Bosonization and the Shear Sound of Two-Dimensional Fermi Liquids

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We study the appearance of a sharp collective mode which features transverse current fluctuations within the bosonization approach to interacting two-dimensional Fermi liquids. This mode is analogous to the shear sound modes in elastic media, and, unlike the conventional zero sound mode, it is damped in weakly interacting Fermi liquids and only separates away from the particle-hole continuum when the quasiparticle mass becomes twice the transport mass \( m^* \gtrsim 2m \). The shear sound should be present in a large class of interacting Fermi liquids especially those proximate to critical points where the quasiparticle mass diverges. We also detail a quick path between bosonization and classical Landau’s Fermi liquid theory by constructing a mapping between the solutions of the classical kinetic equation and the quantized bosonic eigenmodes. By further mapping the kinetic equation into a 1D tight-binding model we solve for the entire spectrum of collective and incoherent particle-hole excitations of Fermi liquids with non-zero \( F_0 \) and \( F_1 \) Landau parameters.

**Introduction.** When Landau introduced his theory of the Fermi liquid more than 60 years ago [1, 2] it was not immediately clear the extent to which it was an approximate description. Subsequent developments, such as the theorem of Luttinger [3] asserting the adiabatic invariance of the Fermi volume, contributed to strengthen the belief on the essential validity of Landau’s theory. At the dawn of the twentieth century the advent of modern approaches like the renormalization group of fermions [4] and higher dimensional bosonization [5–8] contributed to cement the agreement that in two-dimensions and higher Landau’s Fermi liquid theory (LFLT) captures the essential long wavelength and low energy behavior of a large class of interacting systems with a Fermi surface known as Landau Fermi liquids (LFL).

In this letter we describe a short path between higher dimensional bosonization and LFLT which highlights their intimate connection. We will demonstrate that the harmonic nature of the bosonized theory leads to an equivalence between classical and quantum equations of motion, analogous to how the Ehrenfest theorem relates the dynamics of classical and quantum Harmonic oscillators. This connection is more transparent when parametrizing the theory in non-canonical variables in which the equation of motion reduces to Landau’s kinetic equation. Our approach is inspired by and in close connection to top-down approaches to bosonization which start from LFLT viewed as a classical field theory and construct from it a quantum field theory by inferring the quantization relations of its classical variables [9, 10].

We apply this formalism to study an unconventional collective mode in Fermi liquids that has the characteristics of a shear sound wave resembling the transverse excitations of an elastic medium. This mode is absent in classical fluids due to their vanishing shear modulus.

**Formalism.** The long-wavelength and low-energy description of most phases of matter involves a finite number of continuum fields related to conservation laws and order parameters in the case of broken symmetry phases. However, LFL depart radically from this, in that they have an infinite number of slow degrees of freedom which parametrize the shape of the Fermi surface [6]. The state of a LFL can be parametrized by the Fermi radius at any point in space \( x, \mathbf{p}_F = \mathbf{p}^0_F + \mathbf{u}_x, \theta \), where \( \theta \) is the angle on the Fermi surface. In bosonization the Fermi radius becomes a quantum mechanical operator whose algebra is given by [6–10]:

\[
\left[ \hat{u}_{x,\theta}, \hat{u}_{x',\theta'} \right] = \frac{(2\pi)^2}{ip_F} \delta(\theta - \theta') \delta_n (\mathbf{x} - \mathbf{x}') + O(\hat{u}), \tag{1}
\]

where \( \partial_n = \hat{p}_\theta \cdot \partial_x \) is the derivative along the normal \( \hat{p}_\theta \)

In weakly interacting Fermi liquids only the zero sound mode is a sharp collective excitation [11, 12]. However, as we will demonstrate, in 2D a well separated shear sound mode emerges from the continuum when the quasiparticle mass, \( m^* \), becomes twice the transport mass, \( m \). Such enhancement could be readily accessible in systems proximate to a critical point at which \( m^* \) diverges.

**FIG. 1:** (a) Zero and (b) shear sound with wavevector \( \mathbf{q} \) parallel to the x-axis. The color scale represents the density and the arrows the current fluctuations.
of the Fermi surface. We introduce a matrix notation for \( \theta \) that will compactify our formulae, by defining:

\[
\psi^\dagger_\theta G_{\theta,\theta'} w_{\theta'} \equiv \int d\theta d\theta' v^*(\theta) G(\theta, \theta') w(\theta').
\]

(2)

With this notation, the Hamiltonian governing the dynamics of the Fermi surface can be written as:

\[
\hat{H} = \int d^2x \hat{u}_{x,\theta}^\dagger \hat{h}_{x,\theta} \hat{u}_{x,\theta'},
\]

(3)

where \( h(\theta, \theta') = v_F p_F (2\pi \delta(\theta' - \theta) + F(\theta' - \theta))/(2\pi)^3 \). \( F(\theta' - \theta) \) is the Landau function characterizing the interactions between quasiparticles [13]. Notice that LFLT has an infinite number of conserved quantities which measure the spatially averaged shape of the Fermi surface. Formally, any operator of the form \( \hat{g}(\theta) = \int d^2x \ g(\theta) \hat{u}_{x,\theta} \) is a conserved quantity.

To exploit translational invariance we introduce the Fourier modes of the Fermi surface deformations \( \hat{u}_{q,\theta} \equiv \int d^2x \ \hat{u}_{x,\theta} e^{-iq\cdot x} \). These operators can be interpreted as bare particle-hole creation operators \( c_{q+1/2}^\dagger p_{q-1/2} \) with \( p \) coarse grained over a region near the angle \( \theta \) on the Fermi surface [8, 14]. The equation of motion following from Eqs. (1) and (3) for these operators is:

\[
i_\partial \hat{u}_{q,\theta} = \left[ \hat{u}_{q,\theta}, \hat{H} \right] = K_{\theta,\theta'} \hat{u}_{q,\theta'},
\]

(4)

\[
K(\theta, \theta') = v_F q \cdot \hat{p}_0 \left( \delta(\theta - \theta') + \frac{1}{2\pi} F(\theta - \theta') \right).
\]

(5)

The equation above can be recognized to be an operator version of the classic Landau’s linearized kinetic equation [11, 12]. Notice that \( \hat{u}_{q,\theta} \) do not satisfy canonical bosonic commutation relations and that the kinetic matrix, \( K_{\theta,\theta'} \), is non-Hermitian. However, there exists a simple similarity transformation between \( K \) and its Hermitian conjugate:

\[
K = T K^\dagger T^{-1}, \quad T_{\theta,\theta'} = \frac{(2\pi)^2 q \cdot \hat{p}_0}{p_F} \delta(\theta - \theta').
\]

(6)

We are now in a position to state a mapping between the classical solutions of Landau’s kinetic equation and their quantum counterpart. For each classical eigenmode of the kinetic equation, \( \psi_{\lambda, q, \theta} \), there is a quantum eigenmode, \( \hat{\psi}_{\lambda, q, \theta} \), given by:

\[
\hat{\psi}_{\lambda, q, \theta} = \psi^\dagger_{\lambda, q, \theta} T_{\theta, \theta'}^{-1} \hat{u}_{q, \theta'}.
\]

(7)

where \( K_{\theta,\theta'} \hat{\psi}_{\lambda, q, \theta'} = E_{\lambda} \hat{\psi}_{\lambda, q, \theta} \) and \( i_\partial \hat{\psi}_{\lambda, q, \theta} = E_{\lambda} \hat{\psi}_{\lambda, q, \theta} \). By choosing a suitable normalization for the classical solutions, \( \psi^\dagger_{\lambda, q, \theta} T_{\theta, \theta'}^{-1} \hat{u}_{q, \theta'} = \text{sgn}(E_{\lambda}) \delta_{\lambda, \lambda'} \), we arrive at canonical bosonic eigenmodes describing the fluctuations of the shape of the Fermi surface:

\[
\left[ \hat{\psi}_{\lambda, q, \theta}, \hat{\psi}^\dagger_{\lambda', q', \theta'} \right] = (2\pi)^3 \delta(q - q') \text{sgn}(E_{\lambda}) \delta_{\lambda, \lambda'}.
\]

(8)

where the sign of the eigenvalue \( E_{\lambda} \) dictates which one of the pair \( \psi_{\lambda, q}, \psi^\dagger_{\lambda, q', \theta'} \) is the raising and which one is lowering operators. These eigenmodes describe both collective oscillations such as the zero sound and also the continuum of particle-hole excitations. Any two-body operator can be represented as a linear combination of these modes and in particular: \( \hat{u}_{x, \theta} = \sum_{\lambda} \text{sgn}(E_{\lambda}) \psi_{\lambda, q} \psi_{\lambda, q, \theta} \).

**Mapping to a chain.** As we have seen, the quantum problem reduces to the eigenvalue problem of the classic kinetic equation. We begin by simplifying the classical eigenvalue problem by exploiting its symmetries. Rotational symmetry allows us to restrict \( q = q^\perp \). We measure the angle along the Fermi surface, \( \theta \), from this axis. Additionally, we assume a mirror symmetry \( F(\theta) = F(-\theta), \ K_{\theta, \theta'} = K_{-\theta, -\theta'} \), which decouples the even and odd parity eigenmodes, which we label with a superscript \( \sigma = \pm \) denoting: \( \psi^\sigma_{\lambda, q, \theta} = \sigma \psi^\sigma_{\lambda, q, -\theta} \).

There is also a time-reversal symmetry \( K^T_{\theta, \theta'} = K_{-\theta, -\theta'} \) which implies that the eigenfunctions can be taken to be purely real [15]. The kinetic equation also has a particle-hole-like symmetry which follows from an inversion in momentum space: \( K_{\theta + \pi, \theta' + \pi} = -K_{\theta, \theta'} \). Therefore, the eigenfunctions, \( \psi^\sigma_{\lambda, q, \theta} \), come in pairs with opposite eigenvalues. Namely, if \( \psi^\sigma_{\lambda, q, \theta} \) is an eigenfunction with eigenvalue \( E^\sigma_{\lambda} \), then \( \psi_{\lambda, q, \theta + \pi} \) is an eigenfunction with eigenvalue \(-E^\sigma_{\lambda} \). For fixed \( q \) these two solutions describe physically distinct modes. The one with positive (negative) eigenvalue will be a creation (destruction) operator, and, its destruction (creation) operator partner will live in the space of excitations with momentum \(-q\). This feature can be traced back to the property that particle-hole excitations with small momentum \( q \) can only be created in one of the halves of the Fermi surface satisfying \( q \cdot \hat{p}_0 > 0 \).

We now describe a convenient representation of the kinetic equation in a similar spirit to a recent treatment of spin orbit coupled systems [16]. We begin by decomposing into angular momentum channels (\( q \) implicit below):

\[
F(\theta) = F_0 + \sum_{l=1}^{\infty} 2 F_l \cos(l \theta)
\]

(9)

\[
\psi^\dagger_{\lambda, \theta} = \psi^\dagger_{\lambda, 0} + \sum_{l=1}^{\infty} 2 \psi^\dagger_{\lambda, l} \cos(l \theta),
\]

(10)

\[
\psi^\dagger_{\lambda, \theta} = \sum_{l=1}^{\infty} 2 \psi^\dagger_{\lambda, l} \sin(l \theta).
\]

(11)

With this the kinetic equation takes the form of a non-
Hermitian tight-binding model in which the sites are the angular momentum channels:

\[ E_l^{\sigma} \psi_{\lambda,l+1} = t_l \psi_{\lambda,l} + t_{l+2} \psi_{\lambda,l+2}, \]

where \( t_l = v_F q (1 + F_l) / 2 \) for \( l \neq 0 \) and \( t_0 = v_F q (1 + F_0) \) and the coefficients \( \psi_{\lambda,l} \) are understood to vanish when \( l < 0 \) for \( \sigma = + \) and when \( l < 1 \) for \( \sigma = - \). We see that the Landau parameters play the role of bond-disorder in the effective tight binding model. Notice that the eigenvalue problem for the odd modes is completely independent of \( F_0 \). A remarkable property which becomes transparent in this way of writing the problem is that there exists a simple relation between the eigenvalue problem in the odd and even subspaces. Namely, the eigenvalue problem in the even sector for a set of Landau parameters \( \{ F_l \} \) can be mapped into the problem in the odd sector with modified Landau parameters \( \{ F'_l \} \) by relabelling sites as \( l \rightarrow l + 1 \), such that the Landau parameters are related by \( F'_{l+1} = F_l \) for \( l \geq 1 \) and \( F'_1 = 1 + 2 F_0 \). Since the kinetic equation is customarily solved by truncating the Landau parameters up to some \( l \), then, one only needs to solve for the even sector and apply this mapping to obtain the odd sector solution.

**Shear sound.** We begin by considering the simplest interacting Fermi liquid with only a non-zero s-wave Landau parameter, \( F_0 \neq 0 \) and \( F_{l>0} = 0 \). In this case the tight binding chain has only one defective bond connecting the \( l = 0 \) site at the end of the chain in the even sector \( \sigma = + \). As detailed in the supplementary material \[17\], one can solve Eq. 12 recursively. There are two kinds of solutions. The first kind form the analogue of “band” and describe excitations in the particle-hole continuum \((E < v_F q)\), and are found to be (up to global constant):

\[ \psi_{l \geq 1}^+ = \cos l \theta_E - F_0 \frac{\sin (l - 1) \theta_E}{\sin \theta_E}, \]

where \( \cos \theta_E = E / v_F q \) and parametrizes the angle on the Fermi where the particle-hole pair is created. The second kind are isolated solutions analogous to bound states created by the “bond-disorder”. The \( F_0 \) model has a single isolated bound state that is present only for \( F_0 > 0 \) and corresponds to the celebrated zero sound mode. Its dispersion is found to be \[18\]:

\[ \frac{E_0}{v_F q} = \frac{1 + F_0}{\sqrt{1 + 2 F_0}}, \quad F_0 > 0, \]

And the wavefunction of the zero-sound is:

\[ \psi_{l \geq 1}^+ = \psi_0^+ \frac{2(F_0 + 1)}{(1 + 2 F_0)^{1/2}}, \]

\[ \psi_0^+ = \left( \frac{\pi q}{2 p v} \frac{F_0}{(1 + 2 F_0)^{1/2} (1 + 2 F_0 + 2 F_0^2)} \right)^{1/2}. \]

In the \( F_0 \) model the odd parity modes are identical to the non-interacting Fermi gas, and, hence there is no transverse collective modes and only the particle-hole continuum. We will now consider a more realistic model of the Landau Fermi liquid which has non-vanishing \( \{ F_0, F_1 \} \) Landau parameters. The mapping described in the previous section between odd and even parity sectors immediately implies that this model can support an undamped collective odd mode. The dispersion and wavefunction of this model can be obtained from the zero sound solutions by replacing \( l \rightarrow l + 1 \), \( F_0 \rightarrow (F_1 - 1) / 2 \) and read as \[19\]:

\[ \frac{E_1}{v_F q} = \frac{1 + F_1}{2 \sqrt{F_1}}, \quad F_1 > 1, \]

\[ \psi_{l \geq 2}^- = \psi_1^- \frac{F_1 + 1}{F_1^{1/2}}, \quad \psi_1^- = \left( \frac{\pi q}{4 p v} \frac{F_1 - 1}{F_1^{3/2}} \right)^{1/2}. \]

The shape of the Fermi surface deformations associated with shear and zero sound modes are illustrated in Fig. 2. As we will see this extra collective mode features transverse current fluctuations with no density oscillations in analogy with the shear sound of elastic media. The eigenmodes in the continuum can be obtained by the same mapping. The even sector gets modified by the introduction of a finite \( F_1 \) but not in an essential way \[20\].

The study of shear fluctuations of interacting electrons has an important precedent in the work of Conti and Vignale \[21\] (see also \[22\]). Our expression for the shear sound velocity differs from theirs \[21\]. The origin of this discrepancy is presently unclear to us, but, we emphasize that our results are expected to be exact in the long-wavelength limit of a LFL provided that higher angular momentum Landau parameters \((l \geq 2)\) are negligible.

**FIG. 2:** Fermi surface deformations for zero (a) and shear (b) sound eigenmodes \((F_0 = 1, F_1 = 3)\).

**Density and current responses.** Any two body operator has a linear expansion in bosonic eigenmodes:
\begin{equation}
\dot{O}_q = \int d\theta O(q, \theta) \dot{u}_{q, \theta} = \sum_{\lambda} O_{\lambda, q} \dot{\psi}_{\lambda, q}.
\end{equation}

This expansion allows to quantify the amplitude of the oscillation of any physical quantity in any given eigenmode and to compute linear response functions. In particular the density and current operators of the liquid take the following form: \( \dot{\rho}_q = \rho_F \int d\theta \hat{u}_{q, \theta} / (2\pi)^2 \), \( \dot{J}_q = \rho_F^2 / m \int d\theta \hat{u}_{q, \theta} \dot{\rho}_q / (2\pi)^2 \), where \( m \) is the transport mass that controls the Drude weight. The amplitudes of these quantities is found to be:

\begin{equation}
\rho_{\lambda, q} = \text{sgn}(E_{\lambda}) \frac{\rho_F}{2\pi} \psi_{\lambda, 0}^+.
\end{equation}

\begin{equation}
J_{\lambda, q} = \text{sgn}(E_{\lambda}) \frac{\rho_F^2}{2\pi m} (\psi_{\lambda, 0}^+ q + \psi_{\lambda, 0}^- q).\n\end{equation}

Notice that the density and the longitudinal component of the current only have weight in the even parity sector, whereas the transverse component of the current only has weight in the odd parity sector. Therefore the shear sound, which has odd parity, will have purely transverse current oscillations with no accompanying density fluctuations. The imaginary part of the transverse current-current correlation will feature a sharp peak at the energy of the shear sound mode when it separates from the particle-hole continuum for \( F_1 > 1 \) (for details of correlation functions see [17]). The spectral weight of this peak vanishes as \( F_1 \to 1 \) and is found to be [17]:

\begin{equation}
w_{j, j, \perp} = \rho_F^4 \frac{F_1 - 1}{16 m^2 F_1^{3/2}.}
\end{equation}

Discussion. We begin by discussing the applicability of our results. For brevity we have focused on spinless fermions but our results apply as well to the case of the symmetric modes of spin unpolarized systems in which spin up and down Fermi surfaces oscillate identically. Also, we have focused on Fermi liquids interacting via short range forces. LFLT in metals requires accounting for the long ranged Coulomb interaction. However, to a good approximation, the Coulomb interaction modifies only the behavior of the modes in the even sector which involve longitudinal current-density fluctuations, e.g. transforming the zero sound into a plasma mode [11]. Modes in the odd sector, like the shear sound, remain unaltered by the Coulomb interaction because they do not involve charge fluctuations and hence our discussion of these modes is applicable to metals [20].

Although we have focused on two-dimensions similar phenomena can occur in three-dimensions. In fact, the possibility of a shear sound mode in \(^3\text{He}\) was long ago recognized. In 3D a critical Landau parameter \( F_1 \gtrsim 6 \) is required [11, 12, 23–25]. Although the Landau parameters of \(^3\text{He}\) are believed to be above this value [12] experimental observation [26] of this collective mode has remained elusive [27] because it remains close to the particle-hole continuum even at largest attainable values of \( F_1 \) [12].

LFLT is parametrized by an infinite number of dimensionless parameters, \( \{F_l\} \), whose determination for specific microscopic models can only typically be done approximately. Fortunately, the leading Landau parameters have simple relations to common experimental probes. In particular, \( F_1 \), controls the ratio of the quasi-particle mass to the transport mass [28] \( m^*/m = 1 + F_1 \). Notice that the transport mass only equals the bare mass \( m_0 \) in Galilean invariant systems [29–31]. \( m^* \) can be obtained from specific heat measurements, or quantum oscillations, while \( m \) can be inferred from the Drude weight, or the London penetration length [32].

Thus, we expect that in systems where interactions have rendered \( m^* \gtrsim 2m \) \( (F_1 > 1) \) the shear sound will emerge out of the particle-hole continuum as a sharp excitation [33]. We suspect that such relatively moderate renormalization should be accessible in a variety of two-dimensional LFL. For example, in \(^3\text{He}\) films on graphite [34, 35] where \( m^* \) diverges on approaching a Mott transition [36, 37]. Also in quasi-2D metals near criticality such as the Iron based superconductors which have a diverging \( m^* \) [38, 39]. It is under debate if both or only one of the masses is enhanced at such critical point [40–44]. The finite and smooth behavior of the residual conductivity near the critical point [45] suggests that \( m^* \) has greater enhancement than \( m \) as required for the appearance of the shear sound [46]. Additional candidates include quasi-two-dimensional heavy-Fermion materials with large enhancements of the quasi-particle mass [47–49], and ultracold fermionic gases with enhanced p-wave interactions [50, 51].

Finally we would like to comment on potential experimental probes. One way to study this collective mode is to measure ultra-sound attenuation as attempted in \(^3\text{He}\) films on graphite [26, 27]. Alternatively, in metals, devices like the Corbino viscometer [52] or multi-terminal devices that could generate vorticity of current flow [53, 54], such as those studied in the hydrodynamic approach to electron transport [55, 56], could be used to excite shear sound provided they can be operated in a sufficiently fast dynamical regime to minimize the excitation of particle-hole pairs. It would also be interesting to study the behavior of the shear sound under magnetic fields, which recent studies have incorporated within the bosonization formalism [57, 58].

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We focus on circularly symmetric Fermi surfaces and note that the Fermi radius field $\hat{u}_{x,\theta}$ is understood to be a Hermitian operator.

We are assuming the eigenvalues to be real, which requires the Hamiltonian in Eq. 3 to be positive definite, which is the usual condition to avoid Pomeranchuk-like instabilities.

Provided that the higher angular momentum Landau parameters ($l \geq 2$) remain small.

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SUPPLEMENTARY MATERIAL

Derivation of Kinetic Equation Solutions

We derive in detail the solution to the kinetic equation Eq. 12 in the main text, which is a recursion relation. In particular, we solve for the even mode wavefunction

$$\psi_{\lambda,0}^{+} = \psi_{\lambda,0}^{-} + \sum_{l=1}^{\infty} 2\psi_{\lambda,l}^{+} \cos(l\theta),$$

(23)

for the example of a Fermi liquid with \( F_0 \neq 0 \) and \( F_{l>0} = 0 \). We will suppress the energy index label \( \lambda \) and the parity label + to avoid clutter and reintroduce them at the appropriate discussions that follow. In this case, Eq. 14 takes on the simple form

$$\psi_{l+2} = 2s\psi_{l+1} - \psi_l, \quad l \geq 1,$$

(24)

with the “initial conditions” \( \psi_1 = 2s\psi_0 \) and \( \psi_2 = (4s^2 - 2a_0)\psi_0 \), where \( \psi_0 \) becomes an overall constant fixed by normalization. Here we have introduced the reduced energy \( s = \frac{E_0}{\sqrt{\psi_q}} \). Note that the two initial conditions are obtained directly from Eq. 14. Rewriting the above recursion relation Eq. 24 as \( 0 = \sum_{l=1}^{\infty} (\psi_l - 2s\psi_{l-1} + \psi_{l-2}) r^l \) gives rise to the explicit solution for the generating function

$$\psi(r) = \sum_{l=0}^{\infty} \psi_l r^l = \frac{\psi_0 (1 + \alpha r^2)}{(r_+ - r)(r_- - r)},$$

(25)

where \( \alpha = 1 + 2F_0 \) and \( r_{\pm} = s \pm \sqrt{s^2 - 1} \). The wavefunction is then given by the coefficients obtained from expanding the generating function in powers of \( r \),

$$\psi_{l \geq 1} = \frac{\psi_0}{2\sqrt{s^2 - 1}} \left( \frac{1 - \alpha r^2}{r_+^{l+1}} - \frac{1 - \alpha r^2}{r_-^{l+1}} \right).$$

(26)

The behavior of \( \psi \) is different for modes in the particle-hole continuum, \( 0 < s < 1 \), and excitations above the continuum, \( s > 1 \). When \( 0 < s < 1 \), we write \( r_{\pm} = e^{\pm i\theta_F} \) and find that the wavefunctions of such excitations are oscillatory and therefore do not diverge,

$$\psi_{l \geq 1} = 2\psi_0 \left( \cos l\theta_F - \frac{F_0}{\sin\theta_F} \sin(l - 1)\theta_F \right).$$

(27)

The system therefore supports excitations of any energy \( E_\lambda < \psi_q \) and moreover, such excitations are all-ways localized on the Fermi surface and correspond to the quasiparticle excitations of the system.

On the other hand, the wavefunction generally diverges for any arbitrary value \( s > 1 \) such that solutions do not generally exist. This divergence can be seen from \( \lim_{l \to \infty} \psi_l \sim \lim_{l \to \infty} \frac{1 - \alpha r^2}{r_-^{l+1}} \to \infty \) because \( 0 < r_- < 1 \) when \( s > 1 \). However, solutions can exist under specific conditions when the numerator of this divergent term vanishes, i.e., when the condition \( 1 - (1 + 2F_0)(s - \sqrt{s^2 - 1})^2 = 0 \) is satisfied. The solution \( s_0 > 0 \) to this condition is given by Eq. 14 in the main text,

$$s_0 = \frac{E_0}{\psi_q} = \frac{1 + F_0}{\sqrt{1 + 2F_0}}, \quad F_0 > 0,$$

(28)

where we recover the exact zero sound velocity \( v_0 = \frac{E_0}{\psi_q} \) obtained from the classical Khalatnikov/Abrikosov approach [18]. The corresponding zero sound wavefunction simplifies to Eq. 15 of the main text,

$$\psi_{l \geq 1}^+ = \psi_0 \frac{2(F_0 + 1)}{(1 + 2F_0)^{1/2}}, \quad F_0 > 0,$$

(29)

Unlike modes in the particle-hole continuum, such excitations are always delocalized over the Fermi surface and correspond to the collective modes of the system (see Fig.2 of main text). The criteria of non-divergence of \( \psi_l \) explains why at least \( F_0 > 0 \) is required for the zero sound mode to exist and exactly determines the value of its velocity as a function of \( F_0 \).

The normalization constant \( \psi_q^+ \) is determined from the condition \( \psi_{l,a,q}^+ T_{l,a,q}^{-1} \psi_{l,a,q} = \text{sgn}(E_\lambda)\delta_{\lambda,\lambda'} \) for the case when \( \lambda = \lambda' = s_0 \),

$$1 = \psi_{s_0,0}^+ T_{0,0}^{-1} \psi_{s_0,0}^+$$

$$= \psi_0^+ \frac{p_F}{(2\pi)^2q} \sum_{l,m=0}^{\infty} a_{l,m} \psi_{s_0,l}^+ \psi_{s_0,m} \gamma_{lm},$$

(30)

where

$$\gamma_{lm} = \int d\theta \cos(l \theta) \cos(m \theta)$$

(31)

$$= \begin{cases} 0 & \text{if } l \text{ is even or } m < l \text{ is odd} \\ 2\pi l^{l+m-1} & \text{otherwise and } m < l \end{cases}$$

(32)

can be evaluated via contour integration. Substituting Eq. 29 into Eq. 30 and evaluating the sum explicitly, one eventually arrives at the normalization constant given by Eq. 16 in the main text,
\[
\psi_0^+ = \left( \frac{\pi q}{2\rho_F (1 + 2F_0)^{1/2}(1 + 2F_0 + 2F_0^2)} \right)^{1/2} F_0. \tag{32}
\]

Repeating the steps above to solve for the eigenmodes of an LFL with non-trivial \(F_{i<2}\) and \(F_{i\geq2} = 0\), we find the solution to the odd collective mode, i.e. the shear sound mode:

\[
\psi_{s_1,\theta} = \sum_{i=1}^{\infty} 2\psi_{s_1,l} \sin(l\theta), \tag{33}
\]

\[
s_1 = \frac{E_1}{q \nu_{\chi}} = \frac{1 + F_1}{2\sqrt{F_1}}, \quad F_1 > 1, \tag{34}
\]

\[
\psi_{s_1,l>2} = \psi_{s_1,l}^*-\frac{1}{F_1^{1/2}}, \quad F_1 > 1, \tag{35}
\]

as per Eq. 17 and Eq. 18 in the main text where \(v_{\chi} = \frac{E_1}{q \nu_{\chi}}\) is the shear sound velocity. Equations 34 and 35 can alternatively be obtained from the zero sound solutions (Eqs. 28 and 29) by replacing \(l \to l + 1\), \(F_0 \to (F_1 - 1)/2\).

The normalization constant \(\psi_1^+\) however cannot be obtained simply from this mapping and has to be worked out separately due to \(\psi_{s_0,0}^+\) having a unit prefactor in \(\psi_{s_1,\theta}^+\) in Eq. 23 compared to \(\psi_{s_1,1}^+\) having a prefactor of \(2\) in \(\psi_{s_1,\theta}^+\) in Eq. 33. The normalizing condition for the odd modes read,

\[
1 = \psi_{s_1,\theta}^+ T_{\theta,\theta}^{-1} \psi_{s_1,\theta}^+ \tag{36}
\]

\[
= \frac{\rho_F}{(2\pi)^2} \sum_{l,m=1}^\infty a_{lm} \psi_{s_1,l}^p \psi_{s_1,m}^p \tilde{\gamma}_{lm},
\]

\[
\tilde{\gamma}_{lm} = \int d\theta \frac{\sin(l\theta) \sin(m\theta)}{\cos(\theta)}
\]

\[
\begin{cases} 0 & \text{if } l \text{ is odd or } m < l \text{ is even} \\ 2\pi l^2 + m^2 & \text{otherwise and } m < l \end{cases}
\]

(37)

which can be evaluated to obtain the result given in Eq. 18 in the main text,

\[
\psi_{s_1}^+ = \left( \frac{\pi q}{4\rho_F F_1} \right)^{1/2} F_0. \tag{39}
\]

Introducing a non-trivial \(F_1\) does however affect the even modes. It can be shown that \(s_0(F_0,F_1)\) and \(s_1(F_1,F_2)\) can also be derived explicitly by repeating the above procedure, the details of which will be discussed in Ref [20].

### Derivation of Density and Current Response Functions

In this section, we provide a detailed derivation of the response functions given in the main text. The operator corresponding to an observable \(\mathcal{O}(q) = \langle \hat{O}_q \rangle\) can be explicitly expressed in terms of its quantum eigenmodes,

\[
\hat{O}_q = \int d\theta \mathcal{O}(q,\theta) \hat{u}(q,\theta) = \sum_{\lambda} O_{\lambda,q} \psi_{\lambda,q}. \tag{40}
\]

\[
O_{\lambda,q} = \text{sgn}(E_\lambda) \int d\theta \mathcal{O}(q,\theta) \psi_{\lambda,q,\theta}. \tag{41}
\]

This operator evolves as \(\hat{O}_q(t) = \sum_{\lambda} O_{\lambda,q} \psi_{\lambda,q}(t)e^{-iE_\lambda t}\), which simplifies the computation of response functions, \(\chi_{AB}(q,t) = -i\Theta(t) \langle \hat{A}_q(t),\hat{B}_\lambda - q \rangle\). It can be shown that its imaginary part in frequency domain reads,

\[
\text{Im}\chi_{AB}(q,\omega) = -\pi A \sum_{\lambda} \text{sgn}(E_\lambda) A_{\lambda,q} B_{\lambda,-q} \delta(\omega - E_\lambda). \tag{42}
\]

The density \(\rho(x)\) and current \(j(x)\) of a LFL with a distribution function \(n(t,x,p) = \Theta(p_F(t,x,\theta) - p)\) are

\[
\rho(x) = \int \frac{d^2p}{(2\pi)^2} n(x,p) = \rho_0 + p_F \int \frac{d\theta}{(2\pi)^2} u(x,\theta),
\]

\[
j(x) = (j^\parallel(x), j^\perp(x)) = p_F^2 \int \frac{d\theta}{(2\pi)^2} u(x,\theta)p_\theta.
\]

to first order in \(u(x,\theta)\). We define the current via its transport mass \(m\) and explicitly separate the current into its longitudinal \(j^\parallel\) and transverse \(j^\perp\) components. The corresponding operator coefficients are

\[
\rho_{\lambda,q} = \text{sgn}(E_\lambda) \frac{p_F}{2\pi} \psi_{\lambda,0}^+,
\]

\[
j_{\parallel,\lambda,q} = \text{sgn}(E_\lambda) \frac{p_F^2}{2\pi m} \psi_{\lambda,0}^+,
\]

\[
j_{\perp,\lambda,q} = \text{sgn}(E_\lambda) \frac{p_F^2}{2\pi m} \psi_{\lambda,1}^-,
\]

From Eq. 42, we can simply read off the density-density and current-current correlation functions,
\[ \text{Im} \chi_{\rho \rho}(\mathbf{q}, \omega) = -A \frac{p_e^2}{4\pi} \sum_i |\psi_{i,0}^+|^2 \text{sgn}(E_i) \delta(\omega - E_i^+), \]  
(46) 

\[ \text{Im} \chi_{j\parallel j\parallel}(\mathbf{q}, \omega) = -A \frac{p_e^4}{4\pi m^2} \sum_i |\psi_{i,1}^+|^2 \text{sgn}(E_i) \delta(\omega - E_i^+), \]  
(47) 

\[ \text{Im} \chi_{j\perp j\perp}(\mathbf{q}, \omega) = -A \frac{p_e^4}{4\pi m^2} \sum_i |\psi_{i,1}^-|^2 \text{sgn}(E_i) \delta(\omega - E_i^-). \]  
(48) 

In particular, for a system with only non-trivial \( F_0 \), the density-density and longitudinal current-current correlation functions will exhibit a sharp peak at the zero-sound energy \( E_0 = qv_F \frac{1+F_0}{\sqrt{1+2F_0}} \) with the following spectral weights,

\[ w_{\rho \rho,0} = \frac{p_v q}{8 \sqrt{1+2F_0}} \frac{F_0}{(1+F_0+2F_0^2)}, \]  
(49) 

\[ w_{j\parallel j\parallel,0} = \frac{p_v^3 q}{2m^2} \frac{F_0(1+F_0)^2}{(1+2F_0+2F_0^2)}. \]  
(50) 

On the other hand, for a system with non-trivial \( F_0 \) and \( F_1 \), the transverse current-current correlation function will exhibit a sharp peak at the shear sound energy \( E_1 = qv_F \frac{1+F_1}{2\sqrt{F_1}} \) with spectral weight

\[ w_{j\perp j\perp,1} = \frac{p_v^3 q}{16m^2} \frac{F_1 - 1}{F_1^{\frac{3}{2}}} \]  
(51)

\[ = \frac{1}{2} \left( w_{j\parallel j\parallel,0} - 2 \frac{p_e^2}{m^2} w_{\rho \rho,0} \right) \bigg|_{F_0 \to \frac{1}{2}(F_1 - 1)}. \]  
(52)

where in the last line, we found an interesting mapping between the even and odd sector spectral weights between these correlation functions.