Pseudoscalar Decay Constants $f_D$ and $f_{Ds}$ in Lattice QCD with Exact Chiral Symmetry

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Abstract

We determine the masses and decay constants of pseudoscalar mesons $D$, $D_s$, and $K$ in quenched lattice QCD with exact chiral symmetry. For 100 gauge configurations generated with single-plaquette action at $\beta = 6.1$ on the $20^3 \times 40$ lattice, we compute point-to-point quark propagators for 30 quark masses in the range $0.03 \leq m_q a \leq 0.80$, and measure the time-correlation functions of pseudoscalar and vector mesons. The inverse lattice spacing $a^{-1}$ is determined with the experimental input of $f_\pi$, while the strange quark bare mass $m_s a = 0.08$, and the charm quark bare mass $m_c a = 0.80$ are fixed such that the masses of the corresponding vector mesons are in good agreement with $\phi(1020)$ and $J/\psi(3097)$ respectively. Our results of pseudoscalar-meson decay constants are $f_K = 152(6)(10)$ MeV, $f_D = 235(8)(14)$ MeV, and $f_{Ds} = 266(10)(18)$ MeV.

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1 Introduction

The pseudoscalar-meson decay constants (e.g., $f_D$, $f_{Ds}$, $f_B$ and $f_{Bs}$) play an important role in extracting the CKM matrix elements (e.g., the leptonic decay width of $D_s^+ \rightarrow l^+\nu_l$ is proportional to $f_{Ds}^2|V_{cs}|^2$), which are crucial for testing the flavor sector of the standard model via the unitarity of CKM matrix. Experimentally\(^1\), precise determination of $f_{Ds}$ and $f_D$ will soon result from the high-statistics program of CLEO-c, however, the determination of $f_B$ and $f_{Bs}$ remains beyond the reach of current experiments.

Theoretically, lattice QCD provides a solid framework to compute the masses and decay constants of pseudoscalar mesons (as well as other physical observables) nonperturbatively from the first principles of QCD. Thus reliable lattice QCD determinations of $f_B$ and $f_{Bs}$ are of fundamental importance, in view of their experimental determinations are still lacking. Obviously, the first step for lattice QCD is to check whether the lattice determinations of $f_D$ and $f_{Ds}$ will agree with those coming soon from the high-statistics charm program of CLEO-c. This motivates our present study.

In this paper, we compute quenched quark propagators for 30 quark masses in the range $0.03 \leq m_qa \leq 0.80$, in the framework of optimal domain-wall fermion proposed by Chiu [10]-[12]. Then we determine the inverse lattice spacing $a^{-1} = 2.237(76)$ GeV from the pion time-correlation function, with the experimental input of pion decay constant $f_\pi = 132$ MeV. The strange quark bare mass $m_s a = 0.08$ and the charm quark bare mass $m_c a = 0.80$ are fixed such that the corresponding masses extracted from the vector meson correlation function agree with $\phi(1020)$ and $J/\psi(3097)$ respectively. Then the masses and decay constants of any hadrons containing $c,s$, and $u(d)$ quarks\(^2\) are predictions of QCD from the first principles, with the understanding that chiral extrapolation to physical $m_{u,d} \simeq m_s/25$ (or equivalently $m_\pi = 135$ MeV) is required for any observables containing $u(d)$ quarks.

For pseudoscalar and vector mesons, we measure their time correlation functions for the following three categories: (i) two quarks have the same mass; (ii) one quark mass is fixed at $m_s$; (iii) one quark mass is fixed at $m_c$. Note that for mesons which are composed of strange and/or charm quarks, their masses and decay constants can be measured directly without chiral extrapolation.

The outline of this paper is as follows. In section 2, we outline our formulation of exact chiral symmetry on the lattice, and our computation of quark propagators. In section 3, we determine the inverse lattice spacing, the strange quark bare mass, and the charm quark bare mass. In section 4, we present our results of $m_K$ and $f_K$. In section 5, we present our results of $m_D$, $m_{Ds}$, $f_D$, and $f_{Ds}$. In section 6, we summarize our results and conclude with some remarks.

\(^1\)See, for example, Refs. [11][2][3][4], and other experimental results compiled by PDG [5].

\(^2\)In this paper, we work in the isospin limit $m_u = m_d$. 

1
2 Lattice quarks with exact chiral symmetry

To implement exact chiral symmetry on the lattice [6, 7, 8, 9], we consider the optimal domain-wall fermion proposed by Chiu [10]-[12]. The action of optimal domain-wall fermion can be written as [12]

\[ A_F = \sum_{s,s'=0}^{N_s+1} \sum_{x,x'} \bar{\psi}(x,s) \left\{ (\omega_s D_w(x,x') + \delta_{x,x'}) \delta_{ss'} 
+ (\omega_{s'} D_w(x,x') - \delta_{x,x'}) (P_+ \delta_{s',s-1} + P_- \delta_{s',s+1}) \right\} \psi(x',s') \] (1)

with boundary conditions

\[ P_+ \psi(x,-1) = -r m_q P_+ \psi(x,N_s + 1), \]
\[ P_- \psi(x,N_s + 2) = -r m_q P_- \psi(x,0), \quad r = \frac{1}{2m_0}, \]

where \( m_q \) is the bare quark mass, and \( \{\omega_s, s = 0, \ldots, N_s + 1\} \) are specified by the exact formula derived in Ref. [10]. Here \( H_w = \gamma_5 D_w \), and \( D_w \) is the standard Wilson Dirac operator plus a negative parameter \(-m_0 (0 < m_0 < 2)\). The quark fields are constructed from the boundary modes at \( s = 0 \) and \( s = N_s + 1 \) with \( \omega_0 = \omega_{N_s + 1} = 0 \) [12]:

\[ q(x) = \sqrt{r} [P_- \psi(x,0) + P_+ \psi(x,N_s + 1)], \]
\[ \bar{q}(x) = \sqrt{r} \left[ \bar{\psi}(x,0) P_+ + \bar{\psi}(x,N_s + 1) P_- \right]. \] (3)

After introducing pseudofermions with \( m_q = 2m_0 \), the generating functional for \( n \)-point Green’s function of the quark fields can be derived as [12],

\[ Z[J, \bar{J}] = \frac{\int [dU] e^{-A_g \text{det} \left[ (D_c + m_q)(1 + rD_c)^{-1} \right]} \exp \left\{ \bar{J} (D_c + m_q)^{-1} J \right\}}{\int [dU] e^{-A_g \text{det} \left[ (D_c + m_q)(1 + rD_c)^{-1} \right]}} \] (4)

where \( A_g \) is the action of the gauge fields, \( \bar{J} \) and \( J \) are the Grassman sources of \( q \) and \( \bar{q} \) respectively, and

\[ D_c = 2m_0 \frac{1 + \gamma_5 S_{\text{opt}}}{1 - \gamma_5 S_{\text{opt}}}, \]
\[ S_{\text{opt}} = \frac{1 - \prod_{s=1}^{N_s} T_s}{1 + \prod_{s=1}^{N_s} T_s}, \]
\[ T_s = \frac{1 - \omega_s H_w}{1 + \omega_s H_w}. \] (7)

Using the exact formula of \( \omega_s \) [10], one immediately obtains

\[ S_{\text{opt}} = \begin{cases} H_w R_Z^{(n,n)}(H_w^2), & N_s = 2n + 1, \\
H_w R_Z^{(n-1,n)}(H_w^2), & N_s = 2n, \end{cases} \] (8)
where \( R_Z(H^2_w) \) is the Zolotarev optimal rational polynomial [13] for the inverse square root of \( H^2_w \),

\[
R_Z^{(n,n)}(H^2_w) = \frac{d_0}{\lambda_{\text{min}}} \prod_{l=1}^{n} \frac{1 + h^2_w/c_{2l}}{1 + h^2_w/c_{2l-1}} \\
= \frac{1}{\lambda_{\text{min}}} (h^2_w + c_{2n}) \sum_{l=1}^{n} \frac{b_l}{h^2_w + c_{2l-1}}, \quad h^2_w = H^2_w/\lambda_{\text{min}}^2 \tag{9}
\]

and

\[
R_Z^{(n-1,n)}(H^2_w) = \frac{d'_0}{\lambda_{\text{min}}} \prod_{l=1}^{n} \frac{1 + h^2_w/c'_l}{1 + h^2_w/c'_{2l-1}} = \frac{1}{\lambda_{\text{min}}} \sum_{l=1}^{n} \frac{b'_l}{h^2_w + c'_{2l-1}}, \tag{10}
\]

where the coefficients \( d_0, d'_0, c_l \) and \( c'_l \) are expressed in terms of elliptic functions [13] with arguments depending on \( N_s \) and \( \lambda_{\text{max}}^2/\lambda_{\text{min}}^2 \), and \( \lambda_{\text{min}} (\lambda_{\text{max}}) \) is fixed to be the greatest lower bound (least upper bound) of the eigenvalues of \( |H_w| \) for the set of gauge configurations under investigation.

From (9), the effective 4D lattice Dirac operator for the fermion determinant is

\[
D(m_q) = (D_c + m_q)(1 + rD_c)^{-1} = m_q + (m_0 - m_q/2) \left[ 1 + \gamma_5 H_w R_Z(H^2_w) \right] \tag{11}
\]

and the quark propagator in background gauge field is

\[
\langle q(x) \bar{q}(y) \rangle = - \frac{\delta^2 Z[J,J]}{\delta J(x) \delta J(y)} \bigg|_{J=\bar{J}=0} = (D_c + m_q)^{-1} = (1 - rm_q)^{-1}[D_{x,y}^{-1}(m_q) - r \delta_{x,y}] \tag{12}
\]

Note that \( D_c \) is exactly chirally symmetric (i.e. \( D_c \gamma_5 + \gamma_5 D_c = 0 \)) in the limit \( N_s \to \infty \), and its deviation from exact chiral symmetry due to finite \( N_s \) is the minimal provided that the weights \( \{ \omega_s \} \) are fixed according to the formula derived in Ref. [10]. Further, the bare quark mass \( m_q \) (whether heavy or light) in the quark propagator \( (D_c + m_q)^{-1} \) is well-defined for any gauge configurations.

In practice, we have two ways to evaluate the quark propagator [12] in background gauge field:

(i) To solve the linear system of the 5D optimal DWF operator;
(ii) To solve \( D_{x,y}^{-1}(m_q) \) from the system

\[
D(m_q) Y = \left[ m_q + (m_0 - m_q/2) \left( 1 + \gamma_5 H_w R_Z(H^2_w) \right) \right] Y = I, \tag{13}
\]

with nested conjugate gradient [14], and then substitute the solution vector \( Y \) into [12].

Since either (i) or (ii) yields exactly the same quark propagator, in principle, it does not matter which linear system one actually solves. However, in
practice, one should choose the most efficient scheme for one’s computational system (hardware and software). For our present system (a Linux PC cluster of 100 nodes \[15\]), it has been geared to the scheme (ii), and it attains the maximum efficiency if the inner conjugate gradient loop of \([13]\) is iterated with Neuberger’s 2-pass algorithm \[16\]. So we use the scheme (ii) to compute the quark propagator, with the quark fields \((2)-(3)\) defined by the boundary modes at \(s = 0\) and \(s = N_s + 1\). Note that Neuberger’s 2-pass algorithm not only provides very high precision of chiral symmetry with fixed amount of memory, but also is faster than the single pass algorithm for \(n > 12 \sim 25\) (where \(n\) is the order of the rational polynomial \(R^{(n-1,n)}\)) for most computer platforms, as discussed by Chiu and Hsieh \[17\].

We generate 100 gauge configurations with single plaquette gauge action at \(\beta = 6.1\) on the \(20^3 \times 40\) lattice. Fixing \(m_0 = 1.3\), we project out 16 low-lying eigenmodes of \(|H_w|\) and perform the nested conjugate gradient in the complement of the vector space spanned by these eigenmodes. For \(N_s = 128\), the weights \(\{\omega_s\}\) are fixed with \(\lambda_{\min} = 0.18\) and \(\lambda_{\max} = 6.3\), where \(\lambda_{\min} \leq \lambda(|H_w|) \leq \lambda_{\max}\) for all gauge configurations. For each configuration, point to point quark propagators are computed for 30 bare quark masses in the range \(0.03 \leq m_qa \leq 0.8\), with stopping criteria \(10^{-11}\) and \(2 \times 10^{-12}\) for the outer and inner conjugate gradient loops respectively. Then the norm of the residual vector of each column of the quark propagator is less than \(2 \times 10^{-11}\)

\[||(D_c + m_q)Y - I|| < 2 \times 10^{-11},\]

and the chiral symmetry breaking due to finite \(N_s\) is less than \(10^{-14}\),

\[\sigma = \left| \frac{Y^\dagger S^2 Y}{Y^\dagger Y} - 1 \right| < 10^{-14},\]

for every iteration of the nested conjugate gradient. Further details of our scheme have been described in Refs. \[18, 17\].

In this paper, we measure the time-correlation functions for pseudoscalar (P(S)) and vector (V) mesons,

\[C_{PS}(t) = \left\langle \sum_x \text{tr}\{\gamma_5(D_c + m_Q)^{-1}(D_c + m_q)^{-1}\} \right\rangle_U \tag{14}\]

\[C_{V}(t) = \left\langle \frac{1}{3} \sum_{\mu=1}^{3} \sum_x \text{tr}\{\gamma_\mu(D_c + m_Q)^{-1}(D_c + m_q)^{-1}\} \right\rangle_U \tag{15}\]

where the subscript \(U\) denotes averaging over gauge configurations. Here \(C_{PS}(t)\) and \(C_{V}(t)\) are measured for the following three categories:

(i) Symmetric masses \(m_Q = m_q\),

(ii) Asymmetric masses with fixed \(m_Q = m_s = 0.08a^{-1}\),

(iii) Asymmetric masses with fixed \(m_Q = m_c = 0.80a^{-1}\),

where \(m_q\) is varied for 30 masses in the range \(0.03 \leq m_qa \leq 0.80\).
3 Determination of $a^{-1}$, $m_s$, and $m_c$

![Graph of $f_\pi a$ versus $m_q a$]

Figure 1: The pion decay constant $f_\pi a$ versus the bare quark mass $m_q a$. The solid line is the linear fit.

For symmetric masses $m_Q = m_q$, the pseudoscalar time-correlation function $C_\pi(t)$ is measured, and is fitted to the usual formula

$$\frac{Z}{2m_\pi a}[e^{-m_\pi at} + e^{-m_\pi a(T-t)}]$$  \hspace{1cm} (16)

to extract the pion mass $m_\pi a$ and the pion decay constant

$$f_\pi a = 2m_q a \frac{\sqrt{Z}}{m_\pi^2 a^2}.$$  \hspace{1cm} (17)

In Figs. 1 and 2 we plot the decay constant $f_\pi a$ and pion mass square $(m_\pi a)^2$ versus bare quark mass $m_q a$, respectively.
\[ (m_\pi^2 a)^2 = A_1 (m_q a)^{1/(1+\delta)} + B (m_q a)^2 \]

Figure 2: The pion mass square \((m_\pi a)^2\) versus the bare quark mass \(m_q a\). The solid line is the fit of Eq. (19).

The data of \(f_\pi a\) (see Fig. 1) is well fitted by the straight line
\[ f_\pi a = 0.059(2) + 0.235(38) \times (m_q a) \, . \]

Then taking \(f_\pi a\) at \(m_q a = 0\) equal to 0.132 GeV times the lattice spacing \(a\), we can determine the lattice spacing \(a\) and its inverse,
\[ a^{-1} = \frac{0.132}{f_0} \text{ GeV} = 2.237(76) \text{ GeV} , \]
\[ a = 0.088(3) \text{ fm} \, . \quad (18) \]

Thus the size of our lattice is about \((1.8 \text{ fm})^3 \times 3.6 \text{ fm}\). Since the smallest pion mass is 439 MeV, the lattice size is about \((3.9)^3 \times 7.8\), in units of the Compton wavelength \((\sim 0.45 \text{ fm})\) of the smallest pion mass.

The data of \(m_\pi^2\) (see Fig. 2) can be fitted by the form \[ m_\pi^2 a^2 = A_1 (m_q a) \frac{1}{1+\delta} + B (m_q a)^2 \]

(19)
in quenched chiral perturbation theory \((q\chi PT)\). The fitted parameters are

\[
\begin{align*}
\delta &= 0.187(21) \\
A_1 &= 0.669(45) \\
B &= 2.666(357)
\end{align*}
\]

with \(\chi^2/{\text{d.o.f.}}=0.54\). Evidently, the coefficient of quenched chiral logarithm \(\delta = 0.187(21)\) is in good agreement with the theoretical estimate \(\delta \simeq 0.176\) in \(q\chi PT\).

The bare mass of strange quark is determined by extracting the mass of vector meson from the time-correlation function \(C_V(t)\). At \(m_qa = 0.08\), \(m_Va = 0.460(4)\), which gives \(m_V = 1029(10)\) MeV, in good agreement with the mass of \(\phi(1020)\). Thus we take the strange quark bare mass to be \(m_s a = 0.08\).

Similarly, at \(m_qa = 0.80\), \(m_Va = 1.368(2)\), which gives \(m_V = 3060(5)\) MeV, in good agreement with the mass of \(J/\Psi(3097)\). Thus, we fix the charm quark bare mass to be \(m_c a = 0.80\).

\section*{4 \(f_K\) and \(m_K\)}

Next we measure the time-correlation function of kaon \(C_K(t)\) with \(m_Q\) fixed at \(m_s = 0.08a^{-1}\), while \(m_q\) is varied for 30 masses in the range \(0.03 \leq m_qa \leq 0.80\). Then the data of \(C_K(t)\) is fitted by the formula analogous to \((16)\) to extract the kaon mass \(m_Ka\) and the kaon decay constant \(f_Ka\).

In Fig. 3, the kaon mass \(m_K\) is plotted versus \(m_\pi\), for 15 quark masses in the range \(0.03 \leq m_qa \leq 0.10\). The data of \(m_Ka\) can be fitted by

\[
m_Ka = 0.197(1) + 0.255(4)(m_\pi a) + 0.389(8)(m_\pi a)^2.
\]

At the physical limit \(m_\pi = 135\) MeV, it gives \(m_K = 478(16)\) MeV, in good agreement with the experimental value of kaon mass (495 MeV).

In Fig. 4, \(f_K a\) is plotted versus bare quark mass \(m_qa\). The data is well fitted by the straight line

\[
f_Ka = 0.068(0) + 0.116(1) \times (m_qa)
\]

At \(m_qa = 0\), it gives \(f_K = 152(6)\) MeV, in agreement with the value \(f_K^+ = 159.8 \pm 1.4 \pm 0.44\) MeV complied by PDG.

\section*{5 \(f_D, f_{Ds}, m_D, \text{ and } m_{Ds}\)}

Now we turn to charmed pseudoscalar mesons. We measure the time-correlation function \(C_D(t)\) with \(m_Q\) fixed at \(m_c = 0.80a^{-1}\), while \(m_q\) is varied for 30 different masses in the range \(0.03 \leq m_qa \leq 0.80\). Then the data of \(C_D(t)\) is
Figure 3: The kaon mass $m_K$ versus the pion mass $m_\pi$ for 15 bare quark masses in the range $0.03 \leq m_q a \leq 0.10$. The solid line is the quadratic fit.

fitted by the formula analogous to (16) to extract the mass $m_D a$ and decay constant $f_D a$.

In Fig. 5, $m_D a$ is plotted versus $m_\pi a$, for 15 quark masses in the range $0.03 \leq m_q a \leq 0.10$. The data of $m_D a$ can be fitted by

$$m_D a = 0.816(0) + 0.101(3)(m_\pi a) + 0.298(6)(m_\pi a)^2$$

At $m_\pi = 135$ MeV, it gives $m_D = 1842(15)$ MeV, in good agreement with the mass of $D$ meson (1865 MeV). In Fig. 6, the decay constant $f_D a$ is plotted versus bare quark mass $m_q a$. The data is well fitted by the straight line

$$f_D a = 0.105(1) + 0.172(1) \times (m_q a)$$

At $m_q a = 0$, it gives $f_D = 235(8)$ MeV, which serves as a prediction of lattice QCD with exact chiral symmetry.
The pseudoscalar meson of $c\bar{s}$ or $s\bar{c}$ corresponds to $m_Qa = m_c a = 0.80$ and $m_qa = m_s a = 0.08$. Its mass and decay constant are extracted directly from the time-correlation function, which are plotted as the eleventh data point (counting from the smallest one) in Figs. 4 and 5 respectively. The results are $m_{D_s}a = 0.878(2)$ and $f_{D_s}a = 0.119(2)$. The mass gives $m_{D_s} = 1964(5)$ MeV, in good agreement with the mass of $D_s(1968)$. The decay constant gives $f_{D_s} = 266(10)$ MeV, which agrees with the value $f_{D^+_s} = 267\pm33$ MeV complied by PDG.

6 Summary and Concluding Remarks

In this paper, we have determined the masses and decay constants of pseudoscalar mesons $K$, $D$ and $D_s$, in quenched lattice QCD with exact chiral
Figure 5: The mass of $D$ meson $m_D a$ versus the pion mass $m_\pi a$ for 15 bare quark masses in the range $0.03 \leq m_q a \leq 0.10$. The solid line is the quadratic fit.

Our results are:

$$m_K = 478 \pm 16 \pm 20 \text{ MeV},$$
$$m_D = 1842 \pm 15 \pm 21 \text{ MeV},$$
$$m_{D_s} = 1964 \pm 5 \pm 10 \text{ MeV},$$
$$f_K = 152 \pm 6 \pm 10 \text{ MeV},$$
$$f_D = 235 \pm 8 \pm 14 \text{ MeV},$$
$$f_{D_s} = 266 \pm 10 \pm 18 \text{ MeV},$$

where in each case, the first error is statistical, while the second is our crude estimate of combined systematic uncertainty. It is interesting to see whether the values of $f_D$ and $f_{D_s}$ coming soon from the high-statistics charm program of CLEO-c would agree with our values determined by lattice QCD with exact chiral symmetry. Further, we note that in a recent 3-flavor unquenched
lattice QCD study [20] with $O(a^2)$ improved staggered light quarks and $O(a)$-improved charm quark, their results of $f_D$ and $f_{Ds}$ agree with our values.

Obviously, our next task is to determine $f_B$ and $f_{Bs}$ which are of fundamental importance, in view of their experimental determinations are still lacking. Since we will not use any approximations for the heavy b quark, our lattice spacing must be small enough such that $m_b a < 1$. Even though this does not seem to be formidable for $f_{Bs}$, it is unclear whether we can determine $f_B$ reliably via chiral extrapolation. Presumably, $f_B$ would behave like a function linear in $m_q$ for a wide range of $m_q$, similar to $f_D$ (Fig 6) and $f_K$ (Fig. 4), then one should be able to obtain a reliable chiral extrapolation even for data points with $m_q > m_s$. 

Figure 6: The D-meson decay constants $f_D a$ versus the bare quark mass $m_q a$. The solid line is the linear fit.
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References

[1] G. Bonvicini et al. [CLEO Collaboration], Phys. Rev. D 70, 112004 (2004)

[2] M. Ablikim et al. [BES Collaboration], Phys. Lett. B 610, 183 (2005)

[3] J. Z. Bai et al. [BES Collaboration], Phys. Rev. Lett. 74, 4599 (1995).

[4] K. Kodama et al. [Fermilab E653 Collaboration], Phys. Lett. B 382, 299 (1996)

[5] S. Eidelman et al. [Particle Data Group Collaboration], Phys. Lett. B 592, 1 (2004).

[6] D. B. Kaplan, Phys. Lett. B 288, 342 (1992); Nucl. Phys. Proc. Suppl. 30, 597 (1993).

[7] R. Narayanan and H. Neuberger, Nucl. Phys. B 443, 305 (1995)

[8] H. Neuberger, Phys. Lett. B 417, 141 (1998)

[9] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25, 2649 (1982)

[10] T. W. Chiu, Phys. Rev. Lett. 90, 071601 (2003)

[11] T. W. Chiu, Phys. Lett. B 552, 97 (2003)

[12] T. W. Chiu, hep-lat/0303008; Nucl. Phys. Proc. Suppl. 129, 135 (2004).

[13] N. I. Akhiezer, *Theory of approximation* (Dover, New York, 1992); *Elements of the theory of elliptic functions*, Translations of Mathematical Monographs, 79, (American Mathematical Society, Providence, RI. 1990).

[14] H. Neuberger, Phys. Rev. Lett. 81, 4060 (1998)

[15] T. W. Chiu, T. H. Hsieh, C. H. Huang and T. R. Huang, Int. J. Mod. Phys. C 14, 723 (2003)

[16] H. Neuberger, Int. J. Mod. Phys. C 10, 1051 (1999)

[17] T. W. Chiu and T. H. Hsieh, Phys. Rev. E 68, 066704 (2003)

[18] T. W. Chiu and T. H. Hsieh, Phys. Rev. D 66, 014506 (2002)

[19] S. R. Sharpe, Phys. Rev. D 46, 3146 (1992)

[20] J. N. Simone et al. [The Fermilab Lattice, MILC and HPQCD Collaborations], Nucl. Phys. Proc. Suppl. 140, 443 (2005)