Admissible Measurements and Robust Algorithms for Ptychography

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Abstract
We study an approach to solving the phase retrieval problem as it arises in a phaseless imaging modality known as ptychography. In ptychography, small overlapping sections of an unknown sample (or signal, say \( x_0 \in \mathbb{C}^d \)) are illuminated one at a time, often with a physical mask between the sample and light source. The corresponding measurements are the noisy magnitudes of the Fourier transform coefficients resulting from the pointwise product of the mask and the sample. The goal is to recover the original signal from such measurements. The algorithmic framework we study herein relies on first inverting a linear system of equations to recover a fraction of the entries in \( x_0 x_0^* \) and then using non-linear techniques to recover the magnitudes and phases of the entries of \( x_0 \). Thus, this paper’s contributions are three-fold. First, focusing on the linear part, it expands the theory studying which measurement schemes (i.e., masks, shifts of the sample) yield invertible linear systems, including an analysis of the conditioning of the resulting systems. Second, it analyzes a class of improved magnitude recovery algorithms and, third, it proposes and analyzes algorithms for phase recovery in the ptychographic setting where large shifts—up to 50% the size of the mask—are permitted.

Keywords Phase retrieval · Ptychography · Efficient algorithms · Measurement design

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1 Introduction

Phase retrieval is the problem of solving a system of equations of the form

$$y = |Ax_0|^2 + \eta,$$  \hspace{1cm} (1)

where $x_0 \in \mathbb{C}^d$ is the objective signal, $A \in \mathbb{C}^{D \times d}$ is a known measurement matrix, $\eta \in \mathbb{R}^D$ is an unknown perturbation vector, and $y \in \mathbb{R}^D$ is the vector of measurement data. Here $|\cdot|^2$ acts componentwise so that for $v \in \mathbb{C}^n$ we have $|v|^2 = |v_j|^2$. In phase retrieval, the goal is to recover an estimate of $x_0$ from knowledge of $y$ and $A$. We sometimes rephrase the system (1) as

$$y_j = |\langle a_j, x_0 \rangle|^2 + \eta_j,$$  \hspace{1cm} (2)

where the $a_j^\ast$ stand for the rows of $A$ and are referred to as the measurement vectors.

The name phase retrieval comes from viewing the $|\cdot|^2$ operation as erasing the phases of the complex-valued measurements $\langle a_j, x_0 \rangle$ and leaving only their magnitudes; solving for $x_0$ may be considered as a way of retrieving this phase information. We immediately note that this problem contains an unavoidable phase ambiguity, in the sense that, for any solution $x$ and any $\theta \in [0, 2\pi)$, we will have that $e^{i\theta} x$ is also a solution, as $|\langle a_j, e^{i\theta} x \rangle|^2 = |\langle a_j, x \rangle|^2$.

1.1 Phase Retrieval in Imaging Science

The phase retrieval problem appears in a multitude of imaging systems, since most optical sensors – most significantly, charge-coupled devices and photographic film – do not respond to the phase of an incoming light wave. Rather they respond only to the number and energy of photons arriving at its surface, so they indicate only the intensity (absolute value squared), and not the phase, of the electromagnetic waves to which they are exposed. This corresponds to our model in (1) by imagining that the $i^{th}$ entry $a_i^\ast x$ of $Ax \in \mathbb{C}^D$ corresponds to the magnitude and phase of the light arriving at the $i^{th}$ pixel in an array of sensors. Areas of optics that encounter this problem include astronomy [19,66], diffraction imaging [54,58], laser pulse characterization [9,29], electron microscopy [52], and x-ray crystallography [14,18,20,30]. Non-optical disciplines that can benefit from solutions to phase retrieval include speech recognition and audio processing [4,42,53], blind channel estimation [40], and self-calibration [41].

The practice of these disciplines has produced many creative solutions to particular instances of the phase retrieval problem, and throughout the 20th century the field largely evolved by the invention of ad hoc solutions that resolved the data at hand. Notably, however, Gerchberg and Saxton [24] proposed an algorithm that can be applied to fairly general data, with remarkably minimal assumptions made on the structure of the object $x$ being detected. This result inspired numerous variants (e.g., [6,7,22,23,60–62]), each of which empirically improved performance, but none of which produced a solid mathematical theory to explain why or when they would succeed. Physicists, chemists, and biologists made astounding scientific achievements.
in this fashion, but even with all this progress, the community remained largely in want of such a theoretical foundation that could offer reliable solutions in general settings until recent decades.

There are three main questions about phase retrieval problems that the scientific community would wish to answer theoretically: first, in an ideal, noiseless case where \( \eta = 0 \), for what matrices \( A \in \mathbb{C}^{D \times d} \) does the system of equations (1) possess a unique solution (up to the known phase ambiguity)? Second, given a case where a unique solution exists, is there an algorithm that can recover it? Third, when a recovery process exists, is it stable so that in the presence of noise \( \eta \neq 0 \), the estimate \( \hat{x} \) does not differ much (or differs to a known degree, as a function of \( ||\eta|| \)) from \( x_0 \)?

This paper expands upon the theory of phase retrieval by studying a new class of matrices, that is of particular interest to ptychographic imaging, and an associated recovery algorithm that is proven to solve the system (1) with guaranteed stability to noise and with known, competitive computational cost.

Thus, we begin with a description of the application/setting which forms the subject of our analysis, along with a brief description of the phase retrieval strategy whose components we will study in more detail in later sections.

### 1.2 Local Measurements and Ptychography

Consider the case where the vectors \( a_j \) represent shifts of compactly-supported vectors \( m_j, j = 1, \ldots, K \) for some \( K \in \mathbb{N} \). Using the notation \([n]_k := [k, k+n) \subseteq \mathbb{N}\) and defining \([n] := [n]_1\), we take \( x_0, m_j \in \mathbb{C}^d \) with \( \text{supp}(m_j) \subseteq [\delta] \subseteq [d] \) for some \( \delta \in \mathbb{N} \). We also denote the space of Hermitian matrices in \( \mathbb{C}^k \times k \) by \( \mathcal{H}^k \). Now we have measurements of the form

\[
(y_{\ell})_j = |\langle x_0, S^\ell m_j \rangle|^2, \quad (j, \ell) \in [K] \times P,
\]

where \( P \subseteq [d]_0 \) is arbitrary and \( S \in \mathbb{C}^{d \times d} \) is the discrete circular shift operator, namely \((Sx)_i = x_{i-1}\). One can see that (3) represents the modulus squared of the correlation between \( x_0 \) and locally supported measurement vectors so we refer to the entries of \( y \) as local correlation measurements.

To see the connection to (a discretized version of) ptychography, consider \( \gamma, x_0 \in \mathbb{C}^d \), denoting discretized versions of a known physical mask and unknown sample, respectively. In ptychographic imaging, small regions of a specimen are illuminated one at a time, often with a physical mask between the specimen and the light source, and an intensity detector captures each of the resulting diffraction patterns. Thus each of the ptychographic measurements is a local measurement, which under certain assumptions (see, e.g., [20,25,33]) can be modeled by

\[
(y_{\ell})_j = \left| \sum_{n=1}^d \gamma_n (x_0)_{n-\ell} e^{-2\pi i (j-1)(n-1)/d} \right|^2, \quad (j, \ell) \in [d] \times [d]_0,
\]

where indexing is considered modulo-\( d \). So, \((y_{\ell})_j\) is a diffraction measurement corresponding to the \( j^{th} \) Fourier mode of a circular \( \ell \)-shift of the specimen. Note that the
use of circular shifts is for convenience only as one can zero-pad $x_0$ and $\gamma$ to obtain the same $\langle y_\ell \rangle_j$. In practice, one may not need to use all the shifts $\ell \in [d]$ as a subset may suffice, and we also consider this case in this paper. Now, defining $m_j \in \mathbb{C}^d$ by

$$(m_j)_n = \overline{\gamma_n} \frac{2\pi i(j-1)(n-1)}{d}$$

and rearranging (4), we obtain

$$\langle y_\ell \rangle_j = \left| \sum_{n=1}^{d} (x_0)_n - \ell (m_j)_n \right|^2 = \left| \langle S^{-\ell}x_0, m_j \rangle \right|^2 = \left| \langle x_0, S^\ell m_j \rangle \right|^2.$$  

Thus (6) shows that ptychography (with $\ell$ ranging over any subset of $[d]$) represents a case of the general system seen in (3). Returning to (3) and following [4,15,31], the problem may be lifted to a linear system on the space of $\mathbb{C}^{d \times d}$ matrices. In particular, we observe that

$$\langle y_\ell \rangle_j = \langle x_0^*x_0^*, S^\ell m_j m_j^* S^{\ell*} \rangle,$$  

where the inner product above is the Hilbert-Schmidt inner product. Restricting, for now, to the case $P = [d]$, for every matrix $A \in \text{span}\{S^\ell m_j m_j^* S^{\ell*}\}_{\ell,j}$ we have $A_{ij} = 0$ whenever $|i-j| \mod d \geq \delta$. Therefore, we introduce the family of operators $T_k : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d}$ given by

$$T_k(A)_{ij} = \begin{cases} A_{ij}, & |i-j| \mod d < k \\ 0, & \text{otherwise}. \end{cases}$$

Note that $T_\delta$ is simply the orthogonal projection operator onto $T_\delta(\mathbb{C}^{d \times d})$, of which $\text{span}\{S^\ell m_j m_j^* S^{\ell*}\}_{\ell,j}$ is a subspace; therefore,

$$\langle y_\ell \rangle_j = \langle x_0^*x_0^*, S^\ell m_j m_j^* S^{\ell*} \rangle = \langle T_\delta(x_0^*x_0^*), S^\ell m_j m_j^* S^{\ell*} \rangle, \quad (j, \ell) \in [K] \times P. \quad (8)$$

For convenience, we set $D := K|P|$ to be the total number of measurements and define the map $\mathcal{A} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^D$

$$\mathcal{A}(X)_{(\ell,j)} = \langle X, S^\ell m_j m_j^* S^{\ell*} \rangle.$$  

With this in hand, we are prepared to consider our reconstruction strategy, which follows the outline laid out in [31,33]. Namely, we will first consider the restriction $\mathcal{A}|_{T_\delta(\mathbb{C}^{d \times d})}$ of $\mathcal{A}$ to the domain $T_\delta(\mathbb{C}^{d \times d})$, the largest domain on which $\mathcal{A}$ may be injective. Initially, the framework we consider consists of designing measurements $\{m_j\}$ (via the masks $\gamma$) such that $\mathcal{A}|_{T_\delta(\mathbb{C}^{d \times d})}$ is invertible and then recovering an estimate of $x_0$ from

$$T_\delta(x_0^*x_0^*) =: X_0.$$  

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This recovery process, in turn, is performed by deducing the magnitudes $|x_0|$ and phases $\text{sgn}(x_0) := \frac{x_0}{|x_0|}$ of $x_0$ separately. This pseudo-algorithm is stated in Algorithm 1.

**Algorithm 1** Outline for our phase retrieval algorithm

**Input:** Measurements $y \in \mathbb{R}^D$ as in (3)  
**Output:** An estimate $x$ of $x_0$.  
1: Compute the matrix $X = A^{-1}_{T_\delta} (\mathbb{C}^{d \times d}) (y) \in T_\delta (\mathbb{H}^d)$.  
2: Compute the magnitudes $|x|$ from $X$ (e.g., by methods described in Section 4.1)  
3: Compute the phases $\text{sgn}(x)$ from $X$ (e.g., by methods described in Section 4.2)  
4: Return $x = |x| \circ \text{sgn}(x)$.

To give an example within the framework of Algorithm 1, similar to the algorithm studied in [31], one method of recovering $x_0$ from $X_0 = T_\delta (x_0 x_0^*)$ in the noiseless case would be to simply write $|x|_i = \sqrt{(X_0)_{ii}}$. To obtain the phases (up to a global shift), we could consider $\text{sgn}(X_0)$ as a matrix of relative phases, in the sense that $\text{sgn}(X_0)_{ij} = (x_0)_i (x_0)_j$, allowing us to inductively set $\text{sgn}(x)_1 = 1$ and $\text{sgn}(x)_i = \text{sgn}(x)_{i-1} \text{sgn}(X_{i,i-1})$ for $i = 2, \ldots, d$. Since we will deal with the noisy scenario in this paper, we will strive to develop more sophisticated techniques than these. Indeed, having broken down our main model and recovery algorithm in this manner, we are prepared to chart out the structure and contributions of this paper, keeping in mind that we will generalize the framework of Algorithm 1 to handle shifts $\ell$ that don’t cover all $[d]_0$, and hence scenarios whereby $\text{span}(S^\ell m_j m_j^* S^\ell^*)_{\ell, j}$ is a strict subspace of $T_\delta (\mathbb{C}^{d \times d})$.

1.3 Organization and Contributions

From Algorithm 1, we can identify three main areas of study. The first is the design of $\{m_j\}$ that permit invertible – and well conditioned – linear systems $A$. The second and third areas relate to the magnitude and phase recovery steps of lines 2 and 3. We wish to propose provably efficient and robust algorithms for these sub-tasks, and then combine them to obtain a robust method.

This paper presents contributions in each of the three areas. Indeed, in the first part of the paper, we focus on the case where the full set of shifts is used, and we build on a paper by Iwen, Preskitt, Saab, and Viswanathan [33] that in-turn improves upon the previous work by Iwen, Viswanathan, and Wang in [31] concerning the framework in Sect. 1.2. In Sect. 2, we derive a quickly calculable and exact expression for the condition number of the linear system $A$, and we leverage this to expand our collection of known spanning families. Specifically, we will discover that—in line with ptychographic imaging—setting $m_j = \gamma \circ f_j^d$, $j \in [2\delta - 1]$, where $f_j^d$ is the $j^{\text{th}}$ Fourier vector in $\mathbb{C}^d$ and $\gamma \in \mathbb{R}^d$ has support $[\delta]$ produces an invertible system under a very mild condition. We further prove that this condition holds for almost all $\gamma$; this result is particularly interesting considering that $\gamma \in \mathbb{C}^d$ has $\delta$ degrees of freedom, but must generate a spanning set for a subspace of dimension $d(2\delta - 1)$ when all shifts are taken in (7).
Algorithm 2 Phase Retrieval from Local Ptychographic Measurements

\textbf{Input:} A family of masks \(\{m_j\}_{j=1}^{D} \) of support \(\delta\); \(s,d,d \in \mathbb{N}\) satisfying \(d = d_s \geq 2\delta - 1\). A \((T_{\delta,s},d)\)-covering \(\{J_l\}_{l \in [N]}\), defined in (40), Section 4.1. Measurements \(y = A(x_0x_0^*) + n \in \mathbb{R}^{\mathbb{N}^D}\), as in (33) (see Section 3).

\textbf{Output:} \(x \in \mathbb{C}^d\) with \(x \approx e^{i\theta}x_0\) for some \(\theta \in [0, 2\pi]\).

1. Compute the matrix \(X = A^{-1}_{T_{\delta,s},(C^d \times d)}\), \(v \in T_{\delta,s}(\mathcal{H}^d)\) as an estimate of \(T_{\delta,s}(x_0x_0^*)\) (as in Section 3).
2. Form the banded matrix of phases, \(\tilde{X} = \text{sgn}(X) \in T_{\delta,s}(\mathcal{H}^d)\), by normalizing the non-zero entries of \(X\) (replacing any zero entries in the band with 1’s).
3. Compute \(v\), the eigenvector of \(\tilde{X}\) corresponding to the largest eigenvalue and set \(\tilde{x} = \text{sgn}(v)\) (as in Section 4.2).
4. Return \(x = \text{BlockMag}(X,\{J_l\}) \circ \tilde{x}\) (see Section 4.1).

In the second part of the paper, we focus on increasing the match between our model and the laboratory practices of ptychographers. To that end, we devise algorithms and derive theory that handle the practical setup where large shifts are used. The algorithmic framework we propose and analyze is summarized in Algorithm 2. Specifically, in Sect. 3 we study the conditioning of the linear system arising from a set of shifts that is smaller than \([d]/0\). In particular, we consider taking shifts \(\ell = sk, k \in [d/s]\), where \(s\) is a fixed step size. This leads to a new subspace \(T_{\delta,s}(\mathbb{C}^d \times d) \subseteq T_{\delta}(\mathbb{C}^d \times d)\), for which we derive condition number estimates in the spirit of Sect. 2. In Sect. 4.1, we propose and analyze a magnitude estimation step for Algorithm 2 and prove that it is robust to noise. In Sect. 4.2, we extend the phase-estimation technique of [33] to the setting of large shifts, and show that this technique is robust. Finally, in Sect. 4.3, we put our results together and prove that Algorithm 2 comprises a stable phase retrieval method in the setting of large ptychographic shifts. This result is summarized in the theorem below, which is a slightly weaker but more streamlined version of Theorem 5.

\textbf{Theorem 1} Let \(A\) be the linear system arising from a set of measurements

\[\{S^{\ell}\, m_j^* S^{-\ell}\}_{(\ell,j) \in [d/s]_0 \times [D]}\]

which spans \(T_{\delta,s}(\mathbb{C}^d \times d)\), where the vectors \(m_j \in \mathbb{C}^d\) are of support \(\delta\), and where \(T_{\delta,s}\) is defined in Eq. (34). Let \(\sigma^{-1}_{\min}\) be the smallest singular value of \(A\) restricted to \(T_{\delta,s}\). The error associated with recovering \(x_0\) from the noisy measurements \(y_{j,\ell} = |\langle S^{\ell}m_j, x_0 \rangle|^2 + n_{j,\ell}\), using Algorithm 2 satisfies

\[\min_{\theta \in [0, 2\pi]} ||x - e^{i\theta}x_0||_2 \leq C \cdot \sigma^{-1}_{\min} \left(\frac{d^2 \cdot s}{\delta^{3/2}} \cdot \frac{\|x_0\|_\infty}{\min_j |(x_0)_j|^2} + \frac{\delta^{-1/2}}{\min_j |(x_0)_j|}\right) \cdot \|n\|_2.\]  

(11)

We remark that Theorem 1 requires the entries of \(x_0\) to all be non-zero, which is a fairly strong assumption. As noted in [31], this can be achieved by applying a randomizing matrix to the objective vector prior to taking the measurements modeled by \(A\). Alternatively, Theorem 4 of [33] bounds the error according to the number of...
entries below a certain magnitude threshold, and also cites numerical results to indicate that zero entries are tolerable under fairly general conditions. Their proof technique could well be applied in Corollary 2, but in our opinion it would add complexity without fundamentally changing our main results.

1.4 Related Work

The history of modern algorithmic phase retrieval begins in the 1970’s with [24] by Gerchberg and Saxton, where the measurement data corresponded to knowing the magnitude of both the image \( x_0 \) and its Fourier transform. This result was famously expanded upon by Fienup [23] later that decade, one significant improvement being that only the magnitude of the Fourier transform of \( x_0 \) must be known in the case of a signal \( x_0 \) belonging to some fixed convex set \( C \) (typically, \( C \) is the set of non-negative, real-valued signals restricted to a known domain). Though these techniques work well in practice and have been popular for decades, they are notoriously difficult to analyze. These are iterative methods that work by improving an initial guess until they stagnate. In 2015, Marchesini et al. proved that alternating projection schemes using generic measurements are guaranteed to converge to the correct solution if provided with a sufficiently accurate initial guess and algorithms for ptychography were explored in particular [43]. Waldspurger then proved that a spectral initialization reaches this basin of attraction with high probability using a simple Gaussian suite of measurements [64]. The application of alternating minimizations to sparse phase retrieval has received considerable attention, as well [35,46], although results in [32] suggest that virtually any phase retrieval method may easily be composed with compressed sensing techniques to target sparse signals. However, despite this impressive body of work, no global recovery guarantees currently exist for alternating projection techniques using local measurements (i.e., finding a sufficiently accurate initial guess is not generally easy).

Other works have proved probabilistic recovery guarantees when provided with globally supported Gaussian measurements. Methods for which such results exist vary in their approach, and include convex relaxations [15,56,65], gradient descent strategies [17], graph-theoretic [1,55] and frame-based approaches [5,11,12], and variants on conventional alternating minimization ideas [45,64]. The approach of non-convex optimization by gradient descent, named Wirtinger Flow in its first application to phase retrieval [17], has enjoyed recent success in a variety of phase retrieval applications [13] as well as blind deconvolution [40] and low-rank matrix recovery [63].

Several recovery algorithms achieve theoretical recovery guarantees while using at most \( D = \mathcal{O}(d \log^4 d) \) masked Fourier coded diffraction pattern measurements, including both PhaseLift [16,27], and Wirtinger Flow [17]. However, until recently, there has been no construction of these measurements that were not randomized, and – to our knowledge – the theory has not studied locally supported measurements of the type considered here. Kueng, Gross, and others have tried to derandomize the constructions for PhaseLift in particular by drawing the measurements from certain matrix groups [37,38], but the first completely deterministic, albeit non-local (and
possibly non-physical), construction of a measurement system with provable, global recovery via PhaseLift appeared in [36].

Among the first treatments of local measurements are [8,21,34], in which it is shown that STFT (short-time Fourier transform [2,49]) measurements with specific properties can allow (sparse) phase retrieval in the noiseless setting, and several recovery methods have been proposed [10,28]. Similarly, the phase retrieval approach from [1] was extended to STFT measurements in [55] in order to produce recovery guarantees in the noiseless setting. More recently, randomized robustness guarantees were developed for time-frequency measurements in [48]. However, no deterministic robust recovery guarantees have been proven in the noisy setting for any of these approaches. Furthermore, none of the algorithms developed in these papers are demonstrated to be empirically competitive with standard alternating projection techniques for large signals when utilizing windowed Fourier and/or correlation-based measurements. In [31], the authors first propose a deterministic measurement scheme and prove the first deterministic robustness results in the recent literature, although these results treat a “greedy” recovery algorithm, different from the one developed herein, and they obtain weaker recovery guarantees. Very recently, excellent work by Pearlmutter et al. [47] studied the related problem of inverting spectrogram measurements when the mask is locally supported and the signal is bandlimited. Finally, [44] studied the ptychographic setup we consider in the second part of the paper, and proved an analogous (but slightly different) result to our Theorem 4 using the magnitude estimation techniques of [33], rather than the more sophisticated techniques we consider herein.

In the midst of such an active and diverse field of research, the major contributions of our work are that it takes into account the local measurements that match the models for key applications such as ptychography. In this setting, we have produced a provably fast and stable recovery algorithm for a deterministically stated class of measurement systems that corresponds well to ptychographic imaging.

1.5 Notation

We take a moment to gather some of the notation that is used throughout the paper. Table 1 displays some of the most commonly used objects. We remark that, in this table and throughout this work, indices of a vector $x \in \mathbb{C}^n$ or matrix $A \in \mathbb{C}^{m \times n}$ are always taken modulo the appropriate dimension. For example, $x_{n+1} := x_1$ and $A_{00} = A_{mn}$. Given matrices $V_j \in \mathbb{C}^{m_j \times n_j}$ for $j \in [n]$

$\text{diag}(V_j)_{j=1}^n = \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_n \end{bmatrix} \in \mathbb{C}^{\sum m_j \times \sum n_j}.$

To conveniently switch between matrices and vectors of different sizes, $\mathcal{R}_d : \bigcup_{k=1}^\infty \mathbb{C}^k \rightarrow \mathbb{C}^d$ is a resizing map, which truncates or zero-pads as appropriate. For
Similarly, \( R \ell \in \mathbb{R} \) as in (2). Thus, by design, it is clear that the vectors of \( \mathbb{R}^m \times n \) truncated or zero-pads matrices to size \( m \times n \). For \( A \in \mathbb{C}^{m \times n} \), we define vec\((A) \in \mathbb{C}^{mn} \) with vec\((A)_{(j-1)m+i} = A_{ij} \) for \( i, j \in [m] \times [n] \). To invert vec, we use mat\((m, n) : \mathbb{C}^{mn} \to \mathbb{C}^{m \times n} \), such that mat\((m, n)(v)_{ij} = v_{(j-1)m+i} \).

### 2 Invertible Local Measurement Systems

Herein we focus on the setup described in Sect. 1.2, while also accounting for noise as in (2). Thus, by design, it is clear that the vectors \( \{S^\ell m_j m_j^*S^{-\ell} \} \) are all contained in the subspace \( T_\delta(\mathcal{H}^d) \) of \( \mathcal{H}^d \) (defined in (7)), where \( d \) is the ambient dimension.

| Parameters | Name and Type | Definition | Comments |
|------------|---------------|------------|----------|
| \( A \subseteq [d] \) | \( 1_d \in \mathbb{C}^d \) | \((1_d)_i = 1 \) for \( i \in [d] \) | \( 1 := 1_d \) |
| \( x \in \mathbb{C}^d, k \in [d] \) | \( \text{circ}_k(x) \in \mathbb{C}^{d \times k} \) | \( \text{circ}_k(x) = [s^0 s \cdots s^{k-1}] \) | \( \text{circ}(x) = \text{circ}_d(x) \) |
| \( \ell \in \mathbb{Z}, A \in \mathbb{C}^{m \times n} \) | \( \text{diag}(A, \ell) \in \mathbb{C}^m \) | \( \text{diag}(A, \ell)_i = A_{i,i+\ell} \) | Notation overloaded with diag(). |
| \( x \in \mathbb{C}^d \) | \( \text{diag}(x) \in \mathbb{C}^{d \times d} \) | \( \text{diag}(x)_i = x_i x_i \) | Also written \( D_x \) or \( \text{diag}(x)_{j=1}^d \). |
| \( e_i^a \in \mathbb{R}^n \) | \( e_i^a = I_n e_i \) | \( F_d \) is unitary. \( F := F_d^\dagger \) |
| \( f_d \in \mathbb{C}^{d \times d} \) | \( f_d = F_d e_j \) | \( f_j := f_d^\dagger \) |
| \( \mathcal{T}^d \subseteq \mathbb{R}^{d \times d} \) | \( A \in \mathcal{T}^d \) iff \( A = A^* \) | Hermitian matrices |
| \( I_d \in \mathbb{R}^{d \times d} \) | \( I_d = I_{d}^\dagger \) | We define \([k] = [k]_1\). |
| \( n \in \mathbb{Z} \) | \([k]_n \subseteq \mathbb{N} \) | \([n, n+k) \cap \mathbb{Z} \) |
| \( p \in [n]_k \) satisfying \( p \equiv m \, \text{mod} \, n \) | \( m \, \text{mod} \, n \equiv m \, \text{mod} \, n \) |
| \( S^d \subseteq \mathbb{R}^{d \times d} \) | \( S^d = \mathbb{R}^{d \times d} \) \( \cap \mathcal{T}^d \) | Symmetric matrices |
| \( R_d \subseteq \mathbb{R}^{d \times d} \) | \( R_d x)_i = x_{2-i} \) | \( R := R_d^\dagger \) |
| \( x, y \in \mathbb{C}^d \) | \( x \circ y \in \mathbb{C}^d \) | \( x \circ y)_i = x_{1+y} \) |
| \( A \subseteq B \) | \( \chi_A : B \to \{0, 1\} \) | \( \chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases} \) |
| \( \omega_d \in \mathbb{C} \) | \( \omega_d = e\frac{2\pi i}{d} \) | \( \omega := \omega_d^\dagger \) |

\( \dagger \)We omit the subscript (or superscript) when it is obvious from context.
(meaning $x_0, m_j \in \mathbb{C}^d$) and $\delta$ is the support size of the masks (so $\text{supp}(m_j) \subseteq [\delta]$). Following the framework of Algorithm 1 we now focus on the linear system (9). In [31,33], a total of two examples of collections of vectors $\{m_j\}_{j \in \mathbb{Z}} \subseteq \mathbb{C}^d$ such that this linear system was invertible on $T_\delta(\mathcal{H}^d)$ were given. Considering that one of the major contributions of Algorithm 1 is that it admits measurement models that intend to replicate laboratory conditions, our knowledge of which vectors are compatible with our algorithm and with the theory built for it is critical in promoting applicability. In Sect. 2.2, we study the conditioning of the linear system in (9) as a function of the set of masks $\{m_j\}_{j=1}^D$, resulting in Theorem 2. This result also gives us a description of all sets of masks $\{m_j\}_{j=1}^D$ (and $\gamma \in \mathbb{R}^d$ that generate such families) that are capable of spanning $T_\delta(\mathcal{H}^d)$, in the sense that we have a checkable condition that indicates whether the linear system of (9) is invertible. In Sect. 2.3, we provide a few examples of explicit $\gamma \in \mathbb{R}^d$ that are proven to satisfy the conditions to span $T_\delta(\mathcal{H}^d)$. We consider the act of inverting $\mathcal{A}$ from a practical perspective in Sect. 2.4. We write its inverse explicitly and analyze the runtime of calculating $\mathcal{A}^{-1}(y)$ in Sect. 2.4.

2.1 Preliminaries

Before we begin these analyses, we introduce some definitions. We say that $\{m_j\}_{j=1}^D \subseteq \mathbb{C}^d$ is a local measurement system or family of masks of support $\delta$ if $1 \in \text{supp}(m_j)$ and $\text{supp}(m_j) \subseteq [\delta]$ for each $j$. If we further have that each $m_j$ satisfies $m_j = \mathcal{R}_d(\sqrt{K} f_j^K) \circ \gamma$ for some $K \geq \max(\delta, D)$, $\gamma \in \mathbb{C}^d$ satisfying $\text{supp}(\gamma) = [\delta]$, then we call $\{m_j\}_{j=1}^D$ a local Fourier measurement system of support $\delta$ with mask $\gamma$ and modulation index $K$. If $K = D = 2\delta - 1$, then we simply refer to $\{m_j\}_{j=1}^D$ as a local Fourier measurement system of support $\delta$ with mask $\gamma$. We add that, if we say that $\{m_j\}_{j=1}^D$ is a local Fourier measurement system with support $\delta$ and mask $\gamma$, this implies an assertion that $\text{supp}(\gamma) = [\delta]$. Given a local measurement system $\{m_j\}_{j=1}^D$ in $\mathbb{C}^d$, the associated lifted measurement system is the set $\mathcal{L}_{\{m_j\}} = \{S^\ell m_j^* S^{-\ell}\}_{(\ell, j) \in \mathbb{Z} \times \mathbb{Z}} \subseteq \mathbb{C}^{d \times d}$. We then say that a family of masks $\{m_j\}_{j=1}^D \subseteq \mathbb{C}^d$ of support $\delta$ is a spanning family if $\text{span}_\mathbb{R} \mathcal{L}_{\{m_j\}} = T_\delta(\mathcal{H}^d)$. The measurement operator associated with a local measurement system is the operator

$$\mathcal{A} : T_\delta(\mathbb{C}^{d \times d}) \rightarrow \mathbb{C}^{\mathbb{Z} \times \mathbb{Z}}$$

$$\mathcal{A}(X)_{(\ell, j)} = \langle S^\ell m_j^* S^{-\ell}, X \rangle. \quad (12)$$

The canonical matrix representation of $\mathcal{A}$ is the matrix $A \in \mathbb{C}^{dD \times (2\delta - 1)}$, defined by

$$\begin{pmatrix}
\text{diag}(X, 1 - \delta) \\
\vdots \\
\text{diag}(X, \delta - 1)
\end{pmatrix}_{(j-1)d + \ell} = \mathcal{A}(X)_{(\ell-1, j)}. \quad (13)$$
For convenience, we define the diagonal vectorization operator \( D_I : \mathbb{C}^{d \times d} \to \mathbb{C}^{|I|^d} \) for any collection \( \ell_i \) for any integer \( k \leq \frac{d + 1}{2} \) by

\[
D_I(X) = \begin{bmatrix}
\text{diag}(X, \ell_1) \\
\vdots \\
\text{diag}(X, \ell_{|I|})
\end{bmatrix}
\]

and

\[
D_k(X) = D_{[2k-1]_{1-k}}(X) = \begin{bmatrix}
\text{diag}(X, 1 - k) \\
\vdots \\
\text{diag}(X, k - 1)
\end{bmatrix},
\]

so that (13) becomes \( AD_\delta(X)_{(j-1)d+\ell} = A(X)_{(\ell-1, j)} \). We remark that, when \( 2k - 1 \leq d \), \( D_k \) is invertible on \( T_k(\mathbb{C}^{d \times d}) \), and for \( v \in \mathbb{C}^{d(2k-1)} \), we use \( D_k^{-1}(v) \) or \( D_k^*(v) \) to represent the matrix in \( T_k(\mathbb{C}^{d \times d}) \) whose diagonals are given by the \( 2k - 1 \) distinct \( d \)-length blocks of \( v \).

2.2 Invertibility and Condition Numbers

We now calculate the singular values, and therefore the condition number, of the measurement operator \( A \) for an arbitrary local measurement system \( \{m_j\}_{j=1}^d \) in Theorem 2, the main result of this section. We remark that this is an important element in using the error bounds proven in [33], since they all unavoidably rely on the condition number \( \kappa \) and singular values of the linear system solved in line 1 of Algorithm 1, and leaving this quantity unknown and unestimated renders these bounds far less useful. We also remark that, in this section, as in Sect. 2.3, we focus on calculations of the condition numbers \( \kappa \), rather than the smallest singular value \( \sigma_{\text{min}} \) of the linear systems, since \( \kappa \) is scale invariant. Indeed, if \( \{m_j\}_{j \in [D]} \) is a local measurement system with \( \sigma_{\text{min}}^{-1} = s \), then \( \{tm_j\}_{j \in [D]} \) has \( \sigma_{\text{min}}^{-1} = \frac{s}{r} \), so it would appear that simply making \( r \) large could arbitrarily improve how well the estimate \( X \) arising from line 1 of Algorithm 1 approximates \( T_\delta(x_0 x_0^*) \); of course, this action would correspond to simply multiplying the observed measurements by the scalar \( t^2 \), and slips in its advantage by ignoring that \( ||n||_2 \) would also scale as \( t^2 \) in such a case. This process clearly buys us nothing, so the \( \sigma_{\text{min}}^{-1} \) inequalities fail, in this way, to consider a sense of scale between \( ||n||_2 \) and \( ||A(x_0 x_0^*)||_2 \). Therefore, by referencing \( \kappa \) and SNR = \( \frac{||A(x_0 x_0^*)||}{||n||_2} \) we have inequalities that more accurately describe the relationship between the design of the linear system and the accuracy of the estimate produced by Algorithm 1.

We emphasize further that, perhaps equally or even more significantly, this result gives a description of all local measurement systems that are usable for phase retrieval in Algorithm 1, since we may simply check whether \( \{m_j\}_{j \in [D]} \) leads to a system with any singular values of 0. This is an important addition to the framework of [33], since previously we only possessed two examples of families of masks that produced invertible linear systems. Before stating the result, we introduce the operator \( R \) which maps \( x \in \mathbb{C}^d \) to \( (x_2 - i)_{i=1}^d \) (where indexing is mod \( d \) as always). We also introduce
the interleaving operators \( P^{(d,N)} : \mathbb{C}^{dN} \to \mathbb{C}^{dN} \) for any \( d, N \in \mathbb{N} \), each of which is a permutation defined by

\[
(P^{(d,N)} v)_{(i-1)N+j} = v_{(j-1)d+i}.
\] (16)

Informally, we can view the action of \( P^{(d,N)} \) as beginning with \( v \in \mathbb{C}^{dN} \), transforming it into a \( d \times N \) matrix, then transposing this to an \( N \times d \) matrix and re-vectorizing it.

**Theorem 2** Given a family of masks \( \{ m_j \}_{j \in [D]} \) of support \( \delta \leq \frac{d+1}{2} \), we define \( g_j = \text{diag} (m_j m_j^*, m) \),

\[
H = P^{(d,D)} \begin{bmatrix} R_{g_1}^{1-\delta} & \cdots & R_{g_1}^{\delta-1} \\ \vdots & \ddots & \vdots \\ R_{g_D}^{1-\delta} & \cdots & R_{g_D}^{\delta-1} \end{bmatrix},
\]

and \( M_j = \sqrt{d} \left( f_j^d \otimes I_D \right)^* H \). Then the singular values of \( A \) as defined in (12) are \( \{ \sigma_i(M_j) \}_{(i,j) \in [D] \times [d]} \) and its condition number is

\[
k(A) = \frac{\max_{i \in [d]} \sigma_{\max}(M_i)}{\min_{i \in [d]} \sigma_{\min}(M_i)}.
\]

Although Theorem 2 is satisfyingly general, perhaps the most useful result in this section is the strictly narrower Proposition 1 which greatly generalizes the mask construction in [31]. A similar result, derived in the context of signal recovery from spectrogram measurements, was presented in [3].

**Proposition 1** Let \( \{ m_j \}_{j=1}^D \subseteq \mathbb{C}^d \) be a local Fourier measurement system with support \( \delta \), mask \( \gamma \), and modulation index \( K \), where \( D = 2\delta - 1 \leq d \). Let \( A \) be the associated measurement operator as in (12), with canonical matrix representation \( A \) as in (13).

If \( K = D \), then the condition number of \( A \) is

\[
k(A) = \frac{d^{-1/2} \| \gamma \|_2^2}{\min_{m \in [\delta], j \in [d]} \left| F_d^* (\gamma \circ S^{-m} \gamma) j \right|}.
\] (17)

If \( K > D \), the condition number is bounded by

\[
k(A) \leq \frac{d^{-1/2} \| \gamma \|_2^2}{\min_{m \in [\delta], j \in [d]} \left| F_d^* (\gamma \circ S^{-m} \gamma) j \right|} \kappa(F_K),
\] (18)

where \( \tilde{F}_K \in \mathbb{C}^{D \times D} \) is the \( D \times D \) principal submatrix of \( F_K \).
We recall that the design of local Fourier measurement systems is motivated by the application of ptychography – in this type of laboratory setup, \( \gamma \) can represent a mask or “illumination function,” describing the intensity of radiation applied to each segment of the sample – so it is appropriate to the end user that our simplest and most conveniently applied result pertains to a realistic, broad class of local measurement systems. In particular, we can now determine masks/illumination functions that are admissible for our phase retrieval algorithm: following almost immediately from Proposition 1, we have sufficient conditions for a local Fourier measurement system to be a spanning family, which we state in Corollary 1.

**Corollary 1** Let \( \{m_j\}_{j \in D} \) be a local Fourier measurement system of support \( \delta \) with mask \( \gamma \in \mathbb{R}^d \) and modulation index \( K \), where \( D = 2\delta - 1 \). Then \( \{m_j\}_{j \in [D]} \) is a spanning family if \( F_d^*(\gamma \circ S^{-m}\gamma) \neq 0 \) for all \( m \in [\delta]_0 \), \( j \in [d] \) and \( K \geq D \).

**Remark** The condition in Corollary 1 for a local Fourier measurement system to be a spanning family is generic, in the sense that it fails to hold only on a subset of \( \mathbb{R}^d \) with Lebesgue measure zero, except possibly when \( \delta > \frac{d}{2} \). We consider that, whenever \( m \neq \frac{d}{2} \), the set of \( \gamma \in \mathbb{R}^d \) giving at least one zero in \( F_d^*(\gamma \circ S^{-m}\gamma) \) is a finite union of zero sets of non-trivial quadratic polynomials and hence a set of zero measure. Indeed, when \( m \neq \frac{d}{2} \), we have that

\[
F_d^*((e_1 + e_{m+1}) \circ Sm(e_1 + e_{m+1}))_k = f_k^d e_{m+1} = \omega^{-m(k-1)},
\]

so \( \gamma \mapsto F_d^*(\gamma \circ S^m\gamma)_k \) is a non-zero, homogeneous quadratic polynomial and therefore has a zero locus of measure zero.

The proofs of Theorem 2 and Proposition 1 require some preliminary work in defining and studying a few new operators pertaining to the structure of (12). These definitions and a number of results concerning them are contained in Appendix A.1. We begin with the proof of Theorem 2, of which Proposition 1 is a special case.

**Proof of Theorem 2** We consider the rows of the matrix \( A \) representing the measurement operator \( A \), defined in (13) and (12). We vectorize \( X \in T_\delta(\mathbb{C}^{d \times d}) \) by its diagonals with \( D_\delta \), as in (15) and set \( \chi_m = \text{diag}(X, m), m = 1 - \delta, \ldots, \delta - 1 \) and recall that \( g_m^j = \text{diag}(m_j m_j^*, m) \). Then

\[
A(X)_{(\ell, j)} = \langle S_\ell m_j m_j^* S^{-\ell}, X \rangle = \sum_{m=1-\delta}^{\delta-1} \langle S_\ell g_m^j, \chi_m \rangle,
\]

so that the definition of \( A \) in (13) immediately yields its \( (j - 1)d + \ell \)th row as

\[
\begin{bmatrix}
g_{1-\delta}^j S_{1-\ell}^j \cdots g_{\delta-1}^j S_{1-\ell}^j
\end{bmatrix}.
\]

Transposing this expression, we see that the \( (j - 1)d + 1 \)st through \( (j - 1)d + d \)th rows of \( A \) compose

\[
\begin{bmatrix}
circ(g_{1-\delta}^j)^* \cdots \circirc(g_{\delta-1}^j)^*
\end{bmatrix}.
\]

Together
with circ$(v)^* = \text{circ}(R\bar{v})$, $A$ is the block matrix given by

$$
A = \begin{bmatrix}
circ(g_1^{1-\delta})^* & \cdots & \circ(g_{\delta-1}^{1-\delta})^* \\
\vdots & \ddots & \vdots \\
circ(g_1^D)^* & \cdots & \circ(g_{\delta-1}^D)^*
\end{bmatrix} = \begin{bmatrix}
circ(Rg_1^{1-\delta}) & \cdots & \circ(Rg_{\delta-1}^{1-\delta}) \\
\vdots & \ddots & \vdots \\
circ(Rg_1^D) & \cdots & \circ(Rg_{\delta-1}^D)
\end{bmatrix},
$$

which may be transformed, by (52) of Lemma 3, to

$$
P^{(d, D)} A P^{(d, 2\delta-1)^*} = \text{circ}^D \left( P^{(d, D)} \begin{bmatrix} Rg_1^{1-\delta} & \cdots & Rg_{\delta-1}^{1-\delta} \\
\vdots & \ddots & \vdots \\
Rg_1^D & \cdots & Rg_{\delta-1}^D
\end{bmatrix} \right) = \text{circ}^D (H). \tag{19}
$$

Quoting Corollary 3 establishes the theorem. \hfill \Box

We are now able to prove Proposition 1.

**Proof of Proposition 1** We begin by remarking that, for any $j, m \in [d]$, the Cauchy-Schwarz inequality gives us

$$
f_j^{d^*}(\gamma \circ S^{-m} \gamma) = \left( D_{f_j^{d^*}} \gamma \right) \cdot (S^{-m} \gamma) \leq \frac{1}{\sqrt{d}} ||\gamma||_2^2.
$$

Observing that $f_j^{d^*}(\gamma \circ \gamma) = \frac{1}{\sqrt{d}} ||\gamma||_2^2$, we have that

$$
\max_{(j, m) \in [d] \times [\delta]} F_d^*(\gamma \circ S^{-m} \gamma) = \frac{1}{\sqrt{d}} ||\gamma||_2^2.
$$

Recall that $D = 2\delta - 1 \leq d$ and set $\tilde{F}_K \in \mathbb{C}^{2\delta-1 \times 2\delta-1}$, $(\tilde{F}_K)_{ij} = \frac{1}{\sqrt{K}} \omega_{(i-1)(j-\delta)}^K$ to be the principal submatrix of $\text{diag}(\sqrt{K} f_j^{K} f_j^{K})$, let $v_j = R_d(\sqrt{K} f_j^{K})$. For a local Fourier measurement system (i.e., $K = 2\delta - 1$), we have

$$
g_m^j = \text{diag}(m_j m_j^*, m) = \text{diag}(\gamma \circ v_j(\gamma \circ v_j)^*, m)
$$

$$
= \text{diag}(D_{v_j} \gamma \gamma^* D_{v_j}, m) = \omega_K^{-m(j-1)} \text{diag}(\gamma \gamma^*, m), \tag{21}
$$

so, setting $g_m = \text{diag}(\gamma \gamma^*, m)$, we have $g_m^j = \text{diag}(m_j m_j^*, m) = \omega_K^{-m(j-1)} g_m$. Therefore, we label the $2\delta - 1 \times 2\delta - 1$ blocks of $H$ (where $H$ is as in Theorem 2) by $H^* = [H_1^* \cdots H_d^*]$, so that

$$
(H\ell)_{ij} = (Rg_{j-\delta}^i)_{ij} = \omega_K^{(i-1)(j-\delta)} (Rg_{j-\delta})_{ij}
$$

and $M_{\ell} = \sqrt{d} (f_j^{d} \otimes I_D) H = \sum_{k=1}^{d} \omega_d^{-(\ell-1)(k-1)} H_k$, giving

$$
(M\ell)_{ij} = \sum_{k=1}^{d} \omega_d^{-(\ell-1)(k-1)} (H_k)_{ij} = \omega_K^{(i-1)(j-\delta)} \sum_{k=1}^{d} \omega_d^{-(\ell-1)(k-1)} (Rg_{j-\delta})_{k}
$$

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\[ = \sqrt{d} \omega_K^{(i-1)(j-\delta)} (\tilde{F}_d g_{j-\delta}) \ell. \]

In other words,

\[ M_\ell = \sqrt{d K} \tilde{F}_K \text{diag}(f_{2-\ell g 1-\delta}, \ldots, f_{2-\ell g \delta-1}). \quad (22) \]

If \( K = 2\delta - 1 \), then \( \tilde{F}_K \) is unitary, and the singular values of \( M_\ell \) are \( \{ \sqrt{d K} \mid f_{\ell g j} \mid \}_{\delta-1}^{1} \) (since \( |f_{2-\ell g j}| = |\tilde{F}_d g j| = |f_{\ell g j}| \)). Recognizing that \( S_j g_j = g_{-j} \), then Theorem 2 and (20) take us to (17). If \( D = 2\delta - 1 < K \), then the argument remains unchanged, except that the singular values of \( M_\ell \), instead of being known explicitly, are bounded above and below by \[ \max_{|j|<\delta} |f_{\ell g j}| \sigma_{\max}(\tilde{F}_K) \] and \[ \min_{|j|<\delta} |f_{\ell g j}| \sigma_{\min}(\tilde{F}_K) \] respectively, which gives (18). \( \square \)

### 2.3 Explicit Examples of Spanning Families

In this section, we analyze three explicit examples of masks \( \gamma \in \mathbb{R}^d \) and their corresponding local Fourier measurement systems, and prove under what conditions these constitute spanning families. The goal is to constructively provide examples of spanning families that are well-conditioned, and which are scalable in the sense that they may be used for any choice of \( d \) and \( \delta \). Specifically, we analyze Example 1 of [31] (also known as “exponential masks,” as we take \( \gamma_i = C a_i \) for some \( C, a \in \mathbb{R} \)) with the new results above, and find improvements on the bounds of its condition number, which scales roughly like \( \kappa \approx \delta^2 \). Then, a new set of masks is proposed and studied. These masks, referred to as “near-flat masks,” are constructed by taking \( \gamma = a e_1 + \mathbb{1}_{[\delta]} \in \mathbb{R}^d \), and we provide a choice of \( a \) that achieves a condition number that is asymptotically linear in \( \delta \) – a notable improvement over the conditioning of the exponential masks. Finally, we note the somewhat curious case of a constant mask, \( \gamma = \mathbb{1}_{[\delta]} \). Here, \( \gamma \) produces a spanning local Fourier measurement system – with poor conditioning – when \( d \)’s prime divisors are each greater than \( \delta \).

#### Example 1: Exponential Masks

We first briefly review an example initially proposed in [31]; in fact, this family of masks is the first spanning family to have been studied in the relevant literature. Here, we will take \( d \in \mathbb{N} \) to be the ambient dimension and \( \delta \leq \frac{d+1}{2} \). Then, we let \( \{ m_j(a) \}_{j=1}^{2\delta-1} \) be the local Fourier measurement system with mask \( \gamma(a) \in \mathbb{R}^d \) defined by \( \gamma(a)_i = a^{i-1} \), for some \( 0 < a \in \mathbb{R}, a \neq 1 \). The authors in [31] show that the measurement operator \( \mathcal{A} \) associated with this family has a condition number bounded by \( \kappa \leq \max\{144e^2, (\frac{3e(\delta-1)}{2})^2\} \) when \( a \) is chosen to be \( \max\{4, \frac{\delta-1}{2}\} \). This bound was slightly strengthened in [50] through a different choice of \( a \), but it was shown to be optimal up to a constant in the sense that, for optimal \( a(\delta) \), \( \kappa \) still grows as \( O(\delta^2) \).
Example 2: Near-Flat Masks

We now analyze masks of the form $\gamma = a e_1 + 1_{[\delta]}$ in Proposition 2, which, for certain choices of $a(\delta)$, are shown in Proposition 2 to produce linear systems with a condition number that is asymptotically less than that of the exponential masks by a factor of $\delta$.

Proposition 2  Let $\{m_j\}_{j \in [D]} \subseteq \mathbb{C}^d$ be the local Fourier measurement system of support $\delta \leq \frac{d+1}{2}$ with mask $\gamma \in \mathbb{R}^d$ given by $\gamma = a e_1 + 1_{[\delta]}$ where $a > \delta - 1$. Then this is a spanning family with condition number bounded above by

$$\kappa \leq \frac{a^2 + 2a + \delta}{a - \delta + 1}. \quad (23)$$

If we choose $a = 2\delta - 1$, we have $\kappa \leq 4\delta + 1$.

Proof of Proposition 2 We calculate the condition number directly. We immediately have $||\gamma||^2_2 = (a + 1)^2 + (\delta - 1) = a^2 + 2a + \delta$, which is the numerator of (23), so it remains only to provide a lower bound on $\sqrt{d} f_j^{d*}(\gamma \circ S^{-m} \gamma)$. To achieve this, we remark that, for $m \geq 1$,

$$\sqrt{d} f_j^{d*}(\gamma \circ S^{-m} \gamma) = a + \sum_{i=1}^{\delta-m} \omega_d^{(j-1)(i-1)} = \begin{cases} a + \frac{\delta-m}{a+\omega_d^{(j-1)(\delta-m)}} & j = 1 \\ a + \frac{\delta-m}{1-\omega_d^{(j-1)}} & \text{otherwise.} \end{cases}$$

Clearly, this expression has its maximum absolute value when $j = 1$, as $|a + \sum_{i=1}^{\delta-m} \omega_d^{(j-1)(i-1)}| \leq a + \sum_{i=1}^{\delta-m} |\omega_d^{(j-1)(i-1)}| = a + \delta - m$, so we consider that, for $j \neq 1$, we have

$$|\sqrt{d} f_j^{d*}(\gamma \circ S^{-m} \gamma)| \geq \left| \text{Re} \left( a + \frac{\delta-m}{1-\omega_d^{(j-1)}} \right) \right|. \quad (24)$$

We then reduce the term $\text{Re} \left( \frac{1-\omega_d^{(j-1)(\delta-m)}}{1-\omega_d^{j-1}} \right)$ by setting $e^{i\theta} := \omega_d^{j-1}$ and $k := \delta - m$ and finding

$$\text{Re} \left( 1 - e^{ik\theta} \right) = \text{Re} \left( \frac{(1-e^{ik\theta})(1-e^{-\delta i})}{2-2\cos \theta} \right) = \frac{(1-\cos \theta) + \cos(\delta-1)\theta - \cos k\theta}{2-2\cos \theta} = \frac{1}{2} + \frac{\sin(k - \frac{1}{2}) \theta \sin \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}, \quad (25)$$

where the third line comes from $\sin^2 x = (1 - \cos(2x))/2$ and $\cos a - \cos b = \frac{1}{2} \sin \left( \frac{a+b}{2} \right) \sin \left( \frac{b-a}{2} \right)$. Using $|\sin n\theta| \leq n|\sin \theta|$, this gives us that $\text{Re} \left( \frac{1-e^{ik\theta}}{1-e^{i\theta}} \right) \geq$
\[ \frac{1}{2} - \frac{2k-1}{2} = -k, \] and hence
\[ \left| \sqrt{d} f^d_j (\gamma \circ S^{-m} \gamma) \right| \geq a - \delta + m \geq a - \delta + 1 \quad (26) \]
for all \( 1 \leq m < \delta \). For \( m = 0 \), a similar calculation gives that
\[ \left| \sqrt{d} f^d_j (\gamma \circ \gamma) \right| \geq a^2 + 2a - \delta, \quad (27) \]
and we notice that this bound is greater than the bound for the case \( 1 \leq m \), stated in (26) whenever \( a \geq 1+\sqrt{2} \). This means (26) is tighter than (27) whenever \( a > \delta - 1 \), which is necessary for these bounds to be positive and meaningful. Therefore, we may restrict to choices of \( a > \delta - 1 \) and use the bound \( \left| \sqrt{d} f^d_j (\gamma \circ S^{-m} \gamma) \right| \geq a - \delta + 1 \). This completes the proof. \[
\square
\]
This result is a new contribution to the collection of well-conditioned local measurement systems. Example 2 of [33] constituted a local measurement system with a condition number that scales as \( O(\delta) \), but it was a somewhat cumbersome construction. Each of its members \( m_j \) was quite sparse, having either 1 or 2 nonzero entries, and did not correspond to a local Fourier measurement system (which corresponds to the diffraction process that we expect to govern our measurement apparatus). By contrast, the example in Proposition 2, while not necessarily simple to achieve in the lab, at least has the merit of accommodating the Fresnel diffraction model which motivates the study of local Fourier measurement systems, and its conditioning asymptotically equals that of the sparse construction that was previously the most well-conditioned measurement system known.

**Example 3: Constant Masks**

After these two examples, we also remark that the simplest type of mask – a constant mask, where \( \gamma = \mathbb{1}_{[\delta]} \) – can actually produce a spanning family, albeit a badly conditioned one. Moreover, the conditions required of \( \delta \) and \( d \) to admit this are upsettingly number theoretical, so we present this result in Proposition 3 as a negative, though relevant, result.

**Proposition 3** Take \( d \in \mathbb{N} \) and \( \delta \leq d \). Then, with \( D = \min(d, 2\delta - 1) \), the local Fourier measurement system \( \{m_j\}_{j=1}^D \) of support \( \delta \) and mask \( \gamma = \mathbb{1}_{[\delta]} \) is a spanning family if and only if \( d \) is strictly \( \delta \)-rough, in the sense that \( k \mid d \implies k > \delta \). In this event, and if we additionally take \( d > 4 \), the condition number of \( A \) is bounded by \( \kappa \leq \delta d^2/8 \).

**Proof of Proposition 3** We begin by remarking that \( \gamma \circ S^{-(\delta-k)} \gamma = \mathbb{1}_{[k]} \), such that
\[
\sqrt{d} f^d_j (\gamma \circ S^{-(\delta-k)} \gamma) = \sum_{i=1}^{k} \omega_d^{(j-1)(i-1)} = \begin{cases} k, & j = 1 \\ 1 - \omega_d^{(j-1)k}, & \text{otherwise} \end{cases}, \quad (28)
\]
and hence $f_j^{d_\ast} (\gamma \circ S^{-(\delta-k)} \gamma) = 0$ iff $(j - 1)k = nd$ for some positive integer $n$. By Corollary 1, this means that $\gamma$ produces a spanning family iff there does not exist a pair $(j, k) \in [d] \times [\delta]$ such that $jk = nd$ for some positive integer $n$. This condition occurs iff there is no pair $(j, k) \in [d] \times [\delta]$ such that $jk = d$, which is to say that $\gamma$ produces a spanning family iff $d$ is strictly $\delta$-rough. To get the condition number, consider that $||\gamma||^2 = |\mathbb{E}(1 - \omega d)| = 1 - \cos \left(\frac{2\pi}{d}\right)$. For $d > 4$, we use $1 - \cos(x) \geq (2x/\pi)^2$ to get that $(1 - \cos(\frac{2\pi}{d}))/2 \geq 8/d^2$, which completes the proof. The only possible case when $d \leq 4$ is $d = 3, \delta = 2$, and we can find by exhaustive calculation that $\kappa = \frac{2}{2 - \sqrt{3}}$. 

This result shows that, while constant masks can produce spanning families in some circumstances, the condition number of the resulting linear system is remarkably unstable as a function of the parameters of the discretization, $d$ and $\delta$. At the very least, we have that if $(\delta, d)$ admits a constant spanning local Fourier measurement system, then $(\delta, d + 1)$ will not. Since $d$ is intended to represent the number of pixels in the sensor array, this is a prohibitively specific requirement to be made of the discretization of the phase retrieval problem, so we emphasize that the result of Proposition 3 is of primarily mathematical interest.

2.4 Inverting $\mathcal{A}$

In this section, we use the results of Sect. 2.2 to explicitly state the inverse of the measurement operator $\mathcal{A}$ and remark that it can be used to easily deduce the computational complexity of calculating its inverse (initially calculated in [31], see also [33]). Fix a local measurement system $\{m_j\}_{j=1}^D$ with support $\delta$, with associated measurement operator $\mathcal{A}$ and canonical matrix representation $A$, as in Eqs. (12), and (13). Then, (19) and Lemma 6 give

$$A = P(D, d)(F_d \otimes I_D) \text{diag}(M_{\ell})_{\ell=1}^d (F_d \otimes I_D)^* P(d, D),$$

where we recall $M_{\ell}$ from (60). In the case where $\{m_j\}_{j=1}^D$ is a local Fourier measurement system with mask $\gamma$ and modulation index $K$, we define $Z \in \mathbb{C}^D \times d$ by

$$Z_{m\ell} = \sqrt{dK} f_{\ell}^{d_\ast} g_{m-\delta}.$$  

Setting $z_{\ell} = Ze_{\ell}$, we have

$$M_{\ell} = \tilde{F}_K D_{z_{\ell}} \text{ and diag}(M_{\ell})_{\ell=1}^d = (I_d \otimes \tilde{F}_K) \text{diag}(\text{vec}(Z)),$$  

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by which we further reduce $A$ to

$$A = P^{(D,d)}(F_d \otimes I_D)(I_d \otimes \tilde{F}_K) \text{diag}(\text{vec}(Z))(F_d \otimes I_D)^* P^{(d,D)}.$$ 

This reasoning immediately produces the inverse of $A$, which we state in Proposition 4.

**Proposition 4** Let $A \in \mathbb{C}^{dD \times dD}$ be the canonical representation of the measurement operator $A$ associated with a local Fourier measurement system $\{m_j\}_{j=1}^d$ of support $\delta \leq \frac{d+1}{2}$ with mask $\gamma \in \mathbb{R}^d$. Defining $Z$ as in (29), we have

$$A^{-1} = P^{(D,d)}(F_d \otimes I_D)(\text{diag}(\text{vec}(Z)))^{-1}(I_d \otimes \tilde{F}_K^*)(F_d \otimes I_D)^* P^{(d,D)}. \quad (31)$$

If $\{m_j\}_{j=1}^D$ is a general local measurement system, and $A$ is invertible, then its inverse is given by

$$A^{-1} = P^{(D,d)}(F_d \otimes I_D) \text{diag}(M_{\ell}^{-1})_{\ell=1}^d (F_d \otimes I_D)^* P^{(d,D)}.$$ 

This formulation makes it straightforward to deduce the computational complexity of inverting $A$. In the case of local Fourier measurement systems, the dominant cost is that of computing $D = O(\delta)$ Fourier transforms of size $d$, so the cost of inverting $A$ comes out to $O(\delta d \log d)$, as in [31].

### 3 Phtchographic Model

In our model for the ptychographic setup of (3), we have so far assumed that measurements are taken corresponding to all shifts $\ell \in [d]_0$; in the notation of (3), this is equivalent to taking $P = [d]_0$. Herein we present a useful generalization to the case where $P = s[d/s]_0$, and $s \in \mathbb{N}$ divides $d$.

The motivation for studying this case is that, unfortunately, in practice taking $P = [d]_0$ is usually an impossibility, since in many cases an illumination of the sample can cause damage to the sample [59], and applying the illumination beam (which can be highly irradiative) repeatedly at a single point can destroy it. In ptychography as it is usually performed in the lab, the beam is shifted by a far larger distance than the width of a single pixel—instead of overlapping on $\delta - 1$ of $\delta$ pixels, adjacent illumination regions will typically overlap on a percentage of their support, on the order of 50% (or even less sometimes) [18,57]. Considering the risks to the sample and the costs of operating the measurement equipment, there are strong incentives to reduce the number of illuminations applied to an object, so our theory ought to address a model that reflects this.

---

1 The difficulty of moving the illumination apparatus at a scale equal to the desired optical resolution is another reason taking $P = [d]_0$ is a cumbersome assumption.
3.1 Measurement Operator and Its Domain

Towards this model, instead of using all shifts in our lifted measurement system, we fix a shift size $s \in \mathbb{N}, s < \delta$, where $d = \overline{d}s$ with $\overline{d} \in \mathbb{N}$ and use $S^{s \ell} m_j m^*_j S^{-s \ell}$ for $\ell \in [\overline{d}]_0$. We introduce the following generalization of the lifted measurement system: given a family of masks of support $\delta, \{m_j\}_{j \in [D]} \subseteq \mathbb{C}^d$, and $s, \overline{d} \in \mathbb{N}$ with $\overline{d} = d/s$, the associated lifted measurement system of shift $s$ is

$$L^s_{\{m_j\}} := \{S^{s \ell} m_j m^*_j S^{-s \ell}\}_{(\ell, j) \in [\overline{d}]_0 \times [D]} \subseteq \mathbb{C}^{d \times d}.$$  

(32)

This leads to an obvious redefinition of the measurement operator, now $\mathcal{A} : \mathbb{C}^{d \times d} \rightarrow \mathbb{R}^{[\overline{d}]_0 \times [D]}$:

$$\mathcal{A}(X)(\ell, j) = \langle S^{s \ell} m_j m^*_j S^{-s \ell}, X \rangle, \quad (\ell, j) \in [\overline{d}]_0 \times [D].$$  

(33)

This will also force us to reconsider the subspace of $\mathbb{C}^{d \times d}$ with which we are working in the domain of $\mathcal{A}$, since it is clearly impossible, by inspection of Fig. 1, for $L^s$ to span $T_\delta(\mathbb{C}^{d \times d})$ with a shift size $s > 1$. In an effort to define the subspace analogous to $T_\delta(\mathbb{C}^{d \times d})$ in the ptychographic case, we let $J_{\delta, s} = \bigcup_{\ell \in [\overline{d}]_0} \text{supp}(S^{s \ell} \mathbb{1}_{[\delta]} \mathbb{1}_{[\delta]}^* S^{-s \ell})$ be the set of indices “reached” by this system, and we let

$$T_{\delta, s}(X) = \begin{cases} X_{ij}, & (i, j) \in J_{\delta, s} \\ 0, & \text{otherwise} \end{cases}$$

(34)

be the projection onto the associated subspace of $\mathbb{C}^{d \times d}$. $T_{\delta, s}$ is visualized in Fig. 1.

3.2 Conditioning of $\mathcal{A}$ for Ptychography

With this setup in hand, we begin our analysis of the linear system $\mathcal{A}(X) = y$ with a number of lemmas that unravel the structure of this operator. Our goal will be to proceed similarly to Sect. 2.2 by rewriting $\mathcal{A}$ as a product of a block-circulant matrix with certain permutations, at which point we will be able to cite Corollary 3, which
renders a convenient expression for the condition number. In service of this strategy, in this section we introduce a few new operators that are useful in the analysis of \( \mathcal{A} \). For \( N \in \mathbb{N} \), we define \( T_N : \bigcup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell N \times m} \to \bigcup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell m \times N} \), the blockwise transpose operator, defined by

\[
T_N \left( \begin{bmatrix} V_1 \\ \vdots \\ V_\ell \end{bmatrix} \right) = \begin{bmatrix} V_1^* \\ \vdots \\ V_\ell^* \end{bmatrix}
\]

for \( V_1, \ldots, V_\ell \in \mathbb{C}^{N \times m} \). We also define, for \( (k_j)_{j=1}^n \) and permutation \( P \in \{0,1\}^{n \times n} \), the blockwise permutation operator \( \mathcal{P}(P, (k_j)) : \mathbb{C}^{K \times K} \to \mathbb{C}^{K \times K} \), where \( K = \sum_{j=1}^n k_j \). Our intention will be to permute the blocks of a block vector \( [v_1^T \cdots v_n^T]^T \), where \( v_j \in \mathbb{C}^{k_j} \). In order to specify \( \mathcal{P}(P, (k_j)) \) precisely, we permit an overloading of notation on permutations: namely, if \( P \in \{0,1\}^{m \times m} \) is a permutation, then we identify \( P \) with the mapping \( \pi : [m] \to [m] \) where \( \pi(i) = j \) whenever \( Pe_i = e_j \). In particular, if we write \( P(i) \), we mean “\( j \) such that \( Pe_i = e_j \)”.

With this in mind, \( \mathcal{P}(P, (k_j)) \) is defined, for \( v_j \in \mathbb{C}^{k_j} \), by

\[
\mathcal{P}(P, (k_j)) \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_{P(1)} \\ \vdots \\ v_{P(n)} \end{bmatrix}.
\]

### 3.3 Matrix Representation and Conditioning of \( \mathcal{A} \)

To begin the discussion of the matrix representation of \( \mathcal{A} \), we refresh our notation: \( d, \delta, \overline{d}, s \in \mathbb{N} \) satisfy \( d = \overline{d}s \) and \( 2\delta - 1 \leq d \). We have \( D \in \mathbb{N} \) (arbitrary for now) measurement vectors \( \{m_j\}_{j \in [D]} \in \mathbb{C}^d \) satisfying \( 1 \in \text{supp}(m_j) \subseteq [\delta] \), and we set \( g^k_m = \text{diag}(m_k m^*_k, m) \) for all \( 1 - \delta \leq m \leq \delta - 1 \) and all \( k \in [D] \). The expressions \( \mathcal{L}^s_{\{m_j\}}, T_{\delta,s}, \) and \( \mathcal{A} \) are defined in Eqs. (32)–(34).

We now consider the question of when span \( \mathcal{L}^s_{\{m_j\}} = T_{\delta,s} \) and what the condition number of \( \mathcal{A} \) will be. As in (15), we vectorize \( X \) by its diagonals \(^2\) with \( D_\delta(X) \in \mathbb{C}^{d(2\delta - 1)} \) and write \( A \in \mathbb{C}^{\overline{D}D \times (2\delta - 1)d} \) such that

\[
(AD_\delta(X))_{(j-1)\overline{D}+\ell} = \begin{pmatrix} \text{diag}(X, 1 - \delta) \\ \vdots \\ \text{diag}(X, \delta - 1) \end{pmatrix}_{(j-1)\overline{D}+\ell} = \mathcal{A}(X)_{(\ell-1,j)}
\]

\[
= \langle S^{(\ell-1)}m_j m^*_j S^{(\ell-1)}, X \rangle = \sum_{m=1-\delta}^{\delta-1} S^{(\ell-1)} m^*_j g^m \text{diag}(X, m),
\]

\(^2\) Notice that this will force \( A \), the matrix representing \( \mathcal{A} \), to be singular. We expand on this later.
which gives the \((j - 1)d + \ell\)th row of \(A\) as

\[
\begin{bmatrix}
S^\ell(\ell - 1) g_{1-\delta}^j \\
\vdots \\
S^\ell(\ell - 1) g_{\delta-1}^j
\end{bmatrix}^* = \begin{bmatrix}
\text{circ}(R_d T_s g_{1-\delta}^j) \\
\vdots \\
\text{circ}(R_d T_s g_{\delta-1}^j)
\end{bmatrix} = \begin{bmatrix}
\text{circ}(R_d T_s g_{1-\delta}^D) \\
\vdots \\
\text{circ}(R_d T_s g_{\delta-1}^D)
\end{bmatrix}.
\]

so that, by Lemma 7, we have

\[
A = \begin{bmatrix}
\text{circ}^s(g_{1-\delta}^1) & \cdots & \text{circ}^s(g_{1-\delta}^D) \\
\vdots & \ddots & \vdots \\
\text{circ}^s(g_{\delta-1}^1) & \cdots & \text{circ}^s(g_{\delta-1}^D)
\end{bmatrix}^* = \begin{bmatrix}
\text{circ}(R_d T_s g_{1-\delta}^1) & \cdots & \text{circ}(R_d T_s g_{\delta-1}^1) \\
\vdots & \ddots & \vdots \\
\text{circ}(R_d T_s g_{1-\delta}^D) & \cdots & \text{circ}(R_d T_s g_{\delta-1}^D)
\end{bmatrix}.
\]

However, because \(T_{\delta,s} \subseteq T_\delta\) when \(s > 1\), this operator can never be invertible. In fact, when \(s > 1\), \(A\) has several completely zero columns, corresponding to the coordinates of entries in \(T_{\delta}/T_{\delta,s}\), so that, by Lemma 7, we have

\[
\text{circ}(R_d T_s g_{1-\delta}^j) = \text{circ}(R_d T_s g_{\delta-1}^j).
\]

To compute the condition number of \(A|_{T_{\delta,s}(\mathbb{C}^{d \times d})}\), we may take an orthogonal basis matrix \(N\) for \(D_{\delta}(T_{\delta,s})\) and analyze the singular values of \(AN\). We construct \(N\) by considering that, for each \(m = 1 - \delta, \ldots, \delta - 1\), \(R_d T_s g_m^j \in \mathbb{C}^{d \times s}\) has \(\min\{s, \delta - |m|\}\) non-zero columns; specifically, these are columns \([1, \delta - m + 1]\) for \(m \geq 0\) and \([|m| + 1, \delta + 1]\) mod \(m < 0\). We denote and enumerate these intervals by \(J_m = \{j_1^m, \ldots, j_{|m|}^m\}\) and set \(N_m := \left[ e_{j_1^m} \cdots e_{j_{|m|}^m} \right] \) such that \(R_d T_s g_m^j N_m\) has no zero columns. The identity \(\text{circ}(AB) = \text{circ}(A)(I \otimes B)\) gives

\[
\text{circ}(R_d T_s g_m^j N_m) = \text{circ}(R_d T_s g_m^j)(I_s \otimes N_m).
\]

so setting \(N := \text{diag}(I_s \otimes N_m)_{m=1-\delta}^{\delta-1}\), the columns of \(N\) form a basis for \(\text{Row}(A)\) as desired. This result is summarized in Proposition 5.

**Proposition 5** Fix \(s, \delta, d \in \mathbb{N}\) satisfying \(s < \delta, s \mid d, \delta \leq \frac{d+1}{2}\). Define \(J_m := [1, \delta - m + 1]\) for \(m \in [0, \delta]\) and \(J_m := [|m| + 1, \delta + 1]\) mod \(m \in [1 - \delta, 0]\). Further setting \(N_m := \left[ e_{j_1^m} \cdots e_{j_{|m|}^m} \right]\) and \(N := \text{diag}(I_s \otimes N_m)_{m=1-\delta}^{\delta-1}\), we have \(\text{Col}(N) = D_{\delta}(T_{\delta,s}(\mathbb{C}^{d \times d}))\).

To prove a result analogous to that of Theorem 2 for \(AN\), we will need to show that the restriction matrix \(N\) commutes well with the permutations used in the condition number analysis of Sect. 2.2, preserving the block-circulant structures that made the

\[\text{circ}^s(g_m^k) \in \mathbb{C}^{d \times s}\] and \(\text{circ}(R_d T_s g_m^k) \in \mathbb{C}^{d \times d}\).

4 By \(V/W\), where \(W \subseteq V\), we mean \(V \cap W^\perp\).
analysis possible. Thankfully it does; following the intuition of (19), referring to our expression of $A$ in (37), and making use of Lemmas 3 and 8, we can arrive at

\[
A' := \text{circ} \left( p(\overline{\mathcal{G}, D}) \left( p(\overline{\mathcal{G}, 2\delta - 1} \otimes I_s) \right)^* \right)
\]

This may be reduced further by applying Lemma 9, which gives us that, setting

\[
P_1 = \mathcal{P}(P(2\delta - 1, \overline{\mathcal{J}}), (s)_{j=1}^{\overline{\mathcal{J}}} (2\delta - 1)) = P(2\delta - 1, \overline{\mathcal{J}}) \otimes I_s
\]

we will have $N = \text{diag}(I_{\overline{\mathcal{J}}} \otimes N_m)_{m=1-\delta}^{{\delta-1}} = P_1(I_{\overline{\mathcal{J}}} \otimes N') P_2^*$. This gives

\[
A'(I_{\overline{\mathcal{J}}} \otimes N') = P(\overline{\mathcal{G}, D}) A P_1(I_{\overline{\mathcal{J}}} \otimes N') = P(\overline{\mathcal{J}, D}) A N P_2.
\]

We refer to (38) to obtain

\[
A'(I_{\overline{\mathcal{J}}} \otimes N') = \text{circ} \left( p(\overline{\mathcal{G}, D}) (I_D \otimes R_{\overline{\mathcal{J}}}) \left( \begin{array}{cccc}
T_{s g_{1-\delta}} & \cdots & T_{s g_{1-1}} \\
\vdots & \ddots & \vdots \\
T_{s g_{1-\delta}} & \cdots & T_{s g_{1-1}} \\
\end{array} \right) \right) N',
\]

which, along with (39) and Corollary 3, gives us Theorem 3.

**Theorem 3** Take $A$ as in (37), $N$ and $N_m$ as in Proposition 5,

\[
H = P(\overline{\mathcal{J}, D}) (I_D \otimes R_{\overline{\mathcal{J}}}) \left( \begin{array}{cccc}
T_{s g_{1-\delta}} & \cdots & T_{s g_{1-1}} \\
\vdots & \ddots & \vdots \\
T_{s g_{1-\delta}} & \cdots & T_{s g_{1-1}} \\
\end{array} \right) \text{diag}(N_m)_{m=1-\delta}^{{\delta-1}}
\]

and $M_j = \sqrt{d}(f_j^D \otimes I_D)^* H$ for $j \in [\overline{\mathcal{J}}]$. The condition number of $AN$ is given by

\[
\max_{i \in [\overline{\mathcal{J}}]} \sigma_{\text{max}}(M_i) \quad \min_{i \in [\overline{\mathcal{J}}]} \sigma_{\text{min}}(M_i).
\]

In particular, $A|_{T_{s, t}(\mathbb{C}^d \times d)}$ is invertible if and only if each of the $M_i$ are of full rank.

It remains an open problem to find an explicit construction of masks $\gamma$ that may scale with $d$, $\delta$, and $s$, and which provably yields measurement operators of bounded
condition number. Nonetheless, Theorem 3 provides a certificate for any fixed collection of masks at a specific choice of dimensions, and the interested reader can experiment with various constructions using resources at [51].

4 Analysis of the Recovery Algorithm

In this section, we focus on algorithms by which we can estimate $x_0$ from $T_{\delta,s}(A^{-1}(y))$. We begin with Sect. 4.1, which discusses some improvements over the results of [33] that can be made to the magnitude estimation step of Algorithm 1 (line 2). Indeed, these improvements, which we call “blockwise” magnitude estimation, were first implemented (without theoretical analysis) in the numerical study of [33]. While they empirically delivered better results, no proof was provided to quantify the improvement, and we remedy this in Sect. 4.1. Having dealt with the magnitude estimation step in the ptychographic setting, we then handle the phase estimation step in Sect. 4.2. Finally, Sect. 4.3 describes and proves the robustness bounds for an algorithm analogous to that of [33], albeit fully capable of handling ptychographic shifts $s > 1$, and taking advantage of the results in Sect. 4.1 and Sect. 4.2 for improved magnitude and phase estimation.

4.1 Blockwise Magnitude Estimation

For the moment, we restrict our discussion to the special case of dense shifts in our measurements, where $s = 1$ as in [33] and Sect. 2. In [33], a functional but rudimentary technique to calculate the magnitudes of the entries of $x_0$ from $X \approx T_{\delta}(x_0 x_0^*)$ was proposed. It relied, as we do here, on first computing $X = A^{-1}(A(x_0 x_0^*) + n)$, where $x_0 \in \mathbb{C}^d$ is the ground truth objective vector, $A : \mathbb{C}^{d \times d} \to \mathbb{R}^{d \times d}$ is the linear measurement operator determined by the masks $\{ m_j \}_{j \in [D]}$, as defined, for example, in (12) of Sect. 2, and $n \in \mathbb{R}^{d \times d}$ is arbitrary noise. Then $X = X_0 + A^{-1}(n)$, where $X_0 = T_{\delta}(x_0 x_0^*)$, and the magnitude of $(x_0)_i$ was estimated by simply taking $|x_i| = \sqrt{X_{ii}} \approx \sqrt{X_{0ii}} = |x_{0i}|$. This technique works, as was proven in [33], but it was also seen that empirically, a more sophisticated technique does a much better job. Next, we will prove stability bounds for the improved technique, which we now describe.

We notice that taking $|x_i| = X_{ii}$ is equivalent to taking $|x_i|$ to be the rank-1 approximation of the $1 \times 1, i^{th}$ diagonal block matrix of $X$, namely $[X_{ii}]$, since these diagonal blocks are equal to the diagonal blocks of the untruncated $x_0 x_0^*$ when there is no noise. However, given the width of the diagonal band in $T_{\delta}(\mathbb{C}^{d \times d})$, we could just as easily take blocks of size up to $\delta \times \delta$ and calculate their top eigenvectors; this would give us $2\delta - 1$ estimates for each entry’s magnitude, so we can combine them by averaging them together. To denote these blocks, we will set $|X|^{(\ell)} = \text{diag}(\mathbb{1}_{[\delta]}^\perp)X\text{diag}(\mathbb{1}_{[\delta]}^\perp)^5$ and $u^{(\ell)}$ to be the top eigenvector of $|X|^{(\ell)}$, normalized such that $||u^{(\ell)}||_2 = |||X|^{(\ell)}||_2$. We then produce our estimate of the magnitudes by taking $|x| = \frac{1}{\delta} \sum_{\ell=1}^{d} u^{(\ell)}$.

Here, and in the remainder of this section, we emphasize that all indices of objects in $\mathbb{C}^d$ and $\mathbb{C}^{d \times d}$ are taken modulo $d$. 
Before formally stating this algorithm, we observe how it may be generalized. Firstly, we notice that this method can easily handle arbitrary block sizes \( m \) for the blockwise, eigenvector-based magnitude estimations by simply taking \( |X|^{(\ell, m)} = \text{diag}(\mathbb{1}_{[m]})|X| \text{diag}(\mathbb{1}_{[m]}) \), \( m \leq \delta \) — although this would require changing the denominator in the averaging step, using \( \frac{1}{m} \sum_{\ell=1}^{d} u^{(\ell, m)} \). Proceeding further, we can generalize this technique to use any collection \( \{J_i\}_{i=1}^{N} \), \( J_i \subseteq [d] \) satisfying
\[
[d] \subseteq \bigcup_{i=1}^{N} J_i \quad \text{and} \quad \mathbb{1}_{J_i} \mathbb{1}_{J_i}^{*} \in T_{\delta}(\mathbb{C}^{d \times d}).
\] (40)

We will call any collection satisfying (40) a \((T_{\delta}, d)\)-covering (or just covering when \( T_{\delta} \) and \( d \) are clear from context), and the process of estimating magnitudes of \( x_0 \) from \( X \) with respect to a \((T_{\delta}, d)\)-covering is described in Algorithm 3. It is worth remarking that the “averaging step,” specified in line 3, is optimal in the least-squares sense. We set \( P_{J_i} = \text{diag}(\mathbb{1}_{J_i}) \) to be the orthogonal projections onto the coordinate subspace associated with \( J_i \) and consider that the vectors \( u^{(J_i)} \) (in line 2) represent estimates of the projections \( P_{J_i} |x_0| \). The least squares solution to \( P_{J_i} u = u^{(J_i)} \), or
\[
\begin{bmatrix}
P_{J_1} \\
\vdots \\
P_{J_N}
\end{bmatrix}
\begin{bmatrix}
u^{(J_1)} \\
\vdots \\
u^{(J_N)}
\end{bmatrix} = \begin{bmatrix}
u^{(J_1)} \\
\vdots \\
u^{(J_N)}
\end{bmatrix}
\]
is obtained by taking the pseudoinverse. In this case, we have
\[
\begin{bmatrix}
P_{J_1} \\
\vdots \\
P_{J_N}
\end{bmatrix}^{*} \begin{bmatrix}
P_{J_1} \\
\vdots \\
P_{J_N}
\end{bmatrix} = \sum_{i=1}^{N} P_{J_i}^{*} P_{J_i} = \sum_{i=1}^{N} P_{J_i} = \text{diag}(\mu),
\]
with \( \mu_j = |\{i : j \in J_i\}| \) as in line 1 of Algorithm 3. Considering that \( P_{J_i} u^{(J_i)} = u^{(J_i)} \), we have
\[
u = \begin{bmatrix}
P_{J_1} \\
\vdots \\
P_{J_N}
\end{bmatrix}^{*} \begin{bmatrix}
u^{(J_1)} \\
\vdots \\
u^{(J_N)}
\end{bmatrix} = \text{diag}(\mu)^{-1} \begin{bmatrix}
P_{J_1} \\
\vdots \\
P_{J_N}
\end{bmatrix}^{*} \begin{bmatrix}
u^{(J_1)} \\
\vdots \\
u^{(J_N)}
\end{bmatrix} = D_{\mu}^{-1} \left( \sum_{i=1}^{N} u^{(J_i)} \right).
\]

We will denote the output of Algorithm 3 by \(|x| = \text{BlockMag}(X, \{J_i\})\). In an overloading of notation, when the covering consists of intervals of length \( m \), in the sense that \( J_i = [m]_i \) for \( i = 1, \ldots, d \), we will also write it as \( \text{BlockMag}(X, \{[m]_i\}_{i \in [d]}) = \text{BlockMag}(X, m) \), see Fig 2. To include the case of ptychography, where these intervals are shifted by more than 1, we write

\[6\] We remark that this definition and the recovery algorithm are very obviously extensible to the use of \( T_{\delta,s} \) instead of \( T_{\delta} \). In fact, this is a restriction, if we consider in (40) that \( T_{\delta,s} \subseteq T_{\delta} \). The definition, therefore, of a \((T_{\delta,s}, d)\)-covering, is made by analogy to (40).
Algorithm 3 Blockwise Magnitude Estimation

Input: $X \in T_{\delta,d}(\mathcal{H}^d)$, typically assumed to be an approximation $X \approx T_{\delta,s}(x_0x_0^*)$. A $(T_{\delta,s}, d)$-covering \{J_i\}_{i \in [N]}.

Output: An estimate $|x|$ of $|x_0|$.

1: For $j \in [d]$, set $\mu_j = \{|i : j \in J_i|\}$ to be the number of appearances the index $j$ makes in $\{J_i\}$.

2: For $i \in [N]$, set $u^{(J_i)}$ to be the leading eigenvector of $|X^{(J_i)}| = \text{diag}(1_{J_i})|X|\text{diag}(1_{J_i})$, normalized such that $||u^{(J_i)}||_2 = \sqrt{||X^{(J_i)}||_2}$.

3: Return $|x| = D^{-1}_\mu \left( \sum_{i=1}^N u^{(J_i)} \right)$.

\begin{align*}
\text{BlockMag}(X, m) &= \text{BlockMag}(X, \{|m|_i\}_{i \in [d]}), \quad \text{and} \\
\text{BlockMag}(X, (m, s)) &= \text{BlockMag}(X, \{|m|_{1+s-\ell-1}\}_{\ell \in [d]}), \\
\end{align*}

where $d = \frac{d'}{s}$ is an integer. In this way, the improved magnitude estimation technique used in the numerical experiments of [33] is $|x| = \text{BlockMag}(X, \delta)$, while $|x_i| = \sqrt{x_{ii}} = \text{BlockMag}(X, 1)$. We can prove a bound on the error of the estimate.

**Proposition 6** Let $\{J_i\}_{i \in [N]}$ be a $(T_{\delta,s}, d)$-covering, and suppose $X_0 = T_{\delta,s}(x_0x_0^*)$ for some $x_0 \in \mathbb{C}^d$. Using the notation of Algorithm 3 (in particular, $\mu_j$ is as in line 1, and $u^{(J_i)} = |\text{diag}(1_{J_i})x_0|$), given $X \in T_{\delta,s}(\mathcal{H}^d)$, we have that the output $|x|$ satisfies

$$||\text{BlockMag}(X, \{J_i\}) - |x_0||_2 \leq \frac{\max_j \mu_j}{\min_j \mu_j} \frac{1 + 2\sqrt{2}}{\min_i ||u^{(J_i)}||_2} ||X_0 - X||_F.$$  \hspace{1cm} (42)

As special cases for $T_{\delta}(\mathbb{C}^{d\times d})$, we have

$$||\text{BlockMag}(X, m) - x_0||_2 \leq \frac{1 + 2\sqrt{2}}{\min_i ||u^{[m]}||_2} ||X - X_0||_F$$

$$||\text{BlockMag}(X, \delta) - x_0||_2 \leq \frac{1 + 2\sqrt{2}}{\min_i ||u^{[\delta]}||_2} ||X - X_0||_F$$

$$||\text{BlockMag}(X, 1) - x_0||_2 \leq \frac{||\text{diag}(X - X_0)||_F}{\min_i |x_0|} \hspace{1cm} (43)$$
Proof of Proposition 6  

The first inequality of (42) is clear, since line 2 of Algorithm 3 will always return \( u^{(J_i)} = 1 \) \( \forall x_0 \), so line 3 will give

\[
\left( D_\mu^{-1} \left( \sum_{i=1}^N u^{(J_i)} \right) \right)_j = \frac{1}{\mu_j} \sum_{i=1}^N [x_0]_j \mathbb{1}_{j \in J_i} = [x_0]_j,
\]

The second comes by writing

\[
\left\| D_\mu^{-1} \left( \sum_{i=1}^N u^{(J_i)} - u^{(J_i)} \right) \right\|_2^2 \leq \left( \frac{1}{\min_j \mu_j} \right)^2 \left\| \sum_{i=1}^N (u^{(J_i)} - u^{(J_i)}) \right\|_2^2.
\]

From there, we consider that the \( j \)th term in the summation

\[
\left\| \sum_{i=1}^N (u^{(J_i)} - u^{(J_i)}) \right\|_2^2 = \sum_{j=1}^d \left( \sum_{i=1}^N (u^{(J_i)} - u^{(J_i)}) \right)_j^2.
\]

has at most \( \max_k \mu_k \) nonzero summands, so, by \( (\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2 \), we have

\[
\left\| \sum_{i=1}^N (u^{(J_i)} - u^{(J_i)}) \right\|_2^2 \leq \max_j \mu_j \sum_{i=1}^N (u^{(J_i)} - u^{(J_i)})^2.
\]

We then apply Lemma A.2 of [33] to get

\[
\sum_{i=1}^N (u^{(J_i)} - u^{(J_i)})^2 \leq (1 + 2\sqrt{2}) \frac{||u^{(J_i)} u^{(J_i)*} - X^{(J_i)}||^2_F}{||u^{(J_i)}||^2_2}
\]

In this expression, we consider that, in the summation \( \sum_{i=1}^N ||u^{(J_i)} u^{(J_i)*} - X^{(J_i)}||^2_F \), the term \( (X_{0ij} - X_{ij})^2 \) appears at most \( \max \{\mu_i, \mu_j\} \) times, such that \( \sum_{i=1}^N ||u^{(J_i)} u^{(J_i)*} - X^{(J_i)}||^2_F \leq \max_j \mu_j ||X_0 - X||^2_F \). Using this substitution, combining (44)–(47), and taking the square root of both sides gives

\[
\left\| D_\mu^{-1} \left( \sum_{i=1}^N u^{(J_i)} - u^{(J_i)} \right) \right\|_2 \leq \frac{\max_j \mu_j}{\min_j \mu_j} \frac{1 + 2\sqrt{2}}{\min_i ||u^{(J_i)}||_2} ||X_0 - X||_F
\]

\footnote{Here, we use the substitution \( \eta ||x_0||_2 = \frac{||X - x_0||_F}{||x_0||_2} \).}
as desired. The first two inequalities of (43) are immediate by observing that \( \mu_j = m \) when the covering is \( \{[m]_i\}_{i \in [d]} \). The third comes from setting \( \epsilon_i = X_{ii} - (X_0)_{ii} \) and writing

\[
\| |x| - |x_0| \|_2^2 = \sum_{i=1}^{d} \left( \sqrt{X_{ii}} - |x_0| \right)^2 = \sum_{i=1}^{d} \left( \sqrt{|x_0|^2 + \epsilon_i} - \sqrt{|x_0|^2} \right)^2 \\
= \sum_{i=1}^{d} \left( \frac{(|x_0|^2 + \epsilon_i) - |x_0|^2}{\sqrt{|x_0|^2 + \epsilon_i} + |x_0|} \right)^2 \leq \sum_{i=1}^{d} \frac{\epsilon_i^2}{\min_i |x_0|^2} = \frac{|\text{diag}(X - X_0)|^2}{\min_i |x_0|^2}.
\]

One immediate benefit from Eq. (43) is that the estimation error from BlockMag \((X, 1)\) no longer scales poorly with \( d^{1/4} \) as in Theorem 5 of [33]. In the error bound for BlockMag \((X, m)\) in Eq. (43), we notice that the bound is strictly decreasing with \( m \), since, for \( m_1 > m_2 \), we have

\[
\min_i \|x_0^{[m_1]}\|_2^2 \geq (m_1 - m_2) \min_i \|x_0\|_2^2 + \min_i \|x_0^{[m_2]}\|_2^2 \geq m_1 \min_i \|x_0\|_2^2.
\]

Also, considering (42), it is clear that BlockMag \((X, \delta)\) gives the absolute best bound over all \( \{J_i\} \), since any \((T_\delta, d)\)-covering \( \{J_i\}_{i \in [N]} \) satisfies \( J_i \subseteq \delta \) for some \( \ell \). This gives that \( \min_i \|u(J_i)\|_2 \leq \min_i \|u(\delta)\|_2 \), and obviously \( \max_i \mu_j \mu_j \geq 1 \), so the bound for BlockMag \((X, \{J_i\})\) in (42) can never be better than that for BlockMag \((X, \delta)\) in (43). We also remark that two easy ways to ensure \( \frac{\max_i \mu_j}{\min_i \mu_j} = 1 \) is minimized are to take \( J_0 \subseteq \delta \) and let \( J_\ell = J_0 + \ell \) be a “cyclic” covering, or to let \( \{J_i\} \) be a partition of \([d]\). These strategies will be relevant in Sect. 4.3, but in the case of \( s = 1 \), BlockMag \((X, \delta)\) always has the optimal bound for magnitude estimation error.

4.2 Phase Estimation for Ptychography

We now consider a simple algorithm for estimating the relative phase between the entries of \( x_0 \), from an estimate of \( T_{\delta, s}(x_0 x_0^*) \) obtained, say, by inverting \( A \) on \( T_{\delta, s}(\mathbb{C}^{d \times d}) \). This algorithm is a generalization of an analogous one in [33], and is similar to another algorithm, albeit with slightly different bounds, that was very recently presented in [44]. For simplicity, we restrict our attention to the case where \( s \) divides \( \delta \), and \( \delta \) divides \( d \), so that \( \delta = s \delta \) and \( d = s \delta \). We defer an analysis of the general case with arbitrary \( s, \delta, d \), as well as a study of more sophisticated algorithms (e.g., based on semidefinite programming, as in [50]) to other work. The following simple, yet helpful, lemma holds.

**Lemma 1** If \( \delta, s, d \in \mathbb{N} \) are such that \( s \) divides \( \delta \), and \( \delta \) divides \( d \), then

\[
T_{\delta, s}(1_d 1_s^*) = T_{\delta/s}(1_{d/s} 1_{s/d}^*) \otimes (1_s^* 1_s^*).
\]

**Proof of Lemma 1** The proof consists of simply enumerating the non-zero indices of each of \( T_{\delta, s}(1_d 1_s^*) \) and \( T_{\delta/s}(1_{d/s} 1_{d/s}^*) \otimes (1_s^* 1_s^*) \). In both cases, these are
\[ \bigcup_{t \in [d_0]}^2 \delta_{\ell+1} \text{ by definition for the first and by the definition of the Kronecker product for the second.} \]

It then follows from the properties of Kronecker products that the eigenvalues of \( T_{\delta,s}(\mathbb{1}_d \mathbb{1}_d^*) \) are simply the pairwise products of the eigenvalues of \( T_{\delta/s}(\mathbb{1}_{d/s} \mathbb{1}_{d/s}^*) \) and \( \mathbb{1}_s \mathbb{1}_s^* \). Equally importantly, the eigenvectors of \( T_{\delta,s}(\mathbb{1}_d \mathbb{1}_d^*) \) are the pairwise Kronecker products of those of \( T_{\delta/s}(\mathbb{1}_{d/s} \mathbb{1}_{d/s}^*) \) and \( \mathbb{1}_s \mathbb{1}_s^* \). So, the entire eigendecomposition of \( T_{\delta,s}(\mathbb{1}_d \mathbb{1}_d^*) \) is fully known, as \( T_{\delta/s}(\mathbb{1}_{d/s} \mathbb{1}_{d/s}^*) \) is a circulant matrix, hence diagonalized by the discrete Fourier transform, and \( \mathbb{1}_s \mathbb{1}_s^* \) is a rank-one matrix. In particular, the normalized eigenvector of \( T_{\delta,s}(\mathbb{1}_d \mathbb{1}_d^*) \) corresponding to the leading eigenvalue is simply \( \mathbb{1}_d / \sqrt{d} \). Now, consider that \( \text{diag} \left( \frac{x_0}{|x_0|} \right) \) is unitary and that

\[
T_{\delta,s} \left( \frac{x_0 x_0^*}{|x_0 x_0^*|} \right) = \text{diag} \left( \frac{x_0}{|x_0|} \right) T_{\delta,s} \left( \mathbb{1}_d \mathbb{1}_d^* \right) \text{diag} \left( \frac{x_0}{|x_0|} \right)^* \]

so \( T_{\delta,s} \left( \frac{x_0 x_0^*}{|x_0 x_0^*|} \right) \) and \( T_{\delta,s} \left( \mathbb{1}_d \mathbb{1}_d^* \right) \) are similar. Given the eigendecomposition \( T_{\delta,s} \left( \mathbb{1}_d \mathbb{1}_d^* \right) = \mathbf{V} \Lambda \mathbf{V}^* \),

\[
T_{\delta,s} \left( \frac{x_0 x_0^*}{|x_0 x_0^*|} \right) = \left( \text{diag} \left( \frac{x_0}{|x_0|} \right) \mathbf{V} \right) \Lambda \left( \text{diag} \left( \frac{x_0}{|x_0|} \right) \mathbf{V} \right)^* \tag{48}
\]

is itself an eigendecomposition. Together, these observations imply that the eigenvector corresponding to the leading eigenvalue of \( T_{\delta,s} \left( \frac{x_0 x_0^*}{|x_0 x_0^*|} \right) \) is \( \text{diag} \left( \frac{x_0}{|x_0|} \right) \mathbb{1}_d / \sqrt{d} \), i.e., it is the vector of phases \( \frac{x_0}{|x_0|} \), up to a harmless normalization! This implies that in the absence of noise, we can obtain \( \text{sgn}(x_0) \) easily via an eigendecomposition. It remains to show that this procedure is robust to noise.

**Theorem 4** Fix \( s < \delta < d \in \mathbb{N} \) such that \( \delta \) divides \( d \) and \( s \) divides \( \delta \). Let \( \tilde{X}_0 = T_{\delta,s} \left( \frac{x_0 x_0^*}{|x_0 x_0^*|} \right) \), and let \( \tilde{X} \) be as in line 2 from Algorithm 2. Let \( \tilde{\tau} = \text{sgn}(v) \), with \( v \) being the leading eigenvector of \( \tilde{X} \). If \( \| \tilde{X}_0 - \tilde{X} \|_F \leq \tilde{\eta} \) for some \( \tilde{\eta} > 0 \), then there exists a positive constant \( C \) such that

\[
\min_{\theta \in [0,2\pi]} \| \text{sgn} (x_0) - e^{i\theta} \tilde{\tau} \|_2 \leq C \frac{\tilde{\eta} \delta^{s/2}}{\delta^2} \]

**Proof of Theorem 4** Given an undirected graph \( G = (V,E) \) with \( d \) vertices, let \( D \) be the diagonal matrix of its degrees and \( W \) be its adjacency matrix, and define its connection Laplacian \( L := I - D^{-1/2}WD^{-1/2} \). Denoting by \( 0 \leq \lambda_1 \leq \ldots \leq \lambda_d \) the spectral gap associated with \( G \) is \( \tau := \lambda_2 \). By Theorem 4 of [33], noting that the graph associated with the “adjacency matrix” \( T_{\delta,s}(\mathbb{1}_{d/s} \mathbb{1}_{d/s}^*) \) is \((2\delta - s)\)-regular, there exists a constant \( C' > 0 \), such that

\[
\min_{\theta \in [0,2\pi]} \| \text{sgn} (x_0) - e^{i\theta} \tilde{x} \|_2 \leq C' \frac{\tilde{\tau} \| \tilde{X} - \tilde{X}_0 \|_F}{\tau \sqrt{2\delta - s}} \leq C' \frac{\tilde{\eta}}{\tau \sqrt{2\delta - s}}.
\]
It remains to bound $\tau$ from below and to that end we denote by $v_1 \geq v_2 \geq \ldots v_d$ the eigenvalues of $T_{d/s}(\mathbb{I}_{d/s} \circ \mathbb{I}_{d/s})$. We then invoke Lemma 2 of [33] to conclude that there exists a constant $C'' > 0$, such that $\min(v_1 - |v_j|) \geq C''(d/s)^3 = C \frac{d^2}{d^2 s}$, and the same conclusion holds for $T_{d,s}(\mathbb{I}_{d} \circ \mathbb{I}_{d})$ by Lemma 1. Now, since the eigenvector of $\tilde{X}$ corresponding to its leading eigenvalue is an eigenvector of $L$ corresponding to its smallest eigenvalue, we have $\tau \geq C'' \frac{d^2}{d^2 s}$, so that

$$\min_{\theta \in [0,2\pi]} \| \text{sgn} \cdot \frac{x_0}{\|x_0\|} - e^{i\theta} \tilde{x} \|_2 \leq \frac{C' \tilde{\eta} d^2 s}{C'' d^2 \sqrt{2\delta - s}} \leq C \tilde{\eta} \frac{d^2}{\delta^2/2 s}.$$

\[\Box\]

**Corollary 2** Let $\tilde{X}_0 = T_{d,s} \left( \frac{x_0 s^2}{\|x_0\|} \right)$, and let $\tilde{X}$ be as in line 2 from Algorithm 2. Let $\tilde{x} = \text{sgn}(v)$, with $v$ being the leading eigenvector of $\tilde{X}$. If $\sigma_{\text{min}} := \sigma_{\text{min}}(A|_{T_{d,s}(\mathbb{C}^{d \times d})})$ and $\kappa := \kappa(A|_{T_{d,s}(\mathbb{C}^{d \times d})})$ denote the smallest singular value and condition number of $A|_{T_{d,s}(\mathbb{C}^{d \times d})}$, respectively, then

$$\min_{\theta \in [0,2\pi]} \| \text{sgn} \cdot \frac{x_0}{\|x_0\|} - e^{i\theta} \tilde{x} \|_2 \leq C \cdot \frac{d^2 \cdot \|X\|_F}{\delta^2/2} \cdot \frac{\sigma_{\text{min}}^{-1} \cdot \|n\|_2}{\min_j \|x_0\|_j^2}$$

and

$$\min_{\theta \in [0,2\pi]} \| \text{sgn} \cdot \frac{x_0}{\|x_0\|} - e^{i\theta} \tilde{x} \|_2 \leq C \cdot \frac{d^2 \cdot \|X\|_F}{\delta^2/2} \cdot \frac{\kappa \cdot \|X\|_F}{\text{SNR} \cdot \min_j \|x_0\|_j^2}$$

**Proof** We proceed as in the proof of Lemma 6 in [33]. Setting $N = X - X_0$ we have

$$\left| (\tilde{X}_0)_{j,k} - (\tilde{X})_{j,k} \right| = \left| (\tilde{X}_0)_{j,k} - \text{sgn} \left( \frac{X_{j,k}}{|(X_0)_{j,k}|} \right) \right| \leq \left| (\tilde{X}_0)_{j,k} - \frac{X_{j,k}}{|(X_0)_{j,k}|} \right| + \left| \frac{X_{j,k}}{|(X_0)_{j,k}|} - \text{sgn} \left( \frac{X_{j,k}}{|(X_0)_{j,k}|} \right) \right| \leq 2 \left| (\tilde{X}_0)_{j,k} - \frac{X_{j,k}}{|(X_0)_{j,k}|} \right| = 2 \left| \frac{N_{j,k}}{|(X_0)_{j,k}|} \right|.$$

This gives $\|\tilde{X} - \tilde{X}_0\|_F \leq \frac{2\|N\|_F}{\min_j \|x_0\|_{j,k}} \leq \frac{2\|N\|_F}{\min_j \|x_0\|_{j,k}}$, where $\|N\|_F$ can be bounded by $\sigma_{\text{min}}(A)^{-1} \|n\|_2$ or by $\kappa(A) \frac{\|X\|_F}{\text{SNR}}$. Combining this with Theorem 4 yields the result.

\[\Box\]

### 4.3 Recovery Algorithm for Ptychography

We are now ready to prove that Algorithm 2 stably produces an estimate of $x_0$; Theorem 5 provides a bound on the accuracy of the output of Algorithm 2.
\textbf{Theorem 5} Let \( d, d', \delta, s \in \mathbb{N} \) be such that \( s \) divides \( \delta \), and \( \delta \) divides \( d \) with \( d = \delta s \). Suppose we have a family of masks \( \{ m_j \}_{j \in [D]} \in \mathbb{C}^d \) of support \( \delta \) and \( a \) \((T_{\delta,s}, d)\)-covering \( J_\ell := \{ \{ m \}_{1 + s (\ell - 1)} \}_{\ell \in [d]} \). Denote by \( A \) the corresponding linear operator, and by \( \sigma_{\min} \) and \( \kappa \) the smallest singular value and condition number of \( A |_{T_{\delta,s}(\mathbb{C}^d \times \mathbb{C}^d)} \).

Further let \( x_0 \in \mathbb{C}^d, n \in \mathbb{R}^{dD} \) be arbitrary and set \( X_0 = T_{\delta,s}(x_0x_0^*) \), \( X = A^{-1}(A(X_0) + n) \). Define SNR = \( \frac{||A(X_0)||_2}{ ||n||_2 } \). Then the output \( x \) of Algorithm 2 satisfies

\[
\min_{\theta \in [0,2\pi]} ||x - e^{i\theta}x_0||_2 \leq C \cdot \sigma_{\min}^{-1} \left( \frac{d^2 \cdot s}{\delta^{3/2}} \cdot \frac{||x_0||_\infty}{\min_j |(x_0)_j|^2} + \frac{1}{\min_\ell \kappa} \frac{||\text{diag}(1_{J_\ell})x_0||_2}{||n||_2} \right) \cdot ||n||_2
\]

\[
\min_{\theta \in [0,2\pi]} ||x - e^{i\theta}x_0||_2 \leq C \cdot \kappa \cdot \left( \frac{d^2 \cdot s}{\delta^{3/2}} \cdot \frac{||x_0||_\infty}{\min_j |(x_0)_j|^2} + \frac{1}{\min_\ell \kappa} \frac{||\text{diag}(1_{J_\ell})x_0||_2}{||n||_2} \right) \cdot \frac{||X_0||_F}{\text{SNR}}. \tag{49}
\]

\textbf{Proof of Theorem 5} We set \( \tilde{x} = \text{sgn}(x) \), \( \tilde{x}_0 = \text{sgn}(x_0) \) and use the triangle inequality to obtain

\[
\min_{\theta \in [0,2\pi]} ||x - e^{i\theta}x_0||_2 \leq ||x_0||_\infty \min_{\theta \in [0,2\pi]} ||\tilde{x} - e^{i\theta}\tilde{x}_0||_2 + ||x||_2 - ||x_0||_2.
\]

We bound each of the summands separately. Specifically, we directly use Corollary 2 to bound \( \min_{\theta \in [0,2\pi]} ||\tilde{x} - e^{i\theta}\tilde{x}_0||_2 \) and Proposition 6, Eq. \eqref{eq:42}, to bound \( ||x||_2 - ||x_0||_2 \).

For the latter, we use the fact that our choice of \( d, s, \delta \) yields \( \frac{\max_j \mu_j}{\min_j \mu_j} = 1 \), and we use the stability of the linear system \( A \) to bound \( ||X - X_0||_F \) above by \( \sigma_{\min}^{-1} ||n||_2 \) and \( \kappa \frac{||X_0||_F}{\text{SNR}} \).

For example, when \( s = 1 \), the error bounds of Theorem 5 can be made more explicit by simply choosing from among the constructions of masks in Sect. 2.3 and plugging in the relevant bounds on \( \kappa \) or \( \sigma_{\min} \). However, when \( s > 1 \), one must settle for using the condition numbers computed by Theorem 3, which for now do not have a simpler closed-form expression. Since the condition number is at least computable from our results, a practitioner may still derive error bounds for any fixed measurement system given by a choice of \( d, \delta, s \), and measurement vectors \( \{ m_j \}_{j \in [D]} \).

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\section*{A Appendix}

\subsection*{A.1 Interleaving Operators and Circulant Structure}

To set the stage for the proof of Theorem 2, we introduce a certain collection of permutation operators and study their interactions with circulant and block-circulant matrices. The structure we identify here will be of much use to us in unraveling the linear systems we encounter in our model for phase retrieval with local correlation measurements. For \( \ell, N_1, N_2 \in \mathbb{N}, v \in \mathbb{C}^{\ell N_1}, k \in [\ell], \) and \( H \in \mathbb{C}^{\ell N_1 \times N_2} \), we define
the block circulant operator $\text{circ}^N_1$ by
\[
\text{circ}^N_1(v) = \begin{bmatrix} v & S^N_1 v & \cdots & S^{(k-1)}N_1 v \end{bmatrix},
\]
\[
\text{circ}^N_1(H) = \begin{bmatrix} H & S^N_1 H & \cdots & S^{(k-1)}N_1 H \end{bmatrix},
\]
where, as with $\text{circ}(\cdot)$, when we omit the subscript we define $\text{circ}^N_1(H) = \text{circ}^N_1(H)$ and $\text{circ}^N_1(v) = \text{circ}^N_1(v)$. We now proceed with the following lemmas; the first establishes the inverse of $P^{(d,N)}$.

**Lemma 2** For $d, N \in \mathbb{N}$, we have
\[
(P^{(d,N)})^{-1} = P^{(d,N)*} = P^{(N,d)}.
\]

**Proof of Lemma 2** Simply take $v \in \mathbb{C}^{dN}$ and calculate, for $i \in [d], j \in [N],$
\[
(P^{(d,N)}P^{(N,d)}v)_{(i-1)N+j} = (P^{(d,N)}(P^{(N,d)}v))_{(i-1)N+j} = (P^{(N,d)}v)_{(j-1)d+i} = v_{(i-1)N+j},
\]
with these equalities coming from the definition in (16).

We now observe some useful ways in which the interleaving operators commute with the construction of circulant matrices.

**Lemma 3** Suppose $V_i \in \mathbb{C}^{k \times n}, v_{ij} \in \mathbb{C}^k, w_j \in \mathbb{C}^{N_1}$ for $i \in [N_1], j \in [N_2]$ and
\[
M_1 = \begin{bmatrix} \text{circ}(V_1) \\ \vdots \\ \text{circ}(V_{N_1}) \end{bmatrix}, \quad M_2 = \begin{bmatrix} \text{circ}^N_1(w_1) & \cdots & \text{circ}^N_1(w_{N_2}) \end{bmatrix}, \quad \text{and}
\]
\[
M_3 = \begin{bmatrix} \text{circ}(v_{11}) & \cdots & \text{circ}(v_{1N_2}) \\ \vdots & \ddots & \vdots \\ \text{circ}(v_{N_11}) & \cdots & \text{circ}(v_{N_1N_2}) \end{bmatrix}.
\]

Then
\[
P^{(k,N_1)}M_1 = \text{circ}^N_1 \left( P^{(k,N_1)} \begin{bmatrix} V_1 \\ \vdots \\ V_{N_1} \end{bmatrix} \right), \quad (50)
\]
\[
M_2 P^{(k,N_2)*} = \text{circ}^N_1 \left( \begin{bmatrix} w_1 & \cdots & w_{N_2} \end{bmatrix} \right), \quad (51)
\]
\[
P^{(k,N_1)}M_3 P^{(k,N_2)*} = \text{circ}^N_1 \left( P^{(k,N_1)} \begin{bmatrix} v_{11} & \cdots & v_{1N_2} \\ \vdots & \ddots & \vdots \\ v_{N_11} & \cdots & v_{N_1N_2} \end{bmatrix} \right). \quad (52)
\]
Proof of Lemma 3  We index the matrices to check the equalities. For (50), we take \((a, b, \ell, j) \in [d] \times [N_1] \times [k] \times [n]\) and have

\[
(P^{(k,N_1)} M_1)_{(a-1)N_1+b,(\ell-1)n+j} = (M_1)_{(b-1)k+a,(\ell-1)n+j} = \begin{bmatrix}
S^{\ell-1} V_1 \\
\vdots \\
S^{\ell-1} V_{N_1}
\end{bmatrix}_{(b-1)k+a,j}
\]

and

\[
\text{circ}^N_1 \left( P^{(k,N_1)} \begin{bmatrix}
V_1 \\
\vdots \\
V_{N_1}
\end{bmatrix}_{(a-1)N_1+b,(\ell-1)n+j} \right) = (V_b)_{a,\ell-1,j} = (V_b)_{a+\ell-1,j}
\]

For (51), we take \((a, b, j) \in [k] \times [N_2] \times [k N_1]\) and have

\[
(P^{(k,N_2)} M_2^*)_{(a-1)N_2+b,j} = (M_2)_{j,(b-1)k+a} = (w_b)_{j+(a-1)N_1}
\]

and

\[
\left( \text{circ}^N_1 \left( [w_1 \ldots w_{N_2}] \right) \right)_{j,(a-1)N_2+b} = (S_{N_1(a-1)} w_b)_j = (w_b)_{j+N_1(a-1)},
\]

and (52) follows immediately by combining (50) and (51).

Lemma 4 introduces useful identities relating interleaving operators to kronecker products.

Lemma 4  For \(v \in \mathbb{C}^N, V = [V_1 \ldots V_{\ell}] \in \mathbb{C}^{N \times \ell}, A = [A_1 \ldots A_m] \in \mathbb{C}^{d \times m}, \) and \(B_i \in \mathbb{C}^{m \times k}, i \in [\ell],\) we have

\[
P^{(d,N)} (v \otimes A) = A \otimes v 
\]

(53)

\[
P^{(d,N)} (V \otimes A) = \begin{bmatrix}
A \otimes V_1 & \cdots & A \otimes V_{\ell}
\end{bmatrix}
\]

(54)

\[
P^{(d,N)} (V \otimes A) P^{(\ell,m)} = A \otimes V
\]

(55)

\[
(V \otimes A) \begin{bmatrix}
B_1 \ldots 0 \\
\vdots \\
0 \ldots B_{\ell}
\end{bmatrix} = [V_1 \otimes A B_1 \ldots V_{\ell} \otimes A B_{\ell}]
\]

(56)

Proof of Lemma 4  For (53), we see that, for \(i, j, k \in [d] \times [N] \times [m],\) we have

\[
(P^{(d,N)} v \otimes A)_{(i-1)N+j,k} = (v \otimes A)_{(j-1)d+i,k} = v_j A_{ik}, \text{ while}
\]

\[
(A \otimes v)_{(i-1)N+j,k} = A_{ik} v_j,
\]

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and (54) follows by considering that $V \otimes A = [V_1 \otimes A \ldots V_\ell \otimes A]$. To get (55), we trace the positions of columns, considering that $(V \otimes A)e_{(i-1)m+j} = A_j \otimes V_i$, so

$$P^{(d,N)}(V \otimes A)e_{(i-1)m+j} = A_j \otimes V_i = (A \otimes V)e_{(j-1)\ell+i}.$$ 

As for (56), we remark that

$$(V \otimes A) \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_\ell \end{bmatrix} = (V \otimes A) \begin{bmatrix} e_1^\ell \otimes B_1 \cdots e_\ell^\ell \otimes B_\ell \\ (V \otimes A)(e_1^\ell \otimes B_1) \cdots (V \otimes A)(e_\ell^\ell \otimes B_\ell) \\ V_1 \otimes AB_1 \cdots V_\ell \otimes AB_\ell \end{bmatrix}.$$

The following lemma on the Kronecker product is standard (e.g., Theorem 13.26 in [39]).

**Lemma 5** We have $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$ for any $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times p}, C \in \mathbb{C}^{p \times k}$. In particular, for $a, b \in \mathbb{C}^d$, $\text{vec}(ab^*) = \overline{b} \otimes a$, and

$$\text{vec}E_{jk} (\text{vec}E_{j'k'})^* = E_{kk'} \otimes E_{jj'}.$$ 

(57)

The next lemma covers the standard result concerning the diagonalization of circulant matrices, as well as a generalization to block-circulant matrices.

**Lemma 6** For any $v \in \mathbb{C}^d$, we have

$$\text{circ}(v) = F_d \text{diag}(\sqrt{d} F_d^* v) F_d^* = \sqrt{d} \sum_{j=1}^{d} (f_j^d)^* f_j^d f_j^d f_j^d \quad (58)$$

Suppose $V \in \mathbb{C}^{kN \times m}$, then $\text{circ}^N(V)$ is block diagonalizable by

$$\text{circ}^N(V) = (F_k \otimes I_N)(\text{diag}(M_1, \ldots, M_k))(F_k \otimes I_m)^*.$$ 

(59)

where

$$\sqrt{k} (F_k \otimes I_N)^* V = \begin{bmatrix} M_1 \\ \vdots \\ M_k \end{bmatrix}, \quad \text{or} \quad M_j = \sqrt{k} (f_j^k \otimes I_N)^* V$$ 

(60)
Proof of Lemma 6 The diagonalization in (58) is a standard result: see, e.g., Theorem 7 of [26].

To prove (59), we set \( V_i \) to be the \( k \times m \) blocks of \( V \) such that \( V^* = [V^*_1 \cdots V^*_k] \) and begin by observing that, for \( u \in \mathbb{C}^k \) and \( W \in \mathbb{C}^{m \times p} \), the \( \ell \)th \( k \times p \) block of \( \text{circ}^N(V)(u \otimes W) \) is

\[
\left( \text{circ}^N(V)(u \otimes W) \right)[\ell] = \sum_{i=1}^{k} u_i \left( S^{N(i-1)} V \right)_{\ell} W = \sum_{i=1}^{k} u_i V_{\ell-i+1} W.
\]

Taking \( u = f_j^k \) and \( W = I_m \), this gives

\[
\left( \text{circ}^N(V)(f_j^k \otimes I_m) \right)[\ell] = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \omega_k^{(j-1)(i-1)} V_{\ell-i+1} I_m = \frac{1}{\sqrt{k}} \omega_k^{(j-1)(\ell-1)} \sum_{i=1}^{k} \omega_k^{-(j-1)(i-1)} V_i
\]

\[= (f_j^k)_{\ell} \left( \sqrt{k} (f_j^k \otimes I_N)^* V \right) = (f_j^k)_{\ell} M_j.
\]

This relation is equivalent to the statement of the lemma, i.e., having

\[
\text{circ}^N(V)(f_j^k \otimes I_m) = (f_j^k \otimes M_j) = (f_j^k \otimes I_N)M_j.
\]

\( \square \)

Lemma 6 immediately gives the following corollary regarding the conditioning of \( \text{circ}^N(V) \), with which we return to considering spanning families of masks.

Corollary 3 With notation as in Lemma 6, the condition number of \( \text{circ}^N(V) \) is

\[
\frac{\max_{i \in [k]} \sigma_{\max}(M_i)}{\min_{i \in [k]} \sigma_{\min}(M_i)}.
\]

A.2 Lemmas on Block Circulant Structure

We begin with Lemma 7, which describes the transposes of block circulant matrices. For this lemma and the remainder of this section, the reader is advised to recall the definitions of \( R_k \) and \( P^{(d,N)} \) from Sect. 1.5 and (16), as well as \( \mathcal{P}(P, \{k_i\}) \) and \( \mathcal{T}_N \) from Sect. 3.2.

Lemma 7 Given \( k, N, m \in \mathbb{N} \) and \( V \in \mathbb{C}^{kN \times m} \), we have

\[
\text{circ}^N(V)^* = \text{circ}^m ((R_k \otimes I_m)\mathcal{T}_N(V)).
\]

Proof of Lemma 7 Suppose \( V_i \) are the \( N \times m \) blocks of \( V \), such that \( V = [V_1^T \cdots V_k^T]^T \). Indexing blockwise, we have \( \text{circ}^N(V)[i,j] = V_{i-j+1} \), so that \( \text{circ}^N(V)^*[i,j] = V_{j-i+1}^* \). In other words,
\[
\text{circ}^N (V)^* = \begin{bmatrix}
V_1^* & V_2^* & \cdots & V_N^*
V_N^* & V_1^* & \cdots & V_{N-1}^*
\vdots & \vdots & \ddots & \vdots \\
V_2^* & V_3^* & \cdots & V_1^*
\end{bmatrix} = \text{circ}^m ((R_k \otimes I_m) T_N(V)).
\]

\[\square\]

Lemmas 8 and 9 provide identities for a few block matrix structures that will be of interest.

**Lemma 8**

Given \( N_1, N_2, k, m \in \mathbb{N} \) and \( V_i \in \mathbb{C}^{kN_1 \times m} \) for \( i \in [N_2] \), we have

\[
\left[ \text{circ}^{N_1}(V_1) \cdots \text{circ}^{N_1}(V_{N_2}) \right] (P^{(k,N_2)} \otimes I_m)^* = \text{circ}^{N_1} \left( [V_1 \cdots V_{N_2}] \right).
\]

**Proof of Lemma 8**

We quote (51) from Lemma 3 and consider that \( P^{(k,N_2)} \otimes I_m \) is a permutation that changes the blockwise indices of \( m \times p \) blocks (or, acting from the right, \( p \times m \) blocks) exactly the way that \( P^{(k,N_2)} \) changes the indices of a vector. \[\square\]

**Lemma 9**

Given \( k, n \in \mathbb{N} \) and \( V_j \in \mathbb{C}^{m_j \times n_j} \) for \( j \in [n] \) and setting \( M = \sum_{j=1}^n m_j, N = \sum_{j=1}^N n_j \), and \( D = \text{diag}(I_k \otimes V_j)_{j=1}^n \in \mathbb{C}^{kM \times kN} \), we have

\[
D = \begin{bmatrix}
I_k \otimes V_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & I_k \otimes V_n
\end{bmatrix} = P_1 (I_k \otimes \text{diag}(V_j)_{j=1}^n) P_2^*
\]

where \( P_1 = \mathcal{P}(P^{(n,k)}, (m_{j \mod 1n})_{j=1}^{kn}) \) and \( P_2 = \mathcal{P}(P^{(n,k)}, (n_{j \mod 1n})_{j=1}^{kn}). \)

**Proof of Lemma 9**

We immediately reduce to the case \( m_j = n_j = 1 \) for all \( j \) by observing that \( P_1 \) and \( P_2 \) will act on blockwise indices precisely as \( P^{(n,k)} \) acts on individual indices. Here, we replace \( V_j \) with \( v_j \in \mathbb{C} \), and note that \( \text{diag}(V_j)_{j=1}^n = \text{diag}(v) \). Hence, we need only remark that

\[
(\text{diag}(I_k \otimes v_{i_\ell})_{\ell=1}^n)_{((i_1-1)k+i_2,((j_1-1)k+j_2))} = \begin{cases} v_{i_1}, & i_1 = j_1 \text{ and } i_2 = j_2 \\
0, & \text{otherwise} \end{cases},
\]

while

\[
(P^{(n,k)}(I_k \otimes \text{diag}(v))P^{(n,k)^*})_{((i_1-1)k+i_2,((j_1-1)k+j_2))} = (I_k \otimes \text{diag}(v))_{((i_2-1)n+i_1,((j_2-1)n+j_1))} = \begin{cases} v_{i_1}, & i_1 = j_1 \text{ and } i_2 = j_2 \\
0, & \text{otherwise} \end{cases}.
\]

\[\square\]
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