Possibilities of using probabilistic models in theoretical physics

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Abstract. The article presents a historical overview of the path to understanding the probabilistic nature of the laws governing the behavior of microobjects and the creation of quantum mechanics. Its postulates are given and directions in the mathematization of quantum theory are described. The fundamental differences of the quantum probabilistic model from the classical one are presented in an overview.

1. Introduction
Theoretical physics does not operate with the direct structures of the material world, but explores their theoretical models, which adequately and rationally reflect the properties of real objects and their relationships. Its theories are a form of reflection of the laws of functioning, development of processes of objective reality. The physical states of systems are described by a set of parameters that are presented in theory as some quantities or their distribution. Descriptions of the laws of functioning and development of microsystems, which are characterized by both wave and particle properties, demonstrate a fundamental difference in the probabilistic nature of these descriptions.

The descriptions of the states of objects of quantum systems are based on the principle of uncertainty. The essence of the Heisenberg’s uncertainty principle is that trying to determine one of the conjugate quantities, the value of another quantity cannot be determined with the same accuracy. So, for example, it is impossible to simultaneously accurately determine the location of a particle and its momentum. Therefore, when studying the states of physical microsystems, the methods and concepts of probability theory are applied. But at the same time, a need arises for a more definite consideration from the mathematical point of view of the structure of quantum theory and the establishment of correspondence between formal elements and physical reality.

2. On the creation and mathematization of quantum theory
The problem of the mathematical presentation of the axioms of physics was formulated by D. Hilbert in 1900 in his report at the II International Mathematical Congress in Paris. In particular, D. Hilbert’s speech touched on problems “on the axiomatic construction of those physical disciplines in which mathematics now plays an outstanding role: it is primarily a theory of probability and mechanics”. Along with the logical justification of probability theory, D. Hilbert noted the need for the development of the mean value method in mathematical physics, namely, the kinetic theory of gases, thereby touching on one of the most profound problems of mathematical physics, the study of which subsequently led to the appearance of mathematical methods of statistical mechanics and the theory of dynamical systems. This problem is known as the sixth Hilbert problem.
The search for the axiomatic foundation of probability theory ended in 1933, when A.N. Kolmogorov built a system of axioms, representing a set of formally simple and intuitively clear provisions on which the entire mathematical structure of the theory is based.

The first formal, independent and consistent quantum theories were recognized as matrix mechanics, developed in 1925 by W. Heisenberg, M. Born and P. Jordan, and wave mechanics, developed in 1926 by E. Schrödinger, which arose in the course of unique guesses and the successful selection of mathematical objects ready to express a peculiar combination of discreteness in continuity in describing the functioning of quantum objects.

The fundamental work in the field of mathematics of quantum mechanics was the monograph by J. von Neumann “Mathematical Foundations of Quantum Mechanics” (1932) [1] J. von Neumann showed that the states of quantum systems can be considered as elements of a Hilbert space. As a result, characteristic physical quantities, for example, momenta, can be represented as linear operators over a Hilbert space, and then the uncertainty principle is expressed in the non-commutativity of the operators corresponding to physical quantities. In this case, operators play the same fundamental role as functions in classical mechanics. But, if in classical physical theories the mathematical apparatus of functions is used on the basis of intuitive assumptions that seem so obvious that classical mechanics seems to form a single whole with mathematics underlying it, then in quantum theory the situation exhibits a different character. Thus, the study of problems of quantum mechanics was reduced to the study of algebras of linear Hermitian operators over a Hilbert space. The ideas of J. von Neumann served as the starting point for a number of attempts to construct a system of simple, physically interpreted postulates, from which the formalism of Hilbert space would logically follow.

Currently, there are three main directions in the mathematization of the axiomatics of quantum theory. The most fruitful approach in mathematics is the algebraic one, which takes as the basis the "algebra of observables" of the physical system [2, 3]. The physical applications of this approach relate mainly to the structural issues of the theory of systems with an infinite number of degrees of freedom - quantum fields and media. The starting element of the quantum-logical approach is the lattice of utterances - two-digit observables taking the values 0 and 1 [4-6]. An important result in this direction was the creation of a system of axioms characterizing the lattice of projectors of Hilbert space in a quantum mechanical system. The introduction of an algebraic structure in these approaches required certain assumptions that did not have direct physical motivation. In 1963, in his monograph Lectures on the Mathematical Foundations of Quantum Mechanics, the American mathematician J. Mackey set forth the concepts of quantum mechanics from a purely mathematical point of view, starting from some of the primary properties of the statistical description of a physical system [26]. The author used the concepts of abstract algebra, set-theoretic topology, measure theory, and self-adjoint operators in Hilbert space. The work of J. Mackey was not completed, but had a strong influence on subsequent scientific research in this direction. So in the 70s of the twentieth century, a third approach stood out in the presentation of the foundations of quantum mechanics, called operational or “convex”. With this approach, the concept of “state” plays a decisive or equivalent role with respect to the concept of “observable” or “measurement” [7-10]. One of the main elements of the proposed mathematical scheme is a convex set of states of a physical system. This approach is also called statistical because it represents the logical development of a statistical interpretation of quantum mechanics.

3. Axioms of quantum theory
The quantum theory is based on seven axioms:

1) Each quantum system is assigned a Hilbert space over the field of complex numbers, that is, a linear space with a scalar product that has the completeness property with respect to this scalar product.

The state of a quantum system is the unit vector of a Hilbert space, the observed one is a self-adjoint operator, which in the final case corresponds to a Hermitian matrix (invariant with respect to
transposition and complex conjugation). The average value is calculated by the formula
\[ \langle A \rangle = \langle \psi, A \psi \rangle. \]

2) As a result of measuring the observable, one of the values belonging to the spectrum of the corresponding operator can be obtained.

3) The evolution of a quantum system in time: if \( \psi_0 \) is the initial state, \( \psi_t \) is the final state, then they are connected by a unitary transformation:

\[ \psi_t = U_t \psi_0. \] (1)

These transformations form a unitary group: if we shift by time \( t \), and then by time \( s \), then \( U_{t+s} = U_s U_t \).

By Stone's theorem, a continuous unitary group can be represented as:

\[ U_t = e^{-\frac{i}{\hbar} \int_0^t H dt}, \] (2)

where \( H \) is a self-adjoint operator, \( U(q) \) is the potential energy operator.

If we substitute the expression for operator (2) into formula (1) and differentiate, we obtain the Schrödinger equations (stationary - formula (3), non-stationary - formula (4)):

\[ H \psi = E \psi, \] (3)

\[ i\hbar \frac{d\psi_t}{dt} = H \psi_t. \] (4)

4) The existence of three-dimensional physical space (for example, in a Hilbert space, a unitary representation of the translation group and the rotation group of the three-dimensional space should be realized).

5) If we have two quantum systems, one of which corresponds to the Hilbert space \( \mathcal{H}_1 \), and the second to \( \mathcal{H}_2 \), then the composite system corresponds to their tensor product

\[ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \]

6) Identical particles in quantum mechanics must obey Bose and Fermi statistics. For example, the wave function of a joint system must be either symmetric with respect to the permutation of the arguments (Bose statistics) or antisymmetric (Fermi statistics).

7) The possibility of the existence of internal symmetries in quantum systems. In particular, the fundamental group of symmetries for the standard model of elementary particles:

\[ SU(3) \times SU(2) \times U(1). \]

4. Fundamental differences between quantum and classical probabilities

A scientific experiment conventionally consists of two main stages. The first stage is the establishment of initial conditions, the determination of the input data of the experiment. At the second stage, measurements are made when, in a certain way, the object under study or a system of objects interacts with the measuring device. The result in each individual experiment is a specific outcome - the output of the experiment. It is necessary to ensure the possibility of unlimited repetition of the measurement under given conditions, that is, conditions are created for a sequence of identical and independent repetitions of the experiment. The results of a scientific experiment are determined by both stages. This dependence has a statistical character, since the results are subject to random scatter, the amplitude of which varies depending on the properties of the studied object and the characteristics of the experiment.

Classical physical theory is based on the assumption of the fundamental compatibility of any measurement procedures. This assumption is justified by the fact that classical mechanics studies macroscopic objects whose physical interaction with measuring instruments is negligible.

In quantum mechanics, the incompatibility of measurements is revealed, which is due to the existence of two different ways of organizing a macroscopic spatio-temporal medium, which can be mutually exclusive. “In quantum physics, data on atomic objects obtained using various experimental facilities are in a kind of additional relation to each other” ([11], p. 170).
These circumstances arising in the study of the objects of the macrocosm and the microworld are described in the language of probability theory and mathematical statistics by means of two statistical models: classical and quantum. The statistical model and its main elements, such as states and measurements, are mathematical objects. The probability in both classical and quantum physics is a number distributed between the numbers 0 and 1, but the difference is in how these numbers are parameterized. In this case, there is a certain parallelism between the two types of parameterization.

A classical statistical model is a model in which the simplex of probability distributions in the phase space serves as a set of states, and the measurement class contains all kinds of deterministic measurements [7]. A statistical model of quantum mechanics is a model in which states are described by various density matrices, and measurements are described by various affine mappings of density matrices in probability distributions on the space of measurement results [7].

The quantum statistical model has two fundamental differences from the classical statistical model.

4.1. The matrix representation of the structural objects of the classical statistical model (probabilistic probability distribution, random variable, mathematical expectation) allows you to move on to the quantum probabilistic description. For a finite quantum system, which can be considered as a non-commutative analogue of a finite probability system, the state is described by a density matrix consisting of complex elements. The density matrix is a positive Hermitian matrix, and its trace is equal to unity. The set of such matrices is a convex subset of the real linear space of all $n \times n$ Hermitian matrices. They describe pure states. Quantum observables (Hermitian operators), which are described by commuting matrices, are called joint. Quantum observables that are described by non-commuting matrices are called complementary. Complementarity, expressed by the non-commutativity of the corresponding observable algebra, is the first fundamental difference between a quantum probabilistic description and a classical one.

4.2. In the classical world, there are two types of correlations: when one event is the cause of another, or when both events have a common cause. In quantum theory, a third type of correlation arises, associated with the nonlocal properties of coupled (entangled) states of several particles.

In 1935, A. Einstein, B. Ya. Podolsky, N. Rosen drew attention to the unusual properties of composite quantum systems [12]. Subsequently, based on this article, already in the 60s, the Swiss theoretical physicist D.S. Bell formulated and proved Bell's inequalities (Bell's theorem), which laid the theoretical basis for experimental studies of the EPR paradox (paradox A of Einstein, B.Ya. Podolsky and N. Rosen), which consists in measuring the parameters of a microobject in an indirect way, without directly affecting this object. D.S. Bell noted that one can quantitatively characterize the entangled states of a composite quantum system so that one can detect the difference between the classical and quantum cases.

Correlation inequalities:

Classic (Bell). Let $X_j$, $Y_k$ $(j, k = 1,2)$ be random variables on one probability space $\Omega$, such that $|X_j| \leq 1$, $|Y_k| \leq 1$. Then for any probability distribution $P$ on $\Omega$

\[
|E_pX_1Y_1 + E_pX_1Y_2 + E_pX_2Y_1 - E_pX_2Y_2| \leq 2.
\]

Quantum (Zirelson). Let $X_j$, $Y_k$ $(j, k = 1,2)$ be quantum observables, such that $|X_j| \leq 1$, $|Y_k| \leq 1$, $X_jY_k = Y_kX_j$. Then for any quantum state $S$

\[
|E_SX_1Y_1 + E_SX_1Y_2 + E_SX_2Y_1 - E_SX_2Y_2| \leq 2\sqrt{2}.
\]

The second fundamental difference is entanglement - inseparability:

Consider a composite system of two q-bits $AB$. Equality in the quantum case is achieved for the observables $X_j = \sigma(a_j) \otimes I_B$, $Y_k = I_A \otimes \sigma(b_k)$ and the Bell state $= |\psi\rangle AB$: $= 1/\sqrt{2} (|\uparrow\rangle A \otimes |\uparrow\rangle B + |\downarrow\rangle A \otimes |\downarrow\rangle B) = 1/\sqrt{2} (|\uparrow\rangle + |\downarrow\rangle).$

Bell's inequalities provided a theoretical basis for conducting physical experiments. The first successful experiments to test Bell's inequalities were carried out by Clauser and Friedman in 1972. Clauser's group experiments were followed by Aspe's experiments in 1981. In his classical
experiment, two fluxes of photons with zero total spin emitted from source S were directed to Nicolas A and B prisms. In them, due to birefringence, polarization of each of the photons was divided into elementary ones, after which the beams were directed to the D + and D– detectors. The signals from the detectors through the photomultipliers entered the recording device R, where Bell's inequality was calculated.

The death blow to locality was inflicted in 1989 by the multiply connected Greenberger – Horn – Zeilinger states, which laid the basis for quantum teleportation — the transmission of a quantum state over a distance using a concatenated (entangled) pair disconnected in space and a classical communication channel in which the state is destroyed at the point of departure at taking measurements and recreated at the point of reception.

Using the concatenated state of q-bits formed the basis of super-dense coding - a method that allows you to transfer two bits of classical information using only one q-bit.

The development of this theory led subsequently to the emergence of a quantum computer and quantum algorithms, which are classical algorithms that specify a sequence of unitary operations (gates) indicating which q-bits they need to be performed on. A quantum algorithm is specified either in the form of a verbal description of such commands, or by means of their graphical notation in the form of a gate system. The result of the quantum algorithm is probabilistic. Due to a small increase in the number of operations in the algorithm, you can arbitrarily bring the probability of obtaining the correct result to unity.

5. Conclusions

Concluding the review of the applicability of probabilistic models in studies of theoretical physics, it should be noted that the classical model, derived from classical mechanics, a priori is not the only possible and most adequate for probabilistic modeling of quantum objects. Undoubtedly, the adoption of a particular model should take into account the data of various physical experiments.

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References
[1] J von Neuman 1996 Mathematical Foundations of Quantum Mechanics, (Beyer, R. T., trans., Princeton Univ. Press.)
[2] Mathematical Problems of Relativistic Physics by Irving E. Segal with an appendix on Group Representations in Hilbert Space by George W. Mackey (Lectures in Applied Mathematics Series, 2), (Providence, R.I. American Mathematical Society, 1963.140 p.) (Irving E. Segal. Matematicheskie problemy relyativistskoy fiziki [Mathematical problems of relativistic physics]. Moscow, Mir Publ.,1968. 190 p.(in Russian))
[3] Emeh, Gerard G. 1972 Algebraic methods in statistical and quantum field theory (New York a. o.) 423
[4] J M Jauch 1968 Foundations of Quantum Mechanics (Addison-Wesley, Reading, MA).
[5] V S Varadarajan 1968 Geometry of Quantum Theory (Van Nostrand, New York).
[6] C Piron 1976 Foundations of Quantum Physics (W. A. Benjamin Inc., London).
[7] A S Kholevo 1976 Issledovaniya po obschey teorii statisticheskikh resheniy (Moscow, Nauka Publ.,) 124 (in Russ.)
[8] E B Davies 1976 Quantum Theory of Open Systems (Academic Press, London).
[9] A Hartk¨amper and H. Neumann (eds.), 1973 Foundation of quantum mechanics and ordered linear spaces (Advanced Study Institute, Marburg) 29.
[10] G Ludwig 1983 Foundations of Quantum Mechanics (Springer -Verlag, Berlin).
[11] L V Tarasov 2017 Osnovy kvantovoy mekhaniki (Moscow, URSS Publ. (in Russ.)) 278
[12] A Einstein, B Podolsky, N Rosen 1935 American Phys. Soc. 47 (10) 777 DOI: 10.1103/PHYSREV.47.777
[13] G W Mackey 1963 *Mathematical foundations of quantum mechanics* (W A Benjamin, New York) x-137 DOI:10.1016/0029-5582(64)90052-5
[14] A S Kholevo 2020 *Veroyatnostnye i statisticheskie aspekty kvantovoy teorii* (Moscow, MCCME Publ.) 364 (in Russ.)
[15] B V Gnedenko 1969 *Kurs teorii veroyatnostey* (Moscow, Nauka Publ.) 445 (in Russ.)
[16] A N Shiryaev 2004 *Veroyatnost’* (Moscow, MCCME Publ.) 552 (in Russ.)
[17] A N Kolmogorov 1998 *Osnovnye ponyatiya teorii veroyatnostey* (Moscow, Fazis Publ.) 228 (in Russ.)
[18] V K Nevolin 2013 *Kvantovaya fizika i nanotekhnologii*. (Moscow, Teknosfera Publ.) 128 (in Russ.)