The shape of \( \mathbb{Z}/\ell\mathbb{Z} \)-number fields.

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Abstract

Let \( \ell \) be a prime and let \( L/\mathbb{Q} \) be a Galois number field with Galois group isomorphic to \( \mathbb{Z}/\ell\mathbb{Z} \). We show that the shape of \( L \), see definition 1.2, is either \( \frac{1}{2}\mathbb{A}_{\ell-1} \) or a fixed sub lattice depending only on \( \ell \); such a dichotomy in the value of the shape only depends on the type of ramification of \( L \). This work is motivated by a result of Bhargava and Shnidman, and a previous work of the first named author, on the shape of \( \mathbb{Z}/3\mathbb{Z} \) number fields.

1 Introduction

Let \( L \) be a number field and let \( O_L \) be its maximal order. Let \( O^0_L \) be the trace zero module of \( O_L \) i.e., the set \( \{ x \in O_L : \text{tr}_{L/\mathbb{Q}}(x) = 0 \} \). Let us now consider the symmetric \( \mathbb{Z} \)-bilinear form obtained by restricting the trace pairing to \( O^0_L \times O^0_L \rightarrow \mathbb{Z} \)

\[
(x, y) \mapsto \text{tr}_{L/\mathbb{Q}}(xy);
\]

we will denote by \( q_L \) the integral trace zero form i.e., the associated integral quadratic form. In [Bha-Sha], the authors use a sub-lattice of the binary quadratic form \( \langle O^0_L, q_L \rangle \) to count cubic fields. From their work, if \( L \) is a Galois cubic field, one can deduce that after scaling the form \( \langle O^0_L, q_L \rangle \) in such a way that the form is primitive, one obtains an integral binary quadratic form that is independent on the field. A straightforward calculation shows that the scaling factor is \( 2 \cdot \text{rad}(d_L) \), where \( d_L \) is the discriminant of \( L \) and \( \text{rad}(\cdot) \) denotes the usual radical of an integer. Throughout the paper we will use the notation \( \text{rad}_L := \text{rad}(d_L) \).

The following result and the explicit calculation of the scaling factor can be found in [Man, Theorem 3.1].

**Theorem 1.1.** Let \( L \) be a Galois cubic field. Then, the rational binary quadratic form \( \frac{1}{2\text{rad}_L}q_L \) is integral, primitive, and does not depend on the field \( L \). In particular, any two cubic fields of the same discriminant have isometric integral trace zero forms. Furthermore,

\[
\langle O^0_L, \frac{1}{\text{rad}_L}q_L \rangle \cong 2x^2 - 2xy + 2y^2.
\]
In the general case, given a number field \( L \) of degree \( n \), the form \( q_L \) is an integral quadratic form of rank \( n - 1 \). By scaling the form \( q_L \) by a suitable positive integer \( n_L \) one can write \( q_L = n_L Q_L \) where \( Q_L \) is an integral primitive quadratic form of rank \( n - 1 \). We define the \textit{shape} of a number field as follows:

\textbf{Definition 1.2.} Let \( L \) be a number field. The \textit{shape} of \( L \) is the equivalence class of the quadratic form \( Q_L \) under the natural \( \text{GL}_{n-1}(\mathbb{Z}) \) action.

The study of the shape has been of interest mostly for cubic fields: In [Man, Theorem 6.5] and [Man1, Theorem 1.3] it is proved that, under certain ramification hypotheses, the shape is a complete invariant. See also [Bha-Har].

Suppose now that \( L \) is a quadratic number field with discriminant either odd or divisible by \( 8 \). An elementary calculation shows that the form \( \frac{1}{\text{rad}_L} q_L \) is an integral primitive quadratic form independent of the field \( L \). In particular, it is equivalent to \( Q_L \). Moreover,

\[ \langle O_L^0, \frac{1}{\text{rad}_L} q_L \rangle \cong 2x^2 \text{ or, equivalently, } Q_L \cong x^2. \]

Let us denote by \( \mathbb{A}_n \) the usual \( n \)-dimensional root lattice i.e., the lattice associated to the integral quadratic form

\[ \sum_{1 \leq i \leq n} 2x_i^2 - \sum_{1 \leq i, j \leq n \atop |i-j|=1} x_i x_j. \]

Then, if we look at the shape of quadratic and Galois cubic fields, we notice a clear similarity. This can be made more explicit by observing that \( 2x^2 - 2xy + 2y^2 \) and \( 2x^2 \) are the quadratic forms associated to the root lattices \( \mathbb{A}_2 \) and \( \mathbb{A}_1 \) respectively. A natural question arises: Can this be generalized to higher dimensions? More concretely, let \( \ell \) be an odd prime and let \( L \) be an \( \mathbb{Z}/\ell\mathbb{Z} \)-extension of \( \mathbb{Q} \) of discriminant \( d_L \).

(a) Is the form \( \frac{1}{\text{rad}_L} q_L \) integral and independent of the field \( L \)?

(b) Is the lattice \( \langle O_L^0, \frac{1}{\text{rad}_L} q_L \rangle \) isometric to \( \mathbb{A}_{\ell-1} \)?

The purpose of this paper is to answer questions (a) and (b). In the absence of wild ramification, it turns out that both (a) and the first part of (b) are answered positively. If there is wild ramification, question (a) still has a positive answer, but the isometry with \( \mathbb{A}_{\ell-1} \) exists only in the case \( \ell = 3 \); if \( \ell > 3 \), the lattice \( \langle O_L^0, \frac{1}{\text{rad}_L} q_L \rangle \) can be realized as a proper sub-lattice of \( \mathbb{A}_{\ell-1} \). Furthermore, such a lattice is isometric to a scaled Craig’s lattice independent of the field \( L \).

The main result of this paper is:
**Theorem** (cf. Theorem 2.17). Let $\ell$ be an odd prime and let $L$ be a Galois extension of $\mathbb{Q}$ with $\text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/\ell\mathbb{Z}$. Then, the lattice $\langle O_L^0, \frac{1}{\text{rad}(d_L)} q_L \rangle$ is an integral even lattice, which after scaling by a factor of $1/2$ is equivalent to $Q_L$. Moreover, there is a lattice embedding

$$\langle O_L^0, \frac{1}{\text{rad}_L} q_L \rangle \hookrightarrow \mathbb{A}_{\ell-1},$$

which is an isometry if and only if $\ell$ is tame in $L$. Furthermore, in the case of wild ramification, also the image of the embedding depends only on $\ell$.

**Remark 1.3.** In Theorem 2.17, see below, we give an explicit description of $\langle O_L^0, \frac{1}{\text{rad}(d_L)} q_L \rangle$ in terms of Craig’s lattices. In polynomial terms Theorem 2.17 says:

$$Q_L \cong \begin{cases} \sum_{1 \leq i \leq \ell-1} x_i^2 - \sum_{1 \leq i,j \leq \ell-1} x_i x_j & \text{if } L/\mathbb{Q} \text{ is tame} \\ \sum_{1 \leq i \leq \ell-1} (\frac{\ell-1}{2}) x_i^2 - \sum_{1 \leq i,j \leq \ell-1, i < j} x_i x_j & \text{if } L/\mathbb{Q} \text{ has wild ramification} \end{cases}$$

and the second integral quadratic form can be embedded in the first for every $\ell$. Notice that for $\ell = 3$, and only in this case, these two forms are equivalent.

### 1.0.1 Our definition of shape

Even though our definition of shape is inspired by the definition of shape of cubic rings given in [Bha-Har] the two forms are not equivalent in general. In the definition given in [Bha-Har], the authors replace the lattice $O_L^0$ by the sub-lattice of it given by

$$\widehat{O}_L^0 := \{ x \in \mathbb{Z} + [L : \mathbb{Q}]O_L \mid \text{tr}_{L/\mathbb{Q}}(x) = 0 \}.$$

For a Galois cubic field $L$, Theorem 1.11 says that

$$Q_L \cong x^2 - xy + y^2;$$

hence, by the results in [Bha-Har] on $\mathbb{Z}/3\mathbb{Z}$-extensions, the two notions of shape are the same for such fields. More generally, if $L$ is a $\mathbb{Z}/\ell\mathbb{Z}$-number field in which $\ell$ ramifies, one can verify that $\widehat{O}_L^0 = \ell O_L^0$, which implies that the two notions of shape are the same for wild $\mathbb{Z}/\ell\mathbb{Z}$-number fields. For tame $\mathbb{Z}/\ell\mathbb{Z}$-extensions the two forms are equivalent only for $\ell = 3$. However, for such extensions, the change of basis between the modules $O_L^0$ and $\widehat{O}_L^0$ is canonical and only depends on $\ell$. In particular, in the appropriate setting, all the results in this paper can be written in terms of the shape as defined by Bhargava and Shindman.
2 Proofs of results

2.1 Facts about $\mathbb{Z}/\ell\mathbb{Z}$-extensions

In this section, we will prove that the shape $Q_L$, as defined above, only depends on the discriminant of the field $L$. To achieve this, we show that the scaling factor that transforms the trace into the shape can be canonically written in terms of the discriminant (see Proposition 2.6).

Proposition 2.1. Let $\ell$ be an odd prime and let $L/\mathbb{Q}$ be a Galois $\mathbb{Z}/\ell\mathbb{Z}$-extension. Let $p \neq \ell$ be a prime that ramifies in $L$ and let $\mathcal{P}$ the unique prime ideal of $O_L$ lying above $p$. Then, $O_L^0 + p\mathbb{Z}$ is contained in $\mathcal{P}$ as a $\mathbb{Z}$-module.

Proof. It is enough to show that $O_L^0 \subseteq \mathcal{P}$. Since $p$ is totally ramified we have that $[O_L : \mathcal{P}] = p$. In particular, $\mathcal{P}$ is a maximal $\mathbb{Z}$-submodule of $O_L$. If we suppose that there exists an element $a \in O_L^0$ such that $a \notin \mathcal{P}$ then we would be able to write $O_L = a\mathbb{Z} + \mathcal{P}$; hence, there should exist $\beta \in \mathcal{P}$ and $n \in \mathbb{Z}$ such that

$1 = an + \beta$.

Since $\mathcal{P}$ is invariant by the action of $\text{Gal}(L/\mathbb{Q})$ we know that $\text{tr}_{L/\mathbb{Q}}(\beta) \in \mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$ for any $\beta \in \mathcal{P}$. By applying the trace operator, one can see that the above equality contradicts that $p \neq \ell$; hence such an element $a$ does not exist and $O_L^0 \subseteq \mathcal{P}$.

Corollary 2.2. Let $\ell$ be an odd prime and let $L/\mathbb{Q}$ be a Galois $\mathbb{Z}/\ell\mathbb{Z}$-extension. Then, for all $a, b \in O_L^0$ and for all ramified prime $p$ in $L$, we have that $p$ divides $\text{tr}_{L/\mathbb{Q}}(ab)$. In other words $\langle O_L^0, \frac{1}{\text{rad}_L} Q_L \rangle$ is an integral lattice.

Proof. Let $p$ be a ramified prime. If $p \neq \ell$ we know by Proposition 2.1 that $ab \in \mathcal{P}$ for all $a, b \in O_L^0$, where $\mathcal{P}$ is the unique prime ideal in $O_L$ lying over $p$. In particular, $\text{tr}_{L/\mathbb{Q}}(ab) \in \text{tr}_{L/\mathbb{Q}}(\mathcal{P}) \subseteq \mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$. If $p = \ell$ then it follows from [Man, Proposition 2.6] that $\text{tr}_{L/\mathbb{Q}}(ab) \subseteq \ell\mathbb{Z}$ for all $a, b \in O_L$ and, a fortiori, for all $a, b \in O_L^0$.

Definition 2.3. Let $L$ be a number field of discriminant $d_L$. The radical discriminant of $L$, denoted by $\text{rad}_L$, is the square free integer divisible by only ramified primes in $L$ and that has the same sign than $d_L$.

Lemma 2.4. Let $\ell$ be an odd prime and let $L/\mathbb{Q}$ be a Galois $\mathbb{Z}/\ell\mathbb{Z}$-extension. Let $n_L$ be the product of primes not equal to $\ell$ that ramify in $L$ and let

$$\delta_\ell(L) = \begin{cases} 1 & \text{if } \ell \text{ ramifies in } L, \\ 0 & \text{otherwise}. \end{cases}$$

Then, $\text{disc}(L) = n_L^{\ell-1} (\ell \delta_\ell(L))^2 (\ell-1)$ and $\text{rad}_L = \ell \delta_\ell(L) n_L$. 

In particular, any two degree ℓ Galois number fields K and L have the same discriminant if and only if they have the same radical discriminant.

Proof. For an integer prime p, let \( v_p \) be the standard p-adic valuation. If p is a prime that ramifies in L then it is totally ramified. Moreover, if \( p \neq ℓ \) then it is tamely ramified, hence we know from [S, Chapter III, Proposition 13] that

\[
v_p(\text{disc}(L)) = ℓ - 1.
\]

If ℓ is ramified in L then it has wild ramification, and the wild ramification group at ℓ is the whole Galois group \( \text{Gal}(L/Q) \). In the notation of [S, Chapter IV] we have that \( G_i = \text{Gal}(L/Q) \) for \( i = -1, 0, 1 \). Since all the ramification groups are either trivial or of order ℓ we have by [S, Chapter IV, Proposition 4] that

\[
v_ℓ(\text{disc}(L)) = (N_L + 2)(ℓ - 1),
\]

where \( N_L = \# \{ i > 1 : G_i \neq 1 \} \). Thanks to [S, Chapter III, Remark to Proposition 13] we have that \( v_ℓ(\text{disc}(L)) \leq 2(ℓ - 1) \), which by the above equation implies that \( N_L = 0 \). Since \( L/Q \) is an odd Galois extension it’s discriminant is positive. Hence,

\[
\text{disc}(L) = \prod_{p|\text{disc}(L)} p^{v_p(\text{disc}(L))} = n_L^{ℓ-1}(ℓ^{δ_ℓ(L)})^{2(ℓ-1)}.
\]

It follows that \( \text{rad}_L = ℓ^{δ_ℓ(L)}n_L \).

Lemma 2.5. Let ℓ be an odd prime and let \( L/Q \) be a Galois \( \mathbb{Z}/ℓ\mathbb{Z} \)-extension. The determinant of the integral lattice \( \langle O_L^0, q_L \rangle \) is given by

\[
\det(\langle O_L^0, q_L \rangle) = \begin{cases} 
\text{disc}(L) & \text{if } L/Q \text{ is wild,} \\
ℓ^{δ_ℓ(L)} & \text{otherwise.}
\end{cases}
\]

Proof. This follows from [Man, Lemma 2.3] and [Man, Proposition 2.6].

Proposition 2.6. Let ℓ be an odd prime and let \( L/Q \) be a Galois \( \mathbb{Z}/ℓ\mathbb{Z} \)-extension. Then,

\[
Q_L \cong \frac{1}{2\text{rad}_L} q_L.
\]

Proof. Thanks to Corollary 2.2, Lemma 2.4 and Lemma 2.5 we have that \( \langle O_L^0, \frac{1}{\text{rad}_L} q_L \rangle \) is an integral lattice with determinant equal to

\[
\det(\langle O_L^0, \frac{1}{\text{rad}_L} q_L \rangle) = \begin{cases} 
ℓ^{δ_ℓ(L)} & \text{if } L/Q \text{ is wild,} \\
ℓ & \text{otherwise.}
\end{cases}
\]

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If $M$ is the Gram matrix representing $\langle O_L^0, \frac{1}{\text{rad}_L}q_L \rangle$ in any given basis then all its entries are relatively prime. Otherwise, there would be a positive integer $d \neq 1$ such that $d^{\ell-1} \mid \det(M)$, and this is a contradiction since $\det(M) \mid \ell^{\ell-2}$. Since $\langle O_L^0, q_L \rangle$ is an even integral lattice, and since 2 is unramified in $L$, we have that $\frac{1}{\text{rad}_L}q_L$ is an integral quadratic form. By the analysis on $M$ we conclude that $\frac{1}{\text{rad}_L}q_L$ is a primitive integral quadratic form hence, by definition, it is the shape $Q_L$.

2.2 Tamely ramified extensions

Since the integral structure of tamely ramified abelian fields is well behaved, we begin dealing with the tame case.

**Proposition 2.7.** Let $\ell$ be a prime and let $L$ be a tame $\mathbb{Z}/\ell\mathbb{Z}$-extension of $\mathbb{Q}$. Then, there exists $e_1 \in O_L$, a generator of $O_L$ as a $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$-module, such that

$$\text{tr}_{L/\mathbb{Q}}(e_1 \sigma(e_1)) = \text{tr}_{L/\mathbb{Q}}(e_1 \tau(e_1))$$

for all $\sigma, \tau \in \text{Gal}(L/\mathbb{Q}) \setminus \{\text{Id}\}$.

**Proof.** This follows from the existence of a Lagrangian basis proven in [C-P, pg 193-195].

**Theorem 2.8.** Let $\ell$ be a prime and let $L$ be a tame $\mathbb{Z}/\ell\mathbb{Z}$-extension of $\mathbb{Q}$. Then,

$$\langle O_L^0, \frac{1}{\text{rad}_L}q_L \rangle \cong \mathbb{A}_{\ell-1}.$$  

**Proof.** Let $\sigma \in \text{Gal}(L/\mathbb{Q})$ be a generator and let $e_1 \in O_L$ and such that the set

$$\mathcal{B} = \{ e_i := \sigma^{i-1}(e_1) \mid 1 \leq i \leq \ell \}$$

is an integral basis for $O_L$. By Proposition 2.7 we can assume that for all $2 \leq i, j \leq \ell$ we have that

$$\text{tr}_{L/\mathbb{Q}}(e_1 e_i) = \text{tr}_{L/\mathbb{Q}}(e_1 e_j). \tag{1}$$

Let $\Gamma = \{ e_i e_j \mid 1 \leq i < j \leq \ell \}$ be the set of all possible products of two elements in $\mathcal{B}$.

**Claim:** for all $\gamma \in \Gamma$ we have that $\text{Tr}_{L/\mathbb{Q}}(\gamma) = \text{Tr}_{L/\mathbb{Q}}(e_1 e_2)$.

**Proof of the claim:** Consider the following subsets of $\Gamma$:

$$
\begin{align*}
\Gamma_1 &= \{ e_1 e_2, e_2 e_3, ..., e_\ell e_1 \}, \\
\Gamma_2 &= \{ e_1 e_3, e_2 e_4, ..., e_\ell e_2 \}, \\
&\vdots \\
\Gamma_{\ell-1} &= \{ e_1 e_{\ell-1},..., e_{\ell-1} e_1, e_1 e_{\ell-1} \}. \\
\end{align*}
$$

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are conjugate we have that for all $i$.

Since $L$ is totally real we have that $a > b$. Suppose now that $i < j$; if $j \neq i + 1$, then

$$\text{Tr}_{L/Q}(w_i w_j) = \text{Tr}_{L/Q}(e_i e_j - e_i e_{i+1} - e_i e_{j+1} + e_{i+1} e_{j+1}) = b - b + b = 0.$$  

For $j = i + 1$ we have that

$$\text{Tr}_{L/Q}(w_i w_{i+1}) = \text{Tr}_{L/Q}(e_i e_{i+1} - e_i e_{i+1} - e_i e_{i+2} + e_{i+1} e_{i+2}) = b - a + b = b - a.$$  

The above calculations can be summarized by writing

$$M = \frac{(a - b)}{-\text{rad}_L} A$$

where $A$ is the Gram matrix of the root lattice $A_{\ell-1}$ in its standard basis. Since $\det(M) = \ell$ (cf proof of Proposition 2.6) and also $\det(A) = \ell$, we have that $\left(\frac{(a-b)}{-\text{rad}_L}\right)^{\ell-1} = 1$. On the other hand since $a > b$ we conclude that $M = A$, hence the result.  

Remark 2.9. By looking at $(\text{Tr}_{L/Q}(e_1))^2$ in the above proof, one can show that $1 = a + (\ell - 1)b$. This, together with the value obtained above for $a - b$, allows us to conclude that $a = 1 + (\ell - 1)\text{rad}_L$ and that $b = 1 - \text{rad}_L$. These explicit values for $a$ and $b$, and the proof above, are a generalization to $\ell > 3$ of the proof of [Man, Theorem 3.1].

2.3 Wild ramification

In the case wild ramification we use the theory of ideal lattices, in fact only cyclotomic ones. For an introduction, background and terminology on ideal lattices see [Ba], [Ba1] and [Ba2].
2.3.1 Ideal lattices

Let $\ell$ be an odd prime and let $\zeta_\ell \in \mathbb{C}$ be a primitive $\ell$-root of unity. For a totally real element $\beta \in \mathbb{Q}(\zeta_\ell)$, we denote by $I_\beta$ the ideal lattice $\langle \mathbb{Z}[\zeta_\ell], \beta \rangle$. In other words, $I_\beta$ is the positive definite lattice obtained by considering the $\mathbb{Z}$-module $\mathbb{Z}[\zeta_\ell]$ endowed with the bilinear pairing defined by

\[
\langle \cdot, \cdot \rangle_\beta : \mathbb{Z}[\zeta_\ell] \times \mathbb{Z}[\zeta_\ell] \to \mathbb{Q} \quad (x, y) \mapsto \text{tr}_{\mathbb{Q}(\zeta_\ell)/\mathbb{Q}}(\beta xy).
\]

Let $\mathcal{D}_\ell^{-1}$ be the inverse different of $\mathbb{Q}(\zeta_\ell)$. Whenever $\beta \in \mathcal{D}_\ell^{-1}$, the ideal lattice $I_\beta$ is an integral lattice i.e., the bilinear pairing $\langle \cdot, \cdot \rangle_\beta$ is $\mathbb{Z}$-valued.

**Lemma 2.10.** Let $\alpha, \beta \in \mathcal{D}_\ell^{-1}$ be totally real elements. Suppose that there exists $\gamma \in \mathbb{Z}[\zeta_\ell] \setminus \{0\}$ such that $\frac{\alpha}{\beta} = \gamma$. Then, the $\mathbb{Z}$-module homomorphism

\[
\phi_\gamma : \mathbb{Z}[\zeta_\ell] \to \mathbb{Z}[\zeta_\ell] \quad x \mapsto \gamma x
\]

is an injective morphism of lattices from $I_\alpha$ to $I_\beta$. Moreover, if $\gamma \in (\mathbb{Z}[\zeta_\ell])^*$, the morphism $\phi_\gamma$ is an isometry.

**Proof.** The map is clearly additive and since $\gamma \neq 0$ it is injective. Let $x, y \in \mathbb{Z}[\zeta_\ell]$. Then,

\[
\langle \phi_\gamma(x), \phi_\gamma(y) \rangle_\beta = \langle \gamma x, \gamma y \rangle_\beta = \text{tr}_{\mathbb{Q}(\zeta_\ell)/\mathbb{Q}}(\beta \gamma x\gamma y) = \text{tr}_{\mathbb{Q}(\zeta_\ell)/\mathbb{Q}}(\beta \gamma x\gamma y) = \text{tr}_{\mathbb{Q}(\zeta_\ell)/\mathbb{Q}}(\alpha x\gamma y) = \langle x, y \rangle_\alpha.
\]

Thus, $\phi_\gamma$ is a lattice morphism. Furthermore, If $\gamma$ is a unit then $\phi_\gamma$ is an isometry with inverse given by $\phi_{\gamma^{-1}} : I_\beta \to I_\alpha$. \(\square\)

**Proposition 2.11.** Let $\ell$ be an odd prime. We define $\alpha_\ell, \beta_\ell$ and $\delta_\ell$ in the following way:

\[
\alpha_\ell = \frac{(\zeta_\ell - \zeta_\ell^{-1})^2(\ell-1)}{\ell^2}, \beta_\ell = \frac{(2 - \zeta_\ell^2 - \zeta_\ell^{-2})}{\ell}, \quad \text{and} \quad \delta_\ell = \frac{(2 - \zeta_\ell - \zeta_\ell^{-1})}{\ell}.
\]

Then,

(i) The elements $\alpha_\ell, \beta_\ell$ and $\delta_\ell$ belong to $\mathcal{D}_\ell^{-1}$ and are all totally real.

(ii) There exist $\gamma \in (\mathbb{Z}[\zeta_\ell])^*$ and $\eta \in (\mathbb{Z}[\zeta_\ell]) \setminus \{0\}$ such that

\[
\frac{\delta_\ell}{\beta_\ell} = \gamma \overline{\gamma} \quad \text{and} \quad \frac{\alpha_\ell}{\beta_\ell} = \eta \overline{\eta}.
\]

**Proof.**
(i) Since $\alpha_{\ell}$, $\beta_{\ell}$ and $\delta_{\ell}$ are invariant under complex conjugation they are totally real elements of $\mathbb{Q}(\zeta_{\ell})$. Notice that $\ell\beta_{\ell} = (\zeta_{\ell} - \zeta_{\ell}^{-1})^2 \in (1 - \zeta_{\ell})$, the unique maximal ideal in $\mathbb{Z}[\zeta_{\ell}]$ lying over $\ell$. In particular, for any $x \in \mathbb{Z}[\zeta_{\ell}]$ we have that $\ell\beta_{\ell}x \in (1 - \zeta_{\ell})$. Since $\text{tr}_{\mathbb{Q}(\zeta_{\ell})/\mathbb{Q}}((1 - \zeta_{\ell})) = \ell \mathbb{Z}$ we have that

\[ \text{tr}_{\mathbb{Q}(\zeta_{\ell})/\mathbb{Q}}(\ell\beta_{\ell}x) \in \ell \mathbb{Z} \] or equivalently \( \text{tr}_{\mathbb{Q}(\zeta_{\ell})/\mathbb{Q}}(\beta_{\ell}x) \in \mathbb{Z} \) i.e., $\beta_{\ell} \in \mathcal{D}_{\ell}^{-1}$. Since $\beta_{\ell}$ and $\delta_{\ell}$ are conjugated we also have that $\delta_{\ell} \in \mathcal{D}_{\ell}^{-1}$. Since $\ell = u(1 - \zeta_{\ell})^{\ell-1}$ for some unit $u$, we have that $\alpha_{\ell} \in \mathbb{Z}[\zeta_{\ell}]$ so in particular we have that $\alpha_{\ell} \in \mathcal{D}_{\ell}^{-1}$.

(ii) Notice that

\[ \frac{\delta_{\ell}}{\beta_{\ell}} = \frac{(\zeta_{\ell}^{\ell+1} - \zeta_{\ell}^{-(\ell+1)})}{(\zeta_{\ell} - \zeta_{\ell}^{-1})^2} \]

and that

\[ \gamma := \frac{\zeta_{\ell}^{\ell+1} - \zeta_{\ell}^{-(\ell+1)}}{\zeta_{\ell} - \zeta_{\ell}^{-1}} \in (\mathbb{Z}[\zeta_{\ell}])^*. \]

Because $\overline{\gamma} = \gamma$, we have that

\[ \frac{\delta_{\ell}}{\beta_{\ell}} = \gamma \overline{\gamma}. \]

Since $N_{\mathbb{Q}(\zeta_{\ell})/\mathbb{Q}}(1 - \zeta_{\ell}) = \ell$, and $\ell$ is odd, there is some $\eta_0 \in \mathbb{Z}[\zeta_{\ell}]$ such that $\ell = \eta_0 \overline{\eta_0}$. Furthermore, there exits a unit $u_0$ such that $\eta_0 = u_0(1 - \zeta_{\ell})^{\ell-1}/2$. Let $\eta_1 := (\zeta_{\ell} - \zeta_{\ell}^{-1})^{\ell-2}$. Since $3 \leq \ell$ we have that $\eta := \frac{\eta_1}{\eta_0} \in \mathbb{Z}[\zeta_{\ell}]$. The result follows, because

\[ \frac{\alpha_{\ell}}{\beta_{\ell}} = \frac{(\zeta_{\ell} - \zeta_{\ell}^{-1})^{2(\ell-2)}}{\ell} = \gamma \overline{\gamma}. \]

\[ \square \]

**Corollary 2.12.** Let $\ell$ be an odd prime and let $\alpha_{\ell}$ and $\delta_{\ell}$ be as in the above proposition. Then, there exits an injective morphism of lattices:

\[ I_{\alpha_{\ell}} \hookrightarrow I_{\delta_{\ell}}. \]

**Proof.** Thanks to Proposition 2.11 we have that

\[ \frac{\alpha_{\ell}}{\delta_{\ell}} = \gamma_1 \overline{\gamma_1} \]

for some $\gamma_1 \in (\mathbb{Z}[\zeta_{\ell}]) \setminus \{0\}$. It follows from Lemma 2.10 that

\[ \phi_{\gamma_1} : I_{\alpha_{\ell}} \rightarrow I_{\delta_{\ell}} \]

is an embedding of ideal lattices.

\[ \square \]
The following result of Conner and Perlis emphasizes the connection between the wild ramification case and the results on ideal lattices. See [C-P, Lemma IV.9.3 + pg 199 3d formula + §IV.14].

**Theorem 2.13** (Conner-Perlis). Let \( \ell \) be an odd prime and let \( L \) be a \( \mathbb{Z}/\ell\mathbb{Z} \)-extension of \( \mathbb{Q} \) which is ramified at \( \ell \). Let \( m_L \) be the product of all the integer primes different from \( \ell \) that are ramified in \( L \). Let \( \mu_L := \frac{m_L}{\ell}(\zeta_\ell - \zeta^{-1}_\ell)^{2(\ell-1)} \). Then,

\[
\langle \mathbb{Z}[\zeta_\ell], \mu_L \rangle \cong \langle O_L^0, q_L \rangle.
\]

**Corollary 2.14.** Let \( L \) be a number field as in Theorem 2.13. Then, \( Q_L \) is the quadratic form associated to the lattice

\[
\left\langle \mathbb{Z}[\zeta_\ell], \frac{(\zeta_\ell - \zeta^{-1}_\ell)^{2(\ell-1)}}{2\ell^2} \right\rangle.
\]

In particular, the shape of a wildly ramified \( \mathbb{Z}/\ell\mathbb{Z} \)-extension of \( \mathbb{Q} \) only depends on the prime \( \ell \).

**Proof.** Since \( \ell \) is ramified in \( L \) we have that \( \text{rad}_L = \ell m_L \). Therefore the result follows from Theorem 2.13 and Proposition 2.6.

### 2.3.2 Craig’s lattices

For positive integers \( k \) and \( n \) the lattice \( A^{(k)}_n \), known as Craig’s lattice, denotes the lattice defined in [C-S, Chapter 8 §6]. Recall that whenever \( \ell \) is an odd prime and \( n = \ell - 1 \) then a Craig’s lattice can be realized as a cyclotomic ideal lattice:

\[
A^{(k)}_{\ell-1} \cong \left\langle (1 - \zeta_\ell)^k, \text{tr}_{\mathbb{Q}(\zeta_\ell)}/\mathbb{Q}(\frac{1}{\ell}x\bar{y}) \right\rangle.
\]

See also [Bac-Bat, §4] for background and main properties of Craig’s lattices.

**Lemma 2.15.** Let \( \ell \) be an odd prime, and let \( \zeta_\ell \in \mathbb{C} \) be a primitive \( \ell \)-root of unity. We define \( \alpha_\ell := \frac{(\zeta_\ell - \zeta^{-1}_\ell)^{2(\ell-1)}}{\ell} \). Then,

\[
I_{\alpha_\ell} \cong \frac{1}{\ell} A^{(\ell-1)}_{\ell-1}.
\]

**Proof.** Since \( (1 - \zeta_\ell)^{\ell-1} = \ell \mathbb{Z}[\zeta_\ell] \), we have that \( \frac{1}{\ell} A^{(\ell-1)}_{\ell-1} \cong \langle \mathbb{Z}[\zeta_\ell], 1 \rangle = I_1 \). On the other hand, since \( \gamma := \frac{(\zeta_\ell - \zeta^{-1}_\ell)^{\ell-1}}{\ell} \in (\mathbb{Z}[\zeta_\ell])^* \) and \( \alpha_\ell = \gamma \gamma \), we have, thanks to Lemma 2.10, that

\[
I_{\alpha_\ell} \cong I_1.
\]
Given a lattice $\Lambda$ we denote by $q(\Lambda)$ the equivalence class of the quadratic form associated to it. Combining Corollary 2.14 and Lemma 2.15 we obtain the following result:

**Theorem 2.16.** Let $\ell$ be an odd prime and let $L$ be a $\mathbb{Z}/\ell\mathbb{Z}$-extension of $\mathbb{Q}$ which is ramified at $\ell$. Then,

$$Q_L \cong q\left(\frac{1}{2\ell} \mathbb{A}_{\ell-1}\right).$$

### 2.4 Proof of the main result

We are now able to state and to prove our main result (see §1)

**Theorem 2.17.** Let $\ell$ be a prime and let $L$ be a $\mathbb{Z}/\ell\mathbb{Z}$-extension of $\mathbb{Q}$. Then, the lattice $\left\langle O_L, \frac{1}{\rad(d_L)} q_L \right\rangle$ is an integral lattice isometric to a sub-lattice of $\mathbb{A}_{\ell-1}$. Moreover, if the type of ramification of $\ell$ is fixed then such lattice depends only on $\ell$. Specifically,

$$\frac{1}{2 \cdot \rad_L} q_L \cong Q_L \cong \begin{cases} \frac{1}{2\ell} \mathbb{A}_{\ell-1} & \text{if } L/\mathbb{Q} \text{ is tame} \\ \frac{1}{2\ell} \mathbb{A}_{\ell-1}^{(\ell-1)} & \text{if } L/\mathbb{Q} \text{ has wild ramification.} \end{cases}$$

**Proof.** The isometry on the left hand has been obtained in Proposition 2.6. Using Theorem 2.8 and the isometry on the left, one obtains the second isometry in the case $L/\mathbb{Q}$ is a tame extension. The isometry on the right in the wildly ramified case is ensured by Theorem 2.16. To conclude, we only have to show that in the case of wild ramification we have an embedding

$$\left\langle O_L, \frac{1}{\rad(d_L)} q_L \right\rangle \hookrightarrow \mathbb{A}_{\ell-1}.$$

Using Lemma 2.15 and the part of this Theorem we already proved, this is equivalent to verify the existence of an embedding

$$I_{\alpha_{\ell}} \hookrightarrow \mathbb{A}_{\ell-1}.$$

Recall the notation introduced in subsection 2.3.1.

$$\alpha_{\ell} = \left(\zeta_{\ell} - \frac{1}{\zeta_{\ell}}\right)^{2(\ell-1)} \ell^2$$

and

$$\delta_{\ell} = \frac{2 - \zeta_{\ell} - \zeta_{\ell}^{-1}}{\ell}.$$

Since the root lattice $\mathbb{A}_{\ell-1}$ is isometric to $I_{\delta}$ (see [Ba-Su, §3 The root lattice $\mathbb{A}_{p-1}$]) the existence of an isometric embedding (2) is equivalent to the existence of an embedding

$$I_{\alpha_{\ell}} \hookrightarrow I_{\delta_{\ell}}.$$

The above is ensured thanks to Corollary 2.12.  

\[\square\]
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