A note on the Voigt profile function

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Abstract

A Voigt profile function emerges in several physical investigations (e.g. atmospheric radiative transfer, astrophysical spectroscopy, plasma waves and acoustics) and it turns out to be the convolution of the Gaussian and the Lorentzian densities. Its relation with a number of special functions has been widely derived in literature starting from its Fourier type integral representation. The main aim of the present paper is to introduce the Mellin-Barnes integral representation as a useful tool to obtain new analytical results. Here, starting from the Mellin-Barnes integral representation, the Voigt function is expressed in terms of the Fox H-function which includes representations in terms of the Meijer G-function and previously well-known representations with other special functions.

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1 Introduction

During the recent past, representations of the Voigt profile function in terms of special functions have been discussed in this Journal [1, 2, 3, 4, 5, 6]. With the present paper we aim to continue those researches. In particular, we derive the representation in terms of Fox H-function, which includes previously well-known results.

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The Voigt profile function emerges in several physical investigations as atmospheric radiative transfer, astrophysical spectroscopy, plasma waves and acoustics and molecular spectroscopy in general. Mathematically, it turns out to be the convolution of the Gaussian and the Lorentzian densities. Here we are studying the ordinary Voigt function and not its mathematical generalizations, for example, see the papers [7, 8, 9]. The computation of the Voigt profile is an old issue in literature and many efforts are directed to evaluate this function with different techniques. In fact, an analytical explicit representation in terms of elementary functions does not exist and it can be considered a special function itself. Moreover it is strictly related to the plasma dispersion function [10] and to a number of special functions as, for example, the confluent hypergeometric function, the complex complementary error function, the Dawson function, the parabolic cylinder function and the Whittaker function, see e.g. [1, 2, 3, 11, 12, 13, 14]. All previous representations are derived starting from the integral formula due to Reiche in 1913 [15] that is actually a Fourier type integral.

The Voigt profile function remains nowadays a mathematically and computationally interesting problem because computing profiles with high accuracy is still an expensive task. The actual interest on this topic is proven by several recent papers (2007) on mathematical [16, 17, 18, 7, 19, 20] and numerical aspects [21, 22, 23]. A huge collection exists of published works on this topic, but instead to report it, we give the significative datum that searching in Google Scholar the strings “voigt profile function” and “voigt function” the number of files found is ∼ 24,600 and ∼ 70,100, respectively.

The Mellin-Barnes integrals are a family of integrals in the complex plane whose integrand is given by the ratio of products of Gamma functions. Despite of the name, the Mellin-Barnes integrals were initially studied in 1888 by the Italian mathematician S. Pincherle [24, 25] in a couple of papers on the duality principle between linear differential equations and linear difference equations with rational coefficients. The Mellin-Barnes integrals are strongly related with the Mellin transform, in particular with the inverse transformation. As shown by O.I. Marichev [26], the problem to evaluate integrals can be successfully faced with a powerful method mainly based on their reduction to functions whose Mellin transform is the ratio of product of Gamma functions and then, after the inversion, the problem consists in the evaluation of Mellin-Barnes integrals. Moreover, they are also the essential tools for treating higher transcendental functions as Fox H-function and Meijer G-function and a useful representation to compute asymptotic behaviour of functions [27].

The main object of the present paper is to introduce the Mellin-Barnes integral representation as a useful tool to obtain new analytical results that in the future can lead to efficient numerical algorithms for the Voigt function. A successful application of such approach has been shown in [28], where the parametric evolution equation of the Voigt function (and its probabilistic
generalization) is derived and the scaling laws, with respect to the parameter, in the asymptotic regimes are computed using the Mellin-Barnes integral representation. After all, this work can be seen also as an interesting exercise per se of application of the Mellin-Barnes integral method.

The rest of the paper is organized as follows. In section 2 the basic definition of the Voigt profile function is given and some classical and recent representations are reviewed. In section 3 the Mellin-Barnes integral representation of the Voigt function is derived and, in section 4 starting from this result, first the Voigt function is expressed in terms of the Fox H-function and later, in cascade, the representation with the Meijer G-function and other special functions are obtained. Finally in section 5 the summary and conclusions are given.

2 The Voigt profile function

2.1 Basic definition

The Gaussian \( G(x) \) and the Lorentzian \( L(x) \) profiles are defined as

\[
G(x) = \frac{1}{\sqrt{\pi \omega_G}} \exp \left[ -\frac{x^2}{\omega_G^2} \right], \quad L(x) = \frac{1}{\pi \omega_L} \frac{\omega_L^2}{x^2 + \omega_L^2},
\]

where \( \omega_G \) and \( \omega_L \) are the corresponding widths. The variable \( x \) is a wavenumber and then its physical dimension is a length raised up to \(-1\). The Voigt profile \( V(x) \) is obtained by the convolution of \( G(x) \) and \( L(x) \)

\[
V(x) = \int_{-\infty}^{+\infty} L(x-\xi)G(\xi) \, d\xi = \frac{\omega_L/\omega_G}{\pi^{3/2}} \int_{-\infty}^{+\infty} e^{-(\xi/\omega_G)^2} \frac{e^{-\xi^2 + \omega_L^2}}{(x-\xi)^2 + \omega_L^2} \, d\xi.
\]

The comparison between the Voigtian, Gaussian and Lorentzian profile is shown in figure 1 for different values of the width ratio \( \omega_L/\omega_G \).

Let \( \hat{f}(\kappa) \) be the Fourier transform of \( f(x) \) so that

\[
\hat{f}(\kappa) = \int_{-\infty}^{+\infty} e^{+i\kappa x} f(x) \, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\kappa x} \hat{f}(\kappa) \, d\kappa,
\]

then

\[
\hat{V}(\kappa) = \hat{G}(\kappa)\hat{L}(\kappa) = e^{-\omega_G^2\kappa^2/4} e^{-\omega_L|\kappa|},
\]

and

\[
V(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\kappa x} e^{-\omega_G^2\kappa^2/4 - \omega_L|\kappa|} \, d\kappa = \frac{1}{\pi} \int_{0}^{+\infty} e^{-\omega_L \kappa - \omega_G^2\kappa^2/4} \cos(\kappa x) \, d\kappa.
\]

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Figure 1: Comparison between the Voigtian, Gaussian and Lorentzian profiles with $a = \omega_L / \omega_G = 0.01, 0.1, 1, 2$. 
Formula (5) is the integral representation due to Reiche [15].

2.2 Some classical and recent representations

Let \( x' \) be the dimensionless variable \( x' = x/\omega_G \), the Voigt function can be re-arranged in the form

\[
V(x) = \frac{1}{\sqrt{\pi \omega_G}} K(x', y), \quad K(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-\xi^2}}{(x - \xi)^2 + y^2} d\xi,
\]

(6)

where \( y = \omega_L/\omega_G \) and from (3) it follows that

\[
K(x, y) = \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} e^{-y\xi - \xi^2/4} \cos(x\xi) d\xi.
\]

(7)

The Voigt function does not possess an explicit representation in terms of elementary functions and several alternatives to (2) have been given in literature, mainly with the intention to obtain a more efficient numerical computation.

Combining \( x \) and \( y \) in the complex variable \( z = x - iy \), the function \( K(x, y) \) (6) is

\[
K(x, y) = \text{Re}[W(z)], \quad W(z) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-\xi^2}}{z - \xi} d\xi.
\]

(8)

where \( W(z) \) is strongly related to the plasma dispersion function [10] and some classical representations can be found, see e.g. [11, 12, 13]. In fact, form the relation of \( W(z) \) with the complex complementary error function \( \text{Erf} c(-iz) \) and the Dawson function \( F(z) = e^{-z^2} \int_{0}^{z} e^{\xi^2} d\xi \) then it follows that

\[
K(x, y) = \text{Re}[W(z)], \quad W(z) = e^{-z^2} \text{Erf} c(-iz), \quad y > 0,
\]

(9)

\[
K(x, y) = \text{Re}[W(z)], \quad W(z) = e^{-z^2} + \frac{2i}{\sqrt{\pi}} F(z).
\]

(10)

More recent representations are, for example, those derived in 2001 by Di Rocco et al [29]

\[
K(x, y) = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{\Gamma(n + 1)} F_1 \left( \frac{2n + 1}{2} ; \frac{1}{2} ; y^2 \right) \right. \left. - \frac{2a}{\Gamma \left( \frac{2n + 1}{2} \right)} F_1 \left( n + 1, \frac{3}{2} ; y^2 \right) \right\} x^{2n},
\]

(11)
where $_1F_1(\alpha, \beta; z)$ is the confluent hypergeometric function, and in 2007 by Zaghloul [20], which “completed” a previous formula given by Roston & Obaid [30],

$$K(x, y) = (1 - \text{erf}(y))e^{(-x^2+y^2)} \cos(2xy) + \frac{2}{\sqrt{\pi}} \int_0^x e^{(-x^2+\xi^2)} \sin[2y(x-\xi)] \, d\xi.$$  \hspace{1cm} (12)

Recently (2007), He & Zhang [17] claimed to have derived an exact calculation of the Voigt profile that is proportional to the product of an exponential and a cosine function. However this representation assumes negative values in contrast with the non-negativity of the Voigt function. For this reason, this result is not correct.

Further representations are given in terms of special functions, see for example the one involving the confluent hypergeometric function $_1F_1$ [2, 3]

$$K(x, y) = e^{(y^2 - x^2)} \cos(2xy) - \frac{1}{\sqrt{\pi}} \left\{ (y + ix)_{1F_1}(1; 3/2; (y + ix)^2) + (y - ix)_{1F_1}(1; 3/2; (y - ix)^2) \right\}, \hspace{1cm} (13)$$

and others involving the Whittaker function $W_{k,m}$, the Erfc function and the parabolic cylinder function [14, formulae (17,13,16)]

$$K(x, y) = \frac{1}{2\sqrt{\pi}} \left\{ (y - ix)^{-1/2} e^{(y-ix)^2/2} W_{-1/4,-1/4}((y - ix)^2) + (y + ix)^{-1/2} e^{(y+ix)^2/2} W_{-1/4,-1/4}((y + ix)^2) \right\}, \hspace{1cm} (14)$$

$$K(x, y) = \frac{1}{2} \left\{ e^{(y-ix)^2} \text{Erfc}(y - ix) + e^{(y+ix)^2} \text{Erfc}(y + ix) \right\}, \hspace{1cm} (15)$$

$$K(x, y) = \frac{e^{(y^2-x^2)/2}}{\sqrt{2\pi}} \left\{ e^{-ixy} D_{-1}[\sqrt{2}(y - ix)] + e^{+iyx} D_{-1}[\sqrt{2}(y + ix)] \right\}. \hspace{1cm} (16)$$

### 3 The Mellin-Barnes integral representation

Let us consider again dimensional variables and the Gauss, Lorentz and Voigt functions defined as in (1-2). From (4) we have

$$V(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} e^{-\omega_0^2 \kappa^2/4 - \omega_L \kappa} \, d\kappa = \frac{1}{2\pi} \left\{ I_+(x) + I_-(x) \right\}, \hspace{1cm} (17)$$
where

\[ I_\pm(x) = \int_0^{+\infty} e^{-\omega_L \pm ix} \kappa^{-\omega_G \kappa^2/4} d\kappa. \]

Following [27], the Mellin-Barnes integral representations of \( I_\pm(x) \) can be obtained from the definition of the Gamma function

\[ \Gamma(z) = \int_0^{+\infty} \xi^{z-1} e^{-\xi} d\xi, \]

and the Mellin-Barnes integral representation of \( e^{-z} \) (\( z \neq 0 \))

\[ e^{-z} = \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) z^{-s} ds = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n, \]

where \( \mathcal{L} \) denotes a loop in the complex \( s \) plane which encircles the poles of \( \Gamma(s) \) (in the positive sense) with endpoints at infinity in \( \text{Re}(s) < 0 \) and with no restrictions on \( \text{arg} \ z \) [27]. The functions \( I_\pm(x) \) have the following Mellin-Barnes integral representations

\[
I_\pm(x) = \int_0^{+\infty} e^{-\omega_G \kappa^2/4} \left\{ \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s)[(\omega_L \pm ix)\kappa]^{-s} ds \right\} d\kappa
\]

\[
= \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \left\{ \int_0^{+\infty} e^{-\omega_G \kappa^2/4} \kappa^{-s} d\kappa \right\} (\omega_L \pm ix)^{-s} d\kappa
\]

\[
= \frac{1}{\omega_G} \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \Gamma\left( \frac{1}{2} - \frac{s}{2} \right) \left[ \frac{2}{\omega_G} (\omega_L \pm ix) \right]^{-s} ds. \quad (18)
\]

Hence the Mellin-Barnes integral representation of the Voigt function is given by

\[
V(x) = \frac{1}{2\pi \omega_G} \left\{ \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \Gamma\left( \frac{1}{2} - \frac{s}{2} \right) \left[ \frac{2}{\omega_G} (\omega_L + ix) \right]^{-s} ds + \right.
\]

\[
\left. \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \Gamma\left( \frac{1}{2} - \frac{s}{2} \right) \left[ \frac{2}{\omega_G} (\omega_L - ix) \right]^{-s} ds \right\}. \quad (19)
\]

However we note that for a complex number \( z = |z|e^{i\theta}, \theta = \arctan(Im(z)/Re(z)) \), the following rule holds

\[
z^n + \bar{z}^n = |z|^n e^{in\theta} + |z|^n e^{-in\theta}
\]

\[
= |z|^n (e^{in\theta} + e^{-in\theta})
\]

\[
= 2|z|^n \cos(n\theta) = 2|z|^n \cos(n \arctan(Im(z)/Re(z)))
\]
and formula (19) becomes

\[ V(x) = \frac{1}{\pi \omega_G} \frac{1}{2\pi i} \int_\mathcal{L} \Gamma(s) \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \cos \left[ s \arctan \left( \frac{x}{\omega_L} \right) \right] \left( 4\frac{\omega_G^2 + x^2}{\omega_G^2} \right)^{-s/2} ds. \] (20)

Consider again (18), changing \( s \to -s \) and taking the corresponding integration path \( \mathcal{L} \) as a loop in the complex plane that encircles the poles of \( \Gamma(-s) \), an other Mellin-Barnes integral representation of the Voigt function equivalent to (19) is

\[ V(x) = \frac{1}{2\pi \omega_G} \left\{ \frac{1}{2\pi i} \int_\mathcal{L} \Gamma(-s) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right) \left[ \frac{2}{\omega_G} (\omega_L + ix) \right]^s ds + \frac{1}{2\pi i} \int_\mathcal{L} \Gamma(-s) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right) \left[ \frac{2}{\omega_G} (\omega_L - ix) \right]^s ds \right\}, \] (21)

and in more compact form

\[ V(x) = \frac{1}{\pi \omega_G} \frac{1}{2\pi i} \int_\mathcal{L} \Gamma(-s) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right) \cos \left[ s \arctan \left( \frac{x}{\omega_L} \right) \right] \left( 4\frac{\omega_G^2 + x^2}{\omega_G^2} \right)^{s/2} ds. \] (22)

4 The Fox H- and Meijer G-function representations

4.1 The Fox H-function representation

The Voigt function can be represented in terms of well known special function, see §2.2, but its representation in terms of Fox H-function is still not known. The expression in terms of Fox H-function is important because it is the most modern representation method and, actually, it is the most compact form to represent higher transcendental functions. The definition of the Fox H-function is given in Appendix.

For \( \omega_G \) and \( \omega_L \) fixed, \( 2(\omega_L + ix)/\omega_G \neq 0 \) and \( 2(\omega_L - ix)/\omega_G \neq 0 \), from (21) and (19) we have, respectively,

\[ V(x) = \frac{1}{2\pi \omega_G} \left\{ H_{11}^{11} \left[ \frac{2}{\omega_G} (\omega_L + ix) \left| \frac{(1/2, 1/2)}{(0, 1)} \right. \right] + H_{11}^{11} \left[ \frac{2}{\omega_G} (\omega_L - ix) \left| \frac{(1/2, 1/2)}{(0, 1)} \right. \right] \right\}, \] (23)
As a consequence of the fact that this is the most comprehensive representation, in cascade, the others with less general functions can be obtained.

### 4.2 The Meijer G-function representation

The first \( H \)-function in (23) can be rewritten in terms of the Meijer G-function. Setting \( Z = \frac{2(\omega_L + ix)}{\omega_G} \), we have

\[
H_{11}^{11} \left[ Z \left| \begin{array}{c} (1/2, 1/2) \\ (0, 1) \end{array} \right. \right] = \frac{1}{2\pi i} \int_L \Gamma(-s) \Gamma \left( \frac{1}{2} + \frac{s}{2} \right) Z^s ds
\]

\[
= \frac{2}{2\pi i} \int_L \Gamma(-2s) \Gamma \left( \frac{1}{2} + s \right) Z^{2s} ds
\]

\[
= \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_L \Gamma(-s) \Gamma \left( \frac{1}{2} - s \right) \Gamma \left( \frac{1}{2} + s \right) \left( \frac{Z^2}{4} \right)^s ds
\]

\[
= \frac{1}{\sqrt{\pi}} G_{12}^{21} \left[ \frac{Z^2}{4} \left| \begin{array}{c} 1/2 \\ 0, 1/2 \end{array} \right. \right],
\]

where the change of variable \( s \to 2s \) and the duplication rule for the Gamma function, \( \Gamma(2z) = \Gamma(z) \Gamma(1/2 + z) 2^{2z-1} \pi^{-1/2} \), are applied. The second \( H \)-function in (23) follows from the first with \( \bar{Z} = \frac{2(\omega_L - ix)}{\omega_G} \). Finally, the Voigt function in terms of the Meijer G-function is given by

\[
V(x) = \frac{1}{2\pi^{3/2} \omega_G} \left\{ G_{12}^{21} \left[ \frac{Z^2}{4} \left| \begin{array}{c} 1/2 \\ 0, 1/2 \end{array} \right. \right] + G_{12}^{21} \left[ \frac{Z^2}{4} \left| \begin{array}{c} 1/2 \\ 0, 1/2 \end{array} \right. \right] \right\}.
\]

The Meijer G-function can be reduced to other special functions and, for example, representations given in (14, 15, 16) are straightforwardly recovered.

In fact, the Meijer G-function and the Whittaker function \( W_{k,m} \) are related by the formula [31, p. 435]

\[
G_{12}^{21} \left[ \begin{array}{c} a \\ b, c \end{array} \right] = \Gamma(b - a + 1) \Gamma(c - a + 1) x^{(b+c-1)/2} e^{x/2} W_{k,m}(k),
\]

where \( k = a - (b + c + 1)/2 \) and \( m = (b - c)/2 \). Thus the G-functions in (26) are expressed in terms of the Whittaker function as
\[ G_{12}^{21} \left[ \frac{Z^2}{4} \left| \begin{array}{c} 1/2 \\ 0,1/2 \end{array} \right. \right] = \sqrt{\pi} \left( \frac{Z}{2} \right)^{-1/2} e^{Z^2/8} W_{-1/4,-1/4}(Z^2/4). \]

Hence, the Voigt function in terms of the Whittaker function is

\[ V(x) = \frac{1}{2\pi \omega_G} \left\{ e^{Z^2/4} W_{-1/4,-1/4}(Z^2/4) + e^{\bar{Z}^2/4} W_{-1/4,-1/4}(\bar{Z}^2/4) \right\}. \] (27)

Moreover, from the identity

\[ G_{12}^{21} \left[ \frac{Z^2}{4} \left| \begin{array}{c} 1/2 \\ 0,1/2 \end{array} \right. \right] = \pi e^{Z^2/4} \text{Erf}c(Z/2), \]

we have

\[ V(x) = \frac{1}{2\sqrt{\pi} \omega_G} \left\{ e^{Z^2/4} \text{Erf}c(Z/2) + e^{\bar{Z}^2/4} \text{Erf}c(\bar{Z}/2) \right\}, \] (28)

and from

\[ G_{12}^{21} \left[ \frac{Z^2}{4} \left| \begin{array}{c} 1/2 \\ 0,1/2 \end{array} \right. \right] = \sqrt{2\pi} e^{Z^2/8} D_{-1}(Z/\sqrt{2}), \]

we have

\[ V(x) = \frac{1}{\sqrt{2\pi \omega_G}} \left\{ e^{Z^2/8} D_{-1}(Z/\sqrt{2}) + e^{\bar{Z}^2/8} D_{-1}(\bar{Z}/\sqrt{2}) \right\}. \] (29)

Setting \( \omega_G = 1 \) in (27,28,29) formulae (14,15,16) are recovered, respectively.

### 5 Summary and conclusions

In the present paper the Mellin-Barnes integral representation of the Voigt profile function is derived. We think that this integral representation is a useful tool to have new analytical and numerical results in the subject. Starting from this, the Voigt function has been expressed in terms of the Fox H-function, which is the most comprehensive representation. In cascade, the expression in terms of the Meijer G-function is obtained and the previous well-known representations with the Whittaker, the \( \text{Erf}c \) and the Parabolic cylinder functions are recovered.
Appendix. The Fox H- and the Meijer G- functions

The Fox H-function is defined as [32, 33]
\[ H^{m n}_{p q} \left[ \frac{z}{\left| \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right|} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} h(s) z^s ds, \]
where
\[ h(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j s)}{\prod_{j=n+1}^{p} \Gamma(1 - b_j + B_j s) \prod_{j=m+1}^{q} \Gamma(a_j - A_j s)}, \]
an empty product is interpreted as unity, \( \{m, n, p, q\} \) are nonnegative integers so that \( 0 \leq m \leq q, 0 \leq n \leq p \), \( \{A_j, B_j\} \) are positive numbers and \( \{a_j, b_j\} \) complex numbers. The H-function is an analytic function of \( z \) and makes sense if [32, 33]
\[ i) \quad \forall z \neq 0, \quad \mu > 0 \]
\[ ii) \quad 0 < |z| < \beta^{-1}, \quad \mu = 0 \]
where
\[ \mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \quad \beta = \prod_{j=1}^{p} A_j^{A_j} \prod_{j=1}^{q} B_j^{-B_j}. \]

The Meijer G-function corresponds to the special case \( A_j = B_k = 1 \) \( (j = 1, \ldots, p; k = 1, \ldots, q) \)
\[ H^{m n}_{p q} \left[ \frac{z}{\left| \begin{array}{c} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{array} \right|} \right] = G^{m n}_{p q} \left[ \frac{z}{\left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right|} \right]. \]

References

[1] Exton H 1981 J. Phys. A: Math. Gen. 14 L75
[2] Katriel J 1982 J. Phys. A: Math. Gen. 15 709
[3] Fettis H E 1983 J. Phys. A: Math. Gen. 16 663
[4] Keshavamurthy R S 1987 J. Phys. A: Math. Gen. 20 L273
[5] Sajo E 1993 J. Phys. A: Math. Gen. 26 2977
[6] Gubner J A 1994 J. Phys. A: Math. Gen. 27 L745
[7] Sampoorna M, Nagendra K N and Frisch H 2007 J. Quant. Spectrosc. Radiat. Transfer 104 71
[8] Srivastava H M and Miller E A 1987 Astrophy. Space Sci. 135 111
[9] Srivastava H M and Chen M P 1992 Astrophy. Space Sci. 192 63
[10] Fried B D and Conte S D 1961 The Plasma Dispersion Function (New York: Academic)
[11] Armstrong B H 1967 J. Quant. Spectrosc. Radiat. Transfer 7 61
[12] Schreier F 1992 J. Quant. Spectrosc. Radiat. Transfer 48 743
[13] Shippony Z and Read W G 1993 J. Quant. Spectrosc. Radiat. Transfer 50 635
[14] Yang S 1994 Int. J. Math. Educ. Sci. Technol. 25 845
[15] Reiche F 1913 Ber. Deutsch. Phys. Ges. 15 3
[16] Dulov E N and Khripunov D M 2007 J. Quant. Spectrosc. Radiat. Transfer 107 421
[17] He J and Zhang Q 2007 J. Opt. A 9 565
[18] Petrov E P 2007 J. Quant. Spectrosc. Radiat. Transfer 103 272
[19] Srivastava H M and Pogány T K 2007 Russian J. Math. Phys. 14 194
[20] Zaghoul M R 2007 Mon. Not. R. Astron. Soc. 375 1043
[21] Leiweke R J 2007 J. Quant. Spectrosc. Radiat. Transfer 103 597
[22] Letchworth K L and Benner D C 2007 J. Quant. Spectrosc. Radiat. Transfer 107 173
[23] Mendenhall M H 2007 J. Quant. Spectrosc. Radiat. Transfer 105 519
[24] Pincherle S 1888 Atti R. Accad. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. 4 (4) 694 and 792 [Reprinted in UMI (Unione Matematica Italiana) 1954 Salvatore Pincherle: Opere Scelte vol 1 (Roma: Cremonese) p 223 and 231]
[25] Mainardi F and Pagnini G 2003 J. Comput. Appl. Math. 153 331
[26] Marichev O I 1982 Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables (Chichester: Ellis Horwood Limited)
[27] Paris R B and Kaminski D 2001 Asymptotics and Mellin-Barnes Integrals (Cambridge: Cambridge University Press)
[28] Pagnini G and Mainardi F 2007 arXiv:0711.4246 Submitted
[29] Di Rocco H O, Iriarte D I and Pomarico J 2001 *Applied Spectroscopy* **55** 822

[30] Roston G D and Obaid F S 2005 *J. Quant. Spectrosc. Radiat. Transfer* **94** 255

[31] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F 1954 *Tables of Integral Transforms* vol 2 New York: McGraw-Hill)

[32] Mathai A M and Saxena R K 1978 *The H-function with Applications in Statistics and Other Disciplines* (New Delhi: Wiley Eastern Limited)

[33] Srivastava H M, Gupta K C and Goyal S P 1982 *The H-functions of One and Two Variables with Applications* (New Delhi: South Asian)