Some Relations on Paratopisms and an Intuitive Interpretation on the Adjugates of a Latin Square*

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Abstract

This paper will present some intuitive interpretation of the adjugate transformations of arbitrary Latin square. With this trick, we can generate the adjugates of arbitrary Latin square directly from the original one without generating the orthogonal array. The relations of isotopisms and adjugate transformations in composition will also be shown. It will solve the problem that when $F_1*I_1=I_2*F_2$ how can we obtain $I_2$ and $F_2$ from $I_1$ and $F_1$, where $I_1$ and $I_2$ are isotopisms while $F_1$ and $F_2$ are adjugate transformations and $**$ is the composition of transformations. These methods could distinctly simplify the computation on a computer for the issues related to main classes of Latin squares. This will improve the efficiency apparently in computation for some related problems.

Key Words: Latin square, Adjugate, Conjugate, Isotopism, Paratopism, Intuitive interpretation, Orthogonal array, Main class, Computational Complexity

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1 Introduction

Latin squares play an important role in experimental design in combinatorics and statistics. They are wildly used in the manufacturing of industry, production of agriculture, etc. Besides the application in orthogonal experiment in agriculture, Latin square could also be used to minimize experiment errors in the design of experiments in agronomic research (refer [23]). Any Latin square is a multiplication table of a quasi-group, which is an important structure in algebra. Mutually orthogonal Latin squares are closely linked with finite projective planes ([25, 24]). Sets of orthogonal Latin squares can be applied in error correcting codes in communication (refer [6, 19]). It is also used in Mathematical puzzles such as Sudoku ([8]), KenKen (refer [35, 36]), etc.

The name “Latin square” originated for the first time in the 36 officer problem introduced by Leonhard Euler (refer [12] or [13]). He used Latin characters as symbols, which gave the name of “Latin square”. (Now we usually use Hindu-Arabic numerals instead). Ever since then, a lot of results have been obtained on this discipline.

One of the main clue in the development of this subject is the enumeration problem, such as the number $L_n$ of reduced Latin squares of order $n$ (a positive integer) or the number of equivalence classes (such as isotopy classes or main classes) of Latin squares of small orders. It is clear that the number $N_n$ of Latin squares of order $n$ is $n! \cdot (n - 1)!$ times of the number $L_n$ of the reduced Latin squares of order $n$. But there is no explicit relation (formula) for $L_n$ and the number $S_n^{(1)}$ of isotopy classes of Latin squares of order $n$, which could be used to compute $S_n^{(1)}$ from $L_n$ or $n$ in practical. Nor can the number $S_n^{(2)}$ of main classes of Latin squares of order $n$ be calculated directly from $L_n$ or $n$. 

1 Introduction

Until nowadays, there is no practical computation formula for \( L_n \) which could easily obtain \( L_n \). Although Jia-yu Shao and Wan-di Wei gave a simple and explicit formula (in form) for \( L_n \) in 1992 (refer [34]), \( L_n = n! \sum_{A \in B_n} (-1)^{\sigma_0(A)} \left( \frac{\text{Per}A}{n} \right) \), where \( B_n \) is the set of all the 0-1 square matrices of order \( n \), \( \sigma_0(A) \) is the number of “0” appeared in the matrix \( A \), “Per” is the permanent operator, but this formula is still not so efficiency in practical. There is no practical asymptotic formula for \( L_n \), either (refer [30]). The difference for the most accurate upper bounds and lower bounds of \( L_n \) is so huge, \( \left( \frac{(n!)^2}{n^2} \right) \leq L_n \leq \prod_{k=1}^{n} (k!)^{n/k} \) (mentioned in [37], pp.161-162), which made it impossible to estimate the value of \( L_n \) by this formula. The upper bound was inferred from the van der Waerden permanent conjecture by Richard M. Wilson in 1974 ([39]). Later around 1980, the van der Waerden conjecture was solved independently by G. P. Egorichev ([11, 10]) and D. I. Falikman (refer [14, 15]) almost at the same time.

James R. Nechvatal (mentioned in [18]) gave a general asymptotic formula for generalized Latin Rectangles in 1981, and Ira. Gessel ([17]) gave a general asymptotic formula for Latin Rectangles in 1987, although they could result asymptotic formulae of \( L_n \) in formal, but neither seems suitable for asymptotic analysis. Chris D. Godsil and Brendan D. McKay obtained a better asymptotic formula in 1990 ([18]).

When the order \( n \) is less than 4, the number \( L_n \) of reduced Latin squares of order \( n \) is obvious, i.e., \( L_1 = L_2 = L_3 = 1 \). When \( n = 4 \) or 5, Euler found that \( L_4 = 4 \) and \( L_5 = 56 \) in 1782 (refer [12]), together with the values of \( L_1, L_2 \) and \( L_3 \). Cayley re-found these results (up to 5) in 1890 (refer [5]). M. Frolov found the value 9,408 of \( L_6 \) in 1890 (mentioned in [28]), later in 1901 Tarry re-found it (mentioned in [28]). The number \( S_6^{(1)} = 22 \) was first obtained by Erich Schönhardt [33] in 1930. Ronald A. Fisher and Frank Yates [16] re-found \( S_6^{(1)} \) independently in 1934, they also found the values of \( S_n^{(1)} \) when \( n \leq 5 \). In 1966, D. A. Preece [32] found that there are 564 isotopy classes of Latin squares of order 7. Mark B. Wells [38] acquired \( L_6 = 535, 281, 401, 856 \) in 1967. In 1990, Galina Kolesova, Clement W. H. Lam and Larry Thiel [22] gained \( S_8^{(1)} = 1,676,267, S_8^{(2)} = 283,657 \) and confirmed Wells’ result. The value of \( L_9 = 377,597,570,964,258,816 \) was found by S. E. Bammel and Jerome Rothstein [2] in 1975. Brendan D. McKay and Eric Rogozski [29] found that \( L_{10} = 7,580,721,483,160,132,811,489,280 \) and \( S_{10}^{(1)} = 208,904,371,354,363,006 \) and \( S_{10}^{(2)} = 34,817,397,894,749,939 \) in 1995. Brendan D. McKay and Ian M. Wanless [30] found the value of \( L_{11} = 5,363,937,773,277,371,298,119,673,540,771,840 \) and \( S_{11}^{(1)} = 12,216,177,315,369,229,261,482,540 \) and \( S_{11}^{(2)} = 2,036,029,552,582,883,134,196,099 \) in 2005.

Before the invention of computers, mathematician could do the enumeration work by hand with some theoretical tools, which would probably involve some errors. It was also very difficult to verify a known result, hence some excellent experts obtained false values even if the corrected one had been found. In 1915, Percy A. MacMahon gained an incorrect value of \( L_5 \) in a different way from other experts (refer [27]). In 1930, Savarimuthu M. Jacob ([20]) obtained a wrong value of \( L_6 \) after Frolov and Tarry had already found the
corrected one. The value of $L_7$ counted by Frolov is incorrect, either.

It was mentioned in [31] that Clausen, an assistant of a German astronomer Schumacher, found 17 "basic forms" of Latin squares of order 6. This information was described in a letter from Schumacher to Gauss dated August 10, 1842. This letter was quoted by Gunther in 1876 and by Ahrens in 1901. Tarry also found 17 isotopy classes of Latin squares of order 6. Until E. Schönhardt gained the correct values of $L_n$, $S_n^{(1)}$ and $S_n^{(2)}$ when $n \leq 6$ in 1930.

In 1939, H. W. Norton ([31]) obtained the wrong values of $S_7^{(1)}$ and $S_7^{(2)}$. After Preece gained the correct value of $S_7^{(1)}$ in 1966, James W. Brown [3] announced another incorrect value 563 of $S_7^{(1)}$ and this result was widely quoted as the accepted value for several decades ([7] [9]).

In recent decades, with the application of computers, the efficiency in the enumeration of equivalence classes of Latin squares has been improved greatly. However, it is still very difficult to avoid errors because of the complexity in huge amount of computation.

In the case of order 8, J. W. Brown [3] also provided a wrong value of $S_8^{(1)}$ in 1968, and Arlazarov et al. provided a false value of $S_8^{(2)}$ in 1978 (mentioned in [28, 22]).

In some cases, the number of some types of equivalence classes of Latin squares of a certain order would likely to be believed correct after at least two times of independent computation. (refer [28])

Technically, according to some conclusions in group theory, especially the enumeration method of orbits when a group acts on a set, by generating the representatives of all the equivalence classes of a certain order and the enumerating the members in every invariant group of a representative, we will know the number of objects in every equivalence class, hence the total number of Latin squares of a certain order will be obtained. But the process contains too much equivalence classes, which costs too much time.  

In logical, the structure of the relations of the Latin squares in the same main class is a graph, a powerful graph isomorphism program “Nauty” is very useful to compute the invariant group of the Latin square within main class transformations.

In practical, when computing these objects (to find the invariant group of a Latin square), the process in computation is similar to visiting a tree with some of its branches being isomorphic where only one of the isomorphic branches will be visited so as to improve the efficiency.

When considering the topics related to the main classes of Latin squares, we will usually generate the adjugates (or conjugates) of an arbitrary Latin square. The routine procedure

\[1\] Translated from the German word “Grundformen”. According to the context, it probably means the isotopy classes.

\[2\] The number of isotopy classes or main classes of Latin squares of order $n$ is much less than the number of reduced Latin squares of order $n$. In general, we can not afford the time to visit all the reduced Latin squares of order $n$ (when $n > 7$) even in some super computers.
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is to generate the orthogonal array of the Latin square first, then permute the rows (or columns in some literature) of the array and turn the new orthogonal array into a new Latin square at last, which seems not so convenient. Among the adjugates of a Latin square $Y$, the author has not found the detailed descriptions on the adjugates other than the orthogonal array, except two simple cases, $Y$ itself and the transpose $Y^T$. In Sec. 3 intuitive explanation of the adjugates of an arbitrary Latin square will be described. With this method, the other 4 types adjugates of arbitrary Latin square could be generated by transpose and/or replacing the rows/columns by their inverse.

Sometimes we need to consider the composition of adjugate transformations and isotopic transformations, especially when generating the invariant group of a Latin square in main class transformation. It is necessary to exchange the priority order of the these two types of transformations for simplification. But in general cases, an adjugate transformation and an isotopic transformation do not commute. So we want to find the relations of isotopic transformations and adjugate transformations in composition when their positions are interchanged. In other words, for any adjugate transformation $F_1$ and any isotopic transformation $I_1$, how can we find the adjugate transformation $F_2$ and the isotopic transformation $I_2$, s.t., $F_1 \circ I_1 = I_2 \circ F_2$? The answer will be presented in Sec. 4.

2 Preliminaries

Here a few notions appeared in this paper will be recalled so as to avoid ambiguity.

Suppose $n$ be a positive integer and it is greater than 1.

A permutation is the reordering of the sequence $1, 2, 3, \cdots, n$. An element $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ in the symmetry group $S_n$ is also called a permutation.

For convenience, in this paper, these two concepts will not be distinguished rigorously. They might even be mixing used. When referring “a permutation $\alpha$” here, sometimes it will be a bijection of the set $\{1, 2, 3, \cdots, n\}$ to itself, sometimes it will standard for the sequence $[\alpha(1), \alpha(2), \cdots, \alpha(n)]$. For a sequence $[b_1, b_2, \cdots, b_n]$ which is a rearrangement of $[1, 2, \cdots, n]$, in some occasions it may also stand for a transformation $\beta \in S_n$, such that $\beta(i) = b_i$ ($i = 1, 2, \cdots, n$). The actual meaning can be inferred from the contexts. In computer, these two kinds of objects are stored almost in the same way.

We are compelled to accept this ambiguity. Otherwise, it will cost too much to avoid this ambiguity because we have to use much more words to describe a simple operation and much more symbols to show a concise expression. An example will be shown after Lemma 1.

Let $\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \in S_n$, here we may call $[a_1, a_2, \cdots, a_n]$ the one-row form of the permutation $\alpha$, and call $\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$ the two-row form of the permutation $\alpha$. 
A matrix with every row and every column being permutations of 1, 2, · · ·, \( n \), \(^3\) is called a Latin square of order \( n \). Latin squares with both the first row and the first column being in natural order are said to be reduced or in standard form (refer [9] page 128). For example, the matrix below is a reduced Latin square of order 5,

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 1 & 4 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 5 & 3 \\
5 & 4 & 1 & 3 & 2
\end{bmatrix}.
\]

For convenience, when referring the inverse of a row (or a column) of a Latin square square, we mean the one-row form of the inverse of the permutation presented by the row (or the column), not the sequence in reverse order. For instance, the inverse of the third row \([3 \ 5 \ 4 \ 2 \ 1]\) of the Latin square above, is believed to be \([5 \ 4 \ 1 \ 3 \ 2]\), not the reverse \([1 \ 2 \ 4 \ 5 \ 3]\) of it, since \([3 \ 5 \ 4 \ 2 \ 1]\) of the Latin square above, is believed to be \([5 \ 4 \ 1 \ 3 \ 2]\), not the reverse (or the column), not the sequence in reverse order. For instance, the inverse of the third square, we mean the one-row form of the permutation presented by the row

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 5 & 3 \\
5 & 4 & 1 & 3 & 2
\end{bmatrix}, \]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 1 & 3 \\
3 & 4 & 1 & 2 & 3 \\
4 & 1 & 2 & 3 & 4
\end{bmatrix}, \]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 1 & 3 \\
3 & 4 & 1 & 2 & 3 \\
4 & 1 & 2 & 3 & 4
\end{bmatrix} \]

From now on, let \( Y_i = [y_{i1}, y_{i2}, \cdots, y_{in}] \) be the \( i \)'th row of a Latin square \( Y \) and \( Z_i \)

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 1 & 4 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 5 & 3 \\
5 & 4 & 1 & 3 & 2
\end{bmatrix}.
\]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 3 & 4 \\
5 & 4 & 1 & 3 & 2
\end{bmatrix}.
\]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 1 & 3 \\
3 & 4 & 1 & 2 & 3 \\
4 & 1 & 2 & 3 & 4
\end{bmatrix}, \]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 1 & 3 \\
3 & 4 & 1 & 2 & 3 \\
4 & 1 & 2 & 3 & 4
\end{bmatrix} \]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 3 & 4 \\
5 & 4 & 1 & 3 & 2
\end{bmatrix}.
\]

\[\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 3 & 4 \\
5 & 4 & 1 & 3 & 2
\end{bmatrix}.
\]
be the \(i\)’th column of \(Y\) (\(i=1, 2, \cdots, n\)). In this paper, we do not distinguish the permutation transformation \(\zeta : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}\), \(k \mapsto \zeta(k)\) (\(k=1, 2, \cdots, n\)) and the column sequence \([\zeta(1), \zeta(2), \cdots, \zeta(n)]^T\). Here the super-script “\(T\)” means transpose. \(Y_i^{-1}\) and \(Z_i^{-1}\) are the inverse of \(Y_i\) and \(Z_i\), respectively (as the one row form of the inverse of the corresponding transformations, while \(Y_i^{-1}\) is a row and \(Z_i^{-1}\) is a column). \(^5\) Whether a permutation symbol \(\alpha\) stands for a row sequence or column sequence will be inferred from the contexts.

\(\forall \alpha, \beta, \gamma \in S_n\), we know the definition of the composition \(\alpha \cdot \beta\) as a permutation transformation, here \((\alpha \cdot \beta)(i)\) is defined by \(\alpha(\beta(i))\), \(i=1, 2, \cdots, n\). We define \(\gamma Y_i\) as the one-row form of the permutation \(\gamma\) and the permutation with the one-row form \(Y_i\) (\(i=1, 2, \cdots, n\)), i.e, \(\gamma Y_i\) will stand for the sequence \([\gamma(y_{i1}), \gamma(y_{i2}), \cdots, \gamma(y_{in})]\), or some times the permutation transformation \(\left(\begin{array}{cccc} 1 & 2 & \cdots & n \\ \gamma(y_{i1}) & \gamma(y_{i2}) & \cdots & \gamma(y_{in}) \end{array}\right)\) according to the contexts. So are \(Z_i\alpha^{-1}\) and \(\gamma Z_i\), except that \(Z_i\) is a column, so both are columns.

It is not difficult to verify that

**Lemma 1.**

\[
\mathcal{R}_\alpha(Y) = \begin{pmatrix} Y_{\alpha^{-1}(1)} \\ \vdots \\ Y_{\alpha^{-1}(n)} \end{pmatrix} = (Z_1\alpha^{-1}, \cdots, Z_n\alpha^{-1}), \\
\mathcal{C}_\beta(Y) = \begin{pmatrix} Y_{\beta^{-1}} \\ \vdots \\ Y_{\beta^{-1}(n)} \end{pmatrix} = (Z_{\beta^{-1}(1)}, \cdots, Z_{\beta^{-1}(n)}), \\
\mathcal{L}_\gamma(Y) = \begin{pmatrix} \gamma Y_1 \\ \vdots \\ \gamma Y_n \end{pmatrix} = (\gamma Z_1, \cdots, \gamma Z_n).
\]

If we distinguish the two kinds of permutations strictly, it will make the above equation much more complex. For example, in order to show that the transformation \(\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}\)

\(^5\) Actually, \(Z_i^{-1} = \begin{pmatrix} \zeta_i^{-1}(1) \\ \vdots \\ \zeta_i^{-1}(n) \end{pmatrix}\) is the transpose the sequence \([\zeta_i^{-1}(1), \zeta_i^{-1}(2), \cdots, \zeta_i^{-1}(n)]\), where \(\zeta_i^{-1}\) is the inverse of the permutation \(\zeta_i = \begin{pmatrix} y_{1i} & y_{2i} & \cdots & y_{ni} \end{pmatrix}\). \(Y_i^{-1}\) is obtained in the same way, except that \(Y_i^{-1}\) is a row sequence.
is derived from a sequence $R = [a_1, a_2, \cdots, a_n]$ which is a reordering of $[1, 2, \cdots, n]$, we
will use a symbol such as $\mathcal{S}(R)$ to denote the transformation
$$\begin{pmatrix}
1 & 2 & \cdots & n \\
\frac{a_1}{a_2} & \frac{a_2}{a_n} & \cdots & a_n
\end{pmatrix};
$$
f for a transformation $\beta = \begin{pmatrix}
1 & 2 & \cdots & n \\
b_1 & b_2 & \cdots & b_n
\end{pmatrix}$, and a sequence $[b_1, b_2, \cdots, b_n]$, in order to show
the relation of $\beta$ and the corresponding sequence, we will denote the sequence by $\mathcal{S}(\beta)$.
Therefore Lemma 1 should be stated as
$$\mathcal{S}_\alpha(Y) = \begin{pmatrix}
Y_{\alpha^{-1}(1)} \\
\vdots \\
Y_{\alpha^{-1}(n)}
\end{pmatrix} = \left( (\mathcal{S} (Z_1^T \cdot \alpha^{-1}))^T, \cdots, (\mathcal{S} (Z_n^T \cdot \alpha^{-1}))^T \right),$$
$$\mathcal{C}_\beta(Y) = \begin{pmatrix}
\mathcal{S} (Y_1 \cdot \beta^{-1}) \\
\vdots \\
\mathcal{S} (Y_n \cdot \beta^{-1})
\end{pmatrix} = (Z_{\beta^{-1}(1)}, \cdots, Z_{\beta^{-1}(n)}),$$
$$\mathcal{L}_\gamma(Y) = \begin{pmatrix}
\mathcal{S} (\gamma \cdot Y_1) \\
\vdots \\
\mathcal{S} (\gamma \cdot Y_n)
\end{pmatrix} = (\left( (\mathcal{S} (\gamma \cdot Z_1^T))^T, \cdots, (\mathcal{S} (\gamma \cdot Z_n^T))^T \right).$$

The two transformation symbols $\mathcal{S}$ and $\mathcal{F}$ here will make the above equations not as
concise as the previous ones, and they will distract our attention. Also will it result a big
trouble to demonstrate some other propositions which are more complicated than Lemma 1. We will pay too much price to distinguish strictly these two kinds of permutations. Therefore we will ignore it in this paper.

Later, the two symbols $\mathcal{S}$ and $\mathcal{F}$ will stand for some other injective maps.

For $\forall \alpha, \beta, \gamma \in S_n$, and any Latin square $Y$, $\mathcal{S}_\alpha \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma$ will be called an isotopism, $(\mathcal{S}_\alpha \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma)(Y)$ will be called isotopic to $Y$.

Let $\mathcal{S}_\alpha$ be the set of all the isotopy transformations of Latin squares of order $n$.

Let $Y = (y_{ij})_{n \times n}$ be an arbitrary Latin square with elements belonging to the set
$\{1, 2, \cdots, n\}$. With regard to the set $T = \{(i, j, y_{ij}) \mid 1 \leq i, j \leq n\}$, we will have
$$(i, j) \mid 1 \leq i, j \leq n = \{(j, y_{ij}) \mid 1 \leq i, j \leq n\} = \{(i, y_{ij}) \mid 1 \leq i, j \leq n\},$$
so each pair of different triplets $(i, j, y_{ij})$ and $(r, t, y_{rt})$ in $T$ will share at most one identical
entry in the same position. The set $T$ is also called the orthogonal array representation
of the Latin square $Y$.

From now on, each triplets $(i, j, y_{ij})$ in the orthogonal array set $T$ of a Latin square
$Y = (y_{ij})_{n \times n}$ will be written in the form of a column vector $\begin{pmatrix} i \\
j \\
y_{ij}\end{pmatrix}$ so as to save some
space (while in a lot of papers, the triplets are denoted in a row vector). The orthogonal
array set $T$ of the Latin square $Y$ can be denoted by a matrix
$$V = \begin{bmatrix}
1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \bullet & \bullet & \bullet & n & n & \cdots & n \\
1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \bullet & \bullet & \bullet & 1 & 2 & \cdots & n \\
y_{11} & y_{12} & \cdots & y_{1n} & y_{21} & y_{22} & \cdots & y_{2n} & \bullet & \bullet & \bullet & y_{n1} & y_{n2} & \cdots & y_{nn}
\end{bmatrix}.$$

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of size $3 \times n^2$, with every column being a triplets consists of the indices of a position in a Latin square and the element in that position. The matrix $V$ will also be called the *orthogonal array* (matrix). From now on, when referring the orthogonal array of a Latin square, it will always be the matrix with every column being a triplets related to a position of the Latin square, unless otherwise specified.

The definition of “orthogonal array” in general may be found in reference [9] (page 190). In some references, such as [4], the orthogonal array is defined as an $n^2 \times 3$ array.

Every row of the orthogonal array of a Latin square consists of the elements $1, 2, \cdots, n$, and every element appears exact $n$ times in every row.

For example, the orthogonal arrays of the Latin squares $A_1$ are

$$V_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5
2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4
\end{bmatrix}.$$  

Obviously, reordering the columns of an orthogonal array will not change the Latin square it corresponds to.

On the other hand, if we can construct a matrix $V = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n^2 \\ b_1 & b_2 & b_3 & \cdots & b_n^2 \\ c_1 & c_2 & c_3 & \cdots & c_n^2 \end{bmatrix}$ of size $3 \times n^2$ satisfying the following conditions:

- (O1) Every row of the array is comprised of the elements $1, 2, \cdots, n$;
- (O2) Every element $k$ $(1 \leq k \leq n)$ appears exact $n$ times in every row;
- (O3) The columns of this array are orthogonal pairwise, that is to say, any pair of columns share at most one element in the same position;

then we can construct a matrix $Y_2$ of order $n$ from this orthogonal array by putting the number $c_t$ in the position $(a_t, b_t)$ of an empty matrix of order $n$ $(t=1, 2, 3, \cdots, n^2)$. It is clear that $Y_2$ is a Latin square. Hence this array $V$ is the orthogonal array of a certain Latin square $Y_2$.

So there is a one to one correspondence between the Latin squares of order $n$ and the arrays of size $3 \times n^2$ satisfying the conditions before (ignoring the ordering of the columns of the orthogonal array). Hence there is no problem to call a $3 \times n^2$ matrix an orthogonal array if it satisfies the 3 conditions mentioned here.

This means that an orthogonal array of size $3 \times n^2$ corresponding to a Latin square $Y$, will still be an orthogonal array after permuting its rows, but the new orthogonal array
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will correspond to another Latin square (called a conjugate or adjugate, which will be mentioned later) closely related to $Y$.

The operation of permuting the rows of a Latin square $Y$ according to a permutation $\alpha$ will correspond to the action of $\alpha$ on the members in the 1st row of its orthogonal array. Permuting the columns of a Latin square $Y$ according to a permutation $\beta$ will correspond to replacing the number $i$ in the 2nd row of its orthogonal array by $\beta(i)$ ($i = 1, 2, \cdots, n$). The transformation $\mathcal{L}_\gamma$ acting on the Latin square $Y$ corresponds to the action of substituting the number $i$ by $\gamma(i)$ in the 3rd row of the orthogonal array ($i = 1, 2, \cdots, n$).

Here we define the representative of an isotopy class or a main class as the minimal one in lexicographic order. Obviously, there are some other standards for an isotopy class representative or a main class representative. For some purpose, some other definition of canonical form of an equivalence class will be more efficient in practical. These definitions will not be discussed here, but the related algorithms derived from the conclusions in this paper, may rely on them.

3 Adjugates of a Latin square

It will not be difficult to understand that the permutation of rows of an orthogonal array will not interfere the orthogonality of the columns, and the array after row permutation will still be an orthogonal array of a certain Latin square. The new Latin square is firmly related to the original one. As the number of the reorderings of a sequence of 3 different entries is 6, there are exact 6 transformations for permuting the rows of an orthogonal array.

Every column of an orthogonal array consists of 3 elements, the first is the row index, the second is the column index, and the third is the entry of the Latin square in the position determined by the two numbers before it. Let $[r, c, e]$ or (1) denote the transformation to keep the original order of the rows of an orthogonal array; $[c, r, e]$ or (1 2) denote the transformation to interchange the rows 1 and 2 of an orthogonal array; $[r, e, c]$ or (2 3) denote the transformation to interchange the rows 2 and 3 of an orthogonal array, etc. (A similar notation may be found in reference [28].) There are 5 transformations that will indeed change the order of the rows of an orthogonal array. For convenience, denote the $i$'th row of the Latin square $Y$ by $Y_i$ ($i = 1, 2, \cdots, n$). According to the convention on page 5, $Y_i = \left[ y_{i1}, y_{i2}, \cdots, y_{in} \right]$ will sometimes represent the transformation that sends $j$ to $y_{ij}$ ($j = 1, 2, \cdots, n$), and a permutation $\alpha \in S_n$ will sometimes denote the sequence $[\alpha(1), \alpha(2), \cdots, \alpha(n)]$ which relies on the contexts.

(1) $[r, e, c]$ or $\alpha(3)

If the 2nd row and the 3rd row of an orthogonal array $V$ are interchanged, it will result

$$V^{(l)} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \bullet & \bullet & \bullet & n & n & \cdots & n \\
1 & 2 & \cdots & n & y_{11} & y_{12} & \cdots & y_{1n} & y_{21} & y_{22} & \cdots & y_{2n} & y_{n1} & y_{n2} & \cdots & y_{nn} \\
1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \bullet & \bullet & \bullet & 1 & 2 & \cdots & n
\end{bmatrix} \quad (3.1)$$
Sort the columns of $V^{(I)}$ in lexicographical order, such that the submatrix consists of the 1st row and 2nd row will be

$$V_0 = \begin{bmatrix}
1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\
1 & 2 & \cdots & n & 2 & 2 & \cdots & n & \cdots & 1 & n & \cdots & n
\end{bmatrix}. \quad (3.2)$$

It results

$$V^{(IA)} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\
y'_{11} & y'_{12} & \cdots & y'_{1n} & y'_{21} & y'_{22} & \cdots & y'_{2n} & \cdots & y'_{n1} & y'_{n2} & \cdots & y'_{nn}
\end{bmatrix}. \quad (3.3)$$

Of course $V^{(IA)}$ is essentially the same as $V^{(I)}$ as they correspond to the same Latin square. There are $n-1$ vertical lines in the arrays $V$, which divide it into $n$ segments. Every segment of $V$ represents a row of $Y$ as the members in a segment have the same row index. It is clear that the columns of $V^{(I)}$ in any segment will keep staying in that segment after sorting. That is to say, the entries in the same row of the original Latin square $Y$ will still be in the same row in the new Latin square (denoted by $Y^{(I)}$) corresponds to the orthogonal array $V^{(I)}$, because the row index of every entry is not changed in the process of interchanging the 2nd row and the 3rd row of the orthogonal array. In every segment, the 2nd row and the 3rd row will make up a permutation in two-row form. It is widely known that to exchange the two rows of a permutation $\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\
a_1 & a_2 & \cdots & a_n \end{pmatrix}$ will result its inverse $\alpha^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\
a_1' & a_2' & \cdots & a_n' \end{pmatrix}$. So the sequence $[y'_{11}, y'_{12}, \cdots, y'_{in}]$ is the inverse of $[y_{11}, y_{12}, \cdots, y_{in}]$ as both are reorderings of $[1, 2, \cdots, n]$ ($i = 1, 2, \cdots, n$). Hence, to interchange the 2nd row and 3rd row of the orthogonal array of a Latin square corresponds to substitute every row of the Latin square by its inverse, i.e., $Y^{(I)} = \begin{pmatrix} Y_1^{-1} \\ \vdots \\ Y_n^{-1} \end{pmatrix}$. This operation is mentioned implicitly in reference [22].

(2) $[c, r, e]$ or $(1\ 2)$

With regard to the orthogonal array $V$ in (2.1) of a Latin square $Y = (y_{ij})_{n \times n}$, if the 1st row and the 2nd row of $V$ are interchanged, it will result

$$V^{(II)} = \begin{bmatrix}
1 & 1 & \cdots & n & 2 & 2 & \cdots & n & \cdots & n & 1 & n & \cdots & n \\
y_{11} & y_{12} & \cdots & y_{1n} & y_{21} & y_{22} & \cdots & y_{2n} & \cdots & y_{n1} & y_{n2} & \cdots & y_{nn}
\end{bmatrix}. \quad (3.4)$$

It means that every entry $(i, j, y_{ij})^T$ will become $(j, i, y_{ij})^T$, that is to say, the entry in the $(j, i)$ position of the new Latin squares (denoted by $Y^{(II)}$) corresponds to $V^{(II)}$ is the entry $y_{ij}$ in the $(i, j)$ position of the original Latin square $Y$. Hence the new Latin square $Y^{(II)}$ is the transpose of the original one.

In order to understand the transformation $[c, e, r]$, we should first introduce the operation $[e, c, r]$.
(3) \([e, r, c]\) or \((1 \ 3 \ 2)\)
First, interchange the rows 1 and 2 of \(V\), then interchange the rows 2 and 3. This operation corresponds to replacing the \(i\)'th row of \(Y\) with the inverse of the \(i\)'th row of its transpose \(Y^T\), or substituting the \(i\)'th row of \(Y\) by the inverse of the \(i\)'th column of \(Y\). Denote the result by \(Y^{(III)}\). \(Y^{(III)} = \begin{pmatrix} (Z_1^{-1})^T \\ \vdots \\ (Z_n^{-1})^T \end{pmatrix}\). Here the sup-index “T” means the transpose, as \(Z_i\) is a column of \(Y^{(i)} = 1, 2, \ldots, n\).

(4) \([e, c, r]\) or \((1 \ 3)\)
If the 1st row and the 3rd row of \(V\) are interchanged, it will result
\[
V^{(IV)} = \begin{bmatrix}
y_{11} & y_{12} & \cdots & y_{1n} \\
1 & 2 & \cdots & n \\
1 & 1 & \cdots & 1 \\
y_{21} & y_{22} & \cdots & y_{2n} \\
2 & 2 & \cdots & 2 \\
2 & 2 & \cdots & 2 \\
y_{31} & y_{32} & \cdots & y_{3n} \\
3 & 3 & \cdots & 1 \\
3 & 3 & \cdots & 1 \\
y_{n1} & y_{n2} & \cdots & y_{nn} \\
n & n & \cdots & n \\
n & n & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
\end{bmatrix}.
\]

It is difficult to find the relation of the original Latin square \(Y\) and the new Latin square \(V^{(IV)}\) corresponds to array \(V^{(IV)}\) by sorting the columns of \(V^{(IV)}\) in lexicographic order. But another way will work. To interchange column \((l-1)n+k\) and column \((k-1)n+l\) of \(V\), \((l = 1, 2, \ldots, n-1; k = l+1, \ldots, n)\), which means to sort the columns of \(V\) such that the submatrix consists of the 1st row and the 2nd row is
\[
V^{(A)}_0 = \begin{bmatrix}
1 & 2 & \cdots & n \\
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
2 & 2 & \cdots & 2 \\
2 & 2 & \cdots & 2 \\
\end{bmatrix},
\]

will result
\[
V^{(A)} = \begin{bmatrix}
1 & 2 & \cdots & n \\
1 & 1 & \cdots & 1 \\
z_{11} & z_{12} & \cdots & z_{1n} \\
z_{21} & z_{22} & \cdots & z_{2n} \\
z_{n1} & z_{n2} & \cdots & z_{nn} \\
z_{11} & z_{12} & \cdots & z_{1n} \\
z_{21} & z_{22} & \cdots & z_{2n} \\
z_{n1} & z_{n2} & \cdots & z_{nn} \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
2 & 2 & \cdots & 2 \\
2 & 2 & \cdots & 2 \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
\end{bmatrix}.
\]

\(V^{(A)}\) corresponds to the Latin square \(Y\), too. The vertical lines in the array \(V^{(A)}\) divide it into \(n\) segments. Every segment corresponds to a column of \(Y\) since all the entries in a segment of \(V^{(A)}\) share the same column index. That is to say, \([z_{i1}, z_{i2}, \ldots, z_{in}]\) is the \(i\)'th column of \(Y\).

Interchanging the 1st row and the 3rd row of \(V^{(A)}\) will result
\[
V^{(IVA)} = \begin{bmatrix}
z_{11} & z_{12} & \cdots & z_{1n} \\
z_{21} & z_{22} & \cdots & z_{2n} \\
z_{n1} & z_{n2} & \cdots & z_{nn} \\
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & n \\
2 & 2 & \cdots & 2 \\
2 & 2 & \cdots & 2 \\
3 & 3 & \cdots & 1 \\
3 & 3 & \cdots & 1 \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n \\
\end{bmatrix}.
\]

Obviously, \(V^{(IVA)}\) and \(V^{(IV)}\) correspond to the same Latin square. Sorting the columns in each segment of \(V^{(IVA)}\) such that the 1st row and the 2nd row are the same as \(V^{(A)}_0\), it
Adjugates of a Latin square

will result

\[ V^{(IVB)} = \begin{bmatrix}
1 & 2 & \cdots & n \\
1 & 1 & \cdots & 1 \\
 \varepsilon'_{11} & \varepsilon'_{12} & \cdots & \varepsilon'_{1n} \\
 \varepsilon'_{21} & \varepsilon'_{22} & \cdots & \varepsilon'_{2n} \\
 \varepsilon'_{31} & \varepsilon'_{32} & \cdots & \varepsilon'_{3n} \\
 \cdots & \cdots & \cdots & \cdots \\
 \varepsilon'_{m1} & \varepsilon'_{m2} & \cdots & \varepsilon'_{mn}
\end{bmatrix} \]

(3.9)

So \([z'_{11}, z'_{12}, \ldots, z'_{nn}]\) is the inverse of \([z_{11}, z_{12}, \ldots, z_{nn}]\). Let the Latin square corresponds to \(V^{(IVB)}\) (or \(V^{(IVA)}\), \(V^{(IV)}\)) be \(Y^{(IV)}\). Therefore, the Latin square \(Y^{(IV)}\) related to \(V^{(IV)}\) generated by interchange the 1st row and 3rd row of the orthogonal array \(V\), consists of the columns which are the inverse of the columns of \(Y\). \(Y^{(IV)} = (Z_{1}^{-1} \cdots Z_{n}^{-1})\).

(5) \([c, e, r]\) or \((1 2 3)\)

First, interchange the rows 1 and 3 of \(V\), then interchange the rows 2 and 3, or equivalently, first interchange rows 1 and 2, then interchange rows 1 and 3. This operation corresponds to substituting the \(i\)'th column of \(Y\) by the inverse of the \(i\)'th column of its transpose \(Y^T\), or replacing the \(i\)'th column of \(Y\) with the inverse of the \(i\)'th row of \(Y\). Denote the result by \(Y^{(V)}\). \(Y^{(V)} = \left( (Y_{1}^{-1})^T \cdots (Y_{n}^{-1})^T \right)\).

Lemma 2. Let \(\eta \in S_3\), denote by \(F_\eta\) the transformation mentioned above, i.e., \(F_{(1)}(Y) = Y\), \(F_{(12)}(Y) = Y^{(II)} = Y^T\), \(F_{(23)}(Y) = Y^{(I)}\), \(F_{(13)}(Y) = Y^{(IV)}\), etc. By definition, it is clear that \(F_{\eta_1} \circ F_{\eta_2} = F_{\eta_1 \eta_2} \quad (\forall \eta_1, \eta_2 \in S_3)\).

(3.10)

The 6 Latin squares correspond to the orthogonal arrays obtained by permuting the rows of the orthogonal array \(V\) of the Latin square \(Y\) are called the conjugates or adjugates or parastrophes of \(Y\). Of course, \(Y\) is a conjugate of itself. The set of the Latin squares that are isotopic to any conjugate of a Latin square \(Y\) is called the main class or specy or paratopy class of \(Y\). If a Latin square \(Z\) belongs to the main class of \(Y\), then \(Y\) and \(Z\) are called paratopic or main class equivalent. Sometimes, the set of the Latin squares that are isotopic to \(Y\) or \(Y^T\) is called the type of \(Y\). (refer [1] or [28])

In this paper, a special equivalence class inverse type is defined, for the convenience in the process of generating the representatives of all the main classes of Latin squares of a certain order. The inverse type of a Latin square \(Y\) is the set of the Latin squares that are isotopic to \(Y\) or \(Y^{(I)}\), where \(Y^{(I)}\) is the Latin square corresponds to the orthogonal array \(V^{(I)}\) obtained by interchanging the 2nd row and the 3rd row of the orthogonal array \(V\) of \(Y\) as described before. (This idea is from the notion “row inverse” in reference [22]) The inverse type mentioned here may be called “row inverse type” for the sake of accuracy since \(Y^{(IV)}\) is another type of inverse (column inverse) of \(Y\). The reason for choosing row inverse type is that the Latin squares are are generated by rows in the following papers by the author. (In some papers, Latin squares are generated by columns, then the column inverse type will be useful.)
4 Relations on Paratopisms

The transformation that send a Latin square to another one paratopic to it is called a paratopism or a paratopic transformation. Let \( \mathcal{P}_n \) be the set of all the paratopic transformations of Latin squares of order \( n \), sometimes denoted by \( \mathcal{P} \) for short if it results no ambiguity. It is obvious that \( \mathcal{P} \) together with the composition operation “\( \circ \)” will form a group, called the paratopic transformation group or paratopism group. Obviously, all the isotopy transformations are paratopisms.

It will not be difficult to find out by hand the following relations of paratopic transformation group a group, called the paratopic transformation group or paratopism group. On the transformation that send a Latin square to another one paratopic to it is called a paratopism or a paratopic transformation. Let \( \mathcal{P}_n \) be the set of all the paratopic transformations of Latin squares of order \( n \), sometimes denoted by \( \mathcal{P} \) for short if it results no ambiguity. It is obvious that \( \mathcal{P} \) together with the composition operation “\( \circ \)” will form a group, called the paratopic transformation group or paratopism group. Obviously, all the isotopy transformations are paratopisms.

It will not be difficult to find out by hand the following relations of \( \mathcal{F}_n \) and \( \mathcal{C}_\alpha, \mathcal{R}_\beta, \mathcal{L}_\gamma \) when exchange their orders if we are familiar with how the transformations \( \mathcal{C}_\alpha, \mathcal{R}_\beta, \mathcal{L}_\gamma \) change the rows and columns in detail.

**Theorem 3.** For \( \forall \mathcal{F} \in \mathcal{I}_n, \forall \alpha, \beta, \gamma \in S_n \), these equalities hold,

\[
\mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F}, \\
\mathcal{R}_\alpha \circ \mathcal{F}_{(12)} = \mathcal{F}_{(12)} \circ \mathcal{C}_\alpha, \\
\mathcal{C}_\beta \circ \mathcal{F}_{(12)} = \mathcal{F}_{(12)} \circ \mathcal{R}_\beta, \\
\mathcal{L}_\gamma \circ \mathcal{F}_{(12)} = \mathcal{F}_{(12)} \circ \mathcal{L}_\gamma.
\]

The equalities in the 1st line and the 2nd line are obvious. Here we explain some equalities in the 3rd line and the 4th line. The readers may get the proof of other equalities in the same way without difficulties.

By the convention on page 6,

\[
Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \xrightarrow{\mathcal{F}_{(23)}} \begin{pmatrix} Y_1^{-1} \\ \vdots \\ Y_n^{-1} \end{pmatrix} \xrightarrow{\mathcal{R}_\alpha} \begin{pmatrix} Y_1^{-1(1)} \\ \vdots \\ Y_n^{-1(n)} \end{pmatrix} \xrightarrow{\mathcal{F}_{(23)}} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix},
\]

\[
\mathcal{R}_\alpha \circ \mathcal{F}_{(23)} = \mathcal{F}_{(23)} \circ \mathcal{R}_\alpha \text{ holds.}
\]

\[
Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \xrightarrow{\mathcal{F}_{(23)}} \begin{pmatrix} Y_1^{-1} \\ \vdots \\ Y_n^{-1} \end{pmatrix} \xrightarrow{\mathcal{C}_\beta} \begin{pmatrix} Y_1^{-1\beta^{-1}} \\ \vdots \\ Y_n^{-1\beta^{-1}} \end{pmatrix} \xrightarrow{\mathcal{F}_{(23)}} \begin{pmatrix} \beta Y_1 \\ \vdots \\ \beta Y_n \end{pmatrix} \xrightarrow{\mathcal{L}_\beta} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix},
\]

\[
\mathcal{C}_\beta \circ \mathcal{F}_{(23)} = \mathcal{F}_{(23)} \circ \mathcal{L}_\beta.
\]

\[
Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \xrightarrow{\mathcal{F}_{(23)}} \begin{pmatrix} Y_1^{-1} \\ \vdots \\ Y_n^{-1} \end{pmatrix} \xrightarrow{\mathcal{L}_\gamma} \begin{pmatrix} \gamma Y_1^{-1} \\ \vdots \\ \gamma Y_n^{-1} \end{pmatrix} \xrightarrow{\mathcal{F}_{(23)}} \begin{pmatrix} Y_1 \gamma^{-1} \\ \vdots \\ Y_n \gamma^{-1} \end{pmatrix} \xrightarrow{\mathcal{C}_\gamma} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix},
\]

\[
\mathcal{L}_\gamma \circ \mathcal{F}_{(23)} = \mathcal{F}_{(23)} \circ \mathcal{L}_\gamma.
\]

\[
Y = ( Z_1, \ldots, Z_n ) \xrightarrow{\mathcal{F}_{(13)}} ( Z_1^{-1}, \ldots, Z_n^{-1} ) \xrightarrow{\mathcal{R}_\alpha} ( Z_1^{-1\alpha^{-1}}, \ldots, Z_n^{-1\alpha^{-1}} ),
\]

\[
Y = ( Z_1, \ldots, Z_n ) \xrightarrow{\mathcal{L}_\alpha} ( \alpha Z_1, \ldots, \alpha Z_n ) \xrightarrow{\mathcal{F}_{(13)}} ( Z_1^{-1\alpha^{-1}}, \ldots, Z_n^{-1\alpha^{-1}} ),
\]

then \( \mathcal{R}_\alpha \circ \mathcal{F}_{(13)} = \mathcal{F}_{(13)} \circ \mathcal{L}_\alpha. \)

With the equalities before, it will be easy to obtain the properties when \( \mathcal{F}_{(132)} \) or \( \mathcal{F}_{(123)} \) composited with \( \mathcal{R}_\alpha, \mathcal{C}_\beta, \) or \( \mathcal{L}_\gamma. \) And we will have,
Theorem 4. For \( \forall \alpha, \beta, \gamma \in S_n, \forall \mathcal{T} \in \mathcal{J}_n \),
\[ \mathcal{T} \circ \mathcal{F}(1) = \mathcal{F}(1) \circ \mathcal{T}, \]
\[ (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \circ \mathcal{F}(12) = \mathcal{F}(12) \circ (\mathcal{R}_\beta \circ \mathcal{C}_a \circ \mathcal{L}_\gamma), \]
\[ (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \circ \mathcal{F}(13) = \mathcal{F}(13) \circ (\mathcal{R}_\beta \circ \mathcal{C}_\gamma \circ \mathcal{L}_a), \]
\[ (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \circ \mathcal{F}(23) = \mathcal{F}(23) \circ (\mathcal{R}_a \circ \mathcal{C}_\gamma \circ \mathcal{L}_\beta). \]

Since \( \mathcal{F}(123) = \mathcal{F}(12) \circ \mathcal{F}(12), \mathcal{F}(132) = \mathcal{F}(12) \circ \mathcal{F}(13) \), so

Theorem 5. For \( \forall \alpha, \beta, \gamma \in S_n \),
\[ (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \circ \mathcal{F}(123) = \mathcal{F}(123) \circ (\mathcal{R}_\beta \circ \mathcal{C}_\gamma \circ \mathcal{L}_a), \]
\[ (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \circ \mathcal{F}(132) = \mathcal{F}(132) \circ (\mathcal{R}_\gamma \circ \mathcal{C}_a \circ \mathcal{L}_\beta). \]

Here we denote an isotopy by \( \mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma \) not \( \mathcal{C}_\beta \circ \mathcal{R}_a \circ \mathcal{L}_\gamma \) (although \( \mathcal{R}_a, \mathcal{C}_\beta, \mathcal{L}_\gamma \) commute pairwise), the reason is, when exchanging the position of \( (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \) and \( \mathcal{F}_\eta \), the result is to permute the subscript of the three transformation according to the permutation \( \eta \in S_3 \), as shown above. (Just substitute 1, 2, 3 by \( \alpha, \beta, \gamma \), respectively. For instance, \( (123) \) will become \( (\alpha \beta \gamma) \), so moving \( \mathcal{F}(123) \) from the right side of \( (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \) to the left side, \( \alpha, \beta, \gamma \) will become \( \beta, \gamma, \alpha \), respectively.) So we can denote these formulas as below.

Theorem 6. (main result 1) \( \forall \alpha_1, \alpha_2, \alpha_3 \in S_n, \forall \beta_1, \beta_2, \beta_3 \in S_n, \forall \eta \in S_3 \),
\[ (\mathcal{R}_{\alpha_1} \circ \mathcal{C}_{\alpha_2} \circ \mathcal{L}_{\alpha_3}) \circ \mathcal{F}_\eta = \mathcal{F}_\eta \circ (\mathcal{R}_{\alpha_{n(1)}} \circ \mathcal{C}_{\alpha_{n(2)}} \circ \mathcal{L}_{\alpha_{n(3)}}), \tag{4.1} \]
\[ \mathcal{F}_\eta \circ (\mathcal{R}_{\beta_1} \circ \mathcal{C}_{\beta_2} \circ \mathcal{L}_{\beta_3}) = (\mathcal{R}_{\beta_{\eta^{-1}(1)}} \circ \mathcal{C}_{\beta_{\eta^{-1}(2)}} \circ \mathcal{L}_{\beta_{\eta^{-1}(3)}}) \circ \mathcal{F}_\eta. \tag{4.2} \]

With Lemmas 2 and 2, it is not difficult to explain the six theorems described above, although not so intuitive.

It is clear that any paratopic transformation is the composition of an isotopy \( (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \) \( (\alpha, \beta, \gamma \in S_n) \) and a certain conjugate transformation \( \mathcal{F}_\eta \ (\eta \in S_3) \).

Since \( \mathcal{J}_n = \{ \mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma \mid \alpha, \beta, \gamma \in S_n \} \cong S_3^3 \), it is convenient to denote \( \mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma \circ \mathcal{F}_\eta \) by \( \mathcal{P} (\alpha, \beta, \gamma, \eta) \) for short. So we have \( |\mathcal{P}_n| = |S_3^3| \times |S_3| = |\mathcal{F}_n| \times |S_3| = 6 (n!)^3 \). Let \( \mathcal{P} : \mathcal{S}_n \times S_3 \to \mathcal{P}_n, (\alpha, \beta, \gamma, \eta) \mapsto \mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma \circ \mathcal{F}_\eta \); it is clear that \( \mathcal{P} \) is a bijection. As a set, \( \mathcal{P}_n \) is isomorphic to \( \mathcal{J}_n \times S_3 \) or \( S_3^3 \times S_3 \), but as a group, \( \mathcal{P}_n \) is not isomorphic to \( \mathcal{J}_n \times S_3 \), (refer [28]) because \( \mathcal{P} \) is not compatible with the multiplications in \( \mathcal{P}_n \) as \( (\mathcal{R}_a \circ \mathcal{C}_\beta \circ \mathcal{L}_\gamma) \) and \( \mathcal{F}_\eta \) do not commute, unless \( \eta = (1) \).

Theorem 7. (main result 2) In general, \( \forall \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in S_n, \forall \eta, \zeta \in S_3 \),
\[ \mathcal{P} (\alpha_1, \alpha_2, \alpha_3, \eta) \circ \mathcal{P} (\beta_1, \beta_2, \beta_3, \zeta) = \mathcal{P} (\alpha_1 \beta_{\eta^{-1}(1)} \alpha_2 \beta_{\eta^{-1}(2)} \alpha_3 \beta_{\eta^{-1}(3)}, \eta \zeta). \]

Usually, \( \mathcal{P} (\alpha_1 \beta_{\eta^{-1}(1)} \alpha_2 \beta_{\eta^{-1}(2)} \alpha_3 \beta_{\eta^{-1}(3)}, \eta \zeta) \) differs from \( \mathcal{P} (\alpha_1 \beta_1, \alpha_2 \beta_2, \alpha_3 \beta_3, \eta \zeta) \).
5 Application

With these theorems mentioned above, we can avoid generating the orthogonal array when producing the adjugates of a Latin square, which will be helpful for efficiency in computation especially when writing source codes. Also will it save a lot of time when generating all the representatives of main classes of Latin squares of a certain order if we using these relations together with some properties of cycle structures and properties of isotopic representatives, as we can avoid generating a lot of Latin squares when testing main class representatives.

When generating the invariant group of a Latin square in paratopic transformations, it will be more convenient to simplify the composition of two or more paratopisms by the relations described in Sec. 2 (although this benefit for improving the efficiency in computation is not so conspicuous for a single Latin square, but it is remarkable for a large number of Latin squares).

Some new algorithms for related problems, such as the algorithms for testing or generating a representative of an equivalence class (a main class or an isotopic class, etc.), the algorithms for generating the invariant group of a Latin square in some transformation (isotopisms and main class transformations), will be presented in the near future. They are different from these described in [21].

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