Directed Width Parameters and Circumference of Digraphs

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Abstract

We prove that the directed treewidth, DAG-width and Kelly-width of a digraph are bounded above by its circumference plus one.

Keywords: arboreal decomposition, directed treewidth, DAG-decomposition, DAG-width, Kelly decomposition, Kelly-width.

1 Introduction

The circumference of an undirected graph (resp. digraph) $G$, denoted by $\text{circ}(G)$, is the length of a longest simple undirected (resp. directed) cycle in $G$. The circumference of a DAG is defined to be one. The circumference of an undirected tree is defined to be two. Birmele [Bir03] proved that the treewidth of an undirected graph $G$, denoted by $\text{tw}(G)$, is at most its circumference minus one.

**Theorem 1.** (Birmele [Bir03]) For an undirected graph $G$, $\text{tw}(G) \leq \text{circ}(G) - 1$.

Motivated by the success of treewidth in algorithmic and structural graph theory, efforts have been made to generalize treewidth to digraphs. Johnson et al. [JRST01] introduced the first directed analogue of treewidth called directed treewidth. Berwanger et al. [BDHK06] and independently Obdrzalek [Obd06] introduced DAG-width. Hunter and Kreutzer [HK08] introduced Kelly-width. For a digraph $G$, let $\text{dtw}(G)$, $\text{dgw}(G)$ and $\text{kw}(G)$ denote its directed treewidth, DAG-width and Kelly-width respectively. All these directed width measures are generalizations of undirected treewidth i.e., for an undirected graph $G$, let $\bar{G}$ be the digraph obtained by replacing each edge $\{u, v\}$ of $G$ by two directed edges $(u, v)$ and $(v, u)$, then:

- $\text{dtw}(\bar{G}) = \text{tw}(G)$ [JRST01, Theorem 2.1]
- $\text{dgw}(\bar{G}) = \text{tw}(G) + 1$ [BDHK06, Proposition 5.2]
- $\text{kw}(\bar{G}) = \text{tw}(G) + 1$ [HK08]
We prove that the directed treewidth, DAG-width and Kelly-width of a digraph are bounded above by its circumference plus one. Our proofs generalize Birmele’s idea of constructing a tree decomposition using a depth-first search tree. For the directed treewidth we construct an arboreal decomposition from the depth-first search tree very naturally. The underlying arborescence is the depth-first search tree itself. For the DAG-width and Kelly-width we construct the underlying DAG using the depth-first search tree and some carefully chosen additional edges. Constructing the corresponding “bags” requires some additional work to satisfy the strict guarding conditions of DAG-decompositions and Kelly-decompositions. Our main theorem is as follows:

**Theorem 2.** For a digraph $G$,

- $\text{dtw}(G) \leq \text{circ}(G) + 1$
- $\text{dgw}(G) \leq \text{circ}(G) + 1$
- $\text{kw}(G) \leq \text{circ}(G) + 1$

Birmele’s theorem is tight as $\text{tw}(K_n) = n - 1$ and $\text{circ}(K_n) = n$. Since $\text{dtw}(\overrightarrow{K_n}) = n - 1$, $\text{dgw}(\overrightarrow{K_n}) = n$ and $\text{kw}(\overrightarrow{K_n}) = n$, we conjecture that Theorem 2 can be improved with the following tight bounds:

**Conjecture 3.** For a digraph $G$,

- $\text{dtw}(G) \leq \text{circ}(G) - 1$
- $\text{dgw}(G) \leq \text{circ}(G)$
- $\text{kw}(G) \leq \text{circ}(G)$

Birmele’s theorem does not hold for pathwidth since complete binary trees have unbounded pathwidth. Nesetril and Ossona de Mendez [NdM12] showed that the pathwidth of a 2-connected graph $G$ is at most $(\text{circ}(G) - 2)^2$. Marshall and Wood [MW13] improved this bound to $\lfloor \text{circ}(G)/2 \rfloor (\text{circ}(G) - 1)$. Generalizing these results to directed pathwidth, under a suitable directed connectivity assumption is an interesting open problem.

### 1.1 Notation

We use standard graph theory notation and terminology (see [Die05]). All digraphs are finite and simple (i.e. no self loops and no multiple arcs). For a digraph $G$, we write $V(G)$ for its vertex set and $E(G)$ for its arc set. For $S \subseteq V(G)$ we write $G[S]$ for the subdigraph induced by $S$, and $G \setminus S$ for the subdigraph induced by $V(G) - S$.

We use the term DAG when referring to directed acyclic graphs. A node is a root if it has no incoming arcs. The DAG $T$ is an arborescence if it has a unique root $r$ such that for every node $i \in V(T)$ there is a unique directed walk from $r$ to $i$. Note that every arborescence arises from an undirected tree by selecting a root and directing all edges away from the root.

Let $T$ be a DAG. For two distinct nodes $i$ and $j$ of $T$, we write $i \prec_T j$ if there is a directed walk in $T$ with first node $i$ and last node $j$. For convenience, we write $i \prec j$ whenever $T$ is clear from the context. For nodes $i$ and $j$ of $T$, we write $i \preceq j$ if either $i = j$ or $i \prec j$. For an arc $e = (i, j)$
and a node $k$ of $T$, we write $e \prec k$ if either $j = k$ or $j \prec k$. We write $e \sim i$ (and $e \sim j$) to mean that $e$ is incident with $i$ (and $j$ respectively). We define $T_{\geq v} = T\{x \mid x \geq v\}$.

Let $W = (W_i)_{i \in V(T)}$ be a family of finite sets called node bags, which associates each node $i$ of $T$ to a node bag $W_i$. We write $W_{\preceq i}$ to denote $\bigcup_{j \preceq i} W_j$. For an arc $e$ of $T$, we write $W_{\succ e}$ to denote $\bigcup_{j \succ e} W_j$. Let $A = (A_e)_{e \in E(T)}$ be a family of finite sets called arc bags, which associates each arc $e$ of $T$ to an arc bag $A_e$. We write $A_{\sim i}$ to denote $\bigcup_{e \sim i} A_e$.

### 1.2 Guarding, $X$-normal and Directed unions

Width measures like DAG-width and Kelly-width are based on the following notion of guarding:

**Definition 4.** [Guarding] Let $G$ be a digraph and $W, X \subseteq V(G)$. We say $X$ guards $W$ if $W \cap X = \emptyset$, and for all $(u, v) \in E(G)$, if $u \in W$ then $v \in W \cup X$.

In other words, $X$ guards $W$ means that there is no directed path in $G \setminus X$ that starts from $W$ and leaves $W$. The notion of directed treewidth is based on a weaker condition:

**Definition 5.** [$X$-normal] Let $G$ be a digraph and $W, X \subseteq V(G)$. We say $W$ is $X$-normal if $W \cap X = \emptyset$, and there is no directed path in $G \setminus X$ with first and last vertices in $W$ that uses a vertex of $G \setminus (W \cup X)$.

In other words, $W$ is $X$-normal means that there is no directed path in $G \setminus X$ that starts from $W$, leaves $W$ and then returns to $W$. A digraph $D$ is a directed union of digraphs $D_1$ and $D_2$ if $D_1$ and $D_2$ are induced subgraphs of $D$, $V(D_1) \cup V(D_2) = V(D)$, and no edge of $D$ has head in $V(D_1)$ and tail in $V(D_2)$. The directed treewidth, DAG-width and Kelly-width are closed under directed unions (see [JRST01, BDH12, MTV10]). The following theorem is immediate.

**Theorem 6.** ([JRST01, BDH12, MTV10]) The directed treewidth (resp. DAG-width, Kelly-width) of a digraph $G$ is equal to the maximum directed treewidth (resp. DAG-width, Kelly-width) taken over the strongly-connected components of $G$.

Also, the circumference of a digraph $G$ is equal to the maximum circumference taken over the strongly-connected components of $G$. Hence, we may assume that all digraphs are strongly-connected in the rest of this paper.

### 1.3 Depth-first search tree

Let $G$ be a strongly-connected digraph. Let $T$ be a depth-first search tree of $G$ starting at an arbitrary root $r \in V(G)$. The tree $T$ is an arborescence rooted at $r$. The edges of $G$ are classified into one of the four types: tree edges, forward edges, back edges and cross edges (see [CLRS01]). For a vertex $v \in V(G)$, let $dfs(v)$ be the “dfs number” of $v$ i.e., the time-stamp assigned to $v$ when $v$ is visited for the first time during the construction of $T$. 
2 Directed treewidth and Circumference

Definition 7. [Arboreal decomposition and directed treewidth][JGST01] An arboreal decomposition of a digraph $G$ is a triple $\mathcal{D} = (T, W, A)$, where $T$ is an arborescence, and $W = (W_i)_{i \in V(T)}$ is a family of subsets (node bags) of $V(G)$, and $A = (A_e)_{e \in E(T)}$ is a family of subsets (arc bags) of $V(G)$, such that:

- $W$ is a partition of $V(G)$. \hspace{1cm} (DTW-1)
- For each arc $e \in E(T)$, $W_{s_{e}}$ is $A_{e}$-normal. \hspace{1cm} (DTW-2)

The width of an arboreal decomposition $\mathcal{D} = (T, W, A)$ is defined as $\max\{|W_i \cup A_{i,i} : i \in V(T)| - 1\}$. The directed treewidth of $G$, denoted by $dtw(G)$, is the minimum width over all possible arboreal decompositions of $G$.

Theorem 8. For a digraph $G$, $dtw(G) \leq circ(G) + 1$.

Proof. Let $T$ be the depth-first search tree constructed in Section 1.3. Let $W = (W_i)_{i \in V(T)}$ be a partition of $V(G)$ defined as $W_i = \{i\}$ for each $i \in V(T)$. For every edge $e = (r, v) \in E(T)$, we define $A_e = \{r\}$. For every edge $e = (u, v) \in E(T)$ such that $u \neq r$ we define $A_e$ as follows:

- if there are no back edges from $W_{s_{e}}$, we define $A_e = \{r\}$.
- if there are back edges from $W_{s_{e}}$, let $B$ be the set of all vertices $b \preceq u$ such that there is a back edge from some vertex in $W_{s_{e}}$ to $b$. Let $b_0$ be the minimal element in $B$ with respect to $\preceq$. Let $A_e = \{r\} \cup \{x | b_0 \preceq x \preceq u\}$. Note that $|\{x | b_0 \preceq x \preceq u\}| \leq l - 1$ and hence $|A_e| \leq l$.

Let $A = (A_e)_{e \in E(T)}$. We claim that $\mathcal{D} = (T, W, A)$ is an arboreal decomposition of $G$ of width at most $l + 1$. By construction, $W = (W_i)_{i \in V(T)}$ is a partition of $V(G)$ so $\mathcal{D}$ satisfies (DTW-1). To show that $\mathcal{D}$ satisfies (DTW-2) we must show that for each arc $e \in E(T)$, $W_{s_{e}}$ is $A_{e}$-normal. For every edge $e = (r, v) \in E(T)$, every directed path that leaves $W_{s_{e}}$ and returns to $W_{s_{e}}$ must go through the root $r$. Hence, $W_{s_{e}}$ is $A_{e}$-normal. For every edge $e = (u, v) \in E(T)$ such that $u \neq r$ we consider the following cases:

- if there are no back edges from $W_{s_{e}}$, every directed path that leaves $W_{s_{e}}$ and returns to $W_{s_{e}}$ must go through the root $r$. Hence, $W_{s_{e}}$ is $A_{e}$-normal.
- if there are back edges from $W_{s_{e}}$, every directed path that leaves $W_{s_{e}}$ and returns to $W_{s_{e}}$ must go through the root $r$ (or) go through a vertex in $\{x | b_0 \preceq x \preceq u\}$. Hence, $W_{s_{e}}$ is $A_{e}$-normal.

The size of each arc bag is at most $l$. Let $e_1 = (u, v), e_2 = (v, w) \in E(T)$. Let $B_1$ be the set of all vertices $b \preceq u$ such that there is a back edge from some vertex in $W_{s_{e_1}}$ to $b$. Let $B_2$ be the set of all vertices $b' \preceq v$ such that there is a back edge from some vertex in $W_{s_{e_2}}$ to $b'$. Note that $B_2 \subseteq B_1 \cup \{v\}$. Hence, for every $i \in V(T)$ the number of vertices in $A_{i,i}$ is at most $l + 1$ and the number of vertices in $W_i \cup A_{i,i}$ is at most $l + 2$. Hence, the width of $\mathcal{D} = (T, W, A)$ is at most $l + 1$. \qed
3 DAG-width and Circumference

**Definition 9.** [DAG-decomposition and DAG-width [BDHK06, Obd06, BDH+12]] A DAG decomposition of a digraph \( G \) is a pair \( \mathcal{D} = (T, \mathcal{X}) \) where \( T \) is a DAG, and \( \mathcal{X} = (X_i)_{i \in V(T)} \) is a family of subsets (node bags) of \( V(G) \), such that:

- \( \bigcup_{i \in V(T)} X_i = V(G) \). \hfill (DGW-1)
- For all nodes \( i, j, k \in V(T) \), if \( i \preceq j \preceq k \), then \( X_i \cap X_k \subseteq X_j \). \hfill (DGW-2)
- For all arcs \((i, j) \in E(T), X_i \cap X_j \) guards \( X_{\preceq j} \setminus X_i \). For any root \( r \in V(T), X_{\geq r} \) is guarded by \( \emptyset \). \hfill (DGW-3)

The width of a DAG-decomposition \( \mathcal{D} = (T, \mathcal{X}) \) is defined as \( \max \{|X_i| : i \in V(T)\} \). The **DAG-width** of \( G \), denoted by \( \text{dgw}(G) \), is the minimum width over all possible DAG-decompositions of \( G \).

**Theorem 10.** For a digraph \( G \), \( \text{dgw}(G) \leq \text{circ}(G) + 1 \).

**Proof.** Let \( T \) be the depth-first search tree constructed in Section 1.3. We construct a DAG \( \tilde{T} \) by adding more edges to \( T \). For a vertex \( v \in V(T) \) let \( S_v = \{u \mid \text{dfs}(u) < \text{dfs}(v) \text{ and } u \not\in T \} \). Add new edges from \( v \) to every vertex in \( S_v \). We do this for every \( v \in V(T) \). The graph \( \tilde{T} \) obtained in this way is a DAG.

We now define the set of node bags \( \mathcal{X} = (X_v)_{v \in V(\tilde{T})} \). Let \( X_r = \{r\} \). For every vertex \( v \neq r \), we define \( X_v \) as follows:

- if there are no back edges from \( T_{\geq v} \), we define \( X_v = \{r\} \).
- if there are back edges from \( T_{\geq v} \), let \( B \) be the set of all vertices \( b \preceq_T v \) such that there is a back edge from some vertex in \( T_{\geq v} \) to \( b \). Let \( b_0 \) be the minimal element in \( B \) with respect to \( \preceq_T \). Let \( X_v = \{r\} \cup \{x \mid b_0 \preceq x \preceq v\} \). Note that \(|\{x \mid b_0 \preceq x \preceq v\}| \leq l \) and hence \(|X_v| \leq l+1 \).

The size of each node bag is at most \( l + 1 \). We claim that \( \mathcal{D} = (\tilde{T}, \mathcal{X}) \) is a DAG decomposition of \( G \). Note that \( V(G) = V(\tilde{T}) \) and \( v \in X_v \) for every vertex \( v \in V(\tilde{T}) \). Hence, (DGW-1) is satisfied. Consider two vertices \( i \neq j \) such that \( i \in X_j \). There exist \( b \preceq i \) and \( a \preceq j \) such that \( (a, b) \in E(G) \) is a back edge. Every vertex \( k \) such that \( i \preceq k \preceq j \) satisfies \( b \preceq k \preceq j \), and hence by our construction \( k \in X_j \). So, (DGW-2) is satisfied.

All the out-going edges from \( X_{\preceq j} \setminus X_i \) are either back edges (or) edges going through the root \( r \). All the heads of the back edges from \( X_{\preceq j} \setminus X_i \) are in \( X_i \cap X_j \). Also, \( r \in X_v \) for every \( v \in V(\tilde{T}) \). Hence, \( X_i \cap X_j \) guards \( X_{\preceq j} \setminus X_i \) and (DGW-3) is satisfied. \( \square \)

4 Kelly-width and Circumference

Kelly-decomposition and Kelly-width were introduced by Hunter and Kreutzer [HK08].

**Definition 11.** [Kelly-decomposition and Kelly-width [HK08]] A Kelly-decomposition of a digraph \( G \) is a triple \( \mathcal{D} = (T, \mathcal{W}, \mathcal{X}) \) where \( T \) is a DAG, and \( \mathcal{W} = (W_i)_{i \in V(T)} \) and \( \mathcal{X} = (X_i)_{i \in V(T)} \) are families of subsets (node bags) of \( V(G) \), such that:
\begin{itemize}
\item $W$ is a partition of $V(G)$. \hfill (KW-1)
\item For all nodes $i \in V(T)$, $X_i$ guards $W_{\geq i}$. \hfill (KW-2)
\item For each node $i \in V(T)$, the children of $i$ can be enumerated as $j_1, \ldots, j_s$ so that for each $j_q$, $X_{j_q} \subseteq W_i \cup X_i \cup \bigcup_{p < q} W_{j_p}$. Also, the roots of $T$ can be enumerated as $r_1, r_2, \ldots$ such that for each root $r_q$, $X_{r_q} \subseteq \bigcup_{p < q} W_{r_p}$. \hfill (KW-3)
\end{itemize}

The width of a Kelly-decomposition $D = (T, W, \mathcal{X})$ is defined as $\max \{|W_i \cup X_i| : i \in V(T)\}$. The Kelly-width of $G$, denoted by $kw(G)$, is the minimum width over all possible Kelly-decompositions of $G$.

\begin{theorem}
For a digraph $G$, $kw(G) \leq circ(G) + 1$.
\end{theorem}

\begin{proof}
Let $\hat{T}$ be the DAG constructed in the proof of Theorem 10. Let $W = (W_i)_{i \in V(T)}$ be a partition of $V(G)$ defined as $W_i = \{i\}$ for each $i \in V(T)$. We now define the set of node bags $\mathcal{X} = (X_v)_{v \in V(\hat{T})}$. Let $X_v = \emptyset$. For every vertex $v \neq r$, we define $X_v$ as follows:

\begin{itemize}
\item if there are no back edges from $T_{\geq v}$, we define $X_v = \{r\}$.
\item if there are back edges from $T_{\geq v}$, let $B$ be the set of all vertices $b \preceq_T v$ such that there is a back edge from some vertex in $T_{\geq v}$ to $b$. Let $b_0$ be the minimal element in $B$ with respect to $\preceq_T$. Let $X_v = \{r\} \cup \{x \mid b_0 \preceq x \preceq v\} \setminus v$. Note that $|\{x \mid b_0 \preceq x \preceq v\} \setminus v| \leq l - 1$ and hence $|X_v| \leq l$. We call $b_0$ the “hook” of $v$ and denote it by $hook(v)$.
\end{itemize}

The size of each node bag is at most $l$, so the size of each $|W_i \cup X_i|$ is at most $l + 1$. We claim that $D = (\hat{T}, W, \mathcal{X})$ is a Kelly decomposition of $G$. By construction, $W = (W_i)_{i \in V(T)}$ is a partition of $V(G)$ so $D$ satisfies (KW-1).

All the out-going edges from $W_{\geq i}$ are either back edges (or) edges going through the root $r$. All the heads of the back edges from $W_{\geq i}$ are in $X_i$. Also, $r \in X_v$ for every $v \in V(\hat{T})$. Hence, $X_i$ guards $W_{\geq i}$ and (KW-2) is satisfied.

Recall the definition of “hook”. For a vertex $v \in V(T)$, if there are no back edges from $T_{\geq v}$, we define $hook(v) = v$. For a node $i \in V(T)$, we enumerate the children of $i$ as $j_1, \ldots, j_s$ such that $hook(j_1) \geq hook(j_2) \geq \cdots \geq hook(j_s)$. With this ordering, (KW-3) is satisfied.
\end{proof}

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