Views, variety and quantum mechanics

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Abstract

A non-local hidden variables theory for non-relativisitic quantum theory is presented, which gives a realist completion of quantum mechanics, in the sense of a complete description of individual events. The proposed fundamental theory is an extension of an energetic causal set theory, ([12]- [15]), which assumes that time, events, causal structure, momentum and energy are fundamental. But space and the wave function are emergent.

The beables of the theory are the views of the events, which are a subset of their causal pasts. Thus, this theory asserts that the universe is a causal network of events, which consists of partial views of itself as seen by looking backwards from each event.

The fundamental dynamics is based on an action whose potential energy is proportional to the variety, which is a measure of the diversity of the views of the events, while the kinetic energy is proportional to its rate of change. The Schrodinger equation is derived to leading order in an expansion in density of the events of the fundamental histories. To higher order, there are computable corrections, non-linear in the wave function, from which new physical effects may be predicted.

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1 Introduction

"Inconsistencies challenge the legitimacy of the theories they afflict [34]"

In this paper we further our understanding of a class of realistic theories which are posited to be satisfactory completions of both general relativity and quantum theory. Unlike an interpretation, a completion requires the introduction of new dynamics. These theories are based on a number of ideas, some of which are new here, others of which have appeared in papers over the last several years [16, 12, 13].
The belief that quantum-mechanics is incomplete is as old as the theory itself. The first completion of quantum mechanics, was proposed, by Louise de Broglie, even before quantum mechanics was fully formed[1]. David Bohm rediscovered it 25 years later[2]. Other realist completions of quantum mechanics include Nelson’s stochastic mechanics[3]; meanwhile Steve Adler[4] Artem Starodubtsev[5] and Markopoulou and the author constructed hidden variables theories from matrix models[9, 10]

In earlier proposals for completions of quantum mechanics, I introduced the concept of non-local and relational hidden variables theories[6, 27]. These theories were developed in a number of versions[9, 10, 7, 8].

The class of theories we study here are called the causal theory of views. (CTV) They were introduced in [16], and were built on a wider framework called energetic causal sets, which were developed with Marina Cortes[12]-[15]. They share some features with causal set models[11] - although also some differences. Other models which embellish causal set models with charges and other conserved quantaties were introduced by Cohl Furey[20] and Fotini Markopoulou[19].

The role of variety[28] in the formulation of completions of quantum mechanics was introduced in [26] and developed in [16]. Some related ideas concerned with the real ensemble formulation were introduced in [25] and developed in[26] and [16].

1.1 The main ideas

• The first idea is that space is emergent, but time in the sense of causality is fundamental[1]

• But if space is emergent, at the fundamental level there is no space, hence no notion of distance, hence no notion of locality. And no notion of non-locality, either. Local- ity and non-locality are both emergent. Hence so must be the distinction between quantum and classical dynamics.

But dynamics is about moving in space. Without space, there are no distances to measure the fall-off of forces, there are no fields, no derivatives etc. What can dy- namics involve?

Also, without space, what is a reference system? What is the purpose of a symme- try?

Einstein had several different motivations in his search for general relativity. Rather late in the process, he understood the role of gauge invariance under active diffeo- morphisms. It took a good think through the hole experiment-which he initially misunderstood. Once he had that he was essentially done. But another motivation was to relativize the concept of inertia so that there was an expansion of the relativ- ity principle from an equivalence of inertial frames to a general principle of relativity

\[1\]The first person to propose this option forcefully was Fotini Markopoulou[21].
under which all frames would be equivalent. In this he failed, which is good because the premise is wrong.

There seems to be no equivalence between inertial and accelerating motion. But if space takes a walk that distinction also disappears and there is a path to start with a general equivalence of observers. This is the motivation for what follows.

- We democratize and universalize the notion of a frame of reference by replacing it with the notion of a view of an event \[16\].

In most experiments we deploy just one or two frames of reference. But every event has a view, which is short for its view of its causal past. That is the information available from physical degrees of freedom whose coincidence constitutes that event, about the past causal progenitors of that event.

- We then replace measures of distance, measured according to a fixed metric, \(q_{ab}\), in background dependent formulations of physics, with the measures of differences between views.

We postulate that nature has an innate capability of judging differences between views. This may seem strange, but it is no stranger than the familiar metaphysical assumptions that underlie standard, background dependent physics. Namely to assume that particles of matter automatically know their distances to all other particles, as is necessary if the particle is to be able to figure out how much and in what direction it should accelerate. Or, alternatively, how do particles “know” what representations they are members of and what their couplings should be to each of the other fields.

- We propose to construct dynamics for a background independent theory with no space to be embedded in, directly in terms of differences between views. If we posit that the relations that define individual events are causal relations, then the views are views not just of the nearby neighbourhood but of the causal past.

- To some extent, increasing distances may have the effect of increasing a difference of views, as new items come into view as you travel away from your starting point. But there may also be two atoms or molecules, far away from each other in physical space, which have very similar views, given that being simple systems they have only a few qubits of information. These are then both members of an ensemble containing similar views, which includes them and a myriad of other particles. We will see that, by virtue of having similar views, they interact strongly with each other.

- Nature can then distinguish unique views, which have no copies anywhere in the universe.
• Unique events (ie events with unique views), have definite values, they are the be-
ables.

• Unique views, we postulate, have definite causal relations amongst themselves. This is part of the resolution of the question of whether causal structure is definite.

• We want to express the postulate that the dynamics of the particles in the universe act to increase the total diversity of their views. A measure of that diversity is called the variety; roughly speaking it is defined as the sum over all pairs of views of a function of the distance between those views. Increasing diversity is accomplished by setting the potential energy equal to a constant times the variety \[28, 33\].

Our theory is intrinsically relativistic, as must be any theory whose ontology includes discrete events related by causal processes\[11, 12, 13\]. Our theory generates causal sets that, when they embed in an emergent space, embed into a Minkowski spacetime\[11, 12, 13\].

But at the dynamical level the theory breaks lorentz invariance in that the Hamiltonian consists of a sum of kinetic energy and potential energy terms, and their is no simple boost generator that mixes them.

In this paper we discuss mainly the non-relativistic limit, developing ideas and tools that we hope will be of use in more realistic theories. So we will take the non-relativistic limit earlier than in the previous presentations of this idea\[26, 16\].

2 The kinematics of views

First, we give a reformulation of the causal theory of views, which has been described previously in \[16\].

2.1 Kinematics without space

We begin by defining the components of a causal theory of views.

We posit that an history is an energetic causal set, \(ECS\), which consists of a set of events, \(E = \{I, J, K, ..\}\), related by a set of direct causal links, \(L^I_J\). Each event \(I\) has a set of \(n^I_0\) parents and a set of \(n^I_c\) children. A single event may have be up to \(n^I_{pre}\) predecessors, and each of these may have up to \(n^I_c\) progeny.

Note that the causal links are a subset of the causal relations, which are inferred from the causal links by transitivity.

To each link, \(J^I\), there is attached an energy-momentum \(p^I_{0,J} \in P\) that links event \(K\) to event \(J\). When the event \(J\) occurs, that much energy-momentum is transferred from event \(K\) to event \(J\).

The \(P\) live in an energy-momentum space \(P\), which has a norm and may have a non-trivial connection. In the rest of this paper we shall assume that this energy - momentum
space is flat. The question of how curving the momentum space affects the physics is closely related to the relative locality theories\cite{23, 24}, which indeed was one of the inspirations for EC\cite{22}.

At each event, $I$ there is a conservation law for energy-momentum. This is expressed as:

$$P_{aI} = \sum_{K \in IPast(I)} P_{aK}^I - \sum_{L \in IFut(I)} P_{aL}^I = 0$$

(1)

Here $IPast$ and $IFut$ are the immediate past and immediate futures of the event $I$. We also denote this by $J| > K$.

The collection of information about the incoming energy and momentum transferred to an event from an event’s immediate predecessors is called the view of the event, $\mathcal{V}_I$. The simplest way to represent it is by means of an unordered list.

$$\mathcal{V}_I = \{p_{aJ}^I, \ldots\}.$$  

(2)

3 Dynamics for theories of views

3.1 Taking the non-relativistic limit.

The energy-momentum transferred, $p_{aK}^I$, have dimensions of momentum, so

$$p_{0K}^J = \frac{1}{c} E_K^J$$

(3)

We define the non-relativistic limit by the limit $c \to \infty$, which implies that all the zero components vanish $p_{0K}^I \to 0$. Similarly

$$x_0 = \frac{t}{c} \to 0$$

(4)

3.2 Half-quantum theory: the half integral

The dynamics is defined by a path integral, but of a rather unusual sort. Normally one defines the canonical path integral as an integral over phase space-seen as a bundle over the configuration space. In many simple cases, the integrals over momentum are Gaussian and can be done directly. The result is an integral over configuration space.

The theory we are presenting here is structured a bit differently. One of our core ideas is that the fundamental theory is defined only in terms of momenta variables plus causal structure. Spacetime is to emerge.

\textsuperscript{2}Notation: $\alpha = 0, 1, 2, 3$ is a spacetime index, $a=1,2,3$ is a spatial index, while $I, J, K = 1 \ldots N$ is a label on the events.
So we begin with what we call a half integral\[13].

\[
Z[\Gamma] = \prod_{J|>K \in \Gamma} \int dP_a^K \prod_I \delta(\mathcal{P}_I^a) e^{-i\mathcal{H}(P)}
\] (5)

We call this half-quantum theory, defined by the half integral, because we integrate over momentum and energy but not position, which indeed nowhere appears. Since only half of the canonical pair appears, there are no canonical commutation relations. Hence, if \(\hbar\) appears at some point, there will have to be a different role for it.

We define the half-integral for a specific causal network, denoted by \(\Gamma\), so all events, \(I\) are in \(\Gamma\). We will here focus on these integrals, with \(\Gamma\) very large and complex, in the sense of halving large values of the variety \(\mathcal{V}[\Gamma]\). In this paper \(\Gamma\) is fixed as we are interested in the sums over labels on \(\Gamma\). We will consider varying \(\Gamma\) later.

### 3.3 Exponentiating the constraints

To give the full definition of the theory (5) we next want to specify the Hamiltonian, \(\mathcal{H}\).

The first step is to exponentiate the constraints that generate energy-momentum conservation in (6). We introduce the dual space to the momentum space \(\mathcal{P}^*\), which has elements \(z_I^a\). We need one for every event.

\[
Z = \prod_{J|>K} \int dP_a^K \prod_I \int dz_I^a e^{-i \sum I z_I^a \mathcal{P}_I^a - \mathcal{H}(P)}
\] (6)

Next we specify the form of the Hamiltonian \(\mathcal{H}(P)\) which we will take to be the sum of kinetic and potential energy terms.

\[
\mathcal{H} = gT + g'U
\] (7)

Note that the dimensions of \([g] = \text{time/mass}\), while the dimensions of \([g'] = \left[ \frac{\hbar^2}{8mg^2} \right]\).

So the effective action is given by

\[
S^{\text{eff}} = \sum_I z_I^a \mathcal{P}_I^a - gT - g'U
\] (8)

### 3.4 Difference and variety

The next step is to construct detailed kinetic and potential energy terms. Both are based on the idea of variety, which is a measure of how diverse the views are. The kinetic energy, \(T\), measures the rate of change of views among causally related events. \(U\) is a potential energy that measures the diversity of views among causally unrelated events.
These are both based on measures of difference between pairs of views, which is gener-
ically denoted $D(I, J)$. The sum over these gives the total variety:

$$V = \frac{2}{N(N-1)} \sum_{I,J} D(I, J)$$  \hspace{1cm} (9)

A convenient form for $D(I, J)$ is

$$D(I, J) = (W_I - W_J)^2$$  \hspace{1cm} (10)

where $W_I$ is the view seen by or at the event, $I$.

The different theories in this class come from three choices which must be made.

1. how the views are defined and represented,
2. how the differences $D(I, J)$ between views are defined and
3. which pairs of views are summed over to define the kinetic and potential energies.

In the theories discussed here, we take the views to be a set of incoming energy mo-
momentum vectors, generally density weighted, where the weight is $w$.

$$W^I_a = \sum_{K \in \text{Past}(I)} \frac{p^I_{aK}}{|p^I_{aK}|^w}$$  \hspace{1cm} (11)

I believe it may be the case that there is a single universality class (or just a few) among
these theories, but at the present state of knowledge I look for simple choices that lead to
simple derivations of the recovery of quantum mechanics.

3.5 Constructing the kinetic energy

Consider an event $I$ and an event $J$ in its immediate causal past. We indicate this by

$$I \mid > J$$  \hspace{1cm} (12)

We define the surprise at an event $I$ to be a measure of how much it differs from its imme-
diate predecessors.

$$\text{Surprise}(I) = | \sum_{K \in \text{IPast}(I)} D(I, K)|^2$$  \hspace{1cm} (13)

So one simple definition of kinetic energy is

$$T_{\text{surprise}} = \sum_I \text{Surprise}(I)$$  \hspace{1cm} (14)
Here we will make use of a similar form. We define the causal variety with $N$ events by

$$ T = \left( \sum_{I>j} (W^{(p=0)}_a I - W^{(p=0)}_a J)^2 \right) $$

(15)

Written out this is

$$ T = \left( \sum_{I>j} \left( \sum_{l>K} p^I_{aK} - \sum_{j>L} p^J_{aL} \right) \right)^2 $$

(16)

3.6 Constructing the potential energy

The potential energy is

$$ U = \sum_{I<>J} D(I, J)_{w=2} $$

(17)

where $I <> J$ means $I$ and $J$ have no causal relation, i.e. analogous to a spacelike separation.

Written out this becomes

$$ U = U_{<>} = \sum_{I<>J} \left( \sum_{l>K} \frac{p^I_{aK}}{\left| p^I_{aK} \right|^2} - \sum_{j>L} \frac{p^J_{aL}}{\left| p^J_{aL} \right|^2} \right)^2 $$

(18)

3.7 Summary of the theory

We define the Hamiltonian:

$$ \mathcal{H} = gT + g'U $$

(19)

Note that the only geometry referenced in the action is the geometry of momentum space $\mathcal{P}$.

In the course of deriving the Schrödinger equation to leading order in $g'$, we will find also that to zeroth order in $g'$, the Hamilton Jacobi equations are satisfied. This implies the conservation in time of the Hamiltonian, (19).

The corresponding half integral is

$$ Z = \prod_{J|>K} \int dp^K_{aJ} \prod_I \delta(\mathcal{P}^I_{a}) e^{i(gT + g'U)} $$

(20)

This defines the theory\(^3\)

---

\(^3\)It is easy to check that the dimensions of $g$ are $\text{time}^{-1} \text{mass}$ while the dimensions of $g'$ are those of $\frac{\text{energy}^2}{\text{mass}}$. Note that we do not use units in which $\hbar$ or $c$ are equal to one.
In fact, we will see as we go along that we will want to choose

\[
g' = \frac{\hbar^2 m}{8ZV}
\]

to get quantum mechanics to emerge at the end. Indeed, it is only through this choice for \( g' \) that \( \hbar \) enters the theory at all. \( ZV \) is a dimensionless constant, defined below.

4 Evaluating the half integral

What are we going to do with this theory? There is no space or spacetime. The only time ordering is causal order.

How do we get physics out of it? We start our analysis with the form resulting from exponentiating the constraints.

\[
Z = \prod_j^K \int dp_{aM}^K \prod_I \int dz_I^a e^{(\sum_I z_I^a p_I^a + gT + g'U)}
\]

The above expression, with the effective action

\[
S_{\text{eff}} = -\sum_I z_I^a p_I^a + gT + g'U
\]

gives us a somewhat different, but equally precise definition for the theory; we then proceed to evaluate the partition function. We do this in a series of steps.

We have picked the order of integration to integrate over the momentum variables, \( p_{aM} \) first, for fixed values of the \( z_I^a \). Then at the end we will integrate over the \( z_I^a \) in a certain course grained approximation.

4.1 Observables

Let \( \mathcal{A}[z^a_I] \) be a functional of the \( N \) \( z^a_I \)'s, as well as the choice of an \( N \)-event causal set. Then the expectation value is defined by

\[
< \mathcal{A}(z^a_I) > = \frac{1}{Z} \prod_j^K \int dp_{aJ}^K \prod_I \int dz_I^a \mathcal{A}(z^a_I) e^{S_{\text{eff}}}
\]

\[
= \frac{1}{Z} \prod_I \int dz_I^a \mathcal{A}(z^a_I) \prod_j^K \int dp_{aJ}^K e^{S_{\text{eff}}}
\]

where we have exchanged the order of integration.
4.2 Integrating over the transverse momenta

We are going to approximate these integrals by the stationary phase approximation. To do this it will be convenient to divide the momentum fluctuations into transverse and longitudinal parts and then begin with the former.

The result of integrating over the momenta is very simple, so I state it up front. We will be able to satisfy the condition of stationary phase for \( p_{aK}^L \) when \( z_K^b \) and \( z_L^b \) are located so that

\[
p_{aK}^L g^{ab} = \frac{1}{gn_{pre}[z_L^b - z_K^b]} + \mathcal{O}(g'^2).
\]

(25)

here \( g^{ab} \) is the metric on momentum space. That is going to tie down the momenta transferred from event \( K \) to event \( L \) to be a function of the differences between the two \( z_K^b \) of the events they connect.

Recalling that \( g \) is proportional to a time per mass, this is just the ordinary,

\[
p = m \frac{\Delta z}{\Delta t}
\]

(26)

Recall that there is one of these relations for every pair of connected events. Each can be thought of as a little fragment of a three dimensional space. But to assert that space has emerged we have to find consistent solutions for all \( J \mid > K \).

- **Split the integrals over momentum transferred, \( p_{aJ}^I \), for each pair of causally connected events, \( I \mid > J \), into transverse and longitudinal fluctuations.**

In the neighbourhood around a particular \( p_{aJ}^K \) (technically the cotangent space of \( \mathcal{P} \)) there are two kinds of fluctuations (or small variations): transverse and longitudinal. These may be defined in terms of projection operators:

\[
\mathcal{T}_{ac} = [\delta_{ac} - \frac{p_{aK}^L p_{aJ}^L}{|p_{aK}^L|^2}],
\]

(27)

\[
\mathcal{L}_{ac} = \frac{p_{aJ}^K p_{aJ}^K}{|p_{aJ}^K|^2}
\]

(28)

\[
\mathcal{L}_{ac} + \mathcal{T}_{ac} = \delta_{ac}
\]

(29)

independently for each \( J \) and \( K \).

We can then, for each \( J \mid < |K \) divide the fluctuations in the momenta \( \delta p_{aJ}^K \) into transverse and longitudinal modes

\[
\delta p_{aJ}^{LK} = \mathcal{L}_a \delta p_{cJ}^{LK}, \quad \delta p_{aJ}^{TK} = \mathcal{T}_a \delta p_{cJ}^{TK}
\]

(30)

We will write

\[
p_{aL}^M = l_{aL}^M + t_{aL}^M
\]

(31)
• Find equations of motion for transverse momentum fluctuations.

We are going to carry out the integrals over the transverse momenta first. We make use of the stationary phase approximation, which means we have to work out the equations of motion for the transverse momentum modes.

\[
\frac{\delta S^{\text{eff}}}{\delta t_c M} = 0
\]  

(32)

Each of the three terms in the effective action contributes to the fluctuations of the momentum modes.

• Compute the gradient of the potential energy.

We first compute the term from the gradient of the potential energy

\[
\Pi_{cK} = \frac{\delta U}{\delta p_{cK}} = 2 \frac{1}{|p_{dK}|} \mathcal{T}^{cd}(\mathcal{W}_d^I - \mathcal{W}_d^K)
\]  

(33)

So we find that \( U \) is pure transverse. Of course might have known this from the fact that it is composed of transverse functions, \( \tilde{p}_{aJ} = \frac{p_{aJ}}{|p_{aJ}|^2} \).

• Compute the gradient of the kinetic energy

For the kinetic energy it is easiest to choose the density weight \( w = 0 \). Together with the fact that we only compare views one causal step apart, it is much simpler to evaluate. It mostly comes down to separating cases.

Consider an event \( I \) and its view, which is made up of the events in its immediate causal past.

Notice that there can be multiple paths between two events, which implies that there may be closed paths on the momentum space. For example, suppose \( I \) has two direct antecedents, which are \( J \) and \( K \). Lets further consider \( L \) which is an antecedent of both \( I \) and \( J \), but not \( K \).

There are two paths to get from \( L \) to \( I \), one direct via the link \( p_{aL}^I \). Or the energy may travel from \( L \) to \( J \) via \( p_{aL}^J \) and then From \( J \) to \( I \) on \( p_{aJ}^I \).

We posit that two momentum vectors are added together by parallel transporting the first along the second. Then they must satisfy the linear equations.

\[
p_{aL}^I = p_{aL}^J + p_{aJ}^I
\]  

(34)

Now the kinetic energy is proportional to square of the difference between \( \mathcal{W}_I \) and \( \mathcal{W}_J \).
More precisely
\[ T = \sum_I \left( \sum_{J \in \text{Past}(I)} p_a^I - \sum_{K < \text{Past}(I)} p_a^J \right)^2 \] (35)

It is easy to see in this case this is just
\[ T = \sum_{I|>J} D_I^2 (p_a^I)^2 = \sum_{I|>J} D_I \left( \left( t_a^I \right)^2 + \left( l_a^I \right)^2 \right) \] (36)

So the contribution to the propagation or equation of motion of the momenta from the kinetic energy is both longitudinal and transverse

Here \( D_I \) is the number of immediate predecessors event \( I \) has. Sometimes this is set
\[ D_I = n_{\text{pre}} \] (37)

We can then put all three terms the variation of the action together to find
\[
\left[ p_a^K \right] = \frac{1}{g n_{\text{pre}}^2} (z_a^I - z_a^K) - \frac{g'}{g n_{\text{pre}}^2} T^{ab} (W_b^I - W_b^K) \] (38)

At a stationary phase point, \( \Pi_{aM}^N = 0 \), we can project the dual vector on momentum space into longitudinal and transverse modes with respect to each \( p_{aN}^M \).

The result is
\[
(z_a^M - z_a^M)_L = g n_{\text{pre}}^2 I_{aM}^N \] (39)

We also have
\[
(z_a^N - z_a^M)_T = g n_{\text{pre}}^2 I_{aM}^N + g' T_{aM}^N (W_b^N - W_b^M) \] (40)

Note that this inversion from a functional of the \( p_a^K \) to functional of the \( z_a^I \) always exists because the dynamics ensures that there is no ambiguity coming from the possibility that the map between the cotangent spaces of \( V \) and the tangent spaces. But there cannot be because such a singular configuration would have infinite energy.

• Integration of the potential energy term.

We use the information we now have about the stationary phase approximation to integrate out the transverse small fluctuations. The present situation is
\[
Z = \prod_I \int dz_I^a \prod_J^K \int dt_a^K_J \prod_J^K \int dt_a^K_J e^{t \left( -\sum_I z_I^a p_a^I + g n_{\text{pre}}^2 (l^2 + t^2) + g' t \right) + g' \int dt}
= \prod_I \int dz_I^a \prod_J^K \int dt_a^K_J \mathcal{Y}(z_I^a, t_a^K) \] (41)
We start with (41). We are next going to do the integral over $\int dt^J_a K$.

This is of the form

$$\mathcal{Y} = \prod_{J|>K} \int dt^J_a K e^{i(z^a_J - z^a_K)(t^J_a + t^J_K) - g n_{pre}^2 \left\{(t^J_a)^2 + (t^J_K)^2\right\} + g' U(p^N_{aM})}$$

(42)

If we set $g' = 0$ the integrals simplify to pure Gaussian integrals.

$$\mathcal{Y}_{g'=0} = \prod_{J|>K} \int dt^J_a K e^{i(z^a_J - z^a_K)(t^J_a + t^J_K) - g n_{pre}^2 \left\{(t^J_a)^2 + (t^J_K)^2\right\}}$$

(43)

We perform the integral over the $t^J_a K$, yielding

$$\mathcal{Y}_{g'=0} \approx e^{i\left((z^a_J - z^a_K)(t^J_a + t^J_K) + \frac{1}{g'} (z^a_J - z^a_K)^2\right)}$$

(44)

The stationary phase condition for the transverse integral is

$$t^J_a K = \frac{1}{2 g n_{pre}} \left(z^a_J - z^a_K\right) + \mathcal{O}(g'^2)$$

(45)

Note that the whole of $(z^a_J - z^a_K)$ is involved in the stationary phase condition (45).

We can then write, an expansion in powers of $g'$

$$\mathcal{Y} \approx \prod_{J|>K} \int dt^J_a K e^{i\left((z^a_J - z^a_K)(t^J_a + t^J_K) - g(t^J_a)^2\right)} [1 + g' U(p^N_{aM})]$$

$$\approx e^{i\left((z^a_J - z^a_K)(t^J_a + t^J_K) + \frac{1}{g'} (z^a_J - z^a_K)^2\right)} [1 + g' < U > + \mathcal{O}(g'^2)]$$

(46)

The potential energy is then a function of the $z^a_I$.

$$U(z) = < U > = \frac{1}{g^2} \sum_{I|<J} \left( \sum_{K \in \text{IPast}(I)} \frac{z^a_I - z^a_K}{|z^a_I - z^a_K|^2} - \sum_{L \in \text{IPast}(J)} \frac{z^a_J - z^a_L}{|z^a_J - z^a_L|^2} \right)^2$$

(47)

The effective action is then a functional of the $z^a_I$ and the longitudinal part of the momentum variables

$$S^{eff}(z^a_I, l_a) = \frac{1}{g^2} \sum_{JK} (z^a_J - z^a_K)^2 + \sum_{I} z^a_I L^I_a + g n_{pre}^2 g |l_{aJ}|^2 + g' U^{eff}$$

(48)
where the effective potential is,

\[ U_{\text{eff}} = \frac{g'}{g^2} \sum_{I<J} \left( \sum_{K \in \text{Past}(I)} \frac{z_I^a - z_K^a}{|z_I^a - z_K^a|^2} - \sum_{N \in \text{Past}(J)} \frac{z_J^a - z_N^a}{|z_J^a - z_N^a|^2} \right)^2 + C + \mathcal{O}(g'^2) \] (49)

Here we have defined the constant,

\[ C = \sum_{JK} \frac{g'}{g^2} (z_J^a - z_K^a)^2 \] (50)

This is a divergent constant and we will remove it in what follows.

To summarize, we have carried out the integration over the transverse parts of the momentum variables. We will proceed to carry out integration of the longitudinal momentum variables, saving the integration over the embedding coordinates, \( z_I^a \) for last.

### 4.3 Integration over the longitudinal momenta

- We are left with integrals to do in the \( z_I^a \) and the longitudinal momenta, \( l_{aJ}^K \). We have

\[ Z = \prod_I \int dz_I^a \prod_{JK} \int (dl_{aJ}^K) L e^{-i \sum_I z_I^a \mathcal{L}_a^I - gD(l_{aJ}^J)^2 + g'U(l)} \] (51)

where the potential energy (49) is now a functional of the \( z \)'s.

- Integrating by coarse graining.

We now want to integrate over the embedding coordinates, \( z_I^a \) and the longitudinal parts of the momentum transferred, \( l_{aJ}^K \). We will do these integrals by coarse graining. That is, we are adopting a Wilsonian point of view over which the path integral over short scale fluctuations, with an infrared cutoff, \( L \), is equivalent to doing a coarse graining which replaces sums over microscopic variables by averages over coarse grained collective degrees of freedom.

In the present case that means we replace the sums over the positions of the embedding coordinates, \( z_K^a \) by an integral over probability distributions, \( \rho(z) \). We assume that the number of events in the causal network, which we denote by \( N \), is large. We emphasize again that these \( N \) events and their causal relations become a single quantum particle under coarse graining.

The details of how a single classical (in this case) relativistic particle emerge from a causal network of events, are in the first two papers on energetic causal sets [12, 13].
For example, consider an observable $A(z^a)$ which is a function of the $z_i^a$'s. The basic definition of the average or expectation value of $A$ is

$$\langle A \rangle = \frac{1}{N} \sum_{K=1}^{N} A(z_K)$$

(52)

where $N$ is the total number of events in $\Gamma$.

We will use $\rho(z)$ to replace this with

$$\langle A \rangle = \frac{1}{N} \sum_{K=1}^{N} A(z_K) \rightarrow \int_{a}^{R} d^{d}z \rho(z) A(z)$$

(53)

In the limit $N \rightarrow \infty$ this defines the probability density for configurations, $\rho(z)$.

Let’s pause and make a few observations. First, we are here always discussing ensembles of events or of particles at a given, fixed time, $t$.

Notice also that the precise matching between the microscopic and macroscopic degrees of freedom depends critically on the boundary conditions which are assumed. The manifold $\Sigma$ with metric $g^{\mu\nu}$ describes the configuration space $C$ of the system we are describing.

If we ask that $\langle I \rangle = 1$ this implies that

$$\int_{c} \rho = 1$$

(54)

so $\rho(z)$ is a normalized probability density, for finding the position of a single quantum particle. We will see shortly that it is conserved.

We could impose instead

$$\int_{c} \rho = M$$

(55)

in which case the same probability distribution functions can describe $N$ non-interacting particles.

This is an important point. The elementary events are part of a microscopic description. At this level there are no particles, and also no waves (or wave functions.) A large number of elementary event correspond to a single (quantum) particle.

More precisely, $x^a$ are coordinates on a configuration space, which might be three dimensional space for a single particle or a $M = 3N$ dimensional space for a system of $M$ particles.

Similarly we turn the sums on $I$ to an integral

$$\frac{1}{N} \sum_{i} \phi(x_{k+i}, x_{k}) \rightarrow Z \int_{a}^{R} d^{d}x \rho(z + x) \phi(z + x, z)$$

(56)
The possibility that the integral only approximates the sum for finite \( N \), because of the roughness of the estimate for the limits on the integral, is accounted for by an adjustable normalization factor \( Z \). Note that to complete the definition of the coarse graining we need to choose the limits of integration. More on this below.

The use of probability here suggests that we are introducing an ensemble of similar systems to represent the probabilities for the distribution of events in the single system we are studying. Depending on the context, this could be an imagined ensemble, as in the case of much equilibrium statistical physics, an ensemble consisting of many similar classical universes\[22\], or a \textit{real ensemble} of many similar regions within our universe, as I suggested in several earlier papers (\[25, 26\].)

For the purposes of showing the emergence of quantum mechanics from a realistic theory, any of these can work as part of a background story that is put forward to connect these equations to nature. So we will not touch these issues till the summary chapter.

- **Introduce coarse grained observables.**

Before we can represent the final \( dz^a \) integrals, we have first to integrate over the longitudinal momenta, \( l_a^N \).

How do we coarse grain the momentum variables? We define a coarse grained function, which we can think of as a kind of momentum-energy current, \( p_a(z) \) on \( z^a \) such that

\[
I = \int dp_a \delta \left( p_a - \sum_{K \geq L} \left[ p_a(z_K^a) - p_a^L \right] \right)
\]  

(57)

If we insert this in the integral (51) and integrate over the remaining longitudinal modes, \( dt^K_{aJ} \), we will find only functionals of \( \rho(z) \) and \( p(z)_a \).

- **Integrating over the longitudinal fluctuations:**

That \( p_a \) is longitudinal means that

\[
\epsilon^{ijk} \partial_j p_k = 0
\]  

(58)

The solution to which is that there exists a scalar phase \( S(z) \) such that,

\[
\partial_a S = p_a
\]  

(59)

so we express this as

\[
I = \int dS(z) \delta \left( \partial_a S - p_a \right)
\]  

(60)

Again, we insert this form of \( I \) into the integrand, integrate over the \( p_a(z) \) and we end up with a functional integral over the density \( \rho \) and the \( S \). The result is an effective action that depends only on the emergent large scale collective degrees of
freedom, $\rho(z)$ and $S(z)$. These are not only emergent degrees of freedom, they are functions of the $z^a$ which are themselves emergent degrees of freedom.

- **The terms from the conservation of momentum**

  We study lastly the constraint term that governs energy momentum conservation,

  $$\mathcal{X} = < \sum_I P^I_{a,z^a_I} > = < \sum_{I,J} p^I_{a,j} (z^a_I - z^a_J) >$$

  (61)

  We can consider, first the classical limit, in which we insert infinite numbers of events into the trajectory[12,13],

  $$\mathcal{X} \to \sum_i \int_{\gamma_i} ds \ p(s) \dot{z}^a$$

  (62)

  Instead we will assume that the events are spread with a density limited by (), so we find instead

  $$\sum_{I,J} P^I_{a}(z^a_{I+1} - z^a_I) \to \int d^d z \rho(z) \dot{z}^a \partial_a S$$

  (63)

  We now use current conservation,

  $$\dot{\rho} = -\partial_a V^a$$

  (64)

  where $V^a = \rho \dot{z}^a$ to find that

  $$\mathcal{X} = \int_C \rho \dot{S}$$

  (65)

  **4.4 Assembling the pieces**

- **Putting together all these pieces, the action becomes simply,**

  $$S^{cg}(\rho(z), S(z)) = \int dt \int_C \rho(z,t) \left[ \dot{S} + gn^2_{pre} g^{ab}(\partial_a S(z,t))(\partial_b S(z,t)) + \frac{g'}{g^2} \mathcal{U}(z) \right]$$

  (66)

  At the end we have an ordinary path integral expressed in terms of the emergent variables, $\rho(z)$ and $S(z)$.

  $$Z(\rho, S) = \int d\rho dS e^{S^{cg}}.$$  

  (67)

  Let us stop and have a look at the action, (66).
Treating it as a classical action (or just looking at the stationary phase approximation, we can take the variation by $\rho$. This yields immediately the Hamilton-Jacobi equation plus one new term proportional to $g'$.

$$\dot{S} + g\gamma g^{ab}(\partial_a S(z,t)) (\partial_b S(z,t) + \frac{g'}{g^2}U(z)) = 0$$ (68)

We will shortly see that to correspond with the Schrodinger we have to set

$$\frac{g'}{g^2} = \frac{\hbar^2}{8m} Z_V$$ (69)

Where $Z_V$ is a constant which absorbs various factors.

The variation of $S$ by $S$ yields probability conservation.

$$\dot{\rho} = -\partial_a (\rho g^{ab} \partial_b S)$$ (70)

Thus, the action (66) has the Schrodinger equation as its classical equations of motion.

Considering it an action for classical physics, the action (66) has an intriguing structure. Notice that independent of the presence or absence of the potential energy proportional to $\hbar^2$, the probability density, $\rho$ is conjugate to the phase variable $S$. If we were to start with a standard canonical analysis, one finds that the non-vanishing Poisson bracket is.

$$[S(x), \rho(y)] = \delta^n(x, y)$$ (71)

• Normalizations, limits and cutoffs

There is one last detail which is to specify the limits on the remaining integrals to ensure probability conservation.

That is, for finite $N$ and finite densities of events, the $\rho(z)$ has upper and lower limits that reflect the fact that the integral over $\rho$ stands for a sum over a finite set, which consists of $N$ microscopic events, whose coarse description is a single quantum particle.

• The uv cutoff

Note that we have to be careful to impose limits on the integral to avoid unphysical divergences in $\frac{1}{x}$. These divergences are unphysical because for finite $N$ two configuration variables, $x_k^a$ and $x_j^a$, cannot come closer than a limit which varies inversely with the density at $x_k^a$ and $N$. This is because if $x_k^a$ and $x_j^a$ are nearest neighbours in the distribution, the density at one of their locations is related to their separation.

$$\rho(x_k^a) \approx \frac{1}{N|x_k - x_j|^d}$$ (72)
where $d$ is the dimension of the configuration space. Hence, for finite $N$ they are very unlikely to coincide. When we approximate the sums by integrals, the integrals representing intervals between configurations must then be cut off by a short distance cutoff $a$ that scales inversely like a power of $N\rho(z)$. The short distance cutoff $a(z)$ on the integral above in $d^dx$ then expresses this fact that there is a limit to $\frac{1}{x}$ related to the density. Hence the short distance cutoff is at

$$a(z) = \frac{1}{(N\rho(z))^{\frac{1}{d}}}$$  \hspace{1cm} (73)

- **The infrared cutoff**

There is also an infrared cutoff, $R$. This tells us that two systems further than $R$ in configuration space do not figure in each other’s views. A key question turns out to be how the physical cutoff scales with $N$. To ensure that the large $N$ limit is uniform, we want to scale $R$ the same way as $a$. Certainly the density $\rho$ is real and well defined, so when we define the uv cutoff by (73) we also have to scale the ir cutoff $R$, so that the ratio

$$L = \frac{R}{a}$$  \hspace{1cm} (74)

is fixed. So the long distance coordinate cutoff $R$ must scale like

$$L = \frac{R}{(N)^{\frac{1}{d}}}$$  \hspace{1cm} (75)

We will define

$$L = \frac{R}{(N)^{\frac{1}{d}}}$$  \hspace{1cm} (76)

to represent a fixed physical lengths scale which is held fixed when we take the limit of large $N$ at the end of the calculation. That way, the physical ultraviolet and infrared cutoffs scale the same way with $N$. But the large scale, infrared cutoff, $L$ can’t know about the value of the probability distribution at some far off point $z$, so while $a$ scales with $\rho$, $r'$ doesn’t.

As a result when we scale $x$ and $d^dx$ with $a$ to make the integrals dimensionless, we define $r$, such that, $R = ar$. But we then hold fixed

$$L = \frac{1}{\rho^{\frac{1}{d}}} r = N^{\frac{1}{d}} R$$  \hspace{1cm} (77)

as we take $N$ large. $L$, unlike $r$, is a length. We shall see that $L$ defines a new physical length scale at which the linearity of quantum mechanics gives way to a non-linear theory.
4.5 Emergence of the Bohmian potential

Once we get the scalings and the limits right, the potential energy term (49) can be expanded in powers of $\frac{1}{N^{\sigma}}$. The leading term reproduces David Bohm’s quantum potential[].

The continuum approximation to the spacelike or acausal variety is,

$$V = \int d^d z \rho(z) Z_V \int_0^R d^d x \int_0^R d^d y \left[ \frac{x^a}{x^2} - \frac{y^a}{y^2} \right] \rho(z + x) \rho(z + y)$$

(78)

We do a scale transformation by writing $x^a = a \alpha^a$ and $y^a = a \beta^a$. To get a single integral over a local function we can expand

$$\rho(z + a \alpha) = \rho(z) + a \alpha^a \partial_a \rho(z) + \frac{1}{2} a^2 \alpha^a \alpha^b \partial_{ab} \rho(z) + \ldots$$

(79)

and similarly for $\rho(z + a \beta)$ and perform the integrations, holding the upper limit $r'$ fixed.

The normalization factor is

$$Z_V = \frac{1}{N^3 \Omega^2 (r'^d - 1)^2} \approx \frac{d^2}{2N \Omega^2 r'^{2d}}$$

(80)

The result is

$$V = \int d^d z \rho \left( \frac{1}{R^2} - \left( \frac{1}{\rho} \partial \rho \right)^2 + \frac{g'}{d + 2} r'^2 \frac{(\nabla^2 \rho)^2}{\rho^2} + \ldots \right)$$

(81)

Here we ignore total derivatives, which don’t contribute to the potential energy. The first term is an ignorable constant. The second term is what we want; its variation gives the Bohmian quantum potential.

We must recall that in the action the $U$ is multiplied by $\frac{g'}{g^2}$. We choose this to be equal to

$$\frac{g'}{g^2} = \frac{\hbar^2}{8m} Z_V$$

(82)

The higher order terms are suppressed by powers of $\frac{1}{N^{\sigma}}$. The result is

$$U = -\frac{\hbar^2}{8m} V = \frac{\hbar^2}{8m} \int d^d z \rho \left( \frac{1}{\rho} \partial \rho \right)^2 + O \left( \frac{1}{N^{\sigma}} \right)$$

(83)

which we recognize as the term whose variation gives the Bohmian quantum potential.

The leading correction is

$$U^\Delta V = -\frac{\hbar^2}{8m} \Delta V = -\frac{1}{N^\sigma} \frac{\hbar^2 r'^2}{8m} \int d^d z \rho \left( \frac{1}{\rho} \nabla^2 \rho \right)^2$$

(84)

which contributes non-linear corrections to the Schrödinger equation.
5 Why variety leads to Bohm

There is a simple effective field theory argument that a term proportional to Bohm’s potential must appear. We organize an expansion of the effective action for a particle moving stochastically in space in a probabilistic theory, in powers of derivatives.

This effective action should have dimensions of energy, and be lorentzian or galilean invariant as well as parity even. It can depend on a probability density for the position of the particle, \( z^a(t) \), which is \( \rho(z, t) \) and spatial derivatives of \( \rho \), notated \( \partial_a \rho \). It could also depend on the momentum, \( p_a \), which we have seen may be re-expressed as \( \partial_a S \). Note that \( \rho \) is the only density in the problem. The only place that derivatives of \( \rho \) can then appear is in functions of \n\[
\frac{\partial_a \rho}{\rho} \tag{85}
\]

Thus the general form is

\[
< U > = \int d^d z \rho(z) F[S, \frac{\partial_a \rho}{\rho}, \partial_a S, \dot{S}] \tag{86}
\]

where \( F \) is a scalar functional on the configuration space. Through second order in derivatives we have only,

\[
< U > = \int d^d z \rho(z) \left( \alpha \dot{S} + \beta \partial_a S \partial_b S h^{ab} + \gamma \frac{\partial_a \rho}{\rho} \frac{\partial_b \rho}{\rho} h^{ab} + \ldots \right) \tag{87}
\]

The main failure mode of this kind of consideration is that the signs may not work out for the energy to be bounded from below; in particular if \( \beta < 0 \) the equations give us a description of classical diffusion.

The opposite sign leads to Schrodinger quantum mechanics, as we are about to see.

5.1 Emergence of quantum mechanics

We can now put all three terms together,

\[
S^{III} = \int d^d z \rho(z) \left[ \dot{S} - g \partial_a S h^{ab} \partial_b S - \frac{g'}{g^2} \sqrt{\rho} \frac{\partial_a \rho}{\rho} \frac{\partial_b \rho}{\rho} \right] + O\left( \frac{1}{N^2} \right) \tag{88}
\]

This has a very interesting structure.

When we vary the action (88) by the probability density, \( \rho \), you get the real part of the Schrodinger equation (89),

\[
\dot{S} = \frac{1}{2m} (\partial_i S)^2 + \frac{g'}{g^2} \nabla \sqrt{\rho} \tag{89}
\]

whereas varying the phase, \( S \) verifies the law of current conservation, (70).

The wave function

\[
\psi = \sqrt{\rho} e^{i\tilde{S}} \tag{90}
\]
then satisfies the Schrodinger equation,

\[ i\hbar \frac{d\psi}{dt} = \left[ -\frac{\hbar^2}{2m} \nabla^2 \right] \psi \]  

This enforces our constants to be

\[ \frac{g'}{g^2} = \frac{\hbar^2}{8m} Z_V \]  

6 Looking ahead

6.1 The interpretation

Bryce de Witt used to like to say that we should let the mathematical structure of a theory dictate its interpretation. Whatever the merits, generally, of that advise, this would seem to be a good case for that practice. The reason is that the quantum mechanics is revealed to be just a first order phenomenon in an expansion of a small parameter around the classical Hamilton Jacobi equation.

Let us start with \( \hbar = 0 \). We have an action principle

\[ S^{III} = \int d^dz \rho(z) \left[ \dot{S} - \frac{1}{m} \partial_a \dot{S} h^{ab} \partial_{b}S \right] \]  

Varying this by the probability density \( \rho \)

we reach the Hamilton-Jacobi equation,

\[ \dot{S} - \frac{1}{m} \partial_a S g^{ab} \partial_{b}S \]  

together with probability conservation.

\[ \dot{\rho} = -\partial_a (\rho g^{ab} \partial_{b}S) \]

The presence of the conservation law doesn’t give us room for varying the interpretation at \( \hbar = 0 \). And neither does it when we turn on the potential energy by increasing \( g' \approx \hbar^2 \).

The theory at \( \hbar = 0 \) has a straightforward interpretation. We have a probabilistic description of the motion of a single particle which follows the gradients of the Hamilton-Jacobi equation. \( \rho(z, t) \) is an evolving functional that gives probabilities that the particle can be detected at different points in the emergent space coordinatized by the \( z_i \).

The trajectories of the emergent particles are guided by a pilot wave, which is the Hamilton-Jacobi function. Thus even at \( \hbar = 0 \) we have a wave particle duality?

Where did that come from? Its been there all along-prior to \( \hbar \), prior to quantum theory, reflected in the Poisson bracket (71).
Furthermore we understand that theory as an emergent description of a fundamental theory of events and their causal relations.

Now we turn on \( \hbar \). The interpretation doesn't change. It can’t as it was already complete at \( \hbar = 0 \).

We still have a fundamental theory of events, causal relations, energy and momentum. We still have an emergent coarse grained description, in terms of emergent spacetime on which there are excitations which behave like the particles of quantum mechanics. All that changes is that new, non-local forces are introduced, given by the variety potential.

### 6.2 Summary

We have given a new formulation of the causal theory of views which improves on earlier versions in several ways.

The fundamental Hamiltonian is simpler and both the kinetic and potential energy terms express directly the idea of increasing variety, or diversity of causal neighbours, or views. The potential energy measures the diversity of views of events acausally related to each other, whereas the kinetic energy measures the change in views between causally closely related events. This makes the derivation of quantum mechanics much simpler, and it may also be hoped that this is a step towards a specially covariant theory.

We also can mention that we have accomplished here something many approaches to quantum gravity aspire to: the emergence of space from a theory without space.

This program raises some very interesting issues related to the reality of time\[17\] and the physics of the brain, which are addressed in some of the papers indicated\[18\].

### 6.3 Celestial spheres as views

In the near future we hope to be back to present a fully relativistic version of the present theory. That will likely incorporate the insights of recent work on celestial spheres\[35\] as sites of clarity for understanding the asymptotic limits of Minkowski space. Our expectation is that celestial spheres are simply fully relativistic views. In this construction, an incoming massless quanta is represented by a point on the \( S^2 \) which gives the direction from which the quanta came. The label on the point then gives the energy. The group of conformal transformations is \( SL(2, C) \) which gives a representation of the lorentz group.

In the relativistic case we will also make use of the notion of best matching from the work of Julian Barbour and collaborators\[29\].

Finally, an obvious approach to try is to copy the form of the action of interacting relativistic particles used in relative locality\[23, 24\].
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