Let $d \geq 1$ be an integer, $\mathbb{T}^d$ the $d$-dimensional torus, and

$$F : T^*\mathbb{T}^d \longrightarrow T^*\mathbb{T}^d$$

a $C^\infty$ twist map. Twist maps are examples of symplectic diffeomorphisms; they will be defined more precisely in part 1. We assume that $F$ is without conjugate points. This means that

$$\forall n \in \mathbb{Z} \setminus \{0\}, \forall (\dot{x}, p) \in T^*\mathbb{T}^d, DF^n(V(\dot{x}, p)) \cap V(F^n(\dot{x}, p)) = \{0\},$$

where $V(\dot{x}, p)$ denotes the vertical space at the point $(\dot{x}, p)$. It is proved in [1] that this implies that $F$ is $C^0$-integrable, i.e. there exists a continuous foliation of $T^*\mathbb{T}^d$, each leaf being a Lipschitz Lagrangian graph that is $F$-invariant.

It could be that all the leaves of this foliation are in fact smooth, but this is still an open question. However, some of them are indeed smooth: it is shown in [1] (proposition 3.1) that if $\overline{F}$ is a lift of $F$ to $T^*\mathbb{R}^d$ then for every $n \in \mathbb{N}$ and every $r \in \mathbb{Z}^d$, the set

$$\overline{G}^*_{N,r} = \{(x, p) \in T^*\mathbb{R}^d \text{ s.t. } \overline{F}^N(x, p) = (x + r, p)\}$$

is a $C^\infty$ Lagrangian $\overline{F}$-invariant graph. Its projection $G^*_{N,r}$ on $T^*\mathbb{T}^d$ is one of the leaves of the foliation. It is by definition a union of periodic orbits sharing the same period $N$.

Here we study the dynamics of $F$ in a neighborhood of $G^*_{N,r}$. We use a KAM theorem to show the existence of a rich family of $F$-invariant Lagrangian graphs accumulating on $G^*_{N,r}$, on which $F$ is conjugated to a translation of non-resonant vector. In fact, we have the following result:

**Theorem:** Let $\overline{\omega} \in \mathbb{R}^d$ be strongly Diophantine vector, i.e. there are real numbers $\gamma > 0$ and $\tau > 0$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \forall l \in \mathbb{Z}, |k \cdot \overline{\omega} + l| \geq \frac{\gamma}{|k|\tau}.$$

For every large integer $m$, there is a $C^\infty$ Lagrangian embedding $i_m : \mathbb{T}^d \longrightarrow T^*\mathbb{T}^d$ such that

1) $i_m(\dot{x}) = (\psi_m(\dot{x}), f_m(\dot{x}))$, where $\psi_m$ is a $C^\infty$ diffeomorphism of $\mathbb{T}^d$, isotopic to the identity, and

$$\mathcal{T}_m = i_m(\mathbb{T}^d) = \{(\dot{x}, (f_m \circ \psi_m^{-1})(\dot{x})); \dot{x} \in \mathbb{T}^d\}$$
is a Lagrangian graph; the sequence \((T_m)\) converges to \(G^*_N,r\) in \(C^\infty\) topology.

ii) The sequence \((\psi_m)\) converges in \(C^\infty\) topology to a diffeomorphism \(\psi_\infty\) (independant of \(\pi\)), isotopic to the identity.

iii) The tori \(T_m\) are \(F\)-invariant and the restriction of \(F\) to \(T_m\) is conjugated to a non-resonant translation. More precisely,

\[
\forall n \in \mathbb{Z}, \forall \dot{x} \in \mathbb{T}^d, F^n(i_m(\dot{x})) = i_m(\dot{x} + \frac{n}{N}r + \frac{n}{mN}\omega).
\]

Note that this gives us some insight into the dynamics of \(F\) restricted to \(G^*_N,r\).

**Corollary**: The diffeomorphism \(\psi_\infty\) conjugates the action of \(F\) on \(G^*_N,r\) to a translation of vector \(\frac{r}{N}\) on \(\mathbb{T}^d\).

In the case of a continuous flow associated to a Tonelli Hamiltonian, a similar result is established in [2]. Our strategy is to mimic the proof given in this article, but many problems arise when we switch from the continuous to the discrete case. For example, we can no more make use of a quantity that is constant along the orbits (as the Hamiltonian in the continous case), or derivate along the flow. As a result, some key parts of the proof need totally different arguments.

The paper is organized as follows. In section 1, we briefly recall some basic facts on twist maps and some results of [1]. In section 2, we explain how to find a normal form for \(F\) in the neighborhood of \(G^*_N,r\). This requires two lemmas which are proved in section 3. We then apply a KAM theorem in section 4 and explain the end of the proof of the theorem.

## 1. Twist maps without conjugate points

Here we give a brief introduction to the theory of twist maps. We refer the reader to [6] for a complete study. Let \(d \geq 1\) be an integer. Denote by \(\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d\) the \(d\)-dimensional torus. Let \(\mathbb{T}^*\mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^*\) be the cotangent space of \(\mathbb{R}^d\). Consider a generating function, that is a map \(S: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}\) of class \(C^\infty\) which satisfies the following two conditions:

(C1) \(\forall r \in \mathbb{Z}^d, \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, S(x + r, y + r) = S(x, y)\);

(C2) (‘uniform twist condition’, see [3]) There is a real number \(A > 0\) for which

\[
\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \forall \xi \in \mathbb{R}^d, \sum_{i,j} \frac{\partial^2 S(x, y)}{\partial x_i \partial y_j} (x, y) \xi_i \xi_j \leq -A||\xi||^2.
\]

A sequence \((x_n)_{n \in \mathbb{Z}}\) with values in \(\mathbb{R}^d\) is said to be extremal if it satisfies

\[
\forall n \in \mathbb{Z}, \partial_2 S(x_{n-1}, x_n) + \partial_1 S(x_n, x_{n+1}) = 0.
\]
Extremal sequences are the critical points of the (formal) action functionnal which assigns to each sequence \( (x_n)_{n \in \mathbb{Z}} \) the sum of the serie \( \sum_{n \in \mathbb{Z}} S(x_n, x_{n+1}) \). The generating function also gives rise to a symplectic diffeomorphism \( F \) of \( T^* \mathbb{T}^d \). Let \( \overline{F} : T^* \mathbb{R}^d \rightarrow T^* \mathbb{R}^d \) be the diffeomorphism implicitly defined by

\[
\overline{F}(x, p) = (x', p') \iff p = -\partial_1 S(x, x') \text{ and } p' = \partial_2 S(x, x').
\]

The diffeomorphism \( \overline{F} \) is exact symplectic, which means that \( \overline{F}^* \alpha - \alpha = dS \), where \( \alpha = \sum_{i=1}^d x_i dq_i \) is the Liouville 1-form on \( T^* \mathbb{R}^d \). Note that condition \( (C1) \) implies that \( \overline{F} \) is the lift to \( T^* \) of a symplectic diffeomorphism \( F \) of \( T^* \mathbb{T}^d \).

Let \( pr_1 : (\dot{x}, p) \in T^* \mathbb{T}^d \mapsto \dot{x} \in \mathbb{T}^d \) be the canonical projection. The vertical space at \((\dot{x}, p) \in T^* \mathbb{T}^d \) is \( V(\dot{x}, p) = \text{Ker} Dpr_1(\dot{x}, p) \). We say that the twist map \( F \) is without conjugate points if

\[
\forall n \in \mathbb{Z} \setminus \{0\}, \forall (\dot{x}, p) \in T^* \mathbb{T}^d, \ DF^n(V(\dot{x}, p)) \cap V(F^n(\dot{x}, p)) = \{0\}.
\]

This hypothesis has strong consequences on the behaviour of extremal sequences. It is shown in [1] (corollary 1.5) that if \( F \) is without conjugate points, then for every \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d \) and every integer \( N \geq 1 \), there is a unique extremal sequence \( (x_n)_{n \in \mathbb{Z}} \) with \( x_0 = x \) et \( x_N = y \). Moreover, this extremal sequence minimizes the action in the following sense. If \( k \) and \( l \) are two integers with \( l - k \geq 2 \), then, letting \( y_k = x_k \) and \( y_l = x_l \), one has

\[
\forall (y_{k+1}, \ldots, y_{l-1}) \in (\mathbb{R}^d)^{l-k-1}, \sum_{i=k}^{l-1} S(x_i, x_{i+1}) \leq \sum_{i=k}^{l-1} S(y_i, y_{i+1}),
\]

and equality holds if and only if \( y_i = x_i \) for every \( i \in \{k+1, \ldots, l-1\} \).

Let \( r \in \mathbb{Z}^d, N \geq 1 \) an integer, and \( x \in \mathbb{R}^d \). Consider the extremal sequence \( (x_n)_{n \in \mathbb{Z}} \) with \( x_0 = x \) and \( x_N = x_0 + r \). It is a non-trivial fact (see [1], proposition 2.1) that we have

\[
\forall n \in \mathbb{Z}, x_{n+N} = x_n + r.
\]

We can use this to construct periodic orbits of \( F \). In fact, define a sequence \( (p_n) \) with values in \((\mathbb{R}^d)^*\) as follows:

\[
\forall n \in \mathbb{Z}, p_n = -\partial_1 S(x_n, x_{n+1}).
\]

It is periodic (this is a consequence of \( (C1) \)). Moreover, \( (x_n, p_n)_{n \in \mathbb{Z}} \) is an orbit of \( \overline{F} \) whose projection to \( T^* \mathbb{T}^d \) is periodic with period \( N \). Letting \( x \) vary in \( \mathbb{R}^d \), we obtain a subset \( G_{N, r}^* \) of \( T^* \mathbb{T}^d \) that is a graph over the whole of \( \mathbb{T}^d \). It is a union of periodic orbits of \( F \) and is therefore \( F \)-invariant. A result of [1] (proposition 3.1) is that this graph is of class \( C^\infty \) and Lagrangian.

The aim of this paper is to study the dynamics of \( F \) in a neighborhood of \( G_{N, r}^* \). We will make use of the following tool. Assume that \( \Phi \) is a symplectic diffeomorphism of
that leaves invariant the null section \(0_{T^d} = \{ (\dot{x},0), \dot{x} \in \mathbb{T}^d \} \). For \( \varepsilon > 0 \), consider the map 
\[
\mathcal{R}_\varepsilon : (x,p) \in T^*\mathbb{T}^d \mapsto (x,\varepsilon p) \in T^*\mathbb{T}^d.
\]
Since \( \mathcal{R}_\varepsilon^*\alpha = \varepsilon\alpha \), \( \Phi_\varepsilon = \mathcal{R}_\varepsilon^{-1} \circ \Phi \circ \mathcal{R}_\varepsilon \) is, just as \( \Phi \), a symplectic diffeomorphism of \( T^*\mathbb{T}^d \) with \( \Phi_\varepsilon(0_{T^d}) = 0_{T^d} \). The study of \( \Phi_\varepsilon \) when \( \varepsilon \) goes to 0 gives us an insight into how \( \Phi \) behaves near \( 0_{T^d} \). Note that when \( \Phi = F \) is the twist map associated to the generating function \( S \), then \( \Phi_\varepsilon \) is the twist map associated to the generating function \( S_\varepsilon \).

We shall use the following notations. \( x \) always denotes a point in \( \mathbb{R}^d \), while \( \dot{x} \) refers to an element of \( T^d \). \(< \cdot,\cdot > : \mathbb{R}^d \times (\mathbb{R}^d)^* \rightarrow \mathbb{R} \) is the duality bracket. If \( M \) is a matrix or a linear operator, we note \( M^T \) its transpose and (if \( M \) is invertible) \( M^{-1} \) the inverse of \( M^T \).

2. A normal form for \( F^N \)

We fix once and for all an integer \( N \geq 1 \) and \( r \in \mathbb{Z}^d \). The Lagrangian graph \( \mathcal{G}^*_N,r \) may be written as 
\[
\mathcal{G}^*_N,r = \{ (\dot{x},p_\infty + du(\dot{x})), \dot{x} \in \mathbb{T}^d \},
\]
with \( p_\infty \in (\mathbb{R}^d)^* \) and \( u : \mathbb{T}^d \rightarrow \mathbb{R} \) a \( C^\infty \) map. In this section, we explain how to obtain a normal form for \( F^N \) in the neighborhood of \( \mathcal{G}^*_N,r \). The precise statement is as follows.

**Proposition 1** : There exists a symplectic \( C^\infty \) diffeomorphism \( G \) of \( T^*\mathbb{T}^d \) of the form 
\[
G(\dot{x},p) = (\psi(\dot{x}),p_\infty + Du(\psi(\dot{x})) + D\psi(\dot{x})^{-T}p),
\]
where \( \psi \) is a diffeomorphism of \( \mathbb{T}^d \) isotopic to the identity, such that \( G(0_{T^d}) = \mathcal{G}^*_N,r \), and
\[
G^{-1} \circ F^N \circ G(\dot{x},p) = (\dot{x} + \overline{B}p + O(p^2), p + O(p^3)),
\]
where \( \overline{B} \in L((\mathbb{R}^d)^*,\mathbb{R}^d) \) is symmetric positive definite.

**Proof** : Consider the symplectic change of variables 
\[
G_0(\dot{x},p) = (\dot{x}, p + p_\infty + du(\dot{x})),
\]
and the generating function \( S_0(x, y) = S(x, y) - u(x) + u(y) + < p_\infty, y - x > \). It satisfies the conditions (C1) and (C2). The associated exact symplectic diffeomorphism of \( T^*\mathbb{T}^d \) is \( F_0 = G_0^{-1} \circ F \circ G_0 \). Since \( G_0 \) preserves the fibers and its restriction to each fiber is a translation, \( F_0 \) is without conjugate points. By definition of \( \mathcal{G}^*_N,r \), we have 
\[
\forall \dot{x} \in \mathbb{T}^d, F_0^N(\dot{x},0) = (\dot{x},0).
\]
This implies that the differential of \( F_0^N \) at \( (\dot{x},0) \) takes the form 
\[
DF_0^N(\dot{x},0)[\delta x, \delta p] = (\delta x + B(\dot{x})\delta p, D(\dot{x})\delta p),
\]

4
with $B(\dot{x}) \in L((\mathbb{R}^d)^*, \mathbb{R}^d)$ and $D(\dot{x}) \in L((\mathbb{R}^d)^*, (\mathbb{R}^d)^*)$. Moreover $DF_0^N(\dot{x}, 0)$ is a symplectic linear map, so $D(\dot{x}) = i\text{d}_{(\mathbb{R}^d)^*}$, and $B(\dot{x})$ is symmetric. In fact, we can say more about $B(\dot{x})$:

**Lemma 1:** $B(\dot{x})$ is symmetric positive definite.

This lemma will be proved in the next section. As a consequence, we can define a Riemannian metric $g$ on $T^d$:

$$\forall \dot{x} \in T^d, \forall v, v' \in \mathbb{R}^d, g((\dot{x}, v), (\dot{x}, v')) = <B(\dot{x})^{-1}v, v'>.$$ 

The next step is to prove that $g$ enjoys a rather strong property:

**Lemma 2:** The metric $g$ is without conjugate points.

Once again, we postpone the proof to the next section. D. Burago and S. Ivanov proved (see [4]) that any Riemannian metric on the torus that is free of conjugate points must be flat. So $g$ is flat: there exists a $C^\infty$ diffeomorphism $\psi$ of $T^d$ isotopic to the identity and a symmetric positive definite $\overline{A} \in L(\mathbb{R}^d, (\mathbb{R}^d)^*)$ such that

$$\forall x \in T^d, \forall v, v' \in \mathbb{R}^d, \psi(\dot{x})^{-1}D\psi(\dot{x}) \cdot v, D\psi(\dot{x}) \cdot v' = \overline{A}v, v'.$$

Let $B = \overline{A}^{-1}$. Then

$$\forall \dot{x} \in T^d, B = D\psi^{-1}(\dot{x})B(\dot{x})D\psi(\dot{x})^{-T}.$$ 

Consider the symplectic diffeomorphism of $T^d$: $G_1(\dot{x}, p) = (\psi(\dot{x}), D\psi(\dot{x})^{-T})$ and the composition $G = G_0 \circ G_1$. We have

$$G(\dot{x}, p) = (\psi(\dot{x}), p_\infty + Du(\psi(\dot{x})) + D\psi(\dot{x})^{-T}p).$$

Let $F_1 = G_1^{-1} \circ F_0 \circ G_1 = G^{-1} \circ F \circ G$. It satisfies

$$\forall \dot{x} \in T^d, F_1^N(\dot{x}, 0) = (\dot{x}, 0) \text{ and } D_pF_1^N(\dot{x}, 0) = \overline{B}. $$

Then the next lemma applied to $F_1^N$ implies that $F_1^N(\dot{x}, p) = (\dot{x} + \overline{A}p + O(p^2), p + O(p^3))$ as desired.

**Lemma 3:** Let $F : T^*T^d \longrightarrow T^*T^d$ be an exact symplectic diffeomorphism. Assume that $F$ fixes every point of $0_{T^d}$. Let $\overline{B}(\dot{x}) = D_pF(\dot{x}, 0)$ for every $\dot{x} \in T^d$. Then

$$F(\dot{x}, p) = (\dot{x} + \overline{B}(\dot{x})p + O(p^3), p - \frac{1}{2}D_p < B(\dot{x})p, p > + O(p^3)).$$

The proof of this lemma is given in [2] (page 182). Note that it uses in a crucial way the maps $\Phi_\varepsilon$ introduced at the end of the first section.
3. Proof of lemma 1 and lemma 2

We begin with the proof of lemma 1. Let \( \dot{x}_0 \in \mathbb{T}^d \) and \((\dot{x}_n)\) the sequence of points in \( \mathbb{T}^d \) defined as \( F^n_0(\dot{x}_0, 0) = (\dot{x}_n, 0) \). As \( F_0 \) preserve \( 0_{\mathbb{T}^d} \), the matrix of \( D F_0(\dot{x}_i, 0) \) in the canonical basis is a symplectic matrix of the form

\[
m_i = \begin{pmatrix} a_i & b_i \\ O & d_i \end{pmatrix},
\]

hence the matrix \( s_i := b_i d_i \) symmetric. It is straightforward to check that if two matrices of the type \( m_i \) are such that the \( s_i \) are positive definite, then the same property holds for their product. By an immediate induction, the same is true for a product of any number of such matrices.

For all \( \dot{x} \in \mathbb{T}^d \), \( F_0^N(\dot{x}, 0) = (\dot{x}, 0) \), so that the matrix of \( D F_0^N(\dot{x}_0, 0) \) may be written as

\[
M_N = \begin{pmatrix} I_d & B_N \\ O_d & D_N \end{pmatrix}.
\]

\( M \) is symplectic, hence \( D_N = I_d \) and \( B_N \) is symmetric. As a consequence of the chain rule, \( M = m_{N-1}m_{N-2}\ldots m_1m_0 \). So it only remains to prove that each \( s_i \) is positive definite to get the conclusion that \( B_N \) is also positive definite. As \( \dot{x}_0 \) is an arbitrary point in \( \mathbb{T}^d \), we only need to check that \( s_0 \) is positive definite.

It is possible to express the differential of a twist map in terms of its generating function: the result is that

\[
m_0 = \begin{pmatrix} -\partial_{12}S(x_0, x_1)^{-1}\partial_{11}S(x_0, x_1) & -\partial_{12}S(x_0, x_1)^{-1} \\ O_d & -\partial_{22}S(x_0, x_1)\partial_{12}S(x_0, x_1)^{-1} \end{pmatrix}.
\]

Therefore \( s_0 = \partial_{12}S(x_0, x_1)^{-T}\partial_{22}S(x_0, x_1)\partial_{12}S(x_0, x_1)^{-1} \), and we have to show that \( \partial_{22}S(x_0, x_1) \) is positive definite to finish the proof.

Let \( \overline{F}_0 \) be a lift of \( F_0 \) to \( T^*\mathbb{R}^d \) and \( x_0 \in \mathbb{R}^d \). For any integer \( n \), we have \( \overline{F}_0^n(x_0, 0) = (x_n, 0) \) for some \( x_n \in \mathbb{R}^d \). For every \( p \in (\mathbb{R}^d)^* \), let \( x_n(p) = \text{pr}_1(\overline{F}_0^n(x_0, p)) \). Clearly \( x_0(p) = x_0 \) for all \( p \), and \( x_n(x_n(0)) \) for all \( n \). Consider the action functional

\[
\mathcal{A}_n(p) = S(x_0, x_1(p)) + \sum_{i=1}^{n-1} S(x_i(p), x_{i+1}(p)) + S(x_n(p), x_{n+1}).
\]

Since \( \overline{F} \) is without conjugate points, we know (see part 1) that \( \mathcal{A}_n \) admits a global strict minimum at \( p = 0 \). We compute

\[
D \mathcal{A}_n(p) = \partial_2 S(x_0, x_1(p)) Dx_1(p) + \sum_{i=1}^{n-1} D_i + \partial_1 S(x_n(p), x_{n+1}) Dx_{n+1}(p),
\]

where \( D_i = \partial_1 S(x_i(p), x_{i+1}(p)) Dx_i(p) + \partial_2 S(x_i(p), x_{i+1}(p)) Dx_{i+1}(p) \).
As \((x_0, x_1(p), x_2(p), \ldots, x_{n+1}(p))\) is an extremal sequence, we have
\[
\partial_2 S(x_{i-1}(p), x_i(p)) + \partial_1 S(x_i(p), x_{i+1}(p)) = 0
\]
for all \(i\), so almost all terms in \(D\mathcal{A}_n(p)\) cancel out and we are left with
\[
D\mathcal{A}_n(p) = [\partial_2 S(x_{n-1}(p), x_n(p)) + \partial_1 S(x_n(p), x_{n+1}(p))]Dx_n(p).
\]
This may be rewritten as
\[
D\mathcal{A}_n(p) = [-\partial_1 S(x_n(p), x_{n+1}(p)) + \partial_1 S(x_n(p), x_{n+1}(p))]Dx_n(p),
\]
whence the following expression for the second differential of \(\mathcal{A}_n\):
\[
D^2\mathcal{A}_n(0) : (v, v') \mapsto -\partial_{12} S(x_n, x_{n+1})(Dx_n(0) \cdot v, Dx_{n+1}(0) \cdot v').
\]
This implies that \(S_n = -B_n^T \partial_{12} S(x_n, x_{n+1})B_{n+1}\) is the (symmetric positive definite) matrix of \(D^2\mathcal{A}_n(0)\).

From now on, we assume that \(n = kN\) for some integer \(k\). Hence we have \(x_n = x_0 + kr, x_{n+1} = x_1 + kr\) and condition (C1) implies \(\partial_{12} S(x_n, x_{n+1}) = \partial_{12} S(x_0, x_1)\). Using the same argument and the chain rule, the matrix of \(DF^0_0(x_0, 0) = DF^{kN}_0(x_0, 0)\) is
\[
M_{kN} = \begin{pmatrix} I_d & B_{kN} \\ O_d & I_d \end{pmatrix} = \begin{pmatrix} I_d & kB_N \\ O_d & I_d \end{pmatrix},
\]
so that \(B_n = B_{kN} = kB_N\).

To compute \(B_{kN+1}\), we use once again the matrix \(m_0\) introduced above. The relation \(DF^{kN+1}_0(x_0, 0) = DF^0_0(x_{kN}, 0) \circ DF^{kN}_0(x_0, 0)\) implies that \(M_{kN+1} = m_0 M_{kN}\), so we get
\[
B_{kN+1} = -k\partial_{12} S(x_0, x_1)^{-1}\partial_{11} S(x_0, x_1)B_N - \partial_{12} S(x_0, x_1)^{-1},
\]
and finally
\[
S_{kN} = k^2 B_N \partial_{11} S(x_0, x_1)B_N + kB_N.
\]

Dividing by \(k^2\) and letting \(k\) go to infinity implies that \(B_N \partial_{11} S(x_0, x_1)B_N\) is positive semi-definite. But \(F_0\) has no conjugate points, so \(B_N\) is invertible and \(\partial_{11} S(x_0, x_1)\) is positive semi-definite. The matrix \(m_0\) being symplectic, we have
\[
\partial_{11} S(x_0, x_1)^t \partial_{12} S(x_0, x_1)^{-1} \partial_{22} S(x_0, x_1) \partial_{12} S(x_0, x_1)^{-1} = I_d.
\]
This implies that \(\partial_{11} S(x_0, x_1)^t\) is invertible, so it has to be positive definite, as well as \(\partial_{22} S(x_0, x_1)\).

The proof of lemma 2 is close to the one given [2] (page 182 and 183), so we will only sketch it, trying to put into perspective the main ideas, and referring to [2] for
technical details. We lift $g$ to a $\mathbb{Z}^d$-periodic Riemannian metric on $\mathbb{R}^d$ and consider the corresponding Hamiltonian function

$$H : (x, p) \in T^* \mathbb{R}^d \mapsto \frac{1}{2} < B(x)p, p > \in \mathbb{R},$$

with Hamiltonian vector field $X_H$ and Hamiltonian flow $(\tilde{\phi}_t^H)_{t \in \mathbb{R}}$. To prove the absence of conjugate points, we argue by contradiction. As explained in [2], if $g$ had conjugate points then we could find two points $x$ and $y$ in $\mathbb{R}^d$ connected in time $S > 0$ by two distinct non-degenerate geodesics. So there would be $p_1$ and $p_2$ in $(\mathbb{R}^d)^*$ such that

$$\text{pr}_1 \circ \tilde{\phi}_S^H(x, p_1) = \text{pr}_1 \circ \tilde{\phi}_S^H(x, p_2) = y$$

(1)

with $D_p \text{pr}_1 \circ \tilde{\phi}_S^H(x, p_1)$ and $D_p \text{pr}_1 \circ \tilde{\phi}_S^H(x, p_2)$ invertible.

We now consider $\mathfrak{T}_0 : T^* \mathbb{R}^d \rightarrow T^* \mathbb{R}^d$ a lift of $F_0$. If we could replace $\tilde{\phi}_S^H$ by $\mathfrak{T}_0^N$ in (1), we would get a contradiction since $F_0$ is without conjugate points. So we try to find some link between $\tilde{\phi}_S^H$ and $\mathfrak{T}_0^N$. Once again, consider the maps

$$\Phi_\varepsilon = R^{-1}_\varepsilon \circ F_0^N \circ R_\varepsilon : T^* \mathbb{T}^d \rightarrow T^* \mathbb{T}^d$$

and a lift $\overline{\Phi}_\varepsilon : T^* \mathbb{R}^d \rightarrow T^* \mathbb{R}^d$ (chosen in such a way that $\overline{\Phi}_\varepsilon$ is close to the identity when $\varepsilon \rightarrow 0$). Then lemma 3 implies that

$$\overline{\Phi}_\varepsilon(x, p) = (x + \varepsilon B(x)p + O(\varepsilon^2), p - \frac{\varepsilon}{2} D_x < B(x)p, p > + O(\varepsilon^2))$$

and hence

$$\overline{\Phi}_\varepsilon(x, p) = (x, p) + \varepsilon X_H(x, p) + O(\varepsilon^2).$$

This is reminiscent of the Euler method used for numerical integration of ordinary differential equations. So we may hope that if $\varepsilon$ is small enough and $n$ not too large, $\overline{\Phi}^n_\varepsilon(x, p)$ won’t be very far from $\tilde{\phi}_m^H(x, p)$. This turns out to be true, we refer to Lemma 2.1 of [2] for a rigorous statement. Applied to our case, it implies that when $m$ goes to infinity, the sequence $(\overline{\Phi}_{S/m}^m)$ converges to $\tilde{\phi}_S^H$ on coimpact sets in topology $C^1$. As $D_p \text{pr}_1 \circ \tilde{\phi}_S^H(x, p_1)$ and $D_p \text{pr}_1 \circ \tilde{\phi}_S^H(x, p_2)$ are invertible, we may use the implicit function theorem to obtain that when $m$ is large enough, we can find $p'_1$ close to $p_1$ and $p'_2$ close to $p_2$ such that

$$\text{pr}_1 \circ \overline{\Phi}_{S/m}^m(x, p'_1) = \text{pr}_1 \circ \overline{\Phi}_{S/m}^m(x, p'_2) = y.$$ 

It is easy to check that we then have

$$\text{pr}_1 \circ \mathfrak{T}_0^N(x, p'_1) = y + mr = \text{pr}_1 \circ \mathfrak{T}_0^N(x, p'_2),$$

and hence a contradiction.
4. Construction of the non-resonant tori

The existence of these tori is given by the following proposition. Its proof is not complicated and is very similar to the proof of Proposition 7 in [2], so we will not repeat it here. The main idea is to apply a KAM theorem (theorem 1.2.3 in [5]) to the family of symplectic symplectic maps $U_m = \Phi_{1/m}^m = R_{1/m}^{-1} \circ F^m \circ R_{1/m}$.

**Proposition 2**: Let $F : T^*\mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ be a $C^\infty$ symplectic diffeomorphism. Assume that on a neighborhood of 0$_{\mathbb{T}^d}$,

$$F(\dot{x}, p) = (\dot{x} + \overline{B}p + O(p^2), p + O(p^3)),$$

where $\overline{B} \in L((\mathbb{R}^d)^*, \mathbb{R}^d)$ is symmetric non-degenerate. Let $\overline{\omega} \in \mathbb{R}^d$ be strongly Diophantine. Then for any large $m$ there is a $C^\infty$ Lagrangian embedding $j_m : \mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ such that

$$\forall \dot{x} \in \mathbb{T}^d, F^m(j_m(\dot{x})) = j_m(\dot{x} + \overline{\omega}).$$

Moreover, $j_m$ is of the following form:

$$j_m(\dot{x}) = (\dot{x} + u_m(\dot{x}), \overline{B}^{-1}\left(\frac{\overline{\omega}}{m}\right) + v_m(\dot{x})),$$

with $u_m : \mathbb{T}^d \rightarrow \mathbb{R}^d$ and $v_m : \mathbb{T}^d \rightarrow (\mathbb{R}^d)^*$ of class $C^\infty$, and, for any $k$,

$$||u_m||_{C^k(\mathbb{T}^d)} = o(1) \text{ and } ||v_m||_{C^k(\mathbb{T}^d)} = o\left(\frac{1}{m}\right) \text{ as } m \rightarrow \infty.$$

Let $F_1 = G^{-1} \circ F \circ G$, where $G$ is the symplectic diffeomorphism given by proposition 1. According to this proposition, we have

$$F_1^N(\dot{x}, p) = G^{-1} \circ F^N \circ G(\dot{x}, p) = (\dot{x} + \overline{B}p + O(p^2), p + O(p^3)),$$

so that we may apply proposition 2 to $F_1^N$. Consider the set $j_m(\mathbb{T}^d)$. It is clearly invariant by $F_1^{Nm}$, but the following stronger result holds.

**Lemma 4**: $j_m(\mathbb{T}^d)$ is $F_1^{-1}$-invariant.

**Proof**: $F$ is without conjugate points, so $T^*\mathbb{T}^d$ is the disjoint union of $F$-invariant graphs $(g_i)_{i \in I}$, and hence a disjoint union of the $F_1$-invariant graphs $(G^{-1}(g_i))_{i \in I}$. Pick $j_m(\dot{x}_0, p_0) \in j_m(\mathbb{T}^d)$, it belongs to some $G^{-1}(g_i)$. As $G^{-1}(g_0)$ is $F_1$-invariant, we only need to show that $j_m(\mathbb{T}^d) = G^{-1}(g_0)$. Consider

$$E = \{\dot{x} \in \mathbb{T}^d \text{ s.t. } j_m(\dot{x}) \in G^{-1}(g_0)\}.$$

Note that if $\dot{x} \in E$, then so does $\dot{x} + \overline{\omega}$, since $j_m(\dot{x} + \overline{\omega}) = F_1^{Nm}(j_m(\dot{x}))$ and $G^{-1}(g_0)$ is $F_1$-invariant. The point $\dot{x}_0$ belongs to $E$, hence $E$ contains $\dot{x}_0 + k\overline{\omega}$ for all integer $k$. 9
These points are dense in $\mathbb{T}^d$ and $E$ is clearly closed, so $E = \mathbb{T}^d$ and $j_m(\mathbb{T}^d) \subset G^{-1}(g_0)$. Then the compact manifold $j_m(\mathbb{T}^d)$ is included in the connected manifold $G^{-1}(g_0)$, and they have the same dimension, so they coincide.

For any integer $n$ and any (large) integer $m$, we may then consider the map

$$\alpha_{m,n} = j_m^{-1} \circ F_1^n \circ j_m : \mathbb{T}^d \longrightarrow \mathbb{T}^d.$$  

According to proposition 2, $\alpha_{m,Nm}$ is the translation $\tau_{\omega}$ of vector $\omega$. Since $F_1^n$ commutes with $F_1^{Nm}$, $\alpha_{m,m}$ commutes with $\alpha_{m,Nm} = \tau_{\omega}$. Using the same topological arguments as in the proof of lemma 4, we may conclude that every $\alpha_{m,n}$ is a translation. Moreover we clearly have $\alpha_{n+1,m} = \alpha_{n,m} \circ \alpha_{1,m}$ for any integer $n$, so if $\alpha_{m,1}$ is the translation of vector $\beta_m$, then $\alpha_{m,n}$ is the translation $n\beta_m$. As $\alpha_{m,Nm} = \tau_{\omega}$, we get

$$\exists k_m \in \mathbb{Z}^d \text{ s.t. } Nm\beta_m = \omega + k_m. \quad (2)$$

We are going to show that if $m$ is large enough, then $k_m = mr$. When $m$ goes to infinity, $j_m$ converges uniformly to $j_\infty : \hat{x} \in \mathbb{T}^d \longmapsto (\hat{x}, 0) \in T^*\mathbb{T}^d$. Let $f_1 : \mathbb{T}^d \longrightarrow \mathbb{T}^d$ be the map such that $F_1(\hat{x}, 0) = (f_1(\hat{x}), 0)$ for all $\hat{x} \in \mathbb{T}^d$. Then

$$\text{pr}_1(j_m(\hat{x} + \beta_m)) = \text{pr}_1 \circ j_m \circ \alpha_{m,1}(\hat{x}) = \text{pr}_1 \circ F_1 \circ j_m(\hat{x}).$$

The right-hand side converges to $f_1(\hat{x})$, while the left-hand is equal to

$$\hat{x} + \beta_m + u_m(\hat{x} + \beta_m) = \hat{x} + \beta_m + o(1).$$

This implies that the sequence $(\beta_m)$ converges to some vector $\beta_\infty$ and that $f_1$ is the translation of vector $\beta_\infty$.

Recall that the map $\psi : \mathbb{T}^d \longrightarrow \mathbb{T}^d$ given by proposition 1 is isotopic to the identity. So if $\overline{\psi}$ denotes a lift of $\psi$ to $\mathbb{R}^d$, $\overline{\psi}$ commutes with the translation of vector $r$. Let $\overline{G}$ be a lift of $G$ to $T^*\mathbb{R}^d$ with $\text{pr}_1 \circ \overline{G} = \overline{\psi}$. Then $\overline{F}_1 = \overline{G}^{-1} \circ \overline{F} \circ \overline{G}$ is a lift of $F_1$ to $T^*\mathbb{R}^d$. We now compute $\overline{F}_1^N(x, 0) = \overline{G}^{-1} \circ \overline{F}^N \circ \overline{G}(x, 0)$ for any $x \in \mathbb{R}^d$. To begin with, $\overline{G}(x, 0) = (\overline{\psi}(x), p)$ for some $p \in (\mathbb{R}^d)^*$. As $\overline{G}(0_{\mathbb{R}^d}) = \overline{G}_{N,r}$, $\overline{F}^N(\overline{\psi}(x), p) = (\overline{\psi}(x) + r, p)$. Since $\overline{\psi}$ commutes with the translation of vector $r$, we finally have $\overline{F}_1^N(x, 0) = \overline{G}^{-1}(\overline{\psi}(x) + r, p) = (x + r, 0)$. Since $F_1(\hat{x}, 0) = (\hat{x} + \beta_\infty, 0)$, this implies that $\beta_\infty = \overline{\psi}$, so that

$$N\beta_m = r + o(1) \quad (3)$$

We know that $\alpha_{m,N} = j_m^{-1} \circ F_1^N \circ j_m$ is the translation of vector $N\beta_m$. This implies

$$\forall \hat{x} \in \mathbb{T}^d, \ F_1^N(j_m(\hat{x})) = j_m(\hat{x} + N\beta_m).$$

According to proposition 2, we have

$$F_1^N(j_m(\hat{x})) = F_1^N(\hat{x} + u_m(\hat{x}), B^{-1}(\frac{\omega}{m}) + v_m(\hat{x})).$$
Using the estimates on $u_m$ and $v_m$ given by proposition 2 and proposition 1, we get

$$F_1^N(j_m(\dot{x})) = (\dot{x} + u_m(\dot{x}) + \frac{\omega}{m} + o\left(\frac{1}{m}\right), \overline{B}^{-1}\left(\frac{\omega}{m}\right) + o\left(\frac{1}{m}\right))$$  \hspace{1cm} (4)

On the other hand,

$$j_m(\dot{x} + N\beta_m) = (\dot{x} + N\beta_m + u_m(\dot{x} + N\beta_m), \overline{B}^{-1}\left(\frac{\omega}{m}\right) + o\left(\frac{1}{m}\right))$$  \hspace{1cm} (5)

Comparing (4) and (5) leads to: there is a vector $l_m \in \mathbb{Z}^d$ such that

$$\forall \dot{x} \in \mathbb{T}^d, \ N\beta_m + u_m(\dot{x} + N\beta_m) = u_m(\dot{x}) + \frac{\omega}{m} + l_m + o\left(\frac{1}{m}\right)$$

Taking the mean value when $\dot{x}$ varies in $\mathbb{T}^d$, we obtain

$$N\beta_m = \frac{\omega}{m} + l_m + o\left(\frac{1}{m}\right).$$  \hspace{1cm} (6)

By (3) and (6), $l_m = r + o(1)$. As $l_m$ and $r$ are both vectors of $\mathbb{Z}^d$, $l_m = r$ if $m$ is large enough. Equation (6) becomes

$$Nm\beta_m = \omega + mr + o(1).$$  \hspace{1cm} (7)

Comparing (2) and (7), we have $k_m = mr + o(1)$ and this implies as above that $k_m = mr$ for large $m$. Equation (2) now states that

$$\beta_m = \frac{\omega}{Nm} + \frac{r}{N}.$$  

To finish the proof, simply define $i_m : \mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ as

$$i_m = G \circ j_m = (\psi_m, f_m).$$

Then $\psi_m(\dot{x}) = \psi(\dot{x} + u_m(\dot{x}))$, so that the sequence converges in $C^\infty$ topology to $\psi$. The set $i_m(\mathbb{T}^d)$ is (as $j_m(\mathbb{T}^d)$) a Lagrangian manifold, and it is a graph because $\psi$ is a diffeomorphism.
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