Nijenhuis Geometry III: $gl$-regular Nijenhuis operators

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Abstract

We study Nijenhuis operators, that is, $(1,1)$-tensors with vanishing Nijenhuis torsion under the additional assumption that they are $gl$-regular, i.e., every eigenvalue has geometric multiplicity one. We prove the existence of a coordinate system in which the operator takes first or second companion form, and give a local description of such operators. We apply this local description to study singular points. In particular, we obtain their normal forms in dimension two and discover topological restrictions for the existence of $gl$-regular Nijenhuis operators on closed surfaces.

This paper is an important step in the research programme suggested in [5, 8].

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1 Basic definitions and main results

Given a \((1,1)\) tensor field \(L\) on a manifold \(M^n\), one defines the \textit{Nijenhuis torsion} of \(L\) as

\[
\mathcal{N}_L(\xi, \eta) = L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta],
\]

where \(\xi, \eta\) are arbitrary vector fields. Recall that \(L\) is said to be a \textit{Nijenhuis operator} if its Nijenhuis torsion vanishes.

Nijenhuis geometry studies Nijenhuis operators and their properties, both local and global. A research programme and general strategy for studying such operators were suggested in [5]. This paper is devoted to the next item of our agenda (after [5], [6], [15]) and is focused on Nijenhuis operators satisfying \(\text{gl}\)-regularity condition.

We start with the following equivalent definitions of \(\text{gl}\)-\textit{regular} operators \(L: \mathbb{R}^n \to \mathbb{R}^n\) (the same notation \(L\) will be used for the matrix corresponding to this operator, with appropriate amendments under coordinate transformations if necessary):

- \(L\) is a regular element of the Lie algebra \(\text{gl}(n, \mathbb{R})\) in the sense that the adjoint orbit \(\mathcal{O}(L) = \{P L P^{-1} \mid P \in \text{GL}(n, \mathbb{R})\} \subset \text{gl}(n, \mathbb{R})\) has maximal dimension.
- The operators \(\text{Id}, L, \ldots, L^{n-1}\) are linearly independent.
- For each eigenvalue of \(L\) there is exactly one Jordan block in its Jordan normal form (this includes complex eigenvalues).
- The minimal polynomial of \(L\) coincides with the characteristic polynomial

\[
\chi_L(\lambda) = \det(\lambda \cdot \text{Id} - L) = \lambda^n - c_1 \lambda^{n-1} - \cdots - c_n.
\]

- \(L\) is similar to the \textit{first companion form}

\[
\begin{pmatrix}
  c_1 & 1 & 0 & \ldots & 0 \\
  c_2 & 0 & 1 & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & 0 \\
  c_{n-1} & 0 & \ldots & 0 & 1 \\
  c_n & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

where \(c_i\) are the coefficients of the characteristic polynomial \(\chi_L(\lambda)\).
- \(L\) is similar to the \textit{second companion form}

\[
\begin{pmatrix}
  0 & 1 & 0 & \ldots & 0 \\
  0 & 0 & 1 & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & 0 \\
  0 & 0 & \ldots & 0 & 1 \\
  c_n & c_{n-1} & \ldots & c_2 & c_1
\end{pmatrix},
\]

where \(c_i\) are the coefficients of the characteristic polynomial \(\chi_L(\lambda)\).
We say that a Nijenhuis operator $L$ defined on a smooth manifold $M$ is $\text{gl}$-regular, if it is $\text{gl}$-regular at every point $p \in M$ [5, Definition 2.9]. Many results in our paper are local and in this case $M$ is an open domain in $\mathbb{R}^n$.

Note that the eigenvalues of $\text{gl}$-regular operators are not necessarily smooth as the following example shows. Consider the $\text{gl}$-regular Nijenhuis operator

$$L = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$$

on $\mathbb{R}^2(x, y)$. Its eigenvalues are

$$\lambda_{1,2} = x \pm \sqrt{x^2 + 4y}.$$ 

On the curve $x^2 + 4y = 0$, $L$ is similar to a single Jordan block with eigenvalue $\frac{x}{2}$. If $x^2 + 4y > 0$, then $L$ is semisimple with distinct real eigenvalues (thus, $\mathbb{R}$-diagonalizable) whereas for $x^2 + 4y < 0$ this operator has two complex conjugate eigenvalues. In particular, this shows that $\text{gl}$-regular operators may admit singular points (cf. [5, Definition 2.8]) at which the algebraic structure of $L$ changes.

All the objects we are dealing with are supposed to be real analytic. The first result of the paper is the following theorem which gives a local characterisation of $\text{gl}$-regular Nijenhuis operators of any algebraic type.

**Theorem 1.1.** Consider a real analytic $\text{gl}$-regular operator $L$ with characteristic polynomial

$$\chi_L(\lambda) = \det(\lambda \cdot \text{Id} - L) = \lambda^n - f_1 \lambda^{n-1} - \cdots - f_n$$

for $n \geq 2$ in a sufficiently small neighbourhood of a point $p \in M$. Then the following are equivalent

(i) $L$ is Nijenhuis.

(ii) There exists a local coordinate system $x = (x^1, \ldots, x^n)$ in which $L$ takes the following form

$$L_{\text{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \cdots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \cdots & 0 & 1 \\ f_n & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $f_i = f_i(x)$ are coefficients of the characteristic polynomial in this coordinate system. These coefficients satisfy the following system of PDEs:

$$\frac{\partial f_i}{\partial x^j} = f_{i+1} \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \quad \frac{\partial f_n}{\partial x^j} = f_n \frac{\partial f_1}{\partial x^{j+1}},$$

for $1 \leq i, j \leq n - 1$. 


There exists a local coordinate system \( x = (x^1, \ldots, x^n) \) in which \( L \) takes the following form

\[
L_{\text{comp}2}(x) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
f_n & f_{n-1} & \cdots & f_2 & f_1
\end{pmatrix},
\]

(4)

where \( f_i = f_i(x) \) are coefficients of the characteristic polynomial in this coordinate system. These coefficients satisfy a system of PDEs that can be written in the form

\[
d\omega = 0, \quad d(L^*\omega) = 0,
\]

(5)

where \( \omega = f_n dx^1 + \cdots + f_1 dx^n \).

Following the terminology from Linear Algebra, we will refer to (2) and (4) as the first and second companion forms of \( L \).

**Remark 1.1.** If a Nijenhuis operator \( L \) is differentially non-degenerate at a point \( p \in M \) [5, Definition 2.10], then there are two distinguished coordinate systems in which \( L \) takes the first and second companion form. Namely, if we take the coefficients of the characteristic polynomial of \( L \) as local coordinates, i.e., set \( x^i = f_i \), then in these coordinates \( L \) takes the form

\[
L_{\text{comp}1}(x) = \begin{pmatrix}
x^1 & 1 & 0 & \cdots & 0 \\
x^2 & 0 & 1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
x^{n-1} & 0 & \cdots & 0 & 1 \\
x^n & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

(6)

Similarly, if we set \( x^1 = \text{tr} \, L, x^2 = \frac{1}{2} \text{tr} \, L^2, \ldots, x^n = \frac{1}{n} \text{tr} \, L^n \), then in these coordinates, we have

\[
L_{\text{comp}2}(x) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
f_n(x) & f_{n-1}(x) & \cdots & f_2(x) & f_1(x)
\end{pmatrix},
\]

where \( f_i(x) \) are the so-called Newton-Girard polynomials that express the coefficients of the characteristic polynomial in terms of the traces of powers of \( L \) appropriately rescaled, see [6, Appendix B] for details.

The point of Theorem 1.1, however, is that such a nice companion form exists for any gl-regular Nijenhuis operator so that in the real analytic category the differential non-degeneracy condition is not actually important.

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1Recall that this condition means that the differentials \( df_1(p), \ldots, df_1(p) \) are linearly independent.
Remark 1.2. The existence of the first companion form for an operator \( L \) is equivalent to the existence of a vector field \( \xi \) such that \( \xi, L\xi, L^2\xi, \ldots, L^{n-1}\xi \) pairwise commute and are linearly independent (for \( L_{\text{comp1}} \), this vector field is \( \xi = \partial_{x^n} \)). Similarly, the existence of the second companion form for \( L \) is equivalent to the existence of a vector field \( \xi = \partial_{x^n} \).

Remark 1.3. The reducibility of an operator to a companion form by a coordinate transformation is a non-trivial condition. Indeed, companion forms (2) and (4) are parametrised by \( n \) functions (in \( n \) variables). The coordinate change is also parametrized by \( n \) functions. At the same time, an operator field \( L \) (not necessarily Nijenhuis) is parametrised by \( n^2 \) functions.

For \( n > 2 \) one has \( n^2 > 2n \) and, thus, almost no operator field \( L \) can be brought to companion form.

As a specific example, consider \( L \) such that the coefficients \( f_i \) of its characteristic polynomial \( \chi_L(\lambda) \) are all constant. The companion form for \( L \) will then be a constant matrix. Hence, if \( L \) is reducible to companion form by a suitable coordinate transformation then its Nijenhuis torsion \( N_L \) necessarily vanishes, which is not always the case. Indeed, take

\[
L = \begin{pmatrix}
0 & 1 & 0 \\
-(y^2 + 1) & 0 & 1 \\
0 & (y^2 + 1) & 0
\end{pmatrix}.
\]

This operator is nilpotent, but \( N_L \neq 0 \). Thus, \( L \) cannot be brought to companion form.

Remark 1.4. The set of coordinate systems in which gl-regular Nijenhuis operator \( L \) is in first or second companion form is parametrised by \( n \) functions of one variable which is the maximal number of possible parameters. In more precise terms, the equations defining the corresponding coordinate transformations (see (11) and (14) below) are in involution for a gl-regular operator \( L \) if and only if \( L \) is Nijenhuis (see Propositions 3.2 and 3.3).

Theorem 1.1 characterises gl-regular Nijenhuis operators but, in fact, should not be interpreted as their local description. To get such a description one needs another important step. Namely, one needs to resolve PDE system (3) in order to find functions \( f_i \) from the first column of \( L_{\text{comp1}} \). The second result of our paper is an algebraic method for solving this system for arbitrary initial conditions.

Theorem 1.2. For \( n \) arbitrary real analytic functions \( v_1(t), \ldots, v_n(t) \) defined in a neighbourhood of zero, consider the function

\[
r(\lambda, t) = \lambda^n - v_1(t)\lambda^{n-1} - v_2(t)\lambda^{n-2} - \cdots - v_{n-1}(t)\lambda - v_n(t)
\]

and the matrix relation

\[
r(L, M) = 0,
\]

where \( M = x^1L^{n-1} + x^2L^{n-2} + \cdots + x^{n-1}L + x^n\text{Id} \) and \( L \) is a gl-regular \( n \times n \) matrix. Then

- From this matrix relation, the coefficients \( f_1, \ldots, f_n \) of the characteristic polynomial of \( L \) can be uniquely expressed in a neighbourhood of \( x = 0 \) as real analytic functions in \( x^1, \ldots, x^n \) (by Implicit Function Theorem).
• The functions \( f_1(x), \ldots, f_n(x) \) so obtained are solutions of (3) satisfying the initial condition

\[
\begin{align*}
f_1(0, \ldots, 0, x^n) &= v_1(x^n), \\
f_2(0, \ldots, 0, x^n) &= v_2(x^n), \\
&\quad \ldots \\
f_n(0, \ldots, 0, x^n) &= v_n(x^n).
\end{align*}
\]

This theorem gives local description for all gl-regular Nijenhuis operators and therefore provides a “list” of all possible singularities that can occur for gl-regular operators (Example \( 5.1 \) demonstrates how it works in practice). One should, however, remember that the first companion form for a Nijenhuis operator \( L \) is not unique. In other words, different companion forms can be equivalent. Speaking in rigorous terms, on the space of all (Nijenhuis) companion forms \( L_{\text{comp1}} \) given by (2), we can introduce a natural action of the groupoid that consists of coordinate transformations sending one companion form into another. Local classification of gl-regular operators in proper sense amounts to the orbit classification for this action. For \( n \geq 3 \), we hope to address this problem elsewhere.

In the two-dimensional case, which is somehow rather special, the local classification of gl-regular Nijenhuis operators is obtained in Section 6, Theorem 6.1. In addition to three (algebraically) generic types of gl-regular operators, this theorem describes five types\(^2\) of singular points (series \( L_{\text{nc}}, M, O, P \) and \( S \)) for gl-regular operators in dimension 2. It appears that locally every gl-regular Nijenhuis operator can be reduced to an explicit polynomial canonical form, which is quite different from the companion form. Our choice is explained by the following natural reason. The functions \( f_1 \) and \( f_2 \) involved in \( L_{\text{comp1}} \) are solutions of (3) and Theorem 1.2 suggests that they can be found explicitly only in exceptional cases. Despite its elegance and convenience for various theoretical purposes, the companion form \( L_{\text{comp1}} \) does not provide description in elementary functions. However, such a description can be achieved by an appropriate change of variables and that is what Theorem 6.1 does.

Based on this theorem we obtain the following global description of Nijenhuis operators on closed two-dimensional manifolds.

**Theorem 1.3.** Let \( (M^2, L) \) be a closed connected gl-regular Nijenhuis 2-manifold. Then one of the following holds:

1. \( M^2 \) is orientable and \( L = \alpha \text{Id} + \beta A \), where \( A \) is a complex structure on \( M^2 \) and \( \alpha, \beta \in \mathbb{R} \) are constants, \( \beta \neq 0 \).
2. \( M^2 \) is homeomorphic to either a torus or a Klein bottle and \( L \) has two distinct real eigenvalues on \( M^2 \) at each point.
3. \( M^2 \) is homeomorphic to a torus and \( L \) is similar to a Jordan block at each point of \( M^2 \).
4. \( M^2 \) is homeomorphic to either a torus or a Klein bottle and one of the eigenvalues of \( L \) is constant.

In the first three cases, the algebraic type of \( L \) remains the same at each point of the surface. In other words, the set of singular points is empty. In the forth case, the eigenvalues of

\(^2\)The other two series \( L_{\text{nl}} \) and \( N \) from Theorem 6.1 are not singular as the algebraic type of these operators does not change, at each point the operator is a \( 2 \times 2 \) Jordan block.
L may collide and we show in Proposition \[ \text{Proposition 6.2} \] that the corresponding singular point necessarily belongs to the \( M \)-series, one of five series from Theorem \[ \text{Theorem 6.1} \]. In particular, the other types of singular points cannot occur on compact surfaces.

Theorem \[ \text{Theorem 1.3} \] provides topological obstructions for existence of (non-trivial) gl-regular Nijenhuis operators in dimension 2.

**Corollary 1.1.** Let \( M^2 \) be either a sphere or a closed Riemann surface of genus \( \geq 2 \). Then \( M^2 \) cannot carry any gl-regular Nijenhuis operator \( L \) except for \( L = \alpha \text{Id} + \beta A \), where \( A \) is a complex structure on \( M^2 \) and \( \alpha, \beta \in \mathbb{R} \), \( \beta \neq 0 \).

**Corollary 1.2.** A non-orientable closed 2-manifold different from a Klein bottle cannot carry any gl-regular Nijenhuis operator.

Another result of our paper is description of various scenarios for Nijenhuis perturbations of a Jordan block. Assume that at a given point \( p \), all the coefficients \( f_1, \ldots, f_n \) of the characteristic polynomial of a Nijenhuis operator \( L \) vanish so that \( L(p) \) is similar to a Jordan block with zero eigenvalues. What can we say about the algebraic type of \( L \) at a generic point \( q \in U(p) \)? Formula \[ \text{(6)} \] gives an example when \( L(q) \) typically becomes semisimple, moreover for any prescribed collection of eigenvalues \( \lambda_1, \ldots, \lambda_n \) (with arbitrary multiplicities and including complex conjugate pairs) there exists exactly one point \( q \) that realises this spectrum of \( L \). This scenario coincides with the versal deformation of a Jordan block in terms of V. Arnold \[ \text{[2]} \]. But can \( L \) split into two Jordan blocks? Or, more generally, does there exist a Nijenhuis perturbation of a Jordan block \( J_0 = L(p) \) such that at a generic point \( q \in U(p) \) the operator \( L(q) \) has a prescribed algebraic type?

We use Theorem \[ \text{Theorem 1.2} \] to show that the answer is positive: all scenarios are possible. To state this result in a rigorous way, recall that in the space of all \( n \times n \) matrices, which we interpret as the Lie algebra \( \text{gl}(n, \mathbb{R}) \), we can introduce a natural partition \( \text{gl}(n, \mathbb{R}) = \bigsqcup_{\alpha} W_\alpha \) into families of adjoint orbits having the same algebraic type (Segre characteristic). Such families are sometimes called layers. For regular orbits, their algebraic type is defined by multiplicities \( k_1, \ldots, k_s \) of eigenvalues\(^3\), so that we can write

\[
\text{gl}(n, \mathbb{R})^{\text{reg}} = \bigsqcup_{\sum k_i = n} W_{k_1, \ldots, k_s}, \quad k_1 \leq \cdots \leq k_s, \; s \in \mathbb{N}, \; k_i \in \mathbb{N},
\]

where \( W_{k_1, \ldots, k_s} \subset \text{gl}(n, \mathbb{R}) \) is the subset of gl-regular operators having \( s \) distinct eigenvalues with multiplicities \( k_1, \ldots, k_s \) (regularity will automatically imply that each eigenvalue contributes exactly one Jordan block into the Jordan normal form of the operator). Notice that the Jordan block \( J_0 \) belongs to the closure of each regular layer.

**Theorem 1.4.** For any regular layer \( W_{k_1, \ldots, k_s} \subset \text{gl}(n, \mathbb{R}) \) there exists a Nijenhuis operator \( L \) defined in a small neighbourhood of \( 0 \in \mathbb{R}^n \) such that \( L(0) = J_0 \) and \( L(x) \in \overline{W}_{k_1, \ldots, k_s} \) for all \( x \in U(0) \), where \( \overline{W}_{k_1, \ldots, k_s} \) is the closure of \( W_{k_1, \ldots, k_s} \) (in usual or Zariski topology).

The structure of the paper is as follows. The proofs of Theorems \[ \text{Theorem 1.1} \] and \[ \text{Theorem 1.2} \] are given in Sections 3 and 4 respectively. Section 5 is devoted to Nijenhuis perturbations of a Jordan block and contain the proof of Theorem \[ \text{Theorem 1.3} \]. In Section 6 we obtain local classification of all gl-regular Nijenhuis operators in dimension 2 and prove Theorem \[ \text{Theorem 1.3} \]. These sections are

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\(^3\)Though we deal with real matrices, we make no difference between complex and real roots.
mainly independent on each other and contain no cross references. We conclude the paper with Appendix devoted to some applications of Theorem 1.2 to quasilinear systems of hydrodynamic type \( u_t = L(u)u_x \) in the case when \( L(u) \) is not necessarily diagonalisable Nijenhuis operator.

Acknowledgements. We thank Jenya Ferapontov and Artie Prendergast-Smith for their valuable comments and explanations. The most essential steps resulted in this paper would not have been done without outstanding research environment offered to us by the Institute of Advanced Studies, Loughborough University and Centro Internazionale per la Ricerca Matematica, Trento. We are also grateful to Jena Universität, in particular, Ostpartnerschaft programm for supporting our research on Nijenhuis Geometry for several years. The work of Alexey Bolsinov and Andrey Konyaev was supported by Russian Science Foundation (project 17-11-01303).

2 Outlook and motivation

Our motivation for studying gl-regular Nijenhuis operators was based on a very naive question: “What is the most natural genericity assumption for \((1, 1)\)-tensor fields similar to non-degeneracy of bilinear forms, symmetric or skew-symmetric?” In general algebraic context, the latter condition simply means that a bilinear form belongs to the “largest” orbit of the natural \(GL(n)\)-action and hence is the most typical. As a matter of fact such an orbit, in this case, is open. For operators, there are no open orbits, but we may still consider \(GL(n)\)-orbits of maximal dimension, which is exactly the gl-regularity assumption. In this view, gl-regular operators can be thought of as natural analogs of symplectic forms and (pseudo)-Riemannian metrics.

Another naive way to look at \((1, 1)\)-tensor fields is to think of them as families of matrices depending on parameters (coordinates on the manifold). Then the next natural question would obviously be: “Which bifurcations are typical in such families?”. The most typical bifurcation is a collision of two (or several) eigenvalues resulting in appearance of a Jordan block. That is exactly a singularity which we may observe in the case of gl-regular Nijenhuis operators. One could, of course, avoid collision of eigenvalues by requiring that \(L\) has no multiple eigenvalues, but would make the definition too rigid and exclude many important examples and interesting phenomena. It is worth mentioning that the complement to the set of matrices with no multiple roots has codimension one, whereas the complement to the set of gl-regular matrices is much smaller and has codimension 3.

The “converse” question, naturally appearing in applications, can be stated as follows: “What happens to a Jordan block under a perturbation?”. The answer depends on the number of parameters involved in perturbation and additional assumptions imposed on it. We refer to the famous paper by Arnold devoted to this subject which contains, in particular, an elegant solution in terms of versal deformations. In the context of Nijenhuis geometry, it is quite natural to ask “What are Nijenhuis perturbations of a Jordan block? Can we describe all of them? Which of them are generic (versal in the sense of Arnold)?” This is again a question on gl-regular Nijenhuis operators. It is amazing that the answer turns out to be very similar to that given by Arnold: there is a very simple generic Nijenhuis perturbation of a Jordan block.

\footnote{The non-degeneracy assumption \( \det L \neq 0 \) is much less relevant in Nijenhuis geometry as many problems one has to deal with are invariant w.r.t. shifts \( L \mapsto L + \text{const} \cdot \text{Id} \).}
(see formula (6) and Proposition 5.1), which is unique and coincides exactly with the one given in [2]. All the others can be derived from this canonical one by solving a system of integrable PDEs. We give a purely algebraic algorithm (see Theorem 1.2) how to do it for arbitrary initial condition, i.e., for finding all the solutions.

We also want to emphasise that gl-regular operators share many common properties. There are many facts well known for diagonalisable operators with simple spectrum that still hold true for gl-regular operators. If an operator is diagonalisable almost everywhere and has no multiple eigenvalues, then some (but not all!) of these results can be transferred to gl-regular case by continuity. However, even this procedure is often non-trivial as one needs to show that “transferring objects”, e.g. conservation laws or commuting flows, remain smooth and independent (linearly or functionally or otherwise), i.e., they neither explode nor blow up. Moreover, there are many occasions when a given Nijenhuis operator is not diagonalisable at all, but gl-regularity still guaranties good properties.

For this reason we are trying to use “invariant language” in our proof. This makes things technically a bit more complicated (for Nijenhuis operators written in diagonal form some of our proofs would be just one line) but, as a reward, we manage to cover many different cases by using one universal approach suitable for all Nijenhuis operators satisfying just one additional condition, namely gl-regularity.

We are confident that our results can and will have many applications. Indeed, Nijenhuis operators naturally appear in many unrelated topics in differential geometry and mathematical physics. A possible explanation for this “experimentally observed phenomenon” is as follows. For many geometric systems of partial differential equations, their coefficients are constructed from a certain operator, i.e., (1, 1)–tensor field $L = (L_{ij}(u))$. If such a system is invariant with respect to diffeomorphisms, then the compatibility and involutivity conditions can be invariantly written in terms of $L$. The point is that vanishing of the Nijenhuis torsion of $L$ is, in a certain sense (see e.g. discussion in the introduction of [5]), the simplest non-trivial condition of this kind.

This “experimental observation” suggests that any progress in Nijenhuis geometry might and should be applied in different areas where Nijenhuis operators have appeared, by combining the questions/methods from those topics with new results on Nijenhuis operators.

Until very recently, the list of known results in Nijenhuis geometry was very limited: Haantjes theorem [13], Newlander–Nirenberg theorem [21] and Thompson theorem [24]. These results have been extensively used as a simplifying ansatz in those situations where Nijenhuis operators appear: customary, one works with those coordinates in which the operator takes the “best” possible form provided by these theorems (e.g., in the case of Haantjes theorem, $L$ reduces to diagonal form with diagonal elements $\lambda_i = \lambda_i(u_i)$ and in the case of Thomson and Newlander–Nirenberg Theorems, one works in a coordinate system where $L$ has constant entries).

The assumptions of Haantjes, Newlander–Nirenberg and Thompson theorems essentially limit their applications. They all require that $L$ is algebraically stable, i.e., has the same Segre characteristic at every point. Moreover, they have strong conditions on the Segre characteristic: in Haantjes and Newlander–Nirenberg theorems, the operator $L$ is semi-simple (diagonalisable over complex numbers). Thompson and Newlander–Nirenberg theorems assume that the eigenvalues of $L$ are constant.
This paper, as well as its predecessors [5, 6, 15], aim to repair this situation. An important ingredient of our strategy described in [5] is to develop tools to study and describe Nijenhuis operators near those points where the Segre characteristic changes (singular points in the terminology of [5]) and also on closed manifolds. Any such tool can be applied wherever Nijenhuis operators naturally appear.

Obviously, there are many different types of singularities for Nijenhuis operators. We started our research with two opposite cases: the paper [15] (see also [5, §5]) studies the so-called singular points of scalar type, i.e., those where the operator $L$ vanishes (we may think of them as the most singular points). In the present paper we come from the other side and consider singular points at which the operator $L$ remains $\text{gl}$-regular [5, Definition 2.9] (the least singular points). Our first main result, Theorem 1.1, provides a common framework for studying such singularities: it allows one to assume without loss of generality that $L$ locally takes the first or second companion form (see (2) and (4)). Similar to the diagonal form from the Haantjes theorem, the companion forms (2) and (4) depend on an arbitrary choice of $n$ functions of one variable. In contrast to the Haantjes theorem, they allow bifurcations of the eigenvalues, and in Section 5 we discuss the freedom in such bifurcations.

A demonstration that our strategy works is Theorem 6.1 that describes all possible singularities for $\text{gl}$-regular Nijenhuis operators in dimension 2. As a corollary we have Theorem 1.3 on topological obstructions for the existence of regular Nijenhuis operators on closed two-dimensional surfaces.

We expect many applications of our results. For example, with the help of Theorem 1.3 one can easily reprove most results of the paper [19] devoted to geodesically equivalent metrics on two dimensional semi-riemannian manifolds. By [7], a pair of such metrics allows one to construct a Nijenhuis operator. One can easily show, applying the trick from [20, §3.3], that on a closed surface this operator is always $\text{gl}$-regular provided the metrics are semi-riemannian. Case 1 of Theorem 1.3 corresponds to a trivial geodesic equivalence, and cases 2, 3 and 4, translated to the language of geodesically equivalent metrics, imply most results in [19] and in particular allow to prove the natural generalisation of the projective Obata conjecture for the 2-torus.

We expect that our results may be effectively used in the theory of (infinite-dimensional) integrable systems of hydrodynamic type. They are partial differential equation systems of the form

$$u^i_t = \sum_j A^i_j(u)u^j_x.$$  \hfill (8)

where $u(t, x) = (u^1(t, x), ..., u^n(t, x))$ is an unknown vector-function. In this case the matrix $A = A(u)$ can be seen as an operator on an $n$-dimensional manifold with local coordinates $(u^1, ..., u^n).$ The integrability of this system amounts to a certain condition on the operator $A$ (more general than vanishing of the Nijenhuis torsion, see [25]).

One of the standard methods to work with systems (8) is based on the so-called Riemann invariants which are closely related to finding a polynomial $p$ with coefficients depending on $u$ such that $p(A)$ is a Nijenhuis operator (the eigenvalues of the operator $p(A)$ are precisely the Riemann invariants).

The overwhelming majority of results on integrable systems of hydrodynamic type assume that the operator $A$ is simple (i.e., has $n$ different eigenvalues). Our results allow one to avoid
this assumption. In particular, they can be applied to study stability of solutions of (5) near
the points where the eigenvalues collide. The “proof of concept” is, in fact, the Appendix where
we demonstrate how it works in the simplest case, when the operator $A$ is itself a Nijenhuis
operator.

Notice that not diagonalisable but still gl-regular operators naturally appear in differential
geometry and mathematical physics in the context of integrable PDEs of type (5), see e.g.
[1, 3, 4, 11, 25]. Moreover, they often resemble the companion form discussed in Theorem 1.1.

3 Proof of Theorem 1.1

First of all we observe that every operator $L_{\text{comp1}}$ given by (2) is Nijenhuis if and only if relations
(3) hold. And similarly, every operator $L_{\text{comp2}}$ given by (4) is Nijenhuis if and only if (5) holds.
The verification of this fact is straightforward and we omit it. In terms of Theorem 1.1 this
means, in particular, that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).

It remain to show that every gl-regular Nijenhuis operator $L$ can be (locally) reduced to
either of the companion forms $L_{\text{comp1}}$ and $L_{\text{comp2}}$. Since the proofs for $L_{\text{comp1}}$ and $L_{\text{comp2}}$ are
rather similar, we will do reduction simultaneously for both of them following the same scheme.

Consider a gl-regular Nijenhuis operator $L$ in a neigbourhood $U(p)$ of a point $p \in M$
and choose local coordinates $u = (u^1, \ldots, u^n)$ in this neigbourhood. Our goal is is to find coordinate
transformations bringing $L$ to the first companion form (2) and second companion form (4).

For the first companion form, such a coordinate transformation $u = u(x)$, where $x = (x^1, \ldots, x^n)$ is a new coordinate system, satisfies the following system of PDEs:

$$\left( \frac{\partial u}{\partial x} \right)^{-1} L(u) \left( \frac{\partial u}{\partial x} \right) = L_{\text{comp1}}(x),$$

where $L_{\text{comp1}}$ stands for the first companion form (2) and $\left( \frac{\partial u}{\partial x} \right)$ denotes the Jacobi matrix of the
transformation $u = u(x)$:

$$\left( \frac{\partial u}{\partial x} \right) = \begin{pmatrix}
u^1_{x1} & u^1_{x2} & \cdots & u^1_{xn} \\
u^2_{x1} & u^2_{x2} & \cdots & u^2_{xn} \\
\vdots & \vdots & \ddots & \vdots \\
u^n_{x1} & u^n_{x2} & \cdots & u^n_{xn} 
\end{pmatrix}.$$

Here and throughout the paper, when doing matrix computation, we consider $u$ and $x$ as
column-vectors, also we use $u_{xi}$ or $u^i_x$ for partial derivatives.

Rewriting (9) as

$$\left( \frac{\partial u}{\partial x} \right) L_{\text{comp1}} = L \left( \frac{\partial u}{\partial x} \right)$$

we see that the columns $u_{x^i}$ of $\left( \frac{\partial u}{\partial x} \right)$ satisfy the equations $Lu_{x^i} = u_{x^{i-1}}$ or equivalently

$$u_{x^{n-k}} = L^k u_{x^n}, \quad \text{where } L^k = L \cdot L \cdot \ldots \cdot L, \quad k = 1, \ldots, n - 1.$$  (11)

Lemma 3.1. Systems (10) and (11) are equivalent. In particular, (9) is equivalent to (11)
provided the Jacobi matrix $\left( \frac{\partial u}{\partial x} \right)$ is invertible.
Proof. By construction, (11) simply means that all the columns of the matrices in the left and right hand sides of (10) coincides except for the first column. In other words, system (10), as compared to (11), contains one additional vector relation for the first columns of l.h.s. and r.h.s.

\[ f_1 u_{x^1} + f_2 u_{x^2} + \cdots + f_n u_{x^n} = Lu_{x^1} \]  

(12)

We need to show that this relation follows from (11). This is an easy corollary of the Cayley–Hamilton theorem. Indeed, substituting \( u_{x^{n-k}} = L^k u_{x^n} \) into (12) gives

\[ f_1 L^{n-1} u_{x^n} + f_2 L^{n-2} u_{x^n} + \cdots + f_n u_{x^n} = L^n u_{x^n} \]

or equivalently

\[ (L^n - f_1 L^{n-1} - f_2 L^{n-2} - \cdots - f_n \text{Id}) u_{x^n} = \chi_L(L) u_{x^n} = 0, \]

which holds true automatically by the Cayley–Hamilton theorem. \( \square \)

Similarly, to bring \( L \) to the second companion form, we need to find an invertible transformation \( u = u(x) \) such that

\[ \left( \frac{\partial u}{\partial x} \right)^{-1} L(u) \left( \frac{\partial u}{\partial x} \right) = L_{\text{comp}2}(x), \]

(13)

where \( L_{\text{comp}2} \) is given by (14). Proceeding in a similar way as above, we get \( L \left( \frac{\partial u}{\partial x} \right) = \left( \frac{\partial u}{\partial x} \right) L_{\text{comp}2} \). This gives the following relation on the columns of the Jacobi matrix: \( Lu_{x^i} = u_{x^{i-1}} - f_{n-i} u_{x^n} \). For \( i = n \) we get \( Lu_{x^n} = u_{x^{n-1}} + f_1 u_{x^{n-1}} \), which yields

\[ u_{x^{n-1}} = M_1 u_{x^n} \quad \text{with} \quad M_1 = L - f_1 \text{Id}. \]

Next for \( i = n - 1 \), we get \( Lu_{x^{n-1}} = u_{x^{n-2}} + f_2 u_{x^n} \), yielding

\[ u_{x^{n-2}} = M_2 u_{x^n} \quad \text{with} \quad M_2 = LM_1 - f_2 \text{Id} \]

and so on. Finally we come to the following system of PDEs:

\[ u_{x^{n-k}} = M_k u_{x^n}, \quad \text{where} \quad M_1 = L - f_1 \text{Id}, \quad M_k = L M_{k-1} - f_k \text{Id}, \quad 2 \leq k \leq n, \]

(14)

where \( f_1, \ldots, f_n \) are the coefficients of the characteristic polynomial of \( L \). Equivalently,

\[ M_k = L^k - f_1 L^{k-1} - f_2 L^{k-2} - \cdots - f_{k-1} L - f_k \text{Id}, \quad k = 1, \ldots, n - 1. \]

(15)

This system is equivalent to (13), cf. Lemma 3.1.

Thus, we see that reducing \( L \) to the both first and second companion forms amounts to solving a quasilinear system of PDEs of the form

\[ u_{x^{n-k}} = A_k(u) u_{x^n}, \quad 1 \leq k \leq n - 1, \]

(16)

where for the first companion form we set \( A_k = L^k \), while for the second companion form, \( A_k = M_k \) with \( M_k \) given by (14) or (15).

Notice that (16) is overdetermined and, in general, not necessarily consistent. However the conditions under which local solutions exist for all initial data (in other words, the system is in involution) are well-known.
Proposition 3.1. The following properties of (16) are equivalent

(A) For any real analytic initial condition

\[
\begin{align*}
&u^1(0, \ldots, 0, x^n) = h^1(x^n), \\
&u^2(0, \ldots, 0, x^n) = h^2(x^n), \\
&\ldots \\
&u^n(0, \ldots, 0, x^n) = h^n(x^n),
\end{align*}
\]

or shortly

\[
u(0, \ldots, 0, x^n) = h(x^n),
\]

where \( h \) is a real analytic vector-function of one variable, there exists a unique real analytic solution \( u = u(x) \) of system (16).

(B) Operators \( A_k \)'s pairwise commute (i.e., \( A_k A_j = A_j A_k \)) and

\[
(A_k, A_j)(\xi, \xi) \overset{\text{def}}{=} [A_k \xi, A_j \xi] - A_j[A_k \xi, \xi] - A_k[\xi, A_j \xi] = 0
\]

for any vector field \( \xi \) and \( k, j = 1, \ldots, n-1 \).

Proof. The existence of solutions of (16) for all initial conditions in a more general case is discussed in [6] (and, in fact, can be derived from the Cartan-Kähler theorem [12]). The necessary and sufficient condition is

\[
D_{x^n} - i^1(A_j u_{x^n}) = D_{x^n} - j^1(A_i u_{x^n})
\]

on \( U(p) \), where \( D_{x^n} \) stands for the derivative in virtue of (16). For quasilinear systems this calculation is well-known (see [23], [16]) and leads to (B).

We now apply this Proposition in our special case.

Proposition 3.2. Both systems (11) and (14) satisfy Property (B), and therefore, Property (A) from Proposition 3.1.

Proof. Although verification of (B) for operators \( A_k = L^k \) and \( A_k = M_k \) is a nice exercise in tensor calculus, we prefer to make use of an elegant theory of bidifferential ideals introduced by F. Magri in [18] and then developed by F. Magri and P. Lorenzoni in [16], in particular, to construct hierarchies of commuting flows of hydrodynamic type. They are defined recursively by setting (cf. (14))

\[
A_0 = \text{Id}, \quad A_k = A_{k-1}L - a_k \text{Id}, \quad k = 1, 2, \ldots
\]

for any chain of functions \( a_1, a_2, \ldots \) satisfying relations

\[
da_{k+1} = L^k da_k - a_k da_1.
\]

Under these conditions, the operators \( A_k \) generate commuting flows [16, Proposition 2], i.e. satisfy (B).

Our situation is just a particular case of this construction. Indeed, setting \( a_k = 0 \), we obtain the sequence of operators \( A_k = L^k \). Hence Property (B) holds for (11). Of course, this fact is easy to check independently.

In the case of system (14) we only need to check that the coefficients \( f_k \) of the characteristic polynomial of \( L \) satisfy (18) (we may formally set \( f_k = 0 \) for \( k > n \)), but these are exactly relations from [5, Proposition 2.2]. Hence Property (B) holds for (14). It is worth noticing that (14) can also be understood as an \( \varepsilon \)-system in the sense of M. Pavlov [22] for \( \varepsilon = -1 \).
We have just shown that PDE systems (11) and (14) are both in involution and their (local) solutions $u(x)$ are parametrised by $n$ functions of one variable (initial conditions $h^1(x^n), \ldots, h^n(x^n)$). To make sure that such a solution $u(x)$ defines a desired coordinate transformation, we need to check that the Jacobi matrix $(\frac{\partial u}{\partial x})$ is non-degenerate at least at the initial point. Almost all solutions satisfy this property due to gl-regularity of $L$ (moreover this condition is necessary).

Indeed, for system (11), choose the initial condition $u(0, \ldots, 0, x_n) = h(x_n)$ in such a way that the vector $\xi = u_n(0) = h_n(0)$ is such that $L_n^{-1} \xi, \ldots, L \xi, \xi$ are linearly independent. Since $L$ is gl-regular, almost all vectors $\xi$ satisfy this condition. Due to (11), they form the columns of the Jacobi matrix $(\frac{\partial u}{\partial x})$ at the initial point $x = (0, \ldots, 0, 0)$. Hence, at this point $\det (\frac{\partial u}{\partial x}) \neq 0$ as required.

The same conclusion for solutions of system (14) immediately follows from the fact that $\text{Span}(M_n^{-1} \xi, \ldots, M_1 \xi, \xi) = \text{Span}(L_n^{-1} \xi, \ldots, L \xi, \xi)$. This completes the proof of Theorem 1.1.

We see from this proof that reducibility of $L$ to companion forms (2) and (4) follows from the involutivity (Property (B) from Proposition 3.1) of PDE systems (11) and (14) respectively. This property, in turn, follows from the fact that $L$ is Nijenhuis. It is natural to ask if the latter condition is also necessary for (11) and (14) to be in involution. The answer is positive under the additional assumption that $L$ is gl-regular.

**Proposition 3.3.** Let $n = \dim M > 2$ and $L$ be gl-regular.

1. If $\langle L^i, L^j \rangle = 0$ for $1 \leq i < j \leq n - 1$, i.e., (11) is in involution, then $L$ is a Nijenhuis operator.

2. If $\langle M_i, M_j \rangle = 0$ for $1 \leq i < j \leq n - 1$, where $M_i$ is defined as in (14), i.e., (14) is in involution, then $L$ is a Nijenhuis operator.

**Proof.** 1. It is easily seen that for any three commuting operators $L$, $A$ and $B$ the following (algebraic) identity holds:

$$\mathcal{N}_L(A \xi, B \xi) = (\langle LA, LB \rangle - L\langle LA, B \rangle - L\langle A, LB \rangle + L^2\langle A, B \rangle)(\xi, \xi)$$  \hspace{1cm} (19)

Hence, for $A = L^i$, $B = L^j$, $0 \leq i, j < n - 1$, we have

$$\mathcal{N}_L(L^i \xi, L^j \xi) = (\langle L^{i+1}, L^{j+1} \rangle - L\langle L^{i+1}, L^j \rangle - L\langle L^i, L^{j+1} \rangle + L^2\langle L^i, L^j \rangle)(\xi, \xi) = 0$$

for any $\xi$. Replacing $\xi$ with $\eta = \xi + L \xi$ and setting $j = n - 2$ in this formula, we get

$$0 = \mathcal{N}_L(L^{i+1} \xi, L^{n-2}(\xi + L \xi)) = \mathcal{N}_L(L^i \xi, L^{n-2} \xi) + \mathcal{N}_L(L^i \xi, L^{n-2}(L \xi)) + \mathcal{N}_L(L^{i+1} \xi, L^{n-2} \xi) + \mathcal{N}_L(L^i \xi, L^{n-1} \xi) = 0 + 0 + 0 + \mathcal{N}_L(L^i \xi, L^{n-1} \xi).$$

Thus, $\mathcal{N}_L$ vanishes for any pair of vectors from the set $\xi, L \xi, \ldots, L^{n-1} \xi$. As $L$ is gl-regular, one can choose $\xi$ in a way that $\xi, L \xi, \ldots, L^{n-1} \xi$ form a basis in the tangent space. Hence, $\mathcal{N}_L = 0$, as stated.

2. In what follows we assume that $M_0 = \text{Id}$ and $M_n = 0$ which perfectly agrees with the above definition of $M_i$’s (due to the Cayley-Hamilton theorem). We start with
Lemma 3.2. If \( \langle M_i, M_j \rangle = 0 \) for \( 1 \leq i < j \leq n - 1 \), then the following identities hold
\[
d f_{j+1}(M_i \xi) - d f_{i+1}(M_j \xi) = 0, \quad i, j = 0, \ldots, n - 1.
\]
where \( f_i \) are coefficients of the characteristic polynomial of \( L \) and \( \xi \) is an arbitrary tangent vector.

Proof. In formula (17), the expression \( \langle A, B \rangle \) is treated as a (vector-valued) quadratic form on the tangent bundle (one assumes that \( A \) and \( B \) commute). We can also naturally interpret it as a symmetric bilinear form by setting:
\[
\langle A, B \rangle(\xi, \eta) = \frac{1}{2}([A\xi, B\eta] - A[\xi, B\eta] - B[A\xi, \eta] + [A\eta, B\xi] - A[\eta, B\xi] - B[A\eta, \xi])
\]
Obviously \( \langle A, B \rangle(\xi, \xi) \equiv 0 \) implies \( \langle A, B \rangle(\xi, \eta) \equiv 0 \).

First we observe the following (purely algebraic) identity:
\[
\langle M_i L, M_j \rangle(\xi, \xi) + \langle M_i, M_j L \rangle(\xi, \xi) = M_i(\langle L, M_j \rangle(\xi, \xi) - M_j(\langle L, M_i \rangle(\xi, \xi) - 2(\langle M_j, M_i \rangle(\langle L, M_i \rangle(\xi, \xi).
\]
In our case we have \( L = M_1 + f_1 \Id \) and, in addition, \( \langle M_i, M_j \rangle \equiv 0 \), which gives:
\[
\langle M_i L, M_j \rangle + \langle M_i, M_j L \rangle = M_i(\langle L, M_j \rangle - M_j(\langle L, M_i \rangle = M_i(\langle M_1 + f_1 \Id, M_j \rangle - M_j(\langle M_1 + f_1 \Id, M_i \rangle = M_i(f_1 \Id, M_j) - M_j(\langle f_1 \Id, M_j \rangle - M_j(\langle f_1 \Id, M_i \rangle)
\]
(21)

Using (21) and the definition of \( M_i \)'s, we now compute the right hand side of the identity
\[
0 = \langle M_{i+1}, M_j \rangle + \langle M_i, M_{j+1} \rangle
\]
\[
= \langle M_i L, M_j \rangle + \langle M_i, M_j L \rangle = \langle M_i, L M_j - f_{i+1} \Id \rangle + \langle M_i, L M_j - f_{j+1} \Id \rangle
\]
\[
= \langle M_i L, M_j \rangle + \langle M_i, L M_j \rangle - \langle f_{i+1} \Id, M_j \rangle - \langle f_{j+1} \Id, M_i \rangle
\]
(22)

Notice that
\[
\langle f \Id, A \rangle(\xi, \xi) = [f \xi, A \xi] - A[\xi, A \xi] = d f(A \xi)\xi - d f(\xi) A \xi
\]
for an arbitrary function \( f \) and operator \( A \). Applying this relation to (22) gives
\[
0 = M_i(\langle f_1 \Id, M_j \rangle(\xi, \xi) - M_j(\langle f_1 \Id, M_i \rangle(\xi, \xi) - \langle f_{i+1} \Id, M_j \rangle(\xi, \xi) + \langle f_{j+1} \Id, M_i \rangle(\xi, \xi) =
\]
\[
= (d f_1(M_j \xi) - d f_{i+1}(\xi)) M_i \xi - (d f_1(M_i \xi) - d f_{i+1}(\xi)) M_j \xi +
\]
\[
+ (d f_{i+1}(M_j \xi) - d f_{j+1}(M_i \xi)).
\]
(24)

Recall that \( L \) is gl-regular. Hence \( \xi, L \xi, \ldots, L^{n-1} \xi \) are linearly independent for almost all tangent vectors \( \xi \). By formula (14) for \( M_1 \), this is still true for \( \xi, M_1 \xi, \ldots, M_{n-1} \xi \). Therefore \( \xi, M_i \xi, M_j \xi \) are linearly independent in (24), and the coefficients of this linear combination vanish. Thus, \( d f_{j+1}(M_i \xi) - d f_{i+1}(M_j \xi) = 0 \) for almost all vectors \( \xi \) and by continuity for all vectors. Lemma is proved.

Similar to the first case, our goal is to show that \( \mathcal{N}_L(M_i \xi, M_j \xi) = 0 \) for all \( i, j = 0, \ldots, n - 1 \). Since \( M_0 \xi, M_1 \xi, \ldots, M_{n-1} \xi \) form a basis for a generic vector \( \xi \), this will imply \( \mathcal{N}_L = 0 \).
As above we use (19) with $A = M_i$ and $B = M_j$:

$$\mathcal{N}_L(M_i\xi, M_j\xi) = \left(\langle LM_i, LM_j \rangle - L\langle LM_i, M_j \rangle - L\langle M_i, LM_j \rangle + L^2\langle M_i, M_j \rangle\right)(\xi, \xi). \tag{25}$$

Substituting $LM_i = M_{i+1} + f_{i+1}\text{Id}$ and using the relations $\langle M_i, M_j \rangle = 0$ ($i, j = 0, \ldots, n$) and identity $\langle f_{i+1}\text{Id}, f_{j+1}\text{Id} \rangle = 0$, we can rewrite the vector-valued quadratic form in the right hand side of (25) as follows:

$$\langle LM_i, LM_j \rangle - L\langle LM_i, M_j \rangle - L\langle M_i, LM_j \rangle + L^2\langle M_i, M_j \rangle =$$

$$\langle M_{i+1} + f_{i+1}\text{Id}, M_{j+1} + f_{j+1}\text{Id} \rangle - L\langle M_{i+1} + f_{i+1}\text{Id}, M_j \rangle - L\langle M_i, M_{j+1} + f_{j+1}\text{Id} \rangle =$$

$$\langle f_{i+1}\text{Id}, M_{j+1} \rangle + \langle M_{i+1}, f_{j+1}\text{Id} \rangle - L\langle f_{i+1}\text{Id}, M_j \rangle - L\langle M_i, f_{j+1}\text{Id} \rangle =$$

$$\langle f_{i+1}\text{Id}, LM_j \rangle - f_{j+1}\text{Id} \rangle + \langle LM_i, f_{j+1}\text{Id} \rangle - L\langle f_{i+1}\text{Id}, M_j \rangle - L\langle M_i, f_{j+1}\text{Id} \rangle =$$

$$\langle f_{i+1}\text{Id}, LM_j \rangle + \langle LM_i, f_{j+1}\text{Id} \rangle - L\langle f_{i+1}\text{Id}, M_j \rangle - L\langle M_i, f_{j+1}\text{Id} \rangle$$

Hence, using (23), we get:

$$\mathcal{N}_L(M_i\xi, M_j\xi) = \left(\langle f_{i+1}\text{Id}, LM_j \rangle + \langle LM_i, f_{j+1}\text{Id} \rangle - L\langle f_{i+1}\text{Id}, M_j \rangle - L\langle M_i, f_{j+1}\text{Id} \rangle\right)(\xi, \xi)$$

$$= d f_{i+1}(LM_j\xi) - d f_{i+1}(LM_i\xi) - d f_{j+1}(LM_i\xi) - d f_{j+1}(LM_j\xi) - L\left(d f_{i+1}(M_j\xi) - d f_{i+1}(M_i\xi) \right) - L\left(d f_{j+1}(M_j\xi) - d f_{j+1}(M_i\xi) \right)$$

$$= \left(d f_{i+1}(M_j(L\xi)) - d f_{i+1}(M_j(L\xi))\right) - \left(d f_{j+1}(M_j(L\xi)) - d f_{j+1}(M_j(L\xi))\right)\xi$$

It remains to notice that the coefficients in front of $\xi$ and $L\xi$ vanish by Lemma 3.2, which completes the proof. \hfill \Box

Remark 3.1. The gl-regularity assumption in Proposition 3.3 is essential. Indeed, consider an operator $L$ such that $L^2 = \text{Id}$ or $L^2 = 0$. Then the involutivity conditions $\langle L', L' \rangle = 0$ and $\langle M_i, M_j \rangle = 0$ obviously hold. However, $L$ does not need to be Nijenhuis.

4 Proof of Theorem 1.2

The goal of this section is to study and solve the PDE system:

$$\frac{\partial f_i}{\partial x^j} = f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}},$$

$$\frac{\partial f_n}{\partial x^j} = f_n \frac{\partial f_1}{\partial x^{j+1}}. \tag{28}$$

$1 \leq i, j \leq n - 1$. According to Theorem 1.1, every collection of functions $f_i$ satisfying this system defines a gl-regular Nijenhuis operator of the form

$$L(x) = L_{\text{comp1}}(x) = \begin{pmatrix}
    f_1 & 1 & 0 & \ldots & 0 \\
    f_2 & 0 & 1 & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & 0 \\
    f_{n-1} & 0 & \ldots & 0 & 1 \\
    f_n & 0 & \ldots & 0 & 0
\end{pmatrix}, \tag{29}$$

and vice versa, if this operator is Nijenhuis, then these functions satisfy (28). Throughout this section we deal with a Nijenhuis operator written in first companion form and use $L$ instead of $L_{\text{comp1}}$ to simplify notation.
If we denote \( f = (f_1, \ldots, f_n) \) (in matrix form, we think of \( f \) as a column-vector), then (28) simply means that
\[
 f_{xj} = L f_{x,j+1}, \quad \text{or equivalently} \quad f_{xj} = L^{n-j} f_{xn}, \quad 1 \leq j \leq n - 1. \tag{30}
\]

Observe that (30) coincides with the PDE system (16) with \( A_i = L^i \) that we used above to reduce \( L \) to the first companion form. The difference is that now the operator \( L \) is already in companion form so (30) (equivalently (28)) defines transformations \( f = f(x) \) that preserve this form. (Notice that we are now interested in both invertible and non-invertible transformations.) This observation can be rephrased as follows.

System (28) (or equivalently, (30)) can be written in the following matrix form (see Lemma 3.1):
\[
 \left( \frac{\partial f}{\partial x} \right) L = L \left( \frac{\partial f}{\partial x} \right), \tag{31}
\]

If \( f = f(x) \) defines an invertible transformation, i.e., the Jacobi matrix \( \left( \frac{\partial f}{\partial x} \right) \) is invertible, then (28) is equivalent to
\[
 \left( \frac{\partial x}{\partial f} \right) L = L \left( \frac{\partial x}{\partial f} \right). \tag{32}
\]
which is a linear system of PDEs for unknown functions \( x^i = x^i(f) \). This system can be easily solved in a neighbourhood of any point \( f_1 = c_1, \ldots, f_n = c_n \) for any initial condition and the corresponding solution can be found without integration. We are grateful to E. Ferapontov for explaining us the idea of this method.

**Proposition 4.1.** For any real-analytic initial condition
\[
 x^1(c_1, \ldots, c_{n-1}, c_n + \tau) = v_1(\tau) \\
 \vdots \\
 x^n(c_1, \ldots, c_{n-1}, c_n + \tau) = v_n(\tau)
\]  
where \( \tau \) belongs to a neighborhood of zero, there exists a unique (local) real analytic solution of (32). This solution can be found by using the following procedure. Consider a real analytic function \( F(t) \) constructed from \( v_1, \ldots, v_n \) as follows:
\[
 F(t) = v_1(p(t)) + tv_2(p(t)) + \cdots + t^{n-1}v_n(p(t)), \tag{34}
\]
where \( p(t) = t^n - c_1 t^{n-1} - \cdots - c_{n-1} t - c_n \). Then the solution \( x(f) \) satisfying the initial conditions (33) takes the form
\[
 x(f) = F(L) e_n, \quad \text{where} \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{35}
\]
i.e., \( x(f) \) is the last column of the matrix \( F(L) \).

**Proof.** We first notice that the function (34) is real analytic in a neighbourhood of the spectrum of \( L_0 = L(c_1, \ldots, c_n) \) and therefore locally \( F(L) \) is a real analytic matrix function (see [14]).

\footnote{In a neighbourhood of a pair of complex conjugate eigenvalues \( \lambda, \bar{\lambda} \), real analyticity of \( F \) means, in addition, that in \( F(\lambda) = F(\bar{\lambda}) \).}
To prove formula (35), it is sufficient to check two facts:

- Formula (35) gives a solution of (32) for an arbitrary polynomial \( F(t) \) (then this will be true for any real-analytic functions by continuity as polynomials are everywhere dense in this space).

- The solution defined by (35) indeed satisfies the required initial conditions.

Since the PDE system (35) is linear, instead of an arbitrary polynomial \( F(t) \) it is sufficient to consider only polynomials of the form \( t^k, \ k = 0, 1, 2, \ldots \). We will check this fact by induction, namely, we prove the following

**Lemma 4.1.** Let \( x(f) = \begin{pmatrix} x^1(f) \\ \vdots \\ x^n(f) \end{pmatrix} \) be a solution of (32), then \( \tilde{x}(f) = Lx(f) \) is a solution also.

In other words, multiplication by \( L \) sends solutions to solutions.

**Proof.** It is easy to see that the Jacobi matrix \( \left( \frac{\partial \tilde{x}}{\partial f} \right) \) takes the form

\[
\left( \frac{\partial \tilde{x}}{\partial f} \right) = L \left( \frac{\partial x}{\partial f} \right) + x^1 \cdot \text{Id}
\]

Hence, if \( \left( \frac{\partial x}{\partial f} \right) \) commutes with \( L \), then \( \left( \frac{\partial \tilde{x}}{\partial f} \right) \) commutes also, i.e. \( \tilde{x}(f) \) is a solution of (32), as stated.

It is easy to see that any constant vector-function \( x(f) = a \in \mathbb{R}^n \), and in particular \( x(f) = e_n \), is a solution of (32), then by induction, we have \( Le_n, L^2e_n, \ldots, L^k e_n \) are all solutions too, implying that \( x(f) = F(L)e_n \) is a solution of (32) for any polynomial and hence for any real analytic function \( F \).

To check the initial conditions, we compute \( F(L)e_n \), i.e., the last column of \( F(L) \) for \( f_i = c_i, \ (i = 1, \ldots, n-1) \), \( f_n = c_n + \tau \).

First we notice that the substitution of \( L \) (with the above indicated values of \( f_i \)) into the polynomial \( p(t) = t^n - c_1t^{n-1} - \cdots - c_{n-1}t - c_n \) gives

\[
p(L) = \tau \cdot \text{Id}.
\]

Indeed,

\[
\chi_L(t) = t^n - f_1t^{n-1} - \cdots - f_{n-1}t - f_n = t^n - c_1t^{n-1} - \cdots - c_{n-1}t - (c_n + \tau) = p(t) - \tau.
\]

Hence, \( p(t) = \chi_L(t) + \tau \) and \( p(L) = \chi_L(L) + \tau \cdot \text{Id} = 0 + \tau \cdot \text{Id} \), as stated.

Next,

\[
F(L) = \sum v_{n-k}(p(L))(L)^k = \sum v_{n-k}(\tau \cdot \text{Id})L^k = \sum v_{n-k}(\tau)L^k.
\]

It remains to notice that the last column of \( L^k \) is the \((n - k)\)-th basis vector \( e_{n-k} \) and therefore the last column of \( F(L) \) is \( \sum v_{n-k}(\tau)e_{n-k} = \begin{pmatrix} v_1(\tau) \\ \vdots \\ v_n(\tau) \end{pmatrix} \).
It is worth mentioning that the formulas for solutions of (32) are based on a more general phenomenon (see [25]) related to quasilinear systems of PDEs of type (16) which, in the case of Nijenhus operators, can be explained as follows. Consider the operator

$$M = x_1 L^{n-1}(u) + x_2 L^{n-2}(u) + \cdots + x_{n-1} L(u) + x_n \text{Id}$$

and algebraic relation

$$M = F(L),$$

(36)

where $F$ is an analytic matrix function. If $L$ is a gl-regular Nijenhus operator in coordinates $u^1, \ldots, u^n$, then $x_i$ can be expressed in terms of $u$ and the inverse function $u(x)$ (if it exists!) is a solution of the PDE system

$$u_{x^j} = L^{n-j} u_{x^n}, \quad j = \ldots, n - 1.$$

Our formula is just a particular version of this fact for $L$ being in the first companion form. In this case, one can resolve (36) (this is by the way another advantage of the first companion form) explicitly and choose an appropriate matrix function $F$ for a prescribed initial condition.

Proposition 4.1 describes all the solutions $x = x(f)$ of (32) and therefore all invertible solutions $f = f(x)$ of (28). In other words, we obtain description of all Nijenhus operators in companion form which are differentially non-degenerate (see [2, Definition 2.10] and Remark 1.1). Notice that all these operators can be transformed to each other by a suitable coordinate change and, in particular, each of them can be brought to the form (6). Equivalently, we can say that such operators form the largest (or, generic) orbit of the groupoid that consists of coordinate transformations acting on Nijenhus operators of type (2).

However, our goal is to describe all the solutions $f = f(x)$ of (28), both invertible and non-invertible. Moreover, we would like to be able to construct the solution that corresponds to prescribed initial conditions $f(0, \ldots, 0, x^n) = v(x^n)$. The above method does not allow us to do this and we need to modify it. This is exactly what Theorem 1.2 does by replacing (36) with a more general algebraic relation of the form $r(L, M) = 0$ which, in some sense, interchanges the roles of $L$ and $M$ and, as a result, $x$ and $f$.

We now prove Theorem 1.2. As above, we set $M = x_1 L^{n-1} + x_2 L^{n-2} + \cdots + x_{n-1} L + x_n \text{Id}$ and consider the matrix function

$$r(L, M) = L^n - v_1(M) L^{n-1} - v_2(M) L^{n-2} - \cdots - v_{n-1}(M) L - v_n(M)$$

where $v_i(t)$ are the functions defining the initial conditions for (3) (or equivalently, (30)).

We need to show that the solution $f(x) = (f_1(x), \ldots, f_n(x))$ of (3) with prescribed initial conditions can be obtained by resolving the relation $r(L, M) = 0$ with respect to the coefficients of the characteristic polynomial of $L$.

We first notice that this relation is invariant in algebraic sense so that we may consider the matrices $L$ and $M$ in any basis we like. Of course, we will assume that $L$ is written in companion form (29).

The matrix $\sum v_i(M) L^{n-i}$ commutes with $L$ and its entries are analytic functions in $x$ and $f$. This matrix can be uniquely presented as linear combination

$$\sum v_i(M) L^{n-i} = g_1 L^{n-1} + \cdots + g_{n-1} L + g_n \text{Id},$$

(37)
Comparing with
\[ L^n - f_1 L^{n-1} - \cdots - f_{n-1} L - f_n \text{Id} = 0 \]
(Cayley–Hamilton theorem)
and using gl-regularity of \( L \) we come to the system of algebraic relations
\[ f_i = g_i(x, f). \]
To resolve these relations w.r.t. \( f_i \), i.e. to find \( f_i = f_i(x) \) as a real analytic function of \( x \) (for small \( x \)), it is sufficient to check that \( \frac{\partial g_i}{\partial f_n}(0, \ldots, 0, x^n, f) = 0 \), which is obviously true as
\[ \sum v_i(0 \cdot L^{n-1} + \cdots + 0 \cdot L + x^n \text{Id})L^{n-i} = \sum v_i(x^n)L^{n-i}, \]
implies that \( g_i(0, \ldots, 0, x^n, f) \) coincides with \( v_i(x^n) \) and therefore does not depend on \( f_n \).
This proves the first statement and also shows that the initial conditions are indeed fulfilled: if \( x^1 = \cdots = x^{n-1} = 0 \), then \( f_i(0, \ldots, 0, x^n) = g_i(0, \ldots, 0, x^n, f) = v_i(x^n) \), as required.

The last step is to show that \( f_i(x) \)'s so obtained satisfy (3) or equivalently (30). We start with two lemmas concerning \( g(x, f) \).

**Lemma 4.2.** The vector-function \( g = (g_1 \ldots g_n)^\top \) satisfies (30), i.e.,
\[ g_{x^j} = L g_{x^{j+1}}, \quad j = 1, \ldots, n - 1. \]

**Proof.** Differentiating (37) w.r.t. \( x^j \) we get
\[ L^{n-j}(v'_1(M)L^{n-1} + \cdots + v'_n(M)\text{Id}) = \frac{\partial g_1}{\partial x^j}L^{n-1} + \cdots + \frac{\partial g_n}{\partial x^j}\text{Id}. \] (38)
Similarly, differentiating (37) w.r.t \( x^{j+1} \) we get
\[ L^{n-j-1}(v'_1(M)L^{n-1} + \cdots + v'_n(M)\text{Id}) = \frac{\partial g_1}{\partial x^{j+1}}L^{n-1} + \cdots + \frac{\partial g_n}{\partial x^{j+1}}\text{Id}. \] (39)
Comparing (39) and (38) gives
\[ L \left( \frac{\partial g_1}{\partial x^{j+1}}L^{n-1} + \cdots + \frac{\partial g_n}{\partial x^{j+1}} \right) = \frac{\partial g_1}{\partial x^j}L^{n-1} + \cdots + \frac{\partial g_n}{\partial x^j} \]
Applying the Cayley–Hamilton theorem (i.e., \( L^n = f_1 L^{n-1} + \cdots + f_n \)) we obtain
\[ \left( \frac{\partial g_2}{\partial x^{j+1}} + f_1 \frac{\partial g_1}{\partial x^{j+1}} \right) L^{n-1} + \cdots + \left( \frac{\partial g_n}{\partial x^{j+1}} + f_{n-1} \frac{\partial g_1}{\partial x^{j+1}} \right) L + f_n \frac{\partial g_1}{\partial x^{j+1}} \]
\[ = \frac{\partial g_1}{\partial x^j}L^{n-1} + \cdots + \frac{\partial g_n}{\partial x^j}\text{Id}. \]
Since \( L^{n-1}, \ldots, L, \text{Id} \) are linearly independent, we get \( g_{x^j} = L_{\text{comp1}} g_{x^{j+1}} \) which coincides with the statement of the lemma as \( L = L_{\text{comp1}}. \)

\[ \square \]
Lemma 4.3. Let \( \left( \frac{\partial g}{\partial f} \right) \) be the matrix of partial derivatives \( \frac{\partial g}{\partial f} \), \( 1 \leq i, j \leq n \). Then we have

\[
L \left( \frac{\partial g}{\partial f} \right) = \left( \frac{\partial g}{\partial f} \right) L. \tag{40}
\]

Proof. As already noticed, \( g_i = g_i(x, f) \) are the entries of the last column of \( \sum v_i(M)L^{n-i} \), if \( L \) is written in a companion basis, i.e., \( L = L_{\text{compp}} \). On the other hand for fixed \( x \), the expression \( \sum v_i(M)L^{n-i} \) can be treated as an analytic function \( F(L) \). After this remark, (40) is just a part of the conclusion Proposition 4.1 (with \( x \) replaced by \( g \)).

Consider the implicit equation \( f = g(x, f) \). Differentiating it w.r.t. \( x^j \) and \( x^{j+1} \) we get

\[
f_{x^j} = g_{x^j} + \left( \frac{\partial g}{\partial f} \right) f_{x^j} \quad \text{and} \quad f_{x^{j+1}} = g_{x^{j+1}} + \left( \frac{\partial g}{\partial f} \right) f_{x^{j+1}}.
\]

Multiplying by \( L \) and subtracting we get

\[
f_{x^j} - Lf_{x^{j+1}} = g_{x^j} - Lg_{x^{j+1}} + \left( \frac{\partial g}{\partial f} \right) f_{x^j} - L \left( \frac{\partial g}{\partial f} \right) f_{x^{j+1}}. \tag{41}
\]

Applying Lemma 4.2 and (40), we see that (41) can be written as

\[
\left( \text{Id} - \left( \frac{\partial g}{\partial f} \right) \right) (f_{x^j} - Lf_{x^{j+1}}) = 0.
\]

As already noticed, the matrix \( \left( \frac{\partial g}{\partial f} \right) \) vanishes for \( x = (0, \ldots, 0, x^n) \), therefore locally \( \text{Id} - \left( \frac{\partial g}{\partial f} \right) \) is invertible and we conclude that \( f \) satisfies (41) or, equivalently (3), which completes the proof of Theorem 1.2.

5 Nijenhuis perturbations of a Jordan block

Our next goal is to discuss Nijenhuis perturbations of a Jordan block \( J_0 \), that is, Nijenhuis operators of the form \( L(x) = J_0 + \) higher order terms. Recall that a generic Nijenhuis perturbation of \( J_0 \) is described by the following

Proposition 5.1 ([5], see also Remark 1.1). Let \( L \) be a Nijenhuis operator such that at a point \( p \), the operator \( L(p) \) is similar to the (nilpotent) Jordan block \( J_0 \). Assume that the differentials of the coefficients of the characteristic polynomial of \( L \) are linearly independent at \( p \). Then in a neighbourhood of \( p \) there exist local coordinates \( x^1, \ldots, x^n \) with \( p \approx (0, \ldots, 0) \) in which \( L(x) \) is given by (6).

In this case, it is easily seen that at a generic point \( q \in U(p) \), the operator \( L(q) \) becomes semisimple with distinct eigenvalues. Moreover, for any collection of real and complex conjugate numbers \( S = \{ \lambda_1, \ldots, \lambda_k, \mu_1, \bar{\mu}_1, \ldots, \mu_s, \bar{\mu}_s \} \) \( (k + 2s = n) \) sufficiently close to zero and not necessarily distinct, there exists a unique point \( q \in U(p) \) such that \( S \) is the spectrum of \( L(q) \). In particular, we see that in \( U(p) \) we can find operators of all possible algebraic types that are potentially allowed for gl-regular operators (this means that for repeated eigenvalues there will be only one Jordan block).
It is natural to ask whether there are other scenarios of Nijenhuis perturbations, for instance, with a prescribed algebraic structure of $L$ at a generic point $q$. For instance, can a Jordan block $J_0$ split into two smaller Jordan blocks of prescribed sizes $k_1, k_2, k_1 + k_2 = n$?

The answer is positive. Let us show that all scenarios are possible. According to Theorems 1.1 and 1.2 we may assume that $L(x) = L_{\text{comp1}}(x)$ where $L_{\text{comp1}}$ is given by (2) and the coefficients $f_1(x), \ldots, f_n(x)$ of the characteristic polynomial $\chi_L$ satisfy (3). To construct the corresponding perturbation one just needs to make sure that the desired scenario happens on the initial straight line $x(\tau) = (0, \ldots, 0, \tau)$. Assume that on this initial line at a generic point $\tau \in (-\varepsilon, \varepsilon)$, the characteristic polynomial

$$\chi_L(x(\tau))(t) = t^n - f_1(x(\tau))t^{n-1} - \cdots - f_n(x(\tau)) = t^n - v_1(\tau)t^{n-1} - \cdots - v_n(\tau)$$

factorises as follows

$$\chi_L(x(\tau))(t) = (t - \mu_1(\tau))^{k_1}(t - \mu_2(\tau))^{k_2} \cdots (t - \mu_s(\tau))^{k_s}.$$ 

where $\mu_i(\tau)$ are some real analytic functions in $\tau$ (perhaps complex valued). In other words, at a generic points of the initial line $x(\tau)$, this polynomial has $s$ distinct roots with multiplicities $k_1, \ldots, k_s$.

According to Theorem 1.2, to describe the solution $f = f(x)$ with given initial conditions $f(x(\tau)) = v(\tau)$ we should consider the relation

$$r(L, M) = L^n - v_1(M)L^{n-1} - \cdots - v_{n-1}(M)L - v_n(M) = 0, \quad \text{with} \quad M = \sum_{i=1}^{n} x^i L^{n-i},$$

and then “solve” it to find the coefficients of the characteristic polynomial of $L$ in terms of $x^1, \ldots, x^n$. Notice that $r(L, M)$ is just the polynomial (12) after the substitution $\tau \mapsto M$, $t \mapsto L$.

We know that this polynomial factorises (for scalars $\tau$ and $t$, but here the difference between scalars and matrices is not essential), hence we can write

$$r(L, M) = (L - \mu_1(M))^{k_1}(L - \mu_2(M))^{k_2} \cdots (L - \mu_s(M))^{k_s} = 0$$

where $\mu$ is now treated as an analytic matrix function.

The eigenvalues of $L$ (as functions in $x$) can now be found from relations of the form:

$$\lambda = \mu_i \left( x_1 \lambda^{n-1} + x_2 \lambda^{n-2} + \cdots + x_{n-1} \lambda + x_n \right)$$

By the Implicit Function Theorem this can be done uniquely in a neighbourhood of a point $(0, \ldots, 0, \tau)$ in such a way that $\lambda(0, \ldots, 0, \tau) = \mu_i(\tau)$, as needed.

No other eigenvalues may occur. The multiplicities of these eigenvalues will be as expected since this condition is fulfilled on the initial line. This shows that at a generic point we have

$$\chi_L(x)(t) = (t - \lambda_1(x))^{k_1}(t - \lambda_2(x))^{k_2} \cdots (t - \lambda_s(x))^{k_s}$$

where $\lambda_i(x)$ will be real analytic functions such that $\lambda_i(0, \ldots, 0, \tau) = \mu_i(\tau)$ (these relations hold as soon as $\mu_i(\tau)$ makes sense).

This argument leads us to the following property of the discriminant of the polynomial $\chi_f = t^n - f_1 t^{n-1} - \cdots - f_{n-1} t - f_n$. 

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Proposition 5.2. Let \( f(x) = (f_1(x), \ldots, f_n(x)) \) be a solution of \((28)\). Assume that the discriminant \( D(f_1, \ldots, f_n) \) of the polynomial \( \chi_{f(x)}(t) = t^n - f_1(x)t^{n-1} - \cdots - f_{n-1}(x)t - f_n(x) \) vanishes on the initial straight line \( x(\tau) = (0, \ldots, 0, \tau) \). Then the discriminant vanishes identically for all \( x = (x^1, \ldots, x^n) \).

Similarly, for each partition \( n = k_1 + \cdots + k_s \), consider the algebraic variety \( \overline{W}_{k_1, \ldots, k_s} \subset \mathbb{R}^n(f_1, \ldots, f_n) \) that is the Zariski closure of the set \( W_{k_1, \ldots, k_s} \) of those \( f \in \mathbb{R}^n \) for which \( \chi_f(t) \) has \( s \) distinct roots with multiplicities \( k_1, \ldots, k_s \). If \( f(x(\tau)) \in \overline{W}_{k_1, \ldots, k_s} \) for the initial line \( x(\tau) = (0, \ldots, 0, \tau) \), then \( f(x) \in \overline{W}_{k_1, \ldots, k_s} \) for all \( x = (x^1, \ldots, x^n) \).

Clearly, the second part of this proposition immediately implies Theorem 1.4.

Let us finally discuss an example showing how Theorem 1.2 works in practice to construct explicit examples of Nijenhuis operators with non-trivial singularities.

Example 5.1. Consider the three dimensional case and, in the settings of Theorem 1.2, define the initial conditions in such a way that on the initial line \( x(\tau) = (0, 0, \tau) \) the characteristic polynomial of \( L \) takes the form

\[
\chi_{L(x(\tau))}(\lambda) = (\lambda - \tau)^2(\lambda - 2\tau) = \lambda^3 - 4\tau\lambda^2 + 5\tau^2\lambda - 2\tau^3,
\]

or equivalently

\[
f_1(0, 0, \tau) = 4\tau = v_1(\tau), \quad f_2(0, 0, \tau) = -5\tau^2 = v_2(\tau), \quad f_3(0, 0, \tau) = 2\tau^3 = v_3(\tau).
\]

The algorithm described in Theorem 1.2 allows us to reconstruct the functions \( f_1, f_2, f_3 \). To that end we need to use the matrix relation

\[
L^3 - (4M)L^2 + (5M^2)L - 2M^3 = 0 \quad \text{with} \quad M = x_1L^2 + x_2L + x_3\text{Id},
\]

to express the coefficients of the characteristic polynomial of \( L \) in terms of \( x_1, x_2 \) and \( x_3 \).

Notice that this relation can be rewritten as \((L - M)^2(L - 2M) = 0\) (this follows immediately from factorisation of the characteristic polynomial on the initial line \( x(\tau) \)). But this factorisation immediately allows us to find the eigenvalues of \( L \) by taking the roots of the polynomial

\[
(\lambda - x_1\lambda^2 - x_2\lambda - x_3)^2(\lambda - 2x_1\lambda^2 - 2x_2\lambda - 2x_3) = 0
\]

Since we are working in a neighbourhood of the origin, we are interested in specific roots, namely those which, on the initial curve, coincide with the above prescribed roots, that is,

\[
\lambda_1(0, 0, x_3) = \lambda_2(0, 0, x_3) = x_3, \quad \lambda_3(0, 0, x_3) = 2x_3.
\]

In this particular case we just need to choose the right root (one of the two) of the corresponding quadratic equation. Namely,

\[
\lambda - x_1\lambda^2 - x_2\lambda - x_3 = 0 \quad \Rightarrow \quad \lambda = \frac{2x_3}{(1 - x_2) + \sqrt{(1 - x_2)^2 - 4x_1x_3}}
\]

\[
\lambda - 2x_1\lambda^2 - 2x_2\lambda - 2x_3 = 0 \quad \Rightarrow \quad \lambda = \frac{4x_3}{(1 - 2x_2) + \sqrt{(1 - 2x_2)^2 - 16x_1x_3}}
\]
The root of the first equation is an eigenvalue of $L$ of multiplicity 2, whereas the root of the second equation is an eigenvalue of multiplicity one. As a result we have found explicit expressions for the eigenvalues of the Nijenhuis operator $L$ in coordinates $x_1, x_2, x_3$:

$$\lambda_1 = \lambda_2 = \frac{2x_3}{(1 - x_2) + \sqrt{(1 - x_2)^2 - 4x_1x_3}}, \quad \lambda_3 = \frac{4x_3}{(1 - 2x_2) + \sqrt{(1 - 2x_2)^2 - 16x_1x_3}}$$

(43)

The final conclusion is that the operator

$$L_{\text{comp 1}} = \begin{pmatrix} f_1(x) & 1 & 0 \\ f_2(x) & 0 & 1 \\ f_3(x) & 0 & 0 \end{pmatrix}$$

with

$$f_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad f_2 = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1, \quad f_3 = \lambda_1\lambda_2\lambda_3,$$

where $\lambda_i$ are defined by [13] is a Nijenhuis operator in first companion form. This is an example of a Nijenhuis perturbation of the nilpotent $3 \times 3$ Jordan block $J_0$ under which $J_0$ splits into two Jordan blocks of size 2 and 1 with non-constant eigenvalues.

6 Local classification of gl-regular Nijenhuis operators in dimension two and its applications

The goal of this section is to describe local normal forms for gl-regular Nijenhuis operators at singular points in dimension 2. However, for the sake of completeness we first recall the list of (algebraically) generic types of such operators along with their local canonical forms:

- Two distinct real eigenvalues: $L = \begin{pmatrix} f(x) & 0 \\ 0 & g(y) \end{pmatrix}$, where $f(x)$ and $g(y)$ are smooth functions such that $f(x) \neq g(y)$ for all $(x, y)$. In the real analytic case, $f(x)$ is either constant or can be reduced, by an appropriate local change of coordinates, to $f(x) = f_0 \pm x^{2m}$ or $f(x) = f_0 + x^{2m-1}, m \in \mathbb{N}$, and similarly for $g(y)$.

- Two complex conjugate eigenvalues: $L = \begin{pmatrix} f(x, y) & -g(x, y) \\ g(x, y) & f(x, y) \end{pmatrix}$, where $h = f + ig$ is a holomorphic function of the complex variable $z = x + iy$, $g(x, y) \neq 0$ for all $(x, y)$. This function $h(z)$ is either constant or can be reduced, by an appropriate local change of coordinates, to $h(z) = h_0 + z^m, m \in \mathbb{N}$.

- Jordan block: $L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix}$, where $f(y)$ is a smooth function. As above, in the real analytic case, $f(y)$ is either constant or can be reduced, by an appropriate local change of coordinates, to $f(x) = f_0 \pm x^{2m}$ or $f(x) = f_0 + x^{2m-1}, m \in \mathbb{N}$.

This classification is easy and well known. A non-trivial problem is to describe local behaviour of $L$ near a singular point $p$ at which the algebraic type of $L$ changes. In dimension 2 under the gl-regularity assumption, there is only one possibility for $L(p)$, namely, this operator (after an appropriate chage of coordinates) is a Jordan block:

$$L(p) = \lambda \text{Id} + J_0, \quad \text{where } J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \lambda = \text{const } \in \mathbb{R}.$$
Since \( L - \lambda \text{Id} \) is still a Nijenhuis operator, we will assume w.l.o.g. that \( L(p) = J_0 \) and our problem reduces to classification of Nijenhuis perturbations of the nilpotent Jordan block \( J_0 \). Below we will describe all possible normal forms for such perturbations, i.e., for Nijenhuis operators \( L \) such that \( L(p) = J_0 \). To our great surprise, they are all polynomial. Before stating our classification result, we notice that there are two essentially different cases depending on the coefficients of the characteristic polynomial

\[
\chi_L(\lambda) = \det(\lambda \cdot \text{Id} - L) = \lambda^2 - v\lambda - u, \quad v = \text{tr} L, \; u = -\det L.
\]

In the real analytic case, there are two possibilities: either \( dv \wedge du \equiv 0 \) or \( dv \wedge du \neq 0 \) on an open everywhere dense subset. In the latter case, the operator \( L \) can be completely reconstructed from \( v \) and \( u \) and the relation (see [5, Corollary 2.2]):

\[
L = \left( \begin{array}{cc} v_x & v_y \\ u_x & u_y \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} v_x & v_y \\ u_x & u_y \end{array} \right), \quad v = \text{tr} L, \; u = -\det L. \quad (44)
\]

At those points where the Jacobi matrix is not invertible, we define \( L \) by continuity. In other words, in the above formula we should automatically observe “cancellation of the denominator” \( v_x u_y - v_y u_x \) involved in the formula of the inverse matrix. For this reason in Theorem 6.1 below, when appropriate, instead of the matrix of \( L \) we will give formulas for \( v(x, y) \) and \( u(x, y) \) as they are much simpler and more intuitive. The reader may easily “reconstruct” \( L \) from (44) and, in particular, see the above mentioned cancellation.

If \( dv \wedge du \equiv 0 \), then (44) makes no sense, but we may still use another more general relation (see [5, Proposition 2.2]):

\[
\left( \begin{array}{cc} v_x & v_y \\ u_x & u_y \end{array} \right) \left( \begin{array}{cc} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{array} \right) = \left( \begin{array}{cc} v & 1 \\ u & 0 \end{array} \right) \left( \begin{array}{cc} v_x & v_y \\ u_x & u_y \end{array} \right), \quad v = \text{tr} L, \; u = -\det L, \; L = \left( \begin{array}{cc} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{array} \right). \quad (45)
\]

We will assume that \( L \) is defined in a neighbourhood of the origin \( p = (0, 0) \in \mathbb{R}^2(x, y) \) and coordinate transformations always leave the origin fixed. The theorem below provides the complete list of normal forms for \( L \) which are divided into several series.

**Theorem 6.1.** Let \( L \) be a Nijenhuis operator such that \( L(p) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \). Then in suitable local coordinates \((x, y)\), this operator takes one of the following forms:

1. Series \( L, M \) and \( N \) (for \( k \geq 1, \epsilon = \pm 1 \)):

   \[
   L_{\text{nil}} = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad L_{\text{nd}} = \left( \begin{array}{cc} x & 1 \\ y & 0 \end{array} \right), \quad M_{2k-1} = \left( \begin{array}{cc} 0 & 1 \\ 0 & y^{2k-1} \end{array} \right), \quad M_{2k}^\epsilon = \left( \begin{array}{cc} 0 & 1 \\ 0 & \epsilon y^{2k} \end{array} \right),
   \]

   \[
   N_{2k-1} = \left( \begin{array}{cc} y^{2k-1} & 1 \\ 0 & y^{2k-1} \end{array} \right), \quad N_{2k}^\epsilon = \left( \begin{array}{cc} \epsilon y^{2k} & 1 \\ 0 & \epsilon y^{2k} \end{array} \right). \quad (46)
   \]

2. Series \( O_{k,\epsilon}^d, k \geq 1, d \geq 2k+1, \; \epsilon = \pm 1, \; c = (c_0, \ldots, c_{k-1}) \in \mathbb{R}^k \) and we set \( \epsilon = 1 \), if \( d = 2m+1 \) is odd.

   The operator \( L \) is defined by (44) with \( v = \text{tr} L \) and \( u = -\det L \) given by

   \[
   v = \alpha xy^{2k-1} + y^k(c_{k-1}y^{k-1} + \cdots + c_1y + c_0), \quad u = \epsilon y^d, \quad \alpha = kc_0^2 \left( 1 - \frac{k}{d} \right) \neq 0.
   \]
3. Series $P_{k,c}^{k,c}, k \geq 1, s \geq 2k, \epsilon = \pm 1, c = (c_0, \ldots, c_{k-1}) \in \mathbb{R}^k$.

The operator $L$ is defined by \textbf{(44)} with $v = \text{tr} L$ and $u = -\det L$ given by

$$v = \alpha xy^s + y^{s-k+1}(c_{k-1}y^{k-1} + \cdots + c_1y + c_0) + 2\epsilon y^k, \ u = -y^{2k}, \ \alpha = 2\epsilon kc_0 \neq 0.$$ 

4. Series $S_{k,c}^{2k,c}$ and $S_{c}^{k,c+1}, k \geq 1, c = (c_0, \ldots, c_{k-1}) \in \mathbb{R}^k$.

The operator $L$ is defined by \textbf{(44)} with $v = \text{tr} L$ and $u = -\det L$ given respectively by

$$v = \alpha xy^{2k-1} + y^k(c_{k-1}y^{k-1} + \cdots + c_1y + c_0), \ u = \epsilon y^{2k}, \ \alpha = \frac{k}{2}(c_0^2 + 4\epsilon) \neq 0,$$

$$v = \alpha xy^{2k} + y^{k+1}(c_{k-1}y^{k-1} + \cdots + c_1y + c_0), \ u = y^{2k+1}, \ \alpha = 2k + 1.$$ 

Proof. The idea of the proof is natural: since $L$ is basically defined by its trace and determinant, we will be looking for local coordinates $x, y$ in which $v = \text{tr} L$ and $u = -\det L$ have their “simplest” possible form. We start with two technical lemmas.

**Lemma 6.1.** Under assumptions of Theorem \textbf{6.1} there exist local coordinates $(x, y)$ such that for $u = -\det L$ one of the following holds:

(i) $u \equiv 0,$ \quad (ii) $u = \pm y^{2k},$ \quad (iii) $u = y^{2k-1}, \ k \in \mathbb{N}.$

Proof. In companion coordinates (see \textbf{(3)}), the function $u = -\det L$ satisfies the equation

$$u_x = g(x, y)u,$$ \hfill (47)

where $g(x, y) = \partial_y \text{tr} L$. Hence $u = f(y) \exp \left( \int_0^x g(t, y)dt \right)$ for some real analytic function $f(y)$.

If $f(y) \equiv 0$, we have Case (i). Otherwise, writing $f$ in the form $f(y) = \epsilon y^m h(y)$ with $\epsilon = \pm 1$, $h(0) > 0, m \in \mathbb{N}$, we get for $m = 2k$ and $m = 2k - 1$ respectively:

$$u = \pm \left( y^{2k} h(y) \exp \left( \int_0^x g(t, y)dt \right) \right)^{2k} \text{ or } u = \left( y^{2k-1} \exp \left( \int_0^x g(t, y)dt \right) \right)^{2k-1}.$$ 

Letting $y_{\text{new}}$ be the expression in brackets gives $u = \pm y_{\text{new}}^{2k}$ or $u = y_{\text{new}}^{2k-1}$, as required. \hfill \Box

This lemma brings $\det L$ to its simplest canonical form. After this we keep the $y$-coordinate fixed and simplify $v = \text{tr} L$ by changing the $x$-coordinate only.

**Lemma 6.2.** There exists a coordinate change of the form $(x_{\text{old}}, y) \mapsto (x, y)$ such that the $l_1^1$-component of $L$ in new coordinates equals identically 1.

Proof. Setting $x_{\text{old}} = g(x, y)$ and applying the standard transformation rule for components of an operator, we observe that the required condition is

$$\frac{l_2^1(g, y) + g_y l_1^1(g, y) - g_y l_1^2(g, y) - g_y^2 l_1^2(g, y)}{g_x} = 1,$$ 

where $l_j^i$ are the components of $L$ in the old coordinate system. Writing this relation in the form $g_x = F(g_y, g, y)$, we can locally solve it by Cauchy-Kovalevskaya theorem. It is important that $L$ is gl-regular, this allows us to choose initial conditions in such a way that $g_x(0, 0) \neq 0$ so that the coordinate transformation is invertible. \hfill \Box
Note that where equation (47) for $u$ may also assume that $l_2 = 1$. Now $L = \begin{pmatrix} f(y) & 1 \\ 0 & g(y) \end{pmatrix}$ can be reconstructed from relation (45). This yields series $M_{2k-1}$ and $M_{2k}$ for different $v$ respectively.

Next suppose $u \equiv 0$, while $v$ is not. In companion coordinates (see (3)), $v$ satisfies the Hopf equation $vv_y - v_x = 0$. This equation can be rewritten as

$$v_x = g(x,y)v \quad \text{with} \quad g = v_x,$$

which is similar to the above equation (17) for $u$. Just in the same way as in Lemma 6.1, we find a coordinate system in which $v = y^{2k-1}$ or $v = cy^{2k}$ for $k \geq 1, \epsilon = \pm 1$. By Lemma 6.2, we may also assume that $l_2 = 1$. Now $L = \begin{pmatrix} f(y) \\ 0 \end{pmatrix}$ can be reconstructed from relation (45). This yields series $M_{2k-1}$ and $M_{2k}$ for different $v$ respectively.

Now let $u \not\equiv 0$, but $dv \wedge du \equiv 0$. Combining Lemmas 6.1 and 6.2, we may assume that $u = y^{2m-1}$ or $u = \pm y^{2m}$ for $m \geq 1$, and $l_2 = 1$. Since $dv \wedge du \equiv 0$, we also know that $v_x \equiv 0$. Relation (45) implies that $l_2 = 0$ and we come to the operator of the form

$$L = \begin{pmatrix} f(y) & 1 \\ 0 & g(y) \end{pmatrix} \quad \text{with} \quad v = f + g \quad \text{and} \quad u = -fg.$$

It is straightforward to check that Nijenhuis condition in this case reads $f'_y(f - g) = 0$. In our case $f$ cannot be constant as in this case, since $L$ is nilpotent at the origin, we would necessarily have $f \equiv 0$, which contradicts our assumption that $u = -fg \not\equiv 0$. Therefore, we conclude that $f - g = 0$, meaning that $L$ is a Jordan block at each point. This yields series $N_{2k-1}$ and $N_{2k}$.

If $dv \wedge du \not\equiv 0$ at the point $p$, then $L$ is differentially non-degenerate and its normal form is $L_{nd}$ [2, Theorem 4.4]. Notice that in terms of Lemma 6.1 the non-degeneracy condition corresponds exactly to the case $u = y$ and below we exclude this case.

Finally we consider the most interesting case when $dv \wedge du \not\equiv 0$ (but $dv \wedge du = 0$ at $p$). As previously, we assume that $u = y^{2m+1}$ or $u = cy^{2m}$ for $m \geq 1, \epsilon = \pm 1$ and $l_2 = 1$. Computing $l_2$ from matrix relation (44) yields the following equation on $v$:

$$v_x = vv_y - \frac{1}{d}y(v_y)^2 + u_y,$$

where $d = 2m + 1$ or $2m$. This equation implies the following

**Lemma 6.3.** The function $v(x, y)$ can be written as $v = v_0(y) + y^s(\alpha x + F)$, where $\alpha \not\equiv 0, s \geq 1$ and $F(x, y)$ is a real analytic function with no constant or linear part.

**Proof.** Let $v(x, y) = v_0(y) + v_1(y)x + v_2(y)x^2 + \ldots$ be a solution of (49). Differentiating (49) w.r.t. $x$ we get

$$v_{xx} = v_xv_y + vv_{xy} - \frac{2}{d}yvv_{xy}.$$

Note that $v = v_0(y)$ satisfies this equation for initial conditions $v(0, y) = v_0(y), v_x(0, y) = v_1(y) \equiv 0$. By Cauchy-Kovalevskaya theorem this solution is unique. Hence, if $v_1 \equiv 0$, then $dv \wedge du \equiv 0$ which is wrong. Thus, in our case $v_1(y) = y^sr_1(y)$, where $r_1(0) = \alpha \not\equiv 0$.

Assume that $s = 0$. This means that $dv$ and $dy$ are linearly independent. We can introduce $x_{new} = v = \text{tr} L$, leaving $y$ the same. In these new coordinates, relation (44) gives

$$L = \begin{pmatrix} 1 & 0 \\ 0 & u_y^{-1} \end{pmatrix} \begin{pmatrix} x_{new} & 0 \\ u(y) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_y \end{pmatrix} = \begin{pmatrix} x_{new} & u_y \\ uu_y^{-1} & 0 \end{pmatrix}.$$
It is easy to see that $L$ at the origin $p$ is similar to the nilpotent Jordan block only for $u = y$. But in this case we get $L = L_{nd}$ falling into the previous case.

Thus, we have $s \geq 1$. Equating the coefficients of $x^i$ it the both sides of (49) yields

\[
(i + 1)v_{i+1}(y) = v_0(y)v'_i(y) + v'_0(y)v_i(y) - \frac{2}{d}yv'_0(y)v'_i(y) + \\
+ \sum_{j=1}^{i-1}(v_j(y)v'_{i-j}(y) - \frac{1}{d}yv'_j(y)v'_{i-j}(y)).
\]

(50)

If $v_1, \ldots, v_i$ are divisible by $y^s$, then $v'_1, \ldots, v'_i$ are divisible by $y^{s-1}$. As $v(0,0) = 0$, then $v_0$ is divisible by $y$. By formula (50) the coefficient $v_{i+1}$ is divisible by $y^s$. Thus, by induction all the coefficients $v_1, v_2, \ldots$ are divisible by $y^s$ and one writes $v = v_0 + y^s(xr_1(y) + \ldots) = v_0 + y^s(\alpha x + F)$, where $F$ is analytic and has no constant or linear parts. Lemma is proved.

Using Lemma 6.3 we introduce new coordinates $x_{new} = x + \frac{1}{\alpha}F + \tilde{v}_0$, $y_{new} = y$, where $\tilde{v}_0$ contains all the terms of $v_0$ of order $\geq s + 1$. In this new coordinate system (for which we continue using old notation $x$ and $y$) we have:

\[
v = p_s(y) + \alpha xy^s, \quad u = y^{2m+1} \text{ or } u = \epsilon y^{2m},
\]

(51)

where $\alpha \neq 0$, $m, s \geq 1$ and $p_s$ is polynomial of degree at most $s$.

This coordinate system is optimal in the sense that $v = \text{tr}L$ and $u = -\text{det} L$ cannot be simplified further. The last step is to distinguish those pairs of functions $v(x, y)$ and $u(x, y)$ from family (51) that indeed generate analytic perturbations of the nilpotent Jordan block $J$ via relation (44). The point is that (44) will generate a Nijenhuis operator $L$ for any $v$ and $u$, but we need only those of them which have no singularity at $p = (0, 0)$ and, moreover, such that $L(p)$ is similar to $J_0$.

Straightforward reconstruction of $L$, from (44) with $v$ and $u$ given by (51), shows that all the components of $L$ are non-singular and vanish at the origin except for $l_2^1$:

\[
L = \begin{pmatrix}
  v & \frac{v_{0,0} - \frac{1}{d}yv_s^2 + u'}{v_s} \\
  \frac{v_{0,0}}{d} & \frac{v_{0,0}}{d}
\end{pmatrix}
\]

where $d = 2m + 1$ or $d = 2m$ (power of $y$ in the formula for $u$).

The “troublesome” component, in more detail, reads:

\[
l_2^1 = s\alpha x^2y^{s-1}\left(1 - \frac{s}{d}\right) + x\left(\frac{1}{y}p_s + \left(1 - \frac{2}{d}\right)p'_s + \frac{p_s p'_s - \frac{1}{d}y(p'_s)^2 + u'}{\alpha y^s}\right).
\]

Notice that $p_s(0) = 0$ and therefore $\frac{1}{y}p_s$ is analytic. Hence, we only need to analyse the fraction

\[
\frac{p_s p'_s - \frac{1}{d}y(p'_s)^2 + u'}{\alpha y^s}
\]

(52)

This fraction must define an analytic function having value 1 at the origin (in order for $L(p)$ to be the standard nilpotent Jordan block). Thus, we need to solve a purely algebraic problem: find
all polynomials $p_s$, for which the denominator of (52) is divisible by $\alpha y^s$ so that this fraction is, in fact, a polynomial with free term equal to 1. We rewrite (52) as

$$p_s p'_s - \frac{1}{d} y(p'_s)^2 = -u' + \alpha y^s + \alpha_1 y^{s+1} + \cdots + \alpha_{s-1} y^{2s-1}, \quad (53)$$

where $\alpha \neq 0$ and $\alpha_i$ are, in general, arbitrary.

Let $p_s$ starts with a term of order $k \geq 1$, that is, $p_s = y^k(c_0 + c_1 y + \cdots + c_{s-k} y^{s-k})$. Then the smallest degree term in the l.h.s. of (53) is $k c_0^2 (1 - \frac{k}{2}) y^{2k-1}$. On the other hand, the term of the smallest degree in the r.h.s. is either $u' = \pm dy^{d-1}$ or $\alpha y^s$ (or both of them).

First, assume $s < d - 1$. Then we get $2k - 1 = s$ and furthermore $p_{2k-1} = c_0 y^k + \cdots + c_{k-1} y^{2k-1}$, where $c_1, \ldots, c_{k-1}$ are arbitrary and $c_0 \neq 0$. We also have $\alpha = k c_0^2 (1 - \frac{k}{2})$ obtaining, as a result, the series $O_{k,c}$.

Next, assume $d - 1 < s$. Then we get $d - 1 = 2k - 1$ and, thus, $u = c_0 y^2 k$. Equating the coefficients of $y^{2k-1}$ on both sides of (53) we get $\frac{k}{2} c_0^2 = -2k \epsilon$ and, thus, $u = -y^{2k}$ and $c_0 = \pm 2$. We write $p_s = \pm 2 y^k + c_1 y^{k+1} + \cdots + c_{s-k} y^s$ and substitute it into (53). Equating the coefficients of $y^{2k}, \ldots, y^{s-1}$ in the l.h.s. of (53) to zero we get, step by step, that $c_1 = c_2 = \cdots = c_{s-2k} = 0$. Hence, re-denoting $c_{s-2k+j} \mapsto c_{j-1}$ for $j = 1, \ldots, k$ we have:

$$p_s = y^{s-k+1} (c_{k-1} y^{k-1} + \cdots + c_1 y + c_0) \pm 2 y^k,$$

and equating the coefficients of $y^s$ in both sides of (53), we obtain $\alpha = \pm 2 k c_0 \neq 0$. This yields series $P_{s,c}^{k,c}$.

Finally, consider $d - 1 = s$. We have two possibilities. First, assume that $d = 2m$, i.e., $u = c_0 y^{2m}$. We get that $2k - 1 = 2m - 1$, $k = m$ and $v = \alpha xy^{m-1} + c_0 y^m + \cdots + c_{m-1} y^{2m-1}$ with $\alpha = \frac{m}{2} (c_0^2 + 4\epsilon) \neq 0$. Now assume that $d = 2m + 1$, i.e., $u = y^{2m+1}$. This yields $v = \alpha xy^{2m} + c_0 y^{m+1} + \cdots + c_{m-1} y^{2m}$ and $\alpha = 2m + 1$. This yields $S_c^{2m}\epsilon$ and $S_c^{2m+1}$ respectively (in the statement of the theorem we replace $m$ by $k$).

**Remark 6.1.** For the series $O$, $P$ and $S$ the canonical coordinate system is essentially unique (in some cases on which can simultaneously change the sign of $x$ and $y$). Indeed, these coordinates are those in which $u = -\det L$ and $v = \text{tr} L$ are given by (51). The integer parameters $m$ and $s$ involved in (51) are uniquely defined for given $u$ and $v$. Hence, $y$ can be reconstructed from $u$ (sometimes up to sign), and $x$ is determined, up to a constant factor, by the condition that $v(0, y)$ is a polynomial of degree $\leq s$. Finally, the rescaling of $x$ is chosen in such a way that at the origin we have $L(0, 0) = J_0$.

This implies that Nijenhuis operators from different series (or from the same series but with different parameters) are not equivalent to each other. The only exception is related to the above mentioned “canonical” transformation $(x, y) \mapsto (-x, -y)$ that changes the parameter $c \in \mathbb{R}^k$, but this change is easy to control.

We now apply the local classification of gl-regular Nijenhuis operators to study the existence (and examples) of such operators on closed two-dimensional surfaces.

Let $(M^2, L)$ be a gl-regular Nijenhuis manifold of dimension 2 (recall that we always assume them to be real analytic). Consider the set $\text{Sing}$ of singular points of $L$ where the algebraic type of $L$ changes. In our case, this means that the eigenvalues of $L$ collide, i.e.

$$\text{Sing} = \{ p \in M^2 \mid v^2 + 4u = 0 \}, \quad \text{where } v = \text{tr } L, \ u = -\det L,$$
unless \( v^2 + 4u \equiv 0 \) on \( M^2 \) meaning that \( L \) is similar to a Jordan block at each point.

From Theorem 6.1 we immediately obtain a local description of \( \text{Sing} \) in canonical coordinates \( x, y \):

- for \( L_{\text{nil}} \), \( N_{2k-1} \) and \( N_{2k} \), the singular set is empty;
- for \( L_{\text{nd}} \) the singular set is \( \text{Sing} = \{ x^2 + 4y = 0 \} \);
- for all the other series \( M, O, P \) and \( S \): \( \text{Sing}_{\text{loc}} = \{ y = 0 \} \).

Thus, locally \( \text{Sing} \) is a smooth curve. Since \( \text{Sing} \subset M^2 \) is closed, we may think of it as a submanifold consisting, perhaps, of several connected components:

\[
\text{Sing} = S_1 \cup \cdots \cup S_\ell.
\]

If \( M \) is compact, then each of them is an embedded circle. Next we can easily observe that all points from \( S_i \) relate to the same series (different components may, of course, relate to different series). However, the parameters of the series may change. This happens for series \( O, P \) and \( S \). Indeed, moving along \( \text{Sing}_{\text{loc}} = \{ y = 0 \} \) leads to the shift \( x_{\text{new}} = x - x_0 \) resulting in the following modification for \( v = \text{tr} L \) (whereas \( \det L \) remains unchanged):

\[
v = \alpha xy^s + c_{k-1}y^s + \cdots = \alpha (x_{\text{new}} + x_0)y^s + c_{k-1}y^s + \cdots = \alpha x_{\text{new}}y^s + (c_{k-1} + \alpha x_0)y^s + \cdots.
\]

In other words, all parameters remain fixed except for \( c_{k-1} \) which undergoes the shift \( c_{k-1} \mapsto c_{k-1} + \alpha x_0 \). Notice that if we move along \( S_i \) in a certain direction, then \( c_{k-1} \) is either strictly increasing or strictly decreasing. This leads us to the following conclusion.

**Proposition 6.1.** Singular points from the series \( O, P \) and \( S \) may not occur on closed gl-regular Nijenhuis 2-manifolds.

According to [5, Proposition], the same conclusion holds for differentially non-degenerate singular points (series \( L_{\text{nd}} \)) and therefore we obtain

**Proposition 6.2.** Let \((M^2, L)\) be a closed gl-regular Nijenhuis 2-manifold. Then

- either \( \text{Sing} \) is empty (i.e. all points of \( M^2 \) are of the same algebraic type),
- or each \( p \in \text{Sing} \) belongs to the series \( M \) and then automatically one of the eigenvalues of \( L \) is constant on \( M^2 \).

We are now ready to prove our final result.

**Proof of Theorem 1.3.** Consider the two options from Proposition 6.2. First assume that \( \text{Sing} = \emptyset \). Then \( L \) belongs to one of three generic types listed in the beginning of this Section:

(i) either \( L \) has two distinct real eigenvalues at each point of \( M^2 \);
(ii) or \( L \) has two complex conjugate eigenvalues at each point of \( M^2 \);
(iii) or \( L \) is similar to a Jordan block at each point of \( M^2 \).

In Case (i), at each point \( p \in M^2 \), we have an eigenbasis basis \( e_1, e_2 \in T_p M^2 \) where \( e_1 \)




 corresponds to the maximal eigenvalue at a given point. If we fix some Riemannian metric on \( M^2 \), we may assume that \( e_i \) are normalised so that \( |e_i| = 1 \). Since such \( e_i \) are defined up to \( \pm \), we have 4 different bases at each point. A priori it is not clear whether or not we can chose a smooth “moving frame” field on the whole manifold, but this can obviously be done on a finite sheeted covering \( \tilde{M}^2 \) of \( M \) (number of sheets is at most four). This implies that \( M^2 \) is parallelisable and hence is a torus. Therefore, \( M^2 \) is either a torus or Klein bottle and we obtain Case 2 of Theorem \ref{main-thm}.

In Case (ii), according to \cite[Theorem 6.1]{5} the complex eigenvalues \( \lambda, \bar{\lambda} \) of the Nijenhuis operator \( L \) are constant and we obtain Case 1 from Theorem \ref{main-thm}.

In Case (iii), at each point \( p \in M^2 \) we have a non-zero eigenvector \( e \in T_p M^2 \) and the same argument as above shows that on \( M^2 \) or on its two sheeted covering one can define a smooth vector field with no singular points. Hence \( M^2 \) is either a torus or Klein bottle. However, in this case we have one additional property that the automorphism group of a Jordan block consists of orientation preserving transformations, which allows us to define orientation on \( M^2 \). Hence, the Klein bottle is forbidden and we are lead to Case 3 of Theorem \ref{main-thm}.

Thus, the condition \( \text{Sing} = \emptyset \) necessarily implies one of the first three cases of Theorem \ref{main-thm}.

Finally, we consider the second option from Proposition \ref{prop}. This option implies that one of the eigenvalues of \( L \) is constant allowing us to consider a non-zero eigenvector related to this eigenvalue at each point and, in the same way as above, to construct a smooth vector field with no zeros either on \( M^2 \) or on its two-sheeted covering. This implies that \( M^2 \) is either a torus or Klein bottle and we obtain Case 4 of Theorem \ref{main-thm}.

Thus, the list of possibilities presented in Theorem \ref{main-thm} is complete. \[\square\]

We conclude this section with examples of Nijenhuis operators listed in Theorem \ref{main-thm}.

**Example 6.1.** Let \( T^2 \) be a torus with standard angle coordinates \( \phi_1 \) and \( \phi_2 \) defined modulo \( 2\pi \). For an operator \( L \) with two distinct eigenvalues at each point \( (\phi_1, \phi_2) \), we can distinguish three essentially different possibilities.

- Two constant eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Let \( \xi \) and \( \eta \) be two vector fields on \( T^2 \) that are linearly independent at each point (NB: there are many non-equivalent examples of such vector fields), then we define \( L \) by setting

\[
L(\xi) = \lambda_1 \xi \quad \text{and} \quad L(\eta) = \lambda_2 \eta.
\]

- One constant eigenvalue (w.l.o.g. \( \lambda_1 = 0 \)), the other \( \lambda_2 \) is not. In coordinates \( (\phi_1, \phi_2) \) we define \( L \) as

\[
L = \begin{pmatrix} 0 & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}
\]

with \( f(\phi_2) > 0 \) or \( f(\phi_2) < 0 \). Here \( \xi = \left( 1, \frac{g(\phi_1, \phi_2)}{f(\phi_2)} \right) \) is an eigenvector field related to the non-constant eigenvalue \( \lambda_2 = f(\phi_2) \).
Two non-constant eigenvalues $\lambda_1$ and $\lambda_2$. An obvious example is

$$L = \begin{pmatrix} f(\phi_1) & 0 \\ 0 & g(\phi_2) \end{pmatrix}, \quad f(\phi_1) < c < g(\phi_2). \quad (56)$$

This example can be modified by taking a finite-sheeted covering over this “standard” torus. On the covering torus, the above global diagonalisation of $L$ is not always possible.

Example 6.2. Each of the above examples (54), (55) and (56) can be naturally “transferred” to the Klein bottle $K^2$ that can be thought of as the quotient of $T^2$ with respect to the involution $\sigma : T^2 \to T^2$ given by $(\phi_1, \phi_2) \mapsto (-\phi_1, \phi_2 + \pi)$. We only need to make sure that $L$ is invariant with respect to $\sigma$. Namely, in the above three cases from Example 6.1 we assume in addition that

- $\xi$ is $\sigma$-invariant, whereas $\eta$ changes the direction under the action of $\sigma$, i.e., $d\sigma(\xi) = \xi$ and $d\sigma(\eta) = -\eta$,
- $f(\phi_2)$ is $\pi$-periodic, $g(\phi_1, \phi_2)$ is even w.r.t. $\phi_1$,
- $g(\phi_2)$ is $\pi$-periodic and $f(\phi_1)$ is even.

If these conditions are fulfilled, then the operators $L$ given by (54), (55) and (56) naturally descend to the quotient $K^2 = T^2/\sigma$.

The next is an example of a Nijenhuis operator on $T^2$ of Jordan block type (see Case 3 in Theorem 1.3).

Example 6.3. Assume that $L$ is a gl-regular operator $L$ on $T^2$ with a single eigenvalue $\lambda$ of multiplicity 2. The cases with constant and non-constant $\lambda$ are essentially different. If $\lambda = \text{const}$, then w.l.o.g. we may assume that $\lambda = 0$, i.e., $L$ is nilpotent.

- Consider two vector fields $\xi$ and $\eta$ on $T^2$ which are linearly independent at each point and define $L$ as follows:

$$L(\xi) = 0, \quad L(\eta) = \xi.$$

Then $L$ is a gl-regular nilpotent Nijenhuis operator on $T^2$ (notice that any nilpotent operator in dimension 2 is automatically Nijenhuis).

- The case with a non-constant eigenvalue on $T^2$ can be modelled as follows:

$$\begin{pmatrix} f(\phi_2) & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}, \quad g(\phi_1, \phi_2) > 0,$$

where $\phi_1, \phi_2$ denote usual angle coordinates on the torus as above.

Finally, we notice that examples corresponding to Case 4 of Theorem 1.3 on the torus $T^2$ and Klein bottle $K^2 = T^2/\sigma$ can be defined by the same formula as (55). The only difference is that now $f(\phi_2)$ vanish for some $\phi_2$ (but then $g(\phi_1, \phi_2)$ does not!). The operator $L$ will become nilpotent at such points, which will be automatically singular from series $M$. Notice that the topological structure of the eigenvector field $\xi$ related to the eigenvalue $f(\phi_2)$ may now be rather non-trivial in contrast to the case when $f \neq 0$.

We conjecture that the above list of examples essentially exhausts all possible real-analytic Nijenhuis operators on closed two-dimensional surfaces. In the smooth case, however, there are essentially different possibilities.
Appendix: On integration of some hydrodynamic type systems

Theorem 1.2 is closely related to another fundamental problem of solving systems of quasilinear PDEs of the form

\[ u^i_t = L^i_j(u)u^j_x, \]  

or a more general system

\[ u_{x_j-1} = L(u)u_{x_j}, \quad j = 1, \ldots, m. \]  

If \( L(u) = \begin{pmatrix} L^i_j(u) \end{pmatrix} \) is a Nijenhuis operator, such a system is known to be integrable. If \( L(x) \) is \( \mathbb{R} \)-diagonalisable, then after rewriting \( L \) in diagonal form by Haantjes theorem, the system (57) splits into \( n \) uncoupled Hopf equations \( u^i_t = \lambda(u^i) u^i_x \) that can be easily solved (see e.g. [9, Section 2.3]). However, at those points where \( L \) is not diagonalisable, this obvious idea does not work directly, although local analytic solutions still exist as (57) is a system of Cauchy-Kovalevskaya type.

This Appendix provides a helpful tool to integrate (57) and (58) near singular points where a Nijenhuis operator is not diagonalisable. It can be used most effectively if the operator \( L \) is \( \text{gl-regular} \), but does not require this assumption.

System (57) determines the dynamics of field variables \( u^i \) which, in turn, describe the dynamics of \( f_i(u) \), the coefficients of the characteristic polynomial of \( L \):

\[ \chi_L(\lambda) = \lambda^n - f_1(u)\lambda^{n-1} - \cdots - f_{n-1}(u)\lambda - f_n(u). \]

Unlike eigenvalues of \( L \), these functions are everywhere smooth and for this reason are more suitable for analysis of solutions of (57) near those points where the eigenvalues collide.

Let \( L(u) \) be a Nijenhuis operator (not necessarily differentially non-degenerate or \( \text{gl-regular} \)). We want to solve (57), i.e., find the solution \( u(t, x) \) with given initial conditions

\[ u^i(0, x) = u^i_0(x). \]  

Instead of solving this problem we shall try to solve a “simpler” problem (which is equivalent to it if \( L \) is differentially non-degenerate). Namely, instead of \( u(t, x) \) we will be looking for \( f_i(u(t, x)) \), the coefficients of the characteristic polynomial, with the corresponding initial condition

\[ f_i(u(0, x)) = f_i(u^i_0(x)) = v_i(x). \]  

Notice that the dependence \( f \) of \( u \) is explicit, but the inversion is not always possible. In this setting we have the following theorem.

**Theorem 7.1.** For \( n \) real analytic functions \( v_1(t), \ldots, v_n(t) \) defined from (60), consider the function

\[ r(\lambda, \mu) = \lambda^n - v_1(\mu)\lambda^{n-1} - v_2(\mu)\lambda^{n-2} - \cdots - v_{n-1}(\mu)\lambda - v_n(\mu) \]

and the matrix relation

\[ r(L, M) = 0, \]

where \( M = tL + x \text{Id} \) and \( L \) is a \( \text{gl-regular} \) \( n \times n \) matrix. Then
From this matrix relation, the coefficients $f_1, \ldots, f_n$ of the characteristic polynomial of $L$ can be uniquely expressed in a neighbourhood of $(t, x) = (0, 0)$ as real analytic functions in $t, x$ (by Implicit Function Theorem).

The functions $f_1(t, x), \ldots, f_n(t, x)$ so obtained are the coefficients of the characteristic polynomial of $L(u(t, x))$ where $u(t, x)$ is the solution of (57) with initial condition (59).

Proof. We first derive the system of PDEs that governs the dynamics of $f_1, \ldots, f_n$. These equations are easy to describe. Indeed, every Nijenhuis operator $L(u)$ satisfies the relation (see [5, Proposition 2.2])

$$
\left( \frac{\partial f}{\partial u} \right) L(u) = L_{\text{comp1}}(f) \left( \frac{\partial f}{\partial u} \right).
$$

Multiplying the both sides of this matrix relation with $u_x$ we get

$$
\left( \frac{\partial f}{\partial u} \right) L(u) u_x = L_{\text{comp1}}(f) \left( \frac{\partial f}{\partial u} \right) u_x
$$

and then using (57) and the standard chain rule $(\frac{\partial f}{\partial u}) u_x = f_x, (\frac{\partial f}{\partial u}) u_t = f_t$:

$$
f_t = L_{\text{comp1}}(f) f_x. \tag{61}
$$

If we denote $x = x^n$ and $t = x^{n-1}$ (index, not power!), then (61) coincides with one the equations from (30). Since we know the description for all (local) solutions of (30), we can simply obtain the required solution of (61) by setting $x^1 = \cdots = x^{n-2} = 0$ in the formulas given in Theorem 1.2. This immediately leads to the conclusion of Theorem 7.1.

Remark 7.1. Can we reconstruct the corresponding solution $u(x, t)$ of the original system (57) from $f(x, t)$? If $L$ is differentially non-degenerate, i.e., $f_1(u), \ldots, f_n(u)$ are functionally independent as functions of $u = (u_1, \ldots, u_n)$, then the answer is positive. This can be done just by inverting the map $u \mapsto f = f(u)$. Thus, for differentially non-degenerate Nijenhuis operators, Theorem 7.1 gives a way for solving (57) near those points where the eigenvalues of $L$ collide.

If the differential non-degeneracy condition violates at some points (but not identically), then the relation $f(x, t) = f(u(x, t))$ will still provides strong algebraic restrictions which might be sufficient for unique reconstruction of $u(x, t)$.

Another useful application of Theorem 7.1 might be detection of singularities (like gradient catastrophe) for solutions $u(x, t)$. If we can find $f(x, t)$ and then observe that this solution is singular at a certain point $(x, t)$, then $u(x, t)$ will be singular too.

Remark 7.2. Theorem 7.1 can be naturally adapted for system (58). We simply need to replace $(t, x)$ with $(x^1, \ldots, x^m)$ and set $M = x^1L^{-1} + \cdots + x^{m-1}L + x^m \text{Id}$. The initial conditions should be taken in the form $u'(0, \ldots, 0, x^m) = u_0'(x^m)$.

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