Limit Laws for Extremes of Dependent Stationary Gaussian Arrays

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Abstract: In this paper we show that the componentwise maxima of weakly dependent bivariate stationary Gaussian triangular arrays converge in distribution after normalisation to Hüsler-Reiss distribution. Under a strong dependence assumption, we prove that the limit distribution of the maxima is a mixture of a bivariate Gaussian distribution and Hüsler-Reiss distribution. Another finding of our paper is that the componentwise maxima and componentwise minima remain asymptotically independent even in the settings of Hüsler and Reiss (1989) allowing further for weak dependence. Further we derive an almost sure limit theorem under the Berman condition for the components of the triangular array.

Key Words: Hüsler-Reiss distribution; Brown-Resnick copula; Gumbel Max-domain of attraction; Berman condition; almost sure limit theorem; weak convergence.

1 Introduction and Main Result

An important multivariate distribution of extreme value theory is the so-called Hüsler-Reiss distribution, which in a bivariate setting is given by

$$H_\lambda(x, y) = \exp \left( -\Phi \left( \frac{\lambda + x - y}{2\lambda} \right) \exp(-y) \right) - \Phi \left( \frac{\lambda + y - x}{2\lambda} \right) \exp(-x), \quad x, y \in \mathbb{R},$$

with $\Phi(\cdot)$ the univariate standard Gaussian distribution and $\lambda \in [0, \infty]$ a parameter. When $\lambda = 0$ we have in fact for any $x, y \in \mathbb{R}$ that $H_0(x, y) = \min(\Lambda(x), \Lambda(y))$ and in the other extreme case $\lambda = \infty$ the distribution function $H_\infty$ is a product distribution with Gumbel marginals $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$. It is clear that for any $\lambda \in [0, \infty]$ the marginals of $H_\lambda$ are Gumbel distributions.

A striking property of $H_\lambda$ is that it is a max-stable distribution function (see Resnick (1987) for definition and main properties), and moreover it is a natural model for extremes of Gaussian triangular arrays, as shown first in Hüsler and Reiss (1989), see also Falk et al. (2010). This fact is very important for statistical applications concerned with models for extremes of dependent risks. The parameter $\lambda$ has a nice representation and comes naturally in the setup of Gaussian triangular arrays. Roughly speaking, if $\rho(n) \in (-1, 1)$ is the correlation coefficient of a bivariate triangular array, then under the Hüsler-Reiss condition

$$\lim_{n \to \infty} (1 - \rho(n)) \ln n = \lambda^2 \in [0, \infty]$$

the distribution function $H_\lambda$ appears as the limiting distribution of the normalized maxima.

In their seminal paper Hüsler and Reiss provided the full multivariate result where the condition (1.2) is assumed for any bivariate pair of a multivariate Gaussian array.

Recently, the research interest on Hüsler-Reiss distribution has grown significantly mainly due to the fact that not only Gaussian, but chi-square, elliptical triangular arrays, and some more general models have componentwise maxima attracted by that distribution (see Hashorva (2005), Frick and Reiss (2010), Hashorva (2008,2013), Hashorva et al. (2012)). Additionally, new interesting applications have been proposed in Manjunath et al. (2012), and further crucial links with Brown-Resnick processes have been recently discovered, see the seminal contribution Brown and Resnick (1977) for the first paper where the bivariate Hüsler-Reiss distribution appears,
and see the recent deep contributions Kabluchko et al. (2009), Kabluchko (2011), Oesting et al. (2012), Engelke et al. (2014) for the aforementioned links. We mention in passing that an important contribution somewhat related to the topic of our paper, but not to our techniques and proofs is O’Brien (1987).

So far, primarily due to technical difficulties, the available results in the literature are concerned with the asymptotic behaviour of maxima of triangular arrays where the limit distribution is Hüsler-Reiss distribution, under the assumption that the components of triangular arrays are independent.

This contribution is the first attempt to allow for dependence among the components of triangular array, remaining in a Gaussian framework. Specifically, we deal with \{X_{n,k} = (X_{n,k}^{(1)}, X_{n,k}^{(2)}) \mid 1 \leq k \leq n, n \geq 1\} a triangular array of bivariate Gaussian random vectors with zero-mean, unit-variance and correlation given by

\[
\text{corr} \left( X_{n,k}^{(1)}, X_{n,k}^{(2)} \right) = \rho_0(n), \quad \text{corr} \left( X_{n,k}^{(i)}, X_{n,l}^{(j)} \right) = \rho_{ij}(|k-l|, n),
\]

where \(1 \leq k \neq l \leq n\) and \(i, j \in \{1, 2\}\).

For notational simplicity, in the sequel define the partial maxima

\[
\left( M_{n,m}^{(1)}, M_{n,m}^{(2)} \right) := \left( \max_{m+1 \leq k \leq n} X_{n,k}^{(1)}, \max_{m+1 \leq k \leq n} X_{n,k}^{(2)} \right), \quad \left( M_n^{(1)}, M_n^{(2)} \right) := \left( M_{n,0}^{(1)}, M_{n,0}^{(2)} \right).
\]

Motivated by Berman condition (see Berman (1964), or Berman (1992)) we consider the following weak dependence condition adapted for the triangular array setup of our paper:

**Assumption A1:** Suppose that \(\sigma := \max_{1 \leq k < n, n \geq 2} |\rho_{ij}(k, n)| < 1\). Let \(\alpha \in (0, \frac{1}{1+\sigma})\) and \(I_n := [n^\alpha]\), and further assume that

\[
\lim_{n \to \infty} \max_{\substack{1 \leq n \leq k \leq n \geq 2 \\alpha \in \{1, 2\}}} |\rho_{ij}(k, n)| \ln n = 0. \tag{1.3}
\]

Our first result below shows that the Berman condition does not change the limit distribution of the componentwise maxima, thus its asymptotic behaviour is the same as in the iid setup of Hüsler and Reiss (1989).

**Theorem 1.1.** Let \(\{X_{n,k} \mid 1 \leq k \leq n, n \geq 1\}\) be a bivariate Gaussian triangular array satisfying Assumption A1. If \(\rho_0(n)\) holds for \(\rho_0(n)\) with \(\lambda \in [0, \infty)\), then we have

\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| P \left( M_n^{(1)} \leq u_n(x), M_n^{(2)} \leq u_n(y) \right) - H_\lambda(x, y) \right| = 0, \tag{1.4}
\]

where \(u_n(s) = a_n s + b_n, s \in \mathbb{R}\) with normalized constants \(a_n\) and \(b_n\) given by

\[
a_n = \frac{1}{\sqrt{2 \ln n}} \quad \text{and} \quad b_n = \sqrt{2 \ln n} - \frac{1}{2\sqrt{2 \ln n}} \ln(4\pi \ln n). \tag{1.5}
\]

A natural relaxation of the weak dependence assumption is to allow the limit in (1.3) to be positive which we formulate below as our second main assumption, namely:

**Assumption A2:** Let \(\tau_{ij} \in (0, \infty)\) be constants for \(i, j \in \{1, 2\}\). Suppose that \(\delta_{ij} := \max_{1 \leq k < n, n \geq 2} |\rho_{ij}(k, n)| < 1\), set \(\omega_{ij} \in (0, \frac{1 - \delta_{ij}}{1 + \delta_{ij}})\) and \(K_{n,ij} := [n^{\omega_{ij}}]\), and further assume that

\[
\lim_{n \to \infty} \max_{K_{n,ij} \leq k < n} |\rho_{ij}(k, n)| \ln k - \tau_{ij} = 0. \tag{1.6}
\]

As shown in our second result below, in the case of strong dependence, the limiting distribution of the joint maxima is given by a Gaussian distribution mixture of the Hüsler-Reiss distribution. We set next \(\tilde{\tau} := \tau_{12} - \frac{1}{2}(\tau_{11} + \tau_{22})\) and \(\tilde{\lambda} := \sqrt{\lambda^2 + \tilde{\tau}}\).

**Theorem 1.2.** Under the assumptions of Theorem 1.1, if further Assumption A2 holds, and assume that \(\tau_{12} \leq \sqrt{\tau_{11} + \tau_{22}}\) and \(\lambda^2 \geq -\tilde{\tau}\), then we have

\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| P \left( M_n^{(1)} \leq u_n(x), M_n^{(2)} \leq u_n(y) \right) - E \left\{ H_\lambda \left( x + \tau_{11} - \sqrt{2 \tau_{11}} Z, y + \tau_{22} - \sqrt{2 \tau_{22}} W \right) \right\} \right| = 0, \tag{1.7}
\]

with \((Z, W)\) a standard bivariate Gaussian random vector with correlation \(\tau_{12}/\sqrt{\tau_{11} + \tau_{22}}\).
The rest of the paper is organized as follows: In Section 2 we discuss the novelty and the importance of our results as well as connections with available contributions in the literature. Further we provide two important extensions: first we show that the componentwise minima is asymptotically independent of the componentwise maxima. This result is well-known for the case of iid bivariate Gaussian sequences. This article is the first to consider the asymptotic behavior of maxima and minima of stationary bivariate Gaussian arrays under the Hüsler and Reiss condition and weak dependent condition. Our second result in Section 2 derives the almost sure limit theorem for the case of weak dependence. Proofs and further results are relegated to Section 3.

2 Discussion and Extensions

As mentioned in the Introduction, all the contributions so far have considered only independent triangular arrays. Our motivation to allow dependence comes from practical situations, where due to the presence of some random inflation/deflation or measurement errors (which are always present) the independence of the components of triangular arrays is not an adequate assumption. This is the first contribution in this direction; we have treated the classical Gaussian setup since the dependence in more general models is very difficult to deal with. Both findings displayed above are of interest: in case of weakly dependent bivariate stationary Gaussian triangular arrays the limiting distribution is Hüsler-Reiss distribution, which is identical to the iid case. Our second result shows that this is no longer the case for strongly dependent stationary Gaussian arrays.

An immediate consequence of Theorem 1.2 is that the univariate maxima also converges after appropriate normalization, i.e.,

$$\lim_{n \to \infty} P \left( M_{n}^{(1)} \leq u_{n}(x) \right) = \mathbb{E} \left\{ \Lambda(x + \tau_{11} - \sqrt{2 \tau_{11}} Z) \right\}, \quad x \in \mathbb{R},$$

which is already derived in Corollary 6.5.2 in Leadbetter et al. (1983).

It can be easily seen that our results hold for multivariate setup and not only for the bivariate setup; we refrain ourself to the bivariate setup for ease of presentation.

There are different (interesting) possibilities to continue the investigation under dependence. For instance, as in Hashorva (2011), one direction is to investigate if the convergence in (1.4) can be stated in a stronger form as convergence of corresponding density functions.

Our first result below concerns the joint asymptotic behaviour of the sample maxima and sample minima. For the iid setup, it is well-known that for multivariate Gaussian random sequences, the sample maxima and sample minima are asymptotically independent (see Davis (1979)). In the framework of triangular arrays suggested by Hüsler and Reiss (1989) no investigation in this direction has been done. Our first result below shows that the asymptotic independence is preserved even in the case of weakly dependent Gaussian arrays.

**Theorem 2.1.** Under the assumptions of Theorem 1.1 we have

$$\lim_{n \to \infty} \sup_{x_{1},x_{2},y_{1},y_{2} \in \mathbb{R}} P \left( -u_{n}(y_{1}) < m_{n}^{(1)} \leq M_{n}^{(1)} \leq u_{n}(x_{1}), -u_{n}(y_{2}) < m_{n}^{(2)} \leq M_{n}^{(2)} \leq u_{n}(x_{2}) \right) - H_{\lambda}(x_{1}, x_{2})H_{\lambda}(y_{1}, y_{2}) = 0,$$

with \( \left( m_{n}^{(1)}, m_{n}^{(2)} \right) := \left( \min_{1 \leq k \leq n} X_{n,k}^{(1)}, \min_{1 \leq k \leq n} X_{n,k}^{(2)} \right) \) the componentwise sample minima.

A different direction which we pursue below is to analyse whether the convergence in (1.4) can be strengthen to almost sure limit convergence; some related results in this direction are obtained in Cheng et al. (1998), Fahrner and Stadtmüller (1998), Csáki and Gonchigdanzan (2002), Tan et al. (2007), Tan and Peng (2009), Peng et al. (2010,2011), Tan and Wang (2011,2012), Weng et al. (2012).
Theorem 2.2. Under the assumptions and notation of Theorem 1.7, suppose further that the elements from different rows satisfy
\[
\text{corr} \left( X_{mk}^{(i)}, X_{ni}^{(j)} \right) = \gamma_{ij}(k, l, m, n) = \gamma_{ij}(|k-l|+1, m, n)
\]
with \(1 \leq k \leq m, 1 \leq l \leq n, \) and all \(m < n, i,j \in \{1,2\}.\) Let \(\theta := \max_{1 \leq k \leq n, n \geq 2, i,j \in \{1,2\}} |\gamma_{ij}(k, m, n)| < 1, \beta \in (0, \frac{1}{\theta+\theta})\) and \(J_n := [n^\beta].\) If for some \(\epsilon > 0\)
\[
\max_{l_n \leq s < n} \rho_{ij}(s, n) \ln s (\ln s)^{1+\epsilon} = O(1)
\]
and
\[
\max_{J_n \leq t \leq n} \gamma_{ij}(t, m, n) \ln t (\ln t)^{1+\epsilon} = O(1)
\]
hold as \(n \to \infty,\) then for any \((x,y) \in \mathbb{R}^2\)
\[
\lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{k} \left( M_k^{(1)} \leq u_k(x), M_k^{(2)} \leq u_k(y) \right) = H_\lambda(x,y)
\]
holds almost surely. Furthermore, for any \(x_1, x_2, y_1, y_2 \in \mathbb{R},\) almost surely
\[
\lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{k} \left( -u_k(y_1) < M_k^{(1)} \leq u_k(x_1), -u_k(y_2) < M_k^{(2)} \leq u_k(x_2) \right) = H_\lambda(x_1, x_2)H_\lambda(y_1, y_2).
\]

Strengthening of the result of Theorem 1.2 to almost sure limit theorem requires significantly more efforts and additional technical conditions, therefore we shall not address that point here.

3 Further Results and Proofs

In this section we present the proofs of the main results. Since those proofs depend on some results which are of some independent interest, we formulate several lemmas.

Lemma 3.1. Under Assumption A1 for \(i,j \in \{1,2\}\) and any \(x,y \in \mathbb{R}\) we have
\[
\lim_{n \to \infty} n \sum_{k=1}^{n-1} |\rho_{ij}(k, n)| \exp \left( -\frac{\omega_n^2}{1 + |\rho_{ij}(k, n)|} \right) = 0,
\]
where \(\omega_n := \min(|u_n(x)|, |u_n(y)|).\) If Assumption A2 holds, then for \(i,j \in \{1,2\}\)
\[
\lim_{n \to \infty} n \sum_{k=1}^{n-1} |\rho_{ij}(k, n) - \tau_{ij}(n)| \exp \left( -\frac{\omega_n^2}{1 + \varrho_{ij}(k, n)} \right) = 0,
\]
where \(\tau_{ij}(n) := \tau_{ij} / \ln n\) and \(\varrho_{ij}(k, n) := \max\{|\rho_{ij}(k, n)|, \tau_{ij}(n)|\}.

Proof. Our proof is similar to that of Lemma 4.3.2 and Lemma 6.4.1 in Leadbetter et al. (1983). For notational simplicity, we omit the index and write below simply \(\rho(k, n)\) instead of \(\rho_{ij}(k, n),\) and similarly we write \(\tau, \tau(n), \delta, \varpi, K_n\) for \(\tau_{ij}, \tau_{ij}(n), \delta_{ij}(k, n), \delta_{ij}, \varpi_{ij}, K_{n,ij}\) respectively. First note that in view of (1.2)
\[
\lim_{n \to \infty} \frac{n}{\sqrt{2\pi b_n}} \exp \left( -\frac{u_n^2(s)}{2} \right) = \exp(-s), s \in \mathbb{R}, \text{ and } \lim_{n \to \infty} \frac{\omega_n}{\sqrt{2\ln n}} = \lim_{n \to \infty} \frac{b_n}{\sqrt{2\ln n}} = 1.
\]
Next, we write
\[ n \sum_{k=1}^{n-1} |\rho(k, n)| \exp \left( -\frac{\omega_n^2}{1 + |\rho(k, n)|} \right) = n \left( \sum_{k=1}^{I_n} + \sum_{k=I_n+1}^{n-1} \right) |\rho(k, n)| \exp \left( -\frac{\omega_n^2}{1 + |\rho(k, n)|} \right) =: S_{n1} + S_{n2}. \]
with $I_n$ given by condition formulated in Assumption A1. By the choice of $\alpha$ and $\sigma$ in Assumption A1, for some positive constants $c_1, c_2$
\[ S_{n1} \leq nn^\alpha \exp \left( -\frac{\omega_n^2}{1 + \sigma} \right) = n^{1+\alpha} \left( \exp \left( -\frac{\omega_n^2}{2} \right) \right)^{\frac{\sigma}{2}} \leq c_1 n^{1+\alpha} \left( \frac{\omega_n}{n} \right)^{\frac{\sigma}{2}} \leq c_2 n^{1+\alpha - \frac{\tau}{\ln n}} \rightarrow 0, \quad n \rightarrow \infty. \]
Next, define $\sigma(l, n) := \max_{1 \leq k < n} |\rho(k, n)| < 1$, then by (1.3)
\[ \lim_{n \rightarrow \infty} \sigma(I_n, n) \omega_n = \lim_{n \rightarrow \infty} 2\sigma(I_n, n) \ln n = 0, \]
and hence we have
\[ S_{n2} \leq n \sigma(I_n, n) \exp \left( -\omega_n \right) \sum_{k=I_n+1}^{n-1} \exp \left( \frac{\omega_n^2 |\rho(k, n)|}{1 + |\rho(k, n)|} \right) \leq n^2 \sigma(I_n, n) \exp \left( -\omega_n^2 \right) \exp \left( \sigma(I_n, n) \omega_n^2 \right) \leq O \left( \sigma(I_n, n) \omega_n \exp \left( \sigma(I_n, n) \omega_n^2 \right) \right) \rightarrow 0, \quad n \rightarrow \infty, \]
thus (3.1) follows. The following constant
\[ \delta(l, n) := \max_{1 \leq k < n} \rho(k, n) < 1 \]
plays an important role for proving (3.2). We split the sum in (3.2) into two terms, the first consist of summation over $1 \leq k \leq K_n$ and the second term is the sum over $K_n < k < n$. As in the proof of $S_{n1}$ above, for some positive constants $c, c_1, c_2$ we have
\[ n \sum_{k=1}^{K_n} |\rho(k, n) - \tau(n)| \exp \left( -\frac{\omega_n^2}{1 + \rho(k, n)} \right) \leq cn^{\alpha} \exp \left( -\frac{\omega_n^2}{1 + \max(\delta, \tau(n))} \right) = cn^{1+\alpha} \left( \exp \left( -\frac{\omega_n^2}{2} \right) \right)^{\frac{1}{1 + \max(\delta, \tau(n))}} \leq c_1 n^{1+\alpha} \frac{\omega_n}{n} \exp \left( -\frac{\omega_n^2}{1 + \max(\delta, \tau(n))} \right) \rightarrow 0, \quad n \rightarrow \infty, \]
since $\tau(n) \rightarrow 0$ and $0 < \varpi < \frac{1}{1 + \delta}$. For the second term, note that
\[ n \sum_{k=K_n+1}^{n-1} |\rho(k, n) - \tau(n)| \exp \left( -\frac{\omega_n^2}{1 + \rho(k, n)} \right) \leq n \exp \left( -\frac{\omega_n^2}{1 + \delta(K_n, n)} \right) \sum_{k=K_n+1}^{n-1} |\rho(k, n) - \tau(n)| \leq n \exp \left( -\frac{\omega_n^2}{1 + \delta(K_n, n)} \right) \frac{\ln n}{n} \sum_{k=K_n+1}^{n-1} |\rho(k, n) - \tau(n)|. \]
Since \( \lim_{n \to \infty} \rho(K_n, n) \ln K_n = \tau \), there exists a constant \( L > 0 \) such that \( \rho(K_n, n) \ln K_n \leq L \), also \( \delta(K_n, n) \ln K_n \leq L \). Consequently, for some positive constants \( c_1, c_2 \)

\[
\frac{n^2}{\ln n} \exp \left( -\frac{\omega_n^2}{1 + \delta(K_n, n)} \right) \leq \frac{n^2}{\ln n} \exp \left( -\frac{\omega_n^2}{1 + L/\ln n} \right) \\
\leq c_1 \frac{n^2}{\ln n} \left( \frac{\omega_n}{n} \right)^{2/(1+L/\ln n)} \\
\leq c_2 n^{2L/(L+\ln n)} (\ln n)^{-L/(L+\ln n)} \\
= O(1), \quad n \to \infty.
\]

Further

\[
\frac{\ln n}{n} \sum_{k=K_n+1}^{n-1} |\rho(k, n) - \tau(n)| \leq \frac{\ln n}{n} \sum_{k=K_n+1}^{n-1} \left| \rho(k, n) - \frac{1}{\ln k} \right| + \frac{\ln n}{n} \sum_{k=K_n+1}^{n-1} \left| \frac{1}{\ln k} - \frac{1}{\ln n} \right| \\
\leq \frac{1}{n^{\omega}} \sum_{k=K_n+1}^{n-1} \left| \rho(k, n) \ln k - \tau \right| + \frac{1}{n} \sum_{k=K_n+1}^{n-1} \left| \frac{1}{\ln n} \right| \\
=: T_{n1} + T_{n2}.
\]

By Assumption A2 \( \lim_{n \to \infty} T_{n1} = 0 \), and further

\[
T_{n2} \leq \frac{\tau}{\omega \ln n} \sum_{k=K_n+1}^{n-1} \left| \ln \frac{k}{n} \right| \frac{1}{n} = O \left( \frac{\tau}{\omega \ln n} \int_0^1 \ln x \, dx \right),
\]

thus the proof is complete. \( \square \)

**Lemma 3.2.** Under the conditions of Theorem 2.2 and the notation of Lemma 3.1 for \( i, j \in \{1, 2\} \) and positive constants \( c_1, c_2 \)

\[
n \sum_{k=1}^{n-1} |\rho_{ij}(k, n)| \exp \left( -\frac{\omega_n^2}{1 + |\rho_{ij}(k, n)|} \right) \leq c_1 (\ln \ln n)^{-(1+\epsilon)}, \tag{3.3}
\]

and

\[
\max_{1 \leq m < n} m \sum_{k=1}^{n} |\gamma_{ij}(k, m, n)| \exp \left( -\frac{\omega_n^2 + \omega_m^2}{2(1 + |\gamma_{ij}(k, m, n)|)} \right) \leq c_2 (\ln n)^{-(1+\epsilon)} \tag{3.4}
\]

hold for some \( \epsilon > 0 \).

**Proof.** For \( m < n \), using the notation of the proof of Lemma 3.1, and write simply \( \gamma(k, m, n) \) instead of \( \gamma_{ij}(k, m, n) \), further define the following constant

\[
\theta(l, m, n) := \max_{1 \leq k \leq n} |\gamma(k, m, n)| < 1.
\]

By (2.4) and (2.5), for some small \( \varepsilon > 0 \) and some positive constants \( c_1, c_2 \) and for all large \( n \)

\[
(\sigma(I_n, n) \omega_n^2 \leq (1 + \varepsilon) \frac{2}{\alpha} (I_n, n) \ln n^\alpha \\
\leq (1 + \varepsilon) \frac{2}{\alpha} \max_{1 \leq k \leq n} |\rho(k, n)| \ln k \\
\leq c_1 (\ln \ln n^\alpha)^{-(1+\epsilon)} \sim c_1 (\ln \ln n)^{-(1+\epsilon)},
\]

and for \( 1 \leq m < n \)

\[
\theta(J_n, m, n) \omega_m \omega_n \leq 2(1 + \varepsilon) \theta(J_n, m, n)(\ln m \ln n)^{1/2} \\
\leq (1 + \varepsilon) \frac{2}{\beta} \max_{1 \leq k \leq n} |\gamma(k, m, n)| \ln k
\]
Similarly, for all $n$ large
\[
\theta(J_n, m, n)\omega_m^2 \leq 2(1 + \varepsilon)\theta(J_n, m, n) \ln m \leq (1 + \varepsilon)\frac{2}{\beta}\theta(J_n, m, n) \ln n^\beta \leq c_2(\ln n)^{-1+\varepsilon},
\]
and
\[
\theta(J_n, m, n)\omega_n^2 \leq 2(1 + \varepsilon)\theta(J_n, m, n) \ln n = (1 + \varepsilon)\frac{2}{\beta}\theta(J_n, m, n) \ln n^\beta \leq c_2(\ln n)^{-1+\varepsilon}.
\]
Combining the above inequalities and along the same lines of the proof of Lemma 2.1 in Csáki and Gonchigdanzan (2002), the claim follows.

**Proof of Theorem 1.1** Let \(\{(\tilde{X}^{(1)}_{n,k}, \tilde{X}^{(2)}_{n,k}), 1 \leq k \leq n, n \geq 1\}\) denote the associated iid triangular array of \(\{X_{n,k}\}\), i.e. the correlation satisfy \(\text{corr} (\tilde{X}^{(1)}_{n,k}, \tilde{X}^{(2)}_{n,k}) = \rho_0(n)\) and \(\text{corr} (\tilde{X}^{(i)}_{n,k}, \tilde{X}^{(j)}_{n,l}) = 0\) for \(1 \leq k \neq l \leq n, i, j \in \{1, 2\}\). Since the condition (1.2) holds, using Theorem 1 in Hüsler and Reiss (1989), we have
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P} \left( \mathbf{M}^{(1)}_n \leq u_n(x), \mathbf{M}^{(2)}_n \leq u_n(y) \right) - H_\lambda(x, y) \right| = 0,
\]
where \(\mathbf{M}^{(1)}_n, \mathbf{M}^{(2)}_n := (\max_{1 \leq k \leq n} \tilde{X}^{(1)}_{n,k}, \max_{1 \leq k \leq n} \tilde{X}^{(2)}_{n,k})\). Hence, we only need to prove that
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P} \left( M^{(1)}_n \leq u_n(x), M^{(2)}_n \leq u_n(y) \right) - \mathbb{P} \left( \mathbf{M}^{(1)}_n \leq u_n(x), \mathbf{M}^{(2)}_n \leq u_n(y) \right) \right| = 0
\]
holds. By Berman’s Normal Comparison Lemma (see Piterbarg (1996) for generalised Berman inequality and Corollary 2.1 in Li and Shao (2002), for all \(x, y \in \mathbb{R}\) we have
\[
\left| \mathbb{P} \left( M^{(1)}_n \leq u_n(x), M^{(2)}_n \leq u_n(y) \right) - \mathbb{P} \left( \mathbf{M}^{(1)}_n \leq u_n(x), \mathbf{M}^{(2)}_n \leq u_n(y) \right) \right| \leq \frac{1}{4} \sum_{k=1}^{n-1} |\rho_{11}(k, n)| \exp \left( -u^2(x) \frac{1}{1 + |\rho_{11}(k, n)|} \right) + \frac{1}{2} \sum_{k=1}^{n-1} |\rho_{12}(k, n)| \exp \left( -u^2(y) \frac{1}{2(1 + |\rho_{12}(k, n)|)} \right) + \frac{1}{n} \sum_{k=1}^{n-1} |\rho_{22}(k, n)| \exp \left( -u^2(y) \frac{1}{1 + |\rho_{22}(k, n)|} \right).
\]
Consequently, in view of (3.1) the proof is complete.

**Proof of Theorem 1.2** Let \(\{Z_{n,0} = (Z^{(1)}_{n,0}, Z^{(2)}_{n,0}), n \geq 1\}\) be a sequence of 2-dimensions standard Gaussian random vectors (with mean-zero and unit-variance) and
\[
\text{corr} \left( Z^{(1)}_{n,0}, Z^{(2)}_{n,0} \right) = \frac{\tau_{12}(n)}{\sqrt{\tau_{11}(n)\tau_{22}(n)}}, \quad \tau_{ij}(n) = \frac{\tau_{ij}}{\ln n}, \quad i, j \in \{1, 2\}.
\]
Further, let \(\{Z_{n,k} = (Z^{(1)}_{n,k}, Z^{(2)}_{n,k}), 1 \leq k \leq n, n \geq 1\}\) denote a triangular array of independent standard Gaussian random vectors such that \(\text{corr} \left( Z^{(i)}_{n,k}, Z^{(j)}_{n,l} \right) = 0\) for \(1 \leq k \neq l \leq n, i, j \in \{1, 2\},\) and
\[
\text{corr} \left( Z^{(1)}_{n,k}, Z^{(2)}_{n,k} \right) = \frac{\rho_0(n) - \tau_{12}(n)}{\sqrt{(1 - \tau_{11}(n))(1 - \tau_{22}(n))}} =: \hat{\rho}_0(n).
\]
Suppose that \(Z_{n,0}\) is independent of \(\{Z_{n,k}, 1 \leq k \leq n\}\), and define
\[
Y_{n,k} = \left( Y^{(1)}_{n,k}, Y^{(2)}_{n,k} \right) = \left( \frac{1}{\sqrt{\tau_{11}(n)}} Z^{(1)}_{n,0} + (1 - \tau_{11}(n)) \right) \hat{\rho}_0(n) Z^{(1)}_{n,k} + \frac{1}{\sqrt{\tau_{22}(n)}} Z^{(2)}_{n,0} + (1 - \tau_{22}(n)) \hat{\rho}_0(n) Z^{(2)}_{n,k}
\]
for \(1 \leq k \leq n, n \geq 1\). It follows that \(Y_{n,k}\) are standard Gaussian random vectors and \(\text{corr} \left( Y^{(1)}_{n,k}, Y^{(2)}_{n,k} \right) = \rho_0(n)\),
\[
\text{corr} \left( Y^{(i)}_{n,k}, Y^{(j)}_{n,l} \right) = \tau_{ij}(n) \text{ for } 1 \leq k \neq l \leq n, i, j \in \{1, 2\}.
\]
For notational simplicity, in the sequel denote
\[
\left( \hat{M}^{(1)}_n, \hat{M}^{(2)}_n \right) := \left( \max_{1 \leq k \leq n} Y^{(1)}_{n,k}, \max_{1 \leq k \leq n} Y^{(2)}_{n,k} \right), \quad \left( \hat{M}^{(1)}_n, \hat{M}^{(2)}_n \right) := \left( \max_{1 \leq k \leq n} Z^{(1)}_{n,k}, \max_{1 \leq k \leq n} Z^{(2)}_{n,k} \right).
\]
Since \( b_n \sim \sqrt{2 \ln n} \), condition (1.2) implies
\[
\lim_{n \to \infty} \frac{b_n^2(1 - \hat{\rho}_0(n))}{1 + \hat{\rho}_0(n)} = \tilde{\lambda}^2,
\]
where \( \tilde{\lambda} = \sqrt{\lambda^2 + \tilde{\tau}}, \tilde{\tau} = \tau_{12} - \frac{1}{2}(\tau_{11} + \tau_{22}) \) and \( \lambda^2 \geq -\tilde{\tau} \). Consequently, in view of Hüsler and Reiss (1989)
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P} \left( \tilde{M}_n^{(1)} \leq u_n(x), \tilde{M}_n^{(2)} \leq u_n(y) \right) - H_{\tilde{\lambda}}(x, y) \right| = 0.
\]
Particularly, for \( \tilde{\lambda} = 0 \) case, i.e., \( \lambda^2 = -\tilde{\tau} \), we have 1 - \( \rho_0(n) \sim \frac{1}{2}(\tau_{11}(n) + \tau_{22}(n)) - \tau_{12}(n) \).
\[
\lim_{n \to \infty} \hat{\rho}_0(n) = \lim_{n \to \infty} \frac{1 - \frac{1}{2}(\tau_{11}(n) + \tau_{22}(n)) + \tau_{12}(n)}{1 - \frac{1}{2}(\tau_{11}(n) + \tau_{22}(n))} = 1,
\]
i.e., the asymptotic complete dependence of the components of \( Z_{n,k} \). For \( \tilde{\lambda} = \infty \) case, i.e., \( \lambda = \infty \), we have \( \lim_{n \to \infty} \rho_0(n) = 0 \). Consequently,
\[
\lim_{n \to \infty} \hat{\rho}_0(n) = \lim_{n \to \infty} \frac{-\tau_{12}(n)}{\sqrt{(1 - \tau_{11}(n))(1 - \tau_{22}(n))}} = 0,
\]
and thus the asymptotic independence of the components of \( Z_{n,k} \) follows. For all \( x, y \in \mathbb{R} \)
\[
\lim_{n \to \infty} \mathbb{P} \left( \tilde{M}_n^{(1)} \leq u_n(x), \tilde{M}_n^{(2)} \leq u_n(y) \right) = \lim_{n \to \infty} \mathbb{P} \left( \tilde{M}_n^{(1)} \leq u_n(x), \tau_{11}(n) \tilde{M}_n^{(2)} \leq u_n(y) \right) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{P} \left( \tilde{M}_n^{(1)} \leq u_n(x), \tau_{11}(n) \tilde{M}_n^{(2)} \leq u_n(y) \right) \varphi_n(z_1, z_2) dz_1 dz_2,
\]
where \( \varphi_n(z_1, z_2) \) is the joint probability density of Gaussian vector \( Z_{n,0} \). Using (1.5), and \( \tau_{ii}(n) = \tau_{ii}/\ln n \), for \( i \in \{1, 2\} \), let \( v_1 = x, v_2 = y \), we have
\[
(u_n(v_i) - \tau_{ii}(n)z_i)(1 - \tau_{ii}(n))^{-\frac{1}{2}} = u_n(v_i + \tau_{ii} - \sqrt{2\tau_{ii}z_i}).
\]
Hence by the dominated convergence theorem, for all \( x, y \in \mathbb{R} \) we have
\[
\lim_{n \to \infty} \mathbb{P} \left( \tilde{M}_n^{(1)} \leq u_n(x), \tilde{M}_n^{(2)} \leq u_n(y) \right) = \mathbb{E} \left\{ H_{\lambda}(x + \tau_{11} - \sqrt{2\tau_{11}Z}, y + \tau_{22} - \sqrt{2\tau_{22}W}) \right\},
\]
where \((Z, W)\) is a standard Gaussian vector with correlation \( \tau_{12}/\sqrt{\tau_{11}\tau_{22}} \). Applying Berman’s Normal Comparison Lemma and (3.2), we get
\[
\left| \mathbb{P} \left( M_n^{(1)} \leq u_n(x), M_n^{(2)} \leq u_n(y) \right) - \mathbb{P} \left( \tilde{M}_n^{(1)} \leq u_n(x), \tilde{M}_n^{(2)} \leq u_n(y) \right) \right|
\leq \frac{1}{4} n \sum_{k=1}^{n-1} |\rho_{11}(k, n) - \tau_{11}(n)| \exp \left( -\frac{u_n^2(x)}{1 + g_{11}(k, n)} \right) + \frac{1}{2} n \sum_{k=1}^{n-1} |\rho_{12}(k, n) - \tau_{12}(n)| \exp \left( -\frac{u_n^2(x) + u_n^2(y)}{2(1 + g_{12}(k, n))} \right) + \frac{1}{4} n \sum_{k=1}^{n-1} |\rho_{22}(k, n) - \tau_{22}(n)| \exp \left( -\frac{u_n^2(y)}{1 + g_{22}(k, n)} \right)
\to 0, \quad n \to \infty,
\]
thus the claim follows. \( \square \)
Proof of Theorem 2.3 Let \( \left\{ \left( \hat{X}_{n,k}^{(1)}, \hat{X}_{n,k}^{(2)} \right), 1 \leq k \leq n, n \geq 1 \right\} \) be iid Gaussian triangular array defined in the proof of Theorem 1.1 and set \( \left( m_n^{(1)}, m_n^{(2)} \right) := \left( \min_{1 \leq k \leq n} \hat{X}_{n,k}^{(1)}, \min_{1 \leq k \leq n} \hat{X}_{n,k}^{(2)} \right) \). We write next

\[
\begin{align*}
\lim_{n \to \infty} nP_1(n, x_1, x_2) &= \Phi \left( \lambda + \frac{x_1 - x_2}{2\lambda} \right) \exp(-x_2) + \Phi \left( \lambda + \frac{x_2 - x_1}{2\lambda} \right) \exp(-x_1) =: D_1, \\
\text{and since } (-\hat{X}_{1,1}^{(1)}, -\hat{X}_{1,1}^{(2)}) &\stackrel{d}{=} (\hat{X}_{1,1}^{(1)}, \hat{X}_{1,1}^{(2)})
\end{align*}
\]

Using Theorem 1 in Hüsler and Reiss (1989), we have

\[
\lim_{n \to \infty} nP_2(n, y_1, y_2) = \Phi \left( \lambda + \frac{y_1 - y_2}{2\lambda} \right) \exp(-y_2) + \Phi \left( \lambda + \frac{y_2 - y_1}{2\lambda} \right) \exp(-y_1) =: D_2.
\]

In view of (1.5) we have \( \lim_{n \to \infty} \frac{n a_n}{\sqrt{2\pi}} \exp \left(-\frac{b_n^2}{2}\right) = 1 \), then with \( \varphi = \Phi' \)

\[
nP_3(n, x_1, y_2) = \frac{a_n}{\sqrt{2\pi}} \int_{x_1}^{\infty} \Phi \left( \frac{-u_n(y_2) - \rho_0(n)u_n(t)}{\sqrt{1 - \rho_0^2(n)}} \right) \varphi(u_n(t)) dt \\
\sim \int_{x_1}^{\infty} \Phi \left( \frac{-u_n(y_2) - \rho_0(n)u_n(t)}{\sqrt{1 - \rho_0^2(n)}} \right) \exp \left(-a_n \frac{b_n t - a_n^2 t^2}{2}\right) dt, \quad n \to \infty.
\]

By (1.2) and (1.6)

\[
\frac{-u_n(y_2) - \rho_0(n)u_n(t)}{\sqrt{1 - \rho_0^2(n)}} = -b_n \frac{1 + \rho_0(n)}{\sqrt{1 - \rho_0^2(n)}} - a_n \frac{y_2 + t}{\sqrt{1 - \rho_0^2(n)}} + a_n t \frac{1 - \rho_0(n)}{\sqrt{1 + \rho_0(n)}} \to -\infty
\]
as \( n \to \infty \), hence \( \lim_{n \to \infty} nP_2(n, x_1, y_2) = 0 \), and by similar arguments \( \lim_{n \to \infty} nP_4(n, y_1, x_2) = 0 \). Consequently, for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \)

\[
\lim_{n \to \infty} \frac{1}{n} \left( 1 - D_1 - D_2 \right) = H_{\lambda}(x_1, x_2) H_{\lambda}(y_1, y_2).
\]

By Berman’s Normal Comparison Lemma and (3.1), for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \) with

\[
\omega_n := \min\{ |u_n(x_1)|, |u_n(x_2)|, |u_n(y_1)|, |u_n(y_2)| \}
\]

we have

\[
\begin{align*}
\left| \mathbb{P} \left( -u_n(y_1) < m_n^{(1)} \leq M_n^{(1)} \leq u_n(x_1), -u_n(y_2) < m_n^{(2)} \leq M_n^{(2)} \leq u_n(x_2) \right) \right|
\end{align*}
\]
By Berman’s Normal Comparison Lemma and (3.3), for all

In the light of Theorem 1.1, (2.6) follows if we show that for any

Proof of Theorem 2.2. In the light of Theorem 1.1 (2.6) follows if we show that for any \( x, y \in \mathbb{R} \)

holds almost surely. By Lemma 3.1 in Csáki and Goungidan (2002), in order to prove (3.5), it suffices to show that for some \( \epsilon > 0 \) and some positive constant \( c \)

Straightforward calculations yield

where \( \Sigma_1 < \infty \). Thus, it remains only to estimate \( \Sigma_2 \). Write next

For \( 1 \leq k < l \leq n \), we have

where

By Berman’s Normal Comparison Lemma and (3.3), for all \( l \leq n \) we obtain

\[
P_1 + P_2 \leq c(\ln \ln l)^{-(1+\epsilon)},
\]
with \( c \) some positive constant. The following inequality \( z^{l-k} - z^{l} \leq \frac{k}{l} \) valid for \( 0 \leq z \leq 1 \) yields further

\[
P_3 = H^{l-k}(u_i(x), u_i(y)) - H^l(u_i(x), u_i(y)) \leq \frac{k}{l},
\]

(3.7) where \( H \) is the distribution function of \( \left( \tilde{X}_{11}^{(1)}, \tilde{X}_{11}^{(2)} \right) \). Again, as above by (3.4), we have with \( c \) some positive constant

\[
E_2 \leq \left| \mathbb{P} \left( M_k^{(1)} \leq u_k(x), M_k^{(2)} \leq u_k(y), M_{l,k}^{(1)} \leq u_l(x), M_{l,k}^{(2)} \leq u_l(y) \right) \right. \\
- \left. \mathbb{P} \left( M_k^{(1)} \leq u_k(x), M_k^{(2)} \leq u_k(y) \right) \mathbb{P} \left( M_{l,k}^{(1)} \leq u_l(x), M_{l,k}^{(2)} \leq u_l(y) \right) \right|
\leq \frac{1}{4} \sum_{i,j=1,2} \sum_{k+1 \leq i \leq l} \left| \gamma_{ij}(s,t,k,l) \right| \exp \left( -\frac{u_i^2(z_i) + u_j^2(z_j)}{2(1 + |\gamma_{ij}(s,t,k,l)|)} \right)
\leq \frac{1}{4} \sum_{i,j=1,2} \sum_{k+1 \leq i \leq l} \left| \gamma_{ij}(s,k) \right| \exp \left( -\frac{u_i^2(z_i) + u_j^2(z_j)}{2(1 + |\gamma_{ij}(s,k,l)|)} \right)
\leq c \left( \ln \ln l \right)^{-(1+\epsilon)},
\]

(3.8)

where \( z_1 = x \) and \( z_2 = y \). Combining (3.6), (3.7) and (3.8), for \( 1 \leq k < l \leq n \) we have

\[
\text{Cov} \left( \mathbb{I} \left( M_k^{(1)} \leq u_k(x), M_k^{(2)} \leq u_k(y) \right), \mathbb{I} \left( M_{l,k}^{(1)} \leq u_l(x), M_{l,k}^{(2)} \leq u_l(y) \right) \right) \leq c \left( \frac{k}{l} + \left( \ln \ln l \right)^{-(1+\epsilon)} \right),
\]

hence \( \Sigma_2 \leq c \left( \ln n \right)^2 \left( \ln \ln n \right)^{-(1+\epsilon)} \) with some positive constant \( c \). The proof of (2.6) is complete. By using Theorem 2.1 and arguments similar to the proof of (2.6), we can show that (2.7) holds. The details are omitted here.

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