Unifying the geometric decompositions of full and trimmed polynomial spaces in finite element exterior calculus

Toby Isaac

Abstract

Arnold, Falk, & Winther, in *Finite element exterior calculus, homological techniques, and applications* (2006), show how to geometrically decompose the full and trimmed polynomial spaces on simplicial elements into direct sums of trace-free subspaces and in *Geometric decompositions and local bases for finite element differential forms* (2009) the same authors give direct constructions of extension operators for the same spaces. The two families – full and trimmed – are treated separately, using differently defined isomorphisms between each and the other’s trace-free subspaces and mutually incompatible extension operators. This work describes a single operator $\mathcal{U}_T$ that unifies the two isomorphisms and also defines a weighted-$L^2$ norm appropriate for defining well-conditioned basis functions and dual-basis functionals for geometric decomposition. This work also describes a single extension operator $\mathcal{E}_{s,T}$ that implements geometric decompositions of all differential forms as well as for the full and trimmed polynomial spaces separately.

1 Introduction

This introduction, describing the contribution of the subsequent sections, is intended for those familiar with finite elements but not with exterior calculus.

1.1 Trace-free scalar-valued finite elements

Let $\mathcal{T}_h$ be a triangulation of a domain $\Omega \subset \mathbb{R}^2$, and consider the standard finite element space $\mathcal{P}_3(\mathcal{T}_h)$ of continuous, real-valued functions that are polynomials with degree at most 3 when restricted to each triangle,

$$\mathcal{P}_3(\mathcal{T}_h) := \{ f \in C^0(\Omega) : f|_T \in \mathcal{P}_3(T), T \in \mathcal{T}_h \}.$$

A standard finite element basis for $\mathcal{P}_3(\mathcal{T}_h)$ associates one basis function to each vertex, two to each edge, and one to each triangle, with each basis function supported only on the triangles surrounding its associated mesh object.

Let $T$ be a triangle in $\mathcal{T}_h$ with vertices at $\{v_0, v_1, v_2\}$, and let $\lambda : T \to \mathbb{R}^3$ be its barycentric coordinates, the unique affine map satisfying $\lambda(v_j) = \delta_{ij}$ and $\sum \lambda_i = 1$. The basis function $\phi_T$ associated with $T$, to be fully continuous and supported only on $T$, must be zero on the boundary $T$: we say that $\phi_T$ is trace-free. Because $\phi_T$ is in $\mathcal{P}_3(T)$ and zero on each of the three edges surrounding $T$, $\phi_T$ must be a multiple of the function

$$\lambda_T := \lambda_0 \lambda_1 \lambda_2.$$

This is the bubble function associated with $T$.

Now let $E$ be an edge in the mesh with vertices $\{v_0, v_1\}$, and let $\lambda$ be its barycentric coordinates. The two basis functions $\{\phi_{E,1}, \phi_{E,2}\}$ associated with $E$, to be continuous and supported only on the triangles on either side of $E$, must be zero at $v_0$ and $v_1$. But while the basis function $\phi_T$ was determined up to a constant, there is more freedom in what $\phi_{E,1}$ and $\phi_{E,2}$ can be: they must vanish on the non-shared edges of the surrounding triangles, but when restricted to $E$, may satisfy

$$\phi_{E,1}|_E = \lambda_{E}p_1, \quad \phi_{E,2}|_E = \lambda_{E}p_2, \quad [\lambda_{E} := \lambda_0 \lambda_1]$$

1
for any basis \( \{p_1, p_2\} \) of \( \mathcal{P}_1(E) \).

The pattern that this example demonstrates is that, for scalar-valued polynomials on an \( n \)-dimensional simplex \( T \), there is an isomorphism between \( \mathcal{P}_r(T) \), the polynomials on \( T \) up to degree \( r \), and the trace-free polynomials up to degree \( r + n + 1 \),

\[
\bar{\mathcal{P}}_{r+n+1}(T) := \{ p \in \mathcal{P}_{r+n+1}(T) : p(x)|_{\partial T} = 0 \}.
\]

This isomorphism can be realized by an operator \( h_T : \mathcal{P}_r(T) \to \bar{\mathcal{P}}_{r+n+1}(T) \) defined by pointwise multiplication with the bubble function,

\[
h_T(p) := \lambda_T p.
\]

We point out that because this operator is defined pointwise it can be applied to any smoothly continuous function on \( T \), not just a polynomial, to construct a trace-free function, and it transforms any basis of \( \mathcal{P}_r(T) \) into a basis for \( \bar{\mathcal{P}}_{r+n+1}(T) \).

### 1.2 Scalar-valued finite element extension operators

To complete the definition of the edge-centered basis function \( \phi_{E,1} \) from the above example, we must define its values \( \phi_{E,1}^T \) in each triangle \( T \) adjacent to \( E \): we call this extending \( \phi_{E,1} \) into \( T \). For the basis function to be a part of \( \mathcal{P}_r(\mathcal{T}_h) \), the extension \( \phi_{E,1}^T \) must be in \( \mathcal{P}_3(T) \), and it must be consistent: restricting back to \( E \) should match the definition on the edge,

\[
\phi_{E,1}^T|_E = \phi_{E,1},
\]

and restricting it to any \( \tilde{E} \neq E \) should be identically zero,

\[
\phi_{E,1}^T|_{\tilde{E}} = 0, \quad \tilde{E} \neq E.
\]

Suppose \( E \) is the edge bounded by \( \{v_0, v_1\} \) and \( T \) is the triangle bounded by \( \{v_0, v_1, v_2\} \). Because \( \phi_{E,1}|_E = \lambda_0\lambda_1 p_1 \) for some \( p_1 \in \mathcal{P}_1(E) \), one way we can define a consistent extension \( \phi_{E,1}^T \) is by projecting each point in \( T \) to a point in \( E \) and evaluating \( p_1 \),

\[
\phi_{E,1}^T(\lambda_0, \lambda_1, \lambda_2) := \lambda_1\lambda_2 p_1(\lambda_0 + \frac{1}{2}\lambda_2, \lambda_1 + \frac{1}{2}\lambda_2).
\]

Note that this procedure works not just for extending \( \bar{\mathcal{P}}_3(E) \) into \( \mathcal{P}_3(T) \): it extends \( \bar{\mathcal{P}}_r(E) \) into \( \mathcal{P}_r(T) \) for any \( r \), and its extension of \( \lambda_0\lambda_1 g \) is a consistent extension for any \( g \) that is smooth on \( \mathcal{E} \).

### 1.3 Finite element exterior calculus

Finite element exterior calculus is a framework that unifies the theory behind \( H^1 \)-conforming finite elements (spanned by fully-continuous basis functions like \( \mathcal{P}_3(\mathcal{T}_h) \) in the preceding section) with that of \( H(\text{div}) \)-conforming elements (which are vector-valued and spanned by basis functions whose normal components only must be continuous on the facets between cells) and \( H(\text{curl}) \)-conforming elements (whose tangential components only must be continuous on the facets and edges between cells).

In finite element exterior calculus, for each \( k \in \mathbb{N}_0 \) there is a space \( \Lambda^k(\Omega) := C^\infty(\Omega; \wedge^k \mathbb{R}^n) \) of differential \( k \)-forms, smooth functions that take values in \( \wedge^k \mathbb{R}^n \), the vector space of algebraic \( k \)-forms, the alternating \( k \)-linear maps that map \( \bigotimes_{i=1}^k \mathbb{R}^n \) to \( \mathbb{R} \). For each \( k \) there is a differential operator \( d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega) \) that generalizes \( \text{grad} \ (k = 0) \), \( \text{curl} \ (k = 1) \), and \( \text{div} \ (k = n-1) \), and an associated Sobolev space \( H\Lambda^k(\Omega) \) that is the closure of \( \Lambda^k(\Omega) \) under the norm

\[
\|v\|_{H\Lambda^k}^2 := \int_\Omega |v|^2 + |dv|^2 \ dx.
\]

To construct \( H\Lambda^k(\Omega) \)-conforming finite element spaces, one must construct piecewise smooth functions that are trace-continuous at the boundaries between cells, where trace continuity is the generalization of the normal continuity of \( H(\text{div}) \)-conforming spaces and tangential continuity of \( H(\text{curl}) \)-conforming spaces.
Given a simplex $T$, let $\hat{\Lambda}^k(T)$ be the trace-free $k$-forms that are smoothly continuous on $T$. (This is distinct from $\Lambda^k(T)$ as defined in other works, which is the subspace of $\Lambda^k(T)$ of functions compactly supported in the interior of $T$.) Given a function space $X(T) \subset \hat{\Lambda}^k(T)$, identifying its trace-free subspace $\hat{X}(T) := X(T) \cap \hat{\Lambda}^k(T)$ is important for finite element construction. These are the basis functions that can be extended by zero outside of $T$ while maintaining trace continuity, so these are the basis functions associated with $T$ in a basis for the whole mesh $\mathcal{T}_h$. Likewise, identifying $\hat{X}(E)$ for a lower-dimensional simplex $E$ is important because the basis functions associated with $E$ are extensions of $\hat{X}(E)$ into the cells surrounding $E$. Constructing $X(T)$ as the direct sum of $\hat{X}(T)$ and extensions of $\hat{X}(E)$ for $E$ in the boundary of $T$ is a geometric decomposition of $X(T)$.

There are two primary families of finite elements for the simplex $T$ in finite element exterior calculus, chosen for their homological properties: the full polynomial spaces $\mathcal{P}_r \Lambda^k(T)$ and the trimmed spaces $\mathcal{P}_r^- \Lambda^k(T)$, which are defined in section 2. These two families are intertwined, because each is isomorphic to the family of trace-free subspaces of the other:

$$\mathcal{P}_r \Lambda^k(T) \cong \hat{\mathcal{P}}_{r+k+1} \Lambda^{n-k}(T);$$

$$\mathcal{P}_r^- \Lambda^k(T) \cong \hat{\mathcal{P}}_{r+k} \Lambda^{n-k}(T).$$

The work of Arnold, Falk, and Winther [2] is a standard reference to which we direct the reader for more details on finite element exterior calculus as well as proofs of isomorphisms [11] and [2, 2, theorems 4.16 and 4.22]. In that work, and in subsequent work on the topic of geometric decompositions [3, 4], the two isomorphisms are treated separately, in that two distinct linear operators are defined, $h^k_T : \mathcal{P}_r \Lambda^k(T) \to \hat{\mathcal{P}}_{r+k+1} \Lambda^{n-k}(T)$ and $h^k_{T,-} : \mathcal{P}_r^- \Lambda^k(T) \to \hat{\mathcal{P}}_{r+k} \Lambda^{n-k}(T)$ (though they are anonymous in [2]), and they are proved to be bijections. Furthermore, the operators $h^k_T$ and $h^k_{T,-}$ are defined with respect to particular choices of basis functions, such that is not clear whether $h^k_{T,-}$ acts pointwise or not.

Subsequent works by the same authors and others [3, 4] give direct constructions of the trace-free subspaces that are based on Bernstein polynomials. Bernstein polynomials have desirable symmetry properties and low-complexity evaluation algorithms but are known to be ill-conditioned for finite element operations relative to other bases. Giving these tradeoffs, it is potentially useful to have a uniform method for constructing trace-free $k$-forms that is not tied to any basis. In [2], the authors define two extension operators from a boundary simplex $f$ into a simplex $g$,

$$E^{r,k}_{f,g} : \mathcal{P}_r \Lambda^k(f) \to \mathcal{P}_r \Lambda^k(g),$$

$$E^{r,k,-}_{f,g} : \mathcal{P}_r^- \Lambda^k(f) \to \mathcal{P}_r^- \Lambda^k(g),$$

that have the required consistency for constructing local, trace-continuous basis functions for the two spaces. These two extension operators are not only defined differently, but are truly distinct in that neither is an extension operator for the other: though $\mathcal{P}_r^- \Lambda^k(f) \subset \mathcal{P}_r \Lambda^k(f)$, $E^{r,k}_{f,g} [\mathcal{P}_r^- \Lambda^k(f)] \not\subset \mathcal{P}_r^- \Lambda^k(g)$, and though $\mathcal{P}_r \Lambda^k(f) \subset \mathcal{P}_{r+1} \Lambda^k(f)$, $E^{r+1,k,-}_{f,g} [\mathcal{P}_r \Lambda^k(f)] \not\subset \mathcal{P}_r \Lambda^k(g)$.

### 1.4 Unifying the geometric decompositions of $\mathcal{P}_r^- \Lambda^k(T)$ and $\mathcal{P}_r \Lambda^k(T)$

A unified construction of trace-free $k$-forms

This work defines an operator $\hat{\mathcal{T}}$ from differential $k$-forms to trace-free $(n-k)$-forms that acts pointwise and induces the isomorphisms [11] and [2] and the more general isomorphism for smooth functions on the closure of $T$:

$$\hat{\mathcal{T}} : \Lambda^k(T) \to \Lambda^{n-k}(T);$$

$$\hat{\mathcal{T}} : \mathcal{P}_r \Lambda^k(T) \to \hat{\mathcal{P}}_{r+k+1} \Lambda^{n-k}(T);$$

$$\hat{\mathcal{T}} : \mathcal{P}_r^- \Lambda^k(T) \to \hat{\mathcal{P}}_{r+k} \Lambda^{n-k}(T).$$
The definition of $\hat{\ast}_T$ and proof of these claims are given in the section 5.3. Here we translate $\hat{\ast}_T$ into operators which construct normal-free functions (for $H(\text{div})$-conforming finite elements) and tangent-free functions (for $H(\text{curl})$-conforming finite elements).

Normal-free. Let $\{E_i\}_{i=1}^n$ be the edges around $T$ ($n_e = 3$ if $n = 2$, $n_e = 6$ if $n = 3$), let $e_i = v_{i,1} - v_{i,0}$, where $\{v_{i,0}, v_{i,1}\}$ are the vertices of edge $E_i$, and let $|T|$ be the $n$-dimensional volume of $T$. Then for any $u \in C^\infty(T; \mathbb{R}^n)$,

$$\hat{\ast}_T u = \frac{1}{n|T|} \sum_{i=1}^n (u, e_i) \lambda_{E_i} e_i$$

is a function with normal-free boundary trace. In this translation of $\hat{\ast}_T$, the input vector-valued function is treated as a function in $\Lambda^1(T)$, while the output is treated as a function in $\hat{\Lambda}^{n-1}(T)$. This operator transforms any basis for a Nédélec edge element of the first (second) kind into a basis for the normal-free subspace of a Nédélec face element of the second (first) kind.

Tangent-free. Let $\{F_i\}_{i=0}^n$ be the facets around $T$, let $\nu_i$ be the unit normal vector for facet $i$, and let $\tilde{\nu}_i$ be $\nu_i$ scaled by $(n-1)!|F_i|$, the determinant of a matrix formed from any set of $n-1$ edge vectors around $F_i$ (in 2D, $\tilde{\nu}_i$ is a 90 degree rotation of $v_{i,1} - v_{i,0}$; in 3D, $\tilde{\nu}_i$ is the cross product $(v_{i,1} - v_{i,0}) \times (v_{i,2} - v_{i,0})$). Then for any $u \in C^\infty(T; \mathbb{R}^n)$,

$$\hat{\ast}_T u = \frac{1}{n!|T|} \sum_{i=0}^n (u, \tilde{\nu}_i) \lambda_{F_i} \tilde{\nu}_i$$

is a function with tangent-free boundary trace. This translation of $\hat{\ast}_T$ is the opposite of the last: the input is treated as a function in $\Lambda^{n-1}(T)$ and the output is treated as a function in $\Lambda^1(T)$. This operator transforms any basis for a Nédélec face element of the first (second) kind into a basis for the tangent-free subspace of a Nédélec edge element of the second (first) kind.

A unified extension of trace-free $k$-forms

This work also defines an extension operator $\hat{E}_{f,g} : \hat{\Lambda}^k(T) \rightarrow \Lambda^k(\mathcal{J})$ that acts pointwise and is a consistent extension operator for full and trimmed polynomials and indeed all differential $k$-forms: for every simplex $g$ in the triangulation $\mathcal{T}_h$,

$$\Lambda^k(\mathcal{J}) = \bigoplus_{f \in \Delta(g)} \hat{E}_{f,g}[\hat{\Lambda}^k(T)],$$

$$\mathcal{P}_r\Lambda^k(\mathcal{J}) = \bigoplus_{f \in \Delta(g)} \hat{E}_{f,g}[\mathcal{P}_r\Lambda^k(T)],$$

$$\mathcal{P}_r^-\Lambda^k(\mathcal{J}) = \bigoplus_{f \in \Delta(g)} \hat{E}_{f,g}[\mathcal{P}_r^-\Lambda^k(T)].$$

We do not translate $\hat{E}_{f,g}$ into $H(\text{curl})$ or $H(\text{div})$ notation here, but do note that it is based on a decomposition that generalizes the decomposition present in the hierarchical bases for $\mathcal{P}_r\Lambda^k(T)$ of Ainsworth and Coyle [1]. This work shows that, although trimmed polynomials are not explicitly represented in those bases, their extensions remain trimmed polynomials of the same order.

2 Preliminaries and notation

Integer maps, permutations, and multi-indices. We let $[a:b]$ be the set $\{a, a + 1, \ldots, b\}$ if $b \geq a$ and $[a:b] = \emptyset$ otherwise. For a map $\rho : [a:b] \rightarrow S$ and $i \in [a:b]$ we define $\rho \setminus \rho(i) : [a:b - 1] \rightarrow S$ by

$$(\rho \setminus \rho(i))(a), \ldots, (\rho \setminus \rho(i))(b - 1) = (\rho(a), \ldots, \hat{\rho}(i), \ldots, \rho(b)),$$

where $\hat{\rho}(i)$ is the omitted range index. Given any map $\rho : M \rightarrow S$ we let $[\rho]$ be the range set $\{\rho(i)\}_{i \in M}$. 


Given any two subsets $M$ and $S$ of $\mathbb{N}_0$ we let $\Sigma(M, S)$ be the set of increasing maps from $M$ to $S$. We adopt the convention that $\Sigma(\emptyset, S)$ contains only the empty map $\emptyset \to \emptyset$. If $M$ is a subset of $S$ and $\rho \in \Sigma(M, S)$, we define its complement to be $\rho^* \in \Sigma(S \setminus M, S)$ such that $[\rho] \cup [\rho^*] = S$. The two complementary maps together define a permutation: we let $\sigma(\rho) \in \{-1, 1\}$ be $1$ if that permutation is even and $-1$ if it is odd.

When treating a map into $\alpha : S \to \mathbb{N}_0$ as a multi-index, we use the standard notation $|\alpha| = \sum_{i \in S} \alpha(i)$.

**Exterior algebra.** We let $\operatorname{Alt}^k V$ denote the alternating $k$-linear forms, or algebraic $k$-forms, on the vector space $V$. By “alternating” we mean that $\omega \in \operatorname{Alt}^k V$ satisfies

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k), \quad i, j \in [1:k].$$

Given $\omega \in \operatorname{Alt}^j V$ and $\mu \in \operatorname{Alt}^k V$, the *wedge product* $\omega \wedge \mu \in \operatorname{Alt}^{j+k} V$ is defined by

$$(\omega \wedge \mu)(v_1, \ldots, v_{j+k}) := \sum_{\sigma \in \Sigma([1:j] \cup [1:k])} \operatorname{sign}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(j)}) \mu(v_{\sigma(j+1)}, \ldots, v_{\sigma(j+k)}).$$

The wedge product is bilinear and anti-commuting, $\mu \wedge \omega = (-1)^{jk} \omega \wedge \mu$.

Given a linear map $J : V \to W$, the *pullback* is $J^* : \operatorname{Alt}^k W \to \operatorname{Alt}^k V$ defined by

$$(J^* \omega)(v_1, \ldots, v_k) := \omega(Jv_1, \ldots, Jv_k), \quad \omega \in \operatorname{Alt}^k W,$$

which distributes over the wedge product $J^*(\omega \wedge \mu) = (J^* \omega) \wedge (J^* \mu)$.

Given $\omega \in \operatorname{Alt}^k V$ and $v \in V$, the *interior product* $\omega \lrcorner v \in \operatorname{Alt}^{k-1} V$ is defined by

$$(\omega \lrcorner v)(v_1, \ldots, v_{k-1}) := \omega(v, v_1, \ldots, v_{k-1}).$$

The interior product follows the product rule

$$(\omega \wedge \mu) \lrcorner v = (\omega \lrcorner v) \wedge \mu + (-1)^j \omega \wedge (\mu \lrcorner v), \quad \omega \in \operatorname{Alt}^j V, \mu \in \operatorname{Alt}^k V.$$

The *inner product* on $\operatorname{Alt}^k V$ when $V$ has its own inner product is defined using any orthonormal basis $\{q_k\}_{k=1}^n$ of $V$ as

$$\langle \omega, \mu \rangle := \sum_{\rho \in \Sigma([1:k] \cup [1:n])} \omega(q_{\rho(1)}, \ldots, q_{\rho(k)}) \mu(q_{\rho(1)}, \ldots, q_{\rho(k)}), \quad \omega, \mu \in \operatorname{Alt}^k V.$$

If the space not only has an inner product but a sign convention that the ordering $(q_1, \ldots, q_k)$ is positively oriented, then the volume form $\operatorname{vol} \in \operatorname{Alt}^n V$ is defined by

$$\operatorname{vol}(q_1, \ldots, q_n) := 1.$$

The *Hodge star operator* $\ast : \operatorname{Alt}^k V \tilde{\to} \operatorname{Alt}^{n-k} V$ is an isometry defined by

$$\omega \wedge \mu := \langle \ast \omega, \mu \rangle \operatorname{vol}, \quad \omega \in \operatorname{Alt}^k V, \mu \in \operatorname{Alt}^{n-k} V.$$

**Exterior calculus.** Exterior calculus can be defined for general manifolds: in this work we only need to consider subsets $\Omega$ of $\mathbb{R}^n$ which have the same tangent space $T_x \Omega$ for each $x \in \Omega$, either because they are $n$-dimensional and $T_x \Omega = \mathbb{R}^n$ or because they are $d$-dimensional subsets of $d$-dimensional hyperplanes.

The set of smooth, $\operatorname{Alt}^k T_x \Omega$-valued functions on $\Omega \subset \mathbb{R}^n$ are *differential $k$-forms* and are denoted $\Lambda^k(\Omega)$. The *exterior derivative* on $k$-forms, $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$, is defined by

$$(d\omega)_x(v_1, \ldots, v_{k+1}) := \sum_{j=1}^{k+1} (-1)^{j+1} \nabla_x(\omega_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1})) \cdot v_j,$$
where again $\delta_j$ is the omitted argument. The exterior derivative applied to each of the coordinate functions $x_1, \ldots, x_n$ is a constant 1-form, so we consider $dx_i$ an algebraic 1-form for each $i$. The set $\{dx_i\}_{i=1}^n$ forms a basis of algebraic $k$-forms that are dual to the canonical basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$,

$$dx_i(e_j) = \delta_{ij}, \quad i, j \in [1:n].$$

Given a sequence $\rho : [a:b] \to S$, not necessarily increasing, we let $x_\rho := \prod_{j=a}^b x_{\rho(i)}$ and

$$(dx)_\rho := dx_{\rho(a)} \wedge \cdots \wedge dx_{\rho(b)},$$

with $x_\rho := (dx)_\rho := 1$ if $[a:b] = \emptyset$. With this notation the canonical orthonormal basis of $\text{Alt}^k \mathbb{R}^n$ in terms of the coordinate functions is written as $\{(dx)_\rho\}_{\rho \in \Sigma([1:k],[1:n])}$, which we call the set of coordinate $k$-forms.

If $\phi : T \to U$ is a smooth map, then $\phi^* : \Lambda^k(U) \to \Lambda^k(T)$ is the pullback for differential $k$-forms, defined by

$$(\phi^* \omega)_x := (D\phi)^*_x \omega_{\phi(x)}, \quad \omega \in \Lambda^k(U),$$

where $(D\phi)_x$ is the Jacobian of $\phi$ at $x$.

We let $\kappa_x : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ be the Koszul operator centered at $x$, which is defined by

$$(\kappa_x \omega)_y := \omega_{y,0}(y - x).$$

An important fact for this work is that the Koszul operator commutes with the pullback of an affine map up to a constant interior product: given affine $\phi : T \to T$,

$$\kappa_x (\phi^* \omega) = \phi^* (\kappa_x \omega - \omega_{y,0}(\phi(x) - \hat{x})). \quad (5)$$

**Oriented simplices.** In all that follows $T$ is a positively oriented $n$-simplex in $\mathbb{R}^n$, $|T|$ is its volume and $\{\lambda_i\}_{i=0}^n$ are its barycentric coordinates. We define $\lambda_\rho$ and $(d\lambda)_\rho$ similarly to $x_\rho$ and $(dx)_\rho$, and we refer to $(d\lambda)_\rho$ as a barycentric $k$-form. Given maps $\rho : S \to [0:n]$ and $\alpha : S \to \mathbb{N}_0$ with the same domain $S$, a barycentric monomial is a polynomial in $P_{\alpha} (T)$ defined by

$$(\lambda_\rho)^\alpha := \prod_{i \in S} \lambda_i^{\alpha(i)}.$$ 

A positively oriented simplex has the property [4.1] that for any permutation $\pi : [0:n] \to [0:n]$ and any $i \in [0:n]$,

$$(d\lambda)_\pi \lambda(i) = (-1)^{\text{sign } \pi} \frac{n! |T| \text{ vol}.}$$

**Boundary simplices.** For each $\rho \in \Sigma([0:d],[0:n])$, the $d$-simplex with vertices $\{v_{\rho(0)}, \ldots, v_{\rho(d)}\}$ in the boundary of $T$ is $f_\rho$. If $i_\rho : f_\rho \to T$ is the inclusion map, then the trace operator $\text{Tr}_{i_\rho} : \Lambda^k(T) \to \Lambda^k(f_\rho)$ is its pullback, $\text{Tr}_{i_\rho} := i_{\rho}^*$. In section [4] when we need the trace of $\omega \in \Lambda^k(f_\rho)$ on another boundary simplex $f_\sigma$, we denote it $\text{Tr}_{f_\rho \sigma} \omega$.

The trace operator lets us more formally define the space $\hat{\Lambda}^k(T)$ from the introduction as

$$\hat{\Lambda}^k(T) := \{\omega \in \Lambda^k(T) : \text{Tr}_{f_\rho} u = 0, f_\rho \neq T\}.$$

**Whitney forms and trimmed polynomials.** The Whitney form $\phi_\rho \in P_1 \Lambda^d(T)$ associated with $f_\rho$ is

$$\phi_\rho := \sum_{i=0}^d (-1)^i \lambda_{\rho(i)} (d\lambda)_{\rho(i)},$$

and the trimmed polynomials are spanned by the product of Whitney forms and scalar polynomials,

$$P^- \Lambda^k(T) = \text{span}\{a_\rho \phi_\rho : a_\rho \in P_{r-1}(T), \rho \in \Sigma([0:k],[0:n])\}.$$ 

An important fact about the trimmed polynomials in this work is that $P^- \Lambda^k(T)$ is the largest subspace of $P_\tau \Lambda^k(T)$ such that $\kappa_x \rho \in P_\tau \Lambda^{k-1}(T)$ for every $\rho \in P^- \Lambda^k(T)$ and every $x$. 

6
3 Trace-free operators

3.1 Previous trace-free operators and bases

We can now define the trace-free operators $h^k_T$ and $h^{-k}_T$ given in (2) that induce isomorphisms (11) and (2):

\[ h^k_T : \sum_{\rho \in \Sigma([0:n-k],[0:n])} a_\rho (d\lambda)_\rho \mapsto \sum_{\rho \in \Sigma([0:n-k],[0:n])} a_\rho \lambda^* \phi_\rho \quad [a_\rho \in \mathcal{P}_r(T)]; \quad (6) \]

\[ h^{-k}_T : \sum_{\rho \in \Sigma([0:k],[0:n])} a_\rho \phi_\rho \mapsto \sum_{\rho \in \Sigma([0:k],[0:n])} a_\rho \lambda^* (d\lambda)_\rho \quad [a_\rho = a_\rho(\lambda_{\rho(0)}, \lambda_{\rho(0)+1}, \ldots, \lambda_n) \in \mathcal{P}_{r-1}(T)]. \quad (7) \]

The set \{$(d\lambda)_\rho^*$\}$_{\rho \in \Sigma([0:n-k],[0:n]), \rho(0)=0}$ is a basis for $\Lambda^k \mathbb{R}^n$, so $h^k_T$ can be computed pointwise, but the set \{$(\phi_\rho)^*\}$_{$\rho \in \Sigma([0:k],[0:n])$} has \binom{n+1}{k} non-zero elements for $x$ in the interior of $T$ while $|\Lambda^k \mathbb{R}^n| = \binom{n}{k}$, so the values of the coefficients \{a_\rho\}$_{\rho \in \Sigma([0:k],[0:n])}$ at a point are overdetermined in eq. (7), making it appear that $h^{-k}_T$ is not a pointwise operator. We show that this appearance is deceptive because both $h^k_T$ and $h^{-k}_T$ are equivalent to the pointwise operator $\star_T$ we define in section 3.3.

These maps directly lead to the constructive bases given for the trace-free subspace in (3) in the following way. In (6), we can express $a_\rho \in \mathcal{P}_r(T)$ as a polynomial of $n+1$ arguments in the barycentric coordinates, $a_\rho = a_\rho(\lambda_0, \ldots, \lambda_n)$, so that both (6) and (7) use polynomials of barycentric coordinates. If one expands each $a_\rho$ in terms of a monomial basis of the barycentric coordinates (which is equivalent up to a multiplicative constant to the Bernstein polynomial basis), the right hand side of (6) defines a basis for $\tilde{\mathcal{P}}_r(T)$ where each basis function is the product of a barycentric monomial and a Whitney form, and the right hand side of (7) defines a basis for $\tilde{\mathcal{P}}_r(T)$ which each basis function is the product of a barycentric monomial and an $(n-k)$-form $(d\lambda)_\rho^*$ for some $\rho \in \Sigma([0:k],[0:n])$.

3.2 The $\star_T$ operator

The duality between $\Lambda^k \mathbb{R}^n$ and $\Lambda^{n-k} \mathbb{R}^n$ is clearly relevant to the isomorphisms (11) and (2), but the Hodge star operator $\star : \Lambda^k \mathbb{R}^n \cong \Lambda^{n-k} \mathbb{R}^n$ does not appear to play any role. While this is surprising at first, it becomes clear that the Hodge star cannot be used because it is not affine invariant: given a bijection $\phi : T \rightarrow \tilde{T}$ between simplices, in general $\star \neq \phi^* \circ \star \circ \phi^{-*}$.

For each simplex $T$, we define a bijection between $\Lambda^k \mathbb{R}^n$ and $\Lambda^{n-k} \mathbb{R}^n$ based instead on the barycentric coordinates of $T$:

\[ \star_T : \omega \mapsto \frac{n!|T|}{\sqrt{n+1}} \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \star(\omega \wedge (d\lambda)_\rho)(d\lambda)_\rho. \quad (8) \]

Lemma 3.1. $\star_T$ is a bijection, $\star_T : \Lambda^k \mathbb{R}^n \cong \Lambda^{n-k} \mathbb{R}^n$.

Proof. We can define a symmetric bilinear form on $\Lambda^k \mathbb{R}^n$,

\[ \langle \omega, \mu \rangle_{\star_T} := \star(\omega \wedge \star_T \mu) = \frac{n!|T|}{\sqrt{n+1}} \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \star(\omega \wedge (d\lambda)_\rho) \star(\mu \wedge (d\lambda)_\rho). \]

If $\star_T \omega = 0$, then $\langle \omega, \mu \rangle_{\star_T} = 0$, and so $\star(\omega \wedge (d\lambda)_\rho) = 0$ for all $\rho \in \Sigma([0:n-k-1],[0:n])$. These $(n-k)$-forms span $\Lambda^{n-k} \mathbb{R}^n$, and $\Lambda^{n-k} \mathbb{R}^n \cong (\Lambda^n \mathbb{R}^n)^*$, so $\omega = 0$.

Because $\star_T$ is defined by a simplex instead of being universal, it can have a useful form of affine invariance.

Lemma 3.2. Given an affine bijection $\phi : T \rightarrow \tilde{T}$, $\star_T = (D\phi)^* \circ \star_{\tilde{T}} \circ (D\phi)^{-*}$. 

7
Proof. If \( \hat{\lambda} \) are the barycentric coordinates of \( \hat{T} \), then \((d\hat{\lambda})_\rho = (D\phi)^-*(d\lambda)_\rho \) and \((D\phi)^-*vol = (|T|/|\hat{T}|)vol\). Hence

\[
((D\phi)^* \circ \star_T \circ (D\phi)^{-*})(\omega)
= (D\phi)^* \left( \frac{n!|T|}{\sqrt{n+1}} \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \star((D\phi)^{-*} \omega \wedge (d\hat{\lambda})_\rho) (d\hat{\lambda})_\rho \right)
= (D\phi)^* \left( \frac{n!|\hat{T}|}{\sqrt{n+1}} \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \star((D\phi)^{-*} \omega \wedge (D\phi)^{-*}(d\lambda)_\rho) (D\phi)^{-*}(d\lambda)_\rho \right)
= \frac{n!|\hat{T}|}{\sqrt{n+1}} \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \star(\omega \wedge (d\lambda)_\rho) (d\lambda)_\rho
= \ast_T \omega.
\]

\( \blacksquare \)

**Lemma 3.3.** If \( T_{eq} \) is an equilateral simplex with edge length \( \sqrt{2} \), then \( \ast_{T_{eq}} = \ast \).

**Proof.** If we take the convention that the canonical basis of \( \mathbb{R}^{n+1} \) is numbered \( e_0, \ldots, e_n \), then for every simplex the barycentric \( k \)-forms are pullbacks of the coordinate \( k \)-forms,

\[
(d\lambda)_\rho = (D\lambda)^*(dx)_\rho, \quad \rho \in \Sigma([0:k],[0:n]).
\]

In the case of \( T_{eq} \), the Jacobian \( D\lambda \) is isometric: \( \|(D\lambda)v\|_2 = \|v\| \) for all \( v \in \mathbb{R}^n \). This is clear when we think of \( \lambda \) as mapping \( T \) to the standard barycentric simplex, which is the simplex in the positive orthant of \( \mathbb{R}^{n+1} \) connecting the \( e_i \) unit vectors, which is also equilateral and has the same edge length.

Because \( (D\lambda) \) is isometric, the \( \binom{n+1}{k+1} \times \binom{n+1}{k+1} \) matrix of inner products between barycentric \( k \)-forms,

\[
M_{\rho,\sigma} := \langle (d\lambda)_\rho, (d\lambda)_\sigma \rangle = \langle (D\lambda)^*(dx)_\rho, (D\lambda)^*(dx)_\sigma \rangle, \quad \rho, \sigma \in \Sigma([0:k],[0:n]),
\]

is an orthogonal projection matrix for every \( k \). This implies that the set \( \{(d\lambda)_\rho \}_{\rho \in \Sigma([0:k],[0:n])} \) is a normalized tight frame [1, Theorem 2.5], that is

\[
\omega = \sum_{\rho \in \Sigma([0:k],[0:n])} \langle \omega, (d\lambda)_\rho \rangle (d\lambda)_\rho, \quad \omega \in Alt^k \mathbb{R}^n.
\]

To compute \( \ast_{T_{eq}} \omega \) for \( \omega \in Alt^k \mathbb{R}^n \), we apply the above fact to the \( (n-k) \)-forms, and note that \( |T_{eq}| = \sqrt{n+1}/n! \) so that the leading constant in \( \blacksquare \) is cancelled, to see

\[
\ast_{T_{eq}} \omega = \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \ast(\omega \wedge (d\lambda)_\rho) (d\lambda)_\rho = \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \langle \ast \omega, (d\lambda)_\rho \rangle (d\lambda)_\rho = \ast \omega.
\]

\( \blacksquare \)

These lemmas imply that, like \( \ast \) itself, \( \ast_T \circ \ast_T = (-1)^{k(n-k)} \).

**Corollary 3.4.** \( \ast_T \circ \ast_T = (-1)^{k(n-k)} \).

**Proof.** Let \( \phi : T \sim \rightarrow T_{eq} \) be an affine bijection. Then

\[
\ast_T \circ \ast_T = (D\phi)^* \circ \ast_{T_{eq}} \circ (D\phi)^{-*} \circ (D\phi)^* \circ \ast_{T_{eq}} \circ (D\phi)^{-*} = (D\phi)^* \circ \ast \circ \circ (D\phi)^{-*} = (-1)^{k(n-k)}.
\]

\( \blacksquare \)
3.3 The $\hat{\ast}T$ operator

Having defined an isomorphism $\ast T$ between algebraic $k$- and $(n-k)$-form with the desired variance, we are now prepared to define an similar isomorphism between differential $k$- and trace-free $(n-k)$-forms. A similar construction to (3) defines a linear operator $\hat{\ast}T : \Lambda^k(T) \to \Lambda^{n-k}(T)$ by multiplying the forms summed in (3) by their complementary bubble functions. We define

$$\hat{\ast}T : \omega \mapsto n!|T| \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \ast(\omega \wedge (d\lambda)_{\rho}) \lambda_{\rho^*} (d\lambda)_{\rho}.$$  \tag{9}$$

Lemma 3.5. $\hat{\ast}T$ is affine invariant: given affine bijection $\phi : T \to \hat{T}$, $\hat{\ast}T = \phi^* \circ \hat{\ast}T \circ \phi^{-*}$.

Proof. The proof is essentially the same as for lemma 3.2. □

Lemma 3.6. $\hat{\ast}T : \Lambda^k(T) \to \Lambda^{n-k}(T)$ is an injection.

Proof. To show that the range of $\hat{\ast}T$ is in $\Lambda^{n-k}(T)$ it is sufficient to show that $\lambda_{\rho^*} (d\lambda)_{\rho} \in \Lambda^{n-k}(T)$ for each $\rho \in \Sigma([0:n-k-1],[0:n])$. This is so because $\text{Tr}_\sigma \lambda_{\rho^*} = 0$ if $[\rho^*] \not\subseteq [\sigma]$ and $\text{Tr}_\sigma (d\lambda)_{\rho} = 0$ if $[\rho] \not\subseteq [\sigma]$. Therefore $\text{Tr}_\sigma \lambda_{\rho^*} (d\lambda)_{\rho} = 0$ for any $[\sigma] \not\subseteq [\rho] \cup [\rho^*]$, that is for any $f_\sigma \neq T$.

To show injectivity, as in lemma 3.1 we can define a symmetric bilinear form

$$\langle \omega, \mu \rangle_{\hat{\ast}T} := \int_T \omega \wedge \hat{\ast}T \mu = n!|T| \int_T \text{vol} \sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \lambda_{\rho^*} \ast(\omega \wedge (d\lambda)_{\rho}) \ast(\mu \wedge (d\lambda)_{\rho}). \tag{10}$$

If $\hat{\ast}T \omega = 0$, then $\langle \omega, \omega \rangle_{\hat{\ast}T} = 0$, and for every $x$ in $T$

$$\sum_{\rho \in \Sigma([0:n-k-1],[0:n])} \lambda_{\rho^*} (\ast(\omega_x \wedge (d\lambda)_{\rho}))^2 = 0.$$

For every $\rho \in \Sigma([0:n-k-1],[0:n])$ and every $x$ in the interior of $T$ where $\lambda_{\rho^*} > 0$ this implies that $\ast(\omega_x \wedge (d\lambda)_{\rho}) = 0$ and so $\omega_x = 0$. By continuity, we conclude $\omega = 0$. □

The operator $\hat{\ast}T$ has an important property, analogous to corollary 3.4, that applying $\hat{\ast}T$ twice is like multiplying by $(-1)^{(k(n-k))} \lambda_T$.

Lemma 3.7. $\hat{\ast}T \circ \hat{\ast}T = (-1)^{(k(n-k))} \lambda_T$.

Proof. Let $\omega \in \Lambda^k(T)$ be given and first expand $\hat{\ast}T \hat{\ast}T \omega$:

$$\hat{\ast}T \hat{\ast}T \omega = (n!|T|)^2 \sum_{\rho, \sigma \in \Sigma([0:n-k-1],[0:n])} \lambda_{\rho^*} \lambda_{\sigma^*} \ast(\omega \wedge (d\lambda)_{\rho}) \ast((d\lambda)_{\rho} \wedge (d\lambda)_{\sigma}) (d\lambda)_{\sigma}. $$

If $[\rho] \cap [\sigma] \neq \emptyset$, then $(d\lambda)_{\rho^*} \wedge (d\lambda)_{\sigma} = 0$. So for each nonzero summand there is a map $\tau \in \Sigma([0:n-1],[0:n])$ and a map $\rho \in \Sigma([0:n-k-1],[0:n-1])$ such that $\rho = \tau \circ \hat{\rho}$ and $\sigma = \tau \circ \hat{\rho}^*$. We use this fact to reorganize the sum, noting that the union of the sets $\{(\tau \circ \rho)^*\}$ and $\{(\tau \circ \rho^*)^*\}$ is $[0:n]$ and their intersection is $\{\tau^*\}$:

$$\hat{\ast}T \hat{\ast}T \omega$$

$$= (n!|T|)^2 \sum_{\tau \in \Sigma([0:n-1],[0:n])} \lambda_{(\tau \circ \hat{\rho})^*} \lambda_{(\tau \circ \hat{\rho}^*)^*} \ast(\omega \wedge (d\lambda)_{\tau \circ \hat{\rho}}) \ast((d\lambda)_{\tau \circ \hat{\rho}} \wedge (d\lambda)_{\tau \circ \hat{\rho}^*}) (d\lambda)_{\tau \circ \hat{\rho}^*}.$$  

$$= \lambda_T \sum_{\tau \in \Sigma([0:n-1],[0:n])} \lambda_{\tau^*} \left\{ (n!|T|)^2 \sum_{\rho \in \Sigma([0:n-k-1],[0:n-1])} \ast(\omega \wedge (d\lambda)_{\tau \circ \hat{\rho}}) \ast((d\lambda)_{\tau \circ \hat{\rho}} \wedge (d\lambda)_{\tau \circ \hat{\rho}^*}) (d\lambda)_{\tau \circ \hat{\rho}^*} \right\}.$$  

9
We show that the term in braces is \((-1)^k(n-k)\omega\) for each \(\tau\).

Let us again number the canonical basis of \(\mathbb{R}^n\) starting from zero, \(e_0, \ldots, e_{n-1}\), and let \(T_{\text{unit}}\) be the unit right simplex in the positive orthant, \(T_{\text{unit}} := \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=0}^{n-1} x_i \leq 1\}\). For each \(\tau \in \Sigma([0:n-1], [0:n])\), define the affine bijection \(\phi_{\tau} : T \to T_{\text{unit}}\) such that the \(\tau(i)\)th vertex maps to \(e_i\) for \(i \in [0:n-1]\) and the remaining vertex maps to the origin. We note that because the volume of \(T_{\text{unit}}\) is \(1/n!\), the volume pullback is \((D\phi_{\tau})^*\omega = \pm (1/(n!|T|))\omega\). We also note that under \(\phi_{\tau}\) the barycentric \(k\)-forms are pullbacks of coordinate \(k\)-forms if their indices are subsets of \(\tau\): \((d\lambda)_{\tau\rho} = (D\phi_{\tau})^*(dx)_{\rho}\) for each \(\rho \in \Sigma([0:n-k-1], [0:n-1])\). Last, we note that the Hodge star maps a coordinate \(k\)-forms \((dx)_{\rho}\) to the coordinate \((n-k)\)-forms with complementary indices, \(*((dx)_{\rho}) = \pm (dx)_{\rho^*}\). Thus for each \(\tau \in \Sigma([0:n-k-1], [0:n])\), the term in braces above becomes

\[
\begin{align*}
(n!|T|)^2 & \sum_{\rho \in \Sigma([0:n-k-1], [0:n-1])} \ast_\omega ((d\lambda)_{\tau\rho} \land (d\lambda)_{\tau\rho^*}) (d\lambda)_{\tau\rho^*} \\
& = (-1)^{k(n-k)}(n!|T|)^2 \sum_{\rho \in \Sigma([0:n-k-1], [0:n-1])} \ast((d\lambda)_{\tau\rho^*} \land (d\lambda)_{\tau\rho^*}) (d\lambda)_{\tau\rho^*} \\
& = (-1)^{k(n-k)}(D\phi_{\tau})^* \sum_{\rho \in \Sigma([0:n-k-1], [0:n-1])} \ast((dx)_{\rho} \land (dx)_{\rho^*}) (dx)_{\rho^*} \\
& = (-1)^{k(n-k)}(D\phi_{\tau})^* \sum_{\rho \in \Sigma([0:n-k-1], [0:n-1])} \langle (D\phi_{\tau})^*\omega, (dx)_{\rho^*} \rangle \langle (dx)_{\rho^*}, (dx)_{\rho} \rangle (dx)_{\rho^*} \\
& = (-1)^{k(n-k)}(D\phi_{\tau})^* \omega \\
& = (-1)^{k(n-k)}\omega.
\end{align*}
\]

Summing these contributions over all \(\tau\), we get

\[
\ast T \ast T \omega = (-1)^{k(n-k)}\lambda_T \omega \sum_{\tau \in \Sigma([0:n-1],[0:n])} \lambda_{\tau^*} = (-1)^{k(n-k)}\lambda_T \omega.
\]

\[\Box\]

We are now ready to prove the main results of this section, that \(\ast T\) implements the isomorphisms \([1]\) and \([2]\).

**Theorem 3.8.** \(\ast T : \mathcal{P}_r\Lambda^k(T) \cong \hat{\mathcal{P}}_{r+k+1}\Lambda^{n-k}(T)\).

**Proof.** Applying \(\ast T\) to \((d\lambda)_{\sigma}\) for some \(\sigma \in \Sigma([0:k-1], [0:n])\), each term in \([1]\) where \([\rho] \cap [\sigma] \neq \emptyset\) vanishes. The only nonzero summands are for \(\rho\) where \([\rho] = [\sigma^*] \setminus \{\sigma^*(j)\}\) for some \(j\). Hence

\[
\begin{align*}
\ast T(d\lambda)_{\sigma} &= n!|T| \sum_{j=0}^{n-k-1} \ast((d\lambda)_{\sigma} \land (d\lambda)_{\sigma^*\setminus\sigma^*(j)}) \lambda_{\sigma^*\setminus\sigma^*(j)} (d\lambda)_{\sigma^*\setminus\sigma^*(j)} \\
& = (-1)^k(\text{sign } \sigma) \lambda_{\sigma} \sum_{j=0}^{n-k-1} (-1)^j \lambda_{\sigma^*\setminus\sigma^*(j)} (d\lambda)_{\sigma^*\setminus\sigma^*(j)} \\
& = (-1)^k(\text{sign } \sigma) \lambda_{\sigma} \phi_{\sigma^*}.
\end{align*}
\]

As a result we conclude \(\ast T(d\lambda)_{\sigma} \in \hat{\mathcal{P}}_{k+1}\Lambda^{n-k}(T)\). Each polynomial in \(\mathcal{P}_r\Lambda^k(T)\) has a representation of the form

\[
\sum_{\sigma \in \Sigma([0:k-1],[0:n])} a_{\sigma}(d\lambda)_{\sigma}, \quad a_{\sigma} \in \mathcal{P}_r(T),
\]

so \(\ast T\) maps \(\mathcal{P}_r\Lambda^k(T)\) into \(\hat{\mathcal{P}}_{r+k+1}\Lambda^{n-k}(T)\). By lemma \([3.5]\) \(\ast T\) is injective, and it has already been established that the spaces have the same dimension, so the operator is an isomorphism. \(\Box\)

**Theorem 3.9.** \(\ast T : \mathcal{P}_r\Lambda^k(T) \cong \hat{\mathcal{P}}_{r+k}\Lambda^{n-k}(T)\).
We are now able to present our final theorem of this section. We note that the proofs of theorems 3.8 and 3.9 show that

\[ \lambda a \in P \sigma \]

which is in \( P_{k+1} \Lambda^{n-k}(T) \). Therefore, as every function in \( P_{k-1} \Lambda^k(T) \) has a representation of the form

\[ \sum_{\sigma \in \Sigma([0:k],[0:n])} a_\sigma \phi_\sigma, \]

where \( a_\sigma \in P_{r-1}(T) \), \( \hat{T} \) maps \( P_{r-1} \Lambda^k(T) \) into \( \hat{T}_{r+k} \Lambda^{n-k}(T) \) injectively, which concludes the proof. \( \square \)

We note that the proofs of theorems 3.8 and 3.9 show that \( \hat{T} \) differs from \( h_T \) and \( h_T \) only by sign conventions.

To complete the claim that \( \hat{T} \) induces an isomorphism between \( \Lambda^k(T) \) and \( \hat{T}_{k-1} \Lambda^{n-k}(T) \), we need to prove that \( \hat{T} : \Lambda^k(T) \to \hat{T}_{k-1} \Lambda^{n-k}(T) \) is a surjection, for which we require one additional lemma.

Let us define the space of \( k \)-forms that are not only trace-free but vanish at the boundary,

\[ \hat{C}^\infty(T, \text{Alt}^{n-k} \mathbb{R}^n) := \{ \omega \in \Lambda^{n-k}(T) : \omega|_{\partial T} = 0 \}. \]

It turns out that \( \hat{T} \) maps \( \hat{T}_{k-1}(T) \) into this space.

**Lemma 3.10.** \( \hat{T} : \hat{T}_{k-1}(T) \to \hat{C}^\infty(T, \text{Alt}^{n-k} \mathbb{R}^n) \).

**Proof.** Let \( x \in f_{\sigma} \neq T \) be given. If \( \omega \in \hat{T}_{k-1}(T) \), then \( (\text{Tr}_{\sigma} \omega)_x = (i_{\sigma}^* \omega)_x = (D_{i_{\sigma}})^* \omega_x = 0 \). We use the fact that the nullspace of \( (D_{i_{\sigma}})^* \) is spanned by barycentric \( k \)-forms whose indices are not contained in \( \sigma \), \( \{(d\lambda)_\rho \in \Sigma([0:k-1],[0:n]) : [\rho] \not\subseteq [\sigma]\} \), to say that \( \omega_x \) is a linear combination of these \( k \)-forms. For such \( (d\lambda)_\rho \), the proof of theorem 3.8 shows that

\[ (\hat{T}(d\lambda)_\rho)_x = (-1)^k (\text{sign} \rho)(\lambda_\rho(d\lambda)_\rho)_x = 0, \]

a conclusion we reach because \( (\lambda_\rho)_x = 0 \) for \( x \in f_{\sigma} \) and \( [\rho] \not\subseteq [\sigma] \). Therefore \( (\hat{T}\omega)_x = 0 \) as well. \( \square \)

We are now able to present our final theorem of this section.

**Theorem 3.11.** \( \hat{T} : \Lambda^k(T) \xrightarrow{\sim} \hat{T}_{k-1}(T) \).

**Proof.** Lemma 3.6 already shows that the map is an injection: it remains to show that it is a surjection.

Let \( \omega \in \Lambda^k(T) \) be given. By lemma 3.7 \( \hat{T} \hat{T} \omega = (-1)^k (\Lambda^k(T) \cong \hat{C}^\infty(T, \text{Alt}^{k-1} \mathbb{R}^n)) \). Given \( \mu \in \Lambda^{n-k}(T) \), by lemma 3.11 we can define an element \( \omega := (\hat{T} \hat{T} \omega)^{-1} \hat{T} \omega \in \Lambda^k(T) \). \( \hat{T} \omega \) is in \( \Lambda^{n-k}(T) \), and \( \hat{T} \hat{T} \omega = \hat{T} \omega \). By the injectivity of \( \hat{T} \), \( \hat{T} \omega = \mu \). \( \square \)
3.4 Optimal dual basis functionals via the $\hat{\ast}_T$ inner product

Because the operator $\hat{\ast}_T$ is defined pointwise, it can be used in practice to adapt an unrestricted basis with desirable qualities – some combination of numerical stability and computational efficiency – into a basis for trace-free subspaces.

Implementations of the finite element method may also define the basis functions indirectly, opting instead to define a unisolvent set of functionals and constructing the basis functions by inverting the generalized vandermonde matrix of a numerically stable basis for the primal space. The canonical basis dual basis functionals for the trace free subspace $\mathcal{P}_r \Lambda^k(T)$ given in [2, Sections 4.5] are

$$\phi_i(\omega) = \int_T {\omega} \wedge \eta_i,$$

where $\{\eta_i\}$ is a basis of $\mathcal{P}_{r-(n-k)} \Lambda^k(T)$; for $\hat{\mathcal{P}}_r \Lambda^k(T)$, they are the basis functionals given in [2, Sections 4.6],

$$\phi_i(\omega) = \int_T {\omega} \wedge \eta_i,$$

where $\{\eta_i\}$ is a basis of $\mathcal{P}_{r-(n-k)} \Lambda^k(T)$.

That work does not suggest which dual basis differential forms $\{\eta_i\}$ to use in either case. The inner product $\langle \cdot, \cdot \rangle_{\hat{\ast} T}$ defined in [10] in the proof of lemma~3.5 is useful for this purpose. If a basis $\{\eta_i\}$ is chosen that is orthonormal with respect to this inner product, then the corresponding basis functions can be computed directly as $\{i_{T} \eta_i\}$. For scalar polynomials, such orthonormal polynomials can be evaluated explicitly using Dubiner-type combinations of Gauss-Jacobi polynomials, such as in the basis of Sherwin and Karniadakis [5]. We do not investigate the construction of an explicit orthonormal basis for $\langle \cdot, \cdot \rangle_{\hat{\ast} T}$ further in this work.

4 Extension operators

4.1 Consistent families of extension operators

In [3], the authors define a consistent family of extension operators for geometric decomposition. Letting $X$ stand in for $\mathcal{P}_r \Lambda^k$ or $\mathcal{P}_r^\perp \Lambda^k$, a consistent family of extension operators is a set of operators $\{E_{f \rightarrow g} : X(f) \rightarrow X(g), f, g \in T_h\}$ for any pair of simplices $f$ and $g$ in a mesh $T_h$, for which trace and extension operators commute in the following sense: if $f$ and $g$ are boundary simplices of simplex $h$, and $f \cap g$ is the intersection of their boundaries, then

$$\text{Tr}_{h \rightarrow g} E_{f \rightarrow h} \omega = E_{f \cap g \rightarrow g} \text{Tr}_{f \cap g \rightarrow g} \omega \quad \omega \in X(f).$$

(11)

This property guarantees that the extension of a trace-free $k$-form $\omega \in X(f)$ into neighboring cells has trace continuity at the boundaries between the cells, and so it can serve as a local basis function in an $H\Lambda^k(\Omega)$-conforming finite element space.

There are many ways to define consistent families of extension operators, but in this work we only define extension operators that are affine invariant and respect all of the symmetries of the simplex. This allows us to give our definitions with respect to the single-element mesh made up of the simplex $T$ and its boundary simplices, and the results can be mapped to a general mesh $T_h$ with properties preserved. We also note that, for the purpose of geometric decomposition, it is not necessary to design an extension operator whose domain is all of $X(f)$, because the only $k$-forms that are extended are in $\hat{X}(f)$.

Using these facts, in this section we design a family of extension operators $\{\hat{E}_{\rho, \sigma} : \hat{X}(f) \rightarrow X(f), [\rho], [\sigma] \subseteq [0:n]\}$ such that for any $f$, such that $[\rho], [\sigma] \subseteq [r],$

$$\text{Tr}_{\tau \rightarrow \sigma} \hat{E}_{\rho, \tau} \omega = \hat{E}_{\rho \cap \sigma, \tau} \text{Tr}_{\rho \cap \sigma \rightarrow \tau} \omega, \quad \omega \in X(f).$$

This is sufficient to show that one has the geometric decomposition

$$X(T) = \bigoplus_{\sigma} \hat{E}_{\sigma, T}[\hat{X}(f)],$$

where we can take not only $X(f) = \mathcal{P}_r \Lambda^k(f)$ or $X(f) = \mathcal{P}_r^\perp \Lambda^k(f)$, but even $X = \Lambda^k(\mathcal{T})$. 

12
4.2 Additional notation

We identify $k$-forms whose domain if $f_\sigma$ with a superscript, such as $\lambda^{(\sigma)}, d\lambda^{(\sigma)}, \phi_{\rho}^{(\tau)}$.

Centroid projectors. For each $d$-simplex $f_\sigma$ in the boundary of $T$, the centroid projector $P_{T,\sigma} : T \to f_\sigma$ is an affine map defined in terms of its actions on the vertices $\{v_i\}_{i=0}^d$ of $T$: if $i \in \sigma$, then $P_{T,\sigma}(v_i) = v_i$, otherwise $v_i$ is mapped to the centroid of $f_\sigma$, $P_{T,\sigma}(v_i) = \frac{1}{d} \sum_{i=0}^d v_{\sigma(i)}$. Given two boundary simplices $f_\sigma$ and $f_{\rho}$, with $[\rho] \subset [\sigma]$, then we define the centroid projector $P_{\sigma,\rho} : f_\sigma \to f_{\rho}$ analogously. Centroid projectors compose with each other, in that if $[\rho] \subset [\sigma] \subset [\tau]$, then

$$P_{\sigma,\rho} \circ P_{\tau,\sigma} = P_{\tau,\rho}. \quad (12)$$

The centroid projector $P_{\sigma,\rho}$ is also a left inverse of the inclusion map $i_{\rho,\sigma} : f_{\rho} \to f_\sigma$, which implies that $\text{Tr}_{r,\rho}$ is a left inverse for $P_{\sigma,\rho}$. Combining this fact with (12), we get

$$P_{\sigma,\rho}^* = \text{Tr}_{r,\rho} \circ P_{\sigma,\rho}^*. \quad (13)$$

Centroid Koszul operators. For each simplex $f_\sigma$, we let $\kappa^{(\sigma)}$ be the Koszul operator centered at the centroid of $f_\sigma$. Because the centroid projector $P_{\sigma,\rho}$ maps the centroid of $f_\sigma$ to the centroid of $f_\rho$, (5) implies

$$\kappa^{(\sigma)} P_{\sigma,\rho}^* = P_{\sigma,\rho}^* \kappa^{(\rho)}. \quad (14)$$

4.3 Previous extension operators

In [3], the authors define two extension operators, $E_{\sigma,T}^{r,k} : \mathcal{P}_r \Lambda^k(f_\sigma) \to \mathcal{P}_r \Lambda^k(T)$ and $E_{\sigma,T}^{r,k,-} : \mathcal{P}_r^- \Lambda^k(f_\sigma) \to \mathcal{P}_r^- \Lambda^k(T)$, and show that they are consistent extension operators that can be used in geometric decompositions of their target spaces. Those extension operators are defined as follows:

$$E_{\sigma,T}^{r,k} : (\lambda^{(\sigma)})^\alpha (d\lambda^\tau) \mapsto (\lambda_\sigma)^\alpha P_{T,\sigma,\alpha} (d\lambda^\tau), \quad |\alpha| = r, \tau \in \Sigma([0:k],[\sigma]); \quad (14)$$

$$E_{\sigma,T}^{r,k,-} : (\lambda^{(\sigma)})^\alpha \phi_\tau \mapsto (\lambda^{(\sigma)})^\alpha \phi_\tau, \quad |\alpha| = r - 1, \tau \in \Sigma([0:k-1],[\sigma]). \quad (15)$$

The projector $P_{T,\sigma,\alpha} : T \to f_\sigma$ used in (14) differs from the centroid projector in that the vertices that are not in $f_\sigma$ map not to the centroid of $f_\sigma$, but to $x_\alpha$, a weighted average of the vertices of $f_\sigma$ with weights determined by the multi-index $\alpha$,

$$x_\alpha = \frac{1}{|\alpha|} \sum_{i=0}^d \alpha(i) v_{\sigma(i)}. \quad (16)$$

As was the case with the trace-free maps in section [5.1] these extension operators are defined in terms of their action on the products of scalar polynomials and barycentric $k$-forms (for full polynomials) or Whitney forms (for trimmed). Unlike that case, though, these operators are mutually incompatible, in that neither maps the other into the correct space.

We demonstrate their incompatibility when extending 1-forms from the triangle $\sigma = (v_1, v_2, v_3)$ to the tetrahedron $T = (v_0, v_1, v_2, v_3)$ by finding specific examples where a 1-form extends to the wrong space.

Example showing $E_{\sigma,T}^{k,r,-} \phi_{T-1} \Lambda^k(f_\sigma) \neq \mathcal{P}_{T-1} \Lambda^k(T)$. In this case $k = 1$ and $r = 3$. Let $\omega = \lambda_1^{(\sigma)} \lambda_2^{(\sigma)} d\lambda_3^{(\sigma)} \in \mathcal{P}_{T} \Lambda^1(f_\sigma)$.

We can expand $\omega$ in a Whitney form basis as $\omega = \lambda_1^{(\sigma)} \lambda_2^{(\sigma)} (\phi_2^{(\sigma)} + \phi_{13}^{(\sigma)})$, so applying $E_{\sigma,T}^{3,1,-}$ by the definition in (15) yields

$$E_{\sigma,T}^{3,1,-} \omega = E_{\sigma,T}^{3,1,-} (\lambda_1^{(\sigma)} \lambda_2^{(\sigma)} (\phi_2^{(\sigma)} + \phi_{13}^{(\sigma)})) = \lambda_1 \lambda_2 (\phi_2^{(\sigma)} + \phi_{13}^{(\sigma)}).$$
On the tetrahedron \( \phi_{32} + \phi_{32} \neq d\lambda_3 \), so \( E_{\sigma,T}^{3,1} \omega \notin \mathcal{P}_2 \Lambda^1(T) \).

**Example showing** \( E_{\sigma,T}^{k,r} \mathcal{P}_r \Lambda^k(f_\sigma) \not\subset \mathcal{P}_r \Lambda^k(T) \). Let \( \omega = \lambda_1^{(\sigma)} \lambda_2^{(\sigma)} \phi_{32}^{(\sigma)} \in \mathcal{P}_3 \Lambda^1(f_\sigma) \) (in this case again \( k = 1 \) and \( r = 3 \)). Expanded in a barycentric 1-form basis, \( \omega = \lambda_1^{(\sigma)} (\lambda_2^{(\sigma)})^2 d\lambda_3^{(\sigma)} - \lambda_1^{(\sigma)} \lambda_2^{(\sigma)} \lambda_3^{(\sigma)} d\lambda_2^{(\sigma)} \). According to [11], \( E_{\sigma,T}^{3,1} \) extends the two 1-forms in this representation of \( \omega \) by the pullbacks of two different projections, \( P_{\sigma,(1,2,0)} \) and \( P_{\sigma,(1,1,1)} \), determined by their barycentric monomials. The first projection \( P_{\sigma,(1,2,0)} \) can be described in terms of barycentric coordinates as

\[
(\lambda_1^{(\sigma)}, \lambda_2^{(\sigma)}, \lambda_3^{(\sigma)}) \leftarrow (\lambda_1 + (1/3)\lambda_0, \lambda_2 + (2/3)\lambda_0, \lambda_3),
\]

so \( P_{\sigma,(1,2,0)}^* d\lambda_3^{(\sigma)} = d\lambda_3 \). The second projection \( P_{\sigma,(1,1,1)} \) is

\[
(\lambda_1^{(\sigma)}, \lambda_2^{(\sigma)}, \lambda_3^{(\sigma)}) \leftarrow (\lambda_1 + (1/3)\lambda_0, \lambda_2 + (1/3)\lambda_0, \lambda_3 + (1/3)\lambda_0),
\]

so \( P_{\sigma,(1,1,1)}^* d\lambda_2^{(\sigma)} = d\lambda_2 + (1/3)d\lambda_0 \). All together, this shows that the extension of \( \omega \) is

\[
E_{\sigma,T}^{3,1} \omega = \lambda_1 \lambda_2^2 d\lambda_3 - \lambda_1 \lambda_2 \lambda_3 (d\lambda_2 + (1/3)d\lambda_0).
\]

The easiest way to show that this is not in \( \mathcal{P}_- \Lambda^1(T) \) is to apply the Koszul operator because \( \kappa_x P \in \mathcal{P}_3 \Lambda^0(T) \) for every \( p \in \mathcal{P}_3 \Lambda^1(T) \) and every \( x \). By [2] Theorem 3.1 we have \( \kappa_x P (\lambda^α d\lambda_1) = \lambda^α (\lambda_1 - \lambda_i(x)) \) for each multi-index \( \alpha \) and each \( i \in [0:n] \). Choosing \( x = v_1 \) where \( \lambda_0(x) = \lambda_2(x) = \lambda_3(x) = 0 \) gives us

\[
\kappa_{v_1} E_{\sigma,T}^{3,1} \omega = -(1/3)\lambda_0 \lambda_1 \lambda_2 \lambda_3 \notin \mathcal{P}_3 \Lambda^0(T).
\]

### 4.4 The bubble decomposition of \( \hat{\Lambda}^k(T) \)

Even though each \( \omega \in \hat{\Lambda}^k(T) \) is trace-free, \( \omega_x \) for \( x \in \partial T \) is not necessarily zero because there can be components of \( \omega_x \) that are perpendicular to the trace operator \( T_x \). In this section we define a way to decompose a trace-free \( k \)-form based on these perpendicular traces called the bubble decomposition. This decomposition is the foundation of the unified extension operator we define below. We present the bubble decomposition for the \( n \)-simplex \( T \): the definition extends naturally to \( \hat{\Lambda}^k(\Sigma(T)) \) for each boundary simplex \( f_\sigma \).

Let \( k \in [0:n] \) be given. For each \( d \geq n - k \) and each \( \sigma \in \Sigma([[0:d], [0:n]]) \), let \( \omega_\sigma \) be a \( (k - (n - d)) \)-form in \( \Lambda^{k-(n-d)}(\Sigma(T)) \). Define \( \hat{E}_{\sigma,T} : \Lambda^{k-(n-d)}(\Sigma(T)) \rightarrow \hat{\Lambda}^k(T) \) by

\[
\hat{E}_{\sigma,T} : \omega_\sigma \mapsto P_{\sigma,T}^* \omega_\sigma \wedge (\lambda_\sigma (d\lambda)_{\sigma^*}).
\]

The range of \( \hat{E}_{\sigma,T} \) is trace-free because the form \( \lambda_\sigma (d\lambda)_{\sigma^*} \) appearing in the right hand side of (16) is trace-free and because trace operators distribute over the wedge product. We first show that \( \hat{E}_{\sigma,T} \) is an injection.

**Lemma 4.1.** \( \hat{E}_{\sigma,T} \) defined by (16) is injective.

**Proof.** Let \( \Sigma = \Sigma([[1:k-(n-d)], [\sigma \setminus (0)]) \). We note that \( \{(d\lambda(\sigma))_\hat{\rho}\}_{\hat{\rho} \in \Sigma} \) is a basis for \( \text{Alt}^{k-(n-d)}(T_x f_\sigma) \), so it is sufficient to show that \( \hat{E}_{\sigma,T}^{\alpha} (d\lambda(\sigma))_\hat{\rho} \neq 0 \) for every nonzero \( \alpha \in C^\infty(\Sigma(T)) \) and every \( \hat{\rho} \in \hat{\Sigma} \).

We note that the pullback of a barycentric 1-form by \( (DP_{\sigma,T})^* \) is

\[
(DP_{\sigma,T})^* d\lambda_i^{(\sigma)} = d\lambda_i + \frac{1}{d+1} \sum_{j \in [\sigma^*]} d\lambda_j, \quad i \in [\sigma].
\]

Because \( d\lambda_j \wedge d\lambda_{\sigma^*} = 0 \) if and only if \( j \in [\sigma^*] \), this implies \( (DP_{\sigma,T})^* d\lambda_i^{(\sigma)} \wedge d\lambda_{\sigma^*} = d\lambda_i \wedge d\lambda_{\sigma^*} \) for each \( i \in [\sigma] \), and in general

\[
(DP_{\sigma,T})^* (d\lambda(\sigma))_\hat{\rho} \wedge d\lambda_{\sigma^*} = (d\lambda)_\hat{\rho} \wedge d\lambda_{\sigma^*}, \quad [\hat{\rho}] \subseteq [\sigma].
\]
Therefore
\[ \hat{E}_{\sigma,T}(d\lambda^{(\sigma)})_{\hat{\rho}} = P_{T,\sigma}^{*}(\alpha(d\lambda^{(\sigma)})_{\hat{\rho}}) \wedge (\lambda_{\sigma}(d\lambda))_{\sigma^*} \]
\[ = (\lambda_{\sigma} P_{T,\sigma}(\alpha)) ((D P_{T,\sigma})^{*}(d\lambda^{(\sigma)})_{\hat{\rho}} \wedge (d\lambda))_{\sigma^*} \]
\[ = (\lambda_{\sigma} P_{T,\sigma}(\alpha)) ((d\lambda)_{\hat{\rho}} \wedge (d\lambda))_{\sigma^*}. \]

Neither the scalar term nor the algebraic \( k \)-form term is identically zero, so \( \hat{E}_{\sigma,T}\alpha(d\lambda^{(\sigma)})_{\hat{\rho}} \neq 0. \]

The following lemma shows how \( \hat{E}_{\sigma,T} \) can represent the components of \( \omega \) perpendicular to \( \text{Tr}_{\sigma} \) for \( x \in f_{\sigma} \) and \( \omega \in \hat{\Lambda}^{k}(T) \).

**Lemma 4.2.** Let \( \omega \in \hat{\Lambda}^{k}(T) \) be given and let a \( d \)-simplex \( f_{\sigma} \) be given such that \( d \geq n - k \). Suppose that \( \omega \) vanishes at the boundaries of \( f_{\sigma} \), that is \( \omega|_{f_{\sigma}} = 0 \) for every \( \tau \) such that \([\tau] \not\subset [\sigma]\). Then there is a unique \( \omega_{\sigma} \in \Lambda^{k-(n-d)}(f_{\sigma}) \) such that
\[ \omega|_{f_{\sigma}} = \hat{E}_{\sigma,T}(\omega_{\sigma})|_{f_{\sigma}}. \]

**Proof.** By theorem 3.11 there exists \( \mu \in \Lambda^{n-k}(T) \) such that \( \omega = \iota_{T}\mu \), so for \( x \in f_{\sigma} \),
\[ \omega_{x} = n!|T| \sum_{\rho \in \Sigma([0:k-1],[0:n])} *(\mu_{x} \wedge (d\lambda)_{\rho}) \left(\lambda_{\rho^*}\right)_{x}(d\lambda)_{\rho} \]
\[ = n!|T| \sum_{\rho \in \Sigma([0:k-1],[0:n])} *(\mu_{x} \wedge (d\lambda)_{\rho}) \left(\lambda_{\rho^*}\right)_{x}(d\lambda)_{\rho}, \]
where we have eliminated terms from the sum that are zero because \( \lambda_{\rho^*}|_{f_{\sigma}} = 0 \). But if \( \rho \in \Sigma([0:k-1],[0:n]) \) and \([\rho^*] \not\subset [\sigma]\), then there is \( \hat{\rho} \in \Sigma([0:k-(n-d)-1],[\sigma]) \) such that \( (d\lambda)_{\hat{\rho}} = \pm(d\lambda)_{\rho} \wedge (d\lambda)_{\sigma^*} \). Therefore there exist some smooth functions \( \{\alpha_{\hat{\rho}}\} \subset C^{\infty}(f_{\sigma}) \) such that
\[ \omega|_{f_{\sigma}} = \sum_{\hat{\rho} \in \Sigma([0:k-(n-d)-1],[\sigma])} \alpha_{\hat{\rho}}((d\lambda)_{\hat{\rho}} \wedge (d\lambda)_{\sigma^*}). \]
By the same argument from the proof of lemma 4.1 we have \( (d\lambda)_{\hat{\rho}} \wedge (d\lambda)_{\sigma^*} = P_{T,\sigma}^{*}(d\lambda^{(\sigma)})_{\hat{\rho}} \wedge (d\lambda)_{\sigma^*} \), so
\[ \omega|_{f_{\sigma}} = \sum_{\hat{\rho} \in \Sigma([0:k-(n-d)-1],[\sigma])} \alpha_{\hat{\rho}}(P_{T,\sigma}^{*}(d\lambda^{(\sigma)})_{\hat{\rho}} \wedge (d\lambda)_{\sigma^*}). \]
Finally, we stipulated that \( \omega \) vanishes at the boundaries of \( f_{\sigma} \), so for each \( \alpha_{\hat{\rho}} \) there is \( \hat{\alpha}_{\hat{\rho}} \in C^{\infty}(f_{\sigma}) \) such that \( \alpha_{\hat{\rho}} = \lambda_{\sigma}\hat{\alpha}_{\hat{\rho}} \), and so
\[ \omega|_{f_{\sigma}} = P_{T,\sigma}^{*}\left( \sum_{\hat{\rho} \in \Sigma([0:k-(n-d)-1],[\sigma])} \hat{\alpha}_{\hat{\rho}}(d\lambda^{(\sigma)})_{\hat{\rho}} \wedge (\lambda_{\sigma}(d\lambda))_{\sigma^*} \right) \]
\[ = \hat{E}_{\sigma,T}\left( \sum_{\hat{\rho} \in \Sigma([0:k-(n-d)-1],[\sigma])} \hat{\alpha}_{\hat{\rho}}(d\lambda^{(\sigma)})_{\hat{\rho}} \right)|_{f_{\sigma}}. \]
The uniqueness follows from lemma 4.4.

Having established a way to encode the perpendicular component of \( \omega|_{f_{\sigma}} \) for a single boundary simplex \( f_{\sigma} \) using \( \hat{E}_{\sigma,T} \), we now construct the full bubble decomposition of a trace-free \( k \)-form.

We denote with \( Z^{k}(T) \) the set of increasing maps \( \sigma \) for which \( \hat{E}_{\sigma,T} \) is defined,
\[ Z^{k}(T) := \bigcup_{d=n-k}^{n} \Sigma([0:d],[0:n]), \]
and use \( \sigma \in Z^k(T) \) to index the \( k \)-bubble trace space \( B^k(T) \), the product of the spaces for which \( \tilde{E}_{\sigma,T} \) is defined,

\[
B^k(T) := \bigotimes_{\sigma \in Z^k(T)} \Lambda^{k-(n-\dim(f_\sigma))}(\mathcal{J}_\sigma).
\]

Finally, we define \( \hat{E}_T : B^k(T) \to \hat{\Lambda}^k(T) \) to be the sum of the extensions of each of the components of the product space: given \( W = \otimes_{\sigma \in Z^k(T)} \omega_\sigma \),

\[
\hat{E}_T : W \mapsto \sum_{\sigma \in Z^k(T)} \hat{E}_{\sigma,T} \omega_\sigma.
\]

**Theorem 4.3.** \( \hat{E}_T \) is a bijection, \( \hat{E}_T : B^k(T) \to \Lambda^k(T) \).

**Proof.** Lemma \[4.1\] showed that \( \hat{E}_{\sigma,T} \) is injective for each individual boundary simplex \( f_\sigma \), but we must show that the sum of contributions from different boundary simplices is still injective.

Suppose \( \omega = \hat{E}_T W = \sum_{\sigma \in Z^k(T)} \hat{E}_{\sigma,T} \omega_\sigma = 0 \). Let \( d \) be the smallest dimension such that \( \omega_\sigma = 0 \) for every \( \sigma \in Z^k(T) \) such that \( \dim(f_\sigma) < d \), and choose \( \tilde{\sigma} \in Z^k(T) \) such that \( \dim(f_{\tilde{\sigma}}) = d \). Given \( \tau \in Z^k(T) \) such that \( \tau \neq \sigma \) and \( \dim(f_\tau) \geq d \), we have

\[
(\hat{E}_{\tau,T} \omega_\tau)|_{f_\tau} = (P^{*}_{\tau,T} \omega) \wedge (\lambda_\tau (d \lambda) \tau)|_{f_\tau} = 0,
\]

a conclusion we reach because \([\tau] \not\subseteq [\tilde{\sigma}]\), and so \( \lambda_\tau |_{f_\tau} = 0 \). Therefore

\[
0 = \omega |_{f_\tau} = (\hat{E}_T W)|_{f_\tau} = (\hat{E}_{\sigma,T} \omega_\sigma)|_{f_\tau}.
\]

By the same argument as in lemma \[4.1\] this implies \( \omega_\sigma = 0 \). Because \( \tilde{\sigma} \) was arbitrarily chosen, \( \omega_\sigma = 0 \) for each \( d \)-simplex \( f_\sigma \), but this violates the way \( d \) was chosen, so we must conclude \( W = 0 \).

Now we prove that \( \hat{E}_T \) is surjective. Suppose \( \omega \in \Lambda^k(T) \) vanishes at \( \partial T \). Then there exists \( \tilde{\omega} \in \Lambda^k(T) \) such that \( \omega = \lambda_T \tilde{\omega} = \hat{E}_T \tilde{\omega}_T \).

Now suppose that we have shown that the range of \( \hat{E}_T \) includes all \( k \)-forms that vanish at all \( f_\sigma \) with \( \dim(f_\sigma) \leq d \), and let \( \omega \in \Lambda^k \) be a \( k \)-form that vanishes at each \( f_\sigma \) with \( \dim(f_\sigma) < d \). For each \( \tilde{\sigma} \) such that \( \dim(f_{\tilde{\sigma}}) = d \) there is, by lemma \[4.2\], \( \tilde{\omega}_\sigma \in \Lambda^{k-(n-d)}(\mathcal{J}_\sigma) \) such that \( (\hat{E}_{\sigma,T} \tilde{\omega}_{\tilde{\sigma}})|_{f_\sigma} = \omega |_{f_\sigma} \). Let

\[
\tilde{\omega} = \omega - \sum_{\tilde{\sigma} \in \Sigma([0:d],[0:n])} \hat{E}_{\tilde{\sigma},T} \tilde{\omega}_{\tilde{\sigma}}.
\]

By \[13\], \( \tilde{\omega} \) vanishes at all \( d \) simplices, so there exists \( \tilde{W} = \otimes_{\sigma \in Z^k(T)} \tilde{\omega}_\sigma \) such that \( \tilde{\omega} = \hat{E}_T \tilde{W} \). Hence defining \( W = \otimes_{\sigma \in Z^k(T)} \omega_\sigma \) by

\[
\omega_\sigma = \begin{cases} 
\tilde{\omega}_\sigma, & \dim(f_\sigma) \neq d, \\
\tilde{\omega}_\sigma - \omega_\sigma, & \dim(f_\sigma) = d,
\end{cases}
\]

we have constructed \( W \) such that \( \hat{E}_T W = \omega \). By induction, we conclude \( \hat{E}_T \) is surjective. \( \square \)

The proof of the bijection \( \hat{E}_T : B^k(T) \to \hat{\Lambda}^k(T) \) shows that \( \hat{E}_{\sigma,T} \) operators define a decomposition of \( \hat{\Lambda}(T) \),

\[
\hat{\Lambda}^k(T) = \bigoplus_{\sigma \in Z^k(T)} \hat{E}_{\sigma,T}[\Lambda^{k-(n-d)}(\mathcal{J}_\sigma)].
\]

We define the inverse of \( \hat{E}_T \) to be the operator \( \hat{B}^k_T : \hat{\Lambda}^k(T) \to B^k(T) \), and we let \( \hat{B}^k_{T,\sigma} : \hat{\Lambda}^k(T) \to \Lambda^{k-(n-d)}(\mathcal{J}_\sigma) \) be the projection of \( \hat{B}^k_{T,\sigma} \) onto the \( \sigma \) component. We have two remarks about this operator.
First, the proof of theorem \[4.3\] shows a constructive procedure for evaluating \((B_{T,r}^k \omega)_{r}\). One must evaluate \(\omega\) at \(P_{T,r}(x)\) for each simplex \(f_{\rho}\) such that \([\rho] \subseteq [\sigma]\), and solve the system

\[
\omega_{P_{T,r}(x)} = \lambda_{\rho}(B_{T,r}^k \omega)_{P_{T,r}(x)} + \sum_{\bar{\rho} \in Z^k(T) \atop |\bar{\rho}| = |\rho|} (\bar{E}_{\bar{\rho},T}B_{T,r}^k \omega)_{P_{T,r}(x)} , \quad [\rho] \subseteq [\sigma].
\]

This is a triangular system that can be solved from the smallest dimension to the largest.

The second remark is that, while we have shown that this bubble decomposition exists for all of \(\hat{\Lambda}^k(T)\), it cannot be used to define a basis for every subspace \(X(T) \subseteq \Lambda^k(T)\). On the other hand, it can be used to define a basis for the trace-free full polynomials \(\hat{P}_T \Lambda^k(T)\),

\[
\hat{P}_T \Lambda^k(T) = \bigoplus_{\sigma \in Z^k(T)} \hat{E}_{\sigma,T}[P_{T \dim(f_\sigma) - 1} \Lambda^{k-(\dim(f_\sigma))}(f_\sigma)].
\] (20)

On the other hand, the components of a trace-free trimmed \(k\)-form \(\omega \in \hat{P}_T \Lambda^1(T)\) will not necessarily be in the same space. As a simple example, consider \(\lambda_0 \phi_{12} \in \hat{P}_T^2 \Lambda^1(T)\) for the triangle \(T = \{v_0, v_1, v_2\}\). Its bubble decomposition is

\[
\lambda_0 \phi_{12} = \lambda_0 \lambda_2 d\lambda_1 - \lambda_0 \lambda_1 d\lambda_2 = \underbrace{P_{T,(0,1)}^* (1) \wedge (\lambda_0 \lambda_2 d\lambda_1)}_{\hat{E}_{(0,2),T}(1)} + \underbrace{P_{T,(0,1)}^* (-1) \wedge (\lambda_0 \lambda_1 d\lambda_2)}_{\hat{E}_{(0,1),T}(-1)},
\]

and each of its bubble components is in \(P_T \Lambda^1(T)\) but not \(P_T^2 \Lambda^1(T)\).

### 4.5 The \(\hat{E}_{\sigma,T}\) operator

We can generalize the bubble decomposition from the previous section to boundary simplices. Given \(\sigma\) and \(\xi\) such that \([\sigma] \subseteq [\xi]\), we can define \(\hat{E}_{\sigma,\xi} : \Lambda^{k-(\dim(f_\xi) - \dim(f_\sigma))}(f_\sigma) \rightarrow \hat{\Lambda}^k(f_\xi)\) by

\[
\hat{E}_{\sigma,\xi} : \omega_\sigma \mapsto P_{\xi,\sigma,\omega_\sigma}^* \omega_\sigma \wedge (\lambda_\sigma^{\xi})_\xi \wedge (d\lambda^{\xi})_\xi, \quad (21)
\]

and we have a similar decomposition result to \[19\].

\[
\hat{\Lambda}^k(f_\xi) = \bigoplus_{\sigma \in Z^k(f_\xi)} \hat{E}_{\sigma,\xi}[\Lambda^{k-(\dim(f_\xi) - \dim(f_\sigma))}(f_\sigma)].
\]

We use the bubble decomposition to define the extension operator \(\hat{E}_{\tau,\xi} : \hat{\Lambda}^k(f_\sigma) \rightarrow \hat{\Lambda}^k(f_\xi)\) when \([\tau] \subseteq [\xi]\) by its action on the bubble components:

\[
\hat{E}_{\tau,\xi} : \hat{E}_{\sigma,\tau,\omega_\sigma} \mapsto P_{\xi,\sigma,\omega_\sigma}^* \omega_\sigma \wedge (\lambda_\sigma^{\xi})_\tau \wedge (d\lambda^{\xi})_\tau. \quad (22)
\]

Note the difference between \[21\] and \[22\] is only in the barycentric form wedged with the pullback: for \(\hat{E}_{\tau,\xi} \hat{E}_{\sigma,\tau,\omega_\sigma}\) it is \((d\lambda^{\xi})_\tau \wedge \lambda_\sigma^{\xi}\), so that the product \(\lambda_\sigma^{\xi} (d\lambda^{\xi})_\tau \wedge \lambda_\sigma^{\xi}\) is not trace-free on \(f_\xi\), but has nonzero trace on any boundary simplex of \(f_\xi\) that contains \(f_\tau\).

**Theorem 4.4.** Let a family of extension operators be defined by

\[
\{\hat{E}_{\tau,\xi} : \hat{\Lambda}^k(f_\sigma) \rightarrow \hat{\Lambda}^k(f_\xi), [\tau] \subseteq [\xi]\},
\]

where \(\hat{E}_{\tau,\xi}\) is defined \[22\]. The family is a consistent family, that is for simplices \(f_\rho, f_\tau\) and \(f_\xi\) such that \([\rho], [\tau] \subseteq [\xi]\), we have

\[
\text{Tr}_{\xi,\rho,\tau} \hat{E}_{\tau,\xi,\omega} = \hat{E}_{\rho,(\tau,\rho),\tau} \text{Tr}_{\tau,\rho,\tau,\rho}, \quad \omega \in \hat{\Lambda}^k(f_\sigma). \quad (23)
\]
We now show that the \( \dot{\sigma} \) with which by the inductive assumption has a geometric decomposition in the \( \dot{\Lambda}^k(\dot{f}_\sigma) \). By the inductive assumption, this completes the proof.

First let a simplex \( f_\rho \) be given such that \( \| \tau \| \leq \| \rho \| \). Because \( \dot{E}_{\sigma,\tau} \omega_\sigma \) is trace-free, \( \text{Tr}_{\tau,\rho \cap \tau} \dot{E}_{\sigma,\tau} \omega_\sigma = 0 \), and so

\[
\dot{E}_{\rho,\tau,\rho} \text{Tr}_{\tau,\rho \cap \tau} \dot{E}_{\sigma,\tau} \omega_\sigma = 0.
\]

By the definition in (22), we see that \( \dot{E}_{\xi,\tau,\rho} \dot{E}_{\sigma,\tau} \omega_\sigma = 0 \) as well, because \( \dot{E}_{\xi,\rho} \lambda^\xi (d\lambda^\xi)_{\tau \setminus \sigma} = 0 \).

Now let a simplex \( f_\rho \) be given such that \( \| \tau \| \leq \| \rho \| \), which implies \( \text{Tr}_{\tau,\rho \cap \tau} \dot{E}_{\sigma,\tau} \omega_\sigma = \omega_\sigma \). We compute the left hand side of (23):

\[
\dot{E}_{\rho,\tau,\rho} \text{Tr}_{\tau,\rho \cap \tau} \dot{E}_{\sigma,\tau} \omega_\sigma = \dot{E}_{\tau,\rho} \dot{E}_{\sigma,\tau} \omega_\sigma = \dot{E}_{\rho,\tau,\rho} \text{Tr}_{\tau,\rho \cap \tau} (\dot{E}_{\sigma,\tau} \omega_\sigma),
\]

where we have used (13) between the second and third right hand sides. Because every \( \omega \in \dot{\Lambda}^k(\dot{f}_\sigma) \) has a bubble decomposition, this completes the proof.

### 4.6 Geometric decompositions

With \( \dot{E}_{\sigma,\tau} \) defining a consistent family of extension operators, we can now show that it defines a geometric decomposition of all of \( \Lambda^k(T) \).

**Theorem 4.5.** With \( \dot{E}_{\sigma,\tau} \) defined as in (22) taking \( f_\xi = T \),

\[
\Lambda^k(T) = \bigoplus_{\dim(f_\sigma) \geq k} \dot{E}_{\sigma,\tau}[\dot{\Lambda}^k(\dot{f}_\sigma)].
\]

**Proof.** The proof proceeds along the same lines as the proof of the bubble decomposion in theorem 4.3. Implicit in the fact that \( \dot{E}_{\sigma,\tau} \) defines a consistent family of extension operators is that

\[
\text{Tr}_\sigma \dot{E}_{\sigma,\tau} \omega = \omega, \quad \omega \in \dot{\Lambda}^k(\dot{f}_\sigma).
\]

This means that if \( \omega \in \dot{\Lambda}^k(\dot{T}) \) then \( \omega = \dot{E}_{\tau,\tau} \omega \). From this base case one can inductively assume that every \( \omega \) has a geometric decomposition in the \( \dot{E}_{\sigma,\tau} \) family if it is trace-free on boundary simplices with dimension less than or equal to \( d \). Then taking \( \omega \) that is trace-free on boundary simplices with dimension less than \( d \) and define

\[
\tilde{\omega} = \omega - \sum_{\sigma \in \Sigma([0:d])} \dot{E}_{\sigma,\tau} \text{Tr}_\sigma \omega,
\]

which by the inductive assumption has a geometric decomposition in the \( \dot{E}_{\sigma,\tau} \) family, and so \( \omega \) has one as well. By the inductive assumption, this completes the proof.

We now show that the \( \dot{E}_{\sigma,\tau} \) extension operator defines a geometric decomposition of full polynomial \( k \)-forms.

**Theorem 4.6.** With \( \dot{E}_{\sigma,\tau} \) defined as in (22) taking \( f_\xi = T \),

\[
\mathcal{P}_r \Lambda^k(T) = \bigoplus_{\dim(f_\sigma) \geq k} \dot{E}_{\sigma,\tau}[\dot{P}_r \Lambda^k(\dot{f}_\sigma)].
\]

**Proof.** Let \( f_\tau \) such that \( \dim(f_\tau) \geq k \) be given. By (20), \( \mathcal{P}_r \Lambda^k(\dot{f}_\tau) \) is spanned by bubble functions of the form \( \dot{E}_{\sigma,\tau} \omega_\sigma \) for \( \omega_\sigma \in \mathcal{P}_r \dim(f_\tau) - \dot{\Lambda}^k(\dot{f}_\sigma)(\dot{f}_\tau) \). By inspection of (22) we see that this means \( \dot{E}_{\sigma,\tau} \dot{E}_{\tau,\tau} \omega_\sigma \in \mathcal{P}_r \dot{\Lambda}^k(T) \). The injectivity of \( \dot{E}_{\sigma,\tau} \), the fact that it defines a consistent family of extension operators, and a counting argument are sufficient to complete the proof.
Our final theorem shows that \( \dot{E}_{\sigma,T} \) defines a geometric decomposition of trimmed polynomial \( k \)-forms as well.

**Theorem 4.7.** With \( \dot{E}_{\sigma,T} \) defined as in (22) taking \( f \equiv T \),

\[
P_\tau^{-} \Lambda^k(T) = \bigoplus_{\dim(f_\tau) \geq k} \dot{E}_{\sigma,T}[\dot{P}_\tau^{-} \Lambda^k(f_\sigma)].
\]

**Proof.** It is sufficient to show that \( \dot{E}_{\tau,T} \omega \in P_{\tau}^{-} \Lambda^k(T) \) for each \( f_\tau \) such that \( \dim(f_\tau) \geq k \) and each \( \omega \in \dot{P}_\tau^{-} \Lambda^k(f_\tau) \). By the proof of theorem (21) it is already known that \( \dot{E}_{\tau,T} \omega \in P_{\tau} \Lambda^k(T) \). We shall apply the Koszul operator centered at the centroid of \( f_\tau \) to \( \dot{E}_{\tau,T} \omega \); if we can show that \( \kappa^{(\tau)} \dot{E}_{\tau,T} \omega \in P_{\tau} \Lambda^{k-1}(T) \), that is sufficient to prove that \( \dot{E}_{\tau,T} \omega \in P_{\tau}^{-} \Lambda^k(T) \).

Let the bubble decomposition of \( \omega \) be

\[
\omega = \sum_{\sigma \in 2^k(f_\tau)} \dot{E}_{\sigma,T} \omega_\sigma.
\]

We apply \( \kappa^{(\tau)} \) to the extension of a single bubble component, but shift it to be centered at \( \kappa^{(\sigma)} \), which introduces a term involving the interior product with some constant vector \( \lambda \).

Because the vector \( \dot{b}_{\sigma,T} \) is constant, the \( (k-1) \)-form \( \delta_{\sigma,1} \) is in \( P_{\sigma} \Lambda^{k-1}(T) \).

We now address the term \( \kappa^{(\sigma)} \dot{E}_{\tau,T} \dot{E}_{\sigma,T} \omega_\sigma \). We wish to show that in this case \( \kappa^{(\sigma)} \) and \( \dot{E}_{\tau,T} \) commute. Using the product rule for the Koszul operator, we have

\[
\kappa^{(\sigma)} \dot{E}_{\tau,T} \dot{E}_{\sigma,T} \omega_\sigma = \kappa^{(\sigma)} \dot{E}_{\tau,T} [P_{T_{\sigma,T}} \omega_\sigma \wedge (\lambda_\sigma (d\lambda)_{\tau\sigma})]
\]

\[
= (\kappa^{(\sigma)} P_{T_{\sigma,T}} \omega_\sigma) \wedge (\lambda_\sigma (d\lambda)_{\tau\sigma}) + (-1)^{\hat{k}} P_{T_{\tau,T}} \omega_\sigma \wedge (\lambda_\sigma \kappa^{(\tau)}(d\lambda)_{\tau\sigma}),
\]

where \( \hat{k} = k - (\dim(f_\tau) - \dim(f_\sigma)) \). The centroid of \( f_\sigma \) is fixed by \( P_{T_{\sigma,T}} \), so \( \kappa^{(\tau)} \) commutes with \( P_{T_{\sigma,T}} \) and the first term becomes

\[
(\kappa^{(\sigma)} P_{T_{\tau,T}} \omega_\sigma) \wedge (\lambda_\sigma (d\lambda)_{\tau\sigma}) = P_{T_{\tau,T}} \kappa^{(\tau)}(\omega_\sigma) \wedge (\lambda_\sigma (d\lambda)_{\tau\sigma}) = \dot{E}_{\tau,T} \dot{E}_{\tau,T} (\kappa^{(\tau)} \omega_\sigma).
\]

Because \( \lambda_j \) vanishes at the centroid of \( f_\sigma \) if \( j \in [\tau \setminus \sigma] \), we have \( \kappa^{(\sigma)} d\lambda_j = \lambda_j \) for each such \( j \). Let \( m + 1 \) be the number of indices in \( [\tau \setminus \sigma] \) and let \( \rho \in \Sigma([0:m] \cup [0:n]) \) be an increasing map such that \( \rho([\rho]) = [\tau \setminus \sigma] \). By the product rule

\[
\kappa^{(\tau)}(d\lambda)_{\tau\rho} = \sum_{j=0}^{m} (-1)^{m} \lambda_\rho(j) (d\lambda)_{\rho\rho(j)}.
\]

Putting this into the second term in (25), we get

\[
(-1)^{\hat{k}} P_{T_{\tau,T}} \omega_\sigma \wedge (\lambda_\sigma \kappa^{(\tau)}(d\lambda)_{\tau\rho}) = \sum_{j=0}^{m} (-1)^{j+\hat{k}} P_{T_{\tau,T}} \omega_\sigma \wedge (\lambda_\sigma \rho_{\rho(j)})
\]

\[
= \sum_{j=0}^{m} P_{T_{\tau,T}} \rho_{\rho(j)} P_{T_{\sigma,\rho(j)}}((-1)^{j+\hat{k}} \omega_\sigma) \wedge (\lambda_\sigma \rho_{\rho(j)})
\]

\[
= \dot{E}_{\tau,T} \left[ \sum_{j=0}^{m} \dot{E}_{\sigma,\rho(j)} \rho P_{T_{\sigma,\rho(j)}}((-1)^{j+\hat{k}} \omega_\sigma) \right].
\]

Combining both terms, we have

\[
\kappa^{(\tau)} \dot{E}_{\tau,T} \dot{E}_{\sigma,T} \omega_\sigma = \dot{E}_{\tau,T} \left[ \dot{E}_{\sigma,T} (\kappa^{(\tau)} \omega_\sigma) + \sum_{j=0}^{m} \dot{E}_{\sigma,\rho(j)} \rho P_{T_{\sigma,\rho(j)}}((-1)^{j+\hat{k}} \omega_\sigma) \right].
\]
We now want to show that the term in brackets is $\kappa^{(\sigma)} \tilde{E}_{\sigma,\tau} \omega_{\sigma}$. We essentially follow the steps above in reverse. For the first term,
\[
\tilde{E}_{\sigma,\tau}(\kappa^{(\sigma)} \omega_{\sigma}) = P^{*}_{\tau,\sigma}(\kappa^{(\sigma)} \omega_{\sigma}) \wedge (\lambda^{(\tau)}_{\sigma})(d\lambda^{(\tau)})_{\tau\setminus\sigma} \\
= (\kappa^{(\sigma)} P^{*}_{\tau,\sigma} \omega_{\sigma}) \wedge (\lambda^{(\tau)}_{\sigma})(d\lambda^{(\tau)})_{\tau\setminus\sigma}.
\]

For the second term,
\[
\sum_{j=0}^{m} \tilde{E}_{\sigma,\tau}(P^{*}_{\sigma,\rho(j),\sigma}((-1)^{j+k} \kappa_{\sigma})) = \sum_{j=0}^{m} P^{*}_{\tau,\sigma,\rho(j),\sigma}((-1)^{j+k} \kappa_{\sigma} \wedge (\lambda^{(\tau)}_{\sigma})(d\lambda^{(\tau)})_{\rho\setminus\rho(j)}) \\
= \sum_{j=0}^{m} (-1)^{j+k} P^{*}_{\tau,\sigma} \omega_{\sigma} \wedge (\lambda^{(\tau)}_{\sigma})(d\lambda^{(\tau)})_{\rho\setminus\rho(j)} \\
= (-1)^{k} P^{*}_{\tau,\sigma} \omega_{\sigma} \wedge (\lambda^{(\tau)}_{\sigma}(\kappa^{(\sigma)})(d\lambda^{(\tau)})_{\tau\setminus\sigma}).
\]

Combining these back together we get the desired result,
\[
(\kappa^{(\sigma)} P^{*}_{\tau,\sigma} \omega_{\sigma}) \wedge (\lambda^{(\tau)}_{\sigma})(d\lambda^{(\tau)})_{\tau\setminus\sigma} + (-1)^{k} P^{*}_{\tau,\sigma} \omega_{\sigma} \wedge (\lambda^{(\tau)}_{\sigma}(\kappa^{(\sigma)})(d\lambda^{(\tau)})_{\tau\setminus\sigma}) = \kappa^{(\sigma)} \tilde{E}_{\sigma,\tau} \omega_{\sigma},
\]
that is,
\[
\kappa^{(\sigma)} \tilde{E}_{\sigma,\tau} \omega_{\sigma} = \tilde{E}_{\sigma,\tau} \kappa^{(\sigma)} \tilde{E}_{\sigma,\tau} \omega_{\sigma}.
\]

We now shift the center of the Koszul operator back to $\kappa^{(\tau)}$, which again introduces an interior product with the constant vector which we denote $\delta_{\sigma,2}$:
\[
\tilde{E}_{\tau,T} \kappa^{(\sigma)} \tilde{E}_{\sigma,\tau} \omega_{\sigma} = \tilde{E}_{\tau,T}(\kappa^{(\tau)} \tilde{E}_{\sigma,\tau} \omega_{\sigma} - (\tilde{E}_{\sigma,\tau} \omega_{\sigma}, \delta_{\sigma,2}).
\]

As with the $\delta_{\sigma,1}$, we know that $\delta_{\sigma,2} \in P_{r} \Lambda^{k-1}(f_{\sigma})$.

Why have we gone through the steps of shifting the Koszul operator between the centroids of $f_{\sigma}$ and $f_{\tau}$? Because although $\kappa^{(\sigma)} \tilde{E}_{\sigma,\tau} \omega_{\sigma}$ is trace-free (as demonstrated in the steps that it took to show that $\kappa^{(\sigma)}$ and $\tilde{E}_{\tau,T}$ commuted), $\kappa^{(\tau)} \tilde{E}_{\sigma,\tau} \omega_{\sigma}$ is not necessarily trace-free. We cannot apply $\tilde{E}_{\tau,T}$ to $\kappa^{(\tau)} \tilde{E}_{\sigma,\tau} \omega_{\sigma}$ or $\delta_{\sigma,2}$ separately, because the domain of $\tilde{E}_{\tau,T}$ is trace-free $k$-forms.

Nevertheless, we can now combine into $\kappa^{(\tau)} \tilde{E}_{\tau,T} \omega$ the contributions from the different terms in the bubble decomposition:
\[
\kappa^{(\tau)} \tilde{E}_{\tau,T} \omega = \sum_{\sigma \in Z^2(f_{\tau})} \tilde{E}_{\tau,T}(\kappa^{(\tau)} \tilde{E}_{\sigma,\tau} \omega_{\sigma} + \delta_{\sigma,2}) + \delta_{\sigma,1} \\
= \tilde{E}_{\tau,T} \left[ \kappa^{(\tau)} \omega + \sum_{\sigma \in Z^2(f_{\tau})} \delta_{\sigma,2} \right] + \sum_{\sigma \in Z^{k}(f_{\tau})} \delta_{\sigma,1}.
\]

By the assumption that $\omega \in \tilde{P}_{r} \Lambda^{k}(f_{\sigma})$, we know that $\kappa^{(\tau)} \omega \in \tilde{P}_{r} \Lambda^{k-1}(f_{\sigma})$. Therefore the term in brackets is in $\tilde{P}_{r} \Lambda^{k-1}(f_{\sigma})$. By theorem 14.6 that implies that the first term, and thus the whole expression, is in $P_{r} \Lambda^{k-1}(T)$.

The main results of this work combine to unify the geometric decompositions of full polynomials, trimmed polynomials, and all $k$-forms through the action of just two operators acting on the unrestricted polynomial spaces.
Corollary 4.8. Generalizing the operator $\hat{*}_T$ to $\hat{*}_\sigma : \Lambda^k(f_\sigma) \rightarrow \Lambda^{\dim(f_\sigma) - k}(\overline{f_\sigma})$, we have

$$\Lambda^k(T) = \bigoplus_{\dim(f_\sigma) \geq k} \hat{E}_{\sigma,T} \hat{*}_\sigma [\Lambda^{\dim(f_\sigma) - k}(\overline{f_\sigma})];$$

(26)

$$\mathcal{P}_T \Lambda^k(T) = \bigoplus_{\dim(f_\sigma) \geq k} \hat{E}_{\sigma,T} \hat{*}_\sigma [\mathcal{P}_T \Lambda^{\dim(f_\sigma) - k}(\overline{f_\sigma})];$$

(27)

$$\mathcal{P}_T^{-1} \Lambda^k(T) = \bigoplus_{\dim(f_\sigma) \geq k} \hat{E}_{\sigma,T} \hat{*}_\sigma [\mathcal{P}_T^{-1} \Lambda^{\dim(f_\sigma) - k}(\overline{f_\sigma})].$$

(28)

References

[1] Mark Ainsworth and Joe Coyle. “Hierarchic finite element bases on unstructured tetrahedral meshes”. In: International journal for numerical methods in engineering 58.14 (2003), pp. 2103–2130.

[2] Douglas N Arnold, Richard S Falk, and Ragnar Winther. “Finite element exterior calculus, homological techniques, and applications”. In: Acta numerica 15 (2006), pp. 1–155.

[3] Douglas N Arnold, Richard S Falk, and Ragnar Winther. “Geometric decompositions and local bases for spaces of finite element differential forms”. In: Computer Methods in Applied Mechanics and Engineering 198.21-26 (2009), pp. 1660–1672.

[4] Martin W Licht. “On basis constructions in finite element exterior calculus”. In: arXiv preprint arXiv:1810.01896 (2018).

[5] Spencer J Sherwin and George Em Karniadakis. “A new triangular and tetrahedral basis for high-order (hp) finite element methods”. In: International Journal for Numerical Methods in Engineering 38.22 (1995), pp. 3775–3802.

[6] Shayne FD Waldron. An introduction to finite tight frames. Springer, 2018.