An effective bound in Dvoretzky-type theorem for polynomials

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Abstract

We modify the Dol’nikov-Karasev proof of the Alebraic Dvoretzky theorem in order to obtain an effective bound on the dimension of the spherically-symmetric section. Additionally we prove a weak version of the $\varepsilon$-Dvoretzky conjecture for normed spaces, showing the existence of a subspace of $\mathbb{R}^n$ of dimension at least $c \log n / |\log \varepsilon|$ in which the given norm is $\varepsilon$-close to a norm obeying a large discrete group of symmetries (“1-unconditional norm”).

1 Introduction

Dvoretzky’s theorem [9, 10] is a fundamental result in the theory of high-dimensional normed spaces that was proved circa 1960. Conjectured earlier by Grothendieck [12] and by others (see [9]), this theorem is formulated as follows:

**Theorem 1.1 (Dvoretzky’s theorem).** Let $\| \cdot \|$ be a norm in $\mathbb{R}^n$ and let $0 < \varepsilon < 1/2$. Suppose that $k$ is an integer that satisfies

$$n \geq \exp \left( C k \frac{\log^2 \varepsilon}{\varepsilon} \right),$$

(1)

where $C > 0$ is a certain universal constant. Then there exists a $k$-dimensional subspace $E \subseteq \mathbb{R}^n$ and $r > 0$ such that for all $x \in E$,

$$(1 - \varepsilon)r|x| \leq \|x\| \leq (1 + \varepsilon)r|x|$$

where $| \cdot |$ is the standard Euclidean norm in $\mathbb{R}^n$.

Theorem 1.1 can be reformulated as stating that any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ has a central $k$-dimensional section that is nearly spherical.

The estimate (1) is taken from Schechtman [23], and its proof utilizes an influential approach by V. Milman to Dvoretzky’s theorem which emphasizes the role of the concentration of measure phenomena [19, 21]. However, the dependence on $1/\varepsilon$ in the estimate (1) is exponential. It is conjectured that the actual dependence on $\varepsilon$ in Dvoretzky’s theorem should be polynomial. This is the conjectural *almost-isometric* variant of Dvoretzky’s theorem. This
conjecture holds true in the case when \( k = 2 \) (see \([18]\)) or when the norm \( \| \cdot \| \) is assumed 1-symmetric, i.e., when
\[
\| (\pm x_{\pi(1)}, \ldots, \pm x_{\pi(n)}) \| = \| (x_1, \ldots, x_n) \|
\]
for all vectors \((x_1, \ldots, x_n) \in \mathbb{R}^n\), for any permutation \( \pi \in S_n \) and for any choice of signs. See Bourgain-Lindenstrauss \([4]\), Tikhomirov \([24]\) and Freksen \([11]\) for analysis of the 1-symmetric case.

All proofs of Theorem 1.1 for general \( k \) rely heavily on probability and analysis, for example through the use of P. Lévy’s concentration inequality for Lipschitz functions on the high-dimensional sphere \([20]\). On the other hand, the proof for the case \( k = 2 \), which is attributed to Gromov by Milman \([18]\), is purely topological. As for the case \( k \geq 3 \), there is hope that a proof employing topological tools could lead to better dependence on \( \varepsilon \) in Theorem 1.1. See also Burago, Ivanov and Tabachnikov \([6]\) for a discussion of possible topological approaches to Dvoretzky’s theorem and their shortcomings.

A different line of research, parallel to the developments related to Dvoretzky’s theorem, goes back to Birch \([3]\), following Brauer \([5]\). This time, rather than analyzing a norm on \( \mathbb{R}^n \) for large \( n \), one looks at a \( d \)-homogeneous polynomial \( P : \mathbb{R}^n \to \mathbb{R} \). The regular sub-structure that one expects to find is a \( k \)-dimensional subspace \( E \subseteq \mathbb{R}^n \) such that the restriction \( P|_E \) vanishes when \( d \) is odd, and is proportional to the Euclidean norm taken to the power \( d \) when \( d \) is even.

**Theorem 1.2** (“Algebraic Dvoretzky-type theorem”). Let \( k, d \geq 1 \). Then there exists \( f(k, d) \geq 1 \) such that if \( n \geq f(k, d) \) and \( P : \mathbb{R}^n \to \mathbb{R} \) is a \( d \)-homogeneous polynomial, then there exists a \( k \)-dimensional subspace \( E \subseteq \mathbb{R}^n \) and \( \alpha \in \mathbb{R} \) such that
\[
P(x) = \alpha |x|^d \quad \forall x \in E.
\]
In the case where \( d \) is odd we have \( \alpha = 0 \).

Note that Theorem 1.2 yields a subspace in which the polynomial is exactly spherically-symmetric. In the odd case, Theorem 1.2 goes back to Birch \([3]\), and precedes the proof of Dvoretzky’s theorem. Reasonable estimates for the function \( f(k, d) \) are known in the odd case, according to Aron and Hajek \([1]\) and to Dol’nikov and Karasev \([8, \text{Theorem 4}]\). Aron and Hajek relied on the Borsuk-Ulam theorem, while the argument by Dol’nikov and Karasev \([8]\) is almost a textbook application of the theory of Stiefel-Whitney characteristic classes, a theory that is beautifully presented in Milnor and Stasheff \([17]\). These topological approaches yield the bound proven in \([8]\),
\[
f(k, d) \leq k + \binom{d + k - 1}{d} \leq (Ck)^d
\]
where \( C > 0 \) is a universal constant.

The case where \( d \geq 4 \) is an even number is more challenging (when \( d = 2 \) a trivial argument gives \( f(k, 2) \leq 2k \)). Theorem 1.2 was conjectured by Milman in \([18, \text{Section 1}]\) and in \([21, \text{Section 1.15}]\). Theorem 1.2 was proven by Dol’nikov and Karasev \([8]\), relying on Carlsson’s solution to the Siegel conjecture, and it yields no effective bound on the function...
$f(k, d)$, or equivalently, on the dimension of the subspace in which a given homogeneous polynomial in $\mathbb{R}^n$ is proportional to a power of the Euclidean norm. In this note we modify the Dol’nikov-Karasev argument, replacing the use of Carlsson’s results by more elementary topological tools, and obtain an effective bound for $f(k, d)$.

**Theorem 1.3.** Let $d, k \geq 2$ with $d$ being even. Assume that

$$n \geq \exp \left( (\tilde{C} k) C d^2 \log d \right)$$

for certain universal constants $C, \tilde{C} > 0$. Then for any $d$-homogeneous polynomial $P : \mathbb{R}^n \to \mathbb{R}$, there exists a $k$-dimensional subspace $E \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$P(x) = \alpha |x|^d \quad \text{for all } x \in E.$$

The value of the universal constants $C, \tilde{C} > 0$ from Theorem 1.3 as well as of all universal constants mentioned in this paper, can be extracted from our proofs. Thus in principle these are explicit universal constants whose value may be computed.

Our next result relies on elementary topological tools in order to make advances towards the almost-isometric variant of Dvoretzky’s theorem. Unfortunately, we do not obtain a full rotational symmetry but only the so-called “1-unconditional symmetries”. A norm $\| \cdot \|$ in $\mathbb{R}^n$ is “unconditional” with respect to the orthonormal basis $(e_1, \ldots, e_n)$ if for any $x_1, \ldots, x_n \in \mathbb{R}$ and any choice of signs,

$$\| \sum_{i=1}^n \pm x_i e_i \| = \| \sum_{i=1}^n x_i e_i \|.$$

Thus the norm $\| \cdot \|$ admits a symmetry group with $2^n$ elements. We say that a norm defined in a subspace of $\mathbb{R}^n$ is unconditional if there exists an orthonormal basis in this subspace with respect to which it is unconditional.

**Theorem 1.4.** Let $\| \cdot \|$ be a norm in $\mathbb{R}^n$ and let $0 < \varepsilon < 1/2$. Suppose that $k \geq 2$ is an integer that satisfies

$$n \geq \left( \frac{C}{\varepsilon} \right)^{3(k-1)}$$

for a certain universal constant $C > 0$. Then there exists a $k$-dimensional subspace $E \subseteq \mathbb{R}^n$ and an unconditional norm $\| \| \cdot \||$ in the subspace $E$ such that for all $x \in E$,

$$(1 - \varepsilon) \|x\| \leq \|x\| \leq (1 + \varepsilon) \|x\|.$$

While the dependence on $1/\varepsilon$ in Theorem 1.4 is only polynomial, as desired, the exponent $3(k-1)$ in (3) is non-optimal. It may be replaced by $\alpha(k-1)$ for any $\alpha > 2$ at the expense of modifying the value of the universal constant $C > 0$ from Theorem 1.4 as can be seen from the proof. It is likely possible to replace the “unconditional” symmetries of the norm $\| \cdot \|$ in Theorem 1.4 by other commutative groups of symmetries, such as cyclic.
permutations of the coordinates. See also Makeev [15] and Burago, Ivanov and Tabachnikov [6]. However, we do not know how to obtain a discrete group of symmetries such as the group of permutations $S_n$, or its 2-Sylow subgroup, while keeping the dependence on $1/\varepsilon$ polynomial.

In this paper we write $\langle \cdot , \cdot \rangle$ for the standard Euclidean product in $\mathbb{R}^n$, and $|x| = \sqrt{\langle x, x \rangle}$. By log we refer to the natural logarithm. Throughout this text we use the letters $c, C, \tilde{C}$ etc. to denote positive universal constants, that may be explicitly computed in principle, whose value may change from one line to the next.

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2 Reducing dimension and adding symmetries

In this section we use relatively simple topological tools in order to prove that it is possible to restrict a polynomial to a subspace, and obtain a polynomial invariant under a large, finite group of symmetries. The corresponding step in the Dol’nikov-Karasev argument [8] relies on Carlsson’s solution to the Siegel conjecture, and yields no effective bound on the dimension of the subspace. Our standard references for the topology that we use are Matousek’s book [16] and Hatcher’s book [13].

Let $V$ be a real finite-dimensional linear space with an inner product, and let $p \geq 2$ be an integer. We say that a finite group $G$ acts on $V$ by freely permuting coordinates if its action on $V$ is linear, and there is an orthonormal basis $e_1, \ldots, e_n \in V$ such that the set $A = \{e_1, \ldots, e_n\}$ is invariant under the group action, and moreover the restriction of the $G$-action to the set $A$ is free (i.e., all orbits are of size $|G|$, the cardinality of $G$). Note that when the cyclic group of $p$ elements $\mathbb{Z}/p\mathbb{Z}$ acts freely on a finite set $A$ of size $kp$, it is possible to enumerate $A = (e_{i,j})_{i\in \mathbb{Z}/p\mathbb{Z}, j=1,\ldots,k}$ so that $g.e_{i,j} = e_{g+i,j}$ where $g+i$ is the sum of $g$ and $i$ modulo $p$. In such a case we can say that $\mathbb{Z}/p\mathbb{Z}$ acts by permuting blocks of size $k$.

For a finite set $\Omega$ we write $V^\Omega$ for the linear space of all $(v_i)_{i\in \Omega}$ with $v_i \in V$ for all $i$. Suppose that we are given a linear action of $\mathbb{Z}/p\mathbb{Z}$ on the linear space $V$, and additionally we have a finite group $G$ acting on some finite set $\Omega$. Clearly the group $G$ acts on $V^\Omega$ via

$$g.(v_i)_{i\in \Omega} = (v_{g+i})_{i\in \Omega}.$$  \hspace{1cm} (4)

An important observation is that the action of $G$ on $\Omega$ and the action of $\mathbb{Z}/p\mathbb{Z}$ on $V$ induce an action of the wreath product $(\mathbb{Z}/p\mathbb{Z})^{\Omega} \rtimes G$ on the space $V^\Omega$.

For a group $H$ acting on $V^\Omega$, we say that a function $P: V^\Omega \to \mathbb{R}$ is $H$-symmetric if it is invariant under the action of $H$ on $V^\Omega$.

**Proposition 2.1.** Let $G$ be a finite group acting transitively on a finite set $\Omega$ of size $m$. Let $P: (\mathbb{R}^n)^\Omega \to \mathbb{R}$ be a homogeneous polynomial of degree $d$ which is $G$-symmetric. Let $p \geq 2$ be a prime number and let $k$ be a positive even number such that

$$n \geq kp + \left( mkp + \frac{d-1}{d} \right)(p-1).$$
Then there exists a subspace \( E \subseteq \mathbb{R}^n \) with \( \dim(E) = kp \) and an action of \( \mathbb{Z}/p\mathbb{Z} \) on \( E \) by freely permuting coordinates such that the restriction of \( P \) to the subspace \( E^{\Omega} \subseteq (\mathbb{R}^n)^{\Omega} \) is \((\mathbb{Z}/p\mathbb{Z})^{\Omega} \times G\)-symmetric.

Before turning to the proof of Proposition 2.1 we recall the concept of an \( E_rH \) space, see [16, Section 6.2] for more information. For a finite group \( H \) and an integer \( r \geq 1 \), an \( E_rH \)-space is an \( r \)-dimensional, simplicial \( H \)-complex with a free \( H \)-action, that is additionally \((r-1)\)-connected. For instance, the sphere \( S^n \) is an \( E_1(\mathbb{Z}/2\mathbb{Z}) \)-space, and when \( n \) is odd the sphere \( S^n \) is also an \( E_1(\mathbb{Z}/m\mathbb{Z}) \)-space, where the group \( \mathbb{Z}/m\mathbb{Z} \) acts on the sphere \( S^{2n-1} \subseteq \mathbb{C}^n \) via multiplication by complex roots of one. Given a topological space \( X \) with a continuous \( H \)-action, we define \( \text{Ind}_H(X) \) to be the minimal \( r \) such that there exists an \( H \)-equivariant function from \( X \) to some \( E_rH \)-space.

Write \( Z_{d,k} \) for the space of \( d \)-homogeneous polynomials in \( \mathbb{R}^k \). Its dimension is \( \binom{d+k-1}{d} \). We denote by \( W_{n,k} \) the Stiefel manifold of all \( k \)-frames in \( \mathbb{R}^n \), i.e., all \( k \)-tuples \((v_1, \ldots, v_k)\) where \( v_1, \ldots, v_k \) are orthonormal vectors in \( \mathbb{R}^n \).

**Proof of Proposition 2.1** Let us identify \( \Omega \) with the set \( \{1, \ldots, m\} \). Let \( WO \) be the space of all collections \((U_i)_{i \in \Omega}\), where \( U_i = (U_{i,1}, \ldots, U_{i,kp}) \in (\mathbb{R}^n)^{kp} \) satisfy the following:

1. For any \( i \), the vectors \( U_{i,1}, \ldots, U_{i,kp} \in \mathbb{R}^n \) are orthogonal to each other and are of length one. In other words, these vectors constitute a \( kp \)-frame in \( \mathbb{R}^n \).
2. For any \( i, j \in \Omega \),
   \[
   \text{Span}(U_i) = \text{Span}(U_j)
   \]
   where \( \text{Span}(U_i) \subseteq \mathbb{R}^n \) is the \( kp \)-dimensional subspace spanned by the vectors \( U_{i,1}, \ldots, U_{i,kp} \).
3. For any \( i, j \in \Omega \), the transition matrix from the vectors \( U_{i,1}, \ldots, U_{i,kp} \) to the vectors \( U_{j,1}, \ldots, U_{j,kp} \) is a \( kp \times kp \) matrix of determinant one.

We observe that the space \( WO \) is canonically identified with \( W_{n,kp} \oplus SO(kp)^{(m-1)} \), where \( W_{n,kp} \) is the Stiefel manifold. Indeed, choosing \( U_1 \) is equivalent to choosing an element in \( W_{n,kp} \), while choosing \( U_2, \ldots, U_k \) amounts to finding the transition matrices from \( U_1 \) to \( U_2, \ldots, U_k \), which are elements of the special orthogonal group \( SO(kp) \).

Let us fix a free action of the group \( H = \mathbb{Z}/p\mathbb{Z} \) on the set \( \{1, \ldots, kp\} \) by permuting blocks of size \( k \). This induces an action of \( H \) on \( W_{n,kp} \), permuting the vectors of the \( kp \)-frame in accordance with the \( H \)-action on \( \{1, \ldots, kp\} \). Since \( k \) is even, for any \( U \in W_{n,kp} \) and \( g \in H \), the transition matrix from the frame \( U \) to the frame \( gU \) is of determinant one. The action of \( H \) on \( W_{n,kp} \) induces an action of \( H \) on \( WO = W_{n,kp} \oplus SO(kp)^{(m-1)} \), via

\[
g(U_1, U_2, \ldots, U_n) = (gU_1, U_2, \ldots, U_n) \quad \text{for } g \in H, (U_1, \ldots, U_n) \in WO.
\]

That is, the group \( H \) only acts on the first \( kp \)-frame, and it permutes its elements in blocks of size \( k \). We claim that

\[
\text{Ind}_H(WO) \geq n - kp.
\]

Indeed, assume that there exists an \( H \)-equivariant function from \( WO \) to \( E_tH \). Since the \( H \)-action on \( WO \) only affects the first \( kp \)-frame, for any fixed element in \( SO(kp)^{(m-1)} \), we obtain an \( H \)-equivariant map from \( W_{n,kp} \) to \( E_tH \). Therefore

\[
\text{Ind}_H(WO) \geq \text{Ind}_H(W_{n,kp}).
\]
Since the Stiefel manifold $W_{n,kp}$ is $(n - kp - 1)$-connected (e.g. Example 4.53 in Hatcher [13]), we obtain from [16, Proposition 6.2.4 (iv)] that $Ind_H(W_{n,kp}) \geq n - kp$ and [5] follows.

For $U = (U_i)_{i=1,\ldots,m} \in WO$ we define the polynomial $\hat{P}(U) \in \mathbb{Z}_{d,kpm}$ via

$$\hat{P}(U)(x) = P \left( \sum_{i=1}^{m} \sum_{j=1}^{kp} x_{i,j} U_{i,j} \right)$$

for $x = (x_{i,j})_{i=1,\ldots,m,j=1,\ldots,kp} \in \mathbb{R}_{kpm}$. Note that $\hat{P}(U)$ is a $d$-homogeneous polynomial in $kpm$ real variables. Thus

$$\hat{P} : WO \to \mathbb{Z}_{d,kpm} \cong \mathbb{R}_{(kpm+d-1)}$$

is a well-defined, continuous map. It follows from [5] and [16, Theorem 6.3.3], that if

$$n - kp \geq \left( \frac{mkp + d - 1}{d} \right)(p - 1)$$

then there exists a point $U \in WO$ such that $\hat{P}$ is constant on the orbit of $U$ under the group $H$. Define $E = \text{Span}(U_1)$. The above action of $H = \mathbb{Z}/p\mathbb{Z}$ on $W_{n,kp}$ induces an action of $H$ on $E$ by freely permuting coordinates.

We claim that the restriction of $P$ to $E^\Omega$ is $(\mathbb{Z}/p\mathbb{Z})^\Omega \rtimes G$-symmetric. Indeed, since $\hat{P}(g,U) = \hat{P}(U)$ for any $g \in H$, the polynomial $P(U)$ is invariant under a cyclic permutation in the first $kp$ coordinates, in blocks of size $k$. Since $G$ acts transitively on $\Omega$ and since $P : (\mathbb{R}^n)^\Omega \to \mathbb{R}$ is $G$-symmetric, the same is true not just for the first block of $kp$ coordinates, but also for the $i^{th}$ block for any $i = 1,\ldots,m$. We thus obtain the desired $(\mathbb{Z}/p\mathbb{Z})^\Omega \rtimes G$-symmetry of the restriction of $\hat{P}$ to the subspace $E^\Omega$. \hfill $\square$

Next we iterate Proposition 2.1 specializing to the case $p = 2$. Let us recall some notation from Dol’nikov and Karasev [8]. For $h \geq 1$ we write $T_h$ for the full graded binary tree of height $h$. This tree contains $2^h$ leaves and $2^h - 1$ internal vertices, graded according to their distance from the root vertex. Write $\Omega_h$ for the collection of $2^h$ leaves of the tree $T_h$. Consider the automorphism group $G_h$ of the graded binary tree $T_h$. This group satisfies

$$G_h = (\mathbb{Z}/2\mathbb{Z})^{\omega_{\Omega_{h-1}}} G_{h-1} = (\mathbb{Z}/2\mathbb{Z})^{\Omega_{h-1}} \rtimes G_{h-1}.$$ 

The set $\Omega_h$ of $2^h$ leaves is invariant under the action of the group $G_h$, and moreover the action of $G_h$ on $\Omega_h$ is transitive. In fact, the group $G_h$ is a 2-Sylow subgroup of the permutation group on $2^h$ elements. The group $G_h$ acts on $(\mathbb{R}^k)^{\Omega_h}$ by permuting coordinates in blocks of size $k$. Set

$$G_h' = (\mathbb{Z}/2\mathbb{Z})^{\Omega_h} \rtimes G_h.$$ 

Observe that $G_h'$ acts on $\mathbb{R}^{\Omega_h}$, where $G_h$ acts by permuting the coordinates of the vector and where $(\mathbb{Z}/2\mathbb{Z})^{\Omega_h}$ acts by switching the signs of the coordinates of the vector.

Let $V$ be a linear space of dimension $2^h k$ with an inner product. We say that $f : V \to \mathbb{R}$ is $G_h$-symmetric if there is an action of $G_h$ on $V$ by linear isometries, isomorphic to the above action on $(\mathbb{R}^k)^{\Omega_h}$, such that $f$ is invariant under the $G_h$ action. The notion of a $G_h'$-symmetric function defined on a $2^h$-dimensional linear space is defined similarly.
Indeed, since polynomial \( F \) is the trivial group and hence any polynomial on \( \mathbb{R} \) embeds \( \mathbb{R} \) is found a subspace \( F \) from (7) and Proposition 2.1 with \( n \) we are given a \( d \) to the subspace exists. Karasev [8, Section 9], using combinatorial and geometric arguments, that there exist \( P \) -symmetric. \( k \) -homogeneous polynomial \( P \) is invariant under the above action of \( G_h \) on \( \mathbb{R}^{2h+1} \cong \mathbb{R}^{\Omega_{h+1}} \), then the restriction of \( f \) to the subspace

\[
\tilde{E} = \left\{ (x_1, -x_1, \ldots, x_{2h}, -x_{2h}) ; (x_1, \ldots, x_{2h}) \in \mathbb{R}^{2h} \right\} \subseteq \mathbb{R}^{2h+1}
\]

is \( G'_h \)-symmetric. Next, we claim that (6) guarantees the existence of even integers \( n \geq n_0 \geq n_1 \geq \ldots \geq n_{h+1} = 1 \) such that

\[
n_i \geq 2n_{i+1} + \left( \frac{2^{i+1}n_{i+1} + d - 1}{d} \right) \quad (i = 0, \ldots, h).
\]

Indeed, since \( \binom{n}{k} \leq \left( \frac{n^k}{k!} \right) \), it suffices to set \( k_i = 5i^2 + \log n_i \) and solve

\[
k_i \geq d \cdot k_{i+1} \quad k = 0, \ldots, h - 1.
\]

This has a solution when \( \log n \geq (Cd)^{h+1} \), hence the required sequence of even integers exists.

Let \( i = 0, \ldots, h \) and let \( F_i \subseteq \mathbb{R}^n \) be a subspace with \( \dim(F_i) = 2^i n_i \). Assume that we are given a \( d \)-homogeneous polynomial \( P : F_i \rightarrow \mathbb{R} \) which is \( G_i \)-symmetric. It follows from (7) and Proposition 2.1 with \( p = 2, m = 2^i, n = n_i \) and \( k = n_{i+1} \) that there exists a subspace \( F_{i+1} \subseteq \mathbb{R}^{n_i} \) with dimension \( \dim(F_{i+1}) = 2^{i+1}n_{i+1} \) such that the restriction of the polynomial \( P \) to the subspace \( F_{i+1} \) is \( G_{i+1} \)-symmetric.

We now iterate the procedure from the previous paragraph, starting from \( i = 0 \). We embed \( \mathbb{R}^{n_0} \) in \( \mathbb{R}^{n_i} \) in an arbitrary manner, restrict the polynomial \( P \) to \( \mathbb{R}^{n_0} \), and note that \( G_0 \) is the trivial group and hence any polynomial on \( \mathbb{R}^{n_0} \) is vacuously \( G_0 \)-symmetric. We have thus found a subspace \( F = F_{h+1} \) such that the restriction of \( P \) to \( F \) is \( G_{h+1} \)-symmetric. \( \square \)

Suppose that \( P : \mathbb{R}^{2h} \rightarrow \mathbb{R} \) is a \( d \)-homogeneous polynomial that is \( G'_h \)-symmetric, where \( d \) is even. Thus \( P \) obeys a rather large group of symmetries. It is proven in Dol’nikov and Karasev [8, Section 9], using combinatorial and geometric arguments, that there exist \( \alpha \in \mathbb{R} \) and a \( k \)-dimensional subspace \( E \subseteq \mathbb{R}^{2h} \) such that \( P(x) = \alpha |x|^d \) for any \( x \in E \), provided that

\[
(2h)^{d^2/4 + d/2} \cdot \left( \frac{k + d - 1}{d} \right)^{d+1} \leq 2^h.
\]

In fact, the subspace \( E \subseteq \mathbb{R}^{2h} \) provided by Dol’nikov and Karasev does not depend on the choice of the \( d \)-homogeneous polynomial \( P \), as long as it is \( G'_h \)-symmetric. We remark that the quantity \( C(\delta) \) discussed in [8, Section 9] may be bounded by \( 2^\delta \).
Proof of Theorem 1.3. We will use Corollary 2.2 as well as the results of Dol’nikov and Karasev [8, Section 9] just mentioned. We set \( h = \lceil \tilde{C}d^2 \log d \rceil \) for a universal constant \( \tilde{C} > 0 \) so that \( h^8d^2 \leq 2^h \). By Corollary 2.2 there exists a subspace \( F \subseteq \mathbb{R}^n \) such that the restriction of the polynomial \( P \) to this subspace is \( G'_{h} \)-symmetric. The dimension of \( F \) is \( 2^h \), where we may assume that

\[
\log n \leq (C d)^{h+1}
\]

for a universal constant \( C > 0 \). Thus,

\[
h^4d^2 \cdot (\log n)^{c/\log d} \leq 2^h. \tag{9}
\]

According to (8), if \( k \) satisfies (2), then

\[
(2h)^d/4+d/2 \cdot \left( e^{k + d - 1} \right)^{d(d+4)} \leq (Ckh)^4d^2 \leq h^4d^2 \cdot (\log n)^{c/\log d} \leq 2^h. \tag{10}
\]

Hence there exists a \( k \)-dimensional subspace \( E \subseteq \mathbb{R}^n \) such that the restriction of \( P \) to this subspace is proportional to \( |x|^d \).

\[\square\]

3 Unconditional symmetries

Consider the group \((\mathbb{Z}/2\mathbb{Z})^k \cong \{\pm 1\}^k\). This group acts on the Stiefel manifold \( W_{n,k} \) of \( k \)-frames in \( \mathbb{R}^n \) by switching the signs of the frame vectors, i.e.,

\[
g.(U_1, \ldots, U_k) = (g_1U_1, \ldots, g_kU_k) \quad \text{for} \quad (U_1, \ldots, U_k) \in W_{n,k}, \quad (g_1, \ldots, g_k) \in \{\pm 1\}^k
\]

where \( U_1, \ldots, U_k \in \mathbb{R}^n \) is a \( k \)-frame. A linear action of a group \( G \) on \( \mathbb{R}^\ell \) (i.e., a representation) is fixed-point-free if there is no vector \( 0 \neq x \in \mathbb{R}^\ell \) with \( g.x = x \) for all \( g \in G \).

**Proposition 3.1.** Consider any fixed-point-free representation of \( \{\pm 1\}^k \) in \( \mathbb{R}^\ell \) for \( \ell \leq n-k \). Then any continuous, \( G \)-equivariant map \( F : W_{n,k} \to \mathbb{R}^\ell \) has to vanish somewhere in \( W_{n,k} \).

The difference between Proposition 3.1 and Theorem 1.1 in Chan, Chen, Frick and Hull [7], is that the representation in \( \mathbb{R}^\ell \) can be arbitrary, as long as it is fixed-point-free. We guess that there should be an elegant algebraic-topology proof of Proposition 3.1 perhaps using Stiefel-Whitney classes.

**Proof of Proposition 3.1** For a non-empty subset \( A \subseteq \{1, \ldots, k\} \) we define the one-dimensional representation

\[
w_A(g) = \prod_{i \in A} g_i \quad \text{for} \quad g = (g_1, \ldots, g_k) \in \{\pm 1\}^k,
\]

where the linear action on \( \mathbb{R} \) is given by \( g.x = w_A(g)x \) for \( x \in \mathbb{R}, g \in G \). Any fixed-point-free, irreducible representation of the abelian group \( G = \{\pm 1\}^k \) is isomorphic to one of these \( 2^k - 1 \) one-dimensional representations. Any finite-dimensional representation of
$G$ splits into a direct sum of irreducible representations. Thus every fixed-point-free, finite-dimensional representation of $\{\pm 1\}^k$ is characterized by a formal sum

$$\tau = \sum_{\emptyset \neq A \subseteq \{1, \ldots, k\}} m_A \cdot A \quad (12)$$

for non-negative integers $m_A$, which count the number of times that each irreducible representation occurs in the given finite-dimensional representation. Note that the dimension of the representation is $|\tau| := \sum_A |m_A|$. 

Recall that we are given a certain fixed-point-free representation of the group $G$ in $\mathbb{R}^\ell$. Write the formal sum corresponding to this representation as

$$S_1 + S_2 + \ldots + S_k$$

where $S_i$ is the part of the formal sum that includes all subsets $A \subseteq \{1, \ldots, k\}$, where the maximal element of $A_{i,j}$ is precisely $i$. Since $\ell \leq n - k$ we have

$$|S_i| \leq n - k \quad \text{for } i = 1, \ldots, k.$$ 

Denote $N = nk - k(k + 1)/2 = \dim(W_{n,k})$. Consider a representation of $G$ in the space $\mathbb{R}^N$ corresponding to the formal sum

$$\tilde{\tau} = \sum_{i=1}^k (n - i - |S_i|) \cdot \{i\} + \sum_{i=1}^k S_i \quad (13)$$

This representation in the space $\mathbb{R}^N$ has an invariant subspace isomorphic to the representation in the space $\mathbb{R}^\ell$ that is given to us. Hence it suffices to prove that any continuous, $G$-equivariant map from $X$ to $\mathbb{R}^N$, vanishes somewhere in $W_{n,k}$. In view of [14, Theorem 2.1], it suffices to construct a smooth, $G$-equivariant map $f : W_{n,k} \to \mathbb{R}^N$, of which zero is a regular value, such that $f^{-1}(0)$ is an orbit of $G$. The function $f$ that we will construct takes the form

$$f = (f_{i,j})_{1 \leq i \leq k, 1 \leq j \leq n - i}$$

for scalar functions $f_{i,j} : W_{n,k} \to \mathbb{R}$. These scalar functions are defined as follows: For $1 \leq i \leq k$ and $|S_i| + 1 \leq j \leq n - i$ we set

$$f_{i,j}(U) = U_{i,i+j} \quad (U \in W_{n,k}) \quad (14)$$

where $U = (U_1, \ldots, U_k)$ is a $k$-frame in $\mathbb{R}^n$, and $U_i = (U_{i,1}, \ldots, U_{i,n}) \in \mathbb{R}^n$. We still need to define $f_{i,j}$ for $1 \leq j \leq |S_i|$. Let us write

$$S_i = \sum_{j=1}^{|S_i|} A_{i,j}$$

for non-empty subsets $A_{i,j} \subseteq \{1, \ldots, k\}$ whose maximal element equals $i$. For $1 \leq j \leq |S_i|$ we set

$$f_{i,j}(U) = U_{i,i+j} \cdot \prod_{r \in A_{i,j} \setminus \{i\}} U_{r,r} \quad (15)$$

for scalar functions $f_{i,j} : W_{n,k} \to \mathbb{R}$. These scalar functions are defined as follows: For $1 \leq i \leq k$ and $|S_i| + 1 \leq j \leq n - i$ we set

$$f_{i,j}(U) = U_{i,i+j} \quad (U \in W_{n,k}) \quad (14)$$

where $U = (U_1, \ldots, U_k)$ is a $k$-frame in $\mathbb{R}^n$, and $U_i = (U_{i,1}, \ldots, U_{i,n}) \in \mathbb{R}^n$. We still need to define $f_{i,j}$ for $1 \leq j \leq |S_i|$. Let us write

$$S_i = \sum_{j=1}^{|S_i|} A_{i,j}$$

for non-empty subsets $A_{i,j} \subseteq \{1, \ldots, k\}$ whose maximal element equals $i$. For $1 \leq j \leq |S_i|$ we set

$$f_{i,j}(U) = U_{i,i+j} \cdot \prod_{r \in A_{i,j} \setminus \{i\}} U_{r,r} \quad (15)$$
This completes the definition of the smooth map \( f : W_{n,k} \to \mathbb{R}^N \). Recalling (11), we observe that the map \( f \) is indeed \( G \)-equivariant; its coordinates \( f_{i,j} \) that are defined in (14) correspond to the first summand in (13), while its coordinates that are defined in (15) correspond to the second summand.

Let us now describe the zero set of \( f \). Suppose that \( U \in W_{n,k} \) satisfies \( f(U) = 0 \). The fact that \( f_{1,j}(U) = 0 \) for all \( 1 \leq j \leq n-1 \) implies that

\[
U_1 = e_1 = (\pm 1, 0, \ldots, 0).
\]

Similarly, the facts that \( f_{2,j}(U) = 0 \) for all \( 1 \leq j \leq n-2 \) and that \( U_2 \perp U_1 \) imply that \( U_2 = e_2 = (0, \pm 1, 0, \ldots, 0) \). By a straightforward induction argument, we see that \( U_j = \pm e_j \), where \( e_1, \ldots, e_n \in \mathbb{R}^n \) are the standard unit vectors. Thus

\[
f^{-1}(0) = \{(\delta_1 e_1, \ldots, \delta_k e_k) \mid \delta_i = \pm 1 \text{ for } i = 1, \ldots, k\} \subseteq W_{n,k}.
\]

We see that \( f^{-1}(0) \) is a set of size \( 2^k \) which is an orbit of \( G \). We need to explain why zero is a regular value of \( f \). To this end we consider a smooth regular curve \( U(t) \in W_{n,k} \), defined for \( t \in (-1, 1) \) with \( U(0) \in f^{-1}(0) \). Note that the derivatives with respect to the variable \( t \), denoted by \( \dot{U}_{1,1}(t), \ldots, \dot{U}_{k,k}(t) \) all vanish for \( t = 0 \), because \( |U_{i,i}(0)| = 1 \geq |U_{i,i}(t)| \) for all \( i \) and \( t \). A crucial observation is that the derivative at \( t = 0 \) of the vector

\[
(U_{i,i+j}(t))_{1 \leq i \leq k, 1 \leq j \leq n-i} \in \mathbb{R}^N
\]

does not vanish, because \( \dot{U}(0) \neq 0 \) and because \( U(t) \in W_{n,k} \) for all \( t \). It thus follows from (14), (15) and the Leibnitz rule for differentiation that \( \frac{d}{dt} f(U(t)) \) does not vanish for \( t = 0 \), as required. We have thus verified all of the requirements from [14, Theorem 2.1], thereby completing the proof.

Next we describe the elegant method from Barvinok [2] for approximating a norm by a homogeneous polynomial taken to some power. Fix \( k \geq 1 \). We refer to a vector \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_{\geq 0}^k \) of non-negative integers as a multi-index. The order of the multi-index \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is

\[
|\alpha| = \sum_{i=1}^k \alpha_i.
\]

The collection of multi-indices of order \( d \) is denoted by \( \mathcal{M}_d \). It is a set of cardinality \( \binom{d+k-1}{k-1} \).

For \( x \in \mathbb{R}^k \) and \( \alpha \in \mathcal{M}_d \) we define

\[
x^\alpha = \prod_{i=1}^k x_i^{\alpha_i} \in \mathbb{R}.
\]

(16)

For an integer \( d \geq 1 \) and \( x \in \mathbb{R}^k \) we set

\[
x^\otimes d = (x^\alpha)_{|\alpha|=d} \in \mathbb{R}^{\mathcal{M}_d}
\]
(the reason for this notation is that the linear space \( \mathbb{R}^{M_d} \) may be identified with the symmetric tensor product of order \( d \) of \( \mathbb{R}^k \)). Note that \((-x) \otimes^d = (-1)^d \cdot x \otimes^d \). We use \((b_\alpha)_{\alpha \in M_d}\) as the coordinates of a vector \( b \in \mathbb{R}^{M_d} \), thus for instance we may write
\[
(x \otimes^d) \alpha = x^\alpha \quad (x \in \mathbb{R}^k, \alpha \in M_d).
\]

For \( a, b \in \mathbb{R}^{M_d} \) we define the scalar product
\[
(a, b) = \sum_{\alpha \in M_d} \binom{d}{\alpha} a_\alpha b_\alpha
\]
where \( \binom{d}{\alpha} = \frac{d!}{\prod_{i=1}^k \alpha_i!} \) is a multinomial coefficient. Note that for any \( x, y \in \mathbb{R}^k \), by the multinomial theorem,
\[
(x \otimes^d, y \otimes^d) = \sum_{\alpha \in M_d} \binom{d}{\alpha} x^\alpha y^\alpha = \left( \sum_{i=1}^k x_i y_i \right)^d = (x, y)^d. \tag{17}
\]

Suppose that \( d \) is an odd, positive integer. Given a norm \( \| \cdot \| \) in \( \mathbb{R}^k \), we consider the dual norm
\[
\| x \|_* = \sup_{0 \neq z \in \mathbb{R}^k} \frac{\langle z, x \rangle}{\| z \|}
\]
and the convex set
\[
K_{\| \cdot \|} = \text{conv} \left\{ x \otimes^d ; x \in \mathbb{R}^k, \| x \|_* \leq 1 \right\}, \tag{18}
\]
where \( \text{conv} \) denotes the convex hull of a set. The set \( K = K_{\| \cdot \|} \subseteq \mathbb{R}^{M_d} \) is a centrally-symmetric convex body with a non-empty interior, as explained in \([2]\). Note that by (17) and (18), for any \( x \in \mathbb{R}^k \),
\[
\| x \|_*^d = \sup_{\| y \|_* \leq 1} \langle x, y \rangle^d = \sup_{\| y \|_* \leq 1} (x \otimes^d, y \otimes^d) = \sup_{a \in K} (x \otimes^d, a). \tag{19}
\]

Among all centrally-symmetric ellipsoids that are contained in \( K = K_{\| \cdot \|} \), there is a unique ellipsoid of maximal volume (see e.g. \([22]\) Corollary 3.7). We denote this maximal-volume ellipsoid by \( \mathcal{E} = \mathcal{E}_{\| \cdot \|} \subseteq \mathbb{R}^{M_d} \). By the John theorem (see e.g. \([22]\) Chapter 3),
\[
\mathcal{E} \subseteq K \subseteq \sqrt{|M_d|} \cdot \mathcal{E}, \tag{20}
\]

where \( |M_d| \) is the cardinality of the set \( M_d \). Since \( \mathcal{E} \) is a centrally-symmetric ellipsoid in \( \mathbb{R}^{M_d} \) and since \( \langle \cdot, \cdot \rangle \) is an inner product, the expression \( \sup_{a \in \mathcal{E}} (a, b)^2 \) is a quadratic function of \( b \in \mathbb{R}^{M_d} \) that is positive for all \( 0 \neq b \in \mathbb{R}^{M_d} \). Consequently, there exists a positive-definite, symmetric matrix
\[
A = (A_{\alpha, \beta})_{\alpha, \beta \in M_d} \in \mathbb{R}^{M_d \times M_d}
\]
which satisfies
\[
\sum_{\alpha, \beta \in M_d} A_{\alpha, \beta} b_\alpha b_\beta = \sup_{a \in \mathcal{E}} (a, b)^2 \quad \text{for all } b \in \mathbb{R}^{M_d}. \tag{21}
\]
The symmetric matrix \( A = A_{\|\cdot\|} \) is uniquely determined by the ellipsoid \( \mathcal{E} = \mathcal{E}_{\|\cdot\|} \). As in Barvinok [2], we conclude from (19), (20) and (21) that
\[
\sum_{\alpha,\beta \in \mathcal{M}_d} A_{\alpha,\beta} x^\alpha x^\beta \leq \|x\|^{2d} \leq |\mathcal{M}_d| \cdot \sum_{\alpha,\beta \in \mathcal{M}_d} A_{\alpha,\beta} x^\alpha x^\beta. \tag{22}
\]
Observe also that \( x^\alpha x^\beta = x^{\alpha + \beta} \) for all \( x \in \mathbb{R}^k, \alpha, \beta \in \mathcal{M}_d \). According to (22), the given norm \( \| \cdot \| \) in \( \mathbb{R}^k \), taken to the power \( 2d \), is approximated by a \( 2d \)-homogeneous polynomial in \( \mathbb{R}^k \).

**Lemma 3.2.** The matrix \( A = A_{\|\cdot\|} \) varies continuously with the norm \( \| \cdot \| \), where we equip the space of norms on \( \mathbb{R}^k \) with the topology of uniform convergence on \( S^{k-1} \). Moreover, for \( \delta \in \{\pm 1\}^k \) and a norm \( \| \cdot \| \) on \( \mathbb{R}^k \), denote
\[
\|x\|_\delta = \|(\delta_1 x_1, \ldots, \delta_k x_k)\| \quad \text{for} \; x \in \mathbb{R}^k.
\]
Then the symmetric matrices \( A_{\|\cdot\|} = (A_{\alpha,\beta})_{\alpha,\beta \in \mathcal{M}_d} \) and \( A_{\|\cdot\|_\delta} = (A_{\alpha,\beta}(\delta))_{\alpha,\beta \in \mathcal{M}_d} \) satisfy
\[
A_{\alpha,\beta}(\delta) = \delta^{\alpha + \beta} A_{\alpha,\beta} \quad \text{for} \; \alpha, \beta \in \mathcal{M}_d, \tag{23}
\]
where we recall that \( \delta^{\alpha + \beta} = \prod_{i=1}^k \delta_i^{\alpha_i + \beta_i} \).

**Proof.** Let us continuously vary the norm \( \| \cdot \| \). Then the dual norm \( \| \cdot \| \) also varies continuously, and the convex body \( K = K_{\|\cdot\|} \subseteq \mathbb{R}^{\mathcal{M}_d} \) varies continuously with respect to the Hausdorff metric. Since the maximal volume ellipsoid \( \mathcal{E} = \mathcal{E}_{\|\cdot\|} \) is uniquely determined, it also varies continuously with respect to the Hausdorff metric. This follows from the fact that if \( f(x,y) \) is a continuous function of two variables in a compact metric space, and \( \min_y f(x,y) \) is uniquely attained for any \( x \) at a point \( y_0(x) \), then \( y_0(x) \) is a continuous function of \( x \). The symmetric matrix \( A \) is determined by \( \mathcal{E} \) through (21), and it is elementary to verify its continuity. Next, for any \( \delta \in \{\pm 1\}^k \) the convex set \( K_{\|\cdot\|_\delta} \) is the image of \( K_{\|\cdot\|} \) under the linear map
\[
(y_\alpha)_{\alpha \in \mathcal{M}_d} \mapsto (\delta^\alpha y_\alpha)_{\alpha \in \mathcal{M}_d}. \tag{24}
\]
Similarly, the ellipsoid \( \mathcal{E}_{\|\cdot\|_\delta} \) is the image of the ellipsoid \( \mathcal{E}_{\|\cdot\|} \) under the linear map in (24). Hence the matrix \( A_{\|\cdot\|_\delta} \) is congruent to the matrix \( A_{\|\cdot\|} \) via the linear transformation (24), and relation (23) holds true. 

Consider the lexicographic order on \( \mathcal{M}_d \). That is, for two distinct multi-indices \( \alpha, \beta \in \mathcal{M}_d \) let \( i_0 \in \{1, \ldots, k\} \) be the minimal index with \( \alpha_{i_0} < \beta_{i_0} \). We write that \( \alpha < \beta \) if \( \alpha_{i_0} < \beta_{i_0} \). It is easy to verify that \( < \) is a linear order relation. Consider the subset
\[
E_d = \left\{ (\alpha, \beta) : \alpha, \beta \in \mathcal{M}_d, \alpha < \beta, \alpha + \beta \not\in 2\mathbb{Z}^k \right\} \subseteq \mathcal{M}_d \times \mathcal{M}_d,
\]
where \( 2\mathbb{Z}^k \) is the collection of all vectors of length \( k \) whose coordinates are even integers.
Proof of Theorem 1.4. For \( U = (U_1, \ldots, U_k) \in W_{n,k} \) with \( U_1, \ldots, U_k \in \mathbb{R}^n \) being a \( k \)-frame, we define the norm

\[
\|x\|_U = \left\| \sum_{i=1}^{k} x_i U_i \right\|
\]

for \( x \in \mathbb{R}^k \).

Let \( d \) be an odd, positive integer such that

\[
\frac{1}{2} \left( \frac{d + k - 1}{k - 1} \right)^2 \leq n - k. \quad (25)
\]

Let us abbreviate \( A(U) := A\|\|_U \in \mathbb{R}^{M_d \times M_d} \). Write \( A(U) = (A_{\alpha,\beta}(U))_{\alpha,\beta \in M_d} \) and recall that \( A_{\alpha,\beta}(U) = A_{\beta,\alpha}(U) \). Consider the map

\[
W_{n,k} \ni U \mapsto (A_{\alpha,\beta}(U))_{(\alpha,\beta) \in E_d} \in \mathbb{R}^\ell \quad (26)
\]

for

\[
\ell = |E_d| \leq \frac{|M_d| \cdot (|M_d| - 1)}{2} \leq \frac{1}{2} \left( \frac{d + k - 1}{k - 1} \right)^2 \leq n - k. \quad (27)
\]

The map defined in (26) is continuous, as follows from Lemma 3.2. It is equivariant with respect to the group \( G = \{ \pm 1 \}^k \), where the action of \( G \) on \( W_{n,k} \) is given by (11), and the action on \( \mathbb{R}^\ell \) is described by (23). Observe that the last action is fixed-point-free, as we only consider \( \alpha, \beta \in M_d \) with \( \alpha + \beta \notin 2\mathbb{Z}^k \). We may apply Proposition 3.1 thanks to (27) and conclude that there exists \( U \in W_{n,k} \) such that for any \( \alpha, \beta \in M_d \),

\[
A_{\alpha,\beta}(U) = 0 \quad \text{whenever } \alpha + \beta \notin 2\mathbb{Z}^k. \quad (28)
\]

We fix such \( U \in W_{n,k} \), and show that the norm \( \|x\|_U \) is approximately an unconditional norm in \( \mathbb{R}^k \). For \( x \in \mathbb{R}^k \) consider the \((2d)\)-homogeneous polynomial,

\[
P(x) = \sum_{\alpha,\beta \in M_d} A_{\alpha,\beta}(U)x^{\alpha + \beta} \quad (x \in \mathbb{R}^k).
\]

It follows from (28) that \( P(x_1, \ldots, x_k) = P(\pm x_1, \ldots, \pm x_k) \) for any choice of signs. In view of (22), the \( \{\pm 1\}^k \)-invariant, \((2d)\)-homogeneous polynomial \( P \) satisfies

\[
P(x)^{1/(2d)} \leq \|x\|_U \leq |M_d|^{1/(2d)} \cdot P(x)^{1/(2d)} \quad (x \in \mathbb{R}^k). \quad (29)
\]

Denote

\[
\varepsilon = |M_d|^{1/(2d)} - 1 \quad (30)
\]

and set also \( \|x\| = 2^{-n} \sum_{\delta \in \{\pm 1\}^k} \|\delta_1 x_1, \ldots, \delta_k x_k\|_U \). Then \( \| \cdot \| \) is an unconditional norm, and it follows from (29) that

\[
\frac{1}{1 + \varepsilon} \|x\| \leq \|x\|_U \leq (1 + \varepsilon) \|x\| \quad \text{for } x \in \mathbb{R}^k.
\]

Recall that \( 1 - \varepsilon \leq 1/(1 + \varepsilon) \) provided that \( 0 < \varepsilon < 1 \). Since \( |M_d| = \binom{d+k-1}{k-1} \) we deduce from (30) that assuming \( d \geq k \),

\[
\varepsilon \leq \left( 2e \frac{d}{k} \right)^{\frac{k}{kd}} - 1 \leq C \frac{\log(d/k)}{d/k} \leq \frac{C}{(d/k)^{2/3}}. \quad (31)
\]
We claim that we may choose \( d \geq ckn \frac{1}{2(k-1)} \) so that (25) would hold true. Indeed, we may assume that \( k \leq n/2 \) as otherwise there is nothing to prove. Thus (25) holds true whenever
\[
\left( \frac{d + k - 1}{k - 1} \right)^2 \leq \left( 2e \frac{d}{k - 1} \right)^{2(k-1)} \leq n.
\]
Consequently, from (31), for any \( k \geq 2 \) the conclusion of the theorem holds true with any \( 0 < \varepsilon < 1/2 \) that satisfies
\[
\varepsilon \leq Cn^{-\frac{1}{3(k-1)}}
\]
for a universal constant \( C > 0 \). The number “3” in (32) may be replaced by any number \( \alpha > 2 \), at the expense of increasing the universal constant \( C > 0 \).

**Remarks.**

1. It should not be a problem to derive a result analogous to Theorem 1.3 that applies for non-homogeneous, even polynomials.

2. In Bourgain-Lindenstrauss [4], Tikhomirov [24] and Fresen [11], one assumes that the given norm is invariant under the group \((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n\) and concludes the existence of approximately-spherical sections with good dependence on the degree of approximation. We note here that it is easy to obtain \((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n\)-symmetric norms from \( S_n \)-symmetric norms, by reducing the dimension by a factor of two. Specifically, if \( \| \cdot \| \) is a permutation-invariant norm in \( \mathbb{R}^n \), for even \( n \), then for \( m = n/2 \), the norm
\[
\| (x_1, \ldots, x_m) \| = \| (x_1, -x_1, x_2, -x_2, \ldots, x_m, -x_m) \|
\]
is a \((\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m\)-invariant norm on \( \mathbb{R}^m \).

3. The proofs in Section 2 yield a polynomial invariant under a discrete group of symmetries isomorphic to a 2-Sylow subgroup of \( S_n \). A straightforward adaptation of the argument would yield another group of symmetries, namely the successive wreath product of \( \mathbb{Z}/p_i\mathbb{Z} \) for a finite sequence of prime numbers \( p_1, \ldots, p_h \). Similarly, the Dol’nikov-Karasev geometric argument [8, Section 9] may be generalized to handle this case.

**References**

[1] Aron, R. M., Hájek, P., *Zero sets of polynomials in several variables.* Arch. Math. (Basel), Vol. 86, no. 6, (2006), 561–568.

[2] Barvinok, A., *Approximating a norm by a polynomial. Geometric aspects of functional analysis.* In Geometric aspects of functional analysis (2001–02), volume 1807 of Lecture Notes in Math., pages 20–26. Springer, 2003.

[3] Birch, B. J., *Homogeneous forms of odd degree in a large number of variables.* Mathematika, Vol. 4, (1957), 102–105.
[4] Bourgain J., Lindenstrauss, J., *Almost Euclidean sections in spaces with a symmetric basis*. In Geometric aspects of functional analysis (1987–88), volume 1376 of Lecture Notes in Math., pages 278–288. Springer, 1989.

[5] Brauer, R., *A note on systems of homogeneous algebraic equations*. Bull. Amer. Math. Soc., Vol. 51, (1945), 749–755.

[6] Burago, D., Ivanov, S., Tabachnikov, S., *Topological aspects of the Dvoretzky theorem*. J. Topol. Anal., Vol. 2, no. 4, (2010), 453–467.

[7] Chan, Y. H., Chen, S., Frick, F., Hull, J. T., *Borsuk-Ulam theorems for products of spheres and Stiefel manifolds revisited*. Topol. Methods Nonlinear Anal. 55, no. 2, (2020), 553–564.

[8] Dol’nikov, V. L., Karasev, R. N., *Dvoretzky type theorems for multivariate polynomials and sections of convex bodies*. Geom. Funct. Anal. (GAFA), Vol. 21, no. 2, (2011), 301–318.

[9] Dvoretzky, A., *A theorem on convex bodies and applications to Banach spaces*. Proc. Nat. Acad. Sci. U.S.A., Vol. 45, no. 2, (1959), 223–226 (erratum in Vol. 45, no. 10).

[10] Dvoretzky, A., *Some results on convex bodies and Banach spaces*. In Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), pages 123–160. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.

[11] Fresen, D., *Explicit Euclidean embeddings in permutation invariant normed spaces*. Adv. Math., Vol. 266, (2014), 1–16.

[12] Grothendieck, A., *Sur certaines classes de suites dans les espaces de Banach, et le théorème de Dvoretzky-Rogers*. Resenhas, 3(4):447–477, 1998. With a foreword by Paulo Cordaro, Reprint of the 1953 original.

[13] Hatcher, A., *Algebraic topology*. Cambridge University Press, 2002.

[14] Klartag, B., *Convex geometry and waist inequalities*. Geom. Funct. Anal. (GAFA), Vol. 27, no. 1, (2017), 130–164.

[15] Makeev, V., *On some combinatorial geometry problems for vector bundles*. St. Petersburg Math. J., Vol. 14, (2003), 1017–1032.

[16] Matoušek, J., *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*. Written in cooperation with Anders Björner and Günter M. Ziegler. Springer, 2003.

[17] Milnor, J., Stasheff, J. D., *Characteristic classes*. Princeton University Press, 1974.

[18] Milman, V. D., *A few observations on the connections between local theory and some other fields*. In Geometric aspects of functional analysis (1986/87), volume 1317 of Lecture Notes in Math., pages 283–289. Springer, 1988.

[19] Milman, V. D., *A new proof of A. Dvoretzky’s theorem on cross-sections of convex bodies*. Funkcional. Anal. i Priložen. 5 (1971), no. 4, 28–37.

[20] Milman, V. D., *The heritage of P. Lévy in geometrical functional analysis*. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). Ast’erisque, No. 157-158, (1988), 273–301.
[21] Milman, V., Dvoretzky’s theorem—thirty years later. Geom. Funct. Anal. (GAFA), Vol. 2, no. 4, (1992), 455–479.

[22] Pisier, G., The volume of convex bodies and Banach space geometry. Cambridge University Press, 1989.

[23] Schechtman, G., Euclidean Sections of Convex Bodies. Asymptotic geometric analysis, Fields Inst. Commun., Vol. 68, Springer, New York, (2013), 271—288.

[24] Tikhomirov, K. E., Almost Euclidean sections in symmetric spaces and concentration of order statistics. J. Funct. Anal., Vol. 265, no. 9, (2013), 2074–2088.

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