TWISTED KUPERBERG INVARIANTS OF KNOTS AND 
REIDEMEISTER TORSION VIA TWISTED DRINFELD DOUBLES 

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ABSTRACT. In this paper, we consider the Reshetikhin-Turaev invariants of knots in the three-sphere obtained from a twisted Drinfeld double of a Hopf algebra, or equivalently, the relative Drinfeld center of the crossed product $\text{Rep}(H) \rtimes \text{Aut}(H)$. These are quantum invariants of knots endowed with a homomorphism of the knot group to $\text{Aut}(H)$. We show that, at least for knots in the three-sphere, these invariants provide a non-involutory generalization of the Fox-calculus-twisted Kuperberg invariants of sutured manifolds introduced previously by the author, which are only defined for involutory Hopf algebras. In particular, we describe the $\text{SL}(n, \mathbb{C})$-twisted Reidemeister torsion of a knot complement as a Reshetikhin-Turaev invariant.

1. Introduction

The theory of braided and ribbon monoidal categories is a rich subject at the intersection of various fields, such as representation theory, physics and low dimensional topology. Their relevance in topology stems from the fact that a ribbon monoidal category produces an entire family of topological invariants of links and tangles in the three-sphere, known as Reshetikhin-Turaev invariants [25]. One of the best known invariants in this family is the Jones polynomial, which is obtained from the category of modules over the quantum group of $\mathfrak{sl}_2$. These link invariants can be extended to invariants of closed 3-manifolds as well and even to a powerful object called a topological quantum field theory (TQFT) [26]. Even thirty years after their introduction, many fundamental questions about these invariants remain open, such as their topological and geometrical content, categorification, etc.

The Reshetikhin-Turaev invariants of links and 3-manifolds admit a refinement when the ribbon categories involved are graded by a group $G$. Here, the algebraic input is a $G$-crossed ribbon category, that is, a category $\mathcal{C}$ which is a disjoint union of subcategories $\mathcal{C} = \bigsqcup_{g \in G} \mathcal{C}_g$ admitting a monoidal structure, $G$-action and ribbon structure compatible with the grading. The output is a topological invariant of $G$-tangles, that is, tangles $T \subset \mathbb{R}^2 \times [0, 1]$ endowed with a flat $G$-connection on a principal $G$-bundle of the complement $X_T$, or equivalently, a homomorphism $\pi_1(X_T) \rightarrow G$. This extension was introduced by Turaev in [27]. Special instances (and modifications) of this construction related to quantum groups have been studied in various works [1,3,5,12,19]. When the $G$-crossed ribbon categories are semisimple (or more precisely, $G$-modular), Turaev and Virelizier extended this construction to a “homotopy quantum field theory” (HQFT) [30]. It has to be noted that the theory of $G$-extensions of a given monoidal category $\mathcal{D}$ (i.e. $G$-graded $\mathcal{C}$ with $\mathcal{C}_1 = \mathcal{D}$), including extensions of quantum groups at roots of unity, is by now well-understood [6–9].

The study of quantum invariants of knots and 3-manifolds endowed with a representation of their fundamental groups seems very natural and (possibly) powerful from a topological
point of view. On the one hand, certain classes of 3-manifolds come equipped with a canonical representation \( \pi_1(M) \to G \), such as hyperbolic 3-manifolds (with \( G = PSL(2, \mathbb{C}) \)) or knot complements in the three sphere (with \( G = \mathbb{Z} = H_1(M) \)). On the other hand, classical invariants such as twisted Reidemeister torsion are by definition invariants of pairs \((M, \rho)\) where \( M \) is a 3-manifold and \( \rho : \pi_1(M) \to GL(n, \mathbb{C}) \) is an homomorphism [28]. The fact that this invariant is defined for any \( \rho \) leads to interesting applications in geometric topology (see the surveys [10] for topological applications and [23] for applications in hyperbolic geometry). These considerations lead to the question of whether quantum invariants from \( G \)-crossed ribbon categories may have applications to geometric topology generalizing those of Reidemeister torsion.

A natural first step to study the above question is to realize twisted Reidemeister torsion as a special case of the Reshetikhin-Turaev invariant of \( G \)-tangles of [27]. Some of the aforementioned works do that in certain cases: [1] for multivariable Alexander, [2] for abelian Reidemeister torsion. However, to the knowledge of the author, the works that find (special cases of) non-abelian Reidemeister torsion as a quantum invariant do not rely directly on \( G \)-crossed ribbon categories. For instance, in [19] McPhail-Snyder finds the \( SL(2, \mathbb{C}) \)-twisted Reidemeister torsion of a link complement using the “holonomy braidings” of Kashaev-Reshetikhin [12,13]. Another example is our previous work [21], where, for involutory Hopf algebras \( H \), we extended Kuperberg’s invariant [16] to an invariant \( I_H^\rho(M, \gamma) \) of balanced sutured manifolds \((M, \gamma)\) endowed with a homomorphism \( \rho : \pi_1(M) \to \text{Aut}(H) \), and we showed that the \( SL(n, \mathbb{C}) \)-twisted Reidemeister torsion was the special case when \( H \) is an exterior algebra \( \Lambda(\mathbb{C}^n) \).

The purpose of the present work is to find a general class of \( G \)-crossed ribbon categories for which the associated invariants contain \( SL(n, \mathbb{C}) \)-twisted Reidemeister torsion, in particular twisted Alexander polynomials [15,18,34], as a special case. We achieve this by finding the Reshetikhin-Turaev version of the twisted Kuperberg invariants \( I_H^\rho \) of our previous work [21], in particular extending that work to non-involutory Hopf algebras (at least for knots in the three-sphere). To do this, we need to take an appropriate Drinfeld double of the graded Hopf object considered in [21]. This is motivated by various works that relate Kuperberg or Turaev-Viro invariants to Hennings or Reshetikhin-Turaev invariants of the corresponding Drinfeld double, see [4,31]. Particularly relevant to us is the work of Turaev and Virelizier in the semisimple \( G \)-crossed setting [31], where it is shown that the Turaev-Viro HQFT and the Reshetikhin-Turaev HQFT are related via the relative Drinfeld center construction of Gelaki-Naidu-Nikshych [11]. Since our previous work depended (implicitly) on a crossed product category \( \mathcal{C} = \text{Rep}(H) \rtimes \text{Aut}(H) \) (see Remark 4.1), we consider here its relative Drinfeld center \( Z_{\text{Rep}(H)}(\mathcal{C}) \). This is a braided \( \text{Aut}(H) \)-crossed category and is equivalent to the category of modules over the “twisted Drinfeld double” introduced by Virelizier in [33] (see Proposition 2.4), here denoted \( \mathcal{D}(H) \), which is a quasi-triangular Hopf group-coalgebra in the sense of [27,32].

In light of the above, we consider the Reshetikhin-Turaev invariant of a knot obtained from a twisted Drinfeld double. We will work in the setting of universal quantum invariants. Recall that in the untwisted setting, if \( H \) is a finite dimensional Hopf algebra for which \( D(H) \) is ribbon and if \( K \) is a framed knot presented as the closure of a long knot \( T \), then the Reshetikhin-Turaev invariant of \( T \) colored by the regular \( D(H) \)-module is an invariant of \( K \). This invariant is left multiplication by an element \( Z_{D(H)}(K) \) of the center of \( D(H) \) and
recovers all the Reshetikhin-Turaev invariants of $K$ via the quantum trace (or a modified trace if the color is projective), hence called the universal invariant. In the twisted setting, if $G \subset \text{Aut}(H)$ is a subgroup for which the twisted Drinfeld double is $G$-ribbon, a similar procedure leads to a “twisted” universal quantum invariant

$$Z_{D(H)}^\rho(T) \in D(H)$$

where $\rho : \pi_1(X_T) \to G$ is an homomorphism. If $H$ is $\mathbb{Z}$-graded, one can use the abelianization $h : \pi_1(X_T) \to \mathbb{Z}$ to extend $\rho$ to a representation $\rho \otimes h$ with coefficients in $\mathbb{K}[t^{\pm 1}]$. This leads to a polynomial invariant

$$Z_{D(H)}^{\rho \otimes h}(T) \in D(H)[t^{\pm \frac{1}{2}}].$$

These are not necessarily invariants of the closure $(K, \rho)$ (except for abelian $\rho$), but they are after an appropriate evaluation. For instance, $\epsilon_{D(H)}(Z_{D(H)}^{\rho \otimes h}(T)) \in \mathbb{K}[t^{\pm \frac{1}{2}}]$ is a polynomial invariant of $(K, \rho)$ and after a framing normalization, one obtains a polynomial invariant

$$P_H^\rho(K, t) \in \mathbb{K}[t^{\pm 1}]$$

of unframed $G$-knots. Our main theorem (Theorem 4.2) is that this evaluation recovers the “Fox-calculus twisted” Kuperberg invariant of our previous work [21] (which is defined for involutory $H$) as follows:

$$P_H^\rho(K, t) = I_H^{\rho \otimes h}(M, \gamma).$$

Here $=$ is equality up to multiplication by $\pm r_H(\rho(g))t^{|\Lambda|}$ for some $g \in \pi_1(X_K)$, $k \in \mathbb{Z}$ ($r_H$ is defined in (1) below and $|\Lambda|$ is the $\mathbb{Z}$-degree of the cointegral of $H$) and $(M, \gamma)$ is the sutured manifold associated to $(S^3, K)$. This theorem is stated for involutory $H$ so that the right hand side is a well-defined invariant, but one should still think that, for non-involutory $H$, $\rho$ is twisting via Fox calculus the tensors of the universal invariant $Z_{D(H)}(K)$.

As a corollary of our main theorem and our previous work [21] we get that the twisted Reidemeister torsion $\tau^{\rho \otimes h}$ is a Reshetikhin-Turaev invariant of a twisted Drinfeld double of an exterior algebra $\Lambda(\mathbb{C}^n)$ (Corollary 4.3):

$$P^\rho_{\Lambda(\mathbb{C}^n)}(K, t) = \tau^{\rho \otimes h}(S^3 \setminus K, m)$$

where $m \subset \partial(S^3 \setminus K)$ is a meridian and $\rho : \pi_1(S^3 \setminus K) \to SL(n, \mathbb{C})$. Thus, the twisted polynomial invariant $P^\rho_H(K, t)$ generalizes twisted Alexander polynomials to arbitrary finite-dimensional $\mathbb{Z}$-graded Hopf algebras (with ribbon double).

The plan of the paper is the following. In Section 2 we recall the notions from Hopf algebra theory that we need and we study the twisted Drinfeld double of a Hopf algebra. In Section 3 we recall the construction of invariants of $G$-tangles of [27], though in the universal setting, so the tangles are not colored by modules. Here we define the invariants $Z_{D(H)}^\rho(T), Z_{D(H)}^{\rho \otimes h}(T)$ and the knot polynomial $P_H^\rho(K, t)$ mentioned above. In Section 4 we state and prove our main theorem.

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2. THE TWISTED DRINFELD DOUBLE OF A HOPF ALGEBRA

We begin by recalling some basic notions and notation from Hopf algebra theory and Hopf group-coalgebras, following mainly [24, 32]. The twisted Drinfeld double \(D(H)\) is defined in Subsection 2.5. In Subsection 2.6 we discuss under which conditions \(D(H)\) is ribbon. In Subsection 2.7 we relate \(D(H)\) to the relative Drinfeld center of [11].

In what follows we denote by \(\text{Vect}\) the category of vector spaces over a field \(K\) and by \(\text{SVect}\) that of super-vector spaces and degree-preserving linear maps.

2.1. Tensor network notation. We will use tensor network notation for tensors of the form \(T : V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_m\) where the \(V_i\)'s and \(W_j\)'s are vector spaces. Such a tensor is denoted by a diagram with \(n\) incoming arrows, or inputs, and \(m\) outgoing arrows, or outputs. The diagram is read from left to right, and the arrows are ordered from top to bottom. Thus, a tensor \(T : V_1 \otimes V_2 \to W\) is denoted by

\[
\text{T} \quad \text{where the top leftmost arrow corresponds to } V_1 \text{ and the bottom one to } V_2.
\]

In this paper we will only consider tensors where all the arrows correspond to the same vector space \(V\), so we do not include \(V\) in the notation. The tensor product \(T_1 \otimes T_2\) is obtained by stacking \(T_1\) on top of \(T_2\) and compositions are drawn by joining the outputs of a tensor with the corresponding inputs in another tensor. When an input/output is the dual of a vector space, we will denote it by reversing the orientation of that arrow. For instance, if \(V\) is a finite-dimensional vector space, a tensor \(T : V \otimes V^* \to V^*,\) the pairing \(V^* \otimes V \to K,\) the copairing \(\mathbb{K} \to V \otimes V^*\) and the trace of a map \(f : V \to V\) are denoted by

\[
\text{T} \quad \text{where the top leftmost arrow corresponds to } V_1 \text{ and the bottom one to } V_2.
\]

The above notation is also valid for super-vector spaces. Here we add the convention that a crossing pair of arrows stands for the symmetry \(\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x\) of super-vector spaces:

\[
\tau = \quad \text{crossing}
\]

If \(V\) is a super-vector space, the left pairing/copairing \(V^* \otimes V \to \mathbb{K}\) and \(\mathbb{K} \to V \otimes V^*\) are the usual ones of vector spaces. We will suppose that the category of super-vector spaces has the canonical pivotal structure, that is, the pairing \(V \otimes V^* \to \mathbb{K}\) and the copairing \(\mathbb{K} \to V^* \otimes V\) are defined by

\[
\text{pairing} = \quad \text{copairing} = \quad \text{trace} = \quad \text{supertrace}
\]

Note that if \(V = V_0 \oplus V_1\) is a finite-dimensional super-vector space, the above trace of \(f : V \to V\) is the supertrace \(\text{tr} (f|_{V_0}) - \text{tr} (f|_{V_1}).\)
2.2. Hopf algebras. A Hopf algebra over $K$ is a vector space $H$ endowed with tensors
\[
m \mapsto m, \quad 1 \mapsto 1, \quad \Delta \mapsto \Delta, \quad \epsilon \mapsto \epsilon, \quad S \mapsto S,
\]
such that $(H, m, 1)$ is an algebra, $(H, \Delta, \epsilon)$ is a coalgebra, $\Delta$ and $\epsilon$ are algebra homomorphisms and $S$ satisfies the antipode axiom. Sometimes, we also use Sweedler’s notation for the coproduct, that is, we write $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$ where it is understood that this is a sum of various elements $h_{(1)} \otimes h_{(2)} \in H^\otimes 2$. We will also consider Hopf algebras in SVect, the only difference is that the multiplicative axiom for the coproduct involves a sign coming from the symmetry of the category. We denote by $\text{Aut}(H)$ the group of Hopf algebra automorphisms of $H$. In this paper, we assume Hopf algebras are finite dimensional.

The dual $H^*$ is a Hopf algebra if the product and the coproduct (which we simply denote by $\Delta$ and $m$ respectively in tensor network notation) are defined by
\[
\Delta \Delta = \Delta, \quad m = m.
\]

2.3. Integrals and cointegrals. Let $H$ be a finite dimensional Hopf algebra. Then there exists a unique element (up to scalar) $\lambda_r \in H^*$ satisfying
\[
\rightarrow \Delta \lambda_r = \lambda_r \rightarrow 1.
\]
This is called a right integral of $H$. Similarly, there exists a unique (up to scalar) left cointegral $\Lambda_l \in H$ satisfying
\[
\Lambda_l m = \epsilon \Lambda_l \rightarrow .
\]

From the uniqueness of the integrals and cointegrals, it follows that there exist group-likes $g \in G(H)$ and $\xi \in G(H^*)$ characterized by the following equations:
\[
\rightarrow \Delta \lambda_r = \lambda_r \rightarrow g, \quad \Lambda_l \rightarrow m = \xi \Lambda_l \rightarrow .
\]

If $H$ is a Hopf algebra in Vect, the trace of a linear map $f : H \rightarrow H$ can be computed from a cointegral/integral pair $\Lambda_l, \lambda_r$ as above satisfying $\lambda_r(\Lambda_l) = 1$ by the following formula of Radford [24, Thm. 10.4.1]:
\[
\bigcirc f = \Lambda_l \rightarrow \Delta \lambda_r \rightarrow S f \bigcirc m \rightarrow \lambda_r.
\]
All of the above holds as well if $H$ is a Hopf algebra in SVect [17], except that the right hand side in Radford’s trace formula has to be multiplied by $(-1)^{|A|}$ (this follows e.g. from [17, Lemma 3.4]).

Now let $\alpha \in \text{Aut}(H)$. Then, if $\lambda_r$ is a right integral of $H$, $\lambda_r \circ \alpha$ is also a right integral so by uniqueness,

$$\lambda_r \circ \alpha = r_H(\alpha) \cdot \lambda_r$$

for some scalar $r_H(\alpha) \in \mathbb{K}^\times$. This defines a group homomorphism $r_H : \text{Aut}(H) \to \mathbb{K}^\times$. Note that if $\lambda_l$ is a left integral, then also $\lambda_l \circ \alpha = r_H(\alpha) \lambda_l$ (since any left integral is a multiple of $\lambda_r \circ S$). Similarly, if $\Lambda$ is any left or right cointegral we have $\alpha(\Lambda) = r_H(\alpha)\Lambda$, since $\lambda(\Lambda) \neq 0$ for any nonzero integral (left or right) and any nonzero cointegral (left or right) [24, Thm. 10.2.2].

2.4. Hopf $G$-coalgebras. We now recall some definitions from [32]. Let $G$ be a group. A Hopf $G$-coalgebra over $\mathbb{K}$ is a family $A = \{A_{\alpha}\}_{\alpha \in G}$ where each $A_{\alpha} = (A_{\alpha}, m_{\alpha}, 1_{\alpha})$ is a $\mathbb{K}$-algebra with multiplication $m_{\alpha}$ and unit $1_{\alpha}$, endowed with algebra morphisms $\Delta_{\alpha,\beta} : A_{\alpha} \otimes A_{\beta} \to A_{\alpha \beta}$ for each $\alpha, \beta \in G$, a counit $\epsilon : A_1 \to \mathbb{K}$ and an antipode $S_{\alpha} : A_{\alpha} \to A_{\alpha^{-1}}$ for each $\alpha \in G$ satisfying graded versions of the usual Hopf algebra axioms (see [32] for more details). For instance, the antipode axiom for Hopf $G$-coalgebras is

This definition is also valid in the category of super-vector spaces. When $G = 1$ we recover the usual notion of Hopf algebra and we denote the structure tensors simply by $m$, $1$, $\Delta$, $\epsilon$, $S$.

If $A = \{A_{\alpha}\}_{\alpha \in G}$ is a Hopf $G$-coalgebra, then $A_1$ is a Hopf algebra in the usual sense. The notion of integral and distinguished group-like extend to this setting [32].

A Hopf $G$-coalgebra $A = \{A_{\alpha}\}_{\alpha \in G}$ is said to be crossed if it is endowed with a family of algebra isomorphisms $\varphi = \{\varphi_{\beta,\alpha} : A_{\alpha} \to A_{\beta \alpha^{-1}}\}_{\alpha,\beta \in G}$, called a crossing, satisfying:

1. $(\varphi_{\beta,\alpha} \otimes \varphi_{\gamma,\beta}) \Delta_{\alpha,\gamma} = \Delta_{\beta \alpha^{-1},\beta \gamma^{-1}} \varphi_{\beta,\alpha} \gamma$, 
2. $\epsilon \varphi_{1,1} = \epsilon$, 
3. $\varphi_{\alpha,\beta \gamma^{-1}} \varphi_{\beta,\gamma} = \varphi_{\alpha \beta,\gamma}$,

for each $\alpha, \beta, \gamma \in G$. We will omit the second subscript of $\varphi_{\beta,\alpha}$ and denote $\varphi_{\beta} = \varphi_{\beta,1}$. If $(A, \varphi)$ is crossed, we will say that a family of elements $x = \{x_{\alpha}\}_{\alpha \in G}$, where each $x_{\alpha} \in A_{\alpha}$, is $G$-invariant if $\varphi_{\beta}(x_{\alpha}) = x_{\alpha \beta \beta^{-1}}$ for each $\alpha, \beta, \gamma \in G$.

A crossed Hopf $G$-coalgebra $(A, \varphi)$ is quasi-triangular if it is endowed with a family of elements $R = \{R_{\alpha,\beta}\}_{\alpha,\beta \in G}$ satisfying the following axioms for each $\alpha, \beta, \gamma \in G$:

1. $R_{\alpha,\beta}$ is invertible in $A_{\alpha} \otimes A_{\beta}$, 
2. $(\varphi_{\beta,\alpha} \otimes \varphi_{\gamma,\beta})(R_{\alpha,\beta}) = R_{\alpha,\gamma^{-1} \beta \gamma^{-1}}$, 
3. $R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(x) = (\tau_{\alpha,\beta} \cdot \varphi_{\alpha,\beta^{-1}}) \Delta_{\alpha,\beta^{-1}}(x) \cdot R_{\alpha,\beta}$ for each $x \in A_{\alpha \beta}$, where $\tau_{\alpha,\beta}$ denotes permutation of two factors (with signs in the super-case), 
4. $(\text{id}_{A_{\alpha}} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta,\gamma}) = (R_{\alpha,\beta,\gamma})_{13} \cdot (R_{\alpha,\beta,\gamma})_{12}$, 
5. $(\Delta_{\alpha,\beta} \otimes \text{id}_{A_{\gamma}})(R_{\alpha,\beta,\gamma}) = ((\text{id}_{A_{\alpha}} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta,\gamma^{-1}}))_{13} \cdot (R_{\beta,\gamma})_{12}$.


The last two equalities hold in $A_{\alpha} \otimes A_{\beta} \otimes A_{\gamma}$, where we note $X_{12\gamma} = x \otimes y \otimes 1_{\gamma}$ for any $X = \sum x \otimes y \in A_{\alpha} \otimes A_{\beta}$ and similarly for $Y_{13\beta}, Z_{13\gamma}$ ($Y \in A_{\beta} \otimes A_{\gamma}, Z \in A_{\alpha} \otimes A_{\gamma}$). In the super-case we will suppose that $R$-matrices have degree zero.

The Drinfeld element of a quasi-triangular Hopf $G$-coalgebra $(A, \varphi, R)$ is $u = \{u_{\alpha}\}_{\alpha \in G}$ defined by
\begin{equation}
(2)
 u_{\alpha} = m_\alpha(S_{\alpha^{-1}} \varphi \otimes \text{id}_{A_\alpha})\tau_{\alpha, \alpha^{-1}}(R_{\alpha, \alpha^{-1}}) \in A_\alpha.
\end{equation}
This is invertible with inverse $u_{\alpha}^{-1} = m_\alpha(\text{id}_{A_\alpha} \otimes S_{\alpha^{-1}})\tau_{\alpha, \alpha}(R_{\alpha, \alpha})$ [32, Lemma 6.5].

A quasi-triangular Hopf $G$-coalgebra $(A, \varphi, R)$ is ribbon if it is endowed with a family $v = \{v_{\alpha}\}_{\alpha \in G}$, where each $v_{\alpha} \in A_\alpha^*$, such that
\begin{enumerate}
\item $\varphi_{\beta}(v_{\alpha}) = v_{\beta_\alpha^{-1}}$, \\
\item $\Delta_{\alpha, \beta}(v_{\alpha}) = (v_{\alpha} \otimes v_\beta)(\varphi_{\alpha} \otimes \text{id}_{A_\beta})R_{\alpha, \beta^{-1}} \tau_{\alpha, \alpha^{-1}}(R_{\alpha, \alpha^{-1}}, \alpha), R_{\alpha, \beta}$, \\
\item $\varphi_\alpha(x) = v_{\alpha}^{-1}xv_\alpha$ for all $x \in A_\alpha$, \\
\item $S_\alpha(v_\alpha) = v_{\alpha}^{-1}$.
\end{enumerate}

The following lemma is a generalization of [24, Theorem 12.3.6]. We say that $g = \{g_{\alpha}\}_{\alpha \in G}$ is group-like if $\Delta_{\alpha, \beta}(g_{\alpha \beta}) = g_\alpha \otimes g_\beta$ for each $\alpha, \beta \in G$ and each $g_{\alpha}$ is invertible in $A_{\alpha}$.

**Lemma 2.1.** Let $(A, \varphi, R, v)$ be a quasi-triangular Hopf $G$-coalgebra. The map $\{v_{\alpha}\} \mapsto \{g_{\alpha} := u_{\alpha}v_{\alpha}\}$ is a bijection between ribbon structures and elements $g = \{g_{\alpha}\}$ of $A$ satisfying
\begin{enumerate}
\item[(1')] $\varphi_{\beta}(g_{\alpha}) = v_{\beta_\alpha^{-1}}$, \\
\item[(2')] $g$ is group-like, \\
\item[(3')] $S_{\alpha^{-1}}S_{\alpha}(x) = g_{\alpha}xg_{\alpha}^{-1}$, \\
\item[(4')] $g_{\alpha}^2 = u_{\alpha}S_{\alpha^{-1}}(u_{\alpha}^{-1})$,
\end{enumerate}
for each $\alpha, \beta \in G$ and $x \in A_{\alpha}$.

**Proof.** We will show that each of the above properties is equivalent to the corresponding property in the definition of ribbon element. The equivalence of (1) and (1') follows from $\varphi_{\beta}(u_{\alpha}) = v_{\beta_\alpha^{-1}}$ [32, Lemma 6.5, d)]. Similarly, the equivalence of (2) and (2') follows immediately from [32, Lemma 6.5, f)]. For (3) we have
\[\varphi_{\alpha}(x) = v_{\alpha}^{-1}xv_{\alpha} \Leftrightarrow \varphi_{\alpha}(x) = g_{\alpha}^{-1}u_{\alpha}xu_{\alpha}^{-1}g_{\alpha} \Leftrightarrow \varphi_{\alpha}(x) = g_{\alpha}^{-1}S_{\alpha^{-1}}S_{\alpha}(\varphi_{\alpha}(x))g_{\alpha},\]
which is equivalent to (3'). In the last equality we used [32, Lemma 6.5, b)]. Finally,
\[S_{\alpha^{-1}}(v_{\alpha}^{-1}) = v_{\alpha} \Leftrightarrow S_{\alpha^{-1}}(g_{\alpha}^{-1})S_{\alpha^{-1}}(u_{\alpha}^{-1}) = u_{\alpha}^{-1}g_{\alpha} \Leftrightarrow g_{\alpha}^{-1}S_{\alpha^{-1}}(u_{\alpha}^{-1}) = u_{\alpha}^{-1}g_{\alpha},\]
Since $g$ is a $G$-invariant group-like and $S_{\alpha^{-1}}S_{\alpha}(\varphi_{\alpha}(x)) = u_{\alpha}xu_{\alpha}^{-1}, g_{\alpha}$ and $u_{\alpha}$ commute so that the last equality above is equivalent to (4')).

If $(A, \varphi, R, v)$ is ribbon, we call $\{g_{\alpha} = u_{\alpha}v_{\alpha}\}_{\alpha \in G}$ the pivot of $A$.

**2.5. Twisted Drinfeld doubles.** Let $H = (H, m, 1, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra with automorphism group Aut($H$). Following [33] we define a quasi-triangular Hopf $(H, \varphi, R, v)$-coalgebra $D(H) = \{D(H)_{\alpha}\}_{\alpha \in \text{Aut}(H)}$ as follows: for each $\alpha \in \text{Aut}(H)$ set $D(H)_{\alpha} := H^* \otimes H$ as vector spaces. We define a product $m_\alpha$ on $D(H)_{\alpha}$ and a coproduct $\Delta_{\alpha, \beta} : D(H)_{\alpha \beta} \to D(H)_{\alpha} \otimes D(H)_{\beta}$ by
Note that if $H$ is a Hopf algebra in SVect the above formulas involve various signs coming from the symmetry and the right pairings of SVect. The antipode is defined by the conditions $S_{\alpha}(h) = \alpha^{-1}(S(h))$, $S_{\alpha}(p) = p \circ S^{-1}$ for $h \in H, p \in H^*$ and the fact that it is an algebra antiautomorphism. This defines a Hopf $\text{Aut}(H)$-coalgebra. This is crossed with

\[
\varphi_{\alpha}(p \otimes h) = p \circ \alpha^{-1} \otimes \alpha(h)
\]

and quasi-triangular with $R$-matrix

\[
R_{\alpha,\beta} = \sum (\epsilon \otimes \alpha(h_i)) \otimes (h^i_\alpha \otimes 1) = \sum (\epsilon \otimes h_i) \otimes (h^i \circ \alpha \otimes 1)
\]

where $(h_i)$ is any vector space basis of $H$ and $(h^i)$ is the dual basis of $H^*$ (for the left pairing). It is easy to see that $R_{\alpha,\beta}$ is invertible with inverse

\[
R_{\alpha,\beta}^{-1} = (S_{D(H)} \otimes \text{id}_{D(H)})(R_{\alpha,\beta}).
\]

We call $D(H) = \{D(H)_{\alpha}\}_{\alpha \in \text{Aut}(H)}$ the twisted Drinfeld double of $H$. If $G$ is a subgroup of $\text{Aut}(H)$, then the restriction $\{D(H)_{\alpha}\}_{\alpha \in G}$ to $G$ will be called the $G$-twisted Drinfeld double, denoted $D(H)|_G$. When $\alpha = \text{id}_H$ the quasi-triangular Hopf algebra $D(H) = D(H)_{\text{id}_H}$ is the usual Drinfeld double of $H$.

From (2) we see that the Drinfeld element of $D(H)$ and its inverse are given by

\[
u_{\alpha} = \frac{S^{-1}}{} , \quad \nu_{\alpha}^{-1} = \frac{S^2}{\alpha}
\]

It is easy to see that the right integral of $D(H)$ is given by $\lambda_{\alpha} = \Lambda_l \otimes \lambda_r$ for every $\alpha$. The distinguished group-like of $D(H)$ is

\[
g_{\alpha} = r_H(\alpha)^{-1} \zeta \otimes g
\]

where $r_H(\alpha) \in \mathbb{K}^\times$ is defined by (1) and $\zeta, g$ are the distinguished group-likes of $H$. If $H \in \text{Vect}$, this follows from the following computation

\[
(id_{D(H)} \otimes \lambda_{\beta}) \circ \Delta_{\alpha,\beta}(p \otimes h) = p_{(2)} \otimes h_{(1)} \otimes p_{(1)}(\Lambda_l)\lambda_r(\alpha^{-1}(h_{(2)}))
\]

\[
= r_H(\alpha)^{-1} p_{(2)} p_{(1)}(\Lambda_l) \otimes h_{(1)} \lambda_r(h_{(2)})
\]

\[
= r_H(\alpha)^{-1} p(\Lambda_l) \zeta \otimes \lambda_r(h)g
\]

\[
= \lambda_{\alpha,\beta}(p \otimes h) \cdot r_H(\alpha)^{-1} \zeta \otimes g
\]

where $p \otimes h \in D(H)_{\alpha,\beta}$. A similar computation holds for $H \in \text{SVect}$. 

2.6. Ribbon elements in the twisted double. We now characterize the ribbon structures on $D(H)|_G$, generalizing a theorem of Kauffman and Radford [24, Theorem 13.7.3].

**Lemma 2.2.** The $G$-invariant group-likes of $D(H)|_G$ have the form \( \{ g_\alpha = p(\alpha)^{-1} \beta \otimes b \}_{\alpha \in G} \) for unique $G$-invariant group-likes $\beta \in H^*$, $b \in H$ and a unique homomorphism $p : G \to \mathbb{K}^\times$.

**Proof.** If \( \{ g_\alpha \} \) is a $G$-invariant group-like of $D(H)|_G$, then $g_1$ is a group-like of $D(H)$ so $g_1 = \beta \otimes b$ for unique $\beta \in G(H^*)$, $b \in G(\widehat{H})$. Since $g_1$ is $G$-invariant, $\beta, b$ must be $G$-invariant. Now, we have
\[
g_\alpha \otimes g_{\alpha^{-1}} = \Delta_{\alpha, \alpha^{-1}}(g_1) = (\beta \otimes b) \otimes (\beta \otimes b)
\]
since $\alpha^{-1}(b) = b$. It follows that $g_\alpha = p_\alpha^{-1} \beta \otimes b$ for some scalar $p_\alpha$ (apply $id \otimes f$ where $f : D(H)_{\alpha^{-1}} \to \mathbb{K}$ satisfies $f(g_{\alpha^{-1}}) \neq 0$). It is clear that $p_\alpha$ is unique, and the group-like condition implies that $\alpha \mapsto p_\alpha$ defines an homomorphism $p : G \to \mathbb{K}^\times$. \hfill \Box

**Proposition 2.3.** The ribbon elements of the $G$-twisted Drinfeld double $D(H)|_G$ are in 1:1 correspondence with triples $(\beta, b, p)$ where

1. $\beta \in H^*$, $b \in H$ are $G$-invariant group-likes such that $S_H^2 = ad_{\beta^{-1}} \circ ad_b$ and $\beta^2 = \zeta \cdot b^2 = g$ (where $\zeta, g$ are the distinguished group-likes of $H$).

2. $p : G \to \mathbb{K}^\times$ is an homomorphism such that $p^2 = r_H$.

The ribbon element and pivot of $D(H)|_G$ corresponding to such triple are given by
\[
v_\alpha = r_H(\alpha)^{-\frac{1}{2}}(\beta \otimes b) \cdot u_\alpha^{-1}, \quad g_\alpha = r_H(\alpha)^{-\frac{1}{2}}(\beta \otimes b)
\]
for each $\alpha \in G$, where we denote $p(\alpha) = r_H(\alpha)^{\frac{1}{2}}$ and $u_\alpha$ is the Drinfeld element.

**Proof.** By [32, Theorem 6.9, b)] the distinguished group-like $g_\alpha$ of $D(H)$ satisfies $g_\alpha = u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}}^{-1})$ (this also uses that $D(H)$ is unimodular and that the character of [32, Lemma 6.2] is trivial). This holds in the super-case as well. Thus, by Lemma 2.1 and (5), ribbon elements of $D(H)|_G$ are in bijection with $G$-invariant group-likes \( \{ g_\alpha \} \) satisfying (3') and
\[
g_\alpha^2 = g_\alpha = r_H(\alpha)^{-1} \zeta \otimes g.
\]
If $(\beta, b, p)$ corresponds to \( \{ g_\alpha \} \) by Lemma 2.2 then (4') is equivalent to
\[
\beta^2 = \zeta, \quad b^2 = g, \quad p(\alpha)^2 = r_H(\alpha).
\]
It is easy to see that (3') is equivalent to $S_H^2 = ad_{\beta^{-1}} \circ ad_b$. This proves the first part of the proposition. The ribbon element associated to such \( \{ g_\alpha \} \) is then given by
\[
v_\alpha = g_\alpha u_\alpha^{-1} = p(\alpha)^{-1}(\beta \otimes b) u_\alpha^{-1}
\]
as desired. \hfill \Box

2.7. Representation theoretic interpretation. We now give a more familiar description of the representation categories of the above twisted Drinfeld doubles. Since this is not essential for this paper, we just briefly recall the objects involved. We refer to [27] for definitions of braided $G$-crossed categories and [11] for relative Drinfeld centers. In what follows, $G$ denotes a group (not necessarily finite) with neutral element 1 and Rep($A$) denotes the category of finite dimensional representations of a $\mathbb{K}$-algebra $A$.

Let $H$ be a finite dimensional Hopf algebra (in $\text{Vect}$, for simplicity) and $G = \text{Aut}(H)$. Since $D(H)$ is a quasi-triangular Hopf $G$-coalgebra, Rep $D(H) := \prod_{\alpha \in G} \text{Rep}(D(H)_{\alpha})$ is a
braided $G$-crossed category in a natural way [27]. This category can be defined entirely in terms of the monoidal category $\mathcal{D} = \text{Rep}(H)$ as follows. For each $X \in \mathcal{D}$ and morphism $f$ of $\mathcal{D}$, let $T_\alpha(X)$ be the vector space $X$ with $H$-action given by $h \cdot x := \alpha^{-1}(h)x$ and $T_\alpha(f) = f$. Then $\alpha \mapsto T_\alpha$ defines a monoidal action of $G$ on $\mathcal{D}$. Let $\mathcal{C} = \mathcal{D} \times G$, that is, $\mathcal{C}$ is the category of pairs $(X, \alpha)$ where $X \in \mathcal{D}, \alpha \in G$ and $\text{Hom}_\mathcal{C}((X, \alpha), (Y, \beta))$ equals $\text{Hom}_\mathcal{D}(X, Y)$ if $\alpha = \beta$ and zero otherwise. This is a $G$-graded monoidal category with tensor product on objects given by 

$$(X, \alpha) \otimes (Y, \beta) = (X \otimes T_\alpha(Y), \alpha \beta)$$

and on morphisms $(f, \alpha) \otimes (g, \beta) = (f \otimes T_\alpha(g), \alpha \beta)$. The neutral component is $\mathcal{D}$ as a monoidal category. We consider the relative Drinfeld center $\mathcal{Z}_\mathcal{D}(\mathcal{C})$ of this category. An object of $\mathcal{Z}_\mathcal{D}(\mathcal{C})$ is an object $(Y, \alpha)$ of $\mathcal{C}$ endowed with a half-braiding $c_{(X,1),(Y,\alpha)} : (X,1) \otimes (Y,\alpha) \to (Y,\alpha) \otimes (X,1)$, which is the same as a $\mathcal{D}$-morphism

$$c_{X,Y} : X \otimes Y \to Y \otimes T_\alpha(X)$$

satisfying $c_{X \otimes X',Y} = (c_{X,Y} \otimes \text{id}_{T_\alpha(X')}) (\text{id}_{X} \otimes c_{X',Y})$ for each $X, X' \in \mathcal{D}$. As shown in [11, Example 3.4], $\mathcal{Z}_\mathcal{D}(\mathcal{C})$ is a braided $G$-crossed category in the following way. First note that the group $G$ acts on $\mathcal{C}$ by $T_\beta(Y, \alpha) = (T_\beta(Y), \beta \alpha \beta^{-1})$. This action extends to $\mathcal{Z}_\mathcal{D}(\mathcal{C})$ if the half-braiding of $T_\beta(Y, \alpha)$ is defined, for $X \in \mathcal{D}$, by

$$(6) \quad c_{(X,1),T_\beta(Y,\alpha)} = T_\beta(c_{T_\beta^{-1}(X,1),(Y,\alpha)}).$$

This is the same as the $\mathcal{D}$-morphism $T_\beta(c_{T_\beta^{-1}(X,1),Y})$. The $G$-braiding is defined by the following isomorphisms

$$(X,\alpha) \otimes (Y,\beta) = \begin{array}{l}(X \otimes T_\alpha(Y,\alpha \beta) \xrightarrow{c_{X,Y}(\alpha)} T_\alpha Y \otimes T_{\alpha \beta^{-1}} X, \alpha \beta) \\= (T_\alpha Y, \alpha \beta^{-1} \otimes (X,\alpha) \\= T_\alpha (Y,\beta) \otimes (X,\alpha).\end{array}$$

**Proposition 2.4.** Let $H$ be a finite dimensional Hopf algebra and $G = \text{Aut}(H)$. Then, $\text{Rep} \ D(H)$ is equivalent, as a braided $G$-crossed category, to the relative Drinfeld center of the crossed product $\mathcal{C} = \text{Rep}(H) \rtimes G$.

**Proof.** The proof is very similar to that of the $G = 1$ case (see e.g. [14]) so we only sketch it. Denote $D_\alpha = D(H)_{\alpha}$ for each $\alpha$. The idea is that if $(X, \alpha)$ is an object of $\mathcal{Z}_\mathcal{D}(\mathcal{C})$ then, by naturality, the whole half-braiding is encoded in the half-braiding $c_{H,X} : H \otimes X \to X \otimes T_\alpha H$. Again by naturality, this map is determined by $c_{H,X}(1 \otimes -)$ which is a right $H$-comodule structure on $X$, or equivalently, a left $H^*$-module structure. Noting that the $H$-factor in the target of $c_{H,X}$ has a twisted $H$-action (twisted by $\alpha^{-1}$), the condition that $c_{H,X}$ is an $H$-module map translates into $X$ being a module over $D_\alpha$. This correspondence is clearly an equivalence and it is monoidal: since the tensor product $(X, \alpha) \otimes (Y, \beta) = (X \otimes T_\alpha Y, \alpha \beta)$ twists by $\alpha^{-1}$ the $H$-module structure on $Y$, the comultiplication $\Delta_{\alpha \beta}$ is twisted by $\alpha^{-1}$ in the second $H$-factor, which is exactly the formula we gave for $\Delta_{\alpha \beta}$. Since the $G$-action on $\mathcal{Z}_\mathcal{D}(\mathcal{C})$ is $T_\alpha(Y, \beta) = (T_\alpha Y, \alpha \beta^{-1})$ the $H$-module structure of $T_\alpha(Y, \beta)$ is twisted by $\alpha^{-1}$ so $\varphi_\alpha : D_\beta \to D_{\alpha \beta^{-1}}$ acts by $\alpha$ on $H \subset D_\alpha$. Now, the $H$-comodule structure of $T_\alpha(Y, \beta)$ is given by the half-braiding $c_{T_\alpha^{-1}H,Y}$ (see (6)). Since $\alpha : H \to T_\alpha^{-1} H$ is an $H$-module map, by naturality one finds

$$(7) \quad c_{T_\alpha^{-1}H,Y}(1 \otimes y) = (\text{id} \otimes \alpha)c_{H,Y}(1 \otimes y).$$
Thus, the \( H \)-comodule structure of \( T_\alpha Y \) is twisted by \( \alpha \), so that \( \varphi_\alpha \) acts by \( \varphi_\alpha(h^*) = h^* \circ \alpha \) on \( H^* \). Taken together, this shows the equivalence is a \( G \)-crossed equivalence. Finally, since \( \text{Rep}(D(H)) = \mathcal{Z}_D(C) \) as \( G \)-categories, the braiding of \( \mathcal{Z}_D(C) \) must be represented by an \( R \)-matrix \( R_{\alpha,\beta}' \) of \( D(H) \). By naturality and (7) above, this is given by

\[
P(R_{\alpha,\beta}') = c_{T_{\alpha-1}D_\alpha,D_\beta}(1_\alpha \otimes 1_\beta)
= (\text{id} \otimes \alpha)c_{H,D_\beta}(1_\alpha \otimes 1_\beta)
\]

where \( P \) is the switch map \( P(x \otimes y) = y \otimes x \) and on the bottom we think of \( H \) as a subalgebra of \( T_{\alpha-1}D_\alpha \). But, \( c_{H,D_\beta}(1_\alpha \otimes 1_\beta) = h^i \otimes h_i \) as in the untwisted case, where \( h_i \) is a basis of \( H \) and \( h^i \) is its dual basis. Therefore, \( R_{\alpha,\beta}' = P((\text{id} \otimes \alpha)(h^i \otimes h_i)) = R_{\alpha,\beta} \) where \( R_{\alpha,\beta} \) is as in (3).

\[\square\]

**Remark 2.5.** Suppose \( H \) is a Hopf algebra in \( \text{SVect} \). Then, it is not true that \( \text{Rep}_S(D(H)) \) is the Drinfeld center of \( \text{Rep}_S(H) \) (the \( S \) subscript indicates that modules are super-vector spaces). Indeed, \( \text{Rep}_S(H) \) is equivalent to \( \text{Rep}(H_{\text{bos}}) \) where \( H_{\text{bos}} \) is the bosonization of \( H \). As an algebra, \( H_{\text{bos}} = \mathbb{K}[\mathbb{Z}/2\mathbb{Z}] \ltimes H \) and so \( D(H_{\text{bos}}) \) has two additional group-likes. This double is itself the bosonization of a Hopf algebra \( D \) in \( \text{SVect} \) such that \( \text{Rep}_S(D) = \mathcal{Z}(\text{Rep}_S(H)) \), and this is an extension of \( D(H) \) by a single group-like. The category \( \text{Rep}_S(D(H)) \) turns out to be equivalent to the full (braided) subcategory of \( \mathcal{Z}(\text{Rep}_S(H)) \) consisting of objects \( V \in \text{Rep}_S(H) \) with a half-braiding \( \{ \sigma_{X,V} : X \otimes V \to V \otimes X \} \in \text{Rep}_S(H) \) for which \( \sigma_{I,V} \) coincides with the symmetry \( c_{I,V} : I \otimes V \to V \otimes I \) of \( \text{SVect} \). Here \( I \) denotes the unique non-trivial invertible object of \( \text{SVect} \), this is an \( H \)-module via \( \epsilon \).

### 3. Universal Reshetikhin-Turaev invariants of \( G \)-tangles

In this section we define the universal twisted invariant of a framed, oriented, \( G \)-tangle with no closed components out of a ribbon Hopf \( G \)-coalgebra. In Subsection 3.3 we explain how to get invariants of framed and unframed \( G \)-knots as closures of long \( G \)-knots. In Subsection 3.4 we specialize to twisted Drinfeld doubles and in Subsection 3.5 we explain how to lift our invariants to polynomials invariants whenever the Hopf algebras are \( \mathbb{Z} \)-graded.

#### 3.1. \( G \)-tangles and (long) \( G \)-knots.

Let \( X = \mathbb{R} \times [-1, \infty) \times [0,1] \) and let \( \partial_-X = \mathbb{R} \times [-1, \infty) \times \{0\} \), \( \partial_+X = \mathbb{R} \times [-1, \infty) \times \{1\} \), \( \partial_0X = \mathbb{R} \times \{-1\} \times [0,1] \). We think of the \( x \)-coordinate as an horizontal line in the plane of the page oriented from left to right, the \( y \)-coordinate as a line transversal to the page and oriented towards it (so \( 0, \infty \) lies behind the page), and the \( z \)-coordinate as a vertical line in the plane of the page oriented upwards. By a \( (p,q) \)-tangle we mean a framed, oriented tangle \( T \subset X \) such that \( T \cap \partial_-X = \{(i,0,0)\}_{i=1}^p, \ T \cap \partial_+X = \{(i,0,1)\}_{i=1}^q, \ T \cap \partial_0X = \{X_{\delta} \cap \partial_0X = X_{\delta} \cap \partial_0X \} = \partial T \) and the intersection is transversal. Moreover, the framing of \( T \) near a point \( (i,0,0) \) (resp. \( (i,0,1) \)) is \( (i - \delta, 0, 0) \) (resp. \( (i - \delta, 0, 1) \)) for some small \( \delta > 0 \). Let \( X_T = X \setminus T \) and \( z \in \partial_0X \) a basepoint. A \( (p,q) \)-\( G \)-tangle is a \( (p,q) \)-tangle endowed with a group homomorphism \( \rho : \pi_1(X_T, z) \to G \). Note that since \( \partial_0X \) is contractible, the choice of basepoint \( z \) is irrelevant. Two \( G \)-tangles \( (T, \rho) \) and \( (T', \rho') \) are isotopic if there is an isotopy \( d_t : X \to X \) rel. \( \partial X \) from \( T \) to \( T' \) such that \( \rho' \circ (d_{1-t})_* = \rho \). Isotopy classes of \( G \)-tangles form the morphisms of a \( G \)-crossed ribbon category, see [27] for details.
A \((p,q)\)-\(G\)-tangle \((T, \rho)\) with a single component will be called a \(G\)-knot if \(p = q = 0\) and a long \(G\)-knot if \(p = q = 1\).

3.2. Invariants of \(G\)-tangles. Let \((A = \{A_\alpha\}_{\alpha \in G}, \varphi, R, v)\) be a ribbon Hopf \(G\)-coalgebra with pivot \(\{g_\alpha\}_{\alpha \in G}\). Let \((T = T_1 \cup \cdots \cup T_m, \rho)\) be an ordered, oriented, framed \(G\)-tangle with no closed components.

Let \(D\) be a planar diagram of \(T\). We draw \(D\) in the plane and assume it comes with the blackboard framing. We also suppose \(D\) is oriented upwards at each crossing, so it is built of the following local pieces:

By an edge of \(D\) we mean a subarc of the diagram having no overpasses in its interior and ending at underpasses. For each edge \(e\) of \(D\) we let \(\gamma_e \in \pi_1(X_T, z)\) be the homotopy class of the loop that goes from the basepoint \(z \in \partial v X\) to a point close to \(e\) along a linear path, then encircles \(e\) once with linking number \(-1\) and finally comes back to \(z\) with a linear path in the opposite direction. Thus, the edges of \(D\) become labeled by elements of \(G\), where the label of \(e\) is \(\rho(\gamma_e)\):

![Diagram of a crossing with three edges](image)

**Figure 1.** A diagram of a crossing with three edges \(e, e', e''\). The paths \(\gamma_e, \gamma_{e'}, \gamma_{e''}\) satisfy \(\gamma_{e'} = \gamma_e \gamma_{e''} \gamma_e^{-1}\). On the right we draw the corresponding \(G\)-colored diagram, where \(\alpha = \rho(\gamma_e), \beta = \rho(\gamma_{e''})\).

To define an invariant of \((T, \rho)\) out of \((A, \varphi, R, v)\) we associate black and white beads to each of the local pieces of the \(G\)-colored diagram as follows:

For positive (resp. negative) crossings, the black beads represent \(R_{\alpha, \beta} = \sum s_\alpha \otimes t_\beta\) (resp. \(R_{\alpha, \beta}^{-1} = \sum s_\alpha^{-1} \otimes t_\beta\)) where \(\alpha, \beta\) are the \(G\)-labels of the corresponding edges and the white bead represents \(\varphi_\alpha\) (resp. \(\varphi_\alpha^{-1}\)). Now, for each cap or cup of the diagram labeled by \(\alpha \in G\), we put a black bead representing either \(1_\alpha\) (left caps/cups) or the pivot \(g_\alpha^{\pm 1}\) (right caps/cups). Note that this is where we use the ribbon element \(v_\alpha\) of \(A_\alpha\) as \(g_\alpha = u_\alpha v_\alpha\).

With these conventions, all the black beads lying over an edge \(e\) of \(D\) labeled by \(\beta\) represent an element of \(A_\beta\). Now follow the orientation of \(e\) from its starting point and multiply the
black beads encountered from right to left. This results in an element \( x_e \in A_\beta \). If an edge \( e' \) follows \( e \) in the orientation of \( D \), then there is a white bead \( \varphi_{\alpha \pm 1} \) in between \( e \) and \( e' \), where \( \alpha \) is the label of the overpass. We “multiply” \( x_{e'} \) with \( x_e \) by taking \( x_{e'} \varphi_{\alpha \pm 1}(x_e) \) and we think of this as the new bead of \( e' \). One can think that \( x_e \) is being slid through the crossing and that when it passes through the white bead labeled with \( \varphi_{\alpha \pm 1} \) then it must be evaluated on it. If we do this for all edges of all components of \( T \), this results in an element \( z_D \in A_{\gamma_1} \otimes \ldots \otimes A_{\gamma_m} \) where \( \gamma_i \) is the label of the endpoint of \( T_i \) for each \( i = 1, \ldots, m \). This procedure also makes sense for Hopf \( G \)-coalgebras in \( \text{SVect} \). Here one has to suppose the set of crossings is totally ordered (the order turns out to be irrelevant since the \( R \)-matrices have degree zero). Then, when following the orientation of \( T \), the black beads are reordered and this introduces various signs into our formula.

**Example 3.1.** Consider the diagram of an open right trefoil \( T \) as in Figure 2 and let \( \rho : \pi_1(X_T) \to G \) be an homomorphism. Let \( e_1, e_2, e_3 \) be the edges of this diagram, starting from the second edge from top to bottom. We denote by \( b_1, b_2, b_3 \in G \) the associated labels via \( \rho \), that is \( b_i = \rho(\gamma_{e_i}) \). From top to bottom, the pairs of black beads on the crossings represent the \( R \)-matrices \( R_{b_2,b_1}, R_{b_1,b_3}, R_{b_3,b_2} \) respectively. If we denote \( R_{\alpha,\beta} = \sum s_\alpha \otimes t_\beta \) for each \( \alpha, \beta \in G \) then we get

\[
z_D = \sum \varphi_{b_2}(t_{b_1}s_{b_1}\varphi_{b_1}(t_{b_2}s_{b_2}\varphi_{b_1}(t_{b_3}s_{b_3})))
\]

If \( A \) is a Hopf \( G \)-coalgebra in \( \text{SVect} \), each term in the above sum has to be multiplied by the sign \((-1)^{|s_{b_3}||t_{b_2}|+|s_{b_2}||t_{b_2}|+|s_{b_1}||s_{b_2}|+|s_{b_1}||t_{b_2}|} \).
We will denote this invariant by

\[ z_D = Z_\Delta^\rho(T). \]

We could also have started from the endpoint of a component of \( T \), follow its opposite orientation and multiply the black beads from left to right, applying \( \varphi^{-1}_\alpha \) each time we encounter a \( \varphi_\alpha \). This would result in an element

\[ z'_D \in A_{\delta_1} \otimes \ldots \otimes A_{\delta_m} \]

where \( \delta_i \) is the label of the starting point of \( T_i \) for each \( i = 1, \ldots, m \). These two elements are related by \( \otimes_{i=1}^m \varphi_{[T_i]}(z'_D) = z_D \), where \([T_i] \in \pi_1(X_T)\) is the homotopy class of a path that goes from the basepoint linearly to the endpoint of \( T_i \), then follows the framing of \( T_i \) along the opposite orientation and finally goes back from the starting point of \( T_i \) linearly to the basepoint.

3.3. Invariants of G-knots. It is well-known that the isotopy class of the closure of a long knot determines that of the long knot itself. Therefore, any invariant of long knots defines a knot invariant. We now generalize this to the case of G-knots.

Let \((T, \rho)\) be a long G-knot and \( K \) the closure of \( T \). We suppose \( K \subset X = \mathbb{R} \times [-1, \infty) \times [0, 1] \). It is clear that the G-label of the starting point of \( T \) coincides with that of the endpoint (indeed, the paths \( \gamma_{e_0}, \gamma_{e_t} \) are homotopic, where \( e_0 \) (resp. \( e_t \)) is the first (resp. final) edge of a diagram of \( T \) along its orientation). Thus, \( \rho \) extends to an homomorphism \( \pi_1(X_K, z) \to G \) that we still denote by \( \rho \). We say that \((K, \rho)\) is the G-closure of \((T, \rho)\).

In what follows, \((K, \rho)\) is a (framed, oriented) G-knot and \((T, \rho)\) is a long G-knot whose G-closure is isotopic to \((K, \rho)\). We denote by \( \gamma \) the G-label of the endpoint of \( T \).

**Lemma 3.3.** Let \((T', \rho')\) be another long G-knot whose closure is isotopic to \((K, \rho)\). Then, there is an element \( \beta \in G \) in the image of \( \rho' \) such that \((T, \rho)\) is isotopic to \((T', \rho'\beta)\) as a long G-knot, where \( \rho'(\beta)(\delta) = \beta \rho'(\delta) \beta^{-1} \) for all \( \delta \in \pi_1(X_{T'}). \)

**Proof.** Let \((K', \rho')\) be the closure \((T', \rho')\). For simplicity, let’s suppose that the origin lies in the closure-strand of both \( K \) and \( K' \). Suppose there is an isotopy \( d_t : X \to X \) such that \( d_t(K) = K' \) and \( \rho' \circ (d_1)_* = \rho \) over \( \pi_1(X_K, z) \) where \( z \in \partial_0 X \) is the basepoint. After an isotopy of the identity of \( S^3 \), we can arrange for \( d_1(0) = 0 \) (note that \((d_1)_* \) is unchanged). Then we can take \( d_t - d_t(0) \) and modify it near \( \partial X \) so that it becomes an isotopy \( f_t \) of G-knots such that \( f_t(0) = 0 \) for all \( t \). Thus, it is an isotopy from \( T_0 = K \setminus K \cap B \) to \( T'_0 = K' \setminus K' \cap B \) in \( S^3 \setminus B \) where \( B \) is a small open ball neighborhood of 0. However, if we identify \( S^3 \setminus B \cong \mathbb{D}^2 \times [0, 1] \), the basepoint \( z \) becomes a point in the interior of \( \mathbb{D}^2 \times [0, 1] \). If \( z_0 \) is point in the boundary of \( \mathbb{D}^2 \times [0, 1] \), then we can identify \( \pi_1(X_T, z_0) \cong \pi_1(X_T, z_0) \) only up to conjugation. Thus \((T_0, \rho)\) is equivalent to a conjugate of \((T, \rho)\), and similarly for \((T'_0, \rho'\beta)\), which proves the lemma.

Therefore any invariant of \((T, \rho)\) that depends on \( \rho \) only up to conjugation is an invariant of the closure \((K, \rho)\). The following are particular instances of this.

**Corollary 3.4.** If the image of \( \rho \) is abelian, then \( Z_\Delta^\rho(T) \in A_\gamma \) is an invariant of \((K, \rho)\).

**Proof.** If \( \rho \) is abelian, the above lemma implies that \((T, \rho)\) is G-isotopic to \((T', \rho')\) so any invariant of \((T, \rho)\) is an invariant of \((K, \rho)\).
Corollary 3.5. Let $A = \{A_\alpha\}_{\alpha \in G}$ be a ribbon Hopf $G$-coalgebra and let $\{f_\alpha : A_\alpha \to \mathbb{K}\}_{\alpha \in G}$ be a family of linear functionals with the property that $f_{\beta \alpha \beta} \circ \varphi_\beta = f_\alpha$ for all $\alpha, \beta \in G$. Then the scalar

$$f_\gamma(Z^\rho_A(T)) \in \mathbb{K}$$

is an invariant of $(K, \rho)$.

Proof. By the above lemma, any other long $G$-knot whose closure is $(K, \rho)$ is isotopic to $(T, \rho_\beta)$ for some $\beta \in G$. The invariant of $(T, \rho_\beta)$ is $Z^\rho_A(T) = \varphi_\beta(Z^\rho_A(T)) \in A_{\beta \gamma \beta}$. It follows that

$$f_\gamma(Z^\rho_A(T)) = f_{\beta \gamma \beta}^{-1}(Z^\rho_A(T))$$

so this is an invariant of $(K, \rho)$. \qed

Now, suppose we want to get invariants of unframed $G$-knots. Let $(T, \rho)$ be a framed long $G$-knot whose closure, after forgetting the framing, is an unframed $G$-knot $(K, \rho)$. Let $Z^\rho_A(T) \in A_\gamma$ be the invariant of $(T, \rho)$, and let

$$Z'_A(T, \rho) = Z^\rho_A(T) \cdot v_\gamma^{-u(T)} = v_\gamma^{-u(T)} \cdot \varphi_\gamma^{-u(T)}(Z^\rho_A(T))$$

where the product is in $A_\gamma$ and $u(T)$ is the writhe of $T$. Then $Z'_A(T, \rho)$ is an invariant of the underlying unframed long $G$-knot. Since the ribbon element of $A$ is $G$-invariant, it follows that $\varphi_\beta(Z'_A(T, \rho)) = Z'_A(T, \rho_\beta)$. Thus, if $\{f_\alpha\}_{\alpha \in G}$ is as in the above corollary, then

$$f_\gamma(Z'_A(T, \rho))$$

is an invariant of the unframed $G$-knot $(K, \rho)$.

3.4. Invariants of $G$-knots from twisted Drinfeld doubles. Now let $H$ be a finite dimensional Hopf algebra and $G \subset \text{Aut}(H)$ be a subgroup such that $\overline{D(H)} | G$ is $G$-ribbon. Let $\{v_\alpha\}_{\alpha \in G}$ be a ribbon structure determined by a triple $(\beta, b, p)$ as in Proposition 2.3. From now on, we denote $p(\alpha) = r_H(\alpha)^{\frac{1}{2}}$. Note that since $D(H)_\alpha = H^* \otimes H$ as vector spaces, the functional $\epsilon_{D(H)}(f \otimes h) = f(1)\epsilon(h)$ is defined over $D(H)_\alpha$ for every $\alpha$ (though it is not an algebra morphism if $\alpha \neq \text{id}_H$).

Corollary 3.6. Let $(K, \rho)$ be an (unframed) oriented $G$-knot. Then

$$P^\rho_H(K) := r_H(\rho(m))^{\frac{1}{2}}u(T)\epsilon_{D(H)}(Z^\rho_{D(H)}(T)) \in \mathbb{K}$$

is an invariant of $(K, \rho)$. Here $(T, \rho)$ is a framed, oriented long $G$-knot whose closure is $(K, \rho)$, $u(T)$ is the writhe of $T$ and $m$ is an oriented meridian of $K$.

Proof. Setting $f_\alpha = \epsilon_{D(H)}$ for every $\alpha$, one easily sees that $\{f_\alpha\}$ satisfies the hypothesis of Corollary 3.5. By (9),

$$\epsilon_{D(H)}(v_\gamma^{-u(T)} \cdot \varphi_\gamma^{-u(T)}(Z^\rho_{D(H)}(T)))$$

is an invariant of $(K, \rho)$, where $\gamma$ is the label of the endpoint of $T$. We now show that this coincides with $P^\rho_H(K)$. Since $v_\gamma = r_H(\gamma)^{-\frac{1}{2}}\beta \otimes b \cdot u_\gamma^{-1}$, we only need to show that $\epsilon_{D(H)}(\beta \otimes b \cdot x) = \epsilon_{D(H)}(x)$ and $\epsilon_{D(H)}(u_\gamma^{-1}x) = \epsilon_{D(H)}(x)$ for all $x \in D(H)_\gamma$. Recall that $u_\gamma^{-1} = h^1 \otimes S^2(\gamma(h_i))$. If $x = f \otimes h$, we can compute $\epsilon_{D(H)}(u_\gamma^{-1}x)$ as follows:
Using that $\gamma(S^2(1)) = 1$ it is easily seen the right hand side is $\epsilon_{D(H)}(x)$. The proof that $\epsilon_{D(H)}(\beta \otimes b \cdot x) = \epsilon_{D(H)}(x)$ is similar (this requires that $b$ is $G$-invariant).

\[ \square \]

3.5. **Lifting to polynomials.** Suppose in addition to the above that $H$ is $\mathbb{Z}$-graded. We now show that the invariant $P^H$ of Corollary 3.6 lifts to a polynomial invariant of $K$. This follows the ideas of [21] (which itself mimics an idea from Reidemeister torsion theory), but now $H$ is a possibly non-involutory Hopf algebra.

Let $H' = H \otimes_{\mathbb{K}} \mathbb{K}[t^{\pm 1}]$ where $t$ is a variable, then $H'$ is a $\mathbb{Z}$-graded Hopf algebra over $\mathbb{K}[t^{\pm 1}]$ and $H \subset H'$ as a $\mathbb{K}$-linear Hopf subalgebra. We will identify $D(H')$ with $D(H)[t^{\pm 1}]$.

For any $\alpha \in \text{Aut}(H)$ and $n \in \mathbb{Z}$ we can define a $\mathbb{K}[t^{\pm 1}]$-linear Hopf automorphism $\alpha \otimes n \in \text{Aut}(H')$ by

\[(\alpha \otimes n)(x) := t^n \alpha(x)\]

for any homogeneous element $x \in H$. This defines an homomorphism

\[\theta : \text{Aut}(H) \times \mathbb{Z} \to \text{Aut}(H'), \ (\alpha, n) \mapsto \alpha \otimes n.\]

If $G \subset \text{Aut}(H)$ is a subgroup, we let $G'$ be the image of $G \times \mathbb{Z}$ under this homomorphism. Note that, if $\Lambda_l$ is a (nonzero) left cointegral of $H$ (and also of $H'$), then

\[\alpha \otimes n(\Lambda_l) = t^n |\Lambda_l| r_H(\alpha) \Lambda_l\]

so that $r_{H'}(\alpha \otimes n) = t^n |\Lambda_l| r_H(\alpha)$. Thus, if $r_H$ has a square root over $G$, then $r_{H'}$ has a square root over $G'$, namely,

\[r_{H'}(\alpha \otimes n)^{1/2} = r_H(\alpha)^{1/2} \cdot t^{n |\Lambda_l|/2}\]

so $D(H')$ has an induced ribbon $G'$-coalgebra structure by Proposition 2.3.

Now let $(K, \rho)$ be an oriented, unframed $G$-knot and let $(T, \rho)$ be a framed long $G$-knot with closure $(K, \rho)$ as unframed $G$-knots. Let $h : \pi_1(X_T) \to \mathbb{Z}$ be the map sending an oriented meridian of $T$ to 1. We will denote $\rho \otimes h = \theta \circ (\rho \times h) : \pi_1(X_T) \to G'$, that is, $\rho \otimes h(\delta) = \rho(\delta) \otimes h(\delta)$ for $\delta \in \pi_1(X_T)$. Since $D(H')$ is $G'$-ribbon, we get an invariant

\[Z_{D(H') \rho \otimes h}^{\rho \otimes h}(T) \in D(H') = D(H)[t^{\pm 1}]\]

which we will denote simply as $Z_{D(H') \rho \otimes h}^{\rho \otimes h}(T)$. By Corollary 3.6

\[P^H_K (K, t) := r_H(\rho(m))^{1/2} \cdot \frac{t^{|\Lambda_l| w(T)/2}}{2} \epsilon_{D(H')}(Z_{D(H') \rho \otimes h}^{\rho \otimes h}(T)) \in \mathbb{K}[t^{\pm 1}]\]
is a polynomial invariant of \((K, \rho)\), where \(m\) is an oriented meridian of \(K\).

In principle, the above belongs to \(K[t^{\pm \frac{1}{2}}]\) but one can show that \(P_H^\rho(K, t) \in K[t^{\pm 1}]\) as follows: first note that the fractional powers of \(t\) come from the beads of \(Z_{D(H)}^{\rho \otimes h}(T)\) at the caps and cups (because of the formula for the pivot of Proposition 2.3) and the normalization factor above. The caps/cups contribute a \(t^{-|\lambda(H)|} \frac{r}{2}\) where \(r\) is the rotation number of the diagram, so the total fractional power is \(t^{-|\lambda(H)|} \frac{r}{2} w(T) - r\). But it is easy to see that for a long knot \(w(T) - r\) is even: the parity of \(w(T) - r\) is an invariant of framed tangles and it is unchanged under crossing changes, thus it coincides with the parity of the trivial tangle which is zero.

Note that for trivial \(\rho\), \(Z_{D(H)}^{h(H)}(T)\) is an invariant of the knot \(K\) (Corollary 3.4). Thus, the whole universal invariant of \(K\) can be “deformed” to a polynomial invariant, provided \(H\) is \(\mathbb{Z}\)-graded.

4. Twisted Kuperberg invariants via twisted Drinfeld doubles

In this section we state and prove our main theorem (Theorem 4.2). We begin by recalling the construction of a (sutured) Heegaard diagram from a bridge presentation of a knot. In Subsection 4.3 we briefly recall the (dual of the) construction of [21]. Our main theorem is shown in Subsection 4.4, along with its corollary about twisted Reidemeister torsion.

In all that follows, we let \(H\) be a finite dimensional involutory Hopf algebra over a field \(K\) (in Vect or SVect) with a two-sided integral \(\lambda\) and a two-sided cointegral \(\Lambda\) such that \(\lambda(\Lambda) = 1\). We also let \(K \subset S^3\) be a knot and \(T\) a long knot whose closure is \(K\).

4.1. From planar diagrams to Heegaard diagrams. Let \(D\) be an oriented planar diagram of the (framed, oriented) long knot \(T\). In all that follows we suppose \(D\) is oriented from \(\mathbb{R}^2 \times \{0\}\) to \(\mathbb{R}^2 \times \{1\}\) and that at each crossing both strands are oriented upwards as in Subsection 3.2. For simplicity, we will suppose the last crossing of \(D\) (when following its orientation) is an underpass. Suppose \(D\) has \(g\) crossings. Then \(D\) determines a bridge presentation of the knot, where the underarcs correspond to the underpasses of the crossings and the overarcs correspond to the edges of the diagram. We will suppose the underarcs are in the plane \(\mathbb{R}^2 \times \{0\}\) and the overarcs lie above it. We will denote by \(a_i\) (resp. \(b_i\)) the underarcs (resp. overarcs), numbered so that when following the opposite orientation of \(D\) we encounter \(a_0, b_1, a_1, b_2, \ldots, a_{g-1}, b_g\) respectively, see Figure 3 (left). We denote by \(b_{\pi(i)}\) the overarc above \(a_i\).

A bridge presentation determines a Heegaard diagram of the knot complement (as a sutured manifold with two sutures in its boundary). The diagram consists of the following: consider the sphere \(\mathbb{R}^2 \times \{0\} \cup \{\infty\}\) over which the underarcs lie. For each overarc \(b_i\) with \(1 \leq i \leq g - 1\) attach a 2-dimensional one-handle to \(\mathbb{R}^2 \times \{0\}\) along the feet of \(b_i\). To do all this inside \(\mathbb{R}^3\), we may think of this handle as “following \(b_i\)” above the plane \(\mathbb{R}^2 \times \{0\}\). Now, delete two disks to the surface obtained, one at the endpoint of \(b_g\) and one at the endpoint of \(a_0\). Let \(\Sigma\) be the resulting oriented surface with boundary. For each \(1 \leq i \leq g - 1\) let \(\alpha_i \subset \Sigma\) be the boundary curve of a disk neighborhood of \(a_i\) that contains the feet of the 1-handles at the endpoints of \(a_i\). We will assume \(\alpha_i\) is oriented as the negative boundary of this disk. For each \(1 \leq i \leq g - 1\) let \(\beta_i \subset \Sigma\) be a circle which is the union of two arcs: the arc \(b_i\) and an arc \(b'_i\) which is parallel to \(b_i\) but runs through the corresponding 1-handle. We
will assume that \( \beta_i \) is oriented by extending the orientation of the arc \( b_i \). Then \((\Sigma, \alpha, \beta)\) is a sutured Heegaard diagram of \( S^3 \setminus K \), that is, the manifold obtained from \( \Sigma \times [0,1] \) by attaching 2-handles along \( \alpha_i \times \{0\} \) and \( \beta_j \times \{1\} \) for each \( i, j \) is homeomorphic to \( S^3 \setminus K \), and the boundary \( \partial \Sigma \times \{\frac{1}{2}\} \) correspond to two meridians in the boundary of \( S^3 \setminus K \). We will also consider the arc \( \beta_g \) in \( \Sigma \setminus \beta \) that joins the two punctures of \( \Sigma \) that is obtained by extending the (oriented) arc \( \beta_g \) to the top puncture of the surface and we denote \( \beta_e = \beta \cup \{\beta_g\} \). We will also suppose that for each \( i = 1, \ldots, g-1 \), \( \alpha_i \) and \( \beta_i \) have a basepoint lying right before (with respect to the orientation of \( \alpha_i, \beta_i \)) the “obvious” intersection point between \( \alpha_i \) and \( \beta_i \) (which comes from the intersection \( a_i \cap b_i \) at their endpoints). See Figure 3 (right). Thus, \((\Sigma, \alpha, \beta_e)\) is an oriented, based, extended Heegaard diagram as in [21].

4.2. Fox calculus. A Heegaard diagram \((\Sigma, \{\alpha_i\}_{i=1}^{g-1}, \{\beta_i\}_{i=1}^{g-1})\) of \( S^3 \setminus K \) together with an arc \( \beta_g \subset \Sigma \setminus (\cup_{i=1}^{g-1} \beta_i) \) joining the two punctures of \( \Sigma \) induces a presentation of \( \pi_1(S^3 \setminus K) \) that has generators \( \beta^*_i \) for each \( 1 \leq i \leq g \) and relators \( \overline{\alpha}_i \) for each \( i = 1, \ldots, g-1 \). This is because \( S^3 \setminus K \) is homeomorphic to a handlebody of genus \( g \) to which we attached \( g-1 \) 2-cells along the \( \alpha \) curves. The \( \beta^*_i \)'s are the duals of the \( \beta_1, \ldots, \beta_g \) in \( \Sigma \). Thus, the upper handlebody (corresponding to \((\Sigma, \beta)\)) is a thickening of the wedge of circles \( \vee_{i=1}^{g} \beta^*_i \). The relator \( \overline{\alpha}_i \) is obtained by following \( \alpha_i \) starting from a basepoint and multiplying, from right to left, the \((\beta^*_i)^\epsilon \)'s corresponding to the intersection points of \( \alpha_i \). Here \( \epsilon = +1 \) if \( \beta_i \cdot \alpha = +1 \) in \( \Sigma \) at the given intersection point and \( \epsilon = -1 \) otherwise.

Let \( F \) be the free group on \( \beta_1^*, \ldots, \beta_g^* \). Since each appearance of a generator \( \beta_j^* \) in the word \( \overline{\alpha}_i \) corresponds to an intersection point between \( \alpha_i \) and \( \beta_j \), the Fox derivative \( \partial \overline{\alpha}_i / \partial \beta^*_j \)
is a sum of words in $F$ indexed by the intersection points $\alpha_i \cap \beta_j$. Thus, we can write

$$\frac{\partial \overline{\alpha}_i}{\partial \overline{\beta}_j} = \sum_{x \in \alpha_i \cap \beta_j} m_x w_x$$

for some $w_x \in F$ and $m_x = 1$ if the intersection at $x$ is positive (that is $\beta_j \cdot \alpha_i = +1$ at this point) and $m_x = -1$ if negative.

Now consider the Heegaard diagram coming from a planar projection as above. Then the $\beta_i$’s can be identified with the loops $\gamma_i$’s encircling the arcs $b_i$ once with linking number $-1$. To avoid cumbersome notation, we will simply denote $b_i = \beta_i^*$. With the above choice of orientations and basepoints on the curves, we have

$$\overline{\alpha}_i = b_{\pi(i)} b_{i+1} b_{\pi(i)}^{-1} b_i^{-1} \text{ or } \overline{\alpha}_i = b_{\pi(i)}^{-1} b_{i+1} b_{\pi(i)} b_i^{-1}$$

depending on whether the crossing at $a_i$ is positive or negative. Thus, the presentation $\pi_1(S^3 \setminus K) = \langle b_1, \ldots, b_g \mid \overline{\alpha}_1, \ldots, \overline{\alpha}_g \rangle$ is simply the Wirtinger presentation. From now on, we think of $\partial \nu / \partial b_i$ as an element of $\mathbb{Z} \langle \pi_1 \rangle$ for each $w \in F$ and $b_i$. If the crossing at $a_i$ is positive, the nonzero Fox derivatives of $\overline{\alpha}_i$ are:

$$\frac{\partial \overline{\alpha}_i}{\partial b_{\pi(i)}} = 1 - b_{\pi(i)} b_{i+1} b_{\pi(i)}^{-1}, \quad \frac{\partial \overline{\alpha}_i}{\partial b_{i+1}} = b_{\pi(i)}, \quad \frac{\partial \overline{\alpha}_i}{\partial b_i} = -1.$$

Thus, if $x_1, x_2, x_3, x_4$ are the points of $\alpha_i \cap \beta^e$ as one follows the orientation of $\alpha_i$ starting from its basepoint, then (12) shows that

$$w_1 = 1, w_2 = b_{\pi(i)} b_{i+1} b_{\pi(i)}^{-1}, w_3 = b_{\pi(i)}, w_4 = 1.$$ 

If the crossing at $a_i$ is negative, then

$$w_1 = 1, w_2 = b_{\pi(i)}^{-1} b_{i+1}, w_3 = b_{\pi(i)}^{-1}, w_4 = b_{\pi(i)}.$$ 

### 4.3. Tensors from Heegaard diagrams

Let $H$ be a Hopf algebra as above and let $\rho : \pi_1(S^3 \setminus K) \rightarrow \text{Aut}(H)$ be an homomorphism. Given a sutured Heegaard diagram $(\Sigma, \alpha = \{\alpha_i\}_{i=1}^g, \beta = \{\beta_i\}_{i=1}^g)$ of $S^3 \setminus K$ with an arc $\beta_g$ and with orientations and basepoints as above, the construction of [21] assigns the following tensors of $H$ (this is actually the dual construction, see Remark 4.1 below). First, to each curve $\alpha_i$ we associate the tensor

$$\rho(w_k^{-1})$$

Here, from bottom to top, the outputs correspond to the intersection points, say $x_1, \ldots, x_k$, of $\alpha_i$ with $\beta$ as one follows its orientation starting from its basepoint and we denote $w_i = w_{x_i}$ as in (11). Note that for the Heegaard diagram coming from a diagram projection, each $\overline{\alpha}_i$ intersects $\beta^e = \beta \cup \{\beta_g\}$ in exactly four points, but intersects $\beta$ in either two, three or four points. Thus, the above tensor associated to $\overline{\alpha}_i$ may have two, three or four outputs depending on whether $|\alpha_i \cap \beta| = 2, 3$ or 4. Now, to each closed curve $\beta$ we associate the tensor
As before, from bottom to top, the inputs of this tensor correspond to the intersection points on \( \beta \) as one follows its orientation. Finally, to each intersection point \( x \) we associate the tensor \( S^{\epsilon_x} \) where \( \epsilon_x \in \{0, 1\} \) is defined by \( m_x = (-1)^{\epsilon_x} \) and \( m_x \) is the sign in (11). In this way, each intersection point of the Heegaard diagram corresponds to a unique output of an \( \alpha \)-tensor and a unique input of a \( \beta \)-tensor. The contraction of all these tensors is a scalar

\[
Z_H^p(\Sigma, \alpha, \beta^c) \in \mathbb{K}.
\]

Under our assumptions on \( H \), it is shown in [21] that the scalar \( Z_H^p(\Sigma, \alpha, \beta^c) \) is an invariant of the underlying sutured 3-manifold up to multiplication by an indeterminacy of the form \((\pm 1)^{|\Lambda|}r_H(\rho(\delta))\), where \( \delta \in \pi_1(M) \). We will denote by \( I_H^p(M, \gamma) \) the invariant up to this indeterminacy. Note that if \( \rho : \pi_1(M) \to \text{Ker}(r_H) \) and \(|\Lambda| = 0 \) there is no indeterminacy at all. These indeterminacies can be removed using \text{Spin}^c structures and homology orientations, but we won’t need this.

**Remark 4.1.** The construction of [21] is a special case of [20] where we consider the semidirect product \( \mathbb{K}[\text{Aut}(H)] \rtimes H \) as a graded algebra, where the grading group is \( \text{Aut}(H) \). As explained in [21], if one sets \( H_\alpha = \{\alpha \cdot h \mid h \in H\} \subset \mathbb{K}[\text{Aut}(H)] \rtimes H \), then \( \{H_\alpha\}_{\alpha \in \text{Aut}(H)} \) is a Hopf group-algebra. The dual of this object is a Hopf group-coalgebra whose representation category is equivalent to \( \text{Rep}(H^*) \rtimes \text{Aut}(H^*) \). The above conventions for \( I_H^p \) differ from those of [21] in that we use this dual object with \( H \) in place of \( H^* \). More precisely, the construction of [21] used the dual presentation having \( \alpha_1^* \)'s as generators and \( \beta_1 \)'s as relations. The present version is obtained by taking the diagram \( (-\Sigma, \beta, \alpha) \) and using \( H^* \) tensors. In other words, the present \( I_H^p \) is the \( I_{H^*}^{-1} \) of [21], where for each \( \alpha \in \text{Aut}(H), \alpha^t \in \text{Aut}(H^*) \) is the dual of \( \alpha \) and \( \rho^{-1} : \pi_1(M) \to \text{Aut}(H^*) \) is defined by \( \rho^{-1}(\delta) = (\rho(\delta)^t)^{-1} \).

4.4. **Main theorem.** Let \( H \) be as in the beginning of this section. By the two-sided cointegral/integral condition, the double \( D(H) \) is ribbon with \( b = 1, \beta = \epsilon \). As in Proposition 2.3, we let \( G \subset \text{Aut}(H) \) be a subgroup over which \( \sqrt{r_H} \) exists, so that \( \frac{D(H)}{G} \) is ribbon and we can define the invariant \( P_H^p(K) \) of Subsection 3.4. Then, the main theorem of the present paper is the following:

**Theorem 4.2.** Let \((K, \rho)\) be a \( G \)-knot in \( S^3 \). Then the Reshetikhin-Turaev invariant of \( K \) from the twisted Drinfeld double of \( H \) recovers the twisted Kuperberg invariant as follows:

\[
P_H^p(K) \cong I_H^p(\gamma) = I_H^p(M, \gamma)
\]

where \((M, \gamma)\) is the sutured manifold associated to the complement of the knot \( K \) (\( M = S^3 \setminus K \) and \( \gamma \) consists of two meridians in \( \partial(S^3 \setminus K) \)). Here \( \cong \) denotes equality up to multiplication by \((\pm 1)^{|\Lambda|}r_H(\rho(\delta))\) for some \( \delta \in \pi_1(S^3 \setminus K) \). Similarly, if \( H \) is \( \mathbb{Z} \)-graded then

\[
P_H^p(K, t) \cong I_H^p(\rho, \gamma) = I_H^p(M, \gamma)
\]

where \( \cong \) denotes equality up to multiplication by \((\pm 1)^{|\Lambda|}r_H(\rho(\delta))t^{|\Lambda|}, k \in \mathbb{Z}, \delta \in \pi_1(S^3 \setminus K) \).
Proof. Let $D$ be a diagram of $T$ and take the associated bridge presentation $a_0, b_1, \ldots, b_g$ of $K$ as in Subsection 4.1. Recall that we suppose that all the underarcs $a_i$ are oriented upwards. Place the black/white beads on the diagram as specified in Subsection 3.2. For simplicity, we will suppose first that all crossings are positive, so the black beads at the crossings come from $R_{\alpha,\beta}$’s (and not $R_{\alpha,\beta}^{-1}$’s). Then each $a_i$ has one black and one white bead: one is an element of $H^*$, which we will denote by $A_i$, and the other is $\varphi_{b_{x(i)}}$ where $b_{x(i)}$ is the overarc at that crossing. Note that by (3), $A_i$ has a $b_{x(i)}$. On the other hand, an overarc $b_i$ may have multiple black beads (but no white bead), which are all elements of $H \subset D(H)_{b_i}$. Since we suppose $b = 1, \beta = \epsilon$, the right caps and cups have a $\sqrt{r_H(p(m))}^{-1} \cdot 1_H$ or $\sqrt{r_H(p(m))}^{-2}$ bead respectively (Proposition 2.3), where $m$ is an oriented meridian. For simplicity, we will suppose there is no bead at all at the caps and cups, and we multiply the resulting tensor by $r_H(p(m))^{-2}$ at the end of the proof, where $r$ is the clockwise rotation number of the diagram. We begin by multiplying all the beads of a given overarc $b_i$, this results in a bead $B_i \in H \subset D(H)_{b_i}$ (which could be the unit of $D(H)_{b_i}$ if $b_i$ has no crossing under it). We will actually compute $z'_D = \varphi_{[T]}^{-1}(Z_D^\rho(T))$ as in (8). Thus, we start from the top of $T$ and successively multiply the beads $A_0, B_1, A_1, \ldots, A_{g-1}, B_g$ in this order, taking care of the evaluations on the crossings $\varphi_{b_{x(i)}} : D_{b_i} \rightarrow D_{b_{x(i)}}$ of $D(H)$. First we take the product $A_0 \cdot B_1 = A_0 \otimes B_1$ in $D(H)_{b_1}$ (this product is simply concatenation since $A_0 \in H^*$ and $B_1 \in H$). Before multiplying with $A_1$, we need to slide $A_0 B_1$ through $a_1$, this has the effect of evaluating this product on the (inverse of the) crossing $\varphi_{b_{x(1)}}$ so we get a new bead $\varphi_{b_{x(1)}}^{-1}(A_0 \otimes B_1)$. We now multiply this with $A_1 \otimes B_2$ inside $D(H)_{b_2}$. Using the multiplication rule for $D(H)_{b_2}$ and that $\varphi_b^{-1}(p \otimes h) = p \circ b \otimes b^{-1}(h)$ for any $p \in H^*, h \in H, b \in G$ the product $\varphi_{b_{x(1)}}^{-1}(A_0 \otimes B_1) \cdot (A_1 \otimes B_2)$ can be written as:

![Diagram](image)

As before, this is evaluated over $\varphi_{b_{x(2)}}^{-1}$ and then multiplied with $A_2 \otimes B_3$ inside $D(H)_{b_3}$. We keep doing this until we reach $A_{g-1} \otimes B_g$. Thus $z'_D$ can be written as
This product is an element of $D(H)_{b_g} = H^* \otimes H$. We now evaluate this product on $\epsilon_{D(H)} = 1 \otimes \epsilon$, note that $\epsilon_{D(H)}(z_D') = \epsilon_{D(H)}(Z_{D(H)}^\rho(T))$. This has the effect of killing all the arrows coming from the leftmost $H^*$-tensor and also kills the rightmost arrow coming from $B_g$. Note that, since $A_0$ is supposed to connect to $B_{\pi(0)}$ in the above picture (because the $R$-matrix is a twisted coevaluation) when $A_0$ is killed, so does the input of $B_{\pi(0)}$ and the tensors immediately above it. Similarly, all the $A_i$ beads connected to $B_g$ disappear. The result is a tensor of the form:

To write this only using tensors in $H$ (and traces) we flip all the arrows oriented downwards. We get the following tensor:
Note that there are various traces in the above picture. Using that $A_i$ has the form $h^* \circ b_{\pi(i)}$ by (3), replacing each of the above traces by a cointegral-integral pair using Radford’s trace formula (see Subsection 2.3) and using that $\lambda$ is a trace (which follows from $S^2 = \text{id}_H$ and $\Lambda$ being 2-sided, see e.g. [17, Lemma 3.9]), we see that the above tensor is the same as

Note that if $H$ is a Hopf algebra in SVect, then the above expression has to be multiplied by $\sigma = (-1)^{|\Lambda|(g-1)}$.

We claim that the above tensor is $\sigma \cdot Z_2^p(\Sigma, \alpha, \beta^e)$ where $(\Sigma, \alpha, \beta^e)$ is the (extended) Heegaard diagram of Subsection 4.1, with the orientations and basepoints specified there. To see this, first note that each $A_i$-output is joined to some input of $B_{\pi(i)}$ and the product appearing right below $A_i$ is simply adding a product to the corresponding input in $B_{\pi(i)}$. Since $B_{\pi(i)}$ is itself a product, this implies that the above expression is in the form coproduct.
followed by antipodes, followed by products. Thus, it has the same form as $Z^\rho_H(\Sigma, \alpha, \beta^e)$ and now we have to check that each coproduct/product has the same outputs/inputs as intersection points of the corresponding $\alpha$ or $\beta$ curve and that the $b$’s twisting the tensors come from the Fox derivatives as in (12). Indeed, for $i = 1, \ldots, g-2$ the $i$-th tensor on the bottom of the above tensor is

![Diagram]

which is exactly the tensor corresponding to $\alpha_i$ as in Subsection 4.3 where the $w_i$ are as in (13). The two rightmost legs have an antipode, because the first two crossings of $\alpha_i$ are negative (assuming that all crossings of the diagram are positive, as we did above). Note that we show four legs coming out of the above tensor, but there may be less if those legs correspond to intersection points of $\alpha_i \cap \beta_g$, this is because we applied $\epsilon$ to the output leg corresponding to $\beta_g$. The $(g-1)$-th coproduct above is only a triple coproduct (or less), this is because $\alpha_{g-1}$ intersects the arc $\beta_g$. Similarly, it is easy to see that the $i$-th product on the top of the above tensor corresponds to the curve $\beta_i$. Note that the tensor for $\beta_1$ is slightly different, this is because there is not an $\alpha$ enclosing the top underarc of the diagram. This shows our claim under the assumption that all crossings of the diagram are positive.

Whenever the crossing at $a_i$ is negative, $a_i$ has a black bead $A_i = S(h^i)$ followed by a white bead $\varphi_{b^{-1}_{\pi(i)}}$. This follows from formula (4) and by sliding the white bead through the white bead. Then, after proceeding as above and using that $S$ is an algebra anti-automorphism, the $i$-th coproduct in the last tensor would instead be

![Diagram]

which is again the tensor defined in Subsection 4.3 by (14). Therefore, we have shown that

$$\epsilon_{D(H)}(Z^\rho_{D(H)}(T)) = \epsilon_{D(H)}(z^d_D) = r_H(\rho(m))^{-r/2}(-1)^{1/2} Z^\rho_H(\Sigma, \alpha, \beta^e)$$

so that

$$P^\rho_H(K) = r_H(\rho(m)) \frac{w(T) - r}{2} (-1)^{1/2} Z^\rho_H(\Sigma, \alpha, \beta^e) \doteq I^\rho_H(M, \gamma)$$

where $\doteq$ means equality up to the above indeterminacy (since $w(T) - r \in 2\mathbb{Z}$).
Note that if $\rho : \pi_1(S^3 \setminus K) \to \text{Ker } (r_H)$, then
\begin{equation}
P_H^\rho(K) = (\pm 1)^{[\Lambda]} I_H^\rho(M, \gamma).
\end{equation}
If $H$ is $\mathbb{Z}$-graded and we consider $\rho \otimes h$ instead, then there is still a $(\pm 1)^{[\Lambda]} t^{[\Lambda]}$ indeterminacy.

Now let $\rho : \pi_1(S^3 \setminus K) \to \text{SL}(n, \mathbb{C})$ be a homomorphism. If $H = \Lambda(\mathbb{C}^n)$ is an exterior algebra, then $\text{Aut}(H) = GL(n, \mathbb{C})$, $r_H$ is the determinant and $\text{Ker } (r_H) = \text{SL}(n, \mathbb{C})$.

**Corollary 4.3.** The $\text{SL}(n, \mathbb{C})$-twisted Reidemeister torsion of the complement of $K$ is recovered as a Reshetikhin-Turaev invariant from a twisted Drinfeld double of an exterior algebra $\Lambda(\mathbb{C}^n)$ by
\[ P_{\Lambda(\mathbb{C}^n)}^\rho(K) = (\pm 1)^n \tau^\rho(S^3 \setminus K, m). \]

The twisted Alexander polynomial $\Delta^\rho_K(t)$ of $K$ is obtained as follows:
\[ P_{\Lambda(\mathbb{C}^n)}^\rho(K, t) \overset{\sim}{=} \tau^\rho \otimes h(S^3 \setminus K, m) \overset{\sim}{=} \det(t^\rho(m) - I_n) \frac{\Delta^\rho_K(t)}{\Delta_{K,0}^\rho(t)}, \]
where $\overset{\sim}{=}$ is equality of to multiplication by $(\pm 1)^n t^{kn}, k \in \mathbb{Z}$ and $\Delta_{K,0}^\rho(t)$ is the 0-th twisted Alexander polynomial of $K$.

**Proof.** The first assertion follows from (15) together with [21, Theorem 2]. Note that, as explain in Remark 4.1, our $I_H^\rho$ is the $I_H^\rho^{-1}$ of [21] so with our conventions we get $I_{\Lambda(\mathbb{C}^n)}^\rho = \tau^\rho$. The second assertion follows from our theorem at $\rho \otimes h$ and standard theorems of Reidemeister torsion, see [21].

**References**

1. Yasuhiro Akutsu, Tetsuo Deguchi, and Tomotada Ohtsuki, *Invariants of colored links*, J. Knot Theory Ramifications 1 (1992), no. 2, 161–184.
2. C. Blanchet, F. Costantino, N. Geer, and B. Patureau-Mirand, *Non-semi-simple TQFTs, Reidemeister torsion and Reshetikhin's invariants*, Adv. Math. 301 (2016), 1–78.
3. Christian Blanchet, Nathan Geer, Bertrand Patureau-Mirand, and Nicolai Reshetikhin, *Holonomy braidings, biquandles and quantum invariants of links with SL$_2(\mathbb{C})$ flat connections*, Selecta Math. (N.S.) 26 (2020), no. 2, Paper No. 19.
4. L. Chang and S. X. Cui, *On two invariants of three manifolds from Hopf algebras*, Adv. Math. 351 (2019), 621–652.
5. Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand, *Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories*, J. Topol. 7 (2014), no. 4, 1005–1053.
6. Alexei Davydov, Pavel Etingof, and Dmitri Nikshych, *Autoequivalences of tensor categories attached to quantum groups at roots of 1*, Lie groups, geometry, and representation theory, Progr. Math., vol. 326, Birkhäuser/Springer, Cham, 2018, pp. 109–136.
7. Alexei Davydov and Dmitri Nikshych, *The Picard crossed module of a braided tensor category*, Algebra Number Theory 7 (2013), no. 6, 1365–1403.
8. Alexei Davydov and Dmitri Nikshych, *Braided picard groups and graded extensions of braided tensor categories*, (2020).
9. Pavel Etingof, Dmitri Nikshych, and Victor Ostrik, *Fusion categories and homotopy theory*, Quantum Topol. 1 (2010), no. 3, 209–273, With an appendix by Ehud Meir.
10. S. Friedl and S. Vidussi, *A survey of twisted Alexander polynomials*, The mathematics of knots, Contrib. Math. Comput. Sci., vol. 1, Springer, Heidelberg, 2011, pp. 45–94.
11. Shlomo Gelaki, Deepak Naidu, and Dmitri Nikshych, *Centers of graded fusion categories*, Algebra Number Theory 3 (2009), no. 8, 959–990 (2009).
12. R. Kashaev and N. Reshetikhin, *Invariants of tangles with flat connections in their complements*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 151–172.
13. Rinat Kashaev and Nicolai Reshetikhin, *Braiding for quantum gl2 at roots of unity*, Noncommutative geometry and representation theory in mathematical physics, Contemp. Math., vol. 391, Amer. Math. Soc., Providence, RI, 2005, pp. 183–197.
14. Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.
15. P. Kirk and C. Livingston, *Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants*, Topology 38 (1999), no. 3, 635–661.
16. G. Kuperberg, *Involuntary Hopf algebras and 3-manifold invariants*, Internat. J. Math. 2 (1991), no. 1, 41–66.
17. , *Non-involuntary Hopf algebras and 3-manifold invariants*, Duke Math. J. 84 (1996), no. 1, 83–129.
18. X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) 17 (2001), no. 3, 361–380.
19. C. McPhail-Snyder, *Holonomy invariants of links and nonabelian Reidemeister torsion*, arXiv:2005.01133 (2020).
20. Daniel L´ opez Neumann, *Kuperberg invariants for balanced sutured 3-manifolds*, arXiv:1904.05786 (2019).
21. , *Twisting Kuperberg invariants via Fox calculus and Reidemeister torsion*, Algebr. Geom. Topol. 25 (2022).
22. Tomotada Ohtsuki, *Quantum invariants*, Series on Knots and Everything, vol. 29, World Scientific Publishing Co., Inc., River Edge, NJ, 2002, A study of knots, 3-manifolds, and their sets. MR 1881401
23. J. Porti, *Reidemeister torsion, hyperbolic three-manifolds, and character varieties*, Handbook of group actions. Vol. IV, Adv. Lect. Math. (ALM), vol. 41, Int. Press, Somerville, MA, 2018, pp. 447–507.
24. D. E. Radford, *Hopf algebras*, Series on Knots and Everything, vol. 49, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.
25. N. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990), no. 1, 1–26.
26. V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, De Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994.
27. , *Homotopy field theory in dimension 3 and crossed group-categories*, arXiv:math/0005291 (2000).
28. , *Introduction to combinatorial torsions*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001, Notes taken by Felix Schlenk.
29. , *Homotopy quantum field theory*, EMS Tracts in Mathematics, vol. 10, European Mathematical Society (EMS), Zürich, 2010, Appendix 5 by Michael Müger and Appendices 6 and 7 by Alexis Virelizier.
30. V. G. Turaev and A. Virelizier, *On 3-dimensional homotopy quantum field theory II: The surgery approach*, Internat. J. Math. 25 (2014), no. 4, 1450027, 66.
31. , *On 3-dimensional homotopy quantum field theory III: comparison of two approaches*, arXiv:1911.10257 (2019).
32. A. Virelizier, *Hopf group-coalgebras*, J. Pure Appl. Algebra 171 (2002), no. 1, 75–122.
33. , *Graded quantum groups and quasitriangular Hopf group-coalgebras*, Comm. Algebra 33 (2005), no. 9, 3029–3050.
34. M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology 33 (1994), no. 2, 241–256.

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