Generalized Bundle Quantum Mechanics

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1 Generalized Bundle Quantum Mechanics

1.1 Mathematical Background

In this section, a quick review of the mathematical background shall be presented to the reader. The material includes:

- Vector spaces, duals and double duals;
- Tensors over a vector space;
- Exterior algebras $\Lambda^*$ and $\Lambda$;
- Metric and tensors;
- Clifford algebras and spinors;
- Fibre bundles as special manifolds;
- Geometrical definition of $\mathcal{C}_3(\mathbb{R})$ spinors;
- Differentiation of spinor fields;

The reader that is already familiar with it, can just jump ahead to section 2.

1.2 Vector spaces, duals and double duals

A vector space can be defined as,

**Definition 1.1.** A set $\mathcal{V}$ is called a vector [linear] space, $(\mathcal{V}, +, \cdot, \mathbb{K})$, over a [scalar] field $\mathbb{K}$ if, between two elements of $\mathcal{V}$, there is an addition, $+$, defined and, between and element of $\mathcal{V}$ and an element of $\mathbb{K}$, there is a scalar multiplication, $\cdot$. In addition to these, $\mathcal{V}$ should be closed under addition and scalar multiplication. (Not to mention the other 4 properties of addition and the other 4 properties of scalar multiplication: associativity, distributivity, commutativity and the neutral element.)

On top of the definition of a vector space, an algebra can be defined. Here’s how one does it:

**Definition 1.2.** An algebra, $\mathcal{A}(\mathcal{V}) = (\mathcal{V}, +, \cdot, \circ, \mathbb{K})$, is a mathematical object constructed out of a vector space by endowing it with an internal multiplication rule, i.e., an operation $\circ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ that satisfies:

$$
\vec{v} \circ (a\vec{x} + b\vec{y}) = a(\vec{v} \circ \vec{x}) + b(\vec{v} \circ \vec{y})
$$

$$
(a\vec{x} + b\vec{y}) \circ \vec{v} = a(\vec{x} \circ \vec{v}) + b(\vec{y} \circ \vec{v}), \quad \forall \vec{v}, \vec{x}, \vec{y} \in \mathcal{V} \text{ and } \forall a, b \in \mathbb{K}.
$$

Now, a vector space can have a basis. For that, it suffices to find a set of $n$ linearly independent vectors, $\{\vec{e}_i, 1 \leq i \leq n\}$, such that $\forall \vec{v} \in \mathcal{V}$ can be uniquely written as $\vec{v} = \sum_{i=1}^{n} v^i \vec{e}_i$, $v^i \in \mathbb{K}$. In this case, the $\{v^i\}$ are called the components of $\vec{v}$ in the basis $\{\vec{e}_i\}$ and $n = \dim(\mathcal{V})$. (Our focus will be on finite $n$ vector spaces.)

**Notation.** For convenience sakes, the following will be used:

$$
\vec{v} = \sum_{i=1}^{n} v^i \vec{e}_i = \vec{e}_\bullet v^\bullet
$$

where

$$
\vec{e}_\bullet \equiv (\vec{e}_1, \ldots, \vec{e}_n)
$$

$$
v^\bullet \equiv \begin{pmatrix}
    v^1 \\
    \vdots \\
    v^n
\end{pmatrix}
$$
The concept of a linear mapping [between two vector/linear spaces] is a quite useful one, so let’s get to it.

**Definition 1.3.** A linear mapping between \((V, +, \cdot, K)\) and \((W, +, \cdot, K)\) is a mapping that satisfies:

\[
\mu : \quad V \to W \\
\vec{v}_1 + \vec{v}_2 \mapsto \mu(\vec{v}_1) + \mu(\vec{v}_2) \\
a \vec{v} \mapsto a \mu(\vec{v})
\]

Using this, the set of linear mappings from \(V\) to \(W\), denoted by \(\mathcal{L}(V, W)\), is a vector space (addition is given by the addition of the mappings and scalar multiplication is given by multiplying the map by the scalar). The space \(\mathcal{L}(V, V)\) is an algebra (Why?), where the internal multiplication is given by function composition.

Thus, in this fashion, \(\forall \vec{v} \in V\), \(\vec{v} = \vec{e} \cdot v^*\), the image vector is given by:

\[
\vec{\omega} = \mu(\vec{v}) = \vec{f} \cdot \mu^* v^*
\]

or, in matrix language,

\[
\omega^* = \mu^* v^*
\]

**Definition 1.4.** The space of the linear forms defined on \(V\) is called the dual of \(V\), denoted by \(V^*\). A linear form is a linear mapping that satisfies:

\[
b v + c \omega : \quad V \to K \\
\vec{x} + \vec{y} \mapsto b v[\vec{x}] + b v[\vec{y}] + c \omega[\vec{x}] + c \omega[\vec{y}] \\
a \vec{x} \mapsto a b v[\vec{x}] + a c \omega[\vec{x}] + a \omega[\vec{x} + \vec{y}]
\]

The reader should note that \(V^* = \mathcal{L}(V, K)\).

A basis can, now, be constructed for \(V^*\) with the help of \(\{\vec{e}_i\}\), the basis for \(V\). Thus, \(\{\vec{e}_i^*\}\) will be a basis for \(V^*\), defined by:

\[
\vec{e}_i^*[\vec{e}_j] = \delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
\]

In this way, now one can find oneself in the position to calculate the scalar associated by \(v\) to \(\vec{x}\):

\[
u[\vec{x}] = v_i \vec{e}_i^*[\vec{x}] = v_i x^j \delta^i_j = v_i x^j.
\]

In an analogous manner done before, one can define the following notation.

**Notation.**

\[
v = v \cdot \vec{e}^*
\]

\[
u[\vec{x}] = v \cdot \vec{e}_i^* \cdot \vec{e}_j^* = v \cdot (\vec{e}_i^* \cdot \vec{e}_j^*) x^j = v \cdot \mathbb{1} = v \cdot x^*,
\]
where

\[ 1 = e^* \cdot e = \text{unit matrix}. \]

The nomenclature of contravariant vectors [for elements of \( V \)] and covariant vectors [for elements of \( V^* \)] follows from the fact that, upon a change of basis of \( V \) by a matrix \( A \), the components of the vectors transform with \( A^{-1} \), while the components of the linear forms transform with \( A \).

With all this material, it is possible, now, to define the double dual of \( V \), denoted by \( V^{**} \). This will be given by,

\[ v_{\vec{v}} : V^* \to K \]

\[ x \mapsto v_{\vec{v}}[x] = \vec{x}[\vec{v}] \]

Therefore, it is possible to prove that \( V \) is isomorphic to \( V^{**} \! \! . \)

### 1.3 Tensors over a vector space

Once the concept of vectors and linear forms is settled, the next step is going towards the definition of tensors, which generalize the former. Tensors and spinors will play a fundamental role in the remainder of this work.

First off, let’s start by defining the space of tensors, denoted by \( \Theta^p_q \).

**Definition 1.5.** A tensor \( T_{pq} \), \( p \) times contravariant and \( q \) times covariant, also termed a \((p,q)\)-tensor or a tensor of rank \((p,q)\), is defined as:

\[
T_{pq} : (V^* \times \cdots \times V^*) \times (V \times \cdots \times V) \to K
\]

\[ (\vec{v}_1, \ldots, \vec{v}_p; \vec{\omega}_1, \ldots, \vec{\omega}_q) \mapsto T_{pq}(\vec{v}_1, \ldots, \vec{v}_p; \vec{\omega}_1, \ldots, \vec{\omega}_q) \]

which is linear in each entry.

**Notation.** The set of \((p,q)\)-tensors will be denoted by,

\[ \Theta^p_q \equiv (V \otimes^p) \otimes (V^*) \otimes^q . \]

It is not difficult to see that \( \Theta^p_q \) is a vector space, \((\Theta^p_q, +, \cdot)\) whose dimension is given by: \( \dim V = n \Rightarrow \dim \Theta^p_q = n^{p+q} \).

**Definition 1.6.** The tensor product (distributive and associative), \( \otimes \), is defined as follows:

\[ \otimes : \Theta^p_{q_1} \times \Theta^p_{q_2} \to \Theta^{p_1+p_2}_{q_1+q_2} \]

\[ (S^{p_1}_{q_1} \otimes T^{p_2}_{q_2}) \mapsto \Theta^{p_1+p_2}_{q_1+q_2} \]

**Notation.** As has been made clear by the notation employed previously, a \((1,1)\)-tensor can be written as \( T \) (and so on and so forth). A typical example of a \((1,1)\)-tensor is a linear mapping from a vector space to itself.

The contraction of tensors can also be constructed.
1.4 Exterior algebras $\Lambda^*$ and $\Lambda$

Now, the attention will be restricted to special kinds of tensors, the $q$-linear alternating forms. The exterior product will be defined and, thus, the Exterior Algebra, $(\Lambda^*, +, \cdot, \wedge)$, will be defined.

**Definition 1.8.** An alternating $q$-form is defined over a $q$-linear form $q\alpha \in \Theta^0_q$ when,

$$q\alpha[\bar{v}_{\sigma(1)}, \ldots, \bar{v}_{\sigma(q)}] = (-1)^{|\sigma|} q\alpha[\bar{v}_1, \ldots, \bar{v}_q], \ \forall \bar{v}_i \in \mathbb{V}, \ 1 \leq i \leq q, \ \forall \sigma \in S_q, \ q \geq 1,$$

where $S_q$ is the permutation group of the elements $\{1, \ldots, q\}$, $\sigma$ is one particular permutation and $|\sigma|$ is its parity.

Now, $\Lambda^*_q$ is the subset of $\Theta^0_q$ that comprises those $q$-linear alternating forms. In particular, $\Lambda^*_1 \simeq V^*$ (for convenience, the elements $1\alpha \in \Lambda^*_1$ will be denoted as $\alpha$) and $\Lambda^*_0 \simeq \mathbb{K}$.

The next step towards the construction of an algebra is defining an internal multiplication, that will be called exterior product. This can be done with the aid of the tensor product ($\otimes$), defined above.

**Definition 1.9.** The exterior product is an antisymmetrized version of the tensor product. It is defined as:

$$\wedge: \Lambda^*_q \times \Lambda^*_r \rightarrow \Lambda^*_{q+r}$$

such that

$$(q\alpha \wedge_r \beta)(\bar{v}_1, \ldots, \bar{v}_{q+r}) \equiv \frac{1}{q! r!} \sum_{\sigma \in S_{q+r}} (-1)^{|\sigma|} q\alpha(\bar{v}_{\sigma(1)}, \ldots, \bar{v}_{\sigma(q)}) \cdot r\beta(\bar{v}_{\sigma(q+1)}, \ldots, \bar{v}_{\sigma(q+r)})$$

Thus, it is not difficult to see that this product is associative, distributive and satisfies: $q\alpha \wedge_r \beta = (-1)^q r \cdot \beta \wedge_q \alpha$. Also, $\dim(\Lambda^*_q) = \frac{n!}{q! (n-q)!}$ (remember that, $n$, labels the dimension of the underlying vector space, $\mathbb{V}$, while, $q$, labels the degree of the form).

To make life a bit simpler in the future [when dealing with Clifford Algebras], let’s define the following.

**Definition 1.10.** The interior product, $i_{\bar{v}}$, is given by:

$$i_{\bar{v}}: \Lambda^*_q \rightarrow \Lambda^*_{q-1}$$

$$q\alpha \mapsto i_{\bar{v}}(q\alpha)$$

where

$$(i_{\bar{v}} q\alpha)(\tilde{\omega}_1, \ldots, \tilde{\omega}_{q-1}) \equiv q\alpha(\bar{v}, \tilde{\omega}_1, \ldots, \tilde{\omega}_{q-1})$$

With this, $(i_{\bar{v}} q\alpha)$, is linear in $\bar{v}$ and $q\alpha$, satisfying:

$$i_{\bar{v}} i_{\bar{w}} + i_{\bar{w}} i_{\bar{v}} = 0$$

$$i_{\bar{v}} (q\alpha \wedge_r \beta) = (i_{\bar{v}} q\alpha) \wedge_r \beta + (-1)^q q\alpha \wedge (i_{\bar{v}} r\beta)$$
At this point, the last concept to be introduced (before defining the exterior algebra) is that of a pull-back of a \(q\)-form.

**Definition 1.11.** Consider two vector spaces, \( \mathbb{V}, \mathbb{W} \), and \( \Lambda_q^\ast (\mathbb{V}), \Lambda_q^\ast (\mathbb{W}) \) associated with them. Let \( f : \mathbb{V} \to \mathbb{W} \), linear. Then, it is possible to associate \( \forall \, q \alpha \in \Lambda_q^\ast (\mathbb{W}) \), a \((f \, q \alpha) \in \Lambda_q^\ast (\mathbb{V})\) such that:

\[
f^* : \Lambda_q^\ast (\mathbb{W}) \to \Lambda_q^\ast (\mathbb{V}) \quad q \alpha \mapsto f \, q \alpha
\]

where

\[
(f \, q \alpha)(\vec{v}_1, \ldots, \vec{v}_q) \equiv q \alpha(f(\vec{v}_1), \ldots, f(\vec{v}_q))
\]

In this fashion, given that, at our disposal, there is the vector space \((\Lambda_q^\ast, +, \cdot, \wedge)\) and the exterior product, \(\wedge\). However, the set \(\Lambda_q^\ast\) is not closed under \(\wedge\). On the other hand, \(\Lambda_q^\ast = \{0\}, \, q > n\) (where \(n = \dim(\mathbb{V})\)). Thus, it is possible to define:

**Definition 1.12.** The structure \((\Lambda^\ast \equiv \bigoplus_{q=0}^n \Lambda_q^\ast, +, \cdot, \wedge)\) forms the exterior algebra over \(\mathbb{V}\). Moreover, \(\dim(\Lambda^\ast) = 2^n\).

Now, given that \(\mathbb{V} \simeq \mathbb{V}^{\ast\ast}\) (by virtue of \(\vec{v}[\vec{x}] \equiv \vec{x}[\vec{v}]\)), the very same construction as above can be repeated, *mutatis mutandis*, in order to generate de dual of the above algebra.

**Definition 1.13.** The structure \((\Lambda \equiv \Lambda^{\ast\ast} \equiv \bigoplus_{q=0}^n \Lambda^q, +, \cdot, \wedge)\) forms the exterior algebra underlying the Clifford algebra over \(\mathbb{V}^\ast\). Moreover, \(\dim(\Lambda) = 2^n\). This algebra is also known as the Clifford exterior algebra.

### 1.5 Metric and tensors

Let’s start by restricting \(\mathbb{K} = \mathbb{R}\).

**Definition 1.14.** A bilinear form \(g \in \Theta_2^0\) is called a metric when it satisfies,

- symmetry: \(g(\vec{v}, \vec{\omega}) = g(\vec{\omega}, \vec{v})\)
- regularity: \(g(\vec{v}, \vec{\omega}) = 0, \, \forall \vec{\omega} \in \mathbb{V} \Rightarrow \vec{v} \equiv \vec{0}\).

By expanding \(g\) in a [canonical] basis, the above conditions become:

\[
g_{i \, j} = g_{j \, i}
\]

\[
\det(g_{i \, j}) \neq 0.
\]

The metric further enables the possibility to define a scalar product and a norm.

**Definition 1.15.** The scalar product of two vectors and the norm of a vector are defined via:

\[
\vec{v} \cdot \vec{\omega} = g(\vec{v}, \vec{\omega}) = g(\vec{\omega}, \vec{v}) = \vec{\omega} \cdot \vec{v}
\]

\[
\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}.
\]
The reader should note that nowhere above the restriction that the metric be positive definite was required. Therefore, the metric can, in general, have any signature. Furthermore, by making use of the metric, the spaces $\mathbb{V}$ and $\mathbb{V}^*$ can be related to one another.

In order to accomplish this equivalence, it suffices to say that, for any vector $\vec{v} \in \mathbb{V}$, there is a linear form associated with it, denoted by $\vec{v}^\flat$, defined by:

$$\vec{v} : \mathbb{V} \to \mathbb{R}$$

$$\vec{x} \mapsto \vec{v}(\vec{x}) \equiv g(\vec{x}, \vec{v}) .$$

The form $\vec{v}$ and the vector $\vec{v}$ are called metric duals. This correspondence can be further pursued and it renders the two exterior algebras, $(\Lambda^*, +, \cdot, \wedge)$, $(\Lambda, +, \cdot, \wedge)$, isomorphic, via the metric $g$. Therefore, either of them can be used in order to construct the Clifford algebra.

### 1.6 Clifford algebras and spinors

In order to move on to the definition of Clifford Algebras, firstly one needs to define the Clifford product, denoted by $\vee$.

**Definition 1.16.** The Clifford product is defined via,

$$\vee : \Lambda^1 \times \Lambda^q \to \Lambda^{q+1} \oplus \Lambda^{q-1}$$

$$(\vec{v}, q\omega) \mapsto \vec{v} \vee q\omega \equiv \vec{v} \wedge q\omega + i_{q\omega} \vec{v}$$

*As it can be seen, this definition makes a heavy use of the isomorphism between $\mathbb{V} \simeq \mathbb{V}^*$.*

A good example is the special case in which $q = 1$:

$$\vec{v} \vee \vec{\omega} = \vec{v} \wedge \vec{\omega} + i_{\vec{v}}(\vec{\omega})$$

$$\vec{v} \vee \vec{\omega} + \vec{\omega} \vee \vec{v} = 2 g(\vec{v}, \vec{\omega})$$

Also, noting that $\vee$ is associative and distributive, the definition above is extendable to $\Lambda^r \times \Lambda^q$.

**Definition 1.17.** The Clifford Algebra can be defined by means of the extension of the definition of the Clifford product to the whole of $\Lambda$. Thus, $(\Lambda, +, \cdot, \vee)$ is an algebra over $\mathbb{V}$. The elements of $\Lambda$ are called multivectors.

Note that, at this level, one has that:

$$\vec{e}_i \vee \vec{e}_j + \vec{e}_j \vee \vec{e}_i = 2 g_{ij} . \tag{1}$$

It is not hard to see, then, that the signature of the metric plays an important role in the Clifford algebra game. Thus, Clifford Algebras are classified by the numbers $r$ and $s$, which are, respectively, the number of components of the metric which are $+1$ and $-1$ [note that $r + s = n$, $n = \dim(\mathbb{V})$]. Therefore, $(\Lambda, +, \cdot, \vee) \equiv \bigl( \mathbb{C}_{r,s}(\mathbb{R}), +, \cdot, \vee \bigr)$.

Also, one can make the direct sum decomposition: $\mathbb{C}_{r,s}(\mathbb{R}) = \mathbb{C}_{r,s}(\mathbb{R})^+ \oplus \mathbb{C}_{r,s}(\mathbb{R})^-$, where $\mathbb{C}_{r,s}(\mathbb{R})^+$ is called the even subalgebra and $\mathbb{C}_{r,s}(\mathbb{R})^-$ is called the odd subalgebra.

Now, for the sake of completeness (and to satisfy the curiosity of some!), let me state some results/properties of Clifford Algebras:

- The real quaternions, $\mathbb{H}(\mathbb{R})$, are the Clifford Algebra $\mathbb{C}_{0,2}(\mathbb{R})$;
1.7 Algebraic Definition of \( C_r, s(\mathbb{R}) \)-spinors

The algebraic definition of spinors is very simple, but often obscured in the literature.

Consider a Clifford Algebra \( \langle C_{r, s}(\mathbb{R}), +, \cdot, \lor \rangle \) and its regular representation, \( \rho \), i.e., the mapping from \( C_{r, s}(\mathbb{R}) \) to its endomorphism [algebraic homomorphism from a set, \( \mathcal{S} \), to itself, denoted by: \( \text{End}(\mathcal{S}) \)] algebra:

\[
\rho : C_{r, s}(\mathbb{R}) \to \text{End} \left( C_{r, s}(\mathbb{R}) \right) \\
v \mapsto \rho_v(x) \equiv v \lor x .
\]

In general, \( \rho \) is not irreducible, i.e., \( \exists J : I \subset C_{r, s}(\mathbb{R}) \) which are left invariant under the action of \( \rho : \rho(J) \subset J \). By definition, such invariant spaces satisfy: \( v \lor J \subset J \), i.e., they are left ideals of \( C_{r, s}(\mathbb{R}) \). Thus, \( \rho \) is irreducible when restricted to any minimal left ideal and, then, it is called the spinor representation. Different choices of minimal left ideals lead to different but equivalent representations.

1.8 Fibre bundles as special manifolds

**Definition 1.18.** A fibre bundle \( \mathcal{E} \) over a \( C^\infty \) manifold \( \mathcal{M} \), with typical fibre \( \mathcal{F} \) and structure group \( \mathcal{G} \), is a set \( (\mathcal{E}, \mathcal{M}, \mathcal{F}, \mathcal{G}, \pi) \) such that:

- \( \mathcal{E} \) is a \( C^\infty \) manifold called the bundle (or total) space;
- \( \mathcal{M} \) is a \( C^\infty \) manifold called the base space;
- \( \mathcal{F} \) is a \( C^\infty \) manifold called the typical fibre;
- \( \mathcal{G} \) is a Lie group acting smoothly on \( \mathcal{F} \) on the left, called the structure (or gauge) group;
- \( \pi \) is a smooth surjection from \( \mathcal{E} \) to \( \mathcal{M} \), called the projection. The set \( \pi^{-1}(P) \), \( P \subset \mathcal{M} \), is called the fibre over \( P \), and is denoted by \( \mathcal{F}_P \);
- There exists a covering \( \{U_i\} \) of \( \mathcal{M} \) by open sets, and diffeomorphisms \( \{\phi_i\} \), mapping \( \pi^{-1}(U_i) \) to \( U_i \times \mathcal{F} \);
- For \( Q \in \pi^{-1}(U_i) \), let \( \phi_i(Q) \) be written \( \phi_i(Q) = (\pi(Q), \psi_{i, \pi(Q)}(Q)) \in U_i \times \mathcal{F} \). Then \( \psi_{i, \pi(Q)} : \mathcal{F}_{\pi(Q)} \to \mathcal{F} \) is a diffeomorphism; \( \mathcal{G} \)
- If \( P \in U_i \cap U_j \), the mappings \( t_{ij} \equiv \psi_{i, P} \circ \psi_{j, P}^{-1} : \mathcal{F} \to \mathcal{F} \), are diffeomorphisms called transition functions and are required to be left-actions by elements of \( \mathcal{G} \).
Loosely speaking, one may say that a fibre bundle $E$ over a manifold $M$, with typical fibre $\mathcal{F}$ and structure group $\mathcal{G}$, is a manifold which is locally diffeomorphic to $M \times \mathcal{F}$, in such a way that any two points in the fibre $\mathcal{F}_P$ are related smoothly by an element of $\mathcal{G}$.

Another notion that will be required is that of a cross-section of a fibre bundle.

**Definition 1.19.** Let $E$ be a fibre bundle over $M$ with typical fibre $\mathcal{F}$. By definition/construction, above each point $P \in M$ there is a fibre $\mathcal{F}_P = \pi^{-1}(P)$. A cross-section, $\sigma$, of $E$ is defined as a smooth assignment, for every $P \in M$, of an element $\sigma_P$ of the fibre $\mathcal{F}_P$. Equivalently, it can be defined as being a smooth mapping from $M$ to $E$ such that $\pi \circ \sigma$ is the identity. Furthermore, a cross-section is called local iff it is a smooth mapping from an open subset of $M$ to $E$.

Among fibre bundles, a special type is more “common” than all the others, the ones called principal bundles.

**Definition 1.20.** A principal bundle is a fibre bundle in which the typical fibre $\mathcal{F}$ is identical with the structure group $\mathcal{G}$.

### 1.9 Geometrical definition of $\mathcal{G}_{3,1}(\mathbb{R})$ spinors

Consider the manifold $M$ and is bundle of orthonormal frames with positive orientation, denoted by $PO^+M$. The structure group of $PO^+M$ is $\mathcal{G} = SO_{3,1}$. The transition functions, $t_{i\ j}$, “acts” on an orthonormal frame belonging to a fibre of $PO^+M$. Moreover, the double-valued mapping, $\varphi$, from the spin group $Spin_{r,s}$ to $SO_{r,s}$, can be used in order to construct the set of functions $\tilde{t}_{i\ j}$, elements of $Spin_{3,1}$, defined by: $\varphi(t_{i\ j}) = \tilde{t}_{i\ j}$.

Thus, given that $t_{i\ j} t_{j\ k} t_{k\ i} = e$, $t_{i\ i} = e$, and that $\varphi$ is a representation of the Spin group, there are solutions for the above equation for $\tilde{t}_{i\ j}$ that satisfy: $\tilde{t}_{i\ j} \tilde{t}_{j\ k} \tilde{t}_{k\ i} = \pm e$, $\tilde{t}_{i\ i} = \pm e$. Among them, the positive answer will be chosen. Then, the principal bundle $PSF^+M$ of spin frames over $M$ is defined as the bundle with $M$ as the base space, $Spin_{3,1}$ as the structure group and transition functions given by $\tilde{t}_{i\ j}$ chosen as above.

It should be noted that, because of the 2-to-1 nature of $\varphi$, the bundle $PSF^+M$ is a double covering of $PO^+M$.

Furthermore, an assignment of a family of spin frames over $M$ can be defined as a cross-section of $PSF^+M$. A spinor above a point $P \in M$ is a linear combination of the elements of the spin frame given by this cross-section.

In addition to the notion of spinor at a point $P$, one may construct the spin bundle $\mathcal{G}M$ above $M$, as the bundle which admits as fibre above the point $P \in M$ the set of all spinors at $P$. The structure group of $\mathcal{G}M$ is the spin group. A spinor field over $M$ is defined as a cross-section of $\mathcal{G}M$.

### 1.10 Differentiation of spinor fields

Let’s start by noting that the connection 1-forms will be denoted equivalently by $A_\mu^\nu$ and $\gamma_\mu^\nu$, i.e., $A_\mu^\nu = \gamma_\mu^\nu$. Thus, one can define a covariant derivative for spinor fields in the following way:

$$\hat{\nabla}_\xi \psi = \left[ \xi(\psi^M) - \hat{A}_N^M(\xi) \psi^N \right] \bar{e}_M$$

$$\equiv \xi(\psi^M) \bar{e}_M - D_{\hat{A}}(\psi),$$

where

$$D_{\hat{A}}(\psi) \equiv \hat{A}_N^M(\xi) \psi^N \bar{e}_M,$$

$$\hat{A} = -\frac{1}{2} A_\mu^\nu \Sigma^{\mu\nu},$$

$$D_{\hat{A}} = D_{\hat{A}} \equiv -\frac{1}{2} A_\mu^\nu \sigma^{\mu\nu},$$
where,

\[
4 \sigma^{\mu \nu} \equiv \gamma^{\mu} \circ \gamma^{\nu} - \gamma^{\nu} \circ \gamma^{\mu},
\]

\[
\equiv 4 \sigma^{\mu \nu \mu}_{\nu} \tilde{e}_{\mu} \otimes \tilde{e}^{\nu}.
\]

Therefore, for a metric compatible connection, one has that

\[
\hat{\nabla}_{\vec{x}} \psi = \left[ \vec{x}(\psi^M) + \frac{1}{2} A_{\mu \nu}(\vec{x}) \sigma^{\mu \nu \mu}_{\nu} \psi^N \right] \tilde{e}_{\mu}.
\]

Now, in order to define a Lie derivative for spinor fields, one could go as follows: Just make the identification \( A_{\mu \nu} = \gamma_{\mu \nu} = -L_{\mu \nu} \), where the spinorial Lie derivative is given by \( \tilde{\mathcal{L}}_{\vec{x}} \tilde{e}_{\mu} = -\tilde{e}_{\nu} L_{\mu \nu} \). That gives:

\[
\tilde{\mathcal{L}}_{\vec{x}} \psi = \vec{x}(\psi^M) \tilde{e}_{M} - \frac{1}{2} L_{\mu \nu} \sigma^{\mu \nu} \psi.
\]

## 2 Geometric Calculus and the Ordering Problem in Quantum Mechanics

Given the motivation proposed by [9], one can use the Geometric Calculus in order to prevent the ordering problem that haunts quantum mechanics. Let’s make a quick overview of the method. Given non-commuting numbers, \( \gamma_{\mu} \in \mathcal{C}_{3,1}(\mathbb{R}) \simeq \mathcal{M}_{4}(\mathbb{R}) \), (where \( \mathcal{C}_{3,1}(\mathbb{R}) \) is a real Clifford Algebra\(^1\) and \( \mathcal{M}_{4}(\mathbb{R}) \) is the space of \( 4 \times 4 \) real matrices) it is known that

\[
\{ \gamma_{\mu}, \gamma_{\nu} \} = \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2 g_{\mu \nu}.
\]

(2)

(It should be noted that \( \mathcal{C}_{2,0}(\mathbb{R}) \simeq \mathcal{C}_{1,1}(\mathbb{R}) \simeq \mathcal{M}_{2}(\mathbb{R}) \). Thus, one can choose to work either with a 4- or a 2-spinor, depending only on the choice of the Clifford Algebra desired\(^2\).)

Thus, one can introduce the expansion of an arbitrary vector and the dual basis as follows\(^3\):

\[
\vec{v} = v^{\mu} \gamma_{\mu};
\]

\[
\{ \gamma^{\mu}, \gamma^{\nu} \} = g^{\mu \nu};
\]

\[
\gamma^{\mu} = g^{\mu \nu} \gamma_{\nu}.
\]

In the same fashion [10] [11],

\[
\partial \equiv \gamma^{\mu} \partial_{\mu};
\]

\[
\partial_{\mu} \gamma_{\nu} = \Gamma^{\alpha}_{\mu \nu} \gamma_{\alpha};
\]

\[
\partial_{\mu} \gamma^{\nu} = -\Gamma^{\nu}_{\mu \alpha} \gamma^{\alpha};
\]

\[
\partial \vec{v} = \gamma^{\mu} \gamma^{\nu} D_{\mu} \psi;
\]

\[
\therefore \partial \partial \psi = \gamma^{\mu} \gamma^{\nu} D_{\mu} D_{\nu} \psi;
\]

where \( \Gamma^{\alpha}_{\mu \nu} \) is the connection, \( \psi \in \mathbb{K} \) is a scalar and \( D_{\mu} = (\partial_{\mu} - \Gamma^{\alpha}_{\mu \nu}) \) is the covariant derivative. If the connection is symmetric (vanishing torsion), then

\[
\partial \partial \psi = D_{\mu} D^{\mu} \psi; \tag{3}
\]

\(^1\)The reader should be aware of the adopted notation, where, for a metric \( g \), its signature is given by \( p \) plus and \( q \) minus signs, with \( p + q = n \). The structure of the Clifford Algebra can only depend on \( p \) and \( q \), thus it is denoted \( \mathcal{C}_{p,q}(\mathbb{K}) \), where \( \mathbb{K} \) is a field (usually taken to be either \( \mathbb{R} \) or \( \mathbb{C} \)). Also, it is a well-known fact that, \( \dim(\mathcal{C}_{p,q}(\mathbb{K})) = 2^n \).

\(^2\)This is a subtle point for Quantum Mechanics, given that, once pure spinors are not part of that theory, the type [of spinor] that one wants to introduce in the theory is completely arbitrary.

\(^3\)The reader should note that the common “slash”-notation is not being used here. This should not be a source of [future] confusion.
which is just D’Alembert’s operator in curved spacetime.

Thus, one has that \( \hbar = 1 \),

\[
p \equiv -i \partial = -i \gamma^\mu \partial_\mu ,
\]

which is Hermitian, \( \langle p \rangle = \langle p^\dagger \rangle \), and whose expectation value, \( \langle p \rangle \), follows a geodesic trajectory in our curved spacetime.

## 3 Bundle Quantum Mechanics

### 3.1 Non-Relativistic Quantum Mechanics \[7\]

The mathematical basis for the reformulation of non-relativistic quantum mechanics in terms of fibre bundles is given by Schrödinger’s equation,

\[
\frac{i \hbar}{\text{d}t} \frac{\text{d}\psi(t)}{\text{d}t} = \mathcal{H}(t) \psi(t) ;
\]

\( \psi \) is the system’s state vector in a suitable Hilbert space \( F \) and \( \mathcal{H} \) is its Hamiltonian.

In the bundle description, one has a Hilbert bundle given by \( (F, \pi, M) \), where the total space is \( F \), the projection is \( \pi \), the base manifold is \( M \) and a typical fibre, \( F_x \), which is isomorphic to \( F_x = \pi^{-1}(x) \), \( \forall x \in M \). Thus, \( \exists l_x : F_x \to F \), \( x \in M \), isomorphisms. A state vector, \( \psi \), and the Hamiltonian, \( \mathcal{H} \), are represented respectively by a state section along paths, \( \Psi : \gamma \to \Psi_\gamma \), and a bundle Hamiltonian (morphisms along paths) \( \mathcal{H} : \gamma \to \mathcal{H}_\gamma \), given by:

\[
\Psi_\gamma : t \mapsto \Psi_\gamma(t) = l_{\gamma(t)}^{-1}(\psi(t)) ;
\]

\[
\mathcal{H}_\gamma : t \mapsto \mathcal{H}_\gamma(t) = l_{\gamma(t)}^{-1} \circ \mathcal{H}(t) \circ l_{\gamma(t)} ;
\]

where \( \gamma : I \to M, I \subseteq \mathbb{R} \) is the world-line for some observer. The bundle evolution operator is given by,

\[
U_\gamma(t, s) = l_{\gamma(s)}^{-1} \circ U(t, s) \circ l_{\gamma(s)}^{-1} : F_{\gamma(s)} \to F_{\gamma(t)}
\]

\[
\Psi_\gamma(t) = U_\gamma(t, s) \Psi_\gamma(s) .
\]

Therefore, in order to write down the bundle Schrödinger equation, \( (D \Psi = 0) \), a derivation along paths, \( (D) \), corresponding to \( U \) is needed,

\[
D : \text{PLift}^1(F, \pi, M) \to \text{PLift}^0(F, \pi, M) ;
\]

where \( \text{PLift}^k(F, \pi, M) = \{ \lambda \text{ lifting} : \lambda \in \mathcal{G}^k \} \), is the set of liftings from \( M \) to \( F \), and,

\[
\lambda : \gamma \to \lambda_\gamma, \lambda \in \text{PLift}^1(F, \pi, M) ;
\]

\[
D^\gamma_\lambda(\lambda) = \lim_{\epsilon \to 0} \frac{U_\gamma(s, s + \epsilon) \lambda_\gamma(s + \epsilon) - \lambda_\gamma(s)}{\epsilon} \quad (5)
\]

where

\[
D^\gamma_\lambda(\lambda) = [(D\lambda)(\gamma)](s) = (D\lambda)_\gamma(s) ;
\]

and, in local coords,

\[
D^\gamma_\lambda(\lambda) = \left( \frac{d\lambda^a(s)}{ds} + \Gamma^a_b(s; \gamma) \left( \lambda^b_\gamma(s) \right) e_a(\gamma(s)) \right) ; \quad (6)
\]
where \( \{ e_a(\gamma(x)) \} \), \( s \in I \) is a basis in \( F_{\gamma(s)} \). Thus, one can clearly see what is happening with this structure, namely the bundle evolution transport is giving origin to the linear connection by means of:

\[
\Gamma^b_a(s; \gamma) = \frac{\partial (U_\gamma(s,t))^b}{\partial t} \bigg|_{s=t} = -\frac{\partial (U_\gamma(t,s))^b}{\partial t} \bigg|_{t=s};
\]

\[
U_\gamma(t,s) e_a(\gamma(s)) = \sum_b (U_\gamma(s,t))^b e_b(\gamma(t));
\]

are the local components of \( U_\gamma \) in \( \{ e_a \} \).

In this manner, there is a bijective correspondence between \( D \) and the bundle Hamiltonian,

\[
\Gamma_\gamma(t) = [\Gamma^b_a(s; \gamma)] = \frac{i}{\hbar} H_\gamma(t); \quad H_\gamma(t) = i \hbar \frac{\partial U_\gamma(t,t_0)}{\partial t} U_\gamma^{-1}(t,t_0) = \frac{\partial U_\gamma(t,t_0)}{\partial t} U_\gamma(t_0,t); \quad \text{where } H_\gamma \text{ is the matrix-bundle Hamiltonian.}
\]

### 3.2 Relativistic Quantum Mechanics

#### 3.2.1 Time-dependent Approach

The framework developed so far can be generalized to relativistic quantum theories in a very straightforward way. However, in doing so, it is seen that time plays a privileged role (thus, leaving the relativistic covariance implicit). Proceeding in such a manner, it is found that,

\[
i \hbar \frac{\partial \psi}{\partial t} = D\mathcal{H} \psi; \quad \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T;
\]

where \( D\mathcal{H} \) is the Dirac Hamiltonian (Hermitian), in the space \( \mathcal{F} \) of state spinors \( \psi \). Once \( D\mathcal{H} \) is a first-order differential equation, a \([\text{Dirac] evolution operator}, (D\mathcal{U})\), can be introduced. It is a 4 \times 4-integral matrix operator uniquely defined by the initial-value problem,

\[
i \hbar \frac{\partial \psi}{\partial t} D\mathcal{U}(t,t_0) = D\mathcal{H} \circ D\mathcal{U}(t,t_0) \quad \text{and} \quad D\mathcal{U}(t_0,t_0) = \mathbf{1}_{\mathcal{F}}.
\]

Thus, the formalism developed earlier can be applied to Dirac particles. The spinor lifting of paths has to be introduced and the \textit{Dirac evolution transport} \([\text{along a path, } \gamma]\) is given by,

\[
D\mathcal{U}(t,s) = l_\gamma^{-1}(t) \circ D\mathcal{U}(t,s) \circ l_\gamma(s), \quad s, t \in I.
\]

The \textit{bundle Dirac equation} is given by (see [5]),

\[
D\mathcal{D}_t^\gamma \Psi_\gamma = 0.
\]

#### Klein-Gordon Equation

The spinless, scalar wavefunction \( \phi \in \mathcal{G}^k, \ k \geq 2 \), over spacetime, satisfies the Klein-Gordon equation if (particle of mass \( m \), electric charge \( e \) and in an external electromagnetic field given by \( A_\mu = (\varphi, \vec{A}) \)),

\[
\left( i \hbar \frac{\partial}{\partial t} - e \varphi \right)^2 - c^2 \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 \phi = m^2 c^4 \phi.
\]

In order to solve this, a trick can be used. Just let \( \varphi = (\varphi + \frac{e}{m c^2} \vec{A} \cdot \vec{p} - \frac{e}{m c^2} \vec{A})^T \). This is a particular good choice if one is interested in the non-relativistic limit. After some first-order (Schrödinger-type) representation of the Klein-Gordon equation is chosen, the bundle formalism can be applied to spinless particles.

Thus, the goal is to describe the given equation of motion in terms of some Schrödinger-type operator and then apply the bundle formalism \textit{mutatis mutandis}. 


3.2.2 Covariant Approach

Now, it will be developed an appropriate covariant bundle description of relativistic quantum mechanics. The difference between the time-dependent and the covariant formalism is analogous to the one between the Hamiltonian and the Lagrangian approaches to relativistic wave equations.

Dirac Equation The covariant Dirac equation (for a spin \( \frac{1}{2} \), mass \( m \) and charge \( e \) particle, in an external electromagnetic field \( A_\mu \)) is given by,

\[
\left( i \hbar \gamma^\mu - m c \mathbb{1}_{4\times 4} \right) \psi = 0 ; \\
\gamma^\mu = \gamma^\mu_\mu D_\mu ; \\
D_\mu = \partial_\mu - \frac{e}{i \hbar c} A_\mu .
\]

Since it is a first-order differential equation, it admits an evolution operator \( \mathcal{U} \), whose job is to connect different values at different spacetime points. Thus, for \( x_1, x_2 \in M_0 \), \( (M_0 \) being the Minkowski spacetime),

\[
\psi(x_2) = \mathcal{U}(x_2, x_1) \psi(x_1) ;
\]

where \( \mathcal{U}(x_2, x_1) \) is a \( 4 \times 4 \)-matrix operator, defined as the unique solution to the initial-value problem:

\[
\left( i \hbar \gamma^\mu - m c \mathbb{1}_{4\times 4} \right) \mathcal{U}(x, x_0) = 0 ; \\
\mathcal{U}(x_0, x_0) = \mathbb{1}_\mathcal{F}, \quad x, x_0 \in M_0 ;
\]

where \( \mathcal{F} \) is the space of 4-spinors.

Assume that \( (\mathcal{F}, \pi, M) \) is a vector bundle with total space \( \mathcal{F} \), projection \( \pi : \mathcal{F} \to M \), fibre \( \mathcal{F} \) and isomorphic fibres \( \mathcal{F}_x = \pi^{-1}(x), x \in M \). Then, there exists linear isomorphisms \( l_x : \mathcal{F}_x \to \mathcal{F} \) — which are assumed to be diffeomorphisms — so that \( \mathcal{F}_x = l_x^{-1}(\mathcal{F}) \) are 4-dim vector spaces.

A \( \mathcal{C}^1 \) section\(^4\) is assigned to a state spinor, \( \psi \), i.e., \( \Psi \in \text{Sec}^1(\mathcal{F}, \pi, M) \), in the following manner:

\[
\Psi(x) = l_x^{-1}(\psi(x)) \in \mathcal{F}_x = \pi^{-1}(x), \quad x \in M .
\]

Thus, it follows that:

\[
\begin{align*}
\Psi(x_2) &= U(x_2, x_1) \Psi(x_1), \quad x_1, x_2 \in M ; \\
U(y, x) &= l_y^{-1} \circ \mathcal{U}(y, x) \circ l_x : \mathcal{F}_x \to \mathcal{F}_y ; \quad x, y \in M ; \\
\mathcal{U}(x_3, x_1) &= \mathcal{U}(x_3, x_2) \circ \mathcal{U}(x_2, x_1), \quad x_1, x_2, x_3 \in M ;
\end{align*}
\]

\( \mathcal{U} \) is a linear transport along the identity map, \( (\mathbb{1}_M) \), of \( M \) in the bundle \( (\mathcal{F}, \pi, M) \). Thus, it is not difficult to see that \( \mathbf{3} \mathbf{4} \mathbf{5} \mathbf{6} \),

\[
\begin{align*}
\mathcal{D}_\mu \Psi &= 0, \quad \mu = 0, 1, 2, 3 ; \\
\mathcal{D}_\mu &= \mathcal{D}_\mu^{\mathbb{1}_M} .
\end{align*}
\]

These are called Bundle Dirac Equations.

At this point, local basis could be introduced and a local view of the above could be written down. However, from this knowledge, what will be of future interest, will come from the facts,

\[
\begin{align*}
G_\mu &= l_x^{-1} \circ \gamma^\mu \circ l_x ; \\
\{ G^\mu, G^\nu \} &= 2 \eta^{\mu\nu} \mathbb{1}_\mathcal{F} ;
\end{align*}
\]

\(^4\) \( \Psi \) is simply a section at this time, as opposed to the previous two cases, in which it was a section along paths. This corresponds to the fact that quantum objects do not have trajectories in a classical sense.
where $\eta^{\mu\nu}$ is the Minkowski metric tensor, $[\eta^{\mu\nu}] = \text{diag}(+1,-1,-1,-1)$. This last equation, (4), is the bundle generalization of (1). It is clear that this expression can be put in terms of local coordinates, in which case it would reduce to (in the equation that follows, \textbf{boldface} denotes the matrix — i.e., the expression in local coordinates — of the operator denoted by the same (kernel) symbol),

$$\{ G^\mu, G^\nu \} = 2 \eta^{\mu\nu} I_{4 \times 4}; \quad (9)$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor and $I_{4 \times 4} = \text{diag}(1,1,1,1)$ is the unit matrix in 4-dim. Thus, (9) is the local expression of (8) and a generalization of (1).

4 Putting it all together

Basically, one wants to generalize eqs (3) and (4) to the bundle formalism previously formulated. In order to do so, let’s remember that ($\hbar = 1$, and bringing the “slash” notation back):

**Bundle results:**

$$G^\mu = l_x^{-1} \circ \gamma^\mu \circ l_x ;$$
$$\delta_\mu = l_x^{-1} \circ \partial_\mu \circ l_x ;$$
$$\mathfrak{g} = G^\mu(x) \circ \partial_\mu ;$$
$$\mathfrak{g} = l_x^{-1} \circ \delta \circ l_x .$$

Thus, the bundle version is given by:

$$p = -i \mathfrak{g} = l_x^{-1} \circ (-i \partial) \circ l_x ;$$

or, in local coords (\textbf{boldface} being matrix-notation, see (8) and (9)):

$$p = -i G^\mu \partial_\mu.$$

This works for (non-relativistic) Quantum Mechanics and for Relativistic Quantum Mechanics.

When dealing with Quantum Field Theories, somethings have to be said before conclusions are drawn. Let’s start with a quick overview of the relevant facts. In the functional [Schrödinger’s] representation for a free scalar QFT, one has that:

$$S = \int L d^4 x = \frac{1}{2} \int (\partial^\mu \varphi \partial_\mu \varphi - m^2 \varphi^2) d^4 x ;$$

where, the conjugate field momentum (note the non-covariant formalism) is:

$$\pi(x) = \frac{\partial L}{\partial (\partial_t \varphi)} = \dot{\varphi}(x) ;$$

and, the Hamiltonian is:

$$H = \frac{1}{2} \int \left( \pi^2 + |\nabla \varphi|^2 + m^2 \varphi^2 \right) d^3 x .$$

The equal-time commutation relations are given by,

$$[ \varphi(\mathbf{x},t), \pi(\mathbf{y},t) ] = i \delta(\mathbf{x} - \mathbf{y}) ;$$
$$[ \varphi(\mathbf{x},t), \varphi(\mathbf{y},t) ] = 0 = [ \pi(\mathbf{x},t), \pi(\mathbf{y},t) ] ;$$

\textsuperscript{5}The generalization to an arbitrary number of dimension is quite clear and straightforward from the equations given.
In the coordinate representation, with a basis for the Fock space, where \( \varphi(\vec{x}) \) is (now) time independent and diagonal (note that \( \phi(\vec{x}) \) is just an ordinary scalar function), one has that:

\[
\varphi(\vec{x}) \ket{\phi} = \phi(\vec{x}) \ket{\phi} ;
\[
\therefore \quad \Psi[\phi] = \bra{\phi} \Psi \rangle ;
\[
\delta \frac{\delta}{\delta \phi(\vec{x})} \phi(\vec{y}) = \delta(\vec{x} - \vec{y})
\]

\[
\therefore \quad \left[ \frac{\delta}{\delta \phi(\vec{x})}, \phi(\vec{y}) \right] = \delta(\vec{x} - \vec{y}) .
\]

Thus, the functional representation of the equal-time commutators turns out to be:

\[
\Rightarrow \quad \pi(\vec{x}) = -i \frac{\delta}{\delta \phi(\vec{x})} ;
\]

\[
\bra{\phi'} \pi(\vec{x}) \ket{\phi} = -i \frac{\delta}{\delta \phi(\vec{x})} \delta[\phi' - \phi] ;
\]

and, the momentum operator, \( P_i \), which generates spatial displacements, is:

\[
\left[ P_j, \varphi(\vec{x}, t) \right] = -i \frac{\partial}{\partial \vec{x}_j} \varphi(\vec{x}, t) ;
\]

\[
\therefore \quad P_j = - \int (\varphi(x) \partial_j \varphi(x)) \, d^3 x ;
\]

\[
P_j = i \int \left( \phi(\vec{x}) \partial_j \frac{\delta}{\delta \phi(\vec{x})} \right) \, d^3 x ;
\]

thus, using (4), one has that:

\[
\Psi = \gamma^j P_j .
\]

On the other hand, in the momentum representation, \( \pi(x) \) is diagonal and time independent, which gives:

\[
\pi(\vec{x}) \ket{\varpi} = \varpi(\vec{x}) \ket{\varpi} ;
\]

\[
\Psi[\varpi] = \bra{\varpi} \Psi \rangle ;
\]

\[
\varphi(\vec{x}) = i \frac{\delta}{\delta \varpi(\vec{x})} ;
\]

\[
E \Psi[\varpi] = \frac{1}{2} \int \left( \frac{\delta}{\delta \varpi(\vec{x})} \left( -\nabla^2 + m^2 \right) \frac{\delta}{\delta \varpi(\vec{x})} + \varpi^2(\vec{x}) \right) \, d^3 x \quad \Psi[\varpi] .
\]

Let us introduce a functional version of the Fourier transform given by:

\[
\Psi[\varpi] = \int \Psi[\phi] e^{i \int \varpi(\vec{x}) \phi(\vec{x}) \, d^3 x} \, D\phi .
\]

Thus, for a free spinor QFT, analogous relations are valid:

\[
H = \int \Psi^\dagger(x) \left( -i \gamma^\mu \nabla_\mu + m \right) \Psi(x) \, d^3 x ;
\]

\[
\left\{ \Psi_\alpha(\vec{x}, t), \Psi_\beta^\dagger(\vec{y}, t) \right\} = \delta_{\alpha\beta} \delta^3(\vec{x} - \vec{y}) ;
\]

\[
\left\{ \Psi_\alpha(\vec{x}, t), \Psi_\beta(\vec{y}, t) \right\} = 0 = \left\{ \Psi_\alpha^\dagger(\vec{x}, t), \Psi_\beta^\dagger(\vec{y}, t) \right\} ;
\]
and in the coordinate representation,

$$\Psi(\vec{x}) |\psi\rangle = \psi(\vec{x}) |\psi\rangle ;$$

where $\Psi(\vec{x})$ is an anticommuting field, thus $\psi(\vec{x})$ must be a spinor of Grassmann functions $\Rightarrow \psi^2(\vec{x}) = 0$, which leads us to:

$$\Phi[\psi] = \langle \psi | \Phi \rangle ;$$

$$\Psi^\dagger(\vec{x}) = \frac{\delta}{\delta \psi(\vec{x})} ;$$

$$\therefore E \Phi[\psi] = \int \left( \frac{\delta}{\delta \psi(\vec{x})} (-i \gamma^\mu \nabla_\mu + m) \psi(\vec{x}) \right) d^3x \Phi[\psi] ;$$

$$\therefore \mathcal{E} = H \gamma^0 .$$

Thus, from all of the above, it is not difficult to see that, a four “spin-vector” can be constructed out of:

$$\mathcal{E} = H \gamma^0 ;$$

$$\Psi = P_j \gamma^j ;$$

$$\therefore \mathcal{P} = (\mathcal{E}, \Psi) = (H \gamma^0, P_j \gamma^j) .$$

In a covariant formulation (the conserved quantity being the energy-momentum tensor), one would have that:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\nu \varphi_a)} \partial^\mu \varphi_a - g^{\mu\nu} \mathcal{L} .$$

$$\therefore (T^{\mu\nu} \gamma_\mu \gamma_\nu) : \text{invariant quantity} ;$$

thus, using the original notation:

$$\mathcal{T}^{\mu\nu} = l_x^{-1} \circ T^{\mu\nu} \circ l_x ;$$

$$\therefore \mathcal{T} = G_\mu \mathcal{T}^{\mu\nu} G_\nu = l_x^{-1} \circ (T^{\mu\nu} \gamma_\mu \gamma_\nu) \circ l_x ;$$

The reader should note that, the properties described in [9] will easily generalize to the above cases. This leads us to the following thought.

Conjecture. The basic description of physical quantities should be done in terms of spin-variables, such as spinors, spin-vectors and spin-tensors.

5 Aknowledgements

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