Magic squares with empty cells

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Abstract

A \( k \)-magic square of order \( n \) is an arrangement of the numbers from 0 to \( kn - 1 \) in an \( n \times n \) matrix, such that each row and each column has exactly \( k \) filled cells, each number occurs exactly once, and the sum of the entries of any row or any column is the same. A magic square is called \( k \)-diagonal if its entries all belong to \( k \) consecutive diagonals. In this paper we prove that a \( k \)-diagonal magic square exists if and only if \( n = k = 1 \) or \( 3 \leq k \leq n \) and \( n \) is odd or \( k \) is even.

1 Introduction

A magic square of order \( n \) is an arrangement of the numbers from 0 to \( n^2 - 1 \) in an \( n \times n \) array such that each number occurs exactly once in the array and the sum of the entries of any row or any column is the same, which is called the magic sum. It is well-known that (see [5]) a magic square exists for each \( n \geq 1 \). A \( k \)-magic square of order \( n \) is an arrangement of the numbers from 0 to \( kn - 1 \) in an \( n \times n \) array such that each row and each column has exactly \( k \) filled cells, each number occurs exactly once in the array, and the sum of the entries of any row or any column is the same.

An integer Heffter array \( H(m, n; s, t) \) is an \( m \times n \) array with entries from \( X = \{ \pm 1, \pm 2, \ldots, \pm ms \} \) such that each row contains \( s \) filled cells and each column contains \( t \) filled cells, the elements in every row and column sum to 0 in \( \mathbb{Z} \), and for every \( x \in X \), either \( x \) or \(-x\) appears in the array. The notion of an integer Heffter array \( H(m, n; s, t) \) was first defined in [1]. Integer Heffter arrays with \( m = n \) represent a type of magic square where each number from the set \( \{1, 2, \ldots, n^2\} \) is used once up to sign. A square integer Heffter array \( H(n; k) \) is an integer Heffter array with \( m = n \) and
there are exactly $k$ filled cells in each row and each column. In [2, 3] it is proved that:

**Theorem 1.** There is an integer $H(n; k)$ if and only if $3 \leq k \leq n$ and $nk \equiv 0, 3 \pmod{4}$.

A signed magic array $SMA(m, n; s, t)$ is an $m \times n$ array with entries from $X$, where $X = \{0, \pm 1, \pm 2, \ldots, \pm (ms - 1)/2\}$ if $ms$ is odd and $X = \{\pm 1, \pm 2, \ldots, \pm ms/2\}$ if $ms$ is even, such that precisely $s$ cells in every row and $t$ cells in every column are filled, every integer from set $X$ appears exactly once in the array, and the sum of each row and of each column is zero. The signed magic squares also represent a type of magic square where each number from the set $X$ is used once. The notion of a signed magic array $SMA(m, n; s, t)$ was first defined in [4]. In the case where $m = n$, the array is called signed magic square. The notation $SMS(n; k)$ is used for a signed magic square with $k$ filled cells in each row and in each column. In [4] it is proved that:

**Theorem 2.** There is an $SMS(n; k)$ precisely when $n = k = 1$ or $3 \leq k \leq n$.

A magic square is called $k$-diagonal if its entries all belong to $k$ consecutive diagonals (this includes broken diagonals as well). In this paper, similar to Theorems 1 and 2, we prove that a $k$-diagonal magic square exists if and only if $n = k = 1$ or $3 \leq k \leq n$ and $n$ is odd or $k$ is even.

Often the definition of a magic square requires the additional condition that the sum of the diagonals are also constant and equal to the sum of the rows and columns. Due to the structure of a $k$-diagonal magic square as given in this paper, this requirement is not well defined, as the number of entries is either $n$ or $k$. Thus we do not require this condition.

## 2 Direct constructions

In this section we present direct constructions for $k$-diagonal magic squares of order $n$ for $k = 3, 4, 5$ and 6. It is easy to see that if there exists a magic square of order $n$ with precisely $k$ filled cells in each row and each column, then the magic sum must be $k(kn - 1)/2$. Hence, if $k$ is odd, then $n$ must be odd. We use the notation $(i, j; e)$ for an element of an array $M$ to indicate that the entry $e$ is in row $i$ and column $j$ of $M$.

**Theorem 3.** There exists a 3-diagonal magic square of order $n \geq 3$ and $n$ is odd.

**Proof.** We fill three consecutive diagonals of an empty array $M$ of order $n$ as follows:
Diagonal 1: \[
\begin{cases}
(2i, n + 2i - 3; (n - 2i - 1)/2) & \text{for } 0 \leq i \leq (n - 1)/2, \\
(2i + 1, n + 2i - 2; n - i - 1) & \text{for } 0 \leq i \leq (n - 3)/2.
\end{cases}
\]

Diagonal 2: \[
\begin{cases}
(i, n + i - 2; 2n + i) & \text{for } 0 \leq i \leq n - 1.
\end{cases}
\]

Diagonal 3: \[
\begin{cases}
(2i, n + 2i - 1; 2n - i - 1) & \text{for } 0 \leq i \leq (n - 1)/2, \\
(2i + 1, 2i; (3n - 2i - 3)/2) & \text{for } 0 \leq i \leq (n - 3)/2.
\end{cases}
\]

Addition in the first and second components of each triple \((i, j; e)\) is done modulo \(n\). (See Figure 1 for a 3-diagonal magic square of order 9.) We now prove that \(M\) is an \((n; 3)\) magic square. First note that the numbers in Diagonal 1 are 0, 1, 2, \ldots, \(n - 1\), in Diagonal 2 are \(2n, 2n + 1, \ldots, 3n - 1\), and in Diagonal 3 are \(n, n + 1, \ldots, 2n - 1\).

Second, we calculate the row sums and column sums. In row \(2i\), where \(0 \leq i \leq (n - 1)/2\), we have \((n - 2i - 1)/2 + (2n + 2i) + (2n - i - 1) = 3(3n - 1)/2\) and in row \(2i + 1\), where \(0 \leq i \leq (n - 3)/2\), we have \((n - i - 1) + (2n + 2i + 1) + (3n - 2i - 3)/2 = 3(3n - 1)/2\).

The column sum for column \(2j\), \(0 \leq j \leq (n - 3)/2\), is \((3n - 2j - 3)/2 + (2n + 2j + 2) + (n - j - 2) = 3(3n - 1)/2\). The column sum for column \(2j + 1\), \(0 \leq j \leq (n - 5)/2\), is \((2n - j - 2) + (2n + 2j + 3) + (n - 2j - 5)/2 = 3(3n - 1)/2\).

Finally, the column sum for column \(n - 2\) is \((3n - 1)/2) + (n - 1) + (2n) = 3(3n - 1)/2\) and for column \(n - 1\) is \((n - 3)/2 + (2n + 1) + (2n - 1) = 3(3n - 1)/2\). This completes the proof.

![Figure 1: A 3-diagonal magic square of order 9](image)

**Theorem 4.** There exists a 4-diagonal magic square of order \(n \geq 4\).

**Proof.** We fill four consecutive diagonals of an empty array \(M\) of order \(n\) as follows:

Diagonal 1: \((i, i + 1; i)\) for \(0 \leq i \leq n - 1\).

Diagonal 2: \((i, i + 2; 4n - i - 1)\) for \(0 \leq i \leq n - 1\).

Diagonal 3: \((n + i - 2, i + 1; 3n - i - 1)\) for \(0 \leq i \leq n - 1\).

Diagonal 4: \((n + i - 2, i + 2; n + i)\) for \(0 \leq i \leq n - 1\).
Addition in first and second components of each triple \((i, j; e)\) is done modulo \(n\). (See Figure 2 for a 4-diagonal magic square of order 9.) We now prove the resulting square is magic. First note that the numbers in Diagonal 1 are 0, 1, 2, \ldots, \(n-1\), in Diagonal 2 are 3\(n\), 3\(n\) + 1, \ldots, 4\(n\) - 1, in Diagonal 3 are 2\(n\), 2\(n\) + 1, \ldots, 3\(n\) - 1 and in diagonal 4 are \(n\), \(n\) + 1, \ldots, 2\(n\) - 1.

Second, we calculate the row sums and column sums. In row 0 \(\leq i \leq n-3\) we have \(i + (4n - i - 1) + (3n - i - 3) + (n + i + 2) = 8n - 2\). In column 2 \(\leq j \leq n-1\) we have \((j - 1) + (4n - j + 1) + (3n - j) + (n + j - 2) = 8n - 2\). Row sum for row \(n-1\) is \((n - 1) + (3n) + (3n - 2) + (n + 1) = 8n - 2\) and for row \(n-2\) is \((n - 2) + (3n + 1) + (3n - 1) + (n) = 8n - 2\). Column sum for column zero is \((n - 1) + (3n + 1) + (2n) + (2n - 2) = 8n - 2\) and for column 1 is \(0 + (3n) + (3n - 1) + (2n - 1) = 8n - 2\). This completes the proof.

![Figure 2: A 4-diagonal magic square of order 9](image)

**Theorem 5.** There exists a 5-diagonal magic square of order \(n \geq 5\), where \(n\) is odd.

**Proof.** We fill five consecutive diagonals of an empty array \(M\) of order \(n\) as follows:

Diagonal 1: \((i, i + 1; i)\) for \(0 \leq i \leq n - 1\).

Diagonal 2: \[
\begin{align*}
(n + 2i - 1, 2i + 1; 2n - i - 1) & \quad \text{for} \quad 0 \leq i \leq (n - 1)/2 \\
(2i, 2i + 2; (3n - 3)/2 - i) & \quad \text{for} \quad 0 \leq i \leq (n - 3)/2.
\end{align*}
\]

Diagonal 3: \((i, i + 3; 3n - i - 1)\) for \(0 \leq i \leq n - 1\).

Diagonal 4: \[
\begin{align*}
(n + 2i - 2, 2i + 2; 4n - i - 1) & \quad \text{for} \quad 0 \leq i \leq (n - 1)/2 \\
(n + 2i - 1, 2i + 3; (7n - 3)/2 - i) & \quad \text{for} \quad 0 \leq i \leq (n - 3)/2.
\end{align*}
\]

Diagonal 5: \((n + i - 2, i + 3; 4n + i)\) for \(0 \leq i \leq n - 1\).

Addition in first and second components of each triple \((i, j; e)\) is modulo \(n\). (See Figure 3 for a 5-diagonal magic square of order 9.)

\[\]
Note that the entries in diagonal $d$, $1 \leq d \leq 5$, are \{n(d-1), n(d-1) + 1, \ldots, nd - 1\}.

The row sum for row $2i$, where $0 \leq i \leq (n-3)/2$, is

$$2i + (3n - 3)/2 - i + 3n - 2i - 1 + 4n - i - 2 + 4n + 2i + 2 = 5(5n - 1)/2.$$  

The row sum for row $2i + 1$, where $0 \leq i \leq (n-5)/2$, is

$$2i + 1 + 2n - i - 2 + 3n - 2i - 2 + (7n - 3)/2 - i - 1 + 4n + 2i + 3 = 5(5n - 1)/2.$$  

Finally, the row sum for row $n-1$ is $n-1 + 2n - 1 + 2n + (7n - 3)/2 + 4n + 1 = 5(5n - 1)/2$ and for row $n-2$ is $n-2 + (3n - 1)/2 + 2n + 1 + 4n - 1 + 4n = 5(5n - 1)/2$.

We now calculate the column sums. The column sum for column 2, where $1 \leq j \leq (n-1)/2$, is

$$2j - 1 + (3n-3)/2 - j + 1 + 3n - 2j + 2 + 4n - j + 4n + 2j - 3 = (5n - 1)/2.$$  

The column sum for column 2, where $1 \leq j \leq (n-3)/2$, is

$$2j + 2n - j - 1 + 3n - 2j + 1 + (7n - 3)/2 - j + 1 + 4n + 2j - 2 = 5(5n - 1)/2.$$  

The sum for column zero is $n-1 + (3n-1)/2 + 2n + 2 + 3n + 5n - 3 = 5(5n - 1)/2$, for column 1 is $0 + 2n - 1 + 2n + 1 + (7n - 1)/2 + 5n - 2 = 5(5n - 1)/2$ and for column 2 is $1 + (3n - 3)/2 + 2n + 4n - 1 + 5n - 1 = 5(5n - 1)/2$.

This completes the proof. 

\[ \square \]

**Theorem 6.** There exists a 6-diagonal magic square of order $n \geq 6$.

**Proof.** Let $n \geq 6$. We fill six consecutive diagonals of an empty array $M$ of order $n$ as follows:

- Diagonal 1: $(i, i + 1; i)$ for $0 \leq i \leq n - 1$.
- Diagonal 2: $(n - 1 + i, i + 1; 2n - 1 - i)$ for $0 \leq i \leq n - 1$.
- Diagonal 3: $(i, i + 3; 2n + i)$ for $0 \leq i \leq n - 1$.
- Diagonal 4: $(n - 1 + i, i + 3; 4n - 1 - i)$ for $0 \leq i \leq n - 1$.
- Diagonal 5: $(i, i + 5; 6n - 2 - 2i)$ for $0 \leq i \leq n - 1$.
- Diagonal 6: $(n - 1 + i, i + 5; 4n + 1 + 2i)$ for $0 \leq i \leq n - 1$.

Addition in first and second components of each triple $(i, j; e)$ is modulo $n$.

(See Figure 4 shows a 6-diagonal magic square of order 10.)

Note that the entries in diagonal $d$, $1 \leq d \leq 4$, are $\{n(d-1), n(d-1) + 1, \ldots, nd - 1\}$. The entries in diagonal 5 are $\{4n, 4n + 2, 4n + 4, \ldots, 6n - 2\}$ and the entries in diagonal 6 are $\{4n + 1, 4n + 3, 4n + 5, \ldots, 6n - 1\}$. 

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The row sum for row $n-1$ is $n-1 + 2n-1 + 3n-1 + 4n-1 + 4n+1 = 18n - 3$ and for row $i$, where $0 \leq i \leq n-2$, is

$$(i) + (2n-i-2)+(2n+i)+(4n-i-2)+(6n-2i-2)+(4n+2i+3) = 18n-3.$$  

The column sum for column $j$, where $5 \leq j \leq n-1$, is

$$(6n-2j+8)+(4n-j+2)+(2n+j-3)+(2n-j)+(j-1)+(4n+2j-9) = 18n-3.$$  

It is straightforward to see that the column sum for column $0 \leq j \leq 4$ is also $18n - 3$. This completes the proof.

\[ \square \]
3 Main Theorem

The following two lemmas are crucial for the construction of a \( k \)-diagonal magic square of order \( n \).

**Lemma 7.** Let \( M \) be a \( k \)-diagonal magic square of order \( n \).

(i) Let \( 2k \leq n \). If we shift every \((i, j; e) \in M\) to \((i + k, j; e)\) (or to \((i, j+k; e)\)), addition is modulo \( n \), the resulting square is a \( k \)-diagonal magic square of order \( n \).

(ii) If we add \( m \) to each nonempty cell of \( M \), the resulting square has row sum and column sum \( km + k(nk - 1)/2 \) and its entries are \( m, m+1, \ldots, mk-1+m \).

**Proof.** For Part (i) note that the entries in each row or each column will be the same after the shift. So the resulting array is a \( k \)-diagonal magic square of order \( n \). The proof of Part (ii) is trivial. \(\square\)

**Lemma 8.** If there exist \( \ell \)-diagonal and \( m \)-diagonal magic squares of order \( n \) and \( \ell + m \leq n \), then there exists an \(( \ell + m \))-diagonal magic square of order \( n \).

**Proof.** Let \( A \) and \( B \) be \( \ell \)-diagonal and \( m \)-diagonal magic squares of order \( n \), respectively. Since \( \ell + m \leq n \), without loss of generality, by Part (i) of Lemma 7 we can assume if a cell \((i, j)\) of \( A \) is filled, the cell \((i, j)\) of \( B \) is empty. In addition, if we superimpose \( A \) and \( B \), we will have \( \ell + m \) consecutive diagonals. Now we form a magic square \( C \) of order \( n \) with \( \ell + m \) entries in each row and in each column as follows: If \((i, j; e) \in A\), then \((i, j; e) \in C\) and if \((i, j; e) \in B\), then \((i, j; e + \ell n) \in C\). Then \( C \) is an \(( \ell + m \))-diagonal magic square of order \( n \) by Part (ii) of Lemma 7. \(\square\)

Figure 5 shows a 7-diagonal magic square of order 9 obtained by the construction given in Lemma 8 and the 3-diagonal and 4-diagonal magic squares given in Figures 1 and 2, respectively.

**Lemma 9.** Let \( n \) be odd and \( 3 \leq k \leq n \). Then there exists a \( k \)-diagonal magic square of order \( n \).

**Proof.** It is easy to see that there are nonnegative integers \( a, b, c \) such \( k = 3a + 4b + 5c \). By Theorems 3, 4 and 5, there are 3-diagonal, 4-diagonal and 5-diagonal magic squares of order \( n \). Apply Lemma 8 to obtain a \( k \)-diagonal magic square of order \( n \). \(\square\)

**Lemma 10.** Let \( n, k \) be even and \( 4 \leq k \leq n \). Then there exists a \( k \)-diagonal magic square of order \( n \).
Figure 5: A 7-diagonal magic square of order 9

\[
\begin{array}{cccc}
27 & 62 & 51 & 38 \\
12 & 28 & 61 & 50 & 39 \\
20 & 16 & 29 & 60 & 49 & 40 \\
7 & 21 & 11 & 30 & 59 & 48 & 41 \\
2 & 22 & 15 & 31 & 58 & 47 & 42 \\
6 & 23 & 10 & 32 & 57 & 46 & 43 \\
44 & 1 & 24 & 14 & 33 & 56 & 45 \\
53 & 36 & 5 & 25 & 9 & 34 & 55 \\
54 & 52 & 37 & 0 & 26 & 13 & 35
\end{array}
\]

Proof. It is easy to see that there are nonnegative integers $a$ and $b$ such that $k = 4a + 6b$. By Theorems 4 and 6 there exist 4-diagonal and 6-diagonal magic squares of order $n$. Apply Lemma 8 to obtain a $k$-diagonal magic square of order $n$.

We are now ready to state the main theorem of this paper. Recall that the magic sum of an $(n; k)$ magic square is $k(nk - 1)/2$. Hence, if $n$ is even and $k$ is odd, there is no $(n; k)$ magic square.

Main Theorem. There exists a $k$-diagonal magic square of order $n$ if and only if $n = k = 1$ or $3 \leq k \leq n$ and $n$ is odd or $k$ is even.

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