NAVIER-STOKES EQUATIONS WITH NAVIER BOUNDARY CONDITIONS FOR A BOUNDED DOMAIN IN THE PLANE

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Abstract. We consider solutions to the Navier-Stokes equations with Navier boundary conditions in a bounded domain Ω in \( \mathbb{R}^2 \) with a \( C^2 \) boundary \( \Gamma \). Navier boundary conditions can be expressed in the form \( \omega(v) = (2\kappa - \alpha)v \cdot \tau \) and \( v \cdot n = 0 \) on \( \Gamma \), where \( v \) is the velocity, \( \omega(v) \) the vorticity, \( n \) a unit normal vector, \( \tau \) a unit tangent vector, and \( \alpha \) is in \( L^\infty(\Gamma) \). Such solutions have been considered in [2] and [3], and, in the special case where \( \alpha = 2\kappa \), by J.L. Lions in [10] and by P.L. Lions in [11]. We extend the results of [2] and [3] to non-simply connected domains. Assuming, as Yudovich does in [15], a particular bound on the growth of the \( L^p \)-norms of the initial vorticity with \( p \), and also assuming that for some \( \epsilon > 0 \), \( \Gamma \) is \( C^{2,1/2+\epsilon} \) and \( \alpha \) is in \( H^{1/2+\epsilon}(\Gamma) \), we obtain a bound on the rate of convergence in \( L^\infty([0, T]; L^2(\Omega) \cap L^2(\Gamma)) \) to the solution to the Euler equations in the vanishing viscosity limit. We also show that if the initial velocity is in \( H^{3}(\Omega) \) and \( \Gamma \) is \( C^3 \), then solutions to the Navier-Stokes equations with Navier boundary conditions converge in \( L^\infty([0, T]; L^2(\Omega)) \) to the solution to the Navier-Stokes equations with the usual no-slip boundary conditions as we let \( \alpha \) grow large uniformly on the boundary.

1. Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^2 \) with a boundary \( \Gamma \) consisting of a finite number of connected components. We always assume that \( \Gamma \) is at least as smooth as \( C^2 \), but will assume additional smoothness as needed.

We consider the existence and uniqueness of a solution \( u \) to the Navier-Stokes equations under Navier boundary conditions; namely,

\[
v \cdot n = 0 \quad \text{and} \quad 2D(v)n \cdot \tau + \alpha v \cdot \tau = 0 \quad \text{on} \quad \Gamma,
\]

where \( \alpha \) is in \( L^\infty(\Gamma) \), \( n \) and \( \tau \) are unit normal and tangent vectors, respectively, to \( \Gamma \), and \( D(v) \) is the rate-of-strain tensor,

\[
D(v) = \frac{1}{2} \left[ \nabla v + (\nabla v)^T \right].
\]

We follow the convention that \( n \) is an outward normal vector and that the ordered pair \((n, \tau)\) gives the standard orientation to \( \mathbb{R}^2 \). (We give an equivalent form of Navier boundary conditions in Corollary 4.2.)

1991 Mathematics Subject Classification. Primary 76D05, 76C99.

Key words and phrases. Fluid mechanics, Navier-Stokes Equations.
J.L. Lions in [10] p. 87-98 and P.L. Lions in [11] p. 129-131 consider the following boundary conditions, which we call Lions boundary conditions:

\[ v \cdot \mathbf{n} = 0 \text{ and } \omega(v) = 0 \text{ on } \Gamma, \]

where \( \omega(v) = \partial_1 v^2 - \partial_2 v^1 \) is the vorticity of \( v \). Lions boundary conditions are the special case of Navier boundary conditions in which \( \alpha = 2\kappa \), as we show in Corollary 4.3.

J.L. Lions, in Theorem 6.10 p. 88 of [10], proves existence and uniqueness of a solution to the Navier-Stokes equations in the special case of Lions boundary conditions, but includes the assumption that the initial vorticity is bounded. With the same assumption of bounded initial vorticity, the existence and uniqueness is established in Theorem 4.1 of [2] for Navier boundary conditions, under the restriction that \( \alpha \) is positive (and in \( C^2(\Gamma) \)). This is the usual restriction, which is imposed to insure the conservation of energy. Mathematically, negative values of \( \alpha \) present no real difficulty, so we do not make that restriction (until the last section). The only clear gain from removing the restriction, however, is that it allows us to view Lions boundary conditions as a special case of Navier boundary conditions for more than just convex domains (nonnegative curvature).

P.L. Lions establishes an energy inequality on p. 130 of [11] that can be used in place of the usual one for no-slip boundary conditions. He argues that existence and uniqueness can then be established—with no assumption on the initial vorticity—exactly as was done for no-slip boundary conditions in the earlier sections of his text. As we will show, P.L. Lions’ s energy inequality applies to Navier boundary conditions in general, which gives us the same existence and uniqueness theorem as for no-slip boundary conditions. (P.L. Lions’s comment on the regularity of \( \frac{\partial u}{\partial t} \) does not follow as in [11], though, because (4.18) of [11] is not valid for general Navier boundary conditions.) Another method of proof is to modify in a straightforward manner the classical proofs as they appear in [10] and [12]. In Section 6 we state the resulting existence and uniqueness theorem, but only prove the corresponding energy bound, which we will need later. In Section 7 we extend the existence, uniqueness, regularity, and convergence results of [2] and [3] to non-simply connected domains.

It is shown in [3] that if the initial vorticity is in \( L^p(\Omega) \) for some \( p > 2 \), then after extracting a subsequence, solutions to the Navier-Stokes equations with Navier boundary conditions converge in \( L^\infty([0,T];L^2(\Omega)) \) to a solution to the Euler equations (with the usual boundary condition of tangential velocity on the boundary) as \( \nu \to 0 \). This extends a result in [2] for initial vorticity in \( L^\infty(\Omega) \), and because the solution to the Euler equations is unique in this case, it follows that the convergence is strong in \( L^\infty([0,T];L^2(\Omega)) \)—that is, does not require the extraction of a subsequence.

The convergence in [3] also generalizes the similar convergence established for the special case of Lions boundary conditions on p. 131 of [11] (though not including the case \( p = 2 \)). The main difficulty faced in making this
generalization is establishing a bound on the $L^p$-norms of the vorticity, a task that is much easier for Lions boundary conditions (see p. 91-92 of [10] or p. 131 of [11]). In contrast, nearly all of [2] and [3], including the structure of the existence proofs, is directed toward establishing an analogous bound.

The methods of proof in [2] and [3] do not yield a bound on the rate of convergence. With the assumptions in [4], such a bound is probably not possible. We can, however, make an assumption that is weaker than that of [2] but stronger than that of [3] and achieve a bound on the rate of convergence. Specifically, we assume, as in [15] and [8], that the $L^p$-norms of the initial vorticity grow sufficiently slowly with $p$ (Definition 8.2) and establish the bound given in Theorem 8.4. To achieve this result, we also assume additional regularity on $\alpha$ and $\Gamma$.

The bound on the convergence rate in $L^\infty([0,T];L^2(\Omega))$ in Theorem 8.3 is the same as that obtained for $\Omega = \mathbb{R}^2$ in [8]. In particular, it gives a bound on the rate of convergence for initial vorticity in $L^\infty(\Omega)$ proportional to

$$\left(\nu t\right)^{1/2} \exp\left(-C\|\omega^0\|_{L^2(\Omega)}^2\right),$$

where $C$ is a constant depending on $\Omega$ and $\alpha$, and $\omega^0$ is the initial vorticity. This is essentially the same bound on the convergence rate as that for $\Omega = \mathbb{R}^2$ appearing in [1].

Another interesting question is whether solutions to the Navier-Stokes equations with Navier boundary conditions converge to a solution to the Navier-Stokes equations with the usual no-slip boundary conditions if we let the function $\alpha$ grow large. We show in Section 9 that such convergence does take place for initial velocity in $H^3(\Omega)$ and $\Gamma$ in $C^3$ when we let $\alpha$ approach $+\infty$ uniformly on $\Gamma$. This type of convergence is, in a sense, an inverse of the derivation of the Navier boundary conditions from no-slip boundary conditions for rough boundaries discussed in [6] and [7].

We follow the convention that $C$ is always an unspecified constant that may vary from expression to expression, even across an inequality (but not across an equality). When we wish to emphasize that a constant depends, at least in part, upon the parameters $x_1, \ldots, x_n$, we write $C(x_1, \ldots, x_n)$. When we need to distinguish between unspecified constants, we use $C$ and $C'$.

For vectors $u$ and $v$ in $\mathbb{R}^2$, we alternately write $\nabla vu$ and $u \cdot \nabla v$, by both of which mean $u^i \partial_i v^j e_j$, where $e_1, e_2$ are basis vectors, and we define $\nabla u \cdot \nabla v = u^{ij} v^{ij}$. Here, as everywhere in this paper, we follow the common summation convention that repeated indices are summed—whether or not one is a superscript and one a subscript.

If $X$ is a function space and $k$ a positive integer, we define $(X)^k$ to be

$$\{(f_1, \ldots, f_k) : f_1 \in X, \ldots, f_k \in X\}.$$

For instance, $(H^1(\Omega))^2$ is the set of all vector fields, each of whose components lies in $H^1(\Omega)$. To avoid excess notation, however, we always suppress
the superscript $k$ when it is clear from the context whether we are dealing with scalar-, vector-, or tensor-valued functions.

2. Function Spaces

Let

\[ E(\Omega) = \{ v \in (L^2(\Omega))^2 : \text{div} \, v \in L^2(\Omega) \} , \]  

(2.1)

as in [12], with the inner product,

\[ (u, v)_{E(\Omega)} = (u, v) + (\text{div} \, u, \text{div} \, v) . \]

We will use several times the following theorem, which is Theorem 1.2 p. 7 of [12].

**Lemma 2.1.** There exists a continuous linear operator $\gamma_n$ mapping $E(\Omega)$ into $H^{-1/2}(\Gamma)$ such that

\[ \gamma_n v = \text{the restriction of } v \cdot n \text{ to } \Gamma , \text{ for every } v \text{ in } (D(\Omega))^2 . \]

Also, the following form of the divergence theorem is true for all vector fields $v$ in $E(\Omega)$ and scalar functions $h$ in $H^1(\Omega)$:

\[ \int_{\Omega} v \cdot \nabla h + \int_{\Omega} (\text{div} \, v) h = \int_{\Gamma} \gamma_n v \cdot \gamma_0 h . \]

We always suppress the trace function $\gamma_0$ in our expressions, and we write $v \cdot n$ in place of $\gamma_n v$.

Define the following function spaces as in [2]:

\[ H = \{ v \in (L^2(\Omega))^2 : \text{div} \, v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \Gamma \} , \]

\[ V = \{ v \in (H^1(\Omega))^2 : \text{div} \, v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \Gamma \} , \]

\[ \mathcal{W} = \{ v \in V \cap H^2(\Omega) : v \text{ satisfies (1.1)} \} . \]

(2.2)

We give $\mathcal{W}$ the $H^2$-norm, $H$ the $L^2$-inner product and norm, which we symbolize by $(\cdot, \cdot)$ and $\| \cdot \|_{L^2(\Omega)}$, and $V$ the $H^1$-inner product,

\[ (u, v)_V = \sum_i (\partial_i u, \partial_i v) , \]

and associated norm. This norm is equivalent to the $H^1$-norm, because Poincaré’s inequality,

\[ \| v \|_{L^p(\Omega)} \leq C(\Omega, p) \| \nabla v \|_{L^p(\Omega)} \]  

(2.3)

for all $p$ in $[1, \infty]$, holds for all $v$ in $V$.

Ladyzhenskaya’s inequality,

\[ \| v \|_{L^4(\Omega)} \leq C(\Omega) \| v \|_{L^2(\Omega)}^{1/2} \| \nabla v \|_{L^2(\Omega)}^{1/2} \]  

(2.4)

also holds for all $v$ in $V$, though the constant in the inequality is domain dependent, unlike the constant for the classical space $V$. 

We will also frequently use the following inequality, which follows from the standard trace theorem, Sobolev interpolation, and Poincaré’s inequality:

\[
\|v\|_{L^2(\Gamma)} \leq C(\Omega) \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} \leq C(\Omega) \|v\|_V
\]

for all \(v\) in \(V\).

3. Hodge Decomposition of \(H\)

Only simply connected domains are considered in \([2]\) and \([3]\). To handle non-simply connected domains we will need a portion of the Hodge decomposition of \(L^2(\Omega)\). We briefly summarize the pertinent facts, drawing mostly from Appendix I of \([12]\).

Let \(\Sigma_1, \ldots, \Sigma_N\) be one-manifolds with boundary that generate \(H_1(\Omega, \Gamma; \mathbb{R})\), the one-dimensional real homology class of \(\Omega\) relative to its boundary \(\Gamma\).

We can decompose the space \(H\) into two subspaces, \(H = H_0 \oplus H_c\), where

\[
H_0 = \{v \in H : \text{all internal fluxes are zero}\}, \\
H_c = \{v \in H : \omega(v) = 0\}.
\]

An internal flux is a value of \(\int_{\Sigma_i} v \cdot n\). Then \(H_0 = H^\perp_c\) and there is an orthonormal basis \(\nabla q_1, \ldots, \nabla q_N\) for \(H_c \subseteq C^\infty(\Omega)\) consisting of the gradients of \(N\) harmonic functions, \(q_1, \ldots, q_N\). (Each \(q_i\) is multi-valued in \(\Omega\), but \(\nabla q_i\) is single-valued.)

If \(v\) is in \(V\), then \(v\) is also in \(H\) so there exists a unique \(v_0\) in \(H_0\) and \(v_c\) in \(H_c\) such that \(v = v_0 + v_c\); also, \((v_0, v_c) = 0\). But \(v_c\) is in \(C^\infty(\Omega)\) and so in \(V\); hence, \(v_0\) also lies in \(V\). This shows that \(V = (V \cap H_0) \oplus H_c\), though this is not an orthogonal decomposition of \(V\).

The following is a result of Yudovich’s:

**Lemma 3.1.** For any \(p\) in \([2, \infty)\) and any \(v\) in \(V \cap H_0\),

\[
\|\nabla v\|_{L^p(\Omega)} \leq C(\Omega)p \|\omega(v)\|_{L^p(\Omega)}.
\]

**Proof.** Let \(v\) be in \(V \cap H_0\). Since \(v\) has no harmonic component, \(v = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)\) for some stream function \(\psi\), which we can assume vanishes on \(\Gamma\). Applying Corollary 1 of \([13]\) with the operator \(L = \Delta\) and \(r = 0\) gives

\[
\|\nabla v\|_{L^p(\Omega)} \leq \|\psi\|_{H^{2,p}(\Omega)} \leq C(\Omega)p \|\Delta \psi\|_{L^p(\Omega)} = C(\Omega)p \|\omega(v)\|_{L^p(\Omega)}.
\]

\(\square\)

For \(\Omega\) simply connected, \(H = H_0\), and Lemma 3.1 applies to all of \(V\).

**Corollary 3.2.** For any \(p\) in \([2, \infty)\) and any \(v\) in \(V\),

\[
\|\nabla v\|_{L^p(\Omega)} \leq C(\Omega)p \|\omega(v)\|_{L^p(\Omega)} + C'(\Omega) \|v\|_{L^2(\Omega)},
\]

the constants \(C(\Omega)\) and \(C'(\Omega)\) being independent of \(p\).
Proof. Let \( v \) be in \( V \) with \( v = v_0 + v_c \), where \( v_0 \) is in \( V \cap H_0 \) and \( v_c \) is in \( H_c \), and assume that \( \nabla v \) is in \( L^p(\Omega) \). Let \( v_c = \sum_{i=1}^N c_i \nabla q_i \) and \( r = \| v_c \|_{L^2(\Omega)} = (\sum c_i^2)^{1/2} \). Then

\[
\| \nabla v_c \|_{L^p(\Omega)} \leq r \max \left\{ 1, |\Omega|^{1/2} \right\} \sum_{i=1}^N \| \nabla \nabla q_i \|_{L^\infty(\Omega)} \leq C \| v_c \|_{L^2(\Omega)},
\]

where we used the smoothness of \( \nabla q_i \). But, \( H_0 = H_c^\perp \), so \( \| v \|_{L^2(\Omega)} = \| v_0 \|_{L^2(\Omega)} + \| v_c \|_{L^2(\Omega)} \) and thus \( \| v_c \|_{L^2(\Omega)} = \| v \|_{L^2(\Omega)} \). Therefore,

\[
\| \nabla v \|_{L^p(\Omega)} \leq \| \nabla v_0 \|_{L^p(\Omega)} + \| v_c \|_{L^p(\Omega)} \leq C(\Omega) p \| \omega(v) \|_{L^p(\Omega)} + C'(\Omega) \| v \|_{L^2(\Omega)}
\]

by virtue of Lemma 3.1. \( \square \)

4. VORTICITY ON THE BOUNDARY

If we parameterize each component of \( \Gamma \) by arc length, \( s \), it follows that

\[
\frac{\partial \mathbf{n}}{\partial \tau} := \frac{dn}{ds} = \kappa \mathbf{\tau},
\]

where \( \kappa \), the curvature of \( \Gamma \), is continuous because \( \Gamma \) is \( C^2 \).

The second part of the following theorem is Lemma 2.1 of [2], and the first part is established similarly.

Lemma 4.1. If \( v \) is in \( (H^2(\Omega))^2 \) with \( v \cdot \mathbf{n} = 0 \) on \( \Gamma \), then

\[
\nabla v \mathbf{n} \cdot \mathbf{\tau} = \omega(v) + \nabla v \mathbf{\tau} \cdot \mathbf{n} = \omega(v) - \kappa v \cdot \mathbf{\tau},
\]

and

\[
D(v) \mathbf{n} \cdot \mathbf{\tau} = \frac{1}{2} \omega(v) - \kappa v \cdot \mathbf{\tau}.
\]

Corollary 4.2. A vector \( v \) in \( V \cap H^2(\Omega) \) satisfies Navier boundary conditions (that is, lies in \( \mathcal{W} \)) if and only if

\[
\omega(v) = (2\kappa - \alpha)v \cdot \mathbf{\tau} \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma.
\]

Also, for all \( v \) in \( \mathcal{W} \) and \( u \) in \( V \),

\[
\nabla v \mathbf{n} \cdot u = (\kappa - \alpha)v \cdot u \text{ on } \Gamma.
\]

Proof. Let \( v \) be in \( V \cap H^2(\Omega) \). Then from (4.2),

\[
2D(v) \mathbf{n} \cdot \mathbf{\tau} + 2\kappa(\mathbf{n} \cdot \mathbf{\tau}) = \omega(v).
\]

If \( v \) satisfies Navier boundary conditions, then (4.3) follows by subtracting \( 2D(v) \mathbf{n} \cdot \mathbf{\tau} + \alpha v \cdot \mathbf{\tau} = 0 \) from (4.5). Conversely, substituting the expression for \( \omega(v) \) in (4.3) into (4.5) gives \( 2D(v) \mathbf{n} \cdot \mathbf{\tau} + \alpha v \cdot \mathbf{\tau} = 0 \).
If \( v \) is in \( W \), then from (4.1),
\[
\nabla v \cdot \tau = \omega(v) - \kappa v \cdot \tau = (2\kappa - \alpha)v \cdot \tau - \kappa \cdot \tau = (\kappa - \alpha)v \cdot \tau,
\]
and (4.4) follows from this, since \( u \) is parallel to \( \tau \) on \( \Gamma \).

\[\square\]

**Corollary 4.3.** For initial velocity in \( H^2(\Omega) \), Lions boundary conditions are the special case of Navier boundary conditions where
\[
\alpha = 2\kappa.
\]
That is, any solution of \((\text{NS})\) with Navier boundary conditions where \( \alpha = 2\kappa \) is also a solution to \((\text{NS})\) with Lions boundary conditions.

5. Weak Formulation

For all \( u \) in \( W \) and \( v \) in \( V \),
\[
\int_{\Omega} \Delta u \cdot v = \int_{\Omega} (\text{div } \nabla u) v - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma} (\kappa - \alpha)u \cdot v - \int_{\Gamma} \nabla u \cdot \nabla v = 0,
\]
where we used (4.4) of Corollary 4.2. This motivates our formulation of a weak solution, in analogy with Problem 3.1 p. 190-191 of [12].

**Definition 5.1.** Given a viscosity \( \nu > 0 \) and initial velocity \( u^0 \) in \( H \), \( u \) in \( L^2([0,T]; V) \) is a weak solution to the Navier-Stokes equations (without forcing) if \( u(0) = u^0 \) and
\[
\frac{d}{dt} \int_{\Omega} u \cdot v + \int_{\Omega} (u \cdot \nabla u) \cdot v + \nu \int_{\Omega} \nabla u \cdot \nabla v - \nu \int_{\Gamma} (\kappa - \alpha)u \cdot v = 0
\]
for all \( v \) in \( V \). (We make sense of the initial condition \( u(0) = u^0 \) as in [12].)

Our formulation of a weak solution is equivalent to that in (2.11) and (2.12) of [2]. This follows from the identity,
\[
2 \int_{\Omega} D(u) \cdot D(v) = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma} \kappa u \cdot v,
\]
which holds for all \( u \) and \( v \) in \( V \). This identity can be derived from (4.1) and Lemma 2.1, and the density of \( H^2(\Omega) \cap V \) in \( V \).

6. Existence and Uniqueness

We prove only the energy bound of the following existence and uniqueness theorem (see the comment in Section 1). We observe, however, that Ladyzhenskaya’s inequality, (2.4), is required in the proof of uniqueness.

**Theorem 6.1.** Assume that \( \Gamma \) is \( C^2 \) and \( \alpha \) is in \( L^\infty(\Gamma) \). Let \( u^0 \) be in \( H \) and let \( T > 0 \). Then there exists a solution \( u \) to \((\text{NS})\). Moreover, \( u \) is in
\[ L^2((0, T]; V) \cap C((0, T]; H), \ u' \text{ is in } L^2((0, T]; V'), \text{ and we have the energy inequality,} \]
\[ \|u(t)\|_{L^2(\Omega)} \leq e^{C(\alpha)\nu t}\|u^0\|_{L^2(\Omega)}, \quad (6.1) \]
where the constant \( C(\alpha) = 0 \) if \( \alpha \) is nonnegative on \( \Gamma \).

**Proof.** We prove only (6.1). We proceed with a Galerkin approximation as in the proof of Theorem 3.1 on p. 192-193 of [12], but use the basis of Corollary A.3. Because this basis is also a basis for \( H \), if we let \( u^0_m \) be the projection in \( H \) of \( u^0 \) onto the span of the first \( m \) basis vectors, then \( u^0_m \rightarrow u^0 \) in \( L^2(\Omega) \). Because the basis is in \( H^2(\Omega) \), the approximate solution \( u_m \) is in \( C^1([0, T]; H^2(\Omega)) \).

Definition 5.1 leads to the following replacement for (3.27) p. 193 of [12]:
\[ (u'_m(t), u_m(t)) + \nu \|\nabla u_m(t)\|^2_{L^2(\Omega)} = \nu \int_{\Gamma} (\kappa - \alpha) u_m \cdot u_m. \]
Using (4.4) of Corollary 4.2 and Lemma 1.2 p. 176 of [12], we conclude that
\[ \frac{1}{2} \frac{d}{dt} \|u_m\|^2_{L^2(\Omega)} + \nu \|\nabla u_m\|^2_{L^2(\Omega)} \leq C\nu \|u_m\|^2_{L^2(\Gamma)}, \quad (6.2) \]
where \( C = \sup_{\Gamma} |\kappa - \alpha| \). Except for the value of the constant, (6.2) is identical to the first inequality on p. 130 of [11], which is for the special case of Lions boundary conditions.

Arguing exactly as in [11], it follows that
\[ \frac{d}{dt} \|u_m\|^2_{L^2(\Omega)} + \nu \|\nabla u_m\|^2_{L^2(\Omega)} \leq C\nu \|u_m\|^2_{L^2(\Omega)}. \]
Integrating over time gives
\[ \|u_m(t)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla u_m(s)\|^2_{L^2(\Omega)} \, ds \leq \|u^0_m\|^2_{L^2(\Omega)} + C\nu \int_0^t \|u_m(s)\|^2_{L^2(\Omega)} \, ds. \quad (6.3) \]
The energy bound,
\[ \|u_m(t)\|^2_{L^2(\Omega)} \leq e^{C\nu t}\|u^0_m\|^2_{L^2(\Omega)} \leq e^{C\nu t}\|u^0\|^2_{L^2(\Omega)}, \quad (6.4) \]
then follows from Gronwall’s lemma, and shows that the right side of (6.3) is bounded uniformly in \([0, T]\). We conclude from (6.3) and (6.4) that
\[ \{u_m\} \text{ is bounded in } L^2([0, T]; V) \cap L^\infty([0, T]; H), \]
from which (6.1) will follow. (If \( \alpha \) is nonnegative, then, in fact, energy is conserved—in the absence of forcing—so \( C(\alpha) = 0 \). This follows from the equation preceding (2.16) of [2].) \qed
7. Additional Regularity

In this section we establish an existence theorem suited to addressing the issue of convergence of a solution to \((NS)\) to a solution to the Euler equations, where we always impose stronger regularity on the initial velocity.

If we assume extra regularity on the initial velocity, that regularity will be maintained for all time. Our proof of this is an adaptation of the proof of Theorem 3.5 p. 202-204 of [12] to establish the regularity of \(u'\), combined with the second half of the proof of Theorem 2.3 of [2] to establish the regularity of \(u\).

**Definition 7.1.** A vector field \(v\) in \(\mathcal{W}\) is called *compatible* if \(\omega(v)\) is in \(L^\infty(\Omega)\).

Definition 7.1 is as in [3], except that we define the vector field to be compatible instead of the vorticity.

**Theorem 7.2.** Assume that \(\Omega\) is a bounded domain with a \(C^{2,1/2+\epsilon}\) boundary \(\Gamma\) and that \(\alpha\) is in \(H^{1/2+\epsilon}(\Gamma) + C^{1/2+\epsilon}(\Gamma)\) for some \(\epsilon > 0\). Let \(u^0\) be in \(\mathcal{W}\) with initial vorticity \(\omega^0\), and let \(u\) be the unique solution to \((NS)\) given by Theorem 6.1 with corresponding vorticity \(\omega\). Let \(T > 0\). Then

\[ u' \in L^2([0,T]; V) \cap C([0,T]; H) \]

If, in addition, \(\omega^0\) is in \(L^\infty(\Omega)\) (so \(u^0\) is compatible), then

\[ u \in C([0,T]; H^2(\Omega)), \ \omega \in C([0,T]; H^1(\Omega)) \cap L^\infty([0,T] \times \Omega).\]

**Proof.** We prove the regularity of \(u'\) in three steps as in the proof of Theorem 3.5 p. 202-204 of [12]. The only change in step (i) is that we use the basis of Corollary A.3 rather than the basis in [12].

No change to step (ii) is required, because (3.88) of [12] still holds.

In step (iii), an additional term of

\[ \nu \int_\Gamma (\kappa - \alpha) |u_m'|^2 \]

appears on the right side of (3.94) of Temam’s proof, which we bound by

\[ CV \|u_m'|\|_{L^2(\Omega)} \|\nabla u_m'|\|_{L^2(\Omega)} \leq \frac{\nu}{2} \|\nabla u_m'|\|_{L^2(\Omega)}^2 + CV \|u_m'|^2_{L^2(\Omega)}.\]

Then (3.95) of Temam’s proof becomes

\[ \frac{d}{dt} \|u_m'(t)\|_{L^2(\Omega)}^2 \leq \phi_m(t) \|u_m'(t)\|_{L^2(\Omega)}^2, \]

where

\[ \phi_m(t) = \left( \frac{2}{\nu} + C\nu \right) \|u_m(t)\|_{L^2(\Omega)}^2, \]

and the proof of the regularity of \(u'\) is completed as in [12], along with the observation in [2] that \(u'\) is then in \(C([0,T]; H)\).

To prove the regularity of \(u\) and \(\omega\), we follow the argument in the second half of the proof of Theorem 2.3 in [2] (which does not rely on \(\alpha\) being...
nonnegative). We must, however, impose additional regularity on $\Gamma$ and on $\alpha$ over that assumed in Theorem 6.1. This is to insure that $u$ lying in $C^{1/2}([0, T]; (H^1(\Omega))^2)$ implies that $(\kappa - \alpha/2)u \cdot \tau$ lies in $C^{1/2}([0, T]; H^1(\Omega))$. Our conditions on $\Gamma$ and $\alpha$ are sufficient, though not necessary (see, for instance, Theorem 1.4.1.1 p. 21 and Theorem 1.4.4.2 p. 28 of [5]).

Then, after it is shown that $u$ is in $C([0, T]; (H^{2, \alpha}(\Omega))^2)$, we know by Sobolev embedding that $u$ is in $C([0, T] \times \Omega)$. Thus,

$$\|u \cdot \nabla u(t)\|_H \leq \|u\|_{L^\infty([0, T] \times \Omega)} \|u(t)\|_V,$$

and since we already have $u$ in $C([0, T]; V)$, it follows that $u \cdot \nabla u$ and also $\Phi$ are in $C([0, T]; H)$. Then $\text{curl} \, \Phi$ is in $C([0, T]; H^{-1}(\Omega))$, and another pass through the argument in [2], this time with $q = 2$, gives $u$ in $C([0, T]; (H^2(\Omega))^2)$. Because the increase in regularity of the solution arises from the equation $-\nabla \psi = u$ with the boundary condition $\psi = 0$, no regularity on $\Gamma$ or on $\alpha$ beyond that we have assumed is required.

(The argument in [2] is for a simply connected domain. We can easily adapt it, though, by using the equivalent of Lemma 2.5 p. 26 of [12], which gives a stream function $\psi$ that is constant on each boundary component, which is good enough to apply Grisvard’s result (Theorem 2.5.1.1 p. 128 of [5]) to conclude that $\psi$ is in $C([0, T]; H^{5/4}(\Omega)).$)

With Theorem 7.2, we have a replacement for Theorem 2.3 of [2] that applies regardless of the sign of $\alpha$. Since the nonnegativity of $\alpha$ is used nowhere else in [2] and [3], all the results of both of those papers apply for simply connected domains as well regardless of the sign of $\alpha$, but with the extra regularity assumed on $\Gamma$ (and the lower regularity assumed on $\alpha$).

To remove the restriction on the domain being simply connected, it remains only to show that Lemmas 3.2 and 4.1 of [3] remain valid for non-simply connected domains. We show this for Lemma 3.2 of [3] in Theorem A.2. As for Lemma 4.1 of [3], we need only use Corollary 3.2 to replace the term $\|\omega(\cdot, t)\|_{L^\infty(\Omega)}^{1-\theta}$ with $(\|\omega(\cdot, t)\|_{L^p(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)})^{1-\theta}$ in the proof of Lemma 4.1 in [3]. Lemma 4.1 of [3] then follows with no other changes in the proof—only the value of the constant $C$ changes.

Let $u$ be the unique solution to $(NS)$ given by Proposition 5.2 of [3], and fix $q > 2$. By Lemma 4.1 of [3] and Corollary 3.2,

$$\|u\|_{L^\infty([0, T]; V)} = \|\nabla u\|_{L^\infty([0, T]; L^2(\Omega))} \leq C \|\nabla u\|_{L^\infty([0, T]; L^2(\Omega))} \leq C(\|\omega\|_{L^\infty([0, T]; L^2(\Omega))} + \|u\|_{L^\infty([0, T]; L^2(\Omega))}) \leq C(T, \alpha, \kappa) e^{C(\alpha)u^T}. \quad (7.1)$$

Also, using Sobolev interpolation, (2.3), and Corollary 3.2,

$$\|u(t)\|_{C(\Omega)} \leq C \|u(t)\|_{L^2(\Omega)}^{\theta} \|u(t)\|_{H^{2, \alpha}(\Omega)}^{1-\theta} \leq C \|u(t)\|_{L^2(\Omega)}^{\theta} (\|\omega(t)\|_{L^2(\Omega)} + \|u(t)\|_{L^2(\Omega)})^{1-\theta},$$
where \( \theta = (q - 2)/(2q - 2) \). This norm is finite and bounded over any finite range of viscosity by (6.1). Using Lemma 4.1 of [3], it follows that
\[
\| u \|_{L^\infty([0,T] \times \Omega)} \leq C
\]
for all \( \nu \) in \((0, 1]\), a bound we will use in Section 8.

8. Vanishing Viscosity

To describe Yudovich’s conditions on the initial vorticity, let \( \phi : (1, \infty) \to [0, \infty) \) be any continuous function. We define two functions, \( \beta_{\epsilon,M,\phi} : [0, \infty) \to [0, \infty) \) and \( \beta_{M,\phi} : [0, \infty) \to [0, \infty) \), parameterized by \( \epsilon \) in \((0, 1) \), \( M > 0 \), and \( \phi \):
\[
\beta_{\epsilon,M,\phi}(x) = M^\epsilon x^{-1-\epsilon} \phi(1/\epsilon),
\beta_{M,\phi}(x) = \inf \{ \beta_\epsilon(x) : \epsilon \in (0, 1) \}.
\]
For brevity, we write \( \beta_\epsilon \) for \( \beta_{\epsilon,M,\phi} \) and \( \beta \) for \( \beta_{M,\phi} \), with the choices of \( M \) and \( \phi \) being understood.

For all \( \epsilon \) in \((0, 1) \), \( \beta_\epsilon(x) \) is a monotonically increasing function continuous in \( x \) and in \( \epsilon \), with \( \lim_{x \to 0^+} \beta_\epsilon(x) = 0 \). It follows that \( \beta \) is a monotonically increasing continuous function and that \( \lim_{x \to 0^+} \beta(x) = 0 \). Also, \( \beta(x) \leq \beta_\epsilon(x) \) for all \( \epsilon \) in \((0, 1) \) and \( x \in [0, \infty) \).

**Definition 8.1.** A continuous function \( \theta : (1, \infty) \to [0, \infty) \) is called admissible if
\[
\int_0^1 \frac{ds}{\beta_{M,\phi}(s)} = \infty,
\]
where \( \phi(p) = p\theta(p) \). This condition is independent of the choice of \( M \).

Some examples of admissible functions are given in [15]. Roughly speaking, a function is admissible if it does not grow much faster than \( \log p \).

**Definition 8.2.** We say that a velocity vector \( v \) has Yudovich vorticity if \( p \mapsto \| \omega(v) \|_{L^p(\Omega)} \) is an admissible function.

**Definition 8.3.** Given an initial velocity \( u^0 \) in \( V \), \( u \in L^2([0,T]; V) \) is a weak solution to the Euler equations if \( u(0) = u^0 \) and
\[
\frac{d}{dt} \int_\Omega u \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v = 0
\]
for all \( v \) in \( V \).

The existence of a weak solution to the Euler equations under the assumption that the initial vorticity \( \omega^0 \) is in \( L^p(\Omega) \) for some \( p > 1 \) (a weaker assumption than that of Definition 8.3 when \( 1 < p < 2 \)) was proved in [14]. These solutions have the property that \( \omega(u) \) is in \( L^\infty_{\text{loc}}(\mathbb{R}; L^p(\Omega)) \). It is shown in [15] that Yudovich initial vorticity is enough to insure uniqueness of solutions for which \( \omega(u) \) and \( \partial_t u \) are in \( L^\infty_{\text{loc}}(\mathbb{R}; L^p(\Omega)) \) for all \( p \) in \([1, \infty) \). (Yudovich’s uniqueness result in [15] applies to a bounded domain in \( \mathbb{R}^n \),...
although existence is not known for \( n > 2 \). His approach works, with only very minor changes, when applied to all of \( \mathbb{R}^n \).

In [8], it is shown that Yudovich initial vorticity is sufficient to provide a bound on the rate of convergence in \( L^\infty([0,T];L^2(\mathbb{R}^2)) \) of solutions to the Navier-Stokes equations with no-slip boundary conditions to the unique solution to the Euler equations. In Theorem 8.4 we extend this result to bounded domains when the Navier-Stokes equations have Navier boundary conditions.

**Theorem 8.4.** Assume that \( \Omega \) and \( \alpha \) are as in Theorem 7.2. Fix \( T > 0 \) and let \( u^0 \) be in \( V \) and have Yudovich vorticity \( \omega^0 \). Let \( \{u_\nu\}_{\nu > 0} \) be the solutions to \((NS)\) given by 5.2 of [3] and \( \overline{u} \) be the unique weak solution to the Euler equations for which \( \omega(\overline{u}) \) and \( \partial_t \overline{u} \) are in \( L^\infty_{\text{loc}}(\mathbb{R};L^p(\Omega)) \), \( \overline{u} \) and each \( u_\nu \) having initial velocity \( u^0 \). Then

\[
\nu \int_\Gamma (\kappa - \alpha) u_\nu \cdot w - \nu \int_\Omega \nabla u_\nu \cdot \nabla w.
\]

(8.3)

Both \( \partial_t u_\nu \) and \( \partial_t \overline{u} \) are in \( L^2([0,T];V') \), so (see, for instance, Lemma 1.2 p. 176 of [12]),

\[
\int_\Omega w \cdot \partial_t w = \frac{1}{2} \frac{d}{dt} \|w\|^2_{L^2(\Omega)}.
\]
Applying Lemma 2.1,
\[
\int_\Omega w \cdot (u_\nu \cdot \nabla w) = \frac{1}{2} \int_\Omega u_\nu \partial_j \sum_i (w^i)^2 = \frac{1}{2} \int_\Omega u_\nu \cdot \nabla |w|^2
\]
\[
= \frac{1}{2} \int_\Gamma (u_\nu \cdot n)|w|^2 - \frac{1}{2} \int_\Omega (\text{div } u_\nu)|w|^2 = 0,
\]
since \(u_\nu \cdot n = 0\) on \(\Gamma\) and \(\text{div } u_\nu = 0\) on \(\Omega\). Thus, integrating (8.3) over time,
\[
\|w(t)\|_{L^2(\Omega)}^2 \leq A + 2 \int_0^t \int_\Omega |w|^2 |\nabla \pi|,
\]
where
\[
A = 2\nu \int_0^t \left[ \int_\Gamma (\kappa - \alpha) u_\nu \cdot w - \int_\Omega \nabla u_\nu \cdot \nabla w \right].
\]
Using (2.5), (7.1), and the conservation of the \(L^2\)-norm of vorticity for the Euler equation, we have
\[
|\int_\Gamma (\kappa - \alpha) u_\nu \cdot w| \leq \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|u_\nu \cdot w\|_{L^1(\Gamma)}
\]
\[
\leq \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|\nabla u_\nu\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \leq C(T, \alpha, \kappa) e^{C(\alpha)\nu T}.
\]
By (7.1) we also have
\[
\|\nabla u_\nu \cdot \nabla w\| \leq \|\nabla u_\nu\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \leq C(T, \alpha, \kappa) e^{C(\alpha)\nu T},
\]
so \(A \leq C(T, \alpha, \kappa) e^{C(\alpha)\nu T}\).

By (7.2), \(\|u_\nu\|_{L^\infty([0,T] \times \Omega)} \leq C\) for all \(\nu\) in \((0,1]\). It is also true that \(\pi\) is in \(L^\infty([0,T] \times \Omega)\) (arguing, for instance, exactly as in the derivation of (7.2)). Thus,
\[
M = \sup_{\nu \in (0,1]} \|w(t)\|_{L^\infty([0,T] \times \Omega)}^2
\]
is finite.

Also, because vorticity is conserved for \(\pi\), we have, by Corollary 3.2
\[
2 \|\nabla \pi(t)\|_{L^p(\Omega)} \leq Cp\|\omega^0\|_{L^p(\Omega)} + C\|\pi\|_{L^2(\Omega)} =: \phi(p)
\]
for all \(p \geq 2\). Then, as in (8),
\[
2 \int_\Omega |w|^2 |\nabla \pi| \leq \beta(\|w\|_{L^2}),
\]
where \(\beta = \beta_{M,\phi}\) is the function in (8.1). (The additive constant \(C\|\pi\|_{L^2(\Omega)}\) in (8.7) does not affect the integral condition in Definition 8.1.)

Letting \(L(t) = \|w(t)\|_{L^2(\Omega)}^2\), we have
\[
L(t) \leq A + \int_0^t \beta(L(r)) \, dr.
\]
Using Osgood’s lemma as in [8], we conclude that
\[ \int_A L(t) \frac{dr}{\beta(r)} \leq t, \]  
and that as \( \nu \to 0, A \to 0, \) and \( L(t) \to 0 \) uniformly over any finite time interval. The rate of convergence given in \( L^\infty([0,T];L^2(\Omega)) \) in (8.2) can be derived from (8.9) precisely as in [8].

By (2.5),
\[ \| u_\nu - \tilde{u} \|_{L^2(\Gamma)} = \| w \|_{L^2(\Gamma)} \leq C \| \nabla w \|_{L^2(\Omega)}^{1/2} \| w \|_{L^2(\Omega)}^{1/2} \]
\[ \leq C(T, \alpha, \kappa) e^{C(\alpha)\nu T} L(t)^{1/2}, \]
from which the convergence rate for \( L^\infty([0,T];L^2(\Gamma)) \) in (8.2) follows.

The convergence rate in \( L^\infty([0,T];L^2(\Omega)) \) established in Theorem 8.4 is the same as that established for the entire plane in [8], except for the values of the constants.

9. No-slip Boundary Conditions
As long as \( \alpha \) is non-vanishing, we can reexpress the Navier boundary conditions in (1.1) as
\[ v \cdot n = 0 \] and \[ 2\gamma D(v) n \cdot \tau + v \cdot \tau = 0 \] on \( \Gamma, \)
where \( \gamma = 1/\alpha. \) When \( \gamma \) is identically zero, we have the usual no-slip boundary conditions. An obvious question to ask is whether it is possible to arrange for \( \gamma \) to approach zero in such a manner that the corresponding solutions to the Navier-Stokes equations with Navier boundary conditions approach the solution to the Navier-Stokes equations with the usual no-slip boundary conditions in \( L^\infty([0,T];L^2(\Omega)). \)

Let \( u^0 \) be an initial velocity in \( V, \) and assume that \( \gamma > 0 \) lies in \( L^\infty(\Gamma). \)
Fix a \( \nu > 0 \) and let
\[ u_{\nu,\gamma} = \text{the unique solution to the Navier-Stokes equations with Navier boundary conditions for } \alpha = 1/\gamma \] and
\[ \tilde{u}_\nu = \text{the unique solution to the Navier-Stokes equations with no-slip boundary conditions,} \]
in each case with the same initial velocity \( u^0. \) (In Theorem 8.4 we wrote \( u_{\nu,\gamma} \) as \( u_\nu. \))

If we let \( \gamma \) approach 0 uniformly on the boundary, we automatically have some control over \( u_{\nu,\gamma} \) on the boundary.

**Lemma 9.1.** For sufficiently small \( \| \gamma \|_{L^\infty(\Gamma)}, \)
\[ \| u_{\nu,\gamma} \|_{L^2([0,T];L^2(\Gamma))} \leq \frac{\| u^0 \|_{L^2(\Omega)}}{\sqrt{\nu}} \| \gamma \|_{L^\infty(\Gamma)}^{1/2}, \]  
(9.2)
Proof. Assume that $\|\gamma\|_{L^\infty(\Gamma)}$ is sufficiently small that $\alpha > \kappa$ on $\Gamma$. Then, as in the proof of Theorem 6.1, we have

$$\frac{1}{2} \frac{d}{dt} \|u_{\nu,\gamma}(t)\|^2_{L^2(\Omega)} + \nu \|\nabla u_{\nu,\gamma}(t)\|^2_{L^2(\Omega)} = \nu \int_{\Gamma} (\kappa - \alpha) u_{\nu,\gamma} \cdot u_{\nu,\gamma},$$

so,

$$\|u_{\nu,\gamma}(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} + 2\nu \int_0^t \int_{\Gamma} (\kappa - \alpha) u_{\nu,\gamma} \cdot u_{\nu,\gamma},$$

But,

$$\int_{\Gamma} (\kappa - \alpha) u_{\nu,\gamma} \cdot u_{\nu,\gamma} \leq -\inf_{\Gamma} \{\alpha - \kappa\} \|u_{\nu,\gamma}(t)\|^2_{L^2(\Gamma)},$$

so

$$\|u_{\nu,\gamma}(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} - 2\nu \inf_{\Gamma} \{\alpha - \kappa\} \|u_{\nu,\gamma}\|^2_{L^2([0,t];L^2(\Gamma))}$$

and

$$\|u_{\nu,\gamma}\|^2_{L^2([0,t];L^2(\Gamma))} \leq \|u_0\|^2_{L^2(\Omega)}/(2\nu \inf_{\Gamma} \{\alpha - \kappa\}).$$

Then (9.2) follows because $\|\gamma\|_{L^\infty(\Gamma)} \inf_{\Gamma} \{\alpha - \kappa\} \to 1$ as $\|\gamma\|_{L^\infty(\Gamma)} \to 0$. $$\square$$

If we assume enough smoothness of the initial data and of $\Gamma$, we can use (9.2) to establish convergence of $u_{\nu,\gamma}$ to $\tilde{u}_\nu$ as $\|\gamma\|_{L^\infty(\Gamma)} \to 0$.

Theorem 9.2. Fix $T > 0$, assume that $u^0$ is in $V \cap H^3(\Omega)$ with $u^0 = 0$ on $\Gamma$, and assume that $\Gamma$ is $C^3$. Then for any fixed $\nu > 0$,

$$u_{\nu,\gamma} \to \tilde{u}_\nu \text{ in } L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; L^2(\Gamma))$$

as $\gamma \to 0$ in $L^\infty(\Gamma)$.

Proof. First, $u_{\nu,\gamma}$ exists and is unique by Theorem 6.1 the existence and uniqueness of $\tilde{u}_\nu$ is a classical result. Because $u^0$ is in $H^3(\Omega)$ and $\Gamma$ is $C^3$, $\tilde{u}_\nu$ is in $L^\infty([0,T]; H^3(\Omega))$ by the argument on p. 205 of [12] following the proof of Theorem 3.6 of [12]. Hence, $\nabla \tilde{u}_\nu$ is in $L^\infty([0,T]; H^2(\Omega))$ and so in $L^\infty([0,T]; C(\Omega))$.

Arguing as in the proof of Theorem 8.4 with $w = u_{\nu,\gamma} - \tilde{u}_\nu$, we have

$$\int_{\Omega} \partial_t w \cdot w + \int_{\Omega} w \cdot (u_{\nu,\gamma} \cdot \nabla w) + \int_{\Omega} w \cdot (w \cdot \nabla \tilde{u}_\nu) + \int_{\Omega} \nabla w \cdot \nabla w - \nu \int_{\Gamma} (\kappa - \alpha) u_{\nu,\gamma} \cdot w + \nu \int_{\Gamma} (\nabla \tilde{u}_\nu \cdot n) \cdot w = 0.$$

But $\tilde{u}_\nu = 0$ on $\Gamma$ so $w = u_{\nu,\gamma}$ on $\Gamma$, and

$$\int_{\Omega} \partial_t w \cdot w + \int_{\Omega} w \cdot (w \cdot \nabla \tilde{u}_\nu) + \int_{\Omega} |\nabla w|^2 + \nu \int_{\Gamma} (\alpha - \kappa)|u_{\nu,\gamma}|^2$$

$$+ \nu \int_{\Gamma} (\nabla \tilde{u}_\nu \cdot n) \cdot u_{\nu,\gamma} = 0.$$
Then, for $||\gamma||_{L^\infty(\Gamma)}$ sufficiently small that $\alpha = 1/\gamma > \kappa$ on $\Gamma,$

$$
||w(t)||_{L^2(\Omega)}^2 \leq A + 2 \int_0^t \int_\Omega |w|^2 |\nabla \bar{u}_\nu|, \tag{9.4}
$$

where

$$
A = -2\nu \int_0^t \int_\Gamma (\nabla \bar{u}_\nu)n \cdot u_{\nu,\gamma}.
$$

By (4.1), $(\nabla \bar{u}_\nu)n \cdot \tau = \omega(\bar{u}_\nu) - \kappa \bar{u}_\nu \cdot \tau = \omega(\bar{u}_\nu)$ on $\Gamma.$ But $u_{\nu,\gamma}$ is parallel to $\tau$ on $\Gamma,$ so $(\nabla \bar{u}_\nu)n \cdot u_{\nu,\gamma} = \omega(\bar{u}_\nu)u_{\nu,\gamma} \cdot \tau.$ Thus,

$$
-\int_\Gamma (\nabla \bar{u}_\nu)n \cdot u_{\nu,\gamma} = -\int_\Gamma (\omega(\bar{u}_\nu))u_{\nu,\gamma} \cdot \tau \leq \|\omega(\bar{u}_\nu)\|_{L^2(\Gamma)} \|u_{\nu,\gamma} \cdot \tau\|_{L^2(\Gamma)},
$$

so

$$
A \leq C\nu \|\bar{u}_\nu\|_{L^2([0,T];H^2(\Omega))} \|u_{\nu,\gamma}\|_{L^2([0,T];L^2(\Gamma))}.
$$

By Theorem 3.10 p. 213 of [12], $\|\bar{u}_\nu\|_{L^2([0,T];H^2(\Omega))}$ is finite (though the bound on it in [12] increases to infinity as $\nu$ goes to 0), so by Lemma 9.1,

$$
A \leq C_1(\nu) \|\gamma\|_{L^\infty(\Gamma)}^{1/2}. \tag{9.5}
$$

Because $\nabla \bar{u}_\nu$ is in $L^\infty([0,T];C(\bar{\Omega})),$

$$
\int_0^t \int_\Omega |w|^2 |\nabla \bar{u}_\nu| \leq C_2(\nu) \int_0^t \|w(s)\|_{L^2(\Omega)}^2 \ ds,
$$

where $C_2(\nu) = \|\nabla \bar{u}_\nu\|_{L^\infty([0,T]\times\Omega)},$ and (9.4) becomes

$$
\|w(t)\|_{L^2(\Omega)}^2 \leq C_1(\nu) \|\gamma\|_{L^\infty(\Gamma)}^{1/2} + C_2(\nu) \int_0^t \|w(s)\|_{L^2(\Omega)}^2 \ ds.
$$

By Gronwall's Lemma,

$$
\|w(t)\|_{L^2(\Omega)}^2 \leq C_1(\nu) \|\gamma\|_{L^\infty(\Gamma)}^{1/2} e^{C_2(\nu)t},
$$

and the convergence in $L^\infty([0,T];L^2(\Omega))$ follows immediately. Convergence in $L^2([0,T];L^2(\Gamma))$ follows directly from Lemma 9.1 since $\bar{u}_\nu = 0$ on $\Gamma.$ \hfill \Box

We cannot prove convergence in $L^\infty([0,T];L^2(\Gamma))$ as we did in Theorem 8.4 because we do not have a bound on the vorticity of $u_{\nu,\gamma}$ that is uniform over sufficiently small values of $||\gamma||_{L^\infty(\Gamma)}.$ But if we did have such a bound, we could also establish convergence in $L^\infty([0,T];L^2(\Omega) \cap L^2(\Gamma))$ when $u^0$ in $V \cap H^2(\Omega)$ has Yudovich initial vorticity by combining the approaches in the proofs of Theorem 8.4 and Theorem 9.2.
APPENDIX A. COMPATIBLE SEQUENCES

For $p$ in $(1, \infty)$, define the spaces
\[ X_0^p = H_0 \cap H^{1,p}(\Omega) \quad \text{and} \quad X^p = H \cap H^{1,p}(\Omega) = X_0^p \oplus H_c, \]
each with the $H^{1,p}(\Omega)$-norm.

**Lemma A.1.** Let $p$ be in $(1, \infty]$. For $p < 2$ let $\hat{p} = p/(2-p)$, for $p > 2$ let $\hat{p} = \infty$, and for $p = 2$ let $\hat{p}$ be any value in $[2, \infty]$. Then for any $v$ in $X_0^p$,\[ \|v\|_{L^{\hat{p}}(\Gamma)} \leq C(p) \|\omega(v)\|_{L^p(\Omega)}. \]

Proof. For $p < 2$ and any $v$ in $X_0^p$, we have\[ \|v\|_{L^{\hat{p}}(\Gamma)} \leq C(p) \|v\|_{L^p(\Omega)}^{1-\lambda} \|\nabla v\|_{L^p(\Omega)}^\lambda \leq C(p) \|\nabla v\|_{L^p(\Omega)} \]
where $\lambda = 2(\hat{p} - p)/(p(\hat{p} - 1)) = 1$ if $p < 2$ and $\lambda = 2/p$ if $p \geq 2$. The first inequality follows from Theorem 3.1 p. 42 of [3], the second follows from [2,3], and the third from Lemma 3.1.

Given a vorticity $\omega$ in $L^p(\Omega)$ with $p$ in $(1, \infty)$, the Biot-Savart law gives a vector field $v$ in $H$ whose vorticity is $\omega$. (That $v$ is in $L^2(\Omega)$ follows as in the proof of Lemma A.1, $\Omega$ being bounded.) Let $v = v_0 + v_c$, where $v_0$ is in $H_0$ and $v_c$ is in $H_c$. Then $\omega(v_0) = \omega$ as well, so we can define a function $K_\Omega$: $L^p(\Omega) \rightarrow H_0$ by $\omega \mapsto v_0$ having the property that $\omega(K_\Omega(\omega)) = \omega$. By (2.3) and Lemma A.1, $v_0$ is also in $H^{1,p}(\Omega)$, so in fact, $K_\Omega$: $L^p(\Omega) \rightarrow X_0^p$ and is the inverse of the function $\omega$. It is continuous by the same two lemmas.

**Theorem A.2.** Assume that $\Gamma$ is $C^2$ and $\alpha$ is in $L^\infty(\Gamma)$. Let $\mathbf{v}$ be in $X^p$ for some $p$ in $(1, \infty)$ and have vorticity $\mathbf{\omega}$. Then there exists a sequence $\{v_i\}$ of compatible vector fields (Definition 7.1) whose vorticities converge strongly to $\mathbf{\omega}$ in $L^p(\Omega)$. The vector fields $\{v_i\}$ converge strongly to $\mathbf{v}$ in $X^p$ and, if $p \geq 2$, also in $V$.

Proof. We adapt the proof of Lemma 3.2 of [3]. Suppose that $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_c$ with $\mathbf{v}_0 \in X_0^p$ and $\mathbf{v}_c$ in $H_c$. Define $\beta$ as in Equation (3.1) of [3], but let $v = K_\Omega[\beta] + \mathbf{v}_c$ and start the iteration with $\omega_1 = \mathbf{\omega}$. Then the fixed point argument goes through unchanged because $v_1 - v_2$ is in $X_0^p$ and we can apply Lemma A.1. The only further change is the estimate on $\|G^n\|_{L^p(\Gamma)}$, which becomes\[ \|G^n\|_{L^p(\Gamma)} \leq 2\kappa - \alpha \|\omega_1\|_{L^\infty} \|\mathbf{\omega}\|_{L^p(\Omega)} + \|\mathbf{v}_c\|_{L^p(\Gamma)} \]
\[ \leq C_p(\|\omega\|_{L^p(\Omega)} + \|\mathbf{v}_c\|_{L^p(\Gamma)}) + \frac{1}{2} \|G^n\|_{L^p(\Gamma)}, \]
for $n$ sufficiently large, which is still sufficient to imply the required bound that insures convergence of $\omega_n$ to $\mathbf{\omega}$ in $L^p(\Omega)$.
Letting $v_n = K_\Omega[\omega_n] + \nabla c$, we have
\[
\|\nabla \bar{v} - \nabla v_n\|_{L^p(\Omega)} = \|\nabla \bar{v}_0 + \nabla c - (\nabla K_\Omega[\omega_n] + \nabla c)\|_{L^p(\Omega)}
\]
\[
\leq C_p \|\omega(\bar{v}_0 - K_\Omega[\omega_n])\|_{L^p(\Omega)} = C_p \|\bar{v} - \omega_n\|_{L^p(\Omega)},
\]
where we used Lemma 3.1. Then by (2.3), $v_n$ converges strongly to $\bar{v}$ in $X^p$ as well. Convergence in $V$ for $p \geq 2$ follows since $\Omega$ is bounded. 

We only require Theorem A.2 for $p \geq 2$. We include all the cases, however, for the same reason as in [3]: in the hope that if the vorticity bound in Lemma 4.1 of [3] can be extended to $p$ in $(1,2)$, then the convergence in Proposition 5.2 of [3] can also be extended (for non-simply connected $\Omega$).

**Corollary A.3.** Assume that $\Gamma$ is $C^2$, and $\alpha$ is in $L^\infty(\Gamma)$. Then there exists a basis for $V$ lying in $W$ that is also a basis for $H$.

**Proof.** The space $V = (V \cap H_0) \oplus H_c$ is separable because $V \cap H_0$ is the image under the continuous function $K_\Omega$ of the separable space $L^2(\Omega)$ and $H_c$ is finite-dimensional. Let $\{v_i\}_{i=1}^\infty$ be a dense subset of $V$. Applying Theorem A.2 to each $v_i$ and unioning all the sequences, we obtain a countable subset $\{u_i\}_{i=1}^\infty$ of $W$ that is dense in $V$. Selecting a maximal independent set gives us a basis for $V$ and for $H$ as well, since $V$ is dense in $H$. \qed

**Acknowledgements**

I wish to thank Josef Málek for suggesting that I look at Navier boundary conditions, and Misha Vishik for many useful discussions.

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