Generalized junction conditions for collapsing models

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Abstract. We have constructed the general junction conditions on the surface of a dissipating relativistic star. The stellar exterior is a spacetime described by the generalised Vaidya metric and a two-fluid energy-momentum tensor, and therefore, defines the local atmosphere, which must be a super-position of standard null radiation and a general null fluid. We have highlighted briefly that our result will effect the physics of the dissipation at the stellar boundary.

1. Introduction

A compact star is formed when a massive star \( (M_* \geq 8 \, M_\odot) \) breaks away from a state of hydrostatic equilibrium and collapses under its own gravity. The stellar object that is formed at the end of the collapse process is usually very dense and small in size (typical radius: \( R \approx 7 - 20 \, \text{km} \)) as compared to the initial configuration. Consequently, the gravitational field in the interior and exterior of the star is significantly strong, and this is reflected in the curvature of the spacetime geometries, so general relativity is required. During the contraction of the massive star and the evolution of the compact object, gravitational binding energy is converted into heat energy, which is used for ionization and dissociation in the dense interior. The excess heat energy is dissipated as radiation across the stellar surface. This process is described adequately for a relativistic star by constructing the junction conditions at the surface. These equations relate the interior and exterior matter variables as well as the respective geometries.

We have extended and generalized the Santos [1] junction condition for relativistic dissipation, by matching the spacetime geometry for an interior stellar fluid with isotropic pressure to that of the local exterior, which is defined by the generalised Vaidya radiating metric. The physical consequence, here, is that the local atmosphere must be a two-fluid system consisting of standard null radiation (photons), and an additional null fluid that has a characteristic non-zero pressure and energy density. The effect of our result on the surface radiation parameters is also considered.

2. The interior and exterior stellar geometry

The dynamics of the gravitational field in the stellar interior is given by the shear-free metric:

\[
ds^2 = -A^2(r,t)dt^2 + B^2(r,t)[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)],
\]

where \( A(r,t) \) and \( B(r,t) \) are metric potentials and the material inside the star is defined as a relativistic fluid with heat conduction by:

\[
T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a,
\]
where $\mu$ and $p$ are the fluid energy density and isotropic pressure respectively, $g_{ab}$ is the metric tensor field and $u$ and $q$ are the fluid four-velocity and the radial heat flux. The relevant Einstein’s field equations $G_{ab} = T_{ab}$ for the interior are:

$$
p = \frac{1}{A^2} \left( -\frac{2 \dot{B}}{B} - \frac{\dot{\dot{B}}^2}{B^2} + \frac{2 \dot{A} \dot{\dot{B}}}{AB} \right) + \frac{1}{B^2} \left( \frac{\dot{B}^2}{B^2} + \frac{2 A' B'}{A B} + \frac{2 A'}{r A} + \frac{2 B'}{r B} \right), \quad \text{and} \quad (3a)
$$

$$
q = -\frac{2}{AB^2} \left( -\frac{\dot{B}'}{B} + \frac{\dot{B}' B}{B^2} + \frac{A' \dot{B}}{A B} \right), \quad (3b)
$$

where dots and primes denote differentiation with respect to coordinate time $t$ and radial distance $r$ respectively.

The dynamics of the gravitational field in the local region outside the star is described by the generalised Vaidya outgoing radiation metric:

$$
ds^2 = -\left( 1 - \frac{2m(v, r)}{r} \right) dv^2 - 2dvdr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)
$$

where $m(v, r)$ is the mass function across the surface, and is related to the gravitational energy within a given radius $r$. The characteristic feature about the metric given in Eq. (4) is that the mass function contains a spatial dependence in the radial direction during dissipation, and this is significantly different from the standard scenario, in which the mass at the boundary only has a time dependence. Husain [2], and Wang and Wu [3] have shown that an appropriate energy momentum tensor for a two-fluid atmosphere is:

$$
T_{ab}^+ = \varepsilon l_a l_b + (\rho + p) (l_a n_b + l_b n_a) + pg_{ab}, \quad (5)
$$

which represents a super-position of null radiation and an arbitrary null fluid. $\varepsilon$ is the energy density of the photon radiation and $\rho$ and $p$ are the energy density and pressure of the additional null fluid, respectively. In general, $T_{ab}^+$ represents a Type II fluid as defined by Hawking and Ellis [4]. The null vector $l$ is a double null eigenvector of the energy-momentum tensor in Eq. (5) and the vector $n$ is normal to the spatial hypersurface.

For the spacetime metric in Eq. (4) and the energy momentum tensor in Eq. (5), the Einstein’s field equations for the local two-fluid stellar atmosphere can be written as:

$$
\varepsilon = -2 \frac{m_r}{r} v^2, \quad (6a)
$$

$$
\rho = 2 \frac{m_r}{r^2}, \quad \text{and} \quad (6b)
$$

$$
p = -\frac{m_{rr}}{r}. \quad (6c)
$$

It is interesting to note that in the standard framework, the Santos [1] junction condition tells us that on the boundary of the star, the pressure of the stellar fluid is proportional to the magnitude of the heat flux given as:

$$
(p_r)_{\Sigma}^s = (qB)_{\Sigma}, \quad (7)
$$

where the superscript $s$ denotes a quantity as defined in the standard Santos framework. Furthermore, the local atmosphere outside the star contains only pure radiation in the form
of null photons that have a characteristic non-zero energy density given by the Eq. (6a). In our new framework, the additional null fluid component defined by the two extra field Eqs. (6b) and (6c) should significantly transform Eq. (7), and consequently change the way in which the heat energy is dissipated as radiation across the surface.

3. The matching process
The interior and exterior spacetimes given in Eqs. (1) and (4) are matched across the spatial hypersurface (stellar boundary) \( \Sigma \). The intrinsic metric to the hypersurface \( \Sigma \) is defined by:

\[
ds^2_\Sigma = -d\tau^2 + Y^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( \tau \) is an intermediate timelike variable defined on the surface, \( Y = Y(\tau) \) is a metric function and \( \xi^\alpha = (\tau, \theta, \phi) \) are coordinates on the surface.

The first fundamental form \( (ds^2_\Sigma) = (ds^2_\Sigma) - (ds^2_\Sigma) = ds^2_\Sigma \) gives:

\[
A(r_\Sigma, t) dt = \left(1 - \frac{2m}{r_\Sigma} + 2 \frac{d\tau}{dv}\right)^{1/2} dv, \quad \text{and} \quad \tau(v) = rB(r_\Sigma, t),
\]

and the second fundamental form \( (K^+_{\alpha\beta})_\Sigma = (K^-_{\alpha\beta})_\Sigma \) yields the following equations:

\[
m(v, \tau) = \left[ \frac{rB}{2} \left( 1 + r^2 \frac{B'^2}{A^2} - \frac{1}{B^2} (B + rB')^2 \right) \right]_\Sigma, \quad \text{and} \quad (-1) \frac{A'}{B} = \left(1 + r \frac{B'}{B} + r \frac{\dot{B}}{A} \right)^{-1} \times \left[ \frac{m_r}{r} + \frac{B'}{B} + \frac{r B'^2}{2 B^3} - \frac{r \dot{B}^2}{2 BA^2} - \frac{1}{A} \left( \frac{\dot{B}'}{B} + \frac{\ddot{B}}{A} - \frac{r B'B}{B^2} - r \frac{B \dot{B}}{A^2} \right) \right]_\Sigma,
\]

where Eq. (10) is the mass profile at the boundary. With some algebraic simplification and using the interior field Eqs. (3a), (3b) and the exterior field Eq. (6b), Eq. (11) reduces to:

\[
(p)_\Sigma = (qB - p)_\Sigma,
\]

which is a generalization of the Santos junction condition given in Eq. (7).

4. The effect of the exterior null fluid energy density on the dissipation parameters
Since our generalised junction condition in Eq. (12) holds on the surface of the dissipating star, it is expected that the additional null fluid in the local stellar atmosphere should have a significant effect on the radiation parameters associated with the dissipation. This may be important to our understanding of stellar atmospheres for relativistic stars. Shapiro and Teukolsky [5], and Kippenhan and Weigert [6] have discussed the atmospheres of compact objects extensively. Furthermore, investigating the effect of the null fluid energy density, which in our case could also be crucial for modelling the final collapse phase leading up to the point of black hole formation as demonstrated in a recent investigation by Sarwe and Tikekar [7]. We have reconstructed the general profiles of some of the relevant radiation quantities, taking into consideration our new
result in Eq. (12) as:

\[
(p_r)_{\Sigma}^g = \left[ -\frac{2}{B} \left( \frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} \right) - \rho(r,t) \right]_{\Sigma},
\]

(13)

\[
L_{\Sigma}^g = \frac{1}{2} B^2 \left[ p \left( r \frac{\dot{B}}{A} + \frac{(rB')'}{B} \right) + \rho \left( 1 + r \frac{B'}{B} \right) \right] \left( \frac{\dot{B}}{A} + \frac{(rB')'}{B} \right)^{-1},
\]

(14)

\[
L_{\infty}^g = \frac{1}{2} B^2 \left[ p \left( r \frac{\dot{B}}{A} + \frac{(rB')'}{B} \right) + \rho \left( 1 + r \frac{B'}{B} \right) \right] \left( \frac{\dot{B}}{A} + \frac{(rB')'}{B} \right),
\]

(15)

\[
(T)_{\Sigma}^g = \left[ \left( \frac{1}{r^2 B^2} \right) \left( \frac{L_{\Sigma}^g}{4\pi\delta} \right) \right]^{\frac{1}{2}}, \text{ and}
\]

(16)

\[
z_{\Sigma}^g = \left[ \left( r \frac{\dot{B}}{A} + \frac{(rB')'}{B} \right)^{-1} \right] - 1.
\]

(17)

In the Eqs. (13) through (17), the superscript \( g \) denotes the quantity as defined or measured in the generalized framework with the metric in Eq. (4), \( (p_r)_{\Sigma}^g \) represents the pressure of the stellar fluid on the surface, and is obtained from the boundary condition given in Eq. (12). \( L_{\Sigma}^g \) and \( L_{\infty}^g \) are the luminosity of the emitted radiation at the surface of the star and the luminosity as observed at infinity, respectively. \( (T)_{\Sigma}^g \) is the effective surface temperature of the radiation and \( z_{\Sigma}^g \) is the gravitational redshift. All of the above dissipation parameters are constrained and generalized by the energy density \( \rho \), of the exterior null fluid.

5. Discussion
We have generalized the Santos junction condition for dissipation in a compact relativistic star. The inclusion of a radial spatial dependence in the mass function across the stellar boundary leads to the local atmosphere consisting of a two-fluid system comprised of null radiation (photons) as in the standard scenario and an additional null fluid having a characteristic non-vanishing energy density and pressure. Our new boundary condition on the surface of the star shows that the stellar fluid pressure is now constrained by the energy density of the exterior null fluid and that the physics of the relativistic dissipation process may consequently change. This is clear, since in the standard Santos framework, the exterior of the star contains only null radiation but in the generalized framework the atmosphere contains an additional null fluid, which can be interpreted as another matter field that arises due to the dissipation. Our result is an improvement on the work of Santos, since the outside of the relativistic star is more general and more realistic. The matter field mentioned above has been identified as a string fluid but may also be a field of particles like neutrinos. Consequently, the radiation parameters at the surface could be increased or decreased depending on the form of the metric potentials, which have to be obtained by solving the junction condition in Eq. (12). This work is currently in progress.

The profiles for the radiation parameters in our generalized scenario, have been generated, and we have demonstrated that each of these quantities associated with the dissipation is, indeed, affected by the null fluid density. This was not the case in the standard framework. We conclude that our results may be significant for understanding the atmospheres of relativistic stars, and are relevant to previous investigations by Glass and Krisch [8, 9], and Krisch and Glass [10].

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