Replica Symmetry Breaking and Phase Transitions in a PT Symmetric
Sachdev-Ye-Kitaev Model

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We show that the low temperature phase of a conjugate pair of uncoupled, quantum chaotic,
nonhermitian systems such as the Sachdev-Ye-Kitaev (SYK) model or the Ginibre ensemble of
random matrices are dominated by replica symmetry breaking (RSB) configurations with a nearly
flat free energy that terminates in a first order phase transition. In the case of the SYK model,
we show explicitly that the spectrum of the effective replica theory has a gap. These features are
strikingly similar to those induced by wormholes in the gravity path integral which suggests a close
relation between both configurations. For a non-chaotic SYK, the results are qualitatively different:
the spectrum is gapless in the low temperature phase and there is an infinite number of second order
phase transitions unrelated to the restoration of replica symmetry.

The study of nonhermitian effective Hamiltonians has
a long history [1, 2]. Perhaps the best known example is
the effective Hamiltonian that describes resonances with
a finite width, for example the one that enters in the cal-
culation of the S-matrix of open quantum systems such as
quantum dots [3] or compound nuclei [4]. Another ex-
ample is the Euclidean QCD Dirac operator at nonzero
chemical potential, which is nonhermitian with spectral
support on a two-dimensional domain of the complex
plane [5]. In hermitian theories, a phase transition may
arise due to the formation of a gap. This may also happen
for nonhermitian systems when the domain of eigenvalues
splits into two or more pieces. However, another mecha-
nism is possible. Because of the nonhermiticity, the ac-
tion is generally complex, and the saddle point with the
largest real part of the free energy may get nullified after
ensemble averaging. In QCD at nonzero baryon chemical
potential, the pion condensation phase is nullified so that
the phase transition to nonzero baryon density becomes
visible [6]. The conclusion is that the phase diagram can
be altered dramatically by the nullification of the leading
saddle point [6, 7].

A second point we wish to make is about the nature of
quenched averages in nonhermitian theories. Although
alternatives are possible [4, 8, 9], quenched averages are
often carried out by means of the replica trick [10]. How-
ever, because of Carlson’s theorem [11], a naive applica-
tion of the replica trick is not guaranteed to work [12].
The best known example of the failure of the replica
trick is in the calculation of the quenched free energy
of the Sherrington-Kirkpatrick model [13], a toy model for
spin glasses, which in the low temperature limit yields a
negative entropy [13]. This inconsistency was ultimately
resolved by postulating a ground state that breaks the
replica symmetry [14, 15]. The problems with the replica
trick are more dramatic for nonhermitian theories as was
first demonstrated for QCD at nonzero chemical poten-
tial $\mu$ [16]. In this case, the $n$ replica (or $n$ flavor) parti-
tion function is given by

$$Z_n = \langle \det^n D(\mu) \rangle,$$

where the averaging is over gauge field configurations
weighted by the Euclidean Yang-Mills action. It was
shown that the quenched approximation, where the de-
terminant is put to unity, is not given by $\lim_{n \to 0} Z_n$ but
rather by

$$\lim_{n \to 0} \langle \det^n (D(\mu)D^\dagger(\mu)) \rangle.$$

Because the disconnected part of the partition function
is nullified due to the phase of the fermion determinant,
this partition function is dominated by replica symmetry
breaking (RSB) configurations which in this context are
referred to as Goldstone bosons of a quark and a con-
jugate quark. A similar RSB mechanism has been iden-
tified in the context of random matrix theory, for both
hermitian [17, 18] and nonhermitian [19] random matrix
ensembles.

The possibility of a partition function where the con-
ected part dominates the disconnected part has recently
received a great deal of attention in the analysis of worm-
hole solutions in Jackiw-Teitelboim (JT) [20, 21] gra-
vity and related theories [22–32]. The existence of these
solutions in Lorentzian signature was first observed in
[33] with the discovery of a low temperature traversable
wormhole phase in a near AdS$_2$ background deformed by
weakly coupling the two boundaries. As temperature
increases, the system eventually undergoes a first order
wormhole to black hole transition. By adding complex
sources, it is possible to find Euclidean wormholes solutions of JT gravity that undergo a similar transition at finite temperature.

Interestingly, a replica calculation of the quenched free energy in JT gravity found that, in the low temperature limit, the contribution of replica wormholes is dominant. Likewise, the evaluation of the von Neumann entropy by the replica trick revealed the existence of additional RSB saddle points, wormholes connecting different copies of black holes in this context. These wormhole configurations are crucial to make the process of black hole evaporation consistent with unitarity.

A natural question to ask is whether these replica wormholes have a field theory analogue. RSB configurations have indeed been explored in the SYK model with real couplings. However, there is no evidence that they dominate the partition function.

In this paper, we answer the question posed in the previous paragraph affirmatively by identifying a pair of nonhermitian random Hamiltonians whose sum is PT-symmetric where RSB configurations are the leading saddle points of the action in the low temperature phase. We consider a nonhermitian version of the SYK model and show that it has a phase transition from a phase dominated by the disconnected part of the partition function to a phase dominated by the connected part, namely, a phase dominated by RSB configurations.

The $q$-body SYK Hamiltonian is defined by

$$H_{\text{SYK}} = (i)^{q/2} \sum_{\alpha_1 < \cdots < \alpha_q} J_{\alpha_1 \cdots \alpha_q} \chi_{\alpha_1} \cdots \chi_{\alpha_q}$$

where the $\chi_{\alpha}$ represent $N$ Majorana fermions, satisfying the anti-commutation relations $\{\chi_{\alpha}, \chi_{\beta}\} = \delta_{\alpha\beta}$, and the $J_{\alpha_1 \cdots \alpha_q}$ are real couplings, sampled from a Gaussian distribution having a vanishing mean value and a variance proportional to $1/N^2$. The coupled SYK model introduced by Maldacena and Qi (MQ) in consists of a Right (R) SYK model and a Left (L) SYK model each with $N/2$ Majorana fermions, and a coupling term $i\mu \sum_{k=1}^{N/2} \chi_k \chi_k^L$. Although the left and right couplings, denoted by $J_{\alpha_1 \cdots \alpha_q}$ are chosen to be the same as in the MQ model, it is also possible, as was noted by the same authors, to take them different. One remarkable observation was made: the solution that couples the right and left SYK continues to exist in absence of an explicit coupling ($\mu = 0$) provided that

$$\langle J^L J^R \rangle > \langle J^L J^L \rangle = \langle J^R J^R \rangle$$

where $\langle \ldots \rangle$ stands for ensemble average. Since the covariance matrix is no longer positive, this cannot be realized by real-valued $J_L$ and $J_R$. However, this can be achieved for the complex couplings

$$J^L = J + ikK, \quad J^R = J - ikK$$

with $J, K$ independent real Gaussian stochastic variables with the same variance and zero mean.

Before continuing, let us analyze the quenched free energy of the single-site SYK we have just introduced:

$$\langle \log Z \rangle = \langle \log |Z| \rangle + i\langle \arg Z \rangle,$$

where $Z$ is the partition function for a specific realization of the couplings. Since the phase of the partition function does not have a preferred direction, we expect that $\langle \arg Z \rangle = 0$. We conclude

$$\langle \log Z \rangle = \frac{1}{2} \langle \log Z Z^* \rangle$$

for a theory where $Z$ and $Z^*$ have equal probability. In particular, the quenched free energy is given by the replica limit

$$\frac{1}{2} \lim_{n \to 0} \langle (ZZ^*)^n - 1 \rangle.$$
where we have neglected prefactors that are subleading in $N$. In the strict large $N$ limit, it simplifies to,

$$\frac{F(T)}{N} = -\frac{2E_0}{N} \theta(T_c - T) - \frac{T \log 2}{2} \theta(T - T_c),$$

with $T_c = 4E_0/(N \log 2)$. In Fig. 2, we show the numerical quenched free energy of the SYK Hamiltonian (9) for $N/2 = 30$, $q = 4$ and $k = 1$ (blue curve) compared to the analytical result (16). The deviation in the constant part seems to scale as $1/N$. The free energy can also be worked out for $k < 1$, where we also find a first order phase transition with $T_c \sim k^4$ for small $k$.

We have thus observed that the leading exponent of the disconnected part of the partition function is nullified by the phase of the Boltzmann factor so that the contribution due to the connected part of the two-point correlation function becomes dominant. The free energy behaves as if the system has a gap. Both features are typical of RSB configurations.

In order to make this connection more explicit, we show that these results can also be obtained by solving the Schwinger-Dyson equations, in the $\Sigma G$ formulation of the two-site SYK model [33, 47] which is equivalent to performing the replica trick and then solving the model in the saddle point approximation. For $T > T_c$, and $k = 1$, the solution with the free $G$ and $\Sigma$ is dominant so that only the kinetic term of the Lagrangian remains. As a consequence, the free energy is $-T \log 2/2$ in agreement with the spectral calculation above. For $T < T_c$, a non-trivial RSB solution becomes dominant which results in a constant free energy up to exponentially small corrections. Similar results can be derived for $k < 1$, where, in agreement with the previous analytical calculation, we have also found $T_c \sim k^4$. Indeed, this feature is shared by both Euclidean [50] and traversable [33, 48] wormholes. Details of this and the previous analytical calculation will be given elsewhere [49].

FIG. 1. The eigenvalue density, obtained from exact diagonalization, for one realization of the $q = 4$, $k = 1$ nonhermitian SYK model with $N/2 = 30$, compared to a circle (red curve).

$\langle Z \rangle = \int d^2 z \rho(z) e^{-\beta z}$. Because the eigenvalue density has rotational invariance, we can use the mean value theorem to show that the partition function is independent of $\beta$ and given by the normalization of $\rho(z)$, which we denote by $D$, $\langle Z \rangle = D$.

To evaluate the second term of (11), we use the sum rule

$$\int d^2 z_2 (R_{2c}(z_1, z_2) + \delta^2 (z_1 - z_2) \rho(z_1)) = 0, \quad (12)$$

and the fact that the correlations are short-range [33, 46] with the connected correlator taking the universal form

$$R_{2c}(z_1, z_2) = R^{unc}(\sqrt{\rho(z)}(z_1 - z_2))\rho(z)^2, \quad (13)$$

where $\bar{z} = (z_1 + z_2)/2$. We thus have that $|z_1 - z_2| < 1/\sqrt{D}$ region gives the dominant contribution and we can Taylor expand the exponent in (11) in powers of $\beta \text{Im}(z_1 - z_2)$. The zero order term vanishes because of the sum rule (12), the linear term vanishes because the probability distribution is even under complex conjugation. After performing the integral over $z_1 - z_2$, we obtain the connected partition function

$$\langle ZZ^* \rangle_c = \beta^2 \langle \zeta^2 \rangle \int_{|z| < E_0} d^2 z e^{-\beta(z + z^*)} = \pi \beta E_0 \langle \zeta^2 \rangle I_1(2\beta E_0), \quad (14)$$

where $\langle \zeta \rangle = (\text{Im}(z_1 - z_2))^2 \sqrt{\rho(z)} \sim D^0$, the ground state energy $E_0 \sim N$ and $D = 2^{N/4}$. Including the disconnected part of the partition function and using the asymptotic expression of $I_1$, we obtain a free energy

$$F = -T \log \left[ e^{2\beta E_0} + 2^{N/2} \right] \quad (15)$$

FIG. 2. The temperature dependence of the free energy of the nonhermitian SYK model for $N/2 = 30$, $q = 4$ and $k = 1$ (blue curve) compared to the analytical result (16). The value of $E_0$ is the radius of the circle in Fig. 1.
where the gap $E_g$ is a fitting parameter. The fit is excellent except for very small times \cite{49}, see Fig. 3. This reinforces the picture that RSB configurations mediate tunneling between the two sites even though there is no direct coupling term in the Hamiltonian. We note that $G_{LL}$ and $G_{RR}$ show a similar decay. In Fig. 3 we also show that the gap $E_g$ depends quadratically on $k$. It would be interesting to understand this exponent from the gravity side. We propose $G_{LR}(0)$ as the order parameter of the transition since a non-vanishing $G_{LR}$ is a distinctive feature of RSB configurations. Results depicted in in Fig. 3 confirm that $G_{LR}(0)$ remains almost constant in the wormhole phase and vanishes for $T > T_c$.

The studied $q = 4$ SYK model is quantum chaotic \cite{41}. We expect very similar results in others quantum chaotic systems such as $q > 2$ SYK models and the Ginibre ensemble of random matrices \cite{45} because RSB depends on the connected two-level correlation function which, to some extent, is universal in this case \cite{51}. However, it is unclear whether quantum chaos is a necessary condition for RSB to occur. In order to further elucidate this issue, we study the non quantum chaotic, $q = 2$ nonhermitian SYK model which admits an explicit analytical solution of the Schwinger-Dyson equations. It can also be solved, see \cite{49} for details, by mapping it onto a model of free fermions. The free energy can be expressed, see also \cite{49}, as a sum over Matsubara frequencies. Each time a new Matsubara frequency enters the sum, by lowering the temperature, a second order phase transition occurs. Kinks in $-dF/dT$, depicted in Fig. 4 indicate the positions of the critical temperatures. The propagator $G_{LR}(\tau)$ can be also expressed as a finite sum over Matsubara frequencies so it does not depend exponentially on $\tau$. Because of the absence of a gap, there is no direct relation between RSB and wormholes which

For traversable wormholes \cite{33}, the spectrum is gapped. Physically, it is related to the interaction-driven tunneling between the left and right sites. The existence of the energy gap can be demonstrated \cite{33} directly from the effective boundary gravity action or from the exponential decay of the left-right Green’s function, $G_{LR}(\tau)$, of the two-site SYK model for low temperatures. We study whether a similar gap exists in the two-site non-hermitian SYK. We stress the gap in this case is not a property of the microscopic Hamiltonian \cite{9} but only of the resulting replica field theory after ensemble average. Since $G_{LR}(\beta/2 - \tau) = -G_{LR}(\tau - \beta/2)$ we employ the ansatz \cite{33}

$$G_{LR}(\tau) \sim \sinh E_g(\beta/2 - \tau), \quad (17)$$

![FIG. 3. Top: $G_{LR}(\tau)$ from the solution of the Schwinger-Dyson equations for the SYK model \cite{9} with $q = 4$, $T = 0.0005$ and $k = 0.5$ fitted by \cite{17}. Middle: The order parameter $G_{LR}(0)$, versus temperature for $k = 0.5$. Bottom: The energy gap $E_g$ for $T = 0.0005 \ll T_c$ as a function of $k$ from the fit \cite{17}.](image_url)

![FIG. 4. Derivative of the free energy for the nonhermitian $q = 2, k = 1$ SYK model. On the way to $T = 0$, the system undergoes infinitely many second order phase transitions. For $T > 1/\pi$, it becomes a constant which is a typical feature of free fermions.](image_url)
further suggests that the physics is qualitatively different from the quantum chaotic case. Further research is needed to delimit the importance of quantum chaos in RSB.

In summary, we have provided evidence that RSB configurations dominate the low temperature phase of the partition function of pairs of random non-hermitian, quantum chaotic systems whose sum is PT-symmetric. These field theory configurations mimic the contribution of wormholes in the gravitational path integral. In both cases, a first order transition occurs when replica symmetric saddle points take control of the partition function.

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