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A novel power consensus algorithm for DC microgrids

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ABSTRACT

A power consensus algorithm for DC microgrids is proposed and analyzed. DC microgrids are networks composed of DC sources, loads, and interconnecting lines. They are represented by differential-algebraic equations connected over an undirected weighted graph that models the electrical circuit. The proposed algorithm features a second graph, which represents the communication network over which the source nodes exchange information about the instantaneous powers, and which is used to adjust the injected current accordingly. This gives rise to a nonlinear consensus-like system of differential-algebraic equations that is analyzed via Lyapunov functions inspired by the physics of the system. We establish convergence to the set of equilibria, where weighted power consensus is achieved, as well as preservation of the weighted geometric mean of the source voltages. The results apply to networks with constant impedance, constant current and constant power loads.

1. Introduction

The proliferation of renewable energy sources and storage devices that are intrinsically operating using the DC regime is stimulating interest in the design and operation of DC microgrids, which have the additional desirable feature of preventing the use of inefficient power conversions at different stages. These DC microgrids might have to be deployed in areas where an AC microgrid is already in place, creating what is called a hybrid microgrid (Loh, Li, Chai, & Blaabjerg, 2013), for which rigorous analytical studies are still in their infancy. Furthermore, the envisioned future in which power generation is far away from the major consumption sites raises the problem of how to transmit power with low losses, a problem for which High Voltage Direct Current (HVDC) networks perform comparatively better than AC networks. Finally, also mobile grids on ships, aircrafts, and trains are based on a DC architecture.

With DC and hybrid microgrids, as well as HVDC networks, on the rise, we need to develop a deeper system-theoretic understanding of this interesting class of dynamical networks. In this paper we propose and analyze a control algorithm for a DC microgrid that enforces power sharing among the different power sources.

1.1. Literature review

The literature on DC microgrids is rapidly growing. We summarize below the contributions that share a systems and control-theoretic point of view on these networks. The work (Nasirian, Moayedi, Davoudi, & Lewis, 2015) relies on a cooperative control paradigm for DC microgrids to replace the conventional secondary control by a voltage and a current regulator. In Zhao and Dörfler (2015) a voltage droop controller for DC microgrids inspired by frequency droop in AC powernetworks is analyzed, and a secondary consensus control strategy is added to prevent voltage drift and achieve optimal current injection. The paper (Belk, Inam, Perreault & Turitsyn, 2016) models the DC microgrid via the Brayton-Moser equations and uses this formalism to show that with the addition of a decentralized integral controller voltage regulation to a desired reference value is achieved. Other schemes achieving desirable power sharing properties are proposed but no formal analysis is provided. In Tucci, Meng, Guerrero, and Ferrari-Trecate (2016), a secondary consensus-based control scheme for current sharing and voltage balancing in DC microgrids is designed in a Plug-and-Play fashion to allow for the addition or removal of generation units. A distributed control method to enforce power sharing among a cluster of DC microgrids is proposed in Moayedi and Davoudi (2016). Other work has focused on the challenges in the stability analysis of DC microgrids using consensus-like algorithms due to the interaction between the communication
network and the physical one (Meng, Dragicevic, Roldán-Pérez, Vasquez, & Guerrero, 2016). Finally, feasibility of the nonlinear algebraic equations in DC power circuits is studied by Barabanov, Ortega, Grino, and Polyak (2016), Lavei, Rantzer, and Low (2011), and Simpson-Porco, Dörfler, and Bullo (2015).

A closely related research area is that of multi-terminal HVDC transmission systems. The paper (Sarlette, Dai, Phulpin, & Ernst, 2012) focuses on cooperative frequency control for these networks. In Andréasson et al. (2014) distributed controllers that keep the voltages close to a nominal value and guarantee a fair power sharing are considered, whereas passivity-based decentralized PI control for the global asymptotic stabilization of multi-terminal high-voltage is studied in Zonetti, Ortega, and Benchabaa (2015). The paper (Zonetti, Ortega, & Schiffer, 2016) studies feasibility and power sharing under decentralized droop control. We refer to Zonetti (2016, Chapter 4) for an annotated bibliography of HVDC transmission systems.

1.2. Main contribution

The objective of the paper is to propose a novel control algorithm that exhibits three main features: (i) it makes sources provide power in prescribed ratios for a wide range of load magnitudes; (ii) it simultaneously guarantees that all voltages stay within a compact set around an operating point, and that the geometric voltage of the source voltages is maintained constant for all time; (iii) it tackles so-called “ZIP” (constant impedance, constant current and constant power) loads, which are known to substantially affect the stability of the system.

Power sharing is essential in microgrid operations because changing load conditions may lead to an imbalance situation in which few sources, if not a single one, provide the majority of the power demand. This might result in cases in which the overloaded sources exceed their capacity limit driving the microgrid to instability.

While droop controllers are usually employed to achieve power distribution in DC microgrids, they cannot achieve such a task exactly because they can only strike a tradeoff between power sharing and voltage control. As such, there are no generally acknowledged criteria for the selection of droop gains that provide guaranteed power sharing properties in the presence of unknown or uncertain variable load conditions. Our controller overcomes such limitations, providing a substantial improvement with respect to existing controllers.

The proposed controller is enabled by communicating the instantaneous source power measurements among neighboring source nodes, averaging these measurements and setting the voltage at the source terminals accordingly. An additional feature of the algorithm is that a weighted geometric average of the source voltages is preserved. In absence of a communication environment, our distributed consensus-based algorithm can also be implemented by power talk communication via the DC microgrid (Angelichinowski et al., 2015).

The system dynamics present interesting features. By averaging the power measurements that the sources communicate amongst each other, the system dynamics becomes an intriguing combination of the physical network (the weighted Laplacian of the electrical circuit appearing in the power measurements) and the communication network (over which the information about the power measurements is exchanged). ZIP loads introduce algebraic equations in the system’s dynamics, adding additional complexity and nonlinearity.

To analyze this system of nonlinear differential-algebraic equations, without going through a linearization of the dynamics, Lyapunov-based arguments become very convenient. The Lyapunov functions in this case are constructed starting from the power dissipated in the network that is further shaped to take into account the specifics of the dynamics. In fact, it is shown that the closed-loop system can be written as a weighted gradient of the Lyapunov function (Lemma 2), a form that is crucial to carry out the stability analysis. The presence of the loads, which shift the equilibrium of interest, is taken into account by the so-called Bregman function (De Persis & Monshizadeh, 2018). The level sets of the Lyapunov functions are used to estimate the excursion of the state response of these systems and therefore, combined with the preservation of the geometric average of the source voltages, can be used to obtain an estimate of the voltage at steady state.

Reactive power sharing algorithms have been first suggested by Schiffer, Seel, Raisch, and Sezi (2016) for network-reduced AC microgrids whose voltage dynamics show similar features as in DC grids. In this paper we show that a related idea can be adopted also for network preserved DC microgrids. The novelties of this contribution with respect to Schiffer et al. (2016) are the different dynamics of the system under study, the explicit consideration of algebraic equations in the model and the use of Lyapunov arguments to prove the main results.

1.3. Paper organization

The model of the DC microgrid is introduced in Section 2. The power consensus algorithm is introduced in Section 3. The analysis of the closed-loop system is carried out in Section 4 for the general case of ZIP loads, and then specialized to the case of ZI loads, since the latter permits to obtain stronger results under weaker conditions. The simulations of the algorithm are provided in Section 5. Conclusions are drawn in Section 6.

1.4. Notation

Given a vector $v$, the symbol $[v]$ represents the diagonal matrix whose diagonal entries are the components of $v$. The notation $\text{col}(v_1, v_2, \ldots, v_n)$, with $v_i$ scalars, represents the vector $[v_1 \ v_2 \ \ldots \ v_n]^T$. If $v_i$ are matrices having the same number of columns, then \( \text{col}(v_1, v_2, \ldots, v_n) \) denotes the matrix $[v_1^T \ v_2^T \ \ldots \ v_n^T]^T$. The symbol $s_{\alpha}$ represents the $n$-dimensional vector of all 1’s, whereas $S_{m \times n}$ is the $m \times n$ matrix of all zeros. When the size of the matrix is clear from the context the index is omitted. The $n \times n$ identity matrix is represented as $I_n$. Given a vector $v \in \mathbb{R}^n$, the symbol $\text{ln}(v)$ denotes the element-wise logarithm, i.e., the vector $[\ln(v_1) \ \ldots \ \ln(v_n)]^T$. Given a set $S$, the symbol $|S|$ indicates the cardinality of the set.

2. DC resistive microgrid

The DC microgrid is modeled as an undirected connected graph $G = (V, E)$, with $V = \{1, 2, \ldots, n\}$ the set of nodes (or buses) and $E \subseteq V \times V$ the set of edges. The edges represent the interconnecting lines of the microgrid, which we assume here to be resistive. Associated to each edge is a weight modeling the conductance (or reciprocal resistance) $1/r_k > 0$, with $k \in E$. The set of nodes is partitioned into the two subsets of $n_i$ DC sources $V_i$ and $n_l$ loads $V_l$, with $n_i + n_l = n$.

Let $I \in \mathbb{R}^n$, $V \in \mathbb{R}_{>0}^n$ denote the vectors of currents and potentials respectively at the nodes of $G$. The current-potential relation in a resistive network is given by the identity $I = BF^T V$, with $B \in \mathbb{R}^{n \times |E|}$ being the incidence matrix of $G$ and $F = \text{diag}(r_1^{-1}, \ldots, r_n^{-1})$ the diagonal matrix of conductances. Considering the partition of the nodes in sources and loads, we let $I = \text{col}(i_l, l)$ and $V = \text{col}(V_s, V_l)$ without loss of generality, where $I_i = \text{col}(i_1, \ldots, i_{n_i})$, $l = \text{col}(l_{n_i+1}, \ldots, l_n)$, $V_s = \text{col}(V_{s_1}, \ldots, V_{s_{n_l}})$, $V_l = \text{col}(V_{l_{n_i+1}}, \ldots, V_{l_n})$, and we correspondingly partition the
incidence matrix as \( B = \text{col}(B_c, B_l) \), with \( B_c \in \mathbb{R}^{n \times |E|} \), \( B_l \in \mathbb{R}^{n \times 1} \).

Then, the current-potential relation can be rewritten as

\[
\begin{bmatrix}
I_s \\
\tilde{I}_s
\end{bmatrix} =
\begin{bmatrix}
B_c \Gamma B_s^T & B_c \Gamma B_l^T \\
B_l \Gamma B_s^T & B_l \Gamma B_l^T
\end{bmatrix}
\begin{bmatrix}
V_s \\
Y_{ss} Y_{sl} \tilde{V}_s
\end{bmatrix} =
\begin{bmatrix}
V_{ss} Y_{sl} & V_{sl}
\end{bmatrix}
\begin{bmatrix}
V_s \\
\tilde{V}_s
\end{bmatrix},
\]

(1)

Observe that both \( Y_{ss} \) and \( Y_{ss} \) are positive definite since they are principal submatrices of a Laplacian of a connected undirected graph. This allows us to eliminate the load voltages as \( V_l = Y_{ll}^{-1} \tilde{V}_l \) and reduce the network to the source nodes \( V_s \) with balance equations

\[ I_s = Y_s Y_{ss}^{-1} \tilde{I}_s = Y_{red} V_s, \]

(2)

where \( Y_{red} = Y_s - Y_{sl} Y_{ll}^{-1} Y_{ss} \) is known as the Kron-reduced conductance matrix (Dörfler & Bullock, 2013) and \(-Y_s Y_{ss}^{-1} \tilde{I}_s\) is the mapping of the load current injections to the sources.

3. Power consensus controllers

We propose controllers that force the different sources to share the total power injection in prescribed ratios (Schiffer et al., 2016). For this purpose, a communication network is deployed to connect the source nodes, through which the controllers exchange information about the instantaneous injected powers. This communication network is modeled as an undirected unweighted graph \((\mathcal{V}_c, \mathcal{E}_c)\), where \( \mathcal{V}_c = \mathcal{V}_c \). Associated with the communication graph is the \( n_c \times n_c \) Laplacian matrix \( L_c = D_c - A_c \), where \( D_c = \text{diag}(D_{c1}, \ldots, D_{cn}) \) is the degree matrix, \( D_{ci}, i \in \mathcal{V}_c \), is the degree of node \( i \), and \( A_c \) is the adjacency matrix of the communication graph. Note that the nodes of the communication network (but not necessarily the edges) coincide with the source nodes of the microgrid. For each node \( i \in \mathcal{V}_c \), the set \( \mathcal{V}_{ci} = \{ j \in \mathcal{V}_c : (i, j) \in \mathcal{E}_c \} \) represents the neighbors connected to node \( i \) via the communication graph.

Controllers. We assume that all sources \( V_s \) are controllable voltage sources (e.g., realized by boost converters), which are controlled as a function of the measured local current and power injections \( I_i \) and \( P_i \) as well as the injected power \( P_j \) at neighboring sources that need to be communicated. In the following, we will design powers consensus controllers in such a way that the algorithm achieves weighted proportional power sharing according to ratios \( K_i > 0 \) chosen by the operator, that is, given \( P_i = I_i V_i, i \in \mathcal{V}_c \), the control objective is to guarantee that at steady state the following identities hold:

\[ K_j \frac{P_j}{K_i} = P_i, \quad \forall i, j \in \mathcal{V}_c. \]

(3)

Keeping this in mind, the proposed controllers are of the form

\[ C_i(V_i) \dot{V}_i = -I_i + u_i, \quad i \in \mathcal{V}_c, \]

(4)

where

\[ C_i(V_i) = V_i^{-1} D_c K_i^{-1} K_i^2, \quad i \in \mathcal{V}_c \]

(5)

can be interpreted as a nonlinear capacitance, \( K_i > 0 \), the power sharing coefficient, is of suitable units such that \( C_i(V_i) \) actually has the units of a capacitance, \( I_i \) is the injected current at node \( i \in \mathcal{V}_c \) as defined in (1), and the term

\[ u_i = V_i^{-1} D_c K_i \sum_{j \in \mathcal{V}_{ci}} K_j^{-1} P_j, \quad i \in \mathcal{V}_c \]

(6)

represents an ideal current source that is controlled as a function of the local voltage \( V_i \) and the injected power \( P_i = V_i I_i \) at the neighboring node sources \( j \in \mathcal{V}_{ci} \). The current sources \( u_i \), \( i \in \mathcal{V}_c \), are designed to make the right-hand side of (4) equal to \( V_i^{-1} D_c K_i \sum_{j \in \mathcal{V}_{ci}} (K_j^{-1} P_j - K_i^{-1} P_i) \), which is a weighted average of the power variables, multiplied by the factor \( V_i^{-1} D_c K_i \). As shown in the reminder of the paper, the average term is the key to attain the source power injections in prescribed ratios.

The dynamic controllers (4)–(6) are initialized at positive values of the voltage, that is, \( V_i(0) > 0 \) for all \( i \in \mathcal{V}_c \). It will be made evident in later sections that these controllers render the positive orthant \( \mathbb{R}^n_+ \) positively invariant, thus showing that the positivity of the initial source voltages yields positivity of these variables for all \( t \geq 0 \).

Remark 1 (Digital Implementation and Circuit Realization). The control algorithm (4), (5), (6) can be implemented at each controllable voltage source as follows. The local current and power injections \( I_i \) and \( P_i \) are measured, and the local current measurements \( I_i \) and the power injections \( P_i \) at neighboring sources \( j \in \mathcal{V}_{ci} \) are processed along with the current measurement \( I_i \) to compute the source voltage value \( V_i \) applied at the source terminals as in (4), (5), (6).

Since the signal \( u_i \) has the dimension of amps and appears as a current signal in (4), we have drawn an equivalent circuit realization of Eq. (4) in Fig. 1. Comparing with Belk et al. (2016), (4), the equivalent current source \( u_i \) can be generated also by a voltage source with value \( u_i \) in series with a resistance \( r_i \) provided that \( u_i = r_i u_i + V_i \). Finally, the dynamic droop controller in Zhao and Dörfler (2015) corresponds in our notation to a constant capacitance \( c_i \) and current source \( u_i \). We remark again that this is merely an equivalent circuit that helps interpreting the controller (4). In the end, the most convenient realization of (4), (5), (6) is by means of a converter controlled as a voltage source.

Multiplying both sides of (4) by \( V_i^2 D_c K_i^{-1} \), one arrives at the closed-loop system

\[ K_i \dot{V}_i = -V_i D_c K_i^{-1} P_i + V_i \sum_{j \in \mathcal{V}_{ci}} K_j^{-1} P_j \]

\[ = V_i \sum_{j \in \mathcal{V}_{ci}} (K_j^{-1} P_j - K_i^{-1} P_i), \quad i \in \mathcal{V}_c, \]

(7)

that is, the voltage at the source terminal is updated according to a weighted power consensus algorithm scaled by the voltage. Provided that \( V_i \neq 0 \) (a property that will be established in the next sections), Eq. (7) shows that at steady state the proposed algorithm achieves proportional power sharing as in (3). A detailed characterization of the steady-state power signals is given in the next section (Lemma 1).

For interpretation purposes, we write (7) as

\[ \frac{d}{dt} K_i \ln(V_i) = \sum_{j \in \mathcal{V}_{ci}} (K_j^{-1} P_j - K_i^{-1} P_i), \quad i \in \mathcal{V}_c, \]

In a classic power system analysis (Chiang, 2011), the term \( K_i \ln(V_i) \) is the natural energy representation of a source power of constant value \( K_i \). The interpretation of the closed loop (7) is then that the voltage at this constant power source is adapted according to a power consensus algorithm.
Remark 2 (Alternative Power Sharing Control). A possibly more simplistic and obvious power sharing controller inspired by the current-sharing controller in Zhao and Dörfler (2015) is based on a distributed averaging integral control given by

\[
\begin{align*}
C_i \dot{V}_i &= -I_i + p_i \\
D_i \dot{p}_i &= I_i - p_i + \sum_{j \in N_i,i} (K_j^{-1}V_j p_j - K_i^{-1}V_i p_i), \quad i \in \mathcal{V}
\end{align*}
\]

where \(p_i\) is a control variable in units of currents, \(C_i > 0\) a gain with capacitance units and \(D_i > 0\) a time constant. Note that the power sharing coefficients \(K_i\) in (8) have the units of voltages. Any steady state of this controller would guarantee for all \(i \in \mathcal{V}\) that \(V_i = 0\), and \(p_i = I_i\) is the steady-state current injection, and the vector of power injections \(K_j^{-1}V_j p_j\) has all identical entries (power sharing). Numerical results (see Section 5) show that (7) and (8) perform similarly. Indeed, in the limit \(D_i \to 0\), near steady-state, and for nearly unit voltages (in per unit system), the closed-loops (8) and (7) have similar dynamics. In the rest of the paper we focus on the analysis of (7), since no analytical guarantee on the stability of system (8) is available at this moment.

Loads. Depending on the particular load models, the term \(I_i\) in (1) takes different expression and will henceforth be denoted as \(I_i(V_i)\) to stress the functional dependence on the load voltages. Prototypical load models that are of interest include the following:

1. (i) constant current loads: \(I_i(V_i) = I_i^* \in \mathbb{R}_{>0}^n\),
2. (ii) constant impedance: \(I_i(V_i) = -Y_i^* V_i\), with \(Y_i^* > 0\) a diagonal matrix of load conductances, and \(V_i = \text{col}(V_{n+1}, \ldots, V_{n+n_i})\), and
3. (iii) constant power: \(I_i(V_i) = [V_i]^{-1} P_i^*\), with \(P_i^* \in \mathbb{R}_{>0}^n\).

To refer to the three load cases above, we will use the indices “I”, “Z” and “P” respectively. The analysis of this paper will focus on the more general case of a parallel combination of the three loads, thus on the case of “ZIP” loads, for which

\[
I_i(V_i) = I_i^* - Y_i^* V_i + [V_i]^{-1} P_i^*.
\]

Moreover, additional and stronger results on the “ZI” case will be reported. The following analysis also applies to any other load scenario where components of \(I_i^*, Y_i^*\) and \(P_i^*\) are possibly zero.

Bearing in mind (1), (7), and vectorizing the expressions to avoid cluttered formulas, the closed-loop system is

\[
K \hat{V}_s \begin{bmatrix}
K_i^{-1} - I_i
-\dot{I}_i(V_i)
\end{bmatrix} = -\begin{bmatrix}
[V_i]_l K_i^{-1} P_i
B_i \Gamma B_i^T V
\end{bmatrix},
\]

where \(V = \text{col}(V_{all})\), \(K_i = \text{diag}(K_1, \ldots, K_{n_i})\), \(P_i = \text{col}(P_{1}, \ldots, P_{n_i})\) given by

\[
P_i = [V_i]_l K_i [V_{all} V_i + Y_a V_i]
\]

is the vector of source power injections and \(I_i(V_i)\) as defined in (9) are the load currents. The interconnected closed-loop DC microgrid is then entirely described by Eqs. (10), (11), (9). An example of a simple closed-loop DC microgrid with two sources and one constant impedance load is given in Fig. 2.

Remark 3 (Nonlinear Consensus Algorithms). To compare the control algorithm (7) with related nonlinear consensus algorithms proposed in the literature (Bauso, Giare, & Pesenti, 2006; Cortes, 2008), we neglect the algebraic constraints and the differentiation between sources and loads. This allows us to rewrite (7) as

\[
K \dot{V} = -[V_{all}]_l K^{-1} [V] B_i \Gamma B_i^T V.
\]

The weighted power mean consensus algorithms of Bauso et al. (2006) and Cortes (2008), on the other hand, can be written as

\[
[W]_l \dot{V} = [V]^{-1} W \Gamma B_i \Gamma B_i^T V, \quad \text{where} \quad W = \text{a vector of weights satisfying} \quad 1^T W = 1 \quad \text{and} \quad r \in \mathbb{R}.
\]

In the special case \(r = 0\), we get

\[
[W]_l \dot{V} = [V] B_i \Gamma B_i^T V,
\]

which is known to converge to the consensus value \(V_{all}^w = \ldots V_{all}^w\). The analysis is based on the Lyapunov function \(\sum_{i=1}^n w_{V_i} - \prod_{i=1}^n V_i^w\).

The nonlinear power consensus algorithm presented in this paper is different in that it uses another layer of averaging in addition to the averaging induced by the physical network. This, and the algebraic constraints, requires a different analysis based on physics-inspired Lyapunov functions.

4. Power consensus algorithm with ZIP loads

In this section we analyze the closed-loop system (10), (11), (9). We start by studying its equilibria, namely the set of points \(V \in \mathbb{R}_{>0}^n\) that satisfy (11), (9), and

\[
\begin{bmatrix}
0 \\
-\dot{I}_i(V_i)
\end{bmatrix} = -\begin{bmatrix}
[V_i]_l K_i^{-1} P_i
B_i \Gamma B_i^T V
\end{bmatrix}.
\]

4.1. Steady-state characterization

In the following, we show that the equilibria are fully characterized by power balance equations at the sources and current balance equations at the loads, respectively.

Lemma 1 (System Equilibria). The equilibria of the system (10), (11), (9) are equivalently characterized by

\[
\mathcal{E}_{ZIP} = \{ V \in \mathbb{R}_{>0}^n : \mathcal{I}_{Z}(V) = 0, \mathcal{P}_{ZIP}(V) = 0 \},
\]

where \(\mathcal{I}_{ZIP}(V) = 0\) is the current balance at the loads

\[
\mathcal{I}_{Z}(V) = I(V_i) - Y_a V_i - Y_a V_i,
\]

\(\mathcal{P}_{ZIP}(V) = 0\) depicts the power balance at the sources

\[
\mathcal{P}_{ZIP}(V) = \begin{bmatrix}
[V]_l Y_{red} V_i + [V]_l Y_{red} V_i - \prod_{i=1}^n V_i^w
\end{bmatrix}.
\]

\(Y_{red}\) is the Kron-reduced conductance matrix, \(Y_{red}^{-1} Y_{red} I(V_i)\) is the mapping of the ZIP loads \(I(V_i)\) to the source buses in the Kron-reduced network as in (2), and \(P_i\) is the vector of power injections by the sources written for \(V \in \mathbb{R}_{>0}^n\) as

\[
\begin{align*}
&I_i = -K_i p_i^*, \quad P_i = \frac{1}{e^T} I_i(V_i) \\
&\quad \text{observes that the steady-state injection (13) achieve indeed power sharing, and the asymptotic power value} \quad \text{p_i^* to which the source power injections converge (in a proportional fashion according to the coefficients} \quad \text{K_i,} \quad \text{is the total current demand divided by the weighted sum of the steady-state source voltages.} \quad \text{The latter values and those of the load voltages are entangled by} \quad \text{K_i,} \quad \text{is the total current demand divided by the weighted sum of the steady-state source voltages.} \quad \text{The latter values and those of the load voltages are entangled by}
\end{align*}
\]

Fig. 2. Circuit considered in Example 1.
the power balance at the sources $\mathcal{P}_{ZIP}(V) = 0$ and the current balance equations at the loads $\mathcal{I}_{ZIP}(V) = 0$, similar to the related studies (Zonetti et al., 2015, Proposition 3.3), (Sanchez, Ortega, Bergna, Molinas, & Grilo, 2013, Lemma 2).

**Proof.** Let $V$ be an equilibrium of (10), (11), (9), that is let $V \in \mathbb{R}^{n_0}_+$ satisfy (12). From the first equation, $0 = [V]_l I_l K_l^{-1} P_l$, it immediately follows that $P_l = K_l Z_n p^*_n$ for some scalar $p^*_n$. We rewrite the current balances as

$$
\left[ [V]_l^{-1} K_l Z_n p^*_n \right]_{I_l(V)} = \left[ B_l \Gamma B_l^T V \right]_{I_l(V)}. \tag{14}
$$

Next, we left-multiply (14) by $[Z_n]_n^{-1} [I]_n$ to obtain

$$
t^T [V]_l^{-1} K_l Z_n p^*_n + t^T [I]_l(V) = 0. \tag{15}
$$

The latter equation can be solved for $p^*_n$ as in (13). From $I_l(V) = B_l \Gamma B_l^T V$, we obtain (see (2)) $\mathcal{I}_{ZIP}(V) = 0$ or

$$
V_l = -Y_l^{-1} Y_l V_l + Y_l^{-1} I_l(V),
$$

which replaced in the first equation of (14) returns

$$
Y_n V_l + Y_n(-Y_l^{-1} Y_l V_l + Y_l^{-1} I_l(V)) = [V]_l^{-1} K_l Z_n p^*_n. \tag{16}
$$

By rearranging the terms, we arrive at

$$
Y_n V_l + Y_n Y_n^{-1} I_l(V) - [V]_l^{-1} K_l Z_n p^*_n = 0,
$$

which can be reformulated as $\mathcal{P}_{ZIP}(V) = 0$ after left-multiplying by $[V]_l$ and bearing in mind (13). The latter and (15) show that $V \in \mathcal{E}_{ZIP}$.

Conversely, let $V \in \mathcal{E}_{ZIP}$. Then the equation $I_l(V) = B_l \Gamma B_l^T V$ in (12) is trivially satisfied. From $\mathcal{P}_{ZIP}(V) = 0$, and $I_l(V) = B_l \Gamma B_l^T V$ written as (15), and going backwards through the passages above, we arrive at

$$
Y_n V_l + Y_n Y_n^{-1} I_l(V) - [V]_l^{-1} K_l Z_n p^*_n = 0,
$$

or equivalently at $[V]_l B_l \Gamma B_l^T V = K_l Z_n p^*_n$. Hence, the power vector $P_l = [V]_l B_l \Gamma B_l^T V$ satisfies $L_l K_l^{-1} P_l = 0$, that is, the first equation in (12). Hence, $V \in \mathcal{E}_{ZIP}$ implies that the equilibrium equations (12) are met.

We make the standing assumption that equilibria exist:

**Assumption 1.** $\mathcal{E}_{ZIP} \neq \emptyset$.

**Remark 4 (Existence of the Equilibria $\mathcal{E}_{ZIP}$).** The analytical investigation of the existence of the equilibria $\mathcal{E}_{ZIP}$ is deferred to a future research. This is a topic of interest on its own and similar problems have been dealt with in recent work about the solvability of reactive or DC power flow equations (Barabanov et al., 2016; Bolognani & Zampieri, 2016; Sanchez et al., 2013; Simpson-Porco et al., 2015; Simpson-Porco, Dörfler, & Bullo, 2016). For instance, the problem in Simpson-Porco et al. (2016) boils down to the solution of quadratic algebraic equations of the form $[V]_l Y_l V_l - [V]_l Y_l V_l + Q_l = 0$, where $Q_l$ is the vector of constant power load demands and $V_l$ is the so called vector of open circuit voltages (again constant). Although similarities between these equations and the equations $\mathcal{P}_{ZIP}(V) = 0 = [V]_l Y_n V_l + [V]_l Y_n Y_n^{-1} I_l(V) + P_l$ could be useful to investigate the nature of the set $\mathcal{E}_{ZIP}$, the non-quadratic nature of $\mathcal{P}_{ZIP}(V) = 0$, as well as the presence of the additional equations $Y_l^{-1} I_l(V) = -V_l Y_n^{-1} V_l$ pose additional challenges. Extra insights could come from the convex relaxation of the DC power flow equations in the context of optimal DC power flow dispatch (Lavei et al., 2011).

**Remark 5 (Equilibrium Power Balance and Voltage Inequalities).** To gain further insights into the equilibrium set $\mathcal{E}_{ZIP}$, recall that the vector of power injections is $P = \text{col}(P_i, P_l) = [V]B_l \Gamma B_l^T V$, where $P_l = [V]_l I_l(V)$. Thus, we have the inherent power balance

$$
t^T P_l + t^T P_l = V^T B_l \Gamma B_l^T V \geq 0 \tag{16}
$$

implying that the amount of supplied power has to make up for load demands and resistive losses. In the special case of constant power loads, $I_l(V) = [V]_l^{-1} P_l$, we obtain the total (or average) power inequality $t^T P_l + t^T P_l \geq 0$. Equivalently, after using (13), we arrive at

$$
- t^T K_l^{-1} [V]_l^{-1} P_l + t^T P_l \geq 0.
$$

This inequality can be reformulated as

$$
\sum_{i \in V_l} a_i \geq \sum_{i \in V_l} b_i,
$$

with $a_i = P_i^T / \sum_{i \in V_l} P_i^T$ and $b_i = K_l / \sum_{i \in V_l} K_l$, which relates a convex combination of the reciprocals of the voltages at the loads, with a convex combination of the reciprocals of the voltages at the sources, and represents another relation between $V_l, V_l$ in addition to those in (16). The average voltage inequality (17) implies that the reciprocal of the harmonic average source voltage must be larger than the reciprocal of the harmonic average load voltage so that power can flow from sources to loads.

In a special case reviewed in the example below, an explicit characterization of the equilibria can be given.

**Example 1.** Consider the case of two sources ($n_1 = 2$) and one load ($n_0 = 1$) as in Fig. 2, in which the constant impedance load is replaced by a ZIP load. The equations $\mathcal{P}_{ZIP}(K_l) = 0$, assuming $K_1 = K_2$, are in this case

$$
\frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} V_1(V_1 - V_2) - \frac{\gamma_1}{\gamma_1 + \gamma_2} V_l I_l(V_l) + \frac{I_l(V_l)}{V_1 + V_2} = 0
$$

and

$$
\frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} V_2(V_2 - V_1) - \frac{\gamma_2}{\gamma_1 + \gamma_2} V_l I_l(V_l) + \frac{I_l(V_l)}{V_1 + V_2} = 0.
$$

We study solutions to the algebraic equations on the curve $V_1 V_2 = c$. The reason for this choice will become clear in Section 4.3. On such a curve, the equations simplify as

$$
V_1^4 - r_1 h(V_1) V_1^3 + c r_1 h(V_1) V_1 - c^2 = 0,
$$

$$
V_2^4 - r_2 h(V_2) V_2^3 + c r_2 h(V_2) V_2 - c^2 = 0,
$$

where $r_i = r_i^{-1}, i = 1, 2$ (the resistance of the transmission line $i$ connecting the source $i$ to the load).

We want to study the solutions of these equations as functions of $I_l(V_l)$. Then these can be regarded as two independent quartic functions for which an analytic, although involved, expressions of the solutions exist according to the Ferrari–Cardano’s formula. These expressions simplify if one takes $r_1 = r_2$. Then there is a unique positive solution given by $V_1 = V_2 = \sqrt{c}$, independent of $I_l(V_l)$. The value of $V_l$ is obtained from the algebraic equation

$$
0 = V_l B_l \Gamma B_l^T V_l - V_l I_l(V_l),
$$

solving

$$
0 = V_l(-\gamma_1 V_1 - \gamma_2 V_2 + (\gamma_1 + \gamma_2) V_l) - V_l h(V_l)
$$

with

$$
2 r_1 h(V_1) V_1^2 - 2 V_l h(V_l) V_1 - \gamma_2 V_2 - P_l^* = 0
$$

and

$$
2 r_2 h(V_2) V_2^2 - 2 V_l h(V_l) V_2 - \gamma_1 V_1 - P_l^* = 0.
$$

In the absence of loads, we have two real roots: a root at $V_l = 0$ and a root at $V_l = \sqrt{c} = V_1 = V_2$. Since the roots of a polynomial are continuous in the parameters, the two real-valued roots can vanish and turn to a complex-conjugate pair for large loading. A
classical root-locus analysis shows that, maintaining $P^*_t$, $Y^*_t$ equal to zero and letting $P^*_t$ decrease to $-\infty$, the two roots meet halfway at $\sqrt{C}/2$ and then diverge to infinity along the vertical axis passing by the point $(\sqrt{C}/2, 0)$ of the complex plane. This is known as "voltage collapse" in power systems.

4.2. A Lyapunov function and hidden gradient form

We pursue a Lyapunov-based analysis of the stability of the closed-loop system (10), (11), (9). Inspired by the Lyapunov analysis of the reactive power consensus algorithm in De Persis and Monshizadeh (2018), we consider the total power dissipated through the network resistors, $1/2 V^T B^T B^* V$, as the first natural Lyapunov candidate for our analysis, to which we add the power dissipated through the impedance loads, to obtain the power losses at passive devices as

$$J(V) = \frac{1}{2} V^T \left( B \Gamma B^* + \begin{bmatrix} 0 & 0 \\ 0 & Y^*_t \end{bmatrix} \right) V. \quad (20)$$

Let $\overline{V} \in \mathcal{E}_{ZP}$, and define $P^*_t = [\overline{V}T] B_t^* \overline{V}$ the source power injection corresponding to the equilibrium source voltage $\overline{V}$ (see (13)). To cope with the asymmetry in the dynamics of the sources and loads we add to $J$ the terms

$$H(V) = -\overline{P}^*_t \ln(V_t),$$

and

$$K(V) = -P^*_t v^T \ln(V_t),$$

which is the way classical power systems transient stability analysis absorbs constant power injections (Chiang, 2011) into a so-called energy function defined here as

$$M(V) := J(V) + H(V) + K(V)$$

$$= \frac{1}{2} V^T (B \Gamma B^*) V + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} V - \overline{P}^*_t \ln(V_t) - P^*_t v^T \ln(V_t). \quad (21)$$

The natural "energy function" (21) has its critical points at voltages for which $P^*_t = -[V_t] Y^*_t V_t + P^*_t$, thus different from the power loads prescribed by the ZIP loads. To center the function $M$ with respect to a non-trivial equilibrium $\overline{V} \in \mathcal{E}_{ZP}$, we use the following Bregman function (De Persis & Monshizadeh, 2018)

$$\mathcal{L}(V) = M(V) - M(\overline{V}) - \left. \frac{\partial M}{\partial V} \right|_{V = \overline{V}} \ln(V - \overline{V}). \quad (22)$$

The next result shows a (perhaps surprising) gradient relation between the dynamics of system (10), (11), (9) and the Bregman function (22) above:

**Lemma 2 (Gradient Dynamics).** The following holds

$$\begin{bmatrix} L_t K_t^{-1} P_s \\ B_t \Gamma B^* V - l_t(V_t) \end{bmatrix} = \begin{bmatrix} L_t[V_t] K_t^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \frac{\partial M(V)}{\partial V} \quad (23)$$

for all $V \in \mathbb{R}^n$. Hence the system (10), (11), (9) can be rewritten as a weighted gradient flow

$$\begin{bmatrix} K_t \dot{V}_t \\ 0 \end{bmatrix} = -\begin{bmatrix} [V_t]Y_t[V_t] K_t^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \frac{\partial \mathcal{L}(V)}{\partial V}. \quad (24)$$

**Proof.** The gradient of the function $M(V)$ writes as

$$\frac{\partial M}{\partial V} = B \Gamma B^* V + \begin{bmatrix} 0 \\ [V_t]^{\dagger} V_t \end{bmatrix} - \left[ [V_t]^{\dagger} P^*_t \right] - \left[ [V_t]^{\dagger} Y^*_t V_t \right].$$

Hence, the Bregman function (22) satisfies

$$\frac{\partial M}{\partial V} = \frac{\partial M}{\partial V} - \left[ [V_t]^{\dagger} P^*_t \right] + \left[ [V_t]^{\dagger} Y^*_t V_t \right].$$

$$\begin{bmatrix} \frac{\partial M}{\partial V} \end{bmatrix} = \begin{bmatrix} B_t \Gamma B^* (V - \overline{V}) \end{bmatrix} + \left[ \begin{bmatrix} 0 \\ [V_t]^{\dagger} Y^*_t (V - \overline{V}) \end{bmatrix} \right] - \left( [V_t]^{\dagger} - [V_t]^{\dagger} \overline{P}^*_t \right) \overline{P}^*_t.$$

Bearing in mind the equilibrium condition at the loads

$$B_t \Gamma B^* V = l_t(V_t) = I^*_t - Y^*_t \overline{V}_t + [\overline{V}_t]^{-1} P^*_t,$$

and replacing it in the second line of the identity above describing $\partial M/\partial V$, we obtain

$$\frac{\partial M}{\partial V} = B_t \Gamma B^* V + \overline{P}^*_t V - [V_t]^{\dagger} P^*_t - I^*_t = B_t \Gamma B^* V - l_t(V_t),$$

which equals precisely the second equation in (23).

Analogously, for the first line $\partial M/\partial V_s$, we write

$$\frac{\partial M}{\partial V_s} = B_s \Gamma B^* (V - \overline{V}) - ([V_t]^{\dagger} - [V_t]^{\dagger} \overline{P}^*_t) \overline{P}^*_s,$$

$$= [V_t]^{\dagger} (P_s - \overline{P}^*_t) \overline{P}^*_s, \quad (25)$$

where to write the second equality we have used the identities $P_s = [V_s] B_t \Gamma B^* V$ and $\overline{P}^*_s = [V_s] B_t \Gamma B^* \overline{V}$. Now note that

$$P_s = [V_s] \frac{\partial M}{\partial V_s} + \overline{P}^*_s,$$

and, multiplying both sides by $L_t K_t^{-1}$, we obtain

$$L_t K_t^{-1} P_s = L_t K_t^{-1} [V_s] \frac{\partial M}{\partial V_s} + L_t K_t^{-1} \overline{P}^*_s = L_t K_t^{-1} \frac{\partial M}{\partial V_s},$$

having exploited that $\overline{V} \in \mathcal{E}_{ZP}$ implies $\overline{P}^*_s = K_t \overline{V}_t$. The identity $L_t K_t^{-1} P_s = L_t K_t^{-1} [V_s] \frac{\partial M}{\partial V_s}$ is the first equation in (23).

In view of the dynamics (10), (11), (9), one immediately realizes that

$$\begin{bmatrix} L_t K_t^{-1} P_s \\ B_t \Gamma B^* V - l_t(V_t) \end{bmatrix} = \begin{bmatrix} -[V_t]^{\dagger} K_t \overline{V}_t \\ [V_t]^{\dagger} \overline{V}_t \end{bmatrix} \frac{\partial \mathcal{L}(V)}{\partial V},$$

showing the identity (24) which concludes the proof.

**Remark 6 (Logarithmic Terms of the Lyapunov Function).** As evident from the proof, the logarithmic terms $\overline{P}^*_s \ln(V_t)$ and $\overline{P}^*_s \ln(V_t)$ in the Lyapunov function yield that

$$\frac{\partial M}{\partial V_s} = [V_t]^{\dagger} (P_s - \overline{P}^*_s), \quad \frac{\partial M}{\partial V_s} = l_t(V_t),$$

that is, they make sure that the critical points $V \in \mathbb{R}^n$ of the Lyapunov function are those for which the algebraic equations modeling the loads are satisfied and the vector of injected powers at the sources is equal to the desired one $\overline{P}$, as characterized in Lemma 1 (see Eq. (13)).

4.3. Convergence of solutions

The particular form of the dynamics (10), (11), (9) elucidated in Lemma 2 permits a straightforward analysis of the (24) convergence properties of the solutions.

**Theorem 1 (Main Result).** Assume that there exists $\overline{V} \in \mathcal{E}_{ZP}$ such that

$$Y_t + Y^*_t + [\overline{V}_t]^{-2} [P^*_t] - Y_t (Y_s + [\overline{V}_s]^{-2} [\overline{P}^*_s])^{-1} Y_t > 0, \quad (26)$$

...
where \( Y_{ss}, Y_{sl}, Y_{ls}, Y_{ll} \) are the submatrices of the Laplacian matrix defined in (1), \( P^*_t \) is the constant power load, and \( P_s \) is the constant source power injection defined in (13). Then the following statements hold:

1. There exists a compact sublevel set \( \Lambda_{ZIP} \) of the shifted Lyapunov function \( M \) in (22) contained in \( \mathbb{R}^n_{>0} \) such that any solution to (10), (11), (9) that originates from initial conditions \( V(0) \) belonging to \( \Lambda_{ZIP} \) exists, always remains in \( \Lambda_{ZIP} \) with strictly positive voltages for all times, and

2. Asymptotically converges to the set of equilibria \( \mathcal{E}_{ZIP} \cap \Lambda_{ZIP} \cap \nu_{\text{mean}} \), where \( \nu_{\text{mean}} \) specifies the preserved weighted geometric mean of the source voltages

\[
\nu_{\text{mean}} := \{ V \in \mathbb{R}^n_{>0} : V_{K^1} \cdots V_{K_n} = V_{K^1}(0) \cdots V_{K_n}(0) \}.
\]

Remark 7 (Interpretation of the Main Condition). The main condition (26) guarantees regularity of the algebraic equations and stability of the solutions. Its role is revealed when converting the constant power loads and the asymptotically constant power injections at the sources to the equivalent impedances \( |V|^{-2}[P^*_t] \) and \( |V|^{-2}[P_s] \). In this case, the equivalent conductance matrix in the steady-state current–balance equations (1) read as

\[
Y_{eq} = \begin{bmatrix} Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} \end{bmatrix} + \begin{bmatrix} |V|^{-1}[P^*_s] & 0 \\ 0 & |V|^{-1}[P^*_t] \end{bmatrix} + \begin{bmatrix} 0 & [Y^T \times |V|^{-1}[P^*_t]] \\ [V^{-1}[P^*_t]] & 0 \end{bmatrix}.
\]

By a Schur complement argument, observe that \( Y_{eq} \) is a well-defined (i.e., positive definite) conductance matrix if and only if the main condition (26) holds.

Proof. Existence and boundedness of solutions. Observe first that

\[
\frac{\partial^2 M}{\partial V^2} = B^T B^T + \begin{bmatrix} 0 & 0 \\ 0 & Y_s^* \end{bmatrix} + \begin{bmatrix} |V|^{-1}[P^*_s] & 0 \\ 0 & |V|^{-1}[P^*_t] \end{bmatrix}.
\]

Let \( \bar{V} \in \mathbb{R}^n_{>0} \) be an equilibrium of the system, i.e., \( \bar{V} \in \mathcal{E}_{ZIP} \). Since \( P^*_t, P^*_s \in \mathbb{R}^n_{>0} \), the steady-state power injection at the sources satisfies \( \bar{P}_s, \bar{P}_t \in \mathbb{R}^n_{>0} \) by (13). Hence, \( |V|^{-1}[P^*_s] > 0 \) is positive definite. Then the Bregman function \( M \) has an isolated minimum at the equilibrium \( \bar{V} \), in view of (26), (29) and a standard Schur complement argument. Then there exists a compact sublevel set \( \Lambda_{ZIP} \) of \( M \) around the equilibrium \( V \) contained in the positive orthant. Without loss of generality this compact sublevel set can be taken so that all the solutions to (10), (11), (9) that originate here locally exist.

The algebraic equations (9) written as in Lemma 1 are

\[
0 = \mathcal{E}_{ZIP}(V) = I(V) - Y_s V_s - Y_l V_l.
\]

To study local solvability of these equations, we analyze

\[
\frac{\partial \mathcal{E}_{ZIP}}{\partial V_l} = -(Y_s + Y_l + [V^{-1}P^*_s])
\]

In view of (26), nonsingularity of \( \partial \mathcal{E}_{ZIP} / \partial V_l \) and therefore regularity of the algebraic condition holds in a neighborhood of \( V \in \Lambda_{ZIP} \) from the implicit function theorem (Abraham, Marsden, & Ratiu, 1988). The sublevel set \( \Lambda_{ZIP} \) can be taken sufficiently small such that it is contained in the neighborhood of regularity for the algebraic equations, thus showing the claim that solutions starting from \( \Lambda_{ZIP} \) locally exist in time, see Hill and Mareels (1990, Theorem 1) and Schiffer and Dörfler (2016, Lemma 2.3).

When computed along these solutions, \( \dot{M}(V(t)) \) satisfies

\[
\dot{M}(V(t)) = \frac{\partial M}{\partial V_s} \bigg|_{V=V(t)} V_s(t) + \frac{\partial M}{\partial V_l} \bigg|_{V=V(t)} V_l(t).
\]

Notice that, by the algebraic constraint (23),

\[
\frac{\partial M}{\partial V_s} \bigg|_{V=V(t)} = b_1 r B^T V(t) - I(V(t)) = 0
\]

for all \( t \) for which a solution exists. Hence, we arrive at

\[
\dot{M}(V(t)) = \frac{\partial M}{\partial V_s} \bigg|_{V=V(t)} V_s(t) - \frac{\partial M}{\partial V_l} \bigg|_{V=V(t)} V_l(t)
\]

\[
= -\frac{\partial M}{\partial V_s} \bigg|_{V=V(t)} K_s^{-1} [V_s L_s[V_s]^t K_r^{-1} - I(V(t))] = 0,
\]

where the second equality holds because of (24). The inequality above shows that \( \dot{M}(V(t)) \) is a non-increasing function of time. By the compactness of the sublevel set around \( \bar{V} \), the solutions are bounded, exist and belong to \( \Lambda_{ZIP} \) for all times. Thus, among others the voltages stay positive for all times.

Convergence. Exploiting the regularity of the algebraic equation, the DAE system can be reduced to an ODE system and then the standard LaSalle invariance principle for ODE can be used to infer convergence, see also Schiffer & Dörfler (2016). We argue as follows. Any solution \( (V_s, V_l) \) to the DAE system (10), (11), (9) originating in \( \Lambda_{ZIP} \) is such that its component \( V_l \) is a solution to the system of ODE

\[
\dot{V}_s = -K_s^{-1} [V_s L_s[V_s]^t K_r^{-1} (Y_{ss} V_s + Y_{sl} \delta(V_s))],
\]

where the map \( V_l = \delta(V_s) \) denotes the solution of the algebraic equation \( \mathcal{E}_{ZIP}(V) = 0 \) in \( \Lambda_{ZIP} \). Define

\[
\mathcal{N}(V_s) := \mathcal{M}(V_s, \delta(V_s))
\]

and observe that

\[
\dot{\mathcal{N}}(V_s(t)) = \frac{\partial \mathcal{M}}{\partial V_s} \bigg|_{V=V(t)} V_s(t) + \frac{\partial \mathcal{M}}{\partial V_l} \bigg|_{V=V(t)} V_l(t)
\]

\[
= \frac{\partial \mathcal{M}}{\partial V_s} \bigg|_{V=V(t)} V_s(t) - I(V_l(t)) = \delta(V_s(t))
\]

since

\[
\frac{\partial \mathcal{M}}{\partial V_l} \bigg|_{V=V(t)} V_l(t) = Y_s V_s(t) + Y_{sl} \delta(V_s(t)) - I(V_l(t)) = \delta(V_l(t))
\]

where the second equality holds because \( V_l(t) = \delta(V_s(t)) \) on \( \Lambda_{ZIP} \) and the third equality because of the algebraic equation in (10), (11), (9). It then follows that

\[
\dot{\mathcal{N}}(V_s) = \left( P_s - P_t \right)^T [V_s^{-1} \delta(V_t) - \delta(V_l(t))]
\]

\[
= -\left( P_s - P_t \right)^T K_s^{-1} L_s K_r^{-1} - P_s^T K_r^{-1} L_s K_r^{-1} P_s \leq 0,
\]

where the first equality descends from (25), the second from (10), and the third from (13).

Since \( V_s \) is bounded, then the standard La Salle invariance principle for ODEs yields convergence of \( V_s \) to the largest invariant set where \( L_s K_r^{-1} P_s = 0 \). Moreover, since the solutions evolve in \( \Lambda_{ZIP} \), since they satisfy the algebraic equations, and since \( L_s K_r^{-1} P_s = 0 \), we have from Lemma 1 that at steady state \( (V_s, V_l) \in \mathcal{E}_{ZIP} \).

Since \( (V_s, V_l) \) is a solution to (10), (11), (9) that remains in \( \Lambda_{ZIP} \), convergence to the set \( \mathcal{E}_{ZIP} \cap \Lambda_{ZIP} \) is inferred.
quantity $V_1^{k_1} \ldots V_n^{k_n}$ is conserved, namely $V_1(t)^{k_1} \ldots V_n(t)^{k_n} = V_1(0)^{k_1} \ldots V_n(0)^{k_n}$ for all $t$. In fact, by (30),

$$K_s \frac{d}{dt} \ln V_s = -[\ln|V_s|]K_s^{-1}(Y_sV_s + Y_d\delta(V_s)),$$

and therefore $\frac{d}{dt}^T K_s \ln V_s = 0$. The thesis then follows. ■

Example 2. Consider again the case of two sources ($n = 2$) and one load ($n_l = 1$) interconnected in a “T” configuration, as in Example 1. If $K_1 = K_2$, the result above shows that on the convergence set $\mathcal{E} \cap A_2 \cap \mathcal{V}_{\text{mean}}, V_s \mathcal{V}_2 = V_1(0)V_2(0) =: c$ for all $t \geq 0$. Hence, as discussed in Example 1, the expression of the (real and positive) solution to Eqs. (18) takes on a particularly simple form, namely $V_1 = V_2 = \sqrt{c} = \sqrt{V_1(0)V_2(0)}$, that is on the convergence set each source voltage is the geometric mean of the initial voltage sources. Accordingly, the load voltage $V_L$ must satisfy (19).

Remark 8 (Capacitors at the Loads). If loads are interconnected to the network via capacitors, the load equations are modified as

$$C \frac{d}{dt} V_L = -h(V_L) + B_l \Gamma$$.\$^2\Gamma V_L.$

Notice that the equilibrium of the system remain the same. Bearing in mind (23), the load dynamics read as

$$C \frac{d}{dt} V_L = \frac{\partial M}{\partial V_L}$.\$

It follows that

$$M = \frac{\partial M}{\partial V_L} K_s^{-1}[V_s]C^{-1} \frac{\partial M}{\partial V_L} \cdot \frac{\partial M}{\partial V_L} C^{-1} \frac{\partial M}{\partial V_L}$$

and one can infer convergence to the set $\mathcal{E} \cap \mathcal{A}_2 \cap \mathcal{V}_{\text{mean}}$ similarly as for the differential–algebraic model.

Remark 9 (Constant Voltage Buses). Similarly to Zhao and Dörfler (2015, Remark 3.3), one can also consider voltage-controlled buses. For example, consider the scenario of all load buses having constant (not necessarily identical) voltages $V_l$ (see Zhao and Dörfler, 2015 for a discussion on this load condition). More precisely, a controller adjust the current injection $I_l$ depending on $V_l$ to maintain the value of the voltage at the constant level $V_l$ so that system (10) reads as

$$K_s V_s = -[V_s]L_s K_s^{-1}[V_s]Y_{ss} + Y_d(V_l - V_l)$$.\$

The only relevant equations for stability of (33) are the ordinary differential equations (33a) driven by the constant term $V_l$. We study their stability using a similar Lyapunov argument as before. Since $V_l$ is now constant, we consider a simplified version of the function $M$, namely $M(V_s) = \mathcal{J}(V_s) + h(V_s)$, where

$$\mathcal{J}(V_s) = \frac{1}{2} [V_s - V_l]^T Y_{ss} [V_s - V_l]$$

are the (shifted) network losses so that $\frac{\partial \mathcal{J}}{\partial V_s} = Y_{ss}(V_s - V_l) = (Y_s V_s + Y_d V_l) - (Y_s V_l + Y_d V_l) = [V_s^{-1}]^{-1}P_1 - [V_s^{-1}]^{-1}P_2$. Together with $h(V_s) = -P_2 \ln(V_s) + P_1 \ln(V_l) + P_1 [V_l^{-1}]^{-1}(V_l - V_s)$, we obtain that $\frac{\partial \mathcal{J}}{\partial V_s} = [V_s^{-1}]^{-1}(P_1 - P_2)$ and thus $K_s V_s = -[V_s]L_s K_s^{-1}[V_s] \frac{\partial \mathcal{J}}{\partial V_s}$. The convergence analysis of the solutions of the system (33) is now analogous to the proof of Theorem 1.

4.4. The case of $Z_l$ loads

In the case of $Z_l$ loads the previous results can be strengthened. First, the set of equilibria can be characterized by two systems of equations, one depending on the source voltages only and the other allowing for a straightforward calculation of the load voltages once the source voltages are determined. Second, the convergence result can be established without any extra condition on the equivalent conductance matrix in (28). Finally, the convergence is to a point rather than to a set.

The first result we present concerns the set of equilibria, which follows by adapting the proof of Lemma 1.

Lemma 3 (Equilibria for $Z_l$ Loads). The set of equilibria of system (10), (11), (9) with $h(V_l) = I_l^T - Y_l^T V_l$ is

$$\mathcal{E} = \{ V \in \mathbb{R}_{\infty} : P_2(V_l) = 0 \}

\cap \{ V = (Y_l + Y_1^*)^{-1}(I_l - Y_l V_l) \},$$

where $P_2(V_l) = \text{powers}$.\$\text{balance}.$\$\text{at the sources}

\$\text{a set of \text{equilibrium}}.$\$\text{loads}_l.$\$\text{a set of \text{source}}.$

We remark that in the $Z_l$ case the equations $P_2(V_l) = 0$ depend on the source voltages only, and once a solution to it is determined, the corresponding voltages at the loads are obtained as $V_l = (Y_l + Y_1^*)^{-1}(I_l - Y_l V_l)$ thereby explicitly solving previous $\mathcal{E} = \emptyset$. Similarly to the case of ZIP loads, we introduce the following standing assumption:

Assumption 2. $\mathcal{E} = \emptyset$.

Our second result concerns the convergence of the dynamics. In the case of $Z_l$ loads, convergence can be established without the definiteness condition on the equivalent conductance matrix $Y_{ss}$ in (28). Indeed, for $P_1^T = 0$, the condition (26) is automatically satisfied. Before, this condition was needed to certify strict convexity of the shifted Lyapunov function $M$ (see (29)) as well as the regularity of the algebraic equation $\mathcal{E} = \emptyset$. Additionally, the limit set in case of $Z_l$ loads is $\mathcal{E} \cap A_2 \cap \mathcal{V}_{\text{mean}}$, where the set of equilibria $\mathcal{E} = \emptyset$ is characterized in Lemma 3, $A_2$ is a sublevel set associated with the Lyapunov function $M$ with $P_1^T = 0$, and the set $\mathcal{V}_{\text{mean}}$ is defined as in (27). Finally, a stronger convergence result can be established, namely any trajectory converges to a point depending on the initial condition. This can be formalized as follows:

Theorem 2 (Point Convergence). Assume that there exists $V_l \in \mathcal{E}$.\$\text{a set of \text{equilibrium}}.$\$\text{loads}_l.$\$\text{a set of \text{source}}.$

(1) The solutions to (10), (11), (9) with $P_1^T = 0$ that originate from any initial condition $V(0)$ belonging to a sublevel set $A_2$ of the shifted Lyapunov function $M$ in (22) with $P_1^T = 0$ contained in $\mathbb{R}_{\infty}^n$ always remain in $A_2$, and

(2) converge to an asymptotically stable equilibrium belonging to $\mathcal{E} \cap A_2 \cap \mathcal{V}_{\text{mean}}$.

Proof. First of all we observe that the proof of Theorem 1 holds for the case of $Z_l$ loads (it suffices to set $P_1^T = 0$ and $h(V_l) = I_l^T - Y_l^T V_l$ throughout the proof). As an additional feature of $Z_l$ loads to be exploited below we can explicitly construct $\delta(V_s) = (Y_l + Y_1^*)^{-1}[I_l - Y_l V_l] + Y_d V_l$, so that $\mathcal{E} = \emptyset$.

From the proof of Theorem 1 (specialized to the case of $Z_l$ loads), it is known that any solution $V_s$ of the ODE (30) is bounded. By Birkhoff’s Lemma (Khalil, 1996, Lemma 3.1) the positive limit set $\mathcal{O}(V_s)$ associated with a solution $V_s(t)$ is non-empty, compact,
and invariant. Moreover, it is contained in $\varepsilon_2 \cap A_2 \cap V_{\text{mean}}$. We would like to prove that $\Omega(V_t)$ is a singleton. To this end, and similarly to De Persis & Monshizadeh (2018) we appeal to Haddad & Chellaboina, (2008, Proposition 4.7), which states that if the positive limit set $\Omega(V_t)$ of a trajectory contains a Lyapunov stable equilibrium $\bar{V}_t$, then $\Omega(V_t) = \{\bar{V}_t\}$. To see this first notice that $\bar{V}_t$ being in $\Omega(V_t)$ and hence in $\varepsilon_2 \cap A_2 \cap V_{\text{mean}}$, it is indeed an equilibrium of the system. Thus, following (31), one can construct a shifted function $\nabla(V_t)$ associated to $\bar{V}_t$. The explicit expression of $\nabla(V_t)$ is given by

$$
\nabla(V_t) = -P_t^1 \bar{V}_t + P_t^1 \nabla(\bar{V}_t) + P_t^1 (\bar{V}_t)^{-1}(V_t - \bar{V}_t)
$$

where $P_t^1$ is a constant matrix.

The gradient of $\nabla(V_t)$ is given by

$$
\frac{\partial \nabla}{\partial V_t} = -[V_t]^{-1}P_t^1 + (V_t)^{-1}P_t^1 + \left(Y_{rs} + Y_{ds} \frac{\partial \delta}{\partial V_t}\right)^T (V_t - \bar{V}_t) + \left(Y_{fs} + (Y_{fl} + Y_{fr}) \frac{\partial \delta}{\partial V_t}\right)^T (\delta(V_t) - \delta(\bar{V}_t)).
$$

Since $\frac{\partial \delta}{\partial V_t} = -(Y_{ls} + Y_{r}^*)^{-1}Y_{ls}$, the last summand above vanishes. With the shorthand $\dot{Y}_{\text{red}} = Y_{rs} - Y_{ds}(Y_{fl} + Y_{fr})^{-1}Y_{ls}$, the gradient simplifies as

$$
\frac{\partial \nabla}{\partial V_t} = -[V_t]^{-1}P_t^1 + (V_t)^{-1}P_t^1 + \dot{Y}_{\text{red}}V_t - \bar{V}_t).
$$

Note that the gradient $\frac{\partial \nabla}{\partial V_t}$ vanishes if $V_t = \bar{V}_t$ and $\nabla$ has a strict local minimum at $\bar{V}_t$, since

$$
\frac{\partial^2 \nabla}{\partial V_t^2} = \dot{Y}_{\text{red}} + (V_t)^{-2}P_t^1.
$$

By (32), $\nabla \leq 0$, and these two properties (properness and the nonpositive time derivative) show that $\bar{V}_t$ is a Lyapunov stable equilibrium. Therefore, $\Omega(V_t) = \{\bar{V}_t\}$, and the solution $V_t(t)$ converges to an equilibrium point. Because $V_t(t)$ is the solution component of the solution to the DAE, and since $V_t$ satisfies $V_t = \delta(V_t) = (Y_{ls} + Y_{r}^*)^{-1}(U - V_{\text{red}})$ we also see that the solution $(V_t(t), V_t(t))$ of the DAE (10), (11), (9) converges to a point in $\varepsilon_2 \cap A_2 \cap V_{\text{mean}}$. Since this equilibrium point is Lyapunov stable by (29) (with $P_t^1 = 0$) and (32), the limit point is also asymptotically stable.

5. Simulations

In this section, we present simulation results about the proposed control strategy (7), and compare it to the averaging-based control method (8) introduced in Remark 2. We use an IEEE 37 bus system adapted from Alwala, Feliachi, & Choudhry (2012), Distribution Test Feeder Working Group (0000) and Kersting (2001) adding lines to form a mesh topology, and upsampling the values of the microgrid parameters. The grid topology is sketched in Fig. 3, and the network and control parameters are given in Tables 1 and 2, and 3. It can be checked that condition (26) is satisfied for this network. There are 26 loads and 7 sources. Among these are eight constant power loads, nine constant impedance loads and nine constant current loads. Fifteen of the loads are initially turned off and are turned on gradually between 9.5 and 10.5 ms; see Table 2. This constitutes a load increase of approximately 1.06 MV, from approximately 779 kW to 1.84 MV. The remaining eleven loads are active throughout.

The voltage evolution both at the sources and at the loads for the controllers (7) and (8) is depicted in Fig. 4. Fig. 5 represents a comparison between the evolution of the voltages at node 1 for the two controllers. For a more detailed comparison, the steady state percentage voltage deviation from the nominal value for all the nodes are reported in Table 2. The power injected at the source nodes is shown for both control strategies in Fig. 6. As predicted by the analysis, at steady state proportional power sharing is achieved by the power sources in conformity with (3). Observe that the voltage deviations are rather small even though both the proposed controller (7) and the distributed integral controller (8) do not account for voltage regulation. Notice also that the two controllers perform similarly, though the voltages for the integral controller tend to be slightly lower than those for the power consensus controller.

6. Conclusions

We have proposed controllers for DC microgrids that average power measurement at the sources. The results apply to network preserved model (systems of DAE) of the microgrid in the presence of ZIP loads. Capacitors at the terminals of the grid that model either $lT$-models of lines or power converter components can be included by means of passivity-based analysis.
flow feasibility and approximations; see Bolognani and Zampieri (2016), Barabanov et al. (2016), Simpson-Porco et al. (2016) and references therein. The distributed averaging integral controller (8) discussed in Remark 2 enjoys the nice feature of not requiring power measurements and could be an enthralling algorithm to investigate further. Finally, the power consensus algorithms preserve the weighted geometric mean of the voltages and is thus a compelling application for nonlinear consensus schemes (Bauso et al., 2006; Cortes, 2008). We believe this connection deserves a deeper investigation.

Table 1
Simulation parameter values. The power sharing coefficients $K_i$ used in [8] have the same numerical values as those in the table, but units of volts.

| Parameter | Value |
|-----------|-------|
| Nominal voltage $V^*$ | $4.8$ kV |
| Power sharing coefficients $K_i$ | $i = 1, 3, 4$ $40 \sqrt{kgm/s}$ |
| | $i = 2, 7$ $20 \sqrt{kgm/s}$ |
| | $i = 5$ $10 \sqrt{kgm/s}$ |
| | $i = 6$ $0.2 F$ |
| Power flow equations, whose solvability still needs to be investigated, e.g., starting from recent advances concerning power flow feasibility and approximations; see Bolognani and Zampieri (2016), Barabanov et al. (2016), Simpson-Porco et al. (2016) and references therein. The distributed averaging integral controller (8) discussed in Remark 2 enjoys the nice feature of not requiring power measurements and could be an enthralling algorithm to investigate further. Finally, the power consensus algorithms preserve the weighted geometric mean of the voltages and is thus a compelling application for nonlinear consensus schemes (Bauso et al., 2006; Cortes, 2008). We believe this connection deserves a deeper investigation.

Table 2
Node properties and final percentage voltage deviation from the nominal value for both the proposed controllers (7) and the distributed integral controller (8).

| Node | ZIP | Turned on | Proposed | DAPI |
|------|-----|----------|----------|------|
| 1    | –   | Always   | –0.14%   | –0.25% |
| 2    | –   | Always   | –0.02%   | –0.37% |
| 3    | –   | Always   | –0.23%   | –0.62% |
| 4    | –   | Always   | –0.04%   | –0.43% |
| 5    | –   | Always   | –0.19%   | –0.57% |
| 6    | –   | Always   | –0.25%   | –0.64% |
| 7    | –   | Always   | –0.13%   | –0.26% |
| 8    | Z   | Gradual  | –0.19%   | –0.57% |
| 9    | I   | Always   | –0.17%   | –0.56% |
| 10   | P   | Gradual  | –0.32%   | –0.71% |
| 11   | Z   | Gradual  | –0.34%   | –0.73% |
| 12   | Z   | Gradual  | –0.32%   | –0.70% |
| 13   | P   | Always   | –0.33%   | –0.71% |
| 14   | Z   | Always   | –0.32%   | –0.71% |
| 15   | I   | Always   | –0.30%   | –0.69% |
| 16   | P   | Gradual  | –0.26%   | –0.65% |
| 17   | Z   | Always   | –0.15%   | –0.54% |
| 18   | I   | Gradual  | –0.18%   | –0.57% |
| 19   | P   | Gradual  | –0.25%   | –0.64% |
| 20   | Z   | Always   | –0.21%   | –0.60% |
| 21   | I   | Always   | –0.16%   | –0.35% |
| 22   | P   | Gradual  | –0.22%   | –0.60% |
| 23   | Z   | Always   | –0.27%   | –0.66% |
| 24   | I   | Gradual  | –0.26%   | –0.65% |
| 25   | P   | Gradual  | –0.30%   | –0.68% |
| 26   | Z   | Gradual  | –0.27%   | –0.66% |
| 27   | I   | Gradual  | –0.32%   | –0.71% |
| 28   | P   | Always   | –0.33%   | –0.71% |
| 29   | Z   | Gradual  | –0.35%   | –0.74% |
| 30   | I   | Gradual  | –0.26%   | –0.64% |
| 31   | P   | Always   | –0.18%   | –0.57% |
| 32   | Z   | Always   | –0.08%   | –0.46% |
| 33   | I   | Gradual  | –0.13%   | –0.52% |

Many interesting new research directions can be taken. The first one is to consider more complex scenarios such as the inclusion of dynamical (inductive) lines and loads. Another one is the extensions of the controllers to network preserved AC microgrids. Moreover, although the preservation of the geometric mean of the voltages allows for an estimate of the voltage excursion, no active voltage regulation is present in the proposed scheme. An addition of voltage controllers to the power consensus algorithm is an interesting and important open problem. The sensitivity of the power consensus algorithm to delays in the communication network is an important feature to be assessed. The theoretical analysis of the algorithm under delays, however, is far from trivial and is left for future research. The power consensus algorithms lead to a new set of power flow equations, whose solvability still needs to be investigated, e.g., starting from recent advances concerning power flow feasibility and approximations; see Bolognani and Zampieri (2016), Barabanov et al. (2016), Simpson-Porco et al. (2016) and references therein. The distributed averaging integral controller (8) discussed in Remark 2 enjoys the nice feature of not requiring power measurements and could be an enthralling algorithm to investigate further. Finally, the power consensus algorithms preserve the weighted geometric mean of the voltages and is thus a compelling application for nonlinear consensus schemes (Bauso et al., 2006; Cortes, 2008). We believe this connection deserves a deeper investigation.

Table 3
Line resistances $\Gamma^{-1}$. The last six lines (2–21, 32–12, 3–19, 6–26, 22–26, 24–33) were added to form a mesh topology.

| Line | Resistance | Line | Resistance |
|------|------------|------|------------|
| 1–9  | 0.3032 $\Omega$ | 16–19 | 0.1198 $\Omega$ |
| 2–17 | 0.1281 $\Omega$ | 17–18 | 0.0958 $\Omega$ |
| 3–14 | 0.2074 $\Omega$ | 19–20 | 0.0958 $\Omega$ |
| 4–21 | 0.1115 $\Omega$ | 19–23 | 0.1475 $\Omega$ |
| 5–24 | 0.1475 $\Omega$ | 20–21 | 0.0691 $\Omega$ |
| 6–28 | 0.5106 $\Omega$ | 21–22 | 0.0802 $\Omega$ |
| 7–32 | 0.0896 $\Omega$ | 23–24 | 0.0488 $\Omega$ |
| 8–9  | 0.0479 $\Omega$ | 24–25 | 0.0793 $\Omega$ |
| 9–10 | 0.3668 $\Omega$ | 25–26 | 0.1281 $\Omega$ |
| 10–11| 0.1475 $\Omega$ | 25–27 | 0.0793 $\Omega$ |
| 10–13| 0.1972 $\Omega$ | 27–28 | 0.1382 $\Omega$ |
| 11–12| 0.1115 $\Omega$ | 28–29 | 0.0802 $\Omega$ |
| 13–14| 0.0323 $\Omega$ | 28–30 | 0.1576 $\Omega$ |
| 13–15| 0.1281 $\Omega$ | 30–31 | 0.0986 $\Omega$ |
| 15–16| 0.0885 $\Omega$ | 31–32 | 0.0986 $\Omega$ |
| 16–17| 0.1594 $\Omega$ | 32–33 | 0.0802 $\Omega$ |
| 2–21 | 0.2430 $\Omega$ | 6–26 | 0.1843 $\Omega$ |
| 3–12 | 0.1566 $\Omega$ | 22–26 | 0.1281 $\Omega$ |
| 3–19 | 0.2166 $\Omega$ | 24–33 | 0.3456 $\Omega$ |

Fig. 4. Voltage plots of the simulation.

Fig. 5. Comparison of voltages at node 1 for both controllers.
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