A conforming primal–dual mixed formulation for the 2D multiscale porous media flow problem

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Abstract
In this paper, a new primal–dual mixed finite-element method is introduced, aimed to model multiscale problems with several geometric subregions in the domain of interest. In each of these regions, porous media fluid flow takes place, but governed by physical parameters at a different scale; in addition, a fluid exchange through contact interfaces occurs between neighboring regions. The well-posedness of the primal–dual mixed finite-element formulation on bounded simply connected polygonal domains of the plane is presented. Next, the convergence of the discrete solution to the exact solution of the problem is discussed, together with the convergence rate analysis. Finally, the numerical examples illustrate the method’s capabilities to handle multiscale problems and interface discontinuities as well as experimental rates of convergence.

Keywords Coupled discontinuous Darcy system · Mixed formulations · Multiscale problems

Mathematics Subject Classification 65M60 · 35J50 · 65N12

1 Introduction
Mixed variational formulations are a very important topic of research in applied mathematics. The Babuska–Brezzi theory (see Theorem 1) is remarkably powerful from the theoretical point of view; however, it introduces high complexity in the discrete finite-element spaces approximating the solution, this reflects in numerical stability problems (see Masud and Hughes 2002). The achievements to overcome such difficulty can be on several directions. One of the streams seeks to stabilize the approximation by modifying the bilinear forms involved, namely using symmetric properties of the tensors as in Figueroa et al. (2008) and Gatica (2006), or including terms in the bilinear forms in a “balanced” way as in Brezzi et al.
(1993) and Masud and Hughes (2002). As this technique has proved to be fruitful and rich in terms of the possibilities to stabilize the forms of interest, some other aspects arise by itself, such as the discussion of minimal stabilization procedures (see Brezzi and Fortin 2001), or the a priori, a posteriori error analysis for these new scheme (see Figueroa et al. 2008). A second approach uses discontinuous Galerkin finite elements (DG). The DG methods have several advantages and goals, some of these are: addressing non-conformality in a more flexible way, treating stability issues due to coupling constraints (demanding regularity in the discrete spaces), and computing in a more accurate way the physical quantity that is known to be predominant in specific subregions. The latter is attained in two ways, by local refining of the mesh and by approximating polynomial spaces; see (Arnold et al. 2000, 2002) for a unified vision of the DG Methods.

All the aforementioned works, whichever the problem they may be analyzing (elasticity, heat diffusion, free flow, Darcy flow, etc), treat separately the primal- and dual mixed formulations (see Brezzi and Fortin 1991; Vivette and Pierre-Arnaud 1986; Raviart and Thomas 1977). The present paper is focused on using simultaneously both fundamental versions for the treatment of multiscale problems in Darcy flow (see Problem (1)), it is, therefore, a primal–dual mixed formulation; in a way, this article is the numerical implementation of the formulation introduced in Morales and Naranjo (2017) (see also Morales and Showalter 2012; Morales 2014 for related formulations). The stability aspects become particularly critical when dealing with multiscale problems, as the presence of physical coefficients with different orders of magnitude adds up to the built-in complexity of the mixed variational formulations (coefficient $a(\cdot)$ in Problem (1)). The primal–dual mixed formulation tackles this issue by removing coupling constraints from the discrete trial spaces while satisfying them only on the solution, i.e., the continuous formulation replaces strong coupling conditions by weak coupling conditions (see Eq. (7) and Problem (14)). Replacing the nature of the coupling conditions is a strategy already used in DG methods using penalization techniques; however, this is done only on the discrete version, while the continuous formulation still relies on strong coupling conditions. The latter is, because, in the Darcy flow problem, while the primal mixed formulation can introduce weak coupling conditions on the normal flow exchange, the normal stress has to stay continuous. In contrast, the dual mixed formulation can introduce weak coupling conditions for the normal stress balance, but it requires the normal flow exchange to be continuous. The continuity constraints of the classical mixed formulations reflect later on, in the deep discussions of convergence present in the DG methods.

Another advantage of the discrete primal–dual mixed formulation that we are to introduce in this work is that, according to the regions, the predominant effect can be chosen to be modeled with the discrete space holding the sense of continuity, while the secondary effect is modeled with the discontinuous space. In the case of Darcy flow, the pressure is the dominant effect in regions of low permeability, while the flow velocity is the predominant one in regions of high permeability (see Figs. 4, 6 and 10). This concept has already raised naturally in the previous DG methods coupling advection with diffusion phenomena, due to the discrete spaces involved in the formulations, see Dawson (1993). To the author’s best knowledge, there is no precedent for having this level of flexibility in the analysis of coupling fluid flow phenomena, as the literature analyzing multiscale flow is mainly focused in coupling Stokes flow with Darcy flow [see (Arbogast and Brunson 2007; Arbogast and Lehr 2006; Gatica et al. 2009; Layton et al. 2003; Morales and Showalter 2017)].

The proposed model is to analyze a variation of the classic porous media problem on a connected bounded open region $\Omega \subset \mathbb{R}^2$, that is:
Fig. 1  

(a) Bipartite Map $\mathcal{G}$ of region $\Omega$.  
(b) Grid $\mathcal{T}$ consistent with $\mathcal{G}$.  

(a) Depicts a bipartite map $\mathcal{G}$ example for a given region $\Omega$. Subregions belonging to $\mathcal{G}_i$ have been labeled with $i$ for $i = 1, 2$. The vector $\mathbf{n}$ and the outer normal vectors $\mathbf{\hat{n}}_1, \mathbf{\hat{n}}_2$ are illustrated for a couple of neighboring elements belonging to $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.  

(b) Depicts an example of grid $\mathcal{T}$ consistent with the map $\mathcal{G}$. Some of the triangles belonging to $\mathcal{T}_1$ and $\mathcal{T}_2$ have been labeled with 1 and 2, respectively.

\[ a(\cdot) \mathbf{u} + \nabla p + g = 0, \quad (1a) \]
\[ \nabla \cdot \mathbf{u} = F \quad \text{in } \Omega. \quad (1b) \]
\[ p = 0 \quad \text{on } \Gamma_d. \quad (1c) \]
\[ \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_f = \partial \Omega - \Gamma_d; \quad (1d) \]

more specifically, when $\Omega$ is partitioned in two sub-domains $\Omega_1, \Omega_2$, such that $a(\cdot)|_{\Omega_1} = O(1)$ and $a(\cdot)|_{\Omega_2} = O(\epsilon)$ for $\epsilon > 0$ small (see Fig. 1). Recall that $a(\cdot)$ is the flow resistance, i.e., the viscosity times the inverse of permeability of the porous medium. Systems such as these are suited for the modeling of oil reservoirs and subsurface water, where a network of thin channels embedded in bedrock occurs; therefore, the flow resistance coefficient changes its order of magnitude from one region $\Omega_1$ to the other $\Omega_2$. In this context, the continuity of the solution $[\mathbf{u}, p]$ across the interface between $\Omega_1$ and $\Omega_2$ becomes a liability from the numerical point of view. Therefore, if it is possible to estimate a priori, the magnitude of change that the solution will experience from one sub-domain to the other (see Example 4), it would be a more strategic approach to artificially introduce a discontinuity across the interface, and model it with a system, see Eq. (6), satisfying a balance/coupling condition for both, normal flux and normal stress, see Eq. (7). As mentioned above, these exchange conditions will be introduced weakly in the formulation allowing full decoupling of the underlying function spaces. Moreover, the trial spaces require that the pressure $q$ is only square integrable $L^2$ on one side of the interface, while it belongs to $H^1$ on the other side of the interface (see Fig. 4a); such discontinuity on the test spaces is ideal to handle discontinuities on the normal stress across the interface. The analogous takes place on the velocities modeling spaces; here, the test functions $v$ belong to $H_{\text{div}}$ on one side of the interface, while they are only square integrable $L^2$ on the other (see Fig. 4b). Again, this scenario will be ideal for discontinuities of normal flux across the interface. In summary, the primal–dual mixed formulation method will be able to capture interface discontinuities using uncoupled, conforming, finite-dimensional spaces, presented in Definitions 17 and 18.

We close this section introducing the general notation. In the present work, vectors are denoted by boldface letters as are vector-valued functions and corresponding function spaces. The symbols $\nabla$ and $\nabla \cdot$ represent the gradient and divergence operators, respectively. The
dimension is indicated by $N$ which will be equal to 2 or 3 depending on the context. Given a function $f : \mathbb{R}^N \to \mathbb{R}$, then $\int_M f \, dS$ denotes the integral on the $N-1$ dimensional manifold $M \subseteq \mathbb{R}^N$. Analogously, $\int_A f \, dx$ stands for the integral in the set $A \subseteq \mathbb{R}^N$; whenever the context is clear, we simply write $\int_A f$. Given an open set $G$ of $\mathbb{R}^N$, the symbols $\| \cdot \|_{0,G}$, $\| \cdot \|_{1,G}$, $\| \cdot \|_{1/2,AG}$, $\| \cdot \|_{-1/2,AG}$, and $\| \cdot \|_{H_{\text{div}}(G)}$ denote the $L^2(G)$, $H^1(G)$, $H^{1/2}(\partial G)$, $H^{-1/2}(\partial G)$ and $H_{\text{div}}(G)$ norms, respectively, while $|M|$ represents the Lebesgue measure of $G$ in $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$ depending on the context.

2 Preliminaries

2.1 Geometric setting

In this section, we set the conditions on the domain of reference as well as its gridding.

Definition 1 Given a bounded open set $\omega$ in $\mathbb{R}^2$, we will say that a bipartite map is a finite collection of connected open subsets $\mathcal{G} = \{ G_n : 1 \leq n \leq N \}$, such that

(i) If $n \neq k$, then $G_n \cap G_k = \emptyset$.
(ii) The union satisfies $\omega - \bigcup_{i=1}^{N} G_n = 0$ and $\text{cl}(\omega) = \bigcup_{i=1}^{N} \text{cl}(G_n)$.
(iii) The collection $\mathcal{G} = \{ G_n : 1 \leq n \leq N \}$ is partitioned in two subcollections $\mathcal{G}_1 = \{ G_i^1 : 1 \leq i \leq I \}$ and $\mathcal{G}_2 = \{ G_j^2 : 1 \leq j \leq J \}$, such that

(a) $\{ G_n : 1 \leq n \leq N \} = \{ G_i^1 : 1 \leq i \leq I \} \cup \{ G_j^2 : 1 \leq j \leq J \}$.
(b) If $i \neq k$, then $|\partial G_i^1 \cap \partial G_k^1| = 0$.
(c) If $j \neq \ell$, then $|\partial G_j^2 \cap \partial G_{\ell}^2| = 0$.

The collections $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be the bipartition or the bi-coloring of the map.

Hypothesis 1 The domain of interest $\Omega$ is a polygonal, bounded, connected region of the plane, and it satisfies that:

(i) It has a bipartite map $\mathcal{G} = \{ G_n : 1 \leq n \leq N \}$, such that $G_n$ is a polygon for each $n = 1, \ldots, N$.
(ii) If $\mathcal{G}_1$, $\mathcal{G}_2$ is the bipartition of the map $\mathcal{G}$, then $\text{cl} \left[ \bigcup \{ L : L \in \mathcal{G}_1 \} \right]$ and $\text{cl} \left[ \bigcup \{ L : L \in \mathcal{G}_2 \} \right]$ are connected.

An example of bipartite map is depicted in Fig. 1a, together with some other concepts introduced in the following definition.

Definition 2 Let $\Omega$ satisfy Hypothesis 1 and let $\mathcal{G} = \{ G_n : 1 \leq n \leq N \}$ be its bipartite map with $\mathcal{G}_1$, $\mathcal{G}_2$ the map bipartition.

(i) For each polygon $K \in \mathcal{G}$, denote by $\nu$ the outer normal vector to its boundary $\partial K$.
(ii) For each polygon $K \in \mathcal{G}$, define $\hat{n}$ by the following:

$$
\hat{n}(x) \overset{\text{def}}{=} \begin{cases} 
\hat{\nu}(x) & K \in \mathcal{G}_1 \text{ and } x \in \partial K, \\
-\hat{\nu}(x) & K \in \mathcal{G}_2 \text{ and } x \in \partial K \cap \Omega, \\
\hat{\nu}(x) & K \in \mathcal{G}_2 \text{ and } x \in \partial K \cap \partial \Omega.
\end{cases}
$$

(iii) Define $\Omega_1 \overset{\text{def}}{=} \bigcup \{ L : L \in \mathcal{G}_1 \}$ and $\Omega_2 \overset{\text{def}}{=} \bigcup \{ M : M \in \mathcal{G}_2 \}$.
(iv) Denote by $\Gamma \define \bigcup \{ \partial K : K \in \mathcal{G} \} - \partial \Omega$ the interface of the domain.

Next, we define the type of grids that will be considered in this work, see Fig. 1b for a simple example.

**Definition 3** Let $\Omega$ be as in Definition 2 above, then

(i) A triangulation $\mathcal{T}$ of the domain $\Omega$ is said to be consistent with the map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ if for each triangle $K \in \mathcal{T}$, it holds that $K \cap \Omega_1 = \emptyset$ or $K \cap \Omega_2 = \emptyset$. Equivalently, $\mathbf{1}_{K \cap \Omega_1} \mathbf{1}_{K \cap \Omega_2} = 0$.

(ii) Given two triangulations $\mathcal{T}'$ and $\mathcal{T}$ of the domain $\Omega$, we say that $\mathcal{T}'$ is a refinement of $\mathcal{T}$, denoted by $\mathcal{T}' \leq \mathcal{T}$, if, for each element $K' \in \mathcal{T}'$, there exists a triangle $K \in \mathcal{T}$, such that $K' \subseteq K$.

(iii) A sequence $\{ \mathcal{T}_h : h > 0 \}$ is said to be monotone if $h' < h$ implies that $\mathcal{T}_{h'} \leq \mathcal{T}_h$.

### 2.2 The strong problem and its continuous weak formulation

We begin this section recalling the general abstract setting to be used in this article. Let $X$ and $Y$ be Hilbert spaces and let $\mathcal{A} : X \to X'$, $\mathcal{B} : X \to Y'$ and $\mathcal{C} : Y \to Y'$ be continuous linear operators, we are to work on the following problem:

Find a pair $(x, y) \in X \times Y$:

\[
\begin{align*}
\mathcal{A} x + \mathcal{B} y &= F_1 \quad \text{in } X', \\
-\mathcal{B} x + \mathcal{C} y &= F_2 \quad \text{in } Y',
\end{align*}
\]

where $F_1 \in X'$ and $F_2 \in Y'$. The following is a well-known result (Girault and Raviart 1979).

**Theorem 1** Assume that the linear operators $\mathcal{A} : X \to X'$, $\mathcal{B} : X \to Y'$, $\mathcal{C} : Y \to Y'$ are continuous, and

(i) $\mathcal{A}$ is non-negative and $X$-coercive on $\ker(\mathcal{B})$.

(ii) $\mathcal{B}$ satisfies the inf–sup condition

\[
\inf_{y \in Y} \sup_{x \in X} \frac{\| \mathcal{B} x(y) \|}{\| x \|_X \| y \|_Y} > 0.
\]

(iii) $\mathcal{C}$ is non-negative symmetric.

Then, for every $F_1 \in X'$ and $F_2 \in Y'$, the Problem (3) has a unique solution in $(x, y) \in X \times Y$; in addition, it satisfies the estimate:

\[
\| x \|_X + \| y \|_Y \leq c (\| F_1 \|_{X'} + \| F_2 \|_{Y'}).
\]

Next, we present the strong problem to be approximated. Given a region $\Omega$ verifying Hypothesis 1, we introduce the following generalization of the Darcy flow Problem (1):

\[
\begin{align*}
a(\cdot) u_1 + \nabla p_1 + g &= 0, \\
\nabla \cdot u_1 &= F \quad \text{in } \Omega_1, \\
p_1 &= 0 \quad \text{on } \partial \Omega_1 \cap \partial \Omega, \\
a(\cdot) u_2 + \nabla p_2 + g &= 0, \\
\nabla \cdot u_2 &= F \quad \text{in } \Omega_2, \\
u_2 \cdot \hat{n} &= 0 \quad \text{on } \partial \Omega_2 \cap \partial \Omega.
\end{align*}
\]
Endowed with the following interface exchange balance conditions:

\[ p_2 - p_1 = f^\Sigma, \]  
\[ u_1 \cdot \hat{n} - u_2 \cdot \hat{n} = \beta(\cdot) p_2 + f_{\hat{n}}. \]  

The problem above allows discontinuity jumps of discontinuity across the interface \( \Gamma \); due to the forcing terms in the normal stress (7a) and normal flux balance conditions (7b), both relationships are nothing, but statements normal stress and normal flux balance. The coefficients \( a(\cdot), \beta(\cdot) \) are non-negative and they stand for the medium resistance to the fluid flow and the interface storage rate, respectively. The multiscaling of the coefficient \( a(\cdot) \) will occur when modeling problems such as geological fissured systems (see Morales 2014) where regions of high permeability have to be coupled with regions of low permeability. On the other hand, the coefficient \( \beta(\cdot) \) is meaningful in this context when one of the regions stores fluid and the other does not; due to the difference in the scaling of the problem between regions, its determination/measurement is an active research field, see (Bhunya et al. 2008) for an example of related work. Finally, recall that Darcy’s law relates only pressure–velocity and that the pressure only acts in normal direction with respect to the physical object in contact, and therefore, the interface fluid exchange conditions (7) can only be stated in the normal direction, while it is not possible to reconcile interface tangential velocity conditions with a Darcy system, see (Morales and Showalter 2017) for an example.

To introduce the modeling spaces to be used in the weak variational formulation, first, notice that \( \{ L : L \in \mathcal{T}_1 \}, \{ M : M \in \mathcal{T}_2 \} \) are the simply connected components of \( \Omega_1 \) and \( \Omega_2 \), respectively. Then:

\[ H_{\text{div}}(\Omega_1) = \bigoplus_{L \in \mathcal{T}_1} H_{\text{div}}(L), \quad H^1(\Omega_2) = \bigoplus_{M \in \mathcal{T}_2} H^1(M). \]

The following space is introduced to couple adequately the action of the pressure traces in the variational formulation:

\[ E(\Omega_2) \overset{\text{def}}{=} \{ q \in H^1(\Omega_2) : q \mathbb{1}_{\partial M \cap \partial L} \in H^{1/2}(\partial L) \text{ for all } (L, M) \in \mathcal{T}_1 \times \mathcal{T}_2 \} = \{ q \in H^1(\Omega_2) : q \mathbb{1}_\Gamma \in H^{1/2}(\Gamma) \}. \]  

We endow \( E(\Omega_2) \) with the \( H^1(\Omega_2) \) inner product. It is direct to see that \( E(\Omega_2) \) is a closed subspace of \( H^1(\Omega_2) \) and consequently a Hilbert space. Also define

\[ V(\Omega_2) \overset{\text{def}}{=} \{ v \in L^2(\Omega_2) : v_2 = \nabla q_2 \text{ for some } q_2 \in E(\Omega_2) \} = \nabla(E(\Omega_2)), \]

endowed with the \( L^2(\Omega_2) \) inner product. Next, we recall a necessary result.

**Lemma 1** Let \( E(\Omega_2) \) and \( V(\Omega_2) \) be as defined in (8), (9), respectively; define:

\[ E_0(\Omega_2) \overset{\text{def}}{=} \{ q_2 \in E(\Omega_2) : \int_{\Omega_2} q_2 = 0 \}. \]  

Then,

(i) There exists a constant \( C > 0 \) depending only on the domain \( \Omega_2 \), such that

\[ \| r_2 \|_{1,\Omega_2} \leq C \| \nabla r_2 \|_{0,\Omega_2}, \quad \text{for all } r_2 \in H. \]  

(ii) The space \( V(\Omega_2) \) is Hilbert.

**Proof** See Lemma 4.4 in Morales and Naranjo (2017).
Now, we are ready to introduce the functional setting of the problem, and we define

\[ X \equiv \mathbf{H}_{\text{div}}(\Omega_1) \times E(\Omega_2). \]  

\[ Y \equiv \mathbf{V}(\Omega_2) \times L^2(\Omega_1). \]  

Endowed with their natural norms

\[ \| [v_1, q_2] \|_X \equiv \left\{ \| v_1 \|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2 + \| q_2 \|_{H^1(\Omega_2)}^2 \right\}^{1/2}, \]  

\[ \| [v_2, q_1] \|_Y \equiv \left\{ \| v_2 \|_{L^2(\Omega_2)}^2 + \| q_1 \|_{L^2(\Omega_1)}^2 \right\}^{1/2}. \]  

**Remark 1**

(i) Clearly, \( X \) is a Hilbert space, to see that \( Y \) is a Hilbert space; see Lemma 1 above and/or Lemma 4.4 in Morales and Naranjo (2017).

(ii) To avoid heavy notation, from now on, the following notational convention will be adopted

\[
\int_{\Gamma} (v_1 \cdot \hat{n}) q_2 \, dS \equiv \langle v_1 \cdot \hat{n}, q_2 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}. \]  

From now on, we assume that \( F \in L^2(\Omega), g \in L^2(\Omega_2), f_\Sigma \in H^{1/2}(\Gamma) \) and \( f_\hat{n} \in H^{-1/2}(\Gamma) \).

Finally, the primal–dual mixed formulation for the Problem (6) with interface balance conditions (7) is given by the following:

Find \( ([u_1, p_2], [u_2, p_1]) \in X \times Y \) : 

\[
\int_{\Omega_1} a \, u_1 \cdot v_1 + \int_{\Gamma} (u_1 \cdot \hat{n}) q_2 \, dS - \int_{\Gamma} (u_1 \cdot \hat{n}) q_2 
\]

\[ + \int_{\Gamma} p_2 (v_1 \cdot \hat{n}) \, dS - \int_{\Omega_1} p_1 \nabla \cdot v_1 - \int_{\Omega_2} u_2 \cdot \nabla q_2 \]

\[ = \int_{\Omega_2} F \, q_2 - \int_{\Omega_1} g \cdot v_1 + \int_{\Gamma} f_\Sigma (v_1 \cdot \hat{n}) \, dS - \int_{\Gamma} f_\hat{n} q_2 \, dS, \]  

\[
\int_{\Omega_1} \nabla \cdot u_1 \, q_1 + \int_{\Omega_2} \nabla \cdot u_2 \, v_2 + \int_{\Omega_2} a \, u_2 \cdot v_2 = \int_{\Omega_1} F \, q_1 - \int_{\Omega_2} g \cdot v_2, \]  

for all \( ([v_1, q_2], [v_2, q_1]) \in X \times Y \).

Define the operators \( \mathcal{A} : X \rightarrow X', \mathcal{B} : X \rightarrow Y' \) and \( \mathcal{C} : Y \rightarrow Y' \) by the following:

\[
\mathcal{A}[v_1, q_2], ([w_1, r_2]) \equiv \int_{\Omega_1} a \, v_1 \cdot w_1 + \int_{\Gamma} (v_1 \cdot \hat{n}) r_2 \, dS - \int_{\Gamma} (v_1 \cdot \hat{n}) r_2 \, dS + \int_{\Gamma} q_2 (w_1 \cdot \hat{n}) \, dS, \]  

\[
\mathcal{B}[v_1, q_2], ([w_2, r_1]) \equiv \int_{\Omega_1} \nabla \cdot v_1 \, r_1 + \int_{\Omega_2} \nabla q_2 \cdot w_2, \]  

\[
\mathcal{C}[v_2, q_1], ([w_2, r_1]) \equiv \int_{\Omega_2} a \, v_2 \cdot w_2. \]  

Hence, the Problem (14) is equivalent to the following:

Find a pair \( ([u_1, p_2], [u_2, p_1]) \in X \times Y \) : 

\[
\mathcal{A}[u_1, p_2] + \mathcal{B}[u_2, p_1] = F_1 \quad \text{in} \, X', \]

\[
-\mathcal{B}[u_1, p_2] + \mathcal{C}[u_2, p_1] = F_2 \quad \text{in} \, Y'. \]  

(16)
Here, $F_1 \in X'$ and $F_2 \in Y'$ are the functionals defined by the right-hand side of (14a) and (14b), respectively. To satisfy the required ellipticity conditions for the operator $\mathcal{A}$, some extra hypotheses on the coefficients become necessary.

**Hypothesis 2** It will be assumed that coefficients of storage exchange $\beta : \Gamma \to [0, \infty)$ and porous medium resistance $a : \Omega \to (0, \infty)$ satisfy that $\beta \in L^\infty(\Gamma), \| \beta \mathbb{1}_\Gamma \|_{L^1(\Gamma)} > 0$ and $a \in L^\infty(\Omega), \parallel \frac{1}{a} \parallel_{L^\infty(\Omega)} > 0$, respectively.

**Theorem 2** Let $\Omega$ be a polygonal region and let $\mathcal{G}$ be a bipartite map, then if the Hypothesis 2 is satisfied, the Problem (14) is well posed.

**Proof** See Theorem 4.8 in Morales and Naranjo (2017).

We close this section recalling the next result on recovering the strong problem from the weak variational formulation (14).

**Theorem 3** The solution of the weak variational Problem (14) is a strong solution of the Problem (6) with the forcing gravitation term $g$ in Eq. (6a) replaced by $P g$, which denotes its orthogonal projection onto the space $V(\Omega_2)$. In particular, if $g1_{\partial \Omega_2} \in V(\Omega_2)$, the weak solution is exactly the strong solution.

**Proof** See Theorem 4.9 in Morales and Naranjo (2017).

### 3 The discretization of the problem

In this section, we present a viable discretization of the Problem (14) in the two-dimensional case, from the theoretical point of view. We start introducing the discrete function spaces, we will denote by $P_\ell(K)$ the polynomials of order $\ell$ on the triangle $K$ and $P_\ell(K) = (P_\ell(K))^2$. As usual, $RT_\ell(K)$ indicates the Raviart–Thomas finite element of degree $\ell$ on the triangle $K$. From now on it will be assumed that the domain $\Omega$ satisfies Hypothesis 1 and that any triangulation $T$ of analysis is consistent with the map $\mathcal{G}$, as introduced in Definition 3. Hence, for a fixed consistent triangulation $T$ with size $h \triangleq \max \{ \text{diameter}(K) : T \in T \}$, we denote the following:

$$
RT_0(\Omega_1, T) \triangleq \{ v \in H_{\text{div}}(\Omega_1) : v|_K \in RT_0(K), \text{ for all } K \in T, K \subseteq \Omega_1 \}, \quad (17a)
$$

$$
\mathcal{Q}(\Omega_2, T) \triangleq \{ q_2 \in H^1(\Omega_2) : q_2 = \xi|_{\Omega_2} \text{ for some } \xi \in H^1(\Omega) \text{ and } q_2|_K \in P_1(K), \text{ for all } K \in T, K \subseteq \Omega_2 \}, \quad (17b)
$$

$$
\nabla \mathcal{Q}(\Omega_2, T) \triangleq \{ v_2 \in P_0(\Omega_2) : v_2 = \nabla q_2 \text{ for some } q_2 \in \mathcal{Q}(\Omega_2, T) \}, \quad (17c)
$$

$$
\mathcal{Q}(\Omega_1, T) \triangleq \{ q_1 \in L^2(\Omega_1) : q_1|_K \in P_0(K), \text{ for all } K \in T, K \subseteq \Omega_1 \}. \quad (17d)
$$

Whenever the triangulation $T$ is clear from the context, we simply write $RT_0(\Omega_1) = RT_0(\Omega_1, T)$, $\nabla \mathcal{Q}(\Omega_2) = \nabla \mathcal{Q}(\Omega_2, T)$, and $\mathcal{Q}(\Omega_\ell) = \mathcal{Q}(\Omega_\ell, T)$ for $\ell = 1, 2$. Notice that $RT_0(\Omega_1) \times \mathcal{Q}(\Omega_2) \subseteq X$ and $\nabla \mathcal{Q}(\Omega_2) \times \mathcal{Q}(\Omega_1) \subseteq Y$. Define the following discrete spaces:

$$
X_h \triangleq RT_0(\Omega_1) \times \mathcal{Q}(\Omega_2), \quad (18a)
$$

$$
Y_h \triangleq \nabla \mathcal{Q}(\Omega_2) \times \mathcal{Q}(\Omega_1), \quad (18b)
$$
endowed $X_h, Y_h$ with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. The discrete operators $\mathcal{A}_h : X_h \to X_h, \mathcal{B}_h : X_h \to Y_h$ and $\mathcal{C}_h : Y_h \to Y'_h$ are defined by the respective restriction of the operators $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ introduced in (15a), (15b) and (15c), that is

$$\mathcal{A}_h [v_1, q_2] (\{w_1, r_2\}) \overset{\text{def}}{=} \mathcal{A} [v_1, q_2] (\{w_1, r_2\}), \quad \text{for all } [v_1, q_2], [w_1, r_2] \in X_h. \quad (19a)$$

$$\mathcal{B}_h [v_1, q_2], (\{w_2, r_1\}) \overset{\text{def}}{=} \mathcal{B} [v_1, q_2] (\{w_2, r_1\}), \quad \text{for all } [v_1, q_2] \in X_h, [w_2, r_1] \in Y_h. \quad (19b)$$

$$\mathcal{C}_h [v_2, q_1] (\{w_2, r_1\}) \overset{\text{def}}{=} \mathcal{C} [v_2, q_1] (\{w_2, r_1\}), \quad \text{for all } [v_2, q_1], [w_2, r_1] \in Y_h. \quad (19c)$$

The discretization of Problem (16) is given by the following:

Find a pair $([u^h_1, p^h_2], [u^h_2, p^h_1]) \in X_h \times Y_h : \mathcal{A} [u^h_1, p^h_2] + \mathcal{B} [u^h_2, p^h_1] = F_1 \text{ in } X'_h,$

$$-\mathcal{B} [u^h_1, p^h_2] + \mathcal{C} [u^h_2, p^h_1] = F_2 \text{ in } Y'_h, \quad (20)$$

where $F_1 \in X'_h$ and $F_2 \in Y'_h$ are known functionals. We are to prove that the Problem (20) above is well posed, verifying that the operators $\mathcal{A}_h, \mathcal{B}_h$, and $\mathcal{C}_h$ satisfy the hypotheses of Theorem 1. Before proving the inf–sup condition of the operator $\mathcal{B}_h$, we recall a well-known result.

**Theorem 4** Let $RT_0(\Omega_1, \mathcal{F}), \mathcal{L}(\Omega_1, \mathcal{F})$ be defined in (17a) and (17d), respectively. Then, for every $q_1 \in \mathcal{L}(\Omega_1, \mathcal{F})$, there exist $v_1 \in RT_0(\Omega_1, \mathcal{F})$ and a constant $C > 0$ depending only on $\Omega_1$, such that $\nabla \cdot v_1 = q_1$ and $\|v_1\|_{H^1(\Omega_1)} \leq C \|q_1\|_{L^2(\Omega_1)}$.

**Proof** See Lemma 5.4, Chapter III, pg 151 in Braess (2007).

**Lemma 2** The operator $\mathcal{B}_h : X_h \to Y'_h$ defined in Eq. (19b) is continuous and satisfies the inf–sup condition, i.e., there exists a constant $C > 0$ depending only on the map $\mathcal{A}$, such that for every $[w_2, r_1] \in Y_h$, there exists $[v_1, q_2] \in X_h$ satisfying

$$\mathcal{B}_h [v_1, q_2] (\{w_2, r_1\}) \geq C \|v_1, q_2\|_{X_h} \|w_2, r_1\|_{Y'_h}. \quad (21)$$

Moreover, the constant $C > 0$ is independent from $[w_2, r_1]$ and the triangulation $\mathcal{F}$.

**Proof** The continuity of $\mathcal{B}_h$ follows from the continuity of $\mathcal{B}$. Now, fix $[w_2, r_1] \in Y_h$, due to Theorem 4, there exists $v_1 \in RT_0(\Omega_1)$, such that $V \cdot v_1 = r_1$ and $\|v_1\|_{H^1(\Omega_1)} \leq C \|r_1\|_{L^2(\Omega_1)}$, with $C > 0$ depending only on the domain $\Omega_1$.

Next, by definition of $\nabla \mathcal{L}(\Omega_2)$, there must exist $\eta \in \mathcal{L}(\Omega_2)$, such that $\nabla \eta = w_2$. Define $q_2 \overset{\text{def}}{=} \eta - \frac{1}{|\Omega_2|} \int_{\Omega_2} \eta$, clearly, $q_2 \in E_0(\Omega_2) \cap \mathcal{L}(\Omega_2)$, and due to the Inequality (11), it holds that $\|q_2\|_{1, \Omega_2} \leq C \|w_2\|_{0, \Omega_2}$ with $C > 0$ depending only on the domain $\Omega_2$.

Then, the pair $[v_1, q_2]$ belongs to $X_h$ and satisfies $\|v_1, q_2\|_X \leq C \|w_2, r_1\|_Y$ with $C > 0$ adequate depending only on the domain $\mathcal{F}$. Therefore:

$$\mathcal{B}_h [v_1, q_2] (\{w_2, r_1\}) = \|w_2, r_1\|_Y^2 \geq \frac{1}{C} \|v_1, q_2\|_X \|w_2, r_1\|_Y.$$

This completes the proof. \hfill $\Box$

**Lemma 3** If Hypothesis 2 is satisfied then, the operator $\mathcal{A}_h : X_h \to X'_h$ defined by (19a) is continuous and $X_h$-coercive on $X_h \cap \ker(\mathcal{B}_h)$; that is:

$$\mathcal{A}_h [v_1, q_2] (\{v_1, q_2\}) \geq C \|v_1, q_2\|_{X_h}^2, \quad \text{for all } [v_1, q_2] \in X_h \cap \ker(\mathcal{B}_h), \quad (22)$$

Where $C > 0$ is an adequate constant depending only on the domain $\mathcal{F}$. \hfill $\Box$
Proof The continuity of the operator $\mathcal{A}_h$ follows from the continuity of the operator $\mathcal{A}$. For the coerciveness of the operator, let $\{v_1, q_2\} \in X_h \cap \ker(\mathcal{B}_h)$; then:

$$\mathcal{B}_h[v_1, q_2](\{w_2, r_1\}) = 0 \text{ for all } \{w_2, r_1\} \in Y_h.$$ (23)

Notice that $\nabla \cdot v_1$ is constant for each $L \in \mathcal{T}$ contained in $\Omega_1$ and that $\nabla q_2$ belongs to $\nabla \mathcal{D}(\Omega_2)$ by definition. Therefore, $[\nabla q_2, \nabla \cdot v_1] \in Y_h$, in particular, testing (23) with $[0, r_1] \in Y_h$, we conclude that $\nabla \cdot v_1 = 0$, since $r_1$ is an arbitrary element in $L^2(\Omega_1)$. On the other hand, clearly $\nabla q_2 \in \nabla \mathcal{D}(\Omega_2)$ and the pair $[\nabla q_2, 0] \in Y$ is eligible for testing (23); which yields $\nabla q_2 = 0$, i.e., $q_2$ is constant inside $\Omega_2$. Hence:

$$\int_{\Omega} \beta q_2^2 = \frac{\|\beta \mathbb{1}_{\Omega} \|_{L^1(\Gamma)}}{\|\Omega_2\|} q_2^2 = \frac{\|\beta \mathbb{1}_{\Omega} \|_{L^1(\Gamma)}}{\|\Omega_2\|} q_2^2.$$ (23)

Using the previous observations, we get that

$$\mathcal{B}_h[v_1, q_2](\{v_1, q_2\}) = \int_{\Omega_1} \mathcal{A} v_1 \cdot v_1 + \int_{\Gamma} \beta q_2^2 \, dS \geq \frac{1}{a} \|v_1\|^2_{H^1(\Omega_1)} + \frac{\|\beta \mathbb{1}_{\Omega} \|_{L^1(\Gamma)}}{\|\Omega_2\|} q_2^2 \geq C \|\{v_1, q_2\}\|^2_X,$$

where $C = \min \left\{ \frac{1}{a} \|\mathbb{1}_{L^\infty(\Omega)}\|^{-1} |\Omega_2|^{-1} \|\beta \mathbb{1}_{\Omega} \|_{L^1(\Gamma)} \right\}$. This completes the proof. □

Theorem 5 Let $\Omega$ be a polygonal region and let $\mathcal{T}$ be a triangulation; then, if the Hypothesis 2 is satisfied, the Problem (20) is well posed.

Proof It is direct to see that the operator $\mathcal{A}_h$ is non-negative and symmetric. Due to this fact, Lemmas 2 and 3, the hypotheses of Theorem 1 are satisfied and the result follows. □

3.1 Strong convergence

In this section, we prove rigorously, under mild hypotheses on a sequence of triangulations $\{\mathcal{T}^h : h > 0\}$, the strong convergence of discrete solutions to the continuous one, i.e., $(u^h, p^h) \rightarrow (u, p)$, when $h \rightarrow 0$. To attain a priori estimates, some previous results are necessary.

Proposition 1 Let $\Omega$ be a domain satisfying Hypothesis 1 then, there exists $C > 0$ depending only on the map $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, such that:

$$\|\xi\|^2_{H^1(\Omega_i)} \leq C^2 \left( \|\nabla \xi\|^2_{L^2(\Omega_i)} + \|\sqrt{\beta} \xi\|^2_{L^2(\Gamma)} \right),$$ (24)

for all $\xi \in H^1(\Omega_i)$ and $i = 1, 2$.

Proof Notice that the map $\xi \mapsto \left( \|\nabla \xi\|^2_{L^2(\Omega_i)} + \|\sqrt{\beta} \xi\|^2_{L^2(\Gamma)} \right)^{1/2}$ is a norm; if it is equal to zero, it follows that $\xi$ is constant, and therefore:

$$0 = \int_{\Gamma} \beta |\xi|^2 = |\xi|^2 \|\beta\|_{L^1(\Gamma)}.$$ (24)

Due to the Hypothesis 2, this implies that $\xi = 0$. From here, a standard application of the Rellich–Kondrachov Theorem delivers the result. □
Proposition 2 Let $\mathcal{T}^h$ be a consistent triangulation of $\Omega$ and let $([u^h_1, p^h_1], [u^h_2, p^h_2]) \in X_h \times Y_h$ be the solution of Problem (20); then, there exists $C > 0$ depending only on the domain $\Omega$, such that

$$
\|p^h_1\|_{0, \Omega_1} \leq C \left( \|u^h_1\|_{0, \Omega_1}^2 + \|p^h_2\|_{1, \Omega_2}^2 + \|g\|_{1, \Omega_2}^2 + \|f\|_{1, \Omega_2}^2 \right)^{1/2}.
$$

(25)

Proof Test Problem (20) with $(v_1, 0, [0, 0]) \in X_h \times Y_h$ and add both equations, this gives the following:

$$
\int_{\Omega_1} p^h_1 \nabla \cdot v_1 = -\int_{\Omega_1} a u^h_1 \cdot v_1 - \int_{\Gamma} p^h_2 (v_1 \cdot \hat{n}) \, dS
+ \int_{\Omega_1} g \cdot v_1 - \int_{\Gamma} f \Sigma (v_1 \cdot \hat{n}) \, dS.
$$

(26)

Applying the CBS inequality to each summand, we get the following:

$$
\left| \int_{\Omega_1} p^h_1 \nabla \cdot v_1 \right| \leq C \left( \|u^h_1\|_{0, \Omega_1} \|v_1\|_{1, \Omega_1} + \|p^h_2\|_{1, \Omega_2} \|v_1\|_{1/2, \Gamma} \right)
+ \|g\|_{0, \Omega_1} \|v_1\|_{1, \Omega_1} + \|f\|_{1, \Omega_2} \|v_1\|_{1/2, \Gamma}
\leq C \left( \|u^h_1\|_{0, \Omega_1} + \|p^h_2\|_{1, \Omega_2} + \|g\|_{0, \Omega_1} + \|f\|_{1, \Omega_2} \right) \|v_1\|_{H^1(\Omega_1)}
\leq 2 C \left( \|u^h_1\|_{1, \Omega_1} + \|p^h_2\|_{1, \Omega_2} + \|g\|_{1, \Omega_2} + \|f\|_{1, \Omega_2} \right)^{1/2} \|v_1\|_{H^1(\Omega_1)}.
$$

Here, the generic constant of the second line is large enough. Due to Theorem 4, there exists $v_1 \in RT_0(\Omega_1)$, such that $\nabla \cdot v_1 = p^h_1$ and $\|v_1\|_{H^1(\Omega_1)} \leq C \|p^h_1\|_{0, \Omega_1}$, where the generic bound $C > 0$, depends only on the domain $\Omega_1$. Testing the expression above with this function, the Inequality (25) follows.

Now, we are ready to present an a priori estimate.

Theorem 6 Let $\{\mathcal{T}^h : h > 0\}$ be a monotone sequence of consistent triangulations of $\Omega$. Denote by $([u^h_1, p^h_1], [u^h_2, p^h_2]) \in X_h \times Y_h$ the solution of Problem (20) associated with the triangulation $\mathcal{T}^h$ with the fixed forcing terms $F, g, f, \Sigma, f_n$. Then, there exists $C > 0$, such that:

$$
\|[u^h_1, p^h_1]\|_X + \|[u^h_2, p^h_2]\|_Y \leq C, \quad \text{for all } h > 0.
$$

(27)

Proof Test Problem (20) with $(u^h, p^h)$ and add both equations, this gives the following:

$$
\int_{\Omega_1} a |u^h_1|^2 + \int_{\Omega_2} a |u^h_2|^2 + \int_{\Gamma} \beta |p^h|_2^2 \, dS
= \int_{\Omega} F p^h - \int_{\Omega_1} g \cdot u^h + \int_{\Gamma} \Sigma (u^h \cdot \hat{n}) \, dS - \int_{\Gamma} f_n p^h_2 \, dS.
$$

(28)

On the right-hand side term, we apply first the usual duality bounds and next the CBS inequality for vectors in $\mathbb{R}^4$, this gives the following:

$$
C_0 \left[ \|u^h_1\|_{0, \Omega_1}^2 + \|u^h_2\|_{0, \Omega_2}^2 + \|\sqrt{\beta} p^h_2\|_{0, \Gamma}^2 \right]
\leq \|F\|_{0, \Omega} \|p^h\|_{0, \Omega} + \|g\|_{L^2(\Omega)} \|u^h\|_{0, \Omega} + \|\Sigma\|_{1/2, \Gamma} \|u^h\|_{H^1(\Omega_1)} + \|f_n\|_{0, \Gamma} \|p^h_2\|_{0, \Gamma}
\leq \sqrt{2} \left[ \|F\|_{0, \Omega} + \|g\|_{0, \Omega_2}^2 + \|\Sigma\|_{1/2, \Gamma} + \|f_n\|_{0, \Omega} \right]^{1/2}
\left[ \|p^h\|_{0, \Omega} + \|p^h_2\|_{0, \Gamma} + \|u^h_1\|_{H^1(\Omega_1)} \right]^{1/2}.
$$

(29)
In the expression above, the constant $\sqrt{2}$ appears due to the estimate $\|u_h^1\|_{0,\Omega_1}^2 + \|h_h^1 \|_{H^{2}d}(\Omega_1) \leq 2 \|u_h^1\|_{H^{2}d}(\Omega_1)$. Next, we focus on giving estimates to the second factor of the right-hand side. To bound the pressure, first, split it in two pieces $\|p_h^2\|_{0,\Omega}^2 = \|p_h^2\|_{0,\Omega_1}^2 + \|p_h^2\|_{0,\Omega_2}^2$; now, due to Proposition 1, there exists $C > 0$ depending only on the map $\mathcal{G}$, such that:

$$\frac{1}{C} \|p_h^2\|_{0,\Omega_1}^2 \leq \|\nabla p_h^2\|_{0,\Omega_1}^2 + \|\sqrt{\beta} p_h^2\|_{0,\Gamma}^2 \leq 2 \|u_h^1\|_{0,\Omega_1} + 2 \|g_h^1\|_{0,\Omega_1} + 2 \|\sqrt{\beta} p_h^2\|_{0,\Gamma}^2 + 2 \|\sqrt{\beta} f_\Sigma\|_{1/2,\Gamma}^2 \leq 2 \|u_h^1\|_{0,\Omega_1} + 2 \|g_h^1\|_{0,\Omega_1} + 2 \|\sqrt{\beta} p_h^2\|_{0,\Gamma}^2 + 2 \|\sqrt{\beta} f_\Sigma\|_{L\infty(\Gamma)}^2 \|f_\Sigma\|_{1/2,\Gamma}^2.$$

(30)

The second inequality holds due to the strong discretized Darcy equation (6a), i.e., $u_h^1 + \nabla p_h^1 = g_h^1$, with $g_h^1$ denoting the orthogonal projection of $g$ onto $\nabla \mathcal{G}(\Omega_2, \mathcal{G}^h)$. In addition, $\|g_h^1\|_{0,\Omega_1} \leq \|g\|_{0,\Omega_1}$, which gives the third inequality. On the other hand, combining the estimates (25) and (24) with (30) gives

$$\|p_h^2\|_{0,\Omega_1}^2 \leq C (\|u_h^1\|_{0,\Omega_1} + \|\sqrt{\beta} p_h^2\|_{0,\Gamma}^2 + \|g_h^1\|_{0,\Omega_1} + \|f_\Sigma\|_{1/2,\Gamma}^2),$$

(31)

for $C > 0$ large enough, which depends only on $\Omega$. Next, due to (6b), it holds that $\|u_h^1\|_{H^{2}d}(\Omega_1) = \|u_h^1\|_{0,\Omega_1}^2 + \|f\|_{0,\Omega_1}^2$. Denoting $\kappa = \max \{1, 2 \|\sqrt{\beta} \|_{L\infty(\Gamma)}\} (\|F\|_{0,\Omega} + \|g_h^1\|_{0,\Omega} + \|f_\Sigma\|_{1/2,\Gamma} + \|f_\Sigma\|_{1/2,\Gamma})$ and introducing these observations in (29), we get the following:

$$C_0^2 \left[ \|u_h^1\|_{0,\Omega_1}^2 + \|u_h^1\|_{0,\Omega_2}^2 + \|\sqrt{\beta} p_h^2\|_{0,\Gamma}^2 \right]^2 \leq 2 \max\{C, 1\} \kappa \left[ \|u_h^1\|_{0,\Omega_1}^2 + \|u_h^1\|_{0,\Omega_2}^2 + \|\sqrt{\beta} p_h^2\|_{0,\Gamma}^2 \right] + \kappa^2.$$

The expression above shows that, for all $h > 0$, a square function is controlled by a linear function of the same argument, and therefore, there must exist yet another constant still denoted by $C > 0$, such that:

$$\|u_h^1\|_{0,\Omega_1} + \|u_h^1\|_{0,\Omega_2} + \|\sqrt{\beta} p_h^2\|_{0,\Gamma} \leq C, \quad \text{for all } h > 0.$$

From here, the strong Darcy equation (6d), the Inequality (24), the Inequality (31), and the conservation Statement (6b) give the result. $\square$

From the standard theory of general Hilbert spaces, the following result is trivial.

**Corollary 1** Assuming the hypotheses of Theorem 6 hold, there exist an element $([u_h^1, p_h^2], [u_h^2, p_h^1]) \in X \times Y$ and a subsequence, still denoted the same, such that $\{(u_h^1, p_h^2), [u_h^2, p_h^1]) : h > 0\}$ is weakly convergent to $([u_0^1, p_0^2], [u_0^2, p_0^1])$.

Before proving the strong convergence of the full sequence of solutions, we recall a standard finite-element theory result.
Proposition 3 Let \( \Omega \) be an open polygonal domain of \( \mathbb{R}^2 \) satisfying Hypothesis 1 and let \( \{ \mathcal{T}^h : h > 0 \} \) be a monotone sequence of consistent triangulations with size \( h \to 0 \); then:

\[
\begin{align*}
\text{cl} \left\{ RT_0(\Omega_1, \mathcal{T}^h) : h > 0 \right\} &= H_{\text{div}}(\Omega_1), \\
\text{cl} \left\{ \mathcal{D}(\Omega_2, \mathcal{T}^h) : h > 0 \right\} &= E(\Omega_2), \\
\text{cl} \left\{ \nabla \mathcal{D}(\Omega_2, \mathcal{T}^h) : h > 0 \right\} &= V(\Omega_2), \\
\text{cl} \left\{ \mathcal{D}(\Omega_1, \mathcal{T}^h) : h > 0 \right\} &= L^2(\Omega_2).
\end{align*}
\]

Proof The identities (32a) and (32d) are standard conformal finite-element results. For the identity (32b), it is enough to extend, in a continuous and linear fashion, the elements of \( \mathcal{D}(\Omega_2, \mathcal{T}^h) \) to polynomials of degree one in the whole domain \( \Omega \). This extension yields the classic FEM space of continuous, piecewise linear affine functions (on the whole domain \( \Omega \)) associated with \( \mathcal{T}^h \), which we denote by \( \mathcal{Q}_1(\Omega, \mathcal{T}^h) \). From the standard theory of conformal finite elements, we know that \( \text{cl} \left\{ \mathcal{Q}_1(\Omega, \mathcal{T}^h) : h > 0 \right\} = H^1(\Omega) \), in particular, the statement (32b) holds. Finally, the identity (32c) follows trivially from (32b).

Next, we prove the convergence of the solutions and identify the limiting problem.

Theorem 7 Let \( \{ \mathcal{T}^h : h > 0 \} \), \( F, g, f_\Sigma, f_{\hat{n}}, [u^h_1, p^h_1], [u^h_2, p^h_2] \) be as in Theorem 6. Then, the element \( ([u^1_1, p^1_1], [u^1_2, p^1_2]) \) given by Corollary 1 is the unique solution to Problem (16). Moreover, the whole sequence converges to this point, that is:

\[
([u^1_1, p^1_1], [u^1_2, p^1_2]) \longrightarrow ([u_1, p_1], [u_2, p_1]), \quad \text{weakly in } X \times Y.
\]

Proof To prove the result, it is enough to show that \( ([u^1_1, p^1_1], [u^1_2, p^1_2]) \) satisfies the variational Statement (14). Let \( ([v_1, q_2], [v_2, q_1]) \) be an arbitrary element of \( X \times Y \) and let \( ([v^h_1, q^h_1], [v^h_2, q^h_2]) \) be its orthogonal projection onto \( X_h \times Y_h \). Due to Proposition 3, the sequence \( \{([v^h_1, q^h_1], [v^h_2, q^h_2]) : h > 0 \} \) converges strongly to \( ([v_1, q_2], [v_2, q_1]) \). Now, test the variational formulation associated with Problem (20); this gives the following:

\[
\begin{align*}
\int_{\Omega_1} a u^h_1 \cdot v^1_1 &+ \int_{\Gamma} \beta p^h_2 q^2_2 \, dS - \int_{\Gamma} (u^h_1 \cdot \hat{n}) \, q^2_2 \, dS + \int_{\Gamma} p^h_2 (v^1_1 \cdot \hat{n}) \, dS \\
&\quad - \int_{\Omega_1} p^h_1 \nabla \cdot v^1_1 - \int_{\Omega_2} u^h_2 \cdot \nabla q^h_2 \\
&= \int_{\Omega_2} F q^h_2 - \int_{\Omega_1} g \cdot v^1_1 + \int_{\Gamma} f_\Sigma (v^h_1 \cdot \hat{n}) \, dS - \int_{\Gamma} f_{\hat{n}} q^2_2 \, dS, \\
\int_{\Omega_1} \nabla \cdot u^h_1 q^1_1 &+ \int_{\Omega_2} \nabla p^h_2 \cdot v^2_2 + \int_{\Omega_2} a u^h_2 \cdot v^2_2 = \int_{\Omega_1} F q^1_1 - \int_{\Omega_2} g \cdot v^h_2.
\end{align*}
\]

Notice that, in both expressions, above each summand of the left-hand side converges, since one of the factors is weakly convergent, while the other is strongly convergent. The right-hand side also converges due to the strong convergence of the quantifiers. Consequently, the element \( ([u^1_1, p^1_1], [u^1_2, p^1_2]) \) satisfies the variational Statement (14) for any arbitrary test function \( ([v_1, q_2], [v_2, q_1]) \). It follows that \( ([u^1_1, p^1_1], [u^1_2, p^1_2]) \) is a solution of Problem (16); this concludes the first part of the theorem.

For the second part, the well-posedness of Problem (16) gives the uniqueness of its solution. Consequently, due to the Estimate (27), any subsequence of \( \{([v^h_1, q^h_1], [v^h_2, q^h_2]) : h > 0 \} \) would have yet another subsequence weakly convergent to the solution of Problem (16). Hence, the Statement (33) follows and the proof is complete. \( \square \)
Finally, we have the following.

**Theorem 8** Let \( \mathcal{T}^h : h > 0 \), \( F, g, f_\Sigma, f_n \), \( ([u_1^h, p_2^h], [u_2^h, p_1^h]) \) be as in Theorem 6 above and let \( ([u_1, p_2], [u_2, p_1]) \) be the solution to Problem (16). Then:

\[
([u_1^h, p_2^h], [u_2^h, p_1^h]) \xrightarrow{h \to 0} ([u_1, p_2], [u_2, p_1]), \quad \text{strongly in } X \times Y. \tag{35}
\]

**Proof** We use the standard approach. By testing Problem (16) with \( (u, p) \) and adding both equations, this yields

\[
\int_{\Omega_1} a |u_1|^2 + \int_{\Omega_2} a |u_2|^2 + \int_{\Gamma} \beta |p_2|^2 \, dS = \int_{\Omega} F \cdot u - \int_{\Omega} g \cdot u + \int_{\Gamma} f_\Sigma (u_1^h \cdot \hat{n}) \, dS - \int_{\Gamma} f_n \, p_2 \, dS.
\]

(36)

On the other hand, taking lim sup in the Identity (28), we get the following:

\[
\limsup_{h \to 0} \left[ \int_{\Omega_1} a |u_1|^2 + \int_{\Omega_2} a |u_2|^2 + \int_{\Gamma} \beta |p_2|^2 \, dS \right]
\]

\[
= \int_{\Omega} F \cdot u - \int_{\Omega} g \cdot u + \int_{\Gamma} f_\Sigma (u_1 \cdot \hat{n}) \, dS - \int_{\Gamma} f_n \, p_2 \, dS
\]

\[
\leq \liminf_{h \to 0} \left[ \int_{\Omega_1} a |u_1|^2 + \int_{\Omega_2} a |u_2|^2 + \int_{\Gamma} \beta |p_2|^2 \, dS \right].
\]

In the expression above, the equality of the second line holds due to the Identity (36) and the inequality of the third line holds due to the weak convergence Statement (33). From here, due to the standard Hilbert space theory, it follows that:

\[
\|u_1^h - u_1\|_{0, \Omega} \xrightarrow{h \to 0} 0, \tag{37a}
\]

\[
\|u_2^h - u_2\|_{0, \Omega} \xrightarrow{h \to 0} 0, \tag{37b}
\]

\[
\|\sqrt{\beta} (p_2^h - p_2)\|_{0, \Gamma} \xrightarrow{h \to 0} 0. \tag{37c}
\]

On the other hand, the solution \( u_1^h \) satisfies the discretization of Eq. (6b). Therefore, it holds that \( \nabla \cdot u_1^h = F^h \), where \( F^h \) is the orthogonal projection of \( F \) on the space \( \mathcal{H}(\Omega_1, \mathcal{T}^h) \).

Since \( \|F^h - F\|_{0, \Omega_1} \to 0 \), it follows that \( \|\nabla \cdot u_1^h - \nabla \cdot u_1\|_{0, \Omega_1} \to 0 \), which, combined with the Statement (37a), yields the following:

\[
\|u_1^h - u_1\|_{H_{\text{div}}(\Omega_1)} \xrightarrow{h \to 0} 0. \tag{38}
\]

Next, the solution \( p_2^h \) satisfies the discretized version of Darcy’s law (6d), i.e., \( u_1^h + \nabla p_2^h = g^h \).

Again, \( g^h \) indicates the orthogonal projection of \( g \) onto \( \nabla \mathcal{H}(\Omega_2, \mathcal{T}^h) \), and due to the strong convergence of the orthogonal projections, it follows that \( \|\nabla p_2^h - \nabla p_2\|_{0, \Omega_2} \to 0 \). The latter, combined with the Statement (37c) and the Inequality (24) implies that:

\[
\|p_2^h - p_2\|_{1, \Omega_2} \xrightarrow{h \to 0} 0. \tag{39}
\]

Finally, for the strong convergence of \( \{p_1^h : h > 0\} \), let \( \{v_1^h : h > 0\} \subseteq H_{\text{div}}(\Omega_1) \) be a sequence of functions such that \( \nabla \cdot v_1^h = p_1^h \) and \( \|v_1^h\| \leq C \); which exists because of
Theorem 4. It will be shown that any subsequence of \( \{ p_1^h : h > 0 \} \) has another subsequence denoted with the index \( h' \), such that

\[
\lim_{h' \to 0} \| p_1^{h'} \|_{0, \Omega_1} = \| p_1 \|_{0, \Omega_1}. \tag{40}
\]

Take a subsequence of \( \{ p_1^h : h > 0 \} \) still denoted the same. Due to the boundedness of \( \{ v_1^h : h > 0 \} \), there must exist a convergent subsequence denoted with the index \( h' \) which is weakly convergent in \( \mathbf{H}_{\text{div}}(\Omega_1) \) to an element \( v_1 \). The Identity (26) holds for any element in \( \mathbf{X}_h \), in particular:

\[
\| p_1^{h'} \|^2_{0, \Omega_1} = \int_{\Omega_1} \phi_1^{h'} \phi_1^{h'} = \int_{\Omega_1} \phi_1^{h'} \mathbf{\nabla} \cdot v_1^{h'} = - \int_{\Omega_1} a \mathbf{u}_1^{h'} \cdot v_1^{h'} - \int_{\Gamma} p_2^h (v_1^{h'} \cdot \mathbf{n}) \, dS + \int_{\Omega_1} \mathbf{g} \cdot v_1^{h'} - \int_{\Gamma} f_\Sigma (v_1^{h'} \cdot \mathbf{n}) \, dS.
\]

All the summands of the right-hand side converge, since one of the factors converges strongly, while the other converges weakly, then the left-hand side also converges, that is:

\[
\lim_{h' \to 0} \| p_1^{h'} \|^2_{0, \Omega_1} = - \int_{\Omega_1} a \mathbf{u}_1 \cdot v_1 - \int_{\Gamma} p_2 (v_1 \cdot \mathbf{n}) \, dS + \int_{\Omega_1} \mathbf{g} \cdot v_1 - \int_{\Gamma} f_\Sigma (v_1 \cdot \mathbf{n}) \, dS.
\]

Observe that, \( \mathbf{\nabla} \cdot v_1 = p_1 \), since \( \{ p_1^h : h > 0 \} \) converges weakly to \( p_1 \), now, test the Statement (14a) with \( [v_1, 0] \in \mathbf{X} \) to get:

\[
\| p_1 \|^2_{0, \Omega_1} = \int_{\Omega_1} p_1 p_1 = \int_{\Omega_1} p_1 \mathbf{\nabla} \cdot v_1 = - \int_{\Omega_1} a \mathbf{u}_1 \cdot v_1 - \int_{\Gamma} p_2 (v_1 \cdot \mathbf{n}) \, dS + \int_{\Omega_1} \mathbf{g} \cdot v_1 - \int_{\Gamma} f_\Sigma (v_1 \cdot \mathbf{n}) \, dS.
\]

Equating both expressions above, we get Eq. 40. From elementary real analysis, it follows that the full sequence of \( L^2(\Omega_1) \)-norms converges to \( \| p_1 \|_{0, \Omega_1} \). Finally, from standard Hilbert space theory, it follows that \( \| p_1^{h'} - p_1 \|^2_{0, \Omega_1} \xrightarrow{h \to 0} 0 \) and the proof is complete. \( \square \)

### 3.2 Rate of convergence

In this section, the rate of convergence analysis is presented. It will be done assuming Hypothesis 2 is satisfied. We proceed in the standard way [see (Gatica 2013)].

**Definition 4** Given \( h > 0 \) fixed, define the operator \( A_h : \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y} \), such that \(([v_1, p_2], [v_2, p_1])\) is mapped to the unique solution \(([\phi_1^h, \phi_2^h], [\phi_2^h, \phi_1^h])\) in \( \mathbf{X}_h \times \mathbf{Y}_h \) of the problem:

\[
\mathcal{A}([u_1^h, p_2^h]) + \mathcal{B}([u_2^h, p_1^h]) = \mathcal{A}([u_1, p_2]) + \mathcal{B}([u_2, p_1]) \quad \text{in } \mathbf{X}_h,
\]

\[
-\mathcal{B}([u_1^h, p_2^h]) + \mathcal{C}([u_2^h, p_1^h]) = -\mathcal{B}([u_1, p_2]) + \mathcal{C}([u_2, p_1]) \quad \text{in } \mathbf{Y}_h,
\]

followed by the canonical embedding \( j : \mathbf{X}_h \times \mathbf{Y}_h \hookrightarrow \mathbf{X} \times \mathbf{Y} \).
Remark 2 (i) The operator $A_h$ above is well defined due to Theorem 5.
(ii) Due to Theorem 5, the operator is $A_h$ linear, continuous, and idempotent.

We have the following result.

Theorem 9 (i) There exists $M > 0$, such that
\[ \| A_h \| \leq M, \quad \text{for all } h > 0; \] 
that is: the family $\{ A_h : h > 0 \}$ is globally bounded.

(ii) Let $((u_1, p_2), [u_2, p_1]) \in X \times Y$, $([u_1^h, p_2^h], [u_2^h, p_1^h]) \in X_h \times Y_h$ be the unique solutions to Problems (16) and (20), respectively; then:
\[ \| ([u_1^h, p_2^h], [u_2^h, p_1^h]) - ([u_1^h, p_2^h], [u_2^h, p_1^h]) \|_{X \times Y} \leq (1 + M) \inf \{ ([v_1^h, q_2^h], [v_2^h, q_1^h]) \in X_h \times Y_h \} \| ([u_1, p_2], [u_2, p_1]) - ([v_1^h, q_2^h], [v_2^h, q_1^h]) \|_{X \times Y}, \] 
for every $h > 0$.

Proof (i) Given an arbitrary element $((v_1, q_2), [v_2, q_1]) \in X \times Y$, defining $\tilde{F}_1 \triangleq \mathcal{S}[v_1, q_2] + \mathcal{B}[v_2, q_1]$ and $\tilde{F}_2 = -\mathcal{B}[v_1, q_2] + \mathcal{C}[v_2, q_1]$, it is clear due to Theorem 3 that $((v_1, q_2), [v_2, q_1])$ is the unique solution to Problem 16 with $F_i$ replaced by $\tilde{F}_i$ for $i = 1, 2$. Recalling Definition 4, it is clear that $A_h((v_1, q_2), [v_2, q_1])$ is the unique solution to Problem (20), and due to the strong convergence analysis, Theorem 8, it holds that $\| ([v_1, q_2], [v_2, q_1]) - A_h((v_1, q_2), [v_2, q_1]) \|_{h \to 0} \to 0$. In particular, the sequence $\{ A_h((v_1, q_2), [v_2, q_1]) : h > 0 \}$ is bounded i.e., the family of operators $\{ A_h : h > 0 \}$ is bounded pointwise; due to the Banach–Steinhaus Uniform Boundedness Principle (from standard Functional Analysis theory), the Statement (42) holds.

(ii) Since $A_h$ is idempotent, we observe that:
\[ ([u_1, p_2], [u_2, p_1]) - ([u_1^h, p_2^h], [u_2^h, p_1^h]) = (I - A_h)\left( ([u_1, p_2], [u_2, p_1]) - ([v_1^h, q_2^h], [v_2^h, q_1^h]) \right). \]

From Statement 42 and the expression above, Inequality (43) follows trivially.

Finally, we have the rate of convergence result.

Theorem 10 Let $((u_1, p_2), [u_2, p_1]) \in X \times Y$, $([u_1^h, p_2^h], [u_2^h, p_1^h]) \in X_h \times Y_h$ be the unique solutions to Problem (16) and (20), respectively; then,
\[ \| u_1 - u_1^h \|_{H_{\text{div}}(\Omega_1)} = \mathcal{O}(h), \] 
\[ \| u_2 - u_2^h \|_{0, \Omega_2} = \mathcal{O}(h), \] 
\[ \| p_1 - p_1^h \|_{0, \Omega_1} = \mathcal{O}(h) \]
\[ \| p_2 - p_2^h \|_{0, \Omega_2} = \mathcal{O}(h^2), \] 
\[ \| p_2 - p_2^h \|_{1, \Omega_2} = \mathcal{O}(h). \]
Proof To prove Inequality (44a), define $\vec{F}_1 := \mathcal{A}[\mathbf{u}_1, 0] + B[\mathbf{0}, 0]$ and $\vec{F}_2 = -B[\mathbf{u}_1, 0] + C[\mathbf{0}, 0]$. Again, due to Theorem 3 ([\mathbf{u}_1, 0], \{0, 0\}) is the unique solution to Problem 16 with $F_i$ replaced by $\vec{F}_i$ for $i = 1, 2$. Applying Inequality (43) yields the following:

$$
\| ([\mathbf{u}_1, 0], \{0, 0\}) - ([\mathbf{u}^h_1, 0], \{0, 0\}) \|_{X \times Y} \leq (1 + M) \inf_{(v^h_1,q^h_1), (v^h_2,q^h_2) \in X_h \times Y_h} \| ([\mathbf{u}_1, 0], \{0, 0\}) - ([v^h_1,q^h_1], [v^h_2,q^h_2]) \|_{X \times Y}.
$$

Let $\Pi_h$ be the global Raviart–Thomas interpolation operator, and then, $([\Pi_h \mathbf{u}_1, 0], \{0, 0\}) \in X_h \times Y_h$; recalling the inequality above, it follows:

$$
\| \mathbf{u}_1 - \mathbf{u}^h_1 \|_{H^1(\Omega)} \leq (1 + M) \| \mathbf{u}_1 - \Pi_h \mathbf{u}^h_1 \|_{H^1(\Omega)} \leq O(h).
$$

In the expression above, the last inequality follows from the standard finite-element theory for interpolation operators, see (Gatica 2013). The remaining statements in (44) are shown using the same scheme.

Remark 3 Observe that the rates of convergence summarized in (44) are all the standard ones, no gain or deterioration has been added by the scheme. This is because there were no strong coupling conditions in building the spaces, neither the continuous $X, Y$, nor the discrete ones $X_h, Y_h$. The interface exchange conditions are satisfied weakly, i.e., only by the solution of the problems (14) and (20), respectively.

4 Numerical examples

In this section, we present two numerical examples to illustrate the method: the first showing a case of continuity and the second showing a slight perturbation of the first to illustrate how the method handles discontinuities across interfaces. The numerical examples use the finite-dimensional spaces $X_h, Y_h$ introduced in (18). The experiments are executed in a MATLAB script using the adaptations of the codes EBmefem.m (see, Carstensen and Bahariawati 2005a, b) and fem2d.m (see, Albery et al. 1999, 2005).

For the sake of clarity, we adopt the domain $\Omega$, the interface $\Gamma$, and the sub-domains $\Omega_1$, $\Omega_2$ as follows (see Fig. 2a):

$$
\Omega \overset{\text{def}}{=} (-1, 1) \times (-1, 1), \quad \Gamma \overset{\text{def}}{=} (-1, 1) \times \{0\} \cup \{0\} \times (-1, 1),
\Omega_1 \overset{\text{def}}{=} (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1), \quad \Omega_2 \overset{\text{def}}{=} (-1, 0) \times (0, 1) \cup (0, 1) \times (-1, 0).
$$

(45)

Again, for simplicity, all the experiments run on the uniform Cartesian grid, see Fig. 2b. The sequence of grids $\{\mathcal{F}^i : 0 \leq i \leq 5\}$ has corresponding sizes $h_i = 2^i$ for $0 \leq i \leq 5$; consequently, it is a monotone sequence as described in Definition 3. The experimental computation for the order of convergence $r$ uses the standard approach. Assuming that the error satisfies $e = O(h^r)$, we approximate $r$ by the following:

$$
r \sim \frac{\log e_{k+1} - \log e_k}{\log h_{k+1} - \log h_k} = \frac{\log e_k - \log e_{k+1}}{\log 2}, \quad \text{for all } 0 \leq k \leq 4.
$$

In the expression above, the last equality holds due to the particular nature of the grids’ size.
Fig. 2  a Depicts a bipartite map \( G = (G_1, G_2) \) example for the region \( \Omega = [0, 1] \times [0, 1] \). Subregions belonging to \( G_1 \) are red-colored and the subregions belonging to \( G_2 \) are blue-colored.  

(b) Depicts an example of a grid \( T \) consistent with the map \( G \). Some of the triangles belonging to \( T_1 \) and \( T_2 \) have been labeled with 1 and 2, respectively (color figure online).

Example 1  
The purpose of the present example is to illustrate how the method handles problems free of discontinuities across the interfaces. The exact solution in this case is given by the following:

\[
p : \Omega \rightarrow \mathbb{R}, \quad p(x, y) = x y (x - 1)^2 (y - 1)^2 (x + 1)^2 (y + 1)^2, \tag{46a}
\]

\[
u : \Omega \rightarrow \mathbb{R}^2, \quad u(x, y) = -\nabla p(x, y), \tag{46b}
\]

see Fig. 3. The forcing terms are as follows:

\[
g : \Omega \rightarrow \mathbb{R}^2, \quad g = 0, \tag{47a}
\]

\[
F : \Omega \rightarrow \mathbb{R}, \quad F = -\nabla \cdot \nabla p, \tag{47b}
\]

The convergence results are reported in Tables 1 and 2, while the average convergence rate behavior is displayed in Eq. (48). In most of the cases, the convergence rate behaves as expected, except for the \( L^2 \)-norms of \( p_1 \) and \( p_2 \) presenting values mildly below the expected:

\[
\| p_1^h - p_1 \|_{0, \Omega_1} = \mathcal{O}(h^{1.76}), \quad \| p_2^h - p_2 \|_{0, \Omega_2} = \mathcal{O}(h^{1.87}), \quad \| p_1^h - p_2 \|_{1, \Omega_2} = \mathcal{O}(h^{1.06}). \tag{48a}
\]

\[
\| u_1^h - u_1 \|_{0, \Omega_1} = \mathcal{O}(h^{1.06}), \quad \| u_1^h - u_1 \|_{H_{\text{div}}(\Omega_1)} = \mathcal{O}(h^{1.06}), \quad \| u_2^h - u_2 \|_{0, \Omega_2} = \mathcal{O}(h^{1.04}). \tag{48b}
\]

Finally, the numerical solution for \( h^{-1} = 8 \) is depicted in Fig. 4: the choice of the grid was based on optical clarity to illustrate both: the nature of discrete solution and its convergence to the continuous solution.

Example 2  
The present example is a perturbation of the previous one, to illustrate how the method handles problems with simultaneous discontinuities across the interfaces in both: the
Fig. 3 Example 1. a Depicts the pressure of the exact solution $p(x, y) = xy(x - 1)^2(y - 1)^2(x + 1)^2(y + 1)^2$, see Eq. (46a). b Depicts the flux of the exact solution $u = -\nabla p$, see Eq. (46b). On the upper right corner is depicted the $x$-component, while the lower right corner displays the $y$-component.

Table 1 Pressure convergence table, Example 1

| $h^{-1}$ | $\|p_1^h - p_1\|_{0, \Omega_1}$ | $r$ | $\|p_2^h - p_2\|_{0, \Omega_2}$ | $r$ | $\|p_2^h - p_2\|_{1, \Omega_2}$ | $r$ |
|----------|-------------------------------|-----|-------------------------------|-----|-------------------------------|-----|
| 1        | 0.1836                        |     | 2.8144                        |     | 0.8643                        |     |
| 2        | 0.0261                        | 0.4383| 0.0721                        | 5.2867| 0.1976                        | 2.1289|
| 4        | 0.0091                        | 1.5201| 0.0226                        | 1.6737| 0.0887                        | 1.1556|
| 8        | 0.0026                        | 1.8074| 0.0062                        | 1.8660| 0.0422                        | 1.0717|
| 16       | 0.0007                        | 1.8931| 0.0016                        | 1.9542| 0.0209                        | 1.0137|
| 32       | 0.0002                        | 1.8070| 0.0004                        | 2.0000| 0.0104                        | 1.0069|

Table 2 Velocities’ convergence table, Example 1

| $h^{-1}$ | $\|u_1^h - u_1\|_{0, \Omega_1}$ | $r$ | $\|u_1^h - u_1\|_{H^1(\Omega_1)}$ | $r$ | $\|u_2^h - u_2\|_{0, \Omega_2}$ | $r$ |
|----------|-------------------------------|-----|-------------------------------|-----|-------------------------------|-----|
| 1        | 0.8264                        |     | 0.8264                        |     | 0.9184                        |     |
| 2        | 0.1409                        | 2.5522| 0.1409                        | 2.5522| 0.1840                        | 2.3194|
| 4        | 0.0617                        | 1.1913| 0.0617                        | 1.1913| 0.0857                        | 1.1023|
| 8        | 0.0302                        | 1.0307| 0.0302                        | 1.0307| 0.0417                        | 1.0392|
| 16       | 0.0150                        | 1.0096| 0.0150                        | 1.0096| 0.0208                        | 1.0035|
| 32       | 0.0075                        | 1.0000| 0.0075                        | 1.0000| 0.0104                        | 1.0000|

normal flux and the normal stress. The perturbation is localized on the fourth quadrant of the domain $(0, 1) \times (0, -1)$. The analytic solution in this case is given by the following:
(a) Pressure Approximate Solution.  

(b) Flux Approximate Solution.

Fig. 4 Example 1, approximate solution for a mesh of size $h^{-1} = 8$. The sub-domains are $\Omega_1 = (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1)$ and $\Omega_2 = (-1, 0) \times (0, 1) \cup (1, 0) \times (-1, 0)$, see Identity (45). 

a Depicts the pressure $p_h$ of the approximate solution, it is piecewise constant on the domain $\Omega_1$ and piecewise linear affine on the domain $\Omega_2$. 

b Depicts the flux of the approximate solution $u^h$. On the upper right corner is depicted the $x$-component of the flux, which is continuous across horizontal edges of $\Omega_1$ and piecewise constant on the domain $\Omega_2$. On the lower right corner, we display the $y$-component of the flux, which is continuous across vertical edges of $\Omega_1$ and piecewise constant on the domain $\Omega_2$.

\[ p : \Omega \rightarrow \mathbb{R}, \]
\[ p(x, y) = xy(x - 1)^2(y - 1)^2(x + 1)^2(y + 1)^2 + \frac{1}{20} ((x - 1)^2 - (y + 1)^2) \mathbb{1}_{(1,0) \times (0,1)}(x, y), \quad (49a) \]
\[ u : \Omega \rightarrow \mathbb{R}^2, \quad u(x, y) = -\nabla p(x, y); \quad (49b) \]

see Fig. 5. The forcing terms are acting inside the domains are identical to the previous example:

\[ g : \Omega \rightarrow \mathbb{R}^2, \quad g = 0, \]
\[ F : \Omega \rightarrow \mathbb{R}, \quad F = -\nabla \cdot \nabla p. \quad (50a) \]

In this case, the interface forcing terms account for the jumps of the solution across the interface, i.e., according to the interface exchange conditions (7b), (7a), $f_\Sigma$ and $f_\hat{n}$ are given by the following:

\[ f_\Sigma : \Gamma \rightarrow \mathbb{R}, \]
\[ f_\Sigma(x, y) = \frac{1}{20} ((x - 1)^2 - 1) \mathbb{1}_{(0,1) \times [0]}(x, y) + \frac{1}{20} (1 - (y + 1)^2) \mathbb{1}_{[0] \times (-1,0)}(x, y), \]
\[ f_\hat{n} : \Gamma \rightarrow \mathbb{R}, \]
\[ f_\hat{n}(x, y) = \frac{1}{20} (x - 4) \mathbb{1}_{(0,1) \times [0]}(x, y) + \frac{1}{20} (4 - y)) \mathbb{1}_{[0] \times (-1,0)}(x, y). \quad (50b) \]
Fig. 5 Example 2, discontinuous exact solution. The visualization angles are different for the pressure and the flux; the choice is made focusing on the jumps of discontinuity. The discontinuities take place on the interface subset \( \{0\} \times (-1, 0) \cup (0, 1) \times \{0\} \) for the pressure \( p \) as well as both components of the velocity \( u_x, u_y \). a Depicts the pressure of the exact solution \( p \), see Eq. (49a). b Depicts the flux of the exact solution \( \mathbf{u} = -\nabla p \), see Eq. (49b). On the upper right corner is depicted the \( x \)-component, while the lower right corner displays the \( y \)-component

Table 3 Pressure convergence table, Example 2

| \( h^{-1} \) | \( \| p_1^h - p_1 \|_{0, \Omega_1} \) | \( r \) | \( \| p_2^h - p_2 \|_{0, \Omega_2} \) | \( r \) | \( \| p_2^h - p_2 \|_{1, \Omega_2} \) | \( r \) |
|---|---|---|---|---|---|---|
| 1 | 0.9984 | | 1.6520 | | 2.5140 | |
| 2 | 0.0261 | 5.2575 | 0.0721 | 4.5181 | 0.1980 | 3.6664 |
| 4 | 0.0091 | 1.5201 | 0.0226 | 1.6737 | 0.0889 | 1.1552 |
| 8 | 0.0026 | 1.8074 | 0.0062 | 1.8660 | 0.0423 | 1.0715 |
| 16 | 0.0007 | 1.8931 | 0.0016 | 1.9542 | 0.0209 | 1.0172 |
| 32 | 0.0002 | 1.8074 | 0.0004 | 2.0000 | 0.0104 | 1.0069 |

It is direct to see that [\( \mathbf{u}, p \)] defined by (49) is the exact solution to the Problem (6) on the geometric domain described by (45) with the forcing terms defined in (50). Again, the boundary and interface conditions are satisfied.

The convergence results are displayed in Tables 3 and 4. The convergence behavior is virtually identical to the continuous case with observable differences (Tables 1 and 2) only for the first stages. Consequently, the convergence rate agrees with those presented in Eq. (48). Finally, the numerical solution for \( h^{-1} = 16 \) is depicted in Fig. 6; the choices of grid and display angle were based on optical clarity for the jumps across the interface.

Example 3 The purpose of the present example is to illustrate how the method handles problems with flux discontinuities across the interfaces. Such discontinuities occur, because the flow resistance coefficient \( a(\cdot) \) has different orders of magnitude within regions \( \Omega_1 \) and \( \Omega_2 \).
Table 4  Velocities’ convergence table, Example 2

| $h^{-1}$ | $||u_h^1 - u_1||_{0,\Omega_1}$ | $r$ | $||u_h^1 - u_1||_{H^{\text{div}}(\Omega_1)}$ | $r$ | $||u_h^2 - u_2||_{0,\Omega_2}$ | $r$ |
|----------|-----------------|-----|---------------------------------|-----|-----------------|-----|
| 1        | 19.8825         |     | 19.8825                         |     | 9.8017           |     |
| 2        | 0.1409          |     | 7.1407                          |     | 0.1409           |     |
| 4        | 0.0617          |     | 1.1913                          |     | 0.0617           |     |
| 8        | 0.0302          |     | 1.0307                          |     | 0.0302           |     |
| 16       | 0.0150          |     | 1.0096                          |     | 0.0150           |     |
| 32       | 0.0075          |     | 1.0000                          |     | 0.0075           |     |

(a) Pressure Approximate Solution.  
(b) Flux Approximate Solution.

Fig. 6 Example 2, approximate solution for a mesh of size $h^{-1} = 16$. The visualization angles are different for the pressure and the flux; the choice is made focusing on the jumps of discontinuity. Notice how the method captures the discontinuities for the pressure $p$ and both components of the velocity $u_x, u_y$. The jumps take place on the interface subset $[0] \times (-1, 0) \cup (0, 1) \times [0]$. a Depicts the pressure $p^h$ of the approximate solution. b Depicts the flux of the approximate solution $u^h$. On the upper right corner is depicted the $x$-component of the flux and, on the lower right corner we display the $y$-component of the flux.

For clarity of exposition, we use the same pressure as in Example 1, i.e., the exact solution, see Fig. 7, is given by the following:

\[
p : \Omega \rightarrow \mathbb{R}, \quad p(x, y) = x y (x - 1)^2(y - 1)^2(x + 1)^2(y + 1)^2, \tag{51a}
\]

\[
u : \Omega \rightarrow \mathbb{R}^2, \quad u(x, y) = -\frac{1}{a(x, y)} \nabla p(x, y). \tag{51b}
\]

Here, the flow resistance coefficient is defined as follows:

\[
a(x, y) \overset{\text{def}}{=} \mathbb{1}_{\Omega_1}(x, y) + 5 \mathbb{1}_{\Omega_2}(x, y); \tag{51c}
\]

in particular, it satisfies Hypothesis 2. The forcing terms are as follows:
Fig. 7  Example 3.  a Depicts the pressure of the exact solution \( p(x, y) = x y (x - 1)^2(y - 1)^2(x + 1)^2(y + 1)^2 \), see Eq. (46a).  b Depicts the flux of the exact solution \( u = -a^{-1} \nabla p \), see Eq. (46b). On the upper right corner is depicted the \( x \)-component, while the lower right corner displays the \( y \)-component. Here, discontinuities occur only for the velocity due to the flow resistance coefficient \( a(\cdot) \), see Eq. (51c). The velocity’s \( x \)-component \( u_x \) has a jump across \( \{0\} \times (-1, 1) \), while the \( y \)-component \( u_y \) jumps across \((-1, 1) \times \{0\} \), see Eq. (52b) for the jumps’ exact algebraic expression

\[ g : \Omega \to \mathbb{R}^2, \quad g = 0, \]
\[ F : \Omega \to \mathbb{R}, \quad F = -\nabla \cdot \frac{1}{a} \nabla p, \]
\[ f_\Sigma : \Gamma \to \mathbb{R}, \quad f_\Sigma (x, y) = 0, \]
\[ f_{\hat{n}} : \Gamma \to \mathbb{R}, \quad f_{\hat{n}}(x, y) = \frac{4}{5} x (x^2 - 1)^2 \mathbb{1}_{(-1, 1) \times \{0\}} + \frac{4}{5} y (y^2 - 1)^2 \mathbb{1}_{\{0\} \times (-1, 1)}. \]

A direct calculation shows that \([u, p]\) defined by (51) is the exact solution of Problem (6), on the geometric domain described by (45) with the forcing terms defined in (52). The flux jump \( f_{\hat{n}}(x, y) \) occurs because of the jump in the flow resistance coefficient \( a(\cdot) \) described in (51c); \( a(\cdot) \) should be a continuous function; the interface flux term would be null, i.e., \( f_{\hat{n}}(x, y) \equiv 0 \). Once more, the boundary conditions (6c), (6f), and the interface exchange conditions (7b), (7a) are satisfied.

The convergence results are reported in Tables 5 and 6, while the average behavior is summarized in Eq. (53). In this case, the convergence rates are deteriorated with respect to those of Example 1, having \( p_1 \) as the poorest one:

\[ \|P_1^h - p_1\|_{0, \Omega_1} = O(h^{1.59}), \quad \|p_2^h - p_2\|_{0, \Omega_2} = O(h^{1.74}), \quad \|p_2^h - p_2\|_{1, \Omega_2} = O(h^{1.08}) \]
\[ (53a) \]
\[ \|u_1^h - u_1\|_{0, \Omega_1} = O(h^{0.93}), \quad \|u_1^h - u_1\|_{H^{1, \Omega_1}} = O(h^{0.93}), \quad \|u_2^h - u_2\|_{0, \Omega_2} = O(h^{1.33}) \]
\[ (53b) \]
Table 5: Pressure convergence table, Example 3

| $h^{-1}$ | $\|p_1^h - p_1\|_{0, \Omega_1}$ | $r$ | $\|p_2^h - p_2\|_{0, \Omega_2}$ | $r$ | $\|p_2^h - p_2\|_{1, \Omega_2}$ | $r$ |
|---------|-------------------------------|-----|-------------------------------|-----|-------------------------------|-----|
| 1       | 0.0246                        |     | 0.1252                        |     | 0.4376                        |     |
| 2       | 0.0104                        |     | 1.2421                        |     | 0.0464                        |     |
| 4       | 0.0046                        |     | 1.1769                        |     | 0.0182                        |     |
| 8       | 0.0013                        |     | 2.1155                        |     | 0.0046                        |     |
| 16      | 0.0003                        |     | 1.5850                        |     | 0.0003                        |     |
| 32      | 0.0001                        |     |                               |     |                               |     |

Table 6: Velocities’ convergence table, Example 3

| $h^{-1}$ | $\|u_1^h - u_1\|_{0, \Omega_1}$ | $r$ | $\|u_1^h - u_1\|_{H\text{div}(\Omega_1)}$ | $r$ | $\|u_2^h - u_2\|_{0, \Omega_2}$ | $r$ |
|---------|-------------------------------|-----|---------------------------------|-----|-------------------------------|-----|
| 1       | 0.1884                        |     | 0.1884                          |     | 0.2092                        |     |
| 2       | 0.1212                        |     | 0.6364                          |     | 0.1212                        |     |
| 4       | 0.0567                        |     | 1.0960                          |     | 0.0567                        |     |
| 8       | 0.0292                        |     | 0.9574                          |     | 0.0292                        |     |
| 16      | 0.0148                        |     | 0.9804                          |     | 0.0148                        |     |
| 32      | 0.0074                        |     | 1.0000                          |     | 0.0074                        |     |

Finally, the numerical solution for $h^{-1} = 8$ is depicted in Fig. 8; the choices of grid and display angle were based on optical clarity to illustrate the pressure of Example 1 from a different point of view and to get a neat picture of the flux jumps across the interface.

**Example 4**

The purpose of the present example is twofold: first, to illustrate how the method handles problems whose velocities drastically change across the interface but still are continuous functions; this is done in controlled/lab conditions. Second, to suggest a heuristic method to proceed in practice i.e., when the real solution is not known. Such abrupt change takes place, because the flow resistance coefficient $a(\cdot)$, defined in Eq. (51c), has different orders of magnitude within regions $\Omega_1$ and $\Omega_2$. Although the exact solution is continuous on the velocity from the theoretical point of view, because of the multiscaling introduced by $a(\cdot)$, it is more convenient/strategic to treat it as discontinuous across the interface as the method does (see Fig. 10 below), to avoid numerical instability. In this example, the exact solution is given by (see Fig. 9)

\[
\begin{align*}
p : \Omega & \to \mathbb{R}, \\
p(x, y) & = \sin^2\left(\frac{\pi}{2}(x - 1)\right) \sin^2\left(\frac{\pi}{2}(y - 1)\right), \\
u : \Omega & \to \mathbb{R}^2, \\
u(x, y) & = -\frac{1}{a(x, y)} \nabla p(x, y),
\end{align*}
\]

Here, the flow resistance coefficient $a(\cdot)$ is defined by Eq. (51c). The forcing terms are

\[
\begin{align*}
g : \Omega & \to \mathbb{R}^2, \\
g & = 0, \\
F : \Omega & \to \mathbb{R}, \\
F & = -\nabla \cdot \frac{1}{a} \nabla p, \\
f_{\Sigma} : \Gamma & \to \mathbb{R}, \\
f_{\Sigma}(x, y) & = 0, \\
f_{\hat{n}} : \Gamma & \to \mathbb{R}, \\
f_{\hat{n}}(x, y) & = -\sin^2\left(\frac{\pi}{2}(x - 1)\right) \mathbb{1}_{(-1,1) \times [0]} - \sin^2\left(\frac{\pi}{2}(y - 1)\right) \mathbb{1}_{[0] \times (-1,1)}.
\end{align*}
\]
(a) Pressure Approximate Solution.  
(b) Flux Approximate Solution.

Fig. 8 Example 3, approximate solution for a mesh of size $h^{-1} = 8$. The sub-domains are $\Omega_1 = (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1)$ and $\Omega_2 = (-1, 0) \times (0, 1) \cup (0, 1) \times (-1, 0)$, see Identity (45). 

a Depicts the pressure $p^h$ of the approximate solution, it is piecewise constant on the domain $\Omega_1$ and piecewise linear affine on the domain $\Omega_2$. 
b Depicts the flux of the approximate solution $u^h$. On the upper right corner is depicted the $x$-component of the flux, which is continuous across horizontal edges of $\Omega_1$ and piecewise constant on the domain $\Omega_2$. On the lower right corner, it is displayed the $y$-component of the flux, which is continuous across vertical edges of $\Omega_1$ and piecewise constant on the domain $\Omega_2$. Observe that the jumps across the interface are captured for both components of the velocity and $u_x$ has a jump across $\{0\} \times (-1, 1)$, while $u_y$ jumps across $(-1, 1) \times \{0\}$.

A direct calculation shows that $[u, p]$ defined by (51) is the exact solution of the Problem (6) on the geometric domain described by (45) with the forcing terms defined in (52). Once more, the boundary conditions (6c), (6f) and the interface exchange conditions (7b), (7a) are satisfied.

The interface normal flux forcing term satisfies $f_\hat{n}(x, y) = -\beta(\cdot) p\big|_{\Gamma}$, for $\beta(\cdot) \equiv 1$ (in particular, Hypothesis 2 is verified). Then, the interface normal flux balance condition Eq. (7b) implies $u_1 \cdot \hat{n}\big|_{\Gamma} = u_2 \cdot \hat{n}\big|_{\Gamma}$. Hence, no flux jumps occur despite the change in the order of magnitude between regions, which comes from the flow resistance coefficient introduced in (51c).

The convergence results are reported in the Tables 7 and 8 below, while the average behavior is summarized in Eq. (56). This case presents a slightly better behavior than Example 3 but still the pressure $L^2(\Omega_1)$-norm differs significantly from the expected one.

\[
\begin{align*}
\|p_1^h - p_1\|_{0, \Omega_1} &= \mathcal{O}(h^{1.66}), & \|p_2^h - p_2\|_{0, \Omega_2} &= \mathcal{O}(h^{1.89}), & \|p_2^h - p_2\|_{1, \Omega_2} &= \mathcal{O}(h^{1.03}), \\
\|u_1^h - u_1\|_{0, \Omega_1} &= \mathcal{O}(h^{1.05}), & \|u_2^h - u_2\|_{\text{div}(\Omega_2)} &= \mathcal{O}(h^{1.05}), & \|u_2^h - u_2\|_{0, \Omega_2} &= \mathcal{O}(h^{1.14}).
\end{align*}
\]

The numerical solution for $h^{-1} = 8$ is depicted in Fig. 10; the choices of grid and display angle were based on optical clarity to illustrate both: the nature of discrete solution and the flux numerical jumps across the interfaces.
Fig. 9  Example 4.  a Depicts the pressure of the exact solution $p(x, y) = \sin^2 \left( \frac{\pi}{2} (x - 1) \right) \sin^2 \left( \frac{\pi}{2} (y - 1) \right)$, see Eq. (46a).  b Depicts the flux of the exact solution $\mathbf{u} = -a^{-1} \nabla p$, see Eq. (46b). On the upper right corner is depicted the $x$-component, while the lower right corner displays the $y$-component. Observe the abrupt changes of $u_x$ across $\{0\} \times (-1, 1)$ and $u_y$ across $(-1, 1) \times \{0\}$ due to the multiscaling of the flow resistance coefficient $a(\cdot)$, see Eq. (51c).

Table 7  Pressure convergence table, Example 4, $f_h = -p|_\Gamma$

| $h^{-1}$ | $\|p_h^1 - p_1\|_{0, \Omega_1}$ | $r$ | $\|p_h^2 - p_2\|_{0, \Omega_2}$ | $r$ | $\|\mathbf{u}_h - \mathbf{u}_1\|_{H_{\text{div}}(\Omega_1)}$ | $r$ |
|---|---|---|---|---|---|---|
| 1 | 0.0624 | 0.2574 | 1.991 |
| 2 | 0.0513 | 0.2826 | 0.746 | 1.7868 | 0.5451 | 1.1374 |
| 4 | 0.0143 | 1.8429 | 0.0245 | 1.6064 | 0.2799 | 0.9616 |
| 8 | 0.0037 | 1.9504 | 0.0068 | 1.8492 | 0.1376 | 0.9244 |
| 16 | 0.0009 | 2.0395 | 0.0007 | 2.0000 | 0.0682 | 1.0126 |
| 32 | 0.0002 | 2.1699 | 0.0004 | 2.0875 | 0.0340 | 1.0042 |

Table 8  Velocities’ convergence table, Example 4, $f_h = -p|_\Gamma$

| $h^{-1}$ | $\|\mathbf{u}_h^1 - \mathbf{u}_1\|_{0, \Omega_1}$ | $r$ | $\|\mathbf{u}_h^2 - \mathbf{u}_2\|_{0, \Omega_2}$ | $r$ |
|---|---|---|---|---|
| 1 | 1.0801 | 1.0801 | 0.3538 |
| 2 | 0.3688 | 1.5503 | 0.3688 | 1.5503 | 0.1080 | 1.7119 |
| 4 | 0.2129 | 0.7927 | 0.2129 | 0.7927 | 0.0558 | 0.9527 |
| 8 | 0.1125 | 0.9203 | 0.1125 | 0.9203 | 0.0275 | 1.0028 |
| 16 | 0.0571 | 0.9784 | 0.0571 | 0.9784 | 0.0136 | 1.0158 |
| 32 | 0.0287 | 0.9924 | 0.0287 | 0.9924 | 0.0068 | 1.0000 |
Next, we present an alternative analysis for the same case. In practice, the exact solution is not known, only the forcing terms, namely \( F, g \) from Eq. (55a), but the pressure is not known at the interface, i.e., we ignore the normal flux term \( \hat{f}_n = -p_n \). However, this term can be introduced after the first iteration to correct it. In our next numerical experiment, the normal flux term in Eq. (55b) is replaced by the following:

\[
\hat{f}_n(x, y) = -\frac{1}{\sqrt{2}} \int_{\Gamma} \left( \sin^2 \left( \frac{\pi}{2} (x - 1) \right) \mathbb{1}_{(-1, 1) \times (0)} + \sin^2 \left( \frac{\pi}{2} (y - 1) \right) \mathbb{1}_{(0) \times (-1, 1)} \right) dS
\]

\[
= -\frac{1}{\sqrt{2}}.
\]

The integral above indicates line integral along the interface \( \Gamma \). Notice that this is the first Fourier coefficient of the normal flux term across the interface, i.e., the \( L^2(\Gamma) \)-orthogonal projection of \( \hat{f}_n \) onto the subspace of constant functions. The numerical solution for \( h^{-1} = 8 \) for this case is displayed in Fig. 10; the choices of grid and display angle were based on optical clarity to highlight the errors that the numerical solution contains both pressure and velocity due to \( \hat{f}_n \) as well as the flux numerical jumps across the interfaces. The approximation norms are summarized in Tables 9, 10, and 11. Clearly, in this case, the convergence rate analysis is pointless, since the numerical solution will not converge to the exact solution. However, it makes sense to compute the percentage relative errors to have a measure of the attained accuracy, that is:
### Table 9: Pressures $L^2$-convergence table, Example 4, $f\hat{n} = -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \int_{\Gamma} p |\Gamma\rangle \ dS$

| $k$ | $h^{-1}$ | $\|p_1^{h_k} - p_1^{h_{k-1}}\|_{0, \Omega_1}$ | $\epsilon(p_1, L^2, h_k)$ | $\|p_2^{h_k} - p_2^{h_{k-1}}\|_{0, \Omega_2}$ | $\epsilon(p_2, L^2, h_k)$ |
|-----|--------|-----------------|------------------|-----------------|------------------|
| 0   | 1      | 0.1594          | 0.3785           |                  |                  |
| 1   | 2      | 0.0558          | 5.1081           | 0.1441           | 14.0083          |
| 2   | 4      | 0.0414          | 1.9374           | 0.1046           | 4.9625           |
| 3   | 8      | 0.0420          | 0.9889           | 0.0880           | 2.0776           |
| 4   | 16     | 0.0425          | 0.5005           | 0.0822           | 0.9687           |
| 5   | 32     | 0.0426          | 0.2509           | 0.0801           | 0.4721           |

### Table 10: Velocities’ convergence table, Example 4, $f\hat{n} = -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \int_{\Gamma} p |\Gamma\rangle \ dS$

| $k$ | $h^{-1}$ | $\|u_1^{h_k} - u_1^{h_{k-1}}\|_{0, \Omega_1}$ | $\epsilon(u_1, L^2, h_k)$ | $\|u_2^{h_k} - u_2^{h_{k-1}}\|_{0, \Omega_2}$ | $\epsilon(u_2, L^2, h_k)$ |
|-----|--------|-----------------|------------------|-----------------|------------------|
| 0   | 1      | 1.6157          | 0.3439           |                  |                  |
| 1   | 2      | 0.4454          | 16.3705          | 0.0946           | 182.3322         |
| 2   | 4      | 0.3282          | 6.0312           | 0.0612           | 89.1516          |
| 3   | 8      | 0.2893          | 2.6582           | 0.0439           | 44.9504          |
| 4   | 16     | 0.2752          | 1.2644           | 0.0384           | 22.6303          |
| 5   | 32     | 0.2699          | 0.6200           | 0.0375           | 11.3547          |

### Table 11: Pressure $H^1(\Omega_2)$-convergence and velocity $H_{\text{div}}(\Omega_1)$-convergence table, Example 4, $f\hat{n} = -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \int_{\Gamma} p |\Gamma\rangle \ dS$

| $k$ | $h_k^{-1}$ | $\|p_2^{h_k} - p_2^{h_{k-1}}\|_{1, \Omega_2}$ | $\epsilon(p_2, H^1, h_k)$ | $\|u_1^{h_k} - u_1^{h_{k-1}}\|_{H_{\text{div}}(\Omega_1)}$ | $\epsilon(u_1, H_{\text{div}}, h_k)$ |
|-----|------------|-----------------|------------------|-----------------|------------------|
| 0   | 1          | 1.2474          | 1.6157           |                  |                  |
| 1   | 2          | 0.4945          | 60.9208          | 0.4454           | 4.2843           |
| 2   | 4          | 0.3232          | 29.0909          | 0.3282           | 0.5974           |
| 3   | 8          | 0.2364          | 14.5850          | 0.2893           | 0.7058           |
| 4   | 16         | 0.2090          | 7.3318           | 0.2752           | 0.3360           |
| 5   | 32         | 0.0375          | 3.6772           | 0.2699           | 0.1648           |

\[
\begin{align*}
\epsilon(p_2, H^1, h_k) & \overset{\text{def}}{=} 100 \frac{\|p_2^{h_k} - p_2^{h_{k-1}}\|_{1, \Omega_2}}{\|p_2^{h_k}\|_{1, \Omega_2}}, \\
\epsilon(u_1, H_{\text{div}}, h_k) & \overset{\text{def}}{=} 100 \frac{\|u_1^{h_k} - u_1^{h_{k-1}}\|_{H_{\text{div}}(\Omega_1)}}{\|u_1^{h_k}\|_{H_{\text{div}}(\Omega_1)}}, \\
\epsilon(p_i, L^2, h_k) & \overset{\text{def}}{=} 100 \frac{\|p_i^{h_k} - p_i^{h_{k-1}}\|_{0, \Omega_i}}{\|p_i^{h_k}\|_{0, \Omega_i}}, \\
\epsilon(u_i, L^2, h_k) & \overset{\text{def}}{=} 100 \frac{\|u_i^{h_k} - u_i^{h_{k-1}}\|_{0, \Omega_i}}{\|u_i^{h_k}\|_{0, \Omega_i}}, \quad \text{for } i = 1, 2.
\end{align*}
\]
Fig. 11 Example 4, approximate solution for a mesh of size $h^{-1} = 8$, the sub-domains are $\Omega_1 = (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1)$ and $\Omega_2 = (-1, 0) \times (0, 1) \cup (0, 1) \times (-1, 0)$, see Identity (45).

(a) Pressure Approximate Solution. (b) Flux Approximate Solution.

The relative errors are written on the column to the right of their corresponding absolute errors, as it can be seen after a few steps, the percentage error tends to contract by a half, i.e., $O(h)$.

5 Conclusions and final discussion

The present work yields several conclusions summarized as follows;

(i) A new conforming primal–dual mixed finite-element scheme has been introduced successfully from both points of view: theoretical and numerical.

(ii) The theoretical analysis of the method includes variational formulation and well-posedness of the continuous problem as well as the choice of finite-dimensional spaces, well-posedness (using the LBB theory) and convergence rates for the discrete problem.

(iii) The method is well suited for analyzing multiscale porous media fluid flow problems, such as oil extraction, groundwater flow, and geological fissured systems.

(iv) The main technical advantages of the method are two: it can handle interface discontinuities which are consistent with the choice of the FEM spaces, see Example 2, and it can handle effectively multiscale phenomena, since it can easily introduce numerical jumps.
across the interfaces, see Example 3. The latter is numerically convenient even when the exact solution is continuous, but it has abrupt changes, see Example 4. Of course, the method can handle regular problems, free of multiple scales and discontinuities, see Example 1.

(v) The power of the method lies in the fact that the FEM spaces do not embed strong coupling conditions between regions, on the contrary, they are fully uncoupled and the fluid exchange conditions only hold for the solution (either numerical or theoretical), but not for the test functions.

(vi) Example 4 is composed of two parts. The first part is the usual analysis displaying the performance of the method under controlled/lab conditions (Tables 7, 8; Figs. 10, 11). The second part suggests an iterative heuristic method to attain better numerical results in multiscale problems: start from reasonable (empirical if possible) values of the pressure on the interfaces, use the computed numerical pressure \( p^h \big|_f \) as input for a new iteration, and continue in this fashion, until the results attain a desired level of stability from one iteration to the next one. The primal–dual mixed scheme certainly allows to proceed this way, however, analyzing if such a method is convergent or under which conditions converges is topic for future work.

(vii) Finally, the implementation for the 3D porous media problem of the same method should not pose substantial theoretical challenges, but computational ones due to its complexity. The development of such implementation for general domains and grids in a public domain fashion is the topic of future work.

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