Self-similar Properties of the Proton Structure at low $x$

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Abstract

Self-similar properties of proton structure in the kinematic region of low Bjorken $x$ are introduced and studied numerically. A description of the proton structure function $F_2(x, Q^2)$ reflecting self-similarity is proposed with a few parameters which are fitted to recent HERA data. The specific parameterisation provides an excellent description of the data which cover a region of four momentum transfer squared, $0.045 \leq Q^2 \leq 120 \text{ GeV}^2$, and of Bjorken $x$, $6.2 \cdot 10^{-7} \leq x \leq 0.01$. 
1 Introduction

Recent measurements of the H1 [1] and ZEUS [2] collaborations at HERA enable to study the proton structure in the region of low $Q^2 \lesssim 1 \text{ GeV}^2$ where perturbative QCD has to face computation difficulties arising from the increase of the strong coupling constant $\alpha_s(Q^2)$. Nevertheless, there is a number of approaches to describe the transition to low $Q^2$ at small $x$ together with the region, $Q^2 > 1 \text{ GeV}^2$, which is described by perturbative QCD very well. Such attempts involve Reggeon exchange ideas [3], dipole interactions [4], vector meson dominance (VMD) [5], efficient parametrisations [6] and others.

This letter presents a different point of view, based on the idea that the proton structure at low $x$ is of fractal nature. Using the fractal dimension concept [7], a simple parametrisation of the proton structure function $F_2(x, Q^2)$ is obtained with a few well defined parameters. A numeric study is made using recent small $x$ HERA data, for $Q^2$ between 0.045 GeV$^2$ and 120 GeV$^2$.

2 Fractal Dimension

The concept of fractal dimension requires to understand what is meant by dimension. In non-fractional dimensions a number of dimensions corresponds to a number of independent directions in a corresponding coordinate system. For example, a line has obviously one dimension, a square two and a cube has three dimensions. The dimension of the Sierpinski gasket [8], shown in Fig. [1], needs a more general definition.

![Sierpinski gasket](image_url)

Figure 1: Sierpinski gasket fractal in iterations No. 1, 3 and 6 (from left). Iteration No. 1 corresponds to the seed image which is arbitrary while the iteration always converges to the same object.

The cube, square and line are self-similar objects: when a line is broken in the middle two lines are obtained, each of half length. By magnifying one of them by a factor of two the original line is rebuilt. The same may be done by dividing a square into four small squares or a cube into smaller cubes. For example, when the magnification factor for a square is 3 the number of smaller squares will be $3^2 = 9$, for a cube the same magnification gives $3^3 = 27$ cubes. In general, when $M$ is the magnification factor, then the number of objects will be $M^D$, where $D$ is the dimension of the object. The dimension $D$ can thus be defined as

$$D = \frac{\log M^D}{\log M} = \frac{\log(\text{number of self-similar objects})}{\log(\text{magnification factor})}. \quad (1)$$
According to this formula, the dimension of the Sierpinski gasket is fractional. When the magnification factor \( M \) is 2 there are 3 identical pieces of the gasket, for \( M = 4 \) the gasket consists of 9 small copies of itself. Therefore its fractal dimension is defined by

\[
D = \frac{\log 3}{\log 2} = \frac{\log 9}{\log 4} = 1.58496 \ldots
\]  

(2)

Roughly speaking, the fractal dimension describes how complicated or how large a self-similar object is. A plane is 'larger' than a line. The Sierpinski gasket is not a line but also far from being a plane. Actually, there exist fractals which are constructed from lines but have dimensions 2 or 3, and therefore fill a plane or a space. An example of such a curve is the so-called Hilbert curve.

The definition of a dimension, given in equation (1), may be generalised for the case of non-discrete fractals. In this generalisation, the magnification (scaling) factor is a real number \( z \) and the number of self-similar objects is represented by a density function \( f(z) \). Taking into account that the dimension may change with scaling, a local dimension is defined as

\[
D(z) = \frac{\partial \log f(z)}{\partial \log z}.
\]  

(3)

For ideal mathematical fractals, discussed so far, \( D(z) \) is constant for the whole fractal. Introducing a scale dependent dimension is natural because many fractals in nature (e.g. plants or coastlines) are not mathematically ideal and usually have a fractal structure only for a certain region of magnification. In such a region, the dimension is approximately constant, \( D(z) = D \), and, following eq. (3), the density function \( f(z) \) is given as

\[
\log f(z) = D \cdot \log z + D_0
\]  

(4)

where \( D_0 \) defines the normalisation of \( f(z) \), which thus has a power law behaviour, \( f(z) \propto z^D \).

In general, fractals may have two independent magnification factors, \( z \) and \( y \). In this case the density \( f(z, y) \) is written in the following way

\[
\log f(z, y) = D_{zy} \cdot \log z \cdot \log y + D_z \cdot \log z + D_y \cdot \log y + D_0.
\]  

(5)

Here the dimension \( D_{zy} \) represents the dimensional correlation relating the \( z \) and \( y \) factors. The function \( f(z, y) \) satisfies a power law behaviour in \( z \) for fixed \( y \) and in \( y \) for fixed \( z \).

It is important to mention that there is certain freedom in selecting magnification factors without changing a shape of the function \( f(z, y) \). It is possible to use any non-zero power of a factor multiplied by a constant: \( z \rightarrow a z^\lambda \). The only effect of such a change is a redefinition of the dimensional parameters \( D_{(z,y,zy)} \) and of the normalisation \( D_0 \), respectively.
3 Self-similar Structure of the Proton

Following the dimensional description of the presented fractal structures, it is interesting to study the properties of functions describing proton structure. In quantum chromodynamics the behaviour of the sea quark densities is driven by gluon emissions and splittings. The deeper the proton structure is probed, the more gluon-gluon interactions can be observed. These, in analogy to fractals, may follow self-similarity, i.e. scaling described by a power law. Indeed, there is a number of hints for a self-similar structure. As an example, Figure 2 shows the unintegrated u-quark density for fixed momentum transfer $Q^2$ and fixed Bjorken $x$, respectively.

![Figure 2](image_url)

Figure 2: Logarithm of the unintegrated u-quark density $\partial u(x, Q^2)/\partial Q^2$ as a function of Bjorken $x$ (a) and $Q^2$ (b). The full and dashed lines correspond to GRV parametrisations in LO and NLO [9], respectively.

For $x \lesssim 0.01$ (below the valence quark region) the unintegrated density function in log-log scale is linear. A linear behaviour is also exhibited by the unintegrated density as a function of $Q^2$ for fixed $x$. Referring to eq. (4), this suggests that $x$ and $Q^2$ could be treated as appropriate magnification (scaling) factors. This is supporting the idea that the proton structure exhibits self-similar properties and may be described as a fractal object.

Magnification factors are supposed to fulfill some criteria. They should be positive, non-zero and have no physical dimension. The two latter requirements concern the selection of $Q^2$ as a magnification factor. The physical dimensionality may be removed by dividing $Q^2$ by a constant $Q_0^2$. For the case of $Q^2 = 0$, the non-zero requirement is not fulfilled, however, the access to this region is needed for integration of unintegrated densities. Thus instead of $Q^2$ a choice of $1 + Q^2/Q_0^2$ as a magnification factor is appropriate. According to the freedom in the magnification factor selection, mentioned above, other equivalent choices are also possible, e.g. $Q_0^2/(Q_0^2 + Q^2)$, $(Q_1^2 + Q^2)/1 GeV^2$ or similar combinations. It is also more appropriate to use $1/x$ as a magnification factor rather than $x$ itself: when the structure is probed deeper, $x$ goes to zero while a magnification factor should rise.
4 Structure Function Parametrisation

The concept of self-similarity, when applied to proton confinement structure, leads to a simple parametrisation of quark densities within the proton in a straightforward way based on Eq. (5). Using magnification factors \(1/x\) and \(1 + Q^2/Q_0^2\), an unintegrated quark density may be written in the following general form

\[
\log f_i(x, Q^2) = D_1 \cdot \log \frac{1}{x} \cdot \log (1 + \frac{Q^2}{Q_0^2}) + D_2 \cdot \log \frac{1}{x} + D_3 \cdot \log (1 + \frac{Q^2}{Q_0^2}) + D_i^0
\]  

(6)

where \(i\) denotes a quark flavour. Conventional, integrated quark densities \(q_i(x, Q^2)\) are defined as a sum over all contributions with quark virtualities smaller than that of the photon probe, \(Q^2\). Thus \(f_i(x, Q^2)\) has to be integrated over \(Q^2\),

\[
q_i(x, Q^2) = \int_0^{Q^2} f_i(x, q^2) \, dq^2.
\]

(7)

Solving equation (7), the following analytical parametrisation of a quark density is obtained

\[
q_i(x, Q^2) = \frac{e^{D_0^i} Q_0^2}{1 + D_3 - D_1 \log x} \left( x^{-D_1 \log (1 + \frac{Q^2}{Q_0^2})} (1 + \frac{Q^2}{Q_0^2})^{D_3 + 1} - 1 \right).
\]

(8)

Notice that in this parametrisation only the normalisation parameter \(D_0^i\) depends on the quark flavour while the other parameters are flavour independent. This assumption means that all quarks are following the fractal structure, i.e. the dimensions \(D_i\) and the magnification factors are common for all of them and they differ in normalisation only.

The proton structure function \(F_2(x, Q^2)\) is related directly to the quark densities \(F_2 = x \sum_i e_i^2 (q_i + \bar{q}_i)\). Thus the assumption about the flavour symmetry of Eq. (8) allows to express \(F_2(x, Q^2)\) directly in the form given on the r.h.s. of Eq. (8) with \(x^{-D_2}\) replaced by \(x^{-D_2 + 1}\) and with a common normalisation factor \(e^{D_0}\):

\[
F_2(x, Q^2) = \frac{e^{D_0} Q_0^2}{1 + D_3 - D_1 \log x} \left( x^{-D_1 \log (1 + \frac{Q^2}{Q_0^2})} (1 + \frac{Q^2}{Q_0^2})^{D_3 + 1} - 1 \right).
\]

(9)

5 Fit to the Data

The five parameters \(D_i\) and \(Q_0^2\) are determined using recent data from the HERA experiments H1 [1] and ZEUS [2] in the range \(1.5 \leq Q^2 \leq 120\,\text{GeV}^2\) (H1) and \(0.045 \leq Q^2 \leq 0.65\,\text{GeV}^2\) (ZEUS). Additionally a cut \(x < 0.01\) has been applied to exclude the valence quark region. The fit parameters are given in Table I and the corresponding description of the \(F_2(x, Q^2)\) data is
Figure 3: Virtual photon-proton cross-section $\sigma_{\gamma^*p} \propto F_2(W^2, Q^2)/Q^2$ as a function of $Q^2$ in $W^2$ bins. H1 (points) and ZEUS (triangles) measurements are shown along with the fit to 4 parameters (full line) and to all 5 parameters (dashed line).
|        | $D_0$  | $D_1$  | $D_2$  | $D_3$  | $Q^2_0$ [GeV$^2$] | $\chi^2$ | $\chi^2$/ndf |
|--------|--------|--------|--------|--------|------------------|----------|-------------|
| all fit| 0.339  | 0.073  | 1.013  | -1.287 | 0.062            | 136.6    | 0.82        |
|        | $\pm 0.145$ | $\pm 0.001$ | $\pm 0.01$ | $\pm 0.01$ | $\pm 0.01$ |           |             |
| $D_2$ fixed | 0.523  | 0.074  | 1      | -1.282 | 0.051            | 138.4    | 0.82        |
|        | $\pm 0.014$ | $\pm 0.001$ | const. | $\pm 0.01$ | $\pm 0.002$ |           |             |

Table 1: Results of the fit. The first row corresponds to a fit to all parameters, in the second row parameter $D_2$ was fixed to 1. The number of $F_2$ data points is 172, total errors were used for the $\chi^2$ calculation.

![Figure 4: Measurement of the structure function $F_2(x, Q^2)$ as a function of $x$ in bins of $Q^2$ by the H1 experiment. The curve represents the fit to 4 parameters, which is indistinguishable from the 5 parameter fit in this kinematic region.](image)

The $\chi^2$ was calculated with total errors, adding the statistical and systematical errors in quadrature. When the relative normalisation of the H1 and ZEUS data, which cover different $Q^2$ regions, was fitted no change beyond 1% was imposed by the fits. Thus the normalisations of the data sets were left untouched.

Refering to Figure 3 the ratio $F_2(W^2, Q^2)/Q^2$ is proportional to the virtual photon-proton cross-section $\sigma_{ep}(W^2, Q^2)$. In the limit $Q^2 \to 0$ and fixed $W^2$ the parametrisation (9) behaves like $Q^2$ only for $D_2 = 1$. This may be easily shown e.g. when the unintegrated structure function $f(x, q^2)$ is introduced.
\[ F_2(x, Q^2) = \int_0^{Q^2} f(x, q^2) \, dq^2 \]  

(10)

the parametrisation of which is identical to (9), with \( D_2 \) replaced by \( D_2 - 1 \) and \( D_0^i \) replaced by \( D_0 \). If \( F_2(x, Q^2) \) behaves like \( Q^2 \) for \( Q^2 \to 0 \) then \( f(x, q^2) \) has to behave like a constant for any \( x = Q^2/(W^2 - M_p^2) \to 0 \). That is possible only if the divergent term, involving \( D_2 \), is zero, i.e. for \( D_2 = 1 \). In this case, since other logarithmic terms go to zero, the ratio \( F_2(W^2, Q^2)/Q^2 \) for \( Q^2 \to 0 \) approaches the value \( e^{D_0} \).

In the fit with \( D_2 \) as a free parameter a value very close to 1 is obtained. Thus a second fit was made, where \( D_2 \) is fixed to 1 (see Table [ ], second row). This fit has 4 parameters and gives nearly the same \( \chi^2/\text{ndf} \) as the first fit to all 5 parameters. Within the kinematic range of the \( F_2 \) data, both fits are nearly indistinguishable. As was stated above, the parameter \( D_0 \) determines the virtual photon-proton cross-section in the photoproduction limit. Its value, obtained from the fit, gives

\[ \sigma_{\gamma p} = \left[ \frac{4\pi^2\alpha}{Q^2} F_2(W^2, Q^2) \right]_{Q^2 \rightarrow 0} = 189 \pm 3 \, \mu\text{b}. \]  

(11)

This is in approximate agreement with the total photoproduction cross-sections measured by the H1 [10] and ZEUS [11] collaborations which were not used in the fit.

Figure 5: Measurement of the structure function \( F_2(x, Q^2) \) as a function of \( x \) in bins of \( Q^2 \) by the ZEUS experiment. The curve represents the fit to 4 parameters, which is indistinguishable from the 5 parameter fit in this kinematic region.
6 Summary

The concept of the self-similar structure of the proton was introduced. This leads to a parametrisation of the proton structure function \( F_2(x, Q^2) \) which describes very well the low \( x \) HERA data, both in the non-perturbative and the deep inelastic domain. The introduced formalism uniquely defines the \( x \) and \( Q^2 \) dependence of parton densities, thus this approach is applicable also to other measures of proton structure, like the longitudinal structure function \( F_L \), the diffractive structure function \( F^{D}_2 \) or the spin structure function \( g_1 \).

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