STABILITY OF PERIODIC TRAVELING WAVES
FOR NONLINEAR DISPERSIVE EQUATIONS

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ABSTRACT. We study the stability and instability of periodic traveling waves for Korteweg-de Vries type equations with fractional dispersion and related, nonlinear dispersive equations. We show that a local constrained minimizer for a suitable variational problem is nonlinearly stable to period preserving perturbations. We then discuss when the associated linearized equation admits solutions exponentially growing in time. The proof uses variational properties of the equation.

1. Introduction

We study the stability and instability of periodic traveling waves for a class of nonlinear dispersive equations, in particular, equations of Korteweg-de Vries (KdV) type

\[ u_t - Mu_x + f(u)_x = 0. \]

Here \( t \in \mathbb{R} \) denotes the temporal variable and \( x \in \mathbb{R} \) is the spatial variable in the predominant direction of wave propagation; \( u = u(x, t) \) is real valued, representing the wave profile or a velocity. Throughout we express partial differentiation either by a subscript or using the symbol \( \partial \). Moreover \( M \) is a Fourier multiplier, defined as \( \hat{M}(\xi) = m(\xi)\hat{u}(\xi) \) and characterizing dispersion in the linear limit, while \( f \) is the nonlinearity. In many examples of interest, \( f \) obeys a power law.

Perhaps the best known among equations of the form (1.1) is the KdV equation itself, which was put forward in [Bou77] and [KdV95] to model the unidirectional propagation of surface water waves with small amplitudes and long wavelengths in a channel; it has since found relevances in other situations such as Fermi-Pasta-Ulam lattices. Observe, however, that (1.1) is nonlocal unless the dispersion symbol \( m \) is a polynomial of \( i\xi \); examples include the Benjamin-Ono equation (see [Ben67, Ono75], for instance) and the intermediate long wave equation (see [Jos77], for instance), for which \( m(\xi) = |\xi| \) and \( \xi \coth \xi - 1 \), respectively, while \( f(u) = u^2 \). Another example, proposed by Whitham [Whi74] to argue for breaking of water waves, corresponds to \( m(\xi) = (\tanh \xi)/\xi \) and \( f(u) = u^2 \). Incidentally the quadratic nonlinearity is characteristic of many wave phenomena.

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A traveling wave solution of (1.1) takes the form \( u(x, t) = u(x - ct) \), where \( c \in \mathbb{R} \) and \( u \) satisfies by quadrature that

\[
\mathcal{M}u - f(u) + cu + a = 0
\]

for some \( a \in \mathbb{R} \). In other words, it steadily propagates at a constant speed without changing the configuration. Periodic traveling waves of the KdV equation are known in closed form, namely cnoidal waves; see [KdV95], for instance. Moreover Benjamin [Ben67] calculated periodic traveling waves of the Benjamin-Ono equation. For a broad range of dispersion symbols and nonlinearities, a plethora of periodic traveling waves of (1.1) may be attained from variational arguments. To illustrate, we shall discuss in Section 2 minimization problems for a family of KdV type equations with fractional dispersion.

Benjamin in his seminal work [Ben72] (see also [Bon75]) explained that solitary waves of the KdV equation are nonlinearly stable. By a solitary wave, incidentally, we mean a traveling wave solution which vanishes asymptotically. Benjamin’s proof hinges upon that the KdV “soliton” arises as a constrained minimizer for a suitable variational problem and spectral information of the associated linearized operator. Later it developed into a powerful stability theory in [GSS87], for instance, for a general class of Hamiltonian systems and led to numerous applications. In the case of \( m(\xi) = |\xi|^\alpha, \alpha \geq 1 \), and \( f(u) = u^{p+1}, p \geq 1 \), in (1.1), in particular, solitary waves were shown in [BSS87] (see also [SS90, Wei87]) to arise as energy minimizers subject to constant momentum and to be nonlinearly stable if \( p < 2\alpha \) whereas they are constrained energy saddles and nonlinearly unstable if \( p > 2\alpha \).

We shall take matters further in Section 4 and establish that a periodic traveling wave of a KdV equation with fractional dispersion is nonlinearly stable with respect to period preserving perturbations, provided that it locally minimizes the energy subject to conservations of the momentum and the mass. In the case of generalized KdV equations, i.e., \( m(\xi) = \xi^2 \) in (1.1), the nonlinear stability of a periodic traveling wave to same period perturbations was determined in [Joh09], for instance, through spectral conditions, which were expressed in terms of eigenvalues of the associated monodromy map (or the periodic Evans function); see also [AP07, APBS06, BJK11, DK10, DN11]. Confronted with nonlocal operators, however, spectral problems may be out of reach by Evans function techniques. Instead we make an effort to replace ODE based arguments by functional analytic ones. The program was recently set out in [BH13].

As a key intermediate step we shall demonstrate in Section 3 that the linearized operator associated with the traveling wave equation is nondegenerate at a periodic, local constrained minimizer for a KdV equation with fractional dispersion. That is to say, its kernel is spanned merely by spatial translations. The nondegeneracy of the linearization proves a spectral condition, which plays a central role in the stability of traveling waves (see [Wei87, Lin08] among others) and the blowup (see [KMR11], for instance) for the related, time evolution equation, and therefore it is of independent interest. In the case of generalized KdV equations, the nondegeneracy at a periodic traveling wave was identified in [Joh09], for instance, with that the wave amplitude not be a critical point of the period. Furthermore it was verified in [Kwo89], among others, at solitary waves. These proofs utilize shooting arguments and the Sturm-Liouville theory for ODEs, which may not be applicable to nonlocal operators. Nevertheless, Frank and Lenzmann [FL12] obtained the property at solitary waves for a family of nonlinear nonlocal equations, which we follow. The
idea lies in to find a suitable substitute for the Sturm-Liouville theory to count the number of sign changes in eigenfunctions for linear operators involving fractional Laplacians.

The present development may readily be adapted to other, nonlinear dispersive equations. We shall illustrate this in Section 5 by discussing equations of regularized long wave type. We shall remark in Section 6 about Lin’s recent approach [Lin08] to linear instability.

2. Existence of local constrained minimizers

We shall address the stability and instability mainly for the KdV equation with fractional dispersion

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Lambda^\alpha u_x + \langle u^2 \rangle_x &= 0,
\end{align*}
\]

where \(0 < \alpha \leq 2\) and \(\Lambda = \sqrt{-\partial_x^2}\) is defined via the Fourier transform as \(\hat{\Lambda} u(\xi) = |\xi| \hat{u}(\xi)\), representing fractional derivatives.

In the case of \(\alpha = 2\), notably, (2.1) recovers the KdV equation while in the case of \(\alpha = 1\) it corresponds to the Benjamin-Ono equation. In the case of \(\alpha = -1/2\), furthermore, (2.1) was argued in [Hur12] to approximate up to “quadratic” order the water wave problem in two spatial dimensions in the infinite depth case. Notice that (2.1) is nonlocal for \(0 < \alpha < 2\). Incidentally fractional powers of the Laplacian occur in numerous applications, such as dislocation dynamics in crystals (see [CDLFM07], for instance) and financial mathematics (see [CT04], for instance).

The present treatment may be adapted mutatis mutandis to general power-law nonlinearities; see Remark 2.4, for instance. We focus on the quadratic nonlinearity, however, to simplify the exposition. Incidentally it is characteristic of many wave phenomena; see [Whi74], for instance.

Throughout we work in the periodic, \(L^2\)-based Sobolev spaces over the interval \([0, T]\), where \(T > 0\) is fixed although at times it is treated as a free parameter. We define a periodic Sobolev space of fractional order to be equipped with the norm

\[
\|u\|_{H^{\alpha/2}_{\text{per}}([0, T])}^2 = \int_0^T (u^2 + |\Lambda^{\alpha/2} u|^2) \, dx,
\]

where \(0 < \alpha < 2\). We employ the standard notation \(\langle \cdot, \cdot \rangle\) for the \(L^2_{\text{per}}((0, T))\)-inner product.

Notice that (2.1) may be written in the Hamiltonian form

\[
\frac{\partial u}{\partial t} = J\delta H(u),
\]

where \(J = \partial_x\) is the symplectic form,

\[
H(u) = \int_0^T \left( \frac{1}{2} |\Lambda^{\alpha/2} u|^2 - \frac{1}{3} u^3 \right) \, dx =: K(u) + U(u)
\]

is the Hamiltonian and \(\delta\) denotes variational differentiation; \(K\) and \(U\) correspond to the kinetic and potential energies, respectively. Notice that (2.1) possesses, in addition to \(H\), the two conserved quantities

\[
P(u) = \int_0^T \frac{1}{2} u^2 \, dx
\]

\[
* \text{Note that } \Lambda^\alpha \partial_x \text{ is non singular for } \alpha \gg -1.
\]
and
\[
(2.4) \quad M(u) = \int_0^T u \, dx,
\]
which correspond to the momentum and the mass, respectively. Conservation of \( P \) implies that (2.1) is invariant under spatial translations thanks to Noether’s theorem while \( M \) is a Casimir invariant of the flow induced by (2.1) and is associated with that the kernel of the symplectic form is spanned by a constant. Notice that
\[
(2.5) \quad \delta P(u) = u \quad \text{and} \quad \delta M(u) = 1.
\]
Moreover (2.1) remains invariant under
\[
(2.6) \quad u(x, t) \mapsto \lambda \alpha \, u(\lambda \, (x + x_0), \lambda^{\alpha + 1} t)
\]
for any \( \lambda > 0 \) for any \( x_0 \in \mathbb{R} \).

**Remark 2.1 (Well-posedness).** In the range \( \alpha \geq -1 \), one may work out the local in time well-posedness for (2.1) in \( H^{3/2+}(0, T) \), combining a compactness argument and a priori bounds. Without recourse to dispersive effects, the proof is identical to that for the inviscid Burgers equation \( u_t + (u^2)_x = 0 \). We omit the detail.

With the help of techniques in nonlinear dispersive equations and specific properties of the equation, the global in time well-posedness for (2.1) may be established in \( H^{3/2+}_\text{per}([0, T]) \) in the case of \( \alpha = 2 \), namely the KdV equation (see [CKS+03], for instance), and in \( H^{\alpha+}_\text{per}([0, T]) \) in the case of \( \alpha = 1 \), the Benjamin-Ono equation (see [Mol08], for instance). For non-integer values of \( \alpha \), however, the existence matter for (2.1) seems not properly understood in spaces of low regularities. The global well-posedness in \( H^{\alpha/2}(\mathbb{R}) \) was recently settled in [KMR11] for (2.1), in the case of \( 1 < \alpha < 2 \) and \( u^{\rho+1} \) in place of \( u^2 \), \( \rho \leq 2\alpha \), but the proof seems to break down in the periodic functions setting.

In what follows we shall work in an appropriate subspace, say, \( X \) of \( H^{\alpha/2}_\text{per}([0, T]) \), where the initial value problem associated with (2.1) is well-posed for some interval of time and \( H, P, M : X \to \mathbb{R} \) are smooth.

A periodic traveling wave of (2.1) takes the form \( u(x, t) = u(x - ct - x_0) \), where \( c \in \mathbb{R} \) represents the wave speed, \( x_0 \in \mathbb{R} \) is the spatial translate and \( u \) is \( T \)-periodic, satisfying by quadrature that
\[
(2.7) \quad \Lambda^\alpha u - u^2 + cu + a = 0
\]
for some \( a \in \mathbb{R} \) (in the sense of distributions). Equivalently, it arises as a critical point of
\[
(2.8) \quad E(u; c, a) = H(u) + cP(u) + aM(u).
\]
Indeed
\[
(2.9) \quad \delta E(u; c, a) = 0
\]
agrees with (2.7) since \( \delta H(u) = \Lambda^\alpha u - u^2 \) using (2.5).

Henceforth we shall write a periodic traveling wave of (2.1) as \( u = u(\cdot; c, a) \). In a more comprehensive description, it is specified by four parameters \( c, a \) and \( T, x_0 \). Note, however, that \( T > 0 \) is arbitrary and fixed. Corresponding to translational invariance (see (2.6)), moreover, \( x_0 \) is inconsequential in the present development. Hence we may mod it out.
In the present notation, a solitary wave whose profile vanishes asymptotically corresponds, formally, to $\alpha = 0$ and $T = +\infty$.

In the case of $\alpha = 2$, periodic traveling waves of (2.1), namely the KdV equation, are well known in closed form, involving Jacobi elliptic functions; see [KdV95], for instance. In the case of $\alpha = 1$, moreover, Benjamin [Ben67] exploited the Poisson summation formula and explicitly calculated periodic traveling waves of (2.1). In general, the existence of periodic traveling waves of (2.1) follows from variational arguments, although one may lose an explicit form of the solution. In the energy subcritical case, in particular, a family of periodic traveling waves of (2.1) locally minimizes the Hamiltonian constrained to conservations of the momentum and the mass, generalizing “ground states” in the solitary wave setting.

**Proposition 2.2** (Existence, symmetry and regularity). Let $1/3 < \alpha \leq 2$. A local minimizer $u$ for $H$ subject to that $P$ and $M$ are conserved exists in $H_{\alpha/2}^\text{per}([0, T])$ for each $0 < T < \infty$ and it satisfies (2.7) for some $c \neq 0$ and $a \in \mathbb{R}$. It depends upon $c$ and $a$ in the $C^1$ manner.

Moreover $u = u(\cdot ; c, a)$ may be chosen to be even and strictly decreasing over the interval $[0, T/2]$, and $u \in H_{\alpha/2}^\infty([0, T])$.

Below we develop integral identities that a periodic solution of (2.7), or equivalently (2.9), a priori satisfies and which will be useful in various proofs.

**Lemma 2.3** (Integral identities). If $u \in H_{\alpha/2}^\text{per}([0, T]) \cap L^3_{\text{per}}([0, T])$ satisfies (2.7), or equivalently (2.9), then

$$2P - cM - aT = 0,$$

$$2K + 3U + 2cP + aM = 0.$$  

Proof. Integrating (2.7), or equivalently (2.9), over the periodic interval $[0, T]$ manifests (2.10). Multiplying it by $u$ and integrating over $[0, T]$ lead to (2.11). \qed

**Proof of Proposition 2.2.** We claim that it suffices to take $a = 0$ and $c = 1$. Suppose on the contrary that $a \neq 0$. We then assume without loss of generality that $c$ and $M$ are of opposite sign and $a > 0$. For, in the case when $c$ and $M$ are of the same sign, noting that (2.1) is time reversible, we make the change of variables $t \mapsto -t$ in (2.1) to reverse the sign of $c$ in (2.7) while leaving other components of the equation invariant. Once we accomplish that $c$ and $M$ are of opposite sign, $a \geq 0$ must follow since $P \geq 0$ and $T > 0$ by definition. We shall then devise the change of variables $u \mapsto u + \frac{1}{2}(\sqrt{c^2 + 4a} - c)$ and rewrite (2.7) as

$$\Lambda^\alpha u - u^2 + \gamma u = 0, \quad \text{where} \quad \gamma = \sqrt{c^2 + 4a} > 0.$$  

Therefore, it suffices to take $a = 0$ in (2.7). Incidentally this is reminiscent of that (2.1) enjoys Galilean invariance under $u(x, t) \mapsto u(x, t) + u_0$ for any $u_0 \in \mathbb{R}$. By virtue of scaling invariance in (2.6), we shall further devise the change of variables $u(x) \mapsto 1/\gamma u(x/\gamma^\alpha)$ and rewrite (2.12) as

$$\Lambda^\alpha u - u^2 + u = 0.$$  

To recapitulate, it suffices to take $a = 0$ and $c = 1$ in (2.7) and seek a local minimizer for $H + P$. (But we shall not a priori assume that $a = 0$ or $c = 1$ in the stability proof in Section 4.)
Since $H^{\alpha/2}_{\text{per}}([0, T])$ in the range $\alpha > 1/3$ is compactly embedded in $L^3_{\text{per}}([0, T])$ by a Sobolev inequality, it follows from calculus of variations that for each parameter (abusing notation) $U < [\ ]$ there exists $u \in H^{\alpha/2}_{\text{per}}([0, T])$ such that

$$K(u) + P(u) = \inf \{ K(\phi) + P(\phi) : \phi \in H^{\alpha/2}_{\text{per}}([0, T]), U(\phi) = U \}.$$  

The proof is rudimentary. We merely pause to remark that $K(\phi) + P(\phi)$ amounts to $\| \phi \|^2_{H^{\alpha/2}_{\text{per}}([0, T])}$ and the constraint is compact in $H^{\alpha/2}_{\text{per}}([0, T])$. Moreover, $u$ satisfies

$$\Lambda^2 u + u = \theta u^3$$

for some $\theta \neq 0$ in the sense of distributions. By a scaling argument, we may choose $U$ to ensure that $\theta = 1$. Consequently (abusing notation) $u \in H^{\alpha/2}_{\text{per}}([0, T])$ attains the constrained minimization problem (2.14) and satisfies (2.13). Note from (2.11) that $2K(u) + 3U(u) + 2P(u) = 0$.

Furthermore we claim that

$$E(u) = \inf \{ E(\phi) : \phi \in H^{\alpha/2}_{\text{per}}([0, T]), \phi \neq 0, 2K(\phi) + 3U(\phi) + 2P(\phi) = 0 \}.$$  

Since

$$E(\phi) = H(\phi) + P(\phi) = K(\phi) + U(\phi) + P(\phi) = \frac{1}{3} (K(\phi) + P(\phi)) = -\frac{1}{2} U(\phi)$$

and $2K(\phi) + 2P(\phi) = -3U(\phi) > 0$ whenever $2K(\phi) + 3U(\phi) + 2P(\phi) = 0$, $\phi \neq 0$, it suffices to show that

$$U(u) = \sup \{ U(\phi) : \phi \in H^{\alpha/2}_{\text{per}}([0, T]), \phi \neq 0, 2K(\phi) + 3U(\phi) + 2P(\phi) = 0 \}.$$  

Suppose that $\phi \in H^{\alpha/2}_{\text{per}}([0, T]), \phi \neq 0$ and $2K(\phi) + 3U(\phi) + 2P(\phi) = 0$. We define

$$b = \left( \frac{U(u)}{U(\phi)} \right)^{1/3},$$

and observe that (2.17) follows if $b \leq 1$ so that $0 \geq U(u) > U(\phi)$. Indeed we infer from (2.16) that

$$2K(b\phi) + 3U(b\phi) + 2P(b\phi) = 2b^2 K(\phi) + 3b^3 U(\phi) + 2b^2 P(\phi) = 2b^2 (1 - b)(K(\phi) + P(\phi)).$$

Moreover, since $U(b\phi) = b^3 U(\phi) = U(u)$ and since $u$ attains the constrained minimization problem (2.14), it follows that

$$K(u) + P(u) \leq K(b\phi) + P(b\phi).$$

Consequently

$$0 = 2K(u) + 3U(u) + 2P(u) \leq 2K(b\phi) + 3U(b\phi) + 2P(b\phi) = 2b^2 (1 - b)(K(\phi) + P(\phi)),$$

whence $b \leq 1$. This proves the claim. Since

$$\langle \delta H(\phi) + \delta P(\phi), \phi \rangle = 2K(\phi) + 3U(\phi) + 2P(\phi)$$

for all $\phi \in H^{\alpha/2}_{\text{per}}([0, T])$, furthermore, $u$ solves the constrained minimization problem (2.15) if and only if $u$ minimizes $H + P$ among its critical points. The existence assertion therefore follows. Clearly, $u$ depends upon $c$ and $a$ in the $C^1$ manner.

\[\text{Footnote:} \text{Note from (2.11) that if } u \in H^{\alpha/2}_{\text{per}}([0, T]), \alpha > 1/3, \text{ satisfies (2.13) then } K(u) + P(u) > 0 \text{ and } U(u) < 0 \text{ unless } u \equiv 0.\]
To proceed, since the symmetric decreasing rearrangement of \( u \) does not increase \( \int_0^T |\Lambda^{\alpha/2} u|^2 \, dx \) for \( 0 < \alpha < 2 \) (see [Par11], for instance, for a proof on \( \mathbb{R}^n \)) while leaving \( \int_0^T u^3 \, dx \) invariant, it follows from the rearrangement argument that a local minimizer for \( H \) subject to conservations of \( P \) and \( M \) must symmetrically decrease away from a point of principal elevation. The symmetry and monotonicity assertion then follows from translational invariance in (2.6). (Note that unlike in the solitary waves setting, for which \( a = 0 \) and \( T = +\infty \), a periodic, local constrained minimizer needs not be positive everywhere.)

It remains to address the smoothness of a periodic solution of (2.1) or equivalently,

\[
(2.18) \quad u = (\Lambda^\alpha + 1)^{-1} u^2
\]
after reduction to \( a = 0, \ c = 1 \) and after inversion; the validity of (2.18) is to be specified in the course the proof. We first claim that if \( u \in H^{\alpha/2}_{\text{per}}([0, T]) \) satisfies (2.18) then \( u \in L^\infty_{\text{per}}([0, T]) \). In the case of \( \alpha > 1 \) this follows immediately from a Sobolev inequality, whereas in the case of \( 1/3 < \alpha \leq 1 \) a proof based upon resolvent bounds for \( (\Lambda^\alpha + 1)^{-1} \) is found in [FL12, Lemma A.3], for instance, albeit in the solitary wave setting. Indeed, the Fourier series \( \frac{1}{|\xi|^2 + 1} \) lies in \( \ell^r(\mathbb{Z}) \) for \( 0 < \alpha < 1 \) for \( r > \frac{1}{1-\alpha} \) by the Hausdorff-Young inequality, whence \( u \in L^\infty_{\text{per}}([0, T]) \) after iterating \( (2.18) \) sufficiently many times.

We then promote \( u \in H^{\alpha/2}_{\text{per}}([0, T]) \cap L^\infty_{\text{per}}([0, T]) \) to \( H^\alpha_{\text{per}}([0, T]) \) since the Plancherel theorem manifests that

\[
\|\Lambda^\alpha u\|_{L^2} = \left\| \frac{\Lambda^\alpha}{\Lambda^\alpha + 1} u^2 \right\|_{L^2} = \left\| \frac{|\xi|^\alpha}{|\xi|^\alpha + 1} \hat{u}^2 \right\|_{L^2} \leq \|u^2\|_{L^2} \leq \|u\|_{L^\infty} \|u\|_{L^2} < \infty.
\]

Furthermore the fractional product rule (see [CW91], for instance) leads to that

\[
\|\Lambda^{2\alpha} u\|_{L^2} = \left\| \frac{\Lambda^{2\alpha}}{\Lambda^\alpha + 1} u^2 \right\|_{L^2} \leq \|\Lambda^\alpha u^2\|_{L^2} \leq C\|u\|_{L^\infty} \|\Lambda^\alpha u\|_{L^2} < \infty
\]

for \( C > 0 \) a constant independent of \( u \). After iterations, therefore, \( u \in H^\infty_{\text{per}}([0, T]) \) follows.

**Remark 2.4** (Power-law nonlinearities). One may rerun the arguments in the proof of Proposition 2.2 in the case of the general power-law nonlinearity

\[
(2.19) \quad u_t - \Lambda^\alpha u_x + (u^{p+1})_x = 0
\]

and obtain a periodic traveling wave, where \( 0 < \alpha \leq 2 \) and \( 0 < p < p_{\text{max}} \) is an integer such that

\[
(2.20) \quad p_{\text{max}} := \begin{cases} \frac{2\alpha}{\frac{\alpha}{1-\alpha} + \infty} & \text{for } \alpha < 1, \\ +\infty & \text{for } \alpha \geq 1. \end{cases}
\]

It locally minimizes in \( H^{\alpha/2}_{\text{per}}([0, T]) \) the Hamiltonian

\[
\int_0^T \left( \frac{1}{2} |\Lambda^{\alpha/2} u|^2 - \frac{1}{p+2} u^{p+2} \right) \, dx
\]

subject to conservations of \( P \) and \( M \), defined in (2.3) and (2.4), respectively. Note that \( 0 < p < p_{\text{max}} \), which is vacuous if \( \alpha \geq 1 \), ensures that (2.19) is \( H^{\alpha/2} \)-subcritical.
and $H^{\alpha/2}_{\text{per}}([0,T]) \subset L^{p+2}_{\text{per}}([0,T])$ compactly. In case $p = 1$, it is equivalent to that $\alpha > 1/3$.

**Remark 2.5** (Periodic vs. solitary waves). In the non-periodic functions setting, Weinstein [Wei87] (see also [FL12]) proved that (2.7) in the range $\alpha > 1/3$ admits a solitary wave, for which $a = 0$ and $T = +\infty$. In case $\alpha > 1/2$ so that (2.7) is $L^2$-subcritical, in addition, the solitary wave further arises as an energy minimizer subject to constant momentum. Periodic, local constrained minimizers for (2.7), constructed in Proposition 2.2, are then expected to tend to the solitary wave as their period increases to infinity. This in some sense generalizes the notion of the homoclinic limit in the case of $\alpha = 2$.

In case $1/3 < \alpha < 1/2$, on the other hand, local constrained minimizers for (2.7) exist in the periodic wave setting, but they are unlikely to achieve a limiting state with bounded energy (the $H^{\alpha/2}$-norm) at the solitary wave limit.

For a broad range of dispersion operators and nonlinearities, including $\alpha \geq -1$ in (2.1), one is able to construct periodic traveling waves of (1.1) at least with small amplitudes via perturbation arguments such as the Lyapunov-Schmidt reduction. In the solitary wave setting, in stark contrast, Pohozaev identities techniques dictate that (2.7) ($a = 0$) in the range $\alpha \leq 1/3$ does not admit any nontrivial solutions in $H^{\alpha/2}(\mathbb{R}) \cap L^3(\mathbb{R})$.

### 3. Nondegeneracy of the Linearization

Throughout the section, let $u(\cdot; c, a)$ be a periodic traveling wave of (2.1), whose existence follows from Proposition 2.2. We shall demonstrate the nondegeneracy of the linearization associated with (2.7) at such a local constrained minimizer.

**Proposition 3.1** (Nondegeneracy). Let $1/3 < \alpha \leq 2$. If $u(\cdot; c, a) \in H^{\alpha/2}_{\text{per}}([0,T])$ for some $c \neq 0$, $a \in \mathbb{R}$ and for some $T > 0$ locally minimizes $H$ subject to that $P$ and $M$ are conserved then the associated linearized operator

\[
\delta^2 E(u; c, a) = \Lambda^\alpha - 2u + c
\]

acting on $L^2_{\text{per}}([0,T])$ is nondegenerate. That is to say,

\[
\ker(\delta^2 E(u; c, a)) = \text{span}\{u_x\}.
\]

The nondegeneracy of the linearization is of fundamental importance in the stability of traveling waves and the blowup for the related, time evolution equation; see [Wei87] [Lin08] [KMR11], among others. To establish the property is far from being trivial, though. Actually, one may cook up a polynomial nonlinearity, say, $f$, for which the kernel of $-\partial_x^2 - f'(u)$ at a periodic traveling wave $u$ is two dimensional at isolated points. It is usually imposed in terms of a spectral condition, although it may be proved in few special cases.

In the case of generalized KdV equations, for which $\alpha = 2$ in (2.1) but the nonlinearity is arbitrary, the nondegeneracy of the linearization at a periodic traveling wave was shown in [Joh09], for instance, to be equivalent to that the wave amplitude not be a critical point of the period, using the Sturm-Liouville theory for ODEs.

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\[\text{\footnotesize In the } L^2\text{-critical case, i.e. } \alpha = 1/2, \text{ periodic traveling waves with small energy tend to the solitary wave as their period increases to infinity. Their stability is, however, delicate and outside the scope of the present development. We refer the reader to [KMR11], for instance.}\]
It was similarly verified in [Kwo89], among others, at solitary waves in all dimensions. Amick and Toland [AT91] demonstrated the property for the Benjamin-Ono equation, both in the periodic and solitary wave settings, by relating the nonlocal, traveling wave equation to a fully nonlinear ODE via complex analysis techniques; unfortunately, the arguments are extremely specific to the Benjamin-Ono equation. Angulo Pava and Natali [APN08] made an alternative proof based upon the theory of totally positive operators, but it necessitates an explicit form of the solution. A satisfactory understanding of the nondegeneracy therefore seems largely missing for nonlocal equations. The main obstruction is that shooting arguments and other ODE methods, which seem crucial in the arguments for local equations, may not be applicable to nonlocal operators.

Nevertheless, Frank and Lenzmann [FL12] recently obtained the nondegeneracy of the linearization at solitary waves for a family of nonlinear nonlocal equations with fractional derivatives. Their idea is to find a suitable substitute for the Sturm-Liouville theory to estimate the number of sign changes in eigenfunctions for a fractional Laplacian with potential. Our proof of Proposition 3.1 follows along the same line as the arguments in [FL12, Section 3], but with appropriate modifications to accommodate the periodic nature of the problem.

Lemma 3.2 (Oscillation of eigenfunctions). Under the hypothesis of Proposition 3.1, an eigenfunction in $H^\alpha_{\text{per}}([0,T]) \cap C^{0}_{\text{per}}([0,T])$ corresponding to the $j$-th eigenvalue of $\delta^2 E(u)$ changes its sign at most $2(j - 1)$ times over the periodic interval $[0,T]$.

The regularity of eigenfunctions follows from the last part of Proposition 2.2; see also [FL12]. A thorough proof of Lemma 3.2 is found, for instance, in [FL12] in the solitary wave setting and in [BK04] in the case of $\alpha = 1$. Here we merely hit the main points.

Note that $\Lambda^\alpha$, $0 < \alpha < 2$, may be viewed as the Dirichlet-to-Neumann operator for an appropriate elliptic, boundary value problem set in the periodic half strip $[0,T]_{\text{per}} \times (0,\infty)$. Specifically (see [RS12, Theorem 1.1], for instance)

$$C(\alpha)\Lambda^\alpha u := \lim_{y \to 0^+} y^{1-\alpha} \phi_y(\cdot,y),$$

where $\phi$ solves

$$\Delta \phi + \frac{1-\alpha}{y} \phi_y = 0 \quad \text{in } [0,T]_{\text{per}} \times (0,\infty), \quad \phi = u \quad \text{on } [0,T]_{\text{per}} \times \{0\}$$

and $C(\alpha)$ is an explicit constant. Accordingly one may derive a variational characterization of (eigenvalues and) eigenfunctions of (3.1) in terms of the Dirichlet type functional

$$\int_{[0,T]_{\text{per}} \times (0,\infty)} |\nabla \phi(x,y)|^2 y^{1-\alpha} \, dx \, dy + \int_0^T (-2u(x) + c) |\phi(x,0)|^2 \, dx$$

in a suitable function class. Lemma 3.2 then follows from nodal domain bounds à la Courant.

Below we gather some facts about $\delta^2 E$.

Lemma 3.3 (Properties of $\delta^2 E$). Under the hypothesis of Proposition 3.1, the followings hold:

(L1) $u_x \in \ker(\delta^2 E(u))$ and it corresponds to the lowest eigenvalue of $\delta^2 E(u)$ restricted to the sector of odd functions in $L^2_{\text{per}}([0,T])$, denoted $L^2_{\text{per,odd}}([0,T])$;
implies that 

\[ L \]

\[ u \]

\[ \delta \]

\[ \delta \]

This implies by Courant’s min–max principle that 

\[ n \]

\[ E \]

\[ \delta \]

\[ \delta \]

Therefore 1, \( u \) locally minimizes \( H \), and hence \( E \), subject to conservations of \( P \) and \( M \), necessarily,

\[ \delta^2 E(u)|_{\{\delta P(u), \delta M(u)\}} \geq 0. \]

This implies by Courant’s mini–max principle that \( \delta^2 E(u) \) has at most two negative eigenvalues, implying (L2).

Lastly, differentiating \((2.7)\) with respect to \( c \) and \( a \), respectively, we use \((3.3)\) to obtain that 

\[ \delta^2 E(u)c = -\delta P(u) = -u \quad \text{and} \quad \delta^2 E(u)a = -\delta M(u) = -1. \]

Therefore 1, \( u \) \in \text{range}(\delta^2 E(u)). \)

\[ M_c(u(\cdot; c, a)) = \delta M(u), \quad u_c = (-\delta^2 E(u)u, u_c) \]

\[ = \langle u_c, -\delta^2 E(u)u_c \rangle = \langle u_c, \delta P(u) \rangle = P_a(u(\cdot; c, a)). \]

Since 

\[ \delta^2 E(u)u = \Lambda^u - 2u^2 + cu = -u^2 - a \]

by \((2.7)\), moreover, \( u^2 \in \text{range}(\delta^2 E(u)). \) \hfill \( \Box \)

**Remark 3.4** (Number of negative directions). Unlike in the solitary wave setting, where \( n_-(\delta^2 E) = 1 \) at a ground state, \( \delta^2 E \) may have up to two negative eigenvalues at a periodic, local constrained minimizer, which is characterized by

\[ n_-(\delta^2 E(u; c, a)) = \left[# c, \right. \left. (M_c(u(\cdot; c, a)), P_a(u(\cdot; c, a))) \right] \]

\[ = \# \text{ of sign changes in } 1, M_c, M_aP_c - M_cP_a, \]

providing that \( M_aP_c - M_cP_a \not= 0 \). A proof is found in [\text{BH13}], for instance.

**Proof of Proposition 3.1**. Consider the orthogonal decomposition 

\[ L^2_{\text{per}}([0, T]) = L^2_{\text{per, odd}}([0, T]) \oplus L^2_{\text{per, even}}([0, T]). \]

Since \( u \) may be chosen to be even by Proposition \( \text{[2.2]} \), it follows that \( L^2_{\text{per, odd}}([0, T]) \) and \( L^2_{\text{per, even}}([0, T]) \) are invariant subspaces of \( \delta^2 E(u) \). Since (L1) of Lemma \( \text{[3.3]} \) implies that

\[ \ker(\delta^2 E(u)|_{L^2_{\text{per, odd}}([0, T])}) = \text{span}\{u_x\}, \]
moreover, it remains to show that \( \ker(\delta^2 E(u)|_{L^2_{\text{per,even}}([0,T])}) = \{0\} \).

Suppose on the contrary that there were a nontrivial function \( \phi \in L^2_{\text{per,even}}([0,T]) \) such that \( \delta^2 E(u)\phi = 0 \). Since \( \delta^2 E(u) \) has at most two negative eigenvalues by (L2) of Lemma 3.3, it follows from Lemma 3.2 that \( \phi \) changes its sign at most twice over the half interval \([0,T/2] \). Consequently, unless \( \phi \) is positive (or negative) throughout the periodic interval \([0,T] \), either there exists \( T_1 \in (0,T/2) \) such that \( \phi \) is positive (or negative) for \( 0 < |x| < T_1 \) and negative (or positive, respectively) for \( x \in (-T/2, T_1) \cup (T_1, T/2) \), or there exist \( T_1 < T_2 \in [0,T/2] \) such that \( \phi \) is positive for \( |x| < T_1 \) and \( T_2 < |x| < T/2 \) (with the understanding that the first interval is empty in case \( T_1 = 0 \) and \( \phi \) is negative for \( x \in (-T_2, T_1) \cup (T_1, T_2) \).

Since \( \phi \) lies in the kernel of \( \delta^2 E(u) \), on the other hand, it must be orthogonal to \( \text{range}(\delta^2 E(u)) \) and, in turn, to the subspace \( \{1, u, u^2 \} \) by (L3) of Lemma 3.3. In particular, \( \langle \phi, 1 \rangle = 0 \), whence \( \phi \) cannot be positive (or negative) throughout \([0,T] \). In case \( \phi \) positive for \( 0 < |x| < T_1 \) and negative for \( T_1 < |x| < T/2 \), for instance, since \( u \) is symmetrically decreasing away from the origin over the interval \((-T/2,T/2) \), we find that
\[
u(x) - u(T_1) > 0 \quad \text{for} \quad |x| < T_1, \quad u(x) - u(T_1) < 0 \quad \text{for} \quad T_1 < |x| < T/2.
\]
Consequently \( \langle \phi, u - u(T_1) \rangle > 0 \), and \( \phi \) cannot be orthogonal to \( \{1, u \} \). In case \( \phi \) changes signs at \( x = \pm T_1 \) and \( x = \pm T_2 \), where \( T_1 < T_2 \), correspondingly, we find that \( (u - u(T_1))(u - u(T_2)) \) is positive in \((-T/2,-T_2) \cup (-T_1,T_1) \cup (T_1,T_2) \) and negative in \((-T_2,-T_1) \cup (T_1,T_2) \), deducing that \( \phi \) cannot be orthogonal to \( \{1, u, u^2 \} \).

A contradiction therefore asserts that \( \ker(\delta^2 E(u)|_{L^2_{\text{per,even}}([0,T])}) = \{0\} \). \( \square \)

**Remark 3.5** (Power-law nonlinearities). One may rerun the arguments in the proof of Proposition 3.1 to establish the nondegeneracy of the linearization at a periodic, local constrained minimizer for (2.19) in the case of \( 0 < \alpha \leq 2 \) and \( 0 \leq p < p_{\text{max}} \), where \( p_{\text{max}} \) is in (2.20), provided that
\[
u^{p+1} - \frac{w^{p+1}(T_1) - u^{p+1}(T_2)}{u(T_1) - u(T_2)} u + \frac{u(T_1)u(T_2)(w^p(T_1) - w^p(T_2))}{u(T_1) - u(T_2)}
\]
for \( T_1 < T_2 \in [0,T/2) \) changes signs at \( x = \pm T_1 \) and \( x = \pm T_2 \) but nowhere else over the interval \((-T/2,T/2) \). Indeed \( 1, u, w^{p+1} \in \text{range}(\delta^2 E(u)) \) in place of (L3) of Lemma 3.3 but otherwise the proof is identical to that in the case of the quadratic nonlinearity.

4. **Stability of constrained energy minimizers**

We turn the attention to the stability of a periodic, local constrained minimizer for (2.7) with respect to period preserving perturbations.

Recall from Section 2 that the initial value problem associated with (2.1) is well-posed in \( X \subset H^{\alpha/2}_{\text{per}}([0,T]) \) for some interval of time, where \( H, P, M : X \to \mathbb{R} \) are smooth. It suffices to take \( X = H^{\beta}_{\text{per}}([0,T]), \beta > 3/2 \).

Throughout the section let \( 1/3 < \alpha \leq 2 \), fixed, and let \( u_0(\cdot, c_0, a_0) \in H^{\alpha/2}_{\text{per}}([0,T]) \), locally minimize \( H \) subject to that \( P \) and \( M \) are conserved for some \( c_0 \neq 0, a_0 \in \mathbb{R} \) and for some \( T > 0 \). In light of Proposition 2.2, \( u_0 \in X \) and it makes a \( T \)-periodic, traveling wave of (2.1).

Notice that the evolution of (2.1) remains invariant under a one-parameter group of isometries corresponding to spatial translations. This motivates us to define the
group orbit of \( u \in X \) as
\[
O_u = \{ u(\cdot - x_0) : x_0 \in \mathbb{R} \}.
\]
Roughly speaking, \( u_0(\cdot ; c_0, a_0) \) is said orbitally stable if a solution of \( 2.1 \) remains close to \( O_{u_0} \) under the norm of \( X \) for all future times whenever the initial datum is sufficiently close to the group orbit of \( u_0 \) under the norm of \( X \). We shall elaborate this below in Theorem 4.1.

The present account of orbital stability is inspired by the Lyapunov method. Let
\[
E_0(u) = H(u) + c_0 P(u) + a_0 M(u),
\]
Proposition 2.2 implies that \( \delta E_0(u_0) = 0 \), i.e., \( u_0 \) is a critical point of \( E_0 \). Moreover, Proposition 3.1 implies that the kernel of \( \delta^2 E_0(u_0) \) is spanned by \( u_0 \). Accordingly \( u_0 \) is expected to be orbitally stable if \( E_0 \) is “convex” at \( u_0 \). As a matter of fact, one may easily verify that if the spectrum of \( \delta^2 E_0(u_0) \) except the simple eigenvalue at the origin were positive and bounded away from zero then \( u_0 \) would indeed be orbitally stable.

However, (1.2) of Lemma 3.3 indicates that \( \delta^2 E_0(u_0) \) admits one or two negative eigenvalues. In other words, \( u_0 \) is a nondegenerate saddle of \( E_0 \) on \( X \). Hence the Lyapunov method may not be directly applicable. In order to control potentially unstable directions and achieve stability, we observe that the evolution under \( 2.1 \) does not take place in the entire space \( X \), but rather on a smooth submanifold of codimension two, along which the momentum and the mass are conserved. Specifically let
\[
\Sigma_0 = \{ u \in X : P(u) = P_0, M(u) = M_0 \},
\]
where
\[
P_0 = P(u_0(\cdot ; c_0, a_0)), \quad M_0 = M(u_0(\cdot ; c_0, a_0)).
\]
Note that \( O_{u_0} \subset \Sigma_0 \) and a solution of \( 2.1 \) with initial datum in \( \Sigma_0 \) remains in \( \Sigma_0 \) at all future times. We shall demonstrate the “convexity” of \( E_0 \) on \( \Sigma_0 \).

**Theorem 4.1** (Orbital stability). Let \( 1/3 < \alpha \leq 2 \). If \( u_0(\cdot ; c_0, a_0) \in H_{per}^{\alpha/2}([0, T]) \) for some \( c_0 \neq 0, a_0 \in \mathbb{R} \) and for some \( T > 0 \) locally minimizes \( H \) subject to that \( P \) and \( M \) are conserved then for any \( \varepsilon > 0 \) sufficiently small there exists a constant \( C = C(\varepsilon) > 0 \) such that:

if \( \phi \in X \) and \( \| \phi \|_X \leq \varepsilon \) and if \( u(\cdot, t) \) is a solution of \( 2.1 \) for some interval of time with the initial condition \( u(\cdot, 0) := u_0 + \phi \in \Sigma_0 \) then \( u(\cdot, t) \) may be continued to a solution for all \( t > 0 \) such that
\[
\sup_{t > 0} \inf_{x_0 \in \mathbb{R}} \| u(\cdot, t) - u_0(\cdot - x_0) \|_X \leq C \| \phi \|_X.
\]

If in addition the matrix
\[
\begin{pmatrix}
M_a(u(\cdot ; c, a)) & P_a(u(\cdot ; c, a)) \\
M_c(u(\cdot ; c, a)) & P_c(u(\cdot ; c, a))
\end{pmatrix}
\]
is not singular at \( u_0(\cdot ; c_0, a_0) \) then the same conclusion holds for all \( \phi \in X \) such that \( \| \phi \|_X \leq \varepsilon \).

To interpret, a periodic, local constrained minimizer for \( 2.7 \) is orbitally stable with respect to nearby solutions with the same momentum and the same mass as the underlying wave, and with respect to arbitrary nearby solutions provided that
the constraint set $\Sigma_0$ is nondegenerate, i.e. (4.4) is not singular, at the underlying wave.

A solitary wave $u_0(\cdot; c_0)$ of (2.1) (not necessarily a ground state), in comparison, was shown in [GSS87], for instance, to be orbitally stable, provided that

$$\ker(\delta^2 E_0(u_0)) = \text{span}\{u_0\}, \quad n_-(\delta^2 E_0(u_0)) = 1, \quad P_c(u_0(\cdot; c_0)) > 0.$$

(Note that the assumption in [GSS87] that the symplectic form of a Hamiltonian system be onto is dispensable in the stability proof; see the remark directly following [GSS87, Theorem 2].) Conditions in (4.5) were, in turn, shown in [BSS87] (see also [SS90, Wei87]) to hold if and only if $\alpha > 1/2$. In the range $\alpha > 1/2$, incidentally, a solitary wave of (2.1) arises as an energy minimizer subject to constant momentum; see Remark 2.5. Theorem 4.1 may therefore be regarded to extend the well-known result about solitary waves.

An obvious approach toward Theorem 4.1 is to rerun the arguments in the proof in [GSS87] and derive stability criteria, analogous to (4.5); see [Joh09], for instance, where the last condition in (4.5) was suitably modified in the case of generalized KdV equations. As indicated in Remark 4.1 however, it is in general difficult to count the number of negative eigenvalues of $\delta^2 E_0$ in the periodic wave setting. We instead exploit variational properties of the equation — the underlying wave arises as a local constrained minimizer and the associated linearization is nondegenerate. Our proof of Theorem 4.1 does not require information about $n_-(\delta^2 E_0)$, apart from the upper bound in Lemma 3.3.

In the range $\alpha > 1/2$, recall from Remark 2.5 that periodic, local constrained minimizers for (2.7), which are orbitally stable by Theorem 4.1 are expected to tend to the solitary wave as the period increases to infinity, and the limiting solitary wave is orbitally stable, as well; see [BSS87], for instance. In the range $1/3 < \alpha < 1/2$, on the other hand, Theorem 4.1 indicates that orbitally stable, local constrained minimizers for (2.7) exist in the periodic wave setting, but they are unlikely achieve a limiting wave form with finite energy at the solitary wave limit.

As a key intermediate step, below we establish the coercivity of $E_0$ on $\Sigma_0$ in a neighborhood of the group orbit of $u_0$. We introduce the semi-distance $\rho : X \to \mathbb{R}$, defined by

$$\rho(u, v) = \inf_{x_0 \in \mathbb{R}} \| u - v(\cdot - x_0) \|_X$$

and rewrite (4.3) as $\sup_{t > 0} \rho(u(\cdot, t), u_0) \leq C \| u(\cdot, 0) - u_0 \|_X$.

**Lemma 4.2** (Coercivity) Under the hypothesis of Theorem 4.1 there exist $\varepsilon > 0$ and $C = C(\varepsilon) > 0$ such that if $u \in \Sigma_0$ with $\rho(u, u_0) < \varepsilon$ then

$$E_0(u) - E_0(u_0) \geq C \rho(u, u_0)^2.$$

**Proof.** The proof closely resembles that of [GSS87, Theorem 3.4] or [Joh09, Proposition 4.3]. Here we include the detail for completeness.

Throughout the proof and the following, $C$ means a positive generic constant; $C$ which appears in different places in the text needs not be the same.

---

§ A solitary wave corresponds to $a = 0$ and $T = +\infty$ in (2.1). Hence it depends, up to spatial translations, merely upon the wave speed.
Thanks to the implicit function theorem (see [BSS87], Lemma 4.1, for instance), for \(\varepsilon > 0\) sufficiently small and for an \(\varepsilon\)-neighborhood \(\mathcal{U}_\varepsilon := \{u \in X : \rho(u, u_0) < \varepsilon\}\) of \(O_{u_0}\), we find a unique \(C^1\) map \(\omega : \mathcal{U}_\varepsilon \to \mathbb{R}\) such that

\[
\omega(u_0) = 0 \quad \text{and} \quad \langle u(\cdot + \omega(u)), u_0 \rangle = 0
\]

for all \(u \in \mathcal{U}_\varepsilon\). Since \(E_0\) is invariant under spatial translations, it suffices to show (4.10) along \(u(\cdot + \omega(u))\). Since \(u_0\) locally minimizes \(H\), and hence \(E_0\), constrained to that \(P = P_0\) and \(M = M_0\), necessarily,

\[
\delta^2 E_0(u_0)\big|_{(\delta P(u_0), \delta M(u_0))} \geq 0,
\]

where \(\{\delta P(u_0), \delta M(u_0)\}\) is the tangent space in \(X\) to the sub-manifold \(\Sigma_0\) at \(u_0\). We fix \(u \in \mathcal{U}_\varepsilon \cap \Sigma_0\) and write

\[
(4.8) \quad u(\cdot + \omega(u)) = u_0 + C_1 \delta P(u_0) + \left( C_2 - C_1 \frac{\delta M(u_0)}{\delta M(u_0)} \right) \delta M(u_0) + y,
\]

where \(C_1, C_2 \in \mathbb{R}\) and \(y \in \{\delta P(u_0), \delta M(u_0), u_0\}\). Note that \(C_1 = C_2 = y = 0\) at \(u = u_0\).

Let \(\phi = u(\cdot + \omega(u)) - u_0\). We may assume that \(\|\phi\|_X < \varepsilon\) possibly after replacing \(u_0\) by \(u_0(\cdot - x_0)\) for some \(x_0 \in \mathbb{R}\). Since \(P\) and \(M\) remain invariant under spatial translations, Taylor’s theorem manifests that

\[
(4.9) \quad P(u) = P(u(\cdot + \omega(u))) = P(u_0) + \langle \delta P(u_0), \phi \rangle + O(\|\phi\|_X^2),
\]

\[
M(u) = M(u(\cdot + \omega(u))) = M(u_0) + \langle \delta M(u_0), \phi \rangle + O(\|\phi\|_X^2).
\]

Since \(\langle \delta M(u_0), \phi \rangle = C_2 \delta M(u_0), \delta M(u_0) = C_2 T\) by (4.13) and (2.5), we infer from the latter equation in (4.9) that \(C_2 = O(\|\phi\|_X^2)\). Similarly, since

\[
\langle \delta P(u_0), \phi \rangle = C_1 \left( \langle \delta P(u_0), \delta P(u_0) \rangle - \frac{\langle \delta M(u_0), \delta P(u_0) \rangle^2}{\delta M(u_0)} \right) + C_2 \langle \delta P(u_0), \delta M(u_0) \rangle
\]

\[
= C_1 \left( \|u_0\|_{L^2_{\rho_{\cdot + \omega(u)_0}}(\mathbb{T})}^2 \frac{M_0^2}{T} \right) - C_2 M_0,
\]

the Cauchy-Schwarz inequality, the former equation in (4.9) and (2.5) lead to that \(C_1 = O(\|\phi\|_X^2)\).

Since \(E_0\) remains invariant under spatial translations, furthermore, Taylor’s theorem manifests that

\[
E_0(u) = E_0(u(\cdot + \omega(u))) = E_0(u_0) + \frac{1}{2} \langle \delta^2 E_0(u_0) \phi, \phi \rangle + o(\|\phi\|_X^2).
\]

We then use (4.14) and \(C_1, C_2 = O(\|\phi\|_X^2)\) to find that

\[
E_0(u) - E_0(u_0) = \frac{1}{2} \langle \delta^2 E_0(u_0) \phi, \phi \rangle + o(\|\phi\|_X^2) = \frac{1}{2} \langle \delta^2 E_0(u_0) y, y \rangle + O(\|\phi\|_X^2).
\]

Since \(\text{ker}(\delta^2 E_0(u_0)) = \text{span}\{u_{0x}\}\) by Proposition 3.1, moreover, noting (4.17), it follows from a spectral decomposition argument that

\[
\inf \{ \langle \delta^2 E_0(u_0) v, v \rangle : \|v\|_X = 1, v \in \{\delta P(u_0), \delta M(u_0), u_{0x}\} \} > 0
\]

so that \(\langle \delta^2 E_0(u_0) y, y \rangle \geq C\|y\|_X^2\).

Finally a straightforward calculation reveals that

\[
\|y\|_X \geq \|\phi\|_X - \left| C_1 \delta P(u_0) + \left( C_2 - C_1 \frac{\delta M(u_0)}{\delta M(u_0)} \right) \delta M(u_0) \right|_X
\]

\[
\geq \|\phi\|_X - C\|\phi\|_X^2,
\]
whence
\[ E_0(u) - E_0(u_0) \geq C\|\phi\|_X^2 = C\|u(\cdot + \omega(u)) - u_0\|_X^2 \geq C\rho(u, u_0)^2. \]

Proof of Theorem 4.1. The proof resembles that of [Joh09, Lemma 4.1] in the case of generalized KdV equations.

Let \( \varepsilon > 0 \) be such that Lemma 4.2 holds and let \( \phi \in X \) satisfy \( \rho(u_0 + \phi, u_0) \leq \varepsilon \) for some \( 0 < \varepsilon < \varepsilon_0 \) sufficiently small. By replacing \( \phi \) by \( \phi(-x_0) \) for some \( x_0 \in \mathbb{R} \), if necessary, we may assume without loss of generality that \( \|\phi\|_X \leq \varepsilon \). Since \( u_0 \) is a critical point of \( E_0 \), then, Taylor’s theorem implies that \( E_0(u_0 + \phi) - E_0(u_0) \leq C\varepsilon^2 \).

Furthermore, notice that if \( u_0 + \phi \in \Sigma_0 \) then the unique solution \( u(\cdot, t) \) of (2.1) with the initial condition \( u(\cdot, 0) = u_0 + \phi \) must lie in \( \Sigma_0 \) as long as the solution exists. Since \( E_0(u(\cdot, t)) = E_0(u(\cdot, 0)) = E_0(u_0 + \phi) \) independently of \( t \), on the other hand, Lemma 4.2 implies that \( \rho(u(\cdot, t), u_0)^2 \leq C\varepsilon^2 \) for all \( t \geq 0 \). This proves the first part of Theorem 4.1.

In case \( u_0 + \phi \) is not required to be in \( \Sigma_0 \) but the matrix (4.4) is not singular at \( u_0(c_0, a_0) \), we utilize the nondegeneracy of the constraint set. That is, the mapping
\[
(c, a) \mapsto (P(u(\cdot; c, a)), M(u(\cdot; c, a)))
\]
is a period-preserving diffeomorphism from a neighborhood of \((c_0, a_0)\) onto a neighborhood of \((P_0, M_0)\). We may therefore find \( c, a \in \mathbb{R} \) such that \( |c| + |a| = O(\varepsilon) \) and \( u_\varepsilon(\cdot; c_0 + c, a_0 + a) \) is a \( T \)-periodic traveling wave of (2.1) satisfying that \( P(u_\varepsilon(\cdot; c_0 + c, a_0 + a)) = P(u_0 + \phi) \) and \( M(u_\varepsilon(\cdot; c_0 + c, a_0 + a)) = M(u_0 + \phi) \).

Let
\[ E_\varepsilon(u) = E_0(u) + cP(u) + aM(u). \]

We may furthermore assume that \( u_\varepsilon \) minimizes \( E_\varepsilon \) subject to that \( P \) and \( M \) are conserved. We then rerun the argument in the proof of Lemma 4.2 and show that
\[ E_\varepsilon(u) - E_\varepsilon(u_\varepsilon) \geq C\rho(u, u_\varepsilon)^2 \]
so long as \( \rho(u, u_\varepsilon) \) is sufficiently small. Since \( u_\varepsilon \) is a critical point of \( E_\varepsilon \), moreover, \( E_\varepsilon(u(\cdot, t)) - E_\varepsilon(u_\varepsilon) = E_\varepsilon(u_0 + \phi) - E_\varepsilon(u_\varepsilon) \leq C\varepsilon^2 \) for all \( t \geq 0 \). Finally the triangle inequality implies that
\[
\rho(u(\cdot, t), u_0)^2 \leq C \left( \rho(u(\cdot, t), u_\varepsilon)^2 + \rho(u_\varepsilon, u_0)^2 \right) \\
\leq C \left( E_\varepsilon(u(\cdot, t)) - E_\varepsilon(u_\varepsilon) \right) + \|u_\varepsilon - u_0\|_X \leq C\varepsilon^2
\]
for all \( t \geq 0 \). In other words, \( u_0(\cdot; c_0, a_0) \) is orbitally stable to small perturbations that “slightly” change \( P \) and \( M \). \( \square \)

One may rerun the arguments in the proof of Theorem 4.1 mutatis mutandis to establish the orbital stability of a nondegenerate periodic, local constrained minimizer for (2.19) in the range \( 0 < \alpha \leq 2 \) and \( 0 < p < p_{\text{max}} \), where \( p_{\text{max}} \) is in (2.20).
5. Adaptation to equations of regularized long wave type

The results in Section 2 through Section 4 are readily adapted to other, nonlinear dispersive equations. We shall illustrate this by discussing equations of regularized long wave (RLW) type

\[ u_t - u_x + \Lambda^\alpha u_t + (u^2)_x = 0, \]

where \( 0 < \alpha \leq 2 \).

In the case of \( \alpha = 2 \), notably, (5.1) recovers the Benjamin-Bona-Mahony (BBM) equation, which was advocated in [BBM72] as an alternative to the KdV equation; in particular solutions of the initial value problem associated with the BBM equation were argued to enjoy better smoothness properties than those with the KdV equation, and hence it was named the regularized long wave equation. For other values of \( \alpha \), (5.1) similarly “regularizes” its KdV counterpart in (2.1). In a certain, weakly nonlinear and long wavelengths approximation, for which \( u_x + u_t = o(1) \), it is formally equivalent to (2.1).

The present treatment may extend mutatis mutandis to general power-law nonlinearities (see Remark 2.4), but we choose to work with the quadratic nonlinearity to simplify the exposition.

In the range \( \alpha \geq 0 \), one may work out the local in time well-posedness for (5.1) in \( H^\beta_{per}([0, T]) \), \( \beta > \max(0, (3 - \alpha)/2) \), via the usual energy method, corroborating that (5.1) regularizes (2.1); see Remark 2.1. The proof is rudimentary. Hence we omit the detail. With the help of the smoothing effects of \( (1 - \partial_x^2)^{-1} \), the global in time well-posedness for (5.1) may be established in \( H^{2+}(\mathbb{R}) \) in the case of \( \alpha = 2 \), namely the BBM equation; see [BT09], for instance. In the periodic functions setting, on the other hand, the existence matter for (5.1) seems not properly understood in spaces of low regularities. For the present purpose, however, a short-time well-posedness result suffices.

Throughout the section we’ll work in an appropriate subspace (abusing notation) \( X \) of \( H^\beta_{per}([0, T]) \), where the initial value problem associated with (5.1) is well-posed for some interval of time; \( T > 0 \), the period, is arbitrary but fixed.

Notice that (5.1) possesses three conserved quantities (abusing notation)

\[ H(u) = \int_0^T \left( \frac{1}{2} u^2 - \frac{1}{3} u^3 \right) \, dx \]

\[ P(u) = \int_0^T \frac{1}{2} (u^2 + |\Lambda^{\alpha/2} u|^2) \, dx, \]

\[ M(u) = \int_0^T u \, dx, \]

which correspond to the Hamiltonian and the momentum, the mass, respectively. Throughout the section we shall use \( H \) and \( P, M \) for those in (5.2) and (5.3), (5.4). Notice that \( H, P, M : X \to \mathbb{R} \) are smooth and invariant under spatial translations. Incidentally (5.1) is written in the Hamiltonian form

\[ u_t = J \delta H(u), \]

for which \( J = (1 + \Lambda^\alpha)^{-1} \partial_x \). Recall that \( \delta \) denotes variational differentiation.
We seek a periodic traveling wave \( u(x, t) = u(x - ct - x_0) \) of (5.1), where \( c \in \mathbb{R} \), \( x_0 \in \mathbb{R} \) and \( u \) is \( T \)-periodic, satisfying by quadrature
\[
c(1 + \Lambda^\alpha)u + u - u^2 + a = 0
\]
for some \( a \in \mathbb{R} \), or equivalently (abusing notation)
\[
\delta E(u; c, a) := \delta(H(u) + cP(u) + aM(u)) = 0.
\]
We’ll write a periodic traveling wave of (5.1) as \( u = u(\cdot; c, a) \) with the understanding that \( T > 0 \) is arbitrary but fixed and that we may mod out \( x_0 \). Below we record the existence, symmetry, and regularity properties for a family of periodic traveling waves of (5.1), which arise local energy minimizers subject to conservations of the momentum and the mass.

**Lemma 5.1** (Existence, symmetry and regularity). Let \( 1/3 < \alpha \leq 2 \).

A local minimizer \( u \) in \( H^{\alpha/2}_{\text{per}}([0, T]) \) for \( H \), defined in (5.2), subject to that \( P \) and \( M \), defined in (5.3) and (5.4), respectively, are conserved exists for each \( 0 < T < \infty \) and it satisfies (5.5) for some \( c \neq 0 \) and \( a \in \mathbb{R} \). It depends upon \( c \) and \( a \) in the \( C^1 \) manner.

Moreover \( u = u(\cdot; c, a) \) may be chosen to be even and strictly decreasing over the interval \([0, T/2]\), and \( u \in H^\infty_{\text{per}}([0, T]) \).

Proof. If \( u(x; c, a) \) is \( T \)-periodic and satisfies (5.5) for some \( c \neq 0 \) and \( a \in \mathbb{R} \) then, by a scaling argument,
\[
(c + 1)u\left(\left(\frac{c + 1}{c}\right)^{1/\alpha} x; 1, c^2 a\right)
\]
is \( \left(\frac{c + 1}{c}\right)^{1/\alpha} \) \( T \)-periodic and satisfies
\[
(1 + \Lambda^\alpha) u + u - u^2 + c^2 a = 0.\]
Therefore it suffices to take \( c = 1 \) in (5.5), which brings us to (2.7) with \( c = 2 \). The proof is therefore identical to that of Proposition 2.2. Hence we omit the detail. \( \square \)

We promptly address the nondegeneracy of the linearization associated with (5.6) at a periodic, local constrained minimizer for (5.5).

**Lemma 5.2** (Nondegeneracy). Let \( 1/3 < \alpha \leq 2 \). If \( u(\cdot; c, a) \in H^{\alpha/2}_{\text{per}}([0, T]) \) for some \( c \neq 0 \), \( a \in \mathbb{R} \) and for some \( T > 0 \) locally minimizes \( H \) subject to that \( P \) and \( M \) are conserved then
\[
\delta^2 E(u; c, a) = c(1 + \Lambda^\alpha) + 1 - 2u,
\]
acting on \( L^2_{\text{per}}([0, T]) \), satisfies that
\[
\ker(\delta^2 E(u; c, a)) = \text{span}\{u_x\}.
\]

Proof. Notice that Lemma 5.2 holds for \( \delta^2 E(u) \) in (5.7); see Section 3 or [FL12, Section 3] for the detail. Notice moreover that (L1) and (L2) of Lemma 5.3 hold for (5.7), see Section 3 for the detail.

We claim that (L3) of Lemma 5.3 holds for \( \delta^2 E(u) \) in (5.7). Differentiating (5.6) with respect to \( c \) and \( a \), respectively, we obtain that
\[
\delta^2 E(u) u_c = -\delta P(u) \quad \text{and} \quad \delta^2 E(u) u_a = -\delta M(u).
\]
Since
\[ \delta P(u) = (1 + \Lambda^\alpha)u \quad \text{and} \quad \delta M(u) = 1 \]
moreover, \((1 + \Lambda^\alpha)u \in \text{range}(\delta^2 E(u))\). Unfortunately \((1 + \Lambda^\alpha)u\) may not be strictly monotone over \([0, T/2]\). Hence one cannot use it in the arguments in the proof of Proposition 5.1 Appealing to \((5.3)\), on the other hand, we find that
\[ c(1 + \Lambda^\alpha)u = u^2 - u - a. \]
Therefore \(u^2 - u \in \text{range}(\delta^2 E)\). Since
\[ \delta^2 E(u)u = c(1 + \Lambda^\alpha)u + u - 2u^2 = -u^2 - a, \]
furthermore, \(u, u^2 \in \text{range}(\delta^2 E)\). This proves the claim. The proof is then identical to that of Proposition 3.1 We omit the detail. \(\square\)

Repeating the arguments in the proof of Theorem 4.1 we ultimately establish the orbital stability of a periodic, local constrained minimizer for \((5.5)\). We summarize the conclusion.

**Theorem 5.3 (Orbital stability).** Let \(1/3 < \alpha \leq 2\) and \(u_0(\cdot; c_0, a_0) \in H^{\alpha/2}_{per}(\mathbb{R}, \mathbb{R})\) locally minimize \(H_\alpha\), defined in \((5.2)\), subject to that \(P\) and \(M\), in \((5.3)\) and \((5.4)\), respectively, are conserved for some \(c_0 \neq 0, a_0 \in \mathbb{R}\) and for some \(T > 0\). Then for any \(\varepsilon > 0\) sufficiently small there exists a constant \(C = C(\varepsilon) > 0\) such that:

- if \(\phi \in X\) and \(\|\phi\|_{X} \leq \varepsilon\) and if \(u(\cdot, t)\) is a solution of \((5.1)\) for some time interval with the initial condition \(u(\cdot, 0) = u_0 + \phi \in X\) satisfying
  \[ P(u_0 + \phi) = P(u_0) \quad \text{and} \quad M(u_0 + \phi) = M(u_0), \]
  then \(u(\cdot, t)\) may be continued to a solution for all \(t > 0\) such that
  \[ \sup_{t > 0} \inf_{x_0 \in \mathbb{R}} \|u(\cdot, t) - u_0(\cdot, x_0)\|_{X} \leq C\|\phi\|_{X}. \]

If in addition the matrix
\[ \begin{pmatrix} M_\alpha(u(\cdot; c, a)) & P_\alpha(u(\cdot; c, a)) \\ M_\alpha(u(\cdot; c, a)) & P_\alpha(u(\cdot; c, a)) \end{pmatrix} \]
is not singular at \(u_0(\cdot; c_0, a_0)\) then the same conclusion holds for all \(\phi \in X\) such that \(\|\phi\|_{X} \leq \varepsilon\).

Related stability results in the case of \(\alpha = 1, 2\) are found, respectively, in [ASB11] and [Joh10] among others.

### 6. Remark on linear instability

We complement the nonlinear stability result in Section 4 by discussing the linear instability of periodic traveling waves of the KdV type equation
\[ u_t - \mathcal{M}u_x + f(u)_x = 0. \]
Here \(\mathcal{M}\) is a Fourier multiplier defined as \(\widehat{\mathcal{M}}u(\xi) = m(\xi)\hat{u}(\xi)\), satisfying that
\[ C_1|\xi|^\alpha \leq m(\xi) \leq C_2|\xi|^\alpha, \quad |\xi| \gg 1 \]
for some \(\alpha \geq 1\) and for some \(C_1, C_2 > 0\), while \(f : \mathbb{R} \to \mathbb{R}\) is \(C^1\), satisfying that
\[ f(0) = f'(0) = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{f(u)}{u} = \infty. \]
Clearly \((2.1)\) satisfies \((6.1)\) and \((6.2)\).
We assume that (1.1) possesses two conserved quantities

\[ P(u) = \int_0^T \frac{1}{2} u^2 \, dx \]

and

\[ M(u) = \int_0^T u \, dx, \]

interpreted as the momentum and the mass, respectively.

We assume that (1.1) supports a smooth, four-parameter family of periodic traveling waves, denoted \( u = u(\cdot - x_0; c, a, T) \), where \( c \) and \( a \) form an open set in \( \mathbb{R}^2 \), \( x_0 \in \mathbb{R} \) is arbitrary (and hence we may mod it out), \( T_0 < T < \infty \) for some \( T_0 > 0 \), and where \( u \) is \( T \)-periodic, satisfying by quadrature that

\[ (6.3) \quad Mu - f(u) + cu + a = 0. \]

For a broad range of dispersion symbols and nonlinearities, the existence of periodic traveling waves of (1.1) may follow from variational arguments, e.g., the mountain pass theorem applied to a suitable functional whose critical point satisfies (6.3).

Linearizing (1.1) about a (nontrivial) periodic traveling wave \( u = u(\cdot; c, a, T) \) in the frame of reference moving at the speed \( c \), we arrive at that

\[ (6.4) \quad v_t = \partial_x (M - f'(u) + c)v =: \partial_x \mathcal{L}(u; c, a)v. \]

Seeking solutions of the form \( v(x, t) = e^{\mu t}v(x) \), moreover, we arrive at the spectral problem

\[ (6.5) \quad \mu v = \partial_x \mathcal{L}(u; c, a)v. \]

We then say that \( u \) is linearly unstable if the \( L^2_{\text{per}}([0, T]) \)-spectrum of \( \mathcal{L}(u) \) intersects the open, right half plane of \( \mathbb{C} \).

We shall derive a criterion governing linear instability of periodic traveling waves of (1.1), which do not necessarily arise as local constrained minimizers. In light of Theorem 4.1, a local constrained minimizer for (6.3) is expected to be nonlinearly stable under the flow induced by (1.1) under certain assumptions.

**Theorem 6.1** (Linear instability). Under the assumptions (6.1) and (6.2), let \( u = u(\cdot; c, a, T) \) be a nontrivial, periodic traveling wave of (1.1) for some \( c \neq 0 \), \( a \in \mathbb{R} \) and for some \( T > T_0 > 0 \). Let \( \Pi : L^2_{\text{per}}([0, T]) \to L^2_{\text{per}}([0, T]) \) denote the orthogonal projection onto the subspace of \( L^2_{\text{per}}([0, T]) \) of mean zero functions, defined by

\[ \Pi u = u - \frac{1}{T} \int_0^T u(x) \, dx. \]

Assume that \( \Pi \mathcal{L}(u; c, a) \Pi \), acting on \( L^2_{\text{per}}([0, T]) \), is nondegenerate; that is to say,

\[ (6.6) \quad \ker(\Pi \mathcal{L}(u; c, a) \Pi) = \operatorname{span}\{u_x\}. \]

Then (6.4) admits a nontrivial solution of the form \( e^{\mu t}v(x) \), \( v \in H^\infty_{\text{per}}([0, T]) \) and \( \mu > 0 \), if either

1. \( n_-(\Pi \mathcal{L}(u; c, a) \Pi) \) is odd and \( P_c(u(\cdot; c, a, T)) < 0 \), or
2. \( n_-(\Pi \mathcal{L}(u; c, a) \Pi) \) is even and \( P_c(u(\cdot; c, a, T)) > 0 \).
Recall that \( n_-(\Pi L(u; c, a)\Pi) \) is the number of negative eigenvalues of \( \Pi L(u; c, a)\Pi \) acting on \( L^2_{per}([0, T]) \), or equivalently, of \( \Pi L(u; c, a) \) acting on \( \Pi L^2_{per}([0, T]) \).

A thorough proof of Theorem 6.1 is found in [Lin08, Section 4] in the solitary wave setting and in [APBSO13] for a regularized Boussinesq equation. The arguments in [Lin08] readily extend to the periodic wave setting. Here we merely hit the main points.

Notice that (6.5) has a nontrivial solution in \( H^\alpha_{per}([0, T]) \) for some \( \mu > 0 \), namely a purely growing mode, if and only if

\[
A^\mu := c - \frac{c\partial_x}{\mu - c\partial_x}(\mathcal{M} - f'(u))
\]

has a nontrivial kernel in \( H^\alpha_{per}([0, T]) \). Since

\[
\frac{c\partial_x}{\mu - c\partial_x} \to 0 \quad \text{as} \quad \mu \to +\infty
\]

while

\[
\frac{c\partial_x}{\mu - c\partial_x} \to \Pi \quad \text{as} \quad \mu \to 0^+
\]

strongly in \( L^2_{per}([0, T]) \) (see [Lin08] for the detail), the spectra of \( A^\mu \) lie in the right half plane of \( \mathbb{C} \) for \( \mu > 0 \) sufficiently large while \( A^\mu \) converges to \( \Pi L(u)\Pi \) strongly in \( L^2_{per}([0, T]) \) as \( \mu \to 0^+ \). We then examine eigenvalues of \( A^\mu \) near the origin in the left half plane of \( \mathbb{C} \) from those of \( \Pi L(u) \) via the moving kernel method. Specifically, (6.6) ensures that for \( \mu > 0 \) sufficiently small a unique eigenvalue \( e_\mu \) of \( A^\mu \) exists in the vicinity of the origin that depends upon \( \mu \) analytically. A lengthy but explicit calculation moreover reveals that

\[
\lim_{\mu \to 0^+} \frac{e_\mu}{\mu} = 0 \quad \text{and} \quad \lim_{\mu \to 0^+} \frac{e_\mu}{\mu^2} = -P_c(u(c; c, a, T)).
\]

Theorem 6.1 therefore follows since if \( A^\mu \) admits an odd number of eigenvalues in the left half plane of \( \mathbb{C} \), signaling that the spectrum of \( A^\mu \) crosses the origin at some \( \mu > 0 \), then a purely growing mode is found.

To conclude, we contrast Theorem 6.1 with Theorem 4.1 as it applies to (2.1) near the large period asymptotics. It is not immediately obvious how they complement each other since Theorem 4.1 is variational in nature whereas Theorem 6.1 is stated using spectral information of the associated linearized operator.

Below we relate spectral properties of \( \Pi L(u)\Pi \) to those of \( L(u) \).

**Lemma 6.2.** Let \( 1/3 < \alpha \leq 2 \). If \( L(u) := L(u; c, a) \) is the linearized operator, in (6.3), associated with (2.1), so that it agrees with \( \delta^2 E(u; c, a) \) in (3.1), then

\[
n_-(\Pi L(u)\Pi) = n_-(L(u)) - \begin{cases} 1 & \text{if } M_a \geq 0, \\ 0 & \text{if } M_a < 0. \end{cases}
\]

Moreover

\[
\dim(ker(\Pi L(u)\Pi)) = \dim(ker(L(u))) + \begin{cases} 1 & \text{if } M_a \geq 0, \\ 0 & \text{if } M_a < 0. \end{cases}
\]
Various forms of the above "index formula" are known in the nonlinear wave community; see [KP12, CPV05, GSS87, BH13] among others. Lemma 6.2 follows upon an application of [KP12, Theorem 2.1], for instance, noting that $1 \in \ker(L(u))$ and $(L(u))^{-1} = -u_a$ by Proposition 3.1.

We recall from Remark 2.5 that in the range $\alpha > 1/2$ periodic traveling waves of (2.1), constructed in Proposition 2.2 as local constrained minimizers, tend to the solitary wave as $a \to 0$ and $T \to +\infty$ satisfying $aT \to 0$, namely in the solitary wave limit, which minimizes the Hamiltonian subject to constant momentum. Thanks to scaling invariance (2.6), on the other hand, (2.7) remains invariant under

$$u(\cdot; c, a, T) \mapsto c^{-1}u(\cdot; 1, c^{-2}a, c^{-1/\alpha}T).$$

Accordingly we may take without loss of generality $c = 1$ and we find that

$$P(1, a, T), M(1, a, T), P_c(1, a, T), M_c(1, a, T) = O(1)$$

for $|a|$ sufficiently small and $T > 0$ sufficiently large; see [BH13, Lemma 3.10] for the detail. Differentiating (2.10) with respect to $a$ and evaluating near the solitary wave limit, we invoke (3.3) to obtain that

$$M_a = -T + 2M_c = -T + O(1) < 0$$

for $|a|$ sufficiently small and $T > 0$ sufficiently large. Lemma 6.2 therefore implies that $\ker(\mathcal{L}(u)) = \text{span}\{u_x\}$ for $|a|$ sufficiently small and for $T > 0$ sufficiently large. In other words, (6.6) holds near the solitary wave limit. Since an explicit calculation dictates that $P_c(u(\cdot; 1, a, T)) > 0$, furthermore,

$$M_aP_c - M_cP_a = M_aP_c - M_c^2 = -P_cT + O(1) < 0$$

for $|a|$ sufficiently small and $T > 0$ sufficiently large. Lemma 6.2 therefore implies in view of (3.4) that

$$n_-(\mathcal{L}(u)) = n_-(\mathcal{L}(u)) = 1$$

near the solitary wave limit.

Theorem 6.1 is thus inconclusive of local constrained minimizers for (2.7), in the $L^2$-subcritical case, i.e., $\alpha > 1/2$, near the solitary wave limit, which is consistent with the result in Theorem 4.1. Indeed one may appeal to [GSS87], for instance, to argue for that local constrained minimizers for (2.7) with large periods and small $a$’s are, in the range $\alpha > 1/2$, orbitally stable under the flow induced by (2.1).

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