TRACIAL STABILITY FOR $C^*$-ALGEBRAS

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Abstract. We consider tracial stability, which requires that tuples of elements of a $C^*$-algebra with a trace that nearly satisfy a relation are close to tuples that actually satisfy the relation. Here both "near" and "close" are in terms of the associated 2-norm from the trace, e.g. the Hilbert-Schmidt norm for matrices. Precise definitions are stated in terms of liftings from tracial ultraproducts of $C^*$-algebras. We completely characterize matricial tracial stability for nuclear $C^*$-algebras in terms of certain approximation properties for traces. For non-nuclear $C^*$-algebras we find new obstructions for stability by relating it to Voiculescu’s free entropy dimension. We show that the class of $C^*$-algebras that are stable with respect to tracial norms on real-rank-zero $C^*$-algebras is closed under tensoring with commutative $C^*$-algebras. We show that $C(X)$ is tracially stable with respect to tracial norms on all $C^*$-algebras if and only if $X$ is approximately path-connected.

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Introduction

The notion of stability is an old one. For a given equation $p(x_1, \ldots, x_n) = 0$ of noncommutative variables $x_1, \ldots, x_n$ one can ask if it is "stable", meaning that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $B$ is a $C^*$-algebra with $b_1, \ldots, b_n \in B$ and $\|p(b_1, \ldots, b_n)\| < \delta$, then there exist $c_1, \ldots, c_n \in B$ such that $p(c_1, \ldots, c_n) = 0$ and $\|c_k - b_k\| < \epsilon$ for $1 \leq k \leq n$.

In other words, if some tuple is close to satisfying the equation, it is near to something that does satisfy the equation.

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"Stability under small perturbations" questions depend very much on the norm we consider and the class of $C^*$-algebras $\mathcal{B}$ we allow.

A folklore "stability" result is related to projections. If $x = x^*$ and $\|x - x^2\| < \varepsilon < 1/4$, with the norm being the usual operator norm, then there is a projection $p \in C^*(x)$ with $\|p - x\| < \sqrt{\varepsilon}$. There are easily proved similar results for isometries $1 - x^*x = 0$ and unitaries $(1 - x^*x)^2 + (1 - xx^*)^2 = 0$ using the polar decomposition.

For the property of being normal, $x^*x - xx^* = 0$, the famous question of stability for finite matrices was asked by Halmos (17). He asked whether an almost normal contractive matrix is necessarily close to a normal contractive matrix. This is considered independently of the matrix size and "almost" and "close" are meant with respect to the operator norm. This question was answered positively by Lin’s famous theorem [20] (see also [7] and [19]).

However the result does not hold when matrices are replaced by operators. A classical example is the sequence $\{S_n\}$ of weighted unilateral shifts with weights, $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1, 1, \ldots$. Each $S_n$ is a compact perturbation of the unweighted shift and Fredholm index arguments show that the distance from $S_n$ to the normal operators is exactly 1, but

$$\lim_{n \to \infty} \|S_n^*S_n - S_nS_n^*\| = 0.$$ 

In other words, the relation $\|x\| \leq 1, \ xx^* - x^*x = 0$ is stable with respect to the class of matrix algebras but is not stable with respect to the class of all $C^*$-algebras. Thus "stability" questions depend on the class of $C^*$-algebras you are considering.

The property of being stable with respect to the class of all $C^*$-algebras and the operator norm is called weak semiprojectivity (8). An excellent exposition of weak semiprojectivity can be found in Loring’s book ([21]).

Although almost normal operators need not be close to normal, however, using the remarkable distance formula of Kachkovsky and Safarov in [19], the first author and Ye Zhang [15] proved that there is a constant $C$ such that, for every Hilbert-space operator $T$, the distance from $T \oplus T \oplus \cdots$ to the normal operators is at most $C \|T^*T - TT^*\|^{1/2}$.

Another famous Halmos stability question ([17]) asks whether two almost commuting unitary matrices are necessarily close to two exactly commuting unitary matrices. It was answered by Voiculescu in the negative [26] (see [5] for a short proof). However if the operator norm is replaced by Hilbert-Schmidt norm, then things change dramatically. In the Hilbert-Schmidt norm almost commuting unitary matrices turn out to be close to commuting ones, and almost commuting self-adjoint ones to be close to commuting self-adjoint ones as was shown in [11] by the first author and Weihua Li. Several quantitative results (estimating $\delta(\varepsilon)$) for almost commuting $k$-tuples of self-adjoint, unitary, and normal matrices with respect to this norm have been obtained in [8], [24] and [6]. Much more generally, it was proved in [11] that any polynomial equation of commuting normal variables is stable with respect to the tracial norms on diffuse von Neumann algebras.
However nothing is known for polynomial relations in non-commuting variables. In this paper we initiate a study of stability of non-commutative polynomial relations, which we translate into lifting problems for noncommutative \( C^* \)-algebras.

We consider \( C^* \)-algebras \( B \) that have a tracial state \( \rho \), and we measure "almost" and "close" in terms of the 2-(semi)norm on \( B \) given by

\[
\| x \|_2 = \rho (x^*x)^{1/2}.
\]

Thus we will address a "Hilbert-Schmidt" type of stability that we call tracial stability.

The original \( \epsilon - \delta \)-definition of norm stability can be reformulated in terms of approximate liftings from ultraproducts \( \prod_{i \in I} A_i \) of \( C^* \)-algebras \( A_i \). Similarly tracial stability can be reformulated in terms of approximate liftings from tracial ultraproducts \( \prod_{i \in I} (A_i, \rho_i) \) of tracial \( C^* \)-algebras \((A_i, \rho_i)\). These ideas are made precise in section 2.

Suppose \( C \) is a class of unital \( C^* \)-algebras that is closed under isomorphisms. We say that a separable unital \( C^* \)-algebra \( A \) is \( C \)-tracially stable if every unital \( * \)-homomorphism from \( A \) into a tracial ultraproduct of \( C^* \)-algebras from the class \( C \) is approximately liftable.

If \( A \) is the universal \( C^* \)-algebra of a relation \( p \), this definition is equivalent to the \( \epsilon - \delta \) definition above with the norm being a tracial norm and \( C^* \)-algebras \( B \) being from the class \( C \).

We will be interested here in matricial tracial stability, \( II_1 \)-factor tracial stability, \( W^* \)-factor-tracial stability, \( RR0 \)-tracial stability (that is when \( C \) is the class of real rank zero \( C^* \)-algebras), and \( C^* \)-tracial stability (that is when \( C \) is the class of all \( C^* \)-algebras).

All previous results (\[11\], \[8\], \[24\] and \[6\]) on matrices which almost commute w.r.t. the Hilbert-Schmidt norm can be reformulated as matricial tracial stability of separable commutative \( C^* \)-algebras.

In fact first results related with \( RR0 \)-tracial stability appeared in \[22\] and \[23\], where there were proved some stability results for projections almost commuting with matrix units, and this was applied to deducing the tracial Rokhlin property for an automorphism of a \( C^* \)-algebra from the Rokhlin property for the corresponding automorphism of an associated von Neumann algebra.

In section 2 of this paper we extend substantially all the previous results about matrices almost commuting with respect to the Hilbert-Schmidt norm.

**Theorem 2.7** Suppose the class \( C \subseteq RR0 \) is closed under taking direct sums and unital corners. If \( B \) is separable unital and \( C \)-tracially stable, and if \( X \) is a compact metric space, then \( B \otimes C(X) \) is \( C \)-tracially stable. In particular every separable unital commutative \( C^* \)-algebra is \( C \)-tracially stable.

Of course a natural obstruction for a \( C^* \)-algebra to be tracially stable can be simply a lack of \( * \)-homomorphisms. Say, a \( C^* \)-algebra has enough almost \( * \)-homomorphisms to matrix algebras to separate points, but not enough actual \( * \)-homomorphisms to matrix algebras to separate points, then of course it is not matricially trivially stable. However it turns out to be not the only obstruction.
We show that a certain approximation property for traces have to hold for a C*-algebra to be matricially tracially stable. For nuclear C*-algebras (even tracially nuclear, see definition in section 3) this property is also sufficient.

**Theorem 3.10** Suppose $A$ is a separable tracially nuclear C*-algebra with at least one tracial state. The following are equivalent

1. $A$ is matricially tracially stable
2. for every tracial state $\tau$ on $A$, there is a positive integer $n_0$ and, for each $n \geq n_0$ there is a unital $*$-homomorphism $\rho_n : A \to M_n(\mathbb{C})$ such that, for every $a \in A$,
   \[ \tau(a) = \lim_{n \to \infty} \tau_n(\rho_n(a)). \]
   (here $\tau_n$ is the usual tracial state on $M_n(\mathbb{C})$)
3. $A$ is W*-factor tracially stable.

These conditions are stronger than just a property to have a separating family of $*$-homomorphisms to matrix algebras. Namely,

**Example 3.11** There exists a residually finite-dimensional (RFD) nuclear C*-algebra which has finite-dimensional irreducible representations of all dimensions but is not matricially tracially stable.

However a Type I C*-algebra is W*-factor tracially stable (in particular, matricially tracially stable) when it has sufficiently many matrix representations (for example, to have a 1-dimensional representation is enough):

**Corollary 3.9** Suppose $A$ is a type I separable unital C*-algebra such that for all but finitely many positive integers $n$ it has a unital $n$-dimensional representation. Then $A$ is W*-factor tracially stable. In particular, for any type I C*-algebra $A$, $A \oplus \mathbb{C}$ is W*-factor tracially stable.

In section 4 we find a close relationship between matricial tracial stability and Voiculescu’s free entropy dimension $\delta_0$. The following result shows that a matricially tracially stable algebra may be forced to have a lot of non-unitarily equivalent representations of some given dimension. Below by $\text{Rep}(A,k) / \sim$ we denote the set of all unital $*$-homomorphisms from $A$ into $M_k(\mathbb{C})$ modulo unitary equivalence.

**Theorem 4.2** Suppose $A = C^*(x_1, \ldots, x_n)$ is matricially tracially stable and $\tau$ is an embeddable tracial state on $A$ such that $1 < \delta_0(x_1, \ldots, x_n)$. Then

\[
\limsup_{k \to \infty} \frac{\log \text{Card}(\text{Rep}(A,k)/\sim)}{k^2} = \infty.
\]

This allows us to show that for non-(tracially) nuclear C*-algebras there arise new obstructions, other than in the (tracially) nuclear case, for being matricially tracially stable.

**Theorem 4.6** There exists an RFD C*-algebra which has the approximation property from Theorem 3.10 but which is not matricially tracially stable.

In the last section we consider C*-tracial stability. In contrast to RR0-tracial stability, not all commutative C*-algebras have this property. The main result of section 5 is a characterization of C*-tracial stability for separable commutative C*-algebras. For that we introduce approximately path-connected spaces. We say that a topological space $X$ is approximately path-connected if, for any finitely many
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points $x_1, \ldots, x_n$, one can find arbitrarily close to them points $x'_1, \ldots, x'_n$ which can be connected by a continuous path.

**Theorem 5.3** Suppose $X$ is a compact metric space. The following are equivalent:

1. $C(X)$ is $C^*$-tracially stable.
2. $X$ is approximately path-connected.

As a final remark here it is interesting how things are reversed when norm stability is replaced by $C^*$-tracial stability. For example, being a projection is norm stable, but not $C^*$-tracially stable. Indeed, if $\{f_n\}$ is any sequence of functions in $C[0,1]$ with trace $\rho(f) = \int_0^1 f(x) dx$ and $0 \leq f_n \leq 1$ and $f_n(x) \to \chi_{[0,1/2]}(x)$ a.e., then $\|f_n - f_n^2\|_2 \to 0$, but since $0, 1$ are the only projections, the $f_n$’s are not $\|\cdot\|_2$-close to a projection in $C[0,1]$. On the other hand, for $C^*$-tracial stability, the problems of normality, commuting pairs of unitaries, and commuting triples of selfadjoint operators the answers are all affirmative, in contrast to the norm stability.

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1. **Preliminaries**

1.1. **Ultrproducts.**

If a unital $C^*$-algebra $B$ has a tracial state $\rho$, we denote the 2-norm (seminorm) given by $\rho$ as $\|\cdot\|_2 = \|\cdot\|_{2,\rho}$ defined by

$$\|b\|_2 = \rho(b^*b)^{1/2}.$$  

We also denote the GNS representation for the state $\rho$ by $\pi_\rho$.

Suppose $I$ is an infinite set and $\alpha$ is an ultrafilter on $I$. We say $\alpha$ is nontrivial if there is a sequence $\{E_n\}$ in $\alpha$ such that $\cap_n E_n = \emptyset$. Suppose $\alpha$ is a nontrivial ultrafilter on a set $I$ and, for each $i \in I$, suppose $A_i$ is a unital $C^*$-algebra with a tracial state $\rho_i$. The tracial ultraproduct $\prod_{i \in I} (A_i, \rho_i)$ is the $C^*$-product $\prod_{i \in I} A_i$ modulo the ideal $J_\alpha$ of all elements $\{a_i\}$ in $\prod_{i \in I} A_i$ for which

$$\lim_{i \to \alpha} \|a_i\|_{2,\rho_i} = \lim_{i \to \alpha} \rho_i(a_i^*a_i) = 0.$$  

We denote the coset of an element $\{a_i\}$ in $\prod_{i \in I} A_i$ by $\{a_i\}_\alpha$.

Tracial ultraproducts for factor von Neumann algebras was first introduced by S. Sakai [25] where he proved that a tracial ultraproduct of finite factor von Neumann algebras is a finite factor. More recently, it was shown in [11] that a tracial ultraproduct $\prod_{i \in I} (A_i, \rho_i)$ of $C^*$-algebras is always a von Neumann algebra with a faithful normal tracial state $\rho_\alpha$ defined by

$$\rho_\alpha(\{a_i\}_\alpha) = \lim_{i \to \alpha} \rho_i(a_i).$$
If there is no confusion, we will denote it just by $\rho$.

1.2. Background theorem.
The next theorem is a key tool for some of our results. It says, that two tuples that are both close to the same tuple in a hyperfinite von Neumann algebra are nearly unitarily equivalent to each other.

**Theorem 1.1.** \(\text{[16]}\) Suppose $A = W^* (x_1, \ldots, x_s)$ is a hyperfinite von Neumann algebra with a faithful normal tracial state $\rho$. For every $\varepsilon > 0$ there is a $\delta > 0$ and an $N \in \mathbb{N}$ such that, for every unital C*-algebra $B$ with a factor tracial state $\tau$ and $a_1, \ldots, a_s, b_1, \ldots, b_s \in B$, if, for every $*$-monomial $m (t_1, \ldots, t_s)$ with degree at most $N$,

\[
\| \tau (m (a_1, \ldots, a_s)) - \rho (m (x_1, \ldots, x_s)) \|_2 < \delta,
\]

\[
\| \tau (m (b_1, \ldots, b_s)) - \rho (m (x_1, \ldots, x_s)) \|_2 < \delta,
\]

then there is a unitary element $u \in B$ such that

\[
\sum_{k=1}^s \| u a_k u^* - b_k \|_2 < \varepsilon.
\]

The following lemma is very well known.

**Lemma 1.2.** If $\gamma$ is a $*$-homomorphism from $A$ to a von Neumann algebra $\mathcal{M}$ with a faithful normal trace $\tau$, then $\gamma(A)' \cong \pi_{\tau \gamma}(A)'$.

2. Tracial stability

Suppose $A$ is a unital C*-algebra and $\pi : A \to \prod_{i \in I} (A_i, \rho_i)$. We say that $\pi$ is *approximately liftable* if there is a set $E \in \alpha$ and for every $i \in E$ there is a unital $*$-homomorphism $\pi_i : A \to A_i$ such that, for every $a \in A$,

\[
\pi (a) = \{ \pi_i (a) \}_\alpha
\]

where we arbitrarily define $\pi_i (a) = 0$ when $i \notin E$.

It actually makes no difference how we define $\pi_i (a)$ when $i \notin E$ since the equivalence class $\{ \pi_i (a) \}_\alpha$ does not change.

Suppose $\mathcal{C}$ is a class of unital C*-algebras that is closed under isomorphisms. We say that a separable unital C*-algebra $A$ is $\mathcal{C}$-tracially stable if every unital $*$-homomorphism from $A$ into any tracial ultraproduct $\prod_{i \in I} (A_i, \rho_i)$, with $A_i \in \mathcal{C}$ and $\rho_i$ any trace on $A_i$, is approximately liftable.

Thus $A$ is $C^*$-tracially stable if every unital $*$-homomorphism from $A$ into any tracial ultraproduct is approximately liftable.

We say that $A$ is matricially tracially stable if every unital $*$-homomorphism from $A$ into an ultraproduct of full matrix algebras $\mathcal{M}_n (\mathbb{C})$ is approximately liftable.

We say that $A$ is finite-dimensionally tracially stable if every unital $*$-homomorphism from $A$ into a tracial ultraproduct of finite-dimensional C*-algebras is approximately liftable.

We say that $A$ is $W^*$-tracially stable if every unital $*$-homomorphism from $A$ into a tracial ultraproduct of von Neumann algebras is approximately liftable.

We say that $A$ is $W^*$-factor tracially stable if every unital $*$-homomorphism from $A$ into a tracial ultraproduct of factor von Neumann algebras is approximately liftable.
Remark 2.1. One can also define tracial stability in the non-unital category. For instance we can define the non-unital version of matricial tracial stability by saying that a separable unital C*-algebra $A$ is matricially tracially stable if every *-homomorphism from $A$ into an ultraproduct of full matrix algebras $M_n(\mathbb{C})$ is liftable. It is easy to show that $A$ is matricially tracially stable in the non-unital category iff $\tilde{A}$ is matricially tracially stable in the unital category. Here $\tilde{A} = A^+$ when $A$ is non-unital and $A = A \oplus \mathbb{C}$ when $A$ is unital. Thus the non-unital version can be easily obtained from the unital one.

To see how this ultraproduct formulation represents the $\varepsilon$-$\delta$ definition of tracial stability mentioned in the introduction, suppose $A$ is the universal unital C*-algebra generated by contractions $x_1, \ldots, x_s$ subject to a relation $p(x_1, \ldots, x_s) = 0$. Suppose $\pi : A \to \prod_{i \in I} (A_i, \rho_i)$ is a unital *-homomorphism such that, for each $1 \leq k \leq s$, $\pi(x_k) = \{x_k(i)\}_{\alpha}$. We then have $0 = \|p \left( x_1, \ldots, x_s \right) \|_{2, \rho_n} = \|p \left( \pi(x_1), \ldots, \pi(x_s) \right) \|_{2, \rho_n} = \lim_{i \to \alpha} \|p \left( x_1(i), \ldots, x_s(i) \right) \|_{2, \rho_i}$.

Since $\alpha$ is a nontrivial ultrafilter on $I$, there is a decreasing sequence $E_1 \supset E_2 \supset \cdots$ in $\alpha$ such that $\bigcap_{k \in \mathbb{N}} E_k = \emptyset$. First suppose $A$ is C*-tracially stable. Then, for each positive integer $m$ there is a number $\delta_m > 0$ such that, when

$$\|p \left( x_1(i), \ldots, x_s(i) \right) \|_{2, \rho_i} < \delta_m$$

there is a unital *-homomorphism $\gamma_{m,i} : A \to A_i$ such that

$$\max_{1 \leq k \leq s} \|x_k(i) - \gamma_{m,i}(x_k)\|_{2, \rho_i} < 1/m.$$

Since $\lim_{i \to \alpha} \|p \left( x_1(i), \ldots, x_s(i) \right) \|_{2, \rho_i} = 0$ we can find a decreasing sequence $\{A_n\}$ in $\alpha$ with $A_n \subset E_n$ such that, for every $i \in A_n$,

$$\|p \left( x_1(i), \ldots, x_s(i) \right) \|_{2, \rho_i} \leq \delta_n.$$ 

For $i \in A_n \setminus A_{n+1}$ we define $\pi_i = \gamma_{n,i}$. We then have that $\{\pi_i\}_{i \in A_1}$ is an approximate lifting of $\pi$.

On the other hand, if $A$ is not C*-tracially stable, then there is an $\varepsilon > 0$ such that, for every positive integer $n$ there is a tracial unital C*-algebra $(A_n, \rho_n)$ and $x_1(n), \ldots, x_s(n)$ such that

$$\|p \left( x_1(n), \ldots, x_s(n) \right) \|_{2, \rho_n} < 1/n,$$

but for every unital *-homomorphism $\gamma : A \to A_n$

$$\max_{1 \leq k \leq s} \|x_k(n) - \gamma(x_k)\|_{2, \rho_n} \geq \varepsilon.$$ 

If we let $\alpha$ be any free ultrafilter on $\mathbb{N}$, we have that the map $\pi(x_k) = \{x_k(n)\}_{\alpha}$ extends to a unital *-homomorphism into $\prod_{n \in \mathbb{N}} (A_n, \rho_n)$ that is not approximately liftable.
The following result shows that pointwise $\| \|_\infty$-limits of approximately liftable representations are approximately liftable.

**Lemma 2.2.** Suppose $A = C^* \{ \{b_1, b_2, \ldots \} \}, \{ (A_i, \rho_i) : i \in I \}$ is a family of tracial $C^*$-algebras, $\alpha$ is a nontrivial ultrafilter on $I$, and $\pi : A \to \prod_{i \in I} (A_i, \rho_i)$ is a unital $*$-homomorphism such that, for each $k \in \mathbb{N}$,

$$\pi (b_k) = \{ b_k (i) \}_\alpha .$$

The following are equivalent:

1. $\pi$ is approximately liftable
2. For every $\varepsilon > 0$ and every $N \in \mathbb{N}$, there is a set $E \in \alpha$ and for every $i \in E$ there is a unital $*$-homomorphism $\pi_i : A \to A_i$ such that, for $1 \leq k \leq N$ and every $i \in E$,

$$\| \pi_i (b_k) - b_k (i) \|_{2, \rho_i} < \varepsilon .$$

**Proof.** Obviously 1) implies 2). We need to prove the opposite implication. Since $\alpha$ is nontrivial, there is a decreasing sequence $\{ B_n \}$ of elements of $\alpha$ such that $\cap_{n=1}^\infty B_n = \emptyset$. For each $n \in \mathbb{N}$, let $N = n$ and $\varepsilon = 1/n$ and let $E_n \in \alpha$ and, for each $i \in E_n$, choose a unital $*$-homomorphism $\pi_{n,i} : A \to A_i$ such that

$$\| \pi_{n,i} (b_k) - b_k (i) \|_{2, \rho_i} < 1/n$$

for $1 \leq k \leq n$. We define $F_n = \cap_{k=1}^n (B_k \cap E_k)$ and for $i \in F_{n+1} \setminus F_n$ we define $\pi_i = \pi_{n,i}$. Since $\cap_{n=1}^\infty F_n = \emptyset$, $\pi_i$'s are defined for each $i \in \bigcup_{n=1}^\infty F_n \in \alpha$. Clearly,

$$\{ \pi_i (b_k) \}_\alpha = \{ b_k (i) \}_\alpha = \pi (b_k)$$

for $k = 1, 2, \ldots$. Since the set $S$ of all $\alpha \in A$ such that $\{ \pi_i (a) \}_\alpha = \pi (a)$ is a unital $C^*$-algebra, we see that $S = A$ and that $\pi$ is approximately liftable. \qed

Although we consider all tracial ultraproducts, however when the algebra $A$ is separable, we need to consider only ultraproducts over $\mathbb{N}$ with respect to one non-trivial ultrafilter.

**Lemma 2.3.** Suppose $A = C^* \{ x_1, x_2, \ldots \}$ is a separable unital $C^*$-algebra, $C$ is a class of unital $C^*$-algebras closed under isomorphism and $\alpha$ is a nontrivial ultrafilter on $\mathbb{N}$. The following are equivalent:

1. $A$ is $C$-tracially stable
2. If $\{(B_n, \gamma_n)\}$ is a sequence of tracial $C^*$-algebras in $C$ and $\pi : A \to \prod_{i \in I} (B_i, \gamma_i)$ is a unital $*$-homomorphism, then $\pi$ is approximately liftable.
3. For every $\varepsilon > 0$, for every positive integer $s$, and for every tracial state $\rho$ on $A$, there is a positive integer $N$ such that if $B \in C$ and $\gamma$ is a tracial state on $B$ and $b_1, \ldots, b_N \in B$ such that $\| b_k \| \leq 1 + \| x_k \|$ for $1 \leq k \leq N$ and

$$| \rho (m (x_1, \ldots, x_N)) - \gamma (b_1, \ldots, b_N) | < \frac{1}{N}$$

for every measurable bounded function $m$ on $\mathbb{R}^N$.
for all $*$-monomials $m(t_1, \ldots, t_N)$ with degree at most $N$, then there is a unital $*$-homomorphism $\pi : A \to B$ such that
\[
\sum_{k=1}^s \|\pi(x_k) - b_k\|_{2, \gamma} < \varepsilon.
\]

Proof. (1) $\Rightarrow$ (2). This is obvious.
(2) $\Rightarrow$ (3). Assume (3) is false. Then there is an $\varepsilon > 0$, an integer $s \in \mathbb{N}$ and a tracial state $\rho$ on $A$ and, for each $N \in \mathbb{N}$, there is a $B_N \in \mathcal{C}$ with a tracial state $\gamma_N$ and $\{b_{N,1}, b_{N,2}, \ldots, b_{N,N}\} \subset B_N$ such that
\[
\|b_{N,k}\| \leq \|x_k\| + 1
\]
for $1 \leq k \leq N < \infty$, and
\[
|\rho(m(x_1, \ldots, x_N)) - \gamma_N(m(b_{N,1}, \ldots, b_{N,N}))| < \frac{1}{N}
\]
for all $*$-monomials $m(t_1, \ldots, t_N)$ with degree at most $N$, such that, for each $N$ there is no $*$-homomorphism $\pi : A \to B_N$ such that
\[
\sum_{k=1}^s \|\pi(x_k) - b_{N,k}\|_{2, \gamma_N} < \varepsilon.
\]
For each $1 \leq N < k$ let $b_{N,k} = 0 \in B_N$. For $1 \leq k < \infty$, let
\[
b_k = \{b_{N,k}\}_\alpha \in \prod_{N \in \mathbb{N}} (B_N, \gamma_N).
\]
Let $\gamma$ be the limit trace on $\prod_{N \in \mathbb{N}} (B_N, \gamma_N)$. It follows that for every $*$-monomial $m(t_1, \ldots, t_k)$ we have
\[
(2.1) \quad \rho(m(x_1, \ldots, x_k)) = \gamma(m(b_1, \ldots, b_k)).
\]
Define a unital $*$-homomorphism $\pi : A \to \prod_{N \in \mathbb{N}} (B_N, \gamma_N)$ by
\[
\pi(x_i) = b_i.
\]
Since $\gamma$ is faithful, it follows from (2.1) that $\pi$ is well-defined and $\rho = \gamma \circ \pi$. By (2), $\pi$ is approximately liftable so there is an $E \in \alpha$ and, for each $N \in E$ a unital $*$-homomorphism $\pi_N : A \to B_N$ such that, for $1 \leq k < \infty$
\[
\pi(x_k) = \{\pi_N(x_k)\}_\alpha.
\]
It follows that
\[
\lim_{N \to \alpha} \sum_{k=1}^s \|b_{N,k} - \pi_N(x_k)\|_{2, \gamma_N} = 0,
\]
which means for some $N \in \mathbb{N}$,
\[
\sum_{k=1}^s \|b_{N,k} - \pi_N(x_k)\|_{2, \gamma_N} < \varepsilon.
\]
This contradiction implies (3) must be true.
(3) $\Rightarrow$ (1). Suppose (3) is true and $\{(B_i, \gamma_i) : i \in I\}$ is a collection of tracial unital C*-algebras with each $B_i$ in $\mathcal{C}$ and suppose $\beta$ is a nontrivial ultrafilter on $I$.
\[ \pi : A \rightarrow \prod_{i \in I} (B_i, \gamma_i). \] Let \( \gamma \) be the limit trace along \( \beta \) and define \( \rho = \gamma \circ \pi \). If \( k \in \mathbb{N} \), we let \( \varepsilon = \frac{1}{k} \) and \( s = k \) we can choose \( N_k \) as in (2). Since \( \beta \) is nontrivial there is a decreasing sequence \( \{ E_j \} \) in \( \beta \) whose intersection is empty. For each \( k \in \mathbb{N} \) write

\[ \pi (x_k) = \{ b_{i,k} \}_\beta \]

with \( \| b_{i,k} \| \leq \| x_k \| \) for every \( i \in I \). It follows, for every *-monomial \( m (t_1, \ldots, t_k) \) that

\[ \lim_{i \rightarrow \beta} \gamma_i (m (b_{i,1}, \ldots, b_{i,k})) = \rho (m (x_1, \ldots, x_k)). \]

Thus the set \( F_k \) consisting of all \( i \in I \) such that

\[ |\rho (m (x_1, \ldots, x_N_k)) - \gamma_i (m (b_{i,1}, \ldots, b_{i,N_k}))| < \frac{1}{N_k} \]

for all *-monomials \( m \) with degree at most \( N_k \) must be in \( \beta \). For each \( k \in \mathbb{N} \) let

\[ W_k = E_k \cap F_1 \cap \cdots \cap F_k. \]

For each \( i \in W_k \setminus W_{k+1} \) there is a unital *-homomorphism \( \pi_i : A \rightarrow B_i \) such that

\[ \sum_{j=1}^{k} \| \pi_i (x_j) - b_{i,j} \|_{2,\gamma_i} < \frac{1}{k}. \]

It is clear for every \( k \in \mathbb{N} \) that \( \pi (x_k) = \{ \pi_i (x_k) \}_\beta \). Since \( \{ a \in A : \pi (a) = \{ \pi_i (a) \}_\beta \} \) is a unital C*-subalgebra of \( A \), containing all \( x_1, x_2, \ldots, \), we conclude that \( A = \{ a \in A : \pi (a) = \{ \pi_i (a) \}_\beta \} \) and thus \( \pi \) is approximately liftable. \( \square \)

It is clear that C*-tracially stable implies W*-tracially stable implies factor tracially stable implies matricially tracially stable. Here are some slightly more subtle relationships.

**Lemma 2.4.** Suppose \( A \) is a separable unital C*-algebra. Then \( A \) is matricially tracially stable if and only if every unital *-homomorphism from \( A \) into a tracial ultraproduct of hyperfinite factors is approximately liftable. Also \( A \) is finitely-dimensionally tracially stable if and only if every unital *-homomorphism from \( A \) into a tracial ultraproduct of hyperfinite von Neumann algebras is approximately liftable.

**Proof.** We prove the first statement; the second follows in a similar fashion. The "if" part is clear since \( \mathcal{M}_\infty (C) \) is a hyperfinite factor. Suppose \( A \) is matricially tracially stable, and suppose \( \{(M_i, \tau_i) : i \in I \} \) is a family with each \( M_i \) a hyperfinite factor and \( \tau_i \) is the unique normal faithful tracial state on \( M_i \). Suppose also that \( \alpha \) is a nontrivial ultrafilter on \( I \) and \( \pi : A \rightarrow \prod_{i \in I} (M_i, \tau_i) \) is a unital *-homomorphism.

If \( E = \{ i \in I : \dim M_i < \infty \} \in \alpha \), then \( \prod_{i \in I} (M_i, \tau_i) = \prod_{i \in E} (M_i, \tau_i) \) is a tracial product of matrix algebras and \( \pi \) is approximately liftable. If \( I \setminus E \in \alpha \), we can assume that each \( M_i \) is a hyperfinite \( II_1 \)-factor. Since \( \alpha \) is nontrivial, there is a
decreasing sequence \( \{E_n\} \) in \( \alpha \) such that \( E_1 = I \) and \( \bigcap_{n=1}^{\infty} E_n = \emptyset \). Suppose \( \{a_n\} \)

\[ \pi(a_n) = \{a_n(i)\}_\alpha \]

with \( ||a_n(i)|| \leq ||a_n|| \), for all \( n \) and \( i \). If \( n \in \mathbb{N} \) and \( i \in E_n \setminus E_{n+1} \), we can find a unital subalgebra \( \mathcal{B}_i \subset M_i \) such that \( \mathcal{B}_i \) is isomorphic to \( M_{k_i}(\mathbb{C}) \) for some \( k_i \in \mathbb{N} \) and such that there is a sequence \( \{b_m(i)\} \) in \( \mathcal{B}_i \) with \( ||b_m(i)|| \leq ||a_m|| \) for each \( m \) and such that

\[ ||a_m(i) - b_m(i)||_2 \leq \frac{1}{n} \]

for \( 1 \leq m \leq n \). It follows that

\[ \pi(a_n) = \{a_n(i)\}_\alpha = \{b_n(i)\}_\alpha \in \prod_{i \in I} (\mathcal{B}_i, \tau_i). \]

It follows that \( \pi : A \to \prod_{i \in I} (\mathcal{B}_i, \tau_i) \). Since \( A \) is matricially tracially stable, \( \pi \) is approximately liftable. \( \square \)

Recall that a \( C^* \)-algebra has real rank zero (RR0) if each its self-adjoint element can be approximated by self-adjoint elements with finite spectrum.

**Theorem 2.5.** Suppose every algebra in the class \( C \) is RR0. Then every separable unital commutative \( C^* \)-algebra is \( C \)-tracially stable.

**Proof.** In [14] the authors proved that if \( J \) is a norm-closed two-sided ideal of a real-rank zero \( C^* \)-algebra \( M \) such that \( M/J \) is an AW*-algebra, then for any commutative separable unital \( C^* \)-algebra \( A \), any \( * \)-homomorphism \( \pi : A \to M/J \), lifts to a unital \( * \)-homomorphism \( \rho : A \to M \). Since the direct product of real-rank zero \( C^* \)-algebras is a real-rank zero \( C^* \)-algebra and a tracial ultraproduct of \( C^* \)-algebras is a von Neumann algebra, the statement follows. \( \square \)

**Proposition 2.6.** Suppose \( C \) is a class such that \( C \oplus B \in C \) whenever \( B \in C \). Every \( C \)-tracially stable separable unital \( C^* \)-algebra \( A \) with at least one representation into a tracial ultraproduct of \( C^* \)-algebras in \( C \) must have a one-dimensional unital representation.

**Proof.** Suppose \( \pi : A \to \prod_{i \in I} (\mathcal{B}_i, \gamma_i) \) is a unital \( * \)-homomorphism where each \( \mathcal{B}_i \in C \) and \( \alpha \) is a nontrivial ultrafilter on \( I \). Let \( \mathcal{D}_i = C \oplus \mathcal{B}_i \) for each \( i \in I \). Since \( \alpha \) is nontrivial there is a decreasing sequence \( I = E_1 \supseteq E_2 \supseteq \cdots \) in \( \alpha \) whose intersection is \( \emptyset \). For \( i \in E_n \setminus E_{n+1} \) define a trace \( \rho_i \) on \( \mathcal{D}_i \) by

\[ \rho_i(\lambda \oplus B) = \frac{1}{n} \lambda + \left(1 - \frac{1}{n}\right) \gamma_i(B). \]

Then

\[ \prod_{i \in \mathbb{N}} (\mathcal{D}_i, \rho_i) = \prod_{i \in \mathbb{N}} (\mathcal{B}_i, \gamma_i). \]

The rest is easy. \( \square \)
Theorem 2.7. Suppose the class $C \subseteq RR_0$ is closed under taking direct sums and unital corners. If $B$ is separable unital and $C$-tracially stable, and if $X$ is a compact metric space, then $B \otimes C(X)$ is $C$-tracially stable.

Proof. Suppose $\{(A_n, \rho_n)\}$ is a sequence of $C^*$-algebras in the class $C$, $\alpha$ is a non-trivial ultrafilter on $\mathbb{N}$, and $f : B \otimes C(X) \to \prod_{n \in \mathbb{N}} (A_n, \rho_n)$ is a unital $*$-homomorphism. By Lemma 2.2, it will be enough for any $x_1, \ldots, x_k \in B \otimes C(X)$, $\epsilon > 0$ to find unital $*$-homomorphisms $\tilde{f}_n : B \otimes C(X) \to A_n$, such that

$$\|\{\tilde{f}_n(x_j)\}_\alpha - f(x_j)\|_{2,\rho} < 3\epsilon,$$

$j \leq k$.

There are $b_i^{(j)} \in B, \phi_i^{(j)} \in C(X), N_1, \ldots, N_k$ such that

$$\|x_j - \sum_{i=1}^{N_j} b_i^{(j)} \otimes \phi_i^{(j)}\| < \epsilon,$$

$j \leq k$. Let

$$M = \max_{1 \leq i \leq N_j, 1 \leq j \leq k} \|f(b_i^{(j)} \otimes 1)\|, \quad N = \max_{j \leq k} N_j.$$

Since $f(1 \otimes C(X))$ is commutative, it is isomorphic to some $C(\Omega)$. We can choose a disjoint collection $\{E_1, \ldots, E_m\}$ of Borel subsets of $\Omega$ whose union is $\Omega$ and $\omega_1 \in E_1, \ldots, \omega_m \in E_m$ such that

$$\|f(1 \otimes \phi_i^{(j)}) - \sum_{i=1}^{m} f(1 \otimes \phi_i^{(j)})(w_i)\chi_{E_i}\|_{2,\rho} < \frac{\epsilon}{NM},$$

for $j \leq k$ and $i \leq N_j$. Since pairwise orthogonal projections generate a commutative $C^*$-algebra, by Theorem 2.5 for each $\chi_{E_l}$, $l \leq m$, we can find projections $P_{i,n} \in A_n$ such that $\chi_{E_l} = \{P_{i,n}\}_\alpha$ and $P_{1,n}, \ldots, P_{m,n}$ are pairwise orthogonal, for each $n$. Since the subalgebra $f(1 \otimes C(X))$ is central in the algebra $f(B \otimes C(X))$, each $\chi_{E_l}$ commutes with $f(B \otimes C(X))$. Hence

$$f(B \otimes C(X)) \subseteq \prod_{n \in \mathbb{N}} \left( \sum_{i=1}^{m} P_{i,n} A_n P_{i,n}, \rho_n \right) = \bigoplus_{n \in \mathbb{N}} \left( \prod_{i=1}^{m} (P_{i,n} A_n P_{i,n}, \rho_n) \right).$$

Since $B$ is $C$-tracially stable, we can find unital $*$-homomorphisms $\psi_{n,i} : B \to P_{i,n} A_n P_{i,n}$, such that

$$\{\psi_{n,i}(b)\}_\alpha = p_i \circ f(b \otimes 1),$$

for any $b \in B$. Here $p_i$ is the projection onto the $i$-th summand in $\bigoplus_{i=1}^{m} \left( \prod_{n \in \mathbb{N}} (P_{i,n} A_n P_{i,n}, \rho_n) \right)$.

Hence for $\psi_n = \bigoplus_{i=1}^{m} \psi_{n,i} : B \to \sum_{i=1}^{m} P_{i,n} A_n P_{i,n}$, we have

$$\{\psi_n(b)\}_\alpha = f(b \otimes 1),$$

for any $b \in B$. Define a $*$-homomorphism $\delta_n : C(X) \to \sum_{i=1}^{m} P_{i,n} A_n P_{i,n}$ by

$$\delta_n(\phi) = \sum_{i=1}^{m} f(1 \otimes \phi)(w_i) P_{i,n}.$$
Define $\hat{f}_n : \mathcal{B} \otimes C(X) \to \mathcal{A}_n$ by

$$\hat{f}_n(b \otimes \phi) = \psi_n(b)\delta_n(\phi).$$

Since $\delta_n(C(X))$ and $\psi_n(B)$ commute, $\hat{f}_n$ is a $*$-homomorphism. By (2.2), (2.3), (2.4), (2.6), for each $j \leq k$ we have

$$(2.7)$$

$$\|\{\hat{f}_n(x_j)\}_{\alpha} - f(x_j)\|_{2,\rho} \leq 2\epsilon + \|\{\hat{f}_n \left( \sum_{i=1}^{N_j} b_i^{(j)} \otimes \phi_i^{(j)} \right) \}_{\alpha} - f \left( \sum_{i=1}^{N_j} b_i^{(j)} \otimes \phi_i^{(j)} \right)\|_{2,\rho}

= 2\epsilon + \left\| \sum_{i=1}^{N_j} \{ \psi_n(b_i^{(j)}) \}_{\alpha} \{ \delta_n(\phi_i^{(j)}) \}_{\alpha} - \sum_{i=1}^{N_j} f(b_i^{(j)} \otimes 1) f(1 \otimes \phi_i^{(j)}) \right\|_{2,\rho}

= 2\epsilon + \left\| \sum_{i=1}^{N_j} f(b_i^{(j)} \otimes 1) \left( \sum_{l=1}^{m} f(1 \otimes \phi_i^{(j)})(w_l)\chi_{E_l} - f(1 \otimes \phi_i^{(j)}) \right) \right\|_{2,\rho}

\leq 2\epsilon + N_jM \frac{\epsilon}{N} \leq 3\epsilon.$$

\[ \square \]

**Proposition 2.8.** Suppose the class $\mathcal{C} \subseteq RR0$ is closed under taking finite direct sums. Then the class of $\mathcal{C}$-tracially stable $C^*$-algebras is closed under taking finite direct sums.

**Proof.** We will show it for direct sums with 2 summands, and the general case is similar. Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{C}$-tracially stable, $\pi : \mathcal{A} \oplus \mathcal{B} \to \prod_{n \in \mathbb{N}} (\mathcal{D}_n, \rho_n)$ a unital $*$-homomorphism and all $\mathcal{D}_n \in \mathcal{C}$. Let

$$p = \pi(1_\mathcal{A}), \ q = \pi(1_\mathcal{B}).$$

Then $p + q = 1$. By Theorem 2.5 we can find projections $P_n, Q_n \in \mathcal{D}_n$ such that

$$\{P_n\}_{\alpha} = p, \ \{Q_n\}_{\alpha} = q, \ P_n + Q_n = 1_{\mathcal{D}_n}.$$ 

Then

$$\pi(\mathcal{A}) \subseteq \prod_{n \in \mathbb{N}} (P_n \mathcal{D}_n P_n, \rho_n), \ \pi(\mathcal{B}) \subseteq \prod_{n \in \mathbb{N}} (Q_n \mathcal{D}_n Q_n, \rho_n).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{C}$-tracially stable, there are unital $*$-homomorphisms $\phi : \mathcal{A} \to \prod P_n \mathcal{D}_n P_n$ and $\psi : \mathcal{B} \to \prod Q_n \mathcal{D}_n Q_n$ such that

$$\{\phi(a)_n\}_{\alpha} = \pi(a), \ \{\psi(b)_n\}_{\alpha} = \pi(b),$$

for all $a \in \mathcal{A}, b \in \mathcal{B}$. Then $\tilde{\pi}$, defined by $\tilde{\pi}(a, b) = \phi(a) + \psi(b)$, is an approximate lift of $\pi$. \[ \square \]

The following proposition is obvious.

**Proposition 2.9.** The class of $\mathcal{C}$-tracially stable $C^*$-algebras is closed under finite free products in the unital category for any $\mathcal{C}$. 


Remark. The preceding proposition cannot be extended to countable free products in the unital category. For example, $B_n = M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ is matricially tracially stable for each positive integer $n$ by Lemma 3.5 and Corollary 3.9 in the next section. Let $\alpha$ be a nontrivial ultrafilter containing the set $\{k! \mid k \in \mathbb{N}\}$. Then, for each $n \in \mathbb{N}$, $B_n$ embeds unitally into $\prod_{n \in \mathbb{N}} (M_i, \tau_i)$ and hence there is a unital $*$-homomorphism from the unital free product $*_{n \in \mathbb{N}} B_n$ into $\prod_{n \in \mathbb{N}} (M_i, \tau_i)$. However, $*_{n \in \mathbb{N}} B_n$ has no finite-dimensional representations. Hence $*_{n \in \mathbb{N}} B_n$ is not matricially tracially stable. However, if $A_n$ is a separable tracial-$C^*$-stable $C^*$-algebra for each $n \in \mathbb{N}$, then the unitization of the free product of the $A_n$’s in the nonunital category is tracially-$C^*$-stable.

3. Matricial tracial stability

Notation: $\tau_n$ usual trace on $M_n(\mathbb{C})$. The tracial ultraproduct is denoted

$$\prod_{n \in \mathbb{N}}^\alpha (M_n(\mathbb{C}), \tau_n).$$

3.1. Embeddable traces. Definition Suppose $A$ is a unital separable $C^*$-algebra and $\tau$ is a tracial state on $A$. We say $\tau$ is embeddable if there is a non-trivial ultrafilter $\alpha$ on $\mathbb{N}$ and a unital $*$-homomorphism

$$\pi : A \to \prod_{n \in \mathbb{N}}^\alpha (M_n(\mathbb{C}), \tau_n)$$

such that

$$\tau_\alpha \circ \pi = \tau.$$

If Connes’ embedding theorem holds, then every tracial state is embeddable.

A tracial state $\tau$ is finite-dimensional if there is a finite-dimensional $C^*$-algebra $B$ with a tracial state $\tau_B$ and a unital $*$-homomorphism

$$\pi : A \to B$$

such that

$$\tau_B \circ \pi = \tau.$$

A tracial state $\tau$ is called matricial if there is a positive integer $n$ and a unital $*$-homomorphism

$$\pi : A \to M_n(\mathbb{C})$$

such that

$$\tau = \tau_n \circ \pi.$$

We say that a matricial tracial state is a factor matricial state if the $\pi$ above can be chosen to be surjective.
Lemma 3.1. A C*-algebra $A = C^*(D)$ with a tracial state $\tau$ admits a trace-preserving $*$-homomorphism into an ultraproduct of matrix algebras if and only if, for every $\epsilon > 0$, every finite tuple $(a_1, \ldots, a_n)$ of elements in $D$, and every finite set $F$ of $*$-monomials, there is a positive integer $k$ and a tuple $(A_1, \ldots, A_n)$ of $k \times k$ matrices such that

\[ (1) \quad \|A_j\| \leq \|a_j\| + 1 \text{ for } 1 \leq j \leq n, \]
\[ (2) \quad |\tau(m(a_1, \ldots, a_n)) - \tau_k(m(A_1, \ldots, A_n))| < \epsilon \text{ for every } m \in F. \]

Proof. The "only if" part is obvious. For the other direction, let $\Lambda$ be the set of all triples $\lambda = (\varepsilon, N, E_\lambda)$ with $\varepsilon > 0$, $E_\lambda$ a finite subset of $D$, $N$ a positive integer, partially ordered by $(\geq, \subseteq)$. By the hypothesis, for each $\lambda \in \Lambda$ there is a positive integer $k_\lambda$ and there is a function $f_\lambda : D \to M_{k_\lambda}$ with $f_\lambda(1) = 1$ and such that,

i. $\|f_\lambda(a)\| \leq \|a\| + 1$ for every $a \in A$,
ii. For all monomials $m$ with degree at most $N$, all $a_1, \ldots, a_n \in E_\lambda$,

\[ |\tau(m(a_1, \ldots, a_n)) - \tau_k(m(f_\lambda(a_1), \ldots, f_\lambda(a_n)))| < \varepsilon. \]

(We can define $f(a) = 0$ on $A \setminus E_\lambda$.)

Define $F : D \to \prod_{\lambda \in \Lambda} (M_{k_\lambda}, \tau_{k_\lambda})$ by $F(a)(\lambda) = f_\lambda(a)$. Let $\alpha$ be an ultrafilter on $\Lambda$ containing $\{\lambda : \lambda \geq \lambda_0\}$ for each $\lambda_0 \in \Lambda$. Define

\[ \rho : A \to \prod_{\lambda \in \Lambda} (M_{k_\lambda}, \tau_{k_\lambda}) \]

by $\rho(a) = [F(a)]_\alpha$. Let $A_0$ denote the unital $*$-algebra generated by $D$. It follows from the definition of the $f_\lambda$’s that $\rho$ extends to a unital $*$-homomorphism $\pi_0 : A_0 \to \prod_{\lambda \in \Lambda} (M_{k_\lambda}, \tau_{k_\lambda})$ with $\tau_\alpha \circ \pi_0 = \tau$. We can now uniquely extend, by continuity, $\pi_0$ to a unital $*$-homomorphism on $A$ such that $\tau = \tau_\alpha \circ \pi$. \hfill $\square$

Corollary 3.2. If a separable C*-algebra with a tracial state admits a trace-preserving $*$-homomorphism into $\prod_{n \in \mathbb{N}} (M_n, \tau_n)$, for some non-trivial ultrafilter $\alpha$, then it admits a trace-preserving $*$-homomorphism into $\prod_{n \in \mathbb{N}} (M_n, \tau_n)$, for any non-trivial ultrafilter $\beta$.

Proof. Suppose $A = C^*(a_1, a_2, \ldots)$ with a tracial state $\tau$, and suppose $\rho : A \to \prod_{n \in \mathbb{N}} (M_n, \tau_n)$ is a unital $*$-homomorphism such that $\tau_\alpha \circ \rho = \tau$. We can assume $\|a_n\| = 1$ for all $n$, and we can let $\rho(a_n) = \{b_{n,k}\}_\alpha$ with $\|b_{n,k}\| \leq \|a_n\| = 1$ for all $k \geq 1$. For each positive integer $s$ there is a positive $k_s$ such that, for every $*$-monomial $m$ with degree at most $s$,

\[ |\tau(m(a_1, \ldots, a_s)) - \tau_{k_s}(m(b_{1,k_s}, \ldots, b_{s,k_s}))| < \frac{1}{2^s}. \]

Suppose $t > 2sk_s$. Then by dividing $t$ by $k_s$ we get $t = k_s q + r$ with $0 \leq r < k_s$. We define $c_j = b_{j,k_s}^{(q)} \oplus 0_r$ (where $b^{(q)}$ is a direct sum of $q$ copies of $b$ and $0_r$ is an $r \times r$ zero matrix). A simple computation gives, for any monomial $m$ of degree at most $s$ that

\[ \tau_A(m(c_1, \ldots, c_s)) = \tau_{k_s}(m(b_{1,k_s}, \ldots, b_{s,k_s})) \left( 1 - \frac{r}{t} \right).
\]
Since $r/t < 1/2s$ and $|\tau_{k_n}(m(b_{1,k_n}, \ldots, b_{s,k_n}))| \leq 1$, we see that

$$|\tau(m(a_1, \ldots, a_s)) - \tau_1(m(c_1, \ldots, c_s))| < \frac{1}{s}.$$  

Hence we get representatives like $(c_1, \ldots, c_s)$ for all $t > 2sk_s$. From this we see that there is a function $F : \{a_1, a_2, \ldots\} \to \prod_{n \in \mathbb{N}}(M_n, \tau_n)$ so that if $f(a_n) = (d_{n,1}, d_{n,2}, \ldots)$, we have

$$\lim_{n \to \infty} \tau_n(m(d_{n,1}, d_{n,2}, \ldots)) = \tau(m(a_1, a_2, \ldots))$$

for any *-monomial $m$. Hence, for any free ultrafilter $\beta$,

$$\lim_{n \to \beta} \tau_n(m(d_{n,1}, d_{n,2}, \ldots)) = \tau(m(a_1, a_2, \ldots)).$$

Thus the map $\pi_0 : \{a_1, a_2, \ldots\} \to \prod_{n \in \mathbb{N}}^\beta (M_n, \tau_n)$ defined by

$$\pi_0(a_k) = F(a_k)_{\beta}$$

extends to a unital $*$-homomorphism $\pi : A \to \prod_{n \in \mathbb{N}}^\beta (M_n, \tau_n)$ such that $\tau_\beta \circ \pi = \tau$.

It follows easily from Lemma 3.1 that the set of embeddable tracial states on $A$ is $*$-weak-compact. It is not hard to show that it is also convex, and contains the finite-dimensional tracial states. The set of finite-dimensional tracial states is the convex hull of the set of factor matricial states. Moreover, the set of finite-dimensional tracial states is contained in the weak*-closure of the set of matricial tracial states.

### 3.2. Matricial tracial stability – necessary conditions

Let $A$ be a unital $C^*$-algebra and let $J_{et}(A)$ be the largest ideal of $A$ that is annihilated by every embeddable trace on $A$. If $A$ has no embeddable traces, then $A/J_{et} = \{0\}$. More generally, the embeddable tracial states of $A/J_{et}$ separate the points of $A/J_{et}$, and if $A$ is separable, $A/J_{et}$ always has a faithful tracial embeddable state. Clearly, $A$ is matricially tracially stable if $A/J_{et}$ is matricially tracially stable. $C^*$-algebras without any embeddable tracial states are automatically matricially tracially stable.

**Theorem 3.3.** Suppose $A$ is a separable unital matricially tracially stable $C^*$-algebra with at least one embeddable trace. Then

1. For all but finitely many positive integers $n$ there is a unital representation $\pi_n : A \to M_n(\mathbb{C})$
2. The set of embeddable tracial states on $A$ is the $*$-weak-closed convex hull of the set of matricial tracial states.

**Proof.** (1). Suppose, via contradiction, that the set $E$ of positive integers $n$ for which there is no $n$-dimensional representation of $A$ is infinite. Let $\alpha$ be a non-trivial ultrafilter on $\mathbb{N}$ with $E \in \alpha$. Let $\tau$ be the limit trace on $\prod_{n \in \mathbb{N}}^\alpha (M_n(\mathbb{C}), \tau_n)$. Then, if $\rho$ is an embeddable trace on $A$, by Corollary 3.2 there is a representation $\pi : A \to \prod_{n \in \mathbb{N}}^\alpha (M_n(\mathbb{C}), \tau_n)$ such that $\tau \circ \pi = \rho$. Since there $A$ is matricially stable, the set $\mathbb{N} \setminus E \in \alpha$, which is a contradiction.
(2). If \( \rho \) is an embeddable trace and \( \alpha \) is a non-trivial ultrafilter on \( \mathbb{N} \) and 
\[
\pi : A \rightarrow \prod_{n \in \mathbb{N}} (M_n(\mathbb{C}), \tau_n)
\]
is a representation such that \( \tau \circ \pi = \rho \), then the matricial tracial stability of \( A \) implies there is a set \( F \in \alpha \) and, for each \( n \in F \), there is a representation \( \pi_n : A \rightarrow M_n(\mathbb{C}) \) such that the \( * \)-weak-limit along \( \alpha \) of the finite-dimensional traces \( \tau_n \circ \pi_n \).

\[
\square
\]

**Corollary 3.4.** Suppose \( A \) is separable and matricially tracially stable. Then \( A/J_{ct} \) is RFD. If the collection of embeddable traces on \( A \) separates points of \( A \), then \( A \) is RFD.

The set of positive integers \( n \) for which there is an \( n \)-dimensional representation is an additive semigroup (direct sums) that is generated by the \( k \in \mathbb{N} \) for which there is a \( k \)-dimensional irreducible representation. The following simple lemma, which yields a reformulation of the first condition in Theorem 3.3, should be well-known.

**Lemma 3.5.** Suppose \( n_1 < n_2 < \cdots < n_s \). The following are equivalent:

1. \( \text{GCD}(n_1, \ldots, n_k) = 1 \).
2. There is a positive integer \( N \) such that every integer \( n \geq N \) can be written as

\[
 n = \sum_{k=1}^{s} a_k n_k
\]

with \( a_1, \ldots, a_k \) nonnegative integers.

**Proof.** (2) \( \Rightarrow \) (1). If we write \( N = \sum_{k=1}^{s} a_k n_k \) and \( N + 1 = \sum_{k=1}^{s} b_k n_k \), then

\[
\sum_{k=1}^{s} (b_k - a_k) n_k = 1,
\]

which implies (1).

(1) \( \Rightarrow \) (2). If \( \text{GCD}(n_1, \ldots, n_k) = 1 \), there are integers \( s_1, \ldots, s_k \) such that

\[
s_1 n_1 + \cdots + s_k n_k = 1.
\]

Let \( m = 1 + n_1 \max(|s_1|, \ldots, |s_k|) \). Let \( N = mn_1 + \cdots + mn_k \). Suppose \( n \geq N \). Using the division algorithm we can find integers \( q \geq 0 \) and \( r \) with \( 0 \leq r < n_1 \) such that

\[
n - N = n_1 q + r.
\]

Thus

\[
n = mn_1 + \cdots + mn_k + qn_1 + r (s_1 n_1 + \cdots + s_k n_k) =
\]

\[
(m + rs_1 + q) n_1 + \sum_{j=2}^{k} (m + rs_j) n_j.
\]

However, \( m + rs_j \geq m - n_1 |s_j| \geq 1 \) for \( 1 \leq j \leq k \).

\[
\square
\]

**Corollary 3.6.** A unital \( C^* \)-algebra satisfies condition (1) in Theorem 3.3 if and only if \( \{ n \in \mathbb{N} : A \text{ has an irreducible } n \text{-dimensional representation} \} \) has greatest common divisor equal to 1.
Lemma 3.7. A separable unital C*-algebra $A$ satisfies statements (1) and (2) in Theorem 3.3 if and only if, for every embeddable tracial state $\tau$ on $A$, there is a positive integer $n_0$ and, for each $n \geq n_0$ there is a unital *-homomorphism $\rho_n : A \to \mathcal{M}_n(\mathbb{C})$ such that, for every $a \in A$,

$$\tau(a) = \lim_{n \to \infty} \tau_n(\rho_n(a))$$

Proof. The "if" part is clear. Suppose statements (1) and (2) in Theorem 3.3 are true. Let $\{a_1, a_2, \ldots\}$ be a norm dense subset if the unit ball of $A$. Let $\tau$ be an embeddable tracial state on $A$, and suppose $N$ is a positive integer. Then there is a convex combination $\sigma$, with rational coefficients, of matricial states, and hence even of factor matricial states, such that

$$|\tau(a_j) - \sigma(a_j)| < 1/2N$$

for $1 \leq j \leq N$. Thus there exist a positive integer $m$ and positive integers $s_1, \ldots, s_\nu$ with $s_1 + \cdots + s_\nu = m$ and positive integers $t_1, \ldots, t_\nu$ with surjective representations $\pi_i : A \to \mathcal{M}_{t_i}(\mathbb{C})$ such that

$$\sigma = \sum_{i=1}^\nu \frac{s_i}{m} \pi_i \circ \sigma_i.$$

Let $M \in \mathbb{N}$ be such $\frac{m}{M}M$ is an integer, for all $i \leq \nu$. If $\rho(k)$ denotes a direct sum of $k$ copies of a representation $\rho$, then let

$$\pi = \oplus_i (\frac{m}{M}) : A \to \mathcal{M}_{mM}(\mathbb{C}).$$

It is easy to check that $\sigma = \tau_{mM} \circ \pi$. Since by the assumption the statement (1) in Theorem 3.3 holds, there is $n_0$ such that for any $n \geq n_0$ there is a unital representation into $\mathcal{M}_n(\mathbb{C})$. Since for a positive integer $q$ we have $\sigma = \tau_{qmM} \circ \pi(q)$, we can assume that $mM > n_0$. Suppose $n > 2mM$. We can write $n = amM + b$ with $a \geq 2$ and $0 \leq b < mM$. Since $n_0 \leq mM + b$, there is a unital representation $\rho : A \to \mathcal{M}_{mM+b}(\mathbb{C})$. Let

$$\pi_{N,n} = \pi^{(a-1)} \oplus \rho : A \to \mathcal{M}_n(\mathbb{C}).$$

Then

$$\tau_n \circ \pi_{N,n}(a_k) = \frac{(a-1)mM\tau_{mM} \circ \pi(a_k) + (mM + b)\tau_{mM+b} \circ \rho(a_k)}{n} =$$

$$= \frac{(a-1)mM}{n}\sigma(a_k) + \frac{mM + b}{n}\tau_{mM+b} \circ \rho(a_k) =$$

$$= \frac{(1 - \frac{mM}{n})\sigma(a_k) + \frac{mM + b}{n}\tau_{mM+b} \circ \rho(a_k)}{n}.$$

There is a positive integer $k_\nu$ such that if $n \geq k_\nu$, then $\frac{mM+b}{n} < \frac{1}{4N}$. Then for all $n \geq k_\nu$

$$|\tau_n \circ \pi_{N,n}(a_k) - \sigma(a_k)| < 1/2N$$

for $1 \leq k \leq N$ and hence

$$|\tau(a_k) - \tau_n \circ \pi_{N,n}(a_k)| < 1/N$$

for $1 \leq k \leq N$. We can easily arrange $k_1 < k_2 < \cdots$ and we can define

$$\rho_n = \pi_{N,n} \text{ if } k_N \leq n < k_{N+1}.$$
3.3. Matricial tracial stability for tracially nuclear C*-algebras. Recall that a unital C*-algebra $A$ is nuclear if and only if, for every Hilbert space $H$ and every unital *-homomorphism $\pi : A \to B(H)$ the von Neumann algebra $\pi(A)''$ is hyperfinite. The algebra $A$ is tracially nuclear \[\text{(1)}\] if, for every tracial state $\tau$ on $A$, the algebra $\pi_{\tau}(A)''$ is hyperfinite, where $\pi_{\tau}$ is the GNS representation for $\tau$. It is easy to show that if $A$ is tracially nuclear, then every trace on $A$ is embeddable.

**Theorem 3.8.** Suppose $A$ is a unital separable tracially nuclear C*-algebra with at least one tracial state. The following are equivalent

1. $A$ is matricially tracially stable
2. $A$ satisfies the conditions (1) and (2) in Theorem 3.3.
3. $A$ is $W^*$-factor tracially stable.

**Proof.**
1) $\Rightarrow$ 2) This follows from Theorem 3.3.

2) $\Rightarrow$ 3). Suppose $A = C^*(x_1, x_2, \ldots)$, with all $x_i$'s being contractions, suppose $\alpha$ is a free ultrafilter on $N$, and suppose $\sigma : A \to \prod_{n \in N} (\mathcal{M}_n, \rho_n)$, where $(\mathcal{M}_n, \rho_n)$ is a factor von Neumann algebra with trace $\rho_n$. Denote by $\rho_\alpha$ the limit trace on $\prod_{n \in N} (\mathcal{M}_n, \rho_n)$. Since $\rho_\alpha$ is a faithful trace on $\prod_{n \in N} (\mathcal{M}_n, \rho_n)$ and since $A$ is tracially nuclear, it follows that $\sigma(A)''$ is hyperfinite.

Since $\alpha$ is an ultrafilter, either $\{n \in N : \dim \mathcal{M}_n < \infty\} \in \alpha$ or $\{n \in N : \dim \mathcal{M}_n = \infty\} \in \alpha$. We first consider the latter case. In this case we can assume that every $\mathcal{M}_n$ is a $II_1$-factor. In this case, it follows from (11) that, for each $n \in N$ there is a unital *-homomorphism $\gamma_n : \sigma(A)'' \to \mathcal{M}_n$ such that, for every $b \in \sigma(A)''$

\[
\lim_{n \to \alpha} \rho_n (\gamma_n(b)) = \rho_\alpha(b),
\]

and $b = \{\gamma_n(b)\}_\alpha$. Hence $\{\gamma_n \circ \sigma\}_{n \in N}$ is an approximate lifting of $\sigma$.

Next suppose $E := \{n \in N : \dim \mathcal{M}_n < \infty\} \in \alpha$ and $\mathcal{M}_n = \mathcal{M}_{k_n}(\mathbb{C})$ and $\rho_n = \tau_{k_n}$ for each $n \in E$.

By assumption and Lemma 3.7 there is an $n_0 \in N$ and, for every $n \geq n_0$ there is a unital *-homomorphism $\pi_n : A \to \mathcal{M}_n(\mathbb{C})$ such that, for every $a \in A$,

\[
\tau_\alpha(a) = \lim_{k \to \infty} \tau_n(\pi_n(a)).
\]

For each $k \in N$, write $\sigma(x_k) = \{x_k(n)\}_\alpha$. For any $s \in N$ and any *-monomial $m(t_1, \ldots, t_s)$ we have

\[
\lim_{n \to \alpha} \tau_n(m(x_1(n), \ldots, x_s(n))) = \tau_\alpha(m(\sigma(x_1), \ldots, \sigma(x_s)))
\]

and

\[
\lim_{n \to \infty} \tau_n(m(\pi_n(x_1), \ldots, \pi_n(x_s))) = \tau_\alpha(m(\sigma(x_1), \ldots, \sigma(x_s))).
\]

Since $\sigma(A)''$ is hyperfinite, it follows from Connes' theorem that, for each $s \in N$, $\sigma(C^*(x_1, \ldots, x_s))'' \subset \sigma(A)''$ is also hyperfinite. It now follows from Theorem 1.1 that there is a sequence of unitaries $U_n \in \mathcal{M}_{k_n}(\mathbb{C})$ such that, for every $s \in N$,

\[
\lim_{n \to \alpha} \|x_s(n) - U_n^* \pi_n(x_s) U_n\|_2 = 0.
\]
Hence \( \{ U_n^* \pi_n (\cdot) U_n \} \) is an approximate lifting of \( \sigma \).

3) \( \Rightarrow \) 1) is obvious. \( \square \)

Recall that a C*-algebra \( \mathcal{A} \) is called GCR (or type I) if for any its irreducible representation \( \pi : \mathcal{A} \to B(H) \), \( \pi(\mathcal{A}) \) contains all the compact operators.

**Corollary 3.9.** Suppose \( \mathcal{A} \) is a separable GCR unital C*-algebra satisfying condition (1) in Theorem 3.3. Then \( \mathcal{A} \) is W*-factor tracially stable.

**Proof.** The extreme points of the tracial states are the factor tracial states ([12]). A factor representation of a GCR C*-algebra must yield a factor von Neumann algebra of type I, which must be isomorphic to some \( B(H) \). If it has a trace, then \( H \) must be finite-dimensional. Thus the factorial tracial states are finite-dimensional, and the Krein-Milman theorem gives that all tracial states are in the weak*-closed convex hull of the finite-dimensional states. Thus the condition (2) in Theorem 3.3 holds. \( \square \)

The next statement follows from Theorem 3.8, Theorem 3.3 and Lemma 3.7.

**Theorem 3.10.** Suppose \( \mathcal{A} \) is a separable tracially nuclear C*-algebra with at least one tracial state. The following are equivalent

1. \( \mathcal{A} \) is matricially tracially stable
2. For every tracial state \( \tau \) on \( \mathcal{A} \), there is a positive integer \( n_0 \) and, for each \( n \geq n_0 \) there is a unital *-homomorphism \( \rho_n : \mathcal{A} \to M_n(\mathbb{C}) \) such that, for every \( a \in \mathcal{A} \),
   \[ \tau (a) = \lim_{n \to \infty} \tau_n (\rho_n (a)). \]
   (here \( \tau_n \) is the usual tracial state on \( M_n(\mathbb{C}) \))
3. \( \mathcal{A} \) is W*-factor tracially stable.

Below we give an example of an RFD nuclear C*-algebra which has finite-dimensional irreducible representations of all dimensions but is not matricially tracially stable.

**Example 3.11.** Suppose \( 0 < \theta_1 < \theta_2 < 1 \) are irrational and \( \{ 1, \theta_1, \theta_2 \} \) is linearly independent over \( \mathbb{Q} \). Let \( \lambda_k = e^{2\pi i \theta_k} \) for \( k = 1, 2 \). Let \( \mathcal{A}_{\theta_k} \) be the irrational rotation algebra generated by unitaries \( U_k, V_k \) satisfying \( U_k V_k = \lambda_k V_k U_k \). We know that each \( \mathcal{A}_{\theta_k} \) is simple nuclear and has a unique tracial state \( \rho_k \). Let \( U = U_1 \oplus U_2 \) and \( V = V_1 \oplus V_2 \). Since \( U V U^* V^* = \lambda_1 \oplus \lambda_2 \), we see that \( 1 \oplus 0 \in C^* (U, V) \), so

\[ C^* (U, V) = \mathcal{A}_{\theta_1} \oplus \mathcal{A}_{\theta_2}. \]

By the linear independence assumption, there is an increasing sequence \( \{ n_k \} \) of positive integers such that

\[ \lambda_1^{n_k} \to 1 \text{ and } \lambda_2^{n_k} \to 1 \]
as \( k \to \infty \). We can assume also that

\[ n_1 = 2 \text{ and } n_2 = 3. \]

For each positive integer \( n \) and each \( \lambda \in \mathbb{C} \), let \( \{ e_1, \ldots, e_n \} \) be the standard orthonormal basis for \( \mathbb{C}^n \), let \( U_{n, \lambda} \) and \( V_n \) be the matrices defined by

\[ U_{n, \lambda} e_j = \lambda^{j-1} e_j \]
and

\[ V_n e_j = e_{j+1} \text{ for } 1 \leq j < n; \ V_n e_n = e_1. \]

It follows from (3.1) that

\[ \| U_{n_k, \lambda_s} V_{n_k} - \lambda_s V_{n_k} U_{n_k, \lambda_s} \| \to 0 \]

as \( k \to \infty \) for \( s = 1, 2 \). The simplicity of each \( A_{\theta_s} \) implies that, for every \(*\)-polynomial \( p \) and \( s = 1, 2 \), we have

\[ (3.3) \quad \| p(U_{n_k, \lambda_s}, V_{n_k}) - p(U_s, V_s) \| \to 0. \]

The uniqueness of the trace on each \( A_{\theta_s} \) implies that, for every \(*\)-polynomial \( p \) and \( s = 1, 2 \), we have

\[ \lim_\alpha \tau_{n_k} (p(U_{n_k, \lambda_s}, V_{n_k})) = \rho_s (p(U_s, V_s)) \]

for any non-trivial ultrafilter \( \alpha \), and hence

\[ (3.4) \quad \tau_{n_k} (p(U_{n_k, \lambda_s}, V_{n_k})) \to \rho_s (p(U_s, V_s)) \]

as \( k \to \infty \).

Let \( \hat{U} = \sum_{k \in \mathbb{N}} U_{n_k, \lambda_1} \oplus U_{n_k, \lambda_2} \) and \( \hat{V} = \sum_{k \in \mathbb{N}} V_{n_k}^{(k-1)} \oplus V_{n_k} \), and let

\[ A = C^* (\hat{U}, \hat{V}) + \mathcal{K}, \]

where

\[ \mathcal{K} = \sum_{k \in \mathbb{N}} M_{kn_k}(\mathbb{C}). \]

It follows that \( \mathcal{K} \) is nuclear and by (3.3) \( A/\mathcal{K} \) is isomorphic to \( C^* (U, V) = A_{\theta_1} \oplus A_{\theta_2} \), which is also nuclear. Hence, \( A \) is nuclear. Moreover, the only finite-dimensional irreducible representations of \( A \) are the coordinate representations \( \pi_k \) for \( k \in \mathbb{N} \).

However, by (3.4), for every \(*\)-polynomial \( p \), we have

\[ \lim_{k \to \infty} \tau_{nk} (\pi_k (\hat{U}, \hat{V})) = \lim_{k \to \infty} \left( \frac{k-1}{k} \tau_{nk} (p(U_{n_k, \lambda_1}, V_{n_k})) + \frac{1}{k} \tau_{nk} (p(U_{n_k, \lambda_2}, V_{n_k})) \right) \]

\[ = \lim_{k \to \infty} \tau_{nk} (p(U_{n_k, \lambda_1}, V_{n_k})) = \rho_1 (p(U_1, V_1)). \]

Hence the trace \( \rho \) on \( A \) that annihilates \( \mathcal{K} \) and sends \( p(\hat{U}, \hat{V}) \) to \( \rho_2 (p(U_2, V_2)) \) is embeddable and cannot be approximated by finite-dimensional tracial states. Thus \( A \) is nuclear and RFD and by (3.2) and Corollary 3.6 satisfies condition (1) in Theorem 3.3 but does not satisfy condition (2), hence is not matricially tracially stable.

4. MATRICIAL TRACIAL STABILITY AND FREE ENTROPY DIMENSION

There is a close relationship between matricial stability and D. Voiculescu’s free entropy dimension [27], [28] (via the free orbit dimension in Hadwin-Shen [13]). D. Voiculescu defined his free entropy dimension \( \delta_0 \) [27], [28], and he applied it to show the existence of a \( II_1 \) factor von Neumann algebra without a Cartan MASA, solving a longstanding problem.

Suppose \( A \) is a unital \( C^* \)-algebra with a tracial state \( \tau \). Suppose \( x_1, \ldots, x_n \) are elements in \( A \), \( \varepsilon > 0 \), \( R \geq \max_{1 \leq j \leq n} ||x_j|| \) and \( N, k \in \mathbb{N} \). Voiculescu [27] defines
\[ \Gamma_{R,\tau} (x_1, \ldots, x_n; N, k, \varepsilon) \] to be the set of all \( n \)-tuples \((A_1, \ldots, A_n)\) of matrices in \( \mathcal{M}_k (\mathbb{C}) \) with norm at most \( R \) such that
\[ |\tau (m (x_1, \ldots, x_n)) - \tau_n (m (A_1, \ldots, A_n))| < \varepsilon \]
for all \( * \)-monomials \( m (t_1, \ldots, t_n) \) with degree at most \( N \).

Connes’ embedding problem is equivalent to the assertion that, for every unital \( C^* \)-algebra \( B \) with a tracial state \( \tau \), every \( n \)-tuple \((x_1, \ldots, x_n)\) of selfadjoint contractions in \( B \), for any \( N \), for any \( R > \max_{1 \leq j \leq n} \| x_j \| \) and for every \( \varepsilon > 0 \) there is a positive integer \( k \) such that \( \Gamma_{R,\tau} (x_1, \ldots, x_n; N, k, \varepsilon) \neq \emptyset \).

Let \( \mathcal{M}_k (\mathbb{C})^n = ((A_1, \ldots, A_n) : A_1, \ldots, A_n \in \mathcal{M}_k (\mathbb{C})) \) and define \( \| \cdot \|_2 \) on \( \mathcal{M}_k (\mathbb{C})^n \) by
\[ \| (A_1, \ldots, A_n) \|_2^2 = \sum_{j=1}^n \tau_k (A_j^* A_j). \]

If \( A = (A_1, \ldots, A_n) \in \mathcal{M}_k (\mathbb{C})^n \) and \( U \) is unitary, we define
\[ U^* A U = (U^* A_1 U, \ldots, U^* A_n U). \]

If \( \omega > 0 \), we define the \( \omega \)-orbit ball of \( A = (A_1, \ldots, A_n) \), denoted by \( U (A_1, \ldots, A_n; \omega) \), to be the set of all \( B = (B_1, \ldots, B_k) \in \mathcal{M}_k (\mathbb{C})^n \) such that there is a unitary \( U \in \mathcal{M}_k (\mathbb{C}) \) such that
\[ \| U^* A U - B \|_2 < \omega. \]

If \( \mathcal{E} \subset \mathcal{M}_k (\mathbb{C})^n \), we define the \( \omega \)-orbit covering number of \( \mathcal{E} \), denoted by \( \nu (\mathcal{E}, \omega) \), to be the smallest number of \( \omega \)-orbit balls that cover \( \mathcal{E} \).

Let \( \mathcal{A} = C^* (x_1, \ldots, x_n) \) and for each positive integer \( k \), let \( \text{Rep}(\mathcal{A}, k) / \simeq \) denote the set of all unital \( * \)-homomorphisms from \( \mathcal{A} \) into \( \mathcal{M}_k (\mathbb{C}) \) modulo unitary equivalence. If \( \pi_1, \pi_2 \in \text{Rep}(\mathcal{A}, k) \) with corresponding images \([\pi], [\rho] \in \text{Rep}(\mathcal{A}, k) / \simeq \) we define a metric
\[ d_k ([\pi], [\rho]) = \min \| (\pi (x_1), \ldots, \pi (x_n)) - (U^* \rho (x_1) U, \ldots, U^* \rho (x_n) U) \|_2 \]
as \( U \) ranges over all of the \( k \times k \) unitary matrices.

For each \( 0 < \omega < 1 \), we define
\[ \nu_{d_k} (\text{Rep}(\mathcal{A}, k) / \simeq, \omega) \]
to be the minimal number of \( d_k \)-balls of radius \( \omega \) it takes to cover \( \text{Rep}(\mathcal{A}, k) / \simeq \).

**Lemma 4.1.** Suppose \( \mathcal{A} = C^* (x_1, \ldots, x_n) \) is matricially tracially stable and \( \tau \) is an embeddable tracial state on \( \mathcal{A} \). Let \( R > \max_{1 \leq j \leq n} \| x_j \| \). For each \( 0 < \omega < 1 \) there exists an \( m_\omega \in \mathbb{N} \) such that, for all integers \( k, N \geq m_\omega \) and every \( 0 < \varepsilon < 1/m_\omega \)
\[ \text{Card} (\text{Rep}(\mathcal{A}, k) / \simeq) \geq \nu_{d_k} (\text{Rep}(\mathcal{A}, k) / \simeq, \omega/4) \geq \nu \left( \Gamma_{R,\tau} (x_1, \ldots, x_n; N, k, \varepsilon) \right) \omega. \]

**Proof.** Let \( 0 < \omega < 1 \). It is easy to deduce from the \( \varepsilon - \delta \)-definition of matricial tracial stability that, there is a positive integer \( m_\omega \) such that, for every \( k \in \mathbb{N} \) and every \( N \geq m_\omega \) and \( 0 < \varepsilon < 1/m_\omega \), we have for each \( B = (b_1, \ldots, b_n) \in \Gamma_{R,\tau} (x_1, \ldots, x_n; N, k, \varepsilon) \) a representation \( \pi_B \in \text{Rep}(\mathcal{A}, k) \) such that
\[ \| B - (\pi_B (x_1), \ldots, \pi_B (x_n)) \|_2 < \omega/4. \]
Now suppose \( N \geq m_\omega \) and \( 0 < \varepsilon < 1/m_\omega \). It follows from the definition of \( s = \nu \left( \Gamma_{R,\tau} (x_1, \ldots, x_n; N, k, \varepsilon), \omega \right) \) that there is a collection
\[ \{ B_j \in \Gamma_{R,\tau} (x_1, \ldots, x_n; N, k, \varepsilon) : 1 \leq j \leq s \} \]
so that, for any \( k \times k \) unitary \( U \) and \( 1 \leq i \neq j \leq k \),
\[
\| U^* B_i U - B_j \|_2 \geq \omega.
\]
For each \( j, 1 \leq j \leq s \). It easily follows that, for \( 1 \leq i \neq j \leq s \),
\[
d_k ([\pi_{B_i}], [\pi_{B_j}]) \geq \omega/2.
\]
Hence every \( d_k \)-ball with radius \( \omega/4 \) contains at most one of \( [\pi_{B_j}] \), \( 1 \leq j \leq s \).
Hence,
\[
\text{Card} \left( \text{Rep} \left( \mathcal{A}, k \right) / \simeq \right) \geq n d_k \left( \text{Rep} \left( \mathcal{A}, k \right) / \simeq, \omega/4 \right) \geq \nu \left( \Gamma_{R, \tau} \left( x_1, \ldots, x_n; N, k, \varepsilon \right), \omega \right).
\]
\( \square \)

In [28] the first named author and J. Shen defined the free-orbit dimension
\( \mathfrak{R}_1 \left( x_1, \ldots, x_n; \tau \right) \). First let
\[
\mathfrak{R} \left( x_1, \ldots, x_n; \tau, \omega \right) = \inf_{\varepsilon, N} \limsup_{k \to \infty} \frac{\log \nu \left( \Gamma_{R, \tau} \left( x_1, \ldots, x_n; N, k, \varepsilon \right), \omega \right)}{k^2 \log \omega},
\]
and let
\[
\mathfrak{R}_1 \left( x_1, \ldots, x_n; \tau \right) = \limsup_{\omega \to 0^+} \mathfrak{R} \left( x_1, \ldots, x_n; \tau, \omega \right).
\]
If \( \mathcal{A} = \mathcal{C}^* (x_1, \ldots, x_n) \) is matricially tracially stable and \( \tau \) is an embeddable tracial state on \( \mathcal{A} \), then by Lemma 4.1
\[
\frac{1}{\log \omega} \limsup_{k \to \infty} \frac{\log \text{Card} \left( \text{Rep} \left( \mathcal{A}, k \right) / \simeq \right)}{k^2} \geq \mathfrak{R} \left( x_1, \ldots, x_n; \tau, \omega \right).
\]
It follows that if \( \mathfrak{R}_1 \left( x_1, \ldots, x_n; \tau \right) > 0 \), then \( \limsup_{k \to \infty} \frac{\log \text{Card}(\text{Rep}(\mathcal{A},k)/\simeq)}{k^2} = \infty \).
If \( \delta_0 \left( x_1, \ldots, x_n; \tau \right) \) denotes D. Voiculescu’s free entropy dimension, we know from [28] that
\[
\delta_0 \left( x_1, \ldots, x_n; \tau \right) \leq 1 + \mathfrak{R}_1 \left( x_1, \ldots, x_n; \tau \right).
\]
This gives the following result, which shows that a matricially tracially stable algebra may be forced to have a lot of representations of each large finite dimension.

**Theorem 4.2.** Suppose \( \mathcal{A} = \mathcal{C}^* (x_1, \ldots, x_n) \) is matricially tracially stable and \( \tau \) is an embeddable tracial state on \( \mathcal{A} \) such that \( 1 < \delta_0 \left( x_1, \ldots, x_n \right) \). Then
\[
\limsup_{k \to \infty} \frac{\log \text{Card} \left( \text{Rep} \left( \mathcal{A}, k \right) / \simeq \right)}{k^2} = \infty.
\]

Below using this theorem we give an example which shows that for not tracially nuclear \( \mathcal{C}^* \)-algebras, the conditions 1) and 2) in Theorem 3.3 are not sufficient for being matricially tracially stable.

**Lemma 4.3.** The set \( \{ (U, V) \in \mathcal{U}_n \times \mathcal{U}_n : C^* (U, V) = \mathcal{M}_n \mathbb{C} \} \) is norm dense in \( \mathcal{U}_n \times \mathcal{U}_n \).

**Proof.** Suppose \( (U, V) \in \mathcal{U}_n \times \mathcal{U}_n \). Perturb \( U \) by an arbitrary small amount so that it is diagonal with no repeated eigenvalues with respect to some orthonormal basis \( \{ e_1, \ldots, e_n \} \) and perturb \( V \) by an arbitrary small amount so that its eigenvalues are not repeated and one eigenvector has the form \( \sum_{k=1}^n \lambda_k e_k \) with \( \lambda_k \neq 0 \) for \( 1 \leq k \leq n \).
\( \square \)

The following lemma states that every irreducible representation of \( (\pi_1 \otimes \pi_2) (\mathcal{A}) \) must either factor through \( \pi_1 \) or through \( \pi_2 \).
Lemma 4.4. If \( \pi = \pi_1 \oplus \pi_2 \) is a direct sum of unital representations of a unital C*-algebra \( \mathcal{A} \) and \( \rho \) is an irreducible representation of \( \mathcal{A} \) such that \( \ker \pi \subset \ker \rho \), then either \( \ker \pi_1 \subset \ker \rho \) or \( \ker \pi_2 \subset \ker \rho \).

Proof. Suppose \( \rho : \mathcal{A} \rightarrow B(H) \) is irreducible. Assume, via contradiction, \( a_i \in \ker \pi_i \) and \( a_i \notin \ker \rho \) for \( i = 1, 2 \). Then \( a_1 \mathcal{A} a_2 \subset \ker \pi \subset \ker \rho \), so \( \rho(a_1) \rho(\mathcal{A}) \rho(a_2) = \{0\} \), and since \( \rho(\mathcal{A})^{-\text{wot}} = B(H) \), we have \( \rho(a_1) B(H) \rho(a_2) = \{0\} \), but \( \rho(a_1) \neq 0 \neq \rho(a_2) \), which is a contradiction. \( \square \)

Lemma 4.5. (Hoover [18]) If \( \pi = \pi_1 \oplus \pi_2 \) is a direct sum of unital representations of a unital C*-algebra \( \mathcal{A} \), then \( \pi(\mathcal{A}) = \pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A}) \) if and only if there is no irreducible representation \( \rho \) of \( \mathcal{A} \) such that \( \ker \rho \subset \ker \pi_1 \) and \( \ker \rho \subset \ker \pi_2 \).

We will give a proof of it for the reader’s convenience.

Proof. Let \( \mathcal{J}_1 = \{ b \in \pi_1(\mathcal{A}) : b \oplus 0 \in \pi(\mathcal{A}) \} \) and \( \mathcal{J}_2 = \{ b \in \pi_2(\mathcal{A}) : 0 \oplus b \in \pi(\mathcal{A}) \} \). Clearly,
\[
\pi(\mathcal{A}) = \pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A}) \text{ if and only if } 1 \in \mathcal{J}_1 \text{ if and only if } 1 \in \mathcal{J}_2 \text{ if and only if } 1 \oplus 1 \in \mathcal{J} \text{ (where } \mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2 \). Hence, if \( \pi(\mathcal{A}) \neq \pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A}) \), \( \mathcal{J} \) is a proper ideal of \( \pi(\mathcal{A}) \) and \( \mathcal{J}_k \) is a proper ideal of \( \pi_k(\mathcal{A}) \) for \( k = 1, 2 \). Moreover, if \( \pi(a) = \pi_1(a) \oplus \pi_2(a) \), we have:
\[
\pi(a) \in \mathcal{J} \text{ if and only if } \pi_1(a) \in \mathcal{J}_1 \text{ if and only if } \pi_2(a) \in \mathcal{J}_2.
\]
Thus the algebras \( \pi(\mathcal{A})/\mathcal{J}, \pi_1(\mathcal{A})/\mathcal{J}_1 \) and \( \pi_2(\mathcal{A})/\mathcal{J}_2 \) are isomorphic to a unital C*-algebra \( \mathcal{D} \). Thus there are surjective homomorphisms \( \rho : \pi(\mathcal{A}) \rightarrow \mathcal{D} \) and \( \rho_k : \pi_k(\mathcal{A}) \rightarrow \mathcal{D} \) such that
\[
\rho \circ \pi = \rho_1 \circ \pi_1 = \rho_2 \circ \pi_2.
\]
If we choose an irreducible representation \( \gamma \) of \( \mathcal{D} \) and replace \( \rho, \rho_1, \rho_2 \) with \( \gamma \circ \rho, \gamma \circ \rho_1, \gamma \circ \rho_2 \) we get irreducible representations. \( \square \)

Theorem 4.6. There exists a C*-algebra which satisfies the conditions 1) and 2) of Theorem 4.3 but is not matricially tracially stable.

Proof. Let \( C^*_r(\mathbb{F}_2) \) be the reduced free group C*-algebra. Then \( C^*_r(\mathbb{F}_2) \) has a unique tracial state \( \tau \) and two canonical unitary generators \( U, V \). D. Voiculescu [27] proved that
\[
\delta_0(U,V;\tau) = 2.
\]
It was also shown by U. Haagerup and S. Thorbjørnsen [2] that there are sequences \( \{U_k\}, \{V_k\} \) of unitary matrices with \( U_k, V_k \in \mathcal{M}_k(\mathbb{C}) \) for each \( k \in \mathbb{N} \) such that, for every \( \ast \)-polynomial \( p(x,y) \)
\[
\lim_{k \to \infty} \|p(U_k,V_k)\| = \|p(U,V)\|.
\]
It follows from the uniqueness of a trace on \( C^*_r(\mathbb{F}_2) \) that, for every \( \ast \)-polynomial \( p(s,t) \),
\[
\lim_{k \to \infty} \tau_k(p(U_k,V_k)) = \tau(p(U,V)).
\]
By Lemma 4.3 we can assume that each pair \( (U_k, V_k) \) is irreducible, i.e., \( C^*(U_k, V_k) = \mathcal{M}_k(\mathbb{C}) \). Let \( U_\infty = U_1 \oplus U_2 \oplus \cdots \) and \( V_\infty = V_1 \oplus V_2 \oplus \cdots \), and let \( \mathcal{A} = C^*(U_\infty, V_\infty) \). Clearly \( \mathcal{A} \) is an RFD C*-algebra. Moreover, for each \( k \in \mathbb{N} \), \( \mathcal{A} \) has, up to unitary equivalence, exactly one irreducible representation \( \pi_k \) of dimension \( k \), namely the one with \( \pi_k(U_\infty) = U_k \) and \( \pi_k(V_\infty) = V_k \). Since each \( k \)-dimensional representation
of \( \mathcal{A} \) is unitarily equivalent to a direct sum of at most \( k \) irreducible representations of dimension at most \( k \), it follows that
\[
\text{Card}(\text{Rep}(\mathcal{A}, k) / \sim) \leq k^k.
\]
Thus
\[
\limsup_{k \to \infty} \frac{\log \text{Card}(\text{Rep}(\mathcal{A}, k) / \sim)}{k^2} \leq \limsup_{k \to \infty} \frac{\log k}{k} = 0.
\]
However, there is a unital \(*\)-homomorphism \( \pi : \mathcal{A} \to \mathcal{C}^*_{\mathbb{F}_2} \) such that \( \pi(U_\infty) = U \) and \( \pi(V_\infty) = V \). Thus \( \tau \circ \pi \) is an embeddable tracial state on \( \mathcal{A} \) and since
\[
\delta_0(U_\infty, V_\infty; \tau \circ \pi) = \delta_0(U, V; \tau) = 2,
\]
it follows from Theorem 4.2 that \( \mathcal{A} \) is not matricially tracially stable.

**Claim 1**: \( \pi_n \) does not factor through \( \bigoplus_{k \neq n < \infty} \pi_k \).

**Proof**: Assume, via contradiction, the claim is false. Since \((U_n, V_n)\) is an irreducible pair for \( n \in \mathbb{N} \), it follows that \( \pi_n \) does not factor through \( \bigoplus_{0 < k < N, k \neq n} \pi_k \) for each positive integer \( N \). Hence, by Lemma [1.1], it follows that \( \pi_n \) factors through \( \bigoplus_{N \leq k \neq n < \infty} \pi_k \), for each positive integer \( N \). However, since by [4.1], for every \( a \in \mathcal{A} \), we have
\[
\|\pi_k(a)\| \to \|\pi(a)\|,
\]
we see that, for every \( a \in \mathcal{A} \),
\[
\|\pi_n(a)\| \leq \|\pi(a)\|.
\]
This means that \( \pi_n \) factors through \( \pi \), which contradicts the fact that \( \mathcal{C}^*_\mathbb{F}_2 \) has no finite-dimensional representations. Thus Claim 1 must be true.

**Claim 2**: \( \mathcal{J} = \sum_{k=1}^{\infty} \mathcal{M}_k(\mathcal{U}) \subset \mathcal{A} \).

**Proof**: It is sufficient to show that the sequence \( \{\text{Id}_k \text{ if } k = n \text{ otherwise} \} \) belongs to \( \mathcal{A} \). Since \( \text{id}_\mathcal{A} = \bigoplus \pi_k \), the claim follows from Claim 1 and Lemma [4.1].

Clearly \( \mathcal{A}/\mathcal{J} \) is isomorphic to \( \mathcal{C}^*_\mathbb{F}_2 \). Thus any factor representation of \( \mathcal{A} \) must either be finite-dimensional or be factorable through the representation \( \pi \). Since the extreme tracial states are the factor tracial states [12], we see that the extreme tracial states on \( \mathcal{A} \) are
\[
\{\tau \circ \pi\} \cup \{\tau_k \circ \pi_k : k \in \mathbb{N}\}.
\]
By [4.2] we see that both conditions in Theorem 5.3 are satisfied, but \( \mathcal{A} \) is not matricially tracially stable. \( \square \)

### 5. \( C^* \)-Tracial Stability

The following lemma must be very well known. We give a proof of it because of lack of convenient references.

**Lemma 5.1**: Suppose \((X, d)\) is a compact metric space and \( \tau \) is a state on \( C(X) \). Let \( \sigma \) be the state on \( C[0, 1] \) given by
\[
\sigma(f) = \int_0^1 f(t) \, dt.
\]
Then, for any non-trivial ultrafilter \( \alpha \) on \( \mathbb{N} \) there is a unital \(*\)-homomorphism \( \pi : C(X) \to \prod_{n \in \mathbb{N}} (C[0, 1], \sigma) \), such that \( \sigma_\alpha \circ \pi = \tau \).
Proof. We know from [11] that $\prod_{n \in \mathbb{N}} (C[0,1],\sigma) = \prod_{n \in \mathbb{N}} (L^\infty[0,1],\sigma)$. We know that there is a probability Borel measure $\mu$ on $X$ such that, for every $f \in C(X)$,

$$\tau(f) = \int_X f \, d\mu.$$ 

Choose a dense subset $\{f_1, f_2, \ldots\}$ of $C(X)$. For each $n \in \mathbb{N}$ we can find a Borel partition $\{E_{n,j} : 1 \leq j \leq k_n\}$ of $X$ so that each $E_{n,j}$ has sufficiently small diameter and points $x_{n,j} \in E_{n,j}$ for $1 \leq j \leq k_n$ such that, for $1 \leq m \leq n$ and for every $x \in X$,

$$\left| f_m(x) - \left( \sum_{j=1}^{k_n} f_m(x_{n,j}) \chi_{E_{n,j}} \right)(x) \right| \leq \frac{1}{n}.$$ 

For each $n \in \mathbb{N}$ we can partition $[0,1]$ into intervals $\{I_{n,j} : 1 \leq j \leq k_n\}$ so that

$$\sigma(\chi_{I_{n,j}}) = \mu(E_{n,j}).$$

For each $n \in \mathbb{N}$ we define a unital $\ast$-homomorphism $\pi_n : C(X) \to L^\infty[0,1]$ by

$$\pi_n(f) = \sum_{j=1}^{k_n} f(x_{n,j}) \chi_{I_{n,j}}.$$ 

We define $\pi : C(X) \to \prod_{n \in \mathbb{N}} (L^\infty[0,1],\sigma)$ by

$$\pi(f_m) = \{\pi_n(f_m)\}_\alpha$$

It is clear, for $m \in \mathbb{N}$, that

$$\sigma_\alpha(\pi(f_m)) = \lim_{n \to \alpha} \sigma(\pi_n(f_m)) = \lim_{n \to \alpha} \sum_{j=1}^{k_n} f_m(x_{n,j}) \mu(E_{n,j}) = \tau(f_m).$$

Since $\{f_1, f_2, \ldots\}$ is dense in $C(X)$, we see that $\tau = \sigma_\alpha \circ \pi$. \qed

We say that a topological space $X$ is \textit{approximately path-connected} if, for each collection $\{V_1, \ldots, V_s\}$ of nonempty open subsets of $X$ there is a continuous function $\gamma : [0,1] \to X$ such that

$$\gamma([0,1]) \cap V_k \neq \emptyset$$

for $1 \leq k \leq s$.

Equivalently, traversing the above path $\gamma$ back and forth a finite number of times, given $0 < a_1 < b_1 < \cdots < a_s < b_s < 1$ we can find a path $\gamma$ such that

$$\gamma((a_k, b_k)) \subset V_k$$

for $1 \leq k \leq s$. Indeed, we first find a path $\gamma_1$ and $0 < c_1 < d_1 < \cdots < c_s < d_s < 1$ such that

$$\gamma_1((c_k, d_k)) \subset V_k$$

for $1 \leq k \leq s$, and then compose $\gamma_1$ with a homeomorphism on $[0,1]$ sending $a_k$ to $c_k$ and $b_k$ to $d_k$ for $1 \leq k \leq s$. 
To prove that $X$ is approximately path-connected when $X$ is Hausdorff, it clearly suffices to restrict to the case in which $\{V_1, \ldots, V_s\}$ is disjoint (Choose $x_k \in V_k$ for each $k$ and chose an open set $W_k \subseteq V_k$ with $x_k \in W_k$ such that $x_k = x_j \Rightarrow W_j = W_k$ and such that $\{W_k : 1 \leq k \leq s\}$ (without repetitions) is disjoint, then consider $\{W_k : 1 \leq k \leq s\}$.)

The following facts are elementary:

1. Every approximately path-connected space is connected
2. A continuous image of an approximately path-connected space is approximately path-connected
3. A cartesian product of approximately path-connected spaces is approximately path-connected.

An interesting example of a compact approximately path connected metric space in $\mathbb{R}^2$ is

$$A = \left\{ \left( x, \sin \left( \frac{1}{x} \right) \right) : 0 < x \leq 1 \right\} \cup \{(0, y) : 1^- \leq y \leq 1 \}.$$

Note that $A \cup B$ with $B = \{(x, 0) : -1 \leq x \leq 0 \}$ is not approximately path-connected. In particular, $A$ and $B$ are approximately path-connected and $A \cap B \neq \emptyset$, but $A \cup B$ is not approximately path-connected.

For compact Hausdorff spaces we have a characterization of approximate path-connectedness which later will be used to characterize $C^*$-tracial stability for commutative $C^*$-algebras.

**Theorem 5.2.** Suppose $X$ is a compact Hausdorff space. The following are equivalent:

1. $X$ is approximately path-connected.
2. If $A$ is a unital $C^*$-algebra, $B$ is an $AW^*$-algebra, $\eta : A \to B$ is a surjective unital $*$-homomorphism, and $\pi : C(X) \to B$ is a unital $*$-homomorphism, then there is is a net $\{\pi_\lambda\}$ of unital $*$-homomorphisms from $C(X)$ to $A$ such that, for every $f \in C(X)$,
   $$(\eta \circ \pi_\lambda)(f) \to \pi(f)$$
   in the weak topology on $B$.
3. If $A$ is a unital $C^*$-algebra, $B$ is an $W^*$-algebra, $\eta : A \to B$ is a surjective unital $*$-homomorphism, and $\pi : C(X) \to B$ is a unital $*$-homomorphism, then there is is a net $\{\pi_\lambda\}$ of unital $*$-homomorphisms from $C(X)$ to $A$ such that, for every $f \in C(X)$,
   $$(\eta \circ \pi_\lambda)(f) \to \pi(f)$$
   in the ultra*$*$-strong topology on $B$.
4. For every regular Borel probability measure $\nu$ on $X$, there is a net $\{\rho_\omega\}$ of unital $*$-homomorphisms from $C(X)$ into $C[0, 1]$ such that, for every $f \in C(X)$,
   $$\int_X f d\nu = \lim_\omega \int_0^1 \rho_\omega(f)(x) \, dx.$$

**Proof.** (1) $\Rightarrow$ (2). Suppose $X$ is approximately path-connected and suppose $A, B, \eta,$ and $\pi$ are as in (2). Let $\Lambda$ be the collection of all tuples of the form $\lambda = (S_\lambda, E_\lambda, \varepsilon_\lambda)$ where $S_\lambda$ is a finite set of states on $B$, $E_\lambda$ is a finite subset of $C(X)$ of functions from $X$ to $[0, 1]$, and $\varepsilon_\lambda > 0$. Suppose $\lambda \in \Lambda$. Clearly, $\pi(C(X))$ is an abelian selfadjoint
C*-subalgebra of $B$, and therefore there is a maximal abelian C*-subalgebra $M$ of $B$ such that $\pi (C (X)) \subseteq M$. Since $B$ is an AW*-algebra, there is a commuting family $P$ of projections in $B$ such that $M = C^* (P)$. Since $C^* (E_\lambda)$ is a separable unital C*-subalgebra of $C (X)$, the maximal ideal space of $C^* (E_\lambda)$ is a compact metric space $X_\lambda$ and there is a surjective continuous function $\zeta_\lambda : X \to X_\lambda$. It follows that $X_\lambda$ is approximately path-connected. Note that we identify $C^* (E_\lambda)$ with $C (X_\lambda)$ we can view a function $f \in C^* (E_\lambda)$ as both a function on $X$ and on $X_\lambda$ and we have $f = f \circ \zeta_\lambda$. Now $\pi (C^* (E_\lambda)) = \pi (C (X_\lambda))$ is a separable subalgebra of $C^* (P)$ so there is a countable subset $P_\lambda \subseteq P$ such that $\pi (C (X_\lambda)) \subseteq C^* (P_\lambda)$. If follows from von Neumann’s argument that there is a $w_\lambda = w_\lambda^*$ with $\sigma (w_\lambda)$ is a totally disconnected subset of $(0,1)$ such that $C^* (w_\lambda) = C^* (P_\lambda)$. Let $\psi_\lambda = \sum_{\psi \in S_\lambda} \psi$ and let $n_\lambda = \zeta (S_\lambda)$. Then there is a measure $\mu_\lambda$ on $\sigma (w_\lambda)$ such that, for every $h \in C (\sigma (w_\lambda))$, we have

$$\int_{\sigma (w_\lambda)} h d\mu_\lambda = \psi_\lambda (h (w_\lambda)).$$

For each $f \in E_\lambda$ there is a function $\hat{h}_f \in C (\sigma (w_\lambda))$ such that $\pi (f) = \hat{h}_f (w_\lambda)$ and there is a function $\hat{h}_f \in C [0,1]$ with $0 \leq \hat{h}_f \leq 1$ such that $\hat{h}_f |_{\sigma (w_\lambda)} = h_f$. We can view $\mu_\lambda$ as a Borel measure on $[0,1]$ by defining $\mu_\lambda ([0,1] \setminus \sigma (w_\lambda)) = 0$. Clearly, $\mu_\lambda ([0,1]) = \psi_\lambda (1) = n_\lambda$. Since $\{ \hat{h}_f : f \in E_\lambda \}$ is equicontinuous and $[0,1] \setminus \sigma (w_\lambda)$ is dense in $[0,1]$, we can find $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_s < b_s < 1$ and $r_1, \ldots, r_s \in \sigma (w_\lambda)$ such that, for every $f \in E$ and $1 \leq j \leq s$,

$$\left| \hat{h}_f (t) - \hat{h}_f (r_j) \right| < \frac{\varepsilon_\lambda}{4n_\lambda} \text{ when } t \in (a_j, b_j)$$

and such that

$$\mu_\lambda ([0,1] \setminus \cup_{j=1}^s (a_j, b_j)) < \frac{\varepsilon_\lambda}{4}.$$

We know $\hat{h}_f (r_j) = h_f (r_j)$ for all $f \in E_\lambda$ and $1 \leq j \leq s$. Since $\pi : C (X_\lambda) \to C (\sigma (w_\lambda))$, there is a $y_j \in X_\lambda$ for $1 \leq j \leq s$ such that, for every $f \in E_\lambda$, $f (y_j) = h_f (r_j)$. On the other hand, we see that there is an $x_j \in X$ for $1 \leq j \leq s$ such that $\zeta_\lambda (x_j) = y_j$. Hence, we have

$$\left| \hat{h}_f (t) - f (x_j) \right| < \frac{\varepsilon_\lambda}{4n_\lambda} \text{ when } t \in (a_j, b_j).$$

We next choose an open set $V_j \subseteq X$ with $x_j \in V_j$ such that, for every $f \in E_\lambda$ and every $x \in V$, we have

$$|f (x) - f (x_j)| < \frac{\varepsilon_\lambda}{4n_\lambda}.$$

We now use the fact that $X$ is approximately path-connected to find a continuous function $\gamma_\lambda : [0,1] \to X$ such that, for $1 \leq j \leq s$,

$$\gamma_\lambda ((a_j, b_j)) \subseteq V_j.$$

We have, for each $f \in E_\lambda$, each $1 \leq j \leq s$, and each $t \in (a_j, b_j)$

$$\left| \hat{h}_f (t) - f \circ \gamma_\lambda (t) \right| < \frac{\varepsilon_\lambda}{2n_\lambda}$$

and for $t \in [0,1] \setminus \cup_{j=1}^s (a_j, b_j)$,

$$\left| \hat{h}_f (t) - f \circ \gamma_\lambda (t) \right| \leq 2.$$
Hence, for every \( f \in E_\lambda \)
\[
|\psi_\lambda(\pi(f)) - \psi_\lambda((f \circ \gamma_\lambda)(w_\lambda))| \leq \int |\hat{h}_f - (f \circ \gamma_\lambda)(w_\lambda)| \, d\mu_\lambda < \\
(\varepsilon_\lambda/2n_\lambda) \mu_\lambda([0, 1]) + 2\mu_\lambda([0, 1] \cup \cup_{j=1}^n (a_j, b_j)) < \varepsilon_\lambda.
\]
Since \( \psi_\lambda = \sum_{\psi \in S_\lambda} \psi \), the measure \( \mu_\lambda \) is a sum of probability measures, one corresponding to each \( \psi \in S_\lambda \). We therefore have, for every \( f \in E_\lambda \) and every \( \psi \in S_\lambda \),
\[
|\psi(\pi(f)) - \psi((f \circ \gamma_\lambda)(w_\lambda))| \leq \int |\hat{h}_f - (f \circ \gamma_\lambda)(w_\lambda)| \, d\mu_\lambda < \varepsilon_\lambda.
\]
We can choose \( v_\lambda \in A \) with \( 0 \leq v_\lambda \leq 1 \) such that \( \eta(v_\lambda) = w_\lambda \). We define a unital *-homomorphism \( \pi_\lambda : C(X) \to A \) by \( \pi_\lambda(f) = (f \circ \gamma_\lambda)(v_\lambda) \). Hence, for every \( f \in C(X) \),
\[
(\eta \circ \pi_\lambda)(f) = \eta((f \circ \gamma_\lambda)(v_\lambda)) = (f \circ \gamma_\lambda)(w_\lambda).
\]
Hence, for every \( f \in E_\lambda \) and every \( \psi \in S_\lambda \) we have
\[
|\psi((\eta \circ \pi_\lambda)(f)) - \psi(\pi(f))| < \varepsilon_\lambda.
\]
It follows, for every \( f \in C(X) \) with \( 0 \leq f \leq 1 \) and every state \( \psi \) on \( B \), that
\[
\lim_{\lambda} \psi((\eta \circ \pi_\lambda)(f)) = \psi(\pi(f)).
\]
Since every \( g \in C(X) \) is a linear combination of \( f \)'s with \( 0 \leq f \leq 1 \) and every continuous linear functional on \( B \) is a linear combination of states, we see, for every \( f \in C(X) \), that \( (\eta \circ \pi_\lambda)(f) \to \pi(f) \) in the weak topology on \( B \).

(2) \( \Rightarrow \) (3). Since every \( W^* \)-algebra is an \( AW^* \)-algebra, it is clear that we can find a net \( \{\pi_\lambda\} \) as in (2). Thus, for every \( f \in C(X) \), \( (\eta \circ \pi_\lambda)(f) \to \pi(f) \) and
\[
(\eta \circ \pi_\lambda)(f^*) = (\eta \circ \pi_\lambda)(f^*) \to \pi(f^*) \pi(f)
\]
ultra*-weakly. Hence we have \( (\eta \circ \pi_\lambda)(f) \to \pi(f) \) in the ultra*-strong topology on \( B \).

(3) \( \Rightarrow \) (4). By Lemma 5.1 there exists a unital *-homomorphism \( \pi : C(X) \to \prod_{n \in \mathbb{N}} (C[0, 1], \sigma) \) such that
\[
(5.1) \quad \sigma_n \circ \pi(f) = \int_X f \, d\nu,
\]
for each \( f \in C(X) \). Here a state \( \sigma \) on \( C[0, 1] \) is given by \( \sigma(g) = \int_0^1 g \, dx \). Let
\[
\eta : \prod_{n \in \mathbb{N}} C[0, 1] \to \prod_{n \in \mathbb{N}} (C[0, 1], \sigma)
\]
be the canonical surjection. By 3) there is a net \( \{\pi_\lambda\} \) of unital *-homomorphisms from \( C(X) \) to \( \prod_{n \in \mathbb{N}} C[0, 1] \) such that, for every \( f \in C(X) \),
\[
(\eta \circ \pi_\lambda)(f) \to \pi(f)
\]
ultra*-strongly. By Lemma 2.2 there exist unital *-homomorphisms \( \rho_n : C(X) \to C[0, 1] \) such that
\[
\pi(f) = \eta((\rho_n(f))_{n \in \mathbb{N}}),
\]
for each \( f \in C(X) \). By 5.1
\[
\int_X f \, d\nu = \lim_{\alpha} \int_0^1 \rho_n(f) \, dx.
\]
The sequence $\int_0^1 \rho_n(f) \, dx$ contains a subnet $\int_0^1 \rho_{n_\omega}(f) \, dx$ which is an ultranet. This ultranet has to converge to $\int_X f \, d\nu$. Indeed, if $\lim_\omega t_n = t$ and $t_{n_\omega}$ is an ultranet, then $\lim_\omega t_{n_\omega} = t$. (Proof: for any $\epsilon > 0$ the set $\{n_\omega \mid |t_{n_\omega} - t| < \epsilon\}$ is infinite, otherwise $\{n_\omega \mid |t_{n_\omega} - t| < \epsilon\} \notin \alpha$ and we would have $\emptyset = \{n_\omega \mid |t_{n_\omega} - t| \geq \epsilon\} \cap \{n \mid |t_n - t| < \epsilon\} \in \alpha$. Hence $t$ is an accumulation point for $\{t_{n_\omega}\}$ and since $\{t_{n_\omega}\}$ is an ultranet, $t$ is its limit.) Thus we have

$$
\int_X f \, d\nu = \lim_\omega \int_0^1 \rho_{n_\omega}(f) \, dx.
$$

(4) $\Rightarrow$ (1). Assume (4) is true. Suppose $V_1, V_2, \ldots, V_s$ are nonempty open subsets of $X$. There is no harm in assuming $\{V_1, \ldots, V_s\}$ is disjoint. For each $1 \leq j \leq s$ we can choose $x_k \in V_k$ and a continuous function $h_j : X \to [0, 1]$ such that $h_j(x_j) = 1$ and $h_j|_{X \setminus V_j} = 0$. Let $\mu = \frac{1}{s} \sum_{j=1}^s \delta_{x_j}$. Then $\mu$ is a probability measure with $\int_X h_j \, d\mu = \frac{1}{s}$. It follows from (4) that there is a unital $*$-homomorphism $\rho : C(X) \to C[0, 1]$ such that

$$
\int_0^1 \rho(h_j)(x) \, dx \neq 0
$$

for $1 \leq k \leq s$. However, there must be a continuous map $\gamma : [0, 1] \to X$ such that $\pi(f) = f \circ \gamma$ for every $f \in C(X)$. For each $1 \leq j \leq s$, $0 \neq \int_0^1 (h_j \circ \gamma)(x) \, dx$ implies that there is a $t_j \in [0, 1]$ such that $h_j(\gamma(t_j)) \neq 0$. Thus, by the definition of $h_j$, we have $\gamma(t_j) \in V_j$ for $1 \leq j \leq s$. Therefore, $X$ is approximately path-connected. \hfill \Box

**Remark.** In statement (2) in Theorem 5.2 if we view $B \subseteq B(H)$ as the universal representation (that is the direct sum of all irreducible representations), then the weak operator topology on $B$ is the weak (and the ultraweak) topology on $B$. Thus, if $(\eta \circ \pi_\lambda)(f) = \pi(f)$ and

$$
[(\eta \circ \pi_\lambda)(f)]^* [(\eta \circ \pi_\lambda)(f)] = (\eta \circ \pi_\lambda)(f^* f) = \pi(f^* f) = \pi(f)^* \pi(f)
$$

weakly for every $f \in C(X)$ implies, for each $f \in C(X)$ that

$$
(\eta \circ \pi_\lambda)(f) \to \pi(f)
$$

is the $*$-strong operator topology in $B(H)$.

As a corollary we obtain a characterization of when a separable commutative $C^*$-algebra is $C^*$-tracially stable.

**Theorem 5.3.** Suppose $(X, d)$ is a compact metric space. The following are equivalent:

1. $C(X)$ is $C^*$-tracially stable.
2. $X$ is approximately path-connected
3. For every state $\tau$ on $C(X)$ there is a sequence $\pi_n : C(X) \to C[0, 1]$ such that, for every $f \in C(X)$,

$$
\tau(f) = \lim_{n \to \infty} \int_0^1 \pi_n(f)(x) \, dx.
$$
Proof. 2) \(\Leftrightarrow\) 3) by the equivalence of statements 1) and 4) in Theorem 5.2 and separability of \(C(X)\).

3) \(\Rightarrow\) 1): Let \(\phi : C(X) \to \prod_{\alpha \in I} (A_{\alpha}, \rho_{\alpha})\) be a unital \(*\)-homomorphism. By 11 \(\prod_{\alpha \in I} (A_{\alpha}, \rho_{\alpha})\) is a von Neumann algebra and by the equivalence 3) \(\Leftrightarrow\) 4) in Theorem 5.2 \(\phi\) is a \(*\)-strong pointwise limit of liftable \(*\)-homomorphisms from \(C(X) \to \prod_{\alpha \in I} (A_{\alpha}, \rho_{\alpha})\). By Lemma 2.2 \(\phi\) is liftable.

1) \(\Rightarrow\) 3): By Lemma 5.1 there is a unital \(*\)-homomorphism \(\pi : C(X) \to \prod_{n \in \mathbb{N}} (C[0,1], \sigma)\) such that \(\pi_{\alpha} \circ \pi = \tau\). By 1) we can lift it and obtain a sequence \(\pi_{n} : C(X) \to C[0,1]\) such that, for every \(f \in C(X)\),

\[
\tau(f) = \lim_{\alpha} \int_{0}^{1} \pi_{n}(f)(x) \, dx.
\]

Taking a subnet, which is an ultranet we obtain (by the same arguments as in the proof of the implication 3) \(\Rightarrow\) 4) in Theorem 5.2) that for any \(f \in C(X)\)

\[
\tau(f) = \lim_{\omega} \int_{0}^{1} \pi_{n_{\omega}}(f)(x) \, dx.
\]

Since \(C(X)\) is separable, we can pass to a subsequence. 

\[\square\]

Corollary 5.4. Suppose \(\mathcal{A}\) is a unital commutative \(C^{*}\)-tracially stable \(C^{*}\)-algebra and \(\mathcal{A}_{0}\) is a unital \(C^{*}\)-subalgebra of \(\mathcal{A}\). Then \(\mathcal{A}_{0}\) is \(C^{*}\)-tracially stable.

Proof. We can assume \(\mathcal{A} = C(X)\) with \(X\) an approximately path-connected compact metric space. Also we can write \(\mathcal{A}_{0} = C(Y)\), and, since \(\mathcal{A}_{0}\) embeds into \(\mathcal{A}\), there is a continuous surjective map \(\varphi : X \to Y\). Thus \(Y\) is approximately path-connected, which implies \(\mathcal{A}_{0}\) is \(C^{*}\)-tracially stable. 

\[\square\]

At this point there is little else we can say about \(C^{*}\)-tracially stable algebras, except that they do not have projections when there is a faithful embeddable tracial state.

Theorem 5.5. Suppose \(\mathcal{A}\) is a separable \(C^{*}\)-tracially stable unital \(C^{*}\)-algebra. Then \(\mathcal{A}/\mathcal{J}_{\mathcal{A}}\) has no nontrivial projections.

Proof. Without loss of generality we can assume \(\mathcal{J}_{\mathcal{A}} = \{0\}\). Then \(\mathcal{A}\) has a faithful embeddable tracial state \(\sigma\). Then there is a tracial embedding \(\pi\) of \((\mathcal{A}, \sigma)\) into \(\prod_{n \in \mathbb{N}} (C^{*}(\mathbb{F}_{2}), \tau)\) for some non-trivial ultrafilter \(\alpha\) on \(\mathbb{N}\). Indeed, \(C^{*}(\mathbb{F}_{2})\) has a unique trace \(\tau\) and it is a subalgebra of the factor von Neumann algebra \(L_{\mathbb{F}_{2}}\), so that \(L_{\mathbb{F}_{2}} = W^{*}(C^{*}(\mathbb{F}_{2}))\). It follows from the Kaplansky density theorem that the \(\|\|_{2}\)-closure of the unit ball of \(C^{*}(\mathbb{F}_{2})\) is the unit ball of \(L_{\mathbb{F}_{2}}\) which implies that

\[
\prod_{n \in \mathbb{N}} (C^{*}(\mathbb{F}_{2}), \tau) = \prod_{n \in \mathbb{N}} (L_{\mathbb{F}_{2}}, \tau).
\]

Since \(L_{\mathbb{F}_{2}}\) contains \(M_{n}(\mathbb{C})\) for each \(n \in \mathbb{N}\), \(\prod_{n} (M_{n}(\mathbb{C}), \tau_{n})\) embeds into \(\prod_{\alpha} (C^{*}(\mathbb{F}_{2}), \tau) = \prod_{\alpha} (L_{\mathbb{F}_{2}}, \tau)\).

Now since \(\mathcal{A}\) is tracially stable, there is a sequence \(\{\pi_{n}\}\) of unital \(*\)-homomorphisms from \(\mathcal{A}\) into \(C^{*}(\mathbb{F}_{2})\) such that, for every \(a \in \mathcal{A}\),

\[
\sigma(a) = \lim_{n \to \alpha} \tau(\pi_{n}(a)).
\]
Suppose $P$ is a projection in $A$. Since $C^*_r(F_2)$ contains no non-trivial projections, for each $n \in \mathbb{N}$, $\pi_n(P) = 0$ or $\pi_n(P) = 1$. Since $\alpha$ is an ultrafilter, eventually $\pi_n(P) = 0$ along $\alpha$ or eventually $\pi_n(P) = 1$ along $\alpha$. Thus $\pi(P) = \{\pi_n(P)\}_\alpha$ is either 0 or 1. Hence $P$ is either 0 or 1. \hfill $\square$

**Remark.** As was pointed out in [11], the tracial ultraproducts remain unchanged when you replace the 2-norm by a $p$-norm ($1 \leq p < \infty$). Therefore all results in this paper remain valid if 2-norms are replaced by $p$-norms.

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