A Framework for Similarity Search with Space-Time Tradeoffs using Locality-Sensitive Filtering

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Abstract

We present a framework based on Locality-Sensitive Filtering (LSF) and apply it to show lower bounds and give improved upper bounds for the space-time tradeoff of solutions to the $(r, cr)$-near neighbor problem in high-dimensional spaces. Locality-sensitive filtering was introduced by Becker et al. (SODA 2016) together with a framework yielding a single, balanced, space-time tradeoff that further relies on the assumption of an efficient oracle for the filter evaluation algorithm. We extend their framework to support the full range of space-time tradeoffs, and through a combination of “powering” and “tensoring” techniques, we are able to remove the oracle assumption.

The LSF framework in this paper can be viewed as a generalization of the Indyk-Motwani (STOC 1998) framework for Locality-Sensitive Hashing (LSH) that represents the dominant paradigm for high-dimensional similarity search in both theory and practice. Given a family of filters, defined as a distribution over pairs of subsets of space that satisfies certain locality-sensitivity properties, we can construct a dynamic data structure that solves the $(r, cr)$-near neighbor problem in $d$-dimensional space with expected query time $dn^\rho_q + o(1)$, expected amortized update time $dn^\rho_u + o(1)$, and expected space usage $dn^{1+\rho_u+o(1)}$ where $n$ denotes the number of points in the data structure. The space-time tradeoff is tied to the tradeoff between query time and update time (insertions/deletions), controlled by the parameters $\rho_q, \rho_u$ that are determined by the filter family.

Laarhoven (arXiv 2015), building on Becker et al., introduced a family of filters for the high-dimensional unit sphere that supports the full range of space-time tradeoffs and analyzed it for the important special case of random data. We show that a small modification to the family of filters gives a simpler analysis that we use, together with our framework, to provide guarantees for worst-case data. Through an application of Bochner’s Theorem from harmonic analysis by Rahimi & Recht (NIPS 2007), we are able to extend our solution on the unit sphere to $\mathbb{R}^d$ under the class of similarity measures corresponding to real-valued characteristic functions. For the characteristic functions of $s$-stable distributions we obtain a solution to the $(r, cr)$-near neighbor problem in $\ell^d_s$-spaces with query and update exponents $\rho_q = \frac{c^s(1+\lambda)^2}{(c^s+\lambda)^2}$ and $\rho_u = \frac{c^s(1-\lambda)^2}{(c^s+\lambda)^2}$ where $\lambda \in [-1, 1]$ is a tradeoff parameter. This improves or matches all data-independent LSH-based solutions, an active line of research dating back almost 20 years, and matches the LSH lower bound by O’Donnell et al. (ITCS 2011), and a similar LSF lower bound proposed in this paper. Finally, we show a lower bound for the space-time tradeoff on the unit sphere that matches Laarhoven’s and our own upper bound in the case of random data.
1 Introduction

In this paper we introduce a Locality-Sensitive Filtering (LSF) framework and apply it to give improved upper and lower bounds for similarity search problems in high-dimensional spaces. Let \((X, D)\) denote a space over a set \(X\) equipped with a symmetric dissimilarity measure \(D\), or a distance function in the case of metric spaces. We consider the \((r, cr)\)-near neighbor problem first introduced by Minsky and Papert in the 1960’s [35, p. 222]. A solution to the \((r, cr)\)-near neighbor problem for a set \(P\) of \(n\) points in \((X, D)\) takes the form of a data structure that supports the following query: given a query point \(x \in X\), if there exists a point \(y \in P\) such that \(D(x, y) \leq r\) then report a point \(y' \in P\) such that \(D(x, y') \leq cr\). In some spaces it turns out to be convenient to work with a measure of similarity rather than dissimilarity. We use \(S\) to denote a symmetric measure of similarity and define the \((\alpha, \beta)\)-similarity problem to be the \((-\alpha, -\beta)\)-near neighbor problem in \((X, -S)\).

A solution to the \((r, cr)\)-near neighbor problem can be viewed as a fundamental building block that yields solutions to many other similarity search problems [23] such as the \(x\)-approximate nearest neighbor problem [19]. The \((r, cr)\)-near neighbor problem is particularly well-studied in \(\ell^d_s\)-spaces where \(d\) is the dimensionality, and distances are measured by distance functions such as \(\ell^d_p\), \(\ell^d_\infty\), and \(\ell^d_1\). With the \(\ell^d_1\)-distance function and distances measured by the \(\ell^d_1\), Hamming space \((\{0, 1\}^d, \ell^d_1)\), and the \(d\)-dimensional unit sphere \(S^d = \{x \in \mathbb{R}^d \mid \|x\|_2 = 1\}\) or inner product similarity \(S(x, y) = \langle x, y \rangle = \sum_{i=1}^{d} x_i y_i\).

In \(\ell^d_s\)-spaces all known solutions to the exact near neighbor problem use either space or time that is exponential in \(d\) or linear in \(n\) [19]. This phenomenon is known as the “curse of dimensionality” and has been observed both in theory and practice. For example, Alman and Williams [1] recently showed that the existence of an algorithm for determining whether a set of \(n\) points in \(d\)-dimensional Hamming space contains a pair of points that are exact near neighbors with a running time strongly subquadratic in \(n\) would refute the Strong Exponential Time Hypothesis (SETH) [51]. This result holds even when \(d\) is rather small, \(d = O(\log n)\). On the technical side, Weber et al. [50] showed that the performance of most of the tree-based approaches to similarity search from the field of computational geometry [10] degrades rapidly to a linear scan as the dimensionality increases.

If we allow an approximation factor of \(c > 1\) then there exist solutions to the \((r, cr)\)-near neighbor problem in \(\ell^d_s\)-spaces with query time that is sublinear in \(n\), where both the space and time complexity of the solution depends only polynomially on \(d\). Techniques for overcoming the curse of dimensionality through approximation were discovered independently by Kushilevitz et al. [28] and Indyk and Motwani [22]. This paper can be viewed as an extension of the latter, classical work by Indyk and Motwani [22] that introduces a general framework for solving the \((r, cr)\)-near neighbor problem using families of functions that partition space, known as Locality-Sensitive Hashing (LSH).

**Problem and model** In this paper we consider randomized data structures that solve \((r, cr)\)-near neighbor problem in \((X, D)\). We will provide guarantees on the construction time, query time, update time (insertions/deletions), and space usage of the data structures. In order to obtain a fully dynamic data structure we apply a powerful dynamization result of Overmars and Leeuwen [39] for decomposable searching problems. Their result allows us to turn a partially dynamic data structure into a fully dynamic data structure, supporting arbitrary sequences of queries and updates, at the cost of an only a constant factor in the space and running time guarantees. Suppose we have a partially dynamic data structure that solves the \((r, cr)\)-near neighbor problem on a set of \(n\) points. By partially dynamic we mean that, after initialization on a set \(P\) of \(n\) points, the data structure supports \(\Theta(n)\) updates without changing the query time by more than a constant factor. Let \(T_q(n), T_u(n),\) and \(T_c(n)\) denote the query time, update time, and construction time of such a data structure containing \(n\) points. Then, by Theorem 1 of
Overmars and Leeuwen [39], there exists a fully dynamic version of the data structure with query time $O(T_q(n))$ and update time $O(T_u(n) + T_c(n)/n)$ that uses only a constant factor additional space. The data structures presented in this paper, as well as most related constructions from the literature, have the property that $T_c(n)/n = O(T_u(n))$, allowing us to go from a partially dynamic to a fully dynamic data structure “for free” in big O notation.

In terms of guaranteeing that the query operation solves the $(r, cr)$-near neighbor problem on the set of points $P$ currently inserted into the data structure, we allow a constant failure probability $\delta < 1$, typically around $1/2$, and omit it from our statements. We make the standard assumption that the adversary does not have knowledge of the randomness used by the data structure. Say we have a data structure with constant failure probability and a bound on the expected space usage. Then, for every positive integer $T$ we can create a collection of $O(\log T)$ independent repetitions of the data structure such that for every sequence of $T$ operations it holds with high probability in $T$ that the space usage will never exceed the expectation by more than a constant factor and no query will fail.

For the model of computation we use the standard word RAM model as defined by Hagerup [18] with a word size of $\Omega(\log n)$ bits — enough to store a reference to a point $x \in P$ in $O(1)$ words. We will make the assumption that a point in $(X, D)$ can be stored in $d$ words and that the dissimilarity between two arbitrary points can be computed in $O(d)$ operations where $d$ is a positive integer that corresponds to the dimension in the various well-studied settings mentioned above. The LSH framework takes as input a distribution $\mathcal{F}$ over functions that partition space (Definition 1) while the LSF framework takes as input a distribution $\mathcal{H}$ over pairs of subsets of space (Definition 2). When describing framework-based solutions to the $(r, cr)$-near neighbor problem we will make the assumption that we can sample, evaluate, and represent elements from $\mathcal{F}$ and $\mathcal{H}$ with negligible error using space and time $O(d)$. If we augment the standard word RAM model with an instruction that generates random words then this assumption holds for a number of well-known concrete LSH and LSF families [22, 13, 15, 17, 8], as well as for the Gaussian LSF family analyzed in this paper.

**Locality-Sensitive Hashing** We proceed by giving a description and stating the main results of the LSH framework. The LSH framework is closely related to the LSF framework by Becker et al. [8] as well as the LSF framework with space-time tradeoffs introduced in this paper. The LSH framework takes a distribution $\mathcal{H}$ over hash functions that partition space with the property that the probability of two points landing in the same partition is an increasing function of their similarity.

**Definition 1.** Let $(X, D)$ be a space defined as a set $X$ equipped with dissimilarity measure $D$ and let $\mathcal{H}$ be a probability distribution over functions $h: X \rightarrow R$. We say that $\mathcal{H}$ is a $(r, cr, p, q)$-sensitive hash family for $(X, D)$ if for all pairs of points $x, y \in X$ it satisfies the following conditions:

- If $D(x, y) \leq r$ then $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \geq p$
- If $D(x, y) \leq cr$ then $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq q$
- $p > q$.

The properties of $\mathcal{H}$ determines a parameter $\rho < 1$ that governs the space and time complexity of the solution to the $(r, cr)$-near neighbor problem.

**Theorem 1** ([22] [19]). Suppose we have access to a $(r, cr, p, q)$-sensitive hash family. Then we can construct a fully dynamic data structure that solves the $(r, cr)$-near neighbor problem. Assume that $1/p$ and $1/\log(1/q)$ are $n^{o(1)}$, then the data structure has

- query time $dn^{\rho+o(1)}$, 
- expected update time $n^{\rho+o(1)} + O(d)$,
- space usage $n^{1+\rho+o(1)} + dn$

where

$$\rho = \frac{\log(1/p)}{\log(1/q)} \quad (1)$$

Locality-sensitive hashing has been extremely influential in both theory and practice and has emerged as the state of the art for similarity search in high dimensions. The survey by Wang et al. [49] provides an overview of this line of work and contains references to more than 100 related papers.

Table 1: Overview of data-independent locality sensitive hashing (LSH) and filtering (LSF) results

| Reference         | Setting                        | $\rho_q$                           | $\rho_u$                           |
|-------------------|--------------------------------|------------------------------------|------------------------------------|
| LSH, Theorem 1    | $(X, D)$ or $(X, S)$           | $\frac{\log(p_0/p_1)}{\log(p_0/p_2)}$ | $\frac{\log(p_0/p_1)}{\log(p_0/p_2)}$ |
| LSF, Theorem 2    | $(X, D)$ or $(X, S)$           | $\frac{\log(p_0/p_1)}{\log(p_0/p_2)}$ | $\frac{\log(p_0/p_1)}{\log(p_0/p_2)}$ |
| Bit-sampling LSH  | $(r, cr)$-mn. in $\ell_s^d$    | $1/c$                              |                                    |
| Ball-carving LSH  | $(r, cr)$-mn. in $\ell_s^d$    | $1/e^2$                            |                                    |
| s-stable LSH      | $(r, cr)$-mn. in $\ell_s^d$    | $(1+\varepsilon)\max(1/e^s, 1/c)$ |                                    |
| Ball-search LSH*  | $(r, cr)$-mn. in $\ell_s^d$    | $c^2/(c+\lambda)^2\cdot \frac{(1+\alpha+\lambda)^2}{(1+\alpha)^2}$ | $c^2/(c+\lambda)^2\cdot \frac{(1+\lambda)^2}{(1+\alpha+\lambda)^2}$ |
| Theorem 5         | $(r, cr)$-mn. in $\ell_s^d$    | $c^2/(c+\lambda)^2\cdot \frac{(1+\alpha)^2}{(c+\lambda)^2}$ | $c^2/(c+\lambda)^2\cdot \frac{(1+\lambda)^2}{(c+\lambda)^2}$ |
| Cross-polytope LSH| $(\alpha, \beta)$-sim. in $(S^d, \langle \cdot, \cdot \rangle)$ | $\frac{1-\alpha}{1+\beta}$        | $\frac{1-\alpha}{1+\beta}$        |
| Spherical cap LSF| $(\alpha, 0)$-sim. in $(S^d, \langle \cdot, \cdot \rangle)$ | $\frac{(1-n+\lambda)^2}{1-\alpha+\lambda^2}$ | $\frac{(n+\lambda)^2}{1-\alpha^2}$ |
| Theorem 3         | $(\alpha, \beta)$-sim. in $(S^d, \langle \cdot, \cdot \rangle)$ | $\frac{(1-n+\lambda)^2}{1-\alpha+\lambda^2}$ | $\frac{(n+\lambda)^2}{1-\alpha^2}$ |
| Theorem 4         | $(\alpha, \beta)$-sim. in $(S^d, \langle \cdot, \cdot \rangle)$ | $\frac{(1-n+\lambda)^2}{1-\alpha+\lambda^2}$ | $\frac{(n+\lambda)^2}{1-\alpha^2}$ |
| LSH lower bound   | $\ell_s^d$                      | $\geq 1/e^s$                       |                                    |
| Theorem 7         | Symmetric LSF in $\ell_s^d$    | $\geq 1/e^s$                       |                                    |
| LSH lower bound   | $(S^d, \langle \cdot, \cdot \rangle)$ | $\geq \frac{1}{1-\alpha}$        |                                    |
| Theorem 8         | Regular LSF in $(S^d, \langle \cdot, \cdot \rangle)$ | $\geq \frac{(1-n+\lambda)^2}{1-\alpha+\lambda^2}$ | $\geq \frac{(n+\lambda)^2}{1-\alpha^2}$ |

Table notes: Space-time tradeoffs for dynamic randomized solutions to similarity search problems in the LSH and LSF framework with query time $dn^{\rho_u+o(1)}$, expected amortized update time $dn^{\rho_u+o(1)}$ and space usage $dn + n^{\rho_u+o(1)}$. Lower bounds are for the exponents $\rho_q, \rho_u$ within their respective frameworks. Here $\varepsilon > 0$ denotes an arbitrary constant, $\lambda \in [-1, 1]$ controls the space-time tradeoff, and $k$ denotes a real-valued characteristic function. Upper bounds in $\ell_s^d$-spaces hold for $0 < s \leq 2$ while lower bounds hold for $0 < s < \infty$. We have hidden $o(1)$ terms in the upper bounds and $O(1)$ terms in the lower bounds. Obs: Bound of Thm. 2 matches [22, 19] when $p_0 = p_u = 1$. Obs: Bound of Thm. 3 matches [29] when $\beta = 0$.

*Assumes $c^2 \geq (1+\lambda)^2/2 + \lambda + \varepsilon$. 


Results and previous work  This paper presents a general framework for a technique known as Locality-Sensitive Filtering (LSF) of which special instances have previously appeared in the literature [17, 8, 29]. Whereas the LSH framework offers a single space-time tradeoff that balances update and query complexity, the LSF framework supports the full range of space-time tradeoffs, without compromising the simplicity of the framework or the underlying data structure. Table 1 summarizes our results and compares them to the literature.

The idea behind locality-sensitive filtering is the following approach to constructing a data structure for similarity search: Assume we have access to a distribution over subsets of space, referred to as filters, that has the property that the probability of a pair of points being contained in a random filter is increasing in the similarity of the points. To construct the data structure we sample a collection of filters and for each filter we store the data points that are contained in it. To query the data structure we inspect the data points associated with the filters containing the query point.

The work by Becker et al. [8] introduced a framework for locality-sensitive filtering with a balanced space-time tradeoff, and with the additional assumption of an oracle that, given a point, is able to return the list of filters in the collection of filters used by the data structure that contain it. Having an efficient algorithm for listing the filters that contain a point is key to the performance of the data structure as the number of filters is typically polynomial in the size of the point set and by the pigeonhole principle cannot be much smaller than $n$ if we want to avoid expensive searches in the bucket associated with each filter. To solve this problem, Becker et al. gave a fast evaluation algorithm for a family of filters for the unit sphere under inner product similarity. We are able to remove the oracle assumption by showing that a combination of “powering” and “tensoring” of the filter family yields an efficient filter evaluation algorithm that works for arbitrary filter families, allowing us to state a more general framework.

Building on the filter family and evaluation algorithm by Becker et al. [8], Laarhoven [29] used the idea of having different filters for the query and update algorithm in order to obtain space-time tradeoffs on the unit sphere in the important setting of random data. This paper extends the LSF framework to match the generality of the LSH framework while supporting the full range of space-time tradeoffs. As an application of the LSF framework, we give a simple family of filters for inner product similarity search on the unit sphere in high dimensions, encompassing and extending the results of [17, 29] to worst-case data. An elegant result by Rahimi & Recht [43] based on Bochner’s theorem [44] from harmonic analysis shows that a large class of similarity measures over $R^d$, namely all real-valued characteristic functions [32], supports a simple and explicit approximate embedding on the unit sphere. That is, if $k(x − y)$ is a real characteristic function, then there exists a map onto the unit sphere $v: R^d → S^l$ such that $⟨v(x), v(y)⟩ ≈ k(x − y)$. Combined with the family of filters for inner product similarity search, we get space-time tradeoffs for the near neighbor problem under characteristic function similarity measures, also known as shift-invariant kernels.

As an important special case we consider the class of characteristic functions of $s$-stable distributions [52] that take the form $k(x − y) = e^{−∥x − y∥^s}$ where $0 < s ≤ 2$. We obtain a unified approach to solving the $(r, cr)$-near neighbor problem in $l^2_s$ that supports the full range of space-time tradeoffs. Balancing the query and update time, our upper bound matches the LSH lower bound [40] and matches or improves well-known upper bounds [22, 15, 3, 37]. In the imbalanced case, where the query and update time differs, our work uniformly improves upon a line of work [11, 3, 33, 27] initiated by Panigrahy [11] that investigates different space-time tradeoffs, particularly in $l^2_2$, against the backdrop of the LSH framework. As an example, consider restricting the data structure to use near-linear space. In the setting of $l^2_s$ we provide the first data-independent solution to the $(r, cr)$-near neighbor problem that achieves a sublinear query time for every $c > 1$. Previously, the best known result by Kapralov [27] only achieves sublinear query time for $c > \sqrt{3}$ in Euclidean space.

On the lower bound side we show that in the case where query and update time is balanced,
the LSF framework must obey known LSH lower bounds for \( \ell^d \), implying that our upper bound for \( \ell^d \) is optimal within both frameworks [36, 10, 7]. An application of an isoperimetric inequality described by O’Donnell [38, Ch. 10] yields a lower bound in the LSF framework for Hamming space that simultaneously bounds \( \rho_0 \) and \( \rho_u \), resulting in a lower bound across the range of space-time tradeoffs. The bound matches our own and Laarhoven’s [29] upper bound for random data as \( d \) goes to infinity.

Finally we point out that the proposed solutions to the \((r, cr)\)-near neighbor problem in this paper are data-independent. A recent result by Andoni & Razenshteyn [6] shows how letting the data guide the choice of the partitions of space used in the LSH framework can lead to improvements in the space and time complexity of the solution. We leave it as an open problem to combine these results and to discover simple data-dependent filter families.

## 2 Locality-sensitive filtering

We define the concept of a family of locality sensitive filters which can be viewed as the equivalent of a family of locality sensitive hash functions in the LSH framework.

**Definition 2.** Let \((X, D)\) be a space and let \(F\) be a probability distribution over the set of filters \(\{(Q, U) \mid Q \subseteq X, U \subseteq X\}\). We say that \(F\) is a \((r, cr, p_1, p_2, p_q, p_u)\)-sensitive filter family for \((X, D)\) if for all points \(x, y \in X\) and a filter \(F = (Q, U)\) sampled independently at random from \(F\) it satisfies the following conditions:

- If \(D(x, y) \leq r\) then \(\Pr_F[x \in Q, y \in U] \geq p_1\)
- If \(D(x, y) \geq cr\) then \(\Pr_F[x \in Q, y \in U] \leq p_2\)
- \(\Pr_F[x \in Q] \leq p_q\)
- \(\Pr_F[x \in U] \leq p_u\)
- \(p_q, p_u > p_1, p_2\).

For a filter \(F = (Q, U)\) we refer to \(Q\) as the query filter and \(U\) as the update filter. We say that a pair of points \((x, y)\) is contained in a filter \(F = (Q, U)\) if \(x \in Q\) and \(y \in U\). For a similarity measure \(S\) we say that a family \(F\) is \((\alpha, \beta, p_1, p_2, p_q, p_u)\)-similarity sensitive for \((X, S)\) if it is \((-\alpha, -\beta, p_1, p_2, p_q, p_u)\)-sensitive for \((X, -S)\).

### 2.1 Powering and tensoring techniques

In order to make our LSF data structure efficiently solve the \((r, cr)\)-near neighbor problem for a wide range of filter families we need to combine two techniques that we refer to as powering and tensoring. The powering technique will amplify the locality-sensitivity properties of the filters. For positive integers \(m, \tau\), the tensoring technique will allow us to “simulate” \(\binom{m}{\tau}\) filters from \(m\) filters. Furthermore, these simulated filters can be evaluated efficiently: given a point \(x \in X\) we can list the simulated filters containing \(x\) in time \(O(dm + L)\) where \(L\) denotes the output size. The space and time savings of the tensoring technique are key to an efficient LSF data structure as the number of filters we wish to evaluate usually exceeds \(n\).

The powering technique for filter families was used in the framework by Becker et al. [8] and is almost identical in terms of properties to the powering technique in the LSH framework [22] where it is used to increase the gap between \(p\) and \(q\).

**Lemma 1.** Given a \((r, cr, p_1, p_2, p_q, p_u)\)-sensitive filter family \(F\) for \((X, D)\) and a positive integer \(\kappa\) define the family \(F^\kappa\) as follows: we sample a filter \(F = (Q, U)\) from \(F\) by sampling \((Q_1, U_1), \ldots, (Q_\kappa, U_\kappa)\) independently from \(F\) and setting \((Q, U) = (\bigcap_{i=1}^\kappa Q_i, \bigcap_{i=1}^\kappa U_i)\). The family \(F^\kappa\) is \((r, cr, p_1^\kappa, p_2^\kappa, p_q^\kappa, p_u^\kappa)\)-sensitive for \((X, D)\). To deal with some corner cases we define \(F^0\) as the family containing the single filter \(F = (X, X)\).
Next we describe the tensoring technique and state some of its properties. Let \( \mathbf{F} \) denote a collection (indexed family) of \( m \) filters and let \( \mathbf{Q} \) and \( \mathbf{U} \) denote the corresponding collections of query and update filters, that is, for \( i \in \{1, \ldots, m\} \) we have that \( \mathbf{F}_i = (\mathbf{Q}_i, \mathbf{U}_i) \). Given a positive integer \( \tau \leq m \) (typically \( \tau \ll m \)) we define \( \mathbf{F}^{\otimes \tau} \) to be the collection of filters formed by taking all the intersections of \( \tau \)-combinations of filters from \( \mathbf{F} \), that is, for every \( I \subseteq \{1, \ldots, m\} \) with \( |I| = \tau \) we have that

\[
\mathbf{F}^{\otimes \tau}_I = \left( \bigcap_{i \in I} \mathbf{Q}_i, \bigcap_{i \in I} \mathbf{U}_i \right).
\]

The following properties of the tensoring technique will be used to provide correctness and running time/space usage guarantees for the LSF data structure that will be introduced in the next subsection.

**Lemma 2.** Suppose we have a \((r, cr, p_1, p_2, p_q, p_u)\)-sensitive filter family \( \mathcal{F} \) for \((X, D)\). Let \( \tau \) be a positive integer and let \( \mathbf{F} \) denote a collection of \( m = \lceil \tau/p_1 \rceil \) independently sampled filters from \( \mathcal{F} \). Then the collection \( \mathbf{F}^{\otimes \tau} \) of \( \binom{m}{\tau} \) filters has the following properties:

- Let \((x, y)\) be a pair of points of distance at most \( r \). Then with probability at least 1/2 there exists a filter in \( \mathbf{F}^{\otimes \tau} \) containing \((x, y)\).
- Let \((x, y)\) be a pair of points of distance at least \( cr \). Then the expected number of filters in \( \mathbf{F}^{\otimes \tau} \) containing \((x, y)\) is at most \( p_2^r \binom{m}{\tau} \).
- In expectation, a point \( x \) is contained in at most \( p_q^r \binom{m}{\tau} \) query filters and at most \( p_u^r \binom{m}{\tau} \) update filters in \( \mathbf{F}^{\otimes \tau} \).
- The evaluation time and space complexity of \( \mathbf{F}^{\otimes \tau} \) is dominated by the time it takes to evaluate and store \( m \) filters from \( \mathcal{F} \).

**Proof.** To prove the first property we note that there exists a filter in \( \mathbf{F}^{\otimes \tau} \) containing \((x, y)\) if at least \( \tau \) filters in \( \mathbf{F} \) contain \((x, y)\). The binomial distribution has the property that the median is at least as great as the mean rounded down \([25]\). By the choice of \( m \) we have that the expected number of filters in \( \mathbf{F} \) containing \((x, y)\) is at least \( \tau \) and the result follows. The second and third properties follow from the linearity of expectation and the fourth is trivial. \( \square \)

The technique described here is a more general variation of the tensoring technique introduced by Dubiner \([17]\) in his algorithm to solve all pairs similarity in Hamming space where the tensoring term refers to the outer product of collections of filters (here we take subsets from a single collection instead). A similar technique is also used in the more recent work by Becker et al. \([8]\) to speed up their filter evaluation algorithm.

### 2.2 The LSF data structure

We will introduce a data structure for maintaining a set of points \( P \subseteq X \) while supporting queries that solve the \((r, cr)\)-near neighbor problem. The data structure has access to a \((r, cr, p_1, p_2, p_q, p_u)\)-sensitive filter family \( \mathcal{F} \) in the sense that it knows the parameters of the family and is able to sample, store, and evaluate filters from \( \mathcal{F} \) in time \( O(d) \).

The data structure supports an initialization operation that initializes a collection of filters \( \mathbf{F} \) where for every filter we maintain a (possibly empty) set of points from \( X \). After initialization the data structure supports three operations: insert, delete, and query. The insert (delete) operation takes as input a point \( x \in X \) and adds (removes) the point from the set of points associated with each update filter in \( \mathbf{F} \) that contains \( x \). The query operation takes as input a point \( x \in X \). For each query filter in \( \mathbf{F} \) that contains \( x \) we proceed by computing the dissimilarity \( D(x, y) \) to every point \( y \) associated with the filter. If a point \( y \) satisfying \( D(x, y) \leq cr \) is encountered, then \( y \) is returned and the query algorithm terminates. If no such point is found, the query algorithm returns a special symbol \("\emptyset"\) and terminates.

The data structure will combine the powering and tensoring techniques in order to simulate the collection of filters \( \mathbf{F} \) from two smaller collections: \( \mathbf{F}_1 \) consisting of \( m_1 \) filters from \( \mathcal{F}^{\otimes 1} \) and
\( F_2 \) consisting of \( m_2 \) filters from \( F^{*2} \). The collection of simulated filters \( F \) is formed by taking all filters \( (Q_1 \cap Q_2, U_1 \cap U_2) \) where \((Q_1, U_1)\) is a member of \( F_1^{*r} \) and \((Q_2, U_2)\) is a member of \( F_2 \). Letting \( I_1 \) denote a \( \tau \)-subset of \( \{1, \ldots, m_1\} \) and \( I_2 \) denote a member of \( \{1, \ldots, m_2\} \) we have for every pair \((I_1, I_2)\) that

\[
F_{I_1, I_2} = (Q_1 \cap Q_2, U_1 \cap U_2) \text{ where } (Q_1, U_1) = F_1^{*r} \text{ and } (Q_2, U_2) = F_2. 
\]

The initialization operation takes \( F \) and parameters \( m_1, \kappa_1, \tau, m_2, \kappa_2 \) and samples and stores \( F_1 \) and \( F_2 \). The filter evaluation algorithm used by the insert, delete, and query operation takes a point \( x \in X \) and computes for \( F_1 \) and \( F_2 \), depending on the operation, the list of update or query filters containing \( x \). From these lists we are able to generate the list of filters in \( F \) containing \( x \). Finally, we use standard hashing techniques \([11]\) with worst case expected constant time operations to implement two hash tables: The first hash table contains the set of points \( P \), allowing us to store a reference to an inserted point in constant space. The second, two-level, hash table is used to maintain the set of points associated with each filter in \( F \).

### 2.3 A framework with space-time tradeoffs

Setting the parameters of the data structure to guarantee correctness while balancing the contribution to the query time from the filter evaluation algorithm, the number of filters containing the query point, and the number of distant points examined, we obtain a partially dynamic data structure that solves the \((r, cr)\)-near neighbor problem with failure probability \( \delta \leq 1/2 + 1/e \). Using a standard dynamization technique by Overmars and Leeuwen \([39, \text{Thm. 1}]\) we obtain a fully dynamic data structure, resulting in the following framework theorem:

**Theorem 2.** Suppose we have access to a \((r, cr, p_1, p_2, q, u)\)-sensitive filter family. Then we can construct a fully dynamic data structure that solves the \((r, cr)\)-near neighbor problem. Assume that \(1/p_1, 1/\log(p_2/p_2), \text{ and } \exp(\log(1/p_1)/\log(\min(p_1, p_u)/p_1))\) are \(o(1)\), then the data structure has

- expected query time \( dn^{\rho_q+o(1)} \),
- expected update time \( n^{\rho_u+o(1)} + dn^{o(1)} \),
- expected space usage \( n^{1+\rho_u+o(1)} + dn + dn^{o(1)} \)

where

\[
\rho_q = \frac{\log p_q/p_1}{\log p_q/p_2}, \quad \rho_u = \frac{\log p_u/p_1}{\log p_u/p_2}. 
\]

**Remark.** In the original statement of the LSH framework \([22]\) the probability parameters \( p, q \) of the locality-sensitive family of hash functions are treated as constants in the big O notation while \( n \) goes to infinity. If we treat \( p_1, p_2, q, p_u \) as constant in Theorem 2 then the assumption holds by the requirement that \( p_q, p_u > p_1 > p_2 \) from Definition 2 and the data structure has expected query time \( O(dn^{\rho_q} \log n) \), expected update time \( O(dn^{\rho_u} \log n) \), and expected space usage \( O(dn + n^{1+\rho_u}) \).

To prove Theorem 2 we begin by describing the construction of the partially dynamic LSF data structure for a set of \( n \) points. The data structure is initialized with the following setting of the parameters:

\[
\kappa_1 = \left[ \frac{\min(p_q, p_u) \log n}{\log(1/p_1)} \right] \quad (5)
\]

\[
\tau = \left[ \frac{\log n}{\kappa_1 \log(p_q/p_2)} \right] \leq \frac{\log(1/p_1)}{\log(\min(p_q, p_u)/p_1)} \quad (6)
\]

\[
m_1 = \left[ \frac{\tau/p_1^{\kappa_1}}{1} \right] \quad (7)
\]

\[
\kappa_2 = \max(0, \left[ \log(n)/\log(p_q/p_2) \right] - \tau \kappa_1) \quad (8)
\]

\[
m_2 = \left[ \frac{1/p_1^{\kappa_2}}{1} \right] \quad (9)
\]
We will now briefly explain the reasoning behind the parameter settings. Begin by observing that the powering and tensoring techniques both amplify the filters from $F$. Let $m = \binom{m_1}{m_2}$ denote the number of simulated filters in our collection $F$ and let $a = \tau \kappa_1 + \kappa_2$ be an integer denoting the number of times each filter has been amplified. Ignoring the time it takes to evaluate the filters, the query time is determined by the sum of the number of filters that contain a query point and the number of distant points associated with those filters that the query algorithm inspects. The expected number of activated filters is given by $n mp^a_2$. Balancing the contribution to the query time from these two effects (ignoring the $O(d)$ factor from distance computations) results in a target value of $a = \lceil \log(n) \log(p_q/p_2) \rceil$. Compared to having an oracle that is able to list the filters from a collection that contains a point, there is a small loss in efficiency from using the tensoring technique due to the increase in the number of filters required to guarantee correctness. The parameters of the LSF data structure are therefore set to minimize the use of tensoring such that the time spent evaluating our collection of filters roughly matches the minimum of the query and update time.

Consider the initialization operation of the LSF data structure with the parameters setting from above. We have that $\kappa_2 \leq \kappa_1$ implying that $mp = O(m_1)$. The initialization time and the space usage of the data structure prior to any insertions is dominated by the time and space used to sample and store the filters in $F_1$. By the assumption that a filter from $F$ can be sampled in $O(d)$ operations and stored using $O(d)$ words, we get a space and time bound on the initialization operation of

$$O(d \kappa_1 m_1) = O(d n^{\min(p_q, p_\sigma)} (\log(n)/\log(p_q/p_2))/p_1).$$

Importantly, this bound also holds for the running time of the filter evaluation algorithm, that is, the preprocessing time required for constant time generation of the next element in the list of filters in $F$ containing a point. In the following analysis of the update and query time we will temporarily ignore the running time of the filter evaluation algorithm.

The expected time to insert or delete a point is dominated by the number of update filters in $F$ that contains it. The probability that a particular update filter in $F$ contains a point is given by $p^a_u$. Using a standard upper bound on the binomial coefficient we get that $m = O(e^\tau/p^a_1)$ resulting in an expected update time of

$$O(mp^a_2 + d) = O(n^{\min(p_a, p_1)} e^\tau + d).$$

The worst case expected query time (the case of $n$ points at distance $cr$ from the query point) can be upper bounded by

$$O(mp^b_2 + d n mp^b_2) = O(n^{\min(p_q, p_1)} (p_q/p_1 + d)).$$

With respect to the correctness of the query algorithm, if a near neighbor $y$ to the query point $x$ exists in $P$, then it is found by the query algorithm if $(x, y)$ is contained in a filter in $F_1^{\tau}$ as well as in a filter in $F_2$. By Lemma the first event happens with probability at least $1/2$ and by the choice of $m_2$, the second event happens with probability at least $1 - (1 - p_1^{\kappa_2})p_2^{\kappa_2} \geq 1 - 1/e$. From the independence between $F_1$ and $F_2$ we can upper bound the failure probability $\delta \leq (1/2)(1 + 1/e)$. This completes the proof of Theorem.

3 Gaussian filters on the unit sphere

In this section we give an analysis of a family of filters for the unit sphere $S^d$ under inner product similarity. It is possible that some of the results in this section can be derived from a more careful analysis of existing filter families, nevertheless, we have decided to include the analysis of the Gaussian filter family due to its simplicity and its use as a stepping stone for the
results in later sections of this paper. The Gaussian filter family is closely related to a number of existing LSH and LSF families from the literature \cite{13,15,8,13} in the sense that they work by projecting the data point \(x\) into a lower dimensional space by taking the dot product \(\langle x, z \rangle\) where \(z \sim \mathcal{N}^d(0,1)\) is a random vector of \(d\) i.i.d. standard normal variables. Through the stability properties of the normal distribution \cite{22} the projection \(\langle x, z \rangle\) satisfies certain locality-preserving properties. Let \(x, y \in \mathbb{R}^d\), then \(\langle x, z \rangle - \langle y, z \rangle = \langle x - y, z \rangle \sim \mathcal{N}(0, \|x - y\|^2_2)\), that is, the difference between the projection follows a normal distribution with variance \(\|x - y\|^2_2\) so that points that are close in \(\mathbb{R}^d\) are more likely to project to a close number in \(\mathbb{R}\).

Panigrahy \cite{34} introduced the idea of having asymmetry in the query and update algorithms in the context of the LSH framework resulting in a solution to the \((r, cr)\)-nearest neighbor problem with space-time tradeoffs. Panigrahy’s approach, as well as a line of related practical and theoretical work \cite{2,33,27}, takes a hash function \(h \in \mathcal{H}\) that partitions space and essentially attempts to locate buckets in the “neighborhood” around the bucket associated with a given point. Laarhoven \cite{29} showed how a similar asymmetry between the query and update algorithm can be obtained in the context of locality-sensitive filtering, extending the family of filters introduced by Becker et al. \cite{8} to give a solution with space-time tradeoffs for the \((\alpha, 0)\)-similarity problem on the unit sphere. The asymmetry is inherent in the view of locality-sensitive filtering, extending the family of filters given in Definition \cite{2} where each filter \(F = (Q, U)\) consists of a query and an update filter. We proceed by describing the family of filters \(\mathcal{C}\) based on spherical caps introduced by Laarhoven \cite{29}. Let \(t > 0\) denote a threshold parameter and \(\gamma > 0\) denote a parameter controlling the asymmetry between query and update filters, then we sample a filter \((Q, U)\) from \(\mathcal{C}\) by sampling a point \(\hat{z} = (1/\|z\|_2) \cdot z\) uniformly at random from \(S^d\) and setting

\[
Q = \{x \in \mathbb{R}^d \mid \langle x, \hat{z} \rangle > \gamma t\}, \quad U = \{x \in \mathbb{R}^d \mid \langle x, \hat{z} \rangle > t\}.
\]  

In this paper we introduce a slightly different family of filters that we refer to as the Gaussian filter \(G\) family where, instead of normalizing the vector \(z \sim \mathcal{N}^d(0,1)\), we use \(z\) directly in place of \(\hat{z}\) above. The properties of \(G\) are stated in Lemma \cite{3} below and can be easily be verified with a simple back-of-the-envelope analysis using two facts: First, for a standard normal random variable \(Z\) we have that \(\Pr[Z > t] \approx e^{-t^2/2}\). Secondly, the invariance of Gaussian projections \(\langle x, z \rangle\) to rotations, allowing us to analyze the projection of arbitrary points \(x, y \in S^d\) with inner product \(\langle x, y \rangle = \alpha\) in a two-dimensional setting \(x = (1, 0)\) and \(y = (\alpha, \sqrt{1 - \alpha^2})\) without any loss of generality. Our use of the second property is inspired in particular by the analysis of the cross-polytope LSH by Andoni et al. \cite{4}. The complete and formal analysis is standard and has been deferred to Appendix \cite{A}.

**Lemma 3.** For every positive integer \(d\) and \(\gamma, t > 0\) let \(G\) denote the family of filters defined as follows: we sample a filter \((Q, U)\) from \(G\) by sampling \(z \sim \mathcal{N}^d(0,1)\) and setting

\[
Q = \{x \in \mathbb{R}^d \mid \langle x, z \rangle > \gamma t\}, \quad U = \{x \in \mathbb{R}^d \mid \langle x, z \rangle > t\}.
\]  

Then for every choice of \(0 \leq \beta < \alpha < 1\) such that \(\alpha \leq \gamma \leq 1/\alpha\) the family \(G\) is \((\alpha, \beta, p_1, p_2, p_q, p_u)\)-similarity sensitive on \((S^d, \langle \cdot, \cdot \rangle)\) with probabilities bounded by

\[
p_1 \geq e^{-(1 + (\gamma - \alpha)^2)/(4\alpha^2)t^2/2} \quad \frac{2\pi(1 + t/\alpha)^2}{\alpha^2}
\]

\[
p_2 \leq e^{-(1 + (\gamma - \beta)^2)/(4\beta^2)t^2/2}
\]

\[
p_q \leq e^{-\gamma^2t^2/2}
\]

\[
p_u \leq e^{-t^2/2}.
\]
Applying Theorem 2 we obtain query and update exponents

\[
\rho_q \leq \frac{(1 - \gamma \alpha)^2}{1 - \alpha^2} \left( \frac{\ln 2 \pi (1 + t/\alpha)^2}{t^2/2} \right) \frac{(1 - \gamma \beta)^2}{1 - \beta^2},
\]

(19)

\[
\rho_u \leq \frac{(\gamma - \alpha)^2}{1 - \alpha^2} \left( \frac{\ln 2 \pi (1 + t/\alpha)^2}{t^2/2} \right) \frac{(1 - \gamma \beta)^2}{1 - \beta^2}.
\]

(20)

We combine the Gaussian filters with Theorem 2 to show that we can solve the \((\alpha, \beta)\)-similarity problem efficiently for the full range of space/time tradeoffs, even when \(\alpha, \beta\) are allowed to depend on \(n\), as long as the gap \(\alpha - \beta\) is not too small. The following Theorem provides a solution to the \((\alpha, \beta)\)-similarity problem on the high-dimensional unit sphere for \(0 \leq \beta < \alpha < 1\). Previously, the best known data-independent solution by Laarhoven [29] was only analyzed for \(\alpha, \beta\) solution to the \((r, cr)\)-near neighbor problem over the similarity measure

\(\mathcal{S}(x, y) = k(x - y)\) where \(k\) is a real-valued characteristic function [48], also known in the machine learning literature as a shift-invariant or translation-invariant kernel [43].

We begin by an informal statement of a fundamental result that underlies the widely used kernel method in machine learning. To avoid introducing unnecessary notation we overload \(k\) to represent functions of the form \(k: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\). It can be shown that if \(k\) is positive definite then there exists a map \(\psi\) into a dot product space such that

\(k(x, y) = \langle \psi(x), \psi(y) \rangle\).

(22)

The function \(k\) is usually called a kernel, the mapping \(\psi\) is called a feature map, and the associated dot product space is called a feature space. For an introduction to kernel techniques as used here, see for example [21]. The Gaussian filters combined with the LSF framework yields an algorithm for performing inner product similarity search on unit vectors. Therefore, viewing a positive definite kernel \(k\) as a similarity measure, if we could find an explicit and efficiently computable feature map \(\psi: \mathbb{R}^d \to \mathbb{S}^l\) we would be able to solve the \((\alpha, \beta)\)-similarity problem on \(\mathbb{R}^d\) for the similarity measure.
k. A classical theorem from harmonic analysis by Bochner [43] together with some elementary properties of characteristic functions [35] shows that the class of normalized positive definite real-valued shift-invariant kernels \( k: \mathbb{R}^d \times \mathbb{R}^d \to [0,1] \), i.e., the shift-invariant functions that may be of interest to our application, corresponds exactly to the class of characteristic functions \( k(x,y) = k(x-y) \) of symmetric distributions. Through an elegant combination of Bochner’s Theorem and Euler’s Theorem, Rahimi & Recht [43] show how to construct approximate feature maps \( \langle v(x),v(y) \rangle \approx \langle \psi(x),\psi(y) \rangle = k(x-y) \) for the the class of real-valued characteristic functions. We proceed by stating a slightly modified version of their result where we ensure that the approximate feature map is into the unit sphere. Relevant properties of characteristic functions as well as a statement of Bochner’s Theorem and a detailed derivation of Rahimi & Recht’s result are included in Appendix B.

**Lemma 4.** Let \( k \) be a real-valued characteristic function with associated distribution function \( \mu \) and let \( l \) be a positive integer. Consider the family of functions \( V \subseteq \{ v: \mathbb{R}^d \to \mathbb{S}^l \} \) where a randomly sampled function \( v \) is defined by, independently for \( j = 1, \ldots, l \), sampling \( z \) from \( \mu \) and \( b \) uniformly on \( [0,2\pi] \), letting \( \hat{v}(x)_j = \sqrt{(2/l)} \cos((z_j,x) + b) \) and normalizing \( v(x)_j = \frac{\hat{v}(x)_j}{\|\hat{v}(x)\|} \). Let \( k(x,y) \) be of interest to our application, corresponds exactly to the class of characteristic functions \( k(x,y) = k(x-y) \) of symmetric distributions. Through an elegant combination of Bochner’s Theorem and Euler’s Theorem, Rahimi & Recht [43] show how to construct approximate feature maps \( \langle v(x),v(y) \rangle \approx \langle \psi(x),\psi(y) \rangle = k(x-y) \) for the the class of real-valued characteristic functions. We proceed by stating a slightly modified version of their result where we ensure that the approximate feature map is into the unit sphere. Relevant properties of characteristic functions as well as a statement of Bochner’s Theorem and a detailed derivation of Rahimi & Recht’s result are included in Appendix B.

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**Theorem 4.** Let \( k: \mathbb{R}^d \to \mathbb{R} \) be a characteristic function and define the similarity measure \( S(x,y) = k(x-y) \). Assume that we have access to samples from the distribution associated with \( k \), then Theorem 3 holds with \( \langle S^d,\langle \cdot,\cdot \rangle \rangle \) replaced by \( \langle \mathbb{R}^d,S \rangle \).

**Proof.** According to Lemma 4 we can set \( l = n^{o(1)} \) to obtain a map \( v: \mathbb{R}^d \to \mathbb{S}^l \) such that the the inner product on \( \mathbb{S}^l \) preserves the pairwise similarity between \( n^{O(1)} \) points with additive error \( \varepsilon = o(1) \). This map has a space and time complexity of \( O(dl) = dn^o(1) \). After applying \( v \) to the data we can solve the \( (\alpha,\beta) \)-similarity problem on \( \langle \mathbb{R}^d,k(x-y) \rangle \) by solving the \( (\alpha-\varepsilon,\beta+\varepsilon) \)-similarity problem on \( \langle \mathbb{S}^d,\langle \cdot,\cdot \rangle \rangle \). We can use Theorem 3 to construct a fully dynamic data structure for solving this problem, adjusting the parameter \( \gamma \) so that it continues to lie in the admissible range. The space and time complexities follow.\( \square \)
5 Similarity search in $\ell_s^d$ with space-time tradeoffs

Consider the $(r, cr)$-near neighbor problem in $\ell_s^d$ for $0 < s \leq 2$. It turns out that the class of characteristic functions for $s$-stable distributions [12] in conjunction with the results from Section 3 yields a powerful solution to this problem.

Lévy [30] was the first to study $s$-stable distributions. He showed that for every $0 < s \leq 2$ there exists a distribution with characteristic function $e^{-|x|^s}$. Through vectorization of the characteristic function this implies the following existence result that we include for clarity.

**Lemma 5.** For every positive integer $d$ and $0 < s \leq 2$ there exists a characteristic function $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, 1]$ of the form

$$k(x, y) = k(x - y) = e^{-\|x - y\|^s_s}. \quad (26)$$

In order to apply Theorem 4 we need to have access to samples from the distribution associated with $k$. In general, given a characteristic function $k$, there is no known algorithm that allows us to efficiently sample its distribution. However, for the class of characteristic functions of $s$-stable distributions, a result by Chambers et al. [12] shows how to sample from an $s$-stable distributions using only two samples from a uniform distribution. This result has previously been used in the related work on similarity search in $\ell_s^d$ by Datar et al. [12].

We now turn to the analysis of Theorem 4 when using the characteristic functions from Lemma 5 to solve the $(r, cr)$-near neighbor problem in $\ell_s^d$. In the language of $(\alpha, \beta)$-similarity from Theorem 3 we have $\alpha = e^{-r^s}$, $\beta = e^{-c^sr^s}$, and we will introduce a space-time tradeoff parameter $\lambda \in [-1, 1]$ to set $\gamma = \alpha^\lambda = e^{-\lambda r^s}$. We proceed by bounding the exponents $\rho_q$ and $\rho_u$ where we make use of the following fact that can be derived from the Taylor expansion of the exponential function: for all $x \geq 0$ it holds that $1 - x \leq e^{-x} \leq 1 - x + x^2/2$. Consider

$$\rho_q = \frac{(1 - \gamma \alpha)^2}{1 - \alpha^2} \frac{(1 - \gamma \beta)^2}{1 - \beta^2} = \frac{(1 - \gamma \alpha)^2}{1 - \gamma \beta)^2} \frac{1 - \beta^2}{1 - \alpha^2}. \quad (27)$$

We proceed by upper bounding $\frac{1 - \gamma \alpha}{1 - \gamma \beta}$ and $\frac{1 - \beta^2}{1 - \alpha^2}$. First we have that

$$\frac{1 - \gamma \alpha}{1 - \gamma \beta} = \frac{1 - e^{-(1+\lambda)r^s}}{1 - e^{-(c^s\lambda)r^s}} \leq \frac{1}{1 - e^{s^2r^s}} \frac{1 + \lambda}{c^s + \lambda}. \quad (28)$$

Similarly

$$\frac{1 - \beta^2}{1 - \alpha^2} = \frac{1 - e^{-2c^sr^s}}{1 - e^{-c^sr^s}} \leq \frac{c^s}{1 - r^s}. \quad (29)$$

By scaling the data points such that $r^s = o(1)$ we obtain simple query and update exponents. Finally, as we set $\alpha$ and $\beta$ to depend on $n$ we need to make sure that the gap $\alpha - \beta$ does not become too small. Assuming $c > 1$ and $0 < s \leq 2$ are constant, we can scale the data such that $r^s = 1/(c^s \ln \ln n)$ to ensure that $\alpha - \beta = \Omega(1/(\ln \ln n))$ and Theorem 4 applies. The following theorem summarizes our results for $\ell_s^d$.

**Theorem 5.** For every choice of constants $0 < s \leq 2$, $-1 \leq \lambda \leq 1$, and $c > 1$ we can construct a fully dynamic data structure that solves the $(r, cr)$-near neighbor in $\ell_s^d$. The data structure satisfies the guarantees from Theorem 2 for

$$\rho_q = \frac{c^s(1 + \lambda)^2}{(c^s + \lambda)^2} + o(1), \quad \rho_u = \frac{c^s(1 - \lambda)^2}{(c^s + \lambda)^2} + o(1). \quad (30)$$

Balancing query and update time (setting $\lambda = 0$), the query and update exponent in Theorem 5 $\rho_q = \rho_u = 1/c^s + o(1)$ matches the best known data-oblivious LSH upper bounds [22, 15, 3] that are known to be tight in the LSH framework [40]. Furthermore, to our knowledge Theorem
is also the first result to match the LSH lower bound for \( s \) in the range \( 1 < s < 2 \), although it is possible that a similar result can be obtained through the use of existing embedding techniques as seems to be hinted in \[37\]. Considering space-time tradeoffs, our result uniformly improves upon the result by Kapralov \[27\] that represents the state of the art in Euclidean space. A detailed comparison of our result to the result of Kapralov is given in Appendix \[C\].

6 Lower bounds

In this section we prove two lower bounds in the LSF framework. The first lower bound holds for symmetric filter families, defined as families \( \mathcal{F} \) where for every \( F = (Q, U) \in \mathcal{F} \) we have that \( Q = U \), and follows from existing LSH lower bounds \[40\] using a proof by contradiction that relies on the construction of an LSH family from an LSF family. If there existed a sufficiently powerful LSF family, then we would be able to construct a LSH family contradicting known LSH lower bounds.

The second lower bound holds for asymmetric filter families in Hamming space \( (\{0,1\}^d, \|\cdot\|_1) \). The lower bound places two additional restrictions on the filter families, both of which we believe can be removed by refining the analysis. First, the lower bound only holds for regular families of filters, defined as families \( \mathcal{F} \) where for every pair of filters \( (Q_1, U_1), (Q_2, U_2) \in \mathcal{F} \) we have that \( |Q_1| = |Q_2| \) and \( |U_1| = |U_2| \). Secondly, the filter family cannot be too asymmetric in the sense that we impose some bounds on the ratio \( |Q|/|U| \) for which the lower bound applies. In spite of these restrictions, the lower bound is powerful enough to match our own and Laarhoven’s upper bounds for the case of random data on the unit sphere. To see that the Gaussian filter family matches the second lower bound under the above restrictions we refer to the central limit theorem: as \( d \) goes to infinity the regular filter family of concentric ball in Hamming space behaves like the Gaussian filter family.

6.1 Symmetric lower bound

We begin by stating the lower bound on the parameter \( \rho = \log(1/p)/\log(1/q) \) for LSH families by O’Donnell et al. \[40\]. The techniques behind the lower bound, as well as a related lower bound by Motwani et al. \[30\], derive from the Fourier analysis of boolean functions \[38\].

**Theorem 6** (O’Donnell et al. \[40\]). Fix \( 1 < c < \infty \), \( 0 < s < \infty \), \( \varepsilon > 0 \), and assume \( 2^{-\omega(d)} \leq q \leq 1 - \varepsilon \). Then for a certain choice of \( r = \omega_d(1) \) every \((r,cr,p,q)\)-sensitive hash family \( \mathcal{H} \) for \( \epsilon_s \) must satisfy that

\[
\frac{\log(1/p)}{\log(1/q)} \geq \frac{1}{c^s} - o_d(1).
\]  

(31)

The following lemma shows how to use a filter family \( \mathcal{F} \) to construct a hash family \( \mathcal{H} \). For the purpose of obtaining a lower bound we focus on showing existence, ignoring the space and time complexity of storing and evaluating a function \( h \in \mathcal{H} \). Given a family \( \mathcal{F} \), we sample a hash function \( \mathcal{H} \) by sampling an infinite sequence of filters from \( \mathcal{F} \). The hash function is the partition defined by assigning each point to the index of the first filter that contains. This technique is similar to the construction of the Spherical LSH by Andoni et al. \[5\].

**Lemma 6.** Suppose there is a symmetric \((r,cr,p_1,p_2,p_3,p_4)\)-sensitive filter family. Then there exists a \((r,cr,p_1/(2p_3),p_2/p_4)\)-sensitive hash family.

**Proof.** Given the filter family \( \mathcal{F} \) we sample a random function \( h \) from the hash family \( \mathcal{H} \) taking an infinite sequence of independently sampled filters \( (F_i)_{i=1}^\infty \) from \( \mathcal{F} \) and setting \( h(x) = \min\{i \mid x \in F_i\} \). The probability of collision is given by

\[
Pr_{h \sim \mathcal{H}}[h(x) = h(y)] = \frac{Pr_{\mathcal{F}}[x \in F \land y \in F]}{Pr_{\mathcal{F}}[x \in F \lor y \in F]}
\]

(32)
We say that \((x, y)\) is \((r, cr; p_1, p_2, p_3, p_4)\)-sensitive if \(r \leq \sum_{i=1}^{d} \min\{1, p_i x_i y_i\}\) and \(cr \leq \sum_{i=1}^{d} \min\{1, p_i x_i y_i - 1\}\). Every symmetric \(\mathcal{F}\)-sensitive filter family \(\mathcal{F}\) for \(\ell^d_q\) satisfies the requirements from Theorem 6 with \(p = p_1/p_3\) and \(q = p_2/p_4\). The constructed family \(\mathcal{H}\) is \((r, cr; p, q)\)-sensitive for \(p = (1/2) \cdot (p_1/p_3)\) and \(q = (p_2/p_4)\). By our choice of \(\kappa\) we have that \(\log(1/p)/\log(1/q) = \log(p_1/p_3)/\log(p_2/p_4) + o_d(1)\) and the lower bound on \(\log(1/p)/\log(1/q)\) from Theorem 6 applies.

6.2 Asymmetric lower bound

We now consider a lower bound for regular filter families, as defined in the introduction to this section. Compared to the symmetric lower bound, this bound applies to the more general case where the filters in the family \(F = (Q, U)\) are allowed to be asymmetric \(|Q| \neq |U|\). The lower bound in this section is stated for Hamming space. Because Hamming space embeds into \(\ell^d_q\)-spaces the lower bound can easily be extended as done in the previous work by Motwani et al. and O’Donnell et al. [36, 40]. To avoid introducing more notation, for the remainder of this section we will adopt the convention of letting the inner product \(\langle x, x \rangle\) between two vectors \(x, y \in \{0, 1\}^d\) denote

\[
\langle x, y \rangle = \frac{1}{d} \sum_{i=1}^{d} (-1)^{x_i} (-1)^{y_i}.
\]

This corresponds to the usual inner product applied to the natural embedding of boolean strings onto \(S^d\), allowing us to re-use much of the intuition developed in earlier sections.

The lower bound uses an isoperimetric-type inequality that holds for random, correlated, points in Hamming space.

**Definition 3.** We use \(x \sim \{0, 1\}^d\) to denote a random vector on the \(d\)-dimensional boolean hypercube where each component of \(x\) is i.i.d. according to

\[
x_i = \begin{cases} 1 & \text{with probability } \frac{1-\alpha}{2} \\ 0 & \text{with probability } \frac{1+\alpha}{2} \end{cases}.
\]

We say that \((x, y)\) is (randomly) \(\alpha\)-correlated if we have that \(x \sim \{0, 1\}^d\) and \(y = x + z \pmod{2}\) where \(z \sim \{0, 1\}^d\).

We are now ready to state a powerful result by O’Donnell, referred to as the generalized small-set expansion theorem in [38], which is key to showing the LSF lower bound. We point out the remarkable similarity to the Gaussian filter family described in Section 3.

**Lemma 7** (O’Donnell [38, p. 285]). Let \(0 \leq \alpha < 1\). Every \(Q, U \subseteq \{0, 1\}^d\) with \(|Q|/2^d = (|U|/2^d)^2 \gamma^2\) for some \(\alpha \leq \gamma \leq 1/\alpha\) must satisfy

\[
Pr_{(x, y) \sim \alpha\text{-correlated}}[x \in Q, y \in U] \leq (|U|/2^d)^{-\frac{\gamma^2 - 2\gamma \alpha}{1 - \alpha^2}}.
\]
The argument for the lower bound proceeds by assuming a regular \((r,c,r,p_1,p_2,p_q,p_u)\)-sensitive filter family \(\mathcal{F}\) for Hamming space where we set \(r = (1 - \alpha)d/2\) and \(c = (1 - \beta)d/2\) for some choice of \(0 < \beta < \alpha < 1\). Furthermore, as in Lemma 7 we use the parameter \(\gamma\) to describe the asymmetry between query and update filters \(|Q|/2^d = (|U|/2^d)^\gamma\). We proceed by deriving constraints on \(p_1, p_2, p_q, p_u\), and minimizing \(p_q\) and \(p_u\) subject to those constraints. Consider

\[
\rho_q = \frac{\log(p_q/p_1)}{\log(p_q/p_2)}. 
\]

Subject to the LSF constraint that \(p_q, p_u > p_1 > p_2 > 0\) we see that \(p_q\) is minimized by setting \(p_q, p_2\) as small as possible and \(p_1\) as large as possible. We will therefore derive lower bounds on \(p_q, p_2\) and an upper bound on \(p_1\). For every value of \(p_1\) and \(p_2\) we minimize \(p_q, p_u\) by choosing \(p_q\) as small as possible.

For a random point \(x \in \{0,1\}^d\) it must hold that \(Pr_{\mathcal{F}}[x \in Q] = |Q|/2^d\). This implies the existence of a fixed point \(y \in \{0,1\}^d\) with the property that \(Pr_{\mathcal{F}}[y \in Q] \geq |Q|/2^d\). A regular filter family must therefore satisfy that \(p_q \geq |Q|/2^d\) and \(p_u \geq |U|/2^d\). Let \(\gamma\) be defined as in Lemma 7 then by a similar argument we have that \(p_2 \geq (|U|/2^d)^{1+\gamma}\).

In order to upper bound \(p_1\) we make use of Lemma 7 together with the following lemma that follows directly from an application of Hoeffding’s inequality [20].

**Lemma 8.** For every \(0 < \varepsilon < (1 - \alpha)/2\) we have that

\[
Pr_{\alpha+\varepsilon\text{-correlated}}[(x,y) \leq \alpha] \leq e^{-\varepsilon^2/2}. 
\]

In the following derivation, assume that \(\alpha, \varepsilon_1\) satisfies \(0 < \varepsilon_1 < (1 - \alpha)/2\), let \(x, y\) denote randomly \((\alpha + \varepsilon_1)\)-correlated vectors in \(\{0,1\}^d\), and assume that \(\alpha + \varepsilon_1 \leq \gamma \leq 1/(\alpha + \varepsilon_1)\), then

\[
(|U|/2^d)^{1+\gamma^2-2\gamma(\alpha+\varepsilon_1)} \geq \Pr(x \in Q, y \in U) 
\]

\[
\geq \Pr(x \in Q, y \in U | \langle x, y \rangle \geq \alpha) \Pr(\langle x, y \rangle \geq \alpha) 
\]

\[
\geq p_1(1 - e^{-\varepsilon_1^2/2}) 
\]

Summarizing the bounds:

\[
p_1 \leq \frac{(|U|/2^d)^{1+\gamma^2-2\gamma(\alpha+\varepsilon_1)}}{1 - e^{-\varepsilon_1^2/2}} 
\]

\[
p_2 \geq (|U|/2^d)^{1+\gamma^2} 
\]

\[
p_q \geq |Q|/2^d 
\]

\[
p_u \geq |U|/2^d. 
\]

**Theorem.** Fix \(0 < \beta < \alpha < 1\). Then there exists a constant \(\zeta > 0\) such that every regular \(((1 - \alpha)d/2, (1 - \beta)d/2, p_1, p_2, p_q, p_u)\)-sensitive filter family on \((\{0,1\}^d, \|\cdot\|_1)\) with \(2^{-\zeta d} \leq |U|/2^d \leq 1 - 1/d\) and \(|Q|/2^d = (|U|/2^d)^\gamma\) where \(\gamma\) satisfies \(\alpha + 2\sqrt{\ln(d)/d} \leq \gamma \leq 1/\alpha - 2\sqrt{\ln(d)/d}\) must have

\[
\rho_q = \ln p_q / p_1 \geq \frac{(1 - \gamma\alpha)^2}{1 - \alpha^2} - o_d(1) \quad \text{and} \quad \rho_u = \ln p_u / p_1 \geq (\gamma - \alpha)^2 / (1 - \alpha^2) - o_d(1) 
\]

when \(p_q\) is set to minimize \(\rho_q\).

**Proof.** Fix \(0 < \zeta \leq 8\ln(2)(1+\alpha^2)\). When minimizing \(\rho_q\) we have that \(\log(p_q/p_2) = -\log(|U|/2^d)\). Setting \(\varepsilon = 2\sqrt{\ln(d)/d}\) we have

\[
\ln 1/p_1 \geq -\frac{1 + \gamma^2 - 2\gamma(\alpha + \varepsilon_1)}{1 - \alpha^2} \ln(|U|/2^d) - O(1/d^2). 
\]
Putting things together

\[
\log\left(\frac{p_2}{p_1}\right) \geq \gamma_2 \ln\left(\frac{|U|}{2d}\right) - \frac{1 + \gamma_2^2 - 2\gamma_2\alpha + \varepsilon}{1 - \alpha^2} \ln\left(\frac{|U|}{2d}\right) - O\left(\frac{1}{d^2}\right) - \ln\left(\frac{|U|}{2d}\right) - O\left(\frac{1}{d^2}\right)
\]

(48)

\[
\log\left(\frac{p_2}{p_1}\right) = \frac{(1 - \gamma\alpha)^2}{1 - \alpha^2} - \frac{2\gamma\varepsilon}{1 - \alpha^2} + O\left(\frac{1}{d^2}\right) / \ln\left(\frac{|U|}{2d}\right)
\]

(49)

\[
\log\left(\frac{p_2}{p_1}\right) = \frac{(1 - \gamma\alpha)^2}{1 - \alpha^2} - O\left(\sqrt{\log(d)/d}\right).
\]

(50)

The derivation of the lower bound for \( \rho_u \) is almost the same and the resulting expression is

\[
\frac{\ln p_u/p_1}{\ln p_2/p_1} \geq \frac{(\gamma - \alpha)^2}{1 - \alpha^2} - O\left(\sqrt{\log(d)/d}\right).
\]

(51)

7 Conclusion and open problems

We introduced a framework for locality-sensitive filtering and proved upper and lower bounds for a wide range of settings of the near neighbor problem.

Interesting open problems include finding the shape of provably exactly optimal filters in different spaces. In the random data setting in Hamming space, this problem boils down to maximizing number of pairs of points below a certain distance that is contained in a subset of the space of a certain size. This is a fundamental problem in combinatorics that has been studied by among others [20], but a complete answer remains elusive. The LSH and LSF lower bounds, along with classical isoperimetric inequalities such as Harper’s Theorem and more recent work summarized in the book by O’Donnell [38] hints that the answer is somewhere between a subcube and a generalized sphere.

A recent result by Chierichetti & Kumar [14] characterizes the set of transformations of similarity measures that are LSH-able as the set of probability-generating functions. This seems to have deep connections to result of this paper that uses characteristic functions that allow well-known kernel transformations. It seems possible that this paper can be viewed as a semi-explicit construction of their result, or that their result can described as an application of Bochner’s Theorem.

Finally, we note that it appears to be possible to strengthen the lower bound for the unit sphere by showing that random instances on sub-spheres are hard to solve. In particular, imagine a set of points on \( P \) the unit sphere where every point \( x \in P \) has \( \sqrt{\beta} \) as its first component and the remaining components are randomly sampled from the surface of a sphere of radius \( \sqrt{1 - \beta} \). Given this information, solving the \( (\alpha, \beta) \)-similarity problem amounts to solving the \( ((\alpha - \beta)/(1 - \beta), 0) \)-similarity problem. It therefore appears that the lower bound of \( \rho \geq (1 - \alpha)/(1 + \alpha) \) can be strengthened to \( (1 - \alpha)/(1 + \alpha - 2\beta) \). In terms of the Euclidean distance this lower bound becomes \( 1/(2c^2 - 1) \) which has appeared before in various places in the literature [17, 6, 4]. The implications of this observation and the relationship to the data-dependent lower bounds by Andoni & Razenshteyn [7] remain to be explored.

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A Gaussian filters

Lemma 9 (Follows from Szarek & Werner [37]). Let $Z$ be a standard normal random variable. Then, for every $t \geq 0$ we have that
\[
\frac{1}{\sqrt{2\pi}} \frac{1}{t+1} e^{-t^2/2} \leq \Pr[Z \geq t] \leq \frac{1}{\sqrt{\pi}} \frac{1}{t+1} e^{-t^2/2}.
\]

Lemma 10 (Simplified version of Lu & Li [31]). Let $z$ be a $d$-dimensional vector of i.i.d. standard normal random variables and let $D \subset \mathbb{R}^d$ be a closed convex domain that does not contain the origin. Let $\Delta$ denote the Euclidean distance to the unique closest point in $D$, then we have that
\[
\Pr[z \in D] \leq e^{-\Delta^2/2}.
\]

Upper bound

Lemma 11. For every $\alpha, \gamma, t, \beta$ satisfying $0 < \alpha < 1$, $\alpha \leq \gamma \leq 1/\alpha$, $t > 0$, and $-1 < \beta < \alpha$ every pair of standard normal random variables $(X,Y)$ with correlation $\beta' \leq \beta$ satisfies
\[
\Pr[X \geq t \wedge Y \geq \gamma t] \leq e^{-\Delta^2/2}
\]
where $\Delta^2 = (1 + (\gamma - \beta')^2/t^2).

Proof. For $\beta' = -1$ the result is trivial. For $-1 < \beta' \leq \beta$ we use the $2$-stability of the normal distribution to analyze a tail bound for $(X,Y)$ in terms of a Gaussian projection vector $z = (Z_1, Z_2)$ applied to unit vectors $x, y \in \mathbb{R}^2$. That is, we can define $X = \langle z, x \rangle$ and $Y = \langle z, x \rangle$ for some appropriate choice of $x$ and $y$. Without loss of generality we set $x = (1, 0)$ and note that for $E[XY] = \beta'$ we must have that $y = (\beta', \sqrt{1 - \beta'^2})$. If we consider the region of $\mathbb{R}^2$ where $z$ satisfies $X \geq t \wedge Y \geq \gamma t$ we get a closed domain $D$ defined by $z = (Z_1, Z_2)$ such that $Z_1 \geq t$ and $Z_2 \geq (\gamma t - \beta' Z_1)/\sqrt{1 - \beta'^2}$. The squared Euclidean distance from the origin to the closest point in $D$ at least $\Delta^2$ as can be seen by the fact that $\Delta^2$ decreasing in $\beta$. Combining this observation with Lemma 10 we get the desired result. \hfill \Box

Lower bound

Lemma 12. For every $\alpha, \gamma, t$ satisfying $0 < \alpha < 1$, $\alpha \leq \gamma \leq 1/\alpha$, and $t > 0$ every pair of standard normal random variables $(X,Y)$ with correlation $\alpha' \geq \alpha$ satisfies
\[
\Pr[X \geq t \wedge Y \geq \gamma t] \geq \frac{e^{-\Delta^2/2}}{2\pi(1 + t/\alpha)^2}
\]
where $\Delta^2 = (1 + (\gamma - \alpha')^2/t^2).

Proof. For $\alpha' = 1$ the result follows directly from Lemma 9. For $\alpha' < 1$ we use the trick from the proof of Lemma 11 and define $X = \langle z, x \rangle$ and $Y = \langle z, x \rangle$ where $x = (1, 0)$ and $y = (\alpha, \sqrt{1 - \alpha^2})$ and $z = (Z_1, Z_2)$ is a vector of two i.i.d. standard normal random variables. This allows us to rewrite the probability as follows:
\[
\Pr[Z_1 \geq t \wedge \alpha Z_1 + \sqrt{1 - \alpha^2} Z_2 \geq \gamma t]
= \Pr[Z_1 \geq t] \Pr[\alpha Z_1 + \sqrt{1 - \alpha^2} Z_2 \geq \gamma t \mid Z_1 \geq t]
\geq \Pr[Z_1 \geq t] \Pr[\alpha t + \sqrt{1 - \alpha^2} Z_2 \geq \gamma t]
\]
By the restrictions on $\alpha$ and $\gamma$ we have that $(\gamma - \alpha)t/\sqrt{1 - \alpha^2} \leq t/\alpha$. The result follows from applying the lower bound from Lemma 9 and noting that the bound is increasing in $\alpha$. \hfill \Box
B Characteristic functions, Bochner’s Theorem, and approximate feature maps

We begin by defining what a characteristic function is and listing some properties that are useful for our application. More information about characteristic functions can be found in the books by Lukacs [32] and Ushakov [48].

Lemma 13 ([32, 48]). Let $Z$ denote a random variable with distribution function $\mu$. Then the characteristic function $k(\Delta)$ of $Z$ is defined as

$$k(\Delta) = \int_{-\infty}^{\infty} \mu(t)e^{i\Delta t}dt$$

and it has the following properties:

- A distribution function is symmetric if and only if its characteristic function is real and even.
- Every characteristic function $k(\Delta)$ is uniformly continuous, has $k(0) = 1$, and $|k(\Delta)| \leq 1$ for all real $\Delta$.
- If $k(\Delta)$ is the characteristic function of an absolutely continuous distribution, then $\lim_{\Delta \to \infty}|k(\Delta)| = 0$.
- Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ be independent random vectors with characteristic functions $k_X(x_1, \ldots, x_n)$ and $k_Y(y_1, \ldots, y_m)$. Then the characteristic function of $Z = (X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ is given by $k_Z(z_1, \ldots, z_{n+m}) = k_X(z_1, \ldots, z_n)k_Y(z_{n+1}, \ldots, z_{n+m})$.

Bochner’s Theorem reveals the relation between the class of real-valued functions $k(x, y)$ that admit a feature space representation

$$k(x, y) = \langle \phi(x), \phi(y) \rangle$$

and characteristic functions.

Theorem 9 (Bochner [44]). A function $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ is positive definite if and only if it can be written on the form

$$k(x, y) = \int_{\mathbb{R}^d} \mu(z)e^{i(z,x-y)}dz$$

where $\mu$ is the probability density function of a symmetric distribution.

Rahimi & Recht’s [43] family of approximate feature maps $V$ is constructed from Bochner’s Theorem by making use of Euler’s Theorem as follows:

$$k(x, y) = \int_{\mathbb{R}^d} \mu(z)e^{i(z,x-y)}dz$$

$$= \int_{\mathbb{R}^d} \mu(z)\cos(\langle z, x-y \rangle)dz + i \int_{\mathbb{R}^d} \mu(z)\sin(\langle z, x-y \rangle)dz$$

$$= \mathbb{E}[\cos(\langle z, x-y \rangle)]$$

$$= \mathbb{E}[\cos(\langle z, x-y \rangle) + \cos(\langle z, x \rangle + \langle z, y \rangle + 2b)]$$

$$= 2 \mathbb{E}[\cos(\langle z, x \rangle + b) \cdot \cos(\langle z, y \rangle + b)].$$

Where the third equality makes use of the fact that $k(x, y)$ is real-valued to remove the complex part of the integral and the fifth equality uses that $2\cos(x)\cos(y) = \cos(x + y) + \cos(x - y)$. 

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Now that we have an approximate feature map onto the sphere for the class of shift-invariant kernels, we will take a closer look at what functions this class contains, and what their applications are for similarity search. Given an arbitrary similarity function, we would like to be able to determine whether it is indeed a characteristic function. Unfortunately, there are no known simple techniques for answering this question in general. However, the machine learning literature contains many applications of different shift-invariant kernels [46] and many common distributions have real characteristic functions (see Appendix B in [48] for a long list of examples). Characteristic functions are also well studied from a mathematical perspective [32, 48], and a number of different necessary and sufficient conditions are known. A classical result by Pólya [42] gives simple sufficient conditions for a function to be a characteristic function. Through the vectorization property from Lemma [13], Pólya’s conditions directly imply the existence of a large class of similarity measures on \( \mathbb{R}^d \) that can fit into the above framework.

**Theorem 10** (Pólya [42]). Every even continuous function \( k : \mathbb{R} \to \mathbb{R} \) satisfying the properties

- \( k(0) = 1 \)
- \( \lim_{\Delta \to \infty} k(\Delta) = 0 \)
- \( k(\Delta) \) is convex for \( \Delta > 0 \)

is a characteristic function.

Based on the results of Section 5 one could hope for the existence of characteristic functions of the form \( k(\Delta) = e^{-|\Delta|^{x}} \) for \( s > 2 \) but it is known that such functions cannot exist [9] Theorem D.8]. Furthermore, Marcinkiewicz [34] shows that a function of the form \( k(\Delta) = \exp(-\text{poly}(\Delta)) \) cannot be a characteristic function if the degree of the polynomial is greater than two.

## C Comparison to Kapralov

For every choice of tradeoff parameter \( \alpha \in [0, 1] \), assuming that \( c^2 \geq 3(1 - \alpha)^2 - \alpha^2 + \varepsilon \) for arbitrarily small constant \( \varepsilon > 0 \), Kapralov [27] obtains query and update exponents

\[
\rho_q = \frac{4(1 - \alpha)^2}{c^2 + (1 - \alpha)^2 - 3\lambda^2}, \quad \rho_u = \frac{4\alpha^2}{c^2 + (1 - \alpha)^2 - 3\lambda^2}.
\]

We convert Kapralov’s notation to our own by setting \( \lambda = 1 - 2\alpha \). To compare, Kapralov sets \( \alpha = 0 \) for near-linear space and we set \( \lambda = 1 \). We want to write Kapralov’s exponents on the form

\[
\rho_q = \frac{c^2(1 + \lambda)^2}{(c^2 + \lambda)^2 + x}, \quad \rho_u = \frac{c^2(1 - \lambda)^2}{(c^2 + \lambda)^2 + x}
\]

for some \( x \) that we will proceed to derive. We have that \( (1 - \alpha)^2 = (1 + \lambda)^2/4 \) and \( \alpha^2 = (1 - \lambda)^2/4 \). Multiplying the numerator and denominator in Kapralov’s exponents by \( c^2 \) we can write Kapralov’s exponents as

\[
\rho_q = \frac{c^2(1 + \lambda)^2}{c^4 + c^2(1 + \lambda)^2/4 - 3c^2(1 - \lambda)^2/4}, \quad \rho_u = \frac{c^2(1 - \lambda)^2}{c^4 + c^2(1 + \lambda)^2/4 - 3c^2(1 - \lambda)^2/4}.
\]

We have that

\[
x = c^4 + c^2(1 + \lambda)^2/4 - 3c^2(1 - \lambda)^2/4 - (c^2 + \lambda)^2
\]

\[
= -c^2(1 + \lambda)^2/2 - \lambda^2.
\]

For every choice of \( \lambda \in [-1, 1] \), and under the assumption that \( c^2 \geq (1 + \lambda)^2/2 + \lambda + \varepsilon \) for an arbitrarily small constant \( \varepsilon > 0 \), we can write Kapralov’s exponents as

\[
\rho_q = \frac{c^2(1 + \lambda)^2}{(c^2 + \lambda)^2 - c^2(1 + \lambda)^2/2 - \lambda^2}, \quad \rho_u = \frac{c^2(1 - \lambda)^2}{(c^2 + \lambda)^2 - c^2(1 + \lambda)^2/2 - \lambda^2}.
\]
To compare Kapralov’s result against our own for search in $\ell_s$-spaces we consider the exponents from Theorem 5, ignoring additive $o(1)$ terms:

$$\rho_q = \frac{c^s(1 + \lambda)^2}{(c^s + \lambda)^2}, \quad \rho_u = \frac{c^s(1 - \lambda)^2}{(c^s + \lambda)^2}. \quad (70)$$

Figure 1 compares values of $\rho_q$ and $\rho_u$ between Theorem 5 and Kapralov’s result [27] in $\ell_2$ with $c = 2$ for the entire range of tradeoffs as determined by $\lambda \in [-1, 1]$. Setting $\lambda = 1$ we obtain a data structure that uses near-linear space and we get a query exponent $\rho_q = 16/25$ while Kapralov obtains an exponent of $\rho_q = 16/20$, ignoring $o(1)$ terms. At the other end of the tradeoff, setting $\lambda = -1$, we get a data structure with query time $n^{o(1)}$ and update exponent $\rho_u = 16/9$ while Kapralov gets an update exponent of $\rho_u = 4$, again ignoring additive $o(1)$ terms.

Figure 1: Comparison of query and update exponents in $\ell_2$ for $c = 2$

The assumption made by Kapralov that $c^2 \geq (1 + \lambda)^2/2 + \lambda + \varepsilon$ means that in the case of a near-linear space data structure ($\lambda = 1$) sublinear query time can only be obtained for $c \geq \sqrt{3}$. In contrast, Theorem 5 gives sublinear query time for every constant $c > 1$. Figure 2 compares $\rho_q$ between Kapralov’s result and Theorem 5 in $\ell_2$ when restricting the data structure to near-linear space.

D Extending solutions to the near neighbor problem through embeddings

There exist Johnson-Lindenstrauss [24] type embeddings from other $\ell_d$-spaces into $\ell_2^{O(\log n)}$ as described in [22, Appendix A], making it possible to extend results for $\ell_2$ to $\ell_s$-space for $s \in [1, 2]$. However, applying this technique comes at a cost that we are able to avoid. For example, suppose we have a solution to the $(r, cr)$-near neighbor problem in Euclidean space with $\rho = 1/c^2$. Then embedding result shows that there exists a solution to the $(r, cr)$-near neighbor problem in $\ell_d^d$ with $\rho = 1/c$, similarly to the work by Datar et al. [15].
Figure 2: Comparison of query exponents in $\ell_2$ when using near-linear space

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\end{figure}

References

[1] J. Alman and R. Williams. Probabilistic polynomials and hamming nearest neighbors. In Proc. FOCS ’15, pages 136–150, 2015.

[2] A. Andoni. Nearest neighbor search: the old, the new, and the impossible. PhD thesis, MIT, 2009.

[3] A. Andoni and P. Indyk. Near-optimal hashing algorithms for approximate nearest neighbor in high dimensions. In Proc. FOCS ’06, pages 459–468, 2006.

[4] A. Andoni, P. Indyk, T. Laarhoven, I. Razenshteyn, and L. Schmidt. Practical and optimal lsh for angular distance. In Proc. NIPS ’15, pages 1225–1233, 2015.

[5] A. Andoni, P. Indyk, H. L. Nguyen, and I. P. Razenshteyn. Beyond locality-sensitive hashing. In Proc. SODA ’14, pages 1018–1028, 2014.

[6] A. Andoni and I. Razenshteyn. Optimal data-dependent hashing for approximate near neighbors. In Proc. STOC ’15, pages 793–801, 2015.

[7] A. Andoni and I. P. Razenshteyn. Tight lower bounds for data-dependent locality-sensitive hashing. CoRR, abs/1507.04299, 2015.

[8] A. Becker, L. Ducas, N. Gama, and T. Laarhoven. New directions in nearest neighbor searching with applications to lattice sieving. In Proc. SODA ’16, pages 10–24, 2016.

[9] Y. Benyamini and J. Lindenstrauss. Geometric nonlinear functional analysis, volume 48. American Mathematical Soc., Providence, Rhode Island, 1998.

[10] A. Z. Broder, M. Charikar, A. M. Frieze, and M. Mitzenmacher. Min-wise independent permutations (extended abstract). In Proc. STOC ’98, pages 327–336, 1998.

[11] J. L. Carter and M. N. Wegman. Universal classes of hash functions. J. Comput. Syst. Sci., 18(2):143–154, 1979.
[12] J. M. Chambers, C. L. Mallows, and B. W. Stuck. A method for simulating stable random variables. *Jour. Am. Stat. Assoc.*, 71(354):340–344, 1976.

[13] M. Charikar. Similarity estimation techniques from rounding algorithms. In *Proc. STOC ’02*, pages 380–388, 2002.

[14] F. Chierichetti and R. Kumar. Lsh-preserving functions and their applications. *J. ACM*, 62(5):33, 2015.

[15] M. Datar, N. Immorlica, P. Indyk, and V. S. Mirrokni. Locality-sensitive hashing scheme based on p-stable distributions. In *Proc. SOCG ’04*, pages 253–262, 2004.

[16] M. de Berg, M. van Kreveld, M. Overmars, and O. C. Schwarzkopf. *Computational geometry*. Springer, Berlin, third edition, 2008.

[17] M. Dubiner. Bucketing coding and information theory for the statistical high-dimensional nearest-neighbor problem. *IEEE Trans. Information Theory*, 56(8):4166–4179, 2010.

[18] T. Hagerup. Sorting and searching on the word RAM. In *Proc. STACS ’98*, pages 366–398, 1998.

[19] S. Har-Peled, P. Indyk, and R. Motwani. Approximate nearest neighbor: Towards removing the curse of dimensionality. *Theory of computing*, 8(1):321–350, 2012.

[20] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Jour. Am. Stat. Assoc.*, 58(301):13–30, 1963.

[21] T. Hofmann, B. Schölkopf, and A. J. Smola. Kernel methods in machine learning. *The annals of statistics*, 36(3):1171–1220, 2008.

[22] P. Indyk and R. Motwani. Approximate nearest neighbors: towards removing the curse of dimensionality. In *Proc. STOC ’98*, pages 604–613, 1998.

[23] Piotr Indyk. Nearest neighbors in high-dimensional spaces. In *Handbook of Discrete and Computational Geometry, Second Edition.*, pages 877–892. Chapman and Hall/CRC, 2004.

[24] W. B. Johnson and J. Lindenstrauss. Extensions of lipschitz mappings into a hilbert space. *Contemporary mathematics*, 26(189-206):1, 1984.

[25] R. Kaas and J. M. Buhrman. Mean, median and mode in binomial distributions. *Statistica Neerlandica*, 34(1):13–18, 1980.

[26] J. Kahn, G. Kalai, and N. Linial. The influence of variables on boolean functions (extended abstract). In *Proc. FOCS ’88*, pages 68–80, 1988.

[27] M. Kapralov. Smooth tradeoffs between insert and query complexity in nearest neighbor search. In *Proc. PODS ’15*, pages 329–342, 2015.

[28] E. Kushilevitz, R. Ostrovsky, and Y. Rabani. Efficient search for approximate nearest neighbor in high dimensional spaces. *SIAM J. Comput.*, 30(2):457–474, 2000.

[29] T. Laarhoven. Tradeoffs for nearest neighbors on the sphere. *CoRR*, abs/1511.07527, 2015.

[30] P. Lévy. *Calcul des probabilités*, volume 9. Gauthier-Villars, Paris, 1925.

[31] D. Lu and W. V. Li. A note on multivariate gaussian estimates. *Journal of Mathematical Analysis and Applications*, 354(2):704–707, 2009.

[32] E. Lukacs. *Characteristic Functions*. Griffin, London, second edition, 1970.
[33] Q. Lv, W. Josephson, Z. Wang, M. Charikar, and K. Li. Multi-probe LSH: efficient indexing for high-dimensional similarity search. In Proc. VLDB ’07, pages 950–961, 2007.

[34] J. Marcinkiewicz. Sur une proprieté de la loi de gauss. Mathematische Zeitschrift, 44(1):612–618, 1939.

[35] M. Minsky and S. Papert. Perceptrons. MIT Press, Cambridge, MA, 1969.

[36] R. Motwani, A. Naor, and R. Panigrahy. Lower bounds on locality sensitive hashing. SIAM J. Discrete Math., 21(4):930–935, 2007.

[37] H. L. Nguyen. Approximate nearest neighbor search in $\ell_p$. CoRR, abs/1306.3601, 2013.

[38] R. O’Donnell. Analysis of Boolean Functions. Cambridge University Press, 2014.

[39] M. H. Overmars and J. van Leeuwen. Worst-case optimal insertion and deletion methods for decomposable searching problems. Information Processing Letters, 12(4):168–173, 1981.

[40] R. O’Donnell, Y. Wu, and Y. Zhou. Optimal lower bounds for locality-sensitive hashing (except when $q$ is tiny). ACM Transactions on Computation Theory (TOCT), 6(1):5, 2014.

[41] R. Panigrahy. Entropy based nearest neighbor search in high dimensions. In Proc. SODA ’06, pages 1186–1195, 2006.

[42] G. Pólya. Remarks on characteristic functions. In Proc. Berkeley Symposium on Mathematical Statistics and Probability 1945-1946, pages 115–123, 1949.

[43] A. Rahimi and B. Recht. Random features for large-scale kernel machines. In Proc. NIPS ’07, pages 1177–1184, 2007.

[44] W. Rudin. Fourier Analysis on Groups. Wiley, New York, 1990.

[45] I. R. Savage. Mill’s ratio for multivariate normal distributions. Jour. Res. NBS Math. Sci., 66(3):93–96, 1962.

[46] B. Schölkopf and A. J. Smola. Learning with Kernels. The MIT Press, Cambridge, Massachusetts, 2002.

[47] S. J. Szarek and E. Werner. A nonsymmetric correlation inequality for gaussian measure. Journal of multivariate analysis, 68(2):193–211, 1999.

[48] N. G. Ushakov. Selected Topics in Characteristic Functions. VSP, Utrecht, The Netherlands, 1999.

[49] J. Wang, H. T. Shen, J. Song, and J. Ji. Hashing for similarity search: A survey. CoRR, abs/1408.2927, 2014.

[50] R. Weber, H. Schek, and B. Stephen. A quantitative analysis and performance study for similarity-search methods in high-dimensional spaces. In Proc. VLDB ’98, pages 194–205, 1998.

[51] Ryan Williams. A new algorithm for optimal constraint satisfaction and its implications. In Proc. ICALP ’04, pages 1227–1237, 2004.

[52] V. M. Zolotarev. One-dimensional stable distributions, volume 65. American Mathematical Soc., 1986.