Robust heterodimensional cycles in two-parameter unfolding of homoclinic tangencies

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Abstract. We consider $C^r$ ($r = 3, \ldots, \infty, \omega$) diffeomorphisms with a generic homoclinic tangency to a hyperbolic periodic point, where this point has at least one complex (non-real) central multiplier, and some explicit conditions are satisfied so that the dynamics near the homoclinic tangency is not effectively one-dimensional. We prove that $C^1$-robust heterodimensional cycles of co-index one appear in any generic two-parameter $C^r$-unfolding of such a tangency. These heterodimensional cycles also have $C^1$-robust homoclinic tangencies.

1 Introduction

Homoclinic tangencies and heterodimensional cycles (see precise definitions in Section 1.1) are considered to be fundamental objects in the research of non-uniformly hyperbolic systems. By definition, the existence of either of them prevents a system to be uniformly hyperbolic. Moreover, they bifurcate to be robust (for homoclinic tangencies, see [23, 25], for heterodimensional cycles, see [8, 9] for $C^1$ case, and [19] for $C^r$ case). They exhibit behaviors which are not seizable among uniformly hyperbolic systems such as presence of zero Lyapunov exponents [15], super-exponential growth of number of periodic points [2, 3, 5, 16], coexistence of infinitely many sinks [12, 14, 22].

While their properties are often separately investigated and in principle they are individual objects, there are researches which suggest links among their existences. For instance, it is shown in [11] that homoclinic tangencies can be created from a class of heterodimensional cycles which they called non-connected. One of the authors discussed [27] the creation of heterodimensional cycles when there is no domination, which is tightly related to the existence of homoclinic tangencies. This problem is pursued in [6, 7, 21]. Results in [14] also imply the existence of heterodimensional cycles near homoclinic tangencies. They assert that hyperbolic periodic orbits of different indices (dimension of unstable manifolds), which are the building blocks of heterodimensional cycles, coexist after bifurcations of generic homoclinic tangencies, provided that the dynamics have enough ‘effective dimension’ for these orbits. This naturally raised a question that whether diffeomorphisms having homoclinic tangencies can be approximated by those having heterodimensional cycles.
In the present paper, we answer this question by proving that, for any $C^r$ ($r = 1, \ldots, \infty, \omega$) diffeomorphism, heterodimensional cycles can be obtained by arbitrarily $C^r$-small perturbations from generic homoclinic tangencies whose local dynamics do not admit low-dimensional reductions, namely, those associated to a hyperbolic periodic point with complex central multipliers (see Theorem 1). Using the results in [19], the found heterodimensional cycles can be made $C^1$-robust and such that the two hyperbolic sets involved are blenders (hyperbolic sets on which non-transverse intersections are robust, see Definition 2). With a further application of the results in [10], we show that one of the blenders also has a $C^1$-robust homoclinic tangency (see Corollary 1). Particularly, all these perturbations can be done in a generic two-parameter unfolding family when the diffeomorphism is at least $C^3$ (see Theorems 2 and 3, and Corollary 2).

1.1 General setting

Consider a diffeomorphism $f$ of a $d$-dimensional ($d \geq 3$) manifold $M$ such that it has an orbit of homoclinic tangency, denoted by $\Gamma$, to a hyperbolic periodic point $O$, i.e., the stable invariant manifold $W_s(O)$ and the unstable invariant manifold $W_u(O)$ intersect each other non-transversely at the points of $\Gamma$. We will create heterodimensional cycles from local bifurcations of $\Gamma$. As shown in [14], such bifurcations strongly depend on the central multipliers of $O$.

Let $\tau$ be the period of $O$ and define the local map $T_0 := f^\tau$ on a small neighborhood $U$ of $O$. The multipliers of $O$ are the eigenvalues of the derivative $D T_0$ at $O$, denoted as

$$|\lambda_1| < \cdots < |\lambda_1| < 1 < |\gamma_1| \leq \cdots \leq |\gamma_d|,$$

where $d^s + d^u = d$. The center-stable and center-unstable multipliers are those nearest to the unit circle from inside and, respectively, from outside. Up to an arbitrarily $C^r$-small perturbation, we can always assume that the central multipliers are just $\lambda_1$ and $\gamma_1$ along with their complex conjugates, if any.

It is well-known that unfolding homoclinic tangencies can lead to the coexistence of hyperbolic periodic points of different indices, see e.g. [14]. However, whether those points can form heterodimensional cycles with their invariant manifolds was not clear. In the case where both $\lambda_1$ and $\gamma_1$ are real, generally no heterodimensional cycles can be born from homoclinic bifurcations of $\Gamma$, because for a generic system there is a two-dimensional invariant manifold enclosing $O$ and the orbit of tangency for all nearby systems, while heterodimensional cycles can exist only in dynamics of dimension three or higher. But the situation changes when the two-dimensional reduction is prevented, as shown in [18] that heterodimensional cycles can be born from a pair of homoclinic tangencies. In this paper we focus on the case where at least one of $\lambda_1$ and $\gamma_1$ is complex so that there naturally exist three-dimensional central dynamics, providing enough room for the birth of heterodimensional cycles. We distinguish the following two cases:

- **saddle-focus**:  
  (2,1): $\lambda_1 = \lambda_2 = \lambda e^{i\omega}$ ($0 < \omega < \pi$), and $\gamma_1$ is real, where $|\lambda_3| < \lambda < 1$ and $|\gamma_1| < |\gamma_2|$, or 
  (1,2): $\lambda_1$ is real, and $\gamma_1 = \gamma_2 = \gamma e^{i\omega}$ ($0 < \omega < \pi$), where $|\lambda_1| > |\lambda_2|$ and $1 < \gamma < |\gamma_3|$;

- **bi-focus** (or saddle-focus of type (2,2) in the terminology of [14]):
\[ \lambda_1 = \lambda_2^* = \lambda e^{i\omega_1} \quad (0 < \omega_1 < \pi), \quad \gamma_1 = \gamma_2^* = \gamma e^{i\omega_2} \quad (0 < \omega_2 < \pi), \text{ where } |\lambda_3| < \lambda < 1 \text{ and } 1 < \gamma < |\gamma_3|. \]

For simplicity, we denote \( \lambda_1 \) and \( \gamma_1 \) by \( \lambda \) and \( \gamma \) whenever they are real.

Before formulating our results, let us define heterodimensional cycles. Recall that for any hyperbolic set \( \Lambda \) of \( f \) there exists a neighbourhood \( \mathcal{U} \) of \( f \) in \( \text{Diff}^r(\mathcal{M}) \) (i.e., the space of \( C^r \) diffeomorphisms) such that for any \( g \in \mathcal{U} \) the set \( \Lambda \) admits a unique continuation \( \Lambda_g \), i.e., the restrictions \( f|_{\Lambda} \) and \( g|_{\Lambda_g} \) are conjugate. Let us use \( \text{ind}(\cdot) \) to denote the indices of hyperbolic sets, i.e., the dimension of their unstable invariant manifolds.

**Definition 1.** We say that a diffeomorphism \( f \) has a heterodimensional cycle if there are two compact, transitive hyperbolic invariant sets \( \Lambda_1 \) and \( \Lambda_2 \) such that \( \text{ind}(\Lambda_1) \neq \text{ind}(\Lambda_2) \) and the intersections \( W^s(\Lambda_1) \cap W^u(\Lambda_2) \) and \( W^u(\Lambda_1) \cap W^s(\Lambda_2) \) are non-empty. The value \( |\text{ind}(\Lambda_1) - \text{ind}(\Lambda_2)| \) is called the co-index of the heterodimensional cycle.

The heterodimensional cycle is \( C^1 \)-robust if there exists a \( C^1 \)-neighbourhood \( \mathcal{U} \) of \( f \) such that any \( \text{diff}^r \)-close to \( f \) has a heterodimensional cycle involving the continuations of the sets \( \Lambda_1 \) and \( \Lambda_2 \).

Note that by the Kupka-Smale theorem, at least one of the two sets \( \Lambda_1 \) and \( \Lambda_2 \) in a robust heterodimensional cycle must be non-trivial. Unless otherwise stated, by saying heterodimensional cycles we always mean those of co-index 1.

**Theorem 1.** Let \( f \) be \( C^r \) with \( r = 1, \ldots, \infty, \omega \). Suppose that \( O \) has a homoclinic tangency and is either

- a saddle-focus of type (2,1) with \( |\lambda\gamma| > 1 \), or
- a saddle-focus of type (1,2) with \( |\lambda\gamma| < 1 \), or
- a bi-focus with \( \lambda\gamma \neq 1 \).

Then, there exists a diffeomorphism arbitrarily \( C^r \)-close to \( f \) such that it has a heterodimensional cycle involving the continuation of \( O \) and a hyperbolic periodic point \( Q \), where \( \text{ind}(Q) = d^u + 1 \) if \( |\lambda\gamma| > 1 \) and \( \text{ind}(Q) = d^u - 1 \) if \( |\lambda\gamma| < 1 \).

**Remark 1.** The results of Theorem 1 and Corollary 1 below are also achieved in [4] with an additional assumption that the homoclinic tangencies are connected to pre-existing blenders.

Note that this theorem is also true when \( |\lambda\gamma| = 1 \), no matter \( O \) is a saddle-focus or a bi-focus, since one gets the desired condition on \( |\lambda\gamma| \) by an arbitrarily \( C^r \)-small perturbation. This theorem is a direct consequence of Theorem 2 in Section 1.2.

We only consider the saddle-focus of (2,1) type with \( |\lambda\gamma| > 1 \), since the second case reduces to this after considering \( f^{-1} \). It should be pointed out that, when \( |\lambda\gamma| < 1 \), one does not expect the emergence of heterodimensional cycles from homoclinic bifurcations. Generically, there exists a three-dimensional invariant manifold containing \( O \) and the orbit of tangency, transverse to the strong-stable and strong-unstable directions. The condition \( |\lambda\gamma| > 1 \) basically means that the three-dimensional central dynamics are not sectionally dissipative. As shown in [14], without this condition only sinks
can be born when the tangency unfolds, which cannot be used to form heterodimensional cycles. In the bi-focus case, such problem does not appear.

The heterodimensional cycles obtained in each case of Theorem 1 can be made $C^1$-robust by a $C^r$-small perturbation, using a recently established $C^r$-stabilization theory for heterodimensional cycles in [19]. The stabilization process is essentially based on the emergence of Bonatti-Díaz blenders discovered in [8].

**Definition 2.** For any diffeomorphism $f$ on a smooth manifold $M$, let $f$ have a compact, transitive, uniformly-hyperbolic invariant set $\Lambda$. Let the dimension of its stable manifold be equal to $d^s$. The set $\Lambda$ is called a center-stable (cs) blender if there exists a $C^1$-open neighborhood $U$ of $f$ and a $C^1$-open set $D$ of smooth embeddings of a $(d^s - 1)$-dimensional disk into $M$ such that, for any system from $U$, each disk $D \in D$ intersects the unstable manifold of the hyperbolic continuation of $\Lambda$. The set $\Lambda$ is a center-unstable (cu) blender if it is a cs-blender for $f^{-1}$.

We say that a cs-blender $\Lambda_1$ of index $d_1$ is activated by a hyperbolic set $\Lambda_2$ of index $d_2$ if $W^s(\Lambda_2)$ contains the image of a disc form the set $D$ in Definition 2; similarly for a cu-blender. One readily sees that to obtain a robust heterodimensional cycle it suffices to have one of the two sets in Definition 1 be a blender, and let the other set activates the blender. Theorem 9 in [19] says that for a heterodimensional cycle involving two hyperbolic periodic orbits $L_1$ and $L_2$ with $\text{ind}(L_1) + 1 = \text{ind}(L_2)$, if either $L_1$ has complex center-stable multipliers or $L_2$ has complex center-unstable multipliers, then, by an arbitrarily small $C^r$-perturbation, one can create a robust heterodimensional cycle involving a cs-blender containing $L_1$ and a cu-blender containing $L_2$, activating each other (see the exact description of this theorem in Section 3.7). Thus, the heterodimensional cycles found in Theorem 1 can be stabilized in such a way that the continuation of $O$ is contained in a cs-blender when $|\lambda_\gamma| > 1$, and in a cu-blender when $|\lambda_\gamma| < 1$. For a full description of the perturbations used, see [20].

With the blender connected to $O$, we can further make the robust heterodimensional cycle satisfies that the blender has a $C^1$-robust homoclinic tangency in the sense of

**Definition 3.** We say that a transitive hyperbolic set of $f$ has a $C^1$-robust homoclinic tangency if there exists a $C^1$-neighbourhood of $f$ such that, for any diffeomorphism in it, the continuation of this set also has a homoclinic tangency.

This is done by a direct use of [10, Theorem 4.9] which says that if $f$ has a hyperbolic periodic point with a homoclinic tangency homoclinically related\(^1\) to a cu-blender, then there exists a diffeomorphism arbitrarily $C^r$-close $f$ such that the continuation of the blender exhibits a $C^1$-robust homoclinic tangency. For applying this theorem, one just needs to pre-perturb the system into the Newhouse domain, where for a dense set of systems the continuations of $O$ have homoclinic tangencies. After that, use the blenders obtained above, up to considering $f^{-1}$ if it is a cs-blender containing $O$. In summary, we obtain

**Corollary 1.** Assume the hypotheses of Theorem 1. There exists a diffeomorphism arbitrarily $C^r$-close to $f$ such that it has a $C^1$-robust heterodimensional cycle involving a cs-blender and a cu-blender, one of which particularly contains the continuation of $O$ and exhibits $C^1$-robust homoclinic tangencies.

\(^1\) We say that a hyperbolic periodic point $P$ and a hyperbolic set $\Lambda$ are homoclinically related if they have the same index, and there exists a periodic point $R \in \Lambda$ such that the invariant manifolds of $P$ and $R$ intersect each other transversely.
Remark 2. The statement of [10, Theorem 4.9] is about the so-called blender-horseshoe, which is equivalent to certain iterated functions system of two functions, while the blenders obtained in [19, Theorem 9] can be different in the sense that they may correspond to iterated functions systems of more than two functions. However, it can be checked that the proof of [10, Theorem 4.9] holds also for blenders given by [19, Theorem 9].

1.2 Creating heterodimensional cycles in finite-parameter families

In this section we assume that \( f \) is \( C^r \) with \( r = 3, \ldots, \infty, \omega \), and formulate parametric versions of Theorem 1 and Corollary 1, namely, we show that these results can be obtained within certain two-parameter unfolding families of generic homoclinic tangencies. The genericity conditions C1 - C6 will be described in Section 2, and they can be fulfilled for any homoclinic tangency after an arbitrarily \( C^r \)-small perturbation (see Proposition 1).

Let us first introduce parameters. Let \( \mu \) be a functional on a small neighborhood \( U \) of \( f \) in \( \text{Diff}^r(M) \), which is defined as the (signed) distance between \( W^u(O) \) and \( W^s_{\text{loc}}(O) \) at a certain point in \( W^u(O) \), see Remark 4 for the choice of this point. So, the set \( \{ g \in U : \mu(g) = 0 \} \) defines a codimensional-1 surface in \( U \), consisting of systems having an orbit of homoclinic tangency close to \( \Gamma \) (the one in \( f \)). We call \( \mu \) the splitting parameter. The other parameters are the frequencies, or arguments of the complex central multipliers of \( O \), which are \( \omega \) in the saddle-focus case and \( \omega_1, \omega_2 \) in the double focus. Although the role of the frequencies in homoclinic bifurcations is not as obvious as \( \mu \), they are essential in forming the non-transverse intersection in a heterodimensional cycle. Since the hyperbolic point \( O \) admits unique continuations for all nearby systems and the dependence on systems is smooth, the above-mentioned parameters all depend smoothly on systems.

We will study the generic two-parameter unfolding family of \( \Gamma \), that is, any two-parameter family \( \{ f_\varepsilon \} \) of \( C^r \) diffeomorphism with \( f_{\varepsilon^*} = f \) satisfying in the saddle-focus case that

\[
\det \left. \frac{\partial (\mu(f_\varepsilon), \omega(f_\varepsilon))}{\partial \varepsilon} \right|_{\varepsilon = \varepsilon^*} \neq 0, \tag{2}
\]

and in the bi-focus case that

\[
\det \left. \frac{\partial (\mu(f_\varepsilon), \omega_1(f_\varepsilon))}{\partial \varepsilon} \right|_{\varepsilon = \varepsilon^*} \neq 0 \quad \text{if } \lambda \gamma > 1 \quad \text{and} \quad \det \left. \frac{\partial (\mu(f_\varepsilon), \omega_2(f_\varepsilon))}{\partial \varepsilon} \right|_{\varepsilon = \varepsilon^*} \neq 0 \quad \text{if } \lambda \gamma < 1. \tag{3}
\]

Recall that we denote by C1 - C6 the generic conditions introduced in Section 2. Denote by \( p/q \) the ratio of coprime integers.

Theorem 2. Suppose \( |\lambda \gamma| > 1 \) and conditions C1 - C4 are satisfied. Let \( O \) be either

- a saddle-focus of type (2,1) with condition C5.1 satisfied, or
- a bi-focus with condition C5.2 satisfied and \( \omega_2/2\pi \) being irrational or equal to \( p/q \) with \( q \geq 7 \).

Then, for any generic two-parameter unfolding family \( \{ f_\varepsilon \} \) of \( \Gamma \) with \( f_{\varepsilon^*} = f \), there exists a sequence \( \{ \varepsilon_j \} \) accumulating on \( \varepsilon = \varepsilon^* \) such that \( f_{\varepsilon_j} \) has a heterodimensional cycle involving the continuation of \( O \) and a periodic point \( Q \) of index \( d^u + 1 \).
For the bi-focus case with $|\lambda \gamma| < 1$, we consider $f^{-1}$ and assume that condition C5.2 is satisfied by $f^{-1}$ and that $\omega_1/2\pi$ is irrational or equals $p/q$ with $q \geq 7$. Since the generic conditions can be achieved by an arbitrarily $C^r$-small perturbation (see Proposition 1), Theorem 1 follows immediately from Theorem 2 when $f$ is at least $C^r$ with $r = 3, \ldots, \infty, \omega$; when $f$ has less regularity, one just needs to first perturb $f$ to be at least $C^3$.

We stress that the conditions of this theorem are given explicitly and verifiable, and the perturbations involved can be much more restrictive, for example, those keeping various symmetries, polynomial perturbations, etc..

Theorem 2 for the saddle-focus case is proved in Sections 3.1 - 3.6. We first create the point $Q$ of different index via changing $\mu$ (Section 3.3), then find the non-transverse intersection $W^u(O) \cap W^s(Q)$ by adjusting $\omega$ with $\mu$ fixed (Section 3.5); and finally use the area expansion guaranteed by $|\lambda \gamma| > 1$ to show that $Q$ can be chosen such that $W^s(O) \cap W^u(Q)$ is non-empty (Section 3.6). The theorem for the bi-focus case is proved in Section 4, using a complete reduction to the saddle-focus case.

As mentioned before, we will use [19, Theorem 9] to stabilize heterodimensional cycles found in the preceding theorem. For this, conditions of that theorem need to be verified, which particularly include checking an arithmetic property on the central multipliers of heterodimensional cycles. Importantly, this must be done within our unfolding family. To this aim, we notice that in creating the point $Q$, although its index is fixed, the specific values of its center-unstable multipliers are not. This leaves a freedom of choosing the parameter values $\varepsilon_j$ in Theorem 2. In fact, if a certain inequality on coefficients of some iterations of $f$ holds (condition C6), then each $\varepsilon_j$ corresponds to a curve $c_j(t)$ in the parameter space (see Lemma 6), on which the found heterodimensional cycle persists and, particularly, $\omega$ changes. It will then be shown in Section 3.7 that changing $t$ indeed gives a heterodimensional cycle fulfilling the conditions of [19, Theorem 9] and we thus obtain

**Theorem 3.** If additionally condition C6.1 for the saddle-focus case, and C6.2 for the bi-focus case are satisfied, then the sequence $\{\varepsilon_j\}$ in Theorem 2 can be chosen such that $f_{\varepsilon_j}$ has a $C^1$-robust heterodimensional cycle involving a cs-blender of index $d^u$ containing the continuation of $O$ and a cu-blender of index $d^u + 1$, activating each other.

By definition, each $\varepsilon_j$ lies in an open disc in the parameter plane, where values corresponding to heterodimensional cycle involving the continuations of $O$ are dense. The arguments used to obtain $C^1$-robust homoclinic tangency in Corollary 1 can be adapted for our unfolding families. On the one hand, it can be checked that the perturbations used in the proof of [10, Theorem 4.9] are essentially ‘moving the tangency’, which is what our parameter $\mu$ does. On the other hand, results in [13] show that every one-parameter unfolding $\{f_\mu\}$ of a generic homoclinic tangency there are intervals of $\mu$ values accumulating on $\mu = 0$ such that the corresponding systems belong to the Newhouse domain, i.e., an open set in the space of $C^r$ diffeomorphisms where those having a homoclinic tangency to the continuation of $O$ are dense.

**Corollary 2.** The sequence $\{\varepsilon_j\}$ in Theorem 3 can be further chosen such that the cs-blender involved in the robust heterodimensional cycle at $\varepsilon = \varepsilon_j$ also exhibits a $C^1$-robust homoclinic tangency.
2 Genericity conditions

We now introduce genericity conditions for the homoclinic tangency. For brevity we will omit the dependence of the invariant manifolds on points when there is no ambiguity. Define the local stable invariant manifolds $W^s_{loc}$ of $O$ as the connected component of $W^s \cup U$ that contains $O$ (recall that $U$ is a small neighborhood of $O$). The local unstable invariant manifolds $W^u_{loc}$ is defined in the same way. Take two points $M^+ \in W^s_{loc} \cap \Gamma$ and $M^- \in W^u_{loc} \cap \Gamma$ such that $f^{n_0}(M^-) = M^+$ for some $n_0$, and define the global map as $T_1 := f^{n_0}$ from a neighborhood of $M^-$ to a neighborhood of $M^+$. By assumption, the image $T_1(W^u_{loc})$ is a surface tangent at $M^+$ to $W^s_{loc}$. We require that

C1. the tangent spaces of $T_1(W^u_{loc})$ and $W^s_{loc}$ at $M^+$ have only one common vector (up to scalar multiplications), that is, $\dim(T_{M^+}T_1(W^u_{loc}) \cap T_{M^+}(W^s_{loc})) = 1$.

C2. the tangency between $T_1(W^u_{loc})$ and $W^s_{loc}$ at $M^+$ is quadratic, and

These two conditions are essentially the ones for quasi-transversality in [24]. They allow us to introduce coordinates $(s_1, s_2, t_1, t_2) \in \mathbb{R} \times \mathbb{R}^{d^s-1} \times \mathbb{R} \times \mathbb{R}^{d^u-1}$ centered at $M^+$ such that $W^s_{loc} = \{t_1 = 0, t_2 = 0\}$ and $T_1(W^u_{loc}) = \{t_1 = g(s_1), s_2 = 0\}$ for some function $g$ satisfying $g(0) = 0, g'(0) = 0, g''(0) \neq 0$.

When $O$ has strong-stable or strong-unstable multipliers, one more condition is needed for the tangency. Denote by $d^{cs}$ and $d^{cu}$ the numbers of central stable and central unstable multipliers. Then, $d^{cs} = 2, d^{cu} = 1$ in the saddle-focus case (as mentioned before we only consider the (2,1) case as the (1,2) case follows from symmetry), and $d^{cs} = 2, d^{cu} = 2$ in the saddle-focus case. We now assume $d^s > d^u$. Recall that (see e.g. [26]) there exists on $W^s$ a strong-stable foliation $\mathcal{F}^{ss}$ consisting of $(d^s-d^{cs})$-dimensional leaves. Particularly, the leaf through $O$ is the strong-stable manifold $W^{ss}$ that corresponds to the multipliers $\lambda_{d^s+1}, \ldots, \lambda_{d^s}$. There also exists a $(d^{cs}+1)$-dimensional extended unstable invariant manifold $W^{uu}$, which contains $W^u$ and is tangent to $O$ to the eigenspace corresponding to $\lambda_1, \ldots, \lambda_{d^u}, \gamma_1, \ldots, \gamma_{d^u}$. Such manifolds are not unique, but any two of them are tangent to each other at points of $W^u$. Similarly, when $d^u > d^{cu}$, there exists on $W^u$ a strong-unstable foliation $\mathcal{F}^{uu}$ consisting of $(d^u-d^{cu})$-dimensional leaves, which includes the strong-unstable manifold $W^{uu}$ corresponding to the multipliers $\gamma_{d^{cu}+1}, \ldots, \gamma_{d^u}$. We also have a $(d^{cu}+1)$-dimensional extended stable invariant manifold $W^{sE}$, which contains $W^s$ and is tangent at $O$ to the eigenspace corresponding to $\gamma_1, \ldots, \gamma_{d^{cu}}, \lambda_1, \ldots, \lambda_{d^u}$. It is not unique and any two of such manifolds are tangent to each other at points of $W^s$. We require that

C3. if $d^{cs} < d^u$, then $DT_1(T_{M^-}W^{uE})$ is transverse to the leaf of $\mathcal{F}^{ss}$ through $M^+$; if $d^{cu} < d^u$, then $DT_1^{-1}(T_{M^+}W^{sE})$ is transverse to the leaf of $\mathcal{F}^{uu}$ through $M^-$.

C4. The homoclinic tangency does not belong to strong manifolds, namely, $\Gamma \not\subset W^{ss} \cup W^{uu}$.

Condition C3 means that the tangency happens in the central directions, which in the saddle-focus case implies condition C1. Conditions C3, C4 are $C^1$-open and conditions C1, C2 are $C^1$-open in the space of system having the homoclinic tangency (i.e., on the co-dimension 1 surface $\{\mu = 0\}$); they do not depend on the choice of the points $M^+$ and $M^-$, and condition C3 does not depend on the choice of the extended manifolds.

The next two genericity conditions are about certain derivatives of $T_1$ and the position of $M^+$, which we explain as follows. First introduce (see e.g. [14]) $C^\sigma$ coordinates $(x, y, u, v) \in \mathbb{R}^{d^s} \times \mathbb{R}^{d^s} \times \mathbb{R}^{d^u} \times \mathbb{R}^{d^u}$ that correspond to the multipliers $\lambda_{d^s+1}, \ldots, \lambda_{d^s}$ and $\gamma_{d^{cu}+1}, \ldots, \gamma_{d^u}$. Then, the tangent spaces at $O$ are spanned by the eigenvectors of the linearization of $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ at $O$, respectively. Moreover, the tangency conditions C1 and C2 reduce to linear conditions on the coefficients of the Taylor expansions of $T_1$ at $O$.
straightening all the local invariant manifolds in $U$:

\[ W^{s}_{loc} = \{ y = 0, v = 0 \}, \quad W^{ss}_{loc} = \{ x = 0, y = 0, v = 0 \}, \]
\[ W^{u}_{loc} = \{ x = 0, u = 0 \}, \quad W^{uu}_{loc} = \{ y = 0, x = 0, u = 0 \}, \]

and the leaves of $F^{s}$ and $F^{uu}$ in $U$ so that they take the forms \{ $x = \text{const}, y = 0, v = 0$ \} and \{ $y = \text{const}, x = 0, u = 0$ \}, respectively. Also, the extended manifold $W^{uE}$ is tangent to \{ $u = 0$ \} at \{ $x = 0, u = 0$ \} and $W^{sE}$ is tangent to \{ $v = 0$ \} at \{ $y = 0, v = 0$ \}. In these coordinates we have $M^{+} = (x^{+}, 0, u^{+}, 0)$ and $M^{-} = (0, y^{-}, 0, v^{-})$, and condition C4 reads

\[ x^{+} \neq 0 \quad \text{and} \quad y^{-} \neq 0. \quad (4) \]

Recall that the local map $T_{0}$ is defined as the iteration of $f$ so that $O$ is a fixed point of it, as defined above (1). We will work with the coordinates which additionally satisfy that the restrictions of $T_{0}$ to the local stable and local unstable manifolds are linear in the corresponding center direction, namely, if we denote $T_{0}|_{W^{s}_{loc}} : (x, u) \mapsto (\tilde{x}, \tilde{u})$ and $T_{0}|_{W^{u}_{loc}} : (y, v) \mapsto (\tilde{y}, \tilde{v})$, then we have

\[
\begin{align*}
\text{Saddle-focus case:} & \quad \tilde{x} = \lambda \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} x \quad \text{and} \quad \tilde{y} = \gamma y; \\
\text{Bi-focus case:} & \quad \tilde{x} = \lambda \begin{pmatrix} \cos \omega_{1} & -\sin \omega_{1} \\ \sin \omega_{1} & \cos \omega_{1} \end{pmatrix} x \quad \text{and} \quad \tilde{y} = \gamma \begin{pmatrix} \cos \omega_{2} & -\sin \omega_{2} \\ \sin \omega_{2} & \cos \omega_{2} \end{pmatrix} y. 
\end{align*} \quad (5)
\]

This can be always achieved, see e.g. Lemma 6 of [14].

Now assume $d^{ss} < d^{s}$, $d^{uu} < d^{u}$, and conditions C1 and C3 are satisfied. Take some small neighborhoods $\Pi^{+}$ and $\Pi^{-}$ of $M^{+}$ and $M^{-}$. Since $W^{uE}$ is tangent to \{ $u = 0$ \} at $M^{-}$ and $W^{sE}$ is tangent to \{ $v = 0$ \} at $M^{+}$, points on $W^{sE}_{loc} \cap \Pi^{+}$ and $W^{uE}_{loc} \cap \Pi^{-}$ can be described by coordinates $(x, y, u)$ and, respectively, $(\tilde{x}, \tilde{y}, \tilde{u})$. On the other hand, condition C3 implies that $T_{1}(W^{uE}_{loc} \cap \Pi^{-})$ and $T_{1}^{-1}(W^{sE}_{loc} \cap \Pi^{+})$ can be parametrised by $(x, y, v)$ and, respectively, $(\tilde{x}, \tilde{y}, \tilde{u})$. As a result, if we denote $S_{2} := T_{1}(W^{uE}_{loc} \cap \Pi^{-}) \cap W^{sE}_{loc}$ and $S_{1} := T_{1}^{-1}(S_{2})$, then the restriction $F := T_{1}|_{S_{1}} : (\tilde{x}, \tilde{y}) \mapsto (x, y)$ is a diffeomorphism. Let $\pi_{x}$ be the projection of vectors to their $x$-components. Since $T_{M}W^{uE}_{loc} \subset T_{M}W^{uE}_{loc}$ and $T_{M}+T_{1}(W^{s}_{loc}) \subset T_{M}+T_{1}(W^{sE}_{loc})$, it follows from condition C1 that $\pi_{y} \circ DF|_{\tilde{y}}$ has rank $(d^{uu}-1)$, and, therefore, $\pi_{x} \circ DF|_{\tilde{y}}$ has rank 1 (since $F$ is a diffeomorphism). In the saddle-focus case, this means that

\[ a := \begin{pmatrix} \partial x_{1} \;/\; \partial \tilde{y} \\ \partial x_{2} \;/\; \partial \tilde{y} \end{pmatrix} \neq 0. \quad (6) \]

These derivatives are explicitly given in Section 3.1. We impose for the saddle-focus case the following condition:

**C5.1.** the vector $(x^{+}_{1}, x^{+}_{2})$ is not parallel to $a$ defined above.

This condition is independent of the choice of coordinates that straighten the local manifolds and foliations, and linearise the central dynamics as (5). Indeed, if $T$ is a $C^{r}$ transformation to another set of such coordinates, then its restrictions to $W^{s}_{loc}$ and $W^{u}_{loc}$ are linear in the central coordinates, which means that there exist some constants $c_{1}, c_{2}$ satisfying $a^{new} = c_{1}a$ and $(x^{+}_{1}, x^{+}_{2})^{new} = c_{2}(x^{+}_{1}, x^{+}_{2})$. A
similar argument also shows that condition C4 does not depend on the choice of the points $M^+$ and $M^-$. In the bi-focus case, the map $\pi_y \circ DF|_{\bar{y}}$ has rank 1, and hence $\det \pi_y \circ DF|_{\bar{y}} = 0$. So, up to a rotation in $y$, one can achieve $\partial y_1 / \partial (\bar{y}_1, \bar{y}_2) = 0$ and $\partial y_2 / \partial (\bar{y}_1, \bar{y}_2) \neq 0$. It can be checked that up to replacing $M^-$ by $T_0^{-1}(M^-)$ and $T_1^{-1}(M^-)$, one especially has $\partial y_2 / \partial (\bar{y}_1, \bar{y}_2) \neq 0$. It follows that, if we take $S'_2 = S_2 \cap \{y_2 = 0\}$ then $S'_1 = F_1^{-1}(S'_2)$ can be parametrised by $(\bar{x}, \bar{y}_1)$, and hence the map $F_{S'_1} : (\bar{x}, \bar{y}_1) \mapsto (x, y_1)$ is a diffeomorphism. We then have

$$b := \left( \frac{\partial x_1}{\partial \bar{y}_1}, \frac{\partial x_2}{\partial \bar{y}_1} \right) \neq 0$$

since $dy_1/d\bar{y}_1 = 0$ by construction. These derivatives are explicitly given in Section 4. For a detailed explanation of the above arguments, see the proof of Lemma 8 in [14].

We impose for the bi-focus case the condition:

**C5.2.** the vector $(x_1^+, x_2^+)$ is not parallel to $b$.

To see that these conditions are well-defined, one first notes that different rotations in $y$ and replacements of $M^-$ involved in getting the derivatives in (7) lead to derivatives differ by a multiplied factor, so the slope of the vector $b$ does not change. Besides, arguing as in the saddle-focus case, these two conditions do not depend on the choice of the linearizing coordinates.

The above genericity conditions are needed to create heterodimensional cycles from unfoldings of the homoclinic tangency. There are two additional conditions for making the found heterodimensional cycles robust within the same parametric families, and they are still given by inequalities on certain derivatives of the global map $T_1$. However, they are more subtle and do not have clear geometric interpretations as the above ones. We will introduce them later after we obtain a formula for $T_1$, in (15) as condition C6.1 for the saddle-focus case and in (145) as condition C6.2 for the bi-focus case.

**Proposition 1.** For a general homoclinic tangency, either a (2,1) saddle-focus or a bi-focus, the corresponding generic conditions can be fulfilled after an arbitrarily $C^r$-small perturbation.

**Proof.** This is obvious for conditions C4 - C6, as the violation of any of them is equivalent to have certain equations hold. It is also the same for conditions C1 and C3. Indeed, they are just requirements on the ranks of the derivatives at $M^-$ of the restrictions of $T_1$ to $W^u_{loc} \cap T_1^{-1}(W^s_{loc})$, and, respectively, to $W^u_{loc} \cap T_1^{-1}(W^{sE}_{loc})$.

Condition C2 means that, when C1 is satisfied, there exist local coordinates near the tangency point such that $T_1(W^u_{loc})$ has the form $y = \phi(x)$ with $x, y \in \mathbb{R}$ for some function $\phi$ satisfying $d^2 y/\partial x^2 \neq 0$ and $d^3 y/\partial x^3 = 0$ at the tangency. So, the loss of quadracity amounts to the vanishing of $d^2 y/\partial x^2$, which can be easily recovered by a small perturbation. For a detailed discussion on conditions C1 and C2, see e.g. [14, 24].
3 The saddle-focus case

In this section we prove Theorems 2 and 3 when $O$ is a saddle-focus of type (2,1). The central multipliers are $\lambda_1 = \lambda_2 = \lambda e^{i\omega}$ ($0 < \omega < \pi$), and $\gamma_1 =: \gamma$ being real, with the assumptions $|\lambda\gamma| > 1$ and C1 - C5.1.

Note that when conditions C3 and C4 hold, there is a three-dimensional invariant manifold containing $O$ and the orbit $\Gamma$ of homoclinic tangency. However, we do not consider reducing the problem to a three-dimensional one, since the method developed here will also be applied to the bi-focus case, where no such reduction exists.

3.1 The first return map

Let us start with defining a first return map near $\Gamma$ and derive a formula for it. We embed $f$ into a generic two-parameter unfolding family $\{f_\varepsilon\}$ with $f_{\varepsilon^*} = f$. According to (2), there will be no difference by taking $\varepsilon = (\mu, \omega)$, so $f_{\varepsilon^*} = f_{0,\omega^*} = f$ with $\omega^* = \omega(f)$.

Take two small cubes $\Pi^+$ and $\Pi^-$ of size $\delta$ centered at $M^+$ and $M^-$, respectively. Throughout the rest of this paper, we use $O(\cdot)$ to denote the term which is smaller than the quantity inside the bracket multiplied by a constant; moreover, this constant is independent of the system and the size of $\Pi^\pm$ in the sense that it remains the same for $\varepsilon$ values sufficiently close to $\varepsilon^*$, and for all $\delta_{\text{new}} < \delta$ (i.e. when we replace $\Pi^\pm$ by subcubes of size $\delta_{\text{new}}$). Similarly, we use $o(1)_{k \to \infty}$ to denote the terms which converge to 0 as $k \to \infty$ uniformly for $\varepsilon$ close to $\varepsilon^*$ and $\delta_{\text{new}} < \delta$.

By Lemma 6 in [14], there exists a coordinate transformation, $C^r$ with respect to variables and $C^{r-2}$ with respect to parameters, such that the local map $T_0$ assumes the form

\[
\begin{align*}
\bar{x}_1 &= \lambda(\varepsilon)x_1 \cos \omega - \lambda(\varepsilon)x_2 \sin \omega + g_1(x, y, u, v, \varepsilon), \\
\bar{x}_2 &= \lambda(\varepsilon)x_1 \sin \omega + \lambda(\varepsilon)x_2 \cos \omega + g_2(x, y, u, v, \varepsilon), \\
\bar{y} &= \gamma(\varepsilon)y + g_3(x, y, u, v, \varepsilon), \\
\bar{u} &= A(\varepsilon)u + g_4(x, y, u, v, \varepsilon), \\
\bar{v} &= B(\varepsilon)v + g_5(x, y, u, v, \varepsilon),
\end{align*}
\]

(8)

where all coefficients depend on $\varepsilon$, $x = (x_1, x_2)$, the eigenvalue of $A$ and $B$ are $\lambda_3 \ldots \lambda_{d^u}$ and $\gamma_2 \ldots \gamma_{d^v}$, respectively, and functions $g$ satisfy

\[
\begin{align*}
g_{1,2,4}(0, y, 0, v, \varepsilon) &= 0, & g_{3,5}(x, 0, 0, 0, \varepsilon) &= 0, & g_{1,2}(x, 0, u, 0, \varepsilon) &= 0, & g_3(0, y, 0, v, \varepsilon) &= 0, \\
\frac{\partial g_{1,2,4}}{\partial x}(0, y, 0, v, \varepsilon) &= 0, & \frac{\partial g_{3,5}}{\partial y}(x, 0, u, 0, \varepsilon) &= 0,
\end{align*}
\]

(9)

for all sufficiently small $x, y, u, v$.

By Lemma 7 in [14], the above formula further implies that for any two points $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) \in \Pi^-, (x_1, x_2, y, u, v) \in \Pi^+$, we have $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) = T_0^k(x_1, x_2, y, u, v)$ if and only if they satisfy the
Here all coefficients depend on $\varepsilon$. For all system $g$ close to $f$, we define $\mu(g)$ as the functional measuring the (signed) distance between $T_1(0,0,y^-(\varepsilon),0,v^-)$ and $W^s_{\text{loc}}(O)$, where $y^-(\varepsilon)$ is chosen as $y^-(\varepsilon)$ above. In this case, the coordinates $(x_1, x_2, \tilde{y}, u, \tilde{v}, \varepsilon)$ are chosen such that $y^+(\varepsilon^*) = v^+(\varepsilon^*) = 0$. It follows from [14, Corollary 1] that conditions C1 - C3 allow us to rewrite the above formula as

$$
\begin{align*}
    x_1 - x_1^+ &= a_{11} \tilde{x}_1 + a_{12} \tilde{x}_2 + b_1(\tilde{y} - y^-) + c_1 \tilde{u} + d_1 v + \ldots, \\
    x_2 - x_2^+ &= a_{21} \tilde{x}_1 + a_{22} \tilde{x}_2 + c_2 \tilde{u} + d_2 v + \ldots, \\
    y &= \mu + a_{31} \tilde{x}_1 + a_{32} \tilde{x}_2 + b_3(\tilde{y} - y^-)^2 + c_3 \tilde{u} + d_3 v + \ldots, \\
    u - u^+ &= a_{41} \tilde{x}_1 + a_{42} \tilde{x}_2 + b_4(\tilde{y} - y^-) + c_4 \tilde{u} + d_4 v + \ldots, \\
    \tilde{v} - v^- &= a_{51} \tilde{x}_1 + a_{52} \tilde{x}_2 + b_5(\tilde{y} - y^-) + c_5 \tilde{u} + d_5 v + \ldots,
\end{align*}
$$

where we wrote $\mu$ in place of the constant term by definition (see Remark 4), all coefficients depend on $\varepsilon$ with at least $C^{r-2}$ smoothness, and, particularly,

$$
a_{31}^2 + a_{32}^2 \neq 0, \quad b_1 \neq 0, b_3 \neq 0, \det d_3 \neq 0 \quad \text{and} \quad \varepsilon \neq 0.$$

Remark 3. In such coordinates, the system is $C^{r-2}$ in parameters, for details see [14, Lemma 6]. Our smoothness requirement $r \geq 3$ originates from the use of first order derivatives of coefficients with respect to parameters, in the proofs of Lemma 6 and 8.
way, $\mu$ is just the constant term in the third equation in (12).

We can now formulate the genericity condition C6.1 mentioned in the end of Section 2. Denote by $e_1$ and $e_2$ the coefficients in front of the terms $\bar{x}_1(\bar{y} - y^-)$ and $\bar{x}_2(\bar{y} - y^-)$ in the dots of the third equation of (12).

C6.1. At least one of the following inequalities holds:

$$a_{11} \neq \frac{b_1^2(x_1^+ a_{31} - x_2^+ a_{32})}{2b_3((x_1^+)^2 + (x_2^+)^2)} + \frac{x_1^+}{x_2^+} a_{21}, \quad a_{12} \neq \frac{b_1^2(x_2^+ a_{31} + x_1^+ a_{32})}{2b_3((x_1^+)^2 + (x_2^+)^2)} + \frac{x_2^+}{x_1^+} a_{22}, \quad e_1 x_1^+ + e_2 x_2^+ \neq 0 \quad (15)$$

We define for each large $k$ the first return map $T_k := T_1 \circ T_0^k$. Note that for any point $(x_1, x_2, y, u, v) \in \Pi^+$ we have $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) = T_k(x_1, x_2, y, u, v) \in \Pi^+$ if and only if there is $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) \in \Pi^-$ such that

$$(x_1, x_2, y, u, v) \xrightarrow{T_k} (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) \xrightarrow{T_1} (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}).$$

The domains of $T_k$ are strip-like regions $T_0^k(\Pi^-) \cap \Pi^+$ accumulating on $W^s_{loc}(O)$. Note that formula (10) shows that $y, v$ are uniquely determined by $(x_1, x_2, \bar{y}, u, \bar{v})$, and $\bar{y}, \bar{v}$ are uniquely determined by $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v})$, with $\bar{y}, \bar{v}$ being given by $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) = T_0^k(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v})$. Thus, for each $k$, this defines a coordinate transformation $(x_1, x_2, y, u, v) \rightarrow (x_1, x_2, \bar{y}, u, \bar{v})$. In the new coordinates $T_k : (x_1, x_2, \bar{y}, u, \bar{v}) \rightarrow (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v})$ is defined on

$$(x_1 - x_1^+, x_2 - x_2^+, \bar{y} - y^-, u - u^+, \bar{v} - v^-) \in [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_l-2} \times [-\delta, \delta]^{d_u-1} =: \hat{\Pi}. \quad (16)$$

3.1.1 A normal form for $T_k$

We now apply several coordinate transformations to bring $T_k$ to a form with a possible new expanding direction being indicated, which will be used for creating an index-$(d^n + 1)$ periodic point.

Let us start with

$$X_1 = x_1 - x_1^+, \quad X_2 = x_2 - x_2^+, \quad Y = \bar{y} - y^-, \quad U = u - u^+, \quad V = \bar{v} - v^- \quad (17)$$

Combining (10) and (12) with replacing $(x_1, x_2, y, u, v)$ by $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v})$ in (12), yields that $(\bar{X}_1, \bar{X}_2, \bar{Y}, \bar{U}, \bar{V}) = T_k(X_1, X_2, Y, U, V)$ if and only if

$$\bar{X}_1 = \lambda^k \bar{X}_1 \left( X_1 + x_1^+ \right) + \lambda^k \bar{X}_2 \left( X_2 + x_2^+ \right) + b_1 Y + \hat{h}_1(X, Y, U, V),$$

$$\bar{X}_2 = \lambda^k \bar{X}_2 \left( X_1 + x_1^+ \right) + \lambda^k \bar{X}_2 \left( X_2 + x_2^+ \right) + \hat{h}_2(X, Y, U, V),$$

$$\bar{Y} = \gamma^k \mu - y^- + \lambda^k \gamma^k \alpha^*(X_1 + x_1^+) + \lambda^k \gamma^k \beta^*(X_2 + x_2^+) + b_3 Y^2 + \gamma^k \hat{h}_3(X, Y, U, V),$$

$$\bar{U} = \lambda^k \bar{U}_4 \left( X_1 + x_1^+ \right) + \lambda^k \bar{U}_5 \left( X_2 + x_2^+ \right) + b_4 Y + \hat{h}_4(X, Y, U, V),$$

$$\bar{V} = \lambda^k \bar{V}_5 \left( X_1 + x_1^+ \right) + \lambda^k \bar{V}_6 \left( X_2 + x_2^+ \right) + b_5 Y + \hat{h}_5(X, Y, U, V), \quad (18)$$

where we used the implicit function theorem in the obvious way along with the facts from (10) that $\bar{Y} = \bar{y} - y^- = \gamma^k \bar{y} - y^- + o(1)_{k \rightarrow \infty}$ and that $\bar{v} = O(\hat{\gamma}^-k)$ is a function of $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, and hence of

12
\( \vec{X}, \vec{Y}, \vec{Z}, \vec{V} \). Here the coefficients are

\[
\begin{align*}
\alpha^* &= a_{31} \cos k \omega + a_{32} \sin k \omega, \\
\beta^* &= -a_{31} \sin k \omega + a_{32} \cos k \omega, \\
\hat{\alpha}_i &= a_{1i} \cos k \omega + a_{2i} \sin k \omega, \\
\hat{\beta}_i &= -a_{1i} \sin k \omega + a_{2i} \cos k \omega, 
\end{align*}
\] (19)

for \( i = 1, 2, 4, 5 \). With the assumption \(|\lambda \gamma| > 1\) and (11) the functions \( \hat{h} \) satisfy

\[
\hat{h}_i = O(\hat{\lambda}^k) + O(\bar{v}) + O(\bar{x}_1^2 + \bar{x}_2^2 + (\bar{y} - \bar{y}_0)^2 + \bar{u}^2 + \bar{v}^2)
\]

\[
= O(\hat{\lambda}^k) + O(\hat{\gamma}^{-k}) + O(\lambda^{2k} + Y^2 + \hat{\lambda}^{2k} + \hat{\gamma}^{-2k})
\]

\[
= O(\hat{\lambda}^k) + O(\lambda^k Y) + O(Y^2), \quad i = 1, 2, 4, 5,
\]

and

\[
\hat{h}_3 = O(\hat{\lambda}^k) + O(\hat{\gamma}^{-k}) + O(\lambda^k Y) + O(Y^3) = O(\hat{\lambda}^k) + O(\lambda^k Y) + O(Y^3),
\]

where the terms \( O(\lambda^k Y) \) are of the form \( O(\hat{\lambda}^k)Y + O(\lambda^k)Y^2 + \ldots \) with \( O(\lambda^k) \) being functions of \( X_1, X_2, U, \bar{V} \), and the derivatives are estimated as

\[
\begin{align*}
\frac{\partial \hat{h}}{\partial (X, U)} &= O(\hat{\lambda}^k) + O(\lambda^k Y), \\
\frac{\partial \hat{h}}{\partial V} &= O(\hat{\gamma}^{-k}), \\
\frac{\partial \hat{h}_3}{\partial Y} &= O(\lambda^k) + O(Y^2), \\
\frac{\partial \hat{h}_3}{\partial \bar{V}} &= O(\hat{\lambda}^k) + O(Y).
\end{align*}
\] (21)

Since \( b_1 \neq 0 \), we can kill all the linear-in-\( Y \) terms in the equations for \( \bar{U} \) and \( \bar{V} \) by

\[
U^{\text{new}} = U - \frac{b_4}{b_1} X_1 \quad \text{and} \quad V^{\text{new}} = V - b_5 Y,
\]

and get

\[
\begin{align*}
\dot{X}_1 &= \lambda^k \dot{\alpha}_1 (X_1 + X_1^+) + \lambda^k \dot{\beta}_1 (X_2 + X_2^+) + b_1 Y + \hat{h}_1 (X, Y, U + \frac{b_4}{b_1} X_1, \bar{V} + b_5 \bar{Y}), \\
\dot{X}_2 &= \lambda^k \dot{\alpha}_2 (X_1 + X_1^+) + \lambda^k \dot{\beta}_2 (X_2 + X_2^+) + \hat{h}_2 (X, Y, U + \frac{b_4}{b_1} X_1, \bar{V} + b_5 \bar{Y}), \\
\dot{Y} &= \gamma^k \mu - y^* + \lambda^k \gamma^k \alpha^* (X_1 + X_1^+) + \lambda^k \gamma^k \beta^* (X_2 + X_2^+) + b_3 \gamma^k Y^2 + \gamma^k \hat{h}_3 (X, Y, U + \frac{b_4}{b_1} X_1, \bar{V} + b_5 \bar{Y}), \\
\dot{\bar{U}} &= \lambda^k (\dot{\alpha}_4 - \frac{b_4}{b_1} \dot{\alpha}_1)(X_1 + X_1^+) + \lambda^k (\dot{\beta}_4 - \frac{b_4}{b_1} \dot{\beta}_1)(X_2 + X_2^+) + \hat{h}_4 (X, Y, U + \frac{b_4}{b_1} X_1, \bar{V} + b_5 \bar{Y}) \\
&\quad \quad - \frac{b_4}{b_1} \hat{h}_1 (X, Y, U + \frac{b_4}{b_1} X_1, \bar{V} + b_5 \bar{Y}), \\
V &= \lambda^k \dot{\alpha}_5 (X_1 + X_1^+) + \lambda^k \dot{\beta}_5 (X_2 + X_2^+) + \hat{h}_5 (X, Y, U + \frac{b_4}{b_1} X_1, \bar{V} + b_5 \bar{Y}),
\end{align*}
\] (23)

where \( X = (X_1, X_2) \). By the implicit function theorem one solves out \( \bar{Y} \) from the third equation, with the same form as before and a new function \( \hat{h}_3^{\text{new}}(X, Y, U, \bar{V}) \), which along with its derivatives satisfy the same estimates as \( \hat{h}_3 \). Substituting this express for \( \bar{Y} \) into the remaining equations, one easily finds by the chain rule that the derivatives of \( \hat{h}_i^{\text{new}}(X, Y, U, \bar{V}) := \hat{h}_i(X, Y, U + b_4 X_1/b_1, \bar{V} + b_5 \bar{Y}) \) \( (i = 1, 2, 4, 5) \) have the same forms as (21). Regarding the \( C^0 \) norms, we have

\[
\begin{align*}
\hat{h}_i^{\text{new}} &= O(\hat{\lambda}^k) + O(\lambda^k Y) + O(Y^2) + O(\hat{\gamma}^{-k} \bar{Y}^2) = O(\hat{\lambda}^k) + O(\lambda^k Y) + O(Y^2) \quad (i = 1, 2, 4, 5), \\
\hat{h}_3^{\text{new}} &= O(\hat{\lambda}^k) + O(\lambda^k Y) + O(Y^3) + O(\hat{\gamma}^{-k} \bar{Y}^2) = O(\hat{\lambda}^k) + O(\lambda^k Y) + O(Y^3),
\end{align*}
\] (24)
where the second equality in each line follows from the fact that \( \dot{Y} \) is bounded.

We replace \( \dot{h} \) by \( \dot{h}^{\text{new}} \) in (23) and drop the superscript ‘new’. Next, we seek for a coordinate transformation to bring the estimates for \( \dot{h}_3 \) to

\[
\dot{h}_3 = O(\dot{\lambda}^k) + O(Y^3), \quad \frac{\partial \dot{h}_3}{\partial (X, U)} = O(\dot{\lambda}^k), \quad \frac{\partial \dot{h}_3}{\partial Y} = O(\dot{\lambda}^k) + O(Y^2), \quad \frac{\partial \dot{h}_3}{\partial V} = O(\gamma^{-k}). \tag{25}
\]

It suffices to remove the linear-in-\( Y \) terms of the form \( c\lambda^kY \) and quadratic terms of the form \( c\lambda^kXY \) in \( \dot{h}_3 \), where \( c \)'s stand for constants. Since \( b_3 \neq 0 \), this can be done by making small changes to \( Y \). Before start introducing the coordinate change, we note that such terms are contained in the \( O(\lambda^kY) \) term in (20), and they originate from the terms \( e_1\dot{x}_1(y - y^-) \) and \( e_2\dot{x}_2(y - y^-) \) in the dots of the third equation of (12). Thus, the above-mentioned terms in \( \dot{h}_3 \) that we want to remove are \( (e_1x_1^+ + e_2x_2^+)\lambda^kY \) and \( (e_1X_1 + e_2X_2)\lambda^kY \).

Now, adding the correction \( -b_3^{-1}(e_1X_1 + e_2X_2)\lambda^kY \), we kill the term \((e_1X_1 + e_2X_2)\lambda^kY \) in \( \dot{h}_3 \). This process also produces extra terms in the equations of (23). We call those terms negligible if the estimates of \( \dot{h} \) do not change after absorbing them. The only one cannot be neglected is \( -b_1b_3^{-1}(e_1X_1 + e_2X_2)\lambda^k + O(\dot{\lambda}^k) \) appearing in the first equation of (23).

Then, we kill the term \((e_1x_1^+ + e_2x_2^+)\lambda^kY \) in \( \dot{h}_3 \) to achieve the first and second estimates in (25). This amounts to adding the constant correction \( -b_1b_3^{-1}(e_1x_1^+ + e_2x_2^+)\lambda^k \) to \( Y \). There will be non-negligible terms \(-b_3^{-1}(e_1x_1^+ + e_2x_2^+)\lambda^k \) appearing in the first equation, and \( c\lambda^k\gamma^kY^2 \) appearing in the third equation, where we do not write the constant \( c \) explicitly as it will not be used.

Finally, one observes that the new constant term in the first equation, along with the original constants in all but the third equation of (23), can be removed by adding simultaneously certain constant corrections of order \( O(\dot{\lambda}^k) \) to \( X, U, V \)-coordinates (whose major parts are the constants written explicitly in (23) and the term \(-b_1b_3^{-1}(e_1x_1^+ + e_2x_2^+)\lambda^k \) additionally for \( X_1 \)), and also an extra correction of order \( O(\dot{\lambda}^k) \) to \( Y \). The price is getting some constant of order \( O(\dot{\lambda}^k\gamma^k) \) in the third equation. In summary, there exists a coordinate transformation of the form

\[
X^{\text{new}} = X_1 - \lambda^k(\dot{\alpha}_1x_1 + \dot{\beta}_1x_2) + b_1b_3^{-1}\lambda^k(e_1x_1^+ + e_2x_2^+) + O(\dot{\lambda}^k),
\]
\[
X_2^{\text{new}} = X_2 - \lambda^k(\dot{\alpha}_2x_1 + \dot{\beta}_2x_2) + O(\dot{\lambda}^k),
\]
\[
Y^{\text{new}} = Y - b_3^{-1}(e_1X_1 + e_2X_2)\lambda^k - b_1b_3^{-1}(e_1x_1^+ + e_2x_2^+)\lambda^k + O(\dot{\lambda}^k),
\]
\[
U^{\text{new}} = U - \lambda^k\left((\dot{\alpha}_4 - \frac{b_4}{b_1}\dot{\beta}_1)x_1^+ + (\dot{\beta}_4 - \frac{b_4}{b_1}\dot{\beta}_1)x_2^+\right) + O(\dot{\lambda}^k),
\]
\[
V^{\text{new}} = V - \lambda^k(\dot{\alpha}_5x_1^+ + \dot{\beta}_5x_2^+) + O(\dot{\lambda}^k),
\]

for some constant terms \( O(\dot{\lambda}^k) \), such that formula (18) assumes the form

\[
\dot{X}_1 = \lambda^k\dot{\alpha}_1X_1 + \lambda^k\dot{\beta}_1X_2 + b_1Y + \dot{h}_1(X, Y, U, V),
\]
\[
\dot{X}_2 = \lambda^k\dot{\alpha}_2X_1 + \lambda^k\dot{\beta}_2X_2 + \dot{h}_2(X, Y, U, V),
\]
\[
\dot{Y} = L + \lambda^k\gamma^k\alpha^*X_1 + \lambda^k\gamma^k\beta^*X_2 + b_3\gamma^kY^2 + \gamma^k\dot{h}_3(X, Y, U, V),
\]
\[
\dot{U} = \lambda^k(\dot{\alpha}_4 - \frac{b_4}{b_1}\dot{\beta}_1)X_1 + \lambda^k(\dot{\beta}_4 - \frac{b_4}{b_1}\dot{\beta}_1)X_2 + \dot{h}_4(X, Y, U, V),
\]
\[
V = \lambda^k\dot{\alpha}_5X_1 + \lambda^k\dot{\beta}_5X_2 + \dot{h}_5(X, Y, U, V),
\]

\[\text{(27)}\]
where

\[ L = \gamma^k \mu - y^- + \lambda^k \gamma^k (\alpha^* x_1^+ + \beta^* x_2^+) + O(\hat{\lambda}^k \gamma^k) \quad (28) \]

including all the constants in the third equation, and the coefficients \( \hat{\alpha}_1, \hat{\beta}_1, b_3 \) and functions \( \hat{h} \) are different from the old ones but we keep the notations for brevity. The changed coefficients are given by

\[
\alpha_1 = \hat{\alpha}_1^{old} - b_1 b_3^{-1} e_1 + o(1)_{k \to \infty}, \quad \beta_1 = \hat{\beta}_1^{old} - b_1 b_3^{-1} e_2 + o(1)_{k \to \infty}, \quad b_3 = b_3^{old} + O(\lambda^k),
\]

where the ‘old’ coefficients are given by (19). The functions \( \hat{h} \) now do not contain constant terms, and the estimates for \( \hat{h}_i \) (i = 1, 2, 4, 5) remain the same as (21), and those for \( \hat{h}_3 \) satisfy (25).

Note that \( \alpha^* \) and \( \beta^* \) cannot vanish at the same time since their phases have a difference \( \pi/2 \) by (19). In fact, we will further consider \( k \) and \( \omega \) such that

\[ \alpha^* \neq 0. \quad (30) \]

We apply the last coordinate change

\[ W = (X_2, U)^T, \quad Z = \alpha^* X_1 + \beta^* X_2 + \lambda^{-k}\hat{h}_3(X_1, X_2, 0, U, 0), \quad (31) \]

and rewrite formula (27) as

\[
\begin{align*}
\hat{Z} &= \lambda^k \alpha_1 Z + b_1 \alpha^* Y + \lambda^k \beta_1 W + h_1(Z, Y, W, \hat{V}), \\
\hat{Y} &= L + \lambda^k \gamma^k Z + b_3 \gamma^k Y^2 + \gamma^k b_2(Z, Y, W, \hat{V}), \\
\hat{W} &= \lambda^k \alpha_3 Z + \lambda^k \beta_3 W + h_3(Z, Y, W, \hat{V}), \\
V &= \lambda^k \alpha_4 Z + \lambda^k \beta_4 W + h_4(Z, Y, W, \hat{V}),
\end{align*}
\]

where and the coefficients \( \alpha \) and \( \beta \) are

\[
\begin{align*}
\alpha_1 &= \hat{\alpha}_1 + \hat{\alpha}_2 \frac{\beta^*}{\alpha^*} + O(\hat{\lambda}^k \lambda^{-k}), & \beta_1 &= \left( -\hat{\alpha}_1 \beta^* + \hat{\beta}_1 \alpha^* - \frac{\hat{\alpha}_2 (\beta^*)^2}{\alpha^*} + \hat{\beta}_2 \beta^*, \ 0 \right), \\
\alpha_3 &= \left( \hat{\alpha}_4 - \frac{b_4}{b_1} \hat{\alpha}_1 \right) \frac{1}{\alpha^*}, & \beta_3 &= \left( \hat{\beta}_4 - \frac{b_4}{b_1} \hat{\beta}_1 - \left( \hat{\alpha}_4 - \frac{b_4}{b_1} \hat{\alpha}_1 \right) \frac{\beta^*}{\alpha^*} \right) \frac{1}{\alpha^*}, \\
\alpha_4 &= \frac{\hat{\alpha}_5}{\alpha^*}, & \beta_4 &= \left( \hat{\beta}_5 - \frac{\hat{\alpha}_5}{\alpha^*} \right) \frac{\beta^*}{\alpha^*} \frac{1}{\alpha^*}.
\end{align*}
\]

The functions \( h \) are given by

\[
\begin{align*}
h_1 &= \alpha^* \hat{h}_1 + \beta^* \hat{h}_2 + O(\hat{\lambda}^k), & h_2 &= \hat{h}_3 - \hat{h}_3(X_1, X_2, 0, U, 0), \\
h_3 &= (\hat{h}_2 - \frac{b_2}{b_1} \hat{h}_1, \hat{h}_4 - \frac{b_4}{b_1} \hat{h}_1)^T + O(\hat{\lambda}^k \lambda^k), & h_4 &= \hat{h}_5 + O(\hat{\lambda}^k \lambda^k),
\end{align*}
\]

(34)
and they satisfy
\[
\begin{align*}
    h_i &= O(\bar{\lambda}^k) + O(\lambda^k Y) + O(Y^2) \quad (i = 1, 3, 4), \\
    \frac{\partial h_i}{\partial (Z, W)} &= O(\bar{\lambda}^k) + O(\lambda^k Y), \\
    \frac{\partial h_2}{\partial Y} &= O(\lambda^k) + O(Y), \\
    \frac{\partial h_2}{\partial V} &= O(\bar{\gamma}^k) ,
\end{align*}
\]
(35)

Recall that the domain $\hat{\Pi}$ of $T_k$ is defined in (16). We will consider in the coordinates $(Z, Y, W, V)$ the restriction of $T_k$ to a cube
\[
\Pi := [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_x-1} \times [-\delta, \delta]^{d_y-1} \subset \hat{\Pi},
\]
where for the sake of saving notations we still use $\delta$ to denote the size. Observe that 0 in the new coordinates corresponds to the intersection point $T_0^{-k}(\{y = y^-, v = v^\pm\}) \cap \{x = x^+, u = u^+\}$ in the old coordinates.

### 3.2 Invariant cone fields

Recall that $T_k$ are defined on the $\Pi$. Below we use $(z, y, w, v)$ to denote vectors in the tangent spaces.

**Lemma 1.** Let $\alpha^* \neq 0$. There exist on $\Pi$ cone fields
\[
\begin{align*}
    C^{cs} &= \{(z, y, w, v) : \|v\| < K^{cs}(\|z\| + \|y\| + \|w\|)\}, \\
    C^{ss} &= \{(z, y, w, v) : |z|, |y|, \|v\| < K^{ss}\|w\|\},
\end{align*}
\]
(37)
(38)
which are backward-invariant in the sense that if a point $\hat{M} \in \Pi$ has its pre-image $M = T_k^{-1}(\hat{M})$ in $\Pi$, then the cone at $\hat{M}$ is mapped into the cone at $M$ by $DT_k^{-1}$; and cone fields
\[
\begin{align*}
    C^{cu} &= \{(z, y, w, v) : \|w\| < K^{cu}(\|z\| + \|y\| + \|v\|)\}, \\
    C^{uu} &= \{(z, y, w, v) : |z|, |y|, \|w\| < K^{uu}k^k\bar{\gamma}^{-k}\|v\|\},
\end{align*}
\]
(39)
(40)
which are forward-invariant in the sense that if a point $M \in \Pi$ has its image $\hat{M} = T_k(M)$ in $\Pi$, then the cone at $\hat{M}$ is mapped into the cone at $M$ by $DT_k$. Here $K^{uu}$ is some constant and other $K$’s are of the forms
\[
\begin{align*}
    K^{cs} &= K_1^{cs}|Y| + K_2^{cs}\lambda^k, \\
    K^{ss} &= K_1^{ss}|Y| + K_2^{ss}, \\
    K^{cu} &= K_1^{cu}|Y| + K_2^{cu}\lambda^k,
\end{align*}
\]
(41)
where $K_2^{ss}$ can be taken arbitrarily small with sufficiently large $k$.

**Remark 5.** By comparing $K^{cs}$ and $K^{ss}$, it seems that $C^{ss}$ is larger than $C^{cs}$ in the $w$-directions. In fact, as can be seen in the proof, we have $||v|| < K^{cs}\|w\|$ also in $C^{ss}$, being consistent with the bound in $C^{cs}$.

**Proof.** By (32) and (35), the tangent map $DT_k$ takes the following form:
\[
\bar{z} = \lambda^k\alpha_1(1 + O(\bar{\lambda}^k\lambda^{-k}) + O(Y))z + b_1\alpha^*(1 + O(\lambda^k) + O(Y))y
\]
\[ y = \lambda^k\gamma k (1 + O(\hat{\lambda}^k\lambda^{-k})) + O(\hat{\lambda}^k\gamma k) \]

\[ \bar{w} = \lambda^k\alpha_3(1 + O(\hat{\lambda}^k\lambda^{-k}) + O(Y))z + (O(Y) + O(\lambda k))y + O(\lambda^k(1 + O(\hat{\lambda}^k\lambda^{-k}) + O(Y))w + O(\hat{\gamma}^{-k})\bar{v}, \]

\[ v = \lambda k\alpha_4(1 + O(\hat{\lambda}^k\lambda^{-k}) + O(Y))z + (O(Y) + O(\lambda k))y + O(\lambda^k(1 + O(\hat{\lambda}^k\lambda^{-k}) + O(Y))w + O(\hat{\gamma}^{-k})\bar{v}. \]

For brevity we write \( O(\hat{\lambda}^k\lambda^{-k}) = o(1) \) in the computations below.

(i) The cone field \( C^{ss} \). Take any vector \((\bar{z}, \bar{y}, \bar{w}, \bar{v})^T \in C^{ss}\) and let

\[ (z, y, w, v)^T = DT_k^{-1}(\bar{z}, \bar{y}, \bar{w}, \bar{v})^T. \]

Combining \( ||\bar{v}|| < (K_1^{s*}|Y| + K_2^{s*})||\bar{w}|| \) with (44), yields

\[ ||\bar{w}|| = O(\lambda^k)|z| + (O(Y) + O(\lambda k))|y| + O(\lambda^k)||w|| + O(\hat{\gamma}^{-k})||w||. \]

In what follows, let \( C_i \) be positive constants independent of \( K_1^{s*} \). Given any \( \rho > 0 \), we can take \( k \) sufficiently large such that

\[ ||\bar{w}|| < C_1\lambda^k(||z| + ||w||) + C_1(||Y| + \rho)|y|. \]  

(47)

Since \( |z|, |\bar{y}|, |\bar{v}| < (K_1^{s*}|Y| + K_2^{s*})||\bar{w}|| \), it follows from (42) and (43) that

\[ |b_1\alpha^*(1 + O(\lambda^k) + O(Y))|y| = O(\lambda^k)|z| + O(\lambda^k)||w|| + (O(\lambda^k) + O(1))||\bar{w}||, \]

\[ |\lambda^k\alpha^k(1 + o(1)_{k \to \infty})z| = (O(\gamma Y) + O(\lambda^k\gamma k))|y| + O(\lambda^k\gamma k)||w|| + (O(Y) + O(1))||\bar{w}||. \]

Combining these with (47) and selecting sufficiently small \( \delta > 0 \) (which bounds \( |Y| \)) and sufficiently large \( k \), we have

\[ |y| \leq C_2\lambda k(||z| + ||w||), \]
\[ |z| \leq C_3|Y|\lambda^{-k}|y| + o(1)_{k \to \infty}||w||. \]

(49)

For an arbitrarily small \( \rho > 0 \), take \( k \) large enough such that \( o(1)_{k \to \infty} < \rho \). Substituting the above two inequalities into each other, we obtain

\[ |z| \leq C_4(||Y| + \rho)||w||, \]
\[ |y| \leq C_5\lambda^k||w||. \]

(50)

Finally, it follows from (45) that

\[ |v| = O(\lambda^k)|z| + (O(Y) + O(\lambda k))|y| + O(\lambda^k)||w|| + O(\hat{\gamma}^{-k})||\bar{w}||. \]

Combining this with (47) and (50), we have

\[ |v| < C_6\lambda^k||w||. \]
Thus, let \( K_1^{cs} = 2C_4 \) and \( K_2^{cs} = 2\max\{C_4\rho, C_5\lambda, C_6\lambda^k\} \). By the choice of \( \rho \), it is immediate that \( K_2^{cs} \) can be arbitrarily small when \( k \) is sufficiently large.

(ii) The cone field \( C^{cs} \). Take any vector \((z, y, w, v)^T \in C^{cs}\) with some given constants \( K_1^{cs} \) and \( K_2^{cs} \). Let

\[
(z, y, w, v)^T = DT_k^{-1}(z, y, w, v)^T.
\]

Since \( \|\tilde{v}\| < (K_1^{cs}|Y| + \lambda^k K_2^{cs})(|\tilde{z}| + |\tilde{y}| + \|\tilde{w}\|) \), it follows from (42) - (45) that

\[
\begin{align*}
|\tilde{z}| &= O(\lambda^k)|z| + O(1)|y| + O(\lambda^k)\|w\| + O(\gamma^{-k}(|\tilde{y}| + \|\tilde{w}\|)), \\
|\tilde{y}| &= O(\lambda^k\gamma^k)|z| + (O(\gamma^k Y) + O(\lambda^k\gamma^k))|y| + O(\lambda^k)\|w\| + O(\gamma^{-k})(|\tilde{z}| + |\tilde{y}|), \\
\|\tilde{w}\| &= O(\lambda^k)|z| + O(\gamma Y) + O(\lambda^k)\|w\| + O(\gamma^{-k})(|\tilde{z}| + |\tilde{y}| + \|\tilde{w}\|), \\
\|v\| &= O(\lambda^k)|z| + (O(\gamma Y) + O(\lambda^k))|y| + O(\lambda^k)\|w\| + O(\gamma^{-k})(|\tilde{z}| + |\tilde{y}| + \|\tilde{w}\|).
\end{align*}
\]

Equations (51) and (53) imply

\[
\|\tilde{w}\| = O(\lambda^k)|z| + (O(\gamma Y) + O(\lambda^k))|y| + O(\lambda^k)\|w\| + O(\gamma^{-k})|\tilde{y}|. \tag{55}
\]

Combining this with (51), one gets

\[
|\tilde{z}| = O(\lambda^k)|z| + O(1)|y| + O(\lambda^k)\|w\| + O(\gamma^{-k})|\tilde{y}|. \tag{56}
\]

Substituting (55) and (56) into (52) yields

\[
|\tilde{y}| = O(\lambda^k\gamma^k)|z| + (O(\gamma^k Y) + O(\lambda^k\gamma^k))|y| + O(\lambda^k\gamma^k)\|w\|. \tag{57}
\]

Combining this with (55) and (56), we obtain

\[
\begin{align*}
\|\tilde{w}\| &= O(\lambda^k)|z| + O(\gamma Y) + O(\lambda^k)\|w\|, \\
|\tilde{z}| &= O(\lambda^k)|z| + O(1)|y| + O(\lambda^k)\|w\|.
\end{align*}
\]

Finally, the above two equations together with (54) and (57) give

\[
\|v\| = O(\lambda^k)|z| + (O(\gamma Y) + O(\lambda^k))|y| + O(\lambda^k)\|w\|.
\]

Therefore, by increasing constants \( K_1^{cs} \) and \( K_2^{cs} \) if necessary, for sufficiently large \( k \in \mathbb{N} \), we have

\[
\|v\| < (K_1^{cs}|Y| + K_2^{cs}\lambda^k)(|z| + |y| + \|w\|),
\]

which implies \((z, y, w, v) \in C^{cs}\) as desired.

(iii) The cone field \( C^{cu} \). Take any vector \((z, y, w, v)^T \in C^{cu}\) with some given constants \( K_1^{cu} \) and \( K_2^{cu} \), we will show that \((z, y, w, v)^T = DT_k(z, y, w, v)^T \) is contained in \( C^{cu}\). Since \( \|w\| < (K_1^{cu}|Y| + \lambda^k K_2^{cu})(|z| + |y| + \|v\|) \), equations (42) - (45) imply

\[
|y| = O(\lambda^k)|z| + O(1)|\tilde{z}| + O(\gamma^{-k})\|\tilde{v}\| + O(\lambda^k)\|v\|, \tag{58}
\]

18
\[ |z| = O(\lambda^{-k} \gamma^{-k})|\bar{y}| + (O(\lambda^{-k} Y) + o(1))_{k \to \infty}|y| + o(1)_{k \to \infty} \|v\| + O(\lambda^{-k} \gamma^{-k})\|\bar{v}\|, \quad (59) \]
\[ \|\bar{w}\| = O(\lambda^{k})|z| + (O(Y) + O(\lambda^{k}))|y| + O(\lambda^{k})\|v\| + O(\gamma^{-k})\|\bar{v}\|, \quad (60) \]
\[ \|v\| = O(\lambda^{k})|z| + (O(Y) + O(\lambda^{k}))|y| + O(\gamma^{-k})\|\bar{v}\|. \quad (61) \]

Using (61), one solves out \(|y|\) and \(|z|\) from (58) and (59) as
\[ |y| = O(\gamma^{-k})|\bar{y}| + O(\gamma^{-k})\|\bar{v}\| + O(1)\|\bar{z}\|, \]
\[ |z| = O(\lambda^{-k} \gamma^{-k})|\bar{y}| + O(\lambda^{-k} \gamma^{-k})\|\bar{v}\| + O(\lambda^{-k} Y) + o(1)_{k \to \infty} \|\bar{z}\|. \quad (62) \]

Substituting these equations into (61), we obtain
\[ \|v\| = O(\gamma^{-k})|\bar{y}| + (O(Y) + O(\lambda^{k}))\|\bar{z}\| + O(\gamma^{-k})\|\bar{v}\|. \quad (63) \]

Finally, substituting (62) and (63) into (60), we get
\[ \|\bar{w}\| = O(\gamma^{-k})|\bar{y}| + O(\gamma^{-k})\|\bar{v}\| + (O(Y) + O(\lambda^{k}))\|\bar{z}\|. \]

By noticing that \(|\gamma^{-1}| < |\gamma^{-1}| < |\lambda|\), it suffices to increase \(K^{cu}_{1}\) and \(K^{cu}_{2}\) if necessary such that for sufficiently large \(k\) and sufficiently small \(\delta\), we have
\[ \|\bar{w}\| < (K^{cu}_{1} Y + K^{cu}_{2} \lambda^{k})(\|\bar{z}\| + |\bar{y}| + \|\bar{v}\|) \]
as desired.

(iv) The cone field \(C^{uu}\). Take any vector \((z, y, w, v)^T \in C^{uu}\) with some given constant \(K^{uu}\). By (40) and (45), we obtain
\[ \|v\| = O(\gamma^{-k})\|\bar{v}\|. \quad (64) \]
Combining this with (42), (43) and (44), yields
\[ |\bar{z}| = O(\gamma^{-k})\|\bar{v}\|, \quad |\bar{y}| = O(\gamma^{k} \gamma^{-k})\|\bar{v}\|, \quad \|\bar{w}\| = O(\gamma^{-k})\|\bar{v}\|. \quad (65) \]

Therefore, by increasing \(K^{uu}\) if necessary, we have
\[ |\bar{z}|, |\bar{y}|, \|\bar{w}\| < K^{uu} \gamma^{k} \gamma^{-k}\|\bar{v}\| \]
as desired. The expansion of vectors follows directly from (64).

Remark 6. Vectors in \(C^{ss}\) are uniformly contracted by \(DT_k\) since substituting (50) into (47) gives
\[ \|\bar{w}\| = O(\lambda^{k})\|w\|. \quad (66) \]
This will be used later for proving Lemma 2.

Remark 7. Replacing \(\partial h_j / \partial V\) in (21) by
\[ \frac{\partial \hat{h}_j}{\partial V} = O(\gamma^{-k} Y) + O(\gamma^{-k}) \quad \text{and} \quad \frac{\partial \hat{h}_i}{\partial V} = O(\gamma^{-k}), \quad i = 1, 3, 4. \quad (67) \]
will lead to weaker estimates for $\partial h/\partial V$ in (35) as

$$\frac{\partial h_2}{\partial V} = O(\gamma^{-k} Y) + O(\gamma^{-k}) \quad \text{and} \quad \frac{\partial h_3}{\partial V} = O(\gamma^{-k}).$$

(68)

It can be easily checked that with the above estimates the cone fields $C^{ss}, C^{cs}, C^{cu}$ persist with the same coefficients in (41), while $C^{cu}$ takes the form

$$C^{cu} = \{(z, y, w, v) : |z|, |y|, \|w\| < (K_{1}^{cu}|Y| + K_{2}^{cu}\gamma|\gamma^{-k}|\|v\|).$$

(69)

The above estimates appear in the reduction of bi-focus case to the saddle-focus case in Section 4. The results below will be proved with these weaker bounds so that they also apply to the bi-focus case.

### 3.3 A periodic point with different index

Throughout this section we assume that $\alpha^*$ is bounded away from zero and $|\lambda\gamma| > 1$.

**Lemma 2.** Assume the weaker estimates (68). For all sufficiently large $k$ and any $t \in (-1, 1)$, there exists at

$$\mu_k = -\lambda^k(\alpha^* x_1^* + \beta^* x_2^*) + O(\lambda^k) + O(\gamma^{-k} y^-) + \frac{(b_1 \alpha^*)^2}{2b_3} \lambda^{2k} t - \frac{(b_1 \alpha^*)^2}{4b_3} \lambda^{2k} t^2 + O(\lambda^k \lambda^k),$$

(70)

where the term $O(\lambda^k)$ is independent of $t$, an index-$(d^n + 1)$ fixed point $Q = (Z_Q, Y_Q, W_Q, V_Q)$ of $T_k$ with

$$Z_Q = -\frac{(b_1 \alpha^*)^2}{2b_3} \lambda^k t + O(\lambda^k), \quad Y_Q = -\frac{b_1 \alpha^*}{2b_3} \lambda^k t + O(\lambda^k), \quad W_Q = O(\lambda^k), \quad V_Q = O(\lambda^k).$$

(71)

Moreover, $\mu_k$ and the coordinates are functions of $\omega$ and $t$ satisfying

$$\frac{\partial \mu_k}{\partial t} = \frac{(b_1 \alpha^*)^2}{2b_3} \lambda^{2k} t - \frac{(b_1 \alpha^*)^2}{2b_3} \lambda^{2k} t^2 + O(\lambda^k \lambda^k), \quad \frac{\partial Z_Q}{\partial t} = -\frac{(b_1 \alpha^*)^2}{2b_3} \lambda^k + O(\lambda^k),$$

$$\frac{\partial Y_Q}{\partial t} = -\frac{b_1 \alpha^*}{2b_3} \lambda^k + O(\lambda^k), \quad \frac{\partial (W_Q, V_Q)}{\partial t} = O(\lambda^k).$$

(72)

**Proof.** Let us write $E = E^{cs} \cap E^{cu}$. Vectors in $E$ have the form $(z, y, S_1(z, y), S_2(z, y))$, where $S_i$ ($i = 1, 2$) are linear maps satisfying that for any given $K^* > 0$ we have $\|S_i(z, y)\| < K^*(|z| + |y|)$ for all sufficiently small $\delta$ and all sufficiently large $k$. To see this, one notes that for a vector $(z, y, w, v) \in E$, the $w$-component satisfies

$$\|w\| < K^{cu}(|z| + |y| + \|v\|),$$

by (39)

$$< K^{cu}(|z| + |y| + K^{cs}(|z| + |y| + \|w\|)),$$

by (37)

or, equivalently,

$$\|w\| < (1 - K^{cu}K^{cs})^{-1} K^{cu}(1 + K^{cs})(|z| + |y|) < K^*(|z| + |y|)$$

since $K^{cu}, K^{cs} \to 0$ as $\delta \to 0$ and $k \to \infty$. The $v$-component can be checked in the same way.
Consider \( DT_k|_E \) as the linear transformation of \( \mathbb{R}^2 \) defined in the following way:

\[
DT_k|_E(z, y) = (\bar{z}, \bar{y})
\]

if and only if

\[
DT_k(z, y, S_1(z, y), S_2(z, y)) = (\bar{z}, \bar{y}, S_1(\bar{z}, \bar{y}), S_2(\bar{z}, \bar{y})).
\]

Combining (42) and (43) with the estimates in (35) (with \( \partial h/\partial \bar{V} \) in (68)), and using the boundedness of \( S_i \), we obtain the formula for \( DT_k|_E \) as

\[
\bar{z} = \lambda^k(\alpha_1 + O(\beta_1) + O(\bar{\lambda}k\lambda^{-k}) + O(Y))z + b_1\alpha^*(1 + O(\lambda^k) + O(Y))y,
\]
\[
\bar{y} = \lambda^k\gamma^k(1 + O(\bar{\lambda}k\lambda^{-k}))z + (2b_3\gamma^kY + O(\gamma^kY^2) + O(\lambda^k\gamma^k))y.
\]

It follows that the eigenvalues of \( DT_k|_E \), denoted by \( \nu_1 \) and \( \nu_2 \), satisfy

\[
\nu_1 + \nu_2 = 2b_3\gamma^kY + O(\gamma^kY^2) + O(\bar{\lambda}k\gamma^k),
\]
\[
\nu_1\nu_2 = -b_1\alpha^*\lambda^k\gamma^k(1 + O(\bar{\lambda}k\lambda^{-k}) + O(Y)),
\]

where we used \(|\lambda\gamma| > 1 \) and (11) to sort the \( O(\cdot) \) terms.

Suppose now \( Q = (Z_Q, Y_Q, W_Q, V_Q) \) is a fixed point of \( T_k \). It is easy to verify that \( DT_k(Q)|_E \) has two eigenvalues outside the unit circle if and only if \(|\nu_1\nu_2| > 1 \) and

\[
\nu_1 + \nu_2 = t(1 + \nu_1\nu_2)
\]

for some \( t \in (-1, 1) \) (see e.g. [17, Section 2.3.1]). In our case the inequality \(|\nu_1\nu_2| > 1 \) follows immediately from (74) for sufficiently large \( k \), due to the assumption \( \lambda\gamma > 1 \). Substituting (73) and (74) into (75) yields

\[
2b_3\gamma^kY_Q + O(\gamma^kY_Q^2) + O(\bar{\lambda}k\gamma^k) = t\left(1 - b_1\alpha^*\lambda^k\gamma^k(1 + O(\bar{\lambda}k\lambda^{-k}) + O(Y_Q))\right),
\]

which gives \( Y_Q \) as a function of other coordinates:

\[
Y_Q = -\frac{b_1\alpha^*}{2b_3}\lambda^kt + O(\bar{\lambda}k). \tag{76}
\]

Thus, if the coordinates of \( Q \) satisfy (76), then both of the two eigenvalues of \( DT_k|_E \) lie outside the unit circle. On the other hand, it follows from (66) and (64) that, among the remaining eigenvalues of \( DT_k(Q) \), there are \( d^u - 1 \) eigenvalues inside the unit circle and \( d^v - 1 \) eigenvalues outside the unit circle. Consequently, \( Q \) has index \( d^v + 1 \).

In what follows, we find the point \( Q \) as a solution to the system consisting of (76) and the equations for \( T_k(Z_Q, Y_Q, W_Q, V_Q) = (Z_Q, Y_Q, W_Q, V_Q) \), which, by (32) and (76), are given by

\[
Z_Q = \lambda^k\alpha_1Z_Q + b_1\alpha^*Y_Q + \lambda^k\beta_1W_Q + h_1, \tag{77}
\]
\[
Y_Q = L + \lambda^k\gamma^kZ_Q + b_3\gamma^kY_Q^2 + \gamma^kh_2, \tag{78}
\]
\[
W_Q = \lambda^k\alpha_3Z_Q + \lambda^k\beta_3W_Q + h_3, \tag{79}
\]
\[
V_Q = \lambda^k\alpha_4Z_Q + \lambda^k\beta_4W_Q + h_4. \tag{80}
\]
Using the Implicit Function Theorem, one can readily express \( Z, W, V \) as functions of \( Y \) from (77), (79) and (80) as

\[
\begin{align*}
Z &= b_1 \alpha^* Y_Q + O(\lambda^k) + O(\lambda^k Y_Q) + O(Y_Q^2), \\
W &= O(\lambda^k) + O(\lambda^k Y_Q) + O(Y_Q^2), \\
V &= O(\lambda^k) + O(\lambda^k Y_Q) + O(Y_Q^2).
\end{align*}
\]

Substituting the results into (76) we obtain \( Y_Q \) as a function of \( t \), which together with (28) and (78) gives (70). Substituting this back into the above expressions and noting that \( L \) is constant, we readily obtain (71) and (72).

3.4 Local invariant manifolds of the point with different index

We next find equations for the local stable and unstable manifolds of \( Q \), which will be used later in finding their intersections with \( W^s(0) \) and, respectively, \( W^u(0) \). Denote by \( W^{s}_{\text{loc}}(Q) \) the connected component containing \( Q \) of \( W^s(0) \cap \Pi \).

**Lemma 3.** Let \( \varepsilon = (\mu, \omega) \) and assume the weaker estimates (68). The local stable manifold \( W^{s}_{\text{loc}}(Q) \) is given by

\[
(Z, Y, V) = (Z_Q, Y_Q, V_Q) + \phi^s(W, \varepsilon),
\]

where \( \phi^s \) is some function defined on \([ -\delta, \delta ]^{d-1} \) with \( \phi^s(W_Q) = 0 \). This function is \( C^{r-2} \) with respect to \( (W, \varepsilon) \) and satisfy

\[
\frac{\partial \phi^s}{\partial W} = O(\lambda(\varepsilon)^k) \quad \text{and} \quad \frac{\partial \phi^s}{\partial \varepsilon} = O(k \lambda(\varepsilon)^{k-1}).
\]

**Proof.** Instead of directly using the cones obtained in Lemma 1, we find \( W^{s}_{\text{loc}}(Q) \) in a more subtle way. Let us first move the origin to \( Q \) by applying

\[
Z^{\text{new}} = Z - Z_Q, \quad Y^{\text{new}} = Y - Y_Q, \quad W^{\text{new}} = W - W_Q, \quad V^{\text{new}} = V - V_Q,
\]

which allows us to rewrite \( T_k \) (see (32)) as

\[
\begin{align*}
\tilde{Z} &= \lambda^k \alpha_1 Z + b_1 \alpha^* Y + \lambda^k \beta_1 W + h_1(M + Q) - h_1(Q), \\
\tilde{Y} &= \lambda^k \gamma Z + 2b_3 \gamma^k Y Q Y + b_3 \gamma^k Y^2 + \gamma^k (h_2(M + Q) - h_2(Q)), \\
\tilde{W} &= \lambda^k \alpha_3 Z + \lambda^k \beta_3 W + h_3(M + Q) - h_3(Q), \\
\tilde{V} &= \lambda^k \alpha_4 Z + \lambda^k \beta_4 W + h_4(M + Q) - h_4(Q),
\end{align*}
\]

where \( M = (Z, Y, W, \tilde{V}) \) and the functions \( h \) satisfy the estimates (35). Using (71) and the scaling

\[
Z = \lambda^k Z^{\text{new}}, \quad Y = \lambda^k Y^{\text{new}}, \quad V = \lambda^k V^{\text{new}},
\]

we can rewrite the above formula as

\[
\begin{align*}
Y &= \frac{1}{b_1 \alpha^*} \tilde{Z} - \frac{\alpha_1}{b_1 \alpha^*} \lambda^k Z - \frac{\beta_1}{b_1 \alpha^*} W - \frac{1}{b_1 \alpha^*} \lambda^{-k}(h_1(\hat{M} + Q) - h_1(Q)), \\
Z &= \lambda^{-k} \gamma \tilde{Y} + b_1 \alpha^* t Y - b_3 Y^2 - \lambda^{-2k}(h_2(\hat{M} + Q) - h_2(Q)),
\end{align*}
\]
\[ \hat{W} = \lambda^k \alpha_3 Z + \lambda^k \beta_3 W + h_3(M + Q) - h_3(Q), \]
\[ V = \lambda^k \alpha_4 Z + \beta_4 W + \lambda^{-k} h_4(M + Q) - h_4(Q), \]

where \( \hat{M} = (\lambda^k Z, \lambda^k Y, W, \lambda^k \hat{V}) \). Note that by (31) and (34), one has \( h_2(Z, W, 0, 0) \equiv 0 \), that is, \( W \) appears in \( h_2 \) with either \( Y \) or \( \hat{V} \) in front of it. It then follows from the estimate for \( h_2 \) in (35) that \( \| h_2(\hat{M} + Q) - h_2(Q) \|_{C^1} = O(\lambda^k \hat{k}) \). This along with other estimates in (35) allows one to use the implicit function theorem to obtain

\[ Y = \frac{1}{b_1 \alpha_s} \hat{Z} - \frac{\beta_1}{b_1 \alpha_s} W - \hat{h}_1(\hat{Z}, \hat{Y}, W, \hat{V}), \]
\[ Z = t \hat{Z} - \beta_1 tW - \frac{b_3}{(b_1 \alpha_s)^2}(Z^2 + \beta_1^2 W^2 - 2 \beta_1 ZW) + \lambda^{-k} \gamma^{-k} \hat{Y} + \hat{h}_2(\hat{Z}, \hat{Y}, W, \hat{V}), \]
\[ W = \lambda^k \alpha_3 Z + \lambda^k \beta_3 W + \hat{h}_3(\hat{Z}, \hat{Y}, W, \hat{V}), \]
\[ V = \lambda^k \alpha_4 Z + \beta_4 W + \hat{h}_4(\hat{Z}, \hat{Y}, W, \hat{V}), \]

where \( \| \hat{h}_{1,2,4} \|_{C^1} = o(1)_{k \to \infty} \) and \( \| \hat{h}_3 \|_{C^1} = O(\lambda^k) \).

Therefore, for all sufficiently large \( k \), the local manifold \( W_{\text{loc}}^*(O) \) has the form

\[ Z = -\beta_1 tW - \frac{b_3}{(b_1 \alpha_s)^2} \beta_1^2 W^2 + o(1)_{k \to \infty}, \]
\[ Y = -\frac{\beta_1}{b_1 \alpha_s} W + o(1)_{k \to \infty}, \]
\[ V = \beta_4 W + o(1)_{k \to \infty}, \]

which has bounded derivatives and hence gives the desired formula and estimates for \( W_{\text{loc}}^*(Q) \) after we return to the original coordinates via (83). The estimate (82) immediately shows that \( W \) runs the whole interval \([-\delta, \delta]^d = 1\) before \( W_{\text{loc}}^*(Q) \) getting out of \( \Pi \).

Take any fixed \( q \in (0,1) \) and denote

\[ \Pi' = [-q\delta, q\delta] \times [-q\delta, q\delta] \times [-q\delta, q\delta]^d = 1 \times [-\delta, \delta]^d \subset \Pi. \] (84)

Before we proceed to find \( W_{\text{loc}}^*(Q) \), let us point out that a local center-stable manifold in \( \Pi' \) can be obtained by considering a suitable extension of the \( C^{r-1} \) map \( T_k^\times : (Z, Y, W, \hat{V}) \mapsto (Z, \tilde{Y}, \tilde{W}, \tilde{V}) \) given by (32). We explain as follows. First scale our variables by \((Z, Y, W, V) = \delta(Z_{\text{new}}, Y_{\text{new}}, W_{\text{new}}, V_{\text{new}})\), which brings \( \Pi \) to a region of size 1, and formula (32) to the form

\[ \tilde{Z} = \lambda^k \alpha_3 Z + b_1 \alpha_4 Y + \lambda^k \beta_1 W + \delta^{-1} h_1(\delta Z, \delta Y, \delta W, \delta \hat{V}), \]
\[ \tilde{Y} = L + \lambda^k \gamma^k Z + \delta b_3 \gamma^k Y^2 + \delta^{-1} \gamma^k h_2(\delta Z, \delta Y, \delta W, \delta \hat{V}), \]
\[ \tilde{W} = \lambda^k \alpha_3 Z + \lambda^k \beta_3 W + \delta^{-1} h_3(\delta Z, \delta Y, \delta W, \delta \hat{V}), \]
\[ \tilde{V} = \lambda^k \alpha_4 Z + \lambda^k \beta_4 W + \delta^{-1} h_4(\delta Z, \delta Y, \delta W, \delta \hat{V}). \]

One easily extends \( T_k^\times |_{\Pi'} \) to a map on \( D := \mathbb{R}^{d+1} \times [-1,1]^{d-1} \) by using a smooth cutoff function \( \varphi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \), which equals identity on \([-q, q] \times [-q, q] \times [-q, q]^{d-1} \) and vanishes outside the unit cube in \( \mathbb{R}^{d+1} \), with bounded derivatives. Namely, define the extension as \( \tilde{T}_k^\times := T_k^\times (\varphi(Z, Y, W, \hat{V}) : D \to D \). Note that \( \tilde{T}_k^\times \) corresponds to a map \( \tilde{T}_k \) which is an extension of \( T_k |_{\Pi'} \) and such that we have
\((\bar{Z}, Y, W, V) = \bar{T}_k(Z, Y, W, V)\) if and only if \((\bar{Z}, Y, W, V) = \bar{T}_k(Z, Y, W, V)\). Recalling that there are no constant terms in the functions \(h\) of \((32)\) and using \((35)\) (with \(\partial h/\partial V\) in \((68)\)), one readily finds

\[
\left\| \frac{\partial(\bar{Z}, \bar{Y}, \bar{W})}{\partial(Z, Y, W)} \right\| = O(\lambda^k|\gamma|^k + \delta|\gamma|^k), \quad \left\| \frac{\partial(\bar{Z}, \bar{Y}, \bar{W})}{\partial V} \right\| = O(\delta) + O(\gamma^k\gamma^{-k}),
\]

\[
\left\| \frac{\partial V}{\partial(Z, Y, W)} \right\| = O(\lambda^k + \delta), \quad \left\| \frac{\partial V}{\partial V} \right\| = O(\gamma^{-k}),
\]

where the norm is taken as the supremum of all the matrix norms evaluated at points in \(D\). As a result, for any fixed \(q\), one can choose sufficiently small \(\delta\) and sufficiently large \(k\) such that

\[
\left\| \frac{\partial(\bar{Z}, \bar{Y}, \bar{W})}{\partial(Z, Y, W)} \right\| \left\| \frac{\partial V}{\partial V} \right\| + \left\| \frac{\partial(\bar{Z}, \bar{Y}, \bar{W})}{\partial V} \right\| \left\| \frac{\partial V}{\partial(Z, Y, W)} \right\| < 1.
\]

This implies that the condition of \([26, \text{Theorem } 4.3]\) is satisfied for the inverse \((\bar{T}_k^\times)^{-1}\), and hence guarantees the existence of a center-stable invariant manifold of \(\bar{T}_k\), consisting all the \(\alpha\)-limit points of orbits of \(\bar{T}_k\). One also sees that the cone field \(C^{uu}\) given by \((69)\) exists for \(\bar{T}_k\), which implies the existence of a strong-unstable foliation of \(\bar{T}_k\).

**Lemma 4.** Let \(\Pi'\) be given by \((84)\). For all sufficiently small \(\delta\) and all sufficiently large \(k\), there exist

- a \(C^1\)-smooth manifold \(W^{cs}\) in \(\Pi'\) containing all the \(\alpha\)-limit points of orbits of \(T_k\), which is locally invariant in the sense that for any point \(M \in W^{cs}\) its iterates by \(T_k\) lie on \(W^{cs}\) as long as they belong to \(\Pi'\); moreover, it has the form \(\{V = w^{cs}(Z, Y, W)\}\) for some function \(w^{cs}\) defined on \([-q\delta, q\delta] \times [-q\delta, q\delta] \times [-q\delta, q\delta]^{d+1}\) with tangent spaces lie in \(C^{cs}\) given by \((37)\).

- a strong-unstable invariant foliation \(\bar{F}^{uu}\) of \(\bar{T}_k\) with \(C^1\) leaves of the form \((Z, Y, W) = l^{uu}(V)\) for some function \(l^{uu}\) defined on \([-\delta, \delta]^{d+1}\), and their tangent spaces lie in \(C^{uu}\) given by \((69)\).

**Remark 8.** For any leaf in the intersection of \(\bar{F}^{uu} \cap \Pi'\), only when all its backward iterates stay in \(\Pi'\), it is a true leaf of \(T_k\), i.e. obtained totally from the dynamics of \(T_k\). Such leaves form an invariant lamination of \(\Pi'\), which particularly defines a local strong-unstable foliation of \(W^{uu}_{loc}(Q)\), the connected component of \(W^{uu}(Q) \cap \Pi'\) containing \(Q\).

We are now in the position to investigate \(W^{uu}_{loc}(Q)\). Let us take \(\Pi'\) as the new \(\Pi\) and still denote its size by \(\delta\).

**Lemma 5.** Assume the weaker estimates \((68)\). For each sufficiently large \(k\), there exists a connected piece \(W^{uu}_{loc}(Q)\) of \(W^{uu}(Q) \cap \Pi\) having the form

\[
W = W_Q + \phi^u(Z, Y, V),
\]

where \(\phi^u\) is some \(C^{r-2}\) function (with respect to variables and parameters) defined on some region \(\mathcal{D} \subset [-\delta, \delta] \times [-\delta^2, \delta^2] \times [-\delta, \delta]^{d+1}\) and satisfying

\[
\phi^u(Z_Q, Y_Q, V_Q) = 0 \quad \text{and} \quad \|\phi^u\|_{C^1} = o(1)_{k \to \infty}.
\]

Moreover, the intersection of \(W^{uu}_{loc}(Q)\) with the center-stable manifold \(W^{cs}\) given by Lemma 4 intersects at least one of the two boundaries \(\{Y = \pm \delta^2\}\).
Proof. The forward-invariance of $C^{cu}$ implies that there is a piece $w$ of $W^u(Q)$ has the form (85) with $\phi^u(Z, Y, V)$ defined on a small neighborhood $D_0$ of $(Z_Q, Y_Q, V_Q)$ and satisfying (86). We will find the desired curve in some iterate of $w \cap W^cs =: w^c$. For brevity, we use the same notation for the name of a surface as well as the equation defining it.

By construction, the piece $w^c$ has the form

$$ (W, V) = w^c(Z, Y) =: (w_1^c(Z, Y), w_2^c(Z, Y)), $$

where $(W_Q, V_Q) = w^c(Z_Q, Y_Q)$, and the first partial derivatives of $w^c$ has order $o(1)_{k \to \infty} + O(\delta)$ since by Lemma 1 the cone fields (37) and (39) persist under the weaker estimates (68). Note that, for any iterate of $w^c$ by $T_k$, it has the same form as $w^c$ and the same estimates for derivatives, as long as it lies in $\Pi$. We prove the lemma by showing below that there exists $n$ such that $T_k^n(w^c)$ intersects one of the boundaries $\{Y = \pm \delta^2\}$ and $T_k^n(w^c)$ $(i = 0, \ldots, n - 1)$ intersects neither of $\{Z = \pm \delta\}$ and $\{Y = \pm \delta^2\}$.

For any surface of the form $(W, V) = s(Z, Y) = (s_1(Z, Y), s_2(Z, Y))$, denote its domain by $D(s)$ and the area of its domain by $A(s)$. Now consider the map $F : D(s) \to D(T_k(s)), (Z, Y) \mapsto (\bar{Z}, \bar{Y})$, which by (32) is given by

$$ \bar{Z} = \lambda^k \alpha_1 Z + b_1 \alpha^* Y + \lambda^k \beta_1 s_1(Z, Y) + h_1(Z, Y, s_1(Z, Y), s_2(\bar{Z}, \bar{Y})), $$
$$ \bar{Y} = L + \lambda^k \gamma^k Z + b_3 \gamma^k Y^2 + \gamma^k h_2(Z, Y, s_1(Z, Y), s_2(\bar{Z}, \bar{Y})). $$

Since the derivatives of $w^c$ and its iterates which lie in $\Pi$ have order $o(1)_{k \to \infty}$, for any $i$ such that $T_k^i(w^c) \subset \Pi$ we have

$$ \inf_{(Z, Y) \in D(T_k^i(w^c))} \left| \frac{\partial F}{\partial (Z, Y)} \right| = O(\lambda^{k^i} \gamma^k) > q $$

for some constant $q > 1$. Consequently, we obtain

$$ A(T_k^i(w^c)) = \int_{F(D(T_k^i(w^c)))} dZdY = \int_{D(T_k^i(w^c))} \left| \frac{\partial F}{\partial (Z, Y)} \right| dZdY > q A(T_k^{i-1}(w^c)) > q^i A(w^c). $$

It follows that there exists $n$ such that $T_k^n(w^c)$ intersects for the first time the union of $\{Z = \pm \delta\}$ and $\{Y = \pm \delta^2\}$. Let us show that $T_k^n(w^c)$ intersects at least one of the boundaries $\{Y = \pm \delta^2\}$. Suppose that $T_k^n(w^c)$ intersects $\{Z = \pm \delta\}$. Formula (32) shows that the $Z$-coordinates of points of $T_k(\Pi)$ are less than $\delta$. Hence, $T_k^n(w^c)$ must intersect the boundary of $T_k(\Pi)$, which means that $T_k^{n-1}(w^c)$ intersects one of the boundaries $\{Z = \pm \delta\}$ and $\{Y = \pm \delta^2\}$, contradicting the assumption that $T_k^n(w^c)$ is the first iterate intersecting the boundaries.

$\square$

3.5 The non-transverse intersection

In the section, we build by changing $\omega$ an non-transverse intersection between $W^u(O)$ with $W^s(Q)$.

Lemma 6. Assume the weaker estimates (68). Let $|\lambda \gamma| > 1$ and the periodic point $Q$ and $\mu_k(t)$ be given by Lemma 2. For all sufficiently large $k$, there exist functions $\omega_k(t)$ defined for $t \in (-1, 1)$
converging to \( \omega^* \) as \( k \to \infty \) such that the intersection \( W^u(O) \cap W^s(Q) \) is non-empty when \( \mu = \mu_k(t) \) and \( \omega = \omega_k(t) \), and, for any fixed \( t \), this intersection splits as \( \omega \) changes. Moreover, if condition C6.1 is satisfied, then \( d\omega_k(t)/dt = O(k^{-1}) \) is non-zero.

**Proof.** Denote \( W^u = W^u_{loc}(O) \cap \Pi^- \). By (12), the image \( T_1(W^u) \) satisfies

\[
\begin{align*}
x_1 - x_1^+ &= b_1(\tilde{y} - y^-) + d_1 v + O((\tilde{y} - y^-)^2 + v^2), \\
x_2 - x_2^+ &= d_2 v + O((y - y^-)^2 + v^2), \\
y &= \mu + b_3(\tilde{y} - y^-)^2 + d_3 v + O(|v(\tilde{y} - y^-)| + v^2) + O(|\tilde{y} - y^-|^3 + |v|^3), \\
u - u^+ &= b_4(\tilde{y} - y^-) + d_4 v + O((\tilde{y} - y^-)^2 + v^2), \\
\tilde{v} - v^- &= b_5(\tilde{y} - y^-) + d_5 v + O((\tilde{y} - y^-)^2 + v^2),
\end{align*}
\]

where \( \tilde{y} - y^- \in [-\delta, \delta] \) and \( \tilde{v} - v^- \in [-\delta, \delta]^{d_y-1} \) are parameters. Denote 

\[
l_1 = \tilde{y} - y^- \quad \text{and} \quad l_2 = \tilde{v} - v^-.
\]

After the coordinate transformation (17), the above equations can be rewritten as

\[
\begin{align*}
X_1 &= b_1 l_1 + O(\gamma^{-k}) + O(l_1^2), \\
X_2 &= O(\gamma^{-k}) + O(l_1^2), \\
Y &= \gamma^k \mu - y^- + b_3 \gamma^k l_1^2 + O(\gamma^k \tilde{x}^-) + O(\gamma^k l_1^3) + O(l_1), \\
U &= b_4 l_1 + O(\gamma^{-k}) + O(l_1^2), \\
l_2 &= b_5 l_1 + O(\gamma^{-k}) + O(l_1^2),
\end{align*}
\]

where we used the estimates (21) (with \( \partial \hat{h}/\partial \hat{V} \) in (67)), and the \( O(\gamma^{-k}) \) and \( O(\gamma^k \tilde{x}^-) \) terms are functions of \( l_1 \) and \( V \). Applying the second coordinate transformation (22), we get 

\[
U^{new} = U - \frac{b_4}{b_1} X_1 = O(\gamma^{-k}) + O(l_1^2),
\]

while other equations above remain the same form. Finally, using (26) and (31) we write \( T_1(W^u) \) as

\[
\begin{align*}
Z &= b_1 \alpha^* l_1 + O(l_1^2) - (\alpha^*(\hat{\alpha}_1 x_1^+ + \hat{\beta}_1 x_2^+) + \beta^*(\hat{\alpha}_2 x_1^+ + \hat{\beta}_2 x_2^+)) \lambda^k + O(\lambda^k), \\
Y &= \gamma^k \mu - y^- + b_3 \gamma^k l_1^2 + O(\gamma^k \tilde{x}^-) + O(\gamma^k l_1^3) + O(l_1) + O(\lambda^k), \\
W &= O(l_1^2) + O(\lambda^k), \\
l_2 &= O(l_1^2) + O(\lambda^k),
\end{align*}
\]

where 

\[
A = -(\alpha^*(\hat{\alpha}_1 x_1^+ + \hat{\beta}_1 x_2^+) + \beta^*(\hat{\alpha}_2 x_1^+ + \hat{\beta}_2 x_2^+))
\]

with coefficients in (27).

By Lemma 3, the local stable manifold \( W^s_{loc}(Q) \) has the form

\[
\begin{align*}
Z &= Z_Q + \phi_1^s(W), \\
Y &= Y_Q + \phi_2^s(W), \\
V &= V_Q + \phi_3^s(W),
\end{align*}
\]
where \( \phi^s \) satisfies (82) and the coordinates of \( Q \) are given by Lemma 2.

Since the point \( Q \) exists at \( \mu = \mu_k \) which is a function of \( t \) given by (70), our parameters here are in fact \( \varepsilon = (t, \omega) \). A non-transverse intersection of \( W^u(O) \) and \( W^s(Q) \) corresponds to a solution \((l_1, W, \omega, t)\) to the system consisting of equations in (88) and (90) with \( \mu = \mu_k \). In what follows we solve this system.

Substituting the \( V \)-equation in (90) into the \( W \)-equation in (88), one expresses \( W \) as a function of \( l_1, \omega, t \):

\[
W = O(l_1^2) + O(\lambda^k). \tag{91}
\]

Equating the \( Z \)-equations and, respectively, the \( Y \)-equations in (88) and (90), yields

\[
Z_Q + \phi_1^s(W) = b_1 \alpha^* l_1 + O(l_1^2) + A \lambda^k + O(\hat{\lambda}^k),
\]

\[
Y_Q + \phi_2^s(W) = \gamma^k \mu_k - y^- + b_j \gamma^k l_1 + O(\gamma^k \hat{\gamma}^{-k}) + O(\gamma^k l_1^2) + O(l_1) + O(\lambda^k).
\]

Since the derivatives of coefficients with respect to parameters are bounded and \( \partial \phi^s / \partial \mu = O(k \lambda^{k-1}) \) by (82), it follows from (72) that

\[
\left. \frac{\partial \phi^s}{\partial t} \right|_{\mu=\mu_k} = O(k \lambda^{3k-1}), \quad \left. \frac{\partial \lambda^k}{\partial t} \right|_{\mu=\mu_k} = O(k \lambda^{3k-1}),
\]

and derivatives with respect to \( t \) for other terms can be calculated in the same way. Substituting the (70), (71), (91) and the \( V \)-equation in (90) into the above two equations, yields

\[
0 = b_1 \alpha^* l_1 + O(l_1^2) + \frac{(b_1 \alpha^*)^2}{2 b_3} \lambda^k t + A \lambda^k + O(\hat{\lambda}^k), \tag{92}
\]

\[
0 = -\lambda^k \gamma^k (\alpha^* x_1^+ + \beta^* x_2^+) + \frac{(b_1 \alpha^*)^2}{2 b_3} \lambda^{2k} \gamma^k t - \frac{(b_1 \alpha^*)^2}{4 b_3} \lambda^{2k} \gamma^k t^2 + b_3 \gamma^k l_1^2 + O(\gamma^k l_1^2) + O(l_1) + O(\hat{\lambda}^k \gamma^k), \tag{93}
\]

where the \( O(\cdot) \) are functions of \( l_1, \omega, t \). By the implicit function theorem, one finds from (92) that

\[
l_1 = -\left( \frac{b_1 \alpha^*}{2 b_3} t + \frac{A}{b_1 \alpha^*} \right) \lambda^k + O(\hat{\lambda}^k). \tag{94}
\]

Substituting (94) into (93) and dividing \( \lambda^k \gamma^k \) to both sides, yields

\[
\alpha^* x_1^+ + \beta^* x_2^+ = \frac{(b_1 \alpha^*)^2}{2 b_3} \lambda^k t + A \lambda^k t + O(\hat{\lambda}^k \lambda^{-k}) = O(\hat{\lambda}^k \lambda^{-k}). \Rightarrow g(\omega, t), \tag{95}
\]

which by (19) can be rewritten as

\[
(a_{31} x_1^+ + a_{32} x_2^+) \cos k\omega + (a_{32} x_1^+ - a_{31} x_2^+) \sin k\omega = g(\omega, t) \tag{96}
\]

with

\[
\left. \frac{\partial g}{\partial \omega} \right|_{k \to \infty} = o(1) \quad \text{and} \quad \left. \frac{\partial g}{\partial t} \right|_{k \to \infty} = \left( \frac{(b_1 \alpha^*)^2}{2 b_3} + A \right) \lambda^k + O(k \hat{\lambda}^{k-1} \lambda^k) + o(\lambda^{2k}). \tag{97}
\]
Note that the inequality
\[(a_{31}x_1^+ + a_{32}x_2^+)^2 + (a_{32}x_1^+ - a_{31}x_2^+)^2 \neq 0\]
is, after sorting terms, equivalent to
\[(a_{31}^2 + a_{32}^2)((x_1^+)^2 + (x_2^+)^2) \neq 0,\] (98)
which holds in our case by (4) and (13). So, equation (96) can be further reduced to
\[\sqrt{(a_{31}^2 + a_{32}^2)((x_1^+)^2 + (x_2^+)^2)} \sin(k\omega + \eta) - g(\omega, t) = 0,\] (99)
where \(\eta \in (-\pi/2, \pi/2]\) satisfies
\[
\sin \eta = \frac{a_{31}x_1^+ + a_{32}x_2^+}{\sqrt{(a_{31}^2 + a_{32}^2)((x_1^+)^2 + (x_2^+)^2)}},
\]
\[
\cos \eta = \frac{a_{32}x_1^+ - a_{31}x_2^+}{\sqrt{(a_{31}^2 + a_{32}^2)((x_1^+)^2 + (x_2^+)^2)}}.
\]

Since \(\cos(k\omega + \eta) \to 1\) by (99) and \(\partial g/\partial \omega \to 0\) by (97), one can use the implicit function theorem to find for every sufficiently large \(k\) a solution to (99) as
\[\omega_k(t) = k^{-1}(n_k \pi - \eta) + k^{-1}\hat{g}(t)\] (100)
for some \(n_k \in \mathbb{Z}\) and some function \(\hat{g} = o(1)_{k \to \infty}\). It is easy to see that for all sufficiently large \(k\) the integers \(n_k\) can taken such that \(\omega_k \to \omega^*\) as \(k \to \infty\), where \(\omega^*\) is the \(\omega\) value of \(f\). Substituting \(\omega_k\) back into the previous equations, one readily finds a solution to the system of equations in (88) and (90), and hence, an intersection of \(W^u(O)\) with \(W^s(Q)\). Obviously, for a fixed \(t\) this intersection splits as \(\omega\) changes.

Since Lemmas 2 and 3 are based on the assumption that \(\alpha^*\) is bounded away from zero, we need to verify this for \(\alpha^*(k\omega_k)\) when \(k\) is sufficiently large. This is easy since \(\beta^*(k\omega_k)\) is close to one when \(\alpha^*(k\omega_k) = o(1)_{k \to \infty}\) (see (19)), which along with the fact that \(\omega_k\) is a solution to (95) implies \(x_2^+ = 0\), contradicting (14).

What remains is to compute \(d\omega_k(t)/dt\). Recall that the derivatives of coefficient with respect to parameters are uniformly bounded and \(\partial \mu(\mu_k)/\partial t = O(\lambda^{2k})\) by (72). Thus, using the implicit function theorem one obtains from (99) that \(d\omega_k(t)/dt = O(k^{-1}) \neq 0\) for all sufficiently large \(k\) if the leading term of \(\partial g/\partial t\) does not vanish, which by (89) and (97) means that
\[
\frac{(b_1\alpha^*)^2}{2b_3} \neq \alpha^*(\hat{\alpha}_1x_1^+ + \hat{\beta}_1x_2^+) + \beta^*(\hat{\alpha}_2x_1^+ + \hat{\beta}_2x_2^+).
\]
Since \(\omega_k\) is a solution to (95), the above inequality holds for all sufficiently large \(k\) if
\[
\frac{b_1^2\alpha^*}{2b_3} \neq \hat{\alpha}_1x_1^+ + \hat{\beta}_1x_2^+ - \frac{x_1^+}{x_2^+}(\hat{\alpha}_2x_1^+ + \hat{\beta}_2x_2^+),
\]
which by (19) and (29) is equivalent to
\[ S_1 \cos k \omega_k + S_2 \sin k \omega_k \neq -\frac{b_1}{b_3}(e_1 x_1^+ + e_2 x_2^+), \]  
(101)
where
\[ S_1 = \frac{b_1^2}{2b_3}a_{31} - x_1^+ a_{11} - x_2^+ a_{12} + \frac{(x_1^+)^2}{x_2^+}a_{21} + x_1^+ a_{22} \quad \text{and} \quad S_2 = \frac{b_2^2}{2b_3}a_{32} - x_1^+ a_{12} + x_2^+ a_{11} + \frac{(x_1^+)^2}{x_2^+}a_{22} - x_1^+ a_{21}. \]

Observe that the above inequality can always be achieved if \( S_1^2 + S_2^2 \neq 0 \). Indeed, since \( \omega_k \) is the solution to (99), it can take two different values, and hence the left hand side of (101) can take two values, one of which must satisfy the inequality. We conclude the proof by noting that each of the first two inequalities in (15) guarantees \( x_2^+ S_1 + x_1^+ S_2 \neq 0 \) since they imply \( x_1^+ S_1 - x_2^+ S_2 \neq 0 \) and, respectively, \( x_2^+ S_1 + x_1^+ S_2 \neq 0 \); when we do not have these two inequalities and both \( S_1 \) and \( S_2 \) vanish, (101) reduces to the third inequality in (15).

### 3.6 The transverse intersection

The desired transverse intersection is obtained by using transverse homoclinic points bi-accumulating the tangency point \( M^- \). Let \( b_3 \) be the coefficient in (12) and \( y^- \) be the \( y \)-coordinate of \( M^- \). We call a tangency normal if \( b_3 y^- > 0 \). The following two lemmas are proved in Section 3.6.1.

**Lemma 7.** Let \( \omega / \pi \) be either irrational or equal to \( p/q \) with \( q \geq 4 \). If conditions C1 - C4 are satisfied and the tangency is normal, then there exist sequences \( \{n_i \in 2\mathbb{N}\} \to \infty \) and \( \{N_i = (0, 0, y_i, 0, v_i)\} \) of transverse intersection points of \( W_{loc}^u(O) \) with \( T_1^{-1} \circ T^{-n_i} \circ T_1^{-1}(W_{loc}^s(O)) \) accumulating on \( M^- = (0, 0, y^-, 0, v^-) \). Moreover, we have \( y_i - y^- \to 0 \) from both sides with \( y_i - y^- = O(\gamma - 2^k) \).

**Lemma 8.** Let \( \omega / \pi \) be either irrational or equal to \( p/q \) with \( q \geq 4 \). For any smooth one-parameter family \( f_\mu \) with \( f_0 \) satisfying conditions C1 - C5.1, there exists a sequence \( \{\mu_k\} \) accumulating on \( \mu = 0 \) such that \( f_{\mu_k} \) has a point \( M_k \in W_{loc}^s(O) \) of homoclinic tangency which is normal and also satisfy C1 - C5.1. Moreover, each new tangency splits as \( \mu \) varies.

**Proposition 2.** Assume the weaker estimates (68). If \( M^- \) is accumulated by the transverse homoclinic points in Lemma 7, then the periodic point \( Q \) obtained for all sufficiently large \( k \) in Lemma 2 satisfy \( W^u(Q) \cap W^s(O) \neq \emptyset \).

**Proof of Theorem 2 for the saddle-focus case.** Let \( \omega_k \to \omega^* \) be any sequence of \( \omega \) values satisfying the condition in Lemma 8, that is, \( \omega_k / \pi \) is either irrational or equal to \( p/q \) with \( q \geq 4 \) for all \( k \). This Lemma then gives a sequence \( \mu_k \to 0 \) such that one has a normal tangency at \( (\mu_k, \omega_k) \), and hence transverse homoclinic points bi-accumulating on \( M^- \) by Lemma 7. Now, applying Lemma 6 and Proposition 2 at each sufficiently large \( k \) gives a sequence \( (\mu_k^i(t), \omega_k^i(t)) \to (\mu_k, \omega_k) \) of parameter values corresponding to heterodimensional cycles involving \( O \) and \( Q \). Finally, denoting \( (\mu_k, \omega_k) := (\mu_k^i(t), \omega_k^i(t)) \) for any fixed \( t \in (0, 1) \) gives the desired sequence.

**Proof of Proposition 2.** Let \( N_i \) be the transverse homoclinic points in Lemma 7 accumulating on \( M^- \). For each \( i \) we find a connected piece \( s_i \) of \( W^s(O) \cap \Pi^- \) that goes through \( N_i \). Below we use the same
letter for a surface and its defining function. Since \(N_i\) are transverse homoclinic points, we can assume that, up to shrinking \(\Pi^-\) if necessary, each piece \(s_i\) is the graph of a smooth function of the form

\[
(y, v) = s_i(x_1, x_2, u),
\]

defined for all the \(x_1, x_2, u\) values of \(\Pi^-\) with \(\|s_i\|_{C^1}\) being uniformly bounded. This uniform boundedness implies that

\[
\max_{M_1, M_2 \in s_i \cap T_0^k(\Pi^+)} |y_{M_1} - y_{M_2}| \to 0
\]

uniformly as \(k \to \infty\), and the same holds for the \(v\)-coordinates. Let \(y_i, v_i\) be the \(y\)- and \(v\)-coordinates of \(N_i\). Since \(s_i\) particularly contains the homoclinic point \(N_i\), we have

\[
\max_{M \in s_i \cap T_0^k(\Pi^+)} |y_M - y_i| \to 0 \quad \text{and} \quad \max_{M \in s_i \cap T_0^k(\Pi^+)} |v_M - v_i| \to 0,
\]

as \(k \to \infty\).

Now consider the surfaces \(s_i^k = T_0^{-k}(s_i) \cap \Pi\). In the \((Z, Y, W, V)\)-coordinates this surface is the graph of a smooth function of the form

\[
(Y, V) = s_i^k(Z, W),
\]

defined on \([-\delta, \delta] \times [-\delta, \delta]^{1+d_y}\). Using Lemma 7, we first fix two points \(N_{i_1}\) and \(N_{i_2}\) with \(y_{i_1} > y^-\) and \(y_{i_2} < y^-\). Then, there exists a constant \(C \in (0, \delta^2)\) such that

\[
y_{i_1} - y^- \in (C, \delta^2) \quad \text{and} \quad y_{i_2} - y^- \in (-\delta^2, -C).
\]

This along with the first relation in (103) implies that, in the \((Z, Y, W, V)\)-coordinates, the \(Y\)-coordinates of points of \(s_i^k_{i_1}\) and \(s_i^k_{i_2}\) satisfy

\[
\{Y_M : M \in s_i^k_{i_1}\} \subset (C, \delta^2) \quad \text{and} \quad \{Y_M : M \in s_i^k_{i_2}\} \subset (-\delta^2, -C).
\]

We can now prove the proposition. Using the leaves of the foliation \(\tilde{F}^{uu}\) in Lemma 4, we construct from \(s_i^k\) a surface of the form \(Y = \tilde{s}_i^k(Z, W, V)\), where the partial derivatives with respect to \(Z, W\) are uniformly bounded and those with respect to \(V\) satisfy (40). Recall that we denote by \(w^c\) the intersection \(W^c_{\text{loc}}(Q) \cap W^{cs}\) (see Lemma 5). One easily sees that the proof will be finished if \(\tilde{s}_i^k(Z, W, V) \cap w^c \neq \emptyset\). Indeed, this means that \(\tilde{s}_i^k\) is comprised of true leaves of \(T_k\), since their backward iterates stay inside \(\Pi\) (see Remark 8), and hence is part of \(W^c_\text{loc}(Q)\). Below we prove that \(\tilde{s}_i^k(Z, W, V) \cap w^c \neq \emptyset\) for some \(i \in \{i_1, i_2\}\).

Assume for now that the intersection of \(w^c\) with \(\{y = \pm \delta^2\}\) given by Lemma 5 is in \(\{y = \delta^2\}\). Combining the equations of \(\tilde{s}_i^k\) and \(W^{cs}\), we find their intersection on \(W^{cs}\) has the form

\[
Y = w^c_{i_1}(Z, W)
\]

defined for all the \(Z, W\) values in \(\Pi\). So, it suffices to show \(w^c_{i_1} \cap w^c \neq \emptyset\). Since \(|Y_Q| = |O(\lambda^k)| < C\) by Lemma 2 for all sufficiently large \(k\), we have \(\{Y_M : M \in s_i^k\} \subset (Y_Q, \delta^2)\). Therefore, one has the desired intersection if the deviation in \(Y\) of leaves of \(\tilde{F}^{uu}\) is small enough so that \(Y_M \in (Y_Q, \delta^2)\) for
$M \in w^k_{x_1}$, see Figure 1. In what follows we show that this is indeed the case.

Figure 1: The surface $s^k_{x_1}$ consisting of leaves of $\mathcal{F}^{uu}$ through $s^k_{x_1}$ (red curve) intersects $W^{cs}$ on $w^k_{x_1}$ (blue curve). The leaves in the gray area of $s^k_{x_1}$ intersect $w^c$ and hence are part of $W^u(Q)$.

By (69), one has $dY/dV = O(Y) + o(1)_{k \to \infty}$. So, the total change in $Y$ after $||V||$ runs over $[-\delta, \delta]$ is bounded by $2(\delta^3 - \delta^2) = c\delta^3$ for some constant $c > 0$. Since we can take $Y_Q \to 0$, it then suffices to show that there exists $\delta_0$ such that for any $\delta < \delta_0$, one can find $N_i$ such that its $Y$-coordinate $Y_i$ lies in $(c\delta^3, \delta^2 - c\delta^3)$. By Lemma 7 one has $Y_i = c'/\gamma - \frac{\gamma}{\delta}$ for some constant $c' > 0$. So, the desired $N_i$ correspond to integer solutions to the inequality $c\delta^3 < c'/\gamma - \frac{\gamma}{\delta} < \delta^2 - c\delta^3$ when $\delta$ is small, whose existence becomes obvious after taking logarithm.

3.6.1 Accumulating transverse homoclinic points: Proofs of Lemmas 7 and 8

Proof of Lemma 7. We will use the formulas for $T_0$ and $T_1$ in the original coordinate $(x_1, x_2, y, u, v)$. Let $l_1 = \bar{y} - y^-$, $l_2 = \bar{v} - v^-$. According to (12), the image $T_1(W^u_{loc}(O) \cap \Pi^-)$ is given by

$$
\begin{align*}
x_1 - x_1^+ &= b_1 l_1 + b_1 v + O(l_1^2 + v^2), \\
x_2 - x_2^+ &= d_1 v + O(l_1^2 + v^2), \\
y &= \mu + b_2 l_2 + d_2 v + O(l_1^2 + v^2) + O(|l_1 v| + v^2), \\
u - u^+ &= b_3 l_1 + b_3 v + O(l_1^2 + v^2), \\
l_2 &= b_3 l_1 + d_3 v + O(l_1^2 + v^2).
\end{align*}
$$

(106)

Since $W^u_{loc}(O)$ is given by $\{y = 0, v = 0\}$, a point $(x_1, x_2, y, u, v) \in T_1(W^u_{loc}(O) \cap \Pi^-)$ is a homoclinic point of $O$ if

$$
T^k_0(x_1, x_2, y, u, v) = (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) \quad \text{and} \quad T_1(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) = (\bar{x}_1, \bar{x}_2, 0, \bar{u}, \bar{v})
$$

(107)

for some $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{u}, \bar{v}) \in \Pi^-$, namely, the point corresponds to a solution $(l_1, v)$ to the system consisting of (106) and (107). It follows from (10) and (12) that these coordinates satisfy

$$
\begin{align*}
\bar{x}_1 &= \lambda^k x_1 \cos k\omega - \lambda^k x_2 \sin k\omega + O(\dot{\lambda}^k), \\
\bar{x}_2 &= \lambda^k x_1 \sin k\omega + \lambda^k x_2 \cos k\omega + O(\dot{\lambda}^k), \\
y &= \gamma^{-k} \bar{y} + O(\dot{\gamma}^{-k}), \\
\bar{u} &= O(\dot{\lambda}^k), \quad \bar{v} = O(\dot{\gamma}^{-k}).
\end{align*}
$$

(108)
where the dots denote quadratic terms in the Taylor expansions, except the \((\tilde{y} - y^-)^2\) term in the third equations.

Let us solve the above-mentioned system. First, set \(\bar{v} = 0\) in the fifth equation of \((109)\), and substitute it into the \(y\)- and \(v\)-equations in \((108)\). This gives
\[
v = O(\gamma^{-k})
\]
as a function of \(x_1, x_2, y, u\). Next, we solve out \(l_1\) from the first equation of \((106)\) as
\[
l_1 = b_1^{-1}(x_1 - x_1^+) - b_1^{-1}d_1v + O((x_1 - x_1^+)^2 + v^2).
\]
Combining this with the expression of \(v\) and the remaining equations of \((106)\), we obtain
\[
\begin{align*}
l_1 &= b_1^{-1}(x_1 - x_1^+) + O((x_1 - x_1^+)^2) + O(\gamma^{-k}), \\
x_2 &= x_2^+ + O((x_1 - x_1^+)^2) + O(\gamma^{-k}), \\
y &= \mu + \frac{b_3}{b_1}(x_1 - x_1^+)^2 + O((x_1 - x_1^+)^3) + O(\gamma^{-k}), \\
u &= u^+ + \frac{b_4}{b_1}(x_1 - x_1^+)^2 + O((x_1 - x_1^+)^2) + O(\gamma^{-k}), \\
l_2 &= \frac{b_5}{b_1}(x_1 - x_1^+) + O((x_1 - x_1^+)^2) + O(\gamma^{-k}),
\end{align*}
\]
where the \(O(\gamma^{-k})\) terms are functions of \(x_1\).

After setting
\[
X = x_1 - x_1^+ \quad \text{and} \quad Y = \tilde{y} - y^-,
\]
the third equation of \((112)\) and the \(y\)-equation of \((108)\) gives
\[
Y = \tilde{y} - y^- = \gamma^k y + o(1)_{k \to \infty} - y^- = \gamma^k \mu - y^- + \frac{\gamma^k b_3}{b_1^2} X^2 + O(\gamma^k X^3) + o(1)_{k \to \infty},
\]
or,
\[
0 = \mu - \gamma^{-k} y^- - \gamma^{-k} Y + \frac{b_3}{b_1^2} X^2 + O(X^3) + o(\gamma^{-k}).
\]

To find \(X\) we need one more equation. Note that the \(y\)-coordinates \(\tilde{y}\) of points on \(T_1 \circ T^k_0 \circ T_1(W_{loc}^u(O))\) can be found by directly combining \((108)\), \((109)\) with using \((106)\) and \((19)\). Alternatively, it can be found by using the \(Y\)-equation in \((18)\) and then returning to the original coordinates by the \((17)\). In either way, we obtain
\[
\tilde{y} = \mu + \alpha^* \lambda^k x_1^+ + \beta^* \lambda^k x_2^+ + \alpha^* \lambda X + b_3 Y^2 + O(\lambda^k X^2) + o(\lambda^k) + O(Y^3),
\]

where the left hand side is a function of $X, Y, \bar{v}$. For points in $W^s_{loc}(O)$, namely, those with $\bar{y} = \bar{v} = 0$, one further has

$$0 = \mu + \alpha^* \lambda^k x_1^+ + \beta^* \lambda^k x_2^+ + \alpha^* \lambda^k X + b_3 Y^2 + O(\lambda^k X^2) + o(\lambda^k) + O(Y^3). \quad (115)$$

The sought transverse homoclinic points now correspond to the non-degenerate solutions to the system consisting of (114) and (115).

Let us find the solutions. Note that there are infinitely many $k \in 2\mathbb{N}$ such that $b_3(\alpha^* x_1^+ + \beta^* x_2^+) < 0$. To see this, use (19) to write

$$\alpha^* x_1^+ + \beta^* x_2^+ = \sqrt{\sigma_1^2 + \sigma_2^2} \sin(k \omega + \eta)$$

where $\sigma_1 = a_{31} x_1^+ + a_{32} x_2^+$ and $\sigma_2 = -a_{31} x_2^+ + a_{32} x_1^+$. We need to show that $\sin(k \omega + \eta)$ with $k \in 2\mathbb{N}$ have different signs, and it suffices to show that $k' \omega / \pi$ with $k' \in \mathbb{N}$ can take at least three different values in $[0, 1)$. But this is guaranteed when $\omega / \pi$ is irrational or equals $p/q$ with $q \geq 4$.

We take such $k \in 2\mathbb{N}$ and apply the scaling

$$(X, Y) \mapsto \left( b_1 \sqrt{\frac{y}{b_3}} - \frac{\gamma}{\bar{b}_3} U, \sqrt{\frac{\alpha^* x_1^+ + \beta^* x_2^+}{-b_3}} \lambda^k V \right).$$

Taking $\mu = 0$, we rewrite (114) and (115) as

$$1 = U^2 + o(1)_{k \to \infty}, \quad 1 = V^2 + o(1)_{k \to \infty}.$$ Obviously, the solutions are $(U, V) = (\pm 1 + o(1)_{k \to \infty}, \pm 1 + o(1)_{k \to \infty})$, which lead to four solutions in the $(X, Y)$-variables:

$$
(X_1^k, Y_1^k) = \left( b_1 \gamma - \frac{\beta}{\bar{b}_3} \sqrt{\frac{y}{b_3}} + o(\gamma^{-\frac{\beta}{2}}), \lambda^k \sqrt{\frac{\alpha^* x_1^+ + \beta^* x_2^+}{-b_3}} + o(\lambda^k) \right),

(X_2^k, Y_2^k) = \left( b_1 \gamma - \frac{\beta}{\bar{b}_3} \sqrt{\frac{y}{b_3}} + o(\gamma^{-\frac{\beta}{2}}), -\lambda^k \sqrt{\frac{\alpha^* x_1^+ + \beta^* x_2^+}{-b_3}} + o(\lambda^k) \right),

(X_3^k, Y_3^k) = \left( -b_1 \gamma - \frac{\beta}{\bar{b}_3} \sqrt{\frac{y}{b_3}} + o(\gamma^{-\frac{\beta}{2}}), \lambda^k \sqrt{\frac{\alpha^* x_1^+ + \beta^* x_2^+}{-b_3}} + o(\lambda^k) \right),

(X_4^k, Y_4^k) = \left( -b_1 \gamma - \frac{\beta}{\bar{b}_3} \sqrt{\frac{y}{b_3}} + o(\gamma^{-\frac{\beta}{2}}), -\lambda^k \sqrt{\frac{\alpha^* x_1^+ + \beta^* x_2^+}{-b_3}} + o(\lambda^k) \right) . \quad (116)

All these solutions are non-degenerate, since the Jacobian of the system comprised by (114) and (115) at these solutions is

$$\frac{4b_3^2}{b_1^2} \lambda^k \gamma^{-\frac{\beta}{2}} = \pm 4b_3 \lambda^k \gamma^{-\frac{\beta}{2}} \sqrt{-y(\alpha^* x_1^+ + \beta^* x_2^+) + o(\lambda^k \gamma^{-\frac{\beta}{2}})} ,$$

which is non-zero.
The corresponding transverse homoclinic points in $T_1(W^u_{loc}(O))$ are given by

\[
N^1_k = (x^+_1 + \tilde{X} + o(|\gamma|^{-\frac{1}{2}}), x^+_2, (\tilde{Y} + y^-)\gamma^{-k} + o(\gamma^{-k}), u^1, v^1),
\]

\[
N^2_k = (x^+_1 + \tilde{X} + o(|\gamma|^{-\frac{1}{2}}), x^+_2, (-\tilde{Y} + y^-)\gamma^{-k} + o(\gamma^{-k}), u^2, v^2),
\]

\[
N^3_k = (x^-_1 - \tilde{X} + o(|\gamma|^{-\frac{1}{2}}), x^-_2, (\tilde{Y} + y^-)\gamma^{-k} + o(\gamma^{-k}), u^3, v^3),
\]

\[
N^4_k = (x^-_1 - \tilde{X} + o(|\gamma|^{-\frac{1}{2}}), x^-_2, (-\tilde{Y} + y^-)\gamma^{-k} + o(\gamma^{-k}), u^4, v^4),
\]

where $\tilde{X} = b_1\gamma^{-\frac{1}{2}}\sqrt{\frac{y^-}{b_3}}$ and $\tilde{Y} = \lambda^\frac{1}{2}\sqrt{\frac{\alpha^*x^+_1 + \beta^*x^+_2}{b_3}}$, and the coordinates $x^+_2, u^j, v^j$ ($j = 1, 2, 3, 4$) are not written can be found from (112). Denote $\hat{N}^j_k = T_1^{-1}(N^j_k)$. Since all these points lie on $W^u_{loc}(O)$, their coordinates are $(0, 0, \hat{y}^j_k, 0, \hat{v}^j_k)$, which, by the first and last equations of (112), are given by

\[
\hat{y}^j_k - y^- = \frac{X^j_k}{b_1} + O((X^j_k)^2) + O(\gamma^{-k})
\]

and

\[
\hat{v}^j_k - v^- = \frac{b_5}{b_1}X^j_k + O((X^j_k)^2) + O(\gamma^{-k}).
\]

Thus, we have $\hat{N}^j_k \to M^-$ as $k \to \infty$ and $\hat{y}^j_k$ and $\hat{v}^j_k$ lie on different sides of $y^-$. \hfill \square

**Proof of Lemma 8.** We will first find new homoclinic tangencies which are normal, then show that they unfolds as $\mu$ varies, and finally prove that these tangencies are generic, namely, conditions C1 - C5.1 are satisfied. The proof is accordingly divided into three parts.

(i) **Creation of secondary homoclinic tangencies.** As discussed in the proof of Lemma 7, homoclinic points in $W^u_{loc}(O)$ with the global map $T_1 \circ T^*_1 \circ T_1$ correspond to solutions in the system consisting of (114) and (115). Particularly, quadratic homoclinic tangencies corresponds to solutions with multiplicity two. In what follows we find such solutions.

On the one hand, after eliminating $\mu$ from (114) and (115), we have

\[
\lambda^k(\alpha^*x^+_1 + \beta^*x^+_2) + \gamma^{-k}y^- + \alpha^*\lambda^kX + \gamma^{-k}Y - \frac{b_3}{b_1^2}X^2 + b_3Y^2 + O(\lambda^kX^2) + O(X^3) + O(Y^3) + o(\lambda^k) = 0. \tag{117}
\]

On the other hand, solutions of multiplicity two implies that the Jacobian of the system vanishes. So, we have

\[
-\gamma^{-k}(\alpha^*\lambda^k + O(\lambda^kX)) + O(\gamma^{-k}) - \frac{4b_2}{b_1^2}Y + O(Y^2) + o(\lambda^k)(X + O(X^2) + o(\gamma^{-k})) = 0,
\]

where we denoted the last two brackets in the first line by

\[
\hat{Y} := Y + O(Y^2) + o(\lambda^k)
\]

and

\[
\hat{X} := X + O(X^2) + o(\gamma^{-k}). \tag{119}
\]

With these notations, (117) can be recast as

\[
\lambda^k(\alpha^*x^+_1 + \beta^*x^+_2) + \gamma^{-k}y^- + \alpha^*\lambda^k\hat{X} + \gamma^{-k}\hat{Y} - \frac{b_3}{b_1^2}\hat{X}^2 + b_3\hat{Y}^2 + O(\hat{X}^3) + O(\hat{Y}^3) + o(\lambda^k) = 0. \tag{120}
\]
We now solve the system consisting of (118) and (120). By taking \( k \) such that \( b_3(\alpha x^+_1 + \beta x^+_2) < 0 \) (as we did in the proof of Lemma 7), we apply the scaling

\[
(\hat{X}, \hat{Y}) \mapsto \sqrt{\frac{-(\alpha x^+_1 + \beta x^+_2)}{b_3}} \left( \frac{-\alpha b^2_1}{4b_3(\alpha x^+_1 + \beta x^+_2)} \lambda^2 \gamma^{-k} U, \lambda^2 V \right),
\]

which takes (118) and (120) to (recall that \(|\lambda\gamma| > 1\))

\[
1 + UV = o(1)_{k \to \infty}, \quad 1 - V^2 = o(1)_{k \to \infty}
\]

with solutions \((U^1, V^1) = (1 + o(1)_{k \to \infty}, -1 + o(1)_{k \to \infty})\) and \((U^2, V^2) = (-1 + o(1)_{k \to \infty}, 1 + o(1)_{k \to \infty})\).

Returning to the \((X, Y)\)-variables, we have

\[
(\hat{X}^1_k, \hat{Y}^1_k) = \sqrt{\frac{-(\alpha x^+_1 + \beta x^+_2)}{b_3}} \left( \frac{-\alpha b^2_1}{4b_3(\alpha x^+_1 + \beta x^+_2)} \lambda^2 \gamma^{-k} + o(\lambda^2 \gamma^{-k}), -\lambda^2 \gamma^{-k} + o(\lambda^2) \right),
\]

\[
(\hat{X}^2_k, \hat{Y}^2_k) = \sqrt{\frac{-(\alpha x^+_1 + \beta x^+_2)}{b_3}} \left( \frac{-\alpha b^2_1}{4b_3(\alpha x^+_1 + \beta x^+_2)} \lambda^2 \gamma^{-k} + o(\lambda^2 \gamma^{-k}), \lambda^2 \gamma^{-k} + o(\lambda^2) \right).
\]

It is easy to check that these solutions are non-degenerate so they indeed have multiplicity two. By (114), the corresponding \( \mu \) values to the solutions have the same form as

\[
\mu^{\pm}_k = \gamma^{-k} y^-(1 + o(1)_{k \to \infty}), \quad j = 1, 2.
\]

By (113) and (119), the tangent points on \( T_1(W^u_{loc}(O)) \) are given by

\[
M^j_k = (\hat{X}^j_k + x^j_1 + o(\gamma^{-k}), x^j_{2,k}, \gamma^{-k}(\hat{Y}^j_k + y^- + o(1)_{k \to \infty}), u^j_k, v^j_k)
\]

with \( x^j_{2,k}, u^j_k, v^j_k \) obtained from (110) and (112).

Denote \( M^{-j}_k = T_1^{-1}(M^j_k) \). Since these points lie on \( W^u_{loc}(O) \), they have coordinates \((0, 0, \hat{y}^{-j}_k, 0, \hat{v}^{-j}_k)\).

It follows from the first and fifth equations of (122) that

\[
\hat{y}^{-j}_k - y^- = \frac{\hat{X}^j_k}{b_1} + O((\hat{X}^j_k)^2) + o(\gamma^{-k}) \quad \text{and} \quad \hat{v}^{-j}_k - v^- = O(\lambda^2 \gamma^{-k}), \tag{125}
\]

which shows that \( M^{-j}_k \to M^- \) as \( k \to \infty \).

Now, let us check that at least one of these homoclinic tangency points is normal. Recall that the global map corresponding to the secondary tangencies is \( T_1 \circ T^k_0 \circ T_1 \). By (109), we have

\[
b^j_{3,k} = \frac{1}{2} \frac{\partial^2 \hat{y}}{\partial y^2}(0, 0, \hat{y}^{-j}_k, 0, \hat{v}^{-j}_k)
\]

\[
= \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \hat{y}}{\partial x_1} \frac{\partial \hat{y}}{\partial y} + \frac{\partial \hat{y}}{\partial x_2} \frac{\partial \hat{y}}{\partial y} + \frac{\partial \hat{y}}{\partial u} \frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{y}}{\partial v} \frac{\partial \hat{v}}{\partial y} \right) \bigg|_{(0,0,\hat{y}^{-j}_k,0,\hat{v}^{-j}_k)} \tag{126}
\]
with
\[
\frac{\partial \bar{y}}{\partial y} |_{T_{k}^{y} \circ T_{1}(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} = 2b_{3}(\bar{y} - y^{-}) + O((\bar{y} - y^{-})^{2}) + O(\lambda^{k})
\]
\[
= 2b_{3} \hat{Y}_{k}^{j} + O((\hat{Y}_{k}^{j})^{2}) + O(\lambda^{k}),
\]  
(127)

\[
\frac{\partial^{2} \bar{y}}{\partial y^{2}} |_{T_{k}^{y} \circ T_{1}(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} = 2b_{3} + O(\hat{Y}_{k}^{j}) + O(\lambda^{k}),
\]

where we use \(\hat{Y}_{k}^{j} = \bar{y} - y^{-} + O((\bar{y} - y^{-})^{2}) + O(\lambda^{k})\) by (113) and (119).

To calculate \(\partial \bar{y}/\partial y\), one needs to consider the composition \(T_{k}^{y} \circ T_{1}\). Thus, we substitute (109) into (108) to get
\[
\frac{\partial \bar{y}}{\partial y} |_{(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} = \frac{\partial \bar{y}}{\partial x_{1}} |_{(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} + \frac{\partial \bar{y}}{\partial x_{2}} |_{(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} + \frac{\partial \bar{y}}{\partial u} |_{(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} + \frac{\partial \bar{y}}{\partial v} |_{(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})}
\]
\[
= 2b_{3} \gamma^{k}(\bar{y} - y^{-}) + O((\bar{y} - y^{-})^{2}) + O(1)_{k \to \infty}
\]
\[
= 2b_{3} \gamma^{k} \hat{X}_{k}^{j} + O((\hat{X}_{k}^{j})^{2}) + O(1)_{k \to \infty},
\]  
(128)

where we use \(\hat{X}_{k}^{j} = \bar{y} - y^{-} + O((\bar{y} - y^{-})^{2}) + O(\gamma^{-k})\) by the first equation in (112) and (119).

Similarly, we obtain
\[
\frac{\partial^{2} \bar{y}}{\partial y^{2}} |_{(0,0,\hat{\epsilon}_{k}^{-3},0,\hat{\epsilon}_{k}^{-3})} = 2b_{3} \gamma^{k} + o(\hat{X}_{k}^{j}) + o(1)_{k \to \infty},
\]

where \(\hat{X}_{k}^{j} = \bar{y} - y^{-} + O((\bar{y} - y^{-})^{2}) + O(\gamma^{-k})\) by the first equation in (112) and (119).

Note that by definition \(\partial \bar{v}/\partial y = 0\). Now using (122) and combining (127), (128) and (129) yields
\[
b_{3}^{j} = \frac{1}{2} \left[ \frac{\partial^{2} \bar{y}}{\partial y^{2}} \left( \frac{\partial \bar{y}}{\partial y} \right)^{2} + \frac{\partial \bar{y}}{\partial y} \frac{\partial^{2} \bar{y}}{\partial y^{2}} \right] + o(\lambda^{k})
\]
\[
= (b_{3} + O(Y_{k}^{j}) + O(\lambda^{k})) \left( 2b_{3} \gamma^{k} \hat{X}_{k}^{j} + O((\hat{X}_{k}^{j})^{2}) + O(1)_{k \to \infty} \right)^{2}
\]
\[
+ b_{3} \hat{Y}_{k}^{j} (2b_{3} \gamma^{k} + o(\hat{X}_{k}^{j}) + o(1)_{k \to \infty}) + o(\lambda^{k})
\]
\[
= (-1)^{j} 2b_{3} \gamma^{k} \left( -\alpha^{*} x_{1}^{j} + \beta^{*} x_{2}^{j} \right) / b_{3},
\]  
(130)

which implies that for one of \(M_{k}^{-1}\) and \(M_{k}^{-2}\) we have \(b_{3,k}y^{-} > 0\).

(ii) Unfolding of secondary homoclinic tangencies. Denote \(T_{1}^{\text{new}} = T_{1} \circ T_{k}^{y} \circ T_{1}\). As in Remark 4, for any system \(g\) close to \(f\), we define \(\mu^{\text{new}}(g)\) as the functional measuring the (signed) distance between \(T_{1}^{\text{new}}(0,0,y^{-}(g),0,v^{-})\) and \(W^{s}_{\text{loc}}(O)\). To show that the newly found tangencies unfolds as \(\mu\) varies, it suffices to prove that for each sufficiently large \(k\), the parameter \(\mu(g)\) satisfies
\[
\frac{\partial \mu^{\text{new}}(f_{\mu})}{\partial \mu} \bigg|_{\mu = \mu_{k}} \neq 0,
\]

where \(\mu_{k}\) are given by (123).
Let us find the $y$-equation of $T_{1}^{\text{new}}$. Combining (12) and (108), yields

\[
\ddot{x}_1 = \lambda^k (x_1^+ \cos k\omega - x_2^+ \sin k\omega) + \lambda^k \cos k\omega [a_{11} \ddot{x}_1 + a_{12} \ddot{x}_2 + b_1 (\ddot{y} - y^-)] \\
- \lambda^k \sin k\omega [a_{21} \ddot{x}_1 + a_{22} \ddot{x}_2 + c_2 \ddot{u} + d_2 v] + O(\lambda^k), \\
\ddot{x}_2 = \lambda^k (x_1^+ \sin k\omega + x_2^+ \cos k\omega) + \lambda^k \sin k\omega [a_{11} \ddot{x}_1 + a_{12} \ddot{x}_2 + b_1 (\ddot{y} - y^-)] \\
+ \lambda^k \cos k\omega [a_{21} \ddot{x}_1 + a_{22} \ddot{x}_2 + c_2 \ddot{u} + d_2 v] + O(\lambda^k), \\
\gamma^{-k} \ddot{y} = \mu + a_{31} \ddot{x}_1 + a_{32} \ddot{x}_2 + b_3 (\ddot{y} - y^-)^2 + c_3 \ddot{u} + d_3 v + O(\gamma^{-k}), \\
\ddot{u} = O(\lambda^k), \quad v = O(\gamma^{-k}).
\] (131)

Substituting (131) into the third equation of (109), we obtain

\[
\ddot{y} = \mu + a_{31} \left[ \lambda^k (x_1^+ \cos k\omega - x_2^+ \sin k\omega) + \lambda^k \cos k\omega (a_{11} \ddot{x}_1 + a_{12} \ddot{x}_2 + b_1 (\ddot{y} - y^-) + c_1 \ddot{u}) \right] \\
- \lambda^k \sin k\omega [a_{21} \ddot{x}_1 + a_{22} \ddot{x}_2 + c_2 \ddot{u}] + a_{32} \left[ \lambda^k (x_1^+ \sin k\omega + x_2^+ \cos k\omega) \right] \\
+ \lambda^k \sin k\omega [a_{11} \ddot{x}_1 + a_{12} \ddot{x}_2 + b_1 (\ddot{y} - y^-) + c_1 \ddot{u}] + \lambda^k \cos k\omega (a_{21} \ddot{x}_1 + a_{22} \ddot{x}_2 + c_2 \ddot{u}) \\
+ b_3 (\mu^2 \gamma^k + a_{31} \ddot{x}_1 \gamma^k + a_{32} \ddot{x}_2 \gamma^k + b_3 \gamma^k (\ddot{y} - y^-)^2 + c_3 \gamma^k \ddot{u} + O(\gamma^k \gamma^{-k}) - y^-)^2 + O(\lambda^k) + d_3 \ddot{v}.
\] (132)

By definition, $\mu^{\text{new}}$ equals the right hand side of the above equation after substituting \((\ddot{x}_1, \ddot{x}_2, \ddot{y}, \ddot{u}, v) = (0, 0, y_k^{-j}, 0, 0)\) (see (125)) into it, which is given by

\[
\mu^{\text{new}} = \mu + a_{31} \lambda^k [(x_1^+ \cos k\omega - x_2^+ \sin k\omega) + b_1 \cos k\omega (y_k^{-j} - y^-)] \\
+ a_{32} \lambda^k [(x_1^+ \sin k\omega + x_2^+ \cos k\omega) + b_1 \sin k\omega (y_k^{-j} - y^-)] \\
+ b_3 (\mu^2 \gamma^k + b_3 \gamma^k (y_k^{-j} - y^-)^2 + O(\gamma^k \gamma^{-k}) - y^-)^2 + O(\lambda^k).
\]

The sought derivative is

\[
\frac{\partial \mu^{\text{new}}}{\partial \mu} = 1 + 2b_3 (\mu \gamma^k + b_3 \gamma^k (y_k^{-j} - y^-)^2 + O(\gamma^k \gamma^{-k}) - y^-) (\gamma^k + O(\gamma^k \gamma^{-k})).
\] (133)

Before checking this derivative we notice that, since $\frac{\partial \ddot{y}}{\partial \ddot{y}} = 0$ at the tangency point, one can find from (132) that

\[
0 = \frac{\partial \ddot{y}}{\partial \ddot{y}} \bigg|_{(0, 0, y_k^{-j}, 0, 0)} = b_1 \alpha^* \lambda^k + 4b_3^2 \gamma^k (y_k^{-j} - y^-) (\mu \gamma^k + b_3 \gamma^k (y_k^{-j} - y^-)^2 + O(\gamma^k \gamma^{-k}) - y^-) + O(\lambda^k),
\]

where the last equality used (19). Since $\alpha^* \neq 0, b_1 \neq 0$ by assumption, combining the above equation with (133) and using (125), yields

\[
\frac{\partial \mu^{\text{new}}}{\partial \mu} \bigg|_{\mu = \mu_k} = 1 - \frac{b_1 \alpha^* \lambda^k (\gamma^k + O(\gamma^k \gamma^{-k}))}{2b_3 \gamma^k X^j_k + O(\lambda^k)} \neq 0,
\]

where $X^j_k = O(\lambda^k \gamma^{-k}) \neq 0$ by (122).

(iii) Genericity conditions. We now prove that the secondary homoclinic tangencies satisfy conditions C1 - C5.1 introduced in Section 2. Condition C2 is shown in the first part of the proof. Condition C3 is also fulfilled since $T_1^{\text{new}} = T_1 \circ T_0^{k} \circ T_1$ with $T_1$ satisfying C3 and $DT_0^{k}$ keeping the bundle $E^{cs} \oplus E^{cu}$
invariant around $O$.

For verifying condition C5.1, we need to find the two vectors of the new tangency points $T_{1}^{\text{new}}(M_{k}^{-j})$, where $M_{k}^{-j} \in W_{\text{loc}}(O)$ are given by (125). Let us first find the derivative vector defined in (6), which is $(\partial \bar{x}_1/\partial \bar{y}, \partial \bar{x}_2/\partial \bar{y})$ here. Substituting (131) into the first two equations of (109), we obtain

\[
\bar{x}_1 = x_1^+ + a_{11}[\lambda^k(x_1^+ \cos k\omega - x_2^- \sin k\omega)] - \lambda^k \sin k\omega(a_{21} x_1 + a_{22} \bar{x}_2 + c_2 \bar{u} + d_2 v) + \lambda^k \sin k\omega(a_{21} \bar{x}_1 + a_{22} \bar{x}_2 + c_2 \bar{u} + d_2 v) + O(\lambda^k) + a_{12}[\lambda^k(x_1^+ \sin k\omega + x_2^- \cos k\omega)] + b_1 \gamma \mu + a_{31} x_1 + a_{32} \bar{x}_2 + b_3(\bar{y} - y^-)^2 + c_3 \bar{u} + d_3 v + O(\gamma^{-k}) - \gamma^{-k} y^- + O(\lambda^k) + d_1 \bar{v},
\]

\[
\bar{x}_2 = x_2^+ + a_{21}[\lambda^k(x_1^+ \cos k\omega - x_2^- \sin k\omega)] - \lambda^k \sin k\omega(a_{21} x_1 + a_{22} \bar{x}_2 + c_2 \bar{u} + d_2 v) + \lambda^k \sin k\omega(a_{21} \bar{x}_1 + a_{22} \bar{x}_2 + c_2 \bar{u} + d_2 v) + O(\lambda^k) + a_{22}[\lambda^k(x_1^+ \sin k\omega + x_2^- \cos k\omega)] + b_1 \gamma \mu + a_{31} x_1 + a_{32} \bar{x}_2 + b_3(\bar{y} - y^-)^2 + c_3 \bar{u} + d_3 v + O(\gamma^{-k}) - \gamma^{-k} y^- + O(\lambda^k) + d_1 \bar{v}.
\]

Thus,

\[
\frac{\partial \bar{x}_1}{\partial \bar{y}} = a_{11} b_1 \lambda^k \cos k\omega + a_{12} b_1 \lambda^k \sin k\omega + 2 b_1 b_3 \gamma \mu (\bar{y} - y^-),
\]

\[
\frac{\partial \bar{x}_2}{\partial \bar{y}} = a_{21} b_1 \lambda^k \cos k\omega + a_{22} b_1 \lambda^k \sin k\omega.
\]

Substituting $\bar{y} = y_k^{-j}$ into (135), with using (122) and (125), gives

\[
( \frac{\partial \bar{x}_1}{\partial \bar{y}}, \frac{\partial \bar{x}_2}{\partial \bar{y}} ) \bigg|_{\bar{x}_k^{-j}} = \left( \pm \sqrt{-\frac{(\alpha^* x_1^+ + \beta^* x_2^-)}{b_3}} \frac{\alpha^* b_3^2 \lambda^k}{2(\alpha^* x_1^+ + \beta^* x_2^-)} + o(\lambda^k) + o(\gamma^{-k}), O(\lambda^k) \right).
\]

To obtain the $x$-coordinates $(\bar{x}_1, \bar{x}_2)$ of $T_{1}^{\text{new}}(M_{k}^{-j})$, we substitute in the right hand sides of (134) the $\bar{x}, \bar{y}, \bar{u}$ coordinates of $M_{k}^{-j}$ (see (125)) with setting $\bar{v} = 0$, and get

\[(\bar{x}_1, \bar{x}_2) = (x_1^+ + b_1 y^- + o(1)_{k \to \infty}, x_2^+ + o(1)_{k \to \infty}). \tag{137}
\]

Since $x_2^+ \neq 0$ by (14), the two vectors (136) and (137) are not parallel for all sufficiently large $k$, and hence condition C5.1 is satisfied. Equation (137) immediately implies condition C4.

Finally, let us check condition C1. Denote

\[i = \dim(T_{1}^{\text{new}}(M_{k}^{-j})W_{\text{loc}}(O)) \cap T_{1}^{\text{new}}(M_{k}^{-j})W_{\text{loc}}(O)). \tag{138}
\]

As discussed above (6), for any tangency satisfying condition C3, the map $F$ induced from its global map satisfies $\text{rank}(\pi_\psi \circ DF|_{\bar{y}}) = d_{\text{eu}} - i = 1 - i \geq 0$. Since $T_{1}^{\text{new}}(W_{\text{loc}}^u(O))$ meets $W_{\text{loc}}^s(O)$ non-transversely at $T_{1}^{\text{new}}(M_{k}^{-j})$, we have $i \geq 1$. This means that $i$ must equal 1 as desired, completing the proof of Lemma 8.

3.7 Stabilization of the found heterodimensional cycles

In this section we prove Theorem 3 for the saddle-focus case by applying the stabilization results in [19]. For that, we need to verify that our heterodimensional cycles satisfy the genericity conditions introduced there.

3.7.1 An adjustment to the central multipliers

By Lemmas 2 and 6, our system has a heterodimensional cycle involving O and Q at \((\mu, \omega) = (\mu_k(t), \omega_k(t))\). Recall that we denote the center-unstable multipliers of Q by \(\nu_1\) and \(\nu_2\). By (75) they are functions of \(t\), and they also depend on \(k\) since Q does.

**Lemma 9.** Let \(\nu(t) = \min\{|\nu_1(t)|, |\nu_2(t)|\}\). For all sufficiently large \(k\), there exist uncountably many \(t \in (−1, 1)\) such that \(\nu_1\) and \(\nu_2\) are real, and the quantities \(\ln \lambda(t)/\ln \nu(t), \omega_k(t)/\pi\) and 1 are rationally independent.

**Proof.** Define a partial function \(g : (−1, 1) \rightarrow \mathbb{Z}^2 \setminus \{0\}\) such that \(g(t) = (i_1, i_2, i_3)\) if and only if

\[
i_1 \frac{\ln \lambda(t)}{\ln \nu(t)} + i_2 \frac{\omega_k(t)}{\pi} + i_3 = 0.
\]  

(138)

The required rational independence is equivalent to say that the domain of \(g\) can at most be a countable subset of \((-1, 1)\).

Suppose for the sake of contradiction that \(g(t)\) is defined on some uncountable subset \(\hat{S}\) of \((-1, 1)\). Then, since \(\mathbb{Z}^2 \setminus \{0\}\) is countable, there exists an uncountable subset \(S \subset \hat{S}\) such that \(g(t)\) is constant on \(S\). Note that \(S\) must contain at least one of its accumulation points, for otherwise one can construct an uncountable family of disjoint open intervals in \((-1, 1)\) contradicting that the countable set \(\mathbb{Q} \cap (-1, 1)\) is dense in \((-1, 1)\). Denote this accumulation point by \(t^*\), and let \(t_n \to t^*\) with \(t_n \in S\) and \(g(t) = (i_1, i_2, i_3)\). Then, by subtraction we have

\[
i_1 \left( \frac{\ln \lambda(t^*)}{\ln \nu(t^*)} - \frac{\ln \lambda(t_n)}{\ln \nu(t_n)} \right) + i_2 \left( \frac{\omega_k(t^*)}{\pi} - \frac{\omega_k(t_n)}{\pi} \right) = 0,
\]

which leads to a contradiction if

\[
i_1 \left( \frac{d \ln \lambda(t^*)}{dt} \ln \nu(t^*) \right) + i_2 \left( \frac{d \omega_k(t^*)}{dt} \right) \neq 0.
\]  

(139)

We finish the proof of this lemma by obtaining (139) with \(\nu_1(t^*)\) and \(\nu_2(t^*)\) being real.

Let us first show how \(\nu_1(t)\) and \(\nu_2(t)\) change as \(t\) varies. Denote \(\text{tr} := \nu_1 + \nu_2\) and \(\text{det} := \nu_1\nu_2\).

For certainty let

\[
\nu_1 = \frac{\text{tr} + \sqrt{\text{tr}^2 - 4\text{det}}}{2} \quad \text{and} \quad \nu_2 = \frac{\text{tr} - \sqrt{\text{tr}^2 - 4\text{det}}}{2}.
\]

There are two lines \(l_1 : \{\text{det} = \text{tr} - 1\}\) and \(l_2 : \{\text{det} = -\text{tr} - 1\}\) in the \((\text{tr}, \text{det})\)-plane, and \(\nu_1 = 1\) on \(l_1\) and \(\nu_2 = -1\) on \(l_2\). The values of \(\nu_1\) and \(\nu_2\) with moduli larger than 1 correspond to two connected regions \(D_1\) and \(D_2\), where \(D_1\) is the region below the lines \(l_1, l_2\) and \(\{\text{det} = -1\}\), and \(D_2\) is the region above the lines \(l_1, l_2\) and \(\{\text{det} = 1\}\). All the values inside \(D_1\), and those inside \(D_2\) and above the curve.
det = tr^2/4 correspond to real \( \nu_1 \) and \( \nu_2 \). For a detailed description, see e.g. [17, Figure 2.9]. What matters is that changing \( t \) from \(-1\) to \( 1\) corresponds to a path in \( D_1 \) or \( D_2 \) connecting two points on \( l_1 \) to \( l_2 \). Thus, there exists an interval \( I_0 \subset (-1,1) \) on which \( \nu_1 \) and \( \nu_2 \) are real, and \( d\nu/dt \neq 0 \). Up to considering \( g(t) \) on \( I_0 \) instead of \((-1,1)\), we can assume \( d\nu(t^*)/dt \neq 0 \) is uniformly bounded away from zero for all large \( k \).

On the other hand, since \( d\mu(t^*)/dt = O(\lambda^{2k}) \) by (72) and \( d\omega_k(t^*)/dt = O(k^{-1}) \) by Lemma 6, it follows from the chain rule that \( d\lambda(t^*)/dt = \partial \lambda(\mu_k(t^*),\omega_k(t^*)) / \partial t = o(1)_{k \to \infty} \). So, the derivative of \( \ln \lambda(t)/\ln \nu(t) \) at \( t = t^* \) is non-zero. In fact, this derivative is bounded away from zero for all large \( k \), since \( d\nu(t)/dt \) depends on the coefficients of the system and their values at \( \omega = \omega_k \) converge to those at \( \omega = \omega^* \) as \( k \to \infty \). Thus, using the fact \( d\omega_k(t^*)/dt = O(k^{-1}) \) again, one readily sees that the left hand side of (139) vanishes only if \( i_1 = i_2 = 0 \) leading to \( i_3 = 0 \) by (138), which is impossible since \( g(t) \) takes values from \( \mathbb{Z}^3 \setminus \{0\} \).

3.7.2 Genericity conditions for heterodimensional cycles

We begin with introducing notations. Let \( H^0 \subset W^u(O) \cap W^s(Q) \) and \( H^1 \subset W^s(O) \cap W^u(Q) \) be two heteroclinic orbits. The heterodimensional cycle is viewed as the set

\[
H := \mathcal{O}(O) \cup \mathcal{O}(Q) \cup H^0 \cup H^1,
\]

where \( \mathcal{O}(\cdot) \) denotes the orbits of a point. Take four points \( M_1^+ \in W^s_{loc}(O) \cap H^1, M_1^- \in W^u_{loc}(O) \cap H^0, M_2^+ \in W^u_{loc}(Q) \cap H^0, M_2^- \in W^s_{loc}(Q) \cap H^1 \) such that \( f^{n_0}(M_1^-) = M_2^+ \) and \( f^{n_1}(M_2^-) = M_1^+ \) for some integers \( n_0 \) and \( n_1 \), and define the transition maps \( F_{12} := f^{n_0} \) between small neighbourhoods of \( M_1^- \) and \( M_2^+ \), and \( F_{21} := f^{n_1} \) between small neighbourhoods of \( M_2^- \) and \( M_1^+ \). In Section 2 we show that there exist a strong-stable foliation \( \mathcal{F}^{ss} \) on \( W^s_{loc}(O) \) and a local extended unstable manifold \( W^uE(O) \); similarly, there exist a strong-unstable foliation \( \mathcal{F}^{uu}_{Q} \) on \( W^u_{loc}(Q) \) and a local extended stable manifold \( W^sE(Q) \) corresponding to \( \nu_1, \nu_2 \) and the stable multipliers of \( Q \). We can now state the first three genericity conditions.

HC1. the image \( F_{12}(W^s_{loc}(O)) \) intersects \( W^s_{loc}(Q) \) transversely at \( M_2^+ \), and the preimage \( F_{12}^{-1}(W^s_{loc}(Q)) \) intersects \( W^u_{loc}(O) \) transversely at the point \( M_1^- \);

HC2. the leaf of \( \mathcal{F}^{uu}_{Q} \) at the point \( M_2^- \) is not tangent to \( F_{21}^{-1}(W^s_{loc}(O)) \) and the leaf of \( \mathcal{F}^{ss} \) at the point \( M_1^+ \) is not tangent to \( F_{21}(W^u_{loc}(Q)) \); and

HC3. the heterodimensional cycle does not belong to strong manifolds, namely, \( M_1^+ \notin \mathcal{W}^{ss}(O) \) and \( M_2^- \notin \mathcal{W}^{uu}(Q) \).

First note that by the construction in the proof of Lemma 6, we can take \( M_1^- := M^- \) and the corresponding transition map is \( F_{12} := T_1 \). For the transverse intersection, one sees from the proof of Proposition 2 that, the piece \( s_k^i \) of \( W^s(O) \) intersecting \( W^u_{loc}(Q) \) is the image under \( T_0^{-k} \) of \( s^i \), which by the proof of Lemma 7 is obtained as the preimage \( T_1^{-1} \circ T_0^{-1} \circ T_1^{-1}(s) \) for some small piece of \( W^u_{loc}(O) \) near \( M^+ \). Thus, we take \( F_{21} := T_1 \circ T_0^{-1} \circ T_1 \circ T_0^{-1} \). By construction, for any \( M_2^- \) taken from \( s_k^i \cap W^u_{loc}(Q) \), the corresponding point \( M_1^+ = F_{21}(M_2^-) \) can be taken arbitrarily close to \( M^+ \) as \( i \to \infty \). In what follows we verify the conditions HC1 - HC3.

Note that \( \mathcal{F}^{ss} \) can be extended to a small neighbourhood of \( O \), so we may work within this
neighbourhood from the beginning. As a result, \( W^s_{\text{loc}}(Q) \) becomes a leaf of this foliation and the first part of condition C3 implies the first part of HC1 since \( Q \) is close to \( M^- \); similar for the second part of HC1.

The first part of HC2 follows from the fact that in the proof of Proposition 2 we intersect \( s_i^k \) by a surface consisting of strong-unstable leaves through points on the center-stable manifold \( W^{cu} \).

For the second part, it suffices to consider the leaf through \( M^+ \) since by taking \( i \) large \( M^+_i \) can be taken arbitrarily close to \( M^+ \). One observes that applying \( T_0 \) does not change in the transversality in condition C3. Hence, for the strong-stable leaf \( l^{ss} \) through \( M^+ \), we have that \( T_0^k \circ F_{21}^{-1}(l^{ss}) \) is transverse to the extended unstable manifold \( W^{uE}(O) \) at \( M^- \). On the other hand, one easily sees that by the coordinate transformations made in Section 3.1, the two manifolds \( W^{uE}(O) \) and \( W^u_{\text{loc}}(Q) \) are graphs of functions of \( x, y, v \). It then follows from the first part of condition C3 that \( F_{21}^{-1}(l^{ss}) \) intersects \( W^u_{\text{loc}}(Q) \) transversely.

The fulfillment of HC3 is obvious. Condition C4 immediately gives \( M^+ \not\in W^{ss}(O_1) \), and hence for \( M^+_i \) since it is close to \( M^+ \). We also have \( M^-_2 \not\in W^{uu}(O_2) \) for otherwise in the proof of Proposition 2 the projection of \( W^{ss}(O) \cap W^u_{\text{loc}}(Q) \) along the strong-unstable foliation will be \( Q \) and hence not intersect the curve \( l \subset W^{cs} \cap W^u_{\text{loc}}(Q) \) connecting \( Q \) and the \( Y \)-boundary.

The last genericity condition depends on the the types (i.e., real or complex) of center-stable multipliers of the less-index point, and the center-unstable multiplier of the other point. Since we will consider the \( t \) given in Lemma 9, we only state the condition for this case, for a full description see [19, Section 2].

**HC4.** in the coordinates where the map near \( O \) assumes the form (8), the \( x \)-vector component of the tangent vector \( p \) to \( W^s_{\text{loc}}(O) \cap F_{21}(W^u_{\text{loc}}(Q)) \) at the point \( M^+ \) is not parallel to the vector \( (x_1^+, x_2^+) \).

Recall \( F_{21} = T_1 \circ T_0^k \circ T_1 \circ T_0^k \). Note that \( T_0 \) gives a rotation in the \( x \)-coordinates by (8) and \( T_1 \) preserves this rotation up to a finite change by condition C3. As a result, by taking different values of \( k \), the \( x \)-vector component of the image of any vector under \( DF_{21} \) can have at least three different directions (since \( \omega \in (0, \pi) \)). On the other hand, for any fixed \( i \) and all sufficiently large \( k \), the change of \( DF_{21}^{-1}p \) is small, as can be seen in the proof of Proposition 2. As a result, the \( x \)-vector component \( p_x \) of \( p \) can have at least three different directions by taking different \( k \), and, since \( M^+_i \) is close to \( M^+ \), at least one of this direction is not parallel to \( (x_1^+, x_2^+) \).

Therefore, for all \( t \) given in Lemma 9, our heterodimensional cycles satisfy conditions HC1 - HC4. This together with the fact that those cycles unfold as \( \omega \) varies (see Lemma 6) allows us to obtain Theorem 3 directly from the following result. We say that \( \{f_\varepsilon\} \) is an unfolding family of a heterodimensional cycle of \( f_0 \) if this heterodimensional cycle splits when \( \varepsilon \) changes from 0.

**Theorem.** [19, Theorem 9] Let a \( C^r \) system \( f \) have a heterodimensional cycle \( H \) involving two hyperbolic periodic points \( O_1 \) and \( O_2 \) with \( \text{ind}(O_1) + 1 = \text{ind}(O_2) \) and let the center-stable multiplier of \( O_1 \) be \( \lambda_{1,1} = \lambda_{1,2} = \lambda e^{i\omega} \) and the center-unstable multiplier of \( O_2 \) be real \( \gamma \). Assume that conditions HC1 - HC4 are satisfied and the quantities \( \ln \lambda / \ln |\gamma|, \omega / \pi \) and 1 are rationally independent. Then, in any neighborhood of \( H \), the system \( f \) has a \( C^1 \)-robust heterodimensional cycle associated to a cu-blender and a cs-blender, activating each other. Moreover, the point \( O_2 \) is homoclinically related to the cu-blender for the system \( f \) and for any \( C^r \)-close system \( g \), and for any unfolding family \( \{f_\varepsilon\} \) there exist intervals \( I_j \) converging to \( \varepsilon = 0 \) such that if \( \varepsilon \in I_j \), then the point \( O_1 \) is homoclinically related to the
4 The bi-focus case

In this section we prove Theorem 2 for the bi-focus case. Let \( O \) be a bi-focus with central multipliers \( \lambda_1 = \lambda_2^* = \lambda e^{i\omega_1} \) (\( 0 < \omega_1 < \pi \)) and \( \gamma_1 = \gamma_2^* = \gamma e^{i\omega_2} \) (\( 0 < \omega_2 < \pi \)). We assume that \( \lambda \gamma > 1 \) and conditions \( C1 - C4 \) and \( C6.1 \) are fulfilled. The theorem will be proved through a reduction to the saddle-focus, which is achieved by introducing suitable coordinates such that the first return map has the same form as (32).

4.1 First return map and its normal form

As in the saddle-focus case, there exist \( C^r \) coordinates [14, Lemma 6] such that the local map \( T_0 \) assumes the same form as (8) with taking \( y \in \mathbb{R}^2 \) and multiplying to the linear term \( \gamma y \) a standard rotation matrix, and it satisfies (9). Then, by [14, Lemma 7], for any point \((x_1, x_2, y_1, y_2, u, v) \in \Pi^+\), we have \((\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{u}, \tilde{v}) = T_0^k(x_1, x_2, y_1, y_2, u, v) \in \Pi^- \) if and only if the points satisfy the relation

\[
\begin{align*}
\tilde{x}_1 &= \lambda^k x_1 \cos k\omega_1 - \lambda^k x_2 \sin k\omega_1 + O(\hat{\lambda}^k), \\
\tilde{x}_2 &= \lambda^k x_1 \sin k\omega_1 + \lambda^k x_2 \cos k\omega_1 + O(\hat{\lambda}^k), \\
\tilde{y}_1 &= \gamma^{-k} y_1 \cos k\omega_2 + \gamma^{-k} y_2 \sin k\omega_2 + O(\hat{\gamma}^{-k}), \\
\tilde{y}_2 &= -\gamma^{-k} y_1 \sin k\omega_2 + \gamma^{-k} y_2 \cos k\omega_2 + O(\hat{\gamma}^{-k}), \\
\tilde{u} &= O(\hat{\lambda}^k), \\
\tilde{v} &= O(\hat{\gamma}^{-k}),
\end{align*}
\]

(140)

where \( \hat{\lambda} \in (|\lambda_3|, \lambda) \) and \( \hat{\gamma} \in (\gamma, |\gamma_3|) \) are chosen to satisfy (11) and

\[
\hat{\gamma}^{-1} > \lambda \gamma^{-1}.
\]

(141)

Here the \( O(\cdot) \) terms are functions of \( x_1, x_2, \tilde{y}_1, \tilde{y}_2, u, \tilde{v} \), and their first derivatives with respect to coordinates have the same forms.

Using conditions \( C1 - C3 \), we apply [14, Corollary 1]. It says that the global map \( T_1 \) from \( \Pi^- \) to \( \Pi^+ \) satisfy that for any point \((\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{u}, \tilde{v}) \in \Pi^- \), we have \((x_1, x_2, y_1, y_2, u, v) = T_1(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{u}, \tilde{v}) \) if and only if

\[
\begin{align*}
x_1 - x_1^+ &= a_{11} \tilde{x}_1 + a_{12} \tilde{x}_2 + b_{11} (\tilde{y}_1 - y_1^-) + b_{12} y_2 + c_1 \tilde{u} + d_1 v + \ldots, \\
x_2 - x_2^+ &= a_{21} \tilde{x}_1 + a_{22} \tilde{x}_2 + b_{22} y_2 + c_2 \tilde{u} + d_2 v + \ldots, \\
y_1 &= \mu + a_{31} \tilde{x}_1 + a_{32} \tilde{x}_2 + b_{31} (\tilde{y}_1 - y_1^-)^2 + c_3 \tilde{u} + d_3 v + \ldots, \\
u - u^+ &= a_{41} \tilde{x}_1 + a_{42} \tilde{x}_2 + b_{41} (\tilde{y}_1 - y_1^-) + b_{42} y_2 + c_4 \tilde{u} + d_4 v + \ldots, \\
\tilde{y}_2 - y_2^- &= a_{51} \tilde{x}_1 + a_{52} \tilde{x}_2 + b_{51} (\tilde{y}_1 - y_1^-) + b_{52} y_2 + c_5 \tilde{u} + d_5 v + \ldots, \\
\tilde{v} - v^- &= a_{61} \tilde{x}_1 + a_{62} \tilde{x}_2 + b_{61} (\tilde{y}_1 - y_1^-) + b_{62} y_2 + c_6 \tilde{u} + d_6 v + \ldots,
\end{align*}
\]

(142)

where \( b_{11} \neq 0, b_{31} \neq 0, a_{31}^2 + a_{32}^2 \neq 0, b_{52} \neq 0, \) det \(d_6 \neq 0 \), and the dots denote the second and higher
order terms (except that no $(y_1 - y_1)^2$-term in the dots of the third equation). Condition C3 implies

$$(x_1^+)^2 + (x_2^+)^2 \neq 0 \quad \text{and} \quad (y_1^-)^2 + (y_2^-)^2 \neq 0. \quad (143)$$

Here the derivative vector in (7) is $b = (b_{11}, 0)$, so Condition C5.2 means

$$x_2^+ \neq 0. \quad (144)$$

Denote by $e_1$ and $e_2$ the coefficients in front of the terms $\tilde{x}_1(\tilde{y} - y^-)$ and $\tilde{x}_2(\tilde{y} - y^-)$ in the dots of the third equation of (142). We impose the last genericity condition for the bi-focus case as

**C6.2.** At least one of the following inequalities holds:

$$a_{11} \neq \frac{b_2^2(x_1^+ a_{31} - x_2^+ a_{32})}{2b_3((x_1^+)^2 + (x_2^+)^2)} + \frac{x_1^+}{x_2} a_{21}, \quad a_{12} \neq \frac{b_2^2(x_2^+ a_{31} + x_1^+ a_{32})}{2b_3((x_1^+)^2 + (x_2^+)^2)} + \frac{x_1^+}{x_2} a_{22}, \quad e_1 x_1^+ + e_2 x_2^+ \neq 0 (145)$$

The first return map $T_k := T_1 \circ T_0^k$ is such that $(\bar{x}, \bar{y}, \bar{u}, \bar{v}) = T_k(x, y, u, v)$ if and only if there is $(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \in \Pi^-$ such that

$$(x, y, u, v) \overset{T_0^k}{\mapsto} (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \overset{T_1}{\mapsto} (\bar{x}, \bar{y}, \bar{u}, \bar{v}).$$

As in the saddle-focus case, after the coordinate transformation $(x, y, u, v) \mapsto (x, \tilde{y}, u, \tilde{v})$, the map $T_k : (x, \tilde{y}, u, \tilde{v}) \mapsto (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ is defined on

$$(x - x^+, \tilde{y} - y^-, u - u^+, \tilde{v} - v^-) \in [-\delta, \delta]^2 \times [-\delta, \delta]^2 \times [-\delta, \delta]^d \times [-\delta, \delta]^d =: \hat{\Pi}.$$

Let

$$X = x - x^+, \quad Y = \tilde{y} - y^-, \quad U = u - u^+, \quad V = \tilde{v} - v^-.$$ 

Using the implicit function theorem, we combine (140) and (142), and get that $(X_1, X_2, Y_1, Y_2, U, V) = T_k(X_1, X_2, Y_1, Y_2, U, V)$ if and only if

\begin{align*}
X_1 &= \lambda^k \alpha_1(X_1 + x_1^+) + \lambda^k \beta_1(X_2 + x_2^+) + b_{11}Y_1 + \hat{h}_1(X_1, X_2, Y_1, U, Y_2, \bar{V}), \\
X_2 &= \lambda^k \alpha_2(X_1 + x_1^+) + \lambda^k \beta_2(X_2 + x_2^+) + \hat{h}_2(X_1, X_2, Y_1, U, \bar{Y}_2, \bar{V}), \\
Y_1 \cos k\omega_2 + \bar{Y}_2 \sin k\omega_2 &= \gamma^k \mu - y^* + \lambda^k \gamma^k \alpha^*(X_1 + x_1^+) + \lambda^k \gamma^k \beta^*(X_2 + x_2^+) + b_{31} \gamma^k \gamma Y_1^2 \\
&\quad + \gamma^k \hat{h}_3(X_1, X_2, Y_1, U, \bar{Y}_2, \bar{V}), \quad (147) \\
U &= \lambda^k \alpha_4(X_1 + x_1^+) + \lambda^k \beta_4(X_2 + x_2^+) + b_{41}Y_1 + \hat{h}_4(X_1, X_2, Y_1, U, \bar{Y}_2, \bar{V}), \\
Y_2 &= \lambda^k \alpha_5(X_1 + x_1^+) + \lambda^k \beta_5(X_2 + x_2^+) + b_{51}Y_1 + \hat{h}_5(X_1, X_2, Y_1, U, \bar{Y}_2, \bar{V}), \\
V &= \lambda^k \alpha_6(X_1 + x_1^+) + \lambda^k \beta_6(X_2 + x_2^+) + b_{61}Y_1 + \hat{h}_6(X_1, X_2, Y_1, U, \bar{Y}_2, \bar{V}),
\end{align*}

where

\begin{equation}
\alpha^* = a_{31} \cos k\omega_1 + a_{32} \sin k\omega_1, \quad \beta^* = -a_{31} \sin k\omega_1 + a_{32} \cos k\omega_1, \\
\alpha_i = a_{i1} \cos k\omega_1 + a_{i2} \sin k\omega_1, \quad \beta_i = -a_{i1} \sin k\omega_1 + a_{i2} \cos k\omega_1, \quad (i = 1, 2, 4, 5, 6) \quad (148)
\end{equation}

\begin{align*}
y^* &= y_1^* \cos k\omega_2 + y_2^* \sin k\omega_2,
\end{align*}

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Here with the assumption $|\lambda\gamma| > 1$, and (11) and (141), the functions $\tilde{h}$ satisfy

$$
\hat{h}_i = O(\hat{\lambda}^k) + O(\lambda^k Y_i) + O(Y_i^2), \quad (i = 1, 2, 4, 5, 6),
\hat{h}_3 = O(\hat{\lambda}^k) + O(\lambda^k Y_1) + O(Y_1^3),
$$

$$
\frac{\partial \hat{h}}{\partial (X_1, X_2, U)} = O(\hat{\lambda}^k) + O(\lambda^k Y_1), \quad \frac{\partial \hat{h}_i}{\partial (Y_i, V)} = O(\gamma^{-k}), \quad \frac{\partial \hat{h}_3}{\partial (Y_1, V)} = O(\lambda^k) + O(Y_1),
$$

(149)

We will further consider $k$ such that

$$
\cos k\omega_2, \quad \cos k\omega_2 + b_{51} \sin k\omega_2 \quad \text{are bounded away from zero. (150)}
$$

This can be always achieved since $\omega_2 \in (0, \pi)$ implies $k\omega_2$ take at least three different values, and if the first quantity vanishes at some $k$ then there must be $k$ such that both quantities are bounded from zero. Now set

$$
Y = Y_1 \cos k\omega_2 + Y_2 \sin k\omega_2.
$$

(151)

Solving out $Y_2$ from the fifth equation in (147) as a function of $X_1, X_2, Y, U, \tilde{Y}, \tilde{V}$

$$
Y_2 = \frac{1}{1 + b_{51} \tan k\omega_2} \left( \lambda^k \tilde{\alpha}_5 (X_1 + x_1^+) + \lambda^k \tilde{\beta}_5 (X_2 + x_2^+) + b_{51} Y \cos^{-1} k\omega_2 \right) + \tilde{h}_5,
$$

and substituting it with (151) into the remaining equations, yields

$$
\tilde{X}_1 = \lambda^k \tilde{\alpha}_1 (X_1 + x_1^+) + \lambda^k \tilde{\beta}_1 (X_2 + x_2^+) + \tilde{b}_{11} Y + \tilde{h}_1 (X_1, X_2, Y, U, \tilde{Y}, \tilde{V}),
\tilde{X}_2 = \lambda^k \tilde{\alpha}_2 (X_1 + x_1^+) + \lambda^k \tilde{\beta}_2 (X_2 + x_2^+) + \tilde{h}_2 (X_1, X_2, Y, U, \tilde{Y}, \tilde{V}),
\tilde{Y} = \gamma^k \mu - y^* + \lambda^k \gamma^k \alpha^* (X_1 + x_1^+) + \lambda^k \gamma^k \beta^* (X_2 + x_2^+) + \tilde{b}_{31} \gamma^k Y^2
\quad + \gamma^k \tilde{h}_3 (X_1, X_2, Y, U, \tilde{Y}, \tilde{V}),
\tilde{U} = \lambda^k \tilde{\alpha}_4 (X_1 + x_1^+) + \lambda^k \tilde{\beta}_4 (X_2 + x_2^+) + \tilde{b}_{41} Y + \tilde{h}_4 (X_1, X_2, Y, U, \tilde{Y}, \tilde{V}),
\tilde{Y}_2 = \lambda^k \tilde{\alpha}_5 (X_1 + x_1^+) + \lambda^k \tilde{\beta}_5 (X_2 + x_2^+) + \tilde{b}_{51} Y + \tilde{h}_5 (X_1, X_2, Y, U, \tilde{Y}, \tilde{V}),
\tilde{V} = \lambda^k \tilde{\alpha}_6 (X_1 + x_1^+) + \lambda^k \tilde{\beta}_6 (X_2 + x_2^+) + \tilde{b}_{61} Y + \tilde{h}_6 (X_1, X_2, Y, U, \tilde{Y}, \tilde{V}),
$$

(152)

where

$$
\tilde{b}_{51} = \frac{b_{51}}{\cos k\omega_2 + b_{51} \sin k\omega_2}, \quad \tilde{\alpha}_5 = \frac{\tilde{b}_{51} \cos k\omega_2}{\cos k\omega_2 + b_{51} \sin k\omega_2}, \quad \tilde{\beta}_5 = \frac{\tilde{b}_{51} \cos k\omega_2}{\cos k\omega_2 + b_{51} \sin k\omega_2},
\tilde{b}_{11} = \frac{b_{11}(1 - \tilde{b}_{51} \sin k\omega_2)}{\cos k\omega_2}, \quad \tilde{\alpha}_j = \frac{\tilde{b}_{11} \tilde{\alpha}_5 \sin k\omega_2}{\cos k\omega_2 + b_{51} \sin k\omega_2}, \quad \tilde{\beta}_j = \tilde{\beta}_j = \frac{\tilde{b}_{11} \tilde{\beta}_5 \sin k\omega_2}{\cos k\omega_2 + b_{51} \sin k\omega_2},
$$

(153)

for $i = 1, 4, 6$ and $j = 1, 2, 4, 6$, and functions $\tilde{h}$ satisfy the same estimates as $\hat{h}$, with replacing $Y_1$ by $Y$.

One readily finds that, after denoting $V^{\text{new}} = (Y_2, V)$ in formula (152), it assumes the same form as (18). We can then apply a series of coordinate transformations similar to (22), (26) and (31) to
rewrite (152) as
\begin{equation}
\begin{aligned}
\tilde{Z} &= \lambda^k \alpha_0 Z + b_1 \alpha^* Y + \lambda^k \beta_1 W + h_1(Z, Y, W, \tilde{V}), \\
\dot{Y} &= L + \lambda^k \gamma Z + b_3 \gamma^2 Y^2 + \gamma^k h_2(Z, Y, W, \tilde{V}), \\
\ddot{W} &= \lambda^k \alpha_3 Z + \lambda^k \beta_3 W + h_3(Z, Y, W, \tilde{V}), \\
V &= \lambda^k \alpha_4 Z + \lambda^k \beta_4 W + h_4(Z, Y, W, V),
\end{aligned}
\end{equation}

where \((Z, Y, W, V) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}\), the coefficients \(\alpha\) and \(\beta\), as in the saddle-focus case (33), are linear combinations of the ones in (152) up to corrections of order \(o(1/k)\), and
\[L = \gamma^k \mu_y + \lambda \gamma^k (\alpha^* x_1 + \beta^* x_2) + O(\lambda \gamma^k),\]
and the functions \(h\) satisfy
\begin{equation}
\begin{aligned}
\frac{\partial h_i}{\partial h_2} &= O(\lambda^k) + O(\lambda^k Y), & \frac{\partial h_i}{\partial Y} &= O(\lambda^k) + O(Y), & \frac{\partial h_i}{\partial V} &= O(\gamma^{-k}), \\
\frac{\partial h_1}{\partial h_2} &= O(\lambda^k) + O(Y^2), & \frac{\partial h_1}{\partial Y} &= O(\lambda^k) + O(Y^2), & \frac{\partial h_1}{\partial V} &= O(\gamma^{-k}) + O(\gamma^{-k}).\end{aligned}
\end{equation}

4.2 Reduction to the saddle-focus case

Recall that proofs of all the results for the saddle-focus case, except Lemmas 7 and 8, are based on the analysis of the first return map (32) using estimates (35) with \(\partial h/\partial \bar{V}\) replaced by (68), which take exactly the same forms as (154) and (155). The conditions used in these proofs are just \((x_1^2) + (x_2^2) \neq 0\) and (14), which also hold here by (143) with (144). So, these results also hold for the bi-focus case with the only difference that \(k\) now needs to be chosen such that (150) is satisfied. It follows that Theorem 2 for the bi-focus case will be proved if we can show that the transverse homoclinic points \(\{N_i\}\) used in Proposition 2 also exist in the bi-focus case. In fact, we prove a stronger version of Lemma 7, which no longer has the sign condition \(b_3 y^- > 0\).

**Lemma 10.** Let \(\omega/2\pi\) be either irrational or have the form \(p/q\) for some coprime \(p, q\) with \(q \geq 7\). If conditions \(C1 - C4\) are satisfied, then there exist sequences \(\{n_i \in \mathbb{N}\} \to \infty\) and \(\{N_i = (0, y_i, 0, v_i)\}\) of transverse intersection points of \(W^u_{loc}(O)\) with \(T_{-1} \circ T_{-n_i} \circ T_{-1}(W^s_{loc}(O))\) accumulating on \(M^- = (0, y^-, 0, v^-)\). Moreover, denoting \(y_i = (y_1^i, y_2^i)\), we have \((y_1^i - y_1^-) \cos k\omega_2 + (y_2 - y_2^-) \sin k\omega_2 \to 0\) from both sides with \(y_i - y^- = O(\gamma^{-\frac{\omega}{2\pi}})\).

**Proof.** We follow the exact procedure in the proof of Lemma 7, and the implicit function theorem will be used in the same way as before without further explanations. The focus is on finding and solving a system of equations whose solutions correspond to the desired homoclinic points, which is a counterpart to the one consisting of (114) and (115).

Setting \(l_1 = \bar{y}_1 - y_1^-\), \(l_2 = \bar{y}_2 - y_2^-\), \(l_3 = \bar{v} - v^-\), one finds from (142) that the image \(T_1(W^u_{loc}(O)) \cap \Pi^-\) is given by
\begin{equation}
\begin{aligned}
x_1 - x_1^+ &= b_1 l_1 + O(|v| + |v + l_1^2|), & x_2 - x_2^+ &= O(|v| + y_2^2 + l_2^2), \\
y_1 &= \mu + b_3 l_1^2 + O(l_1^2) + O(|v| + |v l_1| + |y_2 l_1| + y_2^2), \\
\bar{u} - v^+ &= b_4 l_1 + O(|y_2| + |v + l_1^2|), \\
l_2 &= b_5 l_1 + O(|y_2| + |v + l_1^2|), & l_3 &= b_6 l_1 + O(|y_2| + |v + l_1^2|).
\end{aligned}
\end{equation}

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Expressing $l_1$ as a function of $x_1$ from the first equation, we obtain
\begin{equation}
    l_1 = b_{11}^{-1}(x_1 - x_1^+) + O(|y_2| + |v| + (x_1 - x_1^+)^2).
\end{equation}
(157)

Substituting the last two equations of (142), with setting $\tilde{y} = \tilde{v} = 0$, into the $y$- and $v$-equations in (140), yields $(y_2, v) = O(\gamma^{-k})$ as a function of $x_1, x_2, y_1, u$. Combining this with (157) and the $y_1$- and $l_2$-equations in (156), yields
\begin{align}
    y_1 &= \mu + \frac{b_{31}}{b_{11}}(x_1 - x_1^+)^2 + O((x_1 - x_1^+)^2) + O(\gamma^{-k}), \\
    l_1 &= b_{11}^{-1}(x_1 - x_1^+) + O((x_1 - x_1^+)^2) + O(\gamma^{-k}), \\
    l_2 &= b_{11}^{-1}b_{31}(x_1 - x_1^+) + O((x_1 - x_1^+)^2) + O(\gamma^{-k}).
\end{align}
(158)

Applying coordinate transformations (146) and (151), with using the third equation in (140), the first equation in (158) assumes the form
\begin{equation}
    0 = \mu - \gamma^{-k}y^* - \gamma^{-k}Y + \frac{b_{31}}{b_{11}}X^2 + O(X^3) + o(\gamma^{-k}),
\end{equation}
(159)
where $y^*$ is given by (148).

We find the image of $T_1(W^{u}_{\text{loc}}(O))$ under $T_1 \circ T_0^k$ by combining (140) and (142), with using (156) and (157). Equivalently, one can just use the formula (152) and then undo the coordinate transformation $(\tilde{y}, \tilde{v}) \mapsto (\tilde{Y}, \tilde{V})$. Particularly, we obtain
\begin{equation}
    \tilde{y}_1 = \mu + \lambda^k\alpha^*x_1^+ + \lambda^k\beta^*x_2^+ + \lambda^k\alpha^*X_1 + \tilde{b}_{31}Y^2 + o(\lambda^k) + O(Y^3),
\end{equation}
(160)
where the $o(\lambda^k)$ term is a function of $X_1, Y, \tilde{y}_2, \tilde{v}$.

Transverse homoclinic points correspond to non-degenerate solutions to the system consisting of (159) and (160) with setting $\tilde{y}_1 = \tilde{y}_2 = \tilde{v} = 0$. Comparing these two equations with (114) and (115), one sees that the remaining computation can be done as in the saddle-focus case if $y^*b_{31} > 0$ when (150) is satisfied. But this is easy when $\omega_2/2\pi$ is irrational or of the form $p/q$ with $q \geq 7$. To see the latter, one notes that $k\omega \text{ mod } \pi$ can take $\lfloor (q - 1)/2 \rfloor$ different values, and it suffices to have four different values to make $y^*b_{31} > 0$ and (150) hold at the same time.

The last thing to notice is that the desired convergence in the $y$-coordinates is obtained by substituting the $X$-solutions corresponding to (116) into the last two equations in (158).

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