BOUND STATES OF TWO-DIMENSIONAL
SCHRÖDINGER-NEWTON EQUATIONS

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Abstract. We prove an existence and uniqueness result for ground states and for purely angular excitations of two-dimensional Schrödinger-Newton equations. From the minimization problem for ground states we obtain a sharp version of a logarithmic Hardy-Littlewood-Sobolev type inequality.

1. Introduction

We consider the Schrödinger-Newton system
\begin{equation}
\label{eq:1.1}
\begin{aligned}
&iu_t + \Delta u - \gamma Vu = 0, \\
&\Delta V = |u|^2
\end{aligned}
\end{equation}
in two space dimensions which is equivalent to the nonlinear Schrödinger equation
\begin{equation}
\label{eq:1.2}
\begin{aligned}
&iu_t + \Delta u - \frac{\gamma}{2\pi} (\ln(|x|) * |u|^2) u = 0
\end{aligned}
\end{equation}
with nonlocal nonlinear potential
\[ V(x) = \frac{1}{2\pi} (\ln |x| * |u|^2)(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x - y|) |u(t, y)|^2 \, dy. \]

We are interested in the existence of nonlinear bound states of the form
\begin{equation}
\label{eq:1.3}
\begin{aligned}
&u(t, x) = \phi_\omega(x)e^{-i\omega t}. \\
&\phi_\omega \in H^1_0
\end{aligned}
\end{equation}
The Schrödinger-Newton system (1.1) in three space dimensions has a long standing history. With γ designating appropriate positive coupling constants it appeared first in 1954, then in 1976 and lastly in 1996 for describing the quantum mechanics of a Polaron at rest by S. J. Pekar [1], of an electron trapped in its own hole by Ph. Choquard [2] and of selfgravitating matter by R. Penrose [3]. The two-dimensional model is studied numerically in [4]. For the bound state problem there are rigorous results only for the three dimensional model. In [2] the existence of a unique ground state of the form (1.3) is shown by solving an appropriate minimization problem. This ground state solution \( \phi_\omega(x), \omega < 0 \) is a positive spherically symmetric strictly decreasing function. In [5] the existence of infinitely many distinct spherically symmetric solutions is proven and, in [6], a proof for the existence of anisotropic bound states is claimed. So far, there are no results for the two-dimensional model using variational methods. One mathematical difficulty of the two-dimensional problem is that the Coulomb potential in two space dimensions is neither bounded from...
above nor from below and hence does not define a positive definite quadratic form.

However, recently in [7] the existence of a unique positive spherically symmetric stationary solution \((u, V)\) of (1.1) such that \(V(0) = 0\) has been proven by applying a shooting method to the corresponding system of ordinary differential equations.

In the present paper, we are mainly interested in the ground states of the model

\[
(1.4) \quad u(t, x) = \phi_\omega(x)e^{-i\omega t}, \quad \phi_\omega(x) > 0.
\]

We prove the existence of ground states by solving an appropriate minimization problem for the energy functional \(E(u)\). By use of strict rearrangement inequalities we shall prove that there is a unique minimizer (up to translations and a phase factor) which is a positive spherically symmetric decreasing function (theorems 3.2 and theorem 3.3). The existence and uniqueness of solutions of the form (1.4) for given \(\omega\), however, depends on the frequency \(\omega\). It will turn out that for any \(\omega \leq 0\) there is a unique ground state of the form (1.4). In addition, we shall prove that for any positive \(\omega < \omega^*\) there are two ground states of the form (1.4) with different \(L^2\)-norm. In the limit case \(\omega = \omega^*\) there is again a unique ground state (theorem 3.4). We then apply our existence result to obtain a sharp Hardy-Littlewood-Sobolev inequality for the logarithmic kernel (theorem 4.1).

Finally, we prove the existence of non radial solutions of the Schrödinger-Newton system (1.1) which in polar coordinates \((r, \theta)\) are of the form

\[
(1.5) \quad \psi(t, r, \theta) = \phi_{m, \omega}(r)e^{im\theta-i\omega t}, \quad \phi_{m, \omega}(r) \geq 0
\]

for positive integers \(m\). These are eigenfunctions of the angular momentum operator \(L = -i\partial_\theta\). Again, we prove the existence of such \(\phi_{m, \omega}(r)\) by solving an appropriate minimization problem for the corresponding energy functional reduced to functions of the form (1.5) for any given \(m\) (theorem 5.1). The minimizers can be interpreted as purely angular excitations and we also prove their uniqueness in this class of functions (theorem 5.2).

2. Mathematical Framework

2.1. Functional Setting. The natural function space \(X\) for the quasi-stationary problem is given by

\[
(2.1) \quad X = \{ u : \mathbb{R}^2 \to \mathbb{C} : \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 + \ln(1 + |x|) |u|^2 \ dx < \infty \}.
\]

The space \(X\) is a Hilbert space and by Rellich’s criterion (see, e.g. theorem XIII.65 of [8]) the embedding \(X \hookrightarrow L^2\) is compact. We note \(X_r\), the space of radial functions in \(X\). Formally, the energy \(E\) associated to (1.2) is given by

\[
(2.2) \quad E(u) = \int_{\mathbb{R}^2} |\nabla u(x)|^2 \ dx + \frac{\gamma}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) |u(x)|^2 |u(y)|^2 \ dxdy
\]

\[
= T(u) + \frac{\gamma}{2} V(u)
\]

In order to prove that the energy is indeed well defined on \(X\) we decompose the potential energy \(V(u)\) into two parts applying the identity

\[
(2.3) \quad \ln(r) = -\ln(1 + \frac{1}{r}) + \ln(1 + r)
\]
for all \( r > 0 \). We then define the corresponding functionals
\[
V_1(u) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + \frac{1}{|x-y|}) |u(x)|^2 |u(y)|^2 \, dxdy
\]
\[
V_2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + |x-y|) |u(x)|^2 |u(y)|^2 \, dxdy
\]
(2.4)

Lemma 2.1. The energy functional \( E : X \rightarrow \mathbb{R}_0^+ \) is well defined on \( X \) and of class \( C^1 \).

Proof. Since \( 0 \leq \ln(1 + \frac{1}{r}) \leq \frac{1}{r} \) for any \( r > 0 \) we have by the Hardy-Littlewood-Sobolev inequality (see e.g. [9]) and Sobolev interpolation estimates (see e.g. [10]) that
\[
|V_1(u)| \leq C_1 ||u||^4_{L^4} \leq C_2 ||\nabla u||_2 ||u||^3_2
\]
for some constants \( C_1, C_2 > 0 \). To bound the second term of the potential energy we note that
\[
\ln(1 + |x-y|) \leq \ln(1 + |x| + |y|) \leq \ln(1 + |x|) + \ln(1 + |y|)
\]
and therefore
\[
|V_2(u)| \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \ln(1 + |x|) u(x)^2 \int_{\mathbb{R}^2} |u(y)|^2 \, dxdy.
\]
The regularity properties of \( E(u) \) are obvious. \(\square\)

Finally, we consider the particle number (or charge) defined by
\[
N(u) = \int_{\mathbb{R}^2} |u(x)|^2 \, dx,
\]
(2.5)
which is also a well-defined quantity on \( X \).

2.2. Scaling properties. If \( \phi_\omega(x) \) is a solution of the stationary equation
\[
-\Delta \phi_\omega(x) + \frac{\gamma}{2\pi} \left( \int_{\mathbb{R}^2} \ln(|x-y|) |\phi_\omega(y)|^2 \, dy \right) \phi_\omega(x) = \omega \phi_\omega(x),
\]
(2.6)
with finite particle number \( N_\omega = N(\phi_\omega) \) then by the virial theorem,
\[
T(\phi_\omega) = \frac{\gamma}{8\pi} N^2(\phi_\omega).
\]
(2.7)
For any \( \sigma > 0 \), the scaled function \( \phi_{\omega,\sigma}(x) = \sigma^{-2} \gamma^{1/2} \phi_\omega(x/\sigma) \) solves
\[
-\Delta \phi_{\omega,\sigma}(x) + \frac{1}{2\pi} \left( \int_{\mathbb{R}^2} \ln(|x-y|) |\phi_{\omega,\sigma}(y)|^2 \, dy \right) \phi_{\omega,\sigma}(x)
= \sigma^{-2} \left( \omega + \frac{\gamma N_\omega \ln \sigma}{2\pi} \right) \phi_{\omega,\sigma}(x).
\]
(2.8)
If we choose \( \sigma = \sigma_\omega \) such that \( \omega + \frac{\gamma N_\omega \ln \sigma_\omega}{2\pi} = 0 \), then \( \phi := \phi_{\omega,\sigma_\omega} \) is independent of \( \omega, \gamma \) and satisfies the Schrödinger equation
\[
-\Delta \phi(x) + \frac{1}{2\pi} \left( \int_{\mathbb{R}^2} \ln(|x-y|) |\phi(y)|^2 \, dy \right) \phi(x) = 0.
\]
(2.9)
Define
\[
\Lambda_0 := N(\phi).
\]
(2.10)
Then the virial theorem \((2.7)\) reads
\[ T(\phi) = \frac{\Lambda_0^2}{8\pi}. \]

From \((2.9)\) we get \(V(\phi) = -T(\phi)\), hence
\[ E(\phi) = T(\phi) + \frac{1}{2} V(\phi) = \frac{\Lambda_0^2}{16\pi}. \]

The scaling of the particle number
\[ \Lambda_0 = \sigma^{-2} \omega^2 \gamma N \]
yields a relation between frequency \(\omega\) and the particle number \(N\), which we shall discuss in detail in the following section for ground state solutions.

3. Ground states

3.1. Existence of ground states. We consider the following minimization problem:
\[ e_0(\lambda) = \inf \{ E(u), u \in X, N(u) = \lambda \}. \]
We note that the functional \(u \to E(u)\) is not convex since the quadratic form \(f \to \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x-y| f(x)f(y) \, dx \, dy\) is not positive so that standard convex minimization does not apply. We shall prove the following theorem:

**Theorem 3.1.** For any \(\lambda > 0\) there is a spherically symmetric decreasing \(u_\lambda \in X\) such that \(e_0(\lambda) = E(u_\lambda)\) and \(N(u_\lambda) = \lambda\).

**Proof.** Let \((u_n)_n\) be a minimizing sequence for \(e_0(\lambda)\), that is \(N(u_n) = \lambda\) and \(\lim_{n \to \infty} E(u_n) = e_0(\lambda)\). We also may assume that \(E(u_n)\) is uniformly bounded above. Denoting \(u^*\) the spherically symmetric-decreasing rearrangement of \(u\) we have (see e.g. lemma 7.17 in [11])
\[ T(u) \geq T(u^*), \quad N(u^*) = N(u). \]
Applying the decomposition \(V(u) = V_1(u) + V_2(u)\) defined in \((2.4)\) we may apply the strict version of Riesz’s rearrangement inequality (see e.g. theorem 3.9 in [11]) to \(V_1(u)\) since \(\ln(1+1/|x|)\) is positive and strictly symmetric-decreasing. Therefore
\[ V_1(u) \geq V_1(u^*) \]
with equality only if \(u(x) = u^*(x - x_0)\) for some \(x_0 \in \mathbb{R}^2\). For the second term \(V_2(u)\) we apply the following rearrangement inequality:

**Lemma 3.2.** Let \(f, g\) be two nonnegative functions on \(\mathbb{R}\), vanishing at infinity with spherically symmetric-decreasing rearrangement \(f^*, g^*\), respectively. Let \(v\) be a nonnegative spherically symmetric increasing function. Then
\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x)v(x-y)g(y) \, dx \, dy \geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f^*(x)v(x-y)g^*(y) \, dx \, dy \]

**Proof.** The proof follows the same lines as in [12], lemma 3.2, where we proved the corresponding lemma in one space dimension. We give it here for the sake of
completeness. If \( v \) is bounded, \( v \leq C \), then \( (C - v)^* = C - v \) and by Riesz’s rearrangement inequality (lemma 3.6 in [11]) we have
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x)(C - v(x - y))g(y) \, dx \, dy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f^*(x)(C - v(x - y))g^*(y) \, dx \, dy.
\]
Since
\[
\int_{\mathbb{R}^2} f(x) \, dx \int_{\mathbb{R}^2} g(y) \, dy = \int_{\mathbb{R}^2} f^*(x) \, dx \int_{\mathbb{R}^2} g^*(y) \, dy
\]
the claim follows. If \( v \) is unbounded we define a truncation by \( v_n(x) = \sup (v(x), n) \) and apply the monotone convergence theorem.

By the preceding lemma we have
\[
V_2(u) \geq V_2(u^*)
\]
and consequently
\[
V(u) \geq V(u^*)
\]
with equality only if \( u(x) = e^{i\theta} u^*(x - x_0) \) for some \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^2 \). Therefore we may suppose that \( u_n = u_n^* \). We claim that \( u_n^* \in X \). Indeed, by Newton’s theorem (see e.g. theorem 9.7 in [11]) we have
\[
V(u_n^*) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u_n^*(x)^2 (\ln |x|) \left( \int_{B_{|x|}} u_n^*(y)^2 \, dy \right) \, dx
\]
\[
+ \frac{1}{2\pi} \int_{\mathbb{R}^2} u_n^*(x)^2 \left( \int_{\mathbb{R}^2 \setminus B_{|x|}} (\ln |y|) u_n^*(y)^2 \, dy \right) \, dx
\]
where \( B_{|x|} \) denotes the disc of radius \(|x|\) centered at the origin. Since \( \ln |y| \geq \ln |x| \) for all \( y \in \mathbb{R}^2 \setminus B_{|x|} \) we get
\[
V(u_n^*) \geq \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} (\ln |x|) u_n^*(x)^2 \, dx.
\]
Using
\[
\ln |x| \geq \ln(1 + |x|) - \frac{1}{|x|}
\]
and the sharp Sobolev inequality between the linear operators \(-\Delta\) and \( \frac{1}{|x|} \)
\[
\int_{\mathbb{R}^2} \frac{1}{|x|} |u(x)|^2 \, dx \leq 2||u||_2||\nabla u||_2
\]
we finally get
\[
V(u_n^*) \geq \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} \ln(1 + |x|) u_n^*(x)^2 \, dx - \frac{\lambda^{3/2}}{\pi} ||\nabla u_n^*||_2.
\]
Hence
\[
E(u_n^*) \geq ||\nabla u_n^*||_2^2 + \frac{\gamma \lambda}{4\pi} \int_{\mathbb{R}^2} \ln(1 + |x|) u_n^*(x)^2 \, dx - \frac{\gamma \lambda^{3/2}}{2\pi} ||\nabla u_n^*||_2
\]
proving our claim. We may extract a subsequence which we denote again by \((u_n^*)_n\) such that \( u_n^* \to u^* \) weakly in \( X \), strongly in \( L^2 \) and a.e. where \( u^* \in X \) is a nonnegative spherically symmetric decreasing function. Note that \( u^* \neq 0 \) since \( N(u^*) = \lambda \). We want to show that \( E(u^*) \leq \liminf_{n \to \infty} E(u_n^*) \). Since
\[
T(u^*) \leq \liminf_{n \to \infty} T(u_n^*)
\]
it remains to analyze the functional $V(u)$. Let

$$\eta(x) = \int_{B_{|x|}} |u^*(y)|^2 \, dy, \quad \eta_n(x) = \int_{B_{|x|}} |u^*_n(y)|^2 \, dy.$$ 

Then $\eta_n(x) \to \eta(x)$ uniformly since

$$||\eta_n(x) - \eta(x)||_\infty \leq ||u^*_n - u^*||_2 \cdot ||u^*_n + u^*||_2 \leq 2\sqrt{\lambda}||u^*_n - u^*||_2.$$

We note that for any spherically symmetric density $|u(x)|^2$ with $u \in X$ we may simplify (3.3) to

$$V(u^*_n) = \frac{1}{\pi} \int_{\mathbb{R}^2} u^*_n(x)^2 (\ln |x|) \left( \int_{B_{|x|}} u^*_n(y)^2 \, dy \right) \, dx.$$ 

Therefore, by the definition of $\eta_n, \eta$, we have

$$V(u^*_n) - V(u^*) = \frac{1}{\pi} \int_{\mathbb{R}^2} (\ln |x|) |u^*_n(x)|^2 \left( \eta_n(x) - \eta(x) \right) \, dx$$

$$+ \frac{1}{\pi} \int_{\mathbb{R}^2} (\ln |x|) \eta(x) \left( |u^*_n(x)|^2 - |u^*(x)|^2 \right) \, dx$$

(3.5)

As $n \to \infty$ the first integral in (3.5) will tend to zero. In order to analyze the second integral we again decompose $\ln |x|$ according to (2.3). Then

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \ln(1 + |x|) \eta(x) \left( |u^*_n(x)|^2 - |u^*(x)|^2 \right) \, dx$$

will remain nonnegative since the continuous functional $\phi \to \int_{\mathbb{R}^2} \ln(1 + |x|) \eta(x) |\phi(x)|^2 \, dx$ is positive while

$$- \frac{1}{\pi} \int_{\mathbb{R}^2} \ln(1 + \frac{1}{|x|}) \eta(x) \left( |u^*_n(x)|^2 - |u^*(x)|^2 \right) \, dx$$

converges to zero since by Hölder’s inequality and inequality (3.4) we have the estimate

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \ln(1 + \frac{1}{|x|}) \eta(x) \left( |u^*_n(x)|^2 - |u^*(x)|^2 \right) \, dx$$

$$\leq \frac{2}{\pi}||\eta||_\infty ||\nabla (u^*_n + u^*)||_2^{1/2} ||u^*_n + u^*||_2^{1/2} ||\nabla (u^*_n - u^*)||_2^{1/2} ||u^*_n - u^*||_2^{1/2}$$

$$\leq C||u^*_n - u^*||_2^{1/2}$$

for a positive constant $C$. Hence

$$V(u^*) \leq \liminf_{n \to \infty} V(u^*_n)$$

proving the theorem.

\[\square\]

3.2. **Uniqueness of ground states.** Let $u_\lambda$ denote the solution of the minimization problem (5.3) found in theorem .

**Theorem 3.3.** For any $\lambda > 0$ the solution $u_\lambda$ of the minimization problem (5.3) is unique in the following sense: If $v_\lambda \in X$ is such that $e_0(\lambda) = E(v_\lambda)$ and $N(v_\lambda) = \lambda$, then $v_\lambda \in \{e^{i\theta}u^*(x - x_0), \theta \in \mathbb{R}, x_0 \in \mathbb{R}^2\}$. 
We define the quantities and the potential Then \( W_p > 0 \) since, for any \( \lambda > 0 \) there is a Lagrange multiplier \( \omega \) such that, at least in a weak sense
\[
(3.6) \quad -\Delta u_\lambda(x) + \frac{\gamma}{2\pi} \left( \int_{\mathbb{R}^2} \ln(|x-y|) |u_\lambda(y)|^2 \, dy \right) u_\lambda(x) = \omega u_\lambda(x).
\]

We define the quantities
\[
I_\lambda := \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|y|) |u_\lambda(y)|^2 \, dy, \quad E_\lambda := \omega - \gamma I_\lambda
\]
and the potential
\[
W_\lambda(x) := \frac{1}{2\pi} \left( \int_{\mathbb{R}^2} \ln(|x-y|) |u_\lambda(y)|^2 \, dy \right) - I_\lambda.
\]

Then \( W_\lambda(x) \geq 0 \) for all \( x \in \mathbb{R}^2 \), \( W_\lambda(0) = 0 \) and \( \Delta W_\lambda = |u_\lambda|^2 \). The function \( u_\lambda \) is the ground state of the Schrödinger operator \(-\Delta + \gamma W_\lambda\) with eigenvalue \( E_\lambda \) and therefore \( E_\lambda > 0 \). The rescaled functions
\[
(3.7) \quad u(|x|) := \frac{\sqrt{\gamma}}{E_\lambda} u_\lambda(x/\sqrt{E_\lambda}), \quad W(|x|) := \frac{\gamma}{E_\lambda} W_\lambda(x/\sqrt{E_\lambda})
\]
then \( u, W \) satisfy the universal equations
\[
(3.8) \quad \Delta u = (W - 1)u, \quad \Delta W = |u|^2.
\]

In [7] it was shown that (3.3) admits a unique spherically symmetric solution \((u, W)\) such that \( u \) is a positive decreasing function vanishing at infinity. See also theorem 3.2 of the present paper where we consider a more general system. \( \square \)

3.3. Dependence on parameters and admissible frequencies. We determine explicitly \( e_0(\lambda) \) as a function of \( \lambda \) and the range of admissible frequencies \( \omega \) for ground states of the form (1.4). We consider the unique spherically symmetric solution \((u, W)\), \( u > 0 \) and \( u \) vanishing at infinity, of the universal system (3.3). Let
\[
(3.9) \quad N := \int_{\mathbb{R}^2} |u(|x|)|^2 \, dx, \quad I := \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x|) |u(|x|)|^2 \, dx.
\]

These quantities are finite since, for any \( p > 0 \), \( u \) decays faster than \( \exp(-px) \) at infinity and in particular \( u \in X \). The corresponding numerical values are easily computed from the solution of and we give them in the appendix. By the scaling (3.7) we obtain the following identity for the parameters \( \gamma, \lambda \) and the Lagrange multiplier \( \omega \):
\[
(3.10) \quad \omega = \frac{\gamma \lambda}{N} (1 + I - \frac{N}{4\pi} \ln \frac{\gamma \lambda}{N}).
\]

Multiplying the variational equation (3.4) by \( u_\lambda \) and integrating we obtain
\[
2E(u_\lambda) - T(u_\lambda) = T(u_\lambda) + \gamma V(u_\lambda) = \omega \lambda.
\]

Applying the virial relation (2.4) and using (3.10) we finally have
\[
(3.11) \quad e_0(\lambda) = \frac{\gamma \lambda^2}{16\pi} \left( 1 + 8\pi \frac{1+I}{N} - 2 \ln \left( \frac{\gamma \lambda}{N} \right) \right).
\]
with \( N,I \) given in (3.9). Note that (3.11) can be also obtained by integrating \( \omega = \omega(\lambda) \) in (3.10) with respect to \( \lambda \) since, at least formally,

\[
\frac{de_0(\lambda)}{d\lambda} = \omega.
\]

Taking \( \phi \) in (2.9) as the unique ground state solution the relation (3.10) between \( \lambda, \omega \) and \( \gamma \) simplifies to

(3.12)

\[
\omega = -\frac{\gamma \lambda}{4\pi} \ln \frac{\gamma \lambda}{\Lambda_0}
\]

with \( \Lambda_0 = N(\phi) \) as defined in (2.10). The energy is then given by

(3.13)

\[
e_0(\lambda) = \frac{\gamma \lambda^2}{16\pi} (1 - 2 \ln \frac{\gamma \lambda}{\Lambda_0}).
\]

The relations (3.10), respectively (3.12), yield the following result on admissible frequencies:

**Theorem 3.4.** (1) For any \( \omega \leq 0 \) the Schrödinger-Newton system (1.1) admits a unique spherically ground state of the form (1.4). In particular, for the solution \( \phi \) of the stationary equation (2.9) the energy \( e_0(\lambda) \) attains its maximum.

(2) For any \( 0 < \omega < \omega^\ast \) with

\[
\omega^\ast = \frac{N}{4\pi e} \exp \left( \frac{1}{N} \right) = \frac{\Lambda_0}{4\pi e}
\]

the Schrödinger-Newton system (1.1) admits two spherically ground states of the form (1.4) with different particle numbers.

(3) For \( \omega = \omega^\ast \) there is a unique ground state solution of the form (1.4).

4. A LOGARITHMIC HARDY–LITTLEWOOD–SOBOLEV INEQUALITY

In [13], E. Carlen and M. Loss proved the following logarithmic Hardy-Littlewood-Sobolev inequality:

(4.1)

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \ln(|x-y|) f(x)f(y) \, dx \, dy \leq \frac{1}{n} \int_{\mathbb{R}^n} f(x) \ln f(x) \, dx + C_0
\]

for all nonnegative real-valued \( f \) with

\[
\int_{\mathbb{R}^n} f(x) \, dx = 1, \quad \int_{\mathbb{R}^n} f(x) \ln(1 + |x|) \, dx, \quad \int_{\mathbb{R}^n} f(x) \ln f(x) \, dx < \infty
\]

and sharp constant \( C_0 \). We focus on the case \( n = 2 \). Replacing \( f \) by \(|u|^2\) with \( N(u) = \lambda \) this reads as

(4.2)

\[
-2\pi V(u) \leq \frac{\lambda}{2} \int_{\mathbb{R}^2} |u(x)|^2 \ln |u(x)|^2 \, dx + C_0 \lambda^2 - \frac{\lambda^2}{2} \ln \lambda.
\]

Applying the logarithmic Sobolev inequality (13, see also 11, theorem 8.14)

(4.3)

\[
\int_{\mathbb{R}^2} |u(x)|^2 \ln |u(x)|^2 \, dx \leq \lambda \ln(T(u)) - \lambda(1 + \ln \pi)
\]

to the integral on the r.h.s. of (4.2) and using \( 2C_0 = 1 + \ln \pi \) when \( n = 2 \) we get

\[
-V(u) \leq \frac{\lambda^2}{4\pi} \ln \frac{T(u)}{\lambda}.
\]

However, this inequality is not sharp. Here we give the sharp version:
Theorem 4.1. For any \( u \in X \) the following inequality holds:

\[
- V(u) \leq \frac{\lambda^2}{4\pi} \ln \frac{8\pi T(u)}{N\lambda} - \left( \frac{I + 1}{N} - \frac{1}{8\pi} \right) \lambda^2
\]

where \( \lambda = \int_{\mathbb{R}^2} |u|^2 \, dx \) and the constants \( I \) and \( N \) are given in (3.9).

Proof. By theorem 3.2 we have for any \( u \in X \) with \( \lambda = \int_{\mathbb{R}^2} |u|^2 \, dx \) and for any coupling constant \( \gamma > 0 \) the sharp inequality

\[
T(u) + \frac{\gamma}{2} V(u) \geq e_0(\lambda)
\]

with \( e_0(\lambda) \) given in (3.11). We optimize with respect to \( \gamma \). The optimal \( \gamma \) is given by

\[
\frac{1}{4\pi} \ln \frac{\gamma}{N} = -\lambda^{-2} V(u) + \frac{I + 1}{N} - \frac{1}{8\pi}
\]

which yields the desired inequality. \( \square \)

5. Purely angular excitations

In this section we prove the existence of non radial solutions of the Schrödinger-Newton system \( \| \psi(t, r, \theta) \| = 0 \) which in polar coordinates \((r, \theta)\) are of the form

\[
\psi(t, r, \theta) = \phi_\omega(r) e^{im\theta - i\omega t}, \quad \phi_\omega(r) \geq 0
\]

for positive integers \( m \). These are eigenfunctions of the angular momentum operator \( L = -i \partial_\theta \). We define the function space \( X_{m}^{(s)} \) of spherically symmetric functions \( u : \mathbb{R}^2 \to \mathbb{C} \) by

\[
X_{m}^{(s)} = \{ u = u(|x|) : \int_{\mathbb{R}^2} |\partial_r u|^2 + \frac{m^2}{r^2} ||u||^2 + |u|^2 + ln(1 + r) |u|^2 \, dx < \infty \}.
\]

We define the energy functional for purely angular excitations by

\[
E_m(u) = T_m(u) + \frac{\gamma}{2} V(u), \quad T_m(u) = \int_{\mathbb{R}^2} |\partial_r u|^2 + \frac{m^2}{r^2} ||u||^2 \, dx,
\]

We consider the minimization problem

\[
e_m(\lambda) = \inf\{ E_m(u), u \in X_{m}^{(s)}, N(u) = \lambda \}.
\]

The following theorem holds:

Theorem 5.1. For any \( \lambda > 0 \) there is a nonnegative \( u_\lambda \in X_{m}^{(s)} \) such that \( e_m(\lambda) = E_m(u_\lambda) \) and \( N(u_\lambda) = \lambda \).

Proof. Consider any minimizing sequence \( (u_n) \) for \( e_m(\lambda) \). Since \( E_m(u) \) is decreasing under the replacement \( u \mapsto |u| \) we may suppose that \( u_n \geq 0 \). Now we mimic the proof of theorem 3.2 starting from Newton’s theorem (3.6) replacing the energy \( E \) by \( E_m \) since only the spherical symmetry of the functions are required in this part of the proof.\( \square \)

The minimizing \( u_\lambda \) satisfies the Euler-Lagrange equation

\[
- u_\lambda'' - \frac{1}{r} u_\lambda' + \frac{m^2}{r^2} u_\lambda + \frac{\gamma}{2\pi} \left( \int_{\mathbb{R}^2} \ln(|x - y|) |u_\lambda(|y|)|^2 \, dy \right) u_\lambda = \omega u_\lambda.
\]
for \( r \geq 0 \) and some multiplier \( \omega \). By the same rescaling as in the proof of theorem (3.3) we obtain a system of universal equations given by

\[(5.5) \quad u'' + \frac{1}{r} u' - \frac{m^2}{r^2} u = (W - 1)u, \quad W'' + \frac{1}{r} W' = |u|^2\]

such that \( W(0) = W'(0) = 0 \) and \( u \geq 0 \). We note that \( f(r) := r^{-m}u(r) \) then satisfies \( f(0) > 0 \), \( f'(0) = 0 \) and the differential equation

\[f'' + \frac{2m + 1}{r} f' = (W - 1)f\]

from which we easily deduce \( f > 0 \) and \( f' < 0 \) using the facts \( f \geq 0 \) and \( f \to 0 \) as \( r \to \infty \). Also note that for any \( p > 0 \), \( u \) decays faster than \( \exp(-pr) \) at infinity.

Uniqueness of the solution \( u_{\lambda} \in X_m^{(x)} \) follows again from the uniqueness of the solutions of the universal system (5.5) which we prove in the following theorem:

**Theorem 5.2.** For any \( m \geq 0 \) there is a unique solution \((u, W)\) of (5.5) such that \( u > 0 \) for \( r > 0 \) and \( u \to 0 \) as \( r \to \infty \).

**Proof.** Suppose there are two distinct solutions \((u_1, W_1), (u_2, W_2)\) having the required properties. We may suppose \( u_2(r) > u_1(r) \) for \( r \in [0, \bar{r}] \). We consider the Wronskian

\[w(r) = u_2'(r)u_1(r) - u_1'(r)u_2(r)\]

Note that \( w(0) = 0 \) and \( rw(r) \to 0 \) as \( r \to \infty \). It satisfies the differential equation

\[(rw)' = r(W_2 - W_1)u_1u_2\]

Suppose \( u_2(r) > u_1(r) \) for all \( r > 0 \). Then \( W_2(r) > W_1(r) \) for all \( r > 0 \) since \( r(W_2 - W_1)' = \int_0^r (u_2^2 - u_1^2) s \, ds > 0 \) and hence \( (rw)' > 0 \) for all \( r > 0 \) which is impossible. Hence there exists \( \bar{r} > 0 \) such that \( \delta(r) = u_2(r) - u_1(r) > 0 \) for \( r \in [0, \bar{r}] \), \( \delta(\bar{r}) = 0 \) and \( \delta'(\bar{r}) < 0 \). However, then \( w(\bar{r}) = \delta'(\bar{r})u_1(\bar{r}) < 0 \), but \( (rw)'(r) > 0 \) for all \( r < \bar{r} \), that is \( w(r) > 0 \) for all \( r < \bar{r} \) which is again impossible. \( \square \)

**Appendix A. Numerical values**

Let \((u, W)\) be the unique spherically symmetric solution of the universal equations (3.3) such that \( W(0) = W'(0) = 0 \), \( u > 0 \) vanishing at infinity, i.e.

\[(A.1) \quad (ru')' = r(W - 1)u, \quad (rW')' = r|u|^2.\]

By integrating the second equation we get

\[(A.2) \quad N = 2\pi \int_0^\infty u^2(s) s \, ds = 2\pi \lim_{r \to \infty} r W'(r).\]

Multiplying the second equation by \( \ln r \) and using

\[(rW')' \ln r = (rW' \ln r - W)'\]

we obtain after integration

\[(A.3) \quad I = \int_0^\infty u^2(s) s \ln s \, ds = \lim_{r \to \infty} r W'(r) \ln r - W(r).\]

Solving (A.1) numerically we then find

\[(A.4) \quad N = 2\pi \cdot 1.64145 = 10.3135, \quad I = 0.2276\]
and therefore
\begin{equation}
\Lambda_0 = 46.03.
\end{equation}
With these numerical values the logarithmic Hardy-Littlewood-Sobolev inequality \((4.4)\) reads as follows:
\begin{equation}
- V(u) \leq \frac{\lambda^2}{4\pi} \ln \frac{T(u)}{\lambda} - 0.0084\lambda^2.
\end{equation}

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