The variety of commutative additively and multiplicatively idempotent semirings

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Abstract The variety \( \mathcal{Z} \) of commutative additively and multiplicatively idempotent semirings is studied. We prove that \( \mathcal{Z} \) is generated by a single subdirectly irreducible three-element semiring and it has a canonical form for its terms. Hence, \( \mathcal{Z} \) is locally finite despite the fact that it is residually large. The word problem in \( \mathcal{Z} \) is solvable.

Keywords Semiring · Commutative · Additively idempotent · Multiplicatively idempotent · Variety · Locally finite · Residually large · Word problem

The concept of semiring is surprisingly successful in applications both in mathematics and computer science as it was shown by Golan in [6] and Kuich and Salomaa in [8]. In this paper we concentrate on the variety \( \mathcal{C} \) of commutative multiplicatively idempotent semirings which was investigated by the authors already in [2–4] and in the subvariety of this variety satisfying \( x + x \approx x \). The concept of a Boolean semiring was introduced by Guzmán in [7]. It is a commutative multiplicatively idempotent semiring satisfying

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Supporting information

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the identity $1 + x + x \approx 1$. The variety $\mathcal{B}$ of Boolean semirings contains just two subdirectly irreducible members, namely the two-element (distributive) lattice and the two-element Boolean ring. On the contrary, $\mathcal{C}$ has a proper class of subdirectly irreducible members, i.e. it is residually large, as it was proved by the authors in [2]. The natural question arises if by adding another elementary identity to the identities defining $\mathcal{C}$ the number of subdirectly irreducible members can be reduced. In our previous paper we studied members of $\mathcal{C}$ satisfying $x + y + xyz \approx x + y$. In the present paper we investigate the subvariety $\mathcal{Z}$ of $\mathcal{C}$ determined by $x + x \approx x$ and make frequent use of the identity $x + y + xy \approx x + y$. That this identity indeed holds in $\mathcal{Z}$ can be seen as follows:

$$x + y + xy \approx x^2 + xy + yx + y^2 \approx (x + y)^2 \approx x + y.$$  

We show that $\mathcal{Z}$ is locally finite and we provide an explicit upper bound for the cardinalities of finite free algebras in $\mathcal{Z}$. Our main result is to prove that $\mathcal{Z}$ is generated by a single three-element subdirectly irreducible semiring. Moreover, it turns out that there are single semirings of arbitrary cardinality generating $\mathcal{Z}$.

For the reader’s convenience, all the concepts concerning semirings are included. What concerns concepts from Universal Algebra necessary for understanding the paper, the reader is referred to [1].

We start with the definition of a semiring in the sense of the monograph [6] by Golan.

**Definition 1** A semiring is an algebra $S = (S, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that

- $(S, +, 0)$ is a commutative monoid.
- $(S, \cdot, 1)$ is a monoid.
- The operation $\cdot$ is distributive with respect to $+$.
- $x0 = 0x = 0$ for all $x \in S$

$S$ is called commutative if $\cdot$ is commutative, additively idempotent if $+$ is idempotent and multiplicatively idempotent if $\cdot$ is idempotent. Let $\mathcal{Z}$ denote the variety of commutative additively and multiplicatively idempotent semirings.

The corresponding semiring of type $(2, 2)$ has been called a --distributive bisemilattice by Romanowska and a bisemilattice by Pastijn and Zhao. McKenzie and Romanowska [9] showed that the variety of all bisemilattices has exactly 5 subvarieties. Based on this result, Ghosh et al. [5] and Pastijn [10] have shown that the variety of all semirings with type $(2, 2)$ whose additive semigroup is a semilattice and multiplicative semigroup is a band has exactly 78 subvarieties. In this process, the identity $x + y + xy \approx x + y$ plays a role.

Every bounded distributive lattice $L = (L, \lor, \land, 0, 1)$ is a semiring in $\mathcal{Z}$. On the contrary, non-trivial Boolean rings with unit 1 do not belong to $\mathcal{Z}$ since in every non-trivial Boolean ring with 1 we have $1 + 1 + 1 \cdot 1 = 1 \neq 0 = 1 + 1$. In the following we present an example of a semiring in $\mathcal{Z}$ which is not a lattice since it does not satisfy the absorption law.
Example 2 The algebra $S_3 := (S_3, +, \cdot, 0, 1)$ with $S_3 = \{0, a, 1\}$,

\[
\begin{array}{ccc}
+ & 0 & a & 1 \\
0 & 0 & a & 1 \\
a & a & a & a \\
1 & 1 & a & 1 \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\cdot & 0 & a & 1 \\
0 & 0 & 0 & 0 \\
a & 0 & a & a \\
1 & 0 & a & 1 \\
\end{array}
\]

belongs to $\mathcal{Z}$. Because of $1(1 + a) = a \neq 1$, $S_3$ is not a lattice. Let us remark that $S_3$ satisfies the so-called weak-absorption law $x(x + y) \approx x + xy$.

It is easy to see that $S_3$ is the unique semiring in $\mathcal{Z}$ with cardinality less than 4 which is not a lattice.

Now we repeat the definitions of certain members of $\mathcal{Z}$ introduced in [2]:

Definition 3 For every integer $n > 1$ let $S_n$ be the semiring with universe $S_n = \{1, \ldots, n\}$ and operations defined as follows:

\[
x + y := \begin{cases} 
\max(x, y) & \text{for } x \text{ and } y \text{ are odd} \\
y & \text{for } x \text{ is odd and } y \text{ is even} \\
x & \text{for } x \text{ is even and } y \text{ is odd} \\
\min(x, y) & \text{for } x \text{ and } y \text{ are even}
\end{cases}
\]

and

\[
xy := \min(x, y)
\]

($x, y \in S_n$). Here the operation symbols 0 and 1 are interpreted as 1 and $n$, respectively. Moreover, for every infinite cardinal $k$ let $C_k = (C_k, \leq, 0, 1)$ be a bounded chain of cardinality $k$ and $U_k$ be the semiring with universe $U_k := C_k \times \{1, 2\}$ and operations defined as follows:

\[
(x, i) + (y, j) := \begin{cases} 
\max(x, y), 1 & \text{for } (i, j) = (1, 1) \\
y, 2 & \text{for } (i, j) = (1, 2) \\
x, 2 & \text{for } (i, j) = (2, 1) \\
\min(x, y), 2 & \text{for } (i, j) = (2, 2)
\end{cases}
\]

and

\[
(x, i)(y, j) := \begin{cases} 
(x, i) & \text{for } x < y \\
(x, \min(i, j)) & \text{for } x = y \\
y, j & \text{for } x > y
\end{cases}
\]

($(x, i), (y, j) \in U_k$). Here the operation symbols 0 and 1 are interpreted as $(0, 1)$ and $(1, 2)$, respectively.

It is easy to see that the semiring $S_3$ of Example 2 is isomorphic to the semiring $S_3$ of Definition 3.
It was proved in [2] that $S_3$ is subdirectly irreducible. Moreover, in this paper it was shown that for every integer $n > 1$ the semiring $S_n$ is a subdirectly irreducible member of $Z$ of cardinality $n$ and that for every infinite cardinal $k$ the semiring $U_k$ is a subdirectly irreducible member of $Z$ of cardinality $k$. Hence $Z$ is residually large.

We are going to derive a canonical form of terms in $Z$.

In the following let $n$ denote an arbitrary, but fixed non-negative integer and put $N := \{1, \ldots, n\}$.

**Lemma 4** Every term $t(x_1, \ldots, x_n)$ in $Z$ can be written in the form

$$t_A(x_1, \ldots, x_n) = \sum_{I \subseteq A} \prod_{s \in I} x_s$$  \hspace{1cm} (1)

where the empty sum is defined as $0$ and the empty product as $1$.

**Proof** Let $W$ denote the set of all sums of products of elements of $\{x_1, \ldots, x_n\}$. Obviously, the elements of $W$ are terms in $Z$. It is easy to see that $0, 1 \in W$ and that the sum of two elements of $W$ again belongs to $W$. Using the distributive law one can see that also the product of two elements of $W$ again belongs to $W$. Idempotency of addition and multiplication shows that the elements of $W$ can be written in the form (1). \hfill \Box

**Corollary 5** There exist only finitely many different terms in $Z$ of fixed finite arity which means that $Z$ is locally finite. The number of different $n$-ary terms in $Z$ is at most $2^{2^n}$.

In fact, one can show that $Z$ is locally finite by another method. Indeed, if $S$ is a semiring whose additive semigroup is a semilattice, then $S$ is locally finite if and only if $(S, \cdot)$ is locally finite. It is well-known that every semilattice is locally finite.

Within $Z$ we can write terms in a more economic way.

**Definition 6** A subset $A$ of $2^N$ is called reduced if there does not exist an integer $k > 2$ and pairwise different elements $I_1, \ldots, I_k$ of $A$ with $I_1 \cup \cdots \cup I_{k-1} = I_k$. Let $\tilde{A}$ denote the set of all reduced subsets of $2^N$. The term $t_A$ is called reduced if $A \in \tilde{A}$.

At first, we show that the identity $x + y + xy \approx x + y$ can be extended to arbitrary products of variables as follows:

**Lemma 7** We have

$$Z \models \prod_{i \in I} x_i + \prod_{i \in J} x_i \approx \prod_{i \in I} x_i + \prod_{i \in J} x_i + \prod_{i \in I \cup J} x_i$$

for all $I, J \subseteq N$.

**Proof**

$$Z \models \prod_{i \in I} x_i + \prod_{i \in J} x_i \approx \prod_{i \in I} x_i + \prod_{i \in J} x_i + \prod_{i \in I} x_i \prod_{i \in J} x_i \approx \prod_{i \in I} x_i + \prod_{i \in J} x_i + \prod_{i \in I \cup J} x_i$$

\hfill \Box
In the following we are going to show that every term in $Z$ can be reduced.

**Theorem 8** To every $A \subseteq 2^N$ there exists some $B \in \mathbb{A}$ with $Z \models t_A \approx t_B$.

**Proof** Let $A \subseteq 2^N$. If $A \in \mathbb{A}$ then we are done. Hence assume $A \notin \mathbb{A}$. Then there exist an integer $k > 2$ and pairwise different elements $I_1, \ldots, I_k$ of $A$ with $I_1 \cup \cdots \cup I_{k-1} = I_k$. Then by Lemma 7 we have

$$Z \models \prod_{i \in I_1} x_i + \cdots + \prod_{i \in I_k} x_i \approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_3} x_i + \prod_{i \in I_4} x_i + \cdots + \prod_{i \in I_k} x_i \approx \cdots \approx$$

$$\prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_3} x_i + \prod_{i \in I_4} x_i + \cdots + \prod_{i \in I_k} x_i \approx$$

$$\prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_3} x_i + \prod_{i \in I_4} x_i + \cdots + \prod_{i \in I_k} x_i \approx \prod_{i \in I_1} x_i \prod_{i \in I_2} x_i \prod_{i \in I_3} x_i \prod_{i \in I_4} x_i \prod_{i \in I_{k-1}} x_i \prod_{i \in I_k} x_i$$

and hence

$$Z \models t_A \approx t_A \setminus \{I_k\}.$$

Either $A \setminus \{I_k\} \in \mathbb{A}$ or again one element of $A \setminus \{I_k\}$ can be cancelled. After a finite number of steps one finally ends up with some $B \in \mathbb{A}$ satisfying $Z \models t_A \approx t_B$ (every subset of $2^N$ having less than three elements automatically belongs to $\mathbb{A}$).

**Corollary 9** For $n = 0, 1, 2$ resp. $3$ there are exactly $2, 4, 14$ resp. $122$ reduced $n$-ary terms in $Z$.

**Proof** The proof is evident.

Now we state and prove our main theorem saying that $Z$ is generated by $S_3$.

**Theorem 10** If $A, B \in \mathbb{A}$ then $S_3 \models t_A \approx t_B$ if and only if $A = B$.

**Proof** Assume $t_A \approx t_B$ and $A \neq B$. Without loss of generality $A \setminus B \neq \emptyset$. Let $I \in A \setminus B$. Since $I = \emptyset$ would imply $1 = t_A(0, \ldots, 0) = t_B(0, \ldots, 0) = 0$ we have $I \neq \emptyset$. Let $i \in I$. Put

$$a_j := \begin{cases} 0 & \text{if } j \in N \setminus I \\ a & \text{if } j = i \\ 1 & \text{if } j \in I \setminus \{i\} \end{cases}$$
Then \( t_A(a_1, \ldots, a_n) = a \). Hence we have \( t_B(a_1, \ldots, a_n) = a \). From this we conclude that there exists some \( J_i \in B \) with \( i \in J_i \subseteq I \). Since \( I \not\supseteq B \) we have \( J_i \not\supseteq I \). Now define

\[
b_j := \begin{cases} 
0 & \text{if } j \in N \setminus J_i \\
 a & \text{if } j = i \\
 1 & \text{if } j \in J_i \setminus \{i\}
\end{cases}
\]

Then \( t_B(b_1, \ldots, b_n) = a \). Hence \( t_A(b_1, \ldots, b_n) = a \) which shows that there exists some \( K_i \in A \) with \( i \in K_i \subseteq J_i \). Therefore \( i \in K_i \not\supseteq I \). Obviously,

\[
I = \bigcup_{i \in I} K_i.
\]

Let \( K_1, \ldots, K_s \) denote the pairwise distinct elements of \( \{K_i \mid i \in I\} \). Then \( s > 1 \), \( K_1, \ldots, K_s, I \) are pairwise distinct and \( K_1 \cup \cdots \cup K_s = I \). This contradicts \( A \in \mathbb{A} \). Hence \( A = B \).

**Remark 11** It follows from Theorem 10 that \( B \) in Theorem 8 is uniquely determined by \( A \). So we could write \( B = f(A) \) in Theorem 8. For \( A, B \subseteq 2^N \) we then have \( Z \models t_A \cong t_B \) if and only if \( f(A) = f(B) \). Thus, the word problem in \( Z \) is solvable.

**Remark 12** From Corollary 9 and Theorem 10 we conclude that the free algebras in \( Z \) with 0, 1, 2 resp. 3 free generators have 2, 4, 14 resp. 122 elements.

From Theorem 10 we finally obtain

**Theorem 13** \( Z \) is generated by \( S_3 \). Moreover, \( Z \) is generated by every \( S_n \) for each integer \( n \geq 2 \) and by every \( U_k \) for each infinite cardinal \( k \).

**Proof** Let \( A, B \subseteq 2^N \) and assume \( S_3 \models t_A \cong t_B \). According to Theorem 8 there exist \( C, D \in \mathbb{A} \) with \( Z \models t_C \cong t_A \) and \( Z \models t_D \cong t_B \). We conclude \( S_3 \models t_C \cong t_D \). According to Theorem 10 we obtain \( C = D \) and hence \( Z \models t_A \cong t_C = t_D \cong t_B \). Therefore \( S_3 \) generates \( Z \). If \( n > 2 \) is an integer then \( S_n \) has a subalgebra being isomorphic to \( S_3 \), namely the subalgebra with base set \( \{1, 2, n\} \). Hence, every identity holding in \( S_n \) also holds in \( Z \). Analogously, if \( k \) is an infinite cardinal then \( U_k \) has a subalgebra being isomorphic to \( S_3 \), namely the subalgebra with base set \( \{(0, 1), (0, 2), (1, 2)\} \). Hence, every identity holding in \( U_k \) also holds in \( Z \). This completes the proof of the theorem.

**Remark 14** As mentioned in Remark 12 and Corollary 9, the free algebra \( F_Z(x) \) with one free generator \( x \) in \( Z \) consists just of the four elements \( 0, 1, x, 1 + x \). It is easy to see that the equivalence relation \( \Theta \) with classes \( \{0\}, \{x, 1 + x\}, \{1\} \) is a congruence on \( F_Z(x) \) and \( (F_Z(x))/\Theta \cong S_3 \). This shows that also \( F_Z(x) \) generates \( Z \).

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