A REMARK ON THE POTENTIALS OF OPTIMAL TRANSPORT MAPS

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ABSTRACT. Optimal maps, solutions to the optimal transportation problems, are completely determined by the corresponding $c$-convex potential functions. In this paper, we give simple sufficient conditions for a smooth function to be $c$-convex when the cost is given by minimizing a Lagrangian action.

1. Introduction

The theory of optimal transportation starts from the problem of moving one mass to another in the most efficient way. Mathematically, the masses are given by two Borel probability measures $\mu$ and $\nu$ on a manifold $M$. The efficiency is measured by a cost function $c : M \times M \to \mathbb{R}$ and the problem is to find a Borel map which minimizes the following total cost:

$$\int_M c(x, \varphi(x))d\mu(x)$$

among all Borel maps $\varphi : M \to M$ which push $\mu$ forward to $\nu$. Here the push forward $\varphi_*\mu$ of a measure $\mu$ by a Borel map $\varphi$ is a measure defined by $\varphi_*\mu(U) = \mu(\varphi^{-1}(U))$ for all Borel sets $U \subseteq M$.

When the transportation cost $c$ is given by minimizing a Lagrangian action, the existence and uniqueness of solution to the above problem is known. More precisely, assume that $M$ is a compact connected manifold with no boundary and let $L : TM \to \mathbb{R}$ be a smooth function, called Lagrangian, which satisfies the following:

- the second derivative $\frac{\partial^2 L}{\partial v^2}$ is positive definite,
- $L$ is superlinear (i.e. $\lim_{|v| \to \infty} \frac{L(x,v)}{|v|} = \infty$ and $|\cdot|$ denotes the norm of a Riemannian metric).

Let $c$ be the cost function defined by

$$c(x, y) = \inf \int_0^1 L(\gamma(t), \dot{\gamma}(t))dt,$$

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where the infimum is taken over all smooth curves $\gamma(\cdot)$ connecting $x$ and $y$ (i.e. $\gamma(0) = x$ and $\gamma(1) = y$).

Assuming that the first measure $\mu$ is absolutely continuous with respect to the Lebesgue measure, Bernard-Buffoni (generalizing the earlier works of Brenier [6] in the Euclidean case and McCann [11] in the Riemannian case) proved the existence and uniqueness of solution to the above optimal transportation problem (see also [8, 3, 9] for various extensions of this result and [13, 14] for a detail introduction to the theory of optimal transportation problem).

In order to state the precise result, we need a few definitions. First let us consider the smooth function $H : T^*M \rightarrow \mathbb{R}$, called Hamiltonian, defined on the cotangent bundle $T^*M$ by the Legendre transform of the Lagrangian $L$:

$$H(x, p) = \sup_{v \in T_xM} \left[ p(v) - L(x, v) \right].$$

Let $\vec{H}$ be the Hamiltonian vector field on $T^*M$ defined in local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$ by

$$\vec{H} = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i} \right)$$

and let $\phi_t : T^*M \rightarrow T^*M$ be the flow of the Hamiltonian vector field $\vec{H}$.

Another notion that we need is $c$-convexity of a function. A function $f : M \rightarrow \mathbb{R}$ is $c$-convex if there is a function $\bar{f} : M \rightarrow \mathbb{R}$ such that

$$f(x) = \sup_{y \in M} [\bar{f}(y) - c(x, y)].$$

It is known that any $c$-convex function is Lipschitz (in fact locally semi-convex) and hence differentiable almost everywhere.

**Theorem 1.1.** [6, 11, 5] Assume that the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure and the manifold $M$ is compact. Then the optimal transportation problem corresponding to the transportation cost (2) has a unique solution $\varphi : M \rightarrow M$. Moreover, there exists a $c$-convex function $f : M \rightarrow \mathbb{R}$ such that the solution $\varphi$ is given by

$$\varphi(x) = \pi(\phi_1(df_x)),$$

where $\pi : T^*M \rightarrow \mathbb{R}$ is the canonical projection.

The unique solution $\varphi$ in Theorem 1.1 is called the optimal map. One natural question after Theorem 1.1 would be the following: Is there any
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simple condition which guarantees a given function \( f : M \to \mathbb{R} \) to be \( c \)-convex or whether the corresponding map \( \varphi(x) := \pi(\phi_1(df_x)) \) is an optimal map?

In this paper, we give a solution to the above problem using curvature type invariants introduced in [2] (see also Section 2 and Theorem 3.1 of this paper for more detail). We state a simple consequence of the main result (Theorem 3.1) when the Lagrangian is natural mechanical. More precisely, let \( \langle \cdot, \cdot \rangle \) be a Riemannian metric on a compact manifold \( M \) and let \( U : M \to \mathbb{R} \) be a smooth function, called potential. The next theorem is a version of Theorem 3.1 specialized to the Lagrangians of the form \( L(x, v) = \frac{1}{2}|v|^2 - U(x) \), called natural mechanical Lagrangians.

**Theorem 1.2.** Assume that the Lagrangian \( L \) is of the form \( L(x, v) = \frac{1}{2}|v|^2 - U(x) \), the sectional curvature is non-positive, and the Hessian of the potential \( U : M \to \mathbb{R} \) satisfies \( \text{Hess} U \leq kI \) for some constant \( k \). Let \( f \) be a \( C^2 \) function which satisfies

\[
\text{Hess} f > \begin{cases} 
-\sqrt{|k|} \coth(\sqrt{|k|})I & \text{if } k < 0 \\
-I & \text{if } k = 0 \\
-\sqrt{|k|} \cot(\sqrt{|k|})I & \text{if } k > 0.
\end{cases}
\]

Then \( f \) is \( c \)-convex and the map \( \varphi(x) := \pi(\phi_1(df_x)) \) is the unique optimal map pushing any Borel probability measure \( \mu \) forward to \( \varphi_\ast \mu \).

In the Riemannian case where the potential \( U \equiv 0 \), Theorem 3.1 can be improved using homogeneity of the corresponding Hamiltonian. Let \( v_1 = \frac{\nabla f(x)}{||\nabla f(x)||}, v_2, \ldots, v_n \) be an orthonormal basis of a tangent space \( T_x M \). We let \( S \) be the Hessian matrix of \( f \) with respect to this basis:

\[
\text{Hess } f(v_i) = \sum_{j=1}^{n} S_{ij} v_j.
\]

**Theorem 1.3.** Assume that the Lagrangian \( L \) is of the form \( L(x, v) = \frac{1}{2}|v|^2 \) and the sectional curvature \( K \leq k \) for some constant \( k \). Let \( f \) be a \( C^2 \) function and let \( \lambda := \sqrt{|k|} ||\nabla f|| \). Assume that \( f \) satisfies

\[
S > \begin{cases} 
\left( \begin{array}{cc}
-1 & 0 \\
0 & -\lambda \coth(\lambda) I
\end{array} \right) & k < 0 \\
-I & k = 0 \\
\left( \begin{array}{cc}
-1 & 0 \\
0 & -\lambda \cot(\lambda) I
\end{array} \right) & k > 0
\end{cases}
\]

When \( U \equiv 0 \), Theorem 3.1 can be improved using homogeneity of the corresponding Hamiltonian.
Then $f$ is $c$-convex and the map $\varphi(x) := \pi(\phi_1(df_x))$ is the unique optimal map pushing any Borel probability measure $\mu$ forward to $\varphi_*\mu$.

Note that if the manifold $M$ has non-positive sectional curvature, then the condition in Theorem 1.3 is just $\operatorname{Hess} f > -I$. If the manifold is two dimensional and the Riemannian metric $\langle \cdot, \cdot \rangle$ is compatible with an almost complex structure $J$ (i.e. there is an endomorphism $J : TM \to TM$ such that $J^2 = -I$ and $\langle J u, J v \rangle = \langle u, v \rangle$), then the frame $v_1, v_2$ becomes $\frac{\nabla f}{|\nabla f|}$ and $\frac{J \nabla f}{|\nabla f|}$, respectively. Therefore, Theorem 1.3 simplifies to the following.

**Theorem 1.4.** Assume that the manifold $M$ is two dimensional and there is a complex structure $J$ which is compatible with the Riemannian metric $\langle \cdot, \cdot \rangle$. Let $L$ be the Lagrangian $L(x, v) = \frac{1}{2} |v|^2$ and assume that the Gauss curvature $K \leq k$ for some constant $k$. Let $f$ be a $C^2$ function and let $\lambda := \sqrt{|k||\nabla f(x)|}$. We define the following functions:

$$
\xi = \begin{cases} 
\lambda \coth(\lambda) & \text{if } k < 0 \\
1 & \text{if } k = 0 \\
\lambda \cot(\lambda) & \text{if } k > 0,
\end{cases}
$$

$$
h_1 = \langle (\operatorname{Hess} f(\nabla f), \nabla f) \rangle,
$$

$$
h_2 = \langle (\operatorname{Hess} f(J \nabla f), J \nabla f) \rangle.
$$

Assume that the following inequalities hold for all points $x$ for which $\nabla f(x) \neq 0$:

- $\det \operatorname{Hess} f > -|\nabla f|^2 (\xi h_1 + h_2 + \xi |\nabla f|^2)$,
- $h_1 + h_2 > -(\xi + 1)|\nabla f|^2$.

Then $f$ is $c$-convex and the map $\varphi(x) := \pi(\phi_1(df_x))$ is the unique optimal map pushing any Borel probability measure $\mu$ forward to $\varphi_*\mu$.

The paper is organized as follows. In Section 2, various notions about curvature of Hamiltonian system needed in this paper is recalled. The general result (Theorem 3.1) mentioned above together with its consequence (Theorem 1.2), are stated and proved in Section 3. Finally, the proof of Theorem 1.3 is given in Section 4.
2. Curvature of Hamiltonian System

In this section, we recall some definitions and properties about curvature of Hamiltonian system needed in this paper (see [2, 1] for a more detail discussion).

Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian and let $\vec{H}$ be the corresponding Hamiltonian vector field defined by

$$\omega(\vec{H}, \cdot) = -dH(\cdot),$$

where $\omega$ is the standard symplectic structure on the cotangent bundle $T^*M$ (ie. if $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ is a local coordinate system of $T^*M$, then $\omega$ is given by $\omega := \sum_{i=1}^n dp_i \wedge dx_i$).

Recall that a Lagrangian subspace of a 2n-dimensional symplectic vector space is a n-dimensional subspace on which the symplectic form vanishes. Let $\alpha$ be a point of the cotangent bundle $T^*M$ and let $\mathcal{V}_\alpha$ be the subspace, called the vertical space, of the tangent space $T_\alpha T^*M$ defined by

$$\mathcal{V}_\alpha := \{ X \in T_\alpha T^*M | d\pi(X) = 0 \},$$

where $\pi : T^*M \to M$ is the cotangent bundle projection.

The vertical space $\mathcal{V}_\alpha$ is a Lagrangian subspace of the symplectic vector space $T_\alpha T^*M$. If $\phi_t$ is the flow of the Hamiltonian vector field $\vec{H}$, then we can define the following 1-parameter family of Lagrangian subspaces in $T_\alpha T^*M$.

$${J}_\alpha(t) = d\phi_t^{-1}(\mathcal{V}_{\phi_t(\alpha)}) = \{ d\phi_t^{-1}(X) | X \in \mathcal{V}_{\phi_t(\alpha)} \}.$$  

The family $J_\alpha$ defines a curve, called the Jacobi curve at the point $\alpha$, in the space of all Lagrangian subspaces contained in $T_\alpha T^*M$. For each fix time $t$, we can define an inner product $B^t_\alpha$ on each subspace $J_\alpha(t)$ by

$$B^t_\alpha(e, \bar{e}) = -\omega_\alpha(e, \dot{\bar{e}}(t_0)),$$

where $\bar{e}(\cdot)$ is a curve in the cotangent space $T_\alpha T^*M$ such that $\bar{e}(t)$ is contained in $J_\alpha(t)$ for each $t$.

The following proposition shows that if the Hamiltonian is fibrewise strictly convex, then the inner products $B^t$ are positive definite.

**Proposition 2.1.** [1] Assume that the restriction $H|_{T^*_\alpha M}$ of the Hamiltonian $H$ to each fibre $T^*_\alpha M$ is strictly convex. Then the bilinear form $B^t_\alpha$ is positive definite on $J_\alpha(t)$. Moreover, $B^0_\alpha$ is given by

$$B^0_\alpha(\partial_{p_i}, \partial_{p_j}) = H_{p_ip_j}.$$
For the rest of this paper, we assume that the Lagrangian \( L \) satisfies the conditions stated at the beginning of the introduction (i.e. \( L \) is superlinear and the second derivative of \( L \) in the fibre direction is positive definite). Under these assumptions, the Hamiltonian \( H \) is fibrewise strictly convex and \( B^t_\alpha \) are positive definite quadratic forms.

Recall that a basis \( \{e^1, ..., e^n, f^1, ..., f^n\} \) in a symplectic vector space with symplectic form \( \omega \) is a Darboux basis if the following conditions are satisfied:

\[
\omega(e^i, f^j) = \delta_{ij}, \quad \omega(e^i, e^j) = 0, \quad \omega(f^i, f^j) = 0.
\]

We can also pick a special family of basis \( e^1(t), ..., e^n(t) \) of the subspace \( J^\alpha(t) \) which is orthonormal with respect to the inner product \( B^t_\alpha \). More precisely, we have the following proposition (see [1] for the proof).

**Proposition 2.2.** [1] There exists a smooth family of basis \( e^1(t), ..., e^n(t) \) on the vector space \( J^\alpha(t) \) orthonormal with respect to the canonical inner product \( B^t_\alpha \) such that

\[
\{e^1(t), ..., e^n(t), f^1(t) := -\dot{e}^1(t), ..., f^n(t) := -\dot{e}^n(t)\}
\]

forms a Darboux basis of the symplectic vector space \( T^\alpha T^*M \). Moreover, if \( (\bar{e}^1(t), ..., \bar{e}^n(t)) \) is another such family, then there exists a constant orthogonal matrix \( O \) such that \( \bar{e}^i(t) = \sum_{j=1}^n O_{ij} e^j(t) \).

The 1-parameter family of Darboux basis

\[
\{e^1(t), ..., e^n(t), f^1(t), ..., f^n(t)\}
\]

in Proposition 2.2 is called a **canonical frame** at \( \alpha \). A canonical frame defines a splitting of the tangent bundle \( TT^*M \). More precisely, let \( \mathcal{H}_\alpha \) be the subspace of the tangent space \( T_\alpha T^*M \), called the horizontal space at \( \alpha \), defined by

\[
(4) \quad \mathcal{H}_\alpha = \text{span}\{f^1(0), ..., f^n(0)\}.
\]

Then we have the splitting \( T_\alpha T^*M = \mathcal{V}_\alpha \oplus \mathcal{H}_\alpha \) and both the vertical space \( \mathcal{V}_\alpha \) and the horizontal space \( \mathcal{H}_\alpha \) at \( \alpha \) are Lagrangian subspaces. Recall that \( \pi: T^*M \to M \) is the canonical projection. The restriction of the differential \( d\pi \) to the horizontal space \( \mathcal{H}_\alpha \) gives an identification between \( \mathcal{H}_\alpha \) and \( T_{\pi(\alpha)}M \). We let \( \nu^H \) be the unique vector in the horizontal space \( \mathcal{H}_\alpha \) such that \( d\pi(\nu^H) = v \). We call the vector \( \nu^H \) the horizontal lift of \( v \).
Let \((x_1, ..., x_n, p_1, ..., p_n)\) be a local coordinate system of the cotangent bundle \(T^*M\) and let \(v\) be a tangent vector of the manifold which is given by \(\sum v_i \partial_{x_i}\) in this system. Let \(c_{ij}\) be structure constants defined by

\[
v^H = \sum_{i=1}^{n} v_i \left( \partial_{x_i} + \sum_{j=1}^{n} c_{ij} \partial_{p_j} \right).
\]

The following proposition gives a formula for the structure constants \(c_{ij}\).

**Proposition 2.3.** [1] The structure constants \(c_{ij}\) satisfy

\[
2 \sum_{k,l} H_{p_k p_l} c_{kl} H_{p_p p_j} = \sum_k \left( H_{p_k p_i} - H_{p_k x_k} H_{p_p p_j} - H_{p_i x_k} H_{p_k p_j} - H_{p_p p_k} H_{x_k p_j} \right).
\]

The next proposition gives the definition of the curvature.

**Proposition 2.4.** [1] Let \(\{e^1(t), ..., e^n(t), f^1(t), ..., f^n(t)\}\) be a canonical frame at \(\alpha\). Then there is a family of linear maps \(R_\alpha(t) : J_\alpha(t) \rightarrow J_\alpha(t)\) satisfying

\[
\dot{e}^i(t) = -f^i(t), \quad \dot{f} = R_\alpha(t) e^i(t).
\]

The operators \(R_\alpha(t)\) are independent of the choice of canonical frame and it is symmetric with respect to the positive definite quadratic form \(B^t_\alpha\):

\[
B^t_\alpha(R_\alpha(t) u, v) = B^t_\alpha(u, R_\alpha(t) v).
\]

The operator \(R^H_\alpha := R_\alpha(0) : \mathcal{V}_\alpha \rightarrow \mathcal{V}_\alpha\) is called the curvature operator of the Hamiltonian \(H\). It also satisfies the following property

\[
R_\alpha(t) = d\phi_t^{-1} R^H_{\phi_t(\alpha)} d\phi_t.
\]

If \(X\) is a vector in the tangent space \(T_\alpha T^*M\), then the vertical part and the horizontal part of \(X\) are denoted by \(X_V\) and \(X_H\), respectively. The following proposition gives a characterization of the curvature operator \(R^H_\alpha\) without using canonical frame.

**Proposition 2.5.** [1] Assume that \(X\) is a vertical vector field (ie. \(X(\alpha)\) is contained in \(\mathcal{V}_\alpha\) for each \(\alpha\)). Then the curvature operator \(R^H_\alpha\) satisfies

\[
R^H_\alpha(V(\alpha)) = -[\tilde{H}, [\tilde{H}, V]_H]_\mathcal{V}(\alpha).
\]
Next, we consider the special case where the Hamiltonian is natural mechanical. More precisely, let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on $M$. Let $I : TM \to T^*M$ be the identification of the tangent bundle $TM$ and the cotangent bundle $T^*M$ defined by $I(v) = \langle v, \cdot \rangle$. Let $U : M \to \mathbb{R}$ be a smooth function, called potential. A Hamiltonian is natural mechanical if it is of the following form:

$$H(\alpha) = \frac{1}{2} |I^{-1}(\alpha)|^2 + U(x),$$

where $x = \pi(\alpha)$ and $\pi : T^*M \to M$ is the canonical projection.

The structure constants $c_{ij}$ in this case is related to the Christoffel symbols as follows:

**Proposition 2.6.** Assume that the Hamiltonian is natural mechanical and let $\Gamma^k_{ij}$ be the Christoffel symbols corresponding to the given Riemannian metric and a local coordinate system $(x_1, ..., x_n)$ of the manifold $M$. Then the structure constants $c_{ij}$ in the induced coordinate system $(x_1, ..., x_n, p_1, ..., p_n)$ of the cotangent bundle $T^*M$ are given by

$$c_{ij} = \sum_{k=1}^{n} \Gamma^k_{ij} p_k.$$

**Proof.** Assume that the Hamiltonian is given in local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$ by

$$H(x, p) = \frac{1}{2} \sum_{i,j} g^{ij} p_i p_j + U(x),$$

where $g_{ij} = \langle \partial_{x_i}, \partial_{x_j} \rangle$ and $g^{ij}$ denotes the inverse matrix of $g_{ij}$.

By Proposition 2.3, the structure constants $c_{ij}$ is given by

$$2 \sum_{s,r} g^{ir} c_{rs} g^{sj} = \sum_{k,l} \left( g^{kl} \frac{\partial g^{ij}}{\partial x_k} p_l - g^{kj} \frac{\partial g^{il}}{\partial x_k} p_l - g^{ik} \frac{\partial g^{lj}}{\partial x_k} p_l \right).$$

If we rewrite the above equation, then we get the following as claimed.

$$c_{rs} = \frac{1}{2} \sum_{i,j} g^{ij} p_j \left( - \frac{\partial g_{rs}}{\partial x_i} + \frac{\partial g_{ir}}{\partial x_s} + \frac{\partial g_{is}}{\partial x_r} \right) = \sum_{j} \Gamma^j_{rs} p_j.$$

□
For each point $\alpha$ in the cotangent bundle, we can identify the cotangent space $T^*_x M$ with the vertical space $\mathcal{V}_\alpha$ by

$$\bar{\alpha} \mapsto \bar{\alpha}^V := \left. \frac{d}{dt} \right|_{t=0} (\alpha + t\bar{\alpha}).$$

We call $\bar{\alpha}^V$ the vertical lift of $\bar{\alpha}$ at the point $\alpha$. We also recall that $I : TM \to T^* M$ is the identification of the tangent and the cotangent bundle induced by the Riemannian metric. Under these identifications, the curvature $R^H$ defined above and the Riemannian curvature $\mathcal{R}$ are related as follows.

**Proposition 2.7.** Assume that the Hamiltonian $H$ is natural mechanical. Then the curvature $R^H$ of the Hamiltonian $H$ is given by

$$R^H_{Iu}((Iv)^V) = (I(\mathcal{R}(u,v)u + \text{Hess} U(v)))^V,$$

where $\mathcal{R}$ is the Riemannian curvature and Hess denotes the Hessian with respect to the given Riemannian metric.

**Proof.** Let $E : T^* M \to \mathbb{R}$ be the kinetic energy Hamiltonian defined by

$$E(\alpha) = \frac{1}{2}|I^{-1}(\alpha)|^2.$$

Since $H = E + U$, we have

$$\tilde{H} = \tilde{E} - (dU)^V.$$

Let $V$ be a vertical vector field. Since $[(dU)^V, V]$ is vertical, it follows from Proposition 2.5 that

$$R^H(V) = -[\tilde{E} - (dU)^V, [\tilde{E}, V]^H]^V.$$

By Proposition 2.6 the horizontal spaces of the Hamiltonians $H$ and $E$ coincide. It follows that

$$R^H(V) = R^E(V) + [(dU)^V, [\tilde{E}, V]^H]^V.$$

By [2, Theorem 5.1],

$$R^E_{Iu}((Iv)^V) = (I(\mathcal{R}(u,v)u))^V.$$

Therefore, it remains to show that

$$[(dU)^V, [\tilde{E}, (IX)^V]^H]^V = (I(\text{Hess} U(X)))^V.$$

A calculation using local coordinates shows that

$$d\pi([\tilde{E}, (IX)^V]) = -X.$$
Therefore,
$$[(dU)^{V}, [\vec{E}, (IX)^{V}]_{H}]_{V} = -[(dU)^{V}, X^{H}]_{V} = (J\text{Hess } U(X))^{V}. $$

\[\square\]

3. The Main Result

In this section, we will state and prove the main result (Theorem 3.1). Before that, we need a few definitions.

Let \( f : M \to \mathbb{R} \) be a \( C^2 \) function. Its differential defines a map \( P : x \mapsto d f_x \) from the manifold \( M \) to the cotangent bundle \( T^*M \). Therefore, the differential \( dP : T M \to TT^*M \) of this map sends each tangent space \( T_xM \) to a \( n \) dimensional Lagrangian subspace \( dP(T_xM) \) of the tangent space \( T_{df_x}T^*M = V_{df_x} \oplus H_{df_x} \) (see (3) and (4) for the definition of the vertical space \( V_\alpha \) and the horizontal space \( H_\alpha \), respectively). Therefore, \( dP(T_xM) \) is the graph of a linear map \( \text{Hess}^H f : H_{df_x} \to V_{df_x}. \)

We call this map the \( H \)-Hessian of \( f \). When the Hamiltonian is natural mechanical, the \( H \)-Hessian coincides with the usual Hessian defined using the given Riemannian metric (see Proposition 3.3). Let \( e^1(t), ..., e^n(t), f^1(t), ..., f^n(t) \) be a canonical frame at the point \( \alpha \) (see Proposition 2.2 for the definition) and let \( S \) be the matrix representation of the \( H \)-Hessian defined by

$$\text{Hess}^H f(f^i(0)) = \sum_{i=1}^{n} S_{ij} e^j(0).$$

Finally we can state the main result.

**Theorem 3.1.** Assume that the curvature \( R^H \) corresponding to the Hamiltonian \( H \) satisfies \( R^H_\alpha \leq kI \) for some constant \( k \) and all points \( \alpha \) in the cotangent bundle \( T^*M \). Assume that \( f \) is a \( C^2 \) function which satisfies

$$S > \begin{cases} \sqrt{|k|} \coth(\sqrt{|k|}) I & \text{if } k < 0 \\ -I & \text{if } k = 0 \\ -\sqrt{|k|} \cot(\sqrt{|k|}) I & \text{if } k > 0. \end{cases}$$

Then \( f \) is \( c \)-convex and the \( C^1 \) map \( \varphi(x) := \pi(\phi_1(df_x)) \) is the unique optimal map pushing any Borel probability measure \( \mu \) forward to \( \varphi_*\mu. \)

Before giving the proof of Theorem 3.1, we note that Theorem 1.2 is an immediate consequence of Proposition 2.7, Theorem 3.1, and the following two results (Proposition 3.2 and Proposition 3.3).
Proposition 3.2. Assume that the Hamiltonian is natural mechanical. Let $v_1, ..., v_n$ be an orthonormal frame of the tangent space $T_xM$. Then there is a unique canonical frame $\{e^1(t), ..., e^n(t), f^1(t), ..., f^n(t)\}$ satisfying

$$e^i(0) = (Iv_i)^V, \quad f^i(0) = v_i^H, \quad i = 1, ..., n.$$ 

Proof of Proposition 3.2. Since $v_1, ..., v_n$ is orthonormal, we have, by Proposition 2.1,

$$B^0_\alpha(((Iv_i)^V), (Iv_j)^V) = \delta_{ij}.$$ 

It follows from Proposition 2.2 that there exists a unique canonical frame $\{e^1(t), ..., e^n(t), f^1(t), ..., f^n(t)\}$ such that $e^i(0) = (Iv_i)^V$.

Let $(x_1, ..., x_n, p_1, ..., p_n)$ be a local coordinate system of the cotangent bundle $T^*M$ around the point $\alpha$. Assume that the vectors $v$ and $w$ are given in this coordinate system by $v = \sum_{i=1}^n v_i \partial_{x_i}$ and $w = \sum_{i=1}^n w_i \partial_{x_i}$, respectively. Then

$$(Iv)^V = \sum_{i=1}^n g_{ij} v_i dx_j, \quad w^H = \sum_{i=1}^n w_i \left( \partial_{x_i} + \sum_{j=1}^n c_{ij} \partial_{p_j} \right),$$

where $g_{ij} = \langle v_i, v_j \rangle$.

It follows that

$$\omega(v^V, w^H) = \sum_{i,j=1}^n g_{ij} v_i w_j = \langle v, w \rangle.$$ 

Therefore, by the definition of $e^i(0)$ and the fact that $v_1, ..., v_n$ is an orthonormal family, we have

$$\omega(e^i(0), v_j^H) = \omega((Iv_i)^V, v_j^H) = \delta_{ij}.$$ 

Finally since $e^1(0), ..., e^n(0), f^1(0), ..., f^n(0)$ is a Darboux basis, we have $v_j^H = f^j(0)$ as claimed. □

Proposition 3.3. Assume that the Hamiltonian is natural mechanical. Then

$$\text{Hess}^H f(v^H) = (I(\text{Hess} f(v)))^V.$$
Proof of Proposition 3.3. Let \((x_1, \ldots, x_n, p_1, \ldots, p_n)\) be a local coordinate system and recall that \(c_{ij}\) denotes the structure constants. Suppose \(v\) is given, in this coordinate system, by \(v = \sum v_i \partial_{x_i}\). By Proposition 2.6, its horizontal lift is given by
\[
v^H = \sum_i v_i \left( \partial_{x_i} + \sum_{k,j} \Gamma^k_{ij} p_k \partial_{p_j} \right),
\]
where \(\Gamma^k_{ij}\) denotes the Christoffel symbols.

In the above coordinate system, the differential \(df\) is given by \((x_1, \ldots, x_n, \partial f / \partial x_1, \ldots, \partial f / \partial x_n)\). It follows from the definition of \(H\)-Hessian that
\[
\text{Hess}^H f(v^H) = \sum_{i,j} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} p_k \right) v_i \partial_{p_j} = (I(\text{Hess} f(v)))^V.
\]

\[\square\]

Proof of Theorem 3.1. Recall that \(\phi_t\) denotes the flow of the Hamiltonian vector field \(\vec{H}\). We define the maps \(\Phi_t : M \to T^*M\) and \(\varphi_t : M \to M\) by
\[
\Phi_t(x) = \phi_t(df_x), \quad \varphi_t = \pi \circ \Phi_t.
\]

The image \(\Lambda_t\) of the map \(d\Phi_t : T_xM \to T_{\Phi_t(x)}T^*M\) defined a Lagrangian subspace of the symplectic vector space \(T_{\Phi_t(x)}T^*M\).

Let \(e^1(t), \ldots, e^n(t), f^1(t), \ldots, f^n(t)\) be a canonical frame at the point \(\Phi_0(x)\). Let \(a_t\) and \(b_t\) be two family of matrices defines by
\[
d\Phi_0(d\pi(f^i(0))) = \sum_{j=1}^n (a^i_j f^j(t) + b^i_j e^j(t)).
\]

If we let \(F_t = (f_1(t), \ldots, f_n(t))^T\) and \(E_t = (e_1(t), \ldots, e_n(t))^T\), then (7) implies that \(a_t F_t + b_t E_t\) is constant in time \(t\). By Proposition 2.4, if we differentiate the above equation with respect to \(t\), then we get
\[
0 = \dot{a}_t F_t + a_t \dot{F}_t + \dot{b}_t E_t + b_t \dot{E}_t = \dot{a}_t F_t + a_t \tilde{R}_t E_t + \dot{b}_t E_t - b_t F_t,
\]
where \(\tilde{R}_t\) denotes the matrix given by
\[
\tilde{R}_t(e^i(t)) = \sum_{j=1}^n (\tilde{R}_t)_{ij} e^j(t).
\]
It follows that
\begin{equation}
\dot{a}_t = b_t, \quad \dot{b}_t = -a_t \bar{R}_t. \tag{9}
\end{equation}

Note that the Lagrangian subspace $\Lambda_0$, and hence $\Lambda_t$ for all small enough $t$, is transversal to the vertical space $V_{\Phi_t(x)}$. This, in turn, is equivalent to the matrix $a_t$ being invertible for all such $t$. We claim that $\Lambda_t$ is transversal for all time $t$ in the interval $[0, 1]$. In other words, we need to show that $a_t$ is invertible for all $t$ in $[0, 1]$. If we let
\begin{equation}
S_t := a_t^{-1} b_t, \tag{10}
\end{equation}
then it is enough to proof that $S_t$ is bounded for all time $t$ in $[0, 1]$.

From (7), we have that
\begin{equation*}
f^i(0) + \text{Hess}^H f(f^i(0)) = \sum_{j=1}^n (a^{ij}_0 f^j(0) + b^{ij}_0 e^j(0)).
\end{equation*}

It follows that $a_0 = I$ and $\text{Hess}^H f(f^i(0)) = \sum_{j=1}^n b^{ij}_0 e^j(0)$. Therefore, $S_0$ satisfies the initial condition
\begin{equation*}
S_0 = a_0^{-1} b_0 = S.
\end{equation*}

By (9) and (10), the matrix $S_t$ satisfies the following Riccati equation
\begin{equation*}
\dot{S}_t + S_t^2 + \bar{R}_t = 0.
\end{equation*}

Note that the matrix $S_t$ is bounded above. For the lower bound, we need the following comparison principle of matrix Riccati equations. We denote the transpose of a matrix $B$ by $B^T$.

\textbf{Theorem 3.4.} [12] Let $S^i_t$ be the solutions of the matrix Riccati equations
\begin{equation*}
\dot{S}^i_t = A^i_t + B^i_t S^i_t + S^i_t (B^i_t)^T + S^i_t C^i_t S^i_t, \quad i = 1, 2.
\end{equation*}

Assume that
\begin{equation*}
\begin{pmatrix}
A_t^1 & B_t^1 \\
(B_t^1)^T & C_t^1
\end{pmatrix} \leq \begin{pmatrix}
A_t^2 & B_t^2 \\
(B_t^2)^T & C_t^2
\end{pmatrix} \quad \text{and} \quad S_0^1 < S_0^2.
\end{equation*}

Then $S^1_t < S^2_t$.

Therefore, by the assumption and Theorem 3.4, if we consider the equation
\begin{equation}
\dot{S}_t + S_t^2 + k I = 0, \tag{11}
\end{equation}
then we have
\begin{equation}
\bar{S}_t \leq S_t. \tag{12}
\end{equation}
The solution to (11) is given by the following theorem.

**Theorem 3.5.** Let $S_t$ be the solution of the matrix Riccati equation with constant coefficients

$$\dot{S}_t = A + BS_t + S_tD + S_tC S_t.$$

Let $M_t := \left( \begin{array}{cc} M_1^1 & M_2^1 \\ M_3^1 & M_4^1 \end{array} \right)$ be the fundamental solution of the following equation with initial condition $M_0 = I:

$$\dot{z} = \left( \begin{array}{cc} B & A \\ -C & -D \end{array} \right) z.$$

Then $S_t = (M_1 S_0 + M_2)(M_3 S_0 + M_4)^{-1}$.

It follows that the solution to the equation (11) with initial condition $\bar{S}_0 = S$ is given by

$$\bar{S}_t^k = \Gamma_1(t)(\Gamma_2(t))^{-1},$$

where

$$\Gamma_1(t) = \begin{cases} 
\cosh(\sqrt{|k|}t)S + \sqrt{|k|} \sinh(\sqrt{|k|}t)I & \text{if } k < 0 \\
S & \text{if } k = 0 \\
\cos(\sqrt{|k|}t)S - \sqrt{|k|} \sin(\sqrt{|k|}t)I & \text{if } k > 0
\end{cases}$$

and

$$\Gamma_2(t) = \begin{cases} 
\frac{\sinh(\sqrt{|k|}t)}{\sqrt{|k|}}S + \cosh(\sqrt{|k|}t)I & \text{if } k < 0 \\
tS + I & \text{if } k = 0 \\
\frac{\sin(\sqrt{|k|}t)}{\sqrt{|k|}}S + \cos(\sqrt{|k|}t)I & \text{if } k > 0.
\end{cases}$$

Therefore, by (12), $S_t$ is bounded for all $t$ in $[0, 1]$ if $\Gamma_2(t) > 0$. This, in turn, follows from the following

$$(13) \quad S > \begin{cases} 
-\sqrt{|k|} \coth(\sqrt{|k|})I & \text{if } k < 0 \\
-I & \text{if } k = 0 \\
-\sqrt{|k|} \cot(\sqrt{|k|})I & \text{if } k > 0.
\end{cases}$$

This finishes the proof of the claim that $\Gamma_t$ is transversal for all $t$ in the interval $[0, 1]$. It follows from the claim and compactness of the manifold $M$ that the map $\varphi_t$ is a diffeomorphism for each $t$ in $[0, 1]$.

The following theorem is proved by the method of characteristics (see [4, Theorem 17.1, Section 17.2] for a proof).
Theorem 3.6. Assume that $\varphi_t$ is a diffeomorphism for each time $t$ in the interval $[0,1]$. Then the curve

$$t \mapsto \varphi_t(x)$$

is a strict minimizer of the minimization problem in (2) for each point $x$.

Moreover, there exists $C^2$ solution $f_t$ to the Hamilton-Jacobi equation

$$\partial_t f_t + H(x, df_t) = 0, \quad f_0 = f$$

and it satisfies

$$\Phi_t(x) = (df_t)\varphi_t(x).$$

Finally, we show that $\varphi_1$ is the optimal map between any measure $\mu$ and $(\varphi_1)_*\mu$. Let $\gamma(t)$ be a minimizer of (2) which starts from $\gamma(0) = x$ and ends at $\gamma(1) = y$. Then we have

$$f_1(y) - f_0(x) = \int_0^1 \frac{d}{dt} f_t(\gamma(t)) dt = \int_0^1 \dot{f}_t(\gamma(t)) + df_t(\dot{\gamma}(t)) dt.$$ 

By the Hamilton-Jacobi equation in Theorem 3.6, the above equation becomes

$$f_1(y) - f_0(x) = \int_0^1 -H(\gamma(t), (df_t)_*(\gamma(t))) + df_t(\dot{\gamma}(t)) dt.$$ 

By the definition of the Hamiltonian $H$, the above equation gives

$$f_1(y) - f_0(x) \leq \int_0^1 L(\gamma(t), \dot{\gamma}(t)) dt = c(x, y).$$

By Theorem 3.6, we have $\dot{\varphi}_t(x) = \partial_p H \big|_{df_t}(\varphi_t(x))$. Therefore, if we let $\gamma(t) = \varphi_t(x)$, then (14) becomes

$$f_1(\varphi_1(x)) - f_0(x) = \int_0^1 -H(x, (df_t)_*(\varphi_t(x))) + df_t(\partial_p H \big|_{df_t}(\varphi_t(x))) dt.$$ 

Finally, by the definition of the Hamiltonian $H$ and Theorem 3.6, the above equation gives

$$f_1(\varphi_1(x)) - f_0(x) = \int_0^1 L(\varphi_t(x), \varphi_t(x)) dt = c(x, \varphi_1(x)).$$

If we integrate both sides with respect to $\mu$, then we have

$$\int_M c(x, \varphi_1(x)) d\mu = \int_M f_1 d(\varphi_*\mu) - \int_M f_0 d\mu.$$
Therefore, this finishes the proof if we combine (15), (16), and the following standard theorem in the theory of optimal transportation (see [14] for a proof).

**Theorem 3.7.** Let $\pi_1, \pi_2 : M \times M \to M$ be the projections onto the first and the second entry, respectively. Then the following holds:

$$\inf \int_{M \times M} c(x, y) d\Pi = \sup \int_M f_1 d\nu - \int_M f_0 d\mu,$$

where the infimum on the left is taken over all measures $\Pi$ on the product space $M \times M$ satisfying $(\pi_1)_* \Pi = \mu$ and $(\pi_2)_* \Pi = \mu$, and the supremum on the right is taken over all pairs of continuous functions $(f_0, f_1)$ satisfying $f_1(y) - f_0(x) \leq c(x, y)$.

□

4. The Riemannian case

In the section, we specialize to the Riemannian case and give a proof of Theorem 1.3. In this case, we can use the homogeneity of the corresponding Hamiltonian to improve the result in Theorem 3.1.

**Proposition 4.1.** [2, Lemma 5.1] Assume that the Hamiltonian $H$ is homogeneous of degree $\delta + 1$ in the fibre variable. Let $\vec{r}$ be the Reeb field defined by $\vec{r}(\alpha) = \alpha^\gamma$. Then $\vec{r}(\alpha) - t\vec{H}(\alpha)$ is contained in $J_\alpha(t)$ for all $t$. In particular, the Hamiltonian vector field $\vec{H}$ is horizontal.

**Proof of Theorem 1.3.** Let $\varphi$ and $\Phi$ as in the proof of Theorem 3.1. Let $e^1(t), ..., e^n(t), f^1(t), ..., f^n(t)$ be a canonical frame at the point $\alpha = df_x$ of the cotangent bundle $T^*M$. We claim that $e^1(t)$ can be chosen to be $z(t) := \frac{1}{|I^{-1}(\alpha)|} \left( \vec{r}(\alpha) - t\vec{H}(\alpha) \right)$ if $\alpha \neq 0$.

First, note that $z(t)$ has norm one with respect to the inner product $B^t$ of $B^t$. Indeed, by the definition of $B^t$, we have

$$B^t_\alpha(z(t), z(t)) = -\omega(z(t), \dot{z}(t)) = \frac{\omega(\vec{r}(\alpha), \vec{H}(\alpha))}{|I^{-1}(\alpha)|^2}.$$

A calculation using local coordinates, we have

$$\omega(\vec{r}(\alpha), \vec{H}(\alpha)) = |I^{-1}(\alpha)|^2$$
then we have

\[ B_\alpha^t(z(t), z(t)) = 1. \]

By Theorem 4.1,

\[ z(t) = A_tE_t, \]

where \( A_t \) is a \( n \times 1 \) matrix and \( E_t = (e^1(t), ..., e^n(t))^T \).

If we differentiate the above equation twice and note that \( \ddot{z}(t) = 0 \), then we have

\[ \ddot{A_t}E_t - 2\dot{A_t}F_t - A_t\ddot{R_t}E_t = 0, \]

where \( F_t = (f^1(t), ..., f^n(t))^T \) and \( \ddot{R_t} \) is defined as in (5).

It follows that from the definition of the inner product \( B_\alpha^t \) and (5) that

\[ R_t^{ij} = B_\alpha^t(R_\alpha(t)(e^i(t)), e^j(t)) = -\omega_\alpha(d\phi_t^{-1}R_\alpha^H d\phi_t(e^i(t)), f^j(t)). \]

Since the symplectic form \( \omega \) is preserved along the Hamiltonian flow \( \phi_t \), we also have

\[
\begin{align*}
\ddot{R}_t^{ij} &= -\omega_{\phi_t(\alpha)}(R_{\phi_t(\alpha)}^H d\phi_t(e^i(t)), d\phi_t(f^j(t))) \\
&= B_0^0(\alpha)(R_{\phi_t(\alpha)}^H d\phi_t(e^i(t)), d\phi_t(e^j(t))).
\end{align*}
\]

Let \( v^i(t) \) be the tangent vectors defined along the geodesic \( t \mapsto \pi \circ \phi_t \) by the vertical lift: \( d\phi_t(e^i(t)) = (Iv^i(t))^\alpha \). Note that since the frame \( e^1(t), ..., e^n(t), f^1(t), ..., f^n(t) \) is a Darboux frame, \( v^1(t), ..., v^n(t) \) is orthonormal with respect to the Riemannian metric. It follows from Proposition 2.1 and 2.7 that

\[
\begin{align*}
\ddot{R}_t^{ij} &= B_0^0(\alpha)(R_{\phi_t(\alpha)}^H(Iv^i(t))^\alpha, (Iv^j(t))^\alpha) \\
&= B_0^0(\alpha)((I\mathcal{R}(I^{-1}(\phi_t(\alpha)), v^i(t))I^{-1}(\phi_t(\alpha)))^\alpha, (Iv^j(t))^\alpha) \\
&= (\mathcal{R}(I^{-1}(\phi_t(\alpha)), v^i(t))I^{-1}(\phi_t(\alpha)), v^j(t))_{\pi(\phi_t(\alpha))}.
\end{align*}
\]

If we assume that \( \nabla f(x) \neq 0 \), then \( e^1(t) = \frac{1}{|I^{-1}(\alpha)|}r(\alpha) - t\ddot{H}(\alpha) \) and we have, by Proposition 4.1

\[
(Iv^1(t))^\alpha = d\phi_t(e^i(t)) = \frac{1}{|I^{-1}(\alpha)|}r(\alpha) = \frac{1}{|I^{-1}(\alpha)|}\phi_t(\alpha)^\alpha.
\]
Note that $I^{-1} \alpha = I^{-1} df = \nabla f$ and the Riemannian exponential map $\exp$ satisfies $\pi(\phi_t(Iv)) = \exp(v)$. Therefore, it follows from (17) and (18) that

\begin{equation}
\bar{R}^{ij}_t = |\nabla f(x)|^2 \langle \mathfrak{R}(v^1(t), v^i(t))v^1(t), v^j(t) \rangle_{\exp(\nabla f(x))}.
\end{equation}

Note that the equation in (19) holds also in the case $\nabla f(x) = 0$.

By the assumption of the theorem, the sectional curvature is bounded above by $k$. Therefore, the following

\[
\langle \mathfrak{R}(v^1(t), \cdot)v^1(t), \cdot \rangle
\]

defines a bilinear form on the orthogonal complement of $v^1(t)$ which is bounded above by $kI$. It follows from this and (19) that

\begin{equation}
\bar{R}_t \leq \begin{pmatrix} 0 & 0 \\ 0 & k|\nabla f|^2 I \end{pmatrix}.
\end{equation}

Let $\mathcal{S}$ be the Hessian matrix defined by

\[
\mathcal{S}_{ij} = \langle v^i(0), \text{Hess} f(v^j(0)) \rangle.
\]

Note that $v^1(0) = \frac{\nabla f(x)}{|\nabla f(x)|}$ if $\nabla f(x) \neq 0$.

As in the proof of Theorem 3.1, we want to show that the solution to the Riccati equation

\begin{equation}
\dot{S}_t + S_t^2 + \bar{R}_t = 0, \quad S_0 = \mathcal{S}
\end{equation}

is bounded below under the assumptions of the theorem.

We compare the equation in (21) with

\begin{equation}
\dot{\bar{S}} + \bar{S}^2 + \begin{pmatrix} 0 & 0 \\ 0 & k|\nabla f|^2 I \end{pmatrix} = 0, \quad S_0 = \mathcal{S}.
\end{equation}

By Theorem 3.4 and (20), we have $S_t \geq \bar{S}_t$. If we apply Theorem 3.5, the solution of the initial value problem in (22) is given by

\[
\bar{S}_t = \Gamma_1(t)(\Gamma_2(t))^{-1}
\]

where $\lambda = \sqrt{|k||\nabla f|}$,

\[
\Gamma_1(t) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & \cosh(\lambda t)I \end{pmatrix} & k < 0 \\
\mathcal{S} & k = 0 \\
\begin{pmatrix} 1 & 0 \\ 0 & \cos(\lambda t)I \end{pmatrix} & k > 0
\end{cases}
\]

\[
\Gamma_2(t) = \begin{cases} 
\begin{pmatrix} 0 & 0 \\ 0 & \sinh(\lambda t)I \end{pmatrix} & k < 0 \\
\mathcal{S} & k = 0 \\
\begin{pmatrix} 0 & 0 \\ 0 & \sin(\lambda t)I \end{pmatrix} & k > 0
\end{cases}
\]
and

\[
\Gamma_2(t) = \begin{cases}
\begin{pmatrix}
t & 0 \\
0 & \frac{\sinh(\lambda t)}{\lambda} I
\end{pmatrix} & k < 0 \\
t S + I & k = 0 \\
\begin{pmatrix}
t & 0 \\
0 & \frac{\sin(\lambda t)}{\lambda} I
\end{pmatrix} & k > 0.
\end{cases}
\]

Therefore, our assumption

\[
S > \begin{cases}
\begin{pmatrix}
-1 & 0 \\
0 & -\lambda \coth(\lambda) I
\end{pmatrix} & k < 0 \\
-I & k = 0 \\
\begin{pmatrix}
-1 & 0 \\
0 & -\lambda \cot(\lambda) I
\end{pmatrix} & k > 0,
\end{cases}
\]

implies that \( S_t \) is bounded for all \( t \) in \([0, 1]\).

The rest of the proof follows as in the proof of Theorem 3.1.

\[
\square
\]

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