Growth models on the Bethe lattice

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Abstract – I report on an extensive numerical investigation of various discrete growth models
describing equilibrium and nonequilibrium interfaces on a substrate of a finite Bethe lattice. An
unusual logarithmic scaling behavior is observed for the nonequilibrium models describing the
scaling structure of the infinite-dimensional limit of the models in the Kardar-Parisi-Zhang (KPZ)
class. This gives rise to the classification of different growing processes on the Bethe lattice in terms
of logarithmic scaling exponents which depend on both the model and the coordination number of
the underlying lattice. The equilibrium growth model also exhibits a logarithmic temporal scaling
but with an ordinary power law scaling behavior with respect to the appropriately defined lattice
size. The results may imply that no finite upper critical dimension exists for the KPZ equation.

The Kardar-Parisi-Zhang (KPZ) equation [1] is a simple nonlinear Langevin equation that describes the macro-
sscopic properties of a wide variety of nonequilibrium growth processes [2,3]. This equation is also related
to many other important physical problems such as the
Burgers equation [4], dissipative transport in the driven-
diffusion equation [5] and directed polymers in a random
medium [6-8]. The KPZ equation for a stochastically
growing interface described by a single-valued height function $h(x,t)$ on a $d$-dimensional substrate $x$, is

$$\partial_t h(x,t) = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x,t),$$

(1)

where the first term represents relaxation of the interface caused by a surface tension $\nu$, the second describes
the nonlinear growth locally normal to the surface, and the last is an uncorrelated Gaussian white noise in both
space and time with zero average $\langle \eta(x,t) \rangle = 0$ and $\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t')$, mimicking
the stochastic nature of the growth process. The steady-state interface profile is usually described in terms of the rough-
ness: $w = \langle h^2(x,t) \rangle - \langle h(x,t) \rangle^2$ which for a system of
size $L$ behaves like $L^{\alpha} f(t/L^{\beta/\alpha})$, where $f(x) \rightarrow \text{const}$ as $x \rightarrow \infty$ and $f(x) \sim x^\beta$ as $x \rightarrow 0$, so that $w$ grows with time like $t^\beta$ until it saturates to $L^\alpha$ when $t \sim L^{\alpha/\beta}$. $\alpha$ and $\beta$
are the roughness and the growth exponents, respectively, whose exact values are known only for the special case
$d = 1$ as $\alpha = 1/2$ and $\beta = 1/3$. The ratio $\bar{z} = \alpha/\beta$ is
called dynamic exponent. A scaling relation $\alpha + \bar{z} = 2$
follows from the invariance of eq. (1) to an infinitesimal
tilting of the surface which retains only one independent
exponent, say $\alpha$, in the KPZ dynamics.

It is well known that for dimensions $d \leq 2$ the surface is always rough, while for $d > 2$, eq. (1) shows two dif-
f erent regimes in terms of the dimensionless strength of
the nonlinearity coefficient whose critical value $\lambda_c$ separ-
ates flat and rough surface phases. In the weak-coupling
(flat) regime ($\lambda < \lambda_c$) the nonlinear term is irrelevant
and the behavior is governed by the $\lambda = 0$ fixed point,
i.e., the linear Edward-Wilkinson (EW) equation [9], for
which the exponents are known exactly: $\alpha = (2-d)/2$
and $\beta = (2-d)/4$. In the more challenging strong-coupling
(rough) regime ($\lambda > \lambda_c$), where the nonlinear term is rel-
vant, the behavior of the KPZ equation is quite contro-
versial and characterized by anomalous exponents. There
is, however, a long-standing controversy concerning the existence and the value of an upper critical dimension $d_c$
above which, regardless of the strength of the nonlinearity,
the surface remains flat.

At odds with many theoretical discussions supporting
the existence of a finite upper critical dimension [10]
between three and four [11,12], and an analytical evidence

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that \( d_c \) is bounded from above by four \([13]\), or many others suggesting \( d_c \approx 2.5 \) \([14]\) or \( d_c = 4 \) \([15,16]\), there is nevertheless a long list of evidences questioning these suggestions \([17–23]\), some of which concluded that no finite upper critical dimension exists at all (for the most recent study, see \([23]\)).

Here I study the infinite-dimensional properties of growth models from two different KPZ and EW classes and compare them to realize whether the nonlinear term in (1) is relevant in this limit. It is inspired by the fact that if the nonlinear term is irrelevant in infinite dimensions, then one would expect the same statistical behavior for the models coming from each of the two classes. The result would shed a light on the existence of the upper critical dimension for the KPZ equation. To this aim, I investigate two discrete nonequilibrium models, ballistic deposition (BD) and restricted solid-on-solid (RSOS) models which are believed to be in the KPZ class \([2,24–28]\), as well as an equilibrium model, random deposition with surface relaxation (RDSR) \([29]\), which belongs to the EW class, all defined on the Bethe lattice, an effectively infinite-dimensional lattice. I find that the models from different universality classes correspond to different statistical growth properties and scaling behavior, the evidence that questions the existence of a finite upper critical dimension for the KPZ equation.

Due to its distinctive topological structure, several statistical models involving interactions defined on the Bethe lattice \([30]\) are exactly solvable and computationally inexpensive \([31]\). Various systems including magnetic models \([30]\), percolation \([32–35]\), nonlinear conduction \([36]\), localization \([32,37]\), random aggregates \([38,39]\) and diffusion processes \([40–42]\) have been studied on the Bethe lattice whose analytic results gave important physical insights to subsequent developments of the corresponding research fields. The Bethe lattice is defined as a graph of infinite points each connected to \( z \) neighbors (the coordination number) such that no closed loops exist in the geometry (see fig. 1). A finite type of the graph with boundary is also known as a Cayley tree and possesses the features of both one and infinite dimensions: since \( N_k \), the total number of sites in a Bethe lattice with \( k \) shells, is given as \( N_k = [z(z-1)^k - 2]/(z-2) \), the lattice dimension defined by \( d = \lim_{k \to \infty} \ln N_k / \ln k \) is infinite. It is therefore often mentioned in the literature that the Bethe lattice describes the infinite-dimensional limit of a hypercubic lattice. As the lattice grows the number of sites in the surface, or the last shell, grows exponentially \( z(z-1)^{k-1} \). Therefore, as the number of shells tends to infinity, the proportion of surface sites tends to \( (z-2)/(z-1) \). By surface boundary we mean the set of sites of coordination number unity, the interior sites all have a coordination number \( z \). Thus, the vertices of a Bethe lattice can be grouped into shells as functions of the distances \( k \) from the central vertex. Here \( k \) is the number of bonds of a path between the shell and the central site and will be used as a measure of lattice size.

Fig. 1: Part of a Bethe lattice with coordination number \( z = 3 \) embedded in the plane which is considered here as a substrate of different growth models. The vertically incident particles can land at the top of the lattice sites represented by open small circles at different shells \( k = 0, 1, 2, \ldots \). For a given finite lattice of fixed size \( k \), one lattice site is randomly chosen at each step and a particle is added to that site which can either increase the height according to the standard rules of BD and RSOS models, or it can diffuse through the neighboring edges until it finds the column with a local minima in the searched area according to the RDSR model.

I have carried out extensive simulations of the BD, RSOS and RDSR models on a finite Bethe lattice of different size \( k \) and different coordination number \( z \) (fig. 2). I will first compute the surface width \( w(t, k) \) as a function of time \( t \) and examine its various scaling properties. For a given lattice size \( k \), each Monte Carlo time step is defined as the time required for \( N_k \) particles to deposit on the surface. I show that the surface widths for the models feature a normal behavior as for a typical growth model on a regular lattice: \( w \) increases fast and finally saturates to a fixed value \( w_s \). Nevertheless, the best fit to our data at early time before saturation shows that \( w \) does not increase algebraically with time \( (w \sim t^\beta) \), as is usually observed for growth models on ordinary lattices. Rather I find a logarithmic scaling behavior \( w \sim \ln(t)^\beta \) (see footnote1) for all considered models. I also find that the saturated width \( w_s \) of the interface for two BD and RSOS models behaves like a logarithmic scaling law \( w_s \sim \ln(k)^\alpha \), while for the equilibrium RDSR model, it shows an ordinary power law behavior with the lattice size, \( w_s \sim k^\alpha \). The model-dependent exponents are also found to be functions of the coordination number

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1For the sake of simplicity, I use the same symbols for the exponents \( \alpha \) and \( \beta \), in the logarithmic scaling laws.
of the underlying lattice, $\alpha^i(z)$ and $\beta^i(z)$, where $i = *, *$ and $\circ$ denotes for BD, RSOS and RDSR models, respectively. Let me call $\alpha^i(z)$ and $\beta^i(z)$ roughness and growth exponents, respectively. This different scaling form with respect to the finite-dimensional case can be associated to the exponential (instead of polynomial) growth of the volume of a shell as a function of its radius on the Bethe lattice.

I first consider the BD model on a lattice with $z = 3$. Figure 2 shows the surface width $w(t, k)$ as a function of logarithm of time, for the seventeen different sizes, from the 4th to the 20th generation. At early times before saturation, the data falls onto a straight line in a log-log scale indicating that the surface width initially increases algebraically with the logarithm of time as $w \sim \ln(t)^{\beta^*}$, with $\beta^*(z = 3) \approx 0.75(2)$. The best fit to the saturated width $w_s$ as a function of different lattice size, gives a scaling relation $w_s \sim \ln(k)^{\alpha^*}$, with $\alpha^*(z = 3) \approx 0.825(10)$. As shown in the inset of fig. 2, by standard rescaling of the parameteres, all curves collapse onto a single function.

In order to see how these exponents depend on $z$, series of extensive simulations were performed for the BD model on a Bethe lattice with different coordination number $z = 3, 4, 5, 6$ and 7. For each $z$, the surface width $w(t, k)$ was measured for different lattice size $k$. Figure 3 illustrates the saturated surface widths as functions of the lattice size for each coordination number. The solid lines in the figure show the best logarithmic fits of form $w_s \sim \ln(k)^{\alpha^*(z)}$ to the data, assigning a $z$-dependent roughness exponent $\alpha^*$ to each data set. I also find that the growth exponent $\beta^*$ is dependent on the coordination number of the substrate lattice. $\alpha^*(z)$ and $\beta^*(z)$ are plotted in the inset of fig. 3. As can be seen, $\alpha^*$ decreases, while $\beta^*$ increases almost linearly with $z$.

To see whether such a logarithmic scaling behavior is a characteristic feature of the nonequilibrium growth models on the Bethe lattice, I have also measured the surface width for the RSOS model, for the seventeen different...
16th generation. Inset: saturated surface widths as functions of time, for the thirteen different sizes, from the $k = 4$th to the 16th generation. Inset: saturated surface widths as functions of lattice size $k$, for two different coordination numbers $z = 3$ and 4. Unlike the nonequilibrium BD and RSOS models, $w_s$ here shows a power law scaling behavior $w_s \sim k^{\alpha(z)}$, with the lattice size $k$.

Fig. 5: (Color online) Main panel: surface width $w(t, k)$ for the equilibrium RDSR model on a finite Bethe lattice of coordination number $z = 3$, as a function of the logarithm of time, for the thirteen different sizes, from the $k = 4$th to the 16th generation. Inset: saturated surface widths as functions of lattice size $k$, for two different coordination numbers $z = 3$ and 4. Unlike the nonequilibrium BD and RSOS models, $w_s$ here shows a power law scaling behavior $w_s \sim k^{\alpha(z)}$, with the lattice size $k$.

sizes, from the $k = 4$th to the 20th generation, and for $z = 3$. The results are shown in fig. 4. These suggest the same scaling behavior but with different estimated roughness and growth exponents $\alpha^z(z = 3) \simeq 0.90(1)$ and $\beta^z(z = 3) \simeq 0.57(2)$, respectively. The exponents for this model depend again on the coordination number but both are decreasing with $z$. For $z = 4$, the exponents are estimated as $\alpha^z(z = 4) \simeq 0.68(1)$ and $\beta^z(z = 4) \simeq 0.47(2)$ (see inset of fig. 4).

In random deposition with surface relaxation (RDSR) [29,43], each particle is randomly dropped onto the surface, and it is allowed to diffuse around on the surface within a prescribed region about the deposited column, until it finds the column with a local minima in the searched area. The corresponding continuum model is the EW equation [9].

Figure 5 summarizes the results obtained from implementing the RDSR model on a finite Bethe lattice of different size from the $k = 4$th to the 16th generation with $z = 3$. The first remarkable observation is that the crossover time to the steady state is quite higher than that needed for the above-discussed nonequilibrium models, growing exponentially with $k$ in this case. Therefore, the CPU time required for simulations to reach the desired accuracy is orders of magnitude higher. As shown in fig. 5, the temporal logarithmic scaling behavior for this model again holds. I find the scaling relation $w \sim \ln(t)^{\beta^z(z)}$, with $\beta^z(z = 3) \simeq 0.51(2)$.

The most remarkable scaling feature observed in the RDSR model is that, unlike the two BD and RSOS models, the average saturated width $w_s$ has no longer a power law relation with the logarithm of the size, but with the size $k$ itself. The resulting data is plotted in the inset of fig. 5. I find that $w_s \sim k^{\alpha^z(z)}$, with $\alpha^z(z = 3) \simeq 0.60(1)$. The roughness and growth exponents again depend on the coordination number. Simulations for the RDSR model on a finite Bethe lattice of different size from $k = 3$th to the 14th generation for $z = 4$, provide a satisfactory estimation of the exponents: $\alpha^z(z = 4) \simeq 0.562(10)$ (see inset of fig. 5) and $\beta^z(z = 4) \simeq 0.46(2)$.

To summarize, I have studied three different growth models on a substrate of a finite Bethe lattice with different coordination number. A different scaling behavior is seen with respect to the same models on the ordinary lattice or those on the fractal substrates [44]. Two models, i.e., the BD and RSOS models, are chosen from the nonequilibrium growth processes in the KPZ universality class, and the third, i.e., the RDSR model, is an equilibrium model from the EW class. For all considered models, the surface width grows with a power law scaling relation with the logarithm of the size before saturation. The initial growth is characterized by an exponent which depends on the coordination number of the underlying Bethe lattice as well as on the model in question. In the steady-state regime, the scaling behavior distinguishes between equilibrium and nonequilibrium models. The average saturated width for the nonequilibrium models has a power law scaling relationship with the logarithm of the lattice size, while for the equilibrium RDSR model, it shows a usual power law behavior with the lattice size (instead of the logarithm of the size).

If we admit that the Bethe lattice, as a substrate of a growth model, reflects the infinite-dimensional limit properties of the models, the present results would then imply that the nonlinear term in the KPZ equation has a relevant contribution at this limit, consequently questioning the existence of a finite upper critical dimension for the KPZ equation.

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