TORIC RINGS AND IDEALS OF NESTED CONFIGURATIONS

HIDEFUMI OHSUGI AND TAKAYUKI HIBI

ABSTRACT. The toric ring together with the toric ideal arising from a nested configuration is studied, with particular attention given to the algebraic study of normality of the toric ring as well as the Gröbner bases of the toric ideal. One of the combinatorial applications of these algebraic findings leads to insights on smooth $3 \times 3$ transportation polytopes.

INTRODUCTION

Toric rings and toric ideals play a central role in combinatorial and computational aspects of commutative algebra. In [1], from a viewpoint of algebraic statistics, the concept of nested configurations was introduced. In the present paper, the toric ring together with the toric ideal arising from a nested configuration will be studied in detail.

Let $K[t] = K[t_1, \ldots, t_d]$ denote the polynomial ring in $d$ variables over a field $K$. A (point) configuration of $K[t]$ is a finite set $A = \{t^{a_1}, \ldots, t^{a_n}\}$ of monomials belonging to $K[t]$ satisfying that there exists a vector $w \in \mathbb{R}^d$ such that $w \cdot a_i = 1$ for all $1 \leq i \leq n$. We will associate each configuration $A$ of $K[t]$ with the homogeneous semigroup ring $K[A]$, called the toric ring of $A$, which is the subalgebra of $K[t]$ generated by the monomials belonging to $A$. The toric ring $K[A]$ is called normal if $K[A]$ is integrally closed in its field of fractions. It is known that $K[A]$ is normal if and only if $Z_{\geq 0}\{a_1, \ldots, a_n\} = Z\{a_1, \ldots, a_n\} \cap \mathbb{Q}_{\geq 0}\{a_1, \ldots, a_n\}$. See, e.g., [9, Proposition 13.5]. In addition, $K[A]$ is called very ample if

$$(Z\{a_1, \ldots, a_n\} \cap \mathbb{Q}_{\geq 0}\{a_1, \ldots, a_n\}) \setminus Z_{\geq 0}\{a_1, \ldots, a_n\}$$

is a finite set. In particular, $K[A]$ is very ample if $K[A]$ is normal.

Let $K[x] = K[x_1, \ldots, x_n]$ denote the polynomial ring over $K$ in $n$ variables with each $\deg(x_i) = 1$. The toric ideal $I_A$ of $A$ is the kernel of the surjective homomorphism $\pi : K[x] \to K[A]$ defined by setting $\pi(x_i) = t^{a_i}$ for each $1 \leq i \leq n$. It is known (e.g., [9, Section 4]) that the toric ideal $I_A$ is generated by those homogeneous binomials $u - v$, where $u$ and $v$ are monomials of $K[x]$, with $\pi(u) = \pi(v)$. Fix a monomial order $<$ on $K[x]$. The initial monomial $\in_<(f)$ of $0 \neq f \in I_A$ with respect to $<$ is the biggest monomial appearing in $f$ with respect to $<$. The initial ideal of $I_A$ with respect to $<$ is the ideal $\in_<(I_A)$ of $K[x]$ generated by all initial monomials $\in_<(f)$ with $0 \neq f \in I_A$. An initial ideal $\in_<(I_A)$ is called quadratic (resp. squarefree) if $\in_<(I_A)$ is generated by quadratic (resp. squarefree) monomials. Let, in general, $\mathcal{G}$ be a finite subset of $I_A$ and write $\in_<(\mathcal{G})$ for the ideal $\langle \in_<(g) \mid g \in \mathcal{G} \rangle$ of $K[X]$. A finite set $\mathcal{G}$ of $I_A$ is said to be a Gröbner basis of $I_A$ with respect to $<$.

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< if in_<(G) = in_<(I_A). It is known that a Gröbner basis of I_A with respect to <
always exists. Moreover, if G is a Gröbner basis of I_A, then I_A is generated by G. A
Gröbner basis G of I_A is called quadratic if in_<(G) is quadratic. We are interested
in two implications below:

I_A has a squarefree initial ideal \implies K[A] is normal \implies K[A] is very ample;
I_A has a quadratic Gröbner basis \implies K[A] is Koszul \implies I_A is generated by
quadratic binomials.

It is known that each of the converse of them is false in general. See, e.g., [6, 7].

For the sake of simplicity, let A = \{t^{a_1}, \ldots, t^{a_n}\} be a configuration of K[t] with
the following properties:

- |a_j| = r for each 1 \leq j \leq n;
- t_i \text{ divides the monomial } t^{a_1} \cdots t^{a_n} \text{ for each } 1 \leq i \leq d.

(Note that any configuration is isomorphic to such a configuration.) Assume that, for
each 1 \leq i \leq d, a configuration \( B_i = \{m_{1(i)}, \ldots, m_{\lambda(i)}\} \) of a polynomial ring
\( K[u^{(i)}] = K[u_{1(i)}^{(i)}, \ldots, u_{\lambda(i)}^{(i)}] \) in \( \lambda_i \) variables over \( K \) is given. Then the nested configuration [1]
arising from \( A \) and \( B_1, \ldots, B_d \) is the configuration

\[
A(B_1, \ldots, B_d) := \left\{ m_{i_1}^{(i_{1})} \cdots m_{i_r}^{(i_r)} \mid t_{i_1} \cdots t_{i_r} \in A, \ 1 \leq j \leq \lambda_i \text{ for } 1 \leq k \leq r \right\}
\]
on the polynomial ring \( K[u^{(1)}, \ldots, u^{(d)}] \) in \( \sum_{i=1}^{d} \lambda_i \) variables over \( K \). Here, \( t_{i_1} \cdots t_{i_r} \in A \) is not necessarily squarefree. If \( A = \{t_1 t_2\} \), then \( K[A(B_1, B_2)] \) is the Segre product of \( K[B_1] \) and \( K[B_2] \). Moreover, if \( A = \{t_1^m\} \), then \( K[A(B_1)] \) is the \( m \)-th Veronese subring of \( K[B_1] \).

**Example 0.1.** Let \( A = \{t_1^2, t_1 t_2\}, B_1 = \{u_1^2, u_1 u_2, u_2^2\} \) and \( B_2 = \{v_1^2 v_2, v_1 v_2^2\} \). Then,
the nested configuration \( A(B_1, B_2) \) consists of the monomials

\[
\begin{align*}
&u_1^4, u_1^3 u_2, u_2^2 u_2, u_1 u_2^2, u_2^4, \ u_1^2 v_1 v_2, u_1 u_2 v_1^2 v_2, u_2^2 v_1 v_2, u_1^2 v_1^2 v_2, u_1 u_2 v_1^2 v_2, u_2^2 v_1 v_2, u_2^2 v_1 v_2.
\end{align*}
\]

Then, the matrices

\[
M_A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \ M_{B_1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \ M_{B_2} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},
\]

\[
M_{A(B_1, B_2)} = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}
\]
correspond to the configurations \( A, B_1, B_2 \) and \( A(B_1, B_2) \), respectively.

One of the fundamental facts of the nested configuration is

**Theorem 0.2.** If each of the toric ideals \( I_A, I_{B_1}, \ldots, I_{B_d} \) possesses a quadratic
Gröbner basis, then the toric ideal \( I_{A(B_1, \ldots, B_d)} \) possesses a quadratic Gröbner basis.

In Section 1, we study the normality of the toric ring arising from a nested config-
uration. Our first main result is Theorem 1.2 if each of \( K[A], K[B_1], \ldots, K[B_d] \) are
normal then \( K[A(B_1, \ldots, B_d)] \) is also normal. In general – see Example 1.3 – the
converse does not hold. However, Corollary 1.9 guarantees that, when \(A\) consists of squarefree monomials, each of \(K[A], K[B_1], \ldots, K[B_d]\) is normal if and only if \(K[A(B_1, \ldots, B_d)]\) is normal.

In Section 2, we study Gröbner bases of the toric ideal arising from a nested configuration. A natural generalization of Theorem 0.2 will be obtained. In fact, Theorem 2.5 together with Theorem 2.6 guarantees that if each of \(I_A, I_{B_1}, \ldots, I_{B_d}\) possesses a Gröbner basis consisting of binomials of degree at most \(p\), then \(I_{A(B_1, \ldots, B_d)}\) possesses a Gröbner basis consisting of binomials of degree at most \(\max(2, p)\). Moreover, if each of \(I_A, I_{B_1}, \ldots, I_{B_d}\) possesses a squarefree initial ideal, then \(I_{A(B_1, \ldots, B_d)}\) possesses a squarefree initial ideal.

In Section 3, as one of the combinatorial applications of our algebraic theory of nested configurations, we discuss the toric ideal of a multiple of the Birkhoff polytope \(B_3\). Here \(B_3\) is the convex hull of

\[
\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
\sigma_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

in \(\mathbb{R}^{3 \times 3}\). The toric ideal of \(B_3\) is the toric ideal of the configuration

\[B_1 = \{u_{11}u_{22}u_{33}, u_{12}u_{23}u_{31}, u_{13}u_{21}u_{32}, u_{11}u_{23}u_{32}, u_{12}u_{21}u_{33}, u_{13}u_{22}u_{31}\}\]

of polynomial ring \(K[u_{11}, \ldots, u_{33}]\) and it is a principal ideal generated by \(z_1z_2z_3 - z_4z_5z_6\). Given an integer \(m \geq 1\), \(m\) multiple of \(B_3\) is defined by \(mB_3 = \{m\alpha \mid \alpha \in B_3\}\). Since it is well-known (due to Birkhoff) that

\[mB_3 \cap \mathbb{Z}^{3 \times 3} = \{\sigma_{i_1} + \cdots + \sigma_{i_m} \mid 1 \leq i_1, \ldots, i_m \leq 6\},\]

the toric ideal of \(mB_3\) is the toric ideal of the nested configuration \(A(B_1)\) where \(A = \{t_{1}^{m}\}\). In [2], they say that L. Piechnik and C. Haase proved that the toric ideal of the multiple \(2nB_3\) possesses a squarefree quadratic initial ideal for \(n > 1\). This fact is directly obtained by Theorem 2.6 since the toric ideal of the multiple \(2B_3\) possesses a squarefree quadratic initial ideal. Similarly, since the toric ideal of the multiple \(3B_3\) possesses a squarefree quadratic initial ideal, Theorem 2.6 guarantees that the toric ideal of the multiple \(3nB_3\) possesses a squarefree quadratic initial ideal for \(n > 1\). However, since there are infinitely many prime numbers, it is difficult to show the existence of a squarefree quadratic initial ideal of the toric ideal of \(mB_3\) for all \(m > 1\) in this way. In Theorem 3.4, using another monomial order, we will prove that the toric ideal of the multiple \(mB_3\) possesses a quadratic Gröbner basis for all \(m > 1\).

In Section 4, we give a summary of our algebraic theory of nested configurations.

1. Normality of Toric Rings of Nested Configurations

The purpose of this section is to study normality of \(K[A(B_1, \ldots, B_d)]\).
Lemma 1.1. The toric ring $K[A]$ is normal if and only if
\[
\left\{ \frac{M_1}{M_2} \bigg| \begin{array}{l} M_1, M_2 \in K[A] \text{ are monomials and } \\ \left( \frac{M_1}{M_2} \right)^m \in K[A] \text{ for some } 0 < m \in \mathbb{Z} \end{array} \right\}
\]
is a subset of $K[A]$.

Theorem 1.2. If $K[A], K[B_1], \ldots, K[B_d]$ are normal, then $K[A(B_1, \ldots, B_d)]$ is normal.

Proof. Suppose that $K[A], K[B_1], \ldots, K[B_d]$ are normal and that $K[A(B_1, \ldots, B_d)]$ is not normal. Thanks to Lemma 1.1, there exist monomials $M_1, M_2, M_3$ belonging to $K[A(B_1, \ldots, B_d)]$ such that $M_1/M_2 \notin K[A(B_1, \ldots, B_d)]$ and that $(M_1/M_2)^n = M_3$ for some integer $n > 1$.

Let $\psi : K[A(B_1, \ldots, B_d)] \to K[A]$ be the surjective homomorphism defined by $\psi(m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}) = t_{i_1} \cdots t_{i_r} \in A$. Then $\psi(M_1), \psi(M_2) \in K[A]$ and
\[
(\psi(M_1)/\psi(M_2))^n = \psi(M_3) \in K[A].
\]
Since $K[A]$ is normal, we have $\psi(M_1)/\psi(M_2) \in K[A]$. Thus $\psi(M_1)/\psi(M_2) = t^{a_1} \cdots t^{a_p}$ for some $1 \leq i_1, \ldots, i_p \leq n$.

Let $\rho_k : K[A(B_1, \ldots, B_d)] \to K[B_k]$ be the surjective homomorphism defined by
\[
\rho_k(i_j^{(i)}) = \begin{cases} u_j^{(i)} & \text{if } i = k \\ 1 & \text{otherwise}. \end{cases}
\]
Then $\rho_k(M_1), \rho_k(M_2) \in K[B_k]$ and $(\rho_k(M_1)/\rho_k(M_2))^n = \rho_k(M_3) \in K[B_k]$. Since $K[B_k]$ is normal, $\rho_k(M_1)/\rho_k(M_2) \in K[B_k]$. Thus $\rho_k(M_1)/\rho_k(M_2) = m_{j_1}^{(k)} \cdots m_{j_q}^{(k)}$ for some $1 \leq j_1, \ldots, j_q \leq \lambda$. Since $B_k$ is a configuration, it follows that $\rho_k(M_1) = m_{j_1}^{(k)} \cdots m_{j_q}^{(k)}$ and $\rho_k(M_2) = m_{i_1}^{(k)} \cdots m_{i_r}^{(k)}$. Then $\psi(M_1) = t^{a_1+r_1} \cdots t_d^{a_d+r_d}$ and $\psi(M_2) = t_1^{q_1} \cdots t_d^{q_d}$. Thus we have
\[
\frac{\psi(M_1)}{\psi(M_2)} = t^{a_1} \cdots t^{a_p} = t_1^{q_1} \cdots t_d^{q_d}.
\]
Hence $M_1/M_2 \in K[A(B_1, \ldots, B_d)]$ and this is a contradiction. \hfill \Box

The converse of Theorem 1.2 is false in general.

Example 1.3. Let $A = \{ t_1^2 \}$ and $B_1 = \{ v, uv, u^3v, u^4v \}$. Then $K[B_1]$ is not normal. However, $I_{A(B_1)}$ has a squarefree quadratic initial ideal and hence $K[A(B_1)] = K[[u^i v^j | i = 0, 1, \ldots, 8]]$ is normal.

Let $A = \{ t^{a_1}, \ldots, t^{a_n} \}$ be a configuration. Then $K[A]$ is called very ample if
\[
\left( \mathbb{Z}\{a_1, \ldots, a_n\} \cap \mathbb{Q}_{\geq 0}\{a_1, \ldots, a_n\} \right) \setminus \mathbb{Z}_{\geq 0}\{a_1, \ldots, a_n\}
\]
is a finite set. In particular, $K[A]$ is very ample if $K[A]$ is normal. Theorem 1.2 did not hold when we replaced “normal” with “very ample.”
Example 1.4. Let \( A = \{t_1, t_2\} \), \( B_1 = \{v, uv, u^2v, u^3v\} \) and \( B_2 = \{w\} \). Then \( K[A] \) and \( K[B_2] \) are polynomial rings. On the other hand, \( K[B_1] \) is very ample, but not normal. However, \( K[A(B_1, B_2)] = K[v, uv, u^2v, u^3v, w] \) is not very ample. In fact, the monomial \( u^2vw^2 \) does not belong to \( K[A(B_1, B_2)] \) for all \( \alpha \in \mathbb{Z}_{\geq 0} \).

Let \( P_A \) denote the convex hull of \( \{a \in \mathbb{Z}_{\geq 0}^d \mid t^a \in A\} \). For a subset \( B \subset A \), \( K[B] \) is called the combinatorial pure subring (\[5, 4\]) of \( K[A] \) if there exists a face \( F \) of \( P_A \) such that \( \{b \in \mathbb{Z}_{\geq 0}^d \mid t^b \in B\} = \{a \in \mathbb{Z}_{\geq 0}^d \mid t^a \in A\} \cap F \). For example, if \( B = A \cap K[t_{i_1}, \ldots, t_{i_s}] \) for some \( 1 \leq i_1 < \cdots < i_s \leq d \), then \( K[B] \) is a combinatorial pure subring of \( K[A] \). (This is the original definition of a combinatorial pure subring in \([5]\).)

Lemma 1.5. The toric ring \( K[A(B_1, \ldots, B_d)] \) has a combinatorial pure subring which is isomorphic to \( K[A] \).

Proof. For each \( i = 1, 2, \ldots, d \), let \( \sigma_i \) be an arbitrary monomial of \( B_i \) which corresponds to a vertex of \( P_{B_i} \). It follows that \( K[A(\{\sigma_1\}, \ldots, \{\sigma_d\})] \) is a combinatorial pure subring of \( K[A(B_1, \ldots, B_d)] \). Then \( K[A(\{\sigma_1\}, \ldots, \{\sigma_d\})] \simeq K[A] \). \( \square \)

It is known [8, Lemma 1] that every combinatorial pure subring of a normal (resp. very ample) semigroup ring is normal (resp. very ample). Thus we have the following.

Theorem 1.6. If \( K[A(B_1, \ldots, B_d)] \) is normal (resp. very ample), then \( K[A] \) is normal (resp. very ample).

Lemma 1.7. Let \( m = \max(i \mid t_1^{i_1}t_2^{i_2}\cdots t_d^{i_d} \in A) \geq 1 \). Then \( K[A(B_1, \ldots, B_d)] \) has a combinatorial pure subring which is isomorphic to \( K[A'(B_1)] \) where \( A' = \{t_1^m\} \). In particular, if \( m = 1 \), then we have \( K[A'(B_1)] \simeq K[B_1] \).

Proof. Let \( t_1^{m_1}t_2^{a_2}\cdots t_d^{a_d} \) be the largest monomial of \( A \) with respect to a lexicographic order \( t_1 > \cdots > t_d \). Let \( A = \{t^{a_1} = t_1^{m_1}t_2^{a_2}\cdots t_d^{a_d}, t^{a_2}, \ldots, t^{a_n}\} \). Thanks to [9, Proposition 1.11], there exists a nonnegative integer vector \( v \) such that \( v \cdot a_i > v \cdot a_i \), for all \( 2 \leq i \leq n \). Then \( (m, a_2, \ldots, a_d) \) is a \( v \)-vertex of \( P_A \). Hence \( K[A(B_1, \ldots, B_d)] \) has a combinatorial pure subring \( K[A''(B_1, \ldots, B_d)] \) with \( A'' = \{t_1^m t_2^{a_2}\cdots t_d^{a_d}\} \). For each \( i = 2, \ldots, d \), let \( \sigma_i \) be an arbitrary monomial of \( B_i \) which corresponds to a vertex of \( P_{B_i} \). It follows that \( K[A''(B_1, \{\sigma_2\}, \ldots, \{\sigma_d\})] \) is a combinatorial pure subring of \( K[A''(B_1, \ldots, B_d)] \). Then \( K[A''(B_1, \{\sigma_2\}, \ldots, \{\sigma_d\})] \simeq K[A'(B_1)] \) where \( A' = \{t_1^m\} \). \( \square \)

Thanks to Lemma 1.7 we have the following.

Theorem 1.8. If \( A \) has no monomial divided by \( t_i^2 \) and if \( K[A(B_1, \ldots, B_d)] \) is normal (resp. very ample), then \( K[B_i] \) is normal (resp. very ample).

Corollary 1.9. Suppose that a configuration \( A \) consists of squarefree monomials. Then \( K[A], K[B_1], \ldots, K[B_d] \) are normal if and only if \( K[A(B_1, \ldots, B_d)] \) is normal.
2. Gröbner bases of toric ideals of nested configurations

In this section, using the technique (sorting operator) in the proof of [9, Theorem 14.2], we study Gröbner bases of the toric ideal of a nested configuration. The present section has three subsections:

- Gröbner bases for polynomial ring case, i.e., each \( K[B_i] \) is a polynomial ring;
- Gröbner bases for general case;
- Generators.

First, we introduce the sorting operator used in [9]:

**Example 2.1** ([9], Theorem 14.2). Fix positive integers \( r \) and \( s_1, \ldots, s_d \). Let

\[
A = \{ t_1^{i_1} \cdots t_d^{i_d} \mid i_1 + \cdots + i_d = r, \ 0 \leq i_1 \leq s_1, \ \ldots, \ 0 \leq i_d \leq s_d \}.
\]

We define a natural bijection between the element of \( A \) and weakly increasing strings of length \( r \) over the alphabet \( \{1, 2, \ldots, d\} \) having at most \( s_j \) occurrence of the letter \( j \) which maps the monomial \( t_1^{i_1} \cdots t_d^{i_d} \in A \) to the weakly increasing string

\[
\begin{align*}
  u_1 u_2 \cdots u_r &= 1^{i_1 \text{ times}} 2^{i_2 \text{ times}} 3^{i_3 \text{ times}} \cdots d^{i_d \text{ times}}.
\end{align*}
\]

We write \( x_{u_1 u_2 \cdots u_r} \) for the corresponding variable in \( K[x] \). Let \( \text{sort}(\cdot) \) denote the operator which takes any string over the alphabet \( \{1, 2, \ldots, d\} \) and sorts it into weakly increasing order. It is known [9, Theorem 14.2] that there exists a monomial order \( < \) on \( K[x] \) such that

\[
\{ x_{u_1 u_2 \cdots u_r} x_{v_1 v_2 \cdots v_r} - x_{w_1 w_2 \cdots w_{2r-1}} x_{w_{2r-1} w_{2r}} \mid w_1 w_2 w_3 \cdots w_{2r} = \text{sort}(u_1 v_1 u_2 v_2 \cdots u_r v_r) \}
\]

is a quadratic Gröbner basis of \( I_A \) with respect to \( < \) and \( \text{in}_<(I_A) \) is squarefree. For example, \( x_{12} x_{33} - x_{13} x_{23} \) belongs to the Gröbner basis since we have \( 1233 = \text{sort}(1323) \).

Let, as before, \( A = \{ t^{a_1}, \ldots, t^{a_n} \} \) and \( B_i = \{ m_{i1}^{(i)}, \ldots, m_{ik}^{(i)} \} \) for \( 1 \leq i \leq d \). Let \( K[x] \) be a polynomial ring with the set of variables

\[
\left\{ x_{(i_1, j_1), \ldots, (i_r, j_r)}^{(k)} \mid 1 \leq i_1 \leq \cdots \leq i_r \leq d, \ 1 \leq k \leq n, \ t_{i_1} \cdots t_{i_r} = t^{a_k} \in A, \ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \in A(B_1, \ldots, B_d) \right\}
\]

and let \( K[y] = K[y_1, \ldots, y_n] \) and \( K[z^{(i)}] = K[z_1^{(i)}, \ldots, z_{\lambda_i}^{(i)}] \) \( (i = 1, 2, \ldots, d) \) be polynomial rings. The toric ideal \( I_A \) is the kernel of the homomorphism \( \pi_0 : K[y] \to K[t] \) defined by setting \( \pi_0(y_k) = t^{a_k} \). The toric ideal \( I_{B_i} \) is the kernel of the homomorphism \( \pi_i : K[z^{(i)}] \to K[u^{(i)}] \) defined by setting \( \pi_i(z_j^{(i)}) = m_j^{(i)} \). The toric ideal \( I_{A(B_1, \ldots, B_d)} \) is the kernel of the homomorphism \( \pi : K[x] \to K[u^{(1)}, \ldots, u^{(d)}] \) defined by setting \( \pi( x_{(i_1, j_1), \ldots, (i_r, j_r)}^{(k)} ) = m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \).

**Lemma 2.2.** Let \( p_1 = x_{(i_1, j_1), \ldots, (i_r, j_r)}^{(k)} x_{(i_{r+1}, j_{r+1}), \ldots, (i_{2r}, j_{2r})}^{(k)} \) be a quadratic monomial in \( K[x] \) and let \( \text{sort}(\cdot) \) be the sorting operator over the alphabet

\[
\{(1, 1), (1, 2), \ldots, (1, \lambda_1), (2, 1), \ldots, (d, \lambda_d)\}
\]
with respect to the ordering

$$(1,1) > (1,2) > \cdots > (1,\lambda_1) > (2,1) > \cdots > (d,\lambda_d).$$

Then, $p_2 = x_{(k)}^{(i_1',j_1')(i_2',j_2')\cdots(i_{2r-1}',j_{2r-1}')x_{(k)}^{(i_2',j_2')\cdots(i_{2r}',j_{2r})}$ where

$$(i_1',j_1') \cdots (i_{2r}',j_{2r}') = \text{sort}((i_1,j_1) \cdots (i_{2r},j_{2r}))$$

is a monomial belonging to $K[x]$ and, in particular, we have $p_1 - p_2 \in I_{A(B_1,\ldots,B_d)}$.

**Proof.** Suppose that $x_{(k)}^{(i_1',j_1')(i_2',j_2')\cdots(i_{2r-1}',j_{2r-1}')x_{(k)}^{(i_2',j_2')\cdots(i_{2r}',j_{2r})}$ is not a variable in $K[x]$. Then we have $t_{i_1}t_{i_2}\cdots t_{i_{2r-1}} \neq t_{a_k}$ and hence there exist integers $1 \leq i \leq d$ and $\alpha$ such that $t_i^\alpha$ does not divide $t_{a_k}$ and does not divide $t_{i_1}t_{i_2}\cdots t_{i_{2r-1}}$. Since $i_1' \leq \cdots \leq i_{2r}'$, it then follows that $t_i^\alpha$ does not divide $t_{i_1}t_{i_2}\cdots t_{i_{2r}}$. Thanks to $((i_1',j_1') \cdots (i_{2r}',j_{2r})) = \text{sort}((i_1,j_1) \cdots (i_{2r},j_{2r}))$, we have $t_{i_1}t_{i_2}\cdots t_{i_{2r}} = t_{i_1'}t_{i_2'}\cdots t_{i_{2r}'}$. Hence $t_i^\alpha$ does not divide $t_{i_1}t_{i_2}\cdots t_{i_{2r}}$. It follows that $t_i^\alpha$ does not divide either $t_{i_1}t_{i_2}\cdots t_{i_r}$ or $t_{i_{r+1}}t_{i_{r+2}}\cdots t_{i_{2r}}$. Thus either $t_{i_1}t_{i_2}\cdots t_{i_r}$ or $t_{i_{r+1}}t_{i_{r+2}}\cdots t_{i_{2r}}$ is not equal to $t_{a_k}$. This contradicts that $p_1$ is a monomial of $K[x]$.

On the other hand, by virtue of $((i_1',j_1') \cdots (i_{2r}',j_{2r})) = \text{sort}((i_1,j_1) \cdots (i_{2r},j_{2r}))$, we have $\pi(p_1) = \pi(p_2)$ and hence $p_1 - p_2 \in I_{A(B_1,\ldots,B_d)}$ as desired. \qed

**Lemma 2.3.** Let $y_{k_1}' \cdots y_{k_p}' = y_{k_1} \cdots y_{k_p}$ be a binomial in $I_A$ and let

$$\prod_{\ell=1}^{p} x_{(i_\ell,j_\ell)}^{(k_\ell)}$$

be a monomial in $K[x]$. Then, there exists a binomial

$$\prod_{\ell=1}^{p} x_{(i_\ell,j_\ell)}^{(k_\ell)} - \prod_{\ell=1}^{p} x_{(i_\ell',j_\ell')}^{(k_\ell')} \in I_{A(B_1,\ldots,B_d)},$$

where $\text{sort}((i_1,j_1) \cdots (i_{pr},j_{pr})) = \text{sort}((i_1',j_1') \cdots (i_{pr}',j_{pr}'))$.

**Proof.** Let $\pi(0)(y_{k_\ell}') = t_{i_{\ell}}(t_{i_{\ell+1}}') \cdots t_{i_{\ell}}(t_{i_{\ell+1}}')$ for each $1 \leq \ell \leq p$. Since $y_{k_1} \cdots y_{k_p} - y_{k_1}' \cdots y_{k_p}'$ belongs to $I_A$, we have $\prod_{\ell=1}^{p} t_{i_{\ell}} = \prod_{\ell=1}^{p} t_{i_{\ell}'}$. Hence there exist $j_1',\ldots,j_{pr}'$, such that

$$\text{sort}((i_1,j_1) \cdots (i_{pr},j_{pr})) = \text{sort}((i_1',j_1') \cdots (i_{pr}',j_{pr}')).$$

It then follows that

$$\prod_{\ell=1}^{p} x_{(i_\ell,j_\ell)}^{(k_\ell)} - \prod_{\ell=1}^{p} x_{(i_\ell',j_\ell')}^{(k_\ell')} \in I_{A(B_1,\ldots,B_d)}$$

as desired. \qed

Fix a monomial order $<_i$ on $K[z^{(i)}]$ for each $1 \leq i \leq d$. Let $G_i$ be a Gröbner basis of $I_{B_i}$ with respect to $<_i$. For each $M \in A(B_1,\ldots,B_d)$, the expression $M = m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}$ is called standard if

$$\prod_{i_\ell = j_\ell, \ 1 \leq \ell \leq r} z_{j_\ell}^{(i_\ell)}$$
is a standard monomial with respect to \(G_j\) for all \(1 \leq j \leq d\). In order to study the relation among \(I_A, I_B,\) and \(I_{A(B_1, \ldots, B_d)}\), we define homomorphisms

\[
\varphi_0 : K[x] \rightarrow K[y], \quad \varphi_0 \left( x_{i_1,j_1}^{(k)} \ldots x_{i_r,j_r}^{(k)} \right) = y_k,
\]

\[
\varphi_j : K[x] \rightarrow K[z^{(j)}], \quad \varphi_j \left( x_{i_1,j_1}^{(k)} \ldots x_{i_r,j_r}^{(k)} \right) = \prod_{i=\ell}^{d} z_{j\ell}^{(i)}
\]

where \(m_{j_1}^{(i_1)} \ldots m_{j_r}^{(i_r)}\) is the standard expression defined above.

**Lemma 2.4** (H). Let \(f\) be a binomial in \(K[x]\). Then \(f \in I_{A(B_1, \ldots, B_d)}\) if and only if \(\varphi_i(f) \in I_{B_i}\) for all \(1 \leq i \leq d\). Moreover, if \(f\) belongs to \(I_{A(B_1, \ldots, B_d)}\), then we have \(\varphi_0(f) \in I_A\).

**2.1. Polynomial ring case.** First, we study the case when all of \(K[B_i]\) are polynomial rings.

**Theorem 2.5.** Let \(G_0\) be a Gröbner basis of \(I_A\) with respect to a monomial order \(<_0\). If each \(B_i\) is a set of variables, then the toric ideal \(I_{A(B_1, \ldots, B_d)}\) possesses a Gröbner basis consisting of the following binomials:

1. \[
\prod_{\ell=1}^{P} x_{(i_1,j_1) \ldots (i_{r-1},j_{r-1})}^{(k)} - \prod_{\ell=1}^{P} x_{(i_1,j_1) \ldots (i_{r-1},j_{r-1})}^{(k)} \quad \text{where } y_{k_1} \ldots y_{k_P} \in G_0 \text{ and } \text{sort}((i_1,j_1) \ldots (i_{pr},j_{pr})) = \text{sort}((i'_1,j'_1) \ldots (i'_{pr},j'_{pr})).
\]

2. \[
\prod_{\ell=1}^{P} x_{(i_1,j_1) \ldots (i_{r-1},j_{r-1})}^{(k)} - \prod_{\ell=1}^{P} x_{(i_1,j_1) \ldots (i_{r-1},j_{r-1})}^{(k)} \quad \text{where } \text{sort}((i_1,j_1) \ldots (i_{2r},j_{2r})) = (i'_1,j'_1) \ldots (i'_{2r},j'_{2r}) \text{ with respect to the ordering } (1,1) > (1,2) > \cdots > (1,\lambda_1) > (2,1) > \cdots > (d,\lambda_d).
\]

3. \[
\prod_{\ell=1}^{P} x_{(i_1,j_1) \ldots (i_{r-1},j_{r-1})}^{(k)} - \prod_{\ell=1}^{P} x_{(i_1,j_1) \ldots (i_{r-1},j_{r-1})}^{(k)} \quad \text{where } k < k', i_\ell = i'_\ell \text{ and } j_\ell > j'_\ell.
\]

The initial monomial of each binomial is the first (underlined) monomial and, in particular, the initial monomial of each binomial in (2) and (3) is squarefree. Moreover, the initial monomial of each binomial in (1) is squarefree (resp. quadratic) if the corresponding monomial \(y_{k_1} \ldots y_{k_P}\) is squarefree (resp. quadratic).

**Proof.** Let \(G\) denote the set of binomials above. Thanks to Lemmas 2.2 and 2.3, it is easy to see that \(G\) is a (finite) subset of \(I_{A(B_1, \ldots, B_d)}\).

**Claim 1.** There exists a monomial order such that the initial monomial of each binomial in \(G\) is the underlined monomial.

By virtue of [9] Theorem 3.12, it is enough to show that the reduction modulo \(G\) is Noetherian. Suppose that there exists a sequence of reductions modulo \(G\) which
does not terminate. Let \( v \) be a monomial in \( K[x] \) and assume \( v \xrightarrow{g} v' \) with \( g \in \mathcal{G} \). Then we have
\[
\begin{cases}
\varphi_0(v) > \varphi_0(v') & \text{if } g \text{ in (1)}, \\
\varphi_0(v) = \varphi_0(v') & \text{otherwise}.
\end{cases}
\]
Hence the number of binomials in (1) appearing in the sequence is finite. Thus we may assume that the binomials in (1) do not appear in the sequence. Let \( v \) be a monomial in \( K[x] \) and assume \( v \xrightarrow{g} v' \) where \( g \in \mathcal{G} \) belongs to either (2) or (3). Since \( g \) belongs to either (2) or (3), \( v \) and \( v' \) is of the form \( v = \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1}):\cdots:(i_{\ell r-1},j_{\ell r-1})} \) and \( v' = \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1}):\cdots:(i'_{\ell r-1},j'_{\ell r-1})} \). Let
\[
\text{Inversion}(v) = \begin{cases}
(\xi, \xi') & | \ell(r-1) + 1 \leq \xi \leq \ell r \\
i_\xi = i_{\xi'}, j_\xi < j_{\xi'} & k_\ell < k_{\ell'}
\end{cases},
\]
\[
\text{Inversion}(v') = \begin{cases}
(\xi, \xi') & | \ell'(r-1) + 1 \leq \xi' \leq \ell' r \\
i'_{\xi} = i'_{\xi'}, j'_{\xi} > j'_{\xi'} & k_{\ell'} < k_{\ell'}
\end{cases}.
\]
Then the cardinality of these sets satisfies \(|\text{Inversion}(v)| \geq |\text{Inversion}(v')| \) where equality holds if and only if \( g \) belongs to (2). Hence the number of binomials in (3) appearing in the sequence is finite. Thus we may assume that the binomials in (3) do not appear in the sequence. However, any sequence of reductions modulo the set of binomials in (2) corresponds to the sort of the indices and hence it terminates. This is a contradiction.

**Claim 2.** The set \( \mathcal{G} \) is a Gröbner basis of \( I_{A(B_1,\ldots,B_d)} \).

Suppose that \( \mathcal{G} \) is not a Gröbner basis of \( I_{A(B_1,\ldots,B_d)} \). Thanks to Lemmas 2.2 and 2.3 there exists a binomial \( f = p_1 - p_2 \in I_{A(B_1,\ldots,B_d)} \) such that neither \( p_1 \) nor \( p_2 \) is divisible by the initial monomial of any binomial in \( \mathcal{G} \). By virtue of Lemma 2.1 we have \( \varphi_0(f) = \varphi_0(p_1) - \varphi_0(p_2) \in \mathcal{G} \). If \( \varphi_0(p_1) - \varphi_0(p_2) \neq 0 \), then there exists a binomial \( g \in \mathcal{G}_0 \) such that the initial monomial of \( g \) divides either \( \varphi_0(p_1) \) or \( \varphi_0(p_2) \). This contradicts that neither \( p_1 \) nor \( p_2 \) is divisible by the initial monomial of any binomial in (1). Hence we have \( \varphi_0(p_1) = \varphi_0(p_2) \). Thus \( f \) is of the form
\[
f = \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1}):\cdots:(i_{\ell r-1},j_{\ell r-1})} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1}):\cdots:(i'_{\ell r-1},j'_{\ell r-1})}.
\]
Since neither \( p_1 \) nor \( p_2 \) is divisible by the initial monomial of any binomial in either (2) or (3), it follows that \( p_1 = p_2 \) and hence \( f = 0 \). \( \square \)

### 2.2. General case

**Theorem 2.6.** Let \( \mathcal{G}_0 \) be a Gröbner basis of \( I_A \) and let \( \mathcal{G}_i \) be a Gröbner basis of \( I_{B_i} \) with respect to \( <_{i} \). Then the toric ideal \( I_{A(B_1,\ldots,B_d)} \) possesses a Gröbner basis consisting of the binomials (1), (2) and (3) appearing in Theorem 2.7 together with the following binomials:
\[ (4) \prod_{\ell=1}^{p} x_{M_{\ell}(i_{\ell}, j_{\ell,1}, \ldots, j_{\ell,q_{\ell}})}^{(k_{\ell})} - \prod_{\ell=1}^{p} x_{M_{\ell}(i_{\ell}, j'_{\ell,1}, \ldots, j'_{\ell,q_{\ell}})}^{(k_{\ell})} \mathrm{where ~the ~binomial} \\
\quad 0 \neq \prod_{\ell=1}^{p} x_{j_{\ell,1}}^{(i)} \cdots x_{j_{\ell,q_{\ell}}}^{(i)} - \prod_{\ell=1}^{p} x_{j'_{\ell,1}}^{(i)} \cdots x_{j'_{\ell,q_{\ell}}}^{(i)} \mathrm{belongs ~to} \mathcal{G}_i. \]

The initial monomial of each binomial is the first (underlined) monomial and, in particular, the initial monomial of each binomial above is squarefree (resp. quadratic) if the corresponding monomial \( \prod_{\ell=1}^{p} x_{j_{\ell,1}}^{(i)} \cdots x_{j_{\ell,q_{\ell}}}^{(i)} \) is squarefree (resp. quadratic).

**Proof.** Let \( \mathcal{G} \) denote the set of binomials above. Thanks to Lemmas \ref{2.2} \ref{2.3} and \ref{2.4}, \( \mathcal{G} \) is a (finite) subset of \( I_{A(B_1, \ldots, B_d)} \).

**Claim 1.** There exists a monomial order such that the initial monomial of each binomial in \( \mathcal{G} \) is the underlined monomial.

By virtue of [9, Theorem 3.12], it is enough to show that the reduction modulo \( \mathcal{G} \) is Noetherian. Suppose that there exists a sequence of reductions modulo \( \mathcal{G} \) which does not terminate. Let \( v \) be a monomial in \( K[x] \) and assume \( v \xrightarrow{g} v' \) with \( g \in \mathcal{G} \). Then we have

\[
\begin{align*}
\varphi_j(v) > j \varphi_j(v') & \quad \text{if } g \text{ is in (4) and arising from } \mathcal{G}_j, \\
\varphi_j(v) = \varphi_j(v') & \quad \text{otherwise.}
\end{align*}
\]

Hence the number of binomials in (4) appearing in the sequence is finite. Thus we may assume that the binomials in (4) do not appear in the sequence. However, as we proved in the proof of Theorem \ref{2.5} there exists no sequence of reductions modulo the set of binomials in (1), (2) and (3) which does not terminate. This is a contradiction.

**Claim 2.** The set \( \mathcal{G} \) is a Gröbner basis of \( I_{A(B_1, \ldots, B_d)} \).

Suppose that \( \mathcal{G} \) is not a Gröbner basis of \( I_{A(B_1, \ldots, B_d)} \). Thanks to Lemmas \ref{2.2} \ref{2.3} and \ref{2.4}, there exists a binomial \( f = p_1 - p_2 \in I_{A(B_1, \ldots, B_d)} \) such that neither \( p_1 \) nor \( p_2 \) is divisible by the initial monomial of any binomial in \( \mathcal{G} \). By virtue of Lemma \ref{2.4} we have \( \varphi_i(f) = \varphi_i(p_1) - \varphi_i(p_2) \in I_{B_i} \) for all \( 1 \leq i \leq d \). If \( \varphi_i(p_1) - \varphi_i(p_2) \neq 0 \) for some \( i \), then there exists a binomial \( g' \in \mathcal{G}_i \) such that the initial monomial of \( g' \) divides either \( \varphi_i(p_1) \) or \( \varphi_i(p_2) \). This contradicts that neither \( p_1 \) nor \( p_2 \) is divisible by the initial monomial of any binomial in (4). Hence we have \( \varphi_i(p_1) = \varphi_i(p_2) \) for all \( i \). Moreover, thanks to the argument in the proof of Theorem \ref{2.5} we have \( \varphi_0(p_1) = \varphi_0(p_2) \).

Thus \( f \) is of the form

\[
f = \prod_{\ell=1}^{p} x_{(i_{\ell}, j_{\ell,1}, \ldots, j_{\ell,q_{\ell}})}^{(k_{\ell})} - \prod_{\ell=1}^{p} x_{(i'_{\ell}, j'_{\ell,1}, \ldots, j'_{\ell,q_{\ell}})}^{(k_{\ell})},
\]

where sort((i_1, j_1) \cdots (i_{pr}, j_{pr})) = sort((i'_1, j'_1) \cdots (i'_{pr}, j'_{pr})). Since neither \( p_1 \) nor \( p_2 \) is divisible by the initial monomial of any binomial in either (2) or (3), it follows that \( p_1 = p_2 \) and hence \( f = 0 \). \( \square \)

If \( \mathcal{G}_i \) possesses a binomial of degree 3, then we need the following binomials:
Proposition 2.7. \(x^{(k_1)}_{M_1(i_1,j_1)} x^{(k_2)}_{M_2(i_2,j_2)} x^{(k_3)}_{M_3(i_3,j_3)} - x^{(k_1)}_{M_1(i_1',j_1')} x^{(k_2)}_{M_2(i_2',j_2')} x^{(k_3)}_{M_3(i_3',j_3')}\) where \(z^{(i)}_{j_1} z^{(i)}_{j_2} z^{(i)}_{j_3} \in \mathcal{G}_i\).

(b) \(x^{(k_2)}_{M_1(i_1,j_1)M_2(i_2,j_2)M_3(i_3,j_3)} - x^{(k_2)}_{M_1(i_1',j_1')M_2(i_2',j_2')M_3(i_3',j_3')}\) where \(z^{(i)}_{j_1} z^{(i)}_{j_2} z^{(i)}_{j_3} \in \mathcal{G}_i\).

We do not need (b) if \(A\) has no monomial divided by \(t_i^2\). In general, we have

\[
\deg \left( \prod_{\ell=1}^p z^{(i)}_{j_{\ell,1}} \cdots z^{(i)}_{j_{\ell,q_\ell}} \right) = \sum_{\ell=1}^p q_\ell \geq p = \deg \left( \prod_{\ell=1}^p x^{(k_\ell)}_{M_{\ell}(i_{\ell,j_{\ell,1}}) \cdots (i_{\ell,j_{\ell,q_\ell}})} \right).
\]

The binomials of type (a) are not always needed for a minimal Gröbner basis even if \(\mathcal{G}_i\) has a cubic binomial. In such a case, \(I_{A(B_1,\ldots,B_d)}\) may have a quadratic Gröbner basis. In Section 3, we will show an example.

2.3. Generators. Thanks to a part of the argument in the proof of Theorem 2.6 we have the following.

**Proposition 2.7.** Let \(\mathcal{H}_0\) be a set of binomial generators of \(I_A\) and let \(\mathcal{H}_i\) be a set of binomial generators of \(I_{B_i}\). Then, the toric ideal \(I_{A(B_1,\ldots,B_d)}\) is generated by the following binomials:

1. \(\prod_{\ell=1}^p x^{(k_\ell)}_{(t_{(\ell-1,r)+1},j_{(\ell-1,r)+1}) \cdots (t_{r},j_{r})} - \prod_{\ell=1}^p x^{(k'_\ell)}_{(t_{(\ell-1,r)+1}',j_{(\ell-1,r)+1}') \cdots (t_{r}',j_{r}')}\) where \(y_{k_1} \cdots y_{k_p} = y_{k'_1} \cdots y_{k'_p} \in \mathcal{H}_0\) and\n
\[
\text{sort}((i_{1},j_{1}) \cdots (i_{p},j_{p})) = \text{sort}((i'_{1},j'_{1}) \cdots (i'_{p},j'_{p})).
\]

2. \(x^{(k)}_{(i_1,j_1) \cdots (i_r,j_r)} x^{(k)}_{(i_{r+1},j_{r+1}) \cdots (i_{2r},j_{2r})} - x^{(k)}_{(i'_1,j'_1) \cdots (i'_{2r-1},j'_{2r-1}) (i'_{2r},j'_{2r})}\) where \(\text{sort}((i_{1},j_{1}) \cdots (i_{2r},j_{2r})) = (i'_{1},j'_{1}) \cdots (i'_{2r},j'_{2r})\) with respect to the ordering \((1,1) > (1,2) > \cdots > (1,\lambda_1) > (2,1) > \cdots > (d,\lambda_d)\).

3. \(x^{(k)}_{(i_1,j_1) \cdots (i_{r},j_{r})} x^{(k')}_{(i'_1,j'_1) \cdots (i'_{\ell},j'_{\ell})} - x^{(k)}_{(i_1,j_1) \cdots (i'_{\ell},j'_{\ell})} x^{(k')}_{(i_{\ell+1},j_{\ell+1}) \cdots (i_{r},j_{r})}\) where \(k < k', i_\ell = i'_\ell\) and \(j_\ell > j'_\ell\).

4. \(\prod_{\ell=1}^p x^{(k_\ell)}_{M_{\ell}(i_{\ell,j_{\ell,1}}) \cdots (i_{\ell,j_{\ell,q_\ell}})} - \prod_{\ell=1}^p x^{(k_\ell)}_{M_{\ell}(i'_{\ell,j'_{\ell,1}}) \cdots (i'_{\ell,j'_{\ell,q_\ell}})}\) where the binomial \(0 \neq \prod_{\ell=1}^p z^{(i)}_{j_{\ell,1}} \cdots z^{(i)}_{j_{\ell,q_\ell}} \neq \prod_{\ell=1}^p z^{(i)}_{j'_{\ell,1}} \cdots z^{(i)}_{j'_{\ell,q_\ell}} \) belongs to \(\mathcal{H}_i\).

3. Toric ideals of multiples of the Birkhoff polytope

Let \(c = (c_1, c_2, c_3) \in \mathbb{Z}_{\geq 0}^3\) and \(r = (r_1, r_2, r_3) \in \mathbb{Z}_{\geq 0}^3\) be vectors with \(c_1 + c_2 + c_3 = r_1 + r_2 + r_3\). Then \(3 \times 3\) transportation polytope \(\tilde{T}_{rc}\) is the set of all non-negative \(3 \times 3\) matrices \(A = (a_{ij})\) satisfying\n
\[
\sum_{i=1}^{3} a_{ik} = c_k \quad \text{and} \quad \sum_{j=1}^{3} a_{\ell j} = r_\ell.
\]
for $1 \leq k, \ell \leq 3$. It is known that this is a bounded convex polytope of dimension 4 whose vertices are lattice points in $\mathbb{R}^{3 \times 3}$. The toric ideal of $T_{rc}$ is the toric ideal of the configuration $\{t^{\alpha} \mid \alpha \in T_{rc} \cap \mathbb{Z}^{3 \times 3}\}$.

**Example 3.1.** Let $c = r = (1, 1, 1)$. Then the transportation polytope $B_3 := T_{rc}$ is called the *Birkhoff polytope*. The lattice points in $B_3$ are

$$
\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
\sigma_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

The toric ideal of $B_3$ is the toric ideal of the configuration

$$
\{u_{11}u_{22}u_{33}, u_{12}u_{23}u_{31}, u_{13}u_{21}u_{32}, u_{11}u_{23}u_{32}, u_{12}u_{21}u_{33}, u_{13}u_{22}u_{31}\}
$$

and it is a principal ideal generated by $z_1z_2z_3 - z_4z_5z_6$.

The following is proved by Haase–Paffenholz [2]:

- The toric ideal of $3 \times 3$ transportation polytope is generated by quadratic binomials except for $B_3$.
- The toric ideal of $3 \times 3$ transportation polytope possesses a quadratic square-free initial ideal if it is not a multiple of $B_3$.

Thus, it is natural to ask whether the toric ideal of a multiple of $B_3$ possesses a quadratic Gröbner basis except for $B_3$. The following fact is due to Birkhoff:

- Every non-negative integer $p \times p$ matrix with equal row and column sums can be written as a sum of permutation matrices.

Hence, in particular, we have

$$nB_3 \cap \mathbb{Z}^{3 \times 3} = \{\sigma_{i_1} + \cdots + \sigma_{i_n} \mid 1 \leq i_1, \ldots, i_n \leq 6\}.$$

Thus, in order to study the toric ideal of $n$ multiple of $B_3$, we consider the following:

**Example 3.2.** Let $A = \{t^n\}$ and suppose that $B_1$ satisfies $\zeta(B_1) = 6$ and $I_{B_1} = \langle z_1z_2z_3 - z_4z_5z_6 \rangle$. If $n = 1$, then $A(B_1) = B_1$ and $\{x_1x_2x_3 - x_4x_5x_6\}$ is the reduced Gröbner basis of $I_{A(B_1)}$ with respect to any monomial order. If $n > 1$, then, by virtue of Theorem [2.6] $I_{A(B_1)}$ has a Gröbner basis consisting of the following binomials:

(a) $x_{1M_1}x_{2M_2}x_{3M_3} - x_{4M_1}x_{5M_2}x_{6M_3}$,
(b) $x_{j_1j_2j_3}x_{j_4j_5j_6} - x_{j_4j_5j_6}x_{j_1j_2j_3}$, where $\{j_1, j_2, j_3\} = \{1, 2, 3\}$ and $\{j_4, j_5, j_6\} = \{4, 5, 6\}$,
(c) $x_{j_1\cdots j_n}x_{j_{n+1}\cdots j_{2n}} - x_{j_1j_3\cdots j_{2n-1}}x_{j_2j_4\cdots j_{2n}}$, where $\text{sort}(j_1 \cdots j_{2n}) = j_1' \cdots j_{2n}'$.

Since the Gröbner basis in Example 3.2 is not quadratic, we have to consider another monomial order to find a quadratic Gröbner basis.

**Remark 3.3.** In [2], they say that L. Piechnik and C. Haase proved that the toric ideal of the multiple $2nB_3$ possesses a squarefree quadratic initial ideal for $n > 1$. This fact is directly obtained by Theorem [2.6] since the toric ideal of the multiple $2B_3$.
possesses a squarefree quadratic initial ideal. Similarly, since the toric ideal of the multiple \(3\mathcal{B}_3\) possesses a squarefree quadratic initial ideal, 

Theorem 2.4 guarantees that the toric ideal of the multiple \(3n\mathcal{B}_3\) possesses a squarefree quadratic initial ideal for \(n > 1\). However, since there are infinitely many prime numbers, it is difficult to show the existence of a squarefree quadratic initial ideal of the toric ideal of \(m\mathcal{B}_3\) for all \(m > 1\) in this way.

**Theorem 3.4.** Let \(A = \{t^n\}\) with \(n > 1\) and suppose that \(B_1\) satisfies \(|B_1| = 6\) and \(I_{B_1} = (z_1z_2z_3 - z_4z_5z_6)\). Then, \(I_{A(B_1)}\) has a quadratic Gröbner basis consisting of the following binomials:

(i) \(x_{j_1j_2M_1}x_{j_3M_2} - x_{j_4j_5M_1}x_{j_6M_2}\) where \(\{j_1, j_2, j_3\} = \{1, 2, 3\}\) and \(\{j_4, j_5, j_6\} = \{4, 5, 6\}\),

(ii) \(x_{j_1\ldots j_n}x_{j_{n+1}\ldots j_{2n}} - x_{1\ldots j'_1\ldots j'_n}x_{1\ldots j''_1\ldots j''_n}\) where \(\text{sort}(j_1 \cdots j_{2n}) = 1 \cdots j'_1 \cdots j'_{2n}\)

and \(j''_2 > 1\).

**Proof.** Let \(\mathcal{G}\) denote the set of binomials above. Since \(A = \{t^n\}\), each binomial in (ii) and (iii) belongs to \(I_{A(B_1)}\). In addition, thanks to Lemma 2.3, each binomial in (i) belongs to \(I_{A(B_1)}\). Hence \(\mathcal{G}\) is a (finite) subset of \(I_{A(B_1)}\).

**Claim 1.** There exists a monomial order such that the initial monomial of each binomial in \(\mathcal{G}\) is the underlined monomial.

By virtue of [9] Theorem 3.12, it is enough to show that the reduction modulo \(\mathcal{G}\) is Noetherian. Suppose that there exists a sequence of reductions modulo \(\mathcal{G}\) which does not terminate. Let \(v\) be a monomial in \(K[\mathbf{x}]\) and assume \(v \xrightarrow{g} v'\) with \(g \in \mathcal{G}\). Then we have

\[
\begin{align*}
\varphi_1(v) &> \varphi_1(v') & \text{if } g \in (i), \\
\varphi_1(v) &\leq \varphi_1(v') & \text{if } g \in (ii).
\end{align*}
\]

Hence the number of binomials in (i) appearing in the sequence is finite. Thus we may assume that the binomials in (i) do not appear in the sequence. Let \(v = \prod_{i=1}^{d} x_{i_{(i-1)r+1}\ldots i_{tr}}\), \(v' = \prod_{i=1}^{d} x_{i'_{(i-1)r+1}\ldots i'_{tr}}\) and let \(m_{\ell}\) (resp. \(m'_{\ell}\)) denote the number of 1’s appearing in \(i_{(i-1)r+1}\ldots i_{tr}\) (resp. \(i'_{(i-1)r+1}\ldots i'_{tr}\)). Then, we have

\[
\sum_{1 \leq \ell_1 < \ell_2 \leq q} |m_{\ell_1} - m_{\ell_2}| \geq \sum_{1 \leq \ell_1 < \ell_2 \leq q} |m'_{\ell_1} - m'_{\ell_2}|
\]

if \(g \in \mathcal{G}\) belongs to (ii). (The equality holds if and only if \(g = x_{j_1\ldots j_n}x_{j_{n+1}\ldots j_{2n}} - x_{1\ldots j'_1\ldots j'_n}x_{1\ldots j''_1\ldots j''_n}\) satisfies that the difference between the number of 1’s in \(j_1 \cdots j_n\) and that in \(j_{n+1} \cdots j_{2n}\) is at most one.) Hence, we may assume that 1’s in the indices is stable. Then, since the inversion number is strictly decreasing in the sequence of reductions modulo binomials in (ii), the sequence is finite.

**Claim 2.** The set \(\mathcal{G}\) is a Gröbner basis of \(I_{A(B_1)}\).

Suppose that \(\mathcal{G}\) is not a Gröbner basis of \(I_{A(B_1)}\). Then there exists a binomial \(0 \neq g = p_1 - p_2 \in I_{A(B_1)}\) such that neither \(p_1\) nor \(p_2\) is divisible by the initial monomial of any binomial in \(\mathcal{G}\). Let \(p_1 = \prod_{\ell=1}^{pr} x_{i_{(i-1)r+1}\ldots i_{tr}}\), \(p_2 = \prod_{\ell=1}^{pr} x_{i'_{(i-1)r+1}\ldots i'_{tr}}\).

By Lemma 2.4, we have \(\varphi_1(p_1) - \varphi_1(p_2) = \prod_{\ell=1}^{pr} z_{i_{\ell}} - \prod_{\ell=1}^{pr} z_{i'_{\ell}} \in (z_1z_2z_3 - z_4z_5z_6)\).
Suppose that $\prod_{\xi=1}^{pr} z_{i_\xi} - \prod_{\xi=1}^{pr} z_{i_{\xi}'} \neq 0$. We may assume that $\prod_{\xi=1}^{pr} z_{i_\xi}$ is divided by $z_1z_2z_3$. Since $p_1$ is not divided by the initial monomial of any binomial in (i), $p_1$ is divided by a cubic monomial $x_{1M_1}x_{2M_2}x_{3M_3}$ where $2, 3 \not\in M_1$, $1, 3 \not\in M_2$ and $1, 2 \not\in M_3$. Note that $M_i \neq \emptyset$ by $n > 1$. Since $p_1$ is not divided by the initial monomial of any binomial in (ii), the number of 1’s in $iM_i$ is differ by at most one. Since 1 appears in neither $2M_2$ nor $3M_3$, we have $1 \not\in M_1$. Thus $M_1 \subset \{4, 5, 6\}$. Then $p_1$ is divided by the initial monomial of the binomial $g = x_{1M_1}x_{2M_2} - x_{12M'_1}x_{M'_2}$, where sort$(1M_12M_2) = 12M'_1M'_2$ and $g$ belongs to (ii).

Suppose that $\prod_{\xi=1}^{pr} z_{i_\xi} - \prod_{\xi=1}^{pr} z_{i_{\xi}'} = 0$. Since neither $p_1$ nor $p_2$ is divisible by the initial monomial of any binomial in (ii), there exists $0 \leq p' \leq p$ and $0 \leq \beta \leq r$ such that

$$p_1 = p_2 = \prod_{\ell=1}^{p'} x_{\zeta_{(\ell-1)r+1} \cdots \zeta_{r}} \prod_{\ell=p'+1}^{p} x_{\theta_{(\ell-1)r+1} \cdots \theta_{r}}$$

where $\zeta_{(\ell-1)r+1} = 1$ for all $1 \leq \eta \leq \beta$, $\theta_{(\ell-1)r+1} = 1$ for all $1 \leq \eta \leq \beta - 1$ and $\zeta_{\beta+1} \leq \cdots \leq \zeta_{r} \leq \zeta_{r+\beta+1} \leq \cdots \leq \zeta_{2r} \leq \cdots \leq \zeta_{(p'-1)r+\beta+1} \leq \cdots \leq \zeta_{p'r} \leq \theta_{p'r+\beta} \leq \cdots \leq \theta_{(p'-1)r+\beta} \leq \cdots \leq \theta_{(p'-2)r} \leq \cdots \leq \theta_{(p-1)r+\beta} \leq \cdots \leq \theta_{pr}$. Hence $g = p_1 - p_2 = 0$ and this is a contradiction.

Thus, there exists no binomial $0 \neq g = p_1 - p_2 \in I_{A(B_1)}$ such that neither $p_1$ nor $p_2$ is divisible by the initial monomial of any binomial in $G$ and hence $G$ is a Gröbner basis of $I_{A(B_1)}$ as desired. □

4. Observation

Finally, we conclude this paper with a summary of our algebraic theory of nested configurations. For a configuration $A$, let $G_\prec$ denote the reduced Gröbner basis of $I_A$ with respect to a monomial order $\prec$. Let

$$\lambda(A) := \min \left( \max \left( \deg(g) \mid g \in G_\prec \right) \right).$$

(If $I_A = (0)$, then we set $\lambda(A) = 0$.) Thanks to the results in Section 2, if $\lambda(A(B_1, \ldots, B_\delta)) \neq 0$, then

$$\max(2, \lambda(A)) \leq \lambda(A(B_1, \ldots, B_\delta)) \leq \max(2, \lambda(A), \lambda(B_1), \ldots, \lambda(B_\delta)).$$

Moreover, if $\lambda(A(B_1, \ldots, B_\delta)) \neq 0$ and $A$ consists of squarefree monomials then

$$\lambda(A(B_1, \ldots, B_\delta)) = \max(2, \lambda(A), \lambda(B_1), \ldots, \lambda(B_\delta)).$$

Let $n \geq 2$ be an integer and let $X$ be the one of the following algebraic properties:

1. The toric ring is normal;
2. The toric ideal has a squarefree initial ideal;
3. The toric ideal has a quadratic initial ideal;
4. The toric ideal has a squarefree quadratic initial ideal;
5. The toric ideal has an initial ideal of degree $\leq n$;
6. The toric ideal is generated by quadratic binomials;
7. The toric ideal is generated by binomials of degree $\leq n$. 

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Then we have

\[ A, B_1, \ldots, B_d \text{ have the property } X \implies A(B_1, \ldots, B_d) \text{ has the property } X. \]
\[ A(B_1, \ldots, B_d) \text{ has the property } X \implies A \text{ has the property } X. \]

Moreover, if \( A \) consists of squarefree monomials, then we have

\[ A, B_1, \ldots, B_d \text{ have the property } X \iff A(B_1, \ldots, B_d) \text{ has the property } X. \]

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