Behavior rigidity near non-isolated blow-up points for the semilinear heat equation

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Abstract

We consider the semilinear heat equation with Sobolev subcritical power nonlinearity in dimension $N = 2$, and $u(x,t)$ a solution which blows up in finite time $T$. Given a non isolated blow-up point $a$, we assume that the Taylor expansion of the solution near $(a,T)$ obeys some degenerate situation labeled by some even integer $m(a) \geq 4$. If we have a sequence $a_n \to a$ as $n \to \infty$, we show after a change of coordinates and the extraction of a subsequence that either $a_n, 1 - a_1 = o((a_n, 2 - a_2))$ or $|a_n, 1 - a_1| |a_n, 2 - a_2|^{-\beta} \log |a_n, 2 - a_2|^{-\alpha} \to L > 0$ for some $L > 0$, where $\alpha$ and $\beta$ enjoy a finite number of rational values with $\beta \in (0, 2]$ and $L$ is a solution of a polynomial equation depending on the coefficients of the Taylor expansion of the solution. If $m(a) = 4$, then $\alpha = 0$ and either $\beta = 3/2$ or $\beta = 2$.

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1 Introduction

We consider the semilinear heat equation in space dimension $N \geq 1$ with a Sobolev subcritical power nonlinearity:

$$\begin{cases}
\partial_t u = \Delta u + |u|^{p-1} u, \\
u(0) = u_0 \in L^\infty(\mathbb{R}^N)
\end{cases} \quad (1.1)$$

where

$$p > 1, \quad (N-2)p < N + 2. \quad (1.2)$$

Other phenomena arise in the Sobolev critical and supercritical cases (see from example Merle, Raphaël and Szeftel [11], Schweyer [14] and the references therein).

We consider $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ a solution which blows up at time $T > 0$ and introduce the set of its blow-up points

$$\mathcal{S} = \{a \in \mathbb{R}^N \mid |u(a,t)| \to +\infty \text{ as } t \to T\}$$

(note that $u(x,t) \to u^*(x)$ as $t \to T$ whenever $x \notin \mathcal{S}$). In the literature, we know examples of blow-up solutions where $\mathcal{S}$ is finite or the union of some concentric spheres.
In this paper, we are interested in non-isolated blow-up points. No results are available in this context except for curve singularity (see Zaag [17, 18, 19, 20] and Ghoul, Nguyen and Zaag [1]). Our goal is to introduce new techniques to track this kind of question.

When $N = 1$, we know from Chen and Matano [1] and Herrero and Velázquez [8] that all the blow-up points are isolated. As we are interested in the asymptotic behavior near non-isolated blow-up points, we need to assume that $N \geq 2$.

Let us note that all our statements do hold for unsigned solutions (Proposition 1 together with Theorems 2 and 3). However, for simplicity, we reduce in the presentation and the proofs to the case of nonnegative solutions, without loss of generality. Indeed, we know from Corollary 2 page 108 in Merle and Zaag [13] that the solution has a constant sign in some neighborhood of any given blow-up point, which means that in similarity variables (1.3), the unsigned case can be seen as a perturbation of the nonnegative case with arbitrarily small exponential terms. Although the proof needs a crucial blow-up criterion given below in Proposition 3.2 and which is valid only for nonnegative solutions, one should keep in mind that we have a twin version valid for unsigned solutions and given in Proposition 1.2 page 111 in [13].

Given $a \in \mathcal{J}$, it is convenient to study the local behavior of $u(x,t)$ near $(a,T)$ in the similarity variables version $w_a(y,s)$ first introduced by Giga and Kohn in [5] by

$$w_a(y,s) = (T-t)^{1/p} u(x,t) \text{ where } y = \frac{x-a}{\sqrt{T-t}} \text{ and } s = -\log(T-t). \quad (1.3)$$

Using (1.1), we see that $w_a$ (or $w$ for short) satisfies the following PDE for all $s \geq -\log T$ and $y \in \mathbb{R}^N$:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w. \quad (1.4)$$

From Giga and Kohn [6], we know that

$$w_a(y,s) \to \kappa \equiv (p-1)^{1/(p-1)} \text{ as } s \to \infty, \quad (1.5)$$

uniformly on compact sets and also in $L^2_\rho(\mathbb{R}^N)$, the $L^2$ space with respect to the measure density

$$\rho(y) = \exp \left(-\frac{|y|^2}{4}\right) / (4\pi)^{N/2}. \quad (1.6)$$

According to Velázquez [16] (see also Filippas and Kohn [2] together with Filippas and Liu [3]), we may refine that convergence and obtain the following first order classification:
- either
  $$w_a(y,s) - \kappa \sim -\frac{\kappa}{4ps} \sum_{i=1}^{l} h_2(y_i) \quad (1.7)$$
  where $l = 1, \ldots, N$, after a rotation of coordinates;
- or
  $$w_a(y,s) - \kappa \sim e^{-\left(\frac{m}{2}\right)^{s}} \sum_{j_1+\cdots+j_N=m} C_{m,j_1,\ldots,j_N} h_{j_1}(y_1) \cdots h_{j_N}(y_N) \quad (1.8)$$
  as $s \to \infty$, for some even integer $m = m(a) \geq 4$, where $y = (y_1, \ldots, y_N)$, $h_j(\xi)$ is the rescaled Hermite polynomial defined by

$$h_j(\xi) = \sum_{i=0}^{[j/2]} \frac{j!}{i!(j-2i)!} (-1)^i \xi^{j-2i}, \quad (1.9)$$

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and the multilinear form
\[ \sum_{j_1 + \cdots + j_N = m} C_{m,j_2 \ldots j_N} y_1^{j_1} \cdots y_N^{j_N} \] (1.10)
is non zero and nonpositive. Extending the definition of \( m(a) \) by 2 if (1.7) holds, Khenissy, Rebai and Zaag called \( m(a) \) the “profile order at \( a \)” in [10].

From Velázquez [15], we know that this expansion may indicate whether \( a \) is an isolated blow-up point or not. Indeed, from Theorem 2 page 1570 in [15], we have the following:

(i) If (1.7) holds with \( l = N \) or (1.8) holds with the multilinear form in (1.10) negative, then \( a \) is an isolated blow-up point.

(ii) More generally, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( S \cap B(0, \delta) \subset \Omega_\epsilon \) where:

- \( \Omega_\epsilon \equiv \{ y \in \mathbb{R}^N \mid \sum_{i=1}^l y_i^2 \leq \epsilon |y|^2 \} \) if (1.7) holds,
- \( \Omega_\epsilon \equiv \{ y \in \mathbb{R}^N \mid \sum_{j_1 + \cdots + j_N = m} C_{m,j_2 \ldots j_N} y_1^{j_1} \cdots y_N^{j_N} \leq \epsilon |y|^m \} \) if (1.8) holds.

Expressing item (ii) differently, we may say that the set \( \Omega_0 \cap \{|y| = 1\} \) (which is finite) gives indications on the location of neighboring blow-up points. Indeed, if \( \Omega_0 \cap \{|y| = 1\} = \emptyset \), then we are in Case (i), and \( a \) is an isolated blow-up point; if \( \Omega_0 \cap \{|y| = 1\} \neq \emptyset \), then we cannot assert whether \( a \) is isolated or not (in fact, we believe the converse of item (i) to be very hard); if we further assume that \( a \) is non isolated, then for sure the neighboring blow-up points are located “along” the directions of the non-zero elements of \( \Omega_0 \cap \{|y| = 1\} \).

In this paper, our first goal is to refine the expansion (1.7)-(1.8) up to the second order, and more if possible. In fact, we don’t consider the case (1.7), where more refined expansions were obtained in a series of papers (see Zaag [17] and Ghoul, Nguyen and Zaag [4]). We will instead focus on the case (1.8), where no refinement is available, up to our knowledge. In addition, the exponential decay observed in the case (1.8) is more advantageous than the polynomial interaction of case (1.7), which allows us to better handle the interactions between the various components of the solution in Theorem 3 below.

Our first result states that in fact such an expansion is possible up to any order:

**Proposition 1** (Asymptotic expansion in similarity variables for \( m \geq 4 \)). Consider \( w(y, s) \) a solution to equation (1.4) defined for all \( y \in \mathbb{R}^N \) and \( s \geq s_0 \) for some \( s_0 \) and assume that it satisfies the expansion (1.8) for some even integer \( m \geq 4 \). Then, for any integer \( M \geq 2m \),

\[
 w(y, s) = \kappa + \sum_{j = 2m, \ldots, M} \sum_{l = 1, \ldots, n_j} e^{-\frac{s}{2} l^j h_{i_1}(y_1) \cdots h_{i_N}(y_N)} + o(e^{-\frac{M}{2} s \alpha_M})
 \]
as \( s \to \infty \), uniformly on compact sets and in \( L^q \) for any \( q \geq 2 \), where \( \alpha_j \in \mathbb{N} \) and \( E_{j,l} \subset \mathbb{N}^N \) is finite.
If the multilinear form in (1.10) has some degenerate directions, assuming furthermore that \( w = w_a \) where \( a \) is a non isolated blow-up point of (1.1), we may uncover some rigidity in the expansion, in the sense that we show that some coefficients of the Taylor expansion are zero. In order to simplify the presentation, we assume in the following that

\[
N = 2, \quad p = 2, \quad \text{and} \quad u_0 \geq 0.
\]  

(1.11)

The general case where \( N = 2, \ p > 1 \) and \( u_0 \) has no sign follows with the same proof and natural adaptations of the statements. More precisely, we claim the following:

**Theorem 2** (Second order refined asymptotic expansion near a non isolated point, when \( m \geq 4 \))

Consider \( u(x, t) \) a solution of equation (1.1) blowing up at time \( T > 0 \). Assume in addition that the origin is a non isolated blow-up point with \( m(0) = m \geq 4 \). Then:

(i) Up to some rotation of coordinates, it holds that

\[
w_0(y, s) = 1 + e^{(1 - \frac{m}{2})s} \sum_{j=0}^{m-2} C_{m,j} h_{m-j}(y_1) h_j(y_2) + \sum_{j=0}^{m} C_{m+1,j} h_{m+1-j}(y_1) h_j(y_2) + O(s e^{-\frac{m}{2}s})
\]

as \( s \to \infty \) in \( L^2_\rho \) for any \( q \geq 2 \), for some real coefficients \( C_{i,j} \) such that the multilinear form in (1.10) is nonpositive, where \( w_0(y, s) \) is defined in (1.3).

(ii) If \( m = 4 \), then

\[
(C_{4,0}, C_{4,1}, C_{4,2}) \neq (0, 0, 0), \quad C_{4,0} \leq 0, \quad C_{4,2} \leq 0 \quad \text{and} \quad C_{4,1}^2 - 4C_{4,0}C_{4,2} \leq 0.
\]

(1.13)

**Remark.** If the origin is an isolated blow-up point, we expect no rigidity in the coefficients of the Taylor expansion.

**Remark.** Several higher order improvements of (1.12) (showing cancelations of coefficients) are available in the proof.

Our second statement concerns the local geometry of the blow-up set:

**Theorem 3** (Rigidity in the blow-up set near a non-isolated blow-up point when \( m \geq 4 \)). Consider under the hypotheses of Theorem 2 some sequence \( a_n = (a_n,1, a_n,2) \) of non-zero blow-up points converging to the origin. Then,

(i) It holds that

\[
a_n \cdot e^\perp = o(a_n \cdot e) \quad \text{as} \quad n \to \infty,
\]

up to extracting a subsequence, where \( e \in \mathbb{R}^2 \) is a unitary vector of a degenerate direction of the multilinear form (1.10), \( e^\perp \) is unitary and \( e \cdot e^\perp = 0 \).

(ii) Up to a rotation and a symmetry of the axes, and up to a subsequence, it holds that \( a_n,1 \geq 0, \ a_n,2 \geq 0 \) and (1.12) still holds with possibly different constants, with

either \( a_n,1 = o(a_n,2^2) \) or \( a_n,1 \sim L(a_n,2)^\beta |\log a_n,2|^\alpha \),

for some \( \beta \) and \( \alpha \) enjoying a finite number of rational values with \( 0 < \beta \leq 2 \) (see the proof for a finer description of the localization of \( \beta \)).

Moreover, \( L \) is a solution of a polynomial equation involving the coefficients of the Taylor expansion (1.12) of \( w_0(y, s) \) or one of its higher order refinements.
(iii) When \( m = 4 \), the only possibilities for \( e \) in item (i) are \( e = (0,1) \) and \( e = (-2C_{4,1},C_{4,1}) \), the second possibility occurring only if \( C_{4,1}^2 - 4C_{4,0}C_{4,2} = 0 \) and \( C_{4,2} \neq 0 \). In addition, we have the following simple statement for the behavior of \( a_n \) in item (ii):

\[
either a_{n,1} = o(a_{n,2}^2), \text{ or } a_{n,1} \sim La_{n,2}^2 \text{ or } a_{n,1} \sim La_{n,2}^{3/2} \text{ with } L > 0.
\]

**Remark.** The possible values taken by \( \beta \) are such that \( \beta - \frac{1}{2} \in E_1 \cup E_2 \) where the 2 sets \( E_1 \) and \( E_2 \) are defined respectively in (5.18) and (5.19) below in the proof.

**Remark.** Our strategy to find the precise subquadratic regimes when \( m = 4 \) can be carried out for any fixed \( m \geq 6 \). For example, when \( m = 6 \), we found the following values for \( \beta : 4/3, 3/2 \) and \( 5/3 \), with \( \alpha = 0 \) each time.

**Remark.** Note that in the case \( m = 2 \), we think that a similar result also holds. However, the proof should be very different, since we are in polynomial scales of time \( s \) (see (1.7)), unlike the case \( m \geq 4 \) where we are in exponential scales (see (1.8)).

**Remark.** We believe that our techniques should apply in the case \( N \geq 3 \) and yield a similar result.

Let us briefly explain our strategy in this paper. Using Proposition 1, we first make a Taylor expansion of \( w_0 \). Given \( b \) a nearby blow-up point, we derive from this a Taylor expansion for \( w_b \) (use (3.5) given below). Since \( w_b \) is uniformly bounded in \( L^\infty \), its components in the expansion (2.4) cannot grow, which implies some cancelations in the Taylor coefficients of \( w_0 \), justifying (1.12). In a second step, by the same argument, we derive some constraints on the location of the neighboring blow-up points, leading to Theorems 2 and 3.

Note that for simplicity, we give the proofs only under assumption (1.11). The general case follows by the same proof. The only delicate point is to replace the blow-up criterion given below in Proposition 3.2 by its twin version valid for unsigned solutions and given in Proposition 2.1 page 111 of Merle and Zaag [13].

## 2 Existence of an expansion of the solution in similarity variables up to any order

We prove Proposition 1 in this section.

**Proof of Proposition 1.** Note that the case \( m = 4 \) is harder than the case \( m \geq 6 \). Indeed, the quadratic term in the equation induces more interactions in the former than in the latter case. For that reason, we only give the proof in the harder case, namely when \( m = 4 \).

Introducing

\[
v = w - 1,
\]

we see from equation (1.4) that \( v \) satisfies the following PDE for all \( s \geq -\log T \) and \( y \in \mathbb{R}^2 \):

\[
\partial_s v = \mathcal{L} v + v^2
\]

where

\[
\mathcal{L} v = \Delta v - \frac{1}{2} y \cdot \nabla v + \frac{1}{\rho} \text{div}(\rho \nabla v) + v,
\]

(2.3)
and $\rho(y)$ is defined in (1.6). The operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^2)$, and its spectrum is given by the set $\{1 - \frac{k^2}{4} \mid j \in \mathbb{N}\}$, which consists only in eigenvalues, having $h_n(y_1)h_l(y_2)$ as eigenfunctions, in the sense that

$$\mathcal{L}(h_n(y_1)h_l(y_2)) = \left(1 - \frac{n + l}{2}\right) h_n(y_1)h_m(y_2).$$

Expanding $v$ as follows:

$$v(y, s) = \sum_{i \in \mathbb{N}, 0 \leq j \leq i} v_{i,j}(s)h_{i-j}(y_1)h_j(y_2)$$

(2.4)

and introducing for any $l \in \mathbb{N}$, the $L^2_\rho(\mathbb{R}^2)$ projector $P_{-,l}$ defined by

$$v_{-,l}(y, s) = P_{-,l}(v) \equiv \sum_{i \geq l, 0 \leq j \leq i} v_{i,j}(s)h_{i-j}(y_1)h_j(y_2),$$

(2.5)

we write the following equations satisfied by $v_{i,j}$ and $v_{-,l}$:

$$v'_{i,j}(s) = \left(1 - \frac{i}{2}\right) v_{i,j}(s) + \int v(y, s)^2 k_{i-j}(y_1)k_j(y_2)\rho(y)dy,$$

(2.6)

$$\partial_s v_{-,l} = \mathcal{L}v_{-,l} + P_{-,l}(v^2),$$

(2.7)

where

$$k_j(\xi) = h_j(\xi)/\|h_j\|^2_{L^2_\rho}.$$ 

(2.8)

This way, we are in a position to perform the first iteration, in order to refine the asymptotic expansion in (1.8).

**Part 1: The first iteration**

Like the following iterations, we proceed in 3 steps in order to get the next terms in the expansion, starting from (1.8):

- we first use parabolic regularity to improve the convergence in (1.8), from $L^2_\rho(\mathbb{R}^2)$ to $L^2_\rho(\mathbb{R}^2)$ for any $q \geq 2$;
- then, we use the improved convergence to expand the quadratic term in equation (2.2), and write an ODE satisfied by the component $v_{i,j}$ defined in (2.4); solving that ODE gives an estimate on $v_{i,j}$, better than what (1.8) states;
- finally, we use again the estimate on the quadratic term of equation (2.2) and write from (2.7) a differential inequality satisfied by $v_{-,l}$ defined in (2.5); integrating that inequality gives an improved estimated on $v_{-,l}$.

**Step 1: Parabolic regularity to improve the convergence in (1.8)**

The following parabolic regularity estimate is crucial for the improvement:

**Lemma 2.1.** (Regularizing effect of the operator $\mathcal{L}$)

(i) (Herrero and Velázquez [9]) For any $q > 1$, $r > 1$, $\psi_0 \in L^q_\rho(\mathbb{R}^N)$ and $s > \max\left(0, -\log\left(\frac{1}{e^{s-1}}\right)\right)$, it holds that

$$\|e^{s}\psi_0\|_{L^q_\rho(\mathbb{R}^N)} \leq \frac{C(q, r) e^s e^{-\frac{rs}{2r}}}{(1 - e^{-s})^\frac{Nq}{2N}(q - 1 - e^{-s}(r - 1))^{\frac{Nq}{2r}}} \|\psi_0\|_{L^2_\rho(\mathbb{R}^N)}.$$ 

(ii) Consider $r \geq 2$ and $v_0 \in L^q_\rho(\mathbb{R}^N)$ such that $|v_0(y)| + |\nabla v_0(y)| \leq C(1 + |y|^k)$ for some $k \in \mathbb{N}$. Then, for all $s \geq 0$, we have $\|e^{s}\psi_0\|_{L^q_\rho(\mathbb{R}^N)} \leq e^s \|\psi_0\|_{L^q_\rho(\mathbb{R}^N)}$. 

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Remark. Although we are working in two space dimensions, we felt it better to state the result for any $N \geq 1$, for future purpose.

Proof.
(i) See Section 2 page 139 of [9]. Although the proof in that paper was given for $N = 1$, it extends to higher dimensions with no difficulty.
(ii) From Lemma 4 page 555 in Bricmont and Kupiainen, we know that $|e^{L_s}(1 + |y|^k)| \leq Ce^{s}(1 + |y|^k)$. Therefore, if $\psi(y, s) = e^{L_s}v_0(y)$, we see from the hypotheses that $|\psi(y)| + |\nabla \psi(y)| \leq Ce^{s}(1 + |y|^k)$. Using the linear equation satisfied by $\psi$, we justify that

$$
\frac{d}{ds} \int |\psi(y, s)|^r \rho(y)dy = -r(r - 1) \int |\psi(y, s)|^{r-2} |\nabla \psi(y, s)|^2 \rho(y)dy + \int |\psi(y, s)|^r \rho(y)dy \leq \int |\psi(y, s)|^r \rho(y)dy,
$$

and the result follows. □

Now, if

$$\bar{v}(y, s) = e^{-s} \sum_{j=0}^{4} C_{4,j}h_{4-j}(y_1)h_j(y_2) \text{ and } g = v - \bar{v} \tag{2.9}$$

we see from (1.8) and the definition (2.1) of $v(y, s)$ that

$$\|g(s)\|_{L^2_{\rho}} = O(e^{-s}) \text{ as } s \to \infty. \tag{2.10}$$

Moreover, $\bar{v}$ is an approximate solution of equation (2.2), up to some error term, in the sense that

$$\partial_s \bar{v} = L \bar{v} + \bar{v}^2 + O \left(e^{-2s}(1 + |y|^8)\right)$$

(in fact, in this first iteration, unlike the following iterations, $\bar{v}$ solves the linear equation, but we felt it better to write that equation as a perturbation of the nonlinear equation (2.2), since this latter fact will be always true in the following iterations). Using (2.2), we write the following equation satisfied by $g(y, s)$:

$$\partial_s g = L g + (v + \bar{v})g + O \left(e^{-2s}(1 + |y|^8)\right). \tag{2.11}$$

Recalling the following estimate from Giga and Kohn [6] and Giga, Matsui and Sasayama [7]:

$$\|u(t)\|_{L^\infty} \leq \frac{C}{T - t},$$

we see by definitions (1.3) and (2.1) of $w$ and $v$ that

$$\|v(s)\|_{L^\infty} \leq C. \tag{2.12}$$

Furthermore, by definition (2.9) of $\bar{v}$, we see that

$$|\bar{v}(y, s)| \leq C, \text{ if } |y| < e^{s}. \tag{2.13}$$

Consider then $q \geq 2$. Using the Duhamel formulation of equation (2.11) together with (2.12) and (2.13), we write for some $M > 0$ and for some $s^*$ to be taken fixed large enough,

$$\|g(s)\|_{L^q_{\rho}} \leq \|e^{s^*(L + MId)}g(s - s^*)\|_{L^q_{\rho}} + \int_{s-s^*}^{s} \|e^{(s-\sigma)(L + MId)}\|_{L^q_{\rho}} \|\bar{v}g(\sigma)\|_{L^q_{\rho}} d\sigma$$
Fixing $s^* > \log(q - 1)$, we write from Lemma 2.1

$$
\|g(s)\|_{L^q_{\rho}} \leq C\|g(s - s^*)\|_{L^q_{\rho}} + \int_{s-s^*}^{s} \|1_{\{|x| > e^{s^*}\}}\bar{v}(\sigma)\|_{L^q_{\rho}} \ d\sigma + Ce^{-2s}\int_{s-s^*}^{s} \|1 + |x|^8\|_{L^q_{\rho}} \ d\sigma.
$$

(2.14)

Note that $\|g(s - s^*)\|_{L^q_{\rho}} = o(\varepsilon^{-s})$ as $s \to \infty$ from (2.10) and $\|(1 + |x|^8)\|_{L^q_{\rho}} \leq C$. It remains then to estimate the middle term in the right-hand side of (2.14) in order to conclude.

Since for $\sigma \in [s - s^*, s]$ and $|y| > e^\frac{s}{\rho}$, we have by definition (2.9) of $g$ and $\bar{v}$ and estimate (2.12),

$$
|\bar{v}(y, \sigma)g(y, \sigma)| = |\bar{v}(y, \sigma)(v(y, \sigma) - \bar{v}(y, \sigma))| \leq Ce^{-s}(1 + |y|^4) + Ce^{-2s}(1 + |y|^8),
$$

and

$$
0 \leq \rho(y) \leq Ce^{-|y|^2/8} \times e^{-|y|^2/8} \leq Ce^{-\varepsilon|y|^2/8} e^{-|y|^2/8},
$$

it follows that

$$
\|1_{\{|x| > e^{s^*}\}}\bar{v}(\sigma)\|_{L^q_{\rho}} \leq Ce^{-\varepsilon|y|^2/8}.
$$

Gathering the above-mentioned bounds, it follows that

$$
\|g(s)\|_{L^q_{\rho}} = o(\varepsilon^{-s}) \text{ as } s \to \infty,
$$

which is precisely the goal of Step 1.

**Step 2: Refinement of the behavior of $v_{i,j}$**

Since estimate (1.8) holds in $L^4_{\rho}$ by Step 1, we may use it to refine the ODE (2.6) satisfied by $v_{i,j}$ and write:

$$
\left|v_{i,j}'(s) - \left(1 - \frac{i}{2}\right)v_{i,j}(s)\right| \leq C\|v(s)\|_{L^4_{\rho}}^2 \leq Ce^{-2s}.
$$

(2.15)

Since $|v_{i,j}(s)| \leq Ce^{-s}$ by (2.1) and (1.8), using elementary ODE techniques, we derive for $s$ large enough:

$$
\sup_{i \leq 5} |v_{i,j}(s) - C_{i,j}e^{(1 - \frac{i}{2})s}| \leq Cs\varepsilon^{-2s},
$$

(2.16)

with

$$
C_{i,j} = 0 \text{ if } i \leq 3.
$$

Note that the constant $C$ in this step may depend on $(i, j)$.

**Step 3: Refinement of the behavior of $v_{-l}$**

Since $\|P_{-l}(v^2)\|_{L^2_{\rho}} \leq \|v^2\|_{L^2_{\rho}} = \|v\|_{L^4_{\rho}}^2 \leq Ce^{-2s}$ by the regularity in Step 1, we see from equation (2.7) that

$$
\|v_{-l}\|_{L^2_{\rho}} \leq \left(1 - \frac{l}{2}\right)\|v_{-l}\|_{L^2_{\rho}} + Ce^{-2s},
$$

(2.17)

where $C$ may depend on $l$. Taking $l = 6$ and integrating this inequality yields

$$
\|v_{-6}\|_{L^2_{\rho}} \leq Cs\varepsilon^{-2s}.
$$

(2.18)
Gathering (2.16) and (2.18), we see that

\[ v(y,s) = e^{-s} \sum_{j=0}^{4} C_{4,j} h_{4-j}(y_1) h_{j}(y_2) + e^{-\frac{3}{2}s} \sum_{j=0}^{5} C_{5,j} h_{5-j}(y_1) h_{j}(y_2) + O(s e^{-2s}) \]  \hspace{1cm} (2.19)

as \( s \to \infty \), in \( L^2_p(\mathbb{R}^2) \). This gives the first terms of the expansion announced in Proposition 1.

**Part 2: the following iterations**

The strategy developed in the first iteration works here, iteratively! Let us see how the second iteration works: using the parabolic regularity developed in Step 1 of the first iteration, one can show that estimate (2.19) holds also in \( L^q_p \), for any \( q \geq 2 \), and also uniformly on compact sets. Using this, one can refine the quadratic term in equations (2.6) and (2.7) up to \( O(s e^{-3s}) \). Integrating those equations (with \( i \leq 7 \) and \( l = 8 \)), we refine the expansion of \( v_{j} \) and \( v_{-j} \) up to that order, resulting in an expansion of \( v \) in \( L^2_p \), up to \( O(s^2 e^{-3s}) \), and showing terms like \( e^{-s}, e^{-\frac{3}{2}s}, s e^{-2s} \) and \( e^{-2s} \). Iterating the process, we may get an expansion for \( v \) valid up to any order, in any \( L^q_p \) space with \( q \geq 2 \), and also uniformly on compact sets. This concludes the proof of Proposition 1 when \( p = 2 \), the solution is nonnegative and \( m = 4 \). As we explained in the beginning of the proof, this is the harder case, and the adaptation to the general case is straightforward.

### 3 Rigidity in the Taylor expansion for general \( m \geq 4 \)

This section is devoted to the proof of Theorem 2.

**Proof of Theorem 2** Consider \( u(x,t) \) a solution of equation (1.1) blowing up at time \( T > 0 \). Assume that the origin is a non isolated blow-up point with \( m(0) = m \geq 4 \) is even. We proceed in 4 steps:

- First, we recall from Velázquez [16] the first order Taylor expansion of \( w_0(y,s) \) and show that two coefficients are zero.
- Second, we introduce some geometrical transformation as a crucial tool in the proof.
- Third, we give the next order in the Taylor expansion and show that one coefficient is zero.
- Forth, we take \( m = 4 \) and justify (1.13).

**Step 1: First order Taylor expansion**

As stated in (1.8), we know from Velázquez [16] that

\[ w_0(y,s) = 1 + e^{(1-\frac{m}{2})s} \sum_{j=0}^{m} C_{m,j} h_{m-j}(y_1) h_{j}(y_2) + o(e^{(1-\frac{m}{2})s}) \]  \hspace{1cm} (3.1)

for some real coefficients \( C_{m,j} \) for \( j = 0, \ldots, m \) such that the multilinear form

\[ \sum_{j=0}^{m} C_{m,j} y_1^{m-j} y_2^j \]  \hspace{1cm} (3.2)

is non zero and nonpositive. From the alternative due to Velázquez [16] and given on page 3 we know that this multiform has (at least) one direction of degeneracy. Up to
making a rotation of coordinates, we may assume that the degeneracy direction is the axis \( \{ y_1 = 0 \} \), which means that
\[
C_{m,m} = 0.
\] (3.3)
Moreover, in order to guarantee the nonpositivity of the multilinear form, we directly see that
\[
C_{m,m-1} = 0,
\] (3.4)
otherwise, for \(|y_2|\) large enough, the multilinear form may achieve positive values.

**Step 2: A geometrical transformation**

As already written in the introduction, a crucial idea lays at the heart of our strategy. It consists in remarking that any Taylor expansion of \( w_0(y,s) \) can be translated into a Taylor expansion of \( w_b(z,s) \), for any other point \( b \in \mathbb{R}^2 \) (not necessarily a blow-up point), thanks to the following relation which follows from the similarity variables definition (1.4):
\[
w_b(y_b,s) = w_0(y,s) \text{ where } y = be^\frac{s}{2} + y_b.
\] (3.5)
With a suitable choice of \( b \), the uniform boundedness of \( w_b \) in \( L^\infty \) induces some cancellations of the coefficients appearing in the Taylor expansion of \( w_0 \).

**Step 3: Second order Taylor expansion**

Following the first order expansion (3.1) together with (3.3) and (3.4), we may use Proposition 1 to find the next order in the Taylor expansion:
\[
w_0(y,s) = 1 + e^{(1 - \frac{m}{2})s} \sum_{j=0}^{m-2} C_{m,j} h_{m-j}(y_1) h_j(y_2) + e^{\frac{1-2m}{2}s} \sum_{j=0}^{m+1} C_{m+1,j} h_{m+1-j}(y_1) h_j(y_2) + \bar{v}_0(y,s)
\] (3.6)
with
\[
\| \bar{v}_0(s) \|_{L^q} = O(s e^{-\frac{m}{2}s}) \text{ as } s \to \infty,
\] (3.7)
for any \( q \geq 2 \), for some real constants \( C_{m+1,j} \) for \( j = 0, \ldots, m+1 \). We claim that \( C_{m+1,m+1} = 0 \). Indeed, let us assume by contradiction that \( C_{m+1,m+1} \neq 0 \) and find a contradiction. Following Step 2 and choosing
\[
b = (0, Ae^{\frac{s_0}{2}}),
\] (3.8)
where \( A \) and \( s_0 \) will be taken large enough (in modulus for \( A \)), we show the following (note that \( b \) may or may not be a blow-up point):

**Lemma 3.1.** For any \( A \in \mathbb{R} \) such that \(|A| \geq 1\), it holds that
\[
w_{b,0,0}(s_0) - 1 \sim C_{m+1,m+1} A^{m+1} e^{\frac{(1-m)}{2}s_0} \text{ as } s_0 \to \infty,
\]
where \( b \) is given by (3.8), and \( w_{b,0,0}(s_0) \) is the coordinate of \( w_b(y,s_0) \) along the polynomial \( h_0(y_1) h_0(y_2) \equiv 1 \) as in (2.4).
Let us first use this lemma to find a contradiction, then prove the lemma.
Choosing \( A \) of the sign of \( C_{m+1, m+1} \), then taking \( s_0 \) large enough, we see that
\[
w_{b,0,0}(s_0) > 1.
\]
(3.9)
In other words, \( w_b \) satisfies the following blow-up criterion we proved in [12].

**Proposition 3.2** (Blow-up criterion for nonnegative solutions of (1.4)). Let \( w \) be a nonnegative solution of (1.4) which satisfies
\[
w_{0,0}(s_1) > 1
\]
for some \( s_1 \in \mathbb{R} \). Then, \( w \) cannot be defined for all \((y, s) \in \mathbb{R}^2 \times [s_1, \infty)\).

**Remark.** This proposition is valid only for nonnegative solutions. For unsigned solutions, it should be replaced by a twin statement given in Proposition 1.2 page 111 in [13].

**Proof.** See Proposition 3.8 page 164 in [12].

Since \( w_b \) exists by definition (1.3) for all \((y, s) \in \mathbb{R}^2 \times [-\log T, +\infty)\), a contradiction follows from (3.9) and this proposition. Thus, \( C_{m+1, m+1} = 0 \), provided that we prove Lemma 3.1.

**Proof of Lemma 3.1.** Using the transformation (3.10) together with the definition (3.8) of \( b \), we see that
\[
v_b(y_b, s_0) = v_0(y, s_0) \text{ with } y_1 = y_{b,1}, \quad y_2 = y_{b,2} + A,
\]
(3.10)
hence, we write from (3.11)
\[
v_b(y_b, s_0) = e^{(1-m)s_0} \sum_{j=0}^{m-2} C_{m,j} h_{m-j}(y_{b,1}) h_j(y_{b,2} + A)
\]
\[
+ e^{1-m}s_0 \sum_{j=0}^{m+1} C_{m+1,j} h_{m+1-j}(y_{b,1}) h_j(y_{b,2} + A) + \tilde{v}_b(y_b, s_0)
\]
(3.11)
with
\[
\tilde{v}_b(y_b, s_0) = \tilde{v}_0(y, s_0).
\]
(3.12)
Recalling the classical relation
\[
h_j' = j h_{j-1}
\]
related to the Hermite polynomials (1.9), we may use a Taylor expansion to derive the following binomial relation
\[
h_j(\xi + A) = \sum_{m=0}^{j} \binom{j}{m} A^m h_{j-m}(\xi).
\]
(3.13)
Using this relation and projecting the expression (3.12), we derive the following expression
\[
v_{b,0,0}(s_0) = A^{m+1} C_{m+1, m+1} e^{\frac{1-m}{2}s_0} + \tilde{v}_{b,0,0}(s_0)
\]
where we have used the classical orthogonality relation
\[
\int_{\mathbb{R}} h_l(\xi) h_j(\xi) \rho(\xi) d\xi = 2^j j! \delta_{l,j}.
\] (3.15)

We need to estimate \( \int \bar{v}_b(y_b, s_0) \rho(y_b) dy_b \) in order to conclude. By definition, we write
\[
\bar{v}_{b,0,0}(s_0) = \int \bar{v}_0(y_1, y_2, s_0) \rho(y - (0, A)) dy.
\]

Now, let us compute
\[
\rho(y - (0, A)) = \rho(y) \exp\left(\frac{A}{2} y_2\right) \exp\left(-\frac{A^2}{4}\right).
\]

Since \( \int \exp( Ay_2) \rho(y) dy \leq C e^{A^2} \), using the Cauchy-Schwartz inequality together with (3.7), we write
\[
|\bar{v}_{b,0,0}(s_0)| \leq C e^{A^2} \left( \int \bar{v}_0(y, s_0)^2 \rho(y) dy \right)^{1/2} \leq C e^{A^2} s_0 e^{-\frac{m^2}{4}s_0}.
\] (3.16)

This concludes the proof of Lemma 3.1.

This also concludes the proof of the fact that \( C_{m+1,m+1} = 0 \). From (3.6), we see that the Taylor expansion (1.12) holds.

**Step 4: Proof of (1.13) when \( m = 4 \)**

Using (1.12), we see that the 4-form introduced in (3.2) reads as follows:
\[
C_{4,0} x_1^4 + C_{4,1} x_1^3 x_2 + C_{4,2} x_1^2 x_2^2.
\]

Since this form is non-zero and nonpositive, dividing by \( x_1^2 \), we see that (1.13) follows.

This concludes the proof of Theorem 2.

**4 Rigidity in the geometry of the blow-up set when \( m = 4 \)**

This section is devoted to the proof of Theorem 3 when \( m = 4 \). The proof when \( m \geq 6 \) is different and will be given in Section 5.

Consider \( u(x, t) \) a solution of equation (1.1) blowing up at time \( T > 0 \). Assume that the origin is a non isolated blow-up point with \( m(0) = 4 \). Consider \( a_n \) a sequence of non-zero blow-up points \( a_n \) converging to the origin.

We proceed in 4 subsections to give the proof:
- In Subsection 4.1, we prove item (i) and the first part of item (iii). In particular, after a change of variables, we identify 3 possible regimes for the convergence of \( a_n \): superquadratic, quadratic and subquadratic.
- In Subsection 4.2, we show that the only subquadratic regime follows a 3/2 law, and determine its constant.
- In Subsection 4.3, we focus on the quadratic regime, and show that its “constant” enjoys at most 3 possible values. This will give the second part of item (iii), which clearly implies item (ii).
- Finally, in Subsection 4.4, we give the proof of some propositions which were used in the previous subsections, and which are devoted to some refinements of the behavior of the solution.
4.1 Superlinear behavior for neighboring points

Recalling the alternative of Velázquez [16] given on page 3, we see that locally near the origin, the blow-up set is located along some degeneracy directions of the multilinear form (1.10). From the choice of the axes we made in Theorem 2 and (1.12), we know that the axis \( \{ y_1 = 0 \} \) is a degeneracy direction. Moreover, using (1.13), we remark that if

\[
C_{4,1}^2 - 4C_{4,0}C_{4,2} = 0 \quad \text{and} \quad C_{4,2} \neq 0,
\]

then, we may have another (different) degeneracy direction for the multilinear form (1.10), namely the line of equation \( C_{4,1}x_1 + 2C_{4,2}x_2 = 0 \), no more. Thus, item (i) holds for the sequence \( a_n \). Since these two straight lines are orthogonal (respectively) to \((0,1)\) and \((C_{4,1}, 2C_{4,2})\), the first part of item (iii) follows too.

Up to a rotation and symmetry of the axes, together with the extraction of a subsequence still denoted the same, we may assume that \( a_n \) converges along the \( x_2 \)-axis, (which means that

\[
a_{n,1} = o(a_{n,2})
\]

as \( n \to \infty \), and that

\[
\forall n \in \mathbb{N}, \ a_{n,1} \geq 0 \quad \text{and} \quad a_{n,2} \geq 0. \quad (4.2)
\]

Note that after such a change of variables, the Taylor expansion (1.12) remains valid, with possibly different coefficients, denoted \( C_{4,j} \) for \( j = 0,1,2 \) and \( C_{5,j} \) for \( j = 1,\ldots,4 \).

Up to further extracting a subsequence still denoted the same, we are in one of the following cases as \( n \to \infty \):

- **(subquadratic regime)** with \( a_{n,1} \gg \quad a_{n,2}^2 \).
- **(quadratic regime)** with \( a_{n,1} \sim \quad L a_{n,2}^2 \) for some \( L > 0 \).
- **(superquadratic regime)** with \( a_{n,1} \ll \quad a_{n,2}^2 \).

In the two following subsections, we investigate the subquadratic then the quadratic regimes for the sequence \( a_n \), in order to conclude the proof of Theorem 3.

4.2 Subquadratic regimes for neighboring blow-up points

In this subsection, we assume that

\[
a_{n,1} \gg a_{n,2}^2 \quad \text{as} \quad n \to \infty.
\]

We will show that the sequence \( a_n \) necessarily follows a \( 3/2 \) power law. We will also determine the “constant” in front of such a law.

Since \( a_n \neq 0 \), we see from (1.1) and (1.3) that

\[
a_{n,2} > 0 \quad \text{and} \quad a_{n,1} > 0. \quad (4.4)
\]

We proceed in 4 parts to prove the \( 3/2 \) law for \( a_n \):

- **In Part 1**, we prove that \( C_{4,2} = C_{4,1} = 0 \) and \( C_{4,0} < 0 \).
- **In Part 2**, we prove that \( C_{5,4} = 0 \) and \( a_{n,1} = O \left( a_{n,2}^{1/2} \right) \) as \( n \to \infty \).
- **In Part 3**, assuming that \( \frac{a_{n,1}}{a_{n,2}^{1/2}} \) has a non-zero limit, we determine that limit in terms
of the Taylor coefficients of the solution.

- In Part 4, we rule out the case where \( a_{n,1} = o(a_{n,2}) \).

**Part 1: Proof of the fact that** \( C_{4,2} = C_{4,1} = 0 \) **and** \( C_{4,0} < 0 \)

We claim that it is enough to prove that

\[
C_{4,2} = 0. \tag{4.5}
\]

Indeed, if this holds, recalling that \( C_{4,4} = C_{4,3} = 0 \) in the multilinear form (1.10), thanks to (1.12), we get the following simpler expression for the form:

\[
C_{4,0}y_1^4 + C_{4,1}y_1^3y_2.
\]

Since this is nonpositive and non-zero as stated right after (1.10), we necessarily have

\[
C_{4,1} = 0
\]

and

\[
C_{4,0} < 0. \tag{4.6}
\]

Let us then focus on the proof of (4.5). Proceeding by contradiction, we assume that

\[
C_{4,2} \neq 0. \tag{4.7}
\]

Let us reach a contradiction in this case.

We proceed as we did in Step 3 in the Proof of Theorem 2 to show that \( C_{m+1,m+1} = 0 \): we will derive information related to \( w_b \) defined by (3.5) in Step 2 of the Proof of Theorem 2. However, we will not use the expression (3.8) for \( b \). We will instead take

\[
b = a_n. \tag{4.8}
\]

There is also another difference with the proof of the fact that \( C_{m+1,m+1} = 0 \), in the sense that the contradiction will follow from the behavior of \( v_{b,1,0} \) instead of \( v_{b,0,0} \). From (4.4) and (4.1), we introduce \( B_n \) such that

\[
a_{n,1} = B_n a_{n,2} \text{ with } B_n = o(1) \text{ as } n \to \infty. \tag{4.9}
\]

Following (4.4) and the fact that \( a_n \to 0 \) (to be take large enough later), we also introduce

\[
s_{A,n} \to +\infty \text{ as } n \to \infty \tag{4.10}
\]

such that

\[
a_{n,2} = Ae^{-\frac{s_{A,n}}{2}}, \text{ hence } a_{n,1} = B_n a_{n,2} = AB_n e^{-\frac{s_{A,n}}{2}}. \tag{4.11}
\]

In particular, the transformation (3.5) reads here as

\[
w_b(y_b, s) = w_b(y, s) \text{ with } y_1 = y_{b,1} + AB_n e^{\frac{s}{2}}, \ y_2 = y_{b,2} + Ae^{\frac{s}{2}} \text{ and } \tau = s - s_{A,n}. \tag{4.12}
\]

Using (1.12), we write a new version of (3.12) adapted to our new choice of \( b \) in (4.8):

\[
v_b(y_b, s_{A,n}) = e^{-s_{A,n}} \sum_{j=0}^{2} C_{4,j} h_{4-j}(y_{b,1} + AB_n)h_j(y_{b,2} + A)
\]
Let us explain now how we reach a contradiction, first formally, then in a rigorous way.

Since \( C_n \) as \( n \to \infty \). In order to know which term is dominant between the two appearing in the expansion of \( v_b(y_b, s_{A,n}) \), let us introduce the ratio

\[
R_n = \frac{A^3 B_n e^{-s_{A,n}}}{A^4 e^{-\frac{3}{2} s_{A,n}}} = \frac{B_n}{A e^{-\frac{3}{4} s_{A,n}}} = \frac{a_{n,1}}{a_{n,2}^2} \to \infty
\]

from (4.9), (4.11) and (4.13). In particular, it holds that

\[
B_n \gg A e^{-\frac{3}{4} s_{A,n}} \quad \text{as} \quad n \to \infty.
\]

Since \( C_{4,2} \neq 0 \) by the contradiction hypothesis (4.7), using (4.15), we write

\[
v_{b,1,0}(s_{A,n}) \sim 2A^3 B_n C_{4,2} e^{-s_{A,n}} \quad \text{as} \quad n \to \infty.
\]

Let us explain now how we reach a contradiction, first formally, then in a rigorous way.

\[
+ e^{-\frac{3}{2} s_{A,n}} \sum_{j=0}^{4} C_{5,j} h_{5-j}(y_{b,1} + A B_n) h_j(y_{b,2} + A) + \bar{v}_b(y_b, s_{A,n})
\]

where \( \bar{v}_b(y_b, s_{A,n}) \) is defined in (3.13) (with \( s_0 \) replaced by \( s_{A,n} \)). Using the binomial relation (3.13) together with the orthogonality relation given in (3.15), we may derive from (4.13) a long formula for \( v_b(y_b, s_{A,n}) \). In fact, we won’t do that, since our argument uses only the projections of \( v_b(y_b, s_{A,n}) \) on \( h_0 h_0 \), \( h_1 h_0 \) and \( h_0 h_1 \), with the notation

\[
h_l h_j = h_l(y_{b,1}) h_j(y_{b,2}),
\]

i.e. the projections on the expanding modes \( \lambda = 1 \) and \( \lambda = \frac{1}{2} \) of the linear operator \( \mathcal{L} \) (2.3). For that reason, we will introduce a visual transcription of those 3 projections, in the form of a table whose columns are \( v_{b,0,0}(s_{A,n}) \), \( v_{b,1,0}(s_{A,n}) \) and \( v_{b,0,1}(s_{A,n}) \), and whose lines bear the name of the coefficients \( C_{i,j} \) present in (4.13), together with the rest term \( \bar{v}_b(y_b, s_{A,n}) \). More precisely, this is the table:

| \( C_{4,i} \) | \( C_{4,i} A^4 B_n^{4-i} e^{-s_{A,n}} \) | \( (4 - i) C_{4,i} A^3 B_n^{3-i} e^{-s_{A,n}} \) | \( i C_{4,i} A^2 B_n^{2-i} e^{-s_{A,n}} \) |
|---|---|---|---|
| \( C_{5,i} \) | \( C_{5,i} A^5 B_n^{5-i} e^{-\frac{3}{2} s_{A,n}} \) | \( (5 - i) C_{5,i} A^4 B_n^{4-i} e^{-\frac{3}{4} s_{A,n}} \) | \( i C_{5,i} A^3 B_n^{3-i} e^{-\frac{5}{4} s_{A,n}} \) |
| \( \bar{v}_b(y_b, s_{A,n}) \) | \( \bar{v}_{b,0,0}(s_{A,n}) \) | \( \bar{v}_{b,0,1}(s_{A,n}) \) | \( \bar{v}_{b,1,0}(s_{A,n}) \) |

Arguing as with (3.16), we see that

\[
| \bar{v}_{b,i,j}(s_{A,n}) | \leq C(A) s_{A,n} e^{-2s_{A,n}} \text{ with } (i,j) = (0,0), (1,0) \text{ or } (0,1).
\]

Using this, together with the table and the smallness of \( B_n \) written in (4.2), we see that

\[
v_{b,1,0}(s_{A,n}) = 2 A^3 B_n e^{-s_{A,n}} (C_{4,2} + O(B_n)) + A^4 e^{-\frac{3}{2} s_{A,n}} (C_{5,4} + O(B_n))
\]

(4.15)

as \( n \to \infty \). In order to know which term is dominant between the two appearing in the expansion of \( v_{b,1,0}(s_{A,n}) \), let us introduce the ratio

\[
R_n = \frac{A^3 B_n e^{-s_{A,n}}}{A^4 e^{-\frac{3}{2} s_{A,n}}} = \frac{B_n}{A e^{-\frac{3}{4} s_{A,n}}} = \frac{a_{n,1}}{a_{n,2}^2} \to \infty
\]

(4.16)

from (4.9), (4.11) and (4.13). In particular, it holds that

\[
B_n \gg A e^{-\frac{3}{4} s_{A,n}} \quad \text{as} \quad n \to \infty.
\]

(4.17)

Since \( C_{4,2} \neq 0 \) by the contradiction hypothesis (4.7), using (4.15), we write

\[
v_{b,1,0}(s_{A,n}) \sim 2A^3 B_n C_{4,2} e^{-s_{A,n}} \quad \text{as} \quad n \to \infty.
\]

(4.18)
Proceeding formally first, we restart the argument from (4.12) with \( \tau = s - s_{A,n} \geq 0 \) this time. All the estimates run smoothly and we end-up with the following modification of the table (note that we evaluate functions at \( s = s_{A,n} + \tau \), hence exponential factors appear in the estimates, complying with the expanding nature of the considered projections):

| \( C_{4,i} \) | \( C_{4,i}e^\tau A^4B_n^{4-i}e^{-s_{A,n}} \) | \( (4 - i)C_{4,i}e^\tau A^3B_n^{3-i}e^{-s_{A,n}} \) | \( iC_{4,i}e^\tau A^3B_n^{4-i}e^{-s_{A,n}} \) |
|----------------|-----------------|-----------------|-----------------|
| \( C_{5,i} \) | \( C_{5,i}e^\tau A^5B_n^{5-i}e^{-\frac{i}{2}s_{A,n}} \) | \( (5 - i)C_{5,i}e^\tau A^4B_n^{4-i}e^{-\frac{i}{2}s_{A,n}} \) | \( iC_{5,i}e^\tau A^4B_n^{5-i}e^{-\frac{i}{2}s_{A,n}} \) |
| \( \bar{v}_b(y_b,s_{A,n} + \tau) \) | \( \bar{v}_{b,0,0}(s_{A,n} + \tau) \) | \( \bar{v}_{b,0,1}(s_{A,n} + \tau) \) | \( \bar{v}_{b,0,0}(s_{A,n} + \tau) \) |

Selecting the column on \( v_{b,1,0}(s_{A,n} + \tau) \), we write

\[
\bar{v}_{b,1,0}(s_{A,n} + \tau) \sim 2e^{\tau}A^3B_nC_{4,2}e^{-s_{A,n}} \quad \text{as } n \to \infty, \quad (4.19)
\]

which shows an increase in the size of \( v_{b,1,0}(s_{A,n} + \tau) \) with \( \tau \). Taking an arbitrarily small \( \delta_0 > 0 \) then choosing \( \tau = \tau_n \) such that

\[
2e^{\frac{\tau_n}{2}}|A^3B_nC_{4,2}|e^{-s_{A,n}} = \delta_0,
\]

which means that

\[
\tau_n = 2s_{A,n} + 2\log \frac{\delta_0}{2|A^3B_nC_{4,2}|} \to \infty \quad (4.20)
\]

(\text{use (4.10) and (4.9)}), we see that

\[
|v_{b,1,0}(s_{A,n} + \tau_n)| \sim \delta_0, \quad (4.21)
\]

and this violates the following convergence, uniform with the respect to the blow-up point:

**Proposition 4.1** (Uniform convergence of \( v_a(y,s) \) to 0 when \( a \) is a blow-up point). It holds that

\[
\sup_{a \in \mathcal{A}} \|v_a(s)\|_{L^2_\rho} \to 0 \text{ as } s \to \infty. \quad (4.22)
\]

In particular, for any \( i \in \mathbb{N} \) and \( j = 0, \ldots, i \) with \((i,j) \neq (0,0)\), we have

\[
\sup_{a \in \mathcal{A}} |v_{a,i,j}(s)| \to 0 \text{ as } s \to \infty. \quad (4.23)
\]

**Proof.** Clearly, (4.23) directly follows from (4.22), thanks to the \( L^2_\rho \) projection defined in (2.4). As for (4.22), it is a direct consequence of the Liouville theorem of [12] and [13]. For a proof, one may adapt with no difficulty the proof of Proposition 2.2 page 11 in Khenissy, Rebai and Zaag [10]. \( \blacksquare \)
Clearly, a contradiction follows from (4.21) and (4.23) (note that \(s_{A,n} + \tau_n \to \infty\) by (4.10) and (4.20)), justifying that \(C_{1,2} = 0\). Unfortunately, the change of variables starting from (4.12) can be justified only when \(\tau\) stays bound, and this is not the case in (4.20).

Fortunately, there is a rigorous way to justify (4.19). One has simply to take (4.18) as initial data at \(s = s_{A,n}\), then integrate for \(s \geq s_{A,n}\) the following ODE satisfied by \(v_{b,1,0}(s)\) which we recall from (2.6):

\[
v'_{b,1,0}(s) = \frac{1}{2}v_{b,1,0}(s) + \int v_b(y,s)^2k_1(y_1)\rho(y)dy,
\]

where \(k_1(y_1) = \frac{y_1}{y_1^2 + 1}\) was already introduced in (2.8).

In order to do so, we need to evaluate the size of \(v_b(y,s)\), and this is possible if we evaluate it at \(s = s_{A,n}\), then integrate the PDE satisfied by \(v_b(y,s)\) for \(s \geq s_n\). Let us start then by evaluating the size of \(v_b(y,s)\) at \(s = s_{A,n}\).

Before that, recalling that (4.12) holds in \(L^2_{\rho}([0,\infty))\), we may use the trick based on the Cauchy-Schwarz inequality we used for (3.16) to prove that \(\bar{v}_b(s_{A,n})\) defined in (3.13) and appearing in (4.13) satisfies

\[
\|\bar{v}_b(s_{A,n})\|_{L^2_{\rho}} \leq CA_{s_{A,n}}e^{-2s_{A,n}}.
\]

Hence, we may use (4.13) together with the binomial relation (3.14) in order to show that

\[
\|v_b(y_b,s_{A,n})\|_{L^2_{\rho}} \leq CA^2e^{-s_{A,n}},
\]

for \(n\) large enough. Then, integrating the PDE (2.2), we show that for \(s \geq s_{A,n}\), the solution will stay as small as its value at \(s = s_{A,n}\). More precisely, we claim the following:

**Proposition 4.2** (Uniform bound for \(v_{a_n}\)). There exists a universal constant \(C_0 > 0\) such that for all \(A \geq 1\) and \(D \geq 1\), for \(n\) large enough, we have

\[
\forall s \in [s_{A,n}, Ds_{A,n}], \quad \|v_{a_n}(s)\|_{L^2_{\rho}} \leq C_0A^2e^{-s_{A,n}}.
\]

**Proof.** See subsection 4.3 below.

**Remark.** The result can not hold if one replaces \(v_{a_n}\) by \(v_{x_0}\) where \(x_0\) is not a blow-up point. Indeed, from Giga and Kohn [9], we know in that case that \(w_{x_0}(s) \to 0\), hence \(v_{x_0}(s) \to -1\) as \(s \to \infty\).

**Remark.** In fact, our next argument needs the same estimate with better norms, namely the \(L^4_{\rho}\). In order to justify that, we give two facts:

(i) First, given an arbitrary \(\tau_0 > 0\), note that estimate (4.20) actually holds for all \(s \in [s_{A,n} - \tau_0, s_{A,n}]\), by the same argument (based on the table given right before (4.19), which holds uniformly for \(\tau \in [\tau_0, 0]\) with the same proof). This will provide the \(L^4_{\rho}\) bound of Proposition (4.2) for any \(s \in [s_{A,n}, Ds_{A,n}]\).

(ii) Second, using Lemma (2.1) note that the \(L^4_{\rho}\) estimate follows from the \(L^2_{\rho}\) estimate, after some time shift \(\tau^* = \tau^*(4,2)\) (by the way, using the nonlinear equation (2.2) satisfied by \(v_{a_n}\) and the \(L^\infty\) bound (2.12), we see that \(\partial_t|v| \leq (\mathcal{L} + C)|v|\) is the distribution sense, hence Lemma (2.1) applies). Choosing the constant \(\tau_0\) of item (i) equal to \(\tau^*\) gives the \(L^4_{\rho}\) estimate for any \(s \geq s_{A,n}\).
Note first from the definition \((4.20)\) of \(\tau_n\) and \((4.17)\) that there exists \(D \geq 1\) such that \(s_{A,n} + \tau_n \in [s_{A,n}, Ds_{A,n}]\). Plugging the estimate of Proposition \(\ref{prop:estimates} \) in the quadratic term of \((4.21)\) (we need in fact the \(L^4_p\) estimate, which holds thanks to the remark following Proposition \(\ref{prop:estimates} \)), then integrating for \(s' \in [s_{A,n}, s]\) (remember that \(|v_{a_{n,1},0}(s')|\) is uniformly bounded from \((2.12)\)), we obtain (remember that \(b = a_n\) from \((4.3)\)):

\[
\forall s \in [s_{A,n}, Ds_{A,n}], \quad |v_{b_{1,0}}(s) - e^{\frac{v_{1}}{b_{1,0}}(s_{A,n})}| \leq Ce^{\frac{v_{1}}{2}}A^4e^{-2s_{A,n}}, \quad \text{where} \quad \tau = s - s_{A,n}.
\]  

Using \((4.18)\) and \((4.17)\), we see that \((4.19)\) is justified. Taking \(\tau\) to \(\tau_n\) defined in \((4.20)\), we see that \((4.21)\) holds, which contradicts Proposition \(\ref{prop:estimates} \). Therefore, \((4.7)\) doesn’t hold. Thus, \(C_{4,2} = 0\) and also \(C_{4,1} = 0\) and \(C_{4,0} < 0\) by the argument presented at the beginning of this part.

**Part 2: Proof of the fact that** \(a_{n,1} = O\left(\frac{a_{n,2}^4}{a_{n,2}}\right)\) as \(n \to \infty\).

This estimate will be achieved through 4 lemmas below, numbered 4.3 to 4.6 where we successively prove that \(a_{n,1} = O\left(\frac{a_{n,2}^4}{a_{n,2}}\right)\), then \(a_{n,1} = o\left(\frac{a_{n,2}^4}{a_{n,2}}\right)\), then \(C_{5,4} = 0\) and finally that \(a_{n,1} = O\left(\frac{a_{n,2}^4}{a_{n,2}}\right)\) as \(n \to \infty\). Let us state the first lemma:

**Lemma 4.3.** It holds that \(a_{n,1} = O\left(\frac{a_{n,2}^4}{a_{n,2}}\right)\) as \(n \to \infty\).

**Proof.** Since \(C_{4,2} = C_{4,1} = 0\) by Part 1, arguing as in that part, we can get an improved version of \((4.15)\), thanks to the table given right before that estimate, the smallness of \(B_n\) written in \((4.9)\) together with \((4.14)\):

\[
v_{b_{1,0}}(s_{A,n}) = 4A^3C_{4,0}B_n^3e^{-s_{A,n}} + A^4e^{-\frac{2}{3}s_{A,n}}(C_{5,4} + O(B_n)) + O(s_{A,n}e^{-2s_{A,n}})
\]

as \(n \to \infty\). Here, a discussion arises, according to the ratio of the two terms appearing in this expression:

\[
\frac{A^3B_n^3e^{-s_{A,n}}}{A^4e^{-\frac{2}{3}s_{A,n}}} = \frac{B_n^3}{\frac{A^3}{a_{n,2}^4}} = \frac{a_{n,1}^4}{a_{n,2}^4}.
\]

by definitions \((4.9)\) and \((4.11)\) of \(B_n\) and \(s_{A,n}\). Arguing by contradiction to prove Lemma 4.3 and recalling that \(C_{4,0} \neq 0\) by \((4.6)\), we see that we have

\[
B_n^3 \gg Ae^{-\frac{s_{A,n}}{2}} \quad \text{and} \quad v_{b_{1,0}}(s_{A,n}) \sim 4A^3C_{4,0}B_n^3e^{-s_{A,n}} \quad \text{as} \quad n \to \infty,
\]

at least for a subsequence denoted the same. Arguing as we did for \((4.19)\) and using this time the table right before that estimate, we may show that

\[
v_{b_{1,0}}(s_{A,n} + \tau_n) \sim 4e^{\frac{\tau_n}{2}}A^3B_n^3C_{4,0}e^{-s_{A,n}} \quad \text{as} \quad n \to \infty,
\]

then, derive a contradiction as in \((4.21)\), with

\[
\tau_n = -2\log\left(\frac{4|C_{4,0}|A^3B_n^3e^{-s_{A,n}}}{b_0}\right)
\]

which satisfies \(\tau_n \to \infty\) from \((4.3)\), hence \(s_{A,n} + \tau_n \to \infty\) from \((4.10)\) and also \(s_{A,n} + \tau_n \leq Ds_{A,n}\) for some \(D \geq 1\), from \((4.28)\). The question then reduces to justify \((4.29)\), for this choice of \(\tau_n\), and this can be done exactly as in Part 1, relying on Proposition \(\ref{prop:estimates} \). Thus, Lemma 4.3 is proved.
Now, we state the second lemma of this part:

**Lemma 4.4.** It holds that \( a_{n,1} = o\left( a_{n,2}^\frac{4}{3} \right) \) as \( n \to \infty \).

**Proof.** Arguing by contradiction, we assume that up to some subsequence (still denoted the same), we have

\[
\frac{a_{n,1}}{a_{n,2}^\frac{4}{3}} \to L \text{ as } n \to \infty, \tag{4.30}
\]

for some \( L > 0 \). By definitions (4.9) and (4.11) of \( B_n \) and \( s_{A,n} \), we see that

\[
B_n \sim LA^{1/3}e^{-\frac{s_{A,n}}{6}} \text{ as } n \to \infty. \tag{4.31}
\]

Using (4.13) as in Part 1 and Lemma 4.3, we may use (4.14) and the table right before it together with (4.31) to derive the following expression for another coordinate, namely

\[
v_{b,0,1}(s_{A,n}) = 4A^4B_n e^{-\frac{4}{3}s_{A,n}} \left( 4C_{5,4} + O(B_n) \right) + O(C(A)s_{A,n}e^{-2s_{A,n}})
\]

\[
= 4A^4 + \frac{4}{3}L C_{5,4} e^{-\frac{4}{3}s_{A,n}} + O(e^{-\frac{4}{3}s_{A,n}}).
\]

By the same argument as in Part 1 and Claim 4.3, we see that \( C_{5,4} = 0 \) (the justification follows also in the same way, thanks to Proposition 4.2). Repeating the same argument presented here in this step, we derive the following expression for the other coordinate (use the table right before (4.14)):

\[
v_{b,1,0}(s_{A,n}) = e^{\frac{\tau}{2}} \left( 4A^3C_{4,0}B_n e^{-s_{A,n}} + O(A^4B_n e^{-\frac{4}{3}s_{A,n}}) + O(C(A)s_{A,n}e^{-2s_{A,n}}) \right)
\]

\[
\sim 4A^4 + \frac{4}{3}L C_{5,4} e^{-\frac{4}{3}s_{A,n}}.
\]

Again, the growing factor implies that \( C_{4,0} = 0 \), which is a contradiction by (4.6). Thus, (4.30) doesn’t hold and Lemma 4.4 holds. As for the rigorous proof, it follows exactly as in Part 1, thanks to Proposition 4.2. This concludes the proof of Lemma 4.4. \( \square \)

Now, this is the third lemma of the part:

**Lemma 4.5.** It holds that \( C_{5,4} = 0 \).

**Proof.** Proceeding by contradiction, we assume that

\[
C_{5,4} \neq 0. \tag{4.32}
\]

This time, the argument follows from the behavior of \( v_{b,1,0}(s) \). Using (4.17) and Lemma 4.4 we see by definitions (4.9) and (4.11) of \( B_n \) and \( s_{A,n} \) that

\[
Ae^{-\frac{s_{A,n}}{2}} \ll B_n \ll A\frac{1}{2}e^{-\frac{s_{A,n}}{6}} \text{ as } n \to \infty. \tag{4.33}
\]

Proceeding as before, we may use the table right before (4.19) to derive that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim A^4C_{5,4} e^{\frac{\tau}{2}} e^{-\frac{4}{3}s_{A,n}} \text{ as } n \to \infty
\]

(recall that in (4.13), \( C_{4,1} = C_{4,2} = 0 \) from Part 1). The growth factor \( e^{\frac{\tau}{2}} \) allows to get a contradiction as in previous steps. Thus, (4.32) doesn’t hold and \( C_{5,4} = 0 \). The rigorous justification again follows exactly as in Part 1, thanks to Proposition 4.2. This concludes the proof of Lemma 4.5. \( \square \)
Now, this is the final lemma of the part:

**Lemma 4.6.** It holds that \( a_{n,1} = O(a_{n,2}^{3/2}) \) as \( n \to \infty \).

**Proof.** Since \( C_{5,4} = 0 \) from Lemma 4.5, we go back again to the table right before (4.19) to derive the following expansion

\[
v_{b,1,0}(s_{A,n} + \tau) = e^{\frac{2}{3}A} \left\{ 4A^3 C_{4,0} B_n^3 e^{-s_{A,n}} + 2A^4 B_n e^{-\frac{1}{2} s_{A,n}} (C_{5,3} + O(B_n)) + O\left(s_{A,n} e^{-2s_{A,n}}\right) \right\}
\]

(4.34) as \( n \to \infty \). Making the ratio between the coefficients of the first two terms, we get

\[
\frac{A^3 B_n^3 e^{-s_{A,n}}}{A^4 B_n e^{-\frac{1}{2} s_{A,n}}} = \frac{B_n^3}{A e^{-\frac{3}{2} s_{A,n}}} = a_{n,1} a_{n,2}^{-3},
\]

(4.35) by definitions (4.9) and (4.11) of \( B_n \) and \( s_{A,n} \). Assuming by contradiction that Lemma 4.6 doesn’t hold for some subsequence still denoted the same, we see that \( B_n \gg \sqrt{A} e^{-\frac{3}{2} s_{A,n}} \), hence,

\[
v_{b,1,0}(s_{A,n} + \tau) \sim 4e^{\frac{2}{3}A} A^3 C_{4,0} B_n^3 e^{-s_{A,n}} \quad \text{as} \quad n \to \infty,
\]

and the growth factor shows that \( C_{4,0} = 0 \), which is a contradiction from (4.6). The justification of this step follows as in Part 1. This concludes the proof of Lemma 4.6.

**Part 3: Possible limits for \( a_{n,1} a_{n,2} \)**

From Lemma 4.6, we may extract a subsequence (still denoted the same) such that

\[
\frac{a_{n,1}}{a_{n,2}^{3/2}} \to L_0 \quad \text{as} \quad n \to \infty,
\]

(4.36) for some \( L_0 \geq 0 \). Let us assume that \( L_0 > 0 \) (note that the case \( L_0 = 0 \) will be ruled out in Part 4). Using (4.35), we see that

\[
B_n \sim L_0 \sqrt{A} e^{-\frac{3}{2} s_{A,n}} \quad \text{as} \quad n \to \infty.
\]

From (4.31), we see that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim e^{\frac{2}{3}A} A^4 B_n e^{-\frac{1}{2} s_{A,n}} \left(4C_{4,0} L_0^2 + 2C_{5,3} + O\left(s_{A,n} e^{-2s_{A,n}}\right)\right) \quad \text{as} \quad n \to \infty.
\]

As before, we see from the growth factor that

\[
4C_{4,0} L_0^2 + 2C_{5,3} = 0.
\]

Since \( C_{4,0} < 0 \) from (4.6) and \( L_0 > 0 \), we necessarily see that \( C_{5,3} > 0 \) and

\[
L_0 = \sqrt{-\frac{C_{5,3}}{2C_{4,0}}}.
\]

(4.37)

The justification of this part can be done exactly as Part 1.

**Part 4: Ruling out the case where \( a_{n,1} = o(a_{n,2}^{3/2}) \) as \( n \to \infty \)**

20
Let us assume by contradiction that the limit defined in (4.36) is zero. In other words,

\[ L_0 = 0. \]  

(4.38)

From (4.17) and (4.35), we see that

\[ A e^{-sA,n} \ll B_n \ll A^{\frac{1}{2}} e^{-\frac{sA,n}{4}} \text{ as } n \to \infty. \]  

(4.39)

At the end of this part, we will reach a contradiction, ruling this case out. In fact, our argument will follow from 3 steps:

- In Step 1, we show that \( a_{n,1} = O\left(a_{n,2}^2 |\log a_{n,2}|\right) \text{ as } n \to \infty. \)
- In Step 2, we prove that \( C_{6,6} = C_{5,3} = C_{6,5} = 0. \)
- In Step 3, we reach a contradiction and finish the argument of Part 4.

Step 1: Proof of the fact that \( a_{n,1} = O\left(a_{n,2}^2 |\log a_{n,2}|\right) \text{ as } n \to \infty. \)

Let us first show how the ratio between \( a_{n,1} \) and \( a_{n,2}^2 |\log a_{n,2}| \) naturally arises in our argument. Using (4.34) together with (4.39), we write:

\[ v_{b,1,0}(s_{A,n} + \tau) = e^{\frac{s}{2}} \left\{ 2A^4 B_n e^{-\frac{3}{2}sA,n} (C_{5,3} + O(B_n)) + O \left( s_{A,n} e^{-2sA,n} \right) \right\} \text{ as } n \to \infty. \]  

(4.40)

Clearly, the balance between the \( C_{5,3} \) term and the error term is important. Considering the ratio between the two, we write

\[ \frac{A^5 B_n e^{-\frac{3}{2}sA,n}}{s_{A,n} e^{-2sA,n}} \sim \frac{A^5}{2} \frac{a_{n,1}^2}{a_{n,2}^2 |\log a_{n,2}|} \text{ as } n \to \infty \]  

(4.41)

thanks to the definitions (4.9) and (4.11) of \( B_n \) and \( s_{A,n} \). Thus, the ratio in the title of Step 1 appears.

Proceeding by contradiction, we assume that (for a subsequence denoted the same), we have

\[ a_{n,1} \gg a_{n,2}^2 |\log a_{n,2}| \text{ as } n \to \infty. \]  

(4.42)

Using (4.41) and (4.39), we see that

\[ s_{A,n} e^{-\frac{sA,n}{2}} \ll B_n \ll e^{-\frac{sA,n}{4}} \text{ as } n \to \infty. \]  

(4.43)

Using (4.40), we see that

\[ v_{b,1,0}(s_{A,n} + \tau) \sim 2e^{\frac{s}{2}} A^4 B_n e^{-\frac{3}{2}sA,n} (C_{5,3} + o(1)) \text{ as } n \to \infty, \]

and the growth factor implies that

\[ C_{5,3} = 0. \]  

(4.44)

The rigorous justification is exactly the same as in earlier steps, relying on Proposition 4.2.

At this stage, we need to further refine the Taylor expansion of \( v_0(y,s) \) given in (1.12), taking into account that with respect to the expansion (4.13), we already have
\( C_{4,1} = C_{4,2} = C_{5,3} = C_{5,4} = 0 \), from Part 1, Lemma 4.5 and (1.44). Using the strategy described in Proposition 1 we write:

\[
\begin{align*}
v_0(y,s) &= -\frac{e^{-2s}}{3} C_{4,0}^2 \gamma_{4,4,0} h_0 h_0 - \frac{e^{-2s}}{2} C_{4,0}^2 \gamma_{4,4,2} h_2 h_0 + \left[ C_{4,0} e^{-s} - e^{-2s} C_{4,0}^2 \gamma_{4,4,4} \right] h_4 h_0 \\
&\quad + e^{-\frac{s}{2}} \sum_{j=0}^{2} C_{5,j} \gamma_{5} h_5 - s \gamma_{j} h_j + se^{-2s} C_{4,0}^2 \gamma_{4,4,6} h_6 h_0 + e^{-2s} \sum_{j=0}^{6} C_{6,j} \gamma_{6} h_6 - s \gamma_{j} h_j \\
&\quad + e^{-2s} C_{4,0}^2 \gamma_{4,4,0} h_8 h_0 + O \left( se^{-\frac{s}{2}} \right),
\end{align*}
\]

(4.45)

for some real coefficients \( C_{6,j} \) with \( j = 0, \ldots, 6 \), where we have used the following notation

\[
\gamma_{l,m,n} = \int_{\mathbb{R}} h_l(\xi) h_m(\xi) k_n(\xi) \rho(\xi) d\xi,
\]

with the polynomials defined in (1.9) and (2.8), the weight \( \rho \) in (1.6) and the convention

\[
\gamma_{l,m,n} = 0 \text{ if } l, m \text{ or } n \text{ is negative ,}
\]

together with the notation

\[
h_l h_m \text{ which stands for } h_l(y_1) h_m(y_2).
\]

Following this new refined expansion, we give first the following cancelation:

**Claim 4.7.** It holds that \( C_{6,6} = C_{6,5} = 0 \).

**Proof.** Using the expansion (4.45), we give the following table, which is an update of the former table given right before (1.14):

| \( C_{4,0}^2 \gamma_{4,4,0} \) | \( v_{0,0,0}(s_{A,n} + \tau) \) | \( v_{0,1,0}(s_{A,n} + \tau) \) | \( v_{0,0,1}(s_{A,n} + \tau) \) |
|---|---|---|---|
| \( -\frac{C_{4,0}^2}{3} \gamma_{4,4,0} e^{-2s} e^{-2s_{A,n}} \) | 0 | 0 | 0 |
| \( -\frac{C_{4,0}^2}{2} \gamma_{4,4,2} e^{-s} A^2 B_n^2 e^{-2s_{A,n}} \) | \( -C_{4,0}^2 \gamma_{4,4,2} e^{-s} A^2 B_n^2 e^{-2s_{A,n}} \) | 0 | 0 |
| \( C_{4,0} \) | \( C_{4,0} e^{\tau} A^2 B_n^2 e^{-s_{A,n}} \) | \( 4C_{4,0} e^{\tau} A^2 B_n^2 e^{-s_{A,n}} \) | 0 |
| \( C_{4,0} \) | \( -C_{4,0} \gamma_{4,4,4} A^4 B_n^4 e^{-2s_{A,n}} \) | \( -4C_{4,0} \gamma_{4,4,4} e^{-s} A^3 B_n^3 e^{-2s_{A,n}} \) | 0 |
| \( C_{5,i} \) | \( C_{5,i} e^{\tau} A^2 B_n^2 e^{-s_{A,n}} \) | \( (5 - i) C_{5,i} e^{\tau} A^2 B_n^2 e^{-s_{A,n}} \) | \( iC_{5,i} e^{\tau} A^2 B_n^2 e^{-s_{A,n}} \) |
| \( C_{4,0} \) | \( C_{4,0} \gamma_{4,4,6} e^{\tau} A^6 B_n^6 (s_{A,n} + \tau) e^{-2s_{A,n}} \) | \( 6C_{4,0} \gamma_{4,4,6} e^{\tau} A^5 B_n^5 (s_{A,n} + \tau) e^{-2s_{A,n}} \) | 0 |
| \( C_{6,i} \) | \( C_{6,i} \gamma_{4,4} A^6 B_n^6 e^{-2s_{A,n}} \) | \( (6 - i) C_{6,i} \gamma_{4,4} A^6 B_n^6 e^{-2s_{A,n}} \) | \( iC_{6,i} \gamma_{4,4} A^6 B_n^6 e^{-2s_{A,n}} \) |
| \( C_{4,0}^2 \) | \( C_{4,0}^2 \gamma_{4,4,8} e^{2\tau} A^8 B_n^8 e^{-2s_{A,n}} \) | \( 8C_{4,0}^2 \gamma_{4,4,8} e^{2\tau} A^7 B_n^7 e^{-2s_{A,n}} \) | 0 |
| \( O(se^{-\frac{s}{2}}) \) | \( e^{\tau} O((s_{A,n} + \tau) e^{-\frac{s}{2}}) \) | \( e^{\tau} O((s_{A,n} + \tau) e^{-\frac{s}{2}}) \) | \( e^{\tau} O((s_{A,n} + \tau) e^{-\frac{s}{2}}) \) |
Please keep in mind that this table is fully justified if \( \tau \) is zero or bounded. It happens that for our argument, we will need \( \tau = \tau_0 \) to go to infinity as \( n \to \infty \). In that case, note that the terms bearing the coefficients \( e^{2\tau} \) and \( e^{3\tau/2} \) are “artificial” and should not be taken into account in our argument, including the formal one. This way, we will ignore those terms in our formal argument. Fortunately, when it comes to the rigorous argument, this table will be useful for \( \tau = 0 \) or bounded, and the conclusion will follow from the integration of the PDE for \( \tau \) ranging between 0 and suitably large values.

- **Proof of the fact that** \( C_{6,6} = 0 \).

Using (4.43) and the previous table, we may derive the following estimate for large \( \tau \):

\[
v_{b,0,0}(s_{A,n} + \tau) = e^{\tau}(A^6C_{6,6} + o(1))e^{-2s_{A,n}} \text{ as } n \to \infty,
\]

and the growth factor implies that

\[
C_{6,6} = 0.
\]

As we have just written after the previous table, we don’t take into account the term with \( e^{2\tau} \) is the formal derivation of (4.46). Now, as for the rigorous justification, it goes as in previous steps, relying on the previous table only for bounded \( \tau \), together with Proposition 4.2. However, the rigorous proof is a little more delicate than earlier, since we need to take \( A \) large enough in order to conclude, as we explain in the following:

- first, relying on the previous table and (4.43), we write the following expansion for \( v_{b,0,0} \):

\[
v_{b,0,0}(s_{A,n}) = \left[ -\frac{C_{4,0}^2}{3} \gamma_{4,4,0} + C_{6,6}A^6 + o(1) \right] e^{-2s_{A,n}} \text{ as } n \to \infty.
\]

- second, using Proposition 4.2 (with \( D = 4 \)) and proceeding as for (4.27), we integrate the ODE (2.3) satisfied by \( v_{b,0,0} \) and write:

\[
\forall s \in [s_{A,n}, Ds_{A,n}], \quad |v_{b,0,0}(s) - e^{\tau}v_{b,0,0}(s_{A,n})| \leq C e^{\tau}A^4 e^{-2s_{A,n}}, \text{ where } \tau = s - s_{A,n}.
\]

Using (4.48), and taking \( A \) large enough so that \( A^6 \) term in (4.48) dominates its neighbor with \( C_{4,0}^2 \), and also the \( A^4 \) term in the right-hand side of (4.49), we see that (4.46) is justified.

- **Proof of the fact that** \( C_{6,5} = 0 \).

Proceeding as before, we may write the following expansion:

\[
v_{b,0,1}(s_{A,n} + \tau) = e^{\tau} \left\{ 2A^4 B_n^3 e^{-2s_{A,n}} (C_{5,2} + O(B_n)) + 5A^5 B_n e^{-2s_{A,n}} (C_{6,5} + O(B_n)) + O\left(s_{A,n}e^{-\frac{2s_{A,n}}{5}}\right) \right\}.
\]

Assuming by contradiction that \( C_{6,5} \neq 0 \), we see from (4.43) that

\[
v_{b,0,1}(s_{A,n} + \tau) \sim 5e^{\tau}A^5 B_n e^{-2s_{A,n}} C_{6,5},
\]

and the growth factor leads to a contradiction as usual. Thus, \( C_{6,5} = 0 \). As for the rigorous justification, it is a little more complicated than usual. Indeed, when \( \tau = 0 \), we have \( |v_{b,0,1}(s_{A,n})| \ll e^{-2s_{A,n}}, \) less than \( e^{-2s_{A,n}} \), which is the bound Proposition 4.2 gives on the quadratic term in the equation (2.6) satisfied by \( v_{b,0,1} \). In other words, Proposition 4.2 is not enough to show that the linear term dominates the quadratic term when we integrate equation (2.6). Thus, we need to refine Proposition 4.2 in the following:
Proposition 4.8 (Uniform expansion for $v_{an}$). Following Proposition 4.2 and assuming that
\[ v_0(y, s) = C_{4,0}e^{-s}h_4h_0 + O(e^{-\frac{3}{2} s}) \] as $s \to \infty$ \hfill (4.52)
(in $L^q$ for any $q \geq 2$), we claim that for all $A \geq 1$ and $D \geq 1$, for $n$ large enough, we have for all $s \in [s_{A,n}, Ds_{A,n}]$,
\[ \|v_{an}(y, s) - C_{4,0}e^{-s}h_4(y_1)\|_{L^q} \leq C(A)e^{-\frac{5}{2}s_{A,n}}. \]

Remark. As we wrote right after Proposition 4.2, here also, we have the same estimate valid in $L^q$, by the same parabolic regularity argument.

Proof. See Subsection 4.3. \hfill \Box

Let us use this proposition to justify (4.51) (note from (4.43) that (4.52) holds here). Note first from (4.43) that the time $\tau_n$ which allows (4.51) to be equal to some $\delta_0 > 0$ satisfies $\tau_n \in [0, Ds_{A,n}]$, for some $D > 0$. Let us then apply Proposition 4.8 with $D = D + 1$.

First, when $s = s_{A,n}$, we may use (4.50) (which is fully justified when $\tau = 0$) and (4.43) to write (remember that $b = a_n$ from (4.8)):
\[ v_{b,0,1}(s_{A,n}) = 5A^5B_ne^{-2s_{A,n}}C_{6,5} + O(B_n^2e^{-\frac{3}{2}s_{A,n}}) + O(s_{A,n}e^{-\frac{5}{2}s_{A,n}}). \] \hfill (4.53)

Now, we recall the ODE (2.6) satisfied by $v_{b,0,1}$:
\[ \forall s \geq s_{A,n}, \quad v_{b,0,1}'(s) = \frac{1}{2}\tau b_{0,1}(s) + \int v_b(y, s)^2k_1(y_2)\rho(y)dy. \]
Using Proposition 4.8 with the $L^q$ norm (use the remark following the proposition), we may estimate the quadratic term as follows (note that the contribution of the term $[C_{4,0}e^{-s}h_4(y_1)]^2$ vanishes thanks to the orthogonality between $h_4(y_1)^2$ and $h_1(y_2)$ in $L^q$), and write the following differential inequality on $v_{b,0,1}(s)$:
\[ \forall s \geq s_{A,n}, \quad |v_{b,0,1}'(s) - \frac{1}{2}\tau b_{0,1}(s)| \leq C(s - s_{A,n})e^{-s}A^4e^{-2s_{A,n}} + C(A)e^{-s}e^{-\frac{3}{2}s_{A,n}} + C(s - s_{A,n})^2A^8e^{-4s_{A,n}} + C(A)^2e^{-3s_{A,n}}. \]

Integrating this equation, we see that
\[ \forall \tau \geq 0, \quad |v_{b,0,1}(\tau + s_{A,n}) - e^{\frac{\tau}{2}}v_{b,0,1}(s_{A,n})| \leq C(A)e^{\frac{\tau}{2}}e^{-\frac{5}{2}s_{A,n}}, \]
for $n$ large enough. Using (4.53) and (4.43), we see that (4.51) is justified. Thus, $C_{6,5} = 0$. This concludes the proof of Claim 4.7. \hfill \Box

Now, we are ready to give the final argument of Step 1.

- Final argument of Step 1. Focusing on $v_{b,1,0}(s)$, we write the following from the table given right before (4.47):
\[
\begin{align*}
v_{b,1,0}(s_{A,n} + \tau) &= e^{\frac{\tau}{2}} \left\{ 4A^3B_n^3C_{4,0}e^{-s_{A,n}} + 3A^4B_n^2e^{-\frac{3}{2}s_{A,n}}(C_{5,2} + O(B_n)) + 2A^5B_ne^{-2s_{A,n}}(C_{6,4} + O(B_n)) + O\left(s_{A,n}e^{-\frac{3}{2}s_{A,n}}\right) \right\}.
\end{align*}
\]
Here again, the terms coming with $e^{3\tau/2}$ in the table should not be taken into account at the formal level, as we have already explained following that table. Since $C_{4,0} \neq 0$ from (4.6), using (4.43), we see that

$$v_{b,0}(s_{A,n} + \tau) \sim 4e^{\frac{5}{2}}A^3B_n^3C_{4,0}e^{-s_{A,n}}$$

as $n \to \infty$.

hence, a contradiction follows from the growth factor $e^{\frac{5}{2}}$ (the rigorous justification comes exactly as with the proof of the fact that $C_{6,5} = 0$ in Claim 4.7). This concludes the argument of Step 1, proving that $a_{n,1} = O(a_{n,2}|\log a_{n,2}|)$ as $n \to \infty$.

**Step 2: Proof of the fact that $C_{6,6} = C_{5,3} = C_{6,5} = 0$.**

We will prove these 3 cancelations in the order they appear. Before doing that, let us recall that the Taylor expansion (4.13) still holds. Even better, we already have some cancelations in that expansion, namely: $C_{4,2} = C_{4,1} = 0$ and $C_{5,4} = 0$ (use Part 1 and Lemma 4.5). One more thing we should keep in mind: using (4.3) and Step 1 we know that

$$a_{n,2}^2 \leq a_{n,1} \leq Ca_{n,2}\log a_{n,2}$$

as $n \to \infty$.

Using the definitions (4.9) and (4.11) of $B_n$ and $s_{A,n}$, we see that

$$Ac^{-\frac{s_{A,n}}{4}} \leq B_n \leq Ca_{s_{A,n}}e^{-\frac{s_{A,n}}{4}}$$

as $n \to \infty$. (4.54)

With these facts in mind, we may use the strategy of Proposition 4 and refine the Taylor expansion (1.12) written for $v_0(y,s)$ and prove that:

$$v_0(y,s) = \frac{e^{-2s}}{3}C_{4,0}^2\gamma_{4,0}h_0h_0 - \frac{e^{-2s}}{2}C_{4,0}^2\gamma_{4,2}h_2h_0 + C_{4,0}e^{-s} - e^{-2s}C_{4,0}^2\gamma_{4,4}h_4h_0 + e^{-\frac{s_{A,n}}{2}} \sum_{j=0}^{3} C_{5,j}h_{5-j} + se^{-2s}C_{4,0}^2\gamma_{4,6}h_6h_0 + e^{-2s} \sum_{j=0}^{6} C_{6,j}h_{6-j}h_j + e^{-2s}C_{4,0}^2h_8h_0 + O\left(se^{-\frac{s_{A,n}}{4}}\right),$$

for some real coefficients $C_{6,j}$ with $j = 0, \ldots, 6$, with the same notations following (4.45).

Let us note that for the formal arguments based on this expansion, the table given right before (4.37) is valid here.

In the following, we successively prove the 3 cancelations mentioned in the title of this step.

- **Proof of the fact that $C_{6,6} = 0$.**

Using the same strategy as usual, we justify that

$$v_{b,0,0}(s_{A,n} + \tau) = e^\tau \left\{ A^4B_n^4e^{-s_{A,n}}C_{4,0} + A^5B_n^2e^{-\frac{s_{A,n}}{2}}(C_{5,3} + O(B_n)) + A^6e^{-2s_{A,n}}(C_{6,6} + O(B_n)) + O\left(s_{A,n}e^{-\frac{s_{A,n}}{2}}\right) \right\}.$$ 

As we have written following the table given right before (4.37), we ignore the terms coming with $e^{2\tau}$ in the derivation of this expansion.

Assuming by contradiction that $C_{6,6} \neq 0$ and using (4.54), we see that

$$v_{b,0,0}(s_{A,n} + \tau) \sim e^\tau A^6e^{-2s_{A,n}}C_{6,6}$$

as $n \to \infty$.

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Then, the growth factor \( e^\tau \) leads to a contradiction (for the rigorous justification, Proposition 4.2 is enough here, provided that one takes \( A \) large enough; for a similar argument see the proof of the fact that \( C_{6,6} = 0 \) in Claim 4.7 above). Thus, \( C_{6,6} = 0 \).

- **Proof of the fact that \( C_{5,3} = 0 \).**

Using the same strategy as before, and taking into account the size of \( B_n \) given in (3.34) and assuming by contradiction that \( C_{5,3} \neq 0 \), we see that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim 2e^\tau A^4 B_n e^{-\frac{2}{3} A \cdot n} C_{5,3} \quad \text{as} \quad n \to \infty,
\]

and the growth factor \( e^\tau \) leads to a contradiction, as usual. Once again, the terms with \( e^{3\tau/2} \) in the table given right before (4.47) are ignored in this expansion. As for the rigorous argument, Proposition 4.2 is sufficient for the justification. Thus, \( C_{5,3} = 0 \).

- **Proof of the fact that \( C_{6,5} = 0 \).**

By the same strategy, taking into account the size of \( B_n \) given in (4.54) and assuming by contradiction that \( C_{6,5} \neq 0 \), we see that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim e^\tau A^5 e^{-2s_{A,n}} C_{6,5} \quad \text{as} \quad n \to \infty,
\]

and the growth factor \( e^\tau \) leads to a contradiction, as before. Note that we ignored the terms involving \( e^{3\tau/2} \) in the table given right before (4.47). As for the rigorous justification, it goes as in the proof of the fact \( C_{6,6} = 0 \) in Claim 4.7 in particular, the power of \( A \) is crucial. Thus, \( C_{6,5} = 0 \). This concludes the proof of the cancelations in the title of Step 2.

**Step 3: Final argument of Part 4**

Recalling the Taylor expansion (4.55) and the cancelations of Step 2, we may use the strategy of Proposition 1 and get to the next order:

\[
v_0(y, s) = -\frac{e^{-2s}}{3} C_{4,0}^2 \gamma_{4,4,0} h_0 h_0 - \frac{2e^{-\frac{2}{3} s}}{3} C_{4,0} \sum_{j=0}^{1} C_{5,j} \gamma_{4,5-j,1-j} h_{1-j} h_{j} \quad (4.56)
\]

\[
- \frac{e^{-2s}}{2} C_{4,0}^2 \sum_{j=0}^{2} C_{5,j} \gamma_{4,5-j,3-j} h_{3-j} h_{j}
\]

\[
+ \left[ C_{4,0} e^{-s} - e^{-2s} C_{4,0}^2 \right] h_4 h_0 + \sum_{j=0}^{2} C_{5,j} e^{-\frac{2}{3} s} - 2e^{-\frac{2}{3} s} C_{4,0} C_{5,j} \gamma_{4,5-j,5-j} h_{5-j} h_{j}
\]

\[
+ s e^{-2s} C_{4,0}^2 \gamma_{4,4,6} h_6 h_0 + e^{-2s} \sum_{j=0}^{4} C_{6,j} h_{6-j} h_{j} + \sum_{j=0}^{2} 2s e^{-\frac{2}{3} s} C_{4,0} C_{5,j} \gamma_{4,5-j,7-j} h_{7-j} h_{j}
\]

\[
+ \sum_{j=0}^{7} C_{7,j} e^{-\frac{2}{3} s} h_{7-j} h_{j} + e^{-2s} C_{4,0}^2 h_8 h_0 + 2C_{4,0} e^{-\frac{2}{3} s} \sum_{j=0}^{2} C_{5,j} h_{9-j} h_{j} + O (s^2 e^{-3s})
\]

for some real coefficients \( C_{7,j} \) with \( j = 0, \ldots, 7 \), with the notations given after (4.35).

With this expansion, we are ready to find the contradiction which will show that (4.38) doesn’t hold, concluding thus the argument of Part 4.
Starting from the Taylor expansion \(4.56\) and using the usual strategy, we may update the tool-table before \((4.47)\) and write the following (note that we include only the terms generating \(e^{\alpha \tau}\) with \(\alpha > 0\) and that we change the notation by avoiding to reproduce the coefficients of the first column in the following; in other words, the second to forth column have to be multiplied by the first in order to get the desired expression):

| \(v_{b,0,0}(s_{A,n} + \tau)\) | \(v_{b,1,0}(s_{A,n} + \tau)\) | \(v_{b,0,1}(s_{A,n} + \tau)\) |
|-----------------------------|-----------------------------|-----------------------------|
| \(C_{4,0}\)                | \(e^{-2} A^4 B_n^4 e^{-s_{A,n}}\) | \(0\)                       |
| \(C_{5,i}\)                | \(e^{-2} A^5 B_n^{5-i} e^{-\frac{2}{3}s_{A,n}}\) | \((5 - i)e^{-2} A^4 B_n^{4-i} e^{-s_{A,n}}\) |
| \(C_{6,0}\)                | \(e^{-2} A^6 B_n^{6-i} e^{-2s_{A,n}}\) | \((6 - i)e^{-2} A^5 B_n^{5-i} e^{-2s_{A,n}}\) |
| \(C_{4,0} C_{5,0} e^{-2s_{A,n}}\) | \((7 - i)e^{-2} A^6 B_n^{6-i} e^{-2s_{A,n}}\) | \((7 - i)e^{-2} A^6 B_n^{6-i} e^{-2s_{A,n}}\) |
| \(O(s^2 e^{-3s})\)        | \(e^{-2} O(s_{A,n} + \tau)^2 e^{-3s_{A,n}}\) | \(e^{-2} O(s_{A,n} + \tau)^2 e^{-3s_{A,n}}\) |

As we have written right after the previous table (given right before \(4.47\)), the terms coming with \(e^{2\tau}\) and \(e^{3\tau/2}\) are not relevant for our formal argument.

From this table, and focusing only on the terms bearing \(e^{-2}\) (remember, this is formal), we may write the following expansion:

\[
v_{b,1,0}(s_{A,n} + \tau) = e^{-2} \left\{ 4 A^3 B_n^3 C_{4,0} e^{-s_{A,n}} + 3 A^4 B_n^2 e^{-\frac{4}{3}s_{A,n}} (C_{5,2} + O(B_n)) + 2 A^5 B_n e^{-2s_{A,n}} (C_{6,4} + O(B_n)) + A^6 e^{-\frac{7}{2}s_{A,n}} (C_{7,6} + O(B_n)) + O(s_{A,n}^2 e^{-3s_{A,n}}) \right\}.
\]

Since \(B_n \gg e^{-\frac{4}{3}s_{A,n}}\) by \(4.3\) (translated in particular in \(4.54\)), we clearly see that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim 4 e^{-2} A^3 B_n^3 C_{4,0} e^{-s_{A,n}}\] as \(n \to \infty\),

since \(C_{4,0} \neq 0\) by \(4.6\). Choosing a suitable \(\tau\), we reach a contradiction with \(4.23\) as usual, at least formally. Now, on a rigorous level, Proposition \(4.2\) is not enough, and one needs to use Proposition \(4.3\) and the following argument to conclude (note that Proposition \(4.3\) can indeed be applied here, since the hypothesis \(4.32\) is fulfilled thanks to \(4.56\)). Therefore, the limit \(L_0\) given in \(4.36\) is zero.

As a conclusion to Subsection \(4.2\) whenever we are in the superquadratic case \(4.3\), only the 3/2 power regime is allowed, with a constant \(L_0\) given by \(4.37\). In the next subsection, we will investigate the quadratic regime.
4.3  The quadratic regime

This part is devoted to the study of the quadratic regime mentioned on page 13. Precisely, we assume that
\[ a_{n,1} \sim L a_{n,2}^2 \] as \( n \to \infty \), (4.57)
for some \( L > 0 \). We will show that \( L \) can enjoy only a finite number of values, which depend on the Taylor expansion of the solution.

Our strategy in the same as before: use the Taylor expansion of \( v_0(y,s) \) to derive an expansion for \( v_b(y,s) \) with \( b = a_n \). The non-growth condition imposed on the three components \( v_{b,0,0} \), \( v_{b,1,0} \) and \( v_{b,0,1} \) then implies some conditions on the coefficients as usual. It happens that those conditions take the form of polynomial equations on \( L \), whose coefficients are derived from the Taylor expansion.

We proceed in 3 steps, where we successively improve the Taylor expansion of the solution, in order to find the values of \( L \).

**Step 1: Expansion of order \( e^{-\frac{3}{2}s_{A,n}} \) in \( v_{b,1,0} \)**

To begin with, let us recall that the expansion (1.12) is valid here. In particular, the table given right before (4.19) will be useful in our argument. Noting that
\[ B_n \sim L A e^{-\frac{s_{A,n}}{2}} \] as \( n \to \infty \) (4.58)
from (4.57) and (4.11), then, using our usual strategy, we prove formally that
\[ v_{b,1,0}(s_{A,n} + \tau) = e^{\frac{\tau}{2}} A^4 e^{-\frac{3}{2}s_{A,n}} (2C_{4,2} L + C_{5,4} + o(1)) \] as \( n \to \infty \).

Given the exponential growth factor, this implies that
\[ 2C_{4,2} L + C_{5,4} = 0 \] (4.59)
(in order to justify rigorously this estimate, simply note that Proposition 4.2 holds here, since it only uses the fact that (1.12) holds, which is the case here; using the ODE argument given after Proposition 4.2 allows us to conclude).

If \( C_{4,2} \neq 0 \), then \( C_{4,2} < 0 \) from (1.13), and (4.59) implies that
\[ L = -\frac{C_{5,4}}{2C_{4,2}}. \]
Since \( L > 0 \), this implies that \( C_{5,4} > 0 \). Thus, we assume in the following that
\[ C_{4,2} = 0. \] (4.60)
Since the multilinear form in (1.10) is nonnegative, this implies that
\[ C_{4,1} = 0. \] (4.61)
Using (4.59), we also derive that
\[ C_{5,4} = 0. \] (4.62)

**Step 2: Expansion of order \( e^{-2s_{A,n}} \) in the 3 expanding components**
Starting from the second order expansion (1.12) and using (4.60), (4.61) and (4.62), we may use the strategy of Proposition 1 to derive the following third order Taylor expansion, analogous to (4.45) above (with the same notations):

\[ v_0(y,s) = -e^{-2s}C_{2,0}^4\gamma_4,4,0h_0h_0 - \frac{e^{-2s}}{2}C_{2,0}^4\gamma_4,4,2h_0h_0 + \left[C_{4,0}e^{-s} - e^{-2s}C_{2,0}^4\gamma_4,4,4\right]h_4h_0 \\
+ e^{-\frac{3}{2}s} \sum_{j=0}^{3} C_{5,j}h_{5-j}h_j + se^{-2s}C_{4,0}^2\gamma_4,4,6h_6h_0 + e^{-2s} \sum_{j=0}^{6} C_{6,j}h_{6-j}h_j \\
+ e^{-2s}C_{2,0}^4h_8h_0 + O\left(se^{-\frac{7}{2}s}\right), \tag{4.63} \]

for some real coefficients \(C_{6,j}\) with \(j = 0, \ldots, 6\).

Using our usual strategy, we derive the following expansion:

\[ v_{b,0}(s_{A,n} + \tau) = e^{\tau}A^5e^{-2s_{A,n}}(2LC_{5,3} + C_{6,5} + o(1)), \]

at least on a formal level (this time, the table right before (4.40) is useful to derive this, together with the estimate (4.58) on \(B_n\)). The exponential growth factor then implies that

\[ 2LC_{5,3} + C_{6,5} = 0 \tag{4.64} \]

(as for the rigorous justification, again, Proposition 1.2 is sufficient, since the power of \(A\) in the linear term in the ODE solution is 5, larger that 4, which is the power of the quadratic contribution).

If \(C_{5,3} \neq 0\), then we see that

\[ L = -\frac{C_{6,5}}{2C_{5,3}}. \]

Thus, we assume in the following that

\[ C_{5,3} = 0. \tag{4.65} \]

Using (4.64), we see that

\[ C_{6,5} = 0. \tag{4.66} \]

Writing the expansion of \(v_{b,0,0}\) this time, we formally see from (4.58) and the table given before (4.40) that

\[ v_{b,0,0}(s_{A,n} + \tau) = e^{\tau}A^6e^{-2s_{A,n}}(C_{6,6} + o(1)), \]

and the growth factor implies that

\[ C_{6,6} = 0 \tag{4.67} \]

(again, the rigorous justification uses Proposition 1.2 and the difference of the powers of \(A\) between the linear term and the quadratic term in the solution of the ODE (6 against 4)).

**Step 3: Expansion of order \(e^{-\frac{5}{2}s_{A,n}}\) in two expanding components**

At this stage, we need to further improve expansion (4.63) up to the forth order. Starting from (4.63) and using the strategy of Proposition 1 together with (4.65), (4.66) and (4.67), we write a similar expansion to (4.56):

\[ v_0(y,s) = -\frac{e^{-2s}}{3}C_{2,0}^4\gamma_4,4,0h_0h_0 - \frac{2e^{-\frac{5}{2}s}}{3}C_{4,0}^2\gamma_{4,5,1}h_1h_j \]

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For that, we need to refine Proposition 4.8 up to the order \( e^L \).

The growth factors both imply that (at least formally)

\[
\text{See Subsection 4.4.}
\]

**Proof.**

As one may see by comparing the 2 propositions, the new version replaces the \( C(A) \) constant appearing in Proposition 4.8 by \( CA^2 \). With this improvement, the argument given right after that proposition works (thanks to the difference between the powers of \( A \)), confirming the validity of 4.70. Note however that expansion (4.68) is valid for \( \tau \) large, but not for \( \tau = 0 \); for that reason, one has to carefully use (4.68) to derive that

\[
v_{b,0,0}(s_{A,n}) = -e^{-2s_{A,n}}C_{4,0}^{2} \gamma_{4,4,0} + e^{-\frac{2}{3}s_{A,n}}(-\frac{2}{3}AC_{4,0}C_{5,1}^{2} \gamma_{4,4,0} + A^2C_{7,7} + o(1)).
\]

Apart from that, the adaptation of the argument given right after Proposition 4.8 is straightforward.

This concludes the proof of Theorem 3 when \( m = 4 \), if one assumes the result of Propositions 4.2, 4.8 and 4.9. In order to finish the argument, those propositions will be proved in the next subsection.
4.4 Uniform estimates for \( w_{an} \)

This subsection is dedicated to the proofs of Propositions 4.2, 4.8 and 4.9. This is the only missing element to terminate the proof of Theorem 3 in the case \( m = 4 \), given in the previous subsections.

**Proof of Proposition 4.2.** We will in fact sketch the idea, and avoid giving details.

Let us first start by recalling from Khenissy, Rebai and Zaag [10] that

\[
v_{an}(y,s) \sim P_{\{2 \leq i \leq 4\}}(v_{an}) \quad \text{as} \quad s \to \infty,
\]

uniformly for \( n \) large enough (see Theorem 1 page 4 in that paper). We also know from (4.22) that

\[
\|v_{an}(s)\|_{L^2_\rho} \to 0 \quad \text{as} \quad s \to \infty,
\]

uniformly in \( n \). From (4.71), our desired conclusion will follow if we fix \( C_0 > 0 \) such that for any \( A \geq 1 \) and \( D \geq 1 \),

\[
\forall i = 2, 3, 4, \quad \forall j = 0, \ldots, i, \quad \forall s \in [s_{A,n}, Ds_{A,n}], \quad |v_{an,i,j}(s)| \leq C_0 A^2 e^{-s_{A,n}},
\]

whenever \( n \) is large enough.

We proceed in 2 steps to fix such a \( C_0 \): in Step 1, we initialize (4.72) at \( s = s_{A,n} \), then, in Step 2, we proceed by contradiction to prove it for \( s \geq s_{A,n} \) (the integration of the projections of equation (2.2) on the various coordinates will be crucial for our argument).

**Step 1: Initialization of (4.72).** Since there is a universal constant \( C_2 \geq 1 \) such that for any \( v \in L^2_\rho \) and \( i = 2, 3, 4 \),

\[
\frac{1}{C_2} \max_{j=0,\ldots,i} |v_{i,j}| \leq \|P_i(v)\|_{L^2_\rho} \leq C_2 \max_{j=0,\ldots,i} |v_{i,j}|,
\]

using (4.26) together with (4.71), we see that

\[
\forall i = 2, 3, 4, \quad \forall j = 0, \ldots, i, \quad |v_{an,i,j}(s_{A,n})| \leq \tilde{C} A^2 e^{-s_{A,n}},
\]

whenever \( n \) is large enough, for some universal constant \( \tilde{C} > 0 \).

Fixing \( C_0 = 2 \tilde{C} \), we guarantee from (4.74) that (4.72) holds at \( s = s_{A,n} \).

**Step 2: The contradiction argument.** Let us assume by contradiction that (4.72) is true for all \( s \in [s_{A,n}, s_{n}^*] \), for some minimal \( s_{n}^* \) \( < Ds_{A,n} \) and stops from being true at \( s = s_{n}^* \). From Step 1 and continuity, it follows that \( s_{n}^* > s_{A,n} \). This also implies that we have an equality case at \( s = s_{n}^* \), in the sense that

\[
|v_{an,i,j}(s_{n}^*)| = C_0 A^2 e^{-s_{A,n}}, \quad \text{for some } i = 2, 3, 4 \text{ and } j = 0, \ldots, i.
\]

Starting from (4.75), we integrate the ODE (2.6) satisfied by \( v_{an,i,j} \), with \( 2 \leq i \leq 4 \):

\[
\forall s \geq s_{n}, \quad v_{an,i,j}'(s) = (1 - \frac{i}{2})v_{an,i,j}(s) + \int k_{i-j}(y_1)k_j(y_2)v_{an}(y,s)^2 \rho(y) dy.
\]
Since (1.72) is valid for all $s \in [s_{A,n}, s_n^*]$, using (1.71), we have

$$| \int k_{i,j}(y_1)k_{j}(y_2)v_{a,n}(y, s)^2 \rho(y)dy | \leq CC_0^2 A^4 e^{-2s_{A,n}}$$

(in fact, we need here an estimate in the $L^1_{\rho}$ norm and not just in the $L^2_{\rho}$, and this is possible thanks to parabolic regularity, as we have already explained right after the statement of Proposition 4.2). Therefore, for all $s \in [s_{A,n}, s_n^*]$, we have

$$| v'_{a,n,i,j}(s) - (1 - \frac{i}{2})v_{a,n,i,j}(s) | \leq CC_0^2 A^4 e^{-2s_{A,n}}.$$  

Recalling that $s_n^* \leq Ds_{A,n}$, we integrate this differential inequality, ending with

$$| v_{a,n,i,j}(s) | \leq e^{(1-\frac{i}{2})(s-s_{A,n})} | v_{a,n,i,j}(s_{A,n}) | + CC_0^2 A^4 e^{-2s_{A,n}}(s-s_{A,n}),$$

$$\leq \frac{C_0}{2} A^2 e^{-s_{A,n}} + CC_0^2 A^4 e^{-2s_{A,n}}(D-1)s_{A,n} \leq \frac{3C_0}{4} A^2 e^{-s_{A,n}}, \quad (4.77)$$

if $n$ is large enough. This means in particular that (1.76) is not true and concludes the proof of Proposition 4.2.

**Proof of Proposition 4.8.** Following Proposition 4.2, we assume that (4.52) holds. We also recall from (4.9) that $B_n \to 0$ as $n \to \infty$.

We proceed in 2 steps: we first initialize the estimate at $s = s_{A,n}$, then, we use the bound in Proposition 4.2 to derive then integrate differential inequalities satisfied by the various components.

**Step 1: Initialization at $s = s_{A,n}$.** Using the expansion (4.52) and using our geometric transformation as usual, we may write the following estimates for the various components of the solution at $s = s_{A,n}$ (for the last estimate, remember that (4.52) holds also in $L^1_{\rho}$ and use the Cauchy-Schwarz identity as we did for (4.55)):

$$v_{a,n,4,0}(s_{A,n}) = C_{1,0} e^{-s_{A,n}} + O(e^{-\frac{s}{2}s_{A,n}}), \quad (4.78)$$

$$| v_{a,n,i,j}(s_{A,n}) | \leq C_i e^{-\frac{s}{2}s_{A,n}} \text{ if } (i,j) \neq (4,0), \quad (4.79)$$

$$\| P_{(\lambda \leq -4)}(v_{a,n}(s_{A,n})) \|_{L^2_{\rho}} \leq C(A) e^{-\frac{s}{2}s_{A,n}}. \quad (4.80)$$

**Step 2: Differential inequalities satisfied by the various components.** Using Proposition 4.2 whose conclusion also holds in $L^1_{\rho}$, as explained in the remark following its statement, we may write the following differential inequalities, for $n$ large enough and for all $s \in [s_{A,n}, Ds_{A,n}]$,

$$\forall i \in \mathbb{N}, \quad \forall j = 0, \ldots, i, \quad | v'_{a,n,i,j}(s) - (1 - \frac{i}{2})v_{a,n,i,j}(s) | \leq C_i A^4 e^{-2s_{A,n}},$$

$$\frac{d}{ds} \| P_{(\lambda \leq 1-\frac{i}{2})}(v_{a,n}(s)) \|_{L^2_{\rho}}^2 \leq -2(1 - \frac{i}{2}) \| P_{(\lambda \leq 1-\frac{i}{2})}(v_{a,n}(s)) \|_{L^2_{\rho}}^2 + C_i A^4 e^{-4s_{A,n}}. \quad (4.81)$$

Integrating these various differential inequalities, we get the following estimates, for $n$ large enough and for all $s \in [s_{A,n}, Ds_{A,n}]$,

$$\forall i = 0, 1, \quad \forall j = 0, \ldots, i, \quad | v_{a,n,i,j}(s) | \leq C_i A^4 e^{-2s_{A,n}}, \quad (4.81)$$

$$\forall j = 0, 1, 2, \quad | v_{a,n,2,j}(s) - v_{a,n,2,j}(s_{A,n}) | \leq C(s-s_{A,n}) A^4 e^{-2s_{A,n}}. \quad (4.82)$$
\[\forall i = 3,4, \forall j = 0, \ldots, i, \ |v_{an,i,j}(s) - e^{(1 - \frac{q}{2})(s - s_{A,n})}v_{an,i,j}(s_{A,n})| \leq CA^i e^{-2s_{A,n}}, \tag{4.83}\]
\[\|P_{\{\lambda \leq \frac{1}{2}\}}(v_{an}(s))\|_{L^2_p} \leq e^{-\frac{q}{2}(s - s_{A,n})}\|P_{\{\lambda \leq \frac{1}{2}\}}(v_{an}(s_{A,n}))\|_{L^2_p} + CA^4 e^{-2s_{A,n}}. \tag{4.84}\]

Using (4.78)-(4.80), and recalling that \(s \leq DS_{A,n}\), we conclude the proof of Proposition 4.9. \(\square\)

**Proof of Proposition 4.9** Note first that we can use all the information available in Step 3 of Subsection 4.3. For the proof, we follow the proof of Proposition 4.8 with only one modification: we need to refine identities (4.78)-(4.80) so that we reach the orders of Subsection 4.3. For the proof, we follow the proof of Proposition 4.8 with only one

By definition, we will keep only the \(\tau\) with the change of variables (4.12) (with \(\tau = 0\)), we may use the binomial relation (3.14) to write

\[v_0(y, s) = C_{4,0}e^{-s}h_4h_0 + e^{-\frac{q}{2}s}\sum_{j=0}^{2} C_{5,j}h_{5-j}h_j + O(se^{-2s})\]

as \(s \to \infty\), in \(L^2_p(\mathbb{R}^N)\), for any \(q \geq 2\). Then, using the estimate (4.58) on \(B_n\) together with the change of variables (4.12) (with \(\tau = 0\)), we may use the binomial relation (3.14) to write

\[v_b(y, s_{A,n}) = C_{4,0}e^{-s_{A,n}} \left[ h_4(y_{b,1}) + 4h_3(y_{b,1})AB_n + O(B^2_n)\right] + e^{-\frac{q}{2}s_{A,n}}\sum_{j=0}^{2} C_{5,j}h_{5-j}(y_{b,1})h_j(y, 2) + O(C(A)s_{A,n}e^{-2s_{A,n}})\]

in \(L^2_p(\mathbb{R}^N)\) as \(n \to \infty\), thanks to the same justification as for (4.25). This can be translated as follows:

\[v_{b,3,0}(s_{A,n}) = 4C_{4,0}LA^2 e^{-\frac{q}{2}s_{A,n}} + o(e^{-\frac{q}{2}s_{A,n}}),\]
\[v_{b,4,0}(s_{A,n}) = C_{4,0}e^{-s_{A,n}} + O(C(A)s_{A,n}e^{-2s_{A,n}}),\]
\[v_{b,5,i}(s_{A,n}) = C_{5,i}e^{-\frac{q}{2}s_{A,n}} + O(C(A)s_{A,n}e^{-2s_{A,n}}),\]
\[\sup_{(i,j) \notin \{(0,0),(4,0),(5,0),(5,1),(5,2)\}} |v_{b,i,j}(s_{A,n})| + \|P_{\{\lambda \leq \frac{1}{2}\}}(v_{an}(s_{A,n}))\|_{L^2_p} \leq C(A)s_{A,n}e^{-2s_{A,n}}\]

Using these estimates as initial data, together with the estimates (4.81)-(4.84), which hold here, since the hypotheses of Proposition 4.8 hold also (because (4.52) follows from (4.85), which holds like all the information of Step 3 in Subsection 4.3, we conclude the proof of Proposition 4.9 (we also need Proposition 4.2). \(\square\)

We have just proved the Propositions 4.2, 4.8 and 4.9 in this subsection, finishing this way the proof of Theorem 4.8.

**5 Rigidity in the geometry of the blow-up set when \(m \geq 6\)**

This section is devoted to the proof of Theorem 4.8 when \(m \geq 6\). As for \(m = 4\), the proof uses the same strategy based on the geometric transformation (3.5) given in Step 2 of the proof of Theorem 4.8. However, the number of steps depends linearly on \(m\). For that reason, we have to proceed differently, at some point in the proof. In addition, we have more complicated formulas, and of course in the outcome, the result for \(m \geq 6\) is less explicit, hence, less spectacular than for \(m = 4\). Accordingly, we only give the main steps of the proof and don’t insist on details.
Proof of Theorem 3 when \( m \geq 6 \). Consider \( u(x,t) \) a solution of equation (1.1) blowing up at time \( T > 0 \). Assume that the origin is a non isolated blow-up point where \( m(0) = m \geq 6 \) is even and consider an arbitrary sequence of non-zero blow-up points \( a_n = (a_{n,1}, a_{n,2}) \) converging to the origin as \( n \to \infty \).

(i) This item follows exactly as in the case \( m = 4 \) (see Subsection 4.1 above). In particular, after extracting a subsequence and making a suitable change of variables, we may assume that

\[
a_{n,1} \geq 0 \text{ and } a_{n,2} \geq 0 \text{ for all } n \in \mathbb{N}
\]

and

\[
a_{n,1} = o(a_{n,2}).
\]

Note that Theorem 2 holds here, and so does the expansion (1.1 2), and that the multilinear form in (1.10) is non zero and nonnegative.

(ii) Following (5.2), after the extraction of a subsequence still denoted the same, we identify 3 possible regimes for \( a_n \): superquadratic, quadratic and subquadratic, as with \( m = 4 \). We then proceed in 3 parts:
- Part 1 is dedicated to the superquadratic case, where \( a_{n,1} \gg a_{n,2}^2 \) as \( n \to \infty \).
- In Part 2, we deal with the quadratic case, where \( a_{n,1} \sim L a_{n,2}^2 \), for some \( L > 0 \).
- Finally, in Part 3, we gather all the information and conclude the proof of Theorem 3.

**Part 1: Quantified superquadratic regimes for \( a_n \)**

In this part, we assume that

\[
a_{n,1} \gg a_{n,2}^2 \text{ as } n \to \infty.
\]

We then proceed in 4 steps:
- In Step 1, we show that \( C_{m,m-2} = C_{m,m-3} = 0 \), where those coefficients appear in the expansion (1.12).
- In Step 2, we introduce a new parameter \( \theta \) to measure the convergence of \( B_n = a_{n,1}/a_{n,2} \) to 0 in exponential scales of \( s_{A,n} \) defined in (1.11), and show that \( \theta \) enjoys only a finite number of values in \( \mathbb{Q} \setminus \{0\} \).
- In Step 3, we make a refinement of Step 2, by introducing one further parameter \( \alpha \) to measure the convergence of \( B_n \) to 0 in polynomial corrections of exponential scales, and show that \( \alpha \) enjoys only a finite number of values in \( \mathbb{Q} \).
- In Step 4, following Steps 2 and 3, we make one further refinement, by introducing a new variable \( \psi_n \) to quantify the convergence of \( B_n \) to 0, and show that it converges (up to a subsequence) to some value related to some root of a polynomial involving the coefficients of the Taylor expansion of \( v_0(y,s) \).

**Step 1: Proof of the fact that \( C_{m,m-2} = C_{m,m-3} = 0 \)**

Considering the expansion (1.12), we will now prove that \( C_{m,m-2} = C_{m,m-3} = 0 \). As in the case \( m = 4 \), the nonnegativity of the multilinear form (1.10) implies that it is enough to show that \( C_{m,m-2} = 0 \). Proceeding by contradiction, we assume that \( C_{m,m-2} \neq 0 \). Let us reach a contradiction from the behavior of \( w_b \) defined by (1.12), where \( b = a_n \). Using the notations and assumptions in (4.9)-(4.11), we see from (5.3), (4.10) and (5.2) that

\[
1 \gg B_n \gg A e^{-\frac{s_{A,n}}{2}} \text{ as } n \to \infty.
\]
Using the expansion (1.12) and proceeding as in Part 1 of Subsection 1.2 when \( m = 4 \), both formally and rigorously, we readily see that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim 2e^{\frac{s}{2}}A^{m-1}B_nC_{m,m-2}e^{(1-\frac{m}{2})s_{A,n}} \text{ as } n \to \infty.
\]

The growth factor leads to contradiction. Thus, \( C_{m,m-2} = 0 \) and \( C_{m,m-3} = 0 \) too.

**Step 2: Measuring \( B_n \) in exponential scales of \( s_{A,n} \).**

From (5.4), we may assume that

\[
- \frac{\log B_n}{s_{A,n}} \to \theta \in [0, \frac{1}{2}] \quad \text{or equivalently that} \quad B_n = e^{-(\theta + o(1))s_{A,n}}, \text{ as } n \to \infty, \quad (5.5)
\]

up to a extracting a subsequence, still denoted the same. Now, we are going to prove the following three-fold statement:

**Proposition 5.1.**

(i) The limit \( \theta \in [0, \frac{1}{2}] \) defined in (5.5) is rational.

(ii) It holds that \( \theta \neq 0 \).

(iii) In fact, \( \theta \) enjoys only a finite number of rational values in \( E_1 \cup E_2 \) where the two sets \( E_1 \) and \( E_2 \) are defined below respectively in (5.18) and (5.19).

**Remark.** In fact, as we will see from the proof, items (ii) and (iii) are by-products of the proof of item (i). For clarity, we dedicate the next step to the proof of item (i), then, we explain how to derive items (ii) and (iii) in the following steps.

**Remark.** Note that

\[
E_1 \cup E_2 \subset \left\{ \frac{L}{2G} \mid 1 \leq G \leq 2m - 3 \text{ and } 1 \leq L \leq \min(G, m - 2) \right\}.
\]

**Proof.** We will give the proof of the three items successively.

- **Proof of item (i) of Proposition 5.1.**

  Using the strategy of Proposition 1, we may refine the Taylor expansion of \( v_0(y, s) \) (equal to \( w_0(y, s) - 1 \) by definition (2.1)) given in (1.12) (recalling that \( C_{m,m-2} = C_{m,m-3} = 0 \) from Step 1), up to the order \( O(se^{(2-m)s}) \):

\[
v_0(y, s) = e^{(1-\frac{m}{2})s} \sum_{j=0}^{m-4} C_{m,j}h_{m-j}(y_1)h_j(y_2) + e^{\frac{1-m}{2}s} \sum_{j=0}^{m} C_{m+1,j}h_{m+1-j}(y_1)h_j(y_2) + e^{(1-\frac{k}{2})s} \sum_{k=m+2}^{2m-3} \sum_{j=0}^{k} C_{k,j}h_{k-j}(y_1)h_j(y_2) + \bar{v}_0(y, s)
\]

with \( \bar{v}_0(y, s) = O(se^{(2-m)s}) \) as \( s \to \infty \). Note that we stopped in this expansion when the quadratic term in the equation (2.2) satisfied by \( v_0(y, s) \) becomes relevant; this way, we only have "linear terms" in our expansion.

Using the transformation (3.5), we may use the expansion in (5.6) to derive the following expansion for \( v_b(y_b, s_{A,n} + \tau) \), which is analogous to (4.13), with \( \tau \geq 0 \):

\[
v_b(y_b, s_{A,n} + \tau) = e^{(1-\frac{m}{2})(s_{A,n}+\tau)} \sum_{j=0}^{m-2} C_{m,j}h_{m-j}(y_{b,1} + AB_ne^{\tau})h_j(y_{b,2} + Ae^{\tau})
\]
\[ + e^{\frac{1-k}{2}(s_{A,n}+\tau)} \sum_{j=0}^{m} C_{m+1,j}h_{m+1-j}(y_{b,1} + AB_{n}e^{\tau})h_j(y_{b,2} + Ae^{\tau}) \]
\[ + e^{(1-\frac{k}{2})(s_{A,n}+\tau)} \sum_{k=m+2}^{2m-3} C_{k,j}h_{k-j}(y_{b,1} + AB_{n}e^{\tau})h_j(y_{b,2} + Ae^{\tau}) \]
\[ + \tilde{v}_b(y_b, s_{A,n} + \tau). \] (5.7)

As before, for more visibility, we may write the following table giving the expansion for the 3 expanding components \(v_{b,0,0}(s_{A,n} + \tau), v_{b,1,0}(s_{A,n} + \tau)\) and \(v_{b,1,0}(s_{A,n} + \tau)\), for \(\tau \geq 0\):

| \(C_{m,j}\) | \(v_{b,0,0}(s_{A,n} + \tau)\) | \(v_{b,1,0}(s_{A,n} + \tau)\) | \(v_{b,1,0}(s_{A,n} + \tau)\) |
|----------------|----------------|----------------|----------------|
| \(C_{m+1,j}\) | \(C_{m+1,j}e^{\tau}A^{m-j}B_n^{m-j-2\tau}A^{s_{A,n}}\) | \(jC_{m+1,j}e^{\tau}A^{m-j-1}B_n^{m-j-1}e^{(1-\frac{m}{2})s_{A,n}}\) | \(jC_{m+1,j}e^{\tau}A^{m-j}B_n^{m-j}e^{(1-\frac{m}{2})s_{A,n}}\) |
| \(C_{k,j}\) | \(C_{k,j}e^{\tau}A^{k-1}B_n^{k-j-1}e^{(1-\frac{k}{2})s_{A,n}}\) | \(jC_{k,j}e^{\tau}A^{k-1}B_n^{k-j}e^{(1-\frac{k}{2})s_{A,n}}\) | \(jC_{k,j}e^{\tau}A^{k-1}B_n^{k-j}e^{(1-\frac{k}{2})s_{A,n}}\) |
| \(\tilde{v}_b(y_b, s_{A,n} + \tau)\) | \(\tilde{v}_{b,0,0}(s_{A,n} + \tau)\) | \(\tilde{v}_{b,1,0}(s_{A,n} + \tau)\) | \(\tilde{v}_{b,1,0}(s_{A,n} + \tau)\) |

with the rest terms satisfying

\[
|\tilde{v}_{b,i,j}(y_b, s_{A,n})| \leq C(A, i, j)s_{A,n}e^{(2-m)s_{A,n}},
\] (5.8)

with the same argument as for (3.16).

Now, proceeding by contradiction, we assume that \(\theta\) is not rational. Let us first explain our argument. As in the case \(m = 4\), the contradiction will follow from the behavior of one of the 3 components shown in the table above. In fact, since the multi-linear form in (1.10) is non zero, there is \(l = 0, \ldots, m\) such that \(C_{m,l} \neq 0\). Therefore, it is convenient to choose a component involving this \(C_{m,l}\), hoping to reach a contradiction. Since \(l \in [0, m - 4]\) from (1.12) and Step 1, we focus on the first two components, since the third misses \(C_{m,0}\). More precisely, we will choose the second component, namely \(v_{b,1,0}(s_{A,n} + \tau)\), since it involves lower powers of the small parameter \(B_n\). Furthermore, we need to consider the coefficient of the term involving \(C_{m,l}\), namely \(A^{m-1}e^{\tau}B_n^{m-1}e^{(1-\frac{m}{2})s_{A,n}}\), to be dominant with respect to the error term whose size is shown in (5.8). Since we have from (5.6) that

\[
A^{m-1}e^{\tau}B_n^{m-1}e^{(1-\frac{m}{2})s_{A,n}} \geq A^{m-1}e^{\tau}B_n^{m-1}e^{(1-\frac{m}{2})s_{A,n}} = A^{m-1}e^{\tau}e^{(1-\frac{m}{2}-(m-1)\theta+o(1))s_{A,n}}
\] (5.9)
as \(n \to \infty\), a sufficient condition for this is to have

\[
1 - \frac{m}{2} - (m-1)\theta > 2 - m, \text{ i.e. } \theta < \frac{m-2}{2(m-1)}.
\] (5.10)

Accordingly, since \(\theta \in [0, \frac{1}{2}]\) by (5.7), we consider two cases in the following:

**Case 1:** \(\theta \in [0, \frac{m-2}{2(m-1)}]\). From the table above together with (5.8), we consider the
Proof of Lemma 5.2.

It remains then to prove Lemma 5.2. Following expansion:

$$v_{b,1,0}(s_{A,n} + \tau) = e^{\frac{k}{2}} \left( \sum_{(k,j) \in H_1} e^{(1-\frac{k}{2})s_{A,n}} A^{k-1}(k-j)C_{k,j}B_n^{k-1-j} + O(s_{A,n}e^{(2-m)s_{A,n}}) \right),$$

where

$$H_1 = \{(k,j) \mid m \leq k \leq 2m - 3, \ 0 \leq j \leq k - 1, \ C_{k,j} \neq 0 \ \text{and} \ e^{(1-\frac{k}{2})s_{A,n}}B_n^{k-1-j} \gg s_{A,n}e^{(2-m)s_{A,n}} \}.$$  

(5.11)

We first note that $H_1 \neq \emptyset$, since it contains $(m, l)$, by (5.9) and (5.10). Second, the following non-codominance property between all terms in the expansion (5.11) allows us to conclude:

**Lemma 5.2** (No codominance of terms in the expansion of $v_{b,1,0}(s_{A,n} + \tau)$). Consider two terms in the expansion (5.11), say $(k-j)C_{k,j}A^{k-1}B_n^{k-1-j}e^{(1-\frac{k}{2})s_{A,n}}$ and $(k'-j')C_{k',j'}A^{k'-1}B_n^{k'-1-j'}e^{(1-\frac{k'}{2})s_{A,n}}$ with $C_{k,j}C_{k',j'} \neq 0$, $m \leq k, k' \leq 2m - 3$, $0 \leq j \leq k - 1$, $0 \leq j' \leq k' - 1$, and $(k,j) \neq (k',j')$. Then, one of the terms dominates the other as $n \to \infty$.

Indeed, if this lemma holds, recalling that $H_1$ is a non empty finite set, we may consider $(\tilde{k}, \tilde{j}) \in H_1$ such that

$$v_{b,1,0}(s_{A,n} + \tau) \sim e^{\frac{k}{2}}(\tilde{k} - \tilde{j})C_{\tilde{k},\tilde{j}}A^{\tilde{k}-1}B_n^{\tilde{k}-1-j}e^{(1-\frac{\tilde{k}}{2})s_{A,n}}$$

(5.13)

as $n \to \infty$, with $C_{\tilde{k},\tilde{j}} \neq 0$. From the growth factor $e^{\frac{k}{2}}$, the coordinate $v_{b,1,0}(s_{A,n} + \tau)$ will grow, and a contradiction follows as usual, both for the formal and the rigorous argument. It remains then to prove Lemma 5.2.

**Proof of Lemma 5.2** Using the expansion of $B_n$ in (5.5), we see that

$$(k-j)C_{k,j}A^{k-1}B_n^{k-1-j}e^{(1-\frac{k}{2})s_{A,n}} = (k-j)C_{k,j}A^{k-1}e^{(1-\frac{k}{2}-(k-1-j)\theta + o(1))s_{A,n}}$$

(5.14)

as $n \to \infty$, with a similar expansion with $(k',j')$. Since $(k,j) \neq (k',j')$, it follows that $(k,k-j) \neq (k',k'-j')$. Therefore, recalling that $\theta$ is not rational (this is in fact the contradiction hypothesis), it follows that

$$1 - \frac{k}{2} - (k-1-j)\theta \neq 1 - \frac{k'}{2} - (k'-1-j')\theta,$$

and the conclusion follows.

**Remark.** From this proof, we see that the hypothesis that $\theta$ is not rational is too strong. In fact, the argument works whenever $\theta$ avoids the rationals which are of the form $\frac{k'-k}{2(k-j)-(k'-j')}$, which make a finite collection of numbers, due to the boundedness of the ranges where the parameters lay. This remark will show to be crucial below, while adapting the present step in order to derive the proof of item (iii) of Proposition 5.1.

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Case 2: $\theta \in \left[\left\lfloor \frac{m-2}{M-m} \right\rfloor + 1\right]$. It happens that the argument of Case 1 works here, with small natural adaptations.

The first problem is that the terms in the expansion (5.11) may all be dominated by the error term, therefore, we need to go further in the Taylor expansion (5.6), and this is possible thanks to Proposition 1. The question then is to know the order up to which we carry on the Taylor expansion.

To find that, note that focusing on terms with $C_{k,j} \neq 0$ is a convenient way to have a relevant term in the expansion of $v_{b,1,0}(s_{A,n} + \tau)$. Such a term exists with $(k,j) = (m,l)$, since we know that the multilinear form in (1.10) is non-zero. It remains then to guarantee that the order in front of $C_{m,l}$, namely $e^{(1-\frac{1}{2})s_{A,n}B_m^{m-1}-l}$, is dominant with respect to the error term. Since $l \geq 0$, hence $e^{(1-\frac{1}{2})s_{A,n}B_m^{m-1}-l} \geq e^{(1-\frac{1}{2})s_{A,n}B_m^{m-1} = e^{(1-\frac{1}{2})(m-1)}e^{o(1)s_{A,n}}}$ as $n \to \infty$ by (5.6), we simply need to refine the Taylor expansion (5.6) of $v_0(y,s)$ up to the order $O(s^\gamma e^{(1-\frac{1}{2})s})$ for some $\gamma > 0$, with $M \in \mathbb{N}$ given by

$$M = \left\lfloor (2\theta + 1)(m - 1) \right\rfloor,$$

where the notation $\lceil \cdot \rceil$ stands for the ceiling (or upper integer part) of a given number. This is possible, thanks to Proposition 1. However, in comparison with the Taylor expansion given in (5.6), now we will see “resonant” terms, of order $s^\gamma e^{(1-\frac{1}{2})s}$, the first among them occurs at $k = 2m - 2$, and corresponds to the effect of the quadratic term in (2.2). More precisely, that term is of order $se^{2-m}$.

Now, using the geometric transformation given in Step 2 of Section 3 we may write an expansion of $v_b(y_b, s_{A,n} + \tau)$ analogous to (5.7). We may also write a table similar to the one right before (5.8), giving an expansion for $v_{b,1,0}(s_{A,n} + \tau)$ analogous to (5.11), and which shows resonant orders, as follows:

$$v_{b,1,0}(s_{A,n} + \tau) = e^{\tilde{T}} \left( \sum_{(k,j,i) \in H_2} (s_{A,n} + \tau)^i e^{(1-\frac{1}{2})s_{A,n}} A_{k-1}^{k-1}(k-j)\tilde{C}_{k,j,i}B_n^{k-1-j} + O(s_{A,n}^{2}\gamma e^{(1-\frac{1}{2})s_{A,n}}) \right),$$

where $H_2 \subset \mathbb{N}^3$ is a natural adaptation of the set $H_1$ (5.12) as follows:

$$H_2 = \{(k,j,i) \mid m \leq k \leq M, \ 0 \leq j \leq k - 1, \ 0 \leq i \leq i_k, \ \tilde{C}_{k,j,i} \neq 0 \text{ and } s_{A,n}^{i} e^{(1-\frac{1}{2})s_{A,n}} B_n^{k-1-j} \gg s_{A,n}^{2}\gamma e^{(1-\frac{1}{2})s_{A,n}} \},$$

where $i_k \in \mathbb{N}$ is positive only at resonant orders.

As in Case 1, we first note that $H_2$ is non-empty, thanks to the choice of the order $M$ (5.15) up to which we made the Taylor expansion. Furthermore, the non-codominance property holds here too!

Before proving that, we would like to comment on the time interval where we investigate the codominance property. In Case 1, all the terms in the expansion (5.11) are multiples of the sole function $e^{\tilde{T}}$. Comparing them at $\tau = 0$ or on any subinterval of $[0, \infty)$ gives the same order. Here in Case 2, we have resonant terms, namely multiples of $(s_{A,n} + \tau)^i e^{\tilde{T}}$ with $i \in \mathbb{N}$, and the comparison at $\tau = 0$ or on a larger interval may be different, depending on the coefficient in front of the function and also on the size...
of the interval. For that reason, we need to clearly fix some interval where we make the comparison. The most natural choice is simply the time interval of validity of the expansion \( (5.16) \), namely, the interval where all the functions stay less than some fixed small \( \delta_0 > 0 \). More precisely, given some \((k, j, i) \in H_2\), we take \( \tau \in [0, \tau_{k,j,i,n}] \) such that

\[
(s_{A,n} + \tau_{k,j,i,n})^i e^{\frac{\tau_{k,j,i,n}}{2} e^{(1 - \frac{k}{2})s_{A,n}}} A^{k-1}(k-j) \tilde{C}_{k,j,i} B^{k-1-j} = \delta_0.
\]

Since the function \( \tau \mapsto (s_{A,n} + \tau)^i e^{\frac{\tau}{2}} \) is increasing, we clearly have from \( (5.3) \) that \( \tau_{k,j,i,n} \to \infty \) as \( n \to \infty \), and

\[
\tau_{k,j,i,n} \sim [k - 2 + 2\theta(k - 1 - j)]s_{A,n} \text{ as } n \to \infty
\]

(note that the coefficient of \( s_{A,n} \) is positive, since \( \theta \geq 0 \) and \( k \geq m \geq 6 \)).

Since \( 6 \leq m \leq k \leq M \) and \( \theta \in [0, \frac{1}{2}] \), it follows that \( k - 2 + 2\theta(k - 1 - j) \in [k - 2, 2k - 3] \subset [m - 2, 2M - 3] \), hence

\[
\tau_{k,j,i,n} \leq (2M - 2)s_{A,n}
\]

for \( n \) large enough. Thus, we will consider \( \tau \in [0, (2M - 2)s_{A,n}] \). On that interval, we see that the resonant function \((s_{A,n} + \tau)^i e^{\frac{\tau}{2}}\) is comparable to a pure exponential function, in the sense that

\[
\forall \tau \in [0, (2M - 2)s_{A,n}], \quad s_{A,n}^i e^{\frac{\tau}{2}} \leq (s_{A,n} + \tau)^i e^{\frac{\tau}{2}} \leq (2M - 2)^i s_{A,n}^i e^{\frac{\tau}{2}}.
\]

Thus, the codominance property can be checked at \( \tau = 0 \), as we did in Case 1.

Consider then two different \((k, j, i)\) and \((k', j', i')\) in \( H_2 \), and let us show that either \((s_{A,n} + \tau)^i e^{\frac{\tau}{2}} B^{k-1-j} \) or \((s_{A,n} + \tau)^i e^{\frac{\tau}{2}} B^{k'-1-j'} \) dominates the other, for \( n \) large and \( \tau = 0 \), which is legitimate, from the reduction we have just proved above. Note first from \( (5.3) \) that

\[
s_{A,n}^i e^{(1 - \frac{k}{2})s_{A,n}} B^{k-1-j} = s_{A,n}^i e^{(1 - \frac{k}{2} - (k-1-j)\theta + o(1))s_{A,n}},
\]

as \( n \to \infty \), with a similar estimate for \((k', j', i')\).

If \((k, j) \neq (k', j')\), then \((k, k-j) \neq (k', k'-j')\), hence \(1 - \frac{k}{2} - (k-1-j)\theta \neq 1 - \frac{k'}{2} - (k'-1-j')\theta\), since \( \theta \) is not rational. Taking \( n \) large enough, we see that one term dominates the other.

Now, if \((k, j) = (k', j')\), then \( i \neq i'\), and the two terms are different by the power of \( s_{A,n} \) (their ratio is exactly \( s_{A,n}^{i-i'} \)), and this implies that one term dominates the other. Thus, we see that the non-codominance property holds in Case 2 too.

Since the set \( H_2 \) is finite and non empty, we may consider \((k, j, i) \in H_2\) such that

\[
v_{b,1,0}(s_{A,n} + \tau) \sim (s_{A,n} + \tau)^i e^{\frac{\tau}{2}} (k-j) \tilde{C}_{k,j,i} A^{k-1} B^{k-1-j} e^{(1 - \frac{k}{2})s_{A,n}}
\]

as \( n \to \infty \), with \( \tilde{C}_{k,j,i} \neq 0 \). From the growth factor \( e^{\frac{\tau}{2}} \), the coordinate \( v_{b,1,0}(s_{A,n} + \tau) \) will grow, and a contradiction follows as usual.

Of course, our argument in Cases 1 and 2 is formal, however, it can be made rigorous as usual, like we did at the end of Part 1 of Subsection \( 3.2 \) in the case where \( m = 4 \). This finishes the proof of item (i) in Proposition \( 5.1 \).

- Proof of item (ii) of Proposition \( 5.1 \)
The result comes from a small modification of the argument of the proof of item (i). Assume by contradiction that \( \theta \) defined in (5.3) is zero. All the argument of the proof of item (i) holds here, and we naturally fall in Case 1. In particular (5.11) holds and the finite set \( H_1 \) defined in (5.12) is non empty. It remains just to check the non dominance property stated in Lemma 5.2. Let us consider \( (k,j) \) and \((k',j')\) with \( C_{k,j}C_{k',j'} \neq 0, m \leq k, k' \leq 2m - 3, 0 \leq j \leq k - 1, 0 \leq j' \leq k' - 1, \) and \((k,j) \neq (k',j')\), and show that either \((k-j)C_{k,j}A^{k-1}B_n^{k-1-j}e^{(1-\frac{k}{2})s_{A,n}}\) or \((k'-j')C_{k',j'}A^{k'-1}B_n^{k'-1-j'}e^{(1-\frac{k'}{2})s_{A,n}}\) dominates the other.

If \( k \neq k' \), then this is clear from (5.14). If \( k = k' \), then, recalling that \((k,j) \neq (k',j')\), we necessarily see that \( j \neq j' \), hence the power of \( B_n \) is not the same in the two terms. Since \( B_n \to 0 \) from (5.4), one term dominates the other. Thus, Lemma 5.2 holds here too, and one can carry on the argument of the proof of item (i) to derive that (5.13) holds, which yields a contradiction from the exponential factor. Thus, \( \theta \neq 0 \).


- Proof of item (iii) of Proposition 5.1

In this step, we explain how to derive item (iii) of Proposition 5.1 from the proof of item (i).

As announced earlier, our argument is a small adaptation of the argument already used for the proof of item (i). The key idea for the adaptation was already mentioned in the remark following the proof of Lemma 5.2, having a non rational \( \theta \) is a too strong condition to guarantee non-codominance. according to that remark and to the two cases mentioned in the proof of item (i), we immediately see that \( \theta \) should avoid the following two sets, in order for the contradiction argument to work:

\[
E_1 = \left[0, \frac{m - 2}{2(m - 1)}\right) \cap \left\{ \frac{k' - k}{2(k - j) - (k' - j')} \right\} \quad m \leq k, k' \leq 2m - 3, \quad (5.18)
\]

\[
E_2 = \left[ \frac{m - 2}{2(m - 1)}, \frac{1}{2} \right) \cap \left\{ \frac{k' - k}{2(k - j) - (k' - j')} \right\} \quad m \leq k, k' \leq 2m - 2, \quad (5.19)
\]

\[
0 \leq j \leq k - 1, \quad 0 \leq j' \leq k' - 1, \quad (k,j) \neq (k',j') \text{ and } k - j \neq k' - j'.
\]

This concludes the proof of Proposition 5.1

\[
\square
\]

**Step 3: Polynomial corrections to the exponential decay of \( B_n \).**

In Step 2, we showed that

\[
B_n = e^{-\theta s_{A,n}} \varphi_n \text{ where } \varphi_n = e^{o(s_{A,n})} \text{ as } n \to \infty, \quad (5.20)
\]

for some \( \theta \in (0, \frac{1}{2}] \cap \mathbb{Q} \) enjoying only a finite number of values. As one recalls from the proof, our starting point in the proof lays in the Taylor expansion of the solution \( v_0(y,s) \) provided by Proposition 1. As that proposition allows the existence of “resonant” terms, of the type \( s't' e^{(1-\frac{k}{2})s_{A,n}} \), it is natural to expect that the \( o(s_{A,n}) \) in (5.20) is of order \( \alpha \log s_{A,n} \) for some \( \alpha \in \mathbb{R} \), resulting in \( \varphi_n \sim s_{A,n}^{\alpha} \). This is precisely the aim of this step. Let us then assume that for up some subsequence denoted the same, we have

\[
\frac{\log \varphi_n}{\log s_{A,n}} \to \alpha \in [-\infty, +\infty]. \quad (5.21)
\]

We claim the following:

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Proposition 5.3.

(i) If \( \theta \in [0, \frac{m-2}{2(m-1)}] \), then \( \alpha = 0 \).

(ii) If \( \theta \in \left[ \frac{m-2}{2(m-1)}, \frac{1}{2} \right] \), then \( \alpha \in E_3 \) defined by

\[
E_3 \equiv \left\{ \frac{i' - i}{k - j - (k' - j')} \mid m \leq k, k' \leq M, 0 \leq j \leq k - 1, 0 \leq j' \leq k' - 1, \right.
\]
\[
0 \leq i \leq i_k, 0 \leq i' \leq i_k, (k, j, i) \neq (k', j', i') \text{ and } k - j \neq k' - j' \bigg\},
\]

where \( M \) is defined in (5.15) and \( i_k \) right after (5.17).

Proof. We proceed in 2 steps:

- We first show that \( \alpha \) is finite.
- Then, we show that \( \alpha \) enjoys a finite number of rational values.

- **Proof of the fact that \( \alpha \) is finite.**

Let us assume by contradiction that

\[
\log \frac{\varphi_n}{s_{A,n}} \to -\infty \text{ or } \log \frac{\varphi_n}{s_{A,n}} \to \infty \text{ as } n \to \infty.
\]

for a subsequence still denoted the same. In particular, this implies that

\[
\varphi_n \to 0 \text{ or } \varphi_n \to \infty, \text{ as } n \to \infty.
\]

Our idea is to follow the pattern of the proof of item (i) of Proposition 5.1, where we proved that \( \theta \) is rational. Naturally, we consider the two cases mentioned in that step.

**Case 1:** \( \theta \in [0, \frac{m-2}{2(m-1)}] \). Even though \( \theta \) is rational this time, the expansion (5.11) remains valid with the same finite set \( H_1 \) defined in (5.12), which is non empty, for the same reason. If we can show the non-codominance property as in Lemma 5.2, then, we are done. Let us then prove that lemma, in this new setting. Of course, we need a different argument, since \( \theta \) is rational this time, and the issue concerns the following order term, involving the limit \( \alpha \) defined in (5.21). Consider then two terms in the expansion (5.11) of \( v_{n,1,0}(s_{A,n} + \tau) \), say \((k - j)C_{k,j}A^{k-1}B_{n}^{k-1-j}e^{(1-\frac{k}{2})s_{A,n}}\) and \((k' - j')C_{k',j'}A^{k'-1}B_{n}^{k'-1-j'}e^{(1-\frac{k'}{2})s_{A,n}}\) with \(C_{k,j}C_{k',j'} \neq 0, m \leq k, k' \leq 2m - 3, 0 \leq j \leq k - 1, 0 \leq j' \leq k' - 1, \text{ and } (k,j) \neq (k',j')\). By definition (5.20) of \( \varphi_n \), we see that

\[
(k - j)C_{k,j}A^{k-1}B_{n}^{k-1-j}e^{(1-\frac{k}{2})s_{A,n}} = (k - j)C_{k,j}A^{k-1}e^{(1-\frac{k}{2}-(k-1-j)\theta)}s_{A,n} \varphi_n^{k-1-j},
\]

with a similar expansion for \((k',j')\).

If \(1 - \frac{k}{2} - (k - 1 - j)\theta \neq 1 - \frac{k'}{2} - (k' - 1 - j')\theta\), recalling that \( \varphi_n = e^{\alpha(s_{A,n})} \) as stated in (5.20), we see that one of the two terms dominates the other. Now, if \(1 - \frac{k}{2} - (k - 1 - j)\theta = 1 - \frac{k'}{2} - (k' - 1 - j')\theta\), and this may occur since \( \theta \) is rational, recalling that \((k,j) \neq (k',j')\) we necessarily have \(k - 1 - j \neq k' - 1 - j'\). In other words, the power of \( \varphi_n \) is not the same in the two terms. Since \( \varphi_n \to 0 \) or \( \varphi_n \to \infty \), as stated in (5.24), we see from the description (5.25) that one of the two terms has to dominate the other. Thus, the non-codominance property holds, and a contradiction follows as in Case 1 of the proof of item (i) of Proposition 5.1.
Case 2: $\theta \in \left[\frac{m-2}{2(m-1)}, \frac{1}{2}\right]$. Again, the argument of Case 2 in the proof of item (i) of Proposition 5.1 holds, and we may derive an expansion of $v_{n,1,0}(s_{A,n} + \tau)$ like in (5.16), with “resonant” terms. The finite set $H_2$ is non empty, for the same reason. If we can show the non-codominance property as before, then we are done.

Consider then two different $(k,j,i)$ and $(k',j',i')$ in $H_2$, and let us show that either $(s_{A,n} + \tau)^i e^{s_{A,n} (1-\frac{1}{2})} B_n^{k-1-j} s_{A,n}^i \varphi_n^{k-1-j}$ or $(s_{A,n} + \tau)^j e^{s_{A,n} (1-\frac{1}{2})} B_n^{k'-1-j'} s_{A,n}^j \varphi_n^{k'-1-j'}$ dominates the other, for $n$ large. From the reduction we showed in Case 2 of the proof of item (i) of Proposition 5.1 above, it is enough to check the dominance at $\tau = 0$. Note first from (5.20) that

$$s_{A,n}^i e^{(1-\frac{1}{2})s_{A,n}} B_n^{k-1-j} = s_{A,n}^i e^{(1-\frac{1}{2} - (k-1-j)\theta)} s_{A,n} \varphi_n^{k-1-j},$$

as $n \to \infty$, with a similar estimate for $(k',j',i')$.

If $1 - \frac{1}{2} - (k - 1 - j)\theta \neq 1 - \frac{k'}{2} - (k' - 1 - j')\theta$, recalling that $\varphi_n = e^{o(s_{A,n})}$ from (5.20), we see that one term dominates the other.

Assume then that $1 - \frac{1}{2} - (k - 1 - j)\theta = 1 - \frac{k'}{2} - (k' - 1 - j')\theta$.

If $k - j = k' - j'$, then $k = k'$, and since $(k,j,i) \neq (k',j',i')$, it follows that $i \neq i'$.

From the expression (5.26), we see that the powers of $s_{A,n}$ are different, hence, one term dominates the other.

Now, if $k-j \neq k'-j'$, making the ratio between the two terms, we find $s_{A,n}^i e^{(1-\frac{1}{2} - (k-1-j)\theta)} s_{A,n} \varphi_n^{k-1-j}$. Using (5.20) and (5.24), we see that one term dominates the other.

Of course, our argument in Cases 1 and 2 is formal, however, it can be made rigorous as usual, like we did at the end of Part 1 of Subsection 4.2 in the case $m = 4$. Thus, we have just proved that the parameter $\alpha$ defined in (5.21) is finite.

- Conclusion of the proof of Proposition 5.3

In the previous step, we assumed that $\alpha = \pm \infty$ and reached a contradiction. In fact, a careful check reveals that the contradiction can be reached in Case 1, for any $\alpha \neq 0$, whereas in Case 2, we simply need $\alpha$ to avoid the set $E_3$ defined in (5.22). This concludes the proof of Proposition 5.3.

Step 4: One further refinement in the behavior of $B_n$

So far, thanks to Propositions 5.1 and 5.3, we have proved that

$$B_n = \exp(-\theta s_{A,n}) s_{A,n}^\alpha \psi_n$$

for some $\psi_n = s_{A,n}^{o(1)}$ (5.27)

as $n \to \infty$. This is the aim of this step:

Proposition 5.4. Up to a subsequence, $\psi_n$ converges to some $L > 0$, where $L$ enjoys a finite number of values, all solutions of polynomials whose coefficients depend on the coefficients that arise in the Taylor expansion of $v_0(y,s)$.

Proof. Here again, we crucially use the geometric transformation introduced in Step 2 of Section 3. We proceed in 2 steps:

- In Step (i), we show that $\psi_n$ is bounded away from 0 and from infinity.

- In Step (ii), we show that up to a subsequence, $\psi_n$ converges to some $L > 0$, which is a solution of a polynomial whose coefficients depend on $C_{k,j}$.

- Step (i): $\psi_n$ is bounded away from 0 and from infinity.
We proceed by contradiction, and assume that for a subsequence (still denoted the same), we have
\[ \psi_n \to 0 \text{ or } \psi_n \to \infty \text{ as } n \to \infty. \] (5.28)

As before, we follow the strategy of the proof of item (i) of Proposition 5.1 with its two cases.

Starting by Case 1, where \( \theta \in \left[0, \frac{m-2}{2(m-1)}\right) \), we still see that (5.11) holds, with \( H_1 \) which is still a non empty finite set. It remains only to prove the non codominance property. Consider then two terms in the expansion (5.11) of \( v_{b_{1,0}}(s_{A,n} + \tau) \), say \( (k-j)C_{k,j}A^{k-1-j}e^{(1-\frac{j}{k})s_{A,n}} \) and \( (k'-j')C_{k',j'}A^{k'-1-j'}e^{(1-\frac{j'}{k'})s_{A,n}} \) with \( C_{k,j}C_{k',j'} \neq 0, m \leq k,k' \leq 2m-3, 0 \leq j \leq k-1, 0 \leq j' \leq k'-1, \) and \( (k,j) \neq (k',j') \). Using (5.27), and recalling that \( \alpha = 0 \) from Proposition 5.1, we see that
\[ (k-j)C_{k,j}A^{k-1-j}e^{(1-\frac{j}{k})s_{A,n}} = (k-j)C_{k,j}A^{k-1-j}e^{(1-\frac{j}{k})s_{A,n}+\psi_n^{k-1-j}}, \] (5.29)
with a similar expansion for \( (k',j') \).

If \( 1 - \frac{k}{2} - (k-1-j)\theta \neq 1 - \frac{k'}{2} - (k'-1-j')\theta \), recalling that \( \psi_n = s_{A,n}^{\alpha(1)} = e^{o(\log s_{A,n})} \) as stated in (5.27), we see that one of the two terms dominates the other.

Now, if \( 1 - \frac{k}{2} - (k-1-j)\theta = 1 - \frac{k'}{2} - (k'-1-j')\theta \), recalling that \( (k,j) \neq (k',j') \) we necessarily have \( k-1-j \neq k'-1-j' \). In other words, the power of \( \psi_n \) is not the same in the two terms. Since \( \psi_n \to 0 \) or \( \psi_n \to \infty \), as stated in (5.28), we see from the description (5.29) that one of the two terms has to dominate the other. Thus, co-dominance holds in this context, and a contradiction follows as in Case 1 of the proof of item (i) of Proposition 5.1.

Now, moving to Case 2, where \( \theta \in \left[\frac{m-2}{2(m-1)}, \frac{1}{2}\right] \), we may consider two terms in the expansion (5.13) of \( v_{b_{1,0}}(s_{A,n} + \tau) \), say \( (s_{A,n} + \tau)^i e^{\frac{x}{2}} e^{(1-\frac{j}{k})s_{A,n}B^{k-1-j}} \) and \( (s_{A,n} + \tau)^i e^{\frac{x}{2}} e^{(1-\frac{j}{k'})s_{A,n}B^{k'-1-j'}} \), where \( (k,j,i) \) and \( (k',j',i') \) belong to \( H_2 \) defined in (5.17), and prove that one dominates the other. From the reduction we did in Case 2 of the proof of item (i) of Proposition 5.1 we may check the dominance only at \( \tau = 0 \). Using (5.24), we see that
\[ s_{A,n}^i e^{(1-\frac{j}{k})s_{A,n}B^{k-1-j}} = e^{(1-\frac{j}{k}-\alpha(1-j)\theta)s_{A,n}} s_{A,n}^i + \alpha(k-1-j) \psi_n^{k-1-j} \]
as \( n \to \infty \), with a similar estimate for \( (k',j',i') \).

If \( 1 - \frac{k}{2} - (k-1-j)\theta \neq 1 - \frac{k'}{2} - (k'-1-j')\theta \), recalling that \( \psi_n = s_{A,n}^{\alpha(1)} = e^{o(\log s_{A,n})} \) as stated in (5.27), we see that one of the two terms dominates the other.

Assume then that \( 1 - \frac{k}{2} - (k-1-j)\theta = 1 - \frac{k'}{2} - (k'-1-j')\theta \).

If \( i + \alpha(k-1-j) = i' + (k'-1-j') \alpha \), then, using again the fact that \( \psi_n = s_{A,n}^{\alpha(1)} = e^{o(\log s_{A,n})} \), we see that the power of \( s_{A,n} \) is different between the two terms, hence one term dominates the other.

Now, if \( i + \alpha(k-1-j) = i' + (k'-1-j') \alpha \), then, necessarily \( k-1-j \neq k'-1-j' \), otherwise, \( (k,j,i) = (k',j',i') \). Therefore, the power of \( s_{A,n} \) is the same in the two terms, unlike the power of \( \psi_n \). Recalling that \( \psi_n \to 0 \) or \( \psi_n \to \infty \), we see that one term dominates the other.

**Step (ii):** \( \psi_n \) converges (up to a subsequence)

From Step (i), we may assume that for some subsequence (still denoted the same), we have
\[ \psi_n \to A^{2\theta} L \text{ as } n \to \infty \] (5.30)

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for some $L > 0$ (we write the limit in (5.30) as $A^{2g} L$ and not $L$, since in this form, $L$ will be shown to be a solution of a polynomial whose coefficients are independent of $A$).

We will distinguish two cases as in the proof of item (i) of Proposition 5.1.

**Case 1:** $\theta \in \left[0, \frac{m-2}{2(m-1)}\right)$. Note that $\alpha = 0$ by Proposition 5.3. In this case, we see that (5.11) still holds, with $H_1$ which is still a non-empty finite set. However, this time, we will have codominance, as we show in the following. Since the number of terms is finite in the expansion (5.11), we may consider $(\bar{k}, \bar{j}) \in H_1$ defined in (5.12) such that the term corresponding to this parameter dominates all the others. Using (5.11) and (5.27), this term reads

$$e^{(1-\frac{k}{2})s_{A,n}} A^{k-1}(\bar{k} - \bar{j}) C_{k,j} B_n^{k-1-j} = A^{k-1} C_{k,j} (\bar{k} - \bar{j}) e^{(1-\frac{k}{2} - (\bar{k} - \bar{j})\theta)s_{A,n} \psi_n^{k-1-j}}. \quad (5.31)$$

As we wrote earlier, we expect here to have codominance. Let us then characterize the set $\bar{E}$ of all $(k, j) \in H_1$ such that the corresponding term is of the same size as the term corresponding to $(\bar{k}, \bar{j})$. From (5.31) and (5.30), this means that

$$1 - \frac{k}{2} - (\bar{k} - \bar{j})\theta = 1 - \frac{k}{2} - (k - 1 - j)\theta. \quad (5.32)$$

This way, we may keep only the dominant terms in (5.11), namely those coming from $\bar{E}$, and write

$$v_{b,1,0}(s_{A,n} + \tau) = e^{\frac{\tau}{2}} \left( \sum_{(k,j) \in \bar{E}} e^{(1-\frac{k}{2})s_{A,n}} A^{k-1}(k - j) C_{k,j} B_n^{k-1-j} + O(s_{A,n} e^{(2-m)s_{A,n}}) \right).$$

Using (5.31) and (5.32), together with the convergence (5.30), we derive that

$$v_{b,1,0}(s_{A,n} + \tau) = e^{\frac{\tau}{2}} e^{(1-\frac{k}{2} - (\bar{k} - \bar{j})\theta)s_{A,n}} \left( \sum_{(k,j) \in \bar{E}} A^{k-1}(k - j) C_{k,j} (A^{2g} L)^{k-1-j} + o(1) \right)$$

$$= e^{\frac{\tau}{2}} e^{(1-\frac{k}{2} - (\bar{k} - \bar{j})\theta)s_{A,n}} A^{k-1+2g(\bar{k} - \bar{j})} \left( \sum_{(k,j) \in \bar{E}} (k - j) C_{k,j} L^{k-1-j} + o(1) \right)$$

as $n \to \infty$. From the growth factor $e^{\frac{\tau}{2}}$, this implies that

$$\sum_{(k,j) \in \bar{E}} (k - j) C_{k,j} L^{k-1-j} = 0.$$

Since $L \neq 0$ and $C_{k,j} \neq 0$, for all $(k, j) \in \bar{E}$ (remember that $\bar{E} \subset H_1$ defined in (5.12)), this sum contains at least two terms, and this is precisely the desired polynomial relation. Remember that $\bar{E} \neq \emptyset$, since it contains $(\bar{k}, \bar{j})$, and that $C_{k,j} \neq 0$, for any $(k, j) \in \bar{E}$, since $\bar{E} \subset H_1$ defined in (5.12). Note that the degree of this polynomial is bounded by $k - 1 \leq 2m - 4$ by definition (5.12) of $H_1$, hence, we have at most $2m - 4$ possible values for $L$.

**Case 2:** $\theta \in \left[\frac{m-2}{2(m-1)}, \frac{1}{2}\right]$. We are then in the framework of Case 2 of the proof of item (i) of Proposition 5.1 above. In particular, the finite set $H_2$ is still non-empty, for the same reason. However, we may have codominance in this context. Since the number of terms is finite in the expansion (5.16), we may consider $(\bar{k}, \bar{j}, \bar{i}) \in H_2$ defined in (5.17) such
that the term corresponding to this parameter dominates all the others. Following what we wrote in Case 2 of the proof of item (i) of Proposition 5.1 above, we may just take \( \tau \to 0 \) to discuss codominance issues. Using (5.16) and (5.27), this term reads as follows, when \( \tau = 0 \):

\[
\begin{align*}
& s_{A,n} \bar{t} e^{(1-\frac{k}{2}) s_{A,n}} A^{k-1}(\bar{k} - \bar{j}) \tilde{C}_{\bar{k},\bar{j},i} \bar{B}_n^{k-1-j} \\
= & e^{(1-\frac{k}{2} - \theta(\bar{k} - \bar{j})) s_{A,n}} s_{A,n} \bar{t} e^{i(\bar{k} - \bar{j})} A^{k-1}(\bar{k} - \bar{j}) \tilde{C}_{\bar{k},\bar{j},i} \bar{B}_n^{k-1-j}.
\end{align*}
\]

Let us then characterize the set \( \bar{E} \) of all \( (k, j, i) \in H_2 \) such that the corresponding term in (5.16) is of the same size as the term corresponding to \( (\bar{k}, \bar{j}, \bar{i}) \). From (5.33) and (5.30), we see that we need to have

\[
1 - \frac{k}{2} - \theta(\bar{k} - \bar{j}) = 1 - \frac{k}{2} - \theta(k - 1 - j) \quad \text{and} \quad \bar{i} + \alpha(\bar{k} - 1 - \bar{j}) = i + \alpha(k - 1 - j).
\]

This way, we keep only the dominant terms in (5.16), and we write

\[
v_{b,1,0}(s_{A,n} + \tau) = e^{\bar{t} z} \left( \sum_{(k,j,i) \in \bar{E}} (s_{A,n} + \tau) e^{(1-\frac{k}{2}) s_{A,n}} A^{k-1}(k - j) \tilde{C}_{k,j,i} B_n^{k-1-j} \right)
\]

\[
+ O(s_{A,n}^{\gamma} e^{(\frac{1}{2} - \frac{1}{4}) s_{A,n}}).
\]

Using (5.33) and (5.34), together with (5.30), we write for all \( \tau \in [0, (2M - 2)s_{A,n}] \) (note that this interval refers to our discussion in Case 2 of the proof of item (i) of Proposition 5.1):

\[
v_{b,1,0}(s_{A,n} + \tau) = e^{\bar{t} z} e^{(1-\frac{k}{2} - \theta(\bar{k} - \bar{j})) s_{A,n}} s_{A,n} \bar{t} e^{i(\bar{k} - \bar{j})} A^{k-1+\theta(\bar{k} - \bar{j})}
\]

\[
\times \left( \sum_{(k,j,i) \in \bar{E}} \left( 1 + \frac{\tau}{s_{A,n}} \right)^i \right) (k - j) \tilde{C}_{k,j,i} L^{k-1-j} + o(1) \right).\]

Now, for any \( \delta_0 > 0 \), we consider \( \tau_n(\delta_0) > 0 \) such that

\[
e^{\bar{t} z} e^{(1-\frac{k}{2} - \theta(\bar{k} - \bar{j})) s_{A,n}} s_{A,n} \bar{t} e^{i(\bar{k} - \bar{j})} A^{k-1+\theta(\bar{k} - \bar{j})} = \delta_0.
\]

Clearly, it holds that

\[
\tau_n(\delta_0) \sim [\bar{k} - 2 + \theta(\bar{k} - 1 - \bar{j})] s_{A,n} \quad \text{as} \quad n \to \infty.
\]

Therefore, we see that

\[
v_{b,1,0}(s_{A,n} + \tau_n(\delta_0)) = \delta_0 \left( \sum_{(k,j,i) \in \bar{E}} (k - j) \tilde{C}_{k,j,i} L^{k-1-j} + o(1) \right).\]

This forces the coefficient of \( \delta_0 \) to be zero:

\[
\sum_{(k,j,i) \in \bar{E}} (k - j) \tilde{C}_{k,j,i} L^{k-1-j} = 0,
\]

otherwise, \( v_{b,1,0}(s_{A,n} + \tau_n(\delta_0)) \) will be large as in (4.21), which leads to a contradiction. Since \( L \neq 0 \) and \( \tilde{C}_{k,j,i} \neq 0 \), for all \( (k, j, i) \in \bar{E} \) (remember that \( \bar{E} \subset H_2 \) defined in (5.17)),

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this identity contains at least two terms, and this is precisely the desired polynomial relation.

This concludes the proof of Proposition 5.4 and finishes Part 1 dedicated to the super-quadratic case.

**Part 2: The quadratic regime**

In this part, we consider the quadratic regime, where

\[ a_{n,1} \sim L a_{n,2}^2 \quad \text{as} \quad n \to \infty, \tag{5.35} \]

for some \( L > 0 \), and show that \( L \) satisfies a polynomial equation whose coefficients are given by the Taylor expansion of the solution.

Note first from the notation (4.11) and the definition (4.9) of \( B_n \) that we have

\[ B_n \sim L A e^{-\frac{\theta A}{2}} \quad \text{as} \quad n \to \infty. \]

The proof is in fact a simple adaptation of our argument in the super-quadratic regimes, given in Part 1. Let us then follow that part step by step, and see what changes.

In Part 1, the outcome of Step 1 is the following:

\[ C_{m+1,m} + LC_{m,m-2} = 0. \tag{5.36} \]

If \( C_{m,m-2} \neq 0 \), then, we have our polynomial and we are done (in fact, we have more, in the sense that \( L \) enjoys only one value: \( -C_{m+1,m}/C_{m,m-2} \)). If \( C_{m,m-2} = 0 \), then we also have \( C_{m,m-3} = 0 \), because the multilinear form in (1.10) is nonpositive. In other words, we have exactly the same conclusions as in Step 1 of Part 1 in the super-quadratic case (in fact, we have more, since \( C_{m+1,m} = 0 \) from (5.36)). For short, we can carry on all the next steps of the super-quadratic case up to to the end of Part 1, and see that \( L \) satisfies a polynomial equations. For the reader’s convenience, we would like to mention that hypothesis (5.35) makes many steps either non relevant or trivial:

- In Step 2 of Part 1, \( \theta = \frac{1}{2} \) which makes Proposition 5.1 non relevant. Moreover, (5.35) is stronger than (5.5).
- In Step 3, estimate (5.35) is stronger than (5.20), and \( \alpha \) defined in (5.21) is zero. Accordingly, Proposition 5.3 is non relevant.
- As for Step 4, again, estimate (5.35) is stronger than (5.27), and the first assertion of Proposition 5.4 is clear (\( \psi_n \to L \) as \( n \to \infty \)).
- In conclusion, only the second assertion of Proposition 5.4 remains relevant, and provides us with the polynomial relation for \( L \).

**Part 3: Conclusion of the proof of Theorem 3**

From the study of the super-quadratic case in Part 1, and also the quadratic case in Part 2, we see that either we are in the subquadratic case \( a_{n,1} = o(a_{n,2})^2 \), or

\[ B_n \sim L A^{2\theta} e^{-\theta s_{A,n}} s_{A,n}^\alpha \quad \text{as} \quad n \to \infty, \]

where \( \theta \) and \( \alpha \) enjoy only a finite set of rational values, and \( L > 0 \) is a solution of a polynomial equations whose coefficients depend on the Taylor expansion of the solution. By definitions (4.11) and (4.9) of \( s_{A,n} \) and \( B_n \), we see that

\[ a_{n,1} \sim L_0 \alpha n,^{2\theta+1} |2 \log a_{n,2}|^\alpha, \]

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with $2\theta + 1 \in (0, 1]$, which is the desired estimate in Theorem 3. The set where $2\theta + 1$ lives directly follows from item (iii) in Proposition 5.1.

This concludes the proof of Theorem 3.

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