_0} \) is \( x_{j_0} = (0', h_{j_0}(0')) \), and \( \nabla_{x'} h_{j_0}(0') = 0' \). Throughout the paper, we shall use \( x = (x', x^d) \) and \( D_x \) to denote the point and the derivatives, respectively, in this coordinate system. Then at the point \( x_{j_0} \), \( \ell_k = (0, \ldots, 0, 1, 0, \ldots, 0)^T \) and \( n = (0', 1)^T \). See Figure [1]

For any point \( x_0 \in B_{3/4} \cap \mathcal{D}_{j_0} \), \( j_0 = 1, \ldots, m + 1 \), suppose the closest point on \( \partial \mathcal{D}_{j_0} \) to \( x_0 \) is \( y_0 := (y_0', h_{j_0}(y_0')) \). Let

\begin{equation}
n_{y_0} = \left( n_{y_0}^1, \ldots, n_{y_0}^d \right)^T = \left( 1 + |\nabla_{x'} h_{j_0}(y_0')|^2 \right)^{-1/2} \left( -\nabla_{x'} h_{j_0}(y_0'), 1 \right)^T
\end{equation}

be the unit normal vector at \( (y_0', h_{j_0}(y_0')) \) on the surface \( \Gamma_{j_0} \). Define the tangential vectors by

\begin{equation}
\tau_k = \ell_k(y_0), \quad k = 1, \ldots, d - 1,
\end{equation}

where \( \ell_k \) is defined in (2.4). See Figure [1]

In the coordinate system associated with \( x_0 \) with the axes paralleled to \( n_{y_0} \) and \( \tau_k, k = 1, \ldots, d - 1 \), we will use \( y = (y', y^d) \) and \( D_y \) to denote the point and the derivatives, respectively. Moreover, we have \( y = \Lambda x \), where \( \Lambda = (\Lambda^1, \ldots, \Lambda^d)^T = (\Lambda^k_{i,j})_{i,j=1}^d \) is a \( d \times d \) matrix representing the linear transformation from the coordinate system associated with \( 0 \) to the coordinate system associated with \( x_0 \), and \( \tau_k = (\Gamma_{1k}, \ldots, \Gamma_{dk})^T, k = 1, \ldots, d - 1 \), \( n_{y_0} = (\Gamma_{1d}, \ldots, \Gamma_{dd})^T \), where \( \Gamma = \Lambda^{-1} \). Now we introduce \( m + 1 \) “strips” in the \( y \)-coordinates.

\[ \Omega_j := \{ y \in \mathcal{D} : y^d_{j-1} < y^d < y^d_j \}, \quad j = 1, \ldots, m + 1, \]
where \( y_j = (\Lambda y_0, y_j^d) \in \Gamma_j \) and \( \Lambda' = (\Lambda^1, \ldots, \Lambda^{d-1})^\top \). We also have for any \( 0 < r \leq 1/4, \)
\[
|\mathcal{D}_j \setminus \Omega_j| \cap (B_r(\Lambda^0)) | \leq N r^{d+1/2}, \quad j = 1, \ldots, m + 1.
\]
See, for instance, [10, Lemma 2.3].

For a piecewise Hölder continuous function \( f \in C^{\beta/2,\delta}((-1 + \varepsilon, 0) \times \mathcal{D}_j) \), we define
\[
\overline{f}(t, y) = \int_{Q_j^\gamma(\Lambda^0) \cap ((-1 + \varepsilon, 0) \times \mathcal{D}_j)} f(s, z) \, dz \, ds, \quad (t, y) \in Q_j^\gamma(\Lambda^0) \cap ((-1 + \varepsilon, 0) \times \Omega_j)
\]
to be the corresponding piecewise constant function in \( Q_j^\gamma(\Lambda^0) := (t_0 - r^2/4, t_0) \times B_r(\Lambda^0) \). Note that \( \overline{f} \) only depends on \( y^d \) and we will denote \( \overline{f}(y^d) := f(t, y) \) for abbreviation. Finally, by using (2.16), we have
\[
\|f - \overline{f}\|_{L^1(Q_j^\gamma(\Lambda^0))} \leq \sum_{j=1}^{m+1} \|f - \overline{f}\|_{L^1(Q_j^\gamma(\Lambda^0) \cap ((-1 + \varepsilon, 0) \times \mathcal{D}_j \cap \Omega_j))}
\]
(2.18)
\[
+ \sum_{j=1}^{m+1} \|f - \overline{f}\|_{L^1(Q_j^\gamma(\Lambda^0) \cap ((-1 + \varepsilon, 0) \times \mathcal{D}_j \cap \Omega_j))} \leq Nr^{d+5/2}.
\]

3. Second derivative estimates

We present a complete proof of Theorem 1.2 with \( s = 1 \) in this section. As in Section 2.2, we will take \( Q = (-1, 0) \times B_1 \). The equation (1.1) is equivalent to a (homogeneous) transmission problem
\[
\begin{aligned}
-\partial_t u + D_\alpha(A^{\beta\delta}D_\beta u) &= D_\alpha f^\alpha \quad \text{in} \quad \bigcup_{j=1}^{m+1} (-1, 0) \times \mathcal{D}_j, \\
|u|_{(-1,0) \times \Gamma_j}^+ &= |u|_{(-1,0) \times \Gamma_j}^- \quad \text{on} \quad -1 < \cdot < 0, \quad j = 1, \ldots, m,
\end{aligned}
\]
where
\[
|n_j^\alpha(A^{\beta\delta}D_\beta u - f^\alpha)|_{(-1,0) \times \Gamma_j}^+ := n_j^\alpha(A^{\beta\delta}D_\beta u - f^\alpha)|_{(-1,0) \times \Gamma_j}^+ - n_j^\alpha(A^{\beta\delta}D_\beta u - f^\alpha)|_{(-1,0) \times \Gamma_j}^-
\]
$n_j$ is the unit normal vector on $\Gamma_j$ defined in (2.2), $u|_{(-1,0) \times \Gamma_j}$ and $u|_{(-1,0) \times \Gamma_j}$ are the left and right limits of $u$ (its conormal derivatives) on $(-1,0) \times \Gamma_j$, respectively, $j = 1, \ldots, m$.

To show higher regularity, we take the directional derivative of $u$ in the direction $\ell := \ell_k, k = 1, \ldots, d - 1$, to get

$$
\begin{align}
-(D_o u)_t + D_o(A^{\alpha \beta} D_\beta D_o u) &= g + D_o f^\alpha, \quad \text{in} \quad \bigcup_{j=1}^{m+1} (-1,0) \times D_j,

D_o u|_{(-1,0) \times \Gamma_j} &= D_o u|_{(-1,0) \times \Gamma_j}, \quad [n_j^\alpha (A^{\alpha \beta} D_\beta D_o u - f^\alpha)]_{(-1,0) \times \Gamma_j} = \tilde{h}_j,
\end{align}
$$

where

$$
\begin{align}
g &= D_o \ell(A^{\alpha \beta} D_\beta D_o u + DA^{\alpha \beta} D_\beta u - D f^\alpha),

f^\alpha &= D_o \ell f^\alpha + A^{\alpha \beta} D_\beta D_o \ell D_o u - D_o A^{\alpha \beta} D_\beta u,
\end{align}
$$

and

$$
\tilde{h}_j = [D_o n_j^\alpha (A^{\alpha \beta} D_\beta u + f^\alpha)]_{(-1,0) \times \Gamma_j}.
$$

Note that $D_o n_j$ is a tangential direction on $\Gamma_j$ and we may write $\tilde{h}_j = \tilde{h}_j(t, x')$. Furthermore, by a direct calculation using (2.2) and (2.4), we have $D_o n_j^\alpha \in C^\mu$ as a function of $x'$.

The equation (3.1) is also a transmission problem for $D_o u$, but is inhomogeneous. A difficulty is that $D_o \ell$ is singular at any point where two interfacial boundaries touch or are very close to each other. To cancel out the singularity, we consider

$$
u_t := u_t(z; z_0) = D_o u - u_0,
$$

where $z_0 = (t_0, x_0) \in (-9/16, 0) \times (B_{3/4} \cap \overline{D_{j_0}})$,

$$
u_0 := u_0(x; z_0) = \sum_{j=1}^{m+1} \tilde{\ell}_{i,j} D_o u(t_0, P_j x_0),
$$

$$
P_j x_0 = \begin{cases} 
    x_0 & \text{for } x_0 \in \overline{D_{j_0}}, \\
    (x_{j_0}', \tilde{h}_j(x_{j_0}')) & \text{for } j < j_0, \\
    (x_{j_0}', \tilde{h}_{j-1}(x_{j_0}')) & \text{for } j > j_0,
\end{cases}
$$

and the vector field $\tilde{\ell}_j := (\tilde{\ell}_{1,j}, \ldots, \tilde{\ell}_{d,j})$ is a smooth extension of $\ell|_{\mathcal{D}}$ to $\bigcup_{k=1}^{m+1} \mathcal{D}_k$. From (3.1), we have

$$
\begin{align}
-(\partial_t u_t + D_o(A^{\alpha \beta} D_\beta u_t)) &= g + D_o f^\alpha, \quad \text{in} \quad \bigcup_{j=1}^{m+1} (-1,0) \times D_j,

[n_j^\alpha (A^{\alpha \beta} D_\beta u_t - f^\alpha)]_{(-1,0) \times \Gamma_j} &= \tilde{h}_j,
\end{align}
$$

where

$$f^\alpha_2 := f^\alpha_2(z; z_0) = f^\alpha_1 - A^{\alpha \beta} \sum_{j=1}^{m+1} D_\beta \tilde{\ell}_{i,j} D_o u(t_0, P_j x_0)$$
Now by solving a conormal boundary value problem (or simply adding a term
\[
\sum_{j=1}^{m} D_\beta (1_{x^d > h_j(x')} (n_j^d(x'))^{-1} \tilde{h}_j(t, x'))
\]
to the equation; see [11]), where \(1\) is the indicator function, we can get rid of \(\tilde{h}_j\) in the second equation of (3.7) and reduce the problem (3.7) to a homogeneous transmission problem:

\[
\begin{aligned}
-\partial_t u + D_\alpha (A^{\alpha\beta} D_\beta u_i) &= g + D_\alpha f_3^a \\
\left[n_j^d (A^{\alpha\beta} D_\beta u_i - f_3^a) \right] &\mid (-1, 0) \times \Gamma_j = 0,
\end{aligned}
\]

(3.8)

where

\[
f_3^a := f_3^a(z; z_0) = f_2^a + \delta_{ad} \sum_{j=1}^{m} 1_{x^d > h_j(x')} (n_j^d(x'))^{-1} \tilde{h}_j(t, x')
\]

\[
= D_\ell f^a - D_\ell A^{\alpha\beta} D_\beta u + A^{\alpha\beta} \left(D_\beta \ell_i D_i u - \sum_{j=1}^{m} D_\beta \tilde{\ell}_{ij} D_j u(t_0, P_j x_0)\right)
\]

(3.9)

+ \delta_{ad} \sum_{j=1}^{m} 1_{x^d > h_j(x')} (n_j^d(x'))^{-1} \tilde{h}_j(t, x'),

where \(\delta_{ad} = 1\) if \(\alpha = d\), and \(\delta_{ad} = 0\) if \(\alpha \neq d\).

To prove piecewise regularity of \(u\), we need to show the mean oscillation of \(D u\) in cylinders vanishes in a certain order as the radii of the cylinders go to zero, so that Campanato’s approach in [4, 14] can be applied. However, we note that the mean oscillation of

\[
A^{\alpha\beta} \left(D_\beta \ell_i D_i u - \sum_{j=1}^{m} D_\beta \tilde{\ell}_{ij} D_j u(t_0, P_j x_0)\right)
\]

in (3.9) is only bounded. To this end, we choose a cut-off function \(\zeta \in C_0^\infty(B_1)\) satisfying

\[
0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \text{ in } B_{3/4}, \quad |D\zeta| \leq 8.
\]

Denote

\[
A^{\alpha\beta} := \zeta A^{\alpha\beta} + \nu (1 - \zeta) \delta_{\alpha d} \delta_{ij}.
\]

(3.10)
For \( j = 1, \ldots, m + 1 \), denote \( \mathcal{D}_j^c := \mathcal{D} \setminus \mathcal{D}_j \). Let \( u_j(\cdot; z_0) \in \mathcal{H}^1_p(Q_1^-) \) be the weak solution of the problem
\begin{equation}
\begin{cases}
-\partial_t u_j(\cdot; z_0) + D_\alpha (A^{\alpha\beta} D_\beta u_j(\cdot; z_0)) = -D_\alpha (1_{(-0,0) \times \mathcal{D}_j} A^{\alpha\beta} D_\beta \tilde{c}_{i,j} D_i u(t_0, P_j x_0)) \quad & \text{in } Q_1^-,
 u_j(\cdot; z_0) = 0 \quad & \text{on } \partial_p Q_1^-,
\end{cases}
\end{equation}
where \( 1 < p < \infty \). The solvability of (3.11) follows from Lemma A.3. Furthermore, we have
\begin{equation}
\|u_j(\cdot; z_0)\|_{\mathcal{H}^1_p(Q_1^-)} \leq N\|1_{(-0,0) \times \mathcal{D}_j} A^{\alpha\beta} D_\beta \tilde{c}_{i,j} D_i u(t_0, P_j x_0)\|_{L_p(Q_1^-)}
\end{equation}
where we used the local boundedness estimate of (3.12)
\begin{equation}
\|u\|_{L_{1,p}(Q_1^-)} + \|u\|_{(1+\delta)/2,1+\delta,(-1,0) \times \mathcal{D}_j)} \quad \text{for } i = 1, \ldots, m + 1.
\end{equation}

Now we define
\begin{equation}
\bar{u} := \sum_{j=1}^{m+1} u_j(\cdot; z_0).
\end{equation}

Then \( \bar{u} \) satisfies
\begin{equation}
-\bar{u}_t + D_\alpha (A^{\alpha\beta} D_\beta \bar{u}) = g + D_\alpha \tilde{f}^\alpha \quad \text{in } Q^{3/4},
\end{equation}
where
\begin{equation}
\tilde{f}^\alpha := f^\alpha(\cdot; z_0) = f^\alpha + A^{\alpha\beta} \sum_{j=1}^{m+1} 1_{(-0,0) \times \mathcal{D}_j} D_\beta \tilde{c}_{i,j} D_i u(t_0, P_j x_0),
\end{equation}
and
\begin{equation}
\bar{u}_t := \bar{u}(\cdot; z_0) = u_t - u = D_t u - u_0 - u.
\end{equation}
where \( f^\alpha \) is defined in (3.9). The mean oscillation of \( \tilde{f}^\alpha \) vanishes at a certain rate as the radii of the cylinders go to zero; the details can be found in the proof of (3.46) below. Therefore, we deduce from (3.13) and (3.15) that to prove piecewise regularity of \( u_\epsilon \), we only need to prove that of \( \tilde{u} \).

Denote
\[
\tilde{U} := \tilde{U}(z; z_0) = n^\alpha(A^{\alpha\beta}D_\beta \tilde{u} - \tilde{f}^\alpha),
\]
where \( n^\alpha \) is defined in (2.5), \( \alpha = 1, \ldots, d \). The rest part of this section is devoted to deriving piecewise \( C^{\mu'} \)-continuity of \( D_k\tilde{u} \) and \( \tilde{U} \), where \( k' = 1, \ldots, d - 1 \), and \( \mu' = \min \left\{ \frac{1}{2}, \mu \right\} \). For this, we will prove the following proposition.

**Proposition 3.1.** Let \( \epsilon \in (0, 1) \) and \( p \in (1, \infty) \). Suppose that \( A^{\alpha\beta} \) and \( f^\alpha \) satisfy Assumption [\ref{assump:1.1}] with \( s = 1 \). If \( u \in H^1_p((-1, 0) \times B_1) \) is a weak solution to
\[
-u_\epsilon + D_\alpha(A^{\alpha\beta}D_\beta u) = D_\alpha f^\alpha \quad \text{in } (-1, 0) \times B_1,
\]
then the following assertions hold.

(a) For any \( z_0, z_1 \in (-1 + \epsilon, 0) \times B_{1-\epsilon} \), we have
\[
|D_k\tilde{u}(z_0; z_1)| + |\tilde{u}(z_0; z_1) - \tilde{U}(z_0; z_1)|
\]
\[
\leq N|z_0 - z_1|^{\mu'}\left( ||Du||_{L^p((-1, 0) \times B_1)} + \sum_{j=1}^M |f|_{(1+\delta/2,1+\delta(\epsilon,0)\times D_j)} \right),
\]
where \( \tilde{u} \) and \( \tilde{U} \) are defined in (3.15) and (3.18), respectively, \( \mu' = \min \left\{ \frac{1}{2}, \mu \right\} \).

(b) For \( j = 1, \ldots, m+1 \), it holds that \( u \in C^{1+\mu'/2,2+\mu'}((-1+\epsilon, 0) \times (B_{1-\epsilon} \cap D_j)) \).

The proof of Proposition [\ref{prop:3.1}] is based on the idea in the proof of [\ref{prop:4.2}], which is an adaptation of Campanato’s method in [\ref{campanato}]. We shall first establish an a priori estimate of the modulus of continuity of \( \langle D_k\tilde{u}, \tilde{U} \rangle \) by assuming that \( Du \) is \( C^{1/2} \) in \( t \), and piecewise \( C^1 \) in \( x \). The general case follows from an approximation argument together with the technique of locally flattening the boundaries [\ref{local_flattening} p. 2466].

We next derive some auxiliary results in Sections 3.1 and 3.2, and then prove Proposition [\ref{prop:3.1}] in Section 3.3.

### 3.1. Decay estimates.
As in Section 2.2, we take \( z_0 = (t_0, x_0) \in (-9/16, 0) \times (B_{3/4} \cap D_{1/2}) \) and \( r \in (0, 1/4) \).

**Lemma 3.2.** Let \( \tilde{u} \) and \( \tilde{f}^\alpha \), \( \alpha = 1, \ldots, d \), be defined in (3.15) and (3.17), respectively. Then we have
\[
\int_{Q;_i(z_0)} |D\tilde{u}| dz \leq NC_0,
\]
and
\begin{equation}
\int_{Q_i^{-}\left(\epsilon_0\right)} |f_i| \, dz \leq NC_0,
\end{equation}
where \( N \) depends on \( n, d, m, p, \nu, |A|_{(1+\delta)^2,1+\delta;(-1,0)\times \overline{D}_f} \), and the \( C^{2+\mu} \) norm of \( h_j \), and
\begin{equation}
C_0 := \sum_{j=1}^{m+1} [Du]_{1/2,1;Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f)} + \sum_{j=1}^{M} |f|_{(1+\delta)^2,1+\delta;(-1,0)\times \overline{D}_f} + \|Du\|_{L_1(Q)}.
\end{equation}

**Proof.** Step 1. Proof of (3.20). It follows from (3.15) and (3.5) that
\begin{equation}
\int_{Q_i^{-}\left(\epsilon_0\right)} |Du| \, dz \leq \sum_{j=1}^{m+1} \|D^2 u\|_{L_\infty(Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f))} + \|Du\|_{L_\infty(Q_i^{-}\left(\epsilon_0\right))}
\end{equation}
\begin{equation}
+ \int_{Q_i^{-}\left(\epsilon_0\right)} \left| D\ell^k Du - \sum_{j=1}^{m+1} D\ell^k_j Du(t_0, P_j x_0) \right| \, dz.
\end{equation}
For each \( i = 1, \ldots, m + 1 \),
\begin{equation}
\|D\ell^k Du - \sum_{j=1}^{m+1} D\ell^k_j Du(t_0, P_j x_0)\|_{L_1(Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f))}
\end{equation}
\begin{equation}
\leq N\|D\ell^k(Du - Du(t_0, P_j x_0))\|_{L_1(Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f))}
\end{equation}
\begin{equation}
+ N\|\sum_{j=1, j\neq i}^{m+1} D\ell^k_j Du(t_0, P_j x_0)\|_{L_1(Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f))}.
\end{equation}
To estimate the right-hand side of (3.24), we may assume that \( Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f) \neq \emptyset \). It follows from the definition of \( P_j x_0 \) in (3.6) that if \( i > j_0 \), then \( P_i x_0 = (x_0', h_{i-1}(x_0')) \) and
\begin{equation}
|x_0 - P_i x_0| = |x_0' - h_{i-1}(x_0')| \leq Nr.
\end{equation}
The case of \( i \leq j_0 \) is proved similarly. Thus, we obtain
\begin{equation}
|x - P_i x_0| \leq |x - x_0| + |x_0 - P_i x_0| \leq Nr
\end{equation}
and
\begin{equation}
|Du(t, x) - Du(t_0, P_i x_0)| \leq Nr|Du|_{1/2,1;Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f)}.
\end{equation}
With this, we have
\begin{equation}
\|D\ell^k(Du - Du(t_0, P_j x_0))\|_{L_1(Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f))}
\end{equation}
\begin{equation}
\leq Nr^2|Du|_{1/2,1;Q_i^{-}\left(\epsilon_0\right)\cap((-1+\epsilon,0)\times \overline{D}_f)} \int_{B_i(x_0)\cap \overline{D}_f} |D\ell^k| \, dx.
\end{equation}
By using Lemma 2.1(iii) and min\{a, b\} ≤ √ab for a, b ≥ 0, we have
\begin{equation}
(3.26) \quad \int_{B_r(x_0) \cap D_i} |D\tilde{t}^k| \, dx \leq N \int_{B_r(x_0)} \frac{\min\{2r, h_i - h_{i-1}\}}{|h_i - h_{i-1}|^{1/2}} \, dx' \leq Nr^{d-1/2}.
\end{equation}
Thus, substituting (3.26) into (3.25), we obtain
\begin{equation}
(3.27) \quad \|D\tilde{t}^k (Du - Du(t_0, P_j x_0))\|_{L^1(Q_r'(x_0) \cap ((-1+\varepsilon, 0) \times D))} \leq Nr^{d+5/2} \|Du\|_{L^1(Q_r'(x_0) \cap ((-1+\varepsilon, 0) \times D))}.
\end{equation}
Making use of the fact that \(\tilde{t}_{ij}\) is the smooth extension of \(t|D_j\) to \(U_{k=1}^{m+1} D_k\) and the local boundedness of \(Du\) in Lemma A.1, we have for each \(i = 1, \ldots, m+1\),
\begin{equation}
(3.28) \quad \|D^k_{ij} Du(t_0, P_j x_0)\|_{L^1(Q_r'(x_0) \cap ((-1+\varepsilon, 0) \times D))} \leq Nr^{d+2}\left(\|Du\|_{L^1(Q)} + \sum_{j=1}^M |f|_{(1+\delta)/2, 1+\delta((-1,0) \times \overline{D})}\right).
\end{equation}
Substituting (3.27) and (3.28) into (3.24), we obtain
\begin{equation}
(3.29) \quad \|D\tilde{t}^k Du - \sum_{j=1}^{m+1} D\tilde{t}^k_{ij} Du(t_0, P_j x_0)\|_{L^1(Q_r'(x_0) \cap ((-1+\varepsilon, 0) \times D))} \leq Nr^{d+2}\left(\|Du\|_{L^1(Q)} + \sum_{j=1}^M |f|_{(1+\delta)/2, 1+\delta((-1,0) \times \overline{D})}\right).
\end{equation}
Substituting (3.13) and (3.29) into (3.23), we obtain (3.20).

**Step 2. Proof of (3.21).** By (3.9) and (3.17), we have
\begin{equation}
\begin{aligned}
\int_{Q_r'(x_0)} |\tilde{f}| \, dz &\leq N \sum_{j=1}^M |f|_{(1+\delta)/2, 1+\delta((-1,0) \times \overline{D})} + \|Du\|_{L^1(Q)} \\
&\quad + N \int_{Q_r'(x_0)} |D^k_{ij} Du - \sum_{j=1}^{m+1} 1_{(-1,0) \times D_j} D^k_{ij} Du(t_0, P_j x_0)| \, dz \\
&\quad + \int_{Q_r'(x_0)} |D_{ij} Du - \sum_{j=1}^m 1_{(-1,0) \times D_j} D_{ij} Du(t_0, P_j x_0)| \, dz.
\end{aligned}
\end{equation}
Using Lemma 2.1(iii) and similar to the proof of (3.27), we deduce that
\begin{equation}
\begin{aligned}
\int_{Q_r'(x_0)} |D^k_{ij} Du - \sum_{j=1}^{m+1} 1_{(-1,0) \times D_j} D^k_{ij} Du(t_0, P_j x_0)| \, dz
\end{aligned}
\end{equation}
Proposition 3.3. Under the same assumptions as in Proposition 3.1. For any
\( \phi \) proved. The proof of the lemma is complete. Considering (3.30) and using (3.31) and (3.32), the estimate (3.21) is

\[ \int_{Q_r^c(z_0)} \left( \sum_{j=1}^{m+1} \left| D_{\beta_j} \tilde{h}_j(t, x') \right|^2 \right) dz \leq N \left( \| Du \|_{L^4(Q)} + \sum_{j=1}^{M} | f_j(1+\delta)/(2,1+\delta;(-1,0)\times \partial J) | \right), \]

Furthermore, by (2.2) and (3.3), we have

\[ \int_{Q_r^c(z_0)} \left( \sum_{j=1}^{m+1} \left| D_{\beta_j} \tilde{h}_j(t, x') \right|^2 \right) dz \leq N \left( \| Du \|_{L^4(Q)} + \sum_{j=1}^{M} | f_j(1+\delta)/(2,1+\delta;(-1,0)\times \partial J) | \right), \]

Coming back to (3.30) and using (3.31) and (3.32), the estimate (3.21) is

Denote

\[ \Phi(z_0, r) := \inf_{Q_r(z_0)} \left( \int_{Q_r(z_0)} \left( | D_{\beta_k} \tilde{u}(z; z_0) - q^\| \right|^2 + | \tilde{U}(z; z_0) - Q^\| \right) dz \right)^2, \]

where \( \tilde{u} \) and \( \tilde{U} \) are defined in (3.15) and (3.18), respectively. We shall establish a decay estimate of \( \Phi(z_0, r) \). Before this, we first set

\[ \tilde{u}(t, y; \Lambda z_0) = \tilde{u}(t, x; z_0), \quad \mathcal{A}^{\alpha\beta}(t, y) = \Lambda^{\alpha k} A^{k\ell}(t, x) \Lambda^{\ell\beta}, \]

\[ \tilde{\nu}^a(t, y; \Lambda z_0) = \Lambda^{ak} \tilde{\nu}^a(t, x; z_0), \quad g(t, y) = g(t, x), \]

where \( y = \Lambda x \) and \( \Lambda = (\Lambda^{\alpha\beta})_{a,b=1}^d \) is a \( d \times d \) orthogonal matrix representing the linear transform from the coordinate system associated with 0 to the coordinate system associated with \( x_0 \) defined in Section 2.3. Then we have from (3.16) that \( \tilde{u} \) satisfies

\[ -\tilde{u}_t + D_{\alpha} (\mathcal{A}^{\alpha\beta} D_{\beta} \tilde{u}) = g + D_{\alpha} \tilde{\nu}^a \text{ in } \Lambda(Q_{3/4}^c), \]

where \( \Lambda(Q_{3/4}^c) := (-9/16,0) \times \Lambda(B_{3/4}) \). Denote

\[ \phi(\Lambda z_0, r) := \inf_{Q_{\Lambda z_0}} \left( \int_{Q_{\Lambda z_0}} \left( | D_{y^\alpha} \tilde{u}(z; \Lambda z_0) - q^\| \right|^2 + | \mathcal{A}^{\alpha\beta} D_{y^\alpha} \tilde{u}(z; \Lambda z_0) - \tilde{\nu}^a - Q^\| \right) dx \right)^2. \]

Then we have the following decay estimate of \( \phi(\Lambda z_0, r) \).

**Proposition 3.3.** Under the same assumptions as in Proposition 3.1. For any

\[ 0 < \rho \leq r \leq 1/4, \text{ we have} \]

\[ \phi(\Lambda z_0, \rho) \leq N \left( \frac{1}{r^4} \phi(\Lambda z_0, r/2) + N \rho \phi(\Lambda z_0, r/2) \right), \]

where \( C_0 \) is a constant depending only on \( \Lambda \).
where $C_0$ is defined in (3.22), $\mu' = \min \left\{ \frac{1}{x}, \mu \right\}$. $N$ depends on $n, d, m, p, v$, the $C^{2+\mu}$ norm of $h_j$, and $|A|_{(1+\delta)/2,1+\delta(-1,0)\times D_j}$.

The proof of Proposition 3.3 will be given later. We first use it to prove a decay estimate of $\Phi(z_0, r)$.

**Lemma 3.4.** Under the same assumptions as in Proposition 3.3 For any $0 < \rho \leq r \leq 1/4$, we have

$$
\Phi(z_0, \rho) \leq N\left(\frac{\rho}{r}\right)^{\mu'} \Phi(z_0, r/2) + N\rho^\mu C_0,
$$

where $C_0$ is defined in (3.22), $\mu' = \min \left\{ \frac{1}{x}, \mu \right\}$. $N$ depends on $n, d, m, p, v$, the $C^{2+\mu}$ norm of $h_j$, and $|A|_{(1+\delta)/2,1+\delta(-1,0)\times D_j}$.

**Proof.** Let $y_0$ be as in Section 2.3. Note that

$$
D_k\hat{u}(t, x; z_0) - D_k\hat{u}(t, y; \Lambda z_0) = (\ell^k(x) - \tau_k) \cdot D\hat{u}(t, x; z_0),
$$

and

$$
\hat{U}(t, x; 0) = \mathcal{A}^{\beta\mu}(t, y)D_{y\mu}\hat{u}(t, y; \Lambda z_0) + \hat{\nu}(t, y; \Lambda z_0)
= (n^a - n^a_{y_0})(\mathcal{A}^\beta(t, x)D_{\beta}\hat{u}(t, x; z_0) - \hat{\nu}(t, x; z_0)),
$$

where $\tau_k$ and $n^a_{y_0}$ are defined in (2.15) and (2.14), respectively. For any $x \in B_r(x_0) \cap D_j$, where $r \in (|x_0 - y_0|, 1)$ and $j = 1, \ldots, m + 1$, we shall first estimate $|\ell^{k,0}_d(x) - D_k h_{j_0}(y_0')|$ according to the following three cases:

**Case 1.** If $B_r(x_0) \cap \Gamma_j \neq \emptyset$ and $B_r(x_0) \cap \Gamma_{j-1} \neq \emptyset$, then by using (2.6), we obtain

$$
|D_k h_j(x') - D_k h_{j-1}(x')| \leq N|h_j(x') - h_{j-1}(x')|^1/2 \leq N \sqrt{r}.
$$

Thus, by the triangle inequality,

$$
|\ell^{k,0}_d(x) - D_k h_{j_0}(y_0')| \leq |\ell^{k,0}_d(x) - D_k h_j(x')| + |D_k h_j(x') - D_k h_{j_0}(y_0')| \leq \left|\frac{h_j - x^d}{h_j - h_{j-1}} D_k(h_{j-1} - h_j)(x') + |D_k h_j(x') - D_k h_{j_0}(x')| + Nr \leq N \sqrt{r}.
$$

**Case 2.** If $B_r(x_0) \cap \Gamma_j \neq \emptyset$ and $B_r(x_0) \cap \Gamma_{j-1} = \emptyset$ (see Figure 2), then $|h_j(x') - x^d| \leq Nr$, and by (3.36), we have

$$
|\ell^{k,0}_d(x) - D_k h_{j_0}(y_0')| \leq |\ell^{k,0}_d(x) - D_k h_j(x')| + |D_k h_j(x') - D_k h_{j_0}(y_0')| \leq \left|\frac{h_j - x^d}{h_j - h_{j-1}} D_k(h_{j-1} - h_j)(x') + N \sqrt{r} \leq N|h_j - x^d|/2 + N \sqrt{r} \leq N \sqrt{r}.
$$

$$
|\ell^{k,0}_d(x) - D_k h_{j_0}(y_0')| \leq \left|\frac{h_j - x^d}{h_j - h_{j-1}} D_k(h_{j-1} - h_j)(x') + N \sqrt{r} \leq N|h_j - x^d|/2 + N \sqrt{r} \leq N \sqrt{r}.
$$
Figure 2. Illustration of Case 2

Case 3. If \( B_r(x_0) \cap \Gamma_j = \emptyset \) and \( B_r(x_0) \cap \Gamma_{j-1} \neq \emptyset \), then similar to Case 2, by (3.36),

\[
|\ell_d^{k,0}(x) - D_k h_0(y')| \leq N \sqrt{r}.
\]

Combining (3.37)–(3.39), we have

\[
|\ell_d^{k,0}(x) - D_k h_0(y')| \leq N \sqrt{r}.
\]

Thus, recalling (2.4), (2.15), (2.5), and (2.14), we obtain

\[
|\ell^k(x) - \tau_k| \leq N \sqrt{r}, \quad |\nu(x) - n_{y_0}| \leq N \sqrt{r},
\]

where \( k = 1, \ldots, d - 1 \). This together with (3.35) gives

\[
|D_\rho \bar{u}(t, x; z_0) - D_\rho \bar{u}(t, y; \Lambda z_0)| \leq N \sqrt{r}|D\bar{u}(t, x; z_0)|,
\]

(3.40)

\[
|\bar{U}(t, x; z_0) - \mathcal{A}^{d\rho}t (t, y)D_\rho \bar{u}(t, y; \Lambda z_0) + \bar{v}(t, y; \Lambda z_0)|
\]

\[
\leq N \sqrt{r}(|D\bar{u}(t, x; z_0)| + |\bar{v}(t, x; z_0)|).
\]

Now by using the triangle inequality, (3.40), (3.20), and (3.21), we have

\[
\left( \int_{Q^\rho(z_0)} \left( |D_{\rho'} \bar{u}(z; z_0) - q^k| + |\bar{U}(z; z_0) - Q^k| \right) dz \right)^2
\]

\[
\leq \left( \int_{Q^\rho(\Lambda z_0)} \left( |D_{\rho'} \bar{u}(t; y; \Lambda z_0) - q^k| + |\bar{U}(t; y; \Lambda z_0) - Q^k| \right) dt dy \right)^2 + N \sqrt{r}C_0,
\]

where \( 0 < \rho \leq r \leq 1/4 \) and \( C_0 \) is defined in (3.22). Since \( q^k, Q \in \mathbb{R}^n \) are arbitrary, we obtain

\[
\Phi(z_0, \rho) \leq \phi(\Lambda z_0, \rho) + N \sqrt{r}C_0.
\]
This, in combination with Proposition 3.3, leads to that

\[ \Phi(z_0, \rho) \leq N \left( \frac{\rho}{r} \right)^{\mu'} \Phi(\Lambda z_0, r/2) + N \rho^{\mu'} C_0. \]

Similarly, we have

\[ \phi(\Lambda z_0, r/2) \leq \Phi(z_0, r/2) + N \sqrt{C_0}. \]

Substituting it into (3.41) and using \( \mu' \leq 1/2 \), we obtain

\[ \Phi(z_0, \rho) \leq N \left( \frac{\rho}{r} \right)^{\mu'} \Phi(z_0, r/2) + N \rho^{\mu'} C_0. \]

The lemma is proved. \( \square \)

The rest of this section is to prove Proposition 3.3. For this, we define

\[ \tilde{A}_{\alpha\beta}^j = \tilde{\eta} \tilde{A}_{\alpha\beta}^j(y') + \nu(1 - \tilde{\eta}) \delta_{\alpha\beta} \delta_{ij}, \]

where \( \tilde{A}_{\alpha\beta}^j(y') \) are piecewise constant functions corresponding to \( A_{\alpha\beta} \) (cf. p. 13), \( y = \Lambda x, \Lambda = (\Lambda_i^j \delta_{\alpha\beta})_{\alpha, \beta=1} \) is a \( d \times d \) orthogonal matrix defined in Section 2.3 and \( \tilde{\eta} \in C_0^\infty(B_r(x_0)) \) satisfies

\[ 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{2r/3}(x_0), \quad |D\eta| \leq 6/r. \]

We have the following weak type-(1, 1) estimate (cf. [11] Lemma 4.3).

**Lemma 3.5.** Let \( p \in (1, \infty) \). Let \( v \in \mathcal{H}^1_p (Q_{r/2}^{-}(\Lambda z_0)) \) be a weak solution to the problem

\[
\begin{cases}
-\nu t + D_\alpha (\tilde{A}_{\alpha\beta}^j D_\beta v) = G^1_{r/2} Q_{r/2}^{-}(\Lambda z_0) + \text{div}(F^1_{r/2} Q_{r/2}^{-}(\Lambda z_0)) & \text{in } Q_{r/2}^{-}(\Lambda z_0), \\
v = 0 & \text{on } \partial_p Q_{r/2}^{-}(\Lambda z_0),
\end{cases}
\]

where \( F, G \in L_p(Q_{r/2}^{-}(\Lambda z_0)) \). Then for any \( s > 0 \), we have

\[ ||z \in Q_{r/2}^{-}(\Lambda z_0) : |Dv(z)| > s|| \leq \frac{N}{s} \left( ||F||_{L_1(Q_{r/2}^{-}(\Lambda z_0))} + r ||G||_{L_1(Q_{r/2}^{-}(\Lambda z_0))} \right), \]

where \( N = N(n, d, p, v) \).

We shall choose suitable \( F \) and \( G \) in order to apply Lemma 3.5. We first denote

\[ \tilde{f}_1^\alpha(t, x; z_0) := A_{\alpha\beta}^j (D_\beta f_i D_i u - \sum_{j=1}^{m+1} \frac{1_{(-1,0) \times D_j}}{D_\beta f_i D_i u(t_0, P_j x_0)}), \]

and

\[ \tilde{f}_2^\alpha(t, x) := D_\ell f^\alpha - D_\ell A_{\alpha\beta}^j D_\beta u + \delta_{\alpha d} \sum_{j=1}^m \frac{1_{x^j, \delta_b(x')}(h^\ell_j(x'))^{-1} H_j(t, x')} \]

Then \( \tilde{f}_1^\alpha + \tilde{f}_2^\alpha = \tilde{f}^\alpha \) which is defined in (3.17). Moreover, it follows from \( f^\alpha, A_{\alpha\beta}^j \in C_0^{1,1+\delta}((-1+\eta, 0) \times D_j), D_\ell h_1^\ell \in C^\mu \) together with the definition of
\[ \hat{h}_j(t, x') \] in (3.3), and the fact that the vector field \( \ell \) is \( C^{1/2} \), which is proved in Lemma 2.1(i), that

\[ \tilde{f}_2^\alpha \in C^{\mu'/2+\mu'}((-1+\varepsilon, 0) \times \overline{D_j}), \quad \mu' = \min \left\{ \frac{1}{2}, \mu \right\}. \]

Define

\[ \tilde{t}_1(t, y; \Lambda z_0) = \Lambda^\alpha \tilde{f}_1^\alpha(t, x; z_0), \quad \tilde{t}_2(t, y) = \Lambda^\alpha \tilde{f}_2^\alpha(t, x). \]

Then \( \tilde{t}_1 + \tilde{t}_2 = \tilde{t}^\alpha \) which is defined in (3.33) and \( \tilde{t}_2 \) is also piecewise \( C^{\mu'/2+\mu'} \).

Now we choose

\[ F^\alpha := F^\alpha(t, y; \Lambda z_0) = (A^{\alpha\beta}(y^\ell) - A^{\alpha\beta}(t, y))D_yg(t, y; \Lambda z_0) + \tilde{t}_1^\alpha(t, y; \Lambda z_0) \]

(3.44)

\[ + \tilde{t}_2^\alpha(t, y), \quad F = (F^1, \ldots, F^d), \]

and

(3.45)

\[ G = g(t, y) = g(t, x), \]

where \( \tilde{t}_2^\alpha(y^\ell) \) is a piecewise constant function corresponding to \( \tilde{t}_2^\alpha(t, y) \) and \( g(t, x) \) is defined in (3.2). The following result holds.

**Lemma 3.6.** Let \( F \) and \( G \) be defined as in (3.44) and (3.45), respectively. Then we have

(3.46)

\[ \|F\|_{L^1(Q; (\Lambda z_0))} \leq N\rho^{d+2+\mu'} C_0 \]

and

\[ \|G\|_{L^1(Q; (\Lambda z_0))} \leq \sum_{j=1}^{M+1} \|D^2 u\|_{L^1(Q; (\Lambda z_0) \cap ((-1+\varepsilon, 0) \times \overline{D_j}))} \]

\[ + \sum_{j=1}^M \|f(1+\delta)/(2,1+\delta(-1,0) \times \overline{D_j}) + \|Du\|_{L^1(Q)}, \]

where \( C_0 \) is defined in (3.22), \( \mu' = \min \left\{ \frac{1}{2}, \mu \right\} \), \( N \) depends on \( |A(1+\delta)/(2,1+\delta(-1,0) \times \overline{D_j})|, n, d, p, m, v, \) and the \( C^{2+\mu} \) norm of \( h_j \).

**Proof.** Combining (3.42) and \( A^{ks}(t, x) = \Gamma^{ks} A^{\alpha\beta}(t, y) \Gamma^{\beta s} \), where \( \Gamma = \Lambda^{-1} \), we have

\[ \tilde{t}_1^\alpha(t, y; \Lambda z_0) = \Lambda^\alpha \tilde{f}_1^\alpha(t, x; z_0) \]

\[ = \Lambda^\alpha A^{ks}(D_s\ell_iD_iu - \sum_{j=1}^{M+1} \mathbbm{1}_{(-1,0) \times D_j} D_s\ell_iD_iu(t_0, P_jx_0)) \]

(3.47)

\[ = A^{\alpha\beta}(t, y) \Gamma^{\beta s}(D_s\ell_iD_iu - \sum_{j=1}^{M+1} \mathbbm{1}_{(-1,0) \times D_j} D_s\ell_iD_iu(t_0, P_jx_0)). \]
From $\tilde{u}(t; y; \Lambda z_0) = \check{u}(t; x; z_0)$ in (3.33), one has
\[
(\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(t; y))D_\beta \check{u}(t; y; \Lambda z_0) = (\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(t; y))\Gamma^\beta D_\beta \check{u}(t; x; z_0).
\]
By using (3.5) and (3.15), we have
\[
D_\beta \check{u}(t; x; z_0) = \ell_i D_\beta D_i u - D_\beta \ell_i D_i u - \sum_{j=1}^{m+1} D_\beta \tilde{t}_{i,j} D_i u(t_0, P_j x_0).
\]
These together with (3.47) give
\[
(\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(t; y))D_\beta \check{u}(t; y; \Lambda z_0) + \tilde{v}_1(t; y; \Lambda z_0)

= (\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(t; y))\Gamma^\beta D_\beta \check{u}(t; x; z_0) + \Lambda^\alpha \tilde{f}_1(t; x; z_0)

= (\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(t; y))\Gamma^\beta (\ell_i D_\beta D_i u - D_\beta \ell_i D_i u - \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times \Gamma} D_\beta \tilde{t}_{i,j} D_i u(t_0, P_j x_0))

+ (\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(z))\Gamma^\beta (D_\beta \ell_i D_i u - \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times \Gamma} D_\beta \tilde{t}_{i,j} D_i u(t_0, P_j x_0)) =: I_1 + I_2.
\]
Similar to (2.18), in view of $\mathcal{A} \in C^{(1+\delta)/2,1+\delta}((-1+\varepsilon, 0) \times (\mathcal{D}_\varepsilon \cap \overline{\mathcal{D}}))$ and (2.16), we obtain
\[
\|\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(z)\|_{L_1(Q_\varepsilon'(\Lambda z_0))} \leq N_\delta^{d+\frac{1}{\gamma}}.
\]
From this together with (3.13), Lemma A.1 and the fact that $\mathbb{1}_{(-1,0) \times \mathcal{D}} D_\beta \tilde{t}_{i,j}$ is piecewise $C^\delta$, it follows that
\[
\|I_1\|_{L_1(Q_\varepsilon'(\Lambda z_0))}

\leq \|\overline{\mathcal{A}_\beta^g(y)} - \mathcal{A}_\beta^p(z)\|_{L_1(Q_\varepsilon'(\Lambda z_0))}

\cdot \|\ell_i D_\beta D_i u - D_\beta \ell_i D_i u - \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times \Gamma} D_\beta \tilde{t}_{i,j} D_i u(t_0, P_j x_0)\|_{L_2(Q_\varepsilon'(z_0))}

\leq N_\delta^{d+\frac{1}{\gamma}} \left(\sum_{j=1}^{m+1} \|D^2 u\|_{L_2(Q_\varepsilon'(z_0) \cap ((-1+\varepsilon, 0) \times \mathcal{D}_\varepsilon))} + \sum_{j=1}^{M} |\mathcal{J}|_{(1+\delta)/2,1+\delta((-1,0) \times \overline{\mathcal{D}})}\right)

+ \|D u\|_{L_1(\mathcal{D})}.
\]
By using a similar argument that led to (3.27), we have
\[
\|I_2\|_{L_1(Q_\varepsilon'(\Lambda z_0))} \leq N \sum_{j=1}^{m+1} \|D \ell_i (D_i u - D_i u(t_0, P_j x_0))\|_{L_1(Q_\varepsilon'(z_0) \cap ((-1+\varepsilon, 0) \times \mathcal{D}_\varepsilon))}

\leq N_\delta^{d+\frac{1}{\gamma}} \sum_{j=1}^{m+1} [D u]_{1/2,1}(Q_\varepsilon'(z_0) \cap ((-1+\varepsilon, 0) \times \overline{\mathcal{D}})},
\]
where $[D u]_{1/2,1}$ denotes the parabolic Hölder seminorm.
Now coming back to (3.48) and using (3.49) and (3.50), we obtain
\begin{equation}
\| (A_t \beta^\beta (y^\beta) - A_\beta \beta^\beta ) D_{y^\beta} \tilde{u} - \tilde{y}_2 \|_{L_1(Q_r^-(\Lambda z_0))} \leq N r^{d+\frac{5}{2}} C_0.
\end{equation}
Since \( \tilde{y}_2 \in C^{\mu'/2, \mu'}((-1 + \varepsilon, 0) \times \overline{D}) \), the estimate (2.18) also holds for \( \tilde{y}_2 \) and thus
\begin{equation}
\int_{Q_r^-(\Lambda z_0)} | \tilde{y}_2 (t, y) - \tilde{y}_2 (y^\beta) | \leq N r^{d+2+\mu'} \left( \sum_{j=1}^M | f |_{(1+\delta)/2, 1+\delta((-1,0) \times \overline{D})} + \| D u \|_{L_1(q)} \right),
\end{equation}
where \( \mu' = \min \left\{ \frac{1}{2}, \mu \right\} \). Combining (3.51) and (3.52), we have (3.46).

Next we show the estimate of \( \| G \|_{L_1(Q_r^-(\Lambda z_0))} \). We obtain from (3.26) that
\begin{equation}
\| D u \|_{L_1(Q_r^-(\Lambda z_0))} \leq N r^{d+\frac{5}{2}}.
\end{equation}
Then we have
\begin{equation}
\| G \|_{L_1(Q_r^-(\Lambda z_0))} \leq N r^{d+\frac{5}{2}} \left( \sum_{j=1}^{M+1} \| D^2 u \|_{L_\infty(Q_r^-(\Lambda z_0))} \right) + \sum_{j=1}^M | f |_{(1+\delta)/2, 1+\delta((-1,0) \times \overline{D})} + \| D u \|_{L_1(q)} \right).
\end{equation}

The lemma is proved.

\[ \square \]

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. Recall that \( \tilde{v} \) in Lemma 3.5 satisfies
\begin{equation}
\begin{cases}
- \tilde{v}_t + D_{\alpha} \overrightarrow{A}^\beta D_\beta \tilde{v} = G 1_{Q_{r/2}^- (\Lambda z_0)} + \text{div} F 1_{Q_{r/2}^- (\Lambda z_0)} & \text{in } Q_{r/2}^- (\Lambda z_0), \\
\tilde{v} = 0 & \text{on } \partial_p Q_{r/2}^- (\Lambda z_0),
\end{cases}
\end{equation}
where \( F \) and \( G \) are given in (3.44) and (3.45), respectively. Similar to [11 (4.8)], by using Lemmas 3.5 and 3.6 we have
\begin{equation}
\left( \int_{Q_{r/2}^- (\Lambda z_0)} | D \tilde{v} |^2 d z \right)^2 \leq N r^{d'} C_0,
\end{equation}
where \( C_0 \) is defined in (3.22). Let
\begin{equation}
u_1 (y^\beta) = \int_{\Lambda z_0}^{y^\beta} (\overrightarrow{A}^\beta d (s))^{-1} \tilde{y}_2 (s) d s \quad \text{and} \quad w := w (z; \Lambda z_0) = \tilde{u} (z; \Lambda z_0) - u_1 - \tilde{v},
\end{equation}
where \( \tilde{y}_2 \) is the piecewise constant function corresponding to \( \tilde{y}_2 \) defined in (3.43). Then \( w \) satisfies
\begin{equation}
- w_1 + D_{\alpha} \overrightarrow{A}^\beta (y^\beta) D_\beta (w) = 0 \quad \text{in } Q_{r/2}^- (\Lambda z_0),
\end{equation}
Since the coefficient \( \overline{A^{q\beta}}(y^\beta) \) only depends on \( y^\beta \), for any \( \kappa \in (0, 1/2) \) to be fixed later, by Lemma \([A.4]\), we have
\[
\|D_y^{\alpha'}w(\cdot; \Lambda z_0) - (D_y^{\alpha'}w)_{Q_{\nu}(\Lambda z_0)}\|_{L^{1/2}(\Omega_{1/2}^{1/2}(\Lambda z_0))}^{1/2} \\
+ \|W(\cdot; \Lambda z_0) - (W)_{Q_{\nu}(\Lambda z_0)}\|_{L^{1/2}(\Omega_{1/2}^{1/2}(\Lambda z_0))}^{1/2} \\
\leq N(\kappa r)^{d+5/2} \left( \|D_y^{\alpha'}w\|_{1/2,1; Q_{\nu}(\Lambda z_0)}^{1/2} + \|W\|_{1/2,1; Q_{\nu}(\Lambda z_0)}^{1/2} \right) \\
\leq N\kappa^{d+5/2} \int_{Q_{\nu}(\Lambda z_0)} |(D_y^{\alpha'}w(z; \Lambda z_0), W(z; \Lambda z_0))|^{1/2} \, dz,
\]
where \( W = \overline{A^{q\beta}}(y^\beta)D_y^{\alpha'}w(z; \Lambda z_0) \). Define
\[
h(y^\beta) := \int_0^{y^\beta} \left( \overline{A^{d\beta}}(s) \right)^{-1} (Q - \sum_{\beta=1}^{d-1} \overline{A^{q\beta}}(s)q^\beta) \, ds
\]
and
\[
\tilde{w} := w - \sum_{\beta=1}^{d-1} y^\beta q^\beta - h(y^\beta),
\]
where \( q^\beta, Q \in \mathbb{R}^n, \beta = 1, \ldots, d - 1 \). Then
\[
-\tilde{w}_t + D_\alpha \overline{A^{q\beta}}(y^\beta)D_\beta \tilde{w} = 0 \quad \text{in } Q_{\nu/2}(\Lambda z_0),
\]
and
\[
D_\alpha \tilde{w} = D_\alpha^{\alpha'} w - q^\nu, \quad \tilde{W} := \overline{A^{q\beta}}(y^\beta)D_\beta \tilde{w} = W - Q.
\]
Replacing \( w \) and \( W \) with \( \tilde{w} \) and \( \tilde{W} \) in (3.55), respectively, we have
\[
\|D_y^{\alpha'}w(\cdot; \Lambda z_0) - (D_y^{\alpha'}w)_{Q_{\nu}(\Lambda z_0)}\|_{L^{1/2}(\Omega_{1/2}^{1/2}(\Lambda z_0))}^{1/2} \\
+ \|W(\cdot; \Lambda z_0) - (W)_{Q_{\nu}(\Lambda z_0)}\|_{L^{1/2}(\Omega_{1/2}^{1/2}(\Lambda z_0))}^{1/2} \\
\leq N\kappa^{d+5/2} \int_{Q_{\nu}(\Lambda z_0)} |(D_y^{\alpha'}w(z; \Lambda z_0) - q^\nu, W(z; \Lambda z_0) - Q)|^{1/2} \, dz,
\]
which implies
\[
\left( \int_{Q_{\nu}(\Lambda z_0)} \left( |D_y^{\alpha'}w(z; \Lambda z_0) - (D_y^{\alpha'}w)_{Q_{\nu}(\Lambda z_0)}|^{1/2} + |W(z; \Lambda z_0) - (W)_{Q_{\nu}(\Lambda z_0)}|^{1/2} \right) \, dz \right)^2 \\
\leq N\kappa \left( \int_{Q_{\nu}(\Lambda z_0)} \left( |D_y^{\alpha'}w(z; \Lambda z_0) - q^\nu|^{1/2} + |W(z; \Lambda z_0) - Q|^{1/2} \right) \, dz \right)^2.
\]
By the identity \( \tilde{u} = w + u_1 + v \) and (3.54), we have
\[
D_y^{\alpha'} \tilde{u} = D_y^{\alpha'} \tilde{w} + D_y^{\alpha'} v
\]
and
\[ \mathcal{A}^{d} D_{y^d} \tilde{u} - \tilde{r}^d = (\mathcal{A}^{d} - \mathcal{A}^{d}(y^d)) D_{y^d} \tilde{u} + \mathcal{A}^{d}(y^d) D_{y^d} w + \mathcal{A}^{d}(y^d) D_{y^d} v + \tilde{r}^d(y^d) - \tilde{r}^d, \]
where \( F^d = (\mathcal{A}^{d}(y^d) - \mathcal{A}^{d}) D_{y^d} \tilde{u} - \tilde{r}^d(y^d) + \tilde{r}^d. \) Then we have from the triangle inequality, (3.53), and (3.56) that
\[
\left( \int_{Q_{y^d}(\Lambda z_0)} \left( |D_{y^d} \tilde{u}(z; \Lambda z_0) - (D_{y^d} w)_{Q_{y^d}(\Lambda z_0)}| \right)^{2} \right) ^{1/2} + \left| \mathcal{A}^{d} D_{y^d} \tilde{u}(z; \Lambda z_0) - \tilde{r}^d(z; \Lambda z_0) - (W)_{Q_{y^d}(\Lambda z_0)} \right|^{1/2} dz \right) \leq N \kappa \left( \int_{Q_{y^d}(\Lambda z_0)} \left( |D_{y^d} \tilde{u}(z; \Lambda z_0) - (D_{y^d} w)_{Q_{y^d}(\Lambda z_0)}| \right)^{2} \right) ^{1/2} + N \kappa^{-2(d+2)} r^{\mu'} C_0.
\]
Using (3.46), we deduce
\[
\left( \int_{Q_{y^d}(\Lambda z_0)} \left( |D_{y^d} \tilde{u}(z; \Lambda z_0) - (D_{y^d} w)_{Q_{y^d}(\Lambda z_0)}| \right)^{2} \right) ^{1/2} + \left| \mathcal{A}^{d} D_{y^d} \tilde{u}(z; \Lambda z_0) - \tilde{r}^d(z; \Lambda z_0) - (W)_{Q_{y^d}(\Lambda z_0)} \right|^{1/2} dz \right) \leq N \kappa \left( \int_{Q_{y^d}(\Lambda z_0)} \left( |D_{y^d} \tilde{u}(z; \Lambda z_0) - (D_{y^d} w)_{Q_{y^d}(\Lambda z_0)}| \right)^{2} \right) ^{1/2} + N \kappa^{-2(d+2)} r^{\mu'} C_0.
\]
Since \( q^{\mu'}, Q \in \mathbb{R}^n \) are arbitrary, we deduce that
\[
\phi(\Lambda z_0, \kappa r) \leq N_0 \kappa \phi(\Lambda z_0, r/2) + N \kappa^{-2(d+2)} r^{\mu'} C_0.
\]
Choosing \( \kappa \in (0, 1/2) \) small enough so that \( N_0 \kappa \leq \kappa^{\gamma} \) for any fixed \( \gamma \in (\mu', 1) \) and iterating, we have
\[
\phi(\Lambda z_0, \kappa^{j} r) \leq \kappa^{j\mu'} \phi(\Lambda z_0, r/2) + N \kappa^{j(1-\mu')} C_0.
\]
Hence, we have for any \( \rho \) with \( 0 < \rho \leq r \leq 1/4 \) and \( \kappa^{j}/r \leq \rho < \kappa^{j-1} \),
\[
\phi(\Lambda z_0, \rho) \leq N \left( \frac{\rho}{r} \right)^{\mu'} \phi(\Lambda z_0, r/2) + N \rho^{\mu'} C_0.
\]
The lemma is proved. \( \square \)
Lemma 3.7. Under the same assumptions as in Proposition 3.1, we have

\[ u_{\delta t}(3.58) \]

and the boundedness of \([Du]_{C^{2,\gamma}(\Omega)}\) and \([D^2u]_{L^\infty(\Omega)}\).

Proof. Define

\[ \delta_h^\gamma f(t,x) := \frac{f(t,x) - f(t-h,x)}{h}, \]

where \(\gamma \in \left(0, \frac{1+\delta}{2}\right)\) and \(h \in (0, \varepsilon)\). Then from (1.1), we have

\[ -(\delta_h^\gamma u)_t + D_\alpha(A^{\alpha\beta}D_\beta u) = D_\alpha(\delta_h^\gamma A^{\alpha\beta}D_\beta u(t-h,x)). \quad (3.58) \]

For any \((t_1, x_1), (t_2, x_2) \in (-1 + 2\varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{\Omega})\) with \((t_1, x_1) \neq (t_2, x_2)\), we have

\[ \frac{|\delta_h^\gamma f^\alpha(t_1, x_1) - \delta_h^\gamma f^\alpha(t_2, x_2)|}{|t_1 - s_1|^{1+\delta-2\gamma} + |x_1 - x_2|^{1+\delta-2\gamma}} \leq N[f^{\alpha}]_{C^{1+\delta}(\Omega)} \leq N[f^{\alpha}]_{C^{1+\delta}((-1, 0) \times \Omega)} \leq N[f^{\alpha}]_{C^{1+\delta}((-1, -\varepsilon) \times \Omega)} \leq N[f^{\alpha}]_{C^{1+\delta}((-1, -\varepsilon) \times \Omega)}. \]

This means \(\delta_h^\gamma f^\alpha \in C^{1-2\gamma, 1+\delta-2\gamma}((-1 + 2\varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{\Omega})).\) Similarly, we get \(\delta_h^\gamma A^{\alpha\beta} \in C^{1-2\gamma, 1+\delta-2\gamma}((-1 + 2\varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{\Omega})).\) Then by applying Lemma A.1 to (3.58), we get \(\delta_h^\gamma u \in C^{1+\delta_1}(1+\delta_1, 1+\delta_1)\) with \(\delta_1 := \min\{1 + \delta - 2\gamma, \frac{1}{2}\} > 0\). Moreover, (3.59)

\[ |\delta_h^\gamma u|_{C^{1+\delta_1}(1+\delta_1, 1+\delta_1)} \leq N[f^{\alpha}]_{C^{1+\delta}((-1, 0) \times \Omega)} + ||Du||_{L^1(\Omega)}. \]

Therefore, we obtain for any fixed \(x \in B_{1-2\varepsilon} \cap \overline{\Omega}\) and \(\gamma \in \left(\frac{1}{4}, \frac{1+\delta}{2}\right), \)

\[ u_t(\cdot, x) \in C^{\delta_2/2}((-1 + \varepsilon, 0)), \quad \delta_2 := \delta_1 + 2\gamma - 1 = \min\{\delta, 2\gamma - \frac{1}{2}\}, \]

and satisfies

\[ |u_t|_{C^{\delta_2/2}((-1 + \varepsilon, 0))} \leq N(f^{\alpha})_{C^{1+\delta}((-1, 0) \times \Omega)} + ||Du||_{L^1(\Omega)}. \]
We also obtain from $\delta_h^\gamma u \in C^{(1+\delta_1)/2,1+\delta_1}((-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D}))$ that
\[ D\delta_h^\gamma u \in C^{\delta_1/2,\delta_1}((-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D})). \]
If we take $\gamma$ to be sufficiently close to $\frac{1+\delta}{2}$, then $\delta_1 = 1 + \delta - 2\gamma$, $\gamma + \frac{\delta_1}{2} = \frac{1+\delta}{2}$, and
\[
[Du]_{L,(1+\delta)/2,(-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D})} \leq N \left( \sum_{j=1}^M |f|_{(1+\delta)/2,1+\delta(-1,0) \times \overline{D}} + ||Du||_{L_1(\Omega)} \right).
\]
On the other hand, for any $(t,x_1),(t,x_2) \in (-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D})$ with $x_1 \neq x_2$, by using the triangle inequality, Taylor’s formula, (3.59), and (3.60), we have
\[
|u_t(t,x_1) - u_t(t,x_2)| \leq |u_t(t,x_1) - \deltatu(t,x_1)| + |u_t(t,x_2) - \deltatu(t,x_2)| + |\delta tu(t,x_1) - \delta tu(t,x_2)|
\]
\[
\leq 2h^{\frac{\delta_1}{2}} [u]_{L,\delta/2((-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D}))} + h^\gamma |\delta_h^\gamma u(t,x_1) - \delta_h^\gamma u(t,x_2)|
\]
\[
\leq 2h^{\frac{\delta_1}{2}} [u]_{L,\delta/2((-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D}))} + Nh^\gamma |x_1 - x_2||\delta_h^\gamma u|_{(1+\delta)/2,1+\delta(-1,3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D})}.
\]
where $\delta_h^\gamma u := \delta_h^1 u$; the definition of $\delta_h^1 u$ can be found in (3.57). We obtain by optimizing in $h$ that
\[
|u_t(t,x_1) - u_t(t,x_2)| \leq N|x_1 - x_2|^\frac{\delta}{1+\frac{\delta}{2}} |u|_{L,\delta/2((-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D}))}^{\frac{1+\gamma}{1+\delta/2}} |\delta_h^\gamma u|_{(1+\delta)/2,1+\delta(-1,3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D})}^{\frac{1+\gamma}{1+\delta/2}}.
\]
Now letting $\gamma = \frac{1+\delta}{2} - \varepsilon_0$ for any small constant $0 < \varepsilon_0 \leq \delta - \frac{1}{2}$, then $\delta_2 = \delta$ and
\[
\delta_2 = \frac{\delta}{1+\frac{\delta}{2}} \cdot \frac{1}{1+\frac{\delta}{2} - \gamma} = \frac{\delta}{1+2\varepsilon_0} \geq \frac{1}{2},
\]
where we used the assumption that $\delta > \frac{1}{2}$. We thus obtain
\[
|u_t|_{1/4,1/2((-1+3\varepsilon,0) \times (B_{1-2\varepsilon} \cap \overline{D}))} \leq N \left( \sum_{j=1}^M |f|_{(1+\delta)/2,1+\delta(-1,0) \times \overline{D}} + ||Du||_{L_1(\Omega)} \right).
\]
The lemma is proved. \hfill \Box

**Lemma 3.8.** Under the same assumptions as in Proposition 3.7, we have
\[
\sum_{j=1}^{m+1} ||D^2 u||_{L_\infty(Q'_{1/4}((-1+\varepsilon,0) \times D))} \leq N||Du||_{L_1(\Omega''_{1/4})} + N \sum_{j=1}^M |f|_{(1+\delta)/2,1+\delta(-1,0) \times \overline{D}},
\]
where $N > 0$ is a constant depending only on $n, d, m, p, v, \varepsilon, |A|_{(1+\delta)/2, 1+\delta(1,0) \times D_j}$, and the $C^{2+\mu}$ norm of $h_j$.

Proof. We prove this lemma in two steps. Step 1. We first prove the claim: For any $k' = 1, \ldots, d - 1$, $z_0 \in (-1 + \varepsilon, 0) \times (D_\ell \cap D_\iota)$, and $r \in (0, 1/4)$,

$$
|D_{k'} \tilde{u}(z_0; z_0)| + |\bar{U}(z_0; z_0)|
\leq N r' \sum_{j=1}^{m+1} ||D^2 u||_{L_{\infty}(Q_-(z_0) \cap ((-1+\varepsilon, 0) \times D_j))} + N r \left( \sum_{j=1}^{m} |f_j(1+\delta)/2, 1+\delta(1,0) \times D_j| + ||Du||_{L_1(Q_0)} \right),
$$

(3.61)

where $\tilde{u}$ and $\bar{U}$ are defined in (3.15) and (3.18), respectively, and $\mu' = \min \{ \frac{1}{2}, \mu \}$. Indeed, for any $s \in (0, 1)$, let $q'_{20,s}, Q_{20,s} \in \mathbb{R}^n$ be chosen such that

$$
\Phi(z_0, s) = \left( \int_{Q_-(z_0)} |D_{k'} \tilde{u}(z_0)|^\frac{1}{s} + |\bar{U}(z_0)|^\frac{1}{s} \right)^2.
$$

Using the triangle inequality, we have

$$
|q'_{20,s/2} - q'_{20,s/2}|^\frac{1}{s} \leq |D_{k'} \tilde{u}(z_0) - q'_{20,s/2}|^\frac{1}{s} + |D_{k'} \tilde{u}(z_0) - q'_{20,s/2}|^\frac{1}{s}
$$

and

$$
|Q_{20,s/2} - Q_{20,s/2}|^\frac{1}{s} \leq |\bar{U}(z_0) - Q_{20,s/2}|^\frac{1}{s} + |\bar{U}(z_0) - Q_{20,s/2}|^\frac{1}{s}.
$$

Now taking the average over $z \in Q_{s/2}(z_0)$ and then taking the square, we obtain

$$
|q'_{20,s/2} - q'_{20,s/2}| + |Q_{20,s/2} - Q_{20,s/2}| \leq N(\Phi(z_0, s/2) + \Phi(z_0, s)).
$$

By iterating and using the triangle inequality, we deduce

$$
|q'_{20,2^{-i}s} - q'_{20,2^{-i}s}| + |Q_{20,2^{-i}s} - Q_{20,2^{-i}s}| \leq N \sum_{j=0}^{L} \Phi(z_0, 2^{-i}s).
$$

(3.62)

Recalling the definitions of $\tilde{u}$ and $u_i$ in (3.15) and (3.4), respectively, a direct calculation gives

$$
D_{k'} \tilde{u}(z_0) = \ell_{i,j}^{k'} D_j D_i u + D_{k'} \tilde{u} + \sum_{j=1}^{m+1} D_{k'} \ell_{i,j}^{k'} D_j u(t_0, P_j x_0) - D_{k'} u.
$$

Using (3.17), we have

$$
\bar{U}(z_0) = n^a (A_{\alpha} D_{\beta} \tilde{u} - \bar{f}^a)
\begin{align*}
&= n^a \left( A_{\alpha} D_{\beta} D_j u \ell_{i}^{k'} - D_{\epsilon} f^a + D_{\epsilon} A_{\alpha} D_{\beta} u - A_{\alpha} D_{\beta} u \right) \\
&\quad \text{for } a = 1, \ldots, m.
\end{align*}
$$
From the assumption $Du$ is piecewise $C^1$ in $x$, $A^{a\beta}, f^\alpha \in C^{(1+\delta)/2,1+\delta}((-1 + \varepsilon, 0) \times (D_e \cap \overline{D}_j))$, Lemma 2.1(ii), it follows that $D_{\ell^\varepsilon} \tilde{u}(z; z_0), \tilde{U}(z; z_0) \in C^0((-1 + \varepsilon, 0) \times (D_e \cap \overline{D}_j))$. Taking $\rho = 2^{-r} s$ in (3.34), we have
\[
\lim_{L \to \infty} \Phi(z_0, 2^{-r} s) = 0.
\]
Thus, for any $z_0 \in (-1 + \varepsilon, 0) \times (D_e \cap \overline{D}_j)$, we obtain
\[
\lim_{L \to \infty} q_{2^{-r} s} = D_{\ell^\varepsilon} \tilde{u}(z_0; z_0), \quad \lim_{L \to \infty} Q_{z_0, 2^{-r} s} = \tilde{U}(z_0; z_0).
\]
Now taking $L \to \infty$ in (3.62), choosing $s = r/2$, and using Lemma 3.4, we have for $r \in (0, 1/4), k' = 1, \ldots, d - 1$, and $z_0 \in (-1 + \varepsilon, 0) \times (D_e \cap \overline{D}_j)$,
\[
|D_{\ell^\varepsilon} \tilde{u}(z_0; z_0) - q_{z_0, r/2}^{k'} + |\tilde{U}(z_0; z_0) - Q_{z_0, r/2}| \leq N \sum_{j=0}^{\infty} \Phi(z_0, 2^{-r} - 1; r)
\]
\[
\leq N \Phi(z_0, r/2) + Nr^{d-2} \left( \sum_{j=1}^{m+1} |Du|_{1/2,1; Q'(z_0) \cap (-1 + \varepsilon, 0) \times \overline{D}_j} + ||Du||_{L^1(Q)} \right)
\]
(3.65)
\[
+ \sum_{j=1}^{M} f_{(1+\delta)/2,1+\delta,(-1,0) \times \overline{D}_j} + ||Du||_{L^1(Q)}
\]
By averaging the inequality
\[
|q_{z_0, r/2}^{k'}| + |Q_{z_0, r/2}|
\]
\[
\leq |D_{\ell^\varepsilon} \tilde{u}(z; z_0) - q_{z_0, r/2}^{k'}| + |\tilde{U}(z; z_0) - Q_{z_0, r/2}| + |D_{\ell^\varepsilon} \tilde{u}(z; z_0)| + |\tilde{U}(z; z_0)|
\]
over $z \in Q_{r/2}(z_0)$ and then taking the square, we have
\[
|q_{z_0, r/2}^{k'}| + |Q_{z_0, r/2}|
\]
\[
\leq N \Phi(z_0, r/2) + N \left( \int_{Q_{r/2}(z_0)} \left( |D_{\ell^\varepsilon} \tilde{u}(z; z_0)|^\frac{1}{2} + |\tilde{U}(z; z_0)|^\frac{1}{2} \right) dz \right)^2.
\]
This, in combination with (3.65), Lemmas 3.7, 3.8, the triangle inequality, and
\[
\Phi(z_0, r/2) \leq \left( \int_{Q_{r/2}(z_0)} \left( |D_{\ell^\varepsilon} \tilde{u}(z; z_0)|^\frac{1}{2} + |\tilde{U}(z; z_0)|^\frac{1}{2} \right) dz \right)^2
\]
\[
\leq Nr^{d-2} \left( ||D_{\ell^\varepsilon} \tilde{u}(z; z_0)||_{L^1(Q_{r/2}(z_0))} + ||\tilde{U}(z; z_0)||_{L^1(Q_{r/2}(z_0))} \right),
\]
leads to that
\[
|D_{\ell^\varepsilon} \tilde{u}(z_0; z_0)| + |\tilde{U}(z_0; z_0)|
\]
From the definition of problem (3.8) is equivalent to hand side above. Using the definition of weak solutions, the transmission problem

\begin{equation}
\frac{D}{Dt} \tilde{u}(\cdot; z_0) + D_u u(\cdot; z_0) = f + g \quad \text{in } Q_t^-(z_0).
\end{equation}

By using Hölder’s inequality, we have

\begin{equation}
\|D u_j(\cdot; z_0)\|_{L^2_2(Q_t^-(z_0))} \leq N \left( r^{1-\frac{d_2}{2}} \|u_j(\cdot; z_0)\|_{L^1_1(Q_t^-(z_0))} + r\|g\|_{L^2_2(Q_t^-(z_0))} + \|f_j(\cdot; z_0)\|_{L^2_2(Q_t^-(z_0))} \right).
\end{equation}

By using Hölder’s inequality, we have

\begin{equation}
\|D u_j(\cdot; z_0)\|_{L^1_1(Q_t^-(z_0))} \leq N \left( r^{1-\frac{d_2}{2}} \|u_j(\cdot; z_0)\|_{L^1_1(Q_t^-(z_0))} + r\|g\|_{L^2_2(Q_t^-(z_0))} + \|f_j(\cdot; z_0)\|_{L^2_2(Q_t^-(z_0))} \right).
\end{equation}

Using Lemma 2.1(iii), we have

\[ \int_{B_j(\epsilon; z_0) \cap D_j} |D \tilde{u}|^2 \, dx \leq N \int_{B_j(\epsilon; z_0) \cap D_j} \min\{2r, h_i - h_{i-1}\} \frac{1}{|h_i - h_{i-1}|} \, dx' \leq Nr^{d-1}. \]

Together with this, we have from (3.2) and (3.9) that

\begin{equation}
\|g\|_{L^2_2(Q_t^-(z_0))} \leq Nr^{d+1} \left( \sum_{j=1}^{m+1} \|D^2 u\|_{L^\infty(Q_t^-(z_0) \cap \{(-1+\epsilon,0) \times \overline{D_j}\})} + \|Du\|_{L^1_2(Q)} \right)
\end{equation}

(3.69)\]

and

\begin{equation}
\|f_j(\cdot; z_0)\|_{L^2_2(Q_t^-(z_0))} \leq Nr^{d+1} \sum_{j=1}^{m+1} [Du]_{1/2,1; Q_t^-(z_0) \cap \{(-1+\epsilon,0) \times \overline{D_j}\}}
\end{equation}

(3.69)\]
(3.70) \[ + Nr^{d+2} \left( \|Du\|_{L^1(Q)} + \sum_{j=1}^{M} |f_j|_{(1+\delta)/2,1+\delta((-1,0)\times\mathbb{T})} \right). \]

Substituting (3.68)–(3.70) into (3.67), and using Lemma 3.7, we have

\[ \|Du_t(\cdot;z_0)\|_{L^1(Q_{r/2}(z_0))} \leq Nr^{d+2} \sum_{j=1}^{m+1} \|D^2 u\|_{L^\infty(Q_r(z_0)\cap((-1+\varepsilon,0)\times\mathcal{D}))} \]

\[ + Nr^{d+1} \left( \|Du\|_{L^1(Q)} + \sum_{j=1}^{M} |f_j|_{(1+\delta)/2,1+\delta((-1,0)\times\mathbb{T})} \right). \]

Now we obtain from the definition of \( \tilde{u} \) in (3.15) and (3.13) that

\[ \|D\tilde{u}(\cdot;z_0)\|_{L^1(Q_{r/2}(z_0))} \leq \|Du_t(\cdot;z_0)\|_{L^1(Q_{r/2}(z_0))} + \|Du\|_{L^1(Q_{r/2}(z_0))} \]

\[ \leq Nr^{d+2} \sum_{j=1}^{m+1} \|D^2 u\|_{L^\infty(Q_r(z_0)\cap((-1+\varepsilon,0)\times\mathcal{D}))} \]

\[ + Nr^{d+1} \left( \|Du\|_{L^1(Q)} + \sum_{j=1}^{M} |f_j|_{(1+\delta)/2,1+\delta((-1,0)\times\mathbb{T})} \right). \]

Similarly, we have

\[ \|\tilde{U}(\cdot;z_0)\|_{L^1(Q_{r/2}(z_0))} \leq Nr^{d+2} \sum_{j=1}^{m+1} \|D^2 u\|_{L^\infty(Q_r(z_0)\cap((-1+\varepsilon,0)\times\mathcal{D}))} \]

\[ + Nr^{d+1} \left( \|Du\|_{L^1(Q)} + \sum_{j=1}^{M} |f_j|_{(1+\delta)/2,1+\delta((-1,0)\times\mathbb{T})} \right). \]

Coming back to (3.66), we obtain (3.61).

\textbf{Step 2.} We finish the proof of the desired result. Since \( k,k' = 1,\ldots,d-1 \), there are \( \frac{nd(d-1)}{2} \) equations in (3.63) and \( n(d-1) \) equations in (3.64). Thus we have \( \frac{nd(d-1)}{2} + nd(d+1) \) equations, while \( D^2 u \) has \( \frac{nd(d+1)}{2} \) components. A simple calculation gives

\[ \frac{nd(d+1)}{2} - \frac{n(d-1)(d+2)}{2} = n, \]

which implies that we have to consider \( n \) more equations. To this end, we rewrite the equation (1.1) as

\[ A^{\alpha\beta} D_{\alpha\beta} u = D_{\alpha} f^{\alpha} - D_{\alpha} A^{\alpha\beta} D_\beta u + u_t \]

in \((-1+\varepsilon,0)\times(B_{1-\varepsilon} \cap \mathcal{D})\), \( j = 1,\ldots,M \). We shall use Cramer’s rule and (3.63), (3.64), and (3.71) to solve for \( D^2 u \). However, it is not easy to verify that whether the determinant of the coefficient matrix of

\[ (D^2 u, \ldots, D_1 D_{d} u, D^2_{2} u, \ldots, D_2 D_{d} u, \ldots, D^2_{d} u)^{T} \]
is not equal to 0. For this, we introduce the linear transformation \( \Lambda \) from the coordinate system associated with 0 to the coordinate system associated with the fixed point \( x \in B_r(x_0) \), given by \( \Lambda^k = \ell^k(x) \) and \( \Lambda^d = n(x), \ k = 1, \ldots, d - 1 \), where \( \ell^k(x) \) and \( n(x) \) are defined in (2.4) and (2.5), respectively. Now we define

\[
y = \Lambda x, \quad \hat{u}(t, y) = u(t, x), \quad \hat{A}(t, y) = \Lambda A(t, x) \Lambda^T.
\]

Then

\[
\ell^k(x)\hat{\ell}^k(x)D_iD_j u(t, x) = D_k D_k \hat{u}(t, y), \quad n^\alpha A^\alpha \beta D_\beta D_i \hat{\ell}^k_i = \hat{\Lambda}^\alpha \beta (t, y) D_\beta D_k \hat{u}(t, y),
\]

and

\[
A^\alpha \beta D_\alpha \beta u = \hat{\Lambda}^\alpha \beta (t, y) D_\alpha \beta \hat{u}(t, y).
\]

A direct calculation yields the determinant of the coefficient matrix of

\[
(D_1^2 \hat{u}, \ldots, D_1 D_d \hat{u}, D_2^2 \hat{u}, \ldots, D_2 D_d \hat{u}, \ldots, D_d^2 \hat{u})^T
\]

is \( \hat{\Lambda}^d d^d > 0 \). Therefore, by using Cramer’s rule, we can solve for \( D^2 \hat{u} \) and thus \( D^2 u \). In particular, by using Lemma A.1 and (3.3), we have

\[
|D^2 u(z_0)| \leq N \left( |D_{\ell^d} \hat{u}(z_0; z_0)| + |\hat{\Lambda} u(z_0)| + |D_u(z_0)| \right)
+ N \left( \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta;(-1,0)\times D_j} + \|D_u\|_{L_1(Q)} \right).
\]

Now combining (3.6), Lemma 3.7 and (3.13), we deduce

\[
|D^2 u(z_0)| \leq N r^{\mu'} \sum_{j=1}^{m+1} \|D^2 u\|_{L_\infty(Q_r(z_0) \cap ((1+\epsilon,0)\times D_j))}
+ N r^{-1} \left( \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta;(-1,0)\times D_j} + \|D_u\|_{L_1(Q)} \right).
\]

For any \( z_1 \in Q_{1/4}^- \) and \( r \in (0, 1/4) \), by taking supremum with respect to \( z_0 \in Q_{r}^- (z_1) \cap ((1+\epsilon,0)\times D_j) \), we have

\[
\sum_{j=1}^{m+1} \|D^2 u\|_{L_\infty(Q_r(z_1) \cap ((1+\epsilon,0)\times D_j))}
\leq N r^{\mu'} \sum_{j=1}^{m+1} \|D^2 u\|_{L_\infty(Q_r(z_1) \cap ((1+\epsilon,0)\times D_j))}
+ N r^{-1} \left( \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta;(-1,0)\times D_j} + \|D_u\|_{L_1(Q)} \right),
\]
where $\mu' = \min \{\frac{1}{2}, \mu\}$. By an iteration argument which is essentially the same as that in [10, Lemma 3.4], we get
\[
\sum_{j=1}^{m+1} ||D^2 u||_{L^\infty(Q_{1/4} \cap ((-1+\varepsilon,0) \times D_j))} \leq N ||Du||_{L^2(Q_{3/4})} + N \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta,(-1,0) \times D_j} + ||Du||_{L^2(Q)}
\]
and the lemma is proved. \hfill \Box

3.3. Proof of Proposition 3.1 We are in a position to finish the proof of Proposition 3.1.

Proof of Proposition 3.1 (a) By using Lemmas 3.4, 3.7, 3.8 and 3.2 with $Q_{1/4}$ in place of $Q_r(z_0)$, we have for $r \in (0,1/8)$,
\[
\sup_{z_0 \in Q_{1/8}} \Phi(z_0, r) \leq N \mu' \left( ||Du||_{L^2(Q_{1/4})} + ||f||_{L^1(Q_{1/4})} + \sum_{j=1}^{m+1} |f|_{(1+\delta)/2,1+\delta,(-1,0) \times D_j} \right)
\]
\[
+ \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta,(-1,0) \times D_j} + ||Du||_{L^2(Q)}
\]
(3.72)
\[
\leq N \mu' \left( \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta,(-1,0) \times D_j} + ||Du||_{L^2(Q)} \right).
\]
Applying (3.63) with $r$ in place of $r/2$ and using (3.72), we derive
\[
\sup_{z_0 \in Q_{1/8}} \left( |D_{\rho'} \tilde{u}(z_0; z_0) - q_{z_0,r}'| + |\tilde{u}(z_0; z_0) - Q_{z_0,r}| \right)
\]
(3.73)
\[
\leq N \mu' \left( \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta,(-1,0) \times D_j} + ||Du||_{L^2(Q)} \right),
\]
where $q_{z_0,r}', Q_{z_0,r} \in \mathbb{R}^n$ satisfy
\[
\Phi(z_0, r) = \left( \int_{Q_{1/8} r} \left( |D_{\rho'} \tilde{u}(z; z_0) - q_{z_0,r}'|^{1/2} + |\tilde{u}(z; z_0) - Q_{z_0,r}|^{1/2} \right) dz \right)^2.
\]
Suppose that $z_1 = (t_1, x_1) \in Q_{1/8} \cap ((-1+\varepsilon,0) \times D_j)$ for some $j \in \{1,\ldots,m+1\}$. If $|z_0 - z_1|_p \geq 1/16$, then we obtain from (3.63) that
\[
D_{\rho'} \tilde{u}(z_0; z_0)
\]
(3.74)
\[
= \frac{1}{|x_0|} \|x_0\| D_{\rho'} D_{\rho} u(z_0) - \sum_{j=1, j \neq j_0}^{m+1} D_{\rho'} \tilde{u}_{ij}(x_0) D_{\rho} u(t_0, P, x_0) - D_{\rho'} u(z_0).
\]
Thus, by using Lemma A.1, Lemma 3.8 and (3.13), we have
\[
|D_{\rho'} \tilde{u}(z_0; z_0) - D_{\rho'} \tilde{u}(z_1; z_1)|
From (3.64), it follows that

\[ \bar{U}(z_0; z_0) = n^\alpha(x_0)(A^{\alpha\beta}(z_0)D_\beta D_\alpha u(z_0) + D_\alpha f^\alpha(z_0) + D_\alpha A^{\alpha\beta}(z_0)D_\beta u(z_0) \]

\[ - A^{\alpha\beta}(z_0)D_\beta u(z_0) - \delta_{ad} \sum_{j=1}^m \mathbb{1}_{x_0 \in h_j(x')} (n_j^\ell(x'))^{-1} h_j(t_0, x') \]

\[ \leq N \sum_{j=1}^{m+1} \| D^2 u \|_{L^{\infty}(Q_{1/4^c((-1+\varepsilon,0) \times \mathcal{D}_j)) \cap \mathbb{R}^d)} + \| D u \|_{L^{\infty}(Q_{1/4})} \]

\[ + \frac{N}{|z_0 - z_1|^p} \left( \sum_{j=1}^M |f|_{(1+\delta)/2, 1 + \delta, (-1, 0) \times \mathcal{D}_j} + \| D u \|_{L^1(Q)} \right) \]

(3.75)

If \(|z_0 - z_1|^p \leq 1/16\), then we select \( r = |z_0 - z_1|^p \). Without loss of generality, we assume that \( z_1 \) is above \( z_0 \). By the triangle inequality, we have for any \( z \in Q_r(z_0) \),

\[ |\tilde{U}(z_0; z_0) - \tilde{U}(z_1; z_1)| \leq |z_0 - z_1|^p \left( \sum_{j=1}^M |f|_{(1+\delta)/2, 1 + \delta, (-1, 0) \times \mathcal{D}_j} + \| D u \|_{L^1(Q)} \right) \]

(3.76)

Using a similar argument in deriving (3.75), we obtain

\[ |\tilde{U}(z_0; z_0) - \tilde{U}(z_1; z_1)| \leq |z_0 - z_1|^p \left( \sum_{j=1}^M |f|_{(1+\delta)/2, 1 + \delta, (-1, 0) \times \mathcal{D}_j} + \| D u \|_{L^1(Q)} \right) \]

(3.77)

where \( q_{z_1, 2r}^k, Q_{z_1, 2r} \in \mathbb{R}^d, k' = 1, \ldots, d - 1 \), satisfy

\[ \Phi(z_1, 2r) = \left( \int_{Q_{z_1}(z_1)} \left( |D_{k'} \tilde{u}(z; z_1) - q_{z_1, 2r}^k|^{1/2} + |\tilde{U}(z; z_1) - Q_{z_1, 2r}|^{1/2} \right) dz \right)^2. \]
Now we take the average over \( z \in Q_r^- (z_0) \) and take the square in (3.77) to obtain
(3.78)
\[
|D \rho^\nu \tilde{u}(z_0; z_0) - D \rho^\nu \tilde{u}(z_1; z_1)| + |\tilde{U}(z_0; z_0) - \tilde{U}(z_1; z_1)|
\leq |D \rho^\nu \tilde{u}(z_0; z_0) - q_{z_0, r}^K| + |\tilde{U}(z_0; z_0) - Q_{z_0, r}| + \Phi(z_0, r) + \Phi(z_1, 2r)
+ |D \rho^\nu \tilde{u}(z_1; z_1) - q_{z_1, 2r}^K| + |\tilde{U}(z_1; z_1) - Q_{z_1, 2r}|
+ \left( \int_{Q_r^- (z_0)} |D \rho^\nu \tilde{u}(z; z_0) - D \rho^\nu \tilde{u}(z; z_1)|^2 + |\tilde{U}(z; z_0) - \tilde{U}(z; z_1)|^2 \right) dz \right)^{1/2}.
\]

Next we estimate the last term in (3.78). For any \( x_0 \in B_{1/8} \cap \partial \mathcal{D}_h \) and \( x_1 \in B_{1/8} \cap \partial \mathcal{D}_h \), since \( r = |z_0 - z_1|_H \), by using (3.6) and \( h_j \in C^{2+\mu} \), we have
\[
|P_j x_0 - P_j x_1| \leq N|x_0 - x_1|.
\]
This together with Lemmas 3.7 and 3.8 yields
\[
|Du(t_0, P_j x_0) - Du(t_1, P_j x_1)|
\leq Nr[Du]_{1/2,1+\delta,(-1+\epsilon,0) \times \partial \mathcal{D}_j}
\leq Nr\left( \sum_{j=1}^M |f|_{(1+\delta)/2,1+\delta,(-1,0) \times \partial \mathcal{D}_j} + \|Du\|_{L^1(Q)} \right).
\]

It follows from (3.11) that \( \tilde{u}_j := u_j(z; z_0) - u_j(z; z_1) \in \mathcal{H}_1^1(Q_r^-) \) satisfies
\[
\begin{cases}
-\partial_t \tilde{u}_j + Du(A^{ij} D^i \tilde{u}_j) \\
- Du(1_{(-1,0) \times \mathcal{D}_j} A^{ij} D^i \tilde{u}_j (Du(t_0, P_j x_0) - Du(t_1, P_j x_1))) \in Q_r^-,
\end{cases}
\]
\( \tilde{u}_j = 0 \) on \( \partial \mathcal{D}_j \).

Then by using Lemmas A.1 A.3 (3.79), and the fact that \( 1_{(-1,0) \times \mathcal{D}_j} D^i \tilde{u}_j \) is piecewise \( C^\beta \), we obtain
\[
|u(t; z_0) - u(t; z_1)|_{(1+\mu')/2,1+\mu',(-1+\epsilon,0) \times \partial \mathcal{D} \cap B_{1-i}}
\leq N \sum_{j=1}^{M+1} |\tilde{u}_j|_{L^1(Q_r^-)} + N \sum_{j=1}^{M+1} |1_{(-1,0) \times \mathcal{D}_j} A^{ij} D^i \tilde{u}_j (Du(t_0, P_j x_0) - Du(t_1, P_j x_1))|_{\mu'/2,\mu, Q_r^-}
\leq N \sum_{j=1}^{M+1} |1_{(-1,0) \times \mathcal{D}_j} A^{ij} D^i \tilde{u}_j (Du(t_0, P_j x_0) - Du(t_1, P_j x_1))|_{L^1(Q_r^-)}
\leq Nr\left( \sum_{j=1}^M |f|_{(1+\delta)/2,1+\delta,(-1,0) \times \partial \mathcal{D}_j} \right)
\leq Nr\left( \sum_{j=1}^M |f|_{(1+\delta)/2,1+\delta,(-1,0) \times \partial \mathcal{D}_j} \right).
where
\[ u(; z_0) - u(; z_1) := \sum_{j=1}^{m+1} \tilde{u}_j, \quad i = 1, \ldots, m + 1, \]
and \( \mu' = \min\{\mu, \frac{1}{4}\} \). This together with (3.15), (3.5), and (3.79) yields
\[ |D_{\ell'} \tilde{u}(z; z_0) - D_{\ell'} \tilde{u}(z; z_1)| \]
\[ = \left| \sum_{j=1}^{m+1} D_{\ell'} \tilde{\ell}_{i,j}(D_i u(t_0, P_j x_0) - D_i u(t_1, P_j x_1)) + D_{\ell'} u(z; z_0) - D_{\ell'} u(z; z_1) \right| \]
(3.80)
\[ \leq N \rho' \left( \sum_{j=1}^{M} |f|_{1+\delta/2,1+\delta(-1,0)\times D_j} + ||Du||_{L^4(Q)} \right). \]

Similarly,
(3.81)
\[ |\tilde{U}(z; z_0) - \tilde{U}(z; z_1)| \leq N \rho' \left( \sum_{j=1}^{M} |f|_{1+\delta/2,1+\delta(-1,0)\times D_j} + ||Du||_{L^4(Q)} \right). \]

Coming back to (3.78), and using (3.72), (3.73), (3.80), and (3.81), we obtain
\[ |D_{\ell'} \tilde{u}(z_0; z_0) - D_{\ell'} \tilde{u}(z_1; z_1)| + |\tilde{U}(z_0; z_0) - \tilde{U}(z_1; z_1)| \]
\[ \leq N \rho' \left( \sum_{j=1}^{M} |f|_{1+\delta/2,1+\delta(-1,0)\times D_j} + ||Du||_{L^4(Q)} \right). \]

(b) As showed in Step 2 of the proof of Lemma 3.8 from (3.71) with \( z = z_0, \) (3.74), and (3.76), we find that \( D^2 u(z_0) \) is a combination of
(3.82)
\[ D_\alpha f^\alpha(z_0) - D_\alpha A^{\alpha\beta}(z_0) D_\beta u(z_0) + u_1(z_0), \]
(3.83)
\[ D_{\ell'} \tilde{u}(z_0; z_0) + \sum_{j=1, j \neq j_0}^{m+1} D_{\ell'} \tilde{\ell}_{i,j}(x_0) D_i u(t_0, P_j x_0) + D_{\ell'} u(z_0), \]
and
\[ \tilde{U}(z_0; z_0) + n^\alpha(x_0) \left( D_\ell f^\alpha(z_0) - D_\ell A^{\alpha\beta}(z_0) D_\beta u(z_0) + A^{\alpha\beta}(z_0) D_\beta u(z_0) \right) \]
\[ + \delta_{ad} \sum_{j=1}^{m} \mathbb{1}_{x_j > h_j(x_0)} (n_j^\alpha(x_0'))^{-1} \tilde{h}_j(t_0, x_0') \]
\[ + A^{\alpha\beta}(z_0) \sum_{j=1, j \neq j_0}^{m+1} \mathbb{1}_{x_j < h_j(x_0)} D_\beta \tilde{\ell}_{i,j}(x_0) D_i u(t_0, P_j x_0). \]
(3.84)

Similarly, for any \( z_0 \in (-1 + \varepsilon, 0) \times (B_{1-\varepsilon} \cap D_j) \), \( D^2 u(z_0) \) is a combination of (3.82)–(3.84) with \( z_0 \) replaced with \( \tilde{z}_0 \). Combining (3.19), Lemma 3.7, and
we have
\[
[D^2 u]_{\mu'/2,\mu'(-1+\varepsilon,0) \times (\mathcal{D}_j \cap \mathcal{D}_0)} \leq N \left( \sum_{j=1}^{M} |f|_{(1+\delta)/2,1+\delta;(-1,0) \times \mathcal{D}_j} + \|Du\|_{L^1(Q)} \right)
\]
for any \( j_0 = 1, \ldots, m + 1 \). Proposition 3.1 is proved. \( \square \)

4. General \( C^{(s+1+\mu')/2,s+1+\mu'} \) Estimates

In this section, we prove Theorem 1.2 in the general case of \( s \geq 2 \) by showing the key points and the main ingredients. We prove by induction on \( s \) that if \( A^{\alpha\beta} \) and \( f^\alpha \) are piecewise \( C^{(s+1+\delta)/2,s+1+\delta} \) and the interfacial boundaries are \( C^{s+\mu}, \) then any \( \mathcal{H}_p^1(Q) \) weak solution \( u \) to (1.1) is piecewise \( C^{(s+\mu')/2,s+\mu'} \), with the estimate
\[
|u|_{(s+\mu')/2,s+\mu';(-1+\varepsilon,0) \times (\mathcal{D}_j \cap \mathcal{D}_0)} \leq N \left( \|Du\|_{L^1((-1,0) \times \mathcal{D})} + \sum_{j=1}^{M} |f|_{(s+1+\delta)/2,s+1+\delta;(-1,0) \times \mathcal{D}_j} \right),
\]
where \( j_0 = 1, \ldots, m + 1, \mu' = \min \left\{ \frac{1}{2}, \mu \right\}, N \) depends on \( n, d, m, p, \nu, \varepsilon, \) the \( C^{s+\mu} \) characteristic of \( \mathcal{D}_j, \) and \( |A|_{(s+1+\delta)/2,s+1+\delta;(-1,0) \times \mathcal{D}_0}. \) Now suppose that \( A^{\alpha\beta} \) and \( f^\alpha \) are of the class \( C^{(s+\delta)/2,s+\delta}((-T+\varepsilon,0) \times (\mathcal{D}_0 \cap \mathcal{D}_j), \) and the interfacial boundaries are \( C^{s+1+\mu}. \) We shall prove that \( u \) is piecewise \( C^{(s+1+\mu')/2,s+1+\mu'}. \)

As in (3.1), we will first derive a new equation satisfied by \( D_\ell \cdots D_\ell u \) which is the \( s \)-th directional derivative of \( u \) along the vector fields \( \ell^k, k = 1, \ldots, d - 1. \) For this, since \( D_\ell (fg) = gD_\ell f + fD_\ell g, \) by an induction argument for \( s \geq 2, \) we conclude
\[
D_\ell \cdots D_\ell u = \ell_{i_1} \ell_{i_2} \cdots \ell_{i_q} D_{i_1} D_{i_2} \cdots D_{i_q} u + R(u),
\]
where we used the Einstein summation convention over repeated indices, \( \ell_i := \ell_i^i, \ell = 1, \ldots, s, k_\ell = 1, \ldots, d - 1, i_\ell = 1, \ldots, d, \) and
\[
R(u) = D_{\ell_{i_1}} (\ell_{i_2} \cdots \ell_{i_q}) D_{i_2} \cdots D_{i_q} u + D_{\ell_{i_1}} (D_{\ell_{i_2}} (\ell_{i_3} \cdots \ell_{i_q}) D_{i_3} \cdots D_{i_q} u + D_{\ell_{i_1}} (D_{\ell_{i_2}} (\ell_{i_3} \cdots \ell_{i_q}) D_{i_3} \cdots D_{i_q} u + D_{\ell_{i_1}} (D_{\ell_{i_2}} (\ell_{i_3} \cdots \ell_{i_q}) D_{i_3} \cdots D_{i_q} u + \cdots + D_{\ell_{i_1}} (D_{\ell_{i_2}} (\ell_{i_3} \cdots \ell_{i_q}) D_{i_3} \cdots D_{i_q} u)) \right),
\]
which is the summation of the products of directional derivatives of \( \ell_i \) and derivatives of \( u. \) Furthermore, an induction argument gives
\[
D_{i_1} D_{i_2} \cdots D_{i_q} (A^{\alpha\beta} D_{i_1} D_{i_2} \cdots D_{i_q} u) = A^{\alpha\beta} D_{i_1} D_{i_2} \cdots D_{i_q} u + D_{i_1} A^{\alpha\beta} D_{i_2} \cdots D_{i_q} u.
\]
Then we have

\[ \ell_i \ell_i \cdots \ell_i D_{i_1} D_{i_2} \cdots D_{i_s} f^\alpha = D_\alpha (\ell_i \ell_i \cdots \ell_i D_{i_1} D_{i_2} \cdots D_{i_s} (A^{\alpha \beta} D_\beta u)) - D_\alpha (\ell_i \ell_i \cdots \ell_i D_{i_1} D_{i_2} \cdots D_{i_s} (A^{\alpha \beta} D_\beta u)) \]

(4.4)

and

\[ \ell_i \ell_i \cdots \ell_i D_{i_1} D_{i_2} \cdots D_{i_s} \cdot D_{i_1} D_{i_2} \cdots D_{i_s} (A^{\alpha \beta} D_\beta u) \]

(4.5)

Taking \( D_\ell \cdots D_\ell \) to the equation \(-u_\ell + D_\alpha (A^{\alpha \beta} D_\beta u) = D_\alpha f^\alpha\), and using (4.2), (4.4), and (4.5), we obtain the equation

\[ -(D_\ell \cdots D_\ell u)_\ell + D_\alpha (A^{\alpha \beta} D_\beta (D_\ell \cdots D_\ell u)) = D_\alpha \hat{f}^\alpha + \bar{\gamma} \]

in each subdomain \((-1, 0) \times \mathcal{D}_j\), where

\[ \hat{f}^\alpha_\ell := \ell_i \ell_i \cdots \ell_i D_{i_1} D_{i_2} \cdots D_{i_s} f^\alpha + A^{\alpha \beta} D_\beta (\ell_i \ell_i \cdots \ell_i D_{i_1} D_{i_2} \cdots D_{i_s} u + A^{\alpha \beta} D_\beta (R(u))) \]

(4.6)

and

\[ \bar{\gamma} := D_\alpha (\ell_i \ell_i \cdots \ell_i (D_{i_1} D_{i_2} \cdots D_{i_s} (A^{\alpha \beta} D_\beta u - f^\alpha)) + R(D_\alpha (f^\alpha - A^{\alpha \beta} D_\beta u)). \]

Similarly, by taking \( D_\ell \cdots D_\ell \) to \([n^\alpha_j (A^{\alpha \beta} D_\beta u - f^\alpha)]_{(-1, 0) \times \Gamma_j} = 0\), we derive the boundary condition

\[ [n^\alpha_j (A^{\alpha \beta} D_\beta (D_\ell \cdots D_\ell u) - f^\alpha)]_{(-1, 0) \times \Gamma_j} = \bar{h}_j, \]

where

\[ \bar{h}_j := \left[ -\ell_i \ell_i \cdots \ell_i (\sum_{\tau=1}^{s} D_{i_1} n^\alpha_j D_{i_1} \cdots D_{i_{\tau-1}} D_{i_{\tau+1}} \cdots D_{i_s} (A^{\alpha \beta} D_\beta u - f^\alpha)) \right], \]
\[
+ \sum_{1 \leq t_1 < \cdots < t_s \leq \ell} D_{i_1} D_{i_2} n_j^\alpha D_{i_3} \cdots D_{i_{t_1+1}} D_{i_2} \cdots D_{i_{t_2+1}} D_{i_s} (A^{\alpha \beta} D_\beta u - f^\alpha) \\
+ \cdots + D_i D_{i_2} \cdots D_i n_j^\alpha (A^{\alpha \beta} D_\beta u - f^\alpha) \bigg]_{|(-1,0) \times \Gamma_j}
\]

(4.8)

\[
- [R(n_j^\alpha (A^{\alpha \beta} D_\beta u - f^\alpha))]_{|(-1,0) \times \Gamma_j}.
\]

Consequently, the \(s\)-th directional derivative \(D_\ell \cdots D_\ell u\) satisfies

\[
-(D_\ell \cdots D_\ell u)_t + D_\alpha (A^{\alpha \beta} D_\beta (D_\ell \cdots D_\ell u)) = D_\alpha \ell \hat{f}^\alpha + \hat{g} \quad \text{in } \bigcup_{j=1}^{m+1} (-1,0) \times \mathcal{D}_j,
\]

\[
\left[ n_j^\alpha (A^{\alpha \beta} D_\beta (D_\ell \cdots D_\ell u) - \ell \hat{f}^\alpha) \right]_{|(-1,0) \times \Gamma_j} = \hat{h}_j.
\]

Note that the terms \(D_\beta (\ell_{i_1} \ell_{i_2} \cdots \ell_{i_\ell})\) and \(D_\beta (R(u))\) in (4.6) are singular at any point where two interfaces touch or are close to each other. To cancel out the singularity, we choose a function \(u_0\) as follows:

\[
u_0 := u_0(x; z_0) = \sum_{j=1}^{m+1} \ell_{i_1,j} \ell_{i_2,j} \cdots \ell_{i_{\ell_j},j} D_{i_1} D_{i_2} \cdots D_{i_\ell_j} u(t_0, P_j x_0)
\]

\[
+ \sum_{j=1}^{m+1} \sum_{s=1}^{m} \sum_{t=1}^{s-1} D_{\ell_{i_1,j}} D_{\ell_{i_2,j}} \cdots D_{\ell_{i_{t+1},j}} (\ell_{i_{t+1},j} \cdots \ell_{i_{\ell_j},j}) (D_{i_1} \cdots D_{i_\ell_j} u(t_0, P_j x_0))
\]

\[
+ (x - x_0) \cdot DD_{i_{\ell_j+1}} \cdots D_{i_\ell} u(t_0, P_j x_0) + \cdots
\]

\[
+ \sum_{j=1}^{m+1} (D_{\ell_{i_{\ell_j-1},j}} \ell_{i_{\ell_j},j} \ell_{i_{\ell_j+1},j} \cdots \ell_{i_{\ell_j-2},j}) (D_{i_1} D_{i_2} \cdots D_{i_{\ell_j-2}} D_{i_{\ell_j-1}} u(t_0, P_j x_0))
\]

(4.10)

where \(P_j x_0\) is defined in (3.6), \((t_0, x_0) \in (-9/16, 0) \times (B_{3/4} \cap \mathcal{D}_j), \) \(0 \in (0, 1/4), \)

\(\ell_{i,j}\) is the smooth extension of \(\ell|_{\mathcal{D}_j}\) to \(\bigcup_{k=1, k \neq j}^{m+1} \mathcal{D}_k\). Denote

\[
u_\ell := u_\ell(z; z_0) = D_\ell \cdots D_\ell u - u_0.
\]

Then (4.9) is equivalent to

\[
-(\partial_t u_\ell + D_\alpha (A^{\alpha \beta} D_\beta u_\ell)) = D_\alpha (f_1^\alpha - A^{\alpha \beta} D_\beta u_0) + \hat{g} \quad \text{in } \bigcup_{j=1}^{m+1} (-1,0) \times \mathcal{D}_j,
\]

\[
\left[ n_j^\alpha (A^{\alpha \beta} D_\beta u_\ell - f_1^\alpha + A^{\alpha \beta} D_\beta u_0) \right]_{|(-1,0) \times \Gamma_j} = \hat{h}_j.
\]

As before, by adding a term

\[
\sum_{j=1}^{m} D_\alpha (1) \chi_{\{x > \hat{h}_j(t, x')\}} (n_j^\alpha(x'))^{-1} \hat{h}_j(t, x')
\]
to the equation, the problem \((4.11)\) becomes a (homogeneous) transmission problem

\[
\begin{aligned}
-\partial_t u + D_\alpha (A^{\alpha\beta} D_\beta u) &= D_\alpha \hat{f}_2 \quad \text{in } \bigcup_{j=1}^{m+1} (-1,0) \times D_j, \\
|n_j^\alpha (A^{\alpha\beta} D_\beta u - f_2^\alpha)|_{(-1,0) \times \Gamma_j} &= 0,
\end{aligned}
\]

where

\[
\hat{f}_2 := \hat{f}_2^\alpha (\varepsilon; z_0) = \hat{f}_1^\alpha - A^{\alpha\beta} D_\beta u_0 + \delta_{\alpha\beta} \sum_{j=1}^m \mathbb{1}_{x^j > h_j(x')} (n_j^\beta(x'))^{-1} \tilde{\eta}_j(t, x').
\]

With the function \(u_0\), we can bound the mean oscillation of \(\hat{f}_2\) in cylinders. To make it vanish in a certain order, we shall further consider the problem

\[(4.12)\]

\[
\begin{aligned}
-\partial_t u_j + D_\alpha (A^{\alpha\beta} D_\beta u_j) &= -D_\alpha \left( A^{\alpha\beta} F_\beta \right) \quad \text{in } Q^j_1, \\
u_j &= 0 \quad \text{on } \partial \Omega^j_1,
\end{aligned}
\]

where \(u_j := u_j(\cdot; z_0) \in \mathcal{H}^1_\Omega(Q^j_1)\), the coefficient \(A^{\alpha\beta}\) is defined in \((3.10)\), and

\[
F_\beta := \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times D_j} D_\beta \left( \bar{\ell}_{i_1,j} \bar{\ell}_{i_2,j} \cdots \bar{\ell}_{i_{m-1},j} D_1 D_2 \cdots D_i u(t_0, P_j x_0) \right) + \cdots
\]
\[
+ \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times D_j} D_\beta \left( D_{\tilde{\ell}_{i_{m-1},j}} \tilde{\ell}_{i_1,j} \tilde{\ell}_{i_2,j} \cdots \tilde{\ell}_{i_{m-2},j} D_1 D_2 \cdots D_{i_{m-2}} D_i u(t_0, P_j x_0) \right)
\]
\[
+ (x - x_0) \cdot \overline{DD_1 D_2 \cdots D_{i_{m-2}} D_i u(t_0, P_j x_0)}
\]

\[(4.13)\]

\[
\phi
\]

which is the summation of the products of \(\mathbb{1}_{(-1,0) \times D_j}\) and derivatives of \(u_0\) defined in \((4.10)\). Combining Lemma \(A.3\) and \((4.1)\), we have

\[(4.14)\]

\[
||u_j||_{\mathcal{H}^1_\Omega(Q^j_1)} \leq N ||A^{\alpha\beta} F_\beta||_{L^p(Q^j_1)} \leq N \left( ||Du||_{L^1_\Omega} + \sum_{j=1}^M ||f||_{(s-1+\delta)/2, s-1+\delta((\cdot, -1, 0) \times \overline{D_j \cap B_{1-\varepsilon}})} \right).
\]

Using the fact that \(\tilde{\ell}_{j}\) is the smooth extension of \(\ell_1\) to \(\bigcup_{k=1, k \neq j}^{m+1} D_k\), one can verify that the right-hand side of the equation in \((4.12)\) is piecewise \(C^{\mu'/2}\). Then from Lemma \(A.1\) it follows that \(u_j(\cdot; z_0) \in C(1+\mu'; 2, 1+\mu'; (-1, 0) \times (D_j \cap B_{1-\varepsilon}))\), and satisfies

\[
||u_j||_{2, 1+\mu'; (-1, 0) \times (D_j \cap B_{1-\varepsilon})} \leq N ||Du||_{L^1_\Omega(Q^j_1)} + N ||A^{\alpha\beta} F_\beta||_{L^p_\Omega(Q^j_1)}
\]
\[
\leq N\left(\|Du\|_{L^1(Q)} + \sum_{j=1}^{M} |f_j|_{(s-1+\delta)/2,s-1+\delta(-1,0)\times \mathbb{R})}\right),
\]
where we used (4.14), (4.10), and (4.1) in the second inequality, \( \mu' = \min\{\mu, \frac{1}{4}\} \) and \( i = 1, \ldots, m + 1 \). Thus, we obtain

\[
(4.15) \quad u := u(\cdot; z_0) = \sum_{j=1}^{m+1} u_j(\cdot; z_0) \in C^{(1+\mu')/2,1+\mu'}((-1+\varepsilon, 0) \times (\overline{D_i} \cap B_{1-\varepsilon}))
\]
and

\[
|u_j|_{(1+\mu')/2,1+\mu';(-1+\varepsilon,0)\times(\overline{D_i} \cap B_{1-\varepsilon})}
\leq N\left(\|Du\|_{L^1(Q)} + \sum_{j=1}^{M} |f_j|_{(s-1+\delta)/2,s-1+\delta(-1,0)\times \mathbb{R})}\right).
\]

Define

\[
(4.17) \quad \hat{u} := \hat{u}(\cdot; z_0) = u^f - u.
\]
Then \( \hat{u} \) satisfies

\[
(4.18) \quad -\hat{u}_t + D_\alpha (A^{\alpha\beta} D_\beta \hat{u}) = \hat{g} + D_\alpha \hat{f}^\alpha \quad \text{in} \ Q_{3/4}^{\frac{1}{2}}
\]
where

\[
\hat{f}^\alpha := f^\alpha(z; z_0) = f^\alpha_2(z; z_0) + A^{\alpha\beta} F_\beta
\]

\[
= \ell_1 \ell_2 \cdots \ell_i D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_i} f^\alpha + A^{\alpha\beta} D_\beta (\ell_1 \ell_2 \cdots \ell_i_1) D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_i} u + A^{\alpha\beta} D_\beta (R(u))
\]

\[
- \ell_1 \ell_2 \cdots \ell_i (D_{\alpha_1} A^{\alpha\beta} D_\beta D_{\alpha_2} \cdots D_{\alpha_i} u + \sum_{s=1}^{s-1} D_{\alpha_s} \cdots D_{\alpha_i} (D_{\alpha_{s+1}} A^{\alpha\beta} D_\beta D_{\alpha_{s+2}} \cdots D_{\alpha_i} u))
\]

\[
(4.19)
\]

\[
+ \delta_{\alpha\delta} \sum_{j=1}^{m} \mathbb{1}_{x_j > h_i(x')} (h_j^\delta(x'))^{-1} \hat{h}_i(t, x') - A^{\alpha\beta} \hat{f}_\beta,
\]
and

\[
\hat{f}_\beta := \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times \overline{D_j}} D_\beta (\ell_1 \ell_2 \cdots \ell_i_1) D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_i} u(t_0, P_j x_0) + \cdots
\]

\[
+ \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times \overline{D_j}} D_\beta (D_{\alpha_{i-1}} \ell_i_1) (D_1 \ell_{i_2} \cdots D_{i_2-1} D_{i_2} u(t_0, P_j x_0)
\]

\[
+ (x - x_0) \cdot D D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_i-2} D_{\alpha_i} u(t_0, P_j x_0))
\]

\[
(4.20)
\]

\[
+ \sum_{j=1}^{m+1} \mathbb{1}_{(-1,0) \times \overline{D_j}} D_{\alpha_{i-1}} \ell_i \ell_{i_2} \cdots \ell_{i_2-1} D_\beta D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_i-2} D_{\alpha_i} u(t_0, P_j x_0).
\]
The mean oscillation of \( \bar{f}^u \) vanishes at a certain rate (see the proof of (4.25) below). Hence, it suffices to prove the piecewise regularity of \( \bar{u} \).

Denote
\[
\bar{U} := \bar{U}(z_0) = n^\alpha(A^{\alpha\beta}D_{\beta}\bar{u} - \bar{f}^u).
\]

We are going to prove that \( D_{\beta}\bar{u} \) and \( \bar{U} \) are piecewise \( C^{\mu'/2,\mu'} \), \( k = 1, \ldots , d-1 \). Similar to the argument in Section [3], we will prove the following proposition.

**Proposition 4.1.** Let \( \varepsilon \in (0, 1) \) and \( p \in (1, \infty) \). Suppose that \( A^{\alpha\beta} \) and \( f^u \) satisfy Assumption [1,1] with \( s \geq 2 \). If \( u \in \mathcal{H}_p^s((-1, 0) \times B_1) \) is a weak solution to
\[
-\left( \inf_{\bar{Q}^\varepsilon(z_0)} D_{\alpha}(A^{\alpha\beta}D_{\beta}u) = D_{\alpha}f^u \right) \text{ in } (-1, 0) \times B_1,
\]
then the following assertions hold.

(a) For any \( z_0, z_1 \in (-1 + \varepsilon, 0) \times B_{1-\varepsilon} \), we have for \( k' = 1, \ldots , d-1 \),
\[
|\langle D^k \bar{u}(z_0; z_0), \bar{U}(z_0; z_0) \rangle - \langle D^k \bar{u}(z_1; z_1), \bar{U}(z_1; z_1) \rangle| 
\]
\[
\leq N|z_0 - z_1|^{\mu'} \left( \|Du\|_{L^p((-1, 0) \times B_1)} + \sum_{j=1}^M |f|_{(s+\delta)/2, s+\delta(-1, 0) \times \Omega} \right),
\]
where \( \mu' = \min \left\{ \frac{1}{s}, \mu \right\} \), \( N \) depends on \( n, d, m, p, \nu, \varepsilon, |A|_{(s+\delta)/2, s+\delta(-1, 0) \times \Omega} \), and the \( C^{s+1+\mu} \) characteristic of \( D_j \).

(b) It holds that \( u \in C^{(s+1+\mu')/2, s+1+\mu'}((-1+\varepsilon, 0) \times (B_{1-\varepsilon} \cap \Omega)), j_0 = 1, \ldots , m+1, \) and
\[
|u|_{(s+1+\mu')/2, s+1+\mu'(-1+\varepsilon, 0) \times \Omega} 
\]
\[
\leq N \left( \|Du\|_{L^p((-1, 0) \times \Omega)} + \sum_{j=1}^M |f|_{(s+\delta)/2, s+\delta(-1, 0) \times \Omega} \right),
\]
where \( N \) depends on \( n, d, m, p, \nu, \varepsilon, |A|_{(s+\delta)/2, s+\delta(-1, 0) \times \Omega} \), and the \( C^{s+1+\mu} \) characteristic of \( D_j \).

To prove Proposition 4.1, we denote
\[
\Psi(z_0, r) := \inf_{q^\alpha, Q \subseteq \mathbb{R}^d} \left( \int_{Q \cap Q(z_0)} \left| D_{\alpha}(A^{\alpha\beta}D_{\beta}\bar{u}) - q^\alpha \right|^\frac{1}{2} + |\bar{U}(z; z_0) - Q|^\frac{1}{2} \right) dz
\]
and prove a decay estimate of it. We set
\[
\bar{u}(t, y; \Lambda z_0) = \bar{u}(t, x; z_0), \quad \mathcal{A}^{\alpha\beta}(t, y) = \Lambda^{\alpha k}A^{ks}(t, x)\Lambda^{s\beta},
\]
\[
\bar{f}^a(t, y; \Lambda z_0) = \Lambda^{\alpha k}f^a(t, x; z_0), \quad \bar{g}(t, y) = \bar{g}(t, x),
\]
where \( y = \Lambda x \) and \( \Lambda = (\Lambda^{\alpha\beta})_{\alpha, \beta = 1} \) is a \( d \times d \) orthogonal matrix defined in Section [2.3]. Then we have from (4.18) that \( \bar{u} \) satisfies
\[
-\bar{u} + D_{\alpha}(\mathcal{A}^{\alpha\beta}D_{\beta}\bar{u}) = \bar{g} + D_{\alpha}(\bar{f}^{\bar{a}}) \in \Lambda(Q_{3/4}),
\]
where $\Lambda(Q_{3/4}^-) := (-9/16, 0) \times \Lambda(B_{3/4})$. Denote
\[
\varphi(\Lambda z_0, r) := \inf_{Q^1, Q^2 \subset \mathbb{R}^n} \left( \int_{Q^1 \setminus (\Lambda z_0)} \left( |D g_i \hat{u}(z; \Lambda z_0) - q^i|^{2} + |\mathcal{A}^{\beta\delta} D g_i \hat{u}(z; \Lambda z_0) - \tilde{r}^i - Q^i| \right) dz \right)^2.
\]

The following result holds.

**Proposition 4.2.** For any $0 < \rho \leq r \leq 1/4$, we have
\[
\varphi(\Lambda z_0, \rho) \leq N(\frac{\rho}{r})^{\mu'} \varphi(\Lambda z_0, r/2) + N\rho^{\mu''} C_1,
\]
where $\mu' = \min\left\{ \frac{3}{2}, \mu \right\}$, $N$ depends on $n, d, m, p, v, |A|_{(s+\delta)/(2,s+\delta,0,0)}^r$, and the $C^{s+1+\mu}$ norm of $h_j$ and

\[
C_1 := \sum_{j=1}^{m+1} \left[ D^{\delta} u \right]_{1/2,1; Q_j} \cap ((1-\epsilon, 0) \times \mathcal{D}) + \sum_{j=1}^{M} \left[ f \right]_{(s+\delta)/(2,s+\delta,0,0)} + \|Du\|_{L^1(Q)}.
\]

The proof of Proposition 4.2 is similar to that of Proposition 3.3. We shall provide the main ingredients of the proof. We first rewrite (4.19) as
\[
\hat{f}^1(t, x; z_0) = \hat{f}^1_1(t, x; z_0) + \hat{f}^1_2(t, x),
\]
where
\[
\hat{f}^1_1(t, x; z_0) = A^{\alpha \beta} \left( D_\beta (\ell_{i_1} \ell_{i_2} \cdots \ell_{i_s} D_{r_1} D_{r_2} \cdots D_{r_s} u + D_\beta (R(u)) - \tilde{F}_\beta \right),
\]
and
\[
\hat{f}^1_2(t, x) = \ell_{i_1} \ell_{i_2} \cdots \ell_{i_s} D_1 D_2 \cdots D_s f^\alpha + \delta_{al} \sum_{j=1}^{m} \left[ f \right] (x; y_j) (y_j (x) - 1)^{-1} \hat{h}_j(t, x').
\]

Here $R(u)$ is defined in (4.3). Set
\[
\hat{f}^1_1(t, y; \Lambda z_0) = \Lambda^{\alpha k} f^1_k(t, x; z_0), \quad \hat{f}^1_2(t, y) = \Lambda^{\alpha k} f^1_k(t, x).
\]

Then we choose $F^\alpha$ and $G$ as follows:

\[
F^\alpha := F^\alpha(t, y; \Lambda z_0) = (A^{\alpha \beta}(y^\prime) - A^{\alpha \beta}(t, y)) D_\beta \hat{u}(t, y; \Lambda z_0) + \hat{f}^1_1(t, y; \Lambda z_0) + \hat{f}^1_2(t, y) - \hat{f}^1_2(y^\prime), \quad F = (F^1, \cdots, F^s),
\]
and

\[
G = g(t, y) = \tilde{g}(t, x),
\]

where $\hat{f}^1_2(y^\prime)$ is the piecewise constant function corresponding to $\hat{f}^1_2(t, y)$. 



Lemma 4.3. Let $F$ and $G$ be defined in (4.23) and (4.24), respectively. Then we have

$$\|F\|_{L_1(Q_r^c(Az_0))} \leq Nr^{d+2+\mu'}C_1$$

and

$$\|G\|_{L_1(Q_r^c(Az_0))} \leq Nr^{d+\frac{2}{\alpha}}\left(\sum_{j=1}^{m+1}\|D^{s+1}u\|_{L_\infty(Q_{r_0^c}(Az_0)\cap((0,-1+\varepsilon,0)\times\Omega))}\right)^{1/2}$$

$$+ \sum_{j=1}^{M} |f|((s+\delta)/2,\delta(1,0)\times\Omega_j) + \|Du\|_{L_1(Q)}$$,

where $C_1$ is defined in (4.22), $\mu' = \min\left\{\frac{\alpha}{2}, \mu\right\}$, $N$ depends on $|A_{(s+\delta)/2,\delta(1,0)\times\Omega_j}|$, $n, d, m, p, \nu$ and the $C^{\alpha+1+\mu}$ norm of $h_j$.

Proof: We first note that $\tilde{t}_{1/2}$ is piecewise $C^{\mu/2, \mu'}$, and similar to (3.52), we have

$$\int_{Q_r^c(Az_0)} \left| \tilde{t}_{1/2}(t, y) - \tilde{t}_{1/2}(y') \right| \leq Nr^{d+2+\mu'}\left(\sum_{j=1}^{M} |f|((s+\delta)/2,\delta(1,0)\times\Omega_j) + \|Du\|_{L_1(Q)}\right),$$

where $\mu' = \min\left\{\frac{\alpha}{2}, \mu\right\}$. By using the same argument in deriving (3.48), we obtain

$$\left(\tilde{A}^{q\theta}(y') - \tilde{A}^{q\theta}(t, y)\right) D_{y_1} u(t, y; Az_0) + \tilde{t}_{1/2}(t, y; Az_0)$$

$$= \left(\tilde{A}^{q\theta}(y') - \tilde{A}^{q\theta}(t, y)\right) \Gamma^{q\theta} (\ell_1, \ell_2, \ell_3, D_h D_{i_1} D_{i_2} \cdots D_{i_s} u - D_h u - F_k)$$

$$+ \tilde{A}^{q\theta}(y') \Gamma^{q\theta} (D_h (\ell_1, \ell_2, \ell_3, D_h D_{i_1} D_{i_2} \cdots D_{i_s} u + D_h (R(u)) - F_k),$$

where $F_k$ and $\tilde{F}_k$ are defined in (4.13) and (4.20), respectively. Then similar to (3.51), using (4.16) and Lemma 2.1(iii), we have

$$\left\| \left(\tilde{A}^{q\theta}(y') - \tilde{A}^{q\theta}(t, y)\right) D_{y_1} u(t; Az_0) + \tilde{t}_{1/2}(t; Az_0) \right\|_{L_1(Q_r^c(Az_0))} \leq Nr^{d+5/2}C_1,$$

where $C_1$ is defined in (4.22). Thus (4.25) is proved. By (3.26), we have (4.26). The proof of Lemma 4.3 is finished.

With the above preparations, applying Lemmas 3.5 and 4.3 and following the process in the proof of Proposition 3.3 we obtain Proposition 4.2. With Proposition 4.2 at hand, replicating the proof Lemma 3.4 we have the following decay estimate of $\Psi(z_0, r)$.

Lemma 4.4. For any $0 < \rho \leq r \leq 1/4$, we have

$$\Psi(z_0, \rho) \leq N \left(\frac{\rho}{r}\right)^{\mu'} \Psi(z_0, r/2) + N \rho^{\mu'}C_1,$$
where $C_1$ is defined in (4.22), $\mu' = \min \left\{ \frac{1}{2}, \mu \right\}$, $N$ is a constant depending on $n$, $d$, $m$, $p$, $\nu$, $C$, $|A|_{(s+\delta)/2,s+\delta,-1,0} \times \mathcal{D}$, and the $C^{s+1+\mu}$ norm of $h_j$.

We shall further establish the estimates of $[D^s u]_{t,(1+\delta)/2,(-1+3\epsilon,0)} \times (B_{1-2\epsilon} \cap \mathcal{D})$, and $\|D^{s+1} u\|_{\infty}$.

Lemma 4.5. Under the same assumptions as in Proposition 4.1, we have

\[
\sum_{j=1}^{m+1} [D^s u]_{t,(1+\delta)/2,(-1+3\epsilon,0)} \times (B_{1-2\epsilon} \cap \mathcal{D}) + \sum_{j=1}^{m+1} |u_t|_{(s-1+\mu')/2,s-1+\mu',(-1+\epsilon,0)} \times (B_{1-\epsilon} \cap \mathcal{D})
\]

\[
+ \sum_{j=1}^{m+1} \|D^{s+1} u\|_{L^\infty(Q_{1/4} \cap (-1+\epsilon,0) \times \mathcal{D})}
\]

\[
\leq N\|D u\|_{L^1(Q_{3/4})} + N \sum_{j=1}^M |f|_{(s+\delta)/2,s+\delta,-1,0} \times \mathcal{D},
\]

where $N > 0$ is a constant depending only on $n, d, p, m, \nu, C$, $|A|_{(s+\delta)/2,s+\delta,-1,0} \times \mathcal{D}$, and the $C^{s+1+\mu}$ norm of $h_j$.

Proof. We start with the proof of the estimate of $[D^s u]_{t,1/2}$. Since $A^{\alpha\beta}, f^{\alpha} \in C^{(s+\delta)/2,s+\delta}((-1+\epsilon,0) \times (B_{1-\epsilon} \cap \mathcal{D}))$, one can verify that

\[
\delta_t^{\alpha} A^{\alpha\beta}, \delta_t^{\alpha\beta} f^{\alpha} \in C^{(s+\delta)/2,s+\delta,-1+2\epsilon}((-1+2\epsilon,0) \times (B_{1-\epsilon} \cap \mathcal{D}))
\]

where $\gamma \in (0, \frac{1+\delta}{2})$, $\delta_t^{\alpha\beta}$ is defined in (3.57). Applying the inductive assumption to (3.58), we obtain $\delta_t^{\alpha\beta} u \in C^{(s+\gamma')/2,s+\delta}((-1+3\epsilon,0) \times (B_{1-2\epsilon} \cap \mathcal{D}))$ with $\gamma' := \min\{1+\delta-2\gamma, \frac{1}{2}, \mu\} > 0$ and

\[
|\delta_t^{\alpha\beta} u|_{(s+\gamma')/2,s+\delta,(-1+3\epsilon,0) \times (B_{1-2\epsilon} \cap \mathcal{D})} \leq N \left( \sum_{j=1}^M |f|_{(s+\delta)/2,s+\delta,-1,0} \times \mathcal{D} \right) + \|D u\|_{L^2(Q_{3/4})}
\]

Taking $\gamma$ to be sufficiently close to $\frac{1+\delta}{2}$, we have $\gamma' = 1+\delta-2\gamma$, $\gamma + \frac{\gamma'}{2} = \frac{1+\delta}{2}$, and

\[
[D^s u]_{t,(1+\delta)/2,(-1+3\epsilon,0)} \times (B_{1-2\epsilon} \cap \mathcal{D}) \leq N \left( \sum_{j=1}^M |f|_{(s+\delta)/2,s+\delta,-1,0} \times \mathcal{D} \right) + \|D u\|_{L^2(Q_{3/4})}
\]

Next we prove the higher regularity of $u_t$. By differentiating the equation (1.1) with respect to $t$, we get

\[-(u_t)_t + D_\alpha (A^{\alpha\beta} D_\beta u_t) = D_\alpha \partial_t f^{\alpha} - D_\alpha (\partial_t A^{\alpha\beta} D_\beta u).
\]

By the inductive assumption, we have

\[
(4.27) \quad u \in C^{(s+\mu')/2,s+\mu'}((-1+\epsilon,0) \times (B_{1-\epsilon} \cap \mathcal{D})).
\]
where $\mu' = \min \left\{ \frac{1}{2}, \mu \right\}$, and thus $D_\beta u \in C^{(s-1+\mu')/2, s-1+\mu'}((-1 + \varepsilon, 0) \times (B_1 - \varepsilon \cap \overline{D})).$ Then combining
\[
\partial_t f^x, \partial_t A^{\alpha\beta} \in C^{(s-2+\delta)/2, s-2+\delta}((-1 + \varepsilon, 0) \times (B_1 - \varepsilon \cap \overline{D}))
\]
and the inductive assumption (4.1), we obtain
\[
(4.28) \quad u_t \in C^{(s-1+\mu')/2, s-1+\mu'}((-1 + \varepsilon, 0) \times (B_1 - \varepsilon \cap \overline{D})),
\]
and
\[
|u_t|_{(s-1+\mu')/2, s-1+\mu'}((-1 + \varepsilon, 0) \times (B_1 - \varepsilon \cap \overline{D})) \leq N\left(\|D\! u\|_{L^1((-1,0) \times D)} + \sum_{j=1}^M |f|^j_{(s-1+\delta)/2, s+\delta((-1,0) \times \overline{D})}\right).
\]

Finally, we estimate $\sum_{j=1}^{m+1} \|D^{s+1} u\|_{L^\infty(Q_j \cap ((-1,0) \times \overline{D}))}$. Similar to (3.61), we have
\[
\|D_\beta \tilde{u}(z_0; z_0)\| \leq N\|D\! u\|_{L^1((-1,0) \times D)} + \sum_{j=1}^M |f|^j_{(s+1+\delta)/2, s+\delta((-1,0) \times \overline{D})}
\]
for any $k = 1, \ldots, d - 1, z_0 \in (-1 + \varepsilon, 0) \times (D_e \cap D_j)$, and $r \in (0, 1/4)$. By using the definition of $\tilde{u}$ in (4.17), we have
\[
(4.29) \quad D_\beta \tilde{u}(z_0; z_0) = D_\beta D_\ell \cdots D_\ell u - D_\beta u_0 - D_\beta u
\]
and
\[
A^{\alpha\beta} D_\beta \tilde{u} = A^{\alpha\beta} D_\beta (D_\ell \cdots D_\ell u - u_0 - u),
\]
where $u_0$ and $u$ are defined in (4.10) and (4.15), respectively. Then combining (4.2) and (4.19), we obtain
\[
\tilde{u}(z_0) = n^a \left( A^{\alpha\beta} D_\beta \tilde{u} - f^a \right)
\]
\[
= n^a \left( A^{\alpha\ell_1} \ell_{i_1} \cdots \ell_{i_s} (D_{i_1} D_{i_2} \cdots D_{i_s} u - A^{\alpha\beta} D_\beta u - \ell_{i_1} \ell_{i_2} \cdots \ell_{i_s} D_{i_1} D_{i_2} \cdots D_{i_s} f^a
\]
\[
+ \ell_{i_1} \ell_{i_2} \cdots \ell_{i_s} (D_{i_1} A^{\alpha\beta} D_\beta D_{i_2} \cdots D_{i_s} u + \sum_{\tau=1}^{s-1} D_{i_{\tau+1}} \cdots D_{i_s} (D_{i_{\tau+1}} A^{\alpha\beta} D_\beta D_{i_{\tau+2}} \cdots D_{i_s} u))
\]
\[
(4.30) \quad - \delta_{n^a} \sum_{j=1}^m 1_{x^j > h_j(x')} (n^j_{(x')})^{-1} \hat{h}_j(t, x') - A^{\alpha\beta} F_\beta),
\]
where \( \tilde{h}(t, x') \) and \( F_\beta \) are defined in (4.8) and (4.13), respectively. Note that there are \( n_{(s+1)}^{d+s} \) components in \( D^{s+1} u \). From (4.29) and (4.30), we have \( n_{(s+1)}^{d+s} + n_{(s+2)}^{d+s} \) equations. Since
\[
\left( d + s + 1 \right) - \left( d + s - 1 \right) - \left( d + s - 2 \right) = (d + s - 2),
\]
we need another \( n_{(s+2)}^{d+s} \) equations to solve for \( D^{s+1} u \). For this, by taking the \( (s-1) \)-th derivative of the equation (1.1) with respect to \( x \) in each subdomain, we get the following \( n_{(s+2)}^{d+s} \) equations
\[
A^{a\beta} D_{a\beta} D^{s+1} u
\]
(4.31)
\[= D^{s+1} D_a f^a + D^{s-1} t - \sum_{i=1}^{s-1} (s-1) D^i A^{a\beta} D^{s-1-i} D_{a\beta} u - D^{s-1} (D_a A^{a\beta} D_{a\beta} u).
\]
It follows from the assumption \( A^{a\beta} \in C^{(s+\delta)/2, s+\delta}((-1+\varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{D})) \) and (4.27) that
\[
\sum_{i=1}^{s-1} (s-1) D^i A^{a\beta} D^{s-1-i} D_{a\beta} u, D^{s-1} (D_a A^{a\beta} D_{a\beta} u) \in C^{\mu'/2, \mu'}((-1+\varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{D})).
\]
Combining (4.28) and the condition on \( f \), one can see that the right-hand side of (4.31) is of class piecewise \( C^{\mu'/2, \mu'} \). By mimicking the argument in the proof of Lemma 3.8 (see Step 2), we can solve for \( D^{s+1} u \) by using Cramer’s rule and (4.29)–(4.31). Moreover,
\[
\sum_{j=1}^{m+1} \|D^{s+1} u\|_{L_\infty(Q; (z_1))}
\leq Nr^{\mu'} \sum_{j=1}^{m+1} \|D^{s+1} u\|_{L_\infty(Q; (z_1) \cap((-1+\varepsilon, 0) \times \overline{D}))}
+ Nr^{-1} \left( \|D u\|_{L_1(Q)} + \sum_{j=1}^{M} |f|_{(1+\delta)/2, 1+\delta((-1, 0) \times \overline{D})} \right).
\]
With this estimate, we obtain our desired estimate of
\[
\sum_{j=1}^{m+1} \|D^{s+1} u\|_{L_\infty(Q; (z_1/4) \cap((-1+\varepsilon, 0) \times \overline{D}))}
\]
by using a standard iteration argument. See, for instance, [10, Lemma 3.4]. The lemma is proved.

Finally, we finish the proof of Proposition 4.1.
Proof of Proposition 4.1 (a) The proof of (4.21) closely follows the argument in Section 3.3 by using Lemmas 4.4 and 4.5 and thus is omitted.

(b) Note that the directional derivatives of \( \ell \) are in general not piecewise Hölder continuous. However, for any \( z_0 = (t_0, x_0) \in (-1 + \varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{D}_{j_0}) \), by using (4.2) and (4.10), one can verify that the terms containing (directional) derivatives of \( \ell \) at \( x_0 \) in (4.29) and (4.30) are cancelled. Using (4.29)–(4.31) with \( z = z_0 \) and Cramer’s rule, we find that \( D^{s+1}u(z_0) \) is expressed by

\[
D_\ell \tilde{u}(z_0; z_0), \quad \hat{U}(z_0; z_0), \quad D^{s-1}u_t(z_0), \quad Du(z_0),
\]

\[
\delta_{ad} \sum_{j=1}^m 1_{x:j>(x')} (n_j(x'))^{-1} \tilde{h}_j(t_0, x'_0), \quad n^a(x_0), \quad D^{a}f^{a}(z_0),
\]

\[
A^{ab}(z_0), \quad D^j A^{ab}(z_0), \quad D^j u(z_0), \quad D^j u(t_0, P_j x_0),
\]

and (directional) derivatives of \( \tilde{\ell}, j(x_0) \) for \( j \neq j_0 \),

where \( i = 1, \ldots, s \), and \( \tilde{\ell}, j := (\tilde{\ell}, j, \ldots, \tilde{\ell}, d_i) \) is the smooth extension of \( \ell|_{D_j} \) to \( \bigcup_{k=1, k \neq j}^m D_k \). Similarly, for any \( z_0 = (t_0, x_0) \in (-1 + \varepsilon, 0) \times (B_{1-\varepsilon} \cap \overline{D}_{j_0}) \), \( D^{s+1}u(z_0) \) is expressed by (4.32) with \( z_0 \) replaced with \( z_0 \). Thus, combining (4.1), (4.15), (4.21), Assumption 1.1(b), and Lemma 4.5, we obtain

\[
[D^{s+1}u]_{\mu'/2, \mu'/2(-1+\varepsilon,0) \times (B_{1-\varepsilon} \cap \overline{D}_{j_0})} \leq N\left(\|Du\|_{L^1((-1,0) \times D)} + \sum_{j=1}^M |f|_{(s+\delta)/(2, s+\delta)}, (-1,0) \times D_j} \right)
\]

for \( j_0 = 1, \ldots, m + 1 \). The proof of Proposition 4.1 is complete. \( \square \)

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Appendix A.

Since the local boundedness and piecewise Hölder-regularity of \( Du \), and some auxiliary estimates proved in [11] are crucially used in the proofs above, we present them here for reader’s convenience.

Lemma A.1. [11] Lemma 4.5, Theorem 2.3] Let \( Q^{-}_1 = (-1,0) \times B_1, \varepsilon \in (0,1), \)

\( p \in (1,\infty), \quad A^{ab} \) and \( f^a \) satisfy Assumption 1.1 with \( s = 0 \). Let \( u \in \mathcal{H}^1(Q^{-}_1) \) be a weak solution to (1.1) in \( Q^{-}_1 \). Then \( u \in C^{(1+\mu')/2, 1+\mu'}((-1+\varepsilon,0) \times (B_{1-\varepsilon} \cap \overline{D}_{j_0}) \) and it holds that

\[
\|Du\|_{L^{\infty}(Q^{-}_1)} + |u|_{(1+\mu')/2, 1+\mu'}((-1+\varepsilon,0) \times (B_{1-\varepsilon} \cap \overline{D}_{j_0})}
\]
\[ \leq N\left(\|Du\|_{L^2(Q_1^1)} + \sum_{j=1}^{M} |f_{j/2,\delta((-1,0)\times \overline{D_j})}|\right), \]

where \( j_0 = 1, \ldots, m + 1, \mu' = \min\left\{ \frac{1}{2}, \mu \right\} \), \( N > 0 \) is a constant depending only on \( n, d, m, p, v, \epsilon \), \( |A|_{b/2,\delta((-1,0)\times \overline{D_j})} \), and the \( C^{1+\mu} \) norm of \( h_j \).

Let us recall the \( \mathcal{H}^1_p \)-estimate for parabolic equations with partially small bounded mean oscillation coefficients, \( i.e., \) there exists a small enough constant \( \gamma_0 = \gamma_0(d, n, p, v) \in (0, 1/2) \) and a constant \( r_0 \in (0, 1) \) such that for any \( r \in (0, r_0) \) and \( (t_0, x_0) \in Q_1^1 \) with \( B_r(x_0) \subset B_1 \), in a coordinate system depending on \( (t_0, x_0) \) and \( r \), one can find a \( \bar{A} = \bar{A}(x^d) \) satisfying

\[ (A.1) \quad \int_{Q_r(t_0, x_0)} |A(t, x) - \bar{A}(x^d)| \, dx \, dt \leq \gamma_0. \]

**Lemma A.2.** \([\bar{I}] \text{ Lemma 3.4}\) Let \( 1 < q < \infty \). Assume that \( A \) satisfies \((A.1)\) with a sufficiently small constant \( \gamma_0 = \gamma_0(d, n, q, v) \in (0, 1/2) \) and \( u \in \mathcal{H}^1_{1,loc} \) satisfies

\[ -u_t + D_a(A^{a\beta} D_{\beta} u) = g + \text{div} f \quad \text{in} \ Q_1^1, \]

where \( f, g \in L_q(Q_1^1) \). Then

\[ \|u\|_{\mathcal{H}^1_{1,q}(Q_1^1)} \leq N(\|u\|_{L^2(Q_1^1)} + \|g\|_{L_q(Q_1^1)} + \|f\|_{L_q(Q_1^1)}), \]

where \( N \) depends on \( n, d, v, q, \) and \( r_0 \).

The \( \mathcal{H}^1_p \)-solvability for parabolic systems with coefficients which satisfy \((A.1)\) in \( Q_1^1 \) is also used in the proof above.

**Lemma A.3.** \([\bar{I}] \text{ Lemma 3.5}\) For any \( p \in (1, \infty) \), \( f \in L_p(Q_1^1) \), the following hold.

(a) For any \( u \in \mathcal{H}^1_p(Q_1^1) \) satisfying

\[ (A.2) \quad -u_t + D_a(A^{a\beta} D_{\beta} u) = \text{div} f \quad \text{in} \ Q_1^1 \]

and \( u(-1, \cdot) \equiv 0 \) in \( B_1 \), we have

\[ (A.3) \quad \|u\|_{\mathcal{H}^1_p(Q_1^1)} \leq N\|f\|_{L_p(Q_1^1)}, \]

where \( A^{a\beta} \) is defined in \((3.10)\), \( N \) depends on \( d, n, p, v, \) and \( r_0 \).

(b) For any \( f \in L_p(Q_1^1) \), there exists a unique solution \( u \in \mathcal{H}^1_p(Q_1^1) \) of \((A.2)\) with the initial data \( u(-1, \cdot) \equiv 0 \) in \( B_1 \). Furthermore, \( u \) satisfies \((A.3)\).

Denote

\[ \mathcal{P}_0 u := -u_t + D_a(\bar{A}^{a\beta}(x^d) D_{\beta} u). \]

For parabolic systems with coefficients depending only on \( x^d \), we have the following result.
Lemma A.4. \[\text{Lemma 3.6}\] Let \( p \in (0, \infty) \). Assume that \( u \in C^{0,1}_{loc} \) satisfies \( P_0 u = 0 \) in \( Q^- \). Then there exists a constant \( N = N(n, d, p, \nu) \) such that

\[
[D^x u]_{1/2,1;Q^-_1} \leq N\|D^x u\|_{L_p(Q^-_1)}, \quad [\bar{A}^{d\beta}(x^d)D^\beta u]_{1/2,1;Q^-_1/2} \leq N\|Du\|_{L_p(Q^-_1)}.
\]

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