Global Well-posedness of the Stochastic Kuramoto-Sivashinsky Equation with Multiplicative Noise*†

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Abstract

Global well-posedness of the initial-boundary value problem for the stochastic Kuramoto-Sivashinsky equation in a bounded domain $D$ with a multiplicative noise is studied. It is shown that under suitable sufficient conditions, for any initial data $u_0 \in L^2(D \times \Omega)$ this problem has a unique global solution $u$ in the space $L^2(\Omega, C([0, T], L^2(D)))$ for any $T > 0$, and the solution map $u_0 \mapsto u$ is Lipschitz continuous.

Keywords: Kuramoto-Sivashinsky equation; stochastic partial differential equations; multiplicative noise; well-posedness.

1 Introduction

The deterministic Kuramoto-Sivashinsky equation was independently proposed by Kuramoto [16] and Sivashinsky [18] as a model describing the instability and turbulence of wave fronts in chemical reaction and the laminar flames. It also has many applications in other fields of physics, chemistry, and biology. We refer the reader to see [1], [12], [15], [19], [20], [24] and references cited therein for the study on the deterministic Kuramoto-Sivashinsky equation and its generalizations.

In the present paper, we consider the following initial-boundary value problem of the stochastic generalized Kuramoto-Sivashinsky equation driven by a multiplicative noise:

$$\begin{cases}
\partial_t u + \Delta^2 u + \Delta u + \text{div} f(u) = \dot{V}_t, & x \in D, \ t > 0, \\
u|_{\partial D} = \Delta u|_{\partial D} = 0, & t > 0, \\
u|_{t=0} = u_0, & x \in D.
\end{cases}$$ (1.1)

Here $D$ is a bounded domain in $\mathbb{R}^d$ with a smooth boundary, $f$ is a given $d$-vector function, $\dot{V}_t$ is a multiplicative noise (see (1.2) below), and $u_0$ is a given initial $L^2(D)$-valued random variable. The

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noise is defined in a probability space \((\Omega, \mathcal{F}, P)\) and usually we omit the dependence on samples \(\omega \in \Omega\) in various variables.

The stochastic generalized Kuramoto-Sivashinsky equation (1.1) is a natural extension of the deterministic Kuramoto-Sivashinsky equation subject to random influences. In [10] Duan and Ervin considered the stochastic Kuramoto-Sivashinsky equation with an additive noise. They proved global well-posedness of the one-dimensional stochastic Kuramoto-Sivashinsky equation with an additive noise in the \(L^2\) space. The purpose of the present paper is to study the stochastic generalized Kuramoto-Sivashinsky equation in the case of multiplicative noise. By using the truncation method combined with the \(L^2\) conservation law of the Kuramoto-Sivashinsky equation, we shall prove that the stochastic generalized Kuramoto-Sivashinsky equation (1.1) is globally well-posed in the \(L^2\) space. Here we remark that in [10], in order to establish \(L^2\) well-posedness of the stochastic Kuramoto-Sivashinsky equation with additive noise, the authors used a variable transformation to transform the stochastic equation into a deterministic equation with the sample point variable as a parameter. This method clearly does not work for the present multiplicative noise case.

During the past twenty years, great advancement has been made to the study of stochastic partial differential equations. Some general theories for such equations have been well-established, see, for instance, [5], [9], [11] and the references cited therein. We note that despite that the Kuramoto-Sivashinsky equation is a parabolic equation, the general theory of parabolic stochastic partial differential equations developed in the above-mentioned literatures does not apply to the stochastic Kuramoto-Sivashinsky equation with a multiplicative noise. This is because that in such a theory the nonlinearity is required to be of the asymptotically linear type, whereas the Kuramoto-Sivashinsky equation has a quadratic nonlinear term. Note that for the stochastic nonlinear wave equations with certain polynomial nonlinear terms, this difficulty can be overcome with the aid of the \(H^1\) conservation law of the nonlinear wave equations (cf. [6]–[8]). For the Kuramoto-Sivashinsky equation and its generalized forms, we have only the \(L^2\) conservation law but not any other higher-order conservation laws. It follows that growth condition in the generalized Kuramoto-Sivashinsky equations is more restrictive than the nonlinear wave equations. Similar features are possessed by the stochastic Burgers equations (cf. [13], [14] and [21]) and the stochastic Navier-Stokes equations (cf. [2], [3] and [17]). However, for these equations, since they are of the second-order, in order to get well-posedness of the initial and initial-boundary value problems, the space dimension \(d\) is required to be not greater than 2, and for the case \(d \geq 3\) we have only existence of weak solutions but not any well-posedness result (cf. [2], [3] and [17]). For the stochastic Kuramoto-Sivashinsky equation, as we shall see below, since it is a fourth-order parabolic equation, well-posedness can be ensured for \(d \leq 5\). For discussions on other fourth-order parabolic equations, such as the stochastic Cahn-Hilliard equation, we refer the reader to see [4] and the references cited therein. We also refer the reader to see [22] and [23] for the study of long-term behavior of solutions of the stochastic Kuramoto-Sivashinsky equation (with additive noise).

We make the following assumption on the nonlinearity \(f\):

**Assumption (A)** \(f(0) = 0\), and there exist constants \(C > 0\) and \(p \geq 1\) such that

\[
|f(u) - f(v)| \leq C(1 + |u| + |v|)^{p-1}|u - v| \quad \text{for } u, v \in \mathbb{R}.
\]
As for the noise term \( \dot{V}_t \), we assume that it has the following expression:

\[
\dot{V}_t = \sigma(t, x, u, \partial u, \partial^2 u) \dot{W}_t, \tag{1.2}
\]

where \( \sigma \) is a given function, and \( \dot{W}_t \) is a \( L^2(D) \) valued Wiener process (see Section 2.2 for details; top dots denotes the derivatives in \( t \)).

We impose the following assumption on noise intensity \( \sigma \):

**Assumption (B)** There exist constants \( C > 0 \) and \( \varepsilon > 0 \) such that

\[
|\sigma(t, r, u_1, \xi_1, \zeta_1) - \sigma(t, r, u_2, \xi_2, \zeta_2)| \leq C(|u_1 - u_2| + |\xi_1 - \xi_2| + \varepsilon|\zeta_1 - \zeta_2|) \quad \text{for } u, v \in \mathbb{R}.
\]

for \( t \geq 0, x \in D, u, v \in \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}^d \) and \( \zeta_1, \zeta_2 \in \mathbb{R}^{d(d-1)} \).

Let \( R \) be the covariance operator of the Wiener process \( W_t \), and \( r(x, y) \) be its kernel (see Section 2.2 for details). We need the following assumption on the kernel function:

**Assumption (C)** The kernel function \( r \) is in \( L^\infty(D \times D) \), so that there exists a constant \( C > 0 \) such that

\[
r(x, y) \leq C \quad \text{for } x, y \in D.
\]

Let us now present the main result of this paper. We first consider the special case that \( \sigma \) does not depend on derivatives of \( u \). In this case we have the following result:

**Theorem 1.1** Let the assumptions (A), (B) and (C) be satisfied. Suppose further that \( \sigma(t, x, u, \xi, \zeta) = \sigma(t, x, u) \) and \( 1 \leq p \leq 2 \) for \( 1 \leq d \leq 5 \) and \( 1 \leq p < 1 + \frac{6}{d} \) for \( d \geq 6 \). Then the problem (1.1) is globally well-posed in \( L^2(D \times \Omega) \). More precisely, for any \( u_0 \in L^2(D \times \Omega) \) the problem (1.1) has a unique solution \( u \) such that for any \( T > 0 \), \( u \in L^2(\Omega, C([0, T], L^2(D))) \), and the solution map \( u_0 \mapsto u \) is a Lipschitz continuous map from \( L^2(D \times \Omega) \) to \( L^2(\Omega, C([0, T], L^2(D))) \).

Next we consider the general case. In this case our result is as follows:

**Theorem 1.2** Let the assumptions (A), (B) and (C) be satisfied. Suppose further that \( 1 \leq p \leq 2 \) for \( d = 1 \) and \( 1 \leq p < 1 + \frac{2}{d} \) for \( d \geq 2 \). Then there exists \( \varepsilon_0 > 0 \) such that if \( |\varepsilon| \leq \varepsilon_0 \), then the problem (1.1) is globally well-posed in \( L^2(D \times \Omega) \). More precisely, for any \( u_0 \in L^2(D \times \Omega) \) the problem (1.1) has a unique solution \( u \) such that for any \( T > 0 \), \( u \in L^2(\Omega, C([0, T], L^2(D))) \cap L^2([0, T], H^2(D))) \), and the solution map \( u_0 \mapsto u \) is a Lipschitz continuous map from \( L^2(D \times \Omega) \) to \( L^2(\Omega, C([0, T], L^2(D))) \cap L^2([0, T], H^2(D))) \).

**Remarks.** (1) Note that \( L^2(\Omega, C([0, T], L^2(D))) \hookrightarrow L^2([0, T], L^2(D \times \Omega)) \) (for deterministic \( T > 0 \)). This justifies the notion of “well-posedness in \( L^2(D \times \Omega) \”).

(2) Throughout this paper, for simplicity we only consider the noise consisting of a simple term \( \sigma \dot{W} \). The theorems hold true for multiple noise terms \( \sum_{i=1}^m \sigma_i \dot{W}_i \) with independent Wiener fields \( \dot{W}_i \)'s, provided that each \( \sigma_i \) satisfies the same conditions imposed on \( \sigma \).

(3) Our concern in this paper is well-posedness of the problem (1.1) in suitable function spaces. Theorems 1.1 and 1.2 give sufficient conditions to ensure global well-posedness of the problem (1.1) in the space \( L^2(D \times \Omega) \). If one is not concerned with well-posedness but merely interested in
existence of a solution (so that the solution might not be unique and continuously depend on the initial data), then these sufficient conditions can be weakened. We shall discuss this problem in a different paper.

The rest of this paper is organized as follows. In Section 2 we present some preliminary materials. In Section 3 we present the proof of Theorem 1.1, and in Section 4 we present the proof of Theorem 1.2.

2 Preliminaries

In this section we give some fundamental estimates for integrals related to the Green’s function $G(x, y, t)$ of the the linear partial differential equation $\partial_t u + \Delta^2 u + \Delta u + cu = 0$ (in $D$) subject to the boundary value conditions $u|_{\partial D} = \Delta u|_{\partial D} = 0$. We first consider deterministic integrals, and next consider stochastic integrals.

2.1 Estimates for deterministic integrals

Let $\{\lambda_k\}_{k=1}^\infty$ be the sequence of eigenvalues of the minus Laplace $-\Delta$ on $D$ subject to the homogeneous Dirichlet boundary condition, where multiple eigenvalues are counted in their multiplicities. Let $\{\phi_k\}_{k=1}^\infty$ be the corresponding sequence of eigenfunctions. We assume that they are suitably chosen so that they form an orthonormal basis of $L^2(D)$. Since $\lim_{k \to \infty} \lambda_k (\lambda_k - 1) = \infty$, there exists $c \geq 0$ such that $\mu_k := \lambda_k (\lambda_k - 1) + c > 0$ for all $k \in \mathbb{N}$. Choose a such $c$ and fix it. Let

$$G(t, x, y) = \sum_{k=1}^\infty \phi_k(x) \phi_k(y) e^{-\mu_k t}, \quad x, y \in \overline{D}, \; t > 0; \quad G(x, y, 0) = \delta(x - y).$$

$G$ is the Green’s function of the linear partial differential equation $\partial_t u + \Delta^2 u + \Delta u + cu = 0$ (in $D$) subject to the boundary value conditions $u|_{\partial D} = \Delta u|_{\partial D} = 0$. Note that $\min_{k \geq 1} \mu_k > 0$.

Lemma 2.1 For any $\varphi \in L^2(D)$ and $\alpha \in \mathbb{Z}_+^d$, we have

$$\|\partial_2^\alpha \int_D G(t, x, y) \varphi(y) dy\|_{L^2} \leq C t^{-|\alpha|} \|\varphi\|_{L^2} \text{ for } t > 0. \quad (2.1)$$

Proof: For simplicity of the notation we denote $S(t) \varphi(x) = \int_D G(x, y, t) \varphi(y) dy$. We first consider the case $\alpha = 0$. For $\varphi \in L^2(D)$, let $\varphi = \sum_{k=1}^\infty a_k \phi_k$. Then $S(t) \varphi = \sum_{k=1}^\infty a_k e^{-\mu_k t} \phi_k$, so that

$$\|S(t) \varphi\|^2_{L^2} = \sum_{k=1}^\infty a_k^2 e^{-2\mu_k t} \leq e^{-2c_0 t} \sum_{k=1}^\infty a_k^2 = e^{-2c_0 t} \|\varphi\|^2_{L^2} \quad (c_0 = \min_{k \geq 1} \mu_k > 0), \quad (2.2)$$

by which the assertion for the case $\alpha = 0$ follows. Next, since $\Delta S(t) \varphi = -\sum_{k=1}^\infty a_k \lambda_k e^{-\mu_k t} \phi_k$, we have

$$\|\Delta S(t) \varphi\|^2_{L^2} = \sum_{k=1}^\infty a_k^2 \lambda_k^2 e^{-2\mu_k t}.$$
Since $\mu_k > 0$ for all $k$ and $\lim_{k \to \infty} \frac{\lambda_k^2}{\mu_k} = 1$, there exists a positive constant $C$, actually $C = \max_{k \geq 1} \frac{\lambda_k^2}{\mu_k}$, such that $\lambda_k^2 \leq C\mu_k$ for all $k$. Hence

$$\|\Delta S(t)\varphi\|^2_{L^2} \leq C \sum_{k=1}^{\infty} a_k^2 \mu_k e^{-2\mu_k t} \leq Ct^{-1} \sum_{k=1}^{\infty} a_k^2 = Ct^{-1}\|\varphi\|^2_{L^2}. \quad (2.3)$$

In getting the second last relation we used the elementary inequality $xe^{-x} \leq 1/e$ (for $x > 0$). Since $S(t)\varphi \in H^2(D) \cap H^1_0(D)$ for all $t > 0$, by making use of the well-known Agmon-Douglis-Nirenberg inequality (in $L^2$ case) and the estimates (2.2) and (2.3) we have

$$\sum_{|\alpha|=2} \|\partial^\alpha S(t)\varphi\|_{L^2} \leq C (\|\Delta S(t)\varphi\|_{L^2} + \|S(t)\varphi\|_{L^2}) \leq Ct^{-\frac{1}{2}}\|\varphi\|_{L^2}.$$ 

This proves the assertion for the case $|\alpha| = 2$. The case $|\alpha| = 1$ then follows from interpolation. For general $\alpha \in \mathbb{Z}_+^d$, the proof is similar. We omit the details. \hfill \square

**Lemma 2.2** For any $1 \leq q \leq 2$, $\alpha \in \mathbb{Z}_+^d$ and $\varphi \in L^q(D)$ we have

$$\|\partial_x^\alpha \int_D G(t, x, y)\varphi(y)dy\|_{L^2} \leq Ct^{-\frac{\alpha}{2} - \frac{d}{4}} \|\varphi\|_{L^q}, \quad (2.4)$$

where $C$ is a positive constant depending only on $D$, $d$, $q$ and $\alpha$.

**Proof:** We only need to give the proof for the case $q = 1$, because the case $q = 2$ is ensured by Lemma 2.1, and the rest cases follow from these two special cases by interpolation. Moreover, we may assume that $\varphi \in L^1(D) \cap L^2(D)$, because if this is proved then for general $\varphi \in L^1(D)$ the desired assertion then follows from the density of $L^1(D) \cap L^2(D)$ in $L^1(D)$. For $\varphi \in L^1(D) \cap L^2(D)$ we let $\varphi = \sum_{k=1}^{\infty} a_k \phi_k$. Then $S(t)\varphi = \sum_{k=1}^{\infty} a_k e^{-\mu_k t} \phi_k$, so that $\|S(t)\varphi\|^2_{L^2} = \sum_{k=1}^{\infty} a_k^2 e^{-2\mu_k t}$. We have

$$|a_k| = \left| \int_D \varphi(x) \phi_k(x) dx \right| \leq \|\varphi\|_{L^1} |\phi_k|_{L^\infty} \leq C\lambda_k^\frac{d}{2} \|\varphi\|_{L^1}.$$ 

Here we used the inequality $\|\phi_k\|_{L^\infty} \leq C\lambda_k^\frac{d}{2}$, whose simple proof is as follows: Choose an integer $l$ sufficiently large such that $2l > d/2$. Then by making use of the Gagliardo-Nirenberg inequality and the equation $\Delta^l \phi_k = (-\lambda_k)^l \phi_k$ we have

$$\|\phi_k\|_{L^\infty} \leq C\|\phi_k\|_{L^2}^{1 - \frac{d}{2l}} \|\Delta^l \phi_k\|_{L^2}^{\frac{d}{2l}} = C\lambda_k^\frac{d}{2l}.$$ 

Hence

$$\|S(t)\varphi\|^2_{L^2} \leq C\|\varphi\|_{L^1} \left( \sum_{k=1}^{\infty} \lambda_k^\frac{d}{2l} e^{-2\mu_k t} \right)^\frac{1}{2} \leq Ct^{-\frac{1}{2}}\|\varphi\|_{L^1}.$$ 

By a similar argument we see that for any positive integer $l$,

$$\|\Delta^l S(t)\varphi\|^2_{L^2} \leq Ct^{-\frac{1}{2} - \frac{l}{4}}\|\varphi\|_{L^1}.$$ 

By using again the Gagliardo-Nirenberg inequality we see that for any $\alpha \in \mathbb{Z}_+^d$,

$$\|\partial_x^\alpha S(t)\varphi\|_{L^2} \leq Ct^{-\frac{\alpha}{2} - \frac{d}{4}} \|\varphi\|_{L^1}.$$
This proves the desired assertion. □

We shall also need the following preliminary result which follows from the energy identity for the equation \( \partial_t u + \Delta^2 u + \Delta u + cu = f \):

**Lemma 2.3** For \( \varphi \in L^2(D) \) we have

\[
\int_0^t \int_D G(t, x, y) \varphi(y) dy \, dt \leq C \| \varphi \|_{L^2}^2 \quad \text{for } t > 0. \tag{2.5}
\]

**Proof:** We first assume that \( \varphi \in H^2(D) \cap H_0^1(D) \). In this case \( u(t) := S(t) \varphi \in C^\infty([0, \infty), H^2(D) \cap H_0^1(D)) \), so that the following calculations make sense. By multiplying both sides of the equation \( \partial_t u + \Delta^2 u + \Delta u + cu = 0 \) with \( u \) and integrating over \( D \), we see that

\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2}^2 + \| \Delta u(t) \|_{L^2}^2 - \| \nabla u(t) \|_{L^2}^2 + c \| u(t) \|_{L^2}^2 = 0.
\]

It follows that

\[
\| u(t) \|_{L^2}^2 + 2 \int_0^t [\| \Delta u(\tau) \|_{L^2}^2 - \| \nabla u(\tau) \|_{L^2}^2] \, d\tau \leq \| \varphi \|_{L^2}^2.
\tag{2.6}
\]

Since \( u(t) \in H^2(D) \cap H_0^1(D) \) for all \( t > 0 \), we have

\[
\| \nabla u(t) \|_{L^2}^2 \leq \| \Delta u(t) \|_{L^2} \| u(t) \|_{L^2} \leq \frac{1}{2} \| \Delta u(t) \|_{L^2}^2 + \frac{1}{2} \| u(t) \|_{L^2}^2.
\]

Hence, from (2.6) we get

\[
\int_0^t \| \Delta u(\tau) \|_{L^2}^2 \, d\tau \leq \| \varphi \|_{L^2}^2 + \int_0^t \| u(\tau) \|_{L^2}^2 \, d\tau.
\]

It follows by the Agmon-Douglis-Nirenberg inequality and (2.4) that

\[
\int_0^t \| u(\tau) \|_{H^2}^2 \, d\tau \leq C \int_0^t [\| \Delta u(\tau) \|_{L^2}^2 + \| u(\tau) \|_{L^2}^2] \, d\tau \leq C \| \varphi \|_{L^2}^2 + C \int_0^t e^{-c_\tau} \| \varphi \|_{L^2}^2 \, d\tau \leq C \| \varphi \|_{L^2}^2.
\]

For general \( \varphi \in L^2(D) \) we use approximation. □

### 2.2 Estimates for stochastic integrals

Let \( W_t = W_t(x, \omega) \) \( (t \geq 0) \) be a \( L^2(D) \) valued Wiener process on a probability space \( (\Omega, \mathcal{F}, P) \), i.e., there exists a complete normalized orthogonal basis \( \{ e_k \}_{k=1}^\infty \) of \( L^2(D) \), a sequence of positive numbers \( \{ c_k \}_{k=1}^\infty \) satisfying \( \sum_{k=1}^\infty c_k^2 < \infty \), and a sequence of independent, identically distributed standard Brownian motions \( \{ w^k_t \}_{k=1}^\infty \) \( (k = 1, 2, \cdots) \) on \( (\Omega, \mathcal{F}, P) \), such that

\[
W_t(x, \omega) = \sum_{k=1}^\infty c_k w^k_t(\omega)e_k(x).
\]

By convention, later on we will omit the sample point variable \( \omega \) and simply write \( W_t(x, \omega) \) and \( w^k_t(\omega) \) respectively as \( W_t(x) \) and \( w^k_t \). Note that \( W_t \) is the limit of the finite dimensional Wiener
process $W^n_t = \sum_{k=1}^n c_k w^k e_k$ in $L^2(\Omega, C([0, T], L^2(D)))$ (for any $T > 0$, cf. [9]), so that it belongs to $L^2(\Omega, C([0, T], L^2(D)))$ (for any $T > 0$). Let

$$r(x, y) = \sum_{k=1}^{\infty} c_k^2 e_k(x) e_k(y).$$

Then $\int_{D \times D} |r(x, y)|^2 dxdy = \sum_{k=1}^{\infty} c_k^4 \leq \left( \sum_{k=1}^{\infty} c_k^2 \right)^2 < \infty$, i.e., $r \in L^2(D \times D)$. Moreover, it is clear that $r(x, y) = r(y, x)$, and $\int_{D \times D} r(x, y) \varphi(x) \varphi(y) dxdy \geq 0$ for any $\varphi \in L^2(D)$. Hence $r(x, y)$ defines a positive semi-definite self-adjoint Hilbert-Schmidt operator $R$ on $L^2(D)$:

$$(R\varphi)(x) = \int_D r(x, y) \varphi(y) dy \quad \text{for } \varphi \in L^2(D).$$

In fact, $R$ is a self-adjoint trace class operator on $L^2(D)$, with

$$\| R \|_{\mathcal{L}_1} = \text{Tr} R = \int_D r(x, x) dx = \sum_{k=1}^{\infty} c_k^2 < \infty.$$

A simple computation shows that

$$EW_t(x) = 0 \quad \text{and} \quad E \{ W_t(x)W_s(y) \} = (t \wedge s) r(x, y),$$

where $t \wedge s = \min\{t, s\}$. Note also that $r(x, x) = \sum_{k=1}^{\infty} c_k^2 e_k^2(x) \geq 0$ for $x \in D$.

Let $\{ \mathcal{F}_t \}_{t \geq 0}$ be a filtration of the sub $\sigma$-fields of $\mathcal{F}$, and $u_t = u(t, x, \omega)$ be a continuous $L^2(D)$-valued $\mathcal{F}_t$-adapted random field satisfying the condition

$$E \int_0^T \| u_t \|^2_{\mathcal{H}} dt < \infty, \quad \text{where } \| u_t \|^2_{\mathcal{H}} = \int_D r(x, x) |u(t, x, \omega)|^2 dx. \quad (2.7)$$

Again, by convention later on we omit the sample point variable $\omega$ in $u(t, x, \omega)$ and simply write it as $u_t = u(t, x)$.

**Lemma 2.4** Assume that the condition (2.7) is satisfied. Then we have the following estimate:

$$E \left( \sup_{0 \leq t \leq T} \int_0^t \int_D G(t - s, x, y) u(y, s) dy dW_s(y) \right)^2 \leq CE \left( \int_0^T \| u_t \|^2_{\mathcal{H}} dt \right). \quad (2.8)$$

**Proof:** This is a corollary of Theorem 6.10 of [9]. $\square$

**Lemma 2.5** Assume that the condition (2.7) is satisfied. Then we have the following estimate:

$$\sum_{|\alpha| = 2} E \left( \int_0^T \| \partial_\alpha x G(t - s, x, y) u(s, y) dy dW_s(y) \|_{L^2}^2 dt \right) \leq CE \left( \int_0^T \| u_t \|^2_{\mathcal{H}} dt \right). \quad (2.9)$$

**Proof:** Since $G(t, x, y) = \sum_{k=1}^{\infty} \phi_k(x) \phi_k(y) e^{-\mu x t}$, we have

$$\Delta \int_0^t \int_D G(t - s, x, y) u(s, y) dy dW_s(y) = -\sum_{k=1}^{\infty} \lambda_k e^{-\mu x t} \phi_k(x) \left( \int_0^t \int_D e^{\mu x s} \phi_k(y) u(s, y) dy dW_s(y) \right).$$
so that
\[ E \int_0^T \| \Delta \int_0^t \int_D G(t-s,x,y)u(s,y)dydW_s(y) \|_{L^2}^2 dt \]
\[ = E \int_0^T \left\{ \sum_{k=1}^{\infty} \lambda_k^2 e^{-2\mu_k t} \left( \int_0^t \int_D e^{\mu_k s} \phi_k(y)u(s,y)dydW_s(y) \right)^2 \right\} dt \]
\[ = \sum_{k=1}^{\infty} \lambda_k^2 \int_0^T e^{-2\mu_k t} E \left( \int_0^t \int_D e^{2\mu_k s} (R\phi_k u_s, \phi_k u_s) ds \right) dt \]
\[ = \sum_{k=1}^{\infty} \lambda_k^2 \int_0^T \left( \int_s^T e^{-2\mu_k t} dt \right) e^{2\mu_k s} (R\phi_k u_s, \phi_k u_s) ds \]
\[ \leq \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^2 \int_0^T (R\phi_k u_s, \phi_k u_s) ds \leq CE \int_0^T \sum_{k=1}^{\infty} (R\phi_k u_s, \phi_k u_s) ds \]
\[ = CE \int_0^T \int_D r(x,x)u^2(s,x)dx ds \quad \text{(because } \sum_{k=1}^{\infty} \phi_k(x)\phi_k(y) = \delta(x-y)) \]
\[ = CE \int_0^T \| u(t,\cdot) \|_{H_0}^2 dt. \quad (2.10) \]

In getting the third equality we used the following generalized Itô isometry:
\[ E \left( \int_0^t \int_D u(s,y)dydW_s(y) \right)^2 = E \int_0^t (Ru(s,\cdot), u(s,\cdot)) ds, \]
whose proof is an easy exercise of the stochastic integrals. Indeed, by letting \( J_t(x) = \int_0^t u(s,y)dydW_s(y), \) we have, by the stochastic Fubini theorem (see [9]), that
\[ E \left( \int_0^t \int_D u(s,y)dydW_s(y) \right)^2 = E \left( \int_0^t \int_D u(s,y)dydW_s(y) \right)^2 \]
\[ = E \left\{ (J_t,1)(J_t,1) \right\} = E \int_0^t (Ru(s,\cdot), u(s,\cdot)) ds. \]

Having proved (2.10), (2.9) follows immediately from the Agmon-Douglis-Nirenberg inequality. \( \square \)

### 3 The proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. We shall use the truncation method to prove this theorem.

For every integer \( N > 0 \) we consider a truncated problem as follows: First we choose a mollifier \( \eta_N, \) i.e., \( \eta_N : [0, \infty) \rightarrow [0, 1] \) is a \( C^\infty \) function satisfying the condition
\[ \eta_N(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq N, \\ 0 & \text{for } r \geq 2N. \end{cases} \]
For $u \in L^2$, let $S_N u = \eta_N(\|u\|_{L^2}) u$ and $f_N(u) = f(S_N u)$. The truncated problem takes the form:

$$
\begin{cases}
\partial_t u + \Delta^2 u + \Delta u + \text{div} f_N(u) = \sigma(t, x, u) \hat{W}_t, & x \in D, \quad t > 0, \\
u|_{\partial D} = \Delta u|_{\partial D} = 0, & t > 0, \\
u|_{t=0} = u_0, & x \in D.
\end{cases}
(3.1)
$$

Using the Green's function and the Duhamel's formula, we can convert the above problem into the following equivalent stochastic integral equation:

$$
u(t, x) = \int_D G(t, x, y) u_0(y) dy + c \int_0^t \int_D G(t-s, x, y) u(s, y) dy ds 
+ \int_0^t \int_D \nabla G(t-s, x, y) \cdot f_N(u(s, y)) dy ds 
+ \int_0^t \int_D G(t-s, x, y) \sigma(s, y, u(s, y)) dy dW_s(y).
(3.2)
$$

In what follows we use the Banach fixed point theorem to prove that the above problem is globally well-posed in $L^2(D \times \Omega)$.

For any $T > 0$, let $X_T$ be the set of $L^2(D)$-valued $\mathcal{F}_t$-adapted continuous random processes $u$ on $[0, T]$ such that the norm

$$
\|u\|_{X_T} = \left( E \sup_{0 \leq t \leq T} \|u\|^2_{L^2} \right)^{\frac{1}{2}}
$$

is finite, i.e., $X_T$ is the set of $\mathcal{F}_t$-adapted random processes belonging to $L^2(\Omega, C([0, T], L^2(D)))$. It is evident that $(X_T, \|\cdot\|_{X_T})$ is a Banach space. For $u \in X_T$, let $\Gamma u$ be the right-hand side of (3.2). In what follows we prove that for any $u \in X_T$, $\Gamma u$ is well-defined and belongs to $X_T$ as well, and the operator $\Gamma : X_T \to X_T$ defined in this way is a contraction mapping provided $T$ is sufficiently small.

We first note that the assumption $(A)$ ensures that for any $u, v \in L^2(D)$,

$$
\|f(u)\|_{L^2} \leq C(1 + \|u\|^p_{L^2}),
(3.3)
$$

$$
\|f(u) - f(v)\|_{L^2} \leq C(1 + \|u\|_{L^2} + \|v\|_{L^2})^{p-1} \|u - v\|_{L^2}.
(3.4)
$$

Indeed, by using the assumption $(A)$ we have

$$
\|f(u) - f(v)\|_{L^2} \leq C \int_D (1 + |u| + |v|)^{\frac{3}{2}(p-1)} |u - v|^\frac{3}{2} dx 
\leq C \|u - v\|_{L^2} (1 + \|u\|_{L^2} + \|v\|_{L^2})^{\frac{3}{2}(p-1)},
$$

by which (3.3) and (3.4) immediately follow. We also note that the assumptions $(B)$ and $(C)$ ensure that there exists some constant $C > 0$ such that for any $u, v \in L^2(D)$,

$$
\|\sigma(t, x, u)\|_{L^2} \leq C(1 + \|u\|^2_{L^2}),
(3.5)
$$

$$
\|\sigma(t, x, u) - \sigma(t, x, v)\|_{L^2} \leq C \|u - v\|_{L^2}.
(3.6)
$$
Indeed, the assumptions (C) implies that \( \|u\|_R \leq C\|u\|_{L^2} \). Hence, by using the assumptions (B) we immediately obtain these estimates.

By using Lemma 2.1 with \( \alpha = 0 \) we have

\[
E\left( \sup_{0 \leq t \leq T} \| \int_D G(t, x, y)u_0(y)dy \|^2_{L^2} \right) \leq CE\left( \|u_0\|_{L^2}^2 \right),
\]

(3.7)

and

\[
E\left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D G(t, x, y)u(s, y)dyds \|^2_{L^2} \right) \leq CE\left( \int_0^T \|u(s, \cdot)\|_{L^2}^2 ds \right) \leq CTE\left( \sup_{0 \leq t \leq T} \|u\|_{L^2}^2 \right).
\]

(3.8)

Next, note that by (3.3) we have

\[
\|f_N(u)\|_{L^p} = \|f(S_N u)\|_{L^p} \leq C(1 + \|S_N u\|_{L^2}^p) \leq C(N).
\]

(3.9)

Hence, by using Lemma 2.2 with \( |\alpha| = 1 \) and \( q = \frac{2}{p} \), and noticing the fact that the conditions on \( p \) ensures that \( 1 \leq q = \frac{2}{p} \leq 2 \) and \( \frac{1}{4} \leq \frac{2}{8}(p - 1) + \frac{1}{4} < 1 \), we have

\[
\| \int_0^t \int_D \nabla G(t - s, x, y) \cdot f_N(u(s, y))dyds \|_{L^2}
\]

\leq \int_0^t \| \int_D \nabla G(t - s, x, y) \cdot f_N(u(s, y))dy \|_{L^2} ds
\]

\leq C \int_0^t (t - s)^{-\frac{2}{8}(p-1)-\frac{1}{4}} \|f_N(u(s, y))\|_{L^p} ds
\]

\leq C(N) \int_0^t (t - s)^{-\frac{2}{8}(p-1)-\frac{1}{4}} ds
\]

\[= C(N) t^{\frac{2}{8}-\frac{2}{8}(p-1)}, \]

which yields

\[
E\left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D \nabla G(t - s, x, y) \cdot f_N(u(s, y))dyds \|^2_{L^2} \right) \leq C(N) T^{\frac{2}{8}-\frac{2}{8}(p-1)}.
\]

(3.10)

For the stochastic integral, by using Lemma 2.4 and (3.5) we have

\[
E\left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D G(t - s, x, y)\sigma(u(s, y))dydW_s(y) \|^2_{L^2} \right)
\]

\[\leq CE\left( \int_0^T \|\sigma(u)\|_{L^2}^2 dt \right) \leq CE\left( \int_0^T (1 + \|u\|_{L^2}^2) dt \right)
\]

\[\leq CT\{1 + E(\sup_{0 \leq t \leq T} \|u\|_{L^2}^2)\}.
\]

(3.11)

Combining the inequalities (3.7), (3.8), (3.10) and (3.11), we see that there exists constant \( C(N, T) > 0 \) such that

\[
\|\Gamma u\|^2_{X_T} \leq C(N, T)\{1 + E(\|u_0\|_{L^2}^2) + \|u\|_{X_T}^2\}.
\]
Therefore, the operator $\Gamma$ is well-defined and maps $X_T$ into itself.

Next, from (3.2) we see that for $u, v \in X_T$,

$$
\Gamma u - \Gamma v = c \int_0^t \int_D G(t-s, x, y) [u(s, y) - v(s, y)] dy ds
+ \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds
+ \int_0^t \int_D G(t-s, x, y) [\sigma(u(s, y)) - \sigma(v(s, y))] dy dW_s(y).
$$

By making use of (3.8) we have

$$
E \left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D G(t-s, x, y) [u(s, y) - v(s, y)] dy ds \|_{L^2}^2 \right) \leq CTE \left( \sup_{0 \leq t \leq T} \| u - v \|_{L^2}^2 \right).
$$

From (3.4) we see that

$$
\| f_N(u) - f_N(v) \|_{L^2} = \| f(S_N u) - f(S_N v) \|_{L^2}
\leq C(1 + \| S_N u \|_{L^2} + \| S_N v \|_{L^2})^{p-1} \| S_N u - S_N v \|_{L^2}
\leq C(N) \| u - v \|_{L^2}.
$$

Using this inequality and a similar argument as in the proof of (3.9) we get

$$
E \left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds \|_{L^2}^2 \right)
\leq C \left( \sup_{0 \leq t \leq T} \left( \int_0^t (t-s)^{-\frac{3}{2}(p-1)-\frac{1}{4}} \| f_N(u) - f_N(v) \|_{L^2} \right) ds \right)^2
\leq C(N) E \left( \sup_{0 \leq t \leq T} \| u - v \|_{L^2}^2 \right) \left( \int_0^t (t-s)^{-\frac{3}{2}(p-1)-\frac{1}{4}} ds \right)^2
\leq C(N) T^\frac{3}{2} - \frac{3}{4}(p-1) \ E \left( \sup_{0 \leq t \leq T} \| u - v \|_{L^2}^2 \right).
$$

Finally, by Lemma 2.4 and (3.6) we have

$$
E \left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D G(t-s, x, y) [\sigma(u(s, y)) - \sigma(v(s, y))] dy ds \|_{L^2}^2 \right)
\leq CE \left( \int_0^T \| \sigma(u) - \sigma(v) \|_{H^2}^2 dt \right) \leq CE \left( \int_0^T \| u - v \|_{L^2}^2 dt \right)
\leq CTE \left( \sup_{0 \leq t \leq T} \| u - v \|_{L^2}^2 \right).
$$

Combining (3.12), (3.14) and (3.15), we get

$$
\| \Gamma u - \Gamma v \|_{X_T}^2 \leq C(N)(T^{\frac{3}{2} - \frac{3}{4}(p-1)} + T) \| u - v \|_{X_T}^2.
$$

Since $\frac{3}{2} - \frac{3}{4}(p-1) > 0$, we see that if $T$ is sufficiently small so that $C(N)(T^{\frac{3}{2} - \frac{3}{4}(p-1)} + T) < 1$, then the operator $\Gamma$ is a contraction mapping on $X_T$. 

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By the Banach fixed point theorem, it follows that if $T$ is so small that $C(N)(T^{2-\frac{4}{p-1}}+T)<1$, then the equation (3.2) has a unique solution in $X_T$. Since $T$ does not depend on $u_0$, by a classical argument, the solution can be extended over all the right-half line $[0, \infty)$, i.e. the truncated problem (3.1) has a unique global solution $u^N(t, x)$. Moreover, since this solution is obtained by using the Banach fixed point theorem, we see that the solution map $u_0 \mapsto u^N$ is Lipschitz continuous from $L^2(\Omega, L^2(D))$ to $X_T$ for any $T > 0$. Hence the problem (3.1) is globally well-posed in $L^2(\Omega, L^2(D))$.

We now introduce a stopping time $\tau_N$ as follows:

$$\tau_N = \inf\{t > 0 : \|u^N(t, \cdot)\|_{L^2} > N\}$$

if the set on the right-hand side is nonempty, and set $\tau_N = T$ otherwise. Then, for $t < \tau_N$, $u(t, x) = u^N(t, x)$ is the solution of the problem (1.1). Since $\tau_N$ is increasing in $N$, we can define $\tau_\infty = \lim_{N \to \infty} \tau_N$. For $t < \tau_\infty$, we have $t < \tau_N$ for some $N > 0$, and we define $u(t, x) = u^N(t, x)$.

By uniqueness of the solution of the truncated problem (3.1), this definition makes sense. Thus we have proved that there exists an almost everywhere defined function $\tau_\infty : \Omega \to (0, \infty]$ such that the problem (1.1) has a solution on $[0, \tau_\infty) \times D$ almost surely in $\Omega$. This proves local existence of a solution of the problem (1.1). Moreover, from the above argument we easily see that if $\tau_\infty < \infty$, then

$$\lim_{t \uparrow \tau_\infty} \|u(t, \cdot)\|_{L^2} = \infty.$$  

For uniqueness, suppose that there is another solution $\tilde{u}(t, x)$ defined for $t < \tau$ for a stopping time $\tau$, i.e., $\limsup_{t \uparrow \tau} \|\tilde{u}(t, \cdot)\|_{L^2} = \infty$. Then $\tau \geq \tau_N$ for any $N > 0$, and $\tilde{u}(t, x) = u^N(t, x)$ for $t < \tau_N$, by uniqueness of the solution of the problem (3.1). It follows that $\tau \geq \tau_\infty$ and $\tilde{u}(t, x) = u(t, x)$ for $t < \tau_\infty$. This further implies that $\tau = \tau_\infty$. Therefore, the solution of the problem (1.1) is unique.

To obtain a global solution, we only need to prove that for any finite $T > 0$, there exists a corresponding constant $C(T) > 0$ such that

$$E\|u_{T \wedge \tau_N}\|_{L^2}^2 \leq C(T).$$  

(3.16)

Here and hereafter we use the notation $u_{t \wedge \tau_N}$ to denote the value of $u = u^N$ (defined on the time interval $[0, \tau_N)$) at the time $t \wedge \tau_N$. Indeed, by the Doob’s inequality we have

$$E\|u_{T \wedge \tau_N}\|_{L^2}^2 \geq E\{I(\tau_N \leq T)\|u_{T \wedge \tau_N}\|_{L^2}^2\} \geq N^2 P\{\tau_N \leq T\},$$

where $I$ denotes the indicator function. If (3.15) holds, then we get

$$P\{\tau_N \leq T\} \leq \frac{C(T)}{N^2}.$$  

By the Borel-Cantelli lemma, we have

$$P\{\tau_\infty \leq T\} = 0,$$

and, therefore, $P\{\tau_\infty > T\} = 1$ for any $T > 0$. Hence $u(t, x) = \lim_{N \to \infty} u^N(t, x)$ is a global solution to the problem (1.1) as claimed. Therefore, it suffices to prove (3.16).
Since \( u_{t \land \tau_N} \) is the solution of the problem (3.1) in the time interval \([0, T \land \tau_N]\), by noticing the fact that \( f_N(u) = f(u) \) in this time interval and using the Itô’s formula, we get the following equation:

\[
\|u_{t \land \tau_N}\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2 \int_0^{t \land \tau_N} (\Delta^2 u_s + \Delta u_s + \text{div}f(u_s), u_s)ds + 2 \int_0^{t \land \tau_N} \|\sigma(u_s)\|_R^2 ds.
\]

By integral by parts, we have

\[
\|u_{t \land \tau_N}\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2 \int_0^{t \land \tau_N} (\|u_s\|_{L^2}^2 - \|\nabla u_s\|_{L^2}^2 + \|\sigma(u_s)\|_R^2)ds.
\]

Taking the expectation and using (3.5), we get

\[
E\|u_{t \land \tau_N}\|_{L^2}^2 = E\|u_0\|_{L^2}^2 - 2E \int_0^{t \land \tau_N} (\|u_s\|_{L^2}^2 - \|\nabla u_s\|_{L^2}^2 + \|\sigma(u_s)\|_R^2)ds.
\]

Since \( \|u_s\|_{L^2}^2 - \|\nabla u_s\|_{L^2}^2 \geq \lambda_1 (\lambda_1 - 1) \|u_s\|_{L^2}^2 \), we have

\[
E\|u_{t \land \tau_N}\|_{L^2}^2 \leq E\|u_0\|_{L^2}^2 - 2\lambda_1 (\lambda_1 - 1)E \int_0^{t \land \tau_N} \|u_s\|_{L^2}^2 ds + C E \int_0^{t \land \tau_N} (1 + \|u_s\|_{L^2}^2)ds
\]

By the Gronwall’s lemma, this yields the following estimate:

\[
E\|u_{t \land \tau_N}\|_{L^2}^2 \leq (E\|u_0\|_{L^2}^2 + C T) e^{(C + 2\lambda_1 - 2\lambda_1^2) t} \leq C(T),
\]

where \( C(T) \) is a positive constant independent of \( N \). Letting \( t = T \), we see that (3.16) follows. This completes the proof of Theorem 1.1.

4 The proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. Again, we shall use the truncation method to prove this theorem, but we have to use a different work space.

For every integer \( N > 0 \) let \( f_N \) be as before. We consider the following truncated problem:

\[
\begin{aligned}
\partial_t u + \Delta^2 u + \Delta u + \text{div} f_N(u) &= \sigma(t, x, u, \partial_x u, \partial^2_x u) W_t, \quad x \in D, \quad t > 0, \\
\partial_t u|_{\partial D} &= \Delta u|_{\partial D} = 0, \quad t > 0, \\
\partial_x u|_{x=0} &= u_0, \quad x \in D.
\end{aligned}
\]

As before, we can convert the above problem into the following equivalent stochastic integral equation:

\[
u(t, x) = \int_D G(t, x, y)u_0(y)dy + c \int_0^t \int_D G(t-s, x, y)u(s, y)dyds
\]
\[ + \int_0^t \int_D \nabla G(t-s, x, y) \cdot \mathbf{f}_N(u(s, y))dyds \]
\[ + \int_0^t \int_D G(t-s, x, y)\sigma(s, y, u(s, y), \partial_y u(s, y), \partial^2_y u(s, y))dydW_s(y). \] (4.2)

In what follows we use the Banach fixed point theorem to prove that the above problem is globally well-posed in \(L^2(D \times \Omega)\).

For any \(T > 0\), let \(Y_T\) be the set of \(L^2(D)\)-valued \(\mathcal{F}_t\)-adapted continuous random processes \(u\) on \([0, T]\) such that the norm
\[ \|u\|_{Y_T} = \left( E \sup_{0 \leq t \leq T} \|u\|^2_{L^2} + E \int_0^T \|u\|_{H^2}^2 dt \right)^{\frac{1}{2}} \]
is finite, i.e., \(Y_T\) is the set of \(\mathcal{F}_t\)-adapted random processes belonging to \(L^2(\Omega, C([0, T], L^2(D)) \cap L^2([0, T], H^2(D)))\). It is evident that \((Y_T, \| \cdot \|_{Y_T})\) is a Banach space. For \(u \in Y_T\), let \(\Gamma u\) be the right-hand side of (4.2). In what follows we prove that for any \(u \in Y_T\), \(\Gamma u\) is well-defined and belongs to \(Y_T\) as well, and the operator \(\Gamma : Y_T \rightarrow Y_T\) defined in this way is a contraction mapping provided \(T\) is sufficiently small.

We first note that the assumptions \((B)\) and \((C)\) ensure that there exists some constant \(C > 0\) and \(\varepsilon > 0\) such that for any \(u, v \in H^2(D)\),
\[ \|\sigma(t, x, u, \partial_x u, \partial^2_x u)\|^2_{L^2} \leq C(1 + \|u\|^2_{L^2}) + \varepsilon \|u\|^2_{H^2}, \quad (4.3) \]
\[ \|\sigma(t, x, u, \partial_x u, \partial^2_x u) - \sigma(t, x, v, \partial_x v, \partial^2_x v)\|^2_{L^2} \leq C\|u - v\|^2_{L^2} + \varepsilon \|u - v\|^2_{H^2}. \quad (4.4) \]

By using Lemma 2.4 and (4.3) we have
\[ E \left( \sup_{0 \leq t \leq T} \| \int_0^t \int_D G(t-s, x, y)\sigma(u(s, y), \partial_x u(s, y), \partial^2_x u(s, y))dydW_s(y) \|^2_{L^2} \right) \]
\[ \leq CE \left( \int_0^T \|\sigma(u, \partial_x u, \partial^2_x u)\|^2_{L^2} dt \right) \leq E \left( \int_0^T (1 + \|u\|^2_{L^2}) + (C \|u\|^2_{H^2} dt \right) \]
\[ \leq C(T, \varepsilon)(1 + E \sup_{0 \leq t \leq T} \|u\|^2_{L^2} + E \int_0^T \|u\|^2_{H^2} dt) \leq C(T, \varepsilon)(1 + \|u\|^2_{Y_T}). \] (4.5)

Combining this with the estimates (3.7), (3.8) and (3.10) in Section 3, we see that there exists constant \(C(N, T, \varepsilon) > 0\) such that
\[ E \sup_{0 \leq t \leq T} \|\Gamma u\|^2_{L^2} \leq C(N, T, \varepsilon)\{1 + E(\|u_0\|^2_{L^2}) + \|u\|^2_{Y_T}\}. \] (4.6)

Next, by Lemma 2.3 we have
\[ E \int_0^t \| \int_D G(x, y, t)u_0(y)dy \|^2_{H^2} dt \leq CE\|u_0\|^2_{L^2}. \] (4.7)

Moreover, by using Lemma 2.1 with \(|\alpha| = 0, 2\) we have
\[ \| \int_0^t \int_D G(t-s, x, y)u(s, y)dyds \|^2_{H^2} \leq C \int_0^t \| \int_D G(t-s, x, y)u(s, y)dy \|^2_{H^2} ds \]
\[ \leq C \int_0^t (1 + (t-s)^{-\frac{1}{2}}) \|u\|_{L^2} ds \leq C(t + t^2) \sup_{0 \leq s \leq t} \|u\|_{L^2}, \]

so that

\[ E \int_0^T \int_0^t \int_D G(t-s,x,y)u(s,y)dyds\|_{H^2} dt \leq CT^{\frac{3}{2}} (1 + T^{\frac{1}{2}}) E \sup_{0 \leq t \leq T} \|u\|_{L^2}^2. \quad (4.8) \]

Similarly, by using Lemma 2.2 with \(|\alpha| = 1, 3\) and \(q = \frac{2}{p}\), we have

\[ \| \int_0^t \int_D \nabla G(t-s,x,y) \cdot f_N(u(s,y)) dyds \|_{H^2} \]
\[ \leq C \int_0^t \int_D \nabla G(t-s,x,y) \cdot f_N(u(s,y)) dy \|_{H^2} ds \]
\[ = C \int_0^t \int_D (I - \Delta) \nabla G(t-s,x,y) \cdot f_N(u(s,y)) dy \|_{L^2} ds \]
\[ \leq C \int_0^t \left\{ (t-s)^{-\frac{3}{2}(p-1)-\frac{1}{2}} + (t-s)^{-\frac{3}{2}(p-1)-\frac{3}{2}} \right\} \|f_N(u)\|_{L^2} ds \]
\[ \leq C(N)(t^{-\frac{3}{2}(p-1)} + t^{-\frac{3}{2}(p-1)}), \]

so that

\[ E \int_0^T \int_0^t \int_D \nabla G(t-s,x,y) \cdot f_N(u(s,y)) dyds\|_{H^2} dt \leq C(N)(1 + T)T^{\frac{3}{2} - \frac{3}{2}(p-1)}. \quad (4.9) \]

For the stochastic integral, by using Lemma 2.5 and (4.3) we have

\[ E \int_0^T \int_0^t \int_D G(t-s,x,y) \sigma(u(s,y), \partial_x u(s,y), \partial_x^2 u(s,y)) dydW_s(y)\|_{H^2} dt \]
\[ \leq CE \int_0^T \|\sigma(u, \partial_x u, \partial_x^2 u)\|_{H^2}^2 dt \leq E \left( \int_0^T C(1 + \|u\|_{L^2}^2) + C(\varepsilon)\|u\|_{L^2}^2 dt \right) \]
\[ \leq C(T, \varepsilon)(1 + E \sup_{0 \leq t \leq T} \|u\|_{L^2}^2 + E \int_0^T \|u\|_{H^2}^2 dt) \leq C(T, \varepsilon)(1 + \|u\|_{Y_T}^2). \quad (4.10) \]

Combining (4.6)–(4.10), we see that there exists constant \(C(N,T,\varepsilon) > 0\) such that

\[ \|\Gamma u\|_{Y_T}^2 \leq C(N,T,\varepsilon) \{ 1 + E(\|u_0\|_{L^2}^2) + \|u\|_{Y_T}^2 \}. \]

Therefore, the operator \(\Gamma\) is well-defined and maps \(Y_T\) into itself.

Next, from (4.2) we see that for any \(u, v \in Y_T\),

\[ \Gamma u - \Gamma v = c \int_0^t \int_D G(t-s,x,y) [u(s,y) - v(s,y)] dy ds \]
\[ + \int_0^t \int_D \nabla G(t-s,x,y) \cdot (f_N(u(s,y)) - f_N(v(s,y))) dy ds \]
\[ + \int_0^t \int_D G(t-s,x,y) [\sigma(s,y,u, \partial_y u, \partial_y^2 u) - \sigma(s,y,v, \partial_y v, \partial_y^2 v)] dydW_s(y). \]
Thus

\[
E \sup_{0 \leq t \leq T} \| \Gamma u - \Gamma v \|^2_{L^2} = E \sup_{0 \leq t \leq T} \left\| \int_0^t \int_D G(t-s, x, y)[u(s, y) - v(s, y)] dy ds \right\|
\]

\[
+ \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds
\]

\[
+ \int_0^t \int_D (t-s, x, y)\{\sigma(s, y, u, \partial_y u, \partial_y^2 u) - \sigma(s, y, v, \partial_y v, \partial_y^2 v)\} dy dW_s(y)\|_{L^2}^2
\]

\[
\leq CE \sup_{0 \leq t \leq T} \left\| \int_0^t \int_D G(t-s, x, y)[u(s, y) - v(s, y)] dy ds \right\|_{L^2}^2
\]

\[
+ \left\| \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds \right\|_{L^2}^2
\]

\[
+ \left\| \int_0^t \int_D G(t-s, x, y)\{\sigma(s, y, u, \partial_y u, \partial_y^2 u) - \sigma(s, y, v, \partial_y v, \partial_y^2 v)\} dy dW_s(y)\|_{L^2}^2 \right\}.
\] (4.11)

and

\[
E \int_0^T \| \Gamma u - \Gamma v \|^2_{H^2} dt = E \int_0^T \| \int_0^t \int_D G(t-s, x, y)[u(s, y) - v(s, y)] dy ds \|
\]

\[
+ \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds
\]

\[
+ \int_0^t \int_D (t-s, x, y)\{\sigma(s, y, u, \partial_y u, \partial_y^2 u) - \sigma(s, y, v, \partial_y v, \partial_y^2 v)\} dy dW_s(y)\|_{L^2}^2 dt
\]

\[
\leq CE \int_0^T \left\| \int_0^t \int_D G(t-s, x, y)[u(s, y) - v(s, y)] dy ds \right\|_{H^2}^2
\]

\[
+ \left\| \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds \right\|_{H^2}^2
\]

\[
+ \left\| \int_0^t \int_D G(t-s, x, y)\{\sigma(s, y, u, \partial_y u, \partial_y^2 u) - \sigma(s, y, v, \partial_y v, \partial_y^2 v)\} dy dW_s(y)\|_{H^2}^2 \right\} dt.
\] (4.12)

By (3.12) we have

\[
E \sup_{0 \leq t \leq T} \left\{ \| \int_0^t \int_D G(t-s, x, y)[u(s, y) - v(s, y)] dy ds \|_{L^2}^2 \right\} \leq C T E \sup_{0 \leq t \leq T} \| u - v \|^2_{L^2},
\] (4.13)

and by (3.14) we have

\[
E \sup_{0 \leq t \leq T} \left\{ \| \int_0^t \int_D \nabla G(t-s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dy ds \|_{L^2}^2 \right\}
\]

\[
\leq C(N) T^{\frac{1}{2} - \frac{4}{p} - 1} E \sup_{0 \leq t \leq T} \| u - v \|^2_{L^2}.
\] (4.14)

Moreover, by a similar argument as in the proof of (3.15) and but using (4.4) instead of (3.6) we have

\[
E \sup_{0 \leq t \leq T} \left\{ \| \int_0^t \int_D G(t-s, x, y)\{\sigma(s, y, u, \partial_y u, \partial_y^2 u) - \sigma(s, y, v, \partial_y v, \partial_y^2 v)\} dy dW_s(y)\|_{L^2}^2 \right\}
\]
\[
\leq CTE \sup_{0 \leq t \leq T} \|u - v\|_{L^2}^2 + C\varepsilon E \int_0^T \|u - v\|_{H^2}^2 dt,
\]
and by (4.8) we have
\[
E \int_0^T \int_0^t \int_D G(t - s, x, y) u(s, y) - v(s, y) dyds \|_{H^2}^2 dt \leq CT^2(T + 1)E \sup_{0 \leq t \leq T} \|u - v\|_{L^2}^2. \tag{4.16}
\]
In addition, by a similar argument as in the proof of (4.9) but using (3.14) instead of (3.9), we have
\[
E \int_0^T \int_0^t \int_D \nabla G(t - s, x, y) \cdot [f_N(u(s, y)) - f_N(v(s, y))] dyds \|_{H^2}^2 dt \leq C(N)(1 + T)T^{3 - \frac{4}{p} - (p - 1)}E \sup_{0 \leq t \leq T} \|u - v\|_{L^2}^2. \tag{4.17}
\]
Finally, by Lemma 2.5 and (4.4) we have
\[
E \int_0^T \int_0^t \int_D G(t - s, x, y) \sigma(s, y, u, \partial_x u, \partial_y^2 u) - \sigma(s, y, v, \partial_x v, \partial_y^2 v) dydW_s(y) \|_{H^2}^2 dt \leq CTE \int_0^T \int_0^t \int_D \sigma(s, y, u, \partial_x u, \partial_y^2 u) - \sigma(s, y, v, \partial_x v, \partial_y^2 v) dydW_s(y) \|_{H^2}^2 dt \leq CTE \sup_{0 \leq t \leq T} \|u - v\|_{L^2}^2 + C\varepsilon E \int_0^T \|u - v\|_{H^2}^2 dt. \tag{4.18}
\]
Combining (4.11)–(4.18), we get
\[
\begin{aligned}
\|\Gamma u - \Gamma v\|_{Y_T}^2 &\leq C(N)\{T + (1 + T)T^{3 - \frac{4}{p} - (p - 1)} + (1 + T)T^2 + T + T^3 - \frac{4}{p} + (p - 1) + T\} \\
&\quad \times E \sup_{0 \leq t \leq T} \|u - v\|_{L^2}^2 + C\varepsilon E \int_0^T \|u - v\|_{H^2}^2 dt.
\end{aligned}
\]
It follows that if \(T > 0\) and \(\varepsilon > 0\) are so small that
\[
C(N)\{T + (1 + T)T^{3 - \frac{4}{p} - (p - 1)} + (1 + T)T^2 + T + T^3 - \frac{4}{p} + (p - 1) + T\} < 1 \quad \text{and} \quad C\varepsilon < 1,
\]
then we have
\[
\|\Gamma u - \Gamma v\|_{Y_T} \leq \delta\|u - v\|_{Y_T}
\]
for some \(\delta \in (0, 1)\) depending on \(T\) and \(\varepsilon\), i.e., \(\Gamma\) is a contraction mapping in \(Y_T\). Hence, by a similar argument as in the proof of Theorem 1.1, we see that the desired assertion follows. This completes the proof of Theorem 1.2.

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