Stress state of uniformly piecewise homogeneous space with a periodic system of parallel internal cracks

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Abstract. This paper deals with the antiplane stress state of a uniformly piecewise homogeneous space formed by alternate junction of two distinct layers of the same thickness manufactured of different materials which are weakened by periodic parallel tunnel cracks in their middle planes. The governing system of singular integral equations and the ways of solving the system are are given in general form. The closed solution of the problem is obtained in the special case, where there are central cracks of length 2a in heterogenous layers.

Introduction
The periodic and doubly-periodic problems for two elastic homogeneous massive bodies with cracks form one of the developing directions of the theory of contact and mixed boundary-value problems of the mathematical theory of elasticity. The monographs [1, 2] outline the obtained results in this direction. The number of similar problems studied for piecewise homogeneous bodies is very small. This study considers several antiplane and plane periodic problems for a compound two-component space as well as periodic and doubly-periodic problems for uniformly piecewise homogeneous bodies with interphase cracks, which are important for the layered composites. We mention the research studies [3–7] which are more closely related to the problem.

1. Statement of the problem and derivation of discontinuous solutions
Let us consider the antiplane stress state of a piecewise homogeneous space formed by alternate junction of two distinct layers of thickness 2h with shear moduli $G_1$ and $G_2$. On the middle planes $y = (2n+1)h$ ($n \in \mathbb{Z}$) of the layers along the lines $L_1$ and $L_2$ ($-\infty < z < \infty$) consisting of disjoint intervals $(a_j, b_j)$ ($j = 1, N$) and $(c_i, d_i)$ ($i = 1, M$), there is a periodic system of tunnel parallel cracks. It is assumed that the space is deformed by the action of identical but oppositely directed distributed loads $\tau_0^{(1)}(x)$ and $\tau_0^{(2)}(x)$ acting on the edges of cracks in heterogenous layers.

The problem is to construct the solution and to find the laws of changes in the intensity stress factor at the crack end-points and the crack opening depending on the elastic characteristics of heterogenous layers.

It is obvious that, in this case, the planes $y = (2n+1)h$ ($n \in \mathbb{Z}$) are antisymmetric and the stress state in the compound layers located between the symmetry planes $y = (2k-1)h$ and $y = (2k+1)h$ is the same. Therefore, we can consider only the two-component layer.
(basic layer) between the symmetry planes \( y = \pm h \). For the basic layer, the problem can mathematically be formulated as the boundary-value problem

\[
\begin{aligned}
W_{j}(x, (-1)^{j+1}h) &= 0, \quad x \notin L_{j}, \ j = 1, 2, \\
\tau_{yz}(x, 0) &= \tau_{yz}^{(2)}(x, 0), \quad -\infty < x < \infty, \\
W_{1}(x, 0) &= W_{2}(x, 0), \quad -\infty < x < \infty, \\
\tau_{yz}(x, (-1)^{j+1}h) &= \tau_{0}^{(j)}(x), \quad (x \in L_{j}).
\end{aligned}
\tag{1}
\]

Here \( W_{j}(x, y) \ (j = 1, 2) \) are components of displacements of points of the corresponding layers each of which, in the domain of its definition, satisfies the Laplace equation and is related to the stress components \( \tau_{yz}^{(j)}(x, y) \) by the formula

\[
\tau_{yz}^{(j)}(x, y) = G_{j} \frac{\partial W_{j}(x, y)}{\partial y}.
\tag{2}
\]

To solve the problem, let us introduce the unknown displacements of the crack edges in the heterogenous layers \( W_{j}(x, (-1)^{j+1}h) = W_{j}(x) \ (j = 1, 2) \). First, we solve an auxiliary problem with the last condition in (1) replaced by this relation.

To this end, we represent the solutions of the Laplace equation as Fourier integrals

\[
W_{j}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A_{j}(\lambda) \sinh(\lambda y) + B_{j}(\lambda) \cosh(\lambda y) \right] e^{-i\lambda x} d\lambda \quad (j = 1, 2).
\tag{3}
\]

Satisfying the conditions of the auxiliary boundary-value problem, we express the unknown coefficients \( A_{j}(\lambda) \) and \( B_{j}(\lambda) \) \((j = 1, 2)\) in terms of the Fourier transforms of the functions \( W_{j}(x) \ (j = 1, 2) \) and obtain

\[
A_{1} = \frac{A_{2}}{G} = \frac{\bar{w}_{1}(\lambda) - \bar{w}_{2}(\lambda)}{(1 + G)\sinh(\lambda h)}, \quad B_{1} = B_{2} = \frac{G\bar{w}_{1}(\lambda) + \bar{w}_{2}(\lambda)}{(1 + G)\cosh(\lambda h)},
\]

where

\[
\bar{w}_{j}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{j}(x)e^{i\lambda x} dx.
\]

After substitution of the obtained expressions into (3) and then into (2), the stress state field in the heterogenous layers can be determined by the functions \( W_{j}(x) \ (j = 1, 2) \). In particular, the shear stresses on the boundaries of the two-component layer can be written in terms of \( W_{j}(x) \) as

\[
\begin{aligned}
\tau_{yz}^{(1)}(x, h) &= -\frac{G_{1}}{2h(G + 1)} \left\{ \int_{L_{1}} \frac{G + \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W_{1}'(s) \, ds \right. \\
&\quad \left. + \int_{L_{2}} \frac{1 - \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W_{2}'(s) \, ds \right\}, \\
\tau_{yz}^{(2)}(x, -h) &= \frac{G_{2}G}{2h(G + 1)} \left\{ \int_{L_{1}} \frac{1 - \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W_{1}'(s) \, ds \right. \\
&\quad \left. + \int_{L_{2}} \frac{1 + G\cosh[\mu(s - x)]}{G\sinh[\mu(s - x)]} W_{2}'(s) \, ds \right\} \left( G = \frac{G_{1}}{G_{2}}, \quad \mu = \frac{\pi}{2h}, \quad -\infty < x < \infty \right).
\end{aligned}
\tag{4}
\tag{5}
\]

The obtained relations are used to satisfy the last two relations in (1). Finally, we obtain the following governing system of singular integral equations for the derivative of the displacement...
of the crack edges:

\[
\frac{1}{2h} \int_{L_1} \frac{G + \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W'_1(s) \, ds \\
+ \frac{1}{2h} \int_{L_2} \frac{1 - \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W'_2(s) \, ds = \frac{G + 1}{G_1} \tau_0^{(1)}(s) \quad (x \in L_1)
\]

\[
\frac{G}{2h} \int_{L_1} \frac{1 - \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W'_1(s) \, ds \\
+ \frac{1}{2h} \int_{L_2} \frac{1 + G \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} W'_2(s) \, ds = \frac{G + 1}{G_2} \tau_0^{(2)}(s) \quad (x \in L_2).
\]

This system should be considered with the continuity conditions for the displacements at the crack end-points.

\[
\int_{a_j}^{b_j} W'_1(s) \, ds = 0, \quad \int_{c_i}^{d_i} W'_2(s) \, ds = 0 \quad (j = 1, N; i = 1, M).
\]

Solving the problem is reduced to solving the governing system (6) under conditions (7). Note that, in the general case, for arbitrary \(L_1\) and \(L_2\), the governing system can be solved either by the method of Chebyshev orthogonal polynomials or by the method of mechanical quadratures, where the equations are written on each interval contained in \(L_1\) and \(L_2\). However, we shall continue our research by considering a particular problem which allows a closed solution.

2. Uniformly piecewise homogenous space with a periodic system of identical parallel internal cracks

Let us consider a special case of the problem, where the cracks in the heterogenous layers are identical, i.e., \(L_1 = L_2 = L\). In this case, we use some simple calculations and introduce the functions

\[
\varphi_1(x) = W_1(x) - W_2(x), \quad \varphi_2(x) = GW_1(x) + W_2(x),
\]

\[
f_1(x) = -\frac{\tau_0^{(1)}(x) + G \tau_0^{(2)}(x)}{G_1}, \quad f_2(x) = -\frac{\tau_0^{(1)}(x) - \tau_0^{(2)}(x)}{G_2}.
\]

The governing system of equations (6) can be written as the following independent singular integral equations of the first kind:

\[
\frac{1}{2h} \int_L \frac{\cosh[\mu(s - x)]}{\sinh[\mu(s - x)]} \varphi'_1(s) \, ds = f_1(x) \quad (x \in L),
\]

\[
\frac{1}{2h} \int_L \frac{\varphi'_2(s)}{\sinh[\mu(s - x)]} \, ds = f_2(x) \quad (x \in L).
\]

Conditions (7) become

\[
\int_{a_j}^{b_j} \varphi'_1(s) \, ds = 0, \quad \int_{a_j}^{b_j} \varphi'_2(s) \, ds = 0 \quad (j = 1, N).
\]

The closed solutions of equations (9) and (10) can be obtained for any finite value of \(N\). We construct these solutions for \(N = 1\) and \(a_1 = -a, \ b_1 = a\), i.e., when the distinct layers are
weakened by one finite tunnel crack of length $2a$ occupying the interval $(-a, a)$. In this case, introducing new variables by formulas $\xi = \exp(2\mu s)$, $\eta = \exp(2\mu x)$ and using the notation

$$\varphi_1^*(\eta) = \frac{1}{\eta} \varphi_1 \left( \frac{\ln \eta}{2\mu} \right), \quad f_1^*(\eta) = \frac{1}{\eta} f_1 \left( \frac{\ln \eta}{2\mu} \right), \quad a_* = \exp(-2\mu a),$$

$$\varphi_2^*(\eta) = \frac{1}{\sqrt{\eta}} \varphi_2 \left( \frac{\ln \eta}{2\mu} \right), \quad f_2^*(\eta) = \frac{1}{\sqrt{\eta}} f_2 \left( \frac{\ln \eta}{2\mu} \right), \quad b_* = \exp(2\mu a),$$

we obtain the following system of singular integral equations with the Cauchy kernel:

$$\frac{1}{\pi} \int_{a_*}^{b_*} \frac{\varphi_j^*(\xi)}{\xi - \eta} \, d\xi = f_j^*(\eta) \quad (j = 1, 2; \ a_* < \eta < b_*). \quad (12)$$

Conditions (11) become

$$\int_{a_*}^{b_*} \varphi_j^*(s) \, ds = 0 \quad (j = 1, 2). \quad (13)$$

The solution of equations (12) with conditions (13) is given by the formulas [8]

$$\varphi_1^*(\eta) = -\frac{1}{\pi \omega(\eta)} \int_{a_*}^{b_*} \frac{\omega(\xi) f_1^*(\eta)}{\xi - \eta} \, d\xi \quad (a_* < \eta < b_*), \quad (14)$$

$$\varphi_2^*(\eta) = -\frac{1}{\omega(\eta)} \left[ \frac{1}{\pi} \int_{a_*}^{b_*} \frac{\omega(\xi) f_2^*(\eta)}{\xi - \eta} \, d\xi - C \right] \quad (a_* < \eta < b_*). \quad (15)$$

Here

$$C = \frac{I_1}{I_2}, \quad I_1 = \frac{1}{\pi} \int_{a_*}^{b_*} \frac{d\eta}{\sqrt{\omega(\eta)}} \int_{a_*}^{b_*} \frac{\omega(\xi) f_2^*(\xi)}{\xi - \eta} \, d\xi, \quad I_2 = \frac{2}{\sqrt{b_*}} F\left( \frac{\pi}{2}, k_1 \right),$$

$$\omega(\eta) = \sqrt{(\eta - a_*)(b_* - \eta)}, \quad k_1 = \sqrt{1 - \frac{a_*}{b_*}}.$$

The function $F(\alpha, k_1)$ is the elliptic integral of the first kind. Returning to the initial variables, we obtain

$$\varphi_1'(x) = -\frac{1}{2\mu \omega_*(x)} \int_{-a}^{a} \frac{\omega_*(s) f_1(s)}{\sinh[\mu(s - x)]} \, ds \quad (16)$$

$$\varphi_2'(x) = -\frac{1}{2\mu \omega_*(x)} \left\{ \frac{2\mu}{\pi} \int_{-a}^{a} \frac{\omega_*(\xi) e^{\mu(s-x)} f_2(s)}{\sinh[\mu(s - x)]} \, ds \right\} - C \quad (17)$$

$$\omega_*(x) = \sqrt{\cosh(2\mu a) - \cosh(2\mu x)}, \quad |x| < a.$$

Then, using formulas (8), we obtain the following expressions for the derivative of displacements of the crack edges:

$$W_1'(x) = -\frac{1}{2(G+1)\omega_*(x)} \left\{ \frac{1}{h} \int_{-a}^{a} \frac{\omega_*(\xi)(f_1(s) + e^{\mu(s-x)}f_2(s))}{\sinh[\mu(s - x)]} \, ds - C \right\}, \quad (18)$$

$$W_2'(x) = \frac{1}{2(G+1)\omega_*(x)} \left\{ \frac{1}{h} \int_{-a}^{a} \frac{\omega_*(\xi)(Gf_1(s) - e^{\mu(s-x)}f_2(s))}{\sinh[\mu(s - x)]} \, ds + C \right\}. \quad (19)$$

Note that, in the special case, where $\tau_0^{(1)}(x) = \tau_0^{(2)}(x) = \tau_0(x)$, we have

$$f_1(x) = -\frac{1+G}{G_1} \tau_0(x), \quad f_2(x) = 0.$$
Then from formulas (16)–(19) we derive

\[ C = 0, \quad \varphi_2(x) = 0, \]

\[ W_j'(x) = \frac{(-1)^j}{2hG_j\omega_s(x)} \int_{-a}^{a} \frac{\omega_s(\xi)f_1(s)ds}{\sinh[\mu(s - x)]}. \]

Now, let us determine the stress intensity factors at the crack end-points. First, we determine the stress intensity factors at the end-point of the first crack located on the line \( y = h \). For this, we use formula (4) outside the interval \((-a, a)\). Substituting the values of the function \( W_j'(x) \) from (18) into (4) and taking into account that the second term in (4) is a bounded function, we can write

\[ \tau_{y_2}^{(1)}(x, h) = \frac{G_1q_1(x)}{4h(G + 1)} \int_{-a}^{a} \frac{ds}{\omega_s(s)\sinh[\mu(s - x)]} + R_1(x) \quad (|x| > a). \]

Here

\[ q_1(x) = \frac{1}{h} \int_{-a}^{a} \frac{\omega_s(s)[f_1(s) + e^{\mu(s-x)}f_2(s)]ds}{\sinh[\mu(s - x)]} - C, \]

\[ R_1(x) = \frac{G_1}{4h(G + 1)^2} \int_{-a}^{a} (G + 1)[q_1(s) - q_1(x)] + \{\cosh[\mu(s - x)] - 1\}q_1(s)ds - \frac{G_1}{2h(G + 1)} \int_{-a}^{a} \frac{1 - \cosh[\mu(s - x)]}{\sinh[\mu(s - x)]}W_j'(s)ds. \]

Then, taking into account the value of the integral

\[ \int_{-a}^{a} \frac{ds}{\omega_s(s)\sinh[\mu(s - x)]} = \frac{\text{sign}(x)}{\mu\sqrt{\sinh[\mu(a + x)]\sinh[\mu(a - x)]}}, \]

which, after the change of variables \( \xi = \exp(2\mu s) \), \( \eta = \exp(2\mu x) \), can be reduced to calculating the integral [9]

\[ \int_{a}^{b} \frac{d\xi}{\sqrt{(|\xi - a|)(b - \xi)}} = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{(a-\eta)(b-\eta)}}, & \eta < a, \\
\frac{1}{\sqrt{(\eta-a)(\eta-b)}}, & \eta > b,
\end{array} \right. \]

we obtain the following formulas for determining the reduced stress intensity factors:

\[ K_{11}^{(1)}(\pm a) = \sqrt{2\pi} \lim_{x \rightarrow \pm a \pm 0} \sqrt{|x \mp a|}\tau_{y_2}^{(1)}(x, h) \]

\[ = \pm \frac{G_1\sqrt{h}}{\pi(G + 1)\sqrt{\sinh(2\mu a)}} \left\{ \frac{1}{h} \int_{-a}^{a} \frac{\omega_s(s)[f_1(s) + e^{\mu(s-x)}f_2(s)]ds}{\sinh[\mu(s \mp a)]} - C \right\}. \]

Similarly, for the stress intensity factors at the end-points of the second crack located on the line \( y = -h \), we have

\[ K_{11}^{(2)}(\pm a) = \sqrt{2\pi} \lim_{x \rightarrow \pm a \pm 0} \sqrt{|x \mp a|}\tau_{y_2}^{(2)}(x, h) \]

\[ = \pm \frac{G_2\sqrt{h}}{\pi(G + 1)\sqrt{\sinh(2\mu a)}} \left\{ \frac{1}{h} \int_{-a}^{a} \frac{\omega_s(s)[Gf_1(s) - e^{\mu(s-x)}f_2(s)]ds}{\sinh[\mu(s \mp a)]} + C \right\}. \]
Conclusions
The governing equations for antiplane problem of the elasticity theory for uniformly piecewise homogeneous space with tunnel parallel internal cracks are constructed. It is shown that, in the case where the cracks in heterogenous layers are identical, the closed solutions of the equations can be obtained. These solutions are constructed in the case where the identical central cracks are located in heterogenous layers. Simple formulas are obtained for the determining the stress intensity factors at the crack end-points.

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