Lie algebras

**Action of Weyl group on zero-weight space**

Action du groupe de Weyl sur l'espace de poids nul

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**A R T I C L E   I N F O**

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**A B S T R A C T**

For any simple complex Lie group, we classify irreducible finite-dimensional representations $\rho$ for which the longest element $w_0$ of the Weyl group acts non-trivially on the zero-weight space. Among irreducible representations that have zero among their weights, $w_0$ acts by $\pm \text{Id}$ if and only if the highest weight of $\rho$ is a multiple of a fundamental weight, with a coefficient less than a bound that depends on the group and on the fundamental weight.

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**R É S U M É**

Pour tout groupe de Lie complexe simple, nous classifions les représentations irréductibles $\rho$ de dimension finie telles que le plus long mot $w_0$ du groupe de Weyl agisse non trivialement sur l'espace de poids nul. Parmi les représentations irréductibles dont zéro est un poids, $w_0$ agit par $\pm \text{Id}$ si et seulement si le plus haut poids de $\rho$ est un multiple d'un poids fondamental, avec un coefficient plus petit qu'une borne qui dépend du groupe et du poids fondamental.

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1. Introduction and main theorem

Consider a reductive complex Lie algebra $\mathfrak{g}$. Let $\tilde{G}$ be the corresponding simply-connected Lie group.

We choose in $\mathfrak{g}$ a Cartan subalgebra $\mathfrak{h}$. Let $\Delta$ be the set of roots of $\mathfrak{g}$ in $\mathfrak{h}^*$. We call $\Lambda$ the root lattice, i.e. the abelian subgroup of $\mathfrak{h}^*$ generated by $\Delta$. We choose in $\Delta$ a system $\Delta^+$ of positive roots; let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots in $\Delta^+$. Let $\sigma_1, \ldots, \sigma_r$ be the corresponding fundamental weights. Let $W := N_\tilde{G}(\mathfrak{h})/Z_\tilde{G}(\mathfrak{h})$ be the Weyl group, and let $w_0$ be its longest element (defined by $w_0(\Delta^+) = -\Delta^+$).

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For each simple Lie algebra, we call \((e_1, e_2, \ldots)\) the vectors called \((e_1, e_2, \ldots)\) in the appendix to [2], which form a convenient basis of a vector space containing \(h^*\). Throughout the paper, we use the Bourbaki conventions [2] for the numbering of simple roots and their expressions in the coordinates \(e_i\).

In the sequel, all representations are supposed to be complex and finite-dimensional. We call \(\rho_\lambda\) (resp. \(V_\lambda\)) the irreducible representation of \(g\) with highest weight \(\lambda\) (resp. the space on which it acts). Given a representation \((\rho, V)\) of \(g\), we call \(V^\lambda\) the weight subspace of \(V\) corresponding to the weight \(\lambda\).

**Definition 1.1.** We say that a weight \(\lambda \in h^*\) is radical if \(\lambda \in \Lambda\).

**Remark 1.** An irreducible representation \((\rho, V)\) has non-trivial zero-weight space \(V^0\) if and only if its highest weight is radical.

**Definition 1.2.** Let \((\rho, V)\) be a representation of \(g\). The action of \(W = N_G(h)/Z_G(h)\) on \(V^0\) is well-defined, since \(V^0\) is by definition fixed by \(h\), hence by \(Z_G(h)\). Thus \(w_0\) induces a linear involution on \(V^0\). Let \(p\) (resp. \(q\)) be the dimension of the subspace of \(V^0\) fixed by \(w_0\) (resp. by \(-w_0\)). We say that \((p, q)\) is the \(w_0\)-signature of the representation \(\rho\) and that the representation is:

- \(w_0\)-pure if \(pq = 0\) (of sign +1 if \(q = 0\) and of sign −1 if \(p = 0\));
- \(w_0\)-mixed if \(pq > 0\).

**Remark 2.** Replacing \(\tilde{G}\) by any other connected group \(G\) with Lie algebra \(g\) (with a well-defined action on \(V\)) does not change the definition. Indeed the center of \(\tilde{G}\) is contained in \(Z_G(h)\), so acts trivially on \(V^0\).

Our interest in this property originates in the study of free affine groups acting properly discontinuously (see [7]). We prove the following complete classification. To the best of our knowledge, this specific question has not been studied before; see [4] for a survey of prior work on related, but distinct, questions about the action of the Weyl group on the zero-weight space.

**Theorem 1.3.** Let \(g\) be any simple complex Lie algebra; let \(r\) be its rank. For every index \(1 \leq i \leq r\), we denote by \(p_i\) the smallest positive integer such that \(p_i\sigma_i \in \Lambda\). For every such \(i\), let the “maximal value” \(m_i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\) and the “sign” \(\sigma_i \in \{\pm 1\}\) be as given in Table 1 on page 854.

Let \(\lambda\) be a dominant weight.

(i) If \(\lambda \neq \Lambda\), then the \(w_0\)-signature of the representation \(\rho_\lambda\) is \((0, 0)\).

(ii) If \(\lambda = kp_i\sigma_i\) for some \(1 \leq i \leq r\) and \(0 \leq k \leq m_i\), then \(\rho_\lambda\) is \(w_0\)-pure of sign \((\sigma_i)^k\).

(iii) Finally, if \(\lambda \in \Lambda\) but is not of the form \(\lambda = kp_i\sigma_i\) for any \(1 \leq i \leq r\) and \(0 \leq k \leq m_i\), then \(\rho_\lambda\) is \(w_0\)-mixed.

**Example 1.** Any irreducible representation of \(SL(2, \mathbb{C})\) is isomorphic to \(S^k\mathbb{C}^2\) (the \(k\)-th symmetric power of the standard representation) for some \(k \in \mathbb{Z}_{\geq 0}\). Its \(w_0\)-signature is \((0, 0)\) if \(k\) is odd, \((1, 0)\) if \(k\) is divisible by 4 and \((0, 1)\) if \(k = 2\) modulo 4. This confirms the \(A_1\) entries \((p_1, m_1, \sigma_1) = (2, \infty, -1)\) of Table 1.

Table 1 also gives the values of \(p_i\). These are not a new result; they are immediate to compute from the known descriptions of the simple roots and fundamental weights (given e.g. in [2]).

Point (i) is an immediate consequence of Remark 1.

For point (ii), we show in Section 3 that certain symmetric and antisymmetric powers of defining representations of classical groups are \(w_0\)-pure, and that almost all representations listed in point (ii) are sub-representations of these powers. The finitely many exceptions are treated by an algorithm described in Section 2.

For point (iii), we prove in Section 4 that the set of highest weights of \(w_0\)-mixed representations of a given group is an ideal of the monoid of dominant weights. For any fixed group, this reduces the problem to checking \(w_0\)-mixedness of finitely many representations. In Section 5, we immediately conclude for exceptional groups and for low-rank classical groups by the algorithm of Section 2; we proceed by induction on rank for the remaining classical groups.

2. An algorithm to compute explicitly the \(w_0\)-signature of a given representation

**Proposition 2.1.** Any simple complex Lie group \(G\) admits a reductive subgroup \(S\) whose Lie algebra is isomorphic to \(sl(2, \mathbb{C})^t \times \mathbb{C}^t\), where \((t, s)\) is the \(w_0\)-signature of the adjoint representation of \(G\), and whose \(w_0\) element is compatible with that of \(G\), in the sense that some representative of the \(w_0\) element of \(S\) is a representative of the \(w_0\) element of \(G\). This subgroup \(S\) can be explicitly described.

Note that \(s + t = r\) (the rank of \(G\)) and that \(t = 0\) except for \(A_n\) (\(t = \lceil \frac{n}{2} \rceil\)), \(D_{2n+1}\) (\(t = 1\)) and \(E_6\) (\(t = 2\)).
Table 1
Values of $(p_i, m_i, \sigma_i)$ for simple Lie algebras. Theorem 1.3 states that among irreducible representations with a highest weight $\lambda$ that is radical, only those with $\lambda$ of the form $kp_i m_i$, with $k \leq m_i$, are $w_0$-pure, with a sign given by $\sigma_i$. We write N.A. for $\sigma_i$ sign entries that are not defined due to $m_i = 0$. Since $A_1 \cong B_1 \cong C_1$ and $B_2 \cong C_2$ and $A_3 \cong D_3$, the results match up to reordering simple roots (namely reordering $i = 1, \ldots, r$).

| $A_{r \geq 1}$ | | | |
|-----------------|-----------------|-----------------|-----------------|
| $i = 1$ or $r$  | $r + 1$ | $\infty$ | $(-1)^{r+1/2}$ |
| $1 < i < r$     | $r = 3$ | $r = 4$ | $0$ |
| $r > 3$         | $r = 1$ | $-1$ | |
| $r = 1$         | $r = 2$ | $2$ | $1$ |
| $r > 2$         | $r = 3$ | $1$ | $1$ |
| $i$ odd $> 2$   | $i = 3$ | $1$ | $1$ |
| $r > 3$         | $i = 4$ | $2$ | $1$ |
| $i$ even $> 2$  | $i = 4$ | $2$ | $1$ |

| $B_{r \geq 1}$ | | | |
|-----------------|-----------------|-----------------|-----------------|
| $i = 1$         | $r > 1$ | $\infty$ | $(-1)^{r-\frac{1}{2}}$ |
| $i = 2$         | $r = 2$ | $r = 2$ | $2$ |
| $i < r$         | $i = 2$ | $r = 2$ | $2$ |
| $i = r$         | $r = 1.2$ | $2$ | $1$ |
| $i = r$         | $r > 2$ | $1$ | $0$ |
| $i$ odd $> 2$   | $i = 3$ | $2$ | $1$ |
| $r = 3$         | $i = 4$ | $1$ | $1$ |
| $i$ even $> 2$  | $i = 4$ | $2$ | $1$ |

| $C_{r \geq 1}$ | | | |
|-----------------|-----------------|-----------------|-----------------|
| $i = 1$         | $r = 2$ | $2$ | $\infty$ |
| $i = 2$         | $r = 2$ | $2$ | $1$ |
| $i$ odd $> 2$   | $i = 3$ | $2$ | $0$ |
| $r = 3$         | $i = 4$ | $1$ | $1$ |
| $i$ even $> 2$  | $i = 4$ | $2$ | $1$ |

| $D_{1 \geq 3}$ | | | |
|-----------------|-----------------|-----------------|-----------------|
| $r$ odd         | $i = 1$         | $2$ | $\infty$ |
| $i = 1$         | $r - 1$ | $1$ | $0$ |
| $i$ even $> 2$  | $i = 2$ | $2$ | $\infty$ |
| $r = 4$         | $1$ | $0$ | N.A. |
| $i$ odd $> 2$   | $i = 2$ | $2$ | $1$ |
| $r = 4$         | $i = 3$ | $1$ | $1$ |
| $i$ even $> 2$  | $i = 3$ | $2$ | $1$ |

| $D_{1 \geq 4}$ | | | |
|-----------------|-----------------|-----------------|-----------------|
| $r$ even        | $i = 1$         | $2$ | $\infty$ |
| $i = 1$         | $r - 1$ | $1$ | $0$ |
| $i$ even $> 2$  | $i = 2$ | $2$ | $\infty$ |
| $r = 4$         | $1$ | $0$ | N.A. |
| $i$ odd $> 2$   | $i = 2$ | $2$ | $1$ |
| $r = 4$         | $i = 3$ | $1$ | $1$ |
| $i$ even $> 2$  | $i = 3$ | $2$ | $1$ |

Proof. Let $(h^*)^{-w_0}$ be the $-1$ eigenspace of $w_0$. Recall that two roots $\alpha$ and $\beta$ are called strongly orthogonal if $\langle \alpha, \beta \rangle = 0$ and neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. Table 2 exhibits pairwise strongly orthogonal roots $\{\alpha_1, \ldots, \alpha_4\} \subseteq \Delta$ spanning $(h^*)^{-w_0}$ as a vector space. (Our sets are conjugate to those of [1], but these authors did not need the elements $w_0$ to match.) We then set

$$s := h \oplus \bigoplus_{i=1}^{r} (g^{\alpha_i} \oplus g^{-\alpha_i}),$$

where $g^{\alpha}$ denotes the root space corresponding to $\alpha$. This is a Lie subalgebra of $g$, as follows from $[g^{\alpha}, g^{\beta}] \subseteq g^{\alpha + \beta}$ and from strong orthogonality of the $\alpha_i$. It is isomorphic to $\mathfrak{sl}(2, \mathbb{C})^r \times \mathbb{C}^r$, because it has Cartan subalgebra $h$ of dimension $r = s + t$ and a root system of type $A_r$. We define $S$ to be the connected subgroup of $G$ with algebra $s$.

Let $\overline{\sigma} := \exp \left( \frac{X_{\alpha_1} - Y_{\alpha_1}}{2} \right) \in S$, where for every $\alpha$, $X_{\alpha}$ and $Y_{\alpha}$ denote the elements of $g$ introduced in [3, Theorem 719]. We claim that $\overline{\sigma} := \prod_{i} \overline{\sigma_i}$ is a representative of the $w_0$ element of $S$ and of the $w_0$ element of $G$. By [3, Proposition 11.35], $\overline{\sigma_i}$ is a representative of the reflection $s_{\alpha_i}$, which shows the first statement. Now since the $\alpha_i$ are orthogonal, the product of $s_{\alpha_i}$ acts by $-id$ on their span $(h^*)^{-w_0}$ and acts trivially on its orthogonal complement, like $w_0$.

Then the $w_0$-signature of any representation $\rho$ of $G$ is equal to that of its restriction $\rho | _S$ to $S$. We use branching rules to decompose $\rho | _S = \bigoplus \rho_i$ into irreducible representations of $S$. The total $w_0$-signature is then the sum of those of the $\rho_i$.
Each $\rho_i$ is a tensor product $\rho_{i,1} \otimes \cdots \otimes \rho_{i,s} \otimes \rho_{i,Ab}$, where $\rho_{i,j}$ for $1 \leq j \leq s$ is an irreducible representation of the factor $s_j \simeq sl(2, \mathbb{C})$, and $\rho_{i,Ab}$ is an irreducible representation of the abelian factor isomorphic to $\mathbb{C}^n$. The $w_0$-signature of $\rho_i$ is then the “product” of those of these factors, according to the rule $(p, q) \otimes (p', q') = (pp' + qq', pp' - qp')$. The $w_0$-signatures of all irreducible representations of $sl(2, \mathbb{C})$ have been described in Example 1: the $w_0$-signature of $\rho_{i,Ab}$ is just $(1, 0)$ if the representation is trivial and $(0, 0)$ otherwise.

Branching rules are provided by several software packages. We implemented our algorithm separately in LiE [10] and in Sage [8]. In Sage, we used the Branching Rules module [9], largely written by Daniel Bump.

3. Proof of (ii): that some representations are $w_0$-pure

We must prove that representations of highest weight $\lambda = kp_1 \sigma_1, k \leq m_i$ are $w_0$-pure of sign $\sigma_k^3$ (with data $p_i, m_i, \sigma_i$ given in Table 1). We denote by $\square$ the defining representation of each classical group $(C^{n+1} \text{ for } A_n, C^{2n+1} \text{ for } B_n, C^{2n} \text{ for } C_n$ and $D_n)$, and introduce a basis of it: for every $k \in \{-1, 0, 1\}$ and $i$ such that $\epsilon_{ij}$ (or for $A_n$ its orthogonal projection onto $h^+$) is a weight of $\square$, we call $h_{ij}$ some nonzero vector in the corresponding weight space.

For exceptional groups, all $m_i$ are finite, so the algorithm of Section 2 suffices; we also use it for the representations with highest weight $2\sigma_3$ of $C_3$ and $2\sigma_4$ of $C_4$.

Most other cases are subrepresentations of $S^m\square$ of $A_n$ or $D_{2n+1}$, or of $S^m\square$ or $A^m\square$ or $S^2(\Lambda^2\square)$ of $B_n$ or $C_n$ or $D_{2n}$, all of which will prove to be $w_0$-pure. Here $S^m\rho$ and $A^m\rho$ denote the symmetric and the antisymmetric tensor powers of a representation $\rho$. The remaining cases are mapped to these by the isomorphisms $B_2 \simeq C_2$ and $A_3 \simeq D_3$ and the outer automorphisms $Z/2Z$ of $A_n$ and $Z_3$ of $D_4$.

For $A_n = sl(n + 1, \mathbb{C})$, the defining representation is $\square = C^{n+1} = \text{Span}(h_1, \ldots, h_{n+1})$. A representative $\overline{w_0} \in SL(n + 1, \mathbb{C})$ of $w_0$ acts on $\square$ by $h_j \mapsto h_{j-n+j}$ for $1 \leq j < n + 1$ and by $h_{n+1} \mapsto \sigma_1 h_1$ where $\sigma_1 = (-1)^{(n+1)/2}$, the sign being such that det $\overline{w_0} = -1$. We consider the representation $S^{m(1+1)}\square$. Its zero-weight space $V^0$ is spanned by symmetrized tensor products $h_{j_1} \otimes \cdots \otimes h_{j_{n+1}}$, in which each $h_1$ appears equally many times, namely $k$ times. Hence, $V^0$ is one-dimensional (the representation is thus $w_0$-pure) and spanned by the symmetrization of $v = h_{j_1}^k \otimes h_{j_2}^k \otimes \cdots \otimes h_{j_{n+1}}^k$. We compute $\overline{w_0} \cdot v = h_{n+1}^k \otimes h_{j_1}^k \otimes \cdots \otimes h_{j_{n+1}}^k \otimes (\sigma_1 h_1)$, whose symmetrization is equal to $\sigma_1^k$ times that of $v$; this gives the announced sign $\sigma_1^k$.

For $D_{2n+1} = so(4n + 2, \mathbb{C})$, the defining representation is $\square = C^{4n+2} = \text{Span}(h_{\pm j} | 1 \leq j \leq 2n + 1)$ and $\overline{w_0}$ maps $h_{\pm j} \mapsto h_{\mp j}$ for $1 \leq j \leq 2n$, but fixes $h_{2n+1}$. The zero-weight space $V^0$ of $S^{2k}\square$ is spanned by symmetrizations of $h_{j_1} \otimes h_{j_2} \otimes \cdots \otimes h_{j_k} \otimes h_{-j_k}$, each of which is fixed by $\overline{w_0}$. The representation is $w_0$-pure with $\sigma_1 = +1$, as announced.

The cases of $B_n$ = $so(2n + 1, \mathbb{C})$, $C_n$ = $sp(2n, \mathbb{C})$ and $D_{n \text{ even}} = so(2n, \mathbb{C})$ are treated together:

- $B_n$ has $\square = C^{2n+1} = \text{Span}(h_j | -n \leq j \leq n)$ and $\overline{w_0}$ acts by $h_j \mapsto h_{-j}$ for $j \neq 0$ and $h_0 \mapsto (-1)^j h_0$;
- $C_n$ has $\square = C^{2n} = \text{Span}(h_{\pm j} | 1 \leq j \leq n)$ and $\overline{w_0}$ acts by $h_j \mapsto h_{-j}$ and $h_{-j} \mapsto -h_j$ for $j > 0$;
- $D_n$ has $\square = C^{2n} = \text{Span}(h_{\pm j} | 1 \leq j \leq n)$ and, for even $\square$ acts by $h_j \mapsto h_{-j}$ for all $j$.

First consider $A^m\square$ and $S^m\square$. Their zero-weight spaces are spanned by (anti)symmetrizations of $h_{j_1} \otimes h_{-j_1} \otimes \cdots \otimes h_{j_k} \otimes h_{-j_k}$, where $2k + l = m$. Each of these vectors is fixed by $\overline{w_0}$ up to a sign that only depends on the group, the representation, and (on $k$, or equivalently $l, m$). For $C_n$ and $D_n$ we have $l = 0$ so for each $m$ the representation is $w_0$-pure, with a sign $(-1)^k$ for $S^{2k}\square$ of $C_n$ and $\Lambda^{2k}\square$ of $D_n$, and no sign otherwise. For $A^m\square$ of $B_n$ we note that $l \in \{0, 1\}$ is fixed by the parity of $m$ so the representation is $w_0$-pure; its sign is $(-1)^{ml+k} = (-1)^{mm+(m/2)} = \sigma_m$. For $S^m\square$ of $B_n$, only the parity of $l$ is fixed, but the sign $(-1)^m = (-1)^{ml} = \sigma_m$ still only depends on the representation; it confirms the data of Table 1.

Finally, consider the representation $S^2(\Lambda^2\square)$. Its zero-weight space is spanned by symmetrizations of $(h_j \otimes h_{-j}) \otimes (h_k \otimes h_{-k})$ and $(h_j \otimes h_{-k}) \otimes (h_{-j} \otimes h_k)$ all of which are fixed by $\overline{w_0}$.

4. Cartan product: $w_0$-mixed representations form an ideal

Let $G$ be a simply-connected simple complex Lie group and $N$ a maximal unipotent subgroup of $G$. Define $\mathbb{C}[G/N]$ the space of regular (i.e. polynomial) functions on $G/N$. Pointwise multiplication of functions is $G$-equivariant and makes $\mathbb{C}[G/N]$ into a $\mathbb{C}$-algebra without zero divisors (because $G/N$ is irreducible as an algebraic variety).

Theorem 4.1 ([5, (3.20)–(3.21)]). Each finite-dimensional representation of $G$ (or equivalently of its Lie algebra $g$) occurs exactly once as a direct summand of the representation $\mathbb{C}[G/N]$. The $\mathbb{C}$-algebra $\mathbb{C}[G/N]$ is graded in two ways:

- by the highest weight $\lambda$, in the sense that the product of a vector in $V_\lambda$ by a vector in $V_\mu$ lies in $V_{\lambda+\mu}$ (where $V_\lambda$ stands here for the subrepresentation of $\mathbb{C}[G/N]$ with highest weight $\lambda$);
- by the actual weight $\lambda$, in the sense that the product of a weight vector with weight $\lambda$ by a weight vector with weight $\mu$ is still a weight vector, with weight $\lambda + \mu$.
For given $\lambda$ and $\mu$, we call Cartan product the induced bilinear map $\odot : V_\lambda \times V_\mu \to V_{\lambda+\mu}$. Given $u \in V_\lambda$ and $v \in V_\mu$, this defines $u \odot v \in V_{\lambda+\mu}$ as the projection of $u \otimes v \in V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \ldots$. Since $\mathbb{C}[G/N]$ has no zero divisor, $u \odot v \neq 0$ whenever $u \neq 0$ and $v \neq 0$. We deduce the following.

**Lemma 4.2.** The set of highest weights of $w_0$-mixed irreducible representations of $g$ is an ideal $I_g$ of the additive monoid $M$ of dominant elements of the root lattice.

**Proof.** Consider a $w_0$-mixed representation $V_\lambda$ and a representation $V_\mu$ whose highest weight is radical. We can choose $u_+ \in V_\lambda$ and $u_- \in V_\mu$ in the zero-weight space of $V_\lambda$ such that $w_0 \cdot u_+ = u_+$ and $w_0 \cdot u_- = -u_-$. Then, choose $v \in V_\mu$ such that $w_0 \cdot v = \pm v$ for some sign. Then $u_+ \odot v$ and $u_- \odot v$ are non-zero elements of the zero-weight space of $V_{\lambda+\mu}$, which $w_0 \cdot v$ acts by opposite signs.  

5. **Proof of (iii): that other representations are $w_0$-mixed.**

Let $T^\text{Table}_g$ be the set of dominant radical weights that are not of the form $\lambda = kp_m \sigma_i$, $k \leq m_1$ (with data $p_i, m_i$ given in Table 1). Observe that $T^\text{Table}_g$ is an ideal of $M$. In Section 3 we showed $I_g \subseteq T^\text{Table}_g$. We now show that $T^\text{Table}_g \subseteq I_g$, namely that $V_{\lambda}$ is $w_0$-mixed for radical $\lambda$ other than those described by Table 1. By Lemma 4.2, it is enough to show this for the basis of $T^\text{Table}_g$. For any given group, $T^\text{Table}_g$ has a finite basis, so we simply used the algorithm of Section 2 to conclude for $A_{\leq 5}, B_{\leq 4}, C_{\leq 5}, D_{\leq 6}$ and all exceptional groups.

Now let $g$ be one of $A_{\geq 5}, B_{\geq 4}, C_{\geq 5}, D_{\geq 6}$ and $\lambda$ be in $T^\text{Table}_g$. We proceed by induction on the rank of $g$.

Define as follows a reductive Lie subalgebra $f \times g' \subset g$:

- if $g = sl(n, \mathbb{C})$, choose $f \times g' \simeq (gl(1, \mathbb{C}) \times sl(2, \mathbb{C})) \times sl(n-2, \mathbb{C})$, where $f$ has the roots $\pm (e_1 - e_n)$ and $g'$ has the roots $\pm (e_1 - e_j)$ for $1 \leq i < j < n$;
- if $g = so(n, \mathbb{C})$, we choose $f \times g' \simeq so(4, \mathbb{C}) \times so(n-4, \mathbb{C})$, where $f$ has the roots $\pm e_1 \pm e_2$ and $g'$ has the roots $\pm e_i \pm e_j$ for $3 \leq i < j \leq n$;
- if $g = sp(2n, \mathbb{C})$, we choose $f \times g' \simeq sp(2, \mathbb{C}) \times sp(2n-2, \mathbb{C})$, where $f$ has the roots $\pm 2e_1$ and $g'$ has the roots $\pm e_i \pm e_j$ for $2 \leq i < j \leq n$.

In all three cases, $f \times g'$ and $g$ share their Cartan subalgebra, hence restricting a representation $V$ of $g$ to $f \times g'$ does not change the zero-weight space $V^0$. Additionally, consider any connected Lie group $G$ with Lie algebra $g$: then the $w_0$ elements of the connected subgroup of $G$ with Lie algebra $f \times g'$ and of $G$ itself coincide, or more precisely have a common representative in $G$, because the Lie algebras have the same Lie subalgebra $s$ defined in Proposition 2.1. It follows that a representation of $g$ is $w_0$-mixed if and only if its restriction to $f \times g'$ is.

Next, decompose $V_\lambda = \bigoplus_i (V_{\xi_i} \otimes V_{\mu_i})$ into irreducible representations of $f \times g'$, where $\xi_i$ and $\mu_i$ are dominant weights of $f$ and $g'$, respectively. Consider the subspace

$$V_{\lambda}^{(0,*)} := \bigoplus_i (V_{\xi_i}^0 \otimes V_{\mu_i}) \subset V_\lambda$$

fixed by the Cartan algebra of $f$. It is a representation of $g'$ whose zero-weight subspace coincides with that of $V_\lambda$. The direct sum obviously restricts to radical $\xi_i$, and $\dim V_{\xi_i}^0 = 1$ because we chose $f$ to be a product of $sl(2, \mathbb{C})$ and $gl(1, \mathbb{C})$ factors. Thus the $w_0$ element of $g$ acts on $V_{\xi_i}^0 \otimes V_{\mu_i}$ in the same way, up to a sign, as the $w_0$ element of $g'$ acts on $V_{\mu_i}$.

Lemma 5.2 shows that $V_{\lambda}^{(0,*)}$ has an irreducible subrepresentation $V_v$ such that $v \in T^\text{Table}_g$. By the induction hypothesis, $V_v$ is then $w_0$-mixed hence $w_0$ has both eigenvalues $\pm 1$ on the zero-weight space $V_v^0 \subset V_{\lambda}^{(0,*)}$, namely $V_\lambda$ is $w_0$-mixed.

This concludes the proof of Theorem 1.3.

There remains to state and prove two lemmas. Let $g$ be $A_{n-1}, B_n, C_n$ or $D_n$ and let $\lambda$ be a dominant radical weight of $g$. It can then be expressed in the standard basis $e_1, \ldots, e_n$ as $\lambda = \sum_{i=1}^n \lambda_i e_i$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are integers subject to: for $A_{n-1}$, $\sum \lambda_i = 0$; for $B_n, \lambda_n \geq 0$; for $C_n, \lambda_n \geq 0$ and $\sum \lambda_i \in 2\mathbb{Z}$; for $D_n$, $\lambda_{n-1} \geq |\lambda_n|$ and $\sum \lambda_i \in 2\mathbb{Z}$. In addition, let $f \times g' \subset g$ be the subalgebra defined above. We identify weights of $g'$ with the corresponding weights of $g$ (acting trivially on the Cartan subalgebra of $f$). Note that this introduces a shift in their coordinates: the dual of the Cartan subalgebra of $g'$ is spanned by a subset of the vectors $e_1$ (corresponding to $g$) that starts at $e_2$ or $e_3$, not at $e_1$ as expected.

**Lemma 5.1.** Let $\mu$ be the dominant weight of $g'$ defined as follows:

- for $A_{n-1}$, $\mu = \left(\sum_{i=1}^{n-1} \lambda_i e_{i+1}\right) + \lambda_n e_{n+1}$, where $1 < \ell < n$ is an index such that $\lambda_{\ell-1} + \lambda_\ell \geq 0 \geq \lambda_{\ell} + \lambda_{\ell+1}$ (when several $\ell$ obey this, $\mu$ does not depend on the choice);
- for $B_n$, $\mu = \sum_{i=1}^{n+1} \lambda_i e_{i+2}$;
Then $V_{\mu}$ is a sub-representation of the space $V_{\lambda}^{(0,*)}$ defined earlier.

**Proof for $A_{n-1}$.** Let $v = \sum_{i=1}^{n-2} \lambda_i e_i$, be a dominant rational weight of $g'$. The weight $v$ is among weights of $V_{\lambda}^{(0,*)}$ if and only if it is among weights of $V_{\lambda}$. The condition is that $(\lambda - \bar{v}, \sigma_k) \geq 0$ for all $k$, where $\bar{v}$ is the unique dominant weight of $g$ in the orbit of $v$ under the Weyl group of $g$.

Explicitly, $\bar{v} = \left( \sum_{i=1}^{p-1} v_{i+1} \epsilon_i \right) + \sum_{i=1}^{p} v_{i+1} \epsilon_i$, where $p$ is any index such that $v_p \geq 0 \geq v_{p+1}$. Then the condition is $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k-1} \lambda_i + \lambda_k$ for $1 \leq k < p$ and $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k-1} \lambda_i + \lambda_p$ for $p < k < n$. Let us show that this is equivalent to

$$\sum_{i=2}^{k} v_i \leq \min \left( \sum_{i=1}^{k-1} \lambda_i, \lambda_k \right)$$

for all $2 \leq k \leq n - 2$. (2)

In one direction, the only non-trivial statement is that $2 \sum_{i=1}^{p} \lambda_i \geq \sum_{i=1}^{p-1} \lambda_i + \sum_{i=1}^{p+1} \lambda_i \geq 2 \sum_{i=1}^{p} \lambda_i$, where we used $2\lambda_p \geq \lambda_p + \lambda_{p+1}$. In the other direction, we check $\sum_{i=2}^{k} v_i \leq \sum_{i=1}^{k} \lambda_i$ for $k \leq p - 1$ using $v_2 \geq \cdots \geq v_p \geq 0$, and similarly for $p + 1 \leq k$ using $0 \geq v_{p+1} \geq \cdots \geq v_n$.

Now, $\lambda_{\ell-1} + \lambda_{\ell} \geq 0 \geq \lambda_{\ell+1} + \lambda_{\ell+2} \geq \lambda_{\ell-1} \geq \lambda_{\ell-1} + \lambda_{\ell} + \lambda_{\ell+1} \geq \lambda_{\ell+1} \geq \lambda_{\ell+2}$, so $\mu$ is a dominant weight of $g'$. It is radical because $\sum_{i=2}^{k} v_i \leq \sum_{i=1}^{k} \lambda_i$ is $0$. Furthermore, $\mu$ saturates all bounds (2) with $v$ replaced by $\mu$, as seen using $\lambda_{\ell-1} + \lambda_{\ell} \geq 0 \geq \lambda_{\ell+1} + \lambda_{\ell+2}$ respectively. In particular, we deduce that $\mu$ is among the weights of $V_{\lambda}^{(0,*)}$, hence some irreducible summand $V_{\nu} \subset V_{\lambda}^{(0,*)}$. The dominant radical weight $v$ of $g'$ must also obey (2), namely $\sum_{i=2}^{k} v_i \leq \sum_{i=1}^{k} \mu_i$ (due to the aforementioned saturation). Since $\mu$ is dominant and among weights of $V_{\nu}$, we must also have $(\nu - \mu, \sigma_k') \geq 0$ for all fundamental weights $\sigma_k'$ of $g'$. This is precisely the reverse inequality $\sum_{i=2}^{k} v_i \geq \sum_{i=1}^{k} \mu_i$. We conclude that $\nu = v$. $\square$

**Proof for $B_n$.** Let $e = 1$ for $C_n$ and otherwise $e = 2$. Again, a dominant radical weight $v = \sum_{i=1}^{n-2} (v_i e_i)$ of $g'$ is a weight of $V_{\lambda}^{(0,*)}$ if and only if all $(\lambda - \bar{v}, \sigma_k) \geq 0$, where $\bar{v}$ is the unique dominant weight of $g$ in the Weyl orbit of $v$. In all cases, $\bar{v} = \sum_{i=1}^{n-2} |v_{i+1} + e_i|$, where the absolute value is only useful for the $v_n$ component for $D_n$. The condition is worked out to be $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} |v_{i+1} + e_i|$ for $1 \leq k \leq n - e$. It is easy to check that $\mu$ is a dominant radical weight of $g'$ and that it obeys these conditions.

Consider now an irreducible summand $V_{\nu} \subset V_{\lambda}^{(0,*)}$ that has $\mu$ among its weights. On the one hand, $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} |v_{i+1} + e_i|$ for $1 \leq k \leq n - e$, where the absolute value is only useful for $v_n$ for $D_n$. On the other hand, $(\nu - \mu, \sigma_i') \geq 0$ for all dominant weights $\sigma_i'$ of $g'$ (in particular $e_1 + e_2 + \cdots + e_k$), so $\sum_{i=1}^{k} v_{i+1} + e_i \geq \sum_{i=1}^{k} \mu_{i+1} e_i$ for $1 \leq k \leq n - e$. The two inequalities fix $v_1 = \mu_1$ for all $k$, except $i = n$ when $\eta = 1$ for $C_n$ and $D_n$; in these cases, we conclude by using $\sum_{i=1}^{n} v_i - \sum_{i=1}^{n} v_{i+1} = 2$, since both weights are radical. $\square$

**Lemma 5.2.** For any $\lambda \in T_{\bar{g}}^{\text{Table}}$, there exists $v \in T_{\bar{g}}^{\text{Table}}$ such that the representation of $g'$ with highest weight $v$ is a subrepresentation of $V_{\lambda}^{(0,*)}$.

**Proof for $A_{n-1}$.** Let $n \geq 7$. If the weight $\mu$ defined by Lemma 5.1 is in $T_{\bar{g}}^{\text{Table}}$, we are done. Otherwise, $\mu = m(n-2) \sigma_i'$ or $\mu = m(n-2) \sigma_i$ of $\eta = (n-3) m + l + k$ for integers $m \geq 1$ and $\langle k \rangle = -1$, with the exclusion of the case $k = l = m$ because of $\lambda \in T_{\bar{g}}^{\text{Table}}$. For these dominant weights, the particular irreducible summand $V_{\mu} \subset V_{\lambda}^{(0,*)}$ of Lemma 5.1 is $w_0$-pure, but we now determine another summand that is $w_0$-mixed. The branching rules from $g$ to $g'$ can easily be deduced from the classical branching rules from $g(n, C)$ to $g(n-1, C)$ (given for example in [5, Theorem 9.14]).

Namely, consider the representation of $g(n, C)$ on $V_{\lambda}$ such that the diagonal $g(n, C)$ acts by zero. Then $V_{\lambda}^{(0,*)} \subset V_{\lambda}$ is the subspace on which all three $gl(n, C)$ factors of $gl(n, C) \times gl(n-2, C) \times gl(n-1, C)$ act by zero. It decomposes into irreducible representations of $g' \cong gl(n-2, C)$ with highest weights $\lambda' = \sum_{i=1}^{n-2} \lambda_i e_i$ such that $\sum \lambda_i' = 0$ and such that there exists $\lambda_1, \ldots, \lambda_{n-1}$ with $\sum \lambda_i = 0$ and $\lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$. In general, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$. In particular, we have $\lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$.

For $B_n$, $C_n$, and $D_n$, let $e = 1$ for $C_n$ and otherwise $e = 2$. Again, a dominant radical weight $\nu = \sum_{i=1}^{n-2} (\nu_i e_i)$ of $g'$ is a weight of $V_{\lambda}^{(0,*)}$ if and only if all $(\lambda - \bar{v}, \sigma_k) \geq 0$, where $\bar{v}$ is the unique dominant weight of $g$ in the Weyl orbit of $v$. In all cases, $\bar{v} = \sum_{i=1}^{n-2} |v_{i+1} + e_i|$, where the absolute value is only useful for $\nu_n$ for $D_n$. The condition is worked out to be $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} |v_{i+1} + e_i|$ for $1 \leq k \leq n - e$. It is easy to check that $\mu$ is a dominant radical weight of $g'$ and that it obeys these conditions.

Consider now an irreducible summand $V_{\nu} \subset V_{\lambda}^{(0,*)}$ that has $\mu$ among its weights. On the one hand, $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} |v_{i+1} + e_i|$ for $1 \leq k \leq n - e$, where the absolute value is only useful for $\nu_n$ for $D_n$. On the other hand, $(\nu - \mu, \sigma_i') \geq 0$ for all dominant weights $\sigma_i'$ of $g'$ (in particular $e_1 + e_2 + \cdots + e_k$), so $\sum_{i=1}^{k} v_{i+1} + e_i \geq \sum_{i=1}^{k} \mu_{i+1} e_i$ for $1 \leq k \leq n - e$. The two inequalities fix $v_1 = \mu_1$ for all $k$, except $i = n$ when $\eta = 1$ for $C_n$ and $D_n$; in these cases, we conclude by using $\sum_{i=1}^{n} v_i - \sum_{i=1}^{n} v_{i+1} = 2$, since both weights are radical. $\square$
focus on the summand where \((\lambda_1^n)_{i=1}^n\) and \((\lambda_1^{n-1})_{i=1}^{n-1}\) and \((\lambda'_2)^{n-1}\) all take the form \((m, \ldots, m, l, k, -S)\) where \(S\) is the sum of all other entries, with a different number of \(m\) in each case. Given that we started in rank at least 6, the resulting weight \(\lambda''\) cannot be a multiple of a fundamental weight, hence \(\lambda'' \notin \mathcal{T}_0\). □

**Proof for \(B_n\) with \(n \geq 5\), \(C_n\) with \(n \geq 6\), \(D_n\) with \(n \geq 7\).** We recall \(\epsilon = 1\) for \(C_n\) and otherwise \(\epsilon = 2\). If the weight \(\mu\) defined by Lemma 5.1 is in \(\mathcal{T}_0\), we are done. Otherwise, \(\mu\) can take a few possible forms because we took rank \(\gamma = n - \epsilon\) large enough to avoid special values listed in Table 1. Note that, by construction of \(\mu = \sum_{i=1}^n \mu_i e_i\), we have \(\lambda_i = \mu_i + \epsilon\) for \(1 \leq i \leq n - 3\) for \(D_n\) and \(1 \leq i \leq n - 2\) for \(B_n\) and \(C_n\). The possible dominant radical weights not in \(\mathcal{T}_0\) are as follows.

- First, \(\mu = m \sigma_1^\epsilon = m e_1 + \epsilon\), where additionally \(m\) is even for \(C_n\) and \(D_n\). Then \(\lambda_1 = \mu_1 + \epsilon = m\) and \(\lambda_2 = \mu_2 + \epsilon = 0\) fix \(\lambda = m \sigma_1^\epsilon\), which is not in \(\mathcal{T}_0\).
- Second, \(\mu = 2m_2^\epsilon = 2(e_1 + \epsilon e_2 + \epsilon^2 e_3)\), except for \(D_n\) with odd \(n\). Then \(\lambda_1 = \lambda_2 = 2\) and \(\lambda_3 = 0\) fix \(\lambda = 2m_2^\epsilon\), which is not in \(\mathcal{T}_0\).
- Third, \(\mu = \sum_{i=1}^m e_i + \epsilon\) for some \(m \geq 2\), except for \(D_n\) with odd \(n\), and where additionally \(m\) is even for \(D_n\) with even \(n\) and for \(C_n\). Since \(\lambda_1 = \mu_1 + \epsilon = 1\) and \(\lambda\) is dominant, we deduce that either \(\lambda_1 = \cdots = \lambda_p = 1\) for some \(p\) and all other \(\lambda_i = 0\), or (only in the \(D_n\) case) \(\lambda_1 = \cdots = \lambda_{n-1} = 1 = -\lambda_n\). These weights \(\lambda\) are not in \(\mathcal{T}_0\). Note, of course, that \(p\) and \(m\) are not independent; for example for \(m \leq n - 3\) one has \(m = p\).
- Fourth, \(\mu = (\sum_{i=1}^{n-2} e_i + 2^\epsilon) - e_n\) for \(D_n\) with even \(n\). This weight is not of the form of Lemma 5.1 because one would need \(-1 = \lambda_{n-2} - \eta \geq -\eta \geq -1\); hence \(\eta = 1\) and \(\lambda_{n-2} = 0\), so \(\lambda_{n-1} = \lambda_n = 0\) so \(1 = \eta = \lambda_{n-1} + \lambda_n = 0\) (mod 2). □

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