On Well-posedness of Stochastic Anisotropic $p$-Laplace Equation Driven by Lévy Noise

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Abstract
Lions in Lions (1969) solved the anisotropic $p$-Laplace equation in deterministic setting by considering the anisotropic $p$-Laplace operator in $d$-dimensions as a sum of $d$ monotone coercive operators each defined on a different space. Motivated by this example, we prove existence and uniqueness results for a large class of stochastic partial differential equations (SPDEs) driven by Lévy noise when the operator appearing in the bounded variation term is a sum of operators having different analytic and growth properties. Further, the operators are allowed to be locally monotone without explicitly restricting the growth of the operators appearing in the stochastic integrals. This has been done by identifying an appropriate coercivity condition. As a consequence, well-posedness of Lévy driven stochastic Anisotropic $p$-Laplace equation has been shown. Our framework is most general till date. Many popular SPDEs appearing in real world models such as the stochastic Ginzburg–Landau equation and stochastic Swift–Hohenberg equation, both driven by Lévy noise, fit in our setting. These equations are not covered by the corresponding results in the literature.

Keywords Anisotropic $p$-Laplace equation · Stochastic Swift–Hohenberg equation · Stochastic Ginzburg–Landau equation · Stochastic partial differential equations · Coercivity · Local Monotonicity · Lévy noise

Mathematics Subject Classification (2010) 60H15 · 65M60 · 47J35

1 Introduction and Main Result
We establish the well-posedness of stochastic anisotropic $p$-Laplace equation driven by Lévy noise defined by the following equation,

$$
du_t = \sum_{i=1}^{d} D_i (|D_i u_t|^{p_i-2} D_i u_t) dt + \sum_{j=1}^{d} \zeta_j |D_j u_t|^{p_j/2} dW^j_t
$$
\[ + \sum_{j=1}^{\infty} h_j(u_t) dW_j^j + \int_{\mathcal{D}} \gamma_t(u_t, z) \tilde{N}(dt, dz) \\
+ \int_{\mathcal{D}} \gamma_t(u_t, z) N(dt, dz) \quad \text{on} \quad (0, T) \times \mathcal{D}, \tag{1.1} \]

where \( u_t = 0 \) on boundary of an open bounded domain \( \mathcal{D} \subseteq \mathbb{R}^d \) and \( u_0 \) is a given initial condition. Here, for \( i \in \{1, 2, \ldots, d\} \), \( D_i \) denotes the distributional derivative along the \( i \)-th coordinate in \( \mathbb{R}^d \). Further, \( p_i \geq 2 \) are real numbers, \( \xi_j \) are constants and \( W^j \) are independent Wiener processes on a right continuous complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})\). Also, \( N(dt, dz) \) is a Poisson random measure defined on a \( \sigma \)-finite measure space \((Z, \mathcal{Z}, \nu)\) with intensity \( \nu \) and \( \tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt \) is the compensated Poisson random measure. Note that the Poisson random measure \( N(dt, dz) \) is independent of the Weiner processes \( W^j \). Further, \( \mathcal{D} \in \mathcal{Z} \) is such that \( \nu(\mathcal{D}) < \infty \) and \( \mathcal{D}^c = Z \setminus \mathcal{D} \). The term anisotropic signifies that the parameter \( p \) in the \( p \)-Laplace operator takes different values in different directions, which is evident from the drift term of Eq. 1.1 as \( p_i \)'s can be different. The precise assumptions on the functions \( h_j \) and \( \gamma \) are given in Theorem 1.

Solvability of anisotropic \( p \)-Laplace equation in deterministic setting, i.e.

\[ du_t = \sum_{i=1}^{d} D_i (|D_i u_t|^{p_i-2} D_i u_t) \, dt \quad \text{on} \quad (0, T) \times \mathcal{D}, \quad u_t = 0 \quad \text{on} \quad \partial \mathcal{D} \tag{1.2} \]

has been studied in Lions [8]. Note that if \( p_i = p \) for all \( i \), then a solution to Eq. 1.2 can be found in the Banach space defined by

\[ W^{1,p}_0 (\mathcal{D}) := \{ u | u, D_i u \in L^p(\mathcal{D}), i = 1, 2, \ldots, d; \ u = 0 \ \text{on} \ \partial \mathcal{D} \}. \]

By solution we mean a function \( u \in L^p((0, T); W^{1,p}_0(\mathcal{D})) \) such that for every \( t \in [0, T] \) and \( \phi \in W^{1,p}_0(\mathcal{D}) \),

\[ \int_{\mathcal{D}} u_t(x) \phi(x) dx = \int_{\mathcal{D}} u_0(x) \phi(x) dx - \sum_{i=1}^{d} \int_{0}^{t} \int_{\mathcal{D}} |D_i u_s(x)|^{p-2} D_i u_s(x) D_i \phi(x) dx ds. \]

The proof of existence of a solution to PDE (1.2), with \( p_i = p \) for all \( i \), uses the coercivity of the operator \( \sum_{i=1}^{d} D_i (|D_i u|^{p_i-2} D_i u) \), which means there exists a constant \( \theta > 0 \), known as coefficient of coercivity, such that

\[ -\sum_{i=1}^{d} \int_{\mathcal{D}} |D_i u(x)|^p dx \leq -\theta |u|^p_{W^{1,p}_0}. \]

However, when \( p_i \)'s are different, we can not mimic the above argument as we can not find a \( p \) and a space \( X \) such that

\[ -\sum_{i=1}^{d} \int_{\mathcal{D}} |D_i u(x)|^{p_i} dx \leq -\theta |u|^p_X. \]

holds. To tackle this problem, Lions [8] considered the anisotropic \( p \)-Laplace operator \( \sum_{i=1}^{d} D_i (|D_i u|^{p_i-2} D_i u) \) as a sum of \( d \) operators \( D_i (|D_i u|^{p_i-2} D_i u), \ i = 1, 2, \ldots, d \), where each operator satisfies the coercivity condition with different \( p_i, \theta_i \) and the space \( X_i \), let’s call it anisotropic coercivity condition. Then from the appropriate energy equality

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and anisotropic coercivity condition we get the required a priori estimates. The usual compactness and monotonicity arguments lead to existence of a unique solution of Eq. 1.2 in the space \( \cap_{i=1}^{d} \mathcal{L}^{p}_{\Omega} ((0, T); W_{0}^{1, \pi} (\mathcal{D})) \). Paradox [12] generalized the method of monotone coercive operators used by Lions and the results from [12] can be applied to solve anisotropic \( p \)-Laplace equation driven by Wiener process.

In this article, the technique used in [8] is extended to cover the case of anisotropic \( p \)-Laplace equation (1.1) driven by Lévy noise and a unique solution is obtained in the space
\[
W_{0}^{1, \pi} (\mathcal{D}) := \{ u | u \in \mathbb{L}^{2} (\mathcal{D}), D_{i} u \in \mathcal{L}^{p}_{\Omega} (\mathcal{D}), \; i = 1, 2, \ldots, d; \; u = 0 \text{ on } \partial \mathcal{D} \}.
\]
We now describe the result in detail.

Let \( \mathbb{R}^{d} \) be a \( d \)-dimensional Euclidean space and \( \mathcal{D} \subseteq \mathbb{R}^{d} \) be an open bounded domain with smooth boundary. For any \( p \geq 1 \), \( \mathcal{L}^{p} (\mathcal{D}) \) is the Lebesgue space of equivalence classes of real valued measurable functions \( u \) defined on \( \mathcal{D} \) such that the norm
\[
|u|_{\mathcal{L}^{p}} := \left( \int_{\mathcal{D}} |u(x)|^{p} dx \right)^{\frac{1}{p}}
\]
is finite. Further for \( p_{i} \geq 2 \), consider the spaces
\[
W^{x_{i}, p_{i}} (\mathcal{D}) := \{ u | u \in \mathbb{L}^{2} (\mathcal{D}), D_{i} u \in \mathcal{L}^{p}_{\Omega} (\mathcal{D}) \}.
\]
It is then easy to check that the space \( W^{x_{i}, p_{i}} (\mathcal{D}) \) with the norm
\[
|u|_{i, p_{i}} := |u|_{\mathbb{L}^{2}} + |u|_{i, p_{i}}
\]
is a Banach space, where \( |u|_{i, p_{i}} := |D_{i} u|_{\mathcal{L}^{p_{i}}} \) is a semi-norm. Let \( C_{0}^{\infty} (\mathcal{D}) \) be the space of smooth functions with compact support in \( \mathcal{D} \) and \( W^{x_{i}, p_{i}} (\mathcal{D}) \) be its closure in \( W^{x_{i}, p_{i}} (\mathcal{D}) \). It can be seen that each \( W^{x_{i}, p_{i}} (\mathcal{D}) \) is a separable and reflexive Banach space and \( W_{0}^{1, \pi} (\mathcal{D}) = \cap_{i=1}^{d} W^{x_{i}, p_{i}} (\mathcal{D}) \) is embedded continuously and densely in the space \( \mathcal{L}^{2} (\mathcal{D}) \).

Let \( \mathcal{B} \) be the predictable \( \sigma \)-algebra on \( [0, T] \times \Omega \) and \( \mathcal{B}(W_{0}^{1, \pi} (\mathcal{D})) \) be the Borel \( \sigma \)-algebra on \( W_{0}^{1, \pi} (\mathcal{D}) \). Let \( \gamma : [0, T] \times \Omega \times W_{0}^{1, \pi} (\mathcal{D}) \times \mathcal{Z} \rightarrow \mathcal{L}^{2} (\mathcal{D}) \) be a \( \mathcal{P} \times \mathcal{B}(W_{0}^{1, \pi}) \times \mathcal{Z} \)-measurable function. Finally, \( u_{0} \) is assumed to be a given \( \mathcal{L}^{2} (\mathcal{D}) \)-valued, \( \mathcal{F}_{0} \)-measurable random variable.

Throughout the article, \( C \) is a generic constant that may change from line to line. Further, for a given constant \( p \in [1, \infty) \), \( \mathcal{L}^{p} (\Omega; X) \) denotes the Bochner–Lebesgue space of equivalence classes of random variables \( x \) taking values in a Banach space \( X \) such that the norm
\[
|x|_{\mathcal{L}^{p} (\Omega; X)} := (\mathbb{E} |x|_{X}^{p})^{\frac{1}{p}}
\]
is finite and \( \mathcal{L}^{p} ((0, T); X) \) denotes the Bochner–Lebesgue space of equivalence classes of \( X \)-valued measurable functions such that the norm
\[
|x|_{\mathcal{L}^{p} ((0, T); X)} := \left( \int_{0}^{T} |x_{t}|_{X}^{p} dt \right)^{\frac{1}{p}} < \infty.
\]
Again, \( \mathcal{L}^{p} ((0, T) \times \Omega; X) \) denotes the Bochner–Lebesgue space of equivalence classes of \( X \)-valued stochastic processes which are progressively measurable and the norm
\[
|x|_{\mathcal{L}^{p} ((0, T) \times \Omega; X)} := \left( \mathbb{E} \int_{0}^{T} |x_{t}|_{X}^{p} dt \right)^{\frac{1}{p}}
\]
is finite. Finally, \( \mathcal{D} ((0, T]; X) \) denotes the space of \( X \)-valued càdlàg functions.

**Definition 1.1** (Solution) An adapted, càdlàg, \( \mathcal{L}^{2} (\mathcal{D}) \)-valued process \( u \) is called a solution of the stochastic anisotropic \( p \)-Laplace equation (1.1) if
i) $dt \times \mathbb{P}$ almost everywhere $u \in W^{1,p}_{0}(\mathcal{D})$ and
\[
\mathbb{E}\int_{0}^{T}\int_{\mathcal{D}} \left( |u_{t}(x)|^{2} + \sum_{i=1}^{d} |D_{i}u_{t}(x)|^{p_{i}} \right) dx dt < \infty,
\]

ii) for every $t \in [0, T]$ and $\phi \in W^{1,p}_{0}(\mathcal{D})$, almost surely
\[
\int_{\mathcal{D}} u_{t}(x) \phi(x) dx = \int_{\mathcal{D}} u_{0}(x) \phi(x) dx - \sum_{i=1}^{d} \int_{0}^{t} \int_{\mathcal{D}} D_{i}u_{s}(x) \phi(x) dx ds + \sum_{j=1}^{\infty} \int_{0}^{t} \int_{\mathcal{D}} \phi(x) \gamma_{j}(u_{s}(x), z) dx d\tilde{N}(ds, dz) + \sum_{j=1}^{\infty} \int_{0}^{t} \int_{\mathcal{D}} \phi(x) \gamma_{j}(u_{s}(x), z) dx N(ds, dz).
\]

We end this section by formulating the result regarding well-posedness of stochastic anisotropic $p$-Laplace equation (1.1).

**Theorem 1** Assume that there exists constants $p_{0} \geq \max\{p_{1}, p_{2}, \ldots, p_{d}\}$, $\xi^{2} \leq \frac{2(p_{j}-1)}{p_{j}^{2}(p_{0}-1)}$ and $K > 0$ such that almost surely, the following conditions hold for all $t \in [0, T]$.

1. For all $u, v \in W^{1,p}_{0}(\mathcal{D})$,
\[
\int_{\mathcal{D}^{c}} \int_{\mathcal{D}} |\gamma_{1}(u, z) - \gamma_{1}(v, z)|^{2} dx v(dz) \leq K \int_{\mathcal{D}} |u - v|^{2} dx. \tag{1.4}
\]

2. For all $u \in W^{1,p}_{0}(\mathcal{D})$,
\[
\int_{\mathcal{D}^{c}} \int_{\mathcal{D}} |\gamma_{1}(u, z)|^{2} dx v(dz) \leq K \left( 1 + \int_{\mathcal{D}} |u|^{2} dx \right). \tag{1.5}
\]

3. For all $u \in W^{1,p}_{0}(\mathcal{D})$,
\[
\int_{\mathcal{D}^{c}} \left( \int_{\mathcal{D}} |\gamma_{1}(u, z)|^{2} dx \right)^{\frac{p_{0}}{2}} v(dz) \leq K \left( 1 + \left( \int_{\mathcal{D}} |u|^{2} dx \right)^{\frac{p_{0}}{2}} \right). \tag{1.6}
\]

Further, if the initial condition $u_{0} \in L^{p_{0}}(\Omega; L^{2}(\mathcal{D}))$ and $h_{j} : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}$ are Lipschitz continuous functions with Lipschitz constants $M_{j}$ such that the sequence $(M_{j})_{j \in \mathbb{N}} \in \ell^{2}$, then there exists a unique solution of anisotropic $p$-Laplace equation (1.1) in the sense of Definition 1.1. Furthermore, if $u$ and $\tilde{u}$ are two solutions with initial condition $u_{0}$ and $\tilde{u}_{0}$ respectively, $u_{0}, \tilde{u}_{0} \in L^{p_{0}}(\Omega; L^{2}(\mathcal{D}))$ and for all $u, v \in W^{1,p}_{0}(\mathcal{D})$,
\[
\int_{\mathcal{D}^{c}} \left( \int_{\mathcal{D}} |\gamma_{1}(u, z) - \gamma_{1}(v, z)|^{2} dx \right)^{\frac{p_{0}}{2}} v(dz) \leq K \left( \int_{\mathcal{D}} |u - v|^{2} dx \right)^{\frac{p_{0}}{2}}, \tag{1.7}
\]
then,
\[
\mathbb{E}\left( \sup_{t \in [0, T]} |u_{t} - \tilde{u}_{t}|_{L^{2}}^{p_{0}} + \sum_{i=1}^{d} \int_{0}^{T} |D_{i}u_{t} - D_{i}\tilde{u}_{t}|_{L^{p_{i}}}^{p_{i}} dt \right) \leq C \mathbb{E}|u_{0} - \tilde{u}_{0}|_{L^{2}}^{p_{0}}. \tag{1.8}
\]
with \( p = 2 \) in case \( p_0 = 2 \) and with any \( p \in [2, p_0) \) in case \( p_0 > 2 \).

The rest of the article is organized as follows. In Section 2, we formulate and prove our results in abstract framework by considering a large class of SPDEs of the type (2.1) satisfying Assumptions A-1 to A-5. In Section 3, we show that Eq. 1.1 fits in the framework discussed in Section 2 and hence present a proof of Theorem 1. Finally in Section 4, we apply our results to solve two popular real world models namely, stochastic Ginzburg–Landau equation and stochastic Swift–Hohenberg equation which fit into the framework of this article but, to the best of our knowledge, can not be solved by using results available so far.

2 SPDEs in Abstract Framework: Existence & Uniqueness

Let \((H, (\cdot, \cdot), |\cdot|_H)\) be a separable Hilbert space, identified with its dual. For \( i = 1, 2, \ldots, k \), let \((V_i, |\cdot|_{V_i})\) be Banach spaces with duals \((V_i^*, |\cdot|_{V_i^*})\) and \((\cdot, \cdot)_i\) be the notation for duality pairing between \( V_i \) and \( V_i^* \). It is well known that the vector space \( V := V_1 \cap V_2 \cap \ldots \cap V_k \) with the norm given by \(|\cdot|_V := |\cdot|_{V_1} + |\cdot|_{V_2} + \ldots + |\cdot|_{V_k}\) is a Banach space. Assume that \( V \) is separable, reflexive and is embedded continuously and densely in \( H \). Thus we obtain the Gelfand triple

\[ V \hookrightarrow H \equiv H^* \hookrightarrow V^* \]

where \( \hookrightarrow \) denotes continuous and dense embedding.

We consider the stochastic evolution equation (SEE) driven by Lévy noise of the following form:

\[
du_t = \sum_{i=1}^{k} A_i^j(u_t) dt + \sum_{j=1}^{\infty} B_i^j(u_t) dW_t^j + \int_{\mathcal{D}} \gamma_i(u_t, z) \tilde{N}(dt, dz) + \int_{\mathcal{D}} \gamma_i(u_t, z) N(dt, dz), \quad t \in [0, T]
\]

where \( \mathcal{D} \in \mathcal{F} \) is such that \( \nu(\mathcal{D}) < \infty \). Here, \( A_i^j, i = 1, 2, \ldots, k \) are non-linear operators mapping \([0, T] \times \Omega \times V_1 \) into \( V_i^* \), \( B_j^i \in \mathbb{N} \) is a non-linear operator mapping \([0, T] \times \Omega \times V \) into \( \ell^2(H) \) and \( \gamma \) is a non-linear operator mapping \([0, T] \times \Omega \times V \times Z \) into \( H \). Assume that for all \( v, w \in V_i \), the processes \((A_i^j(u), w))_{t \in [0, T]}\) are progressively measurable and for all \( v, w \in V \), \((B_j^i(v)))_{t \in [0, T]}\) are progressively measurable. Since the concept of weak measurability and strong measurability of a mapping coincides if the codomain is separable, we obtain that for all \( v \in V_i, i = 1, 2, \ldots, k \), \((A_i^j(u)))_{t \in [0, T]}\) are progressively measurable. Further, for all \( v \in V \), \( j \in \mathbb{N} \), \((B_j^i(v)))_{t \in [0, T]}\) are progressively measurable. Finally, \( \gamma \) is assumed to be \( \mathcal{F} \times \mathcal{B}(V) \times \mathcal{F} \)-measurable function and \( u_0 \) is assumed to be a given \( H \)-valued, \( \mathcal{F}_0 \)-measurable random variable.

Further, we assume that there exist constants \( \alpha_i > 1 \) for \( i = 1, 2, \ldots, k \), \( \beta \geq 0, p_0 \geq \beta + 2, \theta > 0, K, L', L'' \) and a nonnegative \( f \in L^2 \) \((0, T) \times \Omega; \mathbb{R})\) such that, almost surely, the following conditions hold for all \( t \in [0, T] \).

A - 1 (Hemicontinuity) For \( i = 1, 2, \ldots, k \) and \( y, x, \tilde{x} \in V_i \), the map

\[ \varepsilon \mapsto \langle A_i^j(x + \varepsilon \tilde{x}), y \rangle_i \]

is continuous.
A - 2 (Local Monotonicity) For all \( x, \bar{x} \in V \),
\[
2 \sum_{i=1}^{k} (A_i^j(x) - A_i^j(\bar{x}), x - \bar{x})_i + \sum_{j=1}^{\infty} |B_i^j(x) - B_i^j(\bar{x})|_H^2 \\
+ \int_{D^c} |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|_H^2 v(dz)
\leq \left[ L' + L''(1 + \sum_{i=1}^{k} |\bar{x}|_{V_i}^\alpha_i)(1 + |\bar{x}|_H^\beta) \right]|x - \bar{x}|_H^2.
\]

A - 3 (\( p_0 \)-Stochastic Coercivity) For all \( x \in V \),
\[
2 \sum_{i=1}^{k} (A_i^j(x), x)_i + (p_0 - 1) \sum_{j=1}^{\infty} |B_i^j(x)|_H^2 + \theta \sum_{i=1}^{k} |x|_{V_i}^{p_0} + \int_{D^c} |\gamma_t(x, z)|_H^2 v(dz)
\leq f_t + K|x|_H^{p_0}.
\]

A - 4 (Growth of \( A_i \)) For \( i = 1, 2, \ldots, k \) and \( x \in V_i \),
\[
|A_i^j(x)|_{V_i}^{\alpha_i-1} \leq (f_t + K|x|_{V_i}^{\alpha_i})(1 + |x|_H^\beta).
\]

A - 5 (Integrability of \( \gamma \)) For all \( x \in V \),
\[
\int_{D^c} |\gamma_t(x, z)|_H^{p_0} v(dz) \leq f_t^{p_0} + K|x|_H^{p_0}.
\]

Remark 1 From Assumptions A-3 and A-4, we obtain
\[
\sum_{j=1}^{\infty} |B_i^j(x)|_H^2 + \int_{D^c} |\gamma_t(x, z)|_H^2 v(dz)
\leq C \left( 1 + f_t^{p_0} + |x|_H^{p_0} + \sum_{i=1}^{k} |x|_{V_i}^{\alpha_i} + |x|_H^{\beta} \sum_{i=1}^{k} |x|_{V_i}^{\alpha_i} \right)
\]
almost surely for all \( t \in [0, T] \) and \( x \in V \). Indeed, using Hölder’s inequality, Young’s inequality and Assumption A-4, we obtain that almost surely for all \( x \in V \) and \( t \in [0, T] \),
\[
\sum_{i=1}^{k} |(A_i^j(x), x)_i| \leq \sum_{i=1}^{k} \left[ \frac{\alpha_i - 1}{\alpha_i} |A_i^j(x)|_{V_i}^{\alpha_i-1} + \frac{1}{\alpha_i} |x|_{V_i}^{\alpha_i} \right]
\leq \sum_{i=1}^{k} \left[ \frac{\alpha_i - 1}{\alpha_i} (f_t + K|x|_{V_i}^{\alpha_i})(1 + |x|_H^\beta) + \frac{1}{\alpha_i} |x|_{V_i}^{\alpha_i} \right]
\leq C \left( f_t + \sum_{i=1}^{k} |x|_{V_i}^{\alpha_i} + |x|_H^{\beta} \sum_{i=1}^{k} |x|_{V_i}^{\alpha_i} + f_t^{p_0} + (1 + |x|_H^{p_0}) \right).
\]
The above inequality along with Assumption A-3 gives the result.
Further, in case \( p_0 = 2 \), i.e. \( \beta = 0 \), using the similar argument as above, we get

\[
\sum_{j=1}^{\infty} |B_j^i(x)|_H^2 + \int_{\mathcal{D}^c} |\gamma_t(x, z)|_H^2 v(dz) \leq C \left( f_t + |x|_H^2 + \sum_{i=1}^{k} |x|_{V_i}^{p_0} \right)
\]

almost surely for all \( t \in [0, T] \) and \( x \in V \).

**Remark 2** From Assumptions A-1, A-2 and A-4, we obtain that almost surely for all \( t \in [0, T] \) and \( i = 1, 2, \ldots, k \), the operators \( A^i_t \) are demicontinuous, i.e. \( v_n \to v \) in \( V_t \) implies that \( A^i_t(v_n) \to A^i_t(v) \) in \( V_t^* \). This follows using similar arguments as in the proof of Lemma 2.1 in [7].

One consequence of Remark 2 is that, progressive measurability of some process \((v_t)_{t \in [0, T]} \) implies the progressive measurability of the processes \((A^i_t(v_t))_{t \in [0, T]} \) for all \( i = 1, 2, \ldots, k \).

If the driving noise in Eq. 2.1 is a Wiener process, i.e. intensity \( v \equiv 0 \), then Pardoux [12] has studied such equations when the operators satisfy hemicontinuity condition A-1, monotonicity condition (A-2 with constant \( L'' = 0 \)), coercivity condition (A-3 with \( p_0 = 2 \), i.e. \( \beta = 0 \)), growth assumption (A-4 with \( \beta = 0 \)) and an additional assumption on operator \( B \) appearing in the stochastic integral term. Note that the noise considered in [12] is a cylindrical \( Q \)-Wiener process taking values in a separable Hilbert space. One can see, e.g. in Neelima and Siška [10, Appendix A], that the stochastic Itô integral with respect to cylindrical \( Q \)-Wiener process taking values in a separable Hilbert space can be expressed in the form of infinite sum of stochastic Itô integrals with respect to independent one-dimensional Wiener processes as considered in Eq. 2.1. In view of this fact, the additional condition on operator \( B \) assumed in [12] can be equivalently stated as the following.

For all \( h \in H \) and positive real numbers \( N \), there exists a constant \( M \) such that for almost all \((t, \omega) \in [0, T] \times \Omega \) and \( x, y \in V \) satisfying \(|x|_V, |y|_V \leq N \), it holds that

\[
\sum_{j=1}^{\infty} |(h, B_j^i(x)) - (h, B_j^i(y))| \leq M|x - y|_V .
\]

(2.2)

For the case \( k = 1 \), Krylov and Rozovskii [7] generalized the results in [12] by removing the additional assumption (2.2) on the operator \( B \). These classical results in [7] have been generalised in number of directions. Gyöngy [3] extended the results in [7] to include SPDEs driven by càdlàg semi-martingales and thus allows \( v \) in Eq. 2.1 to be different from zero. Liu and Röckner [9] have extended the framework in [7] to SPDEs with locally monotone operators where the operator \( A \), which is the operator acting in the bounded variation term, satisfies a less restrictive growth condition. Thus, authors in [9] allow constants \( L'' \) and \( \beta \), appearing in Assumptions A-2 and A-4 respectively, to be non-zero. Brzeźniak, Liu and Zhu [2] generalised the results in [9] to include equations driven by Lévy noise (i.e. \( v \neq 0 \)). However, authors in both [2] and [9] have placed an assumption on the growth of the operators appearing under stochastic integrals. Indeed, in the set up of this article, assumption made in [9] can be equivalently stated as: for all \((t, \omega) \in [0, T] \times \Omega \) and \( x \in V \),

\[
\sum_{j=1}^{\infty} |B_j^i(x)|_H^2 \leq C(f_t + |x|_H^2)
\]

(2.3)
for some \( f \in L^{\infty}_{0}((0, T) \times \Omega; \mathbb{R}) \). Further, assumption made in [2] can be stated as: for \( f \in L^{\infty}_{0}((0, T) \times \Omega; \mathbb{R}) \), there exists a constant \( \xi < \theta' \) such that for all \( (t, \omega) \in [0, T] \times \Omega \) and \( x \in V \),

\[
\sum_{j=1}^{\infty} |B^{j}_{t}(x)|^{2}_{H} + \int_{D^{c}} |\gamma_{t}(x, z)|^{2}_{H} v(dz) \leq f_{t} + C|x|^{2}_{H} + \xi|x|^{a}_{V} \tag{2.4}
\]

where \( \theta' \) is the coefficient of coercivity appearing in coercivity assumption made in [2]. In view of Remark 1, the conditions (2.3) and (2.4) clearly place a restriction on the growth of operators appearing in stochastic integrals. Recently, for the case \( v \equiv 0 \), Neelima and Šiška [10] have overcome this problem by identifying the appropriate coercivity assumption as stated in A - 3 and proved the existence and uniqueness of solutions to Eq. 2.1 (in case \( k = 1 \) and \( v \equiv 0 \)) without explicitly restricting the growth of the operator \( B \) given in Eq. 2.3. This article is a generalization of [2] in two senses: (a) we do not require the explicit growth condition (2.4) to establish existence and uniqueness results, (b) the operator acting in the bounded variation term is of the form \( A^{1} + A^{2} + \cdots + A^{k} \), where the operators \( A^{i} \) have different analytic and growth properties. Again, we have generalized the results in [10] by including SPDEs driven by Lévy noise which satisfy condition (b) stated above, i.e. allowing \( k > 1 \) and \( v \not\equiv 0 \).

In all the above mentioned works, the key to prove the results is the use of an appropriate Itô formula for the square of the \( H \)-norm. The formula is an analogue of the energy equality for PDEs which is an essential tool in proving existence and uniqueness theorems for PDEs. The Itô formula helps in obtaining the a priori estimates under the coercivity and growth assumptions. Under additional assumptions of monotonicity and hemicontinuity, it helps in proving the existence and uniqueness of the solution. Further, it provides a càdlàg version of the solution process in the space \( H \).

In this article, using the Itô formula for processes taking values in intersection of finitely many Banach spaces, given recently by Gyöngy and Šiška [4], we extend the available results in the literature to include the SPDEs of the type (2.1) under the above mentioned assumptions.

**Definition 2.1 (Solution)** An adapted, càdlàg, \( H \)-valued process \( u \) is called a solution of the stochastic evolution equation (2.1) if

i) \( dt \times \mathbb{P} \) almost everywhere \( u \in V \) with

\[
\mathbb{E} \int_{0}^{T} (|u_{t}|^{a}_{V} + |u_{t}|^{2}_{H}) dt < \infty, \quad i = 1, 2, \ldots, k,
\]

ii) almost surely

\[
\int_{0}^{T} \left( |u_{t}|^{p}_{H} + |u_{t}|^{a}_{V} |u_{t}|^{p}_{H}^{-2} \right) dt < \infty, \quad i = 1, 2, \ldots, k \text{ and}
\]

iii) for every \( t \in [0, T] \) and \( \phi \in V \),

\[
(u_{t}, \phi) = (u_{0}, \phi) + \sum_{i=1}^{k} \int_{0}^{t} \langle A_{s}(u_{s}), \phi \rangle ds + \sum_{j=1}^{\infty} \int_{0}^{t} \langle \phi, B^{j}_{s}(u_{s}) \rangle dW_{s}^{j}
\]

\[
+ \int_{0}^{t} \int_{D^{c}} \langle \phi, \gamma_{5}(u_{s}, z) \rangle d\tilde{N}(ds, dz) + \int_{0}^{t} \int_{D} \langle \phi, \gamma_{5}(u_{s}, z) \rangle dN(ds, dz)
\]

almost surely.
The existence and uniqueness of solution to Eq. 2.1 can be obtained from the existence of a unique solution to the stochastic evolution equation,
\[ u_t = u_0 + \sum_{i=1}^{k} \int_0^t A_i^i(u_s)ds + \sum_{j=1}^{\infty} \int_0^t B_j^i(u_s)dW_j^i \]
\[ + \int_0^t \int_{\mathcal{D}} \gamma_s(u_s, z)\tilde{N}(ds, dz) \tag{2.5} \]
for \( t \in [0, T] \), i.e. the case when the last integral in Eq. 2.1 vanishes. This is done by means of the interlacing procedure (see e.g. [2, Section 4.2]). For the sake of completeness of argument, the procedure has been explained in the Appendix A. As a consequence, we will now consider the stochastic evolution equation (2.5) in rest of the article and prove the existence and uniqueness of solution to Eq. 2.5 in Theorems 2, 3 and 5 below. Before that we state two lemmas without proof. Lemma 2.2 (see, e.g. Yor [13, Chapter IV, Proposition 4.7] is used to obtain desired a priori estimates. The proof of Lemma 2.3 can be found in [11].

**Lemma 2.2** Let \( Y \) be a positive, adapted, right continuous process and \( A \) be a continuous increasing process. If
\[ \mathbb{E}[Y_\tau | \mathcal{F}_0] \leq \mathbb{E}[A_\tau | \mathcal{F}_0] \]
for any bounded stopping time \( \tau \), then for any \( r \in (0, 1) \),
\[ \mathbb{E} \sup_{t \geq 0} Y_t^r \leq \frac{2 - r}{1 - r} \mathbb{E} \sup_{t \geq 0} A_t^r. \]

**Lemma 2.3** Let \( r \geq 2 \) and \( T > 0 \). There exists a constant \( K \), depending only on \( r \), such that for every real-valued, \( \mathcal{P} \times \mathcal{Z} \)-measurable function \( \gamma \) satisfying
\[ \int_0^T \int_{\mathcal{Z}} |\gamma_t(z)|^2 \nu(dz) dt < \infty \]
almost surely, then the following estimate holds,
\[ \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t \int_{\mathcal{Z}} |\gamma_t(z)|^2 \tilde{N}(ds, dz) \right)^{\frac{r}{2}} \leq K \mathbb{E} \left( \int_0^T \int_{\mathcal{Z}} |\gamma_t(z)|^2 \nu(dz) dt \right)^{\frac{r}{2}} \]
\[ + K \mathbb{E} \int_0^T \int_{\mathcal{Z}} |\gamma_t(z)|^r \nu(dz) dt. \tag{2.6} \]
It is known that if \( 1 \leq r \leq 2 \), then the second term in Eq. 2.6 can be dropped.

We now show the existence and uniqueness of solution to SPDE (2.5).

**2.1 A Periori Estimates**

We begin by obtaining some a priori estimates of a solution to SPDE (2.5).

**Theorem 2** If \( u \) is a solution of Eq. 2.5 and Assumptions A-3 to A-5 hold, then for any \( p_0 > 2 \)
\[ \sup_{t \in [0, T]} \mathbb{E}|u_t|_{H}^{p_0} + \sum_{i=1}^{k} \mathbb{E} \int_0^T |u_t|_{H}^{p_0-2} |u_t|_{V_i}^{\alpha_i} dt \leq C \mathbb{E}\left(|u_0|_{H}^{p_0} + \int_0^T f_s^{p_0} ds \right) \]
and
\[ \sup_{t \in [0, T]} \mathbb{E}|u_t|_{H}^2 + \sum_{i=1}^{k} \mathbb{E} \int_0^T |u_t|_{V_i}^{\alpha_i} dt \leq C \mathbb{E}\left(|u_0|_{H}^2 + \int_0^T f_s ds \right). \tag{2.7} \]
Moreover,
\[ \mathbb{E} \sup_{t \in [0, T]} |u_t|^2_H \leq C \mathbb{E} \left( |u_0|^2_H + \int_0^T f_s ds \right) \]
and
\[ \mathbb{E} \sup_{t \in [0, T]} |u_t|^{2r}_H \leq C \mathbb{E} \left( |u_0|^{2r}_H + \int_0^T f_s^{2r} ds \right)^r \]  \hspace{1cm} (2.8)
for any \( r \in (0, 1) \), where \( C \) depends only on \( p_0, K, T \) and \( \theta \).

**Proof** Let \( u \) be a solution of Eq. 2.5. In order to obtain higher moment a priori estimates for solutions to Eq. 2.5, we define for each \( n \in \mathbb{N} \),
\[ \sigma_n := \inf \{ t \in [0, T] : |u_t|_H > n \} \wedge T. \]  \hspace{1cm} (2.9)
The solution \( u \), being an adapted and càdlàg \( H \)-valued process, is bounded on every compact interval. Thus \( (\sigma_n)_{n \in \mathbb{N}} \) is a sequence of stopping times converging to \( T \), \( \mathbb{P} \)-a.s. and \( \mathbb{P} \{ \sigma_n < T \} = 0 \) as \( n \to \infty \). Applying Itô’s formula for processes taking values in intersection of finitely many Banach spaces to Eq. 2.5, see [4, Theorem 2.1] and replacing \( \sigma_n \) as \( \sigma_n \) depends only on \( 1 \leq \kappa \leq n \), we get almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \),
\[ |u_{t \wedge \sigma_n}|^2_H = |u_0|^2_H + \int_0^{t \wedge \sigma_n} \left( 2 \sum_{i=1}^{k} \langle A^j_i(u_s), u_s \rangle_t + \sum_{j=1}^{\infty} |B^j_s(u_s)|^2_H \right) ds \]
\[ + 2 \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, B^j_s(u_s)) dW^j_s + \int_0^{t \wedge \sigma_n} \int_{\mathcal{D}^c} 2(u_s, \gamma_s(u_s, z)) \tilde{N}(ds, dz) \]
\[ + \int_0^{t \wedge \sigma_n} \int_{\mathcal{D}^c} |\gamma_s(u_s, z)|^2_H N(ds, dz). \]  \hspace{1cm} (2.10)

Using the fact \( \tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt \), we get
\[ |u_{t \wedge \sigma_n}|^2_H = |u_0|^2_H + \int_0^{t \wedge \sigma_n} \left( 2 \sum_{i=1}^{k} \langle A^j_i(u_s), u_s \rangle_t + \sum_{j=1}^{\infty} |B^j_s(u_s)|^2_H \right) ds \]
\[ + \int_{\mathcal{D}^c} |\gamma_s(u_s, z)|^2_H v(dz) ds + 2 \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, B^j_s(u_s)) dW^j_s \]
\[ + \int_0^{t \wedge \sigma_n} \int_{\mathcal{D}^c} \left( 2(u_s, \gamma_s(u_s, z)) + |\gamma_s(u_s, z)|^2_H \right) \tilde{N}(ds, dz) \]
almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \). Notice that this is a 1-dimensional Itô process. Thus, by Itô’s formula,
\[ |u_{t \wedge \sigma_n}|^{p_0}_H = |u_0|^{p_0}_H + \frac{p_0}{2} \int_0^{t \wedge \sigma_n} |u_s|^{p_0-2}_H \left( 2 \sum_{i=1}^{k} \langle A^j_i(u_s), u_s \rangle_t + \sum_{j=1}^{\infty} |B^j_s(u_s)|^2_H \right) ds \]
\[ + \int_{\mathcal{D}^c} |\gamma_s(u_s, z)|^{p_0-2}_H \left( 2(u_s, \gamma_s(u_s, z)) + |\gamma_s(u_s, z)|^2_H \right) v(dz) ds \]
\[ + \frac{p_0}{2} \int_0^{t \wedge \sigma_n} |u_s|^{p_0-2}_H \sum_{j=1}^{\infty} (u_s, B^j_s(u_s)) dW^j_s \]
\[ + \frac{p_0}{2} \int_0^{t \wedge \sigma_n} \sum_{j=1}^{\infty} |u_s|^{p_0-2}_H (u_s, B^j_s(u_s))^2 ds \]
\[ + \int_0^{t \wedge \sigma_n} \sum_{j=1}^{\infty} |u_s|^{p_0-2}_H (u_s, B^j_s(u_s))^2 ds \]
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almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \).

Again, using the fact \( \tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt \), we get

\[
|u_{t, \wedge \sigma_n}|^p_H = |u_0|^p_H + I_1 + I_2 + p_0 \sum_{j=1}^{\infty} \int_0^{t\wedge \sigma_n} |u_s|^p_H - 2(u_s, B_s^j(u_s))dW^j_s
\]

almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \), where

\[
I_1 := \frac{p_0}{2} \int_0^{t\wedge \sigma_n} |u_s|^p_H - 2 \left( \sum_{i=1}^{k} (A_i^j(u_s), u_s)_i + \sum_{j=1}^{\infty} |B_s^j(u_s)|^2_H \right) ds
\]

and

\[
I_2 := \int_0^{t\wedge \sigma_n} \left[ |u_s + \gamma_s(u_s, z)|^p_H - |u_s|^p_H - p_0 |u_s|^p_H - 2(u_s, \gamma_s(u_s, z)) \right] N(ds, dz).
\]

Using Cauchy-Schwarz inequality, Assumption A-3 and Young’s inequality, we get almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \),

\[
I_1 \leq \frac{p_0}{2} \int_0^{t\wedge \sigma_n} |u_s|^p_H - 2 \left( \sum_{i=1}^{k} (A_i^j(u_s), u_s)_i + (p_0 - 1) \sum_{j=1}^{\infty} |B_s^j(u_s)|^2_H \right) ds
\]

\[
\leq \frac{p_0}{2} \int_0^{t\wedge \sigma_n} |u_s|^p_H - 2 \left( f_s + K |u_s|^2_H - \theta \sum_{i=1}^{k} |u_s|^q_{V_i} \right) ds
\]

\[
\leq \int_0^{t\wedge \sigma_n} \left( f_s^\frac{p_0}{2} + \frac{p_0(K + 1) - 2}{2} |u_s|^{p_0}_H - \theta \sum_{i=1}^{k} |u_s|^{p_0 - 2}_H |u_s|^t_{V_i} \right) ds. \tag{2.12}
\]

We now proceed to estimate \( I_2 \). Notice that due to Taylor’s formula on the map \( t \mapsto |x + ty|^p_H \), for any \( x, y \in H \) and \( p \geq 2 \), we get

\[
|x + y|^p_H - |x|^p_H = \int_0^1 \frac{d}{dt} |x + ty|^p_H dt
\]

and therefore,

\[
||x + y|_H^p - |x|_H^p - p|x|_H^{p-2}(x, y)| = p \int_0^1 \left[ |x + ty|_H^{p-2}(x + ty, y) - |x|_H^{p-2}(x, y) \right] dt
\]

\[
\leq C_p \int_0^1 (|x|_H^{p-2} + |y|_H^{p-2}) |y|_H^2 t dt \leq C_p(|x|_H^{p-2} + |y|_H^p). \tag{2.13}
\]
Now, taking \( x = u_s, y = \gamma_s(u_s, z) \) and \( p = p_0 \) in Eq. 2.13, we get
\[
|u_s + \gamma_s(u_s, z)|_{L^p_{H}}^0 - |u_s|_{L^p_{H}}^0 - p_0|u_s|_{L^p_{H}}^{p_0-2}(u_s, \gamma_s(u_s, z)) \\
\leq C\left(|u_s|_{L^p_{H}}^{p_0-2}|\gamma_s(u_s, z)|_{L^2_{H}}^2 + |\gamma_s(u_s, z)|_{L^p_{H}}^0\right)
\]
and hence using Young’s inequality, we get for all \( t \in [0, T] \) and \( n \in \mathbb{N} \),
\[
I_2 \leq C \int_0^{t \wedge \sigma_n} \int_{D^2} \left[ |u_s|_{L^p_{H}}^{p_0-2}|\gamma_s(u_s, z)|_{L^2_{H}}^2 + |\gamma_s(u_s, z)|_{L^p_{H}}^0 \right] N(ds, dz) \\
\leq C \int_0^{t \wedge \sigma_n} \int_{D^2} \left[ |u_s|_{L^p_{H}}^0 + |\gamma_s(u_s, z)|_{L^p_{H}}^0 \right] N(ds, dz) \tag{2.14}
\]
Using Eqs. 2.12 and 2.14, we obtain from Eq. 2.11
\[
|u_{t \wedge \sigma_n}|_{L^p_{H}}^0 + \theta \frac{p_0}{2} \sum_{i=1}^k \int_0^{t \wedge \sigma_n} |u_s|_{L^p_{H}}^{p_0-2} |u_s|_{V_i}^{q_i} ds \\
\leq |u_0|_{L^p_{H}}^0 + \int_0^{t \wedge \sigma_n} f_s^{\frac{p_0}{2}} ds + C \int_0^{t \wedge \sigma_n} |u_s|_{L^p_{H}}^0 ds \\
+ C \int_0^{t \wedge \sigma_n} \int_{D^2} \left[ |u_s|_{L^p_{H}}^0 + |\gamma_s(u_s, z)|_{L^p_{H}}^0 \right] N(ds, dz) \\
+ p_0 \sum_{j=1}^\infty \int_0^{t \wedge \sigma_n} |u_s|_{L^p_{H}}^{p_0-2} (u_s, B^j_s(u_s)) dW^j_s \\
+ p_0 \int_0^{t \wedge \sigma_n} \int_{D^2} |u_s|_{L^p_{H}}^{p_0-2} (u_s, \gamma_s(u_s, z)) \tilde{N}(ds, dz) \tag{2.15}
\]
almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \). We now aim to apply Lemma 2.2. To that end let \( \tau \) be some bounded stopping time. Then in view of Remark 1 and the fact that \( u \) is a solution of Eq. 2.5, it follows that for all \( t \in [0, T] \) and \( n \in \mathbb{N} \),
\[
\mathbb{E} \sum_{j=1}^\infty \int_0^{t \wedge \sigma_n} \mathbf{1}_{[s \leq \tau]} |u_s|_{L^p_{H}}^{p_0-2} (u_s, B^j_s(u_s)) dW^j_s = 0
\]
and
\[
\mathbb{E} \int_0^{t \wedge \sigma_n} \int_{D^2} \mathbf{1}_{[s \leq \tau]} |u_s|_{L^p_{H}}^{p_0-2} (u_s, \gamma_s(u_s, z)) \tilde{N}(ds, dz) = 0.
\]
Therefore, replacing \( t \wedge \sigma_n \) by \( t \wedge \sigma_n \wedge \tau \) in Eq. 2.15, taking expectation and using Assumption A-5, we obtain for all \( t \in [0, T] \) and \( n \in \mathbb{N} \),
\[
\mathbb{E} |u_{t \wedge \sigma_n \wedge \tau}|_{L^p_{H}}^0 + \theta \frac{p_0}{2} \sum_{i=1}^k \mathbb{E} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_{L^p_{H}}^{p_0-2} |u_s|_{V_i}^{q_i} ds \\
\leq \mathbb{E} |u_0|_{L^p_{H}}^0 + \mathbb{E} \int_0^T f_s^{\frac{p_0}{2}} ds + \mathbb{E} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_{L^p_{H}}^0 ds \\
+ C \mathbb{E} \int_0^{t \wedge \sigma_n \wedge \tau} \int_{D^2} \left[ |u_s|_{L^p_{H}}^0 + |\gamma_s(u_s, z)|_{L^p_{H}}^0 \right] \nu(dz) ds \\
\leq \mathbb{E} |u_0|_{L^p_{H}}^0 + C \mathbb{E} \int_0^T f_s^{\frac{p_0}{2}} ds + C \mathbb{E} \int_0^t |u_{s \wedge \sigma_n \wedge \tau}|_{L^p_{H}}^0 ds \tag{2.16}
\]
From this Gronwall’s lemma yields,
\[ \mathbb{E}|u_{t\wedge \sigma_n \wedge \tau}|_{H}^{p_0} \leq C \mathbb{E}\left(|u_0|_{H}^{p_0} + \int_0^T f_s^{p_0} ds\right) \] (2.17)
for all \( t \in [0, T] \) and \( n \in \mathbb{N} \). Letting \( n \to \infty \) and using Fatou’s lemma, we obtain
\[ \mathbb{E}|u_{t\wedge \tau}|_{H}^{p_0} \leq C \mathbb{E}\left(|u_0|_{H}^{p_0} + \int_0^T f_s^{p_0} ds\right) \]
for all \( t \in [0, T] \). Using Lemma 2.2, we get
\[ \mathbb{E}\sup_{t \in [0, T]}|u_t|_{H}^{p_0} \leq \frac{2 - r}{1 - r} C \mathbb{E}\left(|u_0|_{H}^{p_0} + \int_0^T f_s^{p_0} ds\right)^r \]
for any \( r \in (0, 1) \), which proves the second inequality in Eq. 2.8.

In order to prove (2.7), the estimate 2.17 is used in the right-hand side of Eq. 2.16 with \( \tau = T \) and with \( n \to \infty \). We thus obtain,
\[ \mathbb{E}|u_{t}|_{H}^{p_0} + \theta \frac{p_0}{2} \sum_{i=1}^{k} \mathbb{E}\int_0^t |u_s|_{H}^{p_0 - 2} |u_s|_{H}^{q_i} ds \leq C \mathbb{E}\left(|u_0|_{H}^{p_0} + \int_0^T f_s^{p_0} ds\right) \]
for all \( t \in [0, T] \). If Assumption A-3 holds for some \( p_0 \geq \beta + 2 \), then it holds for \( p_0 = 2 \) as well. Thus, from Eq. 2.10 we obtain
\[ \mathbb{E}|u_{t}|_{H}^{2} + \theta \sum_{i=1}^{k} \mathbb{E}\int_0^t |u_s|_{H}^{q_i} ds \leq \mathbb{E}\left(|u_0|_{H}^{2} + \int_0^T f_s ds\right) + K \mathbb{E}\int_0^t |u_s|_{H}^{2} ds \]
for all \( t \in [0, T] \). Application of Gronwall’s lemma yields,
\[ \sup_{t \in [0, T]} \mathbb{E}|u_t|_{H}^{2} \leq C \mathbb{E}\left(|u_0|_{H}^{2} + \int_0^T f_s ds\right), \]
which in turn gives
\[ \theta \sum_{i=1}^{k} \mathbb{E}\int_0^T |u_s|_{H}^{q_i} ds \leq C \mathbb{E}\left(|u_0|_{H}^{2} + \int_0^T f_s ds\right) \]
and hence (2.7) holds.

To complete the proof it remains to show the first inequality in Eq. 2.8. Considering the sequence of stopping times \( \sigma_n \) defined in Eq. 2.9, as before we observe that the stochastic integrals appearing in the right-hand side of Eq. 2.10 are martingales for each \( n \in \mathbb{N} \). Thus using the Burkholder–Davis–Gundy inequality and Cauchy–Schwartz inequality, we obtain for each \( n \in \mathbb{N} \)
\[ \mathbb{E}\sup_{t \in [0, T]} \left| \sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} (u_s, B_s^j (u_s)) dW_s^j \right| \]
\[ \leq 4 \mathbb{E}\left( \sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |(u_s, B_s^j (u_s))|^2 ds \right)^{\frac{1}{2}} \]
\[ \leq 4 \mathbb{E}\left( \sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |u_s|_{H}^2 |B_s^j (u_s)|_{H}^2 ds \right)^{\frac{1}{2}}. \] (2.18)
Similarly, using Lemma 2.3 and Cauchy-Schwartz inequality, we obtain for each \( n \in \mathbb{N} \)

\[
E \sup_{t \in [0,T]} \left| \int_0^{t \wedge \sigma_n} \int_{\mathcal{D}^c} (u_s, \gamma_s(u_s)) \tilde{N}(ds, dz) \right|
\]

\[
\leq C E \left( \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} |(u_s, \gamma_s(u_s))|^2 \nu(dz) ds \right)^{\frac{1}{2}}
\]

\[
\leq C E \left( \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} |u_s|_H^2 |\gamma_s(u_s)|_H^2 \nu(dz) ds \right)^{\frac{1}{2}}.
\] (2.19)

Thus Eqs. 2.18 and 2.19 along with Remark 1 and Young’s inequality give,

\[
E \sup_{t \in [0,T]} \left| \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, B_j^i(u_s)) dW^i_j \right|
\]

\[
+ E \sup_{t \in [0,T]} \left| \int_0^{t \wedge \sigma_n} \int_{\mathcal{D}^c} (u_s, \gamma_s(u_s)) \tilde{N}(ds, dz) \right|
\]

\[
\leq C E \left( \sup_{t \in [0,T]} |u_{t \wedge \sigma_n}|_H^2 \int_0^{T \wedge \sigma_n} \left( f_s + |u_s|_H^2 + \sum_{i=1}^k |u_s|_{V_i} \right) ds \right)^{\frac{1}{2}}
\]

\[
\leq \epsilon E \sup_{t \in [0,T]} |u_{t \wedge \sigma_n}|_H^2 + C E \int_0^{T \wedge \sigma_n} \left( f_s + |u_s|_H^2 + \sum_{i=1}^k |u_s|_{V_i} \right) ds,
\] (2.20)

for each \( n \in \mathbb{N} \). Moreover, taking supremum and then expectation in Eq. 2.10 and using Assumption A-3 along with Eq. 2.20, we obtain for each \( n \in \mathbb{N} \)

\[
E \sup_{t \in [0,T]} |u_{t \wedge \sigma_n}|_H^2 \leq \epsilon E \sup_{t \in [0,T]} |u_{t \wedge \sigma_n}|_H^2
\]

\[
+ C \left( E|u_0|_H^2 + E \int_0^T f_s ds + \sum_{i=1}^k E \int_0^T |u_s|_{V_i} ds + \sup_{t \in [0,T]} E|u_t|_H^2 \right).
\]

Finally, by choosing \( \epsilon \) small and using Eq. 2.7 for \( p_0 = 2 \), we obtain for each \( n \in \mathbb{N} \)

\[
E \sup_{t \in [0,T]} |u_{t \wedge \sigma_n}|_H^2 \leq C \left( E|u_0|_H^2 + E \int_0^T f_s ds \right)
\]

which on allowing \( n \to \infty \) and using Fatou’s lemma finishes the proof. \( \square \)

Note that we can obtain existence and uniqueness results even if Assumption A-3 is replaced by the following assumption.

**A - 6** For all \( x \in V \),

\[
2 \sum_{i=1}^k \langle A^i_t(x), x \rangle_i + (p_0 - 1) \sum_{j=1}^{\infty} |B^j_t(x)|_{H}^2 + \theta \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + \int_{\mathcal{D}^c} |\gamma_t(x, z)|_H^2 \nu(dz)
\]

\[
\leq f_t + K |x|_H^2,
\]

where, \( \alpha_i < p_0 \) for all \( i \) and \( [\cdot]_{V_i} \) is a seminorm on the space \( V_i \) such that

\[
| \cdot |_{V_i} \leq | \cdot |_H + [ \cdot ]_{V_i}.
\]
In next remark we show that we obtain apriori estimates similar to Eq. 2.7 even if Assumption A-3 is replaced by A-6 and then rest of the argument for showing existence and uniqueness of solution to Eq. 2.5 will remain the same.

**Remark 3** If Assumption A-3 is replaced by the A-6, then replacing \(|u_t|_{V_i}^{\alpha_i} \) everywhere in the proof of Theorem 2, we obtain

\[
\sum_{i=1}^{k} \mathbb{E} \int_0^T [u_s]_{V_i}^{\alpha_i} ds \leq C \mathbb{E} \left( |u_0|_H^2 + \int_0^T f_s ds \right) \]

and

\[
\mathbb{E} \int_0^T |u_s|_{H}^{\alpha_i} ds \leq T \mathbb{E} \sup_{s \in [0, T]} |u_s|_H^{\alpha_i} \leq C \mathbb{E} \left( |u_0|_H^{p_0} + \int_0^T f_s^{p_0} ds \right)
\]

since \( \alpha_i < p_0 \) for all \( i \). Thus,

\[
\sum_{i=1}^{k} \mathbb{E} \int_0^T |u_s|_{V_i}^{\alpha_i} ds \leq C \left( \mathbb{E} \int_0^T |u_s|_H^{\alpha_i} ds + \mathbb{E} \int_0^T |u_s|_{V_i}^{\alpha_i} ds \right)
\]

\[
\leq C \mathbb{E} \left( |u_0|_H^{p_0} + \int_0^T f_s^{p_0} ds + |u_0|_H^2 + \int_0^T f_s ds \right)
\]

giving all the desired a priori estimates for the solution.

### 2.2 Uniqueness of Solution

Before stating the result about uniqueness of solution to stochastic evolution (2.5), we observe the following.

We note that right hand side in the Assumption A-2 can be replaced by

\[
L \left( 1 + \sum_{i=1}^{k} |\bar{x}|_{V_i}^{\alpha_i} \right) (1 + |\bar{x}|_H^{\beta}) |x - \bar{x}|_H
\]

for some constant \( L \). We use this \( L \) in the remaining article.

**Definition 2.4** Let \( \Psi \) be defined as the collection of \( V \)-valued and \( \mathcal{F}_t \)-adapted processes \( \psi \) satisfying

\[
\int_0^T \rho(\psi_s) ds < \infty \quad \text{a.s.}
\]

where

\[
\rho(x) := L \left( 1 + \sum_{i=0}^{k} |x|_{V_i}^{\alpha_i} \right) (1 + |x|_H^{\beta})
\]

for all \( x \in V \).

Note that if \( u \) is a solution to Eq. 2.5 then \( u \in \Psi \).

**Remark 4** For any \( \psi \in \Psi \) and \( v \in L^2(\Omega, D([0, T]; H)) \),

\[
\mathbb{E} \left[ \int_0^T e^{-\int_0^s \rho(\psi_dr) \rho(\psi_s) ds} \rho(\psi_s) |v_s|_H^2 ds \right] \leq \mathbb{E} \sup_{s \in [0, T]} |v_s|_H^2 \int_0^T e^{-\int_0^s \rho(\psi_dr) \rho(\psi_s) ds} ds
\]

\[
= \mathbb{E} \sup_{s \in [0, T]} |v_s|_H^2 \int_0^T \left[ 1 - e^{-\int_0^s \rho(\psi_dr) ds} \right] ds \leq \mathbb{E} \sup_{s \in [0, T]} |v_s|_H^2 < \infty.
\]
This remark justifies the existence of the bounded variation integrals appearing in the proof of uniqueness that follows.

**Theorem 3** Let Assumptions A-2 to A-5 hold and \( u_0, \bar{u}_0 \in L^{p_0}(\Omega; H) \). If \( u \) and \( \bar{u} \) are two solutions of Eq. 2.5 with \( u_0 = \bar{u}_0 \) \( \mathbb{P} \)-a.s., then the processes \( u \) and \( \bar{u} \) are indistinguishable, i.e.

\[
\mathbb{P}\left( \sup_{t \in [0, T]} |u_t - \bar{u}_t|_H = 0 \right) = 1.
\]

**Proof** Consider two solutions \( u \) and \( \bar{u} \) of Eq. 2.5. Thus,

\[
u_t - \bar{u}_t = \sum_{i=1}^{k} \int_{0}^{t} \left( A^i_s(u_s) - A^i_s(\bar{u}_s) \right) ds + \sum_{j=1}^{\infty} \int_{0}^{t} \left( B^j_s(u_s) - B^j_s(\bar{u}_s) \right) dW^j_s + \int_{0}^{t} \int_{\mathcal{D}^c} \left( \gamma_s(u_s, z) - \gamma_s(\bar{u}_s, z) \right) \tilde{N}(ds, dz)
\]

almost surely for all \( t \in [0, T] \). Using the product rule and the Itô’s formula from [4], we obtain

\[
d\left( e^{-\int_{0}^{t} \rho(\bar{u}_s) ds} |u_t - \bar{u}_t|_H^2 \right)
= e^{-\int_{0}^{t} \rho(\bar{u}_s) ds} \left[ d|u_t - \bar{u}_t|_H^2 - \rho(\bar{u}_t)|u_t - \bar{u}_t|_H^2 dt \right]
= e^{-\int_{0}^{t} \rho(\bar{u}_s) ds} \left[ \left( 2 \sum_{i=1}^{k} \langle A^i_t(u_t) - A^i_t(\bar{u}_t), u_t - \bar{u}_t \rangle \right) dt + \sum_{j=1}^{\infty} 2(u_t - \bar{u}_t, B^j_t(u_t) - B^j_t(\bar{u}_t))dW^j_t
+ \int_{\mathcal{D}^c} |\gamma_t(u_t, z) - \gamma_t(\bar{u}_t, z)|_H^2 N(dt, dz) - \rho(\bar{u}_t)|u_t - \bar{u}_t|_H^2 dt \right]
\]

almost surely for all \( t \in [0, T] \). For each \( n \in \mathbb{N} \), consider the sequence of stopping times \( \sigma_n \) given by

\[
\sigma_n := \inf\{ t \in [0, T] : |u_t|_H > n \} \land \inf\{ t \in [0, T] : |\bar{u}_t|_H > n \} \land T.
\]

Replacing \( t \) by \( t_n := t \land \sigma_n \) in Eq. 2.22 and taking expectation, we obtain that almost surely for all \( t \in [0, T] \) and \( n \in \mathbb{N} \)

\[
\mathbb{E} \left( e^{-\int_{0}^{t_n} \rho(\bar{u}_s) ds} |u_{t_n} - \bar{u}_{t_n}|_H^2 \right) - \mathbb{E}|u_0 - \bar{u}_0|_H^2
= \mathbb{E} \int_{0}^{t_n} e^{-\int_{0}^{r} \rho(\bar{u}_s) dr} \left( 2 \sum_{i=1}^{k} \langle A^i_s(u_s) - A^i_s(\bar{u}_s), u_s - \bar{u}_s \rangle \right) dt + \sum_{j=1}^{\infty} |B^j_s(u_s) - B^j_s(\bar{u}_s)|_H^2
+ \int_{\mathcal{D}^c} |\gamma_s(u_s, z) - \gamma_s(\bar{u}_s, z)|_H^2 v(dz) - \rho(\bar{u}_s)|u_s - \bar{u}_s|_H^2 ds \leq 0,
\]

where last inequality follows from Assumption A-2. Thus if \( u_0 = \bar{u}_0 \) \( \mathbb{P} \)-a.s., then

\[
\mathbb{E}[e^{-\int_{0}^{t_n} \rho(\bar{u}_s) ds} |u_{t_n} - \bar{u}_{t_n}|_H^2] \leq 0.
\]
Letting \( n \to \infty \) and using Fatou’s lemma we conclude that for all \( t \in [0, T] \), we have \( \mathbb{P}(|u_t - \tilde{u}_t|^2_H = 0) = 1 \). This, together with the fact that \( u - \tilde{u} \) is càdlàg in \( H \), finishes the proof.

In order to get results about continuous dependence of the solution to Eq. 2.5 on the initial data, we consider the following.

**A - 7 (Strong Monotonicity)** There exist constants \( \theta' > 0 \) and \( K \) such that almost surely, for all \( t \in [0, T] \) and \( x, \bar{x} \in V \),

\[
2 \sum_{i=1}^{k} (A^i_t(x) - A^i_t(\bar{x}), x - \bar{x}) + (p_0 - 1) \sum_{j=1}^{\infty} |B^j_t(x) - B^j_t(\bar{x})|^2_H \\
+ \int_{Dc} |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^2_H \nu(dz) \leq -\theta' \sum_{i=1}^{k} |x - \bar{x}|_{V_i}^{\alpha_i} + K|x - \bar{x}|^2_H.
\]

**A - 8** There exist a constant \( K \) such that almost surely, for all \( t \in [0, T] \) and \( x, \bar{x} \in V \),

\[
\int_{Dc} |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^{p_0}_H \nu(dz) \leq K|x - \bar{x}|^{p_0}_H.
\]

**A - 9** There exist a constant \( K \) such that almost surely, for all \( t \in [0, T] \) and \( x, \bar{x} \in V \),

\[
\sum_{j=1}^{\infty} |B^j_t(x) - B^j_t(\bar{x})|^2_H + \int_{Dc} |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^2_H \nu(dz) \\
\leq K\left(|x - \bar{x}|^2_H + \sum_{i=1}^{k} |x - \bar{x}|_{V_i}^{\alpha_i}\right).
\]

If we replace the local monotonicity Assumption A-2 by the strong monotonicity Assumption A-7 and Assumption A-5 by Assumption A-8, then we obtain the following result about the continuous dependence of the solution to Eq. 2.5 on the initial data.

**Theorem 4** Let Assumptions A-3, A-4, A-7 and A-8 hold. Further assume \( u_0, \tilde{u}_0 \in L^p_0(\Omega; H) \). If \( u \) and \( \tilde{u} \) are two solutions of Eq. 2.5 with initial condition \( u_0 \) and \( \tilde{u}_0 \) respectively, then for \( p_0 > 2 \)

\[
\sup_{t \in [0, T]} \mathbb{E}|u_t - \tilde{u}_t|^{p_0}_H + \mathbb{E} \sum_{i=1}^{k} \int_{0}^{T} |u_t - \tilde{u}_t|^{p_0-2}_H |u_t - \tilde{u}_t|_{V_i}^{\alpha_i} dt \leq CE|u_0 - \tilde{u}_0|^{p_0}_H,
\]

\[
\mathbb{E} \sup_{t \in [0, T]} |u_t - \tilde{u}_t|^{p_0r}_H \leq CE|u_0 - \tilde{u}_0|^{p_0r}_H
\]

for any \( r \in (0, 1) \) and

\[
\sup_{t \in [0, T]} \mathbb{E}|u_t - \tilde{u}_t|^2_H + \sum_{i=1}^{k} \mathbb{E} \int_{0}^{T} |u_t - \tilde{u}_t|^{\alpha_i}_{V_i} dt \leq CE|u_0 - \tilde{u}_0|^2_H.
\]

**Proof** The proof is very similar to the proof of Theorem 2. Indeed we apply Itô formula from [4] to Eq. 2.21 and repeat the proof of Theorem 2 for the process \( u_t - \tilde{u}_t \). Here we note that one needs to use the strong monotonicity Assumption A-7 in place of Assumption
A-3, Assumption A-8 in place of Assumption A-5 and work with the sequence of stopping times given by Eq. 2.23.

Remark 5 Assuming Assumption A-9 in addition to assumptions made in Theorem 4 above, we further obtain

\[ \mathbb{E} \sup_{t \in [0,T]} |u_t - \bar{u}_t|^2_H \leq C \mathbb{E} |u_0 - \bar{u}_0|^2_H. \]

2.3 Existence of Solution

We prove the existence of solution to stochastic evolution equation (2.5) by using the Galerkin method. We consider a Galerkin scheme \((\mathcal{V}_m)_{m \in \mathbb{N}}\) for \(V\), i.e. for each \(m \in \mathbb{N}\), \(\mathcal{V}_m\) is an \(m\)-dimensional subspace of \(V\) such that \(\mathcal{V}_m \subset \mathcal{V}_{m+1} \subset V\) and \(\cup_{m \in \mathbb{N}} \mathcal{V}_m\) is dense in \(V\). Let \(\{\phi_l : l = 1, 2, \ldots, m\}\) be a basis of \(\mathcal{V}_m\). Assume that for each \(m \in \mathbb{N}\), \(u^m_0\) is a \(\mathcal{V}_m\)-valued \(F_0\)-measurable random variable satisfying

\[ \sup_{m \in \mathbb{N}} \mathbb{E} |u^m_0|^p_\mathcal{V}_m < \infty \quad \text{and} \quad \mathbb{E} |u^m_0 - u_0|^2_H \to 0 \quad \text{as} \quad m \to \infty. \]  

(2.24)

It is always possible to obtain such an approximating sequence. For example, consider \(\{\phi_l\}_{l \in \mathbb{N}} \subset V\) forming an orthonormal basis in \(H\) and for each \(m \in \mathbb{N}\), take \(u^m_0 = \Pi_m u_0\) where \(\Pi_m : H \to \mathcal{V}_m\) are the projection operators.

For each \(m \in \mathbb{N}\) and \(\phi_l \in \mathcal{V}_m\), \(l = 1, 2, \ldots, m\), consider the stochastic differential equation:

\[ (u^m_t, \phi_l) = (u^m_0, \phi_l) + \sum_{i=1}^k \int_0^t \langle A^i_s(u^m_s), \phi_l \rangle ds \]
\[ + \sum_{j=1}^m \int_0^t \langle \phi_l, B^j_s(u^m_s) \rangle dW^j_s + \int_0^t \langle \phi_l, \gamma_s(u^m_s, z) \rangle \tilde{N}(ds, dz) \]  

(2.25)

almost surely for all \(t \in [0,T]\). Using the results on solvability of stochastic differential equations in finite dimensional space (see, e.g., Theorem 1 in Gyöngy and Krylov [5]), together with Assumptions A-1 to A-5 and Remark 2, there exists a unique adapted and càdlàg (and thus progressively measurable) \(\mathcal{V}_m\)-valued process \(u^m\) satisfying (2.25).

Lemma 2.5 (A priori Estimates for Galerkin Discretization) Suppose that Eq. 2.24 and Assumptions A-3, A-4 and A-5 hold. Then there exists a constant \(C\) independent of \(m\), such that

i) for every \(p_0 \geq \beta + 2\),

\[ \sup_{t \in [0,T]} \mathbb{E} |u^m_t|^{p_0}_H + \sum_{i=1}^k \mathbb{E} \int_0^T |u^m_t|^{\alpha_i}_{\mathcal{V}_i} dt + \sum_{i=1}^k \mathbb{E} \int_0^T |u^m_t|^{p_0-2}_H |u^m_t|^{\alpha_i}_{\mathcal{V}_i} dt \leq C. \]

ii) Further,

\[ \mathbb{E} \sup_{t \in [0,T]} |u^m_t|^2_H \leq C \quad \text{and} \quad \mathbb{E} \sup_{t \in [0,T]} |u^m_t|^p_H \leq C \]

for any \(p \in [2, p_0)\) in case \(p_0 > 2\).

iii) Moreover, for all \(i = 1, 2, \ldots, k\)

\[ \mathbb{E} \int_0^T |A^i_s(u^m_s)|^{\alpha_i}_{V^i} ds \leq C \]
iv) and finally,
\[
\mathbb{E} \sum_{j=1}^{\infty} \int_0^T |B_j(u^m)|_H^2 ds + \mathbb{E} \int_0^T \int_{D^c} |\gamma_s(u^m, z)|_H^2 \nu(dz) ds \leq C.
\]

**Proof** Proof of (i) and (ii) is almost a repetition of the proof of analogous results in Theorem 2. Indeed, for each \( m, n \in \mathbb{N} \), one can define a sequence of stopping times
\[
\sigma_n^m := \inf \{ t \in [0, T] : |u_t^m|_H > n \} \wedge T
\]
and repeat the proof of Theorem 2 by replacing \( u_t \) with \( u_t^m \) and \( \sigma_n \) with \( \sigma_n^m \). There are two main points to be noted. First, the stochastic integrals appearing on the right-hand side of Eq. 2.10, with \( u_s \) replaced by \( u_s^m \), are martingales for each \( m, n \in \mathbb{N} \). Indeed, on a finite dimensional space, all norms are equivalent and hence for each \( m, n \in \mathbb{N} \),
\[
\mathbb{E} \int_0^{T \wedge \sigma_n^m} |u_s^m|_V^q ds \leq C_m \mathbb{E} \int_0^{T \wedge \sigma_n^m} n^q ds < \infty
\]
with some constant \( C_m \). The second point is that, since
\[
\sup_{m \in \mathbb{N}} \mathbb{E} |u_0^m|_{p_0} < \infty,
\]
one can take a constant independent of \( m \) to obtain (i) and (ii). The estimates in (iii) and (iv) can be proved as below. Using Assumption A-4, we obtain
\[
I := \sum_{i=1}^k \mathbb{E} \int_0^T |A_i(u_s^m)|_{V_i}^{\alpha_i} ds \leq \sum_{i=1}^k \mathbb{E} \int_0^T (f_s + K |u_s^m|_{V_i}^\beta (1 + |u_s^m|_H)) ds
\]
\[
= k\mathbb{E} \int_0^T f_s ds + k\mathbb{E} \int_0^T f_s |u_s^m|_{V_i}^\beta ds + K \sum_{i=1}^k \mathbb{E} \int_0^T |u_s^m|_{V_i}^{\alpha_i} ds
\]
\[+ K \sum_{i=1}^k \mathbb{E} \int_0^T |u_s^m|_H^\beta |u_s^m|_{V_i}^{\alpha_i} ds.
\]

Further application of Young’s inequality yields,
\[
f_s + f_s |u_s^m|_{H}^\beta \leq \frac{4}{p_0} f_s^{p_0} + \frac{p_0 - 2}{p_0} + \frac{p_0 - 2}{p_0} |u_s^m|_H^{p_0 - 2}
\]
\[\leq \frac{4}{p_0} f_s^{p_0} + \frac{p_0 - 2}{p_0} + \frac{p_0 - 2 - \beta}{p_0 - 2} + \frac{\beta}{p_0} |u_s^m|_{H}^{p_0 - 2},
\]
where we have used the fact that \( p_0 \geq \beta + 2 \). Note that it also implies \( |u_s^m|_{H}^\beta \leq (1 + |u_s^m|_H)^{p_0 - 2} \). Hence,
\[
I \leq C \left[ \mathbb{E} \int_0^T f_s^{p_0} ds + T + \mathbb{E} \int_0^T |u_s^m|_{H}^{p_0} ds + \sum_{i=1}^k \mathbb{E} \int_0^T |u_s^m|_{V_i}^{\alpha_i} ds \right.
\]
\[\left. + \sum_{i=1}^k \mathbb{E} \int_0^T |u_s^m|_{V_i}^{\alpha_i} (1 + |u_s^m|_H)^{p_0 - 2} ds \right].
\]
\[
\leq C\left[\mathbb{E}\int_0^T f_s^{\mathbb{P}}_0 ds + T + \sup_{0 \leq s \leq T} \mathbb{E}|u^m_s|_{\mathcal{V}_i}^{\mathbb{P}_0} + \sum_{i=1}^{k} \mathbb{E}\int_0^T |u^m_s|_{\mathcal{V}_i}^{\mathbb{P}_0} ds + \sum_{i=1}^{k} \mathbb{E}\int_0^T |u^m_s|_{\mathcal{V}_i}^{\mathbb{P}_0 - 2} ds \right]. \quad (2.26)
\]

By using (i) in Eq. 2.26, we obtain (iii). Furthermore, by Remark 1, we get

\[
\mathbb{E}\int_0^T \sum_{j=1}^{\infty} |B_j^i(u^{m}_s)|_{\mathcal{H}}^2 ds + \mathbb{E}\int_0^T \int_{\mathcal{D}^\mathcal{P}} |\gamma_s(u^{m}_s, z)|_{\mathcal{H}}^2 v(dz) ds
\]

\leq C\left[ T + \mathbb{E}\int_0^T f_s^{\mathbb{P}}_0 ds + \mathbb{E}\int_0^T |u^m_s|_{\mathcal{V}_i}^{\mathbb{P}_0} ds + \sum_{i=1}^{k} \mathbb{E}\int_0^T |u^m_s|_{\mathcal{V}_i}^{\mathbb{P}_0} (1 + |u^m_s|_{\mathcal{H}})_{\mathcal{P}_0 - 2} ds \right]

and hence by using (i), we get (iv).

\[\square\]

Having obtained the necessary a priori estimates, we will now extract weakly convergent subsequences using the compactness argument. After that using the local monotonicity condition, we establish the existence of a solution to Eq. 2.5.

**Lemma 2.6** Let Assumptions A-2 to A-5 together with Eq. 2.24 hold. Then there is a subsequence \((m_q)_{q \in \mathbb{N}}\) and

i) there exists a process \(u \in \bigcap_{i=1}^{k} L^{\alpha_i}((0, T) \times \mathcal{O}; \mathcal{V}_i)\) such that
\[u^{m_q} \rightharpoonup u \text{ in } L^{\alpha_i}((0, T) \times \mathcal{O}; \mathcal{V}_i) \quad \forall i = 1, 2, \ldots, k,\]

ii) there exist processes \(a^i \in L^{\frac{\alpha_i}{n-i}}((0, T) \times \mathcal{O}; V^*)\) such that
\[A^i(u^{m_q}) \rightharpoonup a^i \text{ in } L^{\frac{\alpha_i}{n-i}}((0, T) \times \mathcal{O}; V^*) \quad \forall i = 1, 2, \ldots, k,\]

iii) there exists a process \(b \in L^2((0, T) \times \mathcal{O}; \ell^2(H))\) such that
\[B(u^{m_q}) \rightharpoonup b \text{ in } L^2((0, T) \times \mathcal{O}; \ell^2(H)),\]

iv) there exists \(\Gamma \in L^2((0, T) \times \mathcal{O} \times Z; H)\) such that
\[\gamma(u^{m_q})1_{\mathcal{D}^\mathcal{P}} \rightharpoonup \Gamma1_{\mathcal{D}^\mathcal{P}} \text{ in } L^2((0, T) \times \mathcal{O} \times Z; H).\]

**Proof** Note that the Banach spaces \(L^{\alpha_i}((0, T) \times \mathcal{O}; \mathcal{V}_i), L^{\frac{\alpha_i}{n-i}}((0, T) \times \mathcal{O}; V_i^*), L^2((0, T) \times \mathcal{O}; \ell^2(H))\) and \(L^2((0, T) \times \mathcal{O} \times Z; H)\) are reflexive. Thus, due to Lemma 2.5, there exists a subsequence \(m_q\) (see, e.g., Theorem 3.18 in [1]) such that

i) \(u^{m_q} \rightharpoonup u^i \text{ in } L^{\alpha_i}((0, T) \times \mathcal{O}; \mathcal{V}_i) \quad \forall i = 1, 2, \ldots, k,\)

(ii) \(A^i(u^{m_q}) \rightharpoonup a^i \text{ in } L^{\frac{\alpha_i}{n-i}}((0, T) \times \mathcal{O}; V_i^*) \quad \forall i = 1, 2, \ldots, k,\)
Further, for any $\xi \in V$ and for any adapted and bounded real valued process $\eta_t$, we have for $i, j \in \{1, 2, \ldots, k\}$

$$
E \int_0^T \eta_t (u^i_t - u^j_t, \xi) dt = E \int_0^T \eta_t (u^i_t - u^m_q, \xi) dt + E \int_0^T \eta_t (u^m_q - u^j_t, \xi) dt
$$

with right-hand-side converging to zero as $q \to \infty$. Therefore the processes $u^i, i=1, 2, \ldots, k$ are equal $dt \times P$ almost everywhere and henceforth are denoted by $u$ in the remaining article.

**Lemma 2.7** Let Assumptions A-2 to A-5 together with Eq. 2.24 hold. Then

i) for $dt \times P$ almost everywhere,

$$
u_t = u_0 + \sum_{i=1}^{k} \int_0^t a^i_s ds + \sum_{j=1}^{\infty} \int_0^t b^j_s dW^j_s + \int_0^t \int_{D^c} \Gamma_s(z) \tilde{N}(ds,dz)
$$

and moreover almost surely $u \in D([0, T]; H)$ and for all $t \in [0, T]$,

$$
|u_t|^2_H = |u_0|^2_H + \sum_{i=1}^{k} \int_0^t \left[ 2 \sum_{j=1}^{\infty} a^i_s, u^j_s \right] ds + \sum_{j=1}^{\infty} \int_0^t |b^j_s|^2_H ds + \sum_{j=1}^{\infty} \int_0^t (u^j_s, b^j_s) dW^j_s + \int_0^t \int_{D^c} 2(u^j_s, \Gamma_s(z)) \tilde{N}(ds,dz)
$$

$$
+ \int_0^t \int_{D^c} |\Gamma_s(z)|^2_H N(ds,dz).
$$

(2.27)

ii) Finally, $u \in L^2(\Omega; D([0, T]; H))$.

**Proof** Using Itô’s isometry, we see that the stochastic integral with respect to Wiener process is a bounded linear operator from $L^2((0, T) \times \Omega; \ell^2(H))$ to $L^2((0, T) \times \Omega \times \Omega; H)$ and hence maps a weakly convergent sequence to a weakly convergent sequence. Thus, we obtain

$$
\sum_{j=1}^{q} \int_0^t B^j_s (u^m_q) dW^j_s \to \sum_{j=1}^{\infty} \int_0^t b^j_s dW^j_s
$$

in $L^2([0, T] \times \Omega; H)$, i.e. for any $\psi \in L^2((0, T) \times \Omega; H),

$$
E \int_0^T \left( \sum_{j=1}^{q} \int_0^t B^j_s (u^m_q) dW^j_s, \psi(t) \right) dt \to E \int_0^T \left( \sum_{j=1}^{\infty} \int_0^t b^j_s dW^j_s, \psi(t) \right) dt.
$$

(2.28)

By similar argument, for any $\psi \in L^2((0, T) \times \Omega; H)$ we have

$$
E \int_0^T \left( \int_0^t \int_{D^c} \gamma_s (u^m_q, z) \tilde{N}(ds,dz), \psi(t) \right) dt
$$

$$
\to E \int_0^T \left( \int_0^t \int_{D^c} \Gamma_s(z) \tilde{N}(ds,dz), \psi(t) \right) dt.
$$

(2.29)
Similarly, using Holder’s inequality we see that for each $i = 1, 2, \ldots, k$, the Bochner integral is a bounded linear operator from $L^{\alpha_i}$ to $L^{\alpha_i-1}$ and is thus continuous with respect to weak topologies. Therefore, for any $\psi \in L^{\alpha_i}$, and is thus continuous with respect to weak topologies. Therefore, for any $\psi \in L^{\alpha_i}$, we have for any $q \geq n$,

$$E \int_0^T \left( \int_0^t A^i_s(u^m_s)ds, \psi(t) \right) dt \to E \int_0^T \left( \int_0^t a^i_s ds, \psi(t) \right) dt.$$  \(2.30\)

Fix $n \in \mathbb{N}$. Then for any $\phi \in \mathcal{V}_n$ and an adapted real valued process $\eta_t$ bounded by a constant $C$, we have for any $q \geq n$,

$$E \int_0^T \eta_t(u^m_t, \phi) dt = E \int_0^T \eta_t \left( u^0_t, \phi \right) + \sum_{i=1}^k \int_0^t \langle a^i_s(u^m_s), \phi \rangle ds$$

$$+ \sum_{j=1}^\infty \int_0^t \langle \phi, B^j_s(u^m_s) \rangle dW^j_s + \int_0^t \int_{\mathcal{D}_c} \langle \phi, \gamma_s(u^m_s, z) \rangle \tilde{N}(ds, dz) \right) dt.$$  \(\text{Taking the limit } q \to \infty \text{ and using Eqs. } 2.24, 2.28, 2.29 \text{ and } 2.30, \text{ we obtain}

$$E \int_0^T \eta_t(u_t, \phi) dt = E \int_0^T \eta_t \left( u^0_t, \phi \right) + \sum_{i=1}^k \int_0^t \langle a^i_s, \phi \rangle ds$$

$$+ \sum_{j=1}^\infty \int_0^t \langle \phi, B^j_s \rangle dW^j_s + \int_0^t \int_{\mathcal{D}_c} \langle \phi, \Gamma_s(z) \rangle \tilde{N}(ds, dz) \right) dt$$

with any $\phi \in \mathcal{V}_n$ and any adapted and bounded real valued process $\eta_t$. Since $\cup_{n \in \mathbb{N}} \mathcal{V}_n$ is dense in $\mathcal{V}$, we obtain

$$u_t = u^0_t + \sum_{i=1}^k \int_0^t a^i_s ds + \sum_{j=1}^\infty \int_0^t b^j_s dW^j_s + \int_0^t \int_{\mathcal{D}_c} \Gamma_s(z) \tilde{N}(ds, dz)$$  \(2.31\)

almost everywhere.

Using Theorem 2.1 on Itô’s formula from [4], there exists an $H$-valued càdlàg modification of the process $u$, denoted again by $u$, which is equal to the right hand side of Eq. 2.31 almost surely for all $t \in [0, T]$. Moreover (2.27) holds almost surely for all $t \in [0, T]$. This completes the proof of part (i) of the lemma. It remains to prove part (ii) of the lemma. To that end, consider the sequence of stopping times $\sigma_n$ defined in Eq. 2.9. Using the Burkholder–Davis–Gundy inequality together with Cauchy–Schwartz’s and Young’s inequalities, we obtain

$$E \sup_{t \in [0, T]} \left| \sum_{j=1}^\infty \int_0^{T \wedge \sigma_n} (u_s, b^j_s) dW^j_s \right| \leq 4 E \left( \sum_{j=1}^\infty \int_0^{T \wedge \sigma_n} |(u_s, b^j_s)|^2_H ds \right)^{1/2}$$

$$\leq 4 E \left( \sum_{j=1}^\infty \int_0^{T \wedge \sigma_n} |u_s|^2_H |b^j_s|^2_H ds \right)^{1/2}$$

$$\leq 4 E \left( \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|^2_H \sum_{j=1}^\infty \int_0^{T \wedge \sigma_n} |b^j_s|^2_H ds \right)^{1/2}$$

$$\leq \epsilon E \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|^2_H + C E \sum_{j=1}^\infty \int_0^{T \wedge \sigma_n} |b^j_s|^2_H ds.$$  \(2.32\)
Similarly,
\[
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^{t \wedge \sigma_n} \left( u_s, \Gamma_s(z) \right) \tilde{N}(ds, dz) \right| \\
\leq C \mathbb{E} \left( \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} \left| (u_s, \Gamma_s(z)) \right|^2 \nu(dz) ds \right)^{1/2} \\
\leq C \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} \left| u_s \right| \left| \Gamma_s(z) \right|^2 \nu(dz) ds \right)^{1/2} \\
\leq C \mathbb{E} \left( \sup_{t \in [0,T]} \left| u_{t \wedge \sigma_n} \right|_{H}^2 \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} \left| \Gamma_s(z) \right|^2 \nu(dz) ds \right)^{1/2} \\
\leq \epsilon \mathbb{E} \sup_{t \in [0,T]} \left| u_{t \wedge \sigma_n} \right|_{H}^2 + C \mathbb{E} \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} \left| \Gamma_s(z) \right|^2 \nu(dz) ds.
\] (2.33)

Replace \( t \) by \( t \wedge \sigma_n \) in Eq. 2.27 and take supremum and then expectation. On using Hölder’s inequality along with Eqs. 2.32 and 2.33, we obtain
\[
\mathbb{E} \sup_{t \in [0,T]} \left| u_{t \wedge \sigma_n} \right|_{H}^2 \leq \mathbb{E} \left| u_0 \right|_{H}^2 + 2 \sum_{i=1}^{k} \left( \mathbb{E} \int_0^{T} \left| a_i^j \right|^{\alpha_i} \left| a_i^{j-1} \right| ds \right)^{\frac{\alpha_i-1}{\alpha_i}} \left( \mathbb{E} \int_0^{T} \left| u_s \right|^{\alpha_i} \nu(dz) ds \right)^{1/2} \\
+ \epsilon \mathbb{E} \sup_{t \in [0,T]} \left| u_{t \wedge \sigma_n} \right|_{H}^2 + C \mathbb{E} \sum_{j=1}^{\infty} \int_0^{T} \left| b_j^i \right|_{H}^2 ds + C \mathbb{E} \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} \left| \Gamma_s(z) \right|^2 \nu(dz) ds
\]
which on choosing \( \epsilon \) small enough gives,
\[
\mathbb{E} \sup_{t \in [0,T]} \left| u_{t \wedge \sigma_n} \right|_{H}^2 \leq C \left[ \mathbb{E} \left| u_0 \right|_{H}^2 + \sum_{i=1}^{k} \left( \mathbb{E} \int_0^{T} \left| a_i^j \right|^{\alpha_i} \left| a_i^{j-1} \right| ds \right)^{\frac{\alpha_i-1}{\alpha_i}} \left( \mathbb{E} \int_0^{T} \left| u_s \right|^{\alpha_i} \nu(dz) ds \right)^{1/2} \\
+ \mathbb{E} \sum_{j=1}^{\infty} \int_0^{T} \left| b_j^i \right|_{H}^2 ds + \mathbb{E} \int_0^{T \wedge \sigma_n} \int_{\mathcal{D}^c} \left| \Gamma_s(z) \right|^2 \nu(dz) ds \right]
\]
Finally taking \( n \to \infty \) and using Fatou’s lemma, we obtain
\[
\mathbb{E} \sup_{t \in [0,T]} \left| u_t \right|_{H}^2 < \infty
\]
which finishes the proof. \( \square \)

In order to prove that the process \( u \) is the solution of Eq. 2.5, it remains to show that \( dt \times \mathbb{P} \) almost everywhere \( A^i(u) = a^i \) for \( i = 1, 2, \ldots, k \), \( B^j(u) = b^j \) for all \( j \in \mathbb{N} \) and \( dt \times \mathbb{P} \times \nu \) almost everywhere \( \gamma(u) 1_{\mathcal{D}^c} = \Gamma 1_{\mathcal{D}^c} \). Recall that \( \Psi \) and \( \rho \) were given in Definition 2.4.

**Theorem 5** (Existence of solution) If Assumptions A-1 to A-5 hold and \( u_0 \in L^{p_0}(\Omega; H) \), then the stochastic evolution equation (2.5) has a unique solution. Hence, using interlacing procedure, Eq. 2.1 has a unique solution.
Proof Let $\psi \in \cap_{i=1}^{k} L^{a_{i}}((0, T) \times \Omega; V_{i}) \cap \Psi \cap L^{2}(\Omega; D([0, T]; H))$. Then using the product rule and Itô’s formula, we obtain

$$
\begin{align*}
\mathbb{E}(e^{-\int_{0}^{t} \rho(\psi)ds}|u_{t}^{2}_{H}) & - \mathbb{E}(|u_{0}^{2}_{H}) = \mathbb{E}
\left[ \int_{0}^{t} e^{-\int_{0}^{s} \rho(\psi)dr} \left( \sum_{i=1}^{k} \langle A^{i}_{s}(u^{m_{q}}_{s}), u^{m_{q}}_{s} \rangle_{i}ight) ds + \sum_{j=1}^{\infty} \| B^{j}_{s}(u^{m_{q}}_{s}) \|_{H}^{2} + \int_{D^{c}} |\gamma_{s}(u^{m_{q}}_{s}, z)|_{H}^{2} v(dz) - \rho(\psi_{s})|u_{s}^{2}_{H} | ds \right]
\end{align*}
$$

(2.34)

and

$$
\begin{align*}
\mathbb{E}(e^{-\int_{0}^{t} \rho(\psi)ds}|u_{t}^{m_{q}}^{2}_{H}) & - \mathbb{E}(u_{0}^{m_{q}}^{2}_{H}) = \mathbb{E}
\left[ \int_{0}^{t} e^{-\int_{0}^{s} \rho(\psi)dr} \left( \sum_{i=1}^{k} \langle A^{i}_{s}(u^{m_{q}}_{s}), u^{m_{q}}_{s} \rangle_{i}ight) ds + \sum_{j=1}^{\infty} \| B^{j}_{s}(u^{m_{q}}_{s}) \|_{H}^{2} + \int_{D^{c}} |\gamma_{s}(u^{m_{q}}_{s}, z)|_{H}^{2} v(dz) - \rho(\psi_{s})|u_{s}^{m_{q}}^{2}_{H} | ds \right]
\end{align*}
$$

for all $t \in [0, T]$. Note that in view of Remark 4, all the integrals are well defined in what follows. Moreover,

$$
\begin{align*}
\mathbb{E}
\left[ \int_{0}^{t} e^{-\int_{0}^{s} \rho(\psi)dr} \left( \sum_{i=1}^{k} \langle A^{i}_{s}(u^{m_{q}}_{s}), u^{m_{q}}_{s} \rangle_{i}ight) ds + \sum_{j=1}^{\infty} \| B^{j}_{s}(u^{m_{q}}_{s}) \|_{H}^{2} + \int_{D^{c}} |\gamma_{s}(u^{m_{q}}_{s}, z)|_{H}^{2} v(dz) - \rho(\psi_{s})|u_{s}^{m_{q}}^{2}_{H} | ds \right]
\end{align*}
$$

\begin{align*}
= \mathbb{E}
\left[ \int_{0}^{t} e^{-\int_{0}^{s} \rho(\psi)dr} \left( \sum_{i=1}^{k} \langle A^{i}_{s}(u^{m_{q}}_{s}), u^{m_{q}}_{s} \rangle - \langle A^{i}_{s}(\psi_{s}), u^{m_{q}}_{s} \rangle_{i} + \gamma_{s}(u^{m_{q}}_{s}, z) - \gamma_{s}(\psi_{s}, z) |_{H}^{2} v(dz) - \rho(\psi_{s}) \left( |u_{s}^{m_{q}} - \psi_{s}|_{H}^{2} + 2(\gamma_{s}(u^{m_{q}}_{s}, z) + \gamma_{s}(\psi_{s}, z)) |_{H}^{2} v(dz) \right) \right) ds \right]
\end{align*}

Now applying the local monotonicity Assumption A-2, we see that

$$
\begin{align*}
\mathbb{E}(e^{-\int_{0}^{t} \rho(\psi)ds}|u_{t}^{m_{q}}^{2}_{H}) & - \mathbb{E}(u_{0}^{m_{q}}^{2}_{H}) \\
& \leq \mathbb{E}
\left[ \int_{0}^{t} e^{-\int_{0}^{s} \rho(\psi)dr} \left( \sum_{i=1}^{k} \langle A^{i}_{s}(\psi_{s}), u^{m_{q}}_{s} \rangle_{i} + 2 \sum_{i=1}^{k} \langle A^{i}_{s}(u^{m_{q}}_{s}), u^{m_{q}}_{s} \rangle_{i} - \rho(\psi_{s}) \left( |u_{s}^{m_{q}} - \psi_{s}|_{H}^{2} + 2(\gamma_{s}(u^{m_{q}}_{s}, z) + \gamma_{s}(\psi_{s}, z)) |_{H}^{2} v(dz) \right) \right) ds \right]
\end{align*}
$$
Integrating over $t$ from 0 to $T$, letting $q \to \infty$ and using the weak lower semicontinuity of the norm we obtain,

$$\mathbb{E}\left[ \int_0^T (e^{-\int_0^t \rho(\psi_s) ds} |u_t|^2_H - |u_0|^2_H) dt \right] \leq \liminf_{k \to \infty} \mathbb{E}\left[ \int_0^T (e^{-\int_0^t \rho(\psi_s) ds} |u_{t}^m|^2_H - |u_0^m|^2_H) dt \right]$$

$$\leq \mathbb{E}\left[ \int_0^T \int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left( 2 \sum_{i=1}^k \langle A^i_s(\psi_s), u_s \rangle + 2 \sum_{i=1}^k \langle a^i_s - A^i_s(\psi_s), \psi_s \rangle \right) + \sum_{j=1}^\infty |B^j_s(\psi_s)|_H^2 + 2 \sum_{j=1}^\infty (b^j_s, B^j_s(\psi_s)) - \int_{D^c} |\gamma_s(\psi_s, z)|_H^2 v(dz)$$

$$+ 2 \int_{D^c} |\Gamma_s(z) - \Gamma_s(z)|_H^2 v(dz) + \rho(\psi_s) \left( |\psi_s|^2_H - 2(\psi_s, \psi_s) \right) ds dt \right]. \quad (2.35)$$

Integrating from 0 to $T$ in Eq. 2.34 and combining this with Eq. 2.35 leads to

$$\mathbb{E}\left[ \int_0^T \int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left( 2 \sum_{i=1}^k \langle a^i_s - A^i_s(\psi_s), \psi_s \rangle \right) + \sum_{j=1}^\infty |B^j_s(\psi_s)|_H^2 + 2 \sum_{j=1}^\infty (b^j_s, B^j_s(\psi_s)) - \int_{D^c} |\gamma_s(\psi_s, z)|_H^2 v(dz)$$

$$- \rho(\psi_s) |\psi_s - \psi_s|^2_H ds dt \right] \leq 0. \quad (2.36)$$

Further, using Definition 2.4, Lemma 2.6 and Lemma 2.7 $u \in \cap_{i=1}^k L^{a_i}((0, T) \times \Omega; V_i) \cap \Psi \cap L^2(\Omega; D([0, T]; H)).$

Taking $\psi = u$ in Eq. 2.36, we obtain that $B^j(u) = b^j$ for all $j \in \mathbb{N}$ and $\gamma(u)1_{D^c} = \Gamma 1_{D^c}.$ Let $\eta \in L^\infty((0, T) \times \Omega; \mathbb{R}), \phi \in V, \epsilon \in (0, 1)$ and let $\psi = u - \epsilon \eta \phi.$ Then from Eq. 2.36 we obtain that,

$$\mathbb{E}\left[ \int_0^T \int_0^t e^{-\int_0^s \rho(u_r - \epsilon \eta \phi) dr} \left( 2 \epsilon \sum_{i=1}^k \langle a^i_s - A^i_s(u_s - \epsilon \eta \phi), \eta_s \phi \rangle \right) + \epsilon^2 \rho(u_s - \epsilon \eta \phi) |\eta_s \phi|_H^2 ds dt \right] \leq 0.$$

Now we divide by $\epsilon$ and let $\epsilon \to 0.$ Then, using Lebesgue dominated convergence theorem and Assumption A-1 we get,

$$\mathbb{E}\left[ \int_0^T \int_0^t e^{-\int_0^s \rho(u_r) dr} 2 \sum_{i=1}^k \langle a^i_s - A^i_s(u_s), \phi \rangle ds dt \right] \leq 0.$$

Since this holds for any $\eta \in L^\infty((0, T) \times \Omega; \mathbb{R})$ and $\phi \in V$, we get that $A^i(u) = a^i$ for all $i = 1, 2, \ldots, k$ which concludes the proof.
3 Stochastic Anisotropic p-Laplace Equation

In this section, we prove Theorem 1 by showing that the stochastic anisotropic p-Laplace equation (1.1), in its weak form, fits in the abstract framework discussed in previous section and hence possesses a unique solution which depends continuously on the initial data.

**Proof of Theorem 1** For \( i = 1, 2, \ldots, d \), take \( V_i := W_{0}^{X_{i}, p_{i}}(\mathcal{D}) \) defined in Section 1 so that the space \( V \) is the space \( W_{0}^{1, p}(\mathcal{D}) \) given by Eq. 1.3. Again for \( i = 1, 2, \ldots, d \) and \( t \in [0, T] \), let \( A_{t}^{i} : V_{i} \rightarrow V_{i}^{*} \) be given by,

\[
A_{t}^{i}(u) := D_{i}(\nu_{i}^{-\frac{p_{i}}{2}} D_{i} u).
\]

Further, let \( B_{t}^{i} : V \rightarrow L^{2}(\mathcal{D}) \) be given by,

\[
B_{t}^{i}(u) := \begin{cases} 
\xi_{j} |D_{j} u|^{\frac{p_{j}}{2}} + h_{j}(u) & \text{for } j = 1, 2, \ldots, d, \\
\xi_{j} & \text{otherwise.}
\end{cases}
\]

We note that for \( u, v \in V_{i} \),

\[
\langle A_{t}^{i}(u), v \rangle_{i} = -\int_{\mathcal{D}} |D_{i} u(x)|^{\nu_{i}^{-\frac{p_{i}}{2}} D_{i} u(x) D_{i} v(x) dx}
\]

and thus using Hölder’s inequality,

\[
|\langle A_{t}^{i}(u), v \rangle_{i}| \leq |u|_{V_{i}}^{\nu_{i}^{-1}} |v|_{V_{i}}.
\]

Thus, for every \( u \in V_{i} \), \( A_{t}^{i}(u) \) is a well-defined linear operator on \( V_{i} \) such that

\[
|A_{t}^{i} u|_{V_{i}^{*}} \leq |u|_{V_{i}}^{\nu_{i}^{-1}}
\]

which implies that Assumptions A-1 and A-4 hold with \( \alpha_{i} = p_{i} \) and \( \beta = 0 \).

We now verify the local monotonicity condition. From standard calculations for \( p \)-Laplace operators we obtain for each \( i = 1, 2, \ldots, d \),

\[
\langle D_{i}(|D_{i} u|^{\nu_{i}^{-\frac{p_{i}}{2}} D_{i} u}) - D_{i}(|D_{i} v|^{\nu_{i}^{-\frac{p_{i}}{2}} D_{i} v}), u - v \rangle_{i} + |\xi_{i} |D_{i} u|^{\frac{p_{i}}{2}} - \xi_{i} |D_{i} v|^{\frac{p_{i}}{2}} |L^{2} \leq 0
\]

provided \( \xi_{i}^{2} \leq \frac{4(p_{i}-1)}{p_{i}^{2}} \). Since the functions \( h_{j}, j \in \mathbb{N} \) are given to be Lipschitz continuous with Lipschitz constants \( M_{j} \) such that \( (M_{j})_{j} \in \ell^{2} \), we have

\[
|h_{j}(u) - h_{j}(v)|_{L^{2}} \leq M_{j}^{2} |u - v|_{L^{2}}.
\]

Using Eq. 1.4, we get for all \( t \in [0, T] \),

\[
\int_{D_{c}} |\gamma_{t}(u, z) - \gamma_{t}(v, z)|_{L^{2}}^{2} v(dz) \leq K |u - v|_{L^{2}}^{2}.
\]

Therefore, for all \( t \in [0, T] \),

\[
2 \sum_{i=1}^{d} \langle A_{t}^{i}(u) - A_{t}^{i}(v), u - v \rangle_{i} + \sum_{j=1}^{\infty} |B_{t}^{i}(u) - B_{t}^{i}(v)|_{L^{2}}^{2}
\]

\[
+ \int_{D_{c}} |\gamma_{t}(u, z) - \gamma_{t}(v, z)|_{L^{2}}^{2} v(dz) \leq C |u - v|_{L^{2}}^{2}
\]

and hence Assumption A-2 is satisfied.
We now wish to verify the $p_0$-stochastic coercivity condition A-3. However, in view of Remark 3, it is enough to verify Assumption A-6 instead. Taking $v = u$ in Eq. 3.1, we get
\[ \langle A^0_i(u), u \rangle_i = -\int_{\mathcal{D}} |D_i u(x)|^{p_0} \, dx. \]
Further,
\[ 2(p_0 - 1) |\xi_i |D_i u|^{\frac{p_i}{2}}|^2 L^2 = 2(p_0 - 1) \xi_i^2 \int_{\mathcal{D}} |D_i u(x)|^{p_0} \, dx. \]
Also, Eq. 1.5 gives
\[ \text{As desired. Since } u, v \text{, we have for each } i = 1, 2, \ldots, d, \]
\[ \text{Further as mentioned in Eq. 3.2,} \]
\[ \langle D_i (|D_i u|^{p_i-2} D_i u) - D_i (|D_i v|^{p_i-2} D_i v), u - v \rangle_i \leq -2^{-(p_i-2)} |D_i u - D_i v|^{p_i} |_{L^{p_i}}. \]

Provided $\xi_i^2 \leq \frac{2(p_i-1)}{p_i^2(p_0-1)}$. Thus we have for $u, v \in W^{1,p}_0(\mathcal{D})$ and $t \in [0, T]$,
\[ 2 \sum_{i=1}^d (A_i^j(u) - A_i^j(v), u - v)_i + (p_0 - 1) \sum_{j=1}^\infty |B_j^i(u) - B_j^i(v)|^2 L^2 \]
\[ + \int_{\mathcal{D}^c} |\gamma_t(u, z) - \gamma_t(v, z)|^2 L^2 v(dz) \leq -\theta' \sum_{i=1}^d |D_i u - D_i v|^{p_i} |_{L^{p_i}} + C |u - v|^2 L^2 \]
for any $\theta'$ satisfying $0 < \theta' < 2^{-(p_i-2)}$ for all $i$. Thus, Assumption A-7 is satisfied. Using Eq. 1.7, we obtain
\[ \int_{\mathcal{D}^c} |\gamma_t(u, z) - \gamma_t(v, z)|^{p_0} L^2 v(dz) \leq K |u - v|^{p_0} L^2. \]
showing that Assumption A-8 holds. Finally, for each \( i = 1, 2, \ldots, d \),
\[
|\xi_i| D_i u \bigg|_{L^2}^{2 p_i} - |\xi_i| D_i v \bigg|_{L^2}^{2 p_i} \leq C \big| D_i u - D_i v \big|_{L^2}^{2 p_i} \leq C|u - v|_{V_i}^{p_i}
\]
and therefore using Eqs. 3.3 and 3.4, we obtain
\[
\sum_{j=1}^{\infty} |B_j^i(u) - B_j^i(v)|_{L^2}^2 + \int_{D^c} |\gamma_t(u, z) - \gamma_t(v, z)|_{L^2}^2 \nu(dz)
\]
\[
\leq K \big( |u - v|_{L^2}^2 + \sum_{i=1}^{d} |u - v|_{V_i}^{p_i} \big)
\]
and hence (1.8) follows from Theorem 4 and Remark 5. This concludes the proof of Theorem 1 and hence establishes the well-posedness of stochastic anisotropic \( p \)-Laplace equation (1.1).

4 Examples

Finally, in this section, we present two examples of stochastic evolution equations which fit in the framework of this article and yet do not satisfy the assumptions of [2, 7] or [9]. Note that stochastic Ginzburg–Landau equation is a special case of the SPDE considered in Example 1 below whereas Example 2 addresses the well-posedness of the stochastic Swift–Hohenberg equation which is widely used as a model to understand various phenomena in pattern formation. Before proceeding to examples, we introduce few more notations.

Let \( D \subseteq \mathbb{R}^d \) be an open bounded domain with smooth boundary and \( W^{1,p}(D) \) be the Sobolev space of real valued functions \( u \), defined on \( D \), such that the norm \( |u|_{W^{1,p}} := \left( \int_{D} (|u(x)|^p + |\nabla u(x)|^p) \, dx \right)^{\frac{1}{p}} \) is finite, where \( \nabla := (D_1, D_2, \ldots, D_d) \) denotes the gradient.

The closure of \( C_0^\infty(D) \) in \( W^{1,p}(D) \) with respect to the norm \( |\cdot|_{1,p} \) is denoted by \( W^{1,p}_0(D) \). Friedrichs’ inequality (see, e.g. Theorem 1.32 in [14]) implies that the norm \( |u|_{W^{1,p}_0} := \left( \int_{D} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} \) is equivalent to \( |u|_{1,p} \) and this equivalent norm \( |u|_{W^{1,p}_0} \) will be used in what follows. Moreover, let \( W^{-1,p}(D) \) denote the dual of \( W^{1,p}_0(D) \) and let \( |\cdot|_{W^{-1,p}} \) be the norm on this dual space. It is well known that
\[
W^{1,p}_0(D) \hookrightarrow L^2(D) \equiv (L^2(D))^* \hookrightarrow W^{-1,p}(D),
\]
where \( \hookrightarrow \) denotes continuous and dense embeddings, is a Gelfand triple.

Example 1 (Quasi-linear equation) Let \( p_1, p_2 > 2 \). Assume that there are constants \( r, s \) such that
\[
f^0(x) x \leq K(1 + |x|^{p_1+1}); \quad |f^0(x)| \leq K(1 + |x|^r)
\]
and \( (f^0(x) - f^0(y))(x - y) \leq K(1 + |y|^r)(|x| + |y|^r) \ \forall \ x, y \in \mathbb{R} \).

Let \( h_j : \mathbb{R} \to \mathbb{R}, \ j \in \mathbb{N} \) be Lipschitz continuous functions with Lipschitz constants \( M_j \) such that the sequence \( (M_j)_{j} \in \ell^2 \). Further, let \( Z = \mathbb{R}^d, D^c = \{ z \in \mathbb{R}^d : |z| \leq 1 \} \) and \( v \) be
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a Lévy measure on $\mathbb{R}^d$. Finally assume that $\gamma : [0, T] \times \Omega \times \mathbb{R} \times Z \rightarrow Z$ satisfies

$$|\gamma_t(x, z) - \gamma_t(y, z)| \leq K|x - y||z| \quad \text{and} \quad |\gamma_t(x, z)| \leq K(1 + |x||z|)$$

almost surely, for all $t \in [0, T]$, $x, y \in \mathbb{R}$, $z \in \mathcal{D}^c$.

Consider the stochastic partial differential equation,

$$du_t = \left( \sum_{l=1}^{d} D_l(|D_l u_t|^{p_1-2} D_l u_t) - |u_t|^{p_2-2} u_t + f^0(u_t) \right) dt$$

$$+ \sum_{j=1}^{d} \zeta |D_j u_t|^{p_1} dW^j_t + \sum_{j=1}^{\infty} h_j(u_t) dW^j_t$$

$$+ \int_{\mathcal{D}^c} \gamma_t(u_t, z) \tilde{N}(dt, dz) + \int_{\mathcal{D}} \gamma_t(u_t, z) N(dt, dz)$$

(4.1)

on $(0, T) \times \mathcal{D}$, where $u_t = 0$ on $\partial \mathcal{D}$ and $u_0$ is a given $\mathcal{F}_0$-measurable random variable and $\zeta$ is a constant to be chosen suitably. Moreover, $W^j$ are independent Wiener processes. We will now show that such an equation, in its weak form, fits the assumptions of the present article if any of the following holds:

1. $d < p_1$, $r = p_1 + 1$, $s \leq p_1$, $t = 2$ and $u_0 \in L^6(\Omega; L^2(\mathcal{D}))$.
2. $d > p_1$, $r = \frac{2p_1}{d} + p_1 - 1$, $s \leq \min \left\{ \frac{p_1^2(s-2)}{(d-p_1)(p_1-2)}, \frac{p_1(p_1-t)}{(p_1-2)} \right\}$, $2 < t < p_1$ and $u_0 \in L^6(\Omega; L^2(\mathcal{D}))$.

Case 1. Take $V_1 := W_0^{p_1}(\mathcal{D})$, $V_2 := L^{p_2}(\mathcal{D})$ and $V := V_1 \cap V_2$. Then $(V, |\cdot|_{V_i})$ are reflexive and separable Banach spaces such that

$$V \hookrightarrow L^2(\mathcal{D}) \equiv (L^2(\mathcal{D}))^* \hookrightarrow V^*.$$ 

Let $A^1 : V_1 \rightarrow V_1^*$ and $A^2 : V_2 \rightarrow V_2^*$ be given by,

$$A^1(u) := \sum_{l=1}^{d} D_l(|D_l u|^{p_1-2} D_l u) + f^0(u) \quad \text{and} \quad A^2(u) := -|u|^{p_2-2} u.$$

Moreover, $B^j : V \rightarrow L^2(\mathcal{D})$ be given by

$$B^j(u) := \begin{cases} \zeta |D_j u|^{p_1} + h_j(u) & \text{for } j = 1, 2, \ldots, d, \\ h_j(u) & \text{otherwise.} \end{cases}$$

The next step is to show that these operators satisfy the Assumptions A-1 to A-5. We immediately notice that A-1 holds since $f^0$ is continuous.

We now wish to verify the local monotonicity condition. As discussed in previous section, for each $l = 1, 2, \ldots, d$

$$\langle D_l(|D_l u|^{p_1-2} D_l u) - D_l(|D_l v|^{p_1-2} D_l v), u - v \rangle_1 + |\zeta |D_l u|^{p_1} - \zeta |D_l v|^{p_1}|_{L^2}^2 \leq 0$$

provided $\zeta^2 \leq \frac{4(p_1-1)}{p_1^2}$. Since the function $-|x|^{p_2-2}x$ is monotonically decreasing, we get

$$\langle -|u|^{p_2-2} u + |v|^{p_2-2} v, u - v \rangle_2 \leq 0.$$  

(4.2)
Further for \( d < p_1 \), by Sobolev embedding we have \( V_1 \subset L^\infty (\mathcal{D}) \) and therefore using the assumptions imposed on \( f_0 \) taking \( t = 2 \), we observe that for \( u, v \in V \)

\[
\langle f^0 (u) - f^0 (v), u - v \rangle_1 \leq K \int_\mathcal{D} (1 + |v(x)|^s)|u(x) - v(x)|^2 \, dx
\]

\[
\leq K (1 + |v|_{L^\infty}^s) |u - v|^2 \leq C (1 + |v|_{V_1}^{p_1}) |u - v|^2
\]

for \( s \leq p_1 \).

Using Lipschitz continuity of the functions \( h_j, j \in \mathbb{N} \), we have

\[
|h_j(u) - h_j(v)|^2 \leq M_j^2 |u - v|^2 ,
\]

where \( M_j \) are the Lipschitz constants such that \( (M_j) \in \ell^2 \). Again using assumptions imposed on \( \gamma \) and the fact that \( v \) is a Lévy measure, we have

\[
\int_{D^c} |\gamma(u, z) - \gamma(v, z)|^2_{L^2} v(dz) \leq \int_{D^c} \int_{\mathcal{D}} |u(x) - v(x)|^2 |z|^2 dx v(dz)
\]

\[
= K \int_{D^c} |z|^2 v(dz) \int_{\mathcal{D}} |u(x) - v(x)|^2 dx \leq C |u - v|^2_{L^2} .
\]

Therefore, we have for all \( u, v \in V \)

\[
2 \sum_{i=1}^2 \langle A_i^i (u) - A_i^i (v), u - v \rangle_i + \sum_{j=1}^{\infty} |B_j^j (u) - B_j^j (v)|^2_{L^2}
\]

\[
+ \int_{D^c} |\gamma(u, z) - \gamma(v, z)|^2_{L^2} v(dz) \leq C \left( 1 + |v|_{V_1}^{p_1} \right) |u - v|^2_{L^2}
\]

\[
\leq C \left( 1 + \sum_{i=1}^2 |v|_{V_i}^{p_i} \right) |u - v|^2_{L^2}.
\]

Hence Assumption A-2 is satisfied with \( \alpha_i := p_i \ (i=1, 2) \) and \( \beta := 0 \). Again,

\[
2 \sum_{i=1}^d \langle D_i |D_i u|^{p_i-2} D_i u, u \rangle_1 = -2 \sum_{i=1}^d \int_{\mathcal{D}} |D_i u(x)|^{p_i} \, dx = -2|u|_{V_1}^{p_i}
\]

and similarly,

\[
2 \langle -|u|^{p_i-2} u, u \rangle_2 = -2|u|_{V_2}^{p_i} .
\]

Moreover using assumptions on \( f^0 \) and Sobolev embedding, we get

\[
2 \langle f^0 (u), u \rangle_1 \leq K \int_{\mathcal{D}} (1 + |u(x)|^{\frac{p_i}{2} + 1}) \, dx \leq K (1 + |u|_{L^\infty}^{p_i} |u|_{L^2})
\]

\[
\leq C (1 + |u|_{V_1}^{p_i} |u|_{L^2}) \leq \delta |u|_{V_1}^{p_i} + C (1 + |u|_{L^2}^2),
\]

where last inequality is obtained using Young’s inequality with sufficiently small \( \delta > 0 \). Further, for any \( p_0 > 2 \)

\[
2 (p_0 - 1) \sum_{j=1}^d |\xi |D_j u|^{p_i} |^2_{L^2} = 2 (p_0 - 1) \xi^2 \sum_{j=1}^d \int_{\mathcal{D}} |D_j u(x)|^{p_i} \, dx
\]

\[
= 2 (p_0 - 1) \xi^2 |u|_{V_1}^{p_i} .
\]
Furthermore, using assumptions on $\gamma$ and the fact that $v$ is a Lévy measure on $\mathbb{R}^d$, we get
\[
\int_{D^c} |\gamma(u, z)|^2_{L^2} v(dz) \leq K \int_{D^c} \int_{\mathcal{D}} (1 + |u(x)|)^2 |z|^2 dx v(dz)
= K \int_{D^c} |z|^2 v(dz) \int_{\mathcal{D}} (1 + |u(x)|)^2 dx
\leq C (1 + |u|_{L^2}^2).
\]
(4.8)

Choose $\xi^2 < \frac{2-\delta}{2(p_0-1)}$, so that $\theta := 2 - 2(p_0 - 1)\xi^2 - \delta > 0$. Then we have,
\[
2 \sum_{i=1}^{2} (A^i(u), u)_i + (p_0 - 1) \sum_{j=1}^{\infty} |B^j(u)|^2_{L^2} + \theta \sum_{i=1}^{d} |u|_{V_i}^{p_i} + \int_{D^c} |\gamma(u, z)|^2_{L^2} v(dz)
\leq C (1 + |u|_{L^2}^2).
\]

Hence Assumption A-3 is satisfied with $\alpha_i := p_i$ ($i = 1, 2$). Again, using the assumptions on $\gamma$, we have
\[
\int_{D^c} |\gamma(u, z)|^{p_0}_{L^2} v(dz) = \int_{D^c} \left( \int_{\mathcal{D}} |\gamma(u(x), z)|^2 dx \right)^{p_0} v(dz)
\leq K \int_{D^c} \left( \int_{\mathcal{D}} (1 + |u(x)|)^2 |z|^2 dx \right)^{p_0} v(dz)
= K \int_{D^c} |z|^{p_0} v(dz) \left( \int_{\mathcal{D}} (1 + |u(x)|)^2 dx \right)^{p_0}
\leq C \int_{D^c} |z|^2 v(dz) \left[ 1 + \left( \int_{\mathcal{D}} |u(x)|^2 dx \right)^{p_0} \right] \leq C (1 + |u|_{L^2}^{p_0})
\]
and hence Assumption A-5 is satisfied.

Note that using Hölder’s inequality, we get for $u, v \in V_1$
\[
\int_{D} |D_l u(x)|^{p_l-1} |D_l v(x)| dx
\leq \left( \int_{D} |D_l u(x)|^{p_l} dx \right)^{\frac{p_l-1}{p_l}} \left( \int_{D} |D_l v(x)|^{p_l} dx \right)^{\frac{1}{p_l}}
\leq \left( \prod_{l=1}^{d} \int_{D} |D_l u(x)|^{p_l} dx \right)^{\frac{p_l-1}{p_l}} \left( \prod_{l=1}^{d} \int_{D} |D_l v(x)|^{p_l} dx \right)^{\frac{1}{p_l}} = |u|^{p_l-1}_{V_1} |v|_{V_1}. \quad (4.9)
\]

Further using assumption on $f^0$ taking $r = p_1 + 1$ and Sobolev embedding, we have for $u, v \in V_1$
\[
\int_{D} |f^0(u(x))||v(x)| dx \leq K \int_{D} (1 + |u(x)|^{p_1+1}) |v(x)| dx
\leq K |v|_{L^2} + K |v|_{L^\infty} |u|_{L^{p_1+1}} \leq K |v|_{V_1} (1 + |u|_{L^{p_1+1}}^2)
\leq K |v|_{V_1} (1 + |u|_{V_1}^{p_1-1} |u|_{L^2}^2)
\]
and hence
\[
|A^1(u)|_{V_1} \leq K |u|^{p_1-1}_{V_1} + K (1 + |u|^{p_1-1}_{V_1} |u|_{L^2}^2) \leq K (1 + |u|_{V_1}^{p_1-1})(1 + |u|_{L^2}^2).
\]
Again, using Hölder’s inequality

$$|A^2(u)|_{L^2} \leq K |u|_{W^{2,1}}^{p_2-1}, \quad (4.10)$$

which implies that Assumption A-4 holds with $\alpha_i := p_i$ ($i = 1, 2$) and $\beta = \frac{2p_1}{p_1 - 1} < 4$.

Thus taking $p_0 = 6$ and $u_0 \in L^6(\Omega; L^2(\mathcal{D}))$, in view of Theorems 2, 3 and 5, Eq. 4.1 has a unique solution and moreover for any $p < 6$ we have,

$$\mathbb{E}\left( \sup_{t \in [0,T]} |u_t|_{L^2}^{p_0} + \sum_{i=1}^{2} \int_{0}^{T} |u_t|_{V_i}^{q_i} dt \right) < C\left( 1 + \mathbb{E}|u_0|_{L^6}^{6} \right).$$

**Case 2.**

In the case $d > p_1$, we verify Assumptions A-2 to A-4 using the Sobolev embedding $W_0^{1,p_1}(\mathcal{D}) \subset L^{\bar{p}}(\mathcal{D})$, $\bar{p} = \frac{dp_1}{d-p_1}$. Let

$$t_0 = \frac{p_1(t-2)}{t(p_1-2)}$$

and

$$\frac{1}{q_1} = \frac{1-t_0}{2} + \frac{t_0}{\bar{p}}.$$

Then, $t_0 \in (0, 1)$ and $q_1 \in (2, \bar{p})$. Thus, we obtain the following interpolation inequality:

$$|u|_{L^{q_1}} \leq |u|_{L^2}^{1-\bar{q}} |u|_{L^{\bar{p}}}^{\bar{q}}, \quad u \in W_0^{1,p_1}(\mathcal{D}) \quad (4.11)$$

Using the fact $2 < p_1$, we can see that $t < q_1$. Let $q_2 = \frac{q_1}{q_1-t}$, then with some calculations, we have

$$t \frac{1}{q_1} = \frac{p_1-t}{p_1-2} + \frac{p_1(t-2)}{\bar{p}(p_1-2)}$$

and

$$q_2 = \frac{\bar{p}(p_1-2)}{(\bar{p} - p_1)(t-2)}.$$

Thus $s \leq \frac{p_1^2(t-2)}{(d-p_1)(p_1-2)} = \frac{(\bar{p} - p_1)(t-2)}{(p_1-2)}$ implies $sq_2 \leq \bar{p}$ and hence we have

$$|u|_{L^{q_2}} \leq C|u|_{L^{\bar{p}}} \leq C|u|_{W_0^{1,p_1}}, \quad u \in W_0^{1,p_1}(\mathcal{D}) \quad (4.12)$$

Hence, using assumption on $f^0$, Hölder’s inequality, interpolation inequality (4.11), Young’s inequality, definition of $t_0$, the fact that $s \leq \frac{p_1(p_1-1)}{p_1-2}$ and Eq. 4.12, we obtain

$$\langle f^0(u) - f^0(v), u - v \rangle \leq K \int_{\mathcal{D}} (1 + |v(x)|^t)|u(x) - v(x)|^t dx$$

$$\leq C(1 + |v|_{L^{q_2}}^t)|u - v|_{L^{q_1}}^t \leq C(1 + |v|_{L^{q_2}}^t)|u - v|_{L^{\bar{p}}}^t$$

$$\leq C(1 + |v|_{L^{q_2}}^t)\left| u - v \right|_{L^{\bar{p}}}^{t (1 - \bar{q})}$$

$$\leq C\left| u - v \right|_{W_0^{1,p_i}} + C\left| u - v \right|_{L^2} \left| u - v \right|_{L^2}^2$$

$$\leq C\left| u - v \right|_{W_0^{1,p_i}} + C\left| u - v \right|_{L^2} \left| u - v \right|_{L^2}^2.$$  

(4.13)

Further as observed in Eq. 3.2, we have

$$\left| D_t \left( |D_t u|^{p_1-2} D_t u \right) - D_t \left( |D_t v|^{p_1-2} D_t v \right), u - v \right|$$

$$\leq 2 \gamma |D_t u|^{p_1-2} - 2 \gamma |D_t v|^{p_1-2} \leq 0$$  

(4.14)

provided $\gamma^2 \leq \frac{2(p_1-1)}{p_1}$. Moreover, using the inequality

$$\langle |a|^r a - |b|^r b, (a - b) \rangle \geq 2^{-r} |a - b|^{r+2} \quad \forall \ r \geq 0, \ a, b \in \mathbb{R},$$
we have,
\[
\{D_t(|D_t u|^{p_1-2}D_t u) - D_t(|D_t v|^{p_1-2}D_t v), u - v\} \leq -2^{-\theta} |D_t u - D_t v|_{L_{p_1}}^2. \tag{4.15}
\]
Thus Assumption A-2 follows from Eqs. 4.2, 4.3, 4.4, 4.13, 4.14 and 4.15. Again, using assumption on \( f^0 \), Hölder’s inequality and Young’s inequality, we obtain
\[
2\langle f^0(u), u \rangle \leq K \int_{\mathcal{D}} (1 + |u(x)|^{p_1}) dx \leq K (1 + |u|_{L_{p_1}}^{p_1} |u|_{L_2}),
\]
and thus we have the following interpolation inequality:
\[
\frac{1}{r} \frac{p_1-1}{\bar{p}} = \frac{p_1-1}{r \bar{p}} + \frac{2p_1}{dr},
\]
and thus using Eqs. 4.5, 4.6, 4.7 and 4.8, Assumption A-3 is satisfied with \( \theta := 2 - 2(p_0 - 1)\gamma^2 - \delta \). In order to prove Assumption A-4, we observe that for \( r = \frac{2p_1}{\bar{p}} + p_1 - 1 \) and the Hölder conjugate \( \bar{p}' \) of \( \bar{p} \),
\[
A^1(u)_{W^{-1, p_1}} \leq K |u|_{W^{-1, p_1}}^{p_1-1} + K (1 + |u|_{W^{-1, p_1}}^{p_1-1} |u|_{L_2}^{2p_1}) \leq K (1 + |u|_{W^{-1, p_1}}^{p_1-1} |u|_{L_2}^{2p_1}).
\]
Above equation along with Eq. 4.10 implies that Assumption A-4 holds with \( \beta = 4 \) and as in Case (1.), the desired result is obtained.

**Example 2** (Stochastic Swift–Hohenberg equation) For \( p > 2 \), consider the stochastic partial differential equation,
\[
du_t = (\eta^2 u_t - (1 + \Delta)^2 u_t - |u_t|^{p-2}u_t)dt + \sum_{j=1}^{\infty} h_j(u_t) dW^j_t + \int_{\mathcal{D}^c} \gamma_t(u_t, z) \tilde{N}(dt, dz) + \int_{\mathcal{D}} \gamma_t(u_t, z) N(dt, dz) \tag{4.16}
\]
on \( (0, T) \times \mathcal{D} \) for some open bounded domain \( \mathcal{D} \subset \mathbb{R}^2 \), where \( u_t = 0 \) on \( \partial \mathcal{D} \) and \( u_0 \) is a given \( \mathcal{F}_{0t} \)-measurable random variable and \( \eta \) is a constant. Remaining terms are same as defined in Example 1 above. Then, taking \( V_1 = W^{2, 2}_0(\mathcal{D}) \), \( V_2 = L^p(\mathcal{D}) \) and defining \( A^1 : V_1 \to V_1^* \) and \( A^2 : V_2 \to V_2^* \) by,
\[
A^1(u) := \eta^2 u_t - (1 + \Delta)^2 u_t \quad \text{and} \quad A^2(u) := -|u|^{p-2}u
\]
and \( \mathcal{B}^j : V \to L^2(\mathcal{D}) \) by
\[
\mathcal{B}^j(u) := h_j(u), \quad j \in \mathbb{N},
\]
one can see that Assumptions A-1 to A-5 are satisfied with \( \alpha_1 = 2, \alpha_2 = p, \beta = 0, p_0 = 2 \) and \( \theta = 2 - \epsilon \) for \( 0 < \epsilon < 2 \). Thus taking \( u_0 \in L^2(\Omega, L^2(\mathcal{D})) \), in view of Theorems 2,3
and 5, Eqs. 4.16 has a unique solution and moreover,

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |u_t|^2_{L^2} + \sum_{i=1}^2 \int_0^T |u_t|_{V_i}^2 \, dt \right) < C \left( 1 + \mathbb{E} |u_0|^2_{L^2} \right).
\]

**Appendix A: Interlacing procedure for SPDEs**

In this section, we present how the interlacing procedure can be used to construct the unique solution of SEE (2.1) with large jumps from the unique solution of the corresponding SEE (2.5) with only small jumps. The work presented in this section is based on the interlacing procedure for a class of SDEs presented in Ikeda and Watanabe [6, Chapter 4, Section 9] and for a class of SPDEs in [2, Section 4.2]. Further we refer the reader to [6], for the details of the notions and results used in this section.

Let \( p \) be the Poisson point process associated to the Poisson random measure \( N(dt, dz) \) and \( D(p) \) be its domain. It is well-known that \( p \) is stationary if and only if there exists a non-negative measure \( \nu \) on \((\mathbb{Z}, \mathcal{Z})\) such that

\[
\mathbb{E} N((0, t] \times A) = t \nu(A)
\]

for all \( t > 0 \), \( A \in \mathcal{Z} \). Thus it follows that \( p \) is a stationary \( \mathcal{F}_t \)-Poisson point process. Further, the assumption \( \nu(D) < \infty \) for \( D \in \mathcal{Z} \) implies that the set,

\[
\{ s \in (0, T] \cap D(p) : p(s, \omega) \in D \}
\]

is finite almost surely. Note that the points in this set corresponds to the jump times of the Poisson process \( N((0, t] \times D) \), \( t \in (0, T] \). Let \( \tau_1 < \tau_2 < \ldots < \tau_n < \ldots \) be the enumeration of these points. Then \( (\tau_n)_{n \in \mathbb{N}} \) is a sequence of stopping times converging to \( T \) \( \mathbb{P} \)-a.s. as \( n \to \infty \).

Before explaining the procedure, we recall the strong Markov property of the Brownian motions and Poisson point processes (see, e.g. [6, Chapter 2, Theorems 6.4 and 6.5]).

**Lemma A.1** (Strong Markov Property) Let \( \tau \) be a stopping time which is finite almost surely. Define,

\[
W^\tau_t = W_{t+\tau} - W_\tau, \quad t \in [0, T - \tau];
\]

\[
p^\tau_t = p_{t+\tau}, \quad t \in D(p^\tau) := \{ t \in (0, \infty) : t + \tau \in D(p) \}
\]

and \( \mathcal{F}^\tau_t = \mathcal{F}_{t+\tau} \), \( t \in [0, T - \tau] \).

Then, \( (W^\tau_t)_{t \in [0, T - \tau]} \) is an infinite dimensional Wiener martingale with respect to \((\mathcal{F}^\tau_t)_{t \in [0, T - \tau]}\) and \( p^\tau \) is a stationary \( \mathcal{F}^\tau_t \)-Poisson point process with intensity measure \( \nu \).

Note that \( W^\tau, p^\tau \) have the same properties as \( W, p \). Thus, we have the following result.

**Lemma A.2** Let \( \tau \) be a stopping time taking values in \([0, T] \) and \( u_\tau \) be an \( H \)-valued, \( \mathcal{F}_\tau \)-measurable random variable such that \( u_\tau \in L^{p_0}(\Omega; H) \). Under the assumptions of Theorem 5, there exists a unique adapted, càdlàg, \( H \)-valued process \( u \) such that \( dt \times \mathbb{P} \) almost everywhere \( u \in V \). Further, \( u \in \bigcap_{i=1}^k L^{a_i}((\tau, T) \times \Omega; V_i) \cap L^{p_0}((\tau, T) \times \Omega; H) \) and almost surely,

\[
uot{\int_{\tau}^{T} A^i_s(u_s) \, ds + \sum_{j=1}^{\infty} \int_{\tau}^{T} B^j_s(u_s) \, dW^j_s + \int_{\tau}^{T} \int_{\mathcal{D}} \gamma_s(u_s, z) \tilde{N}(ds, dz)},
\]

for \( t \in [\tau, T] \).
Proof First we assume that $u_\tau = h \in H$. Clearly, $u_\tau \in L^{p_0}(\Omega; H)$. Let $\tilde{N}(dt, dz)$ be the compensated Poisson random measure associated to the Poisson point process $p_\tau$. Using Lemma A. A.1 and working along the same lines of the proof of Theorem 5 replacing all the computations involving the expectations with conditional expectations with respect to $\mathcal{F}_\tau$, there exists a unique $(\mathcal{F}_\tau)$-adapted càdlàg, $H$-valued process $u_{\tau, h}$ such that,

$$u_{\tau, h}^\tau = h + \sum_{i=1}^{k} \int_{0}^{\tau} A_i^j(u_{\tau, h}^\tau) ds + \sum_{j=1}^{\infty} \int_{0}^{\tau} B_j^i(u_{\tau, h}^\tau) dW_j^{\tau,i} + \int_{0}^{\tau} \int_{D^c} \gamma_{s+\tau}(u_{\tau, h}^\tau, z) \tilde{N}(ds, dz)$$

for $t \in [0, T - \tau]$. Since for any $h \in H$, the solution $u_{\tau, h}^\tau$ is a measurable function of $h$, the solution for a general initial value $u_\tau$, where $u_\tau$, $W^\tau$ and $p^\tau$ are mutually independent, is obtained by replacing $h$ with the $\mathcal{F}_\tau$-measurable random variable $u_\tau$. Thus, we obtain a unique $(\mathcal{F}_\tau)$-adapted càdlàg, $H$-valued process $u_\tau$ such that,

$$u_\tau^\tau = u_\tau + \sum_{i=1}^{k} \int_{0}^{\tau} A_i^j(u_\tau^\tau) ds + \sum_{j=1}^{\infty} \int_{0}^{\tau} B_j^i(u_\tau^\tau) dW_j^{\tau,i} + \int_{0}^{\tau} \int_{D^c} \gamma_{s+\tau}(u_\tau^\tau, z) \tilde{N}(ds, dz)$$

for $t \in [0, T - \tau]$.

Substituting $s + \tau = r$, above equation can be rewritten as

$$u_\tau^\tau = u_\tau + \sum_{i=1}^{k} \int_{\tau}^{t+\tau} A_i^j(u_{r-\tau}^\tau) dr + \sum_{j=1}^{\infty} \int_{\tau}^{t+\tau} B_j^i(u_{r-\tau}^\tau) dW_j^{\tau,i} + \int_{\tau}^{t+\tau} \int_{D^c} \gamma_{r}(u_{r-\tau}^\tau, z) \tilde{N}(dr, dz)$$

for $t \in [0, T - \tau]$. Finally, defining $u_t := u_{t-\tau}^\tau$ for $t \in [\tau, T]$, we observe that $u_t$ is the desired solution of SEE (2.5) in the interval $[\tau, T]$ with initial condition $u_\tau$ and hence the result.

The unique solution of SEE (2.5) in the interval $[\tau, T]$ with initial condition $u_\tau$, obtained using Lemma A.A.2 above, will be denoted by $\tilde{u}_{\tau,t}(u_\tau)$ for $t \in [\tau, T]$ in what follows. Further, we use the notation $u_{\tau,t}(u_\tau)$, $t \in [\tau, T]$ to denote the solution of SEE (2.1) in the interval $[\tau, T]$ with initial condition $u_\tau$.

We now construct the unique solution to SEE (2.1) by using Theorem 5 and Lemma A.A.2 repeatedly.

Recall that $(\tau_n)_{n \in \mathbb{N}}$ is the sequence of the jump times of the Poisson process $N((0, t] \times \mathcal{D})$, $t \in (0, T]$. From Theorem 5, there exists a unique solution $\tilde{u}_{0,t}(u_0)$ to SEE (2.5) with initial condition $u_0$ in the interval $[0, T]$. Thus,

$$\tilde{u}_{0,t}(u_0) = u_0 + \sum_{i=1}^{k} \int_{0}^{t} A_i^j(\tilde{u}_{0,s}(u_0)) ds + \sum_{j=1}^{\infty} \int_{0}^{t} B_j^i(\tilde{u}_{0,s}(u_0)) dW_j^{\tau,i} + \int_{0}^{t} \int_{D^c} \gamma_{s}(\tilde{u}_{0,s}(u_0), z) \tilde{N}(ds, dz)$$
for \( t \in [0, T] \). We construct a solution to \( \text{SEE} \ (2.1) \) on \([0, \tau_1] \) as follows:

\[
 u_{0,t}(u_0) = \begin{cases}
 \tilde{u}_{0,t}(u_0) & \text{for } 0 \leq t < \tau_1, \\
 u_{0,\tau_1^-}(u_0) + \gamma_{\tau_1}(\tilde{u}_{0,\tau_1^-}(u_0), p(\tau_1)) & \text{for } t = \tau_1.
\end{cases}
\]

where we note that \( u_{0,\tau_1^-}(u_0) = \tilde{u}_{0,\tau_1^-}(u_0) = \tilde{u}_{0,\tau_1}(u_0) \) as the \( H \)-valued process \( \tilde{u}_{0,t}(u_0) \), \( t \in [0, T] \) has no jump at time \( \tau_1 \). Clearly, the processes \( \tilde{u}_{0,t}(u_0) \) and \( u_{0,t}(u_0) \) are equivalent \( dt \times \mathbb{P} \)-almost everywhere.

Thus we have,

\[
 u_{0,\tau_1}(u_0) = \tilde{u}_{0,\tau_1}(u_0) + \gamma_{\tau_1}(\tilde{u}_{0,\tau_1}(u_0), p(\tau_1))
\]

\[
 = u_0 + \sum_{i=1}^{k} \int_0^{\tau_1} A_i^j(\tilde{u}_{0,s}(u_0))ds + \sum_{j=1}^{\infty} \int_0^{\tau_1} B_j^j(\tilde{u}_{0,s}(u_0))dW_s^j
\]

\[
 + \int_0^{\tau_1} \int_{\mathcal{D}_c} \gamma_s(\tilde{u}_{0,s}(u_0), z) \tilde{N}(ds, dz) + \gamma_{\tau_1}(\tilde{u}_{0,\tau_1}(u_0), p(\tau_1)).
\]

Since \( \tau_1 \) is the first jump time of the Poisson process \( N((0, t] \times \mathcal{D}) \), \( t \in (0, T] \), we have

\[
 \int_0^{t} \int_{\mathcal{D}_c} \gamma_s(\tilde{u}_{0,s}(u_0), z)N(ds, dz) = \begin{cases}
 0 & \text{for } 0 \leq t < \tau_1, \\
 \gamma_{\tau_1}(\tilde{u}_{0,\tau_1}(u_0), p(\tau_1)) & \text{for } \tau_1 \leq t < \tau_2.
\end{cases}
\]

Thus, it follows that for \( t \in [0, \tau_1] \),

\[
 u_{0,t}(u_0) = u_0 + \sum_{i=1}^{k} \int_0^{t} A_i^j(\tilde{u}_{0,s}(u_0))ds + \sum_{j=1}^{\infty} \int_0^{t} B_j^j(\tilde{u}_{0,s}(u_0))dW_s^j
\]

\[
 + \int_0^{t} \int_{\mathcal{D}_c} \gamma_s(\tilde{u}_{0,s}(u_0), z) \tilde{N}(ds, dz) + \int_0^{t} \int_{\mathcal{D}_c} \gamma_s(\tilde{u}_{0,s}(u_0), z)N(ds, dz)
\]

implying that the process \( u_{0,t}(u_0) \) is an \( H \)-valued solution to \( \text{SEE} \ (2.1) \) on \([0, \tau_1] \). Since \( \gamma_{\tau_1}(\tilde{u}_{0,\tau_1}(u_0), p(\tau_1)) = \gamma_{\tau_1}(\tilde{u}_{0,\tau_1^-}(u_0), p(\tau_1)) \), the uniqueness of the solution \( u_{0,t}(u_0) \) on \([0, \tau_1] \) follows from the uniqueness of the solution \( \tilde{u}_{0,t}(u_0) \) on \([0, \tau_1] \).

Further, let \( \tilde{u}_{\tau_1,t}(u_{0,\tau_1}(u_0)) \) be the unique \( H \)-valued solution to \( \text{SEE} \ (2.5) \) in \([\tau_1, T] \) with initial condition \( u_{0,\tau_1}(u_0) \) obtained using Lemma A. A.2. Thus,

\[
 \tilde{u}_{\tau_1,t}(u_{0,\tau_1}(u_0)) = u_{0,\tau_1}(u_0) + \sum_{i=1}^{k} \int_{\tau_1}^{t} A_i^j(\tilde{u}_{\tau_1,s}(u_{0,\tau_1}(u_0)))ds
\]

\[
 + \sum_{j=1}^{\infty} \int_{\tau_1}^{t} B_j^j(\tilde{u}_{\tau_1,s}(u_{0,\tau_1}(u_0)))dW_s^j + \int_{\tau_1}^{t} \int_{\mathcal{D}_c} \gamma_s(\tilde{u}_{\tau_1,s}(u_{0,\tau_1}(u_0)), z) \tilde{N}(ds, dz)
\]

for \( t \in [\tau_1, T] \). We construct a solution to \( \text{SEE} \ (2.1) \) on \([0, \tau_2] \) as follows:

\[
 u_{0,t}(u_0) = \begin{cases}
 u_{0,t}(u_0) & \text{for } 0 \leq t \leq \tau_1, \\
 \tilde{u}_{\tau_1,t}(u_{0,\tau_1}(u_0)) & \text{for } \tau_1 \leq t < \tau_2, \\
 \tilde{u}_{\tau_1,\tau_2}(u_{0,\tau_1}(u_0)) + \gamma_{\tau_2}(\tilde{u}_{\tau_1,\tau_2^-}(u_{0,\tau_1}(u_0)), p(\tau_2)) & \text{for } t = \tau_2.
\end{cases}
\]

Using the similar argument as above and observing that,

\[
 \int_0^{\tau_2} \int_{\mathcal{D}_c} \gamma_s(\tilde{u}_{0,s}(u_0), z)N(ds, dz) = \gamma_{\tau_1}(\tilde{u}_{0,\tau_1}(u_0), p(\tau_1)) + \gamma_{\tau_2}(\tilde{u}_{0,\tau_2}(u_0), p(\tau_2))
\]

\[
 = \gamma_{\tau_1}(\tilde{u}_{0,\tau_1^-}(u_0), p(\tau_1)) + \gamma_{\tau_2}(\tilde{u}_{\tau_1,\tau_2^-}(u_{0,\tau_1}(u_0)), p(\tau_2)).
\]
we obtain that $u_{0,t}(u_0)$ is a unique solution of SEE (2.1) on the interval $[0, \tau_2]$, where
the uniqueness of the solution follows from the uniqueness of the solutions $\tilde{u}_{0,t}(u_0)$ and
$\tilde{u}_{\tau_1,t}(u_0,\tau_1(u_0))$ of SEE (2.5) on the intervals $[0, T]$ and $[\tau_1, T]$ respectively.

Continuing this interlacing procedure successively, a unique solution to SEE (2.1) can be
constructed on the interval $[0, \tau_n]$ for every $n \in \mathbb{N}$ and hence on $[0, T]$.

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