A coupled-channel analysis of $D^+ \to K^-\pi^+\pi^+$ decay

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Abstract

We perform a coupled-channel analysis of pseudo-data for the $D^+ \to K^-\pi^+\pi^+$ Dalitz plot. The pseudo-data are generated from the isobar model of the E791 Collaboration, and are reasonably realistic. We demonstrate that it is feasible to analyze the high-quality data within a coupled-channel framework that describes the final state interaction of $D^+ \to K^-\pi^+\pi^+$ as multiple rescatterings of three pseudoscalar mesons through two-pseudoscalar-meson interactions in accordance with the two-body and three-body unitarity. The two-pseudoscalar-meson interactions are designed to reproduce empirical $\pi\pi$ and $\pi\bar{K}$ scattering amplitudes. Furthermore, we also include mechanisms that are beyond simple iterations of the two-body interactions, i.e., three-meson-force, derived from the hidden local symmetry model. A picture of hadronic dynamics in $D^+ \to K^-\pi^+\pi^+$ described by our coupled-channel model is found to be quite different from those of the previous isobar-type analyses. For example, we find that the $D^+ \to K^-\pi^+\pi^+$ decay width gets almost quadruplicated when the rescattering mechanisms are turned on. Among the rescattering mechanisms, we find that those associated with the $\rho(770)\bar{K}^0$ channel, which contribute to $D^+ \to K^-\pi^+\pi^+$ only through a channel-coupling, play a major role. We also find that the dressed $D^+$ decay vertices have complex phases, induced by the strong rescatterings, that strongly depend on the momenta of the final pseudoscalar mesons. Although the conventional isobar-type analyses have assumed the phases to be constant, this common assumption is not supported from our more microscopic viewpoint.

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I. INTRODUCTION

With the advent of Charm- and B-factories, a large amount of data for charmed-meson decays have been accumulated in the last decades. Among a number of physical interests, one appealing aspect of studying these charmed-meson decays is that we can gain information about interactions between light mesons and resonances thereby. This was particularly highlighted by the E791 Collaboration’s report on their identification of the $\rho$-meson in the Dalitz plot of the $D^+ \rightarrow \pi^-\pi^+\pi^+$ decay [1]. A similar analysis was also made for the $D^+ \rightarrow K^-\pi^+\pi^+$ decay to identify the $\kappa$ resonance [2]. These findings triggered further analyses of the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot data, paying special attention to the $K^-\pi^+$ s-wave amplitude, as follows. Oller [3] analyzed the E791 data [2] using the $K\pi I=1/2$ ($I$: total isospin) s-wave amplitude based on the chiral unitary approach [4], instead of Breit-Wigner functions for the $K^-\pi^+$ s-wave used in the E791 analysis, and obtained a reasonable fit. The $I=3/2$ s-wave was not considered in his analysis. The E791 Collaboration reanalyzed their data without assuming a particular functional form for the $K^-\pi^+$ s-wave amplitude. Rather, they determined it bin by bin, which they call model independent partial wave analysis (MIPWA) [5]. An interesting finding in the MIPWA was that the obtained $K^-\pi^+$ s-wave amplitude has the phase that depends on the $K^-\pi^+$ energy in a manner significantly different from what is expected from the Watson theorem combined with the LASS empirical amplitude [6], assuming the $I=1/2$ dominance. Edera et al. suggested that this difference can be understood as a substantial mixture of the $K\pi I=1/2$ and $3/2$ s-wave amplitudes [7]. This idea was implemented in the Dalitz plot analysis done by the FOCUS Collaboration [8]. They parametrized the $K^-\pi^+$ s-wave amplitude in terms of the $K$-matrix of $I=1/2$ and $3/2$ that had been fitted to the amplitudes of the LASS [6] and of Estabrooks et al. [9]. They found that the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot can be well fitted with the $K^-\pi^+$ s-wave amplitude in which the $I=1/2$ and $3/2$ components interfere with each other in a rather destructive manner. The FOCUS Collaboration also has done a MIPWA in a subsequent work [10] to find results similar to those of the E791 MIPWA. The quasi-MIPWA has also been done by the CLEO Collaboration [11]. Their new finding was that $I=2$ $\pi\pi$ nonresonant amplitude can give a non-negligible contribution. Meanwhile, an analogous decay, $D^+ \rightarrow K_S^0\pi^0\pi^+$, has been analyzed by the BESIII Collaboration [12]. An interesting finding was that the $\rho(770)\bar{K}$ channel gives by far a dominant contribution. This implies that, although the $\rho(770)\bar{K}$ contribution is not directly observed in the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot, it can play a substantial role in the hadronic final state interactions (FSI) of the $D^+ \rightarrow K^-\pi^+\pi^+$ decay, considering that the two $D^+$ decays share the same hadronic dynamics to a large extent.

Many of the previous analyses of the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot have been done with the so-called isobar model in which a $D^+$-meson decays into an excited state $R$ ($\bar{K}, \bar{K}^*, \bar{K}_2^*$, etc.) and a pseudoscalar meson. The $R$ subsequently decays into a pair of light-pseudoscalar-mesons, while the third pseudoscalar meson is treated as a spectator. The propagation of $R$ is commonly parametrized by a Breit-Wigner function. The total decay amplitude is given by a coherent sum of these isobar amplitudes supplemented by a flat interfering background. On the other hand, as mentioned in the above paragraph, some analysis groups modified the conventional isobar model to use the $K$-matrix or chiral unitary model for the $K^-\pi^+$ s-wave amplitude so that the consistency with the LASS data for $K^-\pi^+ \rightarrow K^-\pi^+$ and with the two-body unitarity is maintained. Meanwhile, in (Q)MIPWA, the $K^-\pi^+$ s-wave amplitude is solely determined by the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot data. We will refer to these models, which do not explicitly consider three-meson-rescattering mechanisms, as isobar-
type models. The basic assumptions common to all of these models are that the spectator pseudoscalar meson interacts with the others very weakly, and/or $D^+ \to Rc$ ($c$: spectator meson) vertices with constant complex phases can absorb such effects. Thus multiple-scattering and channel-couplings required by the three-body unitarity are not (explicitly) taken into account.

Although each analysis group has obtained a reasonable fit to its own data, their results are not necessarily in conformity with theoretical expectations. For example, the E791 Collaboration reported that the phase of the $I=1/2 \, \bar{K} \pi$ $p$-wave amplitude used in their MIPWA is, according to the Watson theorem, not consistent with that of the LASS analysis in the elastic region [5]. This may be originated from either or both of two possible reasons below, each of which signals a serious problem in the basic assumptions underlying in the analysis model. One possible reason is that the $I=1/2 \, \bar{K} \pi$ $p$-wave amplitude in the E791 analysis is given by a coherent sum of Breit-Wigner functions for $\bar{K}^*(892)$ and $\bar{K}^*(1680)$, and it is not consistent with the LASS data to the required precision. Or the amplitude does not satisfy the two-body unitarity not only formally but also quantitatively. Another possibility is that the (neglected) rescattering of the spectator meson with the other mesons plays a substantial role to generate an energy-dependent phase so that the Watson theorem does not hold. Not only the E791 analysis but also all the previous analyses mentioned above would share the same problem, and this seems to indicate a need for going beyond the conventional isobar-type analysis; a unitary coupled-channel approach. In order to extract from data a right physics, e.g., $K^-\pi^+$ $s$-wave amplitude from $D^+ \to K^-\pi^+\pi^+$ Dalitz plot data, one needs to use a theoretically sound model so that unnecessary model artifact does not come into play.

Recently, we have developed a unitary coupled-channel framework for describing a heavy-meson decay into three light hadrons [13]; both the two-body and three-body unitarity are maintained. In the reference, we studied the extent to which the isobar-type description of heavy-meson (or excited meson) decays is valid by analyzing simple pseudo-data. We found a significant effect of the channel-couplings and multiple-rescattering on the Dalitz plot distributions. This study has been extended to an analysis of pseudo-data for excited meson photoproductions [14]. In this work, we apply the formalism of Ref. [13] with some modifications to a realistic case. Thus, we will perform a coupled-channel analysis of the $D^+ \to K^-\pi^+\pi^+$ Dalitz plot pseudo-data generated from the E791’s isobar model [5]. To the best of our knowledge, this is the first coupled-channel Dalitz plot analysis of a $D$-meson decay into three pseudoscalar mesons. We will demonstrate that a quantitative coupled-channel partial wave analysis of the $D^+ \to K^-\pi^+\pi^+$ Dalitz plot is feasible. Then we will examine the hadronic dynamics in the FSI of the $D^+ \to K^-\pi^+\pi^+$ decay within the coupled-channel model. We will study how the fraction of each channel’s contribution is different between an isobar-type model and our full model that includes the three-body scattering. We also examine contributions from the rescattering and channel-couplings to the Dalitz plot distribution. Through these investigations, we address the validity of the above-mentioned basic assumptions of the isobar-type model from this more microscopic viewpoint. The three-body FSI for the $D^+ \to K^-\pi^+\pi^+$ decay has been explored by Magalhães et al. [15] and also by Guimarães et al. [16]. However, their models were still in restricted settings, and more extensions are needed to confront their calculations with the data.

In our analysis, we use two-pseudoscalar-meson interactions that generate unitary amplitudes for $\bar{K} \pi$ and $\pi\pi$ scatterings, and consider resonances ($\bar{k}, \bar{K}^*, \rho$, etc.) as poles in the amplitudes. The two-body interactions are fitted to the LASS and CERN-Munich [17–19] data.
With the two-body interactions, we solve the Faddeev equation for a three-pseudoscalar-meson scattering to obtain an amplitude that respects the three-body unitarity and thus channel-couplings. This amplitude is used to describe the FSI of the $D^+$ decay. Although we also add a flat interfering background amplitude that violates the three-body unitarity to some extent, we will see that the background contribution is very small in our full model. The pseudo-data are fitted by adjusting the strengths and phases of (bare) $D^+ \rightarrow Rc$ vertices, while the two-pseudoscalar-meson interactions are fixed as those obtained by fitting the two-body scattering data. This is in contrast with the previous $D^+ \rightarrow K^-\pi^+\pi^+$ analyses where some of the resonance parameters were also adjusted along with the $D^+ \rightarrow Rc$ vertices. Rather, we analyze the $D^+ \rightarrow K^-\pi^+\pi^+$ decay keeping the consistency with the two-body scattering data for all channels considered in our model. Then we will examine the extent to which we can fit the $D^+$-decay pseudo-data. We consider the $I=1/2 \bar K\pi s-$, $p-$, and $d-$waves as commonly included in the previous analyses. We also explicitly include the $I=3/2 \bar K\pi s$-wave ($I=2 \pi\pi s$-wave) as has been done so in the FOCUS [8] (CLEO [11]) analysis. Furthermore, we consider the $I=1 \pi\pi p$-wave where the $\rho(770)$ plays a major role. This channel has not been considered in the previous analyses because it does not directly decay into the $K^-\pi^+\pi^+$ final state. However, this channel can still contribute to $D^+ \rightarrow K^-\pi^+\pi^+$ through the channel-couplings. Considering the BESIII analysis mentioned above, the $\rho(770)K$ channel is expected to play a substantial role also here, and we will see that this is indeed the case, at least within our analysis.

Now we discuss the last piece of our model. The FSI of the $D^+ \rightarrow K^-\pi^+\pi^+$ decay is a three-pseudoscalar-meson scattering. Generally in a three-body scattering, there can exist a mechanism that cannot be expressed by a combination of two-body mechanisms, i.e., a three-body force. Based on the hidden local symmetry (HLS) model [20], in which vector and pseudoscalar mesons are implemented together in a chiral Lagrangian, we can actually derive “three-meson-force” essentially in a parameter-free fashion, up to form factors we include. Thus we consider some of the HLS-based three-meson-force acting on important channels, and examine how they play a role in the $D^+ \rightarrow K^-\pi^+\pi^+$ decay. If two-pseudoscalar-meson interactions are well determined by precise two-body scattering data, the $D \rightarrow \bar K\pi\pi$ decays and also other decay modes such as $D \rightarrow \pi\pi\pi$ could serve as a ground to study the three-meson-force.

The organization of this paper is briefly as follows. In Sec. II, we discuss our coupled-channel model, and present formulae to calculate the $D^+ \rightarrow K^-\pi^+\pi^+$ decay amplitude. Then in Sec. III, we present numerical results from our analyses of the two-pseudoscalar-meson scattering data, and of the $D^+ \rightarrow K^-\pi^+\pi^+$ decay Dalitz plot pseudo-data. Finally, we give a summary and future prospect in Sec. IV. A derivation and the resulting expressions for the three-meson-force, and model parameters are presented in the Appendices.

II. FORMULATION

We have already developed a formalism to describe a heavy-meson decay into three light mesons in Ref. [13]. In the present work, we basically use the same formalism with some modifications. Thus, here we just present expressions that are needed in the following discussions, specifying the modifications we make for this work. For derivations of most of the expressions, see Sec. II of Ref. [13]. Our formalism can be regarded as a three-dimensional reduction of a fully relativistic formulation. Because we deal with scatterings of light particles, i.e. pions, one may question the validity of this approximation. Although
FIG. 1. Diagrammatic representation of two-light-meson partial wave scattering, \( ab \to a'b' \). (a) two-meson interaction potentials; (b) two-meson partial wave scattering amplitude; (c) two-meson partial wave scattering amplitude from separable contact interactions; (d) dressed \( R \to ab \) decay vertex; (e) dressed \( R \) Green function.

A legitimate concern, we made an argument on this along with improvements needed in future in Sec. V of Ref. [13]. In our formalism, we first construct a two-light-meson \((\pi, K)\) interaction model that is subsequently applied to three-light-meson scattering. The following presentations are also given in this order.

**A. Two-light-meson scattering model**

We describe two-light-meson scatterings with a unitary coupled-channel model. For example, we consider \( \pi\pi \) and \( KK \) channels for a \( \pi\pi \) scattering, while \( \pi\bar{K} \) and \( \eta'\bar{K} \) channels for \( I=1/2 \) s-wave \( \pi\bar{K} \) scattering. We will specify channels for each partial wave later. We model the two-light-meson interactions with resonance\((R)\)-excitation mechanisms or contact interactions or both. We choose a parametrization of the two-meson interactions so that it can be easily applied to a three-meson scattering. A difference from Ref. [13] is that we include the contact interactions here but we did not in Ref. [13]. This is because, this time, we need to deal with partial waves that do not have resonances. Besides, we use a parametrization for \( R \to ab \) \((a,b: \text{pseudoscalar mesons})\) vertex functions that are different from those used in the previous work and, with this parametrization, we need to include contact interactions in addition to \( R\)-excitation mechanisms in order to obtain reasonable fits to some empirical partial wave amplitudes. Thus, we use expressions shown below to describe \( \pi\pi \) and \( K\pi \) scatterings.

We consider a partial wave \( ab \to a'b' \) scattering (see Fig. 1 for a diagrammatic representation) with total energy \( E \), total angular momentum \( L \), total isospin \( I \). We will also denote the incoming and outgoing momenta by \( q \) an \( q' \), respectively; \( q = |q| \) throughout this paper, except for the Appendix. When \( q \) is on-shell, it is related to \( E \) by \( E = E_a(q) + E_b(q) \) and \( E_a(q) = \sqrt{q^2 + m_a^2}; m_a \) being the mass of \( a \). First let us consider a \( ab \to a'b' \) scat-
tering that is not accompanied by a resonance excitation. In this case, we use a separable two-light-meson interaction potential [first term of r.h.s. of Fig. 1(a)] as follows:

\[ v_{a'b',ab}^{LI}(q',q) = w_{a'b'}^{LI}(q')h_{a'b',ab}^{LI} w_{ab}^{LI}(q), \]  

where \( h_{a'b',ab}^{LI} \) is a coupling constant, and \( w_{ab}^{LI}(q) \) is a vertex function. We use the following parametrization for \( w_{ab}^{LI}(q) \):

\[ w_{ab}^{LI}(q) = \frac{1}{\sqrt{E_a(q)E_b(q)}} \left[ \frac{1}{1 + (q/b_{ab}^{LI})^2} \right]^{2+L/2} \left( \frac{q}{m_\pi} \right)^L, \]  

where \( b_{ab}^{LI} \) is a cutoff parameter. Meanwhile, we allow an exception for \( L=0, I=2 \pi\pi \) scattering for which we use a different parametrization for the vertex function:

\[ w_{\pi\pi}^{LI}(q) = \frac{1 + h'(q/m_\pi)^2}{E_\pi(q)} \left[ \frac{1}{1 + (q/b_{\pi\pi}^{LI})^2} \right]^{3}, \]  

where an additional coupling constant \( h' \) has been introduced to obtain a reasonable fit to data. For a later convenience, let us define \( \tilde{w}_{ab}^{LI} \) by

\[ w_{ab}^{LI}(q) = \begin{cases} \frac{1}{\sqrt{2}} \tilde{w}_{ab}^{LI}(q) & \text{(if } a \text{ and } b \text{ are identical particles)}, \\ \tilde{w}_{ab}^{LI}(q) & \text{(otherwise)}. \end{cases} \]  

With the above interaction potential, the partial wave amplitude is given as follows [see Fig. 1(c) for a diagrammatic representation]:

\[ i_{a'b',ab}^{LI}(q',q;E) = \sum_{a'=a''} w_{a'a''}^{LI}(q')\tau_{a'a''ab}^{LI}(E) h_{a'a''ab}^{LI} w_{ab}^{LI}(q), \]  

with

\[ \left[(\tau^{LI}(E))^{-1}\right]_{a'b',ab} = \delta_{a'b',ab} - \Sigma_{a'b',ab}^{LI}(E), \]

\[ \Sigma_{a'b',ab}^{LI}(E) = \int_0^\infty dq q^2 \frac{h_{a'a''ab}^{LI}}{E - E_a(q) - E_b(q) + i\epsilon} \left[w_{ab}^{LI}(q)\right]^2. \]

The amplitude of Eq. (5) can contain resonance pole(s), in general.

Now we extend the model to include explicit degrees of freedom for excitations of spin-\( L \) isospin-\( I \) resonances that contribute to the \( ab \rightarrow a'b' \) scattering. In this case, a two-light-meson interaction potential includes bare \( R \)-excitation mechanisms in addition to the contact potential \( v_{a'b',ab}^{LI} \) defined in Eq. (1) as follows [see also Fig. 1(a)]:

\[ V_{a'b',ab}^{LI}(q',q;E) = v_{a'b',ab}^{LI}(q',q) + \sum_R \tilde{f}_{a'b',ab}^{RL}(q') \frac{1}{E - m_R} \tilde{f}_{R,ab}^{LI}(q), \]  

where \( m_R \) is the bare mass of \( R \); \( \tilde{f}_{ab,ab}^{RL}(q) \) denotes a bare \( R \rightarrow ab \) vertex function and \( \tilde{f}_{R,ab}^{LI}(q) = \tilde{f}_{R,ab}^{LI*}(q) \). Following Ref. [13], we introduce \( \tilde{f}_{ab,ab}^{LI}(q) \) that is related to \( \tilde{f}_{ab,ab}^{LI}(q) \) as follows:

\[ \tilde{f}_{ab,ab}^{LI}(q) = \begin{cases} \frac{1}{\sqrt{2}} \tilde{f}_{ab,ab}^{LI}(q) & \text{(if } a \text{ and } b \text{ are identical particles)}, \\ \tilde{f}_{ab,ab}^{LI}(q) & \text{(otherwise)}. \end{cases} \]
Then we employ the following parametrization for the bare vertex function:

\[
\tilde{f}^{LI}_{ab,R}(q) = g_{ab,R} \frac{m_\pi}{\sqrt{E_a(q) E_b(q)}} \left[ \frac{1}{1 + (q/c_{ab,R})^2} \right]^{1+(L/2)} \left( \frac{q}{m_\pi} \right)^L,
\]

where \(g_{ab,R}\) and \(c_{ab,R}\) are coupling constant and cutoff, respectively. This parametrization is different from that used in Eq. (35) of Ref. [13], and has a proper kinematical factor. With the potential in Eq. (8), the resulting scattering amplitude is given by [Fig. 1(b)]:

\[
T^{LI}_{a'b',ab}(q', q; E) = t^{LI}_{a'b',ab}(q', q; E) + \sum_{R',R} \tilde{f}^{LI}_{a'b',R}(q', E) \tau^{LI}_{R',R}(E) \tilde{f}^{LI}_{R,ab}(q; E),
\]

where the first term is the scattering amplitude from the contact interactions only, as has been defined in Eq. (5). The symbol \(\tilde{f}_{ab,R}\) denotes the dressed vertex that describes the bare \(R \to ab\) decay followed by \(ab\) rescattering through the contact interactions. Expressions for \(\tilde{f}_{ab,R}\) and \(\tilde{f}_{R,ab}\) are [Fig. 1(d)]:

\[
\tilde{f}^{LI}_{ab,R}(q; E) = \frac{\tilde{f}^{LI}_{ab,R}(q) + \sum_{a'b'} \int_0^\infty dq' q'^2 t^{LI}_{a'b',R}(q', q; E) \frac{t^{LI}_{a'b',R}(q')}{E - E_a(q') - E_b(q') + i\epsilon}}{E - E_a(q') - E_b(q') + i\epsilon},
\]

\[
\tilde{f}^{LI}_{R,ab}(q; E) = \frac{\tilde{f}^{LI}_{R,ab}(q) + \sum_{a'b'} \int_0^\infty dq' q'^2 \frac{t^{LI}_{R,a'b'}(q')}{E - E_a(q') - E_b(q') + i\epsilon}}{E - E_a(q') - E_b(q') + i\epsilon}.
\]

In Eq. (11), the dressed Green function for \(R\), \(\tau^{LI}_{R',R}(E)\), has been introduced, and is given by [Fig. 1(e)]:

\[
[(\tau^{LI}(E))^{-1}]_{R'R} = (E - m_R) \delta_{R'R} - \bar{\Sigma}^{LI}_{R'R}(E),
\]

where \(\bar{\Sigma}^{LI}_{R'R}(E)\) is the self-energy of \(R\), and is defined by

\[
\bar{\Sigma}^{LI}_{R'R}(E) = \sum_{ab} \int_0^\infty dq q^2 \cdot \frac{\tilde{f}^{LI}_{R',ab}(q) \tilde{f}^{LI}_{ab,R}(q; E)}{E - E_a(q) - E_b(q) + i\epsilon}.
\]

In case \(ab \to a'b'\) interaction is given by only resonant mechanisms [no first term in Eq. (8)], which is the case in Ref. [13], the corresponding scattering amplitude is obtained from Eq. (11) by dropping the first term, and replacing the dressed vertex (\(\tilde{f}\)) with the bare one (\(f\)) in Eqs. (11) and (15).

The partial wave amplitude, \(T^{LI}_{a'b',ab}\) in Eq. (11), is related to the \(S\)-matrix by

\[
s^{LI}_{ab,ab}(E) = \eta_{LI} e^{2i\delta_{LI}} = 1 - 2\pi i \rho_{ab} T^{LI}_{ab,ab}(q_o, q_o; E),
\]

where \(q_o\) is the on-shell momentum that satisfies \(E = E_a(q_o) + E_b(q_o)\), and \(\rho_{ab} = q_o E_a(q_o) E_b(q_o) / E\) is the phase-space factor. The phase shift and inelasticity are denoted by \(\delta_{LI}\) and \(\eta_{LI}\), respectively.

The parameters contained in the two-light-meson potentials such as \(m_R, g_{ab,R}, c_{ab,R}, h^{LI}_{a'b',ab}\), and \(b^{LI}_{ab}\) are determined by fitting experimental data. A particular choice of the potential, such as the number of \(R\) and contact interactions included, will be specified later for each partial wave and for each of \(\pi\pi\) and \(K\pi\) interactions.
B. Three-light-meson scattering model

We now consider a case where three light-mesons are scattering. First, let us assume that the three mesons interact with each other only through the two-meson interactions discussed in the previous subsection. Because our two-meson interaction potential is given in a separable form, we can cast the Faddeev equation into a two-body like scattering equation (the so-called Alt-Grassberger-Sandhas (AGS) equation [21]) for a $cR \to c'R'$ scattering. Here, $R$ stands for either $R$ or $r_{ab}$, and $r_{ab}$ is a spurious “state” that is supposed to live within a contact interaction in a very short time, and decays (going to the left in equations) into the two light-mesons, $ab$. This degree of freedom is introduced merely for extending the AGS-type $cR \to c'R'$ scattering equation used in Ref. [13] to include the contact two-meson interactions. Thus, the scattering equation for a partial wave amplitude, $T^{\ell \ell' \ell''}_{(cR')_{l},(cR)_{l'}}(p',p;E)$, is given as (see Fig. 2 for a diagrammatic representation)

$$T^{\ell \ell' \ell''}_{(cR')_{l},(cR)_{l'}}(p',p;E) = Z^{cR'}_{cR'}(p',p;E) + \sum_{c',cR''_l,cR''_m} \int_{0}^{\infty} q^2 dq Z^{cR''_l}_{cR''_m,cR''_m} (p',q;E) \times G_{cR''_l,cR''_m} (q,E) T^{\ell' \ell''}_{(cR'')_{l'},(cR)_{l'}}(q,p;E),$$

(17)

where $JPT$ are the total angular momentum, parity, and the total isospin of the $cR$ system and they are conserved in the scattering. The $cR$ state with the relative orbital angular momentum $l$ is denoted by $(cR)_l$; the allowed range for $l$ is determined by $JP$ and the spin-parity of $R$. The magnitude of the incoming (outgoing) relative momentum of the $cR$ ($c'R'$) state is denoted by $p$ ($p'$). The driving term of the scattering is a partial wave form of the so-called Z-diagram, $Z^{cR'}_{cR'}(p',p;E)$. The Z-diagram is a process in which $R \to cR'$ decay is followed by $\bar{c}c \to R'$-formation, as pictorialized in Fig. 3. For a more explicit definition as well as the partial wave expansion of the Z-diagram, we refer the readers to Appendix C of Ref. [13]; in particular, Eqs. (C10)-(C12) of Ref. [13] give an explicit expression for $Z^{cR'}_{cR'}(p',p;E)$ (note $\mathcal{R}=\mathcal{R}$ and $\mathcal{R}'=\mathcal{R}'$) in which $\tilde{f}^{L\ell}_{c\bar{c},R}$ and $\tilde{f}^{L\ell}_{c\bar{c},R'}$ defined in Eq. (10) can be directly plugged in. When $\mathcal{R} = r_{c'\bar{c}}$ ($\mathcal{R}' = r_{c''\bar{c}'}$), the corresponding expression for the Z-diagram can be practically obtained by replacing $\tilde{f}^{L\ell}_{c\bar{c},R}$ ($\tilde{f}^{L\ell}_{c\bar{c},R'}$) in $Z^{cR'}_{cR'}(p',p;E)$ with $\tilde{w}^{L\ell}_{c\bar{c}} (l^{R}_{a'b'\bar{c}'\bar{c}'}, \bar{c}^{L\ell}_{\bar{c}c})$ defined in Eq. (4). The Z-diagrams are known to have the moon-shape singularity [22] that prevents us from solving Eq. (17) with the standard subtraction method. Here we employ the spline method (see Ref. [22] for...
detailed explanations) to obtain numerical solutions from Eq. (17).

In Eq. (17), we have also used the Green function, \( G_{(cR'),(cR)}(q, E) \), for \( R \) and \( R' \) which can be coupled through \( R \to ab \to R' \). It is given by

\[
(G^{-1}(q, E))_{(cR'),(cR)} = \left[ E - E_c(q) - E_R(q) \right] \delta_{R',R} - \Sigma_{R',R}^{LI} (q, E - E_c(q)),
\]

for \( (R, R') = (R, R') \), and

\[
(G^{-1}(q, E))_{(cR'),(cR)} = -\Sigma_{R',R}^{LI} (q, E - E_c(q)) \quad \text{for} \quad (R, R') = (R, r_{ab}R'),
\]

\[
(G^{-1}(q, E))_{(cR'),(cR)} = \delta_{r_{ab},r_{a'b'}} - \Sigma_{R',R}^{LI} (q, E - E_c(q)) \quad \text{for} \quad (R, R') = (r_{ab}, R'),
\]

where we have introduced the self-energies, \( \Sigma_{RI',R}^{LI} (q, E) \), which are given by

\[
\Sigma_{RI',R}^{LI} (p, E) = \sum_{ab} \sqrt{\frac{m_{R'}m_R}{E_{R'}(p)E_R(p)}} \int_0^\infty q^2 dq \frac{M_{ab}(q)}{\sqrt{M_{ab}^2(q) + p^2}} \frac{f_{R',ab}^{LI}(q)f_{ab,R}^{LI}(q)}{E - \sqrt{M_{ab}^2(q) + p^2} + i\epsilon},
\]

\[
\Sigma_{R',ab}^{LI} (p, E) = \sum_{ab} \sqrt{\frac{m_R}{E_{R}(p)}} \int_0^\infty q^2 dq \frac{M_{ab}(q)}{\sqrt{M_{ab}^2(q) + p^2}} \frac{f_{R',ab}^{LI}(q)w_{ab}^{LI}(q)}{E - \sqrt{M_{ab}^2(q) + p^2} + i\epsilon},
\]

\[
\Sigma_{R',r_{ab}}^{LI} (p, E) = \int_0^\infty q^2 dq \frac{M_{ab}(q)}{\sqrt{M_{ab}^2(q) + p^2}} \frac{f_{R',ab}^{LI}(q)w_{ab}^{LI}(q)}{E - \sqrt{M_{ab}^2(q) + p^2} + i\epsilon},
\]

with \( M_{ab}(q) = E_a(q) + E_b(q) \), and \( \sum_{ab} \) runs over all two-meson states from \( R \to ab \) decays. The kinematical factors in the expressions are from the Lorentz transformation to boost the \( R \)-at-rest frame to the \( cR \) center-of-mass frame.

So far, we have considered the three-meson scattering due to multiple iterations of the two-meson interactions. Given the two-meson interactions, this is a necessary consequence of the three-body unitarity. In a three-meson system, however, there may be a room for two-meson interactions. Given the two-meson interactions, this is a necessary consequence that is absent in a two-meson system to play a role. We will refer to such mechanisms as three-meson-force hereafter. Diagrams shown in Fig. 4 can work as a three-meson-force. These are interactions between a vector-meson and a pseudoscalar meson via a vector-meson exchange; for our particular application to \( D^+ \to K^-\pi^+\pi^+ \), they are bare \( \rho-K \), and bare \( K^*-\pi \) interactions. These mechanisms are based on the hidden local symmetry (HLS) model [20] in which vector and pseudoscalar mesons are implemented in a Lagrangian that has a symmetry under nonlinear chiral transformations. Expressions for the Lagrangian and the resulting interaction potentials of Fig. 4 are presented in Appendix A. These mechanisms in Fig. 4 along with the Z-diagram in Fig. 3 have been studied by Jansen et al. [23–25] to examine the \( \pi-\rho \) correlation and its relevance to a soft \( \pi NN \) form factor in a \( NN \) potential. There are also other possible mechanisms that can work as a three-meson force. We show some diagrams in Fig. 5 as examples. The diagram in Fig. 5(a) describes an interaction between a pseudoscalar-meson-pair \( (ab) \) in \( s \)-wave and another pseudoscalar meson \( (c) \) via a vector-meson exchange; this mechanism is also from the HLS Lagrangian. Meanwhile, in the diagram of Fig. 5(b), a \( R \) interacts with a pseudoscalar meson to form a resonance \( (M^*) \), which is followed by a decay into a \( R' \) and a pseudoscalar meson. This is a familiar mechanism and often assumed in partial wave analyses for meson spectroscopy.
FIG. 4. Vector ($V$) and pseudoscalar ($P$) mesons interaction potentials based on the hidden local symmetry model [20]. Vector mesons ($V_{ex}$) are exchanged.

FIG. 5. Three-meson-force not considered in this work. (a) Vector-meson exchange between a pseudoscalar-meson-pair ($ab$) in $s$-wave and a pseudoscalar meson ($c$). (b) $Rc$-interaction via meson-resonance ($M^*$) excitation.

In this work, we consider the vector-pseudoscalar interactions shown in Fig. 4 in our analysis of the $D^+ \rightarrow K^- \pi^+\pi^+$ decay to study their relevance. Thus, the scattering equation in Eq. (17) is modified by adding the new mechanisms to the $Z$-diagrams:

$$Z^{e,JPT}_{(c' R')\rho,(c R)\rho}(p', p; E) \rightarrow Z^{e,JPT}_{(c' R')\rho,(c R)\rho}(p', p; E) + V^{e,JPT}_{(c' R')\rho,(c R)\rho}(p', p) \quad \text{in Eq. (17),} \quad (26)$$

where the added term, $V^{e,JPT}_{(c' R')\rho,(c R)\rho}(p', p)$, is in the partial wave form for which we give explicit expressions in Eqs. (A8), (A12), and (A17). In Eq. (26), $R$ and $R'$ are the lightest spin-1 bare states of either ($I, S$ [strangeness]) = (1, 0) or ($I, S$) = (1/2, −1) which we denote “$\rho$” and “$\bar{K}^*$”, respectively. For our particular application to the $D^+ \rightarrow K^- \pi^+\pi^+$ decay, we include ($VP, V'P', V_{ex}$) = ("$\rho$" $\bar{K}$, "$\rho$" $\bar{K}$, $\rho$), ("$\rho$" $\bar{K}$, "$\bar{K}^*$") $\pi$, $K^*$, ("$\bar{K}^*$") $\pi$, "$\bar{K}^*$", $\pi$, $\rho$) for the diagram Fig. 4(a), and ($VP, V'P', V_{ex}$) = ("$\rho$" $\bar{K}$, "$\rho$" $\bar{K}$, $\pi$), ("$\rho$" $\bar{K}$, "$\bar{K}^*$") $\pi$, $\omega$, ("$\bar{K}^*$") $\pi$, "$\bar{K}^*$", $\pi$) for the diagram Fig. 4(b). On the other hand, we leave examination of mechanisms such as those shown in Fig. 5 to future work for the reasons in the following. As we emphasized in the introduction, even effects of multiple scattering due to the two-meson-force on the $D$-decay has still not been studied in a realistic setting. In this situation, studying the relevance of the three-meson-force to the $D$-decay is indeed in an exploratory level, and thus a reasonable starting point would be to include it in the most important channel. As we will see, the vector-pseudoscalar ($\rho$-$\bar{K}$) channel plays the most important role in the rescattering process, and therefore the new mechanisms of Fig. 4 in this channel should be studied at first. Also, regarding the mechanism in Fig. 5(b), no relevant meson-resonance of spin-0, $I=3/2$, $S=1$ is known in the $D$-meson mass region, and thus we do not need to include it for the moment.
FIG. 6. Diagrammatic representation of $D^+$-decay into three pseudoscalar mesons $a, b, c$ in our coupled-channel model. (a) The isobar-type diagram; (b) The rescattering diagram. The amplitude $T'$ is from the scattering equation represented by Fig. 2. The gray blob represents the dressed $\mathcal{R}$ Green function.

C. $D^+ \to K^-\pi^+\pi^+$ decay amplitude

In our coupled-channel model, the decay amplitude for $D^+ \to K^-\pi^+\pi^+$ is given by

$$T_{K^-\pi^+\pi^+,D^+}(p_{K^-}, p_{\pi^+}, p_{\pi^+}; E = m_{D^+}) = \sum_{(a b c)} T_{(a b c),D^+}(p_a, p_b, p_c; E),$$

where we have introduced the cyclic summation that takes the sum over $abc = K^-\pi^+_1\pi^+_2$, $\pi^+_1\pi^+_2 K^-$, $\pi^+_2 K^-\pi^+_1$, and

$$T_{(a b c),D^+}(p_a, p_b, p_c; E) = \sum_{\mathcal{R}, \mathcal{R}', S_R^z} f_{a b, \mathcal{R}}^{S_R^z}(p_a, p_b) G_{c R, c R'}(p_c, E) \bar{\Gamma}_{c R', D^+}^{S_R^z}(p_c, E),$$

where $S_R^z$ is $z$-component of the spin state of $\mathcal{R}$, and $G_{c R, c R'}$ is the Green function that has been defined in Eqs. (18)-(21). This decay amplitude in Eq. (28) is diagrammatically represented in Fig. 6. The symbol $f_{a b, \mathcal{R}}^{S_R^z}$ denotes a $\mathcal{R} \to ab$ decay vertex function which is explicitly given as

$$f_{a b, \mathcal{R}}^{L_z}(p_a, p_b) = \sqrt{m_{R} E_{a}(q) E_{b}(q) \left( t_a t_b I, t_a + t_b \right) Y_{L,L_z}(q) \tilde{f}_{a b, \mathcal{R}}^{L_z}(q)} \quad \text{for} \quad \mathcal{R} = R,$$

$$f_{a b, \mathcal{R}}^{L_z}(p_a, p_b) = \sqrt{E_{a}(q) E_{b}(q) \left( t_a t_b I, t_a + t_b \right) Y_{L,L_z}(q) \tilde{w}_{a b, \mathcal{R}}^{L_z}(q)} \quad \text{for} \quad \mathcal{R} = r_{a b},$$

where $\tilde{f}_{a b, \mathcal{R}}^{L_z}$ and $\tilde{w}_{a b, \mathcal{R}}^{L_z}$ have been defined in Eqs. (9) and (4), respectively; $\langle j_1 m_1 j_2 m_2 | J M \rangle$ is the Clebsch-Gordan coefficient, and $t_a$ is the isospin of meson $a$ and $t^*_a$ is its $z$ component. The kinematical factors in the equations are from the Lorentz transformation to boost the $ab$-pair center-of-mass (CM) frame to the total CM frame. The momentum $q$ is the relative momentum of the $ab$-pair in their CM frame. The dressed $D^+ \to \mathcal{R'}c$ vertex, $\tilde{\Gamma}_{c \mathcal{R}', D^+}^{S_R^z}$, has also been introduced in Eq. (28), and it is explicitly given by

$$\tilde{\Gamma}_{c \mathcal{R}', D^+}^{S_R^z}(p_c, E) = \sum_{p T} \sum_{l_t l_z} \langle l^z s_{R}, s_{R'} | S_D S_D^z | T t_a^z + t_b^z t_c^z | T t_a^z + t_b^z + t_c^z \rangle \times Y_{l_t, l_z}(\hat{p}_c) \bar{F}_{c \mathcal{R}', D^+}^{S_D P T}(p_c, E),$$

(31)
where \( S_D(=0) \) is the \( D \)-meson spin, and \( t_a^x + t_b^x + t_c^x = 3/2 \) for \( D^+ \to K^-\pi^+\pi^+ \). We sum over the parity \( (P) \) and total isospin \( (T) \) of the final hadronic states because the weak \( D \)-decay does not conserve them. The last factor in the above equation is given by

\[
F^\text{JPT}_{(c'R')_l,D^+}(p_c, E) = F^\text{JPT}_{(c'R')_l,D^+}(p_c) + \sum \int_0^\infty dp_{c'} p_{c'}^2 \frac{\partial^2 T^\text{JPT}_{(c'R')_l,c'R''_m}(p_c, p_{c'}; E)}{\partial E_{c'}} \times G_{c'R'',c'R''_m}(p_{c'}, E) F^\text{JPT}_{(c'R''_m)_l,D^+}(p_{c'}) ,
\]

where \( T^\text{JPT}_{(c'R')_l,c'R''_m} \) is the partial wave amplitude for \( c'R'' \to c'R \) scattering obtained by solving the coupled-channel scattering equation, Eq. (17). The first term on the r.h.s. corresponds to the isobar-type contribution [Fig. 6(a)] while the second term is the contribution from the rescattering [Fig. 6(b)]. The quantity \( F^\text{JPT}_{(c'R')_l,D^+}(p_c) \) is the bare \( D^+ \to (c'R)_l \) vertex function for which we choose a parametrization,

\[
F^\text{JPT}_{(c'R')_l,D^+}(p) = \frac{A_{c'}}{(2\pi)^{3/2}} \frac{C^\text{JPT}_{(c'R')_l}}{\sqrt{2E_c(p)}} \exp \left[ i \phi^\text{JPT}_{(c'R')_l} \right] \left( \frac{(\Lambda^\text{JPT}_{(c'R')_l})^2}{(A_{c'}^2 + (\Lambda^\text{JPT}_{(c'R')_l})^2)^2} \right)^{2+(l/2)} \left( \frac{p}{m_\pi} \right)^{l} ,
\]

with

\[
A_{c'} = \sqrt{\frac{m_{c'}}{2E_c(p)}} \quad \text{for} \quad c' = R , \quad A_{c'} = 1 \quad \text{for} \quad c' = r_{ab} .
\]

In Eq. (33), \( C^\text{JPT}_{(c'R')_l} \), \( \phi^\text{JPT}_{(c'R')_l} \), and \( \Lambda^\text{JPT}_{(c'R')_l} \) are the coupling, phase, and cutoff, respectively, and they will be determined by fitting Dalitz plot distribution data. The couplings \( C^\text{JPT}_{(c'R')_l} \) are nonzero only when \( |S_D - s_{c'}| \leq l \leq S_D + s_{c'} \). The parametrization used in this work [Eq. (33)] is different from the one used in Ref. [13] in choosing the kinematical factor.

The isobar-model-type amplitude [Fig. 6(a)] for the \( D^+ \to K^-\pi^+\pi^+ \) decay, \( T^\text{Isobar}_{(ab)c,R^+} \), is obtained from the above equations (27)-(34) by just dropping the second term of r.h.s. of Eq. (32).

The procedure to calculate the Dalitz plot distribution from the decay amplitude of Eq. (27) is explained in detail in Appendix B or Ref. [13], and we do not repeat it here.

III. ANALYSIS AND RESULTS

Now we apply the coupled-channel formalism discussed in the previous section to analyses of data. First we determine parameters in the two-pseudoscalar-meson scattering model by analyzing experimental data for \( \pi\pi \) and \( \pi\bar{K} \) scatterings. Then we extract resonance parameters from amplitudes of the two-meson interaction model. This two-meson interaction model is a basic ingredient for the three-meson scattering model. In the subsequent subsection, we analyze the \( D^+ \to K^-\pi^+\pi^+ \) decay in a realistic setting.

A. Two-pseudoscalar-meson scattering

For studying the \( D^+ \to K^-\pi^+\pi^+ \) decay in our coupled-channel framework, the \( \pi\pi \) and \( \pi\bar{K} \) scatterings of \( E \lesssim 2 \text{ GeV} \) are relevant. We will determine the model parameters of our \( \pi\pi \) and \( \pi\bar{K} \) scattering models, i.e., \( h_{ab}^{LI}, b_{ab}^{LI}, m_{cR}, g_{ab}, \) and \( c_{ab,R} \) in Eqs. (1)-(3), (8), and (10), by fitting empirical scattering amplitudes for \( E \lesssim 2 \text{ GeV} \).
We analyze the $\pi \overline{K}$ scattering amplitudes from the LASS experiment [6, 9]. For our application to the $D^+ \rightarrow K^-+\pi^+\pi^+$ decay in the next section, we determine the model parameters for $\{L, I\} = \{0,1/2\}, \{0,3/2\}, \{1,1/2\}, \{2,1/2\}$ partial waves. We explain details of our $\pi \overline{K}$ scattering model for each partial wave. For the $\{L, I\} = \{0,1/2\}$ wave, we consider $\pi \overline{K}$-$\eta' \overline{K}$ coupled channels because the $\eta' \overline{K}$ channel is known to play a significant role while $\eta \overline{K}$ does not in this partial wave. We include two bare $R$ states supplemented by a contact $\pi \overline{K} \rightarrow \pi \overline{K}$ interaction. For the $\{L, I\} = \{0,3/2\}$ wave, we consider a contact $\pi \overline{K} \rightarrow \pi \overline{K}$ interaction only. For the $\{L, I\} = \{1,1/2\}$ and $\{2,1/2\}$ waves, we consider coupling of $\pi \overline{K}$ and effective inelastic channels; masses of the two “particles” in the inelastic channel, denoted by $m_{1}^{II}$ and $m_{2}^{II}$, are also fitted to the data. We include three bare $R$ states for $\{L, I\} = \{1,1/2\}$ while a single bare $R$ state for $\{2,1/2\}$. We present the $\pi \overline{K}$ model parameters determined by the fits in TABLE VI of Appendix B.

We present the quality of the fits to the empirical partial wave amplitude [6] of $\pi^+K^-$ $L=0$ partial wave and $\{L, I\} = \{1,1/2\}, \{2,1/2\}$ partial waves in Fig. 7 where phases (upper panels) and modulus (lower panels) of the amplitudes are shown. Also the elastic scattering phase shifts for the $\{L, I\} = \{0,3/2\}$ partial wave calculated with our model are compared with the data [9] in Fig. 8(left). The $s$-wave $\pi^+K^-$ amplitude is calculated by linearly combining the $\{L, I\} = \{0,1/2\}, \{0,3/2\}$ partial wave amplitudes. Overall, we obtain a reasonable description of the data included in the fit. However, one notices that the model has a sudden change and deviation from the data in the phase for the $\{L, I\} = \{1,1/2\}$ partial wave at $E \sim 1.3$ GeV. This is perhaps an artifact of our model that has the threshold where the effective inelastic channel opens. Fortunately, the magnitude of the amplitude is rather small around this energy so that the deviation in the phase will not give a significant impact on observables calculated with this model.

FIG. 7. (Color online) Phase (upper) and modulus (lower) of the amplitudes for the $\pi \overline{K}$ scatterings: (Left) $L=0$ for $\pi^+K^-$; (Center) $\{L, I\} = \{1,1/2\}$; (Right) $\{L, I\} = \{2,1/2\}$. Data are taken from Ref. [6].

1. $\pi \overline{K}$ scattering
FIG. 8. (Color online) Phase shifts ($\delta_{LI}$) of the $\pi K$ and $\pi\pi$ scatterings: (Left) $\pi K$ phase shifts for $\{L, I\} = \{0,3/2\}$ partial wave. Data are from Ref. [9]; (Right) $\pi\pi$ phase shifts for $\{L, I\} = \{0,2\}$ partial wave. Data are from Refs. [26, 27].

TABLE I. Pole positions of the $\pi\bar{K}$ partial-wave amplitudes in the complex-energy plane. A partial wave is specified by the orbital angular momentum ($L$) and the isospin ($I$). Poles below Re[$E$] $\leq$ 2 GeV and |Im[$E$]| $\leq$ 0.25 GeV are presented. Roman numerals in the square brackets specify the Riemann sheet on which each of the poles exist. We use the convention defined in, e.g., Ref. [28], to specify each of the Riemann sheets, I–IV. Each of the states is identified with the corresponding particle name used in the PDG listings [29].

| $L$ | $I$ | Pole positions (GeV), [Riemann sheet], Name |
|-----|-----|------------------------------------------|
| 0   | 1/2 | $0.71 - 0.23i$ [II] $\kappa$ |
| 1   | 1/2 | $0.90 - 0.025i$ [II] $K^*(892)$ |
| 2   | 1/2 | $1.42 - 0.055i$ [III] $K^*_2(1430)$ |
|     |     | $1.44 - 0.14i$ [II] $K^*_0(1430)$ |
|     |     | $1.46 - 0.25i$ [III] $K^*_0(1430)$ |
|     |     | $1.28 - 0.058i$ [III] $K^*(1410)$ |
|     |     | $1.66 - 0.088i$ [III] $K^*(1680)$ |

From the $\pi\bar{K}$ amplitudes of the fixed model, we extract resonance pole positions by the analytic continuation [30, 31] as shown in TABLE I. We present poles below Re[$E$] $\leq$ 2 GeV and |Im[$E$]| $\leq$ 0.25 GeV. We can consistently identify most of the extracted poles with the corresponding particles listed by the Particle Data Group (PDG) [29] as shown in the table. For the $\{L, I\} = \{0,1/2\}$ partial wave, our model has a pole at $0.71 - 0.23i$ GeV that corresponds to the so-called $\kappa$ whose mass is $682 \pm 29$ MeV, and width $547 \pm 24$ MeV in the PDG listing. Also we find two poles at Re[$E$] $\sim$ 1.4 GeV on different Riemann sheets. These two poles are associated with a single resonance [$K^*_0(1430)$] that is split by coupling to $\eta\bar{K}$ channel. For the $\{L, I\} = \{1,1/2\}$ partial wave, our model has the well-established $K^*(892)$. Also in the same partial wave, there is a pole at $1.28 - 0.058i$ GeV that is a bit off the $K^*(1410)$ resonance parameters from the PDG.

2. $\pi\pi$ scattering

We perform an analysis of $\pi\pi$ scattering data with our coupled-channel model in a way similar to the analysis of $\pi\bar{K}$ data in the previous section. Although we only need a $\pi\pi$ model for the $\{L, I\} = \{1,1\}, \{0,2\}$ partial waves for our coupled-channel analysis of the $D^+ \to K^- \pi^+ \pi^+$ decay, we present here our $\pi\pi$ model for all major partial waves for a future reference. We consider $\pi\pi$-$K\bar{K}$ coupled channels for all partial waves except for $\{L, I\} = \{0,2\}$
FIG. 9. (Color online) Phase shifts (upper) and inelasticities (lower) for the ππ scattering: (Left) \{L, I\}={0,0}; (Center) \{L, I\}={1,1}; (Right) \{L, I\}={2,0}. Data are taken from Ref. [17–19].

TABLE II. Pole positions of the ππ partial-wave amplitudes in the complex-energy plane. The other features are the same as those in TABLE I.

| L  | I  | Pole positions (GeV), [Riemann sheet], Name                  |
|----|----|-------------------------------------------------------------|
| 0  | 0  | 0.41 – 0.24i [II] σ                                       |
| 1  | 1  | 0.77 – 0.079i [II] ρ(770)                                   |
| 2  | 0  | 1.25 – 0.099i [III] f2(1270)                                |
|    |    | 1.02 – 0.001i [II] f0(980)                                  |
|    |    | 1.01 – 0.083i [III] ρ(1450)                                 |
|    |    | –                          |
|    |    | 1.38 – 0.15i [III] f0(1370)                                 |
|    |    | 1.58 – 0.16i [III] ρ(1570)                                 |

where only the elastic ππ channel is taken into account. Regarding details of our model for each partial wave, we include two bare \(R\) states supplemented by a contact \(ππ→ππ\) interaction for the \{L, I\}={0,0} wave. For the \{L, I\}={1,1} and \{2,0\} waves, we include two bare \(R\) states and a single bare \(R\) state, respectively. Finally for the \{L, I\}={0,2} wave, we consider a contact \(ππ→ππ\) interaction only. We present the ππ interaction model parameters determined by the fits in TABLE VII of Appendix B.

We present the quality of the fits in Fig. 9 where phase shifts (inelasticities) are shown in the upper (lower) panels. As seen in the figures, we obtain reasonable fits to the data from Refs. [17–19]. The nonzero inelasticities from our model are due to the coupling to the \(KK\) channel of our coupled-channel model. We note that the \(KK\) channel in our model effectively simulates all inelastic channels in which the true \(KK\) channel is a major component, because we did not include \(ππ \rightarrow KK\) and \(KK \rightarrow KK\) data in our analysis, and also because we did not include the 4π channel in the model. From the determined ππ partial wave amplitudes of our model, we extract resonance poles as presented in TABLE II.

Most of the extracted poles are consistently identified with the counterparts in the PDG listings, as shown in TABLE II. A difference from the PDG value is found for the width of \(f_0(980)\); our model has a rather small width (\(\sim 2\) MeV) in comparison with the PDG average (40–100 MeV). This difference was also found in the model used in Ref. [13], and possible sources of the difference were discussed there. Also the second resonance in the
\{L, I\} = \{1, 1\} partial wave perhaps does not correspond to \(\rho(1450)\). However, an effect of this resonance pole on the amplitude seems to be very small, and our model reproduces the empirical amplitude very well.

**B. Analysis of \(D^+ \to K^-\pi^+\pi^+\) Dalitz plot**

Now we will perform a partial wave analysis of pseudo-data for \(D^+ \to K^-\pi^+\pi^+\) Dalitz plot distribution using our coupled-channel model. In what follows, we explain setups of our models used in the analysis. Then we discuss how we prepare pseudo-data and our analysis procedure, which is followed by numerical results.

1. Model setup

In our coupled-channel framework, \(D^+\)-meson decays into \(\mathcal{R}c\) channels, followed by multiple scatterings due to the hadronic dynamics, leading to the final \(K^-\pi^+\pi^+\) state. This process is expressed by Eqs. (28), (31), and (32); with the symmetrization, the decay amplitude is given by Eq. (27). We consider the following 11 \(\mathcal{R}c\) coupled channels in our full calculation:

\[
\mathcal{Rc} = \{ R_{i1}^{01}, R_{i1}^{01}, r_{i1}^{01}, R_{i1}^{11}, R_{i1}^{21}, R_{i1}^{31}, R_{i1}^{31}, R_{i1}^{31}, R_{i1}^{31}, R_{i1}^{31}, R_{i1}^{31}, R_{i1}^{31} \},
\]

where \(R_{i1}^{L,2L}\) stands for \(i\)-th bare \(R\) state with the spin \(L\) and the isospin \(I\); when \(I\) is an integer (half-integer), it is understood that this \(R\) state has the strangeness \(S = 0\) \((S = -1)\) in this paper. Thus, \(R_{i1}^{01}, R_{i1}^{11}, R_{i1}^{21}, R_{i1}^{22}\) are seeds of \(K_0^*, K^*, K_2^*, \rho\) resonances, respectively. In our model, these resonances are included as poles in the unitary scattering amplitudes. \(r_{ab}^{L,2L}\) is a “state” associated with a contact interaction in a partial wave of \(L\) and \(I\), as has been introduced in Sec. II.B. Most of these channels have been considered in the previous analyses [2, 3, 5, 8, 10, 11]. However, \(r_{04}^{04}\) channel was considered only in the CLEO analysis [11]. Also, the \(r_{04}^{04}\) channel was explicitly considered only in the FOCUS analysis [8], but other MIPWA implicitly take account of this channel. The \(R_{i1}^{31}\) channel has not been considered in the previous analyses. This channel can contribute to \(D^+ \to K^-\pi^+\pi^+\) only through the coupled-channel dynamics, and therefore it does not show up in isobar-type models. With the coupled channels considered in this work [Eq. (35)], the final hadronic system has the total spin \(J=0\), parity \(P=+1\), total isospin \(T=3/2\), and \(l=s_{\mathcal{R}}\). The \(D^+ \to \mathcal{R}c\) vertex function is parametrized by Eq. (33) that contains parameters, \(C_{(c\mathcal{R})_{i1}}^{J\Pi T}\), \(\phi_{(c\mathcal{R})_{i1}}^{J\Pi T}\), and \(\Lambda_{(c\mathcal{R})_{i1}}^{J\Pi T}\). We will fit Dalitz plot pseudo-data for \(D^+ \to K^-\pi^+\pi^+\) by adjusting these parameters, except that \(C_{(\pi R_{i1}^{01})_{i1}}^{J\Pi T} = 1\) and \(\phi_{(\pi R_{i1}^{01})_{i1}}^{J\Pi T} = 0\) are fixed.

The hadronic rescattering processes are described by the scattering amplitude, \(T_{(c\mathcal{R})_{i1}}^{J\Pi T}\), defined in Eq. (17), and also by the \(\mathcal{R}c\) Green function defined in Eqs. (18)-(21). The main driving force of the rescattering processes is the two-pseudoscalar-meson interactions that have been fixed in the previous section by fitting the empirical \(\pi K\) and \(\pi\pi\) scattering amplitudes. The two-pseudoscalar-meson interactions enter the scattering equation [Eq. (17)] as the \(Z\)-diagrams and also through the \(\mathcal{R}\) self-energies in Eqs. (22)-(25). The coupled channels specified above and the two-meson interactions, we can find all the contributing \(Z\)-diagrams that induce channel-couplings. In our analysis, we consider all of the contributing \(Z\)-diagrams that contain \(\pi\pi\) and \(KK\) intermediate states. In addition, we include the
three-meson-force based on the HLS model. We have totally six diagrams of this type, as mentioned below Eq. (26). As described in Appendix A, once a single coupling constant is fixed by the $\rho$-meson decay width, all the other couplings are fixed by SU(3) and the HLS. We use different cutoffs in form factors [Eq. (A15)] for different diagrams, thereby breaking SU(3). These cutoffs are determined by fitting the pseudo-data.

In our analysis of $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot pseudo-data, we will basically use three models. The first one is the “Full model” that contains all the dynamical contents described above. The second one is the “Z model” for which the rescattering mechanism is solely due to multiple iteration of the Z-diagrams. Thus the Full model is obtained by adding the three-meson-force to the Z model. The third model is the “Isobar model” that does not explicitly contain the rescattering process. The decay amplitude for the Isobar model has been described at the end of Sec. II C. This Isobar model is still different from most of the isobar models used in the previous Dalitz plot analyses of $D^+ \rightarrow K^-\pi^+\pi^+$ in that all two-pseudoscalar partial wave amplitudes are unitary, and fit well the empirical amplitudes in the relevant energy region. Finally, we add a flat and coherently interfering background amplitude to the $D^+$-decay amplitude from each of the above-described three models, as has been done in all the previous Dalitz plot analyses of $D^+ \rightarrow K^-\pi^+\pi^+$. Specifically, we add the background term to the decay amplitude in Eq. (27) as

$$T_{K^-\pi^+\pi^+,D^+}(p_{K^-},p_{\pi^+},p_{\pi^+}; E=m_{D^+}) + a_{BG} e^{i\phi_{BG}},$$

where the modulus and phase of the background are denoted by $a_{BG}$ and $\phi_{BG}$, respectively. This background term breaks the three-body unitarity to some extent, depending on the strength of the background. The parameters associated with the background, $a_{BG}$ and $\phi_{BG}$, are determined by fitting the pseudo-data. We will see that the background contribution turns out to be very small in the Full model. Finally we remark that the two-pseudoscalar-meson interactions, that have been determined in the previous sections, will not be adjusted to fit the $D^+$-decay pseudo-data. This is in contrast with most of the previous analyses where some of Breit-Wigner parameters were also adjusted along with $D^+ \rightarrow Rc$ vertices.

2. Data and analysis method

We create reasonably realistic pseudo-data of the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot from the isobar model of the E791 Collaboration [5]. The E791 Collaboration obtained the isobar model through their partial wave analysis of the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot of 15,079 events, among which 94.4% were determined to be signals. In generating pseudo-data, we take a procedure similar to Sec. V of Ref. [32] where Dedonder et al. created pseudo-data for $D^0 \rightarrow K^0\pi^+\pi^-$ using the isobar model of the BABAR Collaboration. We start with a grid 400×400 squared cells covering all kinematical region of the Dalitz plot distribution with $M^2_{K^-\pi^+}$ as $x$-axis and another $M^2_{K^-\pi^+}$ as $y$-axis; $M^2_{K^-\pi^+}$ denotes the squared invariant mass of the $K^-\pi^+$ pair. Each cell is given by the E791 isobar model a value of the Dalitz plot distribution at the center of the cell. Then, the values of the Dalitz plot distribution in 10×10 adjacent cells are summed to obtain 40×40 cells, each of which has the value of the partially integrated Dalitz plot distribution. The width of each cell is 0.0649 GeV$^2$. 40×40 cells were used in the E791 analysis [5] to perform their MIPWA, and thus are a reasonable size also in our analysis. The Dalitz plot distribution value in each cell is multiplied by a common normalization constant and then is round off to be an integer; the common normalization constant is chosen so that the sum of the round-off values of all the cells coincides with
FIG. 10. (Color online) The Dalitz plot distribution for $D^+ \to K^- \pi^+ \pi^+$ in 40×40 cells. The left panel shows our pseudo-data generated with the isobar model of the E791 Collaboration [5]. The right panel is the counterpart from the Full model that has been fitted to the pseudo-data. An explanation for how the pseudo-data are generated is found in Sec. III B 2.

15,079 × 94.4% ∼14,234, the number of signals of the E791 experiment. In this way, we have generated pseudo-data for the $D^+ \to K^- \pi^+ \pi^+$ Dalitz plot, as presented in the left panel of Fig. 10.

Next task is to analyze the above pseudo-data with our model. Again, Ref. [32] serves as a useful reference to find an analysis procedure. We calculate the Dalitz plot distribution using the decay amplitude of Eq. (27) and the formulae given in Appendix B or Ref. [13]. In each cell of the 40×40 grid, we integrate the Dalitz plot distribution from our model; the overall normalization is chosen so that the integral over all the kinematical region of the Dalitz plot distribution becomes equal to 14,234, the number of signals for the E791 data. In this way, we have the number of events (a real number) in each of the cells, and can compare it with the counterpart in the pseudo-data. If a given cell of the pseudo-data has the number of events less than 5, then the cell is merged with the adjacent cell in the same x-axis to be a larger cell. This grouping is repeated until the cell contains more than or equal to 5 events. The same grouping is also applied to the event samples from our model. Thus, the total number of effective cells, each of which contains more than or equal to 5 events of the pseudo-data, is 749.

The quality of the fit can be quantified by calculating $\chi^2$ defined by

$$\chi^2 = \sum_j \chi_j^2 = \sum_j \left( \frac{N_j^{th} - N_j^{exp}}{\Delta N_j^{exp}} \right)^2,$$

where $j$ labels each cell, and $N_j^{th}$ ($N_j^{exp}$) is the number of events in the $j$-th cell from our model (pseudo-data). The error of the pseudo-data in each cell is assigned by $\Delta N_j^{exp} = \sqrt{N_j^{exp}}$.

We will perform the least $\chi^2$-fit to the pseudo-data. By simply minimizing $\chi^2$ defined in Eq. (37), however, we obtain models that undershoot the sharp peak associated with $K^*(892)$. This is perhaps because a partial $\chi^2$ from the bins for which $K^*(892)$ is important
TABLE III. The total $\chi^2$-values and $\chi^2$/d.o.f. from the Full, Z and Isobar models obtained by fitting the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot pseudo-data. The Z model without the $(\pi^+\pi^0)_{IP}=1\bar{K}^0$ channel is labeled by “Z (without $\rho$)”.

|        | Full   | Z      | Isobar | Z (without $\rho$) |
|--------|--------|--------|--------|-------------------|
| $\chi^2$ | 199.   | 231.   | 318.   | 356.              |
| $\chi^2$/d.o.f. | 0.28   | 0.32   | 0.44   | 0.49              |

is rather small, leading to the models that are not very good in the $\bar{K}^*(892)$ region. This can happen because the $\bar{K}^*(892)$ peak is so sharp that it gives a significant effect on a rather small portion of the total bins. However, the $\bar{K}^*(892)$ peak in the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot distribution is an important characteristic feature, and should be reproduced reasonably well, rather than searching for the least $\chi^2$. This is one example showing that the simple $\chi^2$-fit does not necessarily lead to a physically reasonable model. Thus, we put an additional weight on the partial $\chi^2$ from the $\bar{K}^*(892)$ region in Eq. (37), and perform the least $\chi^2$-fit. We use the same weight for all the models considered in this work. Once the fit is completed, we calculate the original $\chi^2$ defined by Eq. (37), and use it in the following discussion and in TABLE III.

An error estimation is an important part of a data analysis of this kind. In this work, however, we do not attempt to estimate errors of quantities such as the model parameters and a fraction of each channel’s contribution. A reason is that the data we analyze are, albeit rather realistic, still pseudo-data. Therefore, extracted quantities such as the fractions are regarded as such. Considering that, it does not seem very worthwhile taking another substantial effort to estimate errors. Such an effort should be saved until analyzing the real ones. In addition, an error estimation with our coupled-channel model is more difficult than working with an isobar-type model. This is because our model needs rather long computation time in solving the coupled-channel integral equations of Eq. (17), and takes much longer time to calculate the Dalitz plot distribution than an isobar-type model does. Thus in this work, although we are not able to provide the errors, we focus on our primary purpose of studying effects of coupled-channel dynamics that have been missed in the previous analyses.

3. Numerical results and discussions

We performed the least $\chi^2$-fit to the pseudo-data following the procedure explained in the previous subsection. We used the three models; the Full, Z, and Isobar models. All the parameters determined by the fits are tabulated for each of the models in TABLEs VIII and IX. The Dalitz plot distribution from the Full model is shown in the right panel of Fig. 10. Comparing with the left panel of the pseudo-data, a difference is hardly discernible to the eye. The situation is the same for the Z and Isobar models. The quality of the fit can be quantified by calculating the $\chi^2$-values that are presented in the second row of TABLE III. The $\chi^2$-value of the Full model is better than the others. However, this may be due to the fact that the Full model has more fitting parameters. Thus in the third row of the table, we also show $\chi^2$/d.o.f. to assess this point. The number of the fitting parameters for the Full, Z, and Isobar models are 39, 33, and 27, respectively, as can be found in TABLEs VIII and IX. The number of bins at which $\chi^2$ is calculated is 749. Thus we obtain $\chi^2$/d.o.f. = 199./(749-39)=0.28, 231./(749-33)=0.32, 318./(749-27)=0.44, for the Full, Z, and Isobar
FIG. 11. (Color online) The $\chi^2_j$ distributions. The left, middle, right panels are from the Full, Z, and Isobar models, respectively. The bins with $\chi^2_j \geq 4$ are all given the same color (red); the same applies to the bins with $\chi^2_j \leq 1$ (dark blue).

FIG. 12. (Color online) The $K^-\pi^+ (\pi^+\pi^+)$ squared invariant mass spectrum of the Full model compared with the pseudo-data from the E791 isobar model in the left (right) panel.

models, respectively. Therefore, the ranking of the fit quality is still in the same order as far as $\chi^2$ is concerned.

In order to see the quality of the fit more clearly, we show the $\chi^2_j$ distribution at all of the bins in Fig. 11. These figures show that all of our models fit the pseudo-data rather precisely. Most of the bins are fitted with $\chi^2_j < 1$, and $\chi^2_j$ exceeds 4 at only a small number of the bins. Yet again, the Full model clearly shows a better performance in the fit among the three models we consider. The quality of the fit can be also shown in the projection of the Dalitz plot distribution onto the $M^2_{K^-\pi^+}$ or $M^2_{\pi^+\pi^+}$ distributions, as presented in Figs. 12, 13, and 14 for the Full, Z, and Isobar models, respectively. All the three models give reasonable fits to the pseudo-data as expected.

We remark that we obtained the reasonable fits to the pseudo-data without adjusting the parameters associated with the two-pseudoscalar-meson interactions. On the other hand, as mentioned already, most of the previous analyses varied some Breit-Wigner parameters in fitting their $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot data. A common claim in those analyses [2, 5, 8, 11] was that the width of $\bar{K}^*_{0}(1430)$ obtained in their fits, $\sim 170$ MeV, was significantly smaller than those from the PDG and the LASS analysis, $\sim 270$ MeV. In our model, there are two poles associated with $\bar{K}^*_{0}(1430)$ as shown in TABLE I, and they are on different Riemann
FIG. 13. (Color online) The $K^-\pi^+ (\pi^+\pi^+)$ squared invariant mass spectrum of the Z-model. The other features are the same as those in Fig. 12.

FIG. 14. (Color online) The $K^-\pi^+ (\pi^+\pi^+)$ squared invariant mass spectrum of the Isobar model. The other features are the same as those in Fig. 12.

sheets, the branch point of which is the $\eta'\bar K$ threshold. This two-pole structure is a coupled-channel effect. Interestingly, one of the poles is close to the PDG value, while the other one is close to the values from the previous $D^+ \to K^-\pi^+\pi^+$ analyses. This result may be suggesting that $K^-\pi^+ \to K^-\pi^+$ is more sensitive to one of the poles while $D^+ \to K^-\pi^+\pi^+$ is more sensitive to the other pole. Thus, possibly, the coupled-channel framework enables us to describe the two processes in a unified manner.

Next, we look into the fraction of each channel’s contribution (fit fraction). In an isobar model that describes a $D^+$-meson decay as $D^+ \to \sum R_c \to abc$ where $R$ is expressed by a Breit-Wigner function, the fit fractions of different $R_c$ contributions can be straightforwardly defined, and are often presented in the previous analyses. However, in a model like ours where the resonances are described as poles in unitary scattering amplitudes, the fit fraction of a certain resonance contribution is not so straightforwardly defined, because the scattering amplitude generally contains more than a single resonance, as we have seen in TABLEs I and II. Furthermore, the amplitude also contains nonresonant contributions. There is no unambiguous way to single out a certain resonance contribution. Therefore, we present the fit fractions of different $[(ab)_{Lc}]$ channel contributions where a two-pseudoscalar-meson pair $ab$ has the total angular momentum $L$ and the total isospin $I$. When $ab$ is written with a
TABLE IV. Fit fractions (%) from each of the models fitted to the $D^+ \to K^-\pi^+\pi^+$ pseudo-data. See the text for the definition of the fit fraction. The $(K^-\pi^+)_{S}\pi^+$ fit fraction is the coherent sum of the $(K^-\pi^+)_{S}^{I=1/2}\pi^+$ and $(K^-\pi^+)_{S}^{I=3/2}\pi^+$ fit fractions. The numbers in the parentheses are not included in the “Sum”. The central values of the fit fractions from the E791 Isobar model [5], the FOCUS K-matrix model [8], and the CLEO QMIPWA [11] are also presented. The numbers in $\langle \rangle$ are obtained by the incoherent sum of different resonance contributions in the same partial wave.

|                  | Full      | Z         | Isobar    | Z(without $\rho$) | E791 | FOCUS     | CLEO    |
|------------------|-----------|-----------|-----------|-------------------|------|-----------|---------|
| $(K^-\pi^+)_{S}^{I=1/2}\pi^+$ | 156.1     | 171.5     | 22.1      | 173.1             | –    | 207.25    | –       |
| $(K^-\pi^+)_{D}^{I=1/2}\pi^+$ | 14.5      | 15.6      | 24.2      | 11.8              | 16.1 | (15.99)   | (10.076)|
| $(K^-\pi^+)_{S}^{I=3/2}\pi^+$ | 32.0      | 126.0     | 16.4      | 183.9             | –    | 40.50     | –       |
| $(\pi^+\pi^+)_{S}^{I=2}K^-$  | 1.8       | 3.1       | 7.9       | 7.0               | –    | –         | 15.5    |
| Background       | 0.0       | 0.6       | 17.7      | 50.2              | 16.1 | –         | –       |
| $(K^-\pi^+)_{S}\pi^+$         | (80.8)    | (48.5)    | (33.5)    | (36.6)            | 33.6 | (83.23)   | 97.1    |
| **Sum**          | 205.2     | 317.6     | 88.7      | 426.5             | 66.0 | 264.13    | 122.9   |

Charge state, $(ab)^I_L$ is understood to be the total isospin $I$ state projected onto the particular charge state; e.g., $[(K^-\pi^+)_{S}^{I=0}\pi^+]$. In our coupled-channel-model analysis, however, there is an additional complication in defining the fit fraction from the fact that, even in a single diagram of the $D^+ \to K^-\pi^+\pi^+$ decay, different $[(ab)^I_Lc]$ can couple with each other through the rescattering process as a consequence of the three-body unitarity. Thus we define the fit fraction of a given $[(ab)^I_Lc]$ as follows. We retain all the diagrams in which the final two-pseudoscalar-meson pairs $ab$ from $R$-decays have a given $L$ and $I$, and drop the others. Then we refer to the decay width calculated with this truncated decay amplitude as the $[(ab)^I_Lc]$ partial width. The fit fraction of a given $[(ab)^I_Lc]$ is then naturally defined as the $[(ab)^I_Lc]$ partial width divided by the total width. More specifically, the $[(ab)^I_Lc]$ partial width can be calculated with the modified $D^+$-decay amplitude which is obtained by replacing the summation $\sum_{R,R'}$ in Eq. (28) with $\sum_{R,R'} \delta_S \delta_{LR,L'} \delta_{LR,L'} \delta_{LR',L'}. $

The fit fractions defined in the above paragraph are presented in TABLE IV. The sum of the fit fractions is not necessarily 100% because of the interferences between different contributions in the different rows of the table. The Full and Z models are relatively similar in the fit fractions, while the Isobar model is rather different from the two. The sum is 205.2%, 317.6%, and 88.7% for the Full, Z, and Isobar models, respectively, and therefore there are more constructive interferences in the Isobar model while more destructive interferences in the Full and Z models. The similarity between the Full and Z models perhaps reflects the fact that effects of the three-meson-force is not very large; we will examine the effect of the three-meson-force later. In the Full and Z models, $(K^-\pi^+)_{S}^{I=1/2}\pi^+$ fit fraction is dominant, and the $(K^-\pi^+)_{S}^{I=3/2}\pi^+$ fit fraction is the second. These are both $s$-wave $K^-\pi^+$ contributions, and have not been separated in most of the previous analyses of the $D^+ \to K^-\pi^+\pi^+$ decay. Thus we show in the table also the coherent sum of these for comparison with the previous analysis results. We find a rather large destructive interference between the $I=1/2$ and 3/2 $K^-\pi^+$ $s$-waves. The degree of the destructive interference is rather different between the Full and Z models. By coherently adding the background contribution to the $(K^-\pi^+)_{S}\pi^+$ fit fraction, we obtain 80.6%, 55.9%, and 51.7% for the Full, Z, and Isobar models, respectively.
and the difference between the models gets reduced. The background for the Full model is tiny as seen in the table, and thus the violation of the three-body unitarity is also very small for this model.

We compare the fit fractions from our models with those from the previous analyses done by the experimental groups. Although each of the experimental groups obtained several models through their analyses, we do not try to exhaust all the models in the comparison. Rather we pick up some cases that are particularly interesting to compare with our results. Here, we tabulated three analyses results in TABLE IV: the E791 Isobar model [5], the FOCUS K-matrix model [8], and the CLEO QMIPWA [11]; errors are not presented for simplicity. In the original papers, the fit fractions from these analyses were presented for each of resonances considered. Thus for comparing with our results, we sum the resonance contributions in the same partial wave incoherently (coherently) for the FOCUS and CLEO (E791) analyses. The numbers obtained by the incoherent sum, as indicated by ⟨⟩ in the table, are for $K^{-\pi^+}_{P} I=1/2 \pi^+$ where $K^*(892)$ dominates, and thus the interference effect would not be so large. The E791 Isobar model [5] is the model on which our pseudo-data are based. The fit fractions from this model are relatively close to our Isobar model, although there is still a noticeable difference. The difference in the $K^{-\pi^+}_{P} I=1/2 \pi^+$ fit fractions from this model are relatively close to our Isobar model, although there is still a noticeable difference. The difference in the $K^{-\pi^+}_{P} I=1/2 \pi^+$ fit fraction may be due to the fact that the $K^{-\pi^+}_{P} I=1/2$ amplitudes used in our model and the E791 isobar model are significantly different. While our model fits well the $K^{-\pi^+}_{P} I=1/2$ amplitude of the LASS data as shown in Fig. 7, the E791 isobar model does not so as seen in the elastic region of Figs. 3 and 6 of Ref. [5].

The FOCUS Collaboration separated the $(\bar{K}\pi)_S \pi^+$ amplitude into $I=1/2$ and $3/2$ components of $(\bar{K}\pi)_S$ in their K-matrix model analysis. As shown in TABLE IV, they found a rather large destructive interference between the $I=1/2$ and $3/2$ components. Our Full and Z models also show comparably large destructive interferences, while the Isobar model has a much smaller destructive interference. One may find this odd to his/her expectation, because our Isobar model is, among the three models of ours, the most similar to the FOCUS’s K-matrix model; both of them do not have the rescattering processes explicitly, and both use $(\bar{K}\pi)_S I=1/2$ and $(\bar{K}\pi)_S I=3/2$ amplitudes fitted to the same data [6, 9]. Some differences in the other channels might have led to this difference in the interference. The CLEO Collaboration considered the nonresonant $(\pi^+\pi^+)_{S=2} K^+$ amplitude in their analysis, and found its fit fraction to be 15.5%. This fit fraction is significantly larger than those in our models. Finally, all the models shown in TABLE IV agree that the $(K^-\pi^+)_D I=1/2 \pi^+$ has rather small fit fraction, 0.2 ~ 0.8%, but still gives a significant contribution to the Dalitz plot distribution through the interference with the other partial waves.

With our coupled-channel analysis, it is interesting to examine bare fit fractions defined as follows. We first calculate a bare $[(ab)^c_f]$ partial width using the decay amplitude in which all contributions from the rescattering processes [the second term of Eq. (32); Fig. 6(b)] are omitted. In our coupled-channel description of the $D^+ \to K^-\pi^+\pi^+$ decay, we consider not only $K^-\pi^+\pi^+$ but also $K^0\pi^0\pi^+$ in the hadronic intermediate states. Therefore, when calculating the bare partial width, we consider both of the states in the final state sum. (Precisely speaking, $\bar{K}^0\eta'/\pi^+$, $\bar{K}^0\bar{K}^0K^+$, and the effective inelastic channels in the $\pi\bar{K}$ p- and d-waves are also included in the intermediate states; their partial widths are rather small.) Then, the bare fit fraction for a given $[(ab)^c_f]$ state is defined by the bare $[(ab)^c_f]$ partial width divided by the sum of the bare partial widths for all the considered $[(ab)^c_f]$ states. Thus the sum of all the bare fit fractions is 100% by definition. The result is presented in TABLE V. We leave the column for the Isobar model blank. This is because
the $D^+ \to R\pi$ vertices of the Isobar model implicitly includes the rescattering effect, and we cannot eliminate the effect to extract the bare vertices.

A remarkable feature in TABLE V is the dominance of the fit fraction of $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ in which $\rho(770)$ plays a major role. This fit fraction did not appear in TABLE IV because the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ state can appear only in intermediate states of the $D^+ \to K^-\pi^+\pi^+$ decay, and its contribution is genuinely a coupled-channel effect. The previous $D^+ \to K^-\pi^+\pi^+$ analyses cannot see this effect because they did not explicitly consider three-body dynamics. Actually, the dominance of the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ fit fraction (~85%) was also found in a recent analysis of $D^+ \to K^0_S\pi^+\pi^0$ done by the BESIII Collaboration [12]. The $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0_S\pi^+\pi^0$ decays share the same hadronic dynamics, except for additional but much smaller doubly Cabibbo suppressed channels in the latter. Therefore, the large bare fit fraction of $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ for the $D^+ \to K^-\pi^+\pi^+$ decay seems natural in order to understand the decay mechanisms for $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0_S\pi^+\pi^0$ consistently.

In order to see a contribution of the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ channel to the $D^+ \to K^-\pi^+\pi^+$ decay more clearly, we show in Fig. 15 the $K^-\pi^+\pi^0$ squared invariant mass spectrum of the Full model but couplings to the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ channel are turned off. The figures clearly show the large contribution from the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ channel. It increases the decay rate by ~32%, and also significantly changes the shape of the spectra. We found a similar result for

| $I=1/2\pi^+$ | Full | $Z$ | Isobar | $Z$ (without $\rho$) |
|----------------|-------|------|--------|----------------------|
| $(\bar{K}\pi)_S$ | 28.9 | 17.9 | —      | 8.9                  |
| $(\bar{K}\pi)_P$ | 3.4  | 2.2  | —      | 2.3                  |
| $(\bar{K}\pi)_D$ | 0.3  | 0.3  | —      | 0.2                  |
| $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ | 55.7 | 49.9 | —      | —                    |
| $(\pi^+\pi^0)_{S}^{I=3/2} \bar{K}$ | 11.2 | 28.9 | —      | 56.5                 |
| $(\pi^+\pi^0)_{S}^{I=2} \bar{K}$ | 0.5  | 0.4  | —      | 2.1                  |
| Background      | 0.0  | 0.4  | —      | 30.1                 |

FIG. 15. (Color online) Contributions of the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ channel to the $K^-\pi^+\pi^0$ squared invariant mass spectrum in the left [right] panel. The Full model is shown by the red solid curve. The blue dashed curve is also from the Full model but couplings to the $(\pi^+\pi^0)_{P}^{I=1} \bar{K}^0$ channel are turned off.
FIG. 16. (Color online) Contributions of the three-meson-force (Fig. 4) to the $K^-\pi^+ [\pi^+\pi^+]$ squared invariant mass spectrum in the left [right] panel. The Full model is shown by the red solid curve. The blue dashed curve is also from the Full model but the three-meson-force is turned off. The magenta dot-dashed curve is from the Full model with all the rescattering effects turned off.

Because the $(\pi^+\pi^0)_{P=1}\bar{K}^0$ channel does not appear in the final state, it contributes to the Dalitz plot distribution indirectly, and no peak associated with $\rho(770)$ shows up in the Dalitz plot. Therefore, one may suspect that the trace of the $(\pi^+\pi^0)_{P=1}\bar{K}^0$ channel is almost completely washed out by the other channels in the final state. To try to address this question, we fit the pseudo-data with the Z-model but couplings to the $(\pi^+\pi^0)_{P=1}\bar{K}^0$ channel are turned off. The least $\chi^2$-value reached with this model, labeled by “Z (without $\rho$)”, is shown in TABLE III. The $\chi^2$-value is somewhat worse than that of the Isobar model. We note that the Isobar model and the Z(without $\rho$) model have the same number of adjustable parameters in the fits to the pseudo-data. Thus, the quality of the fit is not improved just by including the rescattering due to the Z-diagrams. Comparing the $\chi^2$-value with that of the Z model, the inclusion of the $(\pi^+\pi^0)_{P=1}\bar{K}^0$ channel in the latter reduces $\chi^2$ by $\sim 35\%$; significant improvement. It is interesting to examine the fit fractions of the Z (without $\rho$) model as tabulated in TABLE IV. The fit fractions are more similar to those of the Z model than the Isobar model, implying a large effect of the rescatterings. However, the flat background fraction is much larger in the Z (without $\rho$) model than that of the Z model, which may be signaling that an important physics (the $\rho\bar{K}$ channel) is missing. However, it is still hard to judge if this coupled-channel analysis has revealed the contribution from the $(\pi^+\pi^0)_{P=1}\bar{K}^0$ channel. The improved $\chi^2$ may be simply because we have additional adjustable parameters in the fit. By analyzing only the $D^+ \to K^-\pi^+\pi^+$ decay in which the $(\pi^+\pi^0)_{P=1}\bar{K}^0$ channel contributes indirectly, we cannot go further. A more definite conclusion can be drawn through a combined analysis of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0_S\pi^+\pi^0$ decays with a coupled-channel model. This interesting subject is left as a future work.

Now let us study how much the three-meson-force contributes to the $D^+ \to K^-\pi^+\pi^+$ decay. In Fig. 16, we compare the $K^-\pi^+ (\pi^+\pi^+)$ squared invariant mass spectrum of the Full model with those from the same model but the three-meson-force being turned off. While the effect of the three-meson-force suppresses the decay width by $\sim 3\%$, the change in the spectrum shape is more significant, as seen in Fig. 16. The $\pi^+K^-$ spectrum is suppressed by the three-meson-force at high $M^2_{\pi^+K^-}$, and the $\pi^+\pi^+$ spectrum is suppressed (enhanced) at low (high) $M^2_{\pi^+\pi^+}$. We found that the effect of the diagram in Fig. 4(a) connected to
the \((\pi^+\pi^0)_{I^P=1}K^0\) channel is the most important among the three-body-type diagrams we consider. In the same figure, we also show the spectrum from the Full model with all the rescattering processes [Fig. 6(b); second term in Eq. (32)] being turned off. Effects of the rescattering mechanisms are quite large; \(~75\%\) reduction of the decay width by turning off the rescattering mechanisms. This may be more than what is expected from the previous result in Fig. 15 that showed the effect of the \((\pi^+\pi^0)_{I^P=1}K^0\) channel. The coupled-channel effect due to channels other than \((\pi^+\pi^0)_{I^P=1}K^0\) is also a major one.

In a Dalitz plot analysis with an isobar-type model, it is assumed that \(M^* \rightarrow Rc\) (\(M^*\): heavy meson, meson resonance, etc.) decay vertex implicitly contains effects of rescattering mechanisms that are simulated by a constant complex phase of the vertex. It is interesting to examine the extent to which this assumption is valid. Figure 17 is for this examination. The upper [lower] panels give the phase [modulus] of dressed \(D^+ \rightarrow Rc\) vertices defined in Eq. (32) for the Full model (red solid curves), as a function of the momentum of the unpaired meson, \(p_c\). The left, middle, and right panels are for \(R_i^{L,2I} = R_{01}^{01}\pi^+, \ R_i^{L,2I} = R_{11}^{11}\pi^+, \) and \(R_i^{L,2I} = r_{K\pi}^{03}\pi^+\), respectively. \(R_{01}^{01} \ [R_{11}^{11}\] works as a seed to develop \(\bar{K} \) and \(K_0^{*}(1430)\) \([K^*(892)]\) resonances, while \(r_{K\pi}^{03}\) mediates the nonresonant \(I=3/2\) s-wave \(\bar{K}\pi\) scattering. For comparison, we also show in the same figure the \(D^+ \rightarrow Rc\) vertices from the Isobar model by the blue dashed curves. Even though both of the models have been fitted to the same pseudo-data of the Dalitz plot distribution, they are rather different in the \(D^+ \rightarrow Rc\) vertices, particularly in the phases. The significant phase variations as a function of \(p_c\) are purely due to the three-body hadronic dynamics required to satisfy the three-body unitarity. The constant phase assumed in the isobar-type models is not justified from this viewpoint.
IV. SUMMARY AND FUTURE PROSPECT

In this work, we have performed a coupled-channel analysis of the pseudo-data for the $D^+ \to K^- \pi^+ \pi^+$ Dalitz plot. The pseudo-data are generated from the isobar model of the E791 Collaboration [5], and are reasonably realistic. As far as we know, this is the first coupled-channel analysis of a realistic Dalitz plot distribution for a $D$-meson decay into a three-pseudoscalar-meson state. We have demonstrated that it is indeed possible to analyze this kind of high-quality data within a coupled-channel framework, and found lots of interesting results that are beyond what can be obtained with the conventional isobar-type model analyses. Let us summarize below what we have done and found in this work.

In our bottom-up approach, we started with developing a two-pseudoscalar-meson interaction model. With a suitable combination of contact interactions and bare resonance-excitation mechanisms, our two-pseudoscalar-meson interaction model successfully describes empirical $\pi\pi$ and $\pi\bar{K}$ scattering amplitudes of low partial waves that are needed to analyze the $D^+ \to K^- \pi^+ \pi^+$ Dalitz plot. Poles associated with $\pi\pi$ and $\pi\bar{K}$ resonances have been extracted, and most of them are in agreement with the PDG listings. Then using the two-pseudoscalar-meson interactions as building blocks, we developed a three-pseudoscalar-meson interaction model that describes the FSI of the $D^+ \to K^- \pi^+ \pi^+$ decay. The main driving force for the three-pseudoscalar-meson scattering process is the $Z$-diagrams and the dressed $\mathcal{R}$-propagator that appear as a necessary consequence of the three-body unitarity. These mechanisms do not contain any adjustable parameters once the two-pseudoscalar-meson interactions have been fixed using the $\pi\pi$ and $\pi\bar{K}$ scattering data. In addition, we considered mechanisms that are beyond simple iterations of the two-body interactions, and thus may be called a three-meson-force. We introduced the three-meson-force to the most important channel in the FSI of the $D^+ \to K^- \pi^+ \pi^+$ decay, i.e., the vector-pseudoscalar channels. Guided by the hidden local symmetry model [20] that incorporates vector and pseudoscalar mesons in a chiral Lagrangian, we derived the vector-pseudoscalar interactions that work as a three-meson-force.

In our analysis of the pseudo-data for the $D^+ \to K^- \pi^+ \pi^+$ Dalitz plot distribution, we basically used three models: Full, Z, and Isobar models. In the models, we took account of all of the channels (partial waves) that had been considered in the previous analyses of the same process. Furthermore, in the Full and Z models, we also considered the $(\pi^+ \pi^0)_{P=1} K^0$ channel where $\rho(770) K^0$ plays a dominant role. This channel can contribute to the $D^+ \to K^- \pi^+ \pi^+$ decay only through the rescattering, and thus this contribution is a purely coupled-channel effect. A distinct feature of our models is that all of the two-pseudoscalar-meson resonances are included as poles of the unitary scattering amplitudes that fit well the empirical $\pi\pi$ and $\pi\bar{K}$ amplitudes. The three-body unitarity is also taken care of within the Full and Z models, up to the rather small flat background amplitude.

Our models fit the pseudo-data with a reasonable precision. As far as the $\chi^2$-value is concerned, the Full, Z, and Isobar models are good in this order. Although this may be simply due to the number of adjustable parameters in the fits, it may also be because important mechanisms (e.g., $\rho \bar{K}$ channel, three-meson-force) are considered in the better model. We have seen that the inclusion of the $\rho \bar{K}$ channel (three-meson-force) reduces $\chi^2$ by $\sim 35\%$ ($\sim 14\%$). Thus the importance of the $\rho \bar{K}$ channel seems rather clear, while it is hard to say that the analysis conclusively shows a significant role of the three-meson-force in the FSI of $D^+ \to K^- \pi^+ \pi^+$. Meanwhile, we were not able to improve the Isobar model just by including the rescattering due to the $Z$-diagrams [$Z$(without $\rho$) model]. Another
important point to be noted here is that we achieved the good fits using two-pseudoscalar-meson amplitudes fixed by the empirical $\pi\pi$ and $\pi K$ amplitudes. This is in sharp contrast with the previous analyses where the $p$-wave $\pi K$ amplitude is given by a sum of the Breit-Wigner functions that does not necessarily reproduce the phase of the empirical amplitude. Also, the previous analyses commonly adjusted parameters associated with $K^*_0(1430)$ in their fits, and obtained widths significantly narrower than those in the PDG listings. It is interesting to observe that our model has two poles associated with $\bar{K}^0$ similar fit fractions. They have rather large fit fractions from the ($\pi^+\pi^0$)$_{I^P=1}^I=2K^0$ channels that interfere with each other destructively. A large destructive interference between these channels was also found in the FOCUS analysis [8]. The fit fraction from the ($\pi^+\pi^0$)$_{I^P=2}^I=2K^0$ channel is considerably smaller than what was found in the CLEO analysis [11]. The flat background contributions is very small in the Full and Z models. With the coupled-channel framework, we were able to study the bare fit fractions for which all the hadronic rescattering effects are absent. This quantity could be useful to study the intrinsic quark-gluon substructure of the $D$-meson. Remarkably, the ($\pi^+\pi^0$)$_{I^P=1}^I=2K^0$ channel, that does not show up in the usual fit fraction, gives a dominant bare fit fraction. This result may appear a bit surprising. However, considering that the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0\pi^0\pi^+$ decays share the same hadronic dynamics to a large extent, this finding is actually very much consistent with the recent BESIII analysis [12] that found a dominant fit fraction ($\sim85\%$) of the ($\pi^+\pi^0$)$_{I^P=1}^I=2K^0$ channel in the $D^+ \to K^0\pi^0\pi^+$ decay.

We further studied coupled-channel effects. We found that the contribution from the rescattering mechanisms almost quadruplicates the $D^+ \to K^-\pi^+\pi^+$ decay width. We also found that the phases of the dressed $D^+ \to R\pi^+$ vertices have rather large dependence on the unpaired $\pi^+$ momentum. The phase variation is a consequence of the explicit treatment of the three-body dynamics. In the conventional isobar-type model analyses of heavy or excited meson ($M^*$) decay into three light mesons, it is assumed that the rescattering effects are small and/or the phases of the $M^* \to RC$ vertices can be approximated by constants. Clearly, our analysis indicates that neither of these assumptions are supported from this more microscopic viewpoint, at least for the $D^+ \to K^-\pi^+\pi^+$ decay.

In future, we will apply our coupled-channel model to a combined analysis of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0\pi^0\pi^+$ decays. The strength of the coupled-channel framework is to describe different processes in a unified manner. Because different aspects of the hadron dynamics appear in different processes, the combined analysis is a powerful method to understand the hadron dynamics with smaller model-dependence. This is why a combined analysis has become standard in the study of the baryon spectroscopy. For example, $\pi N, \gamma N \to \pi N, \gamma N, K\Lambda, K\Sigma$ reaction data are analyzed with a coupled-channel model in a unified manner to extract the nucleon resonance properties in Ref. [33]. This direction should also be pursued to better understand the hadron dynamics in heavy-meson decays and meson resonances. Getting back to the future combined analysis of $D^+ \to K^-\pi^+\pi^+$ and $K^0\pi^0\pi^+$, we expect to better understand the role of the ($\pi^+\pi^0$)$_{I^P=1}^I=2K^0$ channel in both of the processes. Although we found the very important contribution of the ($\pi^+\pi^0$)$_{I^P=1}^I=2K^0$ channel to the $D^+ \to K^-\pi^+\pi^+$ decay, this finding is based on the indirect information. A more definite conclusion should be drawn through the combined analysis. Also, it will be
interesting if a three-meson-force plays a crucial role in understanding the $D$-meson decays in a unified manner. If so, this combined analysis would become a promising method to study the three-meson-force. So far, the $D^+ \rightarrow K^-\pi^+\pi^+$ decay analyses such as MIPWA have been largely motivated to determine the $(\pi^+K^-)_S$ amplitude. Being the three-body hadronic decay, MIPWA is possible only when the three-body FSI can be approximated by a coherent sum of two-body scattering amplitudes. However, our analysis strongly suggests that this approximation is not very likely to be good. A good news from our analysis is that we do not need to change the two-pseudoscalar-meson scattering amplitudes determined by the $\pi\pi$ and $\pi\bar K$ scattering data in order to fit the $D^+ \rightarrow K^-\pi^+\pi^+$ decay Dalitz plot distribution. Although there would be still a room for the two-pseudoscalar-meson interaction model to be improved, once they are fixed, we may be able to study the three-meson-force by analyzing $D$-decays into three pseudoscalar mesons, just like the three-nucleon force has been investigated through analyses of three-nucleon system in the nuclear physics. This possibility will be explored in the future work.

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Appendix A: Three-meson-force based on Hidden Local Symmetry Model

In this Appendix, we first present a set of Lagrangians from the hidden local symmetry (HLS) model [20]. Then we present expressions for potentials, derived from the Lagrangians, between vector-mesons and pseudoscalar-mesons. These potentials work as a three-body-force in a three-pseudoscalar-meson system. Finally, we present the potentials in a partial-wave form that is useful for numerical calculations.

1. Lagrangians from the HLS model

We use symbols $P$ and $V$ to denote octet pseudoscalar-mesons and nonet vector-mesons, respectively. The pseudoscalar meson fields in the SU(3) matrix form are

$$P = \frac{1}{2} \sum_{a=1}^{8} P_a \lambda_a = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \frac{1}{\sqrt{2}} \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & K^0 \\ K^- & -\frac{1}{\sqrt{6}} \pi^0 & -\frac{2}{\sqrt{6}} \eta \end{array} \right),$$

(A1)

where $\lambda_a$ is a Gell-Mann matrix, while the vector meson nonet is given by

$$V = \frac{1}{2} \sum_{a=0}^{8} V_a \lambda_a = \frac{1}{2} \sqrt{2} \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \rho^0 + \frac{1}{\sqrt{3}} \omega & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}} \rho^0 + \frac{1}{\sqrt{3}} \omega & K^{*0} \\ K^{*-} & -\frac{1}{\sqrt{3}} \rho^0 & \phi \end{array} \right),$$

(A2)

where $\lambda_0 = \sqrt{\frac{2}{3}} \mathbb{1}$ ($\mathbb{1}$: unit matrix), and the ideal mixing between the neutral vector mesons is assumed. When $P$ and $V$ are enclosed in the trace symbol, they are fields of the SU(3)
matrix form. Otherwise, e.g., they are in brackets, they are understood to be one of particles contained in the SU(3) matrix elements. It is convenient to work with isospin states rather than the charge states used in Eqs. (A1) and (A2). For the relation between the two sets of the basis, we employ a convention where the charge states are the same as their isospin states (|I|) with some exceptions that need additional phases as follows:

\[ |\rho^+\rangle = -|I = 1, I^z = 1\rangle, \quad |K^+\rangle = -|I = 1/2, I^z = -1/2\rangle, \]
\[ |\pi^+\rangle = -|I = 1, I^z = 1\rangle, \quad |K^-\rangle = -|I = 1/2, I^z = -1/2\rangle. \]  

(A3)

In what follows, |P\rangle and |V\rangle are understood to be isospin states rather than charge states. Also, we use curly symbols to denote creation or annihilation operators. For example, \( \mathcal{P} \) is the annihilation operator contained in the field \( P \), and its normalization is \( \langle 0|\mathcal{P}|0\rangle = 1 \).

The meson interaction Lagrangians we use are those from the HLS model [20, 34]. The VPP interactions are (with the Bjorken-Drell convention for the metric)

\[ \mathcal{L}_{VPP} = 2i g \text{Tr} [V^\mu (\partial^\mu PP - P \partial^\mu P)] , \]  

(A4)

where the trace is taken in the SU(3) space. The coupling \( g \) is related to the \( \rho\pi\pi \) coupling by \( g = g_{\rho\pi\pi} \), and we use \( g_{\rho\pi\pi} = 6.0 \) determined from the \( \rho \to \pi\pi \) decay width. The Yang-Mills type Lagrangian, from which we use VVV interactions, is

\[ \mathcal{L}_{YM} = -\frac{1}{2} \text{Tr} [F_{\mu\nu}F^{\mu\nu}] , \]  

(A5)

with

\[ F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ig[V_\mu, V_\nu] . \]  

(A6)

The VVP interactions are given by

\[ \mathcal{L}_{VVP} = g^2 C \epsilon^{\alpha\beta\gamma\delta} \text{Tr} [\partial_\alpha V_\beta \partial_\gamma V_\delta P] , \]  

(A7)

where we use the convention, \( \epsilon^{0123} = +1 \). The coupling is \( C = -3/(4\pi^2 f_\pi) \) with \( f_\pi \) being the pion decay constant. In our numerical calculations, we use \( g^2 C/2 = g_{\omega\pi\rho} \sim 0.012 \) from an analysis by Durso [35] on the decay \( \omega \to \pi\rho \to \pi\gamma \).

2. \( V\rho \to V'P' \) potentials

We consider a process \( V(p_V, \epsilon_V, I_V, I_V^\rho)P(p_P, I_P, I_P^\rho) \to V'(p_{V'}, \epsilon_{V'}, I_{V'}, I_{V'}^\rho)P'(p_{P'}, I_{P'}, I_{P'}^\rho) \) where the variables in the parentheses specify each particle’s state such as the four-momentum (\( p \)), polarization (\( \epsilon \)), isospin (\( I \)) and its \( z \)-component (\( I^z \)). The 0th component of the four-momentum is related to the spatial part by \( p_0 = \sqrt{p_x^2 + m_x^2} \) where \( m_x \) is the mass for a particle \( x \). The potential diagrammatically represented in Fig. 4(a) is derived from the Lagrangians in Eqs. (A4) and (A5) following the unitary transformation method [36], and is given by

\[ V_{\text{Fig.4(a)}} = \frac{g_{VV'VV'} g_{VV'VV'}}{q^2 - m_{VV'}^2} \left( -1 \right)^{I_V + 2I_{V'} + I_{V'x} - 2I_{V} + 1} \sqrt{(2I_P + 1)(2I_V' + 1)} \]
\[ \times \sum_{I} \left\{ (p_V + p_{V'}) \cdot (p_P + p_{P'}) \epsilon_V \cdot \epsilon_{V'}^* - (p_V + q) \cdot \epsilon_V (p_P + p_{P'}) \cdot \epsilon_V^* \right\} , \]  

(A8)
where \( q = p_V - p_{V'} = p_{P'} - p_P \), and \( m_{V_{ex}} \) and \( I_{V_{ex}} \) are the exchanged vector-meson mass and isospin, respectively. The propagator for the exchanged vector-meson is more explicitly written as
\[
\frac{1}{q^2 - m_{V_{ex}}^2} = \frac{1}{2} \left[ \frac{1}{(p_V - p_{V'})^2 - m_{V_{ex}}^2} + \frac{1}{(p_{P'} - p_P)^2 - m_{V_{ex}}^2} \right],
\]
(A9)
as specified by the unitary transformation [36]. The effective coupling \( g_{VV'V_{ex}} \) is given by
\[
g_{VV'V_{ex}} = \frac{g}{\sqrt{2}} \frac{\text{Tr} \left[ \langle V \rangle \langle V' \rangle \langle V_{ex} \rangle \right]}{(I_{V'}I_{V_{ex}}I_PI_{P'}I_{V}^*I_{V_{ex}}^*|II|^2)},
\]
(A10)
with \( \langle V \rangle = \langle 0|V|V \rangle \), \( \langle V' \rangle = \langle V'|V|0 \rangle \) and \( \langle V_{ex} \rangle = \langle V_{ex}|V|0 \rangle \). The effective couplings \( g_{V_{ex}PP'} \) and \( g_{VV_{ex}P'} \) appearing below are independent of isospin z-components. \( g_{V_{ex}PP'} \) is given by
\[
g_{V_{ex}PP'} = \frac{g}{\sqrt{2}} \frac{\text{Tr} \left[ \langle V_{ex} \rangle \langle P' \rangle \langle P \rangle \right]}{(I_{V_{ex}}I_{V_{ex}}^*I_PI_{P'}I_{P}^*|II|^2)},
\]
(A11)
with \( \langle V_{ex} \rangle = \langle 0|V|V \rangle \), \( \langle P \rangle = \langle 0|P|P \rangle \) and \( \langle P' \rangle = \langle P'|P|0 \rangle \).

Another potential diagrammatically represented by Fig. 4(b) is derived from the Lagrangian in Eq. (A7), and is given by
\[
V_{\text{Fig}4(b)} = \frac{g_{VV_{ex}P}g_{V_{ex}P}}{q^2 - m_{V_{ex}}^2} \sqrt{(2I_V + 1)(2I_{V'} + 1)}
\times \sum \text{W}(I_{V}I_{P}I_{P'}I_{V'}; I_{V_{ex}}I) \left( I_{V}I_{P}I_{P'}I_{P'}I_{V}I_{V_{ex}} | II^2 \right) (I_{V'}I_{V}I_{P}I_{P'}I_{V_{ex}} | II^2)
\times \left\{ p_{V'} \cdot p_{P'} (\epsilon_{V'} \cdot p_{V'} - \epsilon_{V} p_{V'} p_{P'} + p_{V'} p_{P'} \epsilon_{V'p} + p_{P'} p_{P'} \epsilon_{V'p} + p_{P'} p_{P'} p_{P'}) \right\},
\]
(A12)
where the propagator for the exchanged vector-meson is
\[
\frac{1}{q^2 - m_{V_{ex}}^2} = \frac{1}{2} \left[ \frac{1}{(p_{V'} - p_{P'})^2 - m_{V_{ex}}^2} + \frac{1}{(p_V - p_P)^2 - m_{V_{ex}}^2} \right].
\]
(A13)
The effective coupling \( g_{VV_{ex}P'} \) is given by
\[
g_{VV_{ex}P'} = 2g_{\omega_{\rho \pi}} \frac{\text{Tr} \left[ \langle P' \rangle \langle V \rangle \langle V_{ex} \rangle + \langle V_{ex} \rangle \langle V \rangle \right]}{(I_{V_{ex}}I_{V}I_{P}I_{P'}I_{P'}I_{V}^*|II|^2)},
\]
(A14)
with \( \langle P' \rangle = \langle P'|P|0 \rangle \), \( \langle V \rangle = \langle 0|V|V \rangle \) and \( \langle V_{ex} \rangle = \langle V_{ex}|V|0 \rangle \). \( g_{VV'V_{ex}} \) is obtained by simply exchanging labels \( V \rightarrow V' \) and \( P' \rightarrow P \) in \( g_{VV_{ex}P'} \).

Following Ref. [24], we multiply the following form factors to the potentials of Eqs. (A8) and (A12):
\[
F_x(q) = \left( \frac{\Lambda_x^2 - m_{V_{ex}}^2}{\Lambda_x^2 + q^2} \right)^2,
\]
(A15)
where \( \Lambda_x \) is a cutoff to be determined by fitting data. The suffix \( x \) specifies a diagram to which the form factor is multiplied: \( x = \text{Fig. 4(a1), Fig. 4(a2), and Fig. 4(a3)} \) for Fig. 4(a) with \( (V_P,V'P',V_{ex})= (\pi^*,\pi^*,\pi,\rho), (\rho^*\pi^*,\pi,\rho,\rho), \) and \( (\rho^*\pi^*,\rho^*\pi^*,\pi,\rho,\rho) \), respectively; \( x = \text{Fig. 4(b1), Fig. 4(b2), and Fig. 4(b3)} \) are for Fig. 4(b) with \( (V_P,V'P',V_{ex})= (\pi^*,\pi^*,\pi^*,\pi,\rho,\rho), (\rho^*\pi^*,\pi^*,\pi^*,\pi,\rho,\rho), \) and \( (\rho^*\pi^*,\rho^*\pi^*,\pi,\rho,\rho,\rho) \), respectively. Here \( \pi \) stands for a bare state of \( V \), as introduced below Eq. (26). We checked numerical values of the potentials by comparing our calculation with Fig. 8 of Ref. [24].
3. Partial wave expansion

In order to implement the \( VP \rightarrow V'P' \) potentials of Eqs. (A8) and (A12) into the scattering equation of Eq. (17), we need to expand the potentials in terms of the partial wave representation as,

\[
\langle V'(p_{V'}, \epsilon_{V'}, I_{V'}, I_{V'}^z) \mid V(p_V, \epsilon_V, I_V, I_V^z) \rangle = \sum_{T, J, I, l, l', I^z, I^z'} \langle I'_{V'} I'_{V'} | T T^z \rangle \langle I_{V} I_{V} | T T^z \rangle^* \times \langle I'_{V'} I'_{V'} | T T^z \rangle \langle I_{V} I_{V} | T T^z \rangle \times \langle l'_{V'} \epsilon_{V'} | J J^z \rangle \langle l_{V} \epsilon_{V} | J J^z \rangle Y_{l'_{V'}}^* \hat{p}_{V'} Y_{l_{V}} \hat{p}_V \rangle
\]

where \( JPT \) are the total angular momentum, parity, and total isospin, respectively, and \( l \) \((l')\) is the orbital angular momentum of the relative motion of \( V P \) \((V'P')\). Inverting this equation, we obtain

\[
V_{(p_{V'}, \epsilon_{V'}, I_{V'}, I_{V'}^z)} \langle | p_V |, I_V \rangle = \sum_{I_{V}, I_{V'}^z, I_{V'}^z} \langle I_{V} I_{V'}^z I_{V'} | T T^z \rangle \langle I'_{V'} I'_{V'}^z | T T^z \rangle^* \times \sum_{\epsilon_V, \epsilon_{V'}} \langle I_{V} I_{V'}^z | T T^z \rangle \langle I'_{V'} I'_{V'}^z | T T^z \rangle \times \langle I_{V} I_{V'}^z | T T^z \rangle \langle I'_{V'} I'_{V'}^z | T T^z \rangle \times \langle l'_{V'} \epsilon_{V'} | J J^z \rangle \langle l_{V} \epsilon_{V} | J J^z \rangle Y_{l'_{V'}}^* \hat{p}_{V'} Y_{l_{V}} \hat{p}_V \rangle
\]

where \( p_V \) is taken along the \( z \)-axis. This partial wave form of the potentials is plugged in Eq. (26).

Appendix B: Model parameters
TABLE VI. Model parameters for the $\pi\bar{K}$ partial wave scattering with the angular momentum $L$ and the isospin $I$. The parameters are defined in Eqs. (1), (2), (8), and (10). For each partial wave specified by $\{L,I\}$, masses ($m_{R_i}$), couplings ($g_{h_1 h_2, R_i}$), and cutoffs ($c_{h_1 h_2, R_i}$) are for the $i$-th bare $R$ states, $R_i$, and $h_1$ and $h_2$ are particles in a two-pseudoscalar-meson channel. Couplings ($h_{h_1 h_2, h_1 h_2}$) and cutoffs ($b_{h_1 h_2}$) are for the contact interactions. Masses ($m_1$, $m_2$) are for two “particles” in the effective inelastic channel, except for the $\{L,I\}=(1/2)$ wave for which $(m_1, m_2)=(m_K, m_{\eta'})$. The superscripts, $LI$, of the parameters are suppressed for simplicity. The masses and cutoffs are given in the unit of MeV. The hyphens indicate unused parameters.

| $R \{L, I\}$ | $K_0^+ \{0, 1/2\}$ | $0, 3/2$ | $K_1^+ \{1, 1/2\}$ | $K_2^+ \{2, 1/2\}$ |
|---------------|----------------|---------|----------------|----------------|
| $m_{R_1}$     | 1391           | —       | 1081           | 3070           |
| $g_{\pi K, R_1}$ | −6.28         | —       | 0.52           | 0.18           |
| $c_{\pi K, R_1}$ | 609            | —       | 1973           | 1954           |
| $g_{h_1 h_2, R_1}$ | 4.30         | —       | −0.00          | 0.97           |
| $c_{h_1 h_2, R_1}$ | 1966         | —       | 1706           | 1035           |
| $m_{R_2}$     | 1767           | —       | 1580           | —              |
| $g_{\pi K, R_2}$ | 8.22          | —       | 1.84           | —              |
| $c_{\pi K, R_2}$ | 395           | —       | 395            | —              |
| $g_{h_1 h_2, R_2}$ | −4.87        | —       | −3.00          | —              |
| $c_{h_1 h_2, R_2}$ | 395           | —       | 411            | —              |
| $m_{R_3}$     | —              | —       | 1750           | —              |
| $g_{\pi K, R_3}$ | —             | —       | 0.23           | —              |
| $c_{\pi K, R_3}$ | —             | —       | 1316           | —              |
| $g_{h_1 h_2, R_3}$ | —             | —       | 2.65           | —              |
| $c_{h_1 h_2, R_3}$ | —             | —       | 395            | —              |
| $h_{\pi K, \pi K}$ | 1.31        | 0.45    | —              | —              |
| $b_{\pi K}$  | 710            | 1973    | —              | —              |
| $m_1$         | 494            | —       | 591            | 100            |
| $m_2$         | 958            | —       | 662            | 1049           |
TABLE VII. Model parameters for $\pi\pi$ partial wave scatterings. The coupling $h'$ is defined in Eq. (3). The other features are the same as those in TABLE VI.

| $R \{L, I\}$ | $f_0 \{0, 0\}$ | $\{0, 2\}$ | $\rho \{1, 1\}$ | $f_2 \{2, 0\}$ |
|---------------|----------------|-------------|----------------|----------------|
| $m_{R_1}$     | 1166           | —           | 850            | 1561           |
| $g_{\pi\pi,R_1}$ | 5.97         | —           | 1.02           | −0.32          |
| $c_{\pi\pi,R_1}$ | 1162         | —           | 805            | 962            |
| $g_{K\bar{K},R_1}$ | −2.19       | —           | −0.18          | 0.19           |
| $c_{K\bar{K},R_1}$ | 1973         | —           | 395            | 1216           |
| $m_{R_2}$     | 1627           | —           | 1551           | —              |
| $g_{\pi\pi,R_2}$ | −5.23        | —           | 0.49           | —              |
| $c_{\pi\pi,R_2}$ | 1973         | —           | 1973           | —              |
| $g_{K\bar{K},R_2}$ | 11.99        | —           | 3.74           | —              |
| $c_{K\bar{K},R_2}$ | 533          | —           | 395            | —              |
| $h_{\pi\pi,\pi\pi}$ | 0.47         | 0.11        | —              | —              |
| $h'$          | —             | 0.21        | —              | —              |
| $b_{\pi\pi}$  | 897           | 913         | —              | —              |
TABLE VIII. Parameters determined by fitting $D^+ \to K^- \pi^+ \pi^+$ Dalitz plot pseudo-data; $D^+ \to hR_{L_i}^{L_{2i}}$ ($h=\pi, K$) bare coupling ($C_{hR_{L_i}^{L_{2i}}}$), phase ($\phi_{hR_{L_i}^{L_{2i}}}$) in degrees, and cutoff ($\Lambda_{hR_{L_i}^{L_{2i}}}$) in MeV, as defined in Eq. (33). $R_{i}^{L_{2i}}$ stands for $i$-th bare $R$ state with the spin $L$ and isospin $I$. $r_{ab}^{L_{2i}}$ stands for the spurious state (see Sec. II B for the definition) that decays into two pseudoscalar mesons with the orbital angular momentum $L$ and total isospin $I$. The total spin, parity, total isospin, and $hR$ relative orbital angular momentum are $J=0$, $P=+1$, $T=3/2$, and $l=L$, respectively for all the parameters, and thus $JPT$ and $l$ labels are suppressed. The second, third, forth, and fifth columns show the parameters for the Full, Z, Z(without $\rho$), and Isobar models, respectively. The hyphens indicate unused parameters.

|       | Full       | Z          | Isobar     | Z(without $\rho$) |
|-------|------------|------------|------------|-------------------|
| $C_{\pi R_1^{01}}$ | 0.78       | 0.70       | 0.79       | 0.92              |
| $\phi_{\pi R_1^{01}}$ | 27         | 2          | 329        | 255               |
| $\Lambda_{\pi R_1^{01}}$ | 1330       | 613        | 610        | 449               |
| $C_{\pi R_2^{01}}$ | 0.95       | 0.86       | 0.84       | 0.75              |
| $\phi_{\pi R_2^{01}}$ | 264        | 244        | 193        | 229               |
| $\Lambda_{\pi R_2^{01}}$ | 1210       | 1546       | 1818       | 1157              |
| $C_{\pi R_1^{11}}$ | 0.03       | 0.07       | 0.08       | 0.06              |
| $\phi_{\pi R_1^{11}}$ | 220        | 187        | 211        | 171               |
| $\Lambda_{\pi R_1^{11}}$ | 1536       | 997        | 2594       | 796               |
| $C_{\pi R_1^{12}}$ | 1 (fixed)  | 1 (fixed)  | 1 (fixed)  | 1 (fixed)         |
| $\phi_{\pi R_1^{12}}$ | 0 (fixed)  | 0 (fixed)  | 0 (fixed)  | 0 (fixed)         |
| $\Lambda_{\pi R_1^{12}}$ | 458        | 422        | 640        | 435               |
| $C_{\pi R_2^{12}}$ | 0.14       | 0.20       | 0.33       | 0.14              |
| $\phi_{\pi R_2^{12}}$ | 25         | 48         | 60         | 1                 |
| $\Lambda_{\pi R_2^{12}}$ | 1183       | 1026       | 1363       | 1998              |
| $C_{\pi R_3^{12}}$ | 0.15       | 0.07       | 0.06       | 0.09              |
| $\phi_{\pi R_3^{12}}$ | 196        | 352        | 9          | 242               |
| $\Lambda_{\pi R_3^{12}}$ | 578        | 677        | 1906       | 1425              |
| $C_{\pi R_4^{12}}$ | 0.03       | 0.03       | 0.04       | 0.02              |
| $\phi_{\pi R_4^{12}}$ | 185        | 159        | 168        | 159               |
| $\Lambda_{\pi R_4^{12}}$ | 2137       | 2236       | 2170       | 1641              |
| $C_{KR_1^{12}}$ | 0.52       | 0.51       | —          | —                 |
| $\phi_{KR_1^{12}}$ | 282        | 266        | —          | —                 |
| $\Lambda_{KR_1^{12}}$ | 1476       | 1452       | —          | —                 |
| $C_{KR_2^{12}}$ | 1.39       | 1.26       | —          | —                 |
| $\phi_{KR_2^{12}}$ | 96         | 62         | —          | —                 |
| $\Lambda_{KR_2^{12}}$ | 1950       | 1779       | —          | —                 |
| $C_{\pi R_5^{03}}$ | 0.13       | 0.23       | 0.17       | 0.31              |
| $\phi_{\pi R_5^{03}}$ | 130        | 131        | 170        | 84                |
| $\Lambda_{\pi R_5^{03}}$ | 1552       | 1487       | 1564       | 1071              |
| $C_{KR_5^{03}}$ | 0.04       | 0.04       | 0.19       | 0.10              |
| $\phi_{KR_5^{03}}$ | 325        | 39         | 82         | 353               |
| $\Lambda_{KR_5^{03}}$ | 1820       | 1333       | 1192       | 1184              |
TABLE IX. Parameters determined by fitting $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot pseudo-data. The modulus (phase) of the flat background is $a_{BG}$ ($\phi_{BG}$), as introduced in Eq. (36). The cutoffs for the vector-meson exchange potentials for $VP \rightarrow V'P'$ are denoted by $\Lambda_x$ [see Eq. (A15) for definition].

|                  | Full   | Z   | Isobar | Z(without $\rho$) |
|------------------|--------|-----|--------|-------------------|
| $a_{BG}$         | 0.07   | 0.52| 5.26   | 4.37              |
| $\phi_{BG}$      | 168    | 256 | 219    | 265               |
| $\Lambda_{\text{Fig.4(a1)}}$ | 1378   | —   | —      | —                 |
| $\Lambda_{\text{Fig.4(a2)}}$ | 1572   | —   | —      | —                 |
| $\Lambda_{\text{Fig.4(a3)}}$ | 1643   | —   | —      | —                 |
| $\Lambda_{\text{Fig.4(b1)}}$ | 1324   | —   | —      | —                 |
| $\Lambda_{\text{Fig.4(b2)}}$ | 1474   | —   | —      | —                 |
| $\Lambda_{\text{Fig.4(b3)}}$ | 1656   | —   | —      | —                 |
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