Sliding signal processing in telecommunication networks based on two-dimensional discrete Fourier transform

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Abstract. A method of vertical sliding processing of two-dimensional discrete signals in the spatial frequency domain is proposed — a method of fast vertically sliding two-dimensional discrete Fourier transform. The mathematical representation of the two-dimensional discrete Fourier transform in algebraic and matrix form is considered. An effective method of vertically sliding two-dimensional discrete Fourier transform is proposed. The algorithm developed in the framework of the proposed method allows calculating the coefficients (bins) of this transformation in real time.

1 Introduction

It is difficult to overestimate the role and place of digital spectral processing of discrete one-dimensional (1-D) and two-dimensional (2-D) signals in telecommunication networks (telephone, computer, television networks and radio networks). The importance and relevance of the development and improvement of the methods of digital spectral processing of 1-D and 2-D signals is increasing due to the creation of telecommunication universal multiservice networks that can equally effectively transmit any type of information: data, sound and video.

The classical method of spatial-frequency processing of two-dimensional discrete signals is the 1-D and 2-D discrete Fourier transform (DFT), which allows to obtain a 1-D and 2-D frequency spectrum [1-13]. At the same time, there are a number of applications [9-12] where it is necessary to find the values of the frequency spectrum not at all 1-D and 2-D frequencies, but at a subset of them. In this case, the application of the full version of 1-D and 2-D DFT, even on the basis of the fast Fourier transform (FFT), becomes ineffective, since most of the obtained 1-D and 2-D DFT coefficients are not used. The solution to this problem for 1-D signals is given in [9].

The article discusses the solution of this problem for 2-D signals, introduces the concept of moving spatial-frequency processing based on 2-D DFT. Specifically, the vertical spatial-frequency processing of 2-D signals is considered.

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2 Direct two-dimensional discrete Fourier transform

Suppose we are given a discrete two-dimensional signal $x(n_1,n_2)$ in the form of a two-dimensional sequence of finite length $0 \leq n_1 \leq (N_1 - 1)$ and $0 \leq n_2 \leq (N_2 - 1)$ or with a matrix of size $n_1 \times n_2$ in a rectangular reference zone (specifically on a plane).

The direct two-dimensional discrete Fourier transform (2-D DFT) of a two-dimensional signal $x(n_1,n_2)$ is a special case of a direct two-dimensional z-transform:

$$S_{N_1,N_2}(k_1,k_2) = X(z_1,z_2) \bigg|_{z_1=W_{N_1}^{n_1}, \ z_2=W_{N_2}^{n_2}};$$  \hspace{1cm} (1)

and can be specified both in algebraic and in matrix form.

**Algebraic form:**

$$S_{N_1,N_2}(k_1,k_2) = \frac{1}{N_1 \cdot N_2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1,n_2) \exp \left[ -j 2\pi \left( \frac{k_1 n_1}{N_1} + \frac{k_2 n_2}{N_2} \right) \right] = \frac{1}{N_1 \cdot N_2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1,n_2) \cdot W_{N_1}^{k_1 n_1} \cdot W_{N_2}^{k_2 n_2};$$  \hspace{1cm} (2)

where $k_1 = 0, (N_1 - 1)$, $k_2 = 0, (N_2 - 1)$ are spatial frequencies; $x(n_1,n_2)$ – two-dimensional signal; $n_1 = 0, N_1 - 1$, $n_2 = 0, N_2 - 1$; $W_{N_2}^{k_2 n_2} = \exp(-j \frac{2\pi}{N_2} (k_2 n_2))$; $S_{N_1,N_2}(k_1,k_2)$ – coefficients (bins) of 2-D DFT (two-dimensional vector spatial-frequency spectrum).

**Matrix form:**

$$S_{N_1 \times N_2} = \frac{1}{N_1 \cdot N_2} F_{N_1 \times N_1}^{(2)} \cdot X_{N_1 \times N_2} \cdot F_{N_2 \times N_2}^{(1)};$$  \hspace{1cm} (3)

where

$$X_{N_1 \times N_2} = \begin{bmatrix} 0 & 1 & \cdots & (N_2 - 1) \\ 0 & x(0,0) & x(0,1) & \cdots & x(0,N_2 - 1) \\ 1 & x(1,0) & x(1,1) & \cdots & x(1,N_2 - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (N_1 - 1) & x(N_1 - 1,0) & x(N_1 - 1,1) & \cdots & x(N_1 - 1,N_2 - 1) \end{bmatrix};$$  \hspace{1cm} (4)

and

$$F_{N_2 \times N_2}^{(1)} = \begin{bmatrix} 0 & 1 & \cdots & (N_2 - 1) \\ 0 & W_{N_2}^{00} & W_{N_2}^{01} & \cdots & W_{N_2}^{0(N_2 - 1)} \\ 1 & W_{N_2}^{10} & W_{N_2}^{11} & \cdots & W_{N_2}^{1(N_2 - 1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (N_2 - 1) & W_{N_2}^{(N_2 - 1)0} & W_{N_2}^{(N_2 - 1)1} & \cdots & W_{N_2}^{(N_2 - 1)(N_2 - 1)} \end{bmatrix};$$  \hspace{1cm} (5)
Hereinafter there is no loss of generality in omitting the multiplier $1/(N_1 \cdot N_2)$ in (3).

Due to the fact that for the product of the matrices (3) holds the associative property:

$$S_{N_1 \times N_2} = F_{N_1 \times N_1}^{(2)} \cdot X_{N_1 \times N_2} \cdot F_{N_2 \times N_2}^{(1)} = F_{N_1 \times N_2}^{(2)} \cdot X_{N_1 \times N_2} \cdot F_{N_2 \times N_2}^{(1)}$$

and the 2-D DFT core is separable, then, according to (7), you can get the 2-D DFT bins $S_{N_1 \times N_2}$, $k_1 = 0, (N_1 - 1)$, $k_2 = 0, (N_2 - 1)$, in two ways, each consists of two stages.

It is easy to find out that for obtaining $S_{N_1 \times N_2}(k_1, k_2)$, $k_1 = 0, (N_1 - 1)$, $k_2 = 0, (N_2 - 1)$ it is necessary to perform $N_1 \cdot N_2$ one-dimensional 2-D DFTs, for calculating which the algorithms of the fast Fourier transform - FFT can be effectively applied.

2. Sliding spatial-frequency processing of discrete signals

Let us consider the spatial-frequency processing of two-dimensional discrete signals in a sliding spatial analysis window. In contrast to the one-dimensional case for the two-dimensional case, there are 4 possible ways of sliding of the spatial analysis window on the original two-dimensional discrete signal:

- 1\textsuperscript{st} way consists of horizontal right shift (HS +) and horizontal left shift (HS –);
- 2\textsuperscript{nd} way consists of vertical shift up (VS +) and vertical shift down (VS –);
- 3\textsuperscript{rd} way consists of right diagonal shift up (RDS +) and right diagonal shift down (RDS –);
- 4\textsuperscript{th} way consists of left diagonal shift up (LDS +) and left diagonal shift down (LDS –).

Fig. 1 shows a star diagram illustrating 4 types of sliding of a spatial analysis window using a discrete two-dimensional signal and examples of its shift.

![Fig. 1. The star chart illustrating 4 kinds of sliding of a spatial window.](image-url)
Suppose we need to find the coefficient (bin) of a two-dimensional discrete transform $S_{N_1,N_2}(k_1,k_2)$ at the spatial frequency $(k_1,k_2)$ from the samples of the input signal $x(n_1,n_2)$. In this case, the matrix equation (3) is converted into the form:

$$S(k_1,k_2) = \left[W_{N_1}^{k_10}, W_{N_1}^{k_11} \ldots W_{N_1}^{k_1(N-1)}\right] \cdot X_{N_1 \times N_2} \cdot \left\{W_{N_2}^{0k_2}, W_{N_2}^{1k_2} \ldots W_{N_2}^{(N-1)k_2}\right\}. \quad (8)$$

At the first stage, according to (8), we multiply the basis function of frequency $k_2$ and duration $N_2$ by a matrix of a discrete two-dimensional signal $x(n_1,n_2)$. As a result, we obtain a columned matrix $S_{N_2}(n_1,k_2)$ of size $N_2$, having spent on this procedure $N_2 \cdot N_1$ complex multiplications and $(N_2 - 1) \cdot N_1$ complex additions. Further, at the second stage, we multiply the basis function of frequency $k_1$ and duration $N_1$ by a columned matrix of size obtained at the first stage, having spent on this procedure $N_1$ complex multiplications and $(N_1 - 1)$ complex additions.

Thus, it is necessary to expend $N_2 \cdot (N_1 + 1)$ complex multiplications and $(N_2 - 1) \cdot (N_1 + 1)$ complex additions to obtain one coefficient of a two-dimensional discrete transform $S_{N_1,N_2}(k_1,k_2)$ at spatial frequency $(k_1,k_2)$. Considering that performing one complex multiplication requires four real multiplications and two real additions, and one complex addition of two real additions, it is necessary to spend $4 \cdot N_1 \cdot (N_2 + 1)$ real multiplications and $4 \cdot N_1 N_2 + 2 \cdot (N_1 - 1)$ real additions to obtain the value of one coefficient of a two-dimensional discrete transformation $S_{N_1,N_2}(k_1,k_2)$.

Note that this amount of computation needs to be performed at each shift of a two-dimensional spatial analysis window using a two-dimensional signal. At the same time, it is easy to see that for any kind of shift of a two-dimensional signal a large number of values $X_{N_1 \times N_2}$ of the complex matrix in the spatial analysis window remains unchanged.

Note that the shift of the spatial window along a two-dimensional discrete signal can be considered as a shift of a two-dimensional discrete signal in the spatial analysis window in the opposite direction to the movement of the spatial window.

3. The algorithm of vertical sliding processing of 2-D signals based on 2-D DFT

Let us need to find one coefficient (bin) of a two-dimensional discrete transformation $S_{N_1,N_2}(k_1,k_2)$ at a spatial frequency $(m_1,m_2)$ from the samples of the input signal $x(n_1,n_2)$. To obtain a bin, matrix equation (8) is converted to the form:

$$S_{N_1,N_2}(m_1,m_2) = \left[W_{N_1}^{m_10}, W_{N_1}^{m_11} \ldots W_{N_1}^{m_1(N-1)}\right] \cdot X_{N_1 \times N_2} \cdot \left\{W_{N_2}^{m_20}, W_{N_2}^{m_21} \ldots W_{N_2}^{m_2(N-1)}\right\}^T. \quad (9)$$

where $^T$ is the transpose symbol.
The sequence of operations in the algorithm of vertical sliding processing of 2-D signals:

1. Find the column matrix \( S_{N_2}(n_1, m_2) \) by size \( N_1 \), by multiplying the basis function duration \( N_2 \) and frequency \( m_2 \) on a matrix of a discrete two-dimensional signal \( x(n_1, n_2) \).

2. Remember the column matrix \( S_{N_2}(n_1, m_2) \) as a column matrix \( S_{N_2}^{(n_1, m_2)}(0) \).

3. Calculate the 2-D coefficient of the discrete transform \( S_{N_1, N_2}(m_1, m_2) \) by multiplying the column matrix \( S_{N_2}(n_1, m_2) \) by a basis function of duration \( N_1 \) and frequency \( m_1 \). This stage ends the output of the algorithm to the operating mode.

4. Carry out the VS–shift of the discrete spatial window by one sample down the two-dimensional signal \( x(n_1, n_2) \) and obtain the matrix of the discrete two-dimensional signal \( x[(n_1-1), n_2] \).

5. Form a column matrix according to the ratio;
\[
S_{N_2}^{(n_1, m_2)}(-1) = S_{N_2}^{((n_1+1), m_2)}(0); \quad n_1 = 0, N_1 - 2
\]

6. Calculate the coefficient value of a two-dimensional discrete transformation \( S_{N_1, N_2}(m_1, m_2) \) by multiplying the column matrix \( S_{N_2}^{(n_1, m_2)}(-1) \) by a basis function of duration \( N_1 \) and frequency \( m_1 \).

7. Go to the implementation of paragraph No. 4.

4. Conclusion

The effectiveness of the proposed algorithm for vertical sliding processing of two-dimensional discrete signals in the spatial frequency domain in comparison with the standard method for obtaining the coefficient of two-dimensional discrete transformation \( S_{N_1, N_2}(m_1, m_2) \):

- the standard algorithm (8) of the sliding processing of two-dimensional discrete signals in the space-frequency domain requires to obtain a coefficient \( S_{N_1, N_2}(m_1, m_2) \) of two-dimensional discrete conversion of \( 4 \cdot N_1 \cdot N_2 \cdot (N_2 + 1) \) real multiplications and \( 4 \cdot N_1 \cdot (N_1 + N_2 - 1) \) real additions;
- the developed algorithm for the sliding processing of two-dimensional discrete signals in the space-frequency region after reaching the operating mode requires \( 4 \cdot (N_1 + N_2) \) real multiplications and \( 4 \cdot (N_1 + N_2 - 1) \) real additions to obtain a two-dimensional discrete transform coefficient \( S_{N_1, N_2}(m_1, m_2) \).

References

1. W.K. Pratt, *Digital image processing* (4th edn. Wiley-Interscience publication, 2007)
2. S.L. Marpl, *Digital Spectral Analysis with Applications* (New Jersey, Prentice-hall, 1986)
3. L.P. Yaroslavsky, *Compression, restoration, resampling, “compressive sensing”*, J. Optics 17(7), 073001 (2015)
4. M.N. Favorskaya, L.C. Jain, *Development of mathematical theory in computer vision*, (Springer International Publishing, Switzerland, 2015)

5. R.C. Gonzalez, R.E. Woods, *Digital image processing* (4th edn. Published by Pearson, 2018)

6. L. Rabiner, L., B. Gold, *Theory and application of digital signal processing* (New Jersey, Prentice-hall, 1975)

7. W. John, *Multidimensional signal, image, and video processing and coding* (Academic Press is imprint of Elsevier, 2006)

8. V.A. Ponomarev, O.V. Ponomareva, A.V. Ponomarev, *Method for effective measurement of a sliding parametric fourier spectrum* Optoelectronics, Instrumentation and Data Process. **50**(2), p.p. 1–7 (2014)

9. O. Ponomareva, A. Ponomarev, V. Ponomarev, *Evolution of forward and inverse discrete fourier transform*, IEEE East-West Design & Test Symp., pp. 313–318 (2018)

10. O. Ponomareva, A. Ponomarev, N. Ponomareva, *Window - presume parametric discrete Fourier transform*, IEEE East-West Design & Test Symp., pp. 364–368 (2018)

11. O.V. Ponomareva, N. Ponomareva, *Increased accuracy and extended functionality of digital filters based on frequency sampling*, Devices and methodsof measuremants, v. **2**(7), p.p. 114-119, (2013)

12. Favorskaya, M.N., Jain, L.C., Savchina, E.I.: *Perceptually tuned watermarking using non-subsampled shearlet transform*, Computer Vision in Control Systems-3: Aerial and Satellite Image Processing, ISRL, v. **135**, pp. 41–69. (Springer International Publishing AG 2018)

13. Petrovsky, N.A., Rybenkov, E.V., Petrovsky, A.A.: *Two-dimensional non-separable quaternionic paraunitary filter banks*, IEEE Int. conf. on Signal Processing: Algorithms, Architectures, Arrangements, and Applications, Poznan, Poland, pp. 120–125 (2018)