METRIC CURRENTS AND ALBERTI REPRESENTATIONS

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Abstract. We relate Ambrosio-Kirchheim metric currents to Alberti representations and Weaver derivations. In particular, given a metric current $T$, we show that if the module $X(\|T\|)$ of Weaver derivations is finitely generated, then $T$ can be represented in terms of derivations; this extends previous results of Williams. Applications of this theory include an approximation of 1-dimensional metric currents in terms of normal currents and the construction of Alberti representations in the directions of vector fields.

Contents

1. Introduction 1
2. Preliminaries 5
3. 1-dimensional currents and derivations 15
4. Currents and Alberti representations 17
5. A representation formula 19
6. Applications 25
7. Technical tools 36
References 54

1. Introduction

Overview. The goal of this paper is to relate metric currents to Alberti representations and Weaver derivations. In particular, it seems that metric currents carry a weak notion of a differentiable structure which we try to describe by using Alberti representations and Weaver derivations. As a first application we prove an approximation result in which a 1-dimensional metric current is approximated by a sequence of normal currents. As a second application we show how to use 1-dimensional normal currents to produce Alberti representations in the directions of vector fields.

Metric currents. Federer and Fleming [FF60] introduced the theory of currents to study the Plateau problem in Euclidean spaces of dimension higher than 2, and overtime currents have proven useful to attack a wide range of problems, see [ABL88, Lin99, GMS89] to cite some examples. In order to study similar problems in general metric spaces, it became desirable to have an analogue of the Federer-Fleming currents and a major obstacle was that the classical definition of currents...

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uses the differentiable structure of $\mathbb{R}^N$. In [AK00] Ambrosio and Kirchheim, inspired by an idea of de Giorgi [DG95], developed a theory of metric currents starting by circumventing the lack of a differentiable structure. Essentially, $k$-dimensional metric currents are defined by duality with $(k+1)$-tuples of Lipschitz functions $(f,\pi_1,\cdots,\pi_k)$, where the first function $f$ is also bounded. The axioms that currents satisfy are then designed so that one can formally treat, to some extent, the $(k+1)$-tuple $(f,\pi_1,\cdots,\pi_k)$ as a $k$-dimensional differential form $f d\pi_1 \wedge \cdots \wedge d\pi_k$.

In [Wil10] Williams showed that in a differentiability space $(X,\mu)$, those metric currents whose masses are absolutely continuous with respect to $\mu$ are dual to the differential $k$-forms defined using the differentiable structure. This result was the starting point of the present work in which, roughly speaking, we remove the assumption that $(X,\mu)$ is a differentiability space.

For a treatment of metric currents we refer the reader to [AK00]; some basic facts are recalled in Subsection 2.1. Note that Lang [Lan11] has formulated an alternative theory of metric currents in which the finite mass axiom is removed; our results have natural counterparts in that setting.

**Alberti representations.** Alberti representations were introduced in [Alb93] to prove the rank-one property for BV functions; they were later applied to study the differentiability properties of Lipschitz functions $f : \mathbb{R}^N \to \mathbb{R}$ [ACP05, ACP10] and have recently been used to obtain a description of measures in differentiability spaces [Bat12]. We give here an informal definition and refer the reader to [Bat12, Sch13] and Subsection 2.2 for further details.

**An Alberti representation** of a Radon measure $\mu$ is a generalized Lebesgue decomposition of $\mu$ in terms of rectifiable measures supported on path fragments; a path fragment in $X$ is a Lipschitz map $\gamma : K \to X$ where $K \subset \mathbb{R}$ is compact; the set of fragments in $X$ will be denoted by Frag$(X)$ and topologized as a subspace of $K(X)$, the set of compact subsets of $X$ with the Vietoris topology. An Alberti representation of $\mu$ is then a decomposition:

$$\mu = \int_{\text{Frag}(X)} \nu_{\gamma} dP(\gamma),$$

where $P$ is a regular Borel probability measure on Frag$(X)$, and $\nu$ associates to each fragment $\gamma$ a finite Radon measure $\nu_{\gamma}$ which is absolutely continuous with respect to the 1-dimensional Hausdorff measure $H^1_{\gamma}$ on the image of $\gamma$. Examples of an Alberti representation are offered by Fubini’s Theorem; however, in general it is necessary to work with path fragments instead of Lipschitz curves because the space $X$ on which $\mu$ is defined might lack any rectifiable curve.

**Weaver derivations and their relationship with Alberti representations.** Weaver derivations, hereafter simply called derivations, were introduced in [Wea00] and provide a quite broad framework to formulate a notion of differentiability on metric measure spaces. To fix the ideas, let Lip$(X)$ denote the set of real-valued Lipschitz functions defined on $X$ and let Lip$_b(X)$ denote the subset of bounded Lipschitz functions. The vector space Lip$_b(X)$ becomes a Banach algebra with norm:

$$\|f\|_{\text{Lip}_b(X)} = \max(\|f\|_{\infty}, L(f)),$$

where $L(f)$ denotes the Lipschitz constant of $f$. It is a fact [Wea99, Ch. 2] that the Banach algebra Lip$_b(X)$ is a dual Banach space and so it has a weak* topology;
for the present work, it is sufficient to consider sequential convergence which is characterized as follows: $f_n \overset{w^*}{\rightarrow} f$ if and only if the global Lipschitz constants of the $f_n$ are uniformly bounded and $f_n \rightarrow f$ pointwise.

Having fixed a Radon measure $\mu$ on $X$, derivations are weak* continuous bounded linear maps $D : \text{Lip}_b(X) \rightarrow L^\infty(\mu)$ which satisfy the product rule $D(fg) = fDg + gDf$. Intuitively, derivations can be interpreted as measurable vector fields and depend only on the measure class of $\mu$. For example, if $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$, one obtains a derivation $\frac{\partial}{\partial x_i} : \text{Lip}_b(X) \rightarrow L^\infty(\mathcal{L}^n)$ by taking the partial derivative of Lipschitz functions in the $x_i$-direction. Note that the set of derivations is an $L^\infty(\mu)$-module.

Even for metric measure spaces $(X, \mu)$ which cannot admit a differentiable structure the module $X(\mu)$ can be nontrivial. Moreover, one can also study the modules $X(\mu)$ and $X(\mu')$ for mutually singular measures $\mu$ and $\mu'$ on the same space $X$. Derivations provide thus a broad definition of differentiability for Lipschitz functions and it is desirable to obtain a characterization of derivations for general metric measure spaces. In [Sch13] the author showed that there is a correspondence between Alberti representations and Weaver derivations which implies, roughly speaking, that derivations are obtained by taking derivatives along fragments. Some results in [Sch13] relevant for the present work are recalled in Subsection 2.4.

**Main results.** We now describe the main results of this paper and refer the reader to the following sections for an explanation of the terminology; we denote by $\text{M}_k(X)$ the Banach space of $k$-dimensional metric currents in the metric space $X$.

It is an observation\(^1\) that there is a close similarity between Weaver derivations and $1$-dimensional metric currents (see Sec. 3). In the light of [Sch13] it is thus natural to ask how this similarity relates to the existence of Alberti representations. We show that the mass $\|T\|$ of a $k$-dimensional metric current $T$ posseses Alberti representations in the directions of $k$-dimensional cone fields. Specifically, in Section 4 we prove the following:

**Theorem 1.3.** Let $T \in \text{M}_k(X) \setminus \{0\}$ for $k > 0$. Then there are disjoint Borel sets $\{V_j\}_j$ and $1$-Lipschitz functions $\pi^j : X \rightarrow \mathbb{R}^k$ (on $\mathbb{R}^k$ we consider the $l^\infty$ norm) such that:

\begin{enumerate}
  \item $\|T\| \left( X \setminus \bigcup V_j \right) = 0.$
  \item For all $\varepsilon > 0$ and for any $k$-dimensional cone field $C$, the measure $\|T\|$ admits a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation $\mathcal{A}$ with $\mathcal{A}L V_j$ in the $\pi^j$-direction of $C$.
\end{enumerate}

In particular, the module $X(\|T\|)$ contains $k$ independent derivations.

Note that the proof of Theorem 1.3 actually does not take full advantage of the joint continuity of $T$ in its last arguments $(\pi_1, \ldots, \pi_k)$ and so applies to a larger class of metric functionals. It might be worth mentioning a connection between Theorem 1.3 and the classical Rademacher Theorem, which asserts that a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $\mathcal{H}^n$-a.e. point, where $\mathcal{H}^n$ denotes the Lebesgue measure. Given a top dimensional current $T \in \text{M}_n(\mathbb{R}^n)$, Theorem 1.3 implies that the mass measure $\|T\|$ posseses $n$-independent Alberti representations, and then it follows that the conclusion of Rademacher’s Theorem holds for the measure $\|T\|$. Recently, M. Csörnyei and P. Jones have announced that Rademacher’s

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\(^1\)Gong [Gon11, pg. 3] attributes it to Wenger
Theorem is sharp in the sense that, if its conclusion holds for the metric measure space \( (\mathbb{R}^n, \mu) \), then \( \mu \) must be absolutely continuous with respect to the Lebesgue measure.

Note also that Theorem 1.3 suggests that metric currents come with some weak notion of a differentiable structure. To make this intuition precise, we prove a representation formula for metric currents in terms of Weaver derivations; essentially, a \( k \)-dimensional metric current \( T \) is of the form \( \omega_T \| T \| \), where \( \omega_T \) is a measurable \( k \)-dimensional vector field (see the next Subsection) and the formal \( k \)-form \( (f, \pi_1, \ldots, \pi_k) \) can be interpreted as a \( k \)-form in the \( k \)-th exterior power of the Weaver’s cotangent bundle (see also the next Subsection). Specifically, in Section 5 we prove:

**Theorem 1.4.** Let \( T \in M_k(X) \) and assume that \( \mathcal{X}(\| T \|) \) is finitely generated with \( N \) generators. Then there is \( \omega_T \in \mathcal{X}(\| T \|) \) (or \( \omega_T \in \text{Ext}^k \mathcal{X}(\| T \|) \) or \( \omega_T \in \text{Ext}^k \mathcal{X}((\| T \|)) \)) such that:

\[
(1.5) \quad T(f, \pi_1, \ldots, \pi_k) = \int_X f(\omega_T, d\pi_1 \wedge \cdots \wedge d\pi_k) \, d\| T \|.
\]

Moreover, \( \omega_T \) has norm at most \( (C(N))^k \binom{N}{k} \).

Note that the assumption that \( \mathcal{X}(\mu) \) is finitely generated is not very restrictive as it holds if \( \| T \| \) is doubling or if the ambient metric space is doubling [Sch13]. Note also how Theorem 1.4 parallels the representation of classical currents ([KP08, Sec. 7.2], [Fed69, Sec. 4.1]).

In Section 6 we provide two applications of this theory. The first application provides an approximation of 1-dimensional metric currents in terms of normal currents:

**Theorem 1.6.** If \( T \in M_1(Z) \) where \( Z \) is a Banach space and if the module \( \mathcal{X}(\| T \|) \) is finitely generated, then there is a sequence of normal currents \( \{N_n\} \subset N_1(Z) \) such that:

\[
(1.7) \quad \lim_{n \to \infty} \| T - N_n \|_{M_1(Z)} = 0.
\]

This provides an affirmative answer to the 1-dimensional case of a question raised in [AK00, pg. 68]. Note that even though we prove the result in Banach spaces, the proof can be adapted to spaces where fragments can be filled-in to give Lipschitz curves. In particular, the structure of 1-dimensional metric currents seems very close to that of normal currents. Note that this is not the case for classical currents.

As a second application we provide a different method to produce Alberti representations which is based on results of Paolini and Stepanov [PS12, PS13] on the structure of 1-dimensional normal currents. This approach allows to gain a better control on the direction of the Alberti representations; in fact, instead of obtaining Alberti representations in the \( \psi \)-direction of a finite dimensional cone field \( C \), one obtains Alberti representations in the \( \psi \)-direction of a vector field \( v \). Moreover, the Lipschitz function \( \psi \) can be taken to be \( l^2 \)-valued, allowing to control countably many functions. The precise result is Theorem 6.31, which is proved in Subsection 6.2. This result is based on identifying a special class of derivations, which we call normal derivations, which have properties closely related to those of normal currents. A further direction related to this result is to extend the action of derivations to Lipschitz functions which take values in Banach spaces with the Radon-Nikodym property: this will be pursued elsewhere.
Technical tools. Section 7 contains some technical results. In Subsection 7.1 we discuss exterior powers in the categories of Banach spaces, $L^\infty(\mu)$-modules and $L^\infty(\mu)$-normed modules. This material is just an adaptation of the treatment in [CLM79, Ch. 2 and 3] of tensor products. The motivation is to give a precise meaning to an exterior product of derivations $D_1 \wedge \cdots \wedge D_k$; as $\mathcal{X}(\mu)$ is an $L^\infty(\mu)$-normed module, the construction can be done in the three aforementioned categories and the results are different. In the author’s opinion, the most natural choice is probably that of $L^\infty(\mu)$-normed modules.

In Subsection 7.2 we prove Theorem 7.97 which is a criterion to produce Alberti representations for measures in Banach spaces when the direction and the speed are specified by linear maps. This result is used in the proof of Theorem 1.6.

In Subsection 7.3 we discuss Theorem 7.101 which is a renorming trick which allows to obtain a strictly convex local norm on $\mathcal{X}(\mu)$ by taking a biLipschitz deformation of the metric on the ambient metric space. This result is used in the proof of Theorem 6.31 and might be of independent interest. It is worth to point out that Theorem 7.101, when specialized to the context of differentiability spaces, gives a stronger conclusion than Cheeger’s renorming Theorem [Che99, Sec. 12] for PI-spaces. In fact, Theorem 7.101 works in general differentiability spaces, does require only a small perturbation of the distance function, and works globally (while Cheeger’s argument works only on a single chart).

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2. Preliminaries

2.1. Metric currents. We recall here some definitions and facts about metric currents and refer the reader to [AK00, Lan11] for more information.

Let $\mathcal{D}^k(X)$ denote the set of Lip$_b(X) \times (\text{Lip}(X))^{k^2}$ of $(k+1)$-tuples of Lipschitz functions where the first one is bounded. Intuitively, we want to think of a $(k+1)$-tuple $(f, \pi_1, \cdots, \pi_k)$ as a $k$-differential form $f \, d\pi_1 \wedge \cdots \wedge d\pi_k$. A map $T : V \to \mathbb{R}$, where $V$ is a vector space over $\mathbb{R}$ is called subadditive if for each $v_1, v_2 \in V$ one has:

\begin{equation}
|T(v_1 + v_2)| \leq |T(v_1)| + |T(v_2)|;
\end{equation}

the map $T$ is called positively 1-homogeneous if for all $(v, \lambda) \in V \times [0, \infty)$ one has:

\begin{equation}
|T(\lambda v)| = \lambda |T(v)|.
\end{equation}

Definition 2.3. A $k$-dimensional metric functional $T$ on the metric space $X$ is a map $T : \mathcal{D}^k(X) \to \mathbb{R}$ which is subadditive and positively 1-homogeneous in each of its arguments $(f, \pi_1, \cdots, \pi_k)$. The boundary $\partial T$ of a $k$-dimensional metric functional $(k \geq 1)$ is the $(k-1)$-dimensional metric functional defined by:

\begin{equation}
\partial T(f, \pi_1, \cdots, \pi_{k-1}) = T(1, f, \pi_1, \cdots, \pi_{k-1}).
\end{equation}

For 0-dimensional metric functionals we convene that the boundary is 0.

\footnote{for $k = 0$ we let $\mathcal{D}^0(X) = \text{Lip}_b(X)$}
Definition 2.5. A $k$-dimensional metric functional $T$ has finite mass if there is a finite Radon measure $\mu$ such that for each $(f, \pi_1, \cdots, \pi_k) \in D^k(X)$:

$$|T(f, \pi_1, \cdots, \pi_k)| \leq \prod_{i=1}^{k} L(\pi_i) \int_X |f| \, d\mu.$$  

In this case there is a minimal $\mu$ satisfying (2.6), called the mass of $T$ and denoted by $\|T\|_\mu$.

Remark 2.7. Note that any metric functional $T$ with finite mass can be uniquely extended to a map $T : B^\infty(X) \times (\text{Lip}(X))^k$ so that the first argument $f$ can be taken to be a bounded Borel function.

Definition 2.8. Let $T$ be a $k$-dimensional metric functional with finite mass. Suppose that $l \leq k$ and that

$$\omega = (\psi, \pi_1, \cdots, \pi_l) \in B^\infty(X) \times (\text{Lip}(X))^l;$$

the restriction $T|\omega$ is the $(k-l)$-dimensional metric functional defined by:

$$T|\omega(f, \tilde{\pi}_1, \cdots, \tilde{\pi}_{k-l}) = T(f\psi, \pi_1, \cdots, \pi_l, \tilde{\pi}_1, \cdots, \tilde{\pi}_{k-l}).$$

In the Introduction we recalled the notion of weak* convergence for sequences in $\text{Lip}_b(X)$. We now introduce a notion of convergence for sequences in $\text{Lip}(X)$ which plays a fundamental role in the definition of metric currents: if $\{f_n\} \subset \text{Lip}(X)$ and $f \in \text{Lip}(X)$, we write $f_n \overset{w*}{\rightharpoonup} f$ if $f_n \to f$ pointwise and $\sup_n L(f_n) < \infty$.

Definition 2.11. A $k$-dimensional metric functional $T$ of finite mass is called a metric current if it satisfies the following additional properties:

1. $T$ is multilinear in its arguments $f, \pi_1, \cdots, \pi_k$;
2. $T$ is alternating in its last $k$-arguments: $\pi_1, \cdots, \pi_k$;
3. $T$ is local in the sense that if some $\pi_i$ is constant on the set $\{x : f(x) \neq 0\}$, then

$$T(f, \pi_1, \cdots, \pi_k) = 0;$$

4. if $f_n \overset{w*}{\rightharpoonup} f$ and for $i \in \{1, \cdots, k\}$, $\pi_{i,n} \overset{w*}{\rightharpoonup} \pi_i$, one has:

$$\lim_{n \to \infty} T(f_n, \pi_{i,1}, \cdots, \pi_{i,k}) = T(f, \pi_1, \cdots, \pi_k).$$

The set of $k$-dimensional metric currents is denoted by $\mathcal{M}_k(X)$ and is a Banach space with norm $\|T\|_{\mathcal{M}_k(X)} = \|T\|(X)$. An important class of metric currents consists of the normal currents:

Definition 2.14. A $k$-dimensional metric current is a normal current if the boundary $\partial T$ is a metric current. The set of $k$-dimensional normal currents is denoted by $\mathcal{N}_k(X)$ and is a Banach space with norm:

$$\|T\|_{\mathcal{N}_k(X)} = \|T\|(X) + \|\partial T\|(X).$$

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3In this formulation some axioms are redundant, see [AK00, Sec. 3].
2. Alberti representations. In this Subsection we recall some facts about Alberti representations.

**Definition 2.16.** Let $\mu$ be a Radon measure on a metric space $X$ and $M(X)$ denote the set of finite Radon measures on $X$; an Alberti representation of $\mu$ is a pair $(P,\nu)$:

1. The measure $P$ is a regular Borel probability measure on $\text{Frag}(X)$;
2. The map $\nu: \text{Frag}(X) \to M(X)$ is Borel 4 and $\nu_\gamma \ll \mathcal{H}^1$;
3. The measure $\mu$ can be represented as $\mu = \int_{\text{Frag}(X)} \nu_\gamma \, dP(\gamma)$;
4. For each Borel set $A \subset X$ and for all real numbers $a \geq b$, the map $\gamma \mapsto \nu_\gamma (A \cap \gamma(\text{dom } \gamma \cap [a,b]))$ is Borel.

**Remark 2.17.** Note that in this paper the definition of fragments is different from that used in [Sch13] because, for a fragment $\gamma: K \to X$, we neither require $\gamma$ to be biLipschitz nor we require $K$ to have positive Lebesgue measure. However, an application of the area formula [Kir94, Cor. 8] shows that the results that we cite from [Sch13] are still valid in this setting.

In order to define notions of speed and direction for Alberti representations we recall the definitions of Euclidean cone and of the metric differential of a fragment.

**Definition 2.18.** Let $\alpha \in (0, \pi/2)$, $w \in \mathbb{S}^{n-1}$; the open cone $C(w, \alpha) \subset \mathbb{R}^n$ with axis $w$ and opening angle $\alpha$ is:

$$C(w, \alpha) = \{ u \in \mathbb{R}^n : \tan \alpha(w, u) > \| \pi_w^\perp u \|_2 \},$$

where $\pi_w^\perp$ denotes the orthogonal projection on the orthogonal complement of the line $\mathbb{R}w$.

**Definition 2.20.** For a fragment $\gamma \in \text{Frag}(X)$, the metric differential $\text{md } \gamma(t)$ of $\gamma$ at $t \in \text{dom } \gamma$ is the limit

$$\lim_{\text{dom } \gamma \ni t' \to t} \frac{d(\gamma(t'), \gamma(t))}{|t' - t|}$$

whenever it exists; if $t$ is an isolated point of $\text{dom } \gamma$ we convene that the limit is 0.

In order to measure the direction of a fragment $\gamma$, one uses a Lipschitz function $f: X \to \mathbb{R}^n$ and studies the direction of $(f \circ \gamma)'$ using cones.

**Definition 2.22.** An $n$-dimensional cone field $C$ is a Borel map from $X$ to the set of open cones in $\mathbb{R}^n$. Alternatively, an $n$-dimensional cone-field $C$ is specified by a pair of Borel maps $\alpha: X \to (0, \pi/2)$ and $w: X \to \mathbb{S}^{n-1}$ by letting $C(x) = C(\alpha(x), w(x))$.

Given a Lipschitz function $f: X \to \mathbb{R}^n$, an Alberti representation $A = (P, \nu)$ is said to be **in the f-direction of the n-dimensional cone-field $C$** if for $P$-a.e. $\gamma \in \text{Frag}(X)$ and $\mathcal{L}^1 \text{dom } \gamma$-a.e. $t$ one has $(f \circ \gamma)'(t) \in C(\gamma(t))$.

**Definition 2.23.** Let $\sigma: X \to [0, \infty)$ be Borel and $f: X \to \mathbb{R}$ be Lipschitz. An Alberti representation $A = (P, \nu)$ is said to be **have f-speed $\geq \sigma$ (resp. $> \sigma$)** if for $P$-a.e. $\gamma \in \text{Frag}(X)$ and $\mathcal{L}^1 \text{dom } \gamma$-a.e. $t$ one has $(f \circ \gamma)'(t) \geq \sigma(\gamma(t)) \text{md } \gamma(t)$ (resp. $(f \circ \gamma)'(t) > \sigma(\gamma(t)) \text{md } \gamma(t)$).

One finally needs also to control the Lipschitz constant of the fragments used to produce Alberti representations.

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4 on $M(X)$ one takes the weak* topology
**Definition 2.24.** An Alberti representation $A = (P, \nu)$ is said to be $C$-Lipschitz (resp. $(C, D)$-biLipschitz) if $P$-a.e. $\gamma$ is $C$-Lipschitz (resp. $(C, D)$-biLipschitz).

Alberti representations are produced using Rainwater’s Lemma [Rai69], which can be regarded as a generalization of the Radon-Nikodym Theorem. In particular, one studies a notion of nullity for sets with respect to a family of measures.

**Definition 2.25.** Let $S \subset X$ and $\Omega \subset \text{Frag}(X)$. The set $S$ is said to be $\Omega$-null if for each $\gamma \in \Omega$ one has $\mathfrak{H}^{1}\gamma(S) = 0$.

We will use the previous notion of nullity mainly for the following families of fragments:

**Definition 2.26.** Let $f : X \to \mathbb{R}^n$ and $g : X \to \mathbb{R}$ be Lipschitz functions, $\sigma : X \to [0, \infty)$ a Borel function and $C$ an $n$-dimensional cone field. We denote by $\text{Frag}(X, f, C, g, \geq \sigma)$ the set of those $\gamma \in \text{Frag}(X)$ satisfying:

\[
(2.27) \quad (f \circ \gamma)'(t) \in C\gamma(t)) \quad \text{for} \quad L^1\text{dom}^{\gamma}\text{-a.e } t;
\]

\[
(2.28) \quad (g \circ \gamma)'(t) > \sigma\gamma(t)) \quad \text{md}^{\gamma}\text{dom}^{\gamma}\text{-a.e } t;
\]

the set $\text{Frag}(X, f, C, g, \geq \sigma)$ is defined by changing the strict inequality in (2.28) to a non-strict inequality.

The following Theorem (Theorem 2.64 in [Sch13]) is a standard criterion to produce Alberti representations:

**Theorem 2.29.** Let $X$ be a complete separable metric space and $\mu$ a Radon measure on $X$. Then the following are equivalent:

1. The measure $\mu$ admits an Alberti representation in the $f$-direction of $C$ with $g$-speed $> \sigma$;
2. For each $\varepsilon > 0$ the measure $\mu$ admits a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation in the $f$-direction of $C$ with $g$-speed $> \sigma$;
3. Any Borel set $S \subset X$ which is $\text{Frag}(X, f, C, g, \geq \sigma)$-null is also $\mu$-null.

In the following we will also use a gluing principle for Alberti representations (compare Theorem 2.46 in [Sch13]). Note that if $A = (P, \nu)$ is an Alberti representation of $\mu$, for a Borel $U \subset X$ one has an Alberti representation $A \cup U = (P, \nu \cup U)$ which is called the restriction of $A$ to $U$.

**Definition 2.30.** A countable collection $\{U_\alpha\}$ of $\mu$-measurable sets with positive $\mu$-measure is called an $L^\infty(\mu)$-partition of unity if $\mu((\bigcup_\alpha U_\alpha)^c) = 0$; note that in this case

\[
(2.31) \quad \sum_\alpha \chi_{U_\alpha} = 1
\]

where convergence of the series is understood in the weak* sense. If the sets $U_\alpha$ are Borel (resp. compact) the $L^\infty(\mu)$-partition of unity is called Borel (resp. compact).

**Theorem 2.32.** Let $X$ be a complete separable metric space and $\mu$ a Radon measure on $X$ and $\{U_\alpha\}$ a Borel $L^\infty(\mu)$-partition of unity. If for each $\alpha$ the measure $\mu\cup U_\alpha$ admits an Alberti representation in the $f_\alpha$-direction of an $N_\alpha$-dimensional cone field $C_\alpha$ with $g_\alpha$-speed $\geq \sigma_\alpha$, then $\mu$ admits an Alberti representation $A$ such that each restriction $A\cup U_\alpha$ is in the $f_\alpha$-direction of an $N_\alpha$-dimensional cone field $C_\alpha$ with $g_\alpha$-speed $\geq \sigma_\alpha$. Moreover, for each $\varepsilon > 0$ the Alberti representation $A$ can be assumed to be $(1, 1 + \varepsilon)$-biLipschitz.
2.3. Derivations. An $L^\infty(\mu)$-module $M$ is a Banach space $M$ which is also an $L^\infty(\mu)$-module and such that for all $(m, \lambda) \in M \times L^\infty(\mu)$ one has:

\begin{equation}
\|\lambda m\|_M \leq \|\lambda\|_{L^\infty(\mu)} \|m\|_M.
\end{equation}

Among $L^\infty(\mu)$-modules a special rôle is played by $L^\infty(\mu)$-normed modules:

**Definition 2.34.** An $L^\infty(\mu)$-module $M$ is said to be an $L^\infty(\mu)$-normed module if there is a map

\begin{equation}
\cdot |_{M,\text{loc}} : M \to L^\infty(\mu)
\end{equation}

such that:

1. For each $m \in M$ one has $|m|_{M,\text{loc}} \geq 0$;
2. For all $c_1, c_2 \in \mathbb{R}$ and $m_1, m_2 \in M$ one has:

\begin{equation}
|c_1 m_1 + c_2 m_2|_{M,\text{loc}} \leq |c_1| |m_1|_{M,\text{loc}} + |c_2| |m_2|_{M,\text{loc}};
\end{equation}
3. For each $\lambda \in L^\infty(\mu)$ and each $m \in M$, one has:

\begin{equation}
|\lambda m|_{M,\text{loc}} = |\lambda| |m|_{M,\text{loc}};
\end{equation}
4. The local seminorm $|\cdot|_{M,\text{loc}}$ can be used to reconstruct the norm of any $m \in M$:

\begin{equation}
\|m\|_M = \|m|_{M,\text{loc}} \|_{L^\infty(\mu)}.
\end{equation}

Let $\mu$ be a Radon measure on the metric space $X$ and denote by $M_k(\mu)$ the set of $k$-dimensional metric currents whose mass in absolutely continuous with respect to $\mu$.

**Lemma 2.39.** The set $M_k(\mu)$ is a Banach space and an $L^\infty(\mu)$-module. It is not an $L^\infty(\mu)$-normed module if

1. $k > 0$ and $M_k(\mu) \neq \{0\}$;
2. $k = 0$ and $\mu$ is not a Dirac measure.

**Proof.** The space $M_k(X)$ is a Banach space with the mass norm. Suppose that

\begin{equation}
\lim_{k \to \infty} \|T_k - T\|(X) = 0,
\end{equation}

and that for each $k$ one has $\|T_k\|(A) = 0$; then one has $\|T\|(A) = 0$. Thus, $M_k(\mu)$ is a closed subspace of $M_k(X)$ and hence a Banach space.

The action of $L^\infty(\mu)$ on $M_k(\mu)$ is given by

\begin{equation}
\lambda \cdot T = T \lambda \lambda,
\end{equation}

and $\|T \lambda \lambda\|(X) \leq \|\lambda\|_{L^\infty(\mu)} \|T\|(X)$; thus $M_k(\mu)$ is an $L^\infty(\mu)$-module.

Let $\delta_x$ denote the Dirac measure concentrated at $x$. Using [AK00, (iii) in Thm. 3.5] it follows that $M_k(\delta_x) = 0$ for $k > 0$. Thus, if $T \in M_k(\mu)$ is non-trivial, there is a Borel $U \subset X$ with

\begin{equation}
\|T\|(U), \|T\|(X \setminus U) > 0;
\end{equation}

in particular,

\begin{equation}
\|T\|(X) > \max(\|T \lambda \lambda\|(X), \|T \lambda (1 - \chi_U)\|(X))
\end{equation}

and so $M_k(\mu)$ is not an $L^\infty(\mu)$-normed module.

The same argument can be applied if $k = 0$ and $\mu$ is not a Dirac measure. □
We now introduce the notion of derivations. In the Introduction we described sequential convergence for the weak* topology on Lipb(X); for further information we refer the reader to [Wea99, Ch. 2].

**Definition 2.44.** A derivation $D : \text{Lip}_b(X) \to L^\infty(\mu)$ is a weak* continuous, bounded linear map satisfying the product rule:

$$D(fg) = fDg + gDf.$$  

(2.45)

Note that the product rule implies that $Df = 0$ if $f$ is constant. The collection of all derivations $\mathcal{X}(\mu)$ is an $L^\infty(\mu)$-normed module [Wea00, Thm. 2] and the corresponding local norm will be denoted by $\| \cdot \|_{\mathcal{X}(\mu),\loc}$. Note also that $\mathcal{X}(\mu)$ depends only on the measure class of $\mu$.

**Remark 2.46.** Consider a Borel set $U \subset X$ and a derivation $D \in \mathcal{X}(\mu|U)$. The derivation $D$ can be also regarded as an element of $\mathcal{X}(\mu)$ by extending $Df$ to be $0$ on $X \setminus U$ (compare Lemma 2.47). In particular, the module $\mathcal{X}(\mu|U)$ can be naturally identified with the submodule $\chi_U \mathcal{X}(\mu)$ of $\mathcal{X}(\mu)$.

Derivations are local in the following sense ([Wea00, Lem. 27]):

**Lemma 2.47.** If $U$ is $\mu$-measurable and if $f, g \in \text{Lip}_b(X)$ agree on $U$, then for each $D \in \mathcal{X}(\mu)$, $\chi_U Df = \chi_U Dg$.

Note that locality allows to extend the action of derivations on Lipschitz functions so that if $f \in \text{Lip}(X)$ and $D \in \mathcal{X}(\mu)$, $Df$ is well-defined (see Remark 2.115 in [Sch13]). We now pass to consider some algebraic properties of $\mathcal{X}(\mu)$. In general, even if the module $\mathcal{X}(\mu)$ is finitely generated, it is not free. Nevertheless, it is possible to obtain a decomposition into free modules over smaller rings [Wea00, Sch]:

**Theorem 2.48.** Suppose that the module $\mathcal{X}(\mu)$ is finitely generated with $N$ generators. Then there is a Borel partition $X = \bigcup_{i=0}^N X_i$ such that, if $\mu(X_i) > 0$, then $\mathcal{X}(\mu|X_i)$ is free of rank $i$ as an $L^\infty(\mu|X_i)$-module. A basis of $\mathcal{X}(\mu|X_i)$ will be called a local basis of derivations.

In many applications in Analysis on metric spaces the assumption that $\mathcal{X}(\mu)$ is finitely generated is not restrictive: for example it holds if either $\mu$ or $X$ are doubling (Corollary 5.136 in [Sch13]).

In practice, to explicitly use the linear independence of some derivations it is useful to construct pseudodual Lipschitz functions:

**Definition 2.49.** We say that Lipschitz functions $\{g_j\}_{j=1}^k \subset \text{Lip}_b(X)$ are pseudodual to $\{D_i\}_{i=1}^k \subset \mathcal{X}(\mu)$ on a Borel set $U$, if $\chi_U(D_ig_j - \delta_{i,j}) = 0$ and $\mu(U) > 0$. In this case, note that the derivations $\{\chi_U D_i\}_{i=1}^k \subset \mathcal{X}(\mu|U)$ are independent.\(^5\)

The following Lemma constructs pseudodual functions given independent derivations. However, it is a slight improvement of similar results [Gon12, Sch] because it controls the norm of the derivations obtained. This improvement is used in the proof of Theorem 1.4.

**Lemma 2.50.** Suppose that the derivations $\{D_i\}_{i=1}^k \subset \mathcal{X}(\mu)$ are independent. Then there are a Borel $L^\infty(\mu)$-partition of unity $V_\alpha$ and there are, for each $\alpha$, derivations $\{D_{\alpha,i}\}_{i=1}^k \subset \chi_{V_\alpha} \mathcal{X}(\mu)$ and $1$-Lipschitz functions $\{g_{\alpha,j}\}_{j=1}^k \subset \text{Lip}_b(X)$ such that:

\(^5\) we consider the ring $L^\infty(\mu|U)$
(1) The submodule of $\mathcal{X}(\mu)$ generated by the derivations $\{D_{\alpha,i}\}_{i=1}^k$ contains the submodule generated by the derivations $\{\chi_{V_n} D\}_{n=1}^k$.

(2) The derivations $\{D_{\alpha,i}\}_{i=1}^k$ have norm at most $C(k)$, a universal constant depending only on $k$.

(3) The functions $\{f_{\alpha,i}\}_{i=1}^k$ are pseudodual to the derivations $\{D_{\alpha,i}\}_{i=1}^k$ on $V_{\alpha}$.

To prove Lemma 2.50 we introduce a notion of normalization for derivations. We first consider the set where a given derivation vanishes:

**Definition 2.51.** Given a derivation $D \in \mathcal{X}(\mu)$, having chosen a Borel representative of $|D|_{\mathcal{X}(\mu), \text{loc}}$, we let

$$N_D = \left\{ x : |D|_{\mathcal{X}(\mu), \text{loc}} (x) = 0 \right\};$$

note that $N_D$ is well-defined up to Borel $\mu$-null sets and that $\lambda D = 0$ if $\lambda \in \chi_{N_D} \mathcal{L}^\infty(\mu)$. If $N_D$ is $\mu$-null, we say that $D$ is **nowhere vanishing**.

**Lemma 2.53.** For a derivation $D \in \mathcal{X}(\mu)$, having chosen a Borel representative of $|D|_{\mathcal{X}(\mu), \text{loc}}$, we let

$$V_n = \left\{ x : |D|_{\mathcal{X}(\mu), \text{loc}} \in \left( \|D\|_{\mathcal{X}(\mu)}/(n + 1), \|D\|_{\mathcal{X}(\mu)}/n \right) \right\};$$

then

$$\hat{D} = \sum_{\mu(V_n) > 0}^{\infty} \chi_{V_n} \frac{|D|_{\mathcal{X}(\mu), \text{loc}}}{\mu(V_n)} D$$

defines a derivation, the **normalization of $D$**, with $|\hat{D}|_{\mathcal{X}(\mu), \text{loc}} = \chi(N_D)$ - . We will, with slight abuse of notation, denote the normalization of $D$ by $D/|D|_{\mathcal{X}(\mu), \text{loc}}$.

**Proof.** The definition of $\hat{D}$ by (2.55) is well-posed because $\chi_{V_n} |\hat{D}|_{\mathcal{X}(\mu), \text{loc}}$ is invertible in $\mathcal{L}^\infty(\mu \mathcal{L} V_n)$ if $\mu(V_n) > 0$. Moreover, the $V_n$ are uniquely determined up to $\mu$-null sets and so $\hat{D}$ does not depend on the choice of a Borel representative for $|D|_{\mathcal{X}(\mu), \text{loc}}$. Note that for $f \in \text{Lip}_b(X)$ one has

$$\chi_{V_n} |Df| \leq \chi_{V_n} |D|_{\mathcal{X}(\mu), \text{loc}} \|f\|_{\text{Lip}_b(X)},$$

and that the sets $\{V_n : \mu(V_n) > 0\}$ are an $\mathcal{L}^\infty(\mu \mathcal{L} V_n)$-Borel partition of unity. Thus (2.55) provides a bounded linear map $\hat{D} : \text{Lip}_b(X) \to \mathcal{L}^\infty(\mu)$ with norm at most 1. Note also that $\hat{D}$ satisfies the product rule because $D$ does.

We show that $\hat{D}$ is weak* continuous; by the Krein-Šmulian Theorem, if suffices to check continuity for bounded nets. Therefore, suppose that $g \in \mathcal{L}^1(\mu)$ and $f_\eta \xrightarrow{\text{w}^*} f$ where the set $\{f_\eta\}_\eta \cup \{f\}$ is contained in the ball of radius $M$ in $\text{Lip}_b(X)$.

For each $\varepsilon > 0$ there is an $N$ such that for all $h$ of norm at most $M$ in $\text{Lip}_b(X)$,

$$\left| \sum_{\eta > N}^{\infty} \int \frac{\chi_{V_n}}{\mu(V_n)} |Dh|_{\mathcal{X}(\mu), \text{loc}} \, Dh \, d\mu \right| \leq \varepsilon;$$

$$\int g \frac{\chi_{V_n}}{\mu(V_n)} |Dh|_{\mathcal{X}(\mu), \text{loc}} \, Dh \, d\mu \leq \varepsilon;$$
We first prove that for each \( \varepsilon > \chi \) (2.61)

Without loss of generality, we can assume that
Proof of Lemma 2.50.

(2.59) \( \lim_{n} \int g\hat{D}_{N}f_{n} \, d\mu = \int g\hat{D}_{N}f \, d\mu \); combining (2.57) and (2.59), we conclude that
which shows that \( \hat{D} \) is a derivation,

(2.58) \( \hat{D}_{N} = \sum_{n \leq N, \mu(V_{n}) > 0} \frac{\chi_{V_{n}}}{\chi_{V_{n}}[D_{\chi}, \text{loc}]} D \)

is a derivation,

(2.59) \( \lim_{n} \int g\hat{D}_{N}f_{n} \, d\mu = \int g\hat{D}_{N}f \, d\mu \);

combining (2.57) and (2.59), we conclude that

(2.60) \( \lim_{n} \int g\hat{D}f_{n} \, d\mu = \int g\hat{D}f \, d\mu \),

which shows that \( \hat{D} \) is weak* continuous.

We observe that \( \chi_{N_{D}} \) annihilates \( \hat{D} \); thus, to show that \( |\hat{D}|_{\chi(\mu), \text{loc}} = \chi_{N_{D}} \), it suffices to show that the subset \( U \subset N_{D} \) has positive measure, then \( \|\chi_{U}\hat{D}\|_{\chi(\mu)} = 1 \). This follows because, for some \( n, \mu(U \cap V_{n}) > 0 \) and

(2.61) \( \chi_{U \cap V_{n}} |\hat{D}|_{\chi(\mu), \text{loc}} = |\chi_{U \cap V_{n}} \hat{D}|_{\chi(\mu), \text{loc}} = \chi_{U \cap V_{n}} \).

\( \square \)

**Proof of Lemma 2.50.** Without loss of generality, we can assume that \( \mu \) is finite.

We first prove that for each \( \varepsilon > 0 \) there is a Borel \( L^{\infty}(\mu) \)-partition of unity \( \{V_{n}\} \) such that:

- For each \( \alpha \) there are 1-Lipschitz functions \( \{g_{\alpha,j}\}_{j=1}^{k} \) and unit norm derivations \( \{\hat{D}_{\alpha,i}\}_{i=1}^{k} \subset \chi_{V_{\alpha}}(\mu) \);
- The submodule generated by the derivations \( \{\hat{D}_{\alpha,i}\}_{i=1}^{k} \subset \chi_{V_{\alpha}}(\mu) \) contains that generated by the derivations \( \{\chi_{V_{\alpha}}D_{i}\}_{i=1}^{k} \);
- The matrix \( (\chi_{V_{\alpha}}\hat{D}_{\alpha,i}g_{j})_{i,j=1}^{k} \), with entries in \( L^{\infty}(\mu|V_{\alpha}) \), is upper triangular;
- Each entry \( \lambda \) on the diagonal of \( (\chi_{V_{\alpha}}\hat{D}_{\alpha,i}g_{j})_{i,j=1}^{k} \) satisfies \( \lambda \geq 1 - \varepsilon \) (in the ring \( L^{\infty}(\mu|V_{\alpha}) \)).

We will refer to this property as \( P(k, \varepsilon) \) and it will be proved by induction on \( k \).

For \( k = 1 \), we first replace \( D_{1} \) by its normalization \( \tilde{D}_{1} \) (Lemma 2.53) to have

(2.62) \( D_{1}g \geq 1 - \varepsilon \) \( \mu \text{-a.e. on } W \),

is not empty. We choose

(2.63) \( \mu(V_{1}) \geq \frac{1}{2} \sup_{W \in \mathcal{C}_{1}} \mu(W) \)

and keep going exhausting \( X \) in \( \mu \)-measure (compare the proof of Theorem 2.43 in [Sch]). The functions \( g_{\alpha} \) are chosen accordingly to the sets \( V_{\alpha} \) so that (2.62) holds. Then one lets \( D_{\alpha,1} = \chi_{V_{\alpha}}\tilde{D}_{1} \). The derivation \( \chi_{V_{\alpha}}D_{1} \) belongs to the submodule generated by \( \tilde{D}_{\alpha,1} \) because \( \chi_{V_{\alpha}}D_{1} = |D_{1}|_{\chi(\mu), \text{loc}} \tilde{D}_{\alpha,1} \).
We now show that $P(k + 1, \varepsilon)$ follows from $P(k, \varepsilon)$. Using $P(k, \varepsilon)$ for the derivations $\{D_i\}_{i=1}^k$ we can assume, by replacing $\mu$ with a restriction $\mu L V$, that there are 1-Lipschitz functions $\{g_j\}_{j=1}^k$ and derivations $\{\tilde{D}_i\}_{i=1}^k$ such that $P(k, \varepsilon)$ holds. We consider the normalization $\tilde{D}_{k+1}$ of

$$D_{k+1} = \sum_{i=1}^k \frac{D_{k+1} g_i}{D_i g_i} \tilde{D}_i,$$

so that

$$D_{k+1} g_j = 0 \quad (1 \leq j \leq k);$$

note that $D_{k+1}$ belongs to the submodule generated by the derivations $\{\tilde{D}_i\}_{i=1}^k$. We now apply the argument used in the case $k = 1$ to the derivation $\tilde{D}_{k+1}$ in order to complete the proof of $P(k + 1, \varepsilon)$.

If $M_{\alpha}$ denotes the matrix $(\tilde{D}_{\alpha,i} g_{\alpha,j})_{i,j=1}^k$, its determinant satisfies the bounds:

$$\beta(k)^k \leq \det M_{\alpha} \leq \beta(k),$$

where $\beta(k)$ is a univerval constant depending only on $k$. In particular, letting

$$D_{\alpha,i} = \sum_{j=1}^k (M_{\alpha}^{-1})_{i,j} \tilde{D}_{\alpha,j},$$

we have $|D_{\alpha,i}|_{L V_{\alpha, \text{loc}}} \leq C(k, \varepsilon)$ and $D_{\alpha,i} g_{\alpha,j} = \delta_{i,j} \chi V_{\alpha}$. Moreover, solving (2.67) for the derivations $\{\tilde{D}_{\alpha,i}\}_{i=1}^k$ shows that the derivations $\{\chi V_{\alpha} D_i\}_{i=1}^k$ belong to the submodule generated by the $\{D_{\alpha,i}\}_{i=1}^k$. \hfill \square

Consider a Lipschitz map $F : X \to Y$ and a Radon measure $\mu$ on $X$; given a derivation $D \in \mathcal{X}(\mu)$ the **push forward** $F_* D \in \mathcal{X}(F_* \mu)$ is the derivation defined by:

$$\int_Y g (F_* D) f d F_* \mu = \int_X g \circ F D(f \circ F) d \mu \quad (\forall (f, g) \in \mathcal{D}^1(Y)).$$

We now recall the notion of $1$-forms which are dual to derivations.

**Definition 2.69.** The **module of $1$-forms** $\mathcal{E}(\mu)$ is the dual module of $\mathcal{X}(\mu)$, i.e. it consists of the bounded module homomorphisms $\chi : L^\infty(\mu) \to \mathcal{X}$. The module $\mathcal{E}(\mu)$ is an $L^\infty(\mu)$-normed module and the local norm will be denoted by $| \cdot |_{\mathcal{E}(\mu), \text{loc}}$.

To each $f \in \text{Lip}_b(X)$ one can associate the $1$-form $df \in \mathcal{E}(\mu)$ by letting:

$$\langle df, D \rangle = D f \quad (\forall D \in \mathcal{X}(\mu));$$

the map $d : \text{Lip}_b(X) \to \mathcal{E}(\mu)$ is a weak* continuous $1$-Lipschitz linear map satisfying the product rule $df (fg) = gdf + fdg$.

Note that because of Lemma 2.47 one can extend the domain of $d$ to $\text{Lip}(X)$ so that if $f$ is Lipschitz, $df$ is a well-defined element of $\mathcal{E}(\mu)$ and $\|df\|_{\mathcal{E}(\mu)} \leq L(f)$. 

2.4. Correspondence between derivations and Alberti representations. In this Subsection we recall some results in [Sch13] about the correspondence between derivations and Alberti representations. Throughout this Subsection \( F : X \to \mathbb{R}^k \) denotes a Lipschitz function, \( \alpha \in (0, \pi/2) \) an angle, \( \delta \) a positive constant, \( w \in \mathbb{S}^{k-1} \) a unit vector and \( \{u_i\}_{i=1}^{k-1} \) an orthonormal basis for the orthogonal complement of \( w \).

We first recall an approximation scheme (Theorem 3.66 in [Sch13]) which relates Alberti representations and the weak* topology on \( \text{Lip}_b(X) \):

**Theorem 2.71.** Let \( X \) be a compact metric space and \( \mu \) a Radon measure on \( X \). Suppose that \( K \subset X \) is compact and \( \text{Frag}(X, F, C(w, \alpha), \langle w, F \rangle, \geq \delta)\)-null. Denoting by \( d_{\delta, \alpha} \) the distance:

\[
d_{\delta, \alpha}(x, y) = \delta d(x, y) + \cot \alpha \sum_{i=1}^{k-1} |\langle u_i, F(x) - F(y) \rangle|,
\]

there is a sequence of real-valued Lipschitz functions \( \{g_n\} \) and a Borel \( S \subset K \) such that:

1. \( \mu(K \setminus S) = 0 \);
2. \( g_n \xrightarrow{w^*} \langle w, F \rangle \);
3. for each \( x \in S \) and each \( n \) there is an \( r_n > 0 \) such that the restriction \( g_n|B(x, r_n) \) is 1-Lipschitz with respect to the distance \( d_{\delta, \alpha} \).

We will use the following consequence of Theorem 2.71 whose proof is contained in the proofs of Lemma 3.68 and Lemma 3.76 in [Sch13].

**Lemma 2.73.** Let \( X \) be a complete separable metric measure space and \( \mu \) a Radon measure on \( X \). Suppose that the compact set \( K \subset X \) is \( \text{Frag}(X, F, C(w, \alpha), \langle w, F \rangle, \geq \delta)\)-null. Then there are bounded Lipschitz functions \( \tilde{f}_n \xrightarrow{w^*} \tilde{f} \) such that:

1. For each \( x \in K \), \( \tilde{f}(x) = \langle w, F(x) \rangle \);
2. For each \( n \) there are countably many disjoint Borel sets \( \{S_{m, \alpha}\} \) with \( \mu(K \setminus \bigcup_{\alpha} S_{m, \alpha}) = 0 \);
3. For each \( (m, \alpha) \) there is a sequence \( \{\tilde{f}_{m, \alpha, n}\} \subset \text{Lip}_b(X) \) with \( \tilde{f}_{m, \alpha, n} \xrightarrow{w^*} \tilde{f}_{m, \alpha} \) and \( \tilde{f}_{m, \alpha} = \tilde{f}_m \) on \( S_{m, \alpha} \);
4. For each \( (m, \alpha, n) \) there are finitely many points \( \{x_{m, \alpha, n,j}\} \subset S_{m, \alpha} \) and finitely many disjoint Borel sets \( \{S_{m, \alpha, n,j}\} \) with \( \bigcup_j S_{m, \alpha, n,j} = S_{m, \alpha} \) and such that for each \( x \in S_{m, \alpha, n,j} \) one has:

\[
\tilde{f}_{m, \alpha, n,j}(x) = \tilde{f}_{m, \alpha, n,j}(x) + \delta d(x, x_{m, \alpha, n,j}) + \cot \alpha \sum_{i=1}^{k-1} |\langle u_i, F(x) - F(x_{m, \alpha, n,j}) \rangle|.
\]

In Theorem 3.11 in [Sch13] it was shown that to a \( C \)-Lipschitz Alberti representation \( A \) of the measure \( \mu \) it is possible to associate a derivation \( D_A \in \mathcal{X}(\mu) \) by using the formula:

\[
\int_X gD_A f \, d\mu = \int_{\text{Frag}(X)} dP(\gamma) \int_{\text{dom } \gamma} (f \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1}(\mu))(t) \quad (g \in L^1(\mu) \cap \mathcal{B}^\infty(X))
\]

to define \( D_A f \); moreover, one has the norm bound \( \|D_A\|_{\mathcal{X}(\mu)} \leq C \) and if the Alberti representation \( A \) is in the \( F \)-direction of the \( k \)-dimensional cone field \( C \), one has \( D_A F(x) \in C(x) \) for \( \mu \)-a.e. \( x \).
In order to compare the derivations associated to different Alberti representations the following notion of independence for cone fields is useful:

**Definition 2.76.** We say that the \( n \)-dimensional cone fields \( \{ C_i \}_{i=1}^k \) are independent if for each \( x \in X \) and each choice of \( v_{i,x} \in C_i(x) \), the vectors \( \{ v_{i,x} \}_{i=1}^k \) are linearly independent.

Note that if the Alberti representations \( \{ A_i \}_{i=1}^k \) are in the \( F \)-directions of independent cone fields, the corresponding derivations \( \{ D_{A_i} \}_{i=1}^k \) are independent. We will use the following result (Corollary 3.94 in [Sch13]):

**Corollary 2.77.** Suppose that the measure \( \mu \) admits Alberti representations in the \( F \)-direction of \( k \) independent cone fields. Then for each \( \varepsilon > 0 \) and each \( k \)-dimensional cone field \( C \), the measure \( \mu \) admits a \((1, 1 + \varepsilon)\)-Alberti representation in the \( F \)-direction of \( C \).

A special case of the previous result is the following:

**Corollary 2.78.** Suppose that the functions \( \{ F_i \}_{i=1}^k \) are pseudodual to the derivations \( \{ D_i \}_{i=1}^k \); then for any \( k \)-dimensional cone field \( C \), the measure \( \mu \) admits an Alberti representation in the \( F \)-direction of \( C \).

### 3. 1-dimensional currents and derivations

The goal of this Section is to make precise the correspondence between 1-dimensional metric currents and derivations via Theorem 3.7.

**Lemma 3.1.** Consider a metric functional \( T \in \text{MF}_k(X) \) with finite mass. If \( B \subset X \) is Borel and \( \| T \|(B) > 0 \), then for each \( \eta \in (0, 1) \) there are disjoint Borel sets \( B_i \subset B \) and 1-Lipschitz functions\(^6\) \( \pi^i : X \to \mathbb{R}^k \):

\[
\| T \| \left( B \setminus \bigcup_i B_i \right) = 0;
\]

\[
\| T \| (\chi_{B_i} \pi_1^i, \ldots, \pi_k^i) > \eta \| T \|(B_i).
\]

**Proof.** The proof uses [AK00, Prop. 2.7] (characterization of mass): for each \( \varepsilon > 0 \) there are disjoint Borel sets \( B_i \subset B \) and 1-Lipschitz functions\(^6\) \( \pi^i : X \to \mathbb{R}^k \):

\[
B = \bigcup_i B_i;
\]

\[
\sum_i \left( \| T \|(B_i) - \| T \| (\chi_{B_i} \pi_1^i, \ldots, \pi_k^i) \right) < \varepsilon;
\]

let \( J_\alpha = \{ i : \| T \| (\chi_{B_i} \pi_1^i, \ldots, \pi_k^i) \leq \alpha \| T \|(B_i) \} \); then one has:

\[
(1 - \alpha) \sum_{i \in J_\alpha} \| T \|(B_i) < \varepsilon;
\]

so

\[
\| T \| \left( \bigcup_{i \in J_\alpha} B_i \right) < \frac{\varepsilon}{1 - \alpha};
\]

therefore the conclusion of the Lemma is true for those \( i \notin J_\alpha \) which cover all but \( \frac{\varepsilon}{1 - \alpha} \) of the \( \| T \| \)-measure of \( B \). The result follows by an exhaustion argument. \( \square \)

\(^6\)with respect to the \( l^\infty \)-norm
Theorem 3.7. Let $\mu$ be a finite Radon measure on $X$. There is a map

$$\text{Der}_\mu : \mathcal{M}_1(\mu) \rightarrow \mathcal{X}(\mu)$$

(3.8)

where $D_T \in \mathcal{X}(\|T\|)$ is the unique derivation satisfying

$$T(f, \pi) = \int f D_T \pi d\|T\| \quad (\forall (f, \pi) \in L^1(\|T\|) \times \text{Lip}(X))$$

(3.9a)

and

$$|D_T|_{\mathcal{X}(\|T\|), \text{loc}} = 1.$$  

(3.9b)

Conversely, there is an $L^\infty(\mu)$-module homomorphism map

$$\text{Cur}_\mu : \mathcal{X}(\mu) \rightarrow \mathcal{M}_1(\mu)$$

(3.10)

$$D \mapsto T_D$$

where $T_D$ is the unique current satisfying

$$T_D(f, \pi) = \int f D\pi d\mu \quad (\forall (f, \pi) \in L^1(\|T\|) \times \text{Lip}(X))$$

(3.11a)

and

$$\|T_D\| = |D|_{\mathcal{X}(\mu), \text{loc}} \mu.$$  

(3.11b)

Proof. Given $T \in \mathcal{M}_1(\mu)$, for a fixed $f \in \text{Lip}_b(X)$ one defines a linear functional on $L^1(\|T\|)$ by:

$$g \mapsto T(g, f) \quad (g \in L^1(\|T\|));$$

the Riesz Representation Theorem gives a unique $D_T f \in L^\infty(\|T\|)$:

$$\int_X g D_T f d\|T\| = T(g, f);$$

the map $D_T : \text{Lip}_b(X) \rightarrow L^\infty(\|T\|)$ is a derivation because:

- It is linear by linearity of currents;
- It is bounded with norm 1 because:

$$\int_X |g| d\|T\| \leq L(f) \int_X |g| d\|T\|;$$

- The product rule follows from [AK00, Eq. 3.1 in Thm. 3.5];
- The weak* continuity follows from the continuity axiom for currents ((4) in Defn. 2.11).

Note that the module $\mathcal{X}(\|T\|)$ can be canonically identified with the submodule $\chi_{U_T} \mathcal{X}(\mu)$ where

$$U_T = \left\{ x \in X : \frac{d\|T\|}{d\mu}(x) > 0 \right\},$$

so $\text{Der}_\mu$ is well-defined.

By Lemma 3.1, for each $\eta \in (0, 1)$ we can find disjoint Borel sets $B_i$ and 1-Lipschitz functions $\pi^i \in \text{Lip}(X)$ with $\|T\|(X \setminus \bigcup_i B_i) = 0$ and

$$T(\chi_{B_i}, \pi^i) > \eta\|T\|(B_i);$$

(3.16)

in particular, for each $n \in \mathbb{N}$ one has $\chi_{S_i} D_T \pi^i \geq \frac{\eta}{n+1} \chi_{S_i}$, where $S_i$ is a subset of $B_i$ of measure at least $\frac{\eta}{n+1} \|B_i\|$; using an exhaustion argument and then letting $\eta \rightarrow 1$ and $n \rightarrow \infty$, we conclude that (3.9b) holds. Note that we have used the fact that each derivation $D \in \mathcal{X}(\mu)$ can be canonically extended to a map $D : \text{Lip}(X) \rightarrow L^\infty(\mu)$ (see Remark 2.115 in [Sch13]).
We now prove the second part of this Theorem; note that for \( D \in \mathcal{X}(\mu) \) (3.11a) uniquely determines a current \( T_D \in M_1(\mu) \) because the axioms of metric currents follow from the corresponding properties of derivations. Note also that \( T_{D_1} + T_{D_2} = T_{D_3} + T_{D_4} \) and \( T_{\lambda D} = T_D \cdot \lambda \), showing that \( \text{Cur}_{\mu} \) is an \( L^\infty(\mu) \)-module homomorphism.

As \( |D\pi| \leq L(\pi) |D|_{X(\mu)}, \|T_D\| \leq |D|_{X(\mu)} \|\mu\| \). On the other hand, for each \( \eta \in (0, 1) \) and each Borel set \( A \), we can find disjoint Borel sets \( B_i \subset A \) and \( 1 \)-Lipschitz functions \( \pi^i \) with \( \|T\|(A \setminus \bigcup B_i) = 0 \) and

\[
\chi_{B_i} D\pi^i \geq \eta \chi_{B_i} |D|_{X(\mu)}, \text{loc};
\]
in particular,

\[
\|T_D\|(A) \geq \eta \int_A |D|_{X(\mu)}, \text{loc} \, d\mu
\]
which implies (3.11b). \( \square \)

**Remark 3.19.** From Theorem 3.7 one obtains the following identities:

\[
\text{Cur}_{\mu}(\text{Der}_{\mu}(T)) \cdot \frac{d\|T\|}{d\mu} = T
\]

\[
\text{Der}_{\mu}(\text{Cur}_{\mu}(D)) = \frac{D}{|D|_{X(\mu)}, \text{loc}}.
\]

### 4. Currents and Alberti representations

The goal of this Section is to prove Theorem 1.3. Throughout this Section we will denote by \( \{e_i\}_{i=1}^k \) the standard basis of \( \mathbb{R}^k \). In the proof of Theorem 1.3 we will use the following consequence of Rainwater’s Lemma [Rai69] (compare Corollary 5.8 in [Bat12] and Lemma 2.56 in [Sch13]):

**Lemma 4.1.** Let \( X \) be a complete separable metric space and \( \mu \) a Radon measure on \( X \). Let \( f : X \rightarrow \mathbb{R}^k \) be a Lipschitz map, \( w \in \mathbb{S}^{k-1} \), \( \alpha \in (0, \pi/2) \) and \( \delta > 0 \). For any Borel subset \( B \subset X \) there are disjoint Borel sets \( A, S \) such that:

1. \( A \cup S = B \);
2. The measure \( \mu \ll A \) admits an Alberti representation in the \( f \)-direction of \( C(w, \alpha) \) with \( (w, f), \gamma \geq \delta \);
3. The set \( S \) is \( \text{Frag}(X, f, C(w, \alpha), (w, f), \geq \delta) \)-null.

The proof of Theorem 1.3 relies on the following Lemma:

**Lemma 4.2.** Suppose that \( T(\chi_B, \pi_1, \ldots, \pi_k) \geq \eta \|T\|(B) \), where \( B \) is Borel and \( \pi : X \rightarrow \mathbb{R} \) is 1-Lipschitz and \( \eta > 0 \); then for all pairs \( (\delta, \alpha) \in (0, \eta) \times (0, \pi/2) \) there is a Borel partition \( B = A_\varepsilon \cap S_\varepsilon \) with \( \|T\| \ll A_\varepsilon \) admitting an Alberti representation in the \( \alpha \)-direction of \( C(e_\varepsilon, \alpha) \) with \( \pi_i \)-speed \( \geq \delta \) and \( \|T\|(A_\varepsilon) \geq (\eta - \delta) \|T\|(B) \).

**Proof.** Without loss of generality, we assume \( i = 1 \). Because of Lemma 4.1 we will obtain an upper bound on \( \mu(K) \), where \( K \subset B \) is compact and \( \text{Frag}(X, \pi, C(e_1, \alpha), \pi_1 \geq \delta) \)-null. We apply Lemma 2.73 and we will use the notation from its statement in the remainder of the proof. In particular, we take \( w = e_1, u_i = e_{1+i} \) and \( F = (\pi_i)_{i=1}^k \).
The following estimate is obtained by using the locality axiom ((3) in Definition 2.11) and (2.74):

\[
\begin{align*}
\left| T\left( \chi_{S_{m,\alpha,n,j}}, \tilde{f}_{m,\alpha,n}, \pi_2, \ldots, \pi_k \right) \right| & \leq \delta \left| T\left( \chi_{S_{m,\alpha,n,j}}, d\left( x_{m,\alpha,n,j}, \pi_2, \ldots, \pi_k \right) \right) \right| \\
& + \cot \alpha \sum_{\beta > 1} \left| T\left( \chi_{S_{m,\alpha,n,j}}, \pi_\beta - \pi_\beta(x_{m,\alpha,n,j}), \pi_2, \ldots, \pi_k \right) \right|
\end{align*}
\]

we now let

\[
\begin{align*}
S_{m,\alpha,n,j,\beta}^+ & = \{ x \in S_{m,\alpha,n,j} : \pi_\beta(x) \geq \pi_\beta(x_{m,\alpha,n,j}) \} \\
S_{m,\alpha,n,j,\beta}^- & = \{ x \in S_{m,\alpha,n,j} : \pi_\beta(x) < \pi_\beta(x_{m,\alpha,n,j}) \},
\end{align*}
\]

and conclude that, for \( \beta > 1 \),

\[
\begin{align*}
T\left( \chi_{S_{m,\alpha,n,j}}, \pi_\beta - \pi_\beta(x_{m,\alpha,n,j}), \pi_2, \ldots, \pi_k \right) & = T\left( \chi_{S_{m,\alpha,n,j}}, \pi_\beta - \pi_\beta(x_{m,\alpha,n,j}), \pi_2, \ldots, \pi_k \right) \\
& - T\left( \chi_{S_{m,\alpha,n,j},\beta}, \pi_\beta - \pi_\beta(x_{m,\alpha,n,j}), \pi_2, \ldots, \pi_k \right) \\
& - T\left( \chi_{S_{m,\alpha,n,j},\beta}, \pi_\beta, \pi_2, \ldots, \pi_k \right) \\
& = 0
\end{align*}
\]

where in the last inequality we used that currents are alternating. Combining (4.3) and (4.5) we obtain:

\[
\left| T\left( \chi_{S_{m,\alpha}}, \tilde{f}_{m,\alpha,n}, \pi_2, \ldots, \pi_k \right) \right| \leq \delta \| T \|(S_{m,\alpha}).
\]

If we let \( n \nearrow \infty \) we obtain:

\[
\left| T\left( \chi_{S_{m,\alpha}}, \tilde{f}_{m,\alpha}, \pi_2, \ldots, \pi_k \right) \right| \leq \delta \| T \|(S_{m,\alpha});
\]

but by Lemma 2.73 \( \tilde{f}_{m,\alpha} = \tilde{f}_m \) on \( S_{m,\alpha} \) and summing in \( \alpha \) we conclude that:

\[
\left| T\left( \chi_{K}, \tilde{f}_m, \pi_2, \ldots, \pi_k \right) \right| \leq \delta \| K \|;
\]

letting \( m \nearrow \infty \) and using that \( \tilde{f} = \pi_1 \) on \( K \) we conclude that

\[
\left| T\left( \chi_{K}, \pi_1, \pi_2, \ldots, \pi_k \right) \right| \leq \delta \| K \|.
\]

The proof is completed by applying Lemma 4.1. \( \square \)

**Proof of Theorem 1.3.** For \( \eta \in (0,1) \) let the sets \( B_j \) and the functions \( \pi_j \) satisfy the conclusion of Lemma 3.1 for \( B = X \). Let \( \alpha \in (0, \pi/2) \) be such that the cone fields \( \{ \mathcal{C}(e_1, \alpha) \}_k \) are independent. For \( \delta > 0 \), Lemma 4.2 gives a partition \( B_j = A_{j,e_1} \cup S_{j,e_1} \) with \( \| T \| \mathcal{L} A_{j,e_1} \) admitting an Alberti representation in the \( \pi_j \)-direction of \( \mathcal{C}(e_1, \alpha) \) with \( \pi_1 \)-speed \( \geq \delta \) and

\[
\| T \|(A_{j,e_1}) \geq (\eta - \delta) \| T \|(B);
\]

proceeding by induction and applying Lemma 3.1, we obtain a partition

\[
B_j = A_{j,e_1,\ldots,e_k} \cup S_{j,e_1,\ldots,e_k}
\]
with \(|T|\|A_j,e_1,...,e_k\) admitting Alberti representations in the \(\pi^j\)-directions of the cone fields \(\{C(e_i,\alpha)\}_{i=1}^k\) and

\[
(4.12) \quad \|T\|(A_j,e_1,...,e_k) \geq \prod_{i=1}^k (\eta - i\delta) \|T\|(B).
\]

If \(\delta \in (0, \eta/k), c > 0\); as \(\|T|A_j,e_1,...,e_k\) admits Alberti representations in the \(\pi^j\)-directions of \(k\) independent cone fields, the proof is completed by applying Corollary 2.77 and an exhaustion argument.

\[\text{Corollary 4.13.} \quad \text{If } X \text{ is a metric space with Assouad dimension } \leq Q, \text{ then}
\]

\[
(4.14) \quad \mathcal{M}_k(X) = \{0\}
\]

for \(k > Q\); moreover, if \(T \in \mathcal{M}_k(X)\), the module \(X(||T||)\) if finitely generated with at most \(Q\) generators.

\[\text{Proof.} \quad \text{It follows by Theorem 1.3 and by Corollary 4.6 in} \ [\text{Sch13}]. \]

Note that a more general result, which fully exploits the alternating property of metric currents, was obtained by Züst [Züst11, Prop. 2.5] who showed that \(\mathcal{M}_k(X) = \{0\}\) for \(k\) strictly larger than the Nagata dimension of the space \(X\). The class of spaces with finite Nagata dimension is larger than the class of spaces with finite Assouad dimension and the Assouad dimension bounds the Nagata dimension from above [LR13, Thm. 1.1].

5. A REPRESENTATION FORMULA

The goal of this Section is to prove Theorem 1.4 and the representation formula (1.5) which expresses metric currents in terms of derivations. We will use some terminology and results from Subsection 7.1 where, roughly speaking, we construct the exterior powers of the modules \(\mathcal{X}(\mu)\) and \(\mathcal{E}(\mu)\). The dispirited reader may just want to think of expressions like \(D_1 \wedge \cdots \wedge D_k\) and \(df_1 \wedge \cdots \wedge df_k\) as analogues of measurable \(k\)-vectors and \(k\)-covectors fields and keep in mind that as \(\mathcal{X}(\mu)\) and \(\mathcal{E}(\mu)\) are \(L^\infty(\mu)\)-normed modules, their exterior products can be constructed in three different categories: Banach spaces, \(L^\infty(\mu)\)-modules and \(L^\infty(\mu)\)-normed modules.

\[\text{Remark 5.1.} \quad \text{We construct a bilinear pairing between the } L^\infty(\mu)\text{-normed modules Ext}_{\mu,\text{loc}}^k \mathcal{X}(\mu) \text{ and Ext}_{\mu,\text{loc}}^k \mathcal{E}(\mu); \text{ for notational simplicity, we will let } \mathcal{X}^k(\mu) = \text{Ext}_{\mu,\text{loc}}^k \mathcal{X}(\mu) \text{ and } \mathcal{E}^k(\mu) = \text{Ext}_{\mu,\text{loc}}^k \mathcal{E}(\mu). \text{ Consider the map:}
\]

\[
(5.2) \quad \Phi : (\mathcal{X}(\mu))^k \times (\mathcal{E}(\mu))^k \to L^\infty(\mu)
\]

\[((D_1, \cdots, D_k), (\omega_1, \cdots, \omega_k)) \mapsto \det((D_i, \omega_j))_{i,j=1}^k.
\]

For a fixed \(k\)-tuple \(\Omega = (\omega_1, \cdots, \omega_k)\), the map

\[
(5.3) \quad \Phi_\Omega : (\mathcal{X}(\mu))^k \to L^\infty(\mu)
\]

\[(D_1, \cdots, D_k) \mapsto \Phi((D_1, \cdots, D_k), \Omega)
\]
is alternating $L^\infty(\mu)$-multilinear and satisfies the bound

\[ |\Phi_\Omega(D_1, \cdots, D_k)| \leq \sum_{\sigma \in \text{Perm}(k)} \prod_{i=1}^k |\langle D_{\sigma(i)}, \omega_i \rangle| \]

\[ \leq k! \prod_{i=1}^k |\langle D_i \chi(\mu), \omega \rangle| \prod_{j=1}^k |\omega_j| \xi(\mu), \text{loc}. \]  

(5.4)

By the universal property of $\mathcal{X}^k(\mu)$ we obtain an $L^\infty(\mu)$-homomorphism $\hat{\Phi}_\Omega : \mathcal{X}^k(\mu) \to L^\infty(\mu)$. Note that the map

\[ \Psi : (\mathcal{E}(\mu))^k \to (\mathcal{X}^k(\mu))^\prime \]

(5.5)

\[ \Omega \mapsto \hat{\Phi}_\Omega \]

is an alternating $L^\infty(\mu)$-multilinear map with norm at most $k!$ (by (5.4)). By the universal property of $\mathcal{E}^k(\mu)$ we obtain a homomorphism $\hat{\Psi} : \mathcal{E}^k(\mu) \to (\mathcal{X}^k(\mu))^\prime$ and thus an $L^\infty(\mu)$-bilinear pairing

\[ \langle \cdot, \cdot \rangle : \mathcal{X}^k(\mu) \times \mathcal{E}^k(\mu) \to L^\infty(\mu) \]

(5.6)

\[ (\xi, \omega) \mapsto \hat{\Psi}(\omega)(\xi), \]

satisfying

\[ |\langle \xi, \omega \rangle| \leq k! |\xi(\mu), \text{loc}| |\omega(\mathcal{E}^k(\mu), \text{loc}). \]  

(5.7)

By a similar argument, we can produce a pairing working in the category of $L^\infty(\mu)$-modules:

\[ \langle \cdot, \cdot \rangle : \text{Ext}_\mu^k \mathcal{X}(\mu) \times \text{Ext}_\mu^k \mathcal{E}(\mu) \to L^\infty(\mu) \]

which is $L^\infty(\mu)$-bilinear and satisfies:

\[ \|\langle \xi, \omega \rangle\|_{L^\infty(\mu)} \leq k! \|\xi\|_{\text{Ext}_\mu^k \mathcal{X}(\mu)} \|\omega\|_{\text{Ext}_\mu^k \mathcal{E}(\mu)}. \]  

(5.8)

Working in the category of Banach spaces we can produce a pairing

\[ \langle \cdot, \cdot \rangle : \text{Ext}^k \mathcal{X}(\mu) \times \text{Ext}^k \mathcal{E}(\mu) \to L^\infty(\mu) \]

which is $\mathbb{R}$-bilinear an satisfies

\[ \|\langle \xi, \omega \rangle\|_{L^\infty(\mu)} \leq k! \|\xi\|_{\text{Ext}^k \mathcal{X}(\mu)} \|\omega\|_{\text{Ext}^k \mathcal{E}(\mu)}. \]  

(5.9)

Note that given $(D_1, \cdots, D_k) \in (\mathcal{X}(\mu))^k$, we can regard $D_1 \wedge \cdots \wedge D_k$ as either an element of $\mathcal{X}^k(\mu)$, or of $\text{Ext}_\mu^k \mathcal{X}(\mu)$ or of $\text{Ext}^k \mathcal{X}(\mu)$. In the sequel, unless specified all three possibilities are admitted. A similar observation can be applied to an expression $df_1 \wedge \cdots \wedge df_k$ where $(f_1, \cdots, f_k) \in (\text{Lip}(\mathcal{X}))^k$ and to a pairing $\langle D_1 \wedge \cdots \wedge D_k, df_1 \wedge \cdots \wedge df_k \rangle$.

We no prove the local version of Theorem 1.4:

**Lemma 5.12.** For $T \in \mathcal{M}_k(X)$, suppose that the module $\mathcal{X}(||T||)$ is free on the derivations $\{D_i\}_{i=1}^N$ which have pseudodual functions $\{g_i\}_{i=1}^N \subset \text{Lip}_b(X)$. Then there are $\{\lambda_a\}_{a \in \Lambda_k, N} \subset L^\infty(||T||)$ such that:

\[ T(f, \pi_1, \cdots, \pi_k) = \sum_{a \in \Lambda_k, N} \int_X f \lambda_a \langle D_{a_1} \wedge \cdots \wedge D_{a_k}, d\pi_1 \wedge \cdots \wedge d\pi_k \rangle \ d||T||. \]  

(5.13)
Proof. Without loss of generality we can assume that $f \in \text{Lip}_b(X)$, $|f| \leq 1$ and each $\pi_i$ is 1-Lipschitz. Let $\omega = (f, \pi_1, \ldots, \pi_{k-1})$ so that the current $T\omega \in \mathcal{M}_1(X)$ satisfies $\|T\omega\| \ll \|\omega\|$ by [AK00, Eq. 2.5]. By Theorem 3.7 we have:

$$(5.14) \quad T(f, \pi_1, \ldots, \pi_k) = T\omega(\pi_k) = \int_X D_{T\omega} \pi_k \, d\|\omega\|$$

where $D_{T\omega} = \text{Der}_{\|\omega\|}(T\omega)$ is the derivation associated to the 1-dimensional current $T\omega$.

By assumption there are bounded Borel functions $\{\lambda_i\}_{i=1}^N \subset \mathcal{B}_\infty(X)$:

$$(5.15) \quad D_{T\omega} = \sum_{i=1}^N \lambda_i D_i.$$ 

Therefore for $\varepsilon > 0$ there is $\delta > 0$ such that if for some Borel set $S$ and real numbers $c_j$ one has

$$(5.16) \quad \max_{i=1,\ldots,N} \left| D_i \left( \pi_k - \sum_{j=1}^N c_j g_j \right) \right| < \delta \quad (\|\omega\| \text{S-a.e.}),$$

then

$$(5.17) \quad \left| D_{T\omega} \left( \pi_k - \sum_{j=1}^N c_j g_j \right) \right| < \varepsilon \quad (\|\omega\| \text{S-a.e.})$$

holds. From Lusin’s Theorem, after choosing Borel representatives for the $\{D_j \pi_k\}_{j=1}^N$, we find disjoint Borel sets $F_1, \ldots, F_p \subset \text{spt} \, T$:

$$(5.18) \quad \|\omega\|(\text{spt} \, T \setminus \bigcup_{\alpha=1}^p F_\alpha) < \varepsilon$$

and points $x_\alpha \in F_\alpha$:

$$(5.19) \quad \chi_{F_\alpha} |D_j \pi_k - D_j \pi_k(x_\alpha)| < \delta \quad (\forall j \in \{1, \ldots, N\}).$$

Then

$$(5.20) \quad T(f, \pi_1, \ldots, \pi_k) = \int_{\text{spt} \, T \setminus \bigcup_{\alpha=1}^p F_\alpha} D_{T\omega} \pi_k \, d\|\omega\|$$

$$+ \sum_{\alpha=1}^p \int_{F_\alpha} D_{T\omega} \left( \pi_k - \sum_{j=1}^N D_j \pi_k(x_\alpha) g_j \right) \, d\|\omega\|$$

$$+ \sum_{j=1}^N \int_X \left( \sum_{\alpha=1}^p \chi_{F_\alpha} D_j \pi_k(x_\alpha) \right) D_{T\omega} g_j \, d\|\omega\|;$$

from which it follows (by the choice of normalization $D_{T\omega}$ has norm $\leq 1$):

$$(5.21) \quad \left| T(f, \pi_1, \ldots, \pi_k) - \sum_{j=1}^N \int_X \left( \sum_{\alpha=1}^p \chi_{F_\alpha} D_j \pi_k(x_\alpha) \right) D_{T\omega} g_j \, d\|\omega\| \right| < (1 + \|\omega\|(\text{spt} \, T)) \varepsilon.$$
On the other hand,

\[(5.22) \sum_{j=1}^{N} \int_{\Omega} \left( \sum_{\beta=1}^{p} \chi_{F_{\beta}} D_{j} \pi_{k} x_{\alpha} \right) D_{T} L_{\alpha} g_{j} d \|T\| = \sum_{j=1}^{N} \int_{\Omega} D_{j} \pi_{k} D_{T} L_{\alpha} g_{j} d \|T\| \]

\[- \sum_{j=1}^{N} \int_{\partial T \setminus \bigcup_{\alpha=1}^{p} F_{\alpha}} D_{j} \pi_{k} D_{T} L_{\alpha} g_{j} d \|T\| \]

\[- \sum_{j=1}^{N} \sum_{\alpha=1}^{p} \int_{F_{\alpha}} (D_{j} \pi_{k} - D_{j} \pi_{k}(x_{\alpha})) D_{T} L_{\alpha} g_{j} d \|T\|;\]

observing that:

\[(5.23) \left| \sum_{j=1}^{N} \int_{\partial T \setminus \bigcup_{\alpha=1}^{p} F_{\alpha}} D_{j} \pi_{k} D_{T} L_{\alpha} g_{j} d \|T\| \right| \leq N \varepsilon \cdot \max_{j=1,\ldots,N} \|D_{j}\| \cdot \max_{j=1,\ldots,N} L(g_{j}) \]

\[(5.24) \left| \sum_{j=1}^{N} \sum_{\alpha=1}^{p} \int_{F_{\alpha}} (D_{j} \pi_{k} - D_{j} \pi_{k}(x_{\alpha})) D_{T} L_{\alpha} g_{j} d \|T\| \right| \leq N \delta \cdot \max_{j=1,\ldots,N} L(g_{j}) \cdot \|T\|(\partial T);\]

letting \(\varepsilon \to 0, \delta \to 0\) one concludes that

\[(5.25) T(f, \pi_{1}, \ldots, \pi_{k}) = \sum_{j=1}^{N} T(f D_{j} \pi_{k}, \pi_{1}, \ldots, \pi_{k-1}, g_{j}).\]

If \(\Lambda_{k,N}'\) denotes the set of \(k\)-tuples on \(\{1, \ldots, N\}\), by using induction in (5.25),

\[(5.26) T(f, \pi_{1}, \ldots, \pi_{k}) = \sum_{a \in \Lambda_{k,N}} T(f D_{a_{1}} \pi_{1} \cdots D_{a_{k}} \pi_{k}, g_{a_{1}}, \ldots, g_{a_{k}});\]

as currents are alternating

\[(5.27) T(f, \pi_{1}, \ldots, \pi_{k}) = \sum_{a \in \Lambda_{k,N}} T(f(D_{a_{1}} \wedge \cdots \wedge D_{a_{k}}, d\pi_{1} \wedge \cdots \wedge d\pi_{k}), g_{a_{1}}, \ldots, g_{a_{k}});\]

the maps \(\psi \in L^{1}(\|T\|) \rightarrow T(\psi, g_{a_{1}}, \ldots, g_{a_{k}})\) defines a linear functional on \(L^{1}(\|T\|)\) which is represented by some \(\lambda_{a} \in L^{\infty}(\|T\|)\) by the Riesz representation theorem. We conclude that:

\[(5.28) T(f, \pi_{1}, \ldots, \pi_{k}) = \sum_{a \in \Lambda_{k,N}} \int_{\Omega} f \lambda_{a} (D_{a_{1}} \wedge \cdots \wedge D_{a_{k}}, d\pi_{1} \wedge \cdots \wedge d\pi_{k}) d\|T\|.\]

We now prove Theorem 1.4:

**Proof of Theorem 1.4.** Suppose that \(\mathcal{X}(\|T\|)\) has \(N\) generators; then by Theorem 2.48 there is an \(L^{\infty}(\|T\|)-\)Borel partition of unity \(\{U_{\beta}\}_{\beta \in J}\) such that \(J\) is finite with at most \(N\) elements and \(\mathcal{X}(\|T\|L_{\beta})\) is free of rank \(N_{\beta} \leq N\). Having selected a local basis of derivations for each \(U_{\beta}\), we can apply Lemma 2.50 to obtain an \(L^{\infty}(\|T\|)-\)Borel partition of unity \(\{V_{a}\}\) such that:

- The module \(\mathcal{X}(\|T\|L_{\alpha})\) has a basis \(\{D_{\alpha,i}\}_{i=1}^{N_{\alpha}}\).
• The norms of the derivations \( \{D_{\alpha,i}\}_{i=1}^{N_{\alpha}} \) are bounded by a universal constant \( C(N) \).

• There are 1-Lipschitz functions \( \{g_{\alpha,j}\}_{j=1}^{N_{\alpha}} \) pseudodual to the derivations \( \{D_{\alpha,i}\}_{i=1}^{N_{\alpha}} \) on \( V_{\alpha} \).

The hypotheses of Lemma 5.12 are met by the currents \( \{TLV_\alpha\} \) and we have local representations:

\[
(T_\alpha)^{(f,\pi_1,\cdots,\pi_k)} = \sum_{a\in\Lambda_{k,N}} \int_{V_\alpha} f \lambda_{\alpha,a} \langle D_{\alpha,a_1} \wedge \cdots \wedge D_{\alpha,a_k}, d\pi_1 \wedge \cdots \wedge d\pi_k \rangle \ d\|T\|,
\]

for any subset \( W \subset V_\alpha \) and any index \( a \in \Lambda_{k,N} \), letting \( \pi_i = g_{\alpha,a_i} \), we obtain from (5.29) the lower bound

\[
\|T\|(W) \geq T_\alpha(\chi_W, g_{\alpha,a_1}, \cdots, g_{\alpha,a_k}) = \int_W \lambda_{\alpha,a} d\|T\|,
\]

which implies the upper bound \( \|\lambda_{\alpha,a}\|_{L^\infty(\|T\|)} \leq 1 \).

Note that we can regard \( \Lambda_{k,N} \) as a subset of \( \Lambda_{k,N} \) and for \( a_i \in \{1, \cdots, N\} \setminus \{1, \cdots, N\} \) we will improperly use the notation \( D_{a,a_i} \) to denote the trivial derivations. Note that

\[
D_{a_1} = \sum_{\alpha} \lambda_{\alpha,a_1} \chi_{V_\alpha} D_{a,a_1},
\]

\[
D_{a_i} = \sum_{\alpha} \chi_{V_\alpha} D_{a,a_i}, \quad (1 < i \leq k)
\]

define elements of \( \mathcal{X}(\|T\|) \) with norm bounded by \( C(N) \). Therefore

\[
\omega_T = \sum_{a \in \Lambda_{k,N}} D_{a_1} \wedge \cdots \wedge D_{a_k}
\]

defines an element of \( \mathcal{X}^k(\|T\|) \) with norm at most \( (C(N))^k \binom{N}{k} \). By Remark 5.1 one can also regard \( \omega_T \) as an element of either \( \operatorname{Ext}^k_{\|T\|} \mathcal{X}(\|T\|) \) or \( \operatorname{Ext}^k X(\|T\|) \).

We now observe that:

\[
T(f,\pi_1,\cdots,\pi_k) = \sum_{a} (TLV_\alpha)(f,\pi_1,\cdots,\pi_k)
\]

\[
= \sum_{a} \sum_{a\in\Lambda_{k,N}} \int_{V_\alpha} f \lambda_{\alpha,a} \langle D_{a,a_1} \wedge \cdots \wedge D_{a,a_k}, d\pi_1 \wedge \cdots \wedge d\pi_k \rangle \ d\|T\|
\]

\[
= \sum_{a} \int_{X} f \chi_{V_\alpha} \langle \omega_T, d\pi_1 \wedge \cdots \wedge d\pi_k \rangle \ d\|T\|
\]

which proves (1.5).

\[\square\]

Remark 5.35. A consequence of Theorem 1.4 is that one can regard a \( k \)-dimensional metric current \( T \) as a map defined on \( L^1(\|T\|) \times \mathcal{E}^k(\|T\|) \). Moreover, noting that if
Remark 5.36. Note that Theorem 1.4 implies [Wil10, Thm. 1.3]. In fact, if \((X, \mu)\) is a differentiability space, by Lemma 4.1 in [Sch13] the module \(\mathcal{X}(\mu)\) can be identified with the set of bounded measurable sections of the Cheeger’s measurable tangent bundle \(T_\mu X\) (defined in [Che99, pg. 463]). Then the module \(\mathcal{X}(\mu)\) coincides with the set of bounded measurable sections of the \(k\)-th exterior power of \(T_\mu X\); in this way, we recover [Wil10, Thm. 1.3].

For \(k \geq 2\), it is not clear how to identify the elements of \(\mathcal{X}(\mu)\) which give rise to currents. However, we have a partial result concerning normal currents. We start by generalizing the notion of precurrents which was introduced by Williams in the context of differentiability spaces.

**Definition 5.37.** Suppose that \(\mu\) is a finite Radon measure on \(X\). Then each \(\xi \in \mathcal{X}(\mu)\) defines a \(k\)-metric functional \(T_\xi\) by:

\[
T_\xi(f, \pi_1, \cdots, \pi_k) = \int_X f(\xi, d\pi_1 \wedge \cdots \wedge d\pi_k) \, d\mu;
\]

moreover, \(T_\xi\) is multilinear in the arguments \((f, \pi_1, \cdots, \pi_k)\) and alternating in the arguments \((\pi_1, \cdots, \pi_k)\). Note also that (5.7) implies that \(T_\xi\) has finite mass:

\[
\|T_\xi\| \leq k! \|\xi\|_{\mathcal{X}(\mu), \text{loc}} \mu.
\]

We also have that \(T_\xi\) is local in the sense that if

\[
\left\{x : \|\xi\|_{\mathcal{X}(\mu), \text{loc}}(x) \neq 0\right\} \subseteq \bigcup_{\alpha=1}^k V_\alpha,
\]

where the \(V_\alpha\) are Borel sets with \(\pi_\alpha\) constant on \(V_\alpha\), then

\[
T_\xi(f, \pi_1, \cdots, \pi_k) = 0.
\]

In fact, by Theorem 7.54, for each \(\varepsilon > 0\) we can find \(\xi' \in \mathcal{X}(\mu)\) of the form

\[
\xi' = \sum_{i \in I_\xi} D_{i_1} \wedge \cdots \wedge D_{i_k}
\]

with \(\|\xi - \xi'\|_{\mathcal{X}(\mu)} \leq \varepsilon\). Then (5.41) follows because for each \(D \in \mathcal{X}(\mu)\), \(\chi_{V_\alpha} \circ \pi_\alpha = 0\).

We will call \(T_\xi\) the \(k\)-precurrent associated to \(\xi\) and we will denote by \(P_h(\mu)\) the set of \(k\)-precurrents.

**Theorem 5.43.** Given \(\xi \in \mathcal{X}(\mu)\), if the metric functional \(\partial T_\xi\) has finite mass, then \(T_\xi\) is a normal current. If \(\mathcal{X}(\mu)\) is finitely generated, the set \(N_k(\mu)\), which consists of the normal currents whose mass is absolutely continuous with respect to \(\mu\), coincides with the set of those \(T_\xi \in P_h(\mu)\) whose boundary \(\partial T_\xi\) has finite mass.

**Proof of Theorem 5.43.** Assume that the metric functional \(\partial T_\xi\) has finite mass. In order to show that \(T_\xi\) is a metric current, it suffices to check the continuity axiom (4) in Definition 2.11. Suppose that \(f_h \overset{w^*}{\longrightarrow} f\) and \(\pi_{i,h} \overset{w^*}{\longrightarrow} \pi_i\) for all \(1 \leq i \leq k\).

Note that:

\[
|T_\xi(f_h, \pi_{1,h}, \cdots, \pi_{k,h}) - T_\xi(f, \pi_{1,h}, \cdots, \pi_{k,h})| \leq \prod_{i=1}^k L(\pi_{i,h}) \int_X |f_h - f| \, d\|T_\xi\|
\]
Let \( \text{Lemma 6.3.} \)

Moreover, we have:

\[
(6.1) \left[ T_\xi(f, \pi_1, \pi_2, \ldots, \pi_k, h) - T_\xi(f, \pi_1, \pi_2, \ldots, \pi_k, h) \right] = 0.
\]

from (5.46) we have:

\[
(5.47) |T_\xi(f|_X - \pi_1, \pi_2, \ldots, \pi_k)| \leq \prod_{i=2}^{k} L(\pi_i, h) \int_X |f|_X - \pi_1| d\|T_\xi\|,
\]

from (5.46) we have:

\[
(5.49) \lim_{h \to 0} |T_\xi(f, \pi_1, \pi_2, \ldots, \pi_k, h) - T_\xi(f, \pi_1, \pi_2, \ldots, \pi_k)| = 0.
\]

Using that \( T_\xi \) is alternating in the last \( k \) arguments and induction in \( i \), the previous argument gives:

\[
(5.50) \lim_{h \to 0} \left| T_\xi(f, \pi_1, \pi_2, \ldots, \pi_k) - T_\xi(f, \pi_1, \pi_2, \ldots, \pi_k) \right| = 0,
\]

which shows that \( T_\xi \) is a metric current. As \( \partial T_\xi \) has finite mass, the current \( T_\xi \) is normal. The second part of this Theorem follows from Theorem 1.4, because, if \( X(\mu) \) is finitely generated, any metric current is a precurrent. \( \square \)

6. Applications

6.1. Approximation of \( 1 \)-currents by Normal currents. The goal of this subsection is to prove Theorem 1.6. We make the set theoretic assumption that the cardinality of any set is an Ulam number so that by [AK00, Lem 2.9] the masses of metric currents are concentrated on countable unions of compact sets. This assumption is not needed if we consider currents in separable Banach spaces.

Let \( \text{Curves}(X) \) denote the set of Lipschitz maps from \([0, 1]\) to \( X \) topologized as a subspace of \( K([0, 1] \times X) \). To each \( \gamma \in \text{Curves}(X) \), one can then associate a normal current \( [\gamma] \) by letting:

\[
(6.1) [\gamma](fd\pi) = \int_0^1 (f \circ \gamma)(t)(\pi \circ \gamma)'(t) dt \quad ((f, \pi) \in B^\infty(X) \times \text{Lip}(X)).
\]

Note that the mass measure of \( [\gamma] \) can be bounded by:

\[
(6.2) \|\gamma\| \leq \gamma_2 \left(\text{md } \gamma \cdot L^1[0, 1]\right).
\]

**Lemma 6.3.** Let \( Z \) be a Banach space and \( \mu \) a \( \sigma \)-finite Radon measure on \( Z \). Suppose that the derivations \( \{ D_i \}_{i=1}^k \subset X(\mu) \) are independent. Then there are a Borel \( L^\infty(\mu) \)-partition of unity \( V_\alpha \) and there are, for each \( \alpha \), derivations \( \{ D_{\alpha, i} \}_{i=1}^k \subset X(\mu) \) and unit norm functionals \( \{ z_{\alpha, j} \}_{j=1}^k \subset Z^* \) such that:

1. The submodule of \( X(\mu) \) generated by the derivations \( \{ D_{\alpha, i} \}_{i=1}^k \) is the same as the submodule generated by the derivations \( \{ \chi_{V_\alpha} D_i \}_{i=1}^k \);
(2) The functionals \( \{z_{\alpha,j}^*\}_{j=1}^k \) are pseudodual to the derivations \( \{D_{\alpha,i}\}_{i=1}^k \) on \( V_\alpha \).

Proof. Note that \( \mu \) is concentrated on a \( K_\alpha \)-set, i.e. a countable union of compact sets; in particular, \( \text{spt} \mu \) is separable and we can assume that \( Z \) is separable by taking the closure of the linear span of \( \text{spt} \mu \). Up to passing to a Borel \( L^\infty(\mu) \)-partition of unity we can assume that \( Z \) is also bounded. Let \( \{z_i\} \subset Z \) be a countable dense set and for \( i \neq j \) choose a unit norm linear functional \( z_{i,j}^* \) with \( \langle z_{i,j}^*, z_i - z_j \rangle = \|z_i - z_j\|_Z \). By the Stone-Weierstrass Theorem for Lipschitz Algebras [Wea99, Cor. 4.1.9], the family of functionals \( \{z_{i,j}^*\}_{i,j} \) is a countable generating set\(^7\) for \( \text{Lip}_b(Z) \). By [Sch, Prop. 2.35] we can find a Borel \( L^\infty(\mu) \)-partition of unity \( \{V_\alpha\} \) and for each \( \alpha \) unit functionals \( \{z_{\alpha,i}^*\}_{i=1}^k \) such that, letting \( M_\alpha = (D_{\alpha,i}z_{\alpha,i}^*)_{i=1}^k \), we have \( \det M_\alpha \neq 0 \) on \( V_\alpha \). Up to passing to a further Borel partition we can assume that for each \( \alpha \) there is a \( \delta_\alpha > 0 \) such that:

\[
(6.4) \quad |\det M_\alpha(x)| \in (\delta_\alpha, 2\delta_\alpha) \quad (\forall x \in V_\alpha);
\]

we then let \( D_{\alpha,i} = \sum_{j=1}^k (M_\alpha^{-1})_{i,j}D_j \).

Proof of Theorem 1.6. We make the following preliminary Observation (Obs1): suppose that \( \sum_k T_k \) is either a finite sum of 1-currents or a series with

\[
(6.5) \quad \sum_k \|T_k\|_{M_1(Z)} < \infty,
\]

and suppose also that for each \( n \) there is a sequence of normal currents \( \{N_{k,n}\} \subset \mathbf{N}_1(Z) \) such that

\[
(6.6) \quad \lim_{n \to \infty} \|T_k - N_{k,n}\|_{M_1(Z)} = 0;
\]

then, if we let \( T = \sum_k T_k \), there is a sequence of normal currents \( \{N_n\} \subset \mathbf{N}_1(Z) \) such that (1.7) holds.

As \( \mathcal{X}(\|T\|) \) is finitely generated, by Theorem 2.48 and (Obs1) we can reduce to the case in which \( \mathcal{X}(\|T\|) \) is free of rank \( N \). Applying Lemma 6.3 and (Obs1), we can assume that \( \mathcal{X}(\|T\|) \) has a basis consisting of derivations \( \{D_i\}_{i=1}^N \) such that there are unit norm linear functionals \( \{z_{i,j}^*\}_{j=1}^N \) which are pseudodual to the \( \{D_i\}_{i=1}^N \). Let \( z^* = (z_{i,j}^*)_{j=1}^N \) and \( \{e_i\}_{i=1}^N \subset \mathbb{R}^N \) the standard basis of \( \mathbb{R}^N \); by Corollary 2.78 for any \( \alpha \in (0, \pi/2) \) the measure \( \|T\| \) admits C-Lipschitz Alberti representations \( \{A_i\}_{i=1}^N \) with \( \lambda_i \) in the \( z^* \)-direction of \( \mathcal{C}(e_j, \alpha) \) (and with positive \( z_{i,j}^* \)-speed); note that, up to taking an \( L^\infty(\|T\|) \)-partition of unity and choosing \( \alpha \) sufficiently small, we can assume that the derivations \( \{D_{A_i}\}_{i=1}^N \) form a basis of \( \mathcal{X}(\|T\|) \). Applying Theorem 7.97, we can assume that \( A_i = (P_i, \nu_i) \) with spt \( P_i \subset \text{Curves}(Z) \) and \( (\nu_i)_\gamma = h_i\Psi_\gamma \), where \( h_i \) is a Borel function on \( Z \) and \( \Psi_\gamma = \gamma_2L[0,1] \). Denoting the derivation \( \text{Der}_{\|T\|}(T) \) by \( D_T \), there are bounded Borel functions \( \{\lambda_i\}_{i=1}^N \) such that \( D_T = \sum_{i=1}^N \lambda_i D_i \); but this implies that

\[
(6.7) \quad T = \sum_{i=1}^N \text{Cur}_{\|T\|}(\lambda_i D_i),
\]

\(^7\) i.e. for each \( f \in \text{Lip}_b(Z) \) there is a sequence of polynomials \( \{P_n\} \subset \text{Lip}_b(Z) \) in the \( z_{i,j}^* \) with \( P_n \overset{w^*}{\longrightarrow} f \).
and by (Obs1) we reduce to the case in which $T = \lambda D_A$ where $\lambda$ is a bounded Borel function and $A = (P, \nu)$ is a $C$-Lipschitz Alberti representation with $\text{spt} P \subset \text{Curves}(Z)$ and $\nu_\gamma = h\Psi_\gamma$. Let $\mu$ denote the measure
\begin{equation}
\mu = \int_{\text{Curves}(Z)} \Psi_\gamma;
\end{equation}
Note that $\|T\| \ll \mu$ and $h\lambda \in L^1(\mu)$; as $\text{Lip}_b(Z)$ is dense in $L^1(\mu)$, we can find, for each $\varepsilon > 0$, a function $g \in \text{Lip}_b(Z)$ such that:
\begin{equation}
\|g - h\lambda\|_{L^1(\mu)} \leq \varepsilon.
\end{equation}
Note that the metric current $N$ defined by
\begin{equation}
N(f d\pi) = \int_{\text{Curves}(Z)} dP(\gamma) \int_\gamma f \partial_\gamma \pi \, d\Psi_\gamma
= \int_{\text{Curves}(Z)} dP(\gamma) \int_{[0,1]} f \circ \gamma(t) (\pi \circ \gamma)'(t) \, dt
\end{equation}
is normal and so $N \mathbf{L} g$ is normal. However, (6.9) implies that
\begin{equation}
\|N \mathbf{L} g - T\|_{M^1(Z)} \leq C \|g - h\lambda\|_{L^1(\mu)} \leq C\varepsilon.
\end{equation}

6.2. Alberti representations with constant directions. In this Subsection we illustrate a different method to produce Alberti representations. This method allows to refine the way in which the direction is specified. In fact, the cone field is replaced by a vector field and one can also use countably many Lipschitz functions. This method relies on results of [PS12, PS13] on the structure of 1-dimensional normal currents.

We state the Paolini-Stepanov decomposition of normal currents using parametrized curves: note, however, that in [PS13] the result is stated using non-parametrized curves. Recall also that the metric space $X$ is assumed Polish.

**Theorem 6.12** (Corollary 3.3 in [PS13]). Let $N$ be a 1-dimensional normal current defined on $X$; then there is a finite Radon measure $\eta$ on the space $K([0,1] \times X)$ of compact subsets of $[0,1] \times X$ which is concentrated on $\text{Curves}(X)$, and such that:
\begin{equation}
N = \int_{\text{Curves}(X)} [\gamma] \, d\eta(\gamma);
\end{equation}
\begin{equation}
\|N\| = \int_{\text{Curves}(X)} \|\gamma\| \, d\eta(\gamma);
\end{equation}
\begin{equation}
\|N\|(X) = \int_{\text{Curves}(X)} l(\gamma) \, d\eta(\gamma),
\end{equation}
where $l(\gamma)$ denotes the length of $\gamma$ which is given by:
\begin{equation}
l(\gamma) = \int_0^1 \text{md} \gamma(t) \, dt.
\end{equation}
Note that the integrals in (6.13) and (6.14) make sense because the maps $\gamma \mapsto [\gamma]$ and $\gamma \mapsto \|\gamma\|$ are Borel in the following sense: for each $(f, \pi) \in B^\infty(X) \times \text{Lip}(X)$ and each Borel $E \subset X$, the maps $\gamma \mapsto [\gamma](f d\pi)$ and $\gamma \mapsto \|\gamma\|(E)$ are Borel. We need to introduce more terminology:
**Definition 6.17.** The set of maps $\gamma \in \text{Curves}(X)$ with Lipschitz constant at most $n$ is a Polish space and is denoted by $\text{Curves}_n(X)$. The set of Lipschitz maps $\gamma : K \to X$, where $K$ is a nonempty compact subset of $[0, 1]$, is denoted by $\text{Pieces}(X)$ and topologized as a subset of $K([0, 1] \times X)$. Note that $\text{Pieces}(X)$ is a subset of $\text{Frag}(X)$ and a Borel subset of $K([0, 1] \times X)$. The subset of maps $\gamma \in \text{Pieces}(X)$ with Lipschitz constant at most $n$ is a Polish space and is denoted by $\text{Pieces}_n(X)$. If $(\gamma, \tilde{\gamma}) \in \text{Curves}(X) \times \text{Pieces}(X)$ and $\gamma| \text{dom} \tilde{\gamma} = \tilde{\gamma}$, we say that $\tilde{\gamma}$ is a **piece** of $\gamma$.

To each $\gamma \in \text{Pieces}(X)$, one can associate a metric current $[\gamma]$ by letting:

\begin{equation}
[\gamma](f d\pi) = \int_{\text{dom} \gamma} (f \circ \gamma)(t)(\pi \circ \gamma)'(t) \, dt \quad ((f, \pi) \in \mathcal{B}^\infty(X) \times \text{Lip}(X));
\end{equation}

a modification of the argument in Lemma 3.1 in [Sch13] shows that, for each $(f, \pi) \in \mathcal{B}^\infty(X) \times \text{Lip}(X)$, the map

$$\text{Pieces}(X) \to \mathbb{R}$$

$$\gamma \mapsto [\gamma](f d\pi)$$

is Borel. Having fixed an open set $U \subset X$, there is a countable collection $\mathcal{F}_U$ of 1-forms $\omega = \sum_i f_i d\pi_i$ such that, for each $\gamma \in \text{Pieces}(X)$,

\begin{equation}
\|\gamma\|(U) = \sup_{\omega \in \mathcal{F}_U} [\gamma](\omega);
\end{equation}

this implies that, for each Borel $E \subset X$, the map:

$$\text{Pieces}(X) \to [0, \infty)$$

$$\gamma \mapsto \|[\gamma]\|(E)$$

is Borel. Note also that the mass of the current associated to $\gamma \in \text{Pieces}(X)$ can be bounded from above similarly as in (6.2):

\begin{equation}
\|[\gamma]\| \leq \gamma_\sharp (\text{md} \gamma \cdot \mathcal{L}^1 \text{dom} \gamma).
\end{equation}

We now discuss the notion of Alberti representations in the direction of a vector field $v$. In greater generality, we consider $l^2$-valued Lipschitz maps, where $l^2$ is the Hilbert space of $l^2$-summable sequences. In the following, we let $\mathbb{R}^\infty$ denote the product of countably many copies of $\mathbb{R}$ with the product topology. Note that any map $F : X \to l^2$ is determined by its components $F_i$; in particular, if $F$ is Lipschitz and $D \in X(\mu)$, we can choose a Borel representative of each $DF_i$ and denote by $DF$ the Borel map $DF : X \to \mathbb{R}^\infty$ whose $i$-th component is $DF_i$. Moreover, we can stipulate that the maps $DF_i : X \to \mathbb{R}$ are uniformly bounded, with the bound independent of $i$. In the following, this will always be assumed when we apply a derivation $D \in X(\mu)$ to a Lipschitz function $F : X \to l^2$. We finally call a Borel map $v : X \to \mathbb{R}^\infty$, such that the components $v_i$ are uniformly bounded by some $C > 0$, a **vector field**.

**Definition 6.23.** Let $F : X \to l^2$ be Lipschitz and $v : X \to \mathbb{R}^\infty$ a vector field. Denote by $N_v$ the set where $v$ vanishes:

\begin{equation}
N_v = \{x \in X : v(x) = 0\}.
\end{equation}

We say that the Alberti representation $A = (P, \nu)$ of $\mu \mathcal{L}(X \setminus N_v)$ is in the $F$-direction of $v$ if for $P$-a.e. $\gamma$ and $\mathcal{L}^1$-a.e. $t \in \text{dom} \gamma$ there is a $\lambda = \lambda(\gamma, t) > 0$ such
that:

\[(F \circ \gamma)'(t) = \lambda v(\gamma(t)).\]

Given a Lipschitz map \(F : X \to l^2\), to produce vector fields \(v\) with \(\mu \mathbb{L}(X \setminus N_\alpha)\) admitting an Alberti representation in the \(F\)-direction of \(v\), we will use a special class of derivations.

**Definition 6.26.** A derivation \(D \in \mathcal{X}(\mu)\) is called **normal** if there is a Borel \(L^\infty(\mu \mathbb{L}(X \setminus N_D))\)-partition of unity \(\{U_\alpha\}\) such that for each \(\alpha\) there are:

1. An isometric embedding \(\iota_\alpha : U_\alpha \to Z_\alpha\) where \(Z_\alpha\) is a Polish space.
2. A normal current \(N_\alpha\) in \(Z_\alpha\) with \(\iota_\alpha^*(\mu \mathbb{L}U_\alpha) \ll \|N_\alpha\|\).
3. Denoting by \(\mathcal{D}_N \in \mathcal{X}(\|N_\alpha\|)\) the derivation associated to \(N_\alpha\) given by
   Theorem 3.7, there is \(\lambda_\alpha \in L^\infty(\|N_\alpha\|)\) with \(\lambda_\alpha \geq 0\) and

\[(6.27) \quad \iota_\alpha^* \chi_{U_\alpha} D = \lambda_\alpha D_{N_\alpha}.\]

Note that in (6.27) we have used that (2) allows to identify \(\iota_\alpha^* D\) with a derivation in \(\mathcal{X}(\|N_\alpha\|)\).

**Remark 6.28.** We want to remark that there are many normal derivations. Suppose that \(\mu\) admits an Alberti representation in the \(f\)-direction of an \(n\)-dimensional cone field \(\mathcal{C}\). The proof of Theorem 2.64 in [Sch13] allows us to assume that there is an \(L^\infty(\mu)\)-partition of unity \(\{K_\alpha\}\) such that, for each \(\alpha\):

1. The set \(K_\alpha\) is compact and embedds isometrically in \(S_\alpha\), which is a convex compact subset of some Banach space;
2. Regarding \(\mu \mathbb{L}K_\alpha\) as a measure on \(S_\alpha\), it admits a 1-Lipschitz Alberti representation \(\mathcal{A}_\alpha\) in the \(f\)-direction of \(\mathcal{C}\);
3. The Alberti representation \(\mathcal{A}_\alpha\) is of the form

\[(6.29) \quad \mu \mathbb{L}K_\alpha = \int_{\text{Frag}(S_\alpha)} g_\alpha \Psi_\gamma \, dP_\alpha;\]

4. \(g_\alpha\) is a bounded Borel function vanishing on \(S_\alpha \setminus K_\alpha\);
5. The probability measure \(P_\alpha\) is concentrated on the set \(\text{Lip}_1([0, \tau_\alpha], S_\alpha)\) of 1-Lipschitz maps \([0, \tau_\alpha] \to S_\alpha\), where \(\tau_\alpha \in (0, 1]\);
6. \(\Psi_\gamma = \gamma \mathcal{L}[\mathbb{L}(0, \tau_\alpha)]\).

We can then define a normal current \(N_\alpha \in \mathcal{N}_1(S_\alpha)\) by:

\[(6.30) \quad N_\alpha = \int_{\text{Frag}(S_\alpha)} |\gamma| \, dP_\alpha,\]

so that \(\mu \mathbb{L}K_\alpha \ll \|N_\alpha\|\) and \(D_{A_\alpha} = \chi_{\{g_\alpha \neq 0\}} D_{N_\alpha}\) for some nonnegative \(\lambda_\alpha \in \mathbb{B}^\infty(S_\alpha)\) which vanishes on \(S_\alpha \setminus K_\alpha\). Thus, the derivation \(D \in \mathcal{X}(\mu)\) defined by \(D = \sum_\alpha \lambda_\alpha K_\alpha D_{A_\alpha}\) is a normal derivation. Moreover, if \(\mathcal{X}(\mu)\) is finitely generated, by choosing Alberti representations in the directions of independent cone fields, we get a generating set for \(\mathcal{X}(\mu)\) consisting of normal derivations. If \(\mathcal{X}(\mu)\) is not finitely generated, Theorem 3.96 in [Sch13] implies that the \(\text{Lip}_b(X)\)-span of the set of normal derivations is weak* dense in \(\mathcal{X}(\mu)\). Note that in this case it is necessary to use the \(\text{Lip}_b(X)\)-span instead of the \(L^\infty(\mu)\)-span. In fact, if \(D_1, D_2\) are normal derivations and if \(\lambda_1, \lambda_2 \in L^\infty(\mu)\), then \(\lambda_1 D_1 + \lambda_2 D_2\) might not be a normal derivation. However, if \(\lambda_1\) and \(\lambda_2\) are Lipschitz\(^8\), then \(\lambda_1 D_1 + \lambda_2 D_2\) is a normal derivation.

\(^8\)more precisely, \(\lambda_1\) and \(\lambda_2\) have Lipschitz representatives
derivation because if $N$ is a normal current and $f$ is Lipschitz, then $NLf$ is still a normal current.

The goal of this Subsection is the proof of the following Theorem:

**Theorem 6.31.** Let $F : X \to l^2$ a Lipschitz map and $D \in X(\mu)$ a normal derivation. Then $\mu l(X \setminus N_{DF})$ admits a 1-Lipschitz Alberti representation in the $F$-direction of $DF$.

The proof of Theorem 6.31 requires some preparation and part of it has been split into some intermediate Lemmas.

**Lemma 6.32.** In proving Theorem 6.31 we can assume that:

1. The metric space $X$ is a compact subset of a Polish space $Z$.
2. The map $F : X \to l^2$ is 1-Lipschitz and extends to a 1-Lipschitz map $F : Z \to l^2$.
3. There is a normal current $N \in N_1(Z)$ with $\mu \ll \|N\|$ and $D = \lambda D_N$, where $D_N$ is the derivation associated to $N$ given by Theorem 3.7, and $\lambda \in L^\infty(\|N\|)$ is nonnegative.
4. There are constants $0 < C_1 \leq C_2$ such that:

\[
C_1 \leq \frac{d\mu}{d\|N\|}(x) \leq C_2 \quad (\forall x \in X). \tag{6.33}
\]

**Proof.** The proof makes repeated use of the gluing principle for Alberti representations, Theorem 2.32. Let $\{U_\alpha, Z_\alpha, N_\alpha, \iota_\alpha\}$ be as in the definition of a normal derivation 6.26. By taking an $L^\infty(\mu \mathcal{L}U_\alpha)$-partition of unity of each $U_\alpha$, we can assume that the $U_\alpha$ are compact. By the gluing principle for Alberti representations (Theorem 2.32), it suffices to show that the desired representation exists for each $\mu \mathcal{L}(U_\alpha \setminus N_{DF})$. In the following we can thus write $X$ for $U_\alpha$ and drop the index $\alpha$ from the notation. Note also that the vector field $DF \circ t^{-1}$ can be extended to a vector field $v : Z \to \mathbb{R}^\infty$. By Theorem 2.14 in [Sch13] one can also show that the desired representation exists for $\iota_2(\mu \mathcal{L}(X \setminus N_{DF}))$; note that in this case the direction is determined by the function $F \circ t^{-1} : \iota(X) \to l^2$. In the following we will then identify $\iota(X)$ with $X$, $\iota_2 \mu$ with $\mu$, and $\iota_2 D$ with $D$. We now take a MacShane extension

\[
\tilde{F}_t : Z \to \mathbb{R}
\]

of $F_t$ with the same Lipschitz constant $\mathcal{L}(F_t)$ and then choose $c_t \in (0, 1)$ such that

\[
\sum_i c_i^2 \mathcal{L}(F_i)^2 \leq 1. \tag{6.35}
\]

In particular, the map $G : Z \to l^2$ with components $G_t = c_t \tilde{F}_t$ is 1-Lipschitz. Recalling the discussion before Definition 6.23, we also have, after choosing appropriate Borel representatives, that the components of the vector field $DG$ satisfy:

\[
DG_t = c_t DF_t. \tag{6.36}
\]

Consider a fragment $\gamma : K \to X$. As $l^2$ has the Radon-Nikodym property, $F \circ \gamma$ and $G \circ \gamma$ are differentiable for $t \in Q \subset K$, where the Borel set $Q$ satisfies $L^1(K \setminus Q) = 0$. Moreover, at each point $t \in Q$ we have that $(F \circ \gamma)'(t)$ and $(G \circ \gamma)'(t)$ are determined by the derivatives $(F_t \circ \gamma)'(t)$ and $(G_t \circ \gamma)'(t)$ which are related by

\[
(F_t \circ \gamma)'(t) = c_t (G_t \circ \gamma)'(t). \tag{6.37}
\]
In particular, for $\lambda > 0$ the following equations are equivalent:

\begin{align}
(6.38) & \quad (F \circ \gamma)'(t) = \lambda DF(\gamma(t)) \\
(6.39) & \quad (G \circ \gamma)'(t) = \lambda DG(\gamma(t)) ,
\end{align}

and so we can replace $F$ with $G$. Finally, we take another $L^\infty(\mu)$-partition of unity to ensure that (4) holds. \hfill \square

The second ingredient in the proof of Theorem 6.31 is the following notion of strict convexity for the local norm in $X(\mu)$.

**Definition 6.40.** The local norm $| \cdot |_{X(\mu),\text{loc}}$ on $X(\mu)$ is called **strictly convex** if the following holds: whenever one has that for derivations $D_1, D_2 \in X(\mu)$ and for a Borel set $U$:

\begin{equation}
(6.41) \quad |D_1 + D_2|_{X(\mu),\text{loc}}(x) = |D_1|_{X(\mu),\text{loc}}(x) + |D_2|_{X(\mu),\text{loc}}(x) \quad (\text{for } \mu\text{-a.e. } x \in U),
\end{equation}

then $\chi_U D_1$ and $\chi_U D_2$, regarded as elements of $X(\mu\mathbb{L}(U))$, are linearly dependent.

In Subsection 7.3 we show (Theorem 7.101) that it is always possible to perturb the metric on $X$ in a biLipschitz way and obtain a strictly convex local norm on $X(\mu)$. Therefore, for $\varepsilon > 0$, we can assume that the metric $d$ on $Z$ has been replaced by a metric $d(\varepsilon)$ such that:

\begin{equation}
(6.42) \quad d \leq d(\varepsilon) \leq (1 + \varepsilon)d,
\end{equation}

and $| \cdot |_{X(||N||),\text{loc}}$ is strictly convex. We now apply Theorem 6.12 to obtain decompositions of $N$ as in (6.13) and (6.14). We also construct countably many vector fields $w_j : Z \to \mathbb{R}^\infty$ such that:

1. For each $j$, there is $M_j \in \mathbb{N}$ such that $i > M_j$ implies $(w_j)_i = 0$, where $(w_j)_i$ is the $i$-th component of $w_j$.
2. If $DF(z) \neq 0$ and $\xi \in \mathbb{R}^\infty \setminus \{0\}$ is not a positive multiple of $DF(z)$, then $\langle w_j(z), \xi \rangle > 0$ for some $j$.
3. For each $z \in Z$, one has $\langle w_j(z), DF(z) \rangle \leq 0$.

We will denote by $w_0 : Z \to \mathbb{R}^\infty$ the null vector field.

We now introduce the set $\Omega_{\text{fail}}$ of those curves which, roughly speaking, meet $X$ in a set of positive measure where the direction of $F \circ \gamma$ fails to be a positive multiple of $DF$. Specifically, we say that a curve $\gamma \in \text{Curves}(Z)$ belongs to $\Omega_{\text{fail}}$ if and only if there is a piece $\hat{\gamma}$ of $\gamma$ such that:

1. $F \circ \gamma$ is differentiable at each point $t \in \text{dom } \hat{\gamma}$.
2. At each point $t \in \text{dom } \hat{\gamma}$, the vector $(F \circ \gamma)'(t)$ is either 0 or, if it is nonzero, it is not a positive multiple of $DF \circ \gamma(t)$.
3. The piece $\hat{\gamma}$ meets $X \setminus N_{DF}$ in positive mass measure: $||[\hat{\gamma}]||(X \setminus N_{DF}) > 0$.

In general, the set $\Omega_{\text{fail}}$ is not Borel, but, after completing $\eta$, we will show that it becomes $\eta$-measurable. The goal is then to show that $\eta(\Omega_{\text{fail}}) = 0$. Note that the set $\Omega_{\text{fail}}$ is a countable union of the sets

\begin{equation}
(6.43) \quad \Omega_n(w_j) \subset \text{Curves}_n(Z)
\end{equation}

defined as follows: $\gamma \in \text{Curves}_n(Z)$ belongs to $\Omega_n(w_j)$ if and only if there is a piece $\hat{\gamma}$ of $\gamma$ such that:

**F1:** $F \circ \gamma$ is differentiable at each point $t \in \text{dom } \hat{\gamma}$. 

F2: At each point \( t \in \text{dom} \, \hat{\gamma} \), if \( j \neq 0 \) one has \( \langle (F \circ \gamma)'(t), w_j(\gamma(t)) \rangle \geq \frac{1}{n} \), and if \( j = 0 \) one has \( (F \circ \gamma)'(t) = 0 \).

F3: The piece \( \hat{\gamma} \) meets \( X \setminus N_{DF} \) in mass measure at least \( 1/n \): \( \|\hat{\gamma}\|(X \setminus N_{DF}) \geq \frac{1}{n} \).

We will thus study the measurability properties of each set \( \Omega_n(w_j) \), which is the projection of

\[
\Omega_n^{(1)}(w_j) = \left\{ (\gamma, \hat{\gamma}) \in \text{Curves}_n(Z) \times \text{Pieces}_n(Z) : \hat{\gamma} \text{ is a piece of } \gamma \right. \\
\left. \quad \text{and } (F1), (F2) \text{ and } (F3) \text{ hold} \right\}
\]

on \( \text{Curves}_n(Z) \).

**Lemma 6.45.** The set \( \Omega_n^{(1)}(w_j) \) is of class \( \Pi^1_1 \), i.e. coanalytic. Thus \( \Omega_n(w_j) \) is of class \( \Sigma^1_2 \) and, moreover, there is a uniformizing function \( \sigma_{j,n} : \Omega_n(w_j) \to \Omega_n^{(1)}(w_j) \) which is universally measurable and whose graph is of class \( \Pi^1_1 \).

**Proof.** We prove the Lemma for \( j \neq 0 \) as the case \( j = 0 \) requires a minor modification of the argument. Consider the set \( \Omega_n^{(2)}(w_j) \subset \text{Curves}_n(Z) \times \text{Pieces}_n(Z) \times [0,1] \) consisting of the triples \((\gamma, \hat{\gamma}, t)\) such that:

G1: \( \hat{\gamma} \) is a piece of \( \gamma \).

G2: \( \|\hat{\gamma}\|(X \setminus N_{DF}) \geq \frac{1}{n} \).

G3: either \( t \notin \text{dom} \, \hat{\gamma} \) or \( t \in \text{dom} \, \hat{\gamma} \) and \( F \circ \gamma \) is differentiable at \( t \) with 

\[
\langle (F \circ \gamma)'(t), w_j(\gamma(t)) \rangle \geq \frac{1}{n}.
\]

We show that \( \Omega_n^{(2)}(w_j) \) is Borel. First note that the set of couples \((\gamma, \hat{\gamma})\) such that \( \hat{\gamma} \) is a piece of \( \gamma \) is closed in \( \text{Curves}_n(Z) \times \text{Pieces}_n(Z) \). Second, as the map \( \hat{\gamma} \mapsto \|\hat{\gamma}\|(X \setminus N_{DF}) \) is Borel (6.21), the set of pieces with \( \|\hat{\gamma}\|(X \setminus N_{DF}) \geq \frac{1}{n} \) is Borel. Third, the set of pairs \((\hat{\gamma}, t)\) with \( t \in \text{dom} \, \hat{\gamma} \) is closed. Therefore, we have only to show that the set

\[
\hat{\Omega} = \left\{ (\gamma, t) \in \text{Curves}_n(Z) \times [0,1] : (F \circ \gamma)'(t) \text{ exists and } \langle (F \circ \gamma)'(t), w_j(\gamma(t)) \rangle \geq \frac{1}{n} \right\}
\]

is Borel. Let \( S \) denote a countable dense set of \( l^2 \). We then have:

\[
\hat{\Omega} = \bigcap_{\varepsilon \in \mathbb{Q} \cap (0,1)} \bigcup_{\delta \in \mathbb{Q} \cap (0,1)} \bigcap_{s_1,s_2 \in \mathbb{Q} \cap (0,1)} \bigcup_{\xi \in S} \left( \text{Curves}_n(Z) \times \{ t \in (0,1) : |t - s_1| \geq \delta \right. \\
\left. \quad \text{or } |t - s_2| \geq \delta \} \cup S(\varepsilon, \delta, s_1, s_2, \xi) \right),
\]
We conclude that \( \Omega^\gamma, t \) are universally measurable. In particular, they are \( \sigma \)-measurable, as we assume that \( \sigma \) is complete. Moreover, by definition of the maps \( \Xi_{j,n} \) and \( \Xi^e_{j,n} \), we have the relation:

\[
[\gamma] = \Xi_{j,n}(\gamma) + \Xi^e_{j,n}(\gamma);
\]

this implies that

\[
\|\Xi(\gamma)\| \leq \|\Xi_{j,n}(\gamma)\| + \|\Xi^e_{j,n}(\gamma)\|;
\]

however, for \( \eta \)-a.e. \( \gamma \), if \( \gamma \in \Omega_n(\omega_j) \), (6.15) implies that:

\[
\|\Xi(\gamma)\| \geq \|\Xi_{j,n}(\gamma)\| + \|\Xi^e_{j,n}(\gamma)\|.
\]
and thus, for \( \eta \)-a.e. \( \gamma \), we have:

\[
\| \gamma \| = \| \Xi_{j,n}(\gamma) \| + \| \Xi_{j,n}^c(\gamma) \|.
\]

**Lemma 6.62.** For each \( n \) and \( j \) we have that \( \eta(\Omega_n(w_j)) = 0 \).

**Proof of Lemma 6.62.** We argue by contradiction assuming that \( \eta(\Omega_n(w_j)) > 0 \).

Note that:

\[ (6.61) \]

\[
N = \int_{\text{Curves}(Z)} \Xi_{j,n}(\gamma) d\eta(\gamma) + \int_{\text{Curves}(Z)} \Xi_{j,n}^c(\gamma) d\eta(\gamma),
\]

and, using (6.61),

\[ (6.64) \]

\[
\| N \|(Z) = \int_{\text{Curves}(Z)} \| \gamma \|(Z) d\eta(\gamma) = \int_{\text{Curves}(Z)} \| \Xi_{j,n}(\gamma) \|(Z) d\eta(\gamma)
\]

\[
+ \int_{\text{Curves}(Z)} \| \Xi_{j,n}^c(\gamma) \|(Z) d\eta(\gamma)
\]

\[
\geq \| T_{j,n} \|(Z) + \| T_{j,n}^c \|(Z),
\]

where we used:

\[ (6.65) \]

\[
\int_{\text{Curves}(Z)} \| \Xi_{j,n}(\gamma) \|(Z) d\eta(\gamma) \geq \| T_{j,n} \|(Z),
\]

\[ (6.66) \]

\[
\int_{\text{Curves}(Z)} \| \Xi_{j,n}^c(\gamma) \|(Z) d\eta(\gamma) \geq \| T_{j,n}^c \|(Z).
\]

In particular, \( T_{j,n} \) and \( T_{j,n}^c \) are complementary subcurrents of \( N \) because (6.64) implies that

\[ (6.67) \]

\[
\| N \|= \| T_{j,n} \| + \| T_{j,n}^c \|.
\]

Moreover, we also have that:

\[ (6.68) \]

\[
\| T_{j,n} \| = \int_{\text{Curves}(Z)} \| \Xi_{j,n}(\gamma) \| d\eta(\gamma),
\]

\[ (6.69) \]

\[
\| T_{j,n}^c \| = \int_{\text{Curves}(Z)} \| \Xi_{j,n}^c(\gamma) \| d\eta(\gamma).
\]

By Theorem 3.7 we find derivations \( D_{j,n}, D_{j,n}^c \in \mathcal{X}(\|N\|) \) such that

\[ (6.70) \]

\[
T_{j,n}(f d\pi) = \int_Z f D_{j,n} \pi d\|N\|
\]

\[ (6.71) \]

\[
T_{j,n}^c(f d\pi) = \int_Z f D_{j,n}^c \pi d\|N\|
\]

\[ (6.72) \]

\[
\| T_{j,n} \| = |D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}} \|N\|
\]

\[ (6.73) \]

\[
\| T_{j,n}^c \| = |D_{j,n}^c|_{\mathcal{X}(\|N\|), \text{loc}} \|N\|.
\]

Note that (6.33) implies that the measures \( \|N\| \mathcal{L} \mathcal{X} \) and \( \mu \) are in the same measure class and we can thus identify the rings \( L^\infty(\|N\| \mathcal{L} \mathcal{X}) \) and \( L^\infty(\mu) \) and the modules \( \mathcal{X}(\|N\| \mathcal{L} \mathcal{X}) \) and \( \mathcal{X}(\mu) \). Having picked a Borel representative of \( |D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}} \) and letting

\[ (6.74) \]

\[
X_{j,n} = \left\{ x \in X \setminus N_{DF} : |D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}} (x) > 0 \right\},
\]
we show that $\mu(X_{j,n}) > 0$ by showing that $\|T_{j,n}(X \setminus N_{DF}) > 0$: 

$$\|T_{j,n}(X \setminus N_{DF}) = \int_{\text{Curves}(Z)} \|\Xi_{j,n}(\gamma)\|(X \setminus N_{DF}) d\eta(\gamma) \geq \frac{1}{n} \eta(\Omega_n(w_j)) > 0. \quad (6.75)$$

We now combine (6.67), (6.72) and (6.73) with the strict convexity of $|\cdot|_{X(\|\cdot\|),\sigma}$ and the fact that $|D_{j,n}|_{X(\|\cdot\|),\sigma} > 0$ on $X_{j,n}$, to conclude that there is a nonnegative $\lambda_{j,n} \in B^\infty(Z)$, which vanishes on $Z \setminus X_{j,n}$ and is such that:

$$\chi_{X_{j,n}}D_{j,n}^c = \lambda_{j,n}D_{j,n}. \quad (6.76)$$

We then conclude that

$$\chi_{X_{j,n}}D_N = (\chi_{X_{j,n}} + \lambda_{j,n})D_{j,n}. \quad (6.77)$$

If $j = 0$ we have $\chi_{X_{j,n}}D_NF = 0$ which contradicts the fact that $\chi_{X_{j,n}}DF \neq 0$. For $j \neq 0$ we argue as follows: let $M_j$ be the maximal index such that $(w_j)_{M_j} \neq 0$; we consider the 1-form $\omega = \sum_{k=1}^{M_j} (w_j)_k \, dF_k$ and let $g$ denote a nonnegative continuous function; we have:

$$\int_{Z} g(\omega_j, D_{j,n}F) \, d\|N\| = T_{j,n}(g\omega) = \int_{\text{Curves}(Z)} \Xi_{j,n}(\gamma)(g\omega) d\eta(\gamma); \quad (6.78)$$

now, if $\gamma \in \Omega_n(w_j)$, $\sum_{k=1}^{M_j} (w_j)_k(\gamma(t)) (F_k \circ \gamma)'(t) \geq 1/n$ for $t \in \text{dom} \sigma_{j,n}$, which implies:

$$\int_{Z} g(\omega_j, D_{j,n}F) \, d\|N\| \geq \frac{1}{n} \int_{\Omega_n(w_j)} d\eta(\gamma) \int_{\text{dom} \sigma_{j,n}} g \circ \gamma(t) \, dt; \quad (6.79)$$

as the curves in $\Omega_n(w_j)$ are $n$-Lipschitz and because of (6.22), we obtain

$$\int_{Z} g(\omega_j, D_{j,n}F) \, d\|N\| \geq \frac{1}{n^2} \int_{\Omega_n(w_j)} d\eta(\gamma) \int_{\text{dom} \sigma_{j,n}} g \circ \gamma(t) \, dt \geq \frac{1}{n^2} \int_{\Omega_n(w_j)} d\eta(\gamma) \int_{Z} g \, d\|\Xi_{j,n}(\gamma)\|$$

$$= \frac{1}{n^2} \int_{Z} g \, d\|T_{j,n}\|$$

$$= \frac{1}{n^2} \int_{Z} g \, d\|D_{j,n}|_{X(\|\cdot\|),\sigma} \, d\|N\|. \quad (6.80)$$

From (6.80) we conclude that $\langle \omega_j, D_{j,n}F \rangle > 0$ on $X_{j,n}$; moreover, from (6.77) we obtain $\langle \omega_j, DF \rangle > 0$ on $X_{j,n}$, but this contradicts the choice of $w_j$. Thus, $\eta(\Omega_n(w_j)) = 0$. \hfill \Box

**Proof of Theorem 6.31.** By Lemma 6.62 we have $\eta(\Omega_n(w_j)) = 0$ which implies $\eta(\Omega_{\text{fail}}) = 0$. Therefore, for $\eta$-a.e. $\gamma$ and $\mathcal{L}^1(\text{dom} \gamma)$-a.e. $t$, $(F \circ \gamma)'(t)$ is a positive multiple of $DF(\gamma(t))$. The desired Alberti representation is then obtained using the measure $\eta$. Specifically, let

$$\text{Rep} : \text{Curves}(Z) \to \text{Frag}(Z)$$

be a Borel map which reparametrizes each $\gamma \in \text{Curves}(Z)$ to a 1-Lipschitz map $\text{Rep} : [0, [\mathcal{L}(\gamma)] \to Z$. Note that up to passing to a Borel $L^\infty(\mu)$-partition of unity
we can assume that the set $X \setminus N_{DF}$ is compact; we now consider the measure:

$$
\nu_1 = \int_{\text{Curves}(Z)} \|[\text{Rep}(\gamma)]\| \, d\eta(\gamma) = \int_{\text{Frag}(Z)} \|\gamma\| \, d(\text{Rep}_2 \eta)(\gamma)
$$

and observe that $\| N \| \ll \nu_1$ and that $\text{Rep}_2 \eta$ is concentrated on the set of 1-Lipschitz fragments. We now let

$$
\text{Frag}(Z, X \setminus N_{DF}) = \{ \gamma \in \text{Frag}(Z) : \gamma^{-1}(X \setminus N_{DF}) \neq \emptyset \}
$$

and note that $\text{Frag}(Z, X \setminus N_{DF})$ is a closed subset of $\text{Frag}(Z)$. An argument similar to that of Lemma 2.21 in [Sch13] shows that the map:

$$
\text{Red}_{X \setminus N_{DF}} : \text{Frag}(Z, X \setminus N_{DF}) \to \text{Frag}(X)
$$

$$
\gamma \mapsto \gamma^{-1}(X \setminus N_{DF})
$$

is Borel. We now consider the measure

$$
\nu_2 = \int_{\text{Frag}(Z, X \setminus N_{DF})} \|\|\text{Red}_{X \setminus N_{DF}}(\gamma)\|\| d(\text{Rep}_2 \eta)(\gamma) = \int_{\text{Frag}(X)} \|\gamma\| \, d(\text{Red}_{X \setminus N_{DF}} \, \text{Rep}_2 \eta)(\gamma)
$$

and note that $\mu \ll \nu_2$: an Alberti representation as in the statement of this Theorem is then:

$$
\mu = \int_{\text{Frag}(X)} (\text{Rep}_2 \eta)(\text{Frag}(Z, X \setminus N_{DF})) \|\|\text{Red}_{X \setminus N_{DF}}(\gamma)\|\| \frac{d\mu(\gamma)}{d\eta_2(\gamma)} \, d\eta_2(\gamma) \cdot (\text{Rep}_2 \eta)(\text{Frag}(Z, X \setminus N_{DF}))
$$

□

7. Technical Tools

7.1. Exterior Products. In this Subsection we define the exterior powers in different categories:

- In the category $\text{Ban}$, whose objects are Banach spaces and whose morphisms are bounded linear maps;
- In the category $\text{Mod}$, whose objects are $L^\infty(\mu)$-modules and whose morphisms are bounded module homomorphisms;
- In the category $\text{Mod}_{\text{loc}}$, whose objects are $L^\infty(\mu)$-normed modules and whose morphisms are bounded module homomorphisms.

In the following, if $Z$ is a Banach space, we will denote by $Z^*$ its dual. If $Z$ is also an $L^\infty(\mu)$-module, we will denote by $Z'$ the dual module; note that $Z^*$ and $Z'$ are, in general, different (Example 7.13).

Definition 7.1. For Banach spaces $Z$ and $W$, let $\text{Alt}_k(Z; W)$ denote the set of alternating multilinear maps $\varphi : Z^k \to W$ which are bounded with respect to the norm:

$$
\|\varphi\|_{\text{Alt}_k(Z; W)} = \sup \left\{ \|\varphi(m_1, \cdots, m_k)\|_W : \max_{i=1, \cdots, k} \|m_i\|_Z \leq 1 \right\}.
$$

Definition 7.3. For $L^\infty(\mu)$-modules $M$ and $N$, let $\text{Alt}_k(M; N)$ denote the set of alternating $L^\infty(\mu)$-multilinear maps $\varphi : M^k \to N$ which are bounded with respect to the norm:

$$
\|\varphi\|_{\text{Alt}_k(M; N)} = \sup \left\{ \|\varphi(m_1, \cdots, m_k)\|_N : \max_{i=1, \cdots, k} \|m_i\|_M \leq 1 \right\}.
$$
Definition 7.5. Let $Z$ be a Banach space. The **projective $k$-th power of $Z$ in the category Ban** is a pair $(\text{Ext}^k Z, \pi)$, where $\text{Ext}^k Z$ is a Banach space and $\pi \in \text{Alt}_k(Z; \text{Ext}^k Z)$, which satisfies the following universal property: for each $\varphi \in \text{Alt}_k(Z; W)$, where $W$ is a Banach space, there is a unique $\hat{\varphi} \in \text{hom}(\text{Ext}^k Z, W)$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
Z \times \text{Ext}^k Z & \xrightarrow{\pi} & \text{Ext}^k Z \\
\downarrow \varphi & & \downarrow \hat{\varphi} \\
W & & \\
\end{array}
$$

(7.6)

and such that $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k Z, W)} = \|\varphi\|_{\text{Alt}_k(Z; W)}$.

Definition 7.7. Let $M$ be an $L^\infty(\mu)$-module. The **projective $k$-th power of $M$ in the category $L^\infty(\mu)$-Mod** is a pair $(\text{Ext}^k M, \pi)$, where $\text{Ext}^k M$ is an $L^\infty(\mu)$-module and $\pi \in \text{Alt}_k(M; \text{Ext}^k M)$, which satisfies the following universal property: for each $\varphi \in \text{Alt}_k(M; N)$, where $N$ is an $L^\infty(\mu)$-module, there is a unique $\hat{\varphi} \in \text{hom}(\text{Ext}^k M, N)$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
M \times \text{Ext}^k M & \xrightarrow{\pi} & \text{Ext}^k M \\
\downarrow \varphi & & \downarrow \hat{\varphi} \\
N & & \\
\end{array}
$$

(7.8)

and such that $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k M, N)} = \|\varphi\|_{\text{Alt}_k(M; N)}$.

Definition 7.9. Let $M$ be an $L^\infty(\mu)$-normed module. The **projective $k$-th power of $M$ in the category $L^\infty(\mu)$-Mod_{loc}** is a pair $(\text{Ext}^k_{\mu, \text{loc}} M, \pi)$, where $\text{Ext}^k_{\mu, \text{loc}} M$ is an $L^\infty(\mu)$-normed module and $\pi \in \text{Alt}_k(M; \text{Ext}^k_{\mu, \text{loc}} M)$, which satisfies the following universal property: for each $\varphi \in \text{Alt}_k(M; N)$, where $N$ is an $L^\infty(\mu)$-normed module, there is a unique $\hat{\varphi} \in \text{hom}(\text{Ext}^k_{\mu, \text{loc}} M, N)$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
M \times \text{Ext}^k_{\mu, \text{loc}} M & \xrightarrow{\pi} & \text{Ext}^k_{\mu, \text{loc}} M \\
\downarrow \varphi & & \downarrow \hat{\varphi} \\
N & & \\
\end{array}
$$

(7.10)

and such that $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k_{\mu, \text{loc}} M, N)} = \|\varphi\|_{\text{Alt}_k(M; N)}$.

We now present some illustrative examples. Recall that an **atom** for a measure $\mu$ is a positive measure set $A$ such that for each proper subset $B$, $\mu(B) = 0$; note that if $A$ is an atom for a Radon measure $\mu$, $A$ is a singleton. A measure without atoms is called **non-atomic**; in particular, a Radon measure $\mu$ is non-atomic if and only if $\mu(\{x\}) = 0$ for each singleton $\{x\}$. We now recall the Sierpiński’s Theorem [Fry04, pg. 39]:
Theorem 7.11. If $\mu$ is a non-atomic measure on a space $X$ with $\mu(X) = c < \infty$ and $\Sigma$ is the $\sigma$-algebra of $\mu$-measurable sets, then there is a function $S : [0,c] \to \Sigma$ which is monotone with respect to inclusion and is a right inverse of $\mu : \Sigma \to [0,c]$.

In the following we will assume $p \in [1,\infty)$.

Example 7.12. If $\mu$ is a finite sum of Dirac masses, $L^p(\mu)$ can be identified with $L^\infty(\mu)$ and so is free of rank 1.

Suppose that $\mu$ is non-atomic; in particular, by Theorem 7.11, given any positive measure set $U$, it is possible to find $f \in L^p(\mu \mathcal{U})$ with $\|f\|_{L^p(\mu)} \leq 1$ and $\forall n \mu(x \in U : |f(x)| > n) > 0$. Suppose that $L^p(\mu)$ was generated by $f_1, \cdots, f_N$; then there would be a set of positive measure $U$ on which the $f_i$, and hence all the element in $L^p(\mu)$ would be uniformly bounded, leading to a contradiction.

However, any two elements of $L^p(\mu)$ are linearly dependent over $L^\infty(\mu)$. If $f \in L^p(\mu)$ vanishes on a set of positive measure $U$, it suffices to note that $f$ is annihilated by $\chi_U$. If $f$ and $g$ are nowhere vanishing, there is a positive measure set $U$ on which $0 < c_0 < |f|, |g| < c_1 < \infty$; then it is possible to find $\lambda \in L^\infty(\mu)$ with $\chi_U f + \lambda g = 0$. In particular, if $f \in L^p(\mu)$ is nowhere vanishing, the algebraic submodule generated by $f$ is dense.

Example 7.13. Given an $L^\infty(\mu)$-module $M$, there are two notions of dual. The dual module of $M$, hom$(M, L^\infty(\mu)) = M'$ is an $L^\infty(\mu)$-normed module. However, the dual Banach space of $M$, $M^*$, is also an $L^\infty(\mu)$-module if we let

$$
\lambda \varphi(m) = \varphi(\lambda m).
$$

For example, if $M = L^p(\mu)$, then $M^* = L^q(\mu)$.

We show that if $\mu$ is non-atomic, then the algebraic dual of $M$ (and hence $M'$) is trivial. By replacing $\mu$ by $\mu \mathcal{U}$, where $U$ is a set of positive measure, we can assume that $\mu$ is finite, so that $L^\infty(\mu) \subset L^p(\mu)$; let $\Phi : L^p(\mu) \to L^\infty(\mu)$ be a module homomorphism; supposing that $\Phi(1) \neq 0$, we can use Theorem 7.11 to find $f \in L^p(\mu)$ and $\mu$-measurable sets $U_n$ such that:

- for each $n$, $\Phi(1) \chi_{U_n} f \in L^\infty(\mu)$;
- for each $n$:

$$
\mu \left( \left\{ x \in U_n : |\Phi(1) \chi_{U_n} f|(x) > n \right\} \right) > 0.
$$

Note that

$$
\chi_{U_n} \Phi(f) = \Phi(\chi_{U_n} f) = \Phi(1) \chi_{U_n} f
$$

shows that $\Phi(f) \notin L^\infty(\mu)$, a contradiction. Thus $\Phi(1) = 0$ implying $\Phi = 0$. In this case, the dual module of $L^p(\mu)$ is trivial.

Suppose now that $\mu$ is a countable sum of Dirac masses: $\mu = \sum_n c_n \delta_{p_n}$, so that a function $f$ is in the unit ball of $L^p(\mu)$ iff

$$
\sum_n |f_n|^p c_n \leq 1 \quad (f_n = f(p_n));
$$

as $\varphi \in M'$ is determined by the values on the functions $\delta_{p_n}$, we can identify it with the module of those sequences $\{\lambda_n\}$ for which there is a $C_\lambda > 0$ with $|\lambda_n| \leq C_\lambda (c_n)^{1/p}$; the norm is $\sup_n |\lambda_n|/|c_n|^{1/p}$.

Example 7.18. For an $L^\infty(\mu)$-module $N$, $\text{Alt}_k(L^p(\mu); N)$ is trivial and the case $k = 1$ was treated in Example 7.13. Note that $\Omega = L^p(\mu) \cap L^\infty(\mu)$ is a dense algebraic submodule of $L^p(\mu)$; in particular, $T \in \text{Alt}_k(L^p(\mu); N)$ is determined by
its values on $\Omega^k$; however, $\Omega$ is also an algebraic submodule of $L^\infty(\mu)$; in particular, there is an alternating multilinear mapping $\hat{T} = (L^\infty(\mu))^k \to N$ which agrees with $T$ on $\Omega^k$ and vanishes elsewhere; for $k > 1$, $\hat{T} = 0$ and so $T = 0$.

Note that the nullity of $\text{Alt}_k(L^p(\mu); N)$ for $N$ an $L^\infty(\mu)$-normed module, implies that $\text{Ext}_{\mu, \text{loc}}^k L^p(\mu) = 0$.

**Example 7.19.** Let $\| \cdot \|$ a norm on $\mathbb{R}^n$; on $\bigwedge^k \mathbb{R}^n$ we consider the norm:

\[
\| \omega \| = \inf \left\{ \sum_{i \in I} \| v_i \| : \omega = \sum_{i \in I} v_i \wedge \cdots \wedge v_k \right\}.
\]

We will denote by $\mu$ a non-atomic Radon measure.

We claim that $\text{Ext}_{\mu, \text{loc}}^k L^p(\mu; \mathbb{R}^n)$ is trivial. By the Hahn-Banach Theorem, it suffices to show that $\text{Alt}_k(L^p(\mu; \mathbb{R}^n); L^\infty(\mu))$ is trivial; suppose that for $U$ a Borel set of finite measure and $\{v_i\}_{i=1}^k \subset \mathbb{R}^n$ independent vectors we had

\[
T(\chi_U v_1, \ldots, \chi_U v_k) \neq 0
\]

where $T \in \text{Alt}_k(L^p(\mu; \mathbb{R}^n); L^\infty(\mu))$; arguing as in Example 7.13, we would reach a contradiction.

However we show that $\text{Ext}_{\mu, \text{loc}}^k L^p(\mu; \mathbb{R}^n)$ can be identified with $L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n)$ under the assumption $p \in [k, \infty)$. By Hölder’s inequality, the multilinear alternating map

\[
E : (L^p(\mu; \mathbb{R}^n))^k \to L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n)
\]

\[
(f_1, \ldots, f_k) \mapsto f_1 \wedge \cdots \wedge f_k
\]

has norm at most 1. For $\psi \in L^{p/k}(\mu)$ define:

\[
T_\psi : (\mathbb{R}^n)^k \to N
\]

\[
(v_1, \ldots, v_k) \mapsto T(\text{sgn} \psi |\psi|^{1/k} v_1, |\psi|^{1/k} v_2, \ldots, |\psi|^{1/k} v_k);
\]

the map $T_\psi$ is multilinear and alternating (as a map of vector spaces); let $\hat{T}_\psi : \bigwedge^k \mathbb{R}^n \to N$ denote the corresponding linear map given by the universal property of $\bigwedge^k \mathbb{R}^n$. For $\omega \in \bigwedge^k \mathbb{R}^n$ note that

\[
\| \hat{T}_\psi(\omega) \| \leq \| T \| \| \psi \|_{p/k} \| \omega \|;
\]

the multilinearity of $T$ can be used to show that $\hat{T}_{\psi_1 + \psi_2} = \hat{T}_{\psi_1} + \hat{T}_{\psi_2}$; the multilinearity of $T$ over $L^\infty(\mu)$ also implies that $\hat{T}_{\lambda \psi} = \lambda \hat{T}_\psi$. Note that any $f \in L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n)$ can be written as

\[
f = \sum_i f_i \omega_i,
\]

where $\{\omega_i\}$ is a basis of $\bigwedge^k \mathbb{R}^n$; in particular, we can define $\hat{T} : L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n) \to N$ by

\[
\hat{T}(f) = \sum_i \hat{T}_{f_i}(\omega_i);
\]

note that the definition is well-posed because any $f$ has a unique expression (7.25). Furthermore, the definition of $\hat{T}$ does not depend on the choice of the basis $\{\omega_i\}$ as
can be verified by linear algebra. Note also that the map $\hat{T}$ is linear over $L^\infty(\mu)$. Using (7.26) we conclude that there is a dimensional constant $C_{n,k}$ such that

$$\|\hat{T}\| \leq C_{n,k} \|T\|. \quad (7.27)$$

As simple functions are dense in $L^p(\mu;\mathbb{R}^n)$ and using (7.27) we conclude that

$$T(f_1, \ldots, f_k) = \hat{T}(f_1 \wedge \cdots \wedge f_k); \quad (7.28)$$

this implies that $\|\hat{T}\| = \|T\|$ and that $L^{p/k}(\mu; \mathbb{R}^n)$ is the exterior $k$-power of $L^p(\mu;\mathbb{R}^n)$.

In the remainder of this section we assume that $\mu$ is a Radon measure. The following Lemma summarizes some properties of the Banach space $\text{Alt}_k(M;N)$.

**Lemma 7.29.** Let $M,N$ be $L^\infty(\mu)$-modules; then $\text{Alt}_k(M;N)$ is an $L^\infty(\mu)$-module and it is an $L^\infty(\mu)$-normed module if $N$ is an $L^\infty(\mu)$-normed module. Moreover if $M$ and $N$ are $L^\infty(\mu)$-normed modules, for $\varphi \in \text{Alt}_k(M;N)$ and $\{m_i\}_{i=1}^\infty \subset M$

$$|\varphi(m_1, \ldots, m_k)|_{N,\text{loc}} \leq |\varphi|_{\text{Alt}_k(M;N),\text{loc}} |m_1|_{M,\text{loc}} \cdots |m_k|_{M,\text{loc}}. \quad (7.30)$$

**Proof of Lemma 7.29.** The fact that $\text{Alt}_k(M;N)$ is a Banach space with the norm $\|\cdot\|_{\text{Alt}_k(M;N)}$ follows from a standard argument. For $(\varphi,\lambda) \in \text{Alt}_k(M;N) \times L^\infty(\mu)$ the product $\lambda \varphi$ can be defined by:

$$\lambda \varphi(m_1, \ldots, m_k) = \varphi(m_1, \ldots, \lambda m_i, \ldots, m_k) \quad \text{(any choice of } i) \quad (7.31)$$

which makes $\text{Alt}_k(M;N)$ an $L^\infty(\mu)$-module.

If $N$ is an $L^\infty(\mu)$-normed module, for a $\mu$-measurable subset $U \subset X$, we have

$$\|\varphi\|_{\text{Alt}_k(M;N)} = \sup_{\|m\|_M \leq 1} \|\varphi(m_1, \ldots, m_k)\|_N \quad (7.32)$$

$$= \sup_{\|m\|_M \leq 1} \max \left( \|\chi_U \varphi(m_1, \ldots, m_k)\|_N, \|\chi_{X \setminus U} \varphi(m_1, \ldots, m_k)\|_N \right)$$

$$= \max \left( \sup_{\|m\|_M \leq 1} \|(\chi_U \varphi)(m_1, \ldots, m_k)\|, \sup_{\|m\|_M \leq 1} \|(\chi_{X \setminus U} \varphi)(m_1, \ldots, m_k)\| \right)$$

$$= \max \left( \|\chi_U \varphi\|_{\text{Alt}_k(M;N)}, \|\chi_{X \setminus U} \varphi\|_{\text{Alt}_k(M;N)} \right);$$

by [Wea00, Thm. 2] $\text{Alt}_k(M;N)$ is an $L^\infty(\mu)$-normed module.

We now show (7.30) under the assumption that $M$ and $N$ are $L^\infty(\mu)$-normed modules. By [Wea00, Cor. 6] we can find $\Phi_{m_1,\ldots,m_k} \in N'$ with $\|\Phi_{m_1,\ldots,m_k}\|_{N'} \leq 1$ and

$$|\varphi(m_1, \ldots, m_k)|_{N,\text{loc}} = \langle \Phi_{m_1,\ldots,m_k}, \varphi(m_1, \ldots, m_k) \rangle; \quad (7.33)$$

let $\xi \in \text{Alt}_k(M;L^\infty(\mu))$ be defined by

$$\xi(m_1, \ldots, \tilde{m}_k) = \langle \Phi_{m_1,\ldots,m_k}, \varphi(m_1, \ldots, \tilde{m}_k) \rangle; \quad (7.34)$$

for $\varepsilon > 0$ we can find an $L^\infty(\mu)$-partition of unity $\{U_\alpha\}$ such that for $x \in U_\alpha$ and $1 \leq i \leq k$,

$$|\xi|_{\text{Alt}_k(M;L^\infty(\mu)),\text{loc}}(x) \in \left( \|\chi_{U_\alpha} \xi\|_{\text{Alt}_k(M;L^\infty(\mu))} - \varepsilon, \|\chi_{U_\alpha} \xi\|_{\text{Alt}_k(M;L^\infty(\mu))} \right); \quad (7.35)$$

$$|m_i|_{M,\text{loc}}(x) \in \left( \|\chi_{U_\alpha} m_i\|_M - \varepsilon, \|\chi_{U_\alpha} m_i\|_M \right). \quad (7.36)$$
Using the definition of norm in $\text{Alt}_k(M; L^{\infty}(\mu; U))$ and (7.35) and (7.36),
\[
\xi(m_1, \cdots, m_k) = \sum_{\alpha} \chi_{U, \alpha} \xi(m_1, \cdots, m_k) \\
= \sum_{\alpha} (\chi_{U, \alpha} (\chi_{U, m_1}, \cdots, \chi_{U, m_k}) \\
\leq \sum_{\alpha} \chi_{U, \alpha} \| \chi_{U, m_1} \|_{\text{Alt}_k(M; L^{\infty}(\mu))} \cdots \| \chi_{U, m_k} \|_M \\
\leq \sum_{\alpha} \chi_{U, \alpha} \left( |\xi|_{\text{Alt}_k(M; L^{\infty}(\mu)), \text{loc}} + \varepsilon \right) \prod_{i=1}^k (\| m_i \|_{M, \text{loc}} + \varepsilon) \\
= \left( |\xi|_{\text{Alt}_k(M; L^{\infty}(\mu)), \text{loc}} + \varepsilon \right) \prod_{i=1}^k (\| m_i \|_{M, \text{loc}} + \varepsilon).
\]
(7.37)

Note that (7.30) follows from (7.37) letting $\varepsilon \searrow 0$ provided we show
\[
|\xi|_{\text{Alt}_k(M; L^{\infty}(\mu)), \text{loc}} \leq |\varphi|_{\text{Alt}_k(M; N), \text{loc}}.
\]
(7.38)

As for each $\mu$-measurable $U$ we have
\[
\| \chi_U \|_{\text{Alt}_k(M; L^{\infty}(\mu))} \leq \| \chi_U \|_{\text{Alt}_k(M; N)},
\]
(7.39)

(7.38) holds. \hfill \square

We now prove the existence of the exterior powers in the category $\text{Ban}$.

**Theorem 7.40.** For $Z$ a Banach space, the $k$-th exterior power in the category $\text{Ban}$ exists and can be realized as a closed subspace of the dual space $\text{Alt}_k(M; \mathbb{R})^*$; moreover, the algebraic $k$-th exterior power $\bigwedge^k Z$ is dense in $\text{Ext}^k Z$.

**proof of Theorem 7.40.** For $\varphi \in \text{Alt}_k(Z; \mathbb{R})$ let $\bar{\varphi} : \bigwedge^k Z \rightarrow \mathbb{R}$ denote the unique linear map corresponding to $\varphi$ given by the universal property of $\bigwedge^k Z$. In particular, we obtain a map $E$ from $\bigwedge^k Z$ to the algebraic dual of $\text{Alt}_k(Z; \mathbb{R})$ by letting $\langle E(w), \varphi \rangle = \bar{\varphi}(w)$. We now show that $E(w)$ is a bounded functional. Let
\[
w = \sum_{i \in I} z_i \wedge \cdots \wedge z_{i_k}
\]
and note that
\[
\left\| \sum_{i \in I} z_i \wedge \cdots \wedge z_{i_k} \right\|_{(\text{Alt}_k(Z; \mathbb{R}))^*} = \sup_{\| \varphi \|_{\text{Alt}_k(Z; \mathbb{R})} \leq 1} \left\| \sum_{i \in I} z_i \wedge \cdots \wedge z_{i_k} \varphi \right\| \\
\leq \sup_{\| \varphi \|_{\text{Alt}_k(Z; \mathbb{R})} \leq 1} \left\| \sum_{i \in I} \varphi(z_{i_1}, \cdots, z_{i_k}) \right\| \\
\leq \sum_{i \in I} \| z_{i_1} \|_{X} \cdots \| z_{i_k} \|_{X}.
\]
(7.42)

We now show that $E$ is injective; suppose $w \neq 0$; let $Z_0$ denote the linear span of $\Omega = \{ z_{i_j} : j = 1, \ldots, k; i \in I \}$ so that $Z_0$ is a finite dimensional vector space of dimension $L \geq k$. Having chosen a basis $\{ v_{j_1} \}_{j_1=1}^L$ of $Z_0$, without loss of generality we can assume that
\[
w = \sum_{j \in \Lambda, L} c_j v_{j_1} \wedge \cdots \wedge v_{j_k}
\]
(7.43)
with \( c_{(1,\ldots,k)} \neq 0 \). If \( \{v_\alpha^*\}_{\alpha=1}^L \) is the dual basis of \( \{v_\alpha\}_{\alpha=1}^L \), by the Hahn-Banach Theorem the functionals \( v_\alpha^* \) can be extended to elements of \( Z^* \); in particular,

\[
\Xi : Z^k \to \mathbb{R}
\]

\[
(z_1, \ldots, z_k) \mapsto \det((v_\alpha^*, z_i)_i)_{\alpha,i=1}^k
\]

defines an element of \( \text{Alt}_k(Z; \mathbb{R}) \) and

\[
\langle E(w), \Xi \rangle = c_{(1,\ldots,k)} \neq 0
\]

showing that \( E \) is injective.

We can thus identify \( \wedge^k Z \) with a linear subspace of \( \text{Alt}_k(Z; \mathbb{R})^* \) and we will denote its completion in the \( \| \cdot \|_{(\text{Alt}_k(Z; \mathbb{R}))^*} \) norm by \( \wedge^k Z \). The map \( \pi \) is defined by

\[
\pi(z_1, \ldots, z_k) = z_1 \wedge \cdots \wedge z_k;
\]

note that \( \pi \) is alternating and multilinear and (7.42) shows that it is bounded. Let \( \varphi \in \text{Alt}_k(Z; W) \) and define \( \hat{\varphi} : \wedge^k Z \to W \) by

\[
\hat{\varphi} \left( \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right) = \sum_{i \in I} \varphi(z_{i_1}, \ldots, z_{i_k});
\]

this is well-defined because \( \varphi \) is alternating multilinear and because of the universal property of \( \wedge^k Z \). In order to show that \( \hat{\varphi} \) has a unique extension \( \hat{\varphi} : \text{Ext}^k Z \to W \), it suffices to show that \( \hat{\varphi} \) is bounded:

\[
\left\| \hat{\varphi} \left( \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right) \right\|_W = \sup_{w^* \in W^*: \|w^*\|_{W^*} \leq 1} \langle w^*, \hat{\varphi} \left( \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right) \rangle
\]

\[
= \|\varphi\|_{\text{Alt}_k(Z; W)} \sup_{w^* \in W^*: \|w^*\|_{W^*} \leq 1} \sum_{i \in I} \left\langle w^*, \frac{1}{\|\varphi\|_{\text{Alt}_k(Z; W)}} \varphi(z_{i_1}, \ldots, z_{i_k}) \right\rangle
\]

\[
\leq \|\varphi\|_{\text{Alt}_k(Z; W)} \sup_{\tau \in \text{Alt}_k(Z; R): \|\tau\|_{\text{Alt}_k(Z; R)} \leq 1} \left\langle \tau, \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\rangle
\]

\[
\leq \|\varphi\|_{\text{Alt}_k(Z; W)} \left\| \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\|_{(\text{Alt}_k(Z; R))^*}.
\]

Note that (7.48) shows that

\[
\|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k Z, W)} \leq \|\varphi\|_{\text{Alt}_k(Z; W)};
\]

for the reverse inequality, observe that for each \( \varepsilon > 0 \), there are \( z_i \in Z \) (\( i \in \{1, \cdots, k\} \)) such that \( \|z_i\|_Z \leq 1 \) and

\[
\|\varphi\|_{\text{Alt}_k(Z; W)} < \varepsilon + \|\varphi(z_1, \cdots, z_k)\|_Z;
\]

but

\[
\varphi(z_1, \cdots, z_k) = \hat{\varphi}(z_1 \wedge \cdots \wedge z_k)
\]

and by (7.42)

\[
\|z_1 \wedge \cdots \wedge z_k\|_{\wedge^k Z} \leq 1;
\]
thus
\begin{equation}
\|\varphi\|_{\text{Alt}_k(Z;W)} < \varepsilon + \|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k_{\mu,\text{loc}} Z;W)}.
\end{equation}
\end{proof}

We now turn to the existence of exterior powers in the category $\text{Mod}_{\mu,\text{loc}}$.

**Theorem 7.54.** For $M$ an $L^\infty(\mu)$-normed module, the $k$-th exterior power in the category $\text{Mod}_{\mu,\text{loc}}$ exists and can be realized as a closed submodule of the dual module $\text{Alt}_k(M;L^\infty(\mu))'$; moreover, the algebraic $k$-th exterior power $L^\infty(\mu) \wedge^k M$ is dense in $\text{Ext}^k_{\mu,\text{loc}} M$.

**Proof of Theorem 7.54.** Part of the proof is similar to the Banach space case (Theorem 7.40). For $\varphi \in \text{Alt}_k(M;L^\infty(\mu))$ let $\tilde{\varphi} : L^\infty(\mu) \wedge^k M \to L^\infty(\mu)$ denote the unique module homomorphism corresponding to $\varphi$ given by the universal property of $L^\infty(\mu) \wedge^k M$. The same estimate (7.42) used in the Banach space case shows that the map:
\begin{equation}
E : L^\infty(\mu) \wedge^k M \to \text{Alt}_k(M;L^\infty(\mu))'
\end{equation}
sending $w \in L^\infty(\mu) \wedge^k M$ to the functional $E(w)$ satisfying
\begin{equation}
(E(w), \varphi) = \tilde{\varphi}(w),
\end{equation}
is well-defined.

We now show that $E$ is injective. Let
\begin{equation}
w = \sum_{i \in I} m_i \wedge \cdots \wedge m_k \neq 0
\end{equation}
and $M_0$ the $L^\infty(\mu)$-submodule of $M$ generated by the finite set
\begin{equation}
\Omega = \{m_i : j = 1, \ldots, k; i \in I\}.
\end{equation}
By [Wea00, Lem. 9] there are disjoint measurable sets $\{U_i\}_{i=1}^{\#\Omega}$ such that
\begin{equation}
1 = \sum_{i=1}^{\#\Omega} \chi_{U_i},
\end{equation}
and if $\mu(U_i) > 0$, then $\chi_{U_i} M_0$, regarded as an $L^\infty(\mu \cup U_i)$-module, is free of rank $i$; as we are assuming $w \neq 0$, $\chi_{U_L} w \neq 0$ for some index $L \geq k$. Let $\{\tilde{m}_i\}_{i=1}^{L}$ a basis of $\chi_{U_L} M_0$ over $L^\infty(\mu \cup U_i)$; without loss of generality, we can assume that
\begin{equation}
\chi_{U_L} w = \sum_{j \in \Lambda_k, N} \lambda_{j} \tilde{m}_{j_1} \wedge \cdots \wedge \tilde{m}_{j_k},
\end{equation}
with $\lambda_{(1, \ldots, k)} \neq 0$. Moreover, by [Wea00, Thm. 10] we can choose a measurable $V \subset U_L$ with $\chi_V \lambda_{(1, \ldots, k)} \neq 0$ and find $C > 0$ such that, if we define for $x \in V$
\begin{equation}
p_{x} : \mathbb{R}^L \to (0, \infty)
\end{equation}
\begin{equation}
v \mapsto \left| \sum_{i=1}^{L} u_{i} \tilde{m}_{i} \right|_{M_{0,\text{loc}}} (x),
\end{equation}
then $p_x$ is a norm satisfying
\begin{equation}
C p_x(v) \geq \|v\|_{\infty} \quad (\forall (x, v) \in V \times \mathbb{R}^L).
\end{equation}
Note that functions in $L^\infty(\mu \mathcal{M} V)$ can be canonically extended to $L^\infty(\mu)$ because we can indentify $L^\infty(\mu \mathcal{M} V)$ with $\chi_V L^\infty(\mu)$; the maps
\[ \xi_i : \chi_V M_0 \to L^\infty(\mu) \quad (i = 1, \ldots, L) \]
are bounded linear functionals by (7.62). By the Hanh-Banach Theorem [Wea00, Thm. 5] the $\{\xi_i\}$ can be extended to elements of $M'$; in particular,
\[ \Xi : M \ni (m_1, \ldots, m_k) \mapsto \det((\langle \xi_i, m_j \rangle)_{i,j=1}^k) \]
defines an element of $\text{Alt}_k(M; L^\infty(\mu))$ and
\[ E(w)(\chi_V \Xi) = \chi_V \lambda(1, \ldots, k) \neq 0 \]
showing that $E$ is injective. The proof is now completed as in Theorem 7.40. □

We now provide a characterization of the norms in the exterior powers.

**Lemma 7.66.** For $Z$ a Banach space, if $w \in \bigwedge^k Z \hookrightarrow \text{Ext}^k Z$
\[ \|w\|_{\text{Ext}^k Z} = \inf \left\{ \sum_{i \in I} \|z_{i_1}\| \cdots \|z_{i_k}\| : w = \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\}. \]

If $M$ is an $L^\infty(\mu)$-normed module, for each
\[ w \in L^\infty(\mu) \bigwedge^k M \hookrightarrow \text{Ext}^k_{\mu, \text{loc}} M, \]
\[ \|w\|_{\text{Ext}^k_{\mu, \text{loc}} M} = \inf \left\{ \left\| \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}} \right\|_{L^\infty(\mu)} : w = \sum_{i \in I} m_{i_1} \wedge \cdots \wedge m_{i_k} \right\}; \]
moreover, if $w = \sum_{i \in I} m_{i_1} \wedge \cdots \wedge m_{i_k}$,
\[ |w|_{\text{Ext}^k_{\mu, \text{loc}} M, \text{loc}} \leq \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}}. \]

**Proof of Lemma 7.66.** For $Z$ a Banach space, define for $w \in \bigwedge^k Z$
\[ \gamma(w) = \inf \left\{ \sum_{i \in I} \|z_{i_1}\| \cdots \|z_{i_k}\| : w = \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\}; \]
then $\gamma(w)$ is a seminorm and (7.42) shows that
\[ \|w\|_{\text{Ext}^k Z} \leq \gamma(w); \]
in particular (7.72) shows that $\gamma$ is a norm on $\bigwedge^k Z$ and the same argument used in the proof of Theorem 7.40 (compare (7.48)) shows that the completion of $\bigwedge^k Z$ in the $\gamma$-norm satisfies the universal property characterizing $\text{Ext}^k Z$; thus $\|w\|_{\text{Ext}^k Z} = \gamma(w).$
Let $M$ an $L^\infty(\mu)$-normed module; we first show (7.70). It suffices to show that for each $U$ $\mu$-measurable,

$$
\|\chi_U w\|_{\text{Ext}^k_{\mu, \text{loc}} M} \leq \left\| \chi_U \sum_{i \in I} |m_i| \cdot |m_{ik}| \right\|_{L^\infty(\mu)} ;
$$

from the definition of $\| \cdot \|_{\text{Ext}^k_{\mu, \text{loc}} M}$ (proof of Theorem 7.54) we can find, for each $\varepsilon > 0$, an alternating map $\varphi \in \text{Alt}_k(\mu; L^\infty(\mu))$ with norm at most 1 and satisfying:

$$
\|\chi_U w\|_{\text{Ext}^k_{\mu, \text{loc}} M} \leq \|\varphi(\chi_U w)\|_{L^\infty(\mu)} + \varepsilon ;
$$

but (7.30) implies

$$
|\varphi(\chi_U w)| \leq \chi_U \sum_{i \in I} |m_i| \cdot |m_{ik}| \cdot |m_{ik}| \cdot |m_{ik}| ;
$$

from which we obtain (7.73) taking the essential sup and letting $\varepsilon \searrow 0$. To show (7.69) let

$$
\gamma(w) = \inf \left\{ \left\| \sum_{i \in I} |m_i| \cdot |m_{ik}| \cdot |m_{ik}| \cdot |m_{ik}| \right\|_{L^\infty(\mu)} : w = \sum m_i \wedge \cdots \wedge m_{ik} \right\} ;
$$

then $\gamma(w)$ is a seminorm on $L^\infty(\mu) \wedge^k M$. Note that (7.69) implies $\left\| \cdot \right\|_{\text{Ext}^k_{\mu, \text{loc}} M} \leq \gamma$, so that $\gamma$ is a norm; the proof of Theorem 7.54 implies that the completion $Y$ of $L^\infty(\mu) \wedge^k M$ in the $\gamma$-norm satisfies the universal property defining $\text{Ext}^k_{\mu, \text{loc}} M$ provided that $Y$ is an $L^\infty(\mu)$-normed module. To show that $Y$ is an $L^\infty(\mu)$-normed module it suffices to show that for a $\mu$-measurable set $U$,

$$
\gamma(w) = \max(\gamma(\chi_U w), \gamma(\chi_U^\varepsilon w)) .
$$

Having shown (7.77), uniqueness of $\text{Ext}^k_{\mu, \text{loc}} M$ will imply that $\left\| \cdot \right\|_{\text{Ext}^k_{\mu, \text{loc}} M} = \gamma$. To show (7.77), for $\varepsilon > 0$ let

$$
\chi_U w = \sum_{i \in I_U} \chi_U m_i \wedge \cdots \wedge \chi_U m_{ik} ,
$$

$$
\chi_U^\varepsilon w = \sum_{i \in I_{U:\varepsilon}} \chi_U^\varepsilon m_i \wedge \cdots \wedge \chi_U^\varepsilon m_{ik} ,
$$

with

$$
\left\| \sum_{i \in I_U} \chi_U m_i \wedge \cdots \wedge \chi_U m_{ik} \right\|_{L^\infty(\mu)} < \gamma(\chi_U w) + \varepsilon ;
$$

$$
\left\| \sum_{i \in I_{U:\varepsilon}} \chi_U^\varepsilon m_i \wedge \cdots \wedge \chi_U^\varepsilon m_{ik} \right\|_{L^\infty(\mu)} < \gamma(\chi_U^\varepsilon w) + \varepsilon ;
$$

without loss of generality (introducing null terms) we can assume that $I_U = I_{U:\varepsilon} = I$ so that (7.77) follows observing that

$$
w = \sum_{i \in I} (\chi_U m_i \wedge \cdots \wedge (\chi_U m_i \wedge (\chi_U m_i \wedge \cdots \wedge (\chi_U m_i \wedge m_{ik} ) ) ) )
$$

and letting $\varepsilon \searrow 0$. 

There are also pairings between exterior powers:
Lemma 7.83. Suppose $Z$ is a Banach space; the bilinear mapping
\begin{equation}
\wedge : \bigwedge^k Z \times \bigwedge^l Z \to \bigwedge^{k+l} Z
\end{equation}
which on pairs of simple multivectors is given by:
\begin{equation}
\wedge : ((z_1, \cdots, z_k), (u_1, \cdots, u_l)) \mapsto z_1 \wedge \cdots \wedge z_k \wedge u_1 \wedge \cdots \wedge u_l,
\end{equation}
extends to a bounded bilinear map
\begin{equation}
\wedge : \text{Ext}^k Z \times \text{Ext}^l Z \to \text{Ext}^{k+l} Z
\end{equation}
satisfying
\begin{equation}
||\omega_1 \wedge \omega_2||_{\text{Ext}^{k+l}} \leq ||\omega_1||_{\text{Ext}^k} ||\omega_2||_{\text{Ext}^l}.
\end{equation}

Suppose $M$ is an $L^\infty(\mu)$-module; for $1 \leq i \leq k$, the bilinear mapping (in the
category $\text{Ban}$)
\begin{equation}
\Phi_i : L^\infty(\mu) \times \bigwedge^k M \to \bigwedge^k M
\end{equation}
\begin{equation}
\left( \lambda, \sum_{j \in J} m_{j_1} \wedge \cdots \wedge m_{j_k} \right) \mapsto \sum_{j \in J} m_{j_1} \wedge \cdots \wedge \lambda m_{j_1} \wedge \cdots \wedge m_{j_k}
\end{equation}
extends to a bounded bilinear map
\begin{equation}
\Phi_i : L^\infty(\mu) \times \text{Ext}^k M \to \text{Ext}^k M
\end{equation}
satisfying
\begin{equation}
||\Phi_i(\lambda, \omega)||_{\text{Ext}^k Z} \leq ||\lambda||_{L^\infty(\mu)} ||\omega||_{\text{Ext}^k Z}.
\end{equation}

Proof of Lemma 7.83. It follows from the first part of Lemma 7.66; in particular, (7.87) and (7.90) follow from (7.67).

We now turn to the existence of the exterior power in the category $\wedge^k \text{Mod}$.

Theorem 7.91. For $M$ an $L^\infty(\mu)$-module, the $k$-th exterior power in the category $\wedge^k \text{Mod}$ exists and can be realized as a quotient space of $\text{Ext}^k M$ (in $\text{Ban}$) by the
closure of the linear span of the set
\begin{equation}
\left\{ \Phi_i(\lambda, \omega) - \Phi_j(\lambda, \omega) : 1 \leq i, j \leq k, \lambda \in L^\infty(\mu), \omega \in \bigwedge^k M \right\}.
\end{equation}

Proof of Theorem 7.91. Let $Q$ denote the linear span of the set (7.92). If $\varphi \in \text{Alt}_k(M;N)$, where $N$ is an $L^\infty(\mu)$-module, let $\tilde{\varphi} : \text{Ext}^k M \to N$ denote the corresponding map given by the universal property of $\text{Ext}^k M$; note that $\tilde{\varphi}$ annihilates $Q$. Moreover, $\text{Ext}^k M/\tilde{Q}$ becomes an $L^\infty(\mu)$-module letting
\begin{equation}
\lambda.[\omega] = [\Phi_i(\lambda, \omega)] \quad ((\lambda, \omega) \in L^\infty(\mu) \times \text{Ext}^k M \text{ and } 1 \leq i \leq k).
\end{equation}
If we let $\pi'$ denote the composition of $\pi : M^k \to \text{Ext}^k M$ with the quotient map $\text{Ext}^k M \to \text{Ext}^k M/\tilde{Q}$, then $\pi' \in \text{Alt}_k(M;\text{Ext}^k M/\tilde{Q})$; similarly, if we let $\tilde{\varphi} : \text{Ext}^k M/\tilde{Q} \to N$ the map induced by $\varphi$, then $\tilde{\varphi} \in \text{hom}(\text{Ext}^k M/\tilde{Q}, N)$. Note that $\varphi \circ \pi' = \varphi$ and that uniqueness of $\tilde{\varphi}$ follows from uniqueness of $\varphi$. Finally, as $||\omega||_{\text{Ext}^k M/\tilde{Q}} \leq ||\omega||_{\text{Ext}^k M}$, $||\tilde{\varphi}||_{\text{hom}(\text{Ext}^k M/\tilde{Q}, N)} = ||\varphi||_{\text{Alt}_k(M;N)}$. \qed
Remark 7.94. Note that if $M$ is an $L^\infty(\mu)$-module, we have an $\mathbb{R}$-linear surjection
\begin{equation}
\text{Ext}^k M \rightarrow \text{Ext}^k_\mu M
\end{equation}
with norm at most 1; similarly, if $M$ is an $L^\infty(\mu)$-normed module, we have an $L^\infty(\mu)$-linear surjection
\begin{equation}
\text{Ext}^k_\mu M \rightarrow \text{Ext}^k_{\mu, \text{loc}} M
\end{equation}
with norm at most 1.

7.2. Alberti representations in Banach spaces. In this Subsection we prove a refinement for the production of Alberti representations in Banach spaces when the speed and direction are specified using bounded linear maps.

Theorem 7.97. Suppose that $Z$ is a separable Banach space, $\mu$ is a Radon measure on $Z$ and suppose that $f : Z \rightarrow \mathbb{R}^q$ and $g : Z \rightarrow \mathbb{R}$ are bounded linear maps. Let $C(w, \alpha)$ be a $q$-dimensional cone field on $Z$ and $\delta : Z \rightarrow (0, \infty)$ a Borel map; then the following are equivalent:

1. The measure $\mu$ admits an Alberti representation in the $f$-direction of $C(w, \alpha)$ with $g$-speed $> \delta$.
2. The measure $\mu$ admits a $(\delta/\|g\|_Z^*, 1)$-biLipschitz Alberti representation $A = (P, \nu)$ in the $f$-direction of $C(w, \alpha)$ with $g$-speed $> \delta$ and such that $\text{spt} P \subset \text{Curves}(Z)$ and $\nu_\gamma = h \Psi_\gamma$ where $h$ is a Borel function on $Z$ and $\Psi_\gamma = \gamma z \mathcal{L}_1\mathcal{L}[0, 1]$.

Proof of Theorem 7.97. It suffices to show that (1) implies (2). For the moment, we assume that the functions $w$, $\alpha$ and $\delta$ are constant and that the set $\text{spt} \mu$ is compact. By rescaling $g$ and $\delta$, we can assume that $\|g\|_{Z^*} = 1$. Note that $\text{spt} \mu$ must contain a fragment $\gamma$ with $(g \circ \gamma)'(t) > \delta \text{ md } \gamma(t)$ and $(f \circ \gamma)'(t) \in C(w, \alpha)$ for $\mathcal{L}_1 \text{dom } \gamma \text{-a.e. } t$. In particular, there is a vector $z \in Z$ in the unit sphere of $Z$ satisfying $g(z) \geq \delta + 1/n_0$ and $f(z) \in \tilde{C}(w, \alpha - 1/n_0)$ for some $n_0$. Let $\mathcal{X}$ denote the closed convex hull of $\text{spt} \mu \cup (\text{spt } \mu + z)$ in $Z$ and note that $\mathcal{X}$ is compact. For $n \in \mathbb{N}$ let $G_n$ denote the compact set of all $(\delta, 1)$-biLipschitz maps $\gamma : [0, 1] \rightarrow \mathcal{X}$ satisfying:

\begin{align}
\text{sgn}(t - s) (f \circ \gamma(t) - f \circ \gamma(s)) &\in \tilde{C}(w, \alpha - 1/n) \\
\text{sgn}(t - s) (g \circ \gamma(t) - g \circ \gamma(s)) &\geq (\delta + 1/n)|t - s|.
\end{align}

Applying Lemma 2.56 in [Sch13] repeatedly, we obtain a decomposition $\mu = \mu' + \mu L F$ where $\mu'$ has an Alberti representation of the desired form and $F \subset \text{spt } \mu$ is an $F_{\alpha, \delta}$ which is $G_n$-null for every $n$.

We now show that for each fragment $\gamma \in \text{Frag}(\text{spt } \mu)$ in the $f$-direction of $C(w, \alpha)$ and with $g$-speed $> \delta$, the set $F = \mathcal{C}_1$ is $\mathcal{N}_{\gamma}$-null; by (1), this will imply that $\mu(F) = 0$.

Let $\gamma$ be such a fragment and assume that it is $L$-Lipschitz. Note that, if we find countably many compact sets $K_n \subset \text{dom } \gamma$ with $\mathcal{N}_{\gamma} K_n(F) = 0$, then $\mathcal{N}_{\gamma} F = 0$. This allows to use Egorov and Lusin’s Theorems to simplify the discussion. In particular, because $\mathcal{X}$ is convex and because the functions $f$ and $g$ are linear, we can use the argument used to prove Theorem 2.64 in [Sch13] to reduce to the case in which, for some $\rho > 0$, the fragment $\gamma$ extends to an $(L + \rho)$-Lipschitz map $\tilde{\gamma} : I_\rho \rightarrow Z$, where $I_\gamma$ denotes the minimal interval containing $\text{dom } \gamma$, such that for some $n_1 \in \mathbb{N}$, the fragment $\tilde{\gamma}$ is in the $f$-direction of $\tilde{C}(w, \alpha - 1/n_1)$ with $g$-speed $\geq \delta + 1/n_1$. By precomposing $\tilde{\gamma}$ with an affine map and dividing $I_\gamma$ into smaller
subintervals, we can reduce to the case in which $\hat{\gamma}$ is 1-Lipschitz and $I_{\gamma} \subset [0,1]$. Letting $t_0$ denote the right extremum of $I_{\gamma}$, we extend $\hat{\gamma}$ to $[t_0,1]$ by letting $\hat{\gamma}(t) \in I_{\gamma}$. Note that $md \hat{\gamma} \leq 1$ and, letting $n_2 = \max(n_0,n_1)$, we have $(g \circ \hat{\gamma})' \geq \delta + 1/n_2$ and $(f \circ \hat{\gamma}) \in \bar{C}(w,\alpha-1/n_2)$. In particular, $\hat{\gamma} \in G_{n_2}$ which implies $H^1(\hat{\gamma})(F) = 0$ and then $H^1(\gamma)(F) = 0$.

The case in which $\text{spt} \mu$ is not compact and the functions $w, \alpha$ and $\delta$ are not constant, is treated by using Egorov and Lusin’s Theorems like in the last part of the proof of Theorem 2.64 in [Sch13].

\section*{7.3. Renorming}

The goal of this Subsection is the proof of the following result about renorming the module $X(\mu)$ by taking a biLipschitz deformation of the metric on $X$.

\begin{theorem}
Let $(X,d)$ be a Polish space and $\mu$ a Radon measure on $X$. For each $\varepsilon > 0$ there is a metric $d^{(\varepsilon)}$ which satisfies
\begin{equation}
d \leq d^{(\varepsilon)} \leq (1 + \varepsilon)d
\end{equation}
and such that the corresponding local norm $\cdot |_{X(\mu),\text{loc}}$ is strictly convex.
\end{theorem}

We now fix some notation that will be used throughout this Subsection. We let $\{\psi_n\}$ be a countable generating set for the Lipschitz algebra Lip$^b(X)$ and we assume that each function $\psi_n$ is 1-Lipschitz. We then introduce the pseudometrics
\begin{equation}
\Psi(x,y) = \left\| \left( \frac{\psi_n(x) - \psi_n(y)}{n} \right)_n \right\|^2
\end{equation}
and
\begin{equation}
\Psi_M(x,y) = \left( \sum_{n=1}^{M} \left( \frac{\psi_n(x) - \psi_n(y)}{n^2} \right)^2 \right)^{1/2},
\end{equation}
and observe that $\Psi_M \leq \Psi \leq \frac{\pi}{\sqrt{6}}d$. We also define functions
\begin{equation}
\Phi : X \to l^2
\end{equation}
\begin{equation*}
x \mapsto \left( \frac{\psi_n(x)}{n} \right)_n
\end{equation*}
and
\begin{equation}
\Phi_M : X \to \mathbb{R}^M
\end{equation}
\begin{equation*}
x \mapsto \left( \frac{\psi_n(x)}{n} \right)_{n=1}^{M},
\end{equation*}
and observe that $\Phi$ and $\Phi_M$ are $\frac{\pi}{\sqrt{6}}$-Lipschitz with respect to the distance $d$. We finally let
\begin{equation}
d^{(\varepsilon)} = d + \varepsilon \Psi
\end{equation}
so that
\begin{equation}
d \leq d^{(\varepsilon)} \leq \left( 1 + \varepsilon \frac{\pi}{\sqrt{6}} \right)d.
\end{equation}
Note that, given a derivation $D$, after choosing a Borel representative for each $D\psi_n$, we obtain Borel maps $^9$

$$D\Phi : X \to l^2$$

$$x \mapsto \left( \frac{D\psi_n(x)}{n} \right)_n,$$

and

$$D\Phi_M : X \to \mathbb{R}^M$$

$$x \mapsto \left( \frac{D\psi_n(x)}{n} \right)^M_{n=1}.$$

We will now prove that the local norm $| \cdot |^{(c)}_{X(\mu), \text{loc}}$ corresponding to the distance $d^{(c)}$ is strictly convex. We start with the following Lemma, which is essentially folklore and whose proof is included for completeness.

**Lemma 7.111.** If $g \in C^1(\mathbb{R}^k)$ and the functions $\{\psi_i\}_{i=1}^k$ are in $\text{Lip}_b(X)$, then for any derivation $D \in X(\mu)$ it follows that

$$Dg(\psi_1, \cdots, \psi_k) = \sum_{i=1}^k \frac{\partial g}{\partial y^i}(\psi_1, \cdots, \psi_k)D\psi_i.$$

**Proof of Lemma 7.111.** The idea of the proof is essentially based on [AK00, Thm. 3.5(i)].

As the functions $\{\psi_i\}_{i=1}^k$ are bounded, letting $\psi : X \to \mathbb{R}^k$ the Lipschitz function whose $i$-th component is $\psi_i$, there is a $k$-dimensional simplex $S^{10}$ centred about the origin such that $\psi(X)$ lies in the interior of $S$. Using that $g \in C^1(\mathbb{R}^k)$, it is possible to construct Lipschitz functions $g_n : S \to \mathbb{R}$ such that:

1. there is $M_n \in \mathbb{N}$ such that, if $S^{M_n}$ denotes the $M_n$-th iterated barycentric subdivision of $S$, the function $g_n$ is affine linear on each simplex $\Delta \in S^{M_n}$:

$$g_n(v) = \langle V_n, \Delta, v \rangle + c_{n,\Delta} \quad (v \in \Delta).$$

2. For each simplex $\Delta \in S^{M_n}$ one has

$$\sup_{v \in \Delta} |g(v) - g_n(v)| \leq \frac{1}{n}$$

$$\sup_{v \in \Delta} \|V_n - \nabla g(v)\|_2 \leq \frac{1}{n}.$$  

We now let

$$f(x) = g(\psi_1(x), \cdots, \psi_k(x))$$

$$f_n(x) = g_n(\psi_1(x), \cdots, \psi_k(x)),$$

and observe that as $f_n|\psi^{-1}(\Delta)$ agrees with the function

$$x \mapsto \langle V_n, \Delta, \psi(x) \rangle + c_{n,\Delta},$$

the locality property of derivations implies that

$$Df_n(x) = \langle V_n, \Delta, D\psi(x) \rangle$$

for $\mu \psi^{-1}(\Delta)$-a.e. $x$. As $f_n \overset{w^*}{\to} f$, (7.112) follows from (7.119) and (7.115).  \[\square\]

$^9$The Borel $\sigma$-algebras for the strong and the weak topologies on $l^2$ coincide.

$^{10}$we take simplices to be closed
The following Lemma is a key step in the proof of Theorem 7.101.

**Lemma 7.120.** Let \( F : X \to \mathbb{R}^M \) be Lipschitz, \( D \in \mathcal{X}(\mu) \) and \( \theta : X \to (0, \pi/2) \) a Borel map. Let

\[
V_F = \{ x : DF(x) \neq 0 \};
\]

then \( \mu \ll V_F \) admits an Alberti representation in the \( F \)-direction of \( C \left( \frac{DF}{\|DF\|_2}, \theta \right) \).

**Proof of Lemma 7.120.** The proof is essentially based on the argument used in Lemma 3.97 in [Sch13] and details are included for completeness. We consider a Borel \( L^\infty(\mu \ll V_F) \)-partition of unity \( \{ V^{(0)}_l \}_{l \in \mathbb{N}} \) such that, for each \( l \), there is a pair \((s_l, \theta_l) \subset (0, \infty) \times (0, \pi/2) \) with:

\[
|D|_{\mathcal{X}(\mu \ll V_F), \text{loc}}(x) \in (s_l, 2s_l) \quad (\forall x \in V^{(0)}_l);
\]

\[
\theta(x) \in (\theta_l, 2\theta_l) \quad (\forall x \in V^{(0)}_l);
\]

we further subdivide the \( \{ V^{(0)}_l \}_{l \in \mathbb{N}} \) to obtain a Borel \( L^\infty(\mu \ll V_F) \)-partition of unity \( \{ V^{(1)}_l \}_{l \in \mathbb{N}} \) such that, for each \( l \), (7.122) and (7.123) hold and there are \( c_l > 0 \) and \( \varepsilon^{(1)}_l \in (0, c_l/2) \) such that:

\[
\|DF(x)\|_2 \in (c_l, c_l + \varepsilon^{(1)}_l) \quad (\forall x \in V^{(1)}_l);
\]

note that the values of each \( \varepsilon^{(1)}_l \) will be chosen later depending on the corresponding values of \( s_l \) and \( \theta_l \) which were obtained in the previous step. We finally subdivide the \( \{ V^{(1)}_l \}_{l \in \mathbb{N}} \) to obtain a Borel \( L^\infty(\mu \ll V_F) \)-partition of unity \( \{ V^{(2)}_l \}_{l \in \mathbb{N}} \) such that, for each \( l \), (7.122), (7.123) and (7.124) hold and there are \( w_l \in S^{M-1} \) and \( \varepsilon^{(2)}_l \in (0, \varepsilon^{(1)}_l) \) such that:

\[
C(w_l, \theta_l/2) \subset C \left( \frac{DF(x)}{\|DF(x)\|_2}, \theta_l \right) \quad (\forall x \in V^{(2)}_l);
\]

\[
\left\| \frac{DF(x)}{\|DF(x)\|_2} - w_l \right\|_2 \leq \varepsilon^{(2)}_l \quad (\forall x \in V^{(2)}_l);
\]

note that the values of each \( \varepsilon^{(2)}_l \) will be chosen later depending on the corresponding values of \( s_l, \theta_l, c_l \) and \( \varepsilon^{(1)}_l \) which were obtained in the previous steps. We now estimate the error in approximating \( DF \) by \( c_l w_l \) on \( V^{(2)}_l \):

\[
\left\| DF - c_l w_l \right\|_2 \leq \left\| DF - \|DF\|_2 w_l + \left\| DF \right\|_2 w_l - c_l w_l \right\|_2 \\
\leq \left\| DF \right\|_2 \left\| \frac{DF(x)}{\|DF(x)\|_2} - w_l \right\|_2 + \left\| DF \right\|_2 - c_l \\
\leq \left( \frac{c_l}{\|DF\|_2} + \varepsilon^{(2)}_l \right) \varepsilon^{(1)}_l.
\]

In particular, if \( u \) is a unit vector orthogonal to \( w_l \),

\[
\chi_{V^{(2)}_l} |D(u,F)| = \chi_{V^{(2)}_l} \left| \langle u, DF - w_l c_l \rangle \right| \leq \frac{\eta_l}{s_l} \left| D|_{\mathcal{X}(\mu), \text{loc}} \right|,
\]
We now suppose that the Borel set $S_l \subset V_l^{(2)}$ is $\text{Frag}(X, F, \delta_l, w_l, \theta_l/2)$-null: using (7.128) and Lemma 2.73 (compare also Lemma 3.68 and Lemma 3.76 in [Sch13] for details) we obtain

(7.129) \[ \chi_{S_l} |D(w_l, F)| \leq \left( \delta_l + (M - 1) \frac{2\theta_l}{s_l} \cot(\theta_l/2) \right) |D|_{X(\mu),\text{loc}}; \]
onumber

on the other hand, we have

(7.130) \[ \chi_{V_l^{(2)}} D(w_l, F) \geq \chi_{V_l^{(2)}} (c_l - \eta_l). \]

In particular, if $\mu(S_l) > 0$ we have

(7.131) \[ \delta_l \geq \frac{c_l - \eta_l}{2s_l} - (M - 1) \frac{\eta_l}{s_l} \cot(\theta_l/2); \]

this implies that $\mu_{\text{LV}_l^{(2)}}$ admits an Alberti representation $\mathcal{A}_l$ in the $F$-direction of $\mathcal{C}(w_l, \theta_l/2)$ with $F$-speed

(7.132) \[ \geq \delta_l = \frac{c_l - 2\eta_l}{2s_l} - (M - 1) \frac{\eta_l}{s_l} \cot(\theta_l/2), \]

provided that $\delta_l$ is positive. Note that

(7.133) \[ \delta_l = \frac{1}{2s_l} \left( c_l - 2(c_l + \varepsilon_l^{(1)}) \varepsilon_l^{(2)} - 2\varepsilon_l^{(1)} - (M - 1) (c_l + \varepsilon_l^{(1)}) \varepsilon_l^{(2)} + \varepsilon_l^{(1)} \right) \cot(\theta_l/2); \]

if at each step the $\varepsilon_l^{(1)}$ and $\varepsilon_l^{(2)}$ are chosen sufficiently small, one can ensure that $\delta_l > 0$. The proof is completed by gluing together the $\{A_i\}$ (Theorem 2.32) and using (7.125).

**Lemma 7.134.** The local norms $|\cdot|_{X(\mu),\text{loc}}$ and $|\cdot|_{X(\mu),\text{loc}}^{(c)}$ are related by the following equation:

(7.135) \[ |D|_{X(\mu),\text{loc}}^{(c)} = |D|_{X(\mu),\text{loc}} + \varepsilon \|D\Phi\|_2 \quad (\forall D \in X(\mu)). \]

**Proof of Lemma 7.134.** We first show that

(7.136) \[ |D|_{X(\mu),\text{loc}}^{(c)} \leq |D|_{X(\mu),\text{loc}} + \varepsilon \|D\Phi\|_2 \]

by showing that, for each $x \in X$, the distance function $d^{(c)}(x, \cdot)$ satisfies

(7.137) \[ |Dd^{(c)}(x, \cdot)| \leq |D|_{X(\mu),\text{loc}} + \varepsilon \|D\Phi\|_2. \]

Without loss of generality, we can assume that $X$ is bounded. Let $d^{(c)}_M = d + \varepsilon \Psi_M$ and observe that the sequence of Lipschitz functions $\{d^{(c)}_M(x, \cdot)\}_{M \in \mathbb{N}}$ converges to $d^{(c)}(x, \cdot)$, in the weak* topology, as $M \to \infty$. As $d(x, \cdot)$ is 1-Lipschitz with respect to $d$, we have:

(7.138) \[ |Dd(x, \cdot)| \leq |D|_{X(\mu),\text{loc}}. \]

On the closed set $C_0 = \{y : \Psi_M(x, y) = 0\}$, one has $D\Psi_M(x, \cdot) = 0$ by locality of derivations. For $\delta > 0$ consider the closed set $C_\delta = \{y : \Psi_M(x, y) \geq \delta\}$. We can find a function $g : \mathbb{R}^M \to (0, \infty)$ of class $C^1(\mathbb{R}^M)$ such that, if for a $v \in \mathbb{R}^M$ one has

(7.139) \[ \left( \sum_{n=1}^M \frac{|v_n|^2}{n^2} \right)^{1/2} \geq \frac{\delta}{2}, \]
then
\[
g(v) = \left( \sum_{n=1}^{M} \frac{|v_n|^2}{n^2} \right)^{1/2}.
\]
In particular, on \(C_\delta\), the function \(\Psi_M(x, \cdot)\) coincides with
\[
g(\psi_1(\cdot) - \psi_1(x), \ldots, \psi_M(\cdot) - \psi_M(x)),
\]
and Lemma 7.111 gives
\[
D\Psi_M(x, y) = \frac{1}{\Psi_M(x, y)} \sum_{n=1}^{M} \frac{\psi_n(y) - \psi_n(x)}{n} D\psi_n(y) \frac{1}{n}
\]
for \(\mu C_\delta\)-a.e. \(y\). Using the Cauchy inequality and a sequence \(\delta_n \searrow 0\), we conclude that
\[
|D\Psi_M(x, \cdot)| \leq \|D\Phi_M\|_2.
\]
Combining (7.138) and (7.143) we obtain (7.137) and so (7.136) is proved.

We now show that
\[
|D(\varepsilon)_{\alpha}| \geq |D|_{\alpha, \text{loc}} + \varepsilon \|D\Phi\|_2,
\]
and we will assume that a Borel representative has been chosen for each \(D\psi_n\). We first consider the Borel set \(V_0\) where \(\|D\Phi\|_2 = 0\). Having fixed \(\eta > 0\), we take a Borel \(L^\infty(\mu \mathcal{L}V_0)\)-partition of unity \{"\(U_\alpha\)\} such that, for each \(\alpha\), there is a function \(f_\alpha\) which is \(1\)-Lipschitz with respect to the distance \(d\) and satisfying:
\[
\chi_{U_\alpha} Df_\alpha \geq (1 - \eta) \chi_{U_\alpha} |D|_{\alpha, \text{loc}};
\]
this implies that
\[
\chi_{V_0} |D(\varepsilon)_{\alpha}| \geq (1 - \eta) \chi_{V_0} |D|_{\alpha, \text{loc}}.
\]
We now consider the Borel set \(V_1\) where \(\|D\Phi\|_2 > 0\). For each \(\eta > 0\), we take an \(L^\infty(\mu \mathcal{L}V_1)\)-partition of unity \{"\(U_\alpha\)\}, where each set \(U_\alpha\) is compact and such that for each \(\alpha\) there is a quadruple \((f_\alpha, M_\alpha, \theta_\alpha, \delta_\alpha)\) satisfying:

**(P1):** The function \(f_\alpha\) is \(1\)-Lipschitz with respect to the distance \(d\), \(M_\alpha\) is a natural number, \(\theta_\alpha \in (0, \pi/2)\), and \(\delta_\alpha > 0\).

**(P2):** The following inequality holds
\[
\chi_{U_\alpha} Df_\alpha \geq (1 - \eta) \chi_{U_\alpha} |D|_{\alpha, \text{loc}}.
\]

**(P3):** The Borel functions \(\|D\Phi\|_2\) and \(\|D\Phi_{M_\alpha}\|_2\) are continuous on \(U_\alpha\) and satisfy
\[
\|D\Phi_{M_\alpha}\|_2 \geq (1 - \eta) \|D\Phi\|_2 \geq \delta_\alpha > 0.
\]

**(P4):** For all \(x, y \in U_\alpha\), if \(u \in C \left( \frac{D\Phi_{M_\alpha}(x)}{\|D\Phi_{M_\alpha}\|_2}, 2\theta_\alpha \right) \cap S^{M_\alpha-1}\), then
\[
\langle u, D\Phi_{M_\alpha}(y) \rangle \geq (1 - \eta) \|D\Phi_{M_\alpha}(y)\|_2.
\]
By Lemma 7.120 the measure \(\mu U_\alpha\) admits an Alberti representation in the \(f_\alpha\)-direction of the cone field \(C \left( \frac{D\Phi_{M_\alpha}}{\|D\Phi_{M_\alpha}\|_2}, \theta_\alpha \right)\); in particular, for \(\mu U_\alpha\)-a.e. \(x\), there is a fragment \(\gamma_x \in \text{Frag}(U_\alpha)\) such that:

1. \(0\) is a Lebesgue density point of \(\text{dom} \gamma_x\) and \(\gamma_x(0) = x\).
(2) There is a \( v_x \in C \left( \frac{D\Phi_{\alpha_n}(x)}{\|D\Phi_{\alpha_n}(x)\|_2}, \theta_{\alpha} \right) \) with
\[
\Phi_{\alpha_n}(\gamma(r)) = \Phi_{\alpha_n}(x) + v_x r + o(r).
\]
In particular, there are \( r_x, R_x > 0 \) such that for each \( y \in B(x, R_x) \cap U_\alpha \), one has
\[
\frac{\Phi_{\alpha_n}(\gamma_x(r_x)) - \Phi_{\alpha_n}(y)}{\|\Phi_{\alpha_n}(\gamma_x(r_x)) - \Phi_{\alpha_n}(y)\|_2} \in C \left( \frac{D\Phi_{\alpha_n}(x)}{\|D\Phi_{\alpha_n}(x)\|_2}, 2\theta_{\alpha} \right).
\] 
Let
\[
\tilde{f}_\alpha = f_\alpha - \varepsilon \Psi_{\alpha_n}(\gamma_x(r_x), \cdot),
\]
and observe that \( \tilde{f}_\alpha \) is 1-Lipschitz with respect to the distance \( d(\varepsilon) \) and that
\[
D\tilde{f}_\alpha = Df_\alpha - \varepsilon D\Psi_{\alpha_n}(\gamma_x(r_x), \cdot);
\]
an argument similar to that used to prove (7.142) shows that for \( \mu(\cup(U_\alpha \cap B(x, R_x))) \)-a.e. \( y \),
\[
D\Psi_{\alpha_n}(\gamma_x(r_x), y) = -\frac{\langle \Phi_{\alpha_n}(\gamma_x(r_x)) - \Phi_{\alpha_n}(y), D\Phi_{\alpha_n}(y) \rangle}{\|\Phi_{\alpha_n}(\gamma_x(r_x)) - \Phi_{\alpha_n}(y)\|_2} \leq -(1 - \eta) \|D\Phi_{\alpha_n}\|_2,
\]
where in the last step we used (7.151) and (P4). Combining (7.154) with (P2) we obtain
\[
\chi_{U_\alpha} D\tilde{f}_\alpha \geq (1 - \eta) \chi_{U_\alpha} |D|_{\chi(\mu), \text{loc}} + \varepsilon(1 - \eta)^2 \chi_{U_\alpha} \|D\Phi\|_2,
\]
which implies
\[
\chi_{V_1} |D|_{\chi(\mu), \text{loc}} \geq (1 - \eta) \chi_{V_1} |D|_{\chi(\mu), \text{loc}} + \varepsilon(1 - \eta)^2 \chi_{V_1} \|D\Phi\|_2;
\]
letting \( \eta \rightarrow 0 \) in (7.156) and (7.147), (7.144) follows. 

**Proof of Theorem 7.101.** Because of (7.108), we just need to show that the local norm \( |\cdot|_{\chi(\mu), \text{loc}} \) associated to \( d(\varepsilon) \) is strictly convex. Consider derivations \( D_1, D_2 \in \chi(\mu) \) and suppose that for \( \mu(U) \) a.e. \( x \in U \) one has:
\[
|D_1 + D_2|_{\chi(\mu), \text{loc}}(x) = |D_1|_{\chi(\mu), \text{loc}}(x) + |D_2|_{\chi(\mu), \text{loc}}(x);
\]
by Lemma 7.134 we have
\[
|D_1 + D_2|_{\chi(\mu), \text{loc}}(x) + \varepsilon \|D_1 \Phi(x) + D_2 \Phi(x)\|_2 = |D_1|_{\chi(\mu), \text{loc}}(x) + \varepsilon \|D_1 \Phi(x)\|_2
\]
\[
+ |D_2|_{\chi(\mu), \text{loc}}(x) + \varepsilon \|D_2 \Phi(x)\|_2;
\]
because
\[
|D_1 + D_2|_{\chi(\mu), \text{loc}} \leq |D_1|_{\chi(\mu), \text{loc}} + |D_2|_{\chi(\mu), \text{loc}}
\]
\[
\|D_1 \Phi + D_2 \Phi\|_2 \leq \|D_1 \Phi\|_2 + \|D_2 \Phi\|_2,
\]
after choosing Borel representatives of \( D_1 \Phi \) and \( D_2 \Phi \), we find a Borel \( V \subset U \) with \( \mu(U \setminus V) = 0 \) and such that:
\[
\|D_1 \Phi(x) + D_2 \Phi(x)\|_2 = \|D_1 \Phi(x)\|_2 + \|D_2 \Phi(x)\|_2 \quad (\forall x \in V).
\]

\[11\]the ball can be taken either with respect to \( d \) or \( d(\varepsilon) \).
The strict convexity of the norm on $l^2$ implies that for each $x \in V$ the vectors $D_1 \Phi(x)$ and $D_2 \Phi(x)$ are linearly dependent. Let

$$\tilde{V}_1 = \{ (x, \lambda) \in V \times [-1,1] : D_1 \Phi(x) = \lambda D_2 \Phi(x) \}$$

(7.162)

$$\tilde{V}_2 = \{ (x, \lambda) \in V \times [-1,1] : D_2 \Phi(x) = \lambda D_1 \Phi(x) \} ;$$

(7.163)

then $\tilde{V}_1$ and $\tilde{V}_2$ are Borel subsets of $X \times [-1,1]$ and, denoting by $V_i$ the projection of $\tilde{V}_i$ on $X$, we have $V = V_1 \cup V_2$. Note that for each $x$ the section $(\tilde{V}_1)_x$ is compact; in particular, by the Lusin-Novikov Uniformization Theorem [Kec95, Thm. 18.10] 18.10, the sets $V_1$ and $V_2$ are Borel and admit Borel uniformizing functions $\sigma_i : V_i \rightarrow [-1,1]$. In particular,

$$\chi_{V_1} D_1 \Phi = \sigma_1 \chi_{V_1} D_2 \Phi$$

(7.164)

$$\chi_{V_2} D_2 \Phi = \sigma_2 \chi_{V_2} D_1 \Phi ;$$

(7.165)

as the $\{\psi_n\}$ generate $\text{Lip}_b(X)$, (7.164) and (7.165) imply that $\chi_U D_1$ and $\chi_U D_2$, regarded as elements of $X(\mu \cup U)$, are linearly dependent. □

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