Martingale solutions of the stochastic
Hall-magnetohydrodynamics equations on $\mathbb{R}^3$

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Abstract

We prove the existence of a global martingale solution of a stochastic Hall-magneto-
hydrodynamics equations on $\mathbb{R}^3$ with multiplicative noise. Using the Fourier analysis we
construct a sequence of approximate solutions. The existence of a solution is proved via
the stochastic compactness method and the Jakubowski generalization of the Skorokhod
theorem for nonmetric spaces, in particular, the spaces with weak topologies. The main
difficulty is caused by the Hall term which makes the equations strongly nonlinear.

1 Introduction

Magnetohydrodynamics describes the motion of electrically conductive fluid in the presence
of a magnetic field with wide range of applications in geophysics and astrophysics. Mathemati-
cally rigorous analysis of the MHD equations started from Duvaut and Lions [16] and
Sermange and Temam [33], where deterministic MHD equations are considered. These equa-
tions are basically obtained by coupling the Navier-Stokes equations with the Maxwell equa-
tions. Stochastic MHD equations with the Gaussian noise were considered, e.g. in [5], [13],
[20], [21], [31], [32], [34], [41].

The Hall-MHD model is important in the physics of plasma. Mathematical derivation of a
model taking into account the Hall effect was introduced in [1]. Moreover, the authors in [1]
prove the existence of a global weak solutions for the incompressible viscous resistive Hall-
MHD equations in $[0,1]^3$. The proof in [1] is based on the Galerkin approximation and the
compactness method. The uniqueness of global solution in general case is an open problem.
Deterministic Hall-MHD equations were also considered in, e.g., [8], [9], [10], [11].

The analysis of the Hall-magnetohydrodynamics equations was developed by Yamazaki [39],
where the stochastic Hall-MHD equations perturbed by a Gaussian random field on the
domain $D = [0,1]^3$ are considered. Using the Galerkin approximation and the tightness
criteria introduced by Flandoli and Gątarek in [18], the author proves the existence of a
global martingale solution. The method depends strongly on the compactness of appropriate
Sobolev embeddings in the case of the domain $D = [0,1]^3$. See also [40] and [41].

Inspired by [39], we consider the stochastic Hall-MHD equations with a multiplicative Gaussian
noise on $\mathbb{R}^3$ and prove the existence of a global martingale solution. The main difficulty
in comparison to [39] is the fact that in the case of an unbounded domain the standard

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We consider the following Hall-MHD system on \([0,T]\times\mathbb{R}^3\)

\[
\begin{align*}
\quad d\mathbf{u} + \left[ (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - s (\mathbf{B} \cdot \nabla)\mathbf{B} + \nu_1 \Delta \mathbf{u} \right] dt &= f_1(t) \\
&\quad + \mathbf{G}_1(t,\mathbf{u})\,dW_1(t), \\
\quad d\mathbf{B} + \left[ (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} + \varepsilon \text{curl} [(\text{curl}\mathbf{B}) \times \mathbf{B}] - \nu_2 \Delta \mathbf{B} \right] dt &= f_2(t) \\
&\quad + \mathbf{G}_2(t,\mathbf{B})\,dW_2(t), \\
\text{div}\,\mathbf{u} &= 0 \quad \text{and} \quad \text{div}\,\mathbf{B} = 0.
\end{align*}
\]  

(1.1)-(1.3)

The equations are supplemented by the following initial conditions

\[
\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \mathbf{B}(0) = \mathbf{B}_0.
\]  

(1.4)

In this problem \(\mathbf{u}(t,x) = (u_1, u_2, u_3)(t,x), \mathbf{B}(t,x) = (B_1, B_2, B_3)(t,x)\) for \((t,x) \in [0,T] \times \mathbb{R}^3\), are three-dimensional vector fields representing velocity and magnetic fields, respectively, and the real valued function \(p(t,x)\) denotes the pressure of the fluid. The positive constants \(\nu_1, \nu_2, s\) represent kinematic viscosity, resistivity and the Hartmann number, respectively. The curl-operator is defined for a vector field \(\phi : \mathbb{R}^3 \to \mathbb{R}^3\) by

\[
\text{curl}\,\phi := \nabla \times \phi.
\]

The expression

\[
\varepsilon \text{curl} [(\text{curl}\mathbf{B}) \times \mathbf{B}],
\]

which makes the system (1.1)-(1.2) strongly nonlinear, represents the Hall-term with the Hall parameter \(\varepsilon > 0\). For simplicity we will assume that \(s = 1\) and \(\varepsilon = 1\). Moreover, \(f = (f_1, f_2)\) stands for the deterministic external forces and \(\mathbf{G}_1(t,\mathbf{u})\,dW_1(t), \mathbf{G}_2(t,\mathbf{B})\,dW_2(t)\), where \(W_1(t), W_2(t)\) are cylindrical Wiener processes, stand for the random forces.

Problem (1.1)-(1.4) can be rewritten as the following initial value problem for the stochastic equation in appropriate functional spaces

\[
\begin{align*}
\quad dX(t) + \left[ AX(t) + \tilde{B}(X(t)) + \tilde{\mathcal{H}}(X(t)) \right] dt &= f(t)\,dt + G(t, X(t))\,dW(t), \quad t \in [0,T]. \\
\quad X(0) &= X_0.
\end{align*}
\]  

(1.5)

Here \(X = (\mathbf{u}, \mathbf{B}), \, X_0 := (\mathbf{u}_0, \mathbf{B}_0)\), and \(A, \tilde{B} \text{ and } \tilde{\mathcal{H}}\) are the maps corresponding to the Stokes-type operators, the MHD-term and the Hall-term, respectively, defined in Section 2.4.

We prove the existence of a global martingale solution of problem (1.5). The main result is stated in Theorem 3.7. Assumptions 3.1 allow to consider the noise term dependent on the unknown process \(X\) and its spatial derivatives. The construction of a solution is based on the approximation motivated by [4, Section 4] and [17, 26, 24, 6]. We consider approximate stochastic equations, called also truncated equations,

\[
\begin{align*}
\quad dX_n(t) + \left[ A_n(X_n(t)) + \tilde{B}_n(X_n(t)) + \tilde{\mathcal{H}}_n(X_n(t)) \right] dt &= f_n(t)\,dt + G_n(t, X_n(t))\,dW(t), \quad t \in [0,T], \\
\quad X_n(0) &= P_nX_0.
\end{align*}
\]
in the infinite dimensional Hilbert spaces $\mathbb{H}_n$, $n \in \mathbb{N}$, defined via the Fourier transform techniques, see Section 4 and Appendix A. The crucial point is to prove suitable uniform a priori estimates for the approximate solutions $(X_n)_{n \in \mathbb{N}}$ stated in Lemma 4.10. To deal with the Hall-term, which is strongly nonlinear, we introduce the tri-linear form $\mathfrak{h}$ and the bilinear map $\mathcal{H}$, see Section 2.3. The results from Remark 2.3 and Lemma 2.5 concerning $\mathfrak{h}$ and $\mathcal{H}$, are very important in our approach. The main idea of the further steps is similar to [7].

The processes $X_n$ generate a tight sequence of probability measures $\{\text{Law}(X_n), n \in \mathbb{N}\}$ on appropriate functional space. Using Jakubowski’s generalization of the Skorokhod theorem for non metrizable spaces and the martingale representation theorem we prove the existence of a martingale solution of problem (1.5).

Let us briefly recall some relevant applications of the Fourier analysis in partial differential equations. In [17] Fefferman and co-authors study deterministic MHD equations on $\mathbb{R}^d$, $d = 2, 3$, and prove the existence of a unique local in time solution in the space $\mathcal{C}([0, T_\star]; H^s)$ for $s > \frac{d}{2}$. By introducing the Fourier truncation the authors approximate the MHD equation by the truncated equations, see [17, p. 1042].

The same idea, referred to as the Friedrichs method, is used by Bahouri, Chemin and Danchin [4, Section 4] to study other class of deterministic differential equations. Defining the cut-off operators, see [4, p. 174], the authors consider appropriate approximate equations.

Analogously to [17], Mohan and Sritharan [26] apply the Fourier truncation method to prove the existence of unique local solution of the stochastic Euler equation on $\mathbb{R}^d$, $d = 2, 3$, in Sobolev spaces $H^s$ for $s > \frac{d}{2} + 1$. The same method is also used by Manna, Mohan and Sritharan [24] to study local solutions of the stochastic MHD equations with Lévy noise. On the other hand, Brzeźniak and Dhariwal [6] apply the truncated approximation in the study of global solutions of the stochastic tamed Navier-Stokes equations.

The paper is organized as follows. In Sections 2.1 and 2.2 we recall some standard notations and results. In Section 2.3 we analyze the Hall-term. Section 2.4 is devoted to the functional setting of the Hall-MHD problem. In Section 3 we formulate the assumptions, definition of a martingale solution and state the main theorem. In Section 4 we consider approximate equations and prove a priori estimates. The proof of the existence of a global martingale solution is established in Section 5. Some auxiliary results related to the Fourier analysis are contained in Appendix A. Results concerning compactness and tightness criteria are presented Appendices B and C. Appendix D contains the proof of some convergence result for the Hall-term, used in the proof of the main theorem.

2 Functional setting

2.1 Basic spaces and notations

Let $\mathcal{C}_c^\infty = \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ denote the space of all $\mathbb{R}^3$-valued functions of class $C^\infty$ with compact supports in $\mathbb{R}^3$, and let

- $\mathcal{V} := \{u \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^3) : \text{div} \, u = 0\}$,
- $H :=$ the closure of $\mathcal{V}$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$,
- $V :=$ the closure of $\mathcal{V}$ in $H^1(\mathbb{R}^3, \mathbb{R}^3)$. 

In the space $H$ we consider the inner product and the norm inherited from $L^2(\mathbb{R}^3, \mathbb{R}^3)$ and denote them by $(\cdot|\cdot)_H$ and $|\cdot|_H$, respectively, i.e.
\[
(u|v)_H := (u|v)_{L^2}, \quad |u|_H := |u|_{L^2}, \quad u, v \in H.
\]
In the space $V$ we consider the inner product inherited from $H^1(\mathbb{R}^3, \mathbb{R}^3)$, i.e.
\[
(u|v)_V := (u|v)_H + (\nabla u|\nabla v)_{L^2}, \quad u, v \in V,
\]
where
\[
(\nabla u|\nabla v)_{L^2} = \sum_{i=1}^3 \int_{\mathbb{R}^3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx,
\]
and the norm induced by $(\cdot|\cdot)_V$, i.e.
\[
\|u\|_V := \left( |u|_H^2 + |\nabla u|_{L^2}^2 \right)^{\frac{1}{2}}.
\]
Let us also, for any $m \geq 0$ consider the following standard scale of Hilbert spaces
\[
V_m := \text{the closure of } V \text{ in } H^m(\mathbb{R}^3, \mathbb{R}^3) \tag{2.1}
\]
with the inner product inherited from the space $H^m(\mathbb{R}^3, \mathbb{R}^3)$. Of course, $V_0 = H$ and $V_1 = V$.
By $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, we denote the space of all Lebesgue measurable functions $v : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\int_K |v(x)|^2 \, dx < \infty$ for every compact subset $K \subset \mathbb{R}^3$. In the space $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, we consider the Fréchet topology generated by the family of seminorms
\[
p_R(v) := \left( \int_{\mathcal{O}_R} |v(x)|^2 \, dx \right)^{\frac{1}{2}}, \quad R \in \mathbb{N}, \tag{2.2}
\]
where $(\mathcal{O}_R)_{R \in \mathbb{N}}$ is an increasing sequence of open bounded subsets of $\mathbb{R}^3$ with smooth boundaries and such that $\bigcup_{R \in \mathbb{N}} \mathcal{O}_R = \mathbb{R}^3$.
By $H_{loc}$, we denote the space $H$ endowed with the Fréchet topology inherited from the space $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$.
Let $T > 0$. By $L^2(0,T;H_{loc})$ we denote the space of measurable functions $u : [0,T] \to H$ such that for all $R \in \mathbb{N}$
\[
p_{T,R}(u) = \left( \int_0^T p_R^2(u(t,\cdot)) \, dt \right)^{\frac{1}{2}} < \infty,
\]
where $p_R$ is defined defined in (2.2). Note that $p_{T,R}$ are seminorms given explicitely by
\[
p_{T,R}(u) = \left( \int_0^T \int_{\mathcal{O}_R} |u(t,x)|^2 \, dx \, dt \right)^{\frac{1}{2}}. \tag{2.3}
\]
In $L^2(0,T;H_{loc})$ we consider the topology generated by the seminorms $(p_{T,R})_{R \in \mathbb{N}}$.

**Notations.** Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two normed spaces. The symbol $\mathcal{L}(X,Y)$ stands for the space of all bounded linear operators from $X$ to $Y$. If $Y = \mathbb{R}$, the $X' := \mathcal{L}(X,\mathbb{R})$ is called the dual space of $X$. The standard duality pairing is denoted by $X' \langle \cdot | \cdot \rangle_X$. If no confusion seems likely we omit the subscripts $X'$ and $X$ and write $\langle \cdot | \cdot \rangle$. If $X$ and $Y$ are separable Hilbert spaces, then by $\mathcal{T}_2(X,Y)$ we will denote the space of all Hilbert-Schmidt operators from $X$ to $Y$ endowed with the standard norm.
2.2 The form \( b \) and the map \( B \)

Let us consider the following tri-linear form

\[
b(u, w, v) = \int_{\mathbb{R}^3} (u \cdot \nabla w) v \, dx.
\]  

(2.4)

We will recall basic properties of the form \( b \), see also Temam [37]. By the Sobolev embedding theorem, see [2], and the Hölder inequality, we obtain the following estimate

\[
|b(u, w, v)| \leq c\|u\|_V\|w\|_V\|v\|_V, \quad u, w, v \in V
\]  

(2.5)

for some positive constant \( c \). Thus the form \( b \) is continuous on \( V \). Moreover, if we define a bilinear map \( B \) by \( B(u, w) := b(u, w, \cdot) \), then by inequality (2.5) we infer that \( B(u, w) \in V' \) for all \( u, w \in V \) and that the following inequality holds

\[
|B(u, w)|_{V'} \leq c\|u\|_V\|w\|_V, \quad u, w \in V.
\]  

(2.6)

Moreover, the mapping \( B : V \times V \to V' \) is bilinear and continuous.

Let us also recall the following properties of the form \( b \), see [37, Lemma II.1.3],

\[
b(u, w, v) = -b(u, v, w), \quad u, w, v \in V.
\]  

(2.7)

In particular,

\[
b(u, v, v) = 0 \quad u, v \in V.
\]  

(2.8)

If \( m > \frac{5}{2} \) then by the Sobolev embedding theorem,

\[
H^{m-1}(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow C_b(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3, \mathbb{R}^3).
\]

Here \( C_b(\mathbb{R}^3, \mathbb{R}^3) \) denotes the space of continuous and bounded \( \mathbb{R}^3 \)-valued functions defined on \( \mathbb{R}^3 \). If \( u, w \in V \) and \( v \in V_m \) with \( m > \frac{5}{2} \) then

\[
|b(u, w, v)| = |b(u, v, w)| \leq |u|_{L^2} |w|_{L^2} |\nabla v|_{L^\infty} \leq c|u|_{L^2} |w|_{L^2} \|v\|_{V_m}
\]

for some constant \( c > 0 \), where \( V_m \) is the space defined by (2.11). Thus, \( b \) can be uniquely extended to the tri-linear form (denoted by the same letter)

\[
b : H \times H \times V_m \to \mathbb{R}
\]

and \( |b(u, w, v)| \leq c|u|_{L^2} |w|_{L^2} \|v\|_{V_m} \) for \( u, w \in H \) and \( v \in V_m \). At the same time the operator \( B \) can be uniquely extended to a bounded bilinear operator

\[
B : H \times H \to V'_m.
\]

In particular, it satisfies the following estimate

\[
|B(u, w)|_{V'_m} \leq c|u|_H |w|_H, \quad u, w \in H.
\]  

(2.9)

We will also use the following notation, \( B(u) := B(u, u) \).

**Lemma 2.1.** The map \( B : V \to V' \) is locally Lipschitz continuous, i.e. for every \( r > 0 \) there exists a constant \( L_r \) such that

\[
|B(u) - B(\tilde{u})|_{V'} \leq L_r \|u - \tilde{u}\|_V, \quad u, \tilde{u} \in V, \quad \|u\|_V, \|\tilde{u}\|_V \leq r.
\]
Proof. The assertion is classical and follows from the following estimates
\[
|B(u,u) - B(\tilde{u},\tilde{u})|_V \leq |B(u,u - \tilde{u})|_V + |B(u - \tilde{u}, \tilde{u})|_V,
\]
\[
\leq \|B\| (\|u\|_V + \|\tilde{u}\|_V) \|u - \tilde{u}\|_V \leq 2r\|B\| \cdot \|u - \tilde{u}\|_V.
\]
Thus the Lipschitz condition holds with \(L = 2r\|B\|\), where \(\|B\|\) stands for the norm of the bilinear map \(B : V \times V \to V'\). The proof is thus complete. \(\square\)

We will use the following version of the convergence result for the map \(B\) proved in [28].

**Lemma 2.2.** (See [28, Lemma 6.1].) Let \(u, w \in L^2(0,T;H)\) and let \((u_n)_n, (w_n)_n \subset L^2(0,T;H)\) be two sequence bounded in \(L^2(0,T;H)\) and convergent to \(u, w\), respectively, in the the space \(L^2(0,T;H_{loc})\). If \(m > \frac{5}{2}\), then for all \(t \in [0,T]\) and all \(\varphi \in V_m(\mathbb{R}^3;\mathbb{R}^3)\):
\[
\lim_{n \to \infty} \int_0^t \langle B(u_n(s), w_n(s))|\varphi \rangle \, ds = \int_0^t \langle B(u(s), w(s))|\varphi \rangle \, ds.
\]
Recall that the space \(L^2(0,T;H_{loc})\) is defined in Section 2.1.

### 2.3 The form \(\mathfrak{h}\) and the map \(\mathcal{H}\)

Let us introduce the following tri-linear form associated with the Hall term and defined by
\[
\mathfrak{h}(u,w,v) := -\int_{\mathbb{R}^3} \text{curl} [u \times \text{curl} w] \cdot v \, dx
\]
for \(u, w, v \in V\). Using the integration by parts formula for the curl-operator, we obtain
\[
\mathfrak{h}(u,w,v) := -\int_{\mathbb{R}^3} [u \times \text{curl} w] \cdot \text{curl} v \, dx.
\]
(2.10)

Since \((a \times b) \cdot c = -(a \times c) \cdot b\) for \(a, b, c \in \mathbb{R}^3\), we infer that
\[
\mathfrak{h}(u,v,w) = -\mathfrak{h}(u,w,v).
\]
(2.11)

In particular,
\[
\mathfrak{h}(u,v,v) = 0.
\]
(2.12)

Note that, if \(u = w\) then using the formula: \(u \times (\text{curl} u) = \nabla (\frac{|u|^2}{2}) - (u \cdot \nabla) u\), we obtain
\[
\mathfrak{h}(u,u,v) = -b(u,u,\text{curl} v).
\]
(2.13)

**Remark 2.3.** (Basic properties of the form \(\mathfrak{h}\).)

(i) There exists a constant \(c > 0\) such that
\[
|\mathfrak{h}(u,w,v)| \leq c \|u\|_{H^1} \|w\|_{H^1} \|v\|_{H^2}, \quad u, w \in V, \quad v \in V_2.
\]
(2.14)

Thus the form \(\mathfrak{h}\) can be extended to the continuous tri-linear form (denoted again by \(\mathfrak{h}\))
\[
\mathfrak{h} : V \times V \times V_2 \to \mathbb{R}.
\]
(ii) Moreover, there exists a continuous bilinear map $\mathcal{H}: V \times V \to V'_2$ such that
\begin{equation}
\langle \mathcal{H}(u, w) \rangle_v = h(u, w, v), \quad u, w \in V, \quad v \in V_2,
\end{equation}
and
\begin{equation}
|\mathcal{H}(u, w)|_{V'_2} \leq c \|u\|_{H^1} \cdot \|w\|_{H^1}, \quad u, w \in V.
\end{equation}

By (2.11)
\begin{equation}
\langle \mathcal{H}(u, w) \rangle_v = -\langle \mathcal{H}(u, v) \rangle_{w}, \quad u, w \in V,
\end{equation}
and, in particular,
\begin{equation}
\langle \mathcal{H}(u, v) \rangle_v = 0, \quad u \in V, \quad v \in V_2.
\end{equation}

Let us also note that for $u = w \in V$, by (2.13), we have
\begin{equation}
\langle \mathcal{H}(u, u) \rangle_v = \langle B(u, u) \rangle_{\text{curl} v}, \quad u \in V, \quad v \in V_2.
\end{equation}

(iii) If $m > \frac{5}{2}$, then there exists a constant $c > 0$ such that
\[ |h(u, w, v)| \leq c \|u\|_{L^2} \cdot \|w\|_{H^1} \cdot \|v\|_{H^m}, \quad u \in H, \quad w \in V, \quad v \in V_m. \]

Thus the form $h$ extends to the continuous tri-linear form (denoted still by $h$)
\[ h: H \times V \times V_m \to \mathbb{R}, \]
\[ |h(u, w, v)|_{V'_m} \leq c \|u\|_{L^2} \cdot \|w\|_{H^1}, \quad u \in H, \quad w \in V. \]

Proof. By the Hölder inequality and the Sobolev embedding theorem we obtain
\[ |h(u, w, v)| = \left| \int_{\mathbb{R}^3} [u \times (\text{curl} w)] : \text{curl} v \, dx \right| \leq |u|_{L^4} \cdot |\text{curl} w|_{L^2} \cdot |\text{curl} v|_{L^4} \]
\[ \leq \tilde{c} \|u\|_{H^1} \cdot \|w\|_{H^1} \cdot \|v\|_{H^1} \leq c \|u\|_{H^1} \cdot \|w\|_{H^1} \cdot \|v\|_{H^2}, \]
for some constants $\tilde{c}, c > 0$. This completes the proof of assertion (i). Assertion (ii) follows from (i) and (2.11) and (2.12).

To prove (iii), let us fix $m > \frac{5}{2}$. Since $H^{m-1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, by the Hölder inequality and the Sobolev embedding theorem we obtain
\[ |h(u, w, v)| = \left| \int_{\mathbb{R}^3} [u \times (\text{curl} w)] : \text{curl} v \, dx \right| \leq |u|_{L^2} \cdot |\text{curl} w|_{L^2} \cdot |\text{curl} v|_{L^\infty} \]
\[ \leq \tilde{c} \|u\|_{L^2} \cdot \|w\|_{H^1} \cdot \|\text{curl} v\|_{H^{m-1}} \leq c \|u\|_{L^2} \cdot \|w\|_{H^1} \cdot \|v\|_{H^m}. \]

The proof of the remark is thus complete.

We will use the following notation
\[ \mathcal{H}(u) := \mathcal{H}(u, u). \]

Lemma 2.4. The map $\mathcal{H}: V \to V'_2$ is locally Lipschitz continuous, i.e. for every $r > 0$ there exists $L_\mathcal{H}(r) > 0$ such that
\[ |\mathcal{H}(u) - \mathcal{H}(\bar{u})|_{V'_2} \leq L_\mathcal{H}(r) \|u - \bar{u}\|_{V}, \quad u, \bar{u} \in V, \quad \|u\|_{V}, \|\bar{u}\|_{V} \leq r. \]
Proof. By (2.16), we infer that for all \( u, \tilde{u} \in V \) such that \( \|u\|_V, \|\tilde{u}\|_V \leq r \),
\[
|\mathcal{H}(u) - \mathcal{H}(\tilde{u})|_{V_2'} = |\mathcal{H}(u) - \mathcal{H}(\tilde{u})|_{V_2'} \leq |\mathcal{H}(u, u - \tilde{u})|_{V_2'} + |\mathcal{H}(u - \tilde{u})|_{V_2'} \\
\leq \|\mathcal{H}\| \|u - \tilde{u}\|_V + \|\mathcal{H}\| \|u - \tilde{u}\|_V \|\tilde{u}\|_V \\
= \|\mathcal{H}\| (\|u\|_V + \|\tilde{u}\|_V) \|u - \tilde{u}\|_V \leq 2r \|\mathcal{H}\| \|u - \tilde{u}\|_V.
\]
This means that the local Lipschitz condition holds with the constant \( L_{\mathcal{H}}(r) := 2r \|\mathcal{H}\| \), where \( \|\mathcal{H}\| \) denotes the norm of the bilinear map \( \mathcal{H} : V \times V \to V_2' \). The proof of the lemma is thus complete. \( \square \)

In the following lemma we will prove some result concerning the convergence of the Hall term \( \mathcal{H} \). This result is analogous to Lemma 2.2.

**Lemma 2.5.** Let \( u \in L^2(0, T; H) \) and \( w \in L^2(0, T; V) \) and let \( (u_n) \subset L^2(0, T; H) \) and \( (w_n) \subset L^2(0, T; V) \) be two sequences such that

- \( (u_n) \) is bounded in \( L^2(0, T; H) \) and \( w_n \to w \) weakly in \( L^2(0, T; V) \),
- \( u_n \to u \) and \( w_n \to w \) in \( L^2(0, T; H_{loc}) \).

If \( m > \frac{5}{2} \), then for all \( t \in [0, T] \) and all \( \psi \in V_m' \):
\[
\lim_{n \to \infty} \int_0^t \langle \mathcal{H}(u_n(s), w_n(s))|\psi\rangle \, ds = \int_0^t \langle \mathcal{H}(u(s), w(s))|\psi\rangle \, ds.
\]

Recall that the space \( L^2(0, T; H_{loc}) \) is defined in Section 2.1.

**Proof.** The proof of Lemma 2.5 is postponed to Appendix D. \( \square \)

### 2.4 Functional setting of the Hall-MHD system

Using the spaces \( H \) and \( V \) defined in Section 2.1, let us consider the spaces
\[
\mathbb{H} := H \times H, \quad \mathbb{V} := V \times V, \quad \mathbb{V}' := \text{the dual space of } \mathbb{V}
\]  
with the following inner products
\[
(\phi|\psi)_H := (u|v)_{L^2} + (B|C)_{L^2}
\]
for all \( \phi = (u, B) \), \( \psi = (v, C) \in \mathbb{H} \), and
\[
(\phi|\psi)_V := (\phi|\psi)_\mathbb{H} + (\phi|\psi)
\]
for all \( \phi = (u, B) \), \( \psi = (v, C) \in \mathbb{V} \), where
\[
(\phi|\psi) := \nu_1 (\nabla u|\nabla v)_{L^2} + \nu_2 (\nabla B|\nabla C)_{L^2}.
\]

In the spaces \( \mathbb{H} \) and \( \mathbb{V} \) we consider the norms induced by the inner products \( (\cdot|\cdot)_H \) and \( (\cdot|\cdot)_V \), respectively, i.e. \( \|\phi\|^2_H := (\phi|\phi)_H \) for \( \phi \in \mathbb{H} \), and
\[
\|\phi\|^2_V = \|\phi\|^2_\mathbb{H} + \|\phi\|^2,
\]  
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where
\[ \|\phi\|^2 := \langle \phi | \phi \rangle, \quad \phi \in \mathbb{V}. \] (2.21)

**The operator \( A \).** We define the operator \( A \) by the following formula
\[ \langle A\phi | \psi \rangle = \langle \phi | \psi \rangle, \quad \phi, \psi \in \mathbb{V}, \] (2.22)
where \( \langle \cdot | \cdot \rangle \) is given by (2.19).

**Remark 2.6.** It is clear that \( A \in \mathcal{L}(\mathbb{V}, \mathbb{V}') \) and
\[ |A\phi|_{2}, \quad \phi \in \mathbb{V}, \] (2.23)
where \( \|\cdot\| \) is given by (2.21).

Indeed, inequality (2.23) follows from (2.20) and the following inequalities
\[ |\langle A\phi | \psi \rangle| = |\langle \phi | \psi \rangle| \leq \|\phi\| \|\psi\| \leq \|\phi\| (|\psi|^2 + \|\psi\|^2)^{\frac{1}{2}} = \|\phi\| \|\psi\|_{2}. \]

For \( m_1, m_2 \geq 0 \) let us define
\[ \mathbb{V}_{m_1, m_2} := \mathbb{V}_{m_1} \times \mathbb{V}_{m_2}, \] (2.24)
where \( \mathbb{V}_{m_1}, \mathbb{V}_{m_2} \) are the spaces defined by (2.1). In \( \mathbb{V}_{m_1, m_2} \) we consider the product norm
\[ \|\phi\|^2_{m_1, m_2} := \|\mathbf{u}\|_{\mathbb{V}_{m_1}}^2 + \|\mathbf{B}\|_{\mathbb{V}_{m_2}}^2 \] (2.25)
for all \( \phi = (\mathbf{u}, \mathbf{B}) \in \mathbb{V}_{m_1, m_2} \). In the case when \( m_1 = m_2 = m \) we denote
\[ \mathbb{V}_{m} := \mathbb{V}_{m} \times \mathbb{V}_{m} \quad \text{and} \quad \|\cdot\|_{m} := \|\cdot\|_{m, m}. \] (2.26)

It is clear that if \( m = 1 \), then \( \mathbb{V}_1 = \mathbb{V} \) and \( \|\cdot\|_1 = \|\cdot\|_{\mathbb{V}} \).

Let \( T > 0 \). By \( L^2(0, T; \mathbb{H}_{\text{loc}}) \) we denote the space of measurable functions \( \phi : [0, T] \to \mathbb{H} \) such that for all \( R \in \mathbb{N} \)
\[ p_{T,R}(\phi) := \left( \int_0^T \left[ p_R^2(\mathbf{u}(t, \cdot)) + p_R^2(\mathbf{B}(t, \cdot)) \right] dt \right)^{\frac{1}{2}} < \infty, \]
where \( \phi = (\mathbf{u}, \mathbf{B}) \), and \( p_R \) are defined in (2.2). Explicitly,
\[ p_{T,R}(\phi) = \left( \int_0^T \int_{\mathcal{O}_R} \left[ |\mathbf{u}(t, x)|^2 + |\mathbf{B}(t, x)|^2 \right] dxdt \right)^{\frac{1}{2}}. \]

In the space \( L^2(0, T; \mathbb{H}_{\text{loc}}) \) we consider the topology generated by the seminorms \( (p_{T,R})_{R \in \mathbb{N}} \).

**The form \( \tilde{b} \) and the operator \( \tilde{B} \).** Using the form \( b \) defined by (2.4) we will consider the tri-linear form \( \tilde{b} \) on \( \mathbb{V} \times \mathbb{V} \times \mathbb{V} \), where \( \mathbb{V} \) is defined by (2.18), see Sermange and Temam [33] and Sango [31]. Namely,
\[ \tilde{b}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) := b(u^{(1)}, u^{(2)}, u^{(3)}) - b(B^{(1)}, B^{(2)}, u^{(3)}) \]
\[ + b(u^{(1)}, B^{(2)}, B^{(3)}) - b(B^{(1)}, u^{(2)}, B^{(3)}), \]
where \( \phi^{(i)} = (u^{(i)}, B^{(i)}) \in \mathbb{V}, \ i = 1, 2, 3. \) By (2.5) we see that the form \( \tilde{b} \) is continuous. Moreover, by (2.7) and (2.5) the form \( \tilde{b} \) has the following properties
\[ \tilde{b}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) = -\tilde{b}(\phi^{(1)}, \phi^{(3)}, \phi^{(2)}), \quad \phi^{(i)} \in \mathbb{V}, \ i = 1, 2, 3 \]
and in particular

\[ \tilde{b}(\phi^{(1)}, \phi^{(2)}, \phi^{(2)}) = 0, \quad \phi^{(1)}, \phi^{(2)} \in \mathbb{V}. \]

Now, let us define a bilinear map \( \tilde{B} \) by

\[ \tilde{B}(\phi, \psi) := \tilde{b}(\phi, \psi, \cdot), \quad \phi, \psi \in \mathbb{V}. \]  \hspace{1cm} (2.27)

We will also use the notation \( \tilde{B}(\phi) := \tilde{B}(\phi, \phi) \).

Let us recall properties of the map \( \tilde{B} \) stated in [28].

**Lemma 2.7.** (See [28, Lemma 6.4])

(i) There exists a constant \( c_B > 0 \) such that

\[ |\tilde{B}(\phi, \psi)|_{\mathbb{V}'} \leq c_B \|\phi\|_{\mathbb{V}} \|\psi\|_{\mathbb{V}}, \quad \phi, \psi \in \mathbb{V}. \]

In particular, the map \( \tilde{B} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}' \) is bilinear and continuous. Moreover,

\[ \langle \tilde{B}(\phi, \psi) | \theta \rangle = -\langle \tilde{B}(\phi, \theta) | \psi \rangle, \quad \phi, \psi, \theta \in \mathbb{V}, \]

and, in particular,

\[ \langle \tilde{B}(\phi) | \phi \rangle = 0, \quad \phi \in \mathbb{V}. \]  \hspace{1cm} (2.28)

(ii) The mapping \( \tilde{B} \) is locally Lipschitz continuous on the space \( \mathbb{V} \), i.e. for every \( r > 0 \) there exists a constant \( L_r > 0 \) such that

\[ |\tilde{B}(\phi) - \tilde{B}(\tilde{\phi})|_{\mathbb{V}'} \leq L_r \|\phi - \tilde{\phi}\|_{\mathbb{V}}, \quad \phi, \tilde{\phi} \in \mathbb{V}, \quad \|\phi\|_{\mathbb{V}}, \|\tilde{\phi}\|_{\mathbb{V}} \leq r. \]

(iii) If \( m > \frac{5}{2} \), then \( \tilde{B} \) can be extended to the bilinear mapping from \( \mathbb{H} \times \mathbb{H} \) to \( \mathbb{V}'_m \) (denoted still by \( \tilde{B} \)) such that

\[ |\tilde{B}(\phi, \psi)|_{\mathbb{V}'_m} \leq c_B(m) \|\phi\|_{\mathbb{H}} \|\psi\|_{\mathbb{H}}, \quad \phi, \psi \in \mathbb{H}, \]  \hspace{1cm} (2.29)

where \( c_B(m) \) is a positive constant.

From (2.27) and Lemma 2.2 we obtain immediately the following corollary.

**Corollary 2.8.** Let \( \phi, \psi \in L^2(0, T; \mathbb{H}) \) and let \( (\phi_n), (\psi_n) \subset L^2(0, T; \mathbb{H}) \) be two sequence bounded in \( L^2(0, T; \mathbb{H}) \) and such that

\[ \phi_n \to \phi \quad \text{and} \quad \psi_n \to \psi \quad \text{in} \quad L^2(0, T; \mathbb{H}_{loc}). \]

If \( m > \frac{5}{2} \), then for all \( t \in [0, T] \) and all \( \varphi \in \mathbb{V}_m \):

\[ \lim_{n \to \infty} \int_0^t \langle \tilde{B}(\phi_n(s), \psi_n(s)) | \varphi \rangle \, ds = \int_0^t \langle \tilde{B}(\phi(s), \psi(s)) | \varphi \rangle \, ds, \]

where \( \mathbb{V}_m \) is the space defined by (2.26).
The form $\tilde{\mathfrak{h}}$ and the map $\tilde{H}$. Using the form $\mathfrak{h}$ defined by (2.10) we will consider the tri-linear form $\tilde{\mathfrak{h}}$ on $\mathbb{V} \times \mathbb{V} \times \mathbb{V}_{1,2}$ defined by

$$\tilde{\mathfrak{h}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) := \mathfrak{h}(B^{(1)}, B^{(2)}, B^{(3)}),$$

where $\phi^{(i)} = (u^{(i)}, B^{(i)}) \in \mathbb{V}$ for $i = 1, 2$, and $\phi^{(3)} = (u^{(3)}, B^{(3)}) \in \mathbb{V}_{1,2}$. Due to (2.24), $\mathbb{V}_{1,2} := V_{1} \times V_{2}$. By (2.14) we see that the form $\mathfrak{h}$ is continuous. Moreover, by (2.11) and (2.12) the form $\tilde{\mathfrak{h}}$ has the following properties

$$\tilde{\mathfrak{h}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) = -\tilde{\mathfrak{h}}(\phi^{(1)}, \phi^{(3)}, \phi^{(2)}), \quad \phi^{(i)} \in \mathbb{V}, \quad \phi^{(i)} \in \mathbb{V}_{1,2}, \quad i = 2, 3,$$

and in particular

$$\tilde{\mathfrak{h}}(\phi^{(1)}, \phi^{(2)}, \phi^{(2)}) = 0, \quad \phi^{(1)} \in \mathbb{V}, \quad \phi^{(2)} \in \mathbb{V}_{1,2}.$$

Now, let us define a bilinear map $\tilde{\mathcal{H}}$ by

$$\tilde{\mathcal{H}}(\phi, \psi) := \tilde{\mathfrak{h}}(\phi, \psi, \cdot), \quad \phi, \psi \in \mathbb{V}. \quad (2.30)$$

We will also use the notation $\tilde{\mathcal{H}}(\phi) := \tilde{\mathcal{H}}(\phi, \phi)$. Using Remark 2.3 and Lemma 2.4 we obtain the following result.

**Lemma 2.9. (Properties of the map $\tilde{H}$.)**

(i) There exists a constant $c_{\tilde{H}} > 0$ such that

$$|\tilde{\mathcal{H}}(\phi, \psi)|_{\mathbb{V}_{1,2}} \leq c_{\tilde{H}} \|\phi\|_{\mathbb{V}} \|\psi\|_{\mathbb{V}}, \quad \phi, \psi \in \mathbb{V}.$$

In particular, the map $\tilde{\mathcal{H}} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}_{1,2}$ is well-defined bilinear and continuous. Moreover,

$$\langle \tilde{\mathcal{H}}(\phi, \psi)|\Theta \rangle = -\langle \tilde{\mathcal{H}}(\phi, \theta)|\psi \rangle, \quad \phi, \Theta \in \mathbb{V}, \quad \psi, \theta \in \mathbb{V}_{1,2},$$

and, in particular,

$$\langle \tilde{\mathcal{H}}(\phi)|\phi \rangle = 0, \quad \phi \in \mathbb{V}_{1,2}. \quad (2.31)$$

(ii) The map $\tilde{\mathcal{H}}$ is locally Lipschitz continuous on the space $\mathbb{V}$, i.e. for every $r > 0$ there exists a constant $L_{r} > 0$ such that

$$|\tilde{\mathcal{H}}(\phi) - \tilde{\mathcal{H}}(\tilde{\phi})|_{\mathbb{V}_{1,2}} \leq L_{r} \|\phi - \tilde{\phi}\|_{\mathbb{V}}, \quad \phi, \tilde{\phi} \in \mathbb{V}, \quad \|\phi\|_{\mathbb{V}}, \|\tilde{\phi}\|_{\mathbb{V}} \leq r.$$

(iii) If $s \geq 0$ and $m > \frac{5}{2}$, then $\tilde{\mathcal{H}}$ can be extended to the bilinear mapping from $\mathbb{H} \times \mathbb{V}$ to $\mathbb{V}'_{s,m}$ (denoted still by $\tilde{\mathcal{H}}$) such that

$$|\tilde{\mathcal{H}}(\phi, \psi)|_{\mathbb{V}'_{s,m}} \leq c_{\tilde{H}}(s, m) \|\phi\|_{\mathbb{H}} \|\psi\|_{\mathbb{V}}, \quad \phi \in \mathbb{H}, \quad \psi \in \mathbb{V},$$

where $c_{\tilde{H}}(s, m)$ is a positive constant.

In particular, if $m > \frac{5}{2}$, then $\tilde{\mathcal{H}}$ can be extended to the bilinear mapping from $\mathbb{H} \times \mathbb{V}$ to $\mathbb{V}'_{m}$ (denoted still by $\tilde{\mathcal{H}}$) such that

$$|\tilde{\mathcal{H}}(\phi, \psi)|_{\mathbb{V}'_{m}} \leq c_{\tilde{H}}(m) \|\phi\|_{\mathbb{H}} \|\psi\|_{\mathbb{V}}, \quad \phi \in \mathbb{H}, \quad \psi \in \mathbb{V}, \quad (2.32)$$

where $c_{\tilde{H}}(m) := c_{\tilde{H}}(m, m)$ and $\mathbb{V}_{m}$ is the space defined by (2.26).
From lemma (2.25) and the definition of $\tilde{H}$, we obtain immediately the following result.

**Corollary 2.10.** Let $\phi \in L^2(0,T;\mathbb{H})$ and $\psi \in L^2(0,T;\mathbb{V})$ and let $(\phi_n) \subset L^2(0,T;\mathbb{H})$ and $(\psi_n) \subset L^2(0,T;\mathbb{V})$ be two sequences such that

- $(\phi_n)$ is bounded in $L^2(0,T;\mathbb{H})$ and $\psi_n \to \psi$ weakly in $L^2(0,T;\mathbb{V})$,
- $\phi_n \to \phi$ and $\psi_n \to \psi$ in $L^2(0,T;\mathbb{H}_{loc})$.

If $s \geq 0$ and $m > \frac{5}{2}$, then for all $t \in [0,T]$ and all $\varphi \in \mathbb{V}_{s,m}$:

$$
\lim_{n \to \infty} \int_0^t \langle \tilde{H}(\phi_n(s),\psi_n(s))|\varphi \rangle \ ds = \int_0^t \langle \tilde{H}(\phi(s),\psi(s))|\varphi \rangle \ ds,
$$

where $\mathbb{V}_{s,m}$ is the space defined by (2.24)-(2.25).

### 3 Martingale solutions of the Hall-magnetohydrodynamics equations

We will formulate assumptions imposed on the noise terms, the deterministic external forces and the initial conditions in problem (1.1)-(1.4).

**Assumption 3.1.** We assume that

**G.1** $\mathbb{K}_1, \mathbb{K}_2$ are separable Hilbert spaces, and

$$ G_i : [0,T] \times V \to \mathcal{T}_2(\mathbb{K}_i,H), \quad i = 1,2, $$

are two measurable map which are Lipschitz continuous, i.e. there exist constants $L_i$, $i = 1,2$, such that

$$ \|G_i(t,\phi_1) - G_i(t,\phi_2)\|_{\mathcal{T}_2(\mathbb{K}_i,H)}^2 \leq L_i \|\phi_1 - \phi_2\|_V^2, \quad \phi_1,\phi_2 \in V, \ t \in [0,T]. \quad (3.1) $$

In addition, there exist $\lambda_i, \nu_i \in \mathbb{R}$ and $\eta \in (0,2]$ such that

$$ \|G_i(t,\phi)\|_{\mathcal{T}_2(\mathbb{K}_i,H)}^2 \leq \nu_i(2-\eta)|\nabla \phi|_L^2 + \lambda_i|\phi|_H^2 + \nu_i, \quad (t,\phi) \in [0,T] \times V. \quad (3.2) $$

**G.2** The maps $G_i$, $i = 1,2$, can be extended to measurable maps

$$ g_i : [0,T] \times H \to \mathcal{L}(\mathbb{K}_i,V') $$

such that for some $C_i > 0$

$$ \sup_{\psi \in V,\|\psi\| \leq 1} \sup_{y \in \mathbb{K}_i,\|y\|_{\mathbb{K}_i} \leq 1} |V'\langle g(t,\phi)(y)|\psi \rangle_V|^2 \leq C_i(1+|\phi|_H^2), \quad (t,\phi) \in [0,T] \times H. \quad (3.3) $$

**G.3** Moreover, for every $\psi \in V$ the maps $\tilde{g}_{i,\psi}$ defined for $\phi \in L^2(0,T;H)$ by

$$ (\tilde{g}_{i,\psi}(\phi))(t) := \{\mathbb{K}_i \ni y \mapsto V'\langle g_i(t,\phi(t))(y)|\psi \rangle_V \in \mathbb{R} \} \in \mathcal{T}_2(\mathbb{K}_i,\mathbb{R}), \quad t \in [0,T], \quad (3.4) $$

are continuous maps from $L^2(0,T;H_{loc})$ into $L^2(0,T;\mathcal{T}_2(\mathbb{K}_i,\mathbb{R}))$.  

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The spaces $H, V$ and $L^2(0, T; H_{\text{loc}})$ are defined in Section 2.1. Recall also that for any Hilbert spaces $\mathbb{K}$ and $Y$ by $\mathcal{T}_2(\mathbb{K}; Y)$ we denote the space of Hilbert-Schmidt operators from $\mathbb{K}$ into $Y$.

**Assumption 3.2.** We assume also that the following objects are given.

**H.1** A real number $p$ such that

$$ p \in [2, 2 + \gamma), $$

where

$$ \gamma := \begin{cases} \frac{\eta}{\gamma}, & \text{if } \eta \in [0, 2), \\ \infty, & \text{if } \eta = 2, \end{cases} $$

and $\eta$ is the parameter from inequality (3.2).

**H.2** $X_0 := (u_0, B_0) \in H \times H$ and $f := (f_1, f_2)$, where $f_i \in L^p(0, T; V')$ for $i = 1, 2$.

**H.3** $\mathfrak{A} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying usual hypotheses and $W_i(t)$ are two cylindrical Wiener processes in a separable Hilbert space $\mathbb{K}_i$, $i = 1, 2$, defined on the stochastic basis $\mathfrak{A}$.

Let $W(t) := (W_1(t), W_2(t))$. Then $W(t)$ is a cylindrical Wiener process on $\mathbb{K} := \mathbb{K}_1 \times \mathbb{K}_2$, on the stochastic basis $\mathfrak{A}$.

**Example 3.3.** Let $\mathbb{K}_1 = \mathbb{K}_2 := \ell^2(\mathbb{N})$, where $\ell^2(\mathbb{N})$ denotes the space of all sequences $(h_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ such that $\sum_{j=1}^{\infty} h_j^2 < \infty$. It is a Hilbert space with the scalar product given by $(h|k)_\ell^2 := \sum_{j=1}^{\infty} h_j k_j$, where $h = (h_j)$ and $k = (k_j)$ belong to $\ell^2(\mathbb{N})$. Let us put

$$ G_i(\phi) h := \sum_{j=1}^{\infty} \left( \left( b_i^{(j)} \cdot \nabla \right) \phi + c_i^{(j)} \phi \right) h_j, \quad \phi \in V, \quad h = (h_j) \in \ell^2(\mathbb{N}), \quad i = 1, 2, $$

where

$$ b_i^{(j)} : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{and} \quad c_i^{(j)} : \mathbb{R}^3 \to \mathbb{R} \quad \text{for} \quad j \in \mathbb{N} \quad \text{and} \quad i = 1, 2 $$

are given functions of class $C^\infty$ and such that for each $i = 1, 2$

$$ C_i := \sum_{j=1}^{\infty} \left( \| b_i^{(j)} \|_{L^\infty} + \| \text{div} b_i^{(j)} \|_{L^\infty} + \| c_i^{(j)} \|_{L^\infty} \right) < \infty $$

and

$$ \sum_{k,l=1}^{3} \left( 2\delta_{kl} - \sum_{j=1}^{\infty} b_{ik}^{(j)}(x) b_{il}^{(j)}(x) \right) \zeta_k \zeta_l \geq a_i |\zeta|^2, \quad \zeta \in \mathbb{R}^d $$

for some $a_i \in (0, \nu_i]$, where $\nu_i > 0$ for $i = 1, 2$ are coefficients given in equations (1.1) and (1.2). Proceeding similarly as in Section 6 we can prove that the maps $G_i, i = 1, 2$, given by (3.7) satisfy Assumption 3.1.

The maps $G_i, i = 1, 2$, define the following noise terms

$$ G_i(\phi)(t, x)dW_i(t) := \sum_{j=1}^{\infty} \left[ \left( b_i^{(j)}(x) \cdot \nabla \right) \phi(t, x) + c_i^{(j)}(x) \phi(t, x) \right] d\beta_i^{(j)}(t), $$

where $\beta_i^{(j)}$ for $j \in \mathbb{N}$ and $i = 1, 2$ are independent standard Brownian motions. □
The functional setting for the Hall-MHD equations \([1.1]-[1.4]\) involves spaces \(H\) and \(V\) being appropriate products of spaces \(H\) and \(V\) (see Section \(2.4\)). Given the maps \(G_i\) and \(g_i\), \(i = 1, 2\), from Assumption \(3.1\) we introduce maps \(G\) and \(g\) defined on appropriate product spaces.

**Remark 3.4. (The maps \(G\) and \(g\) and their properties.)** Let \(G_1\) and \(G_2\) be the maps given in Assumption \(3.1\). Let us define the following map

\[
G(t, \Phi)(y) := (G_1(t, u)(y_1), G_2(t, B)(y_2)),
\]

where \(t \in [0, T]\), \(\Phi := (u, B) \in V\), \(y = (y_1, y_2) \in K := K_1 \times K_2\).

(i) Then

\[G : [0, T] \times V \to \mathcal{T}_2(K, H).\]

The map \(G\) satisfies the Lipschitz condition, i.e. there exists a constant \(L > 0\) such that

\[
\|G(t, \Phi_1) - G(t, \Phi_2)\|_{T_2(K, H)}^2 \leq L \|\Phi_1 - \Phi_2\|_V^2, \quad \Phi_1, \Phi_2 \in V, \ t \in [0, T].
\]

(ii) The map \(G\) satisfies the following inequality

\[
\|G(s, \Phi)\|_{T_2(K, H)}^2 \leq (2 - \eta)\|\Phi\|_V^2 + \lambda\|\Phi\|_H^2 + \rho, \quad (s, \Phi) \in [0, T] \times V,
\]

where \(\lambda := \lambda_1 + \lambda_2\) and \(\rho := \rho_1 + \rho_2\).

(iii) Let \(g_1\) and \(g_2\) are the maps from Assumption \(3.1\) and let us define

\[
g(t, \Phi)(y) := (g_1(t, u)(y_1), g_2(t, B)(y_2)),
\]

where \(t \in [0, T]\), \(\Phi := (u, B) \in V\), \(y = (y_1, y_2) \in K\). Then the map \(g\) is an extension of the map \(G\) to a measurable map

\[g : [0, T] \times H \to \mathcal{L}(K, V'),\]

and by \(3.3\) we obtain

\[
\sup_{\Psi \in V, \|\Psi\|_V \leq 1} \sup_{y \in K, \|y\|_K \leq 1} \|\Psi \langle g(t, \Phi)(y)|\Psi\rangle_{V'}\|_V^2 \leq C(1 + \|\Phi\|_H^2), \quad (t, \Phi) \in [0, T] \times H.
\]

Moreover, for every \(\Psi \in V\) the map \(\tilde{g}_\Psi\) defined for \(\tilde{\Phi} \in L^2(0, T; \mathbb{H})\) by

\[
(\tilde{g}_\Psi(\tilde{\Phi}))(t) := \{K \ni y \mapsto \langle g(t, \Phi(t))(y)|\Psi\rangle_V \in \mathbb{R}\} \in \mathcal{T}_2(K, \mathbb{R}), \quad t \in [0, T],
\]

are continuous maps from \(L^2(0, T; \mathbb{H}_{loc})\) into \(L^2(0, T; \mathcal{T}_2(K, \mathbb{R}))\).

Let us recall that the spaces \(H\), \(V\) and \(L^2(0, T; \mathbb{H}_{loc})\) are defined in Section \(2.4\).

Using the maps introduced in Section \(2.4\) we can rewrite problem \([1.1]-[1.4]\) as the following stochastic equation

\[
dX(t) + [AX(t) + \bar{B}(X(t)) + \tilde{H}(X(t))] dt = f(t) dt + G(t, X(t))dW(t), \quad t \in [0, T],
\]

\[X(0) = X_0.\]

Here \(X_0 := (u_0, B_0)\) and \(A\), \(\bar{B}\) and \(\tilde{H}\) are the maps defined by \(2.22\), \(2.27\) and \(2.30\), respectively.
Definition 3.5. Let Assumptions 3.1 and 3.2 be satisfied. We say that there exists a martingale solution of problem (3.17) iff there exist

- a stochastic basis $\mathfrak{G} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ satisfying the usual conditions,
- a $\mathbb{F}$-cylindrical Wiener process $\mathbb{W}$ over $\mathfrak{G},$
- and another $\mathbb{F}$- progressively measurable process $X : [0, T] \times \Omega \to \mathbb{H}$ with $\mathbb{P}$-a.e. paths satisfying

$$X(\cdot, \omega) \in C([0, T]; \mathbb{H}_w) \cap L^2(0, T; \mathbb{V}),$$

and such that for all $t \in [0, T]$ and $\varphi \in \mathbb{V}_{1,2}$ the following identity holds $\mathbb{P}$-a.s.

$$(X(t)|\varphi)_\mathbb{H} + \int_0^t \langle AX(s)|\varphi \rangle ds + \int_0^t \langle \tilde{B}(X(s))|\varphi \rangle ds + \int_0^t \langle \tilde{H}(X(s))|\varphi \rangle ds = (X_0|\varphi)_\mathbb{H} + \int_0^t \langle f(s)|\varphi \rangle ds + \left\langle \int_0^t G(s, X(s)) d\mathbb{W}(s)|\varphi \right\rangle.$$

(3.18)

and

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |X(t)|^2_\mathbb{H} + \int_0^T \|X(t)\|_{\mathbb{V}}^2 dt \right] < \infty. \quad (3.19)$$

If all the above conditions are satisfied, then the system

$$(\mathfrak{G}, \mathbb{W}, X)$$

is called a martingale solution of problem (3.17).

$\mathbb{H}_w$ denotes the Hilbert space $\mathbb{H}$ endowed with the weak topology and $C([0, T]; \mathbb{H}_w)$ is the space of all weakly continuous functions $\psi : [0, T] \to \mathbb{H},$ i.e., such that for all $h \in \mathbb{H}$ the real valued function

$$[0, T] \ni t \to (\psi(t)|h)_\mathbb{H} \in \mathbb{R}$$

is continuous.

Remark 3.6. Explicitly, if an $\mathbb{H}$-valued process $X = (u, B)$ is a solution of problem (3.17), then the processes $u$ and $B$ satisfy $\mathbb{P}$-a.s. the following identities: for every $t \in [0, T]$ and $\varphi := (\varphi_1, \varphi_2) \in V_1 \times V_2$

$$(u(t)|\varphi_1)_L^2 + \int_0^t \left\{ \nu_1 (\nabla u(s) \nabla \varphi_1)_L^2 + (u(s) \cdot \nabla u(s) - B(s) \cdot \nabla B(s) | \varphi_1)_L^2 \right\} ds$$

$$= (u(0)|\varphi_1)_L^2 + \int_0^t \langle f_1(s)|\varphi_1 \rangle ds + \left\langle \int_0^t G_1(s, u(s)) d\mathbb{W}_1(s)|\varphi_1 \right\rangle,$$

$$(B(t)|\varphi_2)_L^2 + \int_0^t \left\{ \nu_2 (\nabla B(s) \nabla \varphi_2)_L^2 + (u(s) \cdot \nabla u(s) - B(s) \cdot \nabla B(s) | \varphi_2)_L^2$$

$$- (B(s) \times \text{curl } B(s) | \text{curl } \varphi_2)_L^2 \right\} ds$$

$$= (B(0)|\varphi_2)_L^2 + \int_0^t \langle f_2(s)|\varphi_2 \rangle ds + \left\langle \int_0^t G_2(s, B(s)) d\mathbb{W}_2(s)|\varphi_2 \right\rangle,$$

and the energy inequality

$$\mathbb{E} \left[ \sup_{t \in [0,T]} (|u(t)|^2_L + |B(t)|^2_L) + \int_0^T (|\nabla u(t)|^2_L + |\nabla B(t)|^2_L) dt \right] < \infty.$$
Now we formulate the main result concerning the existence of a martingale solution.

**Theorem 3.7.** Let Assumptions 3.1 and 3.2 be satisfied. In particular, we assume that $p$ satisfies (3.5), i.e.
\[ p \in [2, 2 + \gamma), \]
where $\gamma$ is given by (3.6). Then there exists a martingale solution of problem (3.17) such that
(i) for every $q \in [1, p]$ there exists a constant $C_1(p, q)$ such that
\[ \bar{E} \left[ \sup_{t \in [0, T]} |X(t)|^q_H \right] \leq C_1(p, q), \]
(ii) there exists a constant $C_2(p)$ such that
\[ \bar{E} \left[ \int_0^T \|X(t)\|^2_V dt \right] \leq C_2(p). \]

The rest of the paper is devoted to the proof of Theorem 3.7. In the next section we consider equations approximating equation (3.17).

### 4 Approximate SPDEs

We will use the Friedrichs method based on the Fourier transform techniques, see [4, Section 4, p.174]. This method has been also used, e.g., in [17], [26], [24] and [6].

#### 4.1 The subspaces $\mathbb{H}_n$ and the operators $P_n$

Let
\[ \bar{B}_n := \{ \xi \in \mathbb{R}^3 : |\xi| \leq n \}, \quad n \in \mathbb{N} \]
and let
\[ H_n := \{ v \in H : \text{supp} \, v \subseteq \bar{B}_n \}. \]

In the subspace $H_n$ we consider the norm inherited from the space $H$ defined in Section 2.1. For each $n \in \mathbb{N}$ let us define a map $\pi_n$ by
\[ \pi_n v := \mathcal{F}^{-1}(\mathbb{1}_{\bar{B}_n} \hat{v}), \quad v \in H, \]
where $\mathcal{F}^{-1}$ denotes denotes the inverse of the Fourier transform, see Appendix A. Using Remark A.1, we infer that the map $\pi_n : H \to H_n$ is the orthogonal projection onto $H_n$.

Let
\[ \bar{B}_n := \bar{B}_n \times \bar{B}_n \]
and
\[ H_n := H_n \times H_n. \]

In the subspace $\mathbb{H}_n$ we consider the norm inherited from the space $\mathbb{H} = H \times H$ defined by (2.18). Let us define the operator
\[ P_n := \pi_n \times \pi_n : \mathbb{H} \to \mathbb{H}_n. \]
Explicitly, for $\Phi = (u, B) \in \mathbb{H}$

$$P_n(u, B) = (\pi_n u, \pi_n B) = (F^{-1}(1_{\mathcal{B}_n} \hat{u}), F^{-1}(1_{\mathcal{B}_n} \hat{B})).$$

Since the map $\pi_n : H \to H_n$ is the orthogonal projection onto $H_n$, we infer that

$$P_n : \mathbb{H} \to \mathbb{H}_n$$

is the orthogonal projection onto $\mathbb{H}_n$.

Using Lemma A.3 and Corollary A.5 we infer that the subspaces $\mathbb{H}_n$ are embedded in the spaces $\mathbb{V}_{m_1, m_2}$ for $m_1, m_2 \geq 0$, defined by (2.24) with the equivalence of norms. We have the following results.

**Lemma 4.1.** Let $n \in \mathbb{N}$ and $m_1, m_2 \geq 0$. Then

$$\mathbb{H}_n \hookrightarrow \mathbb{V}_{m_1, m_2},$$

and for all $u \in \mathbb{H}_n$:

$$\|u\|_{m_1, m_2}^2 \leq (1 + n^2)^m \|u\|_{\mathbb{H}_n}^2,$$

where $m = \max\{m_1, m_2\}$.

(Note that the norm of the embedding $\mathbb{H}_n \hookrightarrow \mathbb{V}_{m_1, m_2}$ depends on $n$ and $m_1, m_2$.)

**Corollary 4.2.** On the subspace $\mathbb{H}_n$ the norm $\|\cdot\|_{\mathbb{H}_n}$ and the norms $\|\cdot\|_{m_1, m_2}$, for $m_1, m_2 \geq 0$, inherited from the spaces $\mathbb{V}_{m_1, m_2}$ are equivalent (with appropriate constants depending on $m_1, m_2$ and $n$).

Now, we will concentrate on some properties of the operators $P_n$ in the spaces $\mathbb{V}_{m_1, m_2}$ defined by (2.24). Directly from Lemma A.2 we obtain the following lemma.

**Lemma 4.3.** Let us fix $m_1, m_2 \geq 0$. Then for all $n \in \mathbb{N}$:

$$P_n : \mathbb{V}_{m_1, m_2} \to \mathbb{V}_{m_1, m_2}$$

is well defined linear and bounded. Moreover, for every $u \in \mathbb{V}_{m_1, m_2}$:

$$\lim_{n \to \infty} \|P_n u - u\|_{m_1, m_2} = 0.$$

From Lemma 4.3 we obtain the following corollary which will be frequently used in the proofs.

**Corollary 4.4.** In particular,

(i) $P_n \in \mathcal{L}(\mathbb{V}, \mathbb{V})$, and for all $u \in \mathbb{V}$

$$\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}} = 0,$$

(ii) For every $m \geq 0$, $P_n \in \mathcal{L}(\mathbb{V}_m, \mathbb{V}_m)$ and for all $u \in \mathbb{V}_m$

$$\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}_m} = 0,$$

(iii) For every $m \geq 1$, $P_n \in \mathcal{L}(\mathbb{V}_m, \mathbb{V})$ and for all $u \in \mathbb{V}_m$

$$\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}} = 0.$$
The spaces $\mathbb{V}$ and $\mathbb{V}_m$ is defined by (2.18) and (2.26), respectively.

From Lemma A.3 we obtain the following

**Lemma 4.5.** If $k_1, k_2 > 0$, then

$$P_n : \mathbb{V}_{m_1+k_1,m_2+k_2} \to \mathbb{V}_{m_1,m_2}$$

is well defined and bounded. Moreover, for every $u \in \mathbb{V}_{m_1+k_1,m_2+k_2}$:

$$\|P_n u - u\|_{m_1,m_2}^2 \leq \frac{c}{(1+n^2)^{k}} \|u\|_{m_1+k_1,m_2+k_2}^2,$$

where $k = \max\{k_1, k_2\}$ and $c$ is some constant, and hence

$$\lim_{n \to \infty} |P_n - I|_{\mathcal{L}(\mathbb{V}_{m_1+k_1,m_2+k_2}, \mathbb{V}_{m_1,m_2})} = 0,$$

i.e. the sequence $(P_n)$ is convergent to the identity operator in the sense of operator-norm.

### 4.2 Approximating SPDEs

Construction of a solution of problem (3.17) is based on appropriate approximation in the space $\mathbb{H}_n$ defined by (4.1).

**Definition 4.6.** Let $X_0 \in \mathbb{H}_n$. By an approximation of equation (3.17) we mean an $\mathbb{H}_n$-valued continuous, $\mathbb{F}$-adapted process $\{X_n(t)\}_{t \in [0,T]}$ such that for all $t \in [0,T]$ and $\phi \in \mathbb{H}_n$ the following identity holds $\mathbb{P}$-a.s.

$$\begin{align*}
(X_n(t)|\phi)_{\mathbb{H}} + &\int_0^t \langle AX_n(s)|\phi \rangle \ ds + \int_0^t \langle B(X_n(s))|\phi \rangle \ ds + \int_0^t \langle H(X_n(s))|\phi \rangle \ ds \\
= &\ (X_0|\phi)_{\mathbb{H}} + \int_0^t \langle f(s)|\phi \rangle \ ds + \int_0^t \langle G(s,X_n(s)) \ dW(s)|\phi \rangle. \quad (4.3)
\end{align*}$$

Note that the test functions $\phi$ in (4.3) belong the subspace $\mathbb{H}_n$. Using the Riesz representation theorem for continuous linear functionals on $\mathbb{H}_n$ and the fact that in the space $\mathbb{H}_n$ all norms inherited from the space $H^{m_1} \times H^{m_2} := H_1^{m_1}(\mathbb{R}^3, \mathbb{R}^3) \times H_2^{m_2}(\mathbb{R}^3, \mathbb{R}^3)$, where $m_1, m_2 \geq 0$, are equivalent (see Corollary 4.2), identity (4.3) can be written as a stochastic equation in $\mathbb{H}_n$.

Since $P_n : \mathbb{H} \to \mathbb{H}_n$ is the $(\cdot|\cdot)_{\mathbb{H}}$-orthogonal projection, in particular we have

$$(v|P_n \varphi)_{\mathbb{H}} = (P_n v|\varphi)_{\mathbb{H}} \quad \text{for all} \quad \varphi \in \mathbb{H}_n.$$

Thus for a fixed $v \in \mathbb{H}$ the Riesz representation of the functional

$$\mathbb{H}_n \ni \varphi \mapsto (v|\varphi)_{\mathbb{H}} \in \mathbb{R}$$

is equal $P_n v$.

**Remark 4.7.** Let $n \in \mathbb{N}$ be fixed.
(i) For every \( v \in \mathbb{V} \) there exist \( A_n(v), \widetilde{B}_n(v), \widetilde{H}_n(v) \in \mathbb{H}_n \) such that for every \( \varphi \in \mathbb{H}_n \)

\[
\psi'(Av|\varphi)_V = (A_n(v)|\varphi)_{\mathbb{H}}, \tag{4.4}
\]

\[
\psi'(Bv|\varphi)_V = (\widetilde{B}_n(v)|\varphi)_{\mathbb{H}}, \tag{4.5}
\]

\[
\langle \widetilde{H}(v)|\varphi \rangle_{V_{1,2}} = (\widetilde{H}_n(v)|\varphi)_{\mathbb{H}_n}. \tag{4.6}
\]

Moreover, the map \( \mathbb{V} \ni v \mapsto A_n(v) \in \mathbb{H}_n \) is linear.

(ii) For every \( f \in \mathbb{V}' \) there exists \( f_n(v) \in \mathbb{H}_n \) such that

\[
\psi'(f|\varphi)_V = (f_n|\varphi)_{\mathbb{H}} \quad \text{for all} \quad \varphi \in \mathbb{H}_n. \tag{4.7}
\]

**Proof. Ad (i).** Let us fix \( v \in \mathbb{V} \). Consider the following functional

\[
\mathbb{H}_n \ni \varphi \mapsto \psi'(Av|\varphi)_V \in \mathbb{R}. \tag{4.8}
\]

Since, by Remark 2.2 and Corollary 4.2, for all \( \varphi \in \mathbb{H}_n \)

\[
|\psi'(Av|\varphi)_V| \leq |Av|_{\mathbb{V}'} ||\varphi||_V \leq c_n ||v||_V ||\varphi||_{\mathbb{H}}
\]

for some constant \( c_n \) independent of \( \varphi \), the map defined by (4.8) is a continuous linear functional on \( \mathbb{H}_n \). By the Riesz representation theorem there exists \( A_n(v) \in \mathbb{H}_n \) such that

\[
\psi'(Av|\varphi)_V = (A_n(v)|\varphi)_{\mathbb{H}}, \quad \varphi \in \mathbb{H}_n,
\]

i.e. (4.4) holds. Since \( A \) is linear, the map \( \mathbb{V} \ni v \mapsto A_n(v) \in \mathbb{H}_n \) is linear as well.

To prove (4.5), let us consider the following functional

\[
\mathbb{H}_n \ni \varphi \mapsto \psi'(Bv|\varphi)_V \in \mathbb{R}. \tag{4.9}
\]

Since, by Lemma 2.2(i) and Corollary 4.2, for all \( \varphi \in \mathbb{H}_n \)

\[
|\psi'(Bv|\varphi)_V| \leq |Bv|_{\mathbb{V}'} ||\varphi||_V \leq c_n ||v||_V^2 ||\varphi||_{\mathbb{H}}
\]

for some constant \( c_n \) independent of \( \varphi \), the map defined by (4.9) is a continuous linear functional on \( \mathbb{H}_n \). By the Riesz representation theorem there exists \( \widetilde{B}_n(v) \in \mathbb{H}_n \) such that

\[
\psi'(Bv|\varphi)_V = (\widetilde{B}_n(v)|\varphi)_{\mathbb{H}}, \quad \varphi \in \mathbb{H}_n,
\]

i.e., (4.5) holds.

Similarly, let us consider the following functional

\[
\mathbb{H}_n \ni \varphi \mapsto \psi'(\widetilde{H}(v)|\varphi)_V \in \mathbb{R}. \tag{4.10}
\]

Since, by Lemma 2.9(i) and Corollary 4.2, for all \( \varphi \in \mathbb{H}_n \)

\[
|\psi'(\widetilde{H}(v)|\varphi)_{V_{1,2}}| \leq |\widetilde{H}(v)|_{V_{1,2}} ||\varphi||_{V_{1,2}} \leq c_n ||v||_V^2 ||\varphi||_{\mathbb{H}}
\]

for some constant \( c_n \) independent of \( \varphi \), the map defined by (4.10) is a continuous linear functional on \( \mathbb{H}_n \). By the Riesz representation theorem there exists \( \widetilde{H}_n(v) \in \mathbb{H}_n \) such that

\[
\langle \widetilde{H}(v)|\varphi \rangle_{V_{1,2}} = (\widetilde{H}_n(v)|\varphi)_{\mathbb{H}_n}, \quad \varphi \in \mathbb{H}_n,
\]

(4.10) holds.
i.e., (4.6) holds.

Ad. (ii). For a fixed $f \in \mathcal{V}'$ let us consider the functional

$$
\mathbb{H}_n \ni \varphi \mapsto \mathcal{V}(f| \varphi)_\mathcal{V} \in \mathbb{R}.
$$

(4.11)

Note that that the map defined by (4.11) is a restriction of functional $f$ to the subspace $\mathbb{H}_n$. Since for all $\varphi \in \mathbb{H}_n$

$$
|\mathcal{V}(f| \varphi)_\mathcal{V}| \leq |f|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} \leq c_n |f|_{\mathcal{V}} |\varphi|_{\mathbb{H}}
$$

for some constant $c_n$ independent of $\varphi$, the map defined by (4.11) is a continuous linear functional on $\mathbb{H}_n$. Let $f_n(v) \in \mathbb{H}_n$ denote its Riesz representation in $\mathbb{H}_n$. Then we have

$$
\mathcal{V}(f| \varphi)_\mathcal{V} = (f_n| \varphi)_\mathbb{H}, \quad \varphi \in \mathbb{H}_n,
$$

i.e., (4.7) holds. \qed

Remark 4.8. By Remark 4.7, integral identity (4.3) can be written equivalently as the following stochastic equation in $\mathbb{H}_n$

$$
X_n(t) + \int_0^t \left[ A_n(X_n(s)) + \tilde{B}_n(X_n(s)) + \tilde{H}_n(X_n(s)) \right] ds
= P_nX_0 + \int_0^t f_n(s) ds + \int_0^t G_n(s, X_n(s)) dW(s), \quad t \in [0, T],
$$

where $G_n$ is a map defined by

$$
G_n : [0, T] \times \mathbb{H} \ni (s, X) \mapsto [X \ni y \mapsto P_n(G(s, X)(y))] \in \mathcal{T}_2(\mathbb{K}, \mathbb{R}).
$$

(4.12)

Let us note that, by Lemmas 2.7 and 2.9 for every $n \in \mathbb{N}$ the map

$$
\mathbb{H}_n \ni v \mapsto \tilde{B}_n(v) + \tilde{H}_n(v) \in \mathbb{H}_n
$$

is locally Lipschitz continuous. (The Lipschitz constants depend also on $n$.)

Let us consider the $n$-th approximating stochastic partial differential equation in the space $\mathbb{H}_n$ defined by (4.11), i.e.,

$$
\begin{cases}
\quad dX_n(t) + \left[ A_n(X_n(t)) + \tilde{B}_n(X_n(t)) + \tilde{H}_n(X_n(t)) \right] dt \\
\quad = f_n(t) dt + G_n(t, X_n(t)) dW(t), \quad t \in [0, T],
\end{cases}
$$

(4.13)

$X_n(0) = P_nX_0.$

Proposition 4.9. For each $n \in \mathbb{N}$, there exists a unique global solution $(X_n(t))_{t \in [0, T]}$ of problem (4.13).

Proof. The proof is a direct application of Theorem 3.1 from [3]. In fact, by Lemmas 2.7 and 2.9 for every $n \in \mathbb{N}$ the nonlinear terms $\tilde{B}_n$ and $\tilde{H}_n$ are locally Lipschitz. Thus the exists a local solution $X_n$ of problem (4.13) defined on some random interval $[0, \tau_n)$. Since moreover, by (2.28) and (2.31) for all $X_n \in \mathbb{H}_n$

$$
\langle \tilde{B}_n(X_n) + \tilde{H}_n(X_n), X_n \rangle = 0
$$

using the Itô formula for the function $F(x) = |x|^2_{\mathbb{H}}$, $x \in \mathbb{H}$, we can prove that the processes $X_n$, $n \in \mathbb{N}$, satisfy on the intervals $[0, \tau_n)$, the same uniform estimates as in the subsequent Lemma 4.10 for $q = 2$. Local existence together with uniform estimates guaranty that the solutions $X_n$ are global, i.e. defined on the interval $[0, T)$. \qed
4.3 A priori estimates

In the following lemma we will prove some a priori estimates of the solutions of the approximating equation (4.13).

**Lemma 4.10.** Let Assumptions [3.2] and [3.3] be satisfied. In particular, we assume that \( p \) satisfies (3.5), i.e.
\[
 p \in [2, 2 + \gamma),
\]
where \( \gamma \) is given by (3.6). Then the solutions \((X_n)_{n \in \mathbb{N}}\) of equations (4.13) satisfy the following uniform estimates:

(i) For every \( q \in [2, p] \) there exist positive constants \( C_1(p, q) \) and \( C_2(p, q) \) such that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0, T]} |X_n(s)|^q_{\mathbb{H}} \right] \leq C_1(p, q) \tag{4.14}
\]
and
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |X_n(s)|^{q-2}_{\mathbb{H}} \|X_n(s)\|^2 \, ds \right] \leq C_2(p, q). \tag{4.15}
\]

(ii) In particular, there exists a positive constant \( C_2(p) \) such that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \|X_n(s)\|^2_{\mathbb{V}} \, ds \right] \leq C_2(p). \tag{4.16}
\]

**Proof of Lemma 4.10.** For any \( R > 0 \) we define the stopping time
\[
\tau_R^n := \inf \{ t \in [0, T] : |X_n(t)|_{\mathbb{H}} \geq R \}.
\]
Let us fix \( q \in [2, p] \), where \( p \) satisfies condition (3.5). We apply the Itô formula to the function \( F \) defined by
\[
F : \mathbb{H} \ni x \mapsto |x|^q_{\mathbb{H}} \in \mathbb{R}.
\]
In the sequel we will often omit the subscript \( \mathbb{H} \) and write \(| \cdot | := | \cdot |_{\mathbb{H}}\). Note that
\[
F'(x)(h) = d_x F(h) = q \cdot |x|^{q-2} \cdot \langle x|h \rangle_{\mathbb{H}}, \quad h \in \mathbb{H},
\]
\[
\|F''(x)\| = \|d_x^2 F\| \leq q(q-1) \cdot |x|^{q-2}, \quad x \in \mathbb{H}.
\]
By the Itô formula
\[
|X_n(t \wedge \tau_R^n)|^q - |P_n X_0|^q = \int_0^{t \wedge \tau_R^n} \left\{ q |X_n(s)|^{q-2} \langle X_n(s) | A_n X_n(s) - \tilde{B}_n(X_n(s)) - \tilde{H}_n(X_n(s)) + f_n(s) \rangle + \frac{q(q-1)}{2} |X_n(s)|^{q-2} \|G_n(s, X_n(s))\|_{\mathbb{H}}^2 \right\} \, ds
\]
\[
+ q \int_0^{t \wedge \tau_R^n} |X_n(s)|^{q-2} \langle X_n(s) | G_n(s, X_n(s)) \rangle \, dW(s).
\]
Since \( X_n \) is an \( \mathbb{H}_n \)-valued process,

- by (4.4) and (2.22) we have \( (A_n X_n | X_n) = \langle AX_n | X_n \rangle = \|X_n\|^2 \),
by (4.5) and (2.28), \((\tilde{B}_n X_n | X_n) = (\tilde{B} X_n | X_n) = 0\),
by (4.6) and (2.31), \((\tilde{H}_n X_n | X_n) = (\tilde{H} X_n | X_n) = 0\),
and by (4.7), \((f_n(s) | X_n) = (f(s) | X_n)\),

we infer that
\[
\left| X_n(t \wedge \tau^n_R) \right|^q + q \int_0^{t \wedge \tau^n_R} \left| X_n(s) \right|^{q-2} \left| X_n(s) \right|^2 ds
= \left| P_n X_0 \right|^q + \int_0^{t \wedge \tau^n_R} \left\{ q \left| X_n(s) \right|^{q-2} \langle X_n(s) | f(s) \rangle \right\} ds
+ \frac{q(q-1)}{2} \left| X_n(s) \right|^{q-2} \left\| G_n(s, X_n(s)) \right\|^2_{T_2(K, \mathbb{H})} ds
+ q \int_0^{t \wedge \tau^n_R} \left| X_n(s) \right|^{q-2} \langle X_n(s) | G_n(s, X_n(s)) dW(s) \rangle.
\]

Using estimates (4.18) and (4.19) in (4.17), we obtain
\[
\left| X_n(t \wedge \tau^n_R) \right|^q + q \int_0^{t \wedge \tau_R^n} \left| X_n(s) \right|^{q-2} \left| X_n(s) \right|^2 ds
\leq \left| P_n X_0 \right|^q + \int_0^{t \wedge \tau^n_R} \left\{ q \left( \varepsilon_0 + \frac{1}{2} - \frac{1}{q} \right) \left| X_n(s) \right|^{q-2} \left\| X_n(s) \right\|_{T_2(K, \mathbb{H})}^{q-2} \langle X_n(s) | f(s) \rangle \right\} ds
+ \left( q - 1 \right) \theta t + \frac{q}{2} \int_0^{t \wedge \tau^n_R} \left| f(s) \right|^q_{\mathbb{H}} ds
+ q \int_0^{t \wedge \tau^n_R} \left| X_n(s) \right|^{q-2} \left\| X_n(s) \right\|_{T_2(K, \mathbb{H})}^{q-2} \left\| G_n(s, X_n(s)) \right\|^2_{T_2(K, \mathbb{H})} ds \cdot \left| X_n(s) \right|^{q-2} \langle X_n(s) | f(s) \rangle \right\} ds.
\]

Using estimates (4.18) and (4.19) in (4.17), we obtain
\[
\left| X_n(t \wedge \tau^n_R) \right|^q + q \int_0^{t \wedge \tau^n_R} \left| X_n(s) \right|^{q-2} \left| X_n(s) \right|^2 ds
\leq \left| P_n X_0 \right|^q + \int_0^{t \wedge \tau^n_R} \left\{ q \left( \varepsilon_0 + \frac{1}{2} - \frac{1}{q} \right) \left| X_n(s) \right|^{q-2} \left\| X_n(s) \right\|_{T_2(K, \mathbb{H})}^{q-2} \langle X_n(s) | f(s) \rangle \right\} ds
+ \left( q - 1 \right) \theta t + \frac{q}{2} \int_0^{t \wedge \tau^n_R} \left| f(s) \right|^q_{\mathbb{H}} ds
+ q \int_0^{t \wedge \tau^n_R} \left| X_n(s) \right|^{q-2} \left\| X_n(s) \right\|_{T_2(K, \mathbb{H})}^{q-2} \left\| G_n(s, X_n(s)) \right\|^2_{T_2(K, \mathbb{H})} ds \cdot \left| X_n(s) \right|^{q-2} \langle X_n(s) | f(s) \rangle \right\} ds.
\]

Let us choose \(\varepsilon_0 \in (0, 1)\) such that
\[
\delta = \delta(q, \eta) := q \left[ 1 - \varepsilon_0 - \frac{1}{2} (q - 1)(2 - \eta) \right] > 0,
\]
or equivalently, \( \varepsilon_0 < 1 \wedge \left[ 1 - \frac{1}{2}(q - 1)(2 - \eta) \right] \). Notice that under condition (3.5) such \( \varepsilon_0 \) exists. Denote also

\[
K_q(\lambda, \varrho) := q \left( \varepsilon_0 + \frac{1}{2} - \frac{1}{q} \right) + \frac{1}{2} \tilde{K}_q(\lambda, \varrho).
\]

Then, using the fact that \( \int_0^t |f(s)|^{p_v} ds \leq t^{1 - \frac{q}{p_v}} \cdot \int_0^t |f(s)|^p ds t^{\frac{q}{p}} \), we obtain

\[
|X_n(t \wedge \tau_R^n)|^q + \delta \int_0^{t \wedge \tau_R^n} |X_n(s)|^{q-2} \|X_n(s)\|^2 ds \\
\leq |X_0|^q + K_q(\lambda, \varrho) \int_0^{t \wedge \tau_R^n} |X_n(s)|^q ds + \varrho(q - 1)t + (2\varepsilon_0)^{-\eta} t^{1 - \frac{q}{p}} \left( \int_0^{t \wedge \tau_R^n} |f(s)|^{p_v} ds \right)^{\frac{q}{p}} \\
+ q \int_0^{t \wedge \tau_R^n} |X_n(s)|^{q-2} \langle X_n(s), G_n(s, X_n(s)) \rangle dW(s), \quad t \in [0, T].
\]

(4.21)

Since \( X_n \) is the solutions of the approximating equation (4.13), we infer that the process

\[
\mathcal{M}_n(t) := \int_0^{t \wedge \tau_R^n} |X_n(s)|^{q-2} \langle X_n(s), G_n(s, X_n(s)) \rangle dW(s), \quad t \in [0, T],
\]

is a square integrable martingale. Indeed, by (3.13) and the fact that \( P_n \) is the orthogonal projection in \( \mathbb{H} \) we infer that for every \( t \in [0, T] \),

\[
\int_0^{t \wedge \tau_R^n} \| |X_n(s)|^{q-2} \langle X_n(s), G_n(s, X_n(s)) \rangle \|_{\mathbb{H}}^2 ds \\
\leq \int_0^t \| |X_n(s)|^q \| G_n(s, X_n(s)) \|_{\mathbb{H}}^2 ds \\
\leq \int_0^t \| |X_n(s)|^q \left( (2 - \eta) \|X_n(t)\| + \lambda_0 |X_n(t)|^2 + \varrho \right) ds.
\]

Since \( X_n \) is a solution of the approximate equation (4.13) and, by Corollary 4.2 the norms \( \| \cdot \|_{\mathbb{H}} \) and \( \| \cdot \|_{\mathbb{V}} \) are equivalent on the subspace \( \mathbb{H}_n \), we infer that

\[
\mathbb{E} \left[ \int_0^{t \wedge \tau_R^n} \| |X_n(s)|^{q-2} \langle X_n(s), G_n(s, X_n(s)) \rangle \|_{\mathbb{H}}^2 ds \right] < \infty, \quad t \in [0, T].
\]

and thus we infer, as claimed, that the process \( \mathcal{M}_n \) is a square integrable martingale. Hence, \( \mathbb{E}[\mathcal{M}_n(t)] = 0 \).

By taking expectation in inequality (4.21) we infer that for all \( t \in [0, T] \):

\[
\mathbb{E} \left[ |X_n(t \wedge \tau_R^n)|^q \right] + \delta \mathbb{E} \left[ \int_0^{t \wedge \tau_R^n} |X_n(s)|^{q-2} \|X_n(s)\|^2 ds \right] \leq |X_0|^q \\
+ K_q(\lambda, \varrho) \int_0^{t \wedge \tau_R^n} \mathbb{E} \left[ |X_n(s)|^q \right] ds + \varrho(q - 1)t + (2\varepsilon_0)^{-\eta} t^{1 - \frac{q}{p}} \left( \int_0^{t \wedge \tau_R^n} |f(s)|^{p_v} ds \right)^{\frac{q}{p}}.
\]

(4.22)

In particular,

\[
\mathbb{E} \left[ |X_n(t \wedge \tau_R^n)|^q \right] \leq |X_0|^q + K_q(\lambda, \varrho) \int_0^{t \wedge \tau_R^n} \mathbb{E} \left[ |X_n(s)|^q \right] ds \\
+ \varrho(q - 1)t + (2\varepsilon_0)^{-\eta} T^{1 - \frac{q}{p}} \left( \int_0^{t \wedge \tau_R^n} |f(s)|^{p_v} ds \right)^{\frac{q}{p}}, \quad t \in [0, T].
\]
Using the Gronwall lemma and passing to the limit as $R \to \infty$, we infer that

$$
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[ |X_n(t)|^q \right] \leq \tilde{C}_1(p, q) 
$$

(4.23)

for some constant $\tilde{C}_1(p, q) = \check{C}_1(p, q, T, \eta, \lambda, \varrho, |X_0|, \|f\|_{L^p(0, T; V')} > 0$. Hence by the Fubini theorem

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |X_n(s)|^q \, ds \right] \leq T \tilde{C}_1(p, q).
$$

Using this bound in (4.22) we also obtain

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |X_n(s)|^{q-2} \|X_n(s)\|^2 \, ds \right] \leq C_2(p, q) 
$$

(4.24)

for a new constant $C_2(p, q)$ dependent also on $T, \eta, \lambda, \varrho, |X_0|$, and $\|f\|_{L^p(0, T; V')}$. This completes the proof of estimates (4.13). Putting $q := 2$, by (2.20) we infer that (4.16) holds.

Let us move to the proof of estimate (4.14). By the Burkholder-Davis-Gundy inequality, see eg. [14] or [29], and the Schwarz inequality, there exists a constant $c$ such that

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq T \land \tau^n_R} |X_n(s)|^{q-2} (X_n(s)|G_n(\sigma, X_n(\sigma)) \, dW(\sigma)) \right]
\leq c_q \cdot \mathbb{E} \left[ \left( \int_0^{T \land \tau^n_R} |X_n(\sigma)|^{2q-2} \cdot \|G_n(\sigma, X_n(\sigma))\|_{L^q(K_x, \sigma)}^{2} \, d\sigma \right)^{\frac{1}{2}} \right] 
\leq c_q \cdot \mathbb{E} \left[ \left( \int_0^{T \land \tau^n_R} |X_n(\sigma)|^{q-2} \cdot \|G(\sigma, X_n(\sigma))\|_{L^q(K_x, \sigma)}^{2} \, d\sigma \right)^{\frac{1}{2}} \right] 
\leq \mathbb{E} \left[ \sup_{0 \leq s \leq T \land \tau^n_R} |X_n(s)|^q + \frac{1}{2} c_q^2 \int_0^{T \land \tau^n_R} |X_n(\sigma)|^{q-2} \cdot \|G(\sigma, X_n(\sigma))\|_{L^q(K_x, \sigma)}^{2} \, d\sigma \right].
$$

Using (4.25) and (3.13) in (4.21), we obtain

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq T \land \tau^n_R} |X_n(s)|^q \right] \leq |X_0|^q + \left( K_q(\lambda, \varrho) + \frac{1}{2} c_q^2 \left( \lambda + \varrho \left( 1 - \frac{2}{q} \right) \right) \right) \mathbb{E} \left[ \int_0^{T \land \tau^n_R} |X_n(s)|^q \, ds \right] + \left( (q - 1) \varrho + \frac{2}{q} \right) \int_0^{T \land \tau^n_R} |X_n(s)|^{q-2} \cdot \|G(\sigma, X_n(\sigma))\|_{L^q(K_x, \sigma)}^{2} \, d\sigma \right].
$$

Hence by inequalities (4.23) and (4.24) we infer that there exists a constant $C_1(p, q) = C_1(p, q, T, \eta, \lambda, \varrho, |X_0|, \|f\|_{L^p(0, T; V')} > 0$ such that for every $n \in \mathbb{N}$

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq T \land \tau^n_R} |X_n(s)|^q \right] \leq C_1(p, q),
$$

Passing to the limit as $R \to \infty$, we infer that

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{0 \leq s \leq T \land \tau^n_R} |X_n(s)|^q \right] \leq C_1(p, q),
$$

This completes the proof of estimate (4.14) and of the lemma. \qed
4.4 Auxiliary remarks

Let us consider the sequence \((X_n)_{n \in \mathbb{N}}\) of solutions of equations (4.13). These solutions satisfy uniform estimates stated in Lemma 4.10. Since we consider the Hall-MHD equations on \(\mathbb{R}^3\), the continuous embedding
\[ V \hookrightarrow H \]
is not compact. However, using Lemma 2.5 from [19] (see [7, Lemma C.1]) we can find a separable Hilbert space \(U\) such that
\[ U \subset V_* \subset V_{1,2} \subset V \subset H, \]
where \(V_* = V_m\) for fixed \(m > \frac{5}{2}\), and the embedding
\[ U \hookrightarrow V_* \]
is compact, see Appendix [13]. Using this structure, where the space \(U\) is of crucial importance, we can prove appropriate tightness criterion, see Appendix [C] which we apply to prove the tightness of the sequence of laws of \((X_n)_{n \in \mathbb{N}}\) in Section 5.1. The subsequent reasoning is a preparation for Section 5.1.

Considering also the dual spaces, and identifying \(H\) with its dual \(H'\), we have the following system
\[ U \subset V_* \subset V_{1,2} \subset V \subset H \cong H' \rightarrow V' \rightarrow V'_* \rightarrow U'. \]

Let us recall that the map
\[ P_n : H \rightarrow H_n \]
defined by (4.2) is \((\cdot, \cdot)_H\)-orthogonal projection on \((H, (\cdot, \cdot)_H)\). Further properties of the map \(P_n\) are stated in Corollary [4.3].

**Remark 4.11.** From Corollary [4.3] it follows that we may consider the adjoint operators \(P'_n\) in appropriate spaces.

1. Since \(P_n \in \mathcal{L}(V, V)\), the adjoint operator \(P'_n \in \mathcal{L}(V', V')\) by definition satisfies
\[ \langle \xi | P_n \varphi \rangle_V = \langle P'_n \xi | \varphi \rangle_{V'}, \quad \text{for all} \quad \xi \in V', \quad \varphi \in V. \]

2. Since \(P_n \in \mathcal{L}(V_*, V)\), the adjoint operator \(P'_n \in \mathcal{L}(V'_*, V'_*)\) satisfies
\[ \langle \xi | P_n \varphi \rangle_V = \langle P'_n \xi | \varphi \rangle_{V_*'}, \quad \text{for all} \quad \xi \in V'_*, \quad \varphi \in V_* \]

3. Since \(P_n \in \mathcal{L}(V_*, V_*)\), the adjoint operator \(P'_n \in \mathcal{L}(V'_*, V'_*)\) satisfies
\[ \langle \xi | P_n \varphi \rangle_{V_*} = \langle P'_n \xi | \varphi \rangle_{V'_*}, \quad \text{for all} \quad \xi \in V'_*, \quad \varphi \in V_* \]

Using Definition [4.6] and Remark [4.11] we will show that the approximating equation can be rewritten as an equation in the space \(V'_*\), and by the injection \(V'_* \rightarrow U'\) - also as an equation in \(U'\).
Remark 4.12. (i) If the \( \mathbb{H}_n \)-valued process \( X_n \) satisfies identity (4.3), then in particular, for all \( t \in [0,T] \) and \( \varphi \in \mathbb{V}_* \) we have \( P_n \varphi \in \mathbb{H}_n \) and

\[
(X_n(t)|P_n \varphi)_{\mathbb{H}} + \int_0^t \nu^v(A X_n(s)|P_n \varphi) \nu \, ds + \int_0^t \nu^v(\bar{B}(X_n(s))|P_n \varphi) \nu \, ds \\
+ \int_0^t \nu^v(\bar{H}(X_n(s))|P_n \varphi) \nu \, ds \\
= (X_0|P_n \varphi)_\mathbb{H} + \int_0^t \nu^v(f(s)|P_n \varphi) \nu \, ds + \left( \int_0^t G(s, X_n(s)) \, dW(s)|P_n \varphi \right)_{\bar{\mathbb{H}}}. \tag{4.26}
\]

We used also the properties of the maps \( \bar{B} \) and \( \bar{H} \) stated in Lemmas 2.7 and 2.9 respectively. Since \( P_n : \mathbb{H} \to \mathbb{H}_n \) is an \((\cdot | \cdot)_{\mathbb{H}}\)-orthogonal projection,

\[
(X_n(t)|P_n \varphi)_\mathbb{H} = (P_n X_n(t)|\varphi)_\mathbb{H} = (X_n(t)|\varphi)_{\mathbb{H}}.
\]

(ii) Using the operators \( P'_n \) introduced in Remark 4.11, (4.26) can be written in the form

\[
(X_n(t)|\varphi)_{\mathbb{H}} + \int_0^t \nu^v(P'_n A X_n(s)|\varphi) \nu \, ds + \int_0^t \nu^v(P'_n \bar{B}(X_n(s))|\varphi) \nu \, ds \\
+ \int_0^t \nu^v(P'_n \bar{H}(X_n(s))|\varphi) \nu \, ds \\
= (P_n X_0|\varphi)_{\mathbb{H}} + \int_0^t \nu^v(P'_n f(s)|\varphi) \nu \, ds + \left( \int_0^t P_n G(s, X_n(s)) \, dW(s)|\varphi \right)_{\mathbb{H}} \tag{4.27}, \quad t \in [0,T].
\]

where

\[
P_n \circ G := G_n, \tag{4.28}
\]

and \( G_n \) is the map defined by (4.12).

(iii) Let for \( \varphi \in \mathbb{V}_* \) the map \( \varphi^{**} \) be defined by

\[
\varphi^{**}(\xi) := \xi(\varphi), \quad \xi \in \mathbb{V}'_*.
\]

Note that since

\[
|(X_n(t)|\varphi)_{\mathbb{H}}| \leq |X_n(t)|_{\mathbb{H}} \cdot |\varphi|_{\mathbb{H}} \leq |X_n(t)|_{\mathbb{H}} \cdot \|\varphi\|_{\mathbb{V}_*},
\]

we infer that

\[
\mathbb{V}_* \ni \varphi \mapsto (X_n(t)|\varphi)_{\mathbb{H}} \in \mathbb{R} \in \mathbb{V}'_*.
\]

In particular,

\[
(X_n(t)|\varphi)_{\mathbb{H}} = \varphi^{**}((X_n(t)|\cdot)_{\mathbb{H}})
\]

Thus (4.27) can be rewritten in the form

\[
\varphi^{**}((X_n(t)|\cdot)_{\mathbb{H}}) + \varphi^{**} \left( \int_0^t \left[ P'_n A u_n(s) + P'_n \bar{B}(X_n(s)) + P'_n \bar{H}(X_n(s)) \right] \, ds \right) \\
= \varphi^{**}((P_n X_0|\cdot)_{\mathbb{H}}) + \varphi^{**} \left( \int_0^t P'_n f(s) \, ds \right) + \varphi^{**} \left( \int_0^t P_n G(s, X_n(s)) \, dW(s) \right).
\]
(iv) Using the identification $\mathbb{H} \cong \mathbb{H}'$ and the fact that $\mathbb{H}' \hookrightarrow \mathbb{V}' \hookrightarrow \mathbb{V}'_*$, we identify $X_n(t)$ with the functional induced by $X_n(t)$ on the space $\mathbb{V}_*$. Since the family $\{\varphi^*; \varphi \in \mathbb{V}_*\}$ separates elements of $\mathbb{V}'_*$, we infer that $X_n(t)$ satisfies the following equation

$$X_n(t) + \int_0^t \left[ P_n^* AX_n(s) + P_n^* B(X_n(s)) + P_n^* \mathcal{H}(X_n(s)) \right] ds$$

$$= P_n X_0 + \int_0^t P_n^* f(s) ds + \int_0^t P_n^* G(s, X_n(s)) dW(s), \quad t \in [0, T],$$

where $P_n G$ is defined by (1.28).

5 Existence of a martingale solution. Proof of Theorem 3.7

Having constructed the sequence $(X_n)$ of solutions of equations (4.11), the idea of further steps of the proof is similar to [7]. First we will prove that the sequence of laws of $X_n$, $n \in \mathbb{N}$, form a tight sequence of probability measures on appropriate functional space. Using the Jakubowski version of the Skorokhod theorem we construct new stochastic bases an new processes. The last step is passing to the limit.

5.1 Tightness

Let us consider the sequence $(X_n)_{n \in \mathbb{N}}$ of solutions of equations (4.11). Using the tightness criterion stated in Corollary C.3 in Appendix C, we will prove that the sequence of laws of $X_n$ is tight in the space $Z$ defined by (C.1), i.e.

$$Z := L^2_w(0, T; \mathbb{V}) \cap L^2(0, T; \mathbb{H}^*_\text{loc}) \cap C([0, T]; \mathbb{H}_w) \cap C([0, T]; \mathbb{V}'),$$

equipped with the Borel $\sigma$-field $\sigma(T)$, see Definition C.1.

Lemma 5.1. The set of probability measures $\{\text{Law}(X_n), n \in \mathbb{N}\}$ is tight on the space $(Z, \sigma(T))$.

Proof. We apply Corollary C.3. Let us note that due to estimates (4.14) and (4.16), conditions (C.2) and (C.3) of Corollary C.3 are satisfied. Thus, it is sufficient to prove that the sequence $(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition [A]. By Lemma C.5 it is sufficient to proof the condition [A']. Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times taking values in $[0, T]$. By Remark 4.12 (iv), we have

$$X_n(t) = P_n X_0 - \int_0^t P_n^* AX_n(s) ds - \int_0^t P_n^* B(X_n(s)) ds - \int_0^t P_n^* \mathcal{H}(X_n(s)) ds$$

$$+ \int_0^t P_n^* f_n(s) ds + \int_0^t P_n^* G(s, X_n(s)) dW(s)$$

$$=: J^1_n + J^2_n(t) + J^3_n(t) + J^4_n(t) + J^5_n(t) + J^6_n(t), \quad t \in [0, T].$$

Let us choose and $\theta > 0$. It is sufficient to show that each sequence $J^i_n$ of processes, $i = 1, \ldots, 6$, satisfies the sufficient condition [A'] from Lemma C.5. Since the term $J^1_n$ is constant in time, it satisfies this condition. In fact, we will check that the terms $J^2_n, J^3_n, J^6_n$ satisfy condition [A'] from Lemma C.5 in the space $E = \mathbb{V}'$ and the terms $J^3_n, J^4_n$ satisfy this
condition in \( E = \mathcal{V}_s \). Since the embeddings \( \mathcal{V}' \subset \mathcal{V} \) and \( \mathcal{V}'_s \subset \mathcal{U}' \) are continuous, we infer that \([A^*]\) holds in the space \( E = \mathcal{U}' \), as well.

**Ad \( J_n^2 \).** By Remark 2.6, the linear operator \( A : \mathcal{V} \to \mathcal{V}' \) is bounded, and by Corollary 4.4 \( \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}, \mathcal{V})} < \infty \). Using the Hölder inequality and (4.10), we obtain

\[
E \left[ |J_n^2(\tau_n + \theta) - J_n^2(\tau_n)|_{\mathcal{V}'} \right] \leq |P_n'|_{L(\mathcal{V}', \mathcal{V})} \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} |A X_n(s)|_{\mathcal{V}} ds \right| \mathcal{V}' \right] \\
\leq |P_n|_{L(\mathcal{V}, \mathcal{V})} \theta^\frac{1}{2} \left( \mathbb{E} \left[ \left. \int_0^T \|X_n(s)\|^2 ds \right| \mathcal{V}' \right] \right)^\frac{1}{2} \leq c_2 \cdot \theta^\frac{1}{2},
\]

where \( c_2 = C^\frac{1}{2}_2(p) \cdot \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}, \mathcal{V})} < \infty \).

**Ad \( J_n^3 \).** By (2.29) in Lemma 2.7, \( \tilde{B} : \mathbb{H} \times \mathbb{H} \to \mathcal{V}'_s \) is bilinear and continuous (and hence bounded so that the norm \( \|\tilde{B}\| \) of \( \tilde{B} : \mathbb{H} \times \mathbb{H} \to \mathcal{V}'_s \) is finite), and by Corollary 4.4 \( \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} < \infty \). Then by (4.11) we have the following estimates

\[
E \left[ |J_n^3(\tau_n + \theta) - J_n^3(\tau_n)|_{\mathcal{V}_s} \right] = |P_n'|_{L(\mathcal{V}_s, \mathcal{V}_s)} \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} \tilde{B}(X_n(r)) \right| \mathcal{V}_s \right] \\
\leq |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} \|\tilde{B}(X_n(r))\|_{\mathcal{V}_s} dr \right| \mathcal{V}_s \right] \\
\leq |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} \|\tilde{B}\| \cdot \mathbb{E} \left[ \sup_{r \in [0, T]} |X_n(r)\|_{\mathcal{H}}^2 \right] \leq c_3 \theta,
\]

where \( c_3 = \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} \|\tilde{B}\| C_1(p, 2) < \infty \).

**Ad \( J_n^4 \).** By (2.32) in Lemma 2.7, \( \tilde{H} : \mathbb{H} \times \mathcal{V} \to \mathcal{V}'_s \) is bilinear and continuous (and hence bounded so that the norm \( \|\tilde{H}\| \) of \( \tilde{H} : \mathbb{H} \times \mathcal{V} \to \mathcal{V}'_s \) is finite) and by Corollary 4.4 \( \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} < \infty \). Then by (4.11) and (4.16) we have the following estimates

\[
E \left[ |J_n^4(\tau_n + \theta) - J_n^4(\tau_n)|_{\mathcal{V}_s} \right] \leq |P_n'|_{L(\mathcal{V}_s, \mathcal{V}_s)} \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} \overline{\tilde{H}}(X_n(r)) \right| \mathcal{V}_s \right] \\
\leq |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} \|\overline{\tilde{H}}\| \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} |X_n(r)|_{\mathcal{H}} dr \right| \mathcal{V}_s \right] \\
\leq |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} \|\overline{\tilde{H}}\| \left( \mathbb{E} \left[ \sup_{r \in [\tau_n, \tau_n + \theta]} |X_n(r)|_{\mathcal{H}}^2 \right] \right)^\frac{1}{2} \left( \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} \|X_n(r)\|^2 dr \right| \mathcal{V}_s \right] \right)^\frac{1}{2} \theta^\frac{1}{2} \leq c_4 \theta^\frac{1}{2},
\]

where \( c_4 = \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}_s, \mathcal{V}_s)} \|\overline{\tilde{H}}\| [C_1(p, 2)]^\frac{1}{2} [C_2(p)]^\frac{1}{2} < \infty \).

**Ad \( J_n^5 \).** Since \( f \in L^p(0, T; \mathcal{V}) \) and by Corollary 4.4 \( \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}, \mathcal{V})} < \infty \), using the Hölder inequality, we have

\[
E \left[ |J_n^5(\tau_n + \theta) - J_n^5(\tau_n)|_{\mathcal{V}} \right] \leq |P_n'|_{L(\mathcal{V}', \mathcal{V})} \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} f(s) ds \right| \mathcal{V} \right] \\
\leq |P_n|_{L(\mathcal{V}, \mathcal{V})} \theta^\frac{p-1}{2} \left( \mathbb{E} \left[ \left. \int_0^T |f(s)|_{\mathcal{V}}^p ds \right| \mathcal{V} \right] \right)^\frac{1}{p} \leq c_5 \cdot \theta^\frac{p-1}{2},
\]

where \( c_5 := \sup_{n \in \mathbb{N}} |P_n|_{L(\mathcal{V}, \mathcal{V})} \|f\|_{L^p(0, T; \mathcal{V})} < \infty \).
\textbf{Ad} $J_n^6$. Since $P_n \sigma G = G_n$ (see (4.12) and (4.28)), by (3.15) and inequality (4.14), we obtain the following inequalities

\[
E \left[ |J_n^6(\tau_n + \theta) - J_n^6(\tau_n)|^2 \right] = E \left[ \left| \int_{\tau_n}^{\tau_n + \theta} G_n(s, X_n(s)) \, dW(s) \right|^2 \right] \\
= E \left[ \sup_{\psi \in \mathcal{V}, \|\psi\| \leq 1} \int_{\tau_n}^{\tau_n + \theta} \langle G_n(s, X_n(s)) \cdot \psi \rangle_{\mathcal{V}} \, ds \right]^2 \\
= E \left[ \sup_{\psi \in \mathcal{V}, \|\psi\| \leq 1} \int_{\tau_n}^{\tau_n + \theta} \sup_{y \in \mathcal{K}, \|y\| \leq 1} \langle G_n(s, X_n(s)) \cdot \psi \rangle_{\mathcal{V}} \, dy \right]^2 \\
= E \left[ \sup_{\psi \in \mathcal{V}, \|\psi\| \leq 1} \int_{\tau_n}^{\tau_n + \theta} \sup_{y \in \mathcal{K}, \|y\| \leq 1} \langle P_n(G(s, X_n(s)) \cdot \psi) \rangle_{\mathcal{H}} \, dy \right]^2 \\
\leq E \left[ \int_{\tau_n}^{\tau_n + \theta} \sup_{\psi \in \mathcal{V}, \|\psi\| \leq 1} \sup_{y \in \mathcal{K}, \|y\| \leq 1} \langle g(s, X_n(s)) \cdot \psi \rangle_{\mathcal{V}} \, dy \right]^2 \\
\leq C \cdot \sup_{\psi \in \mathcal{V}, \|\psi\| \leq 1} \|P_n \psi\|_{\mathcal{V}} \cdot E \left[ \int_{\tau_n}^{\tau_n + \theta} (1 + |X_n(s)|^2) \, ds \right] \\
\leq C \left| P_n \right|_{L(V, \mathcal{V})} \left\{ 1 + E \left[ \sup_{s \in [0, T]} |X_n(s)|^2 \right] \right\} \theta \leq c_6 \cdot \theta.
\]

Here $c_6 := C \sup_{n \in \mathbb{N}} \left| P_n \right|_{L(V, \mathcal{V})} \{1 + C_1(2)\} < \infty$. Thus the proof of Lemma 5.1 is complete. \qed

\section*{5.2 Application of the Skorokhod theorem}

We will use the Jakubowski’s generalization of the Skorokhod theorem, see Theorem C.6 in Appendix C. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of the solutions of the approximate equations (4.13). By Lemma 5.1 the set of laws \{Law$(X_n, n \in \mathbb{N})$\} is tight on the space $(\mathcal{Z}, \sigma(\mathcal{T}))$, where $\sigma(\mathcal{T})$ denotes the topological $\sigma$-field. By Theorem C.6 there exists a subsequence $(n_k)$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, on this space $\mathcal{Z}$-valued random variables $\tilde{X}$, $\tilde{X}_{n_k}$, $k \in \mathbb{N}$ such that

the variables $X_{n_k}$ and $\tilde{X}_{n_k}$ have the same laws on the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{Z})$ (5.1) and

$\tilde{X}_{n_k}$ converges to $\tilde{X}$ in $\mathcal{Z}$, $\mathbb{P}$-a.s., (5.2)

where $\mathcal{Z}$ defined by (C.1). Hence, in particular,

$\tilde{X} \in L^2(0, T; \mathcal{V}) \cap C([0, T]; \mathbb{H}_w) \cap C([0, T]; \mathcal{U})$. (5.3)

We will denote the subsequence $(\tilde{X}_{n_k})$ again by $(\tilde{X}_n)$. Define a corresponding sequence of filtrations by $\mathbb{F}_n = (\mathbb{F}_{n,t})_{t \geq 0}$, where $\mathbb{F}_{n,t} = \sigma\{X_n(s), s \leq t\}$, $t \in [0, T]$. Using Lemma 4.10 we infer that the processes $\tilde{X}_n$, $n \in \mathbb{N}$, satisfy the following inequalities: for every $q \in [2, p]$

\[
\sup_{n \in \mathbb{N}} E \left[ \sup_{s \in [0, T]} |\tilde{X}_n(s)|^q \right] \leq C_1(p, q)
\]

(5.4)
and
\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \| \tilde{X}_n(s) \|_V^2 \, ds \right] \leq C_2(p). \] (5.5)

Let us emphasize that the constants \( C_1(p, q) \) and \( C_2(p) \) are the same as in Lemma 4.10. In particular, by (5.4) with \( q := p \)
\[ \mathbb{E} \left[ \sup_{s \in [0, T]} |\tilde{X}_n(s)|_V^p \right] \leq C_1(p), \] (5.6)
where \( C_1(p) := C_1(p, p) \). Using inequality (5.6) we choose a subsequence, still denoted by \((\tilde{X}_n)\), convergent weak star in the space \( L^p(\tilde{\Omega}; L^\infty(0, T; \mathbb{H})) \) and infer that and that the limit process \( \tilde{X} \) satisfies (5.6), as well. That is,
\[ \mathbb{E} \left[ \sup_{s \in [0, T]} |\tilde{X}(s)|_V^p \right] \leq C_1(p). \] (5.7)
This means that the process \( \tilde{X} \) satisfies inequality (3.20) for \( q := p \). Now, let us fix \( q \in [1, p) \). Since \( \sup_{t \in [0, T]} |\tilde{X}(t)|_H^q \leq \left( \sup_{t \in [0, T]} |\tilde{X}(t)|_V^p \right)^{q/p} \), by the Hölder inequality we obtain
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{X}(t)|_H^q \right] \leq \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |\tilde{X}(t)|_V^p \right)^{q/p} \right] \]
\[ \leq \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{X}(t)|_V^p \right] \right)^{q/p} \leq (C_1(p))^{q/p}, \] (5.8)
which means that process \( \tilde{X} \) satisfies inequality (3.20) with the constant \( C_1(p, q) := (C_1(p))^{q/p} \).

By inequality (5.5) we infer that the sequence \((\tilde{X}_n)\) contains further subsequence, denoted again by \((\tilde{X}_n)\), convergent weakly in the space \( L^2([0, T] \times \tilde{\Omega}; \mathbb{V}) \) to \( \tilde{X} \). Moreover, it is clear that
\[ \mathbb{E} \left[ \int_0^T \| \tilde{X}(s) \|_V^2 \, ds \right] \leq C_2(p), \] (5.9)
which means that the process \( \tilde{X} \) satisfies (5.21).

### 5.3 Continuation of the proof of Theorem 3.7

Passing to the limit

We use the argumentation similar to [7], which is closely related to [13] and [15] Section 8]. For each \( n \geq 1 \), let us consider a process \( \tilde{M}_n \) with trajectories in \( C([0, T]; \mathbb{H}_n) \) (in particular in \( C([0, T]; \mathbb{H}) \)) defined by
\[ \tilde{M}_n(t) = \tilde{X}_n(t) - \tilde{X}_n(0) + \int_0^t \left[ A_n(\tilde{X}_n(s)) + \tilde{B}_n(\tilde{X}_n(s)) + \tilde{H}_n(\tilde{X}_n(s)) - f_n(s) \right] \, ds, \quad t \in [0, T], \] (5.10)
where \( A_n, \tilde{B}_n, \tilde{H}_n \) and \( f_n \) are defined by (4.4), (4.5), (4.6) and (4.7), respectively.

**Lemma 5.2.** \( \tilde{M}_n \) is a square integrable martingale with respect to the filtration \( \tilde{\mathbb{F}}_n = (\tilde{\mathcal{F}}_{n,t}) \), where \( \tilde{\mathcal{F}}_{n,t} = \sigma\{\tilde{X}_n(s), \, s \leq t\} \), with the quadratic variation
\[ \langle \tilde{M}_n \rangle_t = \int_0^t \| G_n(\tilde{X}_n(s)) \|_{T_2(\mathbb{X}, \mathbb{H})}^2 \, ds, \quad t \in [0, T]. \] (5.11)
Proof. Indeed, since the process
\[ M_n(t) = X_n(t) - X_n(0) + \int_0^t [A_n(X_n(s)) + B_n(X_n(s)) + \tilde{H}_n(X_n(s)) - f_n(s)] \, ds, \quad t \in [0, T], \]
is a zero-mean \( \mathbb{H}_n \)-valued square integrable martingale with the quadratic variation
\[ \langle M_n \rangle_t = \int_0^t \| G_n(X_n(s)) \|_{T_2(\mathbb{K}, \mathbb{H})}^2 \, ds, \quad t \in [0, T], \]
and the processes \( \tilde{X}_n \) and \( X_n \) have the same laws, thus for all \( s, t \in [0, T], s \leq t \), all functions \( h \) bounded continuous on \( \mathcal{C}([0, s]; \mathbb{H}_n) \), and all \( \psi, \zeta \in \mathbb{H}_n \), we have
\[ \mathbb{E}\left[ \langle M_n(t) - \tilde{M}_n(s) \rangle | \psi \rangle_{\mathbb{H}} h(\tilde{X}_n|0, s) \right] = 0 \]
and
\[ \mathbb{E}\left[ \langle \tilde{M}_n(t) | \psi \rangle_{\mathbb{H}} \langle \tilde{M}_n(t) | \zeta \rangle_{\mathbb{H}} - \langle \tilde{M}_n(s) | \psi \rangle_{\mathbb{H}} \langle \tilde{M}_n(s) | \zeta \rangle_{\mathbb{H}} \right. \]
\[ - \left. \int_s^t (G_n(\tilde{X}_n(\sigma)) \psi | G_n(\tilde{X}_n(\sigma)) \zeta \rangle_{\mathbb{H}} d\sigma \right] h(\tilde{X}_n|0, s) = 0. \]
This concludes the proof of the lemma.

Let \( \tilde{M} \) be an \( \mathbb{U}' \)-valued process defined by
\[ \tilde{M}(t) = \tilde{X}(t) - \tilde{X}(0) + \int_0^t [A \tilde{X}(s) + \tilde{B}(\tilde{X}(s)) + \tilde{H}(\tilde{X}(s)) - f(s)] \, ds, \quad t \in [0, T]. \] (5.12)

**Lemma 5.3.** The process \( (\tilde{M}(t))_{t \in [0, T]} \) has \( \tilde{P} \)-a.s. trajectories in \( \mathcal{C}([0, T]; \mathbb{U}') \).

**Proof.** Since by \((5.3)\), \( \tilde{X} \in \mathcal{C}([0, T]; \mathbb{U}') \), it is sufficient to show that the remaining terms on the r.h.s of \((5.12)\) are \( \tilde{P} \)-a.s. in \( \mathcal{C}([0, T]; \mathbb{U}') \). In fact, the remaining terms are more regular; they belong to \( \mathcal{C}([0, T]; \mathbb{V}') \) or \( \mathcal{C}([0, T]; \mathbb{V}'_1, 2) \), which follows from the following inequalities. By Remark 2.6 the Hölder (Cauchy-Schwarz) inequality and \((5.9)\) we have
\[ \mathbb{E}\left[ |A(\tilde{X}(s))|_{\mathbb{V}'} \right] \leq T^{\frac{1}{2}} \left( \mathbb{E}\left[ \int_0^T \| \tilde{X}(s) \|_{\mathbb{V}}^2 \, ds \right] \right)^{\frac{1}{2}} < \infty. \]
By Lemma 2.7 (i), \((2.20)\), \((5.8)\) and \((5.9)\) we have
\[ \mathbb{E}\left[ \int_0^T |\tilde{B}(\tilde{X}(s))|_{\mathbb{V}'} \, ds \right] \leq c_B \mathbb{E}\left[ \int_0^T \| \tilde{X}(s) \|_{\mathbb{V}}^2 \, ds \right] < \infty. \]
By Lemma 2.9 (i), \((2.20)\), \((5.8)\) and \((5.9)\) we have
\[ \mathbb{E}\left[ \int_0^T |\tilde{H}(\tilde{X}(s))|_{\mathbb{V}'} \, ds \right] \leq c_H \mathbb{E}\left[ \int_0^T \| \tilde{X}(s) \|_{\mathbb{V}}^2 \, ds \right] < \infty. \]
Finally, by condition \((H.2)\) in Assumption \((3.2)\) \( \int_0^T |f(s)|_{\mathbb{V}'} \, ds < \infty \). The proof of the lemma is thus complete.\( \square \)
Lemma 5.4. For all $s, t \in [0, T]$ such that $s \leq t$ and all $\varphi \in \mathcal{V}_*$

(a) $\lim_{n \to \infty} (\tilde{X}_n(t)|P_n \varphi)_{\mathbb{H}} = (\tilde{X}(t)|\varphi)_{\mathbb{H}}$, $\bar{P}$-a.s.,

(b) $\lim_{n \to \infty} \int_s^t \langle A\tilde{X}_n(\sigma)|P_n \varphi \rangle \, d\sigma = \int_s^t \langle A\tilde{X}(\sigma)|\varphi \rangle \, d\sigma$, $\bar{P}$-a.s.,

(c) $\lim_{n \to \infty} \int_s^t \langle \tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi \rangle \, d\sigma = \int_s^t \langle \tilde{B}(\tilde{X}(\sigma))|\varphi \rangle \, d\sigma$, $\bar{P}$-a.s.,

(d) $\lim_{n \to \infty} \int_s^t \langle \tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi \rangle = \int_s^t \langle \tilde{H}(\tilde{X}(\sigma))|\varphi \rangle \, d\sigma$, $\bar{P}$-a.s.

(e) $\lim_{n \to \infty} \int_s^t \langle f(\sigma)|P_n \varphi \rangle \, d\sigma = \int_s^t \langle f(\sigma)|\varphi \rangle \, d\sigma$, $\bar{P}$-a.s.,

where $\langle \cdot, \cdot \rangle$ denotes appropriate duality pairing.

Proof. Let us fix $s, t \in [0, T]$, $s \leq t$ and $\varphi \in \mathcal{V}_*$.

Ad. (a). Since

$$
(\tilde{X}_n(t)|P_n \varphi)_{\mathbb{H}} = (P_n \tilde{X}_n(t)|\varphi)_{\mathbb{H}} = (\tilde{X}_n(t)|\varphi)_{\mathbb{H}},
$$

and by $\mathcal{B}_2$, $\tilde{X}_n \to \tilde{X}$ in $\mathcal{C}([0, T]; \mathbb{H}_w)$, $\bar{P}$-a.s., we infer that $(\tilde{X}_n(\cdot)|\varphi)_{\mathbb{H}} \to (\tilde{X}(\cdot)|\varphi)_{\mathbb{H}}$ in $\mathcal{C}([0, T]; \mathbb{H})$, $\bar{P}$-a.s. Hence, in particular,

$$
\lim_{n \to \infty} (\tilde{X}_n(t)|\varphi)_{\mathbb{H}} = (\tilde{X}(t)|\varphi)_{\mathbb{H}}, \quad \bar{P} - a.s.,
$$

which completes the proof of (a).

Ad. (b). By $\mathcal{B}_2$, we have

$$
\int_s^t \langle A\tilde{X}_n(\sigma)|P_n \varphi \rangle \, d\sigma = \int_s^t \langle (\tilde{X}_n(\sigma)|P_n \varphi) \rangle \, d\sigma.
$$

By $\mathcal{B}_2$, $\tilde{X}_n \to \tilde{X}$ in $L^2_{w}(0, T; \mathcal{V})$, $\bar{P}$-a.s. Moreover, since $\varphi \in \mathcal{V}_*$, we infer that $P_n \varphi \to \varphi$ in $\mathcal{V}$, (see Corollary 1.14). Thus

$$
\lim_{n \to \infty} \int_s^t \langle (\tilde{X}_n(\sigma)|P_n \varphi) \rangle \, d\sigma = \int_s^t \langle \tilde{X}(\sigma)|\varphi \rangle \, d\sigma = \int_s^t \langle A\tilde{X}(\sigma)|\varphi \rangle \, d\sigma, \quad \bar{P} - a.s.
$$

This completes the proof of (b).

Ad. (c). Let us move to the $\tilde{B}$-term. Let us fix $\varphi \in \mathcal{V}_*$. For every $\sigma \in [0, T]$ we have

$$
\langle \tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi \rangle - \langle \tilde{B}(\tilde{X}(\sigma))|\varphi \rangle = \langle \tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi \rangle + \langle \tilde{B}(\tilde{X}_n(\sigma)) - \tilde{B}(\tilde{X}(\sigma))|\varphi \rangle.
$$

Since by $\mathcal{B}_2$, the sequence $(\tilde{X}_n)$ is $\bar{P}$-a.s., weakly convergent to $\tilde{X}$ in $L^2(0, T; \mathcal{V})$, $(\tilde{X}_n)$ is bounded in $L^2(0, T; \mathcal{V})$, and in particular, by $\mathcal{B}_2$, $(\tilde{X}_n)$ is bounded in $L^2(0, T; \mathbb{H})$, as well. Moreover, by $\mathcal{B}_2$, $\tilde{X}_n \to \tilde{X}$ in $L^2(0, T; \mathbb{H}_{w,loc})$, $\bar{P}$-a.s. By Corollary 2.8 we infer that $\bar{P}$-a.s.

$$
\lim_{n \to \infty} \int_s^t \langle \tilde{B}(\tilde{X}_n(\sigma)) - \tilde{B}(\tilde{X}(\sigma))|\varphi \rangle \, d\sigma = 0.
$$

Using $\mathcal{B}_2$ we have

$$
\int_s^t \langle \tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi \rangle_{\mathcal{V}_*} \, d\sigma \leq \int_s^t \langle \tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi \rangle_{\mathcal{V}_*} \, d\sigma
$$

$$
\leq \|\tilde{B}\| \cdot \int_s^t |\tilde{X}_n(\sigma)|^2_{\mathbb{H}} \, d\sigma \cdot \|P_n \varphi - \varphi\|_{\mathcal{V}_*} \leq \|\tilde{B}\| \cdot \|\tilde{X}_n\|^2_{L^2(0, T; \mathbb{H})} \cdot \|P_n \varphi - \varphi\|_{\mathcal{V}_*}, \quad \bar{P} - a.s.
$$

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Since \( \varphi \in V_* \), then, by Corollary 4.4, \( P_n \varphi \to \varphi \) in \( V_* \), and by the boundedness of the sequence \( (\tilde{X}_n) \) in \( L^2(0, T; \mathbb{H}) \), we infer that \( \tilde{P} \)-a.s.

\[
\lim_{n \to \infty} \int_s^t \nu_\ast (\tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi) \nu_\ast \, d\sigma = 0.
\]

Hence

\[
\lim_{n \to \infty} \int_s^t (\tilde{B}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi) \, d\sigma = 0, \quad \tilde{P}\text{-a.s.}
\]

The proof of (c) is thus complete.

**Ad. (d).** Let us move to the \( \tilde{H} \)-term. Let us fix \( \varphi \in V_* \). For every \( \sigma \in [0, T] \) we have

\[
\langle \tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi - \langle \tilde{H}(\tilde{X}(\sigma))|\varphi \rangle \\
= \langle \tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi \rangle + \langle \tilde{H}(\tilde{X}_n(\sigma)) - \tilde{H}(\tilde{X}(\sigma))|\varphi \rangle.
\]

By [5.2], the sequence \( (\tilde{X}_n) \) is \( \tilde{P}\)-a.s. convergent to \( \tilde{X} \) in \( L^2(0, T; V) \cap L^2(0, T; \mathbb{H}_{loc}) \). In particular, \( (\tilde{X}_n) \) is bounded in \( L^2(0, T; \mathbb{H}) \). By Corollary 2.10 we infer that \( \tilde{P}\)-a.s.

\[
\lim_{n \to \infty} \int_s^t \langle \tilde{H}(\tilde{X}_n(\sigma)) - \tilde{H}(\tilde{X}(\sigma))|\varphi \rangle \, d\sigma = 0.
\]

Using [2.32], we have

\[
\left| \int_s^t \nu_\ast (\tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi) \nu_\ast \, d\sigma \right| \leq \int_s^t \nu_\ast (\tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi) \nu_\ast \, d\sigma
\]

\[
\leq \|	ilde{H}\| \cdot \int_s^t |\tilde{X}_n(\sigma)|_{H} |\tilde{X}_n(\sigma)|_{V} \, d\sigma \leq \|P_n \varphi - \varphi\|_{V_*}, \quad \tilde{P}\text{-a.s.}
\]

Since \( \varphi \in V_* \) then, by Corollary 4.4, \( P_n \varphi \to \varphi \) in \( V_* \). By the boundedness of the sequence \( (\tilde{X}_n) \) in \( L^2(0, T; V) \) (and hence in \( L^2(0, T; \mathbb{H}) \), as well), we infer that \( \tilde{P}\)-a.s.

\[
\lim_{n \to \infty} \int_s^t \nu_\ast (\tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi - \varphi) \nu_\ast \, d\sigma = 0.
\]

Hence

\[
\lim_{n \to \infty} \int_s^t (\langle \tilde{H}(\tilde{X}_n(\sigma))|P_n \varphi - \langle \tilde{H}(\tilde{X}(\sigma))|\varphi \rangle) \, d\sigma = 0, \quad \tilde{P}\text{-a.s.}
\]

The proof of (d) is thus complete.

**Ad. (e).** This assertion follows from the facts that \( f \in L^p(0, T; V') \) and for \( \varphi \in V_* \), \( P_n \varphi \to \varphi \) in \( V \).

The proof of the lemma is thus complete. \( \square \)

**Lemma 5.5.** For all \( s, t \in [0, T] \) such that \( s \leq t \), all \( h \in C_b(\mathbb{C}([0, s]; \mathbb{U}'); \mathbb{R}) \) and all \( \psi \in V_* \):

\[
\lim_{n \to \infty} \tilde{E}[(\tilde{M}_n(t) - \tilde{M}_n(s))|\psi] h(\tilde{X}_n[0, s]) = \tilde{E}[(\tilde{M}(t) - \tilde{M}(s))|\psi] h(\tilde{X}[0, s]),
\]

where \( \langle \cdot \rangle \) denotes appropriate duality pairing.
Here, $C_b(C([0,s];\mathbb{U}');\mathbb{R})$ is the space of $\mathbb{R}$-valued bounded and continuous functions defined on $C([0,s];\mathbb{U}')$.

**Proof.** Let us fix $s,t\in[0,T]$, $s\leq t$ and $\psi\in\mathcal{V}_s$. By Remark 4.7 with $\varphi:=P_n\psi$, we have

\[
\langle \tilde{M}_n(t) - \tilde{M}_n(s) | \psi \rangle = (\tilde{X}_n(t) | P_n \psi)_H - (\tilde{X}_n(s) | P_n \psi)_H + \int_s^t \langle A\tilde{X}_n(\sigma) | P_n \psi \rangle \, d\sigma
\]

\[
+ \int_s^t \langle \tilde{B}(\tilde{X}_n(\sigma)) | P_n \psi \rangle \, d\sigma + \int_s^t \langle \tilde{H}(\tilde{X}_n(\sigma)) | P_n \psi \rangle \nu_* \, d\sigma + \int_s^t \langle f(\sigma) | P_n \psi \rangle \, d\sigma.
\]

By Lemma 5.4 we infer that

\[
\lim_{n\to\infty} \langle \tilde{M}_n(t) - \tilde{M}_n(s) | \psi \rangle = \langle \tilde{M}(t) - \tilde{M}(s) | \psi \rangle, \quad \bar{\mathbb{P}}\text{-a.s.} \quad (5.13)
\]

Let us notice that $\sup_{n\in\mathbb{N}} \|h(\tilde{X}_n([0,s]))\|_{\mathbb{L}_T} < \infty$. Moreover, since by (5.2) $\tilde{X}_n \to \tilde{X}$ in $C([0,T];\mathbb{U}')$, we infer that $\lim_{n\to\infty} h(\tilde{X}_n([0,s])) = h(\tilde{X}_{|[0,s]})$, $\bar{\mathbb{P}}$-a.s.

Let us denote

\[
\xi_n(\omega) := \langle (\tilde{M}_n(t,\omega) | \psi) - (\tilde{M}_n(s,\omega) | \psi) \rangle h(\tilde{X}_{n|[0,s]}), \quad \omega \in \tilde{\Omega}.
\]

We will prove that the functions $\{\xi_n\}_{n\in\mathbb{N}}$ are uniformly integrable. We claim that

\[
\sup_{n\geq 1} \mathbb{E}[|\xi_n|^2] < \infty. \quad (5.14)
\]

Indeed, by the continuity of the embedding $\mathcal{V}_s \hookrightarrow \mathbb{H}$ and the Schwarz inequality, for each $n \in \mathbb{N}$ we have

\[
\mathbb{E}[|\xi_n|^2] \leq 2c\|h\|^2_{\mathbb{L}_\infty} \mathbb{E}[\|\tilde{M}_n(t)\|^2_{\mathbb{H}}] + \mathbb{E}[\|\tilde{M}_n(s)\|^2_{\mathbb{H}}]. \quad (5.15)
\]

Since $\tilde{M}_n$ is a continuous martingale with quadratic variation given in (5.11), by the Burkholder-Davis-Gundy inequality we obtain

\[
\mathbb{E}\left[\sup_{t\in[0,T]} |\tilde{M}_n(t)\|^2_{\mathbb{H}} \right] \leq c\mathbb{E}\left[\left(\int_0^T \|G_n(\tilde{X}_n(\sigma))\|_{\mathbb{T}_2(\mathbb{K},\mathbb{H})}^2 \, d\sigma\right)^{1/2}\right]. \quad (5.16)
\]

Since $P_n : \mathbb{H} \to \mathbb{H}_n$ is an orthogonal projection, by (3.13), we have

\[
\|G_n(\tilde{X}_n(\sigma))\|_{\mathbb{T}_2(\mathbb{K},\mathbb{H})}^2 \leq (2 - \eta)\|\tilde{X}_n(\sigma)\|^2 + \lambda\|\tilde{X}_n(\sigma)\|^2_{\mathbb{L}} + \varrho, \quad \sigma \in [0,T]. \quad (5.17)
\]

By (5.16), (5.17), (5.9) and (5.8) (with $q:=2$), we infer that

\[
\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} |\tilde{M}_n(t)\|^2_{\mathbb{H}} \right] < \infty. \quad (5.18)
\]

Then by (5.15) and (5.18) we see that (5.14) holds. Since the sequence $(\xi_n)_{n\in\mathbb{N}}$ is uniformly integrable and by (5.13) it is $\bar{\mathbb{P}}$-a.s. pointwise convergent, application of the Vitali theorem completes the proof of the lemma.

\[
\square
\]

**Lemma 5.6.** For all $s,t\in[0,T]$ such that $s\leq t$, all $h \in C_b(C([0,s];\mathbb{U}');\mathbb{R})$ and all $\psi, \zeta \in \mathcal{V}_s$:

\[
\lim_{n\to\infty} \mathbb{E}\left[\langle \tilde{M}_n(t) | \psi \rangle \langle \tilde{M}_n(t) | \zeta \rangle - \langle \tilde{M}_n(s) | \psi \rangle \langle \tilde{M}_n(s) | \zeta \rangle \right] h(\tilde{X}_{n|[0,s]})
\]

\[
= \mathbb{E}\left[\langle \tilde{M}(t) | \psi \rangle \langle \tilde{M}(t) | \zeta \rangle - \langle \tilde{M}(s) | \psi \rangle \langle \tilde{M}(s) | \zeta \rangle \right] h(\tilde{X}_{|[0,s]}),
\]

where $\langle \cdot | \cdot \rangle$ denotes appropriate duality pairing.
Proof. Let us fix $s, t \in [0, T]$ such that $s \leq t$ and $\psi, \zeta \in \mathbb{V}_s$ and let us denote

$$
\xi_n(\omega) := \{ \langle \tilde{M}_n(t, \omega) \rangle \langle \tilde{M}_n(t, \omega) \rangle \} - \{ \langle \tilde{M}(s, \omega) \rangle \langle \tilde{M}(s, \omega) \rangle \} h(\tilde{X}_n_{[0, s]}(\omega)),
$$

$$
\xi(\omega) := \{ \langle M(t, \omega) \rangle \langle M(t, \omega) \rangle \} - \{ \langle M(s, \omega) \rangle \langle M(s, \omega) \rangle \} h(\tilde{X}_{[0, s]}(\omega)), \quad \omega \in \tilde{\Omega}.
$$

Let us notice that $\sup_{n \in \mathbb{N}} \| h(\tilde{X}_n_{[0, s]}) \|_{L^\infty} < \infty$. Moreover, since $\tilde{X}_n \rightarrow \tilde{X}$ in $\mathcal{C}([0, T]; \mathbb{U'})$, we infer that $\lim_{n \rightarrow \infty} h(\tilde{X}_n_{[0, s]}) = h(\tilde{X}_{[0, s]})$, $\mathbb{P}$-a.s. From this and Lemma 5.4, we infer that $\lim_{n \rightarrow \infty} \xi_n(\omega) = \xi(\omega)$ for $\mathbb{P}$-almost all $\omega \in \tilde{\Omega}$.

**Uniform integrability.** We will prove that the functions $\{ \xi_n \}_{n \in \mathbb{N}}$ are uniformly integrable. To this end, it is sufficient to show that for some $r > 1$,

$$
\sup_{n \geq 1} \mathbb{E}[|\xi_n|^r] < \infty. \tag{5.19}
$$

In fact, we will show that condition (5.19) holds for any $1 < r \leq \frac{2}{3}$.

Indeed, using the Hölder inequality we have for each $n \geq 1$

$$
\mathbb{E}[|\xi_n|^r] \leq \frac{1}{2} \| h \|_{L^\infty}^r \left\{ \mathbb{E}[|\langle \tilde{M}_n(t) \rangle \cdot \langle \tilde{M}_n(t) \rangle |^{2r}] + \mathbb{E}[|\langle \tilde{M}_n(s) \rangle \cdot \langle \tilde{M}_n(s) \rangle |^{2r}] + \mathbb{E}[|\langle \tilde{M}_n(s) \rangle \cdot \langle \tilde{M}_n(s) \rangle |^{2r}] \right\} \tag{5.20}
$$

Note that

$$
\langle \tilde{M}_n(t) \rangle = \langle \tilde{M}_n(t) \rangle_{\mathbb{H}}.
$$

By Lemma 5.2 $(\tilde{M}_n)_{t \in [0, T]}$ is an $\mathbb{H}$-valued continuous square integrable martingale with the quadratic variation given by (5.11). Thus $(\langle \tilde{M}_n(t) \rangle_{\mathbb{H}})_{t \in [0, T]}$ is a $\mathbb{R}$-valued square integrable martingale with the quadratic variation

$$
\langle \langle \tilde{M}_n \rangle_{\mathbb{H}} \rangle_t = \int_0^t \| (G(s, \tilde{X}_n(s)) \rangle_{\mathbb{H}} \|_{L^2(K, \mathbb{R})}^2 ds
$$

and hence

$$
\| (G(s, \tilde{X}_n(s)) \rangle_{\mathbb{H}} \|_{L^2(K, \mathbb{R})}^2 = \sup_{y \in \mathbb{K}, \| y \|_{\mathbb{K}} \leq 1} | (G(s, \tilde{X}_n(s)) \rangle_{\mathbb{V}} | \leq C(1 + |\tilde{X}_n(\cdot)\rangle_{\mathbb{H}}) \cdot \| P_n \psi \|_{\mathbb{V}}^2
$$

for some constant positive $\tilde{C}(\psi)$.

By the Burkholder-Davis-Gundy inequality and the Hölder inequality

$$
\mathbb{E}_t \left[ \sup_{t \leq \sigma} |\langle \tilde{M}_n(\sigma) \rangle \cdot \langle \tilde{M}_n(\sigma) \rangle |^{2r} \right] \leq \left( \mathbb{E}_t \left[ \int_0^T \| (G(s, \tilde{X}_n(s)) \rangle_{\mathbb{H}} \|_{L^2(K, \mathbb{R})}^2 ds \right] \right)^r
$$

$$
\leq \tilde{C}_r(\psi) \cdot \left( \mathbb{E}_t \left[ \int_0^T (1 + |\tilde{X}_n(s)^2|_{\mathbb{H}}) ds \right] \right)^r \leq 2^{r-1} \tilde{C}_r(\psi) \cdot T^r \left( 1 + \mathbb{E}_t \left[ \sup_{s \in [0, T]} |\tilde{X}_n(s)|^{2r} \right] \right).
$$

(5.21)
Using (5.21) and (5.18) in (5.20) we infer that for any \( r \in (1, \frac{p}{2}) \)
\[
\tilde{E}[|\xi_n|^r] \leq 2^r \tilde{C}^r(\psi) \cdot T^r \|h\|_{L^\infty} \left(1 + C_2(2r, p)\right) < \infty.
\]
Thus condition (5.19) holds.

By the Vitali theorem
\[
\lim_{n \to \infty} \tilde{E}[\xi_n] = \tilde{E}[\xi].
\]
The proof of the lemma is thus complete. \(\square\)

For a fixed \( \psi \in \mathbb{H} \) let us define the following map
\[
\psi^{**}: \mathcal{T}_2(\mathbb{K}, \mathbb{H}) \ni f \mapsto [\mathbb{K} \ni y \mapsto (f(y)|\psi)|_H] \in \mathcal{T}_2(\mathbb{K}, \mathbb{R}).
\]

**Lemma 5.7.** (Convergence of quadratic variations). For all \( s, t \in [0, T] \) such that \( s \leq t \), all \( h \in \mathcal{C}(\mathcal{C}([0, s]; \mathbb{U}'), \mathbb{R}) \) and \( \psi, \zeta \in \mathbb{V} \), we have
\[
\lim_{n \to \infty} \tilde{E} \left[ \left( \int_s^t \left\langle \psi^{**}[G_n(\sigma, \tilde{X}_n(\sigma))]|\zeta^{**}[G_n(\sigma, \tilde{X}_n(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} \right) d\sigma \right] \cdot h(\tilde{X}_{[0, s]})
\]
\[
= \tilde{E} \left[ \left( \int_s^t \left\langle \psi^{**}[G(\sigma, \check{X}(\sigma))]|\zeta^{**}[G(\sigma, \check{X}(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} \right) d\sigma \right] \cdot h(\check{X}_{[0, s]}).
\]

**Proof.** Let us fix \( s, t \in [0, T], s \leq t \) and \( \psi, \zeta \in \mathbb{V} \). Note that \( \sup_{n \in \mathbb{N}} \|h(\tilde{X}_{[0, s]}(\omega))\|_{L^\infty} < \infty \).
Moreover, since \( \tilde{X}_n \to \check{X} \) in \( \mathcal{C}([0, T]; \mathbb{U}') \), we infer that \( \lim_{n \to \infty} h(\tilde{X}_{[0, s]}(\omega)) = h(\check{X}_{[0, s]}(\omega)), \tilde{P}\)-a.s.
Let us denote
\[
\xi_n(\omega) := \int_s^t \left\langle \psi^{**}[G_n(\sigma, \tilde{X}_n(\sigma))]|\zeta^{**}[G_n(\sigma, \tilde{X}_n(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} d\sigma,
\]
\[
\xi(\omega) := \int_s^t \left\langle \psi^{**}[G(\sigma, \check{X}(\sigma))]|\zeta^{**}[G(\sigma, \check{X}(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} d\sigma, \quad \omega \in \tilde{\Omega}.
\]
It is sufficient to prove that
\[
\lim_{n \to \infty} \xi_n(\omega) = \xi(\omega)
\]
for \( \tilde{P}\)-almost all \( \omega \in \tilde{\Omega} \), and that the sequence \( (\xi_n) \) is uniformly integrable.

By Remark 3.4(iii), for every \( y \in \mathbb{K} \) we have
\[
(\psi^{**}[G(\sigma, \check{X}(\sigma))](y) = (G(\sigma, \check{X}(\sigma))|\psi)|_H = \psi^* \langle g(\sigma, \check{X}(\sigma))|\psi\rangle_Y = \langle (\check{g}_\psi(\check{X}))(\sigma)\rangle(y),
\]
where \( \check{g}_\psi \) is a map defined by (3.10), and
\[
(\psi^{**}[G_n(\sigma, \tilde{X}_n(\sigma))](y) = (G_n(\sigma, \tilde{X}_n(\sigma))|\psi)|_H = (P_n G(\sigma, \tilde{X}_n(\sigma))|\psi)|_H
\]
\[
= (G(\sigma, \tilde{X}_n(\sigma))|P_n\psi)|_H = \psi^* \langle g(\sigma, \tilde{X}_n(\sigma))|P_n\psi\rangle_Y.
\]
Pointwise convergence (for $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$). Let us consider

$$
\xi_n - \xi
= \int_s^t \left\langle \psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))] \psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} d\sigma
+ \int_s^t \left\langle \psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))] \psi^*[G(\sigma, \tilde{X}(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} d\sigma
+ \int_s^t \left\langle \psi^*[G(\sigma, \tilde{X}(\sigma))] \psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} d\sigma
=: I_1^n + I_2^n + I_3^n.
$$

We will analyze separately the terms $I_1^n, I_2^n, I_3^n$.

Let us begin with $I_1^n$. Note that by the Schwarz inequality

$$
|I_1^n(\sigma)| = \left| \left\langle \psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))] \psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))] \right\rangle_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} \right|
\leq \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))]\|_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} \cdot \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))]\|_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})}
\leq \frac{1}{2} \left\{ \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))]\|^2_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} + \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))]\|^2_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} \right\} =: \frac{1}{2} \{ |I_{11}^n|^2(\sigma) + |I_{12}^n|^2(\sigma) \}.
$$

By Remark 3.4 (iii), we have

$$
|I_{11}^n(\sigma)| = \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma)) - G(\sigma, \tilde{X}(\sigma))]\|_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})}
\leq \sup_{y \in \mathbb{K}, |y|_{\mathbb{K}} \leq 1} \left| \psi^* (g(\sigma, \tilde{X}_n(\sigma))(y)) |P_n \psi \rangle \psi \psi' \rangle_{V} - \psi^* (g(\sigma, \tilde{X}(\sigma))(y)) |\psi \rangle \psi \psi' \rangle_{V} \right|
\leq \sup_{y \in \mathbb{K}, |y|_{\mathbb{K}} \leq 1} \left| \psi^* (g(\sigma, \tilde{X}_n(\sigma))(y)) |P_n \psi \psi' \rangle_{V} - |\psi \psi' \rangle_{V} \right|
\leq \sup_{y \in \mathbb{K}, |y|_{\mathbb{K}} \leq 1} \left| \psi^* (g(\sigma, \tilde{X}_n(\sigma))(y)) - g(\sigma, \tilde{X}(\sigma))(y) |\psi \rangle \psi \psi' \rangle_{V} \right|
\leq \left[ C(1 + |\tilde{X}_n(\sigma)|^2_{\mathbb{H}^2}) \right]^\frac{1}{2} \cdot \|P_n \psi - \psi\|_{V} + \| (\tilde{g}_\psi(\tilde{X}_n))(\sigma) - (\tilde{g}_\psi(\tilde{X}))(\sigma) \|_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})}.
$$

Thus

$$
|I_{11}^n(\sigma)| \leq \left[ C(1 + |\tilde{X}_n(\sigma)|^2_{\mathbb{H}^2}) \cdot \|P_n \psi - \psi\|_{V} + \| (\tilde{g}_\psi(\tilde{X}_n))(\sigma) - (\tilde{g}_\psi(\tilde{X}))(\sigma) \|_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})} \right] \cdot \frac{1}{2}
\leq \frac{1}{2} \{ |I_{11}^n|^2(\sigma) + |I_{12}^n|^2(\sigma) \}.
$$

We have

$$
|I_1^n| \leq \int_s^t |I_{11}^n(\sigma)| d\sigma \leq \frac{1}{2} \int_s^t (|I_{11}^n(\sigma)|^2 + |I_{12}^n(\sigma)|^2) d\sigma.
$$

By (5.22),

$$
\int_s^t |I_{11}^n(\sigma)|^2 d\sigma \leq \frac{1}{2} \left\{ C \left( \int_0^T (1 + |\tilde{X}_n(\sigma)|^2_{\mathbb{H}^2}) d\sigma \right) \cdot \|P_n \psi - \psi\|_{V}^2
+ \left( \int_0^T \left\| (\tilde{g}_\psi(\tilde{X}_n))(\sigma) - (\tilde{g}_\psi(\tilde{X}))(\sigma) \right\|_{\mathcal{T}_2(\mathbb{K}, \mathbb{R})}^2 d\sigma \right) \right\}
= \frac{1}{2} \left\{ CT \left( 1 + \sup_{\sigma \in [0,T]} |\tilde{X}_n(\sigma)|^2_{\mathbb{H}^2} \right) \cdot \|P_n \psi - \psi\|_{V}^2 + \| (\tilde{g}_\psi(\tilde{X}_n) - \tilde{g}_\psi(\tilde{X})) \|_{L^2(0,T;\mathcal{T}_2(\mathbb{K}, \mathbb{R}))}^2 \right\}.
$$
Since \((\tilde{X}_n) \subset L^2(0,T;\mathbb{H})\) and \(\tilde{X}_n \to \tilde{X}\) in \(L^2(0,T;\mathbb{H}_{loc})\), by Remark 3.4 (iii) we infer that for every \(\psi \in \mathcal{V}\)
\[
\lim_{n \to \infty} \tilde{g}_i(\tilde{X}_n) = \tilde{g}_i(\tilde{X}) \text{ in } L^2(0,T;\mathcal{T}_2(\mathbb{K},\mathbb{R})),
\]
Using additionally Corollary 4.4, we obtain
\[
\lim_{n \to \infty} \int_s^t |I_{11}^n(\sigma)|^2 d\sigma = 0.
\]
Analogously,
\[
\lim_{n \to \infty} \int_s^t |I_{12}^n(\sigma)|^2 d\sigma = 0.
\]
Thus
\[
\lim_{n \to \infty} I_1^n = \lim_{n \to \infty} \int_s^t I_1^n(\sigma) d\sigma = 0.
\]
Let us move to \(I_2^n\). Proceeding analogously as in the analysis of the term \(I_1^n\), we obtain the following inequalities
\[
|I_2^n| \leq \int_s^t \left\{ \left[ C(1 + |\tilde{X}_n(\sigma)|^2) \right]^\frac{1}{2} \cdot \|P_n \psi - \psi\|_{\mathcal{V}} + \|g(\tilde{g}_i(\tilde{X}_n)) - (\tilde{g}_i(\tilde{X}))\|_{\mathcal{T}_2(\mathbb{K},\mathbb{R})} \right\}
\cdot \left[ C(1 + |\tilde{X}(\sigma)|^2) \right]^\frac{1}{2} \cdot \|\psi\|_{\mathcal{V}} d\sigma
\leq CT \|\psi\|_{\mathcal{V}} \left( 1 + \sup_{\sigma \in [0,T]} |\tilde{X}_n(\sigma)|^2 \right)^\frac{1}{2} \cdot \left( 1 + \sup_{\sigma \in [0,T]} |\tilde{X}(\sigma)|^2 \right)^\frac{1}{2} \cdot \|P_n \psi - \psi\|_{\mathcal{V}}
+ \|\psi\|_{\mathcal{V}} \left( 1 + \sup_{\sigma \in [0,T]} |\tilde{X}(\sigma)|^2 \right)^\frac{1}{2} \cdot \int_0^T \|g(\tilde{g}_i(\tilde{X}_n)) - (\tilde{g}_i(\tilde{X}))\|_{\mathcal{T}_2(\mathbb{K},\mathbb{R})} d\sigma.
\]
Using Corollary 4.4 and Remark 3.4 (iii), we infer that
\[
\lim_{n \to \infty} I_2^n = \lim_{n \to \infty} \int_s^t I_2^n(\sigma) d\sigma = 0.
\]
Analogously,
\[
\lim_{n \to \infty} I_3^n = \lim_{n \to \infty} \int_s^t I_3^n(\sigma) d\sigma = 0.
\]
This concludes the proof of the pointwise convergence.

**Uniform integrability.** We will prove that the sequence \((\xi_n)\) is uniformly integrable. To this end it is sufficient to show that for some \(r > 0\)
\[
\sup_{n \in \mathbb{N}} \tilde{\mathbb{E}}[|\xi_n|^r] < \infty.
\]
In fact we will show that the above condition holds for every \(r \in (1, \frac{p}{2})\).
We have \(\tilde{\mathbb{P}}\)-a.s
\[
|\xi_n| \leq \int_0^T \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma, \omega))]\|_{\mathcal{T}_2(\mathbb{K},\mathbb{R})} \|\xi^*[G_n(\sigma, \tilde{X}_n(\sigma, \omega))]\|_{\mathcal{T}_2(\mathbb{K},\mathbb{R})} d\sigma
\leq \int_0^T \|\psi^*[G_n(\sigma, \tilde{X}_n(\sigma, \omega))]\|_{\mathcal{T}_2(\mathbb{K},\mathbb{R})} \|\xi^*[G_n(\sigma, \tilde{X}_n(\sigma, \omega))]\|_{\mathcal{T}_2(\mathbb{K},\mathbb{R})} d\sigma.
\]
By Remark 3.4(iii), we have for every $\sigma \in [0, T]$

$$\|\psi^* [G_n(\sigma, \tilde{X}_n(\sigma))] \|_{T_2(\mathbb{K}, \mathbb{R})} = \sup_{y \in \mathbb{K}, \|y\|_K \leq 1} |(\psi^* [G_n(\sigma, \tilde{X}_n(\sigma))])(y)|$$

$$= \sup_{y \in \mathbb{K}, \|y\|_K \leq 1} |\langle \gamma(\sigma, \tilde{X}_n(\sigma))(y) | P_n \psi \rangle | \leq \left[ C(1 + |\tilde{X}_n(\sigma)|^2) \right]^\frac{1}{2} \cdot |P_n|_{L(V, V)} \|\psi\|_V.$$  

Thus

$$|\xi_n| \leq C \|P_n\|^2_{L(V, V)} \|\psi\|_V \|\zeta\|_V \int_0^T (1 + |\tilde{X}_n(\sigma)|^2) d\sigma$$

$$\leq CT \left( \sup_{n \in \mathbb{N}} |P_n|^2_{L(V, V)} \right) \|\psi\|_V \|\zeta\|_V \left( 1 + \sup_{\sigma \in [0, T]} |\tilde{X}_n(\sigma)|^2 \right),$$

and for $r \in (1, \frac{5}{6}]$

$$\mathbb{E}[|\xi_n|^r] \leq c \left( 1 + \mathbb{E} \left[ \sup_{\sigma \in [0, T]} |\tilde{X}_n(\sigma)|^{2r} \right] \right) \leq c(1 + C_2(2r, p)) < \infty.$$

(Here $c = 2^{-r-1} \left( CT \left( \sup_{n \in \mathbb{N}} |P_n|^2_{L(V, V)} \right) \right)^r \|\psi\|^r_{\mathbb{K}} \|\zeta\|^r_{\mathbb{K}} < \infty$, by Corollary 3.4) This completes the proof of the uniform integrability.

By the Vitali theorem

$$\lim_{n \to \infty} \mathbb{E}[\xi_n] = \mathbb{E}[\xi].$$

The proof of the lemma is thus complete.

**Conclusion of the proof of Theorem 3.7.** We use the idea analogous to the reasoning used by Da Prato and Zabczyk in [15] Section 8.3. Let us consider the operator $L$ defined by (3.4) in Appendix [13]

$$L : U \supset D(L) \to \mathbb{H},$$

which is self-adjoint and bijection. Let $L^{-1}$ be its inverse, i.e.

$$L^{-1} : \mathbb{H} \to U.$$

By Lemmas 5.2, 5.3, 5.6 and 5.7 (with $\psi := L^{-1} \varphi$ and $\zeta := L^{-1} \eta$, where $\varphi, \eta \in \mathbb{H}$), we infer that the process $\{M_t\}_{t \in [0, T]}$ is a $U'$-valued continuous square integrable martingale with respect to the filtration $\mathbb{F} = (\mathbb{F}_t)$, where $\mathbb{F}_t = \sigma\{\tilde{X}(s), s \leq t\}$. Let us consider the dual operator of $L^{-1}$:

$$(L^{-1})' : U' \to \mathbb{H}' \equiv \mathbb{H},$$

where the space $\mathbb{H}'$ is identified with $\mathbb{H}$. Then the process $\{(L^{-1})' \tilde{M}_t\}_{t \in [0, T]}$ is a $\mathbb{H}' \equiv \mathbb{H}$-valued continuous square integrable martingale with respect to the filtration $\mathbb{F} = (\mathbb{F}_t)$, where $\mathbb{F}_t = \sigma\{\tilde{X}(s), s \leq t\}$ with the quadratic variation

$$\langle (L^{-1})' \tilde{M} \rangle_t = \int_0^t \|(L^{-1})' \tilde{G}(s, \tilde{X}(s))\|^2_{T_2(\mathbb{K}, \mathbb{H})} ds,$$

where using (using the identification $\mathbb{H} \equiv \mathbb{H}'$) the injection $j : \mathbb{H}' \ni \xi \to \xi|_U \in U'$

$$\tilde{G} : [0, T] \times \mathbb{H} \ni (t, u) \mapsto \{K \ni y \mapsto j(G(t, u)(y))\} \in \mathcal{L}(\mathbb{K}, U'),$$

and

$$(L^{-1})' \tilde{G} : [0, T] \times \mathbb{K} \ni (t, u) \mapsto \{K \ni y \mapsto (L^{-1})'[\tilde{G}(t, u)(y)]\} \in T_2(\mathbb{K}, \mathbb{H}).$$

(The fact that $(L^{-1})' \tilde{G} \in T_2(\mathbb{K}, \mathbb{H})$ follows from the fact that $G(t, u) \in T_2(\mathbb{K}, \mathbb{H})$.) By the martingale representation theorem, see [15], there exist
• a stochastic basis \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})\),

• a cylindrical Wiener process \(\tilde{W}(t)\) defined on this basis,

• and a progressively measurable process \(\tilde{X}(t)\) such that

\[
(L^{-1})'\tilde{X}(t) - (L^{-1})'\tilde{X}(0) + \int_0^t (L^{-1})'[A\tilde{X}(s) + B(\tilde{X}(s)) + H(\tilde{X}(s)) - f(s)] \, ds = \int_0^t (L^{-1})'\tilde{G}(s, \tilde{X}(s)) \, d\tilde{W}(s), \quad t \in [0, T].
\]

Thus for all \(t \in [0, T]\) and all \(v \in U\)

\[
(\tilde{X}(t)|v)_H - (\tilde{X}(0)|v)_H + \int_0^t \langle A\tilde{X}(s) + B(\tilde{X}(s)) + H(\tilde{X}(s)) - f(s)|v \rangle \, ds = \int_0^t \langle G(s, \tilde{X}(s)) \, d\tilde{W}(s)|v \rangle, \quad t \in [0, T].
\]

Since the space \(U\) is dense in \(V_{1,2}\), we infer that the above equation holds for every \(v \in V_{1,2}\).

In conclusion, the system \((\tilde{\mathfrak{A}}, \tilde{W}, X)\), where \(\tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})\), \(\tilde{W} := \tilde{W}\) and \(X := \tilde{X}\), is a martingale solution of the problem \((\text{3.17})\) in the sense of definition \((\text{3.5})\). The proof of theorem \((\text{3.7})\) is thus complete.

A The spaces \(L^2_n\) and the cut-off operators \(S_n\)

We recall some results concerning the Friedrichs method which is based on the Fourier analysis, see [4, Section 4, p.174]. This presentation is also closely related to [6] and [17].

A.1 Preliminaries

Let us recall that the Fourier transform of a rapidly decreasing function \(\psi \in \mathcal{S}(\mathbb{R}^d)\) is defined by (see [30], [36])

\[
\hat{\psi}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \psi(x) \, dx, \quad \xi \in \mathbb{R}^d,
\]

and the Fourier transform of a tempered distribution is defined via duality, i.e., if \(f \in \mathcal{S}'(\mathbb{R}^d)\) then

\[
\langle f|\psi \rangle := \langle f|\hat{\psi} \rangle, \quad \psi \in \mathcal{S}(\mathbb{R}^d).
\]

Let us recall that for \(s \geq 0\) the Sobolev space is defined by

\[
H^s(\mathbb{R}^d) := \{ u \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^d) \}
\]

and

\[
\|u\|_{H^s} := \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}\|_{L^2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
\]

(See [36], [35].) The spaces \(H^s(\mathbb{R}^d)\) are also called Lebesgue spaces and denoted by \(L^2_s\).
A.2 Subspaces $L^2_n$ and the cut-off operators $S_n$.

Let

$$\tilde{B}_n := \{ \xi \in \mathbb{R}^d : |\xi| \leq n \} \subset \mathbb{R}^d, \quad n \in \mathbb{N},$$

and let

$$L^2_n := \{ u \in L^2(\mathbb{R}^d) : \text{supp } \hat{u} \subset \tilde{B}_n \}. \quad (A.1)$$

On the subspace $L^2_n$ we consider the norm inherited from $L^2(\mathbb{R}^d)$.

The cut-off operator $S_n$ is defined by

$$S_n(u) := \mathcal{F}^{-1}(1_{\tilde{B}_n} \hat{u}), \quad u \in L^2(\mathbb{R}^d),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. (See [4, Section 4, p. 174].)

Remark A.1. (See [4, Section 4, p. 174] and [6].) The map

$$S_n : L^2(\mathbb{R}^d) \to L^2_n$$

is an $(\cdot | \cdot)_{L^2}$-orthogonal projection.

A.3 Properties of the operators $S_n$.

The following lemma is closely related to [17, p. 1042] and [6, Lemma 4.1].

Lemma A.2. Let $s \geq 0$ be fixed. Then for all $n \in \mathbb{N}$:

$$S_n : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$$

is well defined linear and bounded. Moreover, for every $u \in H^s(\mathbb{R}^d)$

$$\|S_n u\|_{H^s} \leq \|u\|_{H^s} \quad (A.3)$$

and

$$\lim_{n \to \infty} \|S_n u - u\|_{H^s} = 0. \quad (A.4)$$

Proof. For every $u \in H^s(\mathbb{R}^d)$ we have

$$\|S_n u\|^2_{H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}^{-1}(1_{\tilde{B}_n} \hat{u})(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}^{-1}(1_{\tilde{B}_n} \hat{u})| \hat{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}^{-1}(1_{\tilde{B}_n} \hat{u})| \hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi)|^2 d\xi = \|u\|^2_{H^s},$$

which completes the proof of (A.3).

To prove (A.4) we proceed as follows

$$\|S_n u - u\|^2_{H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}^{-1}(1_{\tilde{B}_n} \hat{u}) - \hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}^{-1}(1_{\tilde{B}_n} \hat{u}) - \hat{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s [1_{\tilde{B}_n}(\xi) - 1] \hat{u}(\xi)|^2 d\xi = \int_{B_\infty} (1 + |\xi|^2)^s \hat{u}(\xi)|^2 d\xi.$$
Since \( u \in H^s(\mathbb{R}^d) \), we infer that
\[
\lim_{n \to \infty} \int_{B^n_0} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi = 0,
\]
which completes the proof of (A.4) and of the lemma. \( \square \)

The following lemma is based on [17, p. 1042].

**Lemma A.3.** If \( s \geq 0 \) and \( k > 0 \), then
\[
S_n : H^{s+k}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)
\]
is well defined and bounded and \( |S_n|_{L(H^{s+k},H^s)} \leq 1 \). Moreover, for every \( u \in H^{s+k}(\mathbb{R}^d) \):
\[
\|S_n u - u\|^2_{H^s} \leq \frac{1}{(1 + n^2)^k} \|u\|^2_{H^{s+k}}.
\]
(A.5)

Thus
\[
\lim_{n \to \infty} |S_n - I|_{L(H^{s+k},H^s)} = 0,
\]
(A.6)

where \( I \) stands for the identity operator.

**Proof.** Let us fix \( s \geq 0 \) and \( k > 0 \). Let \( u \in H^{s+k}(\mathbb{R}^d) \). We have
\[
\|S_n u\|^2_{H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{S_n u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \|\Pi_{B^n_0}(\xi)\hat{u}(\xi)\|^2 d\xi \\
= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^k} \cdot (1 + |\xi|^2)^{s+k} |\hat{u}(\xi)|^2 d\xi \\
\leq \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s+k} |\hat{u}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s+k} |\hat{u}(\xi)|^2 d\xi = \|u\|^2_{H^{s+k}}.
\]

Thus \( S_n \in L(H^{s+k},H^s) \) and \( |S_n|_{L(H^{s+k},H^s)} \leq 1 \).

Moreover,
\[
\|S_n u - u\|^2_{H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{S_n u}(\xi) - \hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \|\Pi_{B^n_0}(\xi)\hat{u}(\xi) - \hat{u}(\xi)\|^2 d\xi \\
= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \|\Pi_{B^n_0}(\xi) - 1\hat{u}(\xi)\|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \\
= \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^k} \cdot (1 + |\xi|^2)^{s+k} |\hat{u}(\xi)|^2 d\xi \leq \frac{1}{(1 + n^2)^k} \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s+k} |\hat{u}(\xi)|^2 d\xi \\
\leq \frac{1}{(1 + n^2)^k} \cdot \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s+k} |\hat{u}(\xi)|^2 d\xi = \frac{1}{(1 + n^2)^k} \cdot \|u\|^2_{H^{s+k}},
\]
which completes (A.5). Thus
\[
|S_n - I|^2_{L(H^{s+k},H^s)} \leq \frac{1}{(1 + n^2)^k}.
\]

which implies that (A.6) holds. The proof is thus complete. \( \square \)
A.4 Relation between the spaces $L^2_n$ and $H^s(\mathbb{R}^d)$ for $s \geq 0$.

Let us recall that on the spaces $L^2_n$, by definition, we consider the norms inherited from the space $L^2(\mathbb{R}^d)$, see (A.1).

Lemma A.4. For each $n \in \mathbb{N}$

$$L^2_n \hookrightarrow H^s(\mathbb{R}^d) \quad \text{for all } s \geq 0$$

and for every $s \geq 0$ and $u \in L^2_n$:

$$\|u\|^2_{H^s} \leq (1 + n^2)^s \|u\|^2_{L^2_n},$$

(A.7)

Note that the norm of the embedding $L^2_n \subset H^s(\mathbb{R}^d)$ depends on $n$ and $s$.

Proof. Let $u \in L^2_n$. Then $S_n u = u$. Let us fix $s > 0$. We will show that $u \in H^s(\mathbb{R}^d)$. Indeed,

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{S_n u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \mathbb{1}_{B_n}(\xi) |\hat{u}(\xi)|^2 d\xi$$

$$= \int_{B_n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \leq (1 + n^2)^s \int_{B_n} |\hat{u}(\xi)|^2 d\xi$$

$$\leq (1 + n^2)^s \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 d\xi = (1 + n^2)^s |u|^2_{L^2} = (1 + n^2)^s |u|^2_{L^2_n},$$

which complete the proof of (A.7) and of the lemma. \qed

Corollary A.5. On the subspace $L^2_n$ the norms $\cdot_{L^2_n}$ and $\cdot_{H^s}$, for $s > 0$, are equivalent (with appropriate constants depending on $s$ and $n$).

Proof. The assertion is a consequence of the following inequalities: for all $u \in L^2_n$

$$|u|^2_{L^2_n} \leq \|u\|^2_{H^s} \leq (1 + n^2)^s |u|^2_{L^2_n}.$$  \qed

B Auxiliary results from functional analysis: the space $\mathbb{U}$ and the operator $L$

We have the following spaces, defined in Section 2.4, which appear in the functional setting of problem (1.1)-(1.4)

$$\mathbb{V}_{1,2} \subset \mathbb{V} \subset \mathbb{H}.$$  

Recall that $\mathbb{V}_{1,2}$, see (2.24), is the space of test functions used in Definition 3.5. For fixed $m > \frac{5}{2}$, let us consider the space

$$\mathbb{V}_* := \mathbb{V}_m,$$

(B.1)

where $\mathbb{V}_m = V_m \times V_m$ is defined by (2.26). The choice of the space $\mathbb{V}_*$ corresponds to the properties on nonlinear maps $\tilde{B}$ and $\tilde{H}$, see Lemmas 2.7 and 2.9 and Corollaries 2.8 and 2.10 in Section 2.4. In fact, instead of (B.1) the space $\mathbb{V}_*$ can be defined as $\mathbb{V}_{m_1,m_2}$ for fixed
$m_1, m_2 > \frac{5}{2}$, defined by (2.24). However, for us it will be sufficient use the space $V_*$ defined by (B.1).

Space $U$. Since the embeddings of Sobolev space are not compact in the case of an unbounded domain, we introduce some auxiliary space $U$ which will be of crucial importance in the compactness in tightness results.

Since $V_*$ is dense in $H$ and the embedding $V_* \hookrightarrow H$ is continuous, by Lemma 2.5 from [19] (see [7, Lemma C.1]) there exists a separable Hilbert space $U$ such that $U \subset V_*$, $U$ is dense in $V_*$ and the embedding $\iota : U \hookrightarrow V_*$ is compact. (B.2)

Then we have

$$U \hookrightarrow V_* \hookrightarrow V_{1,2} \hookrightarrow V \hookrightarrow H.$$  (B.3)

Operator $L$. We define some auxiliary operator which will be used in the proof of the existence of martingale solutions. By (B.2) and (B.3) we infer that, in particular, $U$ is compactly embedded into the space $H$. Let us denote

$$\iota : U \hookrightarrow H$$

and let

$$\iota^* : H \to U,$$

be its adjoint operator. Since the range of $\iota$ is dense in $H$, the map $\iota^* : H \to U$ is one-to-one. Let us put

$$D(L) := \iota^*(H) \subset U,$$

$$Lu := (\iota^*)^{-1} u, \quad u \in D(L).$$  (B.4)

It is clear that $L : D(L) \to H$ is onto. Let us also notice that

$$(Lu|w)_H = (u|w)_U, \quad u \in D(L), \quad w \in U.$$  (B.5)

Indeed, by (B.4) we have for all $u \in D(L)$ and $w \in U$

$$(Lu|w)_H = ((\iota^*)^{-1} u|w)_H = (\iota^* (\iota^*)^{-1} u|w)_U = (u|w)_U,$$

which proves (B.5). By equality (B.5) and the density of $U$ in $H$, we infer that $D(L)$ is dense in $H$.

C Appendix: Compactness and tightness results. The Skorokhod theorem

In this section we present some compactness and tightness results being a straightforward adaptations to our framework of the results proved in [7] and [28]. We use the spaces $H$ and $V$ which appear in the statement of problem (3.17) (see Definition 3.5) as well as the auxiliary space $U$ constructed in Appendix B. By (B.2) and (B.3), in particular, we have

$$U \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow U',$$

the embedding $U \hookrightarrow V$ being compact.

Let us consider the following functional spaces being the counterparts in our framework of the spaces used in [7] and [28], see also [25]:
- \( C([0,T];\mathbb{U}') := \) the space of continuous functions \( \phi : [0,T] \to \mathbb{U}' \) with the topology \( T_1 \)
  induced by norm \( |\phi|_{C([0,T];\mathbb{U}')} := \sup_{t \in [0,T]} |\phi(t)|_{\mathbb{U}'} \),
- \( L^2_w(0;\mathbb{V}) := \) the space \( L^2(0,T;\mathbb{V}) \) with the weak topology \( T_2 \),
- \( L^2(0,T;\mathbb{H}_{loc}) := \) the space of measurable functions \( \phi : [0,T] \to \mathbb{H} \) such that for all \( R \in \mathbb{N} \)
  \[ p_{T,R}(\phi) := \left( \int_0^T \int_{C_R} \left[ |\phi_1(t,x)|^2 + |\phi_2(t,x)|^2 \right] dxdt \right)^{\frac{1}{2}} < \infty, \]
where \( \phi = (\phi_1,\phi_2) \), with the topology \( T_3 \) generated by the seminorms \( (p_{T,R})_{R \in \mathbb{N}} \).

Let \( \mathbb{H}_w \) denote the Hilbert space \( \mathbb{H} \) endowed with the weak topology. Let us consider the fourth space, see [7],

- \( C([0,T];\mathbb{H}_w) := \) the space of weakly continuous functions \( \phi : [0,T] \to \mathbb{H} \) with the weakest topology \( T_4 \) such that for all \( h \in \mathbb{H} \) the mappings
  \[ \mathcal{C}([0,T];\mathbb{H}_w) \ni \phi \mapsto (\phi(\cdot)|h)_\mathbb{H} \in \mathcal{C}([0,T];\mathbb{R}) \]
are continuous. In particular, \( \phi_n \to \phi \) in \( \mathcal{C}([0,T];\mathbb{H}_w) \) iff for all \( h \in \mathbb{H} \): \( (\phi_n(\cdot)|h)_\mathbb{H} \to (\phi(\cdot)|h)_\mathbb{H} \) in the space \( \mathcal{C}([0,T];\mathbb{R}) \).

### C.1 Compactness and tightness criteria

**Definition C.1.** Let us put
\[ \mathcal{Z} := L^2_w(0,T;\mathbb{V}) \cap L^2(0,T;\mathbb{H}_{loc}) \cap \mathcal{C}([0,T];\mathbb{H}_w) \cap \mathcal{C}([0,T];\mathbb{U}') \quad \text{(C.1)} \]
and let \( \mathcal{T} \) be the supremum of the corresponding four topologies, i.e. the smallest topology on \( \mathcal{Z} \) such that the four natural embeddings from \( \mathcal{Z} \) are continuous. The space \( \mathcal{Z} \) will also be considered with the Borel \( \sigma \)-algebra, denoted by \( \sigma(\mathcal{Z}) \), i.e. the smallest \( \sigma \)-algebra containing the family \( \mathcal{T} \).

The following compactness criterion is a simple modification of Lemma 4.1 from [28]. See also [7, Section 3.1] for the case of the Navier-Stokes equations.

**Lemma C.2.** (see [28, Lemma 4.1]) Let
\[ \mathcal{K} \subset L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap \mathcal{C}([0,T];\mathbb{U}') \]
satisfy the following three conditions
(a) \( \sup_{\phi \in \mathcal{K}} \|\phi\|_{L^\infty(0,T;\mathbb{H})} < \infty \), i.e. \( \mathcal{K} \) is bounded in \( L^\infty(0,T;\mathbb{H}) \),
(b) \( \sup_{\phi \in \mathcal{K}} \int_0^T \|\phi(s)\|_{\mathbb{V}}^2 \, ds < \infty \), i.e. \( \mathcal{K} \) is bounded in \( L^2(0,T;\mathbb{V}) \),
(c) \( \lim_{\delta \to 0} \sup_{\phi \in \mathcal{K}} \sup_{|t-s| \leq \delta} \|\phi(t) - \phi(s)\|_{\mathbb{U}'} = 0 \).

Then \( \mathcal{K} \subset \mathcal{Z} \) and \( \mathcal{K} \) is \( \mathcal{T} \)-relatively compact in \( \mathcal{Z} \) defined by (C.1).

One of the main tools in the construction of a martingale solution is the tightness criterion in the space \( \mathcal{Z} \) defined in identity (C.1). We use a slight modification of the criterion stated in [7, Corollary 3.9], in the framework of the Navier-Stokes equation and in [28, Corollary 4.2], where more general setting is considered.
Corollary C.3. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of continuous \(\mathbb{F}\)-adapted \(U'\)-valued processes such that

(a) there exists a positive constant \(C_1\) such that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0,T]} |X_n(s)|_{U'} \right] \leq C_1, \tag{C.2}
\]

(b) there exists a positive constant \(C_2\) such that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \|X_n(s)\|_{U'}^2 \, ds \right] \leq C_2, \tag{C.3}
\]

(c) \((X_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition in \(U'\).

Let \(\tilde{P}_n\) be the law of \(X_n\) on \(Z\). Then for every \(\varepsilon > 0\) there exists a compact subset \(K_\varepsilon\) of \(Z\) such that
\[
\tilde{P}_n(K_\varepsilon) \geq 1 - \varepsilon.
\]

The proof of Corollary C.3 is essentially the same as the proof of [7, Corollary 3.9] or [28, Corollary 4.2].

Let us recall the Aldous condition in the form given by Métivier [25].

Definition C.4. (M. Métivier) A sequence \((X_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition in the space \(U'\) iff

\([A]\) for every \(\varepsilon > 0\) and \(\eta > 0\) there exists \(\delta > 0\) such that for every sequence \((\tau_n)_{n \in \mathbb{N}}\) of \(\mathbb{F}\)-stopping times with \(\tau_n \leq T\) one has
\[
\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P}\{ |X_n(\tau_n + \theta) - X_n(\tau_n)|_{U'} \geq \eta \} \leq \varepsilon.
\]

Below we recall a sufficient condition for the Aldous condition.

Lemma C.5. (See [27, Lemma 9]) Let \((E, \| \cdot \|_E)\) be a separable Banach space and let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \(E\)-valued random variables such that

\([A']\) there exist \(\alpha, \beta > 0\) and \(C > 0\) such that for every sequence \((\tau_n)_{n \in \mathbb{N}}\) of \(\mathbb{F}\)-stopping times with \(\tau_n \leq T\) and for every \(n \in \mathbb{N}\) and \(\theta \geq 0\) the following condition holds
\[
\mathbb{E} \left[ \|X_n(\tau_n + \theta) - X_n(\tau_n)\|_E^\alpha \right] \leq C\theta^\beta. \tag{C.4}
\]

Then the sequence \((X_n)_{n \in \mathbb{N}}\) satisfies condition \([A]\) in the space \(E\).
C.2 Jakubowski’s generalization of the Skorokhod theorem

In the proof of the theorem on the existence of a martingale solution we use a version of the Skorokhod theorem for nonmetric spaces. For convenience of the reader let us recall the following Jakubowski’s [22] generalization of the Skorokhod theorem.

Theorem C.6. (Theorem 2 in [22]). Let \((\mathcal{X}, \tau)\) be a topological space such that there exists a sequence \((f_m)\) of continuous functions \(f_m : \mathcal{X} \to \mathbb{R}\) that separates points of \(\mathcal{X}\). Let \((X_n)\) be a sequence of \(\mathcal{X}\)-valued Borel random variables. Suppose that for every \(\varepsilon > 0\) there exists a compact subset \(K_\varepsilon \subset \mathcal{X}\) such that

\[
\sup_{n \in \mathbb{N}} \mathbb{P}(\{X_n \in K_\varepsilon\}) > 1 - \varepsilon.
\]

Then there exists a subsequence \((X_{n_k})_{k \in \mathbb{N}}\), a sequence \((Y_k)_{k \in \mathbb{N}}\) of \(\mathcal{X}\)-valued Borel random variables and an \(\mathcal{X}\)-valued Borel random variable \(Y\) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

\[
\mathcal{L}(X_{n_k}) = \mathcal{L}(Y_k), \quad k = 1, 2, ...
\]

and for all \(\omega \in \Omega\):

\[
Y_k(\omega) \overset{\tau}{\to} Y(\omega) \quad \text{as} \quad k \to \infty.
\]

Proceeding analogously to [7, Corollary 3.12] or [28, Remark C.2] it is easy to prove the following result for the space \(\mathcal{Z}\) from Definition C.4.

Lemma C.7. The topological space \(\mathcal{Z}\) satisfies the assumptions of Theorem C.6.

D Proof of Lemma 2.5

Proof. Step 1. Assume first that \(\psi \in \mathcal{V}\). There exists \(R > 0\) such that \(\text{supp} \psi\) is a compact subset of the ball \(K_R := \{x \in \mathbb{R}^3 : |x| < R\}\). We have

\[
\int_0^t \left[ \langle \mathcal{H}(u_n(s), w_n(s))|\psi\rangle - \langle \mathcal{H}(u(s), w(s))|\psi\rangle \right] ds
\]

\[
= \int_0^t \langle \mathcal{H}(u_n(s) - u(s), w_n(s))|\psi\rangle ds + \int_0^t \langle \mathcal{H}(u(s), w_n(s) - w(s))|\psi\rangle ds =: I_1(n) + I_2(n).
\]

We will analyze separately the terms \(I_1(n)\) and \(I_2(n)\).

Let us consider the term \(I_1(n)\). By (2.10), (2.15), the Hölder inequality and the Sobolev embedding theorem we obtain

\[
|\langle \mathcal{H}(u, w)|\psi\rangle| = |\mathfrak{h}(u, w, \psi)| = \left| \int_{K_R} [u \times (\text{curl } w)] \cdot \text{curl } \psi dx \right| \leq |u|_{L^2(K_R)} \cdot |\text{curl } w|_{L^2} \cdot |\text{curl } \psi|_{L^\infty}
\]

\[
\leq \tilde{c} |u|_{L^2(K_R)} \cdot \|w\|_{H^1} \cdot \|\psi\|_{H^{m-1}} \leq c |u|_{L^2(K_R)} \cdot \|w\|_{H^1} \cdot \|\psi\|_{H^m}.
\]

for every \(m > \frac{3}{2}\). Thus by the Schwarz inequality for all \(t \in [0, T]\)

\[
|I_1(n)| \leq c \int_0^t |u_{n}(s) - u(s)|_{L^2(K_R)} \cdot \|w_n\|_{H^1} ds \cdot \|\psi\|_{H^m}
\]

\[
\leq c \|u_n - u\|_{L^2(0, T; L^2(K_R))} \cdot \|w_n\|_{L^2(0, T; V)} \cdot \|\psi\|_{H^m}
\]

\[
= c p_{T, R} \|u_n - u\| \cdot \|w_n\|_{L^2(0, T; V)} \cdot \|\psi\|_{H^m}.
\]
Recall that $p_{T,R}$ denotes the seminorm defined by (2.3). Since $u_n \to u$ in $L^2(0,T;\mathcal{H}_{loc})$ and the sequence $(w_n)$ is bounded in $L^2(0,T;V)$, we infer that

$$\lim_{n \to \infty} I_1(n) = 0.$$ 

Let us move to the term $I_2(n)$. Note that

$$\langle \mathcal{H}(u, w_n - w) \rangle = \mathcal{H}(u, w_n - w, \psi) = \int_{\mathbb{R}^3} [u \times \text{curl} (w_n - w)] \cdot \text{curl} \psi \, dx.$$ 

On the other hand, the map

$$\xi : L^2(0,T; L^2(\mathbb{R}^3, \mathbb{R}^3)) \ni z \mapsto \int_{\mathbb{R}^3} [u \times z] \cdot \text{curl} \psi \, dx \in \mathbb{R}$$

is continuous linear functional on the space $L^2(0,T; L^2(\mathbb{R}^3, \mathbb{R}^3))$. Since $w_n \to w$ weakly in $L^2(0,T;V)$, we infer that $\frac{\partial w_n}{\partial x_i} \to \frac{\partial w}{\partial x_i}$ weakly in $L^2(0,T; L^2(\mathbb{R}^3, \mathbb{R}^3))$ for each $i = 1, 2, 3$. Thus $\text{curl} (w_n - w) \to 0$ weakly in $L^2(0,T; L^2(\mathbb{R}^3, \mathbb{R}^3))$. In conclusion,

$$\lim_{n \to \infty} I_2(n) = \lim_{n \to \infty} \xi[\text{curl} (w_n - w)] = 0.$$ 

**Step 2.** If $\psi \in V_m$, then for every $\epsilon > 0$ there exists $\psi_\epsilon \in \mathcal{V}$ such that $\|\psi - \psi_\epsilon\|_{H^m} < \epsilon$. We have

$$\langle \mathcal{H}(u_n, w_n) - \mathcal{H}(u, w) \rangle = \langle \mathcal{H}(u_n, w_n) - \mathcal{H}(u, w) \rangle - \langle \mathcal{H}(u_n, w_n) - \mathcal{H}(u, w) \rangle \psi_\epsilon.$$ 

Thus, by (2.17)

$$\left| \int_0^t \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi \, ds \right|$$

$$\leq \int_0^t \left| \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi - \psi_\epsilon \right| \, ds$$

$$+ \left| \int_0^t \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi_\epsilon \, ds \right|$$

$$\leq \int_0^t \left[ \mathcal{H}(u_n(s), w_n(s)) \right]_{V_m} + \mathcal{H}(u(s), w(s)) \right]_{V_m} \cdot \|\psi - \psi_\epsilon\|_{H^m} \] ds$$

$$+ \left| \int_0^t \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi_\epsilon \, ds \right|$$

$$\leq c \epsilon \int_0^t \left[ \|u_n(s)\|_{L^2} \|w_n(s)\|_{L^2} \right]_{H^1} + \|u(s)\|_{L^2} \|w(s)\|_{H^1} \] ds$$

$$+ \left| \int_0^t \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi_\epsilon \, ds \right|$$

$$\leq \frac{c \epsilon}{2} \int_0^t \left[ \|u_n(s)\|_{L^2}^2 + \|w_n(s)\|_{H^1}^2 \right] ds$$

$$+ \left| \int_0^t \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi_\epsilon \, ds \right|$$

$$\leq \frac{c \epsilon}{2} \sup_{n \in \mathbb{N}} \left[ \|u_n\|_{L^2(0,T;H)}^2 + \|w_n\|_{L^2(0,T;V)}^2 \right] + \|u\|_{L^2(0,T;H)}^2 + \|w\|_{L^2(0,T;V)}^2$$

$$+ \left| \int_0^t \langle \mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s)) \rangle \psi_\epsilon \, ds \right|.$$
Passing to the upper limit as \(n \to \infty\) and using step 1 we obtain

\[
\limsup_{n \to \infty} \left| \int_0^t (\mathcal{H}(u_n(s), w_n(s)) - \mathcal{H}(u(s), w(s))|\psi) \, ds \right| \leq M \varepsilon,
\]

where \(M = \frac{c_2}{2} \left[ \sup_{n \in \mathbb{N}} (\|u_n\|^2_{L^2(0,T;H)} + \|w_n\|^2_{L^2(0,T;V)}) + \|u\|^2_{L^2(0,T;H)} + \|w\|^2_{L^2(0,T;V)} \right] < \infty.\)

Since \(\varepsilon > 0\) is arbitrary, we infer that the assertion holds for all \(\psi \in V_m\). The proof of Lemma 2.5 is thus complete. \(\square\)

References

[1] M. Acheritogaray, P. Degond, A. Frouvelle and J-G. Liu, Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system, Kinet. Relat. Models, 4, 901–918 (2011).
[2] R. Adams, Sobolev spaces, Academic Press, 1975.
[3] S. Albeverio, Z. Brzeźniak, J-L. Wu, Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients, J. Math. Anal. Appl. 371, 309–322, (2010).
[4] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier analysis and nonlinear partial differential equations, Springer-Verlag, 2011.
[5] V. Barbu, G. Da Prato, Existence and Ergodicity for Two-Dimensional Stochastic Magneto-Hydrodynamic Equations, Appl. Math. Optim 56, 145–168 (2007).
[6] Z. Brzeźniak, G. Dhariwal, Stochastic tamed Navier-Stokes equations on \(\mathbb{R}^3\): the existence and the uniqueness of solutions and the existence of an invariant measure, J. Math. Fluid Mech. 22:23, 1–54, (2020).
[7] Z. Brzeźniak, E. Motyl, Existence of a martingale solution to the stochastic Navier-Stokes equations in unbounded 2D and 3D domains, J. Differential Equations, 254, 1627–1685 (2013).
[8] D. Chae, P. Degond and J.-G. Liu, Well-posedness for Hall-magnetohydrodynamics, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31, 555–565 (2014).
[9] D. Chae and J. Lee, On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics, J. Differential Equations, 256, 3835–3858 (2014).
[10] D. Chae and M. Schonbek, On the temporal decay for the Hall-magnetohydrodynamic equations, J. Differential Equations, 255, 3971–3982 (2013).
[11] D. Chae, R. Wan and J. Wu, Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion, J. Math. Fluid Mech., 17, 627–638 (2015).
[12] D. Chae and S. Weng, Singularity formation for the incompressible Hall-MHD equations without resistivity, Ann. Inst. Henri Poincaré Anal. Non Linéaire, 33, 1009–1022 (2016).
[13] I. Chueshov, A. Millet, Stochastic 2D hydrodynamical type systems: well posedeness and large deviations, Appl. Math. Optim. 61(3), 379–420 (2010).
[14] G. Da Prato, J. Zabczyk, *Ergodicity Infinite Dimensional Systems*, Cambridge University Press, 1996.

[15] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 2014.

[16] G. Duvaut, J.L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin Heidelberg, 1976.

[17] C.L. Fefferman, D.S. McCormick, J.C. Robinson, J.L. Rodrigo, *Higher order commutator estimates and local existence for the non-resistive MHD equations and related models*, Journal of Functional Analysis 267, 1035–1056 (2014).

[18] F. Flandoli, D. Gątarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Prob. Theory Related Fields 102, no. 3, 367–391 (1995).

[19] K. Holly, M. Wiciak, *Compactness method applied to an abstract nonlinear parabolic equation*, Selected problems of Mathematics, Cracow University of Technology, 95–160 (1995).

[20] A. Z. Idriss, *Stochastic generalized magnetohydrodynamics equations: well-posedness*, Appl. Anal., 1–22, (2018).

[21] A. Z. Idriss, P. A. Razafimaandimby, *Stochastic generalized magnetohydrodynamics equations with not regular multiplicative noise: Well-posedness and invariant measure*, J. Math. Anal. Appl. 474, 1404–1440, (2019).

[22] A. Jakubowski, *The almost sure Skorohod representation for subsequences in nonmetric spaces*, Teor. Veroyatnost. i Primenen. 42, no. 1, 209-216 (1997); translation in Theory Probab. Appl. 42 no.1, 167–174 (1998).

[23] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[24] U. Manna, M. T. Mohan and S. S. Sritharan, *Stochastic non-resistive magnetohydrodynamic system with Lévy noise*, Random Oper. Stoch. Equ., 25, 155–194 (2017).

[25] M. Métivier, *Stochastic partial differential equations in infinite dimensional spaces*, Scuola Normale Superiore, Pisa, 1988.

[26] M. T. Mohan, S. S. Sritharan, *Stochastic Euler equations of fluid dynamics with Lévy noise*, Asymptot. Anal., 99, 67–103 (2016).

[27] E. Motyl, *Stochastic Navier-Stokes equations driven by Lévy noise in unbounded 3D domains*, Potential Anal., 38, 863–912 (2013).

[28] E. Motyl, *Stochastic hydrodynamic-type evolution equations driven by Lévy noise in 3D unbounded domains - abstract framework and applications*, Stoch. Processes and their Appl. 124, 2052–2097 (2014).

[29] D. Revuz, M. Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, 1999.

[30] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, New York, 1973.

[31] M. Sango, *Magnetohydrodynamic turbulent flows: Existence results*, Physica D, Vol. 239, 12, 912–923 (2010).
[32] A. Schenke, *The stochastic tamed MHD equations - existence, uniqueness and invariant measures*, Stoch PDE: Anal Comp, doi.org/10.1007/s40072-021-00205-x.

[33] M. Sermange, R. Temam, *Some mathematical questions related to the M.H.D. equations*, Comm. Pure Appl. Math., 36, pp. 634–664 (1983).

[34] S.S. Sritharan, P. Sundar, *The stochastic magneto-hydrodynamic system*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 2, No. 2, 241–265 (1999).

[35] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[36] M.E. Taylor, *Partial Differential Equations I*, Springer, 2011.

[37] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, North Holland Publishing Company, Amsterdam - New York - Oxford, 1979.

[38] M.J. Vishik, A.V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer Academic Publishers, Dordrecht, 1988.

[39] K. Yamazaki, *Stochastic Hall-magneto-hydrodynamics system in three and two and a half dimensions*, J. Stat. Phys., 166, 368–397 (2017).

[40] K. Yamazaki, *Remarks on the three and two and a half dimensional Hall-magnetohydrodynamics system: deterministic and stochastic cases*, Complex Analysis and its Synergies, 5, doi.org/10.1007/s40627-019-0033-5 (2019).

[41] K. Yamazaki, *Ergodicity of a Galerkin approximation of three-dimensional magnetohydrodynamics system forced by a degenerate noise*, Stochastics, Vol. 91, No. 1, 114–142 (2019).