Two generalized Lyapunov-type inequalities for a fractional $p$-Laplacian equation with fractional boundary conditions

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Abstract

In this paper, we investigate the existence of positive solutions for the boundary value problem of nonlinear fractional differential equation with mixed fractional derivatives and $p$-Laplacian operator. Then we establish two smart generalizations of Lyapunov-type inequalities. Some applications are given to demonstrate the effectiveness of the new results.

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1 Introduction

Lyapunov’s inequality [1] has proved to be very useful in various problems related with differential equations; for examples, see [2, 3] and the references therein. Recently, many researchers have given some Lyapunov-type inequalities for different classes of fractional boundary value problems (see [4–10]). In [7], Ferreira investigated a Lyapunov-type inequality for the fractional boundary value problem

$$
\begin{align*}
&D^{\alpha}_{a+} y(t) + q(t)y(t) = 0, \quad a < t < b, \\
y(a) &= y(b) = 0,
\end{align*}
$$

(1.1)

where $D^{\alpha}_{a+}$ is the Riemann-Liouville fractional derivative of order $\alpha$, $1 < \alpha \leq 2$, $a$ and $b$ are consecutive zeros, and $q$ is a real and continuous function. It was proved that if (1.1) has a nontrivial solution, then

$$
\int_a^b |q(s)| \, ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}.
$$

(1.2)

Obviously, if we set $\alpha = 2$ in (1.2), one can obtain the classical Lyapunov inequality [1].

In [8], Jleli and Samet considered the fractional differential equation

$$
C D^{\alpha}_{a+} y(t) + q(t)y(t) = 0, \quad a < t < b, 1 < \alpha \leq 2,
$$

(1.3)
with the mixed boundary conditions

\[ y(a) = y'(b) = 0 \quad (1.4) \]

or

\[ y'(a) = y(b) = 0, \quad (1.5) \]

where \( C D^\alpha_a \) is the Caputo fractional derivative of order \( 1 < \alpha \leq 2 \). For boundary conditions (1.4) and (1.5), two Lyapunov-type inequalities were established, respectively, as follows:

\[
\int_a^b (b-s)^{\alpha-2}|q(s)| \, ds \geq \frac{\Gamma(\alpha)}{\max\{\alpha - 1, 2 - \alpha\} (b-a)} \quad (1.6)
\]

and

\[
\int_a^b (b-s)^{\alpha-1}|q(s)| \, ds \geq \Gamma(\alpha). \quad (1.7)
\]

Recently, we considered in [11] the same equation (1.3) with the fractional boundary condition

\[ y(a) = C D^\beta_a y(b) = 0, \]

where \( 0 < \beta \leq 1 \).

In [12], Arifi et al. considered the following nonlinear fractional boundary value problem with \( p \)-Laplacian operator:

\[
\begin{align*}
D^\alpha_a y(t) + q(t)f(y(t)) &= 0, \quad a < t < b, \\
y(a) &= y(b) = 0,
\end{align*}
\]

(1.8)

where \( 2 < \alpha \leq 3, 1 < \beta \leq 2, D^\alpha_a, D^\beta_a \) are the Riemann-Liouville fractional derivative of orders \( \alpha, \beta \), \( \Phi_p(s) = |s|^{p-2}s, p > 1, \) and \( \chi: [a, b] \to \mathbb{R} \) is a continuous function. It was proved that if (1.8) has a nontrivial continuous solution, then

\[
\int_a^b (b-s)^{\beta-1}(s-a)^{\beta-1}|\chi(s)| \, ds \\
\geq (\Gamma(\alpha))^{p-1} \Gamma(\beta)(b-a)^{\beta-1} \left( \int_a^b (b-s)^{\alpha-2}(s-a) \, ds \right)^{1-p}. \quad (1.9)
\]

More recently, Chidouh and Torres in [13] considered the following boundary value problem:

\[
\begin{align*}
D^\alpha_a y(t) + q(t)f(y(t)) &= 0, \quad a < t < b, \\
y(a) &= y(b) = 0,
\end{align*}
\]

(1.10)

where \( D^\alpha_a \) is the Riemann-Liouville fractional derivative with \( 1 < \alpha \leq 2, \) and \( q: [a, b] \to \mathbb{R}, \) is a nontrivial Lebesgue integrable function. Under the assumption that the nonlinear
term $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a concave and decreasing function, it was proved that if (1.10) has a nontrivial solution, then

$$
\int_a^b |q(t)| \, ds > \frac{4^{a-1} \Gamma(\alpha) \eta}{(b-a)^{a-1} f(\eta)},
$$

(1.11)

where $\eta = \max_{t \in [a, b]} y(t)$. Obviously, if we set $f(y) = y$ in (1.11), one can obtain a Lyapunov inequality (1.2).

Motivated by the above work, in this paper, we consider the fractional boundary value problem

$$
\begin{align*}
D_0^\beta a^+ (\Phi_p(CD_0^\alpha a^+ u(t))) - k(t)f(u(t)) & = 0, \quad a < t < b, \\
u'(a) & = CD_0^\alpha a^+ u(a) = 0, \quad u(b) = CD_0^\alpha a^+ u(b) = 0,
\end{align*}
$$

(1.12)

where $1 < \alpha, \beta \leq 2$, and $k : [a, b] \to \mathbb{R}$ is a continuous function. We write (1.12) as an equivalent integral equation and then, by using some properties of its Green function and the Guo-Krasnoselskii fixed point theorem, we can obtain our first result asserting existence of nontrivial positive solutions to problem (1.12). Then, under some assumptions on the nonlinear term $f$, we are able to get two corresponding Lyapunov-type inequalities. Finally in this paper, two corollaries and an example are given to demonstrate the effectiveness of the obtained results.

### 2 Preliminaries

In this section, we recall the definitions of the Riemann-Liouville fractional integral, fractional derivative, and the Caputo fractional derivative and give some lemmas which are useful in this article. For more details, we refer to [14, 15].

**Definition 2.1** Let $\alpha \geq 0$ and $f$ be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order $\alpha$ is defined by $I_a^\alpha f \equiv f$ and

$$
(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds, \quad \alpha > 0, t \in [a, b].
$$

**Definition 2.2** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : [a, b] \to \mathbb{R}$ is given by

$$
(D_0^\alpha f)(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{n-\alpha+1}} \, ds,
$$

where $n$ is the smallest integer greater or equal to $\alpha$ and $\Gamma$ denotes the Gamma function.

**Definition 2.3** The Caputo derivative of fractional order $\alpha \geq 0$ is defined by $CD_0^\alpha f \equiv f$ and

$$
(CD_0^\alpha f)(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds, \quad \alpha > 0, t \in [a, b],
$$

where $n$ is the smallest integer greater or equal to $\alpha$. 

Lemma 2.1 (Guo-Krasnoselskii fixed point theorem [16]) Let $X$ be a Banach space and let $P \subset X$ be a cone. Assume $\Omega_1$ and $\Omega_2$ are bounded open subsets of $X$ with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that

(i) $\|Tu\| \geq \|u\|$ for any $u \in P \cap \partial \Omega_1$ and $\|Tu\| \leq \|u\|$ for any $u \in P \cap \partial \Omega_2$; or

(ii) $\|Tu\| \leq \|u\|$ for any $u \in P \cap \partial \Omega_1$ and $\|Tu\| \geq \|u\|$ for any $u \in P \cap \partial \Omega_2$.

Then $T$ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 2.2 (Jensen's inequality [17]) Let $\nu$ be a positive measure and let $\Omega$ be a measurable set with $\nu(\Omega) = 1$. Let $I$ be an interval and suppose that $u$ is a real function in $L^p(d\nu)$ with $u(t) \in I$ for a real $t \in \Omega$. If $f$ is convex on $I$, then

$$f\left(\int_{\Omega} u(t) \, d\nu(t)\right) \leq \int_{\Omega} (f \circ u)(t) \, d\nu(t).$$

(2.1)

If $f$ is concave on $I$, then the inequality (2.1) holds with $\leq$ substituted by $\geq$.

3 Main results

We begin to write problem (1.12) in its equivalent integral form.

Lemma 3.1 If $u \in C[a, b], 1 < \alpha, \beta \leq 2, p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then BVP (1.12) has a unique solution

$$u(t) = \int_{a}^{b} G(t, s) \Phi_{q}\left(\int_{a}^{b} H(s, \tau) k(\tau) f(u(\tau)) \, d\tau\right) \, ds,$$

(3.1)

where

$$G(t, s) = \begin{cases} (b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ (b-s)^{\alpha-1}, & a \leq t \leq s \leq b, \end{cases}$$

(3.2)

and

$$H(s, \tau) = \begin{cases} \left(\frac{\tau-a}{\beta-a}\right)^{\beta-1}(b-\tau)^{\beta-1} - (s-\tau)^{\beta-1}, & a \leq \tau \leq s \leq b, \\ \left(\frac{\tau-a}{\beta-a}\right)^{\beta-1}(b-\tau)^{\beta-1}, & a \leq s \leq \tau \leq b. \end{cases}$$

(3.3)

Proof Set $\Phi_{p}(D_{a}^{\alpha} u(t)) = v(t)$. Then BVP (1.12) can be turned into the following coupled boundary value problems:

$$\begin{cases} D_{a}^{\alpha} v(t) = k(t) f(u(t)), & a < t < b, \\ v(a) = v(b) = 0, \end{cases}$$

(3.4)

and

$$\begin{cases} \, \, D_{a}^{\alpha} u(t) = \Phi_{q}(v(t)), & a < t < b, \\ u'(a) = u(b) = 0. \end{cases}$$

(3.5)
From Lemma 2 of [7], we see that BVP (3.4) has a unique solution, which is given by

$$v(t) = -\int_a^b H(t,s)k(s)f(u(s))\,ds,$$  \hspace{1cm} (3.6)

where $H(t,s)$ is as in (3.3). Moreover, by Lemma 5 of [8], we see that BVP (3.5) has a unique solution, which is given by

$$u(t) = -\int_a^b G(t,s)\Phi_q(v(s))\,ds,$$  \hspace{1cm} (3.7)

where $G(t,s)$ is as in (3.2). Substitute (3.6) into (3.7), we see that BVP (1.12) has a unique solution which is given by (3.1). \hfill \Box

**Lemma 3.2** The Green's function $H$ defined by (3.3) satisfies the following properties:

1. $H(t,s) \geq 0$ for all $a \leq t, s \leq b$;
2. $\max_{t \in [a,b]} H(t,s) = H(s,s), s \in [a,b]$;
3. $H(t,s)$ has a unique maximum given by

$$\max_{s \in [a,b]} H(t,s) = \frac{(b-a)^{\beta-1}}{4^{\beta-1} \Gamma(\beta)};$$

4. $\min_{t \in \left[\frac{3a+b}{4}, \frac{3b+a}{4}\right]} H(t,s) \geq \sigma(s)H(s,s), a < s < b,$

where

$$\sigma(s) = \begin{cases} 
\frac{(b-a)(b-s)(a-s)}{(b-a)\Gamma(1-\beta)(a-s)\Gamma(\beta-1)} & \text{if } s \in (a, c_\beta], \\
\frac{b-a}{(b-a)\Gamma(1-\beta)} & \text{if } s \in [c_\beta, b),
\end{cases}$$

$$c_\beta := \frac{a+b}{2} - bA_{\beta}, \quad A_\beta = \left(\frac{3}{4}\right)^{\beta-1} - \left(\frac{1}{4}\right)^{\beta-1} \frac{1}{\Gamma(\beta)}.$$  \hspace{1cm} (3.8)

**Proof** The first three properties are proved in [7]. For convenience, we set

$$h_1(t,s) = \frac{1}{\Gamma(\beta)} \left(\frac{t-a}{b-a}\right)^{\beta-1} (b-s)^{\beta-1} - (t-s)^{\beta-1}, \quad s \leq t$$

and

$$h_2(t,s) = \frac{1}{\Gamma(\beta)} \left(\frac{t-a}{b-a}\right)^{\beta-1} (b-s)^{\beta-1}, \quad t \leq s.$$  

From [7], we know that $h_1(t,s)$ is decreasing with respect to $t$ for $s \leq t$, and $h_2(t,s)$ is increasing with respect to $t$ for $t \leq s$. Thus

$$\min_{t \in \left[\frac{3a+b}{4}, \frac{3b+a}{4}\right]} H(t,s) = \begin{cases} 
h_1\left(\frac{3a+b}{4}, s\right) & \text{if } s \in (a, \frac{3a+b}{4}], \\
\min\left[h_1\left(\frac{3a+b}{4}, s\right), h_2\left(\frac{3a+b}{4}, s\right)\right] & \text{if } s \in \left[\frac{3a+b}{4}, \frac{3b+a}{4}\right], \\
h_2\left(\frac{3b+a}{4}, s\right) & \text{if } s \in \left[\frac{3a+b}{4}, b\right].
\end{cases}$$
From

\[ h_1\left(\frac{a+3b}{4}, s\right) = h_2\left(\frac{3a+b}{4}, s\right) \]

we have

\[ \left(\frac{\frac{a+3b}{4} - s}{b-s}\right)^{\beta-1} = \left(\frac{3}{4}\right)^{\beta-1} - \left(\frac{1}{4}\right)^{\beta-1}, \]

which implies that

\[ s = \frac{\frac{a+3b}{4} - bA_{\beta}}{1-A_{\beta}} = c_{\beta}, \]

where \( c_{\beta} \) and \( A_{\beta} \) are as in (3.8). It is easy to check that \( A_{\beta} < \frac{3}{4} \) and \( c_{\beta} < \frac{a+3b}{4} \). On the other hand, since

\[ 3^{\beta-1} + 8^{\beta-1} \geq 2\sqrt{3^{\beta-1}8^{\beta-1}} \geq 4^{\beta-1}3^{\frac{\beta-1}{2}}8^{\frac{\beta-1}{2}} = 96^{\frac{\beta-1}{2}} > 9^{\beta-1}, \]

we have

\[ \left(\frac{3}{4}\right)^{\beta-1} < \left(\frac{2}{3}\right)^{\beta-1} + \left(\frac{1}{4}\right)^{\beta-1}, \]

from which we deduce that \( A_{\beta} < \frac{3}{4} \) and \( c_{\beta} > \frac{3a+b}{4} \). So \( c_{\beta} \in \left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \) is the unique solution of the equation \( h_1\left(\frac{a+3b}{4}, s\right) = h_2\left(\frac{a+3b}{4}, s\right) \). Hence

\[
\min_{t \in \left[\frac{a+3b}{4}, \frac{a+3b}{3}\right]} H(t, s) = \begin{cases} 
  h_1\left(\frac{a+3b}{4}, s\right) & \text{if } s \in (a, c_{\beta}], \\
  h_2\left(\frac{a+3b}{4}, s\right) & \text{if } s \in [c_{\beta}, b) 
\end{cases} 
= \frac{1}{\Gamma\left(\frac{\beta}{\alpha}\right)} \left(\frac{\frac{3(a+b)}{4} - s}{b-s}\right)^{\beta-1} - \left(\frac{a+3b}{4} - s\right)^{\beta-1} 
\begin{cases} 
  \text{if } s \in (a, c_{\beta}], \\
  \text{if } s \in [c_{\beta}, b) 
\end{cases} 
\geq \sigma(s)H(s, s). \]
Let \( E = C[a,b] \) be endowed with the norm \( \|x\| = \max_{t \in [a,b]} |x(t)| \). Define the cone \( P \subset E \) by

\[
P = \{ x \in E : |x(t)| \geq 0 \ \forall t \in [a,b] \text{ and } |x| \neq 0 \}.
\]

**Theorem 3.4** Let \( k : [a, b] \to \mathbb{R}_+ = [0, +\infty) \) be a nontrivial Lebesgue integrable function. Suppose that there exist two positive constants \( r_2 > r_1 > 0 \) such that the following assumptions:

1. \( f(x) \geq \rho \Phi_p(r_1) \) for \( x \in [0, r_1] \),
2. \( f(x) \leq \omega \Phi_p(r_2) \) for \( x \in [0, r_2] \),

are satisfied, where

\[
\rho = \left[ \int_a^b \sigma(T)H(t, r)k(r) \, dr \times \Phi_p \left( \int_a^b \mu(s)G(s, s) \, ds \right) \right]^{-1}
\]

and

\[
\omega = \left[ \int_a^b H(t, r)k(r) \, dr \times \Phi_p \left( \int_a^b G(s, s) \, ds \right) \right]^{-1}.
\]

Then FBVP (1.12) has at least one nontrivial positive solution \( u \) belonging to \( E \) such that \( r_1 \leq \|u\| \leq r_2 \).

**Proof** Let \( T : P \to E \) be the operator defined by

\[
Tu(t) = \int_a^b G(t, s)\Phi_q \left( \int_a^b H(s, \tau)k(\tau)f(u(\tau)) \, d\tau \right) \, ds.
\]

By using the Arzela-Ascoli theorem, we can prove that \( T : P \to P \) is completely continuous. Let \( \Omega_i = \{ u \in P : \|u\| \leq r_i \} \), \( i = 1, 2 \). From (H1), and Lemmas 3.2 and 3.3, we obtain for \( t \in [\frac{3a+b}{4}, \frac{a+3b}{4}] \) and \( u \in P \cap \partial\Omega_1 \)

\[
(Tu)(t) \geq \int_a^b \min_{t \in [\frac{3a+b}{4}, \frac{a+3b}{4}]} G(t, s)\Phi_q \left( \int_a^b H(s, \tau)k(\tau)f(u(\tau)) \, d\tau \right) \, ds
\]

\[
\geq \int_a^b \mu(s)G(s, s)\Phi_q \left( \int_a^b H(s, \tau)k(\tau)f(u(\tau)) \, d\tau \right) \, ds
\]

\[
\geq \int_a^b \mu(s)G(s, s) \, ds \cdot \Phi_q \left( \int_a^b \min_{t \in [\frac{3a+b}{4}, \frac{a+3b}{4}]} H(s, \tau)k(\tau)f(u(\tau)) \, d\tau \right)
\]

\[
\geq \int_a^b \mu(s)G(s, s) \, ds \cdot \Phi_q \left( \int_a^b \sigma(T)H(t, r)k(\tau)f(u(\tau)) \, d\tau \right)
\]

\[
\geq \int_a^b \mu(s)G(s, s) \, ds \cdot \Phi_q \left( \int_a^b \sigma(T)H(t, r)k(\tau) \, d\tau \right) \Phi_q(\rho) \cdot r_1
\]

\[
= \|u\|.
\]
Hence, \(|Tu| \geq |u|\) for \(u \in P \cap \partial \Omega_1\). On the other hand, from (H2), Lemmas 3.2 and 3.3, we have

\[
|Tu| = \max_{t \in [a,b]} \int_a^b G(t,s) \Phi_q \left( \int_s^b H(s,\tau) k(\tau) f(u(\tau)) \, d\tau \right) \, ds \\
\leq \int_a^b G(s,s) \, ds \cdot \Phi_q \left( \int_a^b H(\tau,\tau) k(\tau) f(u(\tau)) \, d\tau \right) \\
\leq \int_a^b G(s,s) \, ds \cdot \Phi_q \left( \int_a^b H(\tau,\tau) k(\tau) \, d\tau \right) \Phi_q(M) |u|,
\]

for \(u \in P \cap \partial \Omega_2\). Thus, by Lemma 2.1, we see that the operator \(T\) has a fixed point in \(u \in P \cap (\tilde{\Omega}_2 \setminus \Omega_1)\) with \(r_1 \leq |u| \leq r_2\), and clearly \(u\) is a positive solution for FBVP (1.12).

Next, we will give two Lyapunov inequalities for FBVP (1.12).

**Theorem 3.5** Let \(k : [a, b] \to \mathbb{R}_+\), be a real nontrivial Lebesgue function. Suppose that there exists a positive constant \(M\) satisfying \(0 \leq f(x) \leq M \Phi_p(x)\) for any \(x \in \mathbb{R}_+\). If (1.12) has a nontrivial solution in \(P\), then the following Lyapunov inequality holds:

\[
\int_a^b k(\tau) \, d\tau > \frac{4^{\beta-1} \Gamma(\beta)}{M(b-a)^{\beta-1}} \Phi_p \left( \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \right).
\]

**Proof** Assume \(u \in P\) is a nontrivial solution for (1.12), then \(|u| \neq 0\). From (3.1), and Lemmas 3.2 and 3.3, \(V_t \in [a, b]\), we have

\[
0 \leq u(t) \leq \int_a^b G(s,s) \Phi_q \left( \int_a^b H(\tau,\tau) k(\tau) f(u(\tau)) \, d\tau \right) \, ds \\
< \int_a^b G(s,s) \, ds \cdot \Phi_q \left( \int_a^b H(\tau,\tau) k(\tau) \, d\tau \right) \Phi_q(M) |u| \\
\leq \frac{1}{\Gamma(\alpha+1)} \int_a^b (b-s)^{\alpha-1} \, ds \cdot \Phi_q \left( \int_a^b \frac{(b-a)^{\beta-1}}{4^{\beta-1} \Gamma(\beta)} k(\tau) \, d\tau \right) \Phi_q(M) |u| \\
= \frac{1}{\Gamma(\alpha+1)} (b-a)^\alpha \cdot \Phi_q \left( \frac{(b-a)^{\beta-1}}{4^{\beta-1} \Gamma(\beta)} \right) \Phi_q \left( \int_a^b k(\tau) \, d\tau \right) \Phi_q(M) |u|,
\]

which implies that

\[
\int_a^b k(\tau) \, d\tau > \frac{4^{\beta-1} \Gamma(\beta)}{M(b-a)^{\beta-1}} \Phi_p \left( \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \right).
\]

**Theorem 3.6** Let \(k : [a, b] \to \mathbb{R}_+\), be a real nontrivial Lebesgue function. Assume that \(f \in C(\mathbb{R}_+, \mathbb{R}_+)\) is a concave and nondecreasing function. If (1.12) has a nontrivial solution \(u \in P\), then

\[
\int_a^b k(\tau) \, d\tau > \frac{4^{\beta-1} \Gamma(\beta) \Phi_p(\Gamma(\alpha+1) \Phi_p(\eta))}{(b-a)^{\alpha+\beta-1} f(\eta)},
\]

where \(\eta = \max_{t \in [a,b]} u(t)\).
Proof By (3.1), Lemmas 3.2 and 3.3, we get

\[
\begin{align*}
u(t) & \leq \int_a^b G(s,t)\Phi_q\left(\int_a^b H(\tau,\tau)k(\tau)f(u(\tau)) d\tau\right) ds, \\
\|u\| & < \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} ds \cdot \Phi_q \left(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\right) \Phi_q \left(\int_a^b k(\tau)f(u(\tau)) d\tau\right) \\
& = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \Phi_q \left(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\right) \Phi_q \left(\int_a^b k(\tau)f(u(\tau)) d\tau\right).
\end{align*}
\]

Using Lemma 2.2, and taking into account that \(f\) is concave and nondecreasing, we see that

\[
\begin{align*}
\|u\| & < \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \Phi_q \left(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\right) \Phi_q \left(\int_a^b k(s) ds \cdot \Phi_q \left(\int_a^b \frac{k(\tau)f(u(\tau)) d\tau}{\int_a^b k(s) ds}\right)\right) \\
& < \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \Phi_q \left(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\right) \Phi_q \left(\int_a^b k(s) ds \cdot \Phi_q (f(\eta))\right),
\end{align*}
\]

where \(\eta = \max_{t\in[a,b]} u(t)\). Hence,

\[
\int_a^b k(s) ds > \frac{4^{\beta-1}\Gamma(\beta)\Phi_q(\Gamma(\alpha+1))\Phi_q(\eta)}{(b-a)^{\alpha\beta-1}f(\eta)}.
\]

The proof is completed. \(\square\)

4 Applications

In the following, some applications of the obtained results are presented.

**Corollary 4.1** If \(\lambda \in [0,4^{\beta-1}\Gamma(\beta)\Phi_q(\Gamma(\alpha+1))]\), then the following eigenvalue problem:

\[
\begin{align*}
D_0^\alpha_D (\Phi_p(CD_0^\alpha_D y(t))) - \lambda \Phi_p(y(t)) = 0, & \quad 0 < t < 1, \\
y'(0) = CD_0^\alpha_D y(0) = 0, y(1) = CD_0^\alpha_D y(1) = 0,
\end{align*}
\]

(4.1)

has no corresponding eigenfunction \(y \in P\), where \(1 < \alpha, \beta \leq 2\), and \(p > 1\).

**Proof** Assume that \(y_0 \in P\) is an eigenfunction of (4.1) corresponding to an eigenvalue \(\lambda_0 \in [0,4^{\beta-1}\Gamma(\beta)\Phi_q(\Gamma(\alpha+1))]\). By using Theorem 3.5 with \(a = 0, b = 1, k(s) = \lambda_0\) and \(M = 1\) \((f(y) = \Phi_p(y))\), we get

\[
\lambda_0 > 4^{\beta-1}\Gamma(\beta)\Phi_q(\Gamma(\alpha+1)),
\]

which is a contradiction. \(\square\)

From Theorems 3.4 and 3.6, we have the following.

**Corollary 4.2** For fractional boundary value problem (1.12), let \(k : [a,b] \to \mathbb{R}_+\) be a non-trivial Lebesgue integrable function, and \(f \in C(\mathbb{R}_+,\mathbb{R}_+)\) be a concave and nondecreasing
function. If there exist two positive constants $r_2 > r_1 > 0$ such that the assumptions (H1) and (H2) hold, then

$$\int_a^b k(\tau) d\tau > \frac{4^\beta 1 - \Gamma(\beta) \Phi_p(\Gamma(\alpha + 1)) \Phi_p(r_1)}{(b - a)^{\alpha + 1 - f(r_2)}}.$$ 

**Example 4.3** Consider the following fractional boundary value problem:

$$\begin{cases}
D_{0+}^{\beta/2}(\Phi_{1.8}(C_{0+}^{1/3} y)) - \sqrt{7} \ln(15 + y) = 0, & 0 < t < 1, \\
y'(0) = C_{0+}^{1/3} y(0) = 0, & y(1) = C_{0+}^{4/3} y(1) = 0.
\end{cases}$$

Obviously, we have

(i) $f(y) = \ln(15 + y) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, concave and nondecreasing;

(ii) $k(t) = \sqrt{7} : [0, 1] \rightarrow \mathbb{R}_+$ is a Lebesgue integrable function with $\int_0^1 k(t) dt = \frac{2}{3} > 0$.

We now compute the values of $\rho$ and $\omega$ in (H1) and (H2), respectively.

Since $A_{3/2} = \left(\frac{3}{7}\right)^{1/2} - \left(\frac{1}{9}\right)^{1/2} = 1 - \frac{\sqrt{3}}{2}$, we have $c_{3/2} = \frac{\frac{3}{7} - A_{3/2}}{1 - A_{3/2}} = 1 - \frac{\sqrt{3}}{6}$, where $A_{3/2}$ and $c_{3/2}$ ($\beta = 3/2$) are as in (3.8). Hence

$$\sigma(s) = \begin{cases}
\frac{\sqrt{3}}{2} \left(\frac{1}{(1-s)^{1/2 - 1}}\right) & \text{if } s \in (0, 1 - \frac{\sqrt{3}}{6}), \\
\frac{1}{2^{1/2}} & \text{if } s \in \left[1 - \frac{\sqrt{3}}{6}, 1\right).
\end{cases}$$

Thus, by a simple computation, we obtain

$$\rho \approx 61.7797, \quad \omega \approx 3.8213.$$ 

Choosing $r_1 = 1/50$ and $r_2 = 1$, we obtain

1. $f(y) = \ln(15 + y) \geq \rho \Phi_{1.8}(r_1)$ for $y \in [0, 1/50]$;
2. $f(y) = \ln(15 + y) \leq \omega \Phi_{1.8}(r_2)$ for $y \in [0, 1]$.

Hence, from Corollary 4.2, we obtain

$$\int_0^1 k(t) dt > \frac{2\Gamma(3/2) \Phi_{1.8}(\frac{1}{50} \Gamma(7/3))}{\ln 16} \approx 0.0321.$$ 

**5 Conclusions**

In this paper, we prove existence of positive solutions to a nonlinear fractional boundary value problem involving a $p$-Laplacian operator. Then, under some mild assumptions on the nonlinear term, we present two new Lyapunov-type inequalities. A numerical example shows that the new results are efficient.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

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References
1. Lyapunov, AM: Problème général de la stabilité du mouvement. Ann. Fac. Sci. Univ. Toulouse 2, 203-407 (1907)
2. Brown, RC, Hinton, DB: Lyapunov inequalities and their applications. In: Rassias, TM (ed.) Survey on Classical Inequalities. Math. Appl., vol. 517, pp. 1-25. Kluwer Academic, Dordrecht (2000)
3. Tiryaki, A: Recent developments of Lyapunov-type inequalities. Adv. Dyn. Syst. Appl. 5(2), 231-248 (2010)
4. Ferreira, RAC: On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. J. Math. Anal. Appl. 412, 1058-1063 (2014)
5. Ma, D: A generalized Lyapunov inequality for a higher-order fractional boundary value problem. J. Inequal. Appl. 2016, 261 (2016)
6. Dhar, S, Kong, Q, McCabe, M: Fractional boundary value problems and Lyapunov-type inequalities with fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. 2016, 43 (2016)
7. Ferreira, RAC: A Lyapunov-type inequality for a fractional boundary value problem. Fract. Calc. Appl. Anal. 16(4), 978-984 (2013)
8. Jleli, M, Samet, B: Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. Math. Inequal. Appl. 18(2), 443-451 (2015)
9. Jleli, M, Kirane, M, Samet, B: Lyapunov-type inequalities for fractional partial differential equations. Appl. Math. Lett. 66, 30-39 (2017)
10. Jleli, M, Nieto, JJ, Samet, B: Lyapunov-type inequalities for a higher order fractional differential equation with fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. 2017, 16 (2017)
11. Rong, J, Bai, C: Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions. Adv. Differ. Equ. 2015, 82 (2015)
12. Arifi, NA, Altun, I, Jleli, M, Lashin, A, Samet, B: Lyapunov-type inequalities for a fractional $p$-Laplacian equation. J. Inequal. Appl. 2016, 189 (2016)
13. Chidouh, A, Torres, DFM: A generalized Lyapunov’s inequality for a fractional boundary value problem. J. Comput. Appl. Math. 312, 192-197 (2017)
14. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
15. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
16. Guo, G, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Notes and Reports in Mathematics in Science and Engineering, vol. 5. Academic Press, Boston (1988)
17. Rudin, W: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)