Cascade Hierarchy in SUSY $SU(5)$ GUT

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Abstract

We study cascade hierarchy in supersymmetric $SU(5)$ grand unified theory. The neutrino Dirac mass matrix of the cascade form can lead to the tri-bimaximal generation mixing at the leading order in the seesaw mechanism while the down quark mass matrix of a hybrid cascade form naturally gives the CKM structure. We embed such experimentally favored mass textures into supersymmetric $SU(5)$ GUT, which gives a relation between the down quark and charged lepton mass matrices. Related phenomenologies, such as lepton flavor violating processes and leptogenesis, are also investigated in addition to lepton mixing angles.
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1 Introduction

The current precision measurements of the neutrino oscillation have suggested that there are large mixing angles among three generations in the lepton sector unlike the quark sector. The experimental data of lepton generation mixing angles [1] is well approximated by the tri-bimaximal mixing [2], which is given by

\[ V_{TB} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \] (1.1)

The characteristic features of this mixing matrix are that the second generation of the neutrino mass eigenstate is represented by tri-maximal mixture of all flavor eigenstates, \( \nu_2 = \sum_\alpha \nu_\alpha/\sqrt{3} \), and the third generation is bi-maximal mixture of \( \mu \) and \( \tau \) neutrinos, \( \nu_3 = (-\nu_\mu + \nu_\tau)/\sqrt{2} \), in the diagonal basis of the charged leptons, respectively. The equation (1.1) implies the following forms of neutrino mass matrix in the flavor basis

\[ M_\nu = \frac{m_1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \frac{m_2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \] (1.2)

where \( m_i \ (i = 1 \sim 3) \) are the neutrino mass eigenvalues. Such suggestive forms of the generation mixing and the neutrino mass matrices give us a motivation to look for a flavor structure of the lepton sector. Actually, a number of proposals based on a flavor symmetry to unravel it and related phenomenologies have been elaborated [3].

Recently, it has been pointed out that the neutrino Dirac mass matrix of a cascade form can lead to the tri-bimaximal generation mixing at the leading order [4]. The mass matrix of the cascade form is given by

\[ M_{\text{cas}} = \begin{pmatrix} \delta & \delta & \delta \\ \delta & \lambda & \lambda \\ \delta & \lambda & 1 \end{pmatrix} v, \] (1.3)

which is described by small parameters, \(|\delta| \ll |\lambda| \ll 1\), and the dimension-one parameter \( v \) denotes an overall mass scale. We call this type of hierarchy “cascade hierarchy”, and the mass matrix with such a hierarchy “cascade matrix”. On the other hand, it is well known that the down quark mass matrix of somewhat different hierarchical form, which is given by

\[ M_{\text{hyb}} = \begin{pmatrix} \epsilon' & \delta' & \delta' \\ \delta' & \lambda' & \lambda' \\ \delta' & \lambda' & 1 \end{pmatrix} v', \] (1.4)

can explain experimentally observed values of CKM matrix elements, where \(|\epsilon'| \ll |\delta'| \ll |\lambda'| \ll 1\). The \((1, 1)\) element of the matrix is much smaller than all other elements. However,
hierarchical structure is similar to the cascade form except for the \((1,1)\) element. In this paper, we call this type of hierarchy “hybrid cascade (H.C.) hierarchy”, and the mass matrix with such a hierarchy “hybrid cascade (H.C.) matrix”.

The interesting fact that the neutrino Dirac mass matrix of the cascade form realize the tri-bimaximal generation mixing and the down quark mass matrix of the H.C. form reproduce the CKM structure gives us a strong motivation to understand the quark and lepton sectors, comprehensively. Towards the comprehensive understanding of the quark/lepton sectors, we investigate embedding such hierarchies into a supersymmetric grand unified theory (SUSY GUT). In this paper, a case of SUSY \(SU(5)\) GUT is studied as the simplest example.

The paper is organized as follows: In section 2, we give more detailed explanation about the cascade hierarchies and discuss them with the fermion masses and mixing angles. In section 3, we embed the cascade hierarchies into a SUSY \(SU(5)\) GUT. The texture analyses for the quark/lepton sectors are also presented in the section. In section 4, we show the numerical analyses of our model, which are the generation mixing angles and related phenomenologies such as lepton flavor violating rare decay processes and leptogenesis. In section 5, some comments on realizations of this model is presented. Section 6 is devoted to the summary. The Appendix gives a discussion about constraints on a structure of non-diagonal right-handed neutrino mass matrix.

2 Cascade hierarchies and fermion mass matrices

First we imply the cascade textures to quark and lepton sectors independently. The implication of the cascades to the lepton sector has been discussed in \(\text{[4]}\). In the work, both mass matrices of neutrino and charged lepton have been taken as the cascade form. However, a hybrid type of cascade form would be phenomenologically allowed for the charged lepton mass matrix. We focus on this point and extend our attention to the quark sector in the context of the cascade hierarchies.

The most famous hierarchical mass (Yukawa) structure is realized by the Froggatt-Nielsen (FN) mechanism \(\text{[5]}\), which is one of fascinating approaches to explain the quark mass hierarchy. A lot of works based on the FN model have been presented, which are typically generating a mass matrix as

\[
M_{\text{wat}} = \begin{pmatrix}
\delta^2 & \delta \lambda & \delta \\
\delta \lambda & \lambda^2 & \lambda \\
\delta & \lambda & 1
\end{pmatrix} v. \tag{2.1}
\]

where \(|\delta| \ll |\lambda| \ll 1\) and \(\mathcal{O}(1)\) coefficients for each element have been dropped here. Such a hierarchical form of mass matrix can be easily realized by an abelian flavor symmetry.
mass eigenvalues  |  cascade  |  waterfall  \\
|-----------------|----------|-----------|
| $m_1 : m_2 : m_3 \sim \delta : \lambda : 1$ | $m_1 : m_2 : m_3 \sim \delta^2 : \lambda^2 : 1$ |

mixing angles  |  cascade  |  waterfall  \\
|-----------------|----------|-----------|
| $\theta_{12} \sim \delta / \lambda$, $\theta_{23} \sim \lambda$, $\theta_{13} \sim \delta$ ($\theta_{ij} \sim m_i / m_j$) | $\theta_{12} \sim \delta / \lambda$, $\theta_{23} \sim \lambda$, $\theta_{13} \sim \delta$ ($\theta_{ij} \sim \sqrt{m_i / m_j}$) |

Table 1: Mass eigenvalues and generation mixing angles induced from the cascade and waterfall mass matrices.

On the other hand, the cascade mass matrix shown in (1.3) has been recently proposed for the lepton sector [4]. Such a kind of mass matrix can realize the tri-bimaximal generation mixing at the leading order in the framework of the seesaw mechanism [6] when the hierarchy of Dirac mass matrices for leptons is described as the cascade form. In the work, mass matrices of right-handed neutrino and charged lepton have been taken as diagonal and cascade form, respectively, and some collections from the charged lepton to the tri-bimaximal mixing have also been discussed. However, if a right-handed Majorana mass matrix gives only small mixing collections, it is experimentally allowed, even if it does not have a diagonal form. Therefore, we can conclude for the forms of neutrino mass matrices that the tri-bimaximal generation mixing can be realized at the leading order if the Dirac mass matrices of the cascade form and a Majorana mass matrix leading small mixing collections are taken. This means that the cascade or H.C. form of Majorana mass matrix are also allowed. The form of charged lepton mass matrix does not have to be the cascade form; if the matrix also induces only small collections to the mixing angles such as the H.C. form, it is allowed.

The mass matrix shown in (2.1) has a more rapid stream of hierarchy flow than the cascade one given in (1.3), and is called the “waterfall” mass matrix. We note that these two types of matrices have the same orders of generation mixing angles, while they induce different mass eigenvalues as shown in Tab. 1. It is seen that relation among the mass eigenvalues and mixing angles are roughly estimated as $\theta_{ij} \sim m_i / m_j$ for the cascade matrix and $\theta_{ij} \sim \sqrt{m_i / m_j}$ for the waterfall.

Note that the experimentally observed values of masses and mixing angles in the quark sector are well approximated by

$$
\theta_{12}^q \sim \sqrt{m_{d_1} / m_{d_2}}, \quad \theta_{23}^q \sim m_{d_2} / m_{d_3}, \quad \theta_{13}^q \sim \theta_{12}^q \theta_{23}^q \sim \sqrt{m_{d_1} / m_{d_3}},
$$

where $m_{d_i}$ are the down-type quark masses. We find that $\theta_{12}^q$ and $\theta_{23}^q$ can be realized in the waterfall and cascade matrix, respectively. That motivates us to combine these matrices such that all mixing angles are appropriately obtained. It can be achieved by a hybrid form of mass matrix given in (1.4). The induced mass eigenvalues and mixing angles from the H.C. matrix are shown in Tab. 2. It is remarked that while the structure of the 2nd–3rd
| hybrid cascade | mass eigenvalues | $m_1 : m_2 : m_3 \sim \delta^2 / \lambda : \lambda : 1$ |
| mass eigenvalues | mixing angles | $\theta_{12} \sim \delta / \lambda, \theta_{23} \sim \lambda, \theta_{13} \sim \delta$ |
|                  |               | $(\theta_{12} \sim \sqrt{m_1 / m_2}, \theta_{23} \sim m_2 / m_3, \theta_{13} \sim \sqrt{m_1 m_2 / m_3})$ |

Table 2: Mass eigenvalues and generation mixing angles induced from the hybrid cascade mass matrix, where we replace $\delta'$ and $\lambda'$ in (1.4) with $\delta$ and $\lambda$, respectively.

| mass textures | up-type quark | $M_u$: cascade or H.C. or small mixing |
|              | down-type quark | $M_d$: H.C. |
| neutrino Dirac | $M_{\nu D}$: cascade |
| charged lepton | $M_e$: cascade or H.C. or small mixing |
| right-handed Majorana | $M_R$: cascade or H.C. or small mixing |

Table 3: Experimentally allowed mass textures for the fermion mass matrices based on cascading hierarchies.

sector is same for cascade and H.C mass matrices, the magnitude of hierarchy between mass eigenvalues of the 1st and 2nd generation in the H.C. matrix is larger than that of the cascade matrix. Since the mass hierarchy of the up-type sector is much larger than that of the down-type sector, the CKM matrix is almost determined by the structure of mass matrix for the down-type quarks. Therefore, the mass hierarchy of the down-type quark mass matrix should be taken as the H.C. form. Both cascade and H.C. forms can be taken as the mass matrix for the up-type sector because the contributions from the up-type sector to the CKM mixing angles are small compared with that from the down-type sector. Moreover, an arbitrary form of up quark mass matrix is allowed as long as collections from the matrix is enough small. It is seen that if the mass matrix for the up sector is described as the cascade form, a larger hierarchy between $\delta$ and $\lambda$ than that of the H.C. case is needed. These discussions for the cascading fermion mass matrices are summarized in Tab. 3.

3 Cascade hierarchies in $SU(5)$ GUT

In this section, we consider embedding the cascade textures into SUSY $SU(5)$ GUT and realizations in the theory. One parameter fit for the cascading hierarchies is also discussed.
3.1 Fermion masses in $SU(5)$ GUT

We give a brief review of the fermion masses in $SU(5)$ GUT before considering the cascade textures in the theory.

In the $SU(5)$ GUT model, the SM fermions belong to the following representations,

$$\bar{\psi}_i = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ \nu \end{pmatrix}_L, \quad \psi_{ij} = \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L, \quad \psi_0 = \nu_R^c, \quad (3.1)$$

where the indices $i$ and $j$ $(i, j = 1 \sim 3)$ correspond to the color ones and $i, j = 4, 5$ are the weak isospin $I_3 = +1/2$ and $-1/2$, respectively. The matter fields $\bar{\psi}_i$ and $\psi_{ij}$ are transformed as $\overline{5}$ and $10$ representations of $SU(5)$, respectively. The right-handed neutrino can be introduced as the singlet under the gauge group. The Higgs fields are assigned to $H_{45}, H_5$ and $\bar{H}_5$, which are transformed as the $45, 5$ and $\overline{5}$ representations. It is seen that a relation between mass matrices of the charged lepton and down-type quark, $M_e \simeq M_d^T$, is induced as a characteristic prediction of the $SU(5)$ GUT since the down-type quarks and charged leptons belong to the same representation. We will discuss the cascade hierarchical mass matrices in the $SU(5)$ GUT while considering such a relation in the following section.

3.2 Possible types of the cascade hierarchies

Let us argue embedding the (hybrid) cascade hierarchical mass matrices into $SU(5)$ GUT. The $SU(5)$ GUT predicts a relation between mass matrices for the down-type quark and charged lepton, $M_e \simeq M_d^T$, due to the unification of matter contents. As discussed above, since only the mass matrix of the H.C. form are allowed for $M_d$, the mass matrix for the charged lepton should also have the H.C. form. On the other hand, some hierarchical structure of the mass matrices for the up-type quark $M_u$ and the right-handed neutrino $M_R$ are allowed as long as induced mixing angles from these matrices can be treated as collections for the CKM and PMNS matrices, respectively. It is seen that the $(3,3)$ element of Yukawa matrix for $M_u$ should be of order one, $(Y_u)_{33} \sim \mathcal{O}(1)$, while $(Y_\nu)_{33} \ll 1$ are generally allowed for $Y_\nu (\ast = d, \nu, e)$, which correspond to the Yukawa matrices for the down-type quark, neutrino, and charged lepton, respectively. Therefore, we parametrize the mass matrices of the cascade or H.C. form for the fermions towards the realization of
embedding the cascading texture into $SU(5)$ GUT as

$$M_u \simeq \begin{pmatrix} 
\epsilon_u & \delta_u & \delta_u \\
\delta_u & \lambda_u & \lambda_u \\
\delta_u & \lambda_u & 1 
\end{pmatrix} v_u, \quad \text{with} \quad \begin{cases} 
|\epsilon_u| = |\delta_u| \ll |\lambda_u| \ll 1 : \text{cascade} \\
|\epsilon_u| \ll |\delta_u| \ll |\lambda_u| \ll 1 : \text{H.C.}
\end{cases}, \quad (3.2)
$$

$$M_d \simeq \begin{pmatrix} 
\epsilon_d & \delta_d & \delta_d \\
\delta_d & \lambda_d & \lambda_d \\
\delta_d & \lambda_d & 1 
\end{pmatrix} \xi_d v_d, \quad \text{with} \quad |\epsilon_d| \ll |\delta_d| \ll |\lambda_d| \ll 1 : \text{H.C.}, \quad (3.3)
$$

$$M_{\nu D} \simeq \begin{pmatrix} 
\delta_\nu & \delta_\nu & \delta_\nu \\
\delta_\nu & \lambda_\nu & \lambda_\nu \\
\delta_\nu & \lambda_\nu & 1 
\end{pmatrix} \xi_\nu v_u, \quad \text{with} \quad |\delta_\nu| \ll |\lambda_\nu| \ll 1 : \text{cascade}, \quad (3.4)
$$

$$M_e \simeq \begin{pmatrix} 
\epsilon_e & \delta_e & \delta_e \\
\delta_e & \lambda_e & \lambda_e \\
\delta_e & \lambda_e & 1 
\end{pmatrix} \xi_e v_d, \quad \text{with} \quad |\epsilon_e| \ll |\delta_e| \ll |\lambda_e| \ll 1 : \text{H.C.}, \quad (3.5)
$$

$$M_R \simeq \begin{pmatrix} 
\epsilon_R & \delta_R & \delta_R \\
\delta_R & \lambda_R & \lambda_R \\
\delta_R & \lambda_R & 1 
\end{pmatrix} M, \quad \text{with} \quad \begin{cases} 
|\epsilon_R| = |\delta_R| \ll |\lambda_R| \ll 1 : \text{cascade} \\
|\epsilon_R| \ll |\delta_R| \ll |\lambda_R| \ll 1 : \text{H.C.}
\end{cases}, \quad (3.6)
$$

where overall factor $\xi_\ast (\ast = d, \nu, e)$ is at most $O(1)$, and for the matrix elements $O(1)$ coefficients have been dropped. Vacuum expectation values of up- and down-type Higgs fields in the supersymmetric scenario are shown by $v_u$ and $v_d$. The characteristic relation from $SU(5)$ GUT, $M_e \simeq M_d^T$, is discussed in the next subsection.

### 3.3 One-parameter fit of cascade mass matrices

The cascade parameters in the quark and charged lepton sectors, $\epsilon_y, \delta_y,$ and $\lambda_y (y = u, d, e)$, can be estimated by experimental values. As shown in (3.2), (3.3) and (3.5), the mass matrix of the up-type quark can be described by either the cascade or H.C. types of hierarchies while only the mass matrix of the H.C. form is allowed for the mass matrices of the down-type and charged lepton. Typical magnitudes of cascade parameters at a low-energy regime and the GUT scale are shown in Tab. 4. The $u, d,$ and $s$ quark masses are estimations of current-quark mass in a $\overline{\text{MS}}$ scheme at a scale $\mu = 2$ GeV [7]. The $c$ and $b$ quark masses are the running masses in the scheme. The top quark mass is determined by the direct observation of top events. In the supersymmetric scenario, threshold effects arise from decoupling of the supersymmetric partner of SM particles could play an important role to determine the GUT scale fermion masses (Yukawa couplings) [8,9]. In particular, a wide region of $b$ quark mass has been considered because it strongly depends on the low-energy SUSY threshold effects. In the analysis focusing on the cascade texture, we refer to a typical GUT scale mass parameters listed in [10] where the Georgi-Jarlskog (GJ) factor [11] successfully explain down-type quark and charged lepton mass spectrum, as seen below. Here, we give some comments from the Tab. 4.
Table 4: Typical magnitude of cascade parameters at a low-energy regime and the GUT scale.

- When $M_u$ takes the cascade form, effects on the CKM mixing from the up-type quark sector are little. The CKM mixing is almost determined by the structure of down-type quark mass matrix.

- There are collections of $\mathcal{O}(10\%)$ from the up sector to the CKM mixing when $M_u$ takes the H.C. form.

- It is known that the mass ratio between the down-type quarks and charged leptons for each generation can be written as

$$\left(\frac{m_x}{m_b}, \frac{m_u}{m_s}, \frac{m_e}{m_t}\right) \sim \left(1, \frac{1}{3}, \frac{1}{3}\right)$$

Therefore, $\theta_{e,23}$ is larger than $\theta_{d,23}$ while $\theta_{e,12}$ is smaller than $\theta_{d,12}$. Throughout this paper, we introduce the GJ factor \[11\] in the charged lepton mass matrix. If the
| $M_u$: cascade | Low-energy scale | $\mathcal{O}(10^{16})$ GeV |
|----------------|------------------|-----------------------------|
| $\lambda_u$    | $0.61 \times \lambda^4$ | $0.87 \times \lambda^4$ |
| $\delta_u$     | $0.35 \times \lambda^8$  | $0.85 \times \lambda^8$ |
| $M_u$: H.C.     |                  |                             |
| $\lambda_u$    | $0.61 \times \lambda^4$ | $0.87 \times \lambda^4$ |
| $\delta_u$     | $0.46 \times \lambda^6$  | $0.86 \times \lambda^6$ |
| $M_d$: H.C.     |                  |                             |
| $\lambda_d$    | $0.47 \times \lambda^2$ | $0.35 \times \lambda^2$ |
| $\delta_d$     | $0.46 \times \lambda^3$  | $0.35 \times \lambda^3$ |

Table 5: Typical magnitudes of cascade parameters in an unit of the Cabibbo angle, $\sin \theta_c \simeq \lambda \simeq 0.227$.

The (2,2) element of the Yukawa matrices is generated by the operator $5 \cdot 10 \cdot H_{45}$ with the standard model (SM) Higgs fields contained in the 45-dimensional representation $H_{45}$, where 5 and 10 stand for matters described by the 5- and 10-dimensional representations, respectively. The operator leads to the well-known relation $m_\mu/m_\tau = -3$, which is favored by the experimental data. It can be understood from the fact that the 45-dimensional representation is traceless and the factor of $-3$ for the charged leptons, and thus, it has to compensate the color factor of 3 for the quarks. Hereafter we express the mass matrices of the down-type quark and charged lepton replaced with (3.5) and (3.3) as

$$M_e \simeq \begin{pmatrix} \epsilon_d & \delta_d & \delta_d \\ \delta_d & -3\lambda_d & \lambda_d \\ \delta_d & \lambda_d & 1 \end{pmatrix} \xi_d v_d, \quad M_d \simeq \begin{pmatrix} \epsilon_d & \delta_d & \delta_d \\ \delta_d & \lambda_d & \lambda_d \\ \delta_d & \lambda_d & 1 \end{pmatrix} \xi_d v_d,$$

(3.8)

by using the same cascade parameters $\epsilon_d$, $\delta_d$, and $\lambda_d$. Furthermore, the fact $m_\tau/m_b \sim 1$ leads to $\xi_e \sim \xi_d$.

There is no relation between the hierarchies of up- and down-type quarks (charged lepton) in the context of the $SU(5)$ GUT. However it is natural to expect that such hierarchies are originated from a symmetry and/or some dynamics in a high energy regime rather than solely determined by the magnitudes of Yukawa coupling constants. Here we express the cascade parameters shown in Tab. 4 by an unit of the Cabibbo angle, $\sin \theta_c \simeq \lambda \simeq 0.227$ in Tab. 5. Here we have taken the $\mathcal{O}(1)$ coefficient of (3,3) element in each matrix as one. If the coefficient is taken as $a \sim \mathcal{O}(1)$, other coefficients in the same matrix are multiplied by a factor $a$.

We note that the since $\xi_d$ determines a ratio between (3,3) element of Yukawa matrices for up- and down-type quarks, it is correlated with the $\tan \beta$, which is the ratio between vacuum expectation values of up- and down-type Higgs fields in the supersymmetric sce-
nario. For a suppressed value of $\xi_d$ leads to a small $\tan \beta$ as follows,

$$\tan \beta \simeq \begin{cases} \frac{v_u}{v_d} \sim \mathcal{O}(50) & \text{for } \xi_d \sim \lambda^0 \text{ [large]} \\ \lambda v_u/v_d \sim \mathcal{O}(10) & \text{for } \xi_d \sim \lambda^1 \text{ [moderate]} \\ \lambda^2 v_u/v_d \sim \mathcal{O}(1) & \text{for } \xi_d \sim \lambda^2 \text{ [small]} \end{cases}.$$  (3.9)

It is also remarked that the magnitude of hierarchy for the 1st generation in the up-type quark sector should be large for the cascade form of $M_u$, $\delta_u \sim \mathcal{O}(\lambda^8)$, compared to the down-type quark sector, $\delta_d \sim \mathcal{O}(\lambda^3)$. On the other hand, H.C. form of $M_u$ has milder hierarchy than cascade one, and similar stream of hierarchy flow to $M_d$. From a viewpoint of some flavor model, like FN model, the different hierarchies could be accompanied by properties of 10 and $\bar{5}$ matter fields, since mass matrices $M_u$ and $M_d$ are accompanied by $10 \cdot 10$ and $10 \cdot \bar{5}$ matter complings, respectively. To differ the hierarchy of $M_u$ from that of $M_d$ drastically, a specific (unnatural) mechanism for the difference is expected. The realization may be possible but we focus on the case of the H.C. mass matrix for $M_u$ in the following discussions.

Finally, we can take cascading textures at GUT scale as

$$M_u \simeq \begin{pmatrix} \lambda^{k_u+6} & \lambda^6 & \lambda^6 \\ \lambda^6 & \lambda^4 & \lambda^4 \\ \lambda^6 & \lambda^4 & 1 \end{pmatrix} v_u,$$

where $k_u \geq 2$ and $k_d \geq 1$ are needed to obtain suitable mass eigenvalues after diagonalizing these matrices. It should be remembered that $M_e \simeq M_d^T$ but the additional GJ factor $-3$ is multiplied to the $(2,2)$ element of $M_e$ as discussed in (3.8).

### 3.4 Neutrino sector

Next, we discuss the neutrino mass matrix with cascading form. The neutrino Dirac mass matrix must be taken as the cascade form to lead to a nearly tri-bimaximal generation mixing. In order to realize the tri-bimaximal pattern, mixing angles among the right-handed neutrinos should be small. The Majorana mass matrix of the right-handed neutrinos $M_R$ has been taken to be diagonal in $[4]$. In the case, cascade parameters are constrained as

$$\left| \frac{\delta_{\nu}}{\lambda_{\nu}} \right|^2 \ll \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \simeq 3.19 \times 10^{-2} < \lambda^2,$$  (3.12)
to conserve the tri-bimaximal mixing at the leading order. Here \( \Delta m_{21}^2 \equiv |m_2|^2 - |m_1|^2 \) and \( \Delta m_{31}^2 \equiv |m_3|^2 - |m_1|^2 \) are the mass squared difference of light neutrinos and the current experimental data at the 3\( \sigma \) level [1] are

\[
\begin{align*}
\Delta m_{21}^2 &= (7.695 \pm 0.645) \times 10^{-5} \text{ eV}^2, \\
|\Delta m_{31}^2| &= 2.40^{+0.12}_{-0.11} \times 10^{-3} \text{ eV}^2.
\end{align*}
\]

The equation (3.12) constrains the magnitude of the cascade hierarchies for neutrino Dirac mass matrix to be \( |\delta\nu/\lambda\nu| < \lambda \). If we assume that the hierarchy in the Dirac mass matrix is originated from the \( \lambda \), we can reparametrize the cascade matrix (3.4) as

\[
M_{\nu D} \simeq \begin{pmatrix} 
\lambda^{d_1} & \lambda^{d_2} & \lambda^{d_1} \\
\lambda^{d_1} & \lambda^{d_2} & -\lambda^{d_2} \\
\lambda^{d_1} & -\lambda^{d_2} & 1 
\end{pmatrix} \lambda^{d_1} \nu_u \
\text{with } d_1 > d_2 \geq 1 \text{ and } d \geq 0,
\]

where \( d_1 \) and \( d_2 \) determine the magnitudes of hierarchy, and an opposite sign between (2,2) and (2,3) elements is experimentally required as commented in [4]. Here we constrain \( d \), \( d_1 \), and \( d_2 \) to be integer. The considerations about constraints on the magnitude of cascade hierarchy of the neutrino Dirac mass matrix in this parametrization will be discussed in the following subsections.

### 3.4.1 Diagonal \( M_R \) case

First, let us consider a case of diagonal Majorana mass matrix of the right-handed neutrinos, \( M_R \). We take a diagonal form of Majorana mass matrix as,

\[
M_R \simeq \begin{pmatrix} 
\lambda^{x_1} & 0 & 0 \\
0 & \lambda^{x_2} & 0 \\
0 & 0 & 1 
\end{pmatrix} M \text{ with } x_1 \geq x_2 \geq 0,
\]

where \( M \) is a mass scale of the heaviest right-handed neutrino. After the seesaw mechanism, one obtains the Majorana mass matrix of light neutrinos in low-energy theory,

\[
M_\nu \simeq \left[ \begin{pmatrix} 
\lambda^{2d_1} & -\lambda^{d_1+d_2} & \lambda^{d_1} \\
-\lambda^{d_1+d_2} & \lambda^{2d_2} & -\lambda^{d_2} \\
\lambda^{d_1} & -\lambda^{d_2} & 1 
\end{pmatrix} + \lambda^{2d_1-x_1} \begin{pmatrix} 
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 
\end{pmatrix} \\
+ \lambda^{2d_2-x_2} \begin{pmatrix} 
\lambda^{2(d_1-d_2)} & \lambda^{d_1-d_2} & -\lambda^{d_1-d_2} \\
\lambda^{d_1-d_2} & 1 & -1 \\
\lambda^{d_1-d_2} & -1 & 1 
\end{pmatrix} \right] \frac{\lambda^{2d_1+2}}{M}.
\]

*As commented below, if the origin of hierarchy in the neutrino sector is assumed to be same as one in the quark sector, the cascade parameter in the neutrino sector could also be described by a Cabibbo unit. Such a situation is considered to be natural in a realization of the cascade and H.C. textures due to the FN mechanism with an abelian flavor symmetry such as \( U(1) \).

†This case has been proposed in [4]. Here we give a brief review of the work and constraints on cascade parameters in our notation.
Operating the $V_{TB}$ to $M_\nu$ as $V_{TB}^\dagger M_\nu V_{TB}$, the resultant neutrino mass matrix becomes

$$ M \equiv V_{TB}^\dagger M_\nu V_{TB} $$

$$ \simeq \left[ \frac{1}{6} \begin{pmatrix} c_1^2 & -\sqrt{2} c_1 c_2 & -\sqrt{3} c_1 c_2 \\ -\sqrt{2} c_1 c_2 & 2 c_2^2 & \sqrt{3} c_2 c_+ \\ -\sqrt{3} c_1 c_+ & \sqrt{3} c_2 c_+ & 3 c_+^2 \end{pmatrix} \right] + 3 \lambda^{2d_1-x_1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} 
+ \frac{\lambda^{2d_2-x_2}}{3} \left( \begin{pmatrix} 2 \lambda^{2(d_1-d_2)} & \sqrt{2} \lambda^{2(d_1-d_2)} & -2 \sqrt{3} \lambda^{d_1-d_2} \\ \sqrt{2} \lambda^{2(d_1-d_2)} & \lambda^{2(d_1-d_2)} & -6 \lambda^{d_1-d_2} \end{pmatrix} \right) \frac{\lambda^{2d_u^2}}{M}, \quad (3.18) $$

where

$$ c_1 \equiv 1 - \lambda^{d_2} - 2 \lambda^{d_1}, \quad (3.19) $$
$$ c_2 \equiv 1 - \lambda^{d_2} + \lambda^{d_1}, \quad (3.20) $$
$$ c_3 \equiv 1 - \lambda^{d_2} + 4 \lambda^{d_1}, \quad (3.21) $$
$$ c_+ \equiv 1 + \lambda^{d_2}. \quad (3.22) $$

The cascade form of the neutrino mass matrix requires the normal hierarchy of light neutrino mass spectrum, and mass eigenvalues are estimated as

$$ m_1 \simeq \frac{\lambda^{2d_u^2}}{6M} \equiv \bar{m}_1, \quad (3.23) $$
$$ m_2 \simeq \left( 3 \lambda^{2d_1-x_1} + \frac{1}{3} \right) \frac{\lambda^{2d_u^2}}{M} \equiv \bar{m}_2 + 2 \bar{m}_1, \quad (3.24) $$
$$ m_3 \simeq \left( 2 \lambda^{2d_2-x_2} + \frac{1}{3} \right) \frac{\lambda^{2d_u^2}}{M} \equiv \bar{m}_3 + 3 \bar{m}_1, \quad (3.25) $$

including the leading order corrections of $\bar{m}_1$.

Towards a profound understanding the hierarchical structure of the mass matrix and constraining on the cascade parameters, we rewrite the effective neutrino mass matrix (3.17) as

$$ M_\nu \simeq \frac{\lambda^{2d_u^2}}{M} \left( \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \right) + \frac{\lambda^{2(d_1+d)-x_1} v_u^2}{M} \left( \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) 
+ \frac{\lambda^{2d_2+d-d} v_u^2}{M} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right) 
+ \frac{\lambda^{2d_u^2}}{M} \left( \begin{pmatrix} -4 + \lambda^{2d_1} & 2 - \lambda^{d_1+d_2} & 2 + \lambda^{d_1} \\ 2 - \lambda^{d_1+d_2} & -1 + \lambda^{2d_2} & -1 - \lambda^{d_2} \\ 2 + \lambda^{d_1} & -1 - \lambda^{d_2} & 0 \end{pmatrix} \right) 
+ \frac{\lambda^{2(d_1+d)-x_2} v_u^2}{M} \left( \begin{pmatrix} \lambda^{2(d_1-d_2)} & \lambda^{d_1-d_2} & -\lambda^{d_1-d_2} \\ \lambda^{d_1-d_2} & 0 & 0 \\ \lambda^{d_1-d_2} & 0 & 0 \end{pmatrix} \right). \quad (3.26) $$
In this mass matrix, if the terms in the first and second lines give dominant contribution, the tri-bimaximal mixing can be realized at the leading order. In order that the term in the third line of (3.26) does not spoil the structures in the first line, \( m_1 \ll m_2, m_3 \) is required. Therefore, the neutrino mass spectrum in the cascade model should be the normal mass hierarchy, as stated. Then it is well approximated that \( m_2 \simeq \sqrt{\Delta m_{21}^2} \) and \( m_3 \simeq \sqrt{|\Delta m_{31}^2|} \). Here one can obtain three constraints and/or relations for the cascade parameters. The first one comes from the hierarchy \( m_1 \ll m_2 \). This means that \(-2d_1 + x_1 \geq 1\) for the parameters by using the (3.23) and (3.24). The second one is derived from the current experiments. Since the experimental data suggests that \( r \equiv \sqrt{\Delta m_{21}^2}/|\Delta m_{31}^2| \simeq 0.18\), one obtains a relation among the cascade parameter as \( 2(d_1 - d_2) - (x_1 - x_2) = 1 \) or 2 from (3.24) and (3.25), where the fact \( \lambda \sim r \) is used. We also find that there exists a relation among the cascade parameters, light and heavy neutrino mass scales, that is, we can have a relation, \( M \simeq \lambda^{2(d_1 + d_2) - x_2} v_u^2/\sqrt{|\Delta m_{31}^2|} \), from \( m_3 \simeq \sqrt{|\Delta m_{31}^2|} \). Finally, we should consider the effects from the term in the last line of (3.26). In order that the term does not spoil the democratic structure in the first line, the hierarchy \( m_2 \gg m_3 \lambda^{d_1 - d_2} \) is needed. This constraint can be expressed by the cascade parameters as \( d_1 - d_2 - (x_1 - x_2) \leq -1 \). We conclude the above constraints and relations for the cascade parameters and physical quantities as

\begin{align}
(i) \quad & m_1 \ll m_2 \quad \Rightarrow \quad -2d_1 + x_1 \geq 1, \\
(ii) \quad & m_2/m_3 \simeq r \simeq 0.18 \quad \Rightarrow \quad 2(d_1 - d_2) - (x_1 - x_2) = 1 \text{ or } 2, \\
(iii) \quad & m_3 \simeq \sqrt{|\Delta m_{31}^2|} \quad \Rightarrow \quad M \simeq \lambda^{2(d_1 + d_2) - x_2} v_u^2/\sqrt{|\Delta m_{31}^2|}, \\
(iv) \quad & m_2 \gg m_3 \lambda^{d_1 - d_2} \quad \Rightarrow \quad d_1 - d_2 - (x_1 - x_2) \leq -1. 
\end{align}

(3.27)

They restrict the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the diagonal form to textures shown in Tabs. 6 and 7. It is seen that the minimal model for the neutrino mass matrices is described by \((d, d_1, d_2, x_1, x_2) = (0, 3, 1, 7, 4)\) given in Tab. 6. In this case, mass scale of the heaviest right-handed neutrino should be the order of the unification scale, namely \((M_1, M_2, M_3) \sim (10^{12}, 10^{14}, 10^{16}) \text{ GeV}\).

Let us comment on the mixing angles predicted by this cascade model. Even if the right-handed neutrino mass matrix is diagonal, the mixing angles deviate from the exact tri-bimaximal pattern. We discuss about these deviations predicted from the cascade model. The neutrino mass matrix after operating the tri-bimaximal mixing matrix (3.18) can be rewritten as,

\[
M_0 \simeq \begin{pmatrix}
\bar{m}_1 + \frac{\lambda^{2(d_1 - d_2)}}{3} \bar{m}_3 & -\sqrt{2} \bar{m}_1 + \frac{\sqrt{2} \lambda^{2(d_1 - d_2)}}{6} \bar{m}_3 & -\sqrt{3} \bar{m}_1 - \frac{\lambda^{d_1 - d_2}}{\sqrt{3}} \bar{m}_3 \\
-\sqrt{2} \bar{m}_1 + \frac{\sqrt{2} \lambda^{2(d_1 - d_2)}}{6} \bar{m}_3 & \bar{m}_2 + \frac{\lambda^{2(d_1 - d_2)}}{3} \bar{m}_3 & \bar{m}_3 \\
-\sqrt{3} \bar{m}_1 - \frac{\lambda^{d_1 - d_2}}{\sqrt{3}} \bar{m}_3 & \sqrt{6} \bar{m}_1 - \frac{\lambda^{d_1 - d_2}}{\sqrt{6}} \bar{m}_3 & \bar{m}_3 + 3 \bar{m}_1
\end{pmatrix}, \tag{3.28}
\]

up to leading order in each term of (3.18). Note that if the cascade model realizes the exact tri-bimaximal mixing, this mass matrix should be diagonal. However, finite off-diagonal
| $d_1$ | $d_2$ | $x_1$ | $x_2$ | $M_{\nu D}/(\lambda^d v_u)$ | $M_R/M$ |
|-------|-------|-------|-------|-----------------------------|---------|
| 3     | 1     | 7     | 4     | $\begin{pmatrix} \lambda^3 & \lambda & -\lambda \\ \lambda^3 & \lambda & -\lambda \\ \lambda^3 & -\lambda & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^6 & 0 \\ 0 & \lambda^5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 3     | 1     | 8     | 5     | $\begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda & -\lambda \\ \lambda^3 & -\lambda & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^8 & 0 \\ 0 & \lambda^5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 3     | 1     | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 4     | 1     | 9     | 4     | $\begin{pmatrix} \lambda^4 & \lambda & \lambda^2 \\ \lambda^4 & \lambda & -\lambda \\ \lambda^4 & -\lambda & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^9 & 0 \\ 0 & \lambda^6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 4     | 1     | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 4     | 2     | 9     | 6     | $\begin{pmatrix} \lambda^4 & \lambda^4 & \lambda \\ \lambda & \lambda^2 & -\lambda^2 \\ \lambda & -\lambda^2 & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^9 & 0 \\ 0 & \lambda^6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 6: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the diagonal form. The matrices are constrained by the experimental data of the neutrino masses with the condition $2(d_1 - d_2) - (x_1 - x_2) = 1$.

| $d_1$ | $d_2$ | $x_1$ | $x_2$ | $M_{\nu D}/(\lambda^d v_u)$ | $M_R/M$ |
|-------|-------|-------|-------|-----------------------------|---------|
| 4     | 1     | 9     | 5     | $\begin{pmatrix} \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda & -\lambda \\ \lambda^4 & -\lambda & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^{10} & 0 \\ 0 & \lambda^6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 4     | 1     | 10    | 6     | $\begin{pmatrix} \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda & -\lambda \\ \lambda^4 & -\lambda & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^{10} & 0 \\ 0 & \lambda^6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 4     | 1     | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 5     | 1     | 11    | 5     | $\begin{pmatrix} \lambda^5 & \lambda & \lambda^5 \\ \lambda^5 & \lambda & -\lambda \\ \lambda^5 & -\lambda & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^{11} & 0 \\ 0 & \lambda^5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 5     | 1     | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 5     | 2     | 11    | 7     | $\begin{pmatrix} \lambda^5 & \lambda^5 & \lambda^5 \\ \lambda & \lambda^2 & -\lambda^2 \\ \lambda & -\lambda^2 & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^{11} & 0 \\ 0 & \lambda^7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 7: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the diagonal form. The matrices are constrained by the experimental data of the neutrino masses with the condition $2(d_1 - d_2) - (x_1 - x_2) = 2$. 
elements give deviations from the tri-bimaximal mixing. Let us estimate these deviations.

The nearly diagonal neutrino mass matrix (3.28) can be diagonal by the following mixing matrix up to the next leading order,

\[
V^{(1)} \simeq \begin{pmatrix}
1 & \theta_{12}^{(1)} & \theta_{13}^{(1)} \\
-\theta_{12}^{(1)} & 1 & \theta_{23}^{(1)} \\
-\theta_{13}^{(1)} & -\theta_{23}^{(1)} & 1
\end{pmatrix},
\]

(3.29)

where

\[
\theta_{12}^{(1)} \simeq -\frac{\sqrt{2} \bar{m}_1}{\bar{m}_2},
\]

(3.30)

\[
\theta_{23}^{(1)} \simeq \frac{\sqrt{6} \bar{m}_1}{\bar{m}_3 - \bar{m}_2} - \frac{\lambda^{d_1 - d_2} \bar{m}_3}{\sqrt{6} (\bar{m}_3 - \bar{m}_2)},
\]

(3.31)

\[
\theta_{13}^{(1)} \simeq -\frac{\sqrt{3} \bar{m}_1}{\bar{m}_3} - \frac{\lambda^{d_1 - d_2}}{\sqrt{3}}.
\]

(3.32)

Therefore, the resultant PMNS matrix including these collection from the cascade structure (here we dropped correction from charged lepton sector),

\[
V_{\text{PMNS}} \simeq V_{\text{TB}} V^{(1)} P_M
\]

(3.33)

\[
= \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & \theta_{12}^{(1)} & \theta_{13}^{(1)} \\
-\theta_{12}^{(1)} & 1 & \theta_{23}^{(1)} \\
-\theta_{13}^{(1)} & -\theta_{23}^{(1)} & 1 \end{pmatrix} P_M,
\]

(3.34)

gives

\[
\sin^2 \theta_{12} \simeq \left| \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \theta_{12}^{(1)} \right|^2
\]

(3.35)

\[
\simeq \left| \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3} \bar{m}_2} \right|^2,
\]

(3.36)

\[
\sin^2 \theta_{23} \simeq \left| -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \theta_{13}^{(1)} + \frac{1}{\sqrt{3}} \theta_{23}^{(1)} \right|^2
\]

(3.37)

\[
\simeq -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \bar{m}_1 (3 \bar{m}_3 - \bar{m}_2) - \frac{\lambda^{d_1 - d_2}}{3 \sqrt{2} (\bar{m}_3 - \bar{m}_2)} \bar{m}_2 \right|^2,
\]

(3.38)

\[
\sin^2 \theta_{13} \simeq \left| \frac{2}{\sqrt{6}} \theta_{13}^{(1)} + \frac{1}{\sqrt{3}} \theta_{23}^{(1)} \right|^2
\]

(3.39)

\[
\simeq -\frac{\lambda^{d_1 - d_2}}{\sqrt{2} (\bar{m}_3 - \bar{m}_2)} \bar{m}_3 - \frac{2}{3} \bar{m}_2 + \frac{\sqrt{2} \bar{m}_1 \bar{m}_2}{\bar{m}_3 (\bar{m}_3 - \bar{m}_2)} \right|^2,
\]

(3.40)

where \(P_M\) is a diagonal phase matrix. It is seen that \(\bar{m}_2\) and \(\bar{m}_3\) are well approximated by \(\sqrt{\Delta m_{21}^2}\) and \(\sqrt{|\Delta m_{31}^2|}\) respectively, if \(\bar{m}_1\) is sufficiently tiny.
3.4.2 Non-diagonal $M_R$ case

Next, let us consider a case of non-diagonal $M_R$, which is generally allowed in the context of the cascade textures. Especially, corrections from a non-diagonal $M_R$ to the tri-bimaximal mixing angles are estimated. Some constraints for the corrections, equivalently the structure of $M_R$, are also presented.

The neutrino Dirac mass matrix is taken as the cascade form given in (3.15). We define the diagonalized mass matrix of the right-handed neutrino, $D_R$, as

$$D_R \equiv U_{\nu R}^T M_R U_{\nu R} \equiv \begin{pmatrix} \lambda x_1 & 0 & 0 \\ 0 & \lambda x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad x_1 \geq x_2 \geq 0,$$

(3.41)

where $M_R$ is non-diagonal mass matrix for the right-handed neutrinos but mixing among each generation is assumed to be small in order to conserve the tri-bimaximal mixing at the leading order. If the mixing angles among each generation of the right-handed neutrino are enough small, the $U_{\nu R}$ can be written as,

$$U_{\nu R} \approx \begin{pmatrix} 1 & \theta_{R,12} & \theta_{R,13} \\ -\theta_{R,12} & 1 & \theta_{R,23} \\ -\theta_{R,13} & -\theta_{R,23} & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & \lambda^{q_{12}} & \lambda^{q_{13}} \\ -\lambda^{q_{12}} & 1 & \lambda^{q_{23}} \\ -\lambda^{q_{13}} & -\lambda^{q_{23}} & 1 \end{pmatrix} \quad \text{with} \quad q_{ij} \geq 1,$$

(3.42)

up to the first order of $\theta_{R,ij} \ (i,j = 1 \sim 3)$.

After the seesaw mechanism, one obtains the Majorana mass matrix of light neutrinos in low-energy theory as,

$$M_\nu \simeq M_{\nu D}^T M_R^{-1} M_{\nu D} \simeq M_{\nu D}^T U_{\nu R} D_R^{-1} U_{\nu R}^T M_{\nu D}.$$  

(3.43)

First, we write down $M_R^{-1}$ as

$$M_R^{-1} \simeq U_{\nu R} D_R^{-1} U_{\nu R}^T \approx \begin{pmatrix} \theta_{R,12}^2 h_1 + \theta_{R,13}^2 h_2 + \theta_{R,12}^2 \theta_{R,13} h_3 & \theta_{R,12} h_21 + \theta_{R,23} \theta_{R,13} h_3 & \theta_{R,12} h_31 - \theta_{R,12} \theta_{R,23} h_2 \\ \theta_{R,12} h_21 + \theta_{R,23} \theta_{R,13} h_3 & \theta_{R,12}^2 h_1 + \theta_{R,23}^2 h_3 & \theta_{R,12} h_31 + \theta_{R,12} \theta_{R,23} h_1 \\ \theta_{R,13} h_31 - \theta_{R,12} \theta_{R,23} h_2 & \theta_{R,23} h_31 + \theta_{R,12} \theta_{R,23} h_1 & \theta_{R,23}^2 h_1 + \theta_{R,23}^2 h_2 \end{pmatrix},$$

(3.44)

where $h_1 \equiv (\lambda x_1 M)^{-1}$, $h_2 \equiv (\lambda x_2 M)^{-1}$, $h_3 \equiv M^{-1}$, and $h_{ij} \equiv h_i - h_j$. After the seesaw
mechanism, the effective mass matrix of the light neutrinos is given by

\[
M_{\nu} \simeq M_{\nu D}^T M_R^{-1} M_{\nu D}
\]

\[
\simeq \lambda^{2d_1} (M^{-1}_R)_{11} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \lambda^{2d_2} (M^{-1}_R)_{22} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \lambda^{2d_2} (M^{-1}_R)_{22} \begin{pmatrix} \lambda^{d_1-d_2} & \lambda^{d_1-d_2} & \lambda^{d_1-d_2} \\ \lambda^{d_1-d_2} & 0 & 0 \\ \lambda^{d_1-d_2} & 0 & 0 \end{pmatrix}
\]

\[
+ (M^{-1}_R)_{33} \begin{pmatrix} \lambda^{d_1} & -\lambda^{d_1+d_2} & \lambda^{d_1} \\ -\lambda^{d_1+d_2} & \lambda^{d_1+d_2} & -\lambda^{d_2} \\ \lambda^{d_1} & -\lambda^{d_2} & 1 \end{pmatrix} + (M^{-1}_R)_{23} \begin{pmatrix} 2\lambda^{2d_1} & 0 & \lambda^{d_1} (1 - \lambda^{d_2}) \\ 0 & -2\lambda^{2d_2} & \lambda^{d_2} (1 + \lambda^{d_2}) \\ \lambda^{d_1} (1 - \lambda^{d_2}) & \lambda^{d_2} (1 + \lambda^{d_2}) & -2\lambda^{d_2} \end{pmatrix}
\]

\[
+ \lambda^{d_1} (M^{-1}_R)_{12} \begin{pmatrix} 2 & \lambda^{d_1} + \lambda^{d_2} & \lambda^{d_1} - \lambda^{d_2} \\ \lambda^{d_1} + \lambda^{d_2} & 2\lambda^{d_2} & 0 \\ \lambda^{d_1} - \lambda^{d_2} & 0 & -2\lambda^{d_2} \end{pmatrix} + \lambda^{d_1} (M^{-1}_R)_{13} \begin{pmatrix} 2\lambda^{d_1} & \lambda^{d_1} - \lambda^{d_2} & \lambda^{d_1} + 1 \\ \lambda^{d_1} - \lambda^{d_2} & -2\lambda^{d_2} & 1 - \lambda^{d_2} \\ \lambda^{d_1} + 1 & 1 - \lambda^{d_2} & 2 \end{pmatrix} \lambda^{2d} v_u^2. \tag{3.45}
\]

Note that if the \(M_R\) is diagonal, which means \(\theta_{R,ij} = 0\), \((M^{-1}_R)_{kl} (k \neq l)\) are vanishing, and thus, the neutrino mass matrix (3.45) results in (3.17).
Operating the $V_{TB}$ to $M_\nu$ as $V_{TB}^T M_\nu V_{TB}$, the neutrino mass matrix becomes

$$\mathcal{M} \equiv V_{TB}^T M_\nu V_{TB}$$

$$\approx 3\lambda^{2d_1} (M^{-1}_R)_{11} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\lambda^{2d_2} (M^{-1}_R)_{22} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ \frac{\lambda^{2d_2} (M^{-1}_R)_{22}}{3} \begin{pmatrix} 2\lambda^2(d_1-d_2) & \sqrt{2}\lambda^2(d_1-d_2) & -2\sqrt{3}\lambda^{d_1-d_2} \\ \sqrt{2}\lambda^2(d_1-d_2) & \lambda^2(d_1-d_2) & -\sqrt{6}\lambda^{d_1-d_2} \\ -2\sqrt{3}\lambda^{d_1-d_2} & -\sqrt{6}\lambda^{d_1-d_2} & 0 \end{pmatrix}$$

$$+ \frac{(M^{-1}_R)_{33}}{6} \begin{pmatrix} c_1^2 & -\sqrt{2}c_1c_2 & 2c_2^2 \\ -\sqrt{2}c_1c_2 & \sqrt{6}c_2c_+ & -\sqrt{3}c_1c_+ \\ -\sqrt{3}c_1c_+ & -\sqrt{6}c_2c_+ & 3c_+^2 \end{pmatrix}$$

$$+ \frac{(M^{-1}_R)_{23}^2}{3\sqrt{2}} \begin{pmatrix} -2\sqrt{2}c_1\lambda^{d_1} & c_3\lambda^{d_1} & \sqrt{6}c_-(\lambda^{d_2} + \lambda^{d_1}) \\ c_3\lambda^{d_1} & 2\sqrt{2}c_2\lambda^{d_1} & -\sqrt{3}c_-(2\lambda^{d_2} - \lambda^{d_1}) \\ \sqrt{6}c_-(\lambda^{d_2} + \lambda^{d_1}) & -\sqrt{3}c_-(2\lambda^{d_2} - \lambda^{d_1}) & -6\sqrt{2}c_+\lambda^{d_2} \end{pmatrix}$$

$$+ \lambda^{d_1} (M^{-1}_R)_{12} \begin{pmatrix} 0 & \sqrt{2}\lambda^{d_1} & 0 \\ \sqrt{2}\lambda^{d_1} & 2\lambda^{d_1} & -\sqrt{6}\lambda^{d_2} \\ 0 & -\sqrt{6}\lambda^{d_2} & 0 \end{pmatrix}$$

$$+ \frac{\lambda^{d_1} (M^{-1}_R)_{13}}{\sqrt{2}} \begin{pmatrix} 0 & -c_1 & 0 \\ -c_1 & 2\sqrt{2}c_2 & \sqrt{3}c_+ \\ 0 & \sqrt{3}c_+ & 0 \end{pmatrix} \lambda^{2d_2} v_u^2, \quad (3.46)$$

where

$$c_3 \equiv 1 - \lambda^{d_2} + 4\lambda^{d_1}, \quad (3.47)$$

$$c_- \equiv 1 - \lambda^{d_2}. \quad (3.48)$$

This mass matrix can be written as

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_{\text{off}} \equiv \mathcal{M}_0 + \left( \begin{array}{ccc} m_{1R}^R & m_{12}^R & m_{13}^R \\ m_{12}^R & m_{2R}^R & m_{23}^R \\ m_{13}^R & m_{23}^R & m_3^R \end{array} \right), \quad (3.49)$$

where $\mathcal{M}_0$ comes from the diagonal elements of $M_R$, which is given by (3.28). The matrix $\mathcal{M}_{\text{off}}$ includes effects from off-diagonal elements of $M_R$, whose elements are described by $\theta_{R,ij}$. If the mixing angles among the right-handed neutrinos are small, the neutrino mass matrix $\mathcal{M}$ should be almost diagonal. This means that the off-diagonal elements of $\mathcal{M}$ give collections to the tri-bimaximal generation mixing. These collections should be enough small to explain the MNS matrix, because the nearly tri-bimaximal structure can be realized by the neutrino Dirac mass matrix of the cascade form with the seesaw mechanism in the case of diagonal $M_R$. If a structure of $M_R$ leads to relatively large collections, then unnatural cancellations are needed to predict experimentally favored generation mixing angles of the lepton sector in the context of cascade neutrino Dirac mass matrix. Therefore,
we focus on a case that the collections from the off-diagonal elements of $M_R$ are enough small not to spoil the nearly tri-bimaximal mixing induced from the cascade neutrino Dirac mass matrix and discuss about the magnitude of collections in the case. This means that the structure of resultant neutrino mass matrix given in (3.46) should not be drastically different from the (3.28) in the diagonal $M_R$ case. Thus the magnitude of neutrino mass eigenvalues given in (3.23) $\sim$ (3.25) and the constraints (3.27) must be held at the leading order even in the case of non-diagonal $M_R$. These discussions gives the following neutrino mass eigenvalues up to the next leading order,

\[ m_1 \simeq \frac{\lambda^{2d_1}v^2_u}{6M} + m_1^R = \bar{m}_1 + m_1^R, \]  
\[ m_2 \simeq \left(3\lambda^{2d_1-x_1} + \frac{1}{3}\right) \frac{\lambda^{2d_2}v^2_u}{M} + m_2^R = \bar{m}_2 + m_2^R + 2\bar{m}_1, \]  
\[ m_3 \simeq \left(2\lambda^{2d_2-x_2} + \frac{1}{2}\right) \frac{\lambda^{2d_1}v^2_u}{M} + m_3^R = \bar{m}_3 + m_3^R + 3\bar{m}_1, \]

where $m_i^R$ include effects from the off-diagonal element of $M_R$ described as

\[ m_1^R \equiv \frac{\lambda^{2d_1}v^2_u}{6M} \lambda^{-x_1}\theta_{R,23}^2, \]  
\[ m_2^R \equiv -\frac{2\lambda^{2d_2}v^2_u}{3M} \lambda^{d_1}(3\lambda^{-x_1}\theta_{R,13} + \lambda^{-x_2}\theta_{R,23}^2), \]  
\[ m_3^R \equiv \frac{\lambda^{2d_1}v^2_u}{2M} \lambda^{-x_1}(2\lambda^{d_2}\theta_{R,12} - \theta_{R,13})^2. \]

Typical textures of non-diagonal $M_R$ are given in Tabs. 8 and 9. All the presented textures of $M_R$ preserve tri-bimaximal neutrino mixing at the leading order with relatively small numbers of $(d_1, d_2, x_1, x_2)$.

The collections to the tri-bimaximal mixing are estimated in the perturbative method as in the diagonal $M_R$ case,

\[ \theta_{12}^{(1)} \simeq -\sqrt{2}\bar{m}_1 + m_{12}^R, \]  
\[ \theta_{23}^{(1)} \simeq \sqrt{6}\bar{m}_1 - \frac{\lambda^{d_1-d_2}}{\sqrt{6}}\bar{m}_3 + m_{23}^R, \]  
\[ \theta_{13}^{(1)} \simeq -\sqrt{3}\bar{m}_1 - \frac{\lambda^{d_1-d_2}}{\sqrt{3}}\bar{m}_3 + m_{13}^R. \]

\[ \text{‡Detailed discussions is given in the Appendix.} \]
Table 8: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one constrained by the experimentally observed values of the neutrino masses with the condition $2(d_1 - d_2) - (x_1 - x_2) = 1$.

| $d_1$ | $d_2$ | $x_1$ | $x_2$ | $M_{\nu D}/(\lambda^d v_u)$ | $M_R/M$ |
|-------|-------|-------|-------|-----------------------------|---------|
| 3     | 1     | 7     | 4     | $(\lambda^3  \lambda - \lambda)$ | $(\lambda^9  \lambda^3  \lambda^2)$ |
| 3     | 1     | 8     | 5     | $(\lambda^3  \lambda - \lambda)$ | $(\lambda^8  \lambda^1  \lambda^3)$ |
| 4     | 1     | 9     | 4     | $(\lambda^4  \lambda - \lambda)$ | $(\lambda^6  \lambda^2  \lambda^1)$ |
| 4     | 1     | 9     | 4     | $(\lambda^4  \lambda - \lambda)$ | $(\lambda^6  \lambda^2  \lambda^1)$ |
| 4     | 2     | 9     | 6     | $(\lambda^4  \lambda^2  \lambda^1)$ | $(\lambda^6  \lambda^5  \lambda^1)$ |

Table 9: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one constrained by the experimentally observed values of the neutrino masses with the condition $2(d_1 - d_2) - (x_1 - x_2) = 2$.

| $d_1$ | $d_2$ | $x_1$ | $x_2$ | $M_{\nu D}/(\lambda^d v_u)$ | $M_R/M$ |
|-------|-------|-------|-------|-----------------------------|---------|
| 4     | 1     | 9     | 5     | $(\lambda^4  \lambda - \lambda)$ | $(\lambda^6  \lambda^5  \lambda^1)$ |
| 4     | 1     | 10    | 6     | $(\lambda^4  \lambda - \lambda)$ | $(\lambda^7  \lambda^6  \lambda^1)$ |
| 5     | 1     | 11    | 5     | $(\lambda^5  \lambda - \lambda)$ | $(\lambda^7  \lambda^5  \lambda^1)$ |
| 5     | 2     | 11    | 7     | $(\lambda^5  \lambda^2  \lambda^1)$ | $(\lambda^7  \lambda^6  \lambda^1)$ |
| 5     | 2     | 11    | 7     | $(\lambda^5  \lambda^2  \lambda^1)$ | $(\lambda^7  \lambda^6  \lambda^1)$ |
Finally, the collections to the PMNS mixing angles are

\begin{align*}
\sin \theta_{12} & \approx \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \frac{\bar{m}_1 + m_{12}^R}{\bar{m}_2 + m_{23}^R}, \\
\sin \theta_{23} & \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{\bar{m}_1 [3(\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{23}^R)]}{\bar{m}_3(m_{23}^R) - (\bar{m}_2 + m_{23}^R)} - \frac{1}{\sqrt{6}} \frac{m_{13}^R}{\bar{m}_3 + m_{33}^R}, \\
\sin \theta_{13} & \approx -\frac{\lambda_{13}^d}{\sqrt{2}} \frac{\bar{m}_3 [(\bar{m}_3 + m_{33}^R) - \frac{2}{3}(\bar{m}_2 + m_{23}^R)]}{\bar{m}_3(\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{23}^R)} + \frac{2}{\sqrt{6}} \frac{m_{13}^R}{\bar{m}_3 + m_{33}^R} \\
& + \frac{1}{\sqrt{3}} \frac{m_{23}^R}{(\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{23}^R)}.
\end{align*}

### 3.5 Charged lepton sector

As mentioned above, we explore the possibility that the mass matrix of charged leptons has the H.C. form which is restricted by the GUT relation of $SU(5)$, $M_e \approx M_d^T$. In this subsection, we study the corrections from the charged lepton sector to the lepton generation mixing angles.

We take the charged lepton mass matrix as the following form,

\[ M_e \approx \begin{pmatrix} \epsilon_d & \delta_d & \delta_d \\ \delta_d & -3\lambda_d & \lambda_d \\ \delta_d & \lambda_d & 1 \end{pmatrix} \xi_d v_d, \tag{3.65} \]

Unlike the neutrino sector, the magnitudes of H.C. can be partially evaluated from the experimentally observed values of charged lepton masses and given by

\[ |\lambda_d| \approx \frac{m_\mu}{3m_\tau}, \quad |\delta_d| \approx \frac{3\sqrt{m_e m_\mu}}{m_\tau}. \tag{3.66} \]

The generation mixing is expressed in terms of the cascade hierarchy parameter, $\lambda_d$ and $\delta_d$, as shown in Tab. 2. Therefore, the corrections from the charged lepton sector are found
to be generically small and the total lepton mixing angles are given at the first order of perturbation as

$$\sin \theta_{12} \simeq \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \frac{-\bar{m}_1 + m_{12}^R}{\bar{m}_2 + m_2^R} + \sqrt{3} \frac{m_e}{m_\mu},$$

$$\sin \theta_{23} \simeq \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2} \bar{m}_3 (m_3^R + (m_3^R - (m_2^R + m_2^R))}\lambda_{d_1-d_2} \bar{m}_3 (m_2^R + m_2^R) - \frac{3 \sqrt{2}}{\sqrt{6} \bar{m}_3 + m_3^R} m_{13}^R + \frac{1}{\sqrt{3} \bar{m}_3 + m_3^R} m_{13}^R,$$

$$\sin \theta_{13} \simeq \frac{-\lambda_{d_1-d_2}}{\sqrt{2} \bar{m}_3 (m_3^R + (m_3^R - (m_2^R + m_2^R))} \lambda_{d_1-d_2} \bar{m}_3 [m_3^R (m_3^R + (m_3^R - (m_2^R + m_2^R))]
+ \frac{\sqrt{2} m_1 (m_2^R + m_2^R)}{(2 m_3^R + m_3^R)} + \frac{2}{\sqrt{6} \bar{m}_3 + m_3^R} m_{13}^R + \frac{1}{\sqrt{3} \bar{m}_3 + m_3^R} m_{13}^R + \frac{3}{\sqrt{2} \bar{m}_3 + m_3^R} m_e. \quad (3.69)$$

One can see the effects from the charged lepton sector of the H.C. form from these expressions. The tri-bimaximal solar neutrino mixing is little (about 4% of $\sin^2 \theta_{12}$) affected. For the atmospheric neutrino mixing, the charged lepton effect becomes 4% of the tri-bimaximal atmospheric angle, $\sin^2 \theta_{23} = 1/2$. Finally, magnitude of effect is estimated as 0.02 for the reactor neutrino angle, $\sin^2 \theta_{13}$.

### 3.6 Quark sector

We investigate the quark mass matrices in this subsection. It must be remembered that the mass matrix of the H.C. form is motivated for the mass spectra and mixing angles of quark sector. The cascading mass matrices of down- and up-type quarks are given in (3.10) and (3.11). The mixing matrices for the down and up sector are roughly estimated as

$$V_d = \begin{pmatrix} O(1) & O(\lambda) & O(\lambda^3) \\ O(\lambda) & O(1) & O(\lambda^2) \\ O(\lambda^3) & O(\lambda^4) & O(1) \end{pmatrix}, \quad V_u = \begin{pmatrix} O(1) & O(\lambda^4) & O(\lambda^8) \\ O(\lambda^4) & O(1) & O(\lambda^4) \\ O(\lambda^8) & O(\lambda^4) & O(1) \end{pmatrix}, \quad (3.70)$$

where the mass matrix for the up quarks is assumed to be a symmetric matrix. It is easily seen from the structure of $V_d$ that the experimentally observed values of CKM matrix can be realized at the leading order. The collections from the $V_u$ are generally small, which are estimated as

$$|V_{td}| \simeq |(V_u)_{31} + (V_u)_{21}(V_u^d)_{23}| \simeq |\lambda^3 (1 + \lambda^2)|,$$

$$|V_{tb}| \simeq |(V_u)_{23} + (V_u^d)_{32}| \simeq |\lambda (1 + \lambda^2)|,$$

$$|V_{ts}| \simeq |(V_u)_{32} + (V_u)_{33}^d| \simeq |\lambda^2 (1 + \lambda^2)|,$$

$$|V_{td}| \simeq |(V_u)_{31} + (V_u)_{21}(V_u^d)_{23}| \simeq |\lambda^3 (1 + \lambda^2)|,$$

$$|V_{tb}| \simeq |(V_u)_{23} + (V_u^d)_{32}| \simeq |\lambda (1 + \lambda^2)|,$$

$$|V_{ts}| \simeq |(V_u)_{32} + (V_u)_{33}^d| \simeq |\lambda^2 (1 + \lambda^2)|.$$
up to order $O(\lambda^2)$ of the dominant term, and

$$|V_{us}| \simeq |(V_d)_{12} + (V_{u}^\dagger)_{21}| \simeq |\lambda(1 + \lambda^3)|,$$  \hspace{1cm} (3.74)

$$|V_{ub}| \simeq |(V_d)_{13} + (V_{u}^\dagger)_{21}(V_{d}^\dagger)_{23}| \simeq |\lambda^3(1 + \lambda^3)|,$$  \hspace{1cm} (3.75)

$$|V_{cd}| \simeq |(V_d)_{21} + (V_{u}^\dagger)_{12}| \simeq |\lambda(1 + \lambda^3)|,$$  \hspace{1cm} (3.76)

to order $O(\lambda^3)$ of the leading term. Collections to other elements are negligibly small. Detailed numerical calculations are given in the next section.

\section{4 Related Phenomenology}

In this section, we numerically investigate related phenomenologies based on the above analyses of cascade textures for the quark and lepton sectors: the generation mixing angles, the lepton flavor violation, and the baryon asymmetry of the Universe via thermal leptogenesis.

\subsection{4.1 Generation mixing angles}

Let us start to examine numerical analyses of the generation mixing of the quark and lepton sectors predicted from the cascade model. In these analyses, we focus on two typical types of minimal texture for the neutrino Dirac and right-handed Majorana neutrino mass matrices given in Tabs. 8 and 9 that is,

Model I : $M_{\nu D} \simeq \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda & -\lambda \\ \lambda^3 & -\lambda & 1 \end{pmatrix} \lambda^d v_u, \quad M_R \simeq \begin{pmatrix} \lambda^7 & \lambda^9 & \lambda^5 \\ \lambda^9 & \lambda^4 & \lambda^3 \\ \lambda^5 & \lambda^3 & 1 \end{pmatrix} M,$  \hspace{1cm} (4.1)

for the case of the condition $2(d_1 - d_2) - (x_1 - x_2) = 1$, and

Model II : $M_{\nu D} \simeq \begin{pmatrix} \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda & -\lambda \\ \lambda^4 & -\lambda & 1 \end{pmatrix} \lambda^d v_u, \quad M_R \simeq \begin{pmatrix} \lambda^9 & \lambda^{11} & \lambda^6 \\ \lambda^{11} & \lambda^5 & \lambda^5 \\ \lambda^6 & \lambda^5 & 1 \end{pmatrix} M,$  \hspace{1cm} (4.2)

for the case of the condition $2(d_1 - d_2) - (x_1 - x_2) = 2$. In both models, the following charged lepton, up and down quark mass matrices are adopted:

$$M_e \simeq \begin{pmatrix} \lambda^{k_d+3} & \lambda^3 & \lambda^3 \\ \lambda^3 & -3\lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \xi_d v_d, \quad (4.3)$$

and

$$M_u \simeq \begin{pmatrix} \lambda^{k_u+6} & \lambda^6 & \lambda^6 \\ \lambda^6 & \lambda^4 & \lambda^4 \\ \lambda^6 & \lambda^4 & 1 \end{pmatrix} v_u, \quad M_d \simeq \begin{pmatrix} \lambda^{k_d+3} & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \xi_d v_d. \quad (4.4)$$
Using the above mass hierarchies, we can numerically examine predictions of PMNS angles, and can compare the predictions of model I and model II. The analysis is carried out with the following procedure. At first, we restrict structure in $M_u$ and $M_d$ with the experimental constraints on quark masses and the CKM matrix. Then the GUT relation between charged lepton and down-type quark provides a constrained structure in $M_e$. For the neutrino sector, different contributions to PMNS angles are obtained for each model. Combining contributions from $M_e$ and neutrino sector, predictions of PMNS angles are obtained.

Note that (4.1) and (4.4) include ambiguity from $O(1)$ coefficients of each element of matrices. In the numerical estimation, the definite form of the mass matrices are used. In the neutrino sector, the mass matrices are defined as

$$M_{\nu D} = \begin{pmatrix} c_{\nu} \lambda^3 & c_{\nu} \lambda^3 & c_{\nu} \lambda^3 \\ c_{\nu} \lambda^3 & b_{\nu} \lambda & -b_{\nu} \lambda \\ c_{\nu} \lambda^3 & -b_{\nu} \lambda & a_{\nu} \end{pmatrix} \lambda^d v_u, \quad M_R = \begin{pmatrix} f_R \lambda^7 & e_R \lambda^9 & d_R \lambda^5 \\ e_R \lambda^9 & c_R \lambda^4 & b_R \lambda^3 \\ d_R \lambda^5 & b_R \lambda^3 & a_R \end{pmatrix} M, \quad (4.5)$$

for model I, and

$$M_{\nu D} = \begin{pmatrix} c_{\nu} \lambda^4 & c_{\nu} \lambda^4 & c_{\nu} \lambda^4 \\ c_{\nu} \lambda^4 & b_{\nu} \lambda & -b_{\nu} \lambda \\ c_{\nu} \lambda^4 & -b_{\nu} \lambda & a_{\nu} \end{pmatrix} \lambda^d v_u, \quad M_R = \begin{pmatrix} f_R \lambda^9 & e_R \lambda^{11} & d_R \lambda^6 \\ e_R \lambda^{11} & c_R \lambda^5 & b_R \lambda^5 \\ d_R \lambda^6 & b_R \lambda^5 & a_R \end{pmatrix} M, \quad (4.6)$$

for model II. For the quark and charged lepton mass matrices, we take

$$M_u = \begin{pmatrix} 0 & e_u \lambda^6 & d_u \lambda^6 \\ e_u \lambda^6 & c_u \lambda^4 & b_u \lambda^4 \\ d_u \lambda^6 & b_u \lambda^4 & a_u \end{pmatrix} v_u, \quad M_d = \begin{pmatrix} 0 & e_d \lambda^3 & d_d \lambda^3 \\ e_d \lambda^3 & c_d \lambda^2 & b_d \lambda^2 \\ d_d \lambda^3 & b_d \lambda^2 & a_d \end{pmatrix} \xi_d v_d. \quad (4.7)$$

and

$$M_e = \begin{pmatrix} 0 & e_d \lambda^3 & d_d \lambda^3 \\ e_d \lambda^3 & -3c_d \lambda^2 & b_d \lambda^2 \\ d_d \lambda^3 & b_d \lambda^2 & a_d \end{pmatrix} \xi_d v_d. \quad (4.8)$$

In the matrices, the coefficients represented by $a, \cdots, f$ with subscripts are taken as complex numbers whose absolute values are constrained as $0.4 \sim 1.4$. The $(1, 1)$ elements of $M_{u,d,e}$ are taken as zero. The limitation on the mass matrices is not essential, that is, the following results are little affected by the $(1, 1)$ elements that satisfy the condition shown in section 3.3.

In figure 1, we show the predicted PMNS mixing angles in $\sin^2 \theta_{13} - \sin^2 \theta_{12}$ plane. Left and right plots are derived from model I and II, respectively. Best-fit value with $3\sigma$ interval of solar mixing angle $\sin^2 \theta_{12} = 0.304^{+0.066}_{-0.054}$ in [1] is also shown by horizontal lines. For model I, $\sin^2 \theta_{13}$ can take larger value than 0.01, which is also favored by recent neutrino oscillation data [1][12], and also be much suppressed. On the other hand, for model II, predicted
value of $\sin^2 \theta_{13}$ is rather restricted. Since the contributions from charged lepton sector to the PMNS mixing angles have no difference between both models, the result implies that the other corrections which deviate from tri-bimaximal mixing in $\sin \theta_{13}$ are larger for model I, rather than model II. Predicted range of $\sin^2 \theta_{12}$ has no significant difference between model I and II. In our case, the nearly tri-bimaximal generation mixing in neutrino sector dominates 12 mixing in PMNS matrix, and mixing from charged lepton sector give relatively small correction to $\sin^2 \theta_{12}$. Thus the plots distributed around $\sin^2 \theta_{12} = 1/3$. It should be notified that the lower limit of $\sin^2 \theta_{12}$ appear in the $3\sigma$ interval. There are no particular correlation between predictions of $\sin^2 \theta_{13}$ and $\sin^2 \theta_{12}$. In figure 2, we also show the predicted PMNS mixing angles in $\sin^2 \theta_{13} - \sin^2 2\theta_{23}$ plane. Left and right plots are derived from model I and II, respectively. In both models, prediction of $\sin^2 2\theta_{23}$ is
mostly larger than 0.99, which is quite close to the best fit value. Due to the GJ factor in $M_e$, corrections to the atmospheric angle from charged lepton mixing is rather suppressed than the previous analysis of the cascade matrices [4]. Finally, we present some figures showing correlations between a neutrino mass ratio, $m_1/m_2$, and each mixing angle in figure [3]. Since the cascade model works only in the NH case, $m_2$ and $m_3$ are approximated by the solar and atmospheric scales. Therefore, the ratio $m_1/m_2$, strongly correlated to $m_1$, is chosen as the vertical axis. These analyses would be checked by the future neutrino experiments and cosmological bounds on its absolute mass.

4.2 Lepton flavor violation

Next, we estimate the branching ratios of flavor violating rare decays of charged leptons. Supersymmetric models generically induce sizable magnitudes of lepton flavor violation (LFV) because there exist additional sources of LFV, which are mass parameters of sleptons. Those flavor violating processes are radiatively generated depending on the structure of lepton mass matrices. We investigate the branching ratios of the rare decay processes, $l_i \rightarrow l_j \gamma$, in these cascade lepton mass matrices.

For simplicity, we assume that soft SUSY breaking masses of sleptons are universal at the GUT scale, $\Lambda_G$. Then the off-diagonal matrix elements are generated by radiative corrections from the Yukawa couplings of neutrinos [13]. The one-loop renormalization group evolution induces the left-handed slepton masses, which are estimated as

$$
(m^2_l)_{ij} \sim \frac{1}{8\pi^2 v^2} (3m_0^2 + |a_0|^2) \sum_k (M_{\nu D}^i)_{ik} (M_{\nu D}^j)_{kj} \ln \left(\frac{|M_k|}{\Lambda_G}\right) \quad \text{(for } i \neq j),
$$

(4.9)

where $m_0$ and $a_0$ are the universal SUSY breaking mass and three-point coupling of scalar superpartners given at the GUT scale. The magnitude of these off-diagonal elements depends on the structure of neutrino Dirac mass matrix and the mass scale of right-handed Majorana neutrinos.

The branching ratio of $l_i \rightarrow l_j \gamma$ is roughly given by

$$
\text{Br}(l_i \rightarrow l_j \gamma) \sim \frac{3\alpha}{2\pi} \frac{|(m^2_l)_{ij}|^2 M_W^4}{m_{\text{SUSY}}^8} \tan^2 \beta,
$$

(4.10)

in the mass insertion approximation, where $\alpha$, $M_W$, and $m_{\text{SUSY}}$ are the fine structure constant, the $W$ boson mass, and a typical mass scale of superparticles circulating in
Figure 3: Correlations between neutrino mass ratio and PMNS mixing angles in model I (left) and II (right).
Table 10: Typical magnitudes of branching ratios for lepton flavor violating rare decay process.

| d | $\text{Br}(\mu \to e\gamma)$ | $\text{Br}(\tau \to e\gamma)$ | $\text{Br}(\tau \to \mu\gamma)$ | $M_1$ [GeV] | $M_2$ [GeV] | $M_3$ [GeV] |
|---|------------------|------------------|------------------|-----------|-----------|-----------|
| 0 | $2.66 \times 10^{-9}$ | $7.80 \times 10^{-10}$ | $3.99 \times 10^{-7}$ | $3.61 \times 10^{12}$ | $3.08 \times 10^{13}$ | $1.16 \times 10^{16}$ |
| 1 | $7.40 \times 10^{-12}$ | $7.60 \times 10^{-12}$ | $2.31 \times 10^{-9}$ | $1.80 \times 10^{10}$ | $1.59 \times 10^{12}$ | $5.98 \times 10^{14}$ |
| 2 | $2.06 \times 10^{-14}$ | $1.28 \times 10^{-13}$ | $4.37 \times 10^{-11}$ | $4.22 \times 10^{9}$ | $8.19 \times 10^{10}$ | $3.08 \times 10^{13}$ |

for the model I, and

| d | $\text{Br}(\mu \to e\gamma)$ | $\text{Br}(\tau \to e\gamma)$ | $\text{Br}(\tau \to \mu\gamma)$ | $M_1$ [GeV] | $M_2$ [GeV] | $M_3$ [GeV] |
|---|------------------|------------------|------------------|-----------|-----------|-----------|
| 0 | $1.85 \times 10^{-10}$ | $3.57 \times 10^{-10}$ | $2.67 \times 10^{-6}$ | $3.61 \times 10^{11}$ | $3.08 \times 10^{13}$ | $5.11 \times 10^{16}$ |
| 1 | $4.95 \times 10^{-13}$ | $1.06 \times 10^{-15}$ | $1.66 \times 10^{-11}$ | $1.86 \times 10^{9}$ | $1.59 \times 10^{12}$ | $2.64 \times 10^{15}$ |
| 2 | $1.33 \times 10^{-15}$ | $2.19 \times 10^{-15}$ | $1.54 \times 10^{-11}$ | $9.57 \times 10^{8}$ | $8.19 \times 10^{10}$ | $1.36 \times 10^{14}$ |

for the model II, where $B \equiv (M_W/m_{\text{SUSY}})^4 \tan^2 \beta$. Typical magnitudes of the branching ratios are shown in Table 10. In these analyses, $\Lambda_G = 2 \times 10^{16}$ GeV is taken. These results are compared with the current experimental upper bounds at 90% confidence level [14,15]:

$$\text{Br}(\mu \to e\gamma) \leq 1.2 \times 10^{-11}, \quad \text{Br}(\tau \to e\gamma) \leq 1.2 \times 10^{-7}, \quad \text{Br}(\tau \to \mu\gamma) \leq 4.5 \times 10^{-8}. \tag{4.17}$$

Notice that $\tan \beta = 38$ and $m_{\text{SUSY}} = 500$ GeV have been taken in the numerical fit of quark and charged lepton masses with a high accuracy of those mass relations at GUT scale in the previous subsection but it was just for simplicity. The magnitude of the threshold corrections are important for those relations, and dependence of corrections on the overall SUSY scale, $m_{\text{SUSY}}$, is negligibly small as long as a model is discussed in a low scale SUSY breaking like in our case. In discussions of the LFV processes given here, we extend our consideration to be more general case, that is, we take $\tan \beta$ and $m_{\text{SUSY}}$ as free parameters of the models and estimate constraints on it from the LFV searches.
The magnitude of the branching ratio for the $\tau \rightarrow e\gamma$ process predicted from the cascade model with $d = 0$ is far below the experimental limit. Further, all values of the ratio with $d \geq 1$ are also sufficiently smaller than the current bounds. On the other hand, the $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$ decays are marginal to the present limit and would be observed in future LFV searches with relatively light superparticle spectrum. These experimental limit in turn constraints $\tan \beta$ and the mass scale of superparticles. The most severe constraint on the scale comes from the $\mu \rightarrow e\gamma$ for the model I with $d = 0$ and $\tau \rightarrow \mu\gamma$ for the model II with $d = 0$. They are $B \leq 4.51 \times 10^{-3}$ and $B \leq 1.68 \times 10^{-2}$ for the model I and II, respectively. They mean that the typical SUSY breaking scale has a lower bound as

$$m_{\text{SUSY}} \geq 1912 \left( \frac{\tan \beta}{38} \right)^{1/2} \text{GeV},$$

for the model I, and

$$m_{\text{SUSY}} \geq 1375 \left( \frac{\tan \beta}{38} \right)^{1/2} \text{GeV},$$

for the model II. In figure 4, the constraints are shown for model I and II. The solid (model I) and dashed (model II) lines show lower bounds of $m_{\text{SUSY}}$ for particular values of $\tan \beta$. Above the line, all the LFV constraints (4.17) are satisfied for each model.

### 4.3 Leptogenesis

Next let us study CP violating phenomenology. Especially, we examine whether the thermal leptogenesis [16] works in the cascade model. The CP asymmetry parameter in the right-handed neutrino, $R_i$, decay is given by

$$\epsilon_i = \frac{\sum_j \Gamma(R_i \rightarrow L_j H) - \sum_j \Gamma(R_i \rightarrow L_j^c H^\dagger)}{\sum_j \Gamma(R_i \rightarrow L_j H) + \sum_j \Gamma(R_i \rightarrow L_j^c H^\dagger)},$$

where $L_i$ and $H$ denote the left-handed lepton and Higgs fields. An approximation for $\epsilon_i$ at low temperature is estimated as [17]

$$\epsilon_1 = \frac{1}{8\pi} \sum_{i \neq 1} \frac{\text{Im}[A_{i1}]^2}{|A_{11}|} F(r_i),$$

If we fix the value of $m_{\text{SUSY}}$ as 500 GeV, the both magnitudes of the branching ratio for the $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$ processes in models I and II with $d = 0$ exceed the current experimental upper bound, and thus, this possibility is ruled out. Therefore, $d \geq 1$ is required for our cascade textures. The parameter $d$ can be related with the heaviest right-handed neutrino mass scale through $M_3 = \lambda^{2(d+1)-x_2v^2/\sqrt{|\Delta m^2_{31}|}}$. Then, the constraints on the heaviest right-handed neutrino mass as $M_3 \lesssim 5.98 \times 10^{14}$ GeV for model I and $M_3 \lesssim 2.64 \times 10^{15}$ GeV for model II are obtained. It is expected that the lepton rare decay processes of $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$ predicted from the minimal model I with $d = 1$ would be observed in near future LFV searches. Those branching ratios become $\text{Br}(\mu \rightarrow e\gamma) = 7.14 \times 10^{-12}$ and $\text{Br}(\tau \rightarrow \mu\gamma) = 2.23 \times 10^{-9}$.
Figure 4: Constraints for $\tan \beta$ and $m_{\text{SUSY}}$ from experimental bounds of LFV processes. In the figure, solid (model I) and dashed (model II) lines show lower bounds of $m_{\text{SUSY}}$ for particular values of $\tan \beta$. Above the line, all the LFV constraints (4.17) are satisfied for each model.

| $|A_{11}|$ | $|A_{21}|$ | $|A_{31}|$ | $|M_1/M_3|$ | $|M_1/M_2|$ | $|A_{11}|$ | $|A_{21}|$ | $|A_{31}|$ | $|M_1/M_3|$ | $|M_1/M_2|$ |
|-------|--------|--------|-----------|-----------|-------|--------|--------|-----------|-----------|
| $3\lambda^6$ | $\lambda^6$ | $\lambda^3$ | $\lambda^7$ | $\lambda^3$ | $3\lambda^8$ | $\lambda^8$ | $\lambda^4$ | $\lambda^9$ | $\lambda^4$ |

Table 11: The relevant quantities for the CP asymmetry parameter.

where $r_i \equiv |M_i/M_1|^2$, $A \equiv (DM_{\nu D}M_{\nu D}^\dagger)/v^2_\nu$, and $D$ is the diagonal phase matrix to make the eigenvalues $M_i$ real and positive. The formula is given in the diagonal basis of right-handed Majorana mass matrix with real positive eigenvalues and are reasonably accurate even at higher temperatures. The function $F$ denotes contributions from the one-loop vertex and self-energy corrections,

$$F(x) = \sqrt{x} \left[ \frac{2}{1-x} - \ln \left( 1 + \frac{1}{x} \right) \right],$$

which is well approximated by $-3/\sqrt{x}$ for $x \gg 1$. The relevant quantities for the CP asymmetry parameter are given in Tab. 11. By utilizing these quantities, the CP asymmetry parameter for each model can be calculated as

$$\epsilon_1 \approx \begin{cases} \frac{1}{8\pi} [\lambda^9 \sin(\theta_2 - \theta_1) + \lambda^7 \sin(\theta_3 - \theta_1)] & \text{for model I,} \\ \frac{1}{8\pi} [\lambda^{12} \sin(\theta_2 - \theta_1) + \lambda^9 \sin(\theta_3 - \theta_1)] & \text{for model II.} \end{cases}$$

where $\theta_i = \arg(M_i)$. It is found that the second term is dominant unless the relevant argument is taken as a specific value such as zero. We here define the resultant CP
asymmetry, $\eta_{CP}$, as the ratio of the lepton asymmetry to the photon number density, $n_\gamma$, 

$$\eta_{CP} = \frac{135 \zeta(3) \kappa s \epsilon_1}{4\pi^4 g_* n_\gamma},$$

where $\kappa$, $s$, and $g_*$ are the efficiency factor, entropy density, and the effective number of degrees of freedom in thermal equilibrium. They are given by [18]

$$\kappa^{-1} \approx \frac{3.3 \times 10^{-3} \text{ eV}}{m_{\text{eff}}} + \left(\frac{m_{\text{eff}}}{5.5 \times 10^{-4} \text{ eV}}\right)^{1.16},$$

$$s = 7.04 n_\gamma,$$

$$g_* = 228.75.$$

The $m_{\text{eff}}$ in the efficiency factor is the effective light neutrino mass defined as $m_{\text{eff}} \equiv |(M_{\nu D}^T M_{\nu D})_{11}/M_1|$. It is known that the efficiency depends only on $m_{\text{eff}}$ when $|M_1| \ll 10^{14}$ GeV, which is realized in both models. Finally, the baryon asymmetry of the universe, $\eta_B$, is transferred via spharelon interactions as $\eta_B = -8 \eta_{CP}/23$. As the result, the baryon asymmetry in our model is predicted as

$$\eta_B \sim \begin{cases} 4.7 \times 10^{-11} \sin \theta_B & \text{for model I,} \\ 1.3 \times 10^{-11} \sin \theta_B & \text{for model II,} \end{cases}$$

where $\theta_B \equiv \theta_3 - \theta_1$ and we take account only of the leading term in (4.23). These results are compared with the current observational data at 68% confidence level from the WMAP 7-years mean result in the standard $\Lambda$CDM model, $\eta_B = (6.19 \pm 0.15) \times 10^{-10}$ [19]. Although the prediction seems to be a little small, note that the above our naive estimation of $\eta_B$ does not involve the effects from combination of $O(1)$ coefficients which generally exist in $M_{\nu D}$ and $M_R$. When the combination of $O(1)$ coefficients generates $O(10)$ ($O(50)$) enhancement factor of the asymmetry for model I (II), then $\eta_B$ can be consistent with the experimental bound as long as the relevant CP violation $\theta_B$ is maximally large. It is interesting that our models, which is constrained by only the neutrino mass spectra and PMNS structure, can lead to favored magnitude of the baryon asymmetry via leptogenesis.

5 Discussions

At the end of this paper, we discuss about a realization of our cascade model in SUSY $SU(5)$ GUT. Especially, we give some comments on flavor symmetry behind the model and extensions of SUSY $SU(5)$ GUT.

5.1 Flavour symmetry

We have presented the texture analyses and shown some phenomenological results. These textures are described by the cascade form for the neutrino Dirac mass (Yukawa) matrix
and the H.C. one for the charged lepton and down quark one. The effects from the right-handed Majorana and up quark mass matrices to the PMNS and CKM structures should be enough small not to spoil the experimentally favored mixing angles induced from the cascade form of neutrino Dirac and H.C. form of down quark mass matrices. The cascade (H.C.) contains two (three) step hierarchies. A smaller factor is concerned with the 1st generation and the other with 2nd one. Moreover, the current neutrino experimental data would suggest that the coefficients of effective mass operators are correlated to each other. These non-trivial features imply some implements introduced in fundamental theory beyond the SM.

An introduction of flavor symmetries is one of such implements. Actually, a simple realization of cascade hierarchy has been shown in [4] based on an $U(1)$ flavor symmetry. In the realization, three gauge singlet scalars, $\phi_i$, are introduced in addition to the fundamental SM fields. These field including the SM field are charged under the $U(1)$ flavor symmetry. A simple quantum number assignment of $U(1)$ flavor symmetry and a requirement of the same magnitude of expectation values, $\langle \phi_1 \rangle \simeq \langle \phi_2 \rangle \simeq \langle \phi_3 \rangle \equiv \Lambda$, lead to cascade texture as (1.3), where $\delta = \lambda^{m+1}$ and $m$ is an arbitrary positive integer. A significant feature of the flavor model is $\delta \leq \lambda^2$ in (1.3), which is suitable for the neutrino Dirac mass matrix as discussed in section 3.4. In addition to the cascade form of neutrino Dirac mass matrix, a realization of H.C. form of charged lepton and down-type quark mass matrices could be obtained by extending the gauge singlet Higgs sector and number of $U(1)$ flavor symmetry. It is worth proceeding a study about simple realization of our model in terms of abelian flavor symmetry.

For the analysis, let us introduce $U(1)_F \times U(1)_{F'} \times Z_3$ flavor symmetry and additional fields $\phi_f, \phi'_f, \phi''_f, \phi'_p, \phi''_p$ and $\phi_z$, which are neutral under the $SU(5)$ gauge symmetry. SM fermions and Higgs fields are involved in $SU(5)$ representations; here the matter fields are denoted by $10_i, \bar{5}_i$ and $1_i$ ($i = 1, 2, 3$). We assume that the $U(1)_F \times U(1)_{F'}$ flavor symmetry is broken by the Higgs vacuum extension values $\langle \phi_f \rangle \simeq \langle \phi'_f \rangle \simeq \langle \phi''_f \rangle \simeq \langle \phi'_p \rangle \simeq \langle \phi''_p \rangle \simeq \lambda \Lambda$, where $\Lambda$ is the cutoff scale of the theory. Also the discrete $Z_3$ is broken by $\langle \phi_z \rangle \simeq M' < \lambda \Lambda$. Although the vacuum expectation values would be determined by possible dynamics of the Higgs sector, those are simply adopted in the analysis. To give suitable $U(1)_F \times U(1)_{F'} \times Z_3$ charge assignments for the $\phi$’s and matter fields, one can lead to effective mass matrices of the quark and lepton fields via higher-dimensional operators suppressed by $\Lambda$.

\footnote{Some complicated mechanisms, such as discrete flavor symmetries, could be needed for the alignment of cascade and H.C. forms. Although a complete flavor model would include additional flavor mechanism, we focus only on the realization of hierarchical structure of the mass matrices in this analysis.}

\footnote{We take up- and down-type Higgs fields are neutral under $U(1)_F \times U(1)_{F'}$, and have 1/3 charges of $Z_3 (mod 1)$.}
Table 12: An example of phenomenologically viable charge assignment of $U(1)_F \times U(1)_{F'}$ flavor symmetry.

In Tab. 12, an example of phenomenologically viable $U(1)_F \times U(1)_{F'}$ charge assignment is shown. For $Z_3$ charges, $\phi_{f,f'}$ are neutral and $10_i$, $\bar{5}_i$ and $\phi_z$ have $1/3 \ (mod \ 1)$. In this case, one can obtain the following hierarchical structure in the effective mass matrices:

$$M_u \simeq \begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^6 \\ \lambda^6 & \lambda^4 & \lambda^4 \\ \lambda^6 & \lambda^4 & 1 \end{pmatrix} v_u, \quad M_d \simeq \begin{pmatrix} \lambda^8 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \lambda^2 v_d, \quad (5.1)$$

$$M_{\nu D} \simeq \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda & \lambda \\ \lambda^3 & \lambda^3 & 1 \end{pmatrix} \lambda^3 v_u, \quad M_R \simeq \begin{pmatrix} \lambda^7 & \lambda^7 & \lambda^7 \\ \lambda^7 & \lambda^4 & \lambda^4 \\ \lambda^7 & \lambda^4 & 1 \end{pmatrix} \lambda M'. \quad (5.2)$$

For the quark (and the charged lepton) sector, H.C. mass matrices are obtained. For the neutrino sector, Dirac mass matrix and mass eigenvalues of $M_R$ correspond to model I in section 4.1. Atmospheric neutrino mass scale determines the magnitude of Majorana mass scale as $M' \simeq 10^{13}$ GeV. Compared to model I, mixing between 1st and 2nd generation of $M_R$ is larger in this case. We numerically checked that prediction of the PMNS angles in this case can satisfy the experimental constraints as the case of model I. The analysis implies that the cascade types of hierarchical structure in quark and lepton mass matrices can be obtained by simple flavor mechanisms.

### 5.2 Extension of SUSY $SU(5)$ GUT

Cascade and H.C. textures are suitable for comprehensive description of hierarchical structure in quark and lepton mass matrices. This implies that the matrices are compatible with GUT, where quark and lepton flavor structure is generally related. In $SU(5)$ GUT, unification of down-type quark and charged lepton leads to constraints between elements of $M_d$ and $M_e$. In our texture analysis, the GJ factor is minimally introduced as in (3.8). Let us give complement discussion about the modification of the relation.

A significant feature of the cascade hierarchy in $SU(5)$ GUT is that the CKM and PMNS matrices should be mainly controlled by mixing structure of down-type quark and neutrino sectors, respectively. In other words, mixing structure of $M_d$ is close to the CKM matrix, and neutrino sector leads to tri-bimaximal generation mixing at the leading order. Details of the relation between $M_d$ and $M_e$ controls mixing structure of $M_e$, which gives correction to PMNS mixing angles as studied in section 3.5.
Relation between $M_d$ and $M_e$ is determined by details of high-energy models, and variety of GUT models could be considered. GUT scale fermion masses extrapolated from low-energy experimental data give implications of the relation of the mass matrices and GUT models. As mentioned, threshold effects which depend on the superparticle spectrum could play an important role to determine the GUT scale fermion masses (Yukawa couplings) in supersymmetric scenario \cite{8,9}. In the recent analysis \cite{20}, for example, the possible relation between down-type quark and charged lepton masses (and corresponding GUT model) has been studied with several SUSY breaking scenarios.

Cascade hierarchies is naturally compatible with the several possibilities. Modification of the GJ relation $(m_e/m_d, m_\mu/m_s, m_\tau/m_b) \simeq (1/3, 3, 1)$ leads to different mixing structure of $M_e$ from our study, that is, thus prediction of PMNS angles would be somewhat changed depends on details of the mass relation. For example, mass relation $(m_e/m_d, m_\mu/m_s, m_\tau/m_b) \simeq (3/8, 6, 3/2)$ is compatible with typical SUSY breaking scenarios \cite{20}. With the ordinary matter assignment of $SU(5)$ representation, H.C. structure of $M_d$ and $M_e$ could be minimally modified by CG coefficients through dimension–five interaction involving Higgs fields, as follows:

$$M_d \simeq \begin{pmatrix}
\epsilon_d & \delta_d & \delta_d \\
\delta_d & \lambda_d & \lambda_d \\
\delta_d & \lambda_d & 1
\end{pmatrix} \xi_d v_d, \quad M_e \simeq \begin{pmatrix}
\epsilon_d & 3/2\delta_d & \delta_d \\
3/2\delta_d & 6\lambda_d & \lambda_d \\
\delta_d & \lambda_d & 3/2
\end{pmatrix} \xi_d v_d. \quad (5.3)$$

In this case, mixing structure of charged lepton mass matrix is changed from our analysis: at the leading order, mixing angles between 1st–2nd, 2nd–3rd and 1st–3rd generations are multiplied 3/4, 2/3, and 3/2 by the minimal GJ case (3.8), respectively. As a result, prediction of PMNS mixing angles are slightly modified through mixing structure of $M_e$.

Future progress of experimental searches is expected to give precise information about low-energy flavor structure and also SUSY parameters. Combined analyses of the fermion flavor structure and SUSY parameters would give key ingredients to reveal high-energy flavor origin. Examination of several types of cascade hierarchies and comparison between GUT models are left to our further study.

6 Summary

We have presented texture analyses based on cascade form. The neutrino Dirac mass matrix of a cascade form can lead to the tri-bimaximal generation mixing at the leading order in the framework of seesaw mechanism. On the other hand, the down quark mass matrix of a hybrid cascade form can reproduce the CKM structure. These facts give us a motivation to study cascade textures in a grand unified theory.

We have embedded such experimentally favored mass textures into a SUSY $SU(5)$ GUT,
which gives a relation between the down quark and charged lepton mass matrices. This
relation constrains the structure of charged lepton mass matrix to a hybrid cascade form.
We have taken the right-handed Majorana mass matrix as a form which gives enough small
corrections to the PMNS structure not to spoil generation mixing structures induced from
the neutrino Dirac mass matrix of the cascade form. The mass matrix of up-type quarks
is also supposed to be a hybrid cascade form in our analyses. Related phenomenologies,
such as lepton flavor violating processes and leptogenesis, have been also investigated in
addition to lepton mixing angles in two typical types of model.

For the lepton mixing angles, the both models described by cascade and hybrid cas-
cade textures give an upper bound on the $\sin^2 \theta_{13}$. The value of $\sin^2 \theta_{13}$ is determined by
summation of collections from the charged lepton and right-handed neutrino sectors, and
properties of cascade texture of neutrino Dirac mass matrix. The value of upper bound in
model I, which is $\sin^2 \theta_{13} \lesssim 0.01$, is larger than one in model II. It might be checked in
upcoming neutrino oscillation experiments. Predicted range of $\sin^2 \theta_{12}$ has no significant
difference between model I and II. Since correction from charged lepton sector are rela-
tively small, the generation mixing in the neutrino sector dominates $\theta_{12}$. The predictions
of $\sin^2 2\theta_{23}$ are mostly larger than 0.99 in both models. Finally, some correlations between
a neutrino mass ratio and each mixing angle have been also presented.

In estimations of lepton flavor violating rare decay processes, it has been shown that the
most severe constraint on a typical SUSY scale correlating with $\tan \beta$ comes from $\mu \to e\gamma$
process for the model I and $\tau \to \mu\gamma$ for the model II. We have also examined whether the
thermal leptogenesis works in both cascade models. Enough baryon asymmetry via the
thermal leptogenesis cannot be generated because of a relatively large hierarchy among the
right-handed Majorana masses. Therefore, we need other source of baryon asymmetry in
order to reproduce the observed values.

Acknowledgement

The work of R.T. is supported by the DFG-SFB TR 27.

A Constraints on structure of non-diagonal $M_R$

We discuss constraints on the structure of non-diagonal $M_R$ in this section. In the section
3.4.2, we have considered a non-diagonal $M_R$ and effects from off-diagonal elements.

We have defined the diagonalized mass matrix of the right-handed neutrino, $D_R$, as
in $(3.41)$ and an unitary matrix, $U_{\nu R}$, which diagonalize the $M_R$, as in $(3.42)$. The resultant
neutrino mass after the seesaw mechanism and operating the tri-bimaximal mixing is given
in (3.46). Each matrix element is written down as

\[ M_{ij} \approx \frac{\lambda^{2d_{ij}}}{M} \left[ 1 + 4\lambda^{2d_{ij} - x_{ij}} + 4\lambda^{d_{ij}}\theta_{R,23} + \lambda^{-x_{ij}}\theta^2_{R,23} \right. \\
\left. + \lambda^{-x_{ij}}(4\lambda^{d_{ij}}\theta_{R,12} - 4\lambda^{d_{ij}}\theta_{R,13} + \theta^2_{R,13}) \right], \quad (A.1) \]

\[ M_{22} \approx \frac{\lambda^{2d_{12}}}{M} \left[ 3\lambda^{2d_{12} - x_{12}} + 1 + \frac{\lambda^{2d_{12} - x_{12}}}{3} + \lambda^{-x_{12}}(-\frac{2\lambda^{d_{12}}}{3}\theta_{R,23} + \theta^2_{R,23}) \right. \\
\left. + \lambda^{-x_{12}}(-2\lambda^{d_{12}}\theta_{R,13} + \theta^2_{R,13}) \right], \quad (A.2) \]

\[ M_{33} \approx \frac{\lambda^{2d_{13}}}{M} \left[ 2\lambda^{2d_{13} - x_{13}} + \frac{1}{2} + \lambda^{-x_{13}}(2\lambda^{d_{13}}\theta_{R,23}) + \frac{\theta^2_{R,23}}{2} - 4\lambda^{d_{13}}\theta_{R,12}\theta_{R,13} \right. \\
\left. + \lambda^{-x_{13}}(2\lambda^{2d_{13}}\theta^2_{R,12} + \frac{1}{2}\theta^2_{R,13}) \right], \quad (A.3) \]

\[ M_{12} \approx -\frac{\lambda^{2d_{12}}}{3\sqrt{2}M}[1 + \lambda^{d_{12} - x_{12}}2\theta_{R,23} + 3\lambda^{-x_{12}}(-2\lambda^{d_{12}}\theta_{R,12} + \lambda^{d_{12}}\theta_{R,13})], \quad (A.4) \]

\[ M_{23} \approx \frac{\lambda^{2d_{23}}}{\sqrt{6}M}[1 - 2\lambda^{d_{23} + d_{23} - x_{23}} + 2\lambda^{d_{23} - x_{23}}\theta_{R,23} \right. \\
\left. + 3\lambda^{-x_{23}}(2\lambda^{d_{23}}\theta_{R,12} - \lambda^{d_{23}}\theta_{R,13} - 2\lambda^{d_{23}}\theta_{R,12}\theta_{R,13} + \theta^2_{R,13}) \right], \quad (A.5) \]

\[ M_{13} \approx -\frac{\lambda^{2d_{13}}}{2\sqrt{3}M}[1 + \lambda^{-x_{13}}(4\lambda^{d_{13} + d_{23}} + 2\lambda^{d_{23}}\theta_{R,23} + \theta^2_{R,23}) \right. \\
\left. - \lambda^{-x_{13}}(2\lambda^{d_{13}}\theta_{R,12}\theta_{R,13} - 4\lambda^{d_{13} + d_{23}}\theta^2_{R,12} - \theta^2_{R,13}) \right], \quad (A.6) \]

where we have omitted terms which are trivially small compared with other terms. We require that the magnitudes of leading order of each term in this mass matrix are the same one as in the case of diagonal $M_R$ case, because the tri-bimaximal mixing can be realized at the leading order. This requirement leads to constraints on the mixing angles, $\theta_{R,ij}$, as

\[ \theta_{R,13} < \frac{3}{2}\lambda^{d_{13}}, \quad \frac{1}{3}\lambda^{d_{13} + x_{13}}, \quad 2\lambda^{d_{23} + (x_{13} - x_{23})/2}, \quad \frac{1}{\sqrt{3}}\lambda^{x_{13}/2}, \quad (A.7) \]

\[ \theta_{R,23} < \lambda^{d_{23} + x_{23}}, \quad \lambda^{d_{23}}, \quad \frac{1}{2}\lambda^{d_{13} + d_{23} + x_{23}} \quad \lambda^{x_{23}/2}, \quad (A.8) \]

\[ \theta_{R,12} < \frac{1}{6}\lambda^{d_{13} + d_{23} + x_{13}}, \quad \lambda^{(x_{13} - x_{23})/2}, \quad (A.9) \]

\[ \theta_{R,12}\theta_{R,13} < \lambda^{d_{23} + x_{13} - x_{23}}, \quad \frac{1}{6}\lambda^{d_{13} + d_{23} + x_{13}}. \quad (A.10) \]

If we fix the values of $(d_1, d_2, x_1, x_2)$, which must be satisfied the conditions $(3.27)$, we can determine the structure leading to maximal collections to the PMNS structure and neutrino mass spectra as shown in Tabs. 8 and 9.
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