From scale properties of physical amplitudes to a predictive formulation of the Nambu–Jona-Lasinio model.

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Abstract

The predictive power of the NJL model is considered in the light of a novel strategy to handle the divergences typical of perturbative calculations. The referred calculational strategy eliminates unphysical dependencies on the arbitrary choices for the routing of internal momenta and symmetry violating terms. In the present work we extend a previous one on the same issue by including vector interactions and performing the discussion in a more general context: it is considered the role of scale arbitrariness for the consistency of the calculations. We show that the imposition of arbitrary scale independence for the consistent regularized amplitudes lead to additional properties for the irreducible divergent objects. These properties allow us to parametrize the remaining freedom in terms of a unique constant where resides all the arbitrariness involved. By searching for the best value for the arbitrary parameter we find a critical condition for the existence of an acceptable physical value for the dynamically generated quark mass. Such critical condition fixes the remaining arbitrariness turning the NJL into a predictive model in the sense that its phenomenological consequences do not depend on possible choices made in intermediary steps. Numerical results are obtained for physical quantities like the vector and axial-vector masses and their coupling constants as genuine predictions.

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I. INTRODUCTION

It would be desirable to describe all the phenomenology of interacting quarks from the point of view of quantum chromodynamics (QCD), the fundamental (renormalizable) theory of strong interactions. Even if this is in principle possible, there are many difficulties to be overcome such that effective models, having the main symmetries of the theory, have been frequently used to describe some aspects of the QCD phenomenology. In this context the Nambu–Jona-Lasinio (NJL) model $[1]$ is of indisputable popularity. The model has been used to describe the low energy hadronic phenomenology through observables like hadronic masses spectrum, correlation and structure functions in vacuum since the downing of QCD. Such observables, among others, have been considered at finite densities and temperatures, which stated a large lack of phenomenological aspects associated with systems of quarks and gluons— for a representative list of references see the reviews in Refs. $[2–9]$. The model, from the point of view of quantum field theory (QFT) formalism, is a nonrenormalizable theory, due to the presence of four (1/2 spin) fermion interactions in the model Lagrangian, such that predictions only make sense within the context of a particular level of approximation. Having this limitation in mind, an uncountable number of valuable works have been and are still being produced nowadays. There are many reasons for such interest. Two of them, however, deserve special attention. The first one is the general acceptance that the NJL model captures many of the essential features of chiral symmetry in QCD. In the limit of exact chiral symmetry the fermions of the model are massless and the interaction Lagrangian density contains the chirally symmetric combinations of four-fermion interactions. The second attractive feature is the fact that the model realizes the dynamically breaking of chiral symmetry already at the one-loop (mean field) approximation such that the fermions become massive. The difficulties with the model predictions are related to the nonrenormalizable character of the corresponding relativistic theory such that they are unavoidable intimately compromising with the specific strategy adopted to handle the ultraviolet divergences present in the amplitudes in each particular level of approximation. Consequently, in order to extract physical predictions, we have to specify a procedure for the necessary handling of the divergent amplitudes as the first step. Concerning this aspect this is not different from the procedure adopted in any relativistic QFT. However, while in a renormalizable theory all the parameters alien to the theory which are introduced in the regularization of the
amplitudes, can be completely removed from the physical amplitudes after the divergences are isolated and eliminated through the reparametrization of the theory, this cannot be done in a nonrenormalizable model. This means that the parametrization of the model, which enables us to remove the divergences, must be made in a particular way restricted to the particular level of approximation considered. Within this context, in general, practitioners of the NJL model have adopted the attitude of using it as being a regularization-dependent model, including the regularization procedure as a part of the definition of the model.

The regularization of divergent amplitudes, on the other hand, is a dangerous process since it involves many types of arbitrariness in the manipulation of improper integrals. They may be converted into ambiguities if the results become dependent on the choices involved. Among such ambiguities there are those associated with the arbitrary routing of the momenta in internal lines of divergent loop amplitudes, whose existence is invariably associated with the violation of space-time homogeneity. Another kind of ambiguities with important consequences in the present contribution is the one associated with the choice of the common scale for the divergent and finite parts of amplitudes that lead to the breaking of scale invariance. In general, ambiguous terms lead to violations of symmetry relations of global and local gauge symmetries. The most commonly used regularization procedures, within the context of NJL model, such as three-and four-momentum cutoff, Pauli-Villars and proper-time lead to one or more of such symmetry violations \([11-13]\). Dimensional regularization (DR) \([14]\), on the other hand, leads to ambiguities-free and symmetry-preserving amplitudes but it is not adequate to NJL model due to the fact that the quadratic divergence which appears in almost all one-loop amplitudes must be assumed as zero in the zero-mass limit, which is associated with the chiral symmetry restoration (at high densities and temperatures). Such type of difficulties lead Willey \([15]\) and Gherghetta \([16]\) to conclude that there is no way to make consistent physical predictions with the NJL model using traditional regularization techniques. However, this question was considered in a later work by Battistel and Nemes \([17]\) by using a novel strategy to handle the divergent amplitudes \([18]\). The referred investigation, within the context of the gauged NJL model revealed that it is possible to obtain ambiguities-free and symmetry-preserving amplitudes, which is the first and crucial obstacle to be removed in order to get a predictive model. We said the first obstacle due to the fact that the requirements found as necessary properties for consistent regularizations, in order to remove, in a systematic way, the potentially ambiguous and symmetry violating terms,
do not fix a single complete regularization. The restrictions imposed, in fact, state only a class of regularizations. This is not sufficient to make a nonrenormalizable model predictive because in this case the model predictions, within the context of a particular approximation scheme, may be sensitive to the particular aspects of the regularization employed, even if such regularization belongs to the class of consistent ones. In this scenario the amplitudes are symmetry-preserving and free from ambiguities associated with the choices of routings for the internal lines momenta but the predictions still involve an arbitrariness which is related to the choice of the particular regularization. The practical consequence is that additional phenomenological information must be used in order to parametrize the model. Such input invariably belongs to the phenomenological scope of the model i.e. it is a part of the phenomenology which we want to describe through the model considered. In the context of NJL model, it is common to use the pion mass to select the value of the cutoff $\Lambda$. The pion mass, on the other hand, is a low energy hadronic observable which we would like to predict from the quark properties present at the model Lagrangian. This situation is very frustrating and deserves additional investigations in order to be solved, making the model predictive, which means to obtain its consequences in a way completely independent of the choices involved in intermediary steps of the calculations.

Having this in mind, in a recent contribution, the question of predictability of the NJL model has been again considered \cite{19}. In a complete and detailed investigation, where even the amplitudes having tensor operators have been considered, it was stated that the amplitudes can be obtained preserving their symmetry relations as well as freeing them from the dependences on the choices for the internal lines momenta routing through the identification of general and universal properties for the divergent Feynman integrals, which we denominated consistency relations. An additional step has been performed by using specific properties for the so called irreducible divergent objects due to the fact that, following this procedure, the remaining freedom associated with the choice of the regularization can be put in terms of an arbitrary parameter. Given this situation we, in a first moment, have to recognize that the parametrization of the model and, consequently, the predictions, are really regularization dependent. However, when we search for the best value for the arbitrary parameter, looking for the value of the dynamically generated quark mass, we identified the existence of a critical condition. Only one value for the arbitrary parameter is associated with a unique and real value for the quark mass. Given this fact we concluded
that the predictions of the NJL model are not dependent on choices and we denominated this interpretation as a predictive formulation of the NJL model. Within this formulation, the parametrization of the model requires only three inputs which are chosen as the value for the quark condensate, the pion decay constant and the current quark mass. All the predictions, including the pion mass, are made without additional parameters like the usual regularization parameters (cutoffs). The numerical results obtained within the context of our formulation are in good agreement with the phenomenological ones. This aspect has been confirmed in an independent investigation made by Rochev [20].

Motivated by these surprising results, in the present work we would like to take an additional step in the construction of the predictive formulation of the NJL model, by introducing the most general kind of arbitrariness involved in the perturbative evaluation of physical amplitudes which is the scale arbitrariness. We will show that the scale properties of the physical amplitudes are the adequate ingredient to discuss and understand the consistency in perturbative calculations involving divergences. The scale properties of the irreducible divergent objects are a natural consequence of the imposition of scale independence in the manipulation and calculations involving the intrinsic mathematical indefinities associated with the improper integrals typical of the perturbative calculations. In addition to this aspect, in the present work, we consider a more general analysis of the model parametrization by considering analytical solutions for the equation which generate the critical condition for the dynamically quark mass which, in addition to the more transparent analysis, allows us to obtain clean and sound clarifications through the comparison of our results with those performed with traditional treatments employing regularizations. The phenomenology of vector mesons is also considered.

The work is organized in the following way: In the section II the model is presented and the required definitions are introduced. In the section III we consider the strategy we adopt to handle with the divergences and to explicit the amplitudes which are necessary in the study of the considered phenomenology of mesons presented in the section IV. Finally, in Section V we make our final comments and conclusions.
II. THE MODEL AND DEFINITIONS

A very general Lagrangean constructed with a self interacting Dirac field consistent with chiral symmetry (broken by the mass term) can be adopted as having the form [21]:

\[ L = \bar{\psi}(x) \left( i\gamma^\mu \partial_\mu - m_0 \right) \psi(x) + \frac{G_S}{2} \left[ (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i\gamma_5 \overrightarrow{\tau} \psi(x))^2 \right] \\
- \frac{G_V}{2} \left[ (\bar{\psi}(x) \gamma_\mu \overrightarrow{\tau} \psi(x))^2 + (\bar{\psi}(x) \gamma_\mu \gamma_5 \overrightarrow{\tau} \psi(x))^2 \right] \\
- \frac{G_T}{2} \left[ (\bar{\psi}(x) \sigma_{\mu\nu} \overrightarrow{\tau} \psi(x))^2 - (\bar{\psi}(x) \sigma_{\mu\nu} \psi(x))^2 \right]. \]

If we take \( \psi \) as the quark field (\( u \) and \( d \) flavors), \( m_0 \) as the current quark mass (\( m_u = m_d \)), which explicitly breaks the chiral symmetry, and \( G_S, G_V \) and \( G_T \) as the scalar (pseudo), vector (axial) and tensor coupling strengths, respectively, then the above functional represents the Lagrangian of the extended SU(2) NJL model. Through this model it is possible to describe low energy hadronic observables. It is precisely this challenge we will consider in what follows. In order to emphasize the main aspects involved and to make the discussions transparent, in the present contribution we adopt \( G_T = 0 \). We start by introducing the required definitions.

The nonperturbative quark propagator \( S(p) \) is given in terms of the self-energy \( \Sigma(p) \) as

\[ S^{-1}(p) = \slashed{p} - \Sigma(p). \]

In the mean field approximation, the self-energy is momentum independent \( \Sigma(p) \equiv M \), with \( M \) satisfying a gap equation

\[ M = m_0 - 2 G_S N_f \langle \bar{\psi}\psi \rangle, \]

where \( N_f = 2 \) is the number of flavors, and \( \langle \bar{\psi}\psi \rangle \) is the one-flavor, Lorentz scalar one-point function (the quark condensate) given by

\[ \langle \bar{\psi}\psi \rangle = N_c T^S, \]

where \( N_c = 3 \) is the number of colors and \( T^S \) is the scalar one-point amplitude defined by

\[ T^S = -i \int \frac{d^4k}{(2\pi)^4} t^S, \quad t^S = tr \left\{ \frac{1}{(\slashed{k} + k_1) - M} \right\}. \]

Here \( k_1 \) is an arbitrary internal line momentum (for details please see the next section).
In the context of NJL model, mesons are relativistic quark-antiquark bound states which, in the random-phase approximation (RPA), its propagators can be written as (see for example Ref. \[3, 4, 22\])

\[
D_{M_1M_2}(p^2) = \frac{2G_S}{1 - 2G_S \Pi_{M_1M_2}(p^2)},
\]

where \(\Pi_{M_1M_2}\) is the polarization functions (fermion’s loops) defined by

\[
\Pi_{M_1M_2}(p^2) = -i \int \frac{d^4k}{(2\pi)^4} \pi_{M_1M_2},
\]

where

\[
\pi_{M_1M_2} = \text{Tr} \{\Gamma_{M_1}S(k + k_1)\Gamma_{M_2}S(k + k_2)\},
\]

with \(S\) being the quark propagator defined previously. \(\Gamma_M\) stands for the flavor and Dirac matrices giving the quantum numbers of the meson \(M\). For example, for the neutral pion, \(\Gamma_M = \tau_3\gamma_5\), for the scalar-isoscalar meson, \(\Gamma_M = 1\). In writing the equations above, we assumed the most general labels for the momenta \(k_1\) and \(k_2\) running in the internal lines of the loop integral. The physical momentum \(p\) is defined as the difference \(k_1 - k_2\) as imposed by energy-momentum conservation at each vertex.

The approach used to study the mesonic phenomenology in the NJL context is well known and is described in great details in standard references of this issue (see for example \[4, 17, 23\] and references therein). Basically, besides the gap equation \(2\), we have to solve a Bethe-Salpeter type equation which in nuclear physics language is known as Hartree-Fock plus Random Phase Approximation. Such program is of simple realization when only spin zero mesons are considered (pion and sigma). On the other hand, the model having vector and axial-vector mesons (rho and \(a_1\)) implies in additional contributions for the scalar sector. The same will occur with the vector sector when the tensor coupling is taken into account. As an example, in the pionic sector, by solving the Bethe-Salpeter equation, we find a mixing between the pseudoscalar (\(\pi\)) and the axial-vector (\(a_1\)) mesons. The pion mass \(m_\pi\), is given by solving the condition

\[
D_\pi(p^2)\big|_{p^2 = m_\pi^2} = 0,
\]

where

\[
D_\pi(p^2) = \left[1 - G_S \Pi^{PP}(p^2)\right] \left[1 + G_V \Pi^{AA}_{(L)}(p^2)\right] + G_S G_V \left[\Pi^{AP}(p^2)\right]^2.
\]
Here we have introduced the longitudinal and transverse tensors, defined as [23]:

\[
\Pi^{AA}_L (p^2) = -L^{\mu\nu} \left[ \Pi^{AA}_{\mu\nu} (p^2) \right],
\]
\[
T^{\mu\nu} \left[ \Pi^{AA}_T (p^2) \right] = -T^{\mu \alpha } T^{\nu \beta} \left[ \Pi^{AA}_{\alpha \beta} (p^2) \right],
\]
\[
\Pi^{VV}_L (p^2) = L^{\mu\nu} \left[ \Pi^{VV}_{\mu\nu} (p^2) \right],
\]
\[
T^{\mu\nu} \left[ \Pi^{VV}_T (p^2) \right] = T^{\mu \alpha } T^{\nu \beta} \left[ \Pi^{VV}_{\alpha \beta} (p^2) \right],
\]
\[
\Pi^{AP} (p^2) = -\frac{p^\mu p^\nu}{p^2} \left[ \Pi^{AP}_{\mu\nu} (p^2) \right],
\]

where

\[
L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}, \quad T^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2},
\]

are the longitudinal and transverse parts of the polarization functions.

The pion phenomenology is also characterized by the decay constant \( f_\pi \) and the pion-quark-quark coupling strength \( g_{\pi qq} \). Experimentally, \( f_\pi \) is related to the weak decay \( \pi^\pm \rightarrow \mu^\pm + \nu_\mu \) and is calculated from the vacuum to one-pion axial-vector current matrix element

\[
\langle 0 | \bar{\psi} (x) \gamma_i \gamma_5 \frac{\tau^i}{2} \psi (x) | \pi^j (q) \rangle = i f_\pi q_\mu \delta_{ij} e^{-ipx},
\]

where \( | \pi^j (q) \rangle \) is a pion state with four-momentum \( q \). At one-loop order, one can express this matrix element in terms of the \( \Pi^{AP} \), defined in Eq. (6) by taking \( M_1 = \gamma_\mu \gamma_5 \) and \( M_2 = \gamma_5 \gamma_\mu \), and the longitudinal part of \( \Pi^{AA}_{\mu\nu} \) \( (M_1 = \gamma_\mu \gamma_5 \) and \( M_2 = \gamma_\nu \gamma_5 \) ), as

\[
f_\pi = \frac{g_{\pi qq}}{2m_\pi} \left[ \Pi^{AP} (m_\pi^2) \right] + \frac{\tilde{g}_{\pi qq}}{4M} \left[ \Pi^{AA}_L (m_\pi^2) \right],
\]

where the \( g_{\pi qq} \) and \( \tilde{g}_{\pi qq} \) are the coupling constants between pion and quarks in the effective interaction Lagrangian [24]

\[
L_{\text{eff}}^{(\pi)} = g_{\pi qq} \overline{\psi} (x) i \gamma_5 \gamma_\tau \psi (x) \cdot \nabla \pi^\tau (x) + \frac{\tilde{g}_{\pi qq}}{M} \overline{\psi} (x) \gamma_\mu \gamma_5 \gamma_\tau \psi (x) \cdot \partial^\mu \nabla \pi^\tau (x),
\]

with \( \nabla \pi^\tau (x) \) being the meson field with the quantum numbers of the pion. The \( g_{\pi qq} \) and \( \tilde{g}_{\pi qq} \) coupling constant are related to the residuo of the scattering matrix at the pion pole \( (p^2 = m_\pi^2) \). The corresponding results are

\[
g_{\pi qq}^2 = \left( - \right) \left. \frac{\partial}{\partial p^2} D_\pi (p^2) \right|_{p^2 = m_\pi^2}, \quad (10)
\]
\[
\tilde{g}_{\pi qq} = \left( - \right) \left. \frac{2MG_V}{m_\pi} \left[ \Pi^{AA}_L (m_\pi^2) \right] \right|_{p^2 = m_\pi^2}, \quad (11)
\]
The vector and axial-vector mesons are described in a very similar way. The pole of the scattering matrix in the vector channel gives the condition which determines the rho mass ($m_\rho$), i.e.,

$$D_\rho \left( p^2 \right) \big|_{p^2 = m_\rho^2} = 0,$$

$$D_\rho \left( p^2 \right) = 1 + G_V \left[ \Pi_{T}^{VV} \left( p^2 \right) \right].$$

(12)

The residuum of the scattering matrix at the rho pole ($p^2 = m_\rho^2$) is related to the coupling constants between rho and quarks in the effective interaction Lagrangian

$$L_{\text{eff}}^{(\rho)} = g_{\rho qq} \bar{\psi}(x) \gamma_\mu \frac{\tau_i}{2} \psi(x) \rho_i^\mu(x),$$

where $\rho_i^\mu(x)$ is the vector field. The rho-quark-quark coupling constant is given by

$$g_{\rho qq}^{-2} = \frac{1}{4G_V} \frac{\partial}{\partial p^2} D_\rho \left( p^2 \right) \big|_{p^2 = m_\rho^2}.$$

(13)

On the other hand, the matrix elements of the electromagnetic current between vector meson states and the vacuum defines the rho meson decay constant ($f_\rho$):

$$\left\langle 0 \left| \frac{1}{2} \gamma_\mu \gamma_5 \psi(x) \right| \rho \right\rangle = \frac{m_\rho^2}{f_\rho} \varepsilon_\mu,$$

where $\varepsilon_\mu$ is the polarization vector of the $\rho$ meson field. We have then the following relation

$$\frac{4m_\rho^2}{f_\rho} = g_{\rho qq} \left[ \Pi_{T}^{VV} \left( p^2 \right) \right]_{p^2 = m_\rho^2}.$$

(14)

Finally, the simplest effective interaction Lagrangean describing the coupling of a axial-vector field $a_{i1}^\mu(x)$ with quark fields may be written as

$$L_{\text{eff}}^{(a_1)} = g_{a_{1}qq} \bar{\psi}(x) \gamma_\mu \gamma_5 \tau_i \psi(x) a_{i1}^\mu(x).$$

In the RPA approximation the axial meson ($a_1$) mass $m_{a_1}$ is given by the condition:

$$D_{a_1} \left( p^2 \right) \big|_{p^2 = m_{a_1}^2} = 0,$$

$$D_{a_1} \left( q^2 \right) = 1 + G_V \left[ \Pi_{T}^{AA} \left( p^2 \right) \right].$$

The $a_1$-quark-quark coupling constant is determined through the condition

$$g_{a_{1}qq}^{-2} = 4 \frac{\partial}{\partial p^2} \Pi_{T}^{AA} \left( p^2 \right) \big|_{p^2 = m_{a_1}^2}.$$

In the next section we evaluate the involved amplitudes, within the context of our strategy. After this we will return to the definitions introduced above to calculate the corresponding physical quantities.
III. THE EVALUATION OF PHYSICAL AMPLITUDES

Let us start with the relevant aspects of the adopted strategy to handle the divergences, alternative to the traditional regularization techniques. Such strategy, proposed and developed by O.A. Battistel in Ref. [18], has a central idea which is to avoid the critical step involved in the regularization process: the explicit evaluation of divergent integrals. In the intermediary steps it is assumed the presence of a regularization only in an implicit way. No specific form for the regularization distribution is adopted and only very general properties of an acceptable regularization are assumed. Through the use of an adequate representation for the propagators, the amplitudes are written in a convenient mathematical form such that when the integration is taken, all the dependence on the internal lines (arbitrary momenta) are located in finite integrals and the divergent ones are written in terms of standard mathematical objects. With this attitude it becomes possible to identify the properties required for a regularization in order to eliminate the potentially ambiguous terms as well as those which are potentially symmetry violating because they appear separated from the finite parts in a natural way. Following this strategy we search for the properties that a regularization must have in order to be consistent.

The implementation of the procedure is made by choosing the adequate representation for the involved propagators. The idea is to write the propagators in a way that the momentum structure is emphasized just because it is in the last instance, this structure that is responsible for the behavior which generates the divergences in the amplitudes. We adopt then an adequate representation for a propagator carrying momentum $k + k_i$ and mass $M$:

$$\frac{1}{[(k + k_i)^2 - M^2]} = \sum_{j=0}^{N} (-1)^j \frac{(k_i^2 + 2k_i \cdot k + \lambda^2 - M^2)^j}{(k^2 - \lambda^2)^{j+1}} + \frac{(-1)^{N+1} (k_i^2 + 2k_i \cdot k + \lambda^2 - M^2)^{N+1}}{(k^2 - \lambda^2)^{N+1} [(k + k_i)^2 - M^2]}.
\tag{15}$$

Here $k_i$ is (in principle) an arbitrary routing momentum of an internal line in a loop, and $M$ is the fermion mass running in the loop. In the above identity we have introduced an arbitrary parameter $\lambda$ with dimension of mass. As it will be shown in the next section, this parameter gives a precise connection between the divergent and finite parts. It plays the role of a common scale for the divergent and finite parts of the corresponding Feynman integrals. The value for $N$ in the equation above must be taken as major or equal to the
highest superficial degree of divergence of the considered theory or model, if we want to take a unique representation for all involved propagators. Once this representation is assumed, the integration in the loop momentum can be introduced (the last Feynman rule). All the Feynman integrals containing the internal momenta will be present in finite integrals. On the other hand, the divergent ones will contain only the arbitrary mass scale $\lambda$ assuming then standard mathematical forms. No divergent integrals need, in fact, to be solved. Only the tensor reduction must be specified such that the divergent content of amplitudes will appear as standard irreducible divergent objects. They need not be calculated since in renormalizable theories they are completely absorbed in the reparametrization of the theory and in nonrenormalizable models they will be directly adjusted to phenomenological parameters in the parametrization of the model in each specific level of approximation considered. In the calculation process the regularization plays a secondary role just because it is only necessary to assume its implicit presence. The tensor reduction of purely divergent integrals, the unique assumption involving divergent integrals in the intermediary steps, works like consistency requirements to be imposed over an eventual regularization distribution. More details about the procedure will be presented in a moment when the amplitudes are evaluated.

If we wish to follow the procedure mentioned above to evaluate the amplitudes $T^S$, $T^{PP}$, $T^{AP}_\mu$, $T^{VV}_\mu$, and $T^{AA}_\mu$, adopting a universal form for the involved propagators of the internal fermions, we must take the following representation

$$S(k + k_i; \lambda^2) = \frac{(k + k_i) + M}{[(k + k_i)^2 - M^2]}\left\{ \frac{1}{(k^2 - \lambda^2)} - \frac{(k_i^2 + 2 (k_i \cdot k) + \lambda^2 - M^2)}{(k^2 - \lambda^2)^2} \right\}$$

$$+ \frac{(k_i^2 + 2 (k_i \cdot k) + \lambda^2 - M^2)^2}{(k^2 - \lambda^2)^3} - \frac{(k_i^2 + 2 (k_i \cdot k) + \lambda^2 - M^2)^3}{(k^2 - \lambda^2)^4}$$

$$+ \frac{(k_i^2 + 2 (k_i \cdot k) + \lambda^2 - M^2)^4}{(k^2 - \lambda^2)^4 [(k + k_i)^2 - M^2]} \right\}.$$  \tag{16}

This expression is obtained by taking $N = 3$ in the Eq. (15) and performing the summation over the values of $j$ (0, 1, 2 and 3). Note that the expression above is, in fact, independent of the arbitrary scale parameter $\lambda$. This can be easily checked by verifying that

$$\frac{\partial}{\partial \lambda^2} S(k + k_i; \lambda^2) = 0.$$  

Let us now evaluate the amplitudes starting by taking the one corresponding to the highest
divergence degree: the $T^S$ defined in (1). First we construct the quantity $t^S$ by performing the Dirac traces and substituting the expression (16) for the propagator. We get then

$$t^S = 4M \left\{ \frac{1}{(k^2 - \lambda^2)} - 2k_\alpha \left( \frac{k_\alpha}{(k^2 - \lambda^2)^2} \right) - (\lambda^2 - M^2) \left( \frac{1}{(k^2 - \lambda^2)^2} \right) \right.$$

$$+ k_\alpha k_\beta \left( \frac{4k_\alpha k_\beta}{(k^2 - \lambda^2)^3} - \frac{g_{\alpha \beta}}{(k^2 - \lambda^2)^2} \right) + 4k_\alpha \left( k_1^2 + \lambda^2 - M^2 \right) \left( \frac{k_\alpha}{(k^2 - \lambda^2)^3} \right)$$

$$+ \frac{(k_1^2 + \lambda^2 - M^2)^2}{(k^2 - \lambda^2)^3} - \frac{(k^2 + 2k_1 \cdot k + \lambda^2 - M^2)^3}{(k^2 - \lambda^2)^4} + \frac{(k_1^2 + 2k_1 \cdot k + \lambda^2 - M^2)^4}{(k^2 - \lambda^2)^4 \left[ (k + k_1)^2 - m^2 \right]} \right\}.$$  

After this, integrating over the loop momenta $k$ we get the amplitude $T^S$ given by

$$T^S = -4M \left\{ i \left[ I_{quad} (\lambda^2) \right] + (M^2 - \lambda^2) i \left[ I_{log} (\lambda^2) \right] \right.$$  

$$- \frac{1}{16\pi^2} \left[ M^2 - \lambda^2 - M^2 \ln \left( \frac{M^2}{\lambda^2} \right) \right] \right.$$  

$$- ik_\alpha k_\beta \left[ \Delta^{\alpha \beta} (\lambda^2) \right], \tag{17}$$

where we have introduced the irreducible divergent objects

$$I_{quad} (\lambda^2) = \int d^4k \frac{1}{(2\pi)^4 (k^2 - \lambda^2)}, \tag{18}$$

$$I_{log} (\lambda^2) = \int d^4k \frac{1}{(2\pi)^4 (k^2 - \lambda^2)^2}, \tag{19}$$

and also the object

$$\nabla_{\mu \nu} (\lambda^2) = \int_A \frac{d^4k}{(2\pi)^4} \frac{2k_\mu k_\nu}{(k^2 - M^2)^2} - \int_A \frac{d^4k}{(2\pi)^4} \frac{g_{\mu \nu}}{(k^2 - M^2)}. \tag{20}$$

To obtain the result (17) we have only integrated the finite integrals and left out the integrals having odd integrands. Now we evaluate the two point-functions. First we rewrite the integrand (7) of such amplitudes as

$$\pi_{M_1 M_2} = \frac{1}{D_{12}} \left\{ Tr (\Gamma_{M_1} \gamma_\alpha \Gamma_{M_2} \gamma_\beta) k_\alpha k_\beta \right.$$  

$$+ [k_2 \beta Tr (\Gamma_{M_1} \gamma_\alpha \Gamma_{M_2} \gamma_\beta) + m Tr (\Gamma_{M_1} \gamma_\alpha \Gamma_{M_2})] k_\alpha$$

$$+ [k_1 \alpha Tr (\Gamma_{M_1} \gamma_\alpha \Gamma_{M_2} \gamma_\beta) + m Tr (\Gamma_{M_1} \Gamma_{M_2} \gamma_\beta)] k_\beta$$

$$+ [k_1 \alpha k_2 \beta Tr (\Gamma_{M_1} \gamma_\alpha \Gamma_{M_2} \gamma_\beta) + mk_1 \alpha Tr (\Gamma_{M_1} \gamma_\alpha \Gamma_{M_2})$$

$$+ mk_2 \beta Tr (\Gamma_{M_1} \Gamma_{M_2} \gamma_\beta) + m^2 Tr (\Gamma_{M_1} \Gamma_{M_2})] \right\},$$

where we have introduced the notation $D_{12} = [(k + k_1)^2 - m^2] [(k + k_2)^2 - m^2]$. Now, by using the representation (16) for the fermion-propagators, performing the Dirac traces with
the appropriate $\Gamma_{M_1}$ and $\Gamma_{M_2}$ chosen, integrating over the loop momentum $k$ and, again dropping out the odd integrals, the two point-functions which are necessary in this work are given by

$$T^{PA}_{\mu} = -4 M q_{\mu} \left\{ i \left[ I_{\log} (\lambda^2) \right] + \frac{1}{16\pi^2} \left[ Z_0 \left( q^2, M^2; \lambda^2 \right) \right] \right\}, \quad (21)$$

$$T^{PP} = 4i \left[ I_{quad} (\lambda^2) \right] - 4 \left( \lambda^2 - M^2 \right) i \left[ I_{log} (\lambda^2) \right]$$

$$+ \frac{1}{4\pi^2} \left[ \lambda^2 - M^2 - M^2 \ln \left( \frac{\lambda^2}{M^2} \right) \right]$$

$$- 2q^2 \left\{ i \left[ I_{log} (\lambda^2) \right] + \frac{1}{16\pi^2} \left[ Z_0 \left( q^2, M^2; \lambda^2 \right) \right] \right\}$$

$$+ i (q_\alpha q_\beta + Q_\alpha Q_\beta) \left[ \Delta^{\alpha\beta} (\lambda^2) \right], \quad (22)$$

$$T^{VV}_{\mu\nu} = \frac{4}{3} (g_{\mu\nu} q^2 - q_{\mu} q_{\nu}) i \left[ I_{log} (\lambda^2) \right]$$

$$- \frac{(g_{\mu\nu} q^2 - q_{\mu} q_{\nu})}{2\pi^2} \left[ Z_2 \left( q^2, M^2; \lambda^2 \right) - Z_1 \left( q^2, M^2; \lambda^2 \right) \right] + A_{\mu\nu} (\lambda^2), \quad (23)$$

$$T^{AA}_{\mu\nu} = \frac{4}{3} (g_{\mu\nu} q^2 - q_{\mu} q_{\nu}) i \left[ I_{log} (\lambda^2) \right]$$

$$- \frac{(g_{\mu\nu} q^2 - q_{\mu} q_{\nu})}{2\pi^2} \left[ Z_2 \left( q^2, M^2; \lambda^2 \right) - Z_1 \left( q^2, M^2; \lambda^2 \right) \right]$$

$$- g_{\mu\nu} M^2 \left\{ 8i \left[ I_{log} (\lambda^2) \right] + \frac{1}{2\pi^2} \left[ Z_0 \left( q^2, M^2; \lambda^2 \right) \right] \right\} + A_{\mu\nu} (\lambda^2). \quad (24)$$

In the above expressions we have introduced the $Z_k \left( q^2, M^2; \lambda^2 \right)$ finite structures

$$Z_k \left( q^2, M^2; \lambda^2 \right) = \int_0^1 dz \ z^k \log \frac{q^2 z (1 - z) - M^2}{-\lambda^2}. \quad (25)$$

The quantity $A_{\mu\nu}$ represents the expression

$$A_{\mu\nu} = 4 \left[ \nabla_{\mu\nu} (\lambda^2) \right] + q^\alpha q^\beta \left[ \frac{1}{3} \Box_{\alpha\beta\mu\nu} (\lambda^2) + \frac{1}{3} g_{\alpha\nu} \Delta_{\mu\beta} (\lambda^2) + g_{\alpha\mu} \Delta_{\beta\nu} (\lambda^2) \right.$$

$$- g_{\mu\nu} \Delta_{\alpha\beta} (\lambda^2) - \frac{2}{3} g_{\alpha\beta} \Delta_{\mu\nu} (\lambda^2) \bigg]\$$

$$+ (q^\alpha Q^\beta - Q^\alpha q^\beta) \left[ \frac{1}{3} \Box_{\alpha\beta\mu\nu} (\lambda^2) + \frac{1}{3} g_{\nu\alpha} \Delta_{\mu\beta} (\lambda^2) + \frac{1}{3} g_{\alpha\mu} \Delta_{\beta\nu} (\lambda^2) \right.$$

$$+ Q^\alpha Q^\beta \left[ \Box_{\alpha\beta\mu\nu} (\lambda^2) - g_{\mu\beta} \Delta_{\nu\alpha} (\lambda^2) - g_{\alpha\mu} \Delta_{\beta\nu} (\lambda^2) - 3g_{\mu\nu} \Delta_{\alpha\beta} (\lambda^2) \bigg]. \quad (26)$$
where we have introduced two new divergent structures defined as

\[ \Box_{\alpha\beta\mu\nu}(\lambda^2) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{24k_\mu k_\nu k_\alpha k_\beta}{(k^2 - M^2)^4} - g_{\alpha\beta} \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{4k_\mu k_\nu}{(k^2 - M^2)^3} - g_{\alpha\nu} \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{4k_\beta k_\mu}{(k^2 - M^2)^3} - g_{\alpha\mu} \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{4k_\beta k_\nu}{(k^2 - M^2)^3}, \]

\[ \Delta_{\mu\nu}(\lambda^2) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{4k_\mu k_\nu}{(k^2 - M^2)^3} - \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2 - M^2)^2}. \]

Note that when the integration sign was introduced (the last Feynman rule) we performed the indicated operation only in the finite integrals. On the other hand, for those structures which have emerged divergent, no additional operation is performed. This means that in the expressions for the calculated amplitudes five quantities remain undefined. They are: \( I_{\text{quad}}(\lambda^2), I_{\text{log}}(\lambda^2), \Delta_{\alpha\beta}(\lambda^2), \nabla_{\alpha\beta}(\lambda^2), \) and \( \Box_{\alpha\beta\mu\nu}(\lambda^2). \) The objects \( \Delta_{\alpha\beta}(\lambda^2), \nabla_{\alpha\beta}(\lambda^2), \) and \( \Box_{\alpha\beta\mu\nu}(\lambda^2) \) are differences between integrals having the same degree of divergence. In principle, to obtain a value for these objects, the integrand must be made finite through the assumption of a regularization distribution. This process can be schematically represented as

\[ \int \frac{d^3k}{(2\pi)^3} f(k) \rightarrow \int \frac{d^3k}{(2\pi)^3} f(k) \left\{ \lim_{\Lambda_i^2 \to \infty} G_{\Lambda_i}(k, \Lambda_i^2) \right\} = \int \frac{d^4k}{(2\pi)^4} f(k). \]

For such regularization distribution to be acceptable it must be even in the integrating momentum \( k \) in order to be consistent with the Lorentz invariance maintenance. This is the reason why we have excluded divergent integrals having odd integrands when the integration sign was introduced. The regularization distribution must also possess the well defined limit

\[ \lim_{\Lambda_i^2 \to \infty} G_{\Lambda_i}(k^2, \Lambda_i^2) = 1, \]

which allows us to connect with the original integral. Here \( \Lambda_i^2 \) are parameters characterizing the regularization distribution. In what follows we will note that the evaluation of divergent integrals is in fact not really necessary.

Following our reasoning line we now search for additional properties that a consistent regularization must have in order to be consistent. First we note that there are potentially ambiguous terms in the calculated amplitudes. They are present in the amplitudes \( T^S, T^{PP} \),
$T^{VV}_{\mu\nu}$, and $T^{AA}_{\mu\nu}$ and are given by

$$(T^S)_{ambi} = k_1 a k_1 b \left[ \Delta^{ab} (\lambda^2) \right],$$

$$(T^{PP})_{ambi} = -Q_\alpha Q_\beta \left[ \Delta^{ab} (\lambda^2) \right],$$

$$(T^{VV})_{ambi} = (T^{AA})_{ambi} = (q^\alpha Q^\beta - Q^\alpha q^\beta) \times \left[ \frac{1}{3} \Box_{\alpha\beta\mu\nu} (\lambda^2) + \frac{1}{3} g_{\nu\alpha} \Delta_{\mu\beta} (\lambda^2) + \frac{1}{3} g_{\alpha\mu} \Delta_{\beta\nu} (\lambda^2) \right] + Q^\alpha Q^\beta \left[ \Box_{\alpha\beta\mu\nu} (\lambda^2) - g_{\mu\beta} \Delta_{\nu\alpha} (\lambda^2) - g_{\alpha\mu} \Delta_{\beta\nu} (\lambda^2) - 3 g_{\mu\nu} \Delta_{\alpha\beta} (\lambda^2) \right].$$

These terms are ambiguous due to two reasons. First, the dependence on the momentum $Q = k_1 + k_2$, in the two-point functions, or $k_1$ in the one-point function, implies ambiguity because this quantity is completely undefined and dependent on the choices for the internal momenta routing. For the second, the quantity $\Delta^{ab} (\lambda^2)$ may only be dependent on the arbitrary scale parameter $\lambda$ which is also a choice. On the other hand, there are terms in the $T^{PP}$, $T^{VV}$, and $T^{AA}$ amplitudes which are nonambiguous concerning the internal momenta choices such as $q_\alpha q_\beta \left[ \Delta^{ab} (\lambda^2) \right]$ and $\nabla_{\mu\nu} (\lambda^2)$. This means that, in order to eliminate all the ambiguous terms, we have no option rather than require the following property for a consistent regularization (for an extensive discussion see the Ref. [19]):

$$\Delta^{\alpha\beta}_{reg} (\lambda^2) = \nabla^{\alpha\beta}_{reg} (\lambda^2) = \Box^{\alpha\beta\mu\nu}_{reg} (\lambda^2) = 0,$$

(31)

which we denominate consistency relations (CR). These properties are an unavoidable requirement if we want to take an additional step in the evaluation of the considered physical amplitudes. In fact, these properties are satisfied within the context of DR and Pauli-Villars methods (see Ref. [25]), for example. We can understand these constraints as follows: if we cannot find such regularization distribution, the perturbative evaluation of physical amplitudes does not make any sense since the results will be dependent on intermediary arbitrary choices. Predictions cannot be made since undefined quantities must be chosen before the description of a certain phenomenology. The analysis of symmetry relations reveals that the same conditions are required to avoid symmetry violations.

After this brief discussion we can define the consistently regularized amplitudes by adopting the conditions (31):

$$T^S = -4 M \left\{ i \left[ I_{quad} (\lambda^2) \right] + (M^2 - \lambda^2) i \left[ I_{log} (\lambda^2) \right] - \frac{1}{16 \pi^2} \left[ M^2 - \lambda^2 - M^2 \ln \left( \frac{M^2}{\lambda^2} \right) \right] \right\},$$

(32)

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\[ \mathcal{T}_{AP}^\mu = -4Mq_\mu \left\{ i \left[ I_{\log} (\lambda^2) \right] + \frac{1}{16\pi^2} \left[ Z_0 \left( q^2, M^2; \lambda^2 \right) \right] \right\}, \quad (33) \]

\[ \mathcal{T}^{PP} = 4i \left[ I_{quad} (\lambda^2) \right] - 4 \left( \lambda^2 - M^2 \right) i \left[ I_{\log} (\lambda^2) \right] \]
\[ + \frac{1}{4\pi^2} \left[ \lambda^2 - M^2 - M^2 \ln \left( \frac{\lambda^2}{M^2} \right) \right] \]
\[ - 2q^2 \left\{ i \left[ I_{\log} (\lambda^2) \right] + \frac{1}{16\pi^2} \left[ Z_0 \left( q^2, M^2; \lambda^2 \right) \right] \right\}, \quad (34) \]

\[ \mathcal{T}_{\mu\nu}^{VV} = \frac{4}{3} \left( g_{\mu\nu} q^2 - q_\mu q_\nu \right) i \left[ I_{\log} (\lambda^2) \right] \]
\[ - \frac{\left( g_{\mu\nu} q^2 - q_\mu q_\nu \right)}{2\pi^2} \left[ Z_2 \left( q^2, M^2; \lambda^2 \right) - Z_1 \left( q^2, M^2; \lambda^2 \right) \right], \quad (35) \]

and

\[ \mathcal{T}^{AA}_{\mu\nu} = \frac{4}{3} \left( g_{\mu\nu} q^2 - q_\mu q_\nu \right) i \left[ I_{\log} (\lambda^2) \right] \]
\[ - \frac{\left( g_{\mu\nu} q^2 - q_\mu q_\nu \right)}{2\pi^2} \left[ Z_2 \left( q^2, M^2; \lambda^2 \right) - Z_1 \left( q^2, M^2; \lambda^2 \right) \right] \]
\[ - g_{\mu\nu} M^2 \left\{ 8i \left[ I_{\log} (\lambda^2) \right] + \frac{1}{2\pi^2} \left[ Z_0 \left( q^2, M^2; \lambda^2 \right) \right] \right\}. \quad (36) \]

Even if the property (31) is fulfilled, it remains yet the possibility of the amplitudes being dependent on the choice for the arbitrary scale parameter \( \lambda \) when the quantities \( I_{quad} (\lambda^2) \) and \( I_{\log} (\lambda^2) \) are evaluated within the context of a certain regularization. Then we can ask ourselves about the additional conditions to be satisfied by a regularization in order to produce scale independent amplitudes. For this purpose we take initially the \( \mathcal{T}_{AP}^\mu \) amplitude. Obviously, scale independence must be required for the full amplitude (divergent and finite parts), i.e.,

\[ \frac{\partial \mathcal{T}_{AP}^\mu}{\partial \lambda^2} = 0. \]

The derivative of the finite part can be performed such that the above imposition states the property

\[ \frac{\partial}{\partial \lambda^2} \left[ I_{\log} (\lambda^2) \right] = - \frac{i}{16\pi^2 \lambda^2}. \quad (37) \]

The argument of scale independence can be applied to the \( \mathcal{T}^S \) amplitude resulting in a property relating in a precise way the irreducible divergent objects

\[ \frac{\partial}{\partial \lambda^2} \left[ I_{quad} (\lambda^2) \right] = I_{\log} (\lambda^2). \quad (38) \]
At this point we can ask ourselves the following: what do the requirements mean? (37) and (38)? They represent two additional consistency requirements to be imposed over an eventual regularization distribution in order to classify it as a consistent regularization. These conditions can be viewed in a very clear way when we relate the irreducible divergent object in two different mass scales $\lambda$ and $\lambda_0$

$$i \left[ I_{\log} (\lambda^2) \right] = i \left[ I_{\log} (\lambda_0^2) \right] - \frac{1}{16\pi^2} \ln \left( \frac{\lambda_0^2}{\lambda^2} \right), \quad (39)$$

$$i \left[ I_{\text{quad}} (\lambda^2) \right] = i \left[ I_{\text{quad}} (\lambda_0^2) \right] + (\lambda^2 - \lambda_0^2) i \left[ I_{\log} (\lambda_0^2) \right] - \frac{1}{16\pi^2} \left[ \lambda^2 - \lambda_0^2 + \lambda^2 \ln \left( \frac{\lambda_0^2}{\lambda^2} \right) \right]. \quad (40)$$

These relations can be obtained from the $T^S$ and $T^\mu_{AP}$ amplitudes, Eqs. (32) and (34), by first evaluating $T^S$ at $k_1 = 0$ and $T^{PP}$ at $q^2 = 0$. Note that the imposition of independence with $\lambda$ leads us in a natural way to the properties (37) and (38). Once the above equation allows us to relate the irreducible divergent object in different scales we denominate them as scale relations. The violation of these properties will result in the breaking of scale properties of physical amplitudes which possess the same status of symmetry violations.

The scale relations allow us to see a very important consequence of the consistency requirements for the irreducible divergent objects. If we separate the parts relative to the dependence on the two different mass scale parameters into two different sides of the equation, we can easily conclude that the equality of both sides can be only justified if both sides are simultaneously equal to the same constant, which means that

$$i \left[ I_{\log} (\lambda^2) \right] - \frac{1}{16\pi^2} \ln \left( \lambda^2 \right) \equiv C_1, \quad (41)$$

$$i \left[ I_{\text{quad}} (\lambda^2) \right] - \lambda^2 i \left[ I_{\log} (\lambda^2) \right] + \frac{1}{16\pi^2} \lambda^2 \equiv C_2, \quad (42)$$

where $C_1$ and $C_2$ are arbitrary constants. These two equations will be useful in the next Section where we study the implications of the scale properties to the formulation of the NJL model within our approach.

IV. PHENOMENOLOGICAL PREDICTIONS

After the developments performed in the preceding sections we are at the position of considering the parametrization of the NJL model, at the considered level of approximation,
and after this appreciating the corresponding phenomenological predictions. These aspects must be considered by studying the consequences of the consistency requirements stated in last section.

Let us start by considering the simplest case, the chirally symmetric one \((m_0 = 0)\). As we have discussed above, due to the nonrenormalizable character of the NJL model, the remaining undefined objects, \(I_{quad} (\lambda^2)\) and \(I_{log} (\lambda^2)\), must be related to the physical parameters which are taken as inputs of the model. In the present work, as it is usual, we take the quark condensate \(\langle \bar{\psi} \psi \rangle\), the pion decay \(f_\pi\), and the vector coupling \(G_V\) as being the phenomenological inputs of the model. In this line of reasoning, it is easy to see that \(I_{quad} (M^2)\) can be directly related to the quark condensate. This fact can be noted from Eqs. (3) and (4). The result is

\[
\imath [I_{quad} (M^2)] = -\frac{\langle \bar{\psi} \psi \rangle}{4N_c M},
\]  

(43)

On the other hand, the parametrization of the \(I_{log} (M^2)\) in terms of the inputs parameters is not so obvious and immediate. Due to this let us work out this parametrization in some detail. At first we note the relation

\[
\Pi^{PA} (p^2) = -\frac{\sqrt{p^2}}{2M} \left[ \Pi^{AA} (p^2) \right],
\]

which, by using Eqs. (9) and (11), allows to write

\[
f_\pi = -\frac{\tilde{g}_{\pi qq}}{4M} \left( 1 - \frac{\tilde{g}_{\pi qq}}{g_{\pi qq}} \right) \left[ \Pi^{AA} (0) \right],
\]  

(44)

\[
\frac{\tilde{g}_{\pi qq}}{g_{\pi qq}} = \frac{G_V \left[ \Pi^{AA} (0) \right]}{1 + G_V \Pi^{AA} (0)},
\]  

(45)

remembering that in the chirally symmetric case \(p^2 = m_\pi^2 = 0\). In the same way we may write

\[
D_\pi (p^2) = -\frac{G_S}{4M^2} p^2 \left[ \Pi^{AA} (p^2) \right],
\]

\[
\frac{\partial}{\partial p^2} D_\pi (p^2) = -\frac{G_S}{4M^2} \left[ \Pi^{AA} (p^2) \right] + \frac{N_c}{4\pi^2} G_S p^2 \left[ Y_1 (p^2, M^2) \right],
\]

where we have used

\[
\Pi^{PP} (p^2) = 8N_c \imath \left[ I_{quad} (M^2) \right] + \frac{p^2}{4M^2} \left[ \Pi^{AA} (p^2) \right],
\]

\[
Y_1 (p^2, M^2) = \frac{\partial}{\partial p^2} Z_0 (p^2, M^2; M^2).
\]
The expressions above and the Eq. (10) lead us to write
\[ g_{\pi qq}^{-2} = \frac{\left[ \Pi_{(L)}^{AA} (0) \right]}{4M^2 \left[ 1 + G_V \Pi_{(L)}^{AA} (0) \right]} . \]

From Eqs. (44), (45) we get
\[ \Pi_{(L)}^{AA} (0) = \frac{4f^2_\pi}{1 - 4G_V f^2_\pi} . \]

Since
\[ \Pi_{(L)}^{AA} (0) = -16N_cM^2i \left[ I_{\log} (M^2) \right] , \]
we finally arrived at the searched parametrization for \( I_{\log} (M^2) \). It is given by
\[ i \left[ I_{\log} (M^2) \right] = -\frac{1}{4N_cM^2} \frac{f^2_\pi}{1 - 4G_V f^2_\pi} . \] (46)

It is interesting to note the following simple (and exact) relations involving the effective pion coupling constant
\[ M = g_{\pi qq}f_\pi , \]
\[ \tilde{g}_{\pi qq} = 4G_V M f_\pi . \]

So, the Eqs. (43) and (46) state the required relations between the two remaining undefined quantities and the input parameters.

Next we consider a very important aspect of our formulation. In the last section we have shown that the objects \( I_{\quad} (M^2) \) and \( I_{\log} (M^2) \) are not independent and precise relations between them was stated. Having this in mind we now, by using the results (43) and (46) in Eq. (42), obtain a simple algebraic equation for the quark mass:
\[ M^3 + \alpha M + \beta = 0 , \] (47)

where we have defined
\[ \alpha = \frac{4\pi^2}{3} \frac{f^2_\pi}{1 - 4G_V f^2_\pi} - C , \quad \beta = -\frac{4\pi^2}{3} \left< \bar{\psi} \psi \right> , \quad C = 16\pi^2C_2 . \]

The cubic algebraic equation (47) gives all the possible values for the constituent quark mass \( M \) - given the values for the coefficients \( \alpha \) and \( \beta \) - which are consistent with the scale independence requirements. We note that both the sign and the value of \( \alpha \) are dependent on the value assumed by the constant \( C \). Therefore, at this point one could conclude that
the physical implications of the model are dependent on the choice of an arbitrary constant since the solutions of the Eq. (47) is obviously sensitive to changes of the coefficient $\alpha$. Putting that in different words, at this stage, it seems that the physical implications are definitely regularization dependent since the parameter $C$ is determined by the specific form of the regularization. Remembering that the regularizations considered as acceptable at this stage are only those that satisfy the Consistency Relations, which eliminates the ambiguous and symmetry violating terms, as well as obey the properties for the irreducible divergent objects, which guaranty the scale independence. However, a more careful analysis must be made since the equation above is a polynomial of the third degree and, consequently, a very large lack of possibility for the roots exists depending on the values for the coefficients of the different powers of $M$. So, the conclusion stated above is premature and may be wrong.

In order to see what the real situation is, we note that the Eq. (47), for $\alpha$ and $\beta$ real, has three solutions which we call $M_1$, $M_2$ and $M_3$. They are:

$$M_1 = S + T,$$

$$M_2 = -\frac{1}{2} (S + T) + \frac{i \sqrt{3}}{2} (S - T),$$

$$M_3 = -\frac{1}{2} (S + T) - \frac{i \sqrt{3}}{2} (S - T),$$

where

$$S = 3 \sqrt{-\frac{\beta}{2} + \sqrt{\Delta}}, \quad T = 3 \sqrt{-\frac{\beta}{2} - \sqrt{\Delta}}, \quad \Delta = \frac{\alpha^3}{27} + \frac{\beta^2}{4}.$$

There are then three cases, depending on the value assumed by $\Delta$, which we now study in details. Firstly, for $\Delta > 0$ we see that there are no real positive solutions, thus no physical solution exists if we recognize that only positive values of $M$ make sense. Secondly, for $\Delta < 0$ we have two real positive roots, $M_1$ and $M_3$, which can be written as

$$M_1 = A \cos \left( \frac{\theta}{3} \right),$$

$$M_3 = A \cos \left( \frac{\theta}{3} + \frac{4\pi}{3} \right),$$

where

$$A = 2 \sqrt{\frac{C}{3} - \frac{4\pi^2}{9} \frac{f_\pi^2}{1 - 4G_V f_\pi^2}},$$

$$\cos \theta = \frac{6\pi^2 \langle \bar{\psi} \psi \rangle}{\sqrt{3 \left( C - \frac{4\pi^2}{3} \frac{f_\pi^2}{1 - 4G_V f_\pi^2} \right)^3}}.$$
This is not a desirable situation just because, for this case, the same set of parameters imply two different values for the dynamically generated quark mass. This is unacceptable from the physical point of view. Finally, for \( \Delta = 0 \) there is just one real positive root. This is a very attractive possibility. It remains to verify if the values for the quantities involved in the cancellation of \( \Delta \) are reasonable ones. Looking at the equation for \( \Delta \) we note that, in order to achieve this situation, the value for the arbitrary parameter must be fixed. In this sense we can say that, in order to obtain a unique solution for the quark mass, we have to pay the price of fixing the value for the arbitrary parameter.

Now we can invert the interpretation. There is an arbitrary parameter and we consider the total range of values for it. We note then that for values which are minor than the critical one given (exactly) by

\[
C_{\text{crit}} = \frac{4\pi^2}{3} \frac{f_\pi^2}{1 - 4G_V f_\pi^2} + \sqrt[3]{12\pi^4 \langle \overline{\psi} \psi \rangle^2}.
\]  

(53)

we have no solutions for the constituent quark mass. For values which are major than the one above we have two real values for the mass corresponding to the same set of parameters and, only \( C_{\text{crit}} \) will lead us to a unique value for the mass, which is given by

\[
M_{\text{crit}} = \sqrt[3]{-\frac{2\pi^2}{N_c} \langle \overline{\psi} \psi \rangle}.
\]

(54)

At this point it seems the question of choosing the adequate solution makes no sense. The answer is certainly obvious: the critical condition naturally fixes the constant \( C \), the last arbitrary parameter still remaining in the model, and determines the value of the constituent quark mass. Within this point of view the model becomes predictive in a sense that all the arbitrariness, involved in the manipulations of the divergent integrals, have disappeared due to the consistency relations, the scale properties of the irreducible divergent objects and by the existence of a critical condition.

Now we can obtain the remaining parameter of the model predictions in chirally symmetric case: the coupling \( G_S \). It is given by

\[
G_S = \frac{1}{2} \sqrt{\frac{2\pi^2}{3 \langle \overline{\psi} \psi \rangle^2}}.
\]

(55)

We note that in our approach the constituent quark mass \( M \) and the coupling \( G_S \) values depend on the quark condensate \( \langle \overline{\psi} \psi \rangle \) only. This is a very attractive a new result and
constitutes a relevant difference between our approach and the traditional ones based on cut-off regularizations. If we assume \( \langle \bar{\psi}\psi \rangle = (-250.0 \text{ MeV})^3 \) and \( f_\pi = 93.0 \text{ MeV} \) as the two first inputs we obtain \( M_{\text{crit}} \simeq 468.4 \text{ MeV} \) and \( G_S \simeq 15.0 \text{ GeV}^{-2} \) and \( g_{\pi qq} \simeq 5.0 \). These values are in good accordance with the expected ones.

Let us now turn our attention to the determination of the vector parameters. For this purpose we first consider the vector-vector polarization function evaluated at \( p^2 = m_\rho^2 \). The result is

\[
\Pi_{(T)}^{VV} (m_\rho^2) = \frac{1}{6\pi^2} m_\rho^2 + \frac{1}{2\pi^2} \left( m_\rho^2 + 2M^2 \right) \left[ Z_0 (m_\rho^2, M^2; M^2) \right] - \frac{m_\rho^2}{6M^2} \left[ \Pi_{(L)}^{AA} (0) \right],
\]

which substituted in Eq. (12) furnish a transcendental equation which determines the \( \rho^0 \) mass

\[
m_\rho^2 = -\frac{2\pi^2}{G_V} \left\{ \frac{1}{3} + \left( 1 + \frac{2M^2}{m_\rho^2} \right) \left[ Z_0 (m_\rho^2, M^2; M^2) \right] - \frac{\pi^2}{3M^2} \left[ \Pi_{(L)}^{AA} (0) \right] \right\}^{-1}.
\]

In the same way from Eqs. (13) and (14) we get the equation for the rho-quark coupling \( g_{\rho qq} \)

\[
g_{\rho qq}^{-2} = -\frac{1}{8\pi^2} \left\{ \frac{1}{3} + \left( m_\rho^2 + 2M^2 \right) \left[ Y_1 (m_\rho^2, M^2) \right] + \left[ Z_0 (m_\rho^2, M^2; M^2) \right] - \frac{\pi^2}{3M^2} \left[ \Pi_{(L)}^{AA} (0) \right] \right\},
\]

and the \( f_\rho \)

\[
f_\rho = \frac{4G_V m_\rho^2}{g_{\rho qq}}.
\]

In order to find the axial-vector meson parameters we evaluated axial-axial polarization function at \( p^2 = m_{a_1}^2 \)

\[
\Pi_{(T)}^{AA} (m_{a_1}^2) = \frac{1}{6\pi^2} m_{a_1}^2 + \frac{1}{2\pi^2} \left( m_{a_1}^2 - 4M^2 \right) \left[ Z_0 (m_{a_1}^2, M^2; M^2) \right] + \left( 1 - \frac{m_{a_1}^2}{6M^2} \right) \left[ \Pi_{(L)}^{AA} (0) \right],
\]

and obtain the \( a_1^0 \) mass condition:

\[
m_{a_1}^2 = -\frac{2\pi^2}{G_V} \left\{ \frac{1}{3} + \left( 1 - \frac{4M^2}{m_{a_1}^2} \right) \left[ Z_0 (m_{a_1}^2, M^2; M^2) \right] + 2\pi^2 \left( \frac{1}{m_{a_1}^2} - \frac{1}{6M^2} \right) \left[ \Pi_{(L)}^{AA} (0) \right] \right\}^{-1},
\]
and also the $a_1$-quark coupling $g_{a_1qq}$

$$g_{a_1qq}^{-2} = \frac{1}{8\pi^2} \left\{ \frac{1}{3} + \left( m_{a_1}^2 - 6M^2 \right) \left[ Y_1 \left( m_{a_1}^2, M^2; M^2 \right) \right] \right. $$

$$+ \left. \left[ Z_0 \left( m_{a_1}^2, M^2; M^2 \right) \right] - \frac{\pi^2}{3M^2} \left[ \Pi_{LA}^A(0) \right] \right\}.$$  

We see that the remaining meson parameters are dependent of the vector coupling. Adopting the $G_V \simeq 16.4 \text{ GeV}^{-2}$, as our last input parameter, we can fix all the remaining quantities of the model. In this way we obtain the masses of the vector and the axial-vector mesons, $m_{\rho} \simeq 770.0 \text{ MeV}$ and $m_{a_1} \simeq 1145.0 \text{ MeV}$, and their corresponding effective coupling constants, $g_{\rho qq} \simeq 4.8$ and $g_{a_1qq} \simeq 2.9$, as well as $\sqrt{C_{\text{crit}}} \simeq 0.96 \text{ GeV}$ and $f_{\rho} \simeq 8.0$. Again the values are in good accordance with the expectations. In our description the meson $\rho$ is a bound state while the meson $a_1$, as well known, are not bound state which implies that its mass has imaginary part. This means that the description of the axial-vector mesons within the NJL model are less realistic than that for the pseudoscalar and vector mesons. In the above results we have discarded the imaginary parts of results for $m_{a_1}$ and $g_{a_1qq}$. In the pion sector, the pion coupling constant $\tilde{g}_{\pi qq}$, which is dependent on $G_V$, acquires the value $\tilde{g}_{\pi qq} \simeq 2.9$.

An interesting point is the one relative to the value found for $C_{\text{crit}}$. It is very similar to that obtained for the cutoff parameter in traditional treatments, which is usually located in the range $600 - 1000 \text{ MeV}$. A simple analysis reveals that this similarity is, in fact, expected. If we remember that the constant $C$ has introduced in the general expression for the quadratic divergence as an arbitrary constant, which means independence of the scale parameter in the basic quadratic divergence $I_{\text{quad}}(\lambda^2)$, it is simple to see that it can be associated to the dominant term ($\Lambda^2$) in the regularized version of the $I_{\text{quad}}(\lambda^2)$. Due to this reason the value for the $C_{\text{crit}}$ is expected to be closely related to the regularization parameter $\Lambda^2$ of the proper-time and to the cutoff $\Lambda^2$ of the sharp cutoff regularization.

A similar analysis can be made about the values found for the quark mass $M$. Although we have concluded that the unique acceptable physical prediction for the mass is that dictated by the critical condition of the Eq. (17), we could note that the values for the solutions $M_1$ and $M_3$, and their corresponding partner $C$, are in agreement with the values usually found in the literature. In fact, by choosing the value of $C$ we can obtain a correspondence $(C, M)$ with those $(\Lambda, M)$ frequently presented in the literature, in investigations made within the context of regularizations. This aspect can be clearly observed in the figure (1). Note that above the critical point, for each value of $C$, there are two independent solutions, namely
and $M_3$. The $M_1$ solutions grows up with the increase of $C$ while $M_3$ asymptotically goes to zero.

In order to give a further clarification relative to these aspects we present, in table I, some typical values for $M$ found in the literature obtained in four representative regularizations schemes (see for example Ref. [3]), the values of $\Lambda$ (cutoff parameter) and the associated value for the parameter $C$. We see that both $\Lambda$ and $C$ are always of the same order but for the four-momentum cutoff and proper time schemes the values are strictly the same.

| scheme                        | $M$ (MeV) | $\Lambda$ (MeV) | $\sqrt{C}$ (MeV) |
|-------------------------------|-----------|-----------------|------------------|
| three-momentum cutoff         | 313       | 653             | 932              |
| four-momentum cutoff          | 238       | 1015            | 1017             |
| proper time                   | 200       | 1086            | 1086             |
| Pauli-Villars                 | 241       | 859             | 1012             |

In the next step we include in the discussion a nonvanishing current quark mass ($m_0 \neq 0$) which gives to the pion a nonvanishing mass. Our first task in this case is to found the new parametrization for the $I_{\log}(M^2)$ since the parametrization for $I_{\text{quad}}(M^2)$ does not change.
In this direction we first note that the determinant $D_\pi (p^2)$ becomes

$$D_\pi (p^2) = \frac{m_0}{M} \left[ 1 + G_V \Pi_{(L)}^{AA} (p^2) \right] - \frac{G_S}{4M^2} p^2 \left[ \Pi_{(L)}^{AA} (p^2) \right],$$

where we have used the gap equation. The condition (8) plus equation above determines the expression for the pion mass:

$$m_\pi^2 = \frac{4m_0 M}{G_S} \frac{1 + G_V \Pi_{(L)}^{AA} (m_\pi^2)}{\left[ \Pi_{(L)}^{AA} (m_\pi^2) \right]}.$$

On the other hand the pion decay constant may be written as

$$f_\pi g_{\piqq}^{-1} = \frac{1}{16M^2} \frac{\left[ \Pi_{(L)}^{AA} (m_\pi^2) \right]}{1 + G_V \Pi_{(L)}^{AA} (m_\pi^2)},$$

where the coupling $g_{\piqq}$ is given by

$$g_{\piqq}^{-2} = \frac{4M^2}{\Pi_{(L)}^{AA} (m_\pi^2)} \frac{\left[ \Pi_{(L)}^{AA} (m_\pi^2) \right]^3}{1 + G_V \Pi_{(L)}^{AA} (m_\pi^2)} - 4f_\pi^2 \left[ \Pi_{(L)}^{AA} (m_\pi^2) \right]^2 + \frac{16N_c m_0 M^3 f_\pi^2}{\pi^2 G_S} \left[ Y_1 (m_\pi^2, M^2) \right] = 0,$$

whose solution gives the searched parametrization for $I_{\log} (M^2)$. Among the many solutions of this cubic algebraic equation there is just one which satisfy the physical conditions expected for $I_{\log} (M^2)$. That solution may be written as

$$\Pi_{(L)}^{AA} (m_\pi^2) = \frac{4}{3} \frac{f_\pi^2}{1 - 4G_V f_\pi} \left\{ 1 + 3\sqrt{R - \sqrt{R^2 - Q^3}} + \sqrt{R + \sqrt{R^2 - Q^3}} \right\},$$

where

$$R = 1 - \frac{9N_c m_0 M^3}{8\pi^2 G_S f_\pi^4} \left( 1 - 4G_V f_\pi^2 \right) \left( 3 - 8G_V f_\pi^2 \right) \left[ Y_1 (m_\pi^2, M^2) \right],$$

$$Q = 1 - \frac{3N_c m_0 M^3 G_V}{\pi^2 G_S f_\pi^2} \left( 1 - 4G_V f_\pi^2 \right) \left[ Y_1 (m_\pi^2, M^2) \right].$$

This gives the searched parametrization for $I_{\log} (M^2)$

$$i \left[ I_{\log} (M^2) \right] = -\frac{1}{16\pi^2} \left[ Z_0 (m_\pi^2, M^2; M^2) \right] - \frac{1}{12N_c M^2} \frac{f_\pi^2}{1 - 4G_V f_\pi} \left\{ 1 + 3\sqrt{R - \sqrt{R^2 - Q^3}} + \sqrt{R + \sqrt{R^2 - Q^3}} \right\}.$$

(56)
Replacing the parametrizations (56) and (43) in (42) we get

$$M^3 \left(1 + \left[ Z_0 \left( m^2_\pi, M^2; M^2 \right) \right] \right) + \left( \frac{\pi^2}{3} \left[ \Pi^{AA}_{(L)} \left( m^2_\pi \right) \right] - C \right) M - \frac{4\pi^2}{3} \langle \bar{\psi} \psi \rangle = 0.$$  

This is a nontrivial nonlinear equation for $M$ which may be simplified by using the following (reasonable) approximations

$$Z_0 \left( m^2_\pi, M^2; M^2 \right) \simeq -\frac{m^2_\pi}{6M^2}, \quad Y_1 \left( m^2_\pi, M^2 \right) \simeq -\frac{1}{6M^2},$$

which, among others things, allow us to see clearly the searched solutions for $M$. Then we get

$$M^3 + \left( \frac{\pi^2}{3} \left[ \Pi^{AA}_{(L)} \right] + \frac{4}{3} m_0 \langle \bar{\psi} \psi \rangle \left[ 1 + G_V \Pi^{AA}_{(L)} \right] \right) M - \frac{4\pi^2}{3} \langle \bar{\psi} \psi \rangle = 0, \quad (57)$$

where we have also used that

$$m^2_\pi \simeq -8m_0 \langle \bar{\psi} \psi \rangle \left[ 1 + G_V \Pi^{AA}_{(L)} \right],$$

$$R \simeq 1 - \frac{3N_c m_0 \langle \bar{\psi} \psi \rangle}{8\pi^2 f^4} \left( 1 - 4G_V f^2 \right) \left( 3 - 8G_V f^2 \right),$$

$$Q \simeq 1 - \frac{N_c m_0 \langle \bar{\psi} \psi \rangle}{\pi^2 f^2} G_V \left( 1 - 4G_V f^2 \right).$$

The equation (57) has the form $M^3 + \alpha M + \beta = 0$ where the coefficients $\alpha$ and $\beta$ are given by

$$\alpha = \frac{\pi^2}{3} \left[ \Pi^{AA}_{(L)} \right] + \frac{4}{3} m_0 \langle \bar{\psi} \psi \rangle \left[ 1 + G_V \Pi^{AA}_{(L)} \right] - C,$$

$$\beta = -\frac{4\pi^2}{3} \langle \bar{\psi} \psi \rangle.$$

Then, in order to find its solutions we follows strictly the same steps which we have used for the Eq.(47). The critical solution does not change (see Eq.(54)) while the critical value of the constant $C$ become

$$C_{crit} = \frac{\pi^2}{3} \left[ \Pi^{AA}_{(L)} \right] + \frac{4}{3} m_0 \langle \bar{\psi} \psi \rangle \left[ 1 + G_V \Pi^{AA}_{(L)} \right] + \sqrt{12\pi^4 \langle \bar{\psi} \psi \rangle^2}.$$
The noncritical solutions $M_1$ and $M_2$ are given by Eqs. (51) and (52) with

$$A = 2 \sqrt{\frac{C}{3} - \frac{\pi^2}{9} \left[ \Pi_{(L)}^{AA} \right] - \frac{4}{9} m_0 \left\langle \overline{\psi} \psi \right\rangle \left[ 1 + G_V \Pi_{(L)}^{AA} \right]},$$

$$\cos \theta = \frac{6\pi^2 \left\langle \overline{\psi} \psi \right\rangle}{\sqrt{3 \left( C - \frac{\pi^2}{3} \left[ \Pi_{(L)}^{AA} \right] - \frac{4}{3} m_0 \left\langle \overline{\psi} \psi \right\rangle \left( G_V + \left[ \Pi_{(L)}^{AA} \right]^{-1} \right) \right)^3}}.$$ 

These solutions when plotted as function of $C$ give a graphic similar to that shown in the Fig. (1).

In the vector sector, the $\rho$ mass and its coupling constant are given, respectively, by

$$m_{\rho}^{-2} = -\frac{G_V}{2\pi^2} \left\{ \frac{1}{3} + \left( 1 + \frac{2M^2}{m_{\rho}^2} \right) \left[ Z_0 \left( m_{\rho}^2, M^2; M^2 \right) \right] - \left[ Z_0 \left( m_{\pi}^2, M^2; M^2 \right) \right] - \frac{\pi^2}{3M^2} \left[ \Pi_{(L)}^{AA} \left( m_{\pi}^2 \right) \right] \right\},$$

and

$$g_{\rho qq}^{-2} = -\frac{1}{24\pi^2} - \frac{1}{8\pi^2} \left( m_{\rho}^2 + 2M^2 \right) \left[ Y_1 \left( m_{\rho}^2, M^2 \right) \right] - \frac{1}{8\pi^2} \left[ Z_0 \left( m_{\rho}^2, M^2; M^2 \right) - Z_0 \left( m_{\pi}^2, M^2; M^2 \right) \right] + \frac{1}{24M^2} \left[ \Pi_{(L)}^{AA} \left( m_{\pi}^2 \right) \right],$$

while for the $a_1$ meson the results are

$$m_{a_1}^{-2} = -\frac{N_c G_V}{6\pi^2} \left\{ \frac{1}{3} + \left( 1 - \frac{4M^2}{m_{a_1}^2} \right) \left[ Z_0 \left( m_{a_1}^2, M^2; M^2 \right) \right] - \left( 1 - \frac{6M^2}{m_{a_1}^2} \right) \left[ Z_0 \left( m_{\pi}^2, M^2; M^2 \right) \right] + \frac{6\pi^2}{N_c} \left( \frac{1}{m_{a_1}^2} - \frac{1}{6M^2} \right) \left[ \Pi_{(L)}^{AA} \left( m_{\pi}^2 \right) \right] \right\},$$

and

$$g_{a_1 qq}^{-2} = \frac{N_c}{12\pi^2} + \frac{N_c}{24\pi^2} \left( m_{a_1}^2 - 4M^2 \right) \left[ Y_1 \left( m_{a_1}^2, M^2 \right) \right] + \frac{N_c}{24\pi^2} \left[ Z_0 \left( m_{a_1}^2, M^2; M^2 \right) - Z_0 \left( m_{\pi}^2, M^2; M^2 \right) \right] - \frac{1}{24M^2} \left[ \Pi_{(L)}^{AA} \left( m_{\pi}^2 \right) \right].$$

Now we are ready to obtain the numerical values for the parameter which belong to the meson phenomenology, like we did for the chirally symmetric case, and to compare them...
with the ones obtained both in literature of this issue and in the experiments. Again we assume \( \langle \bar{\psi} \psi \rangle = (-250.0 \text{ MeV})^3 \), \( f_\pi = 93.0 \text{ MeV} \) and \( G_V \simeq 16.4 \text{ GeV}^{-2} \) as our inputs parameters. On the other hand, in the chirally nonsymmetric case we have added a new parameter in the Lagrangian which break explicit the chiral symmetry, the current quark mass \( m_0 \). We consider the \( m_0 \) as our last input parameter with the value \( m_0 = 5.1 \text{ MeV} \).

We now obtain a nonvanishing pion mass with the value \( m_\pi = 136.7 \text{ MeV} \) which are in good agreement with the experimental one. Concerning with the remaining parameters, we have verified that the results obtained for the case of \( m_0 = 0 \) does not change appreciably.

Of course this is due to the fact that our constituent quark mass \( M \) does not change when \( m_0 \neq 0 \). When we compare our results with the ones predicted by the experiments we see that they are globally very good. Its gratifying for us to see that a consistent treatment of the divergencies when applied to a nonrenormalizable model, in the same way what is done for renormalizable theories, furnish good predictions for the phenomenological observables. It is important to emphasize one more time that the number of input parameters are precisely the ones present in the model Lagrangean and that results are completely independent of the intrinsic arbitrariness or choices made in intermediary steps of this type of calculations. In addition to these very attractive features the results obtained are in excellent agreement with the experimental values.

As a last comment we note that when \( G_V = 0 \), the results obtained in this work give the same ones produced by our previous work \[19\], as should be expected. In particular, in the pion sector analytical results may be written as

\[
m_\pi^2 = \frac{3}{\sqrt{\delta_1 - \delta_2}} - \frac{3}{\delta_1 + \delta_2},
\]

\[
g_{\pi qq}^{-2} = \frac{N_c N_f m_\pi^2}{48\pi^2} \left( \frac{2\pi^2}{N_c} \langle \bar{\psi} \psi \rangle \right) + \frac{f_\pi^2}{2\sqrt{\left( \frac{2\pi^2}{N_c} \langle \bar{\psi} \psi \rangle \right)^2}} \left( 1 + \sqrt{1 + \frac{N_c N_f m_\pi^2}{12\pi^2 f_\pi^2}} \right),
\]

where

\[
\delta_1 = \sqrt{-128\pi^6 N_f m_\pi^3 \langle \bar{\psi} \psi \rangle^4 \left( \frac{3}{N_c} \left( m_0 - \frac{3}{\sqrt{2\pi^2 N_c}} \langle \bar{\psi} \psi \rangle \right) \right)^{-3} \left( 32\gamma + \frac{9N_f m_0^3 \left( \frac{2\pi^2}{N_c} \langle \bar{\psi} \psi \rangle \right)^2}{\left( m_0 - \frac{3}{\sqrt{2\pi^2 N_c}} \langle \bar{\psi} \psi \rangle \right) f_\pi^4} \right)},
\]

\[
\delta_2 = \frac{-\frac{6\pi^2}{N_c N_f} m_0^3 \left( \frac{2\pi^2}{N_c} \langle \bar{\psi} \psi \rangle \right)^2 \left( 2N_f \langle \bar{\psi} \psi \rangle \right)^2}{f_\pi^2 \left( m_0 - \frac{3}{\sqrt{2\pi^2 N_c}} \langle \bar{\psi} \psi \rangle \right)^2}.
\]
which complete our calculations.

V. SUMMARY AND CONCLUSIONS

In the preceding sections, we considered the question of predictive power of the NJL model. Traditionally the model has been used to describe low-energy hadronic observables, in spite of its nonrenormalizable character and, for this reason, the corresponding predictions have been constructed in a previously assumed level of approximation and compromising with a particular regularization prescription. Within this context, it is well known that the results, for the evaluated model amplitudes, invariably emerges as dependent on many types of choices made in intermediary steps of the calculations. The first and immediate of such choices is the regularization prescription. Since in nonrenormalizable models the regularization cannot be removed, the results are assumed regularization dependent. In addition, due to the fact that the DR is not adequate for the treatment of the model amplitudes, the results may emerge as dependent on choices for the routing of internal lines momenta in loops or dependent on arbitrary scales used in the separation of terms having different degrees of divergences. The ambiguous terms, on the other hand, are invariable associated with the violation of symmetries implemented in the model construction. In this scenario, it becomes difficult to talk about the model predictions. Among other aspects, the model is overparametrized just because at least one regularization parameter needs to be specified. This means that one observable, which is in the scope of the model predictions, must be used as an additional input in order to parametrize the model. Given this situation, a relevant question can be put: is it possible to make genuine predictions within the NJL model or the facts described above are definitive?

Having this question in mind, we proposed a very general investigation involving all one and two-point function of the SU(2) version of the NJL model. The conclusion stated reveals the possibility of making the predictions free of ambiguities or symmetry violating terms as well as free from the dependence on the specific regularization choice. This conclusion was made possible just because the calculations were performed following a novel strategy of handling the divergences in the perturbative calculations of QFT. Within this strategy, divergent integrals are never really evaluated. Only general properties for standard divergent objects are adopted which are dictated by the consistent and systematic
elimination of ambiguous and symmetry violating terms. The surprising fact that emerged in the above cited investigation, which is particular to the NJL model, refers to a critical condition obtained in the equation stating the dynamically generated quark mass. After imposing all the constraints coming from the consistency of the perturbative calculations, the existence of freedom associated with the specific form of the regularization was observed. However, when the constituent quark mass is searched for, as a function of the model input parameters as well as a function of the arbitrary parameter representing the arbitrariness remaining, it was observed that a unique reasonable physical solution exists. Only one real value for the quark mass emerges through a critical condition, which fixes the arbitrariness remaining. Since, after this, the model predictions are completely independent on choices, this formulation was denominated predictive.

Following this line of reasoning, the present work can be considered as an additional step relative to Ref. [19]. Here we put the formulation within a more general context and, through analytical solutions, we show in a clear way the origin of the critical condition by obtaining the expressions for the critical value of the arbitrary parameter involved as well as the value of the corresponding constituent quark mass in analytical forms. In addition, we show how to explain the values for the quark mass found in the literature by using traditional regularization schemes. The model considered here for the phenomenology predictions was also extended to consider also the vector mesons. The relevant steps of the strategy adopted in the reported investigation to handle the divergences, which allowed our conclusions, can be summarized as follows:

i) Identify the amplitudes, pertinent to the model, having the highest superficial degree of divergence $D$ ($D = 3$ for the present case).

ii) Through the expression (15) specify the convenient representation for the involved propagators, taken $N = D$ and making the summation indicated. For the present case this means to adopt the representation (16).

iii) Through the Feynman rules, construct the amplitudes, performing all the operation like Dirac or other involved traces operation, algebraic manipulation, convenient reorganizations etc..., except the integration in the loop momentum.

iv) Introduce the integration over the loop momentum, performing the integration in all finite Feynman integrals and letting all the divergent ones unchanged. The obtained result can be put in terms of standard finite functions and standard divergent objects. In the
present case the functions (25) and the objects (18), (19), and (20).

v) Remove the divergent objects which are differences among divergent integrals of the same divergences degree guided by the maintenance of fundamental symmetries like the space-time homogeneity in perturbative calculations, which we denominate consistency relations, defining thus the consistent regularized amplitudes. In the present case this means just to require \( \Delta_{\alpha\beta}(\lambda^2) = \nabla_{\alpha\beta}(\lambda^2) = \Box_{\alpha\beta\mu\nu}(\lambda^2) = 0 \).

vi) Impose the scale independence over the full amplitudes (finite and divergent parts) stating then the scale properties for the irreducible divergent objects. In the present case this means to state the properties (39) and (40) which are relations among the irreducible divergent objects at different scales. Such relations imply definite properties for the irreducible divergent objects, which are for the present case shown in Eqs. (41) and (42). These properties will state relations among the divergent objects which allow us to reduce the freedom remaining to a unique arbitrary constant.

vii) The remaining divergent objects must be removed through the reparametrization of the model at the considered level of approximation. This means to relate them to the physical inputs of the model (similar to a renormalization in renormalizable theories).

viii) To find the best physical values for the remaining arbitrariness. This last step will obviously depend on the specific problem. In the NJL model, as we have shown, this search for the best physical value has ended in the existence of a critical condition. Applications of this strategy to the SU(3) case have been considered revealing the same conclusion stated here: within the context of the adopted strategy to handle the divergences the NJL model becomes predictive. Work along this line is presently under way.

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