Abstract

While algorithms for planar graphs have received a lot of attention, few papers have focused on the additional power that one gets from assuming an embedding of the graph is available. While in the classic sequential setting, this assumption gives no additional power (as a planar graph can be embedded in linear time), we show that this is far from being the case in other settings. We assume that the embedding is straight-line, but our methods also generalize to non-straight-line embeddings. Specifically, we focus on sublinear-time computation and massively parallel computation (MPC).

Our main technical contribution is a sublinear-time algorithm for computing a relaxed version of an $r$-division. We then show how this can be used to estimate Lipschitz additive graph parameters. This includes, for example, the maximum matching, maximum independent set, or the minimum dominating set. We also show how this can be used to solve some property testing problems with respect to the vertex edit distance.

In the second part of our paper, we show an MPC algorithm that computes an $r$-division of the input graph. We show how this can be used to solve various classical graph problems with space per machine of $O(n^{2/3+\epsilon})$ for some $\epsilon > 0$, and while performing $O(1)$ rounds. One needs $\Omega(\log n)$ rounds with $n^{1-\Omega(1)}$ space per machine without the embedding to solve these problems, assuming the 1-vs-2-cycles conjecture. Among the problems solved by our approach are: counting connected components, bipartition, minimum spanning tree problem, $O(1)$-approximate shortest paths, $O(1)$-approximate diameter/radius. Our techniques however also apply to other problems. In conjunction with existing algorithms for computing the Delaunay triangulation, our results imply an MPC algorithm for Euclidean minimum spanning three that uses $O(n^{2/3+\epsilon})$ space per machine and performs $O(1)$ rounds.

Finally, we show two generalizations of our approach. First, we show that that our approach also works for non-planar embeddings. Specifically, we get non-trivial results whenever the number of crossings is $\ll n^2$. Second, we show that our algorithms can be modified to work with embeddings consisting of curves that are not “too squiggly” (as formalized by the total absolute curvature). This results in an algorithm parameterized by the total absolute curvature of all edges. We do this via a new lemma which we believe is of independent interest.
1 Introduction

Planar graphs are one of the most commonly occuring graph classes. When dealing with a planar graph, we would usually have available an embedding of the graph, along with the graph itself — after all, the graph is presumably planar exactly because it is embedded in the plane. This is the case, for example, for maps, (single-layer) circuit boards, computational geometry applications, and more. While much work has been done on algorithms for planar graphs, few algorithms make use of the embedding. The embedding provides no additional power in the classic sequential setting with (super-)linear space and memory, as a graph embedding can be computed in linear time; see for example [31] for an overview of many such algorithms. In this paper, we show that this is far from being the case in some other settings. Specifically, we focus on sublinear-time computation and massively parallel computation.

For the sake of simplicity, throughout most of the paper, we assume that all edges are embedded as line segments (we have an straight-line embedding). However, our techniques also extend to other more general embeddings, as we discuss in Section 6. Furthermore, our techniques also generalize to non-planar embeddings with not too many ($\ll n^2$) crossings.

Our results rely on a modified version of the classic algorithm by Clarkson and Shor [8] for computing a $q$-cutting (we define cuttings below). This is a frequently used technique in geometric algorithms. Commonly, this is used to divide the input into subproblems in divide-and-conquer algorithms. We use $q$-cuttings in a very different fashion. Specifically, we use a $q$-cutting to compute a (relaxed version of) an $r$-division, without using divide-and-conquer. We then use this $r$-division to design algorithms, instead of directly relying on the $q$-cutting.

$r$-division is one of the basic tools for designing efficient algorithms for planar graphs. An $r$-division allows us to decompose the graph into regions of size $\leq r$ such that each region only shares few of its vertices with other regions. This often allows us to solve a problem by separately solving it (or some other related problem) on the regions, and than “putting the partial solutions together” to get a solution for the whole graph. We use this approach extensively. The core technical contribution of this paper is an algorithm that computes a (hybrid) $r$-division of a graph in sublinear time. We then show how this can be applied to solve various algorithmic problems.

The first problem that we solve using our sublinear-time $r$-division algorithm is that of estimating graph parameters from a large class of graph parameters, called additive Lipschitz graph parameters. This includes properties such as maximum matching, maximum independent set, or minimum dominating set. While Newman and Sohler [29] shows how to estimate a more general class of problems, they assumed that the graph has degrees bounded by $d$. Our algorithm works in general planar graphs. Moreover, our algorithm is simple and has complexity $O(\sqrt{n \log n}/\epsilon^2)$. The approach of [29] when used in conjunction with the property-testing algorithm of Kumar, Seshadhri, and Stolman [23] results in complexity $O(d^6/\epsilon^{30654})$, and is unlikely to be practical.

In the second part of this paper, we show a Massively Parallel Computation (MPC) algorithm for computing an $r$-division, based on our sublinear-time algorithm for hybrid $r$-divisions. This can be used to solve a variety of different problems including counting connected components, finding a bipartition, minimum spanning tree problem, $O(1)$-approximate shortest path problem, and $O(1)$-approximate diameter/radius. All our algorithms take in expectation (and with high probability) $O(1)$ rounds and work whenever the space per machine is $S \geq n^{2/3+\epsilon}$ for some constant $\epsilon > 0$. No such algorithms were previously known for $S \leq O(n^{1-\epsilon})$ for any $\epsilon > 0$. In fact, under the famous 1-vs-2 cycles conjecture, $\Omega(\log n)$ rounds are needed for any of the mentioned problems if we do not have an embedding or some other extra information about the graph.

Our algorithm works not only for straight lines but for any $x$-monotone curves. We would like to generalize this to all curves that satisfy some mild technical assumptions. In this part of the
paper, we focus only on the query complexity, as the time complexity depends on the operations we are able to efficiently perform on the lines, thus not allowing for such generality. We could subdivide all edges into $x$-monotone pieces, but there could be many such pieces even if we started with “nice” curves. We believe this technique is of independent interest and could also be used for other problems. We introduce a new technique that is applicable whenever we have a problem that we can solve on a set of $x$-monotone curves and have non-monotone curves as the input (we require that the curves satisfy some mild technical assumptions). The complexity of the algorithm is then parameterized by how curly/squiggly the input curves are, on average. This is captured by the notion of total absolute curvature. Let $A(G)$ be the total absolute curvature of all edges of $G$. Our approach then results in an algorithm parameterized by $n + A(G)$. This means that if the lines have on average constant total absolute curvature, our algorithms will have the same query complexity.

Our results imply an exact MPC algorithm for the Euclidean minimum spanning tree problem that uses $O(1)$ rounds and $O(n^{2/3+\epsilon})$ space per machine, for any $\epsilon > 0$. One may use one of the existing algorithms [18, 22] to compute a Delaunay triangulation of the input set. This can be done in $O(1)$ rounds and $O(n^\epsilon)$ space per machine. It is well-known that the EMST of a point set is a subset of its Delaunay triangulation. To get the euclidean minimum spanning tree, we then may use our MST algorithm that uses $O(1)$ rounds and $O(n^{2/3+\epsilon})$ space per machine. The Delaunay triangulation is an embedded planar graph with a straight-line embedding, and the assumptions of our algorithm are thus satisfied. The only previously known algorithm that uses $o(\log n)$ rounds and $n^{1-\Omega(1)}$ space per machine is that of Andoni, Nikolov, Onak, and Yaroslavtsev [1] which only gives an approximate answer.

1.1 Previous work

1.1.1 Algorithms for geometric graphs

We are not aware of any papers specific to the setting where we have an embedded planar graph as the input. The reason for this is presumably that one can get an embedding in linear time and there is thus no need for such assumption.

It is NP-hard to get an embedding with the smallest possible number of crossings [15], and it thus may give us additional power if we assume that we are given an embedding with a small number of crossings. Eppstein, Goodrich, and Strash [13] have shown that in a graph with $O(n/\log^{(c)}n)$ crossings for a constant $c$, they are able to find an $r$-division in linear time, thus allowing them to get linear time algorithms for several problems, including the single source shortest path problem.

Eppstein and Goodrich [11] show how to use some other geometric properties of the road network than being almost planar to get efficient algorithms. We, however, do not want to make such further assumptions. There have been many papers on more restricted graph families that are geometric in nature, such as Euclidean graphs, geometric intersection graphs, and others, but these are out of the scope of this paper.

1.1.2 Partition oracles, property testing, and additive Lipschitz parameters

A series of papers culminating in [23] has considered partition oracles in planar (and more generally minor-free) graphs with bounded degrees. Partition oracles have been first (implicitly) used by Benjamini, Schramm, and Shapira [6]. A partition oracle divides the vertices into sets of size $O(\epsilon n)$ such that there are $\leq \epsilon n$ edges going between different sets (more specifically, given a vertex, the partition oracle answers which set it lies in). Note that one may use a $\Theta(1/\epsilon^2)$-division as the
Partition oracles may be used for estimation of additive Lipschitz parameters and property testing\(^1\). For parameter estimation, we first remove the edges that go between different sets. We then estimate by sampling “how much a connected component contributes on average to the value of the parameter of the whole graph”.

The technically challenging part is implementing the partition oracle. Recently, a polynomial-complexity partition oracle has been obtained by Kumar et al. [23]. Their algorithm has complexity \(O(d^{685}/\epsilon^{30650})\). This result improves upon the previous work on this problem [19, 25].

A characterization of testable properties in general (unbounded degree) planar graphs has been proven by Czumaj and Sohler [9]. They show that, intuitively speaking, a property is testable, if and only if testing it reduces to testing \(H\)-subgraph-freeness for some graphs \(H\).

### 1.1.3 Massively parallel computation

Until recently, no MPC algorithms have been known for planar graphs in the setting with \(n^{1-\Omega(1)}\) memory per machine that would have better performance than the best known algorithm for general graphs. Recently, two papers that focus on bounded-arboricity graphs (which includes planar graphs) appeared. Specifically, in [4] the authors show algorithms for the maximum matching and maximum independent set problems that use \(O(n^\epsilon)\) space for any \(\epsilon > 0\) and \(O(\log^2 \log n)\) rounds in bounded-arboricity graphs. This has been later improved to \(O(\log \log n)\) by Ghaffari, Grunau, and Jin [17]. Still, these algorithms only use the bound on the arboricity which can also be satisfied by graphs that are far from being planar. No algorithm specific to planar graphs is known so far in the setting with \(n^{1-\Omega(1)}\) memory per machine.

The 1-vs-2-cycles conjecture states that for any \(\epsilon > 0\), one needs \(\Omega(\log n)\) rounds to tell apart one cycle of length \(n\) and two cycles of length \(n/2\) when using \(n^{1-\Omega(1)}\) space per machine. This has been proven for some special classes of algorithms; for an overview of these results, see [28, Section 3.2]. This conjecture is known to be equivalent to many common combinatorial problems [28]. Specifically, any of the problems that we consider in Section 5.2 need \(\Omega(\log n)\) rounds with strongly sublinear space if this conjecture holds. We show that for any \(\epsilon > 0\), we can get an algorithm with \(O(1)\) rounds and \(O(n^{2/3+\epsilon})\) space per machine by using the embedding. Our algorithms thus give a conditional separation between the settings with and without an embedding in MPC. Algorithms with constant round complexity for these problems were previously only obtained in a stronger model called Adaptive Massively Parallel Computation [5].

The idea of “compressing a subgraph” of a planar graph has been introduced by Eppstein, Galil, Italiano, and Spencer [12] in the context of dynamic algorithms for planar graphs. We use this idea extensively together with other approaches to get MPC algorithms in Section 5.2.

### 1.2 Techniques

#### 1.2.1 Computing a hybrid \(r\)-division in sublinear time

We partition the plane into polygons (which we will call sectors) such that all vertices that lie in one polygon/sector belong to the same region. We build a point location data structure for this plane partition and for each polygon in the partition, we store which region its vertices belong to. This together allows us to efficiently determine which region a vertex lies in, just based on its coordinates.

\(^1\)In fact, even non-additive properties may be estimated using partition oracles, as shown by Newman and Sohler [29].
Figure 1: Illustration of our algorithm for computing a division oracle in sublinear time. Note that the sector graph is not necessarily equal to the dual graph (see Definition 7). Note also that the input and the sector graph are disjoint (even though it may seem from the illustration that some vertices are the same). For definition of hybrid divisions, see Definition 4.

It remains to specify how to find a suitable partition of the plane and how to choose for each polygon, which region its vertices will be in. We will make sure that the partition divides the vertices into “uniformly-sized chunks”; specifically, we make sure that for some parameter \( s \), each sector contains at most some \( s \) vertices and is intersected by at most \( s \) edges, and at the same time there are \( O(n \log(n)/s) \) sectors. Such a partition of the plane can be computed by a modified version of the classic algorithm of Clarkson and Shor [8] – we sample \( O(n \log(n)/s) \) edges and vertices and compute the trapezoid decomposition.

We decide to which region a polygon’s vertices belong as follows. We build a graph with vertex set being the set of polygons in the plane partition and an edge between vertices corresponding to two polygons if and only if those polygons are in a certain (quite subtle) sense adjacent\(^2\). This “sector” graph is planar and has \( O(n \log(n)/s) \) vertices and as such, we can find an \( r/s \)-division with \( O(n \log(n)/r) \) regions and boundary size per region \( O(\sqrt{r/s}) \). For each region \( R \) in this \( r/s \)-division, we make a region consisting of vertices lying in polygons whose corresponding vertices lie in \( R \). The size of each region is then \( O(s) \cdot r/s = O(r) \). For each vertex in the boundary of \( R \), we get \( O(s) \) vertices and edges in the boundary of the new region. Therefore, the size of boundary of

\(^2\)The formal definition assumes that each polygon contains some subset of its geometric boundary. The relation of two polygons being adjacent then depends on what polygons the boundaries belong to.
any region is $O(\sqrt{sr})$. We also build a point location data structure, which enables us to efficiently tell which regions a vertex lies in.

### 1.2.2 Estimating additive Lipschitz parameters

Suppose we have a graph parameter $\Pi$ that changes by $O(1)$ by removing any single vertex and such that for disjoint graphs $G_1, G_2$, it holds that $\Pi(G_1 \cup G_2) = \Pi(G_1) + \Pi(G_2)$. Given a graph $G$, we want to estimate $\Pi(G)$. We find an $r$-division for appropriately chosen $r$ with $O(\epsilon n)$ vertices and edges in the boundaries. We will want to be able to perform BFS within one region. In order to do this, we remove all vertices with degree $\geq \Theta(1/\epsilon)$. We let $G'$ be a graph obtained from $G$ by removing all boundary vertices, all vertices incident to boundary edges, and all vertices with degree $\geq \Theta(1/\epsilon)$. There are only $O(\epsilon n)$ such vertices and thus $|\Pi(G) - \Pi(G')| \leq O(\epsilon n)$. By additivity, $\Pi(G') = \sum_{H \in cc(G')} \Pi(H)$ where $cc(G')$ denotes the set of connected components of $G'$. We estimate this sum using the Horvitz-Thompson estimator: let $v$ be a random vertex, and let $H_v$ be the connected component of $G'$ that $v$ lies in; $n \Pi(H)/|H_v|$ is then an unbiased estimator of $\Pi(G')$. We take the average of sufficiently many independent copies and use this as our final estimate.

### 1.2.3 MPC algorithms

In this section, we start by showing a Las Vegas algorithm that computes an $r$-division of the input graph in $O(1)$ rounds, both expected and with high probability. We then use this to design recursive algorithms that solve various problems in expected $O(1)$ rounds with space $O(\frac{n^2}{3} + \epsilon)$ for any constant $\epsilon > 0$. We explain why we need this much space below (usually, an algorithm with complexity $O(n^{1-\epsilon})$ can be modified by using recursion to get down to complexity $O(n^\epsilon)$ for any constant $\epsilon > 0$).

**Computing an $r$-division.** To compute an $r$-division, we execute our sublinear-time algorithm for $r$-divisions. Specifically, we sample in parallel an appropriate number of edges and vertices, and we send them all to one machine. If we pick the number of sampled edges and vertices to be large enough, we then may execute the sublinear-time algorithm on this one machine. This results in an oracle that, given a vertex, returns the region the vertex belongs to. Moreover, the oracle does not perform any queries and only needs to know the coordinates of the vertex that we are querying. We set the parameters in such a way that the oracle fits onto one machine. We can now distribute the oracle to all machines using the broadcast trees. We can now, locally on each machine, determine which region some given vertex lies in. This allows us to move the edges around in such a way that each region in the $r$-division is stored on consecutive machines while any machine stores only edges belonging to only one region.

**Sketch of our algorithm for counting connected components.** We then use our algorithm for computing $r$-divisions to get algorithms for several classic graph problems. For illustration, we sketch the simplest of our algorithms – an algorithm for counting connected components in a graph. Note that under the 1-vs-2-conjecture, one needs $\Omega(\log n)$ rounds for this problem without having the embedding.

In order to make the recursion work, we solve a more general problem. Namely, we assume that we are given a graph $G$ with some subset of vertices $\nabla(G)$ that are marked. We compute the number of connected components that have empty intersection with $\nabla(G)$ while we “compress” the other components by computing a graph $G'$ with the following properties: it contains all vertices
in \( \nabla(G) \), it has \( O(\nabla(G)) \) vertices, and two vertices in \( \nabla(G) \) are connected in \( G' \) if and only if they are connected in \( G \).

This problem can be solved recursively. In the base case, the whole graph fits onto one machine, and we may solve the problem by a simple sequential algorithm. We now describe the recursive case.

We find an \( r \)-division \( \mathcal{R} \) for \( r = |G|^{1-\epsilon} \) for some \( \epsilon > 0 \). For each region \( R \in \mathcal{R} \), we mark its boundary in addition to the vertices that are already marked in \( G \). We then recursively find the number of connected components that do not intersect \( \nabla(R) \) and a graph on \( O(\nabla(R)) \) vertices that connects two vertices in \( \nabla(R) \) iff they are connected in \( R \). Taking the union of such graphs over all \( R \in \mathcal{R} \), we get a graph in which two vertices of \( \partial(\mathcal{R}) \cup \nabla(G) \) are connected only if they are connected in \( G \). We call this graph \( H \). If we choose the parameters right, the graph \( H \) will fit onto one machine. This allows us to further compress this graph, resulting in a graph with \( O(\nabla(G)) \) vertices in which two vertices in \( \nabla(G) \) are connected iff they are connected in \( G \). We compute the number of connected components that have empty intersection with \( \nabla(G) \) by adding the respective numbers from the recursive calls with the number of connected components in \( H \) that have empty intersection with \( \nabla(G) \).

In each successive recursive call, the size of the graph is \( |G|^{1-\epsilon} \) for a constant \( \epsilon > 0 \). The recursion ends if \( |G| \leq O(S) \) for \( S \geq n^{2/3} \). We thus get that the depth of recursion is \( O(1) \). The round complexity is thus \( O(1) \) in expectation and with high probability. This also ensures that the sets \( \nabla(G) \) do not get too large, as in each successive call, the size of this set increases, as we will prove, by at most \( |\partial(R)| \leq o(n^{1/3}) \leq O(S) \).

**Why we need \( \Omega(n^{2/3}) \) space.** The requirement that \( S \geq \Omega(n^{2/3+\epsilon}) \) comes from the following two facts. First, we need that \( |\partial(\mathcal{R})| \leq S \) (otherwise we couldn’t fit the compressed graph of size \( O(|\partial(\mathcal{R})|+|\nabla(G)|) \) onto one machine). Second, we need to make sure that we can construct the \( r \)-division oracle on one machine, which requires the space per machine to be \( S \geq \Omega(n \log n) \).

Thus we need \( S \geq \Omega(\max\{\sqrt{srn^2/n}, n \log n \}) \), which is minimized for \( s = \sqrt{r} \). Since we also need \( r = O(n^{-1}) \), this implies that we need \( \Omega(n^{2/3+\epsilon}/3 \log n) \) space per machine, and \( O(n^{2/3+\epsilon}) \) space turns out to be sufficient.

It is usually the case that if one gets an algorithm with \( O(n^{1-\epsilon}) \) space per machine, it is possible to also get an algorithm with the same asymptotic round complexity and \( O(n^\epsilon) \) space per machine by recursively using the same approach. Why is this not straightforwardly possible in our case? We could compute in such small space a smaller graph on which we would need to recurse. However, it is not clear how to efficiently find an embedding of this smaller graph. Our approach fundamentally depends on having this embedding, and it is thus not clear how if this approach is feasible in our situation.

### 1.3 Beyond straight-line embeddings

The above algorithm in fact works whenever the input consists of \( x \) -monotone curves. We pick an angle \( \theta \sim Unif([0,2\pi]) \), rotate the whole input by \( \theta \) angle around the origin, and subdivide all curves so as to make all the resulting curves are \( x \) -monotone. This can be done by dividing the curve whenever its tangent at a point is vertical. (We may perform the rotation lazily, whenever we want to use the coordinates of a vertex.) We then prove the following lemma: Consider a “nice” curve \( C \) and mark all points such that the tangent of the curve forms angle \( \theta \) with the \( x \)-axis; then the expected number of marked points is \( O(A(C)) \) where \( A(C) \) is the total absolute curvature of \( C \). Applying this lemma to the rotated input, we get that the expected number of points at which
the tangent of an edge is vertical, is $O(A(G))$ for $A(G)$ being the total absolute curvature of all the edges.

2 Preliminaries

We now give several definitions. All the definitions in this section are commonly used and are not new to this paper. We then give definitions at the beginning of Section 3 that are either new to this paper or that we use slightly differently from their common usage.

2.1 Balanced separators and $r$-divisions

The following definition comes (in a slightly different form) from Frederickson [14].

Definition 1. A division $\mathcal{R}$ is a system of subsets of $V(G)$ such that each edge's endpoints lie in common $R \in \mathcal{R}$, which we call a region. An $r$-division is a division such that (1) each $R \in \mathcal{R}$ contains $\leq r$ vertices, (2) it holds $|\mathcal{R}| \leq O(n/r)$, and (3) the boundary of each $R \in \mathcal{R}$ has size $|\partial(R)| \leq O(\sqrt{r})$.

It is well known that planar graphs have $r$-divisions for any value of $r$ [26]. Moreover, $r$-divisions can be computed very efficiently:

Fact 2 (Klein, Mozes, and Sommer [21]). Let $G$ be a planar graph. There is an algorithm that computes an $r$-division in linear time, for any given $r \geq \Omega(1)$.

Throughout this paper, we assume that $r$-divisions are stored in a "dual" representation, where we store for each vertex the set of regions it belongs to, instead of storing for each region the set of vertices it contains. We assume that the representation implements the following operations in worst case $O(1)$ time: membership, cardinality, iterate through elements (taking $O(1)$ time per element). These operations are implemented for example by Cuckoo hash tables.

2.2 Trapezoid Decomposition

Trapezoid decomposition is a geometric structure, originally designed to solve the point location problem – one of the fundamental problems in computational geometry. It has also been used in [8] for computing cuttings of the plane (defined in Section 3); this is how we use trapezoid decompositions in this paper. This is usually defined for a set of lines, but we give a more general definition for a set of curves. We will use this in Section 6 where we deal with non-straight-line embeddings.

Definition 3. Given a set $S$ of $x$-monotone curves, for each curve's endpoints add the maximal possible vertical line segment that touches the endpoint and does not intersect any of the curves in $S$. Taking the union of these line segments with $S$, the trapezoid decomposition of $S$ is defined as the set of connected parts of the plane.

A trapezoid decomposition of a set of line segments can be computed in time $O(n \log n)$ using for example the algorithm by Mulmuley [27]. The algorithm also produces a data structure that allows us to determine which connected part of the plane a point lies in in time $O(\log n)$. This is useful for us as it means that we get algorithms that not only perform few queries, but also have good time complexity.

\footnote{Boundary of $R \in \mathcal{R}$, denoted $\partial(R)$, is defined as $R \cap \left( \bigcup_{R' \in \mathcal{R} \setminus \{R\}} R' \right)$}
2.3 Property testing

For most computational problems, there are trivial linear lower bounds. In order to be able to achieve sublinear complexity, it is then necessary to relax the problem specification. One natural approach, which works for graph parameters is that of selecting for an approximate solution. For graph properties, the notion of property testing is commonly used.

A property on graphs of order \( n \) is defined as a subset of all such graphs that is closed under isomorphism. Let us have some metric \( d(\cdot, \cdot) \) on graphs. Commonly used metric in property testing is the hamming distance, normalized by the number of edges. In this paper, we instead consider the vertex edit distance normalized by the number of vertices. This is defined, for graphs \( G_1, G_2 \) as the smallest number of vertices that have to be removed from \( G_1 \) and \( G_2 \) in order to get isomorphic subgraphs. Note that because of the different normalization, these two metrics are incomparable.

Let us have a property \( \Psi \). In property testing, one requires that if \( G \) does have property \( \Psi \), then the algorithm says that this is indeed the case (with high probability). If, on the other hand, the distance from any graph with property \( \Psi \) is at least \( \epsilon \), then the algorithm is required to reject the input. The algorithm is allowed to answer arbitrarily otherwise.

We also consider tolerant property testing. Tolerant property testing makes the requirements stricter in that the algorithm must also accept the input if the distance to \( \Psi \) is \( \leq \epsilon_1 \) while rejecting if it is \( \geq \epsilon_2 \). Property testing is then equivalent to tolerant property testing with \( \epsilon_1 = 0 \) and \( \epsilon_2 = \epsilon \). Tolerant property testing can also be equivalently viewed as approximating the distance to \( \Psi \).

2.4 Sublinear-time models of computation

In the sublinear time regime, we do not have enough time to even read the whole input. This means that it is of crucial importance what queries we may use to access the data. We use a model defined by the following queries: (1) return a vertex picked uniformly at random, (2) pick an edge uniformly at random, (3) given a vertex \( v \) and \( i \leq \deg(v) \), return the \( i \)-th neighbor of \( v \).

While this model has been considered before, a more standard model allows only for queries (1), (3). For us, this difference is not significant. The reason is that random edge queries can be approximately simulated in time \( O(\frac{\alpha n \log^3 n}{\epsilon^2}) \) for \( \alpha \) being the arboricity of the input graph using the algorithm of Eden, Ron, and Rosenbaum [10]. For planar graphs, it holds that \( \alpha \leq 3 \). Conveniently, our algorithms works (after slightly changing some parameters) even when we may sample edges only approximately uniformly, say for \( \epsilon = 1/2 \). This allows us to use the algorithm from [10] to implement the random edge queries.

In Section 6, we deal with the case of non-straight-line embeddings. In that section, we assume that any neighborhood query also returns the connecting edge. Given an edge, we assume that we may (a) given \( \theta(0, 2\pi) \) find all points on the edge whose tangent has angle \( \theta \) with the \( x \)-axis, and (b) for a ray we may find the first intersection of the ray and the edge.

2.5 Massively parallel computation

Massively parallel computation (MPC) is a theoretical model introduced by Karloff, Suri, and Vassilvitskii [20] that attempts to capture the popular MapReduce framework for parallel computation. Since its inception, this model has received a lot of attention. We now briefly introduce the model. For a more in-depth treatment, see the lecture notes by Ghaffari [16].

In MPC, the computation is performed by \( M \) machines in synchronous rounds. Each machine has \( S \) words of local memory. At the beginning of the computation, the input – represented by \( N \)
words – is evenly partitioned on $O(N/S)$ machines. Each round looks as follows. At the beginning of a round, each machine is storing the messages that were sent to it in the previous round, in addition to the contents of the memory that it had at the end of the previous round. Each machine then may perform arbitrary computation. Then, each machine stores in a predefined place in its memory all the messages that it wants to send, each message having one recipient. At the end of the computation, each machine stores part of the input. For example, in the case of the minimum spanning tree, we want that the minimum spanning tree is stored on some given $O(n/S)$ machines. The complexity of MPC algorithms is mainly measured in terms of the round complexity. To simplify the exposition, we assume that one machine can store $O(S)$ words (instead of just $S$).

### 2.5.1 Broadcast and convergence-cast trees

If we have a message of size $k$, we cannot simply broadcast it to all other machines unless $S \geq kM$ as a machine has to store all its outgoing messages at the end of a round. Broadcast trees can be used to accomplish this in $O(1)$ rounds and $kM^\epsilon$ space per machine for some $\epsilon > 0$. Similarly, a machine cannot receive a message of size $k$ from all other machines. However, if we have an associative operation $\circ$ and messages $m_1, \cdots, m_M$, one may compute $\bigcirc_{i=1}^M m_i$ using the converge-cast trees using per-machine space $kM^\epsilon$. See [16] for explanation.

### 2.6 Notation

We use $V(G), E(G)$ to denote the vertex and edge set of a graph $G$ respectively. In a weighted graph, we denote the weight of an edge $e$ by $w(e)$ and the length (the total weight) of a path $P$ by $w(P)$. We use $cc(G)$ to denote the set of connected components of a graph $G$. For $G$ being an embedded graph, we use $cr(G)$ to denote the number of pairwise crossings between edges of the embedding. For $V' \subseteq V(G)$, we use $G[V']$ to denote the subgraph of $G$ induced by $V'$. For a graph $G$ and $V' \subseteq V(G)$, we define the boundary of $V'$, denoted $\partial_G(V')$, as the subset of $V'$ of vertices that have at least one neighbor outside of $V'$. We define $\partial_G^E(V')$ as the set of edges that have one endpoint in $V'$ and the other in $V \setminus V'$. When the graph $G$ is clear from the context, we drop the subscript. We denote the order (number of vertices) of a graph $G$ by $|G|$, and the size (number of edges) by $\|G\|$. We use $d_G(u,v)$ to denote the distance from $u$ to $v$ in the graph $G$. We may drop the subscript if $G$ is clear from the context. We use $D_G$ be the complete graph with $w(uv) = d_G(u,v)$.

### 3 Finding small separators

We start by giving several definitions that we will need in this section. These definitions are either new (such as (hybrid) $r$-pseudodivisions) or they differ slightly from the way they are sometime used (such as $q$-cuttings). We start by defining two relaxed versions of the $r$-division.

**Definition 4.** A $r$-pseudodivision is a division that satisfies condition (1), i.e. that $|R| \leq r$ for $R \in \mathcal{R}$. Bounds on $|\partial(R)|$ and $|\mathcal{R}|$ like those in (2), (3) have to then be specified explicitly.

A hybrid division is an arbitrary system of subsets of $V(G)$. A hybrid $r$-pseudodivision is a hybrid division satisfying condition (1).

The (hybrid) $r$-pseudodivisions have the advantage that they are simpler to compute while they are sufficient for many of the applications that $r$-divisions are commonly used for.

Given a hybrid $r$-division, one may easily get an $O(r)$-division in linear time. This may be done by, for any boundary edge $uv$, adding $v$ to the region of $u$ and symmetrically adding $u$ to the region...
of \( v \). While this may be done by a linear-time sequential algorithm, this cannot be performed in sublinear time, which is the reason we need this notion.

Note that if we have a \( r \)-pseudodivision with \( k \) regions and boundary sizes \( \leq \ell \), we may get a \( rk/k' \)-pseudodivision with \( k' \) regions and boundary sizes \( \leq \ell k/k' \) by merging regions. It is thus possible to assume that the number of regions is always \( \leq n/r \), at the cost of larger \( r \) and larger boundaries. For this reason, we do not focus too much on minimizing the number of regions as we can always ensure that, if we need it.

We now give definitions of plane partitions and good cuttings.

**Definition 5.** An open (convex polygonal) plane partition is a partition of \( \mathbb{R}^2 \) into sectors \( S_1, \ldots, S_k \), such that for every \( i \in [k] \), the closure of \( S_i \) is a polygon and each of its edges (excluding endpoints) and vertices belong to \( S_j \) for some \( j \in [k] \) (that is, one edge (excluding endpoints) or vertex may not lie in multiple \( S_j \)'s).

As we only consider convex polygonal plane partitions in this paper, we leave out the “convex polygonal” and only talk about plane partitions.

We now give a definition of a \( q \)-cutting. This notion is central to our paper – our algorithm for finding \( r \)-divisions works by first finding a cutting of the input into not too many sets, and then using it to find the division. The definition we use differs slightly from the commonly used definition in that we require that it is a plane partition and not some plane partition up to a set of measure zero. Our definition also differs in that we treat vertices and edges separately, whereas in the previous work, a point would be considered a degenerate line segment.

**Definition 6.** We say a plane partition is a \( q \)-cutting if

1. each sector contains \( \leq n/q \) vertices
2. the number of edges intersecting any sector is \( \leq n/q \)

We say that a \( q \)-cutting is good for a parameter \( \delta \) if

3. the total number of sectors is \( O(q(\log n + \log \delta^{-1})) \)

Given a plane partition, we define its sector graph as follows:

**Definition 7.** Let us have a plane partition \( \mathcal{P} \). We define the sector graph \( G_s(\mathcal{P}) \) as follows. The vertex set is the set of sectors. There is an edge between vertices corresponding to two sectors \( s_1, s_2 \) if and only if there exists point \( x \in s_1 \) such that there exist points in \( s_2 \) arbitrarily close to \( x \) or \( x \in s_2 \) and such points in \( s_1 \).

Note that while it may seem at first glance that the sector graph is equal to the dual graph, this is not the case. The reason is that the \( x \in s_1 \) may coincide with a vertex \( v \) of \( \mathcal{P} \), resulting in an edge in the sector graph between vertices corresponding to \( s_1 \) and \( s_2 \) even if the sectors only meet at \( v \) but do not share an edge. The edge set of the sector graph is thus a (potentially strict) superset of the dual graph’s edge set.

\[ \delta \]

5We will be using \( \delta \) as the failure probability for our algorithms.
3.1 Finding a good \( s/n \)-cutting

In this section, we show how to find a good \( s/n \)-cutting. Our algorithm is a modification of the algorithm from \([8]\). We show the algorithm so as to be self-contained, but also because it differs slightly from the algorithm of \([8]\) as we have to deal with vertices separately.

We assume that no two vertices have the same \( x \)-coordinate. This can be satisfied with high probability by rotating (lazily, whenever we query a vertex or edge) the input by an angle picked uniformly at random from a large enough set.

We interpret a trapezoid decomposition as a plane partition, by arbitrarily putting each edge and vertex into one of the sectors. In order to be able to do this consistently in a distributed fashion, we may pick a vector and put each vertex into the sector that is incident in the direction of the vector. We then put each edge into the sector that its left endpoint is in.

**Algorithm 1:** Find a good \( s/n \)-cutting of the plane

1. \( E' \leftarrow \) sample \( 3n(2 \log n + \log \delta^{-1})/s \) edges
2. \( V' \leftarrow \) sample \( n(2 \log n + \log \delta^{-1})/s \) vertices
3. Compute the trapezoidal map of \( E' \cup V' \) using the algorithm from \([27]\), let \( D \) be the returned data structure for point location for this trapezoidal map.
4. return \( D \)

**Lemma 8.** Given \( s \) and \( \delta > 0 \), Algorithm 1 returns a good \( s/n \)-cutting with probability at least \( 1 - O(\delta) \), represented by a data structure \( D \) that answers point location queries in time \( O(\log(n/s)) \). It has query complexity \( O\left(\frac{n(\log(n + \log \delta^{-1}))}{s}\right) \) and time complexity \( O\left(\frac{n(\log(n/s)(\log(n + \log \delta^{-1}))}{s}\right) \).

**Proof.** Let \( \mathcal{P} \) be the trapezoid map of \( E' \cup V' \). We need to prove that \( \mathcal{P} \) satisfies the conditions 1 - 3 with probability \( 1 - O(\delta) \). The property 3 — that the number of sectors is \( O(n(\log n + \log \delta^{-1})/s) \) — amounts to showing that a trapezoidal map has number of faces linear in the number of line segments. This is well known to be true (see e.g. \([27]\) Lemma 6.2)).

Consider a trapezoid \( t \). We show that if \( t \) violates property 1 or 2, then the probability that \( t \) is a trapezoid in \( \mathcal{P} \) is \( \leq O(\delta/n^2) \). Any trapezoid in any possible trapezoidal map is uniquely determined by a pair of edges, namely its lower and upper boundaries. Therefore, there are \( O(n^2) \) potential trapezoids. By the union bound, we thus get that none of the violating trapezoids lie in \( \mathcal{P} \) with probability at least \( 1 - O(\delta) \).

Consider the case that \( t \) has size \( s > \) — it violates property 1. For \( t \) to be in \( \mathcal{P} \), it has to be the case that none of the \( s \) vertices in \( t \) have been sampled to \( V' \). This happens with probability at most

\[
(1 - s/n)^{n(2 \log n + \log \delta^{-1})/s} \leq \exp(-s \log(n^2/\delta)/s) = \delta/n^2
\]

Consider the case that the sector is intersected by \( s \) edges — it violates property 2. For \( t \) to lie in \( \mathcal{P} \), it has to be the case that none of the \( s \) intersecting edges have been sampled into \( E' \). The number of edges is \( \leq 3n - 6 < 3n \), a uniformly sampled edge intersects \( t \) with probability at least \( s/(3n) \). The probability that we sample none of the intersecting edges is thus at most

\[
(1 - s/(3n))^{3n(2 \log n + \log \delta^{-1})/s} \leq \exp(-s \log(n^2/\delta)/s) = \delta/n^2
\]

The query complexity is clearly as claimed. As for the time complexity, the dominant cost is computing the trapezoidal map. For input of size \( n \), the algorithm for computing trapezoidal decomposition from \([27]\) runs in time \( O(n \log n) \). Therefore, the time complexity is \( O\left(\frac{n(\log(n + \log \delta^{-1}))}{s} \log(n/s)^{\frac{n(\log(n + \log \delta^{-1}))}{s}}\right) \leq O\left(\frac{n(\log(n + \log \delta^{-1}))}{s} \log(n/s)^{\frac{n(\log(n + \log \delta^{-1}))}{s}}\right) \).

\( \square \)
Remark. Note that our analysis does not depend on being able to sample edges (or vertices) exactly uniformly. For example, if we may pick an edge from distribution that gives probability $\geq 1/(2m)$ to every vertex, it is sufficient to let $E'$ be a sample of $6n(\log n + \log \delta^{-1})/s$ and basically the same analysis is sufficient to prove correctness of this algorithm.

This means that if we are not able to sample edges uniformly, we may use the algorithm by Eden et al. [10] that allows us to approximately sample an edge in time $O(\frac{sn}{m} \cdot \log^3 n)$ for $\alpha$ being the arboricity of the graph. This is equal to $O(\log^3 n)$ as $\alpha \leq 3$ for planar graphs and we may choose $\epsilon = 1/2$.

3.2 Finding an $r$-pseudodivision, given a good cutting

Algorithm 2: Compute an $r$-pseudodivision oracle

**input:** Graph $G$ with $n$ vertices and $m$ edges, parameter $s$, good $s/n$-cutting $P$, represented by point location data structure $D$

1. Construct the sector graph $G_s$ of $P$
2. $R \leftarrow$ Find a (strong) $r'$-division of $G_s$ for $r' = r/s$
3. return $(R, D)$

The interpretation of the output is the following. For each region in $R \in \mathcal{R}$, we have a region $R'$ of $G$ consisting of the vertices that lie in a sector whose vertex in $G_s(P)$ is in $R$. An edge lies in the separator if it intersects one of the sectors that are in the boundary of $\mathcal{R}$.

**Theorem 9.** Assume we are given $(\mathcal{R}, D)$ as returned by Algorithm 2 with parameters $s$ and $\delta > 0$. There exists an algorithm that, given a vertex $v \in G$ returns in expected time $O(\log((n \log \delta^{-1})/s))$ and without making any queries a set of regions $v$ belongs to. These sets form, with probability at least $1 - O(\delta)$, a hybrid $r$-pseudodivision $R'$ with boundary size $O(\sqrt{sr})$ per region. The number of regions is $O(n(\log n + \log \delta^{-1})/r)$. The set implements in $O(1)$ time the following operations: membership, cardinality, iterate through elements (taking $O(1)$ time per element).

**Proof.** We condition on $P$ being a good $s/n$-cutting. This happens with probability $1 - O(\delta)$ by Lemma 8. We define the division $R'$ as follows. For each region $R \in \mathcal{R}$, we define a region $R' \in \mathcal{R}'$ consisting of the vertices that lie in sectors whose corresponding vertices are in $R$.

In the division $\mathcal{R}$ of $G'$, each region has size $\leq r' = r/s$. As $P$ is a good $s/n$-cutting, each sector has $\leq s$ vertices. The number of vertices in the region $R' \in \mathcal{R}'$ is thus $\leq sr' = r$.

We now argue that $\mathcal{R}'$ is a hybrid $r$-pseudodivision with boundary size per region $O(\sqrt{sr})$. In the division $\mathcal{R}$ of $G'$, each region has boundary of size $O(\sqrt{r'}) = O(\sqrt{rs})$. The vertices that lie in the boundary of $R'$ are exactly the vertices lying in the $O(\sqrt{r}/s)$ boundary sectors. Since $P$ is a good cutting, each sector has $O(s)$ vertices, meaning that $R'$ has $O(\sqrt{sr})$ boundary vertices. The boundary vertices all intersect at least one of these boundary sectors as otherwise there would have to be adjacent sectors whose vertices in $G_s(P)$ do not share a region; this is a contradiction with $\mathcal{R}$ being an $r$-pseudodivision. The number of edges crossing each sector of $P$ is $O(s)$ by the definition of a good cutting. The number of edges that cross the boundary sectors and thus the number of edges from $R'$ to vertices not in $R'$ is $O(s) \cdot O(\sqrt{r}/s) = O(\sqrt{rs})$. This proves that the boundary of $R'$ has size $O(s) \cdot O(\sqrt{r}/s) = O(\sqrt{sr})$.

The number of regions in $\mathcal{R}$ is $\leq \frac{n(\log n + \log \delta^{-1})/s}{r'} = n(\log n + \delta^{-1})/r$ by the definition of an $r'$-division, and this is the case even if $P$ is not a good $s/n$-cutting.
Given a vertex $v$, we may use $D$ to find $S_v$ in time $O\left(\log((n \log \delta^{-1})/s)\right)$. Let $u \in G_s(P)$ be the corresponding vertex of $S_v$. The set of regions $v$ belongs to in $\mathcal{R}'$ is the same as the set of regions of $u \in \mathcal{R}$. As we mentioned in the preliminaries, we are assuming that we are storing this set in a representation which implements the operations mentioned in the statement in $O(1)$ expected time.

4 Estimating additive parameters

In this section, we consider additive $O(1)$-Lipschitz parameters. Recall that a parameter $\Pi$ is $\alpha$-Lipschitz with respect to a metric $d$ if for any $G_1, G_2$, it holds that $|\Pi(G_1) - \Pi(G_2)| \leq \alpha d(G_1, G_2)$. We consider the vertex edit distance metric, which is defined as the smallest number of vertices that needs to be removed from $G_1$ and $G_2$ in order to get two isomorphic subgraphs of $G_1$ and $G_2$. A property is additive if for graphs $G_1, G_2$ on disjoint vertex sets, it holds that $\Pi(G_1 \cup G_2) = \Pi(G_1) + \Pi(G_2)$.

These two conditions are satisfied by a large set of combinatorial optimization problems, including: minimum vertex cover\(^6\), minimum edge cover\(^7\), maximum independent set\(^8\), minimum feedback vertex set\(^9\), minimum feedback edge set\(^10\), minimum dominating set\(^11\), maximum matching\(^12\), minimum maximal matching\(^13\), $H$-packing\(^14\).

Another class of such problems come from (tolerant) property testing problems with respect to the vertex edit distance. For a property $\Psi$ closed under taking subgraphs, we define a parameter $\Pi_\Psi$ as the smallest number of vertices we need to remove from $G$ to get $G'$ such that $G'$ satisfies $\Psi$. $\Pi$ is then decomposable and 1-Lipschitz. Such problems include, for example, $k$-coloring or subgraph freeness.

**Algorithm 3:** Estimate a $\alpha$-Lipschitz parameter $\Pi$

1. Compute a hybrid $r$-pseudodivision $\mathcal{R}$ using Algorithm 2 with $r = \sqrt{\epsilon n \log^3 n}$, $s = ce^{5/2} \sqrt{n/\log n}$ for $c > 0$ small enough constant, and $\delta = 1/12$
2. $A \leftarrow 0$
3. repeat $k = \frac{36 \alpha^2}{\epsilon^2}$ times
   4. $v \leftarrow$ sample a vertex uniformly
   5. if $v$ is an interior vertex of some region $R \in \mathcal{R}$ and $d(v) \leq \alpha 18/\epsilon$ then
   6. $H \leftarrow$ interior vertices of $R$ reachable from $v$ through vertices of degree $\leq \alpha 18/\epsilon$
   7. $A \leftarrow A + \frac{n}{|H|} \Pi(H)$
8. return $A/k$

**Theorem 10.** Let $\Pi$ be an additive $\alpha$-Lipschitz property. Algorithm 3 returns, with probability at least $2/3$, an additive $\pm \epsilon n$ approximation of $\Pi(G)$. It has query complexity $O(\sqrt{n \log^3 n/\epsilon^{5/2}})$.

\(^6\)Smallest subset $S$ of vertices such that each edge is incident to some vertex in $S$
\(^7\)Smallest subset $S$ of edges such that each vertex is incident to some edge in $S$
\(^8\)Largest subset $S$ of vertices such that there are no two adjacent vertices in $S$
\(^9\)Smallest subset of vertices such that removing these vertices results in a graph with no cycles
\(^10\)Smallest subset of edges such that removing these vertices results in a graph with no cycles
\(^11\)Smallest subset $S$ of vertices such that each vertex is adjacent to some vertex in $S$
\(^12\)Largest subset of edges such that no two of the edges share a vertex
\(^13\)Smallest subset of edges such that no two of them share a vertex and adding any other edge would violate this property
\(^14\)Largest set of vertex-disjoint subgraphs of $G$ such that each of these subgraphs is isomorphic to $H$
Proof. Throughout this proof, we condition on Algorithm\textsuperscript{2} succeeding. We begin by analyzing the expectation and variance of $A$. Let $A_i$ be the increment of $A$ in the $i$-th iteration of the loop on line 3. Let $V'$ be the set of vertices of degree $\geq \alpha/\epsilon$. Let $G'$ be the graph obtained from $G[V \setminus (\partial V(R) \cup V')]$ by removing the edges in $\partial(V(R))$. Note that the algorithm can be seen as in each iteration performing breadth-first-search in $G'$ from $v$ when $v \in V(G')$ and letting $H$ to be the connected component that $v$ lies in, and otherwise letting $H$ be the empty graph, in which case $A_i = 0$.

$$E[A_1] = \sum_{H \in cc(G')} \frac{|H|}{n} \Pi(H) = \sum_{H \in cc(G')} \Pi(H) = \Pi(G')$$

We now analyze the variance of $A$. Recall that $\text{sup}(X)$ is defined as the smallest $x$ such that $P(X > x) = 0$.

$$\text{Var}(A) = k \text{Var}(A_1) \leq k E(A_1)^2 \leq k \alpha^2 n^2$$

where we are using that $\text{sup}(A_1) = \frac{n}{|H|} \text{sup}(\Pi(H)) \leq \alpha n$ where we have $\Pi(H) \leq \alpha |H|$ since the parameter is $\alpha$-Lipschitz and it follows by decomposability that $\Pi(\emptyset, \emptyset) = 0$. Therefore $\text{Var}(A/k) \leq \alpha^2 n^2 / k \leq \frac{1}{36} \epsilon^2 n^2$. By the Chebyshev inequality, it now holds that

$$P\left(\left|\frac{A}{k} - \Pi(G')\right| \geq \epsilon n/3\right) \leq \frac{1}{36 \epsilon^2 n^2} = 1/4$$

We now bound $|\Pi(G') - \Pi(G)|$. By Theorem 9 it holds (recall that we are conditioning on Algorithm\textsuperscript{2} succeeding)

$$|\partial R| \leq O\left(\sqrt{\frac{s r}{r \cdot n \log n}}\right) = O\left(\sqrt{\frac{s}{r} \cdot n \log n}\right) = O\left(\frac{1/2}{\epsilon \log n} \cdot n \log n\right) = O\left(\frac{1/2}{\epsilon n}\right) \leq \epsilon n/(6\alpha)$$

where the last inequality holds for $c$ small enough. Since $\Pi$ is $\alpha$-Lipschitz, removing a vertex changes the value by at most $\alpha n$ and removing an edge by removing its endpoints changes the value by at most $2\alpha$. Removing $\partial R$ from $G$ thus changes the parameter $\Pi$ by at most $\epsilon n/3$. Since $G$ is planar, it holds $||G|| \leq 3n - 6$. There are thus less than $\epsilon n/(3\alpha)$ vertices with degree $> \alpha/\epsilon$. Removing these vertices changes the parameter $\Pi$ by at most $\epsilon n/3$. Putting this together, we have $|\Pi(G) - \Pi(G')| \leq \frac{2}{3} \epsilon n$.

By the union bound, the algorithm succeeds and $|A/k - \Pi(G')| \leq \epsilon n/3$ with probability $\geq 3/4$. When the algorithm succeeds, it holds $|\Pi(G) - \Pi(G')| \leq \frac{2}{3} \epsilon n$. Moreover, the probability that Algorithm\textsuperscript{2} succeeded is at least $1 - 1/12$. We thus have that with probability at least $1 - (1/4 + 1/12) = 2/3$, it holds $|A/k - \Pi(G)| \leq \epsilon n$.

The size of any region in $R$ is $\leq r = \sqrt{\epsilon n \log^2 n}$. In each iteration of the loop in line line 3 we perform breadth-first-search on a subset of one region. We only explore neighborhoods of vertices of degrees $\leq O(1/\epsilon)$, and thus perform at most $O(1/\epsilon)$ queries per explored vertex. This means that each iteration of the loop uses $\sqrt{n \log^2 n/\epsilon}$ queries. We perform $O(1/\epsilon^2)$ iterations of the loop and the total complexity is thus $O(\sqrt{n \log^2 n/\epsilon^{5/2}})$. The complexity of Algorithm\textsuperscript{2} is $O(n \log^{3} n) = O(n \log^{3} n/\epsilon^{5/2})$.

Remark. If we only want an algorithm that returns 0 when $\Pi(G) = 0$ and it returns $> 0$ when $\Pi(G) \geq \epsilon n$, we may get a more efficient algorithm. This is useful for non-tolerant property testing. To get this guarantee, it is sufficient to set $k = \Theta(1/\epsilon)$. We then may set $r = \sqrt{n \log^3 n}, s = c \epsilon^2 n \log n$ for $c > 0$ being a small enough constant. This results in query complexity of $O(\sqrt{n \log^3 n/\epsilon^2})$ instead of $O(\sqrt{n \log^3 n/\epsilon^{5/2}})$. \qed
5 Implications for MPC algorithms

We now show how our sublinear-time algorithm can be used to design efficient parallel algorithms for various problems. All the algorithms we show are based on a subroutine which we now describe.

5.1 Computing an $r$-pseudodivision

We now give a parallel algorithm for computing $r$-pseudodivisions. This algorithm will be central to all our MPC algorithms.

Algorithm 4: Compute a non-hybrid strong $r$-pseudodivision in MPC

1. $s \leftarrow n \log n / S$
2. Sample $\Theta(n \log n)$ edges and vertices (for large enough constant in $\Theta$); send them all to machine 1
3. Order the sampled edges and vertices on machine 1 randomly
4. Execute Algorithm 1 and Algorithm 2 on machine 1 with parameters $r, s, \delta = 1/n$; when the algorithm requests a random edge or vertex, use the next in the specified ordering; this gives us oracle $D$
5. Distribute $D$ to all machines
6. On each machine, calculate for each region the number of vertices from that region stored on the machine; aggregate these numbers on machine 1
7. Compute on machine 1 the total size of each region by adding together for each region the numbers computed in the previous step
8. On machine 1, compute a mapping from the set of regions to the machines such that region $R \in D$ is mapped to $\lceil |R| / S \rceil$ consecutive machines
9. For each edge $uv$ such that $D(u) \cap D(v) = \emptyset$, add $v$ to $D(u)$
10. Distribute to all machines the mapping from regions to machines; move the edges accordingly
11. Verify that region sizes are $\leq r$, number of regions $O(n \log(n)/r)$, and boundaries of all regions have size $O(\sqrt{rn \log n / S})$; restart the algorithm if not (note that the total boundary size can be computed as the difference between $n$ and the total size of regions)

Note that the number of samples used by Algorithm 1 and Algorithm 2 on line 4 is $O\left(\frac{n \log n + \log \delta^{-1}}{s}\right)$ in the worst case so there is always enough samples to perform this number of random queries. The algorithm is thus well-defined.

Lemma 11. Let us have $r \geq n \log n / S^{1-\epsilon}$ for $\epsilon > 0$ being some constant. After execution of Algorithm 4, there exists an $r$-pseudodivision of $G$ with $O(n \log(n)/r)$ regions, boundary of size per region $O(\sqrt{rn \log n / S})$ such that the layout of edges in memory satisfies that

1. each machine only stores edges of one region
2. each region is stored on consecutive machines
3. $O(n/S)$ machines are used
4. on the first machine, there is stored for each region the range of machines that it is stored on

The algorithm uses $O(S)$ space per machine, $O(n/S)$ machines, and performs $O(1)$ rounds both expected and with high probability.
Proof. We first focus on correctness of the output. We then argue that the algorithm indeed uses $O(S)$ space per machine and expected $O(1)$ rounds. When the algorithm finished, it means that it has not been restarted on line 11. Therefore, $D$ represents an $r$-pseudodivision with $O(n \log(n)/r)$ regions and with per-region boundary sizes $O(\sqrt{rn \log n}/S)$. The conditions (1), (2) are then satisfied by the mapping computed on line 8 according to which the edges are mapped to the machines (as we move them accordingly on line 10). Moreover, each region $R \in D$ is stored on $\lceil |R|/S \rceil$ machines, meaning that in total, $O(n/S)$ machines are used, thus satisfying point (3). Moreover, the mapping required in point (4) is computed and stored on machine 1 on line 9.

By Theorem 9, $D$ is consistent with an $r$-pseudodivision with $O(n \log(n)/r)$ regions and boundary size per region $\sqrt{sr} = O(\sqrt{rn \log n}/S)$ (the equality holds by our choice of $s$) with probability at least $1 - O(1/n)$. We restart the algorithm only when this is not the case, which happens with probability $1 - O(\delta) = 1 - O(1/n)$; we thus make $O(1)$ restarts in expectation and with high probability. All lines except from line line 7 clearly take $O(1)$ rounds. Line 7 can be implemented in $O(1)$ rounds using convergence-cast trees (see Section 2.5). Thus, each line can be implemented in $O(1)$ rounds, leading to total of $O(1)$ rounds in expectation and with high probability.

It remains to argue that only $O(S)$ space per machine is used. By the assumption $s = n \log n/S$ and $\delta = 1/n$, it holds that the number of sampled edges on line 2 is $O(\frac{(n \log n + \log \delta^{-1})}{s}) = O(S)$. Lines 3,4,5 then do not use asymptotically more space. Specifically, the space complexity of $D$ is $O(S)$. We are assuming $r \geq n \log n/S^{1-\epsilon}$. The number of regions is $O(\frac{n \log n + \log \delta^{-1}}{r}) \leq O(S^{1-\epsilon})$. Line 6 thus uses $O(S^{1-\epsilon})$ space per machine. Line 7 can be implemented in $O(S)$ space per machine using converge-cast trees. Note that the convergence-cast trees need extra $\mathcal{M}'$ space in addition to the size of one machine for some $\epsilon' > 0$; this is the reason we need $r \geq n \log n/S^{1-\epsilon}$ for some $\epsilon > 0$ and not just $r \geq n \log n/S$. The remaining lines can also be implemented in $O(S^{1-\epsilon})$ space per machine.

5.2 Applications of $r$-pseudodivisions in MPC

We now show how an $r$-pseudodivision can be used to give MPC algorithms for several different problems. As a warm-up, we start with counting connected components. We then show algorithms for bipartition and the minimum spanning tree problem. We then give an algorithm for a problem that generalizes the approximate shortest path problem. We use this to also approximate the diameter/radius of a graph.

The solutions to the mentioned problems follow a common theme. Our approach works as follows. We show that when we have a graph $G$, we may replace an induced subgraph of $G$ by some other smaller graph that “compresses” the induced subgraph, and this does not change the structure of the solutions. This is inspired by and similar to the approach of Eppstein et al. [12]. We compute these “compressing graphs” as follows. We find an $|G|^{1-\epsilon}$-pseudodivision $\mathcal{R}$ for some $\epsilon > 0$, recursively compress each region $R$ into a small graph $H_R$. We then use these graphs $H_R$ to find a graph that compresses the whole graph $G$. We then use this to find a solution to the problem on hand. Specifically, we find an $|G|^{1-\epsilon}$-pseudodivision, and for each region, we compress the rest of the graph except for the region, and then use it to recursively find a solution for that region, passing to the recursive call a compressed version of the rest of the graph.

This does not work in a straightforward way for all above-mentioned problems. For example, for computing (approximately) shortest paths, we need to solve a more general problem in order to make the recursion feasible.

Note that since each machine is only storing vertices of a single region (using Lemma 11), it is straightforward to separately execute the algorithm on each group of machines that store one
region. After sending all the graphs $H_R$ for $R \in \mathcal{R}$ to one machine, we use this single machine to perform the rest of the computation.

Throughout this section, we use $|G|$ to denote the number of vertices and $\|G\|$ to denote the number of edges of $G$ in a given recursive call. We use $n, m$ to denote the respective values in the original call. In this section, we need to compute $r$-pseudodivisions of various graphs. This can be done using Algorithm 4; we do not make this explicit in the rest of this section for the sake of brevity.

The proofs in this section are structured such that we always state a theorem, then we state a stronger claim that implies the theorem. Finally, we give a proof of such claim.

### 5.2.1 Counting connected components

**Theorem 12.** Assume that we are given a graph $G = (V, E)$. For any fixed $\epsilon > 0$, there is an algorithm that returns the number of connected components of $G$; it performs $O(1)$ rounds both in expectation and with high probability, and uses $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}, M = n^{1/3-\epsilon}$.

In fact, we prove a stronger claim. We need this in order to be able to give a recursive algorithm. We may recover this theorem by setting $\nabla(G) = \emptyset$ and using $\ell$ as the answer. Note that $O(0) = 0$.

**Claim 13.** Let us have a graph $G$ and a set of marked vertices $\nabla(G)$ with $|\nabla(G)| = k \leq O(S)$. Then, for any fixed $\epsilon > 0$, there is an algorithm that returns a graph $H$ with $V(H) \supseteq \nabla(G)$ on $O(k)$ vertices and a number $\ell$ such that $cc(G) = cc(H) + \ell$ and two vertices in $\nabla(G)$ are connected in $H$ iff they are connected in $G$; it performs $O(1)$ rounds both in expectation and with high probability, and uses $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}, M = n^{1/3-\epsilon}$.

**Proof.** We give a recursive algorithm. The base case uses a straightforward sequential algorithm, while the recursive case is based on Algorithm 4.

**Base case.** We start by the base case when $n \leq O(S)$. In this case, the whole input fits onto one machine. For each maximal subset $S \subseteq \nabla(G)$ of vertices that are connected in $G$, we add to $H$ a star whose ray vertices are exactly the vertices $S$. It is easy to see that two vertices from $\nabla(G)$ are connected in $H$ iff they are connected in $G$. We set $\ell$ to be the number of connected components that do not have any vertex marked. In the rest of the proof, we focus on the recursive case (when $n > S$).

**Recursive case.** We compute an $r$-pseudodivision $\mathcal{R}$ of $G$ for $r = |G|^{1-\epsilon}$. Recursively, we compute for each region $R \in \mathcal{R}$ a graph $H_R$ with $\nabla(R) \subseteq V(H)$ for $\nabla(R) = \partial(R) \cup (\nabla(G) \cap R)$ and $\ell_R$ such that two vertices in $\nabla(R)$ are connected in $H_R$ iff they are connected in $G$ and $\ell_R$ is the number of connected components that have empty intersection with $\nabla(R)$. In order to perform the recursive call, we need that $\nabla(R) \leq O(S)$, which holds as we will argue below.

Let $G' = \bigcup_{R \in \mathcal{R}} H_R$. We now compute the graph $H$ as follows. We compute the connected components of $G'$ and for each maximal set $S$ of vertices of $\nabla(G)$ that are connected in $G'$, we add to $H$ a star with its ray vertices being $S$. Let $\ell_{G'}$ be the number of connected components in $G'$ that have empty intersection with $\nabla(G)$. We set $\ell = \ell_{G'} + \sum_{R \in \mathcal{R}} \ell_R$.

We now prove correctness. We have from each recursive call that two vertices in $\nabla(R)$ are connected in $H_R$ iff they are connected in $G[R]$. Consider two vertices $u, v \in \nabla(G)$. We argue that they are connected in $H$ if and only if they are connected in $G$. 


Consider a uv-path \( P_H \) in \( H \) and let \( v_i \) the \( i \)-th intersection of \( P_H \) with \( \partial(R) \cup \nabla(G) \). Consider the section of \( P_H \) connecting \( v_i \) and \( v_{i+1} \) for some \( i \). This section is contained in \( H_{i,R_i} \) for some \( R_i \in \mathcal{R} \). It holds \( v_i, v_{i+1} \in \nabla(R_i) \), and we thus have from the recursive call that \( v_i \) and \( v_{i+1} \) is also connected in \( G[R_i] \); denote one such connecting path by \( P_{H,i} \). Consider the union \( \bigcup_i P_{H,i} \). This is clearly a connected graph that contains \( u \) and \( v \), and these two vertices are thus connected in \( G \).

Similarly, consider a uv-path \( P_G \) in \( G \) and let \( v_i \) the \( i \)-th intersection of \( P_G \) with \( \partial(R) \cup \nabla(G) \). Consider the section of \( P_G \) connecting \( v_i \) and \( v_{i+1} \) for some \( i \). This path section is contained in one region, which we call \( R_i \). The vertices \( v_i \) and \( v_{i+1} \) are thus connected in \( G[R_i] \). We thus have from the recursive call that \( v_i \) and \( v_{i+1} \) are connected in \( H_{i,R_i} \). Denote one such connecting path by \( P_{G,i} \). Similarly to the above case, \( \bigcup_i P_{G,i} \) is a a connected graph that contains \( u \) and \( v \), and these two vertices are thus connected in \( H \).

The sum \( \sum_{R \in \mathcal{R}} \ell_R \) counts exactly the connected components of \( G \) that have empty intersection with \( \bigcup_{R \in \mathcal{R}} \nabla(R) = V(G') \) (and thus also with \( \nabla(G) \)). \( \ell_{G'} \) counts the connected components that have non-empty intersection with \( G' \) but have empty intersection with \( \nabla(G) \). Therefore, \( \ell_{G'} + \sum_{R \in \mathcal{R}} \ell_R \) counts connected components that have non-empty intersection with \( \nabla(G) \), as we desired.

We now argue that the algorithm uses \( O(S) \) space per machine, that it performs in expectation \( O(1) \) rounds and that in all recursive calls and for all regions \( R \), it holds \( |\nabla(R)| \leq O(S) \).

In each successive level of recursion, we execute the algorithm on a graph of size \( O(|G|^{1-\epsilon}) \). We stop when \( |G| \leq O(S) \). Since \( S = \Omega(n^{2/3+\epsilon}) \) this implies that for any fixed \( \epsilon > 0 \), the depth of recursion is constant. Since \( r = |G|^{1-\epsilon} \), the boundary size per region is \( O(\sqrt{rn \log n/S}) \leq o(n^{1/3}) \). This means that in each level of recursion, the size of \( \nabla(G) \) increases by at most \( o(n^{1/3}) \). Since there are \( O(1) \) levels of recursion, it holds in all recursive calls that \( |\nabla(G)| \leq o(n^{1/3}) \leq O(S) \).

In our algorithm, excluding the call of Algorithm \( 4 \) we perform \( O(1) \) MPC rounds. Algorithm \( 4 \) takes \( O(1) \) rounds both expected and with high probability. There are \( O(1) \) levels of recursions, resulting in round complexity of \( O(1) \) in expectation and with high probability.

It holds \( |H| = O(|\nabla(G)|) \leq O(|G'|) \). To prove that the algorithm uses \( O(S) \) space per machine, it is thus sufficient to prove that \( |G'| \leq O(S) \). It holds \( |G'| \leq |\partial(G)| + \sum_{R \in \mathcal{R}} |\partial(R)| \). As we have already argued, \( |\partial(G)| \leq O(S) \). We have \( |\partial(R)| \leq O(\sqrt{rn \log n/S}) \) by Lemma \( 11 \). The total size of the boundary is thus \( O(\sqrt{rn \log n/S}) \cdot O(n \log 2) \cdot O(\sqrt{\log 2}) \leq O(S) \), as we wanted to show.

### 5.2.2 Bipartition

Given a graph \( G \) and a set \( S \subseteq V(G) \), we let the set of bipartitions \( B_G(S) \) be the set of subsets \( S' \) of \( S \) such that there is a bipartition of \( G \) such that all vertices in \( S' \) are in one part and all vertices \( S \setminus S' \) in the other. Note that if \( G \) is not bipartite, \( B_G(S) = \emptyset \) for all \( S \subseteq V(G) \). We will need the following lemma about \( B_G(S) \). Throughout this section, when we talk of a 2-coloring, we call the colors white and black. Before showing the algorithm, we will need several lemmas.

The following lemma states, intuitively speaking, that we may replace any connected component \( C \) by a simpler graph without changing the set of bipartitions \( B_G(S) \).

**Lemma 14.** Let us have a graph \( G \) and a set \( S \subseteq V(G) \), and consider one connected component \( C \) of \( G \). Let us have a 2-coloring of \( C \). Let \( G' \) be a graph obtained from \( G \) by replacing \( C \) with a star whose ray vertices are \( C \cap S \) and subdividing edges whose ray vertices are black in the 2-coloring. Then \( B_G(S) = B_{G'}(S) \).

**Proof.** First observe that since \( C \) is a connected component of \( G \), \( B_G(S) \) can be expressed in terms of \( B_{G \setminus C}(S \setminus C) \) and \( B_C(S \cap C) \) as

\[
B_G(S) = \{ S_1 \cup S_2 \mid S_1 \in B_{G \setminus C}(S \setminus C) \cap S_2 \in B_C(S \cap C) \}
\]

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Similarly, let \( C' = G' \setminus (G \setminus C) \) be the connected component replacing \( C \) in \( G' \) then
\[
B_{G'}(S) = \{S_1 \cup S_2 \mid S_1 \in B_{G' \setminus C'}(S \setminus C') \land S_2 \in B_{C'}(S \cap C')\}
\]
Now note that \( G' \setminus C' = G \setminus C \) and \( S \setminus C' = S \setminus C \), so trivially \( B_{G' \setminus C'}(S \setminus C') = B_{G \setminus C}(S \setminus C) \).
Also \( S \cap C' = S \cap C \) by construction, so we can name that set \( S' \) and what remains to be proven is \( B_{C'}(S') = B_{C}(S') \). Since \( C \) and \( C' \) are both connected and bipartite, each have a unique 2-coloring (up to naming of colors). Let \( W \subseteq S' \) be the set of vertices that are white in the 2-coloring of \( C \).
Then \( B_{C}(S') = \{W, S' \setminus W\} \), and by construction we also have \( B_{C'}(S') = \{W, S' \setminus W\} \).

**Lemma 15.** Let us have a graph \( G \) and a subset of its vertices \( S \subseteq V(G) \). Let \( G' \) be a graph resulting by replacing \( G[S] \) by a graph \( H \) with \( B_{G[S]}(\partial(S)) = B_{H}(\partial(H)) \). Then for any 2-coloring \( K \) of \( G \), it holds that \( K|_{(G \setminus S) \cup \partial(S)} \) can be extended to a 2-coloring of \( G' \), and for any 2-coloring \( K' \) of \( G' \), it holds that \( K'|_{(G \setminus S) \cup \partial(S)} \) can be extended to a 2-coloring of \( G \).

**Proof.** Observe that \( B_{G[S]}(\partial(S)) \neq \emptyset \) if and only if \( G[S] \) is bipartite, and that \( B_{H}(\partial(H)) \neq \emptyset \) if and only if \( H \) is bipartite. Thus \( B_{G[S]}(\partial(S)) = B_{H}(\partial(H)) \) implies that \( G[S] \) is bipartite if and only if \( H \) is bipartite. If \( G[S] \) and \( H \) are not bipartite, \( G \) and \( G' \) are not bipartite and there is nothing to prove, so suppose they are bipartite. If \( G[S] \) or \( H \) has a connected component that does not intersect \( \partial(S) = \partial(H) \) then this component is colored independently of the rest of the graph so we may assume that each component of \( G[S] \) and \( H \) intersects \( \partial(S) = \partial(H) \).

Then any 2-coloring of \( K \) of \( G \) includes 2-colorings of \( G[S] \) and \( (G \setminus S) \cup \partial(S) \) that agree on \( \partial(S) \). Let \( W \subseteq \partial(S) \) be the white nodes in this coloring. Then \( W \in B_{G[S]}(\partial(S)) = B_{H}(\partial(H)) \), so \( K|_{\partial(S)} \) extends to a 2-coloring \( K'|_{H} \) of \( H \), and then trivially to a 2-coloring \( K' \) of \( G' \) by setting \( K'|_{G \setminus H} = K|_{G \setminus S} \).

Conversely, any 2-coloring \( K' \) of \( G' \) includes 2-colorings of \( H \) and \( (G \setminus S) \cup \partial(S) \) that agree on \( \partial(H) \). Let \( W \subseteq \partial(H) \) be the white nodes in this coloring. Then \( W \in B_{H}(\partial(H)) = B_{G[S]}(\partial(S)) \), so \( K'|_{\partial(H)} \) extends to a 2-coloring \( K'|_{G[S]} \) of \( G[S] \), and then trivially to a 2-coloring \( K \) of \( G \) by setting \( K|_{G \setminus S} = K'|_{G \setminus H} \).

The following lemma states, intuitively speaking, that we may replace an induced subgraph \( G[S] \) by some graph \( H \) which behaves the same as \( G[S] \) with respect to its boundary and some set \( T \), and this does not change how \( G \) behaves with respect to \( T \). This will allow us to construct the graph \( H \) in a recursive fashion.

**Lemma 16.** Let us have a graph \( G \), two sets \( S, T \subseteq V(G) \), and let us have a graph \( H \) with \( B_{H}(\partial(S) \cup (T \cap S)) = B_{G[S]}(\partial(S) \cup (T \cap S)) \). Let \( G' \) be a graph obtained from \( G \) by replacing \( G[S] \) by \( H \). Then \( B_{G'}(T) = B_{G}(T) \).

**Proof.** Let \( X = \partial(S) \cup (T \cap S) \), and observe that by our assumptions, we have \( B_{H}(X) = B_{G[S]}(X) \) and \( X \subseteq V[H] \). In particular, we have \( \partial(H) = \partial(S) \subseteq X \) so \( B_{H}(\partial(H)) = \{Y \cap \partial(H) \mid Y \in B_{H}(X)\} = \{Y \cap \partial(S) \mid Y \in B_{G[S]}(X)\} = B_{G[S]}(\partial(S)) \). Thus by Lemma 15 \( G \) is bipartite if and only if \( G' \) is bipartite. If \( G \) and \( G' \) are not bipartite, \( B_{G}(T) = B_{G'}(T) = \emptyset \) and we are done. Suppose therefore that \( G \) and \( G' \) are bipartite. Then any connected component in \( G \) or \( G' \) that does not intersect \( T \) has no effect on \( B_{G}(T) \) and \( B_{G'}(T) \) so we may assume without loss of generality that every connected component of \( G \) and \( G' \) intersect \( T \). Then \( B_{H}(X) = B_{G[S]}(X) \) implies that the connected components of \( H \) and \( G[S] \) partition \( X \) the same way, and thus the connected components of \( G \) and \( G' \) partition \( T \) the same way. If we can show the result when \( G \) and \( G' \) are connected, then the general case follows easily by induction on the number of connected
components. Suppose therefore that \( G \) and \( G' \) are connected and bipartite, and therefore each have a unique 2-coloring up to the naming of colors.

By Lemma 15 we have \( B_G((G \setminus S) \cup \partial(S)) = B_{G'}((G \setminus S) \cup \partial(S)) \), and for any \( W \in B_G((G \setminus S) \cup \partial(S)) \), we can let \( K \) and \( K' \) be the unique 2-colorings of \( G \) and \( G' \) such that every vertex in \( W \) is white.

Then in particular \( K \) and \( K' \) agree on \( T \cap ((G \setminus S) \cup \partial(S)) \) and on \( \partial(S) \). Since \( G \) and \( G' \) are both connected, every node in \( X \) is connected to a node in \( \partial(S) \), and has its color uniquely determined by the coloring of \( \partial(S) \). Since \( B_H(X) = B_{G[S]}(X) \), \( K \) and \( K' \) must therefore agree on the color for every vertex in \( X \) as well, and in particular they must also agree on \( T \cap X \). Thus \( K \) and \( K' \) agree on \( T \) and \( B_G(T) = B_{G'}(T) \).

**Theorem 17.** Assume that we are given a graph \( G = (V, E) \). For any \( \epsilon > 0 \), there is an algorithm that computes a 2-coloring of \( G \) or determines that no such 2-coloring exists; it performs \( O(1) \) rounds in expectation and with high probability, and uses \( O(S) \) space per machine and \( O(M) \) machines for \( S = n^{2/3+\epsilon} \), \( M = n^{1/3-\epsilon} \).

We again prove a stronger claim, in order to enable us to give a recursive algorithm. We can use it to prove the theorem by setting \( \nabla(G) = \emptyset \) and using the algorithm (2).

**Claim 18.** Let us have a graph \( G \) and a set of marked vertices \( \nabla(G) \) with \( |\nabla(G)| = k \leq O(S) \). Then, for any fixed \( \epsilon > 0 \), there is

\( (1) \) an algorithm that returns a graph \( H \) with \( V(H) \supseteq \nabla(G) \) on \( O(k) \) vertices such that \( B_G(\nabla(G)) = B_H(\nabla(G)) \)

\( (2) \) an algorithm that, given a 2-coloring of \( \nabla(G) \) computes a 2-coloring of \( G \) extending the given 2-coloring of \( \nabla(G) \), or detects that no such 2-coloring exists

and both algorithms perform \( O(1) \) rounds both in expectation and with high probability, and use \( O(S) \) space per machine and \( O(M) \) machines for \( S = n^{2/3+\epsilon} \), \( M = n^{1/3-\epsilon} \).

**Proof.** We first give the algorithm (1) and then use it to get algorithm (2).

**Base case of algorithm (1).** Assume that \( |G| \leq O(S) \). Let us have a connected component \( C \) in \( G \) with non-empty intersection \( S \) with \( \nabla(G) \). Let us have a 2-coloring of \( C \) (we call the colors black and white). We add to \( H \) a star with its ray vertices being the vertices \( S \) and subdivide edges incident to vertices that are black in the 2-coloring. By Lemma 14, the set \( B(\nabla(G)) \) remains unchanged. We repeat this for all connected components and let \( H \) be the resulting graph. We then have \( B_H(\nabla(G)) = B_G(\nabla(G)) \). Each connected component \( C \) is replaced by a graph on \( O(|C \cap \nabla(G)|) \) vertices and the total size of the resulting graph is thus \( O(|\nabla(G)|) \).

**Recursive case of algorithm (1).** We compute an \( r \)-pseudodivision \( R \) of \( G \) for \( r = |G|^{1-\epsilon} \). Let \( \nabla(R) = \partial(R) \cup (\nabla(G) \cap R) \). Recursively, we compute for each region \( R \in R \) a graph \( H_R \supseteq \nabla(R) \) on \( O(|\nabla(R)|) \) vertices, such that \( B_{H_R}(\nabla(R)) = B_{G[R]}(\nabla(R)) \). Note that for this, we need that \( |\nabla(R)| = O(S) \); we discuss this below. Let \( G' = \bigcup_{R \in R} H_R \). The graph \( G' \) can be obtained by replacing, one by one for each region \( R \in R \), the subgraph \( G[R] \) by \( G[H_R] \). By repeatedly applying Lemma 16, we thus have that \( B_{G'}(\nabla(G)) = B_G(\nabla(G)) \).

We compute a 2-coloring of \( G' \) and for each connected component \( C \) of \( G' \), we add to \( H \) a star with its ray vertices being \( C \cap \nabla(G) \) and subdivide edges whose vertex is black in the 2-coloring. It follows by repeatedly applying Lemma 14 that \( B_{G'}(\nabla(G)) = B_H(\nabla(G)) \) and thus \( B_G(\nabla(G)) = B_H(\nabla(G)) \).
Recursive case of algorithm (2). We compute an $r$-pseudodivision $\mathcal{R}$ of $G$ for $r = |G|^{1-\epsilon}$. For each $R \in \mathcal{R}$, we use algorithm (1) with $\nabla(R) = \partial(R) \cup (\nabla(G) \cap R)$ to get $H_R$ with $B_{H_R}(\nabla(R)) = B_{G[R]}(\nabla(R))$. The boundary size per region is $O(\sqrt{rn \log(n)/M} \leq O(S)$ and we thus have $|\nabla(R)| \leq O(S)$, allowing us to use algorithm (1).

We let $H = \bigcup_{R \in \mathcal{R}} H_R$. Again, this graph can be obtained from $G$ by repeatedly replacing $G[R]$ by $H_R$ for all $R \in \mathcal{R}$. By repeated application of Lemma 15, we thus have that the given 2-coloring of $\nabla(G)$ can be extended to a 2-coloring of $G$ if it can be extended to a 2-coloring of $H$, and if it is extendable to some coloring $K$, then $K|_{\partial(R) \cup \nabla(G)}$ is extendable to a 2-coloring of $H$. Specifically, $H$ is 2-colored in a way that extends the given 2-coloring of $\nabla(G)$. We find such 2-coloring $K'$ by the same BFS-based algorithm that we used in the base case of this algorithm. Similarly, we may extend $K'|_{\partial(R) \cup \nabla(G)}$ to a 2-coloring of $G$ and specifically, that we may extend $K'|_{\partial(R) \cup \nabla(G)}$ to a 2-coloring of $G[R]$. We may thus recursively find such 2-coloring of $G[R]$. Since these colorings agree on $H$ and there are no edges between the interiors of regions of $\mathcal{R}$, these 2-colorings together give a 2-coloring of $G$.

Again, by exactly the same argument, this algorithm performs $O(1)$ rounds, it always holds $|\nabla(R)| \leq O(S)$, and the graph $H$ fits onto one machine with memory $S$. \qed

5.2.3 Minimum Spanning Tree

Throughout this section, we assume that the edges are linearly ordered. If we only have a quasi-ordering, we may extend it to a linear order. In the case of a weighted graph, this can be done by replacing the weight $w(e)$ by $(w(e), id(e))$ where $id(e)$ is the unique identifier of $e$, and comparing these ordered pairs lexicographcally. This assumption guarantees uniqueness of the minimum spanning trees. We denote the minimum spanning tree of $G$ by $T(G)$.

For minimum spanning trees, the concept of compressing a graph in a way that retains the structure of the solutions has been formalized in [12]. For sake of completeness, we now show how to compress a graph for MST below.

Definition 19. Let us have a graph $G = (V, E)$ and a set $\nabla(G) \subseteq V$. We say that a graph $H$ with $V(H) \supseteq \nabla(G)$ compresses $G$ with respect to $\nabla(G)$ if for any graph $G' \supseteq G$ with $\partial_{G'}(G) \subseteq \nabla(G)$, then for $T$ being the minimum spanning tree of $G'$ and $T'$ the minimum spanning tree of $(G' \setminus G) \cup H$, it holds $T \cap (G' \setminus G) = T' \cap (G' \setminus G)$.

The following gives an alternative definition that is more useful for some of our proofs.

Lemma 20. Let $G = (V, E)$ be a graph, let $\nabla(G) \subseteq V$, and let $H$ be a graph with $V(H) \supseteq \nabla(G)$. Then the following are equivalent:

For each connected component $C$, we have added to $H$ a graph on $O(|C \cap \nabla(G)|)$ vertices, and we thus have $|H| \leq O(|\nabla(G)|)$. This proves correctness of the output.

It remains to argue that the algorithm uses $O(S)$ space per machine, that it performs $O(1)$ rounds in expectation and with high probability, and that in all recursive calls and for all regions $R$, it holds $|\nabla(R)| \leq O(S)$. The argument is exactly the same as that in the proof of Theorem 12 and we do not repeat it here.

Base case of algorithm (2). We perform a breadth first search starting from $\nabla(G)$ (note that these vertices are already colored) and whenever we see a vertex $v$ for the first time, we give it color opposite to the color of the vertex from which we discovered $v$. We then verify that the resulting 2-coloring is proper. If it is, we return it, otherwise we have detected that no extending 2-coloring exists.

Base case of the algorithm (2): We perform a breadth first search starting from $\nabla(G)$ (note that these vertices are already colored) and whenever we see a vertex $v$ for the first time, we give it color opposite to the color of the vertex from which we discovered $v$. We then verify that the resulting 2-coloring is proper. If it is, we return it, otherwise we have detected that no extending 2-coloring exists.

Recursive case of algorithm (2). We compute an $r$-pseudodivision $\mathcal{R}$ of $G$ for $r = |G|^{1-\epsilon}$. For each $R \in \mathcal{R}$, we use algorithm (1) with $\nabla(R) = \partial(R) \cup (\nabla(G) \cap R)$ to get $H_R$ with $B_{H_R}(\nabla(R)) = B_{G[R]}(\nabla(R))$. The boundary size per region is $O(\sqrt{rn \log(n)/M} \leq O(S)$ and we thus have $|\nabla(R)| \leq O(S)$, allowing us to use algorithm (1).

We let $H = \bigcup_{R \in \mathcal{R}} H_R$. Again, this graph can be obtained from $G$ by repeatedly replacing $G[R]$ by $H_R$ for all $R \in \mathcal{R}$. By repeated application of Lemma 15, we thus have that the given 2-coloring of $\nabla(G)$ can be extended to a 2-coloring of $G$ if it can be extended to a 2-coloring of $H$, and if it is extendable to some coloring $K$, then $K|_{\partial(R) \cup \nabla(G)}$ is extendable to a 2-coloring of $H$. Specifically, $H$ is 2-colored in a way that extends the given 2-coloring of $\nabla(G)$. We find such 2-coloring $K'$ by the same BFS-based algorithm that we used in the base case of this algorithm. Similarly, we get from Lemma 15 that we may extend $K'|_{\partial(R) \cup \nabla(G)}$ to a 2-coloring of $G$ and specifically, that we may extend $K'|_{\partial(R) \cup \nabla(G)}$ to a 2-coloring of $G[R]$. We may thus recursively find such 2-coloring of $G[R]$. Since these colorings agree on $H$ and there are no edges between the interiors of regions of $\mathcal{R}$, these 2-colorings together give a 2-coloring of $G$.

Again, by exactly the same argument, this algorithm performs $O(1)$ rounds, it always holds $|\nabla(R)| \leq O(S)$, and the graph $H$ fits onto one machine with memory $S$. \qed

5.2.3 Minimum Spanning Tree

Throughout this section, we assume that the edges are linearly ordered. If we only have a quasi-ordering, we may extend it to a linear order. In the case of a weighted graph, this can be done by replacing the weight $w(e)$ by $(w(e), id(e))$ where $id(e)$ is the unique identifier of $e$, and comparing these ordered pairs lexicographically. This assumption guarantees uniqueness of the minimum spanning trees. We denote the minimum spanning tree of $G$ by $T(G)$.

For minimum spanning trees, the concept of compressing a graph in a way that retains the structure of the solutions has been formalized in [12]. For sake of completeness, we now show how to compress a graph for MST below.

Definition 19. Let us have a graph $G = (V, E)$ and a set $\nabla(G) \subseteq V$. We say that a graph $H$ with $V(H) \supseteq \nabla(G)$ compresses $G$ with respect to $\nabla(G)$ if for any graph $G' \supseteq G$ with $\partial_{G'}(G) \subseteq \nabla(G)$, then for $T$ being the minimum spanning tree of $G'$ and $T'$ the minimum spanning tree of $(G' \setminus G) \cup H$, it holds $T \cap (G' \setminus G) = T' \cap (G' \setminus G)$.

The following gives an alternative definition that is more useful for some of our proofs.

Lemma 20. Let $G = (V, E)$ be a graph, let $\nabla(G) \subseteq V$, and let $H$ be a graph with $V(H) \supseteq \nabla(G)$. Then the following are equivalent:
(i) $H$ compresses $G$ with respect to $\nabla(G)$.

(ii) For every $u, v \in \nabla(G)$ and weight $w$, $H$ contains a $u \cdots v$ path of maximum edge weight $< w$ if and only $G$ contains such a $u \cdots v$ path.

Proof. (i) $\implies$ (ii) We will prove the contrapositive. So suppose there exists $u, v \in \nabla(G)$ and weight $w$ such that exactly one of $G$ and $H$ contains a $u \cdots v$ path with maximum edge weight $w' < w$. Let $x$ be a new vertex not in either $G$ or $H$, let $ux$ and $xv$ be new edges with weights $w' < w(ux) < w(xv) < w$, and let $G' = G \cup \{ux, xv\}$. Then $xv \in G' \setminus G$ is exactly one of $T(G')$ and $T((G' \setminus G) \cup H)$, and thus $H$ does not compress $G$ with respect to $\nabla(G)$.

(ii) $\implies$ (i) Consider a graph $G' \supset G$ with $\partial G' \subseteq \nabla(G)$, and minimum spanning trees $T = T(G)$ and $T' = T((G' \setminus G) \cup H)$. We want to show that for every edge $e \in G' \setminus G$, $e \notin T$ if and only if $e \notin T'$. Let $uv \in G' \setminus G$. Since $T$ is a minimum spanning tree in $G'$, $uv \notin T$ if and only if there exists a $u \cdots v$ path in $G'$ where every edge has weight $< w(uv)$. Similarly, since $T'$ is a minimum spanning tree in $(G' \setminus G) \cup H$, $uv \notin T'$ if and only if there exists a $u \cdots v$ path in $(G' \setminus G) \cup H$ where every edge has weight $< w(uv)$. By our assumption, such a $u \cdots v$ path exists in $G'$ if and only if it exists in $(G' \setminus G) \cup H$. \square

We will need a few definitions when trying to compress a graph. We say vertices $v_1, \ldots, v_k \in V$ form an isolated path if $v_1v_2 \cdots v_k$ is a path, $d(v_2) = 2, \ldots, d(v_{k-1}) = 2$, and $v_2, \ldots, v_{k-1} \in V \setminus \nabla(G)$. Let $j$ be such that $v_jv_{j+1}$ is the heaviest edge of the isolated path. We say that $G$ results in $G'$ by contracting an isolated path, if $G'$ can be obtained from $G$ by removing $v_1, \ldots, v_k$ and adding an edge $v_1v_k$ whose weight is before $v_jv_{j+1}$ in the order.

We now prove several sufficient conditions under which $G'$ compresses $G$ with respect to a set $\nabla(G)$.

**Lemma 21.** Let us have a graph $G$ and a subset $\nabla(G) \subseteq V(G)$. Then $H$ compresses $G$ with respect to $\nabla(G)$ for:

(i) $H = T(G)$

(ii) $H$ being obtained from $G$ by removing a leaf of $G$ that does not lie in $\nabla(G)$

(iii) $H$ being obtained from $G$ by contracting an isolated path

(iv) there exists $H'$ such that $H$ compresses $H'$ with respect to $\nabla(G)$ and $H'$ compresses $G$ with respect to the same set

(v) for some $S \subseteq V(G)$, $H$ is obtained by replacing $G[S]$ by $H'$ for some $H'$ which compresses $G[S]$ with respect to $\partial(G[S]) \cup (\nabla(G) \cap S)$

Proof. Let $H$ be obtained from $G$ by one of the processes above. In each case, $H$ satisfies the conditions of Lemma 21(ii), so it follows that $H$ compresses $G$ with respect to $\nabla(G)$.

In detail: In case (i), the minimum spanning tree preserves for all $u, v \in V$ the smallest maximum edge weight taken over over all $u \cdots v$ paths, and thus in particular for all $u, v \in \nabla(G)$.

In case (ii), the removed leaf is not on a path between any $u, v \in \nabla(G)$ so removing it changes nothing for the paths we care about.

In case (iii), we explicitly preserve the maximum weight among the replaced edges, again preserving the maximum weight on all paths we care about.

In case (iv), since $H'$ compresses $G$ and $H$ compresses $H'$, for any $u, v \in \nabla(G)$ and weight $w$, some $u \cdots v$ path in $G$ has maximum weight $< w$ if and only if some $u \cdots v$ path in $H'$ has
maximum weight $< w$ if and only if some $u \cdots v$ path in $H$ has maximum weight $< w$. Thus, $H$ compresses $G$ with respect to $\nabla(G)$.

In case (v) we can even prove something slightly stronger, namely that $H$ compresses $G$ with respect to $X = \nabla(G) \cup \partial(G[S]) \supseteq \nabla(G)$. For this, consider $u, v \in X$ and a weight $w$. If some $u \cdots v$ path in $G$ has maximum edge weight $< w$, we can break it into subpaths where each subpath has its endpoints in $X$, and has maximum weight $< w$. Each such subpath is completely contained either in $(G \setminus G[S]) \cup \partial(S)$ (and thus exists unchanged in $H$) or in $G[S]$. Since $H'$ compresses $G[S]$ with respect to $\partial(G[S]) \cup (\nabla(G) \cap S)$, each subpath of $u \cdots v$ contained in $G[S]$ has a replacement in $H'$ whose max weight is $< w$. Thus, $H$ contains an $u \cdots v$ walk (and therefore a $u \cdots v$ path) whose maximum edge weight is $< w$. Conversely, if $H$ contains such a path, a completely symmetric argument gives that $G$ does.

We now show a sequential algorithm that compresses an input graph $G$ with respect to a given set $\nabla(G)$.

**Lemma 22.** There is an algorithm that, given a graph $G$ and a set $\nabla(G) \subseteq V(G)$, computes $H$ that compresses $G$ with respect to $\nabla(G)$ and such that $|H| = O(|\nabla(G)|)$; it uses $O(|G|)$ space.

**Proof.** We compute $T(G)$ using a standard algorithm, repeatedly remove any leaf that is not in $\nabla(G)$ and contract all isolated paths of length at least 2. By Lemma 21 (part (i),(ii),(iii)), the resulting graph compresses $G$ with respect to $\nabla(G)$. The algorithm clearly has linear space complexity. It thus remains to argue that $|H| = O(|\nabla(G)|)$.

Any vertex in $H$ of degree $< 3$ must be in $\nabla(G)$, otherwise it would have been either removed or contracted. Thus, the number of vertices of degree $< 3$ in $H$ is $\leq |\nabla(G)|$. The claim follows as in a tree with $\leq |\nabla(G)|$ nodes of degree $< 3$, the total number of vertices is $O(|\nabla(G)|)$.

We are now ready to prove existence of an algorithm for finding the minimum spanning tree.

**Theorem 23.** Let us have a graph $G$. For any fixed $\epsilon > 0$, there is an algorithm that computes a minimum spanning tree of $G$; it performs $O(1)$ rounds both in expectation and with high probability, and uses $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}$, $M = n^{1/3-\epsilon}$.

It is again more convenient to prove a stronger claim. We can recover the theorem by using part (2) with $\nabla(G) = \emptyset$, and $G'$ being the empty graph.

**Claim 24.** Let us have a graph $G$ and a set of marked vertices $\nabla(G)$ with $|\nabla(G)| = k \leq O(S)$. Then, for any fixed $\epsilon > 0$, there is

1. an algorithm that returns a graph $H$ with $V(H) \supseteq \nabla(G)$ on $O(k)$ vertices that compresses $G$ with respect to $\nabla(G)$

2. an algorithm that, given a graph $G'$ with $G' \cap G \subseteq \nabla(G)$ and $|G'| \leq O(S)$, computes $T(G \cup G') \cap G$

and both algorithms perform $O(1)$ rounds both in expectation and with high probability, and use $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}$, $M = n^{1/3-\epsilon}$.

**Proof.** For algorithm (1), in the case when $|G| \leq O(S)$, we may use Lemma 22 to compress $G$ with respect to $\nabla(G)$.
Recursive case of algorithm (1). We compute an \( r \)-pseudodivision \( \mathcal{R} \) of \( G \) for \( r = |G|^{1-\epsilon} \). Let \( \nabla(R) = \partial(R) \cup (\nabla(G) \cap R) \). We recursively compute \( H_R \) for every \( R \in \mathcal{R} \) that compresses \( G[R] \) with respect to \( \nabla(R) \). For this we need \( |\nabla(R)| \leq O(S) \). Let \( G' = \bigcup_{R \in \mathcal{R}} H_R \). The graph \( G' \) can be obtained from \( G \) by repeatedly replacing a subgraph \( G[R] \) by the graph \( H_R \) for all \( R \in \mathcal{R} \). It follows from Lemma 21, part (v), that \( G' \) compresses \( G \) with respect to \( \nabla(G) \). We then use Lemma 22 to get \( H \) of order \( O(|\nabla(G)|) \) that compresses \( G' \), and thus also \( G \) (by Lemma 21 part (iv)), with respect to \( \nabla(G) \).

It remains to argue that the algorithm uses \( O(S) \) space per machine, that it performs in expectation \( O(1) \) rounds and that in all recursive calls and for all regions \( R \), it holds \( |\nabla(R)| \leq O(S) \). The argument is exactly the same as that in the proof of Theorem 12 and we do not repeat it here.

Base case of algorithm (2). When the graph fits onto one machine, we may compute the minimum spanning tree using a standard sequential algorithm.

Recursive case of algorithm (2). We compute an \( r \)-pseudodivision \( \mathcal{R} \) of \( G \) for \( r = |G|^{1-\epsilon} \). Let \( \nabla(R) = \partial(R) \cup (\nabla(G) \cap R) \). For each region \( R \in \mathcal{R} \), we compute a graph \( H_R \) that compresses \( G[R] \) with respect to \( \nabla(R) \). We let \( H_R = \bigcup_{R' \in \mathcal{R} \setminus \{R\}} H_{R'} \). The graph \( G[R] \cup H_R \cup G' \) can be obtained from \( G \cup G' \) by repeatedly replacing \( G[R] \) by \( H_{R'} \) for all \( R' \in \mathcal{R} \setminus \{R\} \). By the repeated application of Definition 10, we thus have that \( T(G[R] \cup H_R \cup G') \cap (R \cup G') = T(G \cup V(G')) \cap (R \cup V(G')) \).

We distribute all the graphs \( H_R \) for all \( R \in \mathcal{R} \) to \( |\mathcal{R}| \) machines. Note that it holds

\[
O(|\partial(R)|) = r|G| \log |G|/S \cdot |G| \log |G|/r \leq O(n^{2/3+\epsilon/2} \log^{3/2} n) \leq O(S/n^{\epsilon/3})
\]

and we may thus do so in \( O(1) \) rounds by broadcast trees (see Section 2.5). We then for all \( R \in \mathcal{R} \) (on one machine for each such \( R \)) recursively compute \( T(G[R] \cup H_R \cup G') \cap R \) and thus \( T(G \cup G') \cap R \); we may do this as \( |G' \cup H_R| \leq |G'| + |\partial(R)| \leq O(S) \). Taking union over all \( R \in \mathcal{R} \), we have \( T(G) \).

The space per machine is \( O(S) \) since \( |\partial(R)| \leq O(S) \). The expected round complexity is \( O(1) \) by exactly the same argument as in the previous algorithms. \( \square \)

### 5.2.4 Computing spanners and shortest paths with shortcuts

We now show an algorithm which we will use as a subroutine for finding approximate shortest paths, and radius/diameter. This problem is, in some sense, more general than these two problems. The motivation behind this is that we give a recursive algorithm and we need this more general problem to make the recursion work.

In MPC, by computing a path \( P = e_1 \cdots e_k \), we mean that the edges of the path are stored in their order on the machines. That is, edges \( e_{k+1}, \ldots, e_{k+1} \) are stored on machine \( i \), for some \( 0 = \ell_1 < \ell_2 < \cdots < \ell_k = k \) for some value of \( k' \).

**Lemma 25.** Assume that we are given a graph \( G = (V, E) \) together with a set \( \nabla(G) \subseteq V \) of marked vertices such that \( |\nabla(G)| \leq O(n^{2/3}) \). Let us have some fixed \( \epsilon > 0 \). There is an algorithm that

- computes a graph \( A = (\nabla(G), E') \) such that \(|E'| \leq O(n^{3+\epsilon/2}) \) and for any \( s, t \in \nabla(G) \), it holds \( d_G(s, t) \leq d_A(s, t) \leq O(1) d_G(s, t) \) (that is, \( A \) is a \( \Omega(1) \)-spanner of \( D_G[M] \)).

- Given \( s_1, t_1, \cdots, s_k, t_k \in \nabla(G) \), the algorithm computes \( k' \leq k \) disjoint paths \( P_1, \cdots, P_{k'} \) where \( P_j \) is an \( s_{i_j} \cdots t_{i_j} \)-path where \( i_0 = 0, i_{k'} = k \), and \( i_j < i_{j+1} \) such that \( \sum_{i=1}^{k'} w(P_i) \leq O(1) \sum_{i=1}^{k} d_G(s_i, t_i) \).
It performs $O(1)$ rounds both in expectation and with high probability, and uses $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}$, $M = n^{1/3-\epsilon}$.

Proof. We give a recursive algorithm. It has two parts, each solving one part of the lemma statement.

Base case. If the graph fits onto one machine, we compute a $(2k-1)$-spanner $A_G$ of $G$ for $k = [1/\epsilon]$. Such graph can be computed by a standard algorithm (see e.g. [3]) in space $O(|\nabla(G)|^{1+1/k}) \leq O(n^{1+\epsilon/2})$ and this is also the space complexity of the spanner.

For each $i \in [k]$, we find a shortest $s_it_i$-path. Consider the subgraph $Q(G)$ of $G$ created by the union of these paths. For each connected component $K$ in $Q(G)$, we define $i_{\text{max},K} = \max\{i \mid t_i \in K\}$. Let $i_0 = 0$. We set the path $P_j$ for some $j$ as follows. Consider the connected component $K$ containing $s_{i_{j-1}+1}$. We let $P_j$ be the shortest $s_{i_{j-1}+1}t_{i_{\text{max},K}}$-path and set $i_j = i_{\text{max},K}$. We end when $i_j = k$.

It clearly holds $i_0 = 0$. The connected component containing $s_{i_{j-1}+1}$ also contains $t_{i_{j-1}+1}$ and thus $i_j \geq i_{j-1} + 1$. The paths are disjoint as each is a subset of a different connected component of $Q(G)$. Disjointness implies the first inequality of

$$\sum_{i=1}^{k'} w(P_i) \leq w(Q(G)) \leq \sum_{i=1}^{k} d_G(s_i, t_i)$$

where the second inequality holds because $Q(G)$ is the union of shortest $s_it_i$-paths.

Recursive algorithm. We now show the recursive case. It works in two phases, one for each part of the claim. In each successive level of recursion, we execute the algorithm on a graph of size $O(|G|^{1-\epsilon})$. We stop when $|G| \leq n^{2/3+\epsilon}$. This means that for any fixed $\epsilon$, the depth of recursion is constant. Throughout the proof, we consider $\epsilon$ to be a constant and hide multiplicative $f(\epsilon)$ in the $O$-notation.

First phase. We use Algorithm 4 with $r = |G|^{1-\epsilon}$. By Lemma 11, we get an $r$-pseudodivision $\mathcal{R}$ and a memory layout such that each region is stored on consecutive machines and no machine stores edges from multiple regions. This will allow us to run our algorithm recursively, separately on each region, executing it on the group of machines storing the region. For each region $H$, we let the set of marked vertices be $\nabla(H) = (\nabla(G) \cap V(H)) \cup \partial(H)$. It holds that $|\partial(H)| \leq O(\sqrt{r}|G|\log |G|/S) \leq O(n^{2/3})$. This holds on all levels of recursion and the initial set of marked vertices also has size $O(n^{2/3})$.

The total number of marked vertices of $H$ is thus also $O(n^{2/3})$ as the number of levels of recursion is $O(1)$. For each $H \in \mathcal{R}$, we recursively compute a graph $A_H$ on the set of marked vertices $\nabla(H)$ such that for any $u, v \in \nabla(H)$, we have $d_H(u, v) \leq d_{A_H} \leq O(1)d_H(u, v)$.

Consider the graph $A = \bigcup_{H \in \mathcal{R}} A_H$ on the vertex set $V(A) = \bigcup_{H \in \mathcal{R}} \nabla(H) = \partial(\mathcal{R}) \cup \nabla(G)$. The size of this graph is $O(|G|\log |G|) \cdot r^{(1+\epsilon)/2} \leq O(n^{2+\epsilon})$ and it thus fits onto one machine. We now argue that for any $s, t \in V(A)$, it holds $d_G(s, t) \leq d_A(s, t) \leq O(1)d_G(s, t)$. Consider a shortest $st$-path in $A$ and denote it by $P_A = v_1v_2\ldots v_h$ for some integer $h$. For any $i$, we have that $e_i = v_i,v_{i+1} \in A_H$ for some $H \in \mathcal{R}$. Since $A_H$ is a spanner of $D_H[M(H)]$, it holds that $d_H(v_i,v_{i+1}) \leq w_A(e_i)$. We then have by the triangle inequality that

$$d_G(s, t) \leq \sum_{i=1}^{h} d_H(v_i, v_{i+1}) \leq \sum_{i=1}^{h} w_A(e_i) = d(P_A) = d_A(s, t)$$

\footnote{With the way we use the cairm, it will actually have size 2, namely it will be equal to \{s, t\}. We do not, however, need this.}
We now prove the other inequality. Consider a shortest $st$-path $P_G$ in $G$. For $h = |P \cap V(A)|$ and $i \in [h]$, let $v_i$ be the $i$-th vertex of $P_G$ that lies in $V(A)$. (Note the different definition of the $v_i$’s in this case from the definition in the above argument for the other inequality). Note that $s,t$ are marked and they therefore lie in $V(A)$. For any $i$, it holds that $v_i, v_{i+1} \in H$ for some $H \in \mathcal{R}$. It holds $d_{A_H}(v_i, v_{i+1}) \leq O(1)d_H(v_i, v_{i+1})$ as $A_H$ is guaranteed to be a $O(1)$-spanner of $D_H[M(H)]$ and it holds $d_{D_H(M(H))}(v_i, v_{i+1}) = d_H(v_i, v_{i+1})$. By the triangle inequality, we then have that

$$d_{A_H}(s,t) \leq \sum_{i=1}^k d_{A_H}(v_i, v_{i+1}) \leq O(1) \sum_{i=1}^k d_H(v_i, v_{i+1}) = O(1)d(P_G) = O(1)d_G(s,t)$$

This concludes proof of the first part.

**Second phase.** For each $i \in [k]$, we compute a shortest $s_it_i$-path $P_i$ in $A$ (note that $s_i, t_i \in \nabla(G)$ and thus also $s_i, t_i \in A$). Consider a path $P_i$ and let $e$ be the $j$-th edge of path $P_i$. We define $r_i(e) = (i,j)$ and $r_i(e) = 1$ if $e \notin P_i$. In what follows, we will be comparing these values; we use the lexicographic order for the tuples and we let $\perp$ to be the smallest element. Let $G_P$ be the union of these paths. For each edge $e \in E(G_P)$, let $r_{\max}(e) = \max_{i \in [k]} r_i(e)$. Let $i$ be the largest such that $e \in P_i$, we define $t_{\max}(e)$ to be the endpoint of $e$ that comes later in $P_i$. Consider some subgraph $H \in \mathcal{R}$. We let $p_0 = (0,0)$. We set values $p_e, s_{H,e}, t_{H,e}$ as follows. Consider the smallest pair $(i,j)$ such that $(i,j) \geq p_{\ell-1}$ and such that the $j$-th edge of $P_i$ lies in $E(A_H)$ (we stop if there is no such tuple). Let $u$ be the endpoint of $e = uv$ that comes first in $P_i$. If $u \neq t_{\max}(e)$, we set $s_{H,e} = u$, $t_{H,e} = v$, $p_{\ell} = r_{\max}(e)$.

Once we have set the values $s_{H,i}, t_{H,i}$ like this, we recursively compute a $O(1)$-approximately shortest $s_{H,i}t_{H,i}$-path in $H$. We let $P_{H,1}, \cdots, P_{H,k_H}$ be the returned paths, where $k_H$ is the number of the paths returned by the recursive call. We let $A'_H$ be the graph on $\nabla(H)$ where for each of these $k_H$ paths, we add an edge between the endpoints of the path with its weight being the length of that path.

Let $Q(G) = \bigcup_{H \in \mathcal{R}} A'_H$. We set the values $i_j$ as follows. We let $i_0 = 0$. We consider the connected component of $Q(G)$ that contains $s_1, t_1$ and we let $\ell$ be the greatest such that $t_{\ell}$ lies in this component. We set $i_\ell = \ell$. We let $P_{j,\ell}'$ be the shortest $s_{i_{j-1}+1}t_{i_j}$-path in $Q(G)$. We terminate when $i_j = k$. Note that if this is not the case, we may continue this process. Let $k'$ be the number of such paths that we find.

We now consider one path $P_{j,\ell}'$ (recall that this is a path in the graph $Q(G)$) and we show how to compute a $s_jt_{j+1}$-path $P_{j,\ell}'$ in $G$ such that $d_G(P_{j,\ell}') \leq O(1)d_Q(G)(P_{j,\ell}')$. We will then do this separately for each path $P_{j,\ell}'$. Consider $e_i = uv$, the $i$-th edge in $P_{j,\ell}'$. Since this edge is in $A'_H$ for some $H \in \mathcal{R}$, it has to be the case (by the definition of the graph $A'_H$) that we have computed $uv$-path $P_{H,\ell}$ for some value $\ell$ in one of the recursive calls. Let us define $\tilde{P}_{j,\ell} = P_{H,\ell}$. We let $P_{j,\ell}' = \bigcup_{i=1}^{\ell} P_{j,\ell}$. We will show that these paths satisfy the conditions from the statement. First, we show how to rearrange the edges in memory so that each path is stored by consecutive machines in order in which the edges appear in the path.

We compute the prefix sums of the lengths of the paths $P_{j,1}, \cdots, P_{j,|P_{j}|}$ as well as the prefix sums of the lengths of the paths $P_{1}', \cdots, P_{k}'$. We call these prefix sums $S_{j,1}, \cdots, S_{j,|P_{j}|}$ and $S_{1}', \cdots, S_{k}'$. Suppose that the $i$-th edge $e = uv$ of $P_j$ for some $i,j$ lies in $H \in \mathcal{R}$ and $u$ comes before $v$ in the order of $P_j$. We send to the machines storing $H$ the value $|(i', j'), s.t., (i', j') \leq (i,j)| = S_{i-1} + S_{j,i-1}$. The position of an edge $e$, denoted $p(e)$, in the desired order in which the edges are to be stored is equal to this value plus the number of edges in $P_{H,\ell}$ that come before $e$. We then store $e$ on position $p(e) \mod S$ of machine $[p(e)/S]$. This ensures that the paths are stored on the machines in the above-mentioned order.
Theorem 26. Assume that we are given a graph $G$. We now show an algorithm for finding an approximately shortest path between a pair of vertices.

We will now argue that $\sum_{i=1}^{k'} w(P_i) \leq O(1) \sum_{i=1}^{k} d_G(s_i, t_i)$. The paths $P_i$ are disjoint paths in the graph $Q(G)$ (they are disjoint as there is only one path in each connected component). It thus holds

$$\sum_{i=1}^{k'} w(P_i) \leq \sum_{e \in Q(G)} w(e)$$

Let $v_{j,i}$ be the $i$-th vertex of $P_j$. By the inductive hypothesis, we have $w(P_{j,i}) \leq O(1)d_G(v_{j,i}, v_{j,i+1})$. Therefore, we have

$$w(P_j) = \sum_{i=0}^{|P_j|} w(P_{j,i}) \leq O(1) \sum_{i=0}^{|P_j|} d_G(v_{j,i}, v_{j,i+1}) \leq O(1) \sum_{i=0}^{|P_j|} d_A(v_{j,i}, v_{j,i+1}) = O(1)d_A(s_i, t_i) = O(1)w(P_j)$$

Finally, each edge of $Q(G)$ is in at least one path $P_i$. It thus holds

$$\sum_{e \in Q(G)} w(e) \leq \sum_{i=1}^{k} w(P_i) = \sum_{i=1}^{k} d_G(s_i, t_i)$$

Combining these three inequalities, we get

$$\sum_{i=1}^{k'} w(P_i) \leq O(1) \sum_{i=1}^{k'} w(P_i) \leq O(1) \sum_{i=1}^{k'} w(e) \leq O(1) \sum_{i=1}^{k'} d_G(s_i, t_i)$$

as we wanted to show. \(\square\)

### 5.2.5 Shortest st-path problem

We now show an algorithm for finding an approximately shortest path between a pair of vertices.

**Theorem 26.** Assume that we are given a graph $G = (V, E)$ and $s, t \in V$. For every fixed $\epsilon > 0$, there is an algorithm that computes an $O(1)$-approximate st-path; it performs $O(1)$ rounds both in expectation and with high probability, and uses $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}, M = n^{1/3-\epsilon}$.

**Proof.** This is a special case of the shortest paths with shortcuts problem from Lemma 25. Specifically, we let the set $M$ of marked vertices be $\{s, t\}$. We then set $s_1 = s$ and $t_1 = t$. We then have by Lemma 25 the desired algorithm. \(\square\)

### 5.2.6 Diameter/Radius

Since we give $O(1)$-factor approximation, the two problems are equivalent. We focus on giving approximation algorithm to the diameter problem. This problem is somewhat easier than the shortest path problem in that the answer is one number.
Theorem 27. Assume that we are given a graph $G = (V, E)$. For every fixed $\epsilon > 0$, there is an algorithm that computes an $O(1)$-approximation of the diameter and radius of $G$; it performs $O(1)$ rounds both in expectation and with high probability, and uses $O(S)$ space per machine and $O(M)$ machines for $S = n^{2/3+\epsilon}$, $M = n^{1/3-\epsilon}$.

Proof. If the graph fits onto one machine, we compute the diameter exactly. We now deal with the case when this the graph does not fit onto one machine. We use Algorithm 4 with $r = |G|^{1-\epsilon}$. By Lemma 24 we get an $r$-pseudodivision $R$ and a memory layout such that each region is stored on consecutive machines and no machine stores edges from multiple regions. We recursively compute the approximate diameters of all regions. Since the depth of recursion is $O(1)$ (as each time, we recurse on graphs of size $\leq |G|^{1-\epsilon}$ and stop when $|G| \leq O(n^{2/3+\epsilon})$), this does not increase the asymptotic round complexity of the algorithm. We use Lemma 25 to get a $O(1)$-spanner $A$ of $D_G[\partial(R)]$. We find the diameter of $A$. Let $L$ be the maximum of this diameter and the approximate diameters of the regions of $R$. We return $3L$.

We now argue correctness. We first prove that $\text{diam}(G) \leq 3L$. The argument is by induction. When the whole graphs fits into memory, the claim is true as we compute $\text{diam}(G)$ exactly; this proves the base case. Consider $u, v \in V(G)$ such that $d(u, v)$ is maximal and consider a shortest $uv$-path $P$. We divide $P$ into three parts – the first between $u$ and the first vertex of $P$ that lies in $V(A)$, second between the first and last vertex of $P$ that lie in $V(A)$, and third between the last vertex of $P$ that lies in $V(A)$ and $v$. Let $P'$ be the longest of these three parts. It holds $d(P') \geq d(P)/3$. If $P'$ is the middle part of $P$, then we have $L \geq d_A(P') \geq d_G(P)/3$ and thus $3L \geq d_G(P) = \text{diam}(G)$. If $P'$ is either the first or the third part, it is the case that $P'$ is contained in one region $H \in R$. Then the recursive call on $H$ returns value at least $d_H(G)$ by the inductive hypothesis. It then holds $L \geq d_H(G) \geq d(P)/3$ and thus $3L \geq d_G(P) = \text{diam}(G)$.

It remains to prove that $3L \leq O(1)\text{diam}(G)$. The argument is again by induction. The base case is trivial as in the case where the whole graph fits into memory, we compute the diameter exactly. The diameters of the regions of $R$ are no greater than the diameter of $G$. Thus, the recursive calls return answer $\leq O(1)\text{diam}(G)$. Since $A$ is a spanner of $D_G[\partial(R)]$, it holds for any $u, v \in \partial(R)$ that $d_A(u, v) \leq O(1)d_G(u, v)$ and thus $\text{diam}(A) \leq O(1)\text{diam}(G)$. It follows that $L \leq O(1)\text{diam}(G)$, as we wanted to prove.

6 Beyond straight-line planar embeddings

So far, we have only considered planar graphs with planar straight-line embeddings. In this section, we show that both the assumption on having a straight-line embedding, and the assumption of planarity can be relaxed. We show this separately, but it is also possible to combine this, resulting in algorithms for non-planar embeddings with general curves (satisfying some technical assumptions).

6.1 Beyond straight-line embeddings

We now show how the assumption of having a straight-line embedding can be weakened significantly. While we focus in this section on “nice curves”, the same results can also be obtained for piecewise linear embeddings. We discuss this briefly at the end of this section.

Intuitively speaking, the more “cuved” the curves that embed the edges are, the more time and queries we will have to have to perform in order to get an $r$-pseudodivision of the same quality. A natural measure of “how curved a curve is” is the total absolute curvature. For a curve $C$, we denote this $\text{A}(C)$. This value can be interpreted as the “total angle by which the curve turns”.


Specifically, subdividing a curve at its inflection points (under mild technical assumption), \( A(C) \) is equal to the sum over all the parts of the angles by which the curve turned in the respective parts.

We show that a simple modification of our algorithm gives an algorithm that is parameterized by \( n + A(G) \) instead by \( n \), for \( A(G) = \sum_{e \in E(G)} A(e) \), under the mild technical assumption that the curve can be subdivided into countably many curves each without inflection points. Throughout this section, we assume that a neighbor query on \( v \) gives, apart from the neighbor \( u \), also the curve that embeds the edge \( uv \). We assume that for an edge \( e \) and an angle \( \theta \in [0, 2\pi) \), we can find all points \( x \) on the curve embedding \( e \) such that the tangent to \( e \) at \( x \) has angle \( \theta \) with the \( x \)-axis.

We now give a few definitions. We use \( \kappa(x) \) to denote the curvature at \( x \in C \). If the parameterization \( \gamma \) is clear from the context. Let \( \alpha \) be the length at unit speed by \( \alpha(y) = \frac{\operatorname{arc} \gamma(y)}{\operatorname{arc} \gamma(0)} \) at all points of continuity of \( \alpha(\gamma(y)) \).

**Fact 28.** Let us have a doubly differentiable curve \( C \in C^2 \) and let \( \gamma : [0, L(C)] \to \mathbb{R}^2 \) be its parameterization that has unit speed. It holds \( \kappa(y) = \frac{\delta \alpha(\gamma(y))}{\delta y} \) at all points of continuity of \( \alpha(\gamma(y)) \).

We now prove a lemma that we will use in order to analyse of our modified algorithm.

**Lemma 29.** Let us have \( \theta \sim [0, 2\pi) \). Assume we are given a doubly differentiable curve \( C \) parameterized by its length at unit speed by \( \gamma : [0, L(C)] \to \mathbb{R}^2 \) that can be covered up to a set of measure zero by a countable set \( I \) of open intervals such that no interval contains an inflection point. Mark any point \( y \in [0, L(C)] \) such that \( \alpha(\gamma(y)) = \theta \) and for any \( \epsilon > 0 \), there exists \( 0 < \delta < \epsilon \) with \( \alpha(\gamma(y - \delta)) \neq \theta \). Let \( \ell \) be the number of marked points. Then \( E[\ell] = A(C)/(2\pi) \).

**Proof.** We may assume that \( A(C) < \infty \), as otherwise the claim is \( E[\ell] \leq \infty \), which trivially holds. For a point \( x \in C \) and \( y_x = \gamma^{-1}(x) \), we let \( \beta(x) = \alpha(\gamma(0)) + \int_{y_x} y \kappa(y) dy \). It holds \( \alpha(x) = \beta(x) \mod 2\pi \) by Fact 28, second fundamental theorem of calculus and the fact that all points of discontinuity of \( \alpha(\gamma(y)) \) are jump discontinuities where the function value jumps between 0 and 2\( \pi \). We subdivide the intervals by the smallest possible number of numbers, so as to make sure that for any interval \( I = (a, b) \), it holds that \( |\beta(\gamma(a)) - \beta(\gamma(b))| < 2\pi \). We denote the set of these subverted intervals by \( I' \).

We subdivide each \( I \in I \) into \( \left\lfloor \frac{|\beta(\gamma(a)) - \beta(\gamma(b))|}{2\pi} \right\rfloor + 1 \) intervals. At the same time, the interval \( I \) contributes at least \( \left\lfloor \frac{|\beta(\gamma(0)) - \beta(\gamma(y))|}{2\pi} \right\rfloor \) to the total absolute curvature. Since we are assuming that the total absolute curvature is finite, this made only finitely many subdivisions. \( I' \) is thus still countable. Let us have an arbitrary bijection \( f : \mathbb{N} \to I' \). Let \( I_i = f(i) \) and let \( C_i = \gamma(I_i) \) and let \( a \) and \( b \) be the endpoints of \( I_i \); that is \( I_i = (a_i, b_i) \). Let \( \ell_i \) be the number of points \( y \in I_i \) such that \( \alpha(y) = \theta \). Since it holds \( \ell_i \geq 0 \) for all \( i \), we have by the monotone convergence theorem that \( E[\ell] = \sum_{i=1}^{\infty} E[\ell_i] \).

On any interval \( I_i \), the function \( \beta \) is differentiable and (non-strictly) monotonous. Therefore, \( \gamma^{-1}(\beta^{-1}(\theta)) \cap I_i \) is always either empty or an interval \( I \). If this set is empty, we do not mark any point. Otherwise, only the value \( \inf(I) \) may be marked. This specifically means that we mark at most one number in \( I_i \). This proves the first equality in

\[
E[\ell_i] = P(\theta \in \alpha(C_i)) = |\beta(\gamma(a)) - \beta(\gamma(b))|/2\pi = \frac{1}{2\pi} \left| \int_{y=a_i}^{b_i} \kappa(y) dy \right| = \frac{1}{2\pi} \int_{y=a_i}^{b_i} |\kappa(y)| dy
\]

\[16\text{In fact, it contributes exactly this much, but we will not need this.}\]
The second equality holds because $\alpha(C_i) = \beta(C_i) \mod 2\pi$, the third equality holds by the definition of $\beta$ and basic properties of integrals, and the last equality holds because $\alpha(\gamma(\cdot))$ is on $I_i$ monotonous and the $\kappa(y) = \alpha'(\gamma(y))$ thus has the same sign on the whole interval. We thus have

$$E[\ell] = \sum_{i=1}^{\infty} E[\ell_i] = \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{y=a_i}^{b_i} |\kappa(y)| \, dy = \frac{1}{2\pi} \int_{0}^{L(C)} |\kappa(y)| \, dy = A(C)/(2\pi)$$

We are now ready to describe our modified algorithm. The only part of our paper where we have used the straight-line assumption is the algorithm for finding a good cutting. In the rest of the paper, we have been using this as a black box. In fact, more can be said about how we used this assumption. We have used that for any subset of edges, each trapezoid in the trapezoidal decomposition has boundary consisting of (parts of) two edges and two of the vertical line segments that are added (where one of them may be degenerate and have length zero). For this, it is sufficient that all the edges are $x$-monotonous. Since we are assuming that we may for any edge $e$ find all points on $e$ that have angle $\pi/2$ with the $x$-axis (i.e. points where the curve is vertical), we may find all such points and subdivide an edge at these points. Even if we had access to this modified graph, there may be many such points and the graph resulting from subdividing those edge may be arbitrarily large.

For this reason, we pick $\theta \sim [0, 2\pi)$ and rotate the whole input by this angle. A curve is perpendicular to the $x$-axis at a point $x$ iff the angle at that point in the input is $-\theta + \pi/2$ or $-\theta + 3\pi/2$. Since these two angles are uniform on $[0, 2\pi)$, the expected number of such points on one edge is $O(A(e))$ by Lemma 29.

Let us have a parameter $s$. We now give an algorithm that returns a cutting (for this parameter $s$) with probability at least $1 - O(\delta)$, with expected number of regions $O((n + A(G)) \log(n)/s)$. It has query complexity $O(\frac{n \log n}{s})$.

**Algorithm 5:** Find a $s/n$-cutting of a non-straight-line graph

1. $\theta \sim \text{Unif}([0, 2\pi])$
2. Rotate the whole input by angle $\theta$ around the origin; perform the rotation lazily, rotating each vertex or edge when accessed in the rest of the algorithm
3. $E' \leftarrow$ sample $6n(2 \log n + \log \delta^{-1})/s$ edges
4. $E'' = \emptyset$
5. for $e \in E'$ do
6. $\gamma \leftarrow$ parameterization by length at unit speed of the curve that embeds $e$
7. $D \leftarrow \emptyset$
8. For any $y \in [0, L(e)]$ such that $\alpha(\gamma(y))$ is $\pi/2$ or $3\pi/2$, and for any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ with $\alpha(\gamma(y - \delta)) \neq \pi/2, 3\pi/2$, add $x$ to $D$
9. Subdivide the curve embedding $e$ at all points in $D$
10. Add the resulting curves to $E''$
11. $V' \leftarrow$ sample $n(2 \log n + \log \delta^{-1})/s$ vertices
12. Compute the trapezoidal map of $E'' \cup V'$ using the algorithm from [27], let $D$ be a data structure for point location for this trapezoidal map
13. return $D$

The proof of the following claim is very similar to that of Lemma 8. We therefore just sketch the proof, focusing on the parts where the proof differs.
Lemma 30. Algorithm returns a cutting with probability at least \(1 - O(\delta)\) with expected number of sectors \(O((n + A(G))\log(n)/s)\), represented by a data structure \(D\) that for a given vertex answers which region it lies in. It has query complexity \(O\left(\frac{n(\log n + \log \delta^{-1})}{s}\right)\).

Proof. The query complexity is clearly as claimed. Let \(P\) be the trapezoid map of \(E_n \cup V\). We prove that \(P\) has the claimed properties with probability \(1 - O(\delta)\).

We start by arguing that the number of sectors is in expectation \(O((n + A(G))\log(n)/s)\). The expected total absolute curvature of the edges in \(E'\) is \(E[A(E')] = |E'|A(G)/n\). Conditioned on \(E(E')\), we have \(E[[E]]|A(E') = |E'| + A(E')\) by Lemma 29. By the law of total expectation, we have \(E[[E]] = |E'| + A(E') = O((n + A(G))\log(n)/s)\). A trapezoid decomposition has linear number of edges in the size of the input (see e.g. [7] Lemma 6.2). The complexity of \(P\) is thus as claimed. The rest of the proof is exactly the same as that of Lemma 3.

This allows us to prove what is essentialy a non-straight-line version of Theorem 9. In this part of the paper, we focus only on the query complexity. The proof is analogous to that of Theorem 9, so we do not show it in full detail.

Theorem 31. There is an algorithm that preprocesses the input using \(O(n\log(n)/s)\) queries and produces an oracle \(D\). Given a vertex, \(D\) answers region queries in a way that is consistent with an \(r\)-pseudodivision with per-region boundary size \(O(\sqrt{sr})\) and \(O((n + A(G))\log(n)/r)\) regions.

Proof sketch. We use Lemma 30 to find a cutting with probability \(1 - O(\delta)\) and expected number of regions \(O((n + A(G))\log(n)/s)\). We construct the sector graph \(G_s\) of this partition. We find an \(r'\)-pseudodivision of \(G_s\) for \(r' = r/s\). Given a vertex \(v\), we report that it belongs to the region of the vertex corresponding to the sector that \(v\) lies in. The size of any region is then at most \(r' \cdot s = r\). The size of boundary of any region is \(O(\sqrt{r' \cdot s}) = O(\sqrt{sr})\) by the definition of a cutting. The expected number of regions is \(O\left(\frac{(n + A(G))\log(n)/s}{r'}\right) = O((n + A(G))\log(n)/r)\)

Note that if, on average, \(A(e) = O(1)\) then the resulting bound is the same as what we get when we have a straight-line embedding (in terms of query complexity). We believe that this would be very often the case in practice and that this new algorithm thus removes to a large extent the limitations of our algorithm for straight-line embeddings.

We may use Theorem 31 in both the sublinear-time parameter estimation, and in MPC. By changing parameters of the respective algorithms, we get the following two theorems:

Theorem 32. Let \(\Pi\) be an additive \(O(1)\)-Lipschitz property. There is an algorithm that returns, with probability at least \(2/3\), an additive \(\pm\epsilon n\) approximation of \(\Pi(G)\). It has query complexity \(O\left(\frac{(n + A(G))\log^{3/2} n}{\sqrt{m} \epsilon^{2/3}}\right)\).

Theorem 33. Assume that we are given a graph \(G = (V, E)\). For every fixed \(\epsilon > 0\), there is an algorithm that (1) counts the number of connected components, (2) finds a bipartition of \(G\) or report that \(G\) is not bipartite, (3) compute the minimum spanning tree of \(G\); compute an \(O(1)\)-approximation of (4) a shortest \(s, t\)-path for given \(s, t\), (5) the diameter and radius of \(G\); it performs \(O(1)\) rounds both in expectation and with high probability, and uses \(S = O((n + A(G))^{2/3 + \epsilon})\) space per machine and \(O((A(G) + n)^{1/3 - \epsilon})\) machines, assuming each curve is stored in \(O(1)\) space.

Remark. The above approach of dividing the curve at some points can be used to work with piecewise-linear edges. Specifically, we may subdivide the edges at the endpoints of the line segments. Each part of an edge then will be \(x\)-monotonic (as it will be a line segment). \(A(G)\) in the above bounds is then replaced by the total number of line segments that embed the edges.
6.2 Beyond planar embeddings

Assuming that we have a \( s/n \)-cutting with \( k \) sectors, then the approach used in Section 3 leads without any modifications to an \( r \)-pseudodivision with the same boundary sizes but \( k\frac{s}{r} \) regions instead of \( O(n \log(n)/r) \). It thus remains to show how to find a \( s/n \)-cutting in the non-planar case that does not have too many sectors. This was again solved in [8], and we show a modification of their algorithm that deals separately with the vertices.

For a not necessarily planar embedding of a graph, its planarization\(^{17}\) is the graph obtained by adding a vertex \( w \) for each crossing and replacing each edge \( uv \) that passes through a crossing replaced by \( w \), by two edges, \( uw \) and \( wv \). We modify Algorithm 1 to enable it to work with non-planar graphs.

Algorithm 6: Find a good cutting of the plane

1. \( E' \leftarrow \) sample \( 3n(2 \log n + \log \delta^{-1})/s \) edges
2. \( E'' \leftarrow \) planarization of \( E' \)
3. \( V' \leftarrow \) sample \( n(2 \log n + \log \delta^{-1})/s \) vertices
4. Compute the trapezoidal map of \( E'' \cup V' \) using the algorithm from [27], let \( \mathcal{D} \) be the returned data structure for point location for this trapezoidal map.
5. return \( \mathcal{D} \)

By exactly the same argument as in the proof of Theorem 9, we get that the returned trapezoidal map is . It remains to bound the expected number of sectors. Size of the trapezoidal map is linear in the size of \( E'' \cup V' \). It holds \( |V'| = O(n \log(n)/s) \). It holds \( |E''| = |E'| + cr(E') \). Each of the \( cr(G) \) crossings in \( E(G) \) is also a crossing in \( E' \) with probability

\[
O\left(\left(\frac{n \log(n)/s}{m}\right)^2\right) = O((\log(n)/s)^2)
\]

and we thus have that the expected complexity of the trapezoid decomposition is \( O(n \log(n)/s + cr(G) \log^2(n)/s^2) \). We get the following claim:

**Theorem 34.** There is an algorithm that preprocesses the input using \( O(n \log(n)/s) \) queries and produces an oracle \((\mathcal{R}, \mathcal{D})\). Then there exists an algorithm that, given a vertex \( v \in G \) returns in expected time \( O(\log n/s) \) and without making any queries the set of regions \( v \) belongs to. These sets form, with probability at least \( 1 - O(\delta) \), a hybrid \( r \)-pseudodivision \( \mathcal{R}' \) with boundary size \( O(\sqrt{sr}) \) per region. The number of regions is \( O(n \log(n)/r + cr(G) \log^2(n)/(sr)) \). The set implements in expected \( O(1) \) time operations: membership, cardinality, iterate through elements (taking \( O(1) \) time per element).

7 Open Problems

(1.) **Property testing with respect to hamming distance.** In this paper, we have shown results for property testing with respect to the normalized vertex edit distance. Is it possible to get similar property testing results with respect to the more common normalized hamming distance? Czumaj and Sohler [9] have given a characterization of testable properties in planar graphs (without having an embedding). Proving a separation between the settings with and without the embedding would be very interesting.

\(^{17}\)This is different from the notion used in the graph drawing literature where it refers to the problem of removing the smallest possible number of edges to make an input graph planar; see e.g. [31]
(2.) Random rotation trick. Our result on non-straight-line embeddings relies on Lemma 29. What other problems could this be used for? We also believe the assumption of the curve being divisible into countably many curves without inflection points can be removed. Is this so? It also seems plausible that one could find a generalization of this result to non-differentiable curves. This result would then imply both our result and the result on piecewise-linear edges.

(3.) Minimum (st-)cut MPC algorithm. Is it possible to compute an (approximately) minimum cut in $O(1)$ MPC rounds and $n^{1-\Omega(1)}$ space per machine? It would be natural to use planar graph duality. However, it is far from clear how to do this since we would need to get a good cutting for the dual of the input graph.

(4.) Streaming algorithms for embedded planar graphs. Can some of our techniques be used to give efficient streaming algorithms for embedded planar graphs? Our sublinear-time algorithm can be simulated in $O(1)$ passes. We can then uniformly sample regions from the resulting $r$-pseudodivision. This allows us to approximate Lipschitz additive parameters in the streaming setting. Can one improve upon this simple approach? It would be especially interesting if one could prove a separation between the setting with and without the embedding. The lower bound from 2 should be useful in this.

(5.) MPC algorithms with $n^{2/3-\Omega(1)}$ space per machine. This seems especially feasible for problems where we recurse on subproblems on planar graphs: counting connected components, bipartition, and minimum spanning tree. This problem may however be rather hard. In that case, is it at least possible to get such algorithm for the euclidean minimum spanning tree problem?

(6.) $(1 + \epsilon)$-approximation of shortest paths in MPC. There exist efficient $1 + \epsilon$ approximate distance oracles for planar graphs [24]. Can some of the techniques be used to give better approximation for shortest paths in MPC with $O(n^{1-\Omega(1)})$ space per machine? Are exact shortest paths possible?

Acknowledgement

The authors are grateful Krzysztof Nowicki whose comments have helped improve our MPC algorithms and who pointed out implications of our work to Euclidean MST. The authors are also grateful to Mikkel Thorup and Tomáš Gavenčiak for helpful discussions.

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