A Linear Isotropic Cosserat Shell Model Including Terms up to $O(h^5)$. Existence and Uniqueness

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Abstract
In this paper we derive the linear elastic Cosserat shell model incorporating in the variational problem effects up to order $O(h^5)$ in the shell thickness $h$ as a particular case of the recently introduced geometrically nonlinear elastic Cosserat shell model. The existence and uniqueness of the solution is proven in suitable admissible sets. To this end, inequalities of Korn-type for shells are established which allow to show coercivity in the Lax-Milgram theorem. We are also showing an existence and uniqueness result for a truncated $O(h^3)$ model. Main issue is the suitable treatment of the curved reference configuration of the shell. Some connections to the classical Koiter membrane-bending model are highlighted.

Keywords  Cosserat shell · Micropolar shell · 6-parameter resultant shell · In-plane drill rotations · Isotropy · Existence of minimisers · Linear theories

Mathematics Subject Classification  74A05 · 74A60 · 74B20 · 74G65 · 74K20 · 74K25 · 74Q05

1 Introduction

In this paper we consider the linearised formulation of the geometrically nonlinear Cosserat shell model including terms up to order $O(h^5)$ in the shell-thickness $h$ proposed previously...
The Cosserat approach to shell theory (also called micropolar shell theory) was initiated by the Cosserat brothers, who were the first to elaborate a rigorous study on directed media [23, 24]. The idea is to model a shell-like body as a deformable two-dimensional surface endowed with directors assigned to every point. In this respect, there are several approaches in literature. For instance, Green and Naghdi [37] have elaborated a shell theory using two-dimensional surfaces endowed with a single deformable director also called Cosserat surfaces. The theory of Cosserat surfaces has been presented in the monograph [47] and the linearised theory has been investigated in a number of papers [8–10, 25].

Another direct approach to shell theory, also called the theory of simple shells (or directed surfaces), describes the shell-like body as a deformable surface endowed with an independent triad of orthonormal vectors connected to each point of the surface. The triad of directors characterizes the orientation of material points and introduces thus the micro-rotation tensor. The theory of simple shells has been presented by Zhilin and Altenbach in [4, 5, 54, 55] and a mathematical study of the linearised equations for this model is included in the papers [12, 13].

In [29] we have established a novel geometrically nonlinear Cosserat shell model including terms up to order $O(h^5)$ in the shell-thickness $h$. We expect that for some intricate initial geometries, the higher order terms become sensible, but we are not yet able to delineate a simple geometrical configuration where this can be explained. However, for very irregular and curved initial shell configurations it is clear that these higher order terms have an influence, since they do not come with a definite sign: they may locally stiffen or weaken the shell material, depending on the curvature $H$ and $K$. These types of shells are of great interest in the industrial process of the construction of nano bodies at which scale we expect that the terms of order $O(h^5)$ included here will play important roles in increasing the accuracy of analytical and numerical predictions. The dimensional descent was obtained starting with a 3D-parent Cosserat model and assuming an appropriate 8-parameter ansatz for the shell-deformation through the thickness. This is the derivation approach and it has allowed us to arrive at specific novel strain and curvature measures. In this way, we obtained a kinematical model which is equivalent to the kinematical model of 6-parameter shells. Nevertheless, the theory of 6-parameter shells was developed for shell-like bodies made of Cauchy materials, see the monographs [17, 36] or the papers [27, 43], but our model is expressed in terms of the accepted measures for the bending and the change of curvature (cf. the assertion of Acharya [1] and Anicic and Legér [7], respectively, see also [49]). For recent linear Naghdi shell models [34] which do not incorporate dedicated Cosserat effects we refer to [52, 53], while for a two-dimensional model of elastic shell-like body derived from the three-dimensional linearized micropolar elasticity by using the asymptotic expansion method we refer to [3]. In [11], by using a method which extends the dimensional reduction procedure from classical elasticity to the case of Cosserat shells [50, 51], Bîrsan has obtained a $O(h^5)$-Cosserat shell model by considering an ansatz though the thickness up to power 5. The obtained model is similar to our model. This motivates us to develop in the future a model for an ansatz up to power 5, but using our derivation method. However, this is beyond the aim of this paper regarding only the linearisation of an already established model.

For all our proposed new linearised models we show existence and uniqueness. Existence results for the linearised equations of 6-parameter shells have been proved in [26]. We refer to the review paper [6] for a detailed presentation of various approaches and developments concerning Cosserat-type shell theories and to the books [19–21] for shell theories in the classical linear elasticity framework.

In the linearised version of our Cosserat shell model we obtain the same linear strain measures as in the theory of 6-parameter shells, due to the fact that the kinematical structure is equivalent, since there is an explicit dependence of the internal energy density on the
change of curvature measure and on the bending measure. The obtained model is hyperelastic and the variational formulation leads to second-order Euler-Lagrange equations which would make it suitable for finite element implementation. The study of the consistency with classical shell models based on a 3D Cauchy elastic material for infinitesimal deformations is possible for the constrained Cosserat elastic materials (Toupin-couple stress theory) and we have addressed this in [28].

2 The Geometrically Nonlinear Unconstrained Cosserat Shell Model Including Terms up to $O(h^5)$

2.1 Notations

In this paper, for $a, b \in \mathbb{R}^n$ we let $\langle a, b \rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated vector norm $\|a\|_{\mathbb{R}^n} = \langle a, a \rangle_{\mathbb{R}^n}$. The standard Euclidean scalar product on the set of real $n \times m$ second order tensors $\mathbb{R}^{n \times m}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{n \times m}} = \text{tr}(X Y^T)$, and thus the (squared) Frobenius tensor norm is $\|X\|_{\text{Frob}} = \langle X, X \rangle_{\mathbb{R}^{n \times m}}$. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{I}_n$, so that $\text{tr}(X) = \langle X, \mathbb{I}_n \rangle$, and the zero matrix is denoted by $\mathbf{0}_n$. We let $\text{Sym}(n)$ and $\text{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\text{GL}(n) = \{X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0\}$ the general linear group $\text{SO}(n) = \{X \in \text{GL}(n) \mid X^T X = \mathbb{I}_n, \det(X) = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(n) = \{X \in \mathbb{R}^{n \times n} \mid X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(n) = \{X \in \mathbb{R}^{n \times n} \mid \text{tr}(X) = 0\}$ of traceless tensors. For all $X \in \mathbb{R}^{n \times n}$ we set $\text{sym} X = \frac{1}{2}(X + X^T) \in \text{Sym}(n)$, skew $X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(n)$ and the deviatoric part $\text{dev} X = X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{I}_n \in \mathfrak{sl}(n)$ and we have the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(n) = (\mathfrak{sl}(n) \cap \text{Sym}(n)) \oplus \mathfrak{so}(n) \oplus \mathbb{R} \cdot \mathbb{I}_n$, $X = \text{dev} \text{sym} X + \text{skew} X + \frac{1}{n} \text{tr}(X) \cdot \mathbb{I}_n$. For vectors $\xi, \eta \in \mathbb{R}^n$, we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. A matrix having the three column vectors $A_1, A_2, A_3$ will be written as $(A_1 | A_2 | A_3)$. For a given matrix $M \in \mathbb{R}^{2 \times 2}$ we define the lifted quantities $M^\flat = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 3}$ and

$$\hat{M} = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$  

We make use of the operator $\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ associating with a matrix $A \in \mathfrak{so}(3)$ the vector $\text{axl}(A) := (-A_{23}, A_{13}, -A_{12})^T$. The inverse operator will be denoted by $\text{Anti} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$.

For an open domain $\Omega \subseteq \mathbb{R}^3$, the usual Lebesgue spaces of square integrable functions, vector or tensor fields on $\Omega$ with values in $\mathbb{R}$, $\mathbb{R}^3$, $\mathbb{R}^{3 \times 3}$ or $\mathfrak{so}(3)$, respectively will be denoted by $L^2(\Omega; \mathbb{R})$, $L^2(\Omega; \mathbb{R}^3)$, $L^2(\Omega; \mathbb{R}^{3 \times 3})$ and $L^2(\Omega; \mathfrak{so}(3))$, respectively. Moreover, we use the standard Sobolev spaces $H^1(\Omega; \mathbb{R})$ [2, 33, 35] of functions $u$. For vector fields $u = (u_1, u_2, u_3)^T$ with $u_i \in H^1(\Omega)$, $i = 1, 2, 3$, we define $\nabla u := (\nabla u_1 | \nabla u_2 | \nabla u_3)^T$. The corresponding Sobolev-space will be denoted by $H^1(\Omega; \mathbb{R}^3)$. A tensor $Q : \Omega \rightarrow \mathfrak{so}(3)$ having the components in $H^1(\Omega; \mathbb{R}^3)$ belongs to $H^1(\Omega; \mathfrak{so}(3))$. In writing the norm in the corresponding Sobolev-space we will specify the space. The space will be omitted only when the Frobenius norm or scalar product is considered.

2.2 Shell-Kinematics

Let $\Omega_{\xi} \subseteq \mathbb{R}^3$ be a three-dimensional curved shell-like thin domain. Here, the domains $\Omega_{\xi}$ and $\Omega_{\xi}$ are referred to a fixed right Cartesian coordinate frame with unit vectors $e_i$ along
the axes $Ox_i$. A generic point of $\Omega_\xi$ will be denoted by $(\xi_1, \xi_2, \xi_3)$. The elastic material constituting the shell is assumed to be homogeneous and isotropic and the reference configuration $\Omega_\xi$ is assumed to be a natural state. The deformation of the body occupying the domain $\Omega_\xi$ is described by a vector map $\varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathbb{R}^3$ (called deformation) and by a microrotation tensor $\bar{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \text{SO}(3)$ attached at each point. We denote the current configuration (deformed configuration) by $\Omega_c := \varphi_\xi (\Omega_\xi) \subset \mathbb{R}^3$. We consider the fictitious Cartesian (planar) configuration of the body $\Omega_h$. This parameter domain $\Omega_h \subset \mathbb{R}^3$ is a right cylinder of the form

$$\Omega_h = \{(x_1, x_2, x_3) | (x_1, x_2) \in \omega, \ -h/2 < x_3 < h/2 \} = \omega \times (-h/2, h/2),$$

where $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial \omega$ and the constant length $h > 0$ is the thickness of the shell. For shell–like bodies we consider the domain $\Omega_h$ to be thin, i.e. the thickness $h$ is small.

The diffeomorphism $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ describing the reference configuration (i.e., the curved surface of the shell), will be chosen in the specific form

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2), \quad n_0 = \frac{\partial y_0}{\partial x_1} \times \frac{\partial y_0}{\partial x_2}, \quad (2.1)$$

where $y_0 : \omega \to \mathbb{R}^3$ is a function of class $C^2(\omega)$. If not otherwise indicated, by $\nabla \Theta$ we denote $\nabla \Theta(x_1, x_2, 0)$.

Now, let us define the map $\varphi : \Omega_h \to \Omega_c, \ \varphi(x_1, x_2, x_3) = \varphi_\xi (\Theta(x_1, x_2, x_3))$. We view $\varphi$ as a function which maps the fictitious planar reference configuration $\Omega_h$ into the deformed configuration $\Omega_c$. We also consider the elastic microrotation $\bar{Q}_{e,s} : \Omega_h \to \text{SO}(3), \ \bar{Q}_{e,s}(x_1, x_2, x_3) := \bar{R}_\xi (\Theta(x_1, x_2, x_3))$.

The dimensional descent in [29] is done by assuming that the elastic microrotation is constant through the thickness, i.e. $\bar{Q}_{e,s}(x_1, x_2, x_3) = \bar{Q}_{e,s}(x_1, x_2)$, and by considering an 8-parameter quadratic ansatz in the thickness direction for the reconstructed total deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like body, i.e.,

$$\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left(x_3q_m(x_1, x_2) + \frac{x_1^2}{2} q_o(x_1, x_2)\right) \bar{Q}_{e,s}(x_1, x_2) \nabla \Theta.e_3. \quad (2.2)$$

Here $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ represents the deformation of the total midsurface, $q_m, q_o : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ allow in principal for symmetric thickness stretch ($q_m \neq 1$) and asymmetric thickness stretch ($q_o \neq 0$) about the midsurface and which are given by

$$q_m = 1 - \frac{\lambda}{\lambda + 2\mu} \left[\left(\bar{Q}_{e,s}(\nabla m)(\nabla \Theta)^{-1} \mathbb{I}_3\right) - 2\right], \quad (2.3)$$

$$q_o = -\frac{\lambda}{\lambda + 2\mu} \left(\bar{Q}_{e,s}(\nabla(\bar{Q}_{e,s} \nabla \Theta.e_3))(\nabla \Theta)^{-1} \mathbb{I}_3\right) + \frac{\lambda}{\lambda + 2\mu} \left(\bar{Q}_{e,s}(\nabla m)(\nabla \Theta)^{-1} (\nabla n_0)(\nabla \Theta)^{-1} \mathbb{I}_3\right).$$

This allowed us to obtained a fully two-dimensional minimization problem in which the reduced energy density is expressed in terms of the following tensor fields (the same strain...
measures are also considered in [14, 15, 17, 27, 36] but with other motivations of their significance) on the surface $\omega$

$$\mathcal{E}_{m,s} := \overline{Q}^T_{e,s} (\nabla m|\overline{Q}_{e,s} \nabla \Theta \cdot e_j)[\nabla \Theta]^{-1} - \mathbb{1} \notin \text{Sym}(3),$$

(2.4)

elastic shell strain tensor,

$$\mathcal{K}_{e,s} := \left(\text{axl}(\overline{Q}^T_{e,s} \partial_{x_1} \overline{Q}_{e,s}) \mid \text{axl}(\overline{Q}^T_{e,s} \partial_{x_2} \overline{Q}_{e,s}) \mid 0\right)[\nabla \Theta]^{-1} \notin \text{Sym}(3),$$

(2.5)

elastic shell bending–curvature tensor.

Certainly, different kinematic assumptions would yield to different shell models, especially at order $h^5$, the model constructed by us being a well justified continuum model for the kinematic assumption we have considered. This type of analysis is not in contrasts with the well accepted shell models which can be justified asymptotically, since to our knowledge, excepting the paper [40] and our new paper [48], there are no asymptotically derived Cosserat-models including all membrane-bending-curvature effects. In [48] it is proven that the membrane energy can be justified via Gamma-convergence analysis but the same conclusion for the bending-curvature energy is an open problem.

2.3 Geometrically Nonlinear Energy Functional

In [29] we have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \rightarrow \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s} : \omega \rightarrow \text{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \overline{Q}_{e,s})$ the functional

$$I(m, \overline{Q}_{e,s}) = \int_{\omega} \left[ W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s}) \right] \frac{\det(\nabla y_0|n_0)}{\det(\nabla \Theta)} \, da$$

$$- \Pi(m, \overline{Q}_{e,s}),$$

(2.6)

where the membrane part $W_{\text{memb}}(\mathcal{E}_{m,s})$, the membrane–bending part $W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ and the bending–curvature part $W_{\text{bend,curv}}(\mathcal{K}_{e,s})$ of the shell energy density are given by

$$W_{\text{memb}}(\mathcal{E}_{m,s}) = \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}),$$

$$W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s})$$

$$- \frac{h^3}{3} W_{\text{curv}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s})$$

$$+ \frac{h^3}{6} W_{\text{shell}}(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0})$$

$$+ \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}),$$

$$W_{\text{bend,curv}}(\mathcal{K}_{e,s}) = \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0})$$

$$+ \frac{h^5}{80} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0}^2),$$
and

\[ W_{\text{shell}}(X) = \mu \| \text{sym} X \|^2 + \mu_c \| \text{skew} X \|^2 + \frac{\lambda_0 \mu}{\lambda + 2 \mu} [\text{tr}(X)]^2 \]
\[ = \mu \| \text{dev} \text{sym} X \|^2 + \mu_c \| \text{skew} X \|^2 + \frac{2 \mu (2 \lambda_0 + \mu)}{3(\lambda + 2 \mu)} [\text{tr}(X)]^2, \quad (2.7) \]

\[ W_{\text{shell}}(X, Y) = \mu \langle \text{sym} X, \text{sym} Y \rangle + \mu_c \langle \text{skew} X, \text{skew} Y \rangle + \frac{\lambda_0 \mu}{\lambda + 2 \mu} \text{tr}(X) \text{tr}(Y), \]

\[ W_{\text{mp}}(X) = \mu \| \text{sym} X \|^2 + \mu_c \| \text{skew} X \|^2 + \frac{\lambda_0}{2} [\text{tr}(X)]^2 \]
\[ = W_{\text{shell}}(X) + \frac{\lambda_0^2}{2(\lambda + 2 \mu)} [\text{tr}(X)]^2, \]

\[ W_{\text{curv}}(X) = \mu L_c^2 \left( b_1 \| \text{dev} \text{sym} X \|^2 + b_2 \| \text{skew} X \|^2 + b_3 [\text{tr}(X)]^2 \right), \quad \forall X, Y \in \mathbb{R}^{3 \times 3}. \]

In the formulation of the minimization problem we have considered the Weingarten map (or shape operator) defined by \( L_{y_0} = I_{y_0}^{-1} \Pi_{y_0} \in \mathbb{R}^{2 \times 2}, \) where \( I_{y_0} := [\nabla \gamma_{y_0}]^T \nabla \gamma_{y_0} \in \mathbb{R}^{2 \times 2} \) and \( \Pi_{y_0} := -[\nabla \gamma_{y_0}]^T \nabla n_0 \in \mathbb{R}^{2 \times 2} \) are the matrix representations of the first fundamental form (metric) and the second fundamental form of the surface, respectively. Then, the Gauß curvature \( K \) of the surface is determined by \( K := \det L_{y_0} \) and the mean curvature \( H \) through \( 2H := \text{tr}(L_{y_0}) \).

We have also used the tensors defined by

\[ A_{y_0} := (\nabla \gamma_{y_0} | 0) \left[ \nabla \Theta \right]^{-T} \in \mathbb{R}^{3 \times 3}, \quad B_{y_0} := -\left( \nabla n_0 | 0 \right) \left[ \nabla \Theta \right]^{-T} \in \mathbb{R}^{3 \times 3}, \quad (2.8) \]

and the so-called alternator tensor \( C_{y_0} \) of the surface [55]

\[ C_{y_0} := \text{det} \nabla \Theta \left[ \nabla \Theta \right]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left[ \nabla \Theta \right]^{-1}. \quad (2.9) \]

The parameters \( \mu \) and \( \lambda \) are the Lamé constants of classical isotropic elasticity, \( \kappa = \frac{2\mu + 3\lambda}{3} \) is the infinitesimal bulk modulus, \( b_1, b_2, b_3 \) are non-dimensional constitutive curvature coefficients (weights), \( \mu_c \geq 0 \) is called the Cosserat couple modulus and \( L_c > 0 \) introduces an internal length which is characteristic for the material, e.g., related to the grain size in a polycrystal. The internal length \( L_c > 0 \) is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that \( \mu > 0, \kappa > 0, \mu_c > 0, b_1 > 0, b_2 > 0, b_3 > 0, \) All the constitutive coefficients are coming from the three-dimensional Cosserat formulation, without using any a posteriori fitting of some two-dimensional constitutive coefficients.

The potential of applied external loads \( \Pi(m, \overline{q}_{e,s}) \) appearing in (2.5) is expressed by

\[ \Pi(m, \overline{q}_{e,s}) = \Pi_\omega(m, \overline{q}_{e,s}) + \Pi_\gamma(m, \overline{q}_{e,s}), \quad \text{with} \quad (2.10) \]

\[ \Pi_\omega(m, \overline{q}_{e,s}) = \int_m \{ f, u \} da + \Lambda_\omega(\overline{q}_{e,s}) \quad \text{and} \quad \Pi_\gamma(m, \overline{q}_{e,s}) = \int_m \{ t, u \} ds + \Lambda_\gamma(\overline{q}_{e,s}), \]

where \( u(x_1, x_2) = m(x_1, x_2) - y_0(x_1, x_2) \) is the displacement vector of the midsurface, \( \Pi_\omega(m, \overline{q}_{e,s}) \) is the potential of the external surface loads \( f, \) while \( \Pi_\gamma(m, \overline{q}_{e,s}) \) is the potential of the external boundary loads \( t. \) The functions \( \Lambda_\omega, \Lambda_\gamma : L^2(\omega, SO(3)) \to \mathbb{R} \) are expressed in terms of the loads from the three-dimensional parental variational problem,
see [29], and they are assumed to be continuous and bounded operators. Here, \( \gamma_t \) and \( \gamma_d \) are nonempty subsets of the boundary of \( \omega \) such that \( \gamma_t \cup \gamma_d = \partial \omega \) and \( \gamma_t \cap \gamma_d = \emptyset \). On \( \gamma_t \) we have considered traction boundary conditions, while on \( \gamma_d \) we have the Dirichlet-type boundary conditions:

\[
m\big|_{\gamma_d} = m^*, \quad \text{simply supported (fixed, welded):} \quad \overrightarrow{Q}_{e,s}\big|_{\gamma_d} = \overrightarrow{Q}_{e,s}^*, \quad \text{clamped}, \tag{2.11}
\]

where the boundary conditions are to be understood in the sense of traces.

In our model the total energy is not simply the sum of energies coupling the membrane and the bending effect, respectively. Two further coupling energies are still present and they result directly from the dimensional reduction of the variational problem from the geometrically nonlinear three-dimensional Cosserat elasticity. Our model is constructed in [29] under the following assumptions upon the thickness

\[
h \max\{ \sup_{(x_1,x_2) \in \omega} |\kappa_1|, \sup_{(x_1,x_2) \in \omega} |\kappa_2| \} < 2 \tag{2.12}
\]

where \( \kappa_1 \) and \( \kappa_2 \) denote the principal curvatures of the surface.

The model admits global minimizers for materials with positive Cosserat couple modulus \( \mu_c > 0 \) and the Poisson ratio \( \nu = \frac{\lambda}{\lambda + 2\mu} \) and Young’s modulus \( E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \) are such that\(^1\) \(-\frac{1}{2} < \nu < \frac{1}{2} \) and \( E > 0 \) [30]. Under these assumptions on the constitutive coefficients, together with the positivity of \( \mu, \mu_c, b_1, b_2 \) and \( b_3 \), and the orthogonal Cartan-decomposition of the Lie-algebra \( \mathfrak{gl}(3) \) and with the definition

\[
W_{\text{shell}}(X) := W_{\text{shell}}^\infty(\text{sym} \, X) + \mu_c \| \text{skew} \, X \|^2 \quad \forall \, X \in \mathbb{R}^{3 \times 3},
\]

\[
W_{\text{shell}}^\infty(S) := \mu \| S \|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(S)]^2 \quad \forall \, S \in \text{Sym}(3),
\]

it follows that there exists positive constants \( c_1^+, c_2^+, C_1^+ \) and \( C_2^+ \) such that for all \( X \in \mathbb{R}^{3 \times 3} \) the following inequalities hold

\[
C_1^+ \| S \|^2 \geq W_{\text{shell}}^\infty(S) \geq c_1^+ \| S \|^2 \quad \forall \, S \in \text{Sym}(3),
\]

\[
C_1^+ \| \text{sym} \, X \|^2 + \mu_c \| \text{skew} \, X \|^2 \geq W_{\text{shell}}(X) \geq c_1^+ \| \text{sym} \, X \|^2 + \mu_c \| \text{skew} \, X \|^2 \quad \forall \, X \in \mathbb{R}^{3 \times 3},
\]

\[
C_2^+ \| X \|^2 \geq W_{\text{curv}}(X) \geq c_2^+ \| X \|^2 \quad \forall \, X \in \mathbb{R}^{3 \times 3}.
\]

Here, \( c_1^+ \) and \( C_1^+ \) denote the smallest and the largest eigenvalues, respectively, of the quadratic form \( W_{\text{shell}}^\infty(X) \). Hence, they are independent of \( \mu_c \).

### 2.4 Preliminary Results

In the proof of the existence result for geometrically nonlinear model, the condition on the thickness \( h \) is used only in the step where the coercivity of the internal energy density is deduced, see [30, 31]. We recall the results since they will be useful to establish corresponding existence results in the linearised models, too.

\(^1\)These conditions are equivalent to \( \mu > 0 \) and \( 2\lambda + \mu > 0 \).
Proposition 2.1 (Coercivity in the theory including terms up to order $O(h^5)$) For sufficiently small values of the thickness $h$ such that

$$h \max \{ \sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2| \} < \alpha \quad \text{with} \quad \alpha < \sqrt{2 \over 3} (29 - \sqrt{761}) \simeq 0.97083$$

(2.15)

and for constitutive coefficients satisfying $\mu > 0$, $\mu_c > 0$, $2\lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$, the energy density

$$W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb, bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend, curv}}(\mathcal{K}_{e,s})$$

(2.16)

is coercive in the sense that there exists a constant $a_1^+ > 0$ such that $W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq a_1^+ \parallel \mathcal{E}_{m,s} \parallel^2 + \parallel \mathcal{K}_{e,s} \parallel^2$, where $a_1^+$ depends on the constitutive coefficients. The following inequality holds true

$$W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq a_1^+ \left( \parallel \mathcal{E}_{m,s} \parallel^2 + \parallel \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} \parallel^2 + \parallel \mathcal{K}_{e,s} \parallel^2 \right).$$

(2.17)

In the geometrically nonlinear Cosserat shell model up to $O(h^3)$ the shell energy density $W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ is given by

$$W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = \left( h + \frac{K h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \frac{h^3}{12} W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s})$$

$$- \frac{h^3}{12} H W_{\text{shell}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s})$$

$$+ \frac{h^3}{6} W_{\text{shell}}(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0})$$

$$+ \left( h - \frac{K h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \frac{h^3}{12} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0}).$$

(2.18)

Proposition 2.2 (Coercivity in the truncated theory including terms up to order $O(h^3)$) Assume that the constitutive coefficients are such that $\mu > 0$, $2\lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $L_c > 0$ and let $c_2^+$ denote the smallest eigenvalue of $W_{\text{curv}}(S)$, and $c_1^+$ and $C_1^+$ denote the smallest and the largest eigenvalues of the quadratic form $W_{\text{shell}}(S)$. If the thickness $h$ satisfies one of the following conditions:

i) $h \max \{ \sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2| \} < \alpha$ and $h^2 < (5 - 2 \sqrt{5} (\alpha^2 - 12^2))^{2} \max(c_1^+, c_2^+) \quad \text{with} \quad 0 < \alpha < 2\sqrt{3}$;

ii) $h \max \{ \sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2| \} < \frac{1}{a}$ with $a = \max \left\{ 1 + \frac{\sqrt{3}}{2}, 1 + \frac{1 + 3 \max(C_1^+, \mu_c)}{\min(c_1^+, \mu_c)} \right\}$,

then the total energy density $W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ is coercive, in the sense that there exists a constant $a_1^+ > 0$ such that $W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq a_1^+ \left( \parallel \mathcal{E}_{m,s} \parallel^2 + \parallel \mathcal{K}_{e,s} \parallel^2 \right)$, where $a_1^+$ depends on the constitutive coefficients. The following inequality holds true

$$W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq a_1^+ \left( \parallel \mathcal{E}_{m,s} \parallel^2 + \parallel \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} \parallel^2 + \parallel \mathcal{K}_{e,s} \parallel^2 \right).$$

(2.19)
We define the lifted quantities $\hat{I}_{y_0} \in \mathbb{R}^{3 \times 3}$ by $\hat{I}_{y_0} = (\nabla y_0 | n_0)^T (\nabla y_0 | n_0) = I_{y_0}^0 + \hat{\Theta}_3$ and $\hat{\Pi}_{y_0} \in \mathbb{R}^{3 \times 3}$ by $\hat{\Pi}_{y_0} = - (\nabla y_0 | n_0)^T (\nabla n_0 | n_0) = \Pi_{y_0}^0 - \hat{\Theta}_3$. Some useful properties of the tensors involved in the variational formulation of the considered shell models [29, 30] are gathered next:

**Remark 2.3** The following identities are satisfied:

1. $\text{tr}[A_{y_0}] = 2$, $\det[A_{y_0}] = 0$; $\text{tr}[B_{y_0}] = 2H$, $\det[B_{y_0}] = 0$,
2. $A_{y_0} = [\nabla \Theta]^{-T} \Gamma_{y_0} [\nabla \Theta]^{-1} = I_3 - (0|0|\nabla \Theta,e_3) [\nabla \Theta]^{-1} = I_3 - (0|0|n_0) (0|0|n_0)^T$,
3. $B_{y_0} = [\nabla \Theta]^{-T} \Pi_{y_0}^0 [\nabla \Theta]^{-1}$;
4. $B_{y_0}$ satisfies the equation of Cayley-Hamilton type $B_{y_0}^2 = 2HB_{y_0} + KA_{y_0} = 0$;
5. $A_{y_0}B_{y_0} = B_{y_0}A_{y_0} = B_{y_0}$, $A_{y_0}^2 = A_{y_0}$, $C_{y_0} = 0$; $C_{y_0}^2 = -A_{y_0}$, $\|C_{y_0}\|^2 = 2$;
6. $Q_{e,s}(\nabla Q_{e,s} \nabla \Theta) | (0) [\nabla \Theta]^{-1} = C_{y_0} K_{e,s} - B_{y_0}$;
7. $C_{y_0} K_{e,s} A_{y_0} = C_{y_0} K_{e,s} E_{m,s} A_{y_0} = E_{m,s}$.

Further, we can express the strain tensors using the (referential) fundamental forms $I_{y_0}$, $\Pi_{y_0}$ and $L_{y_0}$ (instead of using the matrices $A_{y_0}$, $B_{y_0}$ and $C_{y_0}$), see [29, 31], i.e.,

$$E_{m,s} = [\nabla \Theta]^{-T} \left( \begin{array}{c} (\bar{Q}_{e,s} \nabla y_0)^T \nabla m - I_{y_0} | 0 \end{array} \right) [\nabla \Theta]^{-1}$$

$$= [\nabla \Theta]^{-T} \left( \begin{array}{c} G | 0 \end{array} \right) [\nabla \Theta]^{-1},$$

$$C_{y_0} K_{e,s} = [\nabla \Theta]^{-T} \left( \begin{array}{c} (\bar{Q}_{e,s} \nabla y_0)^T \nabla (\bar{Q}_{e,s} n_0) + \Pi_{y_0} | 0 \end{array} \right) [\nabla \Theta]^{-1}$$

$$= - [\nabla \Theta]^{-T} \left( \begin{array}{c} R | 0 \end{array} \right) [\nabla \Theta]^{-1},$$

$$E_{m,s} B_{y_0}^2 + C_{y_0} K_{e,s} B_{y_0} = - [\nabla \Theta]^{-T} \left( \begin{array}{c} (R - G L_{y_0}) L_{y_0} | 0 \end{array} \right) [\nabla \Theta]^{-1},$$

where

$$G := (\bar{Q}_{e,s} \nabla y_0)^T \nabla m - I_{y_0} \notin \text{Sym}(2)$$

the change of metric tensor,

$$T := (\bar{Q}_{e,s} n_0)^T \nabla m = ((\bar{Q}_{e,s} n_0, \partial_{e_1} m), (\bar{Q}_{e,s} n_0, \partial_{e_2} m))$$

the transverse shear deformation (row) vector,

$$R := - (\bar{Q}_{e,s} \nabla y_0)^T \nabla (\bar{Q}_{e,s} n_0) - \Pi_{y_0} \notin \text{Sym}(2)$$

the bending strain tensor.

In the above, we can replace $L_{y_0}^2 = 2HL_{y_0} - KI_2$ by the Cayley-Hamilton theorem. The nonsymmetric quantity $R - G L_{y_0}$ represents the change of curvature tensor. The choice of this name will be justified in [32]. For now, we just mention that the definition of $G$ is related to the classical change of metric tensor in the Koiter model [20, 50, 51]

$$G_{\text{Koiter}} := \frac{1}{2} ((\nabla m)^T \nabla m - I_{y_0}) = \frac{1}{2} (I_m - I_{y_0}),$$

$$\text{ Springer}$$
while the bending strain tensor may be compared with the classical bending strain tensor in the Koiter model
\[
\mathcal{R}_{\text{Koiter}} := - (\nabla m)^T \nabla n - \Pi_{y_0}.
\] (2.23)

3 Linearized Cosserat Shell Model Including Terms up to Order $O(h^5)$

In this section we develop the linearisation for the elastic Cosserat shell model including terms up to order $O(h^5)$, i.e., for situations of small Cosserat midsurface deformations and small change of curvature.

3.1 Linearized Strain Measures in the Cosserat Shell Model

We express the total midsurface deformation
\[
m(x_1, x_2) = y_0(x_1, x_2) + v(x_1, x_2),
\] (3.1)

with $v : \omega \to \mathbb{R}^3$, the infinitesimal shell-midsurface displacement. For the elastic rotation tensor $Q_{e,s} \in SO(3)$ there is a skew-symmetric matrix
\[
A_\vartheta := \text{Anti}(\vartheta_1, \vartheta_2, \vartheta_3) := \begin{pmatrix} 0 & -\vartheta_3 & \vartheta_2 \\ \vartheta_3 & 0 & -\vartheta_1 \\ -\vartheta_2 & \vartheta_1 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad \text{Anti} : \mathbb{R}^3 \to \mathfrak{so}(3),
\] (3.2)

where $\vartheta = \text{axl}(A_\vartheta)$ denotes the axial vector of $A_\vartheta$, such that $Q_{e,s} := \exp(A_\vartheta) = \sum_{k=0}^{\infty} \frac{1}{k!} A_\vartheta^k = \mathbb{I}_3 + A_\vartheta + \text{h.o.t.}$ The tensor field $A_\vartheta$ is the infinitesimal elastic microrotation. Here, “h.o.t” stands for terms of order higher than linear with respect to $v$ and $A_\vartheta$. Then, we can expand
\[
Q_{e,s}^T \nabla m - \nabla y_0 = (\mathbb{I}_3 + A_\vartheta + \text{h.o.t.})(\nabla v + \nabla y_0) - \nabla y_0 = \nabla v - A_\vartheta \nabla y_0 + \text{h.o.t.}.
\] (3.3)

Since $\nabla y_0$ is given, the first order term in (3.3) is linear in $v$ and $A_\vartheta$. Correspondingly, we get from the non-symmetric shell strain tensor (which characterises both the in-plane deformation and the transverse shear deformation)
\[
\mathcal{E}_{m,s} = (Q_{e,s}^T \nabla m - \nabla y_0 | 0) [\nabla \Theta]^{-1}
\]
the linearization
\[
\mathcal{E}_{m,s}^{\text{lin}} = (\nabla v - A_\vartheta \nabla y_0 | 0) [\nabla \Theta]^{-1}
= (\partial_1 v - \vartheta \times a_1 | \partial_2 v - \vartheta \times a_2 | 0) [\nabla \Theta]^{-1} \notin \text{Sym}(3).
\] (3.4)

Our aim now is to express all the deformation measures and the linearised models in terms of $A_\vartheta$ as well as in terms of its axial vector $\vartheta$. The following definitions are used to express these quantities in terms of $\vartheta$.

For any column vector $q \in \mathbb{R}^3$ and any matrix $M = (M_1 | M_2 | M_3) \in \mathbb{R}^{3 \times 3}$ we define the cross-product
\[
q \times M := (q \times M_1 | q \times M_2 | q \times M_3) \quad (\text{operates on columns})
\] (3.5)
$M^T \times q^T := -(q \times M)^T \quad \text{(operates on rows)}.$

Note that $M$ can also be a $3 \times 2$ matrix, the definition remains the same. Let us note some properties of these operations: for any column vectors $q_1, q_2 \in \mathbb{R}^3$ and any matrices $M, N \in \mathbb{R}^{3 \times 3}$ (or $\in \mathbb{R}^{3 \times 2}$) we have

\[(q_1 \times M)q_2 = q_1 \times (Mq_2), \quad q_1^T (q_2 \times M) = (q_1 \times q_2)^T M = -q_2^T (q_1 \times M) \quad (3.6)\]

and, more general

\[(q_1 \times M)N = q_1 \times (MN), \quad N^T (q_2 \times M) = -(q_2 \times N)^T M = (N^T \times q_2^T) M. \quad (3.7)\]

With these relations, the infinitesimal microrotation $A_{\vartheta}$ can be expressed as

\[A_{\vartheta} := \vartheta \times \mathbb{1}_3 = \mathbb{1}_3 \times \vartheta^T \in \mathbb{R}^{3 \times 3}. \quad (3.8)\]

It is possible to write the linearised strain measures in a simplified form. The relation

\[E_{lin}^{m,s} = \left( \nabla v - A_{\vartheta} \nabla y_0 \big| 0 \right) \left[ \nabla \vartheta \right]^{-1} \]

turns into

\[E_{lin}^{m,s} = \left( \nabla v - \vartheta \times \nabla y_0 \big| 0 \right) \left[ \nabla \vartheta \right]^{-1} = \left[ \left( \nabla v \big| 0 \right) - \vartheta \times \left( \nabla y_0 \big| 0 \right) \right] \left[ \nabla \vartheta \right]^{-1}. \quad (3.9)\]

Doing the same for the elastic shell bending-curvature tensor

\[K_{e,s} := \left( \text{axl}(\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_{e,s}) | \text{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_{e,s}) \big| 0 \right) \left[ \nabla \vartheta \right]^{-1} \quad (3.10)\]

since

\[\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_{e,s} = (\mathbb{1}_3 - \overline{A}_{\vartheta}) \partial_{x_1} \overline{A}_{\vartheta} + \text{h.o.t.} = \partial_{x_1} \overline{A}_{\vartheta} + \text{h.o.t.} = \overline{A}_{\partial_{x_1} \vartheta} + \text{h.o.t.} = \text{Anti} \partial_{x_1} \vartheta + \text{h.o.t.} \quad (3.11)\]

i.e., \(\text{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_{e,s}) = \partial_{x_2} \vartheta + \text{h.o.t.}\) we deduce $K_{e,s}^{lin} = (\text{axl}(\partial_{x_1} \overline{A}_{\vartheta}) | \text{axl}(\partial_{x_2} \overline{A}_{\vartheta}) \big| 0) \left[ \nabla \vartheta \right]^{-1}$ and

\[K_{e,s}^{lin} = \left( \partial_{x_1} \vartheta \big| \partial_{x_2} \vartheta \big| 0 \right) \left[ \nabla \vartheta \right]^{-1} = \left( \nabla \vartheta \big| 0 \right) \left[ \nabla \vartheta \right]^{-1}. \quad (3.12)\]

**Remark 3.1** The matrix $C_{y_0}$ admits the form

\[C_{y_0} = -n_0 \times \mathbb{1}_3 = -n_0 \times A_{y_0}. \quad (3.13)\]

**Proof** To prove (3.13) we put the definition (2.6) in the following form

\[C_{y_0} = \left[ \nabla \vartheta \right]^{-T} C^0 \left[ \nabla \vartheta \right]^{-1}, \quad \text{with} \quad C = \begin{pmatrix} 0 & \sqrt{\det I_{y_0}} \\ -\sqrt{\det I_{y_0}} & 0 \end{pmatrix}. \quad (3.14)\]

Here, and in the rest of the paper, $a_1, a_2, a_3$ denote the columns of $\nabla \vartheta$, while $a^1, a^2, a^3$ denote the rows of $[\nabla \vartheta]^{-1}$, i.e.,

\[\nabla \vartheta = (\nabla y_0 | n_0) = (a_1 | a_2 | a_3), \quad [\nabla \vartheta]^{-1} = (a^1 | a^2 | a^3)^T. \quad (3.15)\]
In fact, $a_1, a_2$ are the covariant base vectors and $a^1, a^2$ are the contravariant base vectors in the tangent plane given by $a_\alpha := \partial_{x_\alpha} y_0$, $\langle a^\beta, a_\alpha \rangle = \delta^\beta_\alpha$, $\alpha, \beta = 1, 2$, and $a_3 = a^3 = n_0$. The following relations hold [20, page 95]: $\|a_1 \times a_2\| = \sqrt{\det I_{y_0}}$, $a_3 \times a_1 = \sqrt{\det I_{y_0}} a^2$, $a_2 \times a_3 = \sqrt{\det I_{y_0}} a^1$.

Next, we calculate

$$\left[ \nabla \Theta \right]^{-T} C^0 = (a^1 \mid a^2 \mid a^3) \left( \begin{array}{ccc} 0 & \sqrt{\det I_{y_0}} & 0 \\ 0 & 0 & 0 \\ \sqrt{\det I_{y_0}} & 0 & 0 \end{array} \right) = (a^2 \sqrt{\det I_{y_0}} a^1 \mid a^1 \sqrt{\det I_{y_0}} a^1 \mid 0)$$

(3.16)

$$= (a_1 \times n_0 \mid a_2 \times n_0 \mid 0) = -n_0 \times \nabla \Theta .$$

Using (3.6) and the decomposition $I_3 = A_{y_0} + (0 \mid 0 \mid n_0 ) (0 \mid 0 \mid n_0 )^T$, see Remark 2.3, we have

$$C_{y_0} = (-n_0 \times \nabla \Theta) [\nabla \Theta]^{-1} = -n_0 \times I_3 = -n_0 \times A_{y_0}$$

and (3.13) is proved.

From the definition $K_{y_0}^{\text{lin}, s} = (\nabla \vartheta \mid 0) [\nabla \Theta]^{-1}$ together with (3.13) we obtain

$$C_{y_0} K_{y_0}^{\text{lin}, s} = -n_0 \times (\nabla \vartheta \mid 0) [\nabla \Theta]^{-1} .$$

(3.17)

In conclusion, the linear strain measures $E_{n, s}^{\text{lin}}$ and $K_{y_0}^{\text{lin}, s}$ are given by (3.4) and (3.12).

In order to obtain a comparison with the classical linear Koiter-shell model, we also deduce the linear approximation of the constitutive variables $G$, $T$ and $R$. The linear approximation of the change of metric tensor from (2.21)

$$G_{y_0}^{\text{lin}} = (\nabla y_0)^T (\nabla v - \overline{A}_{\vartheta} \nabla y_0) ,$$

(3.18)

admits the alternative expression, due to (3.7),

$$G_{y_0}^{\text{lin}} = (\nabla y_0)^T (\nabla v - \vartheta \times \nabla y_0) , \quad \text{or} \quad G_{y_0}^{\text{lin}} = (\nabla y_0)^T \nabla v + (\vartheta \times \nabla y_0)^T \nabla y_0 .$$

(3.19)

Using (3.6), the linear approximation of the transverse shear vector

$$T_{y_0}^{\text{lin}} = n_0^T (\nabla v - \overline{A}_{\vartheta} \nabla y_0)$$

(3.20)

reads

$$T_{y_0}^{\text{lin}} = n_0^T (\nabla v - \vartheta \times \nabla y_0) , \quad \text{or} \quad T_{y_0}^{\text{lin}} = n_0^T \nabla v + \vartheta^T (n_0 \times \nabla y_0) = n_0^T \nabla v + (\vartheta \times n_0)^T \nabla y_0 .$$

(3.21)

Using (3.19) and (3.21) we can form the matrix

$$\begin{pmatrix} G_{y_0}^{\text{lin}} \\ T_{y_0}^{\text{lin}} \end{pmatrix} = \begin{pmatrix} (\nabla y_0)^T \nabla v + (\vartheta \times \nabla y_0)^T \nabla y_0 \\ n_0^T \nabla v + (\vartheta \times n_0)^T \nabla y_0 \end{pmatrix} = [\nabla \Theta]^T \nabla v + [\vartheta \times \nabla \Theta]^T \nabla y_0 ,$$

(3.22)

in accordance with (3.9) and (2.20).
For the bending strain tensor we get the linearisation

\[
R^{\text{lin}} = -(\nabla y_0)^T (\mathbb{I}_3 - \overline{A}_\theta) \nabla ((\mathbb{I}_3 + \overline{A}_\theta) n_0) - \Pi_y = (\nabla y_0)^T (\overline{A}_\theta \nabla n_0 - \nabla (\overline{A}_\theta n_0)) \tag{3.23}
\]

and, see (3.7), its alternative expressions

\[
R^{\text{lin}} = (\nabla y_0)^T (n_0 \times \nabla \vartheta) \quad \text{or} \quad R^{\text{lin}} = -(n_0 \times \nabla y_0)^T \nabla \vartheta. \tag{3.24}
\]

To obtain a comparison with the classical linear Koiter-shell model, let us first present an alternative form of \(G^{\text{lin}}\). From (3.3) we have

\[
G^{\text{lin}} = (\nabla y_0)^T (\nabla v - \overline{A}_\theta \nabla y_0) = (\nabla y_0)^T (\partial x_1 u + a_1 \times \vartheta \mid \partial x_2 u + a_2 \times \vartheta)
\]

\[
= (\nabla y_0)^T (\nabla v) + (a_1 | a_2)^T (a_1 \times \vartheta \mid a_2 \times \vartheta)
\]

\[
= (\nabla y_0)^T (\nabla v) + \left[ \begin{array}{c} 0 \\ \vartheta - [a_1 \times a_2] \\ 0 \end{array} \right]
\]

\[
= (\nabla y_0)^T (\nabla v) + \left[ \begin{array}{c} 0 \\ \vartheta_n \end{array} \right] \left[ \begin{array}{c} 0 \\ \det I \end{array} \right]
\]

\[
= (\nabla y_0)^T (\nabla v) + \vartheta_n \det I_y(\vartheta, n_0) C. \tag{3.25}
\]

Remember that \(C\) defined by (3.14) is a 2 \(\times\) 2 skew-symmetric matrix. From (3.25) we note the relation

\[
sym(G^{\text{lin}}) = \text{sym}[(\nabla y_0)^T (\nabla v)] =: G^{\text{lin}}_{Koiter}, \tag{3.26}
\]

therefore \(G^{\text{lin}}_{Koiter}\) corresponds to the symmetric part of our \(G^{\text{lin}}\).

### 3.2 The Variational Problem of the Linearized \(O(h^5)\)-Cosserat Shell Model

The form of the energy density remains unchanged upon linearization, since the geometrically nonlinear model is physically linear (quadratic in the employed strain and curvature measures). Thus, the variational problem for the linear Cosserat \(O(h^5)\)-shell model is to find a mid-surface displacement vector field \(v: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3\) and the micro-rotation vector field \(\vartheta: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3\) minimizing on \(\omega\):

\[
I(v, \vartheta) = \int_\omega \left[ \left( h + \frac{K}{12} \right) W_{\text{shell}}(\varepsilon_{m,s}^{\text{lin}}) + \left( \frac{h^3}{12} - \frac{K h^5}{80} \right) W_{\text{shell}}(\varepsilon_{m,s}^{\text{lin}} B_{y_0} + C_{y_0} \kappa_{e,s}^{\text{lin}}) 
\right.
\]

\[
- \frac{h^3}{3} H W_{\text{shell}}(\varepsilon_{m,s}^{\text{lin}}, \varepsilon_{m,s}^{\text{lin}} B_{y_0} + C_{y_0} \kappa_{e,s}^{\text{lin}})
\]

\[
+ \frac{h^3}{6} W_{\text{shell}}(\varepsilon_{m,s}^{\text{lin}}, (\varepsilon_{m,s}^{\text{lin}} B_{y_0} + C_{y_0} \kappa_{e,s}^{\text{lin}}) B_{y_0})
\]

\[
+ \frac{h^5}{80} W_{\text{imp}}(\varepsilon_{m,s}^{\text{lin}} B_{y_0} + C_{y_0} \kappa_{e,s}^{\text{lin}}) B_{y_0} \left) + \left( h - \frac{K h^3}{12} \right) W_{\text{curv}}(\kappa_{e,s}^{\text{lin}}) + \left( \frac{h^3}{12} - \frac{K h^5}{80} \right) W_{\text{curv}}(\kappa_{e,s}^{\text{lin}} B_{y_0}) \right] \tag{3.27}
\]
\[ + \frac{h^5}{80} \mathcal{W}_{\text{curv}}(\mathcal{K}_{e,s}^\text{lin} \mathcal{B}_y^2) \det(\nabla y_0|n_0)da \\
- \Pi^\text{lin}(v, \vartheta), \]

where

\[ \varepsilon_{m,s}^\text{lin} := (\nabla v - A_\vartheta \nabla y_0| 0) [\nabla \Theta]^{-1} = [(\nabla v| 0 - \vartheta \times (\nabla y_0| 0)] [\nabla \Theta]^{-1}, \]

\[ \mathcal{K}_{e,s}^\text{lin} := (\text{axl}(\partial_{x_1} A_\vartheta) | \text{axl}(\partial_{x_2} A_\vartheta)| 0) [\nabla \Theta]^{-1} = (\nabla \vartheta| 0) [\nabla \Theta]^{-1}, \]

(3.28)

and \( \Pi^\text{lin}(v, \vartheta) \) is the linearization of \( \Pi(m, \overline{Q}_{e,s}) \).

For simplicity, we consider only \( (\gamma_d = \partial \omega) \) the Dirichlet-type homogeneous boundary conditions:

\[ v|_{\partial \omega} = 0 \quad \text{simply supported (fixed, welded)}, \quad \vartheta|_{\partial \omega} = 0 \quad (A_\vartheta|_{\partial \omega} = 0), \quad \text{clamped}, \]

(3.29)

where the boundary conditions are to be understood in the sense of traces. Therefore, the admissible set \( \mathcal{A}_{\text{lin}} \) of solutions \( (v, \vartheta) \) is defined by

\[ (v, \vartheta) \in \mathcal{A}_{\text{lin}} := H_0^1(\omega, \mathbb{R}^3) \times H_0^1(\omega, \mathbb{R}^3). \]

(3.30)

Using the orthogonal decomposition in the tangential plane and in the normal direction, gives

\[ X = X^\parallel + X^\perp, \quad X^\parallel := A_{y_0} X, \quad X^\perp := (1 - A_{y_0}) X, \]

(3.31)

and we deduce that for all \( X = (\ast| \ast| 0) \cdot [\nabla z \Theta(0)]^{-1} \) we have the following split in the expression of the considered quadratic forms

\[ W_{\text{shell}}(X) = \mu \| \text{sym} X^\parallel \|^2 + \mu_c \| \text{skew} X^\parallel \|^2 + \frac{\mu + \mu_c}{2} \| X^\perp \|^2 + \frac{\lambda \mu}{\lambda + 2 \mu} \left[ \text{tr}(X) \right]^2 \]

(3.32)

and

\[ W_{\text{curv}}(X) = \mu \mathcal{L}_c^2 \left( b_1 \| \text{sym} X^\parallel \|^2 + b_2 \| \text{skew} X^\parallel \|^2 + \frac{12b_3 - b_1}{3} \left[ \text{tr}(X^\parallel) \right]^2 \right. \]

\[ + \left. \frac{b_1 + b_2}{2} \| X^\perp \|^2 \right). \]

(3.33)

Skipping now all bending related \( h^3 \)-terms we note that there is only one difference between the linearised membrane energy obtained via the derivation approach [29] and the membrane energy obtained via \( \Gamma \)-convergence [48], i.e., the weight of the energy term of the type \( \| X^T n_0 \|^2 = \| X^\perp \|^2 \): 

- derivation approach: the algebraic mean of \( \mu \) and \( \mu_c \), i.e., \( \frac{\mu + \mu_c}{2} \);
- \( \Gamma \)-convergence: the harmonic mean of \( \mu \) and \( \mu_c \), i.e., \( \frac{2 \mu \mu_c}{\mu + \mu_c} \).
This difference has already been observed for the Cosserat flat shell \( \Gamma \)-limit [40]. Like in the case for the membrane part, the weight of the energy term \( \| K_{\text{curv}}^{1} \|^{2} = \| K_{\text{curv}}^{2} n_{0} \|^{2} \) are different as following:

- derivation approach: the algebraic mean of \( b_{1} \) and \( b_{2} \), i.e., \( \frac{b_{1} + b_{2}}{2} \);
- \( \Gamma \)-convergence: the harmonic mean of \( b_{1} \) and \( b_{2} \), i.e., \( \frac{2b_{1}b_{2}}{b_{1} + b_{2}} \).

The other terms of the membrane energy and of the descendent of the 3D curvature energy is in complete agreement to the model derived via the derivation approach.

### 3.3 Existence for the Linearised Cosserat Shell Model

#### 3.3.1 Existence Result for the Linearised \( O(h^5) \)-Cosserat Shell Model

We rewrite the minimization problem (3.27) in a weak form. For this we define the operators

\[
\begin{align*}
\mathcal{E}^{\text{lin}} : \mathcal{A}_{\text{lin}} &\rightarrow \mathbb{R}^{3 \times 3}, \\
\mathcal{K}^{\text{lin}} : \mathcal{A}_{\text{lin}} &\rightarrow \mathbb{R}^{3 \times 3}
\end{align*}
\]

\( \mathcal{E}^{\text{lin}}(v, \vartheta) = \left[ (\nabla v \mid 0) - \vartheta \times (\nabla y_{0} \mid 0) \right] [\nabla \varTheta]^{-1}, \)

\( \mathcal{K}^{\text{lin}}(v, \vartheta) = (\nabla \vartheta \mid 0) [\nabla \varTheta]^{-1}, \)

(3.34)

the bilinear form \( \mathcal{B}^{\text{lin}} : \mathcal{A}_{\text{lin}} \times \mathcal{A}_{\text{lin}} \rightarrow \mathbb{R} \),

\[
\mathcal{B}^{\text{lin}}((v, \vartheta), (\tilde{v}, \tilde{\vartheta}))
\]

\[
:= \int_{\omega} \left[ \left( h + K \frac{h^3}{12} \right) \mathcal{W}_{\text{shell}}(\mathcal{E}^{\text{lin}}(v, \vartheta), \mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) \\
+ \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) \mathcal{W}_{\text{shell}}(\mathcal{E}^{\text{lin}}(v, \vartheta) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(v, \vartheta), \mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) \\
- \frac{h^3}{6} H \mathcal{W}_{\text{shell}}(\mathcal{E}^{\text{lin}}(v, \vartheta), \mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) \\
- \frac{h^3}{6} H \mathcal{W}_{\text{shell}}(\mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}), \mathcal{E}^{\text{lin}}(v, \vartheta) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(v, \vartheta)) \\
+ \frac{h^3}{12} \mathcal{W}_{\text{shell}}(\mathcal{E}^{\text{lin}}(v, \vartheta), (\mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) B_{y_{0}}) \\
+ \frac{h^3}{12} \mathcal{W}_{\text{shell}}(\mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}), (\mathcal{E}^{\text{lin}}(v, \vartheta) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(v, \vartheta)) B_{y_{0}}) \\
+ \frac{h^5}{80} \mathcal{W}_{\text{mp}}((\mathcal{E}^{\text{lin}}(v, \vartheta) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(v, \vartheta)) B_{y_{0}}, (\mathcal{E}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_{0}} + C_{y_{0}} \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) B_{y_{0}}) \\
+ \left( h - K \frac{h^3}{12} \right) \mathcal{W}_{\text{curv}}(\mathcal{K}^{\text{lin}}(v, \vartheta), \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) \\
+ \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) \mathcal{W}_{\text{curv}}(\mathcal{K}^{\text{lin}}(v, \vartheta) B_{y_{0}}, \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_{0}}) \\
+ \frac{h^5}{80} \mathcal{W}_{\text{curv}}(\mathcal{K}^{\text{lin}}(v, \vartheta) B_{y_{0}}^2, \mathcal{K}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_{0}}^2) \right] \det(\nabla y_{0} | n_{0}) da,
\]
and the linear operator

\[ \Pi^{\text{lin}} : A_{\text{lin}} \rightarrow \mathbb{R}, \quad \Pi^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) = \Pi^{\text{lin}}(\tilde{v}, \tilde{\vartheta}). \]  

(3.36)

Then the weak form of the equilibrium problem of the linear theory of Cosserat shells including terms up to order \( O(h^5) \) is to find \((u, \vartheta) \in A_{\text{lin}}\) satisfying

\[ B^{\text{lin}}((u, \vartheta), (\tilde{v}, \tilde{\vartheta})) = \Pi^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) \quad \forall (\tilde{v}, \tilde{\vartheta}) \in A_{\text{lin}}. \]  

(3.37)

**Theorem 3.2** (Existence result for the linear theory including terms up to order \( O(h^5) \)) Assume that the linear operator \( \Pi^{\text{lin}} \) is bounded and that the following conditions concerning the initial configuration are satisfied: \( y_0 : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a continuous injective mapping and

\[ y_0 \in H^1(\omega, \mathbb{R}^3), \quad \nabla_x \Theta(0) \in L^\infty(\omega, \mathbb{R}^{3 \times 3}), \quad \det(\nabla_x \Theta(0)) \geq a_0 > 0, \]  

(3.38)

where \( a_0 \) is a constant. Then, for sufficiently small values of the thickness \( h \) such that

\[ h \max\{\sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2|\} < \alpha \quad \text{with} \quad \alpha < \frac{2}{\sqrt{3}} \left(29 - \sqrt{761}\right) \approx 0.97083 \]  

(3.39)

and for constitutive coefficients such that \( \mu > 0, \mu_c > 0, 2\lambda + \mu > 0, b_1 > 0, b_2 > 0 \) and \( b_3 > 0 \), the problem (3.37) admits a unique solution \((u, \vartheta) \in A_{\text{lin}}\).

**Proof** The proof is based on the Lax–Milgram theorem. Since the thickness \( h \) satisfies (3.39), then due to the results established in [30, 31] for the geometrically nonlinear model, the internal energy is automatically coercive in terms of the linearised strain and curvature variables \( E^{\text{lin}}(v, \vartheta), K^{\text{lin}}(v, \vartheta) \in L^2(\omega, \mathbb{R}^{3 \times 3}) \), in the sense that for all \((v, \vartheta) \in A\) there exists a constant \( a_1^+ > 0 \) such that

\[ B^{\text{lin}}((v, \vartheta), (v, \vartheta)) \geq a_1^+ \left(\|E^{\text{lin}}(v, \vartheta)\|^2_{L^2(\omega)} + \|K^{\text{lin}}(v, \vartheta)\|^2_{L^2(\omega)}\right), \]  

(3.40)

where \( a_1^+ \) depends on the given constitutive coefficients.

Since the bilinear form \( B^{\text{lin}} \) is bounded on \( A_{\text{lin}} \), as well as the linear operator \( \Pi^{\text{lin}} \), it remains to prove that the bilinear form \( B^{\text{lin}} \) is coercive on \( A_{\text{lin}} \). To this aim, it only remains to prove that for all \((v, \vartheta) \in A_{\text{lin}}\) there exists \( c > 0 \) such that

\[ \|E^{\text{lin}}(v, \vartheta)\|^2_{L^2(\omega)} + \|K^{\text{lin}}(v, \vartheta)\|^2_{L^2(\omega)} \geq c (\|v\|^2_{H^1(\omega)} + \|\vartheta\|^2_{H^1(\omega)}). \]  

(3.41)

The first step is to use the following alternative form of the linearised strain measures, i.e.,

\[ E^{\text{lin}}(v, \vartheta) = (\nabla v - A_\vartheta \nabla y_0 | 0) [\nabla \Theta]^{-1}, \quad \text{where} \quad A_\vartheta = \text{Anti} \vartheta. \]  

(3.42)

The next step is to remark that, because the lifted \( 3 \times 3 \) quantity \( \tilde{V}_{y_0} = (\nabla y_0 | n_0)^T (\nabla y_0 | n_0) \in \text{Sym}(3) \) is positive definite and also its inverse is positive definite, using also the Cauchy–Schwarz inequality and the inequality of arithmetic and geometric means we obtain

\[ \|E^{\text{lin}}(v, \vartheta)\|^2 = \left\{(\nabla v - A_\vartheta \nabla y_0 | 0) [\nabla \Theta]^{-1}, (\nabla v - A_\vartheta \nabla y_0 | 0) [\nabla \Theta]^{-1}\right\} \]

\[ \times \left\{ (\nabla v - A_\vartheta \nabla y_0 | 0) [\nabla \Theta]^{-1} [\nabla \Theta]^{-T}, (\nabla v - A_\vartheta \nabla y_0 | 0) \right\} \]
Moreover, since the Frobenius norm is sub-multiplicative, we have
\[
\lambda_0 \left( \| \nabla v - \overline{A}_\theta \nabla y_0 \|_0, \, (\nabla v - \overline{A}_\theta \nabla y_0) \right) \geq \lambda_0 \left( \| \nabla v \|_0^2 + \| \overline{A}_\theta \nabla y_0 \|_0^2 - 2 \{ \nabla v, \overline{A}_\theta \nabla y_0 \} \right)
\]
\[
\geq \lambda_0 \left[ (1 - \varepsilon) \| \nabla v \|_0^2 + \left( 1 - \frac{1}{\varepsilon} \right) \| \overline{A}_\theta \nabla y_0 \|_0^2 \right].
\]
for all \( \varepsilon > 0 \), where \( 0 < \lambda_0 < 1 \) is the smallest eigenvalue of the positive definite matrix \( \widehat{I}^{-1} \), over \( \omega \).

Similarly, we deduce that
\[
\| \mathcal{K}^{\text{lin}}(v, \vartheta) \|_2^2 = \| (\nabla \vartheta \ | 0) \|_2^2 \geq \lambda_0 \| (\nabla \vartheta \ | 0) \|_2^2 = \lambda_0 \| \nabla \vartheta \|_2^2
\]
\[
= \frac{1}{2} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \vartheta \|_2^2 = \frac{1}{2} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \vartheta \|_2^2 = \frac{1}{2} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \overline{A}_\theta \|_2^2.
\]

Moreover, since \( \overline{A}_\theta \in H_0^1(\omega; \mathfrak{so}(3)) \), we deduce the estimate
\[
\sum_{\alpha=1,2} \| \partial_{x_\alpha} \overline{A}_\theta \|_{L^2(\omega)}^2 \geq c_p \| \overline{A}_\theta \|_{L^2(\omega)}^2,
\]
where \( c_p > 0 \) is the constant depending only on \( \omega \) coming from the Poincaré inequality. Because the Frobenius norm is sub-multiplicative, we have
\[
\| \overline{A}_\theta \nabla y_0 \|_0^2 \leq \| \overline{A}_\theta \|_2^2 \sup_{(x_1, x_2) \in \omega} \| \nabla y_0 \|_0^2 \leq \| \overline{A}_\theta \|_2^2 \sup_{(x_1, x_2) \in \omega} \| \nabla y_0 \|_0^2
\]
and therefore
\[
\| \overline{A}_\theta \nabla y_0 \|_{L^2(\omega)}^2 \leq \| \overline{A}_\theta \|_{L^2(\omega)}^2 \sup_{(x_1, x_2) \in \omega} \| \nabla y_0 \|_0^2.
\]
Using (3.44), (3.45) and (3.47) we deduce
\[
\| \mathcal{K}^{\text{lin}}(v, \vartheta) \|_{L^2(\omega)}^2 = \frac{1}{2} \| \mathcal{K}^{\text{lin}}(v, \vartheta) \|_{L^2(\omega)}^2 + \frac{1}{2} \| \mathcal{K}^{\text{lin}}(v, \vartheta) \|_{L^2(\omega)}^2
\]
\[
\geq \frac{1}{4} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \overline{A}_\theta \|_2^2 + \frac{c_p \lambda_0}{4} \| \overline{A}_\theta \|_{L^2(\omega)}^2
\]
\[
\geq \frac{1}{4} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \overline{A}_\theta \|_2^2 + \frac{c_p \lambda_0}{4} \sup_{(x_1, x_2) \in \omega} \| \nabla y_0 \|_0^2 \| \overline{A}_\theta \nabla y_0 \|_{L^2(\omega)}^2
\]
\[
= \frac{1}{4} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \overline{A}_\theta \|_2^2 + \frac{c_p \lambda_0}{4} \frac{\| \overline{A}_\theta \nabla y_0 \|_{L^2(\omega)}^2}{\| \overline{A}_\theta \nabla y_0 \|_{L^2(\omega)}}
\]
\[
\geq \frac{1}{4} \lambda_0 \sum_{\alpha=1,2} \| \partial_{x_\alpha} \overline{A}_\theta \|_2^2 + \frac{c_p \lambda_0}{8 \lambda_M} \| \overline{A}_\theta \nabla y_0 \|_{L^2(\omega)}^2,
\]
where $\lambda_M > 1$ is the largest eigenvalue of the given positive definite matrix $\hat{T}_{y_0}$, over $\omega$. Hence with (3.48) we have

$$\| \mathcal{E}_{\text{lin}}(v, \vartheta) \|^2_{L^2(\omega)} + \| \mathcal{K}_{\text{lin}}(v, \vartheta) \|^2_{L^2(\omega)} \geq \lambda_0 \left[ (1 - \varepsilon) \| \nabla v \|^2_{L^2(\omega)} + \left( 1 - \frac{1}{\varepsilon} + \frac{c_p}{8 \lambda_M} \right) \| \hat{A}_\vartheta \nabla y_0 \|^2_{L^2(\omega)} + \frac{1}{2} \sum_{\alpha=1,2} \| \partial_x^\alpha \hat{A}_\vartheta \|^2_{L^2(\omega)} \right]$$

for all $\varepsilon > 0$. Now we are looking for some $\varepsilon > 0$ such that in parallel

$$1 - \varepsilon > 0 \quad \text{and} \quad 1 - \frac{1}{\varepsilon} + \frac{c_p}{8 \text{tr}(I_{y_0})} > 0, \quad \text{i.e.,} \quad 1 > \varepsilon > \frac{1}{1 + \frac{c_p}{8 \text{tr}(I_{y_0})}}. \quad (3.50)$$

Since, $c_p > 0$ and $\lambda_M > 0$, such a constant $\varepsilon > 0$ exists and there is a positive constant $c_1 > 0$ such that the following Korn-type inequality, see [9] for some related Korn-type inequalities in the Cosserat theory with a deformable vector, is satisfied

$$\| \mathcal{E}_{\text{lin}}(v, \vartheta) \|^2_{L^2(\omega)} + \| \mathcal{K}_{\text{lin}}(v, \vartheta) \|^2_{L^2(\omega)} \geq c_1 \left[ \| \nabla v \|^2_{L^2(\omega)} + \| \nabla \vartheta \|^2_{L^2(\omega)} \right], \quad (3.51)$$

i.e., a positive constant $c > 0$ such that

$$\| \mathcal{E}_{\text{lin}}(v, \vartheta) \|^2_{L^2(\omega)} + \| \mathcal{K}_{\text{lin}}(v, \vartheta) \|^2_{L^2(\omega)} \geq c \left[ \| \nabla v \|^2_{L^2(\omega)} + \| \nabla \vartheta \|^2_{L^2(\omega)} \right] \forall (v, \vartheta) \in \mathcal{A}_{\text{lin}}. \quad (3.52)$$

Hence, the bilinear form $B_{\text{lin}}$ is coercive and the Lax-Milgram theorem leads to the conclusion of the theorem. ■

### 3.3.2 Existence Result for the Linearised Truncated $O(h^3)$-Cosserat Shell Model

As a restricted case of the linear Cosserat $O(h^5)$-shell model we obtain the linear Cosserat $O(h^3)$-shell model, by ignoring the terms of order $O(h^5)$, i.e., the variational problem is to find a mid-surface displacement vector field $v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the micro-rotation vector field $\vartheta : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ minimizing on $\omega$:

$$I(v, \vartheta) = \int_{\omega} \left[ \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}^{\text{lin}}) + h^3 \frac{1}{12} W_{\text{shell}}(\mathcal{E}_{m,s}^{\text{lin}} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}^{\text{lin}}) ight. 
- \left. \frac{h^3}{3} H W_{\text{shell}}(\mathcal{E}_{m,s}^{\text{lin}} + \mathcal{K}_{e,s}^{\text{lin}} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}^{\text{lin}}) ight. 
+ \left. \frac{h^3}{6} W_{\text{shell}}(\mathcal{E}_{m,s}^{\text{lin}} (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}^{\text{lin}}) B_{y_0}) ight. 
+ \left. \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}^{\text{lin}}) + h^3 \frac{1}{12} W_{\text{curv}}(\mathcal{K}_{e,s}^{\text{lin}} B_{y_0}) \right] \det(\nabla y_0 | n_0) da 
- \Pi^{\text{lin}}(v, \vartheta),$$
where $\Pi_{\text{lin}}^{\text{lin}}(v, \vartheta)$ is the linearization of $\Pi(m, \overline{Q}_{e,s})$. The weak form of the equilibrium problem of the linear theory of Cosserat shells including terms up to order $O(h^3)$ is to find $(v, \vartheta) \in \mathcal{A}_{\text{lin}}$ satisfying

$$B_h^{\text{lin}}((v, \vartheta), (\tilde{v}, \tilde{\vartheta})) = \Pi_{\text{lin}}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) \quad \forall (\tilde{v}, \tilde{\vartheta}) \in \mathcal{A}_{\text{lin}},$$

(3.54)

where $B_h^{\text{lin}} : \mathcal{A}_{\text{lin}} \times \mathcal{A}_{\text{lin}} \to \mathbb{R}$,

$$B_h^{\text{lin}}((v, \vartheta), (\tilde{v}, \tilde{\vartheta})) := \int_\omega \left[ \left( h + K \frac{h^3}{12} \right) \mathcal{W}_{\text{shell}}(\epsilon^{\text{lin}}(v, \vartheta), \epsilon^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) + \frac{h^3}{12} \mathcal{W}_{\text{shell}}(\epsilon^{\text{lin}}(v, \vartheta) B_{y_0} + C_{y_0} K^{\text{lin}}(v, \vartheta), \epsilon^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_0} + C_{y_0} K^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) - \frac{h^3}{6} H \mathcal{W}_{\text{shell}}(\epsilon^{\text{lin}}(v, \vartheta), \epsilon^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_0} + C_{y_0} K^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) - \frac{h^3}{6} H \mathcal{W}_{\text{shell}}(\epsilon^{\text{lin}}(\tilde{v}, \tilde{\vartheta}), \epsilon^{\text{lin}}(v, \vartheta) B_{y_0} + C_{y_0} K^{\text{lin}}(v, \vartheta)) + \frac{h^3}{12} \mathcal{W}_{\text{shell}}(\epsilon^{\text{lin}}(v, \vartheta), (\epsilon^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_0} + C_{y_0} K^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) B_{y_0}) + \left( h - K \frac{h^3}{12} \right) \mathcal{W}_{\text{curv}}(K^{\text{lin}}(v, \vartheta), K^{\text{lin}}(\tilde{v}, \tilde{\vartheta})) + \frac{h^3}{12} \mathcal{W}_{\text{curv}}(K^{\text{lin}}(v, \vartheta) B_{y_0}, K^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) B_{y_0}) \right] \det(\nabla y_0 | n_0) da,$$

and the linear operator

$$\Pi_{\text{lin}} : \mathcal{A}_{\text{lin}} \to \mathbb{R}, \quad \Pi_{\text{lin}}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}) = \Pi_{\text{lin}}^{\text{lin}}(\tilde{v}, \tilde{\vartheta}).$$

(3.56)

A similar proof as in the case of the $O(h^3)$ model leads us to the following results (see also the proof of the coercivity from [30, 31])

**Theorem 3.3** (Existence result for the truncated linear theory including terms up to order $O(h^3)$). Assume that the linear operator $\Pi_{\text{lin}}$ is bounded and that the following conditions concerning the initial configuration are satisfied: $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a continuous injective mapping and

$$y_0 \in H^4(\omega, \mathbb{R}^3), \quad \nabla x \Theta(0) \in L^\infty(\omega, \mathbb{R}^{3 \times 3}), \quad \det(\nabla x \Theta(0)) \geq a_0 > 0,$$

(3.57)

where $a_0$ is a constant. Then, if the thickness $h$ satisfies either of the following conditions:

i) $h \max\{\sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2|\} < \alpha$ and $h^2 < \frac{(5-2\sqrt{\alpha})(\alpha^2-12)^2 c_1^*}{4 \alpha^2}$ with $0 < \alpha < 2\sqrt{3}$;

ii) $h \max\{\sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2|\} < \frac{1}{a}$ and $a > \max\left\{ 1 + \frac{\sqrt{2}}{2}, \sqrt{1+\frac{1+3}{2} c_1^*} \right\}$.
where $c_2^\pm$ denotes the smallest eigenvalue of $W_{\text{curv}}^\infty(S)$, and $c_1^\pm$ and $C_1^\pm > 0$ denote the smallest and the biggest eigenvalues of the quadratic form $W_{\text{shell}}^\infty(S)$, and for constitutive coefficients such that $\mu > 0$, $2 k + \mu > 0$, $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $L_c > 0$, the problem (3.54) admits a unique solution $(u, \vartheta) \in \mathcal{A}_{\text{lin}}$.

**Proof** The proof is based on the coercivity inequality already proven in [30, 31], see Proposition 2.2, and the same steps as in the proof of Theorem 3.2. 

**Remark 3.4** We observe that the conditions imposed on the thickness are more restrictive in the truncated $O(h^3)$ model than in the $O(h^5)$ model. In other words, the existence results hold true in $O(h^3)$ only for smaller values of the thickness $h$, in comparison to the $O(h^5)$ model. Moreover, while in the $O(h^5)$ model the conditions imposed on the thickness are independent on the constitutive coefficients (the same conditions for all shells, i.e., all materials), in the $O(h^3)$ model the conditions depend on the assumed constitutive coefficients.

## 4 The Linear Model for the Cosserat Flat Shell: A Consistency Check

For the Cosserat flat shell model (flat initial configuration) obtained in [39] we have $\Theta(x_1, x_2, x_3) = (x_1, x_2, x_3)$ which gives $\nabla \Theta = \mathbb{I}_3$ and $y_0(x_1, x_2) = (x_1, x_2) := \text{id}(x_1, x_2)$.

Also $Q_0 = \mathbb{I}_3$, $n_0 = e_3$, $\overline{Q}_{e,0}(x_1, x_2) = \overline{R}(x_1, x_2)$, $B_{id} = 0$, $C_{id} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$, $L_{id} = 0_2$, $K = 0$, and $H = 0$. Hence, see (3.17), $\mathcal{E}_{e,s}^{\text{lin}}$ and $\mathcal{K}_{e,s}^{\text{lin}}$ are

$$
\mathcal{E}_{m,s}^{\text{lin,plate}} := (\nabla v | 0) - (\overline{A}_\vartheta \cdot e_1 | \overline{A}_\vartheta \cdot e_2 | 0) = (\nabla v | \overline{A}_\vartheta \cdot e_3) - \overline{A}_\vartheta \\
= (\nabla v | 0) - \vartheta \times (e_1 | e_2 | 0) = (\partial_{x_1} v - \vartheta \times e_1 | \partial_{x_2} v - \vartheta \times e_2 | 0)
$$

(4.1)

and

$$
\mathcal{K}_{e,s}^{\text{lin,plate}} := \left( \text{axl}(\partial_{x_4} \overline{A}_\vartheta) | \text{axl}(\partial_{x_5} \overline{A}_\vartheta) | 0 \right) = (\nabla \vartheta | 0) ,
$$

(4.2)

respectively. Moreover, due to (3.17), we deduce

$$
C_{y0} \mathcal{K}_{e,s}^{\text{lin}} = -e_3 \times (\nabla \vartheta | 0) .
$$

(4.3)

The variational problem for the linear Cosserat plate model is to find a midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and the micro-rotation vector field $\vartheta : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ minimizing on $\omega$:

$$
I(v, \vartheta) = \int_{\omega} \left[ h W_{\text{shell}}(\mathcal{E}_{m,s}^{\text{lin,plate}}) + \frac{h^3}{12} W_{\text{shell}}(C_{y0} \mathcal{K}_{e,s}^{\text{lin,plate}}) + h W_{\text{curv}}(\mathcal{K}_{e,s}^{\text{lin,plate}}) \right] da \\
- \Pi_{\text{lin,plate}}(v, \vartheta) ,
$$

(4.4)

where $\mathcal{E}_{m,s}^{\text{lin,plate}}$, $\mathcal{K}_{e,s}^{\text{lin,plate}}$, and $C_{id} \mathcal{K}_{e,s}^{\text{lin}}$ have the expressions from above and $\Pi_{\text{lin,plate}}(v, \vartheta)$ is the linearization of $\Pi(m, \overline{Q}_{e,s})$ for a flat initial configuration. Note the automatic absence of $O(h^5)$-terms for the flat shell problem and that the partial derivatives of the third component of $\vartheta$ survive after linearization only in the curvature part of the energy.
Let us consider the following alternative expression of $\mathcal{C}_{m,s}^{\text{lin,plate}}$ and $\mathcal{C}_{id}\mathcal{K}_{e,s}^{\text{lin,plate}}$:

$$
\mathcal{C}_{m,s}^{\text{lin,plate}} = \begin{pmatrix}
G_{\text{lin,plate}}^{\text{lin,plate}} & 0 \\
\mathcal{T}_{\text{lin,plate}}^{\text{lin,plate}} & 0
\end{pmatrix},
\quad
\mathcal{C}_{id}\mathcal{K}_{e,s}^{\text{lin,plate}} = - \begin{pmatrix}
\mathcal{R}_{\text{lin,plate}}^{\text{lin,plate}} & 0 \\
0 & 0
\end{pmatrix},
$$

(4.5)

where

$$
G_{\text{lin,plate}} = (e_1 \mid e_2)^T (\nabla v - \overline{A}_\theta (e_1 \mid e_2)) = \begin{pmatrix}
\partial_{x_1} v_1 - \partial_3 v_3 \\
\partial_{x_1} v_2 \\
\partial_{x_2} v_1 + \partial_3 v_3
\end{pmatrix},
$$

$$
\mathcal{T}_{\text{lin,plate}} = e_3^T (\nabla v - \overline{A}_\theta (e_1 \mid e_2)) = \begin{pmatrix}
\partial_{x_1} v_1 + \partial_2 v_2 \\
\partial_{x_2} v_1 - \partial_3 v_3 \\
\partial_{x_2} v_2 - \partial_1 v_1
\end{pmatrix},
$$

(4.6)

$$
\mathcal{R}_{\text{lin,plate}} = (e_1 \mid e_2)^T (e_3 \times \nabla \vartheta) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{pmatrix}
$$

as well as the decomposition

$$
\mathcal{K}_{e,s}^{\text{lin,plate}} = \mathcal{C}_{id} (-\mathcal{C}_{id}\mathcal{K}_{e,s}^{\text{lin,plate}}) + (0 \mid 0 \mid e_3) (0 \mid 0 \mid (\mathcal{K}_{e,s}^{\text{lin,plate}})^T e_3)^T
$$

(4.7)

$$
= \mathcal{C}_{id} \begin{pmatrix}
\mathcal{R}_{\text{lin,plate}}^{\text{lin,plate}} & 0 \\
0 & 0
\end{pmatrix} + (0 \mid 0 \mid e_3) (0 \mid 0 \mid (\mathcal{N}_{\text{lin,plate}}^{\text{lin,plate}})^T) \mathcal{T},
$$

where

$$
\mathcal{N}_{\text{lin,plate}} := e_3^T \nabla \vartheta = (\partial_{x_1} \vartheta_3, \partial_{x_2} \vartheta_3),
$$

(4.8)

represents the row vector of drilling bendings. Then the variational problem for the linear Cosserat plate model is to find a midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the micro-rotation vector field $\vartheta : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ minimizing on $\omega$:

$$
I(v, \vartheta)
$$

$$
= \int_\omega \left[ h \left( \mu \| \text{sym } G_{\text{lin,plate}} \|^2 + \mu_c \| \text{skew } G_{\text{lin,plate}} \|^2 + \frac{\lambda \mu}{\lambda + 2\mu} \| \text{tr}(G_{\text{lin,plate}}) \|^2 \right) \right]
$$

in-plane deformation

$$
+ h \frac{\mu + \mu_c}{2} \| \mathcal{T}_{\text{lin,plate}} \|^2
$$

transverse shear

$$
+ \frac{h^3}{12} \left( \mu \| \text{sym } \mathcal{R}_{\text{lin,plate}} \|^2 + \mu_c \| \text{skew } \mathcal{R}_{\text{lin,plate}} \|^2 + \frac{\lambda \mu}{\lambda + 2\mu} \| \text{tr}(\mathcal{R}_{\text{lin,plate}}) \|^2 \right)
$$

bending-type terms of order $O(h^3)$

$$
+ h \mu \left[ b_1 \| \text{sym } \mathcal{R}_{\text{lin,plate}} \|^2 + \left( 8 b_3 + \frac{b_1}{3} \right) \| \text{skew } \mathcal{R}_{\text{lin,plate}} \|^2 + \frac{b_2 - b_1}{2} \| \text{tr}(\mathcal{R}_{\text{lin,plate}}) \|^2 \right]
$$

bending-type terms of order $O(h)$
Here, we have used that, since \(A\) shells \([17, 18, 27, 44, 45]\). In the resultant 6-parameter theory of shells, the strain energy them are included as particular cases in the linearised model of the 6-parameter theory of.

There exist many linear shell model derived in the framework of Cosserat theory. Most of

Comparison with the General 6-Parameter Shell Linear Model

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5 A Comparison with the General 6-Parameter Shell Linear Model

There exist many linear shell model derived in the framework of Cosserat theory. Most of

The simplest expression \(W_P(\varepsilon_{m,s}, \kappa_{e,s})\) has been proposed in the papers \([17, 18]\) in the form

\[
2 W_P(\varepsilon_{m,s}^{\text{lin}}, \kappa_{e,s}^{\text{lin}}) = C \left[ v (\varepsilon_{m,s}^{\text{lin}})^2 + (1 - v) \text{tr}((\varepsilon_{m,s}^{\text{lin}})^T \varepsilon_{m,s}^{\text{lin}}) \right] + \alpha_s C (1 - v) \|\varepsilon_{m,s}^{\text{lin}} n_0\|^2
+ D \left[ v (\kappa_{e,s}^{\text{lin}})^2 + (1 - v) \text{tr}((\kappa_{e,s}^{\text{lin}})^T \kappa_{e,s}^{\text{lin}}) \right]
+ \alpha_t D (1 - v) \|\kappa_{e,s}^{\text{lin}} n_0\|^2,
\]

(5.11)

where the decompositions of \(\varepsilon_{m,s}^{\text{lin}}\) and \(\kappa_{e,s}^{\text{lin}}\) into two orthogonal directions (in the tangential plane and in the normal direction)\(^2\) are considered, i.e.,

\[
\varepsilon_{m,s}^{\text{lin},\perp} = A_{y_0} \varepsilon_{m,s}^{\text{lin}} = (\mathbb{I}_3 - n_0 \otimes n_0) \varepsilon_{m,s}^{\text{lin}}, \quad \kappa_{e,s}^{\text{lin},\perp} = A_{y_0} \kappa_{e,s}^{\text{lin}} = (\mathbb{I}_3 - n_0 \otimes n_0) \kappa_{e,s}^{\text{lin}},
\]

(5.12)

\[
\varepsilon_{m,s}^{\text{lin},\parallel} = (\mathbb{I}_3 - A_{y_0}) \varepsilon_{m,s}^{\text{lin}} = n_0 \otimes n_0 \varepsilon_{m,s}^{\text{lin}}, \quad \kappa_{e,s}^{\text{lin},\parallel} = (\mathbb{I}_3 - A_{y_0}) \kappa_{e,s}^{\text{lin}} = n_0 \otimes n_0 \kappa_{e,s}^{\text{lin}}.
\]

The constitutive coefficient \(C = \frac{E h}{1 - \nu^2}\) is the stretching (in-plane) stiffness of the shell, \(D = \frac{E h^3}{12(1 - \nu^2)}\) is the bending stiffness, and \(\alpha_s, \alpha_t\) are two shear correction factors. Also, \(E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\) and \(\nu = \frac{\nu}{2(\lambda+\mu)}\) denote the Young modulus and Poisson ratio of the isotropic and homogeneous material. In the numerical treatment of non-linear shell problems, the values of the shear correction factors have been set to \(\alpha_s = 5/6, \alpha_t = 7/10\) in \([18]\). The value \(\alpha_s = 5/6\) is a classical suggestion, which has been previously deduced analytically by

\(^2\)Here, we have used that, since \(A_{y_0} = \mathbb{I}_3 - n_0 \otimes n_0\), for all \(X \in \mathbb{R}^{3 \times 3}\) it holds \(\|X\|^2 = \|X^T n_0\|^2\).
Reissner in the case of plates [37, 46]. Also, the value $\alpha_t = 7/10$ was proposed earlier in [42, see p. 78] and has been suggested in the work [41]. However, the discussion concerning the possible values of shear correction factors for shells is long and controversial in the literature [37, 38].

The coefficients in (5.11) are expressed in terms of the Lamé constants of the material $\lambda$ and $\mu$ now by the relations

\[
C^I = \frac{4 \mu (\lambda + \mu)}{\lambda + 2 \mu} h, \quad C (1 - \nu) = 2 \mu h, \\
D^I = \frac{4 \mu (\lambda + \mu)}{\lambda + 2 \mu} \frac{h^3}{12}, \quad D (1 - \nu) = \frac{\mu h^3}{6}.
\]

In [27], Eremeyev and Pietraszkiewicz have proposed a more general form of the strain energy density, namely

\[
2 W_{EP} (e_{m,s}^{\text{lin}}, K_{e,s}^{\text{lin}}) = \alpha_1 (\text{tr} e_{m,s}^{\text{lin}})^2 + \alpha_2 \text{tr} (e_{m,s}^{\text{lin}})^2 + \alpha_3 \text{tr} ((e_{m,s}^{\text{lin}})^T e_{m,s}^{\text{lin}}) + \alpha_4 \| e_{m,s}^{\text{lin}} n_0 \|^2 + \\
\beta_1 (\text{tr} K_{e,s}^{\text{lin}})^2 + \beta_2 \text{tr} (K_{e,s}^{\text{lin}})^2 + \beta_3 \text{tr} ((K_{e,s}^{\text{lin}})^T K_{e,s}^{\text{lin}}) + \beta_4 \| K_{e,s}^{\text{lin}} n_0 \|^2.
\]

Already, note the absence of coupling terms involving $K_{e,s}^{\text{lin}}$ and $e_{m,s}^{\text{lin}}$.

The eight coefficients $\alpha_k, \beta_k$ ($k = 1, 2, 3, 4$) can depend in general on the structure curvature tensor $K^0 = Q_0 (\text{axl}(Q_0^T \partial_{s_2} Q_0) | \text{axl}(Q_0^T \partial_{s_1} Q_0) | 0) (\nabla y_0 | n_0)^{-1}$ of the reference configuration, where $Q_0 = \text{polar}(\nabla y_0 | n_0)$.

We conclude

**Remark 5.1**

i) By comparing our $W_{out} (e_{m,s}^{\text{lin}}, K_{e,s}^{\text{lin}})$ with $W_{EP} (e_{m,s}^{\text{lin}}, K_{e,s}^{\text{lin}})$ we deduce the following identification of the constitutive coefficients $\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4$

\[
\alpha_1 = \left(h + K \frac{h^3}{12}\right) \frac{2 \mu \lambda}{2 \mu + \lambda}, \quad \alpha_2 = \left(h + K \frac{h^3}{12}\right) (\mu - \mu_c), \\
\alpha_3 = \left(h + K \frac{h^3}{12}\right) (\mu + \mu_c), \quad \alpha_4 = \left(h + K \frac{h^3}{12}\right) (\mu + \mu_c),
\]

\[
\beta_1 = 2 \left(h - K \frac{h^3}{12}\right) \mu L_c^2 \frac{12 b_3 - b_1}{3}, \quad \beta_2 = \left(h - K \frac{h^3}{12}\right) \mu L_c^2 (b_1 - b_2),
\]

\[
\beta_3 = \left(h - K \frac{h^3}{12}\right) \mu L_c^2 (b_1 + b_2), \quad \beta_4 = \left(h - K \frac{h^3}{12}\right) \mu L_c^2 (b_1 + b_2).
\]

ii) We observe that $\mu_c^{\text{drill}} := \alpha_3 - \alpha_2 = 2 \left(h + K \frac{h^3}{12}\right) \mu_c$, which means that the in-plane rotational couple modulus $\mu_c^{\text{drill}}$ of the Cosserat shell model is determined by the Cosserat couple modulus $\mu_c$ of the 3D Cosserat material.

iii) In our shell model, the constitutive coefficients are those from the three-dimensional formulation, while the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem. Another important property of our derivation is that it is not needed to use some exact solutions in order to fit the parameters of the obtained shell model.
All the coefficients appearing in the final energy are depending on the three dimensional parental constitutive parameters (which is natural since the material is the same, even if the body is thin or not), on the thickness and on the Gauß and mean curvature.

iv) Beside the one-to-one fitting of our constitutive parameters we also notice that in the 6-parameter shell problem, the characteristic length is automatically identified as $L_c = h$.

v) The major difference between our model and the previously considered general 6-parameter shell model is that we include terms up to order $O(h^5)$ and that, even in the case of a simplified theory of order $O(h^3)$, additional mixed terms like the membrane-bending part $W_{\text{memb.bend}}(c_{m,s}^{\text{lin}}, k_{e,s}^{\text{lin}})$ and $W_{\text{curv}}(k_{e,s}^{\text{lin}}, b_{y_0})$ are included, which are otherwise difficult to guess.

vi) Beside the fact that mixed membrane-bending terms are included, the constitutive coefficients in our shell model depend on both the Gauß curvature $K$ and the mean curvature $H$, see item i) and compare to (2.6). Moreover, due to the bilinearity of the density energy, if the final form of the energy density is expressed as a quadratic form in terms of $W_{\text{memb.bend}}(c_{m,s}^{\text{lin}}, k_{e,s}^{\text{lin}})$, as in the $W_{E\text{P}}(c_{m,s}^{\text{lin}}, k_{e,s}^{\text{lin}})$, then we remark that the dependence on the mean curvature is not only the effect of the presence of the mixed terms, due to the Cayley-Hamilton equation $B^2_{y_0} = 2HB_{y_0} - KA_{y_0}$. See for instance the energy term $W_{\text{curv}}(k_{e,s}^{\text{lin}}, b_{y_0}^2)$ or even $W_{\text{mp}}(c_{m,s}^{\text{lin}}, b_{y_0} + C_{y_0}k_{e,s}^{\text{lin}}, b_{y_0})$ from (3.27).

Appendix A: The Classical Linear (First) Koiter Membrane-Bending Model

According to [20, page 344], [22, page 154, ] in the linear (first) Koiter model, the variational problem is to find a midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ minimizing:

$$\int_{\omega} \left\{ \frac{h}{2} \left( \mu \| [\nabla \Theta]^T (G_{\text{Koiter}}^\text{lin})^2 [\nabla \Theta]^{-1} \|^2 + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr} \left[ [\nabla \Theta]^T (G_{\text{Koiter}}^\text{lin})^2 [\nabla \Theta]^{-1} \right]^2 \right) + \frac{h}{12} \left( \mu \| [\nabla \Theta]^T (R_{\text{Koiter}}^\text{lin})^2 [\nabla \Theta]^{-1} \|^2 + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr} \left[ [\nabla \Theta]^T (R_{\text{Koiter}}^\text{lin})^2 [\nabla \Theta]^{-1} \right]^2 \right) \right\} \times \text{det} (\nabla y_0 | n_0) \, da,$$

where $(G_{\text{Koiter}}^\text{lin})^\text{b}$ and $(R_{\text{Koiter}}^\text{lin})^\text{b}$ are the lifted quantities of the strain measures [20] given by

$$G_{\text{Koiter}}^\text{lin} := \frac{1}{2} \left[ I_m - I_{y_0} \right]^{\text{lin}} = \frac{1}{2} \left[ (\nabla y_0)^T (\nabla v) + (\nabla v)^T (\nabla y_0) \right] = \text{sym} \left( (\nabla y_0)^T (\nabla v) \right),$$

and

$$R_{\text{Koiter}}^\text{lin} := [\Pi_m - \Pi_{y_0}]^{\text{lin}} = \left( \left( n_0, \partial_{x_{\alpha \beta}} v \right) - \sum_{\gamma=1,2} \Gamma_{a\beta}^\gamma \partial_{x_{\gamma}} v a_{\alpha} \right)_{a\beta} \in \mathbb{R}^{2 \times 2}.$$ 

The expression of $R_{\text{Koiter}}^\text{lin}$ involves the Christoffel symbols $\Gamma_{a\beta}^\gamma$ on the surface given by $\Gamma_{a\beta}^\gamma = \{ \alpha^\gamma, \partial_{x_{\alpha}} a_{\beta} \} = \{ \partial_{x_{\alpha}} a_{\gamma}, a_{\beta} \} = \Gamma_{\beta\alpha}^\gamma$. Note that, using $m = y_0 + v$ and $(\nabla m)^T \nabla m = (\nabla y_0)^T \nabla y_0 + (\nabla y_0)^T \nabla v + (\nabla v)^T \nabla y_0 + h.o.t \in \mathbb{R}^{2 \times 2}$, the linear approximation of the difference $\frac{1}{2} \left[ I_m - I_{y_0} \right]^{\text{lin}}$ appearing in the Koiter model can easily be obtained [20, page 92], the linear approximation of the difference $[\Pi_m - \Pi_{y_0}]^{\text{lin}}$ needs some more insights from differential geometry [20, page 95] and it is based on formulas of Gauß $\partial_{x_{\alpha}} a_{\beta} = \sum_{\gamma=1,2} \Gamma_{a\beta}^\gamma a_{\gamma} +$...
\( b_{\alpha \beta} a^3 \) and \( \partial_{\alpha} a^\alpha = - \sum_{\gamma=1,2} \Gamma^\gamma_{\alpha \beta} a^\gamma + b_\beta^\gamma n_0 \) and the formulas of Weingarten \( \partial_{\alpha} a^3 = \partial_{\alpha} a^3 = - \sum_{\beta=1,2} b_{\alpha \beta} a^\beta = - \sum_{\beta=1,2} b_{\beta}^\gamma a_\gamma \) together with the relations [20, page 76] \( b_{\alpha \beta} (m) = - (\partial_{\alpha} a_3 (m), a_\beta (m)) = (\partial_{\alpha} a_\beta (m), a_3 (m)) = b_{\alpha \beta} (m) \), where \( b_{\alpha \beta} (m) \) are the components of the second fundamental form corresponding to the map \( m \), \( b_{\beta}^\alpha (m) \) are the components of the matrix associated to the Weingarten map (shape operator), and on the following linear approximation \( n = n_0 + \sqrt{\text{det}(\nabla y_0^T \nabla y_0)} \left( \partial_{\alpha} y_0 \times \partial_{\beta} y_0 + \partial_{\gamma} x_0 \times \partial_{\gamma} x_0 + \text{h.o.t} \right) - \text{tr}((\nabla y_0^T \nabla y_0)^{-1} \text{sym}(\nabla y_0^T \nabla x_0)) n_0 \).

We note that other alternative forms of the change of metric tensor and the change of curvature tensor in [20, Page 181] are

\[
G^{\text{lin}}_{\text{Koiter}} = \left( \frac{1}{2} (\partial_{\beta} v_\alpha + \partial_{\alpha} v_\beta) - \sum_{\gamma=1,2} \Gamma^\gamma_{\alpha \beta} v_\gamma - b_{\alpha \beta} v_3 \right)_{\alpha \beta} \in \mathbb{R}^{2 \times 2}, \tag{A.4}
\]

and

\[
R^{\text{lin}}_{\text{Koiter}} = \left( \partial_{\alpha} x_\beta v_3 - \sum_{\gamma=1,2} \Gamma^\gamma_{\alpha \beta} \partial_{\gamma} x_\alpha v_3 - \sum_{\gamma=1,2} b_{\gamma}^\alpha b_{\gamma \beta} v_3 \right.
\]

\[
+ \sum_{\gamma=1,2} b_{\beta}^\gamma (\partial_{\gamma} x_\beta v_\gamma - \sum_{\tau=1,2} \Gamma^\tau_{\beta \gamma} v_\gamma) + \sum_{\tau=1,2} b_{\beta}^\gamma (\partial_{\gamma} x_\alpha v_\gamma - \sum_{\tau=1,2} \Gamma^\tau_{\alpha \gamma} v_\gamma)
\]

\[
+ \sum_{\tau=1,2} (\partial_{\alpha} x_\beta)^\tau + \sum_{\gamma=1,2} \Gamma^\tau_{\alpha \gamma} b_{\beta}^\gamma - \sum_{\gamma=1,2} \Gamma^\tau_{\alpha \gamma} b_{\beta}^\gamma \right)_{\alpha \beta} \in \mathbb{R}^{2 \times 2}, \tag{A.5}
\]

respectively. Actually, on one hand, the last form of the curvature tensor will be considered when the admissible set of solutions of the variational problem will be defined. On the other hand, as noticed in [21, Page 175] by considering the form (A.3) of the change of metric tensor, we can impose substantially weaker regularity assumptions on the mapping \( y_0 \). For the linear (first) Koiter model the existence results are given in [20, Theorem 7.1.-1 and Theorem 7.1.-2], see also [16].

While the relation between \( G^{\text{lin}} \) and \( G^{\text{lin}}_{\text{Koiter}} \) holds in the general case, we are able to find a simple explicit relation between \( R^{\text{lin}} \) and \( R^{\text{lin}}_{\text{Koiter}} \) only in the case of the constrained Cosserat-shell model. This is not surprising, since only symmetric stress tensors are taken into account in the classical linear Koiter model, i.e., the internal strain energy does not depend on the skew-symmetric part of the considered strain measures (since it is work conjugate to the skew-symmetric part of the stress tensor). In addition, the linear Koiter model does not consider extra degrees of freedom. In [32] we will discuss the choice of the deformation measures in shell models. We will see that the classical strain measure \( R^{\text{lin}}_{\text{Koiter}} \) (the classical bending strain measure, also known as the change of curvature tensor) does not represent the unique choice and that some other modified expressions of the classical bending tensor may be more suitable in the modelling of a shell.

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