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On the first order asymptotics of partial Bergman kernels (*)

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Abstract. — We show that under very general assumptions the partial Bergman kernel function of sections vanishing along an analytic hypersurface has exponential decay in a neighborhood of the vanishing locus. Considering an ample line bundle, we obtain a uniform estimate of the Bergman kernel function associated to a singular metric along the hypersurface. Finally, we study the asymptotics of the partial Bergman kernel function on a given compact set and near the vanishing locus.

Résumé. — Nous montrons, sous des hypothèses très générales, que le noyau de Bergman partiel des sections s’annulant sur une hypersurface analytique décroît exponentiellement dans un voisinage du lieu d’annulation. Pour un fibré ample, nous montrons une estimée uniforme du noyau de Bergman associé à une métrique singulière le long d’une hypersurface. Finalement nous étudions les asymptotiques du noyau de Bergman sur un compact près du lieu d’annulation.

1. Introduction

Partial Bergman kernels were recently studied in different contexts, especially Kähler geometry [11, 12, 13] or random polynomials [2, 15].

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Let us consider the following general setting.

(A) \((X, \omega)\) is a compact Hermitian manifold of dimension \(n\), \(\Sigma\) is a smooth analytic hypersurface of \(X\), and \(t > 0\) is a fixed real number.

(B) \((L, h)\) is a singular Hermitian holomorphic line bundle on \(X\) with singular metric \(h\) which has locally bounded weights.

We define the space

\[
H_0^0(X, L^p) := H^0(X, L^p \otimes \mathcal{O}(-\lfloor tp \rfloor \Sigma))
\]

(1.1)

of holomorphic sections of the \(p\)-th tensor power \(L^p\) vanishing to order at least \(\lfloor tp \rfloor\) along \(\Sigma\), where \(\lfloor x \rfloor\) denotes the integral part of \(x \in \mathbb{R}\). Set \(d_p = \dim H^0(X, L^p)\) and \(d_{0,p} = \dim H_0^0(X, L^p)\). We introduce on \(H^0(X, L^p)\) the \(L^2\) inner product \((\cdot, \cdot)_p\) induced by the metric \(h_p = h^\otimes p\) and the volume form \(\omega^n/n!\), see (2.1). This inner product is inherited by \(H_0^0(X, L^p)\). The (full) Bergman kernel function is defined by taking an orthonormal basis \(\{S^p_j : 1 \leq j \leq d_p\}\) of \((H^0(X, L^p), (\cdot, \cdot)_p)\) and setting

\[
P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2 h_p, \quad |S^p_j(x)|^2 h_p := \langle S^p_j(x), S^p_j(x) \rangle_{h_p}, \quad x \in X .
\]

(1.2)

By considering an orthonormal basis \(\{S^p_j : 1 \leq j \leq d_{0,p}\}\) of \((H_0^0(X, L^p), (\cdot, \cdot)_p)\), we define the partial Bergman kernel function \(P_{0,p}\) by

\[
P_{0,p}(x) = \sum_{j=1}^{d_{0,p}} |S^p_j(x)|^2 h_p, \quad x \in X .
\]

(1.3)

Note that this definition is independent of the choice of basis, cf. (2.2).

The asymptotics of the Bergman kernel function for a positive line bundle \((L, h)\) \([4, 16]\), see also \([10]\) for a comprehensive study, is very important in understanding the Yau–Tian–Donaldson conjecture. On the other hand, partial Bergman kernels are useful in connection to the slope semi-stability with respect to a submanifold \([14]\). On a toric variety \(X\) (and for a toric \(\Sigma\)) this study was carried out in \([11]\). In this context it is shown that the partial Bergman kernel has an asymptotic expansion, having rapid decay of order \(p^{-\infty}\) (see §2.1 for this notation) in a neighborhood \(U(\Sigma)\) of \(\Sigma\), and giving the full Bergman kernel function to order \(p^{-\infty}\) outside the closure of \(U(\Sigma)\). Moreover \([11]\) gives a complete distributional asymptotic expansion on \(X\), whose leading term has an additional Dirac delta measure plus a dipole measure over \(\partial U(\Sigma)\). These results were generalized in \([13]\) and \([17]\) to the case when the data in question are invariant under an \(S^1\)-action.

In general, if no symmetry is assumed, it was shown in \([2, \text{Theorem 4.3}]\) that if the bundle \(L \otimes \mathcal{O}(-\Sigma)\) is ample, there exists a neighborhood \(U(\Sigma)\) of \(\Sigma\),
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such that \( P_{0,p}(x) \) has exponential decay on \( U(\Sigma) \) and \( p^{-n} P_{0,p}(x) \) converges to \( c_1(L,h)^n/\omega^n \) in \( L^1 \) outside the closure of \( U(\Sigma) \).

Our first result is that under the very general hypotheses (A) and (B) above (in particular, without any positivity condition), the partial Bergman kernel function decays exponentially in a neighborhood of the divisor \( \Sigma \).

**Theorem 1.1.** — Assume that conditions (A)–(B) are fulfilled. Then there exist a neighborhood \( U_t \) of \( \Sigma \) and a constant \( a \in (0,1) \) such that \( P_{0,p} \leq a^p \) on \( U_t \) for \( p > 2t^{-1} \). In particular \( P_{0,p} = O(p^{-\infty}) \) as \( p \to \infty \) on \( U_t \).

For more precise statements see Theorem 3.1 and Corollary 3.3. Theorem 1.1 can be formulated for non-compact manifolds \( X \) (see Theorem 3.4), in which case the exponential decay of the partial Bergman kernel holds in a neighborhood of the intersection of \( \Sigma \) with any given compact subset of \( X \). This includes for instance the case of classical Bergman spaces of \( L^2 \)-holomorphic functions on domains in \( \mathbb{C}^n \).

An object which is closely related to the partial Bergman kernel is the Bergman kernel for a singular metric. The full asymptotic expansion on compact subsets of the regular part of the metric was established in [8, Theorem 1.8]. We are here concerned with asymptotics at arbitrary points with dependence on the distance to the singular set. More precisely, we will consider the following situation.

Let \( S_\Sigma \in H^0(X,\mathcal{O}(\Sigma)) \) be a canonical holomorphic section of the line bundle \( \mathcal{O}(\Sigma) \), vanishing to first order on \( \Sigma \). We fix a smooth Hermitian metric \( h_\Sigma \) on \( \mathcal{O}(\Sigma) \) such that \( \varrho := \log |S_\Sigma|_{h_\Sigma} < 0 \) on \( X \).

We consider a function \( \xi : X \to \mathbb{R} \cup \{-\infty\} \), smooth on \( X \setminus \Sigma \), such that \( \xi = t\rho \) in a neighborhood \( U \) of \( \Sigma \). Let \( \text{dist}(\cdot,\cdot) \) be the distance on \( X \) induced by \( \omega \). Our main result is the following:

**Theorem 1.2.** — Let \( (X,\omega), (L,h), \Sigma \) be as in (A)–(B), and assume \( \omega \) is Kähler and \( h \) is smooth. Consider the singular Hermitian metric \( \tilde{h} = he^{-2\xi} \) on \( L \) and assume that \( c_1(L,\tilde{h}) \geq \varepsilon \omega \) for some constant \( \varepsilon > 0 \). Let \( \tilde{P}_p \) be the Bergman kernel function of \( H^0_{(2)}(X,L^p,\tilde{h}_p,\omega^n/n!) \), where \( \tilde{h}_p := \tilde{h}^{\otimes p} \). Then there exists a constant \( C > 1 \) such that for every \( x \in X \setminus \Sigma \) and every \( p \in \mathbb{N} \) with \( p \text{ dist}(x,\Sigma)^{8/3} > C \) we have

\[
\left| \frac{\tilde{P}_p(x)}{p^n} \frac{\omega^n}{c_1(L,h)^n} - 1 \right| \leq Cp^{-1/8}.
\]  

(1.5)

Theorem 1.2 can be interpreted in two ways. First, if \( x \) runs in a compact set \( K \subset X \setminus \Sigma \), we have a concrete bound \( p_0 = C \text{ dist}(K,\Sigma)^{-8/3} \) such that
for \( p > p_0 \) the estimate (1.5) holds. By [8, Theorem 1.8] we have \( \tilde{P}_p(x) = \sum_{r=0}^{\infty} b_r(x) p^{n-r} + O(p^{-\infty}) \) as \( p \to \infty \) locally uniformly on \( X \setminus \Sigma \). Hence, there exists \( p_0(K) \in \mathbb{N} \) and \( C_K \) such that for \( p > p_0(K) \) we have

\[
\left| \frac{\tilde{P}_p(x)}{p^n} \frac{\omega^n_x}{c_1(L,h)^n_x} - 1 \right| \leq C_K p^{-1} \quad \text{on } K.
\]

However, \( p_0(K) \) is not easy to determine.

We can also recast Theorem 1.2 as a uniform estimate in \( p \) for the singular Bergman kernel on compact sets of \( X \setminus \Sigma \) whose distance to \( \Sigma \) decreases as \( p^{-3/8} \). Indeed, set \( K_p = \{ x \in X : \text{dist}(x, \Sigma) \geq (C/p)^{3/8} \} \). Then (1.5) holds on \( K_p \) for every \( p \).

We consider now the global behavior of the partial Bergman kernel. Given a compact set \( K \subset X \setminus \Sigma \) we set

\[
t_0(K) := \sup \left\{ t > 0 : \exists \eta \in \mathcal{C}^\infty(X, [0,1]), \supp \eta \subset X \setminus K, \eta = 1 \text{ near } \Sigma, \right. \\
\left. \quad \text{and } c_1(L,h) + t dd^c(\eta \varrho) \text{ is a Kähler current on } X \right\}. \quad (1.6)
\]

A consequence of Theorems 1.1 and 1.2 is the following result about the asymptotics of the partial Bergman kernel:

**Theorem 1.3.** — Let \( (X,\omega), (L,h), \Sigma \) be as in (A)–(B), and assume \( \omega \) is Kähler, \( h \) is smooth, and \( c_1(L,h) \geq \varepsilon \omega \) for some constant \( \varepsilon > 0 \). Let \( K \subset X \setminus \Sigma \) be a compact set and let \( t \in (0, t_0(K)) \). Then there exist constants \( C > 1, M > 1 \) and a neighborhood \( U_t \) of \( \Sigma \), all depending on \( t \), such that for \( x \in U_t \) we have

\[
M e^{t\varrho(x)} < 1 \quad \text{and} \quad P_{0,p}(x) \leq (Me^{t\varrho(x)})^p \quad \text{for } p > 2/t, \quad (1.7)
\]

\[
P_{0,p}(x) \geq \frac{p^n}{C} \exp(2tpg(x)) \quad \text{for } p \text{dist}(x, \Sigma)^{8/3} > C, \quad (1.8)
\]

where the function \( \varrho \) is defined in (1.4). Moreover, we have uniformly on \( K \),

\[
P_{0,p}(x) = P_p(x) + O(p^{-\infty}), \quad p \to \infty, \quad (1.9)
\]

and in particular,

\[
P_{0,p}(x) = b_0(x)p^n + b_1(x)p^{n-1} + O(p^{n-2}), \quad p \to \infty, \quad (1.10)
\]

where

\[
b_0 = \frac{c_1(L,h)^n}{\omega^n}, \quad b_1 = \frac{b_0}{8\pi} (r^X - 2\Delta \log b_0), \quad (1.11)
\]

and \( r^X, \Delta \), are the scalar curvature, respectively the Laplacian, of the Riemannian metric associated to \( c_1(L,h) \).
Hence, (1.7) and (1.8) show that on $U_t$ the exponential decay estimate for the partial Bergman kernel function is sharp. Moreover, on $K$ the partial Bergman kernel function has the same asymptotics as the full Bergman kernel function up to order $O(p^{-\infty})$. This was established in [13, Theorem 1.1] under the additional assumption that there is an $S^1$-action in a neighborhood of $\Sigma$. Our method is to estimate the partial Bergman kernel $P_{0,p}$ by above and below with the full Bergman kernel $P_p$ and the singular Bergman kernel $\tilde{P}_p$. On the set where the singular metric $\tilde{h}$ equals $h$, the kernels $\tilde{P}_p$ and $P_p$ differ by $O(p^{-\infty})$. This is shown in Theorem 5.1, which gives a general localization result for singular Bergman kernels. Theorem 5.1 is a straightforward consequence of [8].

However, in Theorem 1.3 we do not necessarily obtain a partition of the manifold $X$ in two sets, one with exponential decay (1.7) and one with “full asymptotics” (1.9), since in general $U_t \cup K \neq X$. In [2, 13, 11, 17] a partition with two different regimes was exhibited under further hypotheses. The approach introduced by Berman [2, Section 4.1] was to consider the equilibrium metric (or extremal envelope) $h_t$ of $h$ with poles along $\Sigma$ (see also [13]). The metric $h_t$ exists for $t$ sufficiently small thanks to the positivity of $(L,h)$. The local plurisubharmonic (psh) potentials of $h_t$ have Lelong number $t$ along $\Sigma$. It is shown in [13, Proposition 2.13] that the partial Bergman kernel function $P_{0,p}$ has exponential decay in the forbidden region $\{ h_t > h \}$ which is a neighborhood of $\Sigma$. Moreover, under additional symmetry assumptions, it is shown in [13, Theorem 1.1] that $P_{0,p}$ is essentially equal to the full Bergman kernel outside the forbidden region.

In Theorem 1.1 we show that exponential decay holds near $\Sigma$ with no assumption on the positivity of $(L,h)$. Without positivity assumptions the equilibrium metric might not exist, but we give here a proof in the general case based only on the sub-average inequality for holomorphic functions.

Our approach in Theorem 1.3 differs from the envelope approach above in that we first fix a compact $K$ disjoint from $\Sigma$ and then construct an interval of small $t > 0$ for which the corresponding partial Bergman kernel $P_{0,p}$ decays exponentially near $\Sigma$ and is essentially equal to the full Bergman kernel on $K$. 
2. Preliminaries

2.1. Bergman kernel function

Let \((L, h)\) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold \((X, \omega)\). We denote by \(H^0(X, L^p)\) the space of holomorphic sections of \(L^p := L^\otimes p\).

Let \(H^0(X, L^p) = H^0(X, L, h_p, \omega^n/n!)\) be the Bergman space of \(L^2\)-holomorphic sections of \(L^p\) relative to the metric \(h_p := h^\otimes p\) induced by \(h\) and the volume form \(\omega^n/n!\) on \(X\), endowed with the inner product
\[
(S, S')_p := \int_X \langle S, S' \rangle_{h_p} \frac{\omega^n}{n!}, \quad S, S' \in H^0(X, L^p).
\]

Set \(\|S\|_p = (S, S)_p, \quad d_p = \dim H^0(X, L^p).\) If \(h\) has locally bounded weights (e.g. \(h\) is smooth) we have of course \(H^0(X, L^p) = H^0(X, L^p)\). We have the following variational characterization of the partial Bergman kernel
\[
P_{0,p}(x) = \max \left\{ \|S(x)\|_p^2 : S \in H^0(X, L^p), \|S\|_p = 1 \right\},
\]
and similar characterizations hold for the full and singular Bergman kernel functions \(P_p\) and \(\tilde{P}_p\).

Throughout the paper we also use the following terminology. For a sequence of continuous functions \(f_p\) on a manifold \(M\) we write \(f_p = O(p^{-\infty})\) if for every compact subset \(K \subset M\) and any \(\ell \in \mathbb{N}\) there exists \(C_{K,\ell} > 0\) such that for all \(p \in \mathbb{N}\) we have \(\|f_p\|_K \leq C_{K,\ell} p^{-\ell}\).

2.2. Geometric set-up

We prepare here the geometric set-up needed for the proofs of our results, by constructing a special neighborhood \(W\) of \(\Sigma\).

Let \((X, \omega)\) be a compact Hermitian manifold of dimension \(n\). Let \((U, z)\), \(z = (z_1, \ldots, z_n)\), be local coordinates centered at a point \(x \in X\). For \(r > 0\) and \(y \in U\) we denote by
\[
\Delta^r(y, r) = \{ z \in U : |z_j - y_j| \leq r, \ j = 1, \ldots, n \}
\]
the (closed) polydisk of polyradius \((r, \ldots, r)\) centered at \(y\). If \(\omega\) is a Kähler form, the coordinates \((U, z)\) are called Kähler at \(y \in U\) if
\[
\omega_z = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j + O(|z - y|^2) \quad \text{on } U.
\]
Since $\Sigma$ is compact, we can find an open cover $W = \{W_j\}_{1 \leq j \leq N}$ of $\Sigma$, where $W_j$ are Stein contractible coordinate neighborhoods centered at points $y_j \in \Sigma$, such that

$$\Delta^n(y_j, 2) \subset W_j, \ \Sigma \subset W := \bigcup_{j=1}^N \Delta^n(y_j, 1), \quad (2.3)$$

$$\Sigma \cap W_j = \{z \in W_j : z_1 = 0\}, \quad \text{for } j = 1 \ldots, N,$$

where $z = (z_1, \ldots, z_n)$ are the coordinates on $W_j$. Moreover, if $\omega$ is a Kähler form, we may also ensure that

$$\forall x \in \Delta^n(y_j, 1), \ \exists z = z(x) \text{ coordinates on } \Delta^n(y_j, 2)$$

centered at $x$ and Kähler at $x$. \quad (2.4)

As in [5, §2.5, Lemma 2.7] one can easily prove the following:

**Lemma 2.1.** — Let $(X, \omega), (L, h), \Sigma, \tilde{h}$ be as in Theorem 1.2, and let $W = \{W_j\}_{1 \leq j \leq N}$ be an open cover of $\Sigma$ verifying (2.3) and (2.4). There exist constants $C_1 > 1$, $C_2 > 0$ and $r_1 > 0$ with the following property: if $j \in \{1, \ldots, N\}, \ x \in \Delta^n(y_j, 1)$ and $z = z(x)$ are the coordinates on $\Delta^n(y_j, 2)$ given by (2.4), then:

1. $\Delta^n_z(x, r_1) \subset \Delta^n(y_j, 2)$ and for $r \leq r_1$ we have

$$n! \ dm \leq (1 + C_1 r^2) \omega^n, \ \omega^n \leq (1 + C_1 r^2)n! \ dm \ \text{ on } \Delta^n_z(x, r), \quad (2.5)$$

where $dm = dm(z)$ is the Euclidean volume and $\Delta^n_z(x, \cdot)$ is the open polydisk relative to the coordinates $z$.

2. $(L, \tilde{h})$ has a weight $\varphi_x$ on $W_j$ with

$$\varphi_x = t \log |f| + \psi_x, \ \psi_x \in C^\infty(W_j), \quad (2.6)$$

$$\psi_x(z) = \Re F_x(z) + \psi_x'(z) + \tilde{\psi}_x(z) \ \text{ on } \Delta^n(y_j, 2),$$

where $f$ is a defining function for $\Sigma \cap W_j$, $F_x(z)$ is a holomorphic polynomial of degree $\leq 2$ in $z$, $\psi_x'(z) = \sum_{\ell=1}^n \lambda_\ell |z_\ell|^2$, $\lambda_\ell = \lambda_\ell(x)$, and

$$|\tilde{\psi}_x(z)| \leq C_2 |z|^3, \ z \in \Delta^n_z(x, r_1). \quad (2.7)$$

### 3. Exponential decay

We prove here Theorem 1.1. Let $W = \{W_j\}_{1 \leq j \leq N}$ be the cover of $\Sigma$ and $W \supset \Sigma$ be the neighborhood of $\Sigma$ constructed in Section 2.2 (see (2.3)). For a function $\varphi \in L^\infty_{\text{loc}}(W_j)$ set

$$\|\varphi\|_{\infty, W_j} = \sup \{ |\varphi(w)| : w \in \Delta^n(y_j, 2) \}.$$
Let \((L, h)\) be a singular Hermitian holomorphic line bundle on \(X\), where the metric \(h\) has locally bounded weights. Since \(L|_{W_j}\) is trivial, we fix a holomorphic frame \(e_j\) of \(L|_{W_j}\), and denote by \(\varphi_j\) the corresponding weight of \(h\) on \(W_j\), i.e. \(|e_j|_h = e^{-\varphi_j}\). Set
\[
\|h\|_\infty = \|h\|_{\infty,W} := \max \{1, \|\varphi_j\|_{\infty,W_j} : 1 \leq j \leq N\},
\]
and let \(\rho\) be the function defined in (1.4).

**Theorem 3.1.** — In the setting of Theorem 1.1, there exists a constant \(A \geq 1\) depending only on \(\rho\) and \(W\) such that for any \(S \in H^0(X,L^p)\), \(x \in W\), and \(p \geq 1\), we have
\[
|S(x)|_{h_p}^2 \leq (A e^{\rho(x)})^{2|tp|} e^{4p\|h\|_\infty} \|S\|_p^2.
\]
Therefore, for every \(x \in W\) and \(p \geq 1\),
\[
P_{0,p}(x) \leq (A e^{\rho(x)})^{2|tp|} e^{4p\|h\|_\infty}.
\]

For the proof we need the following elementary lemma.

**Lemma 3.2.** — If \(k \geq 0\) and \(f \in \mathcal{O}(\Delta(0,2))\), where \(\Delta(0,2) \subset \mathbb{C}\) is the closed disk centered at 0 and of radius 2, then
\[
\int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) \leq \frac{k + 1}{2^{2k}} \int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \, dm(\zeta).
\]

**Proof.** — Consider the power expansion \(f(\zeta) = \sum_{j=0}^\infty a_j \zeta^j\) of \(f\) in \(\Delta(0,2)\). Integrating in polar coordinates we obtain
\[
\int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) = 2\pi \sum_{j=0}^\infty |a_j|^2 \int_0^2 r^{2j+1} \, dr = 2\pi \sum_{j=0}^\infty \frac{2^{2j+2}}{2j+2} |a_j|^2.
\]
On the other hand, \(\zeta^k f(\zeta) = \sum_{j=k}^\infty a_{j-k} \zeta^j\), so
\[
\int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \, dm(\zeta)
\geq \frac{2^{2k}}{k+1} \sum_{j=0}^\infty \frac{2^{2j+2}}{2j+2} |a_j|^2 = \frac{2^{2k}}{k+1} \int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta).
\]

**Proof of Theorem 3.1.** — Let \(x \in W\). Fix \(j \in \{1, \ldots, N\}\) such that \(x \in \Delta^n(y_j,1)\) and let \(e_j\) be the local frame of \(L|_{W_j}\) and \(\varphi_j\) be the corresponding weight of \(h\) as considered in (3.1). Let \(S \in H^0(X,L^p)\). On \(W_j\) we write...
$S = s e_j^{\otimes p}$, with $s \in \mathcal{O}(W_j)$. Then we have $s(z) = z_1^{[tp]} \tilde{s}(z)$, with $\tilde{s} \in \mathcal{O}(W_j)$. Using the sub-averaging inequality we get

$$|S(x)|^2_{h_p} = |x_1|^{2[tp]} |\tilde{s}(x)|^2 e^{-2p\varphi_j(x)}$$

$$\leq |x_1|^{2[tp]} e^{-2p\varphi_j(x)} \frac{1}{\pi^n} \int_{\Delta^n(x,1)} |\tilde{s}(z)|^2 \, dm(z)$$

$$\leq |x_1|^{2[tp]} e^{-2p\varphi_j(x)} \int_{\Delta^n(0,2)} |\tilde{s}(z)|^2 \, dm(z).$$

Applying Fubini’s theorem for the splitting $z = (z_1, z')$ and Lemma 3.2 for the variable $z_1$, we obtain

$$\int_{\Delta^n(0,2)} |\tilde{s}(z)|^2 \, dm(z) = \int_{\Delta^{n-1}(0,2)} \int_{\Delta(0,2)} |\tilde{s}(z_1, z')|^2 \, dm(z_1) \, dm(z')$$

$$\leq \frac{[tp] + 1}{2^{2[tp]}} \int_{\Delta^n(0,2)} |z_1|^{2[tp]} |\tilde{s}(z)|^2 \, dm(z)$$

$$\leq C \exp \left( 2p \sup_{\Delta^n(0,2)} \varphi_j \right) \int_{\Delta^n(0,2)} |s(z)|^2 e^{-2p\varphi_j(z)} \frac{\omega^n}{n!},$$

where $C = C(W) \geq 1$ is chosen such that $dm(z) \leq C\omega^n/n!$ on each $\Delta^n(y_j, 2)$ in the local coordinates of $W_j$, for $j = 1, \ldots, N$. Combining (3.2) and (3.3) we get

$$|S(x)|^2_{h_p} \leq C |x_1|^{2[tp]} \exp \left( 2p \sup_{\Delta^n(0,2)} \varphi_j - 2p\varphi_j(x) \right) \|S\|^2_p.$$ 

Note that there exists a constant $A' = A'(\rho, W) > 1$ such that

$$|x_1| \leq A' e^{\rho(x)}, \quad x \in W.$$ 

(3.5)

Set $A = A'C$. The estimates (3.4) and (3.5) yield

$$|S(x)|^2_{h_p} \leq (C|x_1|)^{2[tp]} e^{4p\|h\|_\infty} \|S\|^2_p \leq (Ae^{\rho(x)})^{2[tp]} e^{4p\|h\|_\infty} \|S\|^2_p.$$

Taking into account (2.2) we immediately obtain the conclusion. 

**Corollary 3.3.** — In the setting of Theorem 3.1 we let

$$U_t := \left\{ x \in W : \left( A e^{\rho(x)} \right)^t e^{4\|h\|_\infty} < 1 \right\}.$$ 

Then for any $x \in U_t$ and $p > 2t^{-1}$ we have

$$P_{0,p}(x) \leq \left( (A e^{\rho(x)})^t e^{4\|h\|_\infty} \right)^p.$$

In particular $P_{0,p} = O(p^{-\infty})$ as $p \to \infty$ on $U_t$. 

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**Proof.** — This follows immediately from Theorem 3.1, since $A e^{\rho(x)} < 1$ for $x \in U_t$, and $2[tp] > 2tp - 2 > tp$ for $p > 2/t$. □

We conclude this section by giving a version of Theorem 3.1 in the case when $X$ is not compact. Let $(X, \omega)$ be a Hermitian manifold of dimension $n$, $\Sigma$ be a smooth analytic hypersurface of $X$, $t > 0$ a fixed real number, and $(L, h)$ a singular Hermitian holomorphic line bundle on $X$ with singular metric $h$ which has locally bounded weights.

As in the case of a compact manifold $X$, we introduce the Bergman space $H^0_{(2)}(X, L^p) = H^0_{(2)}(X, L^p, h_p, \omega^n/n!)$ of $L^2$-holomorphic sections of $L^p$ relative to the metric $h_p$ induced by $h$ and the volume form $\omega^n/n!$ on $X$, endowed with the inner product (2.1). Let $H^0_{(2),0}(X, L^p) \subset H^0_{(2)}(X, L^p)$ be the Bergman space of $L^2$-holomorphic sections of $L^p$ vanishing to order at least $\lfloor tp \rfloor$ along $\Sigma$:

$$H^0_{(2),0}(X, L^p) := H^0_{(2)}(X, L^p \otimes O(-\lfloor tp \rfloor \Sigma)).$$

The spaces $H^0_{(2)}(X, L^p)$ and $H^0_{(2),0}(X, L^p)$ are not necessarily finite dimensional but their Bergman kernel functions $P_p$ and $P_{0,p}$ can be defined as in (1.2), (1.3), by means of at most countable orthonormal bases, see [6, Lemma 3.1].

We fix a compact set $E \subset X$ and consider an open cover $\mathcal{W} = \{W_j\}_{1 \leq j \leq N}$ and a neighborhood $W$ of the compact set $\Sigma \cap E$ constructed as in (2.3). Finally define $\|h\|_{\infty} = \|h\|_{\infty, \mathcal{W}}$ as in (3.1), and the function $\varrho$ as in (1.4), such that $\varrho < 0$ in a neighborhood of $E$. The following theorem is proved exactly as Theorem 3.1:

**Theorem 3.4.** — Let $(X, \omega)$ be a Hermitian manifold of dimension $n$, $\Sigma$ be a smooth analytic hypersurface of $X$, $t > 0$ a fixed real number, and $(L, h)$ a singular Hermitian holomorphic line bundle on $X$ with singular metric $h$ which has locally bounded weights. Then for any compact set $E \subset X$ there exists a neighborhood $W$ of the compact set $\Sigma \cap E$ and a constant $A \geq 1$ such that for every $x \in W$ and $p \geq 1$,

$$P_{0,p}(x) \leq (A e^{\rho(x)})^{2[tp]} e^{4p\|h\|_{\infty}}.$$

The constant $A$ depends only on $\rho$ and $\mathcal{W}$ above.

4. Singular Bergman kernel

In this section we prove Theorem 1.2 by using ideas of Berndtsson, who gave in [3, Section 2] a simple proof for the first order asymptotics of the
Bergman kernel function in the case of powers of an ample line bundle (see also [5, Theorem 1.3]).

We start by recalling the following version of Demailly’s estimates for the $\overline{\partial}$ operator [7, Théorème 5.1] (see also [5, Theorem 2.5]) which will be needed in our proofs.

**Theorem 4.1.** — Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$, and let $B > 0$ be a constant such that $\text{Ric}_\omega \geq -2\pi B \omega$ on $X$. Let $(L, h)$ be a singular Hermitian holomorphic line bundle on $X$ such that $c_1(L, h) \geq \varepsilon \omega$, and fix $p_0$ such that $p_0 \varepsilon \geq 2B$. Then for all $p > p_0$ and all $g \in L^2_{0,1}(X, L^p, \text{loc})$ with $\overline{\partial} g = 0$ and $\int_X |g|_{h_p}^2 \omega^n < \infty$ there exists $u \in L^2_{0,0}(X, L^p, \text{loc})$ such that $\overline{\partial} u = g$ and $\int_X |u|_{h_p}^2 \omega^n \leq \frac{2}{p^2} \int_X |g|_{h_p}^2 \omega^n$.

**Proof of Theorem 1.2.** — Let $\mathcal{W} = \{ W_j \}_{1 \leq j \leq N}$ be an open cover of $\Sigma$ verifying (2.3) and (2.4). If $j \in \{ 1, \ldots, N \}$ and $x \in \Delta^n(y_j, 1)$, let $z = z(x)$ be the coordinates on $\Delta^n(y_j, 2)$ given by (2.4), and let $e_{j,x}$ be a holomorphic frame of $L$ on $W_j$ such that $|e_{j,x}|_{\tilde{h}} = e^{-\varphi_x}$, where $\varphi_x$ is given by (2.6).

Assume now that $x \in \Delta^n(y_j, 1) \ \backslash \ \Sigma$ and define

$$r_x := \sup \left\{ r \in (0, r_1] : \Delta^n_z(x, r) \subset \Delta^n(y_j, 2) \ \backslash \ \Sigma \right\}.$$  

We have

$$\omega_x = \frac{i}{2} \sum_{\ell=1}^{n} dz_\ell \wedge d\bar{z}_\ell,$$

$$c_1(L, \tilde{h})_x = dd^c \varphi_x(0) = dd^c \psi_x(0) = \frac{i}{\pi} \sum_{\ell=1}^{n} \lambda_\ell dz_\ell \wedge d\bar{z}_\ell.$$  

(4.1)

Since $c_1(L, \tilde{h})_x \geq \varepsilon \omega_x$ it follows that $\lambda_\ell \geq \varepsilon$, $\ell = 1, \ldots, n$. Moreover, there exists $H_x \in \mathcal{O}(\Delta^n_z(x, r_x))$ such that $\text{Re} H_x = \text{Re} F_x + t \log |f|$. We define a new frame for $L$ over $\Delta^n_z(x, r_x)$ by $e_x = e^{H_x} e_{j,x}$. Hence

$$|e_x|_{\tilde{h}} = \exp(\text{Re} H_x) \exp(-\varphi_x) = \exp(-\psi_x - \tilde{\psi}_x).$$

We fix now $j \in \{ 1, \ldots, N \}$ and $x \in \Delta^n(y_j, 1) \ \backslash \ \Sigma$ and we will estimate $\tilde{p}_p(x)$. Let $r_p \in (0, r_x/2)$ be an arbitrary number which will be specified later. We start by estimating the norm of a section $S \in H^0_{(2)}(X, L^p, \tilde{h}_p, \omega^n/n!)$ at $x$. Writing $S = se^\tilde{\varphi}_p$, where $s \in \mathcal{O}(\Delta^n_z(x, r_x))$, we obtain by the sub-averaging inequality for psh functions on $\Delta^n_z(x, r_p) = \Delta^n(0, r_p)$,

$$|S(x)|_{\tilde{h}_p}^2 = |s(0)|^2 \leq \frac{\int_{\Delta^n(0, r_p)} |s|^2 e^{-2p\psi_x'} \, dm}{\int_{\Delta^n(0, r_p)} e^{-2p\psi_x'} \, dm}.$$  

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We have further by (2.5), (2.7),

$$\int_{\Delta^n(0,r_p)} |s|^2 e^{-2p\psi_s} \, dm \leq (1 + C_1 r_p^2) \exp(2p \sup_{\Delta^n(0,r_p)} \tilde{\psi}_x) \int_{\Delta^n(0,r_p)} |s|^2 e^{-2p(\psi_s + \tilde{\psi}_x)} \frac{\omega^n}{n!} \leq (1 + C_1 r_p^2) \exp(2C_2 p r_p^3) \|S\|_p^2.$$  

Set

$$E(r) := \int_{|\xi| \leq r} e^{-2|\xi|^2} \, dm(\xi) = \frac{\pi}{2} \left(1 - e^{-2r^2}\right).$$

Since $\lambda_\ell \geq \varepsilon$ we obtain

$$\frac{E(r_p \sqrt{p\varepsilon})}{p^n \lambda_1 \ldots \lambda_n} \leq \int_{\Delta^n(0,r_p)} e^{-2p\psi_s} \, dm \leq \int_{\mathbb{C}^n} e^{-2p\psi_s} \, dm = \frac{(\pi/2)^n}{p^n \lambda_1 \ldots \lambda_n}.$$

Combining these estimates it follows that

$$|S(x)|^2_{\tilde{h}_p} \leq \frac{(1 + C_1 r_p^2) \exp(2C_2 p r_p^3)}{E(r_p \sqrt{p\varepsilon})} p^n \lambda_1 \ldots \lambda_n \|S\|_p^2.$$  

(4.2)

The singular Bergman kernel also satisfies a variational formula,

$$\tilde{P}_p(x) = \max \left\{|S(x)|^2_{\tilde{h}_p} : S \in H^0_{(2)}(X, L^p, \tilde{h}_p, \omega^n/n!), \|S\|_p = 1\right\}.$$  

Hence (4.2) implies the following upper estimate for the singular Bergman kernel,

$$\frac{\tilde{P}_p(x)}{p^n \lambda_1 \ldots \lambda_n} \leq \frac{(1 + C_1 r_p^2) \exp(2C_2 p r_p^3)}{E(r_p \sqrt{p\varepsilon})} , \quad \forall r_p \in (0, r_x/2).$$  

(4.3)

For the lower estimate of $\tilde{P}_p$, let $0 \leq \chi \leq 1$ be a smooth cut-off function on $\mathbb{C}^n$ with support in $\Delta^n(0,2)$ such that $\chi \equiv 1$ on $\Delta^n(0,1)$, and set $\chi_p(z) = \chi(z/r_p)$. Then $F = \chi_p e_\psi^p$ is a section of $L^p$ and $|F(x)|_{\tilde{h}_p} = |e_\psi^p(x)|_{\tilde{h}_p} = 1$.

We have

$$\|F\|_p^2 \leq \int_{\Delta^n(0,2r_p)} \frac{e^{-2p(\psi_s + \tilde{\psi}_x)} \omega^n}{n!} \leq (1 + 4C_1 r_p^2) \exp(16C_2 p r_p^3) \int_{\Delta^n(0,2r_p)} e^{-2p\psi_s} \, dm \leq \left(\frac{\pi}{2}\right)^n \frac{(1 + 4C_1 r_p^2) \exp(16C_2 p r_p^3)}{p^n \lambda_1 \ldots \lambda_n}. $$  

(4.4)
Set $\alpha = \partial F$. Since $||\partial_{\chi}p||^2 = ||\partial_{\chi}||^2/r_p^2$, where $||\partial_{\chi}||$ denotes the maximum of $|\partial_{\chi}|$, we obtain as above

$$||\alpha||_p^2 = \int_{\Delta^n(0,2r_p)} |\partial_{\chi}p|^2 e^{-2p(\psi'_x + \tilde{\psi}_x)} \omega^n/n!$$

$$\leq \frac{||\partial_{\chi}||^2}{r_p^2} \left( \frac{\pi}{2} \right)^n \frac{(1 + 4C_1r_p^2) \exp(16C_2p r_p^3)}{p^n\lambda_1 \ldots \lambda_n}.$$ 

There exists $p_0 \in \mathbb{N}$ such that for $p > p_0$ we can solve the $\partial$-equation by Theorem 4.1. We get a smooth section $G$ of $L^p$ with $\partial G = \alpha = \partial F$ and

$$||G||_p^2 \leq \frac{2}{p_\epsilon} ||\alpha||_p^2 \leq \frac{2||\partial_{\chi}||^2}{p_\epsilon r_p^2} \left( \frac{\pi}{2} \right)^n \frac{(1 + 4C_1r_p^2) \exp(16C_2p r_p^3)}{p^n\lambda_1 \ldots \lambda_n}.$$ 

(4.5)

Note that $G$ is holomorphic on $\Delta^n(0, r_p)$ since $\partial G = \partial F = 0$ there. So the estimate (4.2) applies to $G$ on $\Delta^n(0, r_p)$ and gives

$$|G(x)|_{h_p}^2 \leq \frac{(1 + C_1r_p^2) \exp(2C_2p r_p^3)}{E(r_p \sqrt{p_\epsilon})} p^n\lambda_1 \ldots \lambda_n ||G||_p^2$$

$$\leq \frac{2||\partial_{\chi}||^2}{p_\epsilon r_p^2 E(r_p \sqrt{p_\epsilon})} \left( \frac{\pi}{2} \right)^n (1 + 4C_1r_p^2)^2 \exp(18C_2p r_p^3).$$

Let $S = F - G \in H^0_{(2)}(X, L^p, \tilde{h}_p, \omega^{n/n!})$. Then

$$|S(x)|_{h_p}^2 \geq \left( |F(x)|_{h_p} - |G(x)|_{h_p} \right)^2 = \left( 1 - |G(x)|_{h_p} \right)^2$$

$$\geq \left[ 1 - \left( \frac{\pi}{2} \right)^{n/2} \frac{\sqrt{2} ||\partial_{\chi}|| (1 + 4C_1r_p^2)}{r_p \sqrt{p_\epsilon} E(r_p \sqrt{p_\epsilon})^{n/2}} \exp(9C_2p r_p^3) \right]^2 =: K_1(r_p).$$

Moreover, by (4.4) and (4.5)

$$||S||_p^2 \leq (||F||_p + ||G||_p)^2 \leq \left( \frac{\pi}{2} \right)^n \frac{K_2(r_p)}{p^n\lambda_1 \ldots \lambda_n},$$

where

$$K_2(r_p) = (1 + 4C_1r_p^2) \exp(16C_2p r_p^3) \left( 1 + \frac{\sqrt{2} ||\partial_{\chi}||}{r_p \sqrt{p_\epsilon}} \right)^2.$$ 

Therefore

$$\tilde{P}_p(x) \geq \frac{|S(x)|_{h_p}^2}{||S||_p^2} \geq \left( \frac{2}{\pi} \right)^n p^n\lambda_1 \ldots \lambda_n \frac{K_1(r_p)}{K_2(r_p)}.$$ 

(4.6)
Using now (4.1), (4.3) and (4.6) we deduce that for every \( x \in \bigcup_{j=1}^{N} \Delta^{n}(y_{j}, 1) \setminus \Sigma \), \( r_{p} < r_{x}/2 \) and \( p > p_{0} \),
\[
\frac{K_{1}(r_{p})}{K_{2}(r_{p})} \leq \tilde{P}_{p}(x) \frac{\omega_{x}^{n}}{p^{n}c_{1}(L, \tilde{h})_{x}^{n}} \leq K_{3}(r_{p}),
\]
(4.7)
where
\[
K_{3}(r_{p}) = \left( \frac{\pi/2}{E(r_{p}\sqrt{p\varepsilon})} \right)^{n}(1 + C_{1}r_{p}^{2}) \exp(2C_{2}r_{p}^{3}).
\]
We take now \( r_{p} = p^{-3/8} \), so \( pr_{p}^{3} = p^{-1/8} \to 0 \) and \( pr_{p}^{2} = p^{1/4} \to \infty \) as \( p \to \infty \). Note that there exists a constant \( C_{3} > 0 \) such that
\[
K_{1}(p^{-3/8}) \geq 1 - C_{3}p^{-1/8},
\]
\[
K_{2}(p^{-3/8}) \leq 1 + C_{3}p^{-1/8}, \quad K_{3}(p^{-3/8}) \leq 1 + C_{3}p^{-1/8}.
\]
It follows by (4.7) that there exists a constant \( C_{4} > 0 \) such that
\[
1 - C_{4}p^{-1/8} \leq \tilde{P}_{p}(x) \frac{\omega_{x}^{n}}{p^{n}c_{1}(L, \tilde{h})_{x}^{n}} \leq 1 + C_{4}p^{-1/8}
\]
(4.8) holds for every \( x \in \bigcup_{j=1}^{N} \Delta^{n}(y_{j}, 1) \setminus \Sigma \), \( p > p_{0} \). Now \( r_{x} > c \operatorname{dist}(x, \Sigma) \), for some constant \( c > 0 \), so there exists a constant \( C_{5} > 0 \) such that (4.8) holds for \( p > C_{5} \operatorname{dist}(x, \Sigma)^{-8/3} \). This concludes the proof of (1.5) for \( x \in \bigcup_{j=1}^{N} \Delta^{n}(y_{j}, 1) \setminus \Sigma \).

By [8, Theorem 1.8] there exist \( C_{6} > 0 \) and \( p'_{0} \in \mathbb{N} \) such that
\[
\left| \tilde{P}_{p}(x) \frac{\omega_{x}^{n}}{p^{n}c_{1}(L, \tilde{h})_{x}^{n}} - 1 \right| \leq \frac{C_{6}}{p},
\]
for \( x \in X \setminus \bigcup_{j=1}^{N} \Delta^{n}(y_{j}, 1) \) and \( p > p'_{0} \). The proof of Theorem 1.2 is complete. \( \Box \)

5. Estimates for the partial Bergman kernel

In this section we prove Theorem 1.3. Let \( t < t_{0}(K) \). By the definition (1.6) of \( t_{0}(K) \), there exist \( \eta \in C_{0}(X, [0, 1]) \) and \( \delta > 0 \) such that \( \operatorname{supp}\eta \subset X \setminus K \), \( \eta = 1 \) near \( \Sigma \), and \( c_{1}(L, h) + tdd^{c}(\eta \varepsilon) \geq \delta \omega \) in the sense of currents on \( X \). Define
\[
\tilde{h}_{t} = h \exp(-2t\eta), \quad \tilde{h}_{t,p} = \tilde{h}_{t}^{\otimes p}.
\]
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Note that $\tilde{h}_t = h$ in a neighborhood of $K$ and $\tilde{h}_t \geq h$ on $X$. Since $\Sigma$ is smooth, it follows by (1.4) that $H^0_0(X, L^p) = H^0_{(2)}(X, L^p, \tilde{h}_{t,p}, \omega^n/n!)$. We denote the norm on $H^0_{(2)}(X, L^p, \tilde{h}_{t,p}, \omega^n/n!)$ by

$$
\|S\|_{t,p}^2 = \int_X |S|_{h_{t,p}}^2 \frac{\omega^n}{n!} = \int_X |S|_{\tilde{h}_p}^2 \exp(-2t\eta \theta) \frac{\omega^n}{n!}.
$$

Let $\tilde{P}_{t,p}$ be the Bergman kernel function of $H^0_{(2)}(X, L^p, \tilde{h}_{t,p}, \omega^n/n!)$. Recall that $\|S\|_p$ is the norm given by the scalar product (2.1) on $H^0_0(X, L^p)$. Since $\eta < 0$ we have $\|S\|_{t,p}^2 \geq \|S\|_p^2$ for any $S \in H^0_0(X, L^p)$. Let $S \in H^0_0(X, L^p)$ with $\|S\|_{t,p}^2 \leq 1$. Then $\|S\|_p^2 \leq 1$, too, hence

$$
|S|_{h_{t,p}}^2 = |S|_{\tilde{h}_p}^2 \exp(-2t\eta \theta) \leq P_{0,p} \exp(-2t\eta \theta),
$$

and thus

$$
\tilde{P}_{t,p} \leq P_{0,p} \exp(-2t\eta \theta).
$$

Denote now by $P_p$ the Bergman kernel function of $H^0_0(X, L^p)$ endowed with the scalar product (2.1). Since $H^0_0(X, L^p)$ is isometrically embedded in $H^0(X, L^p)$ we have $P_{0,p} \leq P_p$. Consequently we have shown:

$$
\tilde{P}_{t,p} \exp(2t\eta \theta) \leq P_{0,p} \leq P_p \text{ on } X,
$$

$$
\tilde{P}_{t,p} \leq P_{0,p} \leq P_p \text{ near } K.
$$

Let now $W$ be the neighborhood of $\Sigma$ defined in (2.3) and let $U_t$ be defined as in (3.6), so that the exponential estimate (3.7) holds on $U_t$ for $p > 2t^{-1}$. By shrinking $U_t$ we can assume that $\eta = 1$ on $U_t$. Setting $M := e^{4\|h\|}\infty A^t$ we obtain (1.7). By Theorem 1.2 we have

$$
\tilde{P}_{t,p}(x) \geq (1 - C p^{-1/8}) p^n \frac{c_1(L, \tilde{h}_t)}{\omega^n} x^n
$$

for every $p \in \mathbb{N}$ with $p \mathrm{dist}(x, \Sigma)^{8/3} > C$. Note that $c_1(L, \tilde{h}_t) \geq \delta \omega$ in the sense of currents on $X$. Since $c_1(L, \tilde{h}_t)$ is smooth on $X \setminus \Sigma$ we have

$$
\frac{c_1(L, \tilde{h}_t)}{\omega^n} \geq \delta^n \text{ on } X \setminus \Sigma.
$$

By increasing $C$ if necessary, it follows that

$$
\tilde{P}_{t,p}(x) \geq \frac{p^n}{C} \text{ for } p > C \mathrm{dist}(x, \Sigma)^{-8/3}.
$$

Hence

$$
P_{0,p}(x) \geq \frac{p^n}{C} \exp(2t\theta(x)) \text{ for } x \in U_t \text{ and } p > C \mathrm{dist}(x, \Sigma)^{-8/3}.
$$

This proves (1.8). In order to prove (1.9) we need the following localization theorem for the Bergman kernel for singular Hermitian metrics. We refer to [1, Theorem 2.2] for a localization principle in the case of smooth metrics.
THEOREM 5.1. — Let $(X, \omega)$ be a compact Hermitian manifold and $L \to X$ be a holomorphic line bundle. Consider two singular Hermitian metrics $h_1$ and $h_2$ on $L$, which are smooth outside a proper analytic set $\Sigma \subset X$ and such that $c_1(L, h_1)$, $c_1(L, h_2)$ are Kähler currents. Let $P^{(j)}_p$ be the Bergman projection on $H^0(X, L^p, h^p_j, \omega^n/n!)$, $j = 1, 2$. We assume that there exists an open set $U \subset X \setminus \Sigma$ such that $h_1 = h_2$ on $U$. Then the Bergman kernels satisfy $P^{(1)}_p(z, w) - P^{(2)}_p(z, w) = O(p^{-\infty})$ on $U$ in any $C^\ell$-topology, $\ell \in \mathbb{N}$, as $p \to \infty$.

Proof. — The proof follows essentially from the analysis in [8] (see also [9]). Let $h_0$ be any singular Hermitian metric on $L$, smooth on $X \setminus \Sigma$ and satisfying $c_1(L, h_0) \geq \varepsilon \omega$ in the sense of currents on $X$, for some $\varepsilon > 0$. Let $P^{(0)}_p$ be the Bergman projection on $H^0(X, L^p, h^0_p, \omega^n/n!)$.

Consider an open set $D \subset U$ such that $L|_D$ is trivial. Let $s : D \to L$ be a holomorphic frame and let $\varphi \in \mathcal{C}^\infty(D)$ be the weight of $h_0$ corresponding to $s$, that is, $|s|_{h_0} = e^{-\varphi}$. Let us denote by $\mathcal{E}'(D)$ the space of distributions with compact support in $D$ and by $L^2(D)$ the space of square-integrable functions with respect to the volume form $\omega^n/n!$. The localized Bergman projection with respect to $s$ is the operator $P^{(0)}_{p,s} : L^2(D) \cap \mathcal{E}'(D) \to L^2(D)$, defined by $P^{(0)}_{p,s}(ue^{p\varphi} s^{\otimes p}) = P^{(0)}(u)e^{p\varphi} s^{\otimes p}$. It is easy to see that

$$P^{(0)}_p(z, w) = P^{(0)}_{p,s}(z, w)e^{p(\varphi(z) - \varphi(w))} s^{\otimes p}(z) \otimes (s^{\otimes p})^*(w) \in L^p_z \otimes (L^p_w)^*, \quad z, w \in D. \quad (5.2)$$

By [8, Theorem 9.2] the kernel of $P^{(0)}_{p,s}$ satisfies

$$P^{(0)}_{p,s}(z, w) = S_{p}(z, w) + O(p^{-\infty}) \text{ on } D, \quad (5.3)$$

where $S_p$ is the localized approximate Szegő kernel defined in [8, (3.43)]. Note that by [8, Theorem 3.12] we have

$$S_p(z, w) = e^{ip\Psi(z, w)}b(z, w, p) + O(p^{-\infty}) \text{ on } D, \quad (5.4)$$

where $\Psi : D \times D \to \mathbb{C}$ is a phase function depending on the eigenvalues of $c_1(L, h_0)$ with respect to $\omega$ and described precisely in [8, Theorem 3.8]. Moreover, $b(\cdot, \cdot, p) : D \times D \to \mathbb{C}$ is a semi-classical symbol of order $n = \dim X$, depending only on the restriction of $h$ and $\omega$ to $D$.

We apply now these results for $h_0 = h_1$ and $h_0 = h_2$. Since $h_1|_D = h_2|_D$ we deduce that the weight $\varphi$, the phase $\Psi$ and the symbol $b(\cdot, \cdot, p)$ above are the same for $h_1$ and $h_2$. We infer from (5.3) and (5.4) that $P^{(1)}_{p,s}(z, w) - P^{(2)}_{p,s}(z, w) = O(p^{-\infty})$ on $D$. Finally, (5.2) yields $P^{(1)}_p(z, w) - P^{(2)}_p(z, w) = O(p^{-\infty})$ on $D$. The proof of Theorem 5.1 is complete. \qed
We apply now Theorem 5.1 to the metrics $\tilde{h}_t$ and $h$, which are equal on a neighborhood $V$ of $K$ and infer that

$$\tilde{P}_{t,p} - P_p = O(p^{-\infty})$$

locally uniformly on $V$. \hfill (5.5)

This fact combined with (5.1), (5.5) yields (1.9). Finally, (1.10) and (1.11) follow from the expansion of the Bergman kernel $P_p$ (see [10, Theorems 4.1.1–3]) or of the singular Bergman kernel (see [8, Theorem 1.8]).

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