A CONJECTURE ON THE LENGTHS OF FILLING PAIRS

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Abstract. This work proves a conjecture in [1]. A pair \((\alpha, \beta)\) of simple closed geodesics on a closed and oriented hyperbolic surface \(M_g\) of genus \(g\) is called a filling pair if the complementary components of \(\alpha \cup \beta\) in \(M_g\) are simply connected. The length of a filling pair is defined to be the sum of their individual lengths. We show that the length of any filling pair on \(M\) is at least \(m_g^2\), where \(m_g\) is the perimeter of the regular right-angled hyperbolic \((8g-4)\)-gon.

1. Introduction

Let \(M_g\) be a closed and oriented (hyperbolic) surface of genus \(g \geq 2\).

Definition 1.1. A pair \((\alpha, \beta)\) of simple closed (geodesic) curves on \(M_g\) is called a filling pair of \(M_g\) if \(M_g \setminus (\alpha \cup \beta)\) is a disjoint union of topological disks. The curves \(\alpha, \beta\) are assumed to be in minimal position, i.e., the geometric intersection number \(i(\alpha, \beta)\) of \(\alpha\) and \(\beta\) is equal to \(|\alpha \cap \beta|\).

Definition 1.2. A filling pair intersects minimally when the complement of their union is a single disk, and such a filling pair is also called minimal.

For a minimally intersecting filling pair \((\alpha, \beta)\) of \(M_g\), the geometric intersection number is given by \(i(\alpha, \beta) = 2g - 1\) (see [3]).

The set of all closed and oriented surfaces of genus \(g\) up to isometry is called the moduli space of genus \(g\), and is denoted by \(\mathcal{M}_g\).

Definition 1.3. We define length of a filling pair \((\alpha, \beta)\) on a closed hyperbolic surface \(M \in \mathcal{M}_g\) by

\[
L_M(\alpha, \beta) := l_M(\alpha) + l_M(\beta),
\]

where \(l_M(\alpha)\) denotes the length of the geodesic representative in the free homotopy class \([\alpha]\) of \(\alpha\) on \(M\).

When we cut a hyperbolic surface \(M_g\) open along a minimal filling pair, we obtain a hyperbolic \((8g-4)\)-gon with area \(4\pi (g-1)\) (this area is equal to the area of the surface \(M_g\)). The length of the filling pair is equal to half the perimeter of this \((8g-4)\)-gon.

It is known that among all (hyperbolic) \(n\)-gons with fixed area, the regular \(n\)-gon has the least perimeter (for a proof, refer to Bezdek [2]). In particular, we see that the regular right-angled \((8g-4)\)-gon has the least perimeter among all \((8g-4)\)-gons with area \(4\pi (g-1)\). Thus, if \(m_g\) is the perimeter of a regular right-angled \((8g-4)\)-gon, and

\[
\mathcal{F}_g(M) := \min \{l_M(\alpha, \beta) \mid (\alpha, \beta) \text{ is a minimal filling pair of } M\},
\]

then

\[
\mathcal{F}_g(M) \geq \frac{m_g}{2} \text{ for all } M \in \mathcal{M}_g.
\]
This fact was proved rigorously in [1]. Moreover, it was shown in [1] that there are finitely many surfaces where the equality holds. Furthermore, in [1], Aougab and Huang define the \textit{filling pair systole} function

\[ Y_g : \mathcal{M}_g \to \mathbb{R}, \]

\[ Y_g(M) = \min \{ L_M(\alpha, \beta) \mid (\alpha, \beta) \text{ is a filling pair of } M \}, \]

In this paper, we show that (as conjectured by Aougab and Huang in [1])

\textbf{Theorem 1.1 (Main Theorem).} Let \( Y_g \) be the filling pair systole function as defined above. Then

\[ Y_g(M) \geq \frac{m_g}{2}, \]

where

\[ m_g = (8g - 4) \cdot \cosh^{-1} \left( 2 \left[ \cos \left( \frac{2\pi}{8g - 4} \right) + \frac{1}{2} \right] \right) \]

is the perimeter of the regular right-angled hyperbolic \((8g - 4)\)-gon.

Thus, the filling pair systole function has a global minimum over \( \mathcal{M}_g \).

It is a fact that \( Y_g \) is a topological Morse function; the argument in [1] for \( F_g \), which proves that \( F_g \) is a "generalized systole function" (see [6]), generalizes to \( Y_g \). It follows that there are at most finitely many \( M \in \mathcal{M}_g \) s.t. \( Y_g(M) = \frac{m_g}{2} \).

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\section{2. Partitions of polygons}

Let \((\alpha, \beta)\) be a filling pair of \( M_g \). The complement \( M_g \setminus (\alpha \cup \beta) \) is a disjoint union of topological disks, and we write

\[ M_g \setminus (\alpha \cup \beta) = \bigcup_{i=1}^{k} P_i, \]

where \( k \in \mathbb{N} \) and \( P_i \)'s are topological discs, \( i=1,2,\ldots, k \). Note that, if \( M_g \) is a hyperbolic surface \((g \geq 2)\) and \((\alpha, \beta)\) is a filling pair of geodesics, then \( P_i \)'s are hyperbolic polygons.

In another point of view, we can regard the union \( \Gamma(\alpha, \beta) = \alpha \cup \beta \) as a decorated fat graph on \( M_g \), where the intersection points are the vertices, the sub-arcs between vertices are the edges, and the fat graph structure is determined by the orientation of the surface (see [3] for notation). Note that \( \Gamma(\alpha, \beta) \) is a 4-regular graph on the surface \( M_g \). If the number of vertices and edges in \( \Gamma(\alpha, \beta) \) are \( v \) and \( e \) respectively, then we have \( e = 2v \) and \( v = i(\alpha, \beta) \), where \( i(\alpha, \beta) \) is the geometric intersection number of \( \alpha \) and \( \beta \). Furthermore, \( \Gamma(\alpha, \beta) \) has \( k \) boundary components (or equivalently faces) which is equal to the number of components in \( M_g \setminus (\alpha \cup \beta) \).
It is easy to see that $\Gamma(\alpha, \beta)$ is the 1–skeleton of a cellular decomposition of $M_g$. Therefore, by Euler’s formula, we have,

\[
\begin{align*}
  v - e + k &= 2 - 2g \\
  v &= 2g + k - 2 \\
  e &= 4g + 2k - 4
\end{align*}
\]

Each edge in $\Gamma(\alpha, \beta)$ contributes two sides in the set of polygons $P_i$, $i = 1, \ldots, k$. Note that, among every two consecutive edges of $P_i$’s, one comes from $\alpha$ and the other comes from $\beta$. Since $\alpha$ and $\beta$ are in minimal position, they do not form bi-gons on $M_g$, so the number of sides of each $P_i$ is even. Assume that the number of sides of $P_i$ is $2m_i$ for some $m_i \geq 2$, $i = 1, \ldots, k$. Therefore, by Euler’s formula, we have

\[
\sum_{i=1}^{k} m_i = 4g + 2k - 4. \tag{2.1}
\]

Now, if $P$ is a right-angled regular hyperbolic $(8g - 4)$-gon, then by Gauss-Bonnet formula (see Theorem 1.1.7 in [5]), we have area $(P) = area (M_g) = 4\pi (g - 1)$. Thus,

\[
\sum_{i=1}^{k} \text{area} (P_i) = \text{area} (P). \tag{2.2}
\]

Now, we prove some lemmas which are essential for the subsequent sections.

**Lemma 2.1.** Let $P$ be a regular hyperbolic $2n$-gon, $n \geq 2$, with each interior angle $\theta \geq \frac{\pi}{2}$. Consider two hyperbolic regular $2m_i$-gons $P_i$, $i = 1, 2$ and $m_i \geq 2$ satisfying

\[
\text{area} (P_1) + \text{area} (P_2) = \text{area} (P) \quad \text{and} \quad 2m_1 + 2m_2 = 2n + 4. \tag{2.3}
\]

If the interior angles of $P_i$’s are $\theta_i$, $i = 1, 2$, where $\theta_1 \leq \theta_2$, then $\theta_1 \leq \theta$ and $\theta_2 \geq \frac{\pi}{2}$.

**Proof.** From the given equation (2.3), we have $m_1 + m_2 = n + 2$. Now, using Gauss-Bonnet formula, we have

\[
\sum_{i=1}^{2} [(2m_i - 2)\pi - 2m_i\theta_i] = (2n - 2)\pi - 2n\theta
\]

\[
\Rightarrow (m_1 + m_2)\pi - 2\pi - (m_1\theta_1 + m_2\theta_2) = (n - 1)\pi - n\theta
\]

\[
\Rightarrow m_1\theta_1 + m_2\theta_2 = n\theta + \pi
\]

If possible, assume that $\theta < \theta_1$. Then we have $\theta < \theta_2$ as $\theta_1 \leq \theta_2$. Now, we have

\[
\pi + n\theta = m_1\theta_1 + m_2\theta_2
\]

\[
> (m_1 + m_2)\theta
\]

\[
= (n + 2)\theta
\]

\[
\Rightarrow \theta < \frac{\pi}{2},
\]

which contradicts the hypothesis. Thus, we conclude that $\theta_1 \leq \theta$. 

Now,
\[ \pi + n\theta = m_1\theta_1 + m_2\theta_2 \leq (n + 2)\theta_2 \]
\[ \implies (n + 2)\theta_2 \geq \pi + \frac{n\pi}{2} = \frac{\pi}{2}(n + 2) \]
\[ \implies \theta_2 \geq \frac{\pi}{2}. \]
\[ \square \]

In the next lemma (Lemma 2.2), we generalize Lemma 2.1.

2.1. **Setting.** Suppose \( P_i \)'s are the regular hyperbolic \( 2m_i \)-gons, \( i = 1, \ldots, k \), satisfying

\[ \sum_{i=1}^{k} \text{area}(P_i) = \text{area}(P) \]

(2.4)

\[ \sum_{i=1}^{k} m_i = n + 2(k - 1) \]

(2.5)

Suppose the interior angle of \( P_i \) is \( \theta_i \) for each \( i = 1, \ldots, k \). We define

\[ \theta_{\min} := \min\{ \theta_i \mid i = 1, \ldots, k \} , \]
\[ \theta_{\max} := \max\{ \theta_i \mid i = 1, \ldots, k \} . \]

**Lemma 2.2.** In the Setting 2.1, we have

1. \( \theta_{\min} \leq \theta \) and
2. \( \theta_{\max} \geq \frac{\pi}{2} \).

**Proof.** The proof of Lemma 2.2 is similar to the proof of Lemma 2.1. Using Gauss-Bonnet formula, we have

\[ \sum_{i=1}^{k} [(2m_i - 2)\pi - 2m_i\theta_i] = (2n - 2)\pi - 2n\theta \]

\[ \implies \pi \sum_{i=1}^{k} m_i - k\pi - \sum_{i=1}^{k} m_i\theta_i = (n - 1)\pi - n\theta \]

\[ \implies n\pi + (2k - 2)\pi - k\pi - \sum_{i=1}^{k} m_i\theta_i = n\pi - \pi - n\theta \]

\[ \implies k\pi - \pi - \sum_{i=1}^{k} m_i\theta_i = -n\theta \]

\[ \implies \sum_{i=1}^{k} m_i\theta_i = n\theta + k\pi - \pi \]

(1) Using the inequality

\[ \theta_{\min} \left( \sum_{i=1}^{k} m_i \right) \leq \sum_{i=1}^{k} m_i\theta_i , \]
we have
\[ \theta_{\min} (n + 2k - 2) \leq n\theta + (k - 1) \pi \]
\[ \leq n\theta + (2k - 2) \theta \]
\[ \implies \theta_{\min} \leq \theta. \]

(2) Similarly, using the inequality
\[ \theta_{\max} \left( \sum_{i=1}^{k} m_i \right) \geq \sum_{i=1}^{k} m_i \theta_i, \]
we have
\[ \theta_{\max} (n + 2k - 2) \geq n\theta + (2k - 2) \frac{\pi}{2} \]
\[ \geq (n + 2k - 2) \frac{\pi}{2} \]
\[ \implies \theta_{\max} \geq \frac{\pi}{2}. \]

Lemma 2.3. Let \( P \) be a regular hyperbolic \( 2n \)-gon with each interior angle \( \theta = \frac{\pi}{2} + \epsilon \), where \( \epsilon \geq 0 \). Furthermore, consider \( P_i \)’s are regular \( 2m_i \)-gons, where \( m_i \geq 2 \), \( i = 1, 2 \), satisfying
(1) \( \text{area}(P_1) + \text{area}(P_2) = \text{area}(P) \) and
(2) \( 2m_1 + 2m_2 = 2n + 4 \).
If the interior angles of \( P_i \)’s are \( \theta_i, i = 1, 2 \), then
\[ \theta_1 \leq \theta \implies \theta_2 \geq \frac{\pi}{2}. \]

Proof. Using the equation \( 2m_1 + 2m_2 = 2n + 4 \), we have
\[ \text{area}(P_1) + \text{area}(P_2) = 2n\pi - 2m_1\theta_1 - (2n + 4 - 2m_1)\theta_2. \]
Now, the equation \( \text{area}(P_1) + \text{area}(P_2) = \text{area}(P) \) gives
\[ 2n\epsilon = 2m_1\theta_1 + (2n + 4 - 2m_1)\theta_2 - (n + 2)\pi. \]
Using the hypothesis \( \theta_1 \leq \frac{\pi}{2} + \epsilon \), we have
\[ 2n\epsilon \leq 2m_1 \left( \frac{\pi}{2} + \epsilon \right) + (2n + 4 - 2m_1)\theta_2 - (n + 2)\pi. \]
Now, if possible, we assume that \( \theta_2 < \frac{\pi}{2} \). Then we have
\[ 2n\epsilon < m_1\pi + 2m_1\epsilon + (2n + 4 - 2m_1) \frac{\pi}{2} - (n + 2)\pi \]
\[ = m_1\pi + 2m_1\epsilon + n\pi + 2\pi - m_1\pi - n\pi - 2\pi \]
\[ \implies 2n\pi < 2m_1\epsilon \]
\[ \implies m_1 > n. \]
The inequality \( m_1 > n \) implies \( m_2 \leq 1 \), which is a contradiction. □

We prove the theorem stated below (Theorem 2.4) that will be the key step in proving our main result.
**Theorem 2.4.** Let $P$ be a regular $2n$-gon with interior angle $\theta \left( \geq \frac{\pi}{2} \right)$ and $n \geq 2$. If $P_i$’s are regular $2m_i$-gons, $i = 1, 2$, $m_i \geq 2$, satisfying $m_1 + m_2 = n + 2$ and $\text{area}(P_1) + \text{area}(P_2) = \text{area}(P)$, then

$$\text{Perim}(P) \leq \text{Perim}(P_1) + \text{Perim}(P_2),$$

where $\text{Perim}(P)$ denotes the perimeter of the polygon $P$.

3. Proof of Main Theorem

In this section, assume that Theorem 2.4 is true and prove the main theorem (see Theorem 3.3). Note that, the proof of Theorem 2.4 is computational, and is addressed in Section 5.

Suppose $P$ is a regular right angled hyperbolic $2n$-gon, where $n = 4g - 2$, and $P_i$’s are regular $2m_i$-gons, $m_i \geq 2$, for $i = 1, 2, \ldots, k$, satisfying equations (2.4) and (2.5).

Now, we have the proposition below.

**Proposition 3.1.** $\text{Perim}(P) \leq k \sum_{i=1}^{k} \text{Perim}(P_i)$

**Proof.** Let $\theta_i$’s be the interior angles of $P_i$’s, $i = 1, \ldots, k$. After re-indexing, if needed, we may assume that

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_k.$$

We define regular hyperbolic $2\tilde{m}_j$-gons $\tilde{P}_j$, $j = 1, \ldots, k$, inductively as described below.

1. For $j = 1$, $\tilde{P}_1 := P_1$; $\tilde{m}_1 = m_1$, and area $\left( \tilde{P}_1 \right) = \text{area}(P_1)$.
2. In general, for $j \geq 2$, $2\tilde{m}_j = 2\tilde{m}_{j-1} + 2m_j - 4$ and area $\left( \tilde{P}_j \right) = \text{area}(\tilde{P}_{j-1}) + \text{area}(P_j)$.

Note that, these conditions determine $\tilde{P}_j$ uniquely. Now, we prove the lemma below which will be used to complete the proof of Proposition 3.1.

**Lemma 3.2.** The interior angle $\tilde{\theta}_j$ of $\tilde{P}_j$ satisfies $\tilde{\theta}_j \geq \frac{\pi}{2}$ for each $1 \leq j \leq k$.

**Proof of Lemma 3.2.** The proof is by induction on $j$.

For the base case $j = k$, it is straightforward to see that $\tilde{P}_k = P$, as area $\left( \tilde{P}_k \right) = \text{area}(P)$ and $2\tilde{m}_k = \left( \sum_{i=1}^{k} 2m_i \right) - 4(k - 1) = 2n$, so $\tilde{P}_k \cong P$. Therefore, by hypothesis, we have

$$\tilde{\theta}_k = \theta \geq \frac{\pi}{2}.$$

Inductively, assume that $\tilde{\theta}_{k_0} \geq \frac{\pi}{2}$ for some $k_0 \leq k$.

We want to show that $\tilde{\theta}_{k_0} \geq \frac{\pi}{2}$. First, note that the polygons $P_1, \ldots, P_{k_0}$ and $P = \tilde{P}_{k_0}$ satisfy the conditions of Lemma 2.2.

1. Interior angle $\tilde{\theta}_{k_0}$ of $P = \tilde{P}_{k_0}$ satisfies $\tilde{\theta}_{k_0} \geq \frac{\pi}{2}$.
2. By definition, $\sum_{i=1}^{k_0} \text{area}(P_i) = \text{area}(P_{k_0})$. 
(3) As $2\tilde{m}_j = 2\tilde{m}_{j-1} + 2m_j - 4$ for $j = 2, \ldots, k_0$, adding all these equations, we have

$$2\tilde{m}_{k_0} = \sum_{i=1}^{k_0} 2m_i - 4(k_0 - 1)$$

which implies

$$\sum_{i=1}^{k_0} m_i = \tilde{m}_{k_0} + 2(k_0 - 1).$$

Now, the definition $\theta_{k_0} = \min \{ \theta_i \mid i = 1, \ldots, k_0 \}$ and Lemma 2.2 imply $\theta_{k_0} \leq \tilde{\theta}_{k_0}$. Finally, the polygons $\tilde{P}_{k_0}, P_{k_0}, \tilde{P}_{k_0-1}$ satisfy the following:

1. Interior angle $\tilde{\theta}_{k_0}$ of $\tilde{P}_{k_0}$ satisfy $\tilde{\theta}_{k_0} \geq \frac{\pi}{2}$.
2. Area $\left(\tilde{P}_{k_0-1}\right) + \text{Area}(P_{k_0}) = \text{Area}(\tilde{P}_{k_0})$.
3. $2\tilde{m}_{k_0-1} + 2m_{k_0} = 2\tilde{m}_{k_0} + 4$.
4. $\theta_{k_0} \leq \tilde{\theta}_{k_0}$.

Thus, by Lemma 2.3, we conclude that $\tilde{\theta}_{k_0-1} \geq \frac{\pi}{2}$.

Now we complete the proof of Proposition 3.1. By Lemma 3.2, note that for $j \geq 2$, the polygons $\tilde{P}_j, P_j, \tilde{P}_{j-1}$ satisfy:

1. Interior angle $\tilde{\theta}_j$ of $\tilde{P}_j$ satisfy $\tilde{\theta}_j \geq \frac{\pi}{2}$.
2. Area $\left(\tilde{P}_{j-1}\right) + \text{Area}(P_j) = \text{Area}(\tilde{P}_j)$.
3. $2\tilde{m}_{j-1} + 2m_j = 2\tilde{m}_j + 4$.

Thus, by Theorem 2.4, we conclude that

$$\text{Perim}(\tilde{P}_j) \leq \text{Perim}(P_j) + \text{Perim}(\tilde{P}_{j-1}).$$

for each $j \geq 2$. Summing up the inequalities, we get that

$$\text{Perim}(P) = \text{Perim}(\tilde{P}_k) \leq \sum_{i=1}^{k} \text{Perim}(P_i).$$

Theorem 3.3 (Main Theorem). Let $M = M_g$ be a closed hyperbolic surface of genus $g$ and $(\alpha, \beta)$ be a filling pair of simple closed geodesics. Then

$$L_M(\alpha, \beta) = l_M(\alpha) + l_M(\beta) \geq \frac{m_g}{2},$$

where $m_g$ is the perimeter of a regular right angled hyperbolic $(8g - 4)$-gon.

Proof. Let

$$M \setminus (\alpha \cup \beta) = \bigcup_{i=1}^{k} \tilde{P}_i,$$

where $\tilde{P}_i$’s are hyperbolic $2m_i$-gons for some $m_i \geq 2$, $i = 1, \ldots, k$. We denote $P_i$ to be a regular hyperbolic $2m_i$-gon whose area is equal to area $\left(\tilde{P}_i\right)$. It is a fact that
the regular polygon has the least perimeter among all \( n \)-gons of a fixed area. Thus,

\[
\text{Perim} \left( P_i \right) \leq \text{Perim} \left( \hat{P}_i \right)
\]

\[
\implies \sum_{i=1}^{k} \text{Perim} \left( P_i \right) \leq \sum_{i=1}^{k} \text{Perim} \left( \hat{P}_i \right) = 2L_M \left( \alpha, \beta \right)
\]

Now, Proposition 3.1 implies that

\[
m_g = \text{Perim} \left( P \right) \leq \sum_{i=1}^{k} \text{Perim} \left( P_i \right) \leq 2L_M \left( \alpha, \beta \right)
\]

\[
\implies L_M \left( \alpha, \beta \right) = l_M \left( \alpha \right) + l_M \left( \beta \right) \geq \frac{m_g}{2}.
\]

In what follows, we primarily aim at proving Theorem 2.4.

4. Base Cases \( n = 2, 3 \)

The following is a well-known result in Euclidean and hyperbolic geometry -

**Proposition 4.1** (Isoperimetric inequality). Among all (Euclidean/hyperbolic) \( n \)-gons with fixed area \( A > 0 \), the one with the least perimeter is the regular \( n \)-gon. Similarly, among all \( n \)-gons with fixed perimeter \( P > 0 \), the one with the greatest area is the regular \( n \)-gon.

We can generalise this result as follows (for more details and a proof, see [7]) –

**Theorem 4.2** (Isoperimetric inequality for disconnected regions). Let \( P, P_1, P_2 \) be regular hyperbolic \( n \)-gons, for \( n \geq 3 \), with areas \( A, A_1, A_2 \), and interior angles \( \theta, \theta_1, \theta_2 \). Suppose that

1. \( \theta \geq \cos^{-1} \left( -1 + 2 \sin \left( \frac{\pi}{n} \right) \right) \)
2. \( A_1 + A_2 = A \)

Then

\[
\text{Perim} \left( P_1 \right) + \text{Perim} \left( P_2 \right) \geq \text{Perim} \left( P \right)
\]

Using Theorem 4.1 we get –

**Corollary 4.3.** Theorem 2.4 is true for the cases \( n = 2, 3 \).

**Proof.** (Notation as in the statement of Theorem 4.2) When \( n = 2 \), \( m_1 = m_2 = 2 \), so this case is a direct consequence of Theorem 4.2, as \( \frac{\pi}{2} > \cos^{-1} \left( -1 + 2 \sin \left( \frac{\pi}{n} \right) \right) \).

When \( n = 3 \), WLOG \( m_1 \leq m_2 \). Then \( m_1 = 2, m_2 = 3 \). Let \( P_1^* \) be the hyperbolic regular 6-gon with area equal to area \( (P_1) \). Then \( \text{Perim} \left( P_1 \right) \geq \text{Perim} \left( P_1^* \right) \) by the isoperimetric inequality (see [2]). Also, since \( \frac{\pi}{2} = \cos^{-1} \left( -1 + 2 \sin \left( \frac{\pi}{3} \right) \right) \), Theorem 4.2 implies that \( \text{Perim} \left( P \right) \leq \text{Perim} \left( P_1^* \right) + \text{Perim} \left( P_2 \right) \). Thus, \( \text{Perim} \left( P \right) \leq \text{Perim} \left( P_1 \right) + \text{Perim} \left( P_2 \right) \).

Thus, to prove Theorem 2.4, we assume hereon that \( n \geq 4 \). With this established, we proceed to the full proof of Theorem 2.4.
5. PROOF OF THEOREM 2.4

Notation for this section - $P(n, A, p, \alpha)$ is the hyperbolic regular $n$–gon with number of sides $n$, area $A$, perimeter $p$, angle $\alpha$. We may simply write $P(n, \alpha)$ for the hyperbolic regular polygon with $n$ sides and angle $\alpha$ (the area and perimeter are determined by these attributes).

To prove - For a given $P(2m, A, p, \alpha)$, suppose we have $P(2m_1, A_1, p_1, \alpha_1)$ and $P(2m_2, A_2, p_2, \alpha_2)$ s.t. $A_1 + A_2 = A$; and assume that $m_1, m_2 \geq 2$, $m \geq 4$, $m_1 + m_2 = m + 2$, and $\alpha > \frac{\pi}{2}$. Then $p_1 + p_2 \geq p$.

Before we proceed, observe that, for $P(2n, A, p, \alpha)$,

1. As shown in the proof of Theorem 4.2,

$$\text{Perim}(P(2n, \alpha)) = p = 4n \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2n} \right)}{\sin \left( \frac{\alpha}{2} \right)} \right)$$

Thus,

$$p_1 + p_2 - p = 4m_1 \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m_1} \right)}{\sin \left( \frac{\alpha_1}{2} \right)} \right) + 4m_2 \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m_2} \right)}{\sin \left( \frac{\alpha_2}{2} \right)} \right) - 4m \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m} \right)}{\sin \left( \frac{\alpha}{2} \right)} \right)$$

We wish to prove that

$$m_1 \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m_1} \right)}{\sin \left( \frac{\alpha_1}{2} \right)} \right) + m_2 \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m_2} \right)}{\sin \left( \frac{\alpha_2}{2} \right)} \right) - m \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m} \right)}{\sin \left( \frac{\alpha}{2} \right)} \right) \geq 0$$

(2) By Gauss-Bonnet theorem,

$$\text{area}(P(2n, \alpha)) = A = (2n - 2) \pi - 2n\alpha$$

Also, the conditions $A_1 + A_2 = A$ and $m_1 + m_2 = m + 2$ give us that

$$A_1 + A_2 - A = (2m_1 + 2m_2 - 2m - 2) \pi - (2m_1\alpha_1 + 2m_2\alpha_2 - 2m\alpha)$$

$$= (2) \pi - (2m_1\alpha_1 + 2m_2\alpha_2 - 2m\alpha)$$

$$= 0 \implies m_1\alpha_1 + m_2\alpha_2 = m\alpha + \pi$$

5.0.1. The case $m_1 = 2$. We want to show that

$$f_2(m, \alpha_1) = 2 \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2} \right)}{\sin \left( \frac{\alpha_1}{2} \right)} \right) + m \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m} \right)}{\sin \left( \frac{\alpha_1}{m} + \frac{\pi}{2m} \right)} \right) - m \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m} \right)}{\sin \left( \frac{\alpha}{2} \right)} \right) \geq 0$$

\footnote{We have proved the result for $m = 2, 3$}
Now, the derivative
\[
\frac{\partial}{\partial \alpha_1} f_2 (m, \alpha_1) = - \frac{\cos \left( \frac{\alpha_1}{2} \right) \cos \left( \frac{\pi}{2m} \right)}{\sin \left( \frac{\alpha_1}{2} \right) \sqrt{\cos^2 \left( \frac{\pi}{2m} \right) - \sin^2 \left( \frac{\alpha_1}{2} \right)}} - \frac{\cos \left( \frac{\alpha_1}{2} \right) \cos \left( \frac{\pi}{4} \right)}{\sin \left( \frac{\alpha_1}{2} \right) \sqrt{\cos^2 \left( \frac{\pi}{4} \right) - \sin^2 \left( \frac{\alpha_1}{2} \right)}}.
\]

Fix \( m \). Claim -

1. If \( \frac{m\alpha - (m-2)\pi}{2} > 0 \), i.e. if \( \alpha \geq \frac{m-2}{m} \pi \), then \( \frac{\partial}{\partial \alpha_1} f_2 \) has a unique root (a local maximum) in \((0, \frac{\pi}{2})\). (In this case, \( \alpha_1 \in \left( \frac{m\alpha - (m-2)\pi}{2}, \frac{\pi}{2} \right) \) as \( \alpha_1 \to \frac{m\alpha - (m-2)\pi}{2} \implies A_2 \to 0 \))
2. If \( \alpha \leq \frac{m-2}{m} \pi \), then \( \frac{\partial}{\partial \alpha_1} f_2 < 0 \) everywhere. (This means that \( f_2 \) is monotonically decreasing, and attains minimum value at \( \alpha_1 = \frac{\pi}{2} \))

Assuming the claim, in either case if we show that \( f_2 \geq 0 \) at the end points, then \( f_2 \geq 0 \) throughout. \( f_2 \left( \frac{\pi}{2} \right) = 0 \), and
\[
f_2 \left( m, \frac{m\alpha - (m-2)\pi}{2} \right) = 2 \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{4} \right)}{\cos \left( \frac{m\alpha - (m-2)\pi}{2} \right)} \right) - m \cdot \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m} \right)}{\cos \left( \frac{\pi}{2} \right)} \right)
\]
is zero for \( m = 2 \), but greater than 0 as the first term grows faster than the second term.

To prove the claim, we first make some observations -

- Note that \( \alpha_1 \to \frac{m\alpha - (m-2)\pi}{2} = \pi - \frac{m(\pi - \alpha)}{2} \implies \alpha_2 \to \frac{m-1}{m} \pi \implies A_2 \to 0 \); and \( \frac{\partial}{\partial \alpha_1} f_2 \to +\infty \). Also, as \( \alpha_1 \to \frac{\pi}{2} \), \( \frac{\partial}{\partial \alpha_1} f_2 \to -\infty \). Thus, \( \frac{\partial}{\partial \alpha_1} f_2 \) has a root;
  which is a local maximum for \( f_2 \). Now, as \( \alpha_1 \) varies over \( \left( \pi - \frac{m(\pi - \alpha)}{2}, \frac{\pi}{2} \right) \), \( A_1 \) decreases from \( A \) to 0, \( A_2 \) increases from 0 to \( A \), and \( \alpha_2 \) decreases from \( \frac{m-1}{m} \pi \) to \( \alpha \).
- If \( \alpha < \frac{m-2}{m} \pi \), then \( \alpha_1 \) varies over all of \((0, \frac{\pi}{2})\). We see that \( \alpha_1 \) bounded away from \( \frac{m\alpha - (m-2)\pi}{2} \implies A_2 \) is bounded away from 0, and \( \alpha_2 \) is bounded away from \( \frac{m-1}{m} \pi \). Thus, \( \lim_{\alpha_1 \to 0^+} \frac{\partial}{\partial \alpha_1} f_2 = -\infty \).

Now, let us look at the function \( g \left( k, \beta \right) = \frac{\cos(\beta) \cos \left( \frac{\pi}{4} \right)}{\sin(\beta) \sqrt{\cos^2 \left( \frac{\pi}{4} \right) - \sin^2(\beta)}} \). Indeed, \( f_2 \left( m, \alpha \right) = g \left( \frac{m}{2} + \frac{\pi}{2m} - \frac{\alpha}{m} \right) - g \left( 2, \frac{\alpha}{2} \right) \). Fix \( k \), and for different values of \( k \) let us observe the graphs of \( g \)-
Fix the graph of $g\left(2, \frac{a_1}{2}\right)$; we understand how $g\left(m, \frac{a_1}{2}\right)$ behaves. Under the action $\frac{a_1}{2} \mapsto \frac{a_1}{m} \mapsto -\frac{a_1}{m} \mapsto \left(\frac{\pi}{2m} + \frac{a}{2}\right) - \frac{a_1}{m}$ (remember - we are considering $m \geq 4$ fixed).
Figure 2. Transformation of \( g(m, \alpha) \) by \( \alpha_1 \mapsto \frac{\pi}{2m} + \alpha, \alpha \mapsto -\frac{\alpha}{m} \mapsto \left( \frac{\pi}{2m} + \frac{\alpha}{2} \right) - \frac{\alpha}{m} \)

Observe that increasing \( \alpha \) from \( \frac{\pi}{2} \) to \( \frac{m-1}{m} \pi \) is basically shifting the graph of \( g(m, \alpha_2 + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \) to the right. So, we make the following observations -

(1) Suppose the graphs of \( g(m, \alpha_2 + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \) and \( g(2, \frac{\alpha_1}{2}) \) intersect at a point \( \alpha_1 = \alpha_1^* \). First, we note that the value of \( \alpha_1^* \) is less than the local minimum of \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \). To see this, note that for \( m \geq 4, \alpha = \frac{\pi}{2} \), if \( \alpha_1^0 \) is the local minimum for \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \), then \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) < g(2, \frac{\alpha_1}{2}) \) for every \( \alpha_1 > \alpha_1^0 \) (the second curve lies strictly below the first); and as \( \alpha \) increases, so does the difference (again, we are simply moving the graph of \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \) to the right, so relatively decreasing the value of \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \) where \( \alpha_1 > \alpha_1^0 \)). Hence, if the curves do meet, it must be at some \( \alpha_1 < \alpha_1^* \).

Then, by the nature of graphs of \( g(k, \beta) \), for \( \alpha_1 < \alpha_1^* \), \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \) grows much faster than \( g(2, \frac{\alpha_1}{2}) \). Thus, as \( \alpha_1 \) ranges over \( (0, \alpha_1^*) \), \( \frac{\partial}{\partial \alpha_1} f_2 \) is decreasing monotonically till it hits 0. For \( \alpha_1 > \alpha_1^* \), \( g(2, \frac{\alpha_1}{2}) \) will always be greater than \( g(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}) \) - the former grows faster than and grows to \( \infty \) as \( \alpha_1 \to \frac{\pi}{2} \), while the latter is bounded in \( (0, \frac{\pi}{2}) \). That the two curves
don’t intersect again is true for \( m \geq 4 \), can be checked graphically for \( m = 4 \) and will be true for higher values of \( m \) by the nature of curves \( g(k, \beta) \). This corresponds to the first part of the claim.

![Figure 3](image)

**Figure 3.** \( g\left(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}\right) \) for \( m \geq 4 \), \( \alpha = \frac{\pi}{2} \)

(2) Suppose \( \alpha > \frac{\pi}{2} \) is small enough so that the graphs of \( g\left(2, \frac{\alpha}{2}\right) \) and \( g\left(m, \frac{\alpha}{2} + \frac{\pi}{2m} - \frac{\alpha_1}{m}\right) \) don’t intersect. Since both are curve-above-chord, it follows that \( \frac{\partial}{\partial \alpha_1} f_2 < 0 \) everywhere, which corresponds to the second part of the claim.

This completes our argument for the case \( m_1 = 2 \).

5.0.2. Variation with \( \alpha \).

**Lemma 5.1.** Suppose the result is true for \( P(2m, \alpha) \). Pick \( \beta > \alpha \); then the result is true for \( P(2m, \beta) \).

**Proof.** First, note that the area \( A \) of \( P(2m, \alpha) \) is \( (2m - 2)\pi - 2m\alpha \). So,

\[
\alpha = \frac{(m - 1)\pi - A/2}{m}
\]

We have similar expressions for \( \alpha_1 \) and \( \alpha_2 \). Now, note that if \( \beta > \alpha \), then area \( P(2m, \beta) \) < area \( P(2m, \alpha) \). Rewriting \( P(2m, \alpha) \equiv P(2m, A, p, \alpha) \) as \( P(2m, A) \), the lemma equivalently states that if the proposition is true for \( P(2m, A) \), then it is true for \( P(2m, A - \delta) \) for \( \delta > 0 \).

Suppose \( P(2m, A - \delta, q) \) is partitioned as \( P(2m_1, B_1, q_1) \) and \( P(2m_2, B_2, q_2) \) (where \( B_i \) are the respective areas, and \( q, q_i \) are respective perimeters). Then \( P(2m_1, B_1 + \delta = A_1, q_1^* \)
and $P(2m_2, B_2)$ form a valid partition of $P$. By hypothesis, we know that
\[ q_1^* + q_2 \geq p \]
We want to show that
\[ q_1 + q_2 \geq q \]
With this in mind, let us look at the function (as a function of area $B$)
\[ f_m(B) = \text{perim}(P(2m, B)) - \text{perim}(P(2m, B - \delta)) \]
where we recall that $\text{perim}(P(2m, B)) = 2m \cdot \cosh^{-1}\left(\frac{\cos(\pi/2m)}{\sin\left(\frac{\alpha}{2}\right)}\right)$ ($\delta > 0$ is fixed throughout\footnote{We assume $\delta$ to be small enough; s.t. $B - \delta < (2m - 2)\pi$. This $\delta$ will depend on $B, m$ only. Thus, for any $C < B$, we first obtain $\delta = \delta_{B,m}$, and can keep subtracting $\delta$ finitely many times from $B$ to reach $C$; so showing the result for $B - \delta$ for small $\delta$ is sufficient.}). Note that $q = p - f_m(A)$, and $q_1^* = f_m(A_1)$. Thus, we are interested in calculating the minimum value attained (if such a minimum exists) by $f_m(A) - f_m(A_1)$ as $m_1$ varies, keeping $A, A_1$ fixed. Then, if we prove that $q_1 + q_2 \geq q$ for this case, the result will follow.

Now, fix $A$, and consider $f_n(A)$ as $n$ varies; we see that $f_n(A)$ decreases as $n$ increases. To see this, note that as $m$ increases, the difference term $\delta$ decreases in value; from $\frac{\delta}{4m}$ to $\frac{\delta}{4m+1}$ in the sin term, and that the derivative
\[ \frac{\partial}{\partial B} \text{perim}(P(2m, B)) = \frac{\cos\left(\frac{\pi}{2m}\right) \cot\left(\frac{\pi}{2m} - \frac{B}{4m}\right)}{\sqrt{\cos^2\left(\frac{\pi}{2m}\right) - \sin^2\left(\frac{\pi}{2m} - \frac{B}{4m}\right)}} \]
decreases as $n$ increases.

In particular, we see that $f_n(A)$ is maximum at $n = 2$, for $A$ fixed. Thus, the quantity $f_m(A) - f_m(A_1)$ attains minimum at $m_1 = 2$. Hence, if we prove that $q_1 + q_2 \geq q$ for $m_1 = 2$, we would have proved the result for any $m_1$. However, we have already proved the general result for the case $m_1 = 2$. □

As a result, we can restrict our attention to the case $\alpha = \frac{\pi}{2}$.

5.0.3. Variation with $m$. Consider
\[ f(m, m_1, \alpha_1) = m_1 \cosh^{-1}\left(\frac{\cos(\pi/2m_1)}{\sin\left(\frac{\alpha_1}{2}\right)}\right) + m_2 \cosh^{-1}\left(\frac{\cos(\pi/2m_2)}{\sin\left(\frac{\alpha_2}{2}\right)}\right) - m \cdot \cosh^{-1}\left(\frac{\cos(\pi/2m)}{\sin\left(\frac{\pi}{4}\right)}\right) \]
\[ = m_1 \cosh^{-1}\left(\frac{\cos(\pi/2m_1)}{\sin\left(\frac{\alpha_1}{2}\right)}\right) + (m - m_1 + 2) \cosh^{-1}\left(\frac{\cos(\frac{\pi}{2(m-m_1+2)})}{\sin\left(\frac{m_1 \pi}{2} + \pi - m_1 \alpha_1}{2(m-m_1+2)}\right)}\right) \]
\[ - m \cdot \cosh^{-1}\left(\frac{\cos(\pi/2m)}{\sin\left(\frac{\pi}{4}\right)}\right) \]

i.e. $\frac{1}{4}(p_1 + p_2 - p)$. The goal is to show that $f \geq 0$. Fix $m_1, \alpha_1$; and observe how $f$ varies with $m$.

**Lemma 5.2.** As $m$ increases in $(m_1, \infty)$, $f(m, m_1, \alpha_1)$ increases at first till $m = m^*$, and then monotonically decreases.
A conjecture on the lengths of filling pairs

Assuming this for now, we see that we only need to check two cases - \( f(m_1, m_1, \alpha_1) \geq 0 \), and \( \lim_{m \to \infty} f(m, m_1, \alpha_1) \geq 0 \). The first case is already done, as \( m = m_1 \implies m_2 = 2 \), and this case has been covered at the start (note that everything is symmetric in \( m_1 \) and \( m_2 \)).

For the second case, we see that

\[
\lim_{m \to \infty} f(m, m_1, \alpha_1) = m_1 \cosh^{-1} \left( \frac{\cos(\pi/2m_1)}{\sin \left( \frac{\alpha_1}{2} \right)} \right) - (m_1 - 2) \cosh^{-1} \left( \sqrt{2} \right) - \frac{m_1 \sqrt{2}}{4} (\pi - 2\alpha_1)
\]

\[= h(m_1, \alpha_1) \]

If we prove that \( h(m_1, \alpha_1) \geq 0 \), we are done.

Now, fix \( \alpha_1 \). Then \( h(m_1, \alpha_1) \) increases monotonically with \( m_1 \) (this is easy to see, as \( m_1 \cdot \cosh^{-1} \left( \frac{\cos(\pi/2m_1)}{\sin \left( \frac{\alpha_1}{2} \right)} \right) \) grows much faster than \( m_1 \cdot \text{const} \)). So, what remains is to show that \( h(m_1, \alpha_1) \geq 0 \) for \( m_1 = 2 \); and this particular case is easily verified by looking at the graph of \( g \)

![Figure 4. h(2, \alpha_1)](image)

Finally, we complete the proof of Lemma 5.2.

**Proof of Lemma 5.2.** If \( m_2, \alpha_2 \) were independent of \( m \), then \( f \) would be monotonically decreasing, as \( m \cdot \cosh^{-1} \left( \frac{\cos(\pi/2m)}{\sin \left( \frac{\alpha}{2} \right)} \right) \) is monotonically increasing with \( m \). However, we have the relation here that \( m_2 = m - C \) for a constant \( C \). Focusing on the terms (for constants \( C, D \))

\[
(m - C) \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2(m-C)} \right)}{\sin \left( \frac{\pi}{4} \cdot \frac{m}{m-C} + \frac{D}{2(m-C)} \right)} \right) - m \cdot \cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{2m} \right)}{\sin \left( \frac{\pi}{4} \right)} \right)
\]

we see that for \( m \) large, this difference reduces as constants \( C, D \) become less significant. Noting that the term \( m_1 \cosh^{-1} \left( \frac{\cos(\pi/2m_1)}{\sin \left( \frac{\alpha_1}{2} \right)} \right) \) remains constant, we see that for
large \( m, f \) is decreasing as \( m \) increases. The quantity \( m-C \in (2, \infty) \) as \( m \in (m_1, \infty) \); and \( \frac{\pi}{4} < \pi \cdot \frac{m}{m-C} + \frac{D}{2(m-C)} \); so \( f \) increases at the start, as for smaller values of \( m \), \((m-C)^{-1}\left(\frac{\cos\left(\frac{\pi}{2m-C}\right)}{\sin\left(\frac{\pi}{2m-C} + \frac{D}{2(m-C)}\right)}\right)\) grows faster than \( m \cdot \cosh^{-1}\left(\frac{\cos\left(\frac{\pi}{2m-C}\right)}{\sin\left(\frac{\pi}{2m-C} + \frac{D}{2(m-C)}\right)}\right) \). Thus, \( f \) increases with \( m \) at the start, till some \( m = m^* \), during when the difference is positive, after which \( f \) decreases monotonically.

To see this more clearly, look at the individual parts \((m-C)^{-1}\left(\frac{\cos\left(\frac{\pi}{2m-C}\right)}{\sin\left(\frac{\pi}{2m-C} + \frac{D}{2(m-C)}\right)}\right)\) and \( m \cdot \cosh^{-1}\left(\frac{\cos\left(\frac{\pi}{2m-C}\right)}{\sin\left(\frac{\pi}{2m-C} + \frac{D}{2(m-C)}\right)}\right) \). The second term is independent of \( \alpha_1 \) and increases monotonically with \( m \). The first term varies with \( \alpha_1 \) as follows -

![Figure 5. Graph of the first terms for \( m_1 = 6 \)](image)

As we see, when \( m \) increases from \( m_1 \) to \( \infty \), the relative position of \( \alpha_1 \) (fixed) along the graph of \((m-C)^{-1}\left(\frac{\cos\left(\frac{\pi}{2m-C}\right)}{\sin\left(\frac{\pi}{2m-C} + \frac{D}{2(m-C)}\right)}\right)\) is increasing. In particular, one can notice that for a fixed \( \alpha_1 \), the point may lie to he left of the inflection point of the graph at the start but as \( m \) increases, it moves to the right of the critical point; the derivative of \((m-C)^{-1}\left(\frac{\cos\left(\frac{\pi}{2m-C}\right)}{\sin\left(\frac{\pi}{2m-C} + \frac{D}{2(m-C)}\right)}\right)\) decreases at first with \( m \) till the point \( \alpha_1 \) (relatively) moves past the inflection point of the graph, after which the derivative increases. The value of \( m \) for which \( \alpha_1 \) crosses the inflection point is exactly \( m^* \); we can see from the graphs that as the derivative increases, the distance between the two \( f \) terms increases, and hence the (negative) difference decreases. \( \square \)
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