THE MAGNUS REPRESENTATION AND HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGY COBORDISMS OF SURFACES

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ABSTRACT. The set of homology cobordisms from a surface to itself with markings of their boundaries has a natural monoid structure. To investigate the structure of this monoid, we define and study its Magnus representation and Reidemeister torsion invariants by generalizing Kirk-Livingston-Wang’s argument over the Gassner representation of string links. Moreover, by applying Cochran and Harvey’s framework of higher-order (non-commutative) Alexander invariants to them, we extract several pieces of information about the monoid and related objects.

1. INTRODUCTION

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 1$ with one boundary component. A homology cylinder (over $\Sigma_{g,1}$) consists of a homology cobordism from $\Sigma_{g,1}$ to itself with markings of its boundary. We denote by $C_{g,1}$ the set of isomorphism classes of homology cylinders. Stacking two homology cylinders gives a new one, and by this, we can endow $C_{g,1}$ with a monoid structure (see Section 2 for the precise definition). The origin of homology cylinders goes back to Habiro [7], Garoufalidis-Levine [6] and Levine [14], where the clasper (or clover) surgery theory is effectively used to investigate the structure of $C_{g,1}$.

By a standard method, we can assign a homology cylinder to each homology 3-sphere or pure string link. Also, for a given homology cylinder, we can use an element of the mapping class group of $\Sigma_{g,1}$ to construct another one by changing its markings. Since these operations preserve each monoid structure, $C_{g,1}$ can be regarded as a simultaneous generalization of the monoid of homology 3-spheres, that of string links and the mapping class group, any of which plays an important role in the theory of 3-manifolds. On the other hand, there exists a natural way (called closing) to construct a closed 3-manifold from each homology cylinder. Therefore, through its monoid structure, $C_{g,1}$ serves as an effective tool for classifying closed 3-manifolds.

The aim of this paper is to study the structure of $C_{g,1}$ from rather an algebraic point of view. We mainly use non-commutative rings arising from group rings to define some invariants such as the Magnus representation for $C_{g,1}$ and Reidemeister torsion invariants. Note that our Magnus representation extends that for the mapping class group defined by Morita [16], as the Gassner representation for string links due to Le Dimet [11] and Kirk-Livingston-Wang [10] does that for the pure braid group. See Birman’s book [1] for generalities of the ordinary (pre-extended) Magnus representation, including free differentials.

After defining invariants using non-commutative rings, we shall need some devices to extract information from them. For that, we use the framework of higher-order Alexander invariants due to Cochran [2] and Harvey [8, 9]. Higher-order Alexander invariants are those for finitely presentable groups interpreted as degrees of “non-commutative Alexander polynomials”, which...
have some unclear ambiguity except their degrees. Historically, they are first defined for knot groups by Cochran, and then generalized for arbitrary finitely presentable groups by Harvey. Using them, Cochran and Harvey obtained various sharper results than those brought by the ordinary Alexander invariants — lower bounds on the knot genus or the Thurston norm, necessary conditions for realizing a given group as the fundamental group of some compact oriented 3-manifold, and so on. In the process of applying higher-order Alexander invariants to our case, we shall give its slight generalization (called torsion-degree functions) because of the difference of localizations of non-commutative rings used in the Magnus representation and higher-order Alexander invariants. Then we use it to study several properties of our invariants and relationships between them, from which we will obtain some information about the structure of $C_{g,1}$ and related 3-manifolds.

The outline of this paper is as follows. In Section 2, we review the definition of homology cylinders as well as setting up our notation and terminology. Sections 3 and 4, which are the first main part of this paper, are devoted to define the Magnus representation and study its fundamentals, including some examples. In Section 5 we review the theory of higher-order Alexander invariants, following Harvey’s papers [8, 9], and then define its generalization. In Section 6, which is the second main part, we observe several applications of our invariants.

2. HOMOLOGY CYLINDERS

Throughout the paper, we work in PL or smooth category. Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 1$ with one boundary component. We take a base point $p$ on the boundary of $\Sigma_{g,1}$, and take $2g$ loops $\gamma_1$, \ldots, $\gamma_{2g}$ of $\Sigma_{g,1}$ as shown in Figure 1. We consider them to be an embedded bouquet $R_{2g}$ of $2g$-circles tied at the base point $p \in \partial \Sigma_{g,1}$. Then $R_{2g}$ and the boundary loop $\zeta$ of $\Sigma_{g,1}$ together with one 2-cell make up a standard cell decomposition of $\Sigma_{g,1}$.

The fundamental group $\pi_1 \Sigma_{g,1}$ of $\Sigma_{g,1}$ is isomorphic to the free group $F_{2g}$ of rank $2g$ generated by $\gamma_1$, \ldots, $\gamma_{2g}$, in which $\zeta = \prod_{i=1}^{g} [\gamma_i; \gamma_{g+i}]$.

![Figure 1. A standard cell decomposition of $\Sigma_{g,1}$](image-url)

A homology cylinder $(M, i_+, i_-)$ (over $\Sigma_{g,1}$), which has its origin in Habiro [7], Garoufalidis-Levine [6] and Levine [14], consists of a compact oriented 3-manifold $M$ and two embeddings $i_+, i_- : \Sigma_{g,1} \to \partial M$ satisfying that

1. $i_+$ is orientation-preserving and $i_-$ is orientation-reversing,
2. $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial \Sigma_{g,1}) = i_-(\partial \Sigma_{g,1})$,
3. $i_+|_{\partial \Sigma_{g,1}} = i_-|_{\partial \Sigma_{g,1}}$,
4. $i_+, i_- : H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms.
We denote \( i_+(p) = i_-(p) \) by \( p \in \partial M \) again and consider it to be the base point of \( M \). We write a homology cylinder by \( (M, i_+, i_-) \) or simply by \( M \).

Two homology cylinders are said to be isomorphic if there exists an orientation-preserving diffeomorphism between the underlying 3-manifolds which is compatible with the embeddings of \( \Sigma_{g,1} \). We denote the set of isomorphism classes of homology cylinders by \( C_{g,1} \). Given two homology cylinders \( M = (M, i_+, i_-) \) and \( N = (N, j_+, j_-) \), we can construct a new homology cylinder \( M \cdot N \) by

\[
M \cdot N = (M \cup_{i_\circ (j_+)^{-1}} N, i_+, j_-).
\]

Then \( C_{g,1} \) becomes a monoid with the unit \( 1_{C_{g,1}} := (\Sigma_{g,1} \times I, \text{id} \times 1, \text{id} \times 0) \).

From the monoid \( C_{g,1} \), we can construct the homology cobordism group \( \mathcal{H}_{g,1} \) of homology cylinders as in the following way. Two homology cylinders \( M = (M, i_+, i_-) \) and \( N = (N, j_+, j_-) \) are homology cobordant if there exists a compact oriented 4-manifold \( W \) such that

\begin{enumerate}
\item \( \partial W = M \cup (-N)/(i_+(x) = j_+(x), i_-(x) = j_-(x)) \) \( x \in \Sigma_{g,1} \),
\item the inclusions \( M \hookrightarrow W, N \hookrightarrow W \) induce isomorphisms on the homology,
\end{enumerate}

where \( -N \) is \( N \) with opposite orientation. We denote by \( \mathcal{H}_{g,1} \) the quotient set of \( C_{g,1} \) with respect to the equivalence relation of homology cobordism. The monoid structure of \( C_{g,1} \) induces a group structure of \( \mathcal{H}_{g,1} \). In the group \( \mathcal{H}_{g,1} \), the inverse of \((M, i_+, i_-)\) is given by \((-M, i_-, i_+)\).

**Example 2.1.** For each element \( \varphi \) of the mapping class group \( \mathcal{M}_{g,1} \) of \( \Sigma_{g,1} \), we can construct a homology cylinder \( M_\varphi \in C_{g,1} \) by setting

\[
M_\varphi := (\Sigma_{g,1} \times I, \text{id} \times 1, \varphi \times 0),
\]

where collars of \( i_+(\Sigma_{g,1}) \) and \( i_-(\Sigma_{g,1}) \) are stretched half-way along \( \partial \Sigma_{g,1} \times I \). This gives injective homomorphisms \( \mathcal{M}_{g,1} \hookrightarrow C_{g,1} \) and \( \mathcal{M}_{g,1} \hookrightarrow \mathcal{H}_{g,1} \). From this, we can regard \( C_{g,1} \) and \( \mathcal{H}_{g,1} \) as enlargements of \( \mathcal{M}_{g,1} \).

Let \( N_k(G) := G/\langle \Gamma^k G \rangle \) be the \( k \)-th nilpotent quotient of a group \( G \), where we define \( \Gamma^1 G = G \) and \( \Gamma^l G = [\Gamma^{l-1} G, G] \) for \( l \geq 2 \). For simplicity, we write \( \tilde{N}_k(X) \) for \( N_k(\pi_1 X) \) where \( X \) is a connected topological space, and write \( \check{N}_k \) for \( \tilde{N}_k(F_{2g}) = N_k(\Sigma_{g,1}) \).

Let \((M, i_+, i_-)\) be a homology cylinder. By definition, \( i_+, i_- : \pi_1 \Sigma_{g,1} \to \pi_1 M \) are both 2-connected, namely they induce isomorphisms on the first homology groups and epimorphisms on the second homology groups. Then, by Stallings’ theorem \cite{Stallings}, \( i_+, i_- : N_k \overset{\cong}{\to} N_k(M) \) are isomorphisms for each \( k \geq 2 \). Using them, we obtain a monoid homomorphism

\[
\sigma_k : C_{g,1} \longrightarrow \text{Aut} \check{N}_k \qquad ((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-).
\]

It can be easily checked that \( \sigma_k \) induces a group homomorphism \( \sigma_k : \mathcal{H}_{g,1} \to \text{Aut} \check{N}_k \). We define filtrations of \( C_{g,1} \) and \( \mathcal{H}_{g,1} \) by

\[
C_{g,1}[1] := C_{g,1}, \quad C_{g,1}[k] := \text{Ker} \left( \sigma_k : C_{g,1} \to \text{Aut} \check{N}_k \right) \quad \text{for } k \geq 2,
\]

\[
\mathcal{H}_{g,1}[1] := \mathcal{H}_{g,1}, \quad \mathcal{H}_{g,1}[k] := \text{Ker} \left( \sigma_k : \mathcal{H}_{g,1} \to \text{Aut} \check{N}_k \right) \quad \text{for } k \geq 2.
\]
3. The Magnus representation for homology cylinders

We first summarize our notation. For a matrix $A$ with entries in a ring $R$, and a ring homomorphism $\varphi : R \to R'$, we denote by $\varphi A$ the matrix obtained from $A$ by applying $\varphi$ to each entry. $A^T$ denotes the transpose of $A$. When $R = \mathbb{Z}G$ for a group $G$ or its right field of fractions (if exists), we denote by $A$ the matrix obtained from $A$ by applying the involution induced from $(x \mapsto x^{-1}, x \in G)$ to each entry.

For a module $M$, we write $M^n$ and $M_n$ for the modules of column and row vectors with $n$ entries respectively.

For a finite cell complex $X$ and its regular covering $X_\Gamma$ with respect to a homomorphism $\pi_1 X \to \Gamma$, $\Gamma$ acts on $X_\Gamma$ from the right through its deck transformation group. Therefore we regard the $\mathbb{Z}\Gamma$-cellular chain complex $C_\ast(X_\Gamma)$ of $X_\Gamma$ as a collection of free right $\mathbb{Z}\Gamma$-modules consisting of column vectors together with differentials given by left multiplications of matrices. For each $\mathbb{Z}\Gamma$-bimodule $A$, the twisted chain complex $C_\ast(X; A)$ is given by the tensor product of the right $\mathbb{Z}\Gamma$-module $C_\ast(X_\Gamma)$ and the left $\mathbb{Z}\Gamma$-module $A$, so that $C_\ast(X; A)$ and $H_\ast(X; A)$ are right $\mathbb{Z}\Gamma$-modules.

3.1. Definition of the Magnus representation for homology cylinders. In what follows, we fix an integer $k \geq 2$, which corresponds to the class of the nilpotent quotient. The following construction is based on Kirk-Livingston-Wang’s work of the Gassner representation for string links in [10].

Let $(M, i_+, i_-) \in C_{g,1}$ be a homology cylinder. By Stallings’ theorem, $N_k$ and $N_k(M)$ are isomorphic. Since $N_k$ is a finitely generated torsion-free nilpotent group for each $k \geq 2$, we can embed $\mathbb{Z}N_k$ into the right field of fractions $K_{N_k} := \mathbb{Z}N_k(\mathbb{Z}N_k - \{0\})^{-1}$. (See Section 5) Similarly, we have $\mathbb{Z}N_k(M) \to K_{N_k(M)} := \mathbb{Z}N_k(M)(\mathbb{Z}N_k(M) - \{0\})^{-1}$. We consider the fields $K_{N_k}$ and $K_{N_k(M)}$ to be local coefficient systems on $\Sigma_{g,1}$ and $M$ respectively.

By a standard argument using covering spaces (see for instance [10, Proposition 2.1], [18, Lemma 5.11]), we have the following.

**Lemma 3.1.** $i_\pm : H_\ast(\Sigma_{g,1}, p; i_\pm K_{N_k(M)}) \to H_\ast(M, p; K_{N_k(M)})$ are isomorphisms as right $K_{N_k(M)}$-vector spaces.

**Remark 3.2.** The same conclusion as in Lemma 3.1 can be drawn for the homology with coefficients in any $\mathbb{Z}\pi_1(M)$-algebra $A$ satisfying the following property: Every matrix with entries in $\mathbb{Z}\pi_1(M)$ sent to an invertible one by the augmentation map $\mathbb{Z}\pi_1(M) \to \mathbb{Z}$ is also invertible in $A$. Note that $K_{N_k(M)}$ satisfies this property.

Since $R_{2g} \subset \Sigma_{g,1}$ is a deformation retract, we have

$$H_1(\Sigma_{g,1}, p; i_\pm K_{N_k(M)}) \cong H_1(R_{2g}, p; i_\pm K_{N_k(M)}) = C_1(R_{2g}) \otimes_{\pi_1 R_{2g}} i_\pm^* K_{N_k(M)} \cong K_{N_k(M)}^{2g}$$

with a basis

$$\{\tilde{\gamma}_1 \otimes 1, \ldots, \tilde{\gamma}_{2g} \otimes 1\} \subset C_1(R_{2g}) \otimes_{\pi_1 R_{2g}} i_\pm^* K_{N_k(M)}$$

as a right $K_{N_k(M)}$-vector space, where $\tilde{\gamma}_i$ is a lift of $\gamma_i$ on the universal covering $\widetilde{R_{2g}}$.

**Definition 3.3.** (1) For each $M = (M, i_+, i_-) \in C_{g,1}$, we denote by $i_k^* \in GL(2g, K_{N_k(M)})$ the representation matrix of the right $K_{N_k(M)}$-isomorphism

$$K_{N_k(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; i_\ast K_{N_k(M)}) \cong H_1(M, p; K_{N_k(M)}) \cong H_1(\Sigma_{g,1}, p; i_\ast K_{N_k(M)}) \cong K_{N_k(M)}^{2g}$$
The left vertical maps give \( (2) = (\sigma_k(M)) \) while the right ones give \( M, i_+ \). Hence we have the following.

**Theorem 3.4.** For \( M_1, M_2 \in C_g,1 \), we have

\[
r_k(M_1 \cdot M_2) = r_k(M_1) \cdot r_k(M_2).
\]

**Proof.** We write \( M = M_1 \cdot M_2 \) for simplicity. Let \( i : M_1 \to M \) and \( j : M_2 \to M \) be the natural inclusions. Since \( M = (M, i_+ \cdot j_+ \cdot j_-) \) and \( i \circ i_- = j \circ j_+ \), the map

\[
H_1(\Sigma_g, 1, p) ; j^* \to H_1(M, p ; K_{N_k(M)}) \xrightarrow{(i_+)^{-1}} H_1(\Sigma_g, 1, p ; i^* K_{N_k(M)})
\]

is given as the composite of

\[
H_1(\Sigma_g, 1, p ; j^* K_{N_k(M)}) \xrightarrow{j_1} H_1(M_2, p ; j^* K_{N_k(M)}) \xrightarrow{j_1} H_1(\Sigma_g, 1, p ; j_+^* K_{N_k(M)})
\]

and

\[
H_1(\Sigma_g, 1, p ; i^* K_{N_k(M)}) \xrightarrow{i_1} H_1(M_1, p ; i^* K_{N_k(M)}) \xrightarrow{i_1} H_1(\Sigma_g, 1, p ; i_+^* K_{N_k(M)}).
\]

Hence

\[
(i_+)^{-1} r_k(M) = (i_+)^{-1} r_k(M_2) \cdot (i_+)^{-1} r_k(M_2) = (i_+)^{-1} r_k(M_1) \cdot (i_+)^{-1} j_1 r_k(M_2) = (i_+)^{-1} r_k(M_1) \cdot (i_+)^{-1} j_1 r_k(M_2) = r_k(M_1) \cdot \sigma_k(M_1) r_k(M_2).
\]

This completes the proof. \( \square \)

**Theorem 3.5.** \( r_k : C_g,1 \to GL(2g, K_{N_k}) \) factors through \( H_g,1 \).

**Proof.** Suppose \( M_1 = M_1, i_+, i_- \) and \( M_2 = M_2, j_+, j_- \) are homology cobordant by a homology cobordism \( W \). Let \( i : M_1 \to W, j : M_2 \to W \) be the natural inclusions. We may assume that \( M_1 \cup M_2 \) is a subcomplex of \( W \) and that \( W \) has only one 0-cell \( p \). Since \( \mathbb{Z} N_k \cong \mathbb{Z} N_k(W) \) by Stallings’ theorem, we have \( K_{N_k(W)} := \mathbb{Z} N_k(W) (\mathbb{Z} N_k(W) - \{0\})^{-1} \). We write \( I_+ := i \circ i_+ = j \circ j_+ \) and \( I_- := i \circ i_- = j \circ j_- \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
H_1(\Sigma_g, 1, p ; i^* K_{N_k(W)}) & \xrightarrow{i_-} & H_1(\Sigma_g, 1, p ; I_+^* K_{N_k(W)}) & \xrightarrow{j_-} & H_1(\Sigma_g, 1, p ; j_+^* K_{N_k(W)}) \\
I_+ & \xrightarrow{i} & H_1(W, p ; K_{N_k(W)}) & \xrightarrow{j} & H_1(M_2, p ; j^* K_{N_k(W)}) \\
(i_+)^{-1} & \xrightarrow{i} & (I_+)^{-1} & \xrightarrow{j} & (j_+)^{-1}
\end{array}
\]

The left vertical maps give \( i^* r_k(M_1) \) and the right ones give \( j^* r_k(M_2) \). Applying \( I_+^{-1} \), we obtain \( r_k(M_1) = r_k(M_2) \). \( \square \)
Consequently, we obtain the Magnus representation \( r_k : \mathcal{H}_{g,1} \to GL(2g, \mathcal{K}_{N_k}) \), which is a crossed homomorphism. If we restrict \( r_k \) to \( \mathcal{C}_{g,1}[k] \) (resp. \( \mathcal{H}_{g,1}[k] \)), it becomes a monoid (resp. group) homomorphism.

**Example 3.6.** For \( \varphi \in \mathcal{M}_{g,1} \hookrightarrow \text{Aut} \ F_{2g} \), we can obtain

\[
r_k(M_\varphi) = \frac{\partial r_k}{\partial \gamma_i} (\partial \gamma_i)_{i \in I}^j,
\]

where \( \rho_k : \mathbb{Z}F_{2g} \to \mathbb{Z}N_k \subset \mathcal{K}_{N_k} \) is the natural map and \( \partial / \partial \gamma_i \) are free differentials. From this, we see that \( r_k \) generalizes the original Magnus representation for \( \mathcal{M}_{g,1} \) in [16].

### 3.2. Computation of the Magnus matrix.

In [10], the Gassner matrix of a string link is computed from the Wirtinger presentation of the fundamental group of its exterior, which gives a finite presentation whose deficiency coincides with the number of strings. Recall that the deficiency of a finite presentation \( P = \{ x_1, \ldots, x_n \mid r_1, \ldots, r_m \} \) of a finitely presentable group \( G \) is \( n - m \), and the deficiency of \( G \) is the maximum of all over the deficiencies of finite presentations of \( G \). In our context, we do not have such a useful method in general.

**Definition 3.7.** For \( (M, i_+, i_-) \in \mathcal{C}_{g,1} \), a presentation of \( \pi_1 M \) is said to be *admissible* if it is of the form

\[
(i_-(\gamma_1), \ldots, i_-(\gamma_2), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_2) \mid r_1, \ldots, r_{2g+l}).
\]

Note that there does exist an admissible presentation for each homology cylinder \( (M, i_+, i_-) \). Indeed, take a Morse function with no critical points of indices 0 and 3. Then \( M \) can be seen as \( \Sigma_{g,1} \times I \) with some 1- and 2-handles. Since the Euler characteristics of \( \Sigma_{g,1} \times I \) and \( M \) are the same, the numbers of the attached 1- and 2- handles are the same. Therefore the presentation of \( \pi_1 M \) obtained from a presentation of \( \pi_1(\Sigma_{g,1} \times I) = F_{2g} \) with deficiency 2g by adding new generators and relations corresponding to the 1- and 2-handles has deficiency 2g again. Our claim follows from this. (See also Section 6.2.)

Given an admissible presentation of \( \pi_1 M \) as in Definition 3.7, we define \( 2g \times (2g + l) \), \( l \times (2g + l) \) and \( 2g \times (2g + l) \) matrices \( A, B, C \) by

\[
A = \begin{pmatrix}
\frac{\partial r_j}{\partial i_-(\gamma_i)} & 1 \leq i \leq 2g \\
\frac{\partial r_j}{\partial z_i} & 1 \leq j \leq 2g + l
\end{pmatrix},
B = \begin{pmatrix}
\frac{\partial r_j}{\partial i_+(\gamma_i)} & 1 \leq i \leq 2g \\
0 & 1 \leq j \leq 2g + l
\end{pmatrix},
C = \begin{pmatrix}
\frac{\partial r_j}{\partial i_+(\gamma_i)} & 1 \leq i \leq 2g \\
0 & 1 \leq j \leq 2g + l
\end{pmatrix}
\]

at \( \mathbb{Z}N_k(M) \), namely we apply the natural map

\[
\mathbb{Z}(i_-(\gamma_1), \ldots, i_-(\gamma_2), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_2)) \to \mathbb{Z}\pi_1(M) \to \mathbb{Z}N_k(M)
\]
to each entry of the matrices obtained by free differentials.

**Proposition 3.8.** (1) The square matrix \( \begin{pmatrix} A & B \end{pmatrix} \) is invertible as a matrix with entries in \( \mathcal{K}_{N_k}(M) \).

(2) As matrices with entries in \( \mathcal{K}_{N_k}(M) \), we have

\[
(r_k^t(M) \otimes Z) \begin{pmatrix} A \\ B \end{pmatrix} = -C
\]

for some \( 2g \times l \) matrix \( Z \).
Proof. (1) Let \( t : \mathbb{Z}N_k(M) \to \mathbb{Z} \) be the augmentation map. \( t(A_B) \) gives a presentation matrix of \( H_1(M)/\Phi_+ \), where \( \Phi_+ \) is the subgroup of \( H_1(M) \) generated by \( i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \). (See [4] for this fact through the concept of presentations of a pair of groups.) By definition, \( H_1(M)/\Phi_+ = 0 \), and we have an exact sequence

\[
\mathbb{Z}^{2g+l} \xrightarrow{t(A_B)} \mathbb{Z}^{2g+l} \to H_1(M)/\Phi_+ = 0.
\]

By the Hopfian property of \( \mathbb{Z}^{2g+l} \), we see that \( t(A_B) \) is invertible. (1) follows from this. (See Remark [3.2])

(2) Through a standard argument using Eilenberg–MacLane spaces, we can assume that a given admissible presentation is obtained from a cell decomposition of \( M \). Then \( (A_B) \) gives the boundary map \( C_2(M, p; K_{N_k(M)}) \xrightarrow{\partial_2} C_1(M, p; K_{N_k(M)}) \). Considering the correspondence of 1-cycles, we have

\[
\begin{pmatrix}
I_{2g} \\
0_{(1,2g)} \\
0_{2g}
\end{pmatrix} - \begin{pmatrix}
0_{2g} \\
0_{(1,2g)} \\
r_1(M)
\end{pmatrix} = \begin{pmatrix}
A \\
B \\
C
\end{pmatrix} X \in C_1(M, p; K_{N_k(M)})
\]

for some matrix \( X \), where we write \( 0_k \) and \( 0_{(k,l)} \) for the zero matrices of sizes \( k \times k \) and \( k \times l \) respectively. (2) follows from this. \( \square \)

Note that from (2), we have \( r_k(M) = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ 0_{(1,2g)} \end{pmatrix} \), namely the Magnus matrix \( r_k(M) \) can be computed from any admissible presentation of \( \pi_1(M) \).

Next, we derive a formula for changing a generating system of \( \pi_1 \Sigma_{g,1} \). For a homology cylinder \( (M, i_+, i_-) \), we take an admissible presentation of \( \pi_1 M \) as in Definition [3.7] and construct the matrices \( A, B, C \) as before. Let \( \gamma_1', \ldots, \gamma_{2g}' \) be another generating system of \( \pi_1 \Sigma_{g,1} \). We can take \( \varphi \in \text{Aut} \pi_1 \Sigma_{g,1} \) such that \( \gamma_i' = \varphi(\gamma_i) \) for \( i = 1, \ldots, 2g \).

Proposition 3.9. Let \( r_k(M) \) be the Magnus matrix corresponding to the new generating system. Then

\[
r_k'(M) = \left( \frac{\partial \varphi(\gamma_1)}{\partial \gamma_i} \right)^{-1} r_k(M) \left( \frac{\partial \varphi(\gamma_1)}{\partial \gamma_i} \right).
\]

Proof. We have the following admissible presentation of \( \pi_1 M \) with respect to \( \gamma_1', \ldots, \gamma_{2g}' \):

\[
\pi_1 M \cong \left\langle i_-(\gamma_1'), \ldots, i_-(\gamma_{2g}'), i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1'), \ldots, i_+(\gamma_{2g}), i_+(\gamma_1), \ldots, i_+(\gamma_{2g}), i_+(\gamma_1'), \ldots, i_+(\gamma_{2g}) \right| \begin{pmatrix}
i_-(\gamma_1')i_-(\varphi(\gamma_1))^{-1}, \ldots, i_-(\gamma_{2g})i_-(\varphi(\gamma_{2g}))^{-1}, \\
i_+(\gamma_1')i_+(\varphi(\gamma_1))^{-1}, \ldots, i_+(\gamma_{2g})i_+(\varphi(\gamma_{2g}))^{-1}, \\
i_+(\gamma_1)i_+(\varphi(\gamma_1)), \ldots, i_+(\gamma_{2g})i_+(\varphi(\gamma_{2g})), \\
r_1, \ldots, r_{2g+l} \end{pmatrix} .
\]
The matrices $A', B', C'$ corresponding to this presentation are given by

$$A' = \begin{pmatrix} I_{2g} & 0_{2g,2g+1} & 0_{2g} \
- \left( \frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)_{1 \leq i,j \leq 2g} & A & 0_{2g} \\
0_{(2g,2g)} & B & 0_{(2g,2g)} \\
0_{2g} & C & - \left( \frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)_{1 \leq i,j \leq 2g} \end{pmatrix}$$

$$B' = \begin{pmatrix} \cdots \end{pmatrix}$$

$$C' = \begin{pmatrix} 0_{2g} & 0_{2g,2g+1} & I_{2g} \end{pmatrix}$$

Computing $-C' \left( A' \right)^{-1} \begin{pmatrix} I_{2g} \\
0_{(2g,2g+1)} \\
0_{(4g+1,2g)} \end{pmatrix}$, we obtain the formula.

4. Example: Relationship to the Gassner Representation for String Links

In [14], Levine gave a method for constructing homology cylinders from pure string links. By this, we can obtain many homology cylinders not belonging to the subgroup $M_{g,1}$. Also, we can see a relationship between the Gassner representation for string links and our representation.

For a $g$-component pure string link $L \subset D^2 \times I$, we now construct a homology cylinder $M_L \in C_{g,1}$ as follows. Consider a closed tubular neighborhood of the loops $\gamma_{g+1}, \gamma_{g+2}, \ldots, \gamma_{2g}$ in Figure 1 to be the image of an embedding $\iota : D_g \hookrightarrow \Sigma_{g,1}$ of a $g$-holed disk $D_g$ as in Figure 2.

Let $C$ be the complement of an open tubular neighborhood of $L$ in $D^2 \times I$. For each choice of a framing of $L$, a homeomorphism $h : \partial C \xrightarrow{\approx} \partial(\iota(D_g) \times I)$ is fixed. Then the manifold $M_L$ obtained from $\Sigma_{g,1} \times I$ by removing $\iota(D_g) \times I$ and regluing $C$ by $h$ becomes a homology cylinder. This construction gives an injective monoid homomorphism $\mathcal{L}_g \rightarrow C_{g,1}$ from the monoid $\mathcal{L}_g$ of (framed) pure string links to $C_{g,1}$. Moreover it also induces an injective homomorphism $S_g \rightarrow \mathcal{H}_{g,1}$ from the concordance group $S_g$ of (framed) pure string links to $\mathcal{H}_{g,1}$. In particular, the (smooth) knot concordance group, which coincides with $S_1$, is embedded in $\mathcal{H}_{g,1}$. If we restrict these embeddings to the pure braid group, which is a subgroup of $\mathcal{L}_g$ and $S_g$, their images are contained in $M_{g,1}$.

By the Gassner representation, we mean the crossed homomorphism $r_{G,k} : \mathcal{L}_g \rightarrow GL(g, K_{N_k(D_g)})$ or $r_{G,k} : S_g \rightarrow GL(g, K_{N_k(D_g)})$ given by a construction similar to that in the previous section. (In [11] and [10], only $r_{G,2}$ is treated.) Comparing methods for calculating the Gassner and Magnus representations, we obtain the following.
**Theorem 4.1.** For any pure string link \( L \in \mathcal{L}_g \), \( r_k(M_L) = \left( \begin{array}{*{22}{c}} * & 0_g \\ * & r_{G,k}(L) \end{array} \right) \).

We mention two remarks about this theorem. First we identify \( F_g = \pi_1 D_g \) with the subgroup of \( F_{2g} = \pi_1 \Sigma_{g,1} \) generated by \( \gamma_{g+1}, \ldots, \gamma_{2g} \). Then the maps \( F_g = \langle \gamma_{g+1}, \ldots, \gamma_{2g} \rangle \to F_{2g} \to F_g \), where the second map sends \( \gamma_1, \ldots, \gamma_g \) to 1, show that \( N_k(F_g) \subset N_k \) and \( \mathcal{K}_{N_k}(F_g) \subset \mathcal{K}_{N_k} \). Second, the embeddings \( \mathcal{L}_g \hookrightarrow \mathcal{C}_{g,1} \) and \( S_g \hookrightarrow \mathcal{H}_{g,1} \) have ambiguity with respect to framings. However we can check that the lower right part of \( r_k(M_L) \) is independent of the framings.

**Proof of Theorem 4.1.** All we have to do is to give a suitable presentation of \( \pi_1 M_L \). We divide \( M_L \) into two parts \( M \) and \( C \) as follows.

We take \( g \) points \( q_1, \ldots, q_g \) and \( g \) paths \( l_j \) from the base point \( p \) to \( q_j \) as in Figure 3.

**Figure 3.**

Let \( M \) be the union of \( \sum_{g,1} \times I - D_g \times I \) and \( 2g \) paths \( i_+(l_j) \) and \( i_-(l_j) \) \( (j = 1, \ldots, g) \). Then

\[
\pi_1 M \cong \left\langle i_-(\tilde{\gamma}_1), \ldots, i_-(\tilde{\gamma}_g), i_+(\tilde{\gamma}_1), \ldots, i_+(\tilde{\gamma}_g), i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}), \delta_1, \ldots, \delta_g, i_+(\gamma) \right| i_+(\tilde{\gamma}_1) = i_-(\tilde{\gamma}_1)\delta_1, \ldots, i_+(\tilde{\gamma}_g) = i_-(\tilde{\gamma}_g)\delta_g \right\rangle
\]

where \( \tilde{\gamma}_j = [\gamma_1, \gamma_{g+1}] \cdots [\gamma_{j-1}, \gamma_{g+j-1}]\gamma_j \), \( \gamma \) is the loop corresponding to the outer boundary of \( \iota(D_g) \) and \( \delta_j \) is the composite of paths \( i_-(l_j), \overline{i_-(q_j)i_+(q_j)} \) and \( i_+(l_j') \). We denote by \( C \) the complement of an open tubular neighborhood of \( L \) in \( D_g \times I \) as before.

\[
\pi_1 C \cong \langle i_-(\gamma_{g+1}), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \mid r_1, \ldots, r_{g+l} \rangle
\]

is given by the Wirtinger presentation of \( D \times I - L \). We glue \( C \) to \( M \) by using some fixed framing. Then it is easily seen that \( \pi_1(M \cap C) \) is the free group generated by \( \{i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}), \delta_1, \ldots, \delta_g, i_+(\gamma)\} \).

Using the above decomposition, we obtain

\[
\pi_1 M_L \cong \left\langle i_-(\tilde{\gamma}_1), \ldots, i_-(\tilde{\gamma}_g), i_+(\tilde{\gamma}_1), \ldots, i_+(\tilde{\gamma}_g), z_1, \ldots, z_l, i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \right| i_+(\tilde{\gamma}_1) = i_-(\tilde{\gamma}_1)\tilde{\delta}_1, \ldots, i_+(\tilde{\gamma}_g) = i_-(\tilde{\gamma}_g)\tilde{\delta}_g, i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \right\rangle
\]

where \( \tilde{\delta}_i \) are words in \( i_-(\gamma_{g+1}), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \) which depend on the framing. Rewrite the above presentation by using \( i_+(\gamma_{g+j})'s \) and \( i_-(\gamma_j) \)’s instead of \( i_+(\tilde{\gamma}_j) \)’s and \( i_-(\tilde{\gamma}_j) \)’s. This process does not affect generators \( i_-(\gamma_{g+j}), z_j, i_+(\gamma_{g+j}) \) and relations \( r_j \).
From the resulting admissible presentation, we can compute the Magnus matrix of $M_L$. Then our claim follows by comparing it with the method for calculating the Gassner matrix of $L$ from the Wirtinger presentation of $\pi_1 C$, which is given in [10].

**Corollary 4.2.** $M_{g,1}$ is not a normal subgroup of $\mathcal{H}_{g,1}$ for $g \geq 3$.

**Proof.** In [10], they gave 3-component pure string links denoted by $L_5$ and $L_6$ having the condition that $L_5$ is a pure braid, while the conjugate $L_6 L_5 L_6^{-1}$ is not. To show that $L_6 L_5 L_6^{-1}$ is not a pure braid, they use the fact that $r_{G,2}(L_6 L_5 L_6^{-1})$ has an entry not belonging to $\mathbb{Z}N_2(D_3)$. Then our claim follows from Theorem 4.1 with respect to this example. 

**Example 4.3.** Let $L$ be a 2-component pure string link as depicted in Figure 4.

\[
\begin{align*}
\pi_1 M_L &\cong \langle i_-(\gamma_1), \ldots, i_-(\gamma_4), i_+(\gamma_1), \ldots, i_+(\gamma_4), z \mid i_+(\gamma_1) i_-(\gamma_3)^{-1} i_+(\gamma_4) i_-(\gamma_1)^{-1}, \\
&\quad [i_+(\gamma_1), i_+(\gamma_3)] i_+(\gamma_2) i_-(\gamma_2)^{-1} [i_-(\gamma_3), i_-(\gamma_1)], \\
&\quad i_+(\gamma_4) i_-(\gamma_3) i_+(\gamma_4)^{-1} z^{-1}, \\
&\quad i_-(\gamma_3) i_+(\gamma_3)^{-1} i_-(\gamma_3)^{-1} z, \\
&\quad i_-(\gamma_1) z^{-1} i_+(\gamma_4)^{-1} z \rangle,
\end{align*}
\]

where we use the blackboard framing. We now compute $r_2(M_L)$. We identify $N_2$ and $N_2(M_L)$ by using $i_+$. Using the presentation, we have $z = i_-(\gamma_3) = \gamma_3, i_-(\gamma_4) = \gamma_4, i_-(\gamma_2) = \gamma_2 \gamma_3$ and $i_-(\gamma_1) = \gamma_1 \gamma_3^{-1} \gamma_4$ in $N_2$. Then

\[
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix} =
\begin{pmatrix}
1 & -1 & \gamma_3^{-1} & 0 & 0 & 0 \\
0 & \gamma_1^{-1} \gamma_3 & 1 & \gamma_3^{-1} & \gamma_4^{-1} & 1 - \gamma_3 \\
-\gamma_1^{-1} \gamma_3 & 0 & \gamma_2^{-1} & 0 & 0 & 0 \\
0 & \gamma_2^{-1} & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & \gamma_1^{-1} & 0 & 1 & 0 & 0 \\
\gamma_1^{-1} \gamma_3 & 0 & \gamma_2^{-1} & 0 & 0 & 0 \\
\end{pmatrix}.
\]
Hence
\[
r_2(M_L) = \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\gamma_1^{-1} & -\gamma_2^{-1} & \gamma_3^{-1} + \gamma_4^{-1} - 1 & \gamma_3^{-1} + \gamma_4^{-1} - 1 \\
\gamma_1^{-1} & (1-\gamma_2^{-1})(\gamma_3^{-1} - 1) & \gamma_4^{-1} & -\gamma_2^{-1} - \gamma_3^{-1} - 1
\end{array}\right).
\]

5. Higher-order Alexander invariants and torsion-degree functions

Here we summarize the theory of higher-order Alexander invariants along the lines of Harvey’s papers [8, 9]. For our use, we generalize them to functions of matrices called torsion-degree functions.

A group \( \Gamma \) is poly-torsion-free-abelian (PTFA, for short) if \( \Gamma \) has a normal series of finite length whose successive quotients are all torsion-free abelian. In particular, free nilpotent quotients \( N_k \) are PTFA for all \( k \geq 2 \). Note that every subgroup of a PTFA group is also PTFA. For each PTFA group \( \Gamma \), the group ring \( \mathbb{Z}\Gamma \) is known to be an Ore domain, so that it can be embedded in the right field of fractions \( \mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1} \), which is a skew field. We refer to [8], [17] for localizations of non-commutative rings.

We will also use the following localizations of \( \mathbb{Z}\Gamma \) placed between \( \mathbb{Z}\Gamma \) and \( \mathcal{K}_\Gamma \). Let \( \psi : \Gamma \rightarrow \mathbb{Z} \) be an epimorphism. Then we have an exact sequence
\[
1 \longrightarrow (\Gamma^\psi := \text{Ker } \psi) \longrightarrow \Gamma \overset{\psi}{\longrightarrow} \mathbb{Z} \longrightarrow 1.
\]
We take a splitting \( \xi : \mathbb{Z} \rightarrow \Gamma \) and put \( \tilde{t} := \xi(1) \in \Gamma \). Since \( \Gamma^\psi \) is again a PTFA group, \( \mathbb{Z}\Gamma^\psi \) can be embedded in its right field of fractions \( \mathcal{K}_{\Gamma^\psi} = \mathbb{Z}\Gamma^\psi(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \). Moreover, we can also construct \( \Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \). Then the splitting \( \xi \) gives an isomorphism between \( \mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \) and the skew Laurent polynomial ring \( \mathcal{K}_{\Gamma^\psi}[t^\pm] \), in which \( at = t(\tilde{t}^{-1}a\tilde{t}) \) holds for each \( a \in \mathcal{K}_{\Gamma^\psi} \). \( \mathcal{K}_{\Gamma^\psi}[t^\pm] \) is known to be a non-commutative right and left principal ideal domain. By definition, we have inclusions
\[
\mathbb{Z}\Gamma \hookrightarrow \mathcal{K}_{\Gamma^\psi}[t^\pm] \hookrightarrow \mathcal{K}_{\Gamma}.
\]
\( \mathcal{K}_{\Gamma^\psi}[t^\pm] \) and \( \mathcal{K}_{\Gamma} \) are known to be flat \( \mathbb{Z}\Gamma \)-modules. On \( \mathcal{K}_{\Gamma^\psi}[t^\pm] \), we have a map \( \text{deg}^\psi : \mathcal{K}_{\Gamma^\psi}[t^\pm] \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\} \) assigning to each polynomial its degree. We put \( \text{deg}^\psi(0) := \infty \). By this, \( \mathcal{K}_{\Gamma^\psi}[t^\pm] \) becomes a Euclidean domain. The composite \( \mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \overset{\psi}{\rightarrow} \mathcal{K}_{\Gamma^\psi}[t^\pm] \overset{\text{deg}^\psi}{\rightarrow} \mathbb{Z}_{\geq 0} \cup \{\infty\} \) does not depend on the choice of \( \xi \).

Harvey’s higher-order Alexander invariants [9] are defined as follows. Let \( G \) be a finitely presentable group and let \( \varphi : G \rightarrow \mathbb{Z} \) be an epimorphism. For a PTFA group \( \Gamma \) and an epimorphism \( \varphi : G \rightarrow \Gamma, (\varphi_G, \varphi) \) is called an admissible pair for \( G \) if there exists an epimorphism \( \psi : \Gamma \rightarrow \mathbb{Z} \) satisfying \( \varphi = \psi \circ \varphi_G \). For each admissible pair \( (\varphi_G, \varphi) \) for \( G \), we regard \( \mathcal{K}_{\Gamma^\psi}[t^\pm] = \mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \) as a \( \mathbb{Z}G \)-module, and we define the higher-order Alexander
invariant for $(\varphi_\Gamma, \varphi)$ by
\[
\bar{\delta}^\psi_\Gamma(G) = \dim_{K_{\Gamma^0}}(H_1(G; K_{\Gamma^0}[t^\pm])) \in \mathbb{Z}_{\geq 0} \cup \{\infty\},
\]
\[
\delta^\psi_\Gamma(G) = \dim_{K_{\Gamma^0}}(T_{K_{\Gamma^0}[t^\pm]}H_1(G; K_{\Gamma^0}[t^\pm])) \in \mathbb{Z}_{\geq 0},
\]
where $T_{K_{\Gamma^0}[t^\pm]}M$ denotes the $K_{\Gamma^0}[t^\pm]$-torsion part of a $K_{\Gamma^0}[t^\pm]$-module $M$. $\bar{\delta}^\psi_\Gamma(G)$ and $\delta^\psi_\Gamma(G)$ are called the $\Gamma$-degree and the refined $\Gamma$-degree respectively. (Our definition is slightly different from that of [2].) Note that the right $K_{\Gamma^0}[t^\pm]$-module $H_1(G; K_{\Gamma^0}[t^\pm])$ can be decomposed into
\[
H_1(G; K_{\Gamma^0}[t^\pm]) = (K_{\Gamma^0}[t^\pm])^r \oplus \left( \sum_{i=1}^l \frac{K_{\Gamma^0}[t^\pm]}{p_i(t)K_{\Gamma^0}[t^\pm]} \right)
\]
for some $r \in \mathbb{Z}_{\geq 0}$ and $p_i(t) \in K_{\Gamma^0}[t^\pm]$, then
\[
\bar{\delta}^\psi_\Gamma(G) = \left\{ \begin{array}{ll}
\sum_{i=1}^l \deg^\psi(p_i(t)) & (r = 0), \\
\infty & (r > 0),
\end{array} \right.
\]
\[
\delta^\psi_\Gamma(G) = \sum_{i=1}^l \deg^\psi(p_i(t)).
\]

For a connected space $X$ and an admissible pair $(\varphi_\Gamma, \varphi)$ for $\pi_1 X$, we define $\bar{\delta}^\psi_\Gamma(X) := \bar{\delta}^\psi_\Gamma(\pi_1 X)$ and $\delta^\psi_\Gamma(X) := \delta^\psi_\Gamma(\pi_1 X)$.

For a finitely presentable group $G$ and an admissible pair $(\varphi_\Gamma, \varphi)$ for $G$, the (refined) $\Gamma$-degree can be computed from any presentation matrix of the right $K_{\Gamma^0}[t^\pm]$-module $H_1(G; K_{\Gamma^0}[t^\pm])$. Therefore we can consider it to be a function on the set $M(K_{\Gamma^0}[t^\pm])$ of all matrices with entries in $K_{\Gamma^0}[t^\pm]$. Here we extend this function to $M(K_{\Gamma})$ as follows.

First, we extend $\deg^\psi : K_{\Gamma^0}[t^\pm] \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to $\deg^\psi : K_{\Gamma} \to \mathbb{Z} \cup \{\infty\}$ by setting $\deg^\psi(fg^{-1}) = \deg^\psi(f) - \deg^\psi(g)$ for $f \in \mathbb{Z}\Gamma, g \in \mathbb{Z}\Gamma - \{0\}$ (see for instance [3, Proposition 9.1.1]). It induces a group homomorphism $\deg^\psi : (K_{\Gamma}^\times)_{ab} \to \mathbb{Z}$, where $(K_{\Gamma}^\times)_{ab}$ is the abelianization of the multiplicative group $K_{\Gamma}^\times = K_{\Gamma} - \{0\}$.

Since $K_{\Gamma}$ is a skew field, we have the Dieudonné determinant
\[
\det : GL(K_{\Gamma}) \to (K_{\Gamma}^\times)_{ab},
\]
which is a homomorphism characterized by the following three properties:

(a) $\det I = 1$.

(b) If $A'$ is obtained by multiplying a row of a matrix $A \in GL(K_{\Gamma})$ by $a \in K_{\Gamma}^\times$ from the left, then $\det A' = a \cdot \det A$.

(c) If $A'$ is obtained by adding to a row of a matrix $A$ a left $K_{\Gamma}$-linear combination of other rows, then $\det A' = \det A$.

It induces an isomorphism between $K_1(K_{\Gamma}) \xrightarrow{\sim} (K_{\Gamma}^\times)_{ab}$.

The following lemma will be used in our generalization of Harvey’s invariants. We denote by $M(m, n, K_{\Gamma})$ the set of all $m \times n$ matrices with entries in $K_{\Gamma}$.

**Lemma 5.1.** For $A \in M(m, n, K_{\Gamma})$ with $\rank_{K_{\Gamma}} A = k$, let $U \in M(m - k, m, K_{\Gamma})$, $V \in M(n, n - k, K_{\Gamma})$ be matrices satisfying
\[
\begin{align*}
UA = 0, & \quad \rank_{K_{\Gamma}} U = m - k, \\
AV = 0, & \quad \rank_{K_{\Gamma}} V = n - k.
\end{align*}
\]
For each $I \subset \{1, 2, \ldots, m\}$, $J \subset \{1, 2, \ldots, n\}$ with $\#I = m - k$, $\#J = n - k$, let $U_I$ denote the square matrix defined by taking $i$-th columns from $U$ for all $i \in I$, and $V_J$ denote the one defined by taking $j$-th rows from $V$ for all $j \in J$. We also denote by $A_{I^C, J^C}$ the one defined by taking $i$-th rows from $A$ for all $i \in I^C := \{1, 2, \ldots, m\} - I$ and then taking $j$-th columns for all $j \in J^C := \{1, 2, \ldots, n\} - J$.

1. If $U_I$ or $V_J$ is not invertible, then $A_{I^C, J^C}$ is not invertible.

2. Otherwise,

$$\Delta(A; U, V) := \sgn(II^c) \sgn(JJ^c) \frac{\det A_{I^C, J^C}}{\det U_I \det V_J} \in (K^\times_I)_{ab}$$

is independent of the choice of $I$ and $J$ such that $U_I, V_J$ are invertible, where $\sgn(II^c) \in \{\pm 1\}$ (resp. $\sgn(JJ^c)$) is the signature of the juxtaposition of $I$ and $I^c$ (resp. $J$ and $J^c$), and we put $\det \emptyset := 1$.

3. For $P_1 \in GL(m, K_I)$, $P_2 \in GL(n, K_I)$, $Q_1 \in GL(m - k, K_I)$ and $Q_2 \in GL(n - k, K_I)$,

$$\Delta(P_1^{-1}AP_2^{-1}; Q_1UP_1, P_2VQ_2) = \frac{\Delta(A; U, V)}{\det P_1 \det P_2 \det Q_1 \det Q_2}.$$

Proof. (1) and (2) are deduced from easy observation using non-commutative linear algebra. To prove (3), it suffices to show in the cases where $P_1, P_2, Q_1, Q_2$ are matrices of elementary transformations, and it can be easily checked. \qed

Remark 5.2. In the above situation, the sequence

$$0 \longrightarrow K_I^{n-k} \stackrel{V^*}{\longrightarrow} K_I^n \stackrel{A^*}{\longrightarrow} K_I^{m} \stackrel{U^*}{\longrightarrow} K_I^{m-k} \longrightarrow 0$$

is exact. By taking the standard basis for each vector space, we regard the sequence as a based acyclic chain complex. Then we can take its torsion (see [15], [21] for generalities of torsions). This torsion coincides with $\Delta(A; U, V)$ up to sign.

As seen in Lemma 5.1(3), $\Delta(A; U, V)$ does depend on $U$ and $V$. The following definition and lemma give our rule to take $U$ and $V$.

Definition 5.3. Let $A \in M(m, n, K_I)$ with $\text{rank}_{K_I} A = k$. $(U, V)$ is said to be $\psi$-primitive for $A$ if

1. $U, V$ have entries in $K_{I^C}[t^\pm]$.

2. The row vectors $u_1, \ldots, u_{m-k} \in (K_{I^C}[t^\pm])_m$ of $U$ generate $\text{Ker}(\cdot A) \cap (K_{I^C}[t^\pm])_m$ in $(K_I)_m$ as a left $K_{I^C}[t^\pm]$-module.

3. The column vectors $v_1, \ldots, v_{n-k} \in (K_{I^C}[t^\pm])^n$ of $V$ generate $\text{Ker}(A \cdot) \cap (K_{I^C}[t^\pm])^n$ in $(K_{I^C})^n$ as a right $K_{I^C}[t^\pm]$-module.

Lemma 5.4. (1) There exists a pair $(U, V)$ which is $\psi$-primitive for $A$.

(2) Suppose $U \in M(m - k, m, K_{I^C}[t^\pm])$ and $V \in M(n, n - k, K_{I^C}[t^\pm])$ satisfy $UA = 0$ and $AV = 0$. $(U, V)$ is $\psi$-primitive for $A$ if and only if there exist $\tilde{P}_1 \in GL(m, K_{I^C}[t^\pm])$ and $\tilde{P}_2 \in GL(n, K_{I^C}[t^\pm])$ satisfying $U \tilde{P}_1 = (0(m-k,k) \ I_{m-k}), \ \tilde{P}_2 V = (0(n-k,k) \ I_{n-k})^T$.

(3) If $(U, V)$ and $(U', V')$ are $\psi$-primitive for $A$, then there exist $P_1 \in GL(m, K_{I^C}[t^\pm]), \ P_2 \in GL(n, K_{I^C}[t^\pm]), \ Q_1 \in GL(m - k, K_{I^C}[t^\pm])$ and $Q_2 \in GL(n - k, K_{I^C}[t^\pm])$ such that

$$UP_1 = U', \ \ P_2V = V', \ \ Q_1U = U', \ \ VQ_2 = V'.$$
Proof. We prove only for $V$.

(1) For right $\mathcal{K}_{\Gamma^0}[t^\pm]$-homomorphisms $(\mathcal{K}_{\Gamma^0}[t^\pm])^n \xrightarrow{i} \mathcal{K}_\Gamma^n \xrightarrow{A} \mathcal{K}_{\Gamma^m}$, $\text{Ker}((A \cdot) \circ i) = \text{Ker}(A \cdot) \cap (\mathcal{K}_{\Gamma^0}[t^\pm])^n$ is a right free $\mathcal{K}_{\Gamma^0}[t^\pm]$-module of rank $n - k$. We take a basis $v_1, \ldots, v_{n-k}$ and put $V = (v_1, \ldots, v_{n-k})$. Then $V$ satisfies the desired property.

(2) Suppose $V$ generates $\text{Ker}(A \cdot) \cap (\mathcal{K}_{\Gamma^0}[t^\pm])^n$. The quotient module $(\mathcal{K}_{\Gamma^0}[t^\pm])^n / \text{Ker}((A \cdot) \circ i)$ is $\mathcal{K}_{\Gamma^0}[t^\pm]$-torsion free, and hence $\mathcal{K}_{\Gamma^0}[t^\pm]$-free. Taking a splitting, we have a direct sum decomposition $(\mathcal{K}_{\Gamma^0}[t^\pm])^n \cong (\mathcal{K}_{\Gamma^0}[t^\pm])^n / \text{Ker}((A \cdot) \circ i) \oplus \text{Ker}((A \cdot) \circ i)$. We can extend $V$ to obtain a basis $(\tilde{v}_1, \ldots, \tilde{v}_k, V)$ for $(\mathcal{K}_{\Gamma^0}[t^\pm])^n$. Then $\tilde{P}_2 := (\tilde{v}_1, \ldots, \tilde{v}_k, V)^{-1}$ satisfies $\tilde{P}_2 V = (0_{(n-k,k)} \ I_{n-k})^T$. The inverse is clear.

(3) The existence of $\tilde{P}_2$ follows immediately from (2). That of $Q_2$ is also clear since $V$ and $V'$ are bases of the same right $\mathcal{K}_{\Gamma^0}[t^\pm]$-module. \hfill \qedsymbol

Definition 5.5. Let $\Gamma$ be a PTFA group and let $\psi : \Gamma \to \mathbb{Z}$ be an epimorphism.

(1) The torsion-degree function $d^\psi_\Gamma : M(\mathcal{K}_\Gamma) \to \mathbb{Z}$ is defined by

$$d^\psi_\Gamma(A) := \deg^\psi(\Delta(A; U, V))$$

for a pair $(U, V)$ which is $\psi$-primitive for $A$.

(2) The truncated torsion-degree function $\overline{d}^\psi_\Gamma : M(\mathcal{K}_\Gamma) \to \mathbb{Z} \cup \{\infty\}$ is defined by

$$\overline{d}^\psi_\Gamma(A) := \begin{cases} d^\psi_\Gamma(A) & \text{if rank } A \geq m - 1, \\ \infty & \text{otherwise} \end{cases}$$

for $A \in M(m, n, \mathcal{K}_\Gamma)$.

Since $\mathcal{K}_{\Gamma^0}[t^\pm]$ is an Euclidean domain, every $P \in GL(\mathcal{K}_{\Gamma^0}[t^\pm])$ can be decomposed as products of elementary matrices and diagonal matrices in $GL(\mathcal{K}_{\Gamma^0}[t^\pm])$, which shows that $\deg^\psi(\det P) = 0$. Lemmas 5.1 and 5.4 together with this fact show that $d^\psi_\Gamma$ and $\overline{d}^\psi_\Gamma$ are well-defined.

Example 5.6. (1) For $A \in GL(\mathcal{K}_\Gamma)$, we have $d^\psi_\Gamma(A) = \overline{d}^\psi_\Gamma(A) = \deg^\psi(\det A)$.

(2) Let $M$ be a finitely generated right $\mathcal{K}_{\Gamma^0}[t^\pm]$-module, and let $A$ be a presentation matrix of $M$. Then we have $d^\psi_\Gamma(A) = \dim_{\mathcal{K}_{\Gamma^0}}(T_{\mathcal{K}_{\Gamma^0}[t^\pm]}M)$. As for $\overline{d}^\psi_\Gamma(A)$, we can see that $\overline{d}^\psi_\Gamma(A) \in \mathbb{Z}$ if and only if the rank of the $\mathcal{K}_{\Gamma^0}[t^\pm]$-free part of $M$ is less than 2.

(3) Let $G$ be a finitely presentable group. We take a presentation $\langle x_1, \ldots, x_l \mid r_1, \ldots, r_m \rangle$ of $G$.

For an admissible pair $(\varphi, \varphi)$, the matrix $A := \begin{pmatrix} \varphi_j \partial r_i \\ \partial x_i \end{pmatrix}_{1 \leq i \leq l}^{1 \leq j \leq m}$ at $\mathcal{K}_{\Gamma^0}[t^\pm]$ gives a presentation matrix of $H_1(G, \{1\}; \mathcal{K}_{\Gamma^0}[t^\pm])$. Then Harvey’s invariants are given by

$$\delta^\psi_\Gamma(G) = \dim_{\mathcal{K}_{\Gamma^0}}(T_{\mathcal{K}_{\Gamma^0}[t^\pm]}H_1(G; \mathcal{K}_{\Gamma^0}[t^\pm])) = d^\psi_\Gamma(A),$$

$$\overline{d}^\psi_\Gamma(G) = \dim_{\mathcal{K}_{\Gamma^0}}(H_1(G; \mathcal{K}_{\Gamma^0}[t^\pm])) = \overline{d}^\psi_\Gamma(A),$$

where the second equality of each case follows from the direct sum decomposition

$$H_1(G, \{1\}; \mathcal{K}_{\Gamma^0}[t^\pm]) \cong H_1(G; \mathcal{K}_{\Gamma^0}[t^\pm]) \oplus \mathcal{K}_{\Gamma^0}[t^\pm]$$

shown by Harvey in [8].

Remark 5.7. Friedl [5] gave an interpretation of Harvey’s invariants by Reidemeister torsions. The definition of our truncated torsion-degree functions has some overlaps with his description.
6. APPLICATIONS OF TORSION-DEGREE FUNCTIONS TO HOMOLOGY CYLINDERS

In this section, we study some invariants of homology cylinders arising from the Magnus representation and Reidemeister torsions by using torsion-degree functions associated to nilpotent quotients $N_k$ of $\pi_1 \Sigma_{g,1}$. $N_k$ is known to be a finitely generated torsion-free nilpotent group. In particular, it is PTFA.

Note that we can take a primitive element of $H^1(\Sigma_{g,1})$ instead of an epimorphism $N_k \to \mathbb{Z}$ to define a torsion-degree function since $\text{Hom}(N_k, \mathbb{Z}) = H^1(N_k) = H^1(N_2) = H^1(\Sigma_{g,1})$. We denote by $PH_1(\Sigma_{g,1})$ the set of primitive elements of $H^1(\Sigma_{g,1})$.

6.1. The Magnus representation and torsion-degree functions. First, we apply torsion-degree functions to the Magnus matrix. However, it turns out that the result is trivial.

**Theorem 6.1.** Let $M$ be a homology cylinder. For any $\psi \in PH^1(\Sigma_{g,1})$, the torsion-degree $d^\psi_{N_k}(r_k(M))$ is always zero.

**Proof.** By definition, $d^\psi_{N_k}$ is additive for products of invertible matrices, and invariant under taking the transpose and operating the involution. Moreover, it vanishes for matrices in $GL(\mathbb{Z}N_k)$. In [19], we show that there exists a matrix $\tilde{J} \in GL(2g, \mathbb{Z}N_k)$ satisfying the equality

$$\tilde{r}_k(M)^t \tilde{J} r_k(M) = \sigma_k(M) \tilde{J}.$$

By applying $d^\psi_{N_k}$ to it, we obtain $2d^\psi_{N_k}(r_k(M)) = 0$. This completes the proof. $\square$

**Example 6.2.** Consider the homology cylinder $M_L$ in Example 4.3. $d^\psi_{N_k}(r_2(M_L))$ is given by the degree of $\det r_2(M_L) = \frac{\gamma_1^2 + \gamma_2^2 - 1}{\gamma_1 \gamma_2 (\gamma_1^{-1} + \gamma_2^{-1})}$ with respect to $\psi$. It is zero.

To extract some numerical information from $r_k(M)$, we next apply torsion-degree functions to $I_{2g} - r_k(M)$. Here we assume $M \in C_{g,1}[k]$ and consider only $d^\psi_{N_k}$. The function $d^\psi_{N_k}(I_{2g} - r_k(M)) : C_{g,1}[k] \to \mathbb{Z} \cup \{\infty\}$ factors through $\mathcal{H}_{g,1}$ since $r_k$ does. Note that for every $(M, i_+, i_-) \in C_{g,1}[k]$, two inclusions $i_+$ and $i_-$ induce the same isomorphism $i_+ = i_- : N_k \cong N_k(M)$, so that we can naturally identify them. Under this identification, we have the following.

**Lemma 6.3.** Let $M$ be a homology cylinder belonging to $C_{g,1}[k]$.

1. $(1 - \gamma_1^{-1}, \ldots, 1 - \gamma_2^{-1})(I_{2g} - r_k(M)) = 0$.
2. $(I_{2g} - r_k(M)) \left( \frac{\partial \psi}{\partial \gamma_1}, \ldots, \frac{\partial \psi}{\partial \gamma_{2g}} \right)^t = 0$.

**Proof.** We take an admissible presentation of $\pi_1 M$ as in Definition 3.7. We also take the matrices $A, B, C \in \mathbb{Z}N_k$ corresponding to it. For simplicity, we put $\overline{1 - \frac{2}{\gamma}} := (1 - \gamma_1^{-1}, \ldots, 1 - \gamma_2^{-1})$, $\overline{1 - \frac{2}{\gamma^2}} := (1 - z_1^{-1}, \ldots, 1 - z_2^{-1})$ and $\frac{\partial \psi}{\partial \gamma} := \left( \frac{\partial \psi}{\partial \gamma_1}, \ldots, \frac{\partial \psi}{\partial \gamma_{2g}} \right)$.

1. Using Fundamental formula of free calculus (see Proposition 3.4), we have

$$
\begin{pmatrix}
(1 - \frac{\gamma}{2}) & (1 - \frac{2}{\gamma}) & (1 - \frac{2}{\gamma^2})
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
= 0.
$$

Then, by Proposition 3.8

$$
(1 - \frac{\gamma}{2}, 1 - \frac{2}{\gamma}, 1 - \frac{2}{\gamma^2}) = -(1 - \frac{\gamma}{2})C \begin{pmatrix}A \\ B \end{pmatrix}^{-1} = (1 - \frac{\gamma}{2})(r_k(M) \quad Z).
$$
Our claim follows by taking their first 2g columns.

(2) Let $\tau_\zeta \in M_{g,1} \subset C_{g,1}$ be the Dehn twist along $\zeta$. It belongs to the center of $C_{g,1}$ and acts on $N_k$ by conjugation $x \mapsto \zeta^{-1}x\zeta$. Then

$$r_k(M) = r_k(\tau_\zeta^{-1}M\tau_\zeta)$$

$$= r_k(\tau_\zeta^{-1}) \cdot \sigma_k(\tau_\zeta^{-1}) r_k(M)$$

$$= \sigma_k(\tau_\zeta^{-1}) r_k(\tau_\zeta^{-1})^{-1} \cdot \sigma_k(\tau_\zeta^{-1}) (r_k(M) \cdot \sigma_k(M) r_k(\tau_\zeta))$$

$$= \sigma_k(\tau_\zeta^{-1}) (r_k(\tau_\zeta^{-1})^{-1} \cdot r_k(M) \cdot r_k(\tau_\zeta))$$

$$= (\zeta I_{2g}) \cdot r_k(\tau_\zeta)^{-1} \cdot r_k(M) \cdot r_k(\tau_\zeta) \cdot (\zeta^{-1} I_{2g})$$

where the fourth equality follows from the fact that $M$ acts on $N_k$ trivially. On the other hand, it is easily checked that

$$r_k(\tau_\zeta) = \left( I_{2g} - \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) \right) (\zeta I_{2g})$$

by using free differentials. Then

$$r_k(M) = \left( I_{2g} - \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) \right)^{-1} r_k(M) \left( I_{2g} - \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) \right)$$

$$\implies \left( I_{2g} - \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) \right) r_k(M) = r_k(M) \left( I_{2g} - \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) \right)$$

$$\implies \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) r_k(M) = r_k(M) \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}).$$

From (1), we see $\frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) r_k(M) = \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1})$. Comparing first columns, we have $\frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1}) r_k(M) = \frac{\partial}{\partial \gamma}^T (1 - \frac{\gamma}{\gamma_1})$. (2) follows from this. \qed

**Proposition 6.4.** If $M \in C_{g,1}[k]$ satisfies $\text{rank}_{K_{N_k}}(I_{2g} - r_k(M)) = 2g - 1$, then

$$\overline{d}_{N_k}^\psi(I_{2g} - r_k(M)) = \text{deg}^\psi(\Delta(I_{2g} - r_k(M); 1 - \frac{\gamma}{\gamma_1}, \frac{\partial}{\partial \gamma}^T)).$$

Otherwise $\overline{d}_{N_k}^\psi(I_{2g} - r_k(M)) = \infty$.

**Proof.** By Lemma 6.3, the rank of $I_{2g} - r_k(M)$ is at most $2g - 1$. The case where $\text{rank}_{K_{N_k}}(I_{2g} - r_k(M)) < 2g - 1$ is clear by definition. Suppose that $\text{rank}_{K_{N_k}}(I_{2g} - r_k(M)) = 2g - 1$. The task is to show that $(1 - \frac{\gamma}{\gamma_1})$ and $\frac{\partial}{\partial \gamma}^T$ satisfy the conditions (2) and (3) of Definition 5.3 respectively. Suppose $(1 - \frac{\gamma}{\gamma_1})$ does not satisfy (2). Then $(1 - \frac{\gamma}{\gamma_1})$ can be written as $(1 - \frac{\gamma}{\gamma_1}) = f \nu$, where $\nu \in (K_{N_k}[t^\pm])_{2g}$ and $f \in K_{N_k}[t^\pm]$ with $\text{deg}^\psi(f) \geq 1$. Hence each entry of $(1 - \frac{\gamma}{\gamma_1})$ has degree greater than 0. When $A \in GL(\mathbb{Z}N_k)$, the same holds for each entry of $(1 - \frac{\gamma}{\gamma_1})A$ which is non-zero. However, if we choose $f \in \text{Aut} F_{2g}$ such that $\psi(f(\gamma_1)) = 0$, which does exist,

$$(1 - \gamma_1^{-1}, \ldots, 1 - \gamma_2^{-1}) \left( \frac{\partial f(\gamma_i)}{\partial \gamma_i} \right)_{i,j} = (1 - f(\gamma_1)^{-1}, \ldots, 1 - f(\gamma_2)^{-1}),$$

a contradiction. Hence $(1 - \frac{\gamma}{\gamma_1})$ satisfies (2). By a similar argument, we can show that $\frac{\partial}{\partial \gamma}^T$ satisfies (3), where we use $f \in \text{Aut} F_{2g}$ preserving $\zeta$ and the chain rule for free differentials (see for instance [1, Proposition 3.3]). \qed
Note that $\overline{d}_N^\psi(I_{2g} - r_k(M))$ does not depend on the choice of the generating system of $\pi_1 \Sigma_{g,1}$. This follows from the formulas in Proposition 3.9 and Lemma 5.1 (3). In Section 6.3.3, we will see some examples showing that our invariants are non-trivial at least in the case of $k = 2$.

6.2. $N_k$-torsion and torsion-degree functions. For a homology cylinder $M = (M, i_+, i_-) \in C_{g,1}$, we put $\Sigma^+ := i_*(\Sigma_{g,1})$. Since the relative complex $C_\ast(M, \Sigma^+; K_{N_k}(M))$ obtained from any smooth triangulation of $(M, \Sigma^+)$ is acyclic by Lemma 3.1, we can consider its Reidemeister torsion $\tau(C_\ast(M, \Sigma^+; K_{N_k}(M)))$.

**Definition 6.5.** The $N_k$-torsion of a homology cylinder $M = (M, i_+, i_-) \in C_{g,1}$ is given by

$$\tau_{N_k}(M) := i_+^{-1} \tau(C_\ast(M, \Sigma^+; K_{N_k}(M))) \in K_1(K_{N_k})/(\pm N_k).$$

Recall that Reidemeister torsions are invariant under subdivision of the cell complex $(M, \Sigma^+)$ and simple homotopy equivalence.

Now we consider $\tau_{N_k}(M)$ more closely. First we give a cell decomposition of $\partial M \cong \Sigma_{g,1} \cup (-\Sigma_{g,1})$ by pasting two copies of that of $\Sigma_{g,1}$ in Figure 1. We denote by $R_{2g}$ the subcomplex $i_*(R_{2g})$. Take a triangulation of $\partial M$ by refining the cell decomposition, and extend it to one of $M$. Then

$$\tau(C_\ast(M, R_{2g}^+; K_{N_k}(M))) = \tau(C_\ast(\Sigma^+, R_{2g}^+; K_{N_k}(M))) \cdot \tau(C_\ast(M, \Sigma^+; K_{N_k}(M)))$$

by the multiplicativity of torsions and the fact that $\Sigma^+$ is simple homotopy equivalent to $R_{2g}^+$.

Starting from a 3-simplex of $M$ facing the boundary, we can deform $M$ onto a 2-dimensional subcomplex $M'$ by a simple homotopy equivalence keeping the 1-skeleton of $M$ invariant. Then $\tau(C_\ast(M, R_{2g}^+; K_{N_k}(M))) = \tau(C_\ast(M', R_{2g}^+; K_{N_k}(M)))$. Next, we take a maximal tree $T$ of the 1-skeleton of $M'$ and collapse it to a point. This process also preserves the simple homotopy type of $(M', R_{2g}^+)$, so that $\tau(C_\ast(M', R_{2g}^+; K_{N_k}(M))) = \tau(C_\ast(M'/T, R_{2g}^+/(T \cap R_{2g}^+); K_{N_k}(M)))$.

Consequently, $\tau_{N_k}(M) = i_+^{-1} \tau(C_\ast(M'/T, R_{2g}^+/(T \cap R_{2g}^+); K_{N_k}(M)))$. $M'/T$ is a 2-dimensional cell complex with only one 0-cell, and $R_{2g}^+/(T \cap R_{2g}^+)$ is a subcomplex consisting of one 0-cell and $2g$ 1-cells. The pair $(M'/T, R_{2g}^+/(T \cap R_{2g}^+))$ gives an admissible presentation

$$(i_-((\gamma_1), \ldots, i_-((\gamma_{2g}), z_1, \ldots, z_l, i_+((\gamma_1), \ldots, i_+((\gamma_{2g}) \mid r_1, \ldots, r_{2g+1}))$$

of $\pi_1 M$. For this presentation, we take the matrices $A, B, C \in M(\mathbb{Z}N_k)$ as in Section 3.2. Then

$$\tau_{N_k}(M) = i_+^{-1} \tau(C_\ast(M'/T, R_{2g}^+/(T \cap R_{2g}^+); K_{N_k}(M))) = i_+^{-1} \left(\begin{array}{c} A \\ B \end{array}\right).$$

Note that the matrix $(A^B)$ is a presentation matrix of $H_1(M'/T, R_{2g}^+/(T \cap R_{2g}^+); \mathbb{Z}N_k(M)) \cong H_1(M, \Sigma^+; \mathbb{Z}N_k(M))$.

Since multiplying an element of $\pm N_k$ does not contribute to the degree, we have

$$d_{N_k}^\psi(\tau_{N_k}(M)) = d_{N_k}^\psi(i_+^{-1} \left(\begin{array}{c} A \\ B \end{array}\right)) = \dim_{K_{N_k}^\psi} H_1(M, \Sigma^+; (i_+^{-1})^*K_{N_k}^\psi[t^\pm]).$$

for each $\psi \in PH^1(\Sigma_{g,1})$. From this, we see that $d_{N_k}^\psi(\tau_{N_k}(M))$ can be computed from any admissible presentation of $\pi_1 M$. 

THE MAGNUS REPRESENTATION AND HIGHER-ORDER ALEXANDER INVARIANTS 17
Proposition 6.6. Let \( M_1 = (M_1, i_+, i_-), M_2 = (M_2, j_+, j_-) \in C_{g,1} \). Then

\[
d^\psi_{N_1}((\tau_{N_1}(M_1 \cdot M_2)) = d^\psi_{N_1}(\tau_{N_1}(M_1)) + d^\psi_{N_1}(\tau_{N_1}(M_2))
\]

holds for every \( \psi \in PH^1(\Sigma_{g,1}) \).

Proof. We take an admissible presentation of \( \pi_1 M_1 \) and the matrices \( A_{M_1}, B_{M_1}, C_{M_1} \) corresponding to it. We denote this presentation by

\[
\pi_1 M_1 \cong \langle i_-(\bar{i}), z, i_+(\bar{i}) | \bar{r} \rangle,
\]

for short. Similarly, we take an admissible presentation

\[
\pi_1 M_2 \cong \langle j_-(\bar{i}), w, j_+(\bar{i}) | \bar{s} \rangle
\]

of \( \pi_1 M_2 \) and the matrices \( A_{M_2}, B_{M_2}, C_{M_2} \). Then we obtain an admissible presentation

\[
\pi_1(M_1 \cdot M_2) \cong \langle j_-(\bar{i}), w, j_+(\bar{i}), i_-(\bar{i}), z, i_+(\bar{i}) | \bar{s} \rangle
\]

of \( \pi_1(M_1 \cdot M_2) \). The corresponding partial matrix \( \begin{pmatrix} A_{M_1, M_2} & B_{M_1, M_2} \end{pmatrix} \) at \( \mathbb{Z} N_1(M_1 \cdot M_2) \) is given by

\[
\begin{pmatrix}
 jA_{M_2} & 0 & 0 \\
 jB_{M_2} & 0 & 0 \\
 jC_{M_2} & I_{2g} & 0 \\
 0 & -I_{2g} & iA_{M_1} \\
 0 & 0 & iB_{M_1}
\end{pmatrix},
\]

where \( i : M_1 \rightarrow M_1 \cdot M_2 \) and \( j : M_2 \rightarrow M_1 \cdot M_2 \) are the natural inclusions. From this, we have

\[
d^\psi_{N_1}(\tau_{N_1}(M_1 \cdot M_2)) = d^\psi_{N_1}(\tau_{N_1}(M_1)) + d^\psi_{N_1}(\tau_{N_1}(M_2))
\]

\[
= d^\psi_{N_1}(\tau_{N_1}(M_1)) + d^\psi_{N_1}(\tau_{N_1}(M_2))
\]

\[
= d^\psi_{N_1}(\tau_{N_1}(M_1)) + d^\psi_{N_1}(\tau_{N_1}(M_2))
\]

This completes the proof.

Remark 6.7. Proposition 6.6 can be seen as a generalization of [12 Proposition 1.11].

6.3. Factorization formulas.

6.3.1. The \( N_k \)-degree for the closing of a homology cylinder. For each homology cylinder \( M = (M, i_+, i_-) \), we can construct a closed 3-manifold defined by

\[
C_M := M / (i_+(x) = i_-(x)), \quad x \in \Sigma_{g,1}.
\]

We call it the closing of \( M \). It is easily seen that if \( M \in C_{g,1}[k] \), we have the natural isomorphisms \( \cong N_k(M) \cong N_k(C_M) \). Here we identify these groups.
Theorem 6.8. Let $M = (M, i_+, i_-) \in C_{g,1}[k]$. For each $\psi \in PH^1(N_k)$, we have
\[ \partial_{N_k}^\psi(C_M) = d_{N_k}^\psi(\tau_{N_k}(M)) + d_{N_k}^\psi(I_{2g} - r_k(M)) \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}. \]

Proof. Take an admissible presentation of $\pi_1 M$ as in Definition 5.7 and construct the corresponding matrices $A, B, C \in M(ZN_k)$.

Adding $2g$ relations $i_+(\gamma_j) = i_-(\gamma_j)$ ($j = 1, \ldots, 2g$) and deleting the generators $i_+(\gamma_j)$ by using them, we obtain a presentation of $\pi_1 C_M$. From this presentation, we have a presentation matrix $J_{C_M}$ of $H_1(C_M; p; ZN_k)$ given by
\[ J_{C_M} = \begin{pmatrix} A + C & 0 \\ B & 0 \end{pmatrix}, \]
where the second equality follows from Proposition 5.8. Since $(A_B)$ is invertible in $K_{N_k}$,
\[ \text{rank}_{K_{N_k}} J_{C_M} = \text{rank}_{K_{N_k}} \left( \begin{pmatrix} I_{2g} - r_k(M) & -Z \\ 0_{(1,2g)} & I_l \end{pmatrix} \right) = \text{rank}_{K_{N_k}} (I_{2g} - r_k(M)) + l \leq 2g + l - 1. \]
Hence to show our claim, it suffices to prove the case where this value is just $2g + l - 1$ (see Definition 5.5(2)).

By Fundamental formula of free calculus, we have
\[ (1 - A_{ij}) J_{C_M} = (1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1}, 1 - z_1^{-1}, \ldots, 1 - z_l^{-1}) J_{C_M} = 0. \]

On the other hand, we have
\[ J_{C_M} \left( \begin{pmatrix} A & 0 \\ B \end{pmatrix} \right)^{-1} \begin{pmatrix} \partial C \\ 0_{(1,l)} \end{pmatrix}^T = \begin{pmatrix} I_{2g} - r_k(M) & -Z \\ 0_{(1,2g)} & I_l \end{pmatrix} \begin{pmatrix} \partial C \\ 0_{(1,l)} \end{pmatrix} = 0 \]
by Lemma 6.3(2). Then we can define $\Delta (J_{C_M}; \xi, \mu)$, where we put
\[ \xi := (1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1}, 1 - z_1^{-1}, \ldots, 1 - z_l^{-1}), \]
\[ \mu := \left( \begin{pmatrix} A & 0 \\ B \end{pmatrix} \right)^{-1} \begin{pmatrix} \partial C \\ 0_{(1,l)} \end{pmatrix}^T. \]

Lemma 6.9. $\mu$ belongs to $(ZN_k)^{2g+l}$.

Proof. Recall that $(A_B)$ is a presentation matrix of $H_1(M, \Sigma^+; ZN_k)$, so that we have an exact sequence
\[ 0 \longrightarrow (ZN_k)^{2g+l} \xrightarrow{(A_B)} (ZN_k)^{2g+l} \longrightarrow H_1(M, \Sigma^+; ZN_k) \longrightarrow 0, \]
where the injectivity of the second map follows from the fact that $H_1(M, \Sigma^+; K_{N_k}) = 0$. Hence to prove the lemma, it suffices to show that $\begin{pmatrix} \partial C \\ 0_{(1,l)} \end{pmatrix}^T$ in the third term $(ZN_k)^{2g+l} = C_1(M, \Sigma^+; ZN_k)$ is mapped to $0 \in H_1(M, \Sigma^+; ZN_k)$. In the exact sequence
\[ 0 \longrightarrow C_1(\Sigma^+, p; ZN_k) \longrightarrow C_1(M, p; ZN_k) \longrightarrow C_1(M, \Sigma^+; ZN_k) \longrightarrow 0, \]
the cycle $\begin{pmatrix} \partial C \\ 0_{(1,l)} \end{pmatrix}^T$ is attained by $\begin{pmatrix} \partial C \\ 0_{(1,2g)} \end{pmatrix}^T \in C_1(M, p; ZN_k) = (ZN_k)^{2g+l} \oplus (ZN_k)^{2g}$. Then by observing the boundary corresponding to the relation
\[ \prod_{j=1}^g[i_+(\gamma_j), i_+(\gamma_{g+j})] \left( \prod_{j=1}^g[i_-(\gamma_j), i_-(\gamma_{g+j})] \right)^{-1}, \]
we see that \( \left( \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial \mu} \right) \) is homologous to \( \left( \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial \lambda} \right) \), which comes from \( C(\Sigma^+, p; \mathbb{Z}N_k) \). Our claim follows from this.

Now we continue the proof of Theorem 6.8. We can show that \((\xi, \mu)\) is \( \psi \)-primitive for \( J_{C_M} \) as in the proof of Proposition 6.4. Then we have

\[
\Delta(J_{C_M}; \xi, \mu) = \Delta \left( J_{C_M}; \left( 1 - \frac{\gamma}{\tau}, 1 - \frac{\mu}{\tau} \right), \left( \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial \mu} \right) \right)
\]

\[
= \Delta \left( J_{C_M}; \left( I - \frac{\gamma}{\tau}, 1 - \frac{\mu}{\tau} \right), \left( \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial \mu} \right) \right) \cdot \det \left( \begin{array}{cc} A \\ B \end{array} \right)
\]

From the above argument, we obtain

\[
\overline{d}_{\psi}^\psi(C_M) = \overline{d}_{\psi}^\psi(J_{C_M}) = \deg^\psi(\Delta(J_{C_M}; \xi, \mu))
\]

\[
= \deg^\psi \left( \Delta \left( I_{2g} - r_k(M), \frac{\partial}{\partial \gamma} \right) \right) \cdot \det \left( \begin{array}{cc} A \\ B \end{array} \right)
\]

This completes the proof.

**Remark 6.10.** When \( M \in C_{g,1}[k] \cap \mathcal{M}_{g,1} \), \( I_{2g} - r_k(M) \) itself gives a presentation matrix of \( H_1(C_M, p; \mathbb{Z}N_k) \). Hence we have \( \overline{d}_{\psi}^\psi(C_M) = \overline{d}_{\psi}^\psi(I_{2g} - r_k(M)) \), and moreover \( \overline{d}_{\psi}^\psi(C_M) = \overline{d}_{\psi}^\psi(I_{2g} - r_k(M)) \) for this case.

### 6.3.2. The \( N_{k,T} \)-degree for the mapping torus of a homology cylinder

Given a homology cylinder \( M = (M, i_+, i_-) \), we have another method for obtaining a closed 3-manifold \( T_M \) as follows. First we attach a 2-handle \( I \times D^2 \) along \( I \times i_+(\partial \Sigma_{g,1}) \), so that we obtain a homology cylinder \((M', i'_+, i'_-)\) over a closed surface \( \Sigma_g \), which corresponds to the embedding \( \Sigma_{g,1} \hookrightarrow \Sigma_g \). Then we put

\[
T_M := M'/i'_+(x) = i'_-(x), \quad x \in \Sigma_g
\]

and call \( T_M \) the **mapping torus** of \( M \). Indeed, for \( M \in \mathcal{M}_{g,1} \subset C_{g,1} \), the resulting manifold \( T_{M,g} \) is the usual mapping torus of \( \varphi \) extended naturally to the mapping class of \( \Sigma_g \). If we take an admissible presentation of \( \pi_1M \) briefly denoted by \( \langle i_-(\gamma), \varphi, i_+(\gamma) | \varphi \rangle \), then a presentation of \( \pi_1T_M \) is given by

\[
\pi_1T_M \cong \langle i_-(\gamma), \varphi, i_+(\gamma), \lambda, i_-(\gamma + \lambda) | \varphi, \prod_{j=1}^g[i_-(\gamma_j), i_-(\gamma + \lambda)], i_-(\gamma) \lambda_i(\varphi)^{-1} \lambda^{-1} \rangle,
\]

where \( \lambda \) is the loop \( I \times \{ p \} \subset I \times D^2 \subset T_M \). If \( M \in C_{g,1}[k] \), we have natural isomorphisms \( N_k(\Sigma_g) \cong N_k(M') \) and \( N_k(T_M) \cong N_k(\Sigma_g) \times \langle \lambda \rangle \). Note that these groups are torsion-free nilpotent. We consider \( N_k(\Sigma_g) \) to be a subgroup of \( N_k(T_M) \). For simplicity, we denote \( N_k(\Sigma_g) \) by \( N_{k,0} \) and \( N_k(T_M) \) by \( N_{k,T} \).
We can show that $H_*(M, i_* (\Sigma_g); \mathcal{K}_{N_{k,T}}) = 0$ (see Remark 3.2). Hence by a similar argument, the Magnus representation $\tau_{k,T}: C_{g,1} \to GL(2g, \mathcal{K}_{N_{k,T}})$ and the $N_{k,T}$-torsion

$$\tau_{N_{k,t}}(M) := \tau(C_*(M, \Sigma^+; \mathcal{K}_{N_{k,T}})) \in K_1(\mathcal{K}_{N_{k,T}})/(\pm N_{k,T})$$

are defined. Then we obtain the following factorization formula of the $N_{k,T}$-degree for the mapping torus of a homology cylinder.

**Theorem 6.11.** Let $M \in C_{g,1}[k]$. For each primitive element $\psi \in H^1(N_{k,T}) = H^1(T_M)$, the $N_{k,T}$-degree $\delta_{N_{k,T}}(T_M)$ is finite, and we have

$$\delta_{N_{k,T}}^\psi(T_M) = \delta_{N_{k,T}}^\psi(T_M) = d_{N_{k,T}}^\psi(\tau_{N_{k,T}}(M)) + d_{N_{k,T}}^\psi(I_{2g} - \lambda r_{k,T}(M)) - 2|\psi(\lambda)|.$$

**Proof.** The first assertion is a slight generalization of [8, Proposition 8.4], and we now follow the proof. Let $\psi \in H^1(T_M)$ be the Poincaré dual of the surface $\iota_* (\Sigma_g) = \iota' (\Sigma_g)$. This gives an exact sequence $1 \to N_{k,0} \to N_{k,T} \xrightarrow{\psi} \mathbb{Z} \to 1$. Then our claim is proved by showing that $\delta_{N_{k,T}}^\psi(T_M)$ is finite for this $\psi$.

Let $(T_M)_{N_{k,T}}$ be the $N_{k,T}$-cover of $T_M$, and let $(T_M)_\psi$ be the $\mathbb{Z}$-cover of $T_M$ with respect to $\psi$. $(T_M)_\psi$ is the product $\cdots \times M' \times M' \times \cdots$ of countably many copies of $M'$, and $(T_M)_{N_{k,T}}$ can be regarded as the $N_{k,0}$-cover of $(T_M)_{\psi}$. Then

$$H_*(T_M; \mathcal{K}_{N_{k,T}^\psi} [t^\pm]) = H_*(C_*(((T_M)_{N_{k,T}}) \otimes_{N_{k,T}} \mathbb{Z} N_{k,T}(\mathbb{Z} N_{k,0} - \{0\})^{-1})$$

$$\cong H_*(C_*(((T_M)_{N_{k,T}}) \otimes_{N_{k,0}} \mathbb{Z} N_{k,0} - \{0\})^{-1})$$

$$= H_*(C_*(((T_M)_\psi)_{N_{k,0}}) \otimes_{N_{k,0}} \mathcal{K}_{N_{k,T}^\psi})$$

$$= H_*(((T_M)_\psi; \mathcal{K}_{N_{k,T}^\psi}).$$

Here we remark that the image of the composite $\pi_1((T_M)_\psi) \to \pi_1 T_M \to N_{k,T}$ is contained in $N_{k,0}$. The same holds for the composite $\pi_1 M' \to \pi_1 T_M \to N_{k,T}$. We also remark that $\mathcal{K}_{N_{k,T}^\psi} = \mathcal{K}_{N_{k,0}}$.

We denote by $\Sigma$ again for a lift of $\Sigma \subset T_M$ on $(T_M)_\psi$. We divide $(T_M)_\psi$ at $\Sigma$, and obtain two parts $(T_M)_\psi^+$ and $(T_M)_\psi^-$. Then $(T_M)_\psi^\pm = \lim_{l \to \infty} (M')^l$, and the inclusion $\Sigma \hookrightarrow (M')^l$ induces an isomorphism on homology. We can show that $H_*(((M')^l, \Sigma; \mathcal{K}_{N_{k,T}^\psi}) = 0$ by the same way as mentioned in Lemma 3.1. Thus $H_*(((T_M)_\psi^\pm, \Sigma; \mathcal{K}_{N_{k,T}^\psi}) = \lim_{l \to \infty} H_*(((M')^l, \Sigma; \mathcal{K}_{N_{k,T}^\psi}) = 0$, and therefore $H_*(((T_M)_\psi, \Sigma; \mathcal{K}_{N_{k,T}^\psi}) = 0$. This shows that

$$H_*(T_M; \mathcal{K}_{N_{k,T}^\psi} [t^\pm]) \cong H_*(((T_M)_\psi; \mathcal{K}_{N_{k,T}^\psi}) \cong H_*(\Sigma; \mathcal{K}_{N_{k,T}^\psi})$$

is a finite dimensional $\mathcal{K}_{N_{k,T}^\psi}$-vector space, so that $\delta_{N_{k,T}}^\psi(T_M)$ is finite.

To show the second assertion, we take an admissible presentation of $\pi_1 M$, and construct the matrices $A, B, C \in \mathbb{Z} N_{k,T}$ as before. From the presentation, we have a presentation matrix $J_{TM}$.
of $H_1(T_M, p; \mathbb{Z}N_{k,T})$ given by

$$J_{T_M} = \begin{pmatrix} A & \frac{\partial \gamma}{\partial \gamma}^T & I_{2g} \\ B & 0_{(1,1)} & 0_{(1,2g)} \\ 0_{(1,2g+1)} & 0 & -(1 - \frac{\gamma}{\gamma}) \lambda^{-1} I_{2g} \end{pmatrix},$$

where $\gamma^\prime := i^+(\gamma) = i^-(\gamma)$. We remark that $\lambda$ belongs to the center in $\mathcal{N}_{k,T}$. As presentation matrices of $H_1(T_M, p; \mathbb{Z}N_{k,T})$, this matrix is equivalent to the square matrix

$$J'_{T_M} = \begin{pmatrix} A + \lambda C & \frac{\partial \gamma}{\partial \gamma}^T \\ B & 0_{(1,1)} \\ -(1 - \frac{\gamma}{\gamma}) \lambda C & 0 \end{pmatrix}.$$ 

By Proposition 3.3, Lemma 6.3 and Fundamental formula of free calculus, we have

$$J'_{T_M} = \begin{pmatrix} A + \lambda C & \frac{\partial \gamma}{\partial \gamma}^T \\ B & 0_{(1,1)} \\ (I_{2g} - \lambda r_{k,T}(M)) & -\lambda Z \frac{\partial \gamma}{\partial \gamma} \end{pmatrix} \begin{pmatrix} A & 0_{(2g,1)} \\ B & 0_{(1,1)} \\ 0_{(1,2g+1)} & 1 \end{pmatrix}.$$ 

Note that $\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}}$ is invertible in $\mathcal{K}_{N_{k,T}}$. Then it is easily checked that

$$(1 - \frac{\gamma}{\gamma}, 1 - \frac{\gamma}{\gamma}, 1 - \lambda^{-1}) J'_{T_M} = 0,$$

$$J'_{T_M} (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}}) = 0.$$ 

We put $\tilde{\xi} := (1 - \frac{\gamma}{\gamma}, 1 - \frac{\gamma}{\gamma}, 1 - \lambda^{-1})$ and $\tilde{\mu} := (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}})^{-1} (\frac{\partial \gamma}{\partial \gamma} \lambda^{-1} - 1)^T$. As in Proposition 5.4 and Lemma 6.9 we can show that $(\tilde{\xi}, \tilde{\mu})$ is $\psi$-primitive for $J'_{T_M}$. Then

$$\delta^{\psi}_{N_{k,T}} (T_M) = d^{\psi}_{N_{k,T}} (J'_{T_M}) = \deg^\psi \left( \Delta \left( J'_{T_M}; \tilde{\xi}, \tilde{\mu} \right) \right)$$

$$= \deg^\psi \left( \Delta \left( J'_{T_M} (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}})^{-1}; \tilde{\xi}, (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}})^{-1} \end{pmatrix} \right)$$

$$= \deg^\psi \left( \Delta \left( \begin{pmatrix} I_{2g} - \lambda r_{k,T}(M) \text{ mod } 0_{(1,2g)} & -\lambda Z \frac{\partial \gamma}{\partial \gamma} \text{ mod } 0_{(1,1)} \\ \lambda \frac{\partial \gamma}{\partial \gamma} \text{ mod } 0_{(1,2g)} & I_{l} \end{pmatrix}; \tilde{\xi}, (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}})^{-1} \right) \right)$$

$$= \deg^\psi \left( \det \left( I_{2g} - \lambda r_{k,T}(M) \right) \cdot \lambda^{-1} (1 - \lambda^{-1})^{-2} \cdot \det (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}}) \right)$$

$$= \deg^\psi \left( \det (I_{2g} - \lambda r_{k,T}(M)) \cdot \lambda^{-1} (1 - \lambda^{-1})^{-2} \cdot \det (\begin{pmatrix} A \\ B \end{pmatrix}_{0_{(1,2g+1)}}) \right)$$

$$= \deg^\psi (\det (I_{2g} - \lambda r_{k,T}(M))) + \deg^\psi (\det (\tau_{N_{k,T}}(M))) - 2|\psi(\lambda)|.$$
This completes the proof. □

6.3.3. The case of \( k = 2 \) (commutative case). Since \( \mathbb{Z}N_2 = \mathbb{Z}N_2(\Sigma_g) \) and \( \mathcal{K}_2 = \mathcal{K}_2(\Sigma_g) \) are commutative, we can use the ordinary determinant for computation. Moreover, we can obtain some invariants before taking degrees. For example, define

\[
\Delta(M) := (-1)^{i+j} \frac{\det ( (I_{2g} - r_2(M))_{(i,j)})}{(1 - \gamma_i^{-1})(\frac{\partial \mathcal{K}}{\partial \gamma_j})} \in \mathcal{K}_N,
\]

where \( A_{(i,j)} \) is the matrix obtained from a matrix \( A \) by removing its \( i \)-th row and \( j \)-th column. \( \Delta(M) \) is well-defined by Lemma 5.1. Note that this invariant is based on that for string links given in [10], and we call it the Alexander rational function of \( M \).

**Theorem 6.12.** Let \( M \in \mathcal{C}_{g,1}[2] \), and let \( \Delta_{C_M}, \Delta_{T_M} \) be the Alexander polynomials of \( C_M, T_M \), respectively. Then

\[
\Delta_{C_M} := \tau_{N_2}(M) \cdot \Delta(M),
\]

\[
\Delta_{T_M} := \tau_{N_2}(M) \cdot \det (I_{2g} - \lambda r_2(T(M))) \cdot (1 - \lambda^{-1})^{-2},
\]

where \( \equiv \) means that these equalities hold in \( \mathcal{K}_N \) and \( \mathcal{K}_{2N}(T_M) \) up to \( \pm N_2 \) and \( \pm N_2(T_M) \) respectively.

**Proof.** We prove only the first assertion. The proof is almost the same as that for Theorem 6.3 under the following remarks. We follow the notation used there. We may assume that \( \text{rank}_{\mathcal{K}_N} J_{C_M} = \text{rank}_{\mathcal{K}_N} r_2(M) + l = 2g + l - 1 \).

By definition, \( \Delta_{C_M} \) is the greatest common divisor of \( \{ \det \frac{J_{C_M}(i,j)}{T} \}_{1 \leq i,j \leq 2g + l} \). We show that it is nothing other than

\[
\Delta := \Delta \left( J_{C_M}; (1 - \frac{2}{\gamma}, 1 - \frac{2}{\gamma}), \left( \frac{\partial \mathcal{K}}{\partial \gamma} \right)_{(0,1,l)} \right) \equiv \det \left( \begin{array}{c}
A \\
B
\end{array} \right) \cdot \Delta(M).
\]

As seen in Lemma 6.9, \( \left( \begin{array}{c}
A \\
B
\end{array} \right)^{-1} \left( \frac{\partial \mathcal{K}}{\partial \gamma} \right)_{(0,1,l)} \) is a vector in \( (\mathbb{Z}N_2)^{2g+l} \). If \( \Delta \) is in \( \mathbb{Z}N_2 \), it attains the greatest common divisor. To show it, suppose \( \Delta = h_1/h_2 \) where \( h_1 \in \mathbb{Z}N_2 \) and \( h_2 \in \mathbb{Z}N_2 - \{0\} \) are relatively prime. From the definition of \( \Delta \), we have

\[
\frac{(1 - \gamma_i^{-1}) \left( \frac{\partial \mathcal{K}}{\partial \gamma_j} \right)_{h_1}}{h_2} = (-1)^{i+j} \det \frac{J_{C_M}(i,j)}{T} \in \mathbb{Z}N_2.
\]

Hence \( h_2 \) is a common divisor of \( \left\{ (1 - \gamma_i^{-1}) \left( \frac{\partial \mathcal{K}}{\partial \gamma_j} \right) \right\}_{i,j} \)'s, and it is 1. That is, \( h_2 \) is a unit in \( \mathbb{Z}N_2 \).

\[
\det \left( \begin{array}{c}
A \\
B
\end{array} \right) \in \mathcal{K}_N \) (up to \( \pm N_2 \)) does not depend on the choice of an admissible presentation, and it gives \( \tau_{N_2}(M) \). Indeed the matrix \( \left( \begin{array}{c}
A \\
B
\end{array} \right) \) is a presentation matrix of \( H_1(M, \Sigma^+; \mathbb{Z}N_2) \), and its determinant gives a generator of the 0-th elementary ideal, which is principal and invariant under Tietze transformations. This completes the proof. □

The formula in Theorem 6.3 holds as elements of \( \mathbb{Z} \cup \{\infty\} \), so that the additivity loses its meaning when the value is \( \infty \). Note that \( \tau_{N_k}(C_M) = \infty \) if and only if \( \alpha_{N_k}^0(I_{2g} - r_k(M)) = \infty \), and this occurs when \( H_1(C_M; \mathcal{K}_{N_k}^0[\pm]) \) has a non-trivial free part. The following are some examples of homology cylinders which have non-trivial Alexander rational functions. By using
Theorem 6.15 in the next subsection, we obtain many situations where the formula sufficiently works. When $k \geq 3$, the computation becomes quite difficult in general.

**Example 6.13.** Assume that $g = 1$. The Dehn twist $\tau_\zeta \in M_{1,1}$ belongs to $C_{1,1}[3]$. Then, we have

$$r_2(\tau_\zeta) = \begin{pmatrix} \gamma_1^{-1} + \gamma_2^{-1} - \gamma_1^{-1} \gamma_2^{-1} & -1 + 2 \gamma_1^{-1} - \gamma_2^{-2} \\ 1 - 2 \gamma_1^{-1} + \gamma_1^{-2} & 2 - \gamma_1^{-1} - \gamma_2^{-1} + \gamma_1^{-1} \gamma_2^{-1} \end{pmatrix}. $$

Then $\Delta(\tau_\zeta) = 1 \in \mathbb{Z}N_2$, which is non-trivial.

**Example 6.14.** Assume that $g \geq 2$. Let $\tau_1, \tau_2$ and $\tau_3$ be Dehn twists along simple closed curves $c_1, c_2$ and $c_3$ as in Figure 5.

![Figure 5](image)

Then $\tau_1 \tau_2^{-1}, \tau_3 \in C_{g,1}[2]$. By a direct computation, we can check that $\Delta(\tau_1 \tau_2^{-1} \cdot \tau_3) = -(\gamma_1 - 1)^{2g-2}$, although $\Delta(\tau_1 \tau_2^{-1}) = \Delta(\tau_3) = 0$.

### 6.4. $N_k$-torsions and Harvey’s Realization Theorem.

By Proposition 6.6, the degree of the $N_k$-torsion gives a monoid homomorphism

$$d_{N_k}^\psi(\tau_{N_k} \cdot) : C_{g,1}[2] \rightarrow \mathbb{Z}_{\geq 0}$$

for each $\psi \in PH^1(\Sigma_{g,1})$ and an integer $k \geq 2$. To see some properties of these homomorphisms, including their non-triviality, we use a variant of Harvey’s Realization Theorem in [8, Theorem 11.2] which gives a method for performing surgery on a compact orientable 3-manifold to obtain a homology cobordant one having distinct higher-order degrees.

**Theorem 6.15.** Let $M \in C_{g,1}$ be a homology cylinder. For each primitive element $x$ of $H_1(\Sigma_{g,1})$ and any integers $n \geq 2$ and $k \geq 1$, there exists a homology cylinder $M'$ such that

1. $M'$ is homology cobordant to $M$,
2. $d_{N_l}^\psi(\tau_{N_l}(M')) = d_{N_l}^\psi(\tau_{N_l}(M))$ for $2 \leq l \leq n - 1$,
3. $d_{N_n}^\psi(\tau_{N_n}(M')) \geq d_{N_n}^\psi(\tau_{N_n}(M)) + k|p|$

for any $\psi \in PH^1(\Sigma_{g,1})$ satisfying $\psi(x) = p$.

**Proof.** The proof is based on Harvey’s proof of Realization Theorem in [8, Theorem 11.2]. However, since we now use the lower central series instead of the rational derived series, we can shorten the argument.

We take a loop representing $x \in H_1(\Sigma_{g,1})$, and denote it by $x$ again. We also take a loop $\gamma$ whose homology class in $H_1(\Sigma_{g,1})$ is independent of $x$.

We attach a 1-handle to $M \times \{1\} \subset M \times I$, and then attach a 2-handle to obtain a 4-manifold $W$. Here the 2-handle are attached along the loop $\alpha[X_{n-1}, A_{k+1}]$, where $\alpha \in \pi_1 M$ is a loop
corresponding to the added 1-handle, and \( X_{n-1}, A_{k+1} \in \pi_1 M \) are inductively defined by
\[
X_1 = i_+(x), \quad X_l = [i_l(\gamma), X_{l-1}] \quad \text{for} \quad l \geq 2,
A_1 = \alpha, \quad A_l = [i_l(\gamma), A_{l-1}] \quad \text{for} \quad l \geq 2.
\]
It is easily seen that \( X_l \in T^l(\pi_1 M) - T^{l+1}(\pi_1 M) \). \( M' \) is defined as another part of \( \partial W \), namely \( \partial W = M \cup M' \) and \( M \cap M' = \partial M = \partial M' \). From the construction, we have \( H_s(W, M) = 0 \). We also have \( H_s(W, M') = 0 \) by using the Poincaré-Lefschetz duality and the universal coefficient theorem. Hence \( (M', i_+, i_-) \in C_{g,1} \), and it is homology cobordant to \( M \). Stallings’ theorem shows that \( N_l \xrightarrow{\gamma} N_l(M) \to N_l(W) \xleftarrow{\gamma} N_l(W') \xrightarrow{\gamma} N_l \) are all isomorphisms. Using them, we identify \( N_l, N_l(M), N_l(M') \) and \( N_l(W) \).

For simplicity, we put \( K_l := K_{N_l^p}([\gamma]) = \mathbb{Z} N_l(\mathbb{Z} N_l^p - \{0\})^{-1} \). Recall that \( H_s(M, \Sigma^+; K_{N_l}) = H_s(W, \Sigma^+; K_{N_l}) = 0 \) as in Lemma 3.1. By the same proof, we have \( H_s(W, \Sigma^+; K_{N_l}) = 0 \). Hence \( H_s(M, \Sigma^+; K_l), H_s(M', \Sigma^+; K_l) \) and \( H_s(W, \Sigma^+; K_l) \) are all finite dimensional \( K_{N_l^p} \)-vector spaces. As seen in Section 6.2, \( d_{N_l^p}(\tau_N(M)) = \dim_{K_{N_l^p}} H_1(M, \Sigma^+; K_l) \). If we take an admissible presentation of \( \pi_1 M \) and the matrices \( A, B \in \mathbb{Z} N_l \) as before, \( (A, B) \) gives a presentation matrix of \( H_1(M, \Sigma^+; K_l) \). Then one of \( H_1(W, \Sigma^+; K_l) \) is given by
\[
\begin{pmatrix}
A \\
B \\
0_{(1,2^g+1)}
\end{pmatrix}
\]
so that
\[
\dim_{K_{N_l^p}} H_1(W, \Sigma^+; K_l) = d_{N_l^p}(\tau_N(M)) + \deg \left( \frac{\partial_{\alpha} |_{X_{n-1}, A_{k+1}}} {\partial_{\alpha}} \right).
\]
By a direct computation,
\[
\frac{\partial_{\alpha} |_{X_{n-1}, A_{k+1}}} {\partial_{\alpha}} = 1 + \alpha \left\{ (1 - X_{n-1} A_{k+1} X_{n-1}^{-1}) \frac{\partial X_{n-1}} {\partial_{\alpha}} + (X_{n-1} - [X_{n-1}, A_{k+1}]) \frac{\partial A_{k+1}} {\partial_{\alpha}} \right\}.
\]
When \( 2 \leq l \leq n - 1 \), we have \( X_{n-1} = A_{k+1} = 1 \in N_l \), so that \( \frac{\partial_{\alpha} |_{X_{n-1}, A_{k+1}}} {\partial_{\alpha}} = 1 \), and \( H_1(M, \Sigma^+; K_l) \cong H_1(W, \Sigma^+; K_l) \). When \( l = n \), we have \( X_{n-1} \neq A_{k+1} = 1 \in N_l \), so that
\[
\frac{\partial_{\alpha} |_{X_{n-1}, A_{k+1}}} {\partial_{\alpha}} = 1 + (X_{n-1} - 1) \frac{\partial A_{k+1}} {\partial_{\alpha}} = 1 + (X_{n-1} - 1)(x - A_{k+1}) \frac{\partial A_{k}} {\partial_{\alpha}} = \cdots = 1 + (X_{n-1} - 1)(x - A_{k+1})(x - A_{k}) \cdots (x - A_2),
\]
and
\[
\deg \left( \frac{\partial_{\alpha} |_{X_{n-1}, A_{k+1}}} {\partial_{\alpha}} \right) = \deg \left( \frac{\partial_{\alpha} |_{X_{n-1}, A_{k+1}}} {\partial_{\alpha}} \right) = \begin{cases} k |p| & (n \geq 3) \\ (k + 1) |p| & (n = 2) \end{cases}.
\]
In each case, \( \dim_{K_{N_l^p}} H_1(W, \Sigma^+; K_n) \geq d_{N_l^p}(\tau_N(M)) + k |p| \).

By considering the dual handle decomposition, we see that \( W \) is obtained from \( M' \times I \) by attaching a 2-handle and a 3-handle. Hence \( H_1(M', \Sigma^+; K_l) \to H_1(W, \Sigma^+; K_l) \) is an epimorphism. In particular, when \( l = n \),
\[
d_{N_l^p}(\tau_N(M')) \geq \dim_{K_{N_l^p}} H_1(W, \Sigma^+; K_n) \geq d_{N_l^p}(\tau_N(M)) + k |p|.
\]
It remains to proof that the map \( H_1(M', \Sigma^+; K_l) \to H_1(W, \Sigma^+; K_l) \) is injective when \( 2 \leq l \leq n - 1 \). We now show that \( H_2(W, M'; K_l) = 0 \). By the Poincaré-Lefschetz duality, \( H_2(W, M'; K_l) \cong H^2(W, M; K_l) \). On the other hand, it is easily checked that \( H_0(W, M; K_l) = H_1(W, M; K_l) = 0 \). Then the universal coefficient spectral sequence (see
Indeed we have the enlarged Dehn-Nielsen homomorphism. Consequently, \( H_1(M, \Sigma^+; K) \cong H_1(W, \Sigma^+; K) \cong H_1(M', \Sigma^+; K) \) and \( d_{N_k}^\psi(\tau_N(M')) = d_{N_k}^\psi(\tau_N(M)) \). This completes the proof. \( \square \)

**Corollary 6.16.** For any \( \psi \in PH^1(\Sigma_{g,1}) \), the maps \( \{ d_{N_k}^\psi(\tau_N(\cdot)) : C_{g,1}[2] \to \mathbb{Z}_{\geq 0} \}_{k \geq 2} \) are all non-trivial homomorphisms, and independent of each other.

In fact, we can show it by constructing homology cylinders that are homology cobordant to the unit \( 1_{C_{g,1}} \). From this we see that \( C_{g,1}[2], C_{g,1}[3], \ldots, \text{Ker}(C_{g,1} \to \mathcal{H}_{g,1}) \) are not finitely generated monoids. Note that \( d_{N_k}^\psi(\tau_N(M)) = 0 \) if \( M \in \mathcal{M}_{g,1} \), since \( \Sigma_{g,1} \times I \) is simple homotopy equivalent to \( \Sigma_{g,1} \) and hence \( \tau_{N_k}(M) \) is trivial.

**6.5. Appendix: Application of torsion-degree functions to \( \text{Aut } F_n^{acy} \).** In [13], we defined the Magnus representation \( r_k : \text{Aut } F_n^{acy} \to GL(n, K_{N_k}(F_n)) \) for \( \text{Aut } F_n^{acy} \), where \( F_n^{acy} \) is a completion of \( F_n \) in a certain sense and is called the acyclic closure of \( F_n \). The natural map \( F_n \to F_n^{acy} \) is known to be injective and 2-connected. In particular, \( N_k(F_n) = N_k(F_n^{acy}) \), and we denote it briefly by \( N_k \) in this subsection. \( \text{Aut } F_n^{acy} \) can be regarded as an enlargement of \( \text{Aut } F_n \).

Indeed we have the enlarged Dehn-Nielsen homomorphism \( \sigma^{acy} : \mathcal{H}_{g,1} \to \text{Aut } F_{2g}^{acy} \) extending the classical one \( \sigma : \mathcal{M}_{g,1} \to \text{Aut } F_{2g} \). That is, we have the commutative diagram

\[
\begin{array}{ccc}
\text{Aut } F_{2g} & \xrightarrow{\sigma} & \text{Aut } F_{2g}^{acy} \\
\uparrow & & \uparrow \\
\mathcal{M}_{g,1} & \xrightarrow{\sigma^{acy}} & \mathcal{H}_{g,1}
\end{array}
\]

Note that \( \sigma^{acy} \) is not injective. The Magnus representation for homology cylinders is nothing other than the composite \( \mathcal{H}_{g,1} \xrightarrow{\sigma^{acy}} \text{Aut } F_{2g}^{acy} \xrightarrow{r_k} GL(2g, K_{N_k}) \).

We now consider the map \( d_{N_k}^\psi \circ r_k : \text{Aut } F_n^{acy} \to \mathbb{Z} \) for \( \psi \in PH^1(F_n) \), where \( PH^1(F_n) \) denotes the set of primitive elements of \( H^1(F_n) \). Since \( d_{N_k}^\psi(A) = 0 \) for \( A \in GL(ZN_k) \), it follows that \( d_{N_k}^\psi \circ r_k |_{\text{Aut } F_n} \) is trivial. When \( n = 2g \), \( d_{N_k}^\psi \circ r_k |_{\text{Im } \sigma^{acy}} \) is also trivial as seen in Theorem 6.11. On the other hand, \( d_{N_k}^\psi \circ r_k \) is actually non-trivial on \( \text{Aut } F_n^{acy} \) as we will see below. Since \( r_k \) is a crossed homomorphism, we have the following.

**Proposition 6.17.** For \( f, g \in \text{Aut } F_n^{acy} \) and \( \psi \in PH^1(F_n) \), we have

\[
d_{N_k}^\psi(r_k(fg)) = d_{N_k}^\psi(r_k(f)) + d_{N_k}^\psi(r_k(g)).
\]

In particular, if we restrict \( d_{N_k}^\psi \circ r_k \) to \( \text{IAut } F_n^{acy} := \ker(\text{Aut } F_n^{acy} \to N_2 = GL(n, \mathbb{Z})) \), it becomes a homomorphism.

**Remark 6.18.** \( \text{Aut } F_n^{acy} \) acts on \( PH^1(F_n) \) from the right, and hence acts on \( \text{Map}(PH^1(F_n), \mathbb{Z}) \) from the left. We regard \( d_{N_k}(r_k(\cdot)) \) as a map \( \text{Aut } F_n^{acy} \to \text{Map}(PH^1(F_n), \mathbb{Z}) \). Then Proposition 6.17 shows that \( d_{N_k}(r_k(\cdot)) \) is a 1-cocycle in \( C^1(\text{Aut } F_n^{acy}, \text{Map}(PH^1(F_n), \mathbb{Z})) \). We can see that it is non-trivial in \( H^1(\text{Aut } F_n^{acy}, \text{Map}(PH^1(F_n), \mathbb{Z})) \) from the proof of Theorem 6.19 below.

**Theorem 6.19.** For every \( n \geq 2 \), \( \text{IAut } F_n^{acy} \) is not finitely generated. In fact, \( H_1(\text{IAut } F_n^{acy}) \) has infinite rank.

**Proof.** Let \( F_n = \langle x_1, x_2, \ldots, x_n \rangle \). We take \( \psi := x_1^1 \in PH^1(F_n) \). Consider the endomorphism \( f_k \) of \( F_n \) given by

\[
f_k(x_1) = x_1[Y_{k-1}, Y_k], \quad f_k(x_i) = x_i \quad \text{for } i \geq 2,
\]
where we define $Y_1 = x_1$ and $Y_l = [x_2, Y_{l-1}]$ for $l \geq 2$. Since $f_k$ is 2-connected, it induces an automorphism of $F_n^{\text{acy}}$ (see [18 Section 4]). We denote it by $f_k$ again. It belongs to $\text{IAut } F_n^{\text{acy}}$. For such an automorphism, the Magnus matrix $r_l(f_k)$ can be computed by using free differentials. That is, we have

$$r_l(f_k) = \begin{pmatrix}
\frac{\partial f(x_1)}{\partial x_1} & 0_{(1,n-1)} \\
\frac{\partial f(x_1)}{\partial x_2} & I_{n-1} \\
\vdots & \\
\frac{\partial f(x_1)}{\partial x_n} & 
\end{pmatrix}$$

at $\mathbb{Z}N_k$. Then $d_{N_k}^\psi(r_l(f_k)) = \text{deg}^\psi(\text{det}(r_l(f_k))) = \text{deg}^\psi\left(\frac{\partial f(x_1)}{\partial x_1}\right) = \text{deg}^\psi\left(\frac{\partial f(x_1)}{\partial x_1}\right)$. By a direct computation, we have

$$\frac{\partial f(x_1)}{\partial x_1} = 1 + x_1 \left\{(1 - Y_{k-1}Y_kY_{k-1}^{-1})\frac{\partial Y_{k-1}}{\partial x_1} + (Y_{k-1} - [Y_{k-1}, Y_k])\frac{\partial Y_k}{\partial x_1}\right\}.$$ 

When $2 \leq l \leq k - 1$, we have $Y_{k-1} = Y_k = 1 \in N_l$, so that

$$d_{N_k}^\psi(r_l(f_k)) = \text{deg}^\psi\left(\frac{\partial f(x_1)}{\partial x_1}\right) = \text{deg}^\psi(1) = 0.$$ 

When $l = k$, we have $Y_{k-1} \neq 1 \in N_k$, so that

$$\frac{\partial f(x_1)}{\partial x_1} = 1 + x_1(Y_{k-1} - 1)\frac{\partial Y_{k-1}}{\partial x_1} = 1 + x_1(Y_{k-1} - 1)(x_2 - Y_k)\frac{\partial Y_{k-1}}{\partial x_1}$$

$$\cdots = 1 + x_1(Y_{k-1} - 1)(x_2 - Y_k)(x_2 - Y_{k-1})\cdots(x_2 - Y_2),$$

and

$$d_{N_k}^\psi(r_k(f_k)) = \text{deg}^\psi\left(\frac{\partial f(x_1)}{\partial x_1}\right) = \begin{cases} 
1 & (k \geq 3) \\
2 & (k = 2) 
\end{cases}.$$ 

This shows that $\{d_{N_k}^\psi(r_k(\cdot))\}_{k \geq 2}$ are all non-trivial, and independent of each other. Our claim follows from this.

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