DETERMINANT OF HEISENBERG REPRESENTATION OF FINITE GROUPS

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Abstract. In this article we give an explicit formula of determinant of a Heisenberg representation $\rho$ of a finite group $G$ modulo $\text{Ker}(\rho)$. We also show that this formula of determinant is independent of the choice of the maximal isotropic for Heisenberg representation $\rho$.

1. Introduction

Let $G$ be a finite group and $\rho$ be a Heisenberg representation of $G$. Let $Z$ be the scalar group for $\rho$ and $H$ be a maximal isotropic subgroup of $G$ for $\rho$. Let $\chi_H$ be a linear character of $H$ which is an extension of the central character $\chi_Z$ of $\rho$. Then we know that $\rho = \text{Ind}_H^G \chi_H$. The main aim of this article is to give an invariant formula for:

$$\det(\rho) = \det(\text{Ind}_H^G \chi_H).$$

In other words, we will show that this formula is independent of the choice of the maximal isotropic subgroup $H$ because the maximal isotropic subgroups are not unique. From Gallagher’s result (cf. Theorem 2.3) we know that

$$\det(\text{Ind}_H^G \chi_H)(g) = \Delta^G_H(\chi_H(T_{G/H}(g)))$$

for all $g \in G$.

Therefore for explicit computation of determinant of Heisenberg representation $\rho$, we first need to compute the transfer map $T_{G/H}$.

Now let $\rho = (Z, \chi_\rho)$ be a Heisenberg representation of $G$. Then from the definition of Heisenberg representation we have

$$[[G, G], G] \subseteq \text{Ker}(\rho).$$

Now let $\overline{G} := G/\text{Ker}(\rho)$. Then we obtain

$$[[\overline{G}, \overline{G}], \overline{G}] = [G/\text{Ker}(\rho), G/\text{Ker}(\rho)] = [G, G] \cdot \text{Ker}(\rho)/\text{Ker}(\rho) = [G, G]/[G, G] \cap \text{Ker}(\rho).$$

Since $[[G, G], G] \subseteq \text{Ker}(\rho)$, then $[x, g] \in \text{Ker}(\rho)$ for all $x \in [G, G]$ and $g \in G$. Hence we obtain

$$[[\overline{G}, \overline{G}], \overline{G}] = [[G, G]/[G, G] \cap \text{Ker}(\rho), G/\text{Ker}(\rho)] \subseteq \text{Ker}(\rho),$$

This shows that $\overline{G}$ is a two-step nilpotent group.

In general, for Heisenberg setting $G$ need not be two-step nilpotent group. But just we see under modulo $\text{Ker}(\rho)$, i.e., $\overline{G} = G/\text{Ker}(\rho)$ is a two-step nilpotent group. Therefore first we computation $T_{G/H}$ for two-step nilpotent groups and we use those computation for computing $\det(\rho)$ under modulo $\text{Ker}(\rho)$.

2010 Mathematics Subject Classification. 20G05; 22E50

Keywords: Transfer map, Heisenberg representations, Determinant.

The author is partially supported by IMU-Berlin Einstein Foundation, Berlin, Germany and CSIR, Delhi, India.
In Lemma 3.1 we compute $T_{G/H}$, when $H$ is an abelian normal subgroup of a finite group $G$ (of odd index in $G$) with $[G, [G, G]] = \{1\}$, and $G/H$ is an abelian quotient group.

Lemma 1.1. Assume that $G$ is a finite group and $H$ a normal subgroup of $G$ such that

1. $H$ is abelian,
2. $G/H$ is abelian of odd order $d$,
3. $[G, [G, G]] = \{1\}$.

Then we have $T_{G/H}(g) = g^d$ for all $g \in G$.

As a consequence one has $[G, G]^d = \{1\}$, in other words, $G^d$ is contained in the center of $G$.

More generally, combining this above Lemma and the elementary divisor theorem, we obtain the following result (cf. Lemma 3.3).

Lemma 1.2. Assume that $G$ is a finite group and $H$ a normal subgroup of $G$ such that

1. $H$ is abelian
2. $G/H$ is abelian of order $d$, such that (according to the elementary divisor theorem):
   $$G/H \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s$$
   where $m_1 | \cdots | m_s$ and $\prod_i m_i = d$. Moreover, we fix elements $t_1, t_2, \cdots, t_s \in G$ such that $t_i H \in G/H$ generates the cyclic factor $\cong \mathbb{Z}/m_i$, hence $t_i^{m_i} \in H$.
3. $[G, [G, G]] = \{1\}$. In particular $[G, G]$ is in the center $Z(G)$ of $G$.

Then each $g \in G$ has a unique decomposition

(i) $$g = t_1^{a_1} \cdots t_s^{a_s} \cdot h, \quad T_{G/H}(g) = \prod_i T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h),$$

where $0 \leq a_i \leq m_i - 1$, $h \in H$ and

(ii) $$T_{G/H}(t_i) = t_i^d \cdot [t_i^{m_i}, \alpha_i], \quad T_{G/H}(h) = h^d \cdot [h, \alpha],$$

where $\alpha_i \in G/H$ is the product over all elements from $C_i \subset G/H$, the subgroup which is complementary to the cyclic subgroup $< t_i > \mod H$, and where $\alpha \in G/H$ is product over all elements from $G/H$.

Here we mean $[t_i^{m_i}, \alpha_i] := [\hat{t}_i^{m_i}, \hat{\alpha}_i]$, $[h, \alpha] := [\hat{h}, \hat{\alpha}]$ for any representatives $\hat{t}_i, \hat{\alpha} \in G$.

The commutators are independent of the choice of the representatives and are always elements of order $\leq 2$ because $\hat{\alpha}_i^2, \hat{\alpha}_i^2 \in H$, and $H$ is abelian.

As a consequence of (i) and (ii) we can always obtain

(iii) $$T_{G/H}(g) = g^d \cdot \varphi_{G/H}(g),$$

where $\varphi_{G/H}(g) \in Z(G)$ is an element of order $\leq 2$.

As a consequence of the second equality in (ii) combined with $[G, G] \subseteq H \cap \text{Ker}(T_{G/H})$, one has $[G, G]^d = \{1\}$, in other words, $G^d$ is contained in the center $Z(G)$ of $G$. 
If \( \rho = (Z, \chi_Z) \) is a Heisenberg representation of \( G \), then modulo \( \text{Ker}(\rho) \) \( G \) satisfies the Lemmas 3.1 and 3.3 and we obtain the following result (cf. Theorem 4.3).

**Theorem 1.3.** Let \( \rho = (Z, \chi_\rho) \) be a Heisenberg representation of \( G \), of dimension \( d \), and put \( X_\rho(g_1, g_2) := \chi_\rho \circ [g_1, g_2] \). Then we obtain

\[
(1.1) \quad (\det(\rho))(g) = \varepsilon(g) \cdot \chi_\rho(g^d),
\]

where \( \varepsilon \) is a function on \( G \) with the following properties:

1. \( \varepsilon \) has values in \( \{ \pm 1 \} \).
2. \( \varepsilon(gx) = \varepsilon(g) \) for all \( x \in G^2 \cdot Z \), hence \( \varepsilon \) is a function on the factor group \( G / G^2 \cdot Z \), and in particular, \( \varepsilon \equiv 1 \) if \( [G : Z] = d^2 \) is odd.
3. If \( d \) is even then \( \varepsilon \) need not be a homomorphism but:

\[
\varepsilon(g_1 \cdot g_2) = X_\rho(g_1, g_2) \frac{d(d-1)}{2} = \varepsilon(g_1, g_2) \frac{d^2}{2}.
\]

Furthermore,

(a) **When** \( \text{rk}_2(G/Z) \geq 4 \): \( \varepsilon \) is a homomorphism, and exactly \( \varepsilon \equiv 1 \).

(b) **When** \( \text{rk}_2(G/Z) = 2 \): \( \varepsilon \) is not a homomorphism and \( \varepsilon \) is a function on \( G / G^2 Z \) such that

\[
(\det \rho)(g) = \varepsilon(g) \cdot \chi_\rho(g^d) = \begin{cases} 
\chi_\rho(g^d) & \text{for } g \in G^2 Z \\
-\chi_\rho(g^d) & \text{for } g \notin G^2 Z.
\end{cases}
\]

## 2. Notations and preliminaries

### 2.1. Heisenberg Representations

Let \( \rho \) be an irreducible representation of a (pro-)finite group \( G \). Then \( \rho \) is called a Heisenberg representation if it represents commutators by scalar operators. Therefore higher commutators are represented by \( \rho \).

(i.e., \( Z_\rho \subseteq G \) and \( \rho(z) = \text{scalar matrix} \) for every \( z \in Z_\rho \). We can see that the linear characters of \( G \) are Heisenberg representations as the degenerate special case. If \( C^1G = G, C^{i+1}G = [C^iG, G] \) denotes the descending central series of \( G \), the Heisenberg property means \( C^3G \subset \text{Ker}(\rho) \), and therefore \( \rho \) determines a character \( X \) on the alternating square of \( A := G / C^2 G \) such that:

\[
(2.1) \quad \rho([\hat{a}_1, \hat{a}_2]) = X(a_1, a_2) \cdot E
\]

for \( a_1, a_2 \in A \) with lifts \( \hat{a}_1, \hat{a}_2 \in G \). The equivalence class of \( \rho \) is determined by the projective kernel \( Z_\rho \) which has the property that \( Z_\rho / C^2 G \) is the radical of \( X \) and by the character \( \chi_\rho \) of \( Z_\rho \) such that \( \rho(g) = \chi_\rho(g) \cdot E \) for all \( g \in Z_\rho \) and \( E \) being the unit operator. Here \( \chi_\rho \) is a \( G \)-invariant character of \( Z_\rho \) which we call the central character of \( \rho \).

We can prove that the Heisenberg representations \( \rho \) are fully characterized by the corresponding pairs \( (Z_\rho, \chi_\rho) \).

### Proposition 2.1 ([2], Proposition 4.2).

The map \( \rho \mapsto (Z_\rho, \chi_\rho) \) is a bijection between equivalence classes of Heisenberg representations of \( G \) and the pairs \( (Z_\rho, \chi_\rho) \) such that

(a) \( Z_\rho \subseteq G \) is a coabelian normal subgroup,

(b) \( \chi_\rho \) is a \( G \)-invariant character of \( Z_\rho \),

(c) \( X(g_1, g_2) := \chi_\rho(g_1 g_2 g_1^{-1} g_2^{-1}) \) is a nondegenerate alternating character on \( G / Z_\rho \), where \( g_1, g_2 \in G / Z_\rho \) and their corresponding lifts \( g_1, g_2 \in G \).
For pairs \((Z_\rho, \chi_\rho)\) with the properties \((a)-(c)\), the corresponding Heisenberg representation \(\rho\) is determined by the identity:

\[
\sqrt{|G:Z_\rho|} \cdot \rho = \text{Ind}_{Z_\rho}^G \chi_\rho.
\]

Furthermore, if \(H\) is a maximal isotropic subgroup of \(G\) for \(\rho\) and \(\chi_H \in \hat{H}\) is an extension of \(\chi_\rho\), then we have (cf. [2], p. 193, Proposition 5.3)

\[
\rho = \text{Ind}_H^G \chi_H \text{ and } [G:H] = [H:Z_\rho] = \sqrt{|G:Z_\rho|} = \dim(\rho).
\]

2.2. Transfer map. Let \(H\) be a subgroup of a finite group \(G\). If \(\{t_1, t_2, \cdots, t_n\}\) is a left transversal for \(H\) in \(G\), then for any \(g \in G\),

\[
gt_i = t_{g(i)}H,
\]

where the map \(i \mapsto g(i)\) is a permutation of the set \(\{1, 2, \cdots, n\}\). Assume that \(f : H \to A\) is a homomorphism from \(H\) to an abelian group \(A\). Then transfer of \(f\), written \(T_f\), is a mapping

\[
T_f : G \to A \quad \text{given by} \quad T_f(g) = \prod_{i=1}^n f(t_{g(i)}^{-1}gt_i) \quad \text{for all } g \in G.
\]

Since \(A\) is abelian, the order of the factors in the product is irrelevant. Now we take \(f\) the canonical homomorphism, i.e.,

\[
f : H \to H/[H,H], \quad \text{where } [H,H] \text{ is the commutator subgroup of } H.
\]

And we denote \(T_f = T_{G/H}\). Thus by definition of transfer map \(T_{G/H} : G \to H/[H,H]\), we have

\[
T_{G/H}(g) = \prod_{i=1}^n f(t_{g(i)}^{-1}gt_i) = \prod_{i=1}^n t_{g(i)}^{-1}gt_i[H,H],
\]

for all \(g \in G\).

Moreover, if \(H\) is any subgroup of finite index in \(G\), then (cf. [3], Chapter 13, p. 183)

\[
T_{G/HgHg^{-1}}(g') = gT_{G/H}(g')g^{-1},
\]

for all \(g, g' \in G\). Now let \(H\) be an abelian normal subgroup of \(G\). Let \(H^{G/H}\) be the set consisting of the elements which are invariant under conjugation. So it is clear that these elements are central elements and \(H^{G/H} \subseteq Z(G)\), the center of \(G\). When \(H\) is abelian normal subgroup of \(G\), from equation (2.5) we can conclude that (cf. [3], Chapter 13, p. 183) that

\[
\text{Im}(T_{G/H}) \subseteq H^{G/H} \subseteq Z(G).
\]

We also need to mention the generalized Furtwangler’s theorem for this article.

Theorem 2.2 ([6], p. 320, Theorem 10.25). Let \(G\) be a finite group, and let \(T_{G/K} : G \to K/[K,K]\) be the transfer homomorphism, where \([G,G] \subseteq K \subseteq G\). Then \(T_{G/K}(g)[K:G,G] = 1\) for all elements \(g \in G\).

Now if \([K : [G,G]] = 1\), i.e., \(K = [G,G]\), we have \(T_{G/[G,G]}(g) = 1\) for all \(g \in G\), i.e., the transfer homomorphism of a finite group to its commutator is trivial. This is due to Furtwangler. This is also known as Principal Ideal Theorem (cf. [3], p. 194).

To compute the determinant of an induced representation of a finite group, we need the following theorem.
Theorem 2.3 (Gallagher, [5], Theorem 30.1.6). Let $G$ be a finite group and $H$ a subgroup of $G$. Let $\rho$ be a representation of $H$ and denote $\Delta^G_H = \det(\text{Ind}^G_H 1_H)$. Then

\begin{equation}
\det(\text{Ind}^G_H \rho)(g) = (\Delta^G_H)^{\dim(\rho)}(g) \cdot (\det(\rho) \circ T_{G/E})(g), \quad \text{for all } g \in G.
\end{equation}

Let $T$ be a left transversal for $H$ in $G$. Here $\text{Ind}^G_H \rho$ is a block monomial representation (cf. [5], p. 956) with block positions indexed by pairs $(t, s) \in T \times T$. For $g \in G$, the $(t, s)$-block of $\text{Ind}^G_H \rho$ is zero unless $gt \in sH$, i.e., $s^{-1}gt \in H$ and in which case the block equal to $\rho(s^{-1}gt)$. Then we can write for $g \in G$

\begin{equation}
T_{G/E}(g) = \prod_{t \in T} s^{-1}gt[H, H].
\end{equation}

Thus for all $g \in G$

\begin{align*}
\det(\text{Ind}^G_H \rho)(g) &= (\Delta^G_H)^{\dim(\rho)}(g) \cdot (\det(\rho) \circ T_{G/E})(g) \\
&= (\Delta^G_H)^{\dim(\rho)}(g) \cdot (\det(\rho)(\prod_{t \in T} s^{-1}gt[H, H])) \\
&= (\Delta^G_H)^{\dim(\rho)}(g) \cdot \prod_{t \in T} \det(\rho)(s^{-1}gt[H, H]),
\end{align*}

where in each factor on the right, $s = s(t)$ is uniquely determined by $gt \in sH$.

2.3. Some useful results from finite Group Theory. Let $G$ be a finite abelian group and put $\alpha = \prod_{g \in G} g$. By the following theorem we can compute $\alpha$. It is very much essential for our computation. In the Heisenberg setting for computing transfer map we have to deal with abelian group $G/E$ and $\prod_{t \in G/E} t$, where $H$ is a normal subgroup of $G$.

Theorem 2.4 ([10], Theorem 6 (Miller)). Let $G$ be a finite abelian group and $\alpha = \prod_{g \in G} g$.

1. If $G$ has no element of order 2, then $\alpha = e$.
2. If $G$ has a unique element $t$ of order 2, then $\alpha = t$.
3. If $G$ has at least two elements of order 2, then $\alpha = e$.

Let $G$ be a two-step nilpotent group\footnote{Its derived subgroup, i.e., commutator subgroup $[G, G]$ is contained in its center. In other worlds, $[G, [G, G]] = \{1\}$, i.e., any triple commutator gives identity. If $\rho$ is a Heisenberg representation of a finite group $G$, then $G/\ker(\rho)$ is a two-step nilpotent group.}. For this two-step nilpotent group, we have the following lemma.

Lemma 2.5 ([7], p. 77, Lemma 9). Let $G$ be a two-step nilpotent group and let $x, y \in G$. Then

1. $[x^n, y] = [x, y]^n$, and
2. $x^n y^n = (xy)^n [x, y]^{\frac{2(n-1)}{2}}$,

for any $n \in \mathbb{N}$.

We also need the elementary divisor theorem for this article which we take from [3]. Let $G$ be a finite abelian group. So $G$ is finitely generated.
Theorem 2.6 ([3], p. 160, Theorem 3 (Invariant form)). Let $G$ be a finite abelian group. Then

\[(2.10) \quad G \cong \mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \times \cdots \times \mathbb{Z}/n_s.\]

for some integers $n_1, n_2, \ldots, n_s$ satisfying the following conditions:

(a) $n_j \geq 2$ for all $j \in \{1, 2, \ldots, s\}$, and

(b) $n_{i+1} | n_i$ for all $1 \leq i \leq s - 1$.

And the expression in [2.10] is unique: if $G \cong \mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \times \cdots \times \mathbb{Z}/m_r$, where $m_1, m_2, \ldots, m_r$ satisfies conditions (a) and (b), i.e., $m_j \geq 2$ for all $j$ and $m_{i+1} | m_i$ for all $1 \leq i \leq r - 1$, then $s = r$ and $m_i = n_i$ for all $i$.

This theorem is known as the elementary divisor theorem of a finite abelian group. Moreover, since $G$ is direct product of $\mathbb{Z}/n_i$, $1 \leq i \leq s$, then we can write

$|G| = n_1 n_2 \cdots n_s$.

We also need a structure theorem for finite abelian groups which come provided with an alternating character:

Lemma 2.7 ([1], p. 270, Lemma 1(VI)). Let $G$ be a finite abelian group and assume the existence of an alternating bi-character $X : G \times G \to \mathbb{C}^*$ ( $X(g, g) = 1$ for all $g \in G$, hence $1 = X(g_1 g_2, g_1 g_2) = X(g_1, g_2) \cdot X(g_2, g_1)$) which is nondegenerate. Then there will exist elements $t_1, t_1', \ldots, t_s, t_s' \in G$ such that

1. $G = \langle t_1 \rangle \times \langle t_1' \rangle \times \cdots \times \langle t_s \rangle \times \langle t_s' \rangle$
   \[\cong \mathbb{Z}/m_1 \times \mathbb{Z}/m_1' \times \cdots \times \mathbb{Z}/m_s \times \mathbb{Z}/m_s'\text{ and } m_1 | \cdots | m_s.

2. For all $i = 1, 2, \cdots, s$ we have $X(t_i, t_i') = \zeta_{m_i}$, a primitive $m_i$-th root of unity.

3. If we say $g_1 \perp g_2$ if $X(g_1, g_2) = 1$, then $(< t_i > \times < t_i' >)^j = \prod_{j \neq i} (< t_j > \times < t_j' >)$.

3. Explicit computation of the transfer map for two-step nilpotent group

Let $G$ be a finite group with $[G, [G, G]] = \{1\}$. Let $H$ be a normal subgroup of $G$, with abelian quotient group $G/H$ of order $d$. If $d$ is odd, then in the following lemma we compute $T_{G/H}(g)$ for all $g \in G$.

Lemma 3.1. Assume that $G$ is a finite group and $H$ a normal subgroup such that

1. $H$ is abelian,
2. $G/H$ is abelian of odd order $d$,
3. $[G, [G, G]] = \{1\}$.

Then we have $T_{G/H}(g) = g^d$ for all $g \in G$.

As a consequence one has $[G, G]^d = \{1\}$, in other words, $G^d$ is contained in the center of $G$.

Proof. First assume $g = h \in H$ and $T$ is a left transversal for $H$. Then we have

$ht = t \cdot t^{-1}ht \in tH$,
because $H$ is a normal subgroup of $G$. Hence $s = t$, where $s = s(t)$ is a function of $t$ which is uniquely determined by $gt \in sH$, for some $g \in G$. Therefore:

$$T_{G/H}(h) = \prod_{t \in T} s^{-1}ht = \prod_{t \in T} t^{-1}ht = \prod_{t \in T} hh^{-1}t^{-1}ht = \prod_{t \in T} (h \cdot [h^{-1}, t^{-1}])$$

(3.1)

$$= h^d \prod_{t \in T} [h^{-1}, t^{-1}] = h^d[h^{-1}, \prod_{t \in T} t^{-1}].$$

We have used the condition (3) in the last two equalities which means that commutators are in the center. Now we use that $G/H$ is of odd order, hence $x = 1$ if $x \in G/H$ is an element such that $x = x^{-1}$, i.e., $G/H$ has no self-inverse element. Therefore from Theorem 2.4, we have

$$\prod_{t \in T} t^{-1} = \prod_{t \in T} t = 1 \in G/H,$$

hence $T_{G/H}(h) = h^d$. Proceeding with the proof of the Lemma we have now

(3.2)

$$T_{G/H}(th) = T_{G/H}(t) \cdot T_{G/H}(h) = T_{G/H}(t) \cdot h^d.$$

Moreover, from Lemma 2.5 we can write

$$t^d h^d = (th)^d[t^{\frac{d(d-1)}{2}}, h] = (th)^d[e, h] = (th)^d,$$

since $[G, G] \subseteq Z(G)$ and $d$ is odd. So we are left to show that $T_{G/H}(t) = t^d$ for all $t \in T$.

Since $G/H$ is an abelian group of odd order, hence we may write

$$G/H = C \times U,$$

where $C$ is cyclic group of odd order $m | d$, and we assume $t \in T$ such that $tH$ is a generator of $C$. Then our transversal system can be chosen as

$$T = \{t^i u | i = 0, 1, \cdots, m-1, uH \in U\}.$$

Now if $i \leq m-2$ we have $t \cdot t^i u = t^{i+1} \cdot u = s$, hence $s^{-1} \cdot t \cdot t^i u = 1$. But $i = m-1$ we obtain

$$t(t^{m-1} u) = t^m u \in uH, \quad u^{-1} t(t^{m-1} u) = u^{-1} t^m u,$$

hence

$$T_{G/H}(t) = \prod_{u \in U} u^{-1} t^m u = \prod_{u \in U} t^m[t^{-m}, u^{-1}] = t^d \prod_{u \in U} [t^{-m}, u^{-1}]$$

(3.3)

$$= t^d[t^{-m}, \prod_{u \in U} u^{-1}] = t^d[t^{-m}, e] = t^d,$$

since $d$ is odd, then the order of $U$ is also odd and by Theorem 2.4 we have

$$\prod_{u \in U} u^{-1} = \prod_{u \in U} u = e \in U.$$

We also know that any $g \in G$ can uniquely be written as $g = th$, where $t \in T$ and $h \in H$. Then finally we obtain:

(3.4)

$$T_{G/H}(g) = T_{G/H}(th) = T_{G/H}(t) \cdot T_{G/H}(h) = t^d \cdot h^d = (th)^d[t^{\frac{d(d-1)}{2}}, h] = g^d.$$

Moreover, by our assumption (2), we have $G/H$ is an abelian group, therefore $[G, G] \subseteq H$, in particular, $T_{G/H}(h) = h^d$ for $h \in [G, G] \subseteq H$. On the other hand, from Theorem 2.2 $T_{G/[G,G]}$ is

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2Here $d$ divides $\frac{d(d-1)}{2}$ and the order of group $G/H$ is $d$. So for any $t \in G/H$, $t^{\frac{d(d-1)}{2}} = e$, the identity in $G/H$. 
trivial. So under the above Lemma’s conditions we conclude \([G, G]^d = 1\), in other words, \(G^d\) is in the center because due to condition (3) the commutator is bilinear.

\[ \]

Remark 3.2. From Lemma 3.1 we have \(T_{G/H}(g) = g^d\) for all \(g \in G\). This implies for \(g_1, g_2 \in G\)
\[
T_{G/H}(g_1g_2) = (g_1g_2)^d, \text{ on the other hand,}
\]
\[
T_{G/H}(g_1g_2) = T_{G/H}(g_1) \cdot T_{G/H}(g_2) = g_1^d \cdot g_2^d,
\]
because \(T_{G/H}\) is a homomorphism. Hence for all \(g_1, g_2 \in G\) we have
\[
(g_1g_2)^d = g_1^d g_2^d.
\]
This implies \(G^d\) is actually a subgroup of \(G\) not only a subset.

By combining Lemma 3.1 and the elementary divisor theorem, we have the following result.

Lemma 3.3. Assume that \(G\) is a finite group and \(H\) a normal subgroup such that

1. \(H\) is abelian
2. \(G/H\) is abelian of order \(d\), such that (according to the elementary divisor theorem):
\[
G/H \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s
\]
where \(m_1 | \cdots | m_s\) and \(\prod_i m_i = d\). Moreover, we fix elements \(t_1, t_2, \cdots, t_s \in G\) such that \(t_i \in G/H\) generates the cyclic factor \(\cong \mathbb{Z}/m_i\), hence \(t_i^m \in H\).
3. \([G, [G, G]] = \{1\}\). In particular, \([G, G]\) is in the center \(Z(G)\) of \(G\).

Then each \(g \in G\) has a unique decomposition

\[
(i) \quad g = t_1^{a_1} \cdots t_s^{a_s} \cdot h, \quad T_{G/H}(g) = \prod_i T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h),
\]
where \(0 \leq a_i \leq m_i - 1\), \(h \in H\), and

\[
(ii) \quad T_{G/H}(t_i) = t_i^d \cdot [t_i^{m_i}, \alpha_i], \quad T_{G/H}(h) = h^d \cdot [h, \alpha],
\]
where \(\alpha_i \in G/H\) is the product over all elements from \(C_i \subset G/H\), the subgroup which is complementary to the cyclic subgroup \(<t_i>\) mod \(H\), and where \(\alpha \in G/H\) is product over all elements from \(G/H\).

Here we mean \([t_i^{m_i}, \alpha_i] := [t_i^{m_i}, \hat{\alpha}_i], [h, \alpha] := [h, \hat{\alpha}]\) for any representatives \(\hat{\alpha}_i, \hat{\alpha} \in G\).

The commutators are independent of the choice of the representatives and are always elements of order \(\leq 2\) because \(\hat{\alpha}_i^2, \hat{\alpha}^2 \in H\), and \(H\) is abelian. As a consequence of (i) and (ii) we always obtain

\[
(iii) \quad T_{G/H}(g) = g^d \cdot \varphi_{G/H}(g),
\]
where \(\varphi_{G/H}(g) \in Z(G)\) is an element of order \(\leq 2\).

As a consequence of the second equality in (ii) combined with \([G, G] \subseteq H \cap \text{Ker}(T_{G/H})\), one has \([G, G]^d = \{1\}\), in other words, \(G^d\) is contained in the center \(Z(G)\) of \(G\).
Proof. By the given conditions, we have the abelian group $G/H$ of order $d$ with

$$G/H \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s$$

where $m_1 | \cdots | m_s$ and $\prod_i m_i = d$. Moreover, we fix elements $t_1, t_2, \cdots, t_s \in G$ such that $t_i H \in G/H$ generates the cyclic factor $\cong \mathbb{Z}/m_i$, hence $t_i^{-m_i} \in H$. Therefore for a fixed $i \in \{1, 2, \cdots, s\}$ we can define a subgroup $C_i \subset G/H$ such that $C_i$ is complementary to the cyclic subgroup $<t_i>$ of order $m_i$ mod $H$, i.e., $G/H = <t_i> \times C_i$.

Then for a fixed $i \in \{1, 2, \cdots, s\}$, we can choose a transversal system for $H$ in $G$ and which is:

$$T = \{ t^j_i c \mid 0 \leq j \leq m_i - 1, cH \in C_i \}.$$ 

Therefore from equation (3.3) we can write

\begin{equation}
T_{G/H}(t_i) = t_i^d \cdot [t_i^{-m_i}, \prod_{c \in C_i} c] = t_i^d \cdot [t_i^{-m_i}, \alpha_i],
\end{equation}

where $\alpha_i = \prod_{c \in C_i} c$.

For $h \in H$, from equation (3.1) we have

\begin{equation}
T_{G/H}(h) = h^d \cdot [h^{-1}, \alpha],
\end{equation}

where $\alpha = \prod_{t \in T} t$.

We also have $\alpha^2, \tilde{\alpha}^2 \in H$, and the commutator $[,]$ is bilinear by assumption (3), hence $1 = [h, \alpha^2] = [h, \tilde{\alpha}]^2$ and therefore

$$[h, \tilde{\alpha}] = [h, \tilde{\alpha}]^{-1} = [h^{-1}, \tilde{\alpha}].$$

Similarly, we have

$$[t_i^{-m_i}, \tilde{\alpha}_i] = [t_i^{-m_i}, \tilde{\alpha}_i]^{-1} = [t_i^{-m_i}, \tilde{\alpha}_i].$$

Thus we can rewrite the equations (3.5) and (3.6) as:

\begin{equation}
T_{G/H}(t_i) = t_i^d \cdot [t_i^{-m_i}, \prod_{c \in C_i} c] = t_i^d \cdot [t_i^{-m_i}, \alpha_i],
\end{equation}

and

\begin{equation}
T_{G/H}(h) = h^d \cdot [h, \alpha].
\end{equation}

Here $[t_i^{-m_i}, \alpha_i] := [t_i^{-m_i}, \tilde{\alpha}_i]$ and $[h, \alpha] := [h, \tilde{\alpha}]$ for any representatives $\tilde{\alpha}_i, \tilde{\alpha} \in G$.

We also know that every $g \in G$ can be uniquely written as $th$, where $t \in G/H$ and $h \in H$. Again, since $G/H$ is abelian, therefore by using elementary divisor decomposition of $G/H$, we can also uniquely express $t$ as $t = t_1^{a_1} t_2^{a_2} \cdots t_s^{a_s}$, where $0 \leq a_i \leq m_i - 1$. Thus each $g$ has a unique decomposition

$$g = th = t_1^{a_1} t_2^{a_2} \cdots t_s^{a_s} \cdot h.$$
Then we have
\[ T_{G/H}(g) = T_{G/H}(th) = T_{G/H}(t) \cdot T_{G/H}(h) \]
\[ = T_{G/H}(t_1^{a_1}t_2^{a_2} \cdots t_s^{a_s}) \cdot T_{G/H}(h) \]
\[ = \prod_{i=1}^{s} T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h). \]

By the assumption (2), \( G/H \) is an abelian group, hence \([G, G] \subseteq H\). And from equation (3.8) we have \( T_{G/H}(h) = h^d[h, \alpha] \). This implies for \([G, G] \subseteq \ker(T_{G/H})\), hence \([G, G] \subseteq H \cap \ker(T_{G/H})\). On the other hand in general \( T_{G/H} : G \rightarrow H/[H, H] \) is a homomorphism with values in an abelian group, hence it is trivial on commutators. So under the assumptions we can say \([G, G]^d = \{1\}\), in other words, \( G^d \) is in the center \( Z(G) \) because due to assumption (3) the commutator is bilinear. Let \( Z_2 \) be the set of all elements of \( Z(G) \) of order \( \leq 2 \). Since \( G^d \subseteq Z(G) \), then by using Lemma 2.5(2) with \( n = d \), we obtain
\[ \prod_{i=1}^{s} t_i^{a_i \cdot d} \equiv \left( \prod_{i=1}^{s} t_i^{a_i} \right)^d \quad (\text{mod } Z_2) \equiv t^d \quad (\text{mod } Z_2) \]

because combining \( G^d \subseteq Z(G) \) and Lemma 2.5(2) we can write
\[ x^d y^d = (xy)^d \cdot [x, y]^{2d-1} \equiv (xy)^d \quad (\text{mod } Z_2) \text{ for all } x, y \in G. \]

Moreover, since \( \alpha_1^2 = 1 \) and \( \alpha^2 = 1 \), therefore we have \([t_i^{m_i}, \alpha_i]^{a_i} \in Z_2\) for all \( i \in \{1, 2, \cdots, s\}\) and \([h, \alpha] \in Z_2\). Again by using Lemma 2.5(2) we can write
\[ \prod_{i=1}^{s} t_i^{m_i} \cdot [t_i^{m_i}, \alpha_i]^{a_i} \cdot h^d[h, \alpha] \equiv g^d \quad (\text{mod } Z_2). \]

Now by using equations (3.7) and (3.8), we obtain:
\[ T_{G/H}(g) = \prod_{i=1}^{s} T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h) \]
\[ = \prod_{i=1}^{s} t_i^{a_i \cdot d} \cdot [t_i^{m_i}, \alpha_i]^{a_i} \cdot h^d[h, \alpha] \]
\[ \equiv g^d \quad (\text{mod } Z_2) \quad \text{by equation (3.10)} \]
\[ = g^d \cdot \varphi_{G/H}(g), \]

where \( \varphi_{G/H} \) is a correcting function with values in \( Z_2 \).

\[ \square \]

Remark 3.4 (Properties of the correcting function \( \varphi_{G/H} \)). (i) The correcting function \( \varphi_{G/H} \) is a function on \( G/G^2[G, G] \) with values in \( Z_2 \).

Proof. From Lemma 3.3 we have
\[ T_{G/H}(g) = g^d \varphi_{G/H}(g), \]

where \( \varphi_{G/H}(g) \) is a correcting function.

We have here \([G, [G, G]] = \{1\}\). This implies \([g, z] = 1\) for all \( g \in G \) and \( z \in [G, G] \). Since \([G, G] \subseteq \ker(T_{G/H})\), then for all \( x \in [G, G] \) we have
Proof. From Lemma 3.3(iii) we have identity (3.13), hence we will have \( gTG_G = T_{G/H}(g)T_{G/H}(x) = T_{G/H}(g) \) for all \( g \in G \).

Also here we have \([G, G]^d = \{1\}\), then by using Lemma 2.5(2) for \( x \in [G, G] \) we can write
\[
(gx)^d = g^d x^d[g, x]^{-\frac{2(d-1)}{2}} = g^d \quad \text{for all} \quad g \in G.
\]

From Lemma 3.3 we also have \( T_{G/H}(gx)^d = (gx)^d \varphi_{G/H}(gx) \). By comparing these above equations for \( x \in [G, G] \) we obtain
\[
\varphi_{G/H}(gx) = \varphi_{G/H}(g) \quad \text{for all} \quad g \in G.
\]

Moreover, if \( x \in G^2 \), from Lemma 3.3 we have
\[
T_{G/H}(x) = x^d \varphi_{G/H}(x) = x^d \quad \text{since} \quad \varphi_{G/H}(x) = 1 \in Z_2.
\]

So \( T_{G/H}(gx) = T_{G/H}(g) \cdot T_{G/H}(x) = T_{G/H}(g) \cdot x^d \) for all \( g \in G \).

Again from Lemma 2.5(2) we have for \( x \in G^2 \)
\[
(gx)^d = g^d x^d[g, x]^{-\frac{2(d-1)}{2}} = g^d x^d \quad \text{for all} \quad g \in G.
\]

By comparing we can see that \( \varphi_{G/H}(gx) = \varphi_{G/H}(g) \) for all \( g \in G \) and \( x \in G^2 \).

Thus we can conclude that the correcting function \( \varphi_{G/H} \) is a function on \( G/G^2[G, G] \) with values in \( Z_2 \).

\[ \square \]

(ii) \( G^d \subset Z(G) \) if and only if \( \text{Im}(\varphi_{G/H}) \subset Z(G) \).

\textbf{Proof.} From Lemma 3.3(iii) we have \( T_{G/H}(g) = g^d \cdot \varphi_{G/H}(g) \). From relation (2.6) we also know that \( \text{Im}(T_{G/H}) \subset H^{G/H} \subset Z(G) \), hence \( g^d \cdot \varphi_{G/H}(g) \in Z(G) \). Now if \( G^d \subset Z(G) \), then \( \varphi_{G/H}(g) \in Z(G) \) for all \( g \in G \). Hence \( \text{Im}(\varphi_{G/H}) \subset Z(G) \).

Conversely, if \( \text{Im}(\varphi_{G/H}) \subset Z(G) \), then from \( g^d \varphi_{G/H}(g) \in Z(G) \) we can conclude that \( G^d \subset Z(G) \).

\[ \square \]

(iii) When \( d \) is odd (resp. even), \( \varphi_{G/H} \) is a homomorphism (resp. not a homomorphism).

\textbf{Proof.} Since \( T_{G/H} \) is a homomorphism we obtain the identity:
\[
(3.12) \quad (g_1g_2)^d \varphi_{G/H}(g_1g_2) = g_1^d g_2^d \varphi_{G/H}(g_1) \varphi_{G/H}(g_2).
\]

This implies
\[
(3.13) \quad \frac{\varphi_{G/H}(g_1g_2)}{\varphi_{G/H}(g_1) \varphi_{G/H}(g_2)} = \frac{g_1^d g_2^d}{(g_1g_2)^d} = \frac{(g_1g_2)^d[g_1, g_2]^{\frac{d(d-1)}{2}}}{(g_1g_2)^d} = [g_1, g_2]^{\frac{d(d-1)}{2}}.
\]

We also have here \([G, G]^d = 1\). When \( d \) is odd, then \( d \) divides \( \frac{d(d-1)}{2} \), hence the right side of equation (3.13) is equal to 1. Thus when \( d \) is odd, \( \varphi \) is a homomorphism, and exactly \( \varphi \equiv 1 \).

This follows from Lemma 3.1.

But when \( d \) is even \( d \) does not divide \( \frac{d(d-1)}{2} \), hence the right side of equation (3.13) is not equal to 1. This shows that \( \varphi_{G/H} \) is not a homomorphism when \( d \) is even.

\[ \square \]

(iv) If \( H' \subset G \) is another normal subgroup such that \( H' \) is abelian and \( G/H' \) is abelian of order \( d \), then \( \varphi_{G/H'} \) is again a function on \( G/G^2[G, G] \) with values in \( Z_2 \) which satisfies the same identity (3.13), hence we will have
\[
\varphi_{G/H'} = \varphi_{G/H} \cdot I_{H/H'}
\]
for some homomorphism $f_{H,H'} \in \text{Hom}(G/G^2[G,G], \mathbb{Z}_2)$.

4. Invariant formula of determinant for Heisenberg representations

In general, for the Heisenberg setting $G$ need not be two-step nilpotent group. But $\overline{G} = G/\text{Ker}(\rho)$ is always a two-step nilpotent group, where $\rho$ is a Heisenberg representation of $G$. The Lemmas 3.1 and 3.3 hold for two-step nilpotent groups. Therefore to use them in our Heisenberg setting, we have to do our computation under modulo $\text{Ker}(\rho)$. Our determinant computation is under modulo $\text{Ker}(\rho)$. And we drop modulo $\text{Ker}(\rho)$ from our remaining part of this article.

Definition 4.1 (2-rank of a finite abelian group). Let $G$ be a finite abelian group. Then from elementary divisor Theorem 2.6 we can write

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$$

where $m_1|m_2|\cdots|m_s$ and $\prod_{i=1}^{s} m_i = |G|$. We define

the 2-rank of $G$ := the number of $m_i$-s which are even

and we set

$$\text{rk}_2(G) = 2\text{-rank of } G.$$  

When the order of an abelian group $G$ is odd, from the structure of $G$ we have $\text{rk}_2(G) = 0$, i.e., there is no even $m_i$-s for $G$. We also denote

$$G[2] := \{x \in G \mid 2x = 0\},$$ i.e., set of all elements of order at most 2.

If $G = G_1 \times G_2 \times \cdots \times G_r$, $r \in \mathbb{N}$, is an abelian group, then we can show that

$$|G[2]| = \prod_{i=1}^{r} |G_i[2]|.$$  

Proposition 4.2. Let $G$ be an abelian group of $\text{rk}_2(G) = n$. Then $G$ has $2^n - 1$ nontrivial elements of order 2.

Proof. We know that $(\mathbb{Z}_n, +)$ is a cyclic group, where $n \in \mathbb{N}$. If $n$ is odd, $\mathbb{Z}_n$ does not have any nontrivial element of order 2. But when $n$ is even, it is clear that $\frac{n}{2} \in \mathbb{Z}_n$ is the only one nontrivial element of order 2. So this tells us when $n$ is even, $\mathbb{Z}_n$ has a unique element of order 2.

Any given abelian group $G$ of $\text{rk}_2(G) = n$ can be written as

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$$

where $m_1|m_2|\cdots|m_s$ and $m_{s-n+1}, m_{s-n+2}, \cdots, m_s$ are $n$ even, and rest of the $m_i$-s are odd. Therefore from equation (4.2) we conclude that

$$|G[2]| = |\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s}[2]| = \prod_{i=1}^{n} |\mathbb{Z}_{m_{s-n+i}}[2]| = \frac{2\times\cdots\times2}{n\times\text{times}} = 2^n.$$  

Hence we can conclude that when $G$ is abelian with $\text{rk}_2(G) = n$, it has $2^n - 1$ nontrivial elements of order 2.

$\square$
Theorem 4.3. Let $\rho = (Z, \chi_\rho)$ be a Heisenberg representation of $G$, of dimension $d$, and put $X_\rho(g_1, g_2) := \chi_\rho \circ (g_1, g_2)$. Then we obtain

\begin{equation}
(\det(\rho))(g) = \varepsilon(g) \cdot \chi_\rho(g^d),
\end{equation}

where $\varepsilon$ is a function on $G$ with the following properties:

1. $\varepsilon$ has values in $\{\pm 1\}$.
2. $\varepsilon(gx) = \varepsilon(g)$ for all $x \in G^2 \cdot Z$, hence $\varepsilon$ is a function on the factor group $G/G^2 \cdot Z$, and in particular, $\varepsilon \equiv 1$ if $[G : Z] = d^2$ is odd.
3. If $d$ is even then $\varepsilon$ need not be a homomorphism but:

\[
\frac{\varepsilon(g_1)\varepsilon(g_2)}{\varepsilon(g_1g_2)} = X_\rho(g_1, g_2)^{\frac{d(d-1)}{2}} = X_\rho(g_1g_2)^{-\frac{d(d-1)}{2}}.
\]

Furthermore,

(a) when $\text{rk}_2(G/Z) \geq 4$: $\varepsilon$ is a homomorphism, and exactly $\varepsilon \equiv 1$.

(b) when $\text{rk}_2(G/Z) = 2$: $\varepsilon$ is not a homomorphism and $\varepsilon$ is a function on $G/G^2Z$ such that

\[
(\det \rho)(g) = \varepsilon(g) \cdot \chi_\rho(g^d) = \begin{cases} 
\chi_\rho(g^d) & \text{for } g \in G^2Z \\
-\chi_\rho(g^d) & \text{for } g \notin G^2Z.
\end{cases}
\]

Proof. By the given condition, $\rho = (Z, \chi_\rho)$ is a Heisenberg representation of $G$. Let $H$ be a maximal isotopic subgroup for $X_\rho$, then we have $\rho = \text{Ind}_H^G(\chi_H)$, where $\chi_H$ is a linear character of $H$ which extends $\chi_\rho$. Then modulo $\text{Ker}(\chi_\rho) = \text{Ker}(\rho) \subset Z$ the assumptions of the Lemma 3.3 are fulfilled, and therefore:

\[
(\det \rho)(g) = \Delta_H^G(g) \cdot \chi_H(T_{G/H}(g)) = \Delta_H^G(g) \cdot \chi_\rho(T_{G/H}(g)) \quad \text{because the values of } T_{G/H} \text{ are in } Z
\]

\[
= \Delta_H^G(g) \cdot \chi_\rho(g^d) \chi_\rho(\varphi_{G/H}(g)) \quad \text{from Lemma 3.3}
\]

\[
= \varepsilon(g) \cdot \chi_\rho(g^d),
\]

where

\begin{equation}
\varepsilon(g) := \Delta_H^G(g) \cdot \chi_\rho(\varphi_{G/H}(g)).
\end{equation}

Since $\Delta_H^G$ is the quadratic determinant character of $G$, hence for every $g \in G$, we have $\Delta_H^G(g) \in \{\pm 1\}$. And $\varphi_{G/H}(g) \in Z_2$, then $\chi_\rho(\varphi_{G/H}(g)) \in \{\pm 1\}$. Therefore for every $g \in G$,

\[
\varepsilon(g) = \Delta_H^G(g) \cdot \chi_\rho(\varphi_{G/H}(g)) \in \{\pm 1\},
\]

which does not depend on $H$ because $\Delta_H^G = \Delta_1^{G/H}$, and $\chi_\rho$ does not depend on $H$.

Here $Z$ is the scalar group of the irreducible representation $\rho$ of dimension $d$, then by definition of scalar group, elements $z \in Z$ are represented by scalar matrices, i.e.,

\[
\rho(z) = \chi_\rho(z) \cdot I_d, \quad \text{where } I_d \text{ is the } d \times d \text{ identity matrix.}
\]

This implies

\[
(\det \rho)(z) = \chi_\rho(z)^d = \chi_\rho(z^d).
\]

We also know that $Z$ is the radical of $X_\rho$, therefore
From equation (4.3) we obtain
\[ (4.5) \quad (\det \rho)(gz) = (\det \rho)(g) \cdot (\det \rho)(z) = \varepsilon(g)\chi(\rho(g^d))\chi(\rho(z^d)). \]

On the other hand
\[ (4.6) \quad (\det \rho)(gz) = \varepsilon(gz)\chi(\rho((gz)^d)) = \varepsilon(gz)\chi(\rho(g^d)^{z^d}[g, z]^{d(d-1)/2}) = \varepsilon(gz)\chi(\rho(g^d))\chi(\rho(z^d)). \]

On comparing equations (4.5) and (4.6) we get
\[ \varepsilon(gz) = \varepsilon(g) \quad \text{for all} \quad g \in G \quad \text{and} \quad z \in Z. \]

Moreover, since \( \varepsilon(g) \) is a sign, we have
\[ (\det \rho)(g^2) = (\det \rho)(g)^2 = \varepsilon(g)^2\chi(\rho(g^{2d})). \]

Therefore
\[ (4.7) \quad (\det \rho)(gx^2) = (\det \rho)(g) \cdot (\det \rho)(x^2) = \varepsilon(g)\chi(\rho(g^d))\chi(\rho(x^{2d})). \]

On the other hand
\[ (4.8) \quad (\det \rho)(gx^2) = \varepsilon(gx^2)\chi(\rho((gx^2)^d)) = \varepsilon(gx^2)\chi(\rho(g^d))\chi(\rho(x^{2d})). \]

So we see from equations (4.7) and (4.8) \( \varepsilon(gx^2) = \varepsilon(g) \), hence \( \varepsilon \) is a function on \( G/G^2Z \).

In particular, when \( [G: Z] = d^2 \) is odd, i.e., \( |G/H| = d \) is odd, we have \( \varphi_{G/H}(g) = 1 \) as well \( \Delta^G_H(g) = 1 \) because \( H \) is normal subgroup of odd index in \( G \). This shows that \( \varepsilon \equiv 1 \) when \( [G: Z] = d^2 \) is odd.

For checking property (iii), we use equation (4.3) and \( [G, G]^d = \{1\} \). Since \( [G, G]^d = \{1\} \), we have for \( g_1, g_2 \in G \)
\[
(\rho(g_2)^{d-1})^2 = 1, \quad \text{i.e.,} \quad [g_1, g_2]^{d(d-1)/2} = \frac{1}{\chi(\rho(g_2)^{d-1})}. \quad \text{Also,} \\
[g_1, g_2]^{-1} = [g_1, g_2]^{-1} \quad \text{and} \quad [g_1, g_2]^\frac{d}{2} = [g_1, g_2]^{-\frac{d}{2}}. 
\]

From equation (4.3) we obtain
\[
\frac{(\det \rho)(g_1) \cdot (\det \rho)(g_2)}{(\det \rho)(g_1 g_2)} = \frac{\varepsilon(g_1)\chi(\rho(g_1^d)) \cdot \varepsilon(g_2)\chi(\rho(g_2^d))}{\varepsilon(g_1 g_2)\chi(\rho((g_1 g_2)^d))}. 
\]

This implies
\[
\frac{\varepsilon(g_1)\varepsilon(g_2)}{\varepsilon(g_1 g_2)} = \frac{\chi(\rho([g_1, g_2])^{d-1})}{\chi(\rho(g_1^d g_2^d))} = \chi(\rho([g_1, g_2])^{d(d-1)/2}) = X(\rho(g_1^d g_2^d))^{d-1} = X(\rho(g_1, g_2)^d)^{d-1}. 
\]

This shows that \( \varepsilon \) is not a homomorphism when \( d \) is even.

But when \( |G/Z| = d^2 \) and \( d \) is even we can write
\[
G/Z \cong (\mathbb{Z}/m_1 \times \mathbb{Z}/m_1) \times \cdots \times (\mathbb{Z}/m_s \times \mathbb{Z}/m_s) 
\]
\[
\cong \left< t_1 > \times \left< t_1' > \right> \right> \perp \cdots \perp \left< t_s > \times \left< t_s' > \right> \right>, 
\]
such that $m_1 | \cdots | m_s$ and $\prod_{i=1}^{s} m_i^2 = d^2$, $X_\rho(t_i, t'_i) = \chi_\rho([t_i, t'_i]) = \zeta_{m_i}$, a primitive $m_i$-th root of unity because $[t_i, t'_i]^{m_i} = 1$. If $m_{s-1}, m_s$ are both even\footnote{Here $d = m_1 \cdots m_{s-1} m_s$, if both $m_{s-1}, m_s$ are even, then $\frac{d}{m_s} = m_1 \cdots (\frac{m_{s-1}}{2}) \cdot m_s$. This shows that $m_i | \frac{d}{2}$ for all $i \in \{1, \ldots, s\}$. Therefore, $X_\rho(t_i, t'_i) = (\zeta_{m_i})^{\frac{d}{2}} = 1$ for all $i \in \{1, \ldots, s\}$.}, which means 2-rank of $G/Z$ is $\geq 4$ then $\frac{d}{m_s}$ is even and therefore $X_\rho(x, y)^{\frac{d}{2}} \equiv 1$, hence from equation (4.9) we see that $\varepsilon$ is a homomorphism.

Moreover, from the above we see that

$$H/Z \cong <t_1> \times \cdots \times <t_s> \cong H'/Z \cong <t'_1> \times \cdots \times <t'_s>$$

are two maximal isotropic which are isomorphic. We have $H \cap H' = Z$, hence $G$ is not the direct product of $H$ and $H'$ but nevertheless $G = H \cdot H'$. So for any $g \in G$ there must exist a decomposition $g = h \cdot h'$, where $h \in H$ and $h' \in H'$.

Now we assume $\text{rk}_2(G/Z) \neq 2$, hence $\text{rk}_2(H/Z) = \text{rk}_2(H'/Z) \neq 1$. And since $G/H \cong H/Z$ and $G/H' \cong H'/Z$, then $\text{rk}_2(G/H) = \text{rk}_2(G/H') \neq 1$. Then from Proposition 4.2 we can say both $G/H$ and $G/H'$ have at least 3 elements of order 2. Then from Theorem 2.4 we have $\alpha_{G/H} = 1$ and $\alpha_{G/H'} = 1$. Furthermore from formula (3.8) we obtain

$$T_{G/H}(h) = h^d \cdot [h, \alpha_{G/H}] = h^d, \quad \text{and} \quad T_{G/H}(h') = h'^d \cdot [h', \alpha_{G/H'}] = h'^d.$$ 

So we can write

$$(\text{det } \rho)(g) = (\text{det } \rho)(h) \cdot (\text{det } \rho)(h'), \quad \text{here } g = h \cdot h' \text{ is a decomposition of } g \text{ with } h \in H, h' \in H',$$

$$= \chi_\rho(h^d) \cdot \chi_\rho(h'^d), \quad \text{because } \text{rk}_2(G/H') = \text{rk}_2(G/H') \neq 1,$$

$$= \chi_\rho(h^d \cdot h'^d)$$

$$= \chi_\rho((h \cdot h')^d h, h')^{d(d-1)} \quad \text{using Lemma 2.5(2)}$$

$$= \chi_\rho(g^d) \cdot X_\rho(h, h')^{\frac{d(d-1)}{2}}$$

$$= \chi_\rho(g^d),$$

because all $m_i | \frac{d}{2}$, $i \in \{1, 2, \ldots, s\}$, and then

$$X_\rho(h, h')^{\frac{d(d-1)}{2}} = \chi_\rho([h, h'])^{\frac{d(d-1)}{2}} = \zeta_{m}^{\frac{d(d-1)}{2}} = 1,$$

where $\zeta_{m}$ is a primitive $m$-th root of unity and $m$ is some positive integer (which is the order of $[h, h']$) which divides $\frac{d}{2}$. This shows that when $\text{rk}_2(G/Z) \neq 2$ we have $\varepsilon \equiv 1$.

If on the other hand only $m_s$ is even, i.e., $G/Z$ has rank $= 2$, then $\frac{d}{m_s}$ is odd. Therefore $X_\rho(t_s, t'_s)^{\frac{d}{2}} = (\zeta_{m_s})^{\frac{d}{2}} = -1$, since $m_s$ does not divide $\frac{d}{2}$ and $(\zeta_{m_s})^2 = 1$. Therefore $\varepsilon$ cannot be a homomorphism when $\text{rk}_2(G/Z) = 2$.

But since $\varepsilon$ is a function on $G/G^2Z$, hence $\varepsilon|_{G^2Z} \equiv 1$. Therefore when $g \in G^2Z$ we have $(\text{det } \rho)(g) = \chi_\rho(g^d)$. So now we are left to show that for $g \notin G^2Z$, $\varepsilon(g) = -1$, i.e., $(\text{det } \rho)(g) = \chi_\rho(g^d)$.
−χρ(γd). Also, for \( \text{rk}_2(G/Z) = 2 \), \( G/G^2Z \) is Klein’s 4-group\(^4\) and \( \varepsilon \) is a sign function on that group. So up to permutation the possibilities are

\[
\begin{align*}
(1) & \quad + + + + \\
(2) & \quad + + - - \\
(3) & \quad + + - - \\
(4) & \quad + - - -
\end{align*}
\]

The cases (1), (3) can be excluded because we know that \( \varepsilon \) is not a homomorphism. So we have to exclude the case (2) and for this it is enough to see that we must have \( - \) more than once.

If we restrict \( \varepsilon \) to a maximal isotropic subgroup \( H \), then from equation (4.9) we can say \( \varepsilon \) is a homomorphism on \( H \), because \( X^h|_{H \times H} = 1 \). We also have from equation (3.8) \( T_{G/H}(h) = h^d \cdot [h, \alpha_{G/H}] = h^d \varphi_{G/H}(h) \). This implies \( \varphi_{G/H}(h) = [h, \alpha_{G/H}] \) for all \( h \in H \). Moreover, since \( \Delta^G_H|_H \equiv 1 \), then for \( h \in H \) we obtain:

\[
\varepsilon(h) = \Delta^G_H(h) \cdot \chi_\rho([h, \alpha_{G/H}]) = \chi_\rho([h, \alpha_{G/H}]).
\]

If there exists a maximal isotropic subgroup \( H \) of \( \text{rk}_2(H/Z) = 1 \), then from the Proposition 4.2 can say that \( H/Z \) has a unique element of order 2. We also know \( G/H \cong H/Z \) because \( G/H \) and \( H/Z \) are both finite abelian groups of same order \( d \), hence \( \text{rk}_2(H/Z) = \text{rk}_2(G/H) = 1 \). Then from the Proposition 4.2 \( G/H \) has a unique element of order 2, and therefore by Miller’s theorem we have \( \alpha_{G/H} \neq 1 \). Thus for the case \( \text{rk}_2(H/Z) = 1 \) we have

\[
(4.10) \quad \varepsilon(h) = \Delta^G_H(h) \cdot \chi_\rho([h, \alpha_{G/H}]) = \chi_\rho([h, \alpha_{G/H}]) = -1
\]

for all nontrivial \( h \in H \).

Moreover, if \( \text{rk}_2(G/Z) = 2 \), then from the Lemma 2.7 there exists subgroups \( H, H' \) with the following properties

\[
\begin{align*}
(1) & \quad \text{rk}_2(H/Z) = \text{rk}_2(H'/Z) = 1 \\
(2) & \quad G = H \cdot H' \\
(3) & \quad Z = H \cap H'
\end{align*}
\]

Then \( H/G^2Z \) and \( H'/G^2Z \) are two different subgroups of order 2 in Klein’s 4-group. Now take the nontrivial elements of these subgroups are \( h \) and \( h' \) respectively. Then by using equation (4.10) we have

\[
\varepsilon(h) = \varepsilon(h') = -1,
\]

i.e., the nontrivial elements of \( H/G^2Z \) and \( H'/G^2Z \) give the two \( - \) signs for \( \varepsilon \).

Therefore the only possibility is \( + - - - \), i.e., \( \varepsilon \) takes 1 on the trivial coset and \( -1 \) on the three other cosets.

This completes the proof. \( \square \)

\(^4\)Since \( G/Z \) is an abelian group, we have \( G/Z \cong \hat{G}/Z \). When \( \text{rk}_2(G/Z) = 2 \), by Proposition 4.2 there are exactly three elements of order 2 in \( G/Z \), and this each element (i.e., self-inverse element) corresponds a quadratic character of \( G/Z \). Hence the group \( G/G^2Z \) has exactly three quadratic characters. Furthermore, \( G/G^2Z \) is a quotient group of the abelian group \( G/Z \), hence \( G/G^2Z \) is abelian. Therefore \( G/G^2Z \) is isomorphic to the Klein’s 4-group.
Corollary 4.4. (1) Let \( \rho = (\mathbb{Z}, \chi_{\rho}) \in \text{Irr}(G) \) be a Heisenberg representation of odd dimension \( d \). Then \( G^d \subseteq Z \) and
\[
\det(\rho)(g) = \chi_{\rho}(g^d), \quad \text{for all } g \in G.
\]
In particular, \( \det(\rho) \equiv 1 \) if and only if \( \chi_{\rho} \) is a character of \( Z/G^d \). This is only possible if \( [G, G] \not\subseteq G^d \) and if \( \chi_{\rho} \) is a non-trivial character on \( G^d[G, G]/G^d \subseteq Z/G^d \).

(2) Let \( \omega \) be a linear character of \( G \), then \( \rho \otimes \omega = (Z, \chi_{\rho \otimes \omega}) \), where:
\[
\chi_{\rho \otimes \omega} = \chi_{\rho} \cdot \omega_Z, \quad \det(\rho \otimes \omega) = \det(\rho) \cdot \omega^d,
\]
where \( \omega_Z = \omega|_Z \). Therefore it is possible to find \( \omega \) such that \( \det(\rho \otimes \omega) \equiv 1 \), equivalently \( \chi_{\rho} = \omega_Z \) on \( G^d \), if and only if \( \chi_{\rho} \) is trivial on \( G^d \cap [G, G] \).

Proof. (1). We consider \( H \) such that \( Z \subset H \subseteq G \) and \( H \) is maximal isotropic with respect to \( X(g_1, g_2) := \chi_{\rho} \circ [g_1, g_2] \).

By definition, \( Z/[G, G] \) is radical of \( X \), hence \( \text{Ker}(\rho) = \text{Ker}(\chi_{\rho}) \subseteq Z \), and factorizing by \( \text{Ker}(\chi_{\rho}) \) we obtain a group \( \mathcal{C} = G/\text{Ker}(\chi_{\rho}) \) which satisfies the assumptions of the Lemma 3.3. Moreover, \( \rho = \text{Ind}_{\mathcal{C}}^G \chi_H \) for any extension \( \chi_H \) of \( \chi_{\rho} \), hence
\[
\det(\rho) = \Delta^G_H \cdot (\chi_H \circ T_{G/H}) = \chi_H \circ T_{G/H},
\]
because \( H \subseteq G \) is a normal subgroup of odd index \( d \). Applying the Lemma 3.3 we obtain for all \( g \in G \):
\[
(4.11) \quad \det(g) = \chi_H \circ T_{G/H}(g) = \chi_H(g^d) = \chi_{\rho}(g^d), \quad \text{since } g^d \in Z,
\]
because from relation (2.6) we have\(^5\) \( g^d = T_{G/H}(g) \in \text{Im}(T_{G/H}) \subseteq H^G/H \subseteq Z \).

If \( \det(\rho) \equiv 1 \), then from equation (4.11), we have \( \chi_{\rho}(g^d) = 1 \), i.e., \( g^d \in \text{Ker}(\chi_{\rho}) \) for all \( g \in G \). This shows \( G^d \subseteq \text{Ker}(\chi_{\rho}) \). Again if \( G^d \subseteq \text{Ker}(\chi_{\rho}) \), then it is easy to see \( \det \rho \equiv 1 \).

Now if \( \chi_{\rho} : Z/G^d \to \mathbb{C}^\times \) is a character, then we see that \( G^d \subseteq \text{Ker}(\chi_{\rho}) \), i.e., \( g^d \in \text{Ker}(\chi_{\rho}) \). Thus from equation (4.11), we conclude \( \det(\rho) \equiv 1 \).

If \( [G, G] \subseteq G^d \) this would imply \( [G, G] \subseteq \text{Ker}(\chi_{\rho}) \) which means \( Z = G \), hence \( \rho \) is of dimension 1. Also, if \( \chi_{\rho} \) is trivial on \( G^d[G, G]/G^d \), i.e., \( [G, G] \subseteq \text{Ker}(\rho) \), hence dimension of \( \rho \) is 1. Therefore when \( \det \rho \equiv 1 \) and \( [G, G] \not\subseteq G^d \) and \( \chi_{\rho} \) is a nontrivial character on \( G^d[G, G]/G^d \subseteq Z/G^d \), then we can extend \( \chi_{\rho} \) to \( Z/G^d \) because \( \chi_{\rho} \) is \( G \)-invariant and \( G^d[G, G]/G^d \) is a normal subgroup of \( Z/G^d \).

(2) Let \( \omega \) be a linear character of \( G \) and \( \omega_Z = \omega|_Z \). Then we can write (cf. [8], p. 57, Remark (3))
\[
\omega \otimes \text{Ind}_{\mathcal{C}}^G \chi_{\rho} = \text{Ind}_{\mathcal{C}}^G (\chi_{\rho} \otimes \omega_Z) = \text{Ind}_{\mathcal{C}}^G \chi_{\rho \otimes \omega} = d \cdot \rho \otimes \omega,
\]
where \( \chi_{\rho \otimes \omega} = \chi_{\rho} \cdot \omega_Z \) and \( d = \dim(\rho) \). Moreover, it is easy to see \( \chi_{\rho} \otimes \omega_Z \) is a \( G \)-invariant. Therefore we can write \( \rho \otimes \omega = (Z, \chi_{\rho \otimes \omega}) \). Now we are left to compute determinant of \( \rho \otimes \omega \), which follows from the properties of determinant function (cf. [8], p. 955, Lemma 30.1.3). Since \( \dim(\rho) = d \) and \( \omega \) is linear, then we have
\[
\det(\rho \otimes \omega) = \det(\rho)^\omega(1) \cdot \det(\omega)^{\rho(1)} = \det(\rho) \cdot \omega^d.
\]
\(^5\) Here our determinant computation is under modulo \( \text{Ker}(\rho) \). Modulo \( \text{Ker}(\rho) \), \( G \) is a two-step nilpotent group, hence the relation (2.6) is satisfied here.
Here \( d \) is odd and we know \( g^d \in Z \), then for every \( g \in G \), we have
\[
\det(\rho \otimes \omega)(g) = \chi_\rho(g^d) \cdot \omega^d(g) = \chi_\rho \cdot \omega_Z(g^d).
\]
Now if \( \det(\rho \otimes \omega) \equiv 1 \), then we have \( \chi_\rho = \omega_Z^{-1} \) on \( G^d \). This implies, it is possible to find a linear character \( \omega \) such that \( \det(\rho \otimes \omega) \equiv 1 \).

Now let \( \chi_\rho = \omega_Z^{-1} \) on \( G^d \). Since \( G^d \cap [G, G] \subseteq G^d \), \( \chi_\rho \cdot \omega_Z(g) = 1 \) for \( g \in G^d \cap [G, G] \). Then \( \chi_\rho \) is trivial on \( G^d \cap [G, G] \).

Conversely, if \( \chi_\rho \) is trivial on \( G^d \cap [G, G] \), then we are left to show that we can find an \( \omega \) such that \( G^d \subseteq \ker(\chi_\rho \cdot \omega_Z) \).

Put \( Z_1 = G^d \cdot [G, G] \), and \( Z_0 = G^d \cap [G, G] \). Then we have \( Z \supseteq Z_1 \supseteq Z_0 \), and \( Z_1/Z_0 = G^d/Z_0 \times [G, G]/Z_0 \) is a direct product. Now assume that \( \chi_\rho \) is a character of \( Z/Z_0 \). Then the restriction \( \chi_{Z_1} = \chi_1 \cdot \chi_2 \) comes as the product of two characters of \( Z_1 \), where \( \chi_1 \) is trivial on \([G, G]\) and \( \chi_2 \) is trivial on \( G^d \). But then we can find \( \omega \) of \( G/[G, G] \) such that \( \omega_{Z_1} = \chi_1 \), hence \( \omega^{-1} \chi_\rho \) restricted to \( Z_1 \) is equal \( \chi_2 \). In particular, \( \omega^{-1} \chi_\rho \) is trivial on \( G^d \) and therefore \( \omega^{-1} \otimes \rho \) has \( \det(\omega^{-1} \otimes \rho) \equiv 1 \).

**Corollary 4.5.** If \( \rho = (Z, \chi_\rho) \) is a Heisenberg representation (of dimension > 1) for a nonabelian group of order \( p^3 \), \( p \neq 2 \), then \( Z = [G, G] \) is cyclic group of order \( p \), and \( G^p = Z \) or \( G^p = \{1\} \) depending on the isomorphism type of \( G \). So we have \( \det(\rho) \neq 1 \) and \( \det(\rho) \equiv 1 \) depending on the isomorphism type of \( G \).

**Proof.** In this particular case for Heisenberg setting we have \( |Z| = p \) and \( G/Z \) is abelian. This implies \( [G, G] \subseteq Z \). Here \( G \) is nonabelian and \( p \) is prime, therefore \([G, G] = Z \) is a cyclic group of order \( p \). Let \( \Psi : G \to G \) be a \( p \)-power map, i.e., \( g \mapsto g^p \). It can be proved that this map \( \Psi \) is a surjective group homomorphism (by using Lemma 2.5) and the image is in \( Z \) (because from Lemma 3.1 and relation (2.6) we have \( g^p = T_{G/H}(g) \in Z \)), hence from the first isomorphism theorem we have
\[
G/\ker(\Psi) \cong \text{Im}(\Psi) = G^p.
\]
Thus we can write
\[
p^3 = |\ker(\Psi)| \cdot |G^p|.
\]
So we have the possibility
\[
|\ker(\Psi)| = p^3 \text{ or } p^2 \text{ corresponding to } G^p = 1 \text{ or } G^p = Z.
\]
Both the cases are possible depending on the isomorphism type of \( G \). So we can conclude that \( \det(\rho) \neq 1 \) and \( \det(\rho) \equiv 1 \) depending on the isomorphism type of \( G \). \( \square \)

**Remark 4.6.** We know that there are two nonabelian group of order \( p^3 \), up to isomorphism (for details see [9]). Now put
\[
G_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}
\]
We observe that this $G_p$ is a nonabelian group under matrix multiplication of order $p^3$. We also see that $G_p = \{I_3\}$, the identity in $G_p$. Now if $\rho$ is a Heisenberg representation of dimension $\neq 1$ of group $G_p$, then we will have $\det(\rho) \equiv 1$.

And when $G$ is an extraspecial group of order $p^3$, where $p \neq 2$ with $G^p = Z$, we will have $\text{Ker}(\Psi) \cong C_p \times C_p$, where $C_p$ is the cyclic group of order $p$. Therefore $\det(\rho)(g) = \chi_Z(g^p)$. This shows that $\det(\rho) \neq 1$.

From Corollary 4.5, we observe that for nonabelian group of order $p^3$, where $p$ is prime, the determinant of Heisenberg representation of $G$ gives the information about the isomorphism type of $G$.

**Remark 4.7.** When the dimension $d$ of $\rho$ is odd, the 2-rank of $G/Z$ is 0, and we notice that in this case $\varepsilon \equiv 1$. Therefore we could rephrase our above Theorem 4.3 as follows:

1. If the 2-rank of $G/Z$ is different from 2, we have
   \[\det(\rho)(g) = \chi_\rho(g^d).\]

2. If the 2-rank of $G/Z$ is equal to 2, then $G/G^2Z$ is Klein’s 4-group, and we have a sign function $\epsilon$ on $G/G^2Z$ such that
   \[\det(\rho)(g) = \varepsilon(g) \cdot \chi_\rho(g^d).\]

Moreover, the function $\varepsilon$ is not a homomorphism and it takes 1 on the trivial coset and $-1$ on the three other cosets. Thus when $\text{rm}_2(G/Z) = 2$, we can write
\[
\det(\rho)(g) = \begin{cases} 
\chi_\rho(g^d) & \text{for } g \in G^2Z \\
-\chi_\rho(g^d) & \text{for } g \notin G^2Z.
\end{cases}
\]

**Example 4.8.** Let $G$ be a dihedral group of order 8 and we write
\[G = \{e, b, a, a^2, a^3, ab, a^2b, a^3b\} \quad a^4 = b^2 = e, bab^{-1} = a^{-1}\].

It is easy to see that $[G, G] = \{e, a^2\} = Z = Z(G) = G^2$. Thus $G^2Z = \{e, a^2\}$. We also have $G/G^2Z \cong \{G^2Z, aG^2Z, abG^2Z, bG^2Z\}$, and the subgroups of order 4 are:
\[H_1 = \{e, b, a^2, a^2b\}, \quad H_2 = \{e, ab, a^2, a^3b\} \quad \text{and} \quad H_3 = \{e, a, a^2, a^3\}.
\]

And the factor groups are: $G/H_1 = \{H_1, aH_1\}$, $G/H_2 = \{H_2, aH_2\}$ and $G/H_3 = \{H_3, bH_3\}$.

Let $\rho$ be a Heisenberg representation of $G$. The dimension of $\rho$ is 2. In this case the 2-rank of $G/Z$ is 2, then $G/Z = G/G^2Z$ is Klein’s 4-group.

For this group we can see that $\epsilon$ is a function on $G/Z \cong \{Z, aZ, bZ, abZ\}$. When $g = a \in G$, we have $H_1$ for which $g = a \in H_1$, then we have
\[\varepsilon(a) = \Delta_{G/H_1}^a(a) \cdot \chi_\rho([a, \alpha_{G/H_1}]) = \chi_\rho([a, \alpha_{G/H_1}]) = -1.
\]

Similarly, we can see when $g \in bZ$ and $g \in abZ$ we have $\varepsilon(g) = -1$. Thus we can conclude for dihedral group of order 8, that
\[
\det(\rho)(g) = \begin{cases} 
\chi_\rho(g^2) & \text{for } g \in Z \\
-\chi_\rho(g^2) & \text{for } g \notin Z.
\end{cases}
\]

For our next remark we need the following lemma.
Lemma 4.9. Let $\rho = (Z, \chi)$ be a Heisenberg representation of $G$ and put $X_\rho(g_1, g_2) := \chi_\rho([g_1, g_2])$. Then for every element $g \in G$, there exists a maximal isotropic subgroup $H$ for $X_\rho$ such that $g \in H$.

Proof. Let $g$ be a nontrivial element in $G$. Now we take a cyclic subgroup $H_0$ generated by $g$, i.e., $H_0 = \langle g \rangle$. Then $X_\rho(g, g) = 1$ implies $H_0 \subseteq H_0^\perp$. If $H_0$ is not maximal isotropic, then the inclusion is proper and $H_0$ together with some $h \in H_0^\perp \setminus H_0$ generates some larger isotropic subgroup $H_1 \supset H_0$. Again we have $H_1 \subseteq H_1^\perp$, and if $H_1$ is not maximal then the inclusion is proper, then again we proceed same method and will have another isotropic subgroup and we continue this process step by step come to maximal isotropic subgroup $H$.

Therefore for every element $g \in G$, we would have a maximal subgroup $H$ such that $g \in H$. □

Remark 4.10. From equation (4.9) we can say that $\varepsilon$ is a homomorphism when it restricts to $H$ because $X_\rho|_{H \times H} = 1$. Also from Lemma 4.9 if $g \in G$, then there always exists a maximal isotropic subgroup $H$ such that $g = h \in H$. Since $H$ is normal, then $\Delta^G_H = \Delta^{G/H}_1$ is trivial on $H$ and we see in particular for $h \in H$

\begin{equation}
(4.13) \quad (\det \rho)(h) = \chi_H(h)^d \chi_H(\varphi_{G/H}(h)) = \chi_\rho(h^d) \cdot \chi_\rho([h, \alpha_{G/H}]),
\end{equation}

since $h^d, [h, \alpha_{G/H}] \in Z$.

The formula (4.13) reformulates as:

\begin{equation}
(4.14) \quad (\det \rho)(h) = \chi_\rho(h^d) \cdot \chi_\rho([h, \alpha_{G/H}]) = \chi_\rho(h^d) \cdot X_\rho(h, \alpha_{G/H}),
\end{equation}

if $g = h$ sits in some maximal isotropic $H$, and $\alpha_{G/H} \in G/H$ is as above (the product over all elements from $G/H$). Of course some $g \in G$ can sit in several different maximal isotropic $H$.

So it is a little mysterious that the result (=left side of the formula) is independent from that $H$. Moreover, if $H$ is maximal isotropic then

$$G/H \cong \hat{H}/Z, \quad g \mapsto \{h \mapsto X(h, g)\}.$$ 

Now $\alpha_{G/H} \neq 1$ means that $G/H$ has precisely one element of order 2, equivalently $H/Z$ has precisely one character of order 2, equivalently $H/H^2Z$ is of order 2. Therefore equation (4.14) can be reformulated as to say that in the critical case:

$$(\det \rho)(h) = \chi_\rho(h^d) \text{ or } -\chi_\rho(h^d) \text{ depending on } h \in H^2Z \text{ or } h \notin H^2Z.$$
Acknowledgements. I would like to thank Prof E.-W. Zink for suggesting this problem and his constant valuable advices. I also thank to my adviser Prof. Rajat Tandon for his continuous encouragement. I extend my gratitude to Prof. Elmar Grosse-Klönne for providing very good mathematical environment during stay in Berlin. I am also grateful to Berlin Mathematical School for their financial support.

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