Calibrated representations of affine Hecke algebras

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Abstract. This paper introduces the notion of calibrated representations for affine Hecke algebras and classifies and constructs all finite dimensional irreducible calibrated representations. The main results are that (1) irreducible calibrated representations are indexed by placed skew shapes, (2) the dimension of an irreducible calibrated representation is the number of standard Young tableaux corresponding to the placed skew shape and (3) each irreducible calibrated representation is constructed explicitly by formulas which describe the action of each generator of the affine Hecke algebra on a specific basis in the representation space. This construction is a generalization of A. Young’s seminormal construction of the irreducible representations of the symmetric group. In this sense Young’s construction has been generalized to arbitrary Lie type.

0. Introduction

The affine Hecke algebra was introduced by Iwahori and Matsumoto [IM] as a tool for studying the representations of a p-adic Lie group. In some sense, all irreducible principal series representations of the p-adic group can be determined by classifying the representations of the corresponding affine Hecke algebra. Unfortunately, it is not so easy to determine the irreducible representations of the affine Hecke algebra.

Kazhdan and Lusztig [KL] (see also the important work of Ginzburg [CG]) gave a geometric classification of the irreducible representations of the affine Hecke algebra. This classification is a q-analogue of Springer’s construction of the irreducible representations of the Weyl group on the cohomology of unipotent varieties. In the q-case, K-theory takes the place of cohomology and the irreducible representations of the affine Hecke algebra are constructed as quotients of the K-theory of special subvarieties of the flag variety. Although the classification of Kazhdan and Lusztig is an incredible tour-de-force it is difficult to obtain combinatorial information from this geometric construction. For example, it is difficult to determine the dimensions of the irreducible modules.

In this paper I give a new construction of a large family of irreducible modules of the affine Hecke algebra. The basis vectors are labeled by generalized standard Young tableaux and the action of each generator on each basis element is given explicitly. This construction is a generalization of Young’s seminormal construction of the irreducible representations of the symmetric group. In order to obtain this generalization I have had to generalize the concept of standard Young tableaux to arbitrary Lie type.

The modules which I construct I have termed “calibrated” modules. Specifically, a calibrated module is a module which has a basis of simultaneous eigenvectors for all the elements of a large

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commutative algebra inside the affine Hecke algebra. This is analogous to the situation which occurs for representations of complex semisimple Lie algebras where every finite dimensional module is a direct sum of its weight spaces. In contrast to the complex semisimple Lie algebra case, it is never true that all irreducible representations of the affine Hecke algebra are calibrated.

The irreducible really calibrated modules for the affine Hecke algebra are indexed by placed skew shapes, where “placed skew shape” is a generalization of the usual skew shape from combinatorial representation theory and symmetric function theory. As is to be expected, these new generalized skew shapes and standard Young tableaux reduce to the classical objects in the Type A case. This reduction is given in [Ra2].

Remarks on the results in this paper

(1) It is quite a surprise that the seminormal construction of A. Young fits so nicely into general Lie type. Up to now, the general feeling has been that Young’s results are very special to the symmetric group and the Type A case. The direct generalization of Young’s seminormal construction to arbitrary Lie type which is obtained in this paper shows that this is not the case at all. The results of [Ra4] indicate how Young’s natural basis can also be generalized to arbitrary Lie type.

The seminormal representations of A. Young have been previously generalized to Iwahori-Hecke algebras of Type A by Hoefsmit [H] and Wenzl [Wz] independently, to Iwahori-Hecke algebras of types B and D by Hoefsmit [H] and to the cyclotomic Hecke algebras $H_{r,1,n}$ by Ariki and Koike [AK]. All of these generalizations use classical standard Young tableaux and similar formulas for the action of the generators of the Hecke algebra. Using certain surjective homomorphisms [A] from the affine Hecke algebras of type A to the algebras $H_{r,1,n}$ one can easily show that these earlier constructions are type A special cases of the general type construction given in this paper.

In a previous paper [Ra1] I gave a method for generalizing Young’s theory of seminormal representations to general Lie type. I now believe that this earlier idea was not the “proper” way to proceed. The method here is much more natural and yields a cleaner and more beautiful theory. Young’s classical formulas for the seminormal representations of the symmetric group $S_n$ work in general Lie type with no change at all!! The only previously missing ingredient was a good general type definition of standard Young tableaux.

(2) In the classical theory of representations of the symmetric group $S_n$ the “skew shape representations” are particularly well behaved $S_n$-modules. On the other hand there never seemed to be any a priori raison d’etre for skew shape representations via which one could generalize this concept to Weyl groups and Iwahori-Hecke algebras of other Lie types. The results in this paper show that skew shape representations do arise in a perfectly natural way. They correspond to irreducible representations of affine Hecke algebras. In the type A cases one recovers the classical skew shape representations by restricting to the Iwahori-Hecke algebra inside the affine Hecke algebra.

(3) The two main techniques used in this paper are generalizations of the techniques of Matsumoto [Ma] and Rodier [Ro]. In particular, the $\tau$-operators and the calibration graphs $\Gamma(t)$ introduced in Section 2 are generalizations of the intertwining operators of Matsumoto and of the graphs used by Rodier, respectively. I have changed the role of the intertwining operators by having them be “left” operators instead of “right” operators. This means that they are no longer intertwining but there are other benefits to using these operators in this fashion.

(4) Heckman and Opdam [HO1-2] introduced a new “harmonic analysis” approach to the representations of the affine Hecke algebra. In their work they also used the sets $Z(t)$ and $P(t)$ which we use in section 3. These sets arise naturally in their work as the zeros and poles of a certain Harish-Chandra $c$-function. In this paper these sets describe the behaviour of the $\tau$-
operators mentioned in remark (3). This means that there is a strong connection between the $c$-function and these operators. The approach of Heckman-Opdam becomes difficult when one needs to compute the residues of the $c$-function at certain singular points. In some cases these difficulties can be surmounted by using the methods of this paper.

(5) The Kazhdan-Lusztig construction of the irreducible representations of the affine Hecke algebras shows that the structure of these representations is intimately connected with the geometry of certain subvarieties $B_{s,u}$ of the flag variety. It is my hope that the methods which I have used here for studying representations of affine Hecke algebras, which are mostly combinatorial in nature, will be useful for studying the geometry of the $B_{s,u}$ varieties used in the Kazhdan-Lusztig construction. I am hoping that the standard Young tableaux introduced here can be used as index sets for the connected components of these varieties.

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1. The affine Hecke algebra

Let $R$ be a reduced irreducible root system in $\mathbb{R}^n$, fix a set of positive roots $R^+$ and let \{\(\alpha_1, \ldots, \alpha_n\)\} be the corresponding simple roots in $R$. Let $W$ be the Weyl group corresponding to $R$. Let $s_i$ denote the simple reflection in $W$ corresponding to the simple root $\alpha_i$ and recall that $W$ can be presented by generators $s_1, s_2, \ldots, s_n$ and relations

\[
s_i^2 = 1, \quad s_is_js_i\cdots = s_js_is_j\cdots, \quad \text{for } 1 \leq i \leq n, \quad \text{for } i \neq j,\]

where $m_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$.

Fix $q \in \mathbb{C}^*$ such that $q$ is not a root of unity. The *Iwahori-Hecke algebra* $H$ is the associative algebra over $\mathbb{C}$ defined by generators $T_1, T_2, \ldots, T_n$ and relations

\[
T_i^2 = (q - q^{-1})T_i + 1, \quad T_iT_j = T_jT_i, \quad \text{for } 1 \leq i \leq n, \quad \text{for } i \neq j, \tag{1.1}
\]
where $m_{ij}$ are the same as in the presentation of $W$. For $w \in W$ define $T_w = T_{i_1} \cdots T_{i_p}$ where $s_{i_1} \cdots s_{i_p} = w$ is a reduced expression for $w$. By [Bou, Ch. IV §2 Ex. 23], the element $T_w$ does not depend on the choice of the reduced expression. The algebra $H$ has dimension $|W|$ and the set $\{T_w\}_{w \in W}$ is a basis of $H$.

The fundamental weights are the elements $\omega_1, \ldots, \omega_n$ of $\mathbb{R}^n$ given by

$$\langle \omega_i, \alpha_j' \rangle = \delta_{ij}, \quad \text{where} \quad \alpha_i' = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

and $\delta_{ij}$ is the Kronecker delta. The weight lattice is the $W$-invariant lattice in $\mathbb{R}^n$ given by

$$P = \sum_{i=1}^n \mathbb{Z}\omega_i.$$

Let $X$ be the abelian group $P$ except written multiplicatively. In other words,

$$X = \{X^\lambda \mid \lambda \in P\}, \quad \text{and} \quad X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda, \quad \text{for} \lambda, \mu \in P.$$

Let $\mathbb{C}[X]$ denote the group algebra of $X$. There is a $W$-action on $X$ given by

$$wX^\lambda = X^{w\lambda} \quad \text{for} \ w \in W, \ X^\lambda \in X,$$

which we extend linearly to a $W$-action on $\mathbb{C}[X]$.

The affine Hecke algebra $\tilde{H}$ associated to $R$ and $P$ is the algebra given by

$$\tilde{H} = \mathbb{C}\text{-span}\{T_wX^\lambda \mid w \in W, X^\lambda \in X\}$$

where the multiplication of $T_w$ is as in the Iwahori-Hecke algebra $H$, the multiplication of the $X^\lambda$ is as in $\mathbb{C}[X]$ and we impose the relation

$$X^\lambda T_i = T_iX^{s_i\lambda} + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}, \quad \text{for} \ 1 \leq i \leq n \text{ and} \ X^\lambda \in X. \quad (1.2)$$

This formulation of the definition of $\tilde{H}$ is due to Lusztig [Lu2] following work of Bernstein and Zelevinsky. The elements $T_wX^\lambda$, $w \in W, X^\lambda \in X$, form a basis of $\tilde{H}$.

**Theorem 1.3.** (Bernstein, Zelevinsky, Lusztig [Lu1, 8.1]) The center of $\tilde{H}$ is $\mathbb{C}[X]^W = \{f \in \mathbb{C}[X] \mid wf = f\}$. 

### 2. Weight spaces and calibration graphs

**Weights**

Let

$$T = \{\text{group homomorphisms} \ t : X \to \mathbb{C}^*\}.$$ 

The torus $T$ is an abelian group with a $W$-action given by $(wt)(X^\lambda) = t(X^{w^{-1}\lambda})$. Any element $t \in T$ is determined by the values $t(X^{\omega_1}), t(X^{\omega_2}), \ldots, t(X^{\omega_n})$. For any element $t \in T$ define the polar decomposition

$$t = t_r t_c, \quad t_r, t_c \in T \text{ such that} \ t_r(X^\lambda) \in \mathbb{R}_{\geq 0}, \text{ and} \ |t_c(X^\lambda)| = 1,$$
for all $X^\lambda \in X$. There is a unique $\mu \in \mathbb{R}^n$ and a unique $\nu \in \mathbb{R}^n/P$ such that

$$
t_r(X^\lambda) = e^{i(\mu, \lambda)} \quad \text{and} \quad t_c(X^\lambda) = e^{2\pi i (\nu, \lambda)}, \quad \text{for all } \lambda \in P.
$$

(2.1)

In this way we identify the sets $T_r = \{ t \in T \mid t = t_r \}$ and $T_c = \{ t \in T \mid t = t_c \}$ with $\mathbb{R}^n$ and $\mathbb{R}^n/P$, respectively.

**Weight spaces**

Let $M$ be a finite dimensional $\tilde{H}$-module. For each $t \in T$ the $t$-weight space of $M$ and the generalized $t$-weight space are the subspaces

$$
M_t = \{ m \in M \mid X^\lambda m = t(X^\lambda)m \text{ for all } X^\lambda \in X \} \quad \text{and}
$$

$$
M_t^{\text{gen}} = \{ m \in M \mid \text{for each } X^\lambda \in X, (X^\lambda - t(X^\lambda))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0} \}.
$$

respectively. If $M_t^{\text{gen}} \neq 0$ then $M_t \neq 0$. In general $M \neq \bigoplus_{t \in T} M_t$, but we do have

$$
M = \bigoplus_{t \in T} M_t^{\text{gen}}.
$$

This is a decomposition of $M$ into Jordan blocks for the action of $\mathbb{C}[X]$. Define the support of $M$ to be

$$
\text{supp}(M) = \{ t \in T \mid M_t^{\text{gen}} \neq 0 \}.
$$

(2.2)

**Principal series modules**

Let $t \in T$ and let $\mathbb{C}v_t$ be the one dimensional $\mathbb{C}[X]$-module corresponding to the character $t: X \to \mathbb{C}^*$. Specifically, $\mathbb{C}v_t$ is the one dimensional vector space with basis $\{ v_t \}$ and $\mathbb{C}[X]$-action given by

$$
X^\lambda v_t = t(X^\lambda)v_t, \quad \text{for all } X^\lambda \in X.
$$

The principal series representation corresponding to $t$ is

$$
M(t) = \tilde{H} \otimes_{\mathbb{C}[X]} \mathbb{C}v_t.
$$

(2.3)

The set $\{ T_w \otimes v_t \mid w \in W \}$ is a basis for the $\tilde{H}$-module $M(t)$ and $\dim(M(t)) = |W|$.

If $w \in W$ and $X^\lambda \in X$ then the defining relation (1.2) for $\tilde{H}$ implies that

$$
X^\lambda(T_w \otimes v_t) = t(X^{w\lambda})(T_w \otimes v_t) + \sum_{u < w} a_u(T_u \otimes v_t),
$$

where the sum is over $u < w$ in the Bruhat-Chevalley order and $a_u \in \mathbb{C}$. It follows that the eigenvalues of $X$ on $M(t)$ are of the form $wt$, $w \in W$, and by counting the multiplicity of each eigenvalue we have

$$
M(t) = \bigoplus_{w^t \in W^t} M(t)_{w^t}^{\text{gen}} \quad \text{where} \quad \dim(M(t)_{w^t}^{\text{gen}}) = |W_t|, \quad \text{for all } w \in W.
$$

(2.4)
**Theorem 2.5.** (Kato’s irreducibility criterion [Ka]) Let $t \in T$ and define $P(t) = \{ \alpha > 0 \mid t(X^\alpha) = q^{\pm 2} \}$. The principal series module $M(t)$ is irreducible if and only if $P(t) = \emptyset$.

**Remark.** Kato actually proves a more general result and thus needs a further condition for irreducibility. We have simplified matters by specifying the weight lattice $P$ in our construction of the affine Hecke algebra. One can use any $W$-invariant lattice in $\mathbb{R}^n$ and Kato works in this more general situation. When one uses the weight lattice $P$, a result of Steinberg [St, 4.2, 5.3] says that the stabilizer $W_t$ of a point $t \in T$ under the action of $W$ is always a reflection group. Because of this Kato’s criterion takes a simpler form.

**Irreducible modules**

**Proposition 2.6.** Let $M$ be a finite dimensional $\tilde{H}$-module.

(a) For some $t \in T$, $M_t$ is nonzero.

(b) If $M$ is irreducible and $M_t \neq 0$ then $M$ is a quotient of $M(t)$.

(c) If $M$ is irreducible then $\dim(M) \leq |W|$.

**Proof.** (a) As an $X(T)$-module $M$ contains a simple submodule and this submodule must be one-dimensional since all irreducible representations of a commutative algebra are one-dimensional. Thus, there is a nonzero weight vector in $M$.

(b) Let $m_t$ be a nonzero vector in $M_t$. Then there is a unique $\tilde{H}$-module homomorphism determined by

$$
\phi: M(t) \rightarrow M
$$

$$
v_t \mapsto m_t
$$

where $v_t$ is as in the construction of $M(t)$ in (2.3) This map is surjective since $M$ is irreducible. Thus $M$ is a quotient of $M(t)$.

(c) follows from (b) since $\dim(M(t)) = |W|$.

It follows from Proposition (2.6b) and (2.4) that the support $\text{supp}(M)$ of an irreducible $\tilde{H}$-module $M$ is contained in a single Weyl group orbit in $T$. Since $M$ is irreducible and $\tilde{H}$ has countable dimension, Dixmier’s version of Schur’s lemma implies that $Z(\tilde{H})$ acts on $M$ by scalars. Let $t \in T$ be such that

$$
pM = t(p)M, \quad \text{for all } p \in Z(\tilde{H}).
$$

Since $Z(\tilde{H}) = \mathbb{C}[X(T)]^W$ it follows that $t(p(X)) = (wt)(p(X))$ for all $w \in W$. The $W$-orbit $Wt$ of $t$ is the *central character* of $M$. We shall often abuse notation and refer to any weight $s \in Wt$ as “the central character” of $M$.

**Remarks.**

(a) The algebra $\mathbb{C}[X]^W$ is the polynomial ring

$$
\mathbb{C}[X]^W = \mathbb{C}[\chi^{\omega_1}, \ldots, \chi^{\omega_n}],
$$

where $\omega_1, \ldots, \omega_n$ are the fundamental weights in $P$ and $\chi^{\omega_1}, \ldots, \chi^{\omega_n}$, are the corresponding Weyl characters. Thus, in order to specify the $W$-orbit of a weight $t \in T$ it is sufficient to specify the $n$ complex numbers $t(\chi^{\omega_1}), t(\chi^{\omega_2}), \ldots, t(\chi^{\omega_n})$. 
Let Proposition 2.7. Define the $\tau$ as a local operator. Let us describe this operator more precisely.

Proof. (a) Note that $(q-q^{-1})$ is an operator on $M^\text{gen}$. Let $s \in T$ given by

$$s(X^\lambda) = e^{(\gamma^\lambda)}, \quad \text{for all } \lambda \in P,$$

is in $Wt$ and this element is a canonical representative of the $W$-orbit $Wt$. In this way the dominant elements of $\mathbb{R}^n$ index the real central characters.

The $\tau$ operators

The maps $\tau_i : M^\text{gen}_1 \rightarrow M^\text{gen}_{s,t}$ defined below are local operators on $M$ in the sense that they act on each weight space $M^\text{gen}_1$ of $M$ separately. The operator $\tau_i$ is only defined on weight spaces $M^\text{gen}_1$ such that $t(X^{\alpha_i}) \neq 1$.

Proposition 2.7. Let $t \in T$ such that $t(X^{\alpha_i}) \neq 1$ and let $M$ be a finite dimensional $\hat{H}$-module. Define

$$\tau_i : M^\text{gen}_1 \rightarrow M^\text{gen}_{s,t},$$

$$m \mapsto \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right)m.$$

(a) The map $\tau_i : M^\text{gen}_1 \rightarrow M^\text{gen}_{s,t}$ is well defined.

(b) As operators on $M^\text{gen}_1$, $X^\lambda \tau_i = \tau_i X^s \lambda$, for all $X^\lambda \in X$.

(c) As operators on $M^\text{gen}_1$, $\tau_i \tau_i = \frac{(q-q^{-1})X^{\alpha_i}}{(1-X^{\alpha_i})}$. This is in $M^\text{gen}_1$ whenever both sides are well defined operators on $M^\text{gen}_1$.

(d) Both maps $\tau_i : M^\text{gen}_1 \rightarrow M^\text{gen}_{s,t}$ and $\tau_i : M^\text{gen}_{s,t} \rightarrow M^\text{gen}_1$ are invertible if and only if $t(X^{\alpha_i}) \neq q^{\pm 2}$.

(e) Let $1 \leq i \neq j \leq n$ and let $m_{ij}$ be as in (1.1). Then

$$\tau_i \tau_j \tau_i \ldots = \tau_j \tau_i \tau_i \ldots,$$

whenever both sides are well defined operators on $M^\text{gen}_1$.

Proof. (a) Note that $(q-q^{-1})/(1-X^{-\alpha_i})$ is not a well defined element of $\hat{H}$ or $\mathbb{C}[X]$ since it is not a polynomial in $X^{-\alpha_i}$. Because of this we will be careful to view $(q-q^{-1})/(1-X^{-\alpha_i})$ only as a local operator. Let us describe this operator more precisely.

The element $X^{\alpha_i}$ acts on $M^\text{gen}_1$ by $t(X^{\alpha_i})$ times a unipotent transformation. As an operator on $M^\text{gen}_1$, $1 - X^{-\alpha_i}$ is invertible since it has determinant $(1-t(X^{-\alpha_i}))^d$ where $d = \dim(M^\text{gen}_1)$. Since this determinant is nonzero $(q-q^{-1})/(1-X^{-\alpha_i}) = (q-q^{-1})(1-X^{-\alpha_i})^{-1}$ is a well defined operator on $M^\text{gen}_1$. Thus the definition of $\tau_i$ makes sense.

The following calculation shows that $\tau_i$ maps $M^\text{gen}_1$ into $M^\text{gen}_{s,t}$. For the purposes of this calculation we are viewing all elements of $\mathbb{C}[X]$ as local operators on $M^\text{gen}_1$ and we abuse notation and denote the operator $(1-X^{-\alpha_i})^{-1}$ by $1/(1-X^{-\alpha_i})$. We are able to do this without any problems because $\mathbb{C}[X]$ is commutative.

$$X^\lambda \tau_i m = \left(T_i X^{s \lambda} + (q-q^{-1})X^{\lambda} - 1\right) m$$

$$= \left(T_i X^{s \lambda} - (q-q^{-1})X^{s \lambda} \right) m$$

$$= \left(T_i - \frac{q^{-1}}{1-X^{-\alpha_i}}\right) X^{s \lambda} m$$

$$= \tau_i X^{s \lambda} m.$$
(b) follows from the previous calculation.

(c) If $t(X^{-\alpha_i}) \neq 1$ then both $\tau_i: M^\gen_t \rightarrow M^\gen_{s_i,t}$ and $\tau_i: M^\gen_{s_i,t} \rightarrow M^\gen_t$ are well defined. If $m \in M^\gen_t$ then

$$\tau_i \tau_i m = \left( T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} \right) \left( T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} \right) m$$

$$= \left( T_i^2 - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} T_i - T_i^2 \frac{q - q^{-1}}{1 - X^{-\alpha_i}} + \frac{(q - q^{-1})^2}{(1 - X^{-\alpha_i})^2} \right) m$$

$$= \left( (q - q^{-1}) T_i + 1 - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} T_i \right)$$

$$= \left( (q - q^{-1}) T_i + 1 - (q - q^{-1}) T_i + (q - q^{-1})^2 \frac{1 - X^{-\alpha_i} + 1}{(1 - X^{-\alpha_i})^2} \right) m$$

$$= \left( 1 + \frac{(q - q^{-1})^2}{(1 - X^{-\alpha_i})(1 - X^{-\alpha_i})} \right) m$$

$$= 2 - X^\alpha_i - X^{-\alpha_i} + q^2 - 2 + q^{-2} m$$

$$= \frac{(q - q^{-1} X^\alpha_i)(q - q^{-1} X^{-\alpha_i})}{(1 - X^\alpha_i)(1 - X^{-\alpha_i})} m.$$  

(d) The operator $X^\alpha_i$ acts on $M^\gen_t$ as $t(X^\alpha_i)$ times a unipotent transformation. Similarly for $X^{-\alpha_i}$. Thus, as an operator on $M^\gen_t \det((q - q^{-1} X^\alpha_i)(q - q^{-1} X^{-\alpha_i})) = 0$ if and only if $t(X^\alpha_i) = q^{\pm 2}$. Thus part (c) implies that $\tau_i \tau_i$ is invertible if and only if $t(X^\alpha_i) \neq q^{\pm 2}$. The statement follows.

(e) Let us begin with a slight diversion which will be helpful in the proof. Let $t \in T$ be a generic element of $T$ and let $M(t)$ be the corresponding principal series module. Since $t$ is generic, $W_t = \{1\}$ and

$$M(t) = \bigoplus_{w \in W} M(t)_{wt}, \quad \text{and} \quad \dim(M(t)_{wt}) = 1,$$

for all $w \in W$. We have $M(t)_{wt}^\gen = M(t)_{wt}$ since $M(t)_{wt}$ is nonzero whenever $M(t)_{wt}^\gen$ is nonzero and we know that $\dim(M(t)_{wt}^\gen) = 1$. Let $w \in W$ such that $\ell(s_i w) = \ell(w) + 1$. Since $t$ is generic $(wt)(X^\alpha_i) \neq q^{\pm 2}$ for all $w \in W$. Thus by part (d), the map $\tau_i: M(t)_{wt}^\gen \rightarrow M(t)_{s_i wt}^\gen$ is a bijection. Using the vector $v_t \in M(t)_t$ and the maps $\tau_i$ we can construct a basis $\{ v_{wt} \}_{w \in W}$ of $M(t)$ given by

$$v_{s_i wt} = \tau_i v_{wt}, \quad \text{if} \quad \ell(s_i w) = \ell(w) + 1.$$  

This basis is uniquely determined by the conditions

\begin{align*}
(2.8a) \quad X^\lambda v_{wt} &= (wt)(X^\lambda) v_{wt}, & \text{for all } w \in W \text{ and } X^\lambda \in X(T), \\
(2.8b) \quad v_{wt} &= T_w \otimes v_t + \sum_{u < w} a_{w u}(t) (T_u \otimes v_t), & \text{where } a_{w u}(t) \in \mathbb{C}.
\end{align*}

Now we proceed to the proof of the statement. We may assume that $\tilde{H}$ is the affine Hecke algebra corresponding to a rank two root system $R$ generated by simple roots $\alpha_i$ and $\alpha_j$. Let $w_0$ be the longest element of $W$. Every element $w \in W$, $w \neq w_0$ has unique minimal length expression.
as a product of generators $s_i$ and $s_j$. Let $T_w$ be the corresponding product of the $T_i$’s and $T_j$’s. Using the defining relation (1.2) for $\tilde{H}$ we expand to derive

$$\cdots \left( T_i - \frac{q - q^{-1}}{1 - X^{\alpha_i}} \right) \left( T_j - \frac{q - q^{-1}}{1 - X^{\alpha_j}} \right) \left( T_i - \frac{q - q^{-1}}{1 - X^{\alpha_i}} \right) = \cdots (2.8b) T_i T_j + \sum_{w \neq w_0} T_w P_w,$$

(2.9)

where the sum is over $w \in W$ such that $w \neq w_0$ and $P_w$ are rational functions of the $X^\alpha$, $\alpha \in R$. Similarly,

$$\cdots \left( T_j - \frac{q - q^{-1}}{1 - X^{\alpha_j}} \right) \left( T_i - \frac{q - q^{-1}}{1 - X^{\alpha_i}} \right) \left( T_j - \frac{q - q^{-1}}{1 - X^{\alpha_j}} \right) = \cdots T_j T_i T_j + \sum_{w \neq w_0} T_w Q_w,$$

(2.10)

where, as before, the sum is over $w \in W$ such that $w \neq w_0$ and $Q_w$ are rational functions of the $X^\alpha$, $\alpha \in R$. We shall show that $P_w = Q_w$.

Let $t \in T$ be generic and let $M(t)$ be the corresponding principal series module for $\tilde{H}$. By the analysis in the previous paragraph we have

$$v_{w_0 t} = \cdots T_i T_j T_i v_t = \cdots T_i T_j + \sum_{w \neq w_0} T_w P_w v_t$$

and it follows from (2.8b) that $t(P_w) = a_{w_0 w}(t)$ for all $w \in W$, $w \neq w_0$. One shows similarly that $t(Q_w) = a_{w_0 w}(t)$ for all $w \in W$, $w \neq w_0$.

We have shown that, for each $w \in W$, $t(P_w) = t(Q_w)$ for all generic $t \in T$. Since $P_w$ and $Q_w$ are rational functions which coincide on all generic points it follows that

$$P_w = Q_w \quad \text{for all } w \in W, w \neq w_0.$$  

(2.11)

Thus,

$$\cdots T_i T_j T_i = \cdots \left( T_i - \frac{q - q^{-1}}{1 - X^{\alpha_i}} \right) \left( T_j - \frac{q - q^{-1}}{1 - X^{\alpha_j}} \right) \left( T_i - \frac{q - q^{-1}}{1 - X^{\alpha_i}} \right) = \cdots \tau_i \tau_j \tau_i,$$

whenever both sides are well defined operators on $M^\text{gen}_t$. 


The calibration graph

Let \( t \in T \). Define a graph \( \Gamma(t) \) with

\[
\begin{align*}
\text{Vertices:} & \quad W_t, \\
\text{Edges:} & \quad wt \leftrightarrow s_i wt, \quad \text{if} \quad (wt)(X^\alpha) \neq q^{\pm 2}.
\end{align*}
\]

**Proposition 2.12.** If \( M \) is a finite dimensional \( \tilde{H} \)-module then

\[
\dim(M^\text{gen}_t) = \dim(M^\text{gen}_{t'})
\]

if \( t \) and \( t' \) are in the same connected component of the calibration graph.

**Proof.** It follows from Proposition (2.7d) that if there is an edge \( wt \leftrightarrow s_i wt \) in \( \Gamma(t) \) then the map

\[
\tau_i: M^\text{gen}_{wt} \rightarrow M^\text{gen}_{s_i wt}
\]

is a bijection. Thus, \( \dim(M^\text{gen}_t) = \dim(M^\text{gen}_{t'}) \) if \( t \) and \( s_i t \) are connected in the calibration graph \( \Gamma(t) \).

**Corollary 2.13.** If \( M \) is an irreducible \( \tilde{H} \)-module with central character \( t \) then the support \( \text{supp}(M) \) is a union of connected components of the calibration graph \( \Gamma(t) \).

The connected components of \( \Gamma(t) \)

Let \( t \in T \) and define

\[
Z(t) = \{ \alpha > 0 \mid t(X^\alpha) = 1 \}, \quad \text{and} \quad P(t) = \{ \alpha > 0 \mid t(X^\alpha) = q^{\pm 2} \}.
\]

If \( J \subseteq P(t) \) define

\[
\mathcal{F}^{(t,J)} = \{ w \in W \mid R(w) \cap Z(t) = \emptyset, \ R(w) \cap P(t) = J \},
\]

where \( R(w) = \{ \alpha > 0 \mid w\alpha < 0 \} \) is the inversion set of \( w \). Define a placed shape to be a pair \( (t, J) \) such that \( t \in T, J \subseteq P(t) \) and \( \mathcal{F}^{(t,J)} \neq \emptyset \). The elements of the set \( \mathcal{F}^{(t,J)} \) are called standard tableaux of shape \( (t, J) \).

**Theorem 2.14.** The connected components of the calibration graph \( \Gamma(t) \) are given by the partition of the vertices according to the sets

\[
\mathcal{F}^{(t,J)}_t, \quad \text{such that} \ J \subseteq P(t) \text{ and } \mathcal{F}^{(t,J)} \neq \emptyset.
\]

**Proof.** Let us begin by introducing appropriate notation. The chamber

\[
C = \{ x \in \mathbb{R}^n \mid \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in R^+ \}
\]

is a fundamental chamber for the action of \( W \) on \( \mathbb{R}^n \) and the complement \( \mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha) \) of the hyperplanes

\[
H_\alpha = \{ x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = 0 \}, \quad \alpha \in R^+,
\]

where \( R = \{ \alpha > 0 \mid w\alpha < 0 \} \) is the inversion set of \( w \). Define a placed shape to be a pair \( (t, J) \) such that \( t \in T, J \subseteq P(t) \) and \( \mathcal{F}^{(t,J)} \neq \emptyset \). The elements of the set \( \mathcal{F}^{(t,J)} \) are called standard tableaux of shape \( (t, J) \).

**Theorem 2.14.** The connected components of the calibration graph \( \Gamma(t) \) are given by the partition of the vertices according to the sets

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**Proof.** Let us begin by introducing appropriate notation. The chamber

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\]

is a fundamental chamber for the action of \( W \) on \( \mathbb{R}^n \) and the complement \( \mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha) \) of the hyperplanes

\[
H_\alpha = \{ x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = 0 \}, \quad \alpha \in R^+,
\]
in \( \mathbb{R}^n \) is the disjoint union of the chambers \( \{ w^{-1}C \mid w \in W \} \). A chamber \( w^{-1}C \) is on the positive side of the hyperplane \( H_\alpha \) if \( \langle x, \alpha \rangle > 0 \) for all \( x \in w^{-1}C \). The chambers adjacent to \( w^{-1}C \) are the chambers \( w^{-1}s_iC, 1 \leq i \leq n \), and the common face of \( w^{-1}C \) and \( w^{-1}s_iC \) is contained in the hyperplane \( H_{w^{-1}a_i} \).

Now let \( t \) be as in the statement of the Theorem. A result of Steinberg [St, 3.15, 4.2, 5.3] says that the stabilizer of \( t \) is

\[
W_t = \langle s_\alpha \mid \alpha \in Z(t) \rangle,
\]

the subgroup of \( W \) generated by the reflections in the hyperplanes orthogonal to the roots in \( Z(t) \). The elements of the orbit \( Wt \) can be identified with the cosets in \( W/W_t \) and these can be identified with the chambers of \( \mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha) \) which are on the positive side of the hyperplanes \( H_\alpha, \alpha \in Z(t) \). Under this bijection the element \( wt \in Wt \) is identified with the chamber \( w^{-1}C \). The elements \( wt \) and \( s_iwt \) are not connected by an edge in \( \Gamma(t) \) if and only if the hyperplane \( H_{w^{-1}a_i} \) containing the common face of the corresponding (adjacent) chambers \( w^{-1}C \) and \( w^{-1}s_iC \) is the hyperplane \( H_\beta \) for some root \( \beta \in P(t) \). In this way we can identify the graph \( \Gamma(t) \) with the graph with

- **Vertices:** chambers of \( \mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha) \) which are on the positive side of the hyperplanes \( H_\alpha, \alpha \in Z(t) \).
- **Edges:** faces of the chambers which are not contained in the hyperplanes \( H_\beta, \beta \in P(t) \).

The (closures of the) chambers \( w^{-1}C \) which are on the positive side of the hyperplanes \( H_\alpha, \alpha \in Z(t), \) form a convex region in \( \mathbb{R}^n \). This region is a disjoint union of smaller convex regions bounded by the hyperplanes \( H_\beta, \beta \in P(t) \). Each of the smaller regions is on the positive side of some of the hyperplanes \( H_\beta, \beta \in P(t) \), and on the negative side of others. In fact, it is determined by the set \( J \subseteq P(t) \) such that it is on negative side of the hyperplanes \( H_\beta, \beta \in J \). These smaller convex regions correspond to the connected components of \( \Gamma(t) \) and thus the connected components of \( \Gamma(t) \) are given by the sets \( \{ wt \mid w \in F(t,J) \} \) where \( J \subseteq P(t) \) and

\[
F(t,J) = \left\{ w \in W \mid \begin{array}{l}
w^{-1}C \text{ is on the positive side of the hyperplanes } H_\alpha, \alpha \in Z(t), \\
\text{on the positive side of the hyperplanes } H_\alpha, \alpha \in P(t) \setminus J, \\
\text{on the negative side of the hyperplanes } H_\beta, \beta \in J \end{array} \right\}.
\]

Since the chamber \( w^{-1}C \) is on the positive side of a hyperplane \( H_\alpha \) if and only if \( w\alpha > 0 \) it follows that

\[
F(t,J) = \{ w \in W \mid R(w) \cap Z(t) = \emptyset, \ R(w) \cap P(t) = J \}.
\]

3. **An \( \tilde{H} \)-module construction**

Let \( \alpha_i \) and \( \alpha_j \) be simple roots in \( R \) and let \( R_{ij} \) be the rank two root subsystem of \( R \) which is generated by \( \alpha_i \) and \( \alpha_j \). Let \( W_{ij} \) be the Weyl group of \( R_{ij} \), the subgroup of \( W \) generated by the simple reflections \( s_i \) and \( s_j \). A weight \( t \in T \) is calibrated for \( R_{ij} \) if one of the following two conditions holds:

(a) \( t(X_\alpha) \neq 1 \) for all \( \alpha \in R_{ij} \).

(b) \( R_{ij} \) is of type \( C_2 \) or \( G_2 \) (we may assume that \( \alpha_i \) is the long root and \( \alpha_j \) is the short root), \( ut(X_\alpha) = q^2 \) and \( ut(X_\alpha) = 1 \) for some \( u \in W_{ij} \), and \( t(X_\alpha) \neq 1 \) and \( t(X_\alpha) \neq 1 \).
A placed skew shape is a placed shape \((t, J)\) such that for all \(w \in \mathcal{F}(t, J)\) and all pairs \(\alpha_i, \alpha_j\) of simple roots in \(R\) the weight \(wt\) is calibratable for \(R_{ij}\).

**Theorem 3.1.** Let \((t, J)\) be a placed skew shape and let \(\mathcal{F}(t, J)\) be the set of standard tableaux of shape \((t, J)\). Define

\[
\tilde{H}^{(t, J)} = \mathbb{C}\text{-span}\{v_w \mid w \in \mathcal{F}(t, J)\},
\]

so that the symbols \(v_w\) are a labeled basis of the vector space \(\tilde{H}^{(t, J)}\). Then the following formulas make \(\tilde{H}^{(t, J)}\) into an irreducible \(\tilde{H}\)-module: For each \(w \in \mathcal{F}(t, J)\),

\[
X^\lambda v_w = (wt)(X^\lambda)v_w, \quad \text{for } X^\lambda \in X, \text{ and}
\]

\[
T_i v_w = (T_i)_{ww} v_w + (q^{-1} + (T_i)_{ww}) v_{s_iw}, \quad \text{for } 1 \leq i \leq n,
\]

where \((T_i)_{ww} = \frac{q - q^{-1}}{1 - (wt)(X^{-\alpha_i})}\), and we set \(v_{s_iw} = 0\) if \(s_iw \notin \mathcal{F}(t, J)\).

**Proof.** Since \((t, J)\) is a placed skew shape \((wt)(X^{-\alpha_i}) \neq 1\) for all \(w \in \mathcal{F}(t, J)\) and all simple roots \(\alpha_i\). This implies that the coefficient \((T_i)_{ww}\) is well defined for all \(i\) and \(w \in \mathcal{F}(t, J)\).

By construction, the nonzero weight spaces of \(\tilde{H}^{(t, J)}\) are \((\tilde{H}^{(t, J)})_{ww}^{\text{gen}} = (\tilde{H}^{(t, J)})_{wt}\) where \(w \in \mathcal{F}(t, J)\). These weight spaces have dimension 1 and are all in a single connected component of the calibration graph \(\Gamma(t)\). If \(N\) is a proper submodule of \(\tilde{H}^{(t, J)}\) then we would have \(N_{wt} \neq 0\) and \(N_{w't} = 0\) for some \(w \neq w', w, w' \in \mathcal{F}(t, J)\). But since \(wt\) and \(w't\) are in the same connected component of \(\Gamma(t)\) this would contradict Proposition 2.12. Thus \(\tilde{H}^{(t, J)}\) is irreducible if it is an \(\tilde{H}\)-module.

It remains to show that the defining relations for \(\tilde{H}\) are satisfied.

(a) Let \(w \in \mathcal{F}(t, J)\). Then

\[
\left( X^{s_i\lambda} T_i + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}} \right) v_w \\
= \left( (wt)(X^{s_i\lambda}) \frac{(q - q^{-1})}{1 - (wt)(X^{-\alpha_i})} + (q - q^{-1}) \frac{(wt)(X^\lambda) - (wt)(X^{s_i\lambda})}{1 - (wt)(X^{-\alpha_i})} \right) v_w \\
= (s_iwt)(X^{s_i\lambda})(T_i)_{s_iw, w} v_{s_iw} \\
= \frac{(q - q^{-1})}{1 - (wt)(X^{-\alpha_i})} (wt)(X^\lambda)v_w + (T_i)_{s_iw, w}(wt)(X^\lambda)v_{s_iw} \\
= T_i X^\lambda v_w.
\]

(b) Let \(w \in \mathcal{F}(t, J)\). Using the fact that \((T_i)_{ww} + (T_i)_{s_iw, s_iw} = q - q^{-1}\) we have

\[
T_i^2 v_w = ((T_i)_{ww}^2 + (q^{-1} + (T_i)_{ww})(q^{-1} + (T_i)_{s_iw, s_iw})) v_w \\
= (q^{-1} + (T_i)_{ww})(T_i)_{ww} + (T_i)_{s_iw, s_iw} v_{s_iw} \\
= (T_i)_{ww}(T_i)_{ww} + (T_i)_{s_iw, s_iw} v_{s_iw} + (q^{-1} + (T_i)_{ww} + (T_i)_{s_iw, s_iw}) v_{s_iw} \\
= (T_i)_{ww} q-q^{-1} + (q^{-1} + (T_i)_{ww})(q^{-1}) v_{s_iw} + q^{-1}(q^{-1} + q - q^{-1}) v_{s_iw} \\
= ((q - q^{-1}) T_i + 1) v_w.
\]
(c) The braid relation. Let $\alpha_i$ and $\alpha_j$ be simple roots in $R$ and let $w \in F^{(t,J)}$. Since $(t, J)$ is a placed skew shape $wt$ is calibratable. There are two distinct cases to consider.

**Case 1:** When $wt$ is $R_{ij}$-regular, i.e. $(uwt)(X^{\alpha_i}) \neq 1$ and $(uwt)(X^{\alpha_j}) \neq 1$ for all $u \in W_{ij}$.

Let us extend our notation $v_\sigma$ to all $\sigma \in W$ by assuming that $v_\sigma = 0$ whenever $\sigma \not\in F^{(t,J)}$. Then, for all $u \in W_{ij}$, the definition of the action of $T_i$ allows us to write

$$
\left( T_i - \frac{(q - q^{-1})}{1 - (wt)(X^{-\alpha_i})} \right) v_{uw} = \left( q^{-1} + \frac{(q - q^{-1})}{1 - (wt)(X^{-\alpha_i})} \right) v_{s_iuw} = \left( q - q^{-1}(wt)(X^{-\alpha_i}) \right) v_{s_iuw}
$$

whenever $\ell(s_iu) > \ell(u)$ and $(uwt)(X^{-\alpha_i}) \neq 1$. (This is correct because if $\ell(s_iu) > \ell(u)$ and $v_{uw} = 0$ then $v_{s_iuw} = 0$.)

Since $wt$ is $R_{ij}$-regular all of the factors in the following product are well defined and, by [Bou, Ch. VI §1 Cor. 2 to Prop. 17],

$$
\prod_{\alpha \in R_{ij}^+} \frac{q - q^{-1}(wt)(X^{-\alpha})}{1 - (wt)(X^{-\alpha})} = \ldots \left( \frac{q - q^{-1}(s_j s_i wt)(X^{-\alpha_i})}{1 - (s_j s_i wt)(X^{-\alpha_i})} \right) \left( \frac{q - q^{-1}(s_i wt)(X^{-\alpha_i})}{1 - (s_i wt)(X^{-\alpha_i})} \right) \left( \frac{q - q^{-1}(wt)(X^{-\alpha_i})}{1 - (wt)(X^{-\alpha_i})} \right)
$$

where the product is over all positive roots in the root subsystem $R_{ij}$ spanned by $\alpha_i$ and $\alpha_j$. Thus we get that

$$
\left( \prod_{\alpha \in R_{ij}^+} \frac{q - q^{-1}(wt)(X^{-\alpha})}{1 - (wt)(X^{-\alpha})} \right) v_{w_0w} = \ldots \left( T_i - \frac{q - q^{-1}}{1 - (s_j s_i wt)(X^{-\alpha_i})} \right) \left( T_j - \frac{q - q^{-1}}{1 - (s_i wt)(X^{-\alpha_i})} \right) \left( T_i - \frac{q - q^{-1}}{1 - (wt)(X^{-\alpha_i})} \right) v_w
$$

$$
= \ldots \left( T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} \right) \left( T_j - \frac{q - q^{-1}}{1 - X^{-\alpha_j}} \right) \left( T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} \right) v_w
$$

$$
= \ldots \underbrace{T_i T_j T_i}_{m_{ij} \text{ factors}} v_w + \sum_{u < u_0} T_u P_u v_w,
$$

where the notation in the last line is exactly the same as in (2.9). By similar reasoning we obtain

$$
\left( \prod_{\alpha \in R_{ij}^+} \frac{q - q^{-1}(wt)(X^{-\alpha})}{1 - (wt)(X^{-\alpha})} \right) v_{w_0w} = \ldots \underbrace{T_j T_i T_j}_{m_{ij} \text{ factors}} v_w + \sum_{u < u_0} T_u Q_u v_w,
$$

where the $Q_u$ are as in (2.10). By (2.11), $P_u = Q_u$ for all $u \neq u_0$ in $W_{ij}$ and thus

$$
\ldots \underbrace{T_i T_j T_i}_{m_{ij} \text{ factors}} v_w = \ldots \underbrace{T_j T_i T_j}_{m_{ij} \text{ factors}} v_w.
$$
Case 2: Let \( u \in W_{ij} \) be of minimal length such that \((uwt)(X^{\alpha_i}) = q^2\) and \((uwt)(X^{\alpha_j}) = 1\). The only possibilities are the following.

**Type \( C_2 \):**

1. \( u = s_i \): Then
   \[
   X^{\alpha_i} v_w = q^{-2} v_w, \quad T_i v_w = -q^{-1} v_w, \quad X^{\alpha_j} v_w = q^2 v_w, \quad T_j v_w = q v_w.
   \]

2. \( u = s_i s_j \): Then
   \[
   X^{\alpha_i} v_w = q^2 v_w, \quad T_i v_w = q v_w, \quad X^{\alpha_j} v_w = q^{-2} v_w, \quad T_j v_w = -q^{-1} v_w.
   \]

**Type \( G_2 \):**

1. \( u = s_i \): Then
   \[
   X^{\alpha_i} v_w = q^{-2} v_w, \quad X^{\alpha_2} v_w = q^2 v_w, \quad T_i v_w = -q^{-1} v_w, \quad T_j v_w = q v_w.
   \]

2. \( u = s_i s_j \) or \( u = s_i s_j s_i \). Then both \( w \) and \( s_i w \) are in \( \mathcal{F}^{(t,J)} \) and the action of \( X^{\alpha_i}, X^{\alpha_j}, T_i \) and \( T_j \) on \( \mathbb{C}\text{-span}\{v_w, v_{s_i w}\} \) is given by the matrices:
   \[
   T_i = \begin{pmatrix}
   q - q^{-1} & q - q^3 \\
   1 - q^{-4} & 1 - q^4 \\
   q - q^{-5} & q - q^{-1} \\
   1 - q^{-4} & 1 - q^4
   \end{pmatrix}, \quad T_j = \begin{pmatrix}
   -q^{-1} & 0 \\
   0 & q
   \end{pmatrix}.
   \]

3. \( u = s_i s_j s_i \): Then
   \[
   X^{\alpha_i} v_w = q^2 v_w, \quad X^{\alpha_j} v_w = q^{-2} v_w, \quad T_i v_w = q v_w, \quad T_j v_w = -q^{-1} v_w.
   \]

For each of these cases one checks the braid relations by direct computation. For type \( G_2 \) case (2) the calculations can be simplified by observing that \( T_1 T_2 T_1 = T_2 T_1 T_2 \) as operators on \( \mathbb{C}\text{-span}\{v_w, v_{s_i w}\} \). From this it follows that \( T_1 T_2 T_1 T_2 T_1 = T_2 T_1 T_2 T_1 T_2 T_1 \) as operators on \( \mathbb{C}\text{-span}\{v_w, v_{s_i w}\} \).

### 4. Calibrated Representations

A finite dimensional \( \hat{H} \)-module

\[
M \text{ is calibrated if } M_t^{\text{gen}} = M_t, \text{ for all } t \in T.
\]

A calibrated module \( M \) is **really calibrated** if \( t = t_r \) for all \( t \in T \), i.e. \( t_c = 1 \) in the polar decomposition (2.1).

Suppose that \( t \in T \) is regular, i.e. the stabilizer \( W_t \) of \( t \) under the action of \( W \) is trivial. Then \( M(t) = \bigoplus_{w \in W} M_{wt} \), since each \( M_{wt}^{\text{gen}} \) is one dimensional and \( M_{wt} \neq 0 \) whenever \( M_{wt}^{\text{gen}} \neq 0 \). Thus \( M(t) \) is calibrated when \( t \) is regular. So any quotient of \( M(t) \) is calibrated and, by Proposition 2.6b, any irreducible \( \hat{H} \)-module \( M \) with regular central character is calibrated.
Classification of irreducible calibrated modules

We shall eventually show that the modules $\tilde{H}^{(t,J)}$ constructed in Theorem 3.1 are all the irreducible calibrated $\tilde{H}$-modules. The following Proposition shows that the formulas which define the $\tilde{H}$-modules in Theorem 3.1 are more or less forced.

**Proposition 4.1.** Let $M$ be a calibrated $\tilde{H}$-module and assume that for all $t \in T$ such that $M_t \neq 0$,

(A1) $t(X_i^\alpha) \neq 1$ for all $1 \leq i \leq n$, and

(A2) $\dim(M_t) = 1$.

For each $b \in T$ such that $M_b \neq 0$ let $v_b$ be a nonzero vector in $M_b$. The vectors $\{v_b\}$ form a basis of $M$. Let $(T_i)_{cb} \in \mathbb{C}$ and $b(X^\lambda) \in \mathbb{C}$ be given by

$$T_i v_b = \sum_c (T_i)_{cb} v_c \quad \text{and} \quad X^\lambda v_b = b(X^\lambda)v_b.$$ 

Then

(a) $(T_i)_{bb} = \frac{q - q^{-1}}{1 - b(X^{-\alpha_i})}$, for all $v_b$ in the basis,

(b) If $(T_i)_{cb} \neq 0$ then $c = s_i b$,

(c) $(T_i)_{b,s_i b} = (q^{-1} + (T_i)_{bb})(q^{-1} + (T_i)_{s_i b,s_i b}).$

**Proof.** The defining equation for $\tilde{H}$,

$$X^\lambda T_i - T_i X^{s_i \lambda} = (q - q^{-1}) \frac{X^\lambda - X^{s_i \lambda}}{1 - X^{-\alpha_i}},$$

forces

$$\sum_c \left( c(X^\lambda)(T_i)_{cb} - (T_i)_{cb} b(X^{s_i \lambda}) \right) v_c = (q - q^{-1}) \frac{b(X^\lambda) - b(X^{s_i \lambda})}{1 - b(X^{-\alpha_i})} v_b.$$

Comparing coefficients gives

$$c(X^\lambda)(T_i)_{cb} - (T_i)_{cb} b(X^{s_i \lambda}) = 0, \quad \text{if } b \neq c, \text{ and}$$

$$b(X^\lambda)(T_i)_{bb} - (T_i)_{bb} b(X^{s_i \lambda}) = (q - q^{-1}) \frac{b(X^\lambda) - b(X^{s_i \lambda})}{1 - b(X^{-\alpha_i})}.$$

These relations give:

If $(T_i)_{cb} \neq 0$ then $b(X^{s_i \lambda}) = c(X^\lambda)$ for all $X^\lambda \in X$, and

$$(T_i)_{bb} = \frac{q - q^{-1}}{1 - b(X^{-\alpha_i})} \quad \text{if } b(X^{-\alpha_i}) \neq 1 \text{ and } b(X^\lambda) \neq b(X^{s_i \lambda}) \text{ for some } X^\lambda \in X.$$

By assumption (A1), $b(X^{\alpha_i}) \neq 1$ for all $i$. For each fundamental weight $\omega_i$, $X^{\omega_i} \in X$ and $b(X^{s_i \omega_i}) = b(X^{\omega_i - \alpha_i}) \neq b(X^{\omega_i})$ since $b(X^{\alpha_i}) \neq 1$. Thus we conclude that

$$T_i v_b = (T_i)_{bb} v_b + (T_i)_{s_i b,b} v_{s_i b}, \quad \text{with} \quad (T_i)_{bb} = \frac{q - q^{-1}}{1 - b(X^{-\alpha_i})}. $$
This completes the proof of (a) and (b). By the definition of $\tilde{H}$ the vector

$$T_i^2 v_b = ((T_i)_{bb}^2 + (T_i)_{b,s,b}(T_i)_{s,b,b})v_b + ((T_i)_{bb} + (T_i)_{s,b,s,b})(T_i)_{s,b,b}v_{s,b}$$

must equal

$$((q - q^{-1})T_i + 1)v_b = ((q - q^{-1})(T_i)_{bb} + 1)v_b + (q - q^{-1})(T_i)_{s,b,b}v_{s,b}.$$  

Using the formula for $(T_i)_{bb}$ and $(T_i)_{s,b,s,b}$ we find $(T_i)_{bb} + (T_i)_{s,b,s,b} = (q - q^{-1})$. So, by comparing coefficients of $v_b$, we obtain the equation

$$(T_i)_{b,s,b}(T_i)_{s,b,b} = (q - (T_i)_{bb}((T_i)_{bb} + q^{-1}) = (q^{-1} + (T_i)_{bb})(q^{-1} + (T_i)_{s,b,s,b}).$$

\[\Box\]

**Proposition 4.2.** Let $M$ be an irreducible calibrated module. Then, for all $t \in T$ such that $M_t \neq 0$,

(a) $t(X^\alpha_i) \neq 1$ for all $1 \leq i \leq n$, and

(b) $\dim(M_t) = 1$.

**Proof.** (a) The proof is by contradiction. Assume that $t(X^\alpha_i) = 1$. Let $\tilde{H}A_1$ be the subalgebra of $\tilde{H}$ generated by $T_i$ and $X^\alpha_i$ and view $M$ as an $\tilde{H}A_1$-module by restriction. Let $m_t$ be a nonzero element of $M_t$. There is an $\tilde{H}A_1$-module homomorphism

$$\phi: M(t) \rightarrow M$$

$$v_t \mapsto m_t$$

where $M(t)$ is the (two dimensional) principal series module for $\tilde{H}A_1$ and $v_t$ is the generator of $M(t)$. It is easy to check that when $t(X^\alpha_i) = 1$ the module $M(t)$ is an irreducible $\tilde{H}A_1$-module. Thus the map $\phi$ is injective and we can view $M(t)$ as a submodule of $M$. A direct check shows that $M(t)$ is not calibrated and thus it follows that $M$ is not calibrated. This is a contradiction to the assumption that $M$ is calibrated. Thus $t(X^\alpha_i) \neq 1$.

(b) The proof is by contradiction. Assume that $t \in T$ is such that $\dim(M_t) > 1$. Let $m_t$ be a nonzero element of $M_t$. Since $M$ is calibrated, the action of any $\tau_i$ on any weight vector $m$ is a linear combination of the action of $T_i$ and a multiple of the identity. Thus, since $M$ is irreducible, we must be able to generate the rest of $M_t$ by applying $\tau$-operators to $m_t$. Since $\dim(M_t) > 1$ there must be a sequence of $\tau$-operators such that

$$n_t = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}m_t$$

is a nonzero vector in $M_t$ which is not a multiple of $m_t$. Assume that the sequence $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}$ is chosen so that $p$ is minimal.

Let us defer, momentarily, the proof of the following claim.

**Claim:** The element $s_{i_1}s_{i_2}\cdots s_{i_p} = 1$ in $W$.

The claim implies that there is some $1 < k \leq p$ such that $s_{i_1}s_{i_2}\cdots s_{i_k}$ is not reduced and we can use the braid relations to rewrite this word as $s_{i_k}\cdots s_{i_{k-2}}s_{i_k}s_{i_k}$. By Proposition 2.7e the $\tau_i$ operators also satisfy the braid relations and so

$$n_t = \tau_{i_1}'\tau_{i_2}'\cdots\tau_{i_{k-2}}'\tau_{i_k}\tau_{i_k}\cdots\tau_{i_p}m_t.$$
By Proposition 2.7b, the operator $\tau_{i_k}^t \tau_{i_k}$ is equal to a constant times the identity map and thus

$$n_t = c \tau_{i_1}^t \tau_{i_2}^t \cdots \tau_{i_{k-1}}^t \tau_{i_k}^t \cdots \tau_{i_p}^t m_t,$$

where $c$ is some constant. The constant $c$ must be nonzero since $n_t$ is not 0. But the expression

$$c^{-1} n_t = \tau_{i_1}^t \tau_{i_2}^t \cdots \tau_{i_{k-1}}^t \tau_{i_k}^t \cdots \tau_{i_p}^t m_t$$

is shorter than the original expression of $n_t$ and this contradicts the minimality of $p$. It follows that dim$(M_t) \leq 1$.

**Proof of the claim.** By [St, 3.15, 4.2, 5.3] the stabilizer $W_t$ of $t$ under the action of $W$ is

$$W_t = \langle s_\alpha \mid \alpha \in Z(t) \rangle$$

where $Z(t) = \{ \alpha > 0 \mid t(X^\alpha) = 1 \}$. The elements of the orbit $Wt$ can be identified with the cosets of $W/W_t$ and these can be identified with the chambers of $\mathbb{R}^n \setminus \bigcup H_\alpha$ which are on the positive side of all the hyperplanes $H_\alpha$ for $\alpha \in Z(t)$. Specifically, the element $t \in Wt$ corresponds to the chamber $C$ and the element $wt$ of $Wt$ corresponds to the chamber $w^{-1}C$.

For any $1 \leq j \leq p$ we have that $(s_{i_{j+1}} \cdots s_{i_p} t)(X^{\omega_{i_j}}) \neq 1$, since $\tau_{i_j} \cdots \tau_{i_p} m_t$ is well defined. This means that $s_{i_j}(s_{i_{j+1}} \cdots s_{i_p} t)(X^{\omega_{i_j}}) = (s_{i_{j+1}} \cdots s_{i_p} t)(X^{\omega_{i_j} - \alpha_{i_j}}) \neq (s_{i_{j+1}} \cdots s_{i_p} t)(X^{\omega_{i_j}})$ and thus that $s_{i_j}(s_{i_{j+1}} \cdots s_{i_p} t) \neq s_{i_{j+1}} \cdots s_{i_p} t$. So $s_{i_1} \cdots s_{i_p} t$ and $s_{i_{j+1}} \cdots s_{i_p} t$ both correspond to chambers on the positive side of all the hyperplanes $H_\alpha$, $\alpha \in Z(t)$. These two chambers have a common face and this face is contained in the hyperplane $H_{s_{i_{j+1}} \cdots s_{i_p} t}$. In this way we can identify the sequence $t, s_{i_1} t, s_{i_2} t, \cdots, s_{i_{j-1}} t, s_{i_p} t$ with a sequence of chambers where successive chambers in the sequence are adjacent (share a face) and all the chambers in the sequence are on the positive side of all the hyperplanes $H_\alpha$, $\alpha \in Z(t)$. Since $s_{i_1} \cdots s_{i_p} t = t$, the first and last chamber in this sequence are the same. It follows that $s_{i_1} \cdots s_{i_p} = 1$ in $W$. \qed

**Proposition 4.3.** Let $M$ be an irreducible calibrated $\tilde{H}$-module. Suppose that $M_t$ and $M_{s_i t}$ are both nonzero. Then the map $\tau_i : M_t \to M_{s_i t}$ is a bijection.

**Proof.** By Proposition 4.2b, dim$(M_t) = \dim(M_{s_i t}) = 1$, and thus it is sufficient to show that $\tau_i$ is not the zero map. Let $v_t$ be a nonzero vector in $M_t$. Since $M$ is irreducible there must be a sequence of $\tau$ operators such that

$$v_{s_i t} = \tau_{i_1} \cdots \tau_{i_p} v_t$$

is a nonzero element of $M_{s_i t}$. Let $p$ be minimal such that this is the case. We have $\tau_i \tau_{i_1} \cdots \tau_{i_p} v_t \in M_t$. Using the claim which was proved in the proof of Proposition 4.2 we have that $s_i s_{i_1} \cdots s_{i_p} = 1$ in $W$. For notational convenience $i_0 = i$. Let $0 \leq k < p$ be maximal such that $s_{i_k} \cdots s_{i_p} = 1$ is not reduced. If $k \neq 0$ then we can use the braid relations to get

$$v_{s_i t} = \tau_{i_1} \cdots \tau_{i_k} \tau_{i_k} \tau_{i_k+1} \cdots \tau_{i_p} v_t.$$

By Proposition 2.7c $\tau_{i_k} \tau_{i_k}$ is a multiple of the identity and so

$$v_{s_i t} = c \tau_{i_1} \cdots \tau_{i_k-1} \tau_{i_k+2} \cdots \tau_{i_p} v_t.$$
But this contradicts the minimality of \( p \). Thus we must have \( k = 0, p = 1 \) and

\[ v_{s,t} = \tau_i v_t. \]

Thus, since \( v_{s,t} \neq 0, \tau_i \neq 0. \]

**Proposition 4.4.** If \( M \) is a calibrated \( \tilde{H} \)-module and \( t \in T \) is such that \( M_t \neq 0 \) then \( t \) is calibratable for all \( R_{ij} \) generated by simple roots \( \alpha_i \) and \( \alpha_j \) in \( R \).

*Proof.* Let \( \tilde{H}_{ij} \) be the subalgebra of \( \tilde{H} \) generated by \( T_i, T_j, X^{\alpha_i}, \) and \( X^{\alpha_j} \) and view \( M \) as an \( H_{ij} \) module by restriction. The irreducible representations of rank two affine Hecke algebras have been classified and constructed explicitly in [Ra3]. From this classification it is easy to check that the only weights \( t \) which appear in calibrated \( \tilde{H}_{ij} \)-modules are those that are calibratable for \( R_{ij} \). Thus, if \( M_t \neq 0 \), then \( t \) must be calibratable for \( R_{ij} \). □

**Theorem 4.5.** Let \( M \) be an irreducible calibrated \( \tilde{H} \)-module. Let \( t \) be the central character of \( M \) and let \( J = R(w) \cap P(t) \) for any \( w \in W \) such that \( M_{wt} \neq 0 \). Then \( (t, J) \) is a placed skew shape and

\[ M \cong \tilde{H}^{(t,J)}, \]

where \( H^{(t,J)} \) is the module defined in Theorem 3.1.

*Proof.* Proposition 4.3 shows that if \( M_t \) and \( M_{s,t} \) are both nonzero then both \( \tau_i: M_t \to M_{s,t} \) and \( \tau_i: M_{s,t} \to M_t \) are bijections. Thus, by Proposition 2.7d \( t(X^{\alpha_i}) \neq q^{\pm 2} \) and so there is an edge \( t \leftrightarrow s_t \) in the calibration graph. This shows that \( \text{supp}(M) \) is a single connected component of the calibration graph \( \Gamma(t) \). Then \( (t, J) \) (as defined in the statement of the Theorem) is the corresponding placed shape. By Proposition 4.4 \( (t, J) \) must be a placed skew shape. Propositions 4.1 and 4.2 show that there is at most one calibrated \( \tilde{H} \)-module \( M \) such that \( \text{supp}(M) \) is the connected component of \( \Gamma(t) \) labeled by \( (t, J) \). Thus we must have that \( M \cong \tilde{H}^{(t,J)}. \) □
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