Renormalization group approach to anisotropic superconductors at finite temperature.

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Abstract

A renormalization group (RG) analysis of the superconductive instability of an anisotropic fermionic system is developed at a finite temperature. The method appears a natural generalization of Shankar’s approach to interacting fermions and of Weinberg’s discussion about anisotropic superconductors at $T = 0$. The need of such an extension is fully justified by the effectiveness of the RG at the critical point. Moreover the relationship between the RG and a mean-field approach is clarified, and a scale-invariant gap equation is discussed at a renormalization level in terms of the eigenfunctions of the interaction potential, regarded as the kernel of an integral operator on the Fermi surface. At the critical point, the gap function is expressed by a single eigenfunction and no symmetry mixing is allowed. As an illustration of the method we discuss an anisotropic tight-binding model for some classes of high $T_c$ cuprate superconductors, exhibiting a layered structure. Some indications on the nature of the pairing interaction emerge from a comparison of the
model predictions with the experimental data.
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I. INTRODUCTION

In a recent review, Shankar unified a large variety of physical phenomena and their conventional descriptions, such as Landau theory of Fermi liquids, charge density waves and nesting in the Fermi surface, superconductive instabilities and mean-field theories etc., under the scheme of a renormalization group (RG) approach, with the aim of a better understanding of the general nature of the physics involved in such processes. Central to that approach is the idea that the instability towards a phase transition in an interacting fermionic system may be discussed in a very general fashion, using the RG technique, which involves the cut-off and the rescaling of the energies and the momenta close to the Fermi surface. For instance, a charge density wave or a superconducting condensate are recovered owing to very general symmetry properties of the single particle dispersion relation, i.e., the existence of a nesting vector and the validity of time reversal symmetry, respectively.

In an almost contemporary work, Weinberg derived a RG flow equation for superconductors whose Fermi surface satisfies just time reversal invariance, employing the standard field-theoretic scheme. In particular, Weinberg’s work doesn’t claim for a spherical Fermi surface, and therefore shows itself more suitable for taking into account anisotropic materials such as the high $T_c$ layered cuprates.

An extension to a non-zero temperature of the RG analysis is called for, at least for two major reasons: i) the RG approach is more effective around the critical point, which is generally located at a finite temperature $T_c$; ii) in order to make contact with the conventional mean-field approach and with the experimental data, the RG predictions would be required at a finite temperature too. Such an extension is straightforward, from a technical point of view, and even for a generic anisotropic system, the flow equations are decoupled in terms of the eigenvalues of an integral operator $\hat{V}$, whose kernel is the marginal pairing coupling evaluated at the Fermi surface. The critical temperature $T_c$ is then found out to be a simple function of the most negative eigenvalue, which determines the transition point. The standard BCS expression is recovered for a spherical Fermi surface with a constant
pairing coupling, whereas for a generic rotationally invariant system the eigenfunctions reduce to spherical waves, since then the angular momentum and $\hat{V}$ are mutually commuting operators.

In order to make contact with the standard mean-field approximation, we recover again the same analytical expression for $T_c$, though following a quite different path, starting from the usual gap equation. The occurrence in such an expression of just one eigenvalue is a token of the existence of a leading part in the pairing coupling, which leads the transition. In fact, the expansion of the integral operator $\hat{V}$ in terms of its eigenfunctions allows one to determine its relevant part as its projection over the eigenfunction belonging to the most negative eigenvalue. In other words, close to the critical point, the interaction operator $\hat{V}$ acts on the one-dimensional subspace generated by just one eigenfunction. All the physical quantities depend on this eigenfunction, which entirely determines even the symmetry pattern of the gap function. As a consequence, the occurrence of any symmetry mixing in the gap function (such as $s$-$d$ wave mixing) is ruled out at the critical point, since in that limit the gap function reduces to a single eigenfunction with a fixed symmetry (provided that the most negative eigenvalue is not degenerate).

Even far from the critical point, the expansion of $\hat{V}$ in terms of its eigenfunctions may be inserted in the gap equation, thus providing a scale invariant relation, which is of some utility especially when dealing with separable potentials, when the number of eigenvalues is finite and so is the expansion in terms of the eigenfunctions. The resulting set of coupled non-linear equations always admits symmetric solutions, i.e. solutions for the gap function which share the eventual symmetry of the physical system. However, far from the critical point the equations are highly non-linear, so that the uniqueness of the symmetric solutions is not guaranteed, and broken-symmetry solutions are not forbidden.

Such general aspects of the solutions, as their symmetries near the critical point, are more easily discussed by an RG approach than within a mean-field approximation. This feature, together with a major handiness in dealing with the numerical cases, provides the RG approach with more appeal than the mean-field approximation, without spoiling their
A nice illustration of the method is provided by a simple tight-binding model, recently proposed by Spathis et al. for the single-particle dispersion relation of a layered high $T_c$ superconductor, namely Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (BSCCO). In order to compare our results with those already obtained with a conventional BCS procedure, the same dispersion relation is exactly employed, where rotationally non-invariance accounts for the structural anisotropy in BSCCO, and two singlet pairing couplings are considered, namely an on-site and a nearest-neighbour interaction.

In particular, the characteristic dependence of $T_c$ versus the carrier concentration $N$ is recovered, although it seems to be a mere effect of the peaked quasi-bidimensional density of states. Nonetheless, we observe the best qualitative agreement with the experimental data when the two couplings have opposite signs, and precisely in the case of an attractive on-site interaction (negative Hubbard $U$) and an inter-site nearest-neighbour repulsion.

Moreover, while a nearest-neighbour attraction always yields a superconducting ground state at $T = 0$, for any value of the chemical potential $\mu$ (though the critical temperature may be extremely low), an on-site attraction is cancelled out by the presence of a nearest-neighbour repulsion, for a quite large range of values for the chemical potential.

From a physical point of view, a negative Hubbard $U$ could be the effect of a very short ranged coupling interaction, while a nearest-neighbour repulsion could be justified by the Coulomb long-range potential.

We observe, however, that such comments on the nature of the pairing interaction are far from being conclusive, and a larger number of pairing couplings should be retained in the model potential for a full comparison with the experimental data. From this point of view, the method is easily implemented, since adding any other coupling merely implies an increase in the space dimension of the integral operator $\hat{V}$.

The paper is organized as follows. In Section II we outline the RG approach at a finite temperature for a generic anisotropic fermionic system. In Section III very similar results are recovered by use of a standard gap equation, and the possible predictions of the RG approach
about the structure and symmetry of the gap function are discussed. Later in Section IV, as an illustration of the method, we consider a tight-binding model for the single-particle dispersion relation in layered high $T_c$ superconductors, and finally in Section V the numerical results are discussed.

II. RG FLOW AT A FINITE TEMPERATURE

Shankar’s RG approach to the superconductive instability properties of an interacting fermionic system is here generalized to a finite temperature and for a generic anisotropic Fermi surface. No other special symmetry is assumed than time-reversal.

The RG flow for anisotropic superconductors has been first discussed by Weinberg at zero temperature, whereas the RG analysis turns out to be more effective around the critical temperature $T_c$. An extension to finite temperature of Shankar’s derivation of the flow equations is quite straightforward, and we shall focus on the main aspects, thus referring to Shankar’s pedagogical review for the details.

The partition function $Z$ of a many-fermion system can be expressed as the functional integral:

$$ Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{S[\psi,\bar{\psi}]}, $$

being $S = S_0 + S_{\text{int}}$ the action as a functional of the spinor Grassmann fields

$$ \psi \equiv \begin{pmatrix} \psi_\uparrow(k,\omega_n) \\ \psi_\downarrow(k,\omega_n) \end{pmatrix}, \quad \bar{\psi} \equiv \begin{pmatrix} \bar{\psi}_\uparrow(k,\omega_n) & \bar{\psi}_\downarrow(k,\omega_n) \end{pmatrix}, $$

being $k$ a reciprocal lattice vector, $\omega_n = (2n + 1)\pi/\beta$ a fermionic Matsubara frequency, $\beta = T^{-1}$ the inverse temperature (hereafter, we set $\hbar = k_B = 1$). The part of the action which is quadratic in the fields accounts for the free evolution of the system, and may be written as:

$$ S_0 = \frac{1}{\beta} \sum_{\sigma} \sum_n \int \frac{d^3k}{(2\pi)^3} \bar{\psi}_\sigma(k,\omega_n) [i\omega_n + \mu - \varepsilon(k)] \psi_\sigma(k,\omega_n), $$

where $\mu$ is the chemical potential and $\varepsilon(k)$ is the single-particle dispersion relation.
being $\varepsilon(k)$ the free single-particle dispersion relation.

The most general two-particle interaction in $d = 3$ dimensions is seen to contribute to the action through the term:

$$S_{\text{int}} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2} \int d(4)d(3)d(2)d(1) \tilde{\delta}(4321) \bar{\psi}(4)\psi(3)u(4321)\psi(2)\psi(1),$$

being

$$\int d(i) \equiv \frac{1}{\beta} \sum_{\sigma_i} \sum_{n_i} \int \frac{d^3 k_i}{(2\pi)^3}, \quad \psi(i) \equiv \psi_{\sigma_i}(k_i, \omega_{n_i})$$

and

$$\tilde{\delta}(4321) \equiv (2\pi)^3 \beta \tilde{\delta}^{(3)}(k_4 + k_3 - k_2 - k_1)\delta_{n_4 + n_3, n_2 + n_1},$$

where the Dirac $\tilde{\delta}^{(3)}$ enforces momentum conservation up to a vector in the reciprocal lattice. The interaction $u(4321) \equiv u_{\sigma_4,\sigma_3,\sigma_2,\sigma_1}(k_4, k_3, k_2, k_1)$ satisfies the general symmetry properties:

$$u(4321) = -u(3421) = -u(4312) = u(3412)$$

and acts as a $2 \times 2 \times 2 \times 2$ array in the spinor space.

Central to the RG approach is the assumption that all the integrals may be evaluated within the shell $|\varepsilon(k) - \mu| < \Lambda$, with $\Lambda$ a proper energy cut-off, provided all the parameters in the model are renormalized, in order to take into account the elimination of the modes with $|\varepsilon(k) - \mu| > \Lambda$. Thus we suppose that all such parameters are renormalized to the scale $\Lambda$ from the beginning. Of course, a “physical” choice of the cut-off $\Lambda$ would allow for a comparison of the parameters with the “real world” corresponding quantities. A cut-off in energy is preferred, instead of a direct cut-off in momenta, when dealing with a rotationally non-invariant Fermi surface.

When a low-temperature fermionic system is considered, the energy cut-off $\Lambda$ can be chosen small enough to allow a unique decomposition for any vector $k$ belonging to the tiny slice around the Fermi surface $|\varepsilon(k) - \mu| < \Lambda$ as:

$$k = k_0 + \delta k,$$
being \( \mathbf{k}_0 \) the vector belonging to the Fermi surface \((\varepsilon(\mathbf{k}_0) - \mu = 0)\) nearest to \( \mathbf{k} \) and \( \delta \mathbf{k} \) a vector orthogonal to the Fermi surface. Expanding \( \varepsilon(\mathbf{k}) \) around \( \mathbf{k}_0 \), only the linear term shows to be non-irrelevant in the RG flow, and one can safely write:

\[
\varepsilon(\mathbf{k}) \approx \mu + v_F(\mathbf{k}_0) \delta k,
\]

being \( v_F(\mathbf{k}_0) = |\nabla_{\mathbf{k}} \varepsilon|_{\mathbf{k}=\mathbf{k}_0} \) the Fermi velocity. The cut-off condition is then equivalently written as:

\[
|v_F(\mathbf{k}_0) \delta k| < \Lambda.
\]

We may always choose a bidimensional ‘vector’ of parameters \( \theta \equiv (\theta_1, \theta_2) \) on the Fermi surface to individuate \( \mathbf{k}_0 \), together with the energy displacement from the Fermi surface \( \varepsilon = \varepsilon(\mathbf{k}) - \mu = v_F(\mathbf{k}_0) \delta k \) to fix \( \delta k \). Such a change of variables is taken into account within the integrations according to the rule:

\[
\int \frac{d^3k}{(2\pi)^3} \mapsto \int \frac{d^2\theta}{(2\pi)^2} \int_{-\Lambda}^{\Lambda} \frac{d\varepsilon}{2\pi} J(\theta, \varepsilon),
\]

where

\[
J^{-1}(\theta, \varepsilon) = \left| \frac{\partial (\theta, \varepsilon)}{\partial (\mathbf{k})} \right| = |\nabla_{\mathbf{k}} \varepsilon \cdot (\nabla_{\mathbf{k}} \theta_1 \times \nabla_{\mathbf{k}} \theta_2)|
\]

is the inverse Jacobian function. In practice, we shall omit the \( \varepsilon \) dependence in \( J \), since it reveals itself irrelevant in an RG sense, by setting \( J(\theta) \equiv J(\theta, \varepsilon = 0) \), which is the only marginal term in the expansion of \( J \) in powers of \( \varepsilon \).

The free action Eq. (3) now reads:

\[
S_0 = \frac{1}{\beta} \sum_{n,\sigma} \int \frac{d^2\theta}{(2\pi)^2} J(\theta) \int_{-\Lambda}^{\Lambda} \frac{d\varepsilon}{2\pi} \bar{\psi}_{\sigma}(\theta, \varepsilon, \omega_n) [i\omega_n - \varepsilon] \psi_{\sigma}(\theta, \varepsilon, \omega_n).
\]

An elementary step in the RG flow is next defined as that transformation on \( S_0 \) which:

\( i) \) integrates all the modes having \( |\varepsilon| < \Lambda/s \), with \( s \gtrsim 1; \ ii) \) rescales energies and momenta as \( s \omega_n \mapsto \omega'_n \), \( s \varepsilon \mapsto \varepsilon' \), in order to restore the original cut-off; \( iii) \) rescales the fields as \( s^{-3/2} \psi \mapsto \psi' \). At \( T = 0 \), such a transformation leaves the free action invariant, so that the latter can be considered as a fixed point.
At a finite temperature, rescaling the energies as \( s \omega_n \mapsto \omega'_n \) implies a rescaling of the inverse temperature itself as \( \beta/s \mapsto \beta' \). This is what we must pay for having restored the cut-off to its original value. Besides, if any energy scale is associated with the system, such as a non-zero temperature or an energy gap, this one undergoes a renormalization flow, while the cut-off \( \Lambda \) is kept fixed. If one identifies \( s = e^t \), then:

\[
\frac{d \beta_t}{dt} = -\beta_t \quad \Leftrightarrow \quad \beta_t = \beta_0 e^{-t},
\]

which shows that renormalizing the action at a fixed energy cut-off \( \Lambda \) yields a larger effective temperature.

A completely equivalent approach consists in keeping fixed all the energy scales (temperature, gap \textit{etc}), while assuming a flow in the cut-off as:

\[
\Lambda_t = \Lambda_0 e^{-t}.
\]

One approach is recovered from the other by a bare change of variables in the integration over \( \varepsilon \).

Let us now examine how such a RG transformation affects the interaction part of the action functional, Eq. (4). At a tree level (Fig. 1), the only marginal couplings which survive the RG flow are the lowest order terms in the expansion of \( u(4321) \) in powers of \( \varepsilon_i \). Moreover, from momentum conservation and phase space arguments, they restrict to only two contributions:

\[
F(\theta_1, \theta_2) = u(4321) \text{ with } \varepsilon_i = 0, \hat{\Omega}_4 \cdot \hat{\Omega}_3 = \hat{\Omega}_2 \cdot \hat{\Omega}_1, \quad (F \text{ coupling}),
\]

\[
V(\theta_1, \theta_4) = u(4321) \text{ with } \varepsilon_i = 0, \hat{\Omega}_4 + \hat{\Omega}_3 = \hat{\Omega}_2 + \hat{\Omega}_1 = 0, \quad (V \text{ coupling}),
\]

where \( \hat{\Omega}_i \) denotes the unit vector in the direction of \( k_i \equiv (\theta_i, \varepsilon_i = 0) \).

We are interested in the flow of the \( V \)-couplings which are responsible for the superconductive instability of the Fermi surface. We assume the interaction to be spin independent, and we take in consideration only spin singlet pairs, so that we can get rid of all the spin indices, by posing:
\[ V(\theta_1, \theta_4) \equiv V(\theta_4, \theta_1) = \frac{1}{2} \left[ u_{\uparrow \downarrow \uparrow \downarrow}(k_4, -k_4, -k_1, k_1) - u_{\uparrow \downarrow \downarrow \uparrow}(k_4, -k_4, -k_1, k_1) \right] = \]
\[ = \frac{1}{2} \left[ u_{\downarrow \uparrow \uparrow \downarrow}(k_4, -k_4, -k_1, k_1) - u_{\downarrow \uparrow \downarrow \uparrow}(k_4, -k_4, -k_1, k_1) \right], \tag{18} \]

being the \( u \) interactions restricted on the Fermi surface. Besides, the general symmetry properties, Eq. (9), imply that \( V(\theta_1, \theta_4) \) is symmetric with respect to the inversion \( \hat{\Omega}_1 \rightarrow -\hat{\Omega}_1 \) or \( \hat{\Omega}_4 \rightarrow -\hat{\Omega}_4 \).

Following analogous arguments as for \( T = 0 \) we observe that at one-loop level only the BCS diagram contributes to the flow in \( V \) (Fig. 2) with:

\[
dV(\theta_1, \theta_4) = -\frac{1}{2} \int d\Lambda d(6)d(5) G^0(6)G^0(5)V(\theta_1, \theta_5)V(\theta_5, \theta_4)\bar{\delta}(6521), \tag{19} \]

where \( G^0(i) \) denotes the free fermion propagator, which at a finite temperature may be written as:

\[
G^0(i) \equiv G^0(k_i, \omega_{n_i}) = \frac{1}{i\omega_{n_i} - \varepsilon(k_i) + \mu}, \tag{20} \]

and where \( \int d\Lambda d(6)d(5) \) does not contain the sum over spin indices, and the integration is performed over the tiny slice \(|\varepsilon(k) \pm \Lambda| < |d\Lambda|\).

For a generic anisotropic Fermi surface, when an interaction is switched on, both the chemical potential and the shape of the Fermi surface change according to the definition:

\[
\mu' - \varepsilon(k) - \text{Re } \Sigma(k, \mu') = 0, \tag{21} \]

being \( \mu' \) the chemical potential in presence of the interaction and \( \Sigma(k, \omega) \) the self-energy in the interacting fermion propagator. At one-loop level, a contribution to \( \Sigma \) comes from the ‘tadpole’ diagram shown in Fig. 1. However, the propagator in the one-loop correction Eq. (19) is a free propagator, and its denominator in Eq. (20) is determined in terms of the free single-particle dispersion relation and chemical potential. So, no serious problem arises at one-loop from the renormalization of the Fermi surface.

In Eq. (19), the incoming momentum and frequency are zero, and therefore the loop momenta and frequencies are opposite. From time reversal, \( \varepsilon(k) = \varepsilon(-k) \), and taking into account the \( \bar{\delta} \) constraint, one gets:
\[ dV(\theta_1, \theta_4) = -\frac{1}{\beta} \sum_n \int \frac{d^2 \theta}{(2\pi)^3} J(\theta)V(\theta_1, \theta)V(\theta, \theta_4) \int_{\Lambda-d\Lambda}^{\Lambda} \frac{d\varepsilon}{\varepsilon^2 + \omega_n^2} = \]
\[-\int \frac{d^2 \theta}{(2\pi)^3} J(\theta)V(\theta_1, \theta)V(\theta, \theta_4) \int_{\Lambda-d\Lambda}^{\Lambda} \frac{d\varepsilon}{2\varepsilon} \tanh \frac{\beta \varepsilon}{2}, \quad (22) \]

where we made use of the Mittag-Leffler expansion:

\[ \frac{1}{\beta} \sum_n \frac{1}{\omega_n^2 + \varepsilon^2} = \frac{1}{2\varepsilon} \tanh \frac{\beta \varepsilon}{2}. \quad (23) \]

Integrating at the cut-off, with \( dt = |d\Lambda|/\Lambda \), one finally obtains:

\[ \frac{dV_i(\theta_1, \theta_2)}{dt} = -\frac{1}{2} \tanh \left( \frac{\Lambda \beta_i}{2} \right) \int \frac{d^2 \theta}{(2\pi)^3} J(\theta)V_i(\theta, \theta) V_i(\theta, \theta_2), \quad (24) \]

where \( \beta_i \) flows according to Eq. (14).

We notice that the choice of a cut-off in energy allows one to take the factor \( \tanh(\Lambda \beta_i/2) \) out of the integration over the Fermi surface, like in the rotationally invariant case. For \( \beta_0 \to \infty \), Eq. (24) reduces to the flow equation for an anisotropic superconductor recovered by Weinberg at \( T = 0 \). Besides, \( T \) increases monotonically as \( t \) flows down to \( \infty \), so that \( T \to \infty (\beta = 0) \) is always a fixed point, since Eq. (24) would then yield \( dV_i/dt = 0 \).

Introducing the measure

\[ d\tau_\theta \equiv \frac{d^2 \theta}{2(2\pi)^3} J(\theta), \quad (25) \]

one may regard Eq. (24) as the flow equation for the integral operator:

\[ \left( \hat{V}_i \Phi \right)(\theta) = \int d\tau_\theta V_i(\theta, \theta') \Phi(\theta'), \quad (26) \]

where the coupling \( V_i(\theta, \theta') \) plays the rôle of a symmetric Hilbert-Schmidt kernel. Eq. (24) then reads:

\[ \frac{d\hat{V}_i}{dt} = -\tanh \left( \frac{\Lambda \beta_i}{2} \right) \hat{V}_i \cdot \hat{V}_i. \quad (27) \]

The integral operator \( \hat{V}_i \) is Hermitean and admits a complete set of orthonormal eigenfunctions \( \Phi_\alpha \):
\[
\dot{V}_t \Phi_\alpha(\theta) = \lambda_t(\alpha) \Phi_\alpha(\theta) \tag{28}
\]

\[
\int d\tau_0 \Phi_\alpha^*(\theta) \Phi_\beta(\theta) = \delta_{\alpha \beta}. \tag{29}
\]

As suggested by Weinberg, Eq. (27) implies that \([\dot{V}_t, \dot{V}_t] = 0\), so that the eigenfunctions \(\Phi_\alpha(\theta)\) do not flow, whereas the eigenvalues do flow according to the (decoupled) equations:

\[
\frac{d\lambda_t(\alpha)}{dt} = - \tanh \left( \frac{\Lambda \beta_t}{2} \right) \lambda_t^2(\alpha), \tag{30}
\]

which can be integrated together with Eq. (14) to yield:

\[
\lambda_t = \frac{\lambda_0}{1 + \lambda_0 \int_0^t \tanh \left( \frac{\Lambda \beta_t}{2} e^{-\tau} \right) d\tau}, \tag{31}
\]

being \(\lambda_0\) the unrenormalized eigenvalue. The fixed point at \(\beta = 0\) is generally reached unless \(\lambda_t\) diverges for some \(\bar{t}\), where the flow stops. This may only occur if the unrenormalized eigenvalue were negative at the beginning of the flow, \(\lambda_0 < 0\). As shown in Fig. 3, at variance with the \(T = 0\) analysis, this instability may not occur if \(\lambda_0\) is small, and a suitable definition of the critical temperature \(T_c\) can therefore be obtained from Eq. (31) in the limit \(\bar{t} \to \infty\), i.e.,

\[
1 + \lambda_0 \int_0^\infty \tanh \left( \frac{\Lambda \beta_0}{2} e^{-\tau} \right) d\tau = 0, \tag{32}
\]

or, after a change of variables,

\[
\frac{1}{\lambda_0} = - \int_0^{\Lambda \beta_c} \frac{d\xi}{\xi} \tanh \frac{\xi}{2}, \tag{33}
\]

with \(\lambda_0\) the negative eigenvalues being largest in modulus, i.e. the first leading to a divergence in \(\lambda_t\).

Eq. (33) is quite familiar, and in the limit \(\Lambda \beta_c \ll 1\) may be analytically approximated as:

\[
T_c = \frac{2e^\gamma}{\pi} \Lambda e^{1/\lambda_0}, \tag{34}
\]

being \(\gamma \simeq 0.5772\) the Catalan-Euler's constant.
We recover the BCS limit for a constant coupling $V = -2v$, since then:

$$\lambda_0 = -2v \int d\tau_\theta = -nv,$$

being $n$ the density of states per spin at the Fermi energy. In this limit, a natural cut-off is provided by the Debye frequency. Moreover, Eq. (33) is very general, and holds for any anisotropic superconductor. The critical temperature is determined by the most negative eigenvalue of the coupling kernel $V_t(\theta, \theta')$, while the remaining eigenvalues do not play any rôle in the proximity of the critical point. This idea will be developed in the next section, where the rôle of the eigenfunctions will be clarified in connection with the scale invariant gap equation.

III. MEAN FIELD VS RG: A SCALE INVARIANT GAP EQUATION

In the mean-field theory of superconductivity, one is usually led to consider the energy gap function $\Delta(k)$. It is a quantity of central interest, since it is directly measurable and its symmetry patterns are strongly related to the nature of the pairing, as we shall see in the following.

It would be desirable that the RG analysis could give some indication about the nature and the size of the gap function. On the other hand, we would like to throw some light on the connection between a RG approach and a conventional mean-field approximation.

At the end of the previous Section, we noticed that only the most negative eigenvalue of the integral operator $\hat{V}_t$ is responsible for the location of the instability. We are therefore led to expect that the associated eigenfunction should determine the gap structure at the transition point. Indeed, expanding the integral operator $\hat{V}_t$ in terms of its orthonormalized eigenfunctions:

$$\left(\hat{V}_t \cdot \right)(\theta) = \sum_\alpha \Phi^*_{\alpha}(\theta) \lambda_\alpha(\alpha) \int d\tau_\theta' \Phi_{\alpha}(\theta') \cdot ,$$

we may decompose the coupling kernel as $V_t(\theta, \theta') = V_t^c(\theta, \theta') + \tilde{V}_t(\theta, \theta')$, where:
being $\lambda_t(0)$ the most negative eigenvalue of $\hat{V}_t$. The divergence of $\lambda_t(0)$ at the transition point as $t \to \infty$ clearly indicates the relative importance of $V_t^c$ in driving the instability in the proximity of the transition point. In other words, at the transition point, the kernel $V_t(\theta, \theta')$ may be approximated with its ‘critical’ part $V_t^c(\theta, \theta')$, whose structure is entirely determined by the eigenfunction $\Phi_0(\theta)$ belonging to $\lambda_t(0)$.

The very same conclusion may be reached through a completely different path, starting from the standard mean-field gap equation:

$$\Delta(k) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} V(k, k') \frac{\Delta(k')}{E(k')} \tanh \frac{\beta E(k')}{2},$$

$$E(k) = \sqrt{\Delta^2(k) + [\varepsilon(k) - \mu]^2},$$

where $V(k, k')$ is understood as an extension of the kernel $V(\theta, \theta')$ out of the Fermi surface. If $\Delta$ is small, compared to the other proper energies of the system, we may assume that the important contribution to the integral Eq. (38) comes from the region around the Fermi surface. Therefore, introducing a cut-off $\Lambda \gg \Delta$, substituting the renormalized $\hat{V}_t$ kernel for the unrenormalized interaction, and neglecting the energy dependence of $\Delta$ and $V$, we may eventually write Eq. (38) as:

$$\Delta_t(\theta) = -\frac{1}{2} \int d\tau' V_t(\theta, \theta') \Delta_t(\theta') \int_{-\Lambda}^{\Lambda} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta_t^2(\theta')}} \tanh \left[ \frac{\beta}{2} \sqrt{\varepsilon^2 + \Delta_t^2(\theta')} \right].$$

An explicit $t$ dependence had to be attached to $\Delta$ and $\beta$, albeit trivial, being a consequence of choosing to keep $\Lambda$ fixed during the flow: both $\Delta$ and $\beta^{-1}$ are energy scales of the system, and are therefore expanded as $\Delta_t = \Delta e^t$, $\beta_t^{-1} = \beta^{-1} e^t$ during the flow. In the present context, it would be preferable to work with a flowing cut-off $\Lambda_t = \Lambda_0 e^{-t}$ and fixed values of the energy scales $\Delta$, $\beta$, so that changing variables $\varepsilon \mapsto \varepsilon e^t$ we may write Eq. (38) as:

$$\Delta(\theta) = -\frac{1}{2} \int d\tau' V_t(\theta, \theta') \Delta(\theta') \int_{-\Lambda_t}^{\Lambda_t} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta^2(\theta')}} \tanh \left[ \frac{\beta}{2} \sqrt{\varepsilon^2 + \Delta^2(\theta')} \right].$$
We first discuss Eq. (40) near the transition point, $\beta \to \beta_c$, where the RG approach shows itself more effective than at $\beta \gg \beta_c$. In a mean-field framework, the critical point would then be defined just by the vanishing of the gap function. Without loss of generality, we may pose:

$$\Delta(\theta) = \Delta_c \chi(\theta),$$

being $\chi(\theta)$ regular at the critical point and $\Delta_c$ a scale parameter, $\Delta_c = O(\beta - \beta_c)$. Insertion of Eq. (41) into Eq. (40) at the critical point yields:

$$\chi(\theta) = - \int_0^{\Lambda t} \frac{d\varepsilon}{\varepsilon} \tanh \left( \frac{\beta_c \varepsilon}{2} \right) \int d\tau \theta V_t(\theta, \theta') \chi(\theta'),$$

which shows that $\chi(\theta)$ is the eigenfunction of the integral operator $\hat{V}_t$ belonging to the eigenvalue $\lambda_t$ such that:

$$\frac{1}{\lambda_t} = - \int_0^{\Lambda t} \frac{d\xi}{\xi} \frac{\tanh \frac{\xi^2}{2}}{\xi}.$$

The latter result makes sense only if $\lambda_t$ is the most negative eigenvalue of $\hat{V}_t$, in which case Eq. (43) is identical to Eq. (33), thus yielding the same characterization for the critical temperature, Eq. (34).

As we expected, the functional form of the gap function near the critical point is fixed as:

$$\Delta(\theta) = \Delta_c \Phi_0(\theta).$$

We notice that substituting the critical part $V_t^c$ for $V_t$ in the gap equation (10) yields the same result for both the critical temperature and the gap structure. The latter finding is of particular interest when the eigenfunctions display different symmetries, since only one of them is seen to rule over the symmetry pattern of the energy gap, so that no wave mixing, such as $s-d$, may occur as $T \to T_c$.

We remark that Eq. (43) stems from a gap equation linearization near the critical point, so that its agreement with Eq. (33) points forward a substantial equivalence of the RG
approach with the mean-field approximation. However, the former approach displays the remarkable advantage of characterizing the transition in terms of the properties of the system near the Fermi surface. Namely, the shape of the latter together with the interaction kernel evaluated at the Fermi energy entirely determine the RG flow. In this sense, the RG approach allows to classify within a unified scheme the instabilities of fermionic systems sharing the same behaviour near the Fermi surface, although different in nature.

On the other hand, integrating modes far from the Fermi surface provides a simplified description of the system which still contains all its relevant features. Besides, this description will prove itself handier when dealing with the numerical implementations, as in the next Section.

In general, inserting the expansion Eq. (36) in the gap equation (40), we may formally write:

\[
\Delta(\theta) = -\frac{1}{2} \sum_\alpha \Phi^*_\alpha(\theta) \lambda_\alpha(\alpha) \int d\tau \Phi_\alpha(\theta') \Delta(\theta') \int_{-\Lambda}^{\Lambda} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta^2(\theta')}} \tanh \left[ \frac{\beta}{2} \sqrt{\varepsilon^2 + \Delta^2(\theta')} \right],
\]

which means that, at any \( T < T_c \), the gap function is a linear combination of eigenfunctions of \( \hat{V}_t \). This is not a trivial statement, since the eigenfunctions appearing in the expansion belong to non-zero eigenvalues, whose number may be finite in several physically relevant cases, i.e. when the kernel is separable.

Eq. (45) is highly non-linear, and may possess more than one solution. For instance, broken-symmetry solutions may occur as linear combination of eigenfunctions belonging to different invariant subspaces. However, in the limit \( \beta \to \beta_c \), the physical solution tends to a specific eigenfunction with a fixed symmetry, thus preventing the occurrence of a broken-symmetry gap function at the critical point.

Far from the transition point, the present analysis does not add too much to the general comprehension of the problem. For instance, at zero temperature the internal integral can be evaluated in Eq. (45) thus yielding, in the limit \( \Delta \ll \Lambda \):
\[
\Delta(\theta) = - \sum_{\alpha} \Phi_\alpha(\theta) \lambda_t(\alpha) \int d\tau_\theta \Phi_\alpha(\theta') \Delta(\theta') \log \left| \frac{2\Lambda_t}{\Delta(\theta')} \right|.
\] (46)

The latter gap equation, valid in the limit \( \Delta \ll \Lambda \), can be easily proven to be scale invariant according to the flow equation (30), and is completely equivalent to the scale invariant gap equation recovered by Weinberg. Moreover, we write the expansion Eq. (45) as:

\[
\Delta(\theta) = \sum_{\alpha} \Phi_\alpha^*(\theta) \Delta(\alpha),
\] (47)

whose insertion into Eq. (46), making use of the linear independence of the eigenfunctions, yields:

\[
\Delta(\alpha) = -\lambda_t(\alpha) \sum_{\alpha'} \Delta(\alpha') \langle \alpha' | \log \left| \frac{2\Lambda_t}{\Delta(\theta)} \right| | \alpha \rangle,
\] (48)

being \( \langle \alpha' | f(\theta) | \alpha \rangle \equiv \int d\tau_\theta \Phi_\alpha^*(\theta) f(\theta) \Phi_\alpha(\theta) \), which is a set of non-linear coupled equations for the coefficients \( \Delta(\alpha) \). Multiplying times \( \Delta^*(\alpha)/\lambda_t(\alpha) \), summing over \( \alpha \), and introducing the scale \( \Delta_0 \) defined as:

\[
\int d\tau_\theta |\Delta(\theta)|^2 \log \left| \frac{\Delta_0}{\Delta(\theta)} \right| = 0,
\] (49)

one eventually obtains:

\[
\Delta_0 = 2\Lambda_t \exp \left( \frac{\sum_\alpha |\Delta(\alpha)|^2/\lambda_t(\alpha)}{\sum_\alpha |\Delta(\alpha)|^2} \right),
\] (50)

which is valid only in the limit \( \Delta_0 \ll \Lambda_t \). We observe that far from the critical point all the components of the coupling kernel \( \hat{V}_t \) contribute to the determination of the gap function. In practice, the present approach has the advantage of dealing with a set of coefficients \( \Delta(\alpha) \) whose number may be finite, in the case of a separable kernel. However, we cannot escape from a full solution of the set of the non-linear coupled equations (48). The present decomposition in eigenfunctions is the generalization of the standard decomposition in eigenfunctions of the angular momentum, which are simultaneous eigenfunctions of \( \hat{V}_t \) only for a fully rotationally invariant system.
If some symmetry invariance is present in both the interaction and in the Fermi surface, then the set of equations (48) may be partially decoupled: in other words, if \( \{ \Phi_\alpha(\theta) \} \) is an invariant subspace, and \( \hat{U}_g \) is a unitary representation of the symmetry group \( \{ g \} \), i.e. 
\[
[\hat{U}_g, \hat{V}_t] = 0 \quad \text{and} \quad \{ \hat{U}(g) \Phi_\alpha(\theta) \} \equiv \{ \Phi_\alpha(\theta) \}, \quad \forall g \in \{ g \},
\]
then we may require that the gap function shares the same symmetry:
\[
\hat{U}(g) \Delta(\theta) = e^{i\varphi(g)} \Delta(\theta), \quad \forall g \in \{ g \} \quad (|\Delta(\theta)|^2 \text{ invariant}), \quad (51)
\]
which is equivalent to saying that:
\[
\hat{U}(g) \Phi_\alpha(\theta) = e^{i\varphi(g)} \Phi_\alpha(\theta), \quad \forall \Phi_\alpha \in \{ \Phi_\alpha \} \quad (52)
\]
(the eigenfunctions \( \{ \Phi_\alpha \} \) belong to the same eigenvalue \( e^{i\varphi(g)} \)) and \( \Delta(\theta) = \sum_\alpha \Delta(\alpha) \Phi_\alpha^*(\theta) \), with \( \Phi_\alpha \) belonging to the invariant subset \( \{ \Phi_\alpha \} \).

The set of equations (48) does admit such solutions with a given symmetry (eigenfunctions of \( \hat{U} \)) since the matrix elements \( \langle \alpha' \mid \log |2\Lambda_t/\Delta| \mid \alpha \rangle \) vanish if the functions \( \Phi_\alpha, \Phi_{\alpha'} \) belong to different invariant subspaces, provided that \( |\Delta| \) is invariant.

Of course the symmetry invariance of the integral operator \( \hat{V}_t \) does not prevent from the occurrence of broken-symmetry solutions, since the non-linearity of the set of equations (48) does not guarantee the uniqueness of the symmetric solution.

**IV. A TIGHT-BINDING MODEL FOR LAYERED SUPERCONDUCTORS**

The anisotropic flow equations, derived at a finite temperature in the previous Sections, are here employed in the framework of a simple tight-binding model recently proposed for describing the band structure and superconductive properties of Bi\textsubscript{2}Sr\textsubscript{2}CaCu\textsubscript{2}O\textsubscript{8+\delta} (BSCCO).\(^{3,4}\) Alike the majority of the high \( T_c \) cuprate superconductors, BSCCO is characterized by a layered structure, which gives rise to many anisotropic physical properties, and further results in a rotationally non-invariant hole dispersion relation \( \varepsilon(\mathbf{k}) \).

Allowing for nearest-neighbour inter-plane hopping and for nearest-neighbour and next-nearest-neighbour intra-plane hopping, in the usual tight-binding approximation, \( \varepsilon(\mathbf{k}) \) reads
where the components \( k_x, k_y, k_z \) of the wave-vector \( \mathbf{k} \) are measured in units of the respective inverse lattice spacings. The constants \( A, B, C \) have been determined by comparison with photoemission data\(^3\)\(^4\) to be \( A = 0.05 \) eV, \( B = 0.45 \), \( C = 0.1 \) in order to reproduce the observed hole density of states for BSCCO. In particular, the condition \( C \ll B \ll 1 \) denotes a smaller intra-plane next-nearest-neighbour hopping probability than a nearest-neighbour one, and an even smaller inter-plane nearest-neighbour one, due to the large plane separation.

In the following, we shall measure all the energies in units of \( A \).

In order to set up the model, we must assume the existence of a pairing interaction, and the short coherence length would suggest the relevance of the short range part of the interaction. Thus, on very general grounds, neglecting the long range contribution, we may expand any pairing interaction as the sum of an on-site term and of contributions arising from nearest neighbour sites, next-nearest neighbours and so on.

We do not address any question concerning the physical origin of the pairing interaction, but once the free fermion dispersion relation has been fixed on a phenomenological basis, we rather wish to explore the main physical consequences arising from the choice of a pairing interaction. In other words, we keep the dispersion relation fixed, and change the interaction in order to observe the effects on the superconductive instability.

As an illustration of the method developed before, we here retain only two terms in the expansion of the interaction, namely the on-site and the in-plane nearest-neighbour singlet pairing couplings. The effects of only an in-plane nearest-neighbour pairing have been already considered, together with an identical dispersion relation, in the framework of mean-field approximation, by Spathis et al.\(^4\)

The Fourier expansion of the pairing interaction \( V(\mathbf{k}, \mathbf{k}') \) reads as:

\[
V(\mathbf{k}, \mathbf{k}') = u_0 + 2u_x \cos(k_x - k_x') + 2u_y \cos(k_y - k_y'),
\]

(54)
being \( u_0 \) the on-site Hubbard \( U \) and \( u_x, u_y \) the nearest-neighbour interactions in the \( x \) and \( y \) directions, respectively. Enforcing the general symmetry properties Eq. (3) on the interaction matrix, the singlet pairing interaction results in the symmetrized part of Eq. (54) with respect to the inversion \( \mathbf{k}' \rightarrow -\mathbf{k}' \) or \( \mathbf{k} \rightarrow -\mathbf{k} \),

\[
V(\mathbf{k}, \mathbf{k}') = u_0 + u_x [\cos(k_x - k'_x) + \cos(k_x + k'_x)] + u_y [\cos(k_y - k'_y) + \cos(k_y + k'_y)] = u_0 + 2u_x \cos k_x \cos k'_x + 2u_y \cos k_y \cos k'_y.
\] (55)

The Fermi surface, defined as the locus of the points \( \mathbf{k} \) in the momentum space verifying the equation \( \varepsilon(\mathbf{k}) = \mu \), is a quite complicated, definitely rotationally non-invariant surface, whose shape varies furthermore with \( \mu \). For \( \mu_1 = -4 + 4B - 2C \leq \mu \leq -4 + 4B + 2C = \mu_2 \) it is a closed surface, which for small values of the chemical potential \( \mu \) can be approximated by an ellipsoid of square semiaxes \( \bar{k}^2_x = \bar{k}^2_y = (\mu + \mu_1)/(2B - 1), \bar{k}^2_z = -(\mu + \mu_1)/C \), which reduces to a sphere since \( C = 1 - 2B \). The Fermi surface becomes an open surface as \( \mu \) increases up to the full bandwidth, \( \mu_3 = 4 + 4B + 2C \), where it reduces to the eight zone corners. Fig. 4 shows the Fermi surface within the positive octant of the first Brillouin zone for various values of \( \mu \).

It seems convenient to choose \( \theta = (k_x, k_y) \equiv (\theta_x, \theta_y) \) as suitable coordinates upon the Fermi surface, for each value of \( \mu \). Due to its varying shape at increasing \( \mu \), coordinates \( \theta \) will be affected by limitations, depending on \( \mu \), which we shall keep understood in the following, when writing integrals over the Fermi surface. The Jacobian function Eq. (12) may then be straightforwardly worked out as:

\[
J^{-1}(\theta, \varepsilon) = \sqrt{4C^2 - (\mu + \varepsilon + 2(\cos \theta_x + \cos \theta_y) - 4B \cos \theta_x \cos \theta_y)^2}.
\] (56)

The hole dispersion relation Eq. (53) also fixes the density of states of the system, in its normal phase, which may be expressed as:

\[
n(\varepsilon + \mu) = \frac{1}{2\pi} \int \frac{d^2\theta}{(2\pi)^2} J(\theta) = 2 \int d\tau \theta,
\] (57)

A further integration from the band bottom \( \mu = \mu_1 \) yields the total fraction of occupied states:
\[ N(\mu) = \int_{\mu_1}^{\mu} d\varepsilon \ n(\varepsilon), \quad (58) \]

together with the ‘normalization’ condition at the top of the band, \( N(\mu_3) = 1 \).

The quantities \( n(\mu) \) and \( N(\mu) \) have been numerically evaluated, as functions of the chemical potential \( \mu \), \(-\mu_1 \leq \mu \leq \mu_3 \) (Fig. 3, 5). In particular, \( n(\mu) \) displays a pronounced, yet finite, maximum for \( \mu = \mu_2 = -2.0 \). It may be regarded as a token of the system’s quasi-bidimensionality, due to its layered structure, and would have been a true van Hove singularity, resulting in an infinite peak, if no inter-plane hopping had been considered, however small \( (C = 0) \).

For our purposes, the pairing potential Eq. (55) is recognized as the symmetrical kernel \( V(\theta, \theta') \) in Eq. (18), provided that \( k_x, k_y \mapsto \theta_x, \theta_y, k'_x, k'_y \mapsto \theta'_x, \theta'_y \), and that \( \theta, \theta' \) are suitably restricted on the Fermi surface. This kernel may then be put into the ‘separate’ form:

\[ V(\theta, \theta') = \sum_{i,j=0,x,y} \eta_{ij} U_i(\theta) U_j(\theta'), \quad (59) \]

with:

\[
\eta = \begin{pmatrix} u_0 & 0 & 0 \\ 0 & 2u_x & 0 \\ 0 & 0 & 2u_y \end{pmatrix} \quad (60)
\]

and:

\[
U(\theta) = \begin{pmatrix} 1 \\ \cos \theta_x \\ \cos \theta_y \end{pmatrix} \quad (61)
\]

Insertion of Eq. (59) into Eq. (28) for the eigenvalue problem straightforwardly yields:

\[
\Phi_\alpha(\theta) = \sum_{i=0,x,y} \gamma_i(\alpha) U_i(\theta). \quad (62)
\]

Further substitution yields:

\[
\sum_{i=0,x,y} U_i(\theta) \left[ \lambda(\alpha) \gamma_i(\alpha) - \sum_{j,k=0,x,y} \eta_{ij}\gamma_k(\alpha) \int d\tau U_j(\theta) U_k(\theta) \right] = 0, \quad (63)
\]
which, due to the linear independence of the functions $U_i$, yields:

$$\sum_{k=0,x,y} \gamma_k(\alpha) \left[ \lambda(\alpha) \delta_{ik} - \sum_{j=0,x,y} \eta_{ij} \langle U_j U_k \rangle \right] = 0 \quad (64)$$

with:

$$\langle U_j U_k \rangle = \int d\tau_0 U_j(\theta) U_k(\theta), \quad (65)$$

being, in particular, $2\langle U_0^2 \rangle(\mu) = n(\mu)$ (Fig. 3).

The linear homogeneous system Eq. (64) allows then for non trivial solutions $\gamma_i(\alpha) \ (i = 0, x, y)$ if and only if the secular condition is fulfilled:

$$\det \left[ \lambda(\alpha) \delta_{ik} - \sum_{j=0,x,y} \eta_{ij} \langle U_j U_k \rangle \right] = 0, \quad (66)$$

which allows one to determine $\lambda(\alpha)$ as a function of the parameters $u_0, u_x, u_y$ and of the chemical potential $\mu$. These should be regarded as the eigenvalues of the operator $\hat{V}_t$ at the beginning of the RG flow ($t = 0$). Their sign therefore accounts for an instability: the existence of a negative eigenvalue $\lambda(\alpha)$ heralds a later divergence pattern for $t \to t_c$, $t_c$ being determined by the most negative of the $\lambda(\alpha)$. The eigenfunctions may then be determined by standard algebra through the coefficients $\gamma_i(\alpha)$ of their expansion in terms of the $U_i$.

We remark that the main numerical task is to evaluate, once for all, the integrals defined in Eq. (65). The secular condition, Eq. (66), can then be easily discussed by changing at will the values of the coupling parameters. Besides, the overall formalism allows a straightforward generalization, by adding other terms in the development of the potential function, Eq. (55), thus increasing the order of the matrix $\langle U_i U_j \rangle$.

**V. DISCUSSION AND FINAL REMARKS**

The results we have exposed thus far may be discussed with respect to the various possible values in the couplings, $u_0, u_x, u_y$.

In the case we may neglect an on-site pairing ($u_0 = 0$), Eq. (66) factorizes to yield:
\[ \lambda(0) = 0 \]
\[ \lambda(1, 2) = (u_x + u_y)\langle U_x^2 \rangle \pm \sqrt{(u_x - u_y)^2\langle U_x^2 \rangle^2 + 4u_xu_y\langle U_xU_y \rangle^2}. \] (67)

Furthermore, in the symmetric case, \( u_x = u_y \), the non-vanishing eigenvalues reduce to:
\[ \lambda(1, 2) = 2u_x[\langle U_x^2 \rangle \pm \langle U_xU_y \rangle], \] (68)

which are displayed in Fig. 7 as functions of the chemical potential \( \mu \).

Since \( \langle U_x^2 \rangle \geq \langle U_xU_y \rangle \), for any value of \( \mu \), the eigenvalues Eq. (68) are both negative for any attractive inter-site coupling \( u_x < 0 \), however weak. A superconducting ground state is thus predicted for any filling \( N \) of the band, although the critical temperature is undistinguishable from zero even as \( N \gtrsim 0.5 \). This may be regarded as a mere consequence of the very pronounced peak in the density of states (Fig. 5), which determines the behaviour of all the eigenvalues.

As shown in Fig. 8, insertion of the most negative eigenvalue into Eq. (34) yields a maximum for \( T_c \), and the existence of a superconducting phase for \( N \lesssim 0.5 \), with \( u_x = -1.0 \).

Here, the occurrence of one or more peaks is not a serious problem in comparison with the experimental results, which predict a smooth plateau for \( T_c \) vs \( N \). In fact, the experimental data are relative to different samples with different composition, and any eventual peak would be smeared out on the average. The trend in \( T_c \) vs \( N \) predicted by this RG analysis basically confirms the analogous result recovered via direct solution of the standard mean-field gap equation, even if \( T_c \) is never rigorously zero, except at the band edges. At variance with the experimental evidence, for \( u_0 = 0 \), \( T_c \) has a steep increase at the band bottom, starting exactly at \( N = 0 \). We shall comment on this fact later on.

In the case of such a symmetric interaction, \( u_x = u_y \), the exchange symmetry \( x \leftrightarrow y \), which is present in the hole dispersion relation \( \varepsilon(k) \) Eq. (54) and therefore in the Fermi surface, is restored even for the interaction. Thus, the kernel \( V(\theta, \theta') \) is now symmetric under the exchange \( \theta_x \leftrightarrow \theta_y, \theta'_x \leftrightarrow \theta'_y \). The eigenfunctions of the integral operator \( \hat{V} \) are even or odd with respect to such an invariance transformation, and, as discussed at the end
of Section III, the scale invariant gap equation (46) admits even (s wave) or odd (d wave) solutions for the gap function.

Moreover, if \( u_0 = 0 \), there are just two eigenfunctions in the expansion for the gap, Eq. (47), and at \( T = 0 \) the gap equation admits the decoupled solutions:

\[
\Delta(\theta) = \text{const } \Phi_1(\theta); \quad \Delta(\theta) = \text{const } \Phi_2(\theta), \quad (T = 0),
\]

where the proportionality constants are fixed by the scale \( \Delta_0 \) defined by Eq. (49) and explicitly given by Eq. (50):

\[
\Delta_0 = 2\Lambda_t \exp \left( \lambda^{-1}_t(1, 2) \right).
\] (70)

The eigenfunctions \( \Phi_1, \Phi_2 \) follow by direct solution of the system Eq. (64), which yields \( \gamma_y(1) = \gamma_x(1), \gamma_y(2) = -\gamma_x(2) \), whence:

\[
\Phi_1(\theta) = \text{const } [\cos \theta_x + \cos \theta_y] \quad \text{s wave},
\]
\[
\Phi_2(\theta) = \text{const } [\cos \theta_x - \cos \theta_y] \quad \text{d wave}.
\] (71)

The latter equations, together with Eq. (70), fix the two symmetric solutions for the gap function. While in general a broken-symmetry solution may occur (e.g., a mixed s-d wave), a pure s or d wave solution is always expected close to the transition points, which for \( T = 0 \) are \( N = 0 \) and \( N \approx 0.5 \). In Fig. 7 we observe a cross-over around the peaked region of the eigenvalues: at the bottom of the band, \( \mu = \mu_1 \), the most negative eigenvalue corresponds to the even eigenfunction \( \Phi_1 \), while for larger values of \( N \) the most negative eigenvalue corresponds to the odd eigenfunction \( \Phi_2 \). The cross-over explains the occurrence of a double peak for \( T_c \) in Fig. 8, and allows us to predict, at the transition point, the opening of an s-wave gap for \( N < \sim 0.2 \), and of a d-wave gap for \( N \gtrsim 0.2 \). At \( T = 0 \), an intermediate broken-symmetry solution is awaited around the cross-over. All this is in agreement with a previous mean-field analysis.4

Now let us switch on an on-site interaction, \( u_0 \neq 0 \). In general, the cubic equation (66) may be solved analytically, but it is instructive to look a little closer at the case of a
symmetric inter-site interaction, \( u_x = u_y \). Again, the eigenfunctions of \( \hat{V} \) must be even or odd with respect to the exchange \( x \leftrightarrow y \).

The three linearly independent functions \( U_0, U_x, U_y \) may be written in terms of the set \( \{U_0, U_\pm\} \), being \( U_\pm = (U_x \pm U_y)/\sqrt{2} \). Since both \( U_0 \) and \( U_+ \) are even, then the function \( U_- \), which is odd, generates a one-dimensional invariant subspace for \( \hat{V} \), so that \( U_- \equiv \Phi_2 \) must be an eigenfunction for \( \hat{V} \), and its eigenvalue \( \lambda(2) \) cannot be affected by the presence of an on-site coupling. Therefore,

\[
\Phi_2 = \text{const} \left[ \cos \theta_x - \cos \theta_x \right],
\]

\[
\lambda(2) = 2u_x \left[ \langle U_0^2 \rangle - \langle U_x U_y \rangle \right].
\]

In fact, in the symmetric case, the cubic equation (66) factorizes as:

\[
\lambda(0, 1) = \frac{u_0}{2} \langle U_0^2 \rangle + u_x \left[ \langle U_x^2 \rangle + \langle U_x U_y \rangle \right] \pm \sqrt{\left[ \frac{u_0}{2} \langle U_0^2 \rangle - u_x \left( \langle U_x^2 \rangle + \langle U_x U_y \rangle \right) \right]^2 + 4u_0u_x \langle U_0 U_x \rangle^2}.
\]

About the choice of the parameters \( u_0, u_x \), we may anticipate that the overall trend for \( T_c \) vs \( N \) is not relevantly affected by any reasonable change in their values, since it is a direct consequence of the chosen dispersion relation, as it is quite evident by comparison with the density of states pictured in Fig. 5. Nonetheless, the most interesting scenario shows up when there is a competition between on-site and inter-site interactions. Namely, for \( u_0 = -1.0 \), \( u_x = u_y = 1.0 \), the eigenvalues are reported in Fig. 6, and are characterized by some nice features: first of all, one only eigenvalue, \( \lambda(0) \), is negative and leads the phase-transition; we may then notice that at the band bottom, \( N \to 0 \), the negative eigenvalue drops to a very small value at a finite filling \( N > 0 \); finally, the same eigenvalue goes to zero almost linearly inside the band for \( \mu \approx 2.26 \), where a phase transition is expected even at \( T = 0 \). Actually, the eigenvalue never crosses the \( \lambda = 0 \) axis, as it is shown in the insert of Fig. 6. The eigenvalues \( \lambda(0) \) and \( \lambda(1) \), whose corresponding eigenfunctions share the same symmetry, cannot cross and repel each other, as predicted by a general theorem due to E. Wigner and J. von Neumann. However, we notice that a small negative eigenvalue is equivalent to a
vanishing value since, due to the exponential in Eq. (34), the critical temperature becomes extremely low and thus negligible.

Comparing with the case of a pure negative on-site interaction \((u_x = u_y = 0)\), when the eigenvalue coincides (up to a factor) with the density of states \(n\) of Fig. [3], we can assert that the presence of a non-zero nearest-neighbour repulsion reduces the range of the superconductive phase inside the band. This is not trivial since, for instance, if \(u_x < 0\), the presence of an on-site repulsive interaction \(u_0 > 0\) does not produce any relevant effect on the phase diagram. In fact, reversing the sign of all the eigenvalues in Fig. [3], and comparing with Fig. [4], we notice that the presence of the \(d\) wave eigenvalue \(\lambda(2)\), whose value is not affected by \(u_0\), largely reduces the weight of an on-site repulsion against the inter-site attractive coupling.

The progressive reducing of the superconducting phase with the increase of the inter-site repulsion is illustrated in Fig. [4] and Fig. [5]. Here, \(u_0 = -4.5\) and \(u_x\) varies from 0.5 to 3.5. Fig. [4] displays the negative eigenvalue while the critical temperature is reported in Fig. [5]. We notice that, as a consequence of the competition between \(u_0\) and \(u_x\), now at the bottom of the band the critical temperature drops to zero at a finite filling \(N > \sim 0\), ranging from 0 to 0.1, in qualitative agreement with the experimental data.\[5\]

At the critical point the gap function is described by the even eigenfunction:

\[
\Phi(\theta) = \gamma_0 + \gamma_x (\cos \theta_x + \cos \theta_y),
\]

being \((\gamma_0, \gamma_x, \gamma_y = \gamma_x)\) the solution of the linear equations (34).

Up to now we did not make any effort in order to justify the physical origin of the interaction Eq. (55). This simple RG analysis seems to suggest the occurrence of a competition between an on-site negative Hubbard coupling and a repulsive nearest-neighbour interaction, on the basis of the comparison with the experimental data. We notice that, while several microscopic models predict the occurrence of a short-range attractive coupling the nearest-neighbour repulsion could be justified by the presence of a long-range Coulomb interaction. Nonetheless, we must caution, since for a full analysis of the pairing interaction a larger
number of terms should be retained in its definition Eq. (54), thus increasing the order of the coupling matrix, Eq. (60).

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REFERENCES

1 R. Shankar, *Rev. Mod. Phys.*, **66** 1, 129 (1994).

2 S. Weinberg, *Nuclear Phys.* B [FS], **413**, 567 (1994).

3 T. Schneider, M.P. Sörensen, *Z. Phys.*, **B81**, 3 (1990).

4 P.N. Spathis, M.P. Sörensen, N. Lazarides, *Phys. Rev.*, B**45**, 13, 7360 (1992).

5 See, *e.g.*, R. Micnas, J. Ranninger, S. Robaskiewicz, *Rev. Mod. Phys.*, **62** 1, 113 (1990), for a review.

6 J.M. Luttinger, *Phys. Rev.*, **119** 4, 1153 (1960). See also W. Kohn, J.M. Luttinger, *Phys. Rev.*, **118** 1, 41 (1960), and J.M. Luttinger, J.C. Ward, *Phys. Rev.*, **118** 5, 1417 (1960).

7 I.S. Gradshtein, I.M. Ryzhik, *Table of integrals, series, and products*, A. Jeffrey, Ed. (Academic Press, New York, 1980), Eq. 1.421.2.

8 See, *e.g.*, A.N. Kolmogorov, S.V. Fomin, *Elementy teorii funktsij i funktsional’nogo analiza*, (Nauka, Moscow, 1980).

9 See, *e.g.*, A.L. Fetter, J.D. Walecka, *Quantum theory of many-particle systems*, (McGraw-Hill, New York, 1971), p. 439 ff., and P.G. de Gennes, *Superconductivity of Metals and Alloys*, (Addison-Wesley, Redwood City, 1989), p. 141 ff.

10 H. Zhang, H. Sato, *Phys. Rev. Lett.*, **70**, 11, 1697. (1993).

11 L.D. Landau, E.M. Lifshitz, *Quantum Mechanics - Non-relativistic theory*, (Pergamon Press, London, 1958), p. 264 ff.
FIGURES

FIG. 1. The ‘tree’ and the ‘tadpole’ diagrams are shown, which contribute to the interaction part of the action, Eq. (4), at a tree level.

FIG. 2. We show the three second-order one-loop diagrams, together with the factors arising from symmetry and statistics requirements. The only one which gives a relevant contribution is the BCS one.

FIG. 3. RG flow diagram. The flow lines for $(\beta \Lambda, \lambda)$ have been numerically evaluated for different values of the unrenormalized eigenvalue $\lambda_0$. The inverse temperature $\beta$ is accordingly seen to renormalize either to zero or to some finite value, as soon as $\lambda_t$ is able to diverge at a finite $t = \bar{t}$. The dashed line joins the critical points for the different values of $\lambda$, and tends asymptotically to the axis $\beta = 0$ as $t \to \infty$.

FIG. 4. The Fermi surfaces corresponding to a hole dispersion relation Eq. (53) are shown, in correspondence to increasing values of the chemical potential, $\mu = -2.1, -1.8, 0.0, 3.0, 5.8$. The box selects only the positive octant of the first Brillouin zone ($0 \leq k_x, k_y, k_z \leq \pi$), being $\Gamma = (0,0,0)$, $\bar{M} = (\pi,0,0)$, $X = (\pi,\pi,0)$, $Y = (\pi,\pi,\pi)$.

FIG. 5. The density of states $n(\mu)$, Eq. (57), is shown for the normal (non-interacting) Fermi system, as function of the chemical potential $\mu$, $\mu_1 \leq \mu \leq \mu_3$. It displays a pronounced, yet finite, maximum for $\mu = \mu_2 = -2.0$. It is a token of the quasi-bidimensionality of the system, due to its layered structure, and would have been a real van Hove singularity if no inter-plane hopping had been considered ($C = 0$).

FIG. 6. The total fraction of occupied states $N(\mu)$, Eq. (58), is shown for the normal (non-interacting) Fermi system, as functions of the chemical potential $\mu$, $\mu_1 \leq \mu \leq \mu_3$. The normalization condition at the top of the band $N(\mu_3) = 1$ is clearly fulfilled.

FIG. 7. The two non-zero eigenvalues $\lambda(1)$ (dashed line) and $\lambda(2)$ (full line) are here displayed over $\mu$, $\mu_1 \leq \mu \leq \mu_3$, for the values of the parameters $u_0 = 0.0$, $u_x = u_y = -1.0$. 

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FIG. 8. We show here the ratio $T_c/\Lambda$ for the values of the parameters $u_0 = 0.0$, $u_x = u_y = -1.0$, as a function of the fraction of occupied states $N$.

FIG. 9. We show the eigenvalues $\lambda(0)$ (negative), $\lambda(1)$ (positive, dashed) and $\lambda(2)$ (full line) over $\mu$, $\mu_1 \leq \mu \leq \mu_3$, in the symmetrical case corresponding to the values of the parameters $u_0 = -1.0$, $u_x = u_y = 1.0$. The detail focuses on the region around $\mu \simeq 2.262$, where the two eigenvalues corresponding to the same symmetry ($\lambda(0)$ and $\lambda(1)$) run closely without crossing. The eigenvalue $\lambda(0) < 0$ is seen to generate an instability.

FIG. 10. We show the only eigenvalue giving rise to an instability, $\lambda(0)$, as a function of $\mu$, $\mu_1 \leq \mu \leq \mu_3$, for the values of the parameters (from bottom to top): $u_0 = -4.5$, $u_x = u_y = 0.5, 1.5, 2.5, 3.5$.

FIG. 11. We show the ratio $T_c/\Lambda$ as a function of the fraction of occupied states $N(\mu)$, for the values of the parameters (from top to bottom): $u_0 = -4.5$, $u_x = u_y = 0.5, 1.5, 2.5, 3.5$. 

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