EXTENDED EINSTEIN-MAXWELL MODEL

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Abstract

A self-consistent extended Einstein-Maxwell model for relativistic non-stationary polarizable-magnetizable anisotropic media is presented. Based on the analogy with relativistic extended irreversible (transient) thermodynamics, the extended constitutive equations for the electrodynamics of continua are formulated phenomenologically, the convective derivatives of the first, second, etc. orders being taken into account. The master equations for the gravity field contain a modified effective (symmetric) stress-energy tensor of the electromagnetic field in a material medium, the use of this tensor being motivated both by historical analogies and direct variational procedure. By way of example we consider the exact solution of the extended Einstein-Maxwell model, describing the isotropic cosmological model with hidden non-vanishing electromagnetic field, electric polarization and magnetization.

Key words: anisotropic medium, polarization, magnetization, Einstein-Maxwell theory, extended thermodynamics, extended constitutive equations.

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1 Introduction

Non-stationarity of electromagnetically active media is known to lead to complicated non-equilibrium phenomena, and one of the cooperative degrees of freedom, excited by the medium dynamics, relates to the phenomena of polarization and magnetization \[1\]-\[6\]. The Universe is a non-stationary system that contains numerous electrodynamical subsystems of different scales \[7\]-\[9\]. It is natural to expect that the cosmological non-steady background stimulates various evolutionary processes, involving the phenomena of polarization and magnetization. This circumstance attracts our attention to the exact non-stationary solutions to the self-consistent Einstein-Maxwell equations, describing the evolution of gravitating polarizable-magnetizable media. Following basic Einstein’s ideas, the stress and energy of polarization and magnetization themselves act as sources of the gravity field. These sources can not be negligible in comparison with the contribution of pure electromagnetic field: for instance, the plasma susceptibility parameters can be at least of the same order than one (or greater), i.e., the contributions of the electric polarization and pure electric field to the electric induction of the plasma are comparable. Thus, in order to formulate a self-consistent Einstein-Maxwell model one needs to find the adequate energy-momentum tensor of the electromagnetically active medium taking into account the polarization and magnetization of the medium.

In many cosmological models the total stress-energy tensor is presented as a sum of the stress-energy of the fluid (perfect or viscous) and of the energy-momentum tensor of pure electromagnetic field \[10\], \[11\]. To take into account the interacting (cross) terms, which include the polarization and magnetization of the medium, one needs to overcome the so-called “Minkowski-Abraham controversy”, which followed the famous papers of Minkowski \[12\], Einstein and Laub \[13\] and Abraham \[14\]. In the second half of the last century the problem of representation of the stress-energy tensor for the electromagnetically active media revived. The modified versions of the stress-energy tensor, related to the specific properties of the so-called ponderomotive force, have been presented and motivated by de Groot and Mazur \[15\], Grot \[16\], Israel \[17\], Maugin \[18\] and others (see, Refs. \[19\]-\[23\], \[5\], \[6\] for reviews). Some new aspects of the interrelation between the Minkowski and the Abraham versions of the electromagnetic stress-energy tensors and their modifications, can be found in \[24\]-\[31\]. However, the solution of the “Abraham-Minkowski controversy” might lay in a modification of terminology. Particularly, Gordon \[29\] introduced the label “pseudomomentum” in application to the Minkowski momentum. Nelson motivated the use of the term “wave momentum” for the sum of the Abraham momentum and of the Minkowski one \[30\]. Garrison and Chiao \[31\] used the terms “canonical and kinetic form” of electromagnetic momentum. It is clear now, that there are two different aspects in the problem under discussion. The first is connected with the correct definition of the flux four-vector of the electromagnetic field in the material medium as a part of the so-called electromagnetic energy - momentum tensor \[18\], which appears in balance equations. The structure of this tensor can be verified in the laboratory, and a few experiments have been proposed for this purpose (see, e.g., the review \[23\]). The second aspect is connected with the correct construction of the so-called effective stress-energy tensor of the electromagnetic field as an adequate part of the total stress-energy tensor, which appears as the source of the gravity field in the self-consistent Einstein-Maxwell model. It is interesting to clarify the structure of this tensor for the cosmological applications and to check it using the observational data. Here we focus on the problem of representation of the second quantity,
namely, on the effective stress-energy tensor of the electromagnetic field in a polarizable and magnetizable medium.

The paper is organized as follows. In the Section 2 we discuss the reconstruction of the electromagnetic energy-momentum tensor on the base of balance equations, which are the consequences of the Maxwell equations; as well, we concern some historical aspects, which are necessary for further consideration. In the Section 3 we introduce a new version of the effective stress-energy tensor. For this purpose, using Lagrange formalism, we derive directly this tensor for three well-known cases: vacuum, spatially isotropic medium and uniaxial anisotropic medium. Then, based on the obtained exact formulas, we propose an ansatz about the structure of the effective stress-energy tensor for general case and compare it with well-known tensors of this type. In the Section 4 we discuss the stationary and non-stationary phenomenological constitutive equations, linking the polarization - magnetization tensor and its derivatives with the Maxwell tensor and its derivatives. Thus, we introduce the so-called extended constitutive equations for relativistic electrodynamics of continuous media, and in this sense we get an extended Einstein-Maxwell model. In the Section 5 we discuss the example of the exact solution of the Einstein-Maxwell equations, based on the proposed effective stress-energy tensor of the electromagnetic field. To demonstrate the novelty of our approach, we discuss a particular example, when electric and magnetic fields, polarization and magnetization, as well as electric and magnetic inductions in the medium are non-vanishing, nevertheless, their cooperative contribution to the total stress-energy tensor is equal to zero. In this exact model the Einstein equations coincide with the ones from Friedmann-Lemaître-Robertson-Walker (FLRW) model, the electromagnetic field and electromagnetic induction being hidden from the point of view of space-time evolution. Section 6 summarises our findings.

2 Electromagnetic energy-momentum tensor

2.1 Balance equations

The standard phenomenological way to introduce the electromagnetic energy - momentum tensor in a material medium is connected with balance equations (see, e.g., [32] for details). The balance equations can be derived using the Maxwell equations

\[ \nabla_k H^{ik} = -\frac{4\pi}{c} I^i, \]  
\[ \nabla_i F_{kl} + \nabla_l F_{ki} + \nabla_k F_{li} = 0, \]

where \( F^{ik} \) is the Maxwell tensor and \( H^{ik} \) is an induction tensor [1, 2, 33]. The convolution of equation (1) with \( F_i^j \)

\[ F_i^j \nabla_k H^{ik} = -\frac{4\pi}{c} I^i F_i^j \]

can be transformed, using (2), into equations, which have an explicit divergence form

\[ \nabla_k T^{kl} = F^l. \]

The structure of (1) allows one to indicate the pair \( T^{kl} \) and \( F^l \) as a conjugated one, \( T^{kl} \) being an electromagnetic energy-momentum tensor, and \( F^l \) being a ponderomotive force. The choice
for $T^{kl}$ and $F^l$ is not unique. To fix this pair one should use some supplementary motivation (microscopic or phenomenological). Consider now six well-known instances.

2.1.1 Vacuum case

In vacuum ($I^i = 0, H^{ik} = F^{ik}$) equation (4) takes the conservation law form

$$\nabla_k T^{kl}_{(0)} = 0,$$

(5)

where

$$T^{kl}_{(0)} \equiv \frac{1}{4} g^{kl} F_{mn} F^{mn} - F^{km} F^l_m,$$

(6)

is the stress-energy tensor of the electromagnetic field. It is symmetric, traceless, conserved and does not depend explicitly on the velocity four-vector $U^i$ (of an observer or of the medium as a whole). The ponderomotive force in vacuum vanishes.

2.1.2 Minkowski version

By rearranging equation (3) as

$$\nabla_k \left[ \frac{1}{4} g^{kl} H_{mn} F^{mn} - H^{km} F_{lm} \right] = \frac{4}{c} I_i F^{il} + \frac{1}{4} \left[ F_{mn} \nabla^l H^{mn} - H_{mn} \nabla^l F^{mn} \right],$$

(7)

we obtain the conjugated pair $T^{kl}$ and $F^l$

$$T^{kl}_{(\text{Minkowski})} \equiv \frac{1}{4} g^{kl} H_{mn} F^{mn} - H^{km} F^l_m,$$

(8)

$$F^l_{(\text{Minkowski})} = \frac{4}{c} I_i F^{il} + \frac{1}{4} \left[ F_{mn} \nabla^l M^{mn} - M_{mn} \nabla^l F^{mn} \right].$$

(9)

Here $M^{ik} \equiv H^{ik} - F^{ik}$ is the polarization - magnetization tensor of the material medium. The tensor $T^{kl}_{(\text{Minkowski})}$ is traceless, but not symmetric. It does not include explicitly the velocity four-vector of the material medium.

2.1.3 Modified Minkowski tensor

Grot and Eringen [34], Israel [17], Maugin [18], proposed a modified version of $T^{kl}$ and $F^l$ in the balance equation (4). These authors discussed the following electromagnetic energy-momentum tensor and ponderomotive force

$$T^{kl}_{(\text{modifi1})} \equiv \frac{1}{4} g^{kl} F_{mn} F^{mn} - H^{km} F^l_m, \quad F^l = \frac{4}{c} I_m F^{ml} - \frac{1}{2} M^{mn} \nabla^l F_{mn}. $$

(10)

This tensor, whose trace does not vanish, $T^{kl}_{(\text{modifi1})}$, differs from (8) in the first term.
2.1.4 Version of Hehl and Obukhov

Hehl and Obukhov motivated in [28] the following choice for $T_{kl}$ and $F^l$

$$T_{kl}^{(modif2)} \equiv \frac{1}{4} g^{kl} F_{mn} F_{mn} - F^{km} F^l_m, \quad F^l = \frac{4\pi}{c} I_m F^{ml} + F^l_m \nabla_k M^{km}. \quad (11)$$

The structure of this stress-energy tensor coincides with that of the vacuum, thus, it is symmetric, traceless and does not depend on $U^i$. Nevertheless, in contrast to the vacuum case, it is not a conserved quantity, and the ponderomotive force is linear in the divergence of the polarization-magnetization tensor.

2.1.5 Abraham’s version

Abraham [14] proposed to use the symmetric electromagnetic energy-momentum tensor, which depends explicitly on the $U^i$, velocity four-vector of the medium as whole. When the medium is spatially isotropic and homogeneous, the corresponding tensor $T_{ik}^{(Abraham)}$ reads

$$T_{ik}^{(Abraham)} = T_{ik}^{(Minkowski)} + (n^2 - 1) \Omega^i U^k, \quad (12)$$

where $n$ is a refractive index of the medium and the vector $\Omega^i$ is

$$\Omega^i = U_l (H^l U^m + H^m U^l U^i) F_{ms} U^s. \quad (13)$$

Here and below we shall use the normalization $U^k U_k = 1$. Since $\Omega^i U_i = 0$ the tensor $T_{ik}^{(Abraham)}$ is traceless. The corresponding ponderomotive force can be obtained from (4).

2.1.6 Version of de Groot and Suttorp

De Groot and Suttorp [36], based on a microscopic motivation, proposed the following electromagnetic energy-momentum tensor

$$T_{kl}^{(modif3)} \equiv T_{kl}^{(modif1)} - U^l U^m (F^{kn} M_{nm} - M^{kn} F_{nm}) + U^k U^l U^m U_n F_{ms} M^{sn}. \quad (14)$$

This tensor also depends explicitly on the velocity four-vector. The corresponding ponderomotive force can be obtained from (4).

Remark on the microscopic and macroscopic electrodynamics.

As it was emphasized, e.g., in [36, 24], the problem of the choice of the electromagnetic energy-momentum tensor $T^{kl}$ is connected with the basic microscopic model, as well as, with the averaging procedure of the microscopic Maxwell equations (the discussion about averaging procedure see, e.g., in [35]). Indeed, let the microscopic electromagnetic field $f_{ik}$ can be represented as a sum of a mean field $F_{ik}$ and a fluctuation terms $\xi_{ik}$ with vanishing average value $\langle \xi_{ik} \rangle = 0$. Then, one obtains that

$$T^{ik} \equiv \langle \frac{1}{4} g^{ik} f_{mn} f^{mn} - f^{im} f^k_m \rangle = T^{ik}_{(0)} + \langle \tau^{ik} \rangle, \quad (15)$$

where $T^{ik}_{(0)}$ is given by (6). The second term

$$\langle \tau^{ik} \rangle \equiv \langle \frac{1}{4} g^{ik} \xi_{mn} \xi^{mn} - \xi^{im} \xi^k_m \rangle \quad (16)$$

is very sensible to the averaging procedure and essentially depends on the microscopic model of electromagnetic interactions in the medium.
2.2 “DEHB” - representation of the energy-momentum tensor

The tensor $T^{ik}$ can be rewritten in terms of four-vectors $D^i, E^i, H^i$ and $B^i$, which play significant role in the covariant electrodynamics of continuous media [18]. The definitions of these four-vectors are well-known [37]

$$D^i \equiv H^i k U_k, \quad H^i \equiv H^* k U_k, \quad E^i \equiv F^i k U_k, \quad B^i \equiv F^* k U_k.$$  \hspace{1cm} (17)

$D^i, E^i, H^i$ and $B^i$ are orthogonal to the $U^i$, a four-vector of macroscopic velocity of the medium. In its turn, $F^i k$ and $H^i k$ can be represented as

$$F^i k = E^i U^k - E^k U^i - \eta^{ik} j B_j, \quad H^i k = D^i U^k - D^k U^i - \eta^{ik} j H_j,$$ \hspace{1cm} (18)

where

$$\eta^{ik} j \equiv \epsilon^{ik} j s U_s, \quad \epsilon^{ik} j s \equiv E^{ik} j s \sqrt{-g}.$$  \hspace{1cm} (19)

$\epsilon^{ik} j s$ is the Levi-Civita tensor and the term $E^{ik} j s$ is the completely skew-symmetric Levi-Civita symbol with $E^{0123} = 1$. This tensor provides the dualization procedure: $F^{* k} \equiv \frac{1}{2} \epsilon^{ik} j s F^{ij}. \hspace{1cm}$

By means of (18) the tensors $T^{pq}$ (Minkowski), $T^{pq}$ (Abraham), $T^{pq}$ (modif1), $T^{pq}$ (modif2), $T^{pq}$ (modif3) can be represented in terms of four-vectors $D^i, E^i, H^i$ and $B^i$. For instance, the Minkowski tensor has the form

$$T^{pq} \text{(Minkowski)} = \left( \frac{1}{2} g^{pq} - U^p U^q \right) (D^m E_m + H^m B_m) -$$

$$(D^p E^q + B^p H^q) - U^p \eta^{qmn} E_m H_n - U^q \eta^{pmn} D_m B_n.$$ \hspace{1cm} (20)

Taking into account the standard decomposition of this tensor

$$T^{ik} \text{(Minkowski)} = W_{(em)} U^i U^k + U^i I^{(1)}_i + U^k I^{(2)}_i + \mathcal{P}^{ik} \text{(Minkowski)},$$ \hspace{1cm} (21)

one can conclude that the energy density scalar $W_{(em)}$, the first and second flux four-vectors $I^{(1)}_i, I^{(2)}_i$ and the stress tensor $\mathcal{P}^{ik} \text{(Minkowski)}$ read, respectively,

$$W_{(em)} \equiv U_p T^{pq} \text{(Minkowski)} U_q = \frac{1}{2} \left( D^m E_m + H^m B_m \right),$$ \hspace{1cm} (22)

$$I^{(1)}_i \equiv U_p T^{pq} \text{(Minkowski)} \Delta^k_q = -\eta^{k} mn D^m B^n,$$ \hspace{1cm} (23)

$$I^{(2)}_i \equiv \Delta^i q T^{pq} \text{(Minkowski)} U_q = -\eta^{i} mn E^m H^n,$$ \hspace{1cm} (24)

$$\mathcal{P}^{ik} \text{(Minkowski)} \equiv \Delta^i q T^{pq} \text{(Minkowski)} \Delta^k_q = \frac{1}{2} \Delta^{ik} \left( D^m E_m + H^m B_m \right) - \left( D^i E^k + B^i H^k \right).$$ \hspace{1cm} (25)

Note that the Minkowski [8] and Abraham [12] versions of the energy-momentum tensor share the same quantity $W_{(em)}$. For the Abraham (symmetric) version of the electromagnetic energy-momentum tensor, $I^{(1)}_i$ and $I^{(2)}_i$ coincide and

$$I^{(1)}_i = I^{(2)}_i = -\eta^{i} mn E^m H^n.$$ \hspace{1cm} (26)
The stress tensor $P_{ij}^{(Abraham)}$ coincides with the symmetrized one, which was obtained by Minkowski [23]. In terms of three-vectors $\vec{E}$, $\vec{B}$, $\vec{D}$ and $\vec{H}$ the flux three vectors proposed by Minkowski, read, respectively,

$$I^{(1)} = [\vec{D}, \vec{B}], \quad I^{(2)} = [\vec{E}, \vec{H}],$$

where $[\vec{D}, \vec{B}]$ denotes the vectorial product of the three-vectors $\vec{D}$ and $\vec{B}$. In spatially isotropic medium one has $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$, where $\varepsilon$ and $\mu$ are the scalars of electric and magnetic permeability, respectively. Thus, using the standard definition for the Poynting flux three-vector, $\vec{S}_{(Poynting)} = [\vec{E}, \vec{H}]$, one can write for the corresponding three-vectors of the momentum of the electromagnetic field

$$\vec{S}_{(Abraham)} = \vec{S}_{(Poynting)}; \quad \vec{S}_{(Minkowski)} = \varepsilon \mu \vec{S}_{(Poynting)}.$$

### 3 Effective stress-energy tensor

#### 3.1 Lagrange formalism

The Einstein field equations

$$R^{ik} - \frac{1}{2} g^{ik} R = \Lambda g^{ik} + \kappa T^{ik}_{(total)},$$

must have on their right-hand side the so-called total stress-energy tensor $T^{ik}_{(total)}$, which must be symmetric by definition and divergence-free due to the Bianchi identities [38], i.e.,

$$T^{ik}_{(total)} = T^{ki}_{(total)}, \quad \nabla_k T^{ik}_{(total)} = 0.$$

$R^{ik}$ is the Ricci tensor, $R$ is the Ricci scalar, associated with metric $g_{ik}$ and $\Lambda$ is the cosmological constant. Following the standard variation procedure one can define $T^{ik}_{(total)}$ as

$$T^{ik}_{(total)} \equiv - \frac{1}{\sqrt{-g}} \delta \left( \sqrt{-g} \mathcal{L} \right),$$

where the scalar $\mathcal{L}$ denotes the Lagrangian of the whole system, and includes the terms related to the electromagnetic field, the polarization and magnetization of the medium. The main problem is how to separate the contribution of the pure electromagnetic field, the contribution of the polarization and the magnetization and the contribution of the pure matter. This problem seems analogous to the problem of separation of pure gravitational energy-momentum and the energy-momentum of the medium, which is characterized by the gravitational self-interaction. One can extract from the total stress-energy tensor $T^{ik}_{(total)}$ the electromagnetic energy-momentum tensor $T^{ik}$ (see, previous Section), which, in general, is not necessarily symmetric and traceless. On other hand, based on the variation procedure with respect to metric, we can draw from $T^{ik}_{(total)}$ the so-called effective stress-energy tensor of the electromagnetic field $T^{ik}_{(eff)}$, which is symmetric and traceless by definition. Below we will distinguish between $T^{ik}$...
and $T_{\text{eff}}^{ik}$. In order to motivate our ansatz about the effective stress-energy tensor, let us, first, consider the variation procedure of its derivation for the simplest action functional

$$S[F_{mn}, g_{pq}] = \int d^4x \sqrt{-g} \left\{ \frac{R + 2\Lambda}{\kappa} + L_{\text{matter}} + \frac{1}{2} C^{ikmn} F_{ik} F_{mn} \right\}.$$  \hspace{1cm} (32)

Here $C^{ikmn}$ is the linear response tensor, which describes the influence of matter to the electromagnetic field. This tensor has the following symmetries

$$C_{ikmn} = -C_{kimn} = -C_{iknm} = C_{mnik}.$$  \hspace{1cm} (33)

Variation of $S[F_{mn}, g_{pq}]$ with respect to the four-vector electromagnetic potential $A_i$ gives the Maxwell equations

$$\nabla_k H^{ik} = 0, \quad H^{ik} \equiv C^{ikmn} F_{mn},$$  \hspace{1cm} (34)

where $H^{ik}$ is the induction tensor and the current four-vector $I^i$ is absent. The variation with respect to metric tensor $g_{pq}$ yields the Einstein equations (29) with explicit decomposition

$$T_{\text{(total)}}^{pq} = T_{\text{matter}}^{pq} + T_{\text{eff}}^{pq}.$$  \hspace{1cm} (35)

As usual, the symmetric stress-energy tensor of the material medium $T_{\text{matter}}^{pq}$ reads

$$T_{\text{matter}}^{ik} = W U^i U^k + q^i U^k + q^k U^i - P \Delta^{ik} + \Pi^{ik},$$  \hspace{1cm} (36)

where $W$ is an energy density scalar of the matter, $U^i$ is a macroscopic velocity four-vector of the medium as whole, $q^i$ is a heat-flux four-vector, $P$ is the Pascal pressure, $\Delta^{ik} \equiv g^{ik} - U^i U^k$ is a projector and $\Pi^{ik}$ is an anisotropic pressure tensor. The form of the electromagnetic part of the total stress-energy tensor, $T_{\text{eff}}^{pq}$, depends on the suggestions about a structure of $C^{ikmn}$ tensor. When $C^{ikmn}$ incorporates the metric $g_{pq}$ only, the effective stress-energy tensor takes the form

$$T_{\text{eff}}^{pq} = \frac{1}{4} g^{pq} C_{ikmn} F_{ik} F_{mn} - \frac{1}{2} K_{pqikmn} F_{ik} F_{mn},$$  \hspace{1cm} (37)

where the tensor $K_{pqikmn}$ is a formal variation derivative

$$K_{pqikmn} \equiv \frac{\delta}{\delta g_{pq}} C^{ikmn}.$$  \hspace{1cm} (38)

$T_{\text{eff}}^{pq}$ is, by definition, a symmetric tensor, whose trace vanishes when

$$g_{pq} K_{pqikmn} = 2 C^{ikmn}.$$  \hspace{1cm} (39)

In this paper we assume, that the tensor $C^{ikmn}$ contains the metric only. When $C^{ikmn}$ contains the Riemann tensor, the Ricci tensor and the Ricci scalar, the corresponding effective stress-energy tensor includes the covariant derivatives of the Maxwell tensor up to the second order, and we deal with the so-called non-minimal Einstein - Maxwell theory \[39, 40\]. When $C^{ikmn}$ includes the covariant derivative of the velocity four-vector $\nabla_i U_k$, the effective stress-energy tensor involves first covariant derivative of $F_{mn}$ and we deal with dynamo-optical effects \[41\]. But it is worth stressing once again that here we restrict ourselves by the first case only, namely, when $C^{ikmn} = C^{ikmn}[g_{pq}]$. 

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3.2 Structure of $C^{ikjl}$

When $C^{ikjl} = C^{jilk}$, a standard decomposition of the $C^{ikjl}$ tensor in terms of dielectric permeability tensor $\varepsilon^{ik}$, magnetic permeability $(\mu^{-1})_{ik}$ tensor and magneto-electric tensor $\nu^k_i$ is permissible

$$C^{ikmn} = \frac{1}{2} \left[ \varepsilon^{im}U^kU^m - \varepsilon^{in}U^kU^m + \varepsilon^{km}U^iU^m - \varepsilon^{km}U^iU^m \right] -$$

$$\frac{1}{2} \eta^{ik} (\mu^{-1})_{ls} \eta^{nm} - \frac{1}{2} \left[ \eta^{ik}(U^m \nu^i_l - U^m \nu^m_l) + \eta^{lm}(U^i \nu^k_l - U^k \nu^m_l) \right],$$

where

$$\varepsilon^{im} = 2C^{ikmn}U_kU_n, \quad (\mu^{-1})_{pq} = -\frac{1}{2} \eta_{pik}C^{ikmn}\eta_{mnq}, \quad \nu^m_p = \eta_{pik}C^{ikmn}U_n.$$  \hfill (41)

The tensors $\varepsilon_{ik}$ and $(\mu^{-1})_{ik}$ are symmetric, but $\nu^k_i$ is, in general, non-symmetric. These three tensors are orthogonal to $U_i$,

$$\varepsilon_{ik}U^k = 0, \quad (\mu^{-1})_{ik}U^k = 0, \quad \nu^k_iU^i = 0 = \nu^k_iU_k.$$  \hfill (42)

The case $C^{ikjl} \neq C^{jilk}$ is described in detail by Hehl and Obukhov [33]. The decomposition (40) leads to the well-known formulas for the four-vectors of electric induction $D^i$, magnetic field $H_i$, electric field $E^i$ and magnetic induction $B_i$:

$$D^i = \varepsilon^{ik}E_k - \nu^k_iB^k, \quad H_i = \nu^k_iE_k + (\mu^{-1})_{ik}B^k.$$  \hfill (43)

The material tensors $\varepsilon^{ik}$, $(\mu^{-1})_{ik}$, $\nu^k_i$ and $C^{ikmn}$ can be decomposed using the standard tetrad representation

$$S^{i_1i_2\ldots i_n} = X^{i_1}_{(a_1)}X^{i_2}_{(a_2)}\ldots X^{i_n}_{(a_n)}S^{(a_1)(a_2)\ldots(a_n)}.$$  \hfill (44)

Here the symbol $X^{i}_{(a)}$ denotes the set of the four tetrad vectors, whose index $(a)$ runs over $(0), (1), (2), (3)$, and $X^{i}_{(0)} \equiv U^i$. These four four-vectors are assumed to satisfy the orthogonality - normalization rules

$$g_{ik}X^i_{(a)}X^k_{(b)} = \eta_{(a)(b)}, \quad \eta^{(a)(b)}X^p_{(a)}X^q_{(b)} = g^{pq},$$

where $\eta_{(a)(b)}$ denotes the Minkowski matrix, diagonal $(1, -1, -1, -1)$. Since the tetrad four-vectors are linked by the relation containing the metric, for further consideration we have to define the formula for the variation $\frac{\delta X^i_{(a)}}{\delta g^{pq}}$.

3.3 Variation of the tetrad vectors

Varying the relations (45) with respect to the metric, we obtain

$$X_{k(b)}\delta X^k_{(a)} + X_{k(a)}\delta X^k_{(b)} = -X^i_{(a)}X^k_{(b)}\delta g_{ik}.$$  \hfill (47)

The variation of (46) yields

$$\delta g^{pq} = \eta^{(a)(b)} \left[ X^q_{(b)}\delta X^p_{(a)} + X^p_{(a)}\delta X^q_{(b)} \right].$$  \hfill (48)
The relations (47) and (48) are equivalent, since
\[ g^{ik} g_{kj} = \delta_j^i \Rightarrow \delta g_{ik} = -\delta g_{pq} g_{pi} g_{qk}. \] (49)

The variation of arbitrary origin \( \delta X^i_{(a)} \) (not necessarily caused by the metric variation) can be represented as a linear combination of the tetrad four-vectors:
\[ \delta X^i_{(a)} = X^i_{(c)} Y^c_{(a)}. \] (50)

The tetrad tensor \( Y^c_{(a)} \) is not generally symmetric. Using the convolution of (48) with tetrad vectors, we obtain
\[ Y^{(a)(b)} + Y^{(b)(a)} = \delta g_{pq} X^p_{(a)} X^q_{(b)}, \] (51)
where we use the standard rules for the indices, e.g., \( X^f_{(q)} g_{qm} X^m_{(b)} \). Consequently, the symmetric part of the quantity \( Y^{(a)(b)} \), indicated as \( Z^{(a)(b)} \), can be readily found:
\[ Z^{(a)(b)} = \frac{1}{2} \delta g_{pq} X^p_{(a)} X^q_{(b)}, \] (52)
and the law (50) reads now
\[ \delta X^i_{(a)} = \frac{1}{4} \delta g_{pq} \left[ X^p_{(a)} \delta^i_q + X^q_{(a)} \delta^i_p \right] + X^i_{(c)} Z_{(a)}^{(c)}. \] (53)

Here \( Z^{(c)}_{(a)} \) is the skew-symmetric part of \( Y^c_{(a)} \), i.e., \( 2 Z^{(c)}_{(a)} = Y_{(a)(c)} - Y_{(a)(c)} \). Therefore, the variation of the metric produces the variation of the tetrad, described by (53) with vanishing skew-symmetric part \( Z^{(a)(c)} \). Thus, one finally has
\[ \frac{\delta X^i_{(a)}}{\delta g_{pq}} = \frac{1}{4} \left[ X^p_{(a)} \delta^i_q + X^q_{(a)} \delta^i_p \right], \] (54)
and we can use this formula for the variation of the four-velocity vector \( U^i \) and for the variation of the space-like vectors \( X^i_{(a)} \) (\( \alpha = 1, 2, 3 \)).

Note that in [42] the author, considering the stress-energy tensor for the spinor field, has used the formula for the variation of tetrad, which can be easily transformed into (53). The authors of [24] - [27] use a different formula for the variation of the four-velocity vector:
\[ \delta U^i(s) = \delta \left( \frac{dx^i}{ds} \right) = -\frac{1}{2} U^i \delta g_{pq} U^p U^q. \] (55)

This formula is derived on the base of kinematic representation of the four-velocity \( U^i(s) \) as a tangent vector for the observer world-line. Such a representation is adequate to the Lagrangian variation procedure with respect to coordinates. However, it is not possible to use the same method for the calculation of the variation of the space-like vectors \( X^i_{(a)} \).

Our approach is based on the consideration of the four vector fields \( X^i_{(a)}(x) \), subjected to the orthogonality and normalization conditions. This representation is appropriate for the procedure of variation of the Lagrangian [42] with respect to the \( A_i(x) \) and \( g_{pq}(x) \) fields, as well as with respect to spinor field, as it follows from [42].
3.4 Three standard examples

3.4.1 Pure vacuum

The tensor $C^{ikmn}$ of vacuum must be constructed from the metric only,

$$C^{ikmn} = \frac{1}{2} \left( g^{im} g^{kn} - g^{in} g^{km} \right). \quad (56)$$

Then

$$K^{pqikmn} = \frac{1}{2} \left( \delta^i_p \delta^m_q g^{kn} + \delta^k_p \delta^m_q g^{in} - \delta^i_p \delta^m_q g^{km} - \delta^k_p \delta^m_q g^{ik} \right), \quad (57)$$

and the direct variation in (38) gives the standard formula for the symmetric traceless stress-energy tensor of the electromagnetic field in vacuum (6).

3.4.2 Spatially isotropic medium

By contrast to the vacuum case the Lagrangian of electromagnetic field in the spatially isotropic medium should contain one supplementary vector, namely, the velocity four-vector of the medium, $U^i$. The linear response tensor has the following form [18, 5]:

$$C^{ikmn} = \frac{1}{2} \mu \left[ \left( g^{im} g^{kn} - g^{in} g^{km} \right) + \left( \varepsilon - 1 \right) \left( g^{im} U^k U^n g^{kn} + g^{kn} U^i U^m - g^{km} U^i U^n \right) \right], \quad (58)$$

where $\varepsilon$ and $\mu$ are the dielectric and magnetic permeability scalars, respectively. The $\varepsilon^{ik}$, $(\mu^{-1})_{pq}$ and $\nu^i_k$ tensors, entering the linear response tensor (see, (41)) read

$$\varepsilon^{ik} = \varepsilon \Delta^{ik}, \quad (\mu^{-1})_{pq} = \frac{1}{\mu} \Delta_{pq}, \quad \nu^i_k = 0. \quad (59)$$

Since the velocity four-vector is normalized as $U_k U^k = 1$, we should use the variation of $U^i$ with respect to $g_{pq}$ in order to obtain the $K^{pqikmn}$ tensor (38). To do this we employ the formula

$$\delta U^i = \frac{1}{4} \delta g^{pq} \left( U_p \delta_q^i + U_q \delta_p^i \right) \quad (60)$$

as a particular case of (54) with $(a) = (0)$. A straightforward calculation shows that the corresponding effective stress-energy tensor can be written as

$$T^{kl}_{(eff)} \equiv \frac{1}{4} g^{kl} C^{mnpq} F_{mn} F_{pq} - \frac{1}{2} \left( C^{kmpq} F^l_m + C^{lmpq} F^k_m \right) F_{pq}. \quad (61)$$

3.4.3 Medium with uniaxial symmetry

The tensor $C^{ikmn}$ for the uniaxial symmetry contains not only the velocity four-vector $U^i$, but one supplementary space-like four-vector, (say, $X^i$) as well. This vector is normalized according to $g_{ik} X^i X^k = -1$, and orthogonal to $U^i$, i.e., $g_{ik} X^i U^k = 0$. Thus, to calculate the tensor $K^{pqikmn}$ we must find the variation of $X^i$ with respect to the metric $g_{pq}$ in addition to the variation $\delta U^i$. For this purpose we use the formula

$$\delta X^i = \frac{1}{4} \delta g^{pq} \left( X_p \delta_q^i + X_q \delta_p^i \right) \quad (62)$$
as a particular case of (54). To modify the $C^{ikmn}$ tensor for the uniaxial case we follow the procedure described in [43]. First, we modify (59) by

$$
\varepsilon^{ik} = \varepsilon \left( \Delta^{ik} + \xi X^i X^k \right), \quad (\mu^{-1})_{pq} = \frac{1}{\mu} \left( \Delta_{pq} + \zeta X_p X_q \right),
$$

(63)

and consider the medium without magnetoelectric cross-terms, i.e., require the following relation to hold $\nu^{ik} = 0$. Here $\xi$ is a coefficient of anisotropy of the dielectric permeability. In the uniaxial case it is unique, and can be defined, for instance, as $\xi = 3 - \varepsilon^k_k / \varepsilon$. This parameter vanishes when medium is spatially isotropic. $\zeta$ is a corresponding coefficient of anisotropy of the magnetic permeability. Then, we use (63) in (40), and after a long but otherwise straightforward calculation we obtain the expressions for $K^{pqikmn}$ and $T^{pq}_{(eff)}$. The result was to be expected: we recover the formula (61) for the $T^{pq}_{(eff)}$.

### 3.4.4 General case

In general, the tensor of linear response can be represented as

$$
C^{ikmn} = X^i_{(a)} X^k_{(b)} X^m_{(c)} X^n_{(d)} C^{(a)(b)(c)(d)}.
$$

(64)

Our ansatz is that the tetrad quantity $C^{(a)(b)(c)(d)}$ does not depend on the metric, and the variation with respect to $g_{pq}$ reduces to the variation of the tetrad four-vectors only, i.e., to the formula (54). Under such an assumption using (54) and (45) we obtain again the expression (61).

### 3.5 Ansatz

Based on the direct derivation of the tensor $T_{(eff)}^{kl}$ together with the Lagrangian (32), and taking into account the common structure (61) for all three well-known models with the constitutive equations $H^{ik} = C^{ikmn} F_{mn}$, in which $C^{ikmn}$ contains the metric only, we propose to consider the following effective stress-energy tensor of the electromagnetic field with general constitutive equations

$$
T_{(eff)}^{kl} \equiv \frac{1}{4} g^{kl} H_{mn} F_{mn}^{mn} - \frac{1}{2} \left( H^{km} F_{m}^{l} + H^{lm} F_{k}^{m} \right).
$$

(65)

If we use the effective stress-energy tensor $T_{(eff)}^{kl}$ as an electromagnetic energy - momentum tensor $T^{kl}$ in the relation (4), the corresponding ponderomotive force takes the form

$$
F_i^{(eff)} = \frac{4\pi}{c} I^i F_{it} + \frac{1}{4} \left[ F_{mn} \nabla_t M^{mn} - M_{mn} \nabla_t F^{mn} \right] + \frac{1}{2} \nabla_k \left[ M^{km} F_{lm} - F^{km} M_{lm} \right].
$$

(66)

The effective stress-energy tensor (65) coincides with the symmetrized Minkowski electromagnetic energy-momentum tensor (8). Thus, it is manifestly symmetric, traceless and does not depend explicitly on the choice of $U^i$. 

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3.5.1 DEHB - representation of the effective stress-energy tensor

In terms of four-vectors \(D^i, E^i, H^i\) and \(B^i\) the tensor \(T^{pq}_{(\text{eff})}\) can be represented as follows

\[
T^{pq}_{(\text{eff})} = \left( \frac{1}{2} g^{pq} - U^p U^q \right) (D^m E_m + H^m B_m) - \frac{1}{2} (D^p E^q + D^q E^p + H^p B^q + H^q B^p) - \frac{1}{2} (U^p \eta^{mnp} + U^q \eta^{mpn}) (D_m B_n + E_m H_n) .
\] (67)

The energy density scalar \(W_{(\text{eff})}\), the flux four vector \(I^i_{(\text{eff})}\) and the stress tensor \(P^{ik}_{(\text{eff})}\) read, respectively,

\[
W_{(\text{eff})} \equiv U_p T^{pq}_{(\text{eff})} U_q = -\frac{1}{2} (D^m E_m + H^m B_m) ,
\] (68)

\[
I^i_{(\text{eff})} \equiv U_p T^{pq}_{(\text{eff})} \Delta^i_q = \frac{1}{2} \eta^{i mn} (D^m B^n + E^m H^n) ,
\] (69)

\[
P^{ik}_{(\text{eff})} \equiv \Delta^i_p T^{pq}_{(\text{eff})} \Delta^k_q = \frac{1}{2} \Delta^{ik} (D^m E_m + H^m B_m) - \frac{1}{2} \left( D^i E^k + D^k E^i + H^i B^k + H^k B^i \right) .
\] (70)

Note that the energy density scalar \(W_{(\text{eff})}\) coincides with \(W_{(\text{em})}\) given by (22), obtained for the Minkowski and the Abraham tensors. The flux four-vector \(I^i_{(\text{eff})}\) is one half of the sum \(I^i_{(1)}\) and \(I^i_{(2)}\) (23), (24). The stress tensor \(P^{ik}_{(\text{eff})}\) is the symmetrized one, obtained by Minkowski, and coincides with the Abraham stress-tensor. In spatially isotropic medium in the three-vector symbols we have for our definition of the effective stress-energy tensor

\[
\bar{S}_{(\text{eff})} = \frac{1}{2} (\varepsilon \mu + 1) \bar{S}_{(\text{Poynting})} = \frac{1}{2} (\varepsilon \mu + 1) \bar{S}_{(\text{Abraham})} = \frac{1}{2} \frac{\varepsilon \mu + 1}{\varepsilon \mu} \bar{S}_{(\text{Minkowski})} .
\] (71)

4 Constitutive equations

The ansatz (65) concerning the structure of the effective stress-energy tensor of the electromagnetic field in a material medium allows us to consider self-consistently an Einstein-Maxwell model for the evolution of non-stationary electromagnetically active media. This model includes, first, the Einstein field equations (29) with (35), (36) and (65), secondly, the Maxwell equations (1) and (2), thirdly, the so-called constitutive equations, linking the polarization - magnetization tensor \(M^{ik} = H^{ik} - F^{ik}\) and Maxwell tensor \(F_{mn}\). The specific feature of the interaction between a non-stationary material medium and the electromagnetic field is the dynamics of polarization and magnetization, and the constitutive equations must reflect the existence of an inertia in the electromagnetic response of the medium. In order to motivate our ansatz about the constitutive equations, we briefly consider the well-known classical analogs.

4.1 Classical and relativistic extended thermodynamics as a hint for the construction of the covariant extended continuum electrodynamics

4.1.1 Rheological models

The well-known Hook law

\[
\sigma^{\alpha \beta} = C^{\alpha \beta \gamma \rho} e_{\gamma \rho} \equiv \sigma^{\alpha \beta}_{(\text{stationary})} ,
\] (72)
describing the linear relation between the stress tensor $\sigma^{\alpha\beta}$ and the deformation tensor $e_{\gamma\rho}$, is considered to be the simplest stationary linear constitutive law in rheology [4]. Greek indices run over 1, 2, 3 and describe the spatial coordinates. The material tensor $C^{\alpha\beta\gamma\rho}$ is symmetric, i.e.,

$$C^{\alpha\beta\gamma\rho} = C^{\beta\alpha\gamma\rho} = C^{\alpha\beta\rho\gamma} = C^{\gamma\rho\alpha\beta}, \quad (73)$$

and includes the elastic coefficients of the medium [47]. When the medium is non-stationary, the difference $\sigma^{\alpha\beta} - \sigma^{\alpha\beta}_{(\text{stationary})}$ becomes a function of time due to inertia effects. Maxwell’s viscosity model [44] assumes, that this difference is proportional to the time derivative of the stress tensor. In the Kelvin-Voigt model (we follow the terminology of Ref. [45]) the difference $\sigma^{\alpha\beta} - \sigma^{\alpha\beta}_{(\text{stationary})}$ is proportional to the time derivative of the deformation tensor. The Poynting-Thomson model (see, [45]) combines Maxwell and Kelvin-Voigt models and is characterized by the constitutive law

$$\sigma^{\alpha\beta} - C^{\alpha\beta\gamma\rho} e_{\gamma\rho} = -\Gamma^{\alpha\beta\gamma\rho} \frac{\partial}{\partial t} \sigma^{\gamma\rho} + \lambda^{\alpha\beta\gamma\rho} \frac{\partial}{\partial t} e_{\gamma\rho}. \quad (74)$$

Here the first term in the right-hand-side corresponds to the Maxwell model and the second term relates to the Kelvin-Voigt contribution. The quantities $\Gamma^{\alpha\beta\gamma\rho}$ and $\lambda^{\alpha\beta\gamma\rho}$ represent the tensors of relaxation parameters for stresses and strains, respectively. These three basic models provide a rule for the next generalizations of the rheological models (see, e.g., [3]), namely, to introduce the successive derivatives of the stress tensor and/or strain tensor (see, e.g., Jeffreys model [45]), by the same method, used for the first derivative.

### 4.1.2 Heat conduction model

The Fourier law

$$\bar{q} = -\lambda \bar{\nabla} T \equiv \bar{q}_{(\text{stationary})} \quad (75)$$

is a stationary constitutive law for the heat conduction, connecting the heat flux vector $\bar{q}$ with the spatial gradient of the temperature. Taking into account the inertial properties of the heat propagation, Cattaneo [46] considered the difference $\bar{q} - \bar{q}_{(\text{stationary})}$ to be linear in the time derivative of the heat flux vector

$$\bar{q} + \lambda \bar{\nabla} T = -\tau(q) \frac{\partial \bar{q}}{\partial t}. \quad (76)$$

The latter expression, supplemented with the balance equation for the internal energy leads to the hyperbolic equation, governing the temperature evolution

$$\tau(q) \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \chi \bar{\nabla}^2 T, \quad (77)$$

which generalizes the standard parabolic equation. By extending of the constitutive law for the anisotropic media [47] and using Cattaneo’s proposal of the inertia of heat and its extensions [48], one obtains the generalized Fourier-Cattaneo law

$$q^\alpha = -\lambda^{\alpha\beta} \nabla_\beta T - Q^\alpha_\beta \frac{\partial}{\partial t} q^\beta + \chi^{\alpha\beta} \nabla_\beta \frac{\partial T}{\partial t} + \ldots \quad (78)$$
4.1.3 Relaxation of the electric polarization and magnetization

Taking into account the delay in the response of the medium to the applied electromagnetic field, one can use the simplest extended constitutive equations containing the first time derivative

\[ \vec{P} = \chi \vec{E} - \tau_p \frac{\partial}{\partial t} \vec{P}, \quad \vec{M} = \chi \vec{H} - \tau_m \frac{\partial}{\partial t} \vec{M}. \] (79)

These equations are easily transformed into the general relaxation equation

\[ \frac{\partial}{\partial t} \vec{\xi} = -\frac{1}{\tau} (\vec{\xi} - \vec{\xi}_{(\text{stationary})}) \] (80)

for the polarization-magnetization, see, e.g., [49]. By analogy with rheology, Kluitenberg [50, 51, 52] generalized the equations for polarization and magnetization vectors of classical electrodynamics as

\[ P^\alpha = \chi^\alpha_\beta E^\beta - \sum_{(m)=(1)}^{(k)} A_{\beta(m)}^\alpha \left( \frac{\partial}{\partial t} \right)^{(m)} P^\beta + \sum_{(m)=(1)}^{(s)} B_{\beta(m)}^\alpha \left( \frac{\partial}{\partial t} \right)^{(m)} E^\beta, \] (81)

\[ M^\alpha = \zeta^\alpha_\beta H^\beta - \sum_{(m)=(1)}^{(k)} C_{\beta(m)}^\alpha \left( \frac{\partial}{\partial t} \right)^{(m)} M^\beta + \sum_{(m)=(1)}^{(s)} D_{\beta(m)}^\alpha \left( \frac{\partial}{\partial t} \right)^{(m)} H^\beta. \] (82)

If the medium is anisotropic and there are cross-effects, such as pyro - electricity and pyro - magnetism, piezo - electricity and piezo - magnetism [47, 53], magneto - electricity [54], electro - striction and magneto - striction [47, 53], etc., then the "source" \( \chi^\alpha_\beta E^\beta \) in (81) and \( \zeta^\alpha_\beta H^\beta \) in (82) must be supplemented by additional terms

\[ \chi^\alpha_\beta E^\beta \Rightarrow \chi^\alpha_\beta E^\beta + \pi^\alpha (T - T_0) + d^\alpha_{\beta \gamma} \sigma^{\beta \gamma} + \nu^\alpha_\beta H^\beta + Q^\alpha_{\beta \gamma \rho} E^\beta \sigma^{\gamma \rho} + \ldots \] (83)

\[ \zeta^\alpha_\beta H^\beta \Rightarrow \zeta^\alpha_\beta H^\beta + m^\alpha (T - T_0) + h^\alpha_{\beta \gamma} \sigma^{\beta \gamma} + \nu^\alpha_\beta E^\beta + R^\alpha_{\beta \gamma \rho} H^\beta \sigma^{\gamma \rho} + \ldots \] (84)

Here \( \pi^\alpha \) and \( m^\alpha \) are the pyro - electric and pyro-magnetic coefficients, respectively, describing the variation of polarization and magnetization produced by deviation of the temperature from its equilibrium value, \( T_0 \). The coefficients \( d^\alpha_{\beta \gamma} \) and \( h^\alpha_{\beta \gamma} \) describe the linear piezo - electric and piezo - magnetic properties of the medium, respectively. These effects may be induced by the stress tensor \( \sigma^{\beta \gamma} \). The tensor \( \nu^\alpha_\beta \) corresponds to magneto-ionic coefficients; if they are non-vanishing, the medium transforms electric fields into magnetic fields and vice-versa. The electro- and the magneto - striction coefficients \( Q^\alpha_{\beta \gamma \rho} \) and \( R^\alpha_{\beta \gamma \rho} \), describe cross-effects involving the stress tensor and the electric or magnetic field, respectively.

When the electrodynamic system as a whole is under motion, the formulas (81) and (82) have to be supplemented by the terms containing the acceleration vector and the spatial derivatives of the velocity vector [2].

4.1.4 Relativistic fluid in the absence of electromagnetic interactions

The parabolic equations for the temperature evolution run into conflict with the causality principle, since the rate of the temperature propagation is predicted to be unbounded [55, 56]. As an answer to this challenge, the extended relativistic (or causal, or second order, or
transient) thermodynamics was developed. The history, the fundamentals and applications of
the extended thermodynamics are presented in detail in Refs. [45] [57] and [58]-[74].

The extended covariant definition of heat flux four-vector:

\[ q^i - \lambda T \Delta_k^i \left( \frac{1}{T} \nabla^k T - DU^k \right) = \tau(1) \Delta^i_k Dq^k + \frac{\tau(1)}{2} q^i \left[ \Theta + D \left( \log \frac{\tau(1)}{\lambda T^2} \right) \right] \]  

(85)
is a relativistic generalization of the definition (76). Here \( D \equiv U^i \nabla_i \) is the convective derivative, \( \nabla_i \) is a covariant derivative operator; \( \Theta \equiv \nabla_k U^k \) is the fluid expansion. Both parts of the relativistic tensor of non-Pascal pressure

\[ \Pi^{ik} \equiv \Pi^{ik}_{(0)} + \frac{1}{3} \Delta^{ik} \Pi, \quad \Pi \equiv g_{ik} \Pi^{ik} \]  

(86)
obey expressions similar to (85)

\[ \Pi - 3 \zeta \Theta = \tau(0) D \Pi + \frac{\tau(0)}{2} \Pi \left[ \Theta + D \left( \log \frac{\tau(0)}{\zeta T} \right) \right], \]  

(87)

\[ \Pi^{ik}_{(0)} - \eta \Sigma^{ik} = \tau(2) \Delta^i_m \Delta^k_n D \Pi^{mn}_{(0)} + \frac{\tau(2)}{2} \Pi^{ik}_{(0)} \left[ \Theta + D \left( \log \frac{\tau(2)}{\eta T} \right) \right]. \]  

(88)

Here the well-known quantity \( \Sigma^{ik} \) is introduced by

\[ \Sigma^{ik} \equiv \frac{1}{2} \Delta^i_m \Delta^k_n (\nabla^m U^n + \nabla^n U^m) - \frac{1}{3} \Delta^{ik} \Theta. \]  

(89)

The principal novelty of the relativistic formulas (85) - (88) is that they contain the convective derivative of the velocity four-vector \( DU^i \), the expansion scalar \( \Theta \) as a supplementary terms of the inertial origin.

### 4.2 Remark on covariant electrodynamics of continuous media

An obvious analogy exists between the constitutive equations in electrodynamics and rheology. In this sense, \( M^{ik} \) plays in electrodynamics the role of a stress tensor \( \sigma^{\alpha\beta} \) in rheology, and \( F^{ik} \) is an analog of the deformation tensor \( e_{\alpha\beta} \). The main difference is that while electrodynamics deals with skew-symmetric quantities, elastodynamics deals with symmetric ones. Following this analogy, one can see a correspondence between the four-vector potential \( A_i \) in electromagnetism and the displacement vector \( V^\alpha \) in classical elastodynamics. Indeed, the second subsystem of Maxwell equations in (2) leads to the relation

\[ F_{ik} = \partial_i A_k - \partial_k A_i, \]  

(90)

which converts it into an identity. Analogously, from the Saint-Venant conditions (equations) [53] in classical non-relativistic elastodynamics one obtains that the deformation tensor has to be the symmetrized derivative of some three-vector, the displacement vector \( V^\alpha \):

\[ e_{\alpha\beta} = \frac{1}{2} (\partial_\alpha V_\beta + \partial_\beta V_\alpha). \]  

(91)
Likewise, there is an analogy between the constitutive laws in the electrodynamics and elastodynamics. The well-known linear static constitutive equations in electrodynamics says that $H^{ik}$ is proportional to $F_{mn}$

$$H^{ik} = C^{ikjl} F_{jl} \equiv H^{ik}_{(\text{stationary})},$$

(92)

where the tensor $C^{ikjl}$ is the linear response tensor. In contrast to (72) with symmetric tensor $C^{\alpha\beta\rho\sigma}$ (73) the relations (92) contain the tensor $C^{ikjl}$ (33) with skew-symmetric indices in the first and the second pairs. Alternatively, the polarization - magnetization tensor $M^{ik}$ is considered proportional to the Maxwell tensor

$$M^{ik} = \chi^{ikmn} F_{mn} \equiv M^{ik}_{(\text{stationary})}.$$  

(93)

Here $\chi^{ikmn}$ is called the susceptibility tensor, it has the same symmetry of indices, as $C^{ikmn}$. Relation (93) is, in fact, the analog of the Hook expression (72). The covariant phenomenological generalization of the constitutive equations may also be done in terms of polarization and magnetization four-vectors $P^i$ and $M^i$:

$$P^i \equiv M^{ik} U_k, \quad M_i \equiv M_{ik} U^k.$$  

(94)

The equations (93) yield

$$P^i = \alpha^{ik} E_k - \gamma_{ik} B^k, \quad M_i = \gamma_i^k E_k + \beta_{ik} B^k,$$

(95)

where $\alpha^{ik}$, $\beta_{ik}$ and $\gamma_{ik}^i$ can be obtained from (41) by the substitution $C^{ikmn} \Rightarrow \chi^{ikmn}$. Relativistic covariant elastodynamics is much more sophisticated (see, e.g., [5, 6, 75, 76] and references therein), and the described analogy is not so evident. Nevertheless, let us use the main idea of generalization of the elastodynamic constitutive equations in order to obtain extended constitutive equations for the covariant electrodynamics.

### 4.3 Phenomenologically extended constitutive equations for relativistic electrodynamic systems

Based on the analogies described in the previous Section we now introduce generalized phenomenological constitutive equations for non-stationary electromagnetic media. We consider three versions of generalization, which, of course, are equivalent, but can be useful for different applications.

#### 4.3.1 The first version

The first version of the extended electrodynamics of continuous media assumes that the difference between $M^{ik}$ and its stationary value $\chi^{ikjl} F_{jl}$ can be written as

$$M^{ik} - \chi^{ikjl} F_{jl} = \Gamma^{ik}_{mn} D M^{mn} + \Lambda^{ik}_{mn} D F^{mn} + \Omega^{ikmn} D U_m + \ldots (\text{higher derivative terms}).$$

(96)

Here the velocity four-vector enters explicitly the convective derivative $D = U^k \nabla_k$ only. It is clear that the structure of the formula (96) is analogous to the Poynting-Thomson law (74).
4.3.2 The second version

More generally, one can replace the convective derivative, $D$, by the covariant derivative; then we obtain

$$M^{ik} - \chi^{ikj}F_{jl} = \Gamma_{...mn}^{ikj} \nabla_j M^{mn} + \Lambda_{...mn}^{ikj} \nabla_j F^{mn} + \Omega^{ikm} \nabla_j U_m + \text{(higher derivative terms)}. \quad (97)$$

In the last case it is convenient to use the standard decomposition

$$\nabla_j U_l = U_j DU_l + \Sigma_{jl} + \frac{1}{3} \Delta_{jl} \Theta + \omega_{jl}. \quad (98)$$

The shear tensor $\Sigma_{jm}$ was defined in (89), the scalar expansion $\Theta$ is again $\nabla_k U^k$, and the vorticity tensor $\omega_{jl}$ is defined by

$$\omega_{jl} = \frac{1}{2} \Delta_j \Delta_l (\nabla_m U_n - \nabla_n U_m). \quad (99)$$

Obviously, equation (97) reduces to (96) when

$$\Gamma_{...mn}^{ikj} = U_j \Gamma_{...mn}^{ikj}, \quad \Lambda_{...mn}^{ikj} = U^j \Lambda_{...mn}^{ikj}, \quad \Omega^{ikm} = U^j \Omega^{ikm}. \quad (100)$$

In general there exist standard irreducible decompositions of such tensors similar to the decomposition of the $C^{ikmn}$ tensor (40). The constitutive law (97) is applicable not only to the non-stationary models, but to the non-homogeneous media, as well. In the latter case the difference $M^{ik} - M_{\text{(stationary)}}^{ik}$ should be decomposed using the spatial derivatives, $\Delta_i \nabla_k$, in addition to convective derivative $D = U^k \nabla_k$. Moreover, the presence of higher derivative terms in (97) gives the possibility to consider cross-terms of the type $\Theta$, described in (48) for heat conduction.

4.3.3 The third version

Consider now the tetrad $\{U^i, X^i_\alpha\}$, where $(\alpha) = (1), (2), (3)$. Here $X^i_0 \equiv U^i$ is the velocity four-vector of the medium and the tetrad vectors $X^i_\alpha$ are connected with three main directions in the anisotropic medium. Define the tetrad components of the vectors of polarization and magnetization, of the vectors of electric field and magnetic induction, as well the acceleration vector as follows:

$$P_\alpha = P_i X^i_\alpha, \quad M_\alpha = M_i X^i_\alpha, \quad E_\alpha = E_i X^i_\alpha, \quad B_\alpha = B_i X^i_\alpha, \quad (DU)_\alpha = X^i_\alpha DU_i. \quad (101)$$

Since the vectors $P^i$, $M^i$, $E^i$, $B^i$ and $DU^i$ are orthogonal to the velocity four-vector and the vectors $X^i_\alpha$ are space-like, we have the inverse decompositions in the form:

$$P^i = - \sum_\alpha P_\alpha X^i_\alpha, \quad M^i = - \sum_\alpha M_\alpha X^i_\alpha, \ldots. \quad (102)$$

As a main ansatz for the third version we suggest that in the appropriate tetrad the relaxation equations take the form

$$P_\alpha = \alpha_\beta^{(\alpha)} E^{(\beta)} - \gamma_\beta^{(\alpha)} B^{(\beta)} - \tau_\beta^{(\alpha)} (DU)^{(\beta)} + \chi^{(\beta)} D^i E^{(j)} + \xi_\beta^{(\alpha)} D^i B^{(j)} + \eta^{(\alpha)} (DU)^{(\beta)} + \ldots, \quad (103)$$
$$M^{(\alpha)} = \gamma^{(\alpha)}(\beta) E^{(\beta)} + \beta^{(\alpha)}(\beta) B^{(\beta)} - \tau^{(\alpha)}(\beta)(m) DM^{(\beta)} + \kappa^{(\alpha)}(\beta) DE^{(\beta)} + \psi^{(\alpha)}(\beta) DB^{(\beta)} + \zeta^{(\alpha)}(\beta)(DU)^{(\beta)} + \ldots. \quad (104)$$

In such a context we assume, that the Einstein rule for the indices ($\beta$) are valid. The term $\tau^{(\alpha)}(\beta)(p)$ describes the diagonal three-dimensional matrix of the relaxation coefficients for the corresponding tetrad components of the polarization vector. Analogously, three independent coefficients of the diagonal matrix $\tau^{(\alpha)}(\beta)(m)$ describe three relaxation parameters, which differ in general for three tetrad components of the magnetization vector. We assume that different components of the polarization and magnetization vectors evolve with its own relaxation time parameters. The constitutive equations (103) and (104) generalize the formulas (81) and (82), advocated by Kluitenberg [50, 51].

The first and the third versions of the generalization are equivalent, when the law of the tetrad vectors evolution is fixed. To satisfy the conditions of orthogonality-normalization for the tetrad vectors, one can use, e.g., the simplest expression

$$DX^i_{(\alpha)} = \Omega^i_{\ k} X^k_{(\alpha)}, \quad (105)$$

where the tensor $\Omega^i_{\ k}$ is skew-symmetric. Then using

$$M^{ik} = P^i U^k - P^k U^i - \eta^{ikj} M^j, \quad (106)$$

with (102) and (105), and expressing the coefficients $\Gamma^{ik}_{\ mn}, \ldots$, etc., via the tetrad coefficients $\tau^{(\alpha)}(\beta)(p), \tau^{(\alpha)}(\beta)(m), \ldots$, etc., one obtains the third version (103), (104) of the generalized constitutive law from the first one (96).

### 4.3.4 On the electrodynamics of thermo-visco-elastic medium

When thermo-visco-elastic processes take place, the right-hand-sides of the constitutive equations (96), (97), (103), (104) have to be supplemented by the heat-flux vector $q^i$ and its derivatives $D^{(m)}q^i$, the non-equilibrium pressure $\Pi^{ik}$ and its derivatives $D^{(m)}\Pi^{ik}$ as discussed above. In its turn, the constitutive equations for $q^i$ and $\Pi^{ik}$ have to be supplemented by the corresponding electromagnetic terms $M^{ik}, F_m, \ldots, D^{(m)}M^{ik}, D^{(m)}F_m, \ldots$. This problem, of course, requires a special consideration which takes into account the concept of hyperbolicity of master equations (see, e.g., [77]). Thus, in general, one obtains a system of coupled extended constitutive equations for $M^{ik}, q^i$ and $\Pi^{ik}$, describing the covariant extended electrodynamics of continua.

### 5 Example of exact solution of non - stationary Einstein - Maxwell model

#### 5.1 Einstein’s equations

We consider the FLRW cosmological model with line element [10, 38]

$$ds^2 = dt^2 - a^2(t) \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right]. \quad (107)$$
The magnetic field vector, the magnetization vector, the electric field vector and the polarization vector are pointed along the $x^3$ axis. For such a “self-parallel” configuration of the electromagnetic field, the total stress-energy tensor $T^i_k^{\text{(total)}}$, has four non-vanishing components:

\begin{align*}
T^0_0^{\text{(total)}} &= W + X, \quad T^1_1^{\text{(total)}} = -P^{(1)} - X, \quad T^2_2^{\text{(total)}} = -P^{(2)} - X, \quad T^3_3^{\text{(total)}} = -P^{(3)} + X. \quad (108)
\end{align*}

Here

\begin{equation}
X \equiv \frac{1}{2} [H^{12}F_{12} - H^{30}F_{30}] \quad (109)
\end{equation}

is the source term related to the electric and magnetic fields, polarization and magnetization. We do not specify here the relations between the energy density scalar $W$ and the diagonal components of the pressure tensor $P^{(1)}, P^{(2)}$ and $P^{(3)}$. They can describe perfect fluid, viscous fluid, etc. In a co-moving frame with $U^i = \delta^i_0$, the gravity field equations reduce to the following system [38]

\begin{align*}
3 \left( \frac{\dot{a}}{a} \right)^2 &= \Lambda + \kappa(W + X), \quad 2 \dddot{a} + \left( \frac{\dot{a}}{a} \right)^2 = \Lambda - \kappa(P^{(1)} + X), \quad (110) \\
2 \dddot{a} + \left( \frac{\dot{a}}{a} \right)^2 &= \Lambda - \kappa(P^{(2)} + X), \quad 2 \dddot{a} + \left( \frac{\dot{a}}{a} \right)^2 = \Lambda - \kappa(P^{(3)} - X). \quad (111)
\end{align*}

The dot denotes the derivative with respect to time. The Einstein equations are self-consistent when

\begin{equation}
P^{(1)} + X = P^{(2)} + X = P^{(3)} - X = P^{\text{(isotr)}}. \quad (112)
\end{equation}

The formulas (112) guarantee that the global spatial isotropy of the Universe holds. Differentiating (110) - (111) leads to the conservation law

\begin{equation}
\dot{W} + \dot{X} + \left( \frac{\dot{a}}{a} \right) \left[ 4X + 3W + P^{(1)} + P^{(2)} + P^{(3)} \right] = 0. \quad (113)
\end{equation}

In principle, the matter pressure may be anisotropic, but with $P^{(1)} = P^{(2)} = P^{(3)} - 2X$, i.e., the longitudinal pressure $P^{(3)}$ compensates the influence of the electromagnetic pressure. However, the question arises: whether a non-trivial solution exists for which electromagnetic field and electromagnetic induction are non-vanishing, but the matter has an isotropic pressure $P^{(1)} = P^{(2)} = P^{(3)} = P$. Obviously, such a requirement assumes, that $X = 0$. For the well-known models with magnetic field [11] this requirement leads automatically to the absence of the magnetic field. Is it possible to self-consistently consider a non-trivial electromagnetic field in the FLRW background? The answer is yes. Consider such a model.

Given the structure of Einstein’s field equations, in the model of parallel electric and magnetic fields, the longitudinal electromagnetic source governs the evolution of the gravitational field:

\begin{align*}
X &\equiv \frac{1}{2} (H^{12}F_{12} - H^{30}F_{30}) = \frac{1}{2} (M^{12}F_{12} + F^{12}F_{12} - M^{30}F_{30} - F^{30}F_{30}) \\
&= -\frac{1}{2} \left( M^3 B_3 + B^3 B_3 + P^3 E_3 + E^3 E_3 \right) = \frac{1}{2} \left( M^{(3)} B^{(3)} + (B^{(3)})^2 + P^{(3)} E^{(3)} + (E^{(3)})^2 \right). \quad (114)
\end{align*}

When both the polarization vector $P^{(3)}$ and magnetization vector $M^{(3)}$ vanish, then the quantity $X(t)$ is non-negative. Nevertheless, when $P^{(3)}$ and $M^{(3)}$, are non-vanishing, $X(t)$ can be negative for some time interval, or even identically vanish for the special initial conditions. Note that even if $X$ is negative, the total energy density $W + X$ is assumed to be non-negative, since the energy density of the material medium, $W$, is positive. We consider below just the case with $X = 0$. 

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5.2 Maxwell’s equations

The quantities $F_{ik}$ and $M_{ik}$ are considered to be the function of cosmological time only. Thus, it follows from the second subsystem of Maxwell equations that the tensor of electromagnetic field has a few constant components, and using the symmetries of the model, we can choose only $F_{12} = \text{const}$ to be non-vanishing, since the magnetic field points along the $x^3$ axis. Note that the equations (2) do not fix the structure of the electric field, so far an arbitrary function of time. We assume for the simplicity that the current $I^i$ vanishes. The first subsystem of the Maxwell equations

$$\nabla_k H^{ik} = \frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g} H^{ik}) = \frac{1}{a^3} \frac{d}{dt} (a^3 H^{0}) = 0$$

reduces to an identity for $i = 0$ and three differential equations in ordinary derivatives for the three components $H^{\alpha 0}$, whose solution is

$$H^{\alpha 0}(t) \equiv M^{\alpha 0}(t) + F^{\alpha 0}(t) = P^{\alpha}(t) + E^{\alpha}(t) = \frac{\text{const}}{a^3(t)} = D^{\alpha}(t_0) \frac{a^3(t)}{a^3(t_0)}.$$

Note that the last relation includes the polarization vector $P^{\alpha}(t)$ and electric field $E^{\alpha}(t)$ only. Thus, the evolution of the magnetization vector $M^{\alpha}$ is just governed by the constitutive equations.

5.3 Constitutive equations

We use the third version of the generalized constitutive equations (103) and (104). For this case, we specify the tetrad vectors. For the metric (107) the normalization - orthogonality conditions for the tetrad vectors yield

$$U^i = \delta^i_0, \quad X^i_{(\alpha)} = \frac{1}{a(t)} \delta^i_\alpha.$$

(117)

For such a four-velocity vector the acceleration vector $DU^i$ vanishes. Consider the simplest relaxation model without electric conductivity, in which the polarization is coupled to magnetic field by the non-vanishing magnetoelectric coefficients. The master equations for such a model read

$$\tau_{(p)} \dot{P} + P = (\varepsilon_{||} - 1) E - \gamma B,$$

$$\tau_{(m)} \dot{M} + M = \gamma E + \left(\frac{1}{\mu_{||}} - 1\right) B,$$

$$E(t) + P(t) = D(t_0) \left(\frac{a(t_0)}{a(t)}\right)^2,$$

$$B(t) = B(t_0) \left(\frac{a(t_0)}{a(t)}\right)^2, \quad B(t_0) = \frac{F_{12}}{a^2(t_0)}.$$ (121)

Here we use the quantities $P \equiv P^{(3)}$, $E \equiv E^{(3)}$, $B \equiv B^{(3)}$, $D \equiv D^{(3)}$, ... etc., in which, for simplicity, we omit the tetrad indices. Assuming that the relaxation parameters depend on time according to

$$\tau_{(p)} = \xi_1 \left(\frac{\dot{a}}{a}\right)^{-1}, \quad \tau_{(m)} = \xi_2 \left(\frac{\dot{a}}{a}\right)^{-1},$$

(122)
(ξ₁, ξ₂ = const) and introducing the variable \( x = \frac{a(t)}{a(t₀)} \), we obtain that the solutions to (118) - (121) are

\[
P(x) = P(t₀)x^{−\varepsilon/ξ₁} + \Gamma₁ \left( x^{-2} - x^{−\varepsilon/ξ₁} \right),
\]

\[
M(x) = M(t₀)x^{−\varepsilon/ξ₂} + \Gamma₂ \left( x^{-2} - x^{−\varepsilon/ξ₂} \right) + \Gamma₃ \left( x^{−\varepsilon/ξ₁} - x^{−\varepsilon/ξ₂} \right),
\]

where

\[
\Gamma₁ ≡ \left[ \frac{D(t₀)(\varepsilon/ξ₁ - 1) - \gamma B(t₀)}{(\varepsilon/ξ₁ - 2\varepsilon/ξ₂)} \right],
\]

\[
\Gamma₂ ≡ \frac{1}{(1 - 2\varepsilon/ξ₂)(\varepsilon/ξ₁ - 2\varepsilon/ξ₂)} \left\{ B(t₀) \left[ \gamma^2 + \left( \frac{1}{\mu/ξ₁} - 1 \right)(\varepsilon/ξ₁ - 2\varepsilon/ξ₂) \right] + \gamma D(t₀)(1 - 2\varepsilon/ξ₂) \right\},
\]

\[
\Gamma₃ ≡ \frac{\gamma ξ₁}{(ξ₁ - ξ₂\varepsilon/ξ₂)} \left[ Γ₁ - P(t₀) \right].
\]

Thus, the function \( X(t) \) reads

\[
X(t) = \frac{1}{2}x^{(−2−\varepsilon/ξ₂)}B(t₀)[M(t₀) − \Gamma₂ − \Gamma₃] + \frac{1}{2}x^{−4}[B²(t₀) + D²(t₀) + B(t₀)Γ₂ − D(t₀)Γ₁]
\]

\[
+ \frac{1}{2}x^{(−2−\varepsilon/ξ₁)}[B(t₀)Γ₃ + D(t₀)Γ₁ − D(t₀)P(t₀)].
\]

Since \( X(t) \) contains nine free parameters, we can choose three of them, for instance, the initial data, \( P(t₀), M(t₀) \) and \( D(t₀) \), so that \( X(t) = 0 \). In other words, we have a model with “hidden” electric and magnetic fields. For instance, when

\[
P(t₀) = Γ₁, \quad M(t₀) = Γ₂,
\]

it follows that

\[
P(x) = Γ₁ x^{-2}, \quad E(x) = [D(t₀) − Γ₁] x^{-2}, \quad M(x) = Γ₂ x^{-2}, \quad H(x) = [B(t₀) + Γ₂] x^{-2}.
\]

Thus, \( X = 0 \) when

\[
B²(t₀) + D²(t₀) + B(t₀)Γ₂ − D(t₀)Γ₁ = 0.
\]

This condition reduces to the quadratic equation for the ratio \( D(t₀)/B(t₀) \).

\[
\left( \frac{D(t₀)}{B(t₀)} \right)^2 (1 - 2\varepsilon/ξ₁)(1 - 2\varepsilon/ξ₂) + 2\gamma \frac{D(t₀)}{B(t₀)}(1 - ξ₁ - ξ₂) + \left[ γ^2 + \left( \frac{1}{\mu/ξ₁} - 2\varepsilon/ξ₂ \right) \left( \varepsilon/ξ₁ - 2\varepsilon/ξ₂ \right) \right] = 0.
\]

This equation has real roots for a wide choice of the parameters \( ξ₁, ξ₂, γ, μ/ξ₁ \) and \( ε/ξ₁ \). For instance, when \( ξ₁ = ξ₂ = ξ \) and \( γ = 0 \) two real solutions exist when \( \frac{1}{μ/ξ₁} < 2\varepsilon/ξ₂ < ε/ξ₁ \), i.e., when the relaxation parameters \( τ(p) \) and \( τ(m) \) are of the order of Hubble parameter \( H(t) = \dot{a}/a \).

Thus, we obtained an exact solution of the Einstein-Maxwell equations, describing the FLRW-type model, in which there is a non-vanishing magnetic field, the magnetization, the electric field and the polarization of the matter, however, they are hidden, i.e., their total contribution to the stress-energy tensor of the whole system vanishes. This type of behaviour was discussed in [10]. There the stationary magnetic field in vacuum is non-vanishing, nevertheless,
the exact solution to the non-minimal Einstein-Maxwell equations demonstrates the possibility of isotropic FLRW-type expansion. In that case the non-minimal interaction between gravitational and electromagnetic fields inspires some kind of “non-minimal screening” and hiddens magnetic field from the point of view of gravitational dynamics. Here we presented the example of “dynamic screening”, when the polarization and magnetization of the medium compensate the contribution of the electromagnetic field to the total stress-energy tensor. Note that in the proposed model the magnetic induction, \( H^{(3)} = B^{(3)} + M^{(3)} \), and the electric induction, \( D^{(3)} = E^{(3)} + P^{(3)} \), are considered to be non-vanishing. This means, that the electric polarization compensates (partially) the contribution of the magnetic field to the expression for \( X \), and the magnetization compensates (partially) the contribution of the electric field to \( X \) because of the special choice of the initial data. When the electric field and electric polarization are absent, there exists, nevertheless, the possibility that the magnetization compensates the contribution of the magnetic field. It is possible due to the relationship \( 2 \mu |\xi| = 1 \). The magnetic induction is vanishing in this case.

When \( X=0 \) the Universe is expanding isotropically and does not feel the presence of the electric and magnetic fields. The Einstein equations for this case are the standard (see, (110) and (111) with \( X=0 \) and \( P_{(1)}=P_{(2)}=P_{(3)}=P \). We will not specify the equation of state and discuss the solutions. Nevertheless, let us note, that the application of the extended constitutive equations to the cosmic electrodynamics have something in common with inhomogeneous (depending on time) equations of state, introduced in [78] - [81].

6 Conclusions

The fundamentals of covariant phenomenological electrodynamics of relativistic continuous media were elaborated three decades ago, however, the self-consistent description of a gravitating polarizable - magnetizable non-stationary medium is still an open question. In this paper we formulated extended Einstein - Maxwell model appropriated for the non-stationary electromagnetically active relativistic material medium, which has three main ingredients:

(i). The standard Maxwell equations, describing the dydamic phenomena in continuous media in terms of induction tensor and Maxwell tensor (see (1) and (2)).

(ii). A new set of covariant constitutive equations, containing the polarization - magnetization tensor and its first, second, etc., covariant derivatives (see (96), (97), (103) and (104)). In this sense the model can be indicated as an extended one in analogy with extended thermodynamics.

(iii). The Einstein equations for the gravity field, in which we introduced a new effective stress-energy tensor (65), describing the contribution of the electromagnetic field, of the electric polarization and of the magnetization of the medium. This Einstein-Maxwell model can be indicated as self-consistent by two reasons. First, the corresponding extended constitutive equations describe the interaction of matter with electromagnetic field, resulting in the dynamics of polarization - magnetization. Secondly, the polarization and magnetization contribute to the total stress - energy tensor, the source for the gravitational field, via the proposed effective stress - energy tensor of the electromagnetic field.

Concerning our approach, we would like to emphasize three points.

a) This extended model needs verification. In this paper we discuss only one example of the
exact solution to the extended Einstein-Maxwell model, describing the FLRW-type cosmological dynamics with hidden electromagnetic field. We prepared also a paper related to Bianchi-I anisotropic cosmological model with exact solutions of the new type. We hope the extended Einstein-Maxwell model will be also useful in application to the theory of interaction of the gravitational waves with electromagnetically active material media.

b) This model admits also a generalization to gravitating static anisotropic non-homogeneous media. To develop such a model one can introduce the first, second, etc. spatial derivatives of the polarization-magnetization tensor into the constitutive equations in analogy to the convective derivative.

c) We introduced the extended constitutive equations by the phenomenological way. The next step is to confirm this approach by the consideration of the corresponding entropy production scalar and by analyzing the first and the second laws of thermodynamics. We shall present such analyses in future papers.

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