Convex Polytopes, Dihedral Angles, Mean Curvature and Scalar Curvature

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Abstract
We approximate boundaries of convex polytopes $X \subset \mathbb{R}^n$ by smooth hypersurfaces $Y = Y_\epsilon$ with positive mean curvatures and, by using basic geometric relations between the scalar curvatures of Riemannian manifolds and the mean curvatures of their boundaries, establish lower bound on the dihedral angles of $X$.

Contents
1 Combinatorial Spread, $\Box_\gamma$-Spread and $\Box_\gamma$-Inequality .................................................. 1
   1.1 Combinatorially Large Polytopes with Large Complementary Angles ........................................... 4
2 Manifolds with Corners, Mean Convexity and Distance $\text{dist}^b_\gamma$ ........................................... 8
   2.1 $Sc$-Normalized Metric $g^s$, Derivation of 2.A from the LGSL Theorem and $\gamma$-Capillary Problem ........................................... 11
3 Rounding the Corners and $\text{dist}^b_\gamma$-Approximation of Simple Polytopes ..................................... 13
4 Locally Conical Hypersurfaces and the $\text{dist}^b_\gamma$-Approximation Theorem for Non-simple $X$ ........... 15
5 Lipschitz Maps and the Proof of Theorems 1.A. and 2.C. ................................................................. 18
6 Combinatorial Waists and $\gamma^{n-k-1}$-Angles ..................................................................................... 19
7 Surgery with Corners and Related Problems ...................................................................................... 21
8 On Random Polytopes ......................................................................................................................... 22
9 References ............................................................................................................................................. 23

1 Combinatorial Spread, $\Box_\gamma$-Spread and $\Box_\gamma$-Inequality

Let $X \subset \mathbb{R}^n$ be a compact convex polytope let $\partial X$ denote its (topologically spherical) boundary and let $X^\circ$ be the dual convex tessellation of the sphere
\( S^{n-1} \), i.e. where \((n-k-1)\)-cells are the sets of the (unit normal to the) supporting hyperplanes to \( X \) along the interiors of the \( k \)-faces of \( X \).

Let \( E = E(X) \subset S^{n-1} \) be the edge graph of \( X^\circ \). Combinatorially, this is the \((n-2)\)-adjacency graph, where the set of the \((n-1)\)-faces \( F \) of \( X \) is taken for the set vertices and where the edges \( e \) in \( E \) correspond to the pairs of \((n-2)\)-adjacent faces:

vertices \( v_1 \) and \( v_2 \) are joined by an edge \( e = e_{12} \), whenever the corresponding closed faces \( F_1, F_2 \subset X \) meet over a closed \((n-2)\)-face, namely \( F_1 \cap F_2 \subset X \).

Remark on Adjacency and on Simple Polytopes. Recall that a convex polytope \( X \) is simple if

\[
\text{adjacent} \iff (n-2)\text{-adjacent},
\]

where "adjacent" signifies that the intersection \( F_1 \cap F_2 \) is non-empty, i.e. \( F_1 \) and \( F_2 \) meet at a vertex in \( X \).

The combinatorial distance \( \text{dist}_{\text{comb}}(F_1, F_2) \) is the length of the shortest path in \( E \) between the corresponding vertices corresponding to \( F_1 \) and \( F_2 \).

For instance, these distances between opposite faces in the \( n \)-cube \([-1,1]^n\) are equal to 2.

Let \( \varangle_{1,2} = \varangle(F_1, F_2) \) denote the dihedral angle between \((n-2)\)-adjacent faces and let \( \varangle \) stands for the complementary angle,

\[
\varangle_{1,2} = \pi - \varangle_{1,2},
\]

that is the spherical arc length of the edge \( e_{12} \subset S^{n-1} \) dual to the \((n-2)\)-face \( F_{12} = F_1 \cap F_2 \).

\( \varangle \)-Angular Distance. The angular distance or \( \varangle \)-distance \( \text{dist}_\varangle(F_1, F_2) \) between (not necessarily \((n-2)\)-adjacent) \((n-1)\)-faces \( F_1 \) and \( F_2 \) in \( X \) is the minimum of the spherical lengths of edge paths in \( E \) between the vertices of \( X^\circ \) dual to these faces.

Accordingly, the \( \varangle \)- (angular) distance between (unions of) sets of faces, say \( \mathcal{F}_1, \mathcal{F}_2 \subset V \), is the minimum of the \( \varangle \)-distances between the faces in these sets,

\[
\text{dist}_\varangle(\mathcal{F}_1, \mathcal{F}_2) = \min_{F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2} \text{dist}_\varangle(F_1, F_2).
\]

Cubical Example. The \( \varangle \)-distances between opposite faces of the \( n \)-cube \( \square^n = [-1,1]^n \) are equal to \( \pi \).

Combinatorial and Angular Spreads. Let \( \square^k_{\text{comb}}(X) \) be the maximum of the numbers \( d \geq 0 \), such that \( X \) admits a continuous map to the \( k \)-cube,

\[
\Phi : X \to \square^k = [-1,1]^k
\]

with the following properties\(^1\):

- \( \Phi_{\text{comb}} \) The \( \Phi \)-pullbacks of the \((k-1)\)-faces from \( \square^k \) are unions of \((n-1)\)-faces in \( X \).

- \( \Phi_{\text{dist}} \) The combinatorial distances between the pullbacks of the opposite cubical faces \( F_i^+ \subset \square^k \) are \( \geq d \),

\[
\text{dist}_{\text{comb}}(\Phi(F_i^+), \Phi(F_i^-)) \geq d, \quad i = 1, \ldots, k.
\]

\(^1\)If no such map exists, then let \( \square^k_{\text{comb}}(X) = 0 \).
The induced relative homology homomorphism
\[ \Phi_* : H_k(X, \Phi^{-1}(\partial \Box^n)) \to H_k(\Box^n, \partial \Box^n) = \mathbb{Z} \]
doesn’t vanish.

(If \( k = n \), this is equivalent to \( \Phi^{-1}(\partial \Box^n) = \partial X \) and to non-vanishing of the degree of the map \( \Phi : \partial X \to \partial \Box^n \). For instance, homeomorphisms \( \Phi : X \to \Box^n \) satisfy this condition.)

Similarly define the angular spread \( \Box^n_\ell(X) \) with the \( \text{dist}_\ell \) inequality instead of \( \text{dist}_{\text{comb}} \):
\[ \text{dist}_\ell(\Phi(E_{i,\ell}), \Phi(E_{i,\ell})) \geq d \quad i = 1, \ldots, k. \]

Observe that the combinatorial and the angular spreads satisfy
\[ \Box^n(X) \leq \Box^{n-1}(X) \leq \ldots \leq \Box^1(X) = \text{diam}(X), \]
where the diameter refers to the combinatorial and to the angular distances correspondingly,
and that
\[ \Box^n(X) \geq \frac{\Box^{n-1}(X)}{2n+2} \]
for all convex \( n \)-polytopes \( X \).

\( \Box^3_\ell \)-Example. Let \( \Box^3_N \) be the subdivision of the 3-cube \( \Box^3 = \Box^3_1 \), where each 2-face is subdivided into \( N^2 \) equal squares in an obvious way. (If you wish it to became simple, \( \varepsilon \)-perturb with \( \varepsilon \ll 1/N \) the edges of) these small squares, such that the resulting subdivision \( \Box^3_{N,\varepsilon} \) has three squares at each vertex.) Then the combinatorial \( \Box^3 \)-spread of the so subdivided cube is \( N + 1 \).

"Random" Example. Apparently, the combinatorial \( \Box \)-spread of a suitably defined random \( n \)-polytope with \( M \) faces (see section 8) grows, roughly, as \( n \sqrt{n} \).

1.A. Angular Spread Theorem. The top-dimensional \( \{ \} \)-spreads, of all compact convex \( n \)-polytopes \( X \subset \mathbb{R}^n \) are bounded by a universal constant,
\[ \Box^n(X) \leq D = D_n \leq 2(n-1)\sqrt{n}. \]

We shall proof this in section 5 by reduction to the normalized mean curvature mapping theorem (see section 2) the proof of which (see section 2.1) depends on the index theory for Dirac operators on Riemannian spin manifolds with positive scalar curvatures (see sections 3.1.2 and 3.5 in [Gr2021]).

1.B. Corollary. The minimum of the complementary angles of \( X \) is bounded by the combinatorial spread \( \Box^n_{\text{comb}}(X) \) as follows,
\[ \alpha_{\text{min}}(X) \leq D \frac{1}{\Box^n_{\text{comb}}(X)}. \]

for the above constant \( D \).

\(^2\)The topological degree is defined for all continuous equidimensional maps \( f \) between oriented manifolds, e.g. such as our spherical \( \partial X \) and \( \partial \Box^n \), where the non-vanishing condition \( \text{deg}(f) \neq 0 \) doesn’t depend on the orientation for connected (orientable) manifolds. Also the degree is defined for the boundary respecting maps between manifolds with boundaries.
1.C. Conjecture. The above \( D = D_n \) is equal to \( \pi/4 \)

Remark. Probably 1.A, 1.B and 1.C generalize to all convex tessellation of \( S^{n-1} \) (See next section for more about it.)

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1.1 Combinatorially Large Polytopes with Large Complementary Angles

Dirac operators notwithstanding, evaluation of the ranges of possible values of the dihedral angles of polytopes depending on their combinatorial types and/or determination of the combinatorial and metric geometries of polytopes with all complementary angles \( \{X\} \) bounded from below remains problematic.

It is known here (Steinitz?) that if \( \min \{X\} \geq \pi/4 \), then \( X \) is the product of simplices. But – this was pointed out to me by Karim Adiprasito three years ago – there is no bound on the number of faces of \( X \) for \( \min \{X\} \geq \alpha \) for small \( \alpha > 0 \). Later, I found the following on the web.

\[ \pi/4 \text{-Example. Chop off the corners from the prism } \Omega \times [0, \delta] \subset \mathbb{R}^3 \text{, where } \Omega \text{ is the regular } N \text{-gon, } N = 3, 4, \ldots \text{ and } \delta = 10^{-N}, \text{ such that this } \text{"chopping" } \text{fully consumes the } \delta \text{-edges of the prisms, and such that all complementary dihedral angles of the resulting polytopes } X_N \text{ are mutually equal and satisfy } \alpha_N \to \pi/4 \text{ for } N \to \infty. \text{ see [MSE].} \]

Recently, Karim informed me [AK] that infinity of combinatorial types of convex polytopes with \( \min \{X\} \geq \alpha \) exists if and only if \( \alpha < \pi/3 \).

Below, in a similar spirit, we construct \( n \)-polytopes, which "infinitely stretch" in \( n-2 \) directions, while having all complementary dihedral angles bounded from below.

1.1.A. Skyscrapers. Given convex polytopes \( 0 \in X_1 \subset \ldots \subset X_N \subset \mathbb{R}^k \) and numbers \( h_1 > \ldots > h_N \geq 0 \) let

\[
\bigwedge_{(h_i)} \{X_i\} = \bigwedge_{(h_i)} \{X_i\} \subset \mathbb{R}^k \times \mathbb{R}_+ \subset \mathbb{R}^{k+1}
\]

denote the intersection of the cones of heights \( h_i \) over \( X_i \), where the top vertices of these cones lie on the "vertical" axes \( 0 \times \mathbb{R}_+ \subset \mathbb{R}^k \times \mathbb{R}_+ \),

\[
\bigwedge_{(h_i)} \{X_i\} = \bigcap_i \text{cone}_{h_i}(X_i).
\]

Such a \( \mathcal{A} = \mathcal{A}^N = \mathcal{A}\{X_i\} \) is called a skyscraper with the bottom \( X_1 \) and the top \( X_N \) if the following holds:

- the boundary of \( \mathcal{A}\{X_i\} \) has non-empty intersections with all open \( (n-1) \)-faces of the cones \( \text{cone}_{h_i}(X_i) \) and
- two closed side faces \( F_1, F_2 \subset \mathcal{A}\{X_i\} \) do not intersect unless they contained in faces of the cone over some \( X_i \),

\[ F_1, F_2 \subset \partial \text{cone}_{h_i}(X_i). \]

\[ ^3 \text{In view of what Karim Adiprasito recently told me, this } D \text{ is better to be the one from 1.C rather than from 1.B.} \]

\[ ^4 \text{There must exist convex tessellations, which don’t come from convex polytopes, but I failed to find a reference or to find such tessellations by myself.} \]
or in faces of two consecutive cones.

\[ F_1 \subset \partial \text{cone}_{h_i}(X_i) \text{ and } F_2 \subset \partial \text{cone}_{h_{i+1}}(X_{i+1}). \]

Notice that \( \bullet \), implies the following:

\( \bullet \_\text{ } \) the number of the \( k \)-faces of \( \mathbb{A}(X_i) \) satisfies

\[ \#_k \mathbb{A}(X_i) = 1 + \sum_{i=1}^{N} \#_k X_i. \]

Observe that \( \bullet \) can be achieved with suitable \( h_i \) and homothetically scaled \( X_i \).

\( \bullet \_\text{ } \) \( \{ \lambda_i \} \) Given \( X_i \) there exist \( h_1 > ... > h_i > ... > h_N \) and \( 0 < \lambda_1 < ... < \lambda_i < ... < \lambda_N \), such that \( \bullet \) is satisfied by \( \mathbb{A}_{(h_i)} \{ \lambda_i X_i \} \).

The usefulness of his for our purpose is due to the following obvious property of skyscrapers.

1.1.B. Large \( \_\text{ } \) Lemma. Let \( \mathbb{A}_{(h_i)} \{ X_i \} \) be a a skyscraper (with the bottom \( X_1 \) and the top \( X_N \)), such that the complementary angles of all \( X_i \) as well as (by definition acute) angles between the pairs of hyperplanes, which define the faces of consecutive \( X_i \) and of \( X_i + 1 \), are strictly bounded from below by \( \alpha > 0 \).

Then there exist a (large) positive number \( C \) such that vertically \( C \)-stretched \( \mathbb{A} \), that is \( C \uparrow \mathbb{A} = \mathbb{A}_{(C \cdot h_i)} \{ (X_i + 1) \} \) (a true skyscraper) has the complementary dihedral angles between the side faces bounded from below by \( \alpha \)

\[ \text{side}(X \uparrow \mathbb{A}) > \min(\alpha, \pi/2), \]

while these angles at the bottom face are > \( \pi/2 \).

\( \pi/3 \)-Example. Let \( X_1 \subset \mathbb{R}^2 \) be regular triangles, where \( X_1 = X_{i+2} \) and \( X_2 = -X_1 \). Then the complementary side dihedral angles \( \text{side} \) of the corresponding skyscraper

\[ X_{\text{O}} = X_{\text{O}}(N, C) = C \uparrow \mathbb{A} = \bigwedge_{(C \cdot (N-i+1))} \{(N+1)X_i \} \]

satisfy

\[ \text{side}(X_{\text{O}}(C, N)) \rightarrow \pi/3 \text{ for } C \rightarrow \infty, \]

while

\[ \text{bottom}(X_{\text{O}}(C, N)) \rightarrow \pi/2. \]

(This, I guess, must be exactly Adiprasito's example.)

Remarks. (a) The essential difference of \( X_{\text{O}}(C, N) \) from the above "pruned" prism \( \odot_N \times [0, \delta] \) is that

the combinatorial diameters of \( X_{\text{O}}(C, N) \) tend to infinity for \( N \rightarrow \infty \).

In fact,

\[ \text{diam}_{\text{comb}}(X_{\text{O}}(C, N)) = N + 1. \]

(b) The Cartesian products of \( m \) copies of \( X_{\text{O}} \) provide examples of \( 3m \)-polytopes, \( m = 1, 2, ... \), with all complementary angles \( \geq \frac{\pi}{3} - \varepsilon \) for all \( \varepsilon > 0 \) and with arbitrarily large \( \bigwedge_m \) spreads.

\[ \lambda X = \{ \lambda x \}_{x \in X} \subset \mathbb{R}^k. \]
(c) The directional limit set of the faces of the 3-polytopes $X_C(C,N)$ for $N,C \to \infty$, that is the Hausdorff limit of the sets of vertices of the dual tessellations $X^\oplus(C,N)$ of $S^2$, is a 7-point set: a regular hexagon on the equator plus the south pole, while similar $X_N$ with suitably rotated triangles $X^\ominus$ may have arbitrary limit sets on the equator.

Question. Is this limit set always discrete away from an equatorial circle $S^1 \subset S^2$? *(Adiprasito bound $\lambda_{\min} \leq \pi/3$ makes this plausible for $\lambda_{\min} \to \infty \pi/3$.)*

1.1.C. Skyscrapers on Skyscrapers. Finiteness of the directional limit sets of Skyscraper $\mathbb{A}(X_i)$ allows a lower bound on the complementary dihedral angles of double skyscraper $\mathbb{A}(\mathbb{A}(X_i))$, etc.

$\pi/3(2n - 5)$-Example. Let $\rho_{n,k}(\Delta) \subset \mathbb{R}^2$, $n = 3,4,\ldots, k = 0,2n-5$, be the regular triangle rotated by $\rho_{n,k} = k\pi/3(2n - 5)$ and define by induction on $m$ polytopes $X_m = X'_m(n,N_{n,m-2},C_{n,m-2}) \subset \mathbb{R}^m = \mathbb{R}^2 \times \mathbb{R}^{n-2}$, $m = 3,4,\ldots n$, as follows.

Let

$$X_3 = \bigwedge \{\rho_{0,0}(\Delta), \rho_{0,1}(\Delta)|N_{n,1}\} = C_{n,1} \cup 2 \bigwedge \{\lambda_i|\rho_{0,0}(\Delta), \rho_{0,1}(\Delta)|N_{n,1}\},$$

where $\{\lambda_i|A,B|N\}$ stands for $\{\lambda_1A, \lambda_2B, \ldots \lambda_{2n-1}A, \lambda_{2n}B\}$ and where the constants $h_i$ and $\lambda_i$ are chosen as in 1.1.A and where eventually $C_{n,1} \to \infty$ as earlier.

Then we slightly modify $X_3$ by turning the base 2-face $F_{\text{base}} = \Delta = X_3 \cap \mathbb{R}^2 \times 0$ by $\pi/4$, call the result $X'_3$ and inductively define

$$X'_{m+1} = \bigwedge \{\rho_{0,2(m-3)}(X'_m), \rho_{0,2(m-3)+1}(X'_m)|N_{n,m+1}\},$$

where the rotations $\rho$ apply to the $\mathbb{R}^2$-factor in $\mathbb{R}^2 \times \mathbb{R}^{n-2} \supset X'_m$ and where the implicit $h, \lambda$ and $C$-constants are adjusted as earlier.

It is easy to show — we leave checking this to the reader that

\[ \bigoplus \text{ the } \Box^{n-2}_{\text{comb}} \text{ stretch of } X'_n \text{ can be made arbitrarily large with all } N_{n,m} \to \infty \]

and that

\[ \bigoplus \text{ the complementary dihedral angles of } X'_n \text{ satisfy } \]

\[ ) (X'_n) \geq \frac{\pi}{3(2n-5)} - \varepsilon, \]

where $\varepsilon > 0$ can be made arbitrarily small with $C_{n,m} \to \infty$.

Probably a skyscraper pattern is present in all polytopes $X$ with $\boxtimes^{n-2}_{\text{comb}} >> 1$. We partly justify this (conclusively only for $\dim(X) = n = 3$) by looking at the the dual spherical tessellations $X^\ominus$ as follows.

Given a cellular tessellation $T^\ominus$, e.g., a triangulation, of an $(n-1)$-manifold $Y$, define the combinatorial distance between cells, as earlier, by the lengths of minimal chains of cells, denote this by $dist_\ominus$ and define the combinatorial $\Box^k_{\ominus}(Y) = \Box^k_{\text{comb}}(T^\ominus)$, including $diam_\ominus = \Box^1_{\ominus}$, via continuous maps $\Phi: Y \to \Box^k$ by just saying "cell" instead of "face".

The boundary of $X'_n$ (as well as that of $X_n$) with the $dist_{\text{comb}}$-geometry is shaped roughly the same as (properly coarsely homotopy equivalent to) the rectangular solid $[0,N_{n,1}] \times \ldots \times [0,N_{n,n-2}]$.
1.1.D. Large Subdomain Lemma. Let $T^\circ$ be a convex tessellation of the unit sphere $S^{n-1}$, where the cells are called $\triangle$, and let $B^\circ_s \subset S^{n-1}$, $s \in S^{n-1}$, denote the union of closed cells which contain $s$.

Then, given a (small) number $v > 0$,

there exists a connected cellular (i.e. a union of cells) domain $U^\circ$ in the sphere $S^{n-1}$, such that the spherical volumes of the $\odot$-balls $B^\circ_s$ around all points in $U$ are bounded by

$$vol(B^\circ_s) \leq v, s \in U^\circ,$$

and such that the $\odot$-spreads of $U^\circ$ are bounded from below by these of $T^\circ$ as follows:

$$\odot^k(U^\circ) \geq \text{const} \cdot v \cdot \odot^k(U^\circ) - 1, \text{ const } = \text{const}_n > (10n)^{−n}.$$

Indeed, the cardinalities $N = N(v)$ of subsets $S \subset S^{n-1}$ of "$v$-thick" points $s \in S^{n-1}$, i.e. with $vol(B^\circ_s) \geq v$, such that no pair of different points from $S$ is contained in the same closed cell of $T^\circ$, are bounded by $N = \frac{vol(S^{n-1})}{v}$, while the combinatorial diameters of all $\odot$-balls, are at most 2,

$$diam_{\odot}(B^\circ_s) \leq 2 \text{ for all } s \in S^{n-1}.$$

Therefore, given a map $\Phi = \{\Phi_1, \ldots, \Phi_k\} : S^{n-1} \rightarrow [-1,1]^k$ from the definition of $\odot_{\odot}^k$, there exist gaps between pairs of neighbouring mages, say $t_i = \Phi_i(s_i), t'_i = \Phi_i(s'_i) \in [-1,1]$, $i = 1, \ldots, k$, of pairs of "$v$-thick" vertices $s_i$ and $s'_i$, such that

$$\text{dist}_{\odot}(\Phi^{-1}[-1,t_i], \Phi^{-1}[t'_i,1]) \geq \frac{1}{N} \text{dist}_{\odot}(\Phi^{-1}(-1), \Phi^{-1}(1)) - 2(N + 1)$$

and the "$B^\circ$-enlargement" of the intersection $U_n$ of the pullbacks $\Phi^{-1}[t_i,t'_i] \subset S^{n-1}$ is taken for the required $U^\circ$

$$U_n = \bigcap_{i=1}^k \Phi^{-1}[t_i,t'_i] \text{ and } U^\circ = \bigcup_{s \in U_n} B^\circ_s.$$

Here is another obvious observation.

1.1.E. Narrow Band Lemma. If the edges (1-cells) from $T^\circ$, adjacent to a vertex $s \in S^{n-1}$, have lengths $\geq l$ and if $vol(B^\circ_s) \leq v$, then $B^\circ_s \subset S^{n-1}$ is contained in the $\delta$-neighbourhood of an equatorial sphere $S^{n-2} \subset S^{n-1}$,

$$B^\circ_s \subset U_\delta(S^{n-2}),$$

where this $\delta = \delta_n(l, v) > 0$ satisfies for all $n$ and $l > 0$,

$$\delta_n(l, v) \rightarrow 0 \text{ for } v \rightarrow 0.$$

Moreover, if all $(n-2)$-cells $\triangle^{n-2}$ adjacent to $s$ have

$$\text{vol}_{n-2}(\triangle^{n-2}) \geq a > 0,$
then this equatorial $S^{n-2} \subset S^{n-1}$ is unique up to an $\varepsilon$-perturbation, i.e. all equators for which $U_\delta(S^{n-2}) \supset B^\varepsilon$ lie within distance $\varepsilon$ one from another, where

$$\varepsilon = \varepsilon_n(v, \delta, a) \to 0 \text{ for } v \to 0.$$

**1.1.F. Corollary: Elementary Bound on $\square^2_{\text{comb}}$.** A lower bound by $a > 0$ on the $(n-2)$-volumes of $(n-2)$-cells in a convex tessellation $T^\circ$ of $S^{n-1}$ implies an upper bound on the combinatorial $\square^2$-spread of $T^\circ$,

$$\operatorname{vol}_{n-2}(F^{n-2}) \geq a > 0 \implies \square^2(T^\circ) \leq \Theta_n(a).$$

where $\Theta_n$ is a (bounded monotone decreasing) function in $a > 0$.

(If $n = 2$, this is just a qualitative version of 1.B.)

**Proof.** It follows from 1.1.F. and 1.1.E that the above $U^\circ \subset S^{n-1}$ is contained in a $\delta'$-neighbourhood of an equator $S^{n-2} \subset S^{n-1}$, where, for a fixed $a > 0$,

$$\square^k_{\text{comb}}(T^\circ) \to \infty \implies \square^k_{\text{comb}}(U^\circ) \to \infty \text{ for all } k = 1, \ldots.$$

Then one sees that, for $\delta'$ much smaller than $a$, this $U^\circ$ admits a cellular map of degree one from the cylinder $S^{n-2} \times [0, 1]$, which is decomposed into $m \times M$ cells, which are products of cells of some triangulation of $S^{n-2}$ into $n$-simplices and a decompositions of $[0, 1]$ into $M$ segments, where $m$ is bounded by a constant depending on $a$.

It follows that $\square^2(U^\circ)$ is also $\leq m$, hence, it is bounded in terms of $a > 0$.

**QED.**

**Remark.** The above shows that if $\square^1_{\text{comb}}(T^\circ) \to \infty$ with the $(n-2)$ volumes of all $(n-2)$-cells bounded from below by $a$, then the unit sphere $S^{n-2}$ acquires several limit tessellations with the same bound on the volumes of their $(n-2)$-cells and some cells spanned by vertices of different tessellations.

Then, for instance, by looking on pairs of such tessellations, one recovers a special case of Adiprasito’s result for $n = 3$.

**1.1.G. Conjecture.** For all $k = 1, \ldots, n-2$, a lower bound on the $k$-volumes of the $k$-cells in a convex tessellation $T^\circ$ of $S^{n-1}$ implies an upper bound on the combinatorial $\square^{n-k}$-spread of $T^\circ$.

Conversely,

there exist convex tessellations $T^\circ$ of $S^{n-1}$ with arbitrary large $\square^{n-k}_{\text{comb}}(T^\circ)$ and

with the volumes of all $k$-cells bounded away from zero.

Moreover, there are such $T^\circ$, which are dual of convex polytopes $X \subset \mathbb{R}^n$. (A quantitative form of a special case of this is suggested in 6.B.)

## 2 Manifolds with Corners, Mean Convexity and Distance $\operatorname{dist}^k$.

Let $X$ be a smooth $n$-manifold with corners, i.e. locally, at all $x \in X$, it is diffeomorphic to a convex polytope $Q = Q_x \subset \mathbb{R}^n$.

For instance, diffeomorphic images of convex polytopes are manifolds with corners.
Also recall that the mean curvature of a cooriented hypersurface in a Riemannian manifold is the sum of the principal curvatures.

Example. The $R$-sphere $S^{n-1}(R) \subset \mathbb{R}^n$ and the round cylinder $S^{n-2} \times \mathbb{R}^1 \subset \mathbb{R}^n$ satisfy

$$\text{mean.curv}(S^n) = \frac{n-1}{R} \quad \text{and} \quad \text{mean.curv}(S^{n-2}(R) \times \mathbb{R}^1) = \frac{n-2}{R}.$$ 

A Riemannian manifold with corners is called mean convex if all its $(n-1)$-faces $F \subset \partial X$ have non-negative mean curvatures.

For instance, convex domains in $\mathbb{R}^n$ with corners are mean convex.

Given a smooth curve in the boundary of a manifold with corners, say $\gamma \subset \partial X$, which doesn’t intersect $(n-2)$-faces of $X$ and which meets all $(n-2)$-faces of $X$ transversally, say at $x_i \in \partial X$, $i = 1, \ldots, j$, let

$$\text{length}^\gamma_i(\gamma) = \text{length}^\gamma(\gamma) + \sum_{i=1}^{j} \text{length}_{x_i}^\gamma(\gamma),$$

where $\text{length}^\gamma(\gamma) = \int_\gamma \text{mean.curv}(\partial X) d\gamma$, where $\{x_i\}$ are the complementary dihedral angles, $\{x_i\} = \pi - \gamma_{x_i}$, and where the dihedral angle $\gamma_{x_i}$ of $X$ at the point $x_i$ is the angle between the (naturally cooriented) $(n-1)$-dimensional tangent spaces $\gamma_i, \gamma_i' \in T_{x_i}(X)$ to the two $(n-1)$-faces adjacent to the $(n-2)$-face, which contains $x_i$.

Next, assuming $X$ is mean convex and $x_1, x_2 \in \partial X$ are contained inside $(n-1)$-faces, let

$$\text{dist}^\gamma_i(x_1, x_2) = \inf_{\gamma_{1,2}} \text{length}^\gamma_i(\gamma_{1,2}),$$

where the infimum is taken over the above kind of curves $\gamma_{1,2} \subset \partial X$ between $x_1$ and $x_2$.

Although this $\text{dist}^\gamma_i$ is defined not for all points and it may vanish at some pairs of non-equal points, we treat it as a true distance; in particular, we define the corresponding distance between $(n-1)$-faces $F_1, F_2 \subset \partial X$ in the usual way:

$$\text{dist}^\gamma_i(F_1, F_2) = \inf_{x_1, x_2} \text{dist}^\gamma_i(x_1, x_2) \text{ for } x_1 \in F_1 \text{ and } x_2 \in F_2.$$

If the boundary of $X$ contains no corners, i.e. it is smooth, then the corresponding distance is denoted $\text{dist}^\gamma$. This is a true positive distance if $X$ is strictly mean convex, i.e. $\text{mean.curv}(\partial X) > 0$.

Semi(in)stability of $\text{dist}^\gamma$. An arbitrarily $C^1$-small perturbation of a smooth convex hypersurface $Y \subset \mathbb{R}^n$, $n \geq 3$, may significantly diminish the metric $\text{dist}^\gamma$ on $Y$.

For instance,

the unit sphere $S^{n-1} \subset \mathbb{R}^n$, which has $\text{diam}^\gamma(Y) = \frac{(n-1)^2}{2}$, can be $C^1$-approximated by smooth convex hypersurfaces $Y_\varepsilon$ with $\text{diam}^\gamma(Y_\varepsilon) = \pi + \varepsilon$ for all $\varepsilon > 0$ as follows.

Let $A = A_{N, \delta} \subset S^{n-1} \subset \mathbb{R}^n$, where $N > \delta^{-2n}$, be the union of regular equatorial $N$-gons in general position, such for all pairs of points in $S^{n-1}$, there are our

\footnote{An open $k$-face in $X$ is understood as a maximal connected subset where $X$ is locally diffeomorphic to a polyhedral $k$-face.}

\footnote{This can’t be done with $\varepsilon > 0$.}
$N$-gons passing $\varepsilon$ close to both points. Let $B_A \subset \mathbb{R}^n$ be the intersection of subspaces bounded by the hyperplanes tangent to $S^{n-1}$ at the vertices of the $N$-gons and let $Y(N, \delta, \varepsilon)$ be the boundary of the $\varepsilon$-neighbourhood of $B_A$ for $0 < \varepsilon \leq N^{-2n}$.

Then

\[ \text{diam}^h(Y(N, \delta, \varepsilon)) \to \pi \text{ for } \delta \to 0, \]

and, for the same reason,

all $\frac{k}{n}$-spreads of $Y(N, \delta, \varepsilon)$ converge to the ordinary spreads of the unit sphere,

\[ \mathcal{C}^k(Y(N, \delta, \varepsilon)) \to \mathcal{C}^k(S^{n-1}) = \frac{1}{n-1} \mathcal{C}^k(S^{n-1}), \]

where, observe for instance, $\mathcal{C}^n(S^{n-1}) = 2 \arcsin \frac{1}{\sqrt{n}}$.

But the metric $\text{dist}^h$ of a compact convex hypersurface $Y$ can’t everywhere significantly increase under small $C^0$-perturbations of $Y$.

In fact, if $Y = S^{n-1}$, this follows from theorem 2.A below, which is a special case of the normalized mean curvature mapping theorem from section 3.5 of [Gr2021] and which makes the key ingredient of the proof of 1.A.

2.A. Euclidean $\text{dist}^h$-Non-Contraction Theorem. Let $X$ be a compact oriented mean convex Riemannian $n$-manifold with smooth boundary, let $B \subset \mathbb{R}^n$ be a smooth compact convex domain. E.g. the unit ball, and let $f : \partial X \to \partial B$ be a smooth map, which which has non-zero degree.

If $X$ has non-negative scalar curvature, $\text{Sc}(X) \geq 0$, and if $X$ is spin,\(^9\) then $f$ can’t be strictly $\text{dist}^h$-decreasing: there exists a pair of points $x_1, x_2 \in \partial X$, such that

\[ \text{dist}^h(x_1, x_2) \leq \text{dist}^h(f(x_1), f(x_2)). \]

Remarks. (a) As far as the proof of 1.A is concerned, one needs only a very special case of this theorem, namely, where $X$ is also a smooth convex domain in $\mathbb{R}^n$ and $Y$ is the unit ball in $\mathbb{R}^n$. Amazingly, however, even in this case, the only available proof of 2.A relies on the spin geometry and Dirac operators (see below).

(b) The assumption $\text{Sc}(X) \geq 0$ is, obviously, essential: there is no curvature constrains on the boundaries of general Riemannian manifolds.

But what is non-obvious, is how sensitive the geometry of $\partial X$ may be to the sign of the scalar curvature of $X$.

For instance, in agreement with the positive mass theorem in general relativity, there is no Riemannian metric $g$ on the unit ball $B^n \subset \mathbb{R}^n$ with $\text{Sc}(g) > 0$ and with $\text{dist}^h_g$ (non-strictly) greater than the original $\text{dist}^h$ on the unit sphere $S^{n-1} = \partial B^n \subset \mathbb{R}^n$.

(c) It is unknown if the spin condition is essential.

\(^9\)Here $\mathcal{C}^n(S^{n-1})$ means $\mathcal{C}^n(S^n)$, i.e. this is defined via maps $S^{n-1} \to \partial S^n$ of positive degrees.

\(^{10}\)An oriented vector bundle $T \to X$ is spin if the associated principle bundle $G \to X$ with the fibres $G_x = SL(n, \mathbb{R})$, $n = \text{rank}(T)$, admits a double covering $s : G \to G$, such that the pullbacks of the fibers $s^{-1}(G_x)$ are connected; an orientable manifold $X$ is spin if the tangent bundle $T(X)$ is spin.

A necessary and sufficient condition for spin is vanishing of the second Stiefel-Whitney class $w_2(T) \in H^2(X; \mathbb{Z}_2)$; for instance, if $\pi_2(X) = 0$, then the universal covering of $X$ is spin. It is also known that all 3-manifolds are spin, while the complex projective plane $\mathbb{C}P^3$ is non-spin.
The second components of the proof of 1.A - this is an actual contribution of the present paper, is the following.

2.B. \textit{dist}-Approximation Theorem. Let \( X \) be a compact mean convex Riemannian \( n \)-manifold with corners. Then, for all \( \varepsilon > 0 \), there exists a smooth mean convex hypersurface \( Y = Y_\varepsilon \subset X \) and a homeomorphism \( \psi = \psi_\varepsilon : \partial X \to Y \) with the following properties.

\begin{itemize}
  \item 1. The map \( \psi \) is \( \varepsilon \)-close to the identity, \( \text{dist}_Y(\psi_\varepsilon(x), x) \leq \varepsilon \) for all \( x \in \partial X \).
  \item 2. The \( \text{dist}_Y \) in \( Y \) is greater than \( \text{dist}_{\partial X} \) up to an \( \varepsilon \)-error,
\end{itemize}

\[ \text{dist}_Y(\psi(x_1), \psi(x_2)) \geq (1 - \varepsilon)\text{dist}_{\partial X}(x_1, x_2) \]

for all pairs of points positioned within distances \( \geq \varepsilon \) from the corners of \( X \).

We shall proof this in section 4, where we also show that, in the case of convex domains \( X \subset \mathbb{R}^n \), the approximation is possible with \textit{strictly convex} \( Y \).

Then we shall see in section 5 that 2.A and 2.B (trivially) imply the following generalization of 1.A.

2.C. \textit{Riemannian Angular Spread Theorem.} Let \( X \) be a compact orientable mean convex Riemannian \( n \)-manifold with corners and with non-negative scalar curvature, \( \text{Sc}(X) \geq 0 \). If \( X \) is spin, then the cubical \( \rangle \)-spread of \( X \) is universally bounded as follows,

\[ \text{cap}_\rangle^n(X) \leq D = D_n \leq 2(n-1)\sqrt{n}. \]

\textbf{Technical Strictness Remark.} Non-strictness of mean convexity may create inconvenience, e.g. a terminological one in dealing with vanishing "metrics". But this is mainly irrelevant, since, in the cases of our immediate interest, e.g. for compact smooth hypersurfaces in \( \mathbb{R}^n \), strictness of mean convexity, \( \text{mean.curv} \geq 0 \leadsto \text{mean.curv} > 0 \), can is easily achieved by arbitrarily \( C^\infty \)-small perturbations.

In general, with a minor analytic effort, one can \( C^\infty \)-approximate a compact connected mean boundary \( \partial X \) of a Riemannian manifold \( X \) corners by a \textit{strictly convex} hypersurfaces \( Y \subset X \), unless this \( \partial X \) is smooth (no corners) with zero mean curvature.

Thus, one may assume strictness of mean convexity in the present paper whenever this helps to simplify understanding.

2.1 \textit{Sc}-Normalized Metric \( g^\circ \), Derivation of 2.A from the LGSL Theorem and \( \rangle \)-Capillary Problem

The counterpart of \( g^\circ \) for Riemannian manifolds \( X = (X, g) \) with positive scalar curvatures

\[ \text{Sc}(X, x) = \text{Sc}_g(x) > 0 \]

is the \( \text{Sc} \)-normalized Riemannian metric

\[ g^\circ = g^\circ(x) = \text{Sc}_g(x)g(x) \]

on \( X \).

\textsuperscript{11}Smoothing the corners + "Dirac with boundary" is also used by Simon Brendle [Br2023] in the proof of his polyhedral scalar curvature rigidity theorem.
The basic geometric property of this $g^o$ is the following special case of the Llarull-Goette-Semmelmann-Listing theorem (see 3.1.2 in [Gr2021] and references therein):

2.1.A. Euclidean dist$^o$-Area Non-Contraction Theorem. Let $X$ be a connected orientable $n$-dimensional Riemannian manifold with $Sc(X) > 0$ and let $X \subset \mathbb{R}^{n+1}$ a closed convex hypersurface.

Let $f : X \to X$ be a smooth $g^o$-area decreasing map, that is

$$area_{g^o}f(S) < area_{g^o}(S)$$

for all smooth surfaces $S \subset X$, where $g$ is the induced Riemannian metric in $X \subset \mathbb{R}^{n+1}$.

If $X$ is spin, then the map $f$ has degree zero (hence, $f$ is contractible).

Remarks

(a) It is unknown, not even for $n = 4$, if the spin condition is essential.

(b) The proof of 2.1.A depends on the index and vanishing theorems for the Dirac operator on $X$ with coefficients in the vector bundle induced by $f$ from a unitary bundle on $X$.

The simplest kind of result of this kind, where the proof is technically very simple (see [GL1980], says that for no Riemannian metric $g$ on $S^n$ the corresponding $g^o$ can be significantly greater than the spherical metric:

2.1.B. If $dist_{g^o} \geq Cdist_{S^n}$, then $C \leq C_n$ for a universal constant $C_n$.

(In fact, $C_n = \sqrt{n(n-1)}$ by Llarull’s theorem [Ll1998].)

(c) If $g$ has constant scalar curvature, then 2.1.B (but not 2.1.A) can be proven by the technique of the geometric measure theory following ideas from [SY1979].

Moreover:

2.1.C. If a metric $g$ on the unit ball $B^n \subset \mathbb{R}^n$ satisfies $Sc(g) \geq C_n$, then the identity map $id : (B^n, g) \to (B^n, g_{Eucl})$ can’t be distance decreasing.

This is proven in [GL1983] for $n \leq 7$ and extended to all $n$ in [Gr2018] modulo [SY2017], and directly in [Loh2028].

(d) The obvious counterpart of 2.1.A for open manifolds fails to be true.

2.1.D. Example. The Euclidean space $\mathbb{R}^n$, $n \geq 2$, admits a Riemannian metric $g$ with $Sc(g) > 1$ and such that $g^o$ is greater than the Euclidean metric.

$$g^o \geq g_{Eucl}.$$  

(Notice that $g^o$ for such a $g$ is complete, but, (see [GL1983]), $g$ can’t be complete.)

Proof. Recall that the scalar curvature of the metric $g_\phi = dx^2 + \phi^2(x)dy^2$ on the $(x,y)$-plane is

$$Sc(g_\phi(x)) = -2 \frac{d^2\phi(x)}{\phi(x)}.$$  

12 The essential ingredients of the proofs in [Ll1998], [GS2002], [Lis 2010] is a sharp evaluation of eigenvalues of certain operators $R$ in moduli over Clifford algebras, where these $R$ are algebraically associated with the curvature operators $R$ of the underlying Riemannian manifolds $X$. This suggests a direct Clifford algebraic approach to the geometry of convex polytopes, where the complementary dihedral angle play the role of $R$ (compare with [WXY2022]).
Thus, if $\phi(x)$ is a strictly concave positive function on the open interval $(0, 1)$, such that the integrals \( \int_0^{1/2} \frac{\phi'(x)}{\phi(x)} \, dx \) and \( \int_{1/2}^1 \frac{\phi'(x)}{\phi(x)} \, dx \) diverge, then the metric $g_0$ on the band $U = (0, 1) \times (-\infty \times \infty)$ is complete. Moreover, for all $\varepsilon > 0$ there obviously exists a distance decreasing diffeomorphism from $(U, g_0)$ onto $\mathbb{R}^2$.

Now let $\phi(x)$ be equal $x^\alpha$ near $x = 0$ and to $(1 - x)^\alpha$ near $x = 1$ for $0 < \alpha < 1$, observe that these integral diverge and make our example with the obvious distance decreasing diffeomorphism $U \times \mathbb{R}^{n-2} \rightarrow \mathbb{R}^n$.

**On Reduction of 2.A to 2.1.A.** This is achieved for a manifold $X$ with a (mean convex) boundary by applying 2.1.A to the double $\mathcal{D}(X)$ with a a suitably smoothed metric on it (see section 3.5 of [Gr2021]).

A more direct but analytically more involved proof of 2.A based on the the index theorem for manifolds with boundaries was given in [Lott2020].

Then, on the next level of sophistication, the index theory directly applies to manifolds with corners [WXY2022].

This, formally speaking, delivers a two line proof of 1.A, but my unsatisfactory understanding of the techniques developed in [WXY2022] makes me reluctant to make such a shortcut in the proof.

**On Capillary Geometry of $X$.** The above example highlights the difficulty of applying the geometric measure theory to $g^0$ and $g^3$, but it doesn’t fully rule out such applications.

Here is an instance of what one may expect of such an application.

Let $X$ be a mean convex Riemannian $n$-manifold with corners and with positive scalar curvature and let $F_{\gamma}^{n-1} \subset \partial X$ be two faces positioned far away one from another in a suitable sense, where the weakest condition (which may fail to be sufficient) would be a lower bound on the distance $\text{dist}_{1}$ between them: $\text{dist}_{1}(F_{\gamma}^{n-1}, F_{\nu}^{n-1}) \geq \text{const}_{\gamma}$, where, ideally, $\text{const}_{\gamma} = \pi$.

Then one wants to have a smooth hypersurface $Y \subset X$ with $\partial Y \subset \partial X$ transversal to the faces of $X$ and a smooth positive function $\psi(y)$ on $Y$, such that the the $\psi$-warped product of $Y$ with the circle, $X_{\psi} = (Y \times \mathbb{T}^{1}, g_{\psi})$, for $g_{\psi} = g_{Y} + \psi^{2} dt^{2}$, where $g_{Y}$ is the induced Riemannian metric in $Y$, such that the following conditions are satisfied:

- $g_{\psi}$ the metric $g_{\psi}$ has positive scalar curvature,
- $\text{mean}$ the (boundary of the) manifold $X_{\psi}$ is mean convex,
- $\text{dist}$ the $\text{dist}_{1}$-distances between $(n-1)$-faces in $X_{\psi}$ are bounded from below, possibly, times a controlled $(1 + \delta_{\psi})$-factor, by the $\text{dist}_{1}$-distances between the corresponding faces in $X$.

This would allow an inductive proof of (a sharp version?) of 1.A, where, observe, the expected $Y \subset X$, for $\text{dim}(X) = 3$ is a minimal surface (or something of this kind), which, even for 3-polytopes $X \subset \mathbb{R}^{3}$ is by no means flat. (Compare with [Gr2014’], [Gr2018], [Li2019] and section 5.81 in [Gr2021].)

## 3 Rounding the Corners and $\text{dist}_{k}$-Approximation of Simple Polytopes

Let $X \subset \mathbb{R}^{n}$ be a convex polytope and $\nu : \mathbb{R}^{n} \rightarrow X$ be the normal projection, that is $\nu(x) \in X$ is the nearest point point to $X$, i.e.

$$\text{dist}(x, \nu(x)) = \text{dist}(x, X), x \in \mathbb{R}^{n},$$
and let  \( X_\varepsilon = X_{\varepsilon} \supset X, \); \( \varepsilon > 0 \), be the \( \varepsilon \)-neighbourhood of \( X \) that is the set of points \( x \in \mathbb{R}^n \) with \( \text{dist}(x, X) \leq \varepsilon \).

Observe the following (compare with section 5.7 in [Gr2014] and 11.3 in [Gr2018]).

- \( X \cap G_k \) The boundary \( \partial X_\varepsilon \) is equal to the union of closures of the pullbacks of the open \( k \)-faces \( F_k \subset X, k = 0, 1, \ldots, n-1 \) intersected with \( \partial X_\varepsilon \), denoted

\[
G_k = \nu^{-1}(F_k) \cap \partial X_\varepsilon \subset \partial X_\varepsilon,
\]

where such a \( G_k \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \) is isometric to the product of the corresponding face \( F_k \subset \mathbb{R}^k \) by a convex \( \varepsilon \)-spherical polyhedron (dual to the normal section of \( F_k \)) denoted \( F_k \perp \subset S^{n-k-1}(\varepsilon) \subset \mathbb{R}^{n-k} \).

Thus, the principal curvatures of \( G_k \subset \mathbb{R}^n \) are

- \( \text{curv} \)

\[
\frac{1}{n-k}, \ldots, \frac{1}{n-k}
\]

and their mean curvatures satisfy

- \( \text{mean} \)

\[
\text{mean.curv}(G_k) = \frac{1}{\varepsilon^{n-k-1}}.
\]

Different \( G_k \), which intersect across parts of their boundaries, have equal tangent spaces at their common points; thus the boundary \( \partial X_\varepsilon \subset \mathbb{R}^n \) is \( C^1 \)-smooth.

\textbf{Quadratic Form} \( g_{\varepsilon} \) \( \text{and Definition of dist}_{\varepsilon} \): Let \( g_{\varepsilon} \) be the product of the induced Riemannian metric on the hypersurface \( \partial X_\varepsilon \subset \mathbb{R}^n \) by the squared mean curvature of this hypersurface,

\[
g_{\varepsilon} = \left(\text{mean.curv}\right)^2 g_{\text{Eucl}}|\partial X_\varepsilon|
\]

and observe that the metric defined with this Riemannian form \( g_{\varepsilon} \) is exactly our \( \text{dist}_{\varepsilon} \), which is denoted here \( \text{dist}_{\varepsilon} \).

\textbf{3.A. dist}_{\varepsilon} \text{-Convergence Theorem.} If \( X \) is a simple polytope, then the \( \text{dist}_{\varepsilon} \)-distance converges to the \( \gamma \)-distance,

\[
\text{dist}_{\varepsilon}(K_1, K_2) \to \text{dist}_{\gamma}(F_1, F_2)
\]

for all pairs of compact subsets in open \( (n-1) \)-faces \( F_1, F_2 \subset X \).

\( K_1 \subset F_1, K_2 \subset F_2 \subset X. \)

\( \square \text{-Example.} \) If \( X = \square^n = [-1,1]^n \) is the \( n \)-cube, where, as we know, \( \text{dist}_\gamma \) between opposite \( (n-1) \)-faces is \( \pi \), the \( g_{\varepsilon} \)-distance between the corresponding faces in \( \square^n \) is only \( \pi/2 \). To get the full \( \pi \), one needs to go \( \varepsilon \) away from the boundaries of these faces.

\textit{Proof} \[\text{[13]}\] Let \( Q \subset \mathbb{R}^n \) be a convex polyhedral \( n \)-dimensional cone and \( R \subset \partial Q_\varepsilon = \partial Q_{\varepsilon/2} \) be the complement to the flat part of \( \partial Q_\varepsilon \), that is the union of all \( G_k \) with \( k \neq n-1 \).

\[\text{[13]}\] Compare with section 5.7 in [Gr2014] and 3.1.2(8) in [Gr2021]
Observe that this $R$ is a connected $(n - 1)$-manifold with a boundary, where the connected components of this boundary are equal to the boundaries of the $(n - 1)$-faces of $Q$.

**3.B. Minimal Path Lemma.** The shortest paths $\gamma \subset R$ between different connected components $\partial_1, \partial_2 \in \partial R$ are geodesic segments contained in the subsets $G_k = F^k \times F^k \subset R \subset \partial Q_{\varepsilon^{-1}}$, or in the intersection of several such subsets.

Consequently, the Riemannian distance between $\partial_1$ and $\partial_2$ is equal to the spherical distance between the intersection of $\partial_1$ and $\partial_2$ with the spherical polytope $G_0 = Q_{\varepsilon^{-1}} \cap S^{n-1}$.

$$\text{dist}(\partial_1, \partial_2) = \text{dist}_{S^{n-1}}(\partial_1 \cap \Psi S^{n-1}, \partial_1 \cap S^{n-1})$$

*Proof.* A priori, $\gamma$ (which is a $C^1$-smooth curve) is composed of several geodesic segments contained in different $G_k$. But since all geodesic segments in all $G_k$ are distance minimizing, $\gamma$ is equal to the geodesic continuation of its initial segment, say $\gamma_1$ in some $G_k$; thus $\gamma$ stays in this very $G_k$ all along. QED.

Now, let a path $\gamma^b \in \partial X_b$ implement the distance $\text{dist}^b$ between two flat cells in $\partial X$, say between $G_{b-1,1}$ and $G_{b-1,2}$. The length of this path is equal to the sum of $\text{dist}^b$ between components, say $\partial_1$ and $\partial_1$, of the boundary of the non-flat part $R \subset \partial X$, crossed by $\gamma^b$.

If $X$ is simple and and all $G_k = F_k \times \Delta^{n-1-k}$, where $\Delta^{n-1-k}$ are spherical simplices, these distances, because of 3.B, can implemented by geodesic segments in $G_k$ with $k = n - 2$ and 3.A follows.

**About Non-Simple $X$.** Examples show that 3.A fails to be true for non-simple polyhedra $X$, but due to 3.B it allows a a modification applicable to non-simple $X$.

Namely, the $(n - 2)$-adjacency graph $E$ must be replaced by the full adjacency graph $E_s(X) \supset E(X)$, which, similarly to $E$, has the $(n - 1)$-faces for vertices and where the edges correspond to pairs of $(n - 1)$-faces which meet at 0-faces (vertices) of $X$ and where the lengths of these edges are defined by the corresponding angles between these faces.

However the resulting version of 1.A for non-simple $X$ doesn’t bring anything new since it follows from the "simple" case by a generic perturbation of the $(n - 2)$-faces of $X$.

**4 Locally Conical Hypersurfaces and the $\text{dist}^b$-Approximation Theorem for Non-simple $X$**

**4.A. Conical Function Lemma.** Let $Y \subset \mathbb{R}^m$ be a (possibly unbounded, e.g. conical) convex polytope. Then, for all $\varepsilon > 0$ there exists a positive concave, function $\phi = \phi_\varepsilon : Y \to \mathbb{R}_+$, which is piecewise smooth in the interior of $Y$, which vanishes on the boundary $\partial Y$ and which satisfies the following four conditions:

- The directional derivatives of $\phi$ at all boundary points $y \in \partial Y$ are bounded in absolute values by $\varepsilon$, or equivalently $\phi$ is $\varepsilon$-Lipschitz:
  $$|\phi(y_1) - \phi(y_2)| \leq \varepsilon \cdot \text{dist}(y_1, y_2) \text{ for all } y_1, y_2 \in X.$$

- The principal curvatures of the graph $\Gamma_{\phi} \subset \mathbb{R}^m \times \mathbb{R}$, at the smooth points of $\phi$ are everywhere strictly positive.

---

1. With a little extra effort one can make $\phi$ smooth in the interior of $Y$.  

---
The mean curvature of $\Gamma_\phi$ is uniformly positive on compact parts of $Y$ at smooth points $(y, \phi(y)) \in \Gamma_\phi$.

\[
\text{mean.curv}(\Gamma_\phi, (y, \phi(y))) \geq \epsilon(y) > 0.
\]

for a positive continuous function $\epsilon(y)$ on $Y$.

Moreover, for all $(m-2)$-faces $F \subset \partial Y$,

\[
\bullet_1/\partial \quad \text{mean.curv}(\Gamma_\phi(y, \phi(y))) \geq \text{const} \frac{1}{\text{dist}(y, F)}
\]

for some positive constant $\text{const} = \text{const}_{p, \epsilon} > 0$ and all interior points $y \in Y$, where $\phi$ is smooth.

Proof. The existence of $\phi$ is obvious for $m = 1$ and the general case follows by induction in $m$ as follows.

Represent $Y$ by the intersection of the wedges $W_i \subset \mathbb{R}^m$, $i = 1, ..., j$, which are based on the 1-faces $F_i^1 \subset Y$,

\[
Y = \bigcup_{i=1}^j W_i, \quad \text{for } W_i = C_i \times L_i, \quad \text{and } C_i \subset \mathbb{R}_i^{m-1},
\]

where

- $L_i \subset \mathbb{R}^m$ are the straight lines, which extend the 1-faces $F_i^1 \subset Y$;
- $R_i^{m-1} \subset \mathbb{R}^m$ are normal spaces to the faces $F_i^1$ at some points $y_i \in F_i^1$;
- $C_i \subset \mathbb{R}_i^{m-1}$ are the convex tangent cones to $Y$ at the points $y_i$, that are the conical extensions of the intersections of $\mathbb{R}_i^{m-1}$ with small neighbourhoods of $y_i$ in $Y$.

Let $\phi_i(c)$ be concave functions in the cones $C_i$, which satisfy the four conditions $\bullet_{c, \cdot}, \bullet_{\text{curve}, \cdot}, \bullet_{\text{mean}, \cdot}, \bullet_{1/\partial}$.

Let $\tilde{\phi}_i(c, l) = \phi_i(c)$ for $(c, l) \in W_i = C_i \times L_i \subset \mathbb{R}^m$

and let $\phi$ be the minimum of generic $\lambda_i$-perturbations of the functions $\tilde{\phi}_i$ on $Y$,

\[
\tilde{\phi}(y) = \min_i \lambda_i \phi_i(y), \quad y \in Y.
\]

for small generic $\lambda_i > 0$.\footnote{Generic $\lambda_i$ are needed to assure piecewise smoothness of $\tilde{\phi}(y)$ in the interior of $Y$.}

Clearly, the function $\phi$ satisfies $\bullet_{c, \cdot}, \bullet_{\text{curve}, \cdot}, \bullet_{\text{mean}, \cdot}$, but it may fail $\bullet_{1/\partial}$ at the vertices $y_\nu \in Y$.

To correct this, modify $\tilde{\phi}$ at $y_\nu$ as follows. Let $U_\nu \subset Y$ be (very) small (pyramidal) neighbourhoods of $y_\nu \in Y$, which are bounded in $Y$ by hyperplanes cutting $y_\nu$ away from $Y$, let

\[
\bar{Y} = Y \setminus \bigcup_\nu U_\nu
\]

be the correspondingly truncated $Y$ and let $\phi(y)$ be the smallest concave function on $Y$, which is equal to $\tilde{\phi}$ on $\bar{Y}$ and which vanishes on the boundary of $Y$.

In geometric term, the convex body $Y_\phi^+ \subset \mathbb{R}^m \times \mathbb{R}_+$ under the graph $\Gamma_\phi \subset \mathbb{R}^m \times \mathbb{R}_+$ is obtained by firstly cutting away $y_\nu$ from $Y_\phi^+ \subset \mathbb{R}^m \times \mathbb{R}_+$ by vertical half-hyperplanes $H^+_\nu \subset \mathbb{R}^m \times \mathbb{R}_+$ and then adding the cones from $y_\nu$ over the intersection $Y^+_\phi \cap H^+_\nu$ to the resulted truncated $Y^+_\phi$.\footnote{Generic $\lambda_i$ are needed to assure piecewise smoothness of $\tilde{\phi}(y)$ in the interior of $Y$.}
Now, clearly, the mean curvature of $\Gamma_\delta$ does blow-up as $1/d$ for the distance $d$ to the $(n-2)$-faces of $Y$ and the proof of 4.A is concluded.

**Proof of the dist$^3$-approximation theorem 2.B for convex polytopes.**

Let $X \subset \mathbb{R}^m$ be a compact convex polytope and let $X^+_\varepsilon \subset \mathbb{R}^n$ be obtained by adding the subgraphs of the functions $\phi = \phi_\varepsilon$ on all $(n-1)$-faces $Y$ of $X$ to $X$.

The following five properties of $X^+_\varepsilon$ trivially follow from 4.A.

- **pinched**. The set $X^+_\varepsilon$ is pinched between $X$ and a (small) $\delta$-neighbourhood of $X$, $X \subset X^+_\varepsilon \subset U_\delta(X) \subset \mathbb{R}^n$, where $\delta \to 0$ for $\varepsilon \to 0$.

- **convex**. If $\varepsilon > 0$ is sufficiently small, then $X^+_\varepsilon$ is convex.

- **face**. The intersection of the boundary of $X^+$ with $X$ is equal to the union of the closed $(n-2)$-faces of $X$.

$$\partial X^+_\varepsilon \cap X = \partial X^+ \cap \partial X = \bigcup_{i=1, \ldots, k \leq n-2} F^k_i.$$

- **dihedral angles**. The dihedral angles of $X^+_\varepsilon$ along $(n-2)$-faces of $X$ (contained in $\partial X^+_\varepsilon$) are bounded by the dihedral angles of $X$ between these faces as follows,

$$\varepsilon \leq \angle + 2\varepsilon.$$

- **mean curvature**. The mean curvature of $\partial X^+_\varepsilon$ at smooth points $x \in \partial X^+_\varepsilon$ satisfies

$$\text{mean.curv}(\partial X^+_\varepsilon, x) \geq \text{const}{\frac{1}{\text{dist}(x, F^{n-3})}}$$

for some $\text{const} > 0$ and all $(n-3)$-faces $F^{n-3}$ of $X$.

It follows, that paths $\gamma \subset \partial X^+_\varepsilon$, which approach $F^{n-3}$ have infinite $g^1$-lengths; hence $g^1$-shortest paths cross $(n-2)$-faces away from $(n-3)$ faces.

Then an additionally $C^2$-smoothed boundary $Y$ of the $\varepsilon$-neighbourhood $U = U_{\varepsilon,\varepsilon}(X^+_\varepsilon)$ serves as the required approximation of $X$ by the (trivial) argument from 3.A.

**Generalization to Mean Convex Manifolds $X$ with Corners.** Think of $X$ as a mean convex domain with corners in a larger Riemannian manifold, say $W \supset X$ and construct $X^+_\varepsilon \subset W$ in three steps.

1. Make the $(n-1)$-faces $F_i^{n-3}$ of $X$ strictly mean convex by $C^\infty$-perturbations, while keeping these faces unperturbed on the parts of their boundaries which are close to $(n-3)$-faces, i.e. on the intersections $\partial F_i^{n-3} \cap U_\varepsilon(F_j^{n-3})$.

   This is done by linearizing the problem as it is done in the first proof of $(\ast_\varepsilon)$ in section 11.2 of [Gr2018]".

   **Warning.** One can’t, in general, achieve this while keeping the faces fixed everywhere on their boundaries as it was done for convex $X \subset \mathbb{R}^n$.

---

10These are the angles between the pairs of extremal supporting hyperplanes to $X^+_\varepsilon$ at the points $x \in F^{n-2} \subset X \cap X^+_\varepsilon$.

17This argument is outlined in [Gr2018] for simple $X$, i.e. where the $(n-1)$-faces intersect transversally and thus the combinatorial structure of $X$ remains stable under small perturbations, while in the present case, one needs to keep the perturbation fixed on $\partial F^{n-3}$ near the $(n-3)$-faces to preserve the combinatorial structure of $X$.

In any case, all this is a minor matter and one doesn’t lose much by assuming that $X$ is strictly mean convex to start with. Also notice that second "variational proof" of $(\ast_\varepsilon)$ in [Gr2018] is invalid.

17
For instance, if \( X \) is a locally convex geodesic polygon in a Riemannian surface \( W \), then an edge \( F^1 \) in \( X \) can be approximated by a strictly convex curve with the same ends as \( F^1 \), if and only if \( F^1 \), which itself is a geodesic segment, contains no conjugate points.

2. At the second step one make the mean curvature of the faces blow up at the \((n-3)\)-faces with the rate \(1/d\) as in the above convex case. In fact, since this blow-up property is invariant under diffeomorphisms, one can perform it locally in normal geodesic coordinates and then glue these together by a partition of unity argument.

Notice that this is unneeded if \( X \) is simple, where one goes directly to the third step.

3. Once 1 and 2 are done and one arrives at a strictly mean convex \( X_\varepsilon \), which satisfy the above, then, as earlier, one takes the \( C^\infty \)-smoothed boundary of a small \( \varepsilon \)-neighbourhood \( U_{\varepsilon_1}(X_\varepsilon) \) for the required approximation \( Y \) of \( \partial X \) (compare with 5.7 in [Gr2014] and 11.4 in [Gr2018]).

Remark. It would take a couple of extra pages to explicitly write down the (quite boring) details of the above argument but it would add nothing new to what we have already seen in the convex case.

**Convexly Stratified Manifolds.** The step 2 in the above argument takes \( X \) out of the category of manifolds with corners, where the new manifolds are locally diffeomorphic not to convex polytopes but to certain smoothly stratified convex subsets \( X \subset \mathbb{R}^n \), such, for instance, as cones over smooth convex bodies in \( \mathbb{R}^{n-1} \subset \mathbb{R}^n \).

The most general class \( \mathcal{X} \) of such \( X \), where the statement of theorem 2.A makes sense, consists of closed convex domains \( \mathcal{X} \) such that the boundaries of \( X \) are piecewise smooth in the complements of closed \((n-3)\)-dimensional subset \( Z \subset \partial X \).

Probably, the proof of theorem 2.A can be extended to the corresponding class \( \mathcal{X} \) of mean convex Riemannian manifolds \( X \) locally diffeomorphic to such \( \mathcal{X} \).

5 **Lipschitz Maps and the Proof of Theorems 1.A. and 2.C.**

Here is an essential, albeit elementary (and trivial), geometric fact one needs.

**5.A. Lipschitz Mapping Lemma.** Let \( Y \) be a closed orientable Riemannian \((n-1)\)-manifold and \( \phi \) be a continuous map from \( Y \) to the boundary of the \( n \)-cube \( \Box^n = [-1,1]^n \),

\[
\phi : Y \to \partial \Box^n,
\]

such that the distances between the pullbacks of the opposite faces \( \Box^{n-1}_i < \Box^n \), \( i = 1, \ldots, n \), satisfy

\[
dist(\phi^{-1}(\Box^{n-1}_i), \phi^{-1}(\Box^{n-1}_i)) > D.
\]

Then the composition of \( \phi \) with the obvious radial homeomorphism from \( \partial \Box^n \) to the unit sphere

\[
\partial \Box^n \to S^{n-1} \subset \Box^n
\]

is homotopic to a smooth map,

\[
f : Y \to S^n,
\]
such that the differential of $f$ satisfies

$$\|df\| < \frac{2\sqrt{n}}{D}.$$ 

**Proof.** Let $\delta_i(y)$ be the distance functions to $\phi^{-1}(\square_i^{-1}) \subset Y$ truncated by $D' > D$ such that

$$D < D' < \max_i \text{dist}(\phi^{-1}(\square_i^{-1}), \square_i^{-1}),$$

namely,

$$\delta_i(y) = \min(\text{dist}(y, \phi^{-1}(\square_i^{-1})), D')$$

and observe that the map

$$\Delta = \left( \frac{2}{D'} \delta_i(y) - 1, \ldots, \frac{2}{D'} \delta_n(y) - 1 \right)$$

sends

$$Y \to \partial \square^n$$

for $\square^n = [-1, 1]^n \subset \mathbb{R}^n$,

that this map is homotopic to $\phi$ and that it is $\frac{2\sqrt{n}}{D'}$-Lipschitz. Since the radial map radial map $\partial \square^n \to S^{n-1} \to S^{n-1}$ is distance decreasing and $D' > D$, the composed map

$$Y \stackrel{\Delta}{\to} \square^n \to S^n$$

can be approximated by the required $f$.

**5.B. Conclusion of the Proof of Theorem 2.C.** Let $X$ be a Riemannian manifold with corners as in 2.C, let $\square^n(X) > D$ and let $\Phi : X \to \square^n$ be a continuous combinatorial map, which satisfies $\bullet_{\text{dist}_0}$ and $\bullet_{\text{deg}}$ from section 1 and also $\bullet_{\text{dist}_0}$, but now with $D$ instead of $d$.

Then a smooth mean convex hypersurface $Y \subset X$, which approximates $\partial X$ according to 2.B, which we assume strictly mean convex and which we endow with the metric $\text{dist}^k$, and the map $\phi = \Phi : Y \to \partial \square^n$ satisfy the assumptions of 5.A. Hence, $(Y, \text{dist}_k)$ admits a $\lambda$-Lipschitz map to the unit sphere $S^{n-1}$ for $\lambda < \frac{2\sqrt{n}}{D'}$ as in 5.A.

Since the degree of this map doesn’t vanish according to $\bullet_{\text{deg}}$, theorem 2.A says that $\lambda \geq \frac{1}{\text{mean.curv}(S^{n-1})} = \frac{1}{n-1}$, which implies that $\frac{2\sqrt{n}}{D'} \leq \frac{1}{n-1}$ and

$$D < 2(n-1)\sqrt{n}.$$ 

QED

**6 Combinatorial Waists and $\lambda^{n-k-1}$-Angles**

The $P^k$-overlaps of a map from a manifold $X$ with corners, e.g. from a polytope, to some set, say $\alpha : X \to \Xi$, denoted

$$\hat{\mu}_\alpha^k(X) \text{ and } \hat{\nu}_\alpha^k(X),$$

are the maxima of the numbers of open, respectively closed, $l$-faces in $X$ the $\alpha$-images of which in $\Xi$ have a common point $\xi$. 

19
For instance, generic linear maps $\alpha$ from $n$-polytopes $X \subset \mathbb{R}^n$ to $\mathbb{R}^{n-1}$ satisfy $\hat{\#}^{n-1}_\alpha(X) = 2$ and, if $X$ simple, then $\hat{\#}^{n-1}_\alpha(X) = n + 1$.

The spherical $(n-k-1)$-(co)angle of a convex subset $X \subset \mathbb{R}^n$, e.g. a Euclidean $n$-polytope, at a point $x \in \partial X$, denoted

$$\gamma^{n-k-1}_x(X),$$

is the $(n-k-1)$-dimensional spherical volume (Hausdorff measure) of the set of the supporting planes to $X$ at $x$, and where we denote

$$\gamma^{n-k-1}_{F^k} = \gamma^{n-k-1}_x(\bigcup_{x \in F^k} X)$$

for open and closed $k$-faces $F^k$ and for $F^k_\circ \subset F^k$ being the interior parts of these faces.

Then the "angle" $\gamma^{n-k-1}_x(X)$ at a point $x$ in a Riemannian manifold $X$ with corners is defined as the corresponding angle of the tangent cone of $X$ at $x$.

For instance, if $x \in F^{n-2}$, this is the complementary dihedral angle of the face $F^{n-2}$ defined earlier.

Define $\sum^{n-k-1}_\alpha(X)$ and $\sum^{n-k-1}_\alpha(\alpha^{-1}(\xi))$ for Riemannian manifolds $X$ with corners as the supremum over $\xi \in \Xi$ of the sums of these angles over the set of non-empty intersections of the $k$-faces $F^k$ in $X$ with the $\alpha$-pullbacks of points $\xi$ in $\Xi$,

$$\sup_{\xi \in \Xi} \sum_{F^k \in F^{n-k}} (F^k \cap \alpha^{-1}(\xi) \neq \emptyset),$$

i.e. the sum is taken over all open, respectively closed, $k$-faces $F^k \subset X$, which intersect $\alpha^{-1}(\xi)$.

Remark. This definition makes sense for all weight functions $w$ on the faces instead of $\gamma^{n-k-1}_x$, where, e.g. for $w(x) = 1$, one recaptures the numbers $\hat{\#}^{n-k}_\alpha(X)$ and $\hat{\#}^{n-k}_\alpha(X)$.

Problem. Given a class $A$ of function $\alpha$, evaluate possible values $\hat{\#}^{n-k}_\alpha(X)$ and $\sum^{n-k-1}_\alpha(X)$ for convex polytopes and other "interesting" manifolds with corners in terms of other geometric invariants.

Example 6.A. Let $X \subset \mathbb{R}^n$ be a convex polytope and $\alpha : X \to \mathbb{R}^{n-2}$ a continuous map. If $X$ is simple, then the number $\hat{\#}^{n-2}_\alpha(X)$ is bounded from below by the combinatorial $\square$-spread of $X$ as follows

$$\hat{\#}^{n-2}_\alpha(X) \geq \text{cost}_n \cdot \square^{n}_{\text{comb}}(X).$$

Sketch of the Proof. Let $g_e$ be a Riemannian metric on $\partial X$, which distance-wise $\varepsilon$-approximates $\text{dist}_{\text{comb}}$ on $X$. By the argument from the previous section, $(\partial X, g_e)$ admits a 1-Lipschitz map $\Phi$ of non-zero degree to the sphere $S^{n-1}(R)$ of radius $R \geq \text{const}_\alpha \cdot \square^{n}_{\text{comb}}(X)$. It follows by the (quite elementary) $1$-waist inequality for spheres (see Guth2014 and references therein) the $\Phi$-image of the pullback $\alpha^{-1}(\xi)$, $\xi \in \mathbb{R}^{n-2}$, has length $\geq 2\pi R$. Hence, the $g_e$-length of $\alpha^{-1}(\xi)$ is also $\geq 2\pi R$, which, since $X$ is simple, implies the required bound $\hat{\#}^{n-2}_\alpha(X) \geq \text{const}_\alpha R$ for $\varepsilon \to 0$.

Probably, a similar argument applies to continuous maps $\alpha : X \to \mathbb{R}^k$ for all $k = 1, \ldots, n-2$, thus showing, at least for manifolds $X$ with simple corners, that

$$\hat{\#}^{n-2}_\alpha(X) \geq \text{cost}_n \cdot (\square^{n}_{\text{comb}}(X))^{n-k-1}$$

18 This is probably redundant.
for all continuous $\alpha$.

But it is unclear what happens to $\sum_{\alpha}^{k}$.

For instance, let $X \subset \mathbb{R}^n$ be a convex polytope and $k = 1, \ldots, n - 2$.

**Question 6.B.** Is

$$\inf_{\alpha} \sum_{\alpha}^{k}(X) \leq cost_{n}$$

the infimum is taken over all continuous (may be even linear?) maps $\alpha : X \to \mathbb{R}^k$?

**Question 6.C.** Does there exist an $(n-k)$-dimensional affine subspace $A \subset \mathbb{R}^n$, which transversally meets $N > 0$ (open) $k$-dimensional faces $F^k_i \subset X$, $i = 1, \ldots N$, such that

$$\frac{1}{N} \sum_{i=1}^{N} F^k_{\min} (X) \leq \frac{const_{n}}{(\square_{\text{comb}}(X))^{n-k-1}}?$$

The positive answers to this would yield the following generalization of corollary 1.B to $k \leq n - 3$.

**Conjecture 6.D.** If the combinatorial $\square^n$-spread of a convex polytope $X \subset \mathbb{R}^n$ is large, then there exists a $k$-dimensional face $F^k_{\min} \subset X$ with small $\gamma_{n-k-1}$-angle:

$$\gamma_{F^k_{\min}} (X) \leq \text{const}_{n} (\square^n_{\text{comb}}(X))^{-(n-k-1)} ,$$

or, at least,

$$\gamma_{F^k_{\min}} (X) \to 0 \text{ for } \square^n_{\text{comb}} \to \infty.$$

for simple polytopes $X$.

7 Surgery with Corners and Related Problems

It is claimed in section 1.3 of [Gr2014'] that the so called staircase thin surgery of mean convex manifolds with positive scalar curvatures \footnote{See [GL1980], [BaDoSo2018], [Gr2021].} can be also applied to manifolds $X$ with corners. However, I overlooked the difficulty in proving the following.

**7.A. $\prec$-Shrinking Problem.** Let $Y_0 \subset S^{n-1} \subset \mathbb{R}^n$ be a convex spherical polytope.

Does there exist a continuous deformation $Y_t \subset S^{n-1}$, $0 \leq t \leq 1$, of $Y_0$, where all $Y_t$ for $t < 1$ are convex spherical polytopes combinatorially isomorphic to $X_0$ and having their dihedral angles bounded by the corresponding angles of $Y_0$ and where $Y_1$ is a single point?

It is easy to construct such a $Y_t$ for $\text{dim}(Y_0) = 2$, and also for "sufficiently round" spherical polytopes of dimensions $>2$, where such shrinking can be achieved by projective transformations of $Y_0$, but I was unable to prove or disprove it for general $Y \subset S^{n-1}$ if $n \geq 4$.

And granted such a deformation for the spherical base $Y_0$ of the tangent cone $TC_{x_0}(X) \subset T_{x_0}(X) = \mathbb{R}^n$, at a vertex $x_0 \subset X$, say for a strictly mean convex domain $X \subset \mathbb{R}^n$ with corners, the staircase construction delivers another strictly mean convex $X_1 \subset \mathbb{R}^n$, such that
\* cut the domain $X_1$ is diffeomorphic to $X$ with the vertex $x_0$ cut away by a hyperplane $H_0 \subset \mathbb{R}^n$ parallel to a supporting hyperplane of $X$ at $x_0$, where this diffeomorphism moves all points at most by a given $\varepsilon > 0$ and fixes the points $\varepsilon$-far from $x_0$;
\* the dihedral angles at the (old) $(n - 1)$-faces of $X_1$ away from the cut $X \cap H_0$ are bounded by the corresponding angles of $X$;
\* the dihedral angles between the new $(n - 1)$-face corresponding to $X \cap H_0$ with the old ones are equal to $\pi/2$.

Observe that if $n = 3$, this construction, when applied to all vertices of $X$, delivers a simple polytope and thus provides an alternative reduction of the general case of theorem 1.A to that for simple $X$.

However, since 7.A. remains problematic for $\dim(Y) \geq 3$, the thin surgery at the corners remains problematic as well.

Also pondering over 7.A brings to one’s mind the following more general problems.

7.B. Variation Problem. Find the homotopy type of the space $\mathcal{H}(C, \kappa)$ of (possible) dihedral angles of convex $n$-polytopes $X$ of given combinatorial type $C$ in the space of constant curvature $\kappa$ and determine how this space varies depending on $\infty < \kappa < \infty$.

7.C. Scalar Curvature Problem. Let $X$ be a compact connected smooth manifold with corners, let $\infty < \mu_i < \infty$ be numbers associated to the $(n - 1)$-faces of $X$ and $0 < \alpha_j < \pi$ be associated to the $(n - 2)$-faces. Determine the homotopy type of the space $\mathcal{G}(X, \sigma, \mu_i, \alpha_j), \sigma > 0$ of Riemannian metrics $g$ on $X$ such that

\* the scalar curvature of $X$ satisfies:

$$Sc(X) > \sigma;$$

\* the mean curvatures of the $(n - 1)$-faces of $X$ satisfy:

$$\text{mean.curv}_g(F_{n-1}^i) > \mu_i;$$

\* the complementary dihedral angles at the $(n - 2)$-faces satisfy:

$$\gamma_g((F_{n-2}^j) \geq \alpha_j.$$

Also determine how this space varies depending on $(\sigma, \mu_i, \alpha_j)$.

8 On Random Polytopes

Let $\Sigma = \{\sigma_i\}_{i=1}^{N+1} \subset S^{n-1}$ be randomly chosen points on the unit sphere and $X_N = X(\Sigma)$ be the (necessarily simple) convex polyhedron defined by the tangent hyperplanes to the sphere at the points $\sigma_i$.

Let $d_{\text{comb},N}(s_1, s_2), s_1, s_2 \in S^{n-1}$ be the combinatorial distance between the $(n - 1)$-faces $F_1, F_2 \subset X$ of $X_N$ the normal projections of which to $S^{n-1}$ contains the points $s_1$ and $s_2$ respectively. (Never mind the distinction between open and closed faces.)

8.A. Spherical $d_{\text{comb},N}$-Conjecture. There exists a universal constant $\Delta_n$, such that

$$\frac{d_{\text{comb},N}(s_1, s_2)}{N \cdot d_{S^{n-1}}} \rightarrow \Delta_n \text{ for } N \rightarrow \infty.$$
with probability 1 for all pairs of points $s_1, s_2 \in S^{n-1}$:

the probability of the inequality $\left| \frac{\text{dist}_{\text{comb}, N}(s_1, s_2)}{\text{dist}_{S^{n-1}}} - \Delta_n \right| > \varepsilon$ tends to zero for $N \to \infty$ for all $\varepsilon > 0$.

Remark. Probably, this follows by the results/arguments from [BDGHL2021] but I haven’t looked at this closely. In any case an elementary (Poisson) percolation argument shows that, with overwhelming probability,

$$\frac{\text{dist}_{\text{comb}, N}(s_1, s_2)}{N \cdot \text{dist}_{S^{n-1}}} \leq \text{cost}_n$$

and

$$\frac{\text{dist}_{\text{comb}, N}(s_1, s_2)}{N \cdot \text{dist}_{S^{n-1}}} \geq \frac{\text{cost}'}{\log N}$$

for $N \to \infty$.

8.A. Spherical dist$_{1}$-Conjecture. Let $\text{dist}_1(s_1, s_2) = \text{dist}_1(F_1, F_2)$ for the above $F_1, F_2$. Then there exists a universal constant $\Delta_n^1$ such that

$$\frac{\text{dist}_1(s_1, s_2)}{\text{dist}_{S^{n-1}}} \to \Delta_n^1 \text{ for } N \to \infty$$

with probability 1 for all pairs of points $s_1, s_2 \in S^{n-1}$.

Remark. Exact evaluation of $\Delta_n$ and $\Delta_n^1$ may be difficult but the ratio $\Delta_n / \Delta_n^1$ seems computable.

There are other commonly used definition of "random polytope" (see [Schneider2008]); we single out the following.

Let $C(n, M)$ be the set of combinatorial types of simple $n$-polyhedra $X$ with $M$ faces and observe that the cardinality of this set is pinched between two exponentials:

$$A^M \leq \#C(n, M) \leq B^M$$

Cutting $X$ by hyperplanes in two parts suggests that $\log \#C(n, M)$ is (essentially) super-additive, and the limit

$$\lim_{M \to \infty} \frac{\log \#C(n, M)}{M},$$

which seems an interesting number, exists.

Then we assign equal probabilities to all points (combinatorial types) in $C(n, M)$ and conjecture that

the graphs $E = E(X)$ of the so defined random $n$-polytopes $X$ with $M$ faces endowed with metrics $M \to \text{dist}_{\text{comb}}$ Hausdorff converge to the sphere $S^{n-1}(R_n)$ of some radius $R_n$ for $M \to \infty$.

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There is an extensive literature on random polytopes, where much of known estimates of the sizes of random polytopes concern upper bounds on combinatorial edge-diameters, which are motivated by the Hirsch conjecture, while we are interested on lower bounds on the spreads.
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