Light curve of a hotspot on equatorial orbit around Kerr black hole surrounded by reflective firewall

Sudipta Hensh, Jan Schee, and Zdeněk Stuchlík

Research Centre for Theoretical Physics and Astrophysics, Institute of Physics, Silesian University in Opava, Bezručovo náměstí 13, CZ-74601 Opava, Czech Republic

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The light curve of an isolated bright spot in a Keplerian orbit is studied to investigate the signature of the firewall around the event horizon of the black hole. An increase in total observed flux is found. In addition to that, for firewall case comparatively a longer time radiation is observed.

1. INTRODUCTION

Recent technological advancements in getting signatures of astrophysical events has been remarkable in the last decade. Gravitational wave observations by LIGO/VIRGO/KAGRA confirmed the primary existence of the black hole [1]. On the other hand the first image of galactic center black hole of M87 confirmed the existence of the supercompact object but till now not able to resolve completely between black hole and black hole mimickers [2–7]. So, the nature of the black hole is still unknown. Recently, we have also seen first polarized image of the black hole in the center of the M87 galaxy [8, 9]. Next generation gravitational waves detectors and event horizon telescope would be able to shed some light on the strong field signatures of gravity and may test the nature of the black hole.

Vacuum fluctuations produce particle-anti-particle entangled pair and they annihilate very rapidly because of short lifespan. When this pair production occurs in the close vicinity of the event horizon then one of them falls into the black hole and the other one escapes to the infinity as Hawking radiation [10]. Information loss in Hawking radiation is still an active area of research. A proposal to avoid information loss in Hawking radiation is to give up on Einstein’s principle of equivalence by considering that the in-falling observer would observe a pile of high-energy particle in the close vicinity of the horizon; namely firewall [11].

The existence of the firewall would basically appear in the strong field signatures that we can observe due to astrophysical events involving black holes. This can be in two possible ways; either in the gravitational wave emission or in the observed electromagnetic emissions. In this article, our intention is to investigate the latter. We are interested in observing signatures of firewall in the electromagnetic spectrum of a radiating hotspot. Any intrinsic variability in the local emission from the innermost region of the accretion disk could influence the observed X-ray variability [12]. The observational signatures of a corotating spot in accretion disk is studied in [13]. While radiation from hotspots in extreme Reissner-Nordstrom black hole is investigated in [14].

Detection of hot spot orbiting in a stable circular orbit around our galactic center black hole SgrA* by Gravity Collaboration [15] has introduced a lot of curiosity in the scientific community to model hot spot. There are many efforts available in the literature to study the hot spot. While we can not do justice to mention all the efforts, here we refer to some efforts such as [16–23]. A work dedicated to distinguish between the black holes and worm holes using hot spot can be found in [24]. While the model of hot spot from plasma microphysics is still an active area of research, our goal in this article is not to understand the underlying microphysics that governs the formation of hot spot but to focus on observational imprints of the firewall on the electromagnetic spectrum.

In Sec-2, we describe the governing equations of motion that uniquely determine the path traversed by the photon i.e. geodesic. Next in Sec-3, we describe our firewall model considering which this investigation is done and ray-tracing in the presence of the firewall. The constructions of the light curve is described and comparison plot of the light curve is demonstrated in Sec-4. In Sec-5, a summary of the whole analysis is presented. Throughout the paper, we consider \((-,-,+,+)\) signatures and geometrical system of units i.e. \(G = c = 1\).
2. EQUATIONS OF MOTION

The spacetime interval of Kerr geometry in Boyer-Lindquist coordinates read

\[ ds^2 = - \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{A}{\Sigma} \sin^2 \theta d\phi^2 - \frac{4a r}{\Sigma} \sin \theta dt d\phi \]

where

\[ \Delta = r^2 - 2Mr + a^2, \]
\[ \Sigma = r^2 + a^2 \cos^2 \theta, \]
\[ A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta. \]

For numerical advantages we use new radial and new latitudinal coordinates defined by \( u \equiv 1/r \) and \( m \equiv \cos \theta \). Following Hamilton-Jacobi separation of variable procedure one can get easily Carter equations of motion. The equations of motion of photon 4-momentum \( k^\mu \) are given by

\[ \Sigma k^t = aQ(l, m, a) + \frac{(1 + u^2 a^2)P(u, l, a)}{u^2 \Delta}, \]
\[ \Sigma k^u = \pm \sqrt{U(u, l, q, a)}, \]
\[ \Sigma k^m = \pm \sqrt{M(m, l, q, a)}, \]
\[ \Sigma k^\phi = \frac{Q(l, m, a)}{1 - m^2} + \frac{aP(u, l, a)}{\Delta}, \]

with \( l \equiv -k_\phi/k_t \) and \( q \equiv C - l^2 \) (\( C \) is Carter separation constant) being constants of motion and we have defined

\[ Q(l, m, a) \equiv l - a(1 - m^2), \]
\[ P(u, l, a) \equiv \frac{(1 + u^2 a^2) - la}{u^2}, \]
\[ U(u, l, q, a) \equiv 1 + (a^2 - l^2 - q)u^2 + 2[(a - l)^2 + q]u^3 - a^2 q u^4, \]
\[ M(m, l, q, a) \equiv q + (a^2 - l^2 - q)m^2 - a^2 m^4. \]

Introducing new affine parameter by re-scaling as

\[ \frac{d}{d\lambda'} = \Sigma \frac{d}{d\lambda}. \]

With this new affine parameter equations (5)-(8) reads

\[ k^t = aQ(l, m, a) + \frac{(u^2 + a^2)P(u, l, a)}{u^2 \Delta}, \]
\[ k^u = \pm \sqrt{U(u, l, q, a)}, \]
\[ k^m = \pm \sqrt{M(m, l, q, a)}, \]
\[ k^\phi = \frac{Q(l, m, a)}{1 - m^2} + \frac{aP(u, l, a)}{\Delta}. \]

In order to deal with the square root in equations (15) and (16), we make transition to second order differential equations, reading

\[ \frac{d k^u}{d\lambda'} = \frac{1}{2} \frac{dU}{du}, \]
\[ \frac{d k^m}{d\lambda'} = \frac{1}{2} \frac{dM}{dm}. \]
3. RAY-TRACING AND REFLECTION OFF THE FIREWALL

We consider firewall as a reflecting surface situated at \( r_f = r_h + \epsilon \) i.e. just above the event horizon \( (r_h) \) and in our model \( \epsilon = 10^{-4}M \). To keep the problem simple and understand qualitatively we have considered the firewall as a perfect reflector and the source of the radiation is only from the hotspot. In addition to that we also assume that the hotspot is geometrically thin and optically thick. In our model, we are also considering that the hotspot is laying on the equatorial plane maintaining a circular orbit and having a finite extension i.e. \( dr \).

Here we are using so called backward-raytracing method using this technique one can trace the rays from the detector location to the source of emission. At first, we construct a 2-dimensional photographic plate and the coordinate \((\alpha, \beta)\) determines the location of the pixel on the photographic plate.

Each geodesic can be determined by two parameters \((\alpha, \beta)\) tightly connected with impact parameters \((l, q)\). The relation between the photographic plate coordinate and the constants of motion reads \[ \alpha = \frac{l}{\sin \theta_o}, \] \[ \beta^2 = q + \cos^2 \theta_o \left( a^2 - \frac{l^2}{\sin^2 \theta_o} \right), \]

where \( \theta_o \) is the inclination angle of the observer.

In order to determine the origin of the observed photon we construct its geodesics taking following steps:

- First we estimate the range of the photographic plate coordinates by taking projection of the whole black hole-hotspot system onto the observer’s 2-dimensional sky.
- Then we divide the photographic plate \((\alpha - \beta)\) to finite pixels. For a given pixel \( \alpha \in [\alpha_{\min}, \alpha_{\max}] \) and \( \beta \in [\beta_{\min}, \beta_{\max}] \) determine \( l \) and \( q \) from equations (20) and (21).
- Next, we calculate the turning point \( u_t \) i.e. when radial velocity \( U(u_t; l, q, a) = 0 \) and turning points those occur in between the radius of the outer edge of the hotspot, \( r_{\text{rmax}} \) and radius of the event horizon, \( r_h \) are of our interest. This condition can be expressed as \( u_t > u_{\text{max}} = 1/r_{\text{rmax}} \) and \( u_t < u_h = 1/r_h = (1 + \sqrt{1 - a^2})^{-1} \).
- In order to be computationally efficient, assuming the hotspot expands spans between \( r_1 \) and \( r_{\text{rmax}} \), we determine the maximal meaningful value of the affine parameter \( \lambda_{\text{max}} \) as

\[
\lambda_{\text{max}} = \begin{cases} 
\frac{\int_{u_{\alpha}}^{u_{\text{r}}}}{U(u_{\alpha}; l, q, a)} du & \text{for } n_u = 0, \\
\int_{u_{\alpha}}^{u_{\text{r}}} \frac{du}{U(u_{\alpha}; l, q, a)} + \frac{\int_{u_{\text{r}}}^{u_{\text{max}}}}{U(u_{\alpha}; l, q, a)} du & \text{for } n_u = 1,
\end{cases}
\]

where \( n_u \) indicates number of existing turning points along the geodesics (in the case of null geodesics we have two possibilities \( n_u = 0 \), indicating that there is no turning point and \( n_u = 1 \) indicating that there is one turning point).

- To get the geodesic we solve equations (14), (17), (18), (19) with the initial conditions at \( \lambda = 0 \) as

\[
t(0) = 0, \\
u(0) = u_{\alpha}, \\
m(0) = m_{\alpha}, \\
\phi(0) = 0, \\
k^u(0) = \sqrt{U(u_{\alpha}; l, q, a)}, \\
k^m(0) = \text{Sign}(\beta) \sqrt{M(m_{\alpha}; l, q, a)},
\]

where \( u_{\alpha} = 1/r_o \) is the distance at which observer is located, and \( m_{\alpha} = \cos \theta_o \) is the inclination angle of the observer.
Now the procedure splits into two cases: one with no turning point and the other with a turning point.

If there is no turning point i.e. \( n_u = 0 \) then we integrate equations of motion down to the horizon where the solution reads

\[
\begin{align*}
t(\lambda_{\text{max}}) &= t_h, \quad (29) \\
u(\lambda_{\text{max}}) &= u_h, \\
m(\lambda_{\text{max}}) &= m_h, \\
\phi(\lambda_{\text{max}}) &= \phi_h, \\
k^u(\lambda_{\text{max}}) &= k^u_h, \\
k^m(\lambda_{\text{max}}) &= k^m_h. 
\end{align*}
\]

We apply to this propagation vector perfect (energy and angular momentum are conserved) reflection by converting the radial component of the momentum from global coordinate (GC) to the locally non-rotating frame (LNRF) apply reflection and then switch back to the GC frame as given below

\[
k^u_h \xrightarrow{\text{LNRF}} k^u_h \xrightarrow{\text{Reflection}} -k^u_h \xrightarrow{\text{GC}} k'^u_h.
\]

So the new initial conditions read

\[
\begin{align*}
t(0) &= t_h, \\
u(0) &= u_h, \\
m(0) &= m_h, \\
\phi(0) &= \phi_h, \\
k^u(0) &= k'^u_h, \\
k^m(0) &= k^m_h. 
\end{align*}
\]

We determine new maximal value of the affine parameter

\[
\lambda_{\text{max}} = \int_{u_h}^{u_{\text{max}}} \frac{du}{\sqrt{U(u; l, q, a)}}.
\]

Then we integrate the system of differential equations (14), (17), (18), (19) and check the intersection between the rays and the hotspot.

If there is a turning point \( n_u = 1 \) in this case the geodesics is not divided into segments. We integrate system of diff equations (14), (17), (18), (19) down to \( \lambda = \lambda_{\text{max}} \) and check for intersection with the hotspot.

For each intersection radius of emission \( r_e \) is determined along with corresponding frequency shift \( g \) i.e. the ratio of the observed frequency \( \nu_o \) to the emitted frequency \( \nu_e \) using the formula

\[
g = \frac{\nu_o}{\nu_e} = \left[ 1 - \frac{2}{r_e^2} (1 - a\Omega e)^2 - (r_e^2 + a^2)\Omega e^2 \right]^{1/2} (1 - l\Omega e)^{-1},
\]

where \( \Omega e \) is the velocity of the photon orbiting in a Keplarian orbit which can be written as

\[
\Omega e = \frac{1}{r_e^{3/2} + a}.
\]

4. LIGHT CURVE

The local emission profile of the hotspot in our model is given by

\[
I_e = I_0 r_e^{-p},
\]

where \( I_0 \) is a normalization constant and \( r_e \) is the radial location of local emission. As we are assuming that there is no energy loss while travelling from the emission point to the observer, so the observed intensity reads

\[
I_o = g^4 I_e.
\]
| $\theta_o$ | 30° | 45° | 60° | $\theta_o$ | 30° | 45° | 60° |
|----------|-----|-----|-----|----------|-----|-----|-----|
| 0.1      | 2.75311 | 2.78655 | 2.89336 | 0.1      | 2.75723 | 2.85983 | 2.89858 |
| 0.3      | 2.56454 | 2.59181 | 2.59576 | 0.3      | 2.56314 | 2.62253 | 2.57782 |
| 0.5      | 3.22061 | 3.26359 | 3.27170 | 0.5      | 2.79891 | 3.22683 | 3.29580 |
| 0.7      | 2.74059 | 2.76949 | 3.77723 | 0.7      | 2.79994 | 2.80763 | 3.35492 |
| 0.9      | 5.68790 | 5.76954 | 6.41901 | 0.9      | 5.67351 | 5.82698 | 6.04234 |
| 0.998    | 16.16741 | 15.91805 | 16.42907 | 0.998    | 16.18878 | 15.81999 | 16.45374 |

TABLE I. Fractional change in the total flux, \( \frac{(Total\ Flux_{Firewall} - Total\ Flux_{Kerr}) \times 100}{Total\ Flux_{Kerr}} \) due to reflection of the firewall for hotspot location \( r = r_{ISCO} \) with extension, \( dr = 0.5M \) (on the left table) and \( dr = 1M \) (on the right table).

| $\theta_o$ | 30° | 45° | 60° | $\theta_o$ | 30° | 45° | 60° |
|----------|-----|-----|-----|----------|-----|-----|-----|
| 0.1      | 2.70518 | 2.88087 | 3.05934 | 0.1      | 3.03606 | 2.97446 | 3.18057 |
| 0.3      | 2.75398 | 2.88230 | 2.95083 | 0.3      | 2.88109 | 2.80441 | 2.86375 |
| 0.5      | 2.88655 | 3.04380 | 3.20818 | 0.5      | 3.06993 | 3.00101 | 3.20856 |
| 0.7      | 3.01764 | 3.01958 | 2.91420 | 0.7      | 3.00406 | 2.91969 | 2.93161 |
| 0.9      | 3.74769 | 4.19611 | 4.17147 | 0.9      | 3.78371 | 4.09081 | 4.21665 |
| 0.998    | 14.83085 | 14.91613 | 14.91858 | 0.998    | 14.42621 | 14.63764 | 14.84966 |

TABLE II. Fractional change in the total flux, \( \frac{(Total\ Flux_{Firewall} - Total\ Flux_{Kerr}) \times 100}{Total\ Flux_{Kerr}} \) due to reflection of the firewall for hotspot location \( r = 2r_{ISCO} \) with extension, \( dr = 0.5M \) (on the left table) and \( dr = 1M \) (on the right table).

5. DISCUSSION

We performed ray-tracing simulations to construct the geodesic of the photons which travel up to the observer from the hotspot and then considering the local power-law emission profile as given in (45) with the choice \( I_0 = 1 \) and \( p = 1.5 \). In the simulations, four representative values of the orbit of the hot spot \( r = r_{ISCO}, 2r_{ISCO}, 3r_{ISCO}, 4r_{ISCO} \), three values of observer's inclination angle, \( \theta_o = 30^\circ, 45^\circ, 60^\circ \), and two values of radial extension of the hot spot \( dr = 0.5M, 1M \) are considered.

In case of Kerr black hole, some fraction of photons which emit from the hotspot may lost down to the horizon of the black hole forever. If the event horizon is fully screened by the firewall then emitted photon would suffer reflection when it hits the firewall. A fraction of these photons would travel up to the observer and contribute in the observed flux. As expected due to the reflection of the firewall the total flux increases. The comparison between the Kerr...
FIG. 1. Comparison of the light curve for hotspot extension, $dr = 0.5M$ (on the left column) and $dr = 1M$ (on the right column). The hotspot location is at radius $r = r_{\text{ISCO}}$. 
FIG. 2. Comparison of the light curve for hotspot extension, \( dr = 0.5M \) (on the left column) and \( dr = 1M \) (on the right column). The hotspot location is at radius \( r = 2r_{\text{ISCO}} \).
FIG. 3. Comparison of the light curve for hotspot extension, $dr = 0.5M$ (on the left column) and $dr = 1M$ (on the right column). The hotspot location is at radius $r = 3r_{\text{ISCO}}$. 
FIG. 4. Comparison of the light curve for hotspot extension, $dr = 0.5M$ (on the left column) and $dr = 1M$ (on the right column). The hotspot location is at radius $r = 4r_{\text{ISCO}}$. 
TABLE III. Fractional change in the total flux, \( \frac{(\text{Total Flux}_{\text{Firewall}} - \text{Total Flux}_{\text{Kerr}}) \times 100}{\text{Total Flux}_{\text{Kerr}}} \) due to reflection of the firewall for hotspot location \( r = 3r_{\text{ISCO}} \) with extension, \( dr = 0.5\text{M} \) (on the left table) and \( dr = 1\text{M} \) (on the right table).

| \( a \) | \( \theta_o \) | 30° | 45° | 60° | \( a \) | 30° | 45° | 60° |
|-------|-------|-----|-----|-----|-------|-----|-----|-----|
| 0.1   |       | 3.25752 | 3.32353 | 3.26583 | 0.1   | 2.38895 | 2.49516 | 2.93629 |
| 0.3   |       | 2.77288 | 2.87989 | 3.19307 | 0.3   | 2.76849 | 3.06709 | 3.05847 |
| 0.5   |       | 2.82966 | 3.26838 | 3.10464 | 0.5   | 3.20562 | 3.08114 | 3.25260 |
| 0.7   |       | 3.08242 | 3.01541 | 2.92093 | 0.7   | 3.20119 | 2.98717 | 2.97302 |
| 0.9   |       | 3.80323 | 3.93788 | 4.17863 | 0.9   | 3.88292 | 3.87944 | 4.09486 |
| 0.998 |       | 13.10171 | 13.16101 | 13.14078 | 0.998 | 12.74827 | 13.15923 | 13.11430 |

We see that if we increase the spin then for higher value of spin a tail in the observed light curve appears at late time. The possible reason for this is because at larger value of spin the distance between the hotspot and the firewall decreases significantly. It in-fact closes down larger solid angle of the hotspot from which photon may hit the firewall. So one can say that some fraction of the reflected photon in this case would take longer time and appear at the late time in the detector. Effect of firewall with the inclination is not clear from the table at least in our model and further study is required. The fractional change in the total observed flux is presented in the Table III and Table IV for orbital location \( r_{\text{ISCO}}, 2r_{\text{ISCO}}, 3r_{\text{ISCO}}, 4r_{\text{ISCO}} \) respectively. We can see the fractional change in the observed flux is a few percents in our model.

As peak of the light curve both in the case of Kerr black holes and firewall appears almost at the same time so one can think of matching this two curve by choosing tuning normalization factor \( I_0 \), local emission profile and the orbital location of the hotspot. Even if the peak is matched but in the case of firewall observer would not only receive more radiation but also observe radiation for longer time. This feature would break the degeneracy between whether the hotspot does have different microphysics of radiation, orbital location and the presence of a firewall.

We would like to stress that although a sophisticated quantitative study is necessary to investigate real astrophysical scenario but this analysis considering a toy model has the merit to indicate new features which appears due to the presence of firewall in the light curve. A more detail study in this direction would be done in a future work.
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