Mathematical modeling on the force experienced by the wall during the deflection of granular flows

Yu Hui Deng and Yi Zhao

Applied Mathematics Center, Harbin Institute of Technology Shenzhen Graduate School, Shenzhen University Town, XiLi Nanshan, Shenzhen, P.R. China
E-mail: 1.yuhuideng@hitsz.edu.cn 2.zhao.yi@hitsz.edu.cn

Abstract. We develop a mathematical model to describe the system where a dilute granular flow is deflected by an oblique smooth wall. Both particles in the granular flow and the oblique wall experience inelastic collisions with other particles. An analytical solution for the mean force experienced by the wall is obtained through the derivation based on the mathematical model and probability theory. Our theory shows that large angle between the wall and the granular flow may decrease the force experienced by the wall. This is opposite to one’s intuition that large deflected angle implies large force. It is probable particle collisions and scattering after the deflection that prevent the wall from experiencing a further strong impact, thus the wall experience less force.

1. Introduction

Granular flows are commercially important in industrial applications such as pharmaceutical industry, agriculture, and energy production. However, some kinds of granular flows such as rock falls may pose significant natural disaster. In this case, oblique deflectors such as deflecting dams are applied to divert particle flows.

Despite various applications of granular flows and great efforts which have been devoted to the study of them, current understanding in the mechanics of granular flows is limited, and many fundamental difficulties still remain, so there is a great interest in explaining their properties. Behringer [1] gave a general introduction to the focus issue on granular materials, and talked about what we have learned since Jaeger’s article [2] had been published. Pouliquen [3] studied experimentally and deduced a new model of steady flows of particles down a roughened plane. More recently, By using discrete element numerical simulations, Wassgren et al. [4] investigated a two-dimensional dilute granular flow around an immersed cylinder. Hogg et al. [5] investigated more thoroughly the steady deflection of granular flows by obstacles and established a theoretical model of the interaction that may predict quantitatively the flow following deflection.

The purpose of this paper is to investigate the force experienced by the oblique wall, which is very important when diverting granular flows. By mathematical modeling and probability theory we will derive an analytical formula for the mean force. Our theory shows an interesting phenomenon that large angle between the wall and the granular flow may decrease the force experienced by the wall, which may be opposite to one’s intuition. The interpretation for this will be given in this paper.
2. Theoretical approach and results

We consider a two-dimensional system where a dilute horizontal granular flow is deflected by an oblique smooth wall with a deflected angle $\theta$, which is defined as the acute angle between the wall and the moving direction of the flow. We assume that the flow consists of one type of smooth sphere with the same radius $a$ and mass $m$. The particles enter the system with different time intervals. Different spatial distribution along the boundary can be specified for the initial positions of particles. Particles are removed from the system after interactions with other particles and the wall. In this paper, we choose Gaussian distribution with standard deviation $\sigma$ for the initial positions of particles. Particles are removed from the system after interactions with other particles.

In this paper, we aim to investigate the dimensionless mean force experienced by the oblique wall, which is defined as the average impact on the oblique wall by analyzing possible interactions between two particles. For simplicity, we assume, is much larger than any initial thermal energy of the particles. Both collisions between particles and collisions between particles and the wall are inelastic with constant restitution coefficient $e$ once. First let us consider a simple case where the particle-particle collision is the only interaction. With this simplification we can apply statistic average to determine the total collision rate.

We considered a two-dimensional system where a dilute horizontal granular flow is deflected by an oblique smooth wall with a deflected angle $\theta$. We nondimensionalize lengths by $a$, then the system can be described by five dimensionless parameters, the deflected angle $\theta$, the particle-particle restitution coefficient $e$, the particle-wall restitution coefficient $e_w$, and the ratio of the relative velocities of the particle to another particle (or the wall) in the direction along their line of centers $V = \frac{v_p}{a}$ and $S = \frac{v_w}{a}$. In fact, small value of $S$ implies that the particles are close to each other in horizontal (or perpendicular) direction. In this paper, we aim to investigate dimensionless mean force experienced by the oblique wall $f_{\text{mean}} = \frac{F_{\text{mean}}}{mv_0^2/\tau}$.

Since we consider a dilute system where multiple-particle collisions are negligible and pairwise particle collisions dominate, we suppose that there is only one collision between particles, and after the particle-particle collision, one or both of the two particles could hit the oblique wall again only once. With this simplification we can apply statistic average to determine the total average impact on the oblique wall by analyzing possible interactions between two particles. We randomly choose a particle from the system, and denote it by $B_{n+1}$. According to the simplicities, we know that it may collide with anyone of previous particles $B_n, B_{n-1}, \ldots B_1$ only once. First let us consider a simple case where $B_{n+1}$ may only collide with its previous neighbor particle $B_n$. Let $p_1^{(n+1)}$, $p_2^{(n+1)}$ and $p_0^{(1)}$ denote the following probabilities,

\[
\begin{align*}
    p_1^{(n+1)} &= P(B_{n+1} \text{ collides with } B_n), \\
    p_2^{(n+1)} &= P(B_{n+1} \text{ does not collide with } B_n), \\
    p_0^{(1)} &= P(B_{n+1} \text{ collides with } B_n \mid B_n \text{ does not collide with } B_{n-1}).
\end{align*}
\]

By the definition of $p_1^{(n+1)}$, $p_2^{(n+1)}$, $p_0^{(1)}$ and notice that $B_{n+1}$ may collide with $B_n$ if and only if $B_n$ does not collide with $B_{n-1}$, then we can obtain,

\[
p_{1}^{(n+1)} = p_0^{(1)} (1 - p_1^{(n)}) \tag{1}
\]

Similarly, we know that $B_{n+1}$ will not collide with $B_n$ if and only if $B_n$ does collide with $B_{n-1}$ or $B_{n+1}$ misses the rebounding particle $B_n$, when $B_n$ does not collide with $B_{n-1}$, we can also write $p_2^{(n+1)}$ as a function of $p_1^{(n)}$ and $p_2^{(n)}$,

\[
p_2^{(n+1)} = 1 - p_1^{(n+1)} = (1 - p_0^{(1)}) p_2^{(n)} \tag{2}
\]

Combining (1) and (2), we can write the results in a matrix form,

\[
\begin{pmatrix}
    p_{1}^{(n+1)} \\
    p_{2}^{(n+1)}
\end{pmatrix} =
\begin{pmatrix}
    0 & p_0^{(1)} \\
    1 & 1 - p_0^{(1)}
\end{pmatrix}
\begin{pmatrix}
    p_1^{(n)} \\
    p_2^{(n)}
\end{pmatrix} = M^{(1)}
\begin{pmatrix}
    p_1^{(n)} \\
    p_2^{(n)}
\end{pmatrix} \tag{3}
\]
In fact, if \( B_{n+1} \) may only collide with its previous neighbor particle \( B_n \), all the possible states of \( B_{n+1} \) must only depend on the states of \( B_n \). In other words, the states of a single particle in the system have Markov property and the matrix \( M^{(1)} \) in (3) is the transition probability matrix for a single step. When a steady state has been achieved, there will be no difference between the states of \( B_{n+1} \) and the states of \( B_n \), then by letting \( n \to +\infty \) in (3), we can obtain the probability for a single particle experiencing collisions as follows,

\[
p^{(1)} = \lim_{n \to +\infty} p^{(n+1)}_1 = \frac{p^{(1)}_0}{1 + p^{(1)}_0} \quad (4)
\]

Now let us consider a little bit more complicated case where \( B_{n+1} \) may collide with \( B_n \) or \( B_{n-1} \). Let \( p^{(n+1)}_1, p^{(n+1)}_2, p^{(n+1)}_3, p^{(1)}_0 \) and \( p^{(2)}_0 \) respectively denote the following probabilities,

\[
\begin{align*}
p^{(n+1)}_1 &= P(B_{n+1} \text{ collides with } B_n), \\
p^{(n+1)}_2 &= P(B_{n+1} \text{ collides with } B_{n-1}), \\
p^{(n+1)}_3 &= P(B_{n+1} \text{ does not collide with } B_n \text{ and } B_{n-1}), \\
p^{(1)}_0 &= P(B_{n+1} \text{ collides with } B_n \mid B_n \text{ does not collide with } B_{n-1} \text{ and } B_{n-2}), \\
p^{(2)}_0 &= P(B_{n+1} \text{ collides with } B_{n-1} \mid B_{n+1} \text{ does not collide with } B_n, \\
&\quad \text{ and } B_{n-1} \text{ does not collide with } B_n, B_{n-2} \text{ and } B_{n-3}).
\end{align*}
\]

Similar to the previous case but with a little more calculation, we can also describe the system in a matrix form as follows,

\[
\begin{pmatrix}
p^{(n+1)}_1 \\
p^{(n+1)}_2 \\
p^{(n+1)}_3
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & p^{(n)}_{13} \\
0 & p^{(n)}_{22} & p^{(n)}_{23} \\
1 & 1 - p^{(n)}_{22} & 1 - p^{(n)}_{13} - p^{(n)}_{23}
\end{pmatrix}
\begin{pmatrix}
p^{(n)}_1 \\
p^{(n)}_2 \\
p^{(n)}_3
\end{pmatrix} \triangleq M^{(2)}
\]

\[
p^{(n)}_{13} = p^{(1)}_0 (1 - p^{(n)}_{23}), \\
p^{(n)}_{22} p^{(n)}_2 = p^{(2)}_0 (p^{(n)}_2 - p^{(n-1)}_{22} p^{(n-1)}_2), \\
p^{(n)}_{23} p^{(n)}_3 = \frac{1 - p^{(1)}_0}{1 - p^{(2)}_0} p^{(n-1)}_{23} (1 - p^{(n)}_{13} - p^{(n)}_{23}).
\]

The difference between the two cases is that the matrix \( M^{(2)} \) is no longer a constant matrix but a nonconstant matrix with iterative parameters. In fact, if \( B_{n+1} \) may collide with \( B_n \) or \( B_{n-1} \), all the possible states of \( B_{n+1} \) not only depend on the states of \( B_n \) but also depend on the states of \( B_{n-1} \). When a steady state has been achieved, by letting \( n \to +\infty \) in (5), we can obtain

\[
\begin{pmatrix}
p^{(2)}_1 \\
p^{(2)}_2 \\
p^{(2)}_3
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & p^{(2)}_{13} \\
0 & p^{(2)}_{22} & p^{(2)}_{23} \\
1 & 1 - p^{(2)}_{22} & 1 - p^{(2)}_{13} - p^{(2)}_{23}
\end{pmatrix}
\begin{pmatrix}
p^{(2)}_1 \\
p^{(2)}_2 \\
p^{(2)}_3
\end{pmatrix} \triangleq M^{(2)}
\]

\[
p^{(2)}_{22} = \frac{p^{(2)}_0}{1 + p^{(2)}_0}, \\
p^{(2)}_{13} = \frac{p^{(1)}_0 (1 - p^{(2)}_0)}{1 - p^{(2)}_0 + (1 - p^{(2)}_0)^2 p^{(2)}_0}, \\
p^{(2)}_{23} = \frac{(1 - p^{(1)}_0)^2 p^{(2)}_0}{1 - p^{(1)}_0 (p^{(2)}_0 + (1 - p^{(1)}_0)^2 p^{(2)}_0).}
\]

Solving the linear system (6) gives the probability for a single particle experiencing collisions in this case,

\[
p^{(2)} = p_1 + p_2 = \frac{p^{(1)}_0 + B}{1 + p^{(1)}_0 + B}, \\
B = p^{(1)}_{23} (1 - p^{(1)}_0 + p^{(2)}_0) = \frac{p^{(2)}_0 (1 - p^{(1)}_0)^2 (1 - p^{(1)}_0 + p^{(2)}_0)}{1 - p^{(1)}_0 (p^{(2)}_0 + (1 - p^{(1)}_0)^2 p^{(2)}_0)} \quad (7)
\]

Following this procedure we can work out the probabilities for a single particle experiencing collisions with more previous particles. However, by comparing (4) and (7), we find that
including the interaction with more previous particles has a weak effect for the collision probability since \( B \) is relatively small in a dilute system. Therefore, we choose the leading part of \( p = \lim_{n \to +\infty} p^{(n)} \), i.e \( p^{(1)} \) in (4), as the approximate collision probability for a single particle in the system. The force experienced by the wall comes from two aspects. One is the impact, denoted by \( I \), which is caused by the direct particle-wall collision without any collision with previous particles. The other is the impact, denoted by \( J \), which is caused by the particle-wall collision after the particle-particle collision. Thus by letting \( n \to +\infty \), we can write the total average force \( F_{\text{mean}} \) on the wall as follows due to the arbitrariness of \( B_{n+1} \) in the system.

\[
F_{\text{mean}} = \lim_{n \to +\infty} \left( (1 - p_{1}^{(n+1)}) \cdot \mathbf{E}(I|B_{n+1} \text{ does not collide with } B_{n}) + (1 - p_{1}^{(n)}) \cdot \mathbf{E}(J|B_{n} \text{ does not collide with } B_{n-1}) \right) = (1 - p^{(1)})(\hat{I} + \hat{J}) = \frac{1}{1 + p_{0}^{(1)}}(\hat{I} + \hat{J}).(8)
\]

Therefore, by using probability theory and performing integration, we will obtain the analytical formula of dimensionless mean force \( f_{\text{mean}} \) as follows,

\[
\left( \frac{(1 + e_{w}) \sin \theta}{4 + 2\text{erf}(D_{+}) - 4\text{erf}(D_{-})} \right) \left\{ 4 + \left( 1 + e_{w}(1 - e_{w}) \right) \left( \text{erf}(H_{+}) - \text{erf}(H_{-}) \right) + \left( 1 - e_{w} \right) \left( 1 + e_{w} \right) \right\} \left( \text{erf}(D_{+}) - \text{erf}(D_{-}) + M \right) + \left( 1 + e_{w} \right) \left( 1 + e_{w} \right) \left( S^{2} \sin^{2} \theta \right) + \frac{V^{2} \cos^{2} \theta}{4} \right) \left( \text{erf}(D_{+}) - \text{erf}(D_{-}) + M \right) + \frac{V S \sin \theta \cos \theta}{\sqrt{\pi}} \left( e^{-D_{-}^{2}} - e^{-D_{+}^{2}} + e^{-H_{-}^{2}} - e^{-H_{+}^{2}} + N \right) + \frac{S^{2} \sin^{2} \theta}{\sqrt{\pi}} \left( D_{-} e^{-D_{-}^{2}} - D_{+} e^{-D_{+}^{2}} + H_{+} e^{-H_{+}^{2}} - H_{-} e^{-H_{-}^{2}} + R \right) \right\},
\]

where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{t^{2}}/2 \, dt \), and

\[
M = \begin{cases} 0 & \text{erf}(L_{-}) - \text{erf}(L_{+}), \\ \text{erf}(L_{-}) - \text{erf}(L_{+}) & \text{otherwise} \end{cases},
N = \begin{cases} 0 & e^{-L_{+}^{2}} - e^{-L_{-}^{2}}, \\ e^{-L_{+}^{2}} - e^{-L_{-}^{2}} & \text{otherwise} \end{cases},
R = \begin{cases} 0 & \text{erf}(H_{-}) - \text{erf}(H_{+}), \\ \text{erf}(H_{-}) - \text{erf}(H_{+}) & \text{otherwise} \end{cases},
\]

\[
D_{\pm} = \pm 1 - \frac{V \cos \theta}{2 S \sin \theta}, \quad H_{\pm} = \pm \frac{\sqrt{\left( 1 + e_{w} \right) - e_{w} \left( 1 - e_{w} \right)} - \sqrt{(1+e)(1+e_{w})}}{2},
L_{\pm} = \pm \frac{\sqrt{\left( 1 + e_{w} \right) - e_{w} \left( 1 - e_{w} \right)}}{2} - \frac{V \cos \theta}{2 S \sin \theta},
\]

One can show that the mean force \( f_{\text{mean}} \) is a non-monotonic function of the deflected angle \( \theta \). Large angle \( \theta \) may decrease the force experienced by the wall. This is opposite to one’s intuition that large deflected angle implies a large impact perpendicular to the wall, and consequently large force. However, it is the particle-particle collisions and the scattering after the deflection that prevent the wall from experiencing a further strong impact, thus the wall experience less force.

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