Entanglement Assisted Classical Capacity of a Class of Quantum Channels with Long-Term Memory

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Abstract

In this paper we evaluate the entanglement assisted classical capacity of a class of quantum channels with long-term memory, which are convex combinations of memoryless channels. The memory of such channels can be considered to be given by a Markov chain which is aperiodic but not irreducible. This class of channels was introduced in [7], where its product state capacity was evaluated.
1 Introduction

The biggest hurdle in the path of efficient information transmission is the presence of noise, in both classical and quantum channels. This noise causes a distortion of the information sent through the channel. Error-correcting codes are used to overcome this problem. Messages are encoded into code-words, which are then sent through the channel. Information transmission is said to be reliable if the probability of error, in decoding the output of the channel, vanishes asymptotically in the number of uses of the channel (see e.g. [1] and [22]). The aim is to achieve reliable transmission, whilst optimizing the rate, i.e., the ratio between the size of the message and its corresponding codeword. The optimal rate of reliable transmission is referred to as the capacity of the channel.

A classical communications channel has a unique capacity, the formula for which was obtained by Shannon in 1948. A quantum channel, in contrast, has various distinct capacities. This is because there is flexibility in the use of a quantum channel. The particular definition of the capacity which is applicable, depends on the following: (i) whether the information transmitted is classical or quantum; (ii) whether the sender, Alice, is allowed to use inputs entangled over various uses of the channel or whether she is only allowed to use product state inputs; (iii) whether the receiver, Bob, is allowed to make collective measurements over multiple outputs of the channel or whether he is only allowed to measure the output of each channel use separately; (iv) whether Alice and Bob have additional resources e.g. prior shared entanglement.

The different capacities resulting from the different choices mentioned above were evaluated initially for memoryless quantum channels. The capacity of a quantum memoryless channel for transmitting classical information, obtained under the restriction that the inputs are product states and that collective measurements are made on the outputs, is referred to as the product state (classical) capacity of the channel. The formula for this capacity is given by the Holevo-Schumacher-Westmoreland (HSW) Theorem [15, 28]. The formula for the quantum capacity of a memoryless channel, i.e., its capacity for transmitting quantum information, was established through a series of papers [25, 20, 27, 10, 12]. The maximum asymptotic rate of reliable transmission of classical information with the help of unlimited prior entanglement between the sender and the receiver is known as entanglement.

1We follow the normal convention and refer to the sender as Alice, and the receiver as Bob.

2For such a channel, the noise affecting successive input states is assumed to be perfectly uncorrelated.
assisted capacity. The formula for this was first obtained by Bennett, Shor, Smolin and Thapliyal [5, 6] and the proof was later simplified by Holevo [16]. These proofs are based on the HSW Theorem. For an alternative proof, based on a packing argument, see [17].

The assumption of uncorrelated noise in quantum channels cannot be always justified, and memory effects should be accounted for. To our knowledge, the first paper concerning a quantum channel with memory was by Macchiavello and Palma [21]. In [2], an important class of quantum channels with memory, called forgetful channels (cf. [19]) was introduced. In such a channel, the correlation in the noise, acting on inputs to the channel, decays with the number of channel uses. See also [8] and [2].

The capacities of channels with long-term memory (i.e., channels which are “not forgetful”), had remained an open problem until recently. In [7], the classical capacity of a class of quantum channels with long-term memory, which are given by convex combinations of memoryless channels, was evaluated. This is perhaps the simplest class of models of “not forgetful” quantum channels. For further example of such channels see [9]. In this paper we evaluate the entanglement-assisted classical capacity of the same class of channels as in [7]. For a channel $\Phi$ in this class, $\Phi^{(n)} : B(\mathcal{H} \otimes n) \to B(\mathcal{K} \otimes n)$ and the action of $\Phi^{(n)}$ on any state $\rho^{(n)} \in B(\mathcal{H} \otimes n)$ is given as follows:

$$\Phi^{(n)}(\rho^{(n)}) = \sum_{i=1}^{M} \gamma_{i} \phi_{i}^{\otimes n}(\rho^{(n)}),$$

where $\phi_{i} : B(\mathcal{H}) \to B(\mathcal{K}), (i = 1, \ldots, M)$ are completely positive, trace-preserving (CPT) maps and $\gamma_{i} > 0, \sum_{i=1}^{M} \gamma_{i} = 1$. Here $\mathcal{H}$ and $\mathcal{K}$ denote finite-dimensional Hilbert spaces and $B(\mathcal{H})$ denotes the algebra of linear operators acting on $\mathcal{H}$. On using the channel, an initial random choice is made as to which memoryless channel the successive input states are to be transmitted through. A classical version of such a channel was introduced by Jacobs [18] and studied further by Ahlswede [1].

Note that the memory of the class of channels that we study, can be considered to be given by a Markov chain which is aperiodic but not irreducible. This can be seen as follows. Consider a quantum channel (of length $n$) with Markovian correlated noise given by a CPT map $\Phi^{(n)} : B(\mathcal{H} \otimes n) \to B(\mathcal{K} \otimes n)$, which is defined as follows:

$$\Phi^{(n)}(\rho^{(n)}) = \sum_{i_{1}, \ldots, i_{n}=1}^{M} q_{i_{n-1}i_{n}} \cdots q_{i_{1}i_{2}} \gamma_{i_{2}} (\phi_{i_{1}} \otimes \cdots \otimes \phi_{i_{n}})(\rho^{(n)}),$$

Here $(i) q_{ij}$ denote the elements of the transition matrix of a discrete–time Markov chain with a finite state space $I = \{1, 2, \ldots, M\}; (ii) \{\gamma_{i}\}_{i=1}^{M}$ denotes
the invariant distribution of the chain, and (iii) for each \( i \in I, \phi_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K}) \) is a CPT map. Casting the channel defined by (1) in this form yields \( q_{ij} = \delta_{ij} \). Hence the transition matrix of the Markov chain, in this case, is the identity matrix. In other words, once a particular branch \( i \in \{1, \ldots, M\} \) has been chosen, the successive inputs are sent through this branch. Transition between the different branches (which correspond to the different states of the Markov Chain) is not permitted. The Markov chain is therefore aperiodic but not irreducible. Hence the channel has long-term memory and does not lie in the class of forgetful channels.

We start the main body of our paper with some preliminaries in Section 2. Our main result, giving the expression for the entanglement-assisted classical capacity of the channels in question, is stated as a theorem in Section 3. The proofs of the converse and direct parts of this theorem are given in Sections 3.1 and 3.2 respectively. In proving the direct part of the theorem, we make use of the expression for the product state capacity of the channel, which was obtained in [7].

2 Preliminaries

The von Neumann entropy of a state \( \rho \), i.e., a positive operator of unit trace in \( \mathcal{B}(\mathcal{H}) \), is defined as \( S(\rho) := -\text{Tr} \rho \log \rho \), where the logarithm is taken to base 2. A quantum channel is given by a completely positive trace-preserving (CPT) map \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K}) \), where \( \mathcal{H} \) and \( \mathcal{K} \) are the input and output Hilbert spaces of the channel. For any ensemble \( \{p_j, \rho_j\} \) of states \( \rho_j \) chosen with probabilities \( p_j \), the Holevo \( \chi \) quantity is defined as

\[
\chi(\{p_j, \rho_j\}) := S \left( \sum_j p_j \rho_j \right) - \sum_j p_j S(\rho_j).
\]

3 Main Result

As mentioned in the Introduction, in this paper we evaluate the entanglement-assisted classical capacity of the class of channels with long-term memory defined by (1).

Consider the following protocol for the entanglement-assisted transmission of classical information through such a quantum channel. Suppose Alice and Bob share indefinitely many copies of an entangled pure state \( \Psi_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}| \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \). Here the system \( A \) (\( B \)), with Hilbert space \( \mathcal{H}_A \) (\( \mathcal{H}_B \)) is in Alice’s (Bob’s) possession and \( \dim \mathcal{H}_A = \dim \mathcal{H}_B \). Suppose Alice has a set of messages, labelled by the elements of the set \( \mathcal{M}_n = \)
\{1, 2, \ldots, M_n\}, which she would like to communicate via the quantum channel (Ⅰ) to Bob, exploiting this shared entanglement. For this purpose she uses encoding (CPT) maps $\mathcal{E} := \{\mathcal{E}_j\}_{j=1}^J$ (where $J$ is some positive integer) acting on $\mathcal{B}(\mathcal{H}_A)$. In order to transmit her classical messages through the quantum channel, Alice encodes each of her messages in a quantum state in $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$ in the following manner. To each $\alpha \in \mathcal{M}_n$ she assigns a quantum state (or codeword) $\rho_{\alpha}^{AB; n} := \rho_{\alpha, 1} \otimes \cdots \otimes \rho_{\alpha, n} \in \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$. (3)

where

$$\rho_{\alpha, k} = (\mathcal{E}_{j_k} \otimes id_B) \Psi^{AB},$$

for $k = 1, \ldots, n$. Here $j_k \in \{1, \ldots, J\}$ and $id_B$ denotes the identity map in $\mathcal{B}(\mathcal{H}_B)$.

Note that the codewords are states shared between Alice and Bob. Alice then sends her part of these shared states to Bob through $n$ subsequent uses of the quantum channel (Ⅰ). Hence, Bob’s final state corresponding to Alice’s classical message $\alpha$ is

$$\sigma_{\alpha}^{AB; n} := (\Phi^{(n)} \otimes id_B^{\otimes n}) \rho_{\alpha}^{AB; n}.$$ (5)

In order to infer the message that Alice communicated to him, Bob makes a measurement on the state $\sigma_{\alpha}^{AB; n}$, the measurement being described by POVM elements $F_{\alpha}^{AB; n}, \alpha = 1, \ldots, M_n$, with $F_{\alpha}^{AB; n}$ being a positive operator acting in $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$, such that

$$\sum_{\alpha=1}^{M_n} F_{\alpha}^{AB; n} \leq I^{\otimes n}_{AB},$$

and $I_{AB}$ denoting the identity operator acting in $\mathcal{H}_A \otimes \mathcal{H}_B$. Defining $F_0^{AB; n} := (I_A \otimes I_B)^{\otimes n} - \sum_{\alpha=1}^{M_n} F_{\alpha}^{AB; n}$, yields a resolution of identity in $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$. Hence, $\mathcal{F} := \{F_{\alpha}^{AB; n}\}_{\alpha=0}^{M_n}$ defines a POVM. An output $\beta \in \mathcal{M}_n$ of a measurement described by this POVM, would lead Bob to conclude that the codeword was $\rho_{\beta}^{AB; n}$, whereas the output 0 is interpreted as a failure of any inference.

The encoding and decoding operations, employed to achieve reliable transmission of information by means of this protocol, together define a quantum code $\mathcal{C}^{(n)}$ (of length $n$) which is given by the triple $\mathcal{C}^{(n)} := (M_n, \mathcal{E}, \mathcal{F})$, with $M_n$ denoting its size, and $\mathcal{E}, \mathcal{F}$ being the encoding and decoding maps employed.
Assuming equidistribution of messages, the average probability of error for the code $C^{(n)}$ is given by

$$P_e(C^{(n)}) := \frac{1}{M_n} \sum_{\alpha=1}^{M_n} \left( 1 - \text{Tr} \left( F_{\alpha}^{AB;n}(\Phi^{(n)} \otimes \text{id}_B^{\otimes n})(\rho_{\alpha}^{AB;n}) \right) \right).$$ (6)

If for a given $R > 0$ there exists a sequence of $M_n$’s with

$$R \leq \liminf_{n \to \infty} \frac{1}{n} \log M_n,$$

and a sequence of codes $C^{(n)}$ of size $M_n$ such that

$$\lim_{n \to \infty} P_e(C^{(n)}) = 0,$$

then $R$ is said to be an achievable rate.

We define the one-shot entanglement-assisted classical capacity [16] of the long-term memory channel defined by (1) as

$$C_{\text{ea}}^{(1)}(\Phi) := \sup_{\Psi^{AB}} \sup_{R \text{ achievable}} \left[ R : R \text{ achievable} \right],$$ (7)

where the internal supremum is over the rates achievable under the choice of the initial shared state $\Psi^{AB}$.\footnote{In the case of a memoryless channel, this reduces to an alternative expression referring to a single use of the channel [cf., e.g., eq.(1) in [16]].}

More generally Alice and Bob may share indefinitely many copies of a pure state $\Psi^{AB;m}$ in $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes m}$ for some given $m > 1$. In this case Alice can perform a similar construction using encoding CPT maps, $E_{j_k}^{(m)}$, which act in $\mathcal{H}_A^{\otimes m}$. In other words, she uses $m$-block encoding, and encodes a message $\alpha \in \mathcal{M}_n$ by the state

$$\rho_{\alpha,m} := \rho_{\alpha,1}^{(m)} \otimes \ldots \otimes \rho_{\alpha,n}^{(m)} \in \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes mn}),$$

where

$$\rho_{\alpha,k}^{(m)} = (E_{j_k}^{(m)} \otimes \text{id}_B^{\otimes m})\Psi^{AB;m},$$

for $k = 1, \ldots, n$ and $j_k \in \{1, \ldots, J\}$.

As before, Bob uses decoding POVM elements $F_{\alpha,m}^{AB;n}$ which are positive operators acting in $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes mn}$, with $\sum_{\alpha=1}^{M_n} F_{\alpha,m}^{AB;n} \leq I_{AB}^{\otimes mn}$.

The average probability of error for the resultant code (which we denote by $C_{m}^{(n)}$) is given by

$$P_{e,m}^{(n)} \equiv P_e(C_{m}^{(n)}) := \frac{1}{M_n} \sum_{\alpha=1}^{M_n} \left( 1 - \text{Tr} \left( F_{\alpha,m}^{AB;n}(\Phi^{(mn)} \otimes \text{id}_B^{\otimes mn})(\rho_{\alpha,m}^{AB;n}) \right) \right).$$ (8)
This gives rise to the \textit{m-shot} entanglement-assisted classical capacity of the long-term memory channel defined by (1):

\[ C^{(m)}_{ea}(\Phi) := \sup_{\Psi^{AB;m}} \sup[R : R \text{ achievable}], \]

where the internal supremum is over the rates achievable under the choice of the initial shared state \( \Psi^{AB;m} \).

Finally, the full entanglement-assisted classical capacity of \( \Phi \) is given by

\[ C_{ea}(\Phi) := \limsup_{m \to \infty} \frac{1}{m} C^{(m)}_{ea}(\Phi) \]

Our main result is given by the following theorem.

\textbf{Theorem 3.1} \textit{The entanglement assisted classical capacity of a channel} \( \Phi \), \textit{with long-term memory, defined through (1), is given by}

\[ C_{ea}(\Phi) = \max_{\rho} \left[ \bigwedge_{i=1}^{M} I(\rho; \phi_i) \right], \]

\textit{with} \( I(\rho; \phi_i) := S(\rho) + S(\phi_i(\rho)) - S(\rho; \phi_i) \), \textit{where} \( S(\rho; \phi_i) \) \textit{denotes the entropy exchange and is defined as follows:}

\[ S(\rho; \phi_i) := S((\phi_i \otimes \id^R_\rho)^{AR})), \]

\textit{with} \( \psi_\rho^{AR} \) \textit{being a purification of} \( \rho \) \textit{on a reference system} \( R \). \textit{In (11) the maximum is taken over states} \( \rho \in B(\mathcal{H}_A) \).

Here we use the standard notation \( \bigwedge \) to denote the minimum.

\textbf{3.1 Proof of the converse part of Theorem 3.1} \textbf{3.1}

In this section we prove that for any rate \( R > C_{ea}(\Phi) \), with \( C_{ea}(\Phi) \) given by (11), reliable entanglement-assisted transmission of classical information from Alice to Bob via the quantum channel \( \Phi \) (eq.(1)) is impossible, regardless of the encoding used.

Suppose Alice and Bob share multiple copies of an entangled bipartite pure state \( \Psi^{AB;m} \) in \( (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes m} \), where \( m \) is a given positive integer. Then, given \( n \in \mathbb{Z} \), Alice encodes her classical messages by applying chosen \( m \)-block encoding CPT maps, \( n \) times, to her part of the shared state \( (\Psi^{AB;m})^{\otimes n} \).

Here we show that the average error probability of the corresponding code, as defined in (8), does not tend to zero as \( n \to \infty \), for any \( m \) and any choice of encoding maps. For notational simplicity, we will omit the label \( m \) and the superscript \( AB \) in the rest of this section.
Let
\[ \sigma^\alpha_n(i) := \sigma_{\alpha,1}(i) \otimes \ldots \otimes \sigma_{\alpha,n}(i) \]
denote Bob’s final state, if the codeword
\[ \rho^\alpha_n = \rho_{\alpha,1} \otimes \ldots \otimes \rho_{\alpha,n} \in B((H_A \otimes H_B) \otimes^{mn}), \]
(13)
corresponding to the message \( \alpha \), is transmitted through the \( i \)-th branch of the channel. Here \( \sigma_{\alpha,k}(i) = (\phi_i \otimes id_B)\rho_{\alpha,k} \), for \( k = 1, 2, \ldots, n \). Also let
\[ \sigma^\alpha_n := \gamma_i \sigma^\alpha_n(i) \quad \text{and} \quad \bar{\sigma}^\alpha_k(i) = \frac{1}{|M_n|} \sum_{\alpha \in M_n} \sigma^\alpha_k(i) \]
(14)
for \( k = 1, \ldots, n \).

Then the average probability of error (8) equals
\[ \bar{p}_e^{(n)} := 1 - \frac{1}{|M_n|} \sum_{\alpha \in M_n} \text{Tr} \left[ F^\alpha_n \sigma^\alpha_n \right]. \]

We also define the average probability of error corresponding to the \( i \)-th branch of the channel as
\[ \bar{p}_{i,e}^{(n)} := 1 - \frac{1}{|M_n|} \sum_{\alpha \in M_n} \text{Tr} \left[ \sigma^\alpha_n(i) F^\alpha_n \right] \quad \text{so that} \quad \bar{p}_e^{(n)} = \sum_{i=1}^M \gamma_i \bar{p}_{i,e}^{(n)} \]
(15)

Let \( X^{(n)} \) be a random variable with a uniform distribution over the set \( M_n \), characterizing the classical message sent by Alice to Bob. Let \( Y_i^{(n)} \) be the random variable corresponding to Bob’s inference of Alice’s message, when the codeword is transmitted through the \( i \)-th branch of the channel. It is defined by the conditional probabilities
\[ P[Y_i^{(n)} = \beta \mid X^{(n)} = \alpha] = \text{Tr} \left[ F^\beta_n (\phi_i^{\otimes n} \otimes id_B)(\rho^\alpha_n) \right]. \]
(16)

By Fano’s inequality,
\[ h(\bar{p}_{i,e}^{(n)}) + \bar{p}_{i,e}^{(n)} \log(|M_n| - 1) \geq H(X^{(n)} \mid Y_i^{(n)}) = H(X^{(n)}) - H(X^{(n)} : Y_i^{(n)}). \]
(17)

Here \( h(p) := -p \log p - (1-p) \log(1-p) \) denotes the binary entropy, \( H(A) := -\sum_a p_a \log p_a \) denotes the Shannon entropy of a random variable \( A \) with probability mass function \( p_a \), and \( H(A \mid B), H(A : B) \) denote, respectively, the conditional entropy and the mutual information \([4]\) of two random variables
Using the Holevo bound and the subadditivity of the von Neumann entropy we have

\[
H(X^{(n)} : Y^{(n)}_i) \leq S\left(\frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} \sigma^n_\alpha(i)\right) - \frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} S(\sigma^n_\alpha(i))
\]

\[
\leq \sum_{k=1}^{n} \left[ S(\bar{\sigma}_k(i)) - \frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} S(\sigma_{\alpha,k}(i)) \right]
\]

\[
= \sum_{k=1}^{n} \chi \left( \left\{ \frac{1}{|\mathcal{M}_n|} \sigma_{\alpha,k}(i) : \alpha \in \mathcal{M}_n \right\} \right)
\]

\[
= \sum_{k=1}^{n} \frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} S(\sigma_{\alpha,k}(i) \mid \bar{\sigma}_k(i)) := \sum_{k=1}^{n} V_k. \quad (18)
\]

In the above the symbol \( S(\rho \mid \omega) \) denotes the quantum relative entropy of states \( \rho \) and \( \omega \).

The expression \( V_k \) can be rewritten using Donald’s identity [11]:

\[
\sum_{\alpha} p_\alpha S(\omega_\alpha \mid \rho) = \sum_{\alpha} p_\alpha S(\omega_\alpha \mid \bar{\omega}) + S(\bar{\omega} \mid \rho), \quad (19)
\]

where \( \bar{\omega} = \sum_\alpha p_\alpha \omega_\alpha \). We apply this with \( \rho \) replaced by

\[
\bar{\sigma}(i) = \frac{1}{n|\mathcal{M}_n|} \sum_{k=1}^{n} \sum_{\alpha \in \mathcal{M}_n} \sigma_{\alpha,k}(i),
\]

\( \omega_\alpha \) replaced by \( \sigma_{\alpha,k}(i) \), \( p_\alpha \) replaced by \( 1/|\mathcal{M}_n| \), and consequently \( \bar{\omega} \) replaced by \( \bar{\sigma}_k(i) \). Hence,

\[
\frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} S(\sigma_{\alpha,k}(i) \mid \bar{\sigma}_k(i)) = \frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} S(\sigma_{\alpha,k}(i) \mid \bar{\sigma}_i) - S(\bar{\sigma}_k(i) \mid \bar{\sigma}(i))
\]

\[
\leq \frac{1}{|\mathcal{M}_n|} \sum_{\alpha \in \mathcal{M}_n} S(\sigma_{\alpha,k}(i) \mid \bar{\sigma}(i)), \quad (21)
\]

where we have used the non-negativity of the quantum relative entropy. Inserting this into (18) we now have:

\[
\frac{1}{n} H(X^{(n)} : Y^{(n)}_i) \leq \frac{1}{n|\mathcal{M}_n|} \sum_{k=1}^{n} \sum_{\alpha \in \mathcal{M}_n} S(\sigma_{\alpha,k}(i) \mid \bar{\sigma}(i))
\]

\[
= \chi \left( \left\{ \frac{1}{n|\mathcal{M}_n|} \sigma_{\alpha,k}(i) \right\}_{(\alpha,k)} \right). \quad (22)
\]
The inequality (17) now yields (cf. eq.(17) of [16])

\[ h(\pi_{i,e}^{(n)}) + \pi_{i,e}^{(n)} \log |M_n| \geq \log |M_n| - n \chi \left( \left\{ \frac{1}{n|M_n|}, \sigma_{\alpha,k}(i) \right\} \right) \]

\[ \geq \log |M_n| - n I(\rho, \phi_i), \]  

(23)

where

\[ \rho := \sum_{\alpha,k} p_{\alpha,k} \rho_{\alpha,k}^{A} \in B(H_A), \]

with \( p_{\alpha,k} := \frac{1}{n|M_n|} \) for each \( \alpha \) and \( k \), and \( \rho_{\alpha,k}^{A} = \text{Tr}_B(\rho_{\alpha,k}), k = 1, \ldots, n \).

However, since

\[ C_{ea}(\Phi) \geq \bigwedge_{i=1}^{M} I(\rho, \phi_i) \]  

(24)

and \( R = \frac{1}{n} \log |M_n| > C_{ea}(\Phi) \), there must be at least one branch \( i \) such that

\[ \pi_{i,e}^{(n)} \geq 1 - \frac{C_{ea}(\Phi) + 1/n}{R} > 0. \]  

(25)

We conclude from (15) and (25) that

\[ \pi_{i,e}^{(n)} \geq \left(1 - \frac{C_{ea}(\Phi) + 1/n}{R}\right)^{\frac{1}{n}} \bigwedge_{i=1}^{M} \gamma_i. \]  

(26)

Hence \( \pi_{i,e}^{(n)} \) does not tend to zero as \( n \to \infty \), which in turn implies that

\[ C_{ea}(\Phi) \leq \max_{\rho} \left[ \bigwedge_{i=1}^{M} I(\rho; \phi_i) \right]. \]

\[ \square \]

### 3.2 Proof of the direct part of Theorem 3.1

In this section we prove that \( C_{ea}(\Phi) \), defined by (10) satisfies the lower bound

\[ C_{ea}(\Phi) \geq \max_{\rho} \left[ \bigwedge_{i=1}^{M} I(\rho; \phi_i) \right], \]  

(27)

where the maximum is taken over all states \( \rho \in B(H_A) \).

To prove this we employ the following result which we proved in [7]:
Theorem 3.2 The product state capacity of a channel $\Phi$, with long-term memory, defined through (1), is given by
\[
C(\Phi) = \sup_{\{\pi_j, \rho_j\}} \left[ \bigwedge_{i=1}^{M} \chi_i(\{\pi_j, \rho_j\}) \right],
\]
where $\chi_i(\{\pi_j, \rho_j\}) := \chi(\{\pi_j, \phi_i(\rho_j)\})$. The supremum is taken over all finite ensembles of states $\rho_j \in \mathcal{B}(\mathcal{H})$, chosen with probabilities $\pi_j$.

From the definition (7) of the one-shot entanglement assisted capacity and (28) it follows that
\[
C^{(1)}_{ea}(\Phi) = \sup_{\{\pi_j, \rho_j\}, \Psi^{AB}} \left[ \bigwedge_{i=1}^{M} \chi(\{\pi_j, (\phi_i \otimes id_B)^{AB}\}) \right],
\]
where (i) $\Psi^{AB}$ is the bipartite entangled pure state, indefinitely many copies of which are shared by Alice and Bob, and (ii) $\mathcal{E}_j$ are encoding maps acting on $\mathcal{B}(\mathcal{H}_A)$, as described in Section 3, i.e., $\rho_j^{AB} = (\mathcal{E}_j \otimes id_B)\Psi^{AB}$.

Moreover, from the definition (9) of the $m$-shot entanglement assisted capacity it follows that
\[
C^{(m)}_{ea}(\Phi) = \sup_{\{\pi_j^{(m)}, \rho_j^{AB,m}\}, \Psi^{AB,m}} \left[ \bigwedge_{i=1}^{M} \chi(\{\pi_j^{(m)}, (\phi_i^{(m)} \otimes id_B^{\otimes m})^{AB,m}\}) \right].
\]

Now, following [6] and [16], consider a specific encoding ensemble $\{\pi^{(m)}_{j(a,b), \mathcal{E}_{(a,b)}}\}$, where $a, b = 1, 2, \ldots, q$, for some integer $q$, and
\[
\pi^{(m)}_{j(a,b)} = \frac{1}{q^2} \quad \text{and} \quad \mathcal{E}^{(m)}_{(a,b)} = W_{a,b}^{(m)}.
\]
Here $W_{a,b}^{(m)}$ denotes the discrete Weyl-Segal operators (see e.g. [16]) for a $q$-dimensional subspace $\mathcal{Q}_m$ of $\mathcal{H}_A^{\otimes m}$. Further, consider the codewords to be given by
\[
\varrho_{a,b}^{AB,m} = (W_{a,b}^{(m)} \otimes id_B^{\otimes m})(|\psi_{m}^{AB}\rangle \langle \psi_{m}^{AB}|),
\]
where $|\psi_{m}^{AB}\rangle$ denotes a maximally entangled state of rank $q$:
\[
|\psi_{m}^{AB}\rangle := \frac{1}{\sqrt{q}} \sum_{k=1}^{q} |e_k^{(m)}\rangle \otimes |e_k^{(m)}\rangle,
\]
where $\{|e_k^{(m)}\rangle\}_{k=1}^{q}$ is an orthonormal system of vectors in $\mathcal{Q}_m$. Hence,
\[
C^{(m)}_{ea}(\Phi) \geq \bigwedge_{i=1}^{M} \chi(\{\frac{1}{q^2}, (\phi_i^{(m)} \otimes id_B^{\otimes m})^{AB,m}\} \varrho_{a,b}^{AB,m})
\]
From \[16\] it follows that
\[
\chi\left(\frac{1}{q^{r}}\left(\phi_{i}^{\otimes m} \otimes i\mathbb{1}_{B}^{\otimes m}\right)\mathbb{1}_{A}^{\otimes m}\right) = I(\text{Tr}(P^{(m)}); \phi_{i}^{\otimes m}).
\] (32)
where \(P^{(m)}\) is the orthopjection onto \(Q_{m}\). Further, it was proved in \[16\] that if \(Q_{m}\) is chosen to be the strongly \(\delta\)-typical subspace for an arbitrary state \(\rho^{\otimes m} \in \mathcal{B}(\mathcal{H}_{A}^{\otimes m})\), and \(P^{m,\delta}\) is its orthopjection, then
\[
\lim_{\delta \to 0} \lim_{m \to \infty} \frac{1}{m} I\left(\frac{P^{m,\delta}}{\text{Tr}(P^{m,\delta})}; \phi_{i}^{\otimes m}\right) = I(\rho; \phi_{i}).
\] (33)

From (31), (32), (33) and the definition (10) of the full entanglement-assisted capacity, it follows that
\[
C_{ea}(\Phi) \geq \bigwedge_{i=1}^{M} I(\rho; \phi_{i}).
\] (34)

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