On geometric equations and duality for free higher spins

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Abstract

We provide a general scheme for dualizing higher-spin gauge fields in arbitrary irreducible representations of $GL(D,\mathbb{R})$. We also give a recipe for constructing Fronsdal-like field equations and Lagrangians for such exotic fields.

1“Chercheur F.R.I.A.”, Belgium
1 Introduction

Despite several decades of study, higher-spin gauge fields (i.e. spin $S > 2$) are still rather mysterious. For instance, weakly-coupled $\mathcal{N} = 4$ super-Yang-Mills was recently conjectured to be the holographic dual of a theory with infinitely many higher-spin fields in $AdS_5$ [1]. In any case, the old Fronsdal program [2] of constructing interactions of massless higher-spin fields by introducing consistent coupling with sources is far from being achieved. It was formulated in the late seventies when Fang and Fronsdal obtained the covariant Lagrangians for gauge fields of any spin in a flat background [2, 3]. It was soon followed by an alternative approach to free massless higher-spin fields, the so called “gauge approach” introduced by Vasilev [4], which uses geometrical objects generalizing vielbein and spin connection. This approach turned out to be promising for switching on consistent interactions [5]. In a recent work on higher-spin gauge fields, Francia and Sagnotti [6, 7] discovered that forgoing locality allows to relax the trace conditions of the Fang-Fronsdal formulation. For arbitrary spin $S$, gauge invariant field equations were elegantly written in terms of the curvature tensor introduced by de Wit and Freedman [8].

In four dimensions, all the tensorial irreps of the little group $SO(2)$ are completely symmetric and the rank $S$ is equal to the spin. In dimension $D > 4$, other irreps are possible and, in such cases, the “spin” $S$ loosely refers to the number $S$ of columns in the corresponding Young diagram. Tensor fields in arbitrary irreps of the Lorentz group appear in the spectrum of string theory. One may also dualise in the light-cone gauge some of the physical components of a completely symmetric tensor, which naturally leads to “exotic” irreps of the little group $SO(D - 2)$. These duality transformations can be performed covariantly by acting with the (space-time) Levi-Civita tensor on the curvature tensor, exchanging thereby the role played by Bianchi identities and field equations [9]. Guided by the duality symmetry principle, a systematic study led to conjectured field equations for tensor gauge field theories in arbitrary irreps of $GL(D, \mathbb{R})$ [10]. An important object was introduced which generalizes de Wit-Freedman’s curvature for tensor gauge fields in arbitrary irreps of $GL(D, \mathbb{R})$. In [11], de Medeiros and Hull followed another path: they constructed field equations for exotic fields by deriving them from gauge invariant Lagrangians. Doing so, they obtained a higher-derivative version of Francia-Sagnotti’s field equations for any irrep. of $GL(D, \mathbb{R})$. Retrospectively, this part of their work can be seen as the generalization of the work of Francia and Sagnotti for exotic gauge fields.

All known formulations of free massless higher spin fields exhibit new features with respect to spin $S \leq 2$ fields (e.g. trace conditions, non-locality, higher derivative kinetic operators, auxiliary fields, etc). These unavoidable novelties of higher spins are deeply rooted in the fact that the curvature tensor, that might be the central object in higher-spin theory, contains $S$ derivatives. A major progress of the recent approaches was to produce “geometric” field equations, i.e. equations written explicitly in terms of the curvature.

The main result of this paper is to provide an explicit relationship between (1) a duality-symmetric approach to free higher spins fields and (2) the old local approach of Fronsdal (reviewed in section 2), as well as (3) the recent non-local approach of Francia...
and Sagnotti (reviewed in section 3). The equivalence of (1) with (2) and (3) is shown in section 4. As an important by-product of this result, we obtain a covariant method for dualising free higher-spin fields. In section 5, we analyze in detail an exotic tensor gauge theory which is dual to standard spin-three gauge theory in five dimensions. For definiteness, we indeed concentrate on the spin-three field and comment on the general case at the end, section 6.

2 Local approach

The main advantage of the Fang-Fronsdal approach to free massless fields is that it respects two requirements of orthodox quantum field theory:

• (i) Locality, and

• (ii) Second order field equations (for bosonic fields).

The spin-three Fronsdal equation \[ F_{\mu_1 \mu_2 \mu_3} \equiv \Box \phi_{\mu_1 \mu_2 \mu_3} - 3 \partial_{(\mu_1} \partial^{\mu_4} \phi_{\mu_2 \mu_3)\mu_4} + 3 \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3)\mu_4} \mu_4 = 0 \] (1)

where the parenthesis ( ) means the symmetrization with weight one. Indices are raised and lowered with the Minkowski metric \[ \eta_{\mu_1 \mu_2}. \] The spin-3 field \( \phi_{\mu_1 \mu_2 \mu_3} \) is completely symmetric: \( \phi_{(\mu_1 \mu_2 \mu_3)} = \phi_{\mu_1 \mu_2 \mu_3}. \) Its gauge transformations are of the form

\[ \delta \phi_{\mu_1 \mu_2 \mu_3} = 3 \partial_{(\mu_1} \Lambda_{\mu_2 \mu_3)} . \] (2)

But since (2) transforms \( F \) as

\[ \delta F_{\mu_1 \mu_2 \mu_3} = 3 \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Lambda_{\mu_4} \mu_4 , \] (3)

the gauge parameter \( \Lambda_{\mu_1 \mu_2} \) is constrained to be traceless in order to leave the field equation (1) invariant. Eventually, the standard de Donder gauge-fixing condition

\[ \partial^{\mu_4} \phi_{\mu_2 \mu_3 \mu_4} - \partial_{(\mu_2} \phi_{\mu_3)\mu_4} \mu_4 = 0 \] (4)

is used to reduce the Fronsdal equation to its canonical form \( \Box \phi_{\mu_1 \mu_2 \mu_3} = 0. \) As shown in [8], this gauge theory leads to the correct number of physical degrees of freedom: \( \frac{1}{6}(D+2)(D-2)(D-3), \) that is the dimension of the irrep. of the little group \( SO(D-2) \) corresponding to the Young diagram \((1,1,1)\).

3 Non-local approach

The trace condition on the gauge parameter looks simple (even if somewhat unnatural) but, unfortunately, it proved to be technically involved to deal with. Recently, gauge invariant field equations for unconstrained parameters were constructed by Francia and
Sagnotti, at the price of loosing locality. For spin three, a simple form of their field equation is the following \([6]\)

\[
\mathcal{F}_{\mu_1 \mu_2 \mu_3} = \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \mathcal{H}
\]

(5)

where \(\mathcal{H}\) is given by the non-local expression

\[
\mathcal{H} \equiv \frac{1}{\Box^2} \partial^{\mu_1} \mathcal{F}_{\mu_1 \mu_2}^{\mu_2}.
\]

(6)

As one can see, the requirement (i) of section 2 is left over, however the requirement (ii) is still satisfied. The gauge invariance of (5) under unconstrained parameter is easily checked since (3) implies that

\[
\delta \mathcal{H} = 3 \Lambda^\mu_{\mu}.
\]

(7)

Of course, Fronsdal equation is easily recovered by setting \(\mathcal{H} = 0\) by an appropriate gauge transformation. This choice restores locality and fixes the trace of the gauge parameter, as it should be in the Fronsdal approach. In conclusion, the Fronsdal approach is obtained as a gauge-fixing of the Francia-Sagnotti formulation.

As noticed in \([6]\), it turns out that the equation (5), once combined with its trace, can elegantly be expressed in terms of the curvature tensor introduced by de Wit and Freedman \([8]\) (the precise expression is given at the end of the section). The “curvature” is meant for a gauge-invariant object constructed from the field, the vanishing of which implies that the field is pure gauge, i.e. \(\phi_{\mu_1 \mu_2 \mu_3} = \partial_{(\mu_1} \Sigma_{\mu_2 \mu_3)}\). With duality in mind, we consider the most natural gauge-invariant object under (2), which is obtained by taking three curls, i.e. one curl for every index of the field

\[
\mathcal{K}_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3} = \partial_{[\mu_1} \phi_{\nu_1]} [\mu_2, \nu_2] [\mu_3, \nu_3].
\]

(8)

The bracket [ ] stands for the antisymmetrisation (with weight one) over a pair of indices \((\mu_i, \nu_i)\). By construction, this tensor is antisymmetric under the exchange of two indices in any given pair \((\mu_i, \nu_i)\)

\[
\mathcal{K}_{[\mu_1 \nu_1] | \mu_2 \nu_2 | \mu_3 \nu_3} = \mathcal{K}_{\mu_1 \nu_1 | [\mu_2 \nu_2] | \mu_3 \nu_3} = \mathcal{K}_{\mu_1 \nu_1 | \mu_2 \nu_2 | [\mu_3 \nu_3]} = \mathcal{K}_{\mu_1 \nu_1 | [\mu_2 \nu_2] | \mu_3 \nu_3}.\]

(9)

Furthermore, it is obviously symmetric under the exchange of two pairs

\[
\mathcal{K}_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3} = \mathcal{K}_{\mu_2 \nu_2 | \mu_1 \nu_1 | \mu_3 \nu_3} = \mathcal{K}_{\mu_3 \nu_3 | \mu_2 \nu_2 | \mu_1 \nu_1} = \mathcal{K}_{\mu_1 \nu_1 | \mu_3 \nu_3 | \mu_2 \nu_2}.\]

(10)

In other words the curvature tensor belongs to the irrep. of \(GL(D, \mathbb{R})\) corresponding to the Young diagram \((2, 2, 2)\). The curvature tensor also satisfies two types of cyclic identities: algebraic ones (called “first Bianchi identities”) where we antisymmetrize any three indices

\[
\mathcal{K}_{[\mu_1 \nu_1 | \mu_2 \nu_2] | \mu_3 \nu_3} = 0,
\]

(11)

and differential ones (christened as “second Bianchi identities”) where one takes a curl over any pair of indices

\[
\partial_{\rho_1} \mathcal{K}_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3} = 0.
\]

(12)
These properties directly generalize the well-known properties of the linearized Riemann tensor.

In the very inspiring work [12], Damour and Deser proved that the vanishing of curvature (8) indeed implies that the spin-three field is pure gauge. Moreover, they showed that, if in addition Fronsdal equation (1) is satisfied, then the gauge parameter can always be taken to be traceless\(^2\). This last result was one of the first direct manifestation of the curvature tensor relevance in higher-spin gauge theory. Today one can argue that the curvature already plays a significant role in the Fronsdal approach but is somehow “hidden”, due to the conjugated requirements (i) and (ii) of section 2.

The curvature tensor \( R_{\rho_1 \nu_1 \rho_2 ; \rho_2 \nu_2 \rho_3} \) of de Wit and Freedman and the “Riemann tensor” \( K_{\rho_1 \nu_1 | \rho_2 \nu_2 | \rho_3 \nu_3} \) of Damour and Deser are related by acting with the appropriate Young symmetrizers

\[
R^{\mu_1 \nu_1 \rho_1 ; \mu_2 \nu_2 \rho_2} = K^{\mu_1}_{\langle \mu_2 | \nu_2 | \rho_2 \rangle},
\]

and

\[
K_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3} = 2 R_{[\mu_1 \mu_2 \mu_3 ; \nu_1 \nu_2 \nu_3]},
\]

where the three antisymmetrizations are taken over every pair of indices \((\mu_i, \nu_i)\). The de Wit-Freedman tensor is, by construction, symmetric in each of the two sets of three indices

\[
R_{(\mu_1 \nu_1 \rho_1) ; \mu_2 \nu_2 \rho_2} = R_{(\mu_1 \nu_1 \rho_1) ; (\mu_2 \nu_2 \rho_2)} = R_{(\mu_1 \nu_1 \rho_1) ; \mu_2 \nu_2 \rho_2}.
\]

Finally, we mention that (5) is equivalent to the geometric equation [6]

\[
F_{\mu_1 \mu_2 \mu_3} - \frac{1}{\Box} \partial_{(\mu_1} \partial_{\mu_2} F_{\mu_3)} \mu_4 = \frac{1}{\Box} \partial^{\nu_1} R_{\nu_2 \nu_3 ; \mu_1 \mu_2 \mu_3} = 0.
\]

### 4 Duality-symmetric approach

For duality purposes, the curvature tensor \( K_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3} \) is the natural object to consider since it displays the appropriate symmetries. Indeed, one can dualise on every pair of antisymmetric indices.

A decisive step is now to express spin-three field equations as Einstein-like equations in order to follow the scheme developed in [9, 10]. More precisely, the idea was to generalize the approach of [9] for spin-two where linearized Einstein equations were considered as Bianchi identities for the dual theory. On the other hand a deep relationship between \( F_{\mu_1 \mu_2 \mu_3} \) and the trace of the curvature \( K_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3} \) was discovered by Damour and Deser

\[
\partial_{(\mu_1} F_{\nu_2)} \mu_3 = \eta^{\nu_2 \nu_3} K_{\mu_1 \nu_1 | \mu_2 \nu_2 | \mu_3 \nu_3}.
\]

\(^2\)We stress that analogous “generalized Poincaré lemmas” can be shown to hold for rank \( S \) symmetric tensors by using the results of [13]. First, the vanishing of the curvature implies that the field is pure gauge \( \phi_{\mu_1 \rho_2 \ldots \rho_S} = \partial_{(\mu_1} \Lambda_{\rho_2 \ldots \rho_S)} \) with an unconstrained gauge parameter \( \Lambda_{\mu_1 \ldots \mu_{S-1}} \). Second, the Fronsdal equation \( F_{\mu_1 \mu_2 \ldots \rho_S} = \partial_{(\mu_1} \partial_{\mu_2} \Lambda_{\rho_3 \ldots \rho_S)} \mu_{S+1} = 0 \) implies that the trace \( \Lambda_{\mu_1 \ldots \mu_{S-3} \rho_{S-2}} \mu_{S-2} \) of the gauge parameter is a polynomial of order < \( S \). Unfortunately, one may assume without loss of generality that the latter vanishes only in the very specific spin-three case.
Therefore, the “Einstein” equation
\[ K_{\mu_1}^{\nu_1} |_{\mu_2\nu_1} |_{\mu_3\nu_3} = 0 \] (18)
is a consequence of Fronsdal equation (1) as well as Francia-Sagnotti equation (5). Conversely, the field equation (18) states that any curl of \( \mathcal{F}_{\mu_1\mu_2\mu_3} \) vanishes but, according to the analysis of [10, 13], this is equivalent to
\[ \mathcal{F}_{\mu_1\mu_2\mu_3} = \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Sigma . \] (19)
Due to (3), it is clear that one recovers Fronsdal’s approach by performing a gauge transformation with parameter \( \Lambda_{\mu\nu} \), the trace of which is fixed to be \( 3\Lambda^{\mu}_{\mu} = -\Sigma \). On the other hand, from (13) one immediately sees that the field equation (18) implies the vanishing of the trace of de Wit-Freedman’s tensor, thereby ensuring Francia-Sagnotti field equation (16). To summarize, the field equation (18) is an equivalent form of the field equation for the spin-three field. Its drawback is that it does not immediately derive from an action principle due to an inappropriate number of free indices. But from the point of view of duality, it is the most suitable form since it can be read as the following first Bianchi identity
\[ (\ast K)_{[\mu_1...\mu_{D-2} | \mu_{D-1}] | \nu | \rho_1 \rho_2} = 0 \] (20)
for the dual curvature tensor
\[ (\ast K)_{\mu_1...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2} \equiv \frac{1}{2} \epsilon_{\mu_1...\mu_{D-2}\mu_{D-1}\mu_D} K^{\mu_{D-1}\mu_D} | \nu_1 \nu_2 | \rho_1 \rho_2 , \] (21)
where \( \epsilon_{\mu_1...\mu_D} \) is the Levi-Civita tensor. Another first Bianchi identity for the dual curvature follows directly from the first Bianchi identity (11) for the original curvature
\[ (\ast K)_{\mu_1...\mu_{D-2} | \nu | \rho_1 \rho_2 \rho_3} = 0 . \] (22)
Together, these two dual Bianchi identities (20) and (22) imply that the dual curvature tensor is in the irrep. of \( GL(D, \mathbb{R}) \) corresponding to the diagram \((D - 2, 2, 2)\). Of course, in four dimensions, the dual tensor has the same symmetry properties than the original curvature tensor.

Let us now rewrite the second Bianchi (12) in terms of the dual curvature tensor
\[ \partial^{\mu_1} (\ast K)_{\mu_1\mu_2...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2} = 0 \] (23)
Due to the cyclic property (20) we get that the divergence of the curvature tensor vanishes on-shell
\[ \partial^{\mu_2} K_{\mu_1 \nu_1} |_{\mu_2 \nu_2} |_{\mu_3 \nu_3} = 0 . \] (24)
In terms of the dual curvature tensor this translates into the second Bianchi identity
\[ \partial_{[\mu_1} (\ast K)_{\mu_2...\mu_{D-1}] | \nu_1 \nu_2 | \rho_1 \rho_2} = 0 . \] (25)
Another second Bianchi identity

\[(\ast K)_{\mu_1...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2} = 0.\]  

(26)

directly follows from (12). The second Bianchi identities (25) and (26) together with the (on-shell) symmetry property of the dual curvature tensor imply that [10]

\[(\ast K)_{\mu_1...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2} = \partial_{[\mu_1} \tilde{\phi}_{\mu_2...\mu_{D-2}] | [\nu_1, \nu_2] | [\rho_1, \rho_2]}\]  

(27)

for a dual gauge field \(\tilde{\phi}_{[\mu_1...\mu_{D-3}] | \nu | \rho}\) with the symmetry properties corresponding to the Young diagram \((D - 3, 1, 1)\) that is, which satisfies

\[\tilde{\phi}_{[\mu_1...\mu_{D-3}] | \nu | \rho} = \tilde{\phi}_{[\mu_1...\mu_{D-3}] | \nu | \rho}, \quad \tilde{\phi}_{[\mu_1...\mu_{D-3}] | \mu_{D-2}] | \nu} = 0.\]  

(28)

In other words, the tensor \((\ast K)_{\mu_1...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2}\) is indeed a curvature for the dual gauge field \(\tilde{\phi}_{[\mu_1...\mu_{D-3}] | \nu | \rho}\). Finally, the two field equations of the dual field theory are

\[\eta^{\mu_1 \nu_1} (\ast K)_{\mu_1...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2} = 0,\]  

(29)

\[\eta^{\nu_1 \rho_1} (\ast K)_{\mu_1...\mu_{D-2} | \nu_1 \nu_2 | \rho_1 \rho_2} = 0.\]  

(30)

The equations (29) and (30) are respectively obtained from (11) and (18).

In four dimensions, we recover a usual spin-three field \(\tilde{\phi}_{\mu_1 \mu_2 \mu_3}\). In five dimensions, the dual theory is an exotic spin-three theory with gauge field \(\tilde{\phi}_{\mu_1 \mu_2 | \nu | \rho}\). This theory is investigated in detail in the next section. To end up the present section, we mention the two other possibilities: one could have either dualised the curvature over two pairs of antisymmetric indices getting a dual gauge field belonging to the irreducible representation \((D - 3, D - 3, 1)\) of \(GL(D, \mathbb{R})\), or one could have dualised all pairs leading to a \((D - 3, D - 3, D - 3)\) tensor gauge field theory [9, 10].

5 Exotic spin-three gauge theory

The curvature for a gauge field \(\phi_{\mu_1 \nu | \mu_2 | \mu_3}\) with symmetries \((2, 1, 1)\) is given by

\[K_{\mu_1 \nu_1 \rho | \mu_2 \nu_2 | \mu_3 \nu_3} \equiv \partial_{[\mu_1} \phi_{\nu_1 \rho]} | [\mu_2, \nu_2] | [\mu_3, \nu_3]\]  

(31)

and possesses the symmetries \((3, 2, 2)\). Generalizing the analysis of Damour and Deser, the crucial step is to define a Fronsdal tensor \(F_{\mu_1 \nu | \mu_2 | \mu_3}\) in the irrep. \((2, 1, 1)\) as follows

\[F_{\mu_1 \nu_1 | \mu_2 | \mu_3 | \nu_3} \equiv \eta^{\nu_2 \nu_3} K_{\mu_1 \nu_1 \rho | \mu_2 \nu_2 | \mu_3 \nu_3}.\]  

(32)

The result is

\[F_{\mu_1 \nu | \mu_2 | \mu_3} = \Box \phi_{\mu_1 \nu | \mu_2 | \mu_3} - 2 \partial^{\mu_3} \phi_{\mu_1 \nu | \mu_4 | (\mu_2, \mu_3)} + \partial_{\mu_2} \partial_{\mu_3} \phi_{\mu_1 \nu | \mu_4} | \mu_4 \]

\[+ 2 \partial^{\rho} \partial_{[\mu_1} \phi_{\nu] \rho | \mu_2} - 4 \partial_{[\mu_1} \phi_{\nu] \rho | (\mu_2, \mu_3)} | \nu_3\cdot\]  

(33)
We take
\[ \eta^{\mu_1 \mu_2} K_{\mu_1 \nu \rho | \mu_2 \rho_2 \nu_3} = 0 \]  
(34)
as our equation of motion in dimension \( D \geq 5 \). Due to the symmetry properties of the
curvature, the other two traces also vanish. The equations (34) and (32) imply that the
Fronsdal tensor must be written as
\[ F_{\mu_1 \nu | \mu_2 | \mu_3} = \partial_{\mu_2} \partial_{\mu_3} C_{[\mu_1 \nu]} . \]  
(35)
Now, the only way to match the symmetries of both sides of the above equation is through
\[ F_{\mu_1 \nu | \mu_2 | \mu_3} = \partial_{\mu_2} \partial_{\mu_3} \partial_{[\mu_1} C_{\nu]} . \]  
(36)
Another way to obtain this identity is by noticing that setting the curl of a \( GL(D, \mathbb{R}) \)-irreducible tensor (i.e. the Fronsdal tensor) to zero, \( F_{\mu_1 \nu | \mu_2 | \mu_3, \rho} = 0 \), where the curl is
taken on the last column of the tensor, is equivalent to imposing that all the \textit{irreducible}
components with 3 columns of the first derivative of the tensor must be set to zero. This
type of cocycle conditions for \( F_{\mu_1 \nu | \mu_2 | \mu_3} \) directly implies (36), using the results of [10].
The gauge transformations associated to the curvature are [10]
\[ \delta \phi_{\mu_1 \nu | \mu_2 | \mu_3} = 3 \partial_{[\mu_1} S_{\nu \rho] \mu_2 \mu_3} + \frac{3}{2} M_{\mu_1 \nu | (\mu_2, \mu_3)} - \partial_{[\mu_1} M_{\nu \rho] | \mu_2 | \mu_3} \]  
(37)
where \( S_{\mu_1 \mu_2 \mu_3} \) is completely symmetric and \( M_{\mu_1 \nu | \mu_2} \) has the mixed symmetry \( (2, 1) \). The
reducibilities are
\[ \delta \phi_{\mu_1 \nu | \mu_2 | \mu_3} = 0 \iff S_{\mu_1 \mu_2 \mu_3} = \partial_{[\mu_1} S_{\nu \rho] \mu_2 \mu_3} ; \quad M_{\mu_1 \nu | \mu_2} = -\partial_{[\mu_1} S_{\nu \rho] \mu_2} \]  
(38)
for the symmetric reducibility parameter \( S_{\mu_1 \mu_2} \). Then the gauge variation of the Fronsdal
tensor is
\[ \delta F_{\mu_1 \nu | \mu_2 | \mu_3} = \partial_{\mu_2} \partial_{\mu_3} \partial_{[\mu_1} \left( 3 S_{\nu \rho] \rho} - 4 M_{\nu \rho | \mu_2} \right) . \]  
(39)
This, together with (36), shows that an appropriate gauge choice enables us to reach the
Fronsdal gauge
\[ F_{\mu_1 \nu | \mu_2 | \mu_3} = 0 . \]  
(40)
The Fronsdal tensor (33) is gauge-invariant for
\[ 3 S_{\nu \rho} \rho = 4 M_{\nu \rho | \rho} \]  
(41)
which relates the traces of the two gauge parameters. This constraint is preserved by the
"reducibility transformations" (38) if and only if
\[ \eta^{\mu_1 \mu_2} S_{\mu_1 \mu_2} = 0 . \]  
(42)
With the covariant de Donder condition
\[ D_{\mu_1 (\mu_2 \mu_3)} \equiv \phi_{\mu_1 \nu | (\mu_2, \mu_3)} - \partial_{\nu} \phi_{\mu_1 \nu | \mu_2 | \mu_3} = 0 \]  
(43)
which contains as many degrees of freedom as the ones contained in $S_{\mu\nu\rho}$ and $M_{\mu\nu|\rho}$, the field equation takes its canonical form
\[ \square \phi_{\mu_1\nu|\mu_2|\mu_3} = 0. \] (44)

The gauge variation of the de Donder condition (43) gives
\[ \delta D_{\mu_1(\mu_2\mu_3)} = \frac{1}{2} \square (3S_{\mu_1\mu_2\mu_3} - M_{\mu_1(\mu_2|\mu_3)}) \] (45)
provided one imposes the following covariant de Donder condition for the gauge parameters
\[ D_{\mu_1\mu_2} \equiv \partial^\nu \left( S_{\mu_1\mu_2\nu} - \frac{1}{3} M_{\nu(\mu_1|\mu_2)} \right) - \frac{3}{4} \partial_{(\mu_1} S_{\mu_2)} \nu = 0. \] (46)
which is traceless $\eta^{\mu_1\mu_2} D_{\mu_1\mu_2} = 0$ when (41) is satisfied. The variation of $D_{\mu_1\mu_2}$ is given by
\[ \delta D_{\mu_1\mu_2} = \frac{3}{4} \square S_{\mu_1\mu_2}, \] (47)
where one used the consistency condition (42). We then obtained the appropriate gauge conditions which give $\frac{1}{8} (D - 4)(D - 3)(D - 1)(D + 2)$ for the number of physical degrees of freedom [14]. In five dimensions, we indeed have the 7 physical degrees of freedom of the standard spin-three field.

The Fronsdal-like equation (40) can be derived from the action
\[ S[\phi_{\mu_1\nu|\mu_2|\mu_3}] = \frac{1}{2} \int \phi_{\mu_1\nu|\mu_2|\mu_3} G^{\mu_1\nu|\mu_2|\mu_3}, \] (48)
where we introduced the “Einstein” tensor
\[ G_{\mu_1\nu|\mu_2|\mu_3} \equiv F_{\mu_1\nu|\mu_2|\mu_3} - \frac{1}{2} (F^1_{\mu_1\nu} \eta_{\mu_2\mu_3} + F^1_{\mu_1|\mu_2} \eta_{\nu\mu_3} + F^1_{\mu_1|\mu_3} \eta_{\nu\mu_2}) \\
- \eta_{\mu_2|\mu_1} F^2_{\nu\mu_3} - \eta_{\mu_3|\mu_1} F^2_{\nu|\mu_2}, \] (49)
with the two linearly independent traces $F^1_{\mu_1\mu_2} \equiv \eta^{\mu_3\mu_4} F_{\mu_1\mu_2|\mu_3|\mu_4}$ and $F^2_{\mu_1\mu_2} \equiv \eta^{\mu_3\mu_4} F_{\mu_3(\mu_1|\mu_2)|\mu_4}$. The Einstein tensor possesses the symmetries of the field, defines a self-adjoint (second-order) differential operator in (48) and satisfies
\[ \partial^\mu G_{\mu_1\nu|\mu_2|\mu_3} = -\frac{1}{2} \eta_{\mu_2\mu_3} \partial^\mu F^1_{\mu_1\nu} \] (50)
such that the action (48) is invariant under the gauge transformations (37) with constrained parameters satisfying (41). Note that local Lagrangians for tensors in irreps of $GL(D, \mathbb{R})$ corresponding to Young diagrams with two rows were studied in operator form by the authors of [15].
6 Arbitrary spin

The geometric equation (16) is easily generalized for symmetric fields \( \phi_{\mu_1...\mu_S} \): when \( S \) is odd one takes one divergence together with \( \frac{S-1}{2} \) traces of the de Wit-Freedman tensor \( R_{\mu_1...\mu_S;\nu_1...\nu_S} \) and when \( S \) is even one just takes \( \frac{S}{2} \) traces [6]. So one constructs a gauge-invariant object with the symmetries of the spin-\( S \) field but containing \( 2\lfloor \frac{S+1}{2} \rfloor \) derivatives (\( \lfloor \cdot \rfloor \) is the integer part of \( \frac{S+1}{2} \)). Consequently, Francia and Sagnotti further multiplied by \( \Box^{-\lfloor \frac{S-1}{2} \rfloor} \) to get a second order field equation. This natural construction was performed by de Medeiros and Hull for arbitrary irreducible tensors under \( GL(D,\mathbb{R}) \) [11] without dividing by d'Alembertian, i.e. relaxing the requirement (ii) and preferring locality (i).

The classical field equations in the final de Donder gauge take the respective forms

\[ \Box \Phi_{\mu_1...\mu_S} = 0 \]

and

\[ \Box \left[ \frac{S+1}{2} \right] \Phi_{\mu_1...\mu_S} = 0. \]

Their set of solutions is essentially the same since

\[ \ker(\Box) = \ker(\Box^n), \quad \forall n \in \mathbb{N}_0 \]

in the space of "Fourier-transformable" functions, as can be easily checked in momentum space. In this restricted sense, both approaches are thus equivalent at the level of sourceless free field equations. On the one hand the works [5] of Fradkin and Vasiliev on higher-spin interactions put some evidences in favor of the non-local formalism, on the other hand higher derivative theories (like Weyl gravity) are known to be subtle to handle at the quantum level. Nonetheless, duality symmetry could suggest higher derivatives equations since first Bianchi identities can become field equations in the dual picture. The possibility of switching on interactions or quantum consistency might eventually decide between the physical requirements (i) and (ii).

The geometric equation for a rank \( S \) symmetric field is equivalent to [6]

\[ F_{\mu_1\mu_2\mu_3\mu_4...\mu_S} = \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} H_{\mu_4...\mu_S)} \]  

(51)

which generalizes (5). The tensor \( H_{\mu_1...\mu_{S-1}} \) is a non-local function of the field \( \phi_{\mu_1...\mu_S} \) and its derivatives, whose gauge transformation is proportional to the trace of the gauge parameter. The gauge-choice \( H_{\mu_1...\mu_{S-3}} = 0 \) leads to the Fronsdal equation

\[ F_{\mu_1...\mu_S} = 0 . \]  

(52)

Basically, the main supplementary subtlety arising for spin \( S \geq 4 \) is that the usual de Donder condition is reachable with a traceless gauge parameter if and only if the double trace of the field vanishes. Therefore, in the Fronsdal approach the field is constrained to have vanishing double trace (which is consistent with the invariance of the double trace of the field under gauge transformations with traceless parameter). As pointed out in [7], more work is therefore required to obtain the double-trace condition for spin \( S \geq 4 \) in the unconstrained approach. A solution is to take a modified (identically traceless) de Donder gauge which is accessible with constrained gauge parameters [7]. After this further gauge-fixing, the field equation implies the vanishing of the double trace of the field, thereby recovering the usual de Donder condition.
It is now straightforward to generalize all our previous results. The relation (17) is easily generalized by taking $S - 2$ curls of the Fronsdal operator $F_{\mu_1...\mu_S}$ to get the trace of the curvature $K_{\mu_1\nu_1|...|\mu_S\nu_S}$ (as is obvious in the transverse-traceless gauge). Therefore the Fronsdal equation (52) and the Francia-Sagnotti equation (51) both imply the tracelessness of the curvature

$$\eta^{\mu_1\nu_2}K_{\mu_1\nu_1|\mu_2\nu_2|\mu_3\nu_3|...|\mu_S\nu_S} = 0.$$  

Conversely, the Einstein-like equation (53) directly implies the Francia-Sagnotti field equation in its geometric form while, on the other hand, the “Poincaré lemmas” of [10, 13] allow to derive from (53) that

$$F_{\mu_1...\mu_S} = \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Sigma_{\mu_4...\mu_S)}(x).$$  

We then recover the Fronsdal approach by an appropriate (partial) gauge-fixing.

Arbitrary mixed symmetry type tensor gauge fields are studied similarly, but here the tools developed in [10] reveal crucial. The same kind of relationships between the different approaches is expected to hold. A general recipe for constructing local Fronsdal-like equations for any exotic tensor free field is provided by the method followed in section 5: once the generalized curvature corresponding to a given exotic gauge field with $S$ columns is given, its trace (on the first two columns) is identified with $S - 2$ curls of the generalized Fronsdal tensor, the curls being taken on the last columns. As noticed after (36), an Einstein-like equation of motion implies [10] that the corresponding (irreducible) Fronsdal tensor writes in a way generalizing (54). Again, an appropriate partial gauge fixing then brings the Fronsdal tensor to zero.

The duality-symmetric picture sketched in the section 4 applies for arbitrary mixed symmetry type tensor fields [9, 10] and can be summarized as follows. For a start, the first Bianchi identities state that the curvature is irreducible under $GL(D,\mathbb{R})$. Then, the Einstein-like equation means that, on-shell, it is furthermore irreducible under $SO(D - 1, 1)$. Next, the crucial mathematical property is that the Hodge duals of the curvature are therefore also irreducible under $SO(D - 1, 1)$. To conclude, the second Bianchi identities imply that these irreducible tensors are indeed curvatures for some dual gauge fields.

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