K-THEORY OF BERNOULLIhifts OF FInITE GroUPSh ON UHF-ALGEBRAS

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ABSTRACT. We show that the Bernoulli shift and the trivial action of a finite group $G$ on a UHF-algebra of infinite type are $KK^G$-equivalent and that the Bernoulli shift absorbs the trivial action up to conjugacy. As an application, we compute the $K$-theory of crossed products by approximately inner flips on classifiable $C^*$-algebras.

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1. INTRODUCTION

In topological dynamics, a very fertile class of examples is given by Bernoulli shifts, that is, by the shift action of a group $G$ on the product $X^G := \prod_G X$ of $G$-many copies of a given compact space $X$. When the space $X$ is moreover totally disconnected, the $K$-theory of the crossed product $C(X^G) \rtimes_r G$ can be computed in many cases [CEL13]. These computations and the techniques appearing in them are not only of intrinsic interest, but they make possible the computation of the $K$-theory of $C^*$-algebras associated to large classes of (inverse) semigroups, wreath products, and many more examples [CEL13, Li19, Li22]. The simplest non-commutative analogue of a totally disconnected space is a UHF-algebra, that is, a (possibly infinite) tensor product of matrix algebras. The non-commutative version of the Bernoulli shift is the shift action of a group $G$ on the tensor product $A^G = \bigotimes_{g \in G} A$ for a given unital $C^*$-algebra $A$. Our main result computes...
the K-theory of the associated crossed product in the case that \( G \) is finite and that \( A \) is a UHF-algebra:

**Theorem A** (Theorem 2.8). Let \( G \) be a finite group, let \( Z \) be a countable \( G \)-set and let \( M_n \) be a UHF-algebra of infinite type. Then \( M_n \) is \( \text{KK}^G \)-equivalent to \( M_n \hat{\otimes} Z \) where we equip \( M_n \) with the trivial \( G \)-action and \( M_n \hat{\otimes} Z \) with the Bernoulli shift. In particular, we have

\[
K_*(M_n \hat{\otimes} Z \rtimes G) \cong K_*(C^*(G) \otimes M_n) \cong K_*(C^*(G))[1/n].
\]

The proof of Theorem A relies on a representation theoretic argument about invertibility of a certain element in the representation ring \( R_{C^*}(G) \) after inverting sufficiently many primes (see Proposition 2.1). A byproduct of the proof is that the Bernoulli shift absorbs the trivial action not only in \( \text{KK} \)-theory, but up to conjugacy. We point out that this fact may alternatively be extracted from [HW08, Lemma 3.1].

**Theorem B** (Theorem 2.7). With the notation as in Theorem A, there is a \( G \)-equivariant isomorphism

\[
M_n \hat{\otimes} Z \cong M_n \otimes M_n \hat{\otimes} Z.
\]

One immediate consequence of Theorem A and [Zu04, Theorem 3.13] is that the Bernoulli shift \( G \rtimes M_n \hat{\otimes} Z \) as above does not have the Rokhlin property (see Corollary 2.10). Beyond finite group actions, Theorem A also has consequences for infinite groups satisfying the Baum–Connes conjecture with coefficients [BCH94].

**Corollary C** (Corollary 2.11). Let \( G \) be a countable discrete group satisfying the Baum–Connes conjecture with coefficients, let \( Z \) be a \( G \)-set, let \( A \) be a \( G \)-\( C^* \)-algebra and let \( M_n \) be a UHF-algebra. Assume that \( Z \) is infinite or that \( n \) is of infinite type. Then the inclusion \( A \to A \otimes M_n \) induces an isomorphism

\[
K_*(A \rtimes G)[1/n] \cong K_* \left( A \otimes M_n \hat{\otimes} Z \rtimes G \right).
\]

In particular, the right hand side is a \( \mathbb{Z}[1/n] \)-module.

Corollary C will be used in the follow-up paper [CEKN24] together with S. Chakraborty and S. Echterhoff to compute the K-theory of many more general Bernoulli shifts. Another consequence of Theorem A is that the Bernoulli shift of a countable amenable group \( G \) on a strongly self-absorbing (in the sense of [TW07]) \( C^* \)-algebra \( \mathcal{D} \) satisfying the UCT is \( \text{KK}^G \)-equivalent to the trivial \( G \)-action on \( \mathcal{D} \) (see Corollary 2.12 for \( \mathcal{D} = \mathcal{O}_\infty \), this is [Sza18, Corollary 6.9]).

In Section 3 we apply Theorem A and compute \( K_* \left( B \hat{\otimes} C_2 \rtimes C_2 \right) \) whenever \( B \) is a \( C^* \)-algebra with approximately inner flip (in the sense of [ER78]) satisfying the assumptions of the Elliott classification programme.¹ Here,

¹We refer to [Win18, Whi23] and the references therein for an overview of the Elliott programme.
$C_2$ is the cyclic group of order 2. Thanks to Tikuisis’ classification of such C*-algebras \[^{[Tik16, EST24]}\], the computation reduces to the following special case (only the case $n = 1$ is relevant):

**Theorem D** (Theorem 3.2). For supernatural numbers $m$ and $n$ of infinite type, we have

$$K_* \left( q_{C_2}^{\infty \otimes C_2} \rtimes C_2 \right) \cong \begin{cases} \mathbb{Q}_n / \mathbb{Z} \oplus \mathbb{Q}_r / \mathbb{Z}, & * = 0; \\ \mathbb{Q}_m / \mathbb{Z} \oplus \mathbb{Q}_m / \mathbb{Z}, & * = 1, \end{cases}$$

where $r$ is the greatest common divisor of $m$ and $n$.

We refer to Section 3 for the definition of the notation appearing above. Our methods heavily build on Izumi’s computation of the $K$-theory of flip automorphisms on $C^*$-algebras with finitely generated K-theory \[^{[Izu19]}\].

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## 2. KK-theory of Bernoulli shifts

For a finite group $G$, denote by $R_C(G)$ its representation ring, defined as the Grothendieck group of the monoid of isomorphism classes of finite-dimensional complex representations of $G$ with the direct sum as addition and the tensor product as multiplication. The character of a finite-dimensional complex representation $\pi: G \to GL(V_\pi)$ is denoted by

$$\chi_\pi: G \to \mathbb{C}, \quad \chi_\pi(g) := \text{tr} \left( V_\pi \xrightarrow{\pi(g)} V_\pi \right),$$

where tr denotes the (non-normalized) trace. Recall that the map

$$R_C(G) \to C_{\text{class}}(G), \quad \pi \mapsto \chi_\pi$$

is an injective ring homomorphism with values in the algebra $C_{\text{class}}(G)$ of conjugation invariant functions on $G$ with pointwise multiplication. There is a natural isomorphism $R_C(G) \cong KK^G(\mathbb{C}, \mathbb{C})$. We refer to \[^{[Ser77]}\] for an introduction to representation theory of finite groups and to \[^{[Kas88]}\] for the definition of equivariant KK-theory.

**Proposition 2.1.** Let $G$ be a finite group, let $k \geq 1$ and let $Z$ be a finite $G$-set. Denote by $\pi_k: G \to GL(\ell^2(\{1, \ldots, k\}^Z))$ the permutation representation associated to the $G$-set $\{1, \ldots, k\}^Z$. Then the following hold.

1. There exist $\alpha \in R_C(G)$ and $r \geq 1$ such that $[\pi_k]^r = k\alpha$.
2. There exist $\beta \in R_C(G)$ and $l \geq 1$ such that $[\pi_k] \cdot \beta = k^l$. 

Proof. By considering the standard basis in $\ell^2(\{1, \ldots, k\})$, it is easy to see that the trace of $\pi_k(g)$ for $g \in G$ is given by the number of $g$-fixed points in $\{1, \ldots, k\}$, which is the same as the number of $\langle g \rangle$-invariant functions $Z \to \{1, \ldots, k\}$. In other words, the character of $\pi_k$ is given by

$$\chi_{\pi_k}(g) = k^{[Z/\langle g \rangle]}.$$ 

We therefore have

$$\prod_{g \in G} \left(\chi_{\pi_k} - k^{[Z/\langle g \rangle]}\right) = 0 \text{ in } C_{\text{class}}(G).$$

Since the map $\pi \mapsto \chi_\pi$ is injective, we also have

$$\prod_{g \in G} \left([\pi_k] - k^{[Z/\langle g \rangle]}\right) = 0 \text{ in } R_C(G).$$

In particular, there are polynomials $p, q \in \mathbb{Z}[t]$ satisfying

$$[\pi_k]^G = kp([\pi_k]), \quad [\pi_k] \cdot q([\pi_k]) = \prod_{g \in G} k^{[Z/\langle g \rangle]},$$

which proves the proposition. \hfill \Box

**Definition 2.2.** Let $Z$ be a set and let $(A_z)_{z \in Z}$ be a collection of unital $C^\ast$-algebras. The infinite tensor product $\bigotimes_{z \in Z} A_z$ is defined as

$$\bigotimes_{z \in Z} A_z := \lim_{F \subseteq Z} \bigotimes_{z \in F} A_z,$$

where the inductive limit is taken over all finite subsets $F \subseteq Z$ ordered by inclusion, with respect to the connecting maps $a \mapsto a \otimes 1$. Given a discrete group $G$, a unital $C^\ast$-algebra $A$ and a $G$-set $Z$, the *Bernoulli shift* of $G$ on $A^{\otimes Z} := \bigotimes_{z \in Z} A$ is the $G$-action induced by permuting the tensor factors according to the $G$-action on $Z$.

**Definition 2.3.** A *supernatural number* is a formal product $n = \prod_p n_p$ where $p$ runs over all primes and $n_p \in \{0, \ldots, \infty\}$. The *UHF-algebra* associated to $n$ is the infinite tensor product

$$M_n := \bigotimes_p M_{p^{n_p}},$$

with $M_p^{\infty} := M_p^{\otimes \mathbb{N}}$. We call $n$ or $M_n$ of *infinite type* if $n_p \in \{0, \infty\}$ for all $p$. We say that $n = \prod p^{n_p}$ divides $m = \prod p^{m_p}$ if $n_p \leq m_p$ for all $p$.

**Remark 2.4.** Note that the above definition includes natural numbers and matrix algebras as a special case.

**Definition 2.5.** If $M$ is an abelian group, we denote by $M[1/n]$ the inductive limit of the system

$$M \xrightarrow{p_1} M \xrightarrow{p_2} M \xrightarrow{p_3} \ldots,$$

where $(p_1, p_2, \ldots)$ contains each prime dividing $n$ infinitely many times.
Remark 2.6. If \( q = \prod_p p^{n_p} \) with \( n_p \geq 1 \) for all \( p \), then
\[ M[1/q] \cong M \otimes \mathbb{Q}. \]
If \( k \geq 1 \) is a positive integer, then
\[ \mathbb{Z}[1/k] \cong \left\{ \frac{m}{k^n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}_{>0} \right\} \subseteq \mathbb{Q}. \]
In general, the group \( \mathbb{Z}[1/n] \) is different from the closely related group
\[ \mathbb{Q}_n := \left\{ \frac{m}{k} \mid m \in \mathbb{Z}, k \in \mathbb{Z}_{>0} \text{ divides } n \right\}, \]
unless \( n \) is of infinite type.

Theorem 2.7 (cf. [HW08, Lemma 3.1]). Let \( G \) be a finite group, let \( M_n \) be a UHF-algebra and let \( Z \) be a \( G \)-set. Assume that \( Z \) is infinite or that \( n \) is of infinite type. Equip \( M_n \) with the trivial \( G \)-action and \( M_n \otimes Z \) with the Bernoulli shift. Then there is an equivariant isomorphism
\[ M_n \otimes Z \cong M_n \otimes \mathbb{Q}. \]
If \( Z \) is infinite, and \( m < \infty \), there is an equivariant isomorphism
\[ M_m \otimes Z \cong M_m \otimes \mathbb{Q}. \]

Proof. Note that it suffices to prove the statement in the case that \( M_n = M_{p^k} \) (or \( M_m = M_{p^k} \)) for a prime \( p \) and \( k \in \{0, 1, \ldots, \infty\} \), since the general case follows by taking (possibly infinite) tensor products over all primes. As before, if \( Z \) is finite, we denote by \( \pi_p \) the permutation representation of \( G \) on \( V_p := \ell^2([1, \ldots, p]^Z) \), so that \( M_p \otimes Z \) is equivariantly isomorphic to \( \text{End}(V_p) \).

Assume first that \( k = \infty \). We only need to prove the theorem for (any) one \( G \)-orbit of \( Z \) so we may assume that \( Z \) is finite. Let \( \alpha \in R_C(G) \) and \( r \geq 1 \) be as in Proposition 2.1 so that \( [\pi_p]^r = p\alpha \in R_C(G) \). Since \( [\pi_p] \) is a non-negative linear combination of irreducible representations of \( G \), \( \alpha \) has to be the class of a finite-dimensional representation \( \pi_\alpha : G \to \text{GL}(W_\alpha) \). In particular, we have an equivariant isomorphism \( V_p^{\otimes r} \cong \mathbb{C}^p \otimes W_\alpha \). Passing to endomorphisms, we obtain an equivariant isomorphism
\[ \left( M_p \otimes Z \right)^{\otimes r} \cong M_p \otimes \text{End}(W_\alpha) \]
with the trivial \( G \)-action on \( M_p \). By taking the infinite tensor product we obtain an equivariant isomorphism
\[ M_p^{\otimes Z} \cong M_p^{\otimes} \otimes \text{End}(W_\alpha)^{\otimes_\mathbb{N}} \cong M_p^{\otimes} \otimes M_p^{\otimes} \otimes \text{End}(W_\alpha)^{\otimes_\mathbb{N}} \cong M_p^{\otimes} \otimes M_p^{\otimes Z}. \]

Assume now that \( k < \infty \) and that \( Z \) is infinite. Then \( Z \) contains infinitely many orbits of the same type \( G/H \). We may thus assume\(^2\) that \( Z \) is of the form \( Z = \bigsqcup_n G/H \) for some subgroup \( H \subseteq G \). Then there is an equivariant isomorphism
\[ M_n^{\otimes (Z \setminus U_n(G/H))}. \]

\(^2\)The general case follows by taking tensor products with the remaining factor \( M_n^{\otimes (Z \setminus U_n(G/H))} \).
isomorphism $M^\otimes_{p^k} \cong M^\otimes_{p^\infty}/H$. This reduces the proof to the case considered above. □

**Theorem 2.8.** Let $G$ be a finite group, let $Z$ be a countable $G$-set and let $M_n$ be a UHF-algebra of infinite type. Then the canonical inclusions

$$M_n \hookrightarrow M_n \otimes M_n^\otimes Z \hookrightarrow M_n^\otimes Z$$

are $\text{KK}^G$-equivalences, where $M_n$ is endowed with the trivial action and where $M_n^\otimes Z$ is endowed with the Bernoulli shift. If $Z$ is infinite, and $m < \infty$, the same conclusion holds for the inclusions

$$M_m^\otimes \hookrightarrow M_m^\otimes \otimes M_m^\otimes Z \hookrightarrow M_m^\otimes Z.$$  

**Proof.** Since $M_n$ is strongly self-absorbing (in the sense of [TW07]), the map

$$\text{id}_{M_n} \otimes 1: M_n \to M_n \otimes M_n$$

is a $\text{KK}$-equivalence. Using Theorem 2.7, we can identify the map

$$\text{id}_{M_n^\otimes Z} \otimes 1: M_n^\otimes Z \hookrightarrow M_n^\otimes Z \otimes M_n$$

with the map

$$\text{id}_{M_n^\otimes Z} \otimes (\text{id}_{M_n} \otimes 1): M_n^\otimes Z \otimes M_n \to M_n^\otimes Z \otimes M_n \otimes M_n,$$

which is a $\text{KK}^G$-equivalence. Similarly, if $Z$ is infinite and $m < \infty$, the map

$$M_m^\otimes \hookrightarrow M_m^\otimes \otimes M_m^\otimes Z$$

is a $\text{KK}^G$-equivalence. We prove that the map

(2.1) $\text{id}_{M_n^\otimes Z} \otimes 1: M_n \to M_n \otimes M_n^\otimes Z$

is a $\text{KK}^G$-equivalence. Note that this map is the inductive limit of the maps

(2.2) $\text{id}_{M_n^\otimes Y} \otimes 1: M_n \to M_n \otimes M_n^\otimes Y$

where $k$ ranges over all positive integers that divide $n$ and where $Y$ ranges over all finite $G$-subsets of $Z$. It follows from the finiteness of $G$, the nuclearity of the involved algebras and [MN06, Proposition 2.6, Lemma 2.7] that the map in (2.1) is also the homotopy colimit (with respect to the triangulated structure of $\text{KK}^G$) of the maps in (2.2). Since a homotopy colimit of $\text{KK}^G$-equivalences is a $\text{KK}^G$-equivalence, it suffices to show that the maps appearing in (2.2) are $\text{KK}^G$-equivalences.

Note that $\ell^2([1, \ldots, k]^Y)$ implements an equivariant Morita equivalence between $M_k^\otimes Y$ and $\mathcal{C}$ which maps the class of the inclusion $\mathcal{C} \to M_k^\otimes Y$ in $\text{KK}^G_{\mathcal{C}, M_k^\otimes Y}$ to the class $[\pi_k] \in \text{KK}^G_{\mathcal{C}, \mathcal{C}}$ of the permutation representation $\pi_k: G \to \text{GL}(\ell^2([1, \ldots, k]^Y))$. Therefore, the maps in (2.2) can be identified with the elements $[\text{id}_{M_n^\otimes} \otimes \chi_k] \in \text{KK}^G(M_n, M_n)$. 

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3 This follows from the axioms of a triangulated category. The fact that homotopy colimits of maps are not unique does not cause a problem here.
By Proposition 2.1, there is an element \( \beta \in \text{KK}^G(\mathbb{C}, \mathbb{C}) \) and \( l \geq 1 \) such that \( [\pi_k \beta] = k^l \). Thus \( [\text{id}_{M_n} \otimes_G [\pi_k]] \) is invertible with inverse \( \frac{1}{k^l} [\text{id}_{M_n} \otimes_G \beta] \). The same proof shows that, if \( Z \) is infinite and \( m < \infty \), the map

\[
\text{id}_{M_{n \infty}} \otimes_{C^*} \pi_{M_{n \infty}} : M_{n \infty} \rightarrow M_{m \infty} \otimes M_{m} \otimes Z 
\]

is a \( \text{KK} \)-equivalence. \( \square \)

**Remark 2.9.** By [GL21, Theorem B] and [GHV22, Theorem B], a countable discrete group \( G \) is amenable if and only if for some (any) supernatural number \( n \neq 1 \) of infinite type, the Bernoulli shift on \( M_n^{\otimes_G} \) absorbs the trivial action on the Jiang-Su algebra \( Z \) up to cocycle conjugacy. In particular (since \( M_n \cong M_n \otimes Z \)), the conclusion of Theorem 2.7 is false for non-amenable groups. On the other hand, Theorem 2.8 together with the Higson-Kasparov Theorem [HK01] (applied in the form of [MN06, Theorem 8.5]) implies that if \( G \) is a countable amenable group, then \( M_n^{\otimes_G} \) absorbs the trivial action on \( M_n \) up to \( \text{KK} \)-equivalence. It is thus conceivable that a countable discrete group \( G \) is amenable if and only if the Bernoulli shift on \( M_n^{\otimes_G} \) absorbs the trivial action on \( M_n \) up to cocycle conjugacy.

The following observation provides some evidence for this: Let \( G \) be a countable amenable group, \( n \neq 1 \) a supernatural number of infinite type, and \( A \) a \( G \)-\( C^* \)-algebra. By the remarks above, the unital embedding

\[
\text{id} \otimes 1 : (A \otimes M_n^{\otimes_G}) \times G \rightarrow (A \otimes M_n^{\otimes_G}) \times G \otimes M_n 
\]

is a \( \text{KK} \)-equivalence between \( Z \)-stable \( C^* \)-algebras that induces an isomorphism on the trace spaces, in particular it induces an isomorphism on the Elliott invariants. If we additionally assume that \( (A \otimes M_n^{\otimes_G}) \times G \) is simple, separable, nuclear, and satisfies the UCT (which happens in many cases of interest), then the classification of unital, simple, separable, nuclear, \( Z \)-stable \( C^* \)-algebras satisfying the UCT [Phi00, EGLN15, TWW17, CET+21, CGS+23] implies that \( (A \otimes M_n^{\otimes_G}) \times G \cong (A \otimes M_n) \times G \otimes M_n \). This condition is certainly necessary for \( M_n^{\otimes_G} \) to absorb \( M_n \) up to cocycle conjugacy.

**Corollary 2.10.** Let \( G \neq \{e\} \) be a finite group, let \( Z \) be a \( G \)-set and let \( M_n \) be a UHF-algebra of infinite type. Then the Bernoulli shift of \( G \) on \( M_n^{\otimes_Z} \) does not have the Rohlin property.

**Proof.** Assume the contrary. Then [Izu04, Theorem 3.13] yields an isomorphism\(^4\)

\[
K_0 \left( M_n^{\otimes_{Z \times G}} \right) \cong K_0 \left( M_n^{\otimes_Z} \right) = Z/[1/n].
\]

On the other hand, Theorem 2.8 yields an isomorphism\(^5\)

\[
K_0 \left( M_n^{\otimes_{Z \times G}} \right) \cong K_0(C^*(G)) \otimes Z/[1/n] \cong Z/[1/n]^{\otimes \hat{G}},
\]

\(^4\)Theorem 3.13 of [Izu04] is applicable by the combination of [Phi87, Proposition 7.1.3] and [Kis81, Theorem 3.1].

\(^5\)This K-theoretic statement follows from the countable case by taking inductive limits over all countable \( G \)-subsets of \( Z \).
a contradiction.

The following corollary is used in the follow-up paper [CEKN24] with Sayan Chakraborty and Siegfried Echterhoff. We refer to [BCH94] for the formulation of the Baum–Connes conjecture with coefficients. Note that the Baum-Connes conjecture with coefficients holds for many groups, including a-T-menable groups [HK01] and hyperbolic groups [Laf12].

**Corollary 2.11.** Let $G$ be a countable discrete group satisfying the Baum–Connes conjecture with coefficients, let $Z$ be a $G$-set, let $A$ be a $G$-$C^*$-algebra and let $M_n$ be a UHF-algebra. Assume that $Z$ is infinite or that $n$ is of infinite type. Then the inclusion $A \rightarrow A \otimes M_n \otimes Z$ induces an isomorphism

$$K_*(A \rtimes_r G) \cong K_* \left( (A \otimes M_n) \rtimes_r G \right).$$

In particular, the right hand side is a $\mathbb{Z}[1/n]$-module.

**Proof.** By an inductive limit argument, we may assume $Z$ is countable and $A$ is separable. If $G$ is finite, the statement follows from Theorem 2.8 considering the commutative diagram

$$
\begin{array}{ccc}
A & \rightarrow & A \otimes M_n \otimes Z \\
\downarrow_{\phi_1} & & \downarrow_{\phi_2} \\
A \otimes M_n & \rightarrow & A \otimes M_n \otimes M_n \otimes Z
\end{array}
$$

where $\phi_1, \phi_2$ are $KK^G$-equivalences. Assume now that $G$ is infinite. Consider the diagram (2.3). We know that (the restrictions of) $\phi_1, \phi_2$ are $KK^H$-equivalences for every finite subgroup $H \subseteq G$. Since $G$ satisfies the Baum–Connes conjecture with coefficients, the results of [CEO04] (see also [MN06]) imply that $\phi_1$ and $\phi_2$ induce isomorphisms of the $K$-theory groups of reduced crossed products by $G$. The statement follows from this by identifying $K_* \left( (A \otimes M_n) \rtimes_r G \right) \cong K_* \left( A \rtimes_r G \right)[1/n]. \qed$

We end this section with an application to Bernoulli shifts on strongly self-absorbing $C^*$-algebras. Recall that a separable, unital $C^*$-algebra $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing [TW07] if there is an isomorphism $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the first factor inclusion $id_{\mathcal{D}} \otimes 1_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$. Strongly self-absorbing $C^*$-algebras are automatically simple, nuclear [TW07] and $\mathcal{Z}$-stable [Win11]. By the combination of [TW07], Proposition 5.1 and the classification of unital, simple, separable, nuclear, $\mathcal{Z}$-stable $C^*$-algebras in the UCT class [Phi00, EGLN15, TWW17, CET+21, CGS+23], a complete list of strongly self-absorbing $C^*$-algebras satisfying the UCT is given by

$$\mathcal{Z}, M_n, \mathcal{O}_\infty, \mathcal{O}_\infty \otimes M_n, \mathcal{O}_2,$$

where $n \neq 1$ is a supernatural number of infinite type. The following corollary is a generalization of [Sza18 Corollary 6.9]:

...
Corollary 2.12. Let $D$ be a strongly self-absorbing C*-algebra satisfying the UCT and let $G$ be a countable discrete group having a $\gamma$-element equal to $1$. Then, for any countable G-set $Z$, the $G$-C*-algebra $D \otimes Z$ equipped with the Bernoulli shift is $KK^G$-equivalent to $D$ equipped with the trivial $G$-action.

For the proof, we need the following result of Izumi [Izu19] which we spell out here for later reference.

Theorem 2.13 ([Izu19, Theorem 2.1], see also [Sza18, Lemma 6.8]). Let $A, B$ be separable nuclear C*-algebras, let $H$ be a finite group and let $Z$ be a finite $H$-set. Then, there is a map from $KK(A, B)$ to $KK^H(A \otimes Z, B \otimes Z)$ which in particular, sends a class of a $\ast$-homomorphism $\phi$ to the class of $\phi \otimes Z$. Furthermore, this map is compatible with the compositions and in particular sends a $KK$-equivalence to a $KK^H$-equivalence. In particular, the Bernoulli shifts on $A \otimes Z$ and $B \otimes Z$ are $KK^H$-equivalent if $A$ and $B$ are $KK$-equivalent.

Proof of Corollary 2.12. We claim that the unital embeddings

\begin{equation}
D \leftrightarrow D \otimes Z \leftrightarrow D \otimes Z
\end{equation}

are $KK^G$-equivalences. By the assumption on $G$, this amounts to showing that they are $KK^H$-equivalences for every finite subgroup $H \subseteq G$. By the same homotopy co-limit argument as in the proof of Theorem 2.8 it is enough to show that the maps

\begin{equation}
D \leftrightarrow D \otimes Y \leftrightarrow D \otimes Y
\end{equation}

are $KK^H$-equivalences for all finite $H$-subsets $Y$ of $Z$. Now Theorem 2.13 allows us to replace $D$ by a $KK$-equivalent C*-algebra. Thanks to the list (2.4), this reduces the problem to the cases $D = \mathbb{C}$, $D = 0$ and $D = M_n$. The first two cases are trivial and the third one follows from Theorem 2.8. □

3. K-theory of Approximately Inner Flips

In this section we apply Theorem 2.8 to the K-theory of approximately inner flips. Recall that a C*-algebra $A$ is said to have approximately inner flip if the flip automorphism $A \otimes A \to A \otimes A$, $a \otimes b \mapsto b \otimes a$ is approximately inner, i.e. a point-norm limit of inner automorphisms. A C*-algebra $A$ with approximately inner flip must be simple, nuclear and have at most one trace [ER78]. An approximately inner flip necessarily induces the identity map on $K_0(A \otimes A)$ and this largely restricts the class of C*-algebras $A$ with approximately inner flip. Effros and Rosenberg [ER78] showed that if $A$ is AF, then $A$ must be stably isomorphic to a UHF-algebra. Tikuisis [Tik16] determined a complete list of classifiable C*-algebras with approximately

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6See [MN06, Section 8] for a definition of the $\gamma$-element (where $X = \text{pt}$ in our case). By the Higson–Kasparov theorem [HK01], this assumption is satisfied for all a-T-menable groups.

7In the notation of [MN06], this would imply that the mapping cone of (2.5) is in $\mathbb{C} \subseteq KK^G$, but [MN06, Theorem 8.3] implies that $\gamma = 1$ if and only if $\mathbb{C} \subseteq 0$.\"
inner flip. We would like to thank Dominic Enders, André Schemaitat and Aaron Tikuisis for informing us about a corrigendum stated below:

**Theorem 3.1.** ([EST24, Theorem 1.3], Correction to [Tik16, Theorem 2.2]) Let $A$ be a separable, unital C*-algebra with strict comparison, in the UCT class, which is either infinite or quasidiagonal. The following are equivalent.

1. $A$ has approximately inner flip;
2. $A$ is Morita equivalent to one of the following C*-algebras:
   - $C$;
   - $C_{n+1,m}$;
   - $C_{n+1,m} \otimes O_\infty$;
   - $F_{n,m}$.

Here $m, n$ are supernatural numbers with $m$ of infinite type such that $m$ divides $n$, $O_\infty$ is the Cuntz algebra on infinitely many generators, $C_{n+1,m}$ is the simple, separable, unital, $\mathbb{Z}$-stable, quasidiagonal C*-algebra in the UCT class with unique trace satisfying

$$K_0(C_{n+1,m}) \cong \mathbb{Q}_n, \quad [1]_0 = 1; \quad K_1(C_{n+1,m}) \cong \mathbb{Q}_m/\mathbb{Z},$$

and $F_{n,m}$ is the unique unital Kirchberg algebra in the UCT class satisfying

$$K_0(F_{n,m}) \cong \mathbb{Q}_n/\mathbb{Z}, \quad [1]_0 = 0; \quad K_1(F_{n,m}) \cong \mathbb{Q}_m/\mathbb{Z}.$$

Let $C_2$ be the cyclic group of order 2. Our strategy to compute the groups $K_*(A \otimes C_2 \rtimes C_2)$ with $A$ as in Theorem 5.1 builds on Izumi’s remarkable computation of $K_*(B \otimes C_2 \rtimes C_2)$ for all separable nuclear C*-algebras $B$ in the UCT class and with finitely generated K-theory [Izu19]. Izumi’s starting point is his Theorem 2.13 above. This allows him to replace the appearing C*-algebras by finite direct sums of building blocks of the form $C$, $C_0(\mathbb{R})$, $O_{n+1}$ and $D_n$, where $O_{n+1}$ denotes the Cuntz algebra on $n + 1$ generators and where $D_n$ denotes the dimension drop algebra. For $B$ one of these building blocks, Izumi explicitly computes $K_* (B \otimes C_2 \rtimes C_2)$.

We follow the same strategy here. Thanks to Theorems 3.1 and 2.13 above, the following theorem and its corollary determine the K-theory groups $K_* (B \otimes C_2 \rtimes C_2)$ whenever $B$ is a UCT C*-algebra that is KK-equivalent to a C*-algebra in the list in Theorem 5.1 for $m$ and $n$ are of infinite type.

**Theorem 3.2.** Let $m$ and $n$ be supernatural numbers of infinite type. Let $F_{n,m}$ be any C*-algebra satisfying the UCT such that

$$K_*(F_{n,m}) \cong \begin{cases} \mathbb{Q}_n/\mathbb{Z}, & * = 0; \\ \mathbb{Q}_m/\mathbb{Z}, & * = 1. \end{cases}$$

Then we have

$$K_* \left( F_{n,m} \otimes C_2 \right) \cong \begin{cases} \mathbb{Q}_n/\mathbb{Z} \oplus \mathbb{Q}_\tau/\mathbb{Z}, & * = 0; \\ \mathbb{Q}_m/\mathbb{Z} \oplus \mathbb{Q}_m/\mathbb{Z}, & * = 1, \end{cases}$$

where $\tau$ is the greatest common divisor of $m$ and $n$.  


Corollary 3.3. For any supernatural numbers $m$ and $n$ of infinite type, we have

$$K_*(\varepsilon_{n,m} \otimes C_2) \cong \begin{cases} \mathbb{Q}_n \oplus \mathbb{Q}_n, & * = 0; \\ \mathbb{Q}_{mn} / \mathbb{Q}_n \oplus \mathbb{Q}_m / \mathbb{Z} \oplus \mathbb{Z}, & * = 1. \end{cases}$$

Proof. The algebra $\varepsilon_{n,m}$ is KK-equivalent to $M_1 \oplus \mathcal{F}_{1,m}$. Thus, using Theorem 2.13 we see that $\varepsilon_{n,m} \otimes C_2$ is KK-equivalent to

$$(M_n \otimes C_2 \times C_2) \oplus (\mathcal{F}_{1,m} \otimes C_2) \oplus (M_n \otimes \mathcal{F}_{1,m}).$$

Note that $K_0(M_n \otimes \mathcal{F}_{1,m}) \cong 0$ and $K_1(M_n \otimes \mathcal{F}_{1,m}) \cong \mathbb{Q}_n \otimes \mathbb{Q}_m / \mathbb{Z} \cong \mathbb{Q}_{mn} / \mathbb{Q}_n$. The assertion now follows from Theorem 3.2.

We break up the proof of Theorem 3.2 into two lemmas. We denote by $[e_0], [e_1], [1] \in K_0(C^*(\mathbb{Z}/2))$ the classes of the trivial representation, the sign representation and the unit of $C^*(\mathbb{Z}/2)$. We will abuse notation and write KK-elements as arrows $\rightarrow_{KK}$ between $C^*$-algebras, well-aware that they might not be induced by $*$-homomorphisms.

Lemma 3.4. We have

$$K_* (\mathcal{F}_{m,1} \otimes C_2) \cong \begin{cases} \mathbb{Q}_m / \mathbb{Z}, & * = 0; \\ 0, & * = 1. \end{cases}$$

Proof. By the Kirchberg–Phillips classification theorem [Phi00], there is a unital $*$-homomorphism

$$M_m \otimes O_{\infty} \rightarrow \mathcal{F}_{m,1}$$

such that the composition

$$\phi: M_m \xrightarrow{id \otimes 1} M_m \otimes O_{\infty} \xrightarrow{} \mathcal{F}_{m,1}$$

induces the canonical quotient map $\mathbb{Q}_m \rightarrow \mathbb{Q}_m / \mathbb{Z}$ on $K_0$. Denote by

$$M_\phi := \{(a, f) \in M_m \oplus (C[0, 1] \otimes \mathcal{F}_{m,1}) | \phi(a) = f(0)\}$$

the mapping cylinder of $\phi$ and by $C_\phi := \ker(ev_1: M_\phi \rightarrow \mathcal{F}_{m,1})$ the mapping cone of $\phi$. Note that the inclusion $M_m \hookrightarrow M_\phi$ is a KK-equivalence. The short exact sequence

$$0 \rightarrow C_\phi \rightarrow M_\phi \rightarrow \mathcal{F}_{m,1} \rightarrow 0,$$

induces an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q}_m \rightarrow \mathbb{Q}_m / \mathbb{Z} \rightarrow 0$$

on $K_0$ and $0$ on $K_1$. We streamline the notations and re-write the exact sequence (3.1) as

$$0 \rightarrow I \rightarrow B \rightarrow A_0 \rightarrow 0.$$

The only properties that we will use are the induced sequence (3.2) on $K_0$ and that the map $I \rightarrow B$ can be identified with the unital inclusion $\mathbb{C} \hookrightarrow M_m$ in KK-theory. From now on, we follow the beautiful computations of
[Izu19] Theorem 3.4. Writing $I_1 := I \otimes B + B \otimes I \subseteq B^\otimes \mathbb{Z}/2$, we have the following short exact sequences of $C^*$-algebras

(3.3) \[ 0 \longrightarrow I_1 \longrightarrow B^\otimes C_2 \longrightarrow A_0^\otimes C_2 \longrightarrow 0, \]

(3.4) \[ 0 \longrightarrow I^\otimes C_2 \longrightarrow I_1 \longrightarrow (I \otimes A_0) \oplus (A_0 \otimes I) \longrightarrow 0. \]

Taking crossed-products and applying $K$-theory for (3.4) produces the 6-term exact sequence

(3.5) \[ \begin{array}{ccc}
Z \oplus Z & \longrightarrow & K_0(I_1 \times C_2) \longrightarrow Q_m/Z \\
0 & \longleftarrow & K_1(I_1 \times C_2) \longleftarrow 0.
\end{array} \]

Here the generators of $Z \oplus Z$ are the image of $[1]$ and $[e_1]$ via the KK-equivalence $C^\otimes C_2 \times C_2 \overset{\gamma_{KK}}{\longrightarrow} I^\otimes C_2 \times C_2$ obtained from the KK-equivalence $C \overset{\gamma_{KK}}{\longrightarrow} I$ as in Theorem 2.13 and $Q_m/Z$ is identified with $K_0(I \otimes A_0)$. Consider the map from the exact sequence

(3.6) \[ \begin{array}{ccc}
K_0(I \otimes I) \oplus Z & \overset{\cong Z}{\longrightarrow} & K_0(I \otimes B) \oplus Z \overset{\cong Q_m}{\longrightarrow} K_0(I \otimes A_0) \overset{\cong Q_m/Z}{\longrightarrow} \end{array} \]

to the top row of (3.5), which on $K_0$-groups is induced by the canonical algebra inclusions and which sends the generator of $Z$ to the class $[e_1]$. This map clearly is an isomorphism on both the left-hand term and the right-hand term, and therefore also on the middle term. From this we see that $K_0(I_1 \times C_2) \cong Q_m \oplus Z$, with the generator of $Z$ being the image of $[e_1]$ via the KK-element $C^\otimes C_2 \times C_2 \overset{\gamma_{KK}}{\longrightarrow} I^\otimes C_2 \times C_2 \rightarrow I_1 \times C_2$ and with $Q_m$ being the image of $K_0(M_m)$ via the KK-element $M_m \rightarrow B \overset{\gamma_{KK}}{\longrightarrow} I \otimes B \rightarrow I_1$.

Taking crossed products and applying $K$-theory for (3.3) yields the 6-term exact sequence

(3.7) \[ \begin{array}{ccc}
Q_m \oplus Z & \longrightarrow & Q_m \oplus Q_m \longrightarrow K_0 \left( A_0^\otimes C_2 \times C_2 \right) \\
K_1 \left( A_0^\otimes C_2 \times C_2 \right) & \leftarrow & 0 \leftarrow 0,
\end{array} \]

where the generators $Q_m \oplus Q_m$ over $Q_m$ are the images of $[1]$ and $[e_1]$ by the KK-equivalence $M_m^\otimes C_2 \times C_2 \rightarrow B^\otimes C_2 \times C_2$. Here we have used Theorem 2.8. Thus the first arrow in the top row of (3.7) is the natural inclusion. We get $K_1 \left( A_0^\otimes C_2 \times C_2 \right) \cong 0$ and $K_0 \left( A_0^\otimes C_2 \times C_2 \right) \cong Q_m/Z$, generated by the image of $Q_m[e_1]$ in $K_0 \left( M_m^\otimes C_2 \times C_2 \right)$. \qed
Lemma 3.5. We have

\[ K_\ast \left( F_{1,m}^\otimes C_2 \times C_2 \right) \cong \begin{cases} 0, & \ast = 0; \\ \mathbb{Q}_m/\mathbb{Z} \oplus \mathbb{Q}_m/\mathbb{Z}, & \ast = 1. \end{cases} \]

Proof. We write \( A_0 := F_{m,1} \) as in the proof of Lemma 3.4 and use \( A := C_0(\mathbb{R}) \otimes A_0 \) as a model for \( F_{1,m} \). Note that the flip action on \( C_0(\mathbb{R}) \otimes C_2 \) is conjugate to the action on \( C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \) that is trivial on the first factor and reflects at the origin \( 0 \in \mathbb{R} \) on the second factor. We thus have

\[ K_\ast \left( A^\otimes C_2 \times C_2 \right) \cong K_{\ast + 1} \left( \left( C_0(\mathbb{R}) \otimes A_0^\otimes C_2 \right) \times C_2 \right), \tag{3.8} \]

where \( C_2 \) acts via the flip on \( C_0(\mathbb{R}) \) and via the shift on \( A_0^\otimes C_2 \). We consider the short exact sequence

\[ 0 \to \left( C_0(-\infty, 0) \oplus C_0(0, \infty) \right) \otimes A_0^\otimes C_2 \to C_0(\mathbb{R}) \otimes A_0^\otimes C_2 \to A_0^\otimes C_2 \to 0 \]

of \( C^\ast \)-algebras. We have

\[ K_\ast \left( \left( C_0(-\infty, 0) \oplus C_0(0, \infty) \right) \otimes A_0^\otimes C_2 \right) \times C_2 \right) \cong K_{\ast + 1} \left( A_0^\otimes C_2 \right) \]

by the Künneth theorem (since \( \text{Tor}_1^\mathbb{Z}(\mathbb{Q}_m/\mathbb{Z}, \mathbb{Q}_m/\mathbb{Z}) \cong \mathbb{Q}_m/\mathbb{Z} \)). In view of this and Lemma 3.4, taking crossed products and applying K-theory for \( \text{Tor}_1^\mathbb{Z}(\mathbb{Q}_m/\mathbb{Z}, \mathbb{Q}_m/\mathbb{Z}) \) produces the 6-term exact sequence

\[ \begin{array}{c}
\mathbb{Q}_m/\mathbb{Z} \to K_0 \left( \left( C_0(\mathbb{R}) \otimes A_0^\otimes C_2 \right) \times C_2 \right) \to \mathbb{Q}_m/\mathbb{Z} \\
0 \to K_1 \left( \left( C_0(\mathbb{R}) \otimes A_0^\otimes C_2 \right) \times C_2 \right) \to 0.
\end{array} \tag{3.10} \]

By [11k16, Lemma 1.1], the top row of (3.10) splits. Now the Lemma follows from (3.10) and (3.8).

Proof of Theorem 3.2 We use a KK-equivalence

\[ F_{n,m} \sim_{KK} F_{n,1} \oplus F_{1,m}, \]

obtained from the UCT. From this we obtain

\[ K_\ast (F_{n,m}^\otimes C_2 \times C_2) \cong K_\ast (F_{n,1}^\otimes C_2 \times C_2) \oplus K_\ast (F_{1,m}^\otimes C_2 \times C_2) \oplus K_\ast (F_{n,1} \otimes F_{1,m}). \]

The first two summands are computed by Lemma 3.4 and 3.5 respectively, whereas the last summand can be computed using the Künneth theorem as

\[ K_\ast (F_{m,1} \otimes F_{1,n}) \cong \begin{cases} \text{Tor}_1^\mathbb{Z}(\mathbb{Q}_m/\mathbb{Z}, \mathbb{Q}_n/\mathbb{Z}) = \mathbb{Q}_r/\mathbb{Z}, & \ast = 0, \\ \mathbb{Q}_m/\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}_n/\mathbb{Z} = 0, & \ast = 1, \end{cases} \]

where \( r \) denotes the greatest common divisor of \( m \) and \( n \).
By Theorem 3.1, the list of the classifiable $C^*$-algebras with approximately inner flip is up to KK-equivalences given by

$$\varepsilon_{n,1,m} \cong F_{1,m}$$

for supernatural numbers $m$ and $n$ where $m$ is of infinite type such that $m$ divides $n$. Note that $\varepsilon_{n,1,m}$ is KK-equivalent to $M_n \cong F_{1,m}$ and $M_n \cong F_{1,m}$ is KK-equivalent to zero if $m$ divides $n$. In particular, the flip on $\varepsilon_{n,1,m}$ is $KK$-equivalent to the sum of the flips on $M_n \cong F_{1,m}$. Thus, we have

$$K_*(\varepsilon_{n,1,m} \times C_2) \cong K_*(M_n \cong F_{1,m} \times C_2) \cong K_*(M_n \cong F_{1,m} \times C_2) \cong 0.$$  

If both $m$ and $n$ are of infinite type, Theorem 2.8, Theorem 3.2, and Corollary 3.3 compute these $K$-groups:

$$K_0(M_n \cong F_{1,m} \times C_2) \cong \mathbb{Q}_n \oplus \mathbb{Q}_n,$$

and

$$K_1(M_n \cong F_{1,m} \times C_2) \cong 0,$$

$$K_n(M_n \cong F_{1,m} \times C_2) \cong 0.$$

Suppose $n$ is essentially of infinite type, meaning $n = n_0 \cdot n_1$, where $n_0$ is a natural number and $n_1$ is a supernatural number of infinite type. Then, we have $K_*(M_n \cong F_{1,m} \times C_2) \cong K_*(M_{n_1} \cong F_{1,m} \times C_2)$.

Suppose $n = \prod_{i=1}^{\infty} p_i^{n_i}$ where $1 \leq n_i < \infty$ for infinitely many distinct primes $p_i$. Let $q_i = p_i^{n_i}$. Then, $M_n \cong F_{1,m} \times C_2$ is the inductive limit of the system

$$C \times C_2 \rightarrow M_{q_1} \cong F_{1,m} \times C_2 \rightarrow (M_{q_1} \cong F_{1,m} \times C_2 \times C_2 \rightarrow \cdots$$

From this (c.f. Proof of Theorem 2.8), we observe that $K_0(M_n \cong F_{1,m} \times C_2)$ is isomorphic to the inductive limit of the system

$$R_C(C_2) \xrightarrow{\tau_{q_1}} R_C(C_2) \xrightarrow{\tau_{q_2}} R_C(C_2) \xrightarrow{\tau_{q_3}} \cdots$$

where $\tau_k \colon C_2 \rightarrow GL(\ell^2(\{(1,\ldots,k)C_2\}))$ is the permutation representation. We identify $R_C(C_2) \cong \mathbb{Z}^2$ using the trivial representation $\sigma_0$ and the sign representation $\sigma_1$ of $C_2$ as a basis of $R_C(C_2)$. Since $\tau_k = \frac{k}{2}\sigma_0 + \frac{k}{2}\sigma_1$ in $R_C(C_2)$ we see that the system is isomorphic to

$$\mathbb{Z}^2 \xrightarrow{X_k} \mathbb{Z}^2 \xrightarrow{X_k} \mathbb{Z}^2 \xrightarrow{X_k} \cdots$$

where $X_k = \begin{bmatrix} \frac{k}{2} & \frac{k}{2} + 1 \\ \frac{k}{2} & \frac{k}{2} + 1 \end{bmatrix}$, which has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the corresponding eigenvalues $k^2, k$. The system has a subsystem consisting of

$\text{This is because the } C_2 \text{-set } \{(1,\ldots,k)C_2\} \text{ has } k \text{-many fixed points and } \frac{k^2 - 1}{2} \text{-many free orbits. Each fixed point contributes the trivial representation, and each free orbit contributes the regular representation.}$
the span of \[
\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\] in each \(\mathbb{Z}^2\) on which \(X_{q_k}\) acts as \(q_k^2\). The quotient system is isomorphic to
\[
\mathbb{Z} \xrightarrow{q_1} \mathbb{Z} \xrightarrow{q_2} \mathbb{Z} \xrightarrow{q_3} \mathbb{Z} \xrightarrow{q_4} \cdots .
\]
From these, it is not hard to observe that we have the following short exact sequence
\[
0 \longrightarrow \mathbb{Q}_{n^2} \longrightarrow K_0(M_n \otimes C_2) \longrightarrow \mathbb{Q}_n \longrightarrow 0.
\]
Under the assumption on \(n\), we have
\[
K_0(M_n \otimes C_2^\infty) \cong \mathbb{Q}_{n^2} \cong \mathbb{Q}_n \cong K_0(M_n)
\]
and also
\[
K_0(M_n \otimes C_2) \not\cong \mathbb{Q}_{n^2} \oplus \mathbb{Q}_{n^2} \cong K_0(M_n \otimes C_2^\infty) .
\]
These conclusions remain to hold for \(n = n_0 \cdot n_1\) where \(n_0\) is a supernatural number of the type we just considered and \(n_1\) is a supernatural number of infinite type which is coprime to \(n_0\).

From these computations, we can observe the following: let \(A\) be any separable \(C^*\)-algebra that is KK-equivalent to \(E_{n,1,m}\) for supernatural numbers \(m\) and \(n\) where \(m\) is of infinite type such that \(m\) divides \(n\). Then, the following are equivalent:

1. \(n\) is essentially of infinite type;
2. \(K_*(A \otimes \mathbb{Z}) \cong K_*(A)\);
3. \(K_*(A \otimes C_2) \cong K_*(A \otimes C_2^\infty)\).

The conditions (2) and (3) are satisfied for \(A = F_{1,m}\) for any supernatural number \(m\) of infinite type. Note that the condition (3) is a necessary condition for a \(C_2-C^*\)-algebra \(\Lambda \otimes C_2\) equipped with the flip action to be KK\(C_2\)-equivalent to \(\Lambda \otimes \mathbb{Z}\) equipped with the trivial action. These observations naturally lead to the following question.

**Question 3.6.** Let \(A\) be any separable \(C^*\)-algebra that is KK-equivalent to \(E_{n,1,m}\) or \(F_{1,m}\) for supernatural numbers \(m\) and \(n\) where both \(m\) and \(n\) are of infinite type and \(m\) divides \(n\). Is the \(C_2-C^*\)-algebra \(\Lambda \otimes C_2\) equipped with the flip action KK\(C_2\)-equivalent to \(\Lambda \otimes \mathbb{Z}\) equipped with the trivial action?

We note that to answer this question positively, it would suffice to answer it positively for \(A = F_{1,m}\) because the flip on \(E_{n,1,m}\) is KK\(C_2\)-equivalent to the sum of the flips on \(M_n \otimes C_2\) and Theorem 2.8 provides a required KK\(C_2\)-equivalence for \(A = M_n\).

Unfortunately, the methods used to establish the KK\(C_2\)-equivalence between \(M_n \otimes C_2\) and \(M_n\) in Theorem 2.8 do not apply for \(A = F_{1,m}\). Firstly, there is no representation-theoretic argument analogous to Proposition 2.1 for \(F_{1,m}\). Additionally, the diagram
\[
M_n \otimes C_2 \to M_n \otimes C_2 \oplus M_n \to M_n
\]
of KK$^2$-equivalences in Theorem 2.8 does not have a counterpart for $F_1,m$. This is because the unit class $[1]_0 \in K_0(F_1,m) \cong 0$ is trivial.

In a forthcoming article, we will affirmatively answer Question 3.6 utilizing Ralf Meyer’s work [Mey21] on the equivariant bootstrap classes and Manuel Kühler’s equivariant UCT theorem [Köh10].

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