A Kneser-type theorem for backward doubly stochastic differential equations

Yufeng Shi\textsuperscript{a} and Qingfeng Zhu\textsuperscript{b}

\textsuperscript{a}School of Mathematics, Shandong University, Jinan 250100, China
\textsuperscript{b} School of Statistics and Mathematics, Shandong University of Finance, Jinan 250014, China

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Abstract

A class of backward doubly stochastic differential equations (BDSDEs in short) with continuous coefficients is studied. We give the comparison theorems, the existence of the maximal solution and the structure of solutions for BDSDEs with continuous coefficients. A Kneser-type theorem for BDSDEs is obtained. We show that there is either unique or uncountable solutions for this kind of BDSDEs.

\textit{keywords:} Backward doubly stochastic differential equations, comparison theorem, maximal solution, Kneser-type theorem

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) have been independently introduced by Pardoux and Peng \cite{14} and Duffie and Epstein \cite{3}. The comprehensive applications of BSDEs have motivated many efforts to establish the existence and uniqueness of adapted solution under general hypotheses on the coefficients. For instance, for the one-dimensional case, Lepeltier and San Martin \cite{11} proved the existence of a solution to BSDEs under the assumption of continuous coefficient. Recently, Jia and Peng \cite{5} studied the structure of the solutions to BSDEs with continuous coefficients.

The comparison theorem is an important and effective technique in the theory of BSDEs. There are much works concerning the comparison theorems of BSDEs. For instance, for applications to finance, El Karoui, Peng and Quenez \cite{7} have given the comparison theorem of BSDEs with Lipschitz coefficients. And then Liu and Ren \cite{12} have given some comparison theorems of BSDEs with continuous coefficients.

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\textsuperscript{†}Corresponding author, E-mail: yfshi@sdu.edu.cn
In 1923, Kneser [10] proved that the cardinality of the set of solutions for ordinary differential equations (ODEs in short) with continuous coefficients is either one or of continuum. Many interesting generalizations have followed. For example, in 1956, Alexiewicz and Orlicz [11] got the same theorem for a class of partial differential equations (PDEs in short). In 2008, Jia [6] proved Kneser-type theorem for BSDEs with continuous coefficients. For more information about Kneser-type theorems, one can refer to [8, 9, 17, 18].

A class of backward doubly stochastic differential equations (BDSDEs in short) was introduced by Pardoux and Peng [15] in 1994, in order to provide a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs in short). They have proved the existence and uniqueness of solution for BDSDEs under uniformly Lipschitz conditions. Since then, many efforts have been made to relax the assumption on the coefficients. For instance, Gu [4] proved the existence and uniqueness of solution for BDSDEs under the local Lipschitz conditions. Shi, Gu and Liu [16] have relaxed the Lipschitz assumptions to linear growth conditions by virtue of their comparison theorem of BDSDEs with Lipschitz conditions, and showed the existence of the minimal solution of BDSDEs. Bally and Matoussi [2] have given a probabilistic interpretation of the solutions in Sobolev spaces for parabolic semilinear stochastic PDEs in terms of BDSDEs. Zhang and Zhao [19] have proved the existence and uniqueness of solution for BDSDEs on infinite horizon, and described the stationary solutions of SPDEs by virtue of the solutions of BDSDEs on infinite horizon.

Due to their important significance to SPDEs, it is necessary to give intensive investigation to the theory of BDSDEs. The aim of this paper is to study the structure of the solutions to BDSDEs with continuous and linear growth conditions. We firstly generalize the comparison theorems to the case where the coefficients are continuous. As an application of the comparison theorems, we give the existence of the maximal solution of BDSDEs with continuous coefficient by means of our comparison theorems. Finally we will show that there is either unique or uncountable solutions for this kind of BDSDEs. In fact, our result shows the structure of those solutions, that is, we obtain a Kneser-type theorem for BDSDEs.

The rest of the paper is organized as follows. In Section 2, we present the main assumptions and some preliminary results. In Section 3, we generalize the comparison theorem of BDSDEs in [15] to BDSDEs with continuous coefficients. In Section 4, we prove the existence of the maximal solution of BDSDEs. Finally, in Section 5, we discuss the structure of solutions of BDSDEs with continuous coefficients and linear growth conditions.

2 Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space, and \(T > 0\) be an arbitrarily fixed constant throughout this paper. Let \(\{W_t; 0 \leq t \leq T\}\) and \(\{B_t; 0 \leq t \leq T\}\) be two mutually independent standard Brownian Motions with values in \(\mathbb{R}^d\) and \(\mathbb{R}^l\),
respectively, defined on \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{N}\) denote the class of \(P\)-null sets of \(\mathcal{F}\). For each \(t \in [0, T]\), we define \(\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^B\), where for any process \(\{\eta_t\}, \mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}\), \(\mathcal{F}_t^0 = \mathcal{F}_{0,t}^0\). Note that the collection \(\{\mathcal{F}_t; t \in [0, T]\}\) is neither increasing nor decreasing, so it does not constitute a filtration. All the equalities and inequalities mentioned in this paper are in the sense of \(dt \times dP\) almost surely on \([0, T] \times \Omega\).

We use the usual inner product \((\cdot, \cdot)\) and Euclidean norm \(|\cdot|\) in \(\mathbb{R}^k, \mathbb{R}^{k \times l}\) and \(\mathbb{R}^{k \times d}\). All the equalities and inequalities mentioned in this paper are in the sense of \(dt \times dP\) almost surely on \([0, T] \times \Omega\).

For any \(k \in \mathbb{N}\), let \(M^2(0, T; \mathbb{R}^k)\) denote the set of \((\text{classes of } dP \otimes dt \text{ a.e. equal})\) \(k\)-dimensional jointly measurable stochastic processes \(\{\varphi_t; t \in [0, T]\}\) which satisfy:

(i) \(\|\varphi\|^2_{M^2} := E \int_0^T |\varphi_t|^2 dt < \infty\);
(ii) \(\varphi_t\) is \(\mathcal{F}_t\)-measurable, for any \(t \in [0, T]\).

Similarly, we denote by \(S^2([0, T]; \mathbb{R}^k)\) the set of \(k\)-dimensional continuous stochastic processes \(\{\varphi_t; t \in [0, T]\}\) which satisfy:

(iii) \(\|\varphi\|^2_{S^2} := E(\sup_{0 \leq t \leq T} |\varphi_t|)^2 < \infty\);
(iv) \(\varphi_t\) is \(\mathcal{F}_t\)-measurable, for any \(t \in [0, T]\).

For any \(t \in [0, T]\), denote by \(L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^k)\) the set of \(k\)-dimensional random variables \(\xi\), which satisfy:

(v) \(E|\xi|^2 < \infty\);
(vi) \(\xi\) is \(\mathcal{F}_t\)-measurable.

Consider the following BDSDE:

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{1}
\]

We assume that

(H1) \(f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k\) is jointly measurable and such that
\[
f(\cdot, y, z) \in M^2(0, T; \mathbb{R}^k), \quad \forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d},
\]
and there exist a constant \(C > 0\) such that for any \(t \in [0, T]\), \((y_i, z_i) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}, i = 1, 2,\)
\[
|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + |z_1 - z_2|^2).
\]

(H2) \(g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{k \times l}\) is jointly measurable and such that
\[
g(\cdot, y, z) \in M^2(0, T; \mathbb{R}^k), \quad \forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d},
\]
and there exist constants \(C > 0\) and \(0 < \alpha < 1\) such that for any \(t \in [0, T]\), \((y_i, z_i) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}, i = 1, 2,\)
\[
|g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2.
\]

Note that the integral with respect to \(\{B_t\}\) is a “backward Itô integral”, in which the integrand takes values at the right end points of the subintervals in the Riemann type sum (for details refer to [15]), and the integral with respect to \(\{W_t\}\) is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral (see [13] for details).
**Definition 2.1** A pair of processes \((Y, Z) : \Omega \times [0, T] \to \mathbb{R}^k \times \mathbb{R}^{k \times d}\) is called a solution of BDSDE (1), if \((Y, Z) \in S^2([0, T]; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})\) and satisfies BDSDE (1).

**Proposition 2.2** Under assumptions (H1) and (H2), if \(\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)\), then BDSDE (1) has a unique solution \((Y, Z) \in S^2([0, T]; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})\).

This proposition was derived in [15].

### 3 Comparison theorem of BDSDEs with continuous coefficients

In this paper, we only consider one-dimensional BDSDEs, i.e., \(k = 1\). Assume

(H3) for fixed \(\omega\) and \(t\), \(f(\omega, t, \cdot, \cdot)\) is continuous;

(H4) linear growth: \(\forall (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d\), there exists \(0 < K < \infty\), such that

\[
|f(\omega, t, y, z)| \leq |f(\omega, t, 0, 0)| + K|y| + K|z|,
\]

and

\[
\mathbb{E} \int_0^T |f(\omega, t, 0, 0)|^2 dt < \infty.
\]

We consider the following BDSDEs: \((0 \leq t \leq T)\)

\[
Y_t^1 = \xi^1 + \int_t^T f^1(s, Y_s^1, Z_s^1) \, ds + \int_t^T g(s, Y_s^1, Z_s^1) \, dB_s - \int_t^T Z_s^1 \, dW_s, \quad (2)
\]

\[
Y_t^2 = \xi^2 + \int_t^T f^2(s, Y_s^2, Z_s^2) \, ds + \int_t^T g(s, Y_s^2, Z_s^2) \, dB_s - \int_t^T Z_s^2 \, dW_s. \quad (3)
\]

where for \(i = 1, 2\), \(\xi^i \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})\) and \(f^i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) satisfy

(H5) \(\xi^1 \geq \xi^2\), a.s.; \quad f^1(t, y, z) \geq f^2(t, y, z), \ a.s., \quad \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d.

The comparison theorem was established by Shi, Gu and Liu [16] for one-dimensional BDSDEs, where both the coefficients \(f^1\) and \(f^2\) satisfy Lipschitz conditions. In this section, we firstly generalize the comparison theorem to the case where one of the coefficients \(f^1\) and \(f^2\) is only continuous, another is Lipschitz continuous.

**Theorem 3.1** Assume BDSDE (2) satisfies (H1) and (H2), and BDSDE (3) satisfies (H2)–(H4). Let \((Y^1, Z^1)\) and \((Y^2, Z^2)\) be the solutions of BDSDEs (2) and (3), respectively. If (H5) holds, then

\[
P\{Y_t^1 \geq Y_t^2, \ \text{for all } t \in [0, T]\} = 1.
\]
Proof. It is easy to see that \((Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)\) satisfies the following BDSDE

\[
Y_t^1 - Y_t^2 = \xi^1 - \xi^2 + \int_t^T [f^1 (s, Y_s^1, Z_s^1) - f^2 (s, Y_s^2, Z_s^2)] ds \\
+ \int_t^T [g (s, Y_s^1, Z_s^1) - g (s, Y_s^2, Z_s^2)] dB_s \\
- \int_t^T (Z_s^1 - Z_s^2) dB_s, \quad 0 \leq t \leq T.
\]

Applying Itô-Tanaka’s formula (cf. [16]) to \(|(Y_s^1 - Y_s^2)^2|\), we get

\[
| (Y_t^1 - Y_t^2)^2 | = |(\xi^1 - \xi^2)^2| - 2 \int_t^T (Y_s^1 - Y_s^2)^2 [f^1 (s, Y_s^1, Z_s^1) - f^2 (s, Y_s^2, Z_s^2)] ds \\
- 2 \int_t^T (Y_s^1 - Y_s^2)^2 [g (s, Y_s^1, Z_s^1) - g (s, Y_s^2, Z_s^2)] dB_s \\
+ \int_t^T 1_{\{Y_s^2 \leq Y_s^1\}} [g (s, Y_s^1, Z_s^1) - g (s, Y_s^2, Z_s^2)]^2 ds \\
+ 2 \int_t^T (Y_s^1 - Y_s^2)^2 (Z_s^1 - Z_s^2) dW_s - \int_t^T 1_{\{Y_s^1 \leq Y_s^2\}} |(Z_s^1 - Z_s^2)|^2 ds.
\]

From (H5), we have \(\xi^1 - \xi^2 \geq 0\), so

\[
\mathbb{E}|(\xi^1 - \xi^2)^2| = 0.
\]

Since \((Y^1, Z^1)\) and \((Y^2, Z^2)\) are in \(S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)\), by virtue of Lemma 1.3 of [15], it follows that

\[
\mathbb{E} \int_t^T (Y_s^1 - Y_s^2)^2 (Z_s^1 - Z_s^2) dW_s = 0,
\]

\[
\mathbb{E} \int_t^T (Y_s^1 - Y_s^2)^2 [g (s, Y_s^1, Z_s^1) - g (s, Y_s^2, Z_s^2)] dB_s = 0.
\]

Let

\[
\Delta = -2 \int_t^T (Y_s^1 - Y_s^2)^2 [f^1 (s, Y_s^1, Z_s^1) - f^2 (s, Y_s^2, Z_s^2)] ds = \Delta_1 + \Delta_2,
\]

where

\[
\Delta_1 = -2 \int_t^T (Y_s^1 - Y_s^2)^2 [f^1 (s, Y_s^1, Z_s^1) - f^1 (s, Y_s^2, Z_s^2)] ds,
\]

\[
\Delta_2 = -2 \int_t^T (Y_s^1 - Y_s^2)^2 [f^2 (s, Y_s^2, Z_s^2)] ds \leq 0.
\]
From (H1) and Young’s inequality, it follows that

\[
\Delta \leq \Delta_1 \leq 2c \int_t^T (Y^1_s - Y^2_s)^- (|Y^1_s - Y^2_s| + |Z^1_s - Z^2_s|) \, ds
\]

\[
\leq \left(2c + \frac{c^2}{1-\alpha}\right) \int_t^T (|Y^1_s - Y^2_s|)^2 \, ds + (1-\alpha) \int_t^T 1_{\{Y^1_s \leq Y^2_s\}} |Z^1_s - Z^2_s|^2 \, ds,
\]

where \(c > 0\) only depends on the Lipschitz constant \(C\) in (H1). By the assumption (H2), we deduce

\[
\int_t^T 1_{\{Y^1_s \leq Y^2_s\}} |g(s, Y^1_s, Z^1_s) - g(s, Y^2_s, Z^2_s)|^2 \, ds
\]

\[
\leq \int_t^T 1_{\{Y^1_s \leq Y^2_s\}} \left[C|Y^1_s - Y^2_s|^2 + \alpha|Z^1_s - Z^2_s|^2\right] \, ds
\]

\[
= C \int_t^T (|Y^1_s - Y^2_s|)^2 \, ds + \alpha \int_t^T 1_{\{Y^1_s \leq Y^2_s\}} |Z^1_s - Z^2_s|^2 \, ds.
\]

We get

\[
E[(Y^1_t - Y^2_t)^-]^2 \leq \left(C + 2c + \frac{c^2}{1-\alpha}\right) \int_t^T E[(Y^1_s - Y^2_s)^-]^2 \, ds.
\]

By Gronwall’s inequality, it follows that

\[
E[(Y^1_t - Y^2_t)^-]^2 = 0, \forall t \in [0, T].
\]

That is, \(P\{Y^1_t \geq Y^2_t, \text{ for all } t \in [0, T]\} = 1. \quad \square\]

**Remark 3.2** If BDSDE (2) satisfies (H2)–(H4), and BDSDE (3) satisfies (H1) and (H2), similarly to Theorem 3.1, the comparison result is still true.

In next section, we will prove the maximal solution of BDSDE with continuous coefficients as an application of Theorem 3.1. Next, let us generalize the comparison theorem to the case where the coefficients are only continuous.

**Theorem 3.3** Assume BDSDEs (2) and (3) satisfy (H2)-(H4), respectively. Let \((Y^1, Z^1)\) and \((Y^2, Z^2)\) be the minimal solutions of BDSDEs (2) and (3), respectively. If (H5) holds, then

\[
P\{Y^1_t \geq Y^2_t, \text{ for all } t \in [0, T]\} = 1.
\]

Moreover, let \((\bar{Y}^1, \bar{Z}^1)\) and \((\bar{Y}^2, \bar{Z}^2)\) are the maximal solutions of BDSDEs (2) and (3), respectively. If (H5) holds, then

\[
P\{\bar{Y}^1_t \geq \bar{Y}^2_t, \text{ for all } t \in [0, T]\} = 1.
\]
Proof. First, for fixed $t$, we define, as in Lemma 1 of [11], the sequence $f_n^2(t, y, z)$ associated with $f^2$:

$$f_n^2(t, y, z) = \inf_{(y', z') \in Q \times Q^d} \{ f^2(t, y', z') + n(|y - y'| + |z - z'|) \},$$

where $Q$ is the rational number set. Then, for $n \geq K$, $f_n^2$ are measurable and Lipschitz functions, and $f^1 \geq f^2 \geq f_n^2$. Hence, we know that the following BDSDE has a unique solution $(Y_n^2, Z_n^2)$

$$Y_n^2(t) = \xi^2 + \int_t^T f_n^2(s, Y_n^2(s), Z_n^2(s)) \, ds + \int_t^T g(s, Y_n^2(s), Z_n^2(s)) \, dB_s - \int_t^T Z_n^2(s) \, dW_s, \quad n \geq K.$$ 

From Theorem 3.1, it follows that $Y^1 \geq Y_n^2$ and $Y^2 \geq Y_n^2$ a.s. for all $n \geq K$. However, $(Y_n^2, Z_n^2)$ converges uniformly in $t$ to $(\overline{Y}^2, \overline{Z}^2)$ (cf. [10]). Therefore, $Y^1 \geq \overline{Y}^2$ a.s., in particular, $Y^1 \geq Y^2$ a.s.

Next, for fixed $t$, we define the sequence $f_n^1(t, y, z)$ associated with $f^1$,

$$f_n^1(t, y, z) = \sup_{(y', z') \in Q \times Q^d} \{ f^1(t, y', z') - n(|y - y'| + |z - z'|) \},$$

then, by virtue of Lemma 4.2 in next section, for $n \geq K$, $f_n^1$ are measurable and Lipschitz functions, and $f_n^1 \geq f^1 \geq f^2$. Hence, we know that the following BDSDE has a unique solution $(Y_n^1, Z_n^1)$

$$Y_n^1(t) = \xi^1 + \int_t^T f_n^1(s, Y_n^1(s), Z_n^1(s)) \, ds + \int_t^T g(s, Y_n^1(s), Z_n^1(s)) \, dB_s - \int_t^T Z_n^1(s) \, dW_s, \quad n \geq K.$$ 

From Theorem 3.1, it follows that $Y_n^1 \geq Y^1$ and $Y_n^1 \geq Y^2$ a.s. for all $n \geq K$. However, $(Y_n^1, Z_n^1)$ converges uniformly in $t$ to $(\overline{Y}^1, \overline{Z}^1)$ (cf. Lemma 4.2). Therefore, $\overline{Y}^1 \geq Y^2$ a.s., in particular, $\overline{Y}^1 \geq \overline{Y}^2$ a.s. \hfill \Box

Remark 3.4 Suppose that the conditions of Theorem 3.2 hold.

(i) If (2) has a unique solution $(Y^1, Z^1)$, then for any solutions $(Y^2, Z^2)$ of (3) we have

$$P\{Y_t^1 \geq Y_t^2, \text{ for all } t \in [0, T]\} = 1.$$ 

(ii) If (3) has a unique solution $(Y^2, Z^2)$, then for any solutions $(Y^1, Z^1)$ of (2) we have

$$P\{Y_t^1 \geq Y_t^2, \text{ for all } t \in [0, T]\} = 1.$$ 

(iii) If the uniqueness of neither (2) nor (3) holds, we were unable to derive a comparison result for any solutions of (2) and (3).
In particular, we easily see that the Lipschitz condition is a special case of our proposed conditions. In other words, Theorem 3.1 and Theorem 3.2 generalize the comparison result in [16].

Finally, before closing this section, we shall give a comparison result for the case of Remark 2(iii). For this purpose we need now the following conditions:

(H6) \( f(t, y, z) \) is uniformly Lipschitz continuous in \( z \), i.e.,
\[
|f(t, y, z_1) - f(t, y, z_2)| < C|z_1 - z_2|
\]
for all \( t \in [0, T], y \in R, z_1, z_2 \in R^d \);

(H7) For fixed \( t, z \), \( f(t, \cdot, z) \) is equi-continuous in \( y \), i.e., for every \( \varepsilon > 0 \) there exists a constant \( \delta > 0 \) such that if \( y_1, y_2 \in R \), and \( |y_1 - y_2| < \delta \), then \( |f(t, y_1, z) - f(t, y_2, z)| \leq \varepsilon \) for any \( t \in [0, T] \) and \( z \in R^d \);

(H8) \( |\xi| \leq C_0 \), a.s.

where \( C \) and \( C_0 \) are given positive constants.

(H9) \( f(t, y, z) \) is locally Lipschitz continuous in \( y \), i.e., for each \( N > 0 \), there exists a constant \( L_N \) such that
\[
|f(t, y_1, z) - f(t, y_2, z)| \leq L_N|y_1 - y_2|
\]
for all \( t \in [0, T], z \in R^d, y_1, y_2 \in R \) with \( |y_1|, |y_2| \leq N \).

Moreover, by virtue of the results in [4], (1) is uniquely solvable under the conditions: (H2), (H4), (H6), (H8) and (H9). In view of Remark 2, if one of \( f^1 \) and \( f^2 \) satisfies (H4), (H6), (H8) and (H9), and \( g \) satisfies (H2), the conclusion of Theorem 3.2 still holds.

**Theorem 3.5** Assume (H2), (H4), (H8) and one of \( f^1 \) and \( f^2 \) satisfies (H6) and (H7) with

\[
\text{esssup}_{\{t \in [0, T], y \in R, z \in R^d\}} \{f^1(t, y, z) - f^2(t, y, z)\} < \infty,
\]

and let \((Y^1, Z^1)\) and \((Y^2, Z^2)\) be the solutions of BDSDEs (2) and (3), respectively. Then (H5) implies that
\[
P\{Y^1(t) \geq Y^2(t), \text{for all } t \in [0, T]\} = 1.
\]

**Proof.** The proof is divided into two steps. Without loss of generality, we assume that \( f^2 \) satisfies (H6) and (H7). In the first step, we additionally assume that \( f^2(t, y, z) \) satisfies (H9) and shall prove the result of Theorem 3.3 without the assumption (H7).

Under the assumptions (H2), (H4), (H6), (H8) and (H9), (1) with \( (\xi^2, f^2) \) has a unique solution by virtue of Theorem 2.1 of Gu [4]. An application of Remark 2 yields the desired result.

The second step, it remains to remove the additional condition (H9). Fix \( N > 0 \) and choose a Lipschitz continuous mapping \( b(t, y, z) \) in \( y \) and \( z \) such that
\[
f^1(t, y, z) > b(t, y, z) > f^2(t, y, z)
\]
for \( t \in [0, T], y \in R \) with \( |y| < N, z \in R^d \).
From (H10), there exists a constant \( \bar{\varepsilon} > 0 \) such that

\[
\text{esssup}_{t \in [0,T], y \in \mathbb{R}, z \in \mathbb{R}^d} \{ f^1(t, y, z) - f^2(t, y, z) \} = \bar{\varepsilon}.
\]

To show the existence of such a mapping \( b(t, y, z) \), we write \( \bar{b}(t, y, z) := \frac{f^2(t, y, z) + \frac{\bar{\varepsilon}}{2}}{2} \).

\[
J(y) = \begin{cases} 
    k \exp(-(1 - |y|)^{-1}), & \text{for } |y| < 1, \\
    0, & \text{otherwise},
\end{cases}
\]
and \( J_\delta(y) = \delta^{-1}J(y/\delta) \), where the constant \( k \) satisfies \( \int_{\mathbb{R}} J(y)dy = 1 \). Let us smooth out \( \bar{b} \) in \( y \) to obtain \( b_\delta \), i.e., setting

\[
b_\delta(t, y, z) = \int_{\mathbb{R}} \bar{b}(t, x, z)J_\delta(y - x)dx,
\]

where \( \bar{\varepsilon} \) and \( \delta \) are as in (H7). Now any \( b_\delta' \) with \( \delta' \leq \delta \) is our candidate.

Let \( (Y, Z) \) denote a solution of (1) when \( f \) is replaced by \( b \). By the above consideration, we observe

\[
Y^1(t) \geq Y(t) \geq Y^2(t), \text{ a.s. } \forall t \in [0, T].
\]

As a result of that \( N > 0 \) is arbitrary, the desired result is obtained. \( \square \)

**Remark 3.6** (i) Theorem 3.3 gives a comparison result for any solutions of (2) and (3).

(ii) If one of (2) and (3) has a unique solution, Theorem 3.2 implies Theorem 3.3. Otherwise, Theorem 3.3 is not a special case of Theorem 3.2.

### 4 The existence of the maximal solution of BDSDEs

Under the conditions (H2)-(H4), Shi, Gu and Liu [15] have proved that BDSDE (1) has the minimal solution \((Y, Z)\). Now, we will prove that BDSDE (1) has also the maximal solution \((\overline{Y}, \overline{Z})\).

**Theorem 4.1** Assume \( f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) and \( g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^l \) are jointly measurable functions and satisfy (H2)-(H4). Then, if \( \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}) \), BDSDE (1) has a solution \((Y, Z) \in S^2([0,T]; \mathbb{R}) \times M^2(0,T; \mathbb{R}^d) \). Moreover, there is a maximal solution \((\overline{Y}, \overline{Z})\) of BDSDE (1) in the sense that, for any other solution \((Y, Z)\) of BDSDE (1), we have \( \overline{Y} \geq Y \).

In order to prove Theorem 4.1, we need the following lemma which can be obtained by means of the similar arguments in [11], so we omit its proof.

**Lemma 4.2** Let \( f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) be a continuous function with linear growth, that is there exist positive constants \( K, D \), such that \( \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, |f(y, z)| \leq K|y| + K|z| + D \). Then the sequence of functions

\[
f_n(y, z) = \sup_{(y', z') \in Q \times Q^d} \{ f(y', z') - n(|y - y'| + |z - z'|) \}
\]

are in \( L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}) \). The sequence \( f_n \) is uniformly bounded in \( L^1(\Omega, \mathcal{F}_T, P; \mathbb{R}) \) and \( f_n \) is uniformly bounded in \( L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}) \). To prove Theorem 4.1, we need the following lemma which can be obtained by means of the similar arguments in [11], so we omit its proof.
is well defined for $n \geq K$ and it satisfies

(i) linear growth: $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$, $|f_n(y, z)| \leq K|y| + K|z| + D$;

(ii) monotonicity in $n$: $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$, $f_n(y, z) \leq f_{n'}(y, z)$;

(iii) Lipschitz condition: $\forall (y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d$,

$$|f_n(y, z) - f_n(y', z')| \leq n(|y - y'| + |z - z'|);$$

(iv) strong convergence: if $(y_n, z_n) \longrightarrow (y, z)$ then $f_n(y_n, z_n) \longrightarrow f(y, z), n \to \infty$.

Consider, for fixed $(\omega, t)$, the sequence $f_n(\omega, t, y, z)$ associated to $f$ by Lemma 4.2. Also consider $F(\omega, t, y, z) = -|f(\omega, t, 0, 0)| - K|y| - K|z|$. Then $f_n$ and $F(\omega, t, y, z)$ are jointly measurable Lipschitz functions. Given $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, by Proposition 1, there exist two pairs of processes $(Y^n, Z^n)$ and $(U, V)$ which are the solutions of the following BDSDEs (4) and (5), respectively,

$$Y_t^n = \xi + \int_t^T f_n(s, Y^n_s, Z^n_s)ds + \int_t^T g(s, Y^n_s, Z^n_s)dB_s - \int_t^T Z^n_s dW_s, \quad (4)$$
$$U_t = \xi + \int_t^T F(s, U_s, V_s)ds + \int_t^T g(s, U_s, V_s)dB_s - \int_t^T V_s dW_s. \quad (5)$$

From the comparison theorem of [10] and Lemma 4.2, we get

$$\forall n \geq m \geq K, \quad Y^n \geq Y^m \geq U, \quad dt \otimes dP\text{-a.s.} \quad (6)$$

**Lemma 4.3** There exists a constant $A > 0$ depending only on $K$, $C$, $\alpha$, $T$ and $\xi$, such that

$$\forall n \geq K, \quad \|Y^n\|_{S^2} \leq A, \quad \|Z^n\|_{M^2} \leq A, \quad \|U\|_{S^2} \leq A, \quad \|V\|_{M^2} \leq A.$$

**Proof.** First of all, we prove that $\|U\|_{S^2}$ and $\|V\|_{M^2}$ are all bounded. Clearly, from (6), there exists a constant $B$ depending only on $K$, $C$, $\alpha$, $T$ and $\xi$, such that

$$\left( \mathbb{E} \int_0^T |Y^n_s|^2 ds \right)^{1/2} \leq B, \quad \left( \mathbb{E} \int_0^T |U_s|^2 ds \right)^{1/2} \leq B, \quad \|V\|_{M^2} \leq B.$$

Applying Itô’s formula to $|U_t|^2$, we have

$$|U_t|^2 = |\xi|^2 + \int_t^T U_s \cdot F(s, U_s, V_s)ds + \int_t^T U_s \cdot g(s, U_s, V_s)dB_s$$

$$-2 \int_t^T U_s \cdot V_s dW_s + \int_t^T |g(s, U_s, V_s)|^2 ds - \int_t^T |V_s|^2 ds. \quad (7)$$

From (H2), for all $\alpha < \alpha' < 1$, there exists a constant $C(\alpha') > 0$ such that

$$|g(t, u, v)|^2 \leq C(\alpha') (|u|^2 + |g(t, 0, 0)|^2) + \alpha'|v|^2. \quad (8)$$
From (7) and (8), it follows that
\[
|U_t|^2 + \int_t^T |V_s|^2 \, ds \leq |\xi|^2 + 2K \int_t^T |U_s|(|f(s,0,0)| + |U_s| + |V_s|) \, ds \\
+ C(\alpha') \int_t^T (|U_s|^2 + |g(s,0,0)|^2) \, ds + \alpha' \int_t^T |V_s|^2 \, ds \\
+ 2 \int_t^T U_s \cdot g(s,U_s,V_s) \, dB_s - 2 \int_t^T U_s \cdot V_s \, dW_s \\
\leq |\xi|^2 + K^2 D(T-t) + C(\alpha') \int_t^T |g(s,0,0)|^2 \, ds \\
+ \frac{1+\alpha'}{2} \int_t^T |V_s|^2 \, ds \\
+ \left( 1 + 2K + C(\alpha') + \frac{2K^2}{1-\alpha'} \right) \int_t^T |U_s|^2 \, ds \\
+ 2 \int_t^T U_s \cdot g(s,U_s,V_s) \, dB_s - 2 \int_t^T U_s \cdot V_s \, dW_s.
\]
Taking supremum and expectation, by Young’s inequality, we get
\[
\|U\|_{S^2}^2 + \frac{1-\alpha'}{2} \|V\|_{M^2}^2 \leq E \left( |\xi|^2 + K^2 T D + C(\alpha') \int_0^T |g(s,0,0)|^2 \, ds \right) \\
+ \left( 1 + 2K + C(\alpha') + \frac{2K^2}{1-\alpha'} \right) E \int_0^T |U_s|^2 \, ds \\
+ 2E \sup_{0 \leq t \leq T} \left| \int_t^T U_s \cdot g(s,U_s,V_s) \, dB_s \right| \\
+ 2E \sup_{0 \leq t \leq T} \left| \int_t^T U_s \cdot V_s \, dW_s \right|.
\tag{9}
\]
By Burkholder-Davis-Gundy’s inequality (cf. [15], [16]), we deduce
\[
E \left( \sup_{0 \leq t \leq T} \left| \int_t^T U_s \cdot g(s,U_s,V_s) \, dB_s \right| \right) \\
\leq cE \left( \int_0^T |U_s|^2 \cdot |g(s,U_s,V_s)|^2 \, ds \right)^{1/2} \\
\leq cE \left( \left( \sup_{0 \leq t \leq T} |U_t|^2 \right)^{1/2} \left( \int_0^T |g(s,U_s,V_s)|^2 \, ds \right)^{1/2} \right) \\
\leq 2c^2 C(\alpha') E \left( \int_0^T |U_s|^2 \, ds + \int_0^T |g(s,0,0)|^2 \, ds \right) \\
+ \frac{1}{8} \|U\|_{S^2}^2 + 2c^2 \alpha' \|V\|_{M^2}^2.
\tag{10}
\]
In the same way, we have
\[
E \left( \sup_{0 \leq t \leq T} \left| \int_t^T U_s \cdot V_s \, dW_s \right| \right) \leq \frac{1}{8} \|U\|_{S^2}^2 + 2c^2 \|V\|_{M^2}^2.
\tag{11}
\]
From (10), (11) and (9), it follows that

\[ \parallel U \parallel^2_{S^2} + \frac{1 - \alpha'}{2} \parallel V \parallel^2_{M^2} \]

\[ \leq 2 \left( \mathbb{E}[|\xi|^2 + K^2 T D + C(\alpha')(1 + 4c^2)] \mathbb{E} \int_0^T |g(s,0,0)|^2 ds \right) \]

\[ + 2 \left( 1 + 2K + \frac{2K^2}{1 - \alpha'} + C(\alpha')(1 + 4c^2) \right) \mathbb{E} \int_0^T |U_s|^2 ds \]

\[ + 8c^2(1 + \alpha') \parallel V \parallel^2_{M^2} \]

\[ \leq 2 \left( \mathbb{E}[|\xi|^2 + K^2 T D + C(\alpha')(1 + 4c^2)] \mathbb{E} \int_0^T |g(s,0,0)|^2 ds \right) \]

\[ + 2 \left( 1 + 2K + \frac{2K^2}{1 - \alpha'} + C(\alpha')(1 + 4c^2) + 4c^2(1 + \alpha') \right) B^2 \]

\[ := \frac{1 - \alpha'}{2} (B')^2, \]

that is

\[ \parallel U \parallel^2_{S^2} \leq B', \quad \parallel V \parallel^2_{M^2} \leq B'. \]

From (6), it easily follows that

\[ \parallel Y^n \parallel^2_{S^2} \leq B'. \]

Next, we prove the boundedness of \( \parallel Z^n \parallel^2_{M^2} \). Applying Itô’s formula to \( |Y^n_t|^2 \), it follows that

\[ |Y^n_t|^2 = |\xi|^2 + 2 \int_t^T Y^n_s \cdot f_n(s,Y^n_s,Z^n_s) ds + 2 \int_t^T Y^n_s \cdot g(s,Y^n_s,Z^n_s) dB_s \]

\[ - 2 \int_t^T Y^n_s \cdot Z^n_s dB_s + \int_t^T |g(s,Y^n_s,Z^n_s)|^2 ds - \int_t^T |Z^n_s|^2 ds. \]

Taking expectation, we have

\[ \mathbb{E}(|Y^n_t|^2) + \mathbb{E} \int_t^T |Z^n_s|^2 ds = \mathbb{E}[|\xi|^2] + 2\mathbb{E} \int_t^T Y^n_s \cdot f_n(s,Y^n_s,Z^n_s) ds \]

\[ + \mathbb{E} \int_t^T |g(s,Y^n_s,Z^n_s)|^2 ds. \]

From Young’s inequality, it follows that

\[ \mathbb{E}(|Y^n_t|^2) + \mathbb{E} \int_t^T |Z^n_s|^2 ds \]

\[ \leq \mathbb{E}[|\xi|^2] + C' \mathbb{E} \int_t^T |Y^n_t|^2 ds + \frac{1 - \alpha'}{2} \mathbb{E} \int_t^T |Z^n_t|^2 ds \]

\[ + K^2 D (T - t) + C(\alpha') \mathbb{E} \int_t^T |g(s,0,0)|^2 ds + \alpha' \mathbb{E} \int_t^T |Z^n_s|^2 ds, \]
where \( C' = 1 + 2K + C(\alpha') + \frac{2K^2}{1 - \alpha'}, \) and we know \( 0 < \alpha' < 1 \) from (8). Then

\[
\|Z^n\|_{M^2}^2 \leq \frac{2}{1 - \alpha'} \left( C'T(B')^2 + K^2 DT + E|\xi|^2 + C(\alpha')E\int_0^T |g(s,0,0)|^2 ds \right) := (A)^2.
\]

The proof is completed. \( \square \)

**Lemma 4.4** \( \{Y^n, Z^n\}_{n=1}^{+\infty} \) converges in \( S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \).

**Proof.** Let \( n_0 \geq K \). Since \( \{Y^n\} \) is decreasing and bounded in \( S^2([0,T];\mathbb{R}) \), we deduce from the dominated convergence theorem that \( Y^n \) converges in \( S^2([0,T];\mathbb{R}) \).

We shall denote by \( Y \) the limit of \( \{Y^n\} \). Applying Itô’s formula to \( |Y^n_t - Y^m_t|^2 \), we get for \( n, m \geq n_0 \),

\[
E(|Y^n_0 - Y^m_0|^2) + E\int_0^T |Z^n_s - Z^m_s|^2 ds
= 2E\int_0^T (Y^n_s - Y^m_s)(f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s))ds
+ E\int_0^T |g(s, Y^n_s, Z^n_s) - g(s, Y^m_s, Z^m_s)|^2 ds
\leq 2 \left( E\int_0^T |Y^n_s - Y^m_s|^2 ds \right)^{\frac{1}{2}} \left( E\int_0^T |f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)|^2 ds \right)^{\frac{1}{2}}
+ E\int_0^T (C|Y^n_s - Y^m_s|^2 + \alpha|Z^n_s - Z^m_s|^2)ds.
\]

Since \( f_n \) and \( f_m \) are uniformly linear growth and \( \{(Y^n, Z^n)\} \) is bounded, similarly to Lemma 4.3, there exists a constant \( K > 0 \) depending only on \( K, C, \alpha, T \) and \( \xi \), such that

\[
E(|Y^n_0 - Y^m_0|^2) + E\int_0^T |Z^n_s - Z^m_s|^2 ds
\leq E\int_0^T (K|Y^n_s - Y^m_s|^2 + \alpha|Z^n_s - Z^m_s|^2)ds.
\]

So

\[
\|Z^n - Z^m\|_{M^2}^2 \leq \frac{KT}{1 - \alpha'}\|Y^n - Y^m\|_{S^2}^2.
\]

Thus \( \{Z^n\} \) is a Cauchy sequence in \( M^2(0,T;\mathbb{R}^d) \), from which the result follows. \( \square \)

**Proof.** [Proof of Theorem 4.1] For all \( n \geq n_0 \geq K \) we have \( Y^{n_0} \geq Y^n \geq U \), and \( \{Y^n\} \) converges in \( S^2([0,T];\mathbb{R}) \), \( dt \otimes dP \)-a.s. to \( Y \in S^2([0,T];\mathbb{R}) \).

On the other hand, since \( Z^n \) converges in \( M^2(0,T;\mathbb{R}^d) \) to \( Z \), we can assume, choosing a subsequence if needed, that \( Z^n \to Z \) \( dt \otimes dP \)-a.s. and \( G = \sup_n |Z^n| \)
is \( dt \otimes dP \) integrable. Therefore, from (i) and (iv) of Lemma 4.2 we get for almost all \( \omega \),

\[
  f_n(t, Y^n_t, Z^n_t) \to f(t, Y_t, Z_t), \quad (n \to \infty) \quad \text{dt-a.e.}
\]

\[
  |f_n(t, Y^n_t, Z^n_t)| \leq |f(t, 0, 0)| + K \sup_n |Y^n_t| + K \sup_n |Z^n_t|
\]

\[
  = |f(t, 0, 0)| + K \sup_n |Y^n_t| + KG_t.
\]

Thus, for almost all \( \omega \) and uniformly in \( t \), it holds that

\[
  \int_t^T f_n(s, Y^n_s, Z^n_s) ds \to \int_t^T f(s, Y_s, Z_s) ds, \quad (n \to \infty).
\]

From the continuity properties of the stochastic integral, it follows that

\[
  \sup_{0 \leq t \leq T} \left| \int_t^T Z^n_s dW_s - \int_t^T Z_s dW_s \right| \to 0 \quad \text{in probability},
\]

\[
  \sup_{0 \leq t \leq T} \left| \int_t^T g(s, Y^n_s, Z^n_s) dB_s - \int_t^T g(s, Y_s, Z_s) dB_s \right| \to 0 \quad \text{in probability}.
\]

Choosing, again, a subsequence, we can assume that the above convergence is \( P \)-a.s. Finally, we have

\[
  |Y^n_t - Y^m_t| \leq \int_t^T |f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)| ds
\]

\[
  + \left| \int_t^T g(s, Y^n_s, Z^n_s) dB_s - \int_t^T g(s, Y^m_s, Z^m_s) dB_s \right|
\]

\[
  + \left| \int_t^T Z^n_s dW_s - \int_t^T Z^m_s dW_s \right|,
\]

and taking limits on \( m \) and supremum over \( t \), we get

\[
  \sup_{0 \leq t \leq T} |Y^n_t - Y_t| \leq \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| ds
\]

\[
  + \sup_{0 \leq t \leq T} \left| \int_t^T g(s, Y^n_s, Z^n_s) dB_s - \int_t^T g(s, Y_s, Z_s) dB_s \right|
\]

\[
  + \sup_{0 \leq t \leq T} \left| \int_t^T Z^n_s dW_s - \int_t^T Z_s dW_s \right|, \quad P\text{-a.s.}
\]

from which it follows that \( Y^n \) converges uniformly in \( t \) to \( Y \) (in particular, \( Y \) is a continuous process). Note that \( \{Y^n\} \) is monotone; therefore, we actually have the uniform convergence for the entire sequence and not just for a subsequence.

Taking limits in BDSDE (4), we deduce that \( (Y, Z) \) is a solution of BDSDE (1).

Let \((\hat{Y}, \hat{Z}) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)\) be any solution of BDSDE (1). From Theorem 3.1, we get that \( Y^n \geq \hat{Y}, \forall n \in N \) and therefore \( Y \geq \hat{Y} \) proves that \( Y \) is the maximal solution. \( \square \)
5 Knobbe-theorem for BDSDEs

In this section, we shall discuss an interesting question: How many solutions does a one-dimensional BDSDE (1) satisfying (H2)-(H4) have? This is a classical Knobbe-theorem problem. Under some appropriate conditions, we shall prove a Knobbe-theorem for BDSDEs satisfying (H2)-(H4).

Theorem 5.1 (Knobbe-theorem for BDSDEs) We assume (H2)-(H4). Let \((\tilde{Y}_t, \tilde{Z}_t) \in S^2([0, T] ; \mathbb{R}) \times M^2(0, T ; \mathbb{R}^d)\) and \((\bar{Y}_t, \bar{Z}_t) \in S^2([0, T] ; \mathbb{R}) \times M^2(0, T ; \mathbb{R}^d)\) be the minimal and maximal solutions of BDSDE (1) with the terminal condition \(\xi \in L^2(\Omega, \mathcal{F}_T, P ; \mathbb{R})\). We also assume there exists a function \(z = h(t, y, \tilde{z}) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d\), which is the inverse function of \(\tilde{z} = g(t, y, z)\) with respect to \(z\), and \(h(t, y, z)\) satisfies (H2) with respect to \(t, y, \tilde{z}\). Then for any \(t_0 \in [0, T]\) and \(\eta \in L^2(\Omega, \mathcal{F}_{t_0}, P ; \mathbb{R})\) such that

\[ \underline{\eta}_0 \leq \eta \leq \overline{\eta}_0, \quad P\text{-a.s.,} \]

there is at least one solution \((Y_t, Z_t)_{t \in [0, T]} \in S^2([0, T] ; \mathbb{R}) \times M^2(0, T ; \mathbb{R}^d)\) of BDSDE (1) satisfying

\[ Y_{t_0} = \eta, \quad P\text{-a.s.,} \]

Moreover the set of solutions of BDSDE (1) is closed in \(S^2([0, T] ; \mathbb{R}) \times M^2(0, T ; \mathbb{R}^d)\). That is, for any solution \((Y^n_t, Z^n_t)_{t \in [0, T]}\) of BDSDE (1) for \(n = 1, \ldots, \), if \((Y^n_t, Z^n_t) \rightarrow (Y, Z)\) in \(S^2([0, T] ; \mathbb{R}) \times M^2(0, T ; \mathbb{R}^d)\) as \(n \rightarrow \infty\), then \((Y, Z)\) is also a solution of BDSDE (1).

Proof. Let \((Y^1_t, Z^1_t)_{t \in [0, t_0]} \in S^2([0, t_0] ; \mathbb{R}) \times M^2(0, t_0 ; \mathbb{R}^d)\) be a solution of the following BDSDE

\[ Y^1_t = \eta + \int_t^{t_0} f(s, Y^1_s, Z^1_s) ds + \int_t^{t_0} g(s, Y^1_s, Z^1_s) dB_s - \int_t^{t_0} Z^1_s dW_s, \quad t \in [0, t_0]. \]

Consider the following forward doubly stochastic differential equation

\[ Y^2_t = \eta - \int_t^{t_0} f(s, Y^2_s, \bar{Z}^2_s) ds - \int_t^{t_0} \bar{Z}^2_s dB_s + \int_t^{t_0} h(s, Y^2_s, \bar{Z}^2_s) dW_s, \quad t \in [t_0, T]. \tag{12} \]

By the assumptions, it is not difficult to see that (12) is a forward "BDSDE" on \([t_0, T]\), which satisfies the conditions of Theorem 4.1. Consequently, there is a solution \((Y^2_t, \bar{Z}^2_t)_{t \in [t_0, T]} \in S^2([t_0, T] ; \mathbb{R}) \times M^2(t_0, T ; \mathbb{R}^d)\) satisfying (12). We can define a \(\mathcal{F}_t\)-measurable time

\[ \tau = \inf\{t \geq t_0, Y^2_t \notin (\underline{Y}_t, \overline{Y}_t)\}. \]

By \(Y_T = \overline{Y}_T = \xi\), we know that \(\tau \leq T\). Now we define on \([0, T]\)

\[ (Y_t, Z_t) = 1_{[0, \tau]}(t)(Y^1_t, Z^1_t) + 1_{(\tau, T]}(t)(Y^2_t, h(t, Y^2_t, \bar{Z}^2_t)) + 1_{[\tau, T]}(t)(\underline{Y}_t, \underline{Z}_t)1_{\{Y^2_{\tau} = \underline{Y}_\tau\}} + 1_{[\tau, T]}(t)(\overline{Y}_t, \overline{Z}_t)1_{\{Y^2_{\tau} = \overline{Y}_\tau\}}. \]
It is easy to see that \((Y_t, Z_t)_{t \in [0,T]} \in S^2([0,T]; \mathbb{R}) \times M^2(0,T; \mathbb{R}^d)\) is a solution of BDSDE (1) with \(Y_T = \xi\) and \(Y_{t_0} = \eta\).

By the similar arguments in Theorem 4.1, it is easy to check the closedness of the set of solutions of BDSDE (1) by the continuity of \(f\) with respect to \(y\) and \(z\). □

**Corollary 5.2** Indeed, in the case when the solution of BDSDE (1) is not unique, the cardinality of the set of the associated solutions is at least of continuum since we can take

\[
\eta = \alpha Y_{t_0} + (1 - \alpha)Y_{t_0}
\]

for each \(\alpha \in [0,1]\). Thus Theorem 5.1 is a Kneser-type theorem for BDSDEs.

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