A GEOMETRIC CLASSIFICATION OF THE PATH COMPONENTS OF THE SPACE OF LOCALLY STABLE MAPS $S^3 \to \mathbb{R}^4$

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Abstract. Locally stable maps $S^3 \to \mathbb{R}^4$ are classified up to homotopy through locally stable maps. The equivalence class of a map $f$ is determined by three invariants: the isotopy class $\sigma(f)$ of its framed singularity link, the generalized normal degree $\nu(f)$ of any extension of $f$ to a locally stable map of the 4-disk into $\mathbb{R}^5$. Relations between the invariants are described, and it is proved that for any $\sigma$, $\nu$, and $\kappa$ which satisfy these relations, there exists a map $f : S^3 \to \mathbb{R}^4$ with $\sigma(f) = \sigma$, $\nu(f) = \nu$, and $\kappa(f) = \kappa$. It follows in particular that every framed link in $S^3$ is the singularity set of some locally stable map into $\mathbb{R}^4$.

1. Introduction

This paper concerns locally stable codimension one maps from spheres into Euclidean spaces. Locally stable maps are maps with stable map germs, see Section 2. In the nice dimensions of Mather [17], in particular in the case of codimension one maps of spheres of dimensions less than 15, locally stable maps constitute an open and dense subset of $C^\infty(S^k, \mathbb{R}^n)$, the space of smooth maps $S^k \to \mathbb{R}^n$. A homotopy through locally stable maps will be called an $\text{LS}$-homotopy.

Every locally stable map $S^1 \to \mathbb{R}^2$ is an immersion, and the classification of such maps up to $\text{LS}$-homotopy reduces to the well-known classification of immersed plane curves up to regular homotopy, see Whitney [22]. In the case of maps $S^2 \to \mathbb{R}^3$, a locally stable map need not be an immersion. It may have a finite number of singularities, where the rank of the differential equals 1 (so called Whitney umbrellas). Juhász [13] proved that two singular locally stable maps from a closed connected surface into $\mathbb{R}^3$ are $\text{LS}$-homotopic if and only if they have the same number of singularities. The results of [13] have been generalized by Juhász [14] to locally stable maps from any closed smooth $n$-manifold to $\mathbb{R}^{2n-1}$ when $n \neq 3$.

In the present paper we classify locally stable maps $S^3 \to \mathbb{R}^4$ up to $\text{LS}$-homotopy. The classification is more refined than the corresponding classification for maps $S^k \to \mathbb{R}^{k+1}$ when $k \leq 2$. It involves an extension of geometrically defined regular homotopy invariants of immersions $S^3 \to \mathbb{R}^4$ as well as link theory in $S^3$.

A locally stable map $f : S^3 \to \mathbb{R}^4$ is an immersion outside a closed 1-dimensional submanifold $\Sigma(f)$ of $S^3$. Along $\Sigma(f)$ the rank of the differential $df$ of $f$ equals 2. (In fact the restriction of $f$ to a neighborhood of $\Sigma(f)$ can be described as a 1-parameter family of Whitney umbrellas, see Section 2.) The kernel bundle of $df$ is a trivial line bundle.

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bundle along \( \Sigma(f) \) which is nowhere tangential to \( \Sigma(f) \), see Proposition 2.3. Hence \( \Sigma(f) \) is a framed link in \( S^3 \).

Using an idea inspired by a construction in Goryunov [5], we extend the definition of the normal degree for immersions \( S^3 \to \mathbb{R}^4 \) so that it applies also to locally stable maps \( S^3 \to \mathbb{R}^4 \), see Definition 3.3. Following Ekholm and Szücs [5], we consider the algebraic number of cusps (\( \Sigma^1 \)-points) of a locally stable map \( (D^4, \partial D^4) \to (\mathbb{R}^5, \mathbb{R}^4) \) which extends a given locally stable map \( \partial D^4 = S^3 \to \mathbb{R}^4 \) and which agrees with the product extension in some collar neighborhood of \( S^3 \) in \( D^4 \), see Definition 3.3.

To state the classification theorem we introduce the following notation. For a locally stable map \( f : S^3 \to \mathbb{R}^4 \) let \( \sigma(f) \) be the isotopy class of the framed singularity link \( \Sigma(f) \) of \( f \) (if \( f \) is an immersion let \( \sigma(f) \) be the class of the empty link), let \( \nu(f) \) be the (generalized) normal degree of \( f \), and let \( \kappa(f) \) be the algebraic number of cusps of \( f \) to the disk as discussed above.

**Theorem 1.1.** Two locally stable maps \( f, g : S^3 \to \mathbb{R}^4 \) are homotopic through locally stable maps if and only if \( \sigma(f) = \sigma(g), \nu(f) = \nu(g) \), and \( \kappa(f) = \kappa(g) \).

It follows from Theorem 1.1 that two immersions \( f, g : S^3 \to \mathbb{R}^4 \) are regularly homotopic if and only if \( \nu(f) = \nu(g) \) and \( \kappa(f) = \kappa(g) \). This result can also be derived from [4] or [10] together with [5]. When restricted to immersions, the invariant \( \kappa \) takes values in \( 2\mathbb{Z} \), see [3]. Furthermore, if \( f : S^3 \to \mathbb{R}^4 \) is an immersion, then \( \nu(f) \) and \( \frac{1}{2}\kappa(f) \) have different parity, see Proposition 6.1.

**Theorem 1.2.** Let \( \sigma \) be an isotopy class of framed links in \( S^3 \). Then there exists a locally stable map \( f : S^3 \to \mathbb{R}^4 \) such that \( \sigma(f) = \sigma \). Moreover, for any locally stable map \( g : S^3 \to \mathbb{R}^4 \) with \( \sigma(g) = \sigma \) there exists an immersion \( h : S^3 \to \mathbb{R}^4 \) such that \( g \) is homotopic through locally stable maps to the connected sum \( f \# h \). Hence there are integers \( a \) and \( b \) such that

\[
\nu(g) = \nu(f) + a \quad \text{and} \quad \kappa(g) = \kappa(f) + 2a + 4b.
\]

Conversely, if \( a \) and \( b \) are any integers, then there exists a locally stable map \( g : S^3 \to \mathbb{R}^4 \) that satisfies \( \sigma(g) = \sigma \) and Equations (1.1).

Equations (1.1) imply that the parity of \( \kappa(g) \) and the residue class of \( 2\nu(g) + \kappa(g) \) modulo 4 depend only on \( \sigma \). We show that the parity equals the sum of the modulo 2 self linking numbers of all components of the framed link \( \Sigma(g) \), see Proposition 3.6. On immersions the residue class equals 2 modulo 4, see Proposition 6.1. The behavior of the residue class on locally stable maps with nonempty singularity sets will be studied in a forthcoming paper.

The paper is organized as follows. Section 2 contains a short background to stable maps and singularities. Section 3 discusses the invariants \( \nu \) and \( \kappa \). In Section 4 we prove that every framed link in \( S^3 \) can be realized as a framed singularity link of a locally stable map into \( \mathbb{R}^4 \). Sections 5 and 6 are devoted to the proofs of Theorems 1.1 and 1.2.

2. Stable maps and singularities

This section contains background material from the theory of stable maps. In Subsections 2.1 and 2.2 we introduce the concepts of stability and local stability of maps, and review the Thom-Boardman theory of singularities. For convenience most definitions are stated in greater generality than later needed. In Subsections 2.3 and 2.4 we focus on stable and locally stable maps \( S^3 \to \mathbb{R}^4 \), and give a detailed description of
their properties. In particular we define the invariant $\sigma$ mentioned in Section 1. When not included, proofs of the statements in Subsections 2.1,2.4 can be found in [7]. In Subsection 2.2 we fix metric and orientation conventions.

2.1. Stable and locally stable maps. Let $M$ and $N$ be smooth manifolds, with $M$ compact, and denote by $C^\infty(M,N)$ the space of smooth maps $M \to N$ equipped with the Whitney $C^\infty$ topology. We assume, unless otherwise is explicitly mentioned, all manifolds and maps to be smooth.

Recall that a map $f : M \to N$ is stable if each map in a neighborhood of $f$ in $C^\infty(M,N)$ is a composition $\psi \circ f \circ \varphi$ with $\varphi : M \to M$ and $\psi : N \to N$ being diffeomorphisms. Stable maps form an open subset of $C^\infty(M,N)$, and in the nice dimensions of Mather [17] the set of stable maps is dense.

Mather [16] proved that a map $f : M \to N$ is stable if and only if $f$ is infinitesimally stable, that is, if for each vector field $w$ along $f$ there are vector fields $u$ on $M$ and $v$ on $N$ such that

$$w = df \circ u + v \circ f.$$ 

We call a map $f : M \to N$ locally stable if its map germs are infinitesimally stable. That is, if for each $p \in M$ and each germ of a vector field $w$ along $f$ at $p$ there are germs of vector fields $u$ on $M$ at $p$ and $v$ on $N$ at $f(p)$ such that Equation (2.1) is satisfied in a neighborhood of $p$. By [16] every stable map is locally stable.

A homotopy $h_t : M \to N$ will be called an LS-homotopy if each $h_t$ is locally stable. LS-homotopies correspond to continuous paths in the space of locally stable maps, and locally stable maps $f,g : M \to N$ are LS-homotopic if there exists an LS-homotopy from $f$ to $g$.

2.2. Singularities. A point $p \in M$ is a singularity of a map $f : M \to N$ if the differential $df$ of $f$ has positive corank at $p$:

$$\text{rank} \, df(p) < \min\{\dim M, \dim N\}.$$ 

We call $f$ singular if the set of singularities of $f$ is nonempty.

The singularities of a map $f : M \to N$ are classified according to the corank of $df$ and the coranks of the differentials of restrictions of $f$ to submanifolds of singularities. For each integer $r$ let $\Sigma^r(f)$ be the set of points in $M$ at which $df$ has corank $r$. If $\Sigma^r(f)$ is a nonempty manifold consider the restriction $f|_{\Sigma^r(f)}$ and for each integer $s$ define $\Sigma^{r,s}(f)$ to be the set of points in $\Sigma^r(f)$ at which the differential $d(f|_{\Sigma^r(f)})$ has corank $s$. Analogously define for every sequence of integers $r_1, \ldots, r_k$ the set $\Sigma^{r_1,\ldots,r_k}(f)$ to be the set $\Sigma^{r_1}(f|_{\Sigma^{r_2,\ldots,r_k}(f)})$ provided that $\Sigma^{r_1,\ldots,r_k}(f)$ is defined and is a nonempty submanifold of $M$. Boardman [2] associated to every sequence of integers $r_1 \geq \cdots \geq r_k \geq 0$ a fiber subbundle $\Sigma^{r_1,\ldots,r_k}$ of the jet bundle $J^k(M,N)$ and proved for the maps $f$ in a residual subset of $C^\infty(M,N)$ that each jet extension $j^k f$ is transverse to every $\Sigma^{r_1,\ldots,r_k}$, and that $\Sigma^{r_1,\ldots,r_k}(f) = (j^k f)^{-1}(\Sigma^{r_1,\ldots,r_k})$.

The jet extensions of a locally stable map are transverse to every $\Sigma^{r_1,\ldots,r_k}$. Therefore the set of singularities of a locally stable map $f$ is the union of the manifolds $\Sigma^{r_1,\ldots,r_k}(f)$ with $r_1$ positive.

If $\dim M \leq \dim N$, then the restriction of a stable map $f : M \to N$ to each nonempty manifold $\Sigma^{r_1,\ldots,r_k}(f)$ with $r_k = 0$ is an immersion with normal crossings. Hence, if $\Sigma$ is shorthand for $\Sigma^{r_1,\ldots,r_k}(f)$, then for each $q \in N$ and each set of distinct points $p_1, \ldots, p_n \in f^{-1}(q) \cap \Sigma$ the linear subspaces $df(T_{p_1}\Sigma), \ldots, df(T_{p_n}\Sigma)$ of $T_q N$ are in
general position:

\[
\text{codim}\left(\bigcap_{j=1}^{n} df(T_{p_j}\Sigma)\right) = \sum_{j=1}^{n} \text{codim}(df(T_{p_j}\Sigma)).
\]

In particular, \(f|_{\Sigma}\) has transverse self intersections, and the cardinality of the preimage of any point in \(N\) under \(f|_{\Sigma}\) is bounded from above by \(\dim N/(\dim N - \dim \Sigma)\).

2.3. Stable and locally stable maps from \(S^3\) to \(\mathbb{R}^4\). Locally stable maps constitute an open and dense subset of \(C^\infty(S^3,\mathbb{R}^4)\). The singularities of a singular locally stable map \(f : S^3 \to \mathbb{R}^4\) are all of type \(\Sigma^{1,0}\). The set \(\Sigma^{1,0}(f)\) is a closed 1-dimensional submanifold of \(S^3\). From now on we denote \(\Sigma^{1,0}(f)\) by \(\Sigma(f)\) and call \(\Sigma(f)\) the link of singularities of \(f\).

**Proposition 2.1.** Let \(h_t : S^3 \to \mathbb{R}^4\) be an \(\text{LS}-\)homotopy. Then the links \(\Sigma(h_0)\) and \(\Sigma(h_1)\) are isotopic.

**Proof.** Apply [11] Theorem 20.2] to the homotopy \(j^1h_t : S^3 \to J^1(S^3,\mathbb{R}^4)\) and the (closure of) the submanifold \(\Sigma^1\) of \(J^1(S^3,\mathbb{R}^4)\). \(\square\)

A locally stable map \(S^3 \to \mathbb{R}^4\) is characterized by its normal forms in local coordinates. A map \(f : S^3 \to \mathbb{R}^4\) is locally stable if and only if for each \(p \in S^3\) there are charts \(\alpha : U \to \mathbb{R}^3\) on \(S^3\) centered at \(p\) and \(\beta : V \to \mathbb{R}^4\) on \(\mathbb{R}^4\) centered at \(f(p)\) such that \(f(U) \subset V\) and such that \(\beta \circ f \circ \alpha^{-1} : \alpha(U) \to \mathbb{R}^4\) has normal form

\[
(\beta \circ f \circ \alpha^{-1})(x, y, z) = \begin{cases} (x, y, z, 0) & \text{if } p \in S^3 \setminus \Sigma(f), \\ (x, y, y^2, yz) & \text{if } p \in \Sigma(f). \end{cases}
\]

Every locally stable map \(S^3 \to \mathbb{R}^4\) can be deformed into a stable map using an arbitrarily small \(\text{LS}-\)homotopy. The restrictions \(f|_{\Sigma(f)}\) and \(f|_{S^3 \setminus \Sigma(f)}\) of a stable map \(f : S^3 \to \mathbb{R}^4\) are immersions with normal crossings, and the images of these immersions intersect transversely. By (2.2), \(f|_{\Sigma(f)}\) is an embedding, and the preimage \(f^{-1}(q)\) of each \(q \in \mathbb{R}^4\) contains at most 4 points. In fact, if \(f^{-1}(q) \cap \Sigma(f) \neq \emptyset\), then \(f^{-1}(q)\) contains at most 2 points.

**Proposition 2.2.** Let \(f : S^3 \to \mathbb{R}^4\) be a singular stable map, and let \(K\) be a component of \(\Sigma(f)\). Then there are tubular neighborhoods \(\alpha : S^1 \times D_2^2 \to U\) of \(K\) and \(\beta : S^1 \times D_3^3 \to V\) of \(f(K)\) such that \(f(U) \subset V\) and such that \(\beta^{-1} \circ f \circ \alpha : S^1 \times D_2^2 \to S^1 \times D_3^3\) is given by

\[
(\beta^{-1} \circ f \circ \alpha)(\theta, x, y) = (\theta, x, y^2, xy).
\]

**Proof.** This proposition is a special case of a general theorem of Szűcs [20]. Note that in our case the normal bundles of \(\Sigma(f)\) and \(f(\Sigma(f))\) are trivial. \(\square\)

2.4. The kernel bundle and framed links in \(S^3\). Let \(f : S^3 \to \mathbb{R}^4\) be a singular locally stable map. The differential of \(f\) restricts to a linear corank one map \(df : TS^3|\Sigma(f) \to TR^4\). Let \(\text{Ker}(df)\) be its kernel line bundle. Then \(\text{Ker}(df)\) is nowhere tangential to \(\Sigma(f)\).

**Proposition 2.3.** If \(f : S^3 \to \mathbb{R}^4\) is a singular locally stable map, then \(\text{Ker}(df)\) is trivial.
**Proof.** Assume first that $f$ is stable. Let $U$ be an open set in $S^3$ that is properly contained in $S^3 \setminus \Sigma(f)$ and which covers the finitely many points in $S^3 \setminus \Sigma(f)$ with image in $f(\Sigma(f))$. Let $\bar{U}$ be the closure of $U$ in $S^3$, and let $V$ be the set remaining when all triple lines and quadruple points for the restriction $f|_{S^3 \setminus \bar{U}}$ are removed from $S^3 \setminus \bar{U}$. Then $V$ is an open neighborhood of $\Sigma(f)$ such that $|f^{-1}(f(p)) \cap V| \leq 2$ for all $p \in V$ and $f^{-1}(f(p)) \cap V = \{p\}$ for all $p \in \Sigma(f)$.

The restriction $f|_{V \setminus \Sigma(f)}$ is an immersion with normal crossings and at most double points. Hence $\hat{S} = \{p \in V : |f^{-1}(f(p)) \cap V| = 2\}$ and $\hat{S} = f(\hat{S})$ are surfaces in $S^3$ and $\mathbb{R}^4$ respectively, and $f|_{\hat{S}} : \hat{S} \to S$ is a double covering. By [3, Proposition 2.5], which applies to arbitrary connected manifolds, since $\dim \mathbb{R}^4 - \dim S^3$ is odd, the covering $f|_{\hat{S}}$ is the orientation double covering of $S$. Hence $\hat{S}$ is naturally oriented.

Let $Y = \hat{S} \cup \Sigma(f)$. Then $Y$ is a surface in $S^3$, and the orientation of $\hat{S}$ extends to an orientation of $Y$. Moreover, the induced bundle $TY|\Sigma(f)$ equals $\text{Ker}(df) \oplus T\Sigma(f)$. Because $TS(\Sigma(f))$ is orientable so is $\text{Ker}(df)$. Consequently $\text{Ker}(df)$ is trivial.

Now assume $f$ is merely locally stable. Let $h_t : S^3 \to \mathbb{R}^4$ be an $\text{LS}$-homotopy from a stable map $g$ to $f$, and let $J : S^3 \times I \to J^1(S^3, \mathbb{R}^4)$ be the homotopy $J(p,t) = j^1 h_t(p)$. Then $J$ is transverse to $\Sigma^1$ and, by [1, Theorem 20.2], the manifold $\Omega = J^{-1}(\Sigma^1)$ is a product cobordism from $\Sigma(g)$ to $\Sigma(f)$.

Let $H : S^3 \times I \to \mathbb{R}^4 \times I$ be the track of the homotopy $h_t$. The differential of $H$ restricts to a linear corank one map $dH : T(S^3 \times I)|\Omega \to T(\mathbb{R}^4 \times I)$. Let $\text{Ker}(dH)$ be its kernel line bundle. Then $\text{Ker}(dH)$ is trivial since $\text{Ker}(dH)(|\Sigma(g) \times \{0\}|)$ is isomorphic to the trivial bundle $\text{Ker}(df)$ and $\text{Ker}(df)$ is trivial since $\text{Ker}(df)$ is isomorphic to $\text{Ker}(dH)(|\Sigma(f) \times \{1\}|)$.

A **framed link** in $S^3$ is a pair $(L, u)$ where $L$ is a link in $S^3$ and $u : L \to TS^3|L$ is a section such that $u(p)$ is non-tangential to $L$ for every $p \in L$. An **isotopy of framed links** in $S^3$ is a 1-parameter family of framed links $(L_t, u_t)$ such that $L_t = \varphi_t(L_0)$ for an isotopy $\varphi_t : L_0 \to S^3$, and $u_t \circ \varphi_t : L_0 \to TS^3$ is a homotopy. The framed links $(L_0, u_0)$ and $(L_1, u_1)$ are then called **isotopic**, and the family $(L_t, u_t)$ is called an **isotopy from** $(L_0, u_0)$ to $(L_1, u_1)$. It is well known that if $(L_0, u_0)$ and $(L_1, u_1)$ are isotopic framed links in $S^3$, then there is an ambient isotopy $\varphi_t : S^3 \to S^3$ such that $\varphi_t(L_0) = L_1$ and $d\varphi_t \circ u_0 = u_1 \circ \varphi_t|L_0$.

For a singular locally stable map $f : S^3 \to \mathbb{R}^4$ the bundle $\text{Ker}(df)$ is nowhere tangent to the singularity link $\Sigma(f)$. Furthermore $\text{Ker}(df)$ is trivial by Proposition 2.3. Hence $\text{Ker}(df)$ admits a nowhere vanishing section $u : \Sigma(f) \to \text{Ker}(df)$. The pair $(\Sigma(f), u)$ is a framed link in $S^3$, whose isotopy class does not depend on the choice of $u$.

**Definition 2.4.** Let $\sigma(f)$ be the isotopy class of the framed link $(\Sigma(f), u)$.

**Proposition 2.5.** If $h_t : S^3 \to \mathbb{R}^4$ is an $\text{LS}$-homotopy of singular locally stable maps, then $\sigma(h_0) = \sigma(h_1)$.

**Proof.** Let $H : S^3 \times I \to \mathbb{R}^4 \times I$ be the track of $h_t$, and, as in the proof of Proposition 2.3, let $\Omega = J^{-1}(\Sigma^1)$ where $J : S^3 \times I \to J^1(S^3, \mathbb{R}^4)$ is the homotopy $J(p,t) = j^1 h_t(p)$. Then the kernel bundle $\text{Ker}(dH)$ of the restriction of $dH$ to $T(S^3 \times I)|\Omega$ is a trivial line bundle over $\Omega$ which is nowhere tangent to $\Omega$.

Let $q : S^3 \times I \to S^3$ be the projection. Then $q(\Omega \cap (S^3 \times \{t\})) = \Sigma(h_t)$, and $dq$ restricts to an isomorphism from $\text{Ker}(dH)(|\Omega \cap (S^3 \times \{t\})|)$ to $\text{Ker}(dh_t)$. Let $u : \Omega \to \text{Ker}(dH)$ be a nowhere vanishing section. For each $t \in I$ define $u_t : \Sigma(h_t) \to \text{Ker}(dh_t)$
by \( u_t(p) = dq(u(p,t)) \). Then, by [11] Theorem 20.2, \((\Sigma(h_t), u_t)\) is an isotopy from \((\Sigma(h_0), u_0)\) to \((\Sigma(h_1), u_1)\). Hence \( \sigma(h_0) = \sigma(h_1) \).

\[ \square \]

2.5. **Orientations and metrics.** The following conventions will be used throughout. We equip \( \mathbb{R}^n \) with its standard orientation and metric. The closed ball in \( \mathbb{R}^n \) of radius \( r \) centered at the origin is denoted \( D^n_r \). We orient \( S^k \) as the boundary of \( D^{k+1}_1 \), and we equip \( S^k \) with the metric induced from \( \mathbb{R}^{k+1} \).

**Remark.** We use the outward normal first convention to orient the boundary of an oriented manifold. That is, if \( p \) belongs to the boundary \( \partial M \) of an oriented manifold \( M \), then a basis for \( T_p \partial M \) is positively oriented if and only if an outward pointing normal to \( \partial M \) at \( p \) followed by the basis for \( T_p \partial M \) is a positively oriented basis for \( T_p M \).

3. The invariants

An **LS-homotopy invariant** is a locally constant function on the space of locally stable maps. In this section we define two LS-homotopy invariants: \( \nu \), see Subsection [3.1] and \( \kappa \), see Subsection [3.2]. The invariant \( \nu \) generalizes the normal degree for immersions \( S^3 \to \mathbb{R}^4 \), and \( \kappa \) generalizes the \( \mathbb{R}^2 \)-Smale invariant of immersions \( S^3 \to \mathbb{R}^4 \), i.e., the Smale invariant of an immersion \( S^3 \to \mathbb{R}^4 \) composed with the inclusion \( \mathbb{R}^4 \subset \mathbb{R}^5 \), see [5]. In Subsection [3.3] we describe the behavior of \( \nu \) and \( \kappa \) under taking connected sum of locally stable maps, and in Subsection [3.4] we relate \( \nu \) and \( \kappa \) to the Smale invariant for immersions \( S^3 \to \mathbb{R}^4 \).

3.1. The invariant \( \nu \). The normal degree \( \nu \) defined on immersions \( S^3 \to \mathbb{R}^4 \) is a regular homotopy invariant. We extend \( \nu \) to an LS-homotopy invariant.

Let \( f : S^3 \to \mathbb{R}^4 \) be a singular locally stable map. Let \( \Gamma : S^3 \setminus \Sigma(f) \to S^3 \times S^3 \) be the graph of the Gauss map of the immersion \( f|_{S^3 \setminus \Sigma(f)} \), and let \( q_1 : S^3 \times S^3 \to S^3 \) be projection onto the first factor. Then, by [8] Lemma 5.1, the intersection of the closure of the image of \( \Gamma \) with \( q_1^{-1}(\Sigma(f)) \) is a circle bundle \( S \) over \( \Sigma(f) \). The bundle is such that its fiber over \( p \in \Sigma(f) \) is a great circle \( S_p \) in \( \{ p \} \times S^3 \). Denote the closure of the image of \( \Gamma \) by \( M \). Then \( M \) is a manifold with boundary \( S \).

Let \( u : \Sigma(f) \to TS^3|\Sigma(f) \) be a section with no \( u(p) \) in \( \text{Ker}(df) \oplus T\Sigma(f) \). (The section exists by Proposition [2.3].) Define \( U : \Sigma(f) \to S^3 \times S^3 \) by

\[
U(p) = \left( p, \frac{df(u(p))}{|df(u(p))|} \right).
\]

Then, for each \( p \in \Sigma(f) \) there is a unique equator 2-sphere in \( \{ p \} \times S^3 \) which contains \( S_p \) and \( U(p) \). Let \( D_p \) be the closed hemisphere of this 2-sphere that contains \( U(p) \) and has \( S_p \) as its boundary. The hemispheres \( D_p \) form fibers in a disk bundle \( D \) over \( \Sigma(f) \).

Let \( q_2 : S^3 \times S^3 \to S^3 \) be projection onto the second factor. Equip \( M \) with the orientation induced from \( S^3 \setminus \Sigma(f) \) via \( \Gamma \). The orientation on \( M \) defines an orientation on \( S \). Equip the total space of \( D \) with the orientation that induces the opposite orientation on \( S \). (\( D \) is orientable since \( S \) is orientable.) Let \( m_f = q_2(M) \) and \( d_f = q_2(D) \). Then \( m_f + d_f \) is a 3-cycle in \( S^3 \) whose homology class we denote by \([m_f + d_f]\).

**Definition 3.1.** Let \( \nu(f) \) be the integer defined by \([m_f + d_f] = \nu(f)[S^3]\), where \([S^3]\) is the fundamental homology class in \( H_3(S^3; \mathbb{Z}) \).

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Remark. The integer \( \nu(f) \) is independent of the choice of \( u \). Indeed, if \( D' \) is the disk bundle over \( \Sigma(f) \) obtained from a different choice of section \( u' \) of \( TS^3|\Sigma(f) \), and if \( d_f' = q_2(D') \), then \( [m_f + d_f'] = [m_f + d_f] \). The equality is obvious if \( u' \) is homotopic to \( u \) through sections of \( TS^3|\Sigma(f) \) with images in the complement of \( \text{Ker}(df) \oplus T\Sigma(f) \).

On the other hand, if \( u' = u \) and \( p \in \Sigma(f) \), then \( D_p \) and \( D'_p \) together form an equator 2-sphere which is the boundary of a hemisphere \( H_p \) of \( \{p\} \times S^3 \). This can be globalized by Proposition 2.2 and we get that \( d_f - d_f' = \partial q_2(H) \) for a disk bundle \( H \) over \( \Sigma(f) \).

Thus \( d_f - d_f' \) is a zero-homologous cycle in \( S^3 \).

Proposition 3.2. If \( h_t : S^3 \rightarrow \mathbb{R}^4 \) is an LS-homotopy of singular locally stable maps, then \( \nu(h_0) = \nu(h_1) \).

Proof. Define \( J : S^3 \times I \rightarrow J^1(S^3, \mathbb{R}^4) \) by \( J(p,t) = j^1h_t(p) \), and let \( \Omega = J^{-1}(\Sigma^1) \).

Let \( H : S^3 \times I \rightarrow \mathbb{R}^4 \times I \) be the track of \( h_t \), and let \( \text{Ker}(dH) \) be the kernel bundle of \( dH \) over \( \Omega \). Define \( \Gamma : (S^3 \times I) \setminus \Omega \rightarrow S^3 \times I \times S^3 \) by \( \Gamma(p,t) = (p, t, n_t(p)) \), where \( n_t : S^3 \setminus \Sigma(h_t) \rightarrow S^3 \) is the Gauss map of \( h_t|_{S^3 \setminus \Sigma(h_t)} \). Then, as above, the closure of the image of \( \Gamma \) is an oriented manifold \( M \) with oriented boundary a circle bundle \( S \) over \( \Omega \). Fix a section \( u : \Omega \rightarrow T(S^3 \times I)|\Omega \) with no \( u(p,t) \) in \( T\Omega \oplus \text{Ker}(dH) \), and, in analogy with the above, use \( u \) to define a disk bundle \( D \) over \( \Omega \). Orient the total space of \( D \) so that \( D \) induces the opposite orientation on \( S \). Let \( q : S^3 \times I \times S^3 \rightarrow S^3 \) be projection onto the last factor. Then \( m_H = q(M) \) and \( d_H = q(D) \) are 4-chains in \( S^3 \), and \( \partial(m_H + d_H) = m_{h_t} - m_{h_0} + d_{h_t} - d_{h_0} \). Hence \( [m_{h_1} + d_{h_1}] = [m_{h_0} + d_{h_0}] \). \( \square \)

3.2. The invariant \( \kappa \). Let \( D \) be a 4-dimensional disk in \( \mathbb{R}^5_+ \), the upper half space of \( \mathbb{R}^5 \), with boundary \( \partial D = D \cap \mathbb{R}^4 = S^3 \) and such that \( S^3 \times [0, \epsilon) \) is a collar neighborhood of \( S^3 \) in \( D \) for some \( \epsilon > 0 \). Orient \( D \) so that the induced orientation of \( \partial D \) agrees with the fixed orientation on \( S^3 \). Let \( f : S^3 \rightarrow \mathbb{R}^4 \) be a locally stable map, and let \( i : \mathbb{R}^4 \rightarrow \mathbb{R}^5 \) be the standard inclusion. We call an extension \( F : D \rightarrow \mathbb{R}^5 \) of \( i \circ f \) admissible if \( F \) is locally stable and \( F(p,t) = (i(f(p)), t) \) for \( (p, t) \in S^3 \times [0, \epsilon) \). The extension \( F \) has singularities of types \( \Sigma^{1,0} \) and \( \Sigma^{1,1} \). We call the finitely many singularities of type \( \Sigma^{1,1} \) cusps. Ekholm and Szücs [5] proved that each cusp of \( F \) is naturally oriented. Thus each cusp is equipped with a sign.

Definition 3.3. Let \( \kappa(f) \) be the algebraic number of cusps of \( F \).

Remark. The orientation of a cusp \( q \) of \( F \) is defined as follows. The total space of the cokernel bundle \( \xi(F) \) of \( dF \) over the singularity surface \( \Sigma^1(F) \) of \( F \) is oriented, see [5]. Choose a metric for \( \text{Ker}(dF) \) and let \( \pm v \) be the two unit vectors in \( \text{Ker}(dF) \) at \( p \in \Sigma^1(F) \). Then \( s : \Sigma^1(F) \rightarrow \xi(F) \) defined by \( s(p) = d^2F(\pm v(p)) \), where \( d^2F \) is the quadratic differential of \( F \), is a section of \( \xi(F) \) that vanishes exactly at the cusps of \( F \). The orientation of \( q \) is the local intersection number at \( q \) of \( s \) with the zero section.

Lemma 3.4. Let \( N \) be an oriented 4-sphere, and let \( F : N \rightarrow \mathbb{R}^5 \) be a locally stable map. Then the algebraic number of cusps of \( F \) is zero.

Proof. The lemma is a special case of [21] Lemma 3. Here we only sketch the proof. The map \( F \) is null cobordant in the category of maps with at most \( \Sigma^{1,1} \) singularities. That is, there is a 5-manifold \( W \) with boundary \( N \) and a locally stable map \( B : W \rightarrow \mathbb{R}^5 \times I \) whose restriction to \( N \) equals \( F \) and the singularities of \( B \) are of types \( \Sigma^{1,0} \) and \( \Sigma^{1,1} \). The manifold \( \Sigma^{1,1}(B) \) is an oriented cobordism from the negative cusps of \( F \) to the positive cusps of \( F \). Consequently the algebraic number of cusps of \( F \) is zero. \( \square \)
Proposition 3.5. The integer $\kappa(f)$ is independent of the choice of $D$, the admissible extension $F$, and is an $\mathbb{L}$-homotopy invariant of $f$.

Proof: This is a special case of [5, Remark 3.2]. For completeness we include the proof.

Suppose $F : D_1 \to \mathbb{R}^5$ and $G : D_2 \to \mathbb{R}^5$ are admissible extensions of $i \circ f$. Denote the algebraic number of cusps of $F$ and $G$ by $\sharp \Sigma^{1,1}(F)$ and $\sharp \Sigma^{1,1}(G)$ respectively. Let $\rho : \mathbb{R}^5 \to \mathbb{R}^5$ be reflection in $\mathbb{R}^4$, let $\bar{D}_2$ be the manifold $\rho(D_2)$ with orientation inducing the opposite orientation on $\partial \bar{D}_2 = S^3$, and let $\bar{G} : \bar{D}_2 \to \mathbb{R}^5$ be the composition $\rho \circ G \circ |_{\bar{D}_2}$. The algebraic number of cusps of $F \cup \bar{G} : D_1 \cup \bar{D}_2 \to \mathbb{R}^5$ equals $\sharp \Sigma^{1,1}(F) - \sharp \Sigma^{1,1}(G)$. Thus $\sharp \Sigma^{1,1}(F) = \sharp \Sigma^{1,1}(G)$ by Lemma 3.4 which proves the first and second assertion. The third assertion follows immediately. \hfill $\Box$

For a singular locally stable map $f : S^3 \to \mathbb{R}^4$ let $\sharp \text{slk} \Sigma(f)$ be the total self linking number of $\Sigma(f)$. That is, if $K_1, \ldots, K_d$ are the components of $\Sigma(f)$, and if $u : \Sigma(f) \to \text{Ker}(df)$ is a nowhere vanishing section, then let

$$\sharp \text{slk} \Sigma(f) = \sum_{j=1}^{d} \text{slk}(K_j, u|_{K_j})$$

where $\text{slk}(K_j, u|_{K_j})$ is the self linking number of the framed knot $(K_j, u|_{K_j})$.

Proposition 3.6. If $f : S^3 \to \mathbb{R}^4$ is a singular stable map, then $\kappa(f)$ and $\sharp \text{slk} \Sigma(f)$ have the same parity.

Recall that a $k$-parameter deformation of a map $g : S^3 \to \mathbb{R}^4$ is a smooth family of maps $g_\lambda : S^3 \to \mathbb{R}^4$, parameterized by the elements $\lambda$ in an open neighborhood of $0 \in \mathbb{R}^k$, such that $g_0 = g$. The deformation is called versal if every deformation of $g$ is equivalent, up to left-right action of diffeomorphisms, to a deformation induced from $g_\lambda$.

Proof of proposition 3.6. The complement $\Delta$ in $C^\infty(S^3, \mathbb{R}^4)$ of the subspace of locally stable maps is called the discriminant. The discriminant is stratified, $\Delta = \Delta^1 \cup \Delta^2 \cup \cdots \cup \Delta^\infty$, where each $\Delta^k$, for $k < \infty$, is a submanifold of codimension $k$ in $C^\infty(S^3, \mathbb{R}^4)$, and $\Delta^k$ is contained in the closure of $\Delta^1$ for every $k$.

Let $h_t$ be a generic homotopy from the standard embedding $S^3 \to \mathbb{R}^4$ to $f$. Then the path in $C^\infty(S^3, \mathbb{R}^4)$ determined by $h_t$ passes the discriminant only through its top stratum $\Delta^1$, transversely. The finitely many $s \in I$ for which $h_s \in \Delta^1$ we call the critical instances for the homotopy. When $t$ varies between two critical instances the maps $h_t$ are locally stable, and the total self linking number $\sharp \text{slk} \Sigma(h_t)$, when regarded a function of $t$, is constant. Here we assume that $\sharp \text{slk} \Sigma(h_0) = 0$ if $h_0$ is an immersion.

Define the jump $J(s)$ in the total self linking number of $h_t$ at a critical instance $s$ to be the difference $J(s) = \sharp \text{slk} \Sigma(h_{s+\epsilon}) - \sharp \text{slk} \Sigma(h_{s-\epsilon})$, where $\epsilon > 0$ is so small that $s$ is the only critical instance for $h_t$ in the interval $s - \epsilon \leq t \leq s + \epsilon$. If $s_1 < s_2 < \cdots < s_r$ are the critical instances for $h_t$, then

$$\sharp \text{slk} \Sigma(f) = \sum_{i=1}^{r} J(s_i).$$

We show that $\sum_{i=1}^{r} J(s_i)$ is congruent to $\kappa(f)$ modulo 2.

Normal forms for the maps in $\Delta^1$ were computed by Houston and Kirk [9]. Namely, for $h \in \Delta^1$ there is one exceptional point $q \in S^3$, and local coordinates centered at
For each \( p \in S^3 \setminus \{q\} \) there are the usual local coordinates centered at \( p \) and \( h(p) \) respectively, in which \( h \) is given by one of the following two equations:

\[(3.1) \quad h(x, y, z) = (x, y, z, 0) \quad \text{or} \quad h(x, y, z) = (x, y, z^2, yz).\]

Assume \( s \) is a critical instance for the homotopy \( h_t \). The deformation of \( h_s \) determined by \( h_t \) is equivalent to a versal 1-parameter deformation \( v_\lambda \) of \( h_s \). Let \( q \in S^3 \) be the exceptional point for \( h_s \). Then \( v_\lambda \) can be assumed constant in \( \lambda \) outside small coordinate neighborhoods centered at \( q \) and \( h_s(q) \) respectively, and, when expressed in these coordinates, to be given by expression (3.6), . . . , (3.10) respectively. Note that \( J(s) \), or \(-J(s)\), equals the jump in the total self linking number of the versal deformation at \( \lambda = 0 \).

\[(3.6) \quad v_\lambda(x, y, z) = (x, y, z^2, z(z^2 + x^2 + y^2 + \lambda)),\]
\[(3.7) \quad v_\lambda(x, y, z) = (x, y, z^2, z(z^2 + x^2 - y^2 + \lambda)),\]
\[(3.8) \quad v_\lambda(x, y, z) = (x, y, z^2, z(z^2 - x^2 + y^2 + \lambda)),\]
\[(3.9) \quad v_\lambda(x, y, z) = (x, y, z^2, z(z^2 - x^2 - y^2 + \lambda)),\]
\[(3.10) \quad v_\lambda(x, y, z) = (x, y, xz + \lambda z^2 + z^4, yz + z^3).\]

Figure 1 contains bifurcation diagrams for the singularity link under the versal deformations (3.6)–(3.9). The kernel bundle for the differential \( dv_\lambda \) is everywhere parallel to the z-axis (that points towards the observer). Deformations (3.7) and (3.9) are responsible for the vanishing and creation, and deformations (3.7) and (3.8) for the fusion.
and splitting, of components of the singularity link. None of these modifications of the singularity link affects the total self linking number.

Figure 2 contains a bifurcation diagram for the singularity link under the versal deformation (3.10). Outside the box, decorated with the letter $n$, the kernel bundle of $dv_\lambda$ is parallel to the z-axis. Inside the box the kernel bundle is assumed to make $n$ twists about the displayed component of the singularity link. Note that the fiber of the kernel bundle of $dv_0$ over the exceptional point $q$ is tangential to the singularity link. Thus $v_0$ has a cusp at $q$, that is, a singularity of type $\Sigma^{1,1}$. As the diagram indicates, the jump in the total self linking number of $v_\lambda$ at $\lambda = 0$ is $\pm 1$.

Let $H : S^3 \times I \to \mathbb{R}^4 \times I$ be the track of the homotopy $h_t$. Then $H$ is locally stable since $h_t$ is generic. Moreover, by Proposition 3.5, $-\kappa(f)$ is the algebraic number of cusps for $H$. Now $(q,s) \in S^3 \times I$ is a cusp for $H$ if and only if $s$ is a critical instance for $h_t$ and $q$ is an exceptional point as in (3.5) for $h_s$. Thus, if $H$ has $r$ cusps, at critical instances $s_1,s_2,...,s_r$, then $J(s_i) = \pm 1$ for $i = 1,...,r$, and $\kappa(f) \equiv r \equiv \sum_{k=1}^r J(s_i)$ modulo 2. Hence $\kappa(f)$ and $\# \text{slk } \Sigma(f)$ have the same parity.

Remark. An alternative proof of Proposition 3.6 can be obtained from the proof of [12, Proposition 8].

3.3. The connected sum of locally stable maps. Kervaire defined a connected sum operation for immersions $S^k \to \mathbb{R}^n$, see [15] for the explicit construction. This operation can be analogously defined for locally stable maps $S^k \to \mathbb{R}^n$, when $k < n$. We denote the connected sum of two locally stable maps $f,g : S^k \to \mathbb{R}^n$ by $f \natural g : S^k \to \mathbb{R}^n$. The connected sum $f \natural g$ is a locally stable map which, since the singularity sets of $f$ and $g$ have at least codimension 2 in $S^k$, is unique up to $\text{ls}$-homotopy.

Proposition 3.7. If $f,g : S^3 \to \mathbb{R}^4$ are locally stable maps such that $f$ is singular and $g$ is an immersion, then $f \natural g$ is singular and $\sigma(f \natural g) = \sigma(f)$.

Proof. This is immediate from the definition of the connected sum. □

Proposition 3.8. If $f,g : S^3 \to \mathbb{R}^4$ are locally stable maps, then $\nu(f \natural g) = \nu(f) + \nu(g) - 1$.

Proof. Milnor [18] proved that for immersions $f,g : S^3 \to \mathbb{R}^4$ the normal degree of $f \natural g$ is $\nu(f \natural g) = \nu(f) + \nu(g) - 1$. A proof analogous to the proof of Milnor’s shows that the same formula holds for the normal degree of a connected sum of two locally stable maps. □

Proposition 3.9. If $f,g : S^3 \to \mathbb{R}^4$ are locally stable maps, then $\kappa(f \natural g) = \kappa(f) + \kappa(g)$.
\textbf{Proof.} Let \( p, q \in S^3 \) be such that \( f(p) \neq g(p) \) and such that \( f \) and \( g \) are embeddings when restricted to small neighborhoods of \( p \) and \( q \). Normalize \( f \) at \( f(p) \) and \( g \) at \( g(q) \). Thus, \( f \circ i \) and \( i \circ g \) are homotopic to \( f \) and \( g \) respectively.

Let \( N \) be the 4-sphere in \( \mathbb{R}^5 \) obtained by gluing hemispheres to the top and bottom of the cylinder \( S^3 \times (-2, 2) \). Extend \( i \circ f \) and \( i \circ g \) to locally stable maps \( F, G : N \to \mathbb{R}^5 \) such that \( F(x, t) = (t(f(x)), t) \) and \( G(x, t) = (t(g(x)), t) \) for \( (x, t) \in S^3 \times (-1, 1) \). Then \( F \) and \( G \), when restricted to \( N_+ = N \cap \mathbb{R}^5_+ \), are admissible extensions of \( i \circ f \) and \( i \circ g \) respectively.

Let \( H : S^4 \to \mathbb{R}^5 \) be the connected sum of \( F \) and \( G \) obtained by “joining the images \( F(N) \) and \( G(N) \) by a tube from \( F(p) \) to \( G(q) \) with axis \( F(p) + t(g(q) - f(p)) \)”, c.f. [15] page 123. Then \( H \) is locally stable, and \( H|_{N_+} : N_+ \to \mathbb{R}^5 \) is an admissible extension of \( i \circ f \circ g \). The algebraic number of cusps of \( H|_{N_+} \) equals the sum of the algebraic number of cusps of \( F|_{N_+} \) and \( G|_{N_+} \). Hence \( \kappa(f \circ g) = \kappa(f) + \kappa(g) \). \( \square \)

3.4. Immersions. Smale [19] classified immersions of spheres in Euclidean spaces. Associated to each immersion \( f : S^k \to \mathbb{R}^n \) is its Smale invariant \( \Omega_{n,k}(f) \in \pi_k(V_{n,k}) \), where \( V_{n,k} \) is the Stiefel manifold of \( k \)-frames in \( \mathbb{R}^n \), and two immersions \( f, g : S^k \to \mathbb{R}^n \) are regularly homotopic if and only if \( \Omega_{n,k}(f) = \Omega_{n,k}(g) \). Moreover, each element of \( \pi_k(V_{n,k}) \) is the Smale invariant of an immersion \( S^k \to \mathbb{R}^n \). Thus the Smale invariant establish a one-one correspondence between the set of regular homotopy classes of immersions \( S^k \to \mathbb{R}^n \) and \( \pi_k(V_{n,k}) \). Kervaire [15] proved that the set of regular homotopy classes of immersions \( S^k \to \mathbb{R}^n \) is an abelian group under connected sum, and that the Smale invariant establish an isomorphism from this group to \( \pi_k(V_{n,k}) \):

**Proposition 3.10.** If \( f, g : S^k \to \mathbb{R}^n \) are immersions, then \( \Omega_{n,k}(f \circ g) = \Omega_{n,k}(f) + \Omega_{n,k}(g) \).

The group of immersions of \( S^3 \to \mathbb{R}^4 \) is isomorphic to \( \pi_3(V_{4,3}) \). The space \( V_{4,3} \) is homotopy equivalent to \( SO(4) \). A homotopy equivalence is obtained by adding a fourth vector to each 3-frame to make it a positively oriented 4-frame, and then apply the Gram-Schmidt process. We identify \( \pi_3(V_{4,3}) \) with \( \pi_3(SO(4)) \) via the induced isomorphism.

The group \( \pi_3(SO(4)) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \). To define generators for \( \pi_3(SO(4)) \) we identify \( \mathbb{R}^4 \) with the quaternions, \( S^3 \) with the unit quaternions, \( \mathbb{R}^3 \) with the pure quaternions, and define \( \rho : S^3 \to SO(3) \) by \( \rho(x)y = x \cdot y \cdot x^{-1} \) for \( x \in S^3 \), \( y \in \mathbb{R}^3 \). Then the homotopy classes \([\sigma]\) and \([\rho]\) of \( \sigma, \rho : S^3 \to SO(4) \) defined by

\[
\sigma(x)y = x \cdot y \quad \text{and} \quad \rho(x) = \begin{bmatrix} 1 \\ \rho(x) \end{bmatrix} \quad (x \in S^3, y \in \mathbb{R}^4)
\]

respectively, generate \( \pi_3(SO(4)) \), see [11] Chapter 8, Proposition 12.9]. Immersions \( S^3 \to \mathbb{R}^4 \) whose Smale invariants equal \([\sigma]\) and \([\rho]\) were constructed by Hughes [11].

**Lemma 3.11.** If \( f : S^3 \to \mathbb{R}^4 \) is an immersion, and \( \Omega_{4,3}(f) = m[\sigma] + n[\rho] \), then \( \nu(f) = m + 1 \).

**Proof.** For each \( p \in S^3 \) let \( Q(p) \) be the 3-frame \((p \cdot i, p \cdot j, p \cdot k)\) translated to \( T_p S^3 \). Further let \( t(p) \) be the 3-frame \( df(Q(p)) \) translated to the origin in \( \mathbb{R}^4 \), let \( n : S^3 \to S^3 \) be the Gauss map of \( f \), and let \( \Phi(p) \) be the 4-frame obtained by applying the Gram-Schmidt process to \((n(p), t(p))\). Then, by [4] Lemma 3.3.1], the homotopy class of \( \Phi : S^3 \to SO(4) \) equals \( \Omega_{4,3}(f) + [\sigma] \).
Let \( q : SO(4) \to S^3 \) be the fibration that sends each matrix in \( SO(4) \) to its first column vector. Then \( q_*[\sigma] \) generates \( \pi_3(S^3) \) and

\[
\nu(f)q_*[\sigma] = q_*[\Phi] = q_*((\Omega_{4,3}(f) + [\sigma]) = (m + 1)q_*[\sigma] + nq_*[\rho] = (m + 1)q_*[\sigma].
\]

Accordingly \( \nu(f) = m + 1. \) \( \square \)

If \( p : SO(5) \to V_{5,3} \) is the fibration that sends a matrix to its last three column vectors, then \( p_* : \pi_3(SO(5)) \to \pi_3(V_{5,3}) \) is an isomorphism. We identify \( \pi_3(V_{5,3}) \) with \( \pi_3(SO(5)) \) via \( p_* \). Moreover, if \( j : SO(4) \to SO(5) \) is the inclusion

\[
j(A) = \begin{bmatrix} 1 & & & \\ & & A & \end{bmatrix},
\]

then \( j_* : \pi_3(SO(4)) \to \pi_3(SO(5)) \) is an epimorphism with kernel generated by \( 2[\sigma] - [\rho] \), see \([11] \) Chapter 8, Proposition 12.11.

Let \( f : S^3 \to \mathbb{R}^4 \) be an immersion and let \( i : \mathbb{R}^4 \to \mathbb{R}^5 \) be the standard inclusion.

**Lemma 3.12.** \( \Omega_{5,3}(i \circ f) = j_* (\Omega_{4,3}(f)) \).

**Proof.** This is \([3] \) Lemma 3.3.3. \( \square \)

**Lemma 3.13.** \( 2\Omega_{5,3}(i \circ f) = \kappa(f) j_* [\sigma]. \)

**Proof.** This is a consequence of \([3] \) Theorem 1(a)). \( \square \)

**Proposition 3.14.** Two immersions \( f, g : S^3 \to \mathbb{R}^4 \) are regularly homotopic if and only if \( \nu(f) = \nu(g) \) and \( \kappa(f) = \kappa(g) \).

**Proof.** Lemmas 3.11, 3.12 and 3.13 imply that \( \Omega_{4,3}(f) = \Omega_{4,3}(g) \) if and only if \( \nu(f) = \nu(g) \) and \( \kappa(f) = \kappa(g) \). \( \square \)

4. **Realization**

In this section we prove that every framed link in \( S^3 \) is the framed singularity link of a locally stable map into \( \mathbb{R}^4 \).

Let \( V_3(TS^3) \) be the bundle of 3-frames for \( TS^3 \), and let \( Q : S^3 \to V_3(TS^3) \) be an everywhere positively oriented section. If \( (u_1, u_2, u_3) \) is a 3-frame at \( p \in S^3 \), then we denote the matrix of coordinates for \( (u_1, u_2, u_3) \) when expressed in the basis \( Q(p) \) for \( T_pS^3 \) by \( [u_1, u_2, u_3]_Q \). Thus \( (u_1, u_2, u_3) = Q(p)[u_1, u_2, u_3]_Q \). Let \( V_4(T\mathbb{R}^4) \) be the bundle of 4-frames for \( T\mathbb{R}^4 \), and let \( E : \mathbb{R}^4 \to V_4(T\mathbb{R}^4) \) be the section that takes each \( q \in \mathbb{R}^4 \) to the standard basis for \( T_q\mathbb{R}^4 \). As above, if \( (v_1, v_2, v_3, v_4) \) is a 4-frame at \( q \in \mathbb{R}^4 \), then we denote the matrix of coordinates for \( (v_1, v_2, v_3, v_4) \) when expressed in the basis \( E(q) \) for \( T_q\mathbb{R}^4 \) by \( [v_1, v_2, v_3, v_4]_E \).

Let \( V_3(T\mathbb{R}^4) \) be the bundle of 3-frames in \( T\mathbb{R}^4 \), and let \( GL^+(4) \) be the Lie group of real 4-by-4 matrices with positive determinants. The map \( \Psi : V_3(T\mathbb{R}^4) \to GL^+(4) \) defined by \( \Psi(v_1, v_2, v_3) = [v_1, v_2, v_3, v_1 \times v_2 \times v_3]_E \) is a homotopy equivalence. Here, if \( v_1, v_2, v_3 \in T_q\mathbb{R}^4 \), then \( v_1 \times v_2 \times v_3 \) is the unique nonzero vector \( w \in T_q\mathbb{R}^4 \) that is orthogonal to the linear span of \( \{v_1, v_2, v_3\} \) and satisfies \( |w|^2 = \det[v_1, v_2, v_3, w]_E \).

Define \( \Xi : V_3(T\mathbb{R}^4) \to T\mathbb{R}^4 \) by \( \Xi(v_1, v_2, v_3) = v_1 \times v_2 \times v_3 \).

**Lemma 4.1.** Let \( B \) be a proper tubular neighborhood of a link in \( S^3 \). Let \( M = S^3 \setminus B \), and let \( U \) be an open neighborhood of \( \partial M \) in \( M \). Assume \( g : U \to \mathbb{R}^4 \) is an immersion, and assume that the composition \( \Psi \circ dg \circ Q|_U : U \to GL^+(4) \) extends to a continuous map \( G : M \to GL^+(4) \). Then there exists an immersion \( f : M \to \mathbb{R}^4 \) such that \( G \) and \( \Psi \circ df \circ Q|_M \) are homotopic relative \( V \), where \( V \) is an open neighborhood of \( \partial M \).
Let \( \partial R \) of some knot in \( S \) respectively, with \( \alpha = \bigcup_{j=0}^{d} \alpha_j : \bigcup_{j=0}^{d} S^1 \times \mathbb{R}^2 \to \bigcup_{j=0}^{d} U_j = U \) and \( \beta : S^1 \times \mathbb{R}^3 \to V \) be trivializations. On \( U \) and \( V \) we have the coordinate fields \( \partial_\theta, \partial_1, \partial_2, \partial_3 \) induced by \( \alpha \) and \( \beta \) respectively. We assume that \( \alpha \) and \( \beta \) are orientation preserving when \( S^1 \times \mathbb{R}^2 \) and \( S^1 \times \mathbb{R}^3 \) are canonically oriented, and we assume \( \alpha \) to be such that \( \partial_3(p) = u(p) \) for each \( p \in L \).

Define \( f : U \to V \) by \( f|_{U_j} = \beta \circ s \circ \alpha_j^{-1} \), where \( s : S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}^3 \) is the map \( s(\theta, x, y) = (\theta, x, y^2, xy) \). Then \( f \) has singularity link \( \Sigma(f) = L \) and, by the choice of \( \alpha \), the frame \( u \) of \( L \) is a section in \( \text{Ker}(df) \).

Define \( F : U \setminus L \to GL^+(4) \) by \( F = \Psi \circ df \circ Q|_{U \setminus L} \). For \( j = 0, \ldots, d \) let \( \mu_j : S^1 \to U_j \) be the meridian \( \mu_j(\varphi) = \alpha_j(0, \cos \varphi, \sin \varphi) \) of \( L_j \). Then each loop \( F \circ \mu_j \) is homotopically nontrivial. To see this let \( \gamma_j = f \circ \mu_j \) and define \( X : U \setminus L \to T \mathbb{R}^4 \) by \( X = \Xi \circ df \circ Q \). Then \( (F \circ \mu_j)(\varphi) = A_2(\varphi)A_3(\varphi)A_1(\varphi) \) where

\[
A_1(\varphi) = \begin{bmatrix}
[\partial_\theta(\mu_j(\varphi)), \partial_1(\mu_j(\varphi)), \partial_2(\mu_j(\varphi))]_Q^1 \quad 1
\end{bmatrix},
\]

\[
A_2(\varphi) = \begin{bmatrix}
[\partial_\theta(\gamma_j(\varphi)), \partial_1(\gamma_j(\varphi)), \partial_2(\gamma_j(\varphi)), \partial_3(\gamma_j(\varphi))]_E
\end{bmatrix},
\]

\[
A_3(\varphi) = \begin{bmatrix}
1 & 0 & 0 & x_1(\varphi) \\
0 & 1 & 0 & x_2(\varphi) \\
0 & 0 & 2\sin \varphi & x_3(\varphi) \\
0 & 0 & \cos \varphi & x_4(\varphi)
\end{bmatrix},
\]

and \( x_1(\varphi), x_2(\varphi), x_3(\varphi), x_4(\varphi) \) are the coordinates for \( X(\mu_j(\varphi)) \) when expressed in the basis \( (\partial_\theta(\gamma_j(\varphi)), \partial_1(\gamma_j(\varphi)), \partial_2(\gamma_j(\varphi)), \partial_3(\gamma_j(\varphi))) \) for \( T_{\gamma_j(\varphi)} \mathbb{R}^4 \). We conclude, since the loops \( A_1 \) and \( A_2 \) are nullhomotopic, that \( F \circ \mu_j \) and \( A_3 \) are homotopic.

Let \( Y : S^1 \to T \mathbb{R}^4 \) be the vector field \( Y(\varphi) = \cos \varphi \partial_2(\gamma_j(\varphi)) - \sin \varphi \partial_3(\gamma_j(\varphi)) \) along \( \gamma_j \). Then \( Y(\varphi) \) is normal to the hyperplane \( df(T_{\mu_j(\varphi)}S^3) \) in \( \mathbb{R}^4 \), and \( Y(\varphi) \) and \( X(\mu_j(\varphi)) \) points into the same half space. The homotopy \( H(t)(\varphi) = (1-t)X(\mu_j(\varphi)) - tY(\varphi) \) induces a homotopy from \( A_3 \) to the nontrivial loop \( A_4 : S^1 \to GL^+(4) \) given by

\[
A_4(\varphi) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2\sin \varphi & -\cos \varphi \\
0 & 0 & \cos \varphi & \sin \varphi
\end{bmatrix}.
\]

Hence \( A_3 \), and consequently \( F \circ \mu_j \), are homotopically nontrivial.

For \( j = 0, \ldots, d \) let \( \lambda_j : S^1 \to U_j \) be the longitude \( \lambda_j(\theta) = \alpha_j(\theta, 0, 1) \). Further let \( r : S^1 \times \mathbb{R}^3 \to S^1 \times \mathbb{R}^3 \) be fibrewise rotation \( r(\theta, x, y, z) = (\theta, \cos \theta x + \sin \theta y, \cos \theta y - \sin \theta x, z) \), and define \( g : U \to V \) by \( g|_{U_j} = \beta \circ r \circ s \circ \alpha_j^{-1} \). Associated to \( g \) is the map \( G : U \setminus L \to GL^+(4) \) defined by \( G = \Psi \circ dg \circ Q|_{U \setminus L} \). We prove that the loops \( F \circ \lambda_j \) and \( G \circ \lambda_j \) are non-homotopic.
Let $\xi_j = f \circ \lambda_j$ and $\zeta_j = g \circ \lambda_j$. Define $Z_j, W_j : S^1 \to T\mathbb{R}^4$ by $Z_j = \Xi \circ df \circ Q \circ \lambda_j$ and $W_j = \Xi \circ dg \circ Q \circ \lambda_j$. Then the affine homotopies $Z_t(\theta) = (1 - t)Z_j(\theta) + t\partial_3(\xi_j(\theta))$ and $W_t(\theta) = (1 - t)W_j(\theta) + t\partial_3(\zeta_j(\theta))$ induce homotopies between $F \circ \lambda_j$ and $B_1$, respectively $G \circ \lambda_j$ and $B_2$, where $B_1, B_2 : S^1 \to GL^+(4)$ are the loops

$$B_1(\theta) = \left[ df(Q_j(\lambda_j(\theta))), df(Q_2(\lambda_j(\theta))), df(Q_3(\lambda_j(\theta))), \partial_3(\xi_j(\theta)) \right]_E,$$

$$B_2(\theta) = \left[ dg(Q_1(\lambda_j(\theta))), dg(Q_2(\lambda_j(\theta))), dg(Q_3(\lambda_j(\theta))), \partial_3(\zeta_j(\theta)) \right]_E,$$

and $Q_1, Q_2, Q_3 : S^3 \to TS^3$ are the component vector fields of $Q$. Define $C, D_1, D_2 : S^1 \to GL^+(4)$ by

$$C(\theta) = \begin{bmatrix} \partial_0(\xi_j(\theta)), & \partial_1(\lambda_j(\theta)), & \partial_2(\xi_j(\theta)) \end{bmatrix}_{Q}^{-1}_E,$$

$$D_1(\theta) = \begin{bmatrix} \partial_0(\xi_j(\theta)), & \partial_1(\lambda_j(\theta)), & \partial_2(\xi_j(\theta)) \end{bmatrix}_E,$$

$$D_2(\theta) = \begin{bmatrix} \partial_0(\zeta_j(\theta)), & \partial_1(\lambda_j(\theta)), & \partial_2(\zeta_j(\theta)) \end{bmatrix}_E.$$

Then $B_1(\theta) = D_1(\theta)P_1(\theta)C(\theta)$ and $B_2(\theta) = D_2(\theta)P_2(\theta)C(\theta)$, where $P_1, P_2 : S^1 \to GL^+(4)$ are given by

$$P_1(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\
\cos \theta & \cos \theta & 2 \sin \theta & 0 \\
-\sin \theta & -\sin \theta & 2 \cos \theta & 0 \\
0 & 1 & 0 & 1 \end{bmatrix}$$

respectively. The loops $P_1$ and $P_2$ are obviously non-homotopic. Since $D_1$ and $D_2$ are homotopic we conclude that $F \circ \lambda_j$ and $G \circ \lambda_j$ are non-homotopic.

The group $H_1(S^3 \setminus L; \mathbb{Z}_2)$ is freely generated by the homology classes of the meridians $\mu_j$. Each longitude $\lambda_j$ is a cycle in $S^3 \setminus L$. If $\lambda_j$ is homologous to a sum of an odd number of meridians and the loop $F \circ \lambda_j$ is null-homotopic, then replace $f|_{U_j}$ by $g|_{U_j}$ in the definition of $f$. Conversely, if $\lambda_j$ is homologous to a sum of an even number of meridians and $F \circ \lambda_j$ is homotopically non-trivial, then replace $f|_{U_j}$ by $g|_{U_j}$. The redefined map $f : U \to V$ still has singularity link $\Sigma(f) = L$ and section $u \in \text{Ker}(df)$.

Let $B = \bigcup_{j=0}^d \alpha_j(S^1 \times D_1^2)$, and let $M = S^3 \setminus B$. By Morse theory the pair $(M, \partial M)$ has a handle decomposition $\partial M \subset M_0 \subset M_1 \subset M_2 \subset M_3 = M$ with $M_0$ a closed collar neighborhood of $\partial M$ in $M$, $M_1$ connected since $M$ is connected, and $M_0$ obtained from $M_2$ by the attachment of one 3-handle. The map $F_{-1} = F|_{\partial M} : \partial M \to GL^+(4)$ is continuous. Our goal is to extend $F_{-1}$ to a continuous map $F_3 : M \to GL^+(4)$. Lemma \[41\] then assures that the restriction of $F$ to a possibly smaller open neighborhood of $L$ than $U$ extends as an immersion to $M$.

Let $N_0, \ldots, N_d$ be the components of $M_0$, and let $\rho_0 : M_0 \to \partial M$ be a retraction that takes a 2-disk $\Omega \subset \partial N_0$ to a point $b \in \partial M$. Let $H_1, \ldots, H_n$ be the 1-handles attached to $\partial M_0$ to obtain $M_1$. Each handle $H_j$ has a core segment $I_j$, and we may assume, possibly after a handle slide, that for $j = 1, \ldots, d$ the segment $I_j$ has one of its endpoints on $\partial N_0$ and the other on $\partial N_d$, and for $j = d + 1, \ldots, n$ that $I_j$ has both its endpoints in $\Omega$.

Let $F_0 : M_0 \to GL^+(4)$ be the continuous extension $F_{-1} \circ \rho_0$ of $F_{-1}$. Extend $F_0$ arbitrarily but continuously over the segments $I_1, \ldots, I_d$. For $j = d + 1, \ldots, n$ let $\lambda_j : S^1 \to I_j \cup \Omega$ be a simple loop passing through the segment $I_j$. The loop $\lambda_j$ is a cycle in the first homology of $S^3 \setminus L$ with $\mathbb{Z}_2$ coefficients. If $\lambda_j$ is homologous to a sum of an odd number of meridians, then extend $F_0$ over $I_j$ so that $F_0|_{I_j} : I_j \to GL^+(4)$
is a homotopically nontrivial loop, based at \( F_0(b) \). Otherwise extend \( F_0 \) over \( I_j \) as a constant map.

Let \( \rho_1 : M_1 \to M_0 \cup I_0 \cup \cdots \cup I_n \) be a retraction, and let \( F_1 : M_1 \to GL^+(4) \) be the extension \( F_0 \circ \rho_1 \) of \( F_0 \). Consider the induced homomorphism

\[
(F_1)_* : H_1(M_1; \mathbb{Z}_2) \to \pi_1(GL^+(4)) = \mathbb{Z}_2
\]

where we have identified \( \pi_1(GL^+(4)) \) with \( H_1(GL^+(4); \mathbb{Z}_2) \) via the Hurewicz isomorphism. Now \( v \) the homology class of that parameterizes the boundary of a core disk of a 2-handle in \( M_2 \). Moreover, if \( [v] \) extends continuously over \( M_2 \), \( \pi_1(GL^+(4)) \) with \( (F_1)_*[v] = 0 \) for each loop \( v : S^1 \to M_1 \) that parameterizes the boundary of a core disk of a 2-handle in \( M_2 \). Here \([v]\) denotes the homology class of \( v \). So let \( v \) be any such loop. Then, in \( H_1(M_1; \mathbb{Z}_2) \), for some \( a_i, b_j \in \mathbb{Z}_2 \),

\[
[v] = \sum_{i=0}^d a_i [\mu_i] + \sum_{j=0}^n b_j [\lambda_j].
\]

Moreover, if \( [\lambda_j] = \sum_{i=0}^d c_{ij} [\mu_i] \) for \( j = 1, \ldots, n \), with \( c_{ij} \in \mathbb{Z}_2 \), when regarded as homology classes in \( H_1(S^3 \setminus L; \mathbb{Z}_2) \), then by construction

\[
(F_1)_*[v] = \sum_{i=0}^d a_i (F_1)_*[\mu_i] + \sum_{j=0}^n b_j (F_1)_*[\lambda_j] = \sum_{i=0}^d (a_i + \sum_{j=0}^n b_j c_{ij}).
\]

Because the cycle \( v \) is a boundary in \( S^3 \setminus L \) we have that

\[
0 = \sum_{i=0}^d a_i [\mu_i] + \sum_{j=0}^n b_j [\lambda_j] = \sum_{i=0}^d (a_i + \sum_{j=0}^n b_j c_{ij}) [\mu_i]
\]

in \( H_1(S^3 \setminus L; \mathbb{Z}_2) \). Accordingly \( (F_1)_*[v] = 0 \), and we conclude that \( F_1 \) extends to a continuous map \( F_2 : M_2 \to GL^+(4) \). Finally \( F_2 \) extends to a continuous map \( F_3 : M_3 \to GL^+(4) \) because \( \pi_2(GL^+(4)) = 0 \). Hence we have the desired extension of \( F|_{\partial M} \) to \( M \).

\( \square \)

5. Sufficiency

In this section we prove that if locally stable maps \( f \) and \( g \) have isotopic framed singularity links, then there exists an immersion \( k : S^3 \to \mathbb{R}^4 \) such that \( f \) is \( \text{ls} \)-homotopic to \( g^2 k \). Throughout this section let \( Q, E, \Psi, \) and \( \Xi \) be as defined in the introduction to Section 4.

**Lemma 5.1.** Let \( f, g : S^3 \to \mathbb{R}^4 \) be singular stable maps such that \( \sigma(f) = \sigma(g) \). Then there exists a neighborhood \( U \) of \( \Sigma(g) \) and an \( \text{ls} \)-homotopy \( h_1 : S^3 \to \mathbb{R}^4 \) such that \( h_0 = f \) and \( h_1|_U = g|_U \).

**Proof.** For notational simplicity we assume that \( \Sigma(f) \) and \( \Sigma(g) \) are knots. The proof below easily generalizes to the case when \( \Sigma(f) \) and \( \Sigma(g) \) have more than one component.

It follows from the assumption \( \sigma(f) = \sigma(g) \), the Tubular Neighborhood Theorem, Proposition 2.2 and possibly after replacing \( f \) with a map to which \( f \) is \( \text{ls} \)-homotopic, that we can assume that \( \Sigma(f) = \Sigma(g) \), \( f|_{\Sigma(f)} = g|_{\Sigma(g)} \), and that there are tubular neighborhoods \( U \) of \( \Sigma(f) = \Sigma(g) \) and \( V \) of \( f(\Sigma(f)) = g(\Sigma(g)) \), with trivializations \( \alpha^1, \alpha^2 : S^1 \times D_2^\mathbb{R} \to U \) and \( \beta^1, \beta^2 : S^1 \times D_3^\mathbb{R} \to V \), such that \( f(U) \subset V \), \( g(U) \subset V \), and such that \( (\beta^1)^{-1} \circ f \circ \alpha^1 \) and \( (\beta^2)^{-1} \circ g \circ \alpha^2 \) satisfy Equation 2.3. Furthermore we can assume that \( \alpha^1, \alpha^2 \) and \( \beta^1, \beta^2 \) are orientation preserving when \( S^1 \times D_2^\mathbb{R} \) and \( S^1 \times D_3^\mathbb{R} \) are canonically oriented.
Let $\Sigma = \Sigma(f) = \Sigma(g)$, $\hat{\Sigma} = f(\Sigma(f)) = g(\Sigma(g))$, and let $\partial_0^g, \partial_1^g, \partial_2^g$ and $\partial_0^h, \partial_1^h, \partial_2^h, \partial_3^h$ be the coordinate vector fields on $U$ and $V$ induced by $\alpha^i$ and $\beta^i$ for $i = 1, 2$. If $(\partial_1^2|\Sigma, \partial_2^2|\Sigma)$ and $(\partial_1^2|\Sigma, \partial_2^1|\Sigma)$, respectively $(\partial_1^2|\Sigma, \partial_2^1|\Sigma)$ and $(\partial_1^2|\Sigma, \partial_2^2|\Sigma)$, are homotopic frames for $\Sigma$ and $\hat{\Sigma}$, then the identity maps of $U$ and $V$ are ambient isotopic, say via $\varphi_t : S^3 \to S^3$ and $\psi_t : \mathbb{R}^4 \to \mathbb{R}^4$, to $\alpha^1 \circ (\alpha^2)^{-1} : U \to U$ and $\beta^2 \circ (\beta^1)^{-1} : V \to V$ respectively. The homotopy $h_t : S^3 \to \mathbb{R}^4$ defined by $h_t = \psi_t \circ f \circ \varphi_t$ is an $L_3$-homotopy of $f$ such that $h_1|_U = g|_U$.

The frames $(\partial_1^2|\Sigma, \partial_2^2|\Sigma)$ and $(\partial_1^2|\Sigma, \partial_2^1|\Sigma)$ for $\Sigma$ are homotopic because $\sigma(f) = \sigma(g)$. Thus, after precomposing $f$ by the time-one map of an ambient isotopy of $S^3$ which restricts to an isotopy through automorphisms of $U$ from the identity map to $\alpha^1 \circ (\alpha^2)^{-1}$, we can assume that $\alpha^1 = \alpha^2$.

Let $\alpha = \alpha^1 = \alpha^2$. Define $\gamma : S^1 \to S^3$ by $\gamma(\theta) = \alpha(\theta, 0, r)$, and let $\mu : S^1 \to S^3$ be a parameterization of $\Sigma$ which is homotopic in $U$ to $\gamma$. Define $\hat{\gamma}, \hat{\mu} : S^1 \to \mathbb{R}^4$ by $\hat{\gamma} = f \circ \gamma$ and $\hat{\mu} = f \circ \mu$. Let $\hat{\partial}_0, \hat{\partial}_1, \hat{\partial}_2$ be the coordinate vector fields on $U_0$ induced by $\alpha$. The frames $(\partial_1^2|\Sigma, \partial_2^2|\Sigma)$ and $(\partial_1^2|\Sigma, \partial_2^1|\Sigma)$ are homotopic if and only if the loops $A_1, B : S^1 \to GL^+(4)$ defined by

$$A_1(\theta) = [\partial_0^h(\hat{\mu}(\theta)), \partial_1^h(\hat{\mu}(\theta)), \partial_2^h(\hat{\mu}(\theta)), \partial_3^h(\hat{\mu}(\theta))]_E,$$

$$B(\theta) = [\partial_0^h(\hat{\mu}(\theta)), \partial_1^h(\hat{\mu}(\theta)), \partial_2^h(\hat{\mu}(\theta)), \partial_3^h(\hat{\mu}(\theta))]_E,$$

are homotopic. We prove that the homotopy class of $A_1$ is determined by the homology class of $\gamma$ in $H_1(S^3 \setminus \Sigma(f); \mathbb{Z}_2)$ and the homotopy class of the frame $(\partial_1|\Sigma, \partial_1|\Sigma, \partial_2|\Sigma)$. Then the same is true for $B$, and in particular $A_1$ and $B$ are homotopic.

The loop $A_1$ is homotopic to $A_2 : S^1 \to GL^+(4)$ defined by

$$A_2(\theta) = [\partial_0^f(\hat{\gamma}(\theta)), (\partial_1^f + r \partial_2^f)(\hat{\gamma}(\theta)), 2r \partial_2^f(\hat{\gamma}(\theta)), \partial_3^f(\hat{\gamma}(\theta))]_E$$

$$= [df(\partial_0(\gamma(\theta))), df(\partial_1(\gamma(\theta))), df(\partial_2(\gamma(\theta))), df(\partial_3(\gamma(\theta)))]_E,$$

which in turn is homotopic to the loop $A_3 : S^1 \to GL^+(4)$ given by

$$A_3(\theta) = (\Psi \circ df)(\partial_0(\gamma(\theta)), \partial_1(\gamma(\theta)), \partial_2(\gamma(\theta))).$$

Define $F : S^3 \setminus \Sigma(f) \to GL^+(4)$ by $F = \Psi \circ df \circ Q$. Then the homotopy class of $F \circ \gamma$ depends only on the homology class of $\gamma$ in $H_1(S^3 \setminus \Sigma(f); \mathbb{Z}_2)$, see the proof of Proposition 1.2. Now $A_3(\theta) = (F \circ \gamma)(\theta)A_4(\theta)$ where

$$A_4(\theta) = \begin{bmatrix} [\partial_0(\gamma(\theta)), \partial_1(\gamma(\theta)), \partial_2(\gamma(\theta))]_Q & 0 \end{bmatrix}.$$ 

Hence the homotopy class of $A_3$ is the sum of the homotopy classes of $F \circ \gamma$ and $A_4$. Finally, a homotopy from $\gamma$ to $\mu$ in $U$ induces a homotopy from $A_4$ to $A_5 : S^1 \to GL^+(4)$ defined by

$$A_5(\theta) = \begin{bmatrix} [\partial_0(\mu(\theta)), \partial_1(\mu(\theta)), \partial_2(\mu(\theta))]_Q & 0 \end{bmatrix}.$$ 

Since the homotopy class of $A_5$ depends only on the homology class of $(\partial_0|\Sigma, \partial_1|\Sigma, \partial_2|\Sigma)$ we conclude that the homotopy class of $A_1$ depends only on the homology class of $\gamma$ and the homotopy class of $(\partial_0|\Sigma, \partial_1|\Sigma, \partial_2|\Sigma)$.

\textbf{Lemma 5.2.} Let $B$ be a proper tubular neighborhood of a link in $S^3$, Let $M = S^3 \setminus B$, and let $U$ be an open neighborhood of $\partial M$ in $M$. Assume $f, g : M \to \mathbb{R}^4$ are immersions, and assume that the restrictions $f|_U$ and $g|_U$ are regularly homotopic via $h_t : U \to \mathbb{R}^4$. 

\begin{proof}

\end{proof}
Then \( f \) and \( g \) are regularly homotopic via a homotopy that agrees with \( h_t \) in an open neighborhood of \( \partial M \), possibly smaller than \( U \), if and only if the continuous map

\[
(\Psi \circ df \circ Q) \cup H \cup (\Psi \circ dg \circ Q) : M \times \{0\} \cup U \times I \cup M \times \{1\} \to GL^+(4)
\]

extends to a continuous map \( M \times I \to GL^+(4) \). Here \( H : U \times I \to GL^+(4) \) is the homotopy \( H(p,t) = (\Psi \circ dh_t \circ Q)(p) \).

**Proof.** The lemma follows from \([6, \text{Proposition 7.2.1}]\) and \([6, \text{Proposition 8.2.1}]\). \(\square\)

**Proposition 5.3.** Let \( f,g : S^3 \to \mathbb{R}^4 \) be stable maps such that \( \sigma(f) = \sigma(g) \). Then \( f \) is \( \text{LS} \)-homotopic to the connected sum of \( g \) with an immersion \( k : S^3 \to \mathbb{R}^4 \).

**Proof.** By Lemma \(5.4\) we may assume that \( f \) and \( g \) have a common singularity link \( \Sigma = \Sigma(f) = \Sigma(g) \), and that \( f|_U = g|_U \) for an open neighborhood \( U \) of \( \Sigma \).

Let \( W \subset U \) be a tubular neighborhood of \( \Sigma \), let \( B \) be an open proper disk subbundle of \( W \), let \( M = S^3 \setminus B \), and let \( \partial M \subset M_0 \subset M_1 \subset M_2 \subset M_3 = M \) be a handle decomposition of \((M, \partial M)\) such that \( M_0 = W \setminus B \) and \( M_3 \) is obtained from \( M_2 \) by the attachment of one 3-handle. Let \( h_t : U \to \mathbb{R}^4 \) be the constant homotopy \( h_t = f|_U = g|_U \). Define \( F, G : S^3 \setminus \Sigma \to GL^+(4) \) by \( F = \Psi \circ df \circ Q \) and \( G = \Psi \circ dg \circ Q \), define \( H : (M_0 \setminus \Sigma) \times I \to GL^+(4) \) by \( H(p,t) = (\Psi \circ dh_t \circ Q)(p) \), and define \( A_0 : M \times \{0\} \cup M_0 \times I \cup M \times \{1\} \to GL^+(4) \) by \( A_0 = F \cup H \cup G \).

Assume, to begin with, that \( A_0 \) extends to a continuous map

\[
A_2 : M \times \{0\} \cup M_2 \times I \cup M \times \{1\} \to GL^+(4).
\]

Let \( D \) be the 3-handle attached to \( M_2 \) to obtain \( M \), let \( S \) be the topological 3-sphere \( D \times \{0\} \cup \partial D \times I \cup D \times \{1\} \), and consider the restriction \( A_2|_S : S \to GL^+(4) \). Suppose the homotopy class of \( A_2|_S \) equals \( mi_*[\sigma] + ni_*[\rho] \), where \( i : SO(4) \to GL^+(4) \) is the inclusion and \([\sigma]\) and \([\rho]\) are the generators for \( \pi_3(SO(4)) \) defined in Subsection 3.4.

Let \( k : S^3 \to \mathbb{R}^4 \) be an immersion with Smale invariant \( \Omega_{4,3}(k) = -mi_*[\sigma] - ni_*[\rho] \), and let \( g k \) be a connected sum of \( g \) and \( k \) that agrees with \( g \) on \( M_2 \). Then, according to Kervaire [15], if we replace \( g \) by \( g k \) (and let \( A_2 \) be defined by \( A_2(p,1) = (\Psi \circ d(gkt \circ Q)(p) \) on \( D \times \{1\}) \), the homotopy class of \( A_2|_S \) is trivial, and hence \( A_2 \) extends to a continuous map \( M \times I \to GL^+(4) \). This, in turn, implies that \( f \) and \( g k \) are \( \text{LS} \)-homotopic by Lemma 5.2. Thus, the proposition follows if we can arrange it so that \( A_0 \) extends to \( M \times \{0\} \cup M_2 \times I \cup M \times \{1\} \).

Let \( W_0, \ldots, W_d \) be the components of \( W \). Assume \( W \) is so small that the sets \( W_0, \ldots, W_d \) defined by \( W_j = f(W_j) = g(W_j) \) are pairwise disjoint. Postcompose \( f \) and \( g \) by the time-one map of an ambient isotopy of \( \mathbb{R}^4 \) that moves the images of \( f \) and \( g \) so that each \( W_j \) is contained in the closed ball in \( \mathbb{R}^4 \) with radius \( \frac{1}{2} \) centered at \((4j + \frac{3}{2}, 0, 0, 0) \), so that \( a'_j = (4j - 1, 0, 0, 0) \in \hat{W}_{j-1} \) and \( b'_j = (4j + 2, 0, 0, 0) \in \hat{W}_j \) for \( j = 1, \ldots, d \), and so that small open subarcs of the first coordinate axis in \( \mathbb{R}^4 \) centered at the \( a'_j \)s and the \( b'_j \)s are contained in \( f(U) = g(U) \). Let \( B_0, \ldots, B_d \) be the components of \( B \), with \( B_j \subset W_j \), let \( V \) be a disk in the topological boundary of \( M_0 \) in \( M \), and let \( b \in V \).

Let \( H_1, \ldots, H_n \) be the 1-handles attached to \( \partial M_0 \) to obtain \( M_1 \). Each handle \( H_j \) has a core segment \( I_j \) with endpoints \( a_j \) and \( b_j \) on the topological boundary of \( M_0 \) in \( M \). We may assume, possibly after a handle slide, that \( a_j \in W_{j-1}, b_j \in W_j, f(a_j) = g(a_j) = a'_j \), and \( f(b_j) = g(b_j) = b'_j \) for \( j = 1, \ldots, d \), and that \( a_j, b_j \in V \) for \( j = d + 1, \ldots, n \). Moreover we may assume that the segments \( I_j \) are such that \( f(I_1), \ldots, f(I_d) \) is a sequence of pairwise disjoint embedded curves in \( \mathbb{R}^4 \), that each
f(I_j) intersects \( \tilde{W} \) only at \( a_j', b_j' \), and that \( f(I_j) \) is contained in the first coordinate axis close to \( a_j' \) and \( b_j' \). Now postcompose \( f \) with the time-one map of an ambient isotopy of \( \mathbb{R}^4 \) that keeps an open neighborhood of \( f(W) \) fixed, and for \( j = 1, \ldots, d \) moves the curve \( f(I_j) \) onto \( \{(x, 0, 0, 0) : 4j - 1 \leq x \leq 4j + 2\} \).

For \( j = 1, \ldots, d \) let \( \gamma_j : I \to I_j \) be simple paths such that \( \gamma_j(0) = a_j \) and \( \gamma_j(1) = b_j \), and define loops \( u_j : S^1 \to I_j \times \{0\} \cup M_0 \times I \cup I_j \times \{1\} \) by

\[
u_j(\theta) = \begin{cases} (a_j, \frac{2}{\pi} \theta) & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\ (\gamma_j(\frac{2}{\pi} \theta - 1), 1) & \text{if } \frac{\pi}{2} \leq \theta \leq \pi, \\ (b_j, 3 - \frac{2}{\pi} \theta) & \text{if } \pi \leq \theta \leq \frac{3\pi}{2}, \\ (\gamma_j(4 - \frac{2}{\pi} \theta), 0) & \text{if } \frac{3\pi}{2} \leq \theta \leq 2\pi. \end{cases}
\]

For \( j = d + 1, \ldots, n \) let \( \mu_j : S^1 \to V \cup I_j \) be simple loops such that \( \mu_j(0) = b \), and define loops \( u_j : S^1 \to I_j \times \{0\} \cup V \times I \cup I_j \times \{1\} \) by

\[
u_j(\theta) = \begin{cases} (b, \frac{2}{\pi} \theta) & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\ (\mu_j(4\theta - 2\pi), 1) & \text{if } \frac{\pi}{2} \leq \theta \leq \pi, \\ (b, 3 - \frac{2}{\pi} \theta) & \text{if } \pi \leq \theta \leq \frac{3\pi}{2}, \\ (\mu_j(8\pi - 4\theta), 0) & \text{if } \frac{3\pi}{2} \leq \theta \leq 2\pi. \end{cases}
\]

For \( j = 1, \ldots, d \) let \( \lambda_j : \mathbb{R} \to \mathbb{R} \) be a nondecreasing smooth function such that \( \lambda_j(x) = 0 \) for \( x < 4j \) and \( \lambda_j(x) = 1 \) for \( x > 4j + 1 \). Define the isotopy \( \psi^j : \mathbb{R}^4 \to \mathbb{R}^4 \) by

\[
\psi^j(x, y, z, w) = (x, y \cos(2\pi t \lambda(x)) - z \sin(2\pi t \lambda(x)), y \sin(2\pi t \lambda(x)) + z \cos(2\pi t \lambda(x)), w).
\]

Deform \( Q \) so that for \( j = 1, \ldots, d \) we have \( F(a_j) = G(a_j) = G(b_j) = F(b_j) \), where \( F \) and \( G \) are as defined above. Then each pair \( F \circ \gamma_j \) and \( G \circ \gamma_j \) is a pair of loops in \( GL^+(4) \). Inductively, starting with \( j = d \) and counting backwards, deform \( f \) using \( \psi^j \circ f \) into \( \psi^j \circ f \) if \( F \circ \gamma_j \) and \( G \circ \gamma_j \) are not homotopic. Otherwise leave \( f \) unchanged.

The resulting map, which we also denote by \( f \), is such that \( F \circ \gamma_j \) and \( G \circ \gamma_j \) are homotopic for \( j = 1, \ldots, d \).

Recall that \( A_0 : M \times \{0\} \cup M_0 \times I \cup M \times \{1\} \to GL^+(4) \) was defined as \( A_0 = F \cup H \cup G \). The map \( A_0 \) extends continuously to \( M \times \{0\} \cup M_1 \times I \cup M \times \{1\} \) if and only if \( (A_0)_*[u_j] = 0 \) for \( j = 1, \ldots, n \). Here \( (A_0)_* \) is the homomorphism

\[
(A_0)_* : H_1(M \times \{0\} \cup M_0 \times I \cup M \times \{1\}; \mathbb{Z}_2) \to \pi_1(GL^+(4)),
\]

induced by \( A_0 \), where we have identified \( \pi_1(GL^+(4)) \) with \( H_1(GL^+(4); \mathbb{Z}_2) \) via the Hurewicz isomorphism. Now \( (A_0)_* \) satisfies

\[
(A_0)_*[u_j] = \begin{cases} [F \circ \gamma_j] - [G \circ \gamma_j] & \text{for } j = 1, \ldots, d, \\ [F \circ \mu_j] - [G \circ \mu_j] & \text{for } j = d + 1, \ldots, n. \end{cases}
\]

But \( [F \circ \gamma_j] = [G \circ \gamma_j] \) by construction, and the homotopy classes \( [F \circ \mu_j] \) and \( [G \circ \mu_j] \) depend only on the homology class \( [\mu_j] \), which we proved in Proposition 4.2. Hence \( [F \circ \mu_j] = [G \circ \mu_j] \), and \( A_0 \) extends to a continuous map

\[
A_1 : M \times \{0\} \cup M_1 \times I \cup M \times \{1\} \to GL^+(4),
\]

Finally \( A_1 \) extends to \( M \times \{0\} \cup M_2 \times I \cup M \times \{1\} \) because \( \pi_2(GL^+(4)) = 0 \).
6. Proofs of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. The implication from left to right follows from Propositions 2.5, 3.2, and 3.5. To prove the opposite implication let \( f, g : S^3 \to \mathbb{R}^4 \) be locally stable maps, and assume that \( \sigma(f) = \sigma(g) \), \( \nu(f) = \nu(g) \), and \( \kappa(f) = \kappa(g) \). By Proposition 5.3 there exists an immersion \( h : S^3 \to \mathbb{R}^4 \) such that \( f \) is \( \operatorname{LS} \)-homotopic to \( g \circ h \). The equations

\[
\nu(h) = \nu(g) + \nu(h) - 1 - \nu(g) + 1 = \nu(f) - \nu(g) + 1 = 1
\]

\[
\kappa(h) = \kappa(g) + \kappa(h) - \kappa(g) = \kappa(f) - \kappa(g) = 0
\]

together with Proposition 3.14 imply that \( h \) is regularly homotopic to the standard embedding \( S^3 \to \mathbb{R}^4 \). Hence \( f \) and \( g \) are \( \operatorname{LS} \)-homotopic. \( \square \)

**Proposition 6.1.** If \( f : S^3 \to \mathbb{R}^4 \) is an immersion, then the integers \( \nu(f) \) and \( \frac{1}{2} \kappa(f) \) have different parity. Conversely, for each pair \( a, b \in \mathbb{Z} \) such that \( a \) and \( b \) have different parity there exists an immersion \( f : S^3 \to \mathbb{R}^4 \) with \( \nu(f) = a \) and \( \kappa(f) = 2b \).

**Proof.** The proposition follows from Lemmas 3.11, 3.12, and 3.13. \( \square \)

Proof of Theorem 1.2. By Proposition 4.2 there exists a locally stable map \( f : S^3 \to \mathbb{R}^4 \) such that \( \sigma(f) = \sigma \). Let \( g : S^3 \to \mathbb{R}^4 \) be any locally stable map such that \( \sigma(g) = \sigma \). Then, by Proposition 5.3 there exists an immersion \( h : S^3 \to \mathbb{R}^4 \) such that \( g \) is \( \operatorname{LS} \)-homotopic to \( f \circ h \). Let \( a = \nu(h) - 1 \) and \( b = \frac{1}{2} (\kappa(h) - 2a) \). Then \( b \) is an integer by Proposition 6.1 and \( \nu(g) = \nu(f) + a \) and \( \kappa(g) = \kappa(f) + 2a + 4b \) by Propositions 3.8 and 3.9. This proves the first and second assertion.

To prove the last assertion let \( h : S^3 \to \mathbb{R}^4 \) be an immersion with Smale invariant \( a[\sigma] + b[\rho] \). Then \( g = f \circ h \) is a locally stable map that satisfies \( \sigma(g) = \sigma \), \( \nu(g) = \nu(f) + a \), and \( \kappa(g) = \kappa(f) + 2a + 4b \). \( \square \)

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