Self-Adjointness and Polarization of the Fermionic Vacuum in the Background of Nontrivial Topology

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Abstract

Singular configuration of an external static magnetic field in the form of a string polarizes vacuum in the secondly quantized theory on a plane which is orthogonal to the string axis. We consider the most general boundary conditions at the punctured singular point, which are compatible with the self-adjointness of the two-dimensional Dirac Hamiltonian. The dependence of the induced vacuum quantum numbers on the self-adjoint extension parameter and the flux of the string is determined.

In this talk contributing to the celebration of the 70-th birthday of Professor Walter Thirring we would like to report on some exact results concerning the properties of vacuum in the background of nontrivial topology.

As is known, the singular static magnetic monopole background induces fermion number in the vacuum [1-3]

\[
\langle N \rangle = \frac{1}{\pi} \arctan \left( \tan \frac{\Theta}{2} \right),
\]

where \( \Theta \) is the parameter of a self-adjoint extension, which defines the boundary condition at a puncture corresponding to the location of the monopole; this results in the monopole becoming actually the dyon violating the Dirac quantization condition and CP symmetry.

In the present talk we shall be considering quantum numbers which are induced in the fermionic vacuum by the singular static magnetic string background. Since the deletion of a line, as compared to the deletion of a point,
changes the topology of space in a much more essential way (fundamental group becomes nontrivial), the properties of the Θ-vacuum will appear to be much more diverse, as compared to Eq.(1). Restricting ourselves to a surface which is orthogonal to the string axis, let us consider 2+1-dimentional spinor electrodynamics on a plane with a puncture corresponding to the location of the string. We shall show that in this case the induced vacuum fermion number and magnetic flux depend on the self-adjoint extension parameter and the magnetic flux of the string as well.

The pertinent Dirac Hamiltonian has the form

\[ H = -i\vec{\alpha}[\vec{\partial} - i\vec{V}(\vec{x})] + \beta m; \tag{2} \]

where \( \vec{V}(\vec{x}) \) is an external static vector potential. In a flat two-dimensional space \( (\vec{x} = (x^1, x^2)) \) the vacuum fermion number induced by such a background was calculated first in Ref.[4]

\[ \langle N \rangle = -\frac{1}{2}\Phi, \tag{3} \]

where \( m > 0 \) and \( \Phi = \frac{1}{2\pi} \int d^2x B(\vec{x}) \) is the total flux (in the units of \( 2\pi \)) of the external magnetic field strength \( B(\vec{x}) = \vec{\partial} \times \vec{V}(\vec{x}) \) piercing the two-dimensional space (plane).

It should be emphasized, however, that Eq.(3) is valid for regular external field configurations only, i.e. \( B(\vec{x}) = B_{\text{reg}}(\vec{x}) \), where \( B_{\text{reg}}(\vec{x}) \) is a continuous in the whole function that can grow at most as \( O(|\vec{x} - \vec{x}_s|^{-2+\varepsilon}) \) (\( \varepsilon > 0 \)) at separate points; as to a vector potential \( \vec{V}(\vec{x}) = (V_1(\vec{x}), V_2(\vec{x})) \), it is unambiguously defined everywhere on the plane. The regular configuration of an external field polarizes the vacuum locally, and Eq.(3) is just the integrated version of the linear relation between the vacuum fermion number density and the magnetic field strength.

One can ask the following question: whether the nonlocal effects of the external field background are possible, i.e., if the spatial region of nonvanishing field strength is excluded, whether there will be vacuum polarization in the remaining part of space? For the positive answer it is necessary, although not sufficient, that the latter spatial region be of nontrivial topology [5] (see also Ref.[6]). However, the condition on the boundary of the excluded region has not been completely specified. In the present talk this point will be clarified.
by considering the whole set of boundary conditions which are compatible with the self-adjointness of the Dirac Hamiltonian in the remaining region.

We shall be interested in the situation when the volume of the excluded region is shrunk to zero, while the global characteristics of the external field in the excluded region is retained nonvanishing. This implies that singular, as well as regular, configurations of external fields have to be considered. In particular, in two spatial dimensions the magnetic field strength is taken to be a distribution (generalized function)

$$B(\vec{x}) = B_{\text{reg}}(\vec{x}) + 2\pi \Phi(0) \delta(\vec{x}),$$

(4)

where \(\Phi(0)\) is the total magnetic flux (in the units of \(2\pi\)) in the excluded region which is placed at the origin \(\vec{x} = 0\). As to the vector potential, it is unambiguously defined everywhere with the exception of the origin, i.e. the limiting value \(\lim_{|\vec{x}| \to 0} \vec{V}(\vec{x})\) does not exist, or, to be more precise, a singular magnetic vortex is located at the origin

$$\lim_{|\vec{x}| \to 0} \vec{x} \times \vec{V}(\vec{x}) = \Phi(0).$$

(5)

Certainly, a plane has trivial topology, \(\pi_1 = 0\), while a plane with a puncture where the vortex is located has nontrivial topology, \(\pi_1 = \mathbb{Z}\); here \(\mathbb{Z}\) is the set of integer numbers and \(\pi_1\) is the first homotopy group of the surface.

Let us turn now to the boundary condition at the puncture \(\vec{x} = 0\). In the following our concern will be in the case in which the regular part of the magnetic field is absent, \(B_{\text{reg}}(\vec{x}) = 0\). Then, in the representation with \(\alpha_1 = \sigma_1\), \(\alpha_2 = \sigma_2\) and \(\beta = \sigma_3\) (\(\sigma_j\) are the Pauli matrices) the spinor wave function satisfying the Dirac equation has the form

$$\psi(\vec{x}) = \sum_{n \in \mathbb{Z}} \left( \begin{array}{c} f_n(r) \exp(in\varphi) \\ g_n(r) \exp[i(n+1)\varphi] \end{array} \right),$$

(6)

where the radial functions, in general, are

$$\left( \begin{array}{c} f_n(r) \\ g_n(r) \end{array} \right) = \left( \begin{array}{c} C_n^{(1)}(E)J_{n-\Phi(0)}(kr) + C_n^{(2)}(E)Y_{n-\Phi(0)}(kr) \\ \frac{ik}{E+m}[C_n^{(1)}(E)J_{n+1-\Phi(0)}(kr) + C_n^{(2)}(E)Y_{n+1-\Phi(0)}(kr)] \end{array} \right),$$

(7)

\(k = \sqrt{E^2 - m^2}\), \(J_{\mu}(z)\) and \(Y_{\mu}(z)\) are the Bessel and the Neumann functions of the order \(\mu\). It is clear that the condition of regularity at \(r = 0\) can
be imposed on both $f_n$ and $g_n$ for all $n$ in the case of integer values of the quantity $\Phi^{(0)}$ only. Otherwise, the condition of regularity at $r = 0$ can be imposed on both $f_n$ and $g_n$ for all but $n = n_0$, where

$$n_0 = \lfloor \Phi^{(0)} \rfloor,$$

(8)

$\lfloor u \rfloor$ is the integer part of the quantity $u$ (i.e. the integer which is less than or equal to $u$); in this case at least one of the functions, $f_{n_0}$ or $g_{n_0}$, remains irregular, although square integrable, with the asymptotics $r^{-p}$ ($p < 1$) at $r \to 0$. The question arises then, what boundary condition, instead of regularity, is to be imposed on $f_{n_0}$ and $g_{n_0}$ at $r = 0$ in the latter case?

To answer this question, one has to find the self-adjoint extension for the partial Hamiltonian corresponding to the mode with $n = n_0$. If this Hamiltonian is defined on the domain of regular at $r = 0$ functions, then it is Hermitian, but not self-adjoint, having the deficiency index equal to (1,1). Hence the family of self-adjoint extensions is labeled by one real continuous parameter [7] denoted in the following by $\Theta$. It can be shown (see Ref.[8]) that, for the partial Hamiltonian to be self-adjoint, it has to be defined on the domain of functions satisfying the boundary condition

$$\lim_{r \to 0} \cos \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) m r^F f_{n_0}(r) = i \lim_{r \to 0} \sin \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) m r^{1-F} g_{n_0}(r),$$

(9)

where

$$F = \{ \Phi^{(0)} \},$$

(10)

$\{ u \}$ is the fractional part of the quantity $u$, $\{ u \} = u - \lfloor u \rfloor$, $0 \leq \{ u \} < 1$; note here that Eq.(9) implies that $0 < F < 1$, since in the case of $F = 0$ both $f_{n_0}$ and $g_{n_0}$ satisfy the condition of regularity at $r = 0$.

Using the explicit form of the solution to the Dirac equation in the background of a singular magnetic vortex, it is straightforward to calculate the vacuum fermion number induced on a punctured plane. As follows already from the preceding discussion, the vacuum fermion number vanishes in the case of integer values of $\Phi^{(0)}$ ($F = 0$), since this case is indistinguishable from the case of the trivial background, $\Phi^{(0)} = 0$. In the case of noninteger values
of $\Phi^{(0)}$ $(0 < F < 1)$ we get [9]

$$\langle N \rangle = \begin{cases} 
\frac{1}{2}(1 - F), & -1 < A < \infty \\
-\frac{1}{2}(1 + F), & -\infty < A < -1 \\
-\frac{1}{2}F, & A^{-1} = 0
\end{cases}, \quad 0 < F < \frac{1}{2} \quad (11)$$

$$\langle N \rangle = \frac{1}{\pi} \arctan\left(\tan\frac{\Theta}{2}\right), \quad F = \frac{1}{2}, \quad (12)$$

$$\langle N \rangle = \begin{cases} 
-\frac{1}{2}F, & -1 < A^{-1} < \infty \\
\frac{1}{2}(2 - F), & -\infty < A^{-1} < -1 \\
\frac{1}{2}(1 - F), & A = 0
\end{cases}, \quad \frac{1}{2} < F < 1. \quad (13)$$

where

$$A = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan\left(\frac{\Theta}{2} + \frac{\pi}{4}\right), \quad (14)$$

$\Gamma(u)$ is the Euler gamma-function; note that Eq.(12) coincides with Eq.(1).

It is obvious that the vacuum fermion number at fixed values of $\Theta$ is periodic in the value of $\Phi^{(0)}$. This feature (periodicity in $\Phi^{(0)}$) is also shared by the quantum-mechanical scattering of a nonrelativistic particle in the background of a singular magnetic vortex, known as the Aharonov-Bohm effect [10].

The total magnetic flux induced in the fermionic vacuum on a punctured plane is also calculated in Ref. [9]

$$\Phi^{(I)} = -\frac{e^2 F(1 - F)}{2\pi m} \left[ \frac{1}{6} \left( F - \frac{1}{2} \right) + \frac{1}{4\pi} \int_1^{\infty} \frac{dv}{v} \sqrt{v-1} \frac{A v^F - A^{-1} v^{1-F}}{A v^F + 2 + A^{-1} v^{1-F}} \right]; \quad (15)$$

note that the coupling constant $e$ relating the vacuum current to the vacuum magnetic field strength (via the Maxwell equation) has the dimension $\sqrt{m}$ in $2 + 1$-dimensional space-time. At half-integer values of $\Phi^{(0)}$ we get

$$\Phi^{(I)} = -\frac{e^2}{8\pi^2 m} \arctan\left(\tan\frac{\Theta}{2}\right), \quad (F = \frac{1}{2}). \quad (16)$$
Functional dependence of the vacuum fermion number and magnetic flux on $\Theta$ in the range $-\frac{\pi}{2} < \Theta < \frac{3\pi}{2}$ at some fixed values of $F$ is depicted in Figs. 1 and 2. Functional dependence of the vacuum quantum numbers on $F$ at some fixed values of $\Theta$ is depicted in Figs. 3 and 4. Note that, depending on the choice of boundary condition, the vacuum can be either of paramagnetic ($\text{sgn}(\Phi(I))=\text{sgn}(\Phi(0))$) or diamagnetic ($\text{sgn}(\Phi(I))=-\text{sgn}(\Phi(0))$) type.

In conclusion we note that under the charge conjugation,

$$C: \quad \vec{V} \to -\vec{V}, \quad \psi \to \sigma_1 \psi^*, \quad (17)$$

the fermion number operator and its vacuum value, as well as the vacuum magnetic flux, are to be odd. Evidently, the results (11)–(13) and (15) are not, since the boundary condition (9) breaks, in general, the charge conjugation symmetry. However, for certain choices of the parameter $\Theta$ this symmetry can be retained. The most general boundary condition which is compatible both with the periodicity in $\Phi(0)$ and charge conjugation symmetry has the form

$$\Theta = \begin{cases} 
\Theta_C (\text{mod} 2\pi), & 0 < F < \frac{1}{2} \\
0 (\text{mod} 2\pi), & F = \frac{1}{2} \\
-\Theta_C (\text{mod} 2\pi), & \frac{1}{2} < F < 1 
\end{cases}, \quad -\pi < \Theta_C \leq \pi. \quad (18)$$

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Fig. 1. $\langle N \rangle$ as function of $\Theta/\pi$. 
Fig. 2. $2\pi m[e^2 F(1 - F)]^{-1}\Phi^{(j)}$ as function of $\Theta/\pi$. 
Fig. 3. $\langle N \rangle$ as function of $F$. 
Fig. 4. $2\pi m [e^2 F(1 - F)]^{-1} \Phi^{(l)}$ as function of $F$. 