The Dirichlet problem for $p$-harmonic functions on the topologist’s comb

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1 Introduction

In the Dirichlet problem one looks for a $p$-harmonic function $u$ on some bounded domain $\Omega \subset \mathbb{R}^n$ which takes prescribed boundary values $f$. A $p$-harmonic function $u$ is a continuous weak solution of the equation

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = 0.$$  

(And thus for $p = 2$ we obtain the usual harmonic functions.) Here $1 < p < \infty$ is fixed. The nonlinear potential theory associated with $p$-harmonic functions has been studied for half a century, first on $\mathbb{R}^n$ and then in various other situations (manifolds, Heisenberg groups, graphs etc.), and more recently on metric spaces giving a unified treatment covering most of the earlier cases, see the monographs Heinonen–Kilpeläinen–Martio [14] (for weighted $\mathbb{R}^n$) and Björn–Björn [5] (for metric spaces) and the references therein.
If \( f \) is not continuous, then there usually is no \( p \)-harmonic function \( u \) which takes the boundary values as limits (i.e. such that \( \lim_{y \to x} u(y) = f(x) \) for all \( x \in \partial \Omega \)), and even for continuous \( f \) and with \( p = 2 \) this is not always possible. One therefore needs some other precise definition of what is a solution to the Dirichlet problem. For \( p \)-harmonic functions there are at least four different definitions in the literature, of which the Perron method is the most general, see the definitions in Björn–Björn–Shanmugalingam [6,7] and Björn–Björn [4] as well as Theorem 4.2 in [4], or the discussion in the introduction to Chapter 10 in Björn–Björn [5].

For any boundary function \( f : \partial \Omega \to \mathbb{R} := [-\infty, \infty] \), the Perron method produces an upper and a lower Perron solution. When these coincide they give a reasonable solution to the Dirichlet problem, called the Perron solution \( P f \), and \( f \) is said to be resolutive, see Sect. 3 for the precise definition.

In this paper we want to study the Dirichlet problem, or more precisely Perron solutions, for \( p \)-harmonic functions on the topologist’s comb \( \Psi \)

\[
\Psi = ((-1, 1) \times (0, 2)) \setminus \bigcup_{j=0}^{\infty} I_j
\]

in the plane, where \( I_j = (0, 1) \times \{2^{-j}\} \), \( j = 0, 1, \ldots \), see Fig. 1. Let \( I = (0, 1] \times \{0\} \) be the set of inaccessible boundary points of \( \Psi \).

We obtain the following result, which is a special case of Theorem 5.1.

**Theorem 1.1** Let \( f : \partial \Psi \to \mathbb{R} \) be such that \( f|_{\partial \Psi \setminus I} \in C^{\text{bd}}(\partial \Psi \setminus I) \). Then \( f \) is resolutive, and the Perron solution \( P f \) is independent of the values of \( f \) on \( I \), i.e. if \( h = f \) on \( \partial \Psi \setminus I \), then \( P h = P f \).

(In the linear case, \( p = 2 \), this is well known and can be obtained more easily.)

The Perron method was introduced independently by Perron [17] and Remak [18] in the 1920s for harmonic functions. The linear theory was developed further by Wiener and Brelot, and the method is therefore often called the PWB method in the linear case. In the nonlinear case the theory was developed by Granlund–Lindqvist–Martio [13], Kilpeläinen [15] and

![Fig. 1 The topologist’s comb \( \Psi \)](image)
Heinonen–Kilpeläinen–Martio [14] for unweighted and weighted $\mathbb{R}^n$. In particular the resolutivity was obtained for continuous $f : \partial \Omega \to \mathbb{R}$ for arbitrary bounded domains $\Omega \subset \mathbb{R}^n$ (in the unweighted case in [15] and in the weighted case in [14]).

The first invariance result of the kind above (in the nonlinear case) was obtained in Björn–Björn–Shanmugalingam [7] where it was shown that if $f \in C(\partial \Omega)$ and $h = \int_{\partial \Omega} f$ outside a set of $p$-capacity zero, then $h$ is resolutive and $Ph = Pf$. This was obtained for bounded domains $\Omega$ in metric measure spaces (under the usual assumptions that the metric space is complete and the measure is doubling and supports a $p$-Poincaré inequality). In BJörn–Björn [5] this result was improved slightly by allowing for a (sometimes) smaller capacity. More recently, in Björn–Björn–Shanmugalingam [9], it was further improved using again a (sometimes) smaller capacity $\overline{C}_p(\cdot, \Omega)$ introduced therein, which sees the boundary from inside $\Omega$ (see [9] for the precise definition). In particular, it was shown in Example 10.2 in [9] that

$$\overline{C}_p(I, \Psi) = 0,$$

(1.1)

so that Theorem 1.1 was obtained therein for functions $f$ for which there exists $k \in C(\partial \Psi)$ such that $k = f$ on $\partial \Psi \setminus I$, i.e. $f$ such that $f|_{\partial \Psi \setminus I} \in C_{\text{unif}}(\partial \Psi \setminus I)$.

The significance of Theorem 1.1 is that we do not assume any continuity at points in $I$, or more precisely consider functions in $C_{\text{bdd}}(\partial \Psi \setminus I)$. That Theorem 1.1 is not true for unbounded functions in $C(\partial \Psi \setminus I)$ is shown in Example 4.2, as such functions need not be resolutive.

In Theorem 5.1 we obtain a generalization of Theorem 1.1 which is connected with the prime end boundary of $\Psi$. Here it is not the classical prime end boundary of Carathéodory [11] which is used. Instead it is the prime end definition introduced in Adamowicz–Björn–Björn–Shanmugalingam [1] which is the natural choice in this paper. The noncompactness of the prime end closure of the comb leads to some new phenomena, see Sect. 4. In domains which are so-called finitely connected at the boundary, the prime end closure is compact and the theory of Perron solutions with respect to the prime end boundary for such domains was developed in Björn–Björn–Shanmugalingam [9]. Estep–Shanmugalingam [12] are studying similar problems when the prime end closure is noncompact.

Let us compare our result with the unit disc $D$ in the plane and let $x_0 = (1, 0)$. Let also $f : \partial \overline{D} \to \overline{\mathbb{R}}$ be a function such that $f|_{\partial \overline{D}\setminus \{x_0\}}$ is bounded and continuous. If $f$ is semicontinuous then $f$ is resolutive (for this we need to use that $D$ is a regular domain), see Proposition 9.31 in Heinonen–Kilpeläinen–Martio [14] and Proposition 7.3 in Björn–Björn–Shanmugalingam [7] (or Proposition 10.32 in [5]), but if $f(x_0)$ is such that $f$ is not semicontinuous, then it is not known if $f$ is resolutive. Moreover, all choices of $f(x_0)$ which make $f$ upper semicontinuous yield the same Perron solution, by Proposition 7.3 in [7] (or Proposition 10.32 in [5]). Similarly all choices of $f(x_0)$ which make $f$ lower semicontinuous yield the same Perron solution, but we do not know if this Perron solution is the same as the one for upper semicontinuous choices of $f(x_0)$. If $f$ has a jump discontinuity at $x_0$, then we do know that $f$ is resolutive for all choices of $f(x_0)$ and that the Perron solutions all agree [i.e. are independent of $f(x_0)$], by Theorems 6.3 and 7.3 in Björn [3]. (For $p \leq 2$ it is not too difficult to deduce this using the earlier results in Björn–Björn–Shanmugalingam [8].) Thus we have less general invariance results for perturbations on a single point on the boundary of $D$ than those we obtain in this paper for perturbations on $I$ on the boundary of the comb $\Psi$. (Above the regularity of $D$ was important, but there are some results in this direction in [3] which hold also for semiregular sets.)

The outline of the paper is as follows: In Sect. 2 the comb and its various boundaries are introduced, while in Sect. 3 the Perron solutions considered in this paper are defined. In Sect. 4 we obtain some boundary regularity results which will be essential for us. The main
result (Theorem 5.1) is obtained in Sect. 5. Finally in Sect. 6 we combine the ideas in this paper with some ideas in Björn [3] to obtain a generalization of our main result.

2 The topologist’s comb

The aim of this paper is to study the Dirichlet problem on the comb $\Psi$. The boundary points of $\Psi$ are of three different types that will be of interest to us.

A boundary point $x_0 \in \partial \Psi$ is accessible if there is a continuous mapping (a curve) $\gamma : [0, 1] \to \overline{\Psi}$ such that $\gamma(1) = x_0$ and $\gamma([0, 1)) \subset \Psi$. The set $I$ consists of all the inaccessible boundary points.

The boundary points in $I_j$ each have two natural counterparts in the extended boundary we shall define below, one by taking limits from below and one from above. To be more precise, set $\theta_j = 2^{-j}$, $j = 1, 2, \ldots$, and define $F : \Psi \to \mathbb{R}^3$ by letting

$$F(x_1, x_2) = \begin{cases} (x_1, x_2, 0), & \text{if } (x_1, x_2) \in (-1, 0] \times (0, 2), \\ (x_1 \cos \theta_j, x_2, x_1 \sin \theta_j), & \text{if } (x_1, x_2) \in (0, 1) \times (2^{-j}, 2^{1-j}), j = 0, 1, \ldots \end{cases}$$

Let $\Psi^{\text{Ext}}$ be $\Psi$ equipped with the distance $\text{dist}_{\Psi^{\text{Ext}}}(x, y) = |F(x) - F(y)|$, $x, y \in \Psi$. Let also $\partial_{\Psi} \Psi = \Psi^{\text{Ext}} \setminus \Psi$, where $\Psi^{\text{Ext}}$ is the completion of $\Psi^{\text{Ext}}$. Each point in $\bigcup_{j=0}^{\infty} I_j$ corresponds to two points in this extended boundary, whereas all other points in $\partial \Psi$ have one counterpart in $\partial_{\Psi^{\text{Ext}}} \Psi$. Let also $\Phi : \partial_{\Psi^{\text{Ext}}} \Psi \to \partial \Psi$ be the natural map.

The extended boundary is closely related to prime end boundaries. In the Carathéodory prime end theory the only difference is that the closed interval $\overline{T}$ corresponds to one prime end, apart from this there is a one-to-one correspondence between the Carathéodory prime ends and the points in the extended boundary $\partial_{\Psi^{\text{Ext}}} \Psi$ (for this particular set).

In Adamowicz–Björn–Björn–Shanmugalingam [1] a different definition of prime ends was proposed, which in the case of the comb gives a natural one-to-one correspondence between its prime end boundary $\partial P \Psi$ and the points in $\partial_{\Psi^{\text{Ext}}} \Psi \setminus I$, whereas there are no prime ends corresponding to points in $I$, see Example 5.1 in [1]. As we shall see this prime end boundary, and the associated topology, will be of more interest in this paper than the Carathéodory prime end boundary. For us it is enough to know that $\partial P \Psi = \partial_{\Psi^{\text{Ext}}} \Psi \setminus I$, and we refer to [1] for their definition of prime ends.

If we introduce the Mazurkiewicz distance (sometimes called inner diameter distance) $d_M$ on $\Psi$ by letting

$$d_M(x, y) = \inf E \text{ diam } E,$$

where the infimum is taken over all connected sets $E \subset \Psi$ containing $x, y \in \Psi$, then $\partial M \Psi = \partial P \Psi$. Here $\partial M \Psi = \overline{\Psi^M} \setminus \Psi$, where $\overline{\Psi^M}$ is the completion of $(\Psi, d_M)$, and the equality $\partial M \Psi = \partial P \Psi$ is understood in the sense that there is a homeomorphism $H : \overline{\Psi^M} \to \overline{\Psi^P}$ such that $H|_{\Psi}$ is the identity. (For more on the Mazurkiewicz distance see [1] and Björn–Björn–Shanmugalingam [9, 10].)

Note that $\partial_{\Psi^{\text{Ext}}} \Psi \setminus I = \partial P \Psi = \partial M \Psi$ is not compact.

3 Perron solutions

Definition 3.1 A function $u : \Omega \to \mathbb{R} \cup \{\infty\}$ is $p$-superharmonic in a domain (i.e. nonempty open connected set) $\Omega$ if
(a) $u$ is lower semicontinuous;
(b) $u \not\equiv \infty$;
(c) for each domain $G \subseteq \Omega$ and each $h \in C(\overline{G})$ which is $p$-harmonic in $G$ and such that $h \leq u$ on $\partial G$ it is true that $h \leq u$ in $G$.

A function $v : \Omega \to \mathbb{R} \cup \{-\infty\}$ is $p$-subharmonic if $-v$ is $p$-superharmonic.

We will be interested in two types of Perron solutions. We denote the standard Perron solutions using the letter $P$ and the special ones using $S$. For the latter we consider $\Psi_{\text{Ext}}$ with the prime end boundary $\partial_p \Psi := \partial_{\text{Ext}} \Psi \setminus I$. The special Perron solutions are primarily used as a tool in our study of the standard ones.

**Definition 3.2** Given a function $f : \partial_{\text{Ext}} \Psi \to \mathbb{R}$, let $\mathcal{U}_f$ be the set of all $p$-superharmonic functions $u$ on $\Psi$ bounded from below such that

$$\lim_{\Psi_{\text{Ext}} \ni y \to x} \inf \ u(y) \geq f(x) \text{ for all } x \in \partial_{\text{Ext}} \Psi.$$ 

The extended upper Perron solution of $f$ is the function

$$P_{\text{Ext}} f(x) = \inf_{u \in \mathcal{U}_f} u(x), \quad x \in \Psi.$$ 

Let similarly, for $f : \partial_p \Psi \to \mathbb{R}$, $\tilde{\mathcal{U}}_f$ be the set of all $p$-superharmonic functions $u$ on $\Psi$ bounded from below such that

$$\lim_{\Psi_{\text{Ext}} \ni y \to x} \inf \ u(y) \geq f(x) \text{ for all } x \in \partial_p \Psi.$$ 

The special upper Perron solution of $f$ is the function

$$S f(x) = \inf_{u \in \tilde{\mathcal{U}}_f} u(x), \quad x \in \Psi.$$ 

The lower Perron solutions are defined similarly using $p$-subharmonic functions, or equivalently by letting

$$P f = -P_{\text{Ext}}(-f) \quad \text{and} \quad S f = -S(-f).$$

If $P_{\text{Ext}} f = P f$, then we let $P f := P_{\text{Ext}} f$ and $f$ is said to be $P_{\text{Ext}}$-resolutive. We similarly define $S f$ and $S$-resolutivity.

We also similarly define $\overline{P} f$, $\overline{P} f$ and $P f$ for $f : \partial \Psi \to \overline{\mathbb{R}}$, and $\overline{P}_\Omega f$, $\overline{P}_\Omega f$ and $P f$ for $f : \partial \Omega \to \overline{\mathbb{R}}$ for bounded domains $\Omega$.

The proof that standard Perron solutions are $p$-harmonic or identically $\pm \infty$ directly carries over to our special Perron solutions, see Theorem 9.2 in Heinonen–Kilpeläinen–Martio [14].

The following comparison principle shows that $\overline{S} f \leq \overline{S} f$ and $P_{\text{Ext}} f \leq P f$ for all functions $f$. Since it is immediate that $P_{\text{Ext}} f \leq S f$ and $\overline{S} f \leq \overline{P}_{\text{Ext}} f$, we find that

$$P_{\text{Ext}} f \leq \overline{S} f \leq \overline{S} f \leq \overline{P}_{\text{Ext}} f.$$ (3.1)

Moreover, if $f : \partial \Psi \to \mathbb{R}$, then $f$ can naturally be seen as a function on $\partial_{\text{Ext}} \Psi$, and we will do so without further ado. It is easy to see that in this case we always have $\overline{P} f = \overline{P}_{\text{Ext}} f$ and $\overline{P} f = \overline{P}_{\text{Ext}} f$.

Another obvious fact is that if $f_1 \leq f_2$, then $\overline{P}_{\text{Ext}} f_1 \leq \overline{P}_{\text{Ext}} f_2$ and similar inequalities also follow for all the other (lower and upper) types of Perron solutions. We will say that
this inequality holds by simple comparison. This should be seen in relation to the following
important comparison principle.

**Theorem 3.3** (Extended comparison principle) Assume that \( u \) is \( p \)-superharmonic and \( v \) is \( p \)-subharmonic in \( \Psi \). If

\[
\liminf_{\Psi^{\text{Ext}} \ni y \to x} (u(y) - v(y)) \geq 0 \quad \text{for all } x \in \partial_P \Psi, \tag{3.2}
\]

which in particular holds if

\[
\infty \neq \limsup_{\Psi^{\text{Ext}} \ni y \to x} v(y) \leq \liminf_{\Psi^{\text{Ext}} \ni y \to x} u(y) \neq -\infty \quad \text{for all } x \in \partial_P \Psi, \tag{3.3}
\]

then \( v \leq u \) in \( \Psi \).

This can be achieved by a modification of the proof of the corresponding result for standard
Perron solutions, see Theorem 3.1 in Björn [3]. We here instead use the comparison principle
on \( \Psi^k \) as a tool, to give a shorter proof, where \( \Psi_k := \Psi \setminus ([0, 1) \times (0, 2^{-k})) \) equipped with
the Mazurkiewicz distance. The comparison principle on \( \Psi^k \) is given in Proposition 7.2 in
Björn–Björn–Shanmugalingam [9] under the assumption (3.3). The proof therein however
first deduces (3.2) and then proceeds from this assumption.

**Proof** Let \( x_0 \in \Psi \) and \( \varepsilon > 0 \). Then there is \( k \) such that \( x_0 \in \Psi_k \) and such that \( v(y) \leq u(y) + \varepsilon \) if \( |y| < 2^{-k} \). Hence

\[
\liminf_{\Psi^{\text{Ext}} \ni y \to x} (u(y) + \varepsilon - v(y)) \geq 0 \quad \text{for all } x \in \partial^{\text{Ext}} \Psi_k.
\]

By the comparison principle for \( \Psi^k \), see the proof of Proposition 7.2 in [9], we get that \( v \leq u + \varepsilon \) in \( \Psi_k \), and in particular \( v(x_0) \leq u(x_0) + \varepsilon \). Letting \( \varepsilon \to 0 \) completes the proof. \( \square \)

## 4 Boundary regularity

The prime end boundary \( \partial_P \Psi \) is not compact, and thus we have to take extra care when
defining boundary regularity. The three classes \( C(\partial_P \Psi) \), \( C_{\text{bdd}}(\partial_P \Psi) \) (of bounded continuous
functions) and \( C_{\text{unif}}(\partial_P \Psi) \) (of uniformly continuous functions) do not coincide as they do
on compact sets. For the results in this paper it seems that \( C_{\text{bdd}}(\partial_P \Psi) \) is the right choice in
the following definition of boundary regularity.

**Definition 4.1** A point \( x_0 \in \partial_P \Psi \) is \( S \)-regular if

\[
\lim_{\Psi^{\text{Ext}} \ni y \to x_0} Sf(y) = f(x_0) \quad \text{for all } f \in C_{\text{bdd}}(\partial_P \Psi).
\]

That we cannot allow for general \( f \in C(\partial_P \Psi) \) is due to the fact that there are \( f, h \in C(\partial_P \Psi) \) such that \( Sf \equiv \Sigma h \equiv \infty \) and \( \Sigma h \equiv -\infty \), as shown by the following example.

**Example 4.2** Fix \( x_0 \in \Psi \) and let \( f_j(x) = (1 - 2^{j+2}|x - y_j|)_+ \), \( j = 1, 2, \ldots \), where
\( y_j = (1, 3 \cdot 2^{-j-1}) \). By the \( S \)-regularity of \( y_j \) (shown in Proposition 4.3 below) we see that \( Sf_j \neq 0 \). Thus the strong minimum principle, see Theorem 7.12 in Heinonen–Kilpeläinen–
Martio [14], yields \( Sf_j(x_0) > 0 \). Let

\[
f = \sum_{j=1}^{\infty} \frac{2jf_j}{Sf_j(x_0)} \in C(\partial_P \Psi) \quad \text{and} \quad h = \sum_{j=1}^{\infty} \frac{(-1)^jff_j}{Sf_j(x_0)} \in C(\partial_P \Psi).
\]
Then \( Sf(x_0) \geq j \) for all \( j \), and thus \( Sf(x_0) = \infty \). It follows that \( Sf = \infty \).

Moreover, if \( u \in \tilde{\mathcal{U}}_f \) then, by definition, \( u \geq -m \) for some real \( m \geq 0 \). Hence \( 0 \leq u + m \in \tilde{\mathcal{U}}_f \), but this contradicts the fact that \( \overline{S}f = \infty \). Thus there is no such \( u \), i.e. \( \overline{S}h = \infty \).

Similarly \( \overline{S}h = -\infty \).

By (3.1) it follows that \( \overline{P}h \equiv \overline{P}^{ext}h \equiv \infty \) and \( Ph \equiv P^{ext}h \equiv -\infty \). Hence the resolutivity in Theorem 1.1 is not true for arbitrary unbounded continuous functions on \( \partial \Psi \backslash I \).

**Proposition 4.3** Let \( x_0 \in \partial_p \Psi \). Then \( x_0 \) is \( S \)-regular.

**Proof** We will use that all boundary points of \( \partial \Psi \) are regular (with respect to the standard nonextended Perron solutions), which is well-known and e.g. follows from the sufficiency part of the Wiener criterion, see Maz’ya [16]. To do so we need to distinguish those points \( x_0 \) for which \( \Phi(x_0) \) has a unique preimage in \( \partial P \Psi \) and those which have two preimages. (Recall that \( \Phi : \partial \mathcal{E}_\Psi \rightarrow \partial \Psi \) is the natural map.)

**Case 1.** \( \Phi(x_0) \) has the unique preimage \( x_0 \). Let \( f \in C_{\text{bdd}}(\partial P \Psi) \). Then we can find \( h \in C(\partial \Psi) \) such that \( h(x_0) = f(x_0) \) and \( h \geq f \) on \( \partial P \Psi \). By simple comparison, we have \( \overline{S}f \leq \overline{S}h \leq P^{ext}h = Ph \) in \( \Psi \). Thus, using that all boundary points of \( \Psi \) are regular for the standard Perron solutions, we see that

\[
\limsup_{\Psi^{ext}y \rightarrow x_0} \overline{S}f(y) = f(x_0).
\]

Similarly, \( \liminf_{\Psi^{ext}y \rightarrow x_0} \overline{S}f(y) \geq f(x_0) \), which together with the inequality \( \overline{S}f \leq \overline{S}f \) shows that \( \lim_{\Psi^{ext}y \rightarrow x_0} \overline{S}f(y) = \overline{S}f(x_0) \). As \( f \) was arbitrary this yields the \( S \)-regularity of \( x_0 \).

**Case 2.** \( \Phi(x_0) \) has two preimages (of which \( x_0 \) is one). In this case \( x_0 \in I_j \) for some \( j \) and moreover \( x_0 \in \partial G \), where \( G = (2^{-k}, 2^{1-k}) \times (0, 1) \) for some \( k \). Let \( f \in C_{\text{bdd}}(\partial P \Psi) \) and assume that \( 0 \leq f \leq 1 \). Then we can find \( h \in C(\partial G) \) such that \( h(x_0) = f(x_0) \), \( f \leq h \leq 1 \) on \( \partial G \cap \partial P \Psi \) and \( h = 1 \) on \( \partial G \backslash \partial P \Psi \).

Let \( u \) be a \( p \)-superharmonic function competing in the definition of \( \overline{P}Gh \), and set

\[
v = \begin{cases} 
\min\{u, 1\} & \text{in } G, \\
1 & \text{in } \Psi \backslash G.
\end{cases}
\]

Then \( v \) is \( p \)-superharmonic in \( \Psi \), by Pasting lemma 7.9 in Heinonen–Kilpeläinen–Martio [14], and thus \( v \in \tilde{\mathcal{U}}_f \). Hence \( \overline{S}f \leq u \) in \( G \), and since \( u \) was arbitrary, \( \overline{S}f \leq P_Gh \) in \( G \). Thus, using also that all boundary points of \( G \) are regular for the standard Perron solutions, we see that

\[
\limsup_{\Psi^{ext}y \rightarrow x_0} \overline{S}f(y) \leq \lim_{G^{ext}y \rightarrow x_0} P_Gh(y) = h(x_0) = f(x_0).
\]

Continuing exactly as in case 1 we deduce the \( S \)-regularity of \( x_0 \). \( \square \)

We can now use this to deduce \( S \)-resolutivity for \( f \in C_{\text{bdd}}(\partial P \Psi) \).

**Proposition 4.4** Let \( f \in C_{\text{bdd}}(\partial P \Psi) \). Then \( f \) is \( S \)-resolutive.

**Proof** By the regularity of \( x \in \partial P \Psi \) we see that

\[
\lim_{\Psi^{ext}y \rightarrow x} \overline{S}f(y) = f(x) = \lim_{\Psi^{ext}y \rightarrow x} \overline{S}f(y) \text{ for all } x \in \partial P \Psi.
\]
As $Sf$ and $Sf$ are $p$-harmonic we can apply the comparison principle (Theorem 3.3), with $Sf$ as the $p$-subharmonic function and $Sf$ as the $p$-superharmonic function, to deduce that $Sf \leq Sf$. Since we always have $Sf \leq Sf$, we see that $Sf = Sf$. 

**Proposition 4.5** Let $x_0 \in \partial \text{Ext} \Psi$. Then $x_0$ is $P^{\text{Ext}}$-regular, i.e.

$$\lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} f(y) = f(x_0) \text{ for all } f \in C(\partial \text{Ext} \Psi).$$

The proof of this result is almost identical to the proof of Proposition 4.3 above, and we leave it to the interested reader to verify. As in Proposition 4.4 this can be used together with the comparison principle (Theorem 3.3) to obtain the $P^{\text{Ext}}$-resolutivity for all $f \in C(\partial \text{Ext} \Psi)$, which however is merely a special case of our main result (Theorem 5.1) below.

We will need the following consequence of Proposition 4.5.

**Proposition 4.6** Let $x_0 \in \partial \text{Ext} \Psi$ and let $f : \partial \text{Ext} \Psi \to \mathbb{R}$ be a function which is lower semicontinuous at $x_0$ and bounded on $\partial \text{Ext} \Psi$. Then

$$\lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} f(y) \geq \lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} f(y) \geq f(x_0).$$

If $f$ is moreover continuous at $x_0$, then

$$\lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} f(y) = h(x_0) = f(x_0). \quad (4.1)$$

**Proof** (The proof is similar to the corresponding result for standard Perron solutions, see Proposition 7.1 in Björn–Björn–Shanmugalingam [7] or Theorem 10.29 in [5].) We can find a function $h \in C(\partial \text{Ext} \Psi)$ such that $h \leq f$ on $\partial \text{Ext} \Psi$ and $h(x_0) = f(x_0)$. By simple comparison and the Ext-regularity obtained in Proposition 4.5 we get that

$$\lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} f(y) \geq \lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} f(y) \geq \lim_{\Psi^{\text{Ext}} \ni y \to x_0} P^{\text{Ext}} h(y) = h(x_0) = f(x_0).$$

If $f$ is continuous at $x_0$ we apply this also to $-f$ to obtain (4.1). 

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**5 The main result**

The following is the main result of this paper, and Theorem 1.1 is a special case of this result since $Pf = P^{\text{Ext}} f$ if $f : \partial \Psi \to \overline{\mathbb{R}}$.

**Theorem 5.1** Let $f : \partial \text{Ext} \Psi \to \overline{\mathbb{R}}$ be such that $f|_{\partial P \Psi} \in C_{\text{bdd}}(\partial P \Psi)$. Then

$$P^{\text{Ext}} f = Sf.$$ 

In particular, $f$ is $P^{\text{Ext}}$-resolutive and $P^{\text{Ext}} f$ is independent of $f|_I$, i.e. if $h : \partial \text{Ext} \Psi \to \overline{\mathbb{R}}$ is such that $h = f$ on $\partial P \Psi$, then $P^{\text{Ext}} h = P^{\text{Ext}} f = Sh = Sf$.

In the special case when $f$ is bounded on $\partial \text{Ext} \Psi$ and continuous at 0 also from $I$, this can be deduced directly from the comparison principle (Theorem 3.3) and Proposition 4.6. Hence it is to allow for a discontinuity at 0 that we need to work harder.

To prove this we will use a number of results which are available to us for the comb, but not in more general situations. Let us mention the key ingredients which are not generally available, but first we need some more terminology.
A boundary point \( x_0 \in \partial \Omega \) is \textit{semiregular} (with respect to a domain \( \Omega \)) if \( x_0 \) is irregular and the limit
\[
\lim_{\Omega \ni y \to x_0} P_{\Omega} f(y) \quad \text{exists for all} \quad f \in C(\partial \Omega).
\]

An open set is \textit{semiregular} if all its boundary points are either regular or semiregular. There are two types of irregular boundary points, semiregular and strongly irregular boundary points, with very different behaviour, see Björn [2].

(a) We will need the comparison principle (Theorem 3.3) on \( \partial P \Psi \). It needs further investigation to see which sets this can be extended to, see Estep–Shanmugalingam [12].

(b) We will use that all boundary points are regular. However, if there are also some semi-
regular boundary points it should be possible to combine the techniques for proving the comparison principles in Theorem 3.1 in Björn [3] and in Theorem 3.3 to obtain a suitable comparison principle enabling the proof of Theorem 5.1 in such a case.

The situation resembles the one when proving that bounded semicontinuous functions are resolutive, see the discussion in the introduction, although for our new result semiregularity should be possible to handle, whereas strong irregularity is still a serious obstacle.

(c) For \( p > 2 \) we will also need Theorem 6.3 in [3], a result which is only available in unweighted \( \mathbb{R}^n \). Here we apply it for the point 0 and to be able to do so we need to know that 0 is an exterior ray point.

(d) In the unbounded case (i.e. when \( f \) is allowed to be unbounded on \( I \)) we will also need a recent result from Björn–Björn–Shanmugalingam [9].

As there are some extra complications to obtain this result in the unbounded case we first give a proof for the bounded case. It should also be said that Theorem 5.1 is a special case of Theorem 6.1 below (the role of \( f \) in Theorem 5.1 is taken by \( h \) in Theorem 6.1). However, as the proof of Theorem 6.1 is substantially more involved we prefer to give a direct proof of Theorem 5.1 here.

\[ \text{Proof:} \quad (\text{For bounded } f.) \text{ Assume, without loss of generality, that } 0 \leq f \leq 2 \text{ and that } f(0) = 1. \]

Let \( k = \begin{cases} f & \text{on } \partial P \Psi, \\ 1 & \text{on } I, \end{cases} \text{ and } \tilde{k} = k + \chi_{\{0\}}. \)

Let also \( u \in \tilde{U}_\tilde{k} \). Then
\[
\lim \inf_{\Psi \ni y \to 0} u(y) \geq \tilde{k}(0) = 2 \geq \lim \sup_{\Psi \ni y \to 0} P^{\text{Ext}} f(y).
\]

Since \( f \in C_{\text{bdd}}(\partial P \Psi) \), Proposition 4.6 shows that
\[
\lim_{\Psi^{\text{Ext}} \ni y \to x} P^{\text{Ext}} f(y) = f(x) \leq \lim \inf_{\Psi^{\text{Ext}} \ni y \to x} u(y) \quad \text{for all } x \in \partial P \Psi \setminus \{0\}.
\]

Thus, the comparison principle (Theorem 3.3) yields that \( u \geq P^{\text{Ext}} f \). As this is true for all \( u \in \tilde{U}_\tilde{k} \), we obtain that \( P^{\text{Ext}} f \).
We equip $\Psi 1$ with the distance $dG$ and call this space $\Psi G$. Taking the completion of this space we obtain $\partial G\Psi$, in a similar way as when we obtained $\partial G\Psi$. (This time only the points in $I j$, $j = 0, \ldots, m$, are doubled, whereas to each point in $\bigcup_{j=m+1}^{\infty} I j$ there is just one corresponding point in $\partial G\Psi$. The notation follows Björn–Björn–Shanmugalingam [9].)

Next we find $\varphi \in C(\partial G\Psi)$ such that $k \leq \varphi \leq 2$ and $\varphi(x) = k(x) = f(x)$. Let also $\bar{\varphi} = \varphi + \infty\chi I$. By Theorem 11.2 in [9] together with (1.1), we see that $P_{\Psi G}\bar{\varphi} = P_{\Psi G}\varphi$. Thus, by simple comparison, we obtain that

$$P_{\Psi G}\varphi = P_{\Psi G}\bar{\varphi} = P_{\Psi G}\varphi \leq 2. \quad (5.1)$$

That $x$ is a regular boundary point with respect to $\Psi G$ is shown as in Proposition 4.3. Using this we find that

$$\lim_{\Psi G \ni y \to x} P_{\Psi G}\varphi(y) = \varphi(x) = f(x) \leq \liminf_{\Psi G \ni y \to x} u(y).$$

Since $x \in \partial P\Psi \setminus \{0\}$ was arbitrary we have thus shown that

$$\lim_{\Psi G \ni y \to x} P_{\Psi G}\varphi(y) = \varphi(x) = f(x) \leq \liminf_{\Psi G \ni y \to x} u(y) \quad \text{for all } x \in \partial P\Psi \setminus \{0\}.$$
Moreover, using (5.1) again, we see that
\[
\lim_{\Psi(y) \to 0} \overline{P}^f(y) \leq 2 \leq \liminf_{\Psi(y) \to 0} u(y).
\]
Hence, the comparison principle (Theorem 3.3) yields that \( u \geq \overline{P}^f \). As already mentioned, the rest of the proof is exactly as in the bounded case.

\[\Box\]

6 Functions with jumps

In this section we go one step further and combine the technique above with the technique in Björn [3] to deduce the following result.

**Theorem 6.1** Let \( E \subset \partial_p \Psi \setminus \{0\} \) be a countable set. Assume that \( f : \partial_p \Psi \to \mathbb{R} \) is bounded and that \( f|_{\partial_p \Psi} \) is continuous at all points in \( \partial_p \Psi \setminus E \) and has jumps at all points in \( E \).

Let \( h : \partial_p \Psi \to \mathbb{R} \) be such that \( h = f \) on \( \partial_p \Psi \setminus \tilde{E} \), where \( \tilde{E} \subset \partial_p \Psi \) and
\[
C_p(\Phi(\tilde{E})) = 0, \quad \text{if } 1 < p \leq 2,
\]
\[
\tilde{E} \text{ is countable, } \quad \text{if } p > 2. \tag{6.1}
\]
Then both \( f \) and \( h \) are \( P^f \)- and \( S \)-resolutive, and
\[
P^h = P^f = Sh = Sf.
\]

Here \( C_p \) is the Sobolev capacity on \( \mathbb{R}^2 \), see p. 48 in Heinonen–Kilpeläinen–Martio [14]. (Recall also that \( \Phi : \partial_p \Psi \to \partial \Psi \) is the natural map.) By saying that \( f \) has a jump at \( x \in \partial_p \Psi \setminus \{0\} \) we mean that it has limits from the two directions along the boundary, but these limits need not be the same, neither do we impose any condition on the relation between these limits and the value \( f(x) \).

**Remark 6.2** We need to use modifications of Theorems 5.2 and 5.4 in Björn [3] for the \( P^f \)-Perron solutions, and the proofs therein directly generalize to this situation. We also need to use Theorem 5.2 in [3] for jumps at the tips of the comb’s teeth, where the angle is 2\( \pi \). Indeed, in the proof therein we should see \( \tilde{\Omega} \) as a Riemann surface, on whose closure we consider Perron solutions (the theory being the same to the small extent used in the proof). When applying Lemma 7.28 in Heinonen–Kilpeläinen–Martio [14], it is easy to deduce that \( \tilde{v} \) is \( p \)-subharmonic in \( \tilde{\Omega} \) as well as in \( \tilde{\Omega} \), which is enough for the rest of the proof. (It also follows that the other results in [3] are true also for asymptotic corner points with angle 2\( \pi \).)

For \( p > 2 \) we also need the following key lemma. (For simplicity we use some obvious complex notation.)

**Lemma 6.3** Assume that \( p > 2 \) and that \( f, h, E \) and \( \tilde{E} \) are as in Theorem 6.1. Let \( x_0 \in \partial_p \Psi \).

If \( x_0 \neq 0 \), then we let
\[
U(x_0 + re^{i\theta}) = A_1 + (A_2 - A_1) \frac{\theta - \alpha_1}{\alpha_2 - \alpha_1} \quad \text{for } r > 0 \text{ and } \alpha_1 < \theta < \alpha_2,
\]
where \( \alpha_1 < \alpha_2 \leq \alpha_1 + 2\pi \) are the two directions of \( \partial_p \Psi \) near \( x_0 \) chosen so that \( U \) is defined in a neighbourhood of \( x_0 \) in \( \Psi^\pm \), and \( A_j = \lim_{t \to 0^+} f(x_0 + te^{i\alpha_j}), \quad j = 1, 2. \)

If \( x_0 = 0 \) (when we do not just have two directions) we instead let \( A_1 = A_2 = U(x) = f(x_0) \) for all \( x \in \mathbb{R}^2 \).
necessary to use induction in the proof below even in the case when

Without loss of generality assume that

Proof Without loss of generality assume that $A_1 = 0 \leq A_2$ and that $x_0 \in \tilde{E} = E \cup \{0\}$. We can find a nonnegative bounded function $k : \partial_{\text{Ext}} \Psi \to \mathbb{R}$ such that

(a) $k \geq f$ on $\partial_p \Psi$;
(b) $k(x) = k(0)$ for $x \in I$;
(c) $k$ is continuous at all points in $\partial_p \Psi \setminus \{x_0\}$;
(d) $k$ is lower semicontinuous at all points in $I$;
(e) if $x_0 \neq 0$, then $k$ has a jump at $x_0$ with limits $0$ and $A_2$ and $k(x_0) = \sup_{\partial_p \Psi} k$;
(f) while if $x_0 = 0$, we require that $k(0) = \lim_{\partial_p \Psi \ni y \to 0} k(y) = 0$.

Note in particular that $k$ is upper semicontinuous at $x_0$. If $x_0 = 0$, then $k$ is even continuous at $x_0$ (but it need not be continuous at the points in $I$).

Let $z_0 \in \Psi$ and $\varepsilon > 0$. Let also $\{y_j\}_{j=0}^\infty$ be a sequence of points in $\tilde{E}$ such that each point in $\tilde{E}$ appears infinitely many times. We want to construct an increasing sequence $\{k_j\}_{j=0}^\infty$ of bounded functions on $\partial_{\text{Ext}} \Psi$ such that $k_0 = k$ and for each nonnegative integer $j$,

(i) $k_{j+1} - k_j \in C(\partial_{\text{Ext}} \Psi)$;
(ii) $k_j \leq k_{j+1} \leq k_{j} + 1$;
(iii) $\overline{P}^{\text{Ext}} k_{j+1}(z_0) \leq \overline{P}^{\text{Ext}} k_j(z_0) + 2^{-j} \varepsilon$;
(iv) $k_{j+1}(y_j) = k_j(y_j) + 1$;
(v) $k_{j+1}(x) = k_{j+1}(0)$ for $x \in I$.

We proceed by induction and assume that $k_j$ has been constructed for some nonnegative integer $j$. (The initial step is of course to let $k_0 = k$.) Let

$$
\tilde{k}_j = k_j + 2\chi_{E_j}, \text{ where } E_j = \begin{cases} \{y_j\}, & \text{if } y_j \neq 0, \\ \overline{T}, & \text{if } y_j = 0. 
\end{cases}
$$

We want to use the comparison principle (Theorem 3.3) to show that $\overline{P}^{\text{Ext}} \tilde{k}_j = \overline{P}^{\text{Ext}} k_j$. To do so we need to establish that

$$
\lim_{\Psi^{\text{Ext}} \ni y \to x} (\overline{P}^{\text{Ext}} \tilde{k}_j(y) - \overline{P}^{\text{Ext}} k_j(y)) = 0 \quad \text{for all } x \in \partial_p \Psi. \tag{6.3}
$$

If $x_0 \neq 0$, then

$$
\lim_{\Psi^{\text{Ext}} \ni y \to x_0} (\overline{P}^{\text{Ext}} \tilde{k}_j(y) - \overline{P}^{\text{Ext}} k_j(y)) = \lim_{\Psi^{\text{Ext}} \ni y \to x_0} (\overline{P}^{\text{Ext}} k_j(y) - U_j(y)) - \lim_{\Psi^{\text{Ext}} \ni y \to x_0} (\overline{P}^{\text{Ext}} k_j(y) - U_j(y)) = 0, \tag{6.4}
$$

by Theorem 5.2 in Björn [3] and Remark 6.2 (applied to both $\tilde{k}_j$ and $k_j$), where $U_j$ is the function called $U$ in Theorem 5.2 in [3] (translated to $x_0$). Note that the same function $U_j$ applies to both $\tilde{k}_j$ and $k_j$. 

\[ \square \]
Theorem 5.4 in [3] and Remark 6.2 (applied to both $\tilde{k}_j$ and $k_j$) yield
\[
\lim_{\psi_{\text{Ext}} \rightarrow y_j} \overline{P}_{\text{Ext}}^{k_j}(y) = k_j(y_j) = \lim_{\psi_{\text{Ext}} \rightarrow y_j} \overline{P}_{\text{Ext}}^{k_j}(y), \quad \text{if } y_j \notin \{0, 0\}.
\]
Moreover, if $x \in \partial_p \Psi \setminus \{x_0, y_j\}$ or if $x = x_0 = 0 \neq y_j$, then $k_j$ and $\tilde{k}_j$ are continuous at $x$, and thus, by Proposition 4.6,
\[
\lim_{\psi_{\text{Ext}} \rightarrow \tilde{k}_j}(y) = \tilde{k}_j(x) = k_j(x) = \lim_{\psi_{\text{Ext}} \rightarrow \tilde{k}_j}(y).
\]
(6.5)

It remains to handle the case when $x = y_j = 0$ for which we will use the auxiliary function $k_j' = k_j + 2\chi_{\{0\}}$. Let $u' \in \mathcal{U}_{k_j'}$. Then
\[
\liminf_{\psi_{\text{Ext}} \rightarrow 0} (u'(y) - \overline{P}_{\text{Ext}}^{k_j}(y)) \geq \liminf_{\psi_{\text{Ext}} \rightarrow 0} (\overline{P}_{\text{Ext}}^{k_j'}(y) - \overline{P}_{\text{Ext}}^{k_j}(y)) = 0
\]
for all $x \in \partial_p \Psi \setminus \{0\}$, where the equality is obtained as in (6.4) and (6.5). Also
\[
\liminf_{\psi_{\text{Ext}} \rightarrow 0} u'(y) \geq k_j'(0),
\]
while
\[
\limsup_{\psi_{\text{Ext}} \rightarrow 0} \overline{P}_{\text{Ext}}^{k_j}(y) \leq \tilde{k}_j(0) = k_j'(0),
\]
by Proposition 4.6 (applied to $-\tilde{k}_j$) since $\tilde{k}_j$ is upper semicontinuous at 0. By the comparison principle (Theorem 3.3), we see that $u' \geq \overline{P}_{\text{Ext}}^{k_j}$, and since this holds for all $u' \in \mathcal{U}_{k_j'}$, we obtain that $\overline{P}_{\text{Ext}}^{k_j'} \geq \overline{P}_{\text{Ext}}^{k_j}$. The converse inequality holds by simple comparison, and hence $\overline{P}_{\text{Ext}}^{k_j'} = \overline{P}_{\text{Ext}}^{k_j}$. Theorem 5.4 in [3] and Remark 6.2 again (this time applied to $k_j'$ and $k_j$) yield
\[
\lim_{\psi_{\text{Ext}} \rightarrow 0} \overline{P}_{\text{Ext}}^{k_j}(y) = \lim_{\psi_{\text{Ext}} \rightarrow 0} \overline{P}_{\text{Ext}}^{k_j'}(y) = k_j(0) = \lim_{\psi_{\text{Ext}} \rightarrow 0} \overline{P}_{\text{Ext}}^{k_j}(y),
\]
which finally shows (6.3) for all $x \in \partial_p \Psi$ regardless of the values of $x_0$ and $y_j$. We thus conclude that $\overline{P}_{\text{Ext}}^{k_j} \equiv \overline{P}_{\text{Ext}}^{k_j}$, by the comparison principle (Theorem 3.3).

Therefore, we can find $u \in \mathcal{U}_{k_j}$ such that
\[
u(\mathcal{z}_0) < \overline{P}_{\text{Ext}}^{k_j}(\mathcal{z}_0) + \frac{\varepsilon}{2j} = \overline{P}_{\text{Ext}}^{k_j}(\mathcal{z}_0) + \frac{\varepsilon}{2j}.
\]

Extend $u$ to $\partial_{\text{Ext}} \Psi$ by letting
\[
u(x) = \liminf_{\psi_{\text{Ext}} \rightarrow 0} u(y), \quad x \in \partial_{\text{Ext}} \Psi.
\]

Then $u$ is lower semicontinuous on $\overline{\psi}_{\text{Ext}}$ and $u \geq \tilde{k}_j$ on $\partial_{\text{Ext}} \Psi$.

As $u$ is lower semicontinuous, $k_j$ upper semicontinuous, $u \geq \tilde{k}_j = k_j + 2\chi_{E_j}$, and $E_j$ is compact, there is $r > 0$ such that
\[
u(x) > k_j(x) + 1 \quad \text{if } x \in \partial_{\text{Ext}} \Psi \text{ and dist}_{\text{Ext}}(x, E_j) < r.
\]
If $y_j \neq 0$, then we moreover require that $r < \text{dist}_{\text{Ext}}(y_j, I)$. Let
\[
k_{j+1}(x) = k_j(x) + \left(1 - \frac{\text{dist}_{\text{Ext}}(x, E_j)}{r}\right), \quad x \in \partial_{\text{Ext}} \Psi.
\]
Then \( u \geq k_{j+1} \) on \( \partial_{\text{Ext}} \Psi \). Hence \( u \in \mathcal{U}_{k_{j+1}} \) and
\[
\text{Ext}^{\text{Ext}} k_{j+1}(z_0) \leq u(z_0) < \frac{\text{Ext}^{\text{Ext}} k_j(z_0) + \varepsilon}{2^j}.
\]
That the other requirements on \( k_{j+1} \) are fulfilled is clear. We have therefore completed the construction of the sequence \( \{k_j\}_{j=0}^\infty \).

It follows directly that \( \{\text{Ext}^{\text{Ext}} k_j\}_{j=0}^\infty \) is an increasing sequence of \( p \)-harmonic functions in \( \Psi \). Let \( v = \lim_{j \to \infty} \text{Ext}^{\text{Ext}} k_j \). Since
\[
\text{Ext}^{\text{Ext}} k_j(z_0) < \text{Ext}^{\text{Ext}} k(z_0) + \varepsilon \sum_{k=0}^{j-1} 2^{-j} < \text{Ext}^{\text{Ext}} k(z_0) + 2\varepsilon,
\]
we see that \( v(z_0) \leq \text{Ext}^{\text{Ext}} k(z_0) + 2\varepsilon \leq \infty \). Harnack’s convergence theorem (see Theorem 6.14 in Heinonen–Kilpeläinen–Martin [14]) shows that \( v \) is \( p \)-harmonic in \( \Psi \). We next want to show that \( v \in \mathcal{U}_0 \). For \( x \in \partial_p \Psi \setminus E \) we have, by Proposition 4.6, that
\[
\liminf_{y \to x} v(y) \geq \liminf_{y \to x} \text{Ext}^{\text{Ext}} k(y) = k(x) \geq f(x) = h(x).
\]
On the other hand, if \( x \in I \cup \tilde{E} \setminus \{x_0\} \), then \( k_j \) is lower semicontinuous at \( x \), and thus, by Proposition 4.6,
\[
\liminf_{y \to x} v(y) \geq \liminf_{j \to \infty} \liminf_{y \to x} \text{Ext}^{\text{Ext}} k_j(y) \geq \lim_{j \to \infty} k_j(x) = \infty.
\]
Since \( x_0 \) is Ext-regular, by Proposition 4.5, and \( k_j - k \in C(\partial_{\text{Ext}} \Psi) \) we see that
\[
\liminf_{y \to x_0} \text{Ext}^{\text{Ext}} k_j(y) \geq \liminf_{y \to x_0} \text{Ext}^{\text{Ext}} (k_j - k)(y) = (k_j(x_0) - k(x_0)).
\]
Hence
\[
\liminf_{y \to x_0} v(y) \geq \lim_{j \to \infty} \liminf_{y \to x_0} \text{Ext}^{\text{Ext}} k_j(y) \geq \lim_{j \to \infty} (k_j(x_0) - k(x_0)) = \infty.
\]
Thus \( v \in \mathcal{U}_0 \), and in particular
\[
\text{Ext}^{\text{Ext}} h(z_0) \leq v(z_0) \leq \text{Ext}^{\text{Ext}} k(z_0) + 2\varepsilon.
\]
Letting \( \varepsilon \to 0 \) shows that \( \text{Ext}^{\text{Ext}} h(z_0) \leq \text{Ext}^{\text{Ext}} k(z_0) \), and as \( z_0 \in \Psi \) was arbitrary we find that \( \text{Ext}^{\text{Ext}} h \leq \text{Ext}^{\text{Ext}} k \) in \( \Psi \). It follows that
\[
\limsup_{z \to x_0} (\text{Ext}^{\text{Ext}} h(z) - U(z)) \leq \limsup_{z \to x_0} (\text{Ext}^{\text{Ext}} k(z) - U(z)) = 0,
\]
by either Theorem 5.2 in Björn [3] (if \( x_0 \neq 0 \)) or Theorem 5.4 in [3] (if \( x_0 = 0 \)). Applying this also to \(-h\) and using that \( \text{Ext}^{\text{Ext}} h \leq \text{Ext}^{\text{Ext}} k \) give (6.2) and complete the proof.

**Proof of Theorem 6.1** Without loss of generality we may assume that \( 0 \leq f \leq 2 \) and that \( f(0) = 1 \). Let
\[
k = \begin{cases} f & \text{on } \partial_p \Psi, \\ 1 & \text{on } I, \end{cases} \quad \text{and } \tilde{k} = k + \chi_{\{0\}}.
\]
Fix $x \in \partial P \setminus \{0\}$ for the moment. We first observe that it follows, from either Lemma 6.3 (if $p > 2$) or Theorem 5.2 in Björn [3] and Remark 6.2 (if $p \leq 2$), that there is a function $U_x : \Psi \to \mathbb{R}$ such that
\[
\lim_{\Psi^{\text{Ext}} \ni y \to x} (P^{\text{Ext}} f(y) - U_x(y)) = \lim_{\Psi^{\text{Ext}} \ni y \to x} (P^{\text{Ext}} k(y) - U_x(y)) = 0, \quad (6.6)
\]
\[
\lim_{\Psi^{\text{Ext}} \ni y \to x} (P^{\text{Ext}} h(y) - U_x(y)) = 0, \quad (6.7)
\]
We also need that
\[
\lim_{\Psi^{\text{Ext}} \ni y \to x} (P^{\text{Ext}} h(y) - U_x(y)) = 0, \quad (6.8)
\]
which again follows from Lemma 6.3 if $p > 2$.

To establish (6.8) for $p \leq 2$ we proceed as follows: Let $m$ be a positive integer such that $2^{1-m} < |x|$. Let also $G = ((-1, 1) \times (0, 2)) \setminus \bigcup_{j=0}^{m} \tilde{T}_j$, and let $\Psi^G$ and $\partial G^M \Psi$ be as in the proof of the general case of Theorem 5.1. We can then find a function $w : \partial G^M \Psi \to \mathbb{R}$ such that $w \geq f$ on $\partial \Psi$, $w$ is continuous at all points in $\partial G^M \Psi \setminus \{x\}$, and $w - f$ is continuous at $x$. Furthermore, let $E' \subset G^M \Psi$ be the set corresponding to $\tilde{E}$ and $\tilde{w} = w + \infty \chi_{I \cup E'}$. Next, we need to apply Theorem 7.2 in Björn [3] to the function $\tilde{w} \geq h$, but with respect to $\Psi^G$. The proof therein applies also in this case with the following remarks:

(a) The use of Theorem 5.2 in [3] is valid also in our case, see the discussion in Remark 6.2.
(b) Instead of appealing to Theorem 2.4 in [3] (which is Theorem 6.1 in Björn–Björn–Shanmugalingam [7]) we need to use Theorem 11.2 in Björn–Björn–Shanmugalingam [9] and the fact that $\overline{C}_p(E' \cup I, \Psi^G) = 0$ [which follows from (1.1) and (6.1)], where $\overline{C}_p$ is the new capacity introduced in [9].
(c) The proof in [3] is not valid for $p = 2$, but in this case the result follows more easily using linearity. (When $p = 2$ the entire Theorem 6.1 can also be deduced more easily using linearity.)

We thus obtain that
\[
\lim_{\Psi^{\text{Ext}} \ni y \to x} (P_{\Psi^G} \tilde{w}(y) - U_x(y)) = 0.
\]
By simple comparison we have $P^{\text{Ext}} h \leq P^{\text{Ext}} \tilde{w} = P_{\Psi^G} \tilde{w}$, and thus
\[
\lim \sup_{\Psi^{\text{Ext}} \ni y \to x} (P^{\text{Ext}} h(y) - U_x(y)) \leq 0.
\]
Applying this also to $-h$ and using that $P^{\text{Ext}} h \leq P^{\text{Ext}} h$ establishes (6.8) for $p \leq 2$, i.e. for all $p$.

Let next $\varphi := 2 + \infty \chi_{I \cup \Phi}(\tilde{E})$. (Note that $\varphi$ is a function on $\partial \Psi$.) Then $\varphi \geq h$ on $\partial \Psi$. By either Theorem 9.1 in Björn–Björn–Shanmugalingam [9] and (1.1) (if $p \leq 2$) or the comparison principle (Theorem 3.3) and Lemma 6.3 (if $p > 2$), $P^{\text{Ext}} \varphi \equiv P \varphi \equiv 2$. Let $u \in \tilde{U}_k$. Then, by simple comparison
\[
\lim \inf_{\Psi \ni y \to 0} u(y) \geq \tilde{k}(0) = 2 \geq \lim \sup_{\Psi \ni y \to 0} P^{\text{Ext}} \varphi(y) \geq \lim \sup_{\Psi \ni y \to 0} P^{\text{Ext}} h(y).
\]
Moreover, for $x \in \partial P \setminus \{0\}$,
\[
\lim \inf_{\Psi^{\text{Ext}} \ni y \to x} (u(y) - P^{\text{Ext}} h(y)) \geq \lim \sup_{\Psi^{\text{Ext}} \ni y \to x} (P^{\text{Ext}} k(y) - P^{\text{Ext}} h(y)) = 0.
\]
by (6.7) and (6.8). Thus, the comparison principle (Theorem 3.3) yields that \( u \geq \overline{P}^{\text{Ext}} h \).

Since \( u \in \tilde{U} \), we obtain that \( \overline{P}^{\text{Ext}} \tilde{k} \geq \overline{P}^{\text{Ext}} h \).

As \( k \) is continuous at 0 there is \( \psi \in C(\partial \Psi) \) such that \( \psi \geq k \) on \( \partial \text{Ext} \Psi \) and \( \psi(0) = 1 \). (Note that \( \psi \) is a function on \( \partial \Psi \).) Let also \( \tilde{\psi} = \psi + \chi_{\{0\}} \) so that \( \tilde{\psi} \geq \tilde{k} \) on \( \partial \text{Ext} \Psi \). Then \( P \tilde{\psi} = P \psi \) by either Theorem 6.3 in Björn [3] (if \( p > 2 \), note that we apply it to normal Perron solutions) or Theorem 6.1 in Björn–Björn–Shanmugalingam [7] (if \( p \leq 2 \)) (which can also be found as Theorem 10.29 in [5]; the more general Theorem 9.1 in Björn–Björn–Shanmugalingam [9] can also be used). We conclude, using also simple comparison, that

\[
\overline{P}^{\text{Ext}} h \leq \overline{P}^{\text{Ext}} \tilde{k} \leq \overline{P}^{\text{Ext}} \tilde{\psi} = P \psi.
\]

Hence

\[
\limsup_{\Psi \ni y \to 0} \overline{P}^{\text{Ext}} h(y) \leq \lim_{\Psi \ni y \to 0} P \psi(y) = 1,
\]

where the last equality holds because 0 is regular. Applying this to \( 2 - h \) shows that we also have

\[
\liminf_{\Psi \ni y \to 0} \overline{P}^{\text{Ext}} h(y) \geq 1,
\]

which together with the inequality \( \overline{P}^{\text{Ext}} h \leq \overline{P}^{\text{Ext}} h \) gives that

\[
\lim_{\Psi \ni y \to 0} \overline{P}^{\text{Ext}} h(y) = \lim_{\Psi \ni y \to 0} \overline{P}^{\text{Ext}} f(y) = 1.
\]

In particular this holds when \( h = f \).

For \( x \in \partial_p \Psi \setminus \{0\} \), we get from (6.6) and (6.8) that

\[
\lim_{\partial \text{Ext} \Psi \ni y \to x} \left( \overline{P}^{\text{Ext}} h(y) - \overline{P}^{\text{Ext}} f(y) \right) = \lim_{\partial \text{Ext} \Psi \ni y \to x} \left( \overline{P}^{\text{Ext}} h(y) - \overline{P}^{\text{Ext}} f(y) \right)
\]

\[
= \lim_{\partial \text{Ext} \Psi \ni y \to x} \left( \overline{P}^{\text{Ext}} h(y) - \overline{P}^{\text{Ext}} f(y) \right) = 0.
\]

Thus, the comparison principle (Theorem 3.3) yields that \( \overline{P}^{\text{Ext}} h = \overline{P}^{\text{Ext}} h = \overline{P}^{\text{Ext}} f = \overline{P}^{\text{Ext}} f \). The inequalities in (3.1) complete the proof.

\[\square\]

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