STRONGLY LOCAL P-NORM-CUT ALGORITHMS FOR SEMI-SUPERVISED LEARNING AND LOCAL GRAPH CLUSTERING

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Graph based semi-supervised learning is the problem of learning a labeling function for the graph nodes given a few example nodes, often called seeds, usually under the assumption that the graph’s edges indicate similarity of labels. This is closely related to the local graph clustering or community detection problem of finding a cluster or community of nodes around a given seed. For this problem, we propose a novel generalization of random walk, diffusion, or smooth function methods in the literature to a convex p-norm cut function. The need for our p-norm methods is that, in our study of existing methods, we find those principled methods based on eigenvector, spectral, random walk, or linear system often have difficulty capturing the correct boundary of a target label or target cluster. In contrast, 1-norm or maxflow-mincut based methods capture the boundary, but cannot grow from small seed set; hybrid procedures that use both have many hard to set parameters. In this paper, we propose a generalization of the objective function behind these methods involving p-norms. To solve the p-norm cut problem we give a strongly local algorithm – one whose runtime depends on the size of the output rather than the size of the graph. Our method can be thought as a nonlinear generalization of the Anderson-Chung-Lang push procedure to approximate a personalized PageRank vector efficiently. Our procedure is general and can solve other types of nonlinear objective functions, such as p-norm variants of Huber losses. We provide a theoretical analysis of finding planted target clusters with our method and show that the p-norm cut functions improve on the standard Cheeger inequalities for random walk and spectral methods. Finally, we demonstrate the speed and accuracy of our new method in synthetic and real world datasets. Our code is available github.com/MengLiuPurdue/SLQ.

1 INTRODUCTION

Many datasets important to machine learning either start as a graph or have a simple translation into graph data. For instance, relational network data naturally starts as a graph. Arbitrary data vectors become graphs via nearest-neighbor constructions, among other choices. Consequently, understanding graph-based learning algorithms – those that learn from graphs – is a recurring problem. This field has a rich history with methods based on linear systems [Zhou et al., 2003; Zhu et al., 2003], eigenvectors [Joachims, 2003; Hansen and Mahoney, 2012], graph cuts [Blum and Chawla, 2001], and network flows [Lang and Rao, 2004; Andersen and Lang, 2008; Veldt et al., 2016], although recent work in graph-based learning has often focused on embeddings [Perozzi et al., 2014; Grover and Leskovec, 2016] and graph neural networks [Yadati et al., 2019; Klicpera et al., 2019; Li et al., 2019]. Our research seeks to understand the possibilities enabled by a certain p-norm generalization of the standard techniques.

Perhaps the prototypical graph-based learning problems are semi-supervised learning and local clustering. Other graph-based learning problems include role

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discovery and alignments. Semi-supervised learning involves learning a labeling function for the nodes of a graph based on a few examples, often called seeds. The most interesting scenarios are when most of the graph has unknown labels and there are only a few examples per label. This could be a constant number of examples per label, such as 10 or 50, or a small fraction of the total label size, such as 1%. Local clustering is the problem of finding a cluster or community of nodes around a given set of seeds. This is closely related to semi-supervised learning because that cluster is a natural suggestion for nodes that ought to share the same label, if there is a homophily property for edges in the network. If this homophily is not present, then there are transformations of the graph that can make these methods work better [Peel, 2017].

For both problems, a standard set of techniques is based on random walk diffusions and mincut constructions [Zhou et al., 2003; Zhu et al., 2003; Joachims, 2003; Gleich and Mahoney, 2015; Pan et al., 2004]. These reduce the problem to a linear system, eigenvector, random walk, or mincut-maxflow problem, which can often be further approximated. As a simple example, consider solving a seeded PageRank problem that is seeded on the nodes known to be labeled with a single label. The resulting PageRank vector indicates other nodes likely to share that same label. This propensity of PageRank to propagate labels has been used in a many applications and it has many interpretations [Kloumann et al., 2016; Gleich, 2015; Orecchia and Mahoney, 2011; Pan et al., 2004; Lisewski and Lichtarge, 2010; Ghosh et al., 2014], including guilt-by-association [Koutra et al., 2011]. A related class of mincut-maxflow constructions uses similar reasoning [Blum and Chawla, 2001; Veldt et al., 2016, 2019a].

The link between these PageRank methods and the mincut-maxflow computations is that they correspond to 1-norm and 2-norm variations on a general objective function (see [Gleich and Mahoney, 2014] and Equation 1). In this paper, we replace the norm with a general $p$-norm. (For various reasons, we refer to it as a $q$-norm in the subsequent technical sections. We use $p$-norm here as this usage is more common.) The literature on 1 and 2-norms is well established and largely suggests that 1-norm (mincut) objectives are best used for refining large results from other methods – especially because they tend to sharpen boundaries – whereas 2-norm methods are best used for expanding small seed sets [Veldt et al., 2016]. There is a technical reason for why mincut-maxflow formulations cannot expand small seed sets, unless they have uncommon properties, discussed in [Fountoulakis et al., 2020, Lemma 7.2]. The downside to 2-norm methods is that they tend to “expand” or “bleed out” over natural boundaries in the data. This is illustrated in Figure 1(b). The hypothesis motivating this work is that techniques that use a $p$-norm where $1 < p < 2$ should provide a useful alternative – if they can be solved as efficiently as the other cases. This is indeed what we find and a small example of what our methods are capable of is shown in Figure 1(c), where we use a 1.1-norm to avoid the over-expansion from the 2-norm method.
We are hardly the first to notice these effects or propose $p$-norms as a solution. For instance, the $p$-Laplacian [Amghibech, 2003] and related ideas [Alamgir and Luxburg, 2011] have been widely studied as a way to improve results in spectral clustering [Bühler and Hein, 2009] and semi-supervised learning [Brindle and Zhu, 2013]. This has recently been used to show the power of simple nonlinearities in diffusions for semi-supervised learning as well [Ibrahim and Gleich, 2019].

The major rationale for our paper is that our algorithmic techniques are closely related to those used for 2-norm optimization. It remains the case that spectral (2-norm) approaches are far more widely used in practice, partly because they are simpler to implement and use, whereas the other approaches involve more delicate computations. Our new formulations are amenable to similar computation techniques as used for 2-norm problems, which we hope will enable them to be widely used.

To forward the goal of making these techniques useful, we release all of experimental code and the tools necessary to easily use the strongly-local $p$-norm cuts on github:

\[ \text{github.com/MengLiuPurdue/SLQ} \]

This includes related codes for similar purposes as well.

The remainder of this paper consists of a full demonstration of the potential of this idea. We first formally state the problem and review technical preliminaries in Section 2. As an optimization problem the $p$-norm problem is strongly convex with a unique solution. Next, we provide a strongly local algorithm to approximate the solution (Section 3). A strongly local algorithm is one where the runtime depends on the sparsity of the output rather than the size of the input graph. This enables the methods to run efficiently even on large graphs, because, simply put, we are able to bound the maximum output size and runtime independently of the graph size. A hallmark of the existing literature is on these methods is a recovery guarantee called a Cheeger inequality. Roughly, this inequality shows that, if the methods are seeded nearby a good cluster, then the methods will return something that is not too far away from that good cluster. This is often quantified in terms of the conductance of the good cluster and the conductance of the returned cluster. There are a variety of tradeoffs possible here [Andersen et al., 2006; Zhu et al., 2013; Wang et al., 2017]. We prove such a relationship for our methods where the quality of the guarantee depends on the exponent $1/p$, which reproduces the square root Cheeger guarantees [Chung, 1992] for $p = 2$ but gives better results when $p < 2$. Finally, we empirically demonstrate a number of aspects of our methods in comparison with a number of other techniques in Section 5. The goal is to highlight places where our $p$-norm objectives differ.

At the end, we have a number of concluding discussions (Section 6), which highlight dimensions where our methods could be improved, as well as related literature. For instance, there are many ways to use personalized PageRank methods with graph convolutional networks and embedding techniques [Klicpera et al., 2019] – we conjecture that our $p$-norm methods will simply improve on these relationships. Also, and importantly, as we were completing this paper, we became aware of [Yang et al., 2020] which discusses $p$-norms for flow-based diffusions. Our two papers have many similar findings on the benefit of $p$-norms, although there are some meaningful differences in the approaches, which we discuss in Section 6. In particular, our algorithm is distinct and follows a simple generalization of the widely used and deployed push method for PageRank. Our hope is that both papers can highlight the benefits of this idea to improve the practice of graph-based learning.
2 GENERALIZED LOCAL GRAPH CUTS

We consider graphs that are undirected, connected, and weighted with positive edge weights lower-bounded by 1. Let \( G = (V, E, w) \) be such a graph, where \( n = |V| \) and \( m = |E| \). The adjacency matrix \( A \) has non-zero entries \( w(i, j) \) for each edge \((i, j)\), all other entries are zero. This is symmetric because the graph is undirected. The degree vector \( d \) is defined as the row sum of \( A \) and \( D \) is a diagonal matrix defined as \( \text{diag}(d) \). The incidence matrix \( B \in \{0, -1, 1\}^{m \times n} \) measures the differences of adjacent nodes. The \( k \)th row of \( B \) represents the \( k \)th edge and each row has exactly two nonzero elements, i.e. 1 for start node of \( k \)th edge and \(-1\) for end node of \( k \)th edge. For undirected graphs, either node can be the start node or end node and the order does not matter. We use \( \text{vol}(S) \) for the sum of weighted degrees of the nodes in \( S \) and \( \phi(S) = \frac{\text{cut}(S)}{\min(\text{vol}(S), \text{vol}(\bar{S}))} \) for conductance. We use \( i \sim j \) to represent that node \( i \) and node \( j \) are adjacent.

For simplicity, we begin with PageRank, which has been used for all of these tasks in various guises [Zhou et al., 2003; Gleich and Mahoney, 2015; Andersen et al., 2006]. A PageRank vector [Gleich, 2015] is the solution of the linear system \( (I - \alpha AD^{-1})x = (1 - \alpha)v \) where \( \alpha \) is a probability between 0 and 1 and \( v \) is a stochastic vector that gives the seed distribution. This can be easily reworked into the equivalent linear system \( (\gamma D + L)y = \gamma v \) where \( y = Dx \) and \( L = D - A \). The starting point for our methods is a result shown in [Gleich and Mahoney, 2014], where we can further translate this into a 2-norm “cut” computation on a graph called the localized cut graph that is closely related to common constructions in maxflow-mincut computations for cluster improvement [Andersen and Lang, 2008; Fountoulakis et al., 2020].

The localized cut graph is created from the original graph, a set \( S \), and a value \( \gamma \). The construction adds an extra source node \( s \) and an extra sink node \( t \), and edges from \( s \) to the original graph that localize a solution, or bias, a solution within the graph near the set \( S \). Formally, given a graph \( G = (V, E) \) with adjacency matrix \( A \), a seed set \( S \subset V \) and a non-negative constant \( \gamma \), the adjacency matrix of the localized cut graph is:

\[
A_S = \begin{bmatrix}
0 & \gamma d_S^T & 0 \\
\gamma d_S & A & \gamma d_S \\
0 & \gamma d_S & 0
\end{bmatrix}
\]

and a small illustration is

Here \( S \) is the complement of \( S \), \( d_S = De_S \), \( \bar{d}_S = D\bar{e}_S \), and \( e_S \) is an indicator vector for \( S \).

Let \( B, w \) be the incidence matrix and weight vector for the localized cut-graph. Then PageRank is equivalent to the following 2-norm-cut problem (see full details in [Gleich and Mahoney, 2014])

\[
\begin{align*}
\text{minimize} & \quad w^T(Bx)^2 = \sum_{i,j} w_{i,j} (x_i - x_j)^2 = x^T B^T \text{diag}(w) B x \\
\text{subject to} & \quad x_s = 1, x_t = 0
\end{align*}
\]

We call this a cut problem because if we replace the squared term with an absolute value (i.e., \( \sum w_{i,j} |x_i - x_j| \)), then we have the standard \( s,t \)-mincut problem. Our paper proceeds from changing this power of 2 into a more general loss-function \( \ell \) and also adding a sparsity penalty, which is often needed to produce strongly local solutions [Gleich and Mahoney, 2014]. We define this formally now.

\textbf{DEFINITION 1 (Generalized local graph cut)} Fix a set \( S \) of seeds and a value of \( \gamma \). Let \( B, w \) be the incidence matrix and weight vector of the localized cut graph. Then the generalized local graph cut problem is:

\[
\begin{align*}
\text{minimize} & \quad w^T \ell(Bx) + \kappa \gamma d^T x = \sum_{i,j} w_{i,j} \ell(x_i - x_j) + \kappa \gamma \sum_i x_i d_i \\
\text{subject to} & \quad x_s = 1, x_t = 0, x \geq 0.
\end{align*}
\]
Here $\ell(x)$ is an element-wise function and $\kappa \geq 0$ is a sparsity-promoting term.

We compare using power functions $\ell(x) = \frac{1}{q} |x|^q$ to a variety of other techniques for semi-supervised learning and local clustering in Figure 3. If $\ell$ is convex, then the problem is convex and can be solved via general-purpose solvers such as CVX. An additional convex solver is SnapVX [Hallac et al., 2017], which studied a general combination of convex functions on nodes and edges of a graph, although neither of these approaches scale to the large graphs we study in subsequent portions of this paper (65 million edges). To produce a specialized, strongly local solver, we found it necessary to restrict the class of functions $\ell$ to have similar properties to the power function $\ell(x) = \frac{1}{q} |x|^q$ and its derivative $\ell'(x)$.

Reproduction notes for Figure 3. We release the exact code to reproduce this figure. For all methods, for all values above a threshold, we compute 4 quantile lines to give roughly equally spaced regions. (a) PageRank is mathematically non-zero at all nodes in connected graph. Here, we threshold at $10^{-8}$ to focus on the circular contours. This is reproduced by (b) using $q = 2$. The “wiggles” around the edge are because we used CVX to solve this problem and there were minor tolerance issues around the edge. We also boosted the threshold to $5 \cdot 10^{-7}$ because of the tolerance in CVX. (c) Same as (b). (d) we used our SLQ solver as CVXpy with either the ECOS or SCS solver reported an error while using $q = 1.25$. We set $\rho = 0.99$ to get an accurate solution (close to KKT). Here, we used the algorithmic non-zeros as the code introduces elements “sparsely”. (e) This used mathematical non-zeros again because the algorithm from [Kloster and Gleich, 2014] uses the same sparse “push” mechanisms as our SLQ algorithm. (f) CRD returns a set, so we simply display that set. The parameters were chosen to make it look as close to a square as possible. (g and h) We used the forward Euler algorithm from [Ibrahim and Gleich, 2019] with non-zero truncation. $k$ is the number of steps and $h$ is the step-size. These were chosen to make the pictures look like diamonds and squares, respectively to mirror our results. The
entry thresholds were also 5 times the minimum element because the vectors are non-zero everywhere.

**DEFINITION 2** In the $[-1, 1]$ domain, the loss function $\ell(x)$ should satisfy (1) $\ell(x)$ is convex; (2) $\ell'(x)$ is an increasing and anti-symmetric function; (3) For $\Delta x > 0$, $\ell'(x)$ should satisfy either of the following condition with constants $k > 0$ and $c > 0$ (3a) $\ell'(x + \Delta x) \leq \ell'(x) + k\ell'(\Delta x)$ and $\ell'(x) > c$ or (3b) $\ell'(x)$ is c-Lipschitz continuous and $\ell'(x + \Delta x) \geq \ell'(x) + k\ell'(\Delta x)$ when $x \geq 0$.

**REMARK 3** If $\ell'(x)$ is Lipschitz continuous with Lipschitz constant to be $L$ and $\ell''(x) > c$, then constraint 3(a) can be satisfied with $k = L/c$. However, $\ell'(x)$ can still satisfy 3(a) even if it is not Lipschitz continuous. A simple example is $\ell(x) = |x|^{3.5}$, $-1 \leq x \leq 1$. In this case, $k = 1$ but it is not Lipschitz continuous at $x = 0$. On the other hand, when $\ell'(x)$ is Lipschitz continuous, it can satisfy constraint 3(b) even if $\ell''(x) = 0$. An example is $\ell(x) = |x|^{3.5}$, $-1 < x < 1$. In this case $\ell''(x) = 0$ when $x = 0$ but $\ell'(x + \Delta x) \geq \ell'(x) + \ell'\Delta x)$ when $x \geq 0$.

**LEMMA 4** The power function $\ell(x) = \frac{1}{q}|x|^q$, $-1 < x < 1$ satisfies definition 2 for any $q > 1$. More specifically, when $1 < q < 2$, $\ell(x)$ satisfies 3(a) with $c = q - 1$ and $k = 2^{2-q}$, when $q \geq 2$, $\ell(x)$ satisfies 3(b) with $c = q - 1$ and $k = 1$.

**Proof**
First, we know $\ell'(x) = |x|^{q-1}\text{sgn}(x)$ and $\ell''(x) = (q-1)|x|^{q-2}$. And we define $\ell''(0) = \infty$.

For 3(a), since $-1 < x < 1$, $1 < q < 2$, we have $\ell''(x) > (q-1)$. On the other hand

$$\frac{\ell'(x + \Delta x) - \ell'(x)}{\ell'(\Delta x)} = \frac{x}{\Delta x} + 1 \text{ sgn} \left( \frac{x}{\Delta x} + 1 \right) - \text{ sgn} \left( \frac{x}{\Delta x} \right).$$

Define a new function $f(x) = |1 + x|^{q-1}\text{sgn}(1 + x) - |x|^{q-1}\text{sgn}(x)$. $f'(x) = |1 + x|^{q-2} - |x|^{q-2}$. So the maximum of $f(x)$ is achived at $f(-0.5) = 2^{2-q}$.

For 3(b), since $-1 < x < 1$, $q > 2$, we have $\ell''(x) < (q-1)$. And when $x \geq 0$, $(x + \Delta x)^{q-1} \geq x^{q-1} + \Delta x^{q-1}$ is obvious.

Note that the $\ell(x) = |x|$ does not satisfy either choice for property (3). Consequently, our theory will not apply to mincut problems. In order to justify the generalized term, we note that $q$-norm generalizations of the Huber and Berhu loss functions [Owen, 2007] do satisfy these definitions.

**DEFINITION 5** Given $1 < q < 2$ and $0 < \delta < 1$, the “q-Huber” and “Berq” functions are

$$q\text{-Huber} \quad \ell(x) = \begin{cases} \frac{1}{q}x^q & \text{if } |x| \leq \delta, \\ \frac{1}{q}|x|^q + \left(\frac{2-2q}{q}\right)\delta^q & \text{otherwise} \end{cases} \quad \text{and} \quad Berq \quad \ell(x) = \begin{cases} \frac{1}{q}x^q & \text{if } |x| \leq \delta, \\ \frac{1}{q}x^q + \left(\frac{2-2q}{q}\right)\delta^q & \text{otherwise}. \end{cases}$$

**LEMMA 6** When $-1 \leq x \leq 1$, both “q-Huber” and “Berq” satisfy Definition 2. The value of $k$ for both is $2^{2-q}$, the $c$ for $q$-Huber is $q - 1$ while the $c$ for “Berq” is 1.

**Proof**
Obviously, both condition (1) and (2) are satisfied for “q-Huber” and “Berq”. Now we show 3(a) is also satisfied for “q-Huber” based on the proof of lemma 4. The proof of “Berq” is also similar.

When $\Delta x > \delta \ (\Delta x \leq \delta$ is similar)

$$k = \frac{\ell'(x + \Delta x) - \ell'(x)}{\Delta x^{q-1}}$$

$$= \begin{cases} \frac{1}{\Delta x} + 1 \text{ sgn} \left( \frac{x}{\Delta x} + 1 \right) - \text{ sgn} \left( \frac{x}{\Delta x} \right) & |x| > \delta, |x + \Delta x| > \delta \\ \left| \frac{x}{\Delta x} + 1 \right|^{q-1} \text{sgn} \left( \frac{x}{\Delta x} + 1 \right) - \text{ sgn} \left( \frac{x}{\Delta x} \right) & |x| > \delta, |x + \Delta x| \leq \delta \\ \left| \frac{x + \Delta x}{\Delta x} \right|^{q-1} \text{sgn} \left( \frac{x + \Delta x}{\Delta x} \right) - \text{ sgn} \left( \frac{x}{\Delta x} \right) & |x| \leq \delta, |x + \Delta x| > \delta \\ \Delta x^{2-q} & |x| \leq \delta, |x + \Delta x| \leq \delta \\ \end{cases}$$

6
Case 1: Same as the proof of lemma 4.

Case 2: In this case, \( x \) can only be negative, i.e. \( x < -\delta \). After some simplification,

\[
k = \left( \frac{\Delta x}{\delta} \right)^{2-q} - \left( \frac{\Delta x}{\delta} \right)^{2-q} - 1 \left( \frac{-x}{\Delta x} \right)^{q-1}
\]

Note that the right hand side is an increasing function of \( \Delta x \) and \(-\delta - x \leq \Delta x \leq \delta - x\). Replacing \( \Delta x \) by \(-\delta - x \) yields

\[
k = \frac{(-x)^{q-1} - \delta^{q-1}}{(-x - \delta)^{q-1}} > 0
\]

Replacing \( \Delta x \) by \( \delta - x \) yields

\[
k = \frac{\delta^{q-1} + (-x)^{q-1}}{(\delta - x)^{q-1}} \leq 2^{2-q}
\]

Here the last inequality is due to Jensen’s inequality.

Case 3: Its proof is very similar to case 2.

Case 4: Since \( 0 < \Delta x \leq 2\delta \), \( 0 \leq k \leq 2^{2-q} \).

We now state uniqueness.

**Theorem 7** Fix a set \( S \), \( \gamma > 0 \), \( \kappa > 0 \). For any loss function satisfying Definition 2, then the solution \( x \) of (2) is unique. Moreover, define a residual function \( r(x) = -\frac{1}{2} B^T \text{diag}(\ell'(Bx))w \). A necessary and sufficient condition to satisfy the KKT conditions is to find \( x^* \) where \( x^* \geq 0 \), \( r(x^*) = [r_s, g^T, r_t]^T \) with \( g \leq \kappa d \) (where \( d \) reflects the original graph), \( k^* = [0, \kappa d - g, 0]^T \) and \( g^T(\kappa d - g) = 0 \).

**Proof** We first prove uniqueness. The Hessian of the objective in (2) is:

\[
H(i, j) = \begin{cases} 
\ell''(x_i - (e_S)_i) & \text{if } i = j \\
\ell''(x_i - x_j) & \text{if } i \sim j \\
0 & \text{otherwise}
\end{cases}
\]

Thus \( x^T H x = \sum_{i \in V} x_i^2 \ell''(x_i - (e_S)_i) + \sum_{i \sim j} x_i x_j \ell''(x_i - x_j) \). If (3a) is satisfied, we have \( \ell''(x) > 0 \) which means \( x^T H x > 0 \). So the objective 2 is strictly convex and the uniqueness is guaranteed. When (3b) is satisfied, \( \ell'(x + \Delta x) \geq \ell'(x) + k \ell'(\Delta x) \) guarantees that \( \ell''(x) \) can only become zero in a range around zero, i.e. \( \ell''(x) = \ell''(x) = 0 \) when \( x \in [-\psi, \psi] \), where \( 0 \leq \psi \leq 1 \). Then \( x^T H x = 0 \) implies \( x_i \geq 1 - \psi \) when \( i \in S \), \( x_i \leq \psi \) when \( i \notin S \) and \( -\psi \leq x_i - x_j \leq \psi \) or \( x_i x_j = 0 \). In this case, the uniqueness is implied by \( \kappa \gamma d \) in (2), i.e. each \( x_i \) will be the smallest feasible value.

Next, we will show the KKT condition of (2). If we translate problem (2) to add the constraint \( u = Bx \), then the loss is \( \ell'(u) \). The Lagrangian is

\[
\mathcal{L} = w^T \ell(u) + \kappa \gamma d^T x - f^T (Bx - u) - \lambda_s(x_s - 1) - \lambda_t x_t - k^T x
\]

Standard optimality results give the KKT of (2) as

\[
\begin{align*}
\frac{\partial L}{\partial x} &= \kappa d - \frac{1}{\gamma} B^T f - \lambda_s e_s - \lambda_t e_t - k = 0 \\
\frac{\partial L}{\partial u} &= \text{diag}(\ell''(u)) w + f = 0 \\
k^T x &= 0 \\
Bx &= u \\
k &\geq 0, x_s = 1, x_t = 0
\end{align*}
\]

Thus, combining the first and second equations, \( r = \frac{1}{2} B^T f \). Since \( k \geq 0 \), from the first equation, we have \( g \leq \kappa d \). And from \( k^T x = 0 \), we have \( g^T(\kappa d - g) = 0 \). ■
3 STRONGLY LOCAL ALGORITHMS

In this section, we will provide a strongly local algorithm to approximately optimize equation (2) with \(f(x)\) satisfying definition 2. The simplest way to understand this algorithms is as a nonlinear generalization of the Andersen-Chung-Lang push procedure for PageRank [Andersen et al., 2006], which we call ACL. (The ACL procedure has strong relationships with Gauss-Seidel, coordinate solvers, and various other standard algorithms.) The overall algorithm is simple: find a vertex \(i\) where the KKT conditions from Theorem 7 are violated and increase \(x_i\) on that node until we approximately satisfy the KKT conditions. Update the residual, look for another violation, and repeat. The ACL algorithm targets \(q = 2\) case, which has a closed form update, we simply need to replace this with a binary search.

**Algorithm** nonlin-cut\((\gamma, \kappa, \rho, \varepsilon)\) for set \(S\) and graph \(G\) where \(0 < \rho < 1\) and \(0 < \varepsilon\) determine accuracy

1. Let \(x(i) = 0\) except for \(x_s = 1\) and set \(r = -\frac{1}{\rho}B^t\text{diag}[\ell'(Bx)]w\)
2. While there is any vertex \(i\) where \(r_i > \kappa d_i\), or stop if none exists (find a KKT violation)
3. Apply nonlin-push at vertex \(i\), updating \(x\) and \(r\)
4. Return \(x\)

**Algorithm** nonlin-push\((i, \gamma, \kappa, x, r, \rho, \varepsilon)\)

1. Use binary search to find \(\Delta x_i\) such that the \(i\)th coordinate of the residual after adding \(\Delta x_i\) to \(x_i\), \(r'_i = \rho d_i\), the binary search stops when the range of \(\Delta x\) is smaller than \(\varepsilon\) (satisfy KKT at \(i\)).
2. Change the following entries in \(x\) and \(r\) to update the solution and residual
3. (a) \(x_i \leftarrow x_i + \Delta x_i\)
4. (b) For each neighbor \(j\) in the original graph \(G\), \(r_j \leftarrow r_j + \frac{1}{\gamma} w_{i,j} \ell'(x_j - x_i) - \frac{1}{\gamma} w_{i,j} \ell'(x_j - x_i - \Delta x_i)\)

For \(\rho < 1\), we only approximately satisfy the KKT conditions, as discussed further in the Section 3.3. We have the following strongly local runtime guarantee when 3(a) in definition 2 is satisfied. See Section 3.2 for similar guarantee on 3(b). (This ignores binary search, but that only scales the runtime by \(\log \varepsilon\) because the values are in \([0, 1]\).)

**Theorem 8** Let \(\gamma > 0, \kappa > 0\) be fixed and let \(k\) and \(c\) be the parameters from Definition 2 for \(\ell(x)\). For \(0 < \rho < 1\), suppose nonlin-cut stops after \(T\) iterations, and \(d_i\) is the degree of node updated at the \(i\)-th iteration, then \(T\) must satisfy: \(\sum_{i=1}^{T} d_i \leq \text{vol}(S)/c\ell'^{-1}(\gamma(1 - \rho)\kappa/k(1 + \gamma)) = O(\text{vol}(S))\).

The notation \(\ell'^{-1}\) refers to the inverse functions of \(\ell'(x)\). This function must be invertible under the the definition of 3(a). The runtime bound when 3(b) holds is slightly different, see below.

Note that if \(\kappa = 0, \gamma = 0, \rho = 1\), then this bound goes to \(\infty\) and we lose our guarantee. However, if these are not the case, then the bound \(T\) shows that the algorithm will terminate in time that is independent of the size of the graph. This is the type of guarantee provided by strongly local graph algorithms and has been extremely useful to scalable network analysis methods [Leskovec et al., 2009; Jeub et al., 2015; Yin et al., 2017; Veldt et al., 2016; Kloster and Gleich, 2014].

**Lemma 9** During algorithm 1, for any \(i \in \{V\setminus\{s, t\}\}\), \(g_i\) will stay nonnegative and \(0 \leq x_i \leq 1\).

**Proof** We can show this by induction. At the initial step, for node \(i \in S\), \(g_i = d_i\), and for node \(i \in S\), \(g_i = 0\). And after a nonlin-push step, every \(g_i\) will stay nonnegative.
To prove $0 \leq x_i \leq 1$, by expanding $g_i$, we have

$$g_i = -\frac{1}{\gamma} \sum_{j \sim i} w_i \ell'(x_i - x_j) - d_i \ell'(x_i - (e_S)_i)$$

$x_i \geq 0$ is trivial. Suppose $x_i$ is the largest element of $x$ and $x_i > 1$, then we will have $\ell'(x_i - x_j) \geq 0$ for $j \sim i$ and $\ell'(x_i - (e_S)_i) > 0$. Then $g_i < 0$, which is a contradiction.

**3.1 Running Time Analysis When 3(a) Is Satisfied**

**Lemma 10** When 3(a) is satisfied, after calling nonlin-push on node $i$, the decrease of $\|g\|_1$ will be strictly larger than

$$cd_i(\ell')^{-1} \left( \frac{\gamma(1 - \rho)\kappa}{k(1 + \gamma)} \right)$$

**Proof** We use $g'$ to denote $g$ after calling nonlin-push on node $i$. At any intermediate step of nonlin-cut procedure,

$$\|g\|_1 = \sum_{i \in S} g_i = -\sum_{i \in S} d_i \ell'(x_i - 1) - \sum_{i \in S} d_i \ell'(x_i)$$

This is because for any edge $(i, j) \in E$, $g_i$ has a term $\frac{1}{\gamma} w(i, j) \ell'(x_i - x_j)$ while $g_j$ has a term $\frac{1}{\gamma} w(j, i) \ell'(x_j - x_i)$. Since our graph is undirected, $w(i, j) = w(j, i)$, so these two terms will cancel out. What remains are the terms corresponding to the edges connecting to $s$ or $t$. So after calling nonlin-push on node $i$,

$$\|g\|_1 - \|g'\|_1 = d_i \ell'(x_i + \Delta x_i - (e_S)_i) - d_i \ell'(x_i - (e_S)_i)$$

$$\geq d_i \min\{\ell''(x_i + \Delta x_i - (e_S)_i), \ell''(x_i - (e_S)_i)\} \Delta x_i$$

$$\geq cd_i \Delta x_i$$

On the other hand, we need to choose $\Delta x_i$ such that $g'_i = \rho \kappa d_i$. We know

$$g'_i = -\frac{1}{\gamma} \sum_{j \sim i} w(i, j) \ell'(x_i + \Delta x_i - x_j) - d_i \ell'(x_i + \Delta x_i - (e_S)_i)$$

is a decreasing function of $\Delta x_i$. And when $\Delta x_i = 0$, $g'_i = \kappa d_i > \rho \kappa d_i$, when $\Delta x_i = 1$, $g'_i < 0 < \rho \kappa d_i$, since $\ell'(x)$ is a strictly increasing function, there exists a unique $\Delta x_i$ such that $g'_i = \rho \kappa d_i$. Moreover, we can lower bound $\Delta x_i$. To see that,

$$g'_i = \rho \kappa d_i$$

$$= -\frac{1}{\gamma} \sum_{j \sim i} w(i, j) \ell'(x_i + \Delta x_i - x_j) - d_i \ell'(x_i + \Delta x_i - (e_S)_i)$$

$$\geq -\frac{1}{\gamma} \sum_{j \sim i} w(i, j) \ell'(x_i - x_j) - d_i \ell'(x_i - (e_S)_i) - \frac{k(1 + \gamma)}{\gamma} d_i \ell'(\Delta x_i)$$

$$= g_i - \frac{k(1 + \gamma)}{\gamma} d_i \ell'(\Delta x_i)$$

Thus, we have

$$\Delta x_i \geq (\ell')^{-1} \left( \frac{\gamma (g_i - \rho \kappa d_i)}{k(1 + \gamma)d_i} \right) > (\ell')^{-1} \left( \frac{\gamma (1 - \rho)\kappa}{k(1 + \gamma)} \right)$$

which means

$$\|g\|_1 - \|g'\|_1 > cd_i(\ell')^{-1} \left( \frac{\gamma (1 - \rho)\kappa}{k(1 + \gamma)} \right).$$

The only step left to prove Theorem 8 is that at the beginning, we have $\|g\|_1 = \text{vol}(S)$. Then the theorem follows by Lemma 10.
3.2 Running Time Analysis When 3(B) is Satisfied

For the following results, we add an extra strictly increasing condition so that \( \ell'(x) \) is positive. When \( \ell'(x) \) is not strictly increasing, i.e. \( \ell'(x) = 0 \) in a small range round 0, it is our conjecture that the algorithm will still finish in a strongly local time, although we have not yet proven that. Note that this strictly increasing criteria is true for all the loss used in the experiments.

**Lemma 11** When 3(b) is satisfied and \( \ell'(x) \) is strictly increasing, then after calling nonlin-push on node \( i \), the decrease of \( ||g||_1 \) will be strictly larger than

\[
kd_i \ell' \left( \frac{\gamma(1 - \rho)\kappa}{c(1 + \gamma)} \right)
\]

**Proof** Similarly to the proof of lemma 10, after calling nonlin-push on node \( i \),

\[
||g||_1 - ||g'||_1 = d_i \ell'(x_i + \Delta x_i - (e_S)_i) - d_i \ell'(x_i - (e_S)_i) 
\]

\[
\geq kd_i \ell'(\Delta x_i)
\]

On the other hand,

\[
g'_i = \rho \kappa d_i
\]

\[
= -\frac{1}{\gamma} \sum_{j \sim i} w(i,j) \ell'(x_i + \Delta x_i - x_j) - d_i \ell'(x_i + \Delta x_i - (e_S)_i)
\]

\[
\geq -\frac{1}{\gamma} \sum_{j \sim i} w(i,j) \ell'(x_i - x_j) - d_i \ell'(x_i - (e_S)_i) - \frac{c(1 + \gamma)}{\gamma} d_i \Delta x_i
\]

\[
= g_i - \frac{c(1 + \gamma)}{\gamma} d_i \Delta x_i
\]

Thus, we have

\[
\Delta x_i \geq \frac{\gamma(r_i - \rho \kappa d_i)}{c(1 + \gamma)d_i} > \frac{\gamma(1 - \rho)\kappa}{c(1 + \gamma)}
\]

which means

\[
||g||_1 - ||g'||_1 > kd_i \ell' \left( \frac{\gamma(1 - \rho)\kappa}{c(1 + \gamma)} \right).
\]

Lemma 11 along with the same type of analysis as before give the following result when 3(b) is satisfied.

**Theorem 12** Let \( \gamma > 0, \kappa > 0 \) be fixed and let \( k \) and \( c \) be the parameters from Definition 2 for \( \ell(x) \) when 3(b) is satisfied with a strict increase. For \( 0 < \rho < 1 \), suppose nonlin-cut stops after \( T \) iterations, and \( d_i \) is the degree of node updated at the \( i \)-th iteration, then \( T \) must satisfy:

\[
\sum_{i=1}^{T} d_i \leq \frac{\text{vol}(S)}{k\ell'(\gamma(1 - \rho)\kappa/c(1 + \gamma))} = O(\text{vol}(S)).
\]

3.3 More Details on Rho

When \( \rho < 1 \), then we only approximately satisfy the KKT conditions. Here, we do some quick analysis of the difference in the idealized slackness condition \( k^T x = 0 \) compared to what we get from our solver. Note that by choosing \( \rho \) close to 1, we do produce a fairly accurate solution when 3(a) is satisfied.

**Lemma 13** When Algorithm 1 returns, if \( \ell(x) \) satisfies 3(a) we have

\[
k^T x \leq \frac{\kappa k \ell'(1 - \rho)\text{vol}(S)}{c}
\]
Proof We know \( k = [0, \kappa d - r, 0]^T \). Every time algorithm 2 is called at node \( i \), it will set \( g_i = \rho d_i \). In the following iterations, \( g_i \) can only increase until algorithm 2 is called at node \( i \) again. This means \( k \leq (1 - \rho)kd \).

On the other hand, when \( 3(a) \) is satisfied, \( \ell'(1 - x_i) \leq \ell'(x_i) + k\ell'(1) \)

\[
\|g\|_1 = -\sum_{i \in S} d_i \ell'(x_i) - \sum_{i \notin S} d_i \ell'(x_i - 1) \leq -\sum_{i \in V} d_i \ell'(x_i) + k\ell'(1)\text{vol}(S)
\]

\[
\leq -cd^T x + k\ell'(1)\text{vol}(S).
\]

Thus,

\[
d^T x \leq \frac{k\ell'(1)}{c}\text{vol}(S)
\]

Combining the two inequality gives this lemma.

When \( 3(b) \) is satisfied, it is easy to see \( k^T x \leq (1 - \rho)kd^T x \), however, there isn’t a closed form equation on the upper bound of \( k^T x \) in terms of \( \text{vol}(S) \).

### 4 Main Theoretical Results – Cut Quality Analysis

A common use for the results of these localized cut solutions is as localized Fiedler vectors of a graph to induce a cluster [Andersen et al., 2006; Leskovec et al., 2009; Mahoney et al., 2012; Zhu et al., 2013; Orecchia and Zhu, 2014]. This was the original motivation of the ACL procedure [Andersen et al., 2006], for which the goal was a small conductance cluster. One of the most common (and theoretically justified!) ways to convert a real-valued “clustering hint” vector \( x \) into clusters is to use a sweep cut process. This involves sorting \( x \) in decreasing order and evaluating the conductance of each prefix set \( S_j = \{x_1, x_2, ..., x_j\} \) for each \( j \in [n] \). The set with the smallest conductance will be returned. This computation is a key piece of Cheeger inequalities [Chung, 1992; Mihail, 1989]. In the following, we seek a slightly different type of guarantee. We posit the existence of a target cluster \( T \) and show that if \( T \) has useful clustering properties (small conductance, no good internal clusters), then a sweep cut over a \( q \)-norm or \( q \)-Huber localized cut vector seeded inside of \( T \) will accurately recover \( T \). The key piece is understanding how the computation plays out with respect to \( T \) inside the graph and \( T \) as a graph by itself.

### 4.1 Useful Observations

The following two observations are not directly related to the main result. But we still find them useful in understanding the problem in general.

**Lemma 14** For two seed sets \( S_1 \) and \( S_2 \), denote \( x_1 \) and \( x_2 \) to be the solutions of the \( L_q \) norm cut problem using \( S_1 \) and \( S_2 \) correspondingly, if \( S_1 \subseteq S_2 \), then \( x_1 \leq x_2 \).

**Proof** Considering two nonlinear-cut processes \( P_1 \), \( P_2 \) using \( S_1 \) or \( S_2 \) as input correspondingly, suppose we set the initial vector of \( P_2 \) to be the solution of \( P_1 \), i.e. \( x_1 \), then for nodes \( i \notin S_2 \setminus S_1 \), its residual stays zero, while for nodes \( i \in S_2 \setminus S_1 \), its residual becomes positive. This means \( P_2 \) needs more iterations to converge. And each iteration can only add nonnegative values to \( x_1 \). Thus, \( x_1 \leq x_2 \).

**Lemma 15** Suppose that \( \kappa = 0 \). We can compute the exact solution of problem (2) under two extreme cases \( \gamma \to \infty \) and \( \gamma \to 0 \).

- When \( \gamma \to \infty \), \( x_i = 1 \) for \( i \in S \) and \( x_i = 0 \) for \( i \in \bar{S} \).
- When \( \gamma \to 0 \), \( x_i \geq \frac{(\text{vol}(S))^\frac{1}{\gamma}}{(\text{vol}(V))^\frac{1}{\gamma}} \) for any \( i \in V \).
Proof When \( \kappa = 0 \), the objective function of (2) becomes
\[
\sum_{i \sim j} w(i, j) \ell(x_i - x_j) + \gamma \sum_{i \in V} d_i \ell(x_i - (e_S)_i)
\]
When \( \gamma \to \infty \), the first term vanishes, and the second term achieves its smallest value, when \( x_i = 1 \) for \( i \in S \) and \( x_i = 0 \) for \( i \in S \).

When \( \gamma \to 0 \), the second term vanishes, and the first term is minimal with objective zero when every \( x_i \) converges to a fixed constant. Moreover, the KKT condition now becomes
\[
\frac{1}{\gamma} \sum_{j \sim i} w(i, j) \ell'(x_i - x_j) + d_i \ell'(x_i - (e_S)_i) = 0
\]
Summing the KKT condition over all nodes yields:
\[
\sum_{i \in V} d_i \ell'(x_i - (e_S)_i) = 0
\]
So we can compute the constant that \( x_i \) converges to by making \( x_i = c \), which is
\[
c = \frac{(\text{vol}(S))^{\frac{1}{\gamma + 1}}}{(\text{vol}(V) - \text{vol}(S))^{\frac{1}{\gamma + 1}} + (\text{vol}(S))^{\frac{1}{\gamma + 1}}} \geq \frac{(\text{vol}(S))^{\frac{1}{\gamma + 1}}}{(\text{vol}(V))^{\frac{1}{\gamma + 1}}}.
\]

4.2 MAIN THEOREM AND ASSUMPTIONS

As we mentioned before, the key piece is understanding how the computation plays out with respect to \( T \) inside the graph and \( T \) as a graph by itself. We use \( \text{vol}_T(S) \) to be the volume of seed set \( S \) in the subgraph induced by \( T \) and \( \partial T \subset T \) to be the boundary set of \( T \), i.e., nodes in \( \partial T \) has at least one edge connecting to \( T \). Quantities with tildes, e.g., \( \tilde{d} \), reflect quantities in the subgraph induced by \( T \). We assume \( \kappa = 0 \), \( \rho = 1 \) and:

ASSUMPTION 16 The seed set \( S \) satisfies \( S \subseteq T \), \( S \cap \partial T = \emptyset \) and \( \sum_{i \in \partial T}(d_i - \tilde{d}_i)x_i^{q-1} \leq 2\phi(T)\text{vol}(S) \).

We call this the leaking assumption, which roughly states that the solution with the set \( S \) stays mostly within the set \( T \). As some quick justification for this assumption, we note that when when \( q = 2 \), [Zhu et al., 2013] shows by a Markov bound that there exists \( T_q \) where \( \text{vol}(T_q) \geq \frac{1}{2}\text{vol}(T) \) such that any node \( i \in T_q \) satisfies \( \sum_{j \in \partial T}(d_i - \tilde{d}_i)x_i \leq 2\phi(T)d_i \). So in that case, any seed sets \( S \subseteq T_q \) meets our assumption. For \( 1 < q < 2 \), it is trivial to see any set \( S \) with \( \text{vol}(S) \geq \frac{1}{2}\text{vol}(T) \) satisfies this assumption since the left hand side is always smaller than \( \text{cut}(T, \bar{T}) \). However, such a strong assumption is not necessary for our approach. The above guarantee allows for a small \( \text{vol}(S) \) and we simply require Assumption 16 holds. We currently lack a detailed analysis of how many such seed sets there will be.

Our second assumption regards the behavior within only the set \( T \) compared with the entire graph. To state it, we wish to be precise. Consider the localized cut graph associated with the hidden target set \( T \) on the entire graph and let \( B, w \) be the incidence and weights for this graph. We wish to understand how the solution \( x \) on this problem
\[
\begin{align*}
\text{minimize} & \quad w^T \ell(Bx) \\
\text{subject to} & \quad x_s = 1, x_t = 0, x \geq 0
\end{align*}
\]
compares with one where we consider the problem only on the subgraph induced by \( T \). Let \( \tilde{B}, \tilde{w} \) be the incidence matrix of the localized cut graph on the vertex induced subgraph corresponding to \( T \) and seeded on \( T \) (so the tilde-problem is seeded on all nodes). So formally, we wish to understand how \( \tilde{x} \) in
\[
\begin{align*}
\text{minimize} & \quad \tilde{w}^T \ell(\tilde{B}\tilde{x}) \\
\text{subject to} & \quad \tilde{x}_s = 1, \tilde{x}_t = 0, \tilde{x} \geq 0
\end{align*}
\]
and when we uniformly randomly sample points from the largest distance of any node in set $\bar{S}$.

Also, for node $i$, we define $d_{i,j}^\text{in} = \bar{d}_i - d_{i,j}^\text{out}$. Also define $l = (1 + \gamma)^\frac{d_{i,j}^\text{out}}{d_{i,j}^\text{in}} \leq (1 + \gamma)\max(\bar{d}_i)$. Suppose $\bar{x}_i \geq c$ for any node $i$ with distance at most $k - 1$, then we can show for node $i \in L_k$, $\bar{x}_i \geq \frac{c}{1 + \gamma^{k-1}}$. To see this, if $\bar{x}_i < c$, then by the KKT condition,

$$d_{i,j}^\text{in}(c - \bar{x}_i)^{q-1} \leq d_{i,j}^\text{out}x_i^{q-1} + \gamma d_{i,j}x_i^{q-1}$$

Here for $j \sim i$, if $j$ is closer to $S$, we set $\bar{x}_j$ to be $c$, otherwise, we set $\bar{x}_j$ to be 0.

This means

$$\bar{x}_i \geq \frac{c(d_{i,j}^\text{in})^{1-\gamma}}{(d_{i,j}^\text{out} + \gamma d_{i,j})^{1-\gamma} + (d_{i,j}^\text{in})^{1-\gamma}} \geq \frac{c}{l^{\frac{1}{1-\gamma}} + 1}$$

Also, for node $i \in S$, the first iteration of $q$-norm process will add at least $\frac{1}{l^{1-\gamma}}$ to $\bar{x}_i$. (This follows from unrolling the first loop of our algorithm and checking

ASSUMPTION 17 A relatively small $\gamma$ should be chosen such that the solution of localized $q$-norm cut problem in the subgraph induced by target cluster $T$ can satisfy $\min(\bar{x}) \geq \frac{(0.5\text{vol}(S))^{1/(q-1)}}{(\text{vol}(T))^{1/(q-1)}} = M$. \hfill \qed

We will call Assumption 17 a “mixing-well” guarantee.

To better understand this assumption, when $\ell(x) = \frac{1}{q}|x|^q$ and $q = 2$, a solution of the $\text{nonlin-cut}$ process (Algorithm 1) will be equivalent to a Markov process. In this case, one can lower bound $\min(\bar{x})$ by the well known infinity-norm mixing time of Markov chain. In fact, as shown in the proof of lemma 3.2 of [Zhu et al., 2013], when $\gamma \leq O(\phi(T) \cdot \text{Gap})$, they show that $\min(\bar{x}) \geq \frac{0.8\text{vol}(S)}{\text{vol}(T)}$. Here Gap is defined as the ratio of internal connectivity and external connectivity and often assumed to be $\Omega(1)$. We refer to [Zhu et al., 2013] for a detailed definition of this. For $1 < q < 2$, $\text{nonlin-cut}$ is no longer equivalent to the solution of a Markov process and thus it will be more difficult to derive a closed form equation on how small $\gamma$ needs to be so that equation 17 is satisfied. However, lemma 18 (below) shows that for graphs with small diameters, it is easier (i.e. $\gamma$ can be larger) for the solution of (6) to satisfy equation 17. This is reasonable because we expect good clusters and good communities to have small diameters.

LEMMA 18 Assume the subgraph induced by target cluster $T$ has diameter $O(\log |T|)$ and when we uniformly randomly sample points from $T$ as seed sets, the expected largest distance of any node in $\bar{S}$ to $S$ is $O\left(\frac{\log(|T|)}{|S|}\right)$. Also define $\gamma_2$ to be the largest $\gamma$ such that assumption 17 is satisfied at $q = 2$ and assume $\gamma_2 < 1$, if we set $\gamma = \gamma_2^{q-1}$ for $1 < q < 2$, and

$$\text{vol}(S) \leq \left(\frac{\gamma_2}{2 + \gamma_2} \frac{1}{|T|^\frac{1}{(q-1)\log\left(1 + \frac{1}{(q-1)\gamma_2}\right)}}\right)^{q-1}$$

where $l \leq (1 + \gamma)\max(\bar{d}_i)$. Then the solution of 6 can satisfy assumption 17.

Proof Given a seed set $S$, we can partition the $\bar{S}$ into disjoint subsets $L_1 \cup L_2 \cup L_3 \ldots \cup L_n$, where $L_i$ contains nodes that are $i$ distance away from $S$. For any node $i \in L_k$, we denote $d_{i,j}^\text{out}$ to be

$$d_{i,j}^\text{out} = \sum_{j \sim i, j \in L_k \cup L_{k+1}} w(i,j)$$

And $d_{i,j}^\text{in} = \bar{d}_i - d_{i,j}^\text{out}$. Also define $l = (1 + \gamma)^\frac{d_{i,j}^\text{out}}{d_{i,j}^\text{in}} \leq (1 + \gamma)\max(\bar{d}_i)$. Suppose $\bar{x}_i \geq c$ for any node $i$ with distance at most $k - 1$, then we can show for node $i \in L_k$, $\bar{x}_i \geq \frac{c}{1 + \gamma^{k-1}}$. To see this, if $\bar{x}_i < c$, then by the KKT condition,

$$d_{i,j}^\text{in}(c - \bar{x}_i)^{q-1} \leq d_{i,j}^\text{out}x_i^{q-1} + \gamma d_{i,j}x_i^{q-1}$$

Here for $j \sim i$, if $j$ is closer to $S$, we set $\bar{x}_j$ to be $c$, otherwise, we set $\bar{x}_j$ to be 0. This means

$$\bar{x}_i \geq \frac{c(d_{i,j}^\text{in})^{1-\gamma}}{(d_{i,j}^\text{out} + \gamma d_{i,j})^{1-\gamma} + (d_{i,j}^\text{in})^{1-\gamma}} \geq \frac{c}{l^{\frac{1}{1-\gamma}} + 1}$$

Also, for node $i \in S$, the first iteration of $q$-norm process will add at least $\frac{1}{l^{1-\gamma}}$ to $\bar{x}_i$. (This follows from unrolling the first loop of our algorithm and checking

1 The proof of lemma 3.2 in [Zhu et al., 2013] proves that the teleportation probability $\beta = 1 - \alpha$ needs to be smaller than $O(\phi(T) \cdot \text{Gap})$. When $q = 2$, as shown in [Gleich and Mahoney, 2014], $\beta = \frac{\beta_2}{1 + \beta_2}$, which means $\gamma_2 = \frac{2}{1 + \beta_2}$. Since we assume $\gamma_2 < 1$, we have $\beta < \gamma_2 < 2\beta$. In other words, $\gamma_2$ and $\beta$ are only different by a constant factor.
that this satisfies the binary search criteria.), which means \( \bar{x}_i \geq \frac{2^{\frac{q-1}{1+\gamma^{q-1}}}}{1+\gamma^{q-1}} \). Thus, for node \( i \in L_k \),

\[
\bar{x}_i \geq \frac{\gamma^{q-1}}{1+\gamma^{q-1}} \cdot \frac{1}{(1 + l^{\frac{1}{1+\gamma^{q-1}}})^k} = \frac{\gamma_2}{1+\gamma_2} \cdot \frac{1}{(1 + l^{\frac{1}{1+\gamma^{q-1}}})^k}
\]

Since the subgraph induced by target cluster \( T \) has diameter \( O(\log(|T|)) \) and when we uniformly randomly sample points from \( T \) as seed sets, the expected largest distance \( r \) of any node in \( \tilde{S} \) to \( S \) is \( O\left( \frac{\log(|T|)}{|S|} \right) \), we have \( r = O\left( \frac{\log(|T|)}{|S|} \right) \), which means

\[
\min(\bar{x}) \geq \frac{\gamma_2}{1+\gamma_2} \cdot \frac{1}{|T|^\frac{1}{\gamma^{q-1}} \log\left(1 + l^{\frac{1}{1+\gamma^{q-1}}} \right)}
\]

Assumption 17 requires \( \min(\bar{x}) \geq \frac{(0.5 \vol_T(S))^{\frac{1}{1+\gamma^{q-1}}}}{(\vol_T(T))^{\frac{1}{1+\gamma^{q-1}}}} \). So we just need

\[
\frac{\vol_T(S)}{\vol_T(T)} \leq 2 \left( \frac{\gamma_2}{1+\gamma_2} \cdot \frac{1}{|T|^\frac{1}{\gamma^{q-1}} \log\left(1 + l^{\frac{1}{1+\gamma^{q-1}}} \right)} \right)^{q-1},
\]

which was the final assumption.

**LEMMA 19** Under the previous assumptions, define a sweep cut set \( S_c \) as

\[
i \in V \mid x_i \geq \frac{c(0.5 \vol(S))^{\frac{1}{1+\gamma^{q-1}}}}{(\vol(T))^{\frac{1}{1+\gamma^{q-1}}}}
\]

then for any \( 0 < c \leq \frac{1}{2} \),

\[
\vol(S_c \setminus T) = O\left( \frac{\phi(T)}{\gamma c^{q-1}} \right) \vol(T) \quad \vol(T \setminus S_c) = O\left( \frac{\phi(T)}{\gamma} \right) \vol(T)
\]

**Proof** The proof is mostly a generalization to the proof of Lemma 3.4 in [Zhu et al., 2013]. For any \( i \in \bar{T} \), by the KKT condition and Assumption 16

\[
0 = r_i(x)
= -\frac{1}{\gamma} \sum_{j \sim i} w(i, j) \ell''(x_i - x_j) - d_i x_i^{q-1}
= -\frac{1}{\gamma} \sum_{j \sim i, j \in \bar{T}} w(i, j) \ell''(x_i - x_j) - \frac{1}{\gamma} \sum_{j \sim i, j \in T} w(i, j) \ell''(x_i - x_j) - d_i x_i^{q-1}
= -\frac{1}{\gamma} \sum_{j \sim i, j \in \bar{T}} w(i, j) \ell''(x_i - x_j) + \frac{1}{\gamma} \sum_{j \sim i, j \in T} w(i, j) \ell''(x_j - x_i) - d_i x_i^{q-1}
< -\frac{1}{\gamma} \sum_{j \sim i, j \in \bar{T}} w(i, j) \ell''(x_i - x_j) + \frac{1}{\gamma} \sum_{j \sim i, j \in \bar{T}} w(i, j) \ell''(x_j) - d_i x_i^{q-1}.
\]

By summing the inequality above over all nodes in \( \bar{T} \), the first term will all cancel out, it yields that

\[
\sum_{i \in \bar{T}} d_i x_i^{q-1} < \frac{1}{\gamma} \sum_{i \in \partial \bar{T}} (d_i - \tilde{d}_i) x_i^{q-1} \leq \frac{2\phi(T)\vol(S)}{\gamma}.
\]
Now by the definition of our sweep cut set, we know that for $i \in S_c \setminus T$, $x_i^{q-1} \geq \frac{e^{q-1} \text{vol}(S)}{\text{vol}(T)}$, thus

$$\frac{e^{q-1} \text{vol}(S)}{2 \text{vol}(T)} \text{vol}(S_c \setminus T) \leq \sum_{i \in S_c \setminus T} d_i x_i^{q-1} \leq \frac{2 \phi(T) \text{vol}(S)}{\gamma}$$

which means

$$\text{vol}(S_c \setminus T) = O \left( \frac{\phi(T)}{\gamma e^{q-1}} \right) \text{vol}(T).$$

In the following, we define $x_i = \tilde{x}_i + v_i$ and $\ell'(x_i - (e_S)_i) = \ell'(\tilde{x}_i - (e_S)_i) + k_i \ell'(v_i)$. For any node $i \in T$, by KKT condition,

$$0 = r_i(x)$$

$$= -\frac{1}{\gamma} \sum_{j \sim i} w(i, j) \ell'(x_i - x_j) - d_i \ell'(x_i - (e_S)_i)$$

$$= -\frac{1}{\gamma} \sum_{j \sim i \in T} w(i, j) \ell'(x_i - x_j) - \frac{1}{\gamma} \sum_{j \sim i \in T} w(i, j) \ell'(x_i - x_j) - d_i \ell'(x_i - (e_S)_i)$$

$$> -\frac{1}{\gamma} \sum_{j \sim i \in T} w(i, j) \ell'(x_i - x_j) - \frac{1}{\gamma} \sum_{j \sim i \in T} w(i, j) \ell'(x_i - (e_S)_i)_i - (d_i - \tilde{d}_i) \ell'(x_i)$$

$$= -\frac{1}{\gamma} \sum_{j \sim i \in T} w(i, j) \ell'(x_i - x_j) - \frac{1}{\gamma} \sum_{j \sim i \in T} w(i, j) \ell'(\tilde{x}_i - x_j)_i - k_i d_i \ell'(v_i) - (1 + \frac{1}{\gamma})(d_i - \tilde{d}_i) \ell'(x_i)$$

By summing the inequality above over all nodes in $T$, the first and the second terms cancel out, so it yields:

$$\sum_{i \in T} k_i d_i \ell'(v_i) > -\frac{2(1 + \gamma)}{\gamma} \phi(T) \text{vol}(S).$$

For nodes $i \in T \setminus S_c$, $x_i < c \tilde{x}_i$, which means $v_i < (c - 1) \tilde{x}_i$. And $\ell'(v_i) = -(-v_i)^{q-1} = -(1 - c)^{q-1} \frac{0.5 \text{vol}(S)}{\text{vol}(T)} \leq -(1 - c)^{q-1} \frac{0.5 \text{vol}(S)}{\text{vol}(T)}$. (Here we use the fact that $\text{vol}(T) \leq \text{vol}(T)$ and $S \cap \partial T = \emptyset$). From the proof of lemma 18, we know that $S$ will be included in $S_c$. When $i \notin S_c$,

$$k_i = \left( -\frac{\tilde{x}_i}{v_i} + 1 \right)^{q-1} - \left( -\frac{\tilde{x}_i}{v_i} \right)^{q-1} > \frac{2 - c}{(1 - c)^{q-1}} - 1.$$  

Thus, we have

$$\text{vol}(T \setminus S_c) = O \left( \frac{\phi(T)}{\gamma} \right) \text{vol}(T).$$

**Lemma 20** Under the same assumptions as lemma 19, among sweep cut sets $S_c \in \{ S_c \mid \frac{1}{4} \leq c \leq \frac{1}{2} \}$, there exists one $R$ such that $\phi(R) = O \left( \frac{\phi(T)}{\gamma} \frac{\text{Gap}}{\text{Gap}^2} \right)$. 

**Proof** Our proof is mostly a generalization to the proof of Lemma 4.1 in [Zhu et al., 2013]. If $\text{cut}(S_c, S_c) \geq E_0$ holds for all $\frac{1}{4} \leq c \leq \frac{1}{2}$, then we just need to upper bound $E_0$. 

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We introduce values \( k(i, j) \) that allow us to break \( \ell'(x_i - x_j) \) into \( \ell'(x_i) - k(i, j)\ell'(x_j) \). The specific choice \( k(i, j) > 0 \) is uniquely determined by \( x_i \) and \( x_j \). For any node \( i \in S_c \), by KKT condition,

\[
0 = \frac{1}{\gamma} \sum_{j \sim i} w(i, j)\ell'(x_i - x_j) + d_i\ell'(x_i - (e_S)_i)
\]

\[
= \frac{1}{\gamma} \sum_{j \sim i} (w(i, j)\ell'(x_i) - w(i, j)k(i, j)\ell'(x_j)) + d_i\ell'(x_i) - k_i(d_i(e_S)_i).
\]

Define \( K \) to be the matrix induced by \( k(i, j) \). Rearranging the equation above yields:

\[
(K \circ Ax^q - 1)_i = (1 + \gamma)d_i x_i^q - 1 - \gamma k_i d_i(e_S)_i.
\]

Also for two adjacent nodes \( i, j \) that are both in \( S_c \), we have

\[
k(i, j)\ell'(x_j) + k(j, i)\ell'(x_i) = \ell'(x_i) + \ell'(x_j).
\]

This is because \( \ell'(x_i - x_j) + \ell'(x_j - x_i) = 0 \). And for two adjacent nodes \( i, j \) such that \( i \in S_c \) and \( j \notin S_c \), \( x_i > x_j \), \( k(i, j) < 1 \). Define a Lovasz-Simonovits curve \( y \) over \( d_i x_i^q - 1 \), then we have

\[
\sum_{i \in S_c} (K \circ Ax^q - 1)_i + \sum_{i \in S_c} d_i x_i^q - 1
\]

\[
= 2 \sum_{i \in S_c} \sum_{j \sim i, j \notin S_c} w(i, j)x_j^q - 1 + \sum_{i \in S_c} \sum_{j \sim i, j \notin S_c} k(i, j)w(i, j)x_j^q - 1
\]

\[
< 2 \sum_{i \in S_c} \sum_{j \sim i, j \notin S_c} w(i, j)x_j^q - 1 + \sum_{i \in S_c} \sum_{j \sim i, j \notin S_c} w(i, j)x_j^q - 1
\]

\[
\leq y[\text{vol}(S) - \text{cut}(S_c, \tilde{S}_c)] + y[\text{vol}(S) + \text{cut}(S_c, \tilde{S}_c)]
\]

\[
\leq y[\text{vol}(S) - E_0] + y[\text{vol}(S) + E_0]
\]

here the second inequality is due to the definition of Lovasz-Simonovits curve and the third inequality is due to \( y(x) \) is concave. This means

\[
y[\text{vol}(S) - E_0] + y[\text{vol}(S) + E_0] \geq \sum_{i \in S_c} (K \circ Ax^q - 1)_i + \sum_{i \in S_c} d_i x_i^q - 1
\]

\[
\geq (2 + \gamma) \sum_{i \in S_c} d_i x_i^q - 1 - \gamma \sum_{i \in S_c} k_id_i(e_S)_i
\]

\[
\geq (2 + \gamma) \sum_{i \in S_c} d_i x_i^q - 1 - \gamma \sum_{i \in S} k_id_i
\]

\[
= (2 + \gamma) \sum_{i \in S_c} d_i x_i^q - 1 - \gamma \sum_{i \in V} d_i x_i^q - 1
\]

\[
= 2 \sum_{i \in S_c} d_i x_i^q - 1 - \gamma \sum_{i \in S_c} d_i x_i^q - 1
\]

\[
\geq 2y[\text{vol}(S_c)] - O(\phi(T)\text{vol}(S)).
\]

Thus,

\[
y[\text{vol}(S_c)] - y[\text{vol}(S_c - E_0)] \leq y[\text{vol}(S_c + E_0)] - y[\text{vol}(S_c)] + O(\phi(T)\text{vol}(S)).
\]
Similarly to the proof of Lemma 4.1 in [Zhu et al., 2013], we can then derive
\[
\frac{0.5E_0\text{vol}(S)}{4^q-1\text{vol}(T)} \leq y[\text{vol}(S_{1/4})] - y[\text{vol}(S_{1/4}) - E_0]
\]
\[
\leq \frac{\text{vol}(S_{1/8}\backslash S_{1/4})}{E_0} O(\phi(T)\text{vol}(S)) + y[\text{vol}(S_{1/8})] - y[\text{vol}(S_{1/8}) - E_0]
\]
\[
\leq \frac{\text{vol}(S_{1/8}\backslash T) + \text{vol}(T\backslash S_{1/4})}{E_0} O(\phi(T)\text{vol}(S)) + 0.5E_0\text{vol}(S) + \frac{0.5E_0\text{vol}(S)}{8^q-1\text{vol}(T)}.
\]
Hence, \(E_0 \leq O\left(\frac{\phi(T)}{\sqrt{\gamma}}\right)\text{vol}(T)\).

And from lemma 19, we know \(\text{vol}(S_c) = 1 \pm O\left(\frac{\phi(T)}{\gamma}\right)\text{vol}(T)\), since we choose
\[\gamma = (\gamma_2)^{q-1}\] and \(\gamma_2 = \Theta(\phi(T)\cdot \text{Gap}), \text{vol}(S_c) = \Theta(\text{vol}(T))\). So there exists \(R\) such that
\[\phi(R) = O\left(\frac{\phi(T)}{\sqrt{\gamma}}\right) = O\left(\frac{\phi(T)\cdot \text{Gap}}{\text{vol}(T)^{\frac{3-q}{2}}}ight) \leq O\left(\frac{\phi(T)\cdot \text{Gap}}{\text{vol}(T)^{\frac{3-q}{2}}}ight)\).

Here the last inequality uses the fact that \((3 - q)/2 > 1/q) when 1 < q < 2. ■

By combing all these lemmas, we can get the following theorem.

**THEOREM 21** Assume the subgraph induced by target cluster \(T\) has diameter \(O(\log(|T|))\), when we uniformly randomly sample points from \(T\) as seed sets, the expected largest distance of any node in \(S\) to \(T\) is \(O\left(\frac{\log(|T|)}{|S|}\right)\). Assume
\[\frac{\text{vol}^2(S)}{\text{vol}(S)|T|} \leq 2\left((\frac{\gamma_1}{1+\kappa})/|T|\right)^{\frac{1}{2}}\log(1+\gamma_1/(q-1)) + l \leq (1+\gamma)\max(\hat{d}_i),\] then we can set \(\gamma = \gamma_2^{q-1}\) to satisfy assumption 17 for \(1 < q < 2\). Then a sweep cut over \(x\) will find a cluster \(R\) where \(\phi(R) = O\left(\phi(T)\cdot \text{Gap}^{\frac{3-q}{2}}\right)\).

**5 EXPERIMENTS**

We perform three experiments that are designed to compare our method to others designed for similar problems. We call ours SLQ (strongly local q-norm) for \(\ell(x) = (1/q)|x|^q\) with parameters \(\gamma\) for localization and \(\kappa\) for the sparsity. We call it SLQq with the q-Huber loss. Existing solvers are (i) ACL [Andersen et al., 2006], that computes a personalized PageRank vector approximately adapted with the same parameters [Gleich and Mahoney, 2014]; (ii) CRD [Wang et al., 2017], which is hybrid of flow and spectral ideas; (iii) FS is FlowSeed [Veldt et al., 2019a], a 1-norm based method; (iv) HK is the push-based heat kernel [Kloster and Gleich, 2014]; (v) NLD is a recent nonlinear diffusion [Ibrahim and Gleich, 2019]; (vi) GCN is a graph convolutional network [Kipf and Welling, 2016]. Parameters are chosen based on defaults or with slight variations designed to enhance the performance within a reasonable running time. We provide a full Julia implementation of SLQ in Section 5.5. We evaluate the routines in terms of their recovery performance for planted sets and clusters. The bands reflect randomizing seeds choices in the target cluster.

**5.1 CLUSTER RECOVERY IN A SYNTHETIC LFR MODEL**

The first experiment uses the LFR benchmark [Lancichinetti et al., 2008], We vary the mixing parameter \(\mu\) (where larger \(\mu\) is more difficult) and provide 1% of a cluster as a seed, then we check how much of the cluster we recover after a conductance-based sweep cut over the solutions from various methods. Here, we use the F1 score (harmonic mean of precision and recall) and conductance value (cut to volume ratio) of the sets to evaluate the methods. The results are in Figure 4.
Reproduction details. When creating the LFR graphs, we set the power law exponent for the degree distribution to be 2, power law exponent for the community size distribution to be 2, desired average degree to be 10, maximum degree to be 50, minimum size of community to be 200 and maximum size of community to be 500. We create 40 random graphs for each \( \mu \). For SLQ, we set \( \delta = 0, \gamma = 0.1, \rho = 0.5 \) and \( \epsilon = 10^{-8} \). For ACL, we set \( \gamma = 0.1 \). For both SLQ and ACL, \( \kappa \) is automatically chosen from 0.005 and 0.002 based on which will give a cluster with smaller conductance. For HK, we use four different pairs of \( (\epsilon, t) \), which are \((0.0001, 10), (0.001, 20), (0.005, 40)\) and \((0.01, 80)\). And we return the one with the smallest conductance. For CRD, we use default parameters from "localgraphclustering" Python package except \( h \), which is the maximum flow that each edge can handle. We provide results of using \( h = 3 \) and \( h = 5 \). For methods that are using multiple choices of parameters, we report the total running time.

5.2 CLUSTER RECOVERY IN FACEBOOK SCHOOL NETWORKS

The second experiment uses the class-year metadata on Facebook [Traud et al., 2012], which is known to have good conductance structure for at least class year 2009 [Veldt et al., 2019b] that should be identifiable with many methods. Other class years are harder to detect with conductance. Here, we use \( F_1 \) values alone. We use 1\% of the true set as seed. (For GCN, we also use the same number of negative nodes.) (In Section 5.4 we vary the number of seeds.) The results are in Table 1,2 and show SLQ is as good, or better than, CRD and much faster.

Reproduction details. In this experiment, for SLQ, we set \( q = 1.2, \gamma = 0.05, \kappa = 0.005, \epsilon = 10^{-8}, \rho = 0.5 \) and \( \delta = 0 \). For SLQ\( \delta \), the parameters are the same as SLQ except we set \( \delta = 10^{-5} \). For ACL, we set \( \gamma = 0.05 \) and \( \kappa = 0.005 \). For CRD and HK, we use the same parameters as the first experiment. For FS, we set the locality parameter to be 0.5. For NLD, we set the power to be 1.5, step size to be 0.002 and the number of iterations to be 5000. For GCN, we use 5 hidden layers and negative log likelihood loss. We set dropout ratio to be 0.5, learning rate to be 0.01, weight decay to be 0.0005 and the number of iterations to be 200. The feature vector is the 6 different metadata info as described in [Traud et al., 2012]. For each true set, we randomly choose 1\% of the true set as seed 50 times.

5.3 RECALL DURING A SWEEP

The final experiment evaluates a finding from [Kloumann and Kleinberg, 2014] on the recall of seed-based community detection methods. For a group of communities with roughly the same size, we evaluate the recall of the largest \( k \) entries in a diffusion vector. They found PageRank (ACL) outperformed many different methods. In Figure 6, we see the same general result and found that SLQ with \( q > 2 \) gives the best performance.
### TABLE 1 – Cluster recovery results from a set of 7 Facebook networks [Traud et al., 2012]. Students with a specific graduation class year are used as target cluster. We use a random set of 1% of the nodes identified with that class year as seeds. The class year 2009 is the set of incoming students, which form better conductance groups because the students had not yet mixed with the other classes. Class year 2008 is already mixed and so the methods do not do as well there. The values are median F\(_1\) and the violin plots show the distribution over choices of the seeds.

| Year | Alg | UCLA F1 & Med. | MIT F1 & Med. | Duke F1 & Med. | UPenn F1 & Med. | Yale F1 & Med. | Cornell F1 & Med. | Stanford F1 & Med. |
|------|-----|----------------|---------------|----------------|----------------|----------------|-------------------|-------------------|
| 2009 | SLQ | 0.9            | 0.9           | 1.0            | 1.0            | 1.0            | 0.9               | 0.9               |
|      | SLQ\(\delta\) | 0.9           | 0.8           | 1.0            | 0.9            | 0.9            | 0.9               | 0.9               |
|      | CRD-3 | 0.3           | 0.7           | 0.7            | 0.6            | 0.7            | 0.5               | 0.5               |
|      | CRD-5 | 0.9           | 0.9           | 1.0            | 1.0            | 1.0            | 0.9               | 0.9               |
|      | ACL  | 0.9           | 0.8           | 0.9            | 0.9            | 0.9            | 0.9               | 0.9               |
|      | FS   | 0.4           | 0.4           | 0.9            | 0.9            | 0.5            | 0.5               | 0.4               |
|      | HK   | 0.2           | 0.2           | 0.3            | 0.3            | 0.3            | 0.3               | 0.3               |
|      | NLD  | 0.3           | 0.3           | 0.3            | 0.3            | 0.3            | 0.3               | 0.3               |
|      | GCN  | 0.3           | 0.3           | 0.3            | 0.3            | 0.3            | 0.3               | 0.3               |

| Year | Alg | UCLA F1 & Med. | MIT F1 & Med. | Duke F1 & Med. | UPenn F1 & Med. | Yale F1 & Med. | Cornell F1 & Med. | Stanford F1 & Med. |
|------|-----|----------------|---------------|----------------|----------------|----------------|-------------------|-------------------|
| 2008 | SLQ | 0.7            | 0.5           | 0.8            | 0.8            | 0.8            | 0.8               | 0.8               |
|      | SLQ\(\delta\) | 0.6           | 0.5           | 0.7            | 0.7            | 0.7            | 0.7               | 0.7               |
|      | CRD-3 | 0.6           | 0.5           | 0.7            | 0.7            | 0.7            | 0.6               | 0.6               |
|      | CRD-5 | 0.5           | 0.5           | 0.7            | 0.5            | 0.7            | 0.7               | 0.7               |
|      | ACL  | 0.5           | 0.5           | 0.7            | 0.7            | 0.7            | 0.7               | 0.7               |
|      | FS   | 0.5           | 0.5           | 0.5            | 0.7            | 0.7            | 0.7               | 0.7               |
|      | HK   | 0.5           | 0.5           | 0.5            | 0.5            | 0.5            | 0.5               | 0.5               |
|      | NLD  | 0.3           | 0.3           | 0.3            | 0.3            | 0.3            | 0.3               | 0.2               |
|      | GCN  | 0.3           | 0.3           | 0.3            | 0.3            | 0.3            | 0.3               | 0.3               |

| Method | SLQ | SLQ\(\delta\) | CRD-3 | CRD-5 | ACL | FS | HK | NLD | GCN |
|--------|-----|----------------|-------|-------|-----|----|----|-----|-----|
| Time   | 123 | 80             | 3049  | 9378  | 12  | 1593| 106| 10375| 16534|

(seconds)

### 5.4 VARYING SEED SIZE

Finally, we would like to describe an experiment where we study the performance change of different methods when varying the size of the seed set. The dataset we use is the same MIT Facebook dataset and the target cluster is class year 2008. This choice is one where most of the methods in Table 1 did poorly, but ACL did better in some trials. We repeat 50 times for each seed size level. From the previous experiments, we can see that none of the methods works well finding this cluster. In this experiment, we only report results from SLQ, ACL, FS, CRD-3 and HK as they are all strongly local methods and they perform better than global methods as we have seen from previous experiments. Also, we didn’t add CRD-5 because CRD-3 performed better than CRD-5 on this particular cluster as shown in Table 1. The result of this experiment is in Figure 5. When seed size is smaller than 15 nodes, the F1 score of all methods improves as we increase seed size. After 15 nodes, only the F1 score of SLQ and ACL continues to improve when seed size becomes larger, while the performance of other methods stays the same or even slightly worse.

**Reproduction details** For HK and CRD-3, we use the same parameters as the previous Facebook experiment. For ACL and SLQ, we use a coarse binary search (initial region is between 0.001 and 0.1, smallest feasible region is 0.001) to find a good sparsity level such that the total number of nonzero entries is 20% of the total number of nodes. The other parameters are the same as the previous Facebook experiment. We also use a similar coarse binary search (initial region is between 0.4 and 5.0, smallest feasible region is 0.1) to choose \(\epsilon\) for FS. We didn’t implement this procedure for CRD and HK because CRD doesn’t have a standalone parameter to control the sparsity of the solution and HK has already been set up to choose the best cluster from a list of parameters. One thing we would like to mention is that in Table 1, we use 1% nodes of the true cluster as

![FIGURE 5 – This figure shows the performance change (F1 score) of different methods when we vary the size of seed set. The dataset is MIT Facebook with the true cluster to be class year 2008. The envelope represents 20%-80% quantile.](image-url)
seeds which is roughly 32 nodes in this case. So we can see that the performance of both ACL and SLQ is improved upon this extra layer of binary search (i.e. the median F1 score is increased to 0.6). While the performance of FS remains the same.

5.5 OUR FULL JULIA IMPLEMENTATION

Our full implementation is available in the SLQ.jl function on github: github.com/MengLiuPurdue/SLQ and the experiment codes are available too. We verified this Julia implementation of ACL is as efficient as ACL implemented in C++. So there is no appreciable overhead of using Julia compared with C or C++ for this computation.

First we want to mention that in our experiments, we find that we can speed up SLQ by using a slightly modified binary search procedure. The logic is when $q$ is close to 1 and $\text{vol}(S)$ is small, $\Delta x_i$ after each step of “push” procedure is also small. So it doesn’t make sense to set the initial range of binary search to be $[0, 1]$. Instead, we set the initial range to be $[10^{k-1}t, 10^kt]$, where $t$ is chosen from either last $\Delta x_i$ or $(\text{vol}(S)/\text{vol}(A))^{1/(q-1)}$. (Note this is just the lower bound of $x_i$ when $\gamma \to 0$.) Since we can check which side of the bounds we are on, we then determine a value of $k$ by checking $k = 1, 2, ...$ until the residual becomes negative. This strategy is implemented in our code.

6 RELATED WORK AND DISCUSSION

The most strongly related work was posted to arXiv [Yang et al., 2020] contemporaneously as we were finalizing our results. This research applies a $p$-norm function to the flow dual of the mincut problem with a similar motivation. This bears a resemblance to our procedures, but does differ in that we include the localizing set $S$ in our nonlinear penalty. Also, our solver uses the cut values instead of the flow dual on the edges and we include details that enable Huber and Berhu functions for faster computation. In the future, we plan to compare the approaches more concretely.

There also remain ample opportunities to further optimize our procedures. As we were developing these ideas, we drew inspiration from algorithms for $p$-norm regression [Adil et al., 2019]. Also there are faster converging (in theory) solvers using different optimization procedures [Fountoulakis et al., 2017] for 2-norm problems as well as parallelization strategies [Shun et al., 2016].

Our work further contributes to the ongoing research into $p$-Laplacian research [Amghibech, 2003; Bühler and Hein, 2009; Alamgir and Luxburg, 2011; Brindle and Zhu, 2013; Li and Milenkovic, 2018] by giving a related problem that can be solved in a strongly local fashion. We note that our ideas can be easily adapted to the growing space of hypergraph and higher-order graph analysis literature [Benson et al., 2016; Yin et al., 2017; Li and Milenkovic, 2018] where the strategy is to derive a useful hypergraph from graph data to support deeper analysis. We are also excited by the opportunities to combine with generalized
Laplacian perspectives on diffusions [Ghosh et al., 2014]. Moreover, our work contributes to the general idea of using simple nonlinearities on existing successful methods. A recent report shows that a simple nonlinearity on a Laplacian pseudoinverse is competitive with complex embedding procedures [Chanpuriya and Musco, 2020].

Finally, we note that there are more general constructions possible. For instance, differential penalties for $S$ and $\bar{S}$ in the localized cut graph can be used for a variety of effects [Orecchia and Zhu, 2014; Veldt et al., 2019b]. For 1-norm objectives, optimal parameters for $\gamma$ and $\kappa$ can also be chosen to model desirable clusters [Veldt et al., 2019b] – similar ideas may be possible for these $p$-norm generalizations. We view the structured flexibility of these ideas as a key advantage because ideas are easy to compose. This contributed to using personalized PageRank to make graph convolution networks faster [Klicpera et al., 2019].

In conclusion, given the strong similarities to the popular ACL – and the improved performance in practice – we are excited about the possibilities for localized $p$-norm-cuts in graph-based learning.

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