Hochschild homology of preprojective algebras over the integers

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Abstract

We determine the \( \mathbb{Z} \)-module structure and explicit bases for the preprojective algebra \( \Pi \) and all of its Hochschild (co)homology, for any non-Dynkin quiver. This answers (and generalizes) a conjecture of Hesselholt and Rains, producing new \( p \)-torsion elements in degrees \( 2p^\ell, \ell \geq 1 \). We relate these elements by \( p \)-th power maps and interpret them in terms of the kernel of Verschiebung maps from noncommutative Witt theory. We also define a Lie bialgebra structure on \( HH_0(\Pi) \) (from the necklace Lie bialgebra), relate it to Goldman/Turaev’s Lie bialgebra of loops, compute it for extended Dynkin quivers, and compute the Poisson center of \( \text{Sym} \, HH_0(\Pi) \) for all quivers. We then compute the BV algebra structure on Hochschild cohomology, show that the Lie algebra structure \( HH_0(\Pi_Q) \) naturally arises from it, and compute all cyclic homology groups of \( \Pi_Q \). In the process, we define and study related algebraic structures: a “noncommutative BV structure” generalizing the necklace Lie bialgebra, and “free-product” deformations of \( \Pi_Q \), which yield all ordinary deformations as quotients.

Contents

1 Introduction and Main Results
   1.1 \( \mathbb{Z} \)-module structure of preprojective algebras .......................... 3
   1.2 The BV algebra on Hochschild cohomology .............................................. 5
   1.3 New algebraic structures and the McKay correspondence ......................... 6
   1.4 Other results ......................................................................................... 7
   1.5 Outline of paper ..................................................................................... 8
   1.6 Notation and Definitions ......................................................................... 8
   1.7 Acknowledgements .................................................................................. 9

2 Relations to Witt theory and Goldman/Turaev’s algebra of loops
   2.1 Relation to noncommutative Witt theory .................................................. 9
      2.1.1 \( p \)-th power maps and Lie structure of torsion elements .................... 10
   2.2 Relation between \( \Lambda \) and Goldman/Turaev’s Lie algebra ...................... 11
      2.2.1 Partial preprojective algebras and quivers with attached line segments ... 12

3 Hesselholt and Rains’ conjecture and Theorem 1.1.2 for good primes .............. 16
   3.1 Proof of Hesselholt and Rains’ Conjecture 1.1.1 ...................................... 16
   3.2 Proof of Theorem 1.1.2 for good primes: a \( \Gamma \)-equivariant version .......... 21
4 Hilbert series, bases, and Theorem 1.1.2

4.1 Hilbert series and Question 1.1.9 ........................................ 26
4.1.1 The non-Dynkin, non-extended Dynkin and partial preprojective cases . . . 27
4.1.2 The extended Dynkin case ........................................... 28
4.1.3 The Dynkin case ...................................................... 29

4.2 Bases and proof of Theorem 1.1.2 ........................................ 30
4.2.1 Bases of $\Pi_Q$ and $HH_0(\Pi_Q)$ for non-Dynkin, non-extended Dynkin quivers, and for partial preprojective algebras; proof that $r^{(p)} \neq 0$ ........ 33
4.2.2 Bases of $\Pi_Q$ for type $A$ and $D$ quivers and the remainder of Theorem 1.1.2 . 37
4.2.3 Bases of $\Pi_Q$ for type $E$ quivers and presentations of $\Pi_Q$ and $HH_0(\Pi_Q)$ for star-shaped quivers ................................................... 41
4.2.4 Bases of $HH_0(\Pi_Q)$ in the extended Dynkin and Dynkin cases ............ 42

5 The Lie bialgebra $\Lambda_Q$ and connections to noncommutative BV structures and free product deformations on $P_Q$ and $L_Q$ ........................................... 44
5.1 The Lie bialgebra $\Lambda_+$ .............................................. 44
5.2 Lifting brackets to $P$ ..................................................... 46
5.3 Noncommutative BV structure ........................................ 47
5.4 Proof of Propositions 5.1.7 and 5.1.8 .................................... 48
5.5 $P_Q$ as a free product deformation of $\Pi_Q$ and the necklace bracket .......... 50

6 Quivers containing $\tilde{A}_{n-1}$: proof of Theorem 4.2.40 ............. 52
6.1 The case of $\tilde{A}_0$ ..................................................... 52
6.2 Proof of Theorem 1.1.2 for quivers containing $\tilde{A}_{n-1}$ .................. 53
6.3 Hilbert series and (1.1.14) in the $\tilde{A}_n$ case ........................... 54
6.4 Poisson structure on $i_0\Pi_Q i_0$ / Lie structure on $\Lambda_Q$ for $Q = \tilde{A}_{n-1}$ .... 55

7 Quivers containing $\tilde{D}_n$: proof of Theorem 4.2.48 ............. 55
7.1 A basis for $\Pi_{\tilde{D}_n}$: Theorem 4.2.48 (i) .................................. 55
7.2 Proof of Theorem 1.1.2 for quivers containing $\tilde{D}_n$ .................. 55
7.3 Hilbert series for $\tilde{D}_n$ and (1.1.14) ................................... 62
7.4 The Poisson algebra $i_0\Pi_Q i_0$ / the Lie structure on $\Lambda_Q$ for $Q = \tilde{D}_n$ .... 63

8 Quivers of type $\tilde{E}_n$ and completion of the proof of Theorem 1.1.2 ............... 63
8.1 Type $\tilde{E}_6$ ...................................................... 63
8.2 Type $\tilde{E}_7$ ...................................................... 64
8.3 Type $\tilde{E}_8$ ...................................................... 65
8.4 Poisson structure on $i_0\Pi_Q i_0$ / Lie structure on $\Lambda_Q$ for $Q = \tilde{E}_n$ ....... 66
8.5 Proof of Theorem 4.2.30 (completing the proof of Theorem 1.1.2) ......... 68

9 The Lie structure of $\Lambda_{Q,J}$, part II .................................. 69
9.1 The Lie structure on $\Lambda_Q$ for extended Dynkin quivers .................. 69
9.2 Poisson center of Sym $\Lambda_{Q,J}$ .................................... 72
10 Higher Hochschild (co)homology, cyclic homology, and the BV algebra

10.1 Higher Hochschild (co)homology and $Z(\pi \Pi \otimes k)$ ............................. 79
10.2 Outer derivations and deformations of $\Pi_Q \otimes k$ ........................................ 83
  10.2.1 Outer derivations of $\Pi_Q \otimes Q$ (the rational case) .............................. 83
  10.2.2 $HH^1(\Pi_Q \otimes k)$ where $k$ is torsion-free and $Q$ is extended Dynkin .... 85
  10.2.3 $HH^1(\Pi_Q \otimes k)$ when $Q$ is non-Dynkin, non-extended Dynkin, over $Z$ .... 88
  10.2.4 $HH^1(\Pi_Q \otimes k)$ in the case $k$ has torsion .................................... 88
  10.2.5 Summary of $HH^1(\Pi_Q \otimes k)$ ......................................................... 90
10.3 The BV algebra structure ................................................................. 91
10.4 Cyclic homology ............................................................................. 94

A Gröbner Bases and the Diamond Lemma

A.0.1 Gröbner Bases ................................................................. 97
A.0.2 The Diamond Lemma ................................................................. 98

B Loday Poisson, co, and bialgebras ......................................................... 100

1 Introduction and Main Results

There are two main purposes of this paper: first, to establish a new phenomenon wherein Hilbert series of certain graded algebras defined over $\mathbb{Z}$ jumps in characteristic $p$ by 1 in degrees $m \cdot p^\ell$ for fixed $m$, proving and generalizing Hesselholt and Rains’ conjecture; second, to provide a comprehensive study of the Hochschild (co)homology of the non-Dynkin preprojective algebra and its algebraic structures over the integers (extending and explaining characteristic zero results from [CBEG07, EG06]). This includes an interesting BV algebraic structure defined over $\mathbb{Z}$, and new noncommutative structures.

We remark that the Hilbert series and algebraic structures attached to the Hochschild (co)homology in the Dynkin case have been studied in [EE07, En07] (over fields of characteristic zero), and is still in progress. We will explore the relations between the Dynkin and extended Dynkin cases in [ES].

1.1 $\mathbb{Z}$-module structure of preprojective algebras

Given any $g \geq 1$ (thought of as a genus) and any field $k$, one may consider the free algebra $P := \langle x_1 \ldots x_g, y_1 \ldots y_g \rangle$ and its quotient $\Pi := P/\langle r \rangle$ where $r = \sum_i [x_i, y_i]$. The algebra $\Pi$ is a special type of so-called preprojective algebra, and $r$ is the “Lie algebra” version of the relation for the fundamental group of a closed genus-$g$ curve. One may then consider the zeroth Hochschild homology of $\Pi$, $\Lambda := HH_0(\Pi) = \Pi/\Pi, \Pi$, which is an analogue of removing base-point (passing to free homotopy classes).

Over characteristic zero, the Hilbert series of $HH_0(\Pi) = \Pi/\Pi, \Pi$ was computed in [EG06] using matrix integrals. In characteristic $p$, with $g \geq 2$, however, it turns out that the Hilbert series increases slightly. Equivalently, considered over $k = \mathbb{Z}$, $HH_0(\Pi)$ has $\mathbb{Z}/p$-torsion. Motivated by noncommutative Witt theory, Hesselholt proposed in [Hes05] (cf. [Hes97]) certain elements of $HH_0(\Pi)$ that, if nonzero, are $p$-torsion and describe the kernel of his noncommutative “Verschiebung” maps $W_\ell(A) \to W_{\ell+1}(A)$ from $p$-typical noncommutative Witt vectors of length $\ell$ to those of length $\ell + 1$ (over any ring $A$), for any ring $A$. In particular, they show that the Verschiebung is not injective in many cases (which otherwise is difficult to show).
These elements of $HH_0(\Pi)$ are as follows: in the space $L := P/[P,P]$ over $\mathbb{Z}$, $[r^{p^\ell}]$ is a multiple of $p$ (since $p \mid \left((x_i y_i)^{p^\ell} + (-y_i x_i)^{p^\ell}\right)$). One may then consider the image of $[r^{p^\ell}]/p$ in $HH_0(\Pi)$, which must be $p$-torsion if nonzero. Call this element $r^{(p^\ell)} \in HH_0(\Pi)$. The precise connection between these elements and the Verschiebung and Frobenius maps of noncommutative Witt theory is outlined in Section 2.1.

Based on Hesselholt’s ideas and computer calculations of Rains (testing Hilbert series of $HH_0(\Pi)$ over positive characteristic), we have the

**Conjecture 1.1.1.** (Hesselholt and Rains) If $g \geq 2$, the elements $r^{(p^\ell)}$ are nonzero and span the $p$-th power torsion of $HH_0(\Pi)$, for $\Pi = \mathbb{Z}\langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle/((r)), r = \sum_{i=1}^{g} [x_i, y_i]$.

In particular, this claims that there is no $\mathbb{Z}/p^2$-torsion.

We answer positively and generalize this conjecture to the quiver case: free algebras over $\mathbb{Z}^I$ for any set of vertices $I$. In this case, $P$ and $\Pi$ are replaced by the path algebra and preprojective algebra of the quiver. The Rains-Hesselholt case is for a quiver with a single vertex and $g \geq 2$ loops. The correct generalization of $g \geq 2$ for quivers with multiple vertices is then that the quiver be non-Dynkin, non-extended Dynkin; in [EG06], the Hilbert series in characteristic zero was calculated in this generality, using that the non-Dynkin, non-extended Dynkin condition says precisely that $\Pi$ is a “representation complete intersection” (RCI), [EG06].

In more detail, for any quiver $Q$, we let $P$ be the path algebra of the double quiver $\overline{Q}$ (obtained by adding a reverse edge $e^*$ for every $e \in Q$), and let $\Pi$ be the preprojective algebra $\Pi = P/((r))$, where now $r = \sum_{e \in Q} ee^* - e^*e$. (The preprojective algebra $\Pi$ was originally defined by Gelfand and Ponomarev [GP79] in the study of quiver representations.) Then the Hesselholt-Rains conjecture generalizes to:

**Theorem 1.1.2.** If $Q$ is non-Dynkin and non-extended Dynkin, the elements $r^{(p^\ell)}$ are nonzero and span the $p$-torsion of $HH_0(\Pi_Q)$. There is no $p^2$-torsion.

We also present a much more general question (Question 4.1.9), based on conversations with P. Etingof, that asks whether, for finitely-presented graded algebras over $\mathbb{Z}$ (or $\mathbb{Z}^I$), for primes in which they are asymptotic RCI, the new torsion is spanned by elements of the above form, for relations $r$ which lie in the linear span of commutators modulo $p$.

The main strategy for the proof of the theorem is to fix an extended Dynkin subquiver $Q_0 \subseteq Q$, and use and develop facts about extended Dynkin quivers, and the relationship between $\Pi_Q$ and $\Pi_{Q_0}$. The proof is divided into three somewhat overlapping cases:

1. The case of primes $p$ which are good for the extended Dynkin quiver $Q_0$ (i.e., not a factor of the size of the corresponding finite group $\Gamma$ under the McKay correspondence) using the well-known Morita equivalence $\Pi_{Q_0} \simeq k[x,y] \rtimes \Gamma$;

2. The case when $Q_0$ is of type $A$ or $D$, which uses explicit integral bases for $\Pi_{Q_0}, \Pi_Q$ and their zeroth Hochschild homology that we work out in these cases (using the Diamond Lemma over a PID);

3. The remaining cases of $\tilde{E}_n$ and $p \leq 5$, which use the necklace Lie algebra [Gin01, BLB02] and a computation of zeroth Poisson homology in these cases, and $p$-th power maps. (We compute the zeroth Poisson homology for good primes in Section 10.4.)
In the process, we obtain integral, rational, and characteristic-\(p\) bases for \(\Pi_Q\) and \(HH_0(\Pi_Q)\). This relies on the decomposition \(\Pi_Q \cong \Pi_{Q_0} \ast_k \Pi_{Q\setminus Q_0}\), where \(I_0\) is the vertex set of the extended Dynkin subquiver \(Q_0\), and \(\Pi_{Q\setminus Q_0}\) was defined for a similar purpose in \cite{EE05}. For \(HH_0(\Pi_Q)\), we can then (essentially) write elements as cyclic words in \(HH_0(\Pi_{Q_0})\) and \(HH_0(\Pi_{Q\setminus Q_0})\).

The bases should be interesting in their own right. As one simple application of the bases for \(\Pi_Q\), we may deduce that \(\Pi_Q\) is torsion-free (which was proved in \cite{EE05} for non-Dynkin quivers using Gelfand-Kirillov dimension).

In the Dynkin and extended Dynkin cases, we also compute explicit bases for \(\Pi_Q, HH_0(\Pi_Q)\) and the torsion of \(HH_0(\Pi_Q)\), where it turns out that there is finite torsion, and the nonzero elements \(r(p^f)\) only occur in “stably bad primes”: none for \(A_n\); \(p = 2\) for \(D_n\), \(p \in \{2, 3\}\) for \(E_6, E_7, E_8\). The precise result is Theorem \[1.2.60\] (which refines a result of \cite{MOV06}, which established the cases in which \(HH_0(\Pi_Q \otimes \mathbb{F}_p)\) vanishes). Note that there is good reason why the torsion must be as described: it is possible to deduce this result from the proof of Theorem \[1.1.2\] as no other Hilbert series of \(HH_0(\Pi_Q) \otimes \mathbb{F}_p\) would be compatible with the refinement (Theorem \[1.2.30\]) of the main theorem. In view of this and also the nice properties and structure that the Hochschild (co)homology of \(\Pi_Q\) has in the extended Dynkin case, even in bad primes, one can say that \(\Pi_Q\) behaves well in all primes, and thus is a good \(\mathbb{Z}\)-replacement of \(k[x, y] \rtimes \Gamma\).

We then make sense of and prove the following formula (for any quiver):

\[
(r(p^f))^{p^m} = r(p^{f+m}),
\]

which we explain and interpret in terms of noncommutative Witt theory in Section \[2.1\]. The fact that the elements \(r(p^f)\) are nonzero is generalized in Theorem \[9.2.2\] to the statement that the \(p\)-th power maps are injective on all of \(HH_0(\Pi_Q) \otimes \mathbb{F}_p\).

### 1.2 The BV algebra on Hochschild cohomology

Using the above results, we compute the \(\mathbb{Z}\)-module structure of all of the Hochschild homology and cohomology groups for \(\Pi_Q\) when \(Q\) is non-Dynkin (including extended Dynkin), and describe the cyclic homology groups. This extends results of \cite{EG06} to the integral setting. To accomplish this, we need to use the necklace Lie algebra structure on \(\Lambda_Q\), which was defined in \cite{BLB02, Gin01} on the level of \(HH_0(P_Q)\). This was extended to a Lie bialgebra in \cite{Sch05}, and here we show that the Lie bialgebra descends to \(\Lambda_Q = HH_0(\Pi_Q)\) (Proposition \[5.1.7\]).

We then describe the higher Hochschild (co)homology (Corollary \[10.1.2\], which was done in the non-Dynkin, non-extended Dynkin case over fields of characteristic zero in \cite{CBEG07} (along with the interpretations and algebraic structures we will discuss, specialized to that case):

\[
HH^1(\Pi_Q \otimes k) \cong HH^0(\Pi_Q \otimes k) \oplus (HH_0(\Pi_Q)_{\text{free}}(-2) \otimes k) \oplus (HH_0(\Pi_Q)_{\text{tor}}(-2) \otimes \bigoplus_{p \text{ prime}} \text{Hom}_\mathbb{Z}(\mathbb{F}_p, k)),
\]

\[
HH^2(\Pi_Q \otimes k) \cong k \otimes HH_0(\Pi_Q)(-2),
\]

\[
HH^m(\Pi_Q \otimes k) = HH_m(\Pi_Q \otimes k) = 0, \text{ if } m \geq 3,
\]

where, by Theorem \[10.1.1\] the center \(HH^0(\Pi_Q \otimes k)\) is either \(k\) (if \(Q\) is non-Dynkin, non-extended Dynkin), or isomorphic to \(HH^0(\Pi_Q)_{\text{free}} \otimes k\) (when \(Q\) is Dynkin).

\[1\] The Lie algebra direction was proved differently in \cite{CBEG07}.
The summands of $HH^1(\Pi_Q \otimes k)$, as derivations, are given (Section 10.2) by multiples of the (half-)Euler vector field by $HH^0(\Pi_Q \otimes k)$, a lifted adjoint action of the necklace Lie bracket on $HH_0(\Pi_Q \otimes k)$, and derivations that only make sense modulo $p$ (in the extended Dynkin case, these are fractional-$HH^0(\Pi_Q \otimes k)$-multiples of the half-Euler vector field).

Then, $HH^2(\Pi_Q \otimes k) \cong HH_0(\Pi_Q \otimes k)(-2)$ has a simple interpretation in terms of deformed preprojective algebras (Proposition 10.2.1).

We describe the Gerstenhaber algebra structure of the above, where the necklace Lie algebra appears in the cup product $HH^1 \otimes HH^1 \to HH^2$, and in the Gerstenhaber bracket restricted to $HH^1$. In fact, the above structure is part of a Batalin-Vilkovisky (BV) algebra structure, which we describe in Theorem 10.3.1. This follows since, as is well-known, $\Pi$ to $HH$ the dimension of “primitive parts” of $q$ for some constants.

1.3 New algebraic structures and the McKay correspondence

We discuss new non-commutative structures on $\Pi_Q, P_{Q}$ and its homologies, which have served as motivating examples for $Gsb, Gsa$. First, in turns out that the Lie co-bracket on $HH_0(\Pi_Q)$ lifts to a map
\[ \delta_t : P_{\tilde{Q}} \to HH_0(P_{\tilde{Q}}) \otimes P_{\tilde{Q}} \] satisfying a curious noncommutative BV identity:

\[ \delta_t(ab) = \delta_t(a)(1 \otimes b) + (1 \otimes a)\delta_t(b) + (pr \otimes 1)\langle a, b \rangle, \tag{1.3.1} \]

where \( \langle , \rangle : P_{\tilde{Q}} \otimes P_{\tilde{Q}} \to P_{\tilde{Q}} \otimes P_{\tilde{Q}} \) is the Van den Bergh double Poisson bracket \([\text{VdB04}]\). The origin of this identity and more general theory is explained in \([\text{GSb}]\). Here, we use (1.3.1) to prove that \( \Lambda_Q \) is a Lie bialgebra (Proposition 5.1.7), and that the cobracket is trivial on \( \Lambda_Q \otimes \mathbb{Q} \) when \( Q \) is extended Dynkin (Theorem 9.1.1).

Second, we explain that \( P_Q \) is a “free product” deformation of \( \Pi_Q \) in the non-Dynkin case; and furthermore, in the extended Dynkin case, this “quantized” the double bracket. Specifically, for any non-Dynkin quiver \( Q \), we show that the fact that \( \Pi_Q \) is a “noncommutative complete intersection” (NCCI) amounts to the fact that the path algebra \( P_Q \), as a graded \( \mathbb{Z} \)-module, is just a free product of \( \Pi_Q \) with the single parameter \( r \) (just the relation). (This fact is central to this paper since, for any \( \tilde{Q} \supseteq Q \), we may make a similar statement decomposing \( \Pi_{\tilde{Q}} \) as a free product of \( \Pi_Q \) and a “partial preprojective algebra” \([\text{EE05}]\) \( \Pi_{\tilde{Q} Q, r} \)).

To explain, recall \([\text{CBEG07}]\) that the deformed preprojective algebras \( \Pi_Q \) have the form

\[ P_Q[t]/((r - t \cdot f)), \tag{1.3.2} \]

for some \( f \in \Pi_Q \) (which really only depend on \( [f] \in HH_0(\Pi_Q) \approx HH^2(\Pi_Q) \) up to equivalence). The case \( |f| = 0 \), where \( f \) is just a linear combination of vertices, has been extensively studied in the literature for many years. We may replace \( P_Q[t] \) in (1.3.2) by \( P_Q \ast \mathbb{Z}[t] \), thus yielding a “free product” deformation whose quotient is a usual modified preprojective algebra. The flatness is then just the aforementioned fact that \( \Pi_Q \) is an NCCI. Note that, by Proposition 10.2.1 all usual deformations arise from a free product deformation by the above construction, followed by setting \( t \) to be central. For more details, see \([\text{GSa}]\).

Here, we prove (and exploit) the suggestive identity, for \( z_1, z_2 \in HH_0(\Pi_Q \otimes \mathbb{k}) \) with \( Q \) extended Dynkin, and \( \tilde{z}_1, \tilde{z}_2 \in P_Q \otimes \mathbb{k} \) arbitrary lifts:

\[ [z_1, z_2] \equiv \{ z_1, z_2 \} \cdot r \mod [(r)^2 + [(r), P_Q]], \tag{1.3.3} \]

which says that the free product (and hence also usual) deformations of \( \Pi_Q \) are generated, in some sense, by its Poisson bracket (given by the necklace bracket on \( HH_0(\Pi_Q) \) and the BV structure explained in the previous subsection). This is important for this paper since this includes the case \( \Pi_Q \) for quivers \( \tilde{Q} \supseteq Q \).

In terms of the McKay correspondence, we prove (Theorem 9.1.1) that the Poisson algebra

\[ HH^0(\Pi_Q \otimes \mathbb{C}) \]

is isomorphic to \( \mathbb{C}[x, y]^\Gamma \) where \( \Gamma \subset SL_2(\mathbb{C}) \) is the associated finite group. This is already well-known on the level of associative algebras. The proof works for not just \( \mathbb{C} \) but any commutative ring \( \mathbb{k} \) containing \( \frac{1}{\Gamma} \) and \( |\Gamma| \)-th roots of unity.

Then, the previous example just becomes the following: \( \mathbb{C}[x, y] \times \Gamma \) may be deformed by \( \mathbb{C}[x, y] \times \Gamma \), or more generally by \( \mathbb{C}[x, y, t]/(((xy - yx) - t \cdot f) \), which “quantizes” the Poisson bracket on \( \mathbb{C}[x, y]^\Gamma \) when one restricts the commutator to the preimage of the \( \mathbb{C}[x, y]^\Gamma \) under \( \mathbb{C}[x, y] \times \Gamma \rightarrow \mathbb{C}[x, y] \times \Gamma \).

1.4 Other results Other important results we did not already mention include:

- We compute the Poisson center of the algebras \( Sym HH_0(\Pi_Q) \) for any quiver \( Q \), over any base ring (Theorem 9.2.2), which generalizes Theorem 8.6.1 from \([\text{CBEG07}]\) to the extended
Dynkin case and to arbitrary characteristic. Over $\mathbb{Z}$, this center is generated by $HH_0(\Pi_Q)[0]$ and by torsion $(r^{(p^f)})$ in the non-Dynkin, non-extended Dynkin case, as well as products of $p$-torsion and $p$-th powers. Over $\mathbb{F}_p$, it is generated by $p$-th powers, $p$-torsion, and $HH_0(\Pi_Q)[0]$. The $p$-th powers must include $p$-th power operations both on the symmetric algebra and in $HH_0(\Pi_Q) \otimes \mathbb{F}_p$ itself (the latter are “noncommutative” $p$-th power operations that we have already mentioned). Along the way, we also prove that the $p$-th power maps are injective (except on torsion in the extended Dynkin or Dynkin cases). These results are employed in the computation of Hochschild cohomology and cyclic homology of $\Pi_Q$. We generalize these results to the partial preprojective case, where one must also include powers of $r$ in the center.

- We give explicit formulas for the Poisson algebra $HH^0(\Pi_Q) \cong \mathbb{Z}[\Pi_Q]$ when $Q$ is extended Dynkin, and $i_0$ is an extending vertex of $Q$ (Propositions 6.4.2, 7.4.1, and 8.4.1) and describe the zeroth Poisson homology for bad primes for $\tilde{E}_n$ (Proposition 8.4.8), and for good primes in the general case in Section 10.4.

1.5 Outline of paper First, in Section 2 we explain the connections to noncommutative Witt theory and briefly discuss the close relationship between the necklace Lie algebra and Goldman/Turaev’s algebra of loops.

Then, in Section 3 we prove Rains and Hesselholt’s conjecture, and its generalization (the main Theorem 1.1.2) for good primes.

Next, in Section 4 we explain in detail the (suggestive) Hilbert series formulas resulting from Theorem 1.1.2 and describe bases in detail. In the process, we give a simple proof that the elements $r^{(p^f)}$ are all nonzero (Proposition 4.2.37), which is all that was needed for the noncommutative Witt theory. We also generalize Theorem 1.1.2 by describing bases in the $\tilde{A}_n, \tilde{D}_n$ cases, and over $\mathbb{F}_p$ using prime powers, which complete steps (2) and (3) of the proof. The proof of these more general results are completed in the next four sections.

In Section 5 we define and generalize the Lie bialgebra $\Lambda_Q$, obtained as a quotient of the necklace Lie algebra. We explain that it is actually a Poisson algebra in the extended Dynkin case, modulo torsion (by identifying $\Lambda_Q$ with the center of $\Pi_Q$, which is a commutative algebra). We explain how $P_Q$ is a “free product” deformation of $\Pi_Q$, and prove that in the extended Dynkin case it quantizes the Poisson bracket (1.3.3).

In Sections 6, 7, 8 we prove Theorems 4.2.40, 4.2.48, and 4.2.30 using explicit computations in the for the $\tilde{A}_n, \tilde{D}_n$, and $\tilde{E}_n$ cases. This completes the proof of the main Theorem 1.1.2.

In Section 9 we prove Theorem 9.1.1 for an extended Dynkin quiver $Q$, one has $\Lambda_Q \otimes \mathbb{C} \cong \mathbb{C}[x,y]^\Gamma$ with cobracket equal to zero. We then prove Theorem 9.2.2 on the Poisson center of Sym $\Lambda_Q,J$ (for any quiver $Q$ and set of vertices $J$) and the injectivity of the $p$-th power maps.

Finally, in Section 10 we compute the higher Hochschild (co)homology of $\Pi_Q \otimes k$ for non-Dynkin quivers $Q$ and any commutative ring $k$, their interpretation in terms of outer derivations and deformations, the BV algebra structure, and the cyclic homology groups $HC_*(\Pi_Q)$.

In the appendix, we give a version of the Diamond Lemma for PIDs (which we use to compute bases), and define bi, co, and Poisson versions of Loday algebras.

1.6 Notation and Definitions Throughout, we will define the operator $\tau$ to permute components of tensor products. Namely, if $\sigma \in \Sigma_n$, then $\tau_\sigma : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes V_{\sigma^{-1}(2)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ is the result of applying the permutation $\sigma$. 

8
We will take a quiver $Q$ to mean a finite, directed, connected graph, allowing arbitrary multiplicity of edges. Connected here means topologically connected (i.e. this does not depend on the direction of edges). The set of edges will be denoted by the same letter, $Q$, as is standard in the literature. Another letter, usually $I$, will denote the set of vertices. The double $\overline{Q}$ of $Q$ adds a reverse edge for every edge, so that $\#(\overline{Q}) = 2\#(Q)$. The path algebra $kQ$ for a quiver over $k$ (commutative ring or field; usually $\mathbb{Z}$ or $\mathbb{Q}$) is the $k$-module spanned by paths in the quiver (including zero-length paths, namely single vertices), whose multiplication is concatenation. The product $p_1p_2$ of two paths is zero if $p_1$ does not terminate at the same vertex at which $p_2$ begins.

For an edge $e \in Q$, the reverse edge is denoted $e^*$, and we also use the notation $(e^*)^* := e$. If an edge $e$ goes from vertex $i \in I$ to $j \in I$, we say $e : i \to j$, and set $e_s = i; e_t = j$ ("s"="source", "t"="target").

Letting $k^I = \bigoplus_{i \in I} k$ denote the ring spanned by $I$, with product $ij = \delta_{ij} i$, we can consider $\langle Q \rangle$ to be a $k^I$-bimodule with multiplication $iej = \delta_{ie}\delta_{je}e$. One has $kQ = T_k^I Q$.

Finally, we need some further notation throughout:

**Notation 1.6.1.** The "length" of a path in a quiver is the number of edges in the path. Similarly, the length of a line segment of edges is the number of edges in the segment.

**Notation 1.6.2.** If $I_0 \subset I$, and $k$ is any commutative ring, then let $1_{I_0} \in k^I$ denote the matrix which is 1 in entries $(i_0, i_0)$, $\forall i_0 \in I_0$, and zero elsewhere. In terms of idempotents, we have $1_{I_0} = \sum_{i_0 \in I_0} i_0$.

**Notation 1.6.3.** When considering a quotient $A/B$, we will not assume $B$ is an ideal (so, we take the quotient as graded modules by default). The ideal generated by any space will be denoted by placing the space in double parentheses: $\langle r \rangle = \langle r \rangle$ the ideal generated by $r$.

**Notation 1.6.4.** Throughout, we use $F \wedge G := F \otimes G - G \otimes F$.

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2 Relations to Witt theory and Goldman/Turaev’s algebra of loops

2.1 Relation to noncommutative Witt theory In [Hes97], [Hes05], Hesselholt defined $p$-typical Witt vectors $W(A)$ for any noncommutative ring $A$, and the vectors of a given length $\ell$, $W_\ell(A)$. In the case where $A$ is commutative, this reduces to the usual Witt vectors [Wit37].

Hesselholt also generalized the Frobenius ($F$), Verschiebung ($V$), and Restriction ($R$) operators to the noncommutative setting. One has $F, R : W_\ell(A) \to W_{\ell-1}(A)$ and $V : W_\ell(A) \to W_{\ell+1}(A)$.

While, in the commutative case, the Verschiebung operator (which essentially has the form $V(a_0, a_1, \ldots, a_{\ell-1}) = (0, a_0, a_1, \ldots, a_{\ell-1})$) is always injective, it turned out this may not be the case in the noncommutative setting. Hesselholt found exact sequences,

$$HH_1(A) \xrightarrow{\delta^\ell} W_\ell(A) \xrightarrow{\nu^\ell} W_{\ell+1}(A), \quad (2.1.1)$$

which are functorial. In particular, if $\sum_{i=1}^n [x_i, y_i] = 0$ for some $x_i, y_i \in A$, one may consider the map $\phi : \Pi \to A$ mapping $x_i \mapsto \bar{x}_i, y_i \mapsto \bar{y}_i$; then one
has a commutative diagram with exact rows,

\[
\begin{array}{ccc}
HH_1(\Pi) & \xrightarrow{\delta_1} & W_\ell(\Pi) & \xrightarrow{V} & W_{\ell+1}(\Pi) \\
\downarrow{\phi} & & \downarrow{\phi} & & \downarrow{\phi} \\
HH_1(A) & \xrightarrow{\delta_1} & W_\ell(A) & \xrightarrow{V} & W_{\ell+1}(A),
\end{array}
\] (2.1.2)

so that \(\phi \circ \delta_{\ell}[\sum_i x_i \otimes y_i] = \delta_{\ell}[\sum_i \bar{x}_i \otimes \bar{y}_i]\). Hence, the kernel of the Verschiebung map is the span of all elements that can be obtained as the image of the classes \(\delta_{\ell}[\sum_i x_i \otimes y_i]\) under algebra maps \(\Pi \to A\) for all choices of \(n\).

Furthermore, one has a commutative diagram with exact rows,

\[
\begin{array}{ccc}
HH_1(\Pi) & \xrightarrow{\delta_1} & W_\ell(\Pi) & \xrightarrow{V} & W_{\ell+1}(\Pi) \\
\downarrow{\phi} & & \downarrow{\phi} & & \downarrow{\phi} \\
HH_1(A) & \xrightarrow{\delta_1} & W_\ell(A) & \xrightarrow{V} & W_{\ell+1}(A),
\end{array}
\]

so that \(\phi \circ \delta_{\ell}[\sum_i x_i \otimes y_i] = \delta_{\ell}[\sum_i \bar{x}_i \otimes \bar{y}_i]\). Hence, the kernel of the Verschiebung map is the span of all elements that can be obtained as the image of the classes \(\delta_{\ell}[\sum_i x_i \otimes y_i]\) under algebra maps \(\Pi \to A\) for all choices of \(n\).

2.1.1 \(p\)-th power maps and Lie structure of torsion elements

If \(A\) is any commutative ring over characteristic \(p\), one has the well-known formula \((a + b)^p = a^p + b^p\): that is, the \(p\)-th power map is additive. This is no longer true for noncommutative rings, but it is easy to check that the result does still hold in \(A/[A, A]\). That is, \(x^p + y^p - (x + y)^p \in [A, A]\). (In fact, Jacobson discovered in the 1940’s that \(x^p + y^p - (x + y)^p\) can be expressed as a linear combination of successive commutators of \(x\) and \(y\).)

For this section we will let \(Q\) be any quiver. Since \(L = P/[P, P]\) and \(\Lambda = \Pi/[\Pi, \Pi]\), the \(p\)-th power maps \(p : L \otimes F_p \to L \otimes F_p, \Lambda \otimes F_p \to \Lambda \otimes F_p\) are well-defined by the formula \([f]^p := [fp]\) for any \(f \in P\). It thus makes sense to compute the \(p\)-th powers \((r(p^\ell))^p\).

Expanding \(r(p^\ell)\) modulo \(p\), one has

\[
r(p^\ell) = \sum_{(e_1, e_2, \ldots, e_p)}^{(e_1, e_2, \ldots, e_p)} s(e_1, \ldots, e_p)[(e_1 e_1^* \cdots e_p e_p^*)^\ell - 1],
\] (2.1.3)

\[
s(e_1, \ldots, e_p) := \omega(e_1, e_1^*) \cdots \omega(e_p, e_p^*),
\] (2.1.4)

where the ‘ here denotes summing over all distinct cyclic \(p\)-tuples \((e_1, e_2, \ldots, e_p)\) excepting \(p\)-tuples of the form \(e_1 = e_2 = \cdots = e_p\) (although these would cancel anyway).

Hence, one immediately deduces the

**Proposition 2.1.5.** One has \((r(p^\ell))^p = r(p^{\ell+1})\) (which makes sense over any commutative ring \(k\) of characteristic \(p\)).

The Witt-theoretic meaning of this is as follows (cf. Section 2.1): We should think of \(\Pi_Q\) as the “universal” case of the map \(\delta_{\ell} : HH_1(A) \to W_n(A)\) whose image is the kernel of the \(n\)-th Verschiebung map \(V : W_n(A) \to W_{n+1}(A)\), for algebras \(A\) over the semisimple ring \(Z^f\) (or \(k^f\) if we tensor by a field or commutative ring \(k\)).

From Proposition 2.1.5 Hesselholt pointed out that one may deduce \(F^{\ell-1}\delta_{\ell}(\sum_i x_i \otimes y_i) = r(p^\ell) \in W_1(\Pi) = \Pi/[\Pi, \Pi]\)

Then, we immediately deduce from Theorem 1.1.2 the following:

**Proposition 2.1.6.** For any non-Dynkin, non-extended Dynkin quiver \(Q\), the image of the element \(\sum_{e \in Q} e \otimes e^* \in HH_1(\Pi_Q)\) under \(F^{\ell-1}\delta_{\ell}\) is \(r(p^\ell) \in W_1(\Pi_Q) = \Pi_Q/[\Pi_Q, \Pi_Q]\). In particular, none of the higher Verschiebung maps \(W_\ell(\Pi_Q) \to W_{\ell+1}(\Pi_Q)\) are injective.
This only requires one direction of Theorem 4.2.2, that the elements \( r(p^k) \) are nonzero, which is proved in Proposition 4.2.37 as a step in the proof of the theorem.

Hence, Hesselholt and Rains’ conjecture 1.1.1 simply says that, in the “universal” case of \( \Pi \), none of the Verschiebung maps are injective, but have kernels which map to \( r(p^k) \) under the \( \ell - 1 \)-th power of the Frobenius; and for an arbitrary algebra \( A \), the kernel of the Verschiebung map is just the span of the images of these elements \( \delta_i(\sum x_i \otimes y_i) \). In the case \( \ell = 1 \), the kernel of the first Verschiebung map is the space of all elements \( \frac{1}{p}((\sum_i[x_i, y_i])^p) \in W_1(A) = A/\{A, A\} \), such that \( \sum_i[x_i, y_i] = 0 \) in \( A \).

For Dynkin and extended Dynkin quivers, the torsion is finite and listed in Theorem 4.2.60 (and only occurs in “stably bad” primes, as noted in Section 1.1). We thus deduce the

**Proposition 2.1.7.** Let \( A \) be a \( \mathbb{Z}^l \)-algebra and \( p \) a fixed prime. Take any Dynkin or extended Dynkin diagram \( Q \), and any element \( X := \{ \sum_{e \in Q} x_e \otimes y_e \} \in HH_1(A) \) with \( x_e \in iA\ell, y_e \in jA\ell \) for \( e : i \to j \). Then, the map \( F^{\ell - 1} \delta_\ell \) kills \( X \) if \( r(p^k) = 0 \), i.e., if \( r(p^k) \) is not one of the nonzero torsion elements listed in Theorem 4.2.60.

For example, \( F^{\ell - 1} \delta_\ell(X) \) is zero if \( p > 5 \) or \( Q \) is of type \( A \) (or for large enough \( \ell \) for any fixed \( Q \)).

### 2.2 Relation between \( \Lambda \) and Goldman/Turaev’s Lie algebra

In the case where \( Q \) is a quiver with only one vertex and \( g \) loops, \( \Pi := k\langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle / (\sum_{i=1}^{g}(x_1y_i - y_iy_1)) \), and so \( \Lambda = \Pi/[[\Pi, \Pi] \) can be thought of as an analogue of the free homotopy classes of loops on a surface of genus \( g \). In [Gol84], a Lie algebra structure was defined on the vector space with basis the free homotopy classes of loops, and in [Tur91] it was given a Lie bialgebra structure and quantized. If we replace \( \Pi \) with the path algebra \( \overline{\Pi} \) of the double quiver, then the analogous Lie algebra structure was defined (independently) in [Gin01] and the Lie bialgebra structure was defined in [Sch05]. One of the results of this paper (Section 3) shows that the above Lie bialgebra structure descends to \( \Lambda \). In this section, we explain the analogy between quivers and surfaces in more detail.

In the case \( g = 1 \), we have the quiver \( A_0 \); so we suggest that one should think of extended Dynkin quivers as the noncommutative generalizations of closed genus-one surfaces, and the non-Dynkin, non-extended Dynkin quivers as generalizations of higher genus surfaces, or surfaces with punctures.

As a special case, to define an analogue of the genus-\( g \) surface with \( m \) punctures \( p_1, \ldots, p_m \), one may consider the algebra \( \overline{\Pi} := k\langle x_1, \ldots, x_g, y_1, \ldots, y_g, p_1, \ldots, p_m \rangle / (\sum_{i=1}^{g}(x_1y_i - y_iy_1) + \sum_{i=1}^{m} p_i) \), which one may show can be given a Lie structure just like our space. One way to see this is to think not of adjoining loops \( p_1, \ldots, p_m \) to the double quiver \( \overline{Q} \), but rather to think of adjoining \( m \) infinite rays to the quiver \( Q \), all beginning at the initial vertex and going off to infinity. That is, \( \overline{Q} \) is the disjoint union of a star with \( m \) infinite branches and a collection of \( g \) loops at the node of the star. One may see that this makes sense: even though the element \( r \) is no longer defined, the elements \( jrj \) for length-zero paths (vertices) \( j \) are still defined, and we can replace the ideal \( \langle r \rangle \) by the ideal generated by all such \( jrj \). We will show in Proposition 2.2.10 in the next section that \( \Lambda \) for this quiver indeed has the desired form.

In fact, Proposition 2.2.10 has the more general Corollary 2.2.15 if we take the quiver \( Q \) obtained by attaching \( g \) loops and \( m \) line segments of lengths \( \ell_1, \ldots, \ell_m \) (which we allow to be
infinite, thus giving infinite rays described above) to a single vertex $i_s \in I$, then

$$i_s \Pi_Q i_s \cong A := k\langle x_1, \ldots, x_g, y_1, \ldots, y_g, p_1, \ldots, p_m \rangle / \left( \sum_{i=1}^{g} [x_i, y_i] + \sum_{i=1}^{m} p_i p_j^{-1} \right)_{j \in \{1, \ldots, m\}}, \quad (2.2.1)$$

$$\Lambda_Q \cong A / \left( [A, A] + \langle p_j^{-1} \rangle_{j \in \{1, \ldots, m\}, \ell \geq 1} \right), \quad (2.2.2)$$

where by definition we set $p_j^\infty = 0$ (when $\ell_j = \infty$). Thus, we see that the $\Lambda_Q$ for the above quiver is the additive analogue of the free homotopy classes on a surface of genus $g$ with $m$ elliptic points (or punctures) of orders $\ell_1 + 1, \ldots, \ell_m + 1$. In particular, $\Lambda_{\tilde{E}_6}, \Lambda_{\tilde{E}_7}, \text{and } \Lambda_{\tilde{E}_8}$ are the additive analogues of free homotopy classes on the Euclidean orbifolds modeled on a sphere with three elliptic points, of orders $3, 3, 3$ (for $\tilde{E}_6$), $2, 4, 4$ (for $\tilde{E}_7$), and $2, 3, 6$ (for $\tilde{E}_8$). This is exactly as expected, since these are the orbifolds obtained by quotienting the sphere by the group of tetrahedral, octahedral, or icosahedral rotation groups.

In general, the extended Dynkin quivers are the additive analogues of orbifold surfaces of zero Euler characteristic, while the non-Dynkin, non-extended Dynkin quivers are additive analogues of hyperbolic orbifolds (negative Euler characteristic).

One may also consider the statement “the Dynkin quivers are the additive analogues of the spherical orbifolds (positive Euler characteristic)”, although as we will see, $\Lambda_Q$ is a finite abelian group for such quivers (in particular $\Lambda_Q \otimes \mathbb{Q} = 0$), so one cannot obtain much by studying them in this way. It is still interesting to consider all of the higher Hochschild homology for the Dynkin case ([ablo7] for the rational case), which is more complicated than the non-Dynkin case: in particular, (at least over $\mathbb{Q}$) it is periodic of period six, while the non-Dynkin preprojective algebras have global and Hochschild dimension equal to two. We will not discuss the higher Hochschild homology of Dynkin quivers in this paper (in the integral case, this is not yet known, and it would be interesting to study in a future paper). For non-Dynkin quivers, as we will compute, the higher Hochschild homology and cohomology and all of its structure is controlled by the Lie algebra $\Lambda_Q$.

To make the above analogy more precise, one may introduce, for any orbifold $M$ of dimension two with fundamental group $G$, the group algebra $\mathbb{C}[G]$. Then, by [Vai07], Goldman’s Lie algebra appears by studying the Hochschild (co)homology of $G$ (just as $\Lambda$ appears in the Hochschild (co)homology of $\Pi_Q$). Furthermore, as pointed out by P. Etingof, if $i$ is the special vertex of the above quiver $Q$ (where everything is attached), then the pro-nilpotent completion of $i\Pi i$ is isomorphic to $\mathbb{C}[\hat{G}]$ where $\hat{G}$ is the pro-unipotent completion of $G$. It would be interesting to use this to compare the Turaev/Goldman algebra with the necklace Lie algebra.

Finally, it should be noted that the Lie structures of $H H_0(\Pi), H H_0(\mathbb{C}[G])$ discussed above can be explained by studying the Hochschild cohomology: we will show explicitly in Section 10 that the Lie bracket is closely related to the Gerstenhaber bracket on $H H^1(\Pi)$, and still more closely related to the cup product $H H^1(\Pi) \otimes H H^1(\Pi) \rightarrow H H^2(\Pi)$ (and can be obtained from the latter). This is only meaningful for non-Dynkin quivers since $H H_0(\Pi_Q) = 0$ when $Q$ is Dynkin.

In summary, studying the Lie algebra and Hilbert series of $\Lambda$ can be thought of as providing information about the structure of noncommutative surfaces, or of an additive analogues of usual surfaces. In this paper, we concern ourselves with studying the structure of $\Lambda$ in finite characteristic, and of higher Hochschild groups for non-Dynkin quivers.

### 2.2.1 Partial preprojective algebras and quivers with attached line segments

In this section, we explain the related notion of partial preprojective algebras, which we also need throughout this paper.
Definition 2.2.3. [EE05] Given any quiver $Q$ on edge set $I$, and any subset $I_0 \subset I$ of vertices (called the white vertices, with $I \setminus I_0$ the black vertices), define the partial preprojective algebra $\Pi_{Q,I_0}$ by (cf. Notation 1.6.2):

$$\Pi_{Q,I_0} := P_{\overline{Q}} / \langle (1_{I_0}, 1_{I_0}) \rangle.$$  \hfill (2.2.4)

Definition 2.2.5. Define $\Lambda_{Q,I_0} := HH_0(\Pi_{Q,I_0}).$

One way to interpret the partial preprojective algebra is as follows: Take the underlying quiver and adjoin, at each white vertex $i \in I_0$, an infinite ray based at $i$, to form an extended quiver $\hat{Q}_{I_0,\infty}$ Then, the projection $1_I \Pi_{\hat{Q}_{I_0,\infty}} 1_I$ of the preprojective algebra of this extended quiver to $I$ is just the partial preprojective algebra $\Pi_{Q,I_0}$. Precisely,

Definition 2.2.6. For any map $f : I_0 \to \mathbb{Z}_{\geq 0}$, let $\hat{Q}_{I_0,f}$ be the quiver obtained from $Q$ by attaching a segment of length $f(i)$ to each $i \in I_0$ (i.e., the segment has $f(i)$ edges and $f(i)$ vertices).

Definition 2.2.7. For any two functions $f, g : I_0 \to \mathbb{Z}_{\geq 0}$, let $f \leq g$ denote $f(x) \leq g(x)$ for all $x \in I_0$.

Note that, for any $f \geq g$, one has a surjection $\Pi_{\hat{Q}_{I_0,f}} \twoheadrightarrow \Pi_{\hat{Q}_{I_0,g}}$. One may consider the inverse limit $\varprojlim_f \Pi_{\hat{Q}_{I_0,f}}$. We may think of this limit as $\hat{\Pi}_{Q,I_0}$, where $\hat{Q}_{I_0,\infty}$ is the “infinite quiver” described above: this formula is correct if one suitably modifies the definition of preprojective algebra to account for the fact that $r$ no longer exists (it is an infinite sum). Note that Hochschild homology does not commute with the above inverse limit.

One has the following result, which compares the partial preprojective algebra and its zeroth Hochschild homology to the above inverse limit:

Proposition 2.2.8. We have the following natural isomorphisms:

(i) $\Pi_{Q,I_0} \cong \varprojlim_f 1_I \Pi_{\hat{Q}_{I_0,f}} 1_I.$

(ii) $\varprojlim_f HH_0(\Pi_{\hat{Q}_{I_0,f}}) \cong \Lambda_{Q,I_0} / \langle rl^f i \rangle_{i \in I_0, r \geq 1}.$

Proof. (i) This follows from the formula

$$1_I \Pi_{\hat{Q}_{I_0,f}} 1_I \cong \Pi_{Q,I_0} / \langle (r^f(i))i \in I_0 \rangle. \hfill (2.2.9)$$

(ii) Let $\hat{l}_{I_0,f}$ be the vertex set of $\hat{Q}_{I_0,f}$. By the following Proposition 2.2.10, one has an isomorphism $HH_0(1_I \Pi_{\hat{Q}_{I_0,f}} 1_I) / \langle [a_s^f], [i] \rangle_{s \in \hat{l}_{I_0,f}, i \in I_0} \cong HH_0(\Pi_{\hat{Q}_{I_0,f}})$. Here $a_s$ is a loop of length two, obtained by beginning at $i_s$, traversing one edge in $Q_{I_0,f} \setminus \hat{Q}$, and then its reverse back to $i_s$. If we now assume that $f$ is such that $f(i) \geq N$ for all $i \in I_0$, then in degrees $m < 2N$, we evidently have $HH_0(1_I \Pi_{\hat{Q}_{I_0,f}} 1_I)[m] \cong HH_0(\Pi_{Q,I_0})[m]$, using the isomorphism (2.2.9). Under this isomorphism, the image of $a_s^f$ is $i_s r^f i_s$, for any $i_s \in I_0$. Now taking the limit $N \to \infty$ gives the result. \qed
In the above proposition, we used the following more general result for quivers \( \hat{Q} \) obtained from \( Q \) by attaching any number of line segments to its vertices (not limiting to one line segment per vertex). In particular, this explains why adding infinite rays to a vertex is like adding “punctures” to the corresponding “surface” (Section 2.2).

**Proposition 2.2.10.** Let \( Q \) be any quiver on vertex set \( I \) and \( \hat{Q} \supset Q \) a quiver obtained from \( Q \) by attaching some line segments \( L_s, s \in \{1, \ldots, m\} \) of lengths \( d_s \) to vertices \( i_s \in I \): by this we mean \( L_s \cap Q = \{i_s\} \) for any \( s \), and \( L_s \cap L_{s'} = \{i_s\} \cap \{i_{s'}\} \). For each line segment \( L_s \), let \( a_s \) be the loop in \( \hat{Q} \) of length two which begins at \( s \), travels one edge along the segment \( L_s \), and then takes the reverse edge back to \( s \). Then, we have:

(i) The inclusion \( 1_I \hat{\Pi}_Q 1_I \hookrightarrow \hat{\Pi}_Q \) induces a surjection

\[
HH_0(1_I \hat{\Pi}_Q 1_I) \twoheadrightarrow HH_0(\hat{\Pi}_Q)
\]  

(2.2.11)

with kernel \( \langle [a_s^\ell]\rangle_{s \in \{1, \ldots, m\}, \ell \geq 1} \). As a consequence, the inclusion induces an isomorphism

\[
HH_0(\hat{\Pi}_Q) \cong HH_0(1_I \hat{\Pi}_Q 1_I)/\langle [a_s^\ell]\rangle_{s \in \{1, \ldots, m\}, \ell \geq 1}.
\]  

(2.2.12)

(ii) One obtains isomorphisms

\[
HH_0(\hat{\Pi}_Q) \cong HH_0(1_I \hat{\Pi}_Q 1_I)/\langle [p_s^\ell]\rangle_{s \in \{1, \ldots, m\}, \ell \geq 1}, \text{ where } B = \bigoplus_{i \in I_0} B_i, \quad B_i = k(p_s)_{s|i_s=i}/\langle [p_s^d_{l_s}]\rangle_{s \in \{1, \ldots, m\}, l_s \geq 1} \otimes k \langle i \rangle.
\]  

(2.2.13)

Note that it is easy to combine Propositions 2.2.8 and 2.2.10 to describe the case of a quiver with some finite and some infinite line segments added. In this direction, we only state the promised

**Corollary 2.2.15.** Let \( Q \) be the quiver obtained by beginning with one vertex \( i_s \), and attaching \( g \) loops, and \( m \) line segments of lengths \( \ell_1, \ldots, \ell_m \) (allowing for \( \ell_j = \infty \)). Let \( I \) be the vertex set, so that \( i_i \) is a well-defined finite sum for each \( i \in I \). Then, defining \( \Lambda_Q = HH_0(\Pi_Q) \) where \( \Pi_Q = P_Q/((ir_i)_{i \in I}) \), one obtains formulas (2.2.11) and (2.2.2), where we set by definition \( p^\infty_{j_i} : = 0 \) (for any \( j \)). Thus, \( \Lambda_Q \) is the “additive analogue of the orbifold surface of genus \( G \) with orbifold/puncture points \( p_1, \ldots, p_m \) of orders \( \ell_1 + 1, \ldots, \ell_m + 1 \).”

The corollary easily follows from part (ii) of Proposition 2.2.10 using the argument of the proof of Proposition 2.2.8.

The rest of this section is devoted to the proof of Proposition 2.2.10, which the reader may safely skip.

**Proof of Proposition 2.2.10.** (i) First, we show surjectivity in (2.2.11). Any path in \( P_Q \) which lies entirely in one of the line segments \( L_s \) (for any fixed \( s \)) projects to \( HH_0(\hat{\Pi}_Q) \). Then, any other path projects to the same element as a path that begins and ends at a vertex in \( I_0 \). This proves the surjectivity. Also, it is clear that \( [a_s^\ell] \) is in the kernel of (2.2.11) for any \( \ell \geq 1 \) and \( s \in \{1, \ldots, m\} \), so it remains to show that the resulting map (2.2.12) is injective (and hence an isomorphism).

To do this, we construct an explicit inverse. For this, we need to characterize the spaces \( 1_I \Pi_{j_2} \) in terms of \( 1_I \hat{\Pi}_Q 1_I \) (in cases when at least one of \( j_1, j_2 \) are not in \( I \)). For this we have the following lemma:
Lemma 2.2.16. In the situation of Proposition 2.2.10, let \( \tilde{I} \) be the vertex set of \( \tilde{Q} \), and let \( j_1, j_2 \in \tilde{I} \setminus I \), which are on line segments \( L_{s_1} \) and \( L_{s_2} \), respectively. Let \( p_{i_8}j_1 \) be the straight-line path (in \( \tilde{L}_{s_1} \)) from \( i_8 \) to \( j_1 \), and similarly define \( p_{i_9}j_2, p_{j_1}i_{1_0} \), and \( p_{j_2}i_{2_0} \). Suppose the length of \( L_{s} \) is \( d_{s} \) for all \( s \). Let \( d_{i_8}j_1 = \| p_{i_8}j_1 \| \) be the straight-line distance from \( i_8 \) to \( j_1 \), and similarly define \( d_{i_9}j_2 \). Then, we have an exact sequence of \( \mathbb{Z} \)-modules,

\[
0 \rightarrow a_{s_1} - d_{i_8}j_1 + 1 \rightarrow 0 \rightarrow j_1 \Pi Q \rightarrow 0,
\]

(2.2.17)

where \( a' \) is any loop of length two inside \( L_{s_2} \), beginning and ending at \( j_2 \).

Furthermore, if one of \( j_1, j_2 \) is in \( I \) and the other in \( \tilde{I} \setminus I \), then we have the exact sequence

\[
0 \rightarrow a_{s_1} - d_{i_8}j_1 + 1 \rightarrow 0 \rightarrow j_1 \Pi Q \rightarrow 0,
\]

(2.2.19)

\[
0 \rightarrow \Pi Q a_{s_2} - d_{i_9}j_2 + 1 \rightarrow 0 \rightarrow j_1 \Pi Q \rightarrow 0,
\]

(2.2.20)

Proof. The image of any path in \( \Pi Q \) which lies strictly inside a line segment \( L_{s} \) only depends on its endpoints and length. This and the fact that any path from \( j_1 \) to \( j_2 \) must pass through \( \tilde{Q} \) if \( i_8 \neq i_{s_2} \) shows the exactness at \( j_1 \Pi Q j_2 \) above. Exactness (injectivity) at \( a_{s_1} - d_{i_8}j_1 + 1 \Pi Q \) \( \Pi Q \) \( d_{s_2} - d_{i_9}j_2 + 1 \Pi Q \) \( \Pi Q \) \( a_{s_2} - d_{i_9}j_2 + 1 \Pi Q \) \( a_{s_2} - d_{i_9}j_2 + 1 \Pi Q \) and at \( \sum_{\ell=0}^{\min(d_{i_8}j_1, d_{i_9}j_2) - 1} p_{j_{1_2}j_2}(a')^\ell p_{j_1j_2} \) (surjectivity) is obvious. It remains to show exactness at \( i_8 \Pi Q i_{s_2} \). That is, it remains to compute the kernel of \( x \mapsto p_{j_1i_8}x p_{i_9j_2} \). To do this, let us consider

\[
j_1 \Pi Q j_2 \rightarrow j_1 \Pi Q j_2.
\]

(2.2.21)

The kernel of this is \( j_1((r)) j_2 \), which we may rewrite as follows. For all \( s \), let \( 1_{L_{s}} \) be the sum of all vertices on \( L_{s} \) except \( i_{s} \) (so, the vertices from \( \tilde{I} \setminus I \) on \( L_{s} \)). Then, we have

\[
 j_1((r)) j_2 = j_1(1_{L_{s}} + 1_{L_{s}}) j_2 + p_{j_1i_8}((r)) p_{i_9j_2}.
\]

(2.2.22)

We deduce (using the RHS and the observations at the beginning of the proof) that

\[
p_{j_1i_8}P_{i_9j_2}P_{i_9j_2} j_1((r)) j_2 = p_{j_1i_8} a_{s_1} d_{i_8}j_1 + 1 \Pi Q p_{i_9j_2} j_2 + p_{j_1i_8} a_{s_2} d_{i_9}j_2 + 1 \Pi Q p_{i_9j_2} j_2 + p_{j_1i_8}((r)) p_{i_9j_2}.
\]

(2.2.23)

This yields the desired result.

(ii) This easily follows from (i) using the formula

\[
1_{L} Q 1_{L} \cong Q_{I_{0}} \ast_{k_{L}} B.
\]

(2.2.24)
Now, we define the map $HH_0(\hat{\Pi}_Q) \to HH_0(1_I \Pi \hat{\Pi}_Q 1_I)/\langle (a_s^t)_{s \in \{1, \ldots, m\}, t \geq 1} \rangle$ as follows. First, let us define a map

$$\Pi_\hat{\varnothing} : HH_0(1_I \Pi \hat{\Pi}_Q 1_I). \quad (2.2.25)$$

First, the map sends $ixj$ to zero if $i \neq j$ for vertices $i, j \in \hat{I}$. Next, on $1_I \Pi \hat{\Pi}_Q 1_I$, the map is the tautological one. Then, for any $j \in \hat{I} \setminus I$, with $j$ in the line segment $L_s$, we set $p_{j}x_{p_{j}} \mapsto [x(p_{j}, j) p_{j}] = [x a_s^{d_{isj}}]$. To see that this is well-defined, by the lemma it suffices to show that if $x \in a_s^{d_{isj}+1} \Pi_\hat{\varnothing} + \Pi_\varnothing a_s^{d_{isj}+1}$, then $[x a_s^{d_{isj}}] = 0 \in HH_0(1_I \Pi \hat{\Pi}_Q 1_I)$. However, this follows from the fact that $[a_s^{d_{isj}+1} \Pi_\varnothing a_s^{d_{isj}}] + [\Pi_\varnothing a_s^{d_{isj}} a_s^{d_{isj}+1}] \subset [1_I \Pi_\varnothing 1_I, 1_I \Pi_\varnothing 1_I] + [a_s^{d_{isj}+1} 1_I \Pi_\varnothing 1_I]$. Then, the only elements of $j \Pi_\varnothing j'$ which cannot be written in the form $p_{j}x_{p_{j}}$ are (by the Lemma) those elements represented by paths lying entirely in $L_s$, of total length less than $2d_{j}$. Let us define such paths to map to zero. We thus get a well-defined map (2.2.25).

Next, if we post-compose the map with the quotient $HH_0(1_I \Pi \hat{\Pi}_Q 1_I) \to HH_0(1_I \Pi \hat{\Pi}_Q 1_I)/\langle (a_s^t)_{s \in \{1, \ldots, m\}, t \geq 1} \rangle$, then all elements of $\Pi$ represented by paths which lie entirely in $L_s$ (of any length) map to zero. It remains to show that the composite map kills $[\hat{\Pi}_Q, \Pi \hat{\Pi}_Q]$. For this, we need to show that

1. $[p_{j}x_{p_{j}}] = 0$ for any $f, f' \in i_s \Pi \hat{\Pi}_Q i_s$ and $j \in L_s$.
2. $[p_{j}x_{p_{j}} a'] = 0$ for any $f \in i_s \Pi \hat{\Pi}_Q i_s$, $j \in L_s$ ($j \neq i_s$), and where $a'$ is a path of length two beginning and ending at $i_s$.

The image of the element in part (1) is $[f a_s^{d_{isj}} f' a_s^{d_{isj}}] - [f' a_s^{d_{isj}} f a_s^{d_{isj}}] = 0$. The image of the element in part (2) is $[f a_s^{d_{isj}+1}] - [a_s f a_s^{d_{isj}}] = 0$. This proves that our map descends to a map

$$HH_0(\Pi_\hat{\varnothing}) \to HH_0(1_I \Pi_\varnothing 1_I)/\langle (a_s^t)_{s \in \{1, \ldots, m\}, t \geq 1} \rangle. \quad (2.2.26)$$

It is clear from the definition that this map is inverse to the map in (2.2.12) induced by inclusion. □

3 Hesselholt and Rains’ conjecture and Theorem 1.1.2 for good primes

The reader who would like a reminder on Gröbner bases and the Diamond Lemma (and more general versions of them) is referred to Appendix A

3.1 Proof of Hesselholt and Rains’ Conjecture 1.1.1 The purpose of this section is to prove the following main combinatorial result, which in particular implies Conjecture 1.1.1. The proof of this lemma will be generalized later, to compute bases of $\Pi$ and $A$ for quivers containing $\hat{A}_n, \hat{D}_n$.

**Definition 3.1.1.** For a sequence $(x_1, \ldots, x_k) \in X^k$ for any set $X$, let $\text{per}(x)$ be the period (the least positive integer such that $x_i = x_{i + \text{per}(x)}$, with indices taken modulo $k$), and let $\text{rep}(x.) := k/\text{per}(x.)$ be the number of cyclic permutations which fix the sequence (the size of the stabilizer in $\mathbb{Z}/k$ of the sequence), which we can call the “number of times the sequence repeats itself”, hence “rep”.

**Lemma 3.1.2.** Let $A := F/R$ where $F := \mathbb{Z}(x, y, r')$ and $R := \langle (r) \rangle$ with $r := xy - yx + r'$. Then
1. A basis for $A$ is given by monomials in $x, y$, and $r'$ such that the maximal submonomials in $x, y$ are of the type
\[ z_{a,b} := \begin{cases} (xy)^b x^{a-b}, & \text{if } a \geq b, \\ (yx)^a y^{b-a}, & \text{if } a < b, \\ 1, & \text{if } a = b = 0, \end{cases} \tag{3.1.3} \]
i.e. of the form $z_{a_1,b_1} r' z_{a_2,b_2} \cdots r' z_{a_m,b_m+1}$ for $m, a_i, b_i \geq 0$;

2. A basis for $A/[Ar' A, A]$ is given by
   (a) $z_{a,b}$;
   (b) Cyclic monomials $[z_{a_1,b_1} r' z_{a_2,b_2} \cdots z_{a_m,b_m} r']$.

3. The $\mathbb{Z}$-module $A/[A, A]$ can be described as $V/W$, where $V \cong A/[Ar' A, A]$ is described as above, and and $W \subset V$ has as a basis the elements (for $a > b \geq 1$):
\[ W_{a,b} := \sum'_{a,b} \gcd(a,b) \frac{[\{a\}]}{\text{rep}(a,b)} \prod_{\ell=1}^{[\{a\}]} (-r') z_{a_\ell,b_\ell}, \tag{3.1.4} \]
\[ W_{b,a} := \sum'_{a,b} \gcd(a,b) \frac{[\{a\}]}{\text{rep}(a,b)} \prod_{\ell=1}^{[\{a\}]} r' z_{b_\ell,a_\ell}, \tag{3.1.5} \]
\[ W_{a,a} = [(xy + r')^a] - [(xy)^a], \tag{3.1.6} \]
where the sums range over distinct cyclic monomials of the given form, such that $W_{c,d}$ always has bidegree $(c, d)$ in $x, y$ (with $r'$ having bidegree $(1, 1)$), and such that $a_\ell > b_\ell$ for all $\ell$. That is, for $k := |\{a\}| - |\{b\}|$, we require $b_1 + b_2 + \ldots + b_k + k = b$ and $a_1 + a_2 + \ldots + a_k + k = a$, with $a_\ell > b_\ell \geq 0$. The coefficient $\text{rep}(i, j)$ is meant as a period of a $\mathbb{Z}/k$-tuple of elements of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

More precisely, the given subset $V$ maps surjectively to $A/[A, A]$ under the tautological quotient, and the kernel is $W$.

4. None of the elements $W_{a, b}$ are multiples of any non-unit in $\mathbb{Z}$. Modulo $([((r')^m)]_{m \geq 1}$, only the elements $W_{p^\ell, p^\ell}$ are multiples of any non-unit, and its greatest positive integer factor is $p$.

Furthermore, let $\tilde{V} := \tilde{V}[\frac{1}{p} (r')^{p^\ell}]_{p \text{ prime}, \ell \geq 1}$, and $F/[F, F], A/[A, A]$ similarly be the result of adjoining $\frac{1}{p} (r')^{p^\ell}$. Then, the elements $\frac{1}{p} W_{p^\ell, p^\ell} \in \tilde{V}$ and $\frac{1}{p} [r^{p^\ell}] \in F/[F, F]$ have the same image in $A/[A, A]$, are nonzero, and span the torsion of $A/[A, A]$ ($\mathbb{Z}/p$ in degrees $2p^\ell$ and $0$ otherwise).

We note that abstractly understanding $A/[A, A]$ is easy if we wanted to use cyclic words in $x$ and $y$, but the point of finding bases as above is to allow one to obtain bases in a further quotient by a power of $r'$, or in an extension of $A$ (cf. Corollary 3.1.13). In particular, to prove Conjecture 3.1.13, we will view $A$ as a subalgebra of $\Pi = \mathbb{Z}\langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle$ by $x = x_1, y = y_1$, and $r' = \sum_{i=2}^g [x_i, y_i]$, in which case $[(r')^\ell] = 0$ for all $\ell \geq 1$.  

17
Proof. (1) This is not too difficult. Note that it would be easier at this stage to solve the problem if we had defined \( z_{a,b} \) to be \( x^a y^b \) (in which case we could use the Gröbner basis \((xy - yx + r')\) with respect to the graded lexicographical ordering with \(|r'| = |x| = |y| = 1 \) and \( r' < x < y \)), but the choice we made is convenient for the last part of (3) (for more details, see Remark [3.1.17]). One quick proof for \( z_{a,b} \) as defined would be to use the fact that the result is true using \( x^a y^b \)'s instead, while the normal form of \( z_{a,b} \) with respect to the \( x^a y^b \)'s has leading term \( x^a y^b \).

However, we show how to use the Diamond Lemma as a warm-up to the next part. First, define the disorder \( \text{Dis}(M) \) of a monomial \( M \) in \( x, y \) to be the minimal number of swaps of adjacent letters in \( M \) needed to bring it to the form \( z_{a,b} \). For a monomial \( M = M_1 r' M_2 \cdots r' M_{n+1} \) for \( n \geq 1 \), \( \text{Dis}(M) = \text{Dis}(M_1) + \ldots + \text{Dis}(M_{n+1}) \): that is, \( \text{Dis}(M) \) is given by the sum of the above disorder over each maximal monomial in \( x \)'s and \( y \)'s.

Let \( O(M) \) be the maximal nonnegative integer such that \( M \in (r')^{O(M)} \). Then, we define the partial order on monomials such that \( M_1 < M_2 \) iff either \( O(M_1) > O(M_2) \) or \( O(M_1) = O(M_2) \) and \( \text{Dis}(M_1) < \text{Dis}(M_2) \). Every relation \( f(x y - y x + r') g \), where \( f \) and \( g \) are monomials, then has leading term equal to either \( f x y g \) or \( f y x g \) (because of the \( O \) condition), and thus can be viewed as a reduction \( f y x g \leftrightarrow f x y g + f r' g \) or \( f y x g \leftrightarrow f x y g - f r' g \) which reduces the disorder of the leading term. To check confluence, suppose two reductions of a given monomial are possible. Then, it is not difficult to see that the set of reductions \( f(x y - y x + r')\) and \( f y x g \) are monomials, then has leading term equal to either \( f x y g \) or \( f y x g \) (because of the \( O \) condition), and thus can be viewed as a reduction \( f y x g \leftrightarrow f x y g + f r' g \) or \( f y x g \leftrightarrow f x y g - f r' g \) which reduces the disorder of the leading term. To check confluence, suppose two reductions of a given monomial are possible. Then one easily verifies that either the two are in disjoint positions, in which case either can be applied in either order with the same result, or if the two overlap, then one can swap the two pairs of positions in either order, and both will be a sequence of reductions with the same result. (This gives a stronger “diamond” property than the confluence condition, which only requires that the difference of two distinct reductions is the span of some relations of lower order.) So there is a unique reduction to normal form.

With either approach, all reductions appear with leading coefficient 1, so the submodule \( R \) is saturated and the quotient \( F/R \) is free, and has the given \( \mathbb{Z} \)-basis. (2) Let \( F' \subset F \) be spanned by monomials not containing \( r' \) and let \( V' := F' \oplus F r' F/[F r' F, F] \). We have an obvious surjection \( \beta : V' \rightarrow A/[Ar' A, A] \). \( V' \) has a basis consisting of monomials not containing \( r' \) and cyclic monomials containing \( r' \). Let us define a partial order on such monomials: \( f < g \) if either (1) \( O(f) > O(g) \); or (2) \( O(f) = O(g) \) and there are fewer swaps \( x y \leftrightarrow x y \) needed to bring \( f \) to the form of a basis element than for \( g \).

Then, it is not difficult to see that the set of reductions \( f(x y - y x + r') g \) and \( h(x y - y x + r') \) \((f, g, h \text{ monomials, with } h \in (r') \text{ and } f, g \notin (r'))\) form a confluent set \( W' \), so that \( V'/W' \cong A/[Ar' A, A] \), as desired: the normal-form basis of the quotient is given by the elements listed in the statement of (2). One makes the same arguments as before: the main point is that cyclic monomials that contain \( r' \) behave similarly to regular monomials in that one can compute the minimal number of swaps \( x y \leftrightarrow y x \) needed to reduce to a normal-form element, and any two reductions can be performed in either order with the same result. (3) We claim that \([A, A] = [Ar' A, A] + [(z_{a,b-1}), y] + [(z_{a-1,b}), x] \). This follows immediately from the fact that the \( z_{a,b} \) form a basis of \( A \) (Part (1)). This means that \([A, A] = V/W \) where \( W \) is spanned by the relations

\[
w_{a,b,y} := \eta([z_{a,b-1}, y]), w_{a,b,x} := \eta([z_{a-1,b}, x]) \in V,
\]

where

\[
\eta : A \rightarrow A/[Ar' A, A] \cong V \text{ is the quotient}.
\]
It remains to show that
\[ (w_{a,b,x}, w_{a,b,y}) = (W_{a,b}) \]  
(3.1.9)

This will complete the proof of (3).

We will prove the sharper result that
\[ \frac{\gcd(a, b)}{b} w_{a,b,x} = -\frac{\gcd(a, b)}{a} w_{a,b,y} = \pm W_{a,b} \]  
(3.1.10)

using the positive choice of \( \gcd \), where \( \pm \) is plus if \( a \leq b \) and minus if \( a > b \).

First, we note that \( aw_{a,b,x} + bw_{a,b,y} = 0 \), since the left-hand side is equivalent in \( A/[4r' A, A] \) to
\[ \sum_{i=1}^{a+b} \eta([h_{i+1} h_{i+2} \cdots h_{a+b} h_1 h_2 \cdots h_{i-1}, h_i]), \quad h_1 \cdots h_{a+b} = z_{a,b}. \]  
(3.1.11)

Thus, it suffices to consider just one of the \( w_{a,b,x}, w_{a,b,y} \). Let us suppose that \( a > b \). Then, it follows that \( w_{a,b,x} \) is the result of successively commuting the \( x \) on the right of \( (xy)^b x^{a-b} \) all the way to the left, and subtracting the resulting \( x(xy)^b x^{a-b-1} \). So, \( w_{a,b,x} = \sum_{0 \leq b' \leq b-1} [(xy)^b x^{a-b-1}] \).

Each summand can then be reduced to a linear combination of monomials in \( r' \) and the \( z_{-, -} \)'s. Specifically, this is the sum of all possible terms of the form \([r' z_{a_1, b_1} \cdots r' z_{a_k, b_k}]\) such that \( b_1 \geq b - b' - 1 \), and satisfying the conditions of (3.1.1):
\[ a_{\ell} > b_{\ell} \text{ for all } \ell, \text{ and } a_1 + \cdots + a_k + k = a, \]
\[ b_1 + \cdots + b_k + k = b. \]
So, when we add up all the contributions to \( w_{a,b,x} \), we get \(-((b_1 + 1) + \cdots + (b_m + 1) = b)/\text{rep}(a, b)\) copies of each \([(r') z_{a_1, b_1} (r') \cdots (r') z_{a_m, b_m}]\), i.e. just \(-\frac{b}{\gcd(a, b)} W_{a,b}\).

For the same reason, when \( b > a \) we get \( w_{a,b,y} = -\frac{a}{\gcd(a,b)} W_{a,b} \). So it remains to consider \( w_{b,b,x} \).

Here, we get \( \eta((yx)^b - (xy)^b) = [(xy + r')^b] - [(xy)^b] \), as desired.

It is clear that the elements \( W_{a,b} \) are zero if either of \( a \) or \( b \) is zero. This completes the proof of part (3).

(4) Since \( V \) is a free \( \mathbb{Z} \)-module, and each relation (3.1.4)–(3.1.6) lives in a different bigraded degree, we find that the torsion part of \( V/W \) is a direct sum of some cyclic module for each bigraded degree \( a, b \). In fact, the cyclic module is just \( \mathbb{Z}/g_{a,b} \), where \( g_{a,b} \) is the gcd of all the coefficients that appear in (3.1.4)–(3.1.6). To compute this, we claim that the numbers \( \text{rep}(a, b) \) range over all positive factors of \( \gcd(a, b) \): it’s clear that any cyclic monomial of the form \([f^\ell]\) with bidegree \((a, b)\) must have \( \ell \) be a factor of \( a \) and \( b \); on the other hand, for any such factor, we can form the cyclic monomial \([f^\ell]\) where \( f = r' x^a y^b \). So, the gcd of the coefficients \( \frac{\gcd(a, b)}{\text{rep}(a, b)} \) is 1 for all \( a, b \). On the other hand, in the case \( a = b \), we see that the coefficients of \([xy + r')^m - [(xy)^m]\) have \( \gcd \) equal to one, but if we restrict to the terms other than \([r')^m\], then the gcd is equal to \( p \) in the case \( m = p^\ell \) where \( p \) is prime (but still one in any other case). This proves the claims of the first paragraph.

Finally, we need to show that the elements \( \frac{1}{p} W_{p^\ell, p^\ell} \in \hat{V} \) and \( \frac{1}{p}[r^p] \in \hat{F}/[F,F] \) have the same image in \( A/[A,A] \), are nonzero, and span its torsion. First, we have already seen that the image of the elements \( \frac{1}{p} W_{p^\ell, p^\ell} \in \hat{V} \) are nonzero and span the torsion of \( A/[A,A] \), since \( A/[A,A] \) is obtained by modding by the elements \( W_{a,b} \), which now have greatest integer factor = \( p \) if \( a = b = p^\ell \) and greatest integer factor = 1 otherwise.

We can write the image of the element \( \frac{1}{p}[r^p] \) in \( A/[A, A] \) as follows:
\[ \frac{1}{p}[r^p] - (xy)^p r^p - (-yx)^p r^p + \frac{1}{p}(1 + (-1)^p)(xy)^p r^p, \]  
(3.1.12)
and since the first term in square braces is actually $p$ times an integral combination of cyclic words, we can replace terms $(y^x)$ by $(x y + r')$ (without destroying the ability to divide by $p$), and obtain

$$\frac{1}{p}(-(xy)^p - (-xy - r')^p) + \frac{1}{p}(1 + (-1)^p)(xy)^p \equiv \frac{1}{p}([((xy + r')^p] - [(xy)^p]) \quad (\text{mod } p), \hspace{1cm} (3.1.13)$$

which is the image of $\frac{1}{p}W_{p^e, p^f}$ under $F/[F,F] \to A/[A,A]$, as desired. □

From the proof, we immediately deduce the more general

**Corollary 3.1.14.** Let $B$ be any (graded) algebra and $r' \in B \setminus \{0\}$ (of degree 2). Let $r := xy - xy + r'$, and set $F' = \mathbb{Z}[x, y] \ast B$ and $A' = F'/\langle (r) \rangle$. Then

$$HH_0(A') \cong HH_0(\mathbb{Z}[x, y] \ast B)/W,$$

(3.1.15)

where $W$ is as described in Lemma 3.1.2. In particular,

$$\text{torsion}(HH_0(A')) \cong \text{torsion}(HH_0(B)) \oplus \bigoplus_{p \text{ prime, } \ell \geq 1, \text{ such that } (r^\ell)^{p^e} \in pB + [B, B]} \langle r^\ell \rangle.$$

(3.1.16)

The isomorphisms are obtained from $F' \to A'$ by picking a $\mathbb{Z}$-module section of $\mathbb{Z}[x, y]$ into $F'$. The element $r^{(p^e)}$ here is the image of $\frac{1}{p}[r^p] \in HH_0(F')$, coincides with $\frac{1}{p}W_{p^e, p^f}$, and generates a summand of $\mathbb{Z}/p$.

The corollary immediately implies Conjecture 1.1.1 setting $B := \mathbb{Z}[x_2, y_2, \ldots, x_g, y_g]$ and $r' = \sum_{j=2}^g[x_j, y_j]$. (In fact, it implies Theorem 1.1.2 in the case $Q$ contains a loop, i.e., $Q \supset Q_0 \cong \tilde{A}_0$. See Remark 3.2.20.)

**Remark 3.1.17.** As noted, the proof of parts (1)–(3) is slightly shorter if we use $z_{a,b} = x^a y^b$; in this case we have

$$W_{a,b} = \sum_{a,b} \gcd(a, b) \left[ \prod_{\ell = 1}^k (r^\ell x^{a\ell} y^{b\ell}) \right], \hspace{1cm} (3.1.18)$$

and the Diamond Lemma argument is a bit simpler. However, the disadvantage is that the formula for the explicit generator $(r^p) - [(r^p)]/p$ does not follow explicitly from the computation; also, the computation above is more similar to what is needed in the $\tilde{D}_n$ case (note that using $x^a y^b$ is possible even in the $\tilde{A}_{n-1}$ case).

### 3.2 Proof of Theorem 1.1.2 for good primes: a $\Gamma$-equivariant version

In this section, we will prove a “$\Gamma$-equivariant” version of Lemma 3.1.2 which will allow us to prove the main Theorem 1.1.2 for good primes.

Let $Q_0$ be an extended Dynkin quiver, and $\Gamma \subset SL_2(\mathbb{C})$ the corresponding finite subgroup under the McKay correspondence. By [CBH98, §3], for $\mathbb{k} = \mathbb{C}$, we know that there are Morita equivalences

$$P_{Q_0} \otimes \mathbb{k} \simeq \mathbb{k}[x, y] \rtimes \Gamma, \hspace{1cm} (3.2.1)$$

$$\Pi_{Q_0} \otimes \mathbb{k} \simeq \mathbb{k}[x, y] \rtimes \Gamma, \hspace{1cm} (3.2.2)$$
by explicitly realizing
\[ P_{Q_0} \otimes k \cong f(k[x, y] \ltimes \Gamma)f \]  
(3.2.3)
where \( f \in k[\Gamma] \) is a sum of primitive idempotents, one for each irreducible representation of \( \Gamma \). In more detail, \( f = \sum_{i \in I_0} f_i \), where the vertices \( I_0 \) of the extended Dynkin quiver \( Q_0 \) also label the irreducible representations \( U_i \) of \( \Gamma \), and \( f_i \) is chosen so that \( k[\Gamma]f_i \cong U_i \). Then, one may choose elements \( \tilde{e} \in k[x, y] \ltimes \Gamma \) so that the map \( e \mapsto \tilde{e}, i \mapsto f_i \) gives the isomorphism \( (3.2.3) \), and furthermore that the element \( xy - yx \in k[x, y] \ltimes \Gamma \) maps to the element \( r \in P_{Q_0} \), yielding \( (3.2.2) \) (specifically, \( (3.2.3) \) descends to \( \Pi Q_0 \otimes k \cong f(k[x, y] \ltimes \Gamma)f \)).

Then, the idea of the proof of Theorem 3.1.2 for good primes is to repeat the arguments of the previous section, generalizing to a “\( \Gamma \)-equivariant” version. This involves replacing \( A = k[x, y, r']/(xy - yx + r') \cong k[x, y] \) by \( A = (k[x, y, r']/(xy - yx + r')) \ltimes \Gamma \cong k[x, y] \ltimes \Gamma \). Here, we set the \( \Gamma \)-action on \( r' \) to be trivial.

There are a few problems with this. First, in the previous section, we worked over \( \mathbb{Z} \); we cannot get any torsion information setting \( k = \mathbb{C} \) as above. We resolve this problem by restricting to “good characteristic”: primes that do not divide \( |\Gamma| \), where all of the above easily generalizes (as is well-known). As a slight modification, rather than working over an algebraically closed field in good characteristic, we will set \( k := \mathbb{Z}[\mathbb{F}_e, e^{-\frac{m}{2\pi k}}] \), since this allows us to see all of the torsion (except in bad primes) simultaneously.

Second, we will need a way to get from the \( A \) and \( F \) above to actual preprojective algebras of quivers properly containing \( Q_0 \). This will follow from a generalization of Corollary 3.1.14 and some general arguments about (partial) preprojective algebras, further developing some of the ideas of EE05, ECo06.

Third, to understand (a presentation of) \( A/[A, A] \), we first need to understand \( F/[F, F] \) where \( F = k[x, y, r'] \ltimes \Gamma \). For this, we use the general (known) Hochschild theory for skew group algebras, as follows:\footnote{This (known) proposition and proof was explained to the author by P. Etingof}

**Proposition 3.2.4.** Let \( A \) be an associative algebra over a commutative ring \( k \) and \( \Gamma \) a finite group acting on \( A \). Assume that \( k \) contains \( \frac{1}{|\Gamma|} \) (in particular, the characteristic of \( k \) does not divide \( |\Gamma| \)). For any \( \gamma \in \Gamma \), let \( \Gamma_\gamma := \{ \gamma' \mid \gamma' \gamma = \gamma \gamma' \} \) denote the centralizer of \( \gamma \). Then, for any \( A \)-bimodule \( M \), one has

\[
H^\bullet(A \ltimes \Gamma, M) \cong H^\bullet(A, M)^\Gamma, \quad HH^\bullet(A \ltimes \Gamma) \cong \bigoplus_{\text{\gamma representatives of the conjugacy classes in } \Gamma} H^\bullet(A, A^\gamma)^{\Gamma\gamma}; \quad (3.2.5)
\]

\[
H_\bullet(A \ltimes \Gamma, M) \cong H_\bullet(A, M)^\Gamma, \quad HH_\bullet(A \ltimes \Gamma) \cong \bigoplus_{\text{\gamma representatives of the conjugacy classes in } \Gamma} H_\bullet(A, A^\gamma)^{\Gamma\gamma}. \quad (3.2.6)
\]

**Proof.** We prove the second result (for Hochschild homology) since we will use that one more heavily; the first follows from the co-version of the proof.

We write
\[
H_\bullet(A \ltimes \Gamma, M) = \text{Tor}_\bullet^{(A \otimes A^{\text{op}}) \ltimes (\Gamma \times \Gamma^{\text{op}})}((A \ltimes \Gamma, M).
\]
(3.2.7)

For any algebra \( B \), let \( LH^B \) denote the left-derived functor of \( B \) – Bimod \( \rightarrow \) B – Bimod, given by \( M \mapsto M/(mb - bm)_{m \in M, b \in B} \): that is, the derived functor which yields the Hochschild
homology of $B$ with coefficients in the given $B$-bimodule $M$. Let $\Gamma_\Delta := \{(g, g^{-1}) \in \Gamma \times \Gamma^{\text{op}}\}$. By Shapiro’s lemma, since $\text{Ind}_{\Gamma_\Delta}^{\Gamma \times \Gamma^{\text{op}}}(A) = A \otimes \mathbb{k}[\Gamma] \cong A \rtimes \Gamma$ as a $\Gamma \times \Gamma^{\text{op}}$-module, where $A$ (and $M$) are $\Gamma_\Delta \cong \Gamma$-modules by conjugation,

$$\text{Tor}_*^{(A \otimes A^{\text{op}}) \times (\Gamma \times \Gamma^{\text{op}})}(A \rtimes \Gamma, M) = H_*^{\epsilon} \left( L H^{A \otimes A^{\text{op}}} \left( L H^{k[\Gamma]} \otimes k[\Gamma]^{\text{op}} \left( (A \rtimes \Gamma) \otimes M \right) \right) \right) \cong H_*^{\epsilon} \left( L H^{k[\Gamma]} \left( L H^{k[\Gamma]} \right) \otimes k[\Gamma]^{\text{op}} \left( A \rtimes M \right) \right).$$

(3.2.8)

Now, by assumption on characteristic of $\mathbb{k}$, taking $\Gamma$-coinvariants (or invariants) is exact, so the RHS is just

$$\text{Tor}_*^{A \otimes A^{\text{op}}}(A \otimes M)_{\Gamma} = HH_*^{\epsilon}(A, M)_{\Gamma}.$$ 

(3.2.9)

Finally, specializing to $M := A \rtimes \Gamma$, we note that

$$\mathbb{k}[[\Gamma]] = \bigoplus_{\text{conjugacy classes } C} \mathbb{k}[[\Gamma]]_{C} := \bigoplus_{\gamma \in C} \langle \gamma \rangle,$$

(3.2.10)

and $\mathbb{k}[[\Gamma]]_{C}$ is stable under the $\mathbb{k}[[\Gamma]]$-action (given by the conjugation action of $\Gamma$), so that we just end up with the second formula in (3.2.5).

Let us introduce the notation (for $a, b \in A$ and $\gamma \in \Gamma$)

$$[a, b]_{\gamma} := a(\gamma \cdot b) - ba,$$

(3.2.11)

where $\cdot$ denotes the action of $\Gamma$ on $A$ (so $\gamma \cdot b = \gamma b \gamma^{-1} \in A \rtimes \Gamma$). Then, in the case of degree zero, we may rewrite (3.2.5) as

$$HH_0(A \rtimes \Gamma) \cong \bigoplus_{C}(A/[A, A]_{\gamma_{C}})_{\Gamma_{C}},$$

(3.2.12)

where $C$ ranges over the conjugacy classes of $\Gamma$, $\gamma_{C} \in C$ is a fixed choice of representative for each $C$, and $\Gamma_{\gamma} \subset \Gamma$ denotes the centralizer of $\gamma$ in $\Gamma$. Note that this formula may also be obtained directly without using any homological algebra, just the definition $HH_0(B) := B/[B, B]$ and the decomposition of $\mathbb{k}[[\Gamma]]$ into a direct sum of conjugacy classes. However, we will use the above formulas later on, and felt it is better to explain the general result.

We now state our

**Theorem 3.2.13.** Let $\mathbb{k} := \mathbb{Z}[\frac{1}{m}], e^{\frac{2\pi i}{m}}$ and $A = \mathbb{k}\langle x, y, r \rangle / \langle xy - yx + r \rangle$, and let $\Gamma \subset \text{SL}_2(\mathbb{k})$ be a finite subgroup, which acts on $x$ and $y$ by the tautological action on $\langle x, y \rangle$, and fixes $r'$.

Then, the space $HH_0(A \rtimes \Gamma)$ has a canonical decomposition along conjugacy classes $C$ of $\Gamma$,

$$HH_0(A \rtimes \Gamma) = \bigoplus_{C} HH_0(A \rtimes \Gamma)_{C},$$

(3.2.14)

presented as follows:

(i) For $C \neq \{1\}$ and any choice $\gamma_{C} \in C$, one has a presentation $V_C \cong HH_0(A \rtimes \Gamma)_{C}$ where $V_C$ is the space of $\Gamma_{\gamma_{C}}$-invariant cyclic polynomials

$$\sum_{m \geq 1, a, b} \alpha_{a, b} [z_{a_{1}, b_{1}} r' \cdots z_{a_{m}, b_{m}} r'],$$

(3.2.15)

using the obvious map. (We may interpret $V_C$ as $V^{\Gamma_{\gamma_{C}}}$, with $V$ from Lemma 3.1.3.)
(ii) For $C = \{1\}$, one has a presentation $V_1/W_1 \simeq HH_0(A \times \Gamma|_1)$, where $V_1$ is the direct sum of $k[x,y]^\Gamma$ and the space of $\Gamma$-invariant cyclic polynomials of the form (3.2.15), and $W_1$ is spanned by the $\Gamma$-invariant elements in $W$ of Lemma 3.1.2 part (3) (3.1.4 - 3.1.6).

(iii) The projection $W_1 \to (\langle r^i \rangle \otimes k[x,y]^\Gamma)$ modulo $\langle (r^i)^2 \rangle$ is a monomorphism and a $C$-isomorphism, with image

$$(\bigoplus_{a,b}(\gcd(a,b)r^i \otimes x^ay^b))^\Gamma. \quad (3.2.16)$$

(iv) $HH_0(A \times \Gamma)$ is free, but obtains $\mathbb{Z}/p$-torsion in each degree $2p^\ell$ for $p$ prime and $p \nmid |\Gamma|$ after modding by $\langle (r^i)^2 \rangle_{\ell \geq 1}$ (which lives in all $V_C$). The torsion then appears in $V_1/W_1$ and is spanned by the image of $\frac{1}{p}[xy - yx + r^i]^p$ under the map $HH_0(k[x,y]) \to HH_0(A \times \Gamma)/\langle (r^i)^2 \rangle_{\ell \geq 1}$.

The analogue of Corollary 3.1.14 is then

**Corollary 3.2.17.** Let $B$ be any (graded) $k \oplus k[\Gamma]$-algebra, and let $r' \in B \setminus \{0\}$ (of degree two) be such that $(0 \oplus 1_{k[\Gamma]})r' = (0 \oplus 1_{k[\Gamma]})$. Set $F' = (k[x,y] \times \Gamma) \ast k[\Gamma] B$ where $k[\Gamma]$ acts on $B$ by inclusion into the right summand (so $1 \in k[\Gamma]$ does not necessarily act by 1 on $B$). Let $r := xy - yx + r'$, and set $A' := F'/\langle r \rangle$. Then one has isomorphisms

$$HH_0(A') \cong HH_0((k[x,y] \times \Gamma) \ast k[\Gamma] B)/W, \quad (3.2.18)$$

where $W$ is as described in Theorem 3.2.13 (in terms of $x,y$, and $r'$). In particular,

$$\text{torsion}(HH_0(A')) \cong \text{torsion}(HH_0(B)) \oplus \bigoplus_{p \text{ prime}, \ell \geq 1, \text{such that } (r^i)^p \in pB+B} \langle r^i \rangle. \quad (3.2.19)$$

The isomorphisms are obtained from $F' \to A'$ by picking a section of $(k[x,y] \times \Gamma)$ into $F'$. The element $r^i$ is the image of $\frac{1}{p}[r^i]^p \in HH_0(F' \ast k[\Gamma] B)$.

(We omit the proof, which is easy.) This corollary immediately gives us Theorem 3.1.2 for good primes.

**Proof of Theorem 3.1.2 for good primes.** We use the fact that $\Pi_{Q\setminus Q_0} \Pi_{Q\setminus Q_0}$ and $HH_0(\Pi_{Q\setminus Q_0})$ are torsion-free (Proposition 4.2.21). To prove the result in good primes, it suffices to replace $\Pi_Q$ and $\Pi_{Q\setminus Q_0}$ by $\Pi_Q \otimes k$ and $\Pi_{Q\setminus Q_0} \otimes k$ (where $k = \mathbb{Z}[1/|\Gamma| e^{2\pi i}]$ as before). For readability, we make this switch of notation for the proof. We then need to show that the torsion is spanned by nonzero elements $r^i$ for $p \nmid |\Gamma|$.

View $k^I \cong \mathbb{R}[\Gamma]$ as a subalgebra of $k[\Gamma]$. Then, set $B = k[\Gamma] \ast k^I \Pi_{Q\setminus Q_0}$, which is a $k[\Gamma] \ast k[\Gamma]$-algebra. Using the map $k \to k^I$ sending 1 to $1_{\Pi\setminus I}$, we view $B$ as a $k \ast k[\Gamma]$-algebra. (Note that, as a $k[\Gamma] \ast k[\Gamma]$-bimodule, $B$ is just isomorphic to $(k^I \ast k[\Gamma]) \otimes (k^I \ast k[\Gamma])$, so the star-product did not change $B$ from $\Pi_{Q\setminus Q_0}$ all that much.)

Let the element $r^i \in B$ be given by $r' := 1_{I_0}r_0$ and $r_0 \in B$. Now, $F' = (k[x,y] \times k[\Gamma]) \ast k[\Gamma] B$. By definition and the precise version of Morita equivalence from [CBH98] outlined above, we see that $F'f \cong \Pi_{Q\setminus Q_0}$, and that $trf = f(xy - yx + r')$ maps to the element $r$ of $\Pi_{Q\setminus Q_0}$ under the isomorphism.

Then, by the corollary, $HH_0(\Pi_Q \Pi_{Q_1}/\langle r \rangle)$ has torsion spanned by the nonzero elements $r\langle r \rangle$ for $\ell \geq 1$ and primes $p \nmid |\Gamma|$. \[\square\]
Remark 3.2.20. Using the above argument with $\Gamma = \{1\}$, we obtain Theorem \ref{thm:main} in the case of quivers containing a loop, $\tilde{A}_0$, for which all primes are good. This only requires Corollary 3.1.14 and Proposition 4.2.21, and explains the parenthetical comment after Corollary 3.1.14.

Proof of Theorem \ref{thm:main}. Specializing (3.2.12) to our case, we have

$$HH_0(A \rtimes \Gamma) \cong \bigoplus_C (A/([A, Ar'A]_{\gamma_C} + ([z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C})_{a,b \geq 1}))_{\Gamma_{\gamma_C}}. \tag{3.2.21}$$

Next, fix $C$, and let $\lambda_C, \lambda_C^{-1} \in \{e^{2\pi i k}|\Gamma|\}_{1 \leq k \leq |\Gamma|}$ be the eigenvalues of $\gamma_C$ (which has determinant one). Let us choose an eigenbasis $x_C, y_C$ of $\gamma_C$ acting on $\langle x, y \rangle$, which we may obtain using the projections $\frac{\lambda_C - \lambda_C^{-1}}{\lambda_C + \lambda_C^{-1}}(\gamma_C - \lambda_C^{-1})$, unless $\gamma_C = \pm 1$, in which case we can set $x_C = x, y_C = y$. Let us assume $\gamma_C \cdot x_C = \lambda_C x_C$ and $\gamma_C \cdot y_C = \lambda_C^{-1} y_C$. Then, $[x_C, y_C]$ must be a unit multiple of $[x, y] = (-r')$, so we may assume $[x_C, y_C] = [x, y]$ by rescaling. We then have, for any $g \in k(x, y) = k(x_C, y_C)$,

$$[g, x_C]_{\gamma_C} = [g, x_C] + (\lambda_C - 1)gx_C, \quad [g, y_C]_{\gamma_C} = [g, y_C] + (\lambda_C^{-1} - 1)gy_C. \tag{3.2.22}$$

Let us also define $z_{a,b}^C$ as in (3.1.3), but replacing $x$ and $y$ by $x_C$ and $y_C$.

There now remain two steps: (1) to understand $(A/\{A, Ar'A\}_{\gamma_C})_{\Gamma_{\gamma_C}}$, and (2) to compute the needed relations analogous to (3.1.4)–(3.1.6). Both use the $C$-versions of $x, y$, and $z$ and $\gamma_C$-commutators.

We begin with (1), which is the easier step. Since $k[\Gamma]$ is semisimple, we may replace coinvariants by invariants in (3.2.21). Then, the RHS of (3.2.21) is isomorphic to

$$(A^{(\gamma_C)}/([A, Ar'A]_{\gamma_C})^{(\gamma_C)} + ([z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C})_{a,b \geq 1})^{(\gamma_C)})_{\Gamma_{\gamma_C}}, \tag{3.2.23}$$

where $\langle \gamma_C \rangle < \Gamma_{\gamma_C}$ is the cyclic subgroup generated by $\gamma_C$. Invariants under this subgroup are just those polynomials in $x_C$ and $y_C$ such that the bidegree $(a, b)$ satisfies $\lambda_C^{a-b} = 1$ (in other words, letting $|\gamma_C|$ denote the order of $\gamma_C$, we have $|\gamma_C| \mid (a - b)$).

Now, it is not difficult to generalize the argument of Lemma 3.1.2 to show that

$$A^{(\gamma_C)}/([A, Ar'A]_{\gamma_C})^{(\gamma_C)} \tag{3.2.24}$$

is a free graded $k$-module with basis the elements

(a) $z_{a,b}^C$ for $a, b \geq 0$ and $\lambda_C^{a-b} = 1$,

(b) $[z_{a_1,b_1}^C r_1 \cdots z_{a_m,b_m}^C r_m]$, for $m \geq 1$ and $a_i, b_i \geq 0$, except for those elements of the form $[f^r]$, where $f$ has bidegree $(a, b)$ with $\lambda_C^{a-b} \neq 1$ (i.e., $\gamma_C \cdot f = \lambda_C^{a-b} f \neq f$). \tag{3.2.25}

(2) Since taking invariants under $\Gamma_{\gamma_C}$ preserves the property of being torsion-free (it just passes to a submodule), by (3.2.23), it remains only to compute the remaining relations $([z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C})_{a,b \geq 1})^{(\gamma_C)}$, so that modding (3.2.21) by these, and taking $\Gamma_{\gamma_C}$-invariants, yields $HH_0(A \rtimes \Gamma)$.

We claim that

$$[z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C}]_{\gamma_C} = ([z_{a,b}, x]_{\gamma_C}, [z_{a,b}, x]_{\gamma_C} y_{C}, [y_{C}, y_{C}]_{a,b \geq 0}). \tag{3.2.26}$$
This follows from the general formula \([f, gh]_{\gamma_C} = [f(\gamma_C \cdot g), h]_{\gamma_C} + [hf, g]_{\gamma_C}\).

Thus, we will consider commutators of the form
\[ [z_{a-1, b}, xC]_{\gamma_C}(a \geq 1), \quad [z_{a-1, b-1}, xyC]_{\gamma_C}(a, b \geq 1), \quad [y_{b-1}, yC]_{\gamma_C}(b \geq 1). \] (3.2.27)

We would like to eliminate the second commutator above similarly to (3.1.10). There are two complications: the first is that we are now using \(\gamma_C\)-commutators; the second is that we are working in the space of \(\gamma_C\)-twisted cyclic words \((A^{(\gamma)}/([A, ArA]_{\gamma_C})_{(\gamma_C)})\).

Let us assume \(C \neq \{1\}\), since otherwise all our work is done in Lemma 3.1.2. For \(a > b\), let us consider the equation
\[ \sum_{i=0}^{a-b-1} [x^i_C(xCyC)^{b}x^{a-b-i-1}_C, x_C] + \sum_{i=0}^{b-1} [(xCyC)^{i}x^{a-b}_C(xCyC)^{b-i-1}, xCyC] = 0. \] (3.2.28)

To turn this into an identity involving the \(\gamma_C\)-commutators in (3.2.27), we first note the following: if \(f \in A\) is an eigenvector for the action of \(\gamma_C\) (e.g., if \(f\) is monomial in \(x_C, yC\)) with eigenvalue \(\lambda^f_C\), and \(g \in A\) is arbitrary, then
\[ [fg, f] = f[g, f] \equiv \lambda^f_C[g, f] = \lambda^f_C[gf, f] \mod ([A, ArA]_{\gamma_C})_{(\gamma_C)}. \] (3.2.29)

In particular, for \(a > b\) (requiring \(\lambda^a_C = 1\)), we have
\[ [x^i_C(xCyC)^{b}x^{a-b-i-1}_C, x_C] \equiv \lambda^i_C z_{a-1, b} [x_C^i, x_C] \mod ([A, ArA]_{\gamma_C})_{(\gamma_C)}, \] (3.2.30)
\[ [(xCyC)^{i}x^{a-b}_C(xCyC)^{b-i-1}, xCyC] \equiv z_{a-1, b-1} [xCyC], \mod ([A, ArA]_{\gamma_C})_{(\gamma_C)}. \] (3.2.31)

Now, we combine (3.2.22) with (3.2.28), (3.2.30), (3.2.31) to obtain
\[ \frac{1 - \lambda^a_C}{1 - \lambda_C} ([z_{a-1, b}, xC]_{\gamma_C} + (1 - \lambda_C)z_{a, b}) + b[z_{a-1, b-1}, xCyC] \in ([A, ArA]_{\gamma_C})_{(\gamma_C)}. \] (3.2.32)

Since \(\lambda^a_C = 1\) and \(([A, ArA]_{\gamma_C})_{(\gamma_C)}\) is saturated in \(A^{(\gamma)}\), this just says that
\[ [z_{a-1, b-1}, xCyC]_{\gamma_C} \in ([A, ArA]_{\gamma_C})_{(\gamma_C)}. \] (3.2.33)

It should be possible to verify this explicitly without using (3.2.28), but it seems more difficult.

Also, we have the following:
\[ [(xCyC)^{b-1}, xCyC]_{\gamma_C} = 0, \quad [y_{b-1}, yC]_{\gamma_C} = (\lambda_C - 1)y_{b}^{\gamma}. \] (3.2.34)

Thus, we conclude that, for \(C \neq \{1\}\),
\[ ([z_{a, b}, xC]_{\gamma_C}, [z_{a, b}, yC]_{\gamma_C})_{a,b \geq 1}^{(\gamma_C)} = ([z_{a-1, b}, xC]_{\gamma_C}, (\lambda_C - 1)y_{b}^{\gamma})_{a,b \geq 1, \lambda^C = 1}. \] (3.2.35)

Also, we easily see that
\[ [z_{a-1, b}, x]_{\gamma_C} \equiv (\lambda_C - 1)z_{a, b} \mod ((\gamma')). \] (3.2.36)

Finally, note that \(1 - \lambda_C\) is invertible in \(k\), as it is a factor of \(\gamma_C\) (by plugging \(t = 1\) into \(\frac{1 - \lambda_C^{\gamma_C}}{1 - \lambda_C} \)), which is a factor of \(|\Gamma|\). Thus, for \(C \neq \{1\}\), the projection of the relations \(([z_{a, b}, x]_{\gamma_C}, [z_{a, b}, y]_{\gamma_C})_{a,b \geq 1}^{(\gamma_C)}\)
modulo \((r')\) gives an isomorphism with \(\langle [z_{a,b}^C]_{a,b > 0} \rangle\). We thus find that the summand of (3.2.23) corresponding to \(C\) is torsion-free (even if we further mod by \(\langle (r')^\ell \rangle_{\ell \geq 1}\)), and has basis given by elements \([z_{a_1,b_1}^C r_1 \cdots z_{a_m,b_m}^C r_m]\) (part (b) from our list).

In the case \(C = \{1\}\), on the other hand, we are back in the situation of Lemma 3.1.2 except that we have to take invariants at the end. So we get the summand \((V/W)^\Gamma \cong V^\Gamma/W^\Gamma\) where \(V, W\) are given in Lemma 3.1.2 (by construction, \(W\) is \(k[\Gamma]\)-module and a \(\Gamma\)-morphism, hence gives an embedding of \(k[\Gamma]\)-modules (which is a \(\mathbb{C}\)-isomorphism). Thus, the modulo-\((r')^2\) map gives a monomorphism

\[
W^\Gamma \hookrightarrow \langle (r') \otimes k[x,y]^\Gamma \rangle,
\]

with image equal to (3.2.16). Explicitly, \(W^\Gamma\) can be obtained by the inverse of (3.2.16) by pulling back \(\sum a_{a,b} x^a y^b \in k[x,y]^\Gamma\) with \(\gcd(a,b)\) to \(\sum \frac{a_{a,b}}{\gcd(a,b)} W_{a,b}\), which must be in \(W^\Gamma\) by the isomorphism.

Since \(V/W\) is torsion-free, so is \((V/W)^\Gamma\). If we considered instead \(V/(W + \langle (r')^\ell \rangle_{\ell \geq 1}\)), then \(V/W\) has a single copy of \(k/p\) in each degree \(p^\ell\) for \(p\) prime and \(p \mid |\Gamma|\), and \(\ell \geq 1\). To conclude that \((V/W)^\Gamma\) has the same torsion, we need only check that each summand of \(k/p\) is \(\Gamma\)-invariant. This follows because it is generated by the image of \(\frac{1}{p}[r^p'] \in F/[F,F]\), which is \(\Gamma\)-invariant.

The rest of the theorem follows immediately. \(\square\)

## 4 Hilbert series, bases, and Theorem 1.1.2

### 4.1 Hilbert series and Question 4.1.9

In this subsection, we explain the consequences of our results for Hilbert series in positive characteristic, and pose a question (this subsection is not needed for the proof of our main results.)

A main result of [EG06] is a formula for the Hilbert series \(h(A; t)\) of certain \(\mathbb{Z}_{\geq 0}\)-graded algebras \(A := \bigoplus_{m \geq 0} A[m]\) over semisimple rings \(R = \mathbb{C}^I\) which are non-commutative analogues of complete intersections, and also for \(A/[A,A]\). For the latter, it turns out to be more natural to describe the vector space \(\mathcal{O}(A) := \text{Sym}(A/[A,A])_+\), where the + means to pass to the augmentation ideal (i.e. pass to the subspace spanned by elements of positive degree), needed to end up with finite dimension in each graded component. (The reason why \(\mathcal{O}(A)\) is more natural to describe is because it is closely related to the subspace of functions on the representation variety which are invariant under change of basis.)

Here, by Hilbert series for \(A\), we mean a power series in \(t\) with coefficients in \(I\)-by-\(I\) matrices with nonnegative entries, i.e.

\[
h(A; t) := \sum_{m \geq 0} \left[ \dim iA[j|m] \right]_{i,j \in I} t^m \subset \mathbb{Z}_{\geq 0}^{I \times I}[[t]].
\]

(4.1.1)

On the other hand, since \(\mathcal{O}(A)\) is just an \(\mathbb{Z}_{\geq 0}\)-graded vector space, its Hilbert series \(h(\mathcal{O}(A); t) \subset \mathbb{Z}_{\geq 0}[[t]]\) is just a power series in \(t\) with nonnegative coefficients.

Computing \(h(\mathcal{O}(A); t)\) is tantamount to computing \(h(A/[A,A]; t)\): namely, if \(h(A/[A,A]; t) = \sum_m a_m t^m\), then

\[
h(\text{Sym}(A/[A,A])_+; t) = \prod_{m \geq 1} \frac{1}{(1 - t^m)^{a_m}}.
\]

(4.1.2)
4.1.1 The non-Dynkin, non-extended Dynkin and partial preprojective cases

First, let us work over the field $k = \mathbb{C}$. Let $(Q, J)$ be any pair where $Q$ is a quiver, $J \subseteq I$ is a subset of vertices, and either $Q$ is non-Dynkin, non-extended Dynkin, or $J \neq \emptyset$. One then has the formulas [EG06]:

$$h(\Pi_{Q,J}; t) = (1 - t \cdot C + t^2 \cdot 1_{I \setminus J})^{-1}, \quad h(O(\Pi_{Q,J}); t) = \left( \frac{1}{1 - t^2} \right)^{\delta_{J,0}} \prod_{m \geq 1} \frac{1}{\det(1 - t^m \cdot C + t^{2m} \cdot 1_{I \setminus J})},$$

(4.1.3)

where $C$ is the adjacency matrix of $Q$, and $\delta_{J,0} = 1$ if $J = \emptyset$ and 0 otherwise.

As noted in [EG06], the above formulas have representation- and number-theoretic interpretations as follows: Let $V = (\bar{Q})$ and let $L = (1_{I \setminus J} r 1_{I \setminus J})$: thus, $\Pi_Q = T_k V/\langle (L) \rangle$. Let $L^0 = L \cap [T_k V; T_k V]$. Then, (4.1.3) expresses as the following formula [EG06]:

$$h(\Pi_{Q,J}; t) = 1 - h(V; t) + h(L; t), \quad h(O(\Pi_{Q,J})) = \prod_{m \geq 1} \frac{1}{1 - h(L^0; t)} \prod_{m \geq 1} \frac{1}{\det(1 - h(V; t) + h(L; t))}.$$ 

(4.1.4)

Since $L$ is a minimal generating bimodule of $\langle (L) \rangle$, the first formula above (for $h(\Pi; t)$) just says that $\Pi$ is a noncommutative complete intersection (NCCI) (cf. Proposition 4.2.2 from [EG06], Theorem 3.2.4, which was known, cf., e.g., [Ani82]), and both formulas above together say that $\Pi$ is an asymptotic representation-complete intersection (asymptotic RCI), by Theorem 3.7.7 and Proposition 3.7.1 in [EG06]. Note also that, in fact, $\Pi$ is Koszul (by Theorem 2.3.4 in [EG05], in the NCCI case, Koszul is just the requirement that the relations be quadratic ($L \subseteq T_k V[2]$)).

Furthermore, the expression $\prod_{m \geq 1} \frac{1}{1 - h(V; t^m) + h(L; t^m)}$ can be viewed as an analogue of the zeta function, so (following [EG06]) we define

$$\zeta(V, L; t) := \prod_{m \geq 1} \frac{1}{1 - h(V; t^m) + h(L; t^m)}. \quad (4.1.5)$$

Finally, the expression $1 - t \cdot C + t^2 = \frac{1}{t}(t + (\frac{1}{t} + t) - C)$, and $(\frac{1}{t} + t) - C$ is the “$t$-analogue” of the Cartan matrix. Also, $\frac{1}{1 - t \cdot C + t^2}$ is the generating function for (suitably normalized) Chebyshev polynomials $\phi$ of the second type, so that $h(\Pi; t) = 1 + \sum_{m \geq 1} \phi_m(C)$.

Our main result, Theorem 1.1.2, together with the fact that $\Pi_Q$ is torsion-free (which we also prove), generalizes the above formulas:

**Proposition 4.1.6.** Take any pair $(Q, J)$ where $Q$ is a quiver and $J \subseteq I$ is a subset of vertices, such that either $I \neq \emptyset$ or $Q$ is non-Dynkin, non-extended Dynkin. Over any field $k$ of characteristic $p > 0$, the Hilbert series of $\Pi_{Q,J}$ and $O(\Pi_{Q,J})$ are given as follows:

$$h(\Pi_{Q,J}) = (1 - t \cdot C + t^2 \cdot 1_{I \setminus J})^{-1}, \quad h(O(\Pi_{Q,J}); t) = \prod_{\ell \geq 0} \frac{1}{1 - t^{2\ell^2}} \prod_{m \geq 1} \frac{1}{\det(1 - t^m \cdot C + t^{2m} \cdot 1)}, \quad (4.1.7)$$

Using the notation $\zeta, V, L, L^0$ above, the formulas become

$$h(\Pi_{Q,J}) = \frac{1}{1 - t \cdot h(V; t) + t^2 \cdot h(L; t)}, \quad h(O(\Pi_{Q,J}); t) = \zeta(V, L; t) \prod_{\ell \geq 0} \frac{1}{1 - h(L^0; t^{2\ell})}. \quad (4.1.8)$$
P. Etingof and the author have also done computer tests of some finitely presented algebras over $\mathbb{Z}$ which are asymptotic RCI over $\mathbb{Q}$, and in most sufficiently random cases, the above formula has held. This motivates the following generalization of Theorem 1.1.2.

**Question 4.1.9.** (P. Etingof and the author) Let $A = T_{\mathbb{Z}(\varphi)} V/(L)$ be a finitely-presented algebra over $\mathbb{Z}(\varphi)$, with $L$ a minimal generating bimodule. Further suppose $L$ is saturated, and $A$ is an asymptotic RCI (meaning that the Koszul complex of the representation variety is asymptotically exact over $\mathbb{Z}(\varphi)$). Is it then true that the $p$-torsion of $HH_0(A)$ is isomorphic to a graded $F_p$-vector space with basis the elements $r_j^{(p^\ell)}$, for $\ell \geq 1$ or 0? Here, $(r_j)$ is the lift to $L$ of an $F_p$-basis of $(L \otimes F_p) \cap ([T_{\mathbb{Z}(\varphi)}^j V, T_{\mathbb{Z}(\varphi)}^j V] \otimes F_p)$, and $\ell \geq 1$ in the case $r_j \in [T_{\mathbb{Z}(\varphi)}^j V, T_{\mathbb{Z}(\varphi)}^j V]$; otherwise $\ell \geq 0$ (and $r^{(1)}$ is the image of $\frac{1}{p} [r_j]$). Furthermore, there is no $p^2$-torsion.

As before, the elements $r_j^{(p^\ell)}$ must generate $\mathbb{Z}/p$-torsion if nonzero; the question asks whether these are always nonzero in good primes (where the algebra is an asymptotic RCI), and that they span the torsion (there is no higher $(\mathbb{Z}/p^2)$ torsion for any good prime $p$).

### 4.1.2 The extended Dynkin case

The formula for $h(\Pi Q, J)$ above still holds when $Q$ is extended Dynkin and $J = \emptyset$, but the second must be modified (since $\Pi Q$ is still an NCCI but no longer an asymptotic RCI).

Suppose that $i_0 \in I$ is an extending vertex of $Q$, i.e., removing $i_0$ and its incident edges leaves one with the corresponding Dynkin quiver. Over $k = \mathbb{C}$ one has the following isomorphisms of graded vector spaces:

$$ (HH^0(\Pi) \otimes k) \cong (i_0 \Pi i_0 \otimes k) \cong (HH_0(\Pi) \otimes k) \oplus k[0], \tag{4.1.10} $$

where the first map is given by the projection $x \mapsto i_0 x i_0$, and the second is by restriction of the obvious projection $\Pi_+ \twoheadrightarrow HH_0(\Pi)_+$, together with $(i_0 \Pi[0] i_0 \otimes k) \cong k[0]$. The first isomorphism follows from the Morita equivalence; the second was first proved in [MOV06]. Note that the above also generalizes to the case of $k = \mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$ or fields in good characteristic containing $|\Gamma|$-th roots of unity, by the Morita equivalence (cf. Section 3.2). We also may deduce the case $k = \mathbb{Q}$ from the above, since $\mathbb{C}$ is flat over $\mathbb{Q}$.

Using the second isomorphism of (4.1.10), we obtain the following formula:

$$ h(O(\Pi) \otimes \mathbb{Q}; t) = h(\text{Sym} (i_0 \Pi i_0 \otimes \mathbb{Q})_+; t) = \prod_{m \geq 1} \frac{1}{(1 - t^m)^{a_m}}, \tag{4.1.11} $$

where

$$ 1 + \sum_{m \geq 1} a_m t^m = h(i_0 \Pi i_0; t) = \left( \frac{1}{1 - t \cdot C + t^2} \frac{1}{1} \right)_{i_0 i_0} = \phi(C)_{i_0 i_0}. \tag{4.1.12} $$

Here, $i_0 i_0$ denotes the entry of the matrix in the $i_0, i_0$ component. There is a general formula for NCCI algebras [EG06]:

$$ h(O(A); t) = h(\text{Sym} HH_2(A); t) \cdot \zeta(V, L; t). \tag{4.1.13} $$
Proposition 4.1.15. Let $\det(i_0) \phi_m(C)_{i0} \geq 1$ and $\det(1 - t^m \cdot C + t^{2m} - 1)^{-1}$. We provide a new, direct proof:

$$\prod_{m \geq 1} \frac{1}{(1 - t^m)^\varphi_m(C)_{i0}} = \frac{1}{1 - t^2} \prod_{m \geq 1} \frac{1}{\det(1 - t^m \cdot C + t^{2m} - 1)},$$

(4.1.14)

where $\varphi_m := \phi_m - \phi_{m-2}$ is the $m$-th Chebyshev polynomial of the first type. Specifically, we show $i_0 \Pi_Q i_0 \cong HH_0(\Pi_Q)$, compute the Hilbert series over any field:

$$\text{Gröbner bases}.$$
Proposition 4.1.18. Let \( Q \) be a Dynkin quiver. Then \( h(\mathcal{O}(\Pi_Q); t) = hT(\tilde{Q}) \), given in (4.1.17) (for \( \tilde{Q} \) = the extended Dynkin quiver associated to \( Q \)).

4.2 Bases and proof of Theorem 1.1.2

We begin with a description of a basis of \( \Pi_Q \) and \( HH_0(\Pi_Q) \) for any non-Dynkin, non-extended Dynkin quiver, in terms of any basis of the extended Dynkin quiver. Then, we produce bases for \( \Pi_Q \) for extended Dynkin and Dynkin \( Q \) in the \( A, D, \) and \( E \) cases separately. Finally, we describe bases of \( HH_0(\Pi_Q) \) in the Dynkin and extended Dynkin cases.

A basic principle is the following, which gives an interpretation of NCCI for finitely presented algebras:

Proposition 4.2.1. Let \( R = kI \) for any field \( k \), let \( V \) be a finitely-generated, free, \( \mathbb{Z}_+ \)-graded (by “weight”) \( R \)-bimodule, and let \( L \subset T_RV \) be a finitely-generated, homogeneous (with respect to total degree), positive-weight \( R \)-subbimodule, which is a minimal generating \( R \)-subbimodule of \( (L) \). Let \( A = T_RV/(L) \). The following are equivalent:

1. \( A \) is an NCCI.
2. Taking associated graded with respect to the filtration by powers of \( (L) \) (“\( L \)-degree”), one obtains a weight-graded \( R \)-bimodule isomorphism \( T_RV \cong A \star R T_RL \).
3. For any graded \( R \)-bimodule section \( A \hookrightarrow T_RV \), one obtains an isomorphism \( A \star R T_RL \cong T_RV \).

Before we prove this, we first recall from [EG06] (Theorem 3.2.4, which was known from, e.g., [Ani82]) the following characterizations of NCCI (the first of which is (4.1.4) discussed earlier):

Proposition 4.2.2. [Ani82], cf. [EG06] The following are equivalent:

1. \( A \) is an NCCI \((= (L)/(L)^2) \) is a projective \( T_RV \)-bimodule); 
2. \( h(A; t) = (1 - h(V; t) + h(L; t))^{-1} \); 
3. The following sequence is exact (“the Koszul resolution”):
   
   \[
   0 \to L \otimes A \to V \otimes A \to A \to R \to 0, \tag{4.2.3}
   \]

   where the second map is given by restriction of the map
   
   \[
   T_RV \otimes A \to V \otimes A, \quad (v_1 \cdots v_n) \otimes a \mapsto v_1 \otimes (v_2 \cdots v_n a), \tag{4.2.4}
   \]

   and the third map is given by multiplication;

4. The following sequence is exact (“Anick’s resolution,” [Ani86]):
   
   \[
   0 \to A \otimes L \otimes A \to A \otimes V \otimes A \to A \otimes A \to A \to 0, \tag{4.2.5}
   \]

   where the second map is given by restriction of the map
   
   \[
   A \otimes T_RV \otimes A \to A, \quad a \otimes (v_1 \cdots v_n) \otimes b \mapsto \sum_{i=1}^n (av_1 \cdots v_{i-1}) \otimes v_i \otimes (v_{i+1} \cdots v_n b), \tag{4.2.6}
   \]

   and the third map is given by
   
   \[
   a \otimes v \otimes b \mapsto av \otimes b - a \otimes vb. \tag{4.2.7}
   \]
It is actually very easy to prove the equivalence of parts (2), (3), and (4): this is just the Euler-Poincaré principle (the alternating sum of Hilbert series of an exact sequence is zero).

**Proof of Proposition 4.2.1.** It is clear that parts (2) and (3) are equivalent since \( k \) is a field (this would also follow more generally if \( A \) is a free \( k \)-module). We show that (2) is equivalent to exactness of (4.2.3) ((3) of Proposition 4.2.2). It is easy to see that exactness is equivalent to injectivity of \( L \otimes A \to V \otimes A \) (by definition of \( A \)). The latter is equivalent to the following formula in \( TRV \):

\[
(L \otimes TRV) \cap (V \otimes (L)) = L \otimes (L) \tag{4.2.8}
\]

This last equation obviously follows from (2) or (3). Conversely, we may proceed by induction on \( L \)-degree. As the base case, it is clear that \( A \sim gr_0 TRV \). Then, (4.2.8) shows that, in each component of the associated graded of \( TRV \) with respect to powers of \((L)\), \( L \otimes (A^*) \) is linearly independent with \( A^+ L(A^* TRL) \). Then, by induction, we obtain the desired result provided that the multiplication \( L \otimes ((L)) \to ((L)) \subset TRV \) is injective. This follows from minimality of \( L \).

Note that there is a simple alternative proof using the formula (Lemma 2.2.5 of [EE05]):

\[
h(A) = \frac{1}{1-\alpha}, h(B) = \frac{1}{1-\beta} \Rightarrow h(A* B) = \frac{1}{1-\alpha-\beta}, \quad \text{if} \quad \alpha, \beta \in tZ[[t]] \otimes \text{End}(R), \tag{4.2.9}
\]

to show that (2) of Proposition 4.2.1 is equivalent to (2) of Proposition 4.2.2. (Note that (4.2.9) is just \( TRV \star R TRW \sim TR(V \oplus W) \) in the case that \( \alpha, \beta \) are positive, or for arbitrary \( \alpha, \beta \) if one allows \( V \) and \( W \) to be graded super-vector spaces.) We preferred a proof using tensors rather than Hilbert series.

We do not actually need to say much about NCCI in general, but we mention these to explain the meaning of conditions (2), (3) of Proposition 4.2.1 which we will use heavily. Also, we will use Anick’s resolution (4.2.5) later (to study higher Hochschild cohomology: Section 10).

Proposition 4.2.10. (i) Let \( A, B \) be \( R \)-algebras. Then, \( A * R B \) is an NCCI iff \( A \) and \( B \) are NCCIs. More generally, suppose we have a sequence

\[
\begin{array}{cccc}
B & \xleftarrow{g_t} & C & \twoheadrightarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
B & \xleftarrow{g_t} & A * R B & \twoheadrightarrow & A \\
\end{array}
\tag{4.2.11}
\]

of graded \( R \)-algebra morphisms except for the middle vertical arrow, which is an isomorphism of graded \( R \)-bimodules (but not an algebra homomorphism). Suppose that \( C \) is generated as an algebra by an (any) graded \( R \)-bimodule section \( \tilde{A} \subset C \) of \( A \), and that the middle vertical arrow is obtained by taking associated graded with respect to the descending filtration generated by the grading on \( B \) and by \( |A \setminus \{0\}| = 0 \). Then, \( A * R B \) is an NCCI iff \( A \) and \( B \) are.

(ii) Assume \( L \supset L' \) are finitely-generated bimodules, which minimally generate the ideals \((L), (L')\). If \( TRV/(L) \) is an NCCI, then \( TRV/(L') \) is an NCCI.
Proof. (i) Set \( A = TRV/(L) \) and \( B = TRW/(L') \) for minimal, finitely-generated \( L, L' \), so that \( A *_R B = TR(V \oplus W)/(L, L') \), and \( C = TR(V \oplus W)/(L', \tilde{L}) \) for some \( \tilde{L} \) mapping isomorphically to \( L \) under the projection to \( TRV \). The result then follows immediately from (2) of Proposition 4.2.10

(ii) Assume \( TRV/(L) \) is NCCI. Then, we have

\[
TRV \cong TRV/(L) *_R TRL
\]

(4.2.12)
as as graded \( R \)-bimodules (considering some section \( TRV/(L) \hookrightarrow TRV \)). Modding by \( (L') \), we obtain

\[
TRV/(L') \cong TRV/(L) *_R (TRL/(L')), 
\]

(4.2.13)

By part (i), we may conclude the desired result if \( TRL/(L') \) is an NCCI. But, this is obvious since \( L' \subset L \) is just a subbimodule. \( \square \)

More generally, part (ii) of Proposition 4.2.10 shows that, for any NCCI \( TRV/(L) \) (with \( L \) minimal and finitely-generated), if \( L' \subset TRL \), then letting \( \tilde{L}' \) be the image of \( L' \) in \( TRV \), one has that \( TRV/(\tilde{L}') \) is an NCCI iff \( TRL/(\tilde{L}') \) is an NCCI.

Now, we note that all of the above results also hold for \( k = \mathbb{Z} \), dealing with free \( \mathbb{Z} \)-modules (since the definition, in Proposition 4.2.2(i), of an NCCI includes being a free \( \mathbb{Z} \)-module). Then, Hilbert series is computed for free \( \mathbb{Z} \)-modules by tensoring with \( \mathbb{Q} \).

We then deduce the following important result (needed for writing bases), now with \( R = \mathbb{Z}^I \):

**Corollary 4.2.14.**

(i) If \( \Pi_{Q,J} \) is an NCCI and \( J \subset J' \), then \( \Pi_{Q,J'} \) is an NCCI, and

\[
\Pi_{Q,J'} \cong \Pi_{Q,J} *_R TR\langle 1_{J' \setminus J} r 1_{J' \setminus J} \rangle, 
\]

(4.2.15)

by taking associated graded with respect to powers of the ideal \( \langle 1_{J' \setminus J} r 1_{J' \setminus J} \rangle \). One also has an isomorphism \( \sim \) by choosing any graded \( R \)-bimodule section \( \Pi_{Q,J} \twoheadrightarrow \Pi_{Q,J'} \);

(ii) If \( Q = Q_1 \cup Q_2 \) is the disjoint union of two quivers on the same vertex set \( I \), and \( I_0 \subset I \) is the set of vertices incident to edges from both \( Q_1 \) and \( Q_2 \), then if \( \Pi_{Q_1,J} \) is an NCCI, one has

\[
\Pi_{Q,J} \cong \Pi_{Q_1,J} *_R \Pi_{Q_2,J \cup I_0}, 
\]

(4.2.16)

by taking associated graded with respect to the filtration by powers of the ideal \( \langle 1_{Q_2} \rangle \). Similarly, one has

\[
\Pi_{Q,J} \sim \Pi_{Q_1,J} *_R \Pi_{Q_2,J \cup I_0} 
\]

(4.2.17)

by picking any weight-graded \( R \)-bimodule section \( \Pi_{Q_1,J} \subset \Pi_{Q,J} \) of the quotient, and using the composition \( \Pi_{Q_2,J \cup I_0} \twoheadrightarrow \Pi_{Q,J \cup I_0} \twoheadrightarrow \Pi_{Q,J} \) of obvious maps.

(iii) In the situation of (ii), \( \Pi_{Q,J} \) is an NCCI iff \( \Pi_{Q_2,J \cup I_0} \) is an NCCI.

**Proof.** (i) Let \( V = \langle Q \rangle \). We have

\[
TRV = \tilde{\Pi}_{Q,J} *_R TR\langle 1_{J' \setminus J} r 1_{J' \setminus J} \rangle = (\tilde{\Pi}_{Q,J} *_R TR\langle 1_{J' \setminus J} r 1_{J' \setminus J} \rangle) *_R (1_{J' \setminus J} r 1_{J' \setminus J}). 
\]

(4.2.18)

If we mod by \( TR1_{J' \setminus J} r 1_{J' \setminus J} \), we obtain (4.2.15), which together with (4.2.18) proves the desired result.
(ii) We have
\[ \Pi_{Q,J \cup I_0} \cong \Pi_{Q_1,J_\cup I_0} \ast_R \Pi_{Q_2,J \cup I_0}. \]
Now, let \( V_1 = \langle Q_1 \rangle \) and \( V_2 = \langle Q_2 \rangle \). By hypothesis and (4.2.15), we may rewrite (4.2.19) as
\[ \Pi_{Q,J \cup I_0} \cong \Pi_{Q_1,J} \ast_R (1_{I_0} \cup r_Q 1_{I_0} \cup J) \ast_R \Pi_{Q_2,J \cup I_0}. \]
Now, modding by \( (1_{I_0} \cup r_Q 1_{I_0} \cup J) \), we replace all instances of \( 1_{I_0} \cup r_Q 1_{I_0} \cup J \) with \(-1_{I_0} \cup r_Q 2_{I_0} \cup J\), and obtain (4.2.16).

(iii) This follows from (4.2.16) and Proposition 4.2.10(i).

The corollary implies the inductive result used in [EE05] to show that non-Dynkin quivers or partial preprojective algebras with at least one white vertex are Koszul: that is, it allows one to reduce to the case of extended Dynkin quivers and star-shaped quivers whose special vertex is white and whose branches are of unit length.

4.2.1 Bases of \( \Pi_Q \) and \( HH_0(\Pi_Q) \) for non-Dynkin, non-extended Dynkin quivers, and for partial preprojective algebras; proof that \( r^{(p)} \neq 0 \) From now on, we will set \( R = k' \). We begin with the case of partial preprojective algebras:

**Proposition 4.2.21.** Take any quiver \( Q \) on vertex set \( I \), together with a nonempty subset of white vertices, \( J \subset I \). Let \( G \subset Q \) be a forest such that the map \( G \to I, e \mapsto e_0 \) (the source vertex) yields a bijection \( G \cong I \setminus J \). Then, a free \( Z \)-basis of \( \Pi_{Q,J} \) is given by monomials in the edges \( Q \) that do not contain a subword \( ee^* \) for any \( e \in G \). In particular, \( \Pi_{Q,J} \) is an NCCI.

Furthermore, \( \Lambda_{Q,J} \) is a free \( Z \)-module with basis given by cyclic words not containing \( ee^* \) for any \( e \in G \).

**Proof.** A forest satisfying the given condition can be constructed inductively as follows: To begin, for every vertex in \( I \setminus J \) which is adjacent to \( J \), add an edge to \( G \) with source at that vertex and target in \( J \). Inductively, for each vertex of \( I \setminus J \) which is not adjacent to \( G \), add an edge with source at that vertex and target at a vertex adjacent to \( J \). When the process is completed, one clearly arrives at a forest satisfying the desired condition.

The first result follows immediately from the Diamond Lemma (Propositions A.0.28 A.0.30) if we let the partial order on monomials be given by the number of undesirable subwords \( ee^* \) that appear. To see that \( \Pi_{Q,J} \) is an NCCI, we show (2) of Proposition 4.2.1. To do this, we adjoin generators \( r_i (= iiri) \) for all \( i \in I \setminus J \), and apply the Diamond Lemma to reduce any path to a unique sum of monomials in \( \overline{Q} \) and the \( r_i \) not containing \( ee^* \) for any \( e \in G \).

For the final result, we note that one may still use the Diamond Lemma for \( \Lambda_{Q,J} \) because the maximum number of swaps \( ee^* \leftrightarrow e'e \) for \( e \in G \) that may be performed in a cyclic word is still finite, since \( e \in G \) cannot be a loop. This follows reverse-inductively on the distance of an edge \( e \) from \( J \), using that this distance is bounded.

The fact that \( \Pi_{Q,J} \) is an NCCI was first shown in [EE05] using Hilbert series, and in [EG06], \( \Pi_{Q,J} \) was further shown to be a RCI. Note that the fact that \( \Lambda_{Q,J} \) is torsion-free over \( Z \) is new.

As an application of the proposition, by comparing the above basis and the formula (4.1.4) for \( h(\mathcal{O}(\Pi_{Q,J}); t) \) in the asymptotic RCI case (a special case of RCI), one obtains a formula for computing the number of cyclic words in letters \( x_i, y_i, z_i \) not containing \( x_i y_i \) for any \( i \). Conversely, if
Proposition 4.2.25. The algebra $\Pi_Q$ is an NCCI over $\mathbb{Z}$ if $Q$ is non-Dynkin (in particular, it is torsion-free).

Proof. By Corollary 4.2.14 (and the comments afterward), and Proposition 4.2.21 one may reduce to the case that $Q$ is extended Dynkin. For the extended Dynkin case, the easiest proof is to use our bases to show that $\Pi_Q$ is torsion-free; then, after tensoring with $\mathbb{C}$, one may use the Morita equivalence of $\Pi_Q$ with $\mathbb{C}[x,y] \rtimes \Gamma$ from Section 3.2. Alternatively, one could deduce the NCCI property from our computation of bases with some effort (this is only a challenge in the $\tilde{E}_n$ cases: the easiest way to do it is to compute Gröbner bases of $\Pi_Q$ over $\mathbb{Z}$, instead of for $i_s \Pi_Q i_s$ over $\mathbb{Z}$. This is easy to do with MAGMA or even by hand, but the result takes more space to describe, so we did not include it here.)

For any non-Dynkin, non-extended Dynkin quiver $Q$ with vertex set $I$, it is well-known (and easy to check) that $Q \supseteq Q_0$ for some extended Dynkin quiver $Q_0$ with vertex set $I_0$. The results of the previous section then allow us to write

$$\Pi_Q = \Pi_{Q_0} \ast_R \Pi_{Q\setminus Q_0,I_0},$$

(4.2.23)

where $\Pi_{Q_0}$ is an arbitrary graded $R$-bimodule section of $\Pi_Q \twoheadrightarrow \Pi_{Q_0}$, and $\Pi_{Q\setminus Q_0,I_0}$ embeds canonically into $\Pi_Q$ via the sequence $\Pi_{Q\setminus Q_0,I_0} \hookrightarrow \Pi_{Q,I_0} \twoheadrightarrow \Pi_Q$.

Notation 4.2.24. In general, if $Q \supseteq Q_0$ where $Q_0$ is extended Dynkin, then we will fix a graded $R$-bimodule section $\Pi_{Q_0} \subset \Pi_Q$ of $\Pi_Q \twoheadrightarrow \Pi_{Q_0}$, which exists because $\Pi_{Q_0}$ is torsion-free. Then, for any subspace $U \subset \Pi_{Q_0}$, we denote its image under the section by $\Pi_U$.

Now, we proceed to one of our main goals: a description of $\Lambda_Q$ when $Q$ is non-Dynkin and non-extended Dynkin. We begin with the

Proposition 4.2.25. Let $Q \supseteq Q_0$ where $Q_0$ is non-Dynkin. Let $V := \Pi_Q/[[((Q \setminus \tilde{Q}_0))],\Pi_Q]$. Let $B$ be the algebra $B := T_R((\Pi_{Q_0})_+ \otimes_R (\Pi_{Q\setminus Q_0,I_0})_+)$. We have

(i) 

$$V \cong \Pi_{Q_0} \oplus (B/[B,B])_+ \oplus \Lambda_{Q\setminus Q_0,I_0},$$

(4.2.26)

as $\mathbb{Z}$-modules, where the map is given by taking the associated graded with respect to the filtration by powers of the ideal $((Q \setminus \tilde{Q}_0))$. (Note that $B/[B,B]$ has a basis of “alternating cyclic words in $(\Pi_{Q_0})_+$ and $(\Pi_{Q\setminus Q_0,I_0})_+$.”)

(ii) We have $\Lambda_Q \cong V/W$ where $W$ is the image in $V$ of $[\Pi_{Q_0},(\tilde{Q}_0)] + [\Pi_{Q_0},R]$, using Notation 4.2.24.

(iii) Let $\tilde{V} := V/[[\Pi_{Q_0},R]] \cong \bigoplus_{i \in I} iVi$, and let $\Pi_{Q_0} = \bigoplus_{i \in I} i\Pi_{Q_0}$. Then

$$\tilde{V} \cong \Pi_{Q_0} \oplus (B/[B,B])_+ \oplus \Lambda_{Q\setminus Q_0,I_0},$$

(4.2.27)

$$\Lambda_Q \cong \tilde{V}/\tilde{W},$$

(4.2.28)

where $\tilde{W}$ is the image of $[\Pi_{Q_0},(\tilde{Q}_0)] \cap \Pi_Q$. 

34
Proof. The first part follows from (4.2.23) and its proof, together with the Diamond Lemma argument from Lemma 3.1.2 (2). The second part follows from the observation (cf. Lemma 3.1.2 (3)) that \([\Pi_0, \Pi_0] = [\Pi_0(\Pi \setminus \Pi_0) \Pi_0, \Pi_0] + [\Pi_0^\perp(\Pi \setminus \Pi_0^\perp)]\). The third part follows readily from the second.

Now, we describe the space \(W\) away from bad primes:

**Proposition 4.2.29.** Let \(Q \supseteq Q_0\) where \(Q_0\) is extended Dynkin, corresponding to the group \(\Gamma \subset SL_2(\mathbb{C})\). Let us work over \(\mathbb{Z}^{[1]}\). Let \(W_0 \subset W\) be any section of \([\Pi_0, \Pi_0]\) under the composition \(\Pi_0 \rightarrow \Pi_0\) induced by inclusion. Then, for any \([w] \in \mathbb{Z}^{[1]}\),

- (i) \(h(W'; t) = t^2 \cdot h(HH^0(\Pi_0)+; t)\).
- (ii) The composition \(W' \leftarrow V' \rightarrow V'/[[r]]\) is injective, giving an isomorphism \(W' \cong \bigoplus_{m \geq 0} [rHH^0(\Pi_0)+],\) using Notation 4.2.24.

Proof. This follows from Theorem 3.2.13(iii) and the partial proof of Theorem 1.1.2 contained there, if one passes to \(\kappa := \mathbb{Z}^{[1]}_{\frac{1}{m}}\), using the projection \(\kappa[x,y] \times \Gamma \rightarrow f(k[x,y] \times \Gamma)f \cong \Pi_0\). Since \(\mathbb{Z}^{[1]}_{\frac{1}{m}}\) is flat over \(\kappa = \mathbb{Z}^{[1]}_{\frac{1}{m}}\), the result must already have been true over the latter.

To prove Theorem 1.1.2 in full generality, we generalize it and the above result by an analysis in each prime \(p\) using \(p\)-th powers, as follows.

Let us define \(W' := W \cap [[[r]]]\). Let \(W'_p \subset V \otimes F_p\) be the image of the map \(W' \otimes F_p \rightarrow V \otimes F_p\) induced by inclusion. Then, for any \([w] \in W_p',\) we may consider \([w]^p \in (V \cap [[[r]]) \otimes F_p\). We have that \([w]^p \rightarrow 0 \in HH_0(\Pi_0),\) since the same is true for \([w]\). Hence, \([w]^p \in W'_p\) as well. This observation allows us to state the following theorem, which will be proved in Section 8 and refines the main Theorem 1.1.2.

**Theorem 4.2.30.** Let us work over \(\mathbb{Z}^{(p)}\) for some prime \(p\). Let \(Q_0\) be any extended Dynkin quiver and \(Q \supseteq Q_0\). Define \(W \subset V\) as in Proposition 4.2.25. Then, \(W\) has the form

\[W = W_0 \oplus W',\]

where \(W' = W \cap [[[r]]]\), \(W_0\) is a section of \([\Pi_0, \Pi_0]\), satisfying:

- (i) \(W_0 \otimes \mathbb{Z}^{(p)}\) is saturated except in the cases that \((Q_0,p) \in \{(\tilde{D}_n, 2), (\tilde{E}_n, 2), (\tilde{E}_n, 3), (\tilde{E}_8, 5)\},\) when \(W_0 = W_{0,s} \oplus W_{0,r}\) with \(W_{0,s} \otimes \mathbb{Z}^{(p)}\) saturated, and \(W_{0,r}\) has finite rank and will be described in (iii).

- (ii) \(W' = W'_s \oplus W'_r,\) where \(W'_s \otimes \mathbb{Z}^{(p)}\) is saturated (see (iv)) and \(W'_r\) will be described in (iii).

- (iii) \(W_r := W_{0,r} \oplus W'_r\) has a basis of elements \(\{f_\ell\}, \ell \geq 1,\) with \(|f_\ell| = 2p^\ell,\) satisfying

\[\text{ord}_p(f_\ell) = p, \quad \ell \equiv 0 (\text{mod } p),\]

where \(\text{ord}_p\) denotes the greatest power of \(p\) dividing the argument. The Hilbert series of \(W_{0,r}\) is the same as the span of the nonzero elements \(r^{(p)}\) in \(HH_0(\Pi_0)\) (cf. Theorem 4.2.60). Furthermore, the image of \(\frac{1}{p}f_\ell\) in \(\Lambda_\mathbb{Q} \otimes \mathbb{Z}^{(p)}\) is \(\eta^{(p)}\).
(iv) The space $W^t$ has a basis of elements $\{g_{i,t}\}$ as follows:

$$g_{i,t+1} \equiv g_{i,t}^p \pmod{p} \quad (4.2.33)$$

and the $g_{i,0}$ span a subspace which projects isomorphically mod $([(r')]^2) + (p)$ to $[r^p]$, where $U_p \subset HH^0(\Pi_{Q_0}) \otimes \mathbb{F}_p$ is a certain subspace with Hilbert series $h(HH^0(\Pi_{Q_0}) + t) - t^{2p-2}h(HH^0(\Pi_{Q_0})p) + C_p(t)$, where $C_p(t) = 0$ unless $HH_0(\Pi_{Q_0})$ has $p$-torsion, in which case, letting $m$ be the smallest positive integer such that $r^{(p^m)}$ is zero in $HH_0(\Pi_{Q_0})$,

$$t^2 C_p(t) = t^{2p} - t^{2p^m} + \left\{ \begin{array}{ll}
\sum_{\ell=1}^{\frac{t}{2}} t^{4(2\ell+1)} - t^{2\log_2 \left( \frac{2(t-2)}{\ell} \right) + 1} (2\ell+1), & \text{if } p = 2 \text{ and } Q_0 = \tilde{D}_n, \\
\frac{4 r}{t} - t^{56}, & \text{if } p = 2 \text{ and } Q_0 = \tilde{E}_8.
\end{array} \right. \quad (4.2.34)$$

In particular, $i_0 U_p i_0$ contains the image of the Poisson bracket $\{,\} : (i_0 \Pi_{Q_0} i_0 \otimes \mathbb{F}_p)^{\otimes 2} \to (i_0 \Pi_{Q_0} i_0 \otimes \mathbb{F}_p)$ over $\mathbb{F}_p$.

Here, the Poisson algebra $i_0 \Pi_{Q_0} i_0 \otimes \mathbb{F}_p$ will be defined in Section 5, it is an analogue of $\mathbb{C}[x,y]^G$ (and, if $p$ is a good prime, then $i_0 \Pi_{Q_0} i_0 \otimes \mathbb{F}_p \cong \mathbb{F}_p[x,y]^G$ by Theorem 5.1.1). For good primes, the above theorem (except for the statement about Poisson bracket) is not difficult to deduce from Theorem 3.2.13.

Theorem 4.1.12 immediately follows from Theorem 4.2.30, since we deduce that the torsion of $HH_0(\Pi_Q)$ for $Q \supseteq Q_0$ is at most $\mathbb{Z}/p$ in each degree $2p^2$, and this is spanned by $r^{(p^2)}$.

Proposition 4.2.29 (together with Propositions 4.2.25 and 4.2.31) gives us explicit $\mathbb{Q}$-bases of $\Lambda_Q$:

**Proposition 4.2.35.** For any non-Dynkin, non-extended Dynkin quiver $Q$, a $\mathbb{Q}$-basis for $HH_0(\Pi_Q \otimes \mathbb{Q})$ is given as follows, depending on a choice of extended Dynkin subquiver $Q_0 \subsetneq Q$ on vertices $I_0 \subsetneq I$, and a fixed vertex $i' \in I_0$ adjacent to an edge of $Q \setminus Q_0$:

1. A basis of $i_0 \Pi_{Q_0} i_0 \otimes \mathbb{Q} \cong HH_0(\Pi_{Q_0} \otimes \mathbb{Q})$ (given in Theorems 4.2.30 (iii) and 4.2.31 (i), and Proposition 4.2.29 (ii));

2. Alternating cyclic words in a basis of $(\Pi_{Q_0})_+$ (given by the aforementioned theorems and proposition) and a basis of $(\Pi_{Q \setminus Q_0, I_0})_+$ (given by Proposition 4.2.21) such that

   (a) there is at least one basis element of each of $(\Pi_{Q_0})_+$ and $(\Pi_{Q \setminus Q_0, I_0})_+$ included, and

   (b) the source and target vertices of the basis elements match up (to give a nonzero product), excluding elements of the form $[i' e^* e \pi]$, where $e \in G$ and $\pi \in HH_0(\Pi_{Q_0})$, where $G \subset \overline{Q} \setminus Q_0$ is the forest chosen in Proposition 4.2.21 (so $e$ must be the unique edge of $G$ adjacent to $i'$);

3. A basis of $HH_0(\Pi_{Q \setminus Q_0, I_0} \otimes \mathbb{Q})$, given in Proposition 4.2.21.

Theorem 4.2.30 then gives explicit $\mathbb{F}_p$-bases of $\Lambda_Q \otimes \mathbb{F}_p$.

**Proposition 4.2.36.** For any non-Dynkin, non-extended Dynkin quiver $Q$, a $\mathbb{F}_p$-basis for $HH_0(\Pi_Q \otimes \mathbb{F}_p)$ is given as follows, depending on a choice of extended Dynkin subquiver $Q_0 \subsetneq Q$ on vertices $I_0 \subsetneq I$ and a fixed vertex $i' \in I_0$ adjacent to an edge of $Q \setminus Q_0$:
1. A basis of $i_0\Pi_{Q_0}i_0 \otimes \mathbb{F}_p$;

2. alternating cyclic words in a basis of $(\Pi_{Q_0})_+$ and a basis of $(\Pi_{Q_0}, s_0)$, satisfying (a) and (b) in Proposition 4.2.35, excluding elements of the form $[i^t(e^\ell e^t)]$ where $e \in G$ and $\pi \in U_0 \subset P_0$ (using Notation 4.2.24 with $U_0$ defined in Theorem 4.2.30, which we assume is compatible with the chosen basis);

3. A basis of $HH_0(\Pi_{Q_0}, s_0 \otimes \mathbb{F}_p)$.

In the next two subsections, we describe a basis of $\Pi_{Q_0}$ for extended Dynkin quivers (Theorems 4.2.40 and 4.2.48 and Proposition 4.2.50), which is all that remains to obtain bases for $\Pi_Q$ for all non-Dynkin quivers, and for $HH_0(\Pi_Q) \otimes \mathbb{Q}$ using Proposition 4.2.29. We also explicitly describe $W$ in the cases $Q_0 \in \{A_n, D_n\}$, which furnishes simple $\mathbb{Z}$-bases of $HH_0(\Pi_Q)$ for all non-Dynkin, non-extended Dynkin quivers which are not stars with three branches (Theorems 4.2.40 and 4.2.48).

In the final subsection, we compute bases and the abstract structure of $HH_0(\Pi_Q)$ for Dynkin and extended Dynkin $Q$ (Theorem 4.2.60).

We remark that, using just what we know at this point (without requiring the proofs we have postponed), it is easy to give a proof of one direction of Theorem 1.1.2 that the elements $r^{(p^\ell)}$ are nonzero (we will not need this for the proof of the theorem):

**Proposition 4.2.37.** For any non-Dynkin, non-extended Dynkin quiver $Q$, any prime $p > 0$ and any $\ell \geq 1$, the element $r^{(p^\ell)} \in HH_0(\Pi_Q)$ is nonzero.

**Proof.** For any quiver $Q_0$, we may perform the same procedure as in Proposition 1.2.25 to obtain a basis of $A' := P_{Q_0}([r'_i]_{i \in I_0}/([i r_i + r'_i])_{i \in I_0} \cong P_{Q_0}$ (thought of as $\Pi_Q$ for $Q = (Q_0)_{I_0, \infty}$). First, (4.2.23) becomes $A' \cong \Pi_{Q_0} \ast_{k_0} [r'_i]$ where we view $r'_i$ as $k_{Q_0}$-modules by $jr'_ij = \delta_{ij}r'_j$. Let $r' := \sum_i r'_i$. Then, Proposition 4.2.23 presents $V_{A'} := A'/[A', A'r'A']$ as the space $\Pi_{Q_0} \otimes$ alternating cyclic words in a basis of $\Pi_{Q_0}$ and $(r'\ell, \ell \geq 1)$. We may also compute the relations $W_{A'}$ as in the proposition. For any quiver $Q \supseteq Q_0$, we have a canonical map $A' \rightarrow \Pi_Q$ which induces maps $V_{A'} \rightarrow V_Q$ and $W_{A'} \rightarrow W_Q$. It is easy to see that $W_{A'} \rightarrow W_Q$ is a surjection, since the relations are spanned by commutators $[\Pi_{Q_0}, \Pi_{Q_0}]$, which can be taken in $A'$.

On the other hand, by Proposition 4.2.29, the rank of $W_Q$ (which is free) does not depend on the choice of $Q$, but only on $Q_0$ (provided $Q \supseteq Q_0$). Also, if $Q' \supseteq Q_0$ is the quiver obtained from $Q_0$ by adjoining a loop to each vertex in $I_0$, then the map $V_{A'} \rightarrow V_{Q'}$ has kernel equal to $\langle [r'(\ell)] \rangle_{\ell \geq 1}$. By Proposition 4.2.29, this shows that the map $W_{A'} \rightarrow W_{Q'}$ has kernel equal to $\langle [r'] \rangle$ (no higher powers of $r'$ live in the space of relations by Theorem 3.2.13 (iii)). So, we obtain an isomorphism $W_{A'}/\langle [r'] \rangle \cong W_{Q'}$. Since $W_Q, W_{Q'}$ have the same Hilbert series, one also must obtain an isomorphism $W_{A'}/\langle [r'] \rangle \cong W_Q$. Finally, the kernel of $V_{A'} \rightarrow V_Q$ contains the kernel of $V_{A'} \rightarrow V_{Q'}$, since $\langle [r'(\ell)] \rangle$ is zero in $V_Q$.

Hence, in each graded degree $m$, one obtains an isomorphism $W_{Q'}[m] \cong W_Q[m]$, and a monomorphism of their saturations, $Sat(W_{Q'}[m]) \hookrightarrow Sat(W_Q[m])$. In particular, if we lift the element $r^{(p^\ell)}$ in any way to $V_{Q'}$, it lies in $Sat(W_{Q'}[m]) \setminus W_Q[m]$, and hence any lift to $V_Q$ also has this property. \[\square\]

### 4.2.2 Bases of $\Pi_Q$ for type $A$ and $D$ quivers and the remainder of Theorem 1.1.2

In this section, we prove Theorem 1.1.2 in the case of quivers properly containing an extended Dynkin quiver of type $\tilde{A}$ or $\tilde{D}$ (i.e., quivers which are not star-shaped with three branches). The resulting Theorems 4.2.40 and 4.2.48 also provide $\mathbb{Z}$-bases for $\Pi$ for these quivers and for the extended Dynkin
quivers $\tilde{A}_n, \tilde{D}_n$. We also briefly explain how to deduce from this $\mathbb{Z}$-bases of $\Pi$ in the Dynkin case. We postpone computing $HH_0(\Pi)$ in the Dynkin and extended Dynkin cases until Theorem 4.2.60 (at least in the $D$-cases).

We begin with the case $Q = \tilde{A}_{n-1}$. Let us define the following:

$$x = \sum_{e \text{ is counter-clockwise-oriented}} e, \quad y = \sum_{e \text{ is clockwise-oriented}} e. \quad (4.2.38)$$

Let the vertex set $I$ be given the natural structure of a $\mathbb{Z}/n\mathbb{Z}$-torsor, where adding one means moving once counter-clockwise. Finally, we assume that $Q_0$ has counter-clockwise orientation:

![Figure 1: $Q_0 = \tilde{A}_{n-1}$ with the preferred orientation](image)

Remark 4.2.39. This last assumption on orientation is not essential: if we don’t assume it, we write $r = \sum_{i \in I} i(\pm xy \pm yx + r')$ for some choices of signs $\pm$, where $r'$ is still given by $r' = \sum_{e \in \Pi \setminus Q_0} [e,e^*]$. With the above orientation, one has the same formula $r = xy - yx + r'$ as in Lemma 3.1.2. The only part of the Theorem below that changes in the general case is the formula for $W_0 ((iv),(v))$, which only changes by signs.

We then have the

**Theorem 4.2.40.** Let $Q_0 = \tilde{A}_{n-1}$ with the above notation and orientation.

(i) For any $i, j \in I$, a basis of $i\Pi_{Q_0,j}$ is given by $ix^ay^bj$ for $(a-b) \equiv (j-i) \pmod{n}$.

(ii) A basis of $i\Pi_{Q_0,j}$ is also given by the nonzero elements $iz_{a,b}j$ (3.1.3), which are equal to the $ix^ay^bj$ above.

(iii) $HH_0(\Pi_{Q_0})$ is a free $\mathbb{Z}$-module with basis given, for any fixed $i_0 \in I$, by the elements $[i_0x^ay^b] = [i_0z_{a,b}]$ for $a,b \geq 0$ and $n \mid (b-a)$.

(iv) A basis of $HH^0(\Pi_{Q_0})$ is given by $z_{a,b}$ for $a,b \geq 0$ and $n \mid (b-a)$.

(v) For any $Q \supseteq Q_0$, and any fixed vertex $i_0 \in I_0$, the $W$ of Proposition 4.2.25 (ii) has the form $W = W' \oplus W_0$, where $W_0$ is any section of $[\Pi_Q, \Pi_Q]$ under (4.2.26), and $W'$ is a free $\mathbb{Z}$-module with basis the elements $W_{a,b}$ given by (3.1.4)–(3.1.6). As in the case of Lemma 4.1.2, $W_{a,b}$ is a $\mathbb{Z}$-generator of the space $\langle [iz_{a-1,b,x}], [iz_{a,b-1,y}] \rangle_{i \in I}$.
(vi) The elements $W_{a,b}$, for $(a,b) \neq (p^d,p^e)$, span a saturated subspace of $W'$. The order of $W_{p^d,p^e}$ (modulo the other $W_{a,b}$ or otherwise) is $p$.

(vii) The image of $\frac{1}{p}W_{p^d,p^e}$ in $HH_0(\Pi Q)$ is nonzero, equal to $r(p^d)$, and these elements span the torsion of $HH_0(\Pi Q)$.

Parts (i)-(iv) are easy and left to the reader. The remainder of the theorem will be proved in Section 6. As a corollary, one deduces Theorem 1.1.2 in this case, and moreover easily obtains a $\mathbb{Z}$-basis of $HH_0(\Pi Q)$ (i.e., a basis of the free part and $\mathbb{F}_p$-bases of all of the $p$-torsion for all $p$).

Note that one way to obtain bases for $\Pi A_n$, where $A_n$ is the Dynkin quiver, is by discarding the elements of the above basis for $\Pi \tilde{A}_n$ which are zero under the quotient $\Pi \tilde{A}_n \to \Pi A_n$: these are elements that either pass through the extending vertex $i_0$ (which is cut from $\tilde{A}_n$ to obtain $A_n$) or can be made to pass through this vertex by unfolding a power of $xy$. This is true because, for each $i,j \in I$ and $m \geq 1$, there is at most one basis element in $i\Pi \tilde{A}_n[j][m]$ above that projects to a nonzero element of $i\Pi A_n[j][m]$.

Next, we describe the $D$ cases. Let $Q_0 = \tilde{D}_n$. We will need the following notation. Suppose that $Q_0$ is drawn and oriented as follows:

![Figure 2: $Q_0 = \tilde{D}_n$ with the preferred orientation](image)

As in the figure, we let $i_{LU}, i_{LD}, i_{RU}, i_{RD}$ denote the four external vertices ($L, R, U, D$ stand for “left, right, up, down”, respectively). Furthermore, we set $i_L := i_{LU} + i_{LD}$ and $i_R := i_{RU} + i_{RD}$, the sum of the leftmost and rightmost external vertices. We then define

$$1_{in} := \sum_{i \text{ internal}} i. \quad (4.2.41)$$

the sum of internal vertices.

Next, we define

$$R := \sum_{e \in E_0 | e \text{ is oriented rightward}} e, \quad L := \sum_{e \in E_0 | e \text{ is oriented leftward}} e. \quad (4.2.42)$$

Here, $R$ and $L$ should be thought of as similar to $x$ and $y$ in the $\tilde{A}_{n-1}$ case, but there are some differences. The choice of orientation above is not essential: Remark [4.2.39] applies equally well in this case, replacing $x$ by $R$ and $y$ by $L$ (and now only changing the formulas in part (ii) of Theorem [4.2.48] by signs).
We will consider a slightly larger set of paths, called “generalized paths,” \( \text{GP} \subset P_{Q_0}^\bullet \) (we will also use the term for the image of such paths in \( \Pi_Q \) or \( \Pi_Q \)) as follows: Elements of \( \text{GP}[m] \) are products of the form
\[
i_1X_1i_2X_2\cdots i_mX_mi_{m+1}, \quad i_j \in I \cup \{i_L, i_R\}, \quad X_j \in \{L, R\}. \tag{4.2.43}
\]
In other words, we allow products of not just edges, but also the sums of two external edges which are pointed in the same left-right direction. Generalized paths have well-defined “generalized endpoints,” which means either a vertex of \( I \) or one of the elements \( i_L, i_R \) (the sum of the two left or right endpoints). In \( \{4.2.43\} \) these are given by \( i_1 \) and \( i_{m+1} \).

It will be convenient to define “\( F \),” for “forward,” as follows: if \( Y \) is any generalized path, then
\[
YF := \begin{cases} 
YR, & \text{if } Y = Yi_L \text{ or } Y = Y'Ri \text{ where } i \in I \text{ is internal and } Y' \in \text{GP}, \\
YL, & \text{if } Y = Yi_R \text{ or } Y = Y'Li \text{ where } i \in I \text{ is internal and } Y' \in \text{GP}.
\end{cases} \tag{4.2.44}
\]

À priori, in order for symbols \( F \) to become elements of \( P_{Q_0}^\bullet \), they must be multiplied on the left by a generalized path \( Y \). It is easy to check, however, that one obtains a well-defined product \( (P_{Q_0}^\bullet)^+ \times \{F\} \rightarrow P_{Q_0}^\bullet \) by linearity (i.e., it is enough to multiply on the left by any positively-graded element of the path algebra).

It is also convenient to define \( L_U, L_D \) to mean “go left and always take the upward/downward path”:
\[
L_U := \sum_{e \in Q | e \text{ is leftward and not downward}} e, \quad L_D := \sum_{e \in Q | e \text{ is leftward and not upward}} e, \tag{4.2.45}
\]
and similarly define \( R_U, R_D \). Finally, define \( F_U \) by \( \{4.2.44\} \), replacing \( L \) and \( R \) by \( L_U \) and \( R_U \), and similarly define \( F_D \).

We now define our proposed basis elements, for any \( c, C \geq 0 \), and any choice of initial vertex \( i \), terminal vertex \( j \), and initial direction \( R \) or \( L \):
\[
\zeta_{c,C,R,i,j} := \begin{cases} 
i(RL)^c j, & \text{if } C = 0, \\
i(RL)^c Rj, & \text{if } C = 1, \ z_{c,C,L,i,j} := \begin{cases} 
i(RL)^c j, & \text{if } C = 0, \\
i(LR)^c Lj, & \text{if } C = 1, \tag{4.2.46} 
\end{cases}
\end{cases}
\]

Remark 4.2.47. The indices \( c, C \) in the \( \hat{D}_n \) case are analogous to the quantities \( 2 \min(a, b) \) and \( |a - b| \) in the \( \hat{A}_{n-1} \) case: \( c \) gives the number of short loops (\( RL \) or \( LR \)) and \( \frac{1}{n-2} \cdot C \) gives the number of long loops (a power of \( f_U \) or \( f_D \)). We cannot keep the same notation in both cases because in the \( \hat{A}_{n-1} \) case, the winding number is a meaningful quantity (which is \( \frac{1}{n} \cdot (a - b) \)), whereas in the \( \hat{D}_n \) case, the only meaningful quantity is the number of short and long loops: the starting direction \( R \) or \( L \) is only meaningful if beginning at an internal vertex, and the meaning is lost modulo commutators (or when passing to the center).

Theorem 4.2.48. Let \( Q_0 = \hat{D}_n \).

(i) A basis for \( i\Pi_{Q_0}j \) for any \( i, j \in I \) consists of all nonzero elements having the form
(a) \( \zeta_{c,C,R,i,j} \), for \( c, C \geq 0 \) and any \( i, j \), or

...
(b) \(z_{c,C,L,i,j}\), for \(c,C \geq 0, i,j \in I_0\) such that either \(i \in \{i_{RU}, i_{RD}\}\) or \(C \geq 1\).

In particular, for \(i_0 := i_{LU}\), a basis of \(i_0 \Pi_{Q_0} i_0\) is given by \(z_{c,C,R,i_0,j_0}\) for \(c,C \geq 0\).

(ii) For any quiver \(Q \supseteq Q_0\), we have \(W = W' \oplus W_1\), where \(W_1 \subset W\) projects isomorphically under \(V \to \Pi_{Q_0}\) onto the (saturated) submodule

\[
\bigoplus_{i \in I_0, i \notin \{i_{LU}, i_{RU}\}} i \Pi_{Q_0} i \oplus (z_{c,C,L,i_{RU},i_{RU}})_{c \geq 0, C \geq 1},
\]

and \(W'\) has a basis given by elements \(W_{c,C}\) for \(C = 2(n - 2) \cdot C\) and \(c,C' \geq 0\), having the form

\[
W_{c,C} := \sum_{c,C,L,i,j}(\mathcal{g}(c,C)\mathcal{r}(c,C)) \frac{|\{c\}|}{\prod_{m=1} z_{c,m,C,m,i_m,i_{m+1}}} \quad (C > 0),
\]

where \((*)\) is the condition that \(X_m\) always begin going forward:

\[
\prod_{m'=1}^{m-1} z_{0,C,m',X_m,i_{m'},i_{m'+1}} F = \prod_{m'=1}^{m-1} z_{0,C,m',X_m,i_{m'},i_{m'+1}} X_m, \quad \text{and}
\]

\[
c = |\{c\}| + \sum_m c_m, \quad C = \sum_m C_m;
\]

finally,

\[
W_{c,0} := (RL)^c - (LR)^c = i_L(RL)^c - i_R(LR)^c + 1_m((RL)^c - (RL + r)^c)1_{\text{in}}.
\]

(iii) The span of \(W_{c,C}\) for \(C > 0\) or \(c \neq p\) \(\ell\) for any prime \(p\) and \(\ell \geq 1\) is saturated, and the order of \(W_{p^\ell,0}\) is \(p\) (modulo the other \(W_{c,C}\)’s or otherwise).

(iv) The image of \(\frac{1}{p} W_{p^\ell,0}\) is \(r(p^\ell)\) and these elements span the torsion of \(HH_0(\Pi_Q)\).

The theorem will be proved in Section 7. Although we could have included a description of \(HH_0(\Pi_{Q_0})\) above, we relegate this to Theorem 4.2.60.

Note that, as in the \(A_n\) case, one way to obtain bases for \(\Pi_{D_n}\) is from the above basis. One way to do this is to set \(i_0 := i_{LD}\), and use those basis elements \(z_{c,C,X,i,j}\) which are nonzero under the quotient \(\Pi_{D_n} \to \Pi_{D_n}\), and which do not pass through \(i_{LU}\) (but may begin or end at \(i_{LU}\)).

### 4.2.3 Bases of \(\Pi_Q\) for type \(E\) quivers and presentations of \(\Pi_Q\) and \(HH_0(\Pi_Q)\) for star-shaped quivers

In type \(E\), in both the Dynkin and extended Dynkin cases, \(Q\) is a star. We determine bases for \(\Pi_Q\) and \(HH_0(\Pi_Q)\) using general results about preprojective algebras for star-shaped quivers. In particular, we use Corollary 2.2.15 (specialized to the case of finite branches); also, in order to obtain bases for \(iI_j\) for each \(i,j \in I\), we use Lemma 2.2.16. Letting \(i_s\) be the special vertex of a star-shaped quiver \(Q\), the above results allow one to present \(i_s \Pi_{i_s}\) as an algebra, and \(iI_j\) as a quotient of \(i_s \Pi_{i_s}\) by a submodule. Specifically, specializing the aforementioned results to star-shaped quivers, we obtain
Proposition 4.2.54. Let $Q$ be a star-shaped quiver with branches $L_{s'}$ of lengths $d_{s'}$, for $s' \in \{1, \ldots, m\}$. Then one has

(i) $\iota_s \Pi_Q \iota_s \cong A := \mathbb{Z}[x_1, \ldots, x_m]/(x_1 + \cdots + x_m, x_1^{d_1+1}, \ldots, x_m^{d_m+1})$.

(ii) For any $j_1, j_2$ in branches $L_{s_1}, L_{s_2}$, respectively, $i_{i} \Pi_Q j \cong A/(x_{s_1}^{d_{s_1} - d_{s_1,j_1} + 1} + A x_{s_2}^{d_{s_2} - d_{s_2,j_2} + 1})$.

(iii) $HH_0(\Pi_Q) \cong A/([A, A] + \langle x_{s'} \rangle_{1 \leq m \leq s', \ell \geq 1})$.

As a result, using noncommutative Gröbner bases (cf. Appendix A.0.1), we obtain the following bases for type $E$ quivers:

Proposition 4.2.55. Let $Q$ be an extended Dynkin quiver of type $E$, whose longest branch (as a star) has length $p$. Let $A$ be defined as in Proposition 4.2.54 (i). For readability, set $x := x_1, y := x_2, z := x_3$, assume $d_1 \geq d_2 \geq d_3$, and set $d := d_1$.

(i) A basis for $A \cong \iota_3 \Pi_Q \iota_3$ (as a free $\mathbb{Z}$-module) is given by

$$x^e (yx^d)^{e_2} (yx^{d-1})^{e_1} y, \quad Y \in \{yx^{d-2} y, yx^{e_1}, 1\}_{0 \leq e_1 \leq d-2}. \quad (4.2.56)$$

(i.e., $Y$ is an initial subword of $yx^{d-2} y$).

(ii) A basis for $i_0 \Pi_Q i_0$, (as a free $\mathbb{Z}$-module), via Proposition 4.2.54 (ii), is given by

$$p_{i_0 i_3} ((yx^d)^{e_1} y) p_{i_3 i_0} (\ell \geq 0), \quad (4.2.57)$$

$$p_{i_0 i_3} ((yx^d)^{e_1} (yx^{d-1})^{e_2} y) p_{i_3 i_0} (\ell_1, \ell_2 \geq 0). \quad (4.2.58)$$

This proposition will be proved in Section 8 using Gröbner bases (which we obtain using MAGMA).

4.2.4 Bases of $HH_0(\Pi_Q)$ in the extended Dynkin and Dynkin cases In this section, we refine (4.1.10) to the $\mathbb{Z}$-case, thus yielding bases of $HH_0(\Pi_Q)$ (and thus its graded $\mathbb{Z}$-module structure).

Theorem 4.2.60. For any Dynkin quiver $Q$ with corresponding extended Dynkin quiver $\tilde{Q} \supset Q$, with extending vertex $i_0 \in I$, one has the split exact sequence

$$i_0 \Pi_{\tilde{Q}} i_0 \hookrightarrow \Lambda_{\tilde{Q}} \rightarrow \Lambda_Q, \quad \Lambda_Q \cong i_0 \Pi_{\tilde{Q}} i_0 \bigoplus \Lambda_{\tilde{Q}} \quad (4.2.61)$$

with maps given by the natural maps. Furthermore, $i_0 \Pi_{\tilde{Q}} i_0$ is a free $\mathbb{Z}$-module, and $\Lambda_Q$ is finite, given as follows:

1. $(\Lambda_{\tilde{A}})_+ = 0.$

After performing this calculation, the author noticed that in [MOV06] (section 4 in version 2), that $\Lambda_Q \otimes \mathbb{F}_p = 0$ is shown for Dynkin quivers $Q$ and primes $p$ which are not stably bad (which are all cases for which the equality holds). Here we expand on this by explicitly describing $\Lambda_Q$. 

42
2. \((\Lambda_{D_n})[m] = 0\) for all \(m \geq 1\) except when \(4 \mid m\) and \(m \leq 2(n-2)\), in which case \(\Lambda[m] \cong \mathbb{Z}/2\), and is spanned by the element \([i_s(xy)^{m/4}]\), where \(i_s\) is the special vertex, and \(x, y\) are the length-two cycles along the branches of length one (to the endpoint and back), which equals \([r^{m/2}]\) when \(m\) is a power of 2.

In \(\Lambda_{\tilde{D}_n}\), the above element lifts to the torsion element \([i_{LU}(LR)^{m/2}]/2 + [i_{RU}(RL)^{m/2}]/2\), which may also be described as \(X^{m/4-1} \cup r^{(2)}\) for \(X \in HH^0(\Pi_{\tilde{D}_n})\) as described in Proposition 7.4.4. This last formula means \([X^{m/4-1}r^{(2)}]\) for any \(r^{(2)} \in \Pi_{E_{8}}\) such that \([r^{(2)}] = r^{(2)}\) (cf. Example 10.2.36 and Proposition 8.4.1).

3. We have \(\Lambda_{E_6}[m] = 0\) for all \(m \geq 1\) except for \(m \in \{4, 6\}\), where \(\Lambda_{E_6}[4] \cong \mathbb{Z}/2\), and \(\Lambda_{E_6}[6] \cong \mathbb{Z}/3\), spanned by the respective elements \(r^{(2)}, r^{(3)}\).

4. \(\Lambda_{E_7}[m] = 0\) for all \(m \geq 1\) except when \(m\) is 4, 6, 8, or 16, where we get \(\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/2, \mathbb{Z}/2\), respectively, spanned by the elements \(r^{(2)}, r^{(3)}, r^{(4)}, \) and \(r^{(8)}\).

5. \(\Lambda_{E_8}[m] = 0\) for all \(m \geq 1\) except when \(m\) is 4, 6, 8, 10, 16, 18, or 28, where we get \(\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/2, \mathbb{Z}/5, \mathbb{Z}/2, \mathbb{Z}/3, \) and \(\mathbb{Z}/2\), respectively, spanned by the elements \(r^{(2)}, r^{(3)}, r^{(4)}, r^{(5)}, r^{(8)}, r^{(9)}\), and \([i_s x^4 y^4 y x^5 y]\), where \(x, y\) are length-two cycles in the direction of two different branches (say, the longest two branches), and \(i_s\) is the special vertex.

In \(\Lambda_{E_8}[28]\), the last element above lifts to the element \(X^2 \cup r^{(2)} = \left[ X^2 r^{(2)} \right] \) (cf. part (2)).

The elements above also lift to elements which span the torsion of \(\Lambda_{\tilde{Q}}\), which is isomorphic to \(\Lambda_Q\) under the second map of \((4.2.61)\). Explicitly, these elements are the corresponding \(r^{(p^i)}\), and in the \(\Lambda_{\tilde{D}_n}\) and \(\Lambda_{E_8}[28]\) cases, they are as noted above.

We begin with the

**Lemma 4.2.62.** Take any Dynkin quiver \(Q\) with extended Dynkin quiver \(\tilde{Q}\). In each degree \(m \geq 1\) (=total length of paths), the \(\mathbb{Z}\)-module map \(i_0\Pi_{\tilde{Q}}[m]i_0 \rightarrow \Lambda_{\tilde{Q}}[m]\) is an isomorphism iff it is surjective. Furthermore, the cokernel of the map is \(\Lambda_Q[m]\), so the map is surjective iff \(\Lambda_Q[m] = 0\). The last condition holds for all but finitely many \(m\).

**Proof.** Suppose \(i_0\Pi_{\tilde{Q}}[m]i_0 \rightarrow \Lambda_{\tilde{Q}}[m]\) is surjective. Since \(i_0\Pi_{\tilde{Q}}[m]i_0\) is a free \(\mathbb{Z}\)-module (by Proposition 4.2.22), and its rank is the same as the dimension of \(\Lambda_{\tilde{Q}}[m] \otimes \mathbb{Q}\) (by (4.1.10)), it follows that the map is injective and hence an isomorphism.

For the second part, one may consider the homomorphism \(\Pi_{\tilde{Q}} \rightarrow \Pi_Q\) obtained by modding by the extending vertex (and hence the adjacent edges). The kernel is just the span of paths that pass through the extending vertex. The induced map of \(\mathbb{Z}\)-modules \(\Lambda_{\tilde{Q}} \rightarrow \Lambda_Q\) again has kernel equal to the same span as before; and thus the kernel is just the image of the map \(i_0\Pi_{\tilde{Q}}[m]i_0 \rightarrow \Lambda_{\tilde{Q}}[m]\). Since the map is surjective, we get the desired result.

Finally, the statement that \(\Lambda_Q[m] = 0\) for all but finitely many \(m\) follows from the corresponding statement for \(\Pi_Q\), which dates from [GP79] (in fact, \(\Pi_Q\) is the direct sum of all indecomposable representations of the quiver \(Q\), of which there are only finitely many; this was the reason for the definition of \(\Pi\) in the first place.) This is also easy to verify directly using Gröbner bases and Proposition 4.2.54. 

\[\square\]
Proof of Theorem 4.2.60. Since the cokernel of (4.2.61), \( \Lambda_Q \), is always torsion (i.e., \( \Lambda_Q \otimes \mathbb{Q} = 0 \) [MOV06], which is also easy to check using Gröbner bases), and the free rank of the first two terms in (4.2.61), \( i_0 \Pi_Q i_0 \) and \( \Lambda_Q \), are the same [41.1.10], the map \( i_0 \Pi_Q i_0 \rightarrow \Lambda_Q \) is always a monomorphism. It remains only to compute \( \Lambda_Q \) where \( Q \) is Dynkin, and check that the above descriptions indeed give a splitting \( \Lambda_Q \hookrightarrow \Lambda_Q \). We first explain the second step (the splitting).

For the elements \( r(p^t) \), we obviously have \( r(p^t) \mapsto r(q^m) \). In the \( D_n \) case, it is obvious that the first formula \( ([i_{LU}(LR)^{m/2}]/2 + [i_{RU}(RL)^{m/2}]/2) \) gives a splitting, and it is easy to deduce it is torsion by (4.2.53) for \( c = \frac{1}{4} \). For both the \( D_n \) and \( E_8 \) cases, the cup-product formula can be checked using MAGMA (cf. Example [10.2.36]), which immediately implies that they are torsion elements (the section may also be easily computed without using cup product).

It remains to verify the formulas for the torsion of \( \Lambda_Q \) where \( Q \) is Dynkin. This is a finite computation, since \( \Pi_Q \) is finite-dimensional. We do this directly as follows: (1) We note that here \( i_0 \Pi_{\Lambda_n} i_0 \) is one-dimensional for each \( i \), corresponding to the path of length zero. (2) Consider the quiver \( D_n \) obtained from \( \tilde{D}_n \) by cutting the vertex \( i_{LD} \). In this case, \( \Lambda_{D_n} \) is generated by closed paths that sit entirely on the two rightmost external edges (we can apply relations of \( \Pi \) and cyclic rotations to move any loop there). Let \( i \) be the rightmost internal vertex. Then, such elements are zero if they are not a power of the length-four loop of the form \( iR U L R_D L \). Furthermore, the additional relations that one obtains are: (i) any closed path that does not touch the rightmost external edges is zero; (ii) \( 0 = [i(RL)^m] = [((R_U + R_D) L)^m] \); and (iii) \( i(RL)^{n-3} = 0 \) (the latter is an algebra relation from \( \Pi_{D_n} \)). The last condition simply says that any closed loop along the rightmost external edges of length \( \geq 2(n - 2) \) is zero. The second condition says that \( [((R_U L R_D L)^m] + [(R_D L R_U) L)^m] = 0 \), which combined with the fact that these two are equal (which was already true in \( \Lambda_{\tilde{D}_n} \), shows that \( [((L_U L R_D R)^m] \) generates a copy of \( \mathbb{Z}/2 \). Finally, it is straightforward to check that the element \( r(p^{2m}) \) reduces to this element when \( m \) is a power of two.

(3),(4),(5) Using Proposition 4.2.54, we may give an explicit presentation of \( i_0 \Pi_{E_n} i_0 \) and \( \Lambda_{E_n} \) for \( n \in \{6, 7, 8\} \); this immediately cuts off the degree at \( \leq 10 \) for \( \tilde{E}_6, \leq 16 \) for \( \tilde{E}_7 \), and \( \leq 28 \) for \( \tilde{E}_8 \) (these numbers may also be computed using the Coxeter numbers of the corresponding root systems: see e.g. [MOV06, EE07]). The remaining finite computation was completed by hand with the help of MAGMA. In particular, by hand the torsion was computed (using Gröbner bases in terms of edges), and with MAGMA the given elements were shown to span this torsion, and the Hilbert series over finite fields including \( \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5 \) was double-checked.

Theorem 4.2.60 together with the explicit bases we already gave for \( i_0 \Pi_i_0 \) of an extended Dynkin quiver, gives an explicit basis of \( \Lambda \) for all extended Dynkin quivers.

5 The Lie bialgebra \( \Lambda_Q \) and connections to noncommutative BV structures and free product deformations on \( P_Q \) and \( L_Q \)

In this section, we shift from the \( \mathbb{Z} \)-module structure of \( \Lambda_Q \) to its main algebraic structure: a Lie bialgebra structure. We further explain how this comes from a "noncommutative BV structure" on \( P_Q \) and \( L_Q \), and how \( P_Q \) "quantizes" \( \Pi_Q \) in terms of a "free product" deformation. The study of \( \Lambda_Q \) will be continued in Section 6.

5.1 The Lie bialgebra \( \Lambda_+ \) We first recall the necklace Lie bialgebra [BLB02, Gin01, Sch05]. Let \( L := HH_0(P_Q) = P_Q/ [P_Q,P_Q] \). One gives \( L \) the structure of a Lie bialgebra as follows. Let
\(\omega\) be the natural symplectic form on the vector space \(k(Q)\), of dimension \(#(Q) = 2\#(Q)\), with symplectic basis \((e, e^*)_{e \in Q}\). Also, for any edge \(e : i \rightarrow j\) in \(Q\), let \(e_s := i\) and \(e_t := j\). Then one defines the bracket \(\{,\}\) and cobracket \(\delta\) by

\[
\{[a_1a_2\cdots a_m], [b_1b_2\cdots b_n]\} = \sum_{i,j} \omega(a_i, b_j)((a_i)_t a_{i+1}\cdots a_{i-1} b_{j+1}\cdots b_{j-1}),
\]

(5.1.1)

\[
\delta([a_1a_2\cdots a_m]) = \sum_{i<j} \omega(a_i, a_j)((a_j)_t a_{j+1}\cdots a_{i-1}) \wedge [(a_i)_t a_{i+1}\cdots a_{j-1}].
\]

(5.1.2)

Note that the reason for the \((a_i)_t, (a_j)_t\) is in case one has \(n = m = 1\) in the first line, and in case \(j = i + 1\) or \(j = m, i = 1\) in the second line. We have:

**Proposition 5.1.3.** ([Sch05]) The above defines the structure of an involutive Lie bialgebra on \(L\): that is, \(L\) is a Lie bialgebra satisfying \(br \circ \delta = 0\), where \(br : L \otimes L \rightarrow L\) is the bracket.

To define these more suggestively, we may define \([Sch05]\) operators \(\partial_e : L \rightarrow P, D_e : P \rightarrow P \otimes P\) by the formulas

\[
\partial_e([a_1\cdots a_m]) := \sum_i \delta_{a_i,e}(a_i)_t a_{i+1}\cdots a_{i-1},
\]

(5.1.4)

\[
D_e(a_1\cdots a_m) := \sum_i \delta_{a_i,e}a_1\cdots a_{i-1}(a_i)_s \otimes (a_i)_t a_{i+1}\cdots a_m,
\]

(5.1.5)

so that \(\{,\} : L \otimes L \rightarrow L, \delta : L \rightarrow L \otimes L\) are given by (letting \(m : P \otimes P \rightarrow P\) be the multiplication)

\[
\{,\} = \sum_{e \in Q} \Pr \circ (\partial_e \otimes \partial_e^* - \partial_e^* \otimes \partial_e), \quad \delta = \sum_{e \in Q} (\Pr \otimes \Pr)(D_e \partial_e^* - D_e^* \partial_e).
\]

(5.1.6)

(Note that \(\partial_e\) was notated \(\partial_{e_1}\) in \([Sch05]\) and \(D_e\) was notated \(D_{e_1}\).)

Our first result is then

**Proposition 5.1.7.** The subspace \(\langle I \rangle \oplus [Pr] \subset P/[P, P] = L\) is a Lie bi-ideal, where \(\langle I \rangle = L[0]\) is the span of vertices. Hence, \(\Lambda_+ := \Lambda/[\Lambda[0]\) is a Lie bialgebra. Furthermore, \([Pr] \subset L\) is a Lie ideal, so that \(\Lambda\) itself is a Lie algebra.

(The Lie part was proved in \([CBEG07]\) in greater generality.)

At the present time, it is unknown how to quantize \(\Lambda\) even though a quantization of \(L\) was found in \([Sch05]\) (and restated in \([GS06]\)), and by \([EK96]\) theory there should be a functorial quantization of \(L\) which descends to \(\Lambda\); however, it appears impossible to obtain a Hopf ideal corresponding to \([Pr] \subset L\) in the quantized algebra \([Sch05]\) of \(L\).

We can also answer the natural question of how the elements \(r(\varphi^i)\) behave under the Lie structure:

**Proposition 5.1.8.** In \(\Lambda_+, r(\varphi^i)\) is in the kernel of both the bracket and cobracket. More generally, \([ir^m]\) \(\in L\) is in the kernel of the bracket for any \(i \in I, m \geq 1\), and is in the kernel of the cobracket on \(L_+ := L/[L[0]\).

The complete kernel of the Lie bracket is computed in Theorem 9.2.2 (which actually computes the Poisson center of \(\text{Sym} \Lambda\) and generalizes to partial preprojective algebras). Also, in Proposition 9.2.1 we generalize the first statement above to hold for all the torsion of \(\Lambda\).

We will prove Propositions 5.1.7 and 5.1.8 in Section 5.4 after developing the noncommutative BV structure on \(L\).
5.2 Lifting brackets to $P$ In [VdB04], lifts of the Lie bracket on $L$ to $P$ were defined. Namely, one has the double Poisson bracket $\{,\}: P \otimes P \to P \otimes P$:

$$\{a_1 \cdots a_m, b_1 \cdots b_n\} := \sum_i \omega(a_i, b_j) b_1 \cdots b_{j-1}(a_i) a_{i+1} \cdots a_m \otimes a_1 \cdots a_{i-1}(a_i) \otimes b_{j+1} \cdots b_n.$$ (5.2.1)

In terms of the operator $D_e$ of [5.1.5], one has

$$\{,\} := \sum_{e \in Q} (m \otimes m) \circ \tau_{(13)} \circ (D_e \otimes D_{e^*}).$$ (5.2.2)

As mentioned in [VdB04], the formula $m \circ \{,\}: P \otimes P \to P$ defines a Loday bracket. Since this kills $[P, P] \otimes P$, for our purposes it will be more convenient to work with the induced map $\{,\}: L \otimes P \to P$, which we henceforth call the Loday bracket:

$$\{[a_1 \cdots a_m], b_1 \cdots b_n\} := \sum_i \omega(a_i, b_j) b_1 \cdots b_{j-1}(a_i) a_{i+1} \cdots a_{i-1} b_{j+1} \cdots b_m.$$ (5.2.3)

In terms of (5.1.4), (5.1.5),

$$\{,\}_{L \otimes P} := \sum_{e \in Q} m \circ (m \otimes 1) \circ \tau_{(12)} \circ (\partial_e \otimes D_{e^*}).$$ (5.2.4)

Combining this with the Lie bracket $\{,\}: L \otimes L \to L$, one can consider $\{,\}$ to be a Lie bracket on $\hat{P} := L \oplus P$, by defining

$$\{,\}_{P \otimes L} := -\{,\}_{L \otimes P} \circ \tau_{(12)}, \quad \{,\}_{P \otimes P} = 0,$$ (5.2.5)

without losing any information: in fact, this encodes the fact that the bracket $\{,\}$ is skew on $L$. We may now compare the bracket $\{,\}$ with the algebra multiplication on $P$, and it turns out that there is a Poisson structure:

**Proposition 5.2.6.** Defining multiplication $m\big|_{(L \otimes P) \oplus (P \otimes L) \oplus (L \otimes L)} = 0$ by $L$ to be zero, the space $(\hat{P}, \{,\}, m)$ is a noncommutative Poisson algebra:

$$\{a, bc\} = b\{a, c\} + \{a, b\}c.$$ (5.2.7)

**Proof.** The fact that $\hat{P}$ is Lie follows from a straightforward computation, and the Poisson condition follows immediately from the definitions. Alternatively, one may derive this from the double Poisson conditions that $\{,\}$ satisfies [VdB04].

Also, one has the rather obvious identity:

$$\text{pr} \circ \{,\}\big|_{P \otimes L} = \{,\} \circ (\text{pr} \otimes 1), \quad \text{and similarly } \text{pr} \circ \{,\}\big|_{L \otimes P} = \{,\} \circ (1 \otimes \text{pr}).$$ (5.2.8)

It turns out that the cobracket lifts to $P$ as well:

$$\delta_\ell : P \to L \otimes P, \delta : \hat{P} \to \hat{P} \otimes \hat{P},$$

$$\delta_\ell(a_1 \cdots a_n) := \sum_{i<j} -\omega(a_i, a_j) [(a_i) a_{i+1} \cdots a_{j-1}] \otimes a_1 \cdots a_{i-1} (a_i) a_{j+1} \cdots a_n,$$ (5.2.9)

$$\delta_\ell = (m \otimes \text{pr}) \tau_{(23)}(1 \otimes D_e) D_{e^*},$$ (5.2.10)

$$\delta \bigg|_P = \delta_\ell - \tau_{(12)} \delta_\ell,$$ (5.2.11)

with $\delta$ defined on $L$ as usual. One then has
Proposition 5.2.12. The structure $(\tilde{P}, \{\cdot,\cdot\}, \delta)$ is an involutive Lie bialgebra.

Proof. This follows from the same proof as [Sch05], Section 2 (one simply needs to remember where the start and end of pieces that come from $P$ are).

As before, one may view $\delta$ on $P$ as a lift of $\delta$ on $L$: If we define $\left.\text{pr}\right|_L = \text{id}$, one has

$$(\text{pr} \otimes \text{pr})\delta = \delta \circ \text{pr}. \quad (5.2.13)$$

5.3 Noncommutative BV structure

One notices a BV-style connection encompassing the Poisson and Lie bialgebra structures:

Proposition 5.3.1. The following BV-style identity is satisfied by $\tilde{P}$: For any $a, b \in P$, one has

$$\delta (ab) = \delta (a)(1 \otimes b) + (1 \otimes a)\delta (b) + (\text{pr} \otimes 1)\{a, b\}. \quad (5.3.2)$$

To get an equation involving $\delta$, one can apply $(1 - \tau(12))$ to each side (which originally lives in $L \otimes P$).

What this says is that $\delta$ can be defined as the unique cobracket satisfying (5.3.2) which vanishes on $P[0] \oplus P[1]$. Since $\delta$ has total degree $-2$, it must be the unique graded cobracket (or, indeed, graded linear map $P \to L \otimes P$) satisfying (5.3.2).

One could define more general ungraded cobrackets which do not vanish on $P[1]$. For example, one could start with $\delta'_c(e) = [F_e] \otimes e$, for elements $F_e \in L$ assigned to edges $e \in Q$, satisfying: $\delta(F_e) = 0$ (using the old $\delta$ from (5.1.2)), $F_e = -F_e^*$, and $\{f f^*, F_e\} = 0, \forall e, f \in Q$. This extends to a unique $\delta'_{c'}$ on $P$ satisfying (5.3.2), which induces a bialgebra structure on $L$.

However, this does not give an involutive bialgebra structure: to obtain involutivity and the one-cocycle condition, set $\delta_{c'} = \sum_{e \in Q} -F_e \otimes [ee^*] + [ee^*] \otimes \text{ad} F_e$.

More generally, any $\delta'_{c'}$ satisfying (5.3.2) must be of the form

$$\delta'_{c'} = \delta_{c'} + \sum_i F_i \otimes \theta_i, \quad (5.3.3)$$

where $F_i \in L$ and $\theta_i \in \text{Der}(L)$. The condition that $\delta'_{c'}$ induce a cobracket on $L$ (the co-Jacobi condition) is then

$$\text{Skew} \circ \left(\delta\left(\sum_i F_i \otimes \theta_i\right) + \frac{1}{2}\{\sum_i F_i \otimes \theta_i, \sum_j F_j \otimes \theta_j\}\right) = 0, \quad (5.3.4)$$

where

$$\delta (F \otimes \theta) := \delta (F) \otimes \theta - F \otimes \delta (\theta), \quad (5.3.5)$$

$$\delta_{c'} (\theta) := \delta_{c'} (\theta) - (\theta \otimes 1 - 1 \otimes \theta) \circ \delta_{c'}, \quad (5.3.6)$$

$$\{F \otimes \theta, F' \otimes \theta'\} := (F \wedge \theta(F')) \otimes \theta' + (F' \wedge \theta'(F)) \otimes \theta - (F \wedge F') \otimes \{\theta, \theta'\}, \quad (5.3.7)$$

$$\text{Skew} = \sum_{\sigma \in S_3} \text{sign}(\sigma) \cdot \tau_\sigma. \quad (5.3.8)$$
In the case that the \( \theta_t \) are inner derivations (equivalently, they kill \( r \); as a consequence, they descend to \( \Lambda \)), then we may consider maps of the form \( \delta' = \delta + \sum (F_i \otimes \text{ad } G_i - G_i \otimes \text{ad } F_i) \). Then, one has \( \delta_t(\text{ad } F) = (1 \otimes \text{ad } \delta(F). \) Thus, \( (5.3.4) \) says

\[
\text{Skew}((\delta \otimes 1)(\sum F_i \wedge G_i) + \frac{1}{2}(\sum F_i \wedge G_i, \sum F_j \wedge G_j)) = 0, \tag{5.3.9}
\]

where now \( \{F \otimes G, F' \otimes G\} = F \otimes \{G, F'\} \otimes G \), extended linearly.

This is a version of the Maurer-Cartan equation. The explanation is that it is the condition \( (D')^2 = 0 \), where \( D' \) is the operator on the exterior algebra \( \Lambda^*L \) induced by \( \delta' \) and the bracket. Indeed, one interpretation of involutive Lie (super)bialgebras is that \( \Lambda^*L \) forms a BV algebra, using a differential obtained from the sum of bracket and cobracket (cf., e.g., [Gin01], §2.10). As a consequence, solutions of \( (5.3.9) \) will give solutions of \( (5.3.2) \) which descend to involutive Lie bialgebra structures (not just cobrackets). Furthermore, given that \( \delta \) descends to \( \Lambda \), so does the above \( \delta' \).

For example, for any \( F_i, G_i \in [P_Q] \), the element \( \delta'_t = \delta + \sum_i (F_i \otimes \text{ad } G_i - G_i \otimes \text{ad } F_i) \), which gives an involutive Lie bialgebra satisfying \( (5.3.2) \). Note that this still gives a graded Lie bialgebra using the “geometric” grading, given by setting \( |Q^*| = 1 \) and \( |Q| = 0 \): the bracket and cobracket then both have degree \(-1\). The total grading we have been using should be considered the “Bernstein” grading. In [GSD], we use the geometric grading since it exists for much more general algebras (replacing \( P^{\wedge}_0 = T_{PQ} \text{Der}(PQ, P_Q \otimes P_Q) \) by \( T_A \text{Der}(A, A \otimes A) \) for more general \( A \)).

For more details and a general construction of noncommutative BV structures, see [GSD]. For our purposes, we will only need the identity \( (5.3.2) \).

### 5.4 Proof of Propositions 5.1.7 and 5.1.8

We break Proposition 5.1.7 into two lemmas:

**Lemma 5.4.1.** The space \( [Pr] \subset L = P/[P, P] \) is a Lie ideal.

*Proof.* Note that this is actually a special case of Proposition 4.4.3(ii) from [CBEG07]. We give an elementary, independent proof (known to the author before the release of [CBEG07]). Pick \( f = [a_1 \cdots a_m] \in L \) and let \( g \in P \) be arbitrary. We make use of the Poisson bracket \{,\} on \( \overline{P} = L \oplus P \), to obtain

\[
\{[a_1 \cdots a_m], gr\} - \{[a_1 \cdots a_m], g\} r = g\{[a_1 \cdots a_m], r\}
\]

\[
= g \sum_j \omega(a_j, a_j^*_{[a_j]})(-1)^{[a_j]} (a_{j+1} \cdots a_{j-1} a_j - a_j a_{j+1} \cdots a_{j-1})
\]

\[
= g \sum_j (a_{j+1} \cdots a_{j-1} a_j - a_j a_{j+1} \cdots a_{j-1}) = 0. \tag{5.4.2}
\]

Here \{statement\} is defined to equal one if “statement” is true and to equal zero if “statement” is false. Now, the result follows from \( (5.2.8) \). \( \square \)

**Lemma 5.4.3.** The space \( [Pr] + L[0] \subset L \) is a Lie bi-ideal.

*Proof.* It is obvious that \( L[0] \) is in the kernel of the Lie bracket, so we need only show that \( [Pr] \oplus L[0] \) is a Lie coideal. Let \( f \in \overline{P} \) be arbitrary; we compute \( \delta_t(\text{ad } f) \) by means of the BV identity \( (5.3.2) \):

\[
\delta_t(\text{ad } f) = \delta_t(r)(1 \otimes f) + (1 \otimes r)\delta_t(f) + (\text{pr} \otimes 1)[r, f]. \tag{5.4.4}
\]
Now, \( \delta_i(r) = \sum_{e \in Q} (e_t \otimes e_s - e_s \otimes e_t) \), so that the first term on the RHS is in \( L[0] \otimes P \). The second term is obviously in \( L \otimes Pr \). Let us compute the last term on the RHS. Let us assume \( f = a_1 \cdots a_n \) is a single path:

\[
\{ r, a_1 \cdots a_n \} = \sum_i \omega(a^*_i, a_i)(-1)^{|a_i|} \left[ (a_1 \cdots a_{i-1} a_i \otimes (a_i)_t a_{i+1} \cdots a_n - a_1 \cdots a_{i-1}(a_i)_s \otimes a_i \cdots a_n) = a_1 \cdots a_n \otimes (a_n)_t - (a_1)_s \otimes a_1 \cdots a_n, \right. (5.4.5)
\]

since \( \omega(a^*_i, a_i)(-1)^{|a^*_i|} = 1 \) as before. This is in \( P[0] \otimes P + P \otimes P[0] \). The lemma now follows from (5.2.13).

In fact, what we have essentially proved above is the following “refinement” of Proposition 5.1.7:

**Proposition 5.4.6.** 1. The subspace \( PrP \subset \tilde{P} = P \oplus L \) is a Poisson ideal. Hence, \( \Pi \oplus L \) is a Poisson algebra.

2. The subspaces \( Pr \oplus L[0], rP \oplus L[0] \subset P \oplus L = \tilde{P} \) are Lie coideals.

The first part of the proposition should be thought of as saying that “\( L \) acts on \( \Pi \)”. The explanation is that \( L \) acts on \( P \) (see (10.2.3)), and this action kills \( r \) and hence descends to an action on \( \Pi \).

However, we cannot extend \( \Lambda_+ \) to any larger Lie bialgebra:

**Remark 5.4.7.** Note that it is not true that \( PrP \) is contained in any Lie coideal: upon taking cobracket, one obtains some nonzero terms that are not multiples of \( r \). So there is no way to define the cobracket on \( \Pi \).

Also, note that although \( \Lambda \) and \( L \oplus \Pi \) are both Lie algebras, the same is not true of \( \Lambda \oplus \Pi \). Indeed, if we tried to mimic (5.4.2) in computing \( \{ a_1 \cdots a_m, [gr] \} \) where \( a_1 \cdots a_m \in P \) is now considered as a path, and \([gr] \in L \) is the image of \( gr \) under \( P \to L \), then one would obtain an answer that, up to multiples of \( r \), is \( ga_1 \cdots a_m - a_1 \cdots a_m g \). So it is essential to take commutators of \( \Pi \) with \( L \) rather than \( \Lambda \).

On the other hand, we see from the above paragraph that \( \Lambda \) does have a well-defined map to outer derivations of \( \Pi \), since the adjoint action of \( \Lambda \) on \( \Pi \) is well-defined up to inner derivations. We will see in Section 10.2 that this forms the bulk of the outer derivations of \( \Lambda \) (especially in the non-extended Dynkin case).

**Proof of Proposition 5.1.8** We prove the second assertion (which clearly implies the first, since \( L \) is a free module). To show that \([ir^m]\) is in the kernel of the Lie bracket on \( L \) for any \( i \in I \) and \( m \geq 0 \), one simply applies (5.4.2) multiple times, replacing \( r \) with \( ir \) (and hence limiting the sum to those edges \( a_j \) which are adjacent to \( i \)).

Showing that \([ir^m]\) is in the kernel of the cobracket is a bit more difficult. We show more generally that \([ir^m]\) is in the kernel of \( \delta'_i := (q \otimes q') \circ \delta_i : P \to L_+ \otimes P_+ \), where \( q : L \to L/L[0] = L_+, a' : P \to P/P[0] = P_+ \) are the projections. Inductively, we need to show that

\[
0 = \delta'_i([ir^{m+1}]) = ir \delta'_i([ir^m]) + \delta'_i(ir)r^m + (q \circ pr \otimes q')\{ ir, ir^m \} = (q \circ pr \otimes q')\{ ir, ir^m \}. \quad (5.4.8)
\]

Now, considering (5.4.5), one verifies that most of the terms in \( \{ ir, ir^m \} \) cancel, leaving \( ir^m \otimes i - i \otimes ir^m \). This is killed by \( q \circ pr \otimes q' \), verifying the desired result.
5.5 $P_{Q_0}$ as a free product deformation of $\Pi_{Q_0}$ and the necklace bracket

We may consider a different isomorphism from (4.1.10): given an element $z \in HH^0(\Pi_{Q_0})$ for any extended Dynkin quiver $Q_0$, we may consider its image in $HH_0(\Pi_{Q_0})$ under the composition

$$HH^0(\Pi_{Q_0}) \hookrightarrow \Pi_{Q_0} \rightarrow HH_0(\Pi_{Q_0}).$$  \hfill (5.5.1)

Tensoring over $k = \mathbb{Z}[\frac{1}{|\Pi|}, e^{\frac{2\pi i}{|\Pi|}}]$, this composition is an isomorphism, as can be seen by the Morita equivalence $\Pi_{Q_0} \otimes k \cong k[x,y] \rtimes \Gamma$, since in the latter the center is $k[x,y]^\Gamma$, whose projection to $HH_0(\Pi_{Q_0}) \cong k[x,y]^\Gamma$ is the identity under the identifications of Section 3.2 (which are easy to make explicit). Then, in this case, the projection $k[x,y] \rtimes \Gamma \to f(k[x,y] \rtimes \Gamma)f \cong \Pi_{Q_0}$ must induce an isomorphism when passing to the center or to $HH_*$.

Note that the above is not true if we work over $\mathbb{Z}$ or a field in characteristic dividing $|\Gamma|$, even for the cases of $\tilde{A}_{n-1}, \tilde{D}_n$. For example, in the case $\tilde{A}_{n-1}$, a central element involves a sum over all vertices of loops beginning and ending at that vertex; for each vertex, the corresponding summand has the same image in $HH_0(\Pi_{Q_0})$ and hence the sum must be a multiple of $n$, which does not yield an isomorphism when we don’t invert $|\Gamma| = n$.

As a consequence of the above, over $k$, we may identify the necklace Lie algebra for $Q_0$, which we showed to actually be Poisson, with the Poisson algebra $HH^0(\Pi_{Q_0} \otimes k)$ (as an alternative to $k[x,y]^\Gamma \cong i_0\Pi_{Q_0}i_0 \otimes k$, cf. Theorem 3.1.1). This actually works over $k' := \mathbb{Z}[\frac{1}{|\Pi|}]$ since we do not need the roots of unity to express the center of $HH^0(\Pi_{Q_0} \otimes k')$.

Now, let $k := \mathbb{Z}[\frac{1}{|\Pi|}]$. Let $Q \supseteq Q_0$ be any quiver, and as in Section 3.2 let $r' := \sum e \in Q \setminus Q_0 1_{Q_0}(ee^* - e^*e)1_{I_0}$. By Theorem 3.2.13(iii) and Proposition 4.2.29 we know that, in $HH_0(\Pi_Q \otimes k)$, we must have

$$\{[r'z], [h]\} \in \{[(r')^2], [h]\}, \quad \forall z \in HH^0(\Pi_{Q_0} \otimes k), h \in \Pi_Q \otimes k,$$  \hfill (5.5.2)

using Notation 4.2.24. But then, the same statement must be true over $\mathbb{Z}$ since $k$ is flat over $\mathbb{Z}$. Now, for any $f \in \Pi_{Q_0}$, choose any lifts $\tilde{f}, \tilde{z} \in \overline{P_{Q_0}}$. Then, we obtain

$$[\tilde{z}, \tilde{f}] \equiv -\mu_{r'}([\tilde{z}, \tilde{f}]) \pmod{[\{(r')^2\} + \{(r'), f\}],} \hfill (5.5.3)$$

where

$$\mu_{r'}(a \otimes b) := ar'b.$$  \hfill (5.5.4)

Because of Proposition 2.2.8 or by the same argument, one deduces also that the above is true replacing $\Pi_Q$ by $P_{\overline{Q}_0}$, or more generally by $\Pi_{Q,J}$, for $Q \supseteq Q_0$ and either $Q \neq Q_0$ or $J \neq \emptyset$; we then set $r' = -\sum e \in Q_0 (ee^* - e^*e) \in \Pi_{Q,J}$.

We interpret (5.5.3), together with the NCCI property that $P_{\overline{Q}_0} \cong P_{Q_0} * T_k(r')$, as saying that $P_{\overline{Q}_0}$ is a noncommutative or free deformation of $\Pi_{Q_0}$, which “quantizes” the double bracket $\{ \, , \}$ (more precisely, the Poisson bracket on $HH^0(\Pi_{Q_0})$, using the following propositions). One can also say that $P_{Q}$ is a noncommutative deformation of $\Pi_{Q}$ for any non-Dynkin, non-extended Dynkin quiver (the NCCI property yields “flatness”), but without a statement about Poisson bracket. For more details and the general theory of this type of “free product” deformation, see [GSa].

**Example 5.5.5.** In the case $Q_0 = \tilde{A}_0$, say that $Q_0 = \{x\}, \overline{Q_0} = \{x, y\}$. Then, the identity says that, for any (noncommutative) polynomials $P, Q$,

$$[P(x, y), Q(x, y)] \equiv -r'\{P(x, y), Q(x, y)\} \pmod{[\{(r')^2\} + \{(r'), \Pi_{Q_0}\}],} \hfill (5.5.6)$$

50
using the usual Poisson bracket on polynomials in two variables \( x, y \). For example,

\[
[x^n, y] = -\sum_{i=0}^{n-1} x^i r' x^{n-i-1} \equiv -r' x^{n-1} \pmod{[(r')^2] + [(r'), \Pi_Q]}. \tag{5.5.7}
\]

In order to interpret the above over primes where the map \( HH^0(\Pi_{Q_0}) \to HH^0(\Pi_{Q_0}) \) is no longer injective, it is useful to have the following result:

**Proposition 5.5.8.** Let \( Q_0 \) be any extended Dynkin quiver with extending vertex \( i_0 \in I_0 \), and let \( i \in I_0 \) be any vertex. Let \( z \in HH^0(\Pi_{Q_0}) \) be any central element. Then, taking image in \( HH_0(\Pi_{Q_0}) \), we have

\[
[i_0 z] = \dim(\rho_i)[i_0 z], \quad [z] = |\Gamma|[i_0 z]. \tag{5.5.9}
\]

**Proof.** Let us consider the sequence

\[
HH^0(\Pi_{Q_0}) \hookrightarrow HH^0(\Pi_{Q_0}) \otimes \mathbb{C} \hookrightarrow \mathbb{C}[x, y] \rtimes \Gamma \to HH_0(\mathbb{C}[x, y] \rtimes \Gamma) \twoheadrightarrow HH_0(\Pi_{Q_0}) \otimes \mathbb{C}. \tag{5.5.10}
\]

We know that the sequence is injective. By the analysis in Section 3.2, the image of \([i_0 z]\) in \( HH_0(\mathbb{C}[x, y] \rtimes \Gamma) \) consists of projection of \( z \) to \( e \) where \( e \in \Gamma \) is the identity. But this is just the projection of the idempotent \( f_i \) to \( e \), which is taking the trace of the identity element in the representation \( \rho_i \), which is just the dimension. \( \square \)

**Corollary 5.5.11.** For any extended Dynkin quiver \( Q_0 \) and any vertex \( i \), whose corresponding representation of \( \Gamma \) has dimension \( m \), we have a Poisson bracket

\[
i_0 HH^0(\Pi_{Q_0}) \otimes i_0 HH^0(\Pi_{Q_0}) \to i_0 HH^0(\Pi_{Q_0}), \tag{5.5.12}
\]

\[
\{i_0 z_1, i_0 z_2\} = i_0 z_3 \text{ where } \frac{1}{m}[i_0 z_1] + \frac{1}{m}[i_0 z_2] = \frac{1}{m}[i_0 z_3]. \tag{5.5.13}
\]

These Poisson algebras are all isomorphic, for all \( i \), and we may consider the total bracket

\[
HH^0(\Pi_{Q_0}) \otimes HH^0(\Pi_{Q_0}) \to HH^0(\Pi_{Q_0}), \quad \{z_1, z_2\} = \sum_{i' \in I} \{i' z_1, i' z_2\}, \tag{5.5.14}
\]

which is also isomorphic to the Poisson algebra \( iHH^0(\Pi_{Q_0}) \) under the projection map.

Using the above, we can define a Poisson algebra on \( HH^0(\Pi_{Q_0}) \otimes \mathbb{F}_p \) for any prime \( p \), which is isomorphic to \((i_0 \Pi_{Q_0} i_0) \otimes \mathbb{F}_p \) (and \( HH_0(\Pi_{Q_0})_{\text{free}} \otimes \mathbb{F}_p \)). Finally, we can put all of the previous together and deduce the

**Proposition 5.5.15.** Let \( Q \supseteq Q_0 \) where \( Q_0 \) is extended Dynkin. For any \( z \in HH^0(\Pi_{Q_0}) \) and \( x \in \Pi_{Q_0} \), and any lifts \( \tilde{z}, \tilde{x} \) to \( \Pi_Q \), we have

\[
\tilde{z} \tilde{x} - \tilde{x} \tilde{z} \equiv [r' \psi([i_0 z], [x])] \pmod{[(r')^2] + [(r'), \Pi_{Q_0}]}, \tag{5.5.16}
\]

where \( \psi : HH_0(\Pi_{Q_0}) \to HH_0(\Pi_{Q_0}) \) is the map given by composing the projection to \([i_0 \Pi_{Q_0} i_0] \cong HH_0(\Pi_{Q_0})_{\text{free}} \) with the isomorphism to \( HH^0(\Pi_{Q_0}) \).

**Proof.** By Theorem 3.2.13(iii), any commutator \([\tilde{z}, \tilde{x}]\) must be of the form \([r' z']\) modulo \([[(r')^2] + [(r'), \Pi_{Q_0}]]\). Then, we know that \([z'] \in HH_0(\Pi_{Q_0}) \) must be equal to \([z, [x]]\) by (5.5.3). Finally, by Corollary 5.5.11 this means that \([i_0 z' i_0] = \{1, [z], [x]\} = \{i_0 z, [x]\}\). \( \square \)
The above proposition can be interpreted (using \((5.5.3)\)) as saying that, letting
\[
\pi : P_{\tilde{Q}_0} \to \Pi_{Q_0}
\]  
be the projection, we have
\[
\mu \langle \hat{P}_{\tilde{Q}_0}, \pi^{-1}(HH^0(\Pi_{Q_0})) \rangle \subset \pi^{-1}(HH^0(\Pi_{Q_0})), \tag{5.5.17}
\]
where \(\mu\) is the multiplication map. In other words, \(\pi^{-1}(HH^0(\Pi_{Q_0}))\) is a Loday ideal with respect to the Loday bracket \(L \otimes P_{\tilde{Q}_0} \to P_{\tilde{Q}_0}\) of Section 5.2. It is interesting that this result, purely about \(Q_0\), appears in trying to study \(Q \supseteq Q_0\).

6 Quivers containing \(\tilde{A}_{n-1}\): proof of Theorem 4.2.40

6.1 The case of \(\tilde{A}_0\) Although Corollary 3.1.2 already implies Theorem 1.1.2 in the case \(Q_0 = \tilde{A}_0\) (cf. Remark 3.2.20), and in fact Theorem 4.2.40 for \(\tilde{A}_0\), we explain how to prove it using bases, as this will be generalized to \(\tilde{A}_{n-1}\) for general \(n\) in the next subsection.

Let \(Q \supseteq Q_0 = \tilde{A}_0\), i.e., \(Q\) is a quiver containing a loop, say based at \(i_0 \in I\). Fix a maximal tree \(G \subset Q\) as in Proposition 4.2.21: here this just means that all edges of \(G\) are oriented towards the vertex \(i_0\) (one can follow oriented edges of \(G\) to arrive at \(i_0\) from any vertex). We define \(G^* := \{e^* | e \in G\}\), and one evidently has \(G \cap G^* = \emptyset\).

We consider \(Q_0 \subset Q\) to be the quiver \(Q_0 \cong \tilde{A}_0\) given by vertex \(i_0\) together with the loop we assumed to exist at the beginning. Let the loop be \(x\) and its reverse \(y := x^*\). Take \(z_{a,b}\) defined as in (3.1.3).

We recall the isomorphism \(\Lambda_Q \cong V/W\) from Proposition 4.2.26 explicitly in this case. Define \(r_{i_0} = 1_{i_0} r_{i_0} = i_0 r_{i_0}\) as in the setup of Proposition 4.2.25. Let \(F := \Pi_{Q,i_0}\), and set \(r' := r_{i_0} - xy + yx \in \Pi_{Q \setminus Q_0,i_0}\). Set \(A = F/\langle i_0 r_{i_0} \rangle \cong \Pi_Q\).

Then, \(V := A/[A(Q \setminus Q_0)A, A]\) has a basis consisting of:

1. Cyclic monomials containing an edge from \(Q \setminus Q_0\), such that maximal submonomials from \(Q \setminus Q_0\) are as dictated by Proposition 4.2.21 and maximal submonomials from \(Q_0\) are of the form \(z_{a,b}\);

2. Monomials of the form \(z_{a,b}\),

and is free (cf. Proposition 4.2.21). We have \(\Lambda_Q \cong A/[A, A] \cong V/W\) where \(W = \langle W_{a,b} \rangle_{a,b \geq 1, (a,b) \neq 1}\) is as described in Lemma 3.1.2. We have assumed that \(Q \setminus Q_0\) is nonempty, so that \(r' \neq 0\). Also, since \(r'\) is a sum of commutators, we have that \([r']^{p'}\) is a multiple of \(p\) (as an element of \(F/[F,F]\), and hence in \(V\)). Then, the rest of the result follows immediately from (the proof of) Lemma 3.1.2. We note that we could have alternatively proved this result by presenting \(\Pi_Q\) as a case of Corollary 3.1.14 (using that \(\Lambda_{Q \setminus Q_0,i_0}\) is torsion free).

This finishes the proof of Theorem 4.2.40 and hence Theorem 1.1.2 in the \(\tilde{A}_0\) case, which includes Conjecture 1.1.1 as a special case.
6.2 Proof of Theorem 1.1.2 for quivers containing \( \tilde{A}_{n-1} \)

We generalize the previous section to the case of cycles of length \( n > 1 \) (i.e. a copy of \( \tilde{A}_{n-1} \)). We use the notation leading up to Theorem 4.2.40. Recall that \( Q \supseteq Q_0 = \tilde{A}_{n-1} \) according to the Figure 1 there, and the identification of \( I_0 \) with an affine space over the abelian group \( \mathbb{Z}/\#(I_0) \), so that adding one moves in the counter-clockwise direction and subtracting one moves in the clockwise direction. When there is any chance of confusion, if \( i \in I_0 \) and \( m \in \mathbb{Z} \), we write \( i+n \in I_0 \) for the result of adding \( n \) in the affine space. We can describe the path algebra \( P_{Q_0} \) as generated by \( I_0, x, \) and \( y \), with relations/conditions: (1) \( I_0 \) are idempotents of degree zero; (2) \( x \) and \( y \) have degree 1; and (3) \( ix = x(i+1) \), \( (i+1)y = yi \), and \( ixj = jyi = 0 \) if \( j \neq i+1 \). As before, we can define \( z_{a,b} \) by (3.1.3). Here \( x \) and \( y \) are given in (4.2.38).

We define \( r_{I_0} := \sum_{i \in I_0} iri \) and \( r' := r_{I_0}-\sum_{e \in Q}(ee^*-e^*e) \) analogously to the case \( n = 0 \). Let \( F = \Pi_{Q,I_0} \) and \( A = F/((r_{I_0})) = \Pi_{Q} \). Let \( \bar{\Pi}_{Q_0} := \sum_{i \in I_0} i\Pi_{Q_0} \) be the subspace of closed paths beginning and ending at a vertex of \( I_0 \). We compute \( \bar{W} \) and the quotient \( V/\bar{W} \) (cf. Proposition 4.2.25).

In this case, \( W \) is generated by reducing the elements \([z_{a-1,b},ix],[z_{a,b-1},iy] \in P_{Q_0} \) for all \( a,b \in \mathbb{Z}_{\geq 0} \) with \( n | (b-a) \), and all \( i \in I_0 \), modulo \( r_{I_0} = xy -yx + r' \) and commutators of elements which include a multiple of \( \langle Q \setminus Q_0 \rangle \) (including \( r' \)), to obtain a subspace \( W \subset V \).

One finds relations similar to (3.1.4)–(3.1.6), keeping track of the idempotents \( I_0 \), and substituting \( r_{I_0} = xy -yx + r' \) for \( r \). The nice property of our choice of the \( z_{a,b} \)'s is now that \( ix = xy \) for all \( i \in I_0 \). Let us assume that \( a > b \), since essentially the same relations result in the other case (and the torsion must be the same). We may assume that \( n \not| (b-a) \), or else the bidegree \((a,b)\)-part is zero. Let \( \eta : A \to V \) be the projection. Since, writing \( iz_{a,b} = t_1t_2\cdots t_{a+b} \) where \( t_j \in Q_0 \), one has

\[
\sum_{m=1}^{a+b}(t_m t_{m+1} \cdots t_{m-2}, t_{m-1}) = 0, \quad (6.2.1)
\]

it follows (using that \( \eta([Ar'A,A]) = 0 \) that

\[
bn(iz_{a,b-1}, y) + bn((i+1)z_{a-1,b}, x) + \sum_{m=0}^{a-b-1} \eta([i+m]z_{a-1,b}, x) = 0. \quad (6.2.2)
\]

So, we need only compute the \( \eta([iz_{a-1,b}, x]) \), or equivalently, the \( \eta([iz_{a-1,b}, ix]) \): (recall that \( I_0 \) is considered as a \( \mathbb{Z}/\#(I_0) \)-torsor and \( i+n \) is the operation of adding \( n \in \mathbb{Z} \) to \( i \in I_0 \):

\[
\eta([iz_{a-1,b}, ix]) = [(i+1-i)z_{a,b}] + \sum_{c=0}^{b-1} (i-(a-b-2))r'(xy)^c x^{a-b-1}(xy)^{b-1-c}x. \quad (6.2.3)
\]

Using only \( n-1 \) of the above \( n \) relations for each fixed \( a, b \), this can be interpreted as eliminating the elements \([iz_{a,b}] \) for all \( i \in I_0 \) except any fixed vertex \( i_0 \in I_0 \). To add the last relation in, we need only consider the sum of all \( n \) relations, which together with (6.2.2) (allowing us to divide by \( \gcd(\frac{a-b}{n}, b) \)) gives just (3.1.4) for \( a > b \) (the coefficients of \( I_0 \) disappear as we are summing over all translations around the cycle: every vertex of \( I_0 \) becomes \( 1_{i_0} \)). The same argument works for \( a < b \), and so the proof from Section 6.1 shows that \( \Lambda_Q \) has no torsion in these cases. The only detail to note is that, since \( Q \neq Q_0 \), at least some of the elements in the expression for \( W_{a,b} \) having
the necessary coefficients are nonzero (even if there is only an additional edge at one particular vertex \( i \in I_0 \), there are terms in \( W_{a,b} \) where \( r \) only appears adjacent to this vertex \( i \)).

In the case \( a = b = m \), again we see that the quotient \( V/W \) is the same as eliminating \([iz_{m,m}]\) for all \( i \neq i_0 \), and considering only the relation \( W_{a,a} \) from (3.1.4)–(3.1.6). For the same reasons as in Section 6.1, it follows that \( V/W \) has torsion \( \mathbb{Z}/p \) in exactly those bidegrees \((p^\ell, p^\ell)\) for \( p \) prime and \( \ell \geq 1 \), and the torsion is generated by the element \([r^{p^\ell}] / p \in P_{\mathbb{Q}}/[P_{\mathbb{Q}}, P_{\mathbb{Q}}] \). Theorem 4.2.40 is proved.

### 6.3 Hilbert series and \((4.1.14)\) in the \( \tilde{A}_n \) case

In this section, we verify the Hilbert series of \( \Lambda_{\tilde{A}_n} \) using our bases, and give a direct proof of the curious identity (4.1.14) from [EG06] in this case. We provide this since it is an easy consequence of Theorem 4.2.40 (which we just proved), and gives a different proof from what is found elsewhere.

First, from Theorem 4.2.40 we easily deduce

**Proposition 6.3.1.** The Hilbert series of \( \Lambda_{\tilde{A}_n} \) and \( \iota \Pi_{\tilde{A}_n} i \), for any \( i \in I \), are given by

\[
h(\iota \Pi_{\tilde{A}_n} i; t) = h(\Lambda_{\tilde{A}_n}; t) = \frac{1 + t^n}{(1 - t^2)(1 - t^n)} = \frac{1 - t^{2n}}{(1 - t^2)(1 - t^n)^2}.
\]

(6.3.2)

It immediately follows that one has the formula

\[
h(\Lambda_{\tilde{A}_n}; t)(1 - t^2) = 1 + \frac{2t^n}{1 - t^n},
\]

(6.3.3)

which we can use to verify the following formula for Hilbert series from [EG06] (using (4.1.13)):

\[
h(Q(\Pi); t) = \prod_{m \geq 1} \frac{1}{(1 - t^{m})a_m},
\]

(6.3.4)

\[
\prod_{m \geq 1} \frac{1}{(1 - t^{m})a_{m-2}} = \frac{1}{1 - t^2} \cdot \prod_{m \geq 1} \frac{1}{\det(1 - t^{m} \cdot C + t^{2m} \cdot 1)},
\]

(6.3.5)

where \( C \) is the adjacency matrix of \( \tilde{Q} \), and \( a_{-1} = a_0 = 0 \). The element \( 1 - t \cdot C + t^2 \cdot 1 \) is just \( 1/t \) times the so-called “\( t \)-analogue of the Cartan matrix”, \( (1 + \frac{1}{t}) \cdot 1 - C \). For \( \tilde{A}_{n-1} \), one has

\[
\det(1 - t \cdot C + t^2 \cdot 1) = (1 - t^n)^2.
\]

(6.3.6)

To verify (6.3.5), set \( h((\Lambda_{\tilde{A}_{n-1}}); t) = \sum_m a_m t^m \); one then has from (6.3.3)

\[
a_m - a_{m-2} = 2[n \mid m], \quad m \geq 3,
\]

(6.3.7)

which implies the desired identity.

Since \( h(\iota_0 \Pi_{\tilde{A}_{n-1}} i_0; t) = h(\Lambda_{\tilde{A}_{n-1}}; t) \), by (4.1.13), the \( a_m \)’s above satisfy

\[
1 + \sum a_m t^m = \left( \frac{1}{1 - t \cdot C + t^2 \cdot 1} \right)_{00}
\]

(6.3.8)

where 00 indicates the diagonal entry corresponding to \( i_0 \). That is, \( a_m = \phi_m(C)_{00} \), where \( \phi_m \) is the \( m \)-th Chebyshev polynomial of the second type. Since \( \phi_m - \phi_{m-2} = \varphi_m \), a Chebyshev polynomial of the first type, our work above explicitly verifies the identity (4.1.14) from [EG06].

So, from our point of view, this identity is the fact that \( \Lambda_{\tilde{A}_{n-1}} \cong \iota_0 \Pi_{\tilde{A}_{n-1}} i_0 \), together with the similarity between the “partial fractions” identity (6.3.3) and the formula for the determinant of the \( t \)-analogue of the Cartan matrix, (6.3.6).
6.4 Poisson structure on $i_0\Pi_Qi_0$ / Lie structure on $\Lambda_Q$ for $Q = \tilde{A}_{n-1}$ Here we give a description of the Lie structure on $\Lambda_Q$: an alternative description over $\mathbb{C}$ is given in Theorem 9.1.1 using the McKay correspondence.

Note that, by Theorem 4.2.60, for any extended Dynkin quiver $Q$, we have an injection $i_0\Pi_Qi_0 \hookrightarrow \Lambda_Q$, and it is easy to see that the image is a Lie subalgebra under the necklace bracket. As before, it is easy to deduce that the resulting structure on $i_0\Pi_Qi_0$ is that of a Poisson algebra.

In the case of $Q = \tilde{A}_{n-1}$, again by Theorem 4.2.60, $\Lambda_Q$ is torsion-free and the map $i_0\Pi_Qi_0 \hookrightarrow \Lambda_Q$ is in fact an isomorphism onto $(\Lambda_Q)_+ \oplus \langle [i_0] \rangle$.

Then, the simplest way to understand $\Lambda_{\tilde{A}_{n-1}}$ is in terms of the basis $[i_0x^ay^b]$, where $x$ denotes moving clockwise one edge, and $y$ denotes moving counterclockwise, and $i_0$ is a fixed vertex. One requires that $n \mid (a - b)$, and $a,b \in \mathbb{Z}_{\geq 0}$. To compute this, we first compute the bracket in terms of the rational basis $1x^ay^b$, where $1$ is the identity (the sum of all vertices); since there is no torsion in $\Lambda_{\tilde{A}_{n-1}}$, this suffices. Then, it immediately follows that one can compute the bracket (and cobracket) by summing over ways to pair letters $x,y$ that correspond to opposite edges. One easily computes that $\delta$ is zero. The bracket is then

$$[[i_0x^ay^b],[i_0x^cy^d]] = i_0 \frac{ad-bc}{n} x^{a+c-1}y^{b+d-1}. \quad (6.4.1)$$

In other words, the Poisson structure on $i_0\Pi_Qi_0$ is given by $\{i_0z_a,b, i_0z_c,d\} = \frac{ad-bc}{n} i_0z_{a+c-1,b+d-1}$.

In terms of the isomorphism $\Lambda_{\tilde{A}_{n-1}} \cong \Lambda_{\tilde{A}_{n-1}} \otimes 1 \subset \Lambda_{\tilde{A}_{n-1}} \otimes \mathbb{C} \cong \mathbb{C}[x,y]^{\mathbb{Z}/n}$, one has integral basis elements $x^n, y^n, xy$, with Poisson bracket which is $\frac{1}{n}$ times the usual Poisson bracket on $\mathbb{C}[x,y]$, restricted to $\mathbb{C}[x,y]^{\mathbb{Z}/n}$.

Finally, we can give an explicit presentation of $i_0\Pi_Qi_0$ as a graded Poisson algebra over $\mathbb{Z}$ as follows:

**Proposition 6.4.2.** We have the following explicit presentation of $i_0\Pi_Qi_0$ for $Q = \tilde{A}_{n-1}$:

$$X := [i_0x^n], \quad Y := [i_0y^n], \quad Z := [i_0xy], \quad (6.4.3)$$
$$i_0\Pi_Qi_0 \cong \mathbb{Z}[X,Y,Z]/(XY-Z^n), \quad (6.4.4)$$
$$\{X,Y\} = nZ^{n-1}, \quad \{X,Z\} = X, \quad \{Y,Z\} = -Y, \quad (6.4.5)$$
$$|X| = n, \quad |Y| = n, \quad |Z| = 2. \quad (6.4.6)$$

We remark that $i_0\Pi_Qi_0$ can be thought of as $\mathbb{Z}[x,y]^{\mathbb{Z}/n} = \mathbb{C}[x,y]^{\mathbb{Z}/n} \cap \mathbb{C}[x,y]$ (using $\frac{1}{n}$ times the standard bracket), and this also allows one to make sense of $\mathbb{F}_p[x,y]^{\mathbb{Z}/n}$ for primes $p \mid n$. The same comment will apply for other extended Dynkin quivers $Q$, when we explicitly compute $i_0\Pi_Qi_0$ as a Poisson algebra in those cases as well.

7 Quivers containing $\tilde{D}_n$: proof of Theorem 4.2.48

7.1 A basis for $\Pi_{\tilde{D}_n}$: Theorem 4.2.48(i) Our goal is to prove Theorem 4.2.48(i).

Let $1_{\text{in}} = \sum_i \text{internal } i$ be the sum of the internal vertices (which are idempotents in $\Pi_Q$). We define a filtration on $\Pi_Q$ by powers of the bi-ideal $(1_{\text{in}}RL1_{\text{in}}, 1_{\text{in}}LR1_{\text{in}})$. We first show that the associated graded gr $\Pi_Q$ with respect to this is spanned by the given elements, which shows that $\Pi_Q$ is. We begin by showing that $\Pi_Q$ is spanned by a larger set: elements of the form $i(RL)^kY$ or $i(RL)^kY$ where $Y$ is a path that satisfies the following property: The paths change from going to
the left to right or vice-versa only at the four endpoints of $\tilde{D}_n$. This differs from the proposed basis only by allowing $F_U$’s to change to $F_D$’s, and replacing the first $F_U$ or $F$ by any of $L_U, L_D, R_U, R_D$.

First, $\Pi_Q$ is obviously generated by elements $i(RL)^kY, i(RL)^kY$ where $Y$ is a generalized path. We can assume that $Y$ does not begin with $LR, RL$, or any expression equal to one of these, since otherwise we could absorb this into the product $(LR)^m$ or $(RL)^m$: if $Y$ begins at an internal vertex $i$, then $iLR = iRL$, whereas if $Y$ begins at an external vertex, only one of $iLR, iRL$ is nonzero.

Next, we show that if $Y$ includes a change of left-right direction at a vertex other than an external one, then it is equal to an element that makes such a change of direction strictly earlier: since we assume the change cannot happen at the beginning, this will prove the desired result (obviously, there are only finite number of possible expressions $(LR)^kY, (RL)^kY$ of a fixed length to consider).

Suppose that the first change of direction at an internal vertex is of the form $iLR$ where $i, j$ are both internal vertices. Then, it must follow $L$, so that we have $LiLjR = LiLR = LiRL$, giving the desired element which changes direction at an internal vertex earlier. Similarly, we can handle $iRjL$ where $i, j$ are internal.

Next, suppose that the first change of direction is of the form $iLjR$ where $i$ is external (so right external) and $j$ is internal. Then it must be preceded by $RX$ for the appropriate $X \in \{U, D\}$, so we get $RXLjR = RXLRLX = RLRXL$ for $X \neq X \in \{U, D\}$. If these three arrows are the first of $Y$, then we can absorb an additional $RL$ (or $LR$) into the initial power of $RL$ or $LR$, thus discarding $Y$. If not, then these three must be preceded by $R$, since we assumed this was the first change of direction. Then, we get $RiRLRX = RiLRRX$, again yielding an expression that changes direction at an internal vertex earlier.

This proves the desired claim. Now, to show that $\Pi_Q$ is spanned by the given elements, we make two observations: First of all, if we have a nonzero expression $i(RL)^kY$, where $Y$ does not begin with $R_U$ or $R_D$, then $i$ must be an internal vertex and $Y$ must begin with $L_U$ or $L_D$. In this case, we can simply note that $i(LR)^k = i(RL)^k$ and discard this term. Secondly, if $Y$ includes $L_D$ anywhere except at the very end, then it precedes $R$, and we may freely replace $L_D R$ with $LR - L_U R$. Now, if this is the beginning of $Y$, then we can absorb the $RL$ term into $LR$ or $RL$, yielding a term of the desired form. If not, then it must be preceded by $L$ (as it was originally $L_D R$), so the $RLR$ term involves changing direction at an internal vertex, which we showed above is in the span of an expression with shorter $Y$. By induction, one obtains the desired result.

Another way to state the second observation above is by considering the filtration of $\Pi_Q$ by powers of the ideal $(LR, RL)$. Then, by reverse induction on filtration degree, we showed that the given elements span $\Pi_Q$, and moreover the associated graded $\text{gr} \, \Pi_Q$ (since images of words in $LR, RL, L_U, L_D, R_U, R_D$ clearly span $\text{gr} \, \Pi_Q$).

It remains to show linear independence of the given elements over $\mathbb{Z}$. We do this by mirroring the above, but using the Diamond Lemma and a bit more careful analysis. For any generalized path $Y \in GP$, we can construct an alternating sequence

$$E(X) := (t_1, t_2, t_3, \ldots, t_m)_{t_i \in \{L, R\}}, \quad (7.1.1)$$

which we call “the ends of $X$”, as follows: Start with the empty sequence, unless the initial vertex of $X$ is an external one, in which case we start with the side $L$ for left or $R$ for right. Every time we hit the left end (a left external vertex or the superposition of both) we add an $L$ to the sequence $E(X)$, unless $E(X)$ is already nonempty with last term equal to $L$. Every time we hit the right end, we add an $R$, unless the sequence is nonempty with last term $R$. That is, the sequence records the order in which the path hits left and right endpoints, throwing out multiple hits of one side
Before next hitting the opposite side. Put another way, it records which is the first side the path reaches, and the number of times it alternates from one side to another.

Now, for any \( X \in GP \cap \overline{P}_Q[k] \) of length \( k \), with \( E(X) = (t_1, t_2, \ldots, t_m) \), we can write \( X = X_1t_1X_2t_2 \ldots t_mX_{m+1} \) in the unique way such that \( X_1 \in \overline{P}_Q \) has minimal possible length, then for this \( X_1, X_2 \) be of minimal length, etc.

Finally, let us consider the places \( \ell \in \{1, 2, \ldots, k\} \) where \( X \) is “not going in the correct direction.” By “not the correct direction at \( \ell \)”, we mean that the path is headed away from the next endpoint \( t_m \) (=towards the previous endpoint \( t_{m-1} \)) appearing in the path. Precisely, suppose that \( X = a_1a_2 \cdots a_k \) and \( X \) hits the endpoint \( t_m \), at the time corresponding to when we add \( t_m \) to the sequence, between \( a_{\ell-1} \) and \( a_{\ell} \) (i.e., if the last endpoint \( t_m \) is hit at the very end, set \( \ell_m := k + 1 \)). Then, for any \( \ell \leq \ell < \ell_{i+1} \), we say that \( \ell \) is a place where \( X \) heads in the “wrong direction” if \( a_{\ell} \neq t_{\ell+1} \). Now, for any \( X \in GP \cap \overline{P}_Q[k] \) of the form \( X = iXj \) for some \( i, j \in I \), let \( WD(X) \subset \{1, 2, \ldots, k\} \) be the subset of places where \( X \) goes in the wrong direction. One easily sees that the size of \( WD(X) \) is the difference between \( k \) and the shortest path with the same endpoints as \( X \) (this makes sense since we assumed \( X \) had definite endpoints).

Now, we may define \( X < Y \) if either \#\( WD(X) > \#WD(Y) \), or else \#\( WD(X) = \#WD(Y) \) and there is an order-preserving bijection \( \phi : WD(X) \to WD(Y) \) such that \( \phi(c) \leq c \) for all \( c \in C(X) \). That is, the places where \( X \) goes in the “wrong direction” occur strictly before the corresponding places in \( Y \).

Now, we finally have the ordering we need such that the relations on \( \langle GP \cap \overline{P}_Q[k] \rangle \) modulo which one gets \( \Pi_Q[k] \) are confluent and give the desired elements as the normal form (one reduces as in the first part of this proof). So, the desired elements are linearly independent and form a basis of \( \Pi_Q \), and since their images span \( \text{gr} \ \Pi_Q \), they form a basis of \( \text{gr} \ \Pi_Q \) as well.

Similarly to the \( \tilde{A}_{m-1} \) case, it is not difficult to deduce from the above procedure that \( \Pi_Q \) is an NCCI (that one has a unique reduction of \( \overline{P}_Q \) to words in the above basis and the relations \( iri \)). Also, it’s clear from the above proof that for any of the basis elements, we can feel free to replace any of the \( F_U \)’s in the \( z_{c,C,X,i,j} \) by \( F_D \)’s, and we will still be left with a basis.

### 7.2 Proof of Theorem 4.1.2 for quivers containing \( \tilde{D}_n \)

Suppose \( Q \) is any quiver with a proper subquiver \( Q_0 \subseteq Q \) with \( Q_0 \cong \tilde{D}_n \), such that \( Q_0 \) has vertex set \( I_0 \) and \( Q \) has vertex set \( I \). Define \( \text{rt}_0 := \sum_{i \in I_0} iri \). Fix a forest \( G \subset Q \setminus Q_0 \) as in Proposition 4.2.21.

It remains to compute \( W \subset V \), which is spanned by \( \eta([X,e]) \) for \( X \) listed in Theorem 4.2.48(i) and \( e \in Q_0 \) such that \( X \in \overline{P}_Q := \sum_{i \in I} i\Pi_Qi \) (cf. Proposition 4.2.23 (iii)). We can instead let \( e \) be one of \( L_U, L_D, R_U \), or \( R_D \).

First, we need to eliminate some of the commutators \( \eta([X,e]) \). We prefer to eliminate those \( X = z_{c,C,X,i,j} \) with smaller \( c \), so as to maximize the \( c \) of the remaining commutators. Essentially, this means continuing to use the filtration on \( A \) by powers of the ideal \( (LR, RL) \).

First note the basic

**Lemma 7.2.1.** If \( X \) is obtained from \( z_{c,C,X,i,j} \) by changing \( \alpha \) instances of \( R_UL \) to \( R_DL \) or \( L_UR \) to \( L_DR \), then \( X \equiv (-1)^{\alpha} z_{c,C,X,i,j} \) modulo \( (z_{c,c',X,i,j} \mid c' > c) \).

**Proof.** This follows from the fact that \( R_UL + R_DL = RL \), using the ideas in the proof of Theorem 4.2.48(i) (in the last subsection).

We use this to write some relations which are deduced similarly to (6.2.1), (6.2.2). These will
allow us to express certain commutators in terms of commutators involving basis elements living in a smaller power of \((1_{in}RL1_{in}, 1_{in}LR1_{in})\).

**Notation 7.2.2.** We say that an equality holds “modulo commutators with higher powers of \((1_{in}RL1_{in}, 1_{in}LR1_{in})\)” if the equation is true up to \(\eta\) of commutators with elements \(z_{c', C', X', i', j'}\), where \(c'\) is greater than all the indices \(c\) which appear in the equation.

Take \(C\) such that \(2(n - 2) \mid C\) and \(C > 0\). This is equivalent to the condition that a basis element of the form \(z_{c,C,X,i,i}\) has either (1) \(i\) is internal and \(z_{c,C,X,i,i} = z_{c,C-1,X,i,j}X\) for some \(j\) (i.e., it ends in \(X\)); or (2) \(i\) is external. Pick any choice of \(i, X\) such that \(z_{c,C,X,i,i}\) is a basis element; let \(c > 0\). Let \(\bar{X}\) be defined to be \(X\) when \(i\) is internal (first case), and \(\overline{X}\) otherwise (second case). That is, \(\bar{X}\) is the last direction in \(z_{c,C,X,i,i}\). One then computes, using (6.2.1), Lemma 7.2.1, and Notation 7.2.2 for \(c \geq 1,

\[
\sum_j c\eta([z_{c-1,C+1,X,i,j}, j\bar{X}i]) + c\eta([z_{c-1,X,i,i}, j\bar{X}i]) \\
+ \frac{C}{2(n - 2)} \sum_{i', j' \in I_0, \{i_{r:d+e}, y \in \{\ell, r\}} (-1)^{\ell, r}\eta([z_{c-1,Y,i', j'}, j'(L + R)j']) = 0
\]

modulo commutators with higher powers of \((1_{in}RL1_{in}, 1_{in}LR1_{in})\), \(7.2.3\)

which shows that we can eliminate the relations \(\sum_j \eta([z_{c-1,C+1,X,i,j}, j\bar{X}i])\). This says we don’t have to consider commutators of basis elements \(z_{c', C', X', i', j'}\), that end up going in the same direction \(X'\) they began in, with an arrow in the opposite direction \(\overline{X'}\). If \(i\) is adjacent on the \(X\)-side to external vertices, so that the sum contains two terms, then we can choose to eliminate the relation \(\eta([z_{c-1,C+1,X,i,i}, i\bar{X}D\overline{X}i])\). Note in the above that the sums over all \(i', j'\) are effectively only over adjacent pairs \(i', j'\), and the \(j'(L + R)j'\) is just a shorthand for the unique arrow in \(Q_0\) from \(j\) to \(i\).

Also, the relation \(\eta([z_{c-1,C+1,X,i,i}, i\bar{X}U\overline{X}i])\) we left above can be interpreted simply as expressing \(z_{c-1,C+2,X,i,i}\) as \(z_{c,C,X,i\bar{X}U+i\bar{X}U}\) plus some multiple of \(r' = r_{l_0} - \sum_{e \in Q_0} (ee^* - e^*e)\) in the quotient \(V/W\). So we can also eliminate this relation if we also eliminate the generator \(z_{c-1,C+2,X,i,i}\) from \(V\). Actually, this paragraph is still true if \(C = 0\), so we can also eliminate the relation \(\eta([z_{c-1,1,X,i,i}, i\bar{X}D\overline{X}i])\) and the generator \(z_{c-1,2,X,i,i}\).

Now, consider a basis element \(z_{c,C,X,i,i}\) such that \(2(n - 2) \mid C\) and \(2(n - 2) \mid (C - 2)\) (we dealt with the other cases in the last paragraph). In particular, \(i\) is internal and not adjacent to the end on the \(X\)-side. Let \(g = g(X, i)\) be defined to be the distance from \(i\) to the \(X\)-end. One has \(g = \frac{C}{2} - (n - 2) \frac{C}{2(n - 2)}\) since the \(F_0^{-}\)-portion of the element \(z_{c,C,X,i,i}\) goes forward \(g\) units, loops \(\frac{C}{2(n - 2)}\) times around the long distance of \(\tilde{D}_n\), and then moves \(g\) units forward back to \(i\). We will show that (1) \(\eta([z_{c,C-1,X,i,i}, j\bar{X}i]) \equiv -\eta([z_{c,C-1,X,i,i}, i\bar{X}j])\) modulo commutators with higher powers (in particular, modulo terms that begin and end closer to the external vertices on the \(X\)-side than \(i\)), and that (2) either of these (sums of) commutators can be taken to express \(z_{c,C,X,i,i}\) in terms of \(z_{c+e,C-2m,X,j',j'}\) for \(m > 0\) (and \(g(X, j') < g(X, i)\)); eventually this will reduce us to terms beginning and ending at an external vertex.

We see that (1) follows from the identity

\[
[z_{c,C-1,X,i,i}, j\bar{X}i] + [j(XX)c_{z_{0,C-1,X,i,i}, i\bar{X}j}] = [j\bar{X}Xj, z_{c,C-2,X,j,j}]
\]

and (2) follows because \(\eta([z_{c,C-1,X,i,i}, j\bar{X}i]) - (z_{c,C,X,i,i} - z_{c+1,C-2,X,j,j}) \in [r'^A].\)
Similarly to (7.2.4), for $C > 2$ we can also show that $\eta([z_{c,C-1},X,i_{XD},i_{XD}])$ is in the span of other commutators (note that $2(n-2) \mid C$ for this to be nonzero):

$$
[i(XX)^c z_{0,C-1,i_{XD},i_{XD}}] = [i(XX)^c z_{0,C-1,i_{IX},i_{IX}}] + [z_{c,C-1,i_{IX},i_{IX}},i_{IX}] = \sum_{Y \in \{U,D\}} \eta([z_{c,1,X,i_{XY},j_{XY}},i_{XY}]) = \sum_{Y \in \{U,D\}} \eta([z_{c,1,X,i_{XY},i_{XY}},i_{XY}]),
$$

(7.2.5)

which shows that

$$
\eta([z_{c,C-1,i_{XD},i_{XD}}]) = \eta([z_{c,C-1,i_{IX},i_{IX}}]) + \eta([z_{c,C-1,i_{IX},i_{IX}},i_{IX}]) + \eta([z_{c,C-1,i_{IX},i_{IX}},i_{IX}]) + \eta([z_{c,C-1,i_{IX},i_{IX}},i_{IX}]) = \eta([1_{in}RL1_{in},1_{in}LR1_{in}])^{1+1}(\Pi_0),
$$

(7.2.6)

so we can indeed throw out any one of these commutators: we choose $\eta([z_{c,C-1,i_{XD},i_{XD}}])$.

The only other commutators of the form $\eta([X,e])$ with $X$ as in Theorem 4.2.38 (i) and $X \in \Pi_0$ that we have not yet mentioned are those of the form $\eta([z_{c,1,i_{j},i_{j}},i_{i}])$ where $j$ is an internal vertex. Here we can make use of the identity, similar to the above: if $i, j$ are internal, then

$$
\eta([z_{c,1,i_{j},i_{j}},i_{j}]) = \eta([z_{c,1,i_{j},i_{j}},i_{j}]),
$$

(7.2.7)

and otherwise,

$$
\sum_{Y \in \{U,D\}} \eta([z_{c,1,i_{xy},j_{xy}},i_{xy}]) = \sum_{Y \in \{U,D\}} \eta([z_{c,1,i_{xy},j_{xy}},i_{xy}]).
$$

(7.2.8)

Thus, we can eliminate the commutators $\eta([z_{c,1,i_{j},i_{j}},j_{Ri}])$ for $i \neq i_{RD}, j \neq i_{LD}$. Furthermore, in the cases $i = i_{RD}, j = i_{LD}$, these commutators simply allow one to express $z_{c,0,i_{RD},i_{RD}}$ in terms of $z_{c,0,i_{BU},i_{BU}}$ and $z_{c,0,i_{LD},i_{LD}}$ in terms of $z_{c,0,i_{BU},i_{BU}}$, so we can eliminate all commutators $\eta([z_{c,1,i_{j},i_{j}},j_{Ri}])$ along with the generators $z_{c,0,i_{j},i_{j}}$ for $X \in \{L, R\}$. To summarize the result of the computation (with parenthetical English versions):

1. We can eliminate $z_{c,C,i_{j},i_{j}}$ from the basis if $i$ is internal and $2(n-2) \mid C$, thus passing to a subspace $V_0 \subset V$ (that is, we consider only cyclic paths that are a combination of loops at a vertex and long loops around the length of $D_n$);

2. The only commutators we then need to consider, in order to span $W \cap V_0$ (up to dividing certain relations by certain integers), and thus present $\Lambda_0 \cong V_0/(W \cap V_0)$, are

(a) $\eta([z_{c,1},X,i_{j},j_{Ri}])$ for $2(n-2) \mid C$ and $j \neq i_{LD}, i_{RD}$ (we only need consider commutators obtained from the previous elements by separating the last edge off and writing the commutator), and

(b) $\eta([z_{c,1},i_{RD},j_{Li}])$.

In computing these, we use the following basic identities, which are similar to those for $A_0, A_n$:

First, note that, by the choice of orientation of $D_n = Q_0$ in the previous section, in $A = \Pi_0$, one has $1_{in}(LR - RL)1_{in} = 1_{in}(LR - RL + r_{lo})1_{in} \in (Q \setminus Q_0)^2$. So, letting $r' := r_{lo} - \sum_{e \in Q_0}(ee^* - e^*e)$ (as in previous sections), we have

$$
R_X(LR)i_{R} = (R_XLR)i_{R} = (RLR_X + R_XLR_X - R_XLR_X)i_{R} = ((RL)R_X - R_Xr')i_{R} + R_Xr'i_{RX};
$$

(7.2.9)
\[ R(RL)1_{in} = ((RL)R - R'r')1_{in}; \quad (7.2.10) \]

and similarly swapping left with right, and the element \( r' \in \Pi_Q \) with \(-r'\). We can use these to move an \( R \) (or \( L \)) past a power of \( RL \) (or \( LR \)), thus allowing one to compute \( \eta([z_c, C^{-1}, X, i, j, jRi]) \) for \( X = R \) and \( i \) internal, or for \( X = L \) and \( i \) right external vertex (and similarly swapping left with right). For \( i \) internal and \( j \neq i_{LD} \) a vertex adjacent to \( i \) on the left, one obtains

\[
\eta([z_c, C^{-1}, R, i, j, jRi]) = z_{c,C,R,i,j} - z_{c,C,R,j,i} + \sum_{0 \leq c' < c} [r' z_{c', C^{-1}, R, i, j} z_{c - c' - 1, 1, R, j, i}]. \quad (7.2.11)
\]

(We already noted that we can discard the relation in the case \( j = i_{LD} \).)

For \( i \in \{i_{RU}, i_{RD}\} \) right external, one obtains the following. We use the simplified notation \( i^{(c)} := \begin{cases} i, & \text{if } c \text{ is even}, \\ \overline{i}, & \text{if } c \text{ is odd}, \end{cases} \) where \( \overline{i} \) is the other right external vertex from \( i \). We’ll need the same definition for left external vertices later.

\[
\eta([z_c, C^{-1}, L, i, j, jRi]) = z_{c,C,L,i,i} - \eta(z_{c,1,R,j,i^{(c)}} z_{c^{-1}, L, i^{(c)}, j}) + \sum_{c' < c, \epsilon \in \{0,1\}} (-1)^{1+c'+\epsilon}[r' z_{c', C^{-1}, L, i^{(c)}, j} z_{c - c' - 1, 1, R, j, i^{(c)}}]; \quad (7.2.12)
\]

We simplify the RHS by noting

\[
\eta(z_{c,1,R,j,i^{(a)}} z_{c^{-1}, L, i^{(a)}, j}) = \begin{cases} z_{c,C,R,j,j}, & \text{if } i^{(c)} = i_{LU}, \\ -z_{c,C,R,j,j} + Y, & \text{if } i^{(c)} = i_{LD}, \end{cases} \quad (7.2.13)
\]

where \( Y \in \langle z_{c+m(n-2), C^{-2m(n-2)}, R, j, j} \rangle_{m \geq 1} \) (that is, \( Y \) consists of “higher powers of \((1_{in}RL1_{in}, 1_{in}LR1_{in})\)”).

We can thus consider the above relations \((7.2.11), (7.2.12)\) as eliminating for each \( a, b \) the basis elements \( z_{c,C,X,i,i} \) (where \( i \) is not lower external) for all but a single pair \((X_0, i_0)\) where \( i_0 \notin \{i_{RU}, i_{RD}\} \), leaving just a single relation. We can conveniently choose this relation to be the sum of \((7.2.11)\) and \((7.2.12)\) over all \( i \notin \{i_{RU}, i_{LD}\} \) (with \( j \neq i_{LD}\) for \((7.2.11)\)), together with \((-1)^c\) times the sum of the commutators obtained from these by swapping left and right. This is zero if \( c = 0 \), so we can assume \( c \geq 1 \). By \((7.2.13)\), to include all of the commutators we discarded, we need only divide the resulting relation by \( \gcd(c, C, \frac{C}{2(n-2)}). \)

We thus get the single relation (for all choices of \( c, C \geq 1 \) such that \( 2(n-2) \mid C \)):

\[
\frac{\gcd(c, \frac{C}{2(n-2)})}{c} \sum_{i,j \notin \{i_{LD}, i_{RU}\}} \eta([z_{c,C^{-1}, R, i, j}, jLi]) + (-1)^c [z_{c,C^{-1}, L, i, j}, jRi]) =
\frac{\gcd(c, \frac{C}{2(n-2)})}{c} \sum_{c', i, j \notin \{i_{LD}, i_{RU}\}} (-1)^{c X L + (c - c')} (\delta_i X_U + \delta_j X_D + \delta_i X_U + \delta_j X_D) [r' z_{c', C^{-1}, X, i, j} z_{c - c' - 1, X, j, i}] + Y, \quad (7.2.14)
\]

for appropriate choices of the sign \( \pm \) as above, and where \( Y \) is in the span of products of terms \( z_{c', C', X, i', j'} \) and \( r' \), such that the sum of \( C' \)-degrees is strictly less than \( b \). That is, \( Y \) is the image of a greater power of the ideal \((1_{in}RL1_{in}, 1_{in}LR1_{in})\) than \( c \) (as is \( z_{c,C,X,i,j} \) for any \( X, i, j \)). In other
words, if we pass to the associated graded of $\Pi_Q$ with respect to the filtration by powers of the ideal $(1_{in}RL1_{in}, 1_{in}LR1_{in})$ as in the previous section, and the associated graded with respect to the image of this filtration in $\bar{\Pi}_Q/(\Pi_Q(Q \setminus Q_0)\bar{\Pi}_Q, \bar{\Pi}_Q) \cap \bar{\Pi}_Q$, then we can eliminate the $Y$ term and have an equality above. In any case, by induction on filtration degree, we will see that one can neglect the $Y$ portion.

We then expand the RHS in terms of the basis of $V$ previously described: (1) the elements $z_{c,C,X,i,j}$ from Theorem 4.2.48.(i); and (2) cyclic alternating products of elements $z_{c,C,X,i,j}$ and monomials from $Q \setminus Q_0$ not containing $ee^*$ for any $e \in G$ (the forest we picked as in Proposition 4.2.21):\

$$\frac{\gcd(c, \frac{C}{2(n-2)})}{c} \sum_{(c,C,X,i,j)\{\ast\}} (c_1 + 1)s(c,c_i,X)[\prod_{m=1}^{\#\{c\}} r'z_{m,Cm,Xm,i_m,i_{m+1}}] + Y, \quad (7.2.15)$$

$$s(c,c_i,X) := (-1)^{c \delta_{B_r,L}} \prod_{m=1}^{\#\{c\}} (-1)^{(\delta_{B_r,L} + \delta_{B_{r-1},L})} + c_m(\sum_{k \geq m} B_k),$$

where $B_k = \#\{\text{times that an edge of the form } RZi_RZ \text{ or } L Zi_LZ \}$

appears in the factor $z_{c_k,C_k,X_k,i_k,i_{k+1}}$, and $Y \in (1_{in}RL1_{in}, 1_{in}LR1_{in})^{c+1}$. \quad (7.2.16)

where the $(\ast)$ in the first sum indicates that we sum only over distinct $\#\{c\}$-tuples of elements of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \{L,R\} \times I_0 \times I_0$ such that $\sum_{c_m} = c - \#\{c\}, \sum_{C_m} = C, C_m \geq 1, \forall m$, and $x_{m+1}$ is always a forward direction that follows $z_{c_m,C_m,X_m,i_m,i_{m+1}}$ (which can be upward or downward). We take indices modulo the length of the tuples (e.g. $i_1 + 1 = i_1$).

Since $\sum_{m=1}^{\#\{c\}} B_m$ is always even (because $2(n-m) | C$), when we pass to summing only over distinct cyclic $\#\{c\}$-tuples, all of the contributing terms have the same sign, leaving us with

$$\sum_{(c,C,X,i,j)\{\ast\}} \pm \frac{\gcd(c, \frac{C}{2(n-2)})}{\rep(c,c_i,X,i,j)} [\prod_{m \in \mathbb{Z}/\#\{c\}} r'z_{m,Cm,Xm,i_m,i_{m+1}}] + Y, \quad (7.2.17)$$

which now so closely resembles (3.1.13) – (3.1.16) that it is straightforward to conclude that there is no torsion in the portion of $\Lambda_Q$ corresponding to $C \geq 1$. To be precise, consider the grading on $\Pi_Q$ and $\Lambda_Q$ by length of paths, so that $\Pi_Q[m], \Lambda_Q[m]$ denote the subspaces spanned by paths of length $m$. Consider the filtration by powers of the ideal $(1_{in}RL1_{in}, 1_{in}LR1_{in})$ in $\Pi_Q$, and the filtration by the image of these powers in $\Lambda_Q$. In the associated gradeds, $gr \Pi_Q, gr \Lambda_Q$, let us denote the induced grading with parentheses, so $gr \Pi_Q(g), gr \Lambda_Q(g)$ mean the span of multiples of $(1_{in}RL1_{in}, 1_{in}LR1_{in})^{g}$ modulo multiples of $(1_{in}RL1_{in}, 1_{in}LR1_{in})^{g+1}$.

Then, the precise statement we infer from the above is that there is no torsion in $gr \Lambda_Q[m](g)$ except possibly when $2g = m$. For the latter case, we need only consider the span of $[z_{c,0,R,i,i}], [z_{c,0,L,i,RV,i,RY}]$, and multiples of $\bar{Q} \setminus Q_0$ (with each $z_{c,t,0,X_t,i}$ having $X_t = R$ unless $i$ is right external, in which case $X_t = L$). Then, we mod by the relations $\eta([z_{c-1,1,R,i,j}, jL])$ for $j \neq i_{RD}, i \neq i_{LD}, (we can ignore other commutators by our initial reductions)$, and by $(\sum_{i \in I \setminus I_0} ir_{i})$. When $i$ and $j$ are internal vertices, we can view this as eliminating $z_{c,0,R,i,i}$ for all internal vertices except one, and expressing the other in terms of that one and multiples of $r'$. This leaves us with just two relations. Let $j$ be the internal vertex adjacent to the left end. For convenience, let us replace
\[ \eta([z_{c-1,1,1,R,i_{LU}:j}, jLi_{LU}]) \] by \[ \eta([z_{c-1,1,1,R,i_{LU}:j}, jLi_{LU}]) + \eta([z_{c-1,1,1,R,i_{LD}:j}, jLi_{LD}]). \] The added term is just what is used to eliminate the \( z_{c,2,i_{LJ}:i_{LJ}} \), we would otherwise need as a generator (cf. (7.2.3) and the following paragraphs.) Then this relation can be viewed as expressing \( z_{c,0,R,i_{LJ}} \) in terms of \( z_{c,0,R,i_{LU}:i_{LU}} \) (recall we already expressed \( z_{c,0,R,i_{LD},i_{LD}} \) in terms of the latter). This eliminates the generator \( z_{c,0,R,i_{LJ}} \) for the last remaining internal vertex \( i \).

To express the final relation in terms of remaining basis elements, we take the sum over all adjacent \( i, j \) of \( \eta([z_{c-1,1,1,R,i_{LJ}, j}, jLi_{LJ}]) \). The result is

\[ \sum_{i \in I_0} \eta(i((RL)^c-(LR)^c)i) = \sum_{i \in I_0} i((RL)^c-(RL+r^c)i) + \sum_{X \in (L,R)} (i_{LX}(RL)^c i_{LX}-i_{RX}(LR)^c i_{RX}), \tag{7.2.18} \]

and only slight modification is needed to replace the terms \( z_{c,0,R,i_{LD},i_{LD}} \) and \( z_{c,0,R,i_{BD},i_{BD}} \) that appear above.

The result is nonzero, since there exists a vertex \( i \in I_0 \) such that \( i \) is adjacent to an edge of \( Q \setminus Q_0 \); in this case, for every proper factor \( 1 \neq m \mid c \), one has the term \( i((RL)^c-r^c)^m \), nonzero and independent of other terms in the expansion, with coefficient \( m \). Hence, the gcd of all coefficients is one unless \( c \) is a prime power. We conclude that there is only torsion in \( \Lambda_Q \) in prime-power degrees, where it is generated by the single relation (7.2.18). In prime-power degrees \( c = p^k \), for much the same reason as in the \( \tilde{A}_n \) case (by the similarity of (7.2.18) and (3.1.4)–(3.1.6); see the end of the proof for \( \tilde{A}_n \)), the torsion is a single copy of \( \mathbb{Z}/p \), generated by \( \frac{1}{p} [p^{p^k}] \).

This completes the proof of Theorem 4.2.48.

### 7.3 Hilbert series for \( \tilde{D}_n \) and \( \tilde{A}_n \)

From Theorem 4.2.48 we deduce the

**Proposition 7.3.1.** The Hilbert series of \( i_0 \Pi \tilde{D}_n i_0 \) over any field, for \( i_0 \) the extending vertex (i.e. an external vertex) of \( \tilde{D}_n \), and the Hilbert series of \( \Lambda \tilde{D}_n \) over characteristic zero, are given by

\[ h(i_0 \Pi \tilde{D}_{n-1} i_0) = h(\Lambda \tilde{D}_n; t) = \frac{1}{(1-t^{2n-4})(1-t^2)} = \frac{1 + t^{2n-2}}{(1-t^2)(1-t^{2n-4})}. \tag{7.3.2} \]

It immediately follows from the above that one has the formula

\[ h(\Lambda \tilde{D}_n; t)(1-t^2) = 1 + \frac{t^{2n-4}}{1-t^{2n-4}} + \frac{2t^4}{1-t^4} - \frac{t^2}{1-t^2}, \tag{7.3.3} \]

which we can use to verify the formulas (6.3.4), (6.3.5). Namely, letting \( C \) be the adjacency matrix of \( \tilde{D}_n \), one has

\[ \det(1-t \cdot C + t^2 \cdot 1) = \frac{(1-t^{2n-4})(1-t^4)^2}{1-t^2}. \tag{7.3.4} \]

One then has from (7.3.3)

\[ a_m - a_{m-2} = [(2n-4) \mid m] + 2[4 \mid m] - [2 \mid m], \quad m \geq 3, \tag{7.3.5} \]

which implies the desired identity (6.3.5) and verifies (4.1.14) for \( \tilde{D}_n \).
7.4 The Poisson algebra $i_0\Pi_Qi_0$ / the Lie structure on $\Lambda_Q$ for $Q = \tilde{D}_n$

As in Section 6.4, the Lie structure on $\Lambda_Q$ can be described in terms of the Poisson algebra $i_0\Pi_Qi_0$. In particular, by Theorem 4.2.60, we know that $\Lambda_Q \cong \Lambda_Q[0] \oplus \text{torsion}(\Lambda_Q) \oplus (i_0\Pi_Qi_0)_+$, and by Proposition 9.2.1, the second factor is central; the first is obviously central. It is also easy to check that the third factor is a Poisson subalgebra.

We then compute the following:

Proposition 7.4.1. Let $Q = \tilde{D}_n$, and let $i_0 := i_{LU}$ be the upper-left external vertex. Set $X := i_0\varepsilon_{2,0}i_0, Y := i_0\varepsilon_{2,2(n-2)}i_0$, and $Z := i_0\varepsilon_{1,2(n-2)}i_0$. Then, one has

$$i_0\Pi_Qi_0 \cong \mathbb{Z}[X, Y, Z]/\langle (Z^2 + XY^2 - \delta_{2|n}X^n Z) \rangle,$$

(7.4.2)

$$\{X, Y\} = 2Z - \delta_{2|n}X^n Z, \quad \{X, Z\} = 2XY - \delta_{2|n}X^n Z,$$

(7.4.3)

$$\{Y, Z\} = Y^2 - \delta_{2|n}\frac{n}{2}X^n Z - \delta_{2|n}\frac{n-1}{2}X^n Z,$$

(7.4.4)

$$|X| = 2, \quad |Y| = 2(n-2), \quad |Z| = 2(n-1).$$

(7.4.5)

Notice that, over $\mathbb{Z}[\frac{1}{2}], i_0\Pi_Qi_0 \otimes \mathbb{Z}[\frac{1}{2}]$ is generated as a Poisson algebra by $X$ and $Y$.

Proof. This can all be computed using the results of the previous sections. To see that the given relation is the only relation (i.e., the map in (7.4.2) is injective), we can compare Hilbert series. The rest can all be computed using the basis. To compute $\{Y, Z\}$, it is easiest to compute everything else first, and then compute $\{Y, Z^2\}$ two ways: either as $2Z\{Y, Z\}$, or using the formula (7.4.2) for $Z^2$, and then use that $i_0\Pi_Qi_0$ is an integral domain (which is easy to see from the filtration and bases). \qed

8 Quivers of type $\tilde{E}_n$ and completion of the proof of Theorem 1.1.2

We prove Proposition 4.2.55 separately in the cases $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, in the following subsections.

8.1 Type $\tilde{E}_6$

We first compute a basis for the ring $A := \mathbb{Z}[x, y, z]/\langle (x^3, y^3, z^3, x + y + z) \rangle \cong i_3\Pi_i$s (cf. Proposition 4.2.54). We do this by computing a Gröbner basis for the ideal $\langle (x^3, y^3, z^3, x + y + z) \rangle \subset \mathbb{Z}[x, y, z]$, which we can do by computer using Buchberger’s algorithm (we use MAGMA):

Proposition 8.1.1. In the graded lexicographical order with $x < y < z$, the Gröbner basis for the ideal $\langle (x^3, y^3, z^3, x + y + z) \rangle \subset \mathbb{Z}[x, y, z]$ is

$$yxyx^2 - xyxyx + x^2yxy + x^2yx^2,$$

(8.1.2)

$$y^3,$$

(8.1.3)

$$y^2x + yxy + yx^2 + xy^2 + xyx + x^2y,$$

(8.1.4)

$$x^3,$$

(8.1.5)

$$z + y + x.$$  

(8.1.6)

By definition of Gröbner bases we immediately deduce Proposition 4.2.55 (i). We will prove Proposition 4.2.55 (ii) for $\tilde{E}_6, \tilde{E}_7$, and $\tilde{E}_8$ simultaneously in Section 8.3.

As a result of Proposition 4.2.55 (ii), we deduce:
Corollary 8.1.7. We have
\[ h(i_0\Pi i_0; t) = \Phi_{24}(t) = 1 - t^4 + t^8 \]
\[ = \frac{(1 - t^{24})}{(1 - t^4)(1 - t^6)(1 - t^{12})}. \] (8.1.8)

Furthermore, one has the following partial fraction decomposition:
\[ h(i_0\Pi i_0)(1 - t^2) = 1 + \frac{2t^6}{1 - t^6} + \frac{t^4}{1 - t^4} - \frac{t^2}{1 - t^2}. \] (8.1.9)

Proof. For the first part, we note that our basis above shows that
\[ h(i_0\Pi i_0; t) = 1 + t^6(1 - t^6) + t^8(1 - t^6)(1 - t^4). \]
Putting this over the common denominator \((1 - t^4)(1 - t^6)(1 - t^{12})\), we get a numerator of
\[ 1 - t^6 - t^4 + t^6 - t^{10} + t^{10} + t^8 = 1 - t^4 + t^8 = \Phi_{24}(t) \] (the degree-24 cyclotomic polynomial).

For the second part, one may explicitly verify the identity. Note that, since \((1 - t^6)(1 - t^4) = (1 + t^2 + t^4)(1 + t^2)(1 - t^2)^2\) is a decomposition into relatively prime factors, one sees that \(h(i_0\Pi i_0)(1 - t^2)\) must have a partial fraction decomposition with denominators \(1 - t^6, 1 - t^4, 1 - t^2\), and the above is one such. \(\square\)

The meaning of the partial-fraction decomposition (8.1.9) is again the identity (4.1.14) (cf. Section 6.3): setting \(h(i_0\Pi i_0) = \sum a_m t^m\), we have
\[ a_m - a_{m-2} = 2[6 | m] + [4 | m] - [2 | m], \quad m \geq 2. \] (8.1.10)

This bears similarity to the determinant of the \(t\)-analogue of the Cartan matrix:
\[ \det(1 - t \cdot C + t^2 \cdot 1) = \frac{(1 - t^6)^2(1 - t^4)}{1 - t^2}. \] (8.1.11)

Indeed, (8.1.10) just says that
\[ \prod_{m \geq 1} \frac{1}{(1 - t^m)^{a_m - a_{m-2}}} = \prod_{m \geq 1} \frac{1 - t^2}{(1 - t^6)^2(1 - t^4)}, \] (8.1.12)

Then, as in Section 6.3 we verify (4.1.14) in this case.

8.2 Type \(\tilde{E}_7\) We first compute a Gröbner basis for the ideal \(\langle x^4, y^4, z^2, x + y + z \rangle \subset \mathbb{Z}[x, y, z]\) (cf. Proposition 4.2.54), which we can do with MAGMA:

**Proposition 8.2.1.** In the graded lexicographical order with \(a < b < c\), the Gröbner basis for the ideal \(\langle x^4, y^4, z^2, x + y + z \rangle \subset \mathbb{Z}[x, y, z]\) is
\[ \begin{align*}
yx^2yx^3 - xyx^2yx + x^2yx^2yx - x^3yx^2y, & \quad \text{(8.2.2)} \\
yxyx + yx^2y + yx^3 + xyxy + x^3y, & \quad \text{(8.2.3)} \\
x^4, & \quad \text{(8.2.4)} \\
y^2 + yx + xy + y^2, & \quad \text{(8.2.5)} \\
z + y + x. & \quad \text{(8.2.6)}
\end{align*} \]
We immediately deduce Proposition 4.2.55(ii) for $E_7$. See the next subsection for the proof of (ii). As a result of (ii) we deduce the

**Corollary 8.2.7.** One has the formula

\[
    h(i_0 \Pi_{E_7} i_0; t) = \frac{\Phi_{36}(t) = 1 - t^6 + t^{12}}{(1 - t^6)(1 - t^8)} = \frac{1 - t^{36}}{(1 - t^8)(1 - t^{12})(1 - t^{18})}.
\]

(8.2.8)

Additionally, one has the partial fraction decomposition

\[
    h(i_0 \Pi_{E_7} i_0; t)(1 - t^2) = 1 + \frac{t^4}{1 - t^4} + \frac{t^6}{1 - t^6} + \frac{t^8}{1 - t^8} - \frac{t^2}{1 - t^2}.
\]

(8.2.9)

Since, letting $C$ be the adjacency matrix of $\overline{E}_7$, one has the formula

\[
    \det(1 - t \cdot C + t^2 \cdot 1) = \frac{(1 - t^4)(1 - t^6)(1 - t^8)}{1 - t^2},
\]

(8.2.10)

the identity [4.1.14] is verified.

**8.3 Type $\tilde{E}_8$** We first compute a Gröbner basis for the ideal $\langle (x^6, y^3, z^2, x + y + z) \rangle \subset \mathbb{Z}[x, y, z]$ (cf. Proposition 4.2.55), which we can do with MAGMA:

**Proposition 8.3.1.** In the graded lexicographical order with $x \prec y \prec z$, the Gröbner basis for the ideal $\langle (x^6, y^3, z^2, x + y + z) \rangle \subset \mathbb{Z}[x, y, z]$ is

\[
    \begin{align*}
    y^2 & x^4 y^5 - xyx^4 yx^4 + x^2 yx^4 yx^3 - x^3 yx^4 yx^2 + x^4 yx^4 yx - x^5 yx^4 y, \quad (8.3.2) \\
    x^6 & yx^3 yx + yx^4 y + yx^5 + xyx^3 y - x^2 yx^3 - x^3 yx^2 + x^5 y, \quad (8.3.3) \\
    x^{2y} & yx^2 y + yx^2 + 2x^2 y + x^3 y + 2x^4, \quad (8.3.5) \\
    x^3 & yxy - xyx - x^3, \quad (8.3.6) \\
    x^2 & y^2 + yx + xy + x^2, \quad (8.3.7) \\
    x & z + y + x. \quad (8.3.8)
    \end{align*}
\]

We immediately deduce Proposition 4.2.55(i) for the $E_8$ case, which completes the proof of that part. As promised, we now prove part (ii) for $E_6, E_7,$ and $E_8$ simultaneously:

**Proof of Proposition 4.2.55(ii).** First of all, note that the only short path from $i_0$ to itself is the length-zero path, $i_0$ (i.e. $\text{ShortP}_{i_0,i_0} = \langle i_0 \rangle$). Then, by Proposition 2.2.10 the map $i_s \Pi s \to i_0 \Pi_0 f \mapsto p_{i_0,i_s} f_{i_0,i_s}$ has image $(i_0 \Pi_0)_+$ and kernel spanned by elements of the form $x g, g x, g \in i_s \Pi S$. First, the basis elements that begin or end in $x$ are killed. Conversely, for any basis element $g$, expressing $x g$ in terms of basis elements consists only of terms beginning with $x$. On the other hand, $g x$ might not consist only of terms beginning or ending with $x$: this can happen in the case that $g = (y x^d)^{t_1} (y x^{d-1})^t y x^{d-2} y$. In this case, $g x$ is a sum of basis elements beginning or ending in $x$, and the element $(y x^d)^{t_1} (y x^{d-1})^{t_2 + 1}$. Hence, we can simply eliminate the latter element from our basis, and we obtain the stated result.
Corollary 8.3.9. The Hilbert series of $i_0\Pi_{E_8}i_0$ is given by
\[
\begin{align*}
  h(i_0\Pi_{E_8}i_0) &= \frac{\Phi_{12}(t)\Phi_{60}(t) = 1 - t^{10} + t^{20}}{(1 - t^{10})(1 - t^{12})} = \frac{(1 - t^{60})}{(1 - t^{12})(1 - t^{20})(1 - t^{30})}.
\end{align*}
\]
Additionally, one has the partial fraction decomposition
\[
\begin{align*}
  h(i_0\Pi_{E_8}i_0)(1 - t^2) &= 1 + \frac{t^4}{1 - t^2} + \frac{t^6}{1 - t^2} + \frac{t^{10}}{1 - t^2} - \frac{t^2}{1 - t^2}.
\end{align*}
\]
Since, letting $C$ be the adjacency matrix for $E_8$, one has the formula
\[
\det(1 - t \cdot C + t^2 \cdot 1) = \frac{(1 - t^4)(1 - t^6)(1 - t^{10})}{1 - t^2},
\]
the identity (4.1.14) is verified.

### 8.4 Poisson structure on $i_0\Pi Q i_0$ / Lie structure on $\Lambda_Q$ for $Q = \hat{E}_n$

The Poisson structure on $\Lambda_Q$ was already described when tensored over $\mathbb{C}$ (Theorem 9.1.1); here we explain an alternative description over $\mathbb{Z}$ using the bases of Proposition 4.2.55.

**Proposition 8.4.1.** The Poisson structure of $i_0\Pi Q i_0$ is given as follows for $Q = \hat{E}_n$. Let us use the notation of Proposition 4.2.55 and define
\[
\begin{align*}
  X := p_{i_0,i}y_{i,i_0}, \quad Y := p_{i_0,i}y d^2 y_{i,i_0}, \quad Z := p_{i_0,i}y d^2 y_{i,0}.
\end{align*}
\]
Finally, let us assume that all edges of the quiver $Q$ are oriented towards the special vertex.

(i) For $Q = \hat{E}_6$, we have $i_0\Pi Q i_0 \cong \mathbb{Z}[X,Y,Z]/(Z^2 + Y^3 + ZX^2)$, and
\[
\begin{align*}
  \{X,Y\} &= -2Z - X^2, \quad \{X,Z\} = 3Y^2, \quad \{Y,Z\} = -2XZ.
\end{align*}
\]
(ii) For $Q = \hat{E}_7$, we have $i_0\Pi Q i_0 \cong \mathbb{Z}[X,Y,Z]/(Z^2 - X^3Y + Y^3)$, and
\[
\begin{align*}
  \{X,Y\} &= -2Z, \quad \{X,Z\} = 3Y^2 - X^3, \quad \{Y,Z\} = 3X^2Y.
\end{align*}
\]
(iii) For $Q = \hat{E}_8$, we have $i_0\Pi Q i_0 \cong \mathbb{Z}[X,Y,Z]/(Z^2 + X^5 + Y^3)$, and
\[
\begin{align*}
  \{X,Y\} &= -2Z, \quad \{X,Z\} = 3Y^2, \quad \{Y,Z\} = -5X^4.
\end{align*}
\]

In particular, the homogeneous elements $X,Y,Z$ generate $i_0\Pi Q i_0$ as a graded algebra, and after tensoring over any commutative ring containing $\frac{1}{2}$, $X,Y$ generate as a Poisson algebra. We have $|X| = 2(d + 1), |Y| = 4d$, and $|Z| = 6d$.

**Proof.** This can all be verified by an explicit computation. To see that the given relation (e.g., $Z^2 + Y^3 + ZX^2$ for $\hat{E}_6$) is the only relation, we use the Hilbert series computed in the previous sections. To compute the relation and the necklace bracket formulas, we used MAGMA, but it is also tractable by hand (and only requires computations in fairly low degrees). \qed
We note that, in the above, the presentation in (i) of $\tilde{E}_0$ is somewhat more inconvenient than the presentations of $\tilde{E}_7, \tilde{E}_8$ since the latter have the form $Z^2 = P(X, Y)$ for polynomials $P$ in $X, Y$ (i.e., $i_0\Pi_Q i_0$ is a direct sum of the part with odd degree in $Z$ and the part with even degree). Over $\mathbb{Z}[\frac{1}{2}]$, one can fix this:

$$Z' := Z + \frac{1}{2}X^2, \quad i_0\Pi_{\tilde{E}_0}i_0[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][X, Y, Z]/((Z')^2 + Y^3 + \frac{1}{4}X^4)), \quad (8.4.6)$$

$$\{X, Z\}' = 3Y^2, \quad \{Y, Z\}' = X^3, \quad \{X, Y\} = -2Z'. \quad (8.4.7)$$

Finally, we explicitly compute the zeroth Poisson homology of $i_0\Pi_{Q_0}i_0 \otimes \mathbb{F}_p$ where $p$ is a bad prime for $Q_0$, which will be needed to finish the proof of the main Theorem. We will prove the result for good primes in Section 11.4.

**Proposition 8.4.8.** Let $(Q_0, p)$ be one of the seven exceptional cases $(\tilde{E}_6, 2), (\tilde{E}_6, 3), (\tilde{E}_7, 2), (\tilde{E}_7, 3), (\tilde{E}_8, 2), (\tilde{E}_8, 3), (\tilde{E}_8, 5)$. Let $A := i_0\Pi_{Q_0}i_0$ be the Poisson algebra, and set $A_p := A \otimes \mathbb{F}_p$ and $A_Q := A \otimes \mathbb{Q}$. If $(Q_0, p) \neq (\tilde{E}_8, 2)$, then the zeroth Poisson homology $HP_0(A_p) = A_p/\{A_p, A_p\}$ is given, as a graded $A_p$-module, by

$$HP_0(A_p) \cong (A_p^+(2p - 2) \oplus HP_0(A_p)), \quad (8.4.9)$$

where $HP_0(A_p)$ is a trivial $A_p$-module with Hilbert series $\leq h(HP_0(A_Q); t)$.

If $(Q_0, p) = (\tilde{E}_8, 2)$, then we have a correction, related to the torsion element in $HH_0(\Pi_{Q_0})[28]:$

$$HP_0(A_2) \cong ((A_2^+/(X))^2(2) \oplus HP_0(A_2), \quad X = p_{i_0}, y, p_{i_0}.$$

**Proof.** This is done on a case-by-case basis. In all cases, we can assume that the quiver is oriented as in Proposition 8.4.1 because for any $e \in Q_0$, letting $(Q_0)_e$ be the quiver with the edge $e$ reversed, there is an isomorphism $\Pi_{Q_0} \cong \Pi_{(Q_0)_e}$ sending $e$ to $-e$ and fixing all other edges and all vertices, which induces a Poisson isomorphism $i_0\Pi_{Q_0}i_0 \cong i_0\Pi_{(Q_0)_e}$.

Now, we show how to prove the proposition for the case $(\tilde{E}_6, 2)$, and omit the other six cases (which are all similar). Note first that, by Proposition 8.4.1(i), a basis of $i_0\Pi_{Q_0}i_0$ is given by \{X^aY^b, X^aY^bZ\}_{a, b \geq 0}. We wish to compute the image of the Poisson bracket explicitly. Just as in the commutator case (in Lemma 8.4.2), we may use the formula

$$\{f, gh\} = f\{g, h\} + g\{f, h\} = \{f, gh\} + \{g, fh\} \quad (8.4.11)$$

to reduce to computing Poisson brackets of the form $\{X, f\}, \{Y, f\}$, and $\{Z, f\}$. Next, by (8.4.13), we may compute, now over $\mathbb{F}_2$:

$$\{x, x^ay^b\} = bx^{a+2}y^{b-1}, \quad \{y, x^ay^b\} = ax^{a+1}y^b, \quad \{z, x^ay^b\} = ax^{a-1}y^{b+2}, \quad (8.4.12)$$

$$\{x, x^ay^bz\} = bx^{a+2}y^{b-1}z + x^ay^{b+2}, \quad \{y, x^ay^bz\} = ax^{a+1}y^bz, \quad \{z, x^ay^bz\} = ax^{a-1}y^{b+2}z. \quad (8.4.13)$$

Now, it follows from this that the image of the Poisson bracket is spanned by the elements

$$x^ay^b, x^asy^b, \quad \text{where } a \text{ or } b \text{ is even, and } a \geq 2, b \geq 0, \quad (8.4.14)$$

$$x^{2a+1}y^{2b}(y^3 + x^2z), \quad a, b \geq 0, x^{2a+2}, y^{2b+2}, y^{2b+2}z, \quad a, b \geq 0. \quad (8.4.15)$$

67
In other words, the following gives a basis of $HP_0(A_2)$:

$$xy(x^{2a}y^{2b}), x yz(x^{2a}y^{2b}), xz(y^{2b}), yz, z, y, x, 1.$$  \hfill (8.4.16)

It is easy to see that the last five elements are killed by multiplication by $A_2^2$. The first three elements form a $A_2^2$-submodule isomorphic to $((A_2)_2)^2(2)$, which can be seen by formally adjoining an element $r'$ in degree two and setting $r'x^2 = xy, r'y^2 = xz$, and $r'z^2 = xyz$.

Finally, we note that $HP_0(A_2) \cong (xy, yz, z, y, x, 1)$, which can be computed directly, or directly with the help of \cite{BEG01}, which shows that, in general, for a complex symplectic vector space $V$ and a finite subgroup $G \subset Sp(V)$, $\dim HP_0(\mathbb{C}[V]^G)$ is finite-dimensional. \hfill $\square$

### 8.5 Proof of Theorem 4.2.30 (completing the proof of Theorem 1.1.2)

**Proof of Theorem 4.2.30** First, in the cases that $(Q_0, p)$ is a good pair $(p \mid |\Gamma|)$ where $\Gamma \subset SL_2(\mathbb{C})$ is the finite group associated to $Q_0$, the result follows from Theorem 3.2.15; it remains only to verify the statements concerning $p$-th powers and image of Poisson bracket. The statements about $p$-th powers follow from the form of the relations $W_{a,b}$: in particular,

$$W_{pa,pb} \equiv W_{a,b}^p \pmod{p}, \quad \frac{1}{p}W_{p',p''} \equiv (\frac{1}{p}W_{p,p})^{p'} \pmod{p}. \hfill (8.5.1)$$

The statement about Poisson bracket (iv) follows immediately from Proposition 5.5.15.

Next, in the cases that $Q_0 = \tilde{A}_n$ or $Q_0 = \tilde{D}_n$, the results follow from Theorems 4.2.40 and 4.2.48 by a similar analysis of the given relations.

In the seven remaining cases $\{(\tilde{E}_6, 2), (\tilde{E}_7, 2), (\tilde{E}_7, 3), (\tilde{E}_8, 2), (\tilde{E}_8, 3), (\tilde{E}_8, 5)\}$, we make use of the following claim, which follows from Proposition 8.4.8 (as we will explain):

**Claim 8.5.2.** If $I_p$ is the image of the Poisson bracket, then

$$t^2h(i_0\Pi_Q i_0; t) = \sum_{\ell \geq 0} t^{2p^\ell} \left( h(I_p; t^{p^\ell}) + F(t^{p^\ell}) \right), \hfill (8.5.3)$$

where $F(t) = h(HP_0(i_0\Pi_Q i_0 \otimes \mathbb{F}_p) i_0; t) \leq h(HP_0(i_0\Pi_Q i_0 \otimes \mathbb{Q}); t)$.

Let us first use this claim. By Proposition 5.5.15, the image $I_p$ of the Poisson bracket must coincide with the projection of $[HH^0(\Pi_{Q_0}), i_0\Pi_Q i_0] = [i_0\Pi_Q i_0, i_0\Pi_Q i_0] \bmod{\langle [r'], \Pi_Q \rangle}$ to $[r'HH^0(\Pi_{Q_0})]$. So, there exists $I_{p'} \subset W'$ which projects isomorphically to $[r' I_p]$. In view of the observations before the statement of the theorem (given any relation in $W$, its $p$-th power must also be in $W$), it remains to find a space $U_{p'} \subset HH^0(\Pi_{Q_0})$ with Hilbert series $h(U_{p'}; t) = F(t) - t^{2p^{m-2}}$ (where $m$ is the smallest positive integer such that $r^{2p^m}$ is zero in $\Pi_{Q_0}$), such that $U_{p'}$ is independent of $I_p$, and a subspace $\tilde{U}' \subset W$ which projects isomorphically to $[r' U_{p'}]$, as well as to find the space $W_r$ and describe $W_0$, and then we would be done.

The description of $W_0$ is easy: it already has the desired form, except in the case of $(\tilde{E}_8, 2)$, where it is not immediately clear that $W_0[28]$ is saturated. However, this can be verified explicitly (with the help of MAGMA): $[x^4y, x^4y, x^4y]$ projects to $2x^4y^4y^4x^4y^4 \equiv 2x^4y^4x^4y^4y^4 - x^5y^5y^5y^5y^5$...

\footnote{Also, as pointed out by P. Etingof, by a spectral sequence argument, the dimension is at least the dimension of the zeroth Hochschild homology of the Weyl algebra of $V$ smashed with $G$, which by \cite{AFLS00} is the number of conjugacy classes of $G$ which do not have $1$ as an eigenvalue. This is just one less than the number of vertices of the quiver. One in fact has equality in our case, which is probably known.}
\[ [p_{i_{0}i}r_{p_{i_{0}i}}, y p_{i_{0}i}r_{p_{i_{0}i}}y] \text{ in } V, \] and this is not a multiple of 2 (working over \( \mathbb{Z} \), the above relation is in fact not a multiple of any positive integer).

It is clear what \( U'_{p} \) must be, for Hilbert series reasons, and it is a subset of generators of the zeroth Poisson homology of \( i_{0} \Pi_{Q i_{0}} \otimes \mathbb{Q} \) (note that \( i_{0} \Pi_{Q i_{0}} \otimes \mathbb{C} \cong \mathbb{C}[x, y]^\Gamma \)). The degrees of elements of \( U'_{p} \) are \( \leq 20 \) for \( E_{6} \), \( \leq 32 \) for \( E_{7} \), and \( \leq 56 \) for \( E_{8} \). For these low degrees, the needed \( U'_{p} \subset W' \) can be found (or its existence verified) using MAGMA.

Finally, let us find \( W_{r} \). For each \( \ell \geq 1 \), define \( f_{\ell} = p \cdot \bar{f}_{\ell} \), such that \( \bar{f}_{\ell} \) is any noncommutative polynomial in \( x, y \) which projects, modulo commutators, to \( \frac{1}{p}[((x + y)^{p^\ell} - x^{p^\ell} - y^{p^\ell})]. \) Our choice of \( f_{\ell} = p \bar{f}_{\ell} \) makes it clear that, in fact, \( r^{(p)} \) and \( \bar{f}_{\ell} \) have the same image in \( V/W \). In particular, \( p[\bar{f}_{\ell}] = [f_{\ell}] \in W \) and \( [f_{\ell}] \in W' \) for \( \ell \geq m \). Thus, it suffices to show that \( \bar{f}_{\ell} \) is independent of \( W'_{p} \otimes \mathbb{F}_{p} \) for all \( \ell \geq m \); to do this it suffices to show it for \( \ell = m \). In these cases, \( W'_{p} || f_{m} || \otimes \mathbb{Q} \cong HH^{0}(\Pi_{Q}, \mathbb{Q})(2)|| f_{m} || \) is one-dimensional; thus, it suffices to show that \( \bar{f}_{m} \notin pV + [(r')^2] \). This we can explicitly verify with MAGMA.

It remains only to prove the claim. In general, \( G(t) = \sum_{\ell \geq 0} H(t^{p^\ell}) \) iff \( H(t) = G(t) - G(t^{p}) \). This observation, together with Proposition \[8.4.8\] proves the desired result. \( \square \)

9 The Lie structure of \( \Lambda_{Q, J} \), part II

9.1 The Lie structure on \( \Lambda_{Q} \) for extended Dynkin quivers It is interesting to study the Lie structure in the extended Dynkin case. By Propositions \[4.2.25\] and \[4.2.21\] this allows one (in part) to describe the Lie structure on for any quiver. (In the Dynkin case, by Theorem \[4.2.60\] and Proposition \[9.2.1\] the Lie bialgebra is finite and abelian.)

One interesting phenomenon in the extended Dynkin case is that \( \Lambda_{Q} \otimes \mathbb{C} \) obtains not just the structure of a Lie bialgebra, but in fact a Poisson algebra. This can be explained using \( (4.1.10) \), which gives an isomorphism \( HH^{0}(\Pi_{Q} \otimes \mathbb{C}) \to HH_{0}(\Pi_{Q}) \otimes \mathbb{C} \) (one can also just take the projection of \( HH^{0}(\Pi_{Q} \otimes \mathbb{C}) \) mod commutators, as in Section \[5.3\]). Furthermore, it is easy to see that the multiplication in the center endows its image with a Poisson structure together with the necklace bracket (without using anything about the quiver). As will be discussed in Section \[10\] in fact this Poisson structure is part of the BV structure on Hochschild cohomology of \( \Pi_{Q} \).

Using the Morita equivalence \( \Pi_{Q} \otimes \mathbb{C} \cong \mathbb{C}[x, y] \rtimes \Gamma \) and Proposition \[3.2.3\] we know that \( HH^{0}(\Pi_{Q} \otimes \mathbb{C}) \cong \mathbb{C}[x, y]^\Gamma \), which has a Poisson structure obtained from the standard one on \( \mathbb{C}[x, y] \). In this section, we show that the Poisson structure on \( \Lambda_{Q} \otimes \mathbb{C} \) coincides with this one.

To do this, it is more convenient to work with \( i_{0} \Pi_{Q i_{0}} \) rather than \( HH^{0}(\Pi_{Q}) \). The former is still a commutative ring and the projection \( HH^{0}(\Pi_{Q} \otimes \mathbb{C}) \to i_{0} \Pi_{Q i_{0}} \otimes \mathbb{C} \) is an isomorphism. (Also, over \( \mathbb{Z} \), \( i_{0} \Pi_{Q i_{0}} \) is better to work with since the map \( i_{0} \Pi_{Q i_{0}} \to HH_{0}(\Pi_{Q}) \) remains injective after tensoring by a field of any characteristic, unlike for \( HH^{0} \).

**Theorem 9.1.1.** Let \( Q \) be an extended Dynkin quiver.

(i) For any extending vertex \( i_{0} \), the composition \( \mathbb{C}[x, y]^\Gamma \to i_{0} \Pi_{Q i_{0}} \otimes \mathbb{C} \to ((\Lambda_{Q})_{+} \oplus \langle i_{0} \rangle) \otimes \mathbb{C} \) is a Lie algebra isomorphism, where \( \Gamma \subset SL(2, \mathbb{C}) \) is the finite subgroup given by the McKay correspondence. Thus, equipping \( ((\Lambda_{Q})_{+} \oplus \langle i_{0} \rangle) \otimes \mathbb{C} \) with the multiplication from \( i_{0} \Pi_{Q i_{0}} \otimes \mathbb{C} \) (and equipping the latter with the necklace Lie bracket from the former), we obtain Poisson algebra isomorphisms. This remains true when we replace \( \mathbb{C} \) by \( \mathbb{K} = \mathbb{Z} \left[ \frac{1}{1} \right], e^{2\pi i} \).
(ii) Over the complex numbers (or any commutative ring in which the stably bad primes are invertible), $\delta = 0$.

(iii) Over the integers, $\Lambda_Q$ (as a graded Lie algebra over $\mathbb{Z}$) is just the direct sum $(i_0\Pi_Qi_0)_+ \oplus \Lambda_Q[0] \oplus$ torsion, where the latter two summands are abelian, and the first summand is isomorphic to $(\Lambda_Q)_+ \otimes 1 \subset (\Lambda_Q)_+ \otimes \mathbb{C}$.

(iv) Over the integers, the cobracket is nonzero $\tilde{D}, \tilde{E}$. Here, $\delta$ vanishes on the torsion $(\Lambda_Q)_+$, and $\delta((i_0\Pi_Qi_0)_+) \subset (i_0\Pi_Qi_0)_+ \wedge (\Lambda_Q)_{tor}$.

Proof. (i) Kronheimer [Kro89] showed that $\text{Spec } \mathbb{C}[x,y]^\Gamma$ is isomorphic, as a Poisson variety, to the quiver variety $\mu^{-1}(0)/\text{GL}_d$ where $\mu : \text{Rep}_d(Q) \to \mathfrak{gl}_d$ is the moment map, and $d$ is the imaginary root of $Q$ ($d_i$ is equal to the dimension of the irreducible representation corresponding to $i \in I$ under the McKay correspondence). That is, the quiver variety is just the space $\text{Spec } \mathbb{C}[\text{Rep}_d(\Pi_Q)]^{\text{GL}_d}$. Now, one has the trace map $\text{tr} : HH_0(\Pi_Q \otimes \mathbb{C}) \to \mathbb{C}[\text{Rep}_d(\Pi_Q)]^{\text{GL}_d}$, which is a Lie algebra homomorphism. Furthermore, it must be an algebra homomorphism using the identification $i_0\Pi_Qi_0 \otimes \mathbb{C} \rightarrow HH_0(\Pi_Q \otimes \mathbb{C})$, since the dimension $d_{i_0} = 1$ at the extending vertex. Hence, it is a Poisson homomorphism in this case. Composing with Kronheimer’s isomorphism, we get a surjective Poisson homomorphism $HH_0(\Pi_Q \otimes \mathbb{C}) \to \mathbb{C}[x,y]^\Gamma$, which must be an isomorphism since the two spaces have the same dimension.

Alternatively, one can deduce the above fact by showing that there is a unique Poisson structure (up to scaling) on the ring $\mathbb{C}[x,y]^\Gamma$ of degree $-2$, and making use of the isomorphism of rings $i_0\Pi_Qi_0 \otimes \mathbb{C} \cong \mathbb{C}[x,y]^\Gamma$ from [CBH98]. Namely, any Poisson structure on $\mathbb{C}[x,y]^\Gamma$ must give a Poisson structure on $(\mathbb{C}^2 \setminus \{0\})/\Gamma$, which lifts to a $\Gamma$-equivariant Poisson structure on $\mathbb{C}^2 \setminus \{0\}$, which must extend uniquely to 0 by the analogue of Hartogs’ Lemma. Or, algebraically, one can prove directly using the explicit structure of $i_0\Pi_Qi_0$ given in this paper (mostly all that is needed is the Hilbert series, which was already given in [ZG06]) that there is a unique Poisson structure of degree $-2$ up to scaling.

To see that the above generalizes to $k = \mathbb{Z}\left[\frac{1}{d} e^{\frac{2\pi i}{d}}\right]$, we can use the alternative argument from the last paragraph (or generalize Kronheimer’s isomorphism to this case).

(iii) The decomposition, as graded $\mathbb{Z}$-modules, follows from Theorem 4.2.60 and the fact that the torsion is central is postponed to Proposition 9.2.1. To show that the decomposition is one of Lie algebras, we note that the Lie bracket of two elements of $\Lambda$ in the image of $i_0\Pi_Qi_0$ must still be in this image. Thus, the Lie bracket on $\Lambda_Q$ is just obtained by restricting the Lie bracket on $\Lambda_Q \otimes \mathbb{C}$ to the integral form $\Lambda_Q \otimes 1 \subset \Lambda_Q \otimes \mathbb{C}$, and taking the direct sum with the abelian Lie algebra generated by torsion.

(ii) We compute the cobracket on $(\Lambda_Q)_+ \otimes \mathbb{C}$. This immediately gives the more general result, because the image of $\delta$ must be torsion. Hence, if $k$ is any commutative ring in which stably bad primes are invertible, we have $(\Lambda_Q)_{tor} \otimes k = 0$, so $\delta$ must vanish.

We would like to use (5.3.2), which à priori can only be done by first lifting to $P_Q$, where the double bracket is defined. However, let $K := \text{Ker}(\delta) \subset (i_0\Pi_Qi_0)_+$ be the kernel of $\delta$, pulled back to $(i_0\Pi_Qi_0)_+$ using the injection $(i_0\Pi_Qi_0)_+ \hookrightarrow \Lambda_+$ and projection $\Lambda_+ \twoheadrightarrow (i_0\Pi_Qi_0)_+$. Then, we claim

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5Thanks to Pavel Etingof for pointing out this simple proof.
that the map
\[ \{ , \}_{ss,K} : K \otimes K \to K \wedge K, \]
\[ \{a, b\}_{ss,K} = (q \otimes q)(\{a, b\} + \{b, a\}), \] (9.1.2)
where
\[ q : P \to \Pi_Q \] is the projection,
\[ \delta^2 \] yields a well-defined symmetric bilinear operation on \( K \) (note that \( \{ , \} \) preserves \( (i_0 \Pi_Q i_0)^{\otimes 2} \)). This follows because [5.3.2] yields the formula, for \( a, b \in K \),
\[ \{a, b\}_{ss,K} = \delta([ab]), \] (9.1.5)
which is independent of the choice of \( a, b \) since the image of \( ab \) in \((i_0 \Pi_Q i_0)_+\) is independent of this choice.

If \( K_0 \subset K \) is any maximal subspace on which \( \{ , \}_{ss,K} \) vanishes, then pairwise products of elements of \( K_0 \) must also lie in \( K \), and these new elements must also commute with \( K_0 \) under \( \{ , \}_{ss,K} \) using the double Leibniz rule. Hence, \( K_0 \) must be a (nonunital) subalgebra of \( K \). It thus suffices to show that \( K \) contains algebra generators of \((i_0 \Pi_Q i_0)_+\), and that \( \{ , \}_{ss,K} \) vanishes on pairs of such generators. Equivalently, we must show that \( \delta \) vanishes on generators and pairwise products of generators.

We may now show that this is true in each of the cases \( \hat{A}, \hat{D}, \) and \( \tilde{E} \) separately. In the \( \hat{A} \) case, it is actually easy to check that \( \delta \) vanishes on \((i_0 \Pi_Q i_0)_+\) simply by using the basis \( \{ i_0 x^a y^b i_0 \}_{a,b \geq 0} \) from Theorem [4.2.40]. However, we can also conclude this using Proposition [6.4.2] as follows. Clearly, \( \delta(Z) = 0 \) and \( \delta(Z^2) = 0 \). Hence, \( \delta(Z^a) = 0 \) for all \( a \). Also, \( \delta(X) \) and \( \delta(Y) \) must be sums of the form \( \sum a_{ab} Z^a \wedge Z^b \). Then, using the one-cocycle identity, \( 0 = \delta([X,Y]) = \{\delta(X), Y \otimes 1 + 1 \otimes Y \} + \{1 \otimes X + X \otimes 1, \delta(Y)\} \), which allows us to conclude that \( \delta(X) = \delta(Y) = 0 \).

The same argument, replacing \( X \) by \( XZ \), gives \( \delta(XZ) = 0 \), and similarly \( \delta(XY) = 0 \). Equivalently, \( \{X, Z\}_{ss,K} = \{Y, Z\}_{ss,K} = 0 \). Also, we have \( \{X, Y\}_{ss,K} = \delta(XY) = \delta(Z^a) = 0 \). It remains to show \( \delta(X^2) = \delta(Y^2) = 0 \). Using \( \delta(\{X^2, Y\}) = \delta(\{X^2, X\}) = 0 \), we easily conclude as above that \( \delta(X^2) = 0 \), and similarly \( \delta(Y^2) = 0 \).

Next, for \( \hat{D} \), we may also do both approaches as in the \( \hat{A} \) case, but now the second is easier, using Proposition [7.4.1] We must have \( \delta(X) = \delta(X^2) = 0 \), so \( \delta(X^n) = 0 \) for all \( n \). Then, depending on whether \( n \) is even or odd, we have either \( \delta(Y) = 0 \) or \( \delta(Z) = 0 \), simply for degree reasons. This implies that the other is zero, using \( 2\delta(Z) = \delta(\{X, Y\}) = \{X \otimes 1 + 1 \otimes X, \delta(Y)\} \). Then, we again have either \( \delta(XY) = 0 \) or \( \delta(XZ) = 0 \) for degree reasons, and then \( 2\delta(XZ) = \delta(\{XY, X\}) = \{\delta(XY), X \otimes 1 + 1 \otimes X\} \) yields that the other one is zero. Hence, \( \delta(X^a Y) = \delta(X^a Z) = 0 \) for all \( a \geq 0 \). Because of the quadratic relation, it remains only to show that \( \delta(Y^2) = \delta(YZ) = 0 \). By the one-cocycle condition, \( \delta(\{Y, Z\}) = 0 \) (or, \( K \) is closed under the bracket). This implies \( \delta(Y^2) = 0 \). Finally, \( 0 = \delta(\{X, Y^2\}) = 2\delta(YZ) \).

We now consider the \( \tilde{E} \) case, using Proposition [8.4.1] We have \( \delta(X) = \delta(X^2) = 0 \), so \( \delta(X^a) = 0 \) for all \( a \geq 0 \). Since \( |Y| \leq 2|X| - 2 \), we have \( \delta(Y) = 0 \). Hence, \( \delta(Z) = 0 \) (since \( \delta(\{X, Y\}) = 0 \)). Because \( \delta(\{X^2, Y\}) = 0 \), we equally obtain \( \delta(XZ) = 0 \) and hence \( \delta(X^a Z) = 0 \) for all \( a \geq 0 \). Next, \( \{X, Z\} = 3Y^2 - \delta_{a,7} X^3 \) implies that \( \delta(Y^2) = 0 \), and hence \( \delta(X^a Y^b) = 0 \) for all \( a, b \). It remains to show that \( \delta(YZ) = 0 \), which follows from \( \delta(\{X, Y^2\}) = 0 \) (we have \( \delta(Z^2) = 0 \) using the quadratic relation).
(iv) In $\Lambda$ itself, we know from Proposition 9.2.1 (in the next section) that the torsion is central, and vanishes in high enough degree. Then, by definition, the image of $\delta$ on $(i_0\Pi_Qi_0)^+ \cap (\Lambda_Q)_{tor}$ must lie in $(i_0\Pi_Qi_0)^+ \cap (\Lambda_Q)_{tor}$. One may explicitly compute that $\delta$ is not zero in the $D, \tilde{E}$ cases: in the $\tilde{D}_n$ case, either $\delta(Y)$ or $\delta(Z)$ is nonzero, and of the form $\sum (X^a \cup r^{(2)}) \wedge X^b$ for appropriate $a, b$ (cf. Theorem 4.2.60 or Remark 10.2.36 on the notation). In fact, using the severe restriction on the image, it is not difficult to compute $\delta$ completely (one can also use the one-cocycle identity if desired).

\section{Poisson center of Sym $\Lambda_Q,J$}

In this subsection, we compute the Poisson centers of Sym $\Lambda_Q,J$ for any quiver $Q$ and subset of vertices $J$, which in particular includes the cases Sym $\Lambda_Q$ and Sym $L_Q$. In particular, over $\mathbb{F}_p$, the center is entirely generated by vertex idempotents, powers of $r$, and $p$-th power operations (both on $\Lambda$ itself and on the symmetrical algebra).

To prove this, we use reduced $p$-th power constructions: a noncommutative one for $\Lambda$, and (a version of) the usual one on Sym $\Lambda$. In more detail, we enlarge $\Lambda$ to contain the images of maps $[f] \mapsto [f^{p^\ell}]/p^{\ell}$, which make sense as derivations (in fact, on $\Pi$). Then, $\mathbb{F}_p$-linear combinations of such elements act nontrivially on $\Lambda$. In fact, we will see that the usual $p$-th power map on $\Lambda$, and hence the reduced one, is injective, so $[f^{p^\ell}]/p^{\ell}$ is a nontrivial derivation on $\Lambda$ for all nonzero $[f]$.

The noncommutative reduced powers in $\Lambda$ in fact span $HH^1(\Pi_Q)$ (together with the Euler vector fields), at least in the non-Dynkin case, as we will see in the next section. In this section we prove a strong version of nonvanishing of these as derivations: they in fact are nonvanishing when descended to $HH_0(\Pi_Q)$.

We begin by showing that the torsion of $\Lambda_Q$ is central with respect to the necklace bracket.

\begin{proposition}
For any quiver $Q$ and subset $J \subset I$ of vertices, $\text{torsion}(\Lambda_Q,J)$ is central with respect to the necklace bracket and cobracket.
\end{proposition}

\begin{proof}
If $J \neq \emptyset$, then $\Lambda_Q$ has no torsion (Proposition 9.2.1). If $J = \emptyset$ and $Q \neq \tilde{D}_n$ or $\tilde{E}_8$, then the torsion is spanned by elements $r^{(p^\ell)}$ (by Theorems 4.2.1 and 4.2.60) which are central by Proposition 5.1.8. If $Q = \tilde{D}_n$, then the torsion is spanned by elements of the form $[f_L] + [f_R]$, where $[f_L]$ is concentrated on the left side of the quiver (cf. Figure 2) and there is equal to the element $[z_{c,0,R,i_0,i_0,L,U,i_0,L}]$ for some even $c \geq 2$, and $[f_R]$ is concentrated on the right side of the quiver and has the same form. Since $\langle [z_{c,0,R,i_0,i_0,L,U,i_0,L}] \rangle_{c \geq 2, 2|c}$ is abelian (for instance, because $i_0\Pi_{\tilde{D}_n}i_0$ is a Poisson algebra under the necklace bracket), one easily deduces the result. Alternatively, one could say that the bracket must be zero on $\Lambda_{\tilde{D}_n}$, and use that $\Lambda_{\tilde{D}_n} \to \Lambda_{\tilde{D}_n}$ restricts to an isomorphism on torsion. Finally, in the case $Q = \tilde{E}_8$, the torsion is spanned by central elements $r^{(p^\ell)}$ and a single element in top degree of $\Lambda_Q$, which must also be central.

For the statement about cobracket, again we only need to show the statement for the elements $[f_L] + [f_R] \in \Lambda_{\tilde{D}_n}$ and for the element of degree 28 in $\Lambda_{\tilde{E}_8}$. Here, the argument follows for degree reasons, since in the $\tilde{D}_n$ case, $\Lambda_{\tilde{D}_n} \wedge \Lambda_{\tilde{D}_n}$ is zero in degrees $4m - 2$ for $4m \leq 2(n - 2)$, and similarly in the $\tilde{E}_8$-case, $\Lambda_{\tilde{E}_8} \wedge \Lambda_{\tilde{E}_8}$ is zero in degree 26.

\end{proof}

\begin{theorem}
Let $(Q, J, k)$ be any triple of a quiver, a subset of vertices, and a commutative ring $k$.
\end{theorem}

---

\footnote{In [CBEG07], the Poisson centers of Sym $L_Q$ and Sym $\Lambda_Q$ were computed for non-Dynkin, non-extended Dynkin quivers, over a field of characteristic zero, using representation spaces.}
(i) (a) If char(k) = p > 0 where p is prime, then the Poisson center Z(Sym Λ_{Q,J} \otimes k) is given by

\[
Z(\text{Sym} \, \Lambda_{Q,J} \otimes k) = \text{Sym} \, Z(\Lambda_{Q,J} \otimes k) \ltimes \langle F^k \rangle_{F \in \text{Sym} \, \Lambda_{Q,J} \otimes k},
\]

where \([f]^k = [f] \& \cdots \& [f]\) is the symmetric power of \(f\) with itself \(p\) times. Furthermore, the center of the Lie algebra \(\Lambda_{Q,J} \otimes k\) is given by

\[
Z(\Lambda_{Q,J} \otimes k) = \langle ir^f i \rangle + (\Lambda_{Q,J})_{p-tor} + \langle [f^p] \rangle_{f \in \Pi_Q},
\]

where \((\Lambda_{Q,J})_{p-tor}\) is the \(p\)-torsion of \(\Lambda_{Q,J}\), given by Theorem 1.1.2, or (in the extended Dynkin or Dynkin case) Theorem 4.2.60, or \(0\) if \(J \neq \emptyset\).

(b) If char(k) = 0 (again as a commutative ring), then the Poisson center of \(\text{Sym}_k(\Lambda_{Q,J} \otimes k)\) is

\[
\text{Sym}_k(\langle ir^f i \rangle + \text{torsion}(\Lambda_{Q,J}))) \otimes (\bigoplus_{p \text{ prime}} ((\Lambda_{Q,J})_{p-tor} \& \text{Sym}(\langle [f^p] \rangle_{f \in \Pi_Q} + \langle F^k \rangle_{F \in \text{Sym} \, \Lambda_{Q,J} \otimes k})).
\]

(ii) The \(p\)-th power map \(\Lambda_{Q,J} \otimes \mathbb{F}_p \rightarrow \Lambda_{Q,J} \otimes \mathbb{F}_p, [f] \mapsto [f^p]\), is injective if \(Q\) is non-Dynkin, non-extended Dynkin, or if \(J \neq \emptyset\). Also, modulo torsion, the \(p\)-th power map \((\Lambda_{Q,J})_{\text{free}} \otimes \mathbb{F}_p \rightarrow (\Lambda_{Q,J})_{\text{free}} \otimes \mathbb{F}_p\) is injective for all \((Q, J)\).

We will prove (i).(a). Part (i).(b) follows easily from it, in view of the fact that \(\Lambda_{Q,J}\) can never have \(p^2\)-torsion (by Theorem 1.1.2). It suffices to consider the case \(k = \mathbb{F}_p\).

We consider a noncommutative divided-power construction (since the only symmetry is cyclic, we only divide \([z^n]\) by \(n\) rather than \(n!\)). First, note that

\[
([f + g]_p^f) = [f^p] + [g^p] + \sum_{(H(f,g),m)} p^{f-m}[H(f,g)^m],
\]

where the pairs \((H(f,g), m)\) range over cyclic monomials \([H(f,g)]\) of degree \(p^{f-m}\) which have period \(p^{f-m}\) (i.e., are not \(p\)-th powers of other cyclic monomials). This motivates considering the space obtained by adjoining \(\frac{1}{p}\) times the \(p\)-th powers in \(\Lambda_Q\), which will give us more nonzero derivations under the adjoint action.

**Definition 9.2.7.** Suppose \(A\) is any associative algebra over \(\mathbb{Z}\) (or \(\mathbb{Z}_{(p)}\)), and \(A_+ \subset A\) any ideal with \(\cap_{m=1}^{\infty} A_+^m = 0\). Let \((HH_0(A))_{[p]} \subset HH_0(A) \otimes \mathbb{Q}\) be the sub-\(\mathbb{Z}\) (or \(\mathbb{Z}_{(p)}\))-module obtained from \(HH_0(A) \otimes 1\) by adjoining \(\frac{1}{p}[f^p]\) for all \(f \in A_+\). When \(A_+\) is obvious, we will omit the \(A_+\) from the notation, and write \((HH_0(A))_{[p]}\).

Since \(\{[f^p], [g]\} = p^f \mu([\hat{g}, \hat{f}], f^{p-1}]\) where \(\mu\) is the usual multiplication \(\Pi_{Q,J} \otimes \Pi_{Q,J} \rightarrow \Pi_{Q,J},\) and since torsion is central by Proposition 9.2.1, we obtain a natural action

\[
\text{ad} : (\Lambda_{Q,J})_{[p]} \rightarrow \text{Der}_{\text{Lie}}(\Lambda_{Q,J}) \hookrightarrow \text{Der}_{\text{Pois}}(\text{Sym} \, \Lambda_{Q,J}).
\]

The latter subscripts indicate Lie and Poisson algebra derivations, respectively.

**Remark 9.2.9.** We could instead define a \(\mathbb{Z}\)-version of the above, by \(\hat{\Lambda}_{Q,J} := \Lambda_{Q,J} + \bigoplus_{\ell \geq 2} \langle [f^\ell] \rangle_{f \in \Pi_{Q,J}}\).

We do not need this.
Remark 9.2.10. We could equally work over any commutative ring of characteristic $p$, in which case the $p$-th power operation becomes Frobenius-linear.

Next, we do the usual divided power construction for $\text{Sym} \Lambda_{Q,J}$, except modified to only divide by $n$ rather than $n!$ (since we are in the infinitesimal, or derivation, setting):

Definition 9.2.11. For any prime $p$ and $\mathbb{Z}$- or $\mathbb{Z}(p)$-module $V$, define

$$\left(\text{Sym} \ V\right)_{[p]} := \left(\text{Sym} \ V + \sum_{\ell \geq 1} \frac{1}{p^{\ell}} F^{\ell} p^{\ell} \right)_{F \in \text{Sym} \ V},$$

(9.2.12)

where the expression in parentheses makes sense similarly to before (as a sub-$\mathbb{Z}$ or $\mathbb{Z}(p)$-module of $\text{Sym} V \otimes \mathbb{Q}$).

As before, if $L$ is a Lie algebra over $\mathbb{Z}$ whose torsion is central, one has a map $\text{ad} : (\text{Sym} \ L)_{[p]} \to \text{Der}_{\text{Pois}}(\text{Sym} \ L)$; more generally, if $V \to \text{Der}_{\text{Lie}}(L)$ kills torsion, then $(\text{Sym} \ V)_{[p]} \to \text{Der}_{\text{Pois}}(\text{Sym} \ L)$. Now, one may consider the map

$$\text{ad} : (\text{Sym} \ (\Lambda_{Q})_{[p]})_{[p]} \to \text{Der}(\text{Sym} \ A_{Q} \otimes F_{p}).$$

(9.2.13)

Using the following basic lemma, we rephrase the question to one that behaves better under induction:

Definition 9.2.14. For any $\mathbb{Z}_{\geq 0}$-graded Lie algebra $L$, let $Z_{\infty}(L)$ be the subspace of elements commuting with sufficiently-high degree elements:

$$Z_{\infty} = \{ z \in L \mid \exists N \geq 0 \text{ s.t. } \{ z, L[\geq N] \} = 0 \},$$

(9.2.15)

where $L[\geq N] := \sum_{m \geq N} L[m]$. More generally, if $\rho : V \to \text{Der}_{\text{Lie}}(L)$, then let $Z_{\infty,V}(L) := \{ z \in V \mid \exists N \geq 0 \text{ s.t. } \rho(z) \cdot L[\geq N] = 0 \}$. 

Lemma 9.2.16. Let $L$ be any graded Lie algebra over $k := F_{p}$. Then,

$$Z(\text{Sym}_{k} L) \subset \text{Sym}_{k} Z_{\infty}(L) \& \langle F^{\ell} p^{\ell} \rangle_{F \in \text{Sym}_{k} L},$$

(9.2.17)

and $\text{Sym}_{k} Z_{\infty}$ is a Poisson subalgebra.

Proof. For any element $z \in (\text{Sym}_{k} L)_{[p]}$ of total degree $N$, any homogeneous element $f$ of degree $> N$ that commutes with $z$ must also commute with the contraction of $\phi$ with $z$ for any $\phi \in \text{Sym} L[m]^{*}, m \leq N$. Inductively, we find that $z$ lies in the span of products of individual elements that commute with $f$, and $p$-th powers of other elements. Hence, if $z$ is central, it must lie in the given space. 

Using the same proof as above, we have a $[p]$-version, which allows us to get rid of $p$-th powers:

Lemma 9.2.18. Let $L$ be any graded Lie algebra over $\mathbb{Z}$, such that torsion$(L)$ is central. Letting $Z(\text{Sym}_{L})_{[p]} \otimes F_{p}(\text{Sym} L \otimes F_{p})$ denote the elements of $(\text{Sym} L)_{[p]} \otimes F_{p}$ that induce a zero derivation on $\text{Sym} L \otimes F_{p}$, we have

$$Z_{\infty,(\text{Sym} L)_{[p]} \otimes F_{p}(\text{Sym} L \otimes F_{p})} \subset Z(\text{Sym} L)_{[p]} \otimes F_{p}(L \otimes F_{p}),$$

(9.2.19)

74
Note also that, using the same proof as above, in the case that $k = \mathbb{Q}$, we may conclude that $Z(\text{Sym}_q L) \subseteq \text{Sym}_q Z_\infty (L)$.

The theorem then follows from the stronger statement:

**Proposition 9.2.20.** In the case $k = \mathbb{F}_p$,

$$Z_\infty ((\Lambda_Q, J)[p]) = Z((\Lambda_Q, J)[p]) = \left\langle \frac{1}{r^\ell i} \right\rangle.$$

(9.2.21)

Specifically, the proposition above proves the theorem for $k = \mathbb{F}_p$, from which the rest follows. To prove Proposition 9.2.20 we use a series of lemmas. The first one allows us to reduce to the case where $J = \emptyset$ in a precise way:

**Lemma 9.2.22.** If Proposition 9.2.20 holds for $(\hat{Q}, J_1, f, J_2)$ for $f \gg 0$ (meaning when $f$ takes sufficiently large values) and $J_1 \cap J_2 = \emptyset$, then it also holds for $(Q, J_1 \cup J_2)$.

**Proof.** By (an easy generalization of) Proposition 2.2.8 $\Lambda_{Q, J_1 \cup J_2} = \Lambda_{Q, J_1 \cup J_2} / \langle ir \rangle_{i \in J_1}$ is the inverse limit of $\Lambda_{Q, J_1, f, J_2}$ as $f$ ranges over maps $f : J_1 \rightarrow \mathbb{Z}_{\geq 0}$. Also, for any $f$, $\Lambda_{Q, J_1, f, J_2}$ is a quotient of $\Lambda_{Q, J_1 \cup J_2}$. These maps are induced by maps of algebras $\Pi_{Q, J_1 \cup J_2} \rightarrow \Pi_{Q, J_1, f, J_2}$ so preserve all $p$-th power constructions. Hence, $Z_\infty ((\Lambda_{Q, J_1 \cup J_2})[p])$ must map to $Z_\infty ((\Lambda_{Q, J_1, f, J_2})[p])$. Now, fix $m \geq 1$. If we take $f$ such that $f(i) \geq m/2$ for all $i$, then $\Lambda_{Q, J_1, f, J_2}[m] \cong \Lambda_{Q, J_1 \cup J_2}[m] / \langle ir \rangle_{i \in J_1, f \geq 1}$. Now, given the assumption of the lemma, for any $m \geq 1$, the $m$-th graded component of the $Z_\infty ((\Lambda_{Q, J_1 \cup J_2})[p])$ must lie entirely in the desired space. On the other hand, it follows from Propositions 9.2.1 and 5.1.8 that the desired space is central, so we deduce the lemma.

We will use below the following basic fact:

**Proposition 9.2.23.** If $Q$ is non-Dynkin or $J \neq \emptyset$, and $k$ is any field (or $\mathbb{Z}$), then $\Pi_{Q, J} \otimes k$ is prime, i.e., $x \Pi_{Q, J} y \neq 0$ for any $x, y$. If $Q$ is extended Dynkin and $J = \emptyset$, then $i_0 \Pi_{Q, J} \otimes k$ is a (commutative) integral domain.

**Proof.** This can be proved directly from our bases, and is particularly easy if $J \neq \emptyset$ using Proposition 4.2.21. In the case that $Q$ is extended Dynkin, it is also a result of [BGL87]; and the non-Dynkin, non-extended Dynkin case with $J = \emptyset$ may be deduced from the extended Dynkin case using Proposition 4.2.25. The last statement follows from the first.

Next, we establish the base case of the induction (when $Q$ is extended Dynkin):

**Lemma 9.2.24.** If $Q$ is extended Dynkin and $f \in (i_0 \Pi_{Q, J} q^0) \otimes \mathbb{F}_p$, defined by

$$(i_0 \Pi_{Q, J} q^0)_p = \left( i_0 \Pi_{Q, J} q^0 + \sum_{\ell \geq 1} \left\langle \frac{1}{p^\ell f} \right\rangle \right)_{f \in i_0 \Pi_{Q, J} q^0}.$$  (9.2.25)

then $\{ f, i_0 \Pi_{Q, J} q^0 \otimes \mathbb{F}_p \}$ is infinite-dimensional.

**Proof.** We first reduce to showing that $f$ is not central. Since $i_0 \Pi_{Q, J} q^0 \otimes \mathbb{F}_p$ is an integral domain (Proposition 9.2.23), provided that $\{ f, g \} \neq 0$, then $\{ f, \frac{1}{m} q^m \} = g^{m-1} \{ f, g \}$ is nonzero. Also, $i_0 = (i_0 \Pi_{Q, J} q^0)_0$ is central. Hence, if $f \in (i_0 \Pi_{Q, J} q^0)_0$ does not commute with $i_0 \Pi_{Q, J} q^0 \otimes \mathbb{F}_p$ under the bracket, then there exists a $g \in i_0 \Pi_{Q, J} q^0 \otimes \mathbb{F}_p$ of positive degree such that $\{ f, g \} \neq 0$, and then $\{ f, i_0 \Pi_{Q, J} q^0 \otimes \mathbb{F}_p \}$ must be infinite-dimensional.
Now, suppose that \( p \) is a good prime for \( Q \) (that is, \( p \) is not a factor of \( |\Gamma| \) for the corresponding group \( \Gamma \subset SL_2(\mathbb{C}) \)). Letting \( k = \overline{F}_p \), by Theorem 9.1.1(i), we may work with the Poisson algebra \( k[x, y]^\Gamma \). If we remove the singular point, we have an étale covering \( A^2_k \setminus \{(0, 0)\} \to (A^2_k/\Gamma) \setminus \{(0, 0)\} \), where \( A^2_k/\Gamma := \text{Spec}(k[x, y]^\Gamma) \). Since the Poisson bivector for \( \{,\} \) is nondegenerate on \( k[x, y] \), it must also be nondegenerate on this last space, and hence any \( f \in \Gamma(U) \) for any Zariski open set \( U \subset (A^2_k/\Gamma) \) not containing zero, can be in the kernel of \( \{,\} \) iff \( df = 0 \). Using divided powers, \( df \neq 0 \) for any \( f \neq 0 \) in \((i_0\Pi QI_0)_{[p]}\). Hence, we deduce the result.

In the case that \( p \) is not a good prime, we can use the bases of this paper. The result follows easily from Theorems 4.2.40, 4.2.48 in the case that \( Q \) is of type \( A, D \), respectively, and for type \( E \), one need only check primes \( 2, 3 \) (and \( 5 \) for type \( E_8 \)). This was checked by hand using Proposition 8.4.1 less than a page for each of \( E_6, E_7 \) and \( E_8 \) (it suffices to show that an element \( f \in i_0\Pi QI_0 \) which is not a \( p \)-th power is not central, so nothing involving divided powers is needed).

Now, let \( Q \supseteq Q_0 \) where \( Q_0 \) is extended Dynkin (still with \( J = \emptyset \)). We will define some \([p]\) versions of \( W' := W \cap [[[r']]\} \) and \( V' := V \cap [[[r']]\), in the language of Proposition 4.2.25.

**Definition 9.2.26.**

\[
V'_{[p]} := \left( \sum_{\ell \geq 0} \frac{1}{\ell!} \left( ((Q \setminus Q_0) f)^{\ell} \right)_{f \in \Pi Q_0} \right), \tag{9.2.27}
\]

\[
W'_{[p]} := ((Q \cdot W') \cap V'_{[p]}), \tag{9.2.28}
\]

where, as before, the \( \mathbb{Z} \)-modules obtained above are considered as sub-\( \mathbb{Z} \)-modules of \( V \otimes Q \).

To complete the proof of Proposition 9.2.20 we show that the elements of \( V'_{[p]} \otimes \overline{F}_p \) which induce a zero derivation under ad on \( \Lambda_Q \) are spanned by \( W'_{[p]} \otimes \overline{F}_p \) and a subspace of \( [\Pi Q \setminus Q_0, I_0] \) projecting to such central elements of \( (\Lambda_Q \setminus Q_0, I_0)_{[p]} \), which we will then inductively show must be zero.

We will need to consider the associated graded of \( V'_{[p]} \otimes \overline{F}_p \) modulo \( [[[r']]\} \). We have that \( gr_{[[[r']]\}}(W'_{[p]} \otimes \overline{F}_p) \subset \bigoplus_{\ell \geq 0} \frac{1}{\ell!} [r' \Pi Q_0]^o \otimes \overline{F}_p \ (\text{mod } [[[r']]\}^{o+1})]. \tag{9.2.29} \)

**Lemma 9.3.30.** Let \( Q \supseteq Q_0 \) for \( Q_0 \) extended Dynkin, and let us consider \( \Lambda_Q \), as decomposed in the notation of Proposition 4.2.25. If \( z \in W'_{[p]} \otimes \overline{F}_p \) and the projection of \( z \) modulo \( ([r']\Pi Q_0 \setminus Q_0, I_0) \otimes \overline{F}_p \) does not lie in \( ([r']H^0(\Pi Q_0) \cap V'_{[p]} \right) \otimes \overline{F}_p \), then \( z \in Z_{\infty, W'_{[p]} \otimes \overline{F}_p} (\Lambda_Q \otimes \overline{F}_p) \) only if \( z \in [\Pi Q \setminus Q_0, I_0] \otimes \overline{F}_p \).

**Proof.** Consider the following computation of Lie bracket, for \( f_i \in \overline{\Pi Q_0} \) and \( u_i \in \Pi Q \cdot Q_0, I_0 \):

\[
[[r'gr'h], [f_1u_1 \cdots f_nu_n]] = \sum_{i=1}^n [gr'h f_i u_i \cdots f_{i-1}u_{i-1}] + [hr'g f_i u_i \cdots f_{i-1}u_{i-1}] - [f_i gr' h u_i f_{i+1}u_{i+1} \cdots f_{i-1}u_{i-1}] - [f_i hr' gu_i f_{i+1}u_{i+1} \cdots f_{i-1}u_{i-1}], \tag{9.2.31}
\]

as elements of \( gr_{[[[r']]\}}(V'_{[p]} \otimes \overline{F}_p) \).

If any of the following are satisfied:

- \( n \geq 2 \),

76
• at least one of the \( u_i \) is linearly independent with \( r' \), or
• there is an element \( f_{\ell'} \notin HH^{0}(\Pi_Q) \),

then for \( g, h \) of large degree, with \(|g| \gg |h| \gg |f_i| \) for all \( i \) (and not commuting with any fixed noncentral \( f_{\ell'} \)), all of the summands above are linearly independent, unless \((f_1, \ldots, f_n)\), as a cyclic sequence, has period strictly less than \( n \) (using Proposition 9.2.23). Note that it is easy to obtain such \( g \) and \( h \) since they exist in some degree, and then we could multiply by appropriately large-degree central elements.

Suppose now that \((f_1, \ldots, f_n)\) is a cyclic sequence with period \( q \), and let \( n = mq \), so that \([f_1 r' \cdots f_n r']^m\). Then, we can rewrite the above as \( m \cdot \) the sum of \( 4q \) linearly independent terms (at least as elements of \( V \)), by replacing the sums with sums from \( i = 1 \) to \( q \). Furthermore, for any finite collection of distinct cyclic sequences \((f_{j,i})_{i=1}^{n_j}\) of periods \( q_j \), such that \(|g| \gg |h| \gg |f_{j,i}| \) for all \( j, i \), the sum \( \sum_j 4q_j \) resulting distinct summands appearing in the RHS of (9.2.31) are all linearly independent as elements of \( V \). Also, their images must be linearly independent as elements of \( \Lambda Q \otimes \mathbb{F}_p \) since the projection to the associated graded by the filtration \([(r')^*]) \) does not intersect \( \bigoplus_{t \geq 0} ([r'\Pi_Q])^t \) (mod \([(r')^t+1])\), unlike \( W'_p \otimes \mathbb{F}_p \) (9.2.23). Hence, if any \( \mathbb{F}_p \)-linear combination of the images of the elements \([f_1 r' \cdots f_n r']\) in \( V'_p \) has a nonzero projection to \([T((\Pi_Q)_{+} \otimes \mathbb{Z}) (\Pi_Q \setminus Q, \ell, t) + ] \otimes \mathbb{F}_p \), it cannot lie in \( Z_{\infty} V'_p (\Lambda Q \otimes \mathbb{F}_p) \). This amounts to the desired statement.

\[\square\]

Next, we refine the description of the space \( W' \) through \( p \)-th powers given in Theorem 4.2.30.

**Lemma 9.2.32.** We may choose the basis \( \{g_{i,\ell}\} \cup \{f_{\ell}\}_{\ell \geq m} \) of \( W' \) given in Theorem 4.2.30 (iv) (satisfying \( g_{i,\ell} \equiv g^p_{i,0} \) (mod \( p \)), \((\frac{1}{p^\ell} f_{\ell})^p \equiv \frac{1}{p} f_{\ell+1} \) (mod \( p \))) so that we additionally have

\[g_{i,\ell} \in p^\ell V'_p, \quad f_{\ell} \in p^\ell V'_p,\]

where on the RHS we are considering multiplication by \( p^\ell \) to give a map \( V'_p \rightarrow V' \equiv V' \otimes 1 \subset V' \otimes \mathbb{Q} \).

Furthermore, the elements \( \frac{1}{p^\ell} g_{i,\ell}, \frac{1}{p^\ell} f_{\ell} \) project to a basis of \( \mathbb{F}_p \otimes [r' HH^{0}(\Pi_Q)] \) modulo \( \mathbb{F}_p \otimes ([\mathbb{Q} \cdot \cdots \cdots (r')^2]) \cap V'_p \).

**Proof.** Fix any \( g = g_{i,0} \). Note that \( g^p^\ell \) has the desired form (9.2.33), so it suffices to show that

\[g^p^\ell \in W' + (pV' \cap p^\ell V'_p).\]

Suppose that \( g \) is the image of the element

\[\sum_j [h_j, h'_j] \in (r') \]

in \( V' = V \cap (r') \). Then,

\[\left((\sum_j [h_j, h'_j])^p - \sum_j [h_j, h'_j(h_jh'_j)^p - 1]\right),\]

as an element of \( \Lambda Q \), is of the form

\[\sum_{k=1}^\ell p^k (g_k^p)^{p-k},\]

77
for some elements \( g'_k \in \Pi_Q \), and hence also

\[
g^{p'} \equiv \sum_{k=1}^{\ell} p^k ([g'_k]^{p'-k}) \pmod{W'}. \tag{9.2.38}
\]

Next, we have that \( g^{p'} \in \langle ([r']) \rangle \). Considering the above mod \( p^2 \), we find (using Proposition 9.2.23) that, taking the images \( \bar{g}'_k \in V \) of the \( g'_k \) in \( V \), \( \bar{g}'_1 \in \langle ([r']) \rangle + pV \), so we may assume that \( g'_1 \in \langle ([r']) \rangle \) up to modifying \( g'_2 \). Then, inductively, we may assume that all of the \( \bar{g}'_k \in \langle ([r']) \rangle \) except for \( \bar{g}'_\ell \). But then, we also deduce that \( \bar{g}'_\ell \in \langle ([r']) \rangle \). Thus, we obtain (9.2.34).

For the elements \( f_\ell \), the corresponding statement follows from the proof of Theorem 4.2.30 (Section 8.5).

Finally, to see that the elements \( \frac{1}{p^\ell} g_{i,\ell}, \frac{1}{p} f_\ell \) project to a basis of \( \langle r'HH^0(\Pi_{Q_0}) \rangle \otimes \mathbb{F}_p \) modulo \( \mathbb{F}_p \otimes \langle ([r'])^2 \rangle \cap V'_{[p]} \), it suffices to show that they project to linearly independent elements (since the Hilbert series matches, cf. Proposition 4.2.29). But, if their projections were not linearly independent, then some \( \mathbb{F}_p \)-linear combination would give a nonzero element of \( \langle (W'_{[p]} \cap Q : ([r'])^2 \rangle \otimes \mathbb{F}_p \), which would additionally have to induce a zero derivation on \( \Lambda_Q \otimes \mathbb{F}_p \) (by Proposition 9.2.21). But this contradicts Lemma 9.2.30.

We may now deduce the

**Lemma 9.2.39.** We have \( Z_{\infty, V'_{[p]} \otimes \mathbb{F}_p}(\Lambda_Q \otimes \mathbb{F}_p) \subset W'_{[p]} + [\Pi_{Q \setminus Q_0} I_0] \).

**Proof.** If any element of \( V'_{[p]} \otimes \mathbb{F}_p \) commuted with large enough degrees of \( (\Lambda_Q)_{[p]} \), then it would have to lie in \( \langle \langle r'HH^0(\Pi_{Q_0}) \rangle + \langle ([r'])^2 \rangle \cap W'_{[p]} \rangle \otimes \mathbb{F}_p \) (by Lemma 9.2.30), and after adding elements of \( W'_{[p]} \otimes \mathbb{F}_p \), could be made to lie in \( \langle \langle ([r'])^2 \rangle + [\Pi_{Q \setminus Q_0} I_0] \rangle \otimes \mathbb{F}_p \) (by Lemma 9.2.32). By Lemma 9.2.30 again, it would then have to lie in \( [\Pi_{Q \setminus Q_0} I_0] \otimes \mathbb{F}_p \).

Finally, we may complete the proof of Proposition 9.2.20 and hence Theorem 9.2.24 (i):

**Proof of Proposition 9.2.20.** Using Lemma 9.2.22 let us assume that \( J = \emptyset \). By Lemma 9.2.24 we are done if \( Q \) is extended Dynkin, and by Proposition 9.2.21 we are done in the Dynkin case. So let us assume that \( Q \supseteq Q_0 \) with \( Q_0 \) extended Dynkin. Then, let us use the notation of Proposition 4.2.25 and try to compute \( Z_{\infty, (\Lambda_Q)_{[p]}}(\Lambda_Q \otimes \mathbb{F}_p) \). By Lemma 9.2.24 again, it suffices to compute \( Z_{\infty, V'_{[p]}}(\Lambda_Q \otimes \mathbb{F}_p) \). By Lemma 9.2.39 it suffices to show that Proposition 9.2.20 is true for \( (Q \setminus Q_0, I_0) \), or equivalently, for the connected components thereof. So, in this way, we may always reduce from \( (Q, \emptyset) \) to the connected components of \( (Q \setminus Q_0, I_0) \) and hence to the connected components of \( (Q \setminus \overline{Q_0}I_0, f, \emptyset) \), using Lemma 9.2.22. By doing this, we always replace \( Q \) by quivers which are less combinatorially complex: precisely, after trimming the ending branches in \( Q \) and the new quivers to length one (by an ending branch, we mean a line segment which intersects its complement in the quiver at a single vertex of valence \( \geq 3 \)), the new trimmed quivers are always strict subsets of the trimmed \( Q \). Thus, eventually, we can reduce to the case of quivers whose trimming is Dynkin or extended Dynkin. If the former, the quiver must be star-shaped with three branches, or a single line segment (Dynkin of type \( A \)). In the latter case, we can reduce further unless the quiver is star-shaped with three branches. If the resulting quiver is Dynkin, we are done; otherwise, one more application of the above procedure (with \( Q_0 \) of type \( \tilde{E}_n \)) reduces to line segments (quivers of type \( A \)). This completes the proof.

\( \square \)
Proof of Theorem 9.2.2. (ii) This follows from the above construction: inductively, the given p-th power maps are in fact injective, since they are injective on \((\Lambda_Q)_{free}\) for \(Q\) extended Dynkin or Dynkin (the latter is a trivial statement, and the former follows from \((\Lambda_Q)_{free} \cong HH^0(\Pi_Q)\) and that the latter is an integral domain). Then, the inductive argument above gives the desired result.

10 Higher Hochschild (co)homology, cyclic homology, and the BV algebra

10.1 Higher Hochschild (co)homology and \(Z(\Pi i \otimes k)\) Using our computation of \(HH_0(\Pi)\) over \(Z\), one may easily determine the Hilbert series of all Hochschild homology and cohomology groups over any field. To do this, we need one result:

Theorem 10.1.1. Over any commutative ring \(k\), we have

(i) If \(Q\) is extended Dynkin, then \(HH^0(\Pi_Q \otimes k) \rightarrow i_0 \Pi_Q i_0 \otimes k\) under the projection map. In particular, the Hilbert series of the center does not depend on \(k\).

(ii) [CBEG07] If \(Q\) is non-Dynkin, non-extended Dynkin, then the center \(HH^0(\Pi_Q \otimes k) \cong k\). Thus, this Hilbert series also does not depend on \(k\).

Equivalently to the above (given the case of \(k = \mathbb{Q}\), for any non-Dynkin quiver \(Q\), \(HH_1(\Pi_Q)\) is a free \(\mathbb{Z}\)-module.

(iii) More generally, for any non-Dynkin quiver \(Q\) on vertex set \(I\), and any commutative ring \(k\), the projection \(HH^0(\Pi_Q \otimes k) \rightarrow HH^0(i\Pi_Q i \otimes k)\) is an isomorphism for all \(i \in I\) (so \(HH^0(i\Pi_Q i \otimes k)\) does not depend on \(i\)).

(Note that, while [CBEG07] mainly deals with the characteristic zero case, the results used there for part (ii) of the theorem are valid in any characteristic.) We will prove the first part, and as a result give a new proof of the second part, using our bases. In the process of the proof, we will actually prove the more general (iii).

As a consequence of the above and Theorems 1.1.2, 4.2.60, we deduce the structure of all higher Hochschild homology and cohomology over \(k\) when \(Q\) is non-Dynkin:

Corollary 10.1.2. For non-Dynkin quivers, the graded \(k\)-module structure of Hochschild homology and cohomology of \(\Pi_Q \otimes k\) is as follows: \(HH_m(\Pi_Q \otimes k) \cong HH^{2-m}(\Pi_Q \otimes k)(2)\), \(HH^0(\Pi_Q \otimes k)\) is given by Theorem 10.1.1, and

\[
HH^1(\Pi_Q \otimes k) \cong HH^0(\Pi_Q \otimes k) \oplus (HH_0(\Pi_Q)_{free})(-2) \otimes k \oplus (HH_0(\Pi_Q)_{tor})(-2) \oplus \bigoplus_{p \text{ prime}} \text{Hom}_Z(\mathbb{F}_p, k),
\]

(10.1.3)

\[
HH^2(\Pi_Q \otimes k) \cong k \otimes HH_0(\Pi_Q)(-2),
\]

(10.1.4)

\[
HH^m(\Pi_Q \otimes k) = HH_m(\Pi_Q \otimes k) = 0, \text{ if } m \geq 3,
\]

(10.1.5)

where \(HH_0(\Pi_Q)_{tor}\) is the torsion part and \(HH_0(\Pi_Q)_{free} = HH_0(\Pi_Q)/HH_0(\Pi_Q)_{tor}\) is the free part.

Furthermore, the natural maps \(HH^*(\Pi_Q) \rightarrow HH^*(\Pi_Q \otimes k)\) and \(HH_*(\Pi_Q) \rightarrow HH_*(\Pi_Q \otimes k)\) are almost surjective, with cokernel concentrated in degree 1, given by projection onto the third direct summand in (10.1.3).
Note that in the extended Dynkin case, we have that $HH_0(\Pi_Q \otimes k)_+, HH^0(\Pi_Q \otimes k)_+, HH_2(\Pi_Q \otimes k)_+$ are all isomorphic when $\frac{1}{30} \in k$ (or, more generally, when the stably bad primes for $Q$ are invertible in $k$). For the proof, we use the following well-known result ([VdB04], cf. also [Gin05], [VdB04]):

**Proposition 10.1.6.** ([VdB04]) Let $A$ be any associative algebra over a field, of finite Hochschild dimension ($=A$ has finite projective dimension over $A^e = A \otimes A^{op}$), for which

$$HH^n(A, A \otimes A) \cong \begin{cases} A, & \text{if } n=d, \\ 0, & \text{otherwise,} \end{cases}$$ (10.1.7)

for some $d \geq 0$. Then, $HH^n(A, A) \cong HH_{d-n}(A, A)$ for all $n$ (and $HH^n(A, A) = HH_n(A, A) = 0$ for $n > d$).

The algebra $\Pi_Q \otimes k$ is known to satisfy the above property for $d = 2$ whenever $Q$ is non-Dynkin (and $k$ is a field, or actually any commutative ring). In fact, it satisfies a stronger condition: it is Calabi-Yau of dimension 2 (cf. [Gin06], Definition 3.2.3):

**Notation 10.1.8.** For any algebra $A$ let $A^e := A \otimes A^{op}$.

**Definition 10.1.9.** For any algebra $A$ and bimodule $M$, let $M^1 := R\text{Hom}_{A^e}(M, A^e)$. Furthermore, for any map of $A$-bimodules $f : M \to N$, let $f^1 : N^1 \to M^1$ denote the functorially induced map.

**Definition 10.1.10.** ([Gin06]) An algebra $A$ of finite Hochschild dimension is said to be Calabi-Yau of dimension $d$ if there is a quasi-isomorphism (in the derived category) $f : A((d)) \simeq R\text{Hom}_{A^e}(A, A \otimes A)$ such that $f^1 \circ \iota = f((-d))$, where $\iota : A \to R\text{Hom}_{A^e}(R\text{Hom}_{A^e}(A, A \otimes A), A \otimes A)$ is the natural map, and the shifts $((-))$ here are in the derived category.

It is well-known that $\Pi_Q \otimes k$ is Calabi-Yau of dimension 2, but for later use we give an argument. It is enough to show that $\Pi_Q$ has the following special “self-dual” resolution:

$$0 \to \Pi_Q \otimes \langle r \rangle \otimes \Pi_Q \xrightarrow{\phi_1} \Pi_Q \otimes \langle \overline{Q} \rangle \otimes \Pi_Q \xrightarrow{\phi_2} \Pi_Q \otimes \Pi_Q \xrightarrow{(\psi_1^*)^{-1}} \Pi_Q \otimes \langle r \rangle^*, \quad (10.1.11)$$

such that $\psi_2^* = \psi_1^{-1}$. We consider the top horizontal row to be in nonpositive degrees $(-2, -1$, and 0 along the central three positions), and the bottom row to be in nonnegative degrees $(0, 1$, and 2 along the central three positions), so that the isomorphism is indeed an isomorphism of the top complex shifted by two with the bottom complex. The isomorphism immediately gives the quasi-isomorphism $f : A((d)) \simeq R\text{Hom}_{A^e}(A, A \otimes A)$ for $A = \Pi_Q$ and $d = 2$, and the self-dual property shows that $f^1 \circ \iota = f((-d))$. In particular, $\Pi_Q$ is Calabi-Yau of dimension two over $Z$, and the property remains true when applying $\otimes_Z k$ for any commutative ring $k$.

It is easy to verify the properties of the diagram using the fact that $\Pi_Q$ is an NCCI (only the leftmost exactness is not obvious, and follows from the NCCI condition).

Finally, we note that the vertical isomorphisms here are in fact graded isomorphisms of degree $-2$, so that $\Pi_Q$ is Calabi-Yau in the graded sense if by the latter we mean $f : A((d)) \simeq R\text{Hom}_{A^e}(A, A \otimes A)$ is a graded isomorphism of degree $-d$ (the shift here is only in the derived category, and doesn’t
affect the grading). Thus, the diagram immediately gives a graded version of the [VdB04] duality in our case: 
\[ \text{HH}_0(\Pi_Q \otimes \mathbf{k}) \cong \text{HH}^{n-m}(\Pi_Q \otimes \mathbf{k})(2), \]
over any commutative ring \( \mathbf{k} \), as stated at the beginning of Corollary 10.1.2.

Proof of Corollary 10.1.2. First, we note that the above is true in the case \( \mathbf{k} = \mathbb{C} \): if \( Q \) is extended Dynkin, the Hilbert series is well-known, using the Morita equivalence \( \Pi_Q \otimes \mathbb{C} \sim \Gamma \ltimes \mathbb{C}[x,y] \) and Proposition 8.2.4. In the case that \( Q \) is non-Dynkin, non-extended Dynkin and \( \mathbf{k} = \mathbb{C} \), the above follows from [EG06]. Hence it is also true for \( \mathbf{k} = \mathbb{Q} \).

Note that a direct way to obtain (10.1.3) in the case \( \mathbf{k} = \mathbb{Q} \) is using \( HH^1(\Pi_Q \otimes \mathbb{Q}) \cong HH_1(\Pi_Q \otimes \mathbb{Q}) \) (by the Calabi-Yau property) and the sequence \( 0 \rightarrow HH_2(\Pi_Q \otimes \mathbb{Q}) \rightarrow HH_1(\Pi_Q \otimes \mathbb{Q}) \rightarrow HH_0(\Pi_Q \otimes \mathbb{Q}) \) with \( B \) the Connes differential, which is shown to be exact in [EG06], Lemma 3.6.1.

It remains to extend to all other commutative rings \( \mathbf{k} \). By Theorem 10.1.1 the Hilbert series of \( HH^0(\Pi_Q \otimes \mathbf{k}) \cong HH_2(\Pi_Q \otimes \mathbf{k}) \) is independent of the choice of field \( \mathbf{k} \). Since, over \( \mathbb{Z} \), \( HH_* \) is computed by a chain complex of free \( \mathbb{Z} \)-modules (and \( HH^* \) by a cochain complex), by the universal coefficient theorem for homology, \( HH_1 \) and \( HH_2 \) (equivalently, \( HH^0 \) and \( HH^1 \)) must be free \( \mathbb{Z} \)-modules. Also, for any commutative ring \( \mathbf{k} \), \( HH^0(\Pi_Q \otimes \mathbf{k}) \cong HH^2(\Pi_Q \otimes \mathbf{k}) \) must be free \( \mathbf{k} \)-modules and have the same Hilbert series as before. (Note: one may also determine directly that \( HH_1(\Pi_Q) \) is torsion-free using the Anick resolution (4.2.5), and it essentially comes to the same computation as the proof of Theorem 10.1.1.)

The final statement about base change \( \mathbb{Z} \rightarrow \mathbf{k} \) follows immediately from the universal coefficient theorem (more generally, any base change can be assumed to respect the decomposition (10.1.3)).

\[ \square \]

Proof of Theorem 10.1.7. First, we prove (iii) when \( Q \) is extended Dynkin, which also proves (i). We will use the fact that, for any central element \( w \in Z(\Pi_Q) \), the multiplication by \( w \) map \( f \mapsto wf \) is injective, which follows for example from Proposition 9.2.23 even after replacing \( \Pi_Q \) by \( \Pi_Q \otimes \mathbf{k} \) for any field \( \mathbf{k} \). In particular, the projection maps \( Z(\Pi_Q) \rightarrow Z(i\Pi_Qi) \) are injective for all \( i \) (there is also a direct argument to prove the latter using the structure of the extended Dynkin quivers). It remains to show surjectivity, for which we construct an explicit inverse.

For now, we work over \( \mathbb{Z} \). Let \( i \in I \) be any vertex. First, \( \Pi_Q/(i) \) has finite rank, since cutting off any vertex yields a Dynkin quiver (or a union of two such). Now, we would like to say that this makes \( i\Pi_Qi \) “almost” Morita equivalent to \( \Pi_Q \), at least for the purpose of computing center (note though that \( i\Pi_Qi \) is not a projective \( i\Pi_Qi \)-module if \( Q \neq \tilde{A}_0 \)). We consider the map

\[ \pi : Z(\Pi_Qi\Pi_Q) \rightarrow Z(i\Pi_Qi), \quad x \mapsto x\pi. \tag{10.1.12} \]

We would like to construct a right inverse \( \pi^{-1} : Z(i\Pi_Qi) \rightarrow Z(\Pi_Qi\Pi_Q) \): this would be immediate if \( \Pi_Qi\Pi_Q \) were equal to \( \Pi_Q \) using Morita theory, but in our case takes some work. First, the map is injective (again by Proposition 9.2.23), so it suffices for any \( z \in Z(i\Pi_Qi) \) to construct \( z' \in Z(\Pi_Qi\Pi_Q) \) such that \( i\pi z' = z \).

For any \( z \in Z(i\Pi_Qi) \), consider the map

\[ \lambda_z : \Pi_Qi\Pi_Q \rightarrow \Pi_Qi\Pi_Q, \quad f \mapsto f\pi zg. \tag{10.1.13} \]

We show that \( \lambda_z \) is well-defined. For any \( z \in Z(i\Pi_Qi) \), the above gives a well-defined map \( \bar{\lambda}_z : P^\pi Q P^\pi Q \rightarrow \Pi_Qi\Pi_Q \) since \( f\pi g = f\pi (igi)h = f(igi)zh = figzh \). Then, \( \bar{\lambda}_z \) kills \( P^\pi Q (P^\pi Q \cap (r)) \), since,
if $\tilde{f}ig \in \langle (z) \rangle \subset P_{ij}$, then in $\Pi_Q$, $fzg(\Pi i) = 0$, but by Proposition 9.2.23 this shows that $fzg = 0$ in $\Pi_Q$.

Next, we use properties of the center $Z(\Pi_Q)$, which we understand at least after tensoring by $Q$ (e.g., 1.1.10).

Restricting (10.1.3) to $Z(\Pi_Q)' := Z(\Pi_Q) \cap \Pi_Qi\Pi_Q$, we get a map $Z(\Pi_Q)' \to \Pi_Q$. We claim that the image lies in $Z(\Pi_Q)$. First, we note that the image centralizes $\langle (i) \rangle$: if $fig \in Z(\Pi_Q)'$ and $z \in Z(i\Pi_Qi)$, then $(fzg)(hih') = (fig)(hzh') = (hzh')(fig) = (hih')(fzg)$. Next, we use that $\Pi_Q/\langle (i) \rangle$ is finite-dimensional, and that for any $w \in Z(\Pi_Q)$, the multiplication map $\cdot w : \Pi_Q \to \Pi_Q$ is injective. Thus, for any element $f \in \Pi_Q$, there exists $w \in Z(\Pi_Q)$ such that $f w \in (\langle (i) \rangle)$, and then any element $z \in Z(i\Pi_Qi)$ commutes with $fw$, and hence (again using injectivity), with $f$. This proves the claim.

Next, let $k$ be any field, and $w \in Z(\Pi_Q)'$. Let $z' := \frac{\lambda_z(w)}{w} \in$ the fraction field of $Z(\Pi_Q) \otimes k$. Since $\lambda_z(v)w = v\lambda_z(w)$ for all $v \in Z(\Pi_Q)'$, $\lambda_z$ coincides with multiplication by $z'$. Next, since $Z(\Pi_Q)/Z(\Pi_Q)'$ has finite rank, multiplication by $z'$ is well-defined on $Z(\Pi_Q)[N] \otimes k$ for large enough degree $N$. However, if $Z(\Pi_Q) \otimes k$ is smooth in codimension one, this would imply that $z' \in Z(\Pi_Q) \otimes k$, since otherwise the divisor of $z'$ would contain a pole, and there exist elements of arbitrarily large degree which do not vanish on any given divisor. We can easily verify that $Z(\Pi_Q) \otimes k$ is smooth in codimension one by using its presentation as a quotient of $k[X,Y,Z]$ (cf. Propositions 7.4.4, 8.4.1 and $Z^n = XY$ for the $A_{n-1}$ case—these presentations are all well known), or in the case that $k$ is in good characteristic, since this is true for $k[x,y]^\pi$. Thus, $z' \in Z(\Pi_Q) \otimes k$ and multiplication by $z'$ coincides with $\lambda_z$ on $Z(\Pi_Q)' \otimes k$. Hence, $\lambda_z$ coincides with multiplication by $z'$ on all of $\langle (i) \rangle \otimes k$ (again, since $\lambda_z(f)w = fz'$). We deduce that $iz'(z) = z$, and setting $\pi^{-1}(z) := z'$, we have constructed the desired right inverse to $\pi$. Furthermore, this right inverse actually lands in $Z(\Pi_Q)$ (in fact, $Z(\Pi_Q)' = Z(\Pi_Qi\Pi_Q)$ by our reasoning). This proves Theorem 10.1.1 in the extended Dynkin case.

Let us now consider the non-Dynkin, non-extended Dynkin case $Q \supsetneq Q_0$ with $Q_0$ extended Dynkin, and with vertex sets $I \supsetneq I_0$. We show that $Z(i\Pi_Qi \otimes k) = \langle (i) \rangle$ for all $i \in I$. We use the notation of Proposition 1.2.25 which fixes a section $\Pi_{Q_0}$ of $\Pi_Q \to \Pi_{Q_0}$. Suppose $z \in Z(i\Pi_{Q_0}i \otimes k)_+$ is nonzero. Let $w \in Z(\Pi_{Q_0} \otimes k) = Z(\Pi_{Q_0}) \otimes k$ be any element of degree larger than $z$, which lifts to the element $\tilde{w} \in \Pi_Q \otimes k$. Let $x, y \in \Pi_Q \otimes k$ be any elements such that $ix(f'\tilde{w}r)y = 0$ (guaranteed to exist by Proposition 9.2.23). Then, we claim that $F := \langle ix(f'\tilde{w}r)y \rangle$ cannot commute with $z$. We have that $Fz \neq 0$, and may be written in terms of the free product $\Pi_{Q_0} * \Pi_{Q_0 \setminus Q_0, i_0}$ of Proposition 1.2.25 as a sum of terms of the form $f_1f_2 \cdots f_m\tilde{w}f_{m+1} \cdots f_n$ where $f_m, f_{m+1} \in \Pi_{Q_0 \setminus Q_0, i_0}$, and $|f_1| + \cdots + |f_m| = |x| + 2$. On the other hand, if $zF$ is nonzero, its sum in terms of the free product $\Pi_{Q_0} * \Pi_{Q_0 \setminus Q_0, i_0}$ cannot include any terms of the previous form: if it includes anything of the form $fgh$ where $g \in \Pi_{Q_0}[|w|]$, then $|f| \geq |x| + 2 + |z| > |x| + 2$. So $zF$ cannot equal $Fz \neq 0$. This contradicts the assumption that $z \in Z(i\Pi_Qi \otimes k)_+$.

The above does not extend to higher Hochschild cohomology. For example, when $k = F_p$ and $p$ is a stably bad prime for an extended Dynkin quiver $Q$, then one may check that $HH^1(i_0\Pi_Qi_0 \otimes k)$ has some new derivations which do not come from $HH^1(\Pi_Q \otimes k)$. Nonetheless, one has an isomorphism when restricting to positive degree: see Theorem 10.2.39. In general, there seems to be a close link between the Hochschild (co)homology of $\Pi_Q$ and that of $i\Pi_Qi$ when $Q$ is non-Dynkin, which would be interesting to understand better.
10.2 Outer derivations and deformations of $\Pi_Q \otimes k$

Next, we interpret $HH^1(\Pi_Q \otimes k)$ and $HH^2(\Pi_Q \otimes k)$ in terms of outer derivations and deformations, respectively. In the process, we will obtain an explicit version of the identifications in Corollary 10.1.2.

Since $HH^3(\Pi_Q \otimes k) = 0$, formal deformations of $\Pi_Q \otimes k$ are unobstructed and classified (up to gauge equivalence) by $hHH^2(\Pi_Q \otimes k)[[h]]$. The explicit realization is easy and is essentially the same as the $k = \mathbb{C}$ case from [CBEG07], §10:

**Proposition 10.2.1.** One has an explicit identification between $HH_0(\Pi_Q \otimes k)[[h]] \cong \widehat{HH}^2(\Pi_Q \otimes k)[[h]]$ and gauge equivalence classes of deformations of $\Pi_Q \otimes k$ as follows: Let $HH_0(\Pi_Q \otimes k) \subset \Pi_Q \otimes k$ be any set-theoretical section of the map $\Pi_Q \otimes k \to HH_0(\Pi_Q \otimes k)$. For $f \in HH_0(\Pi_Q \otimes k)$ denote by $\tilde{f} \in HH_0(\Pi_Q \otimes k)$ the associated element.

Then, to any series $F := \sum_{m \geq 1} f_m h^m \in hHH_0(\Pi_Q \otimes k)[[h]]$, consider the algebra $(\Pi_Q \otimes k)_F := (P_Q \otimes k)/(r - \tilde{F}))$, where $\tilde{F} := \sum_{m \geq 1} \tilde{f}_m h^m$.

The above map $F \mapsto (\Pi_Q \otimes k)_F$ yields the desired isomorphism.

**Proof.** The result boils down to checking, for infinitesimal deformations, that the induced map $HH_0(\Pi_Q \otimes k) \to HH_2(\Pi_Q \otimes k), F \mapsto \phi_F$, is an isomorphism. First, $HH^2(\Pi_Q \otimes k)$ may be realized (using (10.1.11)) as a quotient of $\Pi_Q \otimes \langle r \rangle^* \otimes k \cong \text{Hom}(\langle r \rangle \otimes k, \Pi_Q \otimes k)$. Then, we see that the isomorphism $HH_0(\Pi_Q \otimes k) \cong HH^2(\Pi_Q \otimes k)$ given by (10.1.11) takes exactly the above form, and in particular the induced map does not depend on the choice of lifting $HH_0(\Pi_Q \otimes k)$. \[\square\]

As in [CBEG07], we can write the above as giving a versal (or miniversal) deformation, but there is nothing new to say.

10.2.1 Outer derivations of $\Pi_Q \otimes \mathbb{Q}$ (the rational case)

To study $HH^1(\Pi_Q \otimes k)$, we will make use of the following map (which we will show in Theorem 10.3.1 is the BV differential, in view of Proposition 10.2.1):

**Definition 10.2.2.** Let $D : HH_0(\Pi_Q \otimes k) \to HH^1(\Pi_Q \otimes k)$ be given by

$$D[a_1a_2 \cdots a_n] = \sum_{i=1}^{n} a_{i+1} \cdots a_{i-1} \iota(a_i), \quad (10.2.3)$$

where

$$\iota(e) = \frac{d}{de^*}, \quad \iota(e^e) = -\frac{d}{de}, \quad \text{for } e \in Q, \quad (10.2.4)$$

and, by definition, for any $e \in \overline{Q}, F \in (e_s P_{\mathbb{Q}^e}e_t) \otimes k$, and any $b_1, \ldots, b_m \in \overline{Q}$,

$$(F \cdot \frac{d}{de})(b_1b_2 \cdots b_m) := \sum_{i=1}^{m} b_1 \cdots b_{i-1}(F \cdot \delta_{b_i,e})b_{i+1} \cdots b_m. \quad (10.2.5)$$

Note that (10.2.3) does not give a well-defined derivation, but does give a well-defined outer derivation (cf. Remark 5.4.7).

Now, we consider the case $k = \mathbb{Q}$. We have
Proposition 10.2.6. Let $k = \mathbb{Q}$ and let $Q$ be non-Dynkin. The decomposition \eqref{10.1.3} can be explicitly realized by

$$HH^1(\Pi_Q \otimes \mathbb{Q}) \cong \langle HH^0(\Pi_Q \otimes \mathbb{Q}) \cdot Eu \rangle \oplus \langle D[f] \rangle_{[f] \in HH_0(\Pi_Q \otimes \mathbb{Q})^+}.$$ \hfill \eqref{10.2.7}

Proof. It is easy to verify that multiples of the Euler vector field are well-defined derivations. We claim that no nonzero element in the span above (for any fixed choice of representatives of $\{D[f]\}$ as derivations) is an inner derivation. Then, since the above span has the correct Hilbert series, we must have the desired equality.

The claim follows from the fact that the above elements act nontrivially on $HH_0(\Pi_Q \otimes \mathbb{Q})$, using the formulas

$$[D[f](g)] = \{[f], [g]\}, \forall f, g \in \Pi_Q,$$ \hfill \eqref{10.2.8}

$$D[i_0 z_1](z_2) = \{z_1, z_2\}, \forall z_1, z_2 \in HH^0(\Pi_Q)$$ \hfill \eqref{10.2.9}

where the second formula uses the (normalized) Poisson bracket we defined on the center (Corollary 5.5.11). The second formula follows from the first. The fact that the $D[f]$ act nontrivially is a consequence of Theorem 9.2.2 and the fact that $D[f] + z \cdot Eu$ acts nontrivially for $z \neq 0$ follows from $\langle D[f] + z \cdot Eu \rangle[f] = zEu([f]) \neq 0$. \hfill \QED

Remark 10.2.10. We could give an alternative argument, using the exactness of the Connes differential over $\mathbb{Q}$ (hence of the BV differential $D$), by Lemma 3.6.1 of [EG06]. Then, the result follows from the explicit computation of $D$ in Theorem 10.3.1, thus bypassing the use of Theorem 9.2.2 and giving a simple argument that $D[f] + z \cdot Eu$ can never be an inner derivation if $[f]$ or $z$ is nonzero. The fact that its restriction to $HH_0(\Pi_Q \otimes \mathbb{Q})$ is nontrivial is stronger and requires Theorem 9.2.2.

Next, we compute the cup product $HH^0(\Pi_Q \otimes \mathbb{Q}) \otimes HH^1(\Pi_Q \otimes \mathbb{Q}) \rightarrow HH^1(\Pi_Q \otimes \mathbb{Q})$ (now we assume $Q$ is extended Dynkin, since otherwise this cup product is just multiplication by scalars):

Proposition 10.2.11. Let $z \in HH^0(\Pi_Q \otimes \mathbb{Q})$ and assume that $z, f$ are homogeneous. We have

$$z \cup z' Eu = zz' Eu, \quad z \cup D[f] = \frac{[f]D[zf] + \{[f], z\} Eu}{|z| + |f|},$$ \hfill \eqref{10.2.12}

where the bracket $\{[f], z\}$ is the Loday bracket $L \otimes P \rightarrow P$ defined in Section 5.2, and $\{[f], z\} \in HH^0(\Pi_Q \otimes \mathbb{Q})$.

Proof. It follows from the definition that, for any derivation $\theta \in HH^1(\Pi_Q \otimes \mathbb{Q})$, $z \cup \theta$ can be represented by the derivation $z \cup \theta(g) = z\theta(g)$ for all $g \in \Pi_Q \otimes \mathbb{Q}$. Thus, the first formula is immediate. For the second, it suffices to show that the two sides are the same when evaluated on the center $HH^0(\Pi_Q \otimes \mathbb{Q})$ (cf. the proof of Proposition 10.2.1). That is, we need to show that, for $A, B, C \in HH^0(\Pi_Q \otimes \mathbb{Q})$,

$$A\{B, C\} = \frac{|B|}{|A| + |B|} \{AB, C\} - \frac{|C|\{A, B\}}{|A| + |B|} \{A, B\} C,$$ \hfill \eqref{10.2.13}

which, using the Leibniz rule, can be rewritten as

$$Eu(A)\{B, C\} + Eu(B)\{C, A\} + Eu(C)\{A, B\} = 0.$$ \hfill \eqref{10.2.14}
It is enough to show that this formula holds after tensoring by \( \mathbb{C} \), when we may identify \( HH^0(\Pi_Q \otimes \mathbb{C}) \) with \( \mathbb{C}[x, y]^\Gamma \) with the usual Poisson bracket. It clearly suffices to show the formula for \( \mathbb{C}[x, y] \) with the usual bracket. Let \( \det(x^ay^b, x^cy^d) := ad - bc \). We then need to show that, for monomials \( A, B, C \in \mathbb{C}[x, y] \),

\[
|A| \det(B, C) + |B| \det(C, A) + |C| \det(A, B) = 0,
\]

which is just the identity (for \( A = x^ay^b, B = x^cy^d, C = x^my^n \))

\[
\det \begin{pmatrix} a + b & c + d & m + n \\ a & c & m \\ b & d & n \end{pmatrix} = 0.
\]

\( \square \)

### 10.2.2 \( HH^1(\Pi_Q \otimes \mathbb{k}) \) where \( \mathbb{k} \) is torsion-free and \( Q \) is extended Dynkin

We may use the above formula to deduce the structure of \( HH^1(\Pi_Q \otimes \mathbb{k}) \) and the cup product with \( HH^0(\Pi_Q \otimes \mathbb{k}) \) in the case that \( \mathbb{k} = \mathbb{Z}[\frac{1}{|\Gamma|}, e^{\pi i}] \) (or a field in good characteristic containing the \(|\Gamma|\)-th roots of unity):

**Definition 10.2.17.** Let the “half Euler derivation” be defined by

\[
HEu := \sum_{e \in Q} e \frac{d}{de}.
\]

Note that \( HEu \) just multiplies a path by the number of edges from \( Q \) that it contains. It is straightforward to verify that \( HEu(r) = 0 \), and hence \( HEu \) defines an element of \( HH^1(\Pi_Q) \) for any quiver. Also, if \( Q \) is a tree, or a bipartite graph with all edges oriented from a fixed subset to the other, then \( Eu = 2HEu \). If \( Q \) is any quiver such that \( HH_0(\Pi_Q)[2] \otimes Q = 0 \), it follows that \( Eu \equiv 2HEu \) as elements of \( HH^1(\Pi_Q) \).

**Proposition 10.2.19.** Let \( Q \) be an extended Dynkin quiver with associated group \( \Gamma \).

(i) Over \( \mathbb{k} = \mathbb{Z}[\frac{1}{|\Gamma|}] \), there exists a (noncanonical) linear map \( \Theta : HH_0(\Pi_Q \otimes \mathbb{k}) \rightarrow HH^0(\Pi_Q \otimes \mathbb{k}) \cdot Eu \) such that we have the decomposition

\[
HH^1(\Pi_Q \otimes \mathbb{k}) \simeq \langle Eu \rangle \oplus \langle \frac{D[f] - \Theta[f]}{|f|} \rangle_{[f] \in HH_0(\Pi_Q \otimes \mathbb{k}) \text{ homogeneous}},
\]

and the formula

\[
z \cup \frac{D[f] - \Theta[f]}{|f|} \equiv \frac{D[zf] - \Theta[zf]}{|z| + |f|} \pmod{HH^0(\Pi_Q \otimes \mathbb{k}) \cup Eu}.
\]

(ii) Over \( \mathbb{Z} \), \( (10.2.20) \) may be corrected to

\[
HH^1(\Pi_Q) \simeq \langle HEu \rangle \oplus \langle \lambda[f] \frac{D[f] - \Theta[f]}{|f|} \rangle,
\]

for some \( \Theta : HH_0(\Pi_Q) \rightarrow HH^1(\Pi_Q) \cdot HEu, \quad \lambda[f] \in \mathbb{Z} \),

and where \( [f] \) ranges over a particular homogeneous \( \mathbb{Z} \)-basis of \( HH_0(\Pi_Q)_{\text{free}} \). Here, if \( Q = \tilde{A}_{n-1} \), then \( \lambda[f] = 1 \) for all but a one-parameter family of basis elements, and for \( Q \) of type \( \tilde{D} \) or \( \tilde{E} \), then \( \lambda[f] = 2 \) for all but a one-parameter family of basis elements (in these cases \( 2 \mid \lambda[f] \) for all \( [f] \)).
Proof. (i) Existence of $\Theta$ such that the above elements lie in $HH^1(\Pi_Q \otimes k)$ (i.e., such that $D[f] - \Theta[f]$ is divisible by $|f|$ in $HH^1(\Pi_Q \otimes k)$) follows in the case that $Q$ is not of type $\tilde{A}$ from (10.2.12) and the fact that generators of $HH^0(\Pi_Q \otimes k)$ as an algebra lie in degrees which are factors of some power of $[\Gamma]$. In the $A_{n-1}$ case, we will say more (and the result already follows if we replace $k$ with $k[z]$).

It remains to show that the above elements span $HH^1(\Pi_Q \otimes k)$ over $k$. To do this, we use the fact that $HH^1(\Pi_Q \otimes k) \cong HH^1(k[x,y])^\Gamma$, by Proposition 3.2.4. So, if we prove the given result for the latter, then the explicit identifications of Proposition 3.2.4 give us the result for the former. But $HH^1(k[x,y]) = k[x,y]\langle \frac{d}{dx}, \frac{d}{dy} \rangle$, and it is easy to compute that the above is a correct decomposition. So we deduce the result. Also, (10.2.21) is an immediate consequence of (10.2.12).

Let us work out the decomposition explicitly on the level of $k[x,y]$, which will give us the $\tilde{A}_{n-1}$ case (and more). Setting $Eu := x \frac{d}{dx} + y \frac{d}{dy}$, and $HEu := x \frac{d}{dx}$ in this case, consider the formula

$$\Theta(z) := \frac{HEu(z)}{xy}, \quad \text{if } y \mid z, \quad \Theta(x^n) = 0, \quad (10.2.23)$$

which we extend linearly to all of $k[x,y]$. We then have the following strengthened version of (10.2.21):

$$z_2 \cup D[z] - \Theta[z] = D[z_2] - \Theta[z_2], \quad \text{if } y \mid z; \quad (10.2.24)$$

$$z_2 \cup D[x^n] = D[z_2] - \Theta[z_2], \quad \frac{HEu(z_2)}{xy} + \delta_{y|z_2} \frac{x^n z_2}{y} \cdot Eu. \quad (10.2.25)$$

It follows immediately from the above (since $D[z]$ has order $|z|$ modulo $Eu$) that $HH^1(k[x,y])$ is spanned by $k[x,y] \cdot Eu$ and $(\frac{D[z]-\Theta[z]}{|z|})_{z \in k[x,y]}$.

We prove the formula (10.2.24), from which (10.2.25) (the only reason (10.2.25) is needed is to avoid terms of the form $\frac{x^n}{y}$). We have, by (10.2.12),

$$z_2 \cup D'(z) = \frac{|z|D[z_2]}{|z|(|z| + |z_2|)} - \frac{HEu(z_2)}{xy} + \frac{\{z, z_2\}}{|z|(|z| + |z_2|)} \cdot Eu, \quad (10.2.26)$$

so it remains to show that (for $z = x^a y^b$, $z_2 = x^c y^d$):

$$a + c \quad \text{if } y \mid z; \quad (10.2.27)$$

$$\frac{a + c}{a + b + c + d} = \frac{ad - bc}{(a + b)(a + b + c + d)},$$

or that

$$\det \begin{pmatrix} a & a + c \\ a + b & a + b + c + d \end{pmatrix} = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad (10.2.28)$$

which follows from a row and a column operation.

It follows that, for any quiver $Q$ with associated group $\Gamma$, setting $k = \mathbb{Z}[\frac{1}{\Gamma}, e^{\frac{2\pi i}{|\Gamma|}}]$, we can compute $HH^1(\Pi_Q \otimes k)$ by finding the $\Gamma$-invariant elements in the span of the above $k[x,y] \cdot y \cup D[z]-\Theta[z]$ with $\Gamma$. We also remark from (10.2.24) that, if we adjoined the elements $HH^0(\Pi_Q \otimes k)^{-1} \cdot Eu$ to $HH^1(\Pi_Q \otimes k)$, then we would be able to pick $\Theta$ so that (10.2.21) is an equality (by extending the first formula of (10.2.23) to the case $y \mid z$, and hence obtaining (10.2.24) for all $z$).
Finally, for $Q = \tilde{A}_{n-1}$, we may use the same formulas as in the $k[x, y]$ case, where $x$ is now the sum of counterclockwise edges (=$\text{edges of } Q$) and $y$ is the sum of clockwise edges (=$\text{edges of } Q^*$), in the notation preceding Theorem 10.2.40. It follows from (10.2.24) that $HH^1(\Pi_Q \otimes k)$ is in this case spanned by $k[x, y] \cdot Eu$ and $\frac{D[z] - \Theta[z]}{|z|}$ for $z \in k[x, y]^\Gamma = k[x^n, y^n, xy]$. This completes the proof.

(ii) We extend to $k = \mathbb{Z}$ as follows. In the $\tilde{A}_{n-1}$ case, restricting $z$ to range over monomials in $x$ and $y$, the only correction needed is that $\frac{D[z] - \Theta[z]}{|z|}$ must be multiplied by $\lambda_{[z]} := n$ in the case that $z \in \{x^{mn}, y^{mn}\}$ is a concatenation of long cycles (and in these cases, $\Theta[z] = 0$). For other basis elements $z = x^ay^b$ with $a, b \neq 0$, we set $\lambda_{[z]} = 1$. It then follows from the reasoning in the proof and the Poisson bracket for $\tilde{A}_{n-1}$ that the resulting elements are linearly independent over $\mathbb{F}_p$ for any prime, including the bad primes, and thus give a $\mathbb{Z}$-basis for $HH^1(\Pi_Q)$. Note that, in this case, we can continue to use $Eu$ rather than $HEu$ (since $HEu = \frac{-D(xy) + Eu}{2}$).

In the $\tilde{D}$ and $\tilde{E}$ cases, we need $\Theta$ to be a map $HH_0(\Pi_Q)_{\text{free}} \to HH^0(\Pi_Q) \cdot HEu$ rather than $Eu$, since all elements have even degree. For the same reason, we need $2 \mid \lambda_{[f]}$ for all $[f]$. In (10.2.12), we will always have $2 \mid \text{gcd}([f], |z|)$, and most elements of $HH^0(\Pi_Q)$ can be written as a product $fz$ with $2 = \text{gcd}([f], |z|)$; so by using cup product of $HH^0(\Pi_Q)$ with $D(HH_0(\Pi_Q))$ we will in general be able to obtain $\lambda_{[z]} = 2$ for most basis elements $[z]$. We specialize to the cases of $\tilde{D}$ and $\tilde{E}$ to complete the description.

For $Q = \tilde{D}_n$, restricting $z$ to be an element of the form $[z_{c, R, i_0, i_0}]$, we multiply $\frac{D[z] - \Theta[z]}{|z|}$ by $4$ in the case that $z = x_{c, 0, R, i_0, i_0}$ and by $2(n - 2)$ in the case $z = z_0, c, R, i_0$ (here $i_0 = i_{LU}$, say). In these exceptional instances, we again have $\Theta[z] = 0$. The other basis elements (as in Theorem 10.2.38) can be equipped with $\lambda_{[z]} = 2$ for suitable choices of $\Theta$.

For $Q = \tilde{E}_n$, we use the notation $X, Y, Z, d$ of Proposition 8.4.1. We restrict $z$ to be an element of the form $X^aY^b$, $X^aY^bZ$, or $X^aY^bZ^2$ (the latter are not needed rationally or in good primes, but we will need them here). Then, we need to multiply $\frac{D[z] - \Theta[z]}{|z|}$ by $|X|$ in the case $z = X^a$, by $|Y|$ in the case $z = Y^b$, by $\text{gcd}(|X|, |Y|)$ in the case $z = X^aY^b$ and $n \neq 6$ (for $n = 6$ this gcd is 2), and by $|Z|$ in the case $z = Z$. Note that the exceptional cases can be limited to a one-parameter family (using the cases $z = X^aY^bZ^2$), as in the $\tilde{A}$ and $\tilde{D}$ cases. Finally, although the above elements do not form a basis (only a spanning set), we can easily reduce it to a basis.

We omit the remaining details of the above verification, which can be done using the explicit formulas for Poisson bracket (Propositions 6.4.2, 7.4.1, and 8.4.1). The only thing to check is that $\Theta$ exists making the above linearly independent derivations on $HH^0(\Pi_Q) \otimes \mathbb{F}_p$ when $p \mid |\Gamma|$.

Note the above also gives a generalization of Lemma 9.2.24 in the proof, we saw not only that the given elements $HEu, \lambda_{[f]} \frac{D[ar{f}] - \Theta[ar{f}]}{|\bar{f}|}$ form a $\mathbb{F}_p$-basis of $HH^1(\Pi_Q) \otimes \mathbb{F}_p$ (and hence any nontrivial $\mathbb{F}_p$-linear combination does not vanish), but that these elements do not vanish when restricted to the center; so they do not kill $i_0\Pi_Qi_0 \otimes \mathbb{F}_p$. As a special case, restricting to the span of fractions of $D[HHe(\Pi_Q)]$, we get Lemma 9.2.24 as follows: the basis elements $[f]$ for which $p \nmid \Theta[f]$ must not be divisible by $p$, and the ones for which $p \mid \Theta[f]$ have the property that $p \mid \frac{|z|}{\lambda_{[z]}}$ iff $z$ is a $p$-th power.

We will further generalize this in Theorem 10.2.39, where we will see that $HH^1(\Pi_Q \otimes k)$ is in fact isomorphic to $HH^1(HH^0(\Pi_Q \otimes k)) \geq 0$ (we don’t need the $\geq 0$ if $k$ does not contain $p$-torsion for $p$ a stably bad prime).
10.2.3 \( HH^1(\Pi_Q \otimes k) \) when \( Q \) is non-Dynkin, non-extended Dynkin, over \( \mathbb{Z} \) Thanks to Theorem 9.2.2(i) and the fact that \( HH^1(\Pi_Q \otimes \mathbb{Q}) \) is mostly spanned by the image of \( D \), the non-Dynkin, non-extended Dynkin case is easier than the extended Dynkin one. We may work over \( \mathbb{Z} \) without exceptions:

**Proposition 10.2.29.** If \( Q \) is non-Dynkin, non-extended Dynkin, one has

\[
HH^1(\Pi_Q) \cong \langle HEu \rangle \oplus \left( \frac{D[f_n]}{n} \right)_{f \in \Pi_Q}.
\]  \hspace{1cm} (10.2.30)

**Proof.** We know that \( \langle HEu \rangle \oplus D[HH_0(\Pi_Q)] \) injects into \( HH^1(\Pi_Q) \) yielding a rational isomorphism. First, the saturation of \( D[HH_0(\Pi_Q)] \) is equal to \( \frac{D[f_n]}{n} \) since these certainly define derivations, and by Proposition 9.2.20, if we tensor by \( F \) (cf. (10.2.4)) such that \( F \) yields a nonzero derivation over \( e \) coefficients, \( \langle HEu \rangle \) injects into \( HH^1(\Pi_Q) \). We may consider the class \( \langle HEu \rangle \oplus \left( \frac{D[f_n]}{n} \right)_{f \in \Pi_Q} \) and \( HEu \) is a multiple of any integer, so the space is saturated.

It remains only to show that \( HH^1(\Pi_Q)/(\frac{D[f_n]}{n}) \) is spanned by the single element \( HEu \). This follows from the computation of the Connes differential in the next section, which just sends \( HEu \) to 1 \( \in HH^0(\Pi_Q) \) and kills \( \left( \frac{D[f_n]}{n} \right) \); however, we give a direct argument: \( HEu + D[f] + \frac{1}{2} D[g^2] \) is not a multiple of any integer \( > 2 \) for any \( f \in \Pi_Q[2], g \in \Pi_Q[1] \), because as a linear operator on \( \langle \overline{Q} \rangle \), it is not a multiple of any integer \( > 2 \). In particular, in terms of the basis \( Q \), the sum of matrix coefficients \( e \mapsto e \) and \( e^* \mapsto e^* \) is 1 for any \( e \in Q \). \( \square \)

10.2.4 \( HH^1(\Pi_Q \otimes k) \) in the case \( k \) has torsion In the case that \( k \) may contain torsion elements, the third direct summand of \( 10.1.3 \) may be nontrivial. In fact, the decomposition is apparently noncanonical: the canonical part is the first two direct summands, which compose the image of the first map in the following exact sequence:

\[
HH^1(\Pi_Q) \otimes k \hookrightarrow HH^1(\Pi_Q \otimes k) \twoheadrightarrow HH^1(\Pi_Q \otimes k)_3 \cong \text{Ext}^1_2(HH_0(\Pi_Q), k),
\]  \hspace{1cm} (10.2.31)

where \( HH^1(\Pi_Q \otimes k)_3 \) is (abstractly) the third direct summand of \( 10.1.3 \). The cokernel of the map \( HH^1(\Pi_Q) \otimes k \hookrightarrow HH^1(\Pi_Q \otimes k) \) is spanned by elements \( F \otimes k \), where \( F \) is an operator of the form

\[
F := \sum_{e \in Q} F_e \cdot \iota(e), \quad F_e \in \Pi_Q
\]  \hspace{1cm} (10.2.32)

(cf. 10.2.31) such that

\[
\sum_{e \in Q} [e, F_e] \in p\Pi_Q,
\]  \hspace{1cm} (10.2.33)

and we need exactly one element in each degree where \( HH_0(\Pi_Q)_{tor} \otimes k \neq 0 \) (degrees \( 2p^k \) for \( Q \) non-Dynkin, non-extended Dynkin), mapping to a nonzero element in the cokernel.

Let \( k = \mathbb{F}_p \) for some prime \( p \). We may explicitly realize the projection

\[
HH^1(\Pi_Q \otimes \mathbb{F}_p) \to HH_0(\Pi_Q)_{tor} \otimes \mathbb{F}_p
\]  \hspace{1cm} (10.2.34)

(the second map of 10.2.31) as follows. For any element \( \theta \in HH^1(\Pi_Q \otimes \mathbb{F}_p) \), we may lift it to a derivation \( \tilde{\theta} of P_{\overline{Q}} \), by picking \( \tilde{\theta}(e) \) for all \( e \in \overline{Q} \). It follows that \( \tilde{\theta}(r) \in \langle (r) \rangle + p\mathbb{F}_p \). Let \( \overline{\theta(r)} \) be its image. We may consider the class \( \left[ p \overline{\theta(r)} \right] \in HH_0(\Pi_Q)_{tor} \otimes \mathbb{F}_p \). This is well-defined, since
It is clear that a splitting exists (by our description of the cokernel, or the Universal Coefficient Theorem). Also, for some $Q,k$, and some choices of degrees $m$, there is a canonical splitting: this is when $HH^1(\Pi_Q)[m] \otimes k = 0$ but $HH^1(\Pi_Q \otimes k)_3[m] \neq 0$. In the extended Dynkin cases, this is true when $Q = \tilde{E}_6$ for all $m$ with $HH^1(\Pi_Q \otimes k)_3[m] \neq 0$ except $m = 6, Q = \tilde{E}_6$ and $m = 8, 16, Q = \tilde{E}_7$.

We may explicitly describe any splitting in the extended Dynkin cases as follows:

**Proposition 10.2.35.** Let $k = \mathbb{F}_p$ for some stably bad prime $p$ for an extended Dynkin quiver $Q$. Any $k$-splitting $\sigma: HH^1(\Pi \otimes k)_3 \hookrightarrow HH^1(\Pi_Q \otimes k)$ of (10.2.31) has the property that, for all $\theta \in HH^1(\Pi_Q \otimes k)_3$, there exists $z_1, z_2 \in HH^0(\Pi_Q \otimes k)$ such that $z_1 \cup \sigma(\theta) = z_2 \cup HEu$.

**Proof.** Given that these elements act nontrivially on the center, the result follows from the fact that, for stably bad primes $p$, exactly one of the generators $X, Y, Z$ of $HH^0(\Pi_Q \otimes k)$ has degree which is not a multiple of $2p$ (i.e., has Q-degree which is not a multiple of $p$). We verify that they act nontrivially in Examples 10.2.36, 10.2.37, and 10.2.38 (where we also compute cup products of $HH^0(\Pi_Q \otimes k)$ with $HH_0(\Pi_Q \otimes k)$). \qed

We now explain in detail how the elements $\theta_{r(2)}$ act on the center $HH^0(\Pi_Q)$ in the extended Dynkin cases. In the process, we also explain the cup product formulas from Theorem 10.2.60. We omit some of the verifications (which are straightforward).

**Example 10.2.36.** One choice of the derivation $\theta_{r(2)}$ associated with the torsion element $r(2)$ acts on $HH^0(\Pi_Q \otimes \mathbb{F}_2)$ for $Q = \tilde{D}_n$ by $\theta_{r(2)}(X) = 0, \theta_{r(2)}(Y) = (n-2)Z$, and $\theta_{r(2)}(Z) = (n-1)XY$; the other choice is this one plus $D[X]$. In the $\tilde{E}$ cases there is only one choice, which acts for $Q = \tilde{E}_6$ by $\theta_{r(2)}(X) = Y, \theta_{r(2)}(Y) = \theta_{r(2)}(Z) = 0$; for $Q = \tilde{E}_7$ and $\tilde{E}_8$ it acts by $\theta_{r(2)}(X) = \theta_{r(2)}(Y) = 0$ and $\theta_{r(2)}(Z) = XY$. That is, in the $\tilde{D}_n$ case, $\theta_{r(2)}$ can be chosen to act on the center as $\frac{X}{Y} HEu$; in the $\tilde{E}_6$ case, it acts on the center as $\frac{X}{Y} HEu$, and in the $\tilde{E}_7, \tilde{E}_8$ cases, $\theta_{r(2)}$ acts as $\frac{X}{Y} HEu$. There is no $\theta_{r(2)}$ in the $\tilde{A}_{n-1}$ cases.

Note: The derivation $\theta_{r(2)}$ can also be simply described (up to inner derivations) on $\Pi_Q \otimes \mathbb{F}_2$, but it does NOT act as a multiple of $HEu$ in general: that is, while $\theta_{r(2)}$ is a fraction of $HEu$ on the nose when restricted to the center, in general it is only true that $z_1 \cup \theta_{r(2)} - z_2 \cup HEu$ is an inner derivation for suitable choices of $z_1, z_2$.

We see that cup products with $\theta_{r(2)}$ also act nontrivially on the center, and it is easy to obtain all the required derivations this way. In the $\tilde{D}_n$ case, the third direct summand is given by $\langle X^m \cup \theta_{r(2)} \rangle 4 \leq m \leq n-2$; in the $\tilde{E}_6$ case, $\langle \theta_{r(2)} \rangle$; for $\tilde{E}_7$, $\langle \theta_{r(2)}, \frac{X}{Y} \cup \theta_{r(2)}, Y \cup \theta_{r(2)} \rangle$; and for $\tilde{E}_8$, $\langle \theta_{r(2)}, \frac{X^2}{Y} \theta_{r(2)}, X \cup \theta_{r(2)}, X \cup \theta_{r(2)}, Y \cup \theta_{r(2)} \rangle$. This explains the necessity of the top-degree torsion element of degree 28 for $\tilde{E}_8$: if it were not for this element (i.e., if it were not true that $X \cup r(2) \neq 0$), then $r(4)$ and $r(2)$ could not be related by the cup product (hence, neither could $\theta_{r(2)}$ and $\theta_{r(4)}$, a contradiction).

**Example 10.2.37.** The derivation $\theta_{r(3)}$ acts on $HH^0(\Pi_Q \otimes \mathbb{F}_3)$ for $Q$ of type $\tilde{E}$ as follows: for $\tilde{E}_6$ it may be chosen to act as $\frac{X}{Y} HEu$; in the $\tilde{E}_7$ it acts as $\frac{Y}{X} HEu$; and in the $\tilde{E}_8$ case it acts as $\frac{X}{Y} HEu$. All of $HH^1(\Pi_Q \otimes k)_3$ can be described in the $\tilde{E}_6, \tilde{E}_7$ cases as $\langle \theta_{r(3)} \rangle$ and in the $\tilde{E}_8$ case as $\langle \theta_{r(3)}, Y \cup \theta_{r(3)} \rangle$.

**Example 10.2.38.** The derivation $\theta_{r(5)}$ acts on $HH^0(\Pi_Q \otimes \mathbb{F}_5)$ for $Q = \tilde{E}_8$ as $\frac{Y}{X} HEu$, and $HH^1(\Pi_Q \otimes \mathbb{F}_5)_3$ is one-dimensional.
We noticed that the restriction of $HH^1(\Pi_Q \otimes k)$ to the center is always injective. In fact, we may deduce the following stronger result:

**Theorem 10.2.39.** For any extended Dynkin quiver $Q$ and any commutative ring $k$, the restriction $HH^1(\Pi_Q \otimes k) \rightarrow HH^1(\Pi_Q)$ (computed over $k$) is an isomorphism if $k$ contains no torsion in stably bad primes. In general, $HH^1(\Pi_Q \otimes k) \simeq HH^1(\Pi_Q \otimes k) \geq 0$, the nonnegatively graded part. Furthermore, $HH^1(\Pi_Q \otimes k)$ is a torsion-free $HH^0(\Pi_Q \otimes k)$-module.

**Proof.** Given the above, it only remains to check that all derivations of $HH^0(\Pi_Q \otimes k)$ come from derivations of $\Pi_Q \otimes k$. Let $k$ contain $\frac{1}{\Gamma}$ and $|\Gamma|$-th roots of unity, this follows from the isomorphism $HH^1(\Pi_Q \otimes k) \simeq HH^1(\Pi_Q)$, where the last isomorphism used the fact that vector fields on $\text{Spec}(k[x, y] / \Gamma)$ are the same as $\Gamma$-invariant vector fields on $\text{Spec}(k[x, y])$ (by the étale quotient map), and the same as vector fields on $\text{Spec}(k[x, y] / \Gamma)$ by the analogue of Hartogs’ Lemma. Furthermore, since $HH^0(\Pi_Q)$ is a free $\mathbb{Z}$-module, we have $HH^1(\Pi_Q) \simeq HH^1(\Pi_Q) \otimes \mathbb{C}$, so the map $HH^1(\Pi_Q) \rightarrow HH^1(\Pi_Q) \otimes \mathbb{C}$ must be a rational isomorphism. Since the map $HH^1(\Pi_Q) \otimes \mathbb{F}_p \rightarrow HH^1(\Pi_Q) \otimes \mathbb{F}_p$ is injective for all $p$, it follows that the map $HH^1(\Pi_Q) \rightarrow HH^1(\Pi_Q) \otimes \mathbb{F}_p$ is injective for all $p$. We may do this as follows: given the presentation $\mathbb{F}_p[X, Y, Z]/\langle \langle F(X, Y, Z) \rangle \rangle$ of $HH^0(\Pi_Q \otimes \mathbb{F}_p)$ (given in Propositions 10.1.3, 10.2.3, and 10.2.4), we may consider linear solutions $(g_X, g_Y, g_Z)$ of the equation

$$g_X \frac{\partial}{\partial X} F(X, Y, Z) + g_Y \frac{\partial}{\partial Y} F(X, Y, Z) + g_Z \frac{\partial}{\partial Z} F(X, Y, Z) = 0. \quad (10.2.40)$$

It is enough to find solutions of (10.2.40) over $\mathbb{F}_p[X, Y, Z]$, since solutions over $\mathbb{F}_p[X, Y, Z]/\langle \langle F(X, Y, Z) \rangle \rangle$ are sums of solutions over $\mathbb{F}_p[X, Y, Z]$ and solutions over $\mathbb{Z}[X, Y, Z]/\langle \langle F(X, Y, Z) \rangle \rangle$; the latter are all obtainable from $HH^1(\Pi_Q)$. These linear solutions may be easily seen to come from elements described above: we may restrict to the case that $p \nmid |\Gamma|$, although this is not necessary. Only in the stably bad prime cases is it possible to get solutions not arising from $HH^0(\Pi_Q) \otimes \mathbb{F}_p$-combinations of $D[X], D[Y]$, and $D[Z]$, and in these cases one of the terms in (10.2.40) vanishes and the other is easy to match possible new solutions with the image of $HH^1(\Pi_Q \otimes \mathbb{F}_p)$ as computed earlier, provided that $|g_X| \geq |X|, |g_Y| \geq |Y|$, and $|g_Z| \geq |Z|$. If we do not require this, then the map is not quite surjective (e.g., the map does not include $\frac{1}{\Gamma} HEu$ for the unique $W \in \{X, Y, Z\}$ such that $p \nmid |W|$ in the stably bad prime cases—these derivations have negative degree.)

**10.2.5 Summary of $HH^1(\Pi_Q \otimes k)$** We described $HH^1(\Pi_Q \otimes k)$ abstractly in (10.1.3), and then explicitly identified the first two direct summands with derivations in (10.2.20), (10.2.22), and (10.2.30). Then, we described the third direct summand as a quotient in (10.2.31) and (10.2.34), and as a summand in the extended Dynkin case using Proposition 10.2.35 and Examples 10.2.36 and 10.2.37, and 10.2.38.

In general, we will use the notation $\theta_{p(f)}$ (or $f$) for a $p$-torsion element $f \in HH_0(\Pi_Q)$ to refer to a chosen generator of $HH^1(\Pi_Q \otimes \mathbb{F}_p)$ mapping to $[f] \otimes \mathbb{F}_p$ under (10.2.34). In the non-Dynkin, non-extended Dynkin cases, we can at least choose this element to be the image of an element $f \in HH_0(\Pi_Q)$ where $HEu(f) = \frac{E_0(f)}{2}$ and $f$ is homogeneous.

Then, for arbitrary $k$, we may write a $\mathbb{F}_p$-basis of $HH^1(\Pi_Q \otimes k)$ by elements $\lambda \theta_{p(f)}$ (in the non-Dynkin, non-extended Dynkin case) and by certain elements $\lambda \theta_{z_2 HEu}$ for $z_1, z_2 \in HH_0(\Pi_Q)$ (in the extended Dynkin case); here $\lambda \in k$. 90
10.3 The BV algebra structure. We describe the BV structure on $HH^\bullet(\Pi_Q \otimes k)$ in terms of Corollary 10.1.2 (cf. Section 1.2 for the definition of BV algebra structure). This includes all of the structure of $D$ (the conjugate of the Connes differential by the duality isomorphisms $HH^\bullet(\Pi_Q \otimes k) \cong HH_{2-*}(\Pi_Q \otimes k)(-2)$), the associative and graded-commutative $\cup$, and the Gerstenhaber bracket (in particular including the necklace bracket in degree one, as we will see).

Theorem 10.3.1. Let $Q$ be a non-Dynkin quiver. The BV structure on $HH^\bullet(\Pi_Q \otimes k)$ is as follows, in terms of Corollary 10.1.2, Proposition 10.2.1, and Section 10.2.5:

1. The differential (obtained from the Connes differential) $D : HH^\bullet(\Pi_Q \otimes k) \to HH^{\bullet-1}(\Pi_Q \otimes k)$ is given as follows:

(a) The map $D : HH^1(\Pi_Q \otimes k) \to HH^0(\Pi_Q \otimes k)$ acts as follows:

\[ D(Eu) = 2, \quad D(HEu) = 1, \quad D(DHH^0(\Pi_Q \otimes k))] = 0, \quad D(HH^1(\Pi_Q \otimes k))_3 = 0. \] (10.3.2)

\[ D(z \cup \theta) = \theta(z) + zD(\theta), \forall z \in HH^0(\Pi_Q \otimes k), \theta \in HH^1(\Pi_Q \otimes k). \] (10.3.3)

In particular, for $k = \mathbb{Z}$, the kernel is spanned by $\langle \frac{D[\ell^m]}{m} \rangle_{m \geq 1}$.

(b) The map $D : HH^2(\Pi_Q \otimes k) \to HH^1(\Pi_Q \otimes k)$ is the formula (10.2.3), mapping to the second direct summand. Over $k = \mathbb{Z}$, the kernel is given by $HH_0(\Pi_Q)_{tor} \otimes k$, and over $Q$ the map is an isomorphism onto the second direct summand.

2. The (graded-commutative) cup product is given as follows:

(a) $HH^0(\Pi_Q \otimes k) \otimes HH^0(\Pi_Q \otimes k) \to HH^0(\Pi_Q \otimes k)$ is just multiplication;

(b) $HH^0(\Pi_Q \otimes k) \otimes HH^1(\Pi_Q \otimes k) \to HH^1(\Pi_Q \otimes k)$ is the map $z \cup \theta(f) = z \cdot \theta(f), \forall f \in \Pi_Q \otimes k$, and is explicitly given in terms of the decomposition (10.1.3) using (10.2.12) and Examples 10.2.36, 10.2.37, and 10.3.2.

(c) $HH^0(\Pi_Q \otimes k) \otimes HH^2(\Pi_Q \otimes k) \to HH^2(\Pi_Q \otimes k)$, via the identification $HH^2(\Pi_Q \otimes k) \cong HH_0(\Pi_Q \otimes k)(-2)$, is the multiplication inside commutators: $z \cup [f] = [zf]$.

(d) $HH^1(\Pi_Q \otimes k) \otimes HH^1(\Pi_Q \otimes k) \to HH^2(\Pi_Q \otimes k)$ is given in terms of the decomposition (10.1.3) by:

1. Multiplication of the first direct summand with itself is zero. In the extended Dynkin case, multiplication of the first and third direct summands with themselves and each other is zero.

2. The multiplication $HH^1(\Pi_Q \otimes k) \otimes HH_0(\Pi_Q)_{tor}(0) \otimes k \to HH_0(\Pi_Q \otimes k)(-2)$ is given by applying the given derivation.

Proof. (1) The differential $D$ is defined to be the dual of the Connes differential $B$ using (10.1.11) (cf. [CBEG07, Gin06]). The results will follow from (10.1.11) and the general formula for the Connes differential $B : HH_0 \to HH_1, HH_1 \to HH_2$ on the level of normalized Hochschild chains:

\[ B(a) = 1 \otimes a, \quad B(a \otimes b) = 1 \otimes a \otimes b - 1 \otimes b \otimes a. \] (10.3.4)
Consider the following inclusion of complexes (the bottom is the normalized bar complex, and the top is (10.1.11)):

\[
\begin{array}{cccc}
0 & \to & \Pi_Q \otimes \langle r \rangle \otimes \Pi_Q & \to \Pi_Q \otimes \langle Q \rangle \otimes \Pi_Q & \to \Pi_Q \otimes \Pi_Q \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \Pi_Q \otimes ((\Pi_Q)_+ \otimes (\Pi_Q)_+) \otimes \Pi_Q & \to \Pi_Q \otimes (\Pi_Q)_+ \otimes \Pi_Q & \to \Pi_Q \otimes \Pi_Q.
\end{array}
\] (10.3.5)

This is a quasi-isomorphism, and there must exist an inverse \( \Phi \), such that \( \Phi_0 = \text{Id} \), which is unique up to chain-homotopy. One explicit realization of \( \Phi \) is given, for any basis of words \( a_1 \cdots a_m \) in \( Q \) of \( \Pi_Q \), by

\[
\Phi_1(x \otimes a_1 \cdots a_m \otimes y) = \sum_{i=1}^{m} x a_1 \cdots a_{i-1} \otimes a_i \otimes a_{i+1} \cdots a_m.
\] (10.3.6)

It is easy to verify that this defines a chain map and hence part of a quasi-isomorphism. To obtain complexes computing the Hochschild homology, we apply the functor \( \otimes \Pi_Q^k \) to \( \Pi_Q \), placing the module part (which now becomes \( \Pi_Q \)) on the left (by convention). We will let \( \Phi \) denote the induced maps on the level of Hochschild chains.

(1b) Using (10.3.4), we may compute

\[
\Phi_1(B(a_1 \cdots a_m)) = \Phi_1(1 \otimes a_1 \cdots a_m) = \sum_i a_{i+1} \cdots a_{i-1} \otimes a_i,
\] (10.3.7)

which goes via the vertical isomorphism in (10.1.11) (after applying \( \otimes \Pi_Q^k \) to the latter) to (10.2.3). The statement about kernel then follows from Theorem 9.2.2 (i).

(1a) We did not compute \( \Phi_2 \) explicitly since this is not so clear. However, we know that, restricted to \( \langle Q \rangle \otimes \langle Q \rangle \), we may express \( \Phi_2 \) by, for the unique \( e \in Q \) such that \( ee^* \) is not in our above basis in degree two,

\[
\Phi_2(a \otimes (v \otimes w) \otimes b) = \delta_{v,e} \delta_{w,e^*} \omega(e, e^*) a \otimes r \otimes b.
\] (10.3.8)

Then, using the second formula in (10.3.4), we have

\[
\Phi_2(B(\sum_{e \in Q} e \otimes e^*)) = \Phi_2(1 \otimes r) = 1,
\] (10.3.9)

so \( HEu = \sum_{e \in Q} e \frac{d}{de} \to \sum_{e} e \otimes e^* \) maps to 1. Similarly, \( Eu \) maps to 2.

Then, (10.3.3) follows immediately from the definition of Gerstenhaber bracket (so that \( \{ \theta, z \} = \theta(z) \)) and the BV identity (1.2.4). Similarly, \( D(D[f]) = 0 \) follows from the fact that \( D^2 = 0 \) (again a BV identity). Next, we show that \( D(HH^1(\Pi_Q \otimes k)_3) = 0 \), i.e., that \( D(\theta[f]) = 0 \) where \( [f] \in HH_0(\Pi_Q)_{tor} \otimes k \) and \( \theta[f] \) is the associated derivation. In the case that \( Q \) is not extended Dynkin, \( HH^0(\Pi_Q \otimes k) = k \), concentrated in degree zero, whereas \( HH^1(\Pi_Q \otimes k)_3 \) lies in degrees \( \geq 2 \), yielding the desired result. Now, assume that \( Q \) is extended Dynkin. We may assume \( k = \mathbb{F}_p \) for some prime \( p \). Using Proposition 10.2.35, we may write \( \theta[f] = \frac{z_1}{z_2} HEu \) for some \( z_1, z_2 \in HH^0(\Pi_Q \otimes k) \), which means that \( \theta[f] \) is uniquely characterized by \( z_2 \cup \theta[f] = z_1 \cup HEu \).

We do this computation using (10.3.3):

\[
D(z_1 HEu) = D(z_2 \cup \left( \frac{z_1}{z_2} HEu \right)) = \{ z_2, \frac{z_1}{z_2} HEu \} + z_2 \cdot D\left( \frac{z_1}{z_2} HEu \right)
= \frac{z_1}{z_2} HEu(z_2) \quad \text{and} \quad z_2 \cdot D\left( \frac{z_1}{z_2} HEu \right),
\] (10.3.10)
but also, $D(z_1 HEu) = HEu(z_1) + z_1$, and also, we have $HEu(z_i) = \frac{|z_i|}{2}z_i$, so we get

$$z_2 \cdot D(\frac{z_1}{z_2} HEu) = (\frac{|z_1|}{2} - \frac{|z_2|}{2} + 1)z_1,$$

(10.3.11)

but we know that $|z_1| - |z_2| + 2 = |[f]|$. But, $p \mid \frac{|f|}{2}$, and thus $z_2 \cdot D(\frac{z_1}{z_2} HEu) = 0$ (here we used that, since $Q$ is a tree, $Eu[g] = 2HEu[g]$ over $\mathbb{Z}$ for all $[g]$). Since multiplication by $z_2$ is injective in $HH^0(\Pi_Q \otimes k)$ (or, the latter is an integral domain), we deduce that $D(\frac{z_1}{z_2} HEu) = 0$.

It remains to prove the claim about the kernel of $D$. Over $\mathbb{Z}$, $HH^1(\Pi_Q)$ is a free $\mathbb{Z}$-module which is rationally spanned by $Eu$ and $D[HH_0(\Pi_Q)_+]$, so the kernel is just the saturation of $D[HH_0(\Pi_Q)_+]$, and this saturation must be $\langle \frac{D([m])}{m} \rangle_{m \geq 1, [f] \in HH_0(\Pi_Q)}$ by Theorem 9.2.2 (i) (cf. Proposition 10.2.29) the same argument holds in the extended Dynkin case).

(2) Parts (a) and (b) are immediate. Part (c) follows from the following formula, for $z \in HH^0(\Pi_Q \otimes k)$ and $\nu \in HH^2(\Pi_Q \otimes k)$:

$$z \cup \nu\left(\sum_{e \in Q} e \otimes e^* - e^* \otimes e\right) = z\nu\left(\sum_{e \in Q} e \otimes e^* - e^* \otimes e\right),$$

(10.3.12)

together with the explicit identification of $HH^2(\Pi_Q \otimes k)$ given in (10.1.11) and Proposition 10.2.1.

We show (2d). First, part (i) is immediate from graded commutativity and associativity, together with Proposition 10.2.35 $z_2w_2 + (\frac{z_1}{z_2} HEu) + (\frac{w_1}{w_2} HEu) = (z_1w_1) \cup (HEu \cup HEu) = 0$, where $w_1, w_2, z_1, z_2$ are central. Note that, if we are working over characteristic two, the graded commutativity does not immediately yield the desired result, but it must still hold since $HEu$ is in the image of $HH^1(\Pi_Q)$ over $\mathbb{Z}$, and there $HEu \cup HEu = 0$. Then, we use the injectivity of the multiplication of any element of $HH^0(\Pi_Q \otimes k)$ on $HH^0(\Pi_Q \otimes k)$, and the isomorphism of the latter with $HH_0(\Pi_Q)_\text{free} \otimes k$ as a $HH^0(\Pi_Q \otimes k)$ module (Corollary 5.5.11 and the following remarks).

Next, for $\theta, \xi \in HH^1(\Pi_Q \otimes k)$, we have, by the definition of cup product and the identification $HH^2(\Pi_Q \otimes k) \cong HH_0(\Pi_Q \otimes k)$,

$$\theta \cup \xi = \sum_{e \in Q}[\theta(e)\xi(e^*) - \theta(e^*)\xi(e)].$$

(10.3.13)

In the case that $\xi = D[f]$, this yields

$$\theta \cup D[f] = [\theta(f)],$$

(10.3.14)
as desired.

We may then write formulas for the Gerstenhaber bracket. All that remains is the bracket with $HH^2(\Pi_Q \otimes k)$:

**Corollary 10.3.15.** We have the following formulas for the Gerstenhaber bracket, for homogeneous $z \in HH^0(\Pi_Q \otimes k), \theta \in HH^1(\Pi_Q \otimes k)$, and $[f] \in HH^2(\Pi_Q \otimes k)$:

$$\{z, [f]\}_\text{Gerst} = -\frac{|z|D[zf] + \{[f], z\}E_u}{|z| + |f|}, \quad \{\theta, [f]\} = -[D(\theta) \cdot f] + D[\theta(f)].$$

(10.3.16)

(Note that the first formula is only nontrivial in the extended Dynkin case.)
As a consequence of the theorem, we immediately deduce that the saturation of \( D[HH_0(\Pi_Q)] \) (i.e., \( \langle D[f^n]/n \rangle \)) is closed under the Gerstenhaber bracket, which just becomes the necklace bracket (by the theorem and (1.2.4)). Precisely, we have

\[
\{[f],[g]\} = D[f] \cup D[g], \quad D\{[f],[g]\} = \{D[f],D[g]\}_{\text{Gerst}},
\]

so that the necklace Lie algebra \( \Lambda \otimes k \) is a central extension of the subalgebra \( D[HH^2(\Pi_Q)] \otimes k \subset HH^1(\Pi_Q \otimes k) \).

We may deduce the Jacobi identity for the necklace Lie algebra using the BV structure as follows. In view of (10.3.17), we have

\[
\{[f],\{[g],[h]\}\} = D[f] \cup \{D[g],D[h]\}_{\text{Gerst}},
\]

which translates the Jacobi identity for the necklace Lie algebra into the following identity for the BV algebra \( HH^*(\Pi_Q) \):

\[
D[f] \cup \{D[g],D[h]\} + D[g] \cup \{D[h],D[f]\} + D[h] \cup \{D[f],D[g]\} = 0,
\]

which follows from \( HH^3(\Pi_Q \otimes k) = 0 \), (1.2.4) and the fact that \( D \) is a second-order differential operator:

\[
D(A \cup B \cup C) = (D(A \cup B) \cup C - D(A) \cup B \cup C + \text{cyc}) + D(1)A \cup B \cup C,
\]

where “cyc” refers to summing over cyclic permutations of \( A, B, \) and \( C \), with the necessary signs so as to be compatible with graded commutativity. The identity (10.3.20) is equivalent to (1.2.4) if one assumes \( D^2 = 0 \) (cf. [Kos85, GSb]), so holds for all BV algebras. So, we deduce (10.3.19), using only that \( D[f], D[g], \) and \( D[h] \) are in the kernel of \( D \) and have degree adding to \( \geq 3 \).

It is also possible to incorporate the other structures we discussed in this paper, such as the \( p \)-th power maps and the Poisson structure on \( HH^0(\Pi_Q) \), into the BV algebra above. The former just gives a \( p \)-th power structure on \( HH^2(\Pi_Q \otimes \mathbb{F}_p)^{(2)} \). The latter may be described in the extended Dynkin case in terms of an additional “duality” map \( \alpha : HH^2(\Pi_Q \otimes k) \to HH^0(\Pi_Q \otimes k) \), given by killing torsion and stipulating that \( \alpha[i_0z] = z, \alpha[i] = d_i \) where \( i_0 \) is an extending vertex and \( i \in I \) is any vertex with \( d_i \) equal to the dimension of the corresponding representation of \( \Gamma \) (cf. Corollary 5.5.11). Then, we obtain that the Poisson structure on \( HH^0(\Pi_Q \otimes k) \) comes from the bracket

\[
\{z_1,z_2\}_{HH^0} := \alpha(D \circ \alpha^{-1}(f) \cup D \circ \alpha^{-1}(g)),
\]

since the map \( D \circ \alpha^{-1} : HH^0(\Pi_Q \otimes k) \to HH^1(\Pi_Q \otimes k) \) is well-defined.

### 10.4 Cyclic homology

Finally, we compute the cyclic homology of \( \Pi_Q \). Let \( Q \) be non-Dynkin. The Connes exact sequence, using the fact that \( HH_m(\Pi_Q) = 0 \) for \( m \geq 3 \), takes the form (cf. e.g., [Lod93]):

\[
\begin{align*}
0 & \to HC_{m+2}(\Pi_Q) \xrightarrow{\sim} HC_m(\Pi_Q) \to 0, \quad m \geq 2 \\
0 & \to HC_3(\Pi_Q) \xrightarrow{\sim} HC_1(\Pi_Q) \xrightarrow{B} HH_2(\Pi_Q) \xrightarrow{I} HC_2(\Pi_Q) \\
& \xrightarrow{\sim} HC_0(\Pi_Q) \xrightarrow{B} HH_1(\Pi_Q) \xrightarrow{I} HC_1(\Pi_Q) \to 0,
\end{align*}
\]

\[m \geq 2\]
Here, the maps $\bar{B}$ and $I$ factorize the Connes differential $B$:

$$0 \to HH_0(\Pi_Q) \xrightarrow{I} HC_0(\Pi_Q) \to 0. \quad (10.4.3)$$

Thus, using the correspondence of $B$ with $D$ via duality, we may compute $HC_\bullet$ using Theorem 10.3.1.

First, tensoring over $\mathbb{Q}$, we know the maps $B : HH_0(\Pi_Q) \xrightarrow{\bar{B}} HC_0(\Pi_Q) \to HH_1(\Pi_Q)$ and $B : HH_1(\Pi_Q) \to HC_1(\Pi_Q) \to HC_2(\Pi_Q)$ become injective (from the previous section), so we deduce that $HC_m(\Pi_Q)$ is torsion for $m \geq 2$—a fact which was also proved in [EG96] using the fact that $B$ forms an exact sequence over $\mathbb{Q}$ with $HC_\bullet(\Pi_Q \otimes \mathbb{Q})$ forming the kernels=images.

Working over $\mathbb{Z}$, the image of $B : HH_0(\Pi_Q) \to HH_1(\Pi_Q)$ is taken via duality to $\langle D[f] \mid f \in HH_0(\Pi_Q) \rangle$. Thus, the right end of (10.4.2) yields an isomorphism

$$HH^1(\Pi_Q)/D[HH^2(\Pi_Q)] \cong \langle HH^0(\Pi_Q) \rangle \oplus \langle D[f^m]/m \rangle / \langle D[f] \rangle \xrightarrow{i} HC_1(\Pi_Q \otimes k)(-2), \quad (10.4.5)$$

where the first term $\langle HH^0(\Pi_Q) \rangle$ is an abstract copy of $HH^0(\Pi_Q)$, which is a free $\mathbb{Z}$-module, and has a rational basis by $HH^0(\Pi_Q) \cdot HEu$. Also, the above map is obtained from $I : HH_1(\Pi_Q) \to HC_1(\Pi_Q)$ by precomposing with the duality isomorphism $HH^1(\Pi_Q)(2) \xrightarrow{\sim} HH_1(\Pi_Q)$ and shifting by $-2$. Using Theorem 10.3.1 (1a) (the kernel of $D$ on $HH^1(\Pi_Q \otimes k)$ is $\langle D[f^n]/n \rangle$), the factorization (10.4.1) and the leftmost part of (10.4.2) then yield

$$HC_3(\Pi_Q) \cong \langle D[f^m]/m \rangle / \langle D[f] \rangle. \quad (10.4.6)$$

Next, we compute $HC_2(\Pi_Q)$. The image of the map $HH_1(\Pi_Q) \xrightarrow{B} HH_2(\Pi_Q) \xrightarrow{\bar{B}} HH^0(\Pi_Q)$ is $\langle \theta(z) + D(\theta) \cdot z, \theta(z) \rangle \in HH^0(\Pi_Q), \theta \in HH^1(\Pi_Q)$. The kernel of the map $HH_0(\Pi_Q) \xrightarrow{\bar{B}} HC_0(\Pi_Q) \to HH_1(\Pi_Q) \xrightarrow{B} HH^1(\Pi_Q)$ is $HH_0(\Pi_Q)_{tor}$. Thus, taking cokernel of the second map of (10.4.2) and kernel of the fifth map gives an exact sequence

$$HH^0(\Pi_Q)/\langle \theta(z) + D(\theta) \cdot z \rangle \in HH^0(\Pi_Q), \theta \in HH^1(\Pi_Q) \to HC_2(\Pi_Q)(-2) \to HH_0(\Pi_Q)_{tor}(-2). \quad (10.4.7)$$

For degree reasons, we see that the above sequence must split.

Taking the above description and computing the abstract graded $\mathbb{Z}$-module structure, we obtain the following result:

**Theorem 10.4.8.** (i) If $Q$ is non-Dynkin, non-extended Dynkin, then we have

$$HC_0(\Pi_Q) \cong HH_0(\Pi_Q), \quad (10.4.9)$$

$$HC_1(\Pi_Q) \cong \mathbb{Z}(2) \oplus HC_3(\Pi_Q), \quad (10.4.10)$$

$$HC_{2n+2}(\Pi_Q) \cong HH_0(\Pi_Q)_{tor}, \forall n \geq 0, \quad (10.4.11)$$

$$HC_{2n+3}(\Pi_Q) \cong \bigoplus_{m \geq 0} \bigoplus_{d \mid m} (\mathbb{Z}/(m/d))^{\oplus q_d(m)}, \forall n \geq 0, \quad (10.4.12)$$

where $q_d$ is given by

$$q_d = \sum_{j \mid d} \mu(d/j) \text{rk}(HH_0(\Pi_Q)_{free}[j]), \quad (10.4.13)$$

95
(ii) If \( Q \) is extended Dynkin and \( k = \mathbb{Z}[\frac{1}{\Gamma}] \), then we have

\[
HC_Q(\Pi_Q) \cong HH_0(\Pi_Q),
\]

\[
HC_1(\Pi_Q) \otimes k \cong k \otimes HH^0(\Pi_Q)(2) + (HC_3(\Pi_Q) \otimes k),
\]

\[
HC_{2n+2}(\Pi_Q) \cong HH_0(\Pi_Q)_{\text{tor}} \oplus \bigoplus_{m \geq 0} \left( \bigoplus_{d|m, 1 < d < m, \gcd(m/d, |\Gamma|) = 1} (k/(m/d))^{\oplus r_m}[m] \right) \oplus (\mathbb{Z}/m)^{\oplus r_m}[m], \forall n \geq 0,
\]

\[
HC_{2n+3}(\Pi_Q) \cong \bigoplus_{m \geq 0} \left( \bigoplus_{d|m, 1 \leq d < m, \gcd(m/d, |\Gamma|) = 1} (k/(m/d))^{\oplus q_d}[m] \right), \forall n \geq 0,
\]

where \( q_d \) and \( r_m \) are given by

\[
q_d = \sum_{j|d, \gcd(d/j, |\Gamma|) = 1} \mu\left( \frac{d}{j} \right) \text{rk}(HH_0(\Pi_Q)_{\text{free}}[j]),
\]

\[
r_m = \text{rk}(HH_0(\Pi_Q)_{\text{free}}[\gcd(|\Gamma|, m)]) - \text{rk}(HH_0(\Pi_Q)_{\text{free}}[m]) + \text{rk}(HH_0(\Pi_Q)_{\text{free}}[m - 2]),
\]

Proof. (i) The first equation is obvious. We need to compute the abstract \( \mathbb{Z} \)-module structure of \( \langle D[f^n]/n \rangle / \langle D[f] \rangle \). We do this using Theorem 9.2.2(ii): since the \( p \)-th power maps are injective on \( HH_0(\Pi_Q)_{\text{free}} \otimes \mathbb{F}_p \), we can form a basis of \( (HH_0(\Pi_Q)_{\text{free}})^+ \otimes \mathbb{F}_p \) such that, if \( [f] \) is in the basis, so is \( [f]^p \). Then, we can lift this basis arbitrarily to \( (HH_0(\Pi_Q)_{\text{free}})^+ \otimes \mathbb{Z}(p) \). This allows us to compute \( \langle D[p^n]/p \rangle / \langle D[f] \rangle \), and hence the odd cyclic homology, and we obtain the above, except localized at \( p \). Since this works for all \( p \), this proves the second equation. The third equation follows immediately from the sequence (10.4.7), since the first term is zero in the non-Dynkin, non-extended Dynkin case.

(ii) This is similar to part (i). The only new difficulty is in computing the first term of (10.4.7). This may be done as follows. First, localizing at a prime \( p \geq 2 \) such that \( p \nmid |\Gamma| \), in view of the fact that \( (Eu(z) + 2z)_{z \in HH^0(\Pi_Q)} \) spans all of \( HH^0(\Pi_Q) \) in degrees not congruent to \(-2\) modulo \( p \), it suffices to consider such degrees. Now, we apply Proposition 10.2.19(i). For any homogeneous basis element \( [f] \in HH_0(\Pi_Q)_{\text{free}} \), the greatest common positive integer divisor of \( D[f] \in HH^1(\Pi_Q) \) and \( [f] \) must equal the greatest common positive divisor of \( \Theta[f] \in HH^0(\Pi_Q) \cdot Eu \) and \( [f] \), if either of these quantities is less than \( [f] \). Let us now pick a \( \mathbb{Z}(p) \)-basis of \( HH_0(\Pi_Q)_{\text{free}} \otimes \mathbb{Z}(p) \) as in part (i), so that, in each degree \( m \) with \( p \mid m \), the basis consists of elements \( [f^p_{m/p^k,s}]^{p^k} \) where \( [f^p_{m/p^k,s}]^{p^k} \) is a basis element in degree \( m/p^{j-k} \) for all \( k \leq j \), and \( s \) is an index. Let us fix this degree \( m \) and the above \( f_{j,s} \). Then, the greatest power of \( p \) that divides \( D[f^p_{m/p^k,s}] \) in \( HH^1(\Pi_Q) \otimes \mathbb{Z}(p) \) is \( p^j \), so the same must be true of \( \Theta[f^p_{m/p^k,s}] \) in the case that \( p^{j+1} \mid m \). Thus, we may form a \( \mathbb{Z}(p) \)-basis of

\[
\Theta[HH_0(\Pi_Q)_{\text{free}}[m] \otimes \mathbb{Z}(p)] + (Eu + 2)(HH^0(\Pi_Q)[m - 2] \otimes \mathbb{Z}(p)) = \{ HH^0(\Pi_Q), HH^0(\Pi_Q) \}[m - 2] + m(HH^0(\Pi_Q)[m - 2] \otimes \mathbb{Z}(p))
\]

from the elements \( \Theta[f^p_{j,s}] \) such that \( p^{j+1} \mid m \), and some number of elements of \( m \cdot HH^0(\Pi_Q \otimes \mathbb{Z}(p))[m - 2] \). This gives us the desired formula localized at \( p \). \( \square \)
We may refine the above to yield a complete abstract description of $HC_\bullet(\Pi_Q)$ over $\mathbb{Z}$ (it remains to consider torsion not coprime to $\Gamma$) using Proposition 10.2.19(ii), which we omit. The above then extends easily to a description of $HC_\bullet$ tensored over any commutative ring $R$, simply by using the universal coefficient theorem for homology (note that $HC_\bullet(\Pi_Q)$ is can be computed by a chain complex of free $\mathbb{Z}$-modules, and $HC_\bullet(\Pi_Q \otimes R)$ can be computed by tensoring this complex by $R$).

Also note that, in part (ii) above, we computed a quotient of the zeroth Poisson homology $HP_0(\text{HH}^0(\Pi_Q \otimes k))$ (for $k = \mathbb{Z}[\frac{1}{p}, e^{\frac{2\pi i}{p}}]$): the quotient by $(Eu + 2)\text{HH}^0(\Pi_Q \otimes k)$. Using this it is not hard to deduce the good-prime analogue of Proposition 8.4.8.

**Corollary 10.4.23.** For $Q$ extended Dynkin and $k = \overline{\mathbb{F}}_p$, an algebraic closure of $\mathbb{F}_p$, where $p \nmid |\Gamma|$, we have

$$HP_0(\text{HH}^0(\Pi_Q \otimes \mathbb{F}_p)) \cong \langle f^p \rangle_{f \in \text{HH}^0(\Pi_Q \otimes \mathbb{F}_p)}(2p - 2) \oplus HP'_0,$$

where $HP'_0$ is finite-dimensional (of dimension $|I| - 1$ where $I$ is the number of vertices), and has Hilbert series equal to that of $\text{HH}^0(\Pi_Q \otimes \mathbb{Q})$.

As pointed out by P. Etingof, the above should also be provable using the method of [GK04].

A consequence of the above corollary is that, in high enough degrees, the relations $W \otimes k \subset V \otimes k$ can actually be obtained by taking commutators of elements of $\text{HH}^0(\Pi_Q \otimes k)$ (rather than merely by commutators of $\Pi_{Q_n}$). More precisely, the only relations we miss by restricting to such commutators are those that project nontrivially to $r' \otimes HP_0(\Pi_Q \otimes \mathbb{F}_p)_0$, which has the same Hilbert polynomial as $HP_0(\Pi_Q \otimes \mathbb{Q})(2)$.

### A Gröbner Bases and the Diamond Lemma

In this section, we briefly recall properties of Gröbner bases, and the more general Diamond Lemma, as tools in verifying that a given set is a basis for a vector space (usually, a quotient of a free noncommutative algebra on finitely many generators by a finitely-generated ideal).

Although these are standard, the results are stated in greater generality than usual (for modules over a PID instead of ideals in a free noncommutative algebra over a field).

**A.0.1 Gröbner Bases** First, suppose that $F = k(x_1, \ldots, x_n)$ is the free noncommutative algebra over the field $k$ generated by indeterminates $x_1, \ldots, x_n$. Consider the graded lexicographical ordering, which means that $M_1 \prec M_2$ if either $|M_1| < |M_2|$ (where $|M|$ denotes the length of a monomial $M$), or $|M_1| = |M_2|$ and $M_1 \ll M_2$ with respect to the lexicographical ordering $\ll$ on $x_1, \ldots, x_n$ where $x_1 < x_2 < \ldots < x_n$. [We can generalize this to replace lexicographical ordering by any total ordering on monomials of a given degree, and generalize the degree $| \cdot |$ to a weighted degree where each $x_i$ is assigned a positive integer, not necessarily 1.]

Given a set of elements $P_i \in F$, a polynomial $P$ is said to be “reducible” if the leading monomial (with respect to $\prec$) $LM(P)$ of $P$ contains as a subword the leading monomial of one of the $P_i$’s. Otherwise, $P$ is said to be “irreducible” with respect to the $P_i$’s. If $P$ is reducible, then a “reduction” of $P$ is an element of the form $P - \lambda X P_i Y$ where $X$ and $Y$ are monomials, $\lambda X M_i Y$ is the leading monomial of $P$, and $M_i$ is the leading monomial of $P_i$.

An **ideal** basis $(P_i)$ for an ideal $I = \langle (P_i) \rangle$ (ideal basis meaning that multiplication on the right or left by $F$, not just $k$, is allowed) is said to be a **Gröbner** basis if any polynomial has a unique
reduction to an irreducible element modulo $P_i$. In other words, the irreducible monomials form a vector space basis of the quotient $F/I$.

The following criterion is well-known (and is the basis for the Buchberger algorithm for computing Gröbner bases):

**Proposition A.0.25.** A set $(P_i)$ forms a Gröbner basis for $I = \langle P_i \rangle$ iff, for any two elements $P_i, P_j$ with leading terms $LM(P_i) = \lambda_i M_i, LM(P_j) = \lambda_j M_j$ and any monomial $M$ with $M = M_i X = Y M_j$ for some monomials $X$ and $Y$, one can reduce the element $\lambda_j P_i X - \lambda_i Y P_j$ to zero (using only the $(P_i)$), meaning that there is a sequence of reductions taking the element to zero.

**Proof.** This follows from the Diamond Lemma in the next subsection. \qed

Note that the above result can be extended to the case of a PID, as in the next subsection.

**A.0.2 The Diamond Lemma** We first state the Diamond Lemma for free modules over a PID, and then specialize to the free algebra case.

Let $R$ be a principal ideal domain (PID). Given any free module $V$ over $R$ with a fixed basis $(v_i)_{i \in I}$ labeled by a partially ordered set (poset) $(I, \prec)$ which satisfies the descending chain condition (dcc) (meaning every descending sequence in $I$ terminates at a finite point), and any submodule $W \subseteq V$, the Diamond Lemma gives a criterion for a set $(w_i)_{i \in I} \subseteq W$ (for some index set $J$) to be a “confluent set” (this is a generalization of being a Gröbner basis, and essentially means that applying relations in any order yields the same result). In this case, any element of $V$ has a unique reduction to an $R$-linear combination of the irreducible monomials $(v_i)_{i \in I'} \subseteq I$ for a certain subset $I' \subseteq I$. In other words, $(\overline{v}_i)_{i \in I'}$ forms an $R$-linear basis of $V/W$.

Let $i \preceq j$ mean $i = j$ or $i < j$. An element $w_j \in V$ defines a “reduction” if, writing $w_j = \sum_{i \in I} \lambda_{ji} v_i$ (all but finitely many $\lambda_{ji}$ are nonzero for each $j$), there exists a unique $\psi(j) \in I$ such that $\lambda_{j\psi(j)} \neq 0$ and $\lambda_{ji} \neq 0$ implies $i \preceq \psi(j)$. Then, the “reduction” associated to $w_j$ sends $\lambda_{j\psi(j)} v_{\psi(j)}$ to $\sum_{i < \psi(j)} \lambda_{ji} v_i$.

For any $i \in I$, let $W_i = \{ w_j \mid j \in J, \psi(j) = i \}$, the set of $w_j$ which reduce a multiple of $v_i$. Also, let $X_i = \text{Span}(W_i) \cap \text{Span}(v_\ell \mid \ell < i)$, which is the submodule consisting of those elements in the span of $(W_i)$ which have zero coefficient of $v_i$ (and hence are a linear combination of monomials $v_\ell$ for $\ell < i$).

**Definition A.0.26.** Suppose that the $w_j, j \in J$ all define reductions, span $W$, and satisfy the condition that, for all $i \in I$, $X_i \subseteq \text{Span}(W_\ell)_{\ell < i}$. Then, the $(w_j)$ is said to be a confluent set of $W$.

**Definition A.0.27.** Let $\gcd_W(i)$ be the gcd of all $\lambda \in R$ such that there is an element $w \in W$ of the form $w = \lambda v_i + \sum_{j < i} \lambda_j v_j$. To be precise, $\gcd_W(i)$ is an ideal of $R$, but we will also use it to refer to any generator of the ideal.

The following version of the Diamond Lemma is straightforward to prove, so the proofs is omitted.

**Proposition A.0.28.** (The Diamond Lemma I) For any confluent set $(w_j)$, and any choices of representatives $R_i \subseteq R$ of $R/\gcd_W(i)$ (i.e. $R_i$ maps bijectively to $R/\gcd_W(i)$) such that $0 \in R_i$ for all (but finitely many) $i$, every element of $V/W$ has a unique expression as a linear combination $\sum_i \mu_i v_i$ where $\mu_i \in R_i$. 

98
**Proposition A.0.29. (The Diamond Lemma II)** For any confluent set \( \{w_j\} \), an \( R \)-linear basis of \( W \) may be obtained by choosing, for each \( i \in I \) such that \( \gcd_W(i) \neq (0) \), an arbitrary element \( w_i' \in \text{Span} W_i \) such that \( w_i' = \gcd_W(i)v_i + \sum_{\ell \prec i} \mu_{i,\ell}v_\ell \). The resulting \( w_i' \) form a basis of \( W \).

**Proposition A.0.30. (The Diamond Lemma III)** Again suppose \( W \) is confluent. Let \( I_{\gcd \neq 1} \subset I \) be the subset such that \( i \in I_{\gcd \neq 1} \) iff \( \gcd_W(i) \neq (1) \). Let \( I_{\gcd = 1} \) be the subset such that \( i \in I_{\gcd = 1} \) iff \( \gcd_W(i) \notin \{0,1\} \). Then \( V/W \) may be presented as \( \langle v_i \rangle_{i \in I_{\gcd 
eq 1}}/\langle w_i'' \rangle_{i \in I_{\gcd = 1}} \) where each \( w_i'' \in W \) is an arbitrary element such that \( w_i'' = \gcd_W(i)v_i + \sum_{\ell \prec i, \ell \in I_{\gcd = 1}} \mu_{i,\ell}v_\ell \) (for example, pick the \( w_i \)'s as in Proposition A.0.29 and write them modulo \( \text{Span}(W_i)_{\ell \prec i} \) as in Proposition A.0.29).

Note that we did **not** actually describe the abstract module structure of \( V/W \) above: for example, one could have \( V = \langle v_1, v_2 \rangle \) and \( W = \langle 3v_1 - v_2, 3v_2 \rangle \), where Proposition A.0.28 says that the quotient is set-theoretically the same as sums \( \lambda_1 v_1 + \lambda_2 v_2 \) for \( \lambda_i \in \{0,1,2\} \). The abstract \( \mathbb{Z} \)-module structure, however, is \( \mathbb{Z}/9 \).

Also, note that \( W \) above is the analogue of multiples of a Gröbner basis by monomials on either side: so \( W \) itself will not form any type of basis.

**Remark A.0.31.** In the case that \( R = \mathbb{k} \) is a field, then the Diamond Lemma says that a basis of \( W/V \) is given by \( \langle \bar{v}_i \rangle_{i \in I'} \) where \( I' = I_{\gcd \neq 1} \) is the set of indices such that \( v_i \) does not appear as the leading term of any of the relations \( (w_j) \) in the confluent set.

**Remark A.0.32.** If we like, we could have taken \( \prec \) to be a well-ordering (e.g. a labeling by \( \mathbb{Z}_{\geq 0} \) if the module is countably generated) with no loss of generality. This is because we could convert any partial ordering satisfying the dcc into an arbitrary well ordering that preserves all relations \( x \prec y \) from the partial order (i.e. such that there is a map of posets from the original set to the totally ordered one). (Note that this requires the Axiom of Choice in general, and it works because dcc is equivalent to saying that any subset has a minimal element.) This would yield exactly the same results and proof. We state it in the poset generality because it is sometimes more convenient to work with only partially-ordered sets.

Now, let us specialize to the free algebra case. Let \( A = R\langle x_1, \ldots, x_m \rangle \) be a free noncommutative algebra generated by indeterminates \( x_i \). Let \( \prec \) be a partial order on the monomials in the \( x_i \)'s satisfying the dcc, such that \( f \prec g \) implies \( h_1f h_2 \prec h_1g h_2 \) for any monomials \( h_1, h_2 \) (this is most easily satisfied by introducing a grading, as in the case of Gröbner bases, so that \( f \prec g \) whenever \( |f| < |g| \)). Suppose \( B \subset A \) is an ideal. Then we can define a set \( \{b_i\} \) to be a confluent ideal basis of \( B \) if the elements \( (fb_i) \), for \( f, g \) ranging over all monomials in the \( x_i \), form a confluent set for \( B \) as an \( R \)-module (with basis the monomials and partial order \( \prec \)). To understand what this means, call the “leading monomial” \( \text{LM}(A) \) of an element of \( A \) the highest monomial which appears with nonzero coefficient with respect to \( \prec \), if such a monomial exists and exceeds all other monomials which have nonzero coefficient. In order to be confluent, we first require that each \( b_i \) have a leading monomial. We then call elements of the form \( fb_i \) “reductions”, and interpret them as reducing the highest monomial modulo \( B \) to lower ones. Then, the confluence condition says: if \( h \in A \) is the leading monomial of multiple reductions, then any linear combination of such reductions which has a lower leading monomial than \( h \) is itself a linear combination of reductions with leading monomial less than \( h \).

Then, in this case, one concludes all of the Diamond Lemma versions. In the case \( R = \mathbb{k} \) is a field, for example, one finds that \( A/B \) has a basis, as a vector space, given by those monomials in “normal form”, which means that they do not contain the leading monomial of any of the \( b_i \)'s as a subset (as a word).
Letting $\prec$ be the graded lexicographical ordering (or a variant as discussed in Section A.0.1) (with $R = k$), we find that a confluent basis is nothing but a Gröbner basis for $B$, and recover Proposition A.0.25. In fact, we recover a version of Gröbner bases over PIDs.

## B Loday Poisson, co, and bialgebras

In this section we introduce (Poisson) Loday bialgebras and coalgebras. Recall [Lod93] that a left Loday algebra (called “Leibniz algebra” by Loday and some others) is a vector space $A$ with a bilinear bracket $\{ , \} : A \otimes A \to A$ satisfying the equation

$$\{ a, \{ b, c \} \} - \{ b, \{ a, c \} \} = \{ \{ a, b \}, c \}.$$  (B.0.33)

### Example B.0.34.

An example one keeps in mind to keep track of the left and rights above is where $A = A_0 \oplus A_1$ and the bracket is a map $A_0 \otimes A_1 \to A_1, A_0 \otimes A_0 \to A_0$. Then, the left Loday conditions make sense, and give $A$ the structure of a Loday algebra if we set $\{ A_1, A \} = 0$.

### Example B.0.35.

In our situation, we take the above with $A_0 := L = P/[P,P]$ and $A_1 := P$. We define the bracket $\{ , \}_L : L \otimes P \to P$ by (5.2.3). The point is that we can choose to view the bracket as one-sided and non-skew (as in [VdB04]) rather than as a Lie bracket by skew-symmetrizing (as we did in Section 5). The difference between $\{ , \}_L$ and $\{ , \}_Lod$, then is that we set $\{ , \}_Lod|_{P \otimes L} = 0$, whereas $\{ , \}_P|_{P \otimes L} := -\{ , \}_Lod|_{P \otimes L} \circ \tau_{(12)}$.

Our Loday bracket additionally satisfies a Poisson condition, as described:

### Definition B.0.36.

If $A$ is an associative algebra equipped with a left Loday bracket $\{ , \}$, then we say that $A$ is a (left) Poisson Loday algebra if

$$\{ a, bc \} = \{ a, b \}c + b\{ a, c \}.$$  (B.0.37)

One may similarly define right Poisson Loday by reading everything right-to-left. (That is, a right Poisson Loday algebra is the same as taking opposite multiplication and opposite bracket on a left Poisson Loday algebra).

Now we define co-Loday algebras by dualizing the Loday condition:

### Definition B.0.38.

If $A$ is a vector space, we say that $A$ is a left Loday coalgebra if it is equipped with a linear map $\delta : A \to A \otimes A$ satisfying

$$(1 - \sigma_{12})(\delta \otimes 1)\delta = (1 \otimes \delta)\delta.$$  (B.0.39)

A right Loday coalgebra is the same as taking opposite cobracket on a left Loday coalgebra:

$$(1 - \sigma_{23})(1 \otimes \delta)\delta = (\delta \otimes 1)\delta.$$  (B.0.40)

The following is straightforward:

### Proposition B.0.41.

A (left or right) Loday coalgebra which is skew-cocommutative ($\delta = -\sigma_{(12)}\delta$) is the same as a co-Lie algebra.

### Example B.0.42.

The bracket $\delta_L : P \to L \otimes P$ (5.2) is a left Loday cobracket on $P$. 

100
Now, we will define the notion of a Loday bialgebra. Recall that a Lie bialgebra is the structure of a Lie algebra and Lie coalgebra which satisfy the one-cocycle condition \( \delta(\{a, b\}) = \{a \otimes 1 + 1 \otimes a, \delta(b)\} + \{\delta(a), b \otimes 1 + 1 \otimes b\} \). We define a Loday bialgebra so that it is a “Lie bialgebra minus skew-symmetry”. That is,

**Definition B.0.43.** A left,left Loday bialgebra is a vector space \( A \) equipped with a left Loday bracket \( \{,\} \) and a left Loday cobracket \( \delta \) satisfying

\[
\delta(\{a, b\}) = \{a \otimes 1 + 1 \otimes a, \delta(b)\} + \{\delta(a), b \otimes 1 + 1 \otimes b\}. \tag{B.0.44}
\]

Similarly, a “\( x,y \)” Loday bialgebra is a \( x \) Loday bracket and a \( y \) Loday cobracket together with the version of (B.0.44) obtained by taking opposite of the structure(s) which become right. For instance, a right,right Loday bialgebra satisfies

\[
\delta(\{a, b\}) = \{a \otimes 1, \delta(b)\} + \{\delta(a), b \otimes 1 + 1 \otimes b\}. \tag{B.0.45}
\]

**Example B.0.46.** The space \((\tilde{P}, \{,\}_{Lod}, \delta_{\ell})\) is a left,left Poisson Loday bialgebra.

We can also consider the “involutivity” condition \( \{,\} \circ \delta = 0 \) for any Loday bialgebra, which is a “noncommutative” version of the condition for Lie algebras (which itself can be viewed as the infinitesimal \( S^2 = Id \) condition).

Finally, we may define the notion of a “BV” Loday bialgebra:

**Definition B.0.47.** If \( L \) is an algebra and an involutive left,left Loday bialgebra, then a map \( \{,\}: L \otimes L \to L \otimes L \) gives \( L \) a noncommutative BV structure if

\[
\delta(ab) = (1 \otimes a)\delta(b) + \delta(a)(1 \otimes b) + \{a, b\}. \tag{B.0.48}
\]

**Example B.0.49.** The space \((\tilde{P}, \{,\}_{Lod}, \delta_{\ell}, (pr \otimes 1)\{,\})\) is a noncommutative BV, Poisson left,left Loday bialgebra, satisfying \( \{,\}_{Lod} \circ \delta = 0 \).

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