REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS AND VISCOSITY SOLUTION OF SECOND ORDER INTEGRO-DIFFERENTIAL EQUATION WITHOUT MONOTONICITY CONDITION: CASE WITH THE MEASURE OF LÉVY INFINITE

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Abstract We consider the problem of viscosity solution of integro-partial differential equation (IPDE in short) with one obstacle via the solution of reflected backward stochastic differential equations (RBSDE in short) with jumps. We show the existence and uniqueness of a continuous viscosity solution of equation with non local terms, if the generator is not monotonous and Levy’s measure is infinite.

Key words Integro-partial differential equation; reflected stochastic differential equations with jumps; viscosity solution; non-local operator

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1 Introduction

We consider the following system of integro-partial differential equation with one-obstacle $\ell$, which is a function of $(t, x)$: $\forall i \in \{1, \cdots, m\}$,

$$
\begin{cases}
\min \left\{ u^i(t,x) - \ell(t,x); -\partial_t u^i(t,x) - b(t,x)^T D_x u^i(t,x) \\
\frac{1}{2} \text{Tr}(\sigma \sigma^T (t,x) D_{xx}^2 u^i(t,x)) - K_i u^i(t,x) - h^i(t,x, u^i(t,x)) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\
u^i(T, x) = g^i(x);
\end{cases}
$$

(1.1)

where the operators $B_i$ and $K_i$ are defined as follows:

$$
B_i u^i(t,x) = \int_{\mathcal{E}} \gamma^i(t,x,e) (u^i(t, x + \beta(t,x,e)) - u^i(t,x)) \lambda(de);
$$

$$
K_i u^i(t,x) = \int_{\mathcal{E}} (u^i(t, x + \beta(t,x,e)) - u^i(t,x) - \beta(t,x,e)^T D_x u^i(t,x)) \lambda(de).
$$

(1.2)

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The resolution of (1.1) is in connection with the following system of backward stochastic differential equations with jumps and one-obstacle $\ell$:

\[
\begin{cases}
(i) & \quad dY^{s,t,x}_s = -f(s, X^{t,x}_s, (Y^{s,t,x}_s)_{i=1,m}, Z^{s,t,x}_s, U^{s,t,x}_s)ds \\
& \quad -dK^{s,t,x}_s + Z^{s,t,x}_s dB_s + \int_E U^{s,t,x}_s(e)\tilde{\mu}(ds, de), \quad s \leq T; \\
(ii) & \quad Y^{s,t,x}_s \geq \ell(s, X^{t,x}_s) \quad \text{and} \quad \int_0^T (Y^{s,t,x}_s - \ell(s, X^{t,x}_s))dK^{s,t,x}_s = 0; \\
(iii) & \quad Y^{s,t,x}_T = g^i(Y^{t,x}_T).
\end{cases}
\tag{1.3}
\]

and the following standard stochastic differential equation of diffusion-jump type:

\[
X^{s,x}_s = x + \int_t^s b(r, X^{t,x}_r) dr + \int_t^s \sigma(r, X^{t,x}_r) dB_r + \int_t^s \int_E \beta(r, X^{t,x}_r, e)\tilde{\mu}(dr, de),
\]

for $s \in [t, T]$ and $X^{s,x}_s = x$ if $s \leq t$.

It is recalled that pioneering work was done for the resolution of (1.1), among these works we can mention those of Barles et al [1] if without obstacle, Harraj et al [6] in the case with two obstacles; with as common point the hypothesis of monotony on the generator and $\gamma \geq 0$. But recently Hamadène and Morlais relaxed these conditions with $\lambda(.)$ finite [4].

In this work, we propose to solve (1.1) by relaxing the monotonicity of the generator and the positivity of $\gamma$ and assuming that $\lambda = \infty$.

This article is organized as follows: in the next section, we give the notations and the assumptions of our objects; in Section 3, we recall a number of existing results; in Section 4, we build estimates and properties for a good resolution of our problem; Section 5 is reserved to give our main result and Section 6 is an extension of our result.

And in the end, classical definition of the concept of viscosity solution is put in Appendix.

2 Notations and Assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be a stochastic basis such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, and $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, $t \geq 0$, and we suppose that the filtration is generated by the two mutually independents processes:

(i) $\mathcal{B} := (\mathcal{B}_t)_{t \geq 0}$ a $d$-dimensional Brownian motion and

(ii) a Poisson random measure $\mu$ on $\mathbb{R}^+ \times \mathcal{E}$, where $\mathcal{E} := \mathbb{R}^\ell - \{0\}$ is equipped with its Borel field $\mathcal{E}$ ($\ell \geq 1$). The compensator $\nu(dt, de) = dt\lambda(de)$ is such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \lambda)([0,t] \times A)\}_{t \geq 0}$ is a martingale for all $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. We also assume that $\lambda$ is a $\sigma$-finite measure on $(\mathcal{E}, \mathcal{E})$, and integrates the function $(1 \wedge |e|^2)$ and $\lambda(\mathcal{E}) = \infty$.

Let us now introduce the following spaces:

(iii) $\mathcal{P}$ (resp. $\mathbb{P}$) the field on $[0, T] \times \Omega$ of $\mathcal{F}_{t \leq T}$-progressively measurable (resp. predictable) sets.

(iv) For $\kappa \geq 1$, $L^2_{\kappa}(\lambda)$ the space of Borel measurable functions $\varphi := (\varphi(e))_{e \in \mathcal{E}}$ from $\mathcal{E}$ into $\mathbb{R}^\kappa$ such that $\|\varphi\|_{L^2_{\kappa}(\lambda)}^2 = \int_{\mathcal{E}} |\varphi(e)|^2_\kappa \lambda(de) < \infty$; $L^2(\lambda)$ will be simply denoted by $L^2(\lambda)$;

(v) $S^2(\mathbb{R}^\kappa)$ the space of rcll (for right continuous with left limits) $\mathcal{P}$-measurable and $\mathbb{R}^\kappa$-valued processes such that $\mathbb{E}[\sup_{s \leq T} |Y^s|^2] < \infty$; $\mathcal{A}^2$ is its subspace of continuous non-decreasing
processes \((K_t)_{t \leq T}\) such that \(K_0 = 0\) and \(\mathbb{E} \left[ |K_T|^2 \right] < \infty\);

(vi) \(\mathbb{H}^2(\mathbb{R}^{k \times d})\) the space of processes \(Z := (Z_s)_{s \leq T}\) which are \(\mathcal{P}\)-measurable, \(\mathbb{R}^{k \times d}\)-valued, and satisfying \(\mathbb{E} \left[ \int_0^T |Z_s|^2 \, ds \right] < \infty\);

(vii) \(\mathbb{H}^2(\mathbb{L}^2(\lambda))\) the space of processes \(U := (U_s)_{s \leq T}\) which are \(\mathcal{P}\)-measurable, \(\mathbb{L}^2(\lambda)\)-valued and satisfying \(\mathbb{E} \left[ \int_0^T |U_s|^2(\omega) \, |\mathbb{L}^2(\lambda)| \, ds \right] < \infty\);

(viii) \(\Pi_g\) the set of deterministics functions \(\varpi: (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \varpi(t, x) \in \mathbb{R}\) of polynomial growth, that is, for which there exists two non-negative constants \(C\) and \(p\) such that for any \((t, x) \in [0, T] \times \mathbb{R}^k\),

\[
|\varpi(t, x)| \leq C(1 + |x|^p).
\]

The subspace of \(\Pi_g\) of continuous functions will be denoted by \(\Pi_g^c\);

(ix) \(\mathcal{U}\) the subclass of \(\Pi_g^c\) which consists of functions \(\Phi: (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \mathbb{R}\) such that for some non-negative constants \(C\) and \(p\), we have

\[
|\Phi(t, x) - \Phi(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ for any } t, x, x'.
\]

(x) For any process \(\theta := (\theta_s)_{s \leq T}\) and \(t \in (0, T], \theta_{t-} = \lim_{s \to t} \theta_s\) and

\[
\Delta \theta = \theta_t - \theta_{t-}.
\]

Now, let \(b\) and \(\sigma\) be the following functions:

\[
b: (t, x) \in [0, T] \times \mathbb{R}^k \mapsto b(t, x) \in \mathbb{R}^k;
\]

\[
\sigma: (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \sigma(t, x) \in \mathbb{R}^{k \times d}.
\]

We assume that they are jointly continuous in \((t, x)\) and Lipschitz continuous w.r.t. \(x\) uniformly in \(t\), that is, there exists a constant \(C\) such that,

\[
\forall(t, x, x') \in [0, T] \times \mathbb{R}^{k + h}, \quad |b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C |x - x'|. \tag{2.1}
\]

Notice that by (2.1) and continuity, the functions \(b\) and \(\sigma\) are of linear growth, that is, there exists a constant \(C\) such that

\[
\forall(t, x, x') \in [0, T] \times \mathbb{R}^{k + h}, \quad |b(t, x)| + |\sigma(t, x)| \leq C |1 + x|. \tag{2.2}
\]

Let \(\beta: (t, x, e) \in [0, T] \times \mathbb{R}^k \times E \mapsto \beta(t, x, e) \in \mathbb{R}^k\) be a measurable function, such that for some real constant \(C\) and for all \(e \in E\), the followings hold:

(i) \(|\beta(t, x, e)| \leq C(1 \wedge |e|)\);

(ii) \(|\beta(t, x, e) - \beta(t, x', e)| \leq C |x - x'| (1 \wedge |e|)\);

(iii) the mapping \((t, x) \in [0, T] \times \mathbb{R}^k \mapsto \beta(t, x, e) \in \mathbb{R}^k\) is continuous for any \(e \in E\).

We are now going to introduce the objects which are specifically connected to the RBSDE with jumps we will deal with. Let \(\ell\) the barrier of (1.3); \((g^i)_{i=1, m}\) and \((h^{(i)})_{i=1, m}\) be two functions defined as follows: for \(i = 1, \cdots, m\),

\[
g^i: \mathbb{R}^k \longrightarrow \mathbb{R}^m
\]

\[
x \longmapsto g^i(x)
\]

and

\[
h^{(i)}: [0, T] \times \mathbb{R}^{k + m + d + 1} \longrightarrow \mathbb{R}
\]
Moreover, we assume that they satisfy the followings:

(H1) The reflecting barrier \( \ell \) is real valued and \( \mathcal{P} \)-measurable process satisfying \( \ell \in \mathcal{U} \), that is, it is continuous and there exists constants \( C \) and \( p \) such that

\[
|\ell(t, x) - \ell(t, x')| \leq C(1 + |x|^p + |x'|^p)|x - x'|, \quad \text{for any } t \geq 0, x, x'.
\]

(H2) For any \( i \in \{1, \cdots, m\} \), the function \( g^i \) belongs to \( \mathcal{U} \).

(H3) For any \( i \in \{1, \cdots, m\} \),

(i) the function \( h^{(i)} \) is Lipschitz in \((y, z, q)\) uniformly in \((t, x)\), that is, there exists a real constant \( C \) such that for any \((t, x) \in [0, T] \times \mathbb{R}^k, (y, z, q) \) and \((y', z', q') \) elements of \( \mathbb{R}^{m+d+1} \),

\[
\left|h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x, y', z', q')\right| \leq C(|y - y'| + |z - z'| + |q - q'|);
\]

(ii) the \((t, x) \mapsto h^{(i)}(t, x, y, z, q)\), for fixed \((y, z, q) \in \mathbb{R}^{m+d+1}\), belongs uniformly to \( \mathcal{U} \), that is, it is continuous and there exists constants \( C \) and \( p \) (which do not depend on \((y, z, q)\)) such that

\[
\left|h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x', y, z, q)\right| \leq C(1 + |x|^p + |x'|^p)|x - x'|, \quad \text{for any } t \geq 0, x, x'.
\]

Note that from [1] by Barles et al, the functions \( h^i(t, x, (y^i)_{i=1,m}, z, q) \) are assumed to be increasing w.r.t \( y^k \) when the other components are fixed. But this assumption does not get involved in this article.

Next, let \( \gamma^i, \ i = 1, \cdots, m \) be Borel measurable functions defined from \([0, T] \times \mathbb{R}^k \times \mathcal{E} \) into \( \mathbb{R} \) and satisfying the followings:

(i) \( |\gamma^i(t, x, e)| \leq C(1 \wedge |e|) \);

(ii) \( |\gamma^i(t, x, e) - \gamma^i(t, x', e)| \leq C(1 \wedge |e|)|x - x'| \ (1 + |x|^p + |x'|^p) \);

(iii) the mapping \( t \in [0, T] \mapsto \gamma^i(t, x, e) \in \mathbb{R} \) is continuous for any \((x, e) \in \mathbb{R}^k \times \mathcal{E} \).

Finally, we introduce the following functions \((f^{(i)})_{i=1,m}\) defined by

\[
\forall(t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^{k+m+d} \times \mathbb{L}^2(\lambda),
\]

\[
f^{(i)}(t, x, y, z, \zeta) := \hat{h}^{(i)}\left(t, x, y, z, \int_{\mathcal{E}} \gamma^i(t, x, e)\zeta(e)\lambda(de)\right).
\]

The functions \((f^{(i)})_{i=1,m}\) enjoy the two following properties:

(a) The function \( f^{(i)} \) is Lipschitz in \((y, z, \zeta)\) uniformly in \((t, x)\), that is, there exists a real constant \( C \) such that

\[
\left|f^{(i)}(t, x, y, z, \zeta) - f^{(i)}(t, x, y', z', \zeta')\right| \leq C(|y - y'| + |z - z'| + \|\zeta - \zeta'\|_{\mathbb{L}^2(\lambda)}),
\]

(b) The function \((t, x) \in [0, T] \times \mathbb{R}^k \mapsto f^{(i)}(t, x, 0, 0, 0)\) belongs to \( \Pi_q \), and then

\[
\mathbb{E}\left[\int_0^T \left|f^{(i)}(r, X^{lx}_r, 0, 0, 0)\right|^2 dr\right] < \infty.
\]

Having defined our data and put our assumptions, we can look at the state of the art.
3 Preliminaries

3.1 A class of diffusion processes with jumps

Let \((t, x) \in [0, T] \times \mathbb{R}^d\) and \((X^{t,x}_s)_{s \leq T}\) be the stochastic process solution of (1.4). Under assumptions (2.1)–(2.3), the solution of Equation (1.4) exists and is unique (see [2] for more details). We state some properties of the process \{\((X^{t,x}_s), \ s \in [0, T]\}\), which can be found in [2].

**Proposition 3.1** For each \(t \geq 0\), there exists a version of \{\((X^{t,x}_s), \ s \in [t, T]\)\} such that \(s \rightarrow X^{t}_s\) is a \(C^2(\mathbb{R}^d)\)-valued rcll process. Moreover, it satisfies the following estimates: \(\forall p \geq 2, \ x, x' \in \mathbb{R}^d\) and \(s \geq 0\),
\[
  \mathbb{E} \left[ \sup_{t \leq r \leq s} |X^{t,x}_r - x|^p \right] \leq M_p(s - t)(1 + |x|^p),
\]
\[
  \mathbb{E} \left[ \sup_{t \leq r \leq s} \left| X^{t,x}_r - Y^{t,x}_r - (x - x')^p \right|^p \right] \leq M_p(s - t)(|x - x'|^p);
\]
for some constant \(M_p\).

3.2 Existence and uniqueness for a RBSDE with jumps

Let \((t, x) \in [0, T] \times \mathbb{R}^d\) and we consider the following \(m\)-dimensional RBSDE with jumps:

\[
  \begin{cases}
    (i) \quad Y^{t,x}_s := (Y^{i,t,x}_s)_{i=1,m}, Z^{t,x}_s := (Z^{i,t,x}_s)_{i=1,m}, \ K^{t,x}_s := (K^{i,t,x}_s)_{i=1,m}, \ U^{t,x}_s := (U^{i,t,x}_s)_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}); \\
    (ii) \quad dY^{i,t,x}_s = -f^{(i)}(s, X^{i,t,x}_s, Y^{i,t,x}_s, Z^{i,t,x}_s, U^{i,t,x}_s)ds + dB^{i} + \int_{\mathcal{L}} U^{i,t,x}_s(e)\tilde{\mu}(ds, de), \quad s \leq T; \\
    (iii) \quad Y^{i,t,x}_T \geq \ell(s, X^{i,t,x}_s) \quad \text{and} \quad \int_0^T (Y^{i,t,x}_s - \ell(s, X^{i,t,x}_s))dK^{i,t,x}_s = 0,
  \end{cases}
\]

where for any \(i \in \{1, \cdots, m\}\), \(Z^{i,t,x}_s\) is the \(i\)th row of \(Z^{t,x}_s\), \(K^{i,t,x}_s\) is the \(i\)th component of \(K^{t,x}_s\), and \(U^{i,t,x}_s\) is the \(i\)th component of \(U^{t,x}_s\).

The following result is related to the existence and uniqueness of a solution for the RBSDE with jumps (3.3).

Its proof is given in [5] using the penalization method (see p.5–12) and the Snell envelope method (see p.14–16).

**Proposition 3.2** Assume that assumptions (H1), (H2), and (H3) hold. Then, for any \((t, x) \in [0, T] \times \mathbb{R}^d\), the RBSDE (3.3) has an unique solution \((Y^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})\).

**Remark 3.3** The solution of this RBSDE with jumps exist and is unique because of the followings:

(i) \(\mathbb{E} \left[ \left| g(X^{t,x}_T) \right|^2 \right] < \infty\), because of polynomial growth of \(g\) and estimate (3.2) on \(X^{t,x}\);

(ii) for any \(i = 1, \cdots, m\), \(f^{(i)}\) verifies the properties (a)–(b) related to uniform Lipschitz w.r.t. \((y, z, \zeta)\) and \(ds \otimes d\mathbb{P}\)-square integrability of the process \((f^{(i)}(s, X^{t,x}_s, 0, 0, 0))_{s \leq T}\).

3.3 Viscosity solutions of integro-differential partial equation with one obstacle

The following result on one obstacle is proved with two obstacles in Harraj et al (see [6], Theorem 4.6, p.47 by using (4.1) p.44), and it establishes the relationship between the solution of (3.3) and the one of system (1.1).
Proposition 3.4 Assume that (H1), (H2), and (H3) are fulfilled. Then, there exists deterministic continuous functions \((u^i(t, x))_{i=1, m}\) which belong to \(\Pi_g\) such that for any \((t, x) \in [0, T] \times \mathbb{R}^k\), the solution of the RBSDE (3.3) verifies that
\[
\forall i \in \{1, \cdots, m\}, \forall s \in [t, T], \quad Y^{i; t,x}_s = u^i(s, X^{t,x}_s).
\] (3.4)
Moreover, if for any \(i \in \{1, \cdots, m\}\),
(i) \(\gamma^i \geq 0\) and
(ii) for any fixed \((t, x, \bar{y}, \bar{z}) \in [0, T] \times \mathbb{R}^{k+m+d}\), the mapping \((q \in \mathbb{R}) \mapsto h^{(i)}(t, x, \bar{y}, \bar{z}, q) \in \mathbb{R}\) is non-decreasing, then the function \((u^i)_{i=1, m}\) is a continuous viscosity solution of (1.1) (in Barles et al's sense, see Definition 6.1 in Appendix).
Finally, the solution \((u^i)_{i=1, m}\) of (1.1) is unique in the class \(\Pi^c_g\).

For the proof, see [6], by adapting the same way.

Remark 3.5 (see [6]) Under the assumptions (H1), (H2), and (H3), there exists a unique viscosity solution of (1.1) in the class of functions satisfying
\[
\lim_{|x| \to +\infty} |u(t, x)| e^{-A[\log(|x|)]^2} = 0
\] (3.5)
uniformly for \(t \in [0, T]\), for some \(A > 0\).

4 Estimates and Properties

In this section, we provide estimates for the functions \((u^i)_{i=1, m}\) defined in (3.4). Recall that
\[
(Y^{i; t,x}_t, Z^{i; t,x}_t, U^{i; t,x}_t, K^{i; t,x}_t) := ((Y^{i; t,x}_i)_{i=1, m}, (Z^{i; t,x}_i)_{i=1, m}, (U^{i; t,x}_i)_{i=1, m}, (K^{i; t,x}_i)_{i=1, m})
\]
is the unique solution of the RBSDE with jumps (3.3).

Lemma 4.1 Under assumptions (H1), (H2), and (H3), for any \(p \geq 2\), there exists two non-negative constants \(C\) and \(\rho\) such that
\[
\mathbb{E} \left[ \int_0^T ds \left( \int_E |U^{i; t,x}_s(e)|^2 \lambda(de) \right)^{\frac{p}{2}} \right] \leq C \left( 1 + |x|^\rho \right). \quad (4.1)
\]

Proof First, let us point out that as \(X^{i; t,x}_s = x\) for \(s \in [0, t]\), then the uniqueness of the solution of RBSDE of (3.3) implies that
\[
Z^{i; t,x}_s = U^{i; t,x}_s = K^{i; t,x}_s = 0, \quad ds \otimes d\mathbb{P}\text{-a.e. on } [0, t] \times \Omega. \quad (4.2)
\]

Next, let \(p \geq 2\) be fixed. Using representation (3.4), for any \(i \in \{1, \cdots, m\}\) and \(s \in [t, T]\), we have
\[
Y^{i; t,x}_s = g^i(X^{t,x}_T) + \int_s^T f^{(i)}(r, X^{t,x}_r, (u^j(X^{t,x}_r))_{j=1, m}, Z^{i; t,x}_r, U^{i; t,x}_r) dr + K^{i; t,x}_T - K^{i; t,x}_s
\]
\[
- \int_s^T Z^{i; t,x}_r dB_r - \int_s^T \int_E U^{i; t,x}_r(e) d\mu(dr, de). \quad (4.3)
\]
This implies that the system of RBSDEs with jumps (3.3) turns into a decoupled one because the equations in (4.3) are not related each other.

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Next, for any $i = 1, \ldots, m$, the functions $u^i$, $g^i$ and $(t, x) \mapsto f^{(i)}(t, x, 0, 0, 0)$ are of polynomial growth and finally $y \mapsto f^{(i)}(t, x, y, 0, 0)$ is Lipschitz uniformly w.r.t. $(t, x)$. Then, for some $C$ and $\rho \geq 0$, we have
\[
\mathbb{E} \left[ \left| g^i(X_T^{t,x}) \right|^p + \left( \int_s^T \left| f^{(i)}(r, X_r^{t,x}, (w^j(X_r^{t,x}))_{j=1,m}, 0, 0) \right|^2 \, dr \right)^{\frac{p}{2}} \right] \leq C \left( 1 + |x|^\rho \right). \tag{4.4}
\]
Let us now fix $i_0 \in \{1, \ldots, m\}$, then for $s \in [t, T]$,
\[
\begin{align*}
(\text{i}) \quad & Y_s^{i_0, t, x} \in \mathcal{S}^2(\mathbb{R}), \ Z_s^{i_0, t, x} \in \mathbb{H}^2(\mathbb{R}^d), \ K_s^{i_0, t, x} \in \mathcal{A}_2^2, \ U_s^{i_0, t, x} \in \mathbb{H}^2(\mathbb{L}^2(\lambda)); \\
(\text{ii}) \quad & Y_s^{i_0, t, x} = g^{i_0}(X_T^{t,x}) + \int_s^T f^{(i_0)}(r, X_r^{t,x}, (w^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) \, dr \\
& \quad + K_T^{i_0, t, x} - K_s^{i_0, t, x} - \int_s^T Z_r^{i_0, t, x} \, dB_r - \int_s^T \int_E U_r^{i_0, t, x} \, \tilde{\mu}(dr, de); \\
(\text{iii}) \quad & Y_s^{i_0, t, x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i_0, t, x} - \ell(s, X_s^{t,x})) \, dK_s^{i_0, t, x} = 0.
\end{align*}
\]
Applying Itô formula to $|Y_s^{i_0, t, x}|^2$ between $s$ and $T$, we have
\[
|Y_s^{i_0, t, x}|^2 + \int_s^T |Z_r^{i_0, t, x}|^2 \, dr + \sum_{r \leq s \leq T} (\Delta Y_r^{i_0, t, x})^2 \leq \mathbb{E} \left[ \left( \int_s^T \left| U_r^{i_0, t, x} \right|_{L^2(\lambda)} \, dr \right)^{\frac{p}{2}} \right] \leq C \mathbb{E} \left[ \left( \sum_{s \leq r \leq T} (\Delta Y_r^{i_0, t, x})^2 \right)^{\frac{p}{2}} \right],
\]
(see [7] p.28–45).
\[
\begin{align*}
\mathbb{E} \left[ |Y_s^{i_0, t, x}|^p \right] & + \mathbb{E} \left[ \left( \int_s^T |Z_r^{i_0, t, x}|^2 \, dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \left( \int_s^T \left| U_r^{i_0, t, x} \right|_{L^2(\lambda)} \, dr \right)^{\frac{p}{2}} \right] \\
& \leq 5^{\frac{p}{2} - 1} \mathbb{E} \left[ |g^{i_0}(X_T^{t,x})|^p \right] \\
& \quad + 5^{\frac{p}{2} - 1} \mathbb{E} \left[ \left( \int_s^T 2 \left| Y_r^{i_0, t, x} f^{(i_0)}(r, X_r^{t,x}, (w^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) \right| \, dr \right)^{\frac{p}{2}} \right] \\
& \quad + 5^{\frac{p}{2} - 1} \mathbb{E} \left[ \left( \int_s^T Y_r^{i_0, t, x} \, dK_r^{i_0, t, x} \right)^{\frac{p}{2}} \right] \\
& \quad + 5^{\frac{p}{2} - 1} \mathbb{E} \left[ \left( \int_s^T \int_E 2 Y_r^{i_0, t, x} U_r^{i_0, t, x} \, \tilde{\mu}(dr, de) \right)^{\frac{p}{2}} \right] + 5^{\frac{p}{2} - 1} \mathbb{E} \left[ \left( \int_s^T 2 Y_r^{i_0, t, x} Z_r^{i_0, t, x} \, dB_r \right)^{\frac{p}{2}} \right]. \tag{4.6}
\end{align*}
\]
For more comprehension, we adopt the following scripture for inequality (4.6);
\[
\mathbb{E} \left[ |Y_s^{i_0, t, x}|^p \right] + \mathbb{E} \left[ \left( \int_s^T |Z_r^{i_0, t, x}|^2 \, dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \left( \int_s^T \left| U_r^{i_0, t, x} \right|_{L^2(\lambda)} \, dr \right)^{\frac{p}{2}} \right] \\
\leq \mathbb{E} \left[ \left( \int_s^T \left| U_r^{i_0, t, x} \right|_{L^2(\lambda)} \, dr \right)^{\frac{p}{2}} \right].
\]
where

\[
T_1(s) = \mathbb{E} \left[ \left( \int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x})))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x} \right| dr \right)^{\frac{p}{2}} \right];
\]

\[
T_2(s) = \mathbb{E} \left[ \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \right];
\]

\[
T_3(s) = \mathbb{E} \left[ \int_s^T \int_\mathbb{R} 2 Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr,de) \right];
\]

\[
T_4(s) = \mathbb{E} \left[ \int_s^T 2 Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r \right]^{\frac{p}{2}}.
\]

We will estimate $T_1(s)$, $T_2(s)$, $T_3(s)$, and $T_4(s)$, $\forall s \in [t, T]$. 

(a) Before starting our estimations, let us linearize $f$ with respect to $(u^j(X_t^{t,x}))_{j=1,m}$ and $Z_t^{i_0,t,x}$, that is,

\[
f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x})))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) = a_r^{t,x} Z_r^{i_0,t,x} + b_r^{t,x} (u^j(X_r^{t,x})))_{j=1,m} + f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x}),
\]

where $a_r^{t,x}$ and $b_r^{t,x}$ are progressively measurable processes respectively bounded by the Lipschitz constants of $f$ in $Z_r^{i_0,t,x}$ and $(u^j(X_r^{t,x})))_{j=1,m}$, that is, $|a_r^{t,x}| \leq C_Z$ and $|b_r^{t,x}| \leq \lambda_1$.

(b) We also take the fact that $f$ is Lipschitz in $U_r^{i_0,t,x}$, that is, there exists a constant Lipschitz $\lambda_2$ such that $|f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x})| \leq |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| + \lambda_2 \|U_r^{i_0,t,x}\|_{L_2(\lambda)}$.

By combining (a) and (b), we have

\[
|f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x})))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x})| \\
\leq |a_r^{t,x} Z_r^{i_0,t,x}| + |b_r^{t,x} (u^j(X_r^{t,x})))_{j=1,m}| + |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| + \lambda_2 \|U_r^{i_0,t,x}\|_{L_2(\lambda)}.
\]

Let us start our estimations.

For $T_1(s)$ By using (4.8), it follows that

\[
\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x})))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x} \right| dr \\
\leq \int_s^T 2 \left| Y_r^{i_0,t,x} (a_r^{t,x} Z_r^{i_0,t,x}) dr + \int_s^T 2 \left| Y_r^{i_0,t,x} (b_r^{t,x} (u^j(X_r^{t,x})))_{j=1,m}) \right| dr \\
+ \lambda_2 \int_s^T 2 \left| Y_r^{i_0,t,x} (U_r^{i_0,t,x}) \right| dr + \int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0) \right| dr \\
\leq C_2^2 C_2 T \epsilon_1^{-1} \Sigma^{2q} + c_1 C_2 \int_s^T \left| Z_r^{i_0,t,x} \right|^2 dr + C_2^2 \lambda_1 T \epsilon_2^{-1} \Sigma^{2q} + \epsilon_2 \lambda_1 C_2^2 T \epsilon_2 \Sigma^{2q} \\
+ C_2^2 T \lambda_2 \epsilon_3^{-1} \Sigma^{2q} + \lambda_2 \epsilon_3 \int_s^T \left| U_r^{i_0,t,x} \right|^2 \|_{L_2(\lambda)} dr + C_2^2 T \epsilon_4^{-1} \Sigma^{2q} + C_2^2 T \epsilon_4 \Sigma^{2q}.
\]

By raising to the power $\frac{p}{2}$ and then taking expectation, it follows that

\[
T_1(s) \leq 8 \frac{p}{2} - 1 C C \left\{ (C_2 T \epsilon_1^{-1})^\frac{p}{2} + (\lambda_1 T \epsilon_2^{-1})^\frac{p}{2} + C_2^p (\epsilon_2 \lambda_1 T)^\frac{p}{2} + (T \lambda_2 \epsilon_3^{-1})^\frac{p}{2} + (T \epsilon_4^{-1})^\frac{p}{2} \right\}
\]

\[\square\]
\begin{equation}
+(T\epsilon_4)^{\frac{\gamma}{2}}\left[1+|x|^p\right]+8^{\frac{\gamma}{2}-1}(\epsilon_1C_Z)^{\frac{\gamma}{2}}E\left[\left(\int_0^T|Z_{r}^{io,t,x}|^2\,dr\right)^{\frac{\gamma}{2}}\right]
+8^{\frac{\gamma}{2}-1}(\lambda_2\epsilon_3)^{\frac{\gamma}{2}}E\left[\left(\int_0^T\|U_{r}^{io,t,x}\|^2_{L^2_m(\lambda)}\,dr\right)^{\frac{\gamma}{2}}\right].
\end{equation}

(4.9)

Before estimating \(T_2(s)\), let us first give an estimate of \(E|K^{io,t,x}_T - K^{io,t,x}_s|^p\), which will serve us in estimating of \(T_2(s)\).

\[K^{io,t,x}_T - K^{io,t,x}_s = Y^{io,t,x}_s - \int_s^T f^{(io)}(r, X^t_{r,x}, \ldots) + \int_s^T Z^{io,t,x}_r dB_r + \int_s^T E_r U^{io,t,x}_r(e)\mu(dr, de).
\]

By (4.8) and Cauchy-Schwartz inequality, it follows that

\[|K^{io,t,x}_T - K^{io,t,x}_s| \leq 2C_q\Sigma^q + C_q\lambda_1T\Sigma^q + C_qT\Sigma^q + C_Z \left(\int_s^T |Z^{io,t,x}_r|^2\,dr\right)^{\frac{\gamma}{2}} + \lambda_2 \left(\int_s^T \|U^{io,t,x}_r\|^2\,dr\right)^{\frac{\gamma}{2}} + \int_s^T Z^{io,t,x}_r dB_r + \int_s^T E_r U^{io,t,x}_r(e)\mu(dr, de).
\]

By raising to the power \(p\), and taking expectation and BDG inequality, we have

\[E\left[K^{io,t,x}_T - K^{io,t,x}_s|^p\right] \leq 5^{p-1}CC_q^p \left\{2^p + (T\lambda_1)p + Tp\right\} |1 + |x|^p|^{\frac{\gamma}{2}} + 5^{p-1}(C_Z^p + C_p)\left(\int_0^T |Z^{io,t,x}_s|^2\,ds\right)^{\frac{\gamma}{2}} + 5^{p-1}(\lambda_2^p + C_p)E\left\{\int_0^T ds\|U^{io,t,x}_s\|^2_{L^2_m(\lambda)}\right\}^{\frac{\gamma}{2}}.
\]

(4.10)

For \(T_2(s)\)

\[\int_s^T |Y^{io,t,x}_s|dK^{io,t,x}_s \leq \int_s^T \|Y^{io,t,x}_s - \ell(s, X^t_{s,x})\|dK^{io,t,x}_s + \int_s^T |\ell(s, X^t_{s,x})|dK^{io,t,x}_s
\]

\[\leq \sup_{s \leq T} |\ell(s, X^t_{s,x})|K^{io,t,x}_s
\]

\[\leq \epsilon_5^{-1}\sup_{s \leq T} |\ell(s, X^t_{s,x})|^2 + \epsilon_5(K^{io,t,x}_T)^2
\]

\[\leq \epsilon_5^{-1}C^2\Sigma^2 + \epsilon_5(K^{io,t,x}_T)^2.
\]

(4.11)

Using (4.10) and (4.11), it follows that

\[T_2(s) \leq 2^{\frac{\gamma}{2}-1}CC_q^p(|\epsilon_5^{-1}|)\frac{\gamma}{2} |1 + |x|^p| + 2^{\frac{\gamma}{2}-1}(\epsilon_3)^{\frac{\gamma}{2}}E\left[(K^{io,t,x}_T)^p\right]
\]

\[\leq 2^{\frac{\gamma}{2}-1}CC_q^p(|\epsilon_5^{-1}|)^{\frac{\gamma}{2}} |1 + |x|^p| + (\epsilon_3)^{\frac{\gamma}{2}}2^{\frac{\gamma}{2}-1}7^{p-1}CC_q^p \left\{2^p + (T\lambda_1)p + Tp\right\} |1 + |x|^p|^{\frac{\gamma}{2}}
\]

\[+ (\epsilon_5)^{\frac{\gamma}{2}}2^{\frac{\gamma}{2}-1}7^{p-1}(C_Z^p + C_p)\left(\int_0^T |Z^{io,t,x}_s|^2\,ds\right)^{\frac{\gamma}{2}}
\]

\[+ \left(\epsilon_5\right)^{\frac{\gamma}{2}}2^{\frac{\gamma}{2}-1}7^{p-1}(\lambda_2^p + C_p)E\left\{\int_0^T ds\|U^{io,t,x}_s\|^2_{L^2_m(\lambda)}\right\}^{\frac{\gamma}{2}}
\]
\[
\leq \left\{ 2^{\frac{p}{2} - 1} C_\nu \left( \epsilon_4^{-1} \right) \frac{2^{\frac{p}{2} - 1}}{2} \Phi_p \left( \epsilon_5 \frac{1}{\Phi_4} - \frac{1}{\Phi_5} \right) + \left( \epsilon_5 \right) \frac{2^{\frac{p}{2} - 1}}{2} \right\} (1 + |x|^p)
\]

For \( T_3(s) \) By BDG inequality, we have
\[
T_3(s) \leq C_p \mathbb{E} \left( \int_0^T |Y_{t,s,t,x}^i|^2 |Z_{t,s,t,x}^i|^2 \, ds \right)^{\frac{p}{2}}
\]
\[
\leq C_p \mathbb{E} \left( \sup_{s \leq T} |Y_{t,s,t,x}^i|^2 \int_0^T |Z_{t,s,t,x}^i|^2 \, ds \right)^{\frac{p}{2}}
\]
\[
\leq C_p C_p \epsilon_6^{\frac{1}{2}} (1 + |x|^p) + C_p \epsilon_6 \left( \int_0^T |Z_{t,s,t,x}^i|^2 \, ds \right)^{\frac{p}{2}}.
\]
(4.12)

For \( T_4(s) \) By BDG inequality, we obtain
\[
T_4(s) \leq C_p \mathbb{E} \left( \int_0^T |Y_{t,s,t,x}^i|^2 \|U_{t,s,t,x}^i\|_{L^2(\lambda)}^2 \, ds \right)^{\frac{p}{2}}
\]
\[
\leq C_p \mathbb{E} \left( \sup_{s \leq T} |Y_{t,s,t,x}^i|^2 \int_0^T \|U_{t,s,t,x}^i\|_{L^2(\lambda)}^2 \, ds \right)^{\frac{p}{2}}
\]
\[
\leq C_p C_p \epsilon_7^{\frac{1}{2}} (1 + |x|^p) + C_p \epsilon_7 \left( \int_0^T \|U_{t,s,t,x}^i\|_{L^2(\lambda)}^2 \, ds \right)^{\frac{p}{2}}.
\]
(4.13)

Finally, by taking estimation of \( T_1(s), T_2(s), T_3(s), \) and \( T_4(s), \) and choosing \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \)
and \( \epsilon_7 \) such that \( \{(\epsilon_5) \frac{2^{\frac{p}{2} - 1}}{2} \Phi_p \left( \epsilon_4^{-1} \right) + \Phi_5 \Phi_4 \} < 1, \}
and the sum of all coefficients of \( (1 + |x|^p) \) was small than 1.

It follows that
\[
\mathbb{E} \left[ \left\{ \int_0^T ds \|U_{t,s,t,x}^i\|_{L^2(\lambda)}^2 \right\}^{\frac{p}{2}} \right] \leq C (1 + |x|^p),
\]
(4.14)

where \( \rho = pq \).

Finally, as \( i_0 \in \{1, \cdots, m\} \) is arbitrary, we then obtain estimate (4.1). \qed

**Proposition 4.2** For any \( i = 1, \cdots, m, u^i \) belongs to \( \mathcal{U} \).

**Proof** Let \( x \) and \( x' \) be elements of \( \mathbb{R}^k \). Let \( (\tilde{Y}^t.x, Z^t.x, U^t.x, K^t.x) \) (resp. \( (\tilde{Y}^t.x', Z^t.x', U^t.x', K^t.x') \)) be the solution of the RBSDE with jumps (3.3) associated with \( f(s, x^t.x, y, \eta, \zeta, g(X^t.x)) \) (resp. \( f(s, X^t.x', y, \eta, \zeta, g(X^t.x')) \)). Applying Itô formula to \( |\tilde{Y}^t.x - \tilde{Y}^t.x'|^2 \) between \( s \) and \( T \), we have
\[
\left| \tilde{Y}^t.x - \tilde{Y}^t.x' \right|^2 + \int_s^T \Delta Z_t \, dt + \sum_{s \leq r \leq T} (\Delta_r \tilde{Y}^t.x)^2
\]
\[
= \left| g(X^t.x) - g(X^t.x') \right|^2 + 2 \int_s^T \left( \tilde{Y}^t.x - \tilde{Y}^t.x' \right), \Delta f(r) \, dr + 2 \int_s^T \left( \tilde{Y}^t.x - \tilde{Y}^t.x' \right) d(\Delta K_r)
\]
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Indeed, by triangle inequality and the fact of \((\Delta U_r(e)) \mu(dr, de) - 2 \int_s^T (\tilde{Y}_{t,x}^{t,x} - \tilde{Y}_{t,x}^{t,x'})(\Delta Z_r) \, dB_r, \quad (4.15)\)
and taking expectation, we obtain the following result: for \(\forall s \in [t, T], \)
\[
E \left[ |\tilde{Y}_{s,x}^{t,x} - \tilde{Y}_{s,x}^{t,x'}|^2 + \int_s^T |\Delta Z_r|^2 \, dr + \int_s^T ||\Delta U_r||_{L^2(\lambda)}^2 \, dr \right] 
\leq E \left[ (g(\Delta f^{t,x}) - g(\Delta f^{t,x'}))^2 + 2 \int_s^T \left( (\tilde{Y}_{s,x}^{t,x} - \tilde{Y}_{s,x}^{t,x'}), \Delta f(r) \right) \, dr \right] + E \left[ 2 \int_s^T \left( \tilde{Y}_{t,x}^{t,x} - \tilde{Y}_{t,x}^{t,x'} \right) \, d(\Delta K_r) \right], \quad (4.16)\]
where the processes \(\Delta X_r, \Delta Y_r, \Delta f(r), \Delta K_r, \Delta Z_r, \Delta U_r, \) and \(\Delta \ell_r\) are defined as follows: for \(\forall r \in [t, T], \)
\[
\Delta f(r) := ((\Delta f^{t,x}(r))_{i=1,m} = (f^{t,x}(r), X_r^{t,x}, \tilde{Y}_{r,x}^{t,x}, Z_r^{t,x}, U_r^{t,x})), \quad \Delta X_r = X_r^{t,x} - X_r^{t,x'}, \quad \Delta Y_r = Y_r^{t,x} - Y_r^{t,x'}, \quad \Delta Z_r = Z_r^{t,x} - Z_r^{t,x'}, \quad \Delta U_r = U_r^{t,x} - U_r^{t,x'}, \quad \Delta \ell_r = (\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x})).
\]
\((\cdot, \cdot)\) is the usual scalar product on \(\mathbb{R}^m\).

Now, we will give an estimation of each three terms of the second member of inequality (4.16).

- \(E \left[ (g(\Delta f^{t,x}) - g(\Delta f^{t,x'}))^2 \right] \)
  \[
  \leq CE \left[ |X_r^{t,x} - X_r^{t,x'}|^2 \left( 1 + |X_r^{t,x'}|^{2p} + |X_r^{t,x'}|^{2p} \right) \right] \quad \text{(for any } i \in \{1, \ldots, m\}, \text{ } g^i \text{ belongs to } \mathcal{U})
  \leq CE \left[ \left( |X_r^{t,x} - X_r^{t,x'} - (x - x') \right) + (x - x') \right]^2 \left( 1 + |(X_r^{t,x} - x) + x|^2 \right.
  + \left. |(X_r^{t,x} - x) + x|^2 \right)^{2p}. \quad (4.17)\]

By triangle inequality and the fact of \((a + b)^p \leq 2^{p-1}(a^p + b^p), \) Cauchy-Schwartz inequality, the relations (3.1) and (3.2) of Proposition 3.1,
\[
E \left[ (g(\Delta f^{t,x}) - g(\Delta f^{t,x'}))^2 \right] \leq C |x - x'|^2 \left( 1 + |x|^{2p} + |x'|^{2p} \right), \quad (4.18)\]

- using (iii) of (3.3), then \(E \left[ 2 \int_s^T \left( \tilde{Y}_{t,x}^{t,x} - \tilde{Y}_{t,x}^{t,x'} \right) \, d(\Delta K_r) \right] \) can be replaced by
  \[
  E \left[ 2 \int_s^T \left( \ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'}) \right) \, d(\Delta K_r) \right].
  \]
Indeed,
\[
E \left[ 2 \int_s^T \left( \tilde{Y}_{t,x}^{t,x} - \tilde{Y}_{t,x}^{t,x'} \right) \, d(\Delta K_r) \right]
\]
By linearity of integral and using (iii) of (3.3), we obtain our result.

Now, by (H1) and Cauchy-Schwartz inequality, we obtain

\[
E \left[ \sup_{0 \leq t \leq T} (\Delta \ell)^2 \right] \leq 2CC' |x - x'|^2 (1 + |x|^p + |x'|^p),
\]

where \( C' = E \left[ (\Delta K_T)^2 \right] \).

- To complete our estimation of (4.16), we need to deal with \( \sup_{0 \leq t \leq T} \Delta f(r) \)dr]. Taking into account the expression of \( f(i) \) given by (2.7), we then split \( \Delta f(r) \) in the following way: for \( r \leq T, \)

\[
\Delta f(r) = (\Delta f(r))_{i=1,m} = \Delta_1(r) + \Delta_2(r) + \Delta_3(r) + \Delta_4(r)
\]

where for any \( i = 1, \ldots, m, \)

\[
\Delta_i^1(r) = h^{(i)} \left( r, X_{t_r}^{i,x}, Y_{l_r}^{i,x}, Z_{r_r}^{i,x}, \int_E \gamma^i(r, X^{i,x}, e) U^{i,x}_{t_r}(e) \lambda(de) \right)
\]

and

\[
\Delta_i^2(r) = h^{(i)} \left( r, X_{t_r}^{i,x}, Y_{l_r}^{i,x}, Z_{r_r}^{i,x}, \int_E \gamma^i(r, X^{i,x}, e) U^{i,x}_{t_r}(e) \lambda(de) \right)
\]

By Cauchy-Schwartz inequality, the inequality \( 2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2 \), relation (2.11), and estimate (3.2), we have

\[
E \left[ 2 \int_s^T (\Delta Y(r), \Delta_1(r)) dr \right]
\]

\[
\leq E \left[ \frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr + C^2 \epsilon \int_s^T |X_{t_r}^{i,x} - X_{t_r}^{i,x'}|^2 (1 + |X_{t_r}^{i,x}|^p + |X_{t_r}^{i,x'}|^p)^2 dr \right]
\]

\[
\leq E \left[ \frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon |x - x'|^2 (1 + |x|^p + |x'|^p)^2.
\]

Besides, as \( h^{(i)} \) is Lipschitz w.r.t. \((y, z, q)\), then

\[
E \left[ 2 \int_s^T (\Delta Y(r), \Delta_2(r)) dr \right] \leq 2CE \left[ \int_s^T |\Delta Y(r)|^2 dr \right]
\]

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and
\[ E \int_s^T (\Delta Y(r), \Delta_3(r))dr \leq E \left[ \frac{1}{2} \int_s^T |\Delta Y(r)|^2 dr + C^2 \epsilon \int_s^T |\Delta Z(r)|^2 dr \right]. \quad (4.22) \]

It remains to obtain a control of the last term. But for any \( s \in [t, T] \), we have
\[
E \left[ 2 \int_s^T (\Delta Y(r), \Delta_4(r))dr \right] 
\leq 2CE \left[ \int_s^T |\Delta Y(r)|dr \times \left( \int_E \left( |U_{r,t,x}^l(e)\Delta \gamma_r(e)| + |\Delta U_r(e)\gamma_r(r, X_{r,t,x}^l, e)| \right) \lambda(de) \right) dr \right]
\leq \frac{2}{\epsilon} E \left[ \int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon E \left[ \int_s^T \left( \int_E \left( |U_{r,t,x}^l(e)\Delta \gamma_r(e)| \right)^2 \lambda(de) \right) dr \right]
\leq \frac{2}{\epsilon} E \left[ \int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon E \left[ \int_s^T \left( \int_E \left( |\Delta U_r(e)\gamma_r(r, X_{r,t,x}^l, e)| \lambda(de) \right)^2 \right) dr \right].
\quad (4.24)

By Cauchy-Schwartz inequality, (2.6) and (3.2), and the result of Lemma 4.1, it holds that
\[
E \left[ \int_s^T \left( \int_E \left( |U_{r,t,x}^l(e)\Delta \gamma_r(e)| \lambda(de) \right)^2 \right) dr \right]
\leq E \left[ \int_s^T dr \left( \int_E \left( |U_{r,t,x}^l(e)|^2 \lambda(de) \right) \left( \int_E \left( |\Delta \gamma_r(e)|^2 \lambda(de) \right) \right) \right]
\leq CE \left\{ \sup_{r \in [t, T]} |X_{r,t,x}^l - X_{r,t,x'}^l| |X_{r,t,x}^l|^2 (1 + \sup_{r \in [t, T]} |X_{r,t,x}^l|^p + |X_{r,t,x'}^l|^p)^2 dr \right\}
\times E \left[ \int_s^T dr \left( \int_E \left( |U_{r,t,x}^l(e)|^2 \lambda(de) \right) \right) \right]
\leq C \left\{ \sup_{r \in [t, T]} |X_{r,t,x}^l - X_{r,t,x'}^l| |X_{r,t,x}^l|^4 (1 + \sup_{r \in [t, T]} |X_{r,t,x}^l|^p + |X_{r,t,x'}^l|^p)^4 \right\}
\times E \left[ \int_s^T dr \left( \int_E \left( |U_{r,t,x}^l(e)|^2 \lambda(de) \right) \right)^2 \right]
\leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}). \quad (4.25)
For some exponent \( p \), on the other hand, using once more Cauchy-Schwartz inequality and (2.6)-(i), we get

\[
\mathbb{E} \left[ \int_s^T \left( \int_E |(\Delta U_t(e))\gamma(r, X_t^{t,x}, e)| \lambda(de) \right)^2 \, dr \right] \\
\leq \mathbb{E} \left[ \int_s^T dr \left( \int_E |(\Delta U_t(e))| \lambda(de) \right) \left( \int_E |\gamma(r, X_t^{t,x}, e)| \lambda(de) \right) \right] \\
\leq C \mathbb{E} \left[ \int_s^T dr \left( \int_E |(\Delta U_t(e))|^2 \lambda(de) \right) \right].
\]

Taking now into account inequalities (4.20)–(4.26), we obtain

\[
\mathbb{E} \left[ |\mathcal{Y}_s^{t,x} - \mathcal{Y}_{s}^{t,x'}|^2 + \int_s^T |\Delta Z_t|^2 \, dt + \int_s^T \|\Delta U_t\|_{L^2(\lambda)} \, dt \right] \\
\leq \mathbb{E} \left[ g(X_T^{t,x}) - g(X_T^{t,x'}) + 2 \int_s^T (\mathcal{Y}_s^{t,x} - \mathcal{Y}_{s}^{t,x'}) \cdot \Delta f(r) \, dr \right] \\
\leq 2 \int_s^T (\mathcal{Y}_s^{t,x} - \mathcal{Y}_{s}^{t,x'}) \, d(\Delta K_r) \\
\leq C^2 \epsilon \mathbb{E} \left[ \int_s^T |\Delta Z(r)|^2 \, dr \right] + C^3 \epsilon \mathbb{E} \left[ \int_s^T dr \left( \int_E |(\Delta U_r(e))|^2 \lambda(de) \right) \right].
\]

Choosing now \( \epsilon \) small enough, we deduce the existence of a constant \( C \geq 0 \), such that for any \( s \in [t, T] \),

\[
\mathbb{E} \left[ |\Delta Y(s)| \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) + \mathbb{E} \left[ \int_s^T |\Delta Y(r)|^2 \, dr \right]
\]

and by Gronwall lemma, this implies that for any \( s \in [t, T] \),

\[
\mathbb{E} \left[ |\Delta Y(s)|^2 \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}).
\]

Finally, taking \( s = t \) and considering (3.4), we obtain the desired result. \( \square \)

**Remark 4.3** This result gives also estimate of \( U \), where we use the function \((h^{(i)})_{1 \leq i \leq m})\), contrary to estimate (4.1).

**Corollary 4.4** For \( u^i \in \mathcal{U} (i = 1, \cdots, m) \), \( B_i u^i \) defined in (1.2) is well posed because the functions \( \beta \) and \((\gamma_i)_{i=1,m} \) verify (2.3) and (2.6), respectively.

**Proof** The main point to notice is that \( \lambda \) integrates \((1 \wedge |e|^p), \forall p \geq 2 \).

We have

\[
|B_i u^i(t, x)| \leq \int_E |\gamma_i(t, x, e)| \times |(u^i(t, x + \beta(t, x, e)) - u^i(t, x))| \lambda(de) \\
\leq \int_E C(1 \wedge |e|) |\beta(t, x, e)| (1 + |x + \beta(t, x, e)|^p + |x|^p) \lambda(de) \\
\leq C^2 (1 + |x|^p (1 + 2^{p-1})) \int_E C(1 \wedge |e|^2) \lambda(de)
\]
Which finish the proof.

Now, by Remark 4.3, the last estimate of $U$ confirm the following result.

**Proposition 4.5** For any $i = 1, \cdots, m$, $(t, x) \in [0, T] \times \mathbb{R}^k$, we have

$$U^{t,i,x}_s(e) = u^i(s, X^{t,i,x}_s + \beta(s, X^{t,i,x}_s, e)) - u^i(s, X^{t,i,x}_s), \quad d\mathbb{P} \otimes ds \otimes d\lambda \text{-a.e. on } \Omega \times [t, T] \times E. \quad (4.27)$$

**Proof** First, note that because the measure $\lambda$ is note finite, then we cannot use the same technique as in [4], where the authors use the jumps of processes and (3.4).

In our case, $U^{t,i,x}$ is only square integrable and not necessarily integrable w.r.t. $d\mathbb{P} \otimes ds \otimes d\lambda$. Therefore, we first begin by truncating the Lévy measure as the same way in [3].

**Step 1** **Truncation of the Lévy measure**

For any $k^* \geq 1$, let us first introduce a new Poisson random measure $\mu_{k^*}$ (obtained from the truncation of $\mu$) and its associated compensator $\nu_{k^*}$ as follows:

$$\mu_{k^*}(ds, de) = 1_{\{|e| \geq \frac{1}{k^*}\}} \mu(ds, de) \quad \text{and} \quad \nu_{k^*}(ds, de) = 1_{\{|e| \geq \frac{1}{k^*}\}} \nu(ds, de).$$

Which means that, as usual, $\mu_{k^*}(ds, de) := (\mu_{k^*} - \nu_{k^*})(ds, de)$ is the associated random martingale measure.

The main point to notice is that

$$\lambda_{k^*}(E) = \int_E \lambda_{k^*}(de) = \int_E 1_{\{|e| \geq \frac{1}{k^*}\}} \lambda(de) = \int_{\{|e| \geq \frac{1}{k^*}\}} \lambda(de) = \lambda\{\{|e| \geq \frac{1}{k^*}\}\} < \infty. \quad (4.28)$$

As in [3], let us introduce the process $k^* X^{t,i,x}$ solving the following standard SDE of jump-diffusion type:

$$k^* X^{t,i,x}_s = x + \int_t^s b(r, k^* X^{t,i,x}_r) dr + \int_t^s \sigma(r, k^* X^{t,i,x}_r) dB_r + \int_t^s \beta(r, k^* X^{t,i,x}_r, e) \tilde{\mu}_{k^*}(dr, de), \quad t \leq s \leq T; \quad k^* X^{t,i,x}_t = x \text{ if } s = t. \quad (4.29)$$

Note that Because of the assumptions on $b, \sigma, \beta$, the process $k^* X^{t,i,x}$ exists and is unique. Moreover, it satisfies the same estimates as in (3.2) because $\lambda_{k^*}$ is just a truncation at the origin of $\lambda$, which integrates $(1 \wedge |e|^2)_{e \in E}$.

On the other hand, let us consider the following Markovian RBSDE with jumps

$$\begin{align*}
(\text{i}) & \quad E \left[ \sup_{s \leq T} \left| k^* Y^{t,i,x}_s \right|^2 + \int_s^T \left| k^* Z^{t,i,x}_r \right|^2 dr + \int_s^T \left| k^* U^{t,i,x}_r \right|_{L^2(\mu_{k^*})}^2 dr \right] < \infty \\
(\text{ii}) & \quad k^* Y^{t,i,x}_s := (k^* Y^{t,i,x}_s)_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad k^* Z^{t,i,x}_s := (k^* Z^{t,i,x}_s)_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
& \quad k^* K^{t,i,x}_s := (k^* K^{t,i,x}_s)_{i=1,m} \in \mathcal{A}_c, \quad k^* U^{t,i,x}_r := (k^* U^{t,i,x}_r)_{i=1,m} \in \mathbb{H}^2(L^2_m(\mu_{k^*})); \\
(\text{iii}) & \quad k^* Y^{t,i,x}_s = g(k^* X^{t,i,x}_s) + \int_s^T f_{\mu_{k^*}}(r, k^* X^{t,i,x}_r, k^* Y^{t,i,x}_r, k^* Z^{t,i,x}_r, k^* U^{t,i,x}_r) dr + k^* K^{t,i,x}_s - k^* K^{t,i,x}_s \\
& \quad - \int_s^T \left\{ k^* Z^{t,i,x}_r dB_r + \int_E k^* U^{t,i,x}_r(e) \tilde{\mu}_{k^*}(dr, de) \right\}, \quad s \leq T; \\
(\text{iv}) & \quad k^* Y^{t,i,x}_s \geq \ell(s, k^* X^{t,i,x}_s) \quad \text{and} \quad \int_0^T (k^* Y^{t,i,x}_s - \ell(s, k^* X^{t,i,x}_s)) d(k^* K^{t,i,x}_s) = 0.
\end{align*} \quad (4.30)\]
Finally, let us introduce the following functions \((f^{(i)})_{i=1,m}\) defined as follows: for \(\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{L}^2_m(\lambda_k),\)

\[
f_{\mu_k^*}(t, x, y, z, \zeta) = (f^{(i)}_{\mu_k^*}(t, x, y, z, \zeta_i))_{i=1,m} := \left( h^{(i)}(t,x,y,z, \int_E \gamma^i(t, x, e)\zeta_i(e)\lambda_k^*(de)) \right)_{i=1,m}.
\]

First, let us emphasize that this latter RBSDE is related to the filtration \((\mathcal{F}_s^k)_{s \leq T}\), generated by the Brownian motion and the independent random measure \(\mu_k^*\). However, this point does not raise major issues because for any \(s \leq T, \mathcal{F}_s^k \subset \mathcal{F}_s\) and by the relationship between \(\mu\) and \(\mu_k^*\).

Next, by the properties of the functions \(b, \sigma, \beta\) and the same opinions of Propositions 3.2 and 3.4, there exists an unique quadruple \((k^* Y^{t, x}, k^* K^{t, x}, k^* Z^{t, x}, k^* U^{t, x})\) solving (4.30) and there also exists a function \(u^{k^*}\) from \([0, T] \times \mathbb{R}^k\) into \(\mathbb{R}^m\) of \(\Pi_g^\ast\) such that

\[\forall s \in [t, T], \quad k^* Y^{t,x} := u^{k^*}(s, k^* X^{t,x}), \quad \mathbb{P} - a.s. \]  \hspace{1cm} (4.31)

Moreover, as in Proposition 4.2, there exists positive constants \(C\) and \(p\) which do not depend on \(k^*\) such that for \(\forall t, x, x',\)

\[|u^{k^*}(t, x) - u^{k^*}(t, x')| \leq C |x - x'| (1 + |x|^p + |x'|^p). \]  \hspace{1cm} (4.32)

Finally, as \(\lambda_{k^*}\) is finite, then we have the following relationship between the process \(k^* U^{t,x} := (k^* U^{t,x})_{i=1,m}\) and the deterministic functions \(u^{k^*} := (u^{k^*})_{i=1,m}\) (see [4]): \(\forall i = 1, \cdots, m,\) we have

\[
k^* U^{t,x}_s(e) = u^{k^*}_i(s, k^* X^{t,x}_s) + \beta(s, k^* X^{t,x}_s, e) - u^{k^*}_i(s, k^* X^{t,x}_s),
\]

\[
d\mathbb{P} \otimes ds \otimes d\lambda_{k^*}, \text{a.e. on } \Omega \times [t, T] \times E.
\]

This is mainly due to the fact that \(k^* U^{t,x}\) belongs to \(L^1 \cap L^2(d\lambda_{k^*} \otimes d\mathbb{P} \otimes d\lambda_{k^*})\) because \(\lambda_{k^*}(E) < \infty\) and then we can split the stochastic integral w.r.t. \(\mu_{k^*}\) in (4.30). Therefore, for any \(i = 1, \cdots, m,\) we have

\[
k^* U^{t,x}_s(e) 1_{\{|e| \geq \frac{1}{2}\}} = (u^{k^*}_i(s, k^* X^{t,x}_s) + \beta(s, k^* X^{t,x}_s, e) - u^{k^*}_i(s, k^* X^{t,x}_s)) 1_{\{|e| \geq \frac{1}{2}\}},
\]

\[
d\mathbb{P} \otimes ds \otimes d\lambda_{k^*}, \text{a.e. on } \Omega \times [t, T] \times E. \]  \hspace{1cm} (4.33)

\[\square\]

**Step 2: Convergence of the auxiliary processes**

Let us now prove the following convergence result:

\[
\mathbb{E} \left[ \sup_{s \leq T} \left| Y^{t,x}_s - k^* Y^{t,x}_s \right|^2 + \left( K^{t,x}_T - k^* K^{t,x}_T \right)^2 + \int_0^T \left| Z^{t,x}_s - k^* Z^{t,x}_s \right|^2 ds \right.
\]

\[
+ \int_0^T ds \int_E \lambda(de) \left| U^{t,x}_s(e) - k^* U^{t,x}_s(e) 1_{\{|e| \geq \frac{1}{2}\}} \right|^2 \xrightarrow{k^* \to +\infty} 0, \]  \hspace{1cm} (4.34)

where \((Y^{t,x}, K^{t,x}, Z^{t,x}, U^{t,x})\) is solution of the RBSDE with jumps (3.3).

First, note that the following convergence result was established in [3]

\[
\mathbb{E} \left[ \sup_{s \leq T} \left| X^{t,x}_s - k^* X^{t,x}_s \right|^2 \right] \xrightarrow{k^* \to +\infty} 0. \]  \hspace{1cm} (4.35)

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We now focus on (4.34). Note that we can apply Itô’s formula, even if the RBSDEs are related to filtrations and Poisson random measures which are not the same, because of the followings:

(i) \( \mathcal{F}_s^k \subset \mathcal{F}_s, \forall s \leq T; \)

(ii) for any \( s \leq T, \int_0^s \mathbb{E} \left[ U_{s,t,x}^k(e) \right] \mu_s^r(\text{dr},\text{de}) = \int_0^s \mathbb{E} \left[ U_{s,t,x}^k(e) 1_{\{|e| \geq \frac{1}{\alpha}\}} \right] \mu_s^r(\text{dr},\text{de}) \) and then the first (\( \mathcal{F}_s^k \))\( _{s \leq T} \)-martingale is also an (\( \mathcal{F}_s \))\( _{s \leq T} \)-martingale.

For \( \forall s \in [0, T] \), we have

\[
\begin{align*}
\left| \mathbb{Y}_{s,T}^t,x - k^* \mathbb{Y}_{s,T}^t,x \right|^2 + \int_0^T \left| \mathbb{Z}_{s,t,x}^t - k^* \mathbb{Z}_{s,T}^t \right|^2 \text{ds} + \sum_{s \leq r \leq T} (k^* \Delta \mathbb{Y}_{r,T}^t,x)^2 \\
= \left[ g(X_{T,T}^t) - g(k^* X_{T,T}^t) \right]^2 + 2 \int_0^T \left( \mathbb{Y}_{r,T}^t,x - k^* \mathbb{Y}_{r,T}^t,x \right) \times k^* \Delta f(r) \text{dr} \\
+ 2 \int_0^T \mathbb{Y}_{r,T}^t,x \left. \left( \mathbb{Y}_{r,T}^t,x - k^* \mathbb{Y}_{r,T}^t,x \right) \right. \times k^* \Delta f(r) \text{dr} \\
- 2 \int_0^T \left( \mathbb{Y}_{r,T}^t,x - k^* \mathbb{Y}_{r,T}^t,x \right) \times k^* \Delta f(r) \text{dr}
\end{align*}
\]

and taking expectation, for \( \forall s \in [t, T] \), we obtain,

\[
\begin{align*}
\mathbb{E} \left[ \left| \mathbb{Y}_{s,T}^t,x - k^* \mathbb{Y}_{s,T}^t,x \right|^2 + \left| k^* \Delta K_T \right|^2 \right] \\
+ \int_0^T \left[ \left| \mathbb{Z}_{s,t,x}^t - k^* \mathbb{Z}_{s,T}^t \right|^2 + \int_E \left| U_{s,T}^t,x - k^* U_{s,T}^t \right| 1_{\{|e| \geq \frac{1}{\alpha}\}} \right]^2 \lambda(\text{de}) \text{ds} \\
\leq \mathbb{E} \left[ \left( g(X_{T,T}^t) - g(k^* X_{T,T}^t) \right)^2 + 2 \int_0^T \left( \mathbb{Y}_{r,T}^t,x - k^* \mathbb{Y}_{r,T}^t,x \right) \times k^* \Delta f(r) \text{dr} \right] \\
+ \mathbb{E} \left[ \sup_{s \leq T} \left( k^* \Delta \ell_s \right) \right]^2 ,
\end{align*}
\]

where the processes \( k^* \Delta X_r, k^* \Delta Y_r, k^* \Delta f(r), k^* \Delta K_r, k^* \Delta Z_r, k^* \Delta U_r \), and \( k^* \Delta \ell_r \) are defined as follows: For \( \forall r \in [0, T] \),

\[
k^* \Delta f(r) := (k^* \Delta f(i)(r))_{i=1,m} = \left( f^{(i)}(r, X_{r,T}^t,x, \mathbb{Y}_{r,T}^t,x, Z_{r,T}^t,x, U_{r,T}^t,x) \\
- f^{(i)}(r, k^* X_{r,T}^t,x, k^* \mathbb{Y}_{r,T}^t,x, k^* Z_{r,T}^t,x, k^* U_{r,T}^t,x) \right)_{i=1,m},
\]

\[
k^* \Delta X_r = X_{r,T}^t,x - k^* X_{r,T}^t,x, \quad k^* \Delta Y_r = \mathbb{Y}_{r,T}^t,x - k^* \mathbb{Y}_{r,T}^t,x = (Y_{r,T}^t,x - k^* Y_{r,T}^t,x)_{j=1,m},
\]

\[
k^* \Delta K_r = K_{r,T}^t,x - k^* K_{r,T}^t,x, \quad k^* \Delta Z_r = Z_{r,T}^t,x - k^* Z_{r,T}^t,x, \quad k^* \Delta U_r = U_{r,T}^t,x - k^* U_{r,T}^t,x 1_{\{|e| \geq \frac{1}{\alpha}\}},
\]

and

\[
k^* \Delta \ell_r = \left( \ell(r, X_{r,T}^t,x) - \ell(r, k^* X_{r,T}^t,x) \right).
\]

Next, for \( r \leq T \), let us set

\[
k^* \Delta f(r) = (f(r, X_{r,T}^t,x, \mathbb{Y}_{r,T}^t,x, Z_{r,T}^t,x, U_{r,T}^t,x) - f_k(r, k^* X_{r,T}^t,x, k^* \mathbb{Y}_{r,T}^t,x, k^* Z_{r,T}^t,x, k^* U_{r,T}^t,x))
\]

\[
= A(r) + B(r) + C(r) + D(r),
\]

where for any \( i = 1, \ldots, m \),

\[
A(r) = \left( h^{(i)} \left( r, X_{r,T}^t,x, \mathbb{Y}_{r,T}^t,x, Z_{r,T}^t,x, \int_E \gamma(r, X_{r,T}^t,x, e) U_{r,T}^t,x(e) \mu_s^r(\text{dr},\text{de}) \right) \\
- h^{(i)} \left( r, k^* X_{r,T}^t,x, k^* \mathbb{Y}_{r,T}^t,x, k^* Z_{r,T}^t,x, \int_E \gamma(r, X_{r,T}^t,x, e) U_{r,T}^t,x(e) \mu_s^r(\text{dr},\text{de}) \right) \right)_{i=1,m},
\]

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\begin{align*}
B(r) &= \left( h^{(i)} \left( r, X_0^{t,x}, Y_0^{t,x}, Z_0^{t,x}, \frac{\gamma_i(r, X_0^{t,x}, e) U_{r_0}^{t,x}(e)}{\lambda(de)} \right) \right)_{i=1,m}, \\
C(r) &= \left( h^{(i)} \left( r, X_0^{t,x}, Y_0^{t,x}, Z_0^{t,x}, \frac{\gamma_i(r, X_0^{t,x}, e) U_{r_0}^{t,x}(e)}{\lambda(de)} \right) \right)_{i=1,m}, \\
D(r) &= \left( h^{(i)} \left( r, X_0^{t,x}, Y_0^{t,x}, Z_0^{t,x}, \frac{\gamma_i(r, X_0^{t,x}, e) U_{r_0}^{t,x}(e)}{\lambda(de)} \right) \right)_{i=1,m}.
\end{align*}

By (4.35) and the fact of \( g \in \mathcal{U} \) and \( \ell \in \mathcal{U} \), we have
\begin{align*}
\mathbb{E} \left[ \left| g(X_T^{t,x}) - g(k^* X_T^{t,x}) \right|^2 \right] & \xrightarrow{k^* \rightarrow +\infty} 0 \quad (4.37) \\
\mathbb{E} \left[ \sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell(k^* X_s^{t,x}) \right|^2 \right] & \xrightarrow{k^* \rightarrow +\infty} 0. \quad (4.38)
\end{align*}

Now, we will interest to \( \mathbb{E} \left[ \int_0^T \left( \gamma_{r^{t,x}} - k^* Y_0^{t,x} \right) \times k^* \Delta f(r) dr \right] \) for getting (4.34).

By (2) and (2.4), for \( \forall r \in [0, T] \), we have
\begin{align*}
|A(r)| & \leq C \left| X_0^{t,x} - k^* X_0^{t,x} \right| \left( 1 + \left| X_0^{t,x} \right|^p + \left| k^* X_0^{t,x} \right|^p \right), \\
|B(r)| & \leq C \left| Y_0^{t,x} - k^* Y_0^{t,x} \right| \quad \text{and} \quad |C(r)| \leq \left| Z_0^{t,x} - k^* Z_0^{t,x} \right|,
\end{align*}
where \( C \) is a constant. Finally let us deal with \( D(r) \), which is more involved. First, note that \( D(r) = (D_i(r))_{i=1,m} \), where
\begin{align*}
D_i(r) &= h^{(i)} \left( r, X_0^{t,x}, Y_0^{t,x}, Z_0^{t,x}, \frac{\gamma_i(r, X_0^{t,x}, e) U_{r_0}^{t,x}(e)}{\lambda(de)} \right) \\
& \quad - h^{(i)} \left( r, X_0^{t,x}, Y_0^{t,x}, Z_0^{t,x}, \frac{\gamma_i(r, k^* X_0^{t,x}, e) U_{r_0}^{t,x}(e)}{\lambda(de)} \right).
\end{align*}

But as \( h^{(i)} \) is Lipschitz w.r.t to the last component \( q \), then,
\begin{align*}
\|D(r)\|^2 & \leq C \left\{ \int_E \left| \gamma_i(r, X_0^{t,x}, e) U_{r_0}^{t,x}(e) - \gamma_i(r, k^* X_0^{t,x}, e) U_{r_0}^{t,x}(e) \right| \lambda(de) \right\}^2 \\
& \quad + C \left\{ \int_E \left| U_{r_0}^{t,x}(e) \right| \lambda(de) \right\}^2 + C \int (1 \wedge |e|) \left| U_{r_0}^{t,x}(e) - k^* U_{r_0}^{t,x}(e) \right| \lambda(de), \quad (4.40)
\end{align*}
and (4.36) become, by using the majorations obtained in (4.39) and (4.40),
\[
\begin{align*}
&\mathbb{E}\left[ \left| Y_{s,t,x} - \kappa^* Y_{s,t,x} \right|^2 \right] \\
&+ \int_0^T \left\{ \left| Z_{s,t,x} - \kappa^* Z_{s,t,x} \right|^2 + \int_E \left| U_{s,t,x} - \kappa^* U_{s,t,x} 1_{\{|x| \geq \frac{1}{2}\}} \right|^2 \lambda(de) \right\} ds \\
&\leq \mathbb{E} \left[ g(X_{s,t}^*, - g(k^* X_{s,t}^*)) \right]^2 + \mathbb{E} \left[ \sup_{s \leq T} \left| \ell(X_{s,t,x}^*) - \ell(k^* X_{s,t,x}^*) \right|^2 \right] \\
&+ C \mathbb{E} \left[ \int_s^T \left| Y_{s,t,x} - \kappa^* Y_{s,t,x} \right|^2 \right] + C \mathbb{E} \left[ \int_0^T \left| X_{r,t,x}^* - \kappa^* X_{r,t,x}^* \right|^2 \right] \\
&+ \int_0^T \int_E \left| \left| U_{r,t,x} - \kappa^* U_{r,t,x} \right| \left( 1 \right| X_{r,t,x}^* \right) + \left| k^* X_{r,t,x}^* \right| \int_E U_{r,t,x}^*(e) \lambda(de) \right]^2. \\
\end{align*}
\]

(4.41)

The two first terms converge to 0 by (4.37) and (4.38).

For the fourth term, we have
\[
\begin{align*}
\mathbb{E} &\left[ \int_0^T \left| X_{r,t,x}^* - \kappa^* X_{r,t,x}^* \right|^2 \left( 1 \right| X_{r,t,x}^* \right) + \left| k^* X_{r,t,x}^* \right| \int_0^T \left( 1 \right| X_{r,t,x}^* \right) + \left| k^* X_{r,t,x}^* \right| \int_0^T \int_E U_{r,t,x}^*(e) \lambda(de) \right]^2. \\
\end{align*}
\]

(4.43)

The first factor in the right-hand side of this inequality goes to 0 when \( k^* \to \infty \) because of (4.35) and the second factor is uniformly bounded by the uniform estimates (3.2) of \( X_{r,t,x}^* \) and \( k^* X_{r,t,x}^* \).

Noting also that the last term converge to 0 when \( k^* \to \infty \), it is a consequence of (4.35), the fact that \( k^* X_{r,t,x}^* \) verifies estimates (3.2) uniformly, the Cauchy-Schwartz inequality (used twice) and finally (4.1) of Lemma 4.1. Then, by Gronwall’s lemma, we deduce first that for any \( s \leq T \),
\[
\mathbb{E} \left[ \left| Y_{s,t,x} - \kappa^* Y_{s,t,x} \right|^2 \right] \to 0 \quad \text{as} \quad k^* \to \infty, \quad (4.42)
\]

and taking \( s = t \), we obtain \( u^k(t,x) \to u(t,x) \). As \( (t,x) \in [0,T] \times \mathbb{R}^k \) is arbitrary, then \( u^k \to u \) pointwisely.

Next, going back to (4.41), taking the limit w.r.t \( k^* \), and using the uniform polynomial growth of \( u^k \) and the Lebesgue dominated convergence theorem as well, we obtain
\[
\mathbb{E} \left[ \int_t^T \int_E \left| U_{s,t,x} - \kappa^* U_{s,t,x} \right|^2 \lambda(de)ds \right] \to 0. \quad (4.43)
\]

**Step 3** Conclusion

First, note that by (4.32) and the pointwise convergence of \((u^k)_{k^*}\) to \( u \), if \((x_{k^*})_{k^*}\) is a sequence of \( \mathbb{R}^k \), which converge to \( x \), then \((u^k(t,x_{k^*}))_{k^*}\) converge to \( u(t,x) \).
Now, let us consider a subsequence which we still denote by \( \{k^*\} \), such that

\[
\sup_{s \leq T} \left| X_{s}^{t,x} - k^* \right|^2 \xrightarrow{k^* \to +\infty} 0,
\]

\( \mathbb{P}\text{-a.s.} \) (and then \( |X_{s}^{t,x} - k^*|^{2} \) \( k^* \to +\infty \) 0 because \( |X_{s}^{t,x} - k^*|^{2} \leq \sup_{s \leq T} |X_{s}^{t,x} - k^*|^{2} \)).

By (4.33) and the fact that \( U_{s}^{t,x}(e) \) is continuous and \( \left| X_{s}^{t,x} - k^* \right| \xrightarrow{k^* \to +\infty} 0 \), and the mapping \( x \mapsto \beta(t,x,e) \) is Lipschitz, then

\[
\left( k^* U_{s}^{t,x}(e)1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^*} = \left( \left| u_{1}^{k^*}(s, k^* X_{s}^{t,x} + \beta(s, k^* X_{s}^{t,x}, e)) - u_{1}^{k^*}(s, k^* X_{s}^{t,x}) \right| 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^* \geq 1}
\]

\[
\xrightarrow{k^* \to +\infty} \left( \left| u_{1}(s, X_{s}^{t,x} + \beta(s, X_{s}^{t,x}, e)) - u_{1}(s, X_{s}^{t,x}) \right| \right)_{k^* \geq 1},
\]

for any \( i = 1, \cdots, m \). Finally, from (4.43), we deduce that

\[
U_{s}^{t,x}(e) = (u_{1}(s, s X_{s}^{t,x} + \beta(s, X_{s}^{t,x}, e)) - u_{1}(s, X_{s}^{t,x})), \quad \text{on } \Omega \times [t,T] \times E,
\]

which is the desired result.

5 Main Result

First, we give the definition of viscosity solution of IPDEs as given in \([3, 4]\). Our main result deals with this definition.

**Definition 5.1** We say that a family of deterministic functions \( u = (u^i)_{i=1,m} \), which belongs to \( \mathcal{U} \), \( \forall i \in \{1, \cdots, m\} \), is a viscosity sub-solution (resp. super-solution) of the IPDE (1.1) if

(i) for \( \forall x \in \mathbb{R}^k \), \( u^i(x,T) \leq g^i(x) \) (resp. \( u^i(x,T) \geq g^i(x) \));

(ii) for \( \forall (t,x) \in [0,T] \times \mathbb{R}^k \) and any function \( \phi \) of class \( C^{1,2}([0,T] \times \mathbb{R}^k) \) such that \( (t,x) \) is a global maximum point of \( u^i - \phi \) (resp. global minimum point of \( u^i - \phi \)) and \( (u^i - \phi)(t,x) = 0 \), one has

\[
\min \left\{ u^i(t,x) - \ell(t,x); -\partial_t \phi(t,x) - \mathcal{L}^X \phi(t,x) - h^i(t,x, \{u^j(t,x)\}_{j=1,m}), \sigma^\top(t,x) D_x \phi(t,x), B_i u^i(t,x) \right\} \leq 0,
\]

\( \text{resp. } \min \left\{ u^i(t,x) - \ell(t,x); -\partial_t \phi(t,x) - \mathcal{L}^X \phi(t,x) - h^i(t,x, \{u^j(t,x)\}_{j=1,m}), \sigma^\top(t,x) D_x \phi(t,x), B_i u^i(t,x) \right\} \geq 0 \).

The family \( u = (u^i)_{i=1,m} \) is a viscosity solution of (1.1) if it is both a viscosity sub-solution and viscosity super-solution.

Note that

\[
\mathcal{L}^X \phi(t,x) = \sigma(t,x)^\top D_x \phi(t,x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t,x) D_x^2 \phi(t,x)) + K \phi(t,x),
\]

where

\[
K \phi(t,x) = \int_E (\phi(t,x + \beta(t,x,e)) - \phi(t,x) - \beta(t,x,e)^\top D_x \phi(t,x)) \lambda(de).
\]

**Theorem 5.2** Under assumptions (H1), (H2), and (H3), IPDE (1.1) has unique solution, which is the \( m \)-tuple of functions \( (u^i)_{i=1,m} \) defined in Proposition 3.4 by (3.4).
**Proof** Step 1 Existence

Assume that assumptions (H1), (H2), and (H3) are fulfilled, then the following multi-
dimensional RBSDEs with jumps

(i) \( Y^{t,x}_{i=1,m} := (Y_{i}^{t,x})_{i=1,m} \in S^2(\mathbb{R}^m) \), \( Z^{t,x}_{i=1,m} := (Z_{i}^{t,x})_{i=1,m} \in H^2(\mathbb{R}^{m \times d}) \),
\( K^{t,x}_{i=1,m} := (K_{i}^{t,x})_{i=1,m} \in A^2 \), \( U^{t,x}_{i=1,m} := (U_{i}^{t,x})_{i=1,m} \in H^2(L^2(\lambda)) \);

(ii) \( Y_{s}^{t,x} = g(X_{T}^{t,x}) + K_{T}^{t,x} - K_{0}^{t,x} - \int_{s}^{T} Z_{r}^{t,x} dB_{r} - \int_{s}^{T} \int_{\mathcal{E}} U_{r}^{t,x}(e) \tilde{\mu}(dr, de) \)
\( + \int_{s}^{T} h^{(i)}(r, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x}) dr \) \( \right(5.3) \)

(iii) \( Y_{s}^{t,x} \geq \ell(s, X_{s}^{t,x}) \) and \( \int_{0}^{T} (Y_{r}^{t,x} - \ell(s, X_{s}^{t,x})) dK_{r}^{t,x} = 0 \)

has the unique solution \( (Y, Z, K, \mu) \). Next, as for any \( i = 1, \cdots, m \), \( u^{i} \) belongs to \( \mathcal{U} \), then by (3.4) in Proposition 3.4, there exists a family of deterministic continuous functions of polynomial growth \( u^{i}_{1=1,m} \), such that for any \( (t, x) \in [0,T] \times \mathbb{R}^{k} \),
\( \forall s \in [t,T], \ Y^{s,t,x}_{i=1,m} = u^{i}(s, X_{s}^{t,x}) \).

Then, by the same proposition, the family \( (u^{i})_{i=1,m} \) is a viscosity solution of the following system:

\[
\begin{align*}
\min \left\{ u^{i}(t, x) - \ell(t, x); -\partial_{t} u^{i}(t, x) - b(t, x) \right\} D_{x} u^{i}(t, x) \\
- \frac{1}{2} \text{Tr} \left( \sigma \sigma^{T} - (t, x) D_{xx}^{2} u^{i}(t, x) \right) - K_{0} u^{i}(t, x) - h^{(i)}(t, x, u^{i}(t, x))_{j=1,m}.
\end{align*}
\]  \( \right(5.4) \)

Now, we know that the family \( (u^{i})_{i=1,m} \) is a viscosity solution; our main objective is to found relation between \( (u^{i})_{i=1,m} \) and \( (u^{i})_{i=1,m} \) defined in (3.4).

For this, let us consider the system of RBSDE with jumps

(i) \( Y^{t,x}_{i=1,m} := (Y_{i}^{t,x})_{i=1,m} \in S^2(\mathbb{R}^m) \), \( Z^{t,x}_{i=1,m} := (Z_{i}^{t,x})_{i=1,m} \in H^2(\mathbb{R}^{m \times d}) \),
\( K^{t,x}_{i=1,m} := (K_{i}^{t,x})_{i=1,m} \in A^2 \), \( U^{t,x}_{i=1,m} := (U_{i}^{t,x})_{i=1,m} \in H^2(L^2(\lambda)) \);

(ii) \( Y_{s}^{t,x} = g(X_{T}^{t,x}) + K_{T}^{t,x} - K_{0}^{t,x} - \int_{s}^{T} Z_{r}^{t,x} dB_{r} - \int_{s}^{T} \int_{\mathcal{E}} U_{r}^{t,x}(e) \tilde{\mu}(dr, de) \)
\( + \int_{s}^{T} h^{(i)}(r, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x}) dr \) \( \right(5.5) \)

(iii) \( Y_{s}^{t,x} \geq \ell(s, X_{s}^{t,x}) \) and \( \int_{0}^{T} (Y_{r}^{t,x} - \ell(s, X_{s}^{t,x})) dK_{r}^{t,x} = 0 \).

By the uniqueness of the solution of the RBSDEs with jumps (5.2), then for \( \forall s \in [t,T] \) and \( \forall i \in \{1 \cdots m\} \), \( Y_{s}^{t,x} = Y_{s}^{t,x} \).

Therefore, \( u^{i} = u^{i} \), such that by (4.43), we obtain
\[ U_{s}^{t,x}(e) = (u_{i}(s, X_{s}^{t,x} + \beta(s, X_{s}^{t,x}, e)) - u_{i}(s, X_{s}^{t,x})) \text{ on } \Omega \times [t,T] \times \mathcal{E} \]
which give the viscosity solution in the sense of Definition 5.1 (see [3]) by plugging (4.44) in $h^{(i)}$ of (5.4).

**Step 2 Uniqueness**

For the uniqueness, let $(\pi^i)_{i=1,m}$ be another family of $U$ which is the viscosity solution of system (1.1) in the sense of Definition 5.1 and we consider RBSDE with jumps defined with $\pi^i$.

\[
\begin{align*}
(i) & \quad Y_{s:t,x}^i := (Y_{s:t,x}^i)_{i=1,m} \in S^2(\mathbb{R}^m), \quad Z_{s:t,x}^i := (Z_{s:t,x}^i)_{i=1,m} \in H^2(\mathbb{R}^m \times d), \\
& \quad K_{s:t,x}^i := (K_{s:t,x}^i)_{i=1,m} \in A^2, \quad U_{s:t,x}^i := (U_{s:t,x}^i)_{i=1,m} \in L^2(\mathbb{R}_m^2(\lambda)); \\
(ii) & \quad Y_{s:t,x}^i = g_i(X_T^i) + K_{s:t,x}^i - K_{s:t,x}^i - \int_s^T Z_{s:t,x}^i \, dB_{r} - \int_s^T \int_E U_{s:t,x}(e) \, \tilde{\mu}(dr, de) \\
& \quad + \int_s^T h^{(i)}(r, X_{r:t,x}^i, Y_{r:t,x}^i, Z_{r:t,x}^i, \\
& \quad \int_E \gamma^{ij}(t, X_{r:t,x}^i, e) (\pi_i(s, X_{s:t,x}^i + \beta(s, X_{s:t,x}^i, e)) - \pi_i(s, X_{s:t,x}^i)) \, \lambda(de) \, dr; \\
(iii) & \quad Y_{s:t,x}^i \geq \ell(s, X_{s:t,x}^i) \text{ and } \int_0^T (Y_{s:t,x}^i - \ell(s, X_{s:t,x}^i)) \, d\tilde{K}_{s:t,x}^i = 0.
\end{align*}
\]

By Feynman Kac formula, $\pi^i(s, X_{s:t,x}^i) = Y_{s:t,x}^i$, where $Y_{s:t,x}^i$ satisfies the RBSDE with jumps (1.3) associated to IPDE (1.1).

Since that the RBSDE with jumps (5.4) has solution and it is unique by assuming that (H1), (H2), and (H3) are verified. By (3.4) in Proposition 3.4, there exists a family of deterministic continuous functions of polynomial growth $(v^i)_{i=1,m}$, such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

\[\forall s \in [t, T], \quad Y_{s:t,x}^i = v^i(s, X_{s:t,x}^i).\]

Thus, by the same proposition, the family $(v^i)_{i=1,m}$ is a viscosity solution of the following system:

\[
\begin{align*}
& \min \left\{ v^i(t, x) - \ell(t, x); -\partial_t v^i(t, x) - b(t, x)^T D_x v^i(t, x) \\
& \quad - \frac{1}{2} \text{Tr}(\sigma \sigma^T (t, x) D^2_{xx} v^i(t, x)) - K_{v^i}(t, x) - h^{(i)}(t, x, v^i(t, x))_{j=1,m}, \\
& \quad (\sigma^T D_x v^i)(t, x), B_i \pi^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\
& v^i(T, x) = g^i(x).
\end{align*}
\]

By the uniqueness of solution of (5.5), $\pi^i$ is the viscosity solution of (5.6), and by Proposition 3.4, $v^i = \pi^i$ for all $i \in \{1, \ldots, m\}$.

Now, for completing our proof, we show that on $\Omega \times [t, T] \times E$, $d\pi \otimes dP \otimes d\lambda$-a.e., for $\forall i \in \{1, \ldots, m\}$, we have

\[\begin{align*}
U_{s:t,x}^i(e) &= (v^i(s, X_{s:t,x}^i + \beta(s, X_{s:t,x}^i, e)) - v^i(s, X_{s:t,x}^i)) \\
&= (\pi_i(s, X_{s:t,x}^i + \beta(s, X_{s:t,x}^i, e)) - \pi_i(s, X_{s:t,x}^i)). \quad (5.8)
\end{align*}\]

By Remark 3.4 in [3], let us consider $(x_k)_{k \geq 1}$, a sequence of $\mathbb{R}^k$, which converges to $x \in \mathbb{R}^k$.
and the two following RBSDE with jumps (adaptation is w.r.t. $\mathcal{F}^k$):

\begin{align}
& \begin{cases}
(i) \quad \bar{Y}^{k^*,t,x}_{s} = (\bar{Y}^{k^*,t,x}_{i=1,m})_{i=1,m} \in S^2(\mathbb{R}^m), \quad \bar{Z}^{k^*,t,x}_{s} := (\bar{Z}^{k^*,t,x}_{i=1,m})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
\bar{K}^{k^*,t,x}_{s} := (\bar{K}^{k^*,t,x}_{i=1,m})_{i=1,m} \in \mathcal{A}_t^2, \quad \bar{U}^{k^*,t,x}_{s} := (\bar{U}^{k^*,t,x}_{i=1,m})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}^2_m(\lambda));
\end{cases} \\
(ii) \quad \bar{Y}^{k^*,t,x}_{s} = g^i(X^k_{T}, t,x) + \bar{K}^{k^*,t,x}_{T} - \bar{K}^{k^*,t,x}_{s} - \int_s^T \bar{Z}^{k^*,t,x}_{r} dB_r \\
- \int_s^T \int_E \bar{U}^{k^*,t,x}_{r}(e) \hat{\mu}(dr, de) + \int_s^T h^{(i)}(r, X^k_{r}, Y^k_{r}, Z^k_{r}, U^k_{r}, e), \\
\int_E \gamma^i(t, X^k_{r}, X^*, e)(\pi_i(s, X^k_{s}, t,x) + \beta(s, X^k_{s}, t,x), e)) \\
- \bar{\omega}_t(s, X^k_{s}, t,x) \lambda(de) dr;
\end{align}

(iii) $\bar{Y}^{k^*,t,x}_{s} \geq \ell(s, X^k_{s}, t,x) + \int_0^T \ell(s, X^k_{s}, t,x))d\bar{K}^{k^*,t,x}_{s} = 0$.

(5.9)

and

\begin{align}
& \begin{cases}
(i) \quad Y^{k^*,t,x}_{s} := (Y^{k^*,t,x}_{i=1,m})_{i=1,m} \in S^2(\mathbb{R}^m), \quad Z^{k^*,t,x}_{s} := (Z^{k^*,t,x}_{i=1,m})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
K^{k^*,t,x}_{s} := (K^{k^*,t,x}_{i=1,m})_{i=1,m} \in \mathcal{A}_t^2, \quad U^{k^*,t,x}_{s} := (U^{k^*,t,x}_{i=1,m})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}^2_m(\lambda));
\end{cases} \\
(ii) \quad Y^{k^*,t,x}_{s} = g^i(X^k_{T}, t,x) + K^{k^*,t,x}_{T} - K^{k^*,t,x}_{s} - \int_s^T Z^{k^*,t,x}_{r} dB_r \\
- \int_s^T \int_E U^{k^*,t,x}_{r}(e) \hat{\mu}(dr, de) + \int_s^T h^{(i)}(r, X^k_{r}, Y^k_{r}, Z^k_{r}, U^k_{r}, e), \\
\int_E \gamma^i(t, X^k_{r}, X^*, e)(\pi_i(s, X^k_{s}, t,x) + \beta(s, X^k_{s}, t,x), e)) \\
+ \beta(s, X^k_{s}, t,x), e) - \pi_i(s, X^k_{s}, t,x) \lambda(de) dr;
\end{align}

(iii) $Y^{k^*,t,x}_{s} \geq \ell(s, X^k_{s}, t,x) + \int_0^T (Y^{k^*,t,x}_{s} - \ell(s, X^k_{s}, t,x))dK^{k^*,t,x}_{s} = 0$.

(5.10)

By Proof of Step 2 of Proposition 4.5, $(\bar{Y}^{k^*,t,x}_{s}, \bar{K}^{k^*,t,x}_{s}, \bar{Z}^{k^*,t,x}_{s}, \bar{U}^{k^*,t,x}_{s})_{k^*} \mathrm{converge~to~} (Y^{k^*,t,x}_{s}, K^{k^*,t,x}_{s}, Z^{k^*,t,x}_{s}, U^{k^*,t,x}_{s})_{k^*} \in S^2(\mathbb{R}) \times \mathcal{A}_t^2 \times \mathbb{H}^2(\mathbb{R}^{m \times d}) \times \mathbb{H}^2(\mathbb{L}^2_m(\lambda))$.

Let $((v^{k^*}_{i=1,m}))_{k^* \geq 1}$ be the sequence of continuous deterministics functions, such that for any $t \leq T$ and $s \in [t,T]$, $Y^{k^*,t,x}_{s} = v^{k^*}(s, X^k_{s}, t,x) \mathrm{\ and \ } Y^{k^*,t,x}_{s} = v^{k^*}(s, X^k_{s}, t,x) \forall i = 1, \ldots, m$.

Thus, by Proof of Proposition 4.5 in Step 1 and Step 2, we have respectively:

(i) $\bar{U}^{k^*,t,x}_{s}(e) = (v^i(s, X^k_{s}, t,x) + \beta(s, X^k_{s}, t,x), e)) - v^i(s, X^k_{s}, t,x)) \mathrm{\ and \ } d\mathbb{P} \otimes d\lambda_{k^*} \mathrm{-a.e \ on \ } [t,T] \times \Omega \times \mathbb{R}$;

(ii) the sequence $((v^{k^*}_{i=1,m}))_{k^* \geq 1}$ converge to $v^i(t, x)$ by using (4.42).

So, $x_k, \longrightarrow k, x$, where we take the following estimation obtained by Ito’s formula and the properties of $h^{(i)}$:

$$
\mathbb{E}\left[ Y^{k^*,t,x}_{s} - Y^{k^*,t,x}_{s} \right]^2 + \left| K^{k^*,t,x}_{T} - K^{k^*,t,x}_{T} \right|^2
$$
\[
+ \int_0^T \left\{ \left| Z^{k^*}-Z^{k^*} \right|^2 + \int_E \left| U^{k^*} - U^{k^*} \right|^2 \lambda_k \left( \, de \right) \right\} ds \\
\leq \mathbb{E} \left[ \left| g(X^{t,x_k^*}) - g(X^{t,x}) \right|^2 \right] + \sup_{s \leq T} \left| \left| \ell \left( k^* \, X^{t,x_k^*} \right) - \ell \left( k^* \, X^{t,x} \right) \right|^{2} \mathbb{E} \left[ \int_0^T \left| \int_E \left( Y^{k^*} - Y^{k^*} \right)^2 \, dr \right| \right] \\
+ C \mathbb{E} \left[ \int_0^T \left| k^* \, X^{t,x_k^*} - k^* \, X^{t,x} \right|^2 \left( 1 + \left| k^* \, X^{t,x_k^*} \right|^p + \left| k^* \, X^{t,x} \right|^p \right) \, dr \right] \\
+ C \sum_{i=1,m} \mathbb{E} \left[ \int_0^T \left| B_i(\phi(r, k^{*} \, X^{t,x_k^*})) - B_i(\phi(r, k^{*} \, X^{t,x})) \right|^2 \, dr \right]. \quad (5.11)
\]

Next, using (4.37) and (4.38), and the fact that the function \((t, x) \mapsto B_i(\phi(t, x))\) belongs to \(\Pi^s_2\), and in the other hand, the majoration of the fourth term of (4.41), we can use Gronwall's lemma, for \(s = t, \forall i = 1, \cdots, m\),
\[
v^{k^{*}}(t, x_k^{*}) \rightarrow v^{k^{*}}(t, x).
\]

Therefore, by (i)–(ii), we have, for any \(i = 1, \cdots, m\),
\[
\bar{U}^{s,t,x}(e) = \left( (v(s, X^{t,x}_s + \beta(s, X^{t,x}_s), e)) - V^i(s, X^{t,x}_s) \right)
\]
\[
ds \otimes d\omega \otimes d\lambda \text{ a.e. in } [t, T] \times \Omega \times E, \quad \forall i \in \{1, \cdots, m\}. \quad (5.12)
\]

By this result, we can replace \((\bar{u}_i(s, X^{t,x}_s + \beta(s, X^{t,x}_s), e)) - \bar{u}_i(s, X^{t,x}_s))\) by \(\bar{U}^{s^{t,x}}(e)\) in (5.8), we deduce that the quadruple \((\bar{Y}^{s^{t,x}}, K^{s^{t,x}}, Z^{s^{t,x}}, U^{s^{t,x}})\) verifies that for \(\forall i \in \{1, \cdots, m\}\),
\[
\begin{align*}
(i) \quad \bar{Y}^{s^{t,x}} := \left( \bar{Y}^{s^{t,x}} \right)_{i=1,m} \in S^2(\mathbb{R}^m), \\
&\quad \bar{K}^{s^{t,x}} := \left( \bar{K}^{s^{t,x}} \right)_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^m), \\
&\quad \bar{Z}^{s^{t,x}} := \left( \bar{Z}^{s^{t,x}} \right)_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^m). \\
(ii) \quad \bar{Y}^{s^{t,x}} = g^i(X^{t,x}_T) + \bar{K}^{s^{t,x}} - \bar{K}^{s^{t,x}} - \int_s^T \bar{Z}^{s^{t,x}} \, dB_r - \int_s^T \int_E \bar{U}^{s^{t,x}}(e) \, \mu(\, dr, de). \\
&\quad + \int_s^T h^{(i)}(r, X^{t,x}_r, Y^{s^{t,x}}_r, Z^{s^{t,x}}_r, U^{s^{t,x}}_r) \, \lambda(\, dr, de) \\
&\quad \int_s^T \gamma^i(r, X^{t,x}_r, e) \, \lambda(\, dr, de) \\
(iii) \quad \bar{Y}^{s^{t,x}} \geq \ell(s, X^{t,x}_s) \text{ and } \int_s^T (\bar{Y}^{s^{t,x}} - \ell(s, X^{t,x}_s)) \, d\bar{K}^{s^{t,x}} = 0.
\end{align*}
\]

It follows that
\[
\bar{Y}^{s^{t,x}} = Y^{s^{t,x}} \quad \text{for } \forall i \in \{1, \cdots, m\}.
\]

With the uniqueness of solution (5.6), we have \(u^i = \bar{u} = u_i^{i^*}\), which means that the solution of (1.1) in the sense of Definition 5.1 is unique inside the class \(\mathcal{U}\).

\[\square\]

6 Extension

In this section, we will redefine the function \(h^{(i)}\) as a function of \(|U^{s^{t,x}}|_{L^2(\lambda)}\) \((\forall i \in \{1, \cdots, m\})\). And we will show that the results of the previous section remain valid; for any \(i \in \{1, \cdots, m\}\), let us consider the functions \(f^{(i)}\) defined by,
\[
\text{for } \forall \, (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{m+d} \times \mathbb{H}^2(\lambda),
\]
\[\square\] Springer
Thus, by the same proposition, the family \((\omega(t, x, y, z, \zeta))_{i=1,m}\) are the same as that defined in Section 2.

We recall that the result of Theorem 5.2 is obtained by having mainly \(U_{s}^{t,x}(e) = (u^{i}(s, X_{s-}^{i,x}) + \beta(s, X_{s-}^{i,x}), e))\); this makes it possible to have Definition 5.1 by passing through a modification of the expression of \(B_{i}w^{i}(\forall i \in \{1, \cdots, m\})\).

We show that \(||U_{s}^{t,x}(e)||_{L^{2}(\lambda)}^{2} = ||u^{i}(s, X_{s-}^{i,x} + \beta(s, X_{s-}^{i,x}), e)) - u^{i}(s, X_{s-}^{i,x}, e))||_{L^{2}(\lambda)}^{2}\) and that in this case, \(B_{i}w^{i}(\forall i \in \{1, \cdots, m\})\) is well.

Now, let \((t, x) \in [0, T] \times \mathbb{R}^{d}\) and let us consider the following \(m\)-dimensional RBSDE with jumps:

\[
\begin{cases}
\text{(i)} & Y_{t,x}^{i} := (Y_{t,x}^{i}x)_{i=1,m} \in S^{2}(\mathbb{R}^{m}), \; Z_{t,x}^{i} := (Z_{t,x}^{i}x)_{i=1,m} \in \mathbb{H}^{2}(\mathbb{R}^{m \times d}), \\
& K_{t,x}^{i} := (K_{t,x}^{i}x)_{i=1,m} \in A^{2}, \; U_{t,x}^{i} := (U_{t,x}^{i}x)_{i=1,m} \in \mathbb{H}^{2}(\mathbb{L}^{2}(\lambda)); \\
& \forall i \in \{1, \cdots, m\} \; Y_{t,x}^{i} = g^{i}(X_{t,x}^{i}) \; \text{and}; \\
\text{(ii)} & dY_{s}^{t,x} = -f^{i}(s, X_{s}^{t,x}, (Y_{s}^{t,x}x)_{i=1,m}, Z_{s}^{t,x}, ||U_{s}^{t,x}(e)||_{L^{2}(\lambda)}) ds \\
& \quad -dK_{s}^{i,t,x} + Z_{s}^{i,t,x} dB_{s} + \int \mathcal{E} U_{s}^{t,x}(e) \mu(ds, de), \; s \leq T; \\
\text{(iii)} & Y_{s}^{t,x} \geq \ell(s, X_{s}^{t,x}) \; \text{and} \; \int_{0}^{T} (Y_{s}^{t,x} - \ell(s, X_{s}^{t,x})) dK_{s}^{i,t,x} = 0.
\end{cases}
\]

Assuming that (H1), (H2), and (H3) are verified and by (3.4) in Proposition 3.4, there exists a family of deterministic continuous functions of polynomial growth \((w^{i})_{i=1,m}\), such that for any \((t, x) \in [0, T] \times \mathbb{R}^{k}\),

\[
\forall s \in [t, T], \quad Y_{s}^{t,x} = w^{i}(s, X_{s}^{t,x}).
\]

Thus, by the same proposition, the family \((w^{i})_{i=1,m}\) is a viscosity solution of the following system:

\[
\begin{cases}
\min \left\{ w^{i}(t, x) - \ell(t, x); -\partial_{i}w^{i}(t, x) - b(t, x)^{T}D_{x}w^{i}(t, x) \\
- \frac{1}{2}Tr(\sigma \sigma^{T}(t, x)D_{x}w^{i}(t, x)) - K_{i}w^{i}(t, x) - h^{i}(t, x, (w^{i}(t, x))_{j=1,m}, \sigma \sigma^{T}(t, x)D_{x}w^{i}(t, x), B_{i}w^{i}(t, x)) \right\} = 0, \; (t, x) \in [0, T] \times \mathbb{R}^{k}; \\
w^{i}(T, x) = g^{i}(x).
\end{cases}
\]

Indeed, using Lemma 4.1 and the fact that \(U_{s}^{t,x}(e) \rightarrow U_{s}^{t,x}(e)\) \((\forall i \in \{1, \cdots, m\})\) when \(x_{k} \rightarrow_{k} x\), we deduce that \(||U_{s}^{t,x}(e)||_{L^{2}(\lambda)} \rightarrow_{k} ||U_{s}^{t,x}(e)||_{L^{2}(\lambda)}\).

Moreover, from property of \(h^{i}\) and Proposition 4.5,

\[
||U_{s}^{t,x}(e)||_{L^{2}(\lambda)} = \left(||w^{i}(s, X_{s}^{t,x} + \beta(s, X_{s}^{t,x}, e)) - w^{i}(s, X_{s}^{t,x}))||_{L^{2}(\lambda)}^{2}\right)^{1/2}.
\]

from where

\[
B_{i}w^{i} = \left\{ \int \mathcal{E} \left(||w^{i}(s, X_{s}^{t,x} + \beta(s, X_{s}^{t,x}, e)) - w^{i}(s, X_{s}^{t,x}))||_{L^{2}(\lambda)}^{2}\right) \lambda(de) \right\}^{1/2}.
\]

Because of Corollary 4.4, we deduce that \(B_{i}w^{i}(\forall i \in \{1, \cdots, m\})\) is well defined.
Appendix Barles et al.’s definition for viscosity solution of IPDE (1.1)

In [1] by Barles et al, the definition of the viscosity solution of system (1.1) is given as follows.

**Definition A.1** We say that a family of deterministic functions \( u = (u^i)_{i=1,m} \) (\( \forall i \in \{1, \cdots, m\} \)) which is continuous is a viscosity sub-solution (resp. super-solution) of the IPDE (1.1) if the followings hold:

(i) \( \forall x \in \mathbb{R}^k, u^i(x, T) \leq g^i(x) \) (resp. \( u^i(x, T) \geq g^i(x) \));

(ii) For any \( (t, x) \in [0, T] \times \mathbb{R}^k \) and any function \( \phi \) of class \( C^{1,2}([0, T] \times \mathbb{R}^k) \) such that \( (t, x) \) is a global maximum point of \( u^i - \phi \) (resp. global minimum point of \( u^i - \phi \)) and \( (u^i - \phi)(t, x) = 0 \), one has

\[
\min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x)))_{j=1,m}, \right. \\
\sigma^\top(t, x) D_x \phi(t, x), B_i \phi(t, x) \right\} \leq 0,
\]

(resp. \( \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x)))_{j=1,m}, \right. \\
\sigma^\top(t, x) D_x \phi(t, x), B_i \phi(t, x)(t, x) \right\} \geq 0 \}).

The family \( u = (u^i)_{i=1,m} \) is a viscosity solution of (1.1) if it is both a viscosity sub-solution and viscosity super-solution.

Note that

\[
\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K\phi(t, x),
\]

where

\[
K\phi(t, x) = \int_{\mathbb{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de).
\]

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