LOJASIEWICZ-TYPE INEQUALITIES FOR NONSMOOTH DEFINABLE
FUNCTIONS IN O-MINIMAL STRUCTURES AND GLOBAL ERROR
BOUNDS

HOÀNG PHI DỪNG

Abstract. In this paper, we give some Lojasiewicz-type inequalities and a nonsmooth slope inequality on non-compact domains for continuous definable functions in an o-minimal structure. We also give a necessary and sufficient condition for which global error bound exists. Moreover, we point out the relationship between the Palais-Smale condition and this global error bound.

1. Introduction

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a real analytic function with $f(0) = 0$. Let $V := \{x \in \mathbb{R}^n | f(x) = 0\}$ and $K$ be a compact subset in $\mathbb{R}^n$. Then the (Classical) Lojasiewicz inequality (see [L1, L2]) asserts that:

- There exist $c > 0, \alpha > 0$ such that
  \[ |f(x)| \geq cd(x, V)^\alpha \quad \text{for } x \in K. \]

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a real analytic function with $f(0) = 0$ and $\nabla f(0) = 0$. The Lojasiewicz gradient inequality (see [L1, L2]) asserts that:

- There exist $C > 0, \rho \in [0,1)$ and a neighbourhood $U$ of 0 such that
  \[ \| \nabla f(x) \| \geq C|f(x)|^\rho \quad \text{for } x \in U. \]

As a consequence, in (1), the order of zero of an analytic function is finite, and if $f(x)$ is close to 0 then $x$ is close to the zero set of $f$. However, if $K$ is not compact, the latter is not always true and the inequality (1) does not always hold (see [DHN, Remark 3.5]). Similarly, in (2), the order of gradient’s zero of an analytic function is smaller than the order of its zero. But if $U$ is not a bounded set, (2) does not always hold (see Example 3.1).

With the Lojasiewicz inequality (1), in the case $K = \mathbb{R}^n$, Hörmander (see [Hor]) substituted the left-hand side by one quantity greater than $|f(x)|$ and he got the following fact

\[ \exists c, \alpha, \beta > 0 \text{ such that } |f(x)|(1 + |x|^\beta) \geq d(x, V)^\alpha, \forall x \in K. \]

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Recently, by replacing $V$ by a large real algebraic set, Ha and Duc (see [HN]), Dinh, Ha and Thao (see [DHN]) gave some versions of Lojasiewicz inequalities in some non-compact cases. Moreover, some necessary and sufficient conditions for which the Lojasiewicz inequality and the global Lojasiewicz inequality exists in some non-compact cases are given.

In the case of differentiable definable functions in an o-minimal structure and $U$ is bounded set, Kurdyka (see [Kur]) proved the Lojasiewicz gradient inequality and Bolte, Daniilidis and Lewis (see [BDL]) proved it in the case of subanalytic functions. In some specific cases of o-minimal structures, other Lojasiewicz-type inequalities have been shown in [Loi].

On the other hand, the Classical Lojasiewicz inequality has the relation with error bounds in Optimization. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous real-valued function. Set

\[(3) \quad S := \{ x \in \mathbb{R}^n | f(x) \leq 0 \}, \]

and set \([f(x)]_+ := \max\{0, f(x)\}\).

We say that (3) has a global Hölderian error bound if there exist $c > 0, \alpha > 0, \beta > 0$ such that

\[(4) \quad d(x, S) \leq c([f(x)]_+^\alpha + [f(x)]_+^\beta) \]

for all $x \in \mathbb{R}^n$, where $d(x, S)$ denotes the Euclidean distance between $x$ and $S$. If, in addition, that $\alpha = \beta = 1$, then we refer (4) as a global Lipschitzian error bound.

Note that $[f(x)]_+ = 0$ if and only if $x \in S$. Hence the existence of the Lojasiewicz inequality with $[f(x)]_+$ is equivalent to the existence of the global Hölderian error bound of $S$.

In the convex case, the first results of error bounds was obtained in the work of Hoffman [H], Robinson [R], Mangasarian [M], Auslender and Crouzeix [AC], Klatte and Li [KL], . . . . The existence of an error bound (Lipschitzian) usually requires the convexity and the so-called Slater condition. When the Slater condition is not satisfied and the set $S$ is defined by one or many polynomial inequalities, global Hölderian error bounds have been shown in [LiG], [LL], [LS], [Y], . . . .

In the non-convex case, Luo and Sturm gave a global Hölderian error bound for polynomial of degree 2 (see [LS, Theorem 3.1]). As far as we know, this is the first result, where a global Hölderian error bound for a non-convex polynomial was established.

Recently, Ha gave a criterion for the existence of a global Hölderian error bound (4) in the case of polynomial of any degree (see [Ha, Theorem A]), without the assumption the convexity and the Slater condition. Moreover, the author pointed out that if a polynomial satisfies the Palais-Smale condition then there exists a global Hölderian error bound.

In this paper, we will give some Lojasiewicz-type inequalities. We will extend some results of [Ha] from polynomial functions to continuous definable functions in an o-minimal structure. We also do not require functions to either be convex or satisfy the Slater condition.
On the other hand, we will establish the Łojasiewicz gradient inequality in a non-compact case with differentiable definable real-valued functions in an o-minimal structure.

The rest of the paper is organized as follows. In Section 2, we recall a short introduction to o-minimal structures and some their properties. In Section 3, a criterion for the existence of Łojasiewicz-type inequalities and Łojasiewicz inequality of gradient will be proved. In Section 4, we give a necessary and sufficient condition for which a global Hölderian error bound exists; moreover, a relation between the Palais-Smale condition and the existence of error bounds will be established in the end.

2. Preliminaries

In this section, we recall some notions and results of geometry of o-minimal structures, which can be found in [DM, D, C].

Definition 2.1. A structure expanding the real field $(\mathbb{R}, +, \cdot)$ is a collection $\mathcal{O} = (\mathcal{O}_n)_{n \in \mathbb{N}}$ where each $\mathcal{O}_n$ is a set of subsets of the affine space $\mathbb{R}^n$, satisfying the following axioms:

1. All algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{O}_n$.
2. For every $n$, $\mathcal{O}_n$ is closed under finite set-theoretical operations.
3. If $A \in \mathcal{O}_n$ and $B \in \mathcal{O}_m$, then $A \times B \in \mathcal{O}_{m+n}$.
4. If $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first $n$ coordinates and $A \in \mathcal{O}_{n+1}$ then $\pi(A) \in \mathcal{O}_n$.

The elements of $\mathcal{O}_n$ are called the *definable subsets* of $\mathbb{R}^n$. Moreover, if $\mathcal{O}$ satisfies:

5. The elements of $\mathcal{O}_1$ are precisely the finite unions of points and intervals.

Then $\mathcal{O}$ is called an *o-minimal structure* on $\mathbb{R}$.

Example 2.1. A semi-algebraic set is finite union of sets $S = \{x \in \mathbb{R}^n | f(x) = 0, g_j(x) < 0, j = 1, \ldots, m\}$ where $f, g_j$ are polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

The collection $\mathcal{O}$ of all semi-algebraic sets in $\mathbb{R}^n$ for all $n \in \mathbb{N}$ is an o-minimal structure on $\mathbb{R}$.

Note that it is usually boring to write down projections in order to show that a subset is definable. We are more used to write down formulas. Let us specify what is meant *first-order formula* (of the language of the o-minimal structure) to . A first-order formula is constructed according to the following rules.

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $P(X_1, \ldots, X_n) = 0$ and $P(X_1, \ldots, X_n) > 0$ are first-order formulas.
2. If $A$ is a definable subset of $\mathbb{R}^n$, then $x \in A$ (where $x = (x_1, \ldots, x_n)$) is a first-order formula.
3. If $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ are the first-order formulas, then $\{\Phi \text{ and } \Psi\}$, $\{\Phi \text{ or } \Psi\}$, $\{$not $\Phi\}$, $\{\Phi \Rightarrow \Psi\}$ are first-order formulas.
(4) If $\Phi(y, x)$ is a first-order formula (where $y = (y_1, \ldots, y_p)$ and $x = (x_1, \ldots, x_n)$) and $A$ is a definable subset of $\mathbb{R}^n$, then $\exists x \in A \Phi(y, x)$ and $\forall x \in A \Phi(y, x)$ are first-order formulas.

**Theorem 2.1** ([C], Theorem 1.13). If $\Phi(x_1, \ldots, x_n)$ is a first-order formula, the set of $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ which satisfy $\Phi(x_1, \ldots, x_n)$ is definable.

**Remark 2.1.** By the rule (4) and the above theorem, the sets $\{x \in \mathbb{R}^n : \exists x_{n+1} (x, x_{n+1}) \in A\}$ (image of $A$ by projection) and $\{x \in \mathbb{R}^n : \forall x_{n+1} (x, x_{n+1}) \in A\}$ (complement of the image of the complement of $A$ by projection) are definable.

**Definition 2.2.** A map $f : A \to \mathbb{R}^p$ (where $A \subset \mathbb{R}^n$) is called **definable** if its graph is a definable subset of $\mathbb{R}^n \times \mathbb{R}^p$.

With any o-minimal structure, we have some elementary properties

**Proposition 2.1.**

(i) The closure, the interior and the boundary of a definable set are definable.

(ii) Compositions of definable maps are definable.

(iii) Images and inverse images of definable sets under definable maps are definable.

(iv) Infimum of a bounded below definable function and supremum of a bounded above definable function are definable functions.

The reader can be found the proofs of these properties in [DM, D].

**Proposition 2.2.** If function $f : \mathbb{R}^n \to \mathbb{R}$ is definable then the set $S = \{x \in \mathbb{R}^n | f(x) \leq 0\}$ is definable.

**Proof.** By definition, $\Gamma_f = \mathbb{R}^n \times f(\mathbb{R}^n)$ is definable.

Let consider the following projection

$$
\pi : \mathbb{R}^{n+1} \to \mathbb{R},
$$

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto x_{n+1}.
$$

By the definition of first-order formula, the set $\pi(\Gamma_f) = \{y \in \mathbb{R} | y = f(x), \text{ for some } x \in \mathbb{R}^n\}$ is definable. Similarly, the set $\{y \in \mathbb{R} | y \leq 0\}$ is definable.

So $S = \pi(\Gamma_f) \cap \{y \leq 0\}$ is definable. \qed

**Proposition 2.3.** If $S$ is a definable set and $S \neq \emptyset$ then the function $d : \mathbb{R}^n \to \mathbb{R}$ defined by

$$
d(x, S) = \inf_{y \in S} \|x - y\|
$$

is well-defined and is a definable function; moreover, it is a continuous function on $\mathbb{R}^n$.  

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Proof. The set \( \{ \| x - y \| : y \in S \} \) is an image of \( S \) by the definable function \( y \mapsto \| x - y \| \), so it is definable subset. Since \( S \neq \emptyset \), \( d \) is well-defined.

Let consider its graph, \( \Gamma_d = \{(x, t) \in \mathbb{R}^{n+1} | t \geq 0 \text{ and } \forall y \in S : t^2 \leq \| x - y \|^2 \text{ and } \forall \epsilon \in \mathbb{R}, \epsilon > 0 \Rightarrow \exists y \in S : t^2 + \epsilon > \| x - y \|^2 \} \).

This set is definable because it is defined by first-order formulas. Hence \( d(x, S) \) is a definable function.

By the triangle inequality, we have \( |d(x, S) - d(x_0, S)| \leq d(x, x_0) \). Therefore \( x \to x_0 \) implies \( d(x, S) \to d(x_0, S) \). Hence \( d(x, S) \) is a continuous function. \( \square \)

**Proposition 2.4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a differentiable, definable function in some o-minimal structure. Then \( \partial f / \partial x_j, j = 1, \ldots, n \) are definable functions and \( \nabla f(x) \) (gradient of \( f \)) is an definable mapping.

**Proof.** By the definition of partial derivatives, we have \( \partial f / \partial x_j \) are defined by

\[
\frac{\partial f}{\partial x_j}(a) = \lim_{x_j \to a_j} \frac{f(x_1, \ldots, x_j, \ldots, x_n) - f(a_1, \ldots, a_j, \ldots, a_n)}{x_j - a_j}, a \in \mathbb{R}^n,
\]

so we have

\[-\epsilon < \frac{f(x_1, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h} - \frac{\partial f}{\partial x_j}(a) < \epsilon, \forall \epsilon > 0, h > 0, j = 1, \ldots, n.\]

This is a first-order formula. By Theorem 2.1, \( \partial f / \partial x_j \) is definable function. This implies that \( \nabla f(x) \) is definable. \( \square \)

The following useful result is a property of semialgebraic functions in one variable.

**Lemma 2.1** (Growth Dichotomy Lemma). Let \( f : (0, \epsilon) \to \mathbb{R} \) be a semi-algebraic function with \( f(s) \neq 0 \) for all \( s \in (0, \epsilon) \). Then there exist constants \( c \neq 0 \) and \( q \in \mathbb{Q} \) such that \( f(s) = cs^q + o(s^q) \) as \( s \to 0^+ \).

The following property is important to our purpose.

**Theorem 2.2** (Monotonicity theorem). Let \( f : (a, b) \to \mathbb{R} \) is a definable function, \( -\infty \leq a < b \leq +\infty \). Then there exist \( a_0, a_1, \ldots, a_{k+1} \) with \( a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b \) such that \( f \) is continuous on each interval \((a_i, a_{i+1})\), moreover \( f \) is either strictly monotone or constant on each \((a_i, a_{i+1})\), \( i = 1, \ldots, k \).

The proof of this theorem can be found in [DM, D, C].

We now recall notion of the subdifferential of a continuous function. This notion plays the role of the usual gradient map, which can be found in [RW, C].

**Definition 2.3.**

(i) The Fréchet subdifferential \( \hat{\partial}f(x) \) of a continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) at \( x \in \mathbb{R}^n \) is given by

\[
\hat{\partial}f(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{\|h\| \to 0, h \neq 0} \frac{f(x + h) - f(x) - \langle v, h \rangle}{\|h\|} \geq 0 \right\}
\]
(ii) The limiting subdifferential at \( x \in \mathbb{R}^n \), denoted by \( \partial f(x) \), is the set of all cluster points of sequences \( \{v^k\}_{k \geq 1} \) such that \( v^k \in \hat{\partial} f(x^k) \) and \( (x^k, f(x^k)) \to (x, f(x)) \) as \( k \to \infty \).

**Remark 2.2.**

(i): It is easy to show that the set \( \hat{\partial} f(x) \) is not empty.

(ii): It is not hard to show that if \( f \) is a definable function then \( \hat{\partial} f(x) \) and \( \partial f(x) \) are definable sets ([I1, Prop 3.1]).

**Definition 2.4.** By using the limiting subdifferential \( \partial f \), we define the nonsmooth slope of \( f \) by

\[
m_f(x) := \inf \{\|v\| : v \in \partial f(x)\}.
\]

By definition, \( m_f(x) = +\infty \) whenever \( \partial f(x) = \emptyset \).

**Definition 2.5.** The strong nonsmooth slope of function \( f \) is defined as follows

\[
|\nabla f|(x) := \lim_{h \to 0} \sup_{h \neq 0} \frac{[f(x) - f(x + h)]_+}{\|h\|},
\]

with \([a]_+ = \max\{a, 0\}\).

The relationship between nonsmooth slope, strong nonsmooth slope and subdifferential is following (see for details in [I2]):

\[
\inf \{\|y\| : y \in \hat{\partial} f(x)\} \geq |\nabla f|(x) \geq m_f(x).
\]

**Remark 2.3.**

(i): It is not hard to show that if \( f \) is a definable function then \( m_f(x) \) and \( |\nabla f|(x) \) are definable ([I1, Prop 3.1]).

(ii): If \( f \) is a differentiable function then the above notions coincide with the usual concept of gradient; that is: \( \partial f(x) = \hat{\partial} f(x) = \{\nabla f(x)\} \) and hence \( m_f(x) = |\nabla f|(x) = \|\nabla f(x)\| \).

3. Main results

3.1. Łojasiewicz-type inequalities.

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous definable function. Assume that \( S := \{x \in \mathbb{R}^n \mid f(x) \leq 0\} \neq \emptyset \). Let \([f(x)]_+ := \max\{f(x), 0\}\). The following results extend the results of [Ha] (see also [DHN]) from polynomial functions to continuous definable functions. The proof follows the steps of proofs of Theorem 2.1 and 2.2 in [Ha], but we use Monotonicity Theorem instead of Growth Dichotomy Lemma.

**Theorem 3.1** (Łojasiewicz-type inequality ”near to the set \( S \)”). The following two statements are equivalent.

(i) For any sequence \( x^k \in \mathbb{R}^n \setminus S \), with \( x^k \to \infty \), it holds that

\[
f(x^k) \to 0 \implies d(x^k, S) \to 0;
\]
(ii) There exist $\delta > 0$ and a function $\mu : [0, \delta] \to \mathbb{R}$ which is definable, continuous and strictly increasing on $[0, \delta]$ with $\mu(0) = 0$ such that
\[
\mu([f(x)]_+) \geq d(x, S), \quad \forall x \in f^{-1}((\infty, \delta]).
\]

Proof.

$(ii) \Rightarrow (i)$: Assume that $x^k \not\in S$, $x^k \to \infty$ and $f(x^k) \to 0$. We have $[f(x^k)]_+ = f(x^k)$. By the continuity of $\mu$ at 0, we get $\mu(f(x^k)) \to 0$. Note that $0 < f(x^k) < \delta$ if $k \gg 1$. Then it follows from the inequality in $(ii)$ that $d(x^k, S) \to 0$.

$(i) \Rightarrow (ii)$: Without loss of generality, we can suppose that $S \neq \mathbb{R}^n$. Then there exists $t_0 > 0$ such that $f^{-1}(t_0) \neq \emptyset$. Because $f$ is continuous, $f^{-1}(t) \neq \emptyset$ for all $0 \leq t \ll 1$.

Let \( \mu(t) := \sup_{x \in f^{-1}(t)} d(x, S), t \geq 0 \). We will show that there exists $\delta > 0$ sufficient small such that $\mu(t)$ have desired properties. Clearly, $\mu(0) = 0$.

We now show that there exists $\delta > 0$ such that $\mu(t) < +\infty$ for all $t \in [0, \delta)$. By contradiction, assume that there exists a sequence $t_k > 0, t_k \to 0$, such that $\mu(t_k) = \infty$ for all $k$. This implies the existence of sequence $x^k \in f^{-1}(t_k)$ such that $d(x^k, S) \to +\infty$ as $k \to \infty$. Hence $x^k \to \infty$. Contradiction.

So $\mu(t) < +\infty$ on $[0, \delta]$ with $\delta > 0$. By Proposition 2.3, we have $\mu(t)$ is definable on $[0, \delta]$.

Using The Monotonicity Theorem, the function $\mu$ is continuous and monotone on $(0, \delta]$ if $0 < \delta \ll 1$.

We now show that $\mu$ is continuous at 0. Suppose $\mu$ is not continuous at 0. That means there exists a sequence $t_k \to 0$ such that $\mu(t_k) = \sup_{x \in f^{-1}(t_k)} d(x, S) \nrightarrow 0$. Hence, there exists a sequence $x^k \in f^{-1}(t_k)$ such that $t_k = f(x^k) \to 0$ and $d(x^k, S) \nrightarrow 0$. On the other hand, $x^k \to \infty$. Indeed, if there exists $x < \infty$ such that $x^k \to x$ then by the continuity of $f$, $f(x^k) \to f(x)$, this implies $f(x) = 0$. That means $d(x^k, S) \to 0$, contradiction. So we have a sequence $x^k \to \infty, f(x^k) \to 0$ and $d(x^k, S) \nrightarrow 0$. This contradicts $(i)$.

Hence $\mu$ is continuous and monotone on $[0, \delta]$.

Note that by $\mu(0) = 0$ and $\mu(t) > 0, \forall t \in (0, \delta)$, if $\delta$ is sufficient small then $\mu(t)$ is strictly increasing on $[0, \delta]$.

For $0 < t < \delta$, let $x \in f^{-1}(t)$, then we have $\mu(t) = \sup_{a \in f^{-1}(t)} d(a, S) \geq d(x, S)$.

Hence $\mu([f(x)]_+) \geq d(x, S), \forall x \in f^{-1}((\infty, \delta])$. \( \square \)

**Remark 3.1.** Note that the condition that $\mu$ is continuous at 0 and $\mu(0) = 0$ in $(ii)$ is necessary.

Let’s consider the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x}{1+x^2}$. The function $f$ is a differentiable semialgebraic function because its graph is the set $\{(x, y) \in \mathbb{R}^2 | (1+x^2)y = x\}$. Then $f$ is a definable function.
We have $S = (-\infty, 0]$. Then we choose $\mu(t) := \sup_{\frac{r}{1+t^2}=t} d(x, S)$ on $0 < t < \frac{1}{2}$. This function is definable, continuous on $(0, \frac{1}{2})$ but not continuous at 0.

Moreover, $x^k \to +\infty$ satisfies $f(x^k) \to 0$ but $d(x^k, S) \to +\infty$, so the statement (i) fails.

**Theorem 3.2** (Lojasiewicz-type inequality "far from the set $S$"). Suppose that for any sequence $x^k \in \mathbb{R}^n \setminus S$, with $x^k \to \infty$ and

$$d(x^k, S) \to \infty$$

Then there exist $r > 0$ and a function $\mu : [r, +\infty) \to \mathbb{R}$ which is definable, strictly increasing and continuous on $[r, +\infty)$ such that

$$\mu([f(x)]_+) \geq d(x, S), \forall x \in f^{-1}([r, +\infty)).$$

**Proof.**

Let us consider two cases:

**Case 1.** The function $f$ is bounded from above, i.e. $r := \sup_{x \in \mathbb{R}^n} f(x) < +\infty$.

By the assumption, there exists $M > 0$ such that $d(x, S) \leq M$ for all $x \in \mathbb{R}^n$. For all $x \in f^{-1}([r, +\infty))$,

$$f(x) \geq r = \frac{r}{M} M \geq \frac{r}{M} d(x, S),$$

Then the function $\mu(t) := \frac{M}{r} t$ with $t \geq r$ have required properties.

**Case 2.** The function $f$ is not bounded from above. By continuity of $f$ and $S \neq \emptyset$, we have $f^{-1}(t) \neq \emptyset$ for all $t \geq 0$. Set $\mu(t) = \sup_{x \in f^{-1}(t)} d(x, S)$.

We claim that there exists $r \gg 1$ such that $\mu(t) = \sup_{x \in f^{-1}(t)} d(x, S) < \infty$ for all $t \geq r$.

By contradiction, assume that $\mu(t) = \infty$ for some $t \gg 1$. Then there exists a sequence $x^k \in f^{-1}(t)$ such that $d(x^k, S) \to \infty$. Of course $x^k \to \infty$, this contradicts the assumption.

So $\mu(t) < +\infty, \forall t \in [r, +\infty)$. This implies that $\mu$ is a definable function on $[r, +\infty)$. By Monotonicity Theorem, $\mu$ is continuous and monotone on $[r, +\infty)$ for $r \gg 1$.

Let

$$M := \sup_{t \in [r, +\infty)} \mu(t).$$

We have two subcases:

**Case 2.1.** $M = +\infty$. Then $\lim_{t \to +\infty} \mu(t) = +\infty$. This means that for $r \gg 1$, the function $\mu$ is strictly increasing on $[r, +\infty)$. Furthermore

$$\mu([f(x)]_+) = \mu(f(x)) \geq d(x, S), \forall x \in f^{-1}([r, +\infty)).$$
Case 2.2. $M < +\infty$. Then, for all $x$ such that $f(x) \geq r$ we have $d(x, S) \leq M$, therefore

$$f(x) \geq r = \frac{r}{M} M \geq \frac{r}{M} d(x, S).$$

The function $\mu := \frac{M}{r} t, t \geq r$, has required properties. \hfill \square

**Remark 3.2.** Note that the converse of the above theorem is false. Indeed, consider the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x}{\sqrt{1 + x^2}}$. The function $f$ is a differentiable semialgebraic function since its graph is the set $\{(x, y) \in \mathbb{R}^2|(1 + x^2)y^2 = x^2\} \cap \{xy > 0\}$. We have $S = (-\infty, 0]$. Then we choose $r < 1$ and let $\mu(t) := \sup_{\frac{x}{\sqrt{1 + x^2}} = t} d(x, S)$ on $(r, 1)$. This function is definable, increasing and continuous.

In the other hand, we have $x^k \to +\infty, d(x^k, S) \to +\infty$ and $f(x^k) \to 1$.

### 3.2. A nonsmooth slope inequality near the fiber for continuous definable functions in an o-minimal structure.

In the case $U$ is not bounded set, the classical Łojasiewicz gradient inequality is not always true. We can see it in the following example

**Example 3.1.** Consider the following example: $f(x, y) = (xy - 1)^2 + (x - 1)^2$ and $U = \mathbb{R}^2$.

Let $x^k$ be $(\frac{1 + k}{1 + k^2}, k)$, we have:

- $x^k \to \infty$.
- $\nabla f(x^k) = (0, 2\frac{k^2 - 1}{(1 + k^2)^2}) \to 0$.
- But $f(x^k) = (\frac{1 + k}{1 + k^2}, k)^2 + (\frac{1 + k}{1 + k^2} - 1)^2 \to 1$.

We prove that $\exists \delta > 0, C > 0, \rho \in \mathbb{R}$ such that $\|\nabla f(x)\| \geq C|f(x)|^\rho$ for $x \in f^{-1}(D_\delta)$. By contradiction, assume that there are $\delta > 0, C > 0$ and $\rho \in \mathbb{R}$ such that the Łojasiewicz gradient inequality holds. We see that $\nabla f(\frac{1 + k}{1 + k^2}, k) \to 0$ and $f(\frac{1 + k}{1 + k^2}, k) \to 0$.

Hence $\rho > 0$. On the other hand, $\nabla f(\frac{1 + k}{1 + k^2}, k) \to 0$ and $f(\frac{1 + k}{1 + k^2}, k) \to 1$; so $\rho \leq 0$, contradiction.

We shall give a criterion for the existence of Łojasiewicz nonsmooth slope inequality on $f^{-1}(D_\delta)$.

Let $\tilde{K}_\infty(f) := \{t \in \mathbb{R} | \exists x^k \to \infty, m_f(x^k) \to 0, f(x^k) \to t\}$ and we call it the set of asymptotic critical values at infinity.

**Theorem 3.3.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous definable function in some o-minimal structure and suppose that $\tilde{K}_\infty(f)$ is finite. Then the following two statements are equivalent:

(i) For any sequence $x^k \to \infty, m_f(x^k) \to 0$ implies $f(x^k) \to 0$. 

II
ii) There exists a function \( \varphi : (0, \delta) \to \mathbb{R} \), which is definable, monotone and continuous such that
\[
m_f(x) \geq \varphi(|f(x)|), \quad \forall x \in f^{-1}(D_\delta).
\]

**Remark 3.3.** The assumption "\( \tilde{K}_\infty(f) \) is finite" is necessary. Indeed, consider the following example
Consider \( f(x, y) = \frac{x}{1 + y^2} \) in the o-minimal structure of all semialgebraic sets, then any \( t \in \mathbb{R} \) is belong to \( \tilde{K}_\infty(f) \), by the sequence \( x^k = (t(1 + k^2), k) \). It is easy to see that \( x^k \to \infty, m_f(x^k) = \|\nabla f(x^k)\| = \sqrt{\left(\frac{1}{1 + k^2}\right)^2 + \left(\frac{2tk}{1 + k^2}\right)^2} \to 0 \) and \( f(x^k) = t \), the (ii) is failed.

**Proof of Theorem 3.3**
(i) \( \Rightarrow \) (ii) : Let \( \varphi(t) := \inf\{m_f(x) : |f(x)| = t\} \), it is easy to see that \( \varphi \) is a definable function (see Propositions 2.3 and 2.4).

**Claim:** There exists \( \delta_1 \) such that \( \varphi(t) > 0, \forall t \in (0, \delta_1) \).

Indeed, assume that for all \( \delta' > 0 \) such that \( (0, \delta') \) has no critical point of \( f \). There exists a value \( t \in (0, \delta') \) such that \( \varphi(t) = 0 \). Then there exists a sequence \( t_k \) such that \( t_k \to t \) implies \( \varphi(t_k) \to 0 \). Therefore there exists a sequence \( x^k \) such that \( f(x^k) = t_k \) and \( m_f(x^k) \to 0 \). So we have \( m_f(x^k) \to 0 \) but \( f(x_k) \to t \neq 0 \), this contradicts with (i). The Claim is proved. So \( \varphi(t) \) is definable.

On the other hand, by Monotonicity Theorem, \( \varphi(t) \) is continuous and monotone on \((0, \delta)\) for \( 0 < \delta \ll 1 \).

By the definition of \( \varphi \), we get \( \varphi(t) \leq m_f(x), \forall x \in f^{-1}(D_\delta) \), which means that \( m_f(x) \geq \varphi(|f(x)|), \forall x \in f^{-1}(D_\delta) \).

(ii) \( \Rightarrow \) (i) : straightforward. \( \square \)

**Remark 3.4.** In Theorem 3.3 if \( f \) is a polynomial then \( \varphi(t) \) is a semialgebraic function in one variable. By Growth Dichotomy Lemma, there exists \( a > 0 \) and \( u \in \mathbb{R}, u > 0 \) such that
\[
\varphi(t) = at^u + o(t^u) \text{ as } t \in (0, \epsilon), \epsilon \ll 1.
\]
This implies \( \varphi(t) \geq ct^u, \forall t \in (0, \epsilon) \). By definition of \( \varphi \), we have \( \|\nabla f(x)\| \geq \varphi(t) \geq ct^u \). Note that \( t = |f(x)| \), so we get the Łojasiewicz gradient inequality on \( f^{-1}(D_\delta) \).

4. Application to error bounds

4.1. Global Hölderian error bound for continuous definable functions in o-minimal structures.

The following criterion extends the error bound result of [Ha] from polynomial functions to definable functions in o-minimal structures.
Theorem 4.1. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous definable function. Assume that \( S := \{ x \in \mathbb{R}^n \mid f(x) \leq 0 \} \neq \emptyset \) and \( [f(x)]_+ := \max\{f(x), 0\} \). Then the following two statements are equivalent

(i) For any sequence \( x^k \in \mathbb{R}^n \setminus S, x^k \to \infty \), we have

1. if \( f(x^k) \to 0 \) then \( d(x^k, S) \to 0 \);
2. if \( d(x^k, S) \to \infty \) then \( f(x^k) \to \infty \).

(ii) There exists a function \( \mu : [0, +\infty) \to \mathbb{R} \), which is definable, strictly increasing and continuous on \([0, +\infty)\) with \( \mu(0) = 0 \), \( \lim_{t \to +\infty} \mu(t) = +\infty \), such that

\[
d(x, S) \leq \mu([f(x)]_+), \quad \forall x \in \mathbb{R}^n.
\]

Proof. The implication (ii) \( \Rightarrow \) (i) is straightforward. We prove the implication (i) \( \Rightarrow \) (ii).

Indeed, by Theorems 3.1 and 3.2 there exist two continuous, strictly increasing, definable functions \( \mu_1 \) on \([0, \delta]\) with \( 0 < \delta \ll 1 \) and \( \mu_2 \) on \([r, +\infty)\) with \( r \gg 1 \) such that

\[
d(x, S) \leq \mu_1([f(x)]_+), \forall x \in f^{-1}((\infty, \delta]).
\]

and

\[
d(x, S) \leq \mu_2([f(x)]_+), \forall x \in f^{-1}([r, +\infty)).
\]

On the other hand, by assumption (i2), there exists \( M > 0 \) such that \( d(x, S) \leq M \) for all \( x \in f^{-1}([\delta, r]) \). Then

\[
f(x) \geq \delta = \frac{\delta}{M} M \geq \frac{\delta}{M} d(x, S)
\]

for all \( x \in f^{-1}([\delta, r]) \). Put \( \mu_3(t) := \frac{M t}{\delta} \) with \( t \in [\delta, r] \), we get \( \mu_3(t) \geq d(x, S) \) and \( \mu_3 \) is a increasing function on \([\delta, r]\).

By definition of \( \mu_3 \) and \( \lim_{t \to 0} \mu_1(t) = 0 \) (Theorem 3.1), we may choose \( \delta \) such that \( \mu_1(t) \leq \mu_3(\delta) = M, \forall t \in [0, \delta] \). Indeed, if \( \exists \delta \) such that \( \mu_1(t) > \mu_3(\delta) \), then we put

\[
M' := \max\{\sup_{t \in [0, \delta]} \mu_1(t), M\} \text{ and } \mu_3(t) := \frac{M'}{\delta} t,
\]

so we have \( \mu_1(t) \leq M' = \mu_3(\delta), \forall t \in [0, \delta] \).

Similarly, by definition of \( \mu_3(t) \) and \( \mu_2(t) \), we may choose \( r \) such that \( \mu_3(r) = \frac{M r}{\delta} \leq \mu_2(t), \forall t \in [r, +\infty) \). Indeed, if \( \exists \delta \in [r, +\infty) \) such that \( \mu_3(r) > \mu_2(t) \), then we may choose \( \mu_3' \) of \( \mu_2(t) + C \) with \( C = \mu_3(r) \), so we have \( d(x, S) \leq \mu_2(t) < \mu_3(t), t \in [r, +\infty) \) and \( \mu_3(t) \leq \mu_3'(t), \forall t \in [r, +\infty) \). Moreover, by definition of \( \mu_2 \), we may choose \( \mu_3'(t) \) as above such that if \( r \gg 1 \) then \( \mu_3'(t) \) is strictly increasing on \([r, +\infty)\) and \( \lim_{t \to +\infty} \mu_3'(t) = +\infty \).
Combine three functions $\mu_1, \mu_2, \mu_3$ and note that we may choose suitable $\delta, r$ and $M$ as above, we get the function $\mu(t) =$ \begin{align*}
\mu_1(t) & \quad \forall t \in [0, \delta] \\
\mu_2(t) & \quad \forall t \in [\delta, r] \\
\mu_3(t) & \quad \forall t \in [r, +\infty)
\end{align*}
. The function $\mu$ is definable, strictly increasing and continuous and $\mu$ satisfies (ii).

4.2. The relation between the Palais-Smale condition and the existence of error bounds.

In this section, we consider continuous functions in an o-minimal structure.

**Definition 4.1.** Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and a real number $t$, we say that $f$ satisfies the Palais-Smale condition at the level $t$, if every sequence $\{x^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ such that $f(x^k) \to t$ and $f(x^k) \to 0$ as $k \to \infty$ possesses a convergence subsequence.

The following theorem also extends Theorem B in [Ha] from polynomial functions to continuous definable functions.

**Theorem 4.2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous definable function. Assume that $S := \{ x \in \mathbb{R}^n \mid f(x) \leq 0 \} \neq \emptyset$. If $f$ satisfies the Palais-Smale condition at each level $t \geq 0$, then there exists a function $\mu : [0, +\infty) \to \mathbb{R}$, which is definable, strictly increasing and continuous $\mu(0) = 0, \lim_{t \to \infty} \mu(t) = \infty$, such that

$$d(x, S) \leq \mu([f(x)]_+), \quad \forall x \in \mathbb{R}^n.$$  

**Proof.** By Theorem 4.1, it is enough to show that $f$ satisfies the Palais-Smale condition at each value $t \geq 0$. In case of continuous definable functions, we use the subdifferential instead of the gradient in [Ha].

By contradiction, first of all, assume that for a sequence $x^k \to \infty, x^k \in \mathbb{R}^n \setminus S$, we have $f(x^k) \to 0$ and $d(x^k, S) \geq \delta > 0$. Similarly to the proof of [Ha] Theorem B, by using Ekeland Variational Principle ([E]), we obtain a sequence $y^k$ such that

$$\frac{1}{\|h\|}(f(y^k + h) - f(y^k)) \geq -\sqrt{\epsilon_k}$$

with $h \in \mathbb{R}^n, 0 < \|h\| < \frac{\delta}{2}$ and $\epsilon_k = f(x^k)$. This implies that

$$\frac{1}{\|h\|}(f(y^k) - f(y^k + h)) \leq \sqrt{\epsilon_k},$$

or

$$\frac{1}{\|h\|}[f(y^k) - f(y^k + h)]_+ \leq \sqrt{\epsilon_k}.$$  

By the definition of the strong slope, we have

$$0 \leq |\nabla f|(y^k) = \limsup_{h \to 0, h \neq 0} \frac{[f(y^k) - f(y^k + h)]_+}{\|h\|} \leq \sqrt{\epsilon_k}.$$
Thus
\[ 0 \leq m_f(y^k) \leq |\nabla f|(y^k) \leq \sqrt{\epsilon_k}. \]

Letting \( k \to \infty \), we get \( m_f(y^k) \to 0 \). So we have found a sequence \( y^k \to \infty, y^k \in \mathbb{R}^n \setminus S \), \( m_f(y^k) \to 0 \) and \( f(y^k) \to 0 \). This means that \( f \) does not satisfy the Palais-Smale condition at the value \( t = 0 \), a contradiction. So we get (i1) of Theorem 4.1.

Now, suppose that for some sequence \( x^k \in \mathbb{R}^n \setminus S \) with \( x^k \to \infty \) such that
\[ d(x^k, S) \to \infty \text{ and } f(x^k) \not\to \infty. \]

Without loss of generality, we may assume that \( f(x^k) \to t_0 \) with \( t_0 \in [0, +\infty) \). Again, by the similar arguments as in [Ha, Theorem B], we have a sequence \( y^k \) such that \( 0 < f(y^k) \leq f(x^k) \) and
\[ \frac{1}{\|h\|} (f(y^k + h) - f(y^k)) \geq -\epsilon_k \cdot \lambda_k \]
with \( h \in \mathbb{R}^n, 0 < \|h\| < \frac{\delta}{2}, \epsilon_k = f(x^k) \) and \( \lambda_k = \frac{2}{d(x^k, S)} \). This implies that
\[ \frac{1}{\|h\|} [f(y^k) - f(y^k + h)]_+ \leq \epsilon_k \lambda_k. \]

By the definition of the strong slope, we have
\[ 0 \leq m_f(y^k) \leq |\nabla f|(y^k) \leq \epsilon_k \lambda_k = \frac{2 \epsilon_k}{d(x^k, S)}. \]

Letting \( k \to \infty \) we have \( \epsilon_k = f(x^k) \to t_0 \) and \( d(x^k, S) \to \infty \). Therefore \( m_f(y^k) \to 0 \).

Consequently, since \( 0 < f(y^k) \leq f(x^k) \), \( y^k \) has a subsequence \( y'^k \) such that \( f(y'^k) \to t_1 \) with \( 0 \leq t_1 \leq t_0 \) which satisfies
\[ y'^k \to \infty, m_f(y'^k) \to 0 \text{ and } f(y'^k) \to t_1. \]

This means that \( f \) does not satisfy the Palais-Smale condition at \( t_1 \), contradiction. So we get (i2) of Theorem 4.1. The theorem is proved. □

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References

[AC] A. Auslander and Crouzeix, Global regularity theorem, Math. Oper. Res., 13 (1988), 243-253.
[BDL] J. Bolte, A. Daniilidis and A. S. Lewis, The Lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems, SIAM J. Optim. 17 (2007), no. 4, 1205-1223.
[C] M. Coste, An Introduction to O-minimal Geometry, Instituto Editoriali e poligrafici internazionali (Università di Pisa, 1999).
[Cl] F. H. Clarke, Optimization and Nonsmooth Analysis, New York et al., John Wiley & Sons 1983.
[DHN] Si Tiep Dinh, Huy Vui Ha, Thi Thao Nguyen, *Lojasiewicz inequality for polynomial functions on non-compact domains*, International Journal of Mathematics, 23, 1250033 (2012), DOI No: 10.1142/S0129167X12500334.

[D] L. van den Dries, *Tame Topology and O-minimal structures*, Cambridge University Press, 1998.

[DM] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J., 84 (1996), 497-540.

[E] I. Ekeland, *Nonconvex minimization problems*, Bull. A.M.S., No.1 (1974), 443-474.

[Ha] Huy Vui Ha, *Global Hölderian error bound for non-degenerate polynomials*, SIAM J. Optim., 2013, to appears.

[HN] Huy Vui Ha and Hong Duc Nguyen, *Lojasiewicz inequality at infinity for polynomial in two real variables*, Math. Z., 266 (2010), 243-264.

[Hor] L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat. 3 N. 53 (1958), 555-568.

[H] A. J. Hoffman, *On approximate solutions of linear inequalities*, Journal of Research of the National Bureau of Standards, 49 (1952), 263-265.

[I1] A. D. Ioffe, *An invitation to tame optimization*, SIAM J. Optim., 19 (2009), No.4, 1894-1917.

[I2] A. D. Ioffe, *Metric regularity and subdifferential calculus*, Uspehi Mat. Nauk., 55 (2000), pp. 103-162 (in Russian), English translation: Russian Math. Surveys, 55 (2000), pp. 501-588.

[KL] D. Klatte and A. Li, *Asymptotic constraint qualifications and global error bounds for convex inequalities*, Math. Progam., 84 (1999), 137-140.

[Kur] K. Kurdyka, *On gradients of functions definable in o-minimal structures*, Ann. Inst. Fourier, 48 (1998), 769-783.

[L1] S. Łojasiewicz, *Sur le problème de la division*, Studia Math.18(1959), 87–136.

[L2] S. Łojasiewicz, *Ensembles semi-analytiques*, Publ. Math. I.H.E.S., Bures-sur-Yvette, France (1964).

[LiG] Guoyin Li, *On the asymptotic well behaved functions and global error bound for convex polynomials*, SIAM J. Optim., 20 (2010), No.4, 1923-1943.

[LL] X. D. Luo and Z. Q. Luo, *Extensions of Hoffman’s Error bound to polynomial systems*, SIAM J. on Optim., 4 (1994), 383-392.

[Loi] T. L. Loi, *Lojasiewicz Inequalities for Sets Definable in the Structure \( \mathbb{R}_{\exp} \)*, Ann. Inst. Fourier, 45 (1995), 951-957.

[LS] Z. Q. Luo and J.F. Sturm, *Error bound for quadratic systems*, in *High Performance Optimization*, H. Frenk, K. Roos, T. Terlaky, and Zhang, eds., Kluwer, Dordrecht, The Netherlands, 2000, pp. 383-404.

[M] O. L. Mangasarian, *A condition number for differentiable convex inequalities*, Math. Oper. Res., 10 (1985), 175-179.

[R] S. Robinson, *An application of error bounds for convex programming in a linear space*, SIAM J. Control 13 (1975), 271-273.

[RW] R. T. Rockafellar, and R. Wets, *Variational Analysis*, Grundlehren Math. Wiss., 317, Springer, New York, 1998.

[Y] W. H. Yang, *Error bounds for convex polynomials*, SIAM J. Optim., 19 (2009), 1633-1647.

Institute of Mathematics, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam

E-mail address: hpdung83@gmail.com