Renormalization-Group Theory of 1D quasiperiodic lattice models with commensurate approximants

Miguel Gonçalves, 1 B. Amorim, 2 Eduardo V. Castro, 3, 4 and Pedro Ribeiro, 5, 4

1 CeFEMA-LaPMET, Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais, 1049-001 Lisbon, Portugal
2 Centro de Física das Universidades do Minho e Porto (CF-UM-UP) and Laboratory of Physics for Materials and Emergent Technologies LaPMET, Universidade do Minho, Campus de Gualtar, 4710-057, Braga, Portugal
3 Centro de Física das Universidades do Minho e Porto, LaPMET, Departamento de Física e Astronomia, Faculdade de Ciências, Universidade do Porto, 4169-007 Porto, Portugal
4 Beijing Computational Science Research Center, Beijing 100193, China
5 CeFEMA, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

We develop a renormalization group (RG) description of the localization properties of one-dimensional (1D) quasiperiodic lattice models. The RG flow is induced by increasing the unit cell of subsequent commensurate approximants. Phases of quasiperiodic systems are characterized by RG fixed points associated with renormalized single-band models. We identify fixed-points that include many previously reported exactly solvable quasiperiodic models. By classifying relevant and irrelevant perturbations, we show that phase boundaries of more generic models can be determined with exponential accuracy in the approximant’s unit cell size, and in some cases analytically. Our findings provide a unified understanding of widely different classes of 1D quasiperiodic systems.

Quasiperiodic systems (QPS) offer a rich playground of exotic physics ranging from interesting localization properties, in one [1–8] or higher [9–15] dimensions, to topological non-trivial phases [16–19]. QPS are of relevance to different platforms, including optical [3, 5, 6, 20–25], cavity-polariton [4, 14, 16, 18, 26], and photonic lattices [27] and electronic moiré systems [28].

The phase diagrams of one-dimensional (1D) QPS include extended, localized, and even critical phases [29–32]. For special fine-tuned models, the phase diagram can be obtained analytically [1, 14, 29, 31, 33–37]. However, generic phase diagrams are only numerically accessible for finite systems [38–46].

Here, we develop a renormalization group (RG) theory of one-dimensional (1D) QPS, where the RG flow is induced by increasing unit cell (UC) sizes of subsequent commensurate approximants (CA). Central to our construction is the dependence of the energy bands of a CA, with UC size L, on the Bloch momentum κ (a flux thread) and on the real-space phase shift φ [Fig. 1(a)]. The energy bands are 2π-periodic in φ = Lφ and κ [47], allowing constant-energy contours to be expressed through an harmonic expansion in φ and κ whose coefficients are renormalized as L increases, defining an RG flow. We show that around criticality and for a given energy, different phases are characterized by RG transformations that flows into fixed-points corresponding to renormalized single-band models that only depend on fundamental harmonics in φ and κ [see Fig. 1(a)]. For these models, the potential (hoppings) become irrelevant at an extended (localized) fixed-point, or both the potential and hoppings are relevant at any scale, signaling a critical fixed-point. Within our theory models are classified as: (i) fixed-point, when the renormalized single-band models are the simplest possible for any CA, only containing fundamental harmonics in φ and κ; (ii) asymptotic, when the latter limit is only approached as the UC size is increased.

Illustrative toy model.— Consider the tight-binding chain $H = -t \sum_j \left( c_j^\dagger c_{j+1} + \text{h.c.} \right) + \sum_j V_j c_j^\dagger c_j$, where

$$V_j = V \cos(2\pi \tau_j + \phi) + V_2 \cos(2(2\pi \tau_j + \phi)).$$

This is the Aubry-André model (AAM) [1] with an additional quasiperiodic second harmonic potential of strength $V_2$. For the irrational τ, we take a series of CAs $\{\tau^{(n)}_c\}$, $n = 1, \ldots, \infty$ defined by $\tau^{(n)}_c = p/L (p, L$ coprime integers), with increasing UC size $L$. We focus on how the characteristic polynomial, $P^{(n)}(\varphi, \kappa) = \det[H^{(n)}(\varphi, \kappa) - E]$, where $H^{(n)}(\varphi, \kappa)$ is the Bloch Hamiltonian for a CA with $\tau^{(n)}_c$, depends on $\kappa$ and $\varphi$. Invariance under $\kappa \to \kappa + 2\pi (\varphi \to \varphi + 2\pi)$, implies that $P^{(n)}$ depends on $\kappa (\varphi)$ only via $\cos(\varphi) [\cos(\varphi)]$ or higher harmonics. For the pure AAM ($V_2 = 0$), only fundamental harmonics in $\varphi$ and $\kappa$ show up,

$$P^{(n)}(\varphi, \kappa) = t^{(n)}_R \cos(\kappa) + V^{(n)}_R \cos(\varphi) + T_R,$$

where $t^{(n)}_R = -2tL$ and $V^{(n)}_R = (-2)^{1-L}V^L$ are the renormalized couplings for the n-th order CA, and $T_R$ is a term independent of $\kappa$ and $\varphi$. As $L \to \infty$, there are three fixed-points for the asymptotic behaviour of $|V^{(n)}_R/t^{(n)}_R|$:

(i) $|V^{(\infty)}_R/t^{(\infty)}_R| = 0$ for $|V| < 2|t|$ (extended, renormalized potential becomes irrelevant); (ii) $|V^{(\infty)}_R/t^{(\infty)}_R| = \infty$...
FIG. 1. (a-left) CA of a quasiperiodic chain. The UC is threaded by a flux $\kappa$, and $\phi$ is the shift between the potential (red) and the lattice sites (blue). The periodicity of the energy with $\phi = \phi/L$ is shown below. (a-right) Effective single-band models to which the constant-energy contours of $E(\phi, \kappa)$ flow as $L$ increases: extended, with no renormalized potential; localized, with no hoppings; and critical, with both hoppings and potential. (b-c) IPR for the model in Eq. (1) with localized, with no hoppings; and critical, with both hoppings and potential. (b-c) IPR for the model in Eq. (1) with $t = 1$ and $\tau_c = 610/987$. Phase boundary computed analytically for $\tau_c = 2/3$ in (b) [full black line], and numerically for $\tau_c = 55/89$ in (c) [black points]. Inset (c) - sketch of the RG flow close to the repulsive fixed-point $V_{2R}/t_R = 1$. (d) RG flow for the sequence $\tau_c = 1, 1/2, 2/3, 3/4, 5/6, 7/8$ [color coded by starting points in (b)]. (e) Generalized Fermi surfaces for $\tau_c = 8/13$ and $\tau_c = 55/89$, for the parameters denoted as cyan points in (c).

for $|V| > 2|t|$ (localized, renormalized hopping becomes irrelevant); and (iii) $|V^R_{c(\infty)}|^2 = 1$ for $|V| = 2|t|$ (critical, both renormalized potential and hopping are relevant at any scale).

For $V_2 \neq 0$, $P^{(n)}$ is given by Eq. 2 with an additional second harmonic in $\phi$, $V_{2R}^C \cos(2\phi)$, where $V_{2R}^C = 2^{1-L}V_{2R}$, and $V_2$ dependent $t_{R}^{(n)}$, $V_{R}^{(n)}$ (see [48] for explicit expressions). The correlation lengths in the extended ($\xi_c$) and localized ($\xi_l$) phases are determined by $V_{R}^{(n)}/t_{R}^{(n)} = e^{-L/\xi_c}$ and $t_{R}^{(n)}/V_{R}^{(n)} = e^{-L/\xi_l}$, which for $V_2 = 0$ yields well-known results. The dependence of $V_{2R}^{(n)}$ with $L$ shows that a small $V_2$ introduces an irrelevant perturbation, since the fixed point of the $L \to \infty$ flow is the same as the AAM. Remarkably, the phase boundary changes smoothly and can be analytically approximated using a CA with small UC, as shown in Fig. 1(b). An example RG flow is shown in Fig. 1(d), color coded by starting points in Fig. 1(b).

A small $V$ term added to the $V_2 \neq 0$ model, however, represents a relevant perturbation. This is shown in the inset of Fig. 1(c) close to the $V = 0$ critical point at $|V_{2R}| = |t_R|$. The phase boundaries only converge when $L$ is large enough so that the model flows to the effective single-band fixed-points. Since $L$ can be significantly large, a complex structure of mobility edges can arise, as shown in Fig. 1(c) (black dashed line). Figure 1(e) shows a level cut of the eigen-energies, $E^{(n)}(\kappa, \phi)$, of the Bloch Hamiltonian of a CA. For $L \to \infty$, a localized (delocalized) fixed point is characterized by the independence of the $E^{(n)}(\kappa, \phi)$, whereas $E^{(n)}(\kappa, \phi) = E^{(n)}(\phi, \kappa)$ at the critical point. While the contribution of $V_{2R}$ is still clear for $L = 13$, it becomes negligible for $L = 89$.

This example illustrates the universal behaviour we observe in all studied models (see also the Supplementary Material (SM) [48]): around criticality, only the fundamental harmonics in $\phi$ and $\kappa$ survive as $L \to \infty$, and the nature of the perturbation (relevant or irrelevant) determines how the phase diagram evolves from the unperturbed case. In particular, we can conclude whether there is one continuously connected mobility edge, as in the example of an irrelevant perturbation shown in Fig. 1(b), or a complex structure of mobility edges, as in the example of a relevant perturbation shown in Fig. 1(c).

In the SM [48] we provide a more in-depth pedagogical study of the simple model in Eq. 1, with additional next-nearest-neighbor hoppings.

**General theory.**— Consider a sequence of Bloch Hamiltonians $H^{(n)}(\phi, \kappa)$ depending on $\phi$ and $\kappa$, for each CA with $\tau_c^{(n)}$. We classify models for which $P^{(n)}$ at energy $E$ may be written as [49]

$$P^{(n)}(E, \phi, \kappa) \equiv \det [H^{(n)}(\phi, \kappa) - E]$$

1. $t_R^{(n)}(E) \cos(\kappa + \kappa_0) + V_R^{(n)}(E) \cos(\phi + \phi_0) + \epsilon_R^{(n)}(E, \phi, \kappa) + T_R^{(n)}(E),$(3)

where $\epsilon_R^{(n)}$ contains all the terms that depend on $\phi$ and $\kappa$, but are not fundamental harmonics, and $T_R^{(n)}(E)$ includes all terms independent of $\phi$ and $\kappa$. Note that $\kappa_0, \phi_0, \kappa'_0, \phi'_0$ may also depend on $E$. Our central conjecture is that close enough to phase boundaries, the term $\epsilon_R^{(n)}$ becomes irrelevant with respect to the fundamental couplings $t_R^{(n)}$, $V_R^{(n)}$ and $C_R^{(n)}$. As $n$ increases, the different phases are then fully characterized:

- Extended: $|C_R^{(n)}|/|t_R^{(n)}|, |V_R^{(n)}|/|t_R^{(n)}| \to 0$
- Localized: $|C_R^{(n)}|/|V_R^{(n)}|, |t_R^{(n)}/V_R^{(n)}| \to 0$
- Critical: $|C_R^{(n)}|/|V_R^{(n)}|, |t_R^{(n)}/V_R^{(n)}| \geq 1$. (4)
The picture in the $n \to \infty$ limit is the following: in the extended (localized) phase only the renormalized hopping (potential), coupled to the $\kappa$-$\phi$-dispersion, is relevant – the energy dispersions flow to the effective extended (localized) single-band model illustrated in Fig. 1(a). In the critical phase, both the $\kappa$- and $\phi$-dispersions are relevant, being characterized by a flow into the critical single-band model of Fig. 1(a). The conditions for the different phase boundaries, including extended-to-localized (E-L), critical-to-extended (C-E), and critical-to-localized (C-L), can be summarized as follows:

$$\text{E-L: } |V_R^{(n)}|/t_R^{(n)} \to 1 \lor |V_R^{(n)}/C_R^{(n)}|, |t_R^{(n)}/C_R^{(n)}| \to 0$$

$$\text{C-E: } |C_R^{(n)}|/t_R^{(n)} \to 1 \land |C_R^{(n)}/V_R^{(n)}| \geq 1$$

$$\text{C-L: } |C_R^{(n)}/V_R^{(n)}| \to 1 \land |C_R^{(n)}/t_R^{(n)}| \geq 1.$$  

(5)

Based on the nature of the $\epsilon_R^{(n)}$ term in Eq. 3, we can classify the 1D QPS in two different groups: fixed-point models, with $\epsilon_R^{(n)}(E, \phi, \kappa) = 0 \forall n$, and asymptotic models, with $\epsilon_R^{(n)} \to 0$ faster than the fundamental couplings $t_R^{(n)}, V_R^{(n)}, C_R^{(n)}$ (as long as they are finite), close to criticality when $n \to \infty$. Remarkably, for fixed-point models the phase boundaries in the limit of $n \to \infty$ (quasiperiodic-limit) may be obtained through the lowest-order CA. As shown below, the solvable models reported in Refs. [1, 29, 34, 37, 50] are all examples of fixed-point models.

**Fixed-point models.**— To exemplify the application of the general description to models beyond the AAM [1], we consider the mosaic models of Ref. [50]. These are tight-binding chains with the same potential as in the AAM, $V_j = V \cos(2\pi j \tau + \phi)$, but only for $j/p \in \mathbb{N}$, with $V_j = 0$ otherwise. We consider the case $p = 3$, for which the simplest possible CA has 3 sites in the UC. The $n^\text{th}$-order CA has $L = 3n$ sites in the UC and $|V_R^{(n)}/t_R^{(n)}| = (|V/2||E^2 - 1|)^n$. The mobility edge is simply defined by $f_{E,V} = |V/2||E^2 - 1| = 1$, in complete agreement with [50]. The correlation lengths are then $\xi_\kappa = 3/\log(f_{E,V}^{-1})$ and $\xi_\ell = 3/\log(f_{E,V})$.

We now consider the following Hamiltonian

$$H = \sum_n \left[ t(a_{n+1}a_n + a_n^\dagger a_{n+1}) + h.c. + \frac{2V \cos(2\pi \tau n + \phi)}{1 + \alpha \cos(2\pi \tau n + \phi)} a_n^\dagger a_n \right].$$

(6)

This is a fixed-point model, which generalizes models previously studied in Refs. [31, 37]. The phase boundary between localized and delocalized regions is given by $E = \left(V \pm \sqrt{V^2 + 2t^2 - 2E^2}\right)/\alpha$, while the critical phase is defined by $|\alpha(E^2 - 2t^2) - 2EV| \leq |2t\alpha| \land |\alpha| \geq 1$ [48]. The analytical phase boundaries, shown in Fig. 2, are in perfect agreement with the numerical results for the IPR and IPR$_k$ [51]. In the extended (localized) phase, the IPR(IPR$_k$) scales as $N^{-1}$, with $N$ the total number of sites, while in the localized (extended) phase it is $N$-independent. At a critical point or phase, the wavefunction is delocalized in real and momentum-space and both the IPR and IPR$_k$ scale down with $N$ [52], as seen in Fig. 2(b). In the SM [48] we also study an additional simpler generalization of the models in Refs. [31, 37] and again simply unveil analytically the phase diagram, also including critical phases.

In the SM [48], we also build different classes of fixed point models. One such class includes the models in Refs. [1, 29, 34, 37, 50]. For this class, we evaluated the correlation length critical exponent $\nu$, obtaining $\nu_{\text{eL}} = 1$ at the extended-localized transition, and $\nu_{\text{eL}(1:3)} = 1/2$ at extended-critical and localized-critical transitions. We also show that our theory can be applied to even more complex cases, by studying a non-abelian Aubry-André model [53]. We do not obtain all the phase boundaries analytically for the first time, to our knowledge, but also analytically compute the correlation length exponent for the localized-critical and extended-critical transitions to be $\nu = 1$ for this model (which was recently numerically calculated for the critical-localized transition [54]). This implies that such transitions belong to a different universality class than the analogous transitions that take place in the abelian class of models just mentioned (for which $\nu = 1/2$), which can be traced back to a clear different scaling behaviour of the renormalized couplings.

**Asymptotic models.**— Consider the model in Eq. (1), with an additional second-nearest-neighbor hopping term of amplitude $t_2$. With respect to the AAM, the small $t_2$ and $V_2$ are irrelevant perturbations as they do not break the original periodicity in $\varphi$ and $\kappa$ and do not change the unit cell. The higher-order harmonics of $P^{(n)}$ have a very simple expression, with $\epsilon_R^{(n)}$ given by

$$\epsilon_R^{(n=L)}(t_2, V_2, \varphi, \kappa) = 2t_2^L \cos(2\kappa) + 2^{n-L} V_2^L \cos(2\varphi).$$

(7)

For small $t_2$ and $V_2$, this is an example where $\epsilon_R^{(n)} \to 0$ exponentially faster than the fundamental couplings as
\( n \to \infty \), as shown in Fig. 1(d) for \( t_2 = 0 \). Therefore, phase boundaries can be accurately predicted by setting \( \epsilon_R^{(n)} = 0 \) [55]. Other terms generating higher order harmonics in \( \varphi \) and \( \kappa \) are also irrelevant, as illustrated in Fig. 3(a) for a model with third-nearest-neighbour hopping \( t_3 \) and potential \( V_4 \cos(2(2\pi j + \phi)) \). The phase boundaries were analytically estimated for a CA with \( L = 4 \) sites in the UC. In Fig. 3(c) we show examples of RG flows for small perturbations and CA up to \( L = 6 \) sites in the unit cell. Using the method just described, in the SM [48] we also analytically capture for the first time a reentrant localization transition unveiled in Ref. [8] (a similar transition was found in Ref. [45]).

Relevant perturbations break the original periodicity in \( \varphi \) and \( \kappa \) of the model. An example of a relevant perturbation is the inclusion of finite \( V_4 \), \( t \) to a model with \( V_2, t_2 \neq 0 \) and \( V, t = 0 \) [56]. In Fig. 3(d), the generalized Fermi surface curves for \( V, t = 10^{-4} \), obtained for \( \tau_c = 34/55 \) (top row), qualitatively differ from those obtained for a larger CA, \( \tau_c = 144/233 \) (bottom row), where only the fundamental harmonics are already visible [48]. The non-trivial transition lines, shown in Fig. 3(b), can still be well approximated using a large enough UC (in this case \( \tau_c = 144/233 \)).

Finally, to illustrate the effects of irrelevant and relevant perturbations on critical phases, we consider a tight-binding chain with potential

\[
V_j = \frac{1}{1 + \alpha \cos(2\pi \tau j + \phi)} + V \cos[(2\pi \tau j + \phi)/q].
\]   

For \( V = 0 \), this model reduces to the one of Refs. [31, 37]. For \( q = 1 (q = 2) \), \( V \) is an irrelevant (relevant) perturbation. An excellent analytical approximation to the phase diagram, including the critical phases, is obtained for the irrelevant perturbation, as shown in Figs. 4(a,c). In Fig. 4(b) we show RG flows in the plane \( t_R/C_R - V_2R/C_R \), where \( \tau_c = 1, 1/2, 2/3, 3/5, 5/8 \). These results are energy independent (only \( V_R \), not shown, depends on energy). (c) Finite-size scaling of IPR and IPRT for the path shown as a dotted line in (a). (d) IPR for \( \tau_c = 610/987 \) and \( \nu = 2 \) (relevant perturbation), with \( V = 0.05 \).

Discussion.— We devise a RG scheme that allows us to identify fixed-point models of 1D QPS described by
their lowest order CA and that include many previously reported exactly solvable examples [1, 31, 34, 37, 50].

Furthermore, we identify asymptotic models whose renormalized couplings flow to one of the fixed-point cases. Here, irrelevant perturbations give rise to higher harmonics in phase shift $\varphi$ and twist angle $\kappa$ whose couplings vanish at the fixed-point. Conversely, perturbations corresponding to lower harmonics are generally relevant and drive the RG flow to a different fixed-point. Hence, in the thermodynamic limit, the properties of the model are determined by the stability of the fixed-point models under different perturbations.

Analyzing the flow of asymptotic models, we show that phase boundaries can be determined to an exponential accuracy in the size of the CA since renormalized irrelevant couplings decay exponentially. The remarkable consequence is that for this broad class of models, the phase diagrams smoothly evolve from the exactly-solvable (fixed-point) cases and can be analytically approximated in a controlled way.

Consequently, our RG scheme explains both why (i) the phase diagram evolves smoothly from the unperurbed case when new irrelevant couplings are added to a model; (ii) phase boundaries acquire a complex fine-structure if the added couplings are relevant. This picture also provides insight into the stability of critical phases, which are typically fragile.

By applying our theory to different models, we unveiled multiple novel analytical results, some capturing interesting recently discovered phenomena [8, 53, 54]. Additionally, using the insights from our theory, we could even create new models with analytically solvable phase diagrams, showing peculiar features including anomalous mobility edges between critical and localized phases, with energy-dependent reentrant behaviours [Figs. 2(a), 4(a)].

More generically, our theory offers a qualitative and quantitative understanding of the localization properties of widely different 1D QPS. It closes the gap between seemingly disconnected results in the literature, provides clear working criteria to create novel models with tailored properties, and is a promising asset to unveil novel universality classes of localization transitions.

MG and PR acknowledge partial support from Fundação para a Ciência e Tecnologia (FCT-Portugal) through Grant No. UID/CTM/04540/2019. BA, EVC and MG acknowledge partial support from FCT-Portugal through Grant No. UIDB/04650/2020. MG acknowledges further support from FCT-Portugal through the Grant SFRH/BD/145152/2019. BA acknowledges further support from FCT-Portugal through Grant No. CEECIND/02936/2017. We finally acknowledge the Tianhe-2JK cluster at the Beijing Computational Science Research Center (CSRC), the Bob/Macc supercomputer through computational project project CPCA/A1/470243/2021 and the OBLIVION supercomputer, through project through project HPCUE/A1/468700/2021 (based at the High Performance Computing Center - University of Évora) funded by the ENGAGE SKA Research Infrastructure (reference POCI-01-0145-FEDER-022217 - COMPETE 2020 and the Foundation for Science and Technology, Portugal) and by the BigData@UE project (reference ALT20-03-0246-FEDER-000033 - FEDER and the Alentejo 2020 Regional Operational Program. Computer assistance was provided by CSRC, CENTRA/IST and the OBLIVION support team.

[1] S. Aubry and G. André, Proceedings, VIII International Colloquium on Group-Theoretical Methods in Physics 3 (1980).
[2] M. Wilkinson and M. V. Berry, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 391, 305 (1984).
[3] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature 453, 895 (2008), arXiv:0804.2609.
[4] Y. Lahini, R. Pugatch, F. Pozzi, M. Sorel, R. Morandotti, N. Davidson, and Y. Silberberg, Physical Review Letters (2009), 10.1103/PhysRevLett.103.013901.
[5] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lütschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science 349, 842 (2015), arXiv:1501.05661.
[6] H. P. Lütschen, S. Scherg, T. Kohlert, M. Schreiber, P. Bordia, X. Li, S. Dasarma, and I. Bloch, Physical Review Letters (2018), 10.1103/PhysRevLett.120.160404.
[7] D. S. Borgia, A. Vishwanath, and R.-J. Slager, “Rational approximations of quasi-periodic problems via projected green’s functions,” (2021).
[8] A. Padhan, M. K. Giri, S. Mondal, and T. Mishra, Phys. Rev. B 105, L220201 (2022).
[9] C. Huang, F. Ye, X. Chen, Y. V. Kartashov, V. V. Konotop, and L. Torner, Scientific Reports 6, 32546 (2016).
[10] J. H. Pixley, J. H. Wilson, D. A. Huse, and S. Gopalakrishnan, Phys. Rev. Lett. 120, 207604 (2018).
[11] M. J. Park, H. S. Kim, and S. Lee, Phys. Rev. B 99, 245401 (2019), arXiv:1812.09170.
[12] B. Huang and W. V. Liu, Phys. Rev. B 100, 144202 (2019).
[13] Y. Fu, E. J. König, J. H. Wilson, Y.-Z. Chou, and J. H. Pixley, npj Quantum Materials 5, 71 (2020).
[14] P. Wang, Y. Zheng, X. Chen, C. Huang, Y. V. Kartashov, L. Torner, V. V. Konotop, and F. Ye, Nature (2020), 10.1038/s41586-019-1851-6, arXiv:2009.08131.
[15] M. Gonçalves, H. Z. Olyaei, B. Amorim, R. Mondaini, P. Ribeiro, and E. V. Castro, 2D Materials 9, 011001 (2021).
[16] Y. E. Kraus, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Physical Review Letters (2012), 10.1103/PhysRevLett.109.06402, arXiv:1109.5983.
[17] Y. E. Kraus and O. Zilberberg, Phys. Rev. Lett. 109, 116404 (2012).
Note that this periodicity also holds if the quasiperiodicity is in hopping (and not on-site) terms, see [59] and [48] for detailed examples. See supplemental material that includes detailed examples for fixed-point and asymptotic models; analytical calculations of correlation length critical exponents across different transitions; similarities between three models with exact phase diagrams in the literature, made clear within the RG framework; more on fixed-point models with critical phases; a generalization of the duality transformation in SciPost Phys. 13, 046 (2022) to include critical phases; and an analytical approximation of the reentrant transition obtained in PRB 105, L220201.

The full characteristic polynomial may factorize into products of characteristic polynomials, each one written in the form of Eq. 3 (see [48] for examples). In this case, each sector has to be studied independently as the energy bands associated to each polynomial are decoupled. Nonetheless, for each sector we can always write $\mathcal{P}$ in the form of Eq. 3.

Note that one should always check whether $\epsilon_R$ is considerably smaller that the renormalized couplings of the fundamental harmonics, before making predictions for the phase diagram under that assumption. For a given CA, it may not be accurate for all regions of the phase diagram, if the value of the perturbation becomes significant. Away from criticality, $\epsilon_R$ may even not become irrelevant as $L \to \infty$.

This system in only well defined for a UC with an even number of sites, in which case it consists of two decoupled models; the results remain unchanged.
[65] Note that the solutions for Eqs. S97 that match true dual points should satisfy \( \lim_{E \to E_c, \lambda \to \lambda_c} E'(E, \lambda) = E_c \) and \( \lim_{E \to E_c, \lambda \to \lambda_c} \lambda'(E, \lambda) = \lambda_c \) at self-dual points \((E_c, \lambda_c)\). There may be other solutions corresponding to points with equal Fermi surfaces \( E(\varphi, \kappa) = E \) at different energy bands. These do not satisfy the previous self-duality conditions.

Supplemental Material for:

Renormalization-Group Theory of 1D quasiperiodic lattice models with commensurate approximants

CONTENTS

Acknowledgments 5

References 5

S1. Fixed-point and asymptotic models: A simple example  SM - 2
  S1.1. Fixed-point models  SM - 2
    \( V = t_2 = 0 \) (\( t - V_2 \) model)  SM - 3
    \( V_2 = t = 0 \) (\( t_2 - V \) model)  SM - 3
    \( V = t = 0 \) (\( t_2 - V_2 \) model)  SM - 4
  S1.2. Asymptotic models: irrelevant perturbations  SM - 5

S2. Some classes of fixed-point models  SM - 6
  S2.1. Aubry-André model  SM - 6
  S2.2. A more general class of fixed-point models  SM - 8
  S2.3. Alternative fixed-point models  SM - 13

S3. Calculating the critical exponent \( \nu \) across different transitions  SM - 17
  S3.1. Class of fixed-point models in Sec. S2 S2.2  SM - 17
  S3.2. Model in Sec. S2 S2.3  SM - 18

S4. Similarities between three models in the literature  SM - 18

S5. More on fixed-point models with critical phases  SM - 20

S6. Dual points: generalization to include critical phases  SM - 21

S7. Reentrant localization transition in PRB 105, L220201 (2022)  SM - 22
S1. FIXED-POINT AND ASYMPTOTIC MODELS: A SIMPLE EXAMPLE

In this section, we study in more detail a simple model to illustrate important concepts introduced in the main text, including the differences between fixed-point and asymptotic models, and between relevant and irrelevant perturbations. We consider the following Hamiltonian:

\[ H = t \sum_n (c_n^\dagger c_{n+1} + \text{h.c.}) + t_2 \sum_n (c_n^\dagger c_{n+2} + \text{h.c.}) + \sum_n \left[V \cos(2\pi \tau_n + \phi) + V_2 \cos(2[2\pi \tau_n + \phi])\right] c_n^\dagger c_n \] (S1)

This model was also considered in the main text, containing additional third-nearest-neighbor hoppings and a quasiperiodic term of the type \( V_3 \cos(3[2\pi \tau n + \phi]) \). Here we drop these terms for simplicity, but the discussion can be generalized to this and more complicated models. The characteristic polynomial for any CA can be written as

\[ P(\varphi, \kappa) = t_R^{(n)} \cos(\kappa) + V_R^{(n)} \cos(\varphi) + V_{2R}^{(n)} \cos(2\varphi) \] (S2)

where \( \varphi = L\phi \), for a CA defined by \( \tau_c = p/L \) (\( p \) and \( L \) co-primes). Below, we provide the renormalized coefficients for CA up to 5 sites in the UC:

| \( \tau_c = p/L \) | \( t_R \) | \( V_R \) | \( t_{2R} \) | \( V_{2R} \) |
|----------------|---------|---------|---------|---------|
| 1              | 2t      | \( V \) | 2t_2    | \( V_2 \) |
| 1/2            | \(-2(t^2 + 2Et_2)\) | \(-\frac{1}{2t}(V^2 + 4EV_2)\) | 2t_2^2  | \( \frac{1}{2t}V_2^2 \) |
| 1/3; 2/3       | \(2t[t^2 + 3t_2(E + t_2)]\) | \(\frac{1}{2t}V(V^2 + 6VE + 3V_2^2)\) | 2t_2^3  | \( \frac{1}{2t^2}V_3^2 \) |
| 1/4; 3/4       | \(-2[t^4 + 4t^2t_2(E + t_2)] + 5t^2(2E^2 - 4t_2^2)\) | \(\frac{1}{2t}(-V^4 - 8E^2V_2 + 16t_2^2V_2^2)\) | 2t_2^4  | \( \frac{1}{2t^2}V_4^2 \) |
| 1/5; 4/5       | \(2t[t^4 + 5t^2(E + t_2)] + 5t_2^2(2E^2 + Et_2 - t_2^2) + 5V_2^2(4E^2 - 2(\sqrt{5} + 1)t_2^2)\) | \(\frac{1}{2t}V(V^4 + 5V^2V_2(2E + V_2))\) | 2t_2^5  | \( \frac{1}{2t^2}V_5^2 \) |

S1.1. Fixed-point models

We first note the cases for which the model is fixed-point. In the following discussion, we present the results in units of the lattice constant \( a \) (see Fig. S1). Note that \( k \) is always the momentum measured in units of \( a \).
FIG. S1. Definition of the lattice constants. $a$ is the lattice constant for $t \neq 0$, while $a'$ is the lattice constant for $t = 0$. In the latter case, the system decouples into two chains.

\[ V = t_2 = 0 \ (t - V_2 \ model) \]

In this case, if $q$ is odd, $V_R = 0$ and only the second harmonic survives. This is a manifestation of $V$ being a relevant perturbation to the $t - V_2$ model: for these CA the unit cell is unchanged but a lower harmonic in $\phi$ is introduced, breaking original the $\phi$-periodicity. For even $L$, the size of the unit cell changes with respect to the $V \neq 0$ case: it contains $L/2$ sites instead of $L$. This can be easily understood because only the potential term $V_2 \cos(2[2\pi \tau_c n + \phi])$ enters the Hamiltonian. The argument of the cosine contains $2\tau_c = 2p/L = p/(L/2)$. If $L$ is even, the irreducible fraction occurs for $L' = L/2$, which sets the size of the UC. This is why $V_R$ is finite for even $L$ in Eq. S3 even when $V = 0$: in this limit the calculation does not consider the true UC which gives rise to a band flattening. As an example, let us take the $\tau_c = 1/2$ case for $t_2 = 0$, for which the Hamiltonian matrix is

\[
\mathcal{H}(\phi, k) = \begin{pmatrix}
 V \cos(\phi) + V_2 \cos(2\phi) & 2t \cos(k) \\
 2t \cos(k) & -V \cos(\phi) + V_2 \cos(2\phi)
\end{pmatrix}
\]

(S4)

Of course, for $V = 0$, the diagonal elements are equal, which indicates that the unit cell has a single site in that case. It is clear that the characteristic polynomial of the above Hamiltonian matrix will have both $\cos(2\phi)$ and $\cos(4\phi)$ terms for $V = 0$. The correct Hamiltonian matrix for $V = 0$ should be

\[
\mathcal{H}(\phi, k) = 2t \cos(k) + V_2 \cos(2\phi)
\]

(S5)

The relevance of the $V \neq 0$ perturbation therefore manifests either in the breaking of the original $\phi$ periodicity if the UC size is unchanged (odd $L$), or in the change of the unit cell (even $L$). The duality symmetries for $V = 0$ and $V \neq 0$ are therefore clearly different.

\[ V_2 = t_2 = 0 \ (t_2 - V \ model) \]

For $t = 0$ the system consists of two decoupled chains, that we label $A$ and $B$ with sites separated by $a' = 2a$ (see Fig. S1). The quasiperiodic potential for each chain, in the usual units with $a = 1$, is now a term $V \cos[2\pi \tau(2n_A + \phi)]$, where $n_A$ and $n_B$ are respectively the site indexes in chains $A$ and $B$. This case is special because the characteristic polynomial decouples into products of polynomials corresponding to each decoupled chain as we will see below.
The smallest-order CA of this system has one site per UC for each chain and can be represented by the following diagonal Hamiltonian matrix (using \( \tau_c = 1 \)):

\[
H_{t=0}(\phi, k) = [2t^2 \cos(2k) + V \cos(\phi)]I_{2 \times 2}
\]  

(S6)

The characteristic polynomial of course factorizes and is given by

\[
P(\phi_1, k_1, \phi_2, k_2) = \prod_{i=1}^{2} P(\phi_i, k_i)
\]  

(S7)

\[
P(\phi_i, k_i) = 2t^2 \cos(2k_i) + V \cos(\phi_i) - E
\]  

(S8)

The well-known Aubry-André critical point, \( |V| = 2|t_2| \), naturally follows if we define \( \kappa_i = 2k_i \) and \( \phi_i = \phi_1 \). The characteristic polynomials for higher-order CA will of course be products of equal polynomials, each corresponding to one Aubry-André chain. For \( t \neq 0 \) we get

\[
H_{t \neq 0}(\phi, k) = \begin{pmatrix} 2t^2 \cos(2k) + V \cos(\phi) & 2t \cos(k) \\ 2t \cos(k) & 2t^2 \cos(2k) + V \cos(\phi) \end{pmatrix}
\]  

(S9)

However, in this case we are not specifying the correct UC. In fact, it has only one site and the energy bands of the Hamiltonian matrix above correspond to a band folding of the single energy band of the correct Bloch Hamiltonian, given by

\[
H_{t \neq 0}(\phi, k) = 2t \cos(k) + 2t^2 \cos(2k) + V \cos(\phi)
\]  

(S10)

Note that the change in the UC between \( t = 0 \) and \( t \neq 0 \) for the same \( \tau_c \) is clearly a manifestation of \( t \) being a relevant perturbation.

\[
V = t = 0 \ (t_2 - V_2 \ \text{model})
\]

This is the most dramatic case, for which both \( V \) and \( t \) are relevant perturbations. The model again consists of two uncoupled chains. \( t \neq 0 \) is clearly relevant because the UC is changed. \( V \neq 0 \) also is because either the UC or the periodicity in \( \phi \) is changed.

The simplest CA, for \( \tau_c = 1 \), corresponds to

\[
H(\phi, k) = [2t^2 \cos(2k) + V_2 \cos(2\phi)]I_{2 \times 2}
\]  

(S11)

In this case, switching on \( V \) changes the periodicity in \( \phi \), adding an additional term \( V \cos(\phi) \). As in this case, we have a term \( V_2 \cos[4\pi \tau_c(2n_A(B)) + \phi] \), the \( V = 0 \) results are the same for \( \tau_c = 1/2 \) and \( \tau_c = 1/4 \). One can also check for instance that the lowest-order CA for odd \( L \) has \( 2L \) sites, \( L \) for each chain.

Another way to see that \( V \) and \( t \) are relevant perturbations is to consider CA with an odd number of sites, as show in the main text. Such CA are only well defined when \( t \neq 0 \). For \( t = 0 \) every CA should have an even number of sites: for each site in one decoupled chain there is a pair site in the other. However, for \( t = \epsilon \), with \( |\epsilon| \ll |t_2| \), CA with odd number of sites become well-defined. If we choose \( t = V = \epsilon \), with \( |\epsilon| \ll |t_2|, |V_2| \), we see from Eq. S3 that the renormalized couplings \( V_R \) and \( t_R \) are very small for low-order CA. However, as the size of the UC is increased, the fundamental harmonics should become dominant near criticality, which implies that \( V_{2R} \) and \( t_{2R} \) will become irrelevant.
S1.2. Asymptotic models: irrelevant perturbations

For small $t_2$ and $V_2$, the couplings $V_{2R}$ and $t_{2R}$ quickly become very small with respect to $V_R$ and $t_R$, as the UC is increased. As long as this is the case, the duality in the fundamental harmonics of $\varphi$ and $\kappa$ may be already almost perfect even for small UC, becoming exponentially better as the UC is increased. For instance, for $\tau_c = 2/3$, the analytical approximation for the mobility edge is (solving $|V_R| = |t_R|$):

$$E_c = \frac{-8t^3 - 24tt^2 \pm V^3 \pm 3VV^2}{24tt^2 \mp 6VV^2} \quad \text{(S12)}$$

which for $t_2 = 0$ corresponds to the analytical curve plotted in Fig.1 of the manuscript. Note that there are two important factors that may affect the accuracy of the analytical approximation. In fact, for a more accurate approximation, we should have:

1. Smaller ratio between the irrelevant and relevant renormalized couplings;
2. Smaller difference between the rational approximant $\tau_c$ and the irrational $\tau$, $|\tau_c - \tau|$.

Note that the second factor is particularly important if the phase boundaries have a strong dependence on $\tau$. In this case, even if the irrelevant renormalized couplings are negligible for a given $\tau_c$, the obtained phase boundaries may not be fully correct if $\tau_c$ is not close enough to $\tau$. Of course, if we choose an irrational $\tau$ very close to $\tau_c$, the approximation should be very accurate regarding that point 1 is satisfied.

In Fig. S2 we show examples of the obtained phase boundaries for low-order CA and the ratio between the irrelevant and relevant couplings. Note that when $V_2$ and $t_2$ are increased, the phase boundaries start to depend significantly on $\tau_c$, even when the irrelevant couplings are negligible. This indicates a true dependence of the phase boundaries of the limiting QPS on $\tau$.

We finish this section by emphasizing that it is always important to make sure that the irrelevant couplings are negligible with respect to the relevant ones in order to make accurate analytical...
predictions of the phase boundaries. In some cases, when the perturbation starts to be significant, the renormalized terms $V_R$ and $t_R$ may become small and higher-order approximants may be needed to describe some phase boundaries.

S2. SOME CLASSES OF FIXED-POINT MODELS

In this section we discuss fixed-point models for which the fundamental renormalized couplings may be computed analytically for any CA. We will start with the Aubry-André model and then define a class of models that includes the latter and the models in [34, 37]. The aim of this section is not only to show explicit examples in which the considerations on the main text regarding fixed-point models apply, but also to provide intuition on the important ingredients behind the existence of these models.

S2.1. Aubry-André model

In this section we provide additional details on fixed-point models and how to identify them. We start with the simplest case: the Aubry-André model. In this case, the Hamiltonian matrix for a given CA is given by:

$$
\mathcal{H} = \begin{pmatrix}
V[\phi] - E & -te^{ik} & 0 & \cdots & 0 & -te^{-ik} \\
-te^{-ik} & V[2\pi\tau_c + \phi] - E & -te^{ik} & \cdots & 0 & 0 \\
0 & -te^{-ik} & V[2\pi(2\tau_c) + \phi] - E & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -te^{-ik} & V[2\pi\tau_c(n_1 - 2) + \phi] - E \\
-te^{ik} & 0 & \cdots & \cdots & -te^{-ik} & V[2\pi\tau_c(n_1 - 1) + \phi] - E \\
\end{pmatrix}
$$

where $V[x] = V\cos(x)$ and $k$ is the momentum measured in units of the UC defined for $V = 0$. Considering a CA defined by $\tau_c = p/q$, with $q = L$ being the number of sites in the UC, we have that:

$$
\mathcal{P}(E, V, \varphi, \kappa) \equiv \det(\mathcal{H} - ET) = A_L \cos(\kappa) + B_L \cos(\varphi) + T_R^{(n)}(E, V)
$$

with $A_L = -2t_L$, $B_L = (-2)^{1-k}V_L$, $\varphi \equiv L\phi$ and $\kappa = Lk$ ($\kappa$ is the momentum measured in units of the true unit cell for $V \neq 0$, with $L$ sites). We will now show how this result can be easily obtained. In [59] we provided a simple physical argument on why the energy bands are periodic in $\varphi$ with period $2\pi/L$ for a given CA: shifts of $\Delta\phi = 2\pi/L$ correspond just to redefinitions of the UC. Therefore $\mathcal{P}$ should be periodic with period $\Delta\varphi \equiv L\Delta\phi = 2\pi$. It is then natural that it has a term proportional to $\cos(\varphi)$. In fact, the coefficient of this term can be computed explicitly. The only way that a term with this periodicity can arise in the determinant of $\mathcal{H}$ is from the product of all the diagonal terms proportional to $V$, that is:

$$
\prod_{n=0}^{L-1} V \cos \left( \frac{2\pi pn}{L} + \frac{\varphi}{L} \right) = \frac{V^L}{(-2)^{L-1}} \left[ \cos(\varphi) - \cos(L\pi/2) \right]
$$

which readily retrieves the correct coefficient $B_L$. All the terms proportional to $\cos(\varphi/r)$, $r \in \mathbb{N} \setminus \{1\}$ should vanish by our previous argument. The only question that remains is whether terms proportional to $\cos(r\varphi)$, $r \in \mathbb{N} \setminus \{1\}$ can be generated. But this is not the case: that would require
a product of more than $L$ terms of the type $\cos[f(n) + \varphi/L]$. Such a term does not exist for the Aubry-André model.

A similar argument can be used to infer the $\kappa$-dependence of $\mathcal{P}$. In this case, we can apply the Aubry-André duality to the eigenfunction $\psi$, defined through $H\psi = E\psi$, to get

$$\mathcal{H}\psi = E\psi \leftrightarrow \mathcal{H}'f = Ef$$

(S16)

$$\mathcal{H}' = UH'U^\dagger, f = U\psi, U_{mn} = e^{\imath 2\pi mn}$$

(S17)

where

$$
\mathcal{H}' = \begin{pmatrix}
T[\kappa/L] - E & \frac{\imath}{2} e^{\imath \phi/L} & \frac{\imath}{2} e^{-\imath \phi/L} & \frac{\imath}{2} e^{\imath \phi/L} & \frac{\imath}{2} e^{-\imath \phi/L} \\
\frac{\imath}{2} e^{-\imath \phi/L} & T[2\pi\tau_c + \kappa/L] - E & 0 & \ldots & 0 \\
0 & \frac{\imath}{2} e^{-\imath \phi/L} & T[2\pi(2\tau_c) + \kappa/L] - E & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\imath}{2} e^{\imath \phi/L} & 0 & \ldots & \ldots & T[2\pi\tau_c(n_1 - 2) + \kappa/L] - E \\
0 & \frac{\imath}{2} e^{\imath \phi/L} & \ldots & \ldots & \frac{\imath}{2} e^{-\imath \phi/L} \end{pmatrix}
$$

(S18)

with $T[x] = -2t \cos(x)$. We only changed the basis and therefore the characteristic polynomial should be the same. The Brillouin zone of a given CA has a period $\Delta k \equiv \Delta \kappa/L = 2\pi/L$ and therefore $\mathcal{P}$ should have this periodicity. A term proportional to $\cos(\kappa)$ should therefore exist. The only way to get such term is, similarly to the case of Eq. S13:

$$\prod_{n=0}^{L-1} (-2t) \cos \left( \frac{2\pi pn}{L} + \frac{\kappa}{L} \right) = (-2t^L)[\cos(\kappa) - \cos(L\pi/2)],$$

(S19)

retrieving the correct coefficient $A_L$. Finally, we note that $|B_L/A_L|$ decays exponentially in the extended phase ($|V| < 2$) and grows exponentially in the localized phase ($|V| > 2$). We can inspect the characteristic decay lengths in each phase. Starting in the extended phase, we have

$$\left|\frac{B_L}{A_L}\right| = \left(\frac{|V|}{2}\right)^L = e^{-\log(2/|V|)L} \equiv e^{-L/\xi_{\text{ext}}},$$

(S20)

where $\xi_{\text{ext}} = 1/\log(2/|V|)$, defined for $|V| < 2$, is precisely the correlation length in the extended phase calculated in [1].

In the localized phase, it is the ratio $|A_L/B_L|$ that decreases exponentially. We have

$$\left|\frac{A_L}{B_L}\right| = \left(\frac{2}{|V|}\right)^L = e^{-\log(|V|/2)L} \equiv e^{-L/\xi_{\text{loc}}},$$

(S21)

where $\xi_{\text{loc}} = 1/\log(|V|/2)$ is the localization length, defined for $|V| > 2$. 
S2.2. A more general class of fixed-point models

To consider more generic fixed-point models, we start with the following generic Schrödinger equation with phase twists $k$:

$$h(\lambda, 2\pi \tau n + \phi) u_n = \sum_m e^{-ik(n-m)} f(\lambda, |n - m|) u_m$$  \hfill (S22)

where $\lambda$ is the set of Hamiltonian parameters, we singled-out $\tau$ as the incommensurability parameter and we define $h(\lambda, 2\pi \tau n + \phi) = E - \chi(\lambda, 2\pi \tau n + \phi)$ for some function $\chi$. We will take CA defined by $\tau = \tau_c = p/L$, with $p$ and $L$ two co-prime integers. In this case we have that $h_n + rL = h_n$; $u_n + rL = u_n$, $n = 0, \cdots, L - 1$, $r \in \mathbb{Z}$.

We start by applying the transformation $\sum_{n=0}^{L-1} e^{i2\pi \tau_n m} = \frac{L}{N} \sum_n e^{i2\pi \tau_m n}$, where $N$ is the total number of sites in the system. The first term in Eq. S22 becomes

$$\sum_{n=0}^{L-1} e^{i2\pi \tau_n m} h(\lambda, 2\pi \tau c n + \phi) u_n = \frac{L}{N} \sum_m e^{i2\pi \tau_m m} h(\lambda, 2\pi \tau c m + \phi) u_m,$$  \hfill (S23)

For the second term we have:

$$\frac{L}{N} \sum_n e^{i2\pi \tau_c n} \sum_m e^{-ik(n-m)} f(\lambda, |n - m|) u_m$$

Using $m' = n - m$:

$$\frac{L}{N} \sum_{m,m'} e^{i2\pi \tau_c m'} e^{-ikm'} f(\lambda, |m'|) e^{i2\pi \tau_m m} u_m$$  \hfill (S24)

Defining $K(\lambda, 2\pi \tau_c \mu - k) = \sum_{m'} e^{im'(2\pi \tau_c - k)} f(\lambda, |m'|)$

$$\frac{L}{N} \sum_{m} K(\lambda, 2\pi \tau_c \mu - k) e^{i2\pi \tau_c m} u_m,$$

Combining, we get

$$\sum_{m} e^{i2\pi \tau_c m} [E - \chi(\lambda, 2\pi \tau_c m + \phi) - K(\lambda, 2\pi \tau_c \mu - k)] u_m = 0$$  \hfill (S25)

We now impose some restrictions in functions $\chi$ and $K$. In particular, they should be such that Eq. S25 can be written as

$$\sum_{m} e^{i2\pi \tau_c m} \left( \frac{A c_{\mu, k} + B c_{m, \phi} + \eta c_{m, \phi} c_{\mu, k} + D}{g_{m, \phi} g'_{\mu, k}} \right) u_m = 0$$  \hfill (S26)

where $c_{m, \phi} = \cos(2\pi \tau_c m + \phi)$ and $c_{\mu, k} = \cos(2\pi \tau_c \mu - k)$, $\mathbf{u} = (u_0, \cdots, u_{L-1})$ and $g_{m, \phi} = g(2\pi \tau_c m + \phi)$, $g'_{\mu, k} = g'(2\pi \tau_c \mu - k)$ are some functions. Note that $P_{\mu m} \equiv A c_{\mu, k} + B c_{m, \phi} + \eta c_{m, \phi} c_{\mu, k} + D$ contains
the coefficients of the characteristic polynomial of the 1-site CA, with \( \tau_c = 1 \). In particular, \( P_{00} \) corresponds exactly to this characteristic polynomial.

We will now compute the \( \phi \)- and \( k \)-dependent parts of the determinant of matrix \( M \) in Eq. S26. Our final aim is to calculate the ratios between renormalized couplings \(|t_L/C_L|, |V_L/C_L| \) and \(|t_L/V_L|\), where these couplings are defined, for the characteristic polynomial of a CA with \( L \) sites in the unit cell, \( \mathcal{P}_L(\varphi, \kappa) \), as

\[
\mathcal{P}_L(\varphi, \kappa) = V_L \cos(\varphi) + t_L \cos(\kappa) + C_L \cos(\varphi) \cos(\kappa) + \cdots
\]

(S27)

where \( \varphi = L\phi \) and \( \kappa = Lk \). The renormalized couplings only depend on the size of the CA’s unit cell for this class of models, as we will see. We will also see why only the fundamental harmonics in \( \varphi \) and \( \kappa \) appear for any CA.

The Leibniz formula for the determinant is

\[
det(M) = \sum_{\sigma \in S_L} \operatorname{sgn}(\sigma) \prod_{\mu=0}^{L-1} M_{\mu,\sigma_\mu}
\]

(S28)

where \( S_L \) is the set of permutations of indexes \( i = 1, \ldots, L \). Using Eq. S26, we have

\[
det(M) = \frac{1}{\prod_{\mu} g_{\mu,0} g_{\mu,k}} \sum_{\sigma \in S_L} \operatorname{sgn}(\sigma) \prod_{\mu=0}^{L-1} e^{i2\pi \tau_{\mu} \sigma_\mu} P_{\mu,\sigma_\mu} \propto \sum_{\sigma \in S_L} \operatorname{sgn}(\sigma) \prod_{\mu=0}^{L-1} e^{i2\pi \tau_{\mu} \sigma_\mu} P_{\mu,\sigma_\mu}
\]

(S29)

We start by focusing on the \(|t_L/C_L| \) ratio. We have

\[
det(M) \propto \sum_{\sigma \in S_L} \operatorname{sgn}(\sigma) \prod_{\mu=0}^{L-1} e^{i2\pi \tau_{\mu} \sigma_\mu} \left( \eta c_{\mu,k}(A/\eta + c_{\mu,\phi}) + B c_{\mu,\phi} + D \right)
\]

(S30)

Only products of terms \( \eta c_{\mu,k}(A/\eta + c_{\mu,\phi}) \) matter to get the terms \( t_L \cos(\kappa) + C_L \cos(\kappa) \cos(\varphi) \) of \( \mathcal{P}_L(\varphi, \kappa) \). This is because to get terms with \( L \) times the original frequency \( k \) or \( \phi \), we need to have \( L \) products of terms \( c_{\mu,k} \) and \( c_{\mu,\phi} \). The only way to accomplish this is by multiplying \( L \) terms of the type \( \eta c_{\mu,k}(A/\eta + c_{\mu,\phi}) \). We can also ask why terms of the type \( \cos(nx) \) with \( x = \phi, k \) and \( n \in \mathbb{N} < L \) do not appear in \( \mathcal{P}_L(\varphi, \kappa) \). This would give rise to periodicity in \( \phi \) and \( k \), \( \Delta \phi, \Delta k > 2\pi/L \), that are forbidden because the Brillouin zone boundaries are separated by \( \Delta k = 2\pi/L \). In the case of phase \( \phi \), shifts of \( \Delta \phi = 2\pi/L \) in a CA with a unit cell with \( L \) sites are just re-definitions of this unit cell [59]. The energy bands are therefore be periodic upon these shifts. Finally, we can also ask why we cannot have terms of the type \( \cos(nx) \) with \( x = \phi, k \) and \( n \in \mathbb{N} > L \). The reason is that such terms would require a number of \( n > L \) products of terms \( c_{\mu,k} \) and \( c_{\mu,\phi} \) that do not appear in the determinant in Eq. S30. Therefore, we only have fundamental harmonics in \( \varphi \) and \( \kappa \), as previously stated.

To make further progress, we will need the following identity for \( p, L \) two co-prime integers [60],

\[
\prod_{\mu=0}^{L-1} \left[ \cos(y) - \cos \left( \frac{2\pi \mu}{L} + x \right) \right] = 2^{1-L} \left[ \cos(Ly) - \cos(Lx) \right],
\]

(S31)
that we can apply in eq. S30 in the case $|A/\eta| < 1$ by identifying $y = A/\eta$. We get

$$
\det(M) \propto \gamma \sigma \left(-\cos(L\pi/2) + \cos(Lk)\right) \left(\cos[L\arccos(A/\eta)] - \cos(L\phi)\right)
\propto \cos[L\arccos(A/\eta)] \cos(Lk) - \cos(Lk) \cos(L\phi) + \cdots
$$

(S32)

where we identified

$$
\gamma \sigma = \sum_{\sigma \in S_L} \text{sgn}(\sigma) \prod_{\mu=0}^{L-1} e^{i2\pi \epsilon_{\mu} \gamma \sigma}
$$

(S33)

We finally have that

$$
|t_L/C_L| = |\cos[L\arccos(A/\eta)]| = |T_L(A/\eta)|
$$

(S34)

where $T_L(x)$ is the Chebyshev polynomial of order $L$. It is very interesting to realize that there is no well-defined limit of $|t_L/C_L|$ for large $L$. In fact, even though we always have $|t_L/C_L| < 1$, its value oscillates with $L$ with a period that becomes larger the closer $A/\eta$ is to 1 (diverging at $A/\eta = 1$, at the extended-to-critical transition). Examples are shown in Fig. S3.

![FIG. S3. $t_L/C_L$ for different $A/\eta < 1$.](image_url)

We now focus in the case $|A| \geq |\eta|$. In this case, it is useful to use the following property [60]:

$$
\prod_{\mu=0}^{L-1} \left[ x \pm y \cos \left(\frac{2\pi \mu}{L} + z\right) \right] = \frac{1}{2^L} \left[ \left( x + \sqrt{x^2 - y^2} \right)^L + \left( x - \sqrt{x^2 - y^2} \right)^L - 2(\mp y)^L \cos(L z) \right]
$$

(S35)

In this case we obtain

$$
|C_L/t_L| = \frac{2}{\left(\frac{A}{\eta} + \sqrt{\left(\frac{A}{\eta}\right)^2 - 1}\right)^L + \left(\frac{A}{\eta} - \sqrt{\left(\frac{A}{\eta}\right)^2 - 1}\right)^L}
$$

(S36)

Note that for $|A/\eta| > 1$ this gives an exponential decay for $|C_L/t_L|$. This is just what we expect in the extended phase, where the dominant coupling is $t_L$, regarding that it also dominates over $V_L$. If we assume that the decay is $|C_L/t_L| \sim e^{-L/\xi_C}$ for large $L$, then the decay length $\xi_{EC}$ is
\[
\xi_{EC} = \frac{1}{\log \left( |A/\eta| + \sqrt{(A/\eta)^2 - 1} \right)} \tag{S37}
\]

which is finite for any \( |A/\eta| > 1 \) and of course diverges for \( |A/\eta| = 1 \). This correlation length characterizes a transition between an extended and a critical phase. For the transition between the localized and critical phase, we should look at the ratio \(|V_L/C_L|\) that can be computed following identical steps as before to give:

\[
|V_L/C_L| = \left\{ \begin{array}{l}
\frac{|T_L(B/\eta)|}{\left[ \frac{1}{2} \left( \frac{|B/\eta| + \sqrt{(B/\eta)^2 - 1}}{L} \right)^L + \left( \frac{|B/\eta| - \sqrt{(B/\eta)^2 - 1}}{L} \right)^L \right], & |B/\eta| < 1 \\
\frac{|A/\eta| + \sqrt{(A/\eta)^2 - 1}}{\log \left( |B/\eta| + \sqrt{(B/\eta)^2 - 1} \right)} \tag{S38}
\end{array} \right.
\]

For \(|B/\eta| \geq 1\) we get the following correlation length characterizing the localized-critical transition:

\[
\xi_{LC} = \frac{1}{\log \left( |B/\eta| + \sqrt{(B/\eta)^2 - 1} \right)} \tag{S39}
\]

We can finally compute \(|t_L/V_L|\) through the previous expressions. For \(|V_L/C_L|, |t_L/C_L| > 1\) (outside the critical phase), we have

\[
|t_L/V_L| = \left( \frac{|A/\eta| + \sqrt{(A/\eta)^2 - 1}}{L} \right)^L + \left( \frac{|B/\eta| - \sqrt{(B/\eta)^2 - 1}}{L} \right)^L
\]

Finally, similarly to what we did in the previous section for the Aubry-André model, we can also define the correlation or localization lengths from the scaling of the ratio \(|t_L/V_L|\) with \(L\), through Eq. S40. We first assume that \(|A| < |B|\) (localized phase). Taking the large \(L\) limit we get

\[
|t_L/V_L| = e^{-L/\xi_{LE}}, \quad \text{with} \quad \xi_{LE} = 1/\log \left[ \frac{|B/\eta| + \sqrt{(B/\eta)^2 - 1}}{|A/\eta| + \sqrt{(A/\eta)^2 - 1}} \right] \tag{S41}
\]

The \(\xi_{LE}\) above is the localization length. We can also compute the correlation length in the extended phase for \(|A| > |B|\). This can be done just by changing \(A\) and \(B\) in the expression above, that is:

\[
|V_L/t_L| = e^{-L/\xi_{EL}}, \quad \text{with} \quad \xi_{EL} = 1/\log \left[ \frac{|A/\eta| + \sqrt{(A/\eta)^2 - 1}}{|B/\eta| + \sqrt{(B/\eta)^2 - 1}} \right] \tag{S42}
\]
In the class of models here studied, the conjectures in Eqs. 4, 5 of the main text can be verified analytically.

We will finally show that the Aubry-André model and the models in Refs. [34, 37] belong to this class of models. For the first, we have (with phase twists $k$):

$$[E + t - V \cos(2\pi \tau_c n + \phi)]u_n = t \sum_m e^{-ik(n-m)}e^{-p|n-m|}u_m, \quad (S43)$$

We can therefore identify $\chi(V, t, 2\pi \tau_c m + \phi) = -t + V \cos(2\pi \tau_c m + \phi)$ and $f(t, |m'|) = te^{-p|m'|}$. We then get

$$K(t, 2\pi \tau_c \mu - k) = t \sum_m e^{im'(2\pi \tau_c \mu - k)}e^{-p|m'|} = \frac{t \sinh p}{\cosh p - \cos(2\pi \tau_c \mu - k)} \quad (S44)$$

Inserting in Eq. S25, we have

$$\sum_m e^{i2\pi \tau_c m \mu} [E + t - Vc_m,\phi - \frac{t \sinh p}{\cosh p - c_{\mu,k}}]u_m = 0$$

$$\sum_m e^{i2\pi \tau_c m \mu} \left[ \frac{(E + t) \cosh p - t \sinh p + (E + t)c_{\mu,k} - V \cosh p c_{m,\phi} + Vc_{m,\phi}c_{\mu,k}}{\cosh p - c_{\mu,k}} \right]u_m = 0 \quad (S45)$$

which is already written in the form of Eq. S26, with $A = E + t$, $B = -V \cosh p$, $\eta = V$ and $D = (E + t) \cosh p - t \sinh p$. The transition between the extended and localized phases occurs for $|A| = |B|$, that is,

$$\cosh p = \left| \frac{E + t}{V} \right| \quad (S46)$$

For this model, critical states only appear when the condition above is satisfied: in particular, even though $\eta \neq 0$, we always have $|B| \geq |\eta|$. Notice that the Aubry-André model can be obtained by making the substitution $t \rightarrow 2te^p$ and taking the large $p$ limit. If we do so, Eq. S46 becomes $|V| = 2|t|$, the critical point of the Aubry-André model.

For the model in Ref. [37] we have [61]

$$\left[ E - 2\lambda \frac{1 - \beta \cos(2\pi \tau_c n + \phi)}{1 + \alpha \cos(2\pi \tau_c n + \phi)} \right]u_n = t(e^{-ik}u_{n+1} + e^{ik}u_{n-1}) \quad (S47)$$

In this case, we have

$$\chi(\lambda, \alpha, 2\pi \tau_c m + \phi) = 2\lambda \frac{1 - \beta \cos(2\pi \tau_c m + \phi)}{1 + \alpha \cos(2\pi \tau_c m + \phi)} \quad (S48)$$

$$f(t, |m'|) = t\delta_{m'|1} \quad (S49)$$
\[ K(t, 2\pi\tau \mu + k) = t \sum_{m'} e^{im'(2\pi\tau \mu - k)} f(t, |m'|) = 2t \cos(2\pi\tau \mu - k) \tag{S50} \]

and therefore, replacing again in Eq. S26, we have

\[
\sum_m e^{2\pi\tau \mu m} \left[ E - 2\lambda \frac{1 - \beta_{m,\phi}}{1 + \alpha_{m,\phi}} - 2tc_{\mu,k} \right] u_m = 0
\]

\[
\sum_m e^{2\pi\tau \mu m} \left[ E - 2\lambda + (\alpha E + 2\lambda \beta)c_{m,\phi} - 2tc_{\mu,k} - 2t\alpha c_{m,\phi}c_{\mu,k} \right] u_m = 0 \tag{S51}
\]

which is again written in the form of Eq. S26, with \( A = -2t, B = \alpha E + 2\lambda \beta, \eta = -2t\alpha \) and \( D = E - 2\lambda \). The mobility edge condition is

\[ 2|t| = |\alpha E + 2\lambda \beta| \tag{S52} \]

Furthermore, critical phases occur for \(|\alpha| > 1 \) (otherwise \(|\eta| \leq |A| \) always), as noticed in Ref. [31].

We can also compute the correlation lengths in Eqs. S39,S41,S42, obtaining the same results that were obtained in [62] using Avila’s theory for computing Lyapunov exponents [63].

### S2.3. Alternative fixed-point models

In order to avoid confusion, we restricted our analysis in the main text to models with the characteristic polynomial in Eq. 3. But there can also be even more generic alternative classes of fixed-point models, given by

\[
\mathcal{P}^{(n)}(E, \varphi, \kappa) =
\]

\[
= t_R^{(n)}(E) \cos[n_1(\kappa + \kappa_0)] + V_R^{(n)}(E) \cos[n_2(\varphi + \varphi_0)]
\]

\[
+ C_R^{(n)}(E) \cos[n_3(\kappa + \kappa'_0)] \cos[n_3(\varphi + \varphi'_0)] + T_R^{(n)}(E) \tag{S53}
\]

with \( n_1 \neq n_2 \neq n_3 \). In this section we show an instructive example of a model of this type, for which the phase diagram can still be exactly obtained.

We consider the non-abelian Aubry-André model in Ref. [53], whose Hamiltonian is given by

\[
H = \sum_n (c_{n+1}^\dagger T_1 c_n + \text{h.c.}) + V \sum_n \cos(2\pi\tau n + \phi) c_{n+1}^\dagger T_2 c_n \tag{S54}
\]

where \( c_n = [c_n^{(1)}, c_n^{(2)}]^T \), \( T_x = -t_0 \sigma_z + it_{so} \sigma_y \) and \( T_2 = V \sigma_z \). This model can also be recast into the Aubry-André model with p-wave pairing [53, 54].

The Schrödinger equation for this model (already with phase twists) is

\[
-\frac{t}{2}(e^{-ik}u_{n+1} + e^{ik}u_{n-1}) + t_{so}(e^{-ik}v_{n+1} + e^{ik}v_{n-1}) + V_n u_n = E u_n
\]

\[
t(e^{-ik}v_{n+1} + e^{ik}v_{n-1}) - t_{so}(e^{-ik}u_{n+1} + e^{ik}u_{n-1}) - V_n v_n = E v_n \tag{S55}
\]
where \( V_n = V \cos(2\pi \tau_c n + \phi) \). Summing and subtracting the two equations, we get respectively:

\[
\begin{align*}
-e^{-ik}(t + t_{so})\psi_{n+1}^{-} + (t + t_{so})e^{ik}\psi_{n-1}^{-} + V_n\psi_{n}^{-} &= E\psi_{n}^{+} \\
e^{-ik}(-t + t_{so})\psi_{n+1}^{+} + (t + t_{so})e^{ik}\psi_{n-1}^{+} + V_n\psi_{n}^{+} &= E\psi_{n}^{-}
\end{align*}
\]

(S56)

where \( \psi_{n}^{\pm} = u_n + v_n \) and \( \psi_{n}^{-} = u_n - v_n \). We now multiply by \( e^{2\pi i \tau_c n \mu} \) and sum over \( n \). Making a change of summation variable we get

\[
\sum_{n} e^{2\pi i \tau_c n \mu} \sum_{\nu} e^{2\pi i \tau_c n \nu} + \sum_{n} e^{2\pi i \tau_c n \mu} + \sum_{n} e^{2\pi i \tau_c n \nu} = 0
\]

(S57)

\[
\sum_{n} e^{2\pi i \tau_c n \mu} \left[ -e^{-i(2\pi \tau_c \mu + k)}(t + t_{so}) + (t + t_{so})e^{i(2\pi \tau_c \mu + k)} + V_n \right] \psi_{n}^{-} - E\psi_{n}^{+} = 0
\]

(S58)

For a given CA with \( L \) sites in the unit cell, we set \( \psi_{n}^{\pm} = \psi_{n+L}^{\pm} \). Moreover, \( \mu = 0, \ldots, L - 1 \), since for \( \mu = L \) we return to the \( \mu = 0 \) equation. Our problem can then be solved by diagonalizing a \( 2L \times 2L \) matrix \( M \), that is, \( M_{\mu \alpha ; n \sigma} \psi_{n}^{\beta} = 0 \), with \( \alpha, \beta = \pm \). Defining \( t_A = -t - t_{so} \) and \( t_B = -t + t_{so} \), we have

\[
M_{\mu n} = e^{2\pi i \tau_c \nu \mu} \begin{pmatrix} t_A e^{-i(2\pi \tau_c \mu + k)} + t_B e^{i(2\pi \tau_c \mu + k)} + V_n & -E \\
-E & t_B e^{-i(2\pi \tau_c \mu + k)} + t_A e^{i(2\pi \tau_c \mu + k)} + V_n \end{pmatrix}
\]

(S59)

The determinant of \( M \) can be explicitly written as

\[
\det(M) = \sum_{\sigma \in S_L} \text{sgn}(\sigma) \prod_{\mu = 0}^{L-1} \prod_{\alpha = \pm} M_{\mu \alpha ; \sigma_{\mu}}
\]

(S60)

where \( S_L \) is the set of permutations of indexes \( i = 1, \ldots, 2L \). The terms that respect the \( 2\pi/L \) periodicity in \( \phi \) and in \( k \) are formed from at least \( L \) products of the \( \phi \)- and \( k \)-dependent terms. A possible contribution arises from the products,

\[
\left( \sum_{\sigma} \text{sgn}(\sigma) \prod_{\mu} e^{2\pi i \tau_c \mu \sigma_{\mu}} \right) \left[ \prod_{\mu} \left( t_A e^{-i(2\pi \tau_c \mu + k)} \right) \prod_{n} \left( V \cos(2\pi \tau_c n + \phi) \right) + \right.
\]

\[
\left. \prod_{\mu} \left( t_A e^{i(2\pi \tau_c \mu + k)} \right) \prod_{n} \left( V \cos(2\pi \tau_c n + \phi) \right) \right]
\]

(S61)

where \( \gamma = \sum_{\sigma} \text{sgn}(\sigma) \prod_{\mu} e^{2\pi i \tau_c \mu \sigma_{\mu}} \) and we used \( \prod_{\mu} e^{i(2\pi \tau_c \mu + k)} = (-1)^L e^{iLk} \) [60]. Notice that \( \prod_{\mu} F_{\mu} = \prod_{\sigma_{\mu}} F_{\sigma_{\mu}} \), with \( F_{\mu} = \left( V \cos(2\pi \tau_c \mu + \phi) \right) \) because irrespectively of the permutation \( \sigma_{\mu} \)
we always get the product of all the possible terms with \( \mu = 0, \cdots, L - 1 \). A similar contribution arises from products of \( t_B \)-dependent and \( V_n \) terms:

\[
\left( \sum_{\sigma} \text{sgn}(\sigma) \prod_{\mu} e^{2\pi i \tau_{c} \mu \sigma_{\mu}} \right) \left[ \prod_{\mu} (t_{BE}^{-i(2\pi \tau_{c} \mu + k)}) \prod_{n} (V \cos(2\pi \tau_{c} n + \phi)) + \prod_{\mu} (t_{BE}^{i(2\pi \tau_{c} \mu + k)}) \prod_{n} (V \cos(2\pi \tau_{c} n + \phi)) \right] = \gamma 2^{2-L}(t_{B}V)^{L}\left( -\cos(L\pi/2) + \cos(L\phi) \right) \cos(Lk)
\]

(S62)

For odd \( L \), this gives rise to a term

\[
2^{2-L}V^{L}[t_{A}^{L} + t_{B}^{L}] \cos(L\phi) \cos(Lk)
\]

(S63)

There is also a term proportional to \( \cos(2Lk) \), which can only be obtained through the product,

\[
\left( \sum_{\sigma} \text{sgn}(\sigma) \prod_{\mu} e^{2\pi i \tau_{c} \mu \sigma_{\mu}} \right) \prod_{\mu} \left[ (t_{AE}^{-i(2\pi \tau_{c} \mu + k)})(t_{BE}^{-i(2\pi \tau_{c} \mu + k)}) + (t_{AE}^{i(2\pi \tau_{c} \mu + k)})(t_{BE}^{i(2\pi \tau_{c} \mu + k)}) \right] = 2\gamma(t_{A}t_{B})^{L} \cos(2Lk)
\]

(S64)

Similarly, there is a term proportional to \( \cos(2L\phi) \), obtained from

\[
\left( \sum_{\sigma} \text{sgn}(\sigma) \prod_{\mu} e^{2\pi i \tau_{c} \mu \sigma_{\mu}} \right) \prod_{\mu} (V \cos(2\pi \tau_{c} \mu + \phi)) \prod_{n} (V \cos(2\pi \tau_{c} n + \phi)) = V^{2L}(-2)^{1-2L}(2\cos^{2}(L\pi/2) - 4 \cos(L\pi/2) \cos(L\phi) + \cos(2L\phi) + 1)
\]

(S65)

From these calculations, we can already see that for odd \( L \), the terms proportional to \( \cos(L\phi) \) and to \( \cos(Lk) \) computed so far vanish. In fact, for odd \( L \), we show below that the characteristic polynomial takes the simple form

\[
P(\varphi, \kappa) = V_{L} \cos(2\kappa) + t_{L} \cos(2\varphi) + C_{L} \cos(\kappa) \cos(\varphi) + \cdots
\]

(S66)

with \( \kappa = Lk \) and \( \varphi = L\phi \). Interestingly, this is a fixed-point model only for odd \( L \), different from all the other models we have discussed. The reason why the expression above is not so straightforward to obtain is because other products that we have not considered so far could also give rise to additional terms proportional to \( \cos(Lk) \) and \( \cos(L\phi) \). We will now argue that this cannot be the case for odd \( L \). We will use the explicit \( M \) matrices for \( L = 2 \) and \( L = 3 \) to get some insight. These are given by

\[
M_{L=2} = \begin{pmatrix}
f_{00} & -E & f_{01} & -E \\
-E & f'_{00} & -E & f'_{01} \\
f_{10} & -E & f_{11} & -E \\
-E & f'_{10} & -E & f'_{11}
\end{pmatrix}
\]

(S67)
with \( f_{\mu n} = t_A e^{-i(2\pi \tau,_{\mu} + k) + t_B e^{i(2\pi \tau,_{\mu} + k)}} + V_n \) and \( f'_{\mu n} = t_B e^{-i(2\pi \tau,_{\mu} + k) + t_A e^{i(2\pi \tau,_{\mu} + k)}} + V_n \). To produce a term proportional to \( \cos(Lk) \) or \( \cos(L\phi) \), we need to have a product of at least \( L \) terms \( f \) or \( f' \). Such product however will have a prefactor \( E_L \). If \( L \) is odd, these terms have to cancel out because the particle-hole symmetry of the model ensures that the energy spectrum is symmetric around \( E = 0 \). We could however have the terms appearing from products of \( L < n < 2L \) terms \( f \) or \( f' \), with \( n \) odd. However, in this case we would have a prefactor \( E^{2L-n} \) that also vanishes because \( 2L-n \) is also odd. Finally, for \( n = 2L \) products, the terms \( \cos(Lk) \) and \( \cos(L\phi) \) vanish as we have seen because \( \cos(L\pi/2) = 0 \) for odd \( L \). We also note that we can only have terms \( \cos(nLx) \), \( n = 0, 1, 2; x = k, \phi \). There aren’t enough products of \( f \) or \( f' \) to generate terms with \( n > 2 \).

Since we have seen that for odd \( L \), the characteristic polynomial acquires the simple fixed-point form in Eq. S66, we can compute the renormalized couplings analytically. Since the results in the thermodynamic limit do not depend on whether \( L \) is even or odd, we can therefore derive all the phase boundaries analytically using the odd \( L \) results. Combining all the results above for odd \( L \), we have

\[
V_L \left| C_L \right| = \frac{2^{1-2L}V^{2L}}{2^{2-L}V^L(t^n_A + t^n_B)} = 1 \left( \frac{V}{2} \right)^L \quad \text{(S69)}
\]

\[
\left| \frac{t_L}{C_L} \right| = \frac{2(t_A t_B)^L}{2^{2-L}V^L(t^n_A + t^n_B)} = 1 \left( \frac{2t_A t_B}{V} \right)^L \quad \text{(S70)}
\]

Assuming \( t, t_{so} > 0 \), we have that \( t^n_B / t^n_A = [(t_{so} - t) / (t_{so} + t)]^L \to 0 \) as \( L \to \infty \). In this limit we have

\[
\left| \frac{V_L}{C_L} \right| \to \frac{1}{2} \left( \left| \frac{V}{2t_A} \right| \right)^L \quad \text{(S71)}
\]

\[
\left| \frac{t_L}{C_L} \right| \to \frac{1}{2} \left( \left| \frac{2t_B}{V} \right| \right)^L \quad \text{(S72)}
\]

The conditions \( \left| V_L / C_L \right| = 1 \) and \( \left| t_L / C_L \right| = 1 \) give respectively the conditions for the critical-to-localized and critical-to-extended transitions. We have

\[
\left| \frac{V_L}{C_L} \right| = 1 \leftrightarrow \left| V \right| = \left| 2t_A 2^{1/L} \right| \to 2|t_A| \leftrightarrow \left| V \right| = 2|t + t_{so}| \quad \text{(S73)}
\]
These expressions are exactly the conditions for the phase transitions obtained numerically in Ref. [53].

**S3. Calculating the Critical Exponent $\nu$ Across Different Transitions**

In this section we show some examples of analytical calculations for the critical exponent $\nu$, for different types of transitions, and show how our theory can be used to unveil different universality classes.

**S3.1. Class of fixed-point models in Sec. S2.2**

We start with the class of models described in Sec. S2.2. The results below apply to all models that fall within this class, including the AAM and the models in Refs. [34, 37]. For the analysis below, we will assume $A, B, \eta > 0$ for simplification, without loss of generality.

*Extended-to-localized and localized-to-extended transitions* We will first consider the extended-to-localized transition. The correlation length that diverges as we approach the critical point from the extended phase is in Eq. S42. At this phase, we have $A > B$. Close to the transition, we have that $A = B + \epsilon$, with $\epsilon = 0^+$. Expanding the correlation length in Eq. S42 for small $\epsilon$, we have

$$\xi_{EL} \propto \epsilon^{-1}$$  \hspace{1cm} (S75)

and therefore we get $\nu = 1$. A similar expansion can be made for the localization length in Eq. S41 for $B = A + \epsilon$ with $\epsilon = 0^+$, which means approaching the critical point from the localized phase. In this case we again see that

$$\xi_{LE} \propto \epsilon^{-1}$$  \hspace{1cm} (S76)

retrieving the same critical exponent $\nu = 1$.

*Localized-to-critical and extended-to-critical transitions* For localized-to-critical and extended-to-critical transitions, the divergent correlation length is in Eqs. S39, S37. Approaching the critical phase from the localized and extended phases using respectively $B = \eta + \epsilon$ and $A = \eta + \epsilon$, and expanding for small $\epsilon = 0^+$, we have that

$$\xi_{LC} \propto \epsilon^{-1/2}$$  \hspace{1cm} (S77)

$$\xi_{EC} \propto \epsilon^{-1/2}$$  \hspace{1cm} (S78)

This therefore retrieves the interesting critical exponents $\nu = 1/2$ for transitions into the critical phase, in agreement with the results in [62] for the localized-to-critical transition using the model in [37].
Critical-to-extended and critical-to-localized transitions At the critical phase, there is no correlation length that can be defined (it is infinite by definition). However, as we have seen in Sec. S2.2 there is also a divergent length corresponding to the period of oscillations of $|t_L/C_L| = |T_L(A/\eta)|$ and $|V_L/C_L| = |T_L(B/\eta)|$, respectively at the critical-to-extended and critical-to-localized transitions. If we expand $A = B = \eta - \epsilon$ for small $\epsilon$, we can easily see that the wavelength of oscillation diverges with $\epsilon^{-1/2}$, which has curiously the same exponent as the correlation length exponent $\nu$ for the transitions into the critical phase.

S3.2. Model in Sec. S2.3

From the ratios between the renormalized couplings in Eqs. S71,S72, we can easily extract the critical exponent $\nu$ for the localized-to-critical and extended-to-critical transitions. For the former, we have:

$$\left| \frac{C_L}{V_L} \right| = 2 \left( \left| \frac{2t_A}{V} \right| \right)^L = 2e^{-L \log(\frac{V}{\pi^2A})}$$

and therefore $\xi = 1/\log(\frac{V}{\pi^2A})$. For $V = 2t_A + \epsilon$, with $\epsilon \to 0^+$ we have $\xi \propto \epsilon^{-1}$ and therefore $\nu = 1$. This exponent was only obtained numerically very recently [54]. Simple calculations show that we have $\nu = 1$ also for the extended-to-critical transition, a result that was not yet obtained, to our knowledge.

S4. SIMILARITIES BETWEEN THREE MODELS IN THE LITERATURE

We here discuss interesting similarities between the models in [29, 34, 37]. The Schrodinger equation for the model in [29] can be written as

$$V_1 \cos \left( 2\pi \tau c n + \phi \right) \psi_{n} + (t + V_2 \cos \left( 2\pi \tau c (n+1/2) + \phi \right)) \psi_{n+1} + (t + V_2 \cos \left( 2\pi \tau c (n-1/2) + \phi \right)) \psi_{n-1} = E \psi_{n}$$

with $\tau_c = p/L$, $n = 0, \ldots, L-1$ and $\psi_{n+L} = \psi_{n}e^{in}$. Below we show the couplings $V_R^{(n)}$, $t_R^{(n)}$ and $C_R^{(n)}$ for different CA with different UC sizes ($L$) [64]:

| $\tau_c = 1/L$ | $t_R^{(n=L)}$ | $V_R^{(n=L)}$ | $C_R^{(n=L)}$ |
|---------------|--------------|--------------|--------------|
| 1             | 2$t$         | $V_1$        | $-2V_2$      |
| 1/2           | $\frac{2}{\pi^2} \left[ -2t^2 + V_2^2 \right]$ | $\frac{1}{2}(-V_1^2 + 2V_2^2)$ | $-\frac{2}{\pi^2} V_2^2$ |
| 1/3           | $\frac{2}{\pi^2} \left[ 4t^3 - 3tV_2^2 \right]$ | $\frac{1}{2\pi} (V_1^3 - 3V_1 V_2^2)$ | $-\frac{2}{\pi^2} V_2^3$ |
| 1/4           | $\frac{2}{\pi^2} \left[ -8t^4 + 8t^2 V_2^2 - V_2^4 \right]$ | $\frac{1}{2\pi} (-V_1^4 + 4V_1^2 V_2^2 - 2V_2^4)$ | $-\frac{2}{\pi^2} V_2^4$ |
| 1/5           | $\frac{2}{\pi^2} \left[ 16t^5 - 20t^3 V_2^2 + 5tV_2^4 \right]$ | $\frac{1}{2\pi} (V_1^5 - 5V_1^3 V_2^2 + 5V_1 V_2^4)$ | $-\frac{2}{\pi^2} V_2^5$ |
| 1/6           | $\frac{2}{\pi^2} \left[ -32t^6 + 48t^4 V_2^2 - 18t^2 V_2^4 + V_2^6 \right]$ | $\frac{1}{2\pi} (-V_1^6 + 6V_1^2 V_2^2 - 9V_1^4 V_2^2 + 2V_2^6)$ | $-\frac{2}{\pi^2} V_2^6$ |
This is a fixed-point model, as only the couplings shown above are generated for each CA. Having realized this, we can apply the conditions of the main text to get the phase boundaries to find the localization-delocalization transition at \(|V_1| = 2|t|\) from \(|y_{(n)}^R/y_{(n)}^R| = 1\) and the critical phase at \((|V_2| > |V_1|/2) \land (|V_2| > 1)\) from \(|C_{(n)}^R/y_{(n)}^R| \geq 1 \land |C_{(n)}^R/y_{(n)}^R| \geq 1\).

In fact, inspection of the coefficients for any CA suggests that we can write \(t_{(n)}^R = Q_{(n)}^1(V_2, -2t), V_{(n)}^R = Q_{(n)}^1(V_2, -V_1)\) and \(|C_{(n)}^R| = |Q_{(n)}^2(V_2)|\), with

\[
Q_L(x, y, \varphi) = -2\sum_{\mu=0}^{L} (y/2)^L - \mu P_{S}^{\mu}(\{v_1, \ldots, v_{L-1}\}) = Q_{L}^1(x, y) + Q_{L}^2(x) \cos(\varphi)
\]  

(S82)

\(v_{\mu} = x \cos(2\pi \tau_{\mu} + \varphi/L), \mu = 0, \ldots, L - 1\)

(S83)

where \(P_{S}^\mu(\{v_1, \ldots, v_{L-1}\})\) is the \(\mu\)-th order symmetric polynomial in the variables \(\{v_1, \ldots, v_{L-1}\}\). \(|Q_{n}^2(x)| = 2^{1-L}x^L\), while \(Q_{n}^1\) is a more complicated polynomial, as seen above for different CA. \(|Q_{n}^1(V_2, 2)/Q_{n}^1(V_2, V_1)| = 1\) implies that \(|V_1| = 2\) for any CA. The critical phases can be found through \(|Q_{n}^2(V_2)/Q_{n}^1(V_2, V_1)| > 1 \land |Q_{n}^2(V_2)/Q_{n}^1(V_2, 2)| > 1\), yielding the correct result.

Remarkably, the polynomial \(Q_L\) appears in more models in the literature whose phase diagrams host exact solutions. These include the models in Refs. [34, 37]. For the model in [37], we have

\[
2\lambda \frac{1 - \cos(2\pi \tau_{\mu} + \phi)}{1 + \alpha \cos(2\pi \tau_{\mu} + \phi)} \psi_n + t(\psi_{n+1} + \psi_{n-1}) = E\psi_n
\]

(S84)

\[
\leftrightarrow \frac{2\lambda - (2\lambda + \alpha E) \cos(2\pi \tau_{\mu} + \phi)}{1 + \alpha \cos(2\pi \tau_{\mu} + \phi)} \psi_n + t(\psi_{n+1} + \psi_{n-1}) \frac{1 + \alpha \cos(2\pi \tau_{\mu} + \phi)}{1 + \alpha \cos(2\pi \tau_{\mu} + \phi)} \frac{E}{1 + \alpha \cos(2\pi \tau_{\mu} + \phi)} \psi_n
\]

Note that the second equation gives rise to a non-hermitian matrix. Nonetheless, its solutions are the same as for the first equation and similarities with Eq. S80 become more apparent. This is the case if we throw away the common denominator, which can be done as long as \(1 + \alpha \cos(2\pi \tau_{\mu} + \phi) \neq 0\). In this case, the coefficients of \(P\) are the same as the ones above up to overall signs if \(V_1 \rightarrow - (2\lambda + \alpha E)\) and \(V_2 \rightarrow \alpha t\). We therefore have that \(t_{(n)}^R = Q_{n}^1(\alpha t, -2t), V_{(n)}^R = Q_{n}^1(\alpha t, 2\lambda + \alpha E)\) and \(|C_{(n)}^R| = |Q_{n}^2(\alpha t)|\) (as long as \(1 + \alpha \cos(2\pi \tau_{\mu} + \phi) \neq 0\)). Note that in this case, \(|C_{(n)}^R/t_{(n)}^R| > 1\) implies that \(|\alpha| > 1\). This is why no critical phases were observed in [37] because there only \(|\alpha| < 1\) was considered.

Finally, for the model in [34], we have

\[
t' \sum_l e^{-pl} \left( e^{ik f_{m-l}} + e^{-ik f_{m+l}} \right) + V \cos(2\pi \tau m + \phi) f_m = E f_m
\]

(S85)

Note that we are considering the phase twists \((k \equiv \kappa/L)\) uniformly distributed in the hoppings for reasons that will become apparent. To check the similarity with the model in Eq. S84, we notice that it can also be written as [37] (again uniformly distributing the phase twists in the hoppings):

\[
t(\psi_{n+1} e^{ik} + \psi_{n-1} e^{-ik}) + 2g(\lambda, \beta) \sum_l e^{-\beta l} \cos[l(2\pi \tau n + \phi)] \psi_n - 4\lambda \cosh(\beta) \psi_n = E \psi_n
\]

(S86)
where \( g = 2\lambda(1 + \cosh \beta)/\sinh \beta \) and \( \cosh \beta = \alpha^{-1} \). Applying the Aubry-André duality,

\[
\psi_n = \sum_m e^{i2\pi mn} f_m \tag{S87}
\]

we arrive at

\[
g(\lambda, \beta) \sum_l e^{-\beta l} \left( e^{i\phi_l} f_{m-l} + e^{-i\phi_l} f_{m+l} \right) + 2t \cos(2\pi \tau mp + k) f_m = [E + 4\lambda \cosh(\beta)] f_m \tag{S88}
\]

We can now see that Eqs. S85,S88 are the same if in the first we make \( t' \to g(\lambda, \beta) \), \( V \to 2t \), \( E \to E + 4\lambda \cosh(\beta) \) and \( k \leftrightarrow \phi \). Note that this last replacement is fundamental because it means that the \( V^{(n)}_R \) coefficient in the model of Eq. S85 is the \( t^{(n)}_R \) coefficient in the model of Eq. S88 and vice-versa.

**S5. MORE ON FIXED-POINT MODELS WITH CRITICAL PHASES**

In this section we provide additional details on the fixed-point model with a critical phase considered in the main text and discuss a related model already considered in the literature.

For the fixed-point model in Eq. 6 of the main text, the lowest-order approximant has 2 sites in the UC. The characteristic polynomial in that case is, defining \( \varphi = \frac{L\phi}{2} \):

\[
P(\varphi, \kappa) = \frac{E^2 - 2t^2 - (2t^2\alpha + 2EV - E^2\alpha) \cos(\varphi) + 2t^2 \cos(\kappa) + 2t^2 \alpha \cos(\kappa) \cos(\varphi)}{1 + \alpha \cos(\varphi)} \tag{S89}
\]

Note that as long as \( 1 + \alpha \cos(\varphi) \neq 0 \), the energy bands are obtained by setting the numerator to zero. The denominator diverges at certain points \( \varphi^* \) when \( |\alpha| > 1 \). But for any other \( \varphi \) we may identify \( P \) with the numerator solely and apply our description. In that case, applying the conditions for the phase boundaries in Eq. 5 of the main text immediately retrieves the analytical expression for the mobility edges given below Eq. 6. Remarkably, for this model, similarly to previously discussed fixed-point models, the phase boundaries obtained in are independent of the CA.

We now also consider the model introduced in [37], whose critical phases were studied in Ref. [31]. The Hamiltonian is

\[
H = -t \sum_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \sum_n \frac{V_1 + V_2 \cos(2\pi \tau n + \phi)}{1 + \alpha \cos(2\pi \tau n + \phi)} c_n^\dagger c_n \tag{S90}
\]

For this model, the phase diagram can again be obtained by studying the lowest-order CA (\( \tau_c = 1 \)), whose characteristic polynomial is given by

\[
P(\varphi, \kappa) = \frac{V_1 - E + (V_2 - E\alpha) \cos(\varphi) + 2t \cos(\kappa) + 2t\alpha \cos(\kappa) \cos(\varphi)}{1 + \alpha \cos(\varphi)} \tag{S91}
\]

This characteristic polynomial gives rise to the following phase boundaries:

\[
\begin{cases}
E = (V_2 \pm 2t)/\alpha \quad \text{, SD points} \\
|V_2 - E\alpha| \leq |2t\alpha| \wedge |\alpha| \geq 1 \quad \text{, Critical phase}
\end{cases} \tag{S92}
\]
We can check that for other approximants the same phase boundaries can be obtained. For instance, for $\tau_c = 1/2$ we have, using $\phi = 2\phi$:

$$P(\varphi, \kappa) = \frac{(V^2_2 - 2t^2\alpha^2 - 2EV_2^{2}\alpha^2)\cos(\varphi) + 4t^2\cos(\kappa) - 2t^2\alpha^2 \cos(\kappa) \cos(\varphi)}{-2 + \alpha^2 + \alpha^2 \cos(\varphi)}$$

(S93)

again giving rise to the phase boundaries in Eqs. S92. In Fig. S4 we exemplify how the results for the phase boundaries obtained here perfectly match the numerical results of the IPR.

**S6. DUAL POINTS: GENERALIZATION TO INCLUDE CRITICAL PHASES**

In [59] we have shown how to define dual points close to criticality for a CA of large enough order in extended and localized phases (and at the self-dual critical point separating them). Here we generalize our definition to include critical phases, by considering also the renormalized coupling $C_R^{(n)}$.

In the main text we have written the characteristic polynomial for a given $n$th-order CA as

$$P(E, \lambda, \varphi, \kappa) = t_R^{(n)}(E, \lambda) \cos(\kappa + \kappa_0) + V_R^{(n)}(E, \lambda) \cos(\varphi + \varphi_0) + C_R^{(n)}(E, \lambda) \cos(\kappa + \kappa_0) \cos(\varphi + \varphi_0) + T_R^{(n)}(E, \lambda) + \epsilon_R^{(n)}(E, \lambda, \varphi, \kappa)$$

(S94)

In the regime that $\epsilon_R^{(n)} \approx 0$, the renormalized couplings $t_R^{(n)}$, $V_R^{(n)}$ and $C_R^{(n)}$ enable us to find dual points. After setting $P$ to zero, we get (dropping the index $n$ for clarity):

$$\cos(\varphi + \varphi_0) = \frac{T_R - t_R \cos(\kappa + \kappa_0)}{V_R + C_R \cos(\kappa + \kappa_0)} = \frac{T_R - t_R \cos(\kappa + \kappa_0)}{1 + \frac{V_R}{T_R} \cos(\kappa + \kappa_0)}$$

(S95)

or
\[
\cos(\kappa + \kappa_0) = \frac{T_R - V_R \cos(\varphi + \varphi_0)}{t_R + C_R \cos(\varphi + \varphi_0)} = \frac{T_R - V_R \cos(\varphi + \varphi_0)}{1 + \frac{C_R}{t_R} \cos(\varphi + \varphi_0)}
\]  

(S96)

Dual points \((E, \lambda)\) and \((E', \lambda')\) should then satisfy:

\[
\begin{align*}
\frac{t_R(E, \lambda)}{V_R(E, \lambda)} &= \frac{V_R(E', \lambda')}{t_R(E', \lambda')} \\
T_R(E, \lambda) &= s T_R(E', \lambda') \\
V_R(E, \lambda) &= s V_R(E', \lambda') \\
C_R(E, \lambda) &= s C_R(E', \lambda') \\
V_R(E, \lambda) &= s V_R(E', \lambda')
\end{align*}
\]

(S97)

with \(s = \pm 1\). For SD points, \((E', \lambda') \equiv (E, \lambda)\) and we simply have \(V_R(E, \lambda) = \pm t_R(E, \lambda)\) as we have seen before. Note that the freedom in the sign \(s\) exists because we may choose \(\varphi_0 = \pi\) or \(\kappa_0 = \pi\) and obtain equally valid conditions for the dual points.

Apart from special cases, solving Eqs. S97 analytically is not possible or impractical [65]. Nonetheless, dual points may still be found numerically similarly to [59], for a given energy band.

**S7. REENTRANT LOCALIZATION TRANSITION IN PRB 105, L220201 (2022)**

An interesting reentrant localization transition was observed in [8] for the Aubry-André model with a staggered potential. We here show that it is possible to capture this transition by inspecting the approximate self-dual points in CA with unit cells that are small enough to allow for an analytical calculation.

The Aubry-André model with a staggered potential was also considered in the Supplementary Materials of [59] and it has an Hamiltonian given by

\[
H = -t \sum_n (c_n^\dagger c_{n+1} + \text{h.c.}) + V \sum_n \cos(2\pi \tau_c n + \phi) c_n^\dagger c_n + \eta \sum_n (-1)^n c_n^\dagger c_n
\]

(S98)

This model is particularly challenging because the staggered potential introduces a relevant perturbation: for finite \(\eta\), a CA is only well defined if the unit cell has an even number of sites, in contrast with the \(\eta = 0\) case. For a CA characterized by \(\tau_c = 5/8\), defining \(\varphi = L\phi/2\) (notice the factor of 2 to account for the periodicity in \(\phi\) for finite \(\eta\)), we have

\[
\mathcal{P}(\varphi, \kappa) = t_R \cos(\kappa) + V_R \cos(\varphi) + V_{2R} \cos(2\varphi)
\]

\[
\approx 2 \cos(\kappa) - \eta E \left[ V^4 (-2\sqrt{2} + E^2 + \eta^2) + \frac{V^6}{2} \right] \cos(\varphi) - \frac{V^8}{128} \cos(2\varphi) + \cdots
\]

(S99)

Note that the fundamental harmonic in \(\varphi\) only arises for \(\eta \neq 0\) (for \(\eta = 0\) we get the Aubry-André model and a fundamental harmonic in \(\varphi' = L\phi\)). The condition \(|t_R| = |V_R|\) is a good approximation for the self-duality condition for \(\tau\) close to \(\tau_c\) as long as \(|t_R|, |V_R| \gg |V_{2R}|\). In Fig. S99 the approximate SD curves obtained through \(|t_R| = |V_R|\) along with the IPR results for \(\tau_i = 575/918\). The reentrant extended phase occurs in the region with smaller IPR that arises at larger \(V\), as observed in [8]. This reentrant transition is captured by the SD curves of the CA with \(\tau_c = 5/8\). The phase transition points do not match exactly the IPR results due to the approximate nature of these curves. In particular we have that \(|V_{2R}/t_R| = |V_{2R}/V_R| \approx 0.2\) for the values of \(V\) around this reentrant transition (the approximation for the smaller \(V\) extended-to-localized transitions is much better as in this case
Furthermore, we also verified that the location of reentrant phase is very sensitive to \( \tau \) and therefore higher-order CAs are needed for more accurate quantitative phase boundaries, for which an analytical treatment becomes more challenging. Nonetheless, it is remarkable to see that the reentrant behaviour is already qualitatively (and almost quantitatively) captured for a CA with only 8 sites in the unit cell. This happens because the values of \( \eta \) considered here are large enough so that the original Aubry-André coupling \( V_{2R} \) may be seen as a perturbation compared to the \( \eta \)-generated coupling \( V_R \), even for such a small unit cell.

\[
|V_{2R}/t_R| = |V_{2R}/V_R| \approx 10^{-5}.
\]

FIG. S5. IPR results for the model in Eq. S98 for \( \eta = 1.8 \), using \( \tau_i = 575/918 \), along with the SD curves obtained through the condition \( |t_R| = |V_R| \) for \( \tau_c = 5/8 \) (see Eq. S99), with the coefficients defined in Eq. S99.