Distributed Location Optimization for Sensors with Limited Range Heterogeneous Capabilities using Generalized Voronoi Partition

K.R. Guruprasad\(^{a,1}\), Debasish Ghose\(^b\)

\(^a\)Department of Mechanical Engineering, National Institute of Technology Karnataka, Surathkal, 575025, India.

\(^b\)Guidance, Control, and Decision Systems Laboratory, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, 560012, India.

Abstract

In this paper we use a generalization of the Voronoi partition to formulate and solve a heterogeneous distributed locational optimization problem for autonomous agents having limited range sensors. Agents equipped with sensors having heterogeneity in their capabilities, communication equipment, and computational capability are to be optimally deployed in a domain of interest. The optimal deployment is found to be a variation of the generalized centroidal Voronoi configuration, where the sensors are located at the centroids of the corresponding generalized Voronoi cells. We provide a few formal results on stability, convergence, and on spatial distributedness of the proposed control laws under some constraints on the agents’ speeds such as limit on maximum speed and constant speed. We support the theoretical results with illustrative simulation results.

Key words: Voronoi partition; Sensor Coverage; Locational Optimization

1 Introduction

1.1 Multi-Agent Systems

Technological advances in areas such as wireless communication, autonomous vehicular technology, computation, and sensors, facilitate the use of large number of agents (UAVs, mobile robots, autonomous vehicles etc.), equipped with sensors, communication equipment, and computation ability, to cooperatively achieve various tasks in a distributed manner. Distributed multi-agent systems have been shown to achieve and maintain formations, move as flocks while avoiding obstacles, etc., thus mimicking their biological counterparts. They can also be used in applications such as search and rescue, surveillance, multiple source identification, and cooperative transportation. The major advantages of distributed systems are immunity to failure of individual agents, their versatility in accomplishing multiple tasks, simplicity of agents’ hardware, and requirement of only minimal local information. At the same time it is important to design distributed control laws that guarantee stability and convergence to the desired collective behavior under limited information and evolving network configurations. One very useful application of the multi-agent systems is sensor network, where a group of autonomous agents perform cooperative sensing of a large geographical area. In this paper, we address the problem of optimal deployment of autonomous agents equipped with sensors, communication equipment, and computational capability.

1.2 Related Literature

Advances in the fields of wireless communication, sensors, computation etc., have led to increased research interest in the area of sensor networks, where a large number of sensors with limited communication capabilities are deployed in the domain of interest. Cassandras and Li [1] provide a survey on sensor networks. An example of such a sensor network is a network of satellites equipped with imaging sensors and used to obtain the map of a large geographical area where each satellite provides the map of a small area. Li and Cassandras [2] represent frequency of occurrence of an event as...
a density function. The network model is probabilistic and assumes that the sensors make observations independently and maximize the expected event detection frequency, incorporating communication costs. Zou and Chakrabarty [3] use the concept of virtual force to solve a similar problem. Hussien and Stipanovic [4]-[5] define the effectiveness of coverage so as to ensure that at least $C^*$ measurements are made at each point in the mission domain, where $C^*$ is a specified threshold on a suitably defined coverage measure. In [5], they ensure collision avoidance, while flocking behavior with collision avoidance is addressed in [6]. Hussein [7] uses Kalman filter for multi-agent coverage, in a setting where the sensors have noise.

One of the main problems addressed in sensor networks is optimally locating the sensors so as to maximize the measurement quality. This class of problem belongs to the problem of locational optimization or facility location [8,9]. A centroidal Voronoi configuration is a standard solution for this class of problems [10]. Voronoi decomposition or Dirichlet tessellation is a widely used partitioning scheme. It finds application in image processing, CAD, sensor coverage, multi-agent search [11] and many more areas. Cortes et al. [12] address the problem of optimal deployment of sensors with limited range in a spatially distributed manner using Voronoi partition. Pimenta et al. [13] follow an approach similar to [12] to address a problem with heterogeneous robots. The authors let the sensors be of different types (in the sense of having different footprints) and relax the point robots assumptions. Power diagram (or Voronoi diagram in Laguerre geometry), a generalization of the standard Voronoi partition, is used to account for different footprints of the sensors (assumed to be discs). Due to assumption of finite size of robots, the robots are assumed to be discs and a free Voronoi region that excludes the robot footprints, is defined and a constrained locational optimization problem is solved. These authors also extend the results to non-convex environments. In [15], authors consider agents with sensors having anisotropic effectiveness, in the sense that effectiveness of the sensor at a point in space depends on the direction along with the Euclidian distance from the sensor. These authors use a non-Euclidian distance measure and use an anisotropic Voronoi partition to solve the problem of optimal sensor deployment.

1.3 Main contributions

Most authors consider sensor network to be homogeneous in nature. Whereas, in practical problems, the sensors may have different capabilities even though they are similar in their functionality. The heterogeneity in capabilities could be due to various reasons, the chief being the difference in specified performance. In this paper we address a locational optimization problem for sensors with heterogeneous capabilities. Voronoi decomposition is one of the tools used in locational optimization problems. But the existing Voronoi decomposition scheme and its variations cannot be used for solving the heterogeneous locational optimization problem. In this paper, we propose a generalization of Voronoi partition, based on the standard Voronoi partition and its variations. Here a concept of node functions is introduced in place of the usual distance measure. We use this generalized Voronoi partition to formulate the heterogeneous locational optimization problem. The mobile sensors are assumed to have heterogeneous capabilities in terms of the sensor effectiveness. A density distribution is used as a measure of probability of occurrence of an event of interest. We show that the optimal deployment configuration is a variation of the centroidal Voronoi configuration. We propose a proportional control law to make the sensors move toward the optimal configuration. Assuming first order dynamics for the mobile sensors, we prove, using LaSalle’s invariance principle, that the trajectories of the sensors converge to the optimal configuration globally and asymptotically. We further analyze the problem in presence of some constraints on the agents’ speeds and with limit on sensor range. Some preliminary results on heterogeneous limited range locational optimization problem were reported in [14].

1.4 Organization of the paper

In Section 2, we provide a few mathematical concepts used in this paper. We propose a generalization of Voronoi partition in Section 3. Section 4 introduces the distributed heterogeneous locational optimization problem along with the objective function, its critical points, the control law and its properties. In Section 5 we analyze the problem with limit on agents’ maximum speed and with a constraint on agents to move with a constant speed initially and slow down as they reach the critical points. We consider the sensor range limits and address the limited range distributed heterogeneous locational optimization problem in Section 6. Section 7 provides illustrative simulation results and the paper is concluded in Section 8 with some discussions.

2 Mathematical preliminaries

In this section we preview a few mathematical concepts from graph theory, spatially distributed functions, LaSalle’s invariance principle and Liebniz theorem used in the present work.

2.1 Concepts from graph theory

Graphs are extensively used in analysis of multi-agent systems and sensor networks. Graph theory [20] provides an excellent tool to represent connectivity of agents or
A graph $G = (\mathcal{U}, \mathcal{E})$ consists of a vertex set $\mathcal{U}$, and an edge set $\mathcal{E} \in \mathbb{R}^{2|\mathcal{U}|}$. A graph $G$ is said to be undirected if $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$. The map $N_G : \mathcal{U} \rightarrow 2^\mathcal{U}$ associates the set $N_G(i)$ of its neighbors in $G$ with vertex $i$. A graph is complete if $\mathcal{E} = \mathcal{U} \times \mathcal{U} \setminus \text{diag}(\mathcal{U} \times \mathcal{U})$, where $\text{diag}(\mathcal{U} \times \mathcal{U}) = \{(u, u) \in \mathcal{U} \times \mathcal{U}\}$. A path connecting vertex $i$ to $j$ is a sequence of vertices $\{i_0 = i, i_1, \ldots, i_k, i_{k+1} = j\}$ with the property that $(i_l, i_{l+1}) \in \mathcal{E}$ for all $l \in \{0, \ldots, k\}$. A graph $G$ is said to be connected if there exists a path connecting any two vertices.

### 2.2 Spatially-distributed functions

One of the desirable properties of a multi-agent system is spatial distributedness. A centralized system requires that all agents are accessible to the central controller and the failure of the central controller leads to failure of the entire system. Here, we formally define spatially-distributed functions.

A function $f : Q^N \rightarrow Y^N$ is **spatially distributed** over graph $G$, if there exist $f_i : Q \times \mathbb{P} \rightarrow Y$, for $i \in \{1, 2, \ldots, N\}$, such that

$$f_i(P) = \hat{f}_i(p_i, N_G(p_i, P))$$

for every $P \in \mathbb{P}_n$. Here $f_i$ is the $i$-th component of $f$ and $Y$ is the range of $f_i$.

For a spatially distributed function, the local information is sufficient to compute its value at any given node in the graph. That is, the information available from the given node itself, and that from its neighbors $N_G(p_i, P)$, is sufficient to evaluate the function at that node.

### 2.3 LaSalle’s invariance principle

Here we state LaSalle’s invariance principle [21,22] used widely to study the stability of nonlinear dynamical systems. We state the theorem as in [23] (Theorem 3.8 in [23]).

Consider a dynamical system in a domain $D$

$$\dot{x} = f(x), \quad f : D \rightarrow \mathbb{R}^d$$  \hspace{1cm} (1)

Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function and assume that

(i) $M \subset D$ is a compact set, invariant with respect to the solutions of (1).

(ii) $\dot{V} \leq 0$ in $M$.

(iii) $E : \{x : x \in M$, and $V(x) = 0\}$; that is, $E$ is set of all points of $M$ such that $\dot{V}(x) = 0$.

(iv) $N$ is the largest invariant set in $E$.

Then, every solution of (1) starting from a point in $M$ approaches $N$ as $t \rightarrow \infty$.

Here, by **invariant set** we mean that if the trajectory is within the set at some time, then it remains within the set for all time. Important differences of the LaSalle’s invariance principle as compared to the Lyapunov theory are (i) $V$ is required to be negative semi-definite rather than negative definite and (ii) the function $V$ need not be positive definite (see Remark on Theorem 3.8 in [23], pp 90-91).

### 2.4 Leibniz theorem and its generalization

The Leibniz theorem is widely used in fluid mechanics [26], and shows how to differentiate an integral whose integrand as well as the limits of integration are functions of the variable with respect to which differentiation is done. The theorem gives the formula

$$\frac{d}{dy} \int_{a(y)}^{b(y)} F(x, y) dx = \int_{a}^{b} \frac{\partial F}{\partial y} dx + \frac{db}{dy} F(b, y) - \frac{da}{dy} F(a, y)$$  \hspace{1cm} (2)

Eqn. (2) can be generalized for a $d$-dimensional Euclidean space as

$$\frac{d}{dy} \int_{V(y)} F(x, y) dV = \int_V \frac{\partial F}{\partial y} dV + \int_S n(x).u(x)F dS$$  \hspace{1cm} (3)

where, $V \subset \mathbb{R}^d$ is the volume in which the integration is carried out, $dV$ is the differential volume element, $S$ is the bounding hypersurface of $V$, $n(x)$ is the unit outward normal to $S$ and $u(x) = \frac{ds}{dy}(x)$ is the rate at which the surface moves with respect to $y$ at $x \in S$.

### 3 Generalization of the Voronoi partition

Voronoi partition [16,17] is a widely used scheme of partitioning a given space and finds applications in many fields such as CAD, image processing and sensor coverage. We briefly preview the standard Voronoi partition and then present a generalized Voronoi partition used in this work.

#### 3.1 Standard Voronoi partition

A collection $\{W_i\}, i \in \{1, 2, \ldots, N\}$ of subsets of a space $X$ with disjoint interiors is said to be a partition of $X$ if $\bigcup_i W_i = X$.

Let $Q \subset \mathbb{R}^d$, be a convex polytope in $d$-dimensional Euclidean space. Let $P = \{p_1, p_2, \ldots, p_N\}, p_i \in Q$, be the
The Voronoi partition generated by $\mathcal{P}$ with respect to an Euclidean norm is the collection \{\(V_i(\mathcal{P})\)\}_{i \in \{1,2,\ldots,n\}} and is defined as,

\[ V_i(\mathcal{P}) = \{ q \in Q | \| q - p_i \| \leq \| q - p_j \|, \forall p_j \in \mathcal{P} \} \quad (4) \]

where, $\| \cdot \|$ denotes the Euclidean norm.

The Voronoi cell $V_i$ is the collection of those points which are closest (with respect to the Euclidean metric) to $p_i$ compared to any other point in $\mathcal{P}$. The boundary of each Voronoi partition is the union of a finite number of line segments forming a closed $C^0$ curve. In $\mathbb{R}^2$, the intersection of any two Voronoi cells is either null, a line segment, or a point. In $d$ dimensional space, the boundaries of the Voronoi cells are unions of convex subsets of at most $d - 1$ dimensional hyperplanes in $\mathbb{R}^d$ and the intersection of two Voronoi cells is either a convex subset of a hyperplane or a null set. Each of the Voronoi cells is a topologically connected non-null set.

Basic components of the Voronoi partition are

i) A space which to be partitioned.
ii) A set of sites, or nodes or generators.
iii) A distance measure such as the Euclidean distance.

### 3.2 Generalization of Voronoi partition

Here we present a generalization of the Voronoi partition considering the heterogeneity in the sensors’ capabilities. We can find several extensions or generalizations of Voronoi partition to suit specific applications [9,18,19]. Herbert and Seidel [24] have introduced an approach in which, instead of the site set, a finite set of real-valued functions $f_i : D \mapsto \mathbb{R}$ is used to partition the domain $D$. Standard Voronoi partition and other known generalizations can be extracted from this abstract general form.

In this paper we define a generalization of the Voronoi partition to suit our application, namely the heterogenous multi-agent search. We use,

i) The domain of interest as the space to be partitioned.
ii) The set site as the set of points in the domain of interest which are the positions of the agents in it.
iii) A set of node functions in place of a distance measure.

Consider a space $Q \subset \mathbb{R}^d$, a set of points called nodes or generators $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$, $p_i \in Q$, with $p_i \neq p_j$, whenever $i \neq j$, and monotonically decreasing analytic functions $f_i : \mathbb{R}^+ \mapsto \mathbb{R}$, where $f_i$ is called a node function for the $i$-th node. Define a collection \{\(V_i\)\}, $i \in \{1,2,\ldots,N\}$, with mutually disjoint interiors, such that

$Q = \bigcup_{i \in \{1,2,\ldots,N\}} V_i$, where $V_i$ is defined as

\[ V_i = \{ q \in Q | f_i(\| p_i - q \|) \geq f_j(\| p_j - q \|) \quad \forall j \neq i, \quad j \in \{1,2,\ldots,N\} \} \quad (5) \]

We call \{\(V_i\)\}, $i \in \{1,2,\ldots,N\}$, as a generalized Voronoi partition of $Q$ with nodes $\mathcal{P}$ and node functions $f_i$. Note that

1) $V_i$ can be topologically non-connected and may contain other Voronoi cells.
2) In the context of the problem discussed in this paper, $q \in V_i$ means that the $i$-th agent/sensor is the most effective in sensing at point $q$. This is reflected in the $\geq$ sign in the definition. In standard Voronoi partition used for the homogeneous case, $\leq$ sign for distances ensured that $i$-th sensor is most effective in $V_i$.
3) The condition that $f_i$ are analytic implies that for every $i,j \in \{1,2,\ldots,N\}$, $f_i - f_j$ is analytic. By the property of real analytic functions [25], the set of intersection points between any two node functions is a set of measure zero. This ensures that the intersection of any two cells is a set of measure zero, that is, the boundary of a cell is made up of the union of at most $d - 1$ dimensional hyperplanes of $\mathbb{R}^d$. Otherwise the requirement that the cells should have mutually disjoint interiors may be violated. Analyticity of the node functions $f_i$ is a sufficient condition to discount this possibility.

**Theorem 1** The generalized Voronoi partition depends at least continuously on $\mathcal{P}$.

**Proof:** If $V_i$ and $V_j$ are adjacent cells, then all the points $q \in Q$ on the boundary common to them is given by $\{ q \in Q | f_i(\| p_i - q \|) = f_j(\| p_j - q \|) \}$, that is, the intersection of corresponding node functions. Let the $j$-th agent move by a small distance $dp$. This makes the common boundary between $V_i$ and $V_j$ move by a distance, say $dx$. Now as the node functions are monotonically decreasing and are continuous, it is easy to see, that $dx \to 0$ as $dp \to 0$. This is true for any two $i$ and $j$. Thus, the Voronoi partition depends continuously on $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$. $\square$

### 3.3 Special cases

The name ‘generalized Voronoi partition’ suggests, that by suitably selecting parameters like the node functions, one should get the standard Voronoi partition and its generalizations as special cases. Below we discuss a few interesting special cases.
Case 1: Weighted Voronoi partition

We consider multiplicatively and additively weighted Voronoi partitions as special cases. Let

\[ f_i(r_i) = -\alpha_i r_i - d_i \]  

where, \( r_i = \| p_i - q \| \) and, \( \alpha_i \) and \( d_i \) take finite positive real values for \( i = 1, 2, \ldots, N \).

Thus,

\[ V_i = \{ q \in Q | \alpha_i r_i + d_i \leq \alpha_j r_j + d_j, \quad \forall j \neq i, \quad j \in \{1, 2, \ldots, N\} \} \]

The partition \( \{V_i\} \) is called a multiplicatively and additively weighted Voronoi partition. \( \alpha_i \) are called multiplicative weights and \( d_i \) are called additive weights. With this generalization, the Voronoi cells no longer possess the nice property of being topologically connected sets. The Voronoi cells could be made up of disjoint patches and one or more Voronoi cells can get embedded inside another cell.

Case 2: Standard Voronoi partition

The standard Voronoi partition can be obtained as a special case of (5) when the node functions are \( f_i(r_i) = -r_i \).

\[ V_i = \{ q \in Q | \| p_i - q \| \leq \| p_j - q \| \quad \forall j \neq i, \quad j \in \{1, 2, \ldots, N\} \} \]

It can be shown that if the node functions are homogeneous \( f_i(.) = f(.) \) for each \( i \in \{1, 2, \ldots, N\} \), then the generalized Voronoi partition gives the standard Voronoi partition.

Case 3: Power diagram

Power diagram PV or Voronoi diagram in Laguerre geometry used in [13] is defined as

\[ PV_i = \{ q \in Q | d_p(q, p_i) \leq d_p(q, p_j), i \neq j \} \]

where, \( d_p(q, p_i) = \| p_i - q \|^2 - R_p^2 \), the power distance between \( q \) and \( p_i \), with \( R_p > 0 \) being a parameter fixed for a given node \( p_i \). In the context of robot coverage problem addressed in [13], \( R_p \) represents the radius of foot print for \( i \)-th robot. It is easy to see that the power diagram can be obtained from the generalized Voronoi partition (5) by setting

\[ f_i(\| q - p_i \|) = \| p_i - q \|^2 - R_p^2 \]

with \( R_p \) as a parameter specific to each node function.

It can be noted that the Voronoi partitions with non-Euclidean metric or pseudo-metric, and with objects such as lines, curves, discs, polytopes, etc., instead of points, as generators, can also be viewed as special cases of the generalized Voronoi partition (5).

Case 4: Voronoi partition based on Non-Euclidean distance

One of the ways in which a standard Voronoi partition can be generalized is use of non-Euclidean distance. The distance measure used depends on the application. As an example, in [15], authors use a non-Euclidian distance measure to incorporate anisotropy in sensor effectiveness. They use \( d(q, p_i) = \| p_i - q \|_{L_i} \), with \( \| p_i - q \|_{L_i} = (q - p_i)^T L_i (q - p_i) \), where

\[ L_i = F_i^T F_i \]

with

\[ F_i = \begin{bmatrix} c/a & 0 & \cos \theta_i \sin \theta_i \\ 0 & c/b & -\sin \theta_i \cos \theta_i \end{bmatrix} \]

By setting the node function as \( f_i(\| p_i - q \|_{L_i}) \), in the generalized Voronoi partition (5), we get the generalization of Voronoi diagram used in [15].

Case 5: Other possible variations

Other possible variations of the Voronoi partition are using objects other than points as sites/nodes and generalization of the space to be partitioned. It is easy to see that these generalizations can be obtained by specific site sets and the spaces.

3.4 Generalized Delaunay graph

Delaunay graph is the dual of Voronoi partition. Two nodes are said to be neighbors (connected by an edge), if the corresponding Voronoi cells are adjacent. This concept can be extended to generalized Voronoi partitioning scheme. For the sake of simplicity we call such a graph a Delaunay graph, \( G_D \). Note that the generalized Delaunay graph, in general, need not have the property of Delaunay triangulation, in fact, it need not even be a triangulation.

Two nodes are said to be neighbors in a generalized Delaunay graph, if the corresponding generalized Voronoi cells are adjacent, that is, \( (i, j) \in E_{G_D} \), the edge set corresponding to the graph \( G_D \), if \( V_i \cap V_j \neq \emptyset \).
4 Heterogeneous locational optimization problem

Here we formulate and solve heterogeneous locational optimization problem (HLOP) for a mobile sensor network. Let \( Q \subseteq \mathbb{R}^d \) be a convex polytope, the space in which the sensors have to be deployed; \( \phi : Q \rightarrow [0, 1] \), be a continuous density distribution function; \( P = \{ p_1, p_2, \ldots, p_N \} \), \( p_i \in Q \) be configuration of \( N \) sensors; \( f_i : \mathbb{R}^+ \rightarrow \mathbb{R} \), \( i \in \{ 1, \ldots, N \} \), be analytic, monotonically decreasing function corresponding to the \( i \)-th node, the sensor effectiveness function of \( i \)-th agent; and \( V_i \subseteq Q \) be the generalized Voronoi cell corresponding to the \( i \)-th node/sensor.

The density \( \phi(q) \) is the probability of an event of interest occurring in \( q \in Q \), indicating the importance of measurement at the given point in \( Q \). As \( \phi(q) \rightarrow 1 \), the importance of measurement at \( q \) increases as the probability of occurrence of an event of interest is higher. The objective of the problem is to deploy the sensors in \( Q \) so as to maximize the probability of detection of an event of interest. Let \( f_i \) be the variation of the sensor effectiveness with the Euclidean distance. It is natural to assume \( f_i \) to be monotonically decreasing.

In case of homogeneous sensors, the sensor located in Voronoi cell \( V_i \) is closest to all the points \( q \in V_i \) and hence, by the strictly decreasing variation of sensor’s effectiveness with distance, the sensor is most effective within \( V_i \). Thus, the Voronoi decomposition leads to optimal partitioning of the space in the sense that, each sensor is most effective within its corresponding Voronoi cell. In the heterogeneous case too, it is easy to see that each sensor is most effective in its own generalized Voronoi cell, by the very definition of the generalized Voronoi decomposition. Now, as the partitioning is optimal, we need to find the location of each sensor within its generalized Voronoi cell.

4.1 The objective function

Consider the objective function to be maximized,

\[
\mathcal{H}(P) = \int_Q \max_i \{ f_i(||q - p_i||) \} \phi(q)dQ
\]

\[
= \sum_i \int_{V_i} f_i(||q - p_i||) \phi(q)dQ
\]

where \( ||\cdot|| \) is the Euclidean distance. Note that the generalized Voronoi decomposition splits the objective function into a sum of contributions from each generalized Voronoi cell. Hence the optimization problem can be solved in a spatially distributed manner, that is, the optimal configuration can be achieved, by each sensor solving the part of objective function corresponding to its cell using only local information.

**Lemma 2** The gradient of the multi-center objective function (12) with respect to \( p_i \) is given by

\[
\frac{\partial \mathcal{H}}{\partial p_i} = \int_{V_i} \phi(q) \frac{\partial f_i(r_i)}{\partial p_i} dQ
\]

where, \( r_i = ||q - p_i|| \).

**Proof.** Let us rewrite (12) as

\[
\mathcal{H} = \sum_i \mathcal{H}_i
\]

where, \( \mathcal{H}_i = \int_{V_i} f_i(r_i)\phi(q)dQ \). Now,

\[
\frac{\partial \mathcal{H}}{\partial p_i} = \sum_j \frac{\partial \mathcal{H}_j}{\partial p_i}
\]

Applying the generalised form of the Leibniz theorem [26]

\[
\frac{\partial \mathcal{H}}{\partial p_i} = \int_{V_i} \phi(q) \frac{\partial f_i(r_i)}{\partial p_i} dQ
\]

\[
+ \sum_{j \in N_i} \int_{A_{ij}} n_{ij}(q).u_{ij}(q)\phi(q)f_i(r_i)dQ
\]

\[
+ \sum_{j \in N_i} \int_{A_{ij}} n_{ji}(q).u_{ji}(q)\phi(q)f_j(r_j)dQ
\]

where,

1. \( N_i \) is the set of indices of agents which are neighbors of the \( i \)-th agent in \( G_D \), the generalized Delaunay graph.
2. \( A_{ij} \) is the part of the bounding surface common to \( V_i \) and \( V_j \).
3. \( n_{ij}(q) \) is the unit outward normal to \( A_{ij} \) at \( q \in A_{ij} \). Note that \( n_{ij}(q) = -n_{ji}(q) \), \( \forall q \in A_{ij} \).
4. \( u_{ij}(q) = \frac{\partial A_{ij}}{\partial p_i}(q) \), the rate of movement of the boundary at \( q \in A_{ij} \) with respect to \( p_i \). Note that \( u_{ij}(q) = u_{ji}(q) \).
5. Note also that \( f_i(r_i) = f_j(r_j) \), \( \forall q \in A_{ij} \), by definition of the generalized Voronoi partition.

By (3)-(5) above, and as \( \phi \) is continuous, it is clear that for each \( j \in N_i \), \( \int_{A_{ij}} n_{ij}(q).u_{ij}(q)\phi(q)f_i(r_i)dQ = -\int_{A_{ij}} n_{ji}(q).u_{ji}(q)\phi(q)f_j(r_j)dQ \)

Hence,

\[
\frac{\partial \mathcal{H}}{\partial p_i} = \int_{V_i} \phi(q) \frac{\partial f_i(r_i)}{\partial p_i} dQ \]

\[ \square \]
4.2 The critical points

The gradient of the objective function (12) with respect to $p_i$, the location of the $i$-th node in $Q$, can be determined using (13) (by Lemma 2) as

$$\frac{\partial H}{\partial p_i} = \int_{V_i} \phi(q) \frac{\partial f_i(r_i)}{\partial p_i} dQ = \int_{V_i} \phi(q) \frac{\partial f_i(r_i)}{\partial (r_i)} (p_i - q) dQ$$

$$= - \int_{V_i} \tilde{\phi}(q)(p_i - q) dQ = - M_{V_i}(p_i - \tilde{C}_{V_i})$$

(17)

where, $r_i = \|q - p_i\|$ and $\tilde{\phi}(q) = - \phi(q) \frac{\partial f_i(r_i)}{\partial (r_i)}$. As $f_i, i \in \{1, \ldots, N\}$ is strictly decreasing, $\tilde{\phi}(q)$ is always non-negative. Here $M_{V_i}$ and $\tilde{C}_{V_i}$ are interpreted as the mass and centroid of the cell $V_i$ with $\tilde{\phi}$ as density. Thus, the critical points are $p_i = \tilde{C}_{V_i}$, and such a configuration $\mathcal{P}$, of agents is called a generalized centroidal Voronoi configuration.

Theorem 3 The gradient, given by (17), is spatially distributed over the Delaunay graph $\mathcal{G}_D$.

Proof. The gradient (17) with respect to $p_i \in \mathcal{P}$, the present configuration, depends only on the corresponding generalized Voronoi cell $V_i$ and values of $\phi$ and the gradient of $f_i$ within $V_i$. The Voronoi cell $V_i$ depends only on the neighbors $\mathcal{N}_{\mathcal{G}_D}(p_i, \mathcal{P})$ of $p_i$. Thus, the gradient (17) can be computed with only local information, that is, the neighbors of $p_i$ in $\mathcal{G}_D$. $\square$

The critical points are not unique, as with the standard Voronoi partition. But in the case of a generalized Voronoi partition, some of the cells could become null and such a condition can lead to local minima.

4.3 The control law

Let us consider the sensor dynamics as

$$\dot{p}_i = u_i$$

(18)

Consider the control law

$$u_i = -k_{prop}(p_i - \tilde{C}_{V_i})$$

(19)

Control law (19) makes the mobile sensors move toward $\tilde{C}_{V_i}$ for positive $k_{prop}$. If, for some $i \in \{1, \ldots, N\}, V_i = \emptyset$, then we define $\tilde{C}_{V_i} = p_i$.

It is not necessary that $\tilde{C}_{V_i} \in V_i$, but $\tilde{C}_{V_i} \in Q$ is true always and this fact ensures that $Q$ is an invariant set for (18) under (19).

It is easy to see, that the control law (19) is spatially distributed in the generalized Delaunay graph.

Theorem 4 The trajectories of the sensors governed by the control law (19), starting from any initial condition $\mathcal{P}(0)$, will asymptotically converge to the critical points of $\mathcal{H}$.

Proof. Here we use LaSalle’s invariance principle discussed earlier. Consider $V(\mathcal{P}) = -\mathcal{H}$, where $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ represents the configuration of $N$ agents/sensors.

$$\dot{V}(\mathcal{P}) = -\frac{d\mathcal{H}}{dt}$$

$$= - \sum_i \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i$$

$$= -2\alpha k_{prop} \sum_i M_{V_i}(p_i - \tilde{C}_{V_i})^2$$

(20)

We observe that, $V: Q \mapsto \mathbb{R}$ is continuously differentiable in $Q$ as $\{V_i\}$ depends continuously on $\mathcal{P}$ by Theorem 1; $M = Q$ is a compact invariant set; $\dot{V}$ is negative definite in $M$: $E = V^{-1}(0) = \{\tilde{C}_{V_i}\}$, which itself is the largest invariant subset of $E$ by the control law (19). Thus by LaSalle’s invariance principle, the trajectories of the agents governed by control law (19), starting from any initial configuration $\mathcal{P}(0)$, will asymptotically converge to set $E$, the critical points of $\mathcal{H}$, that is, the generalized centroidal Voronoi partitions with respect to the density function as perceived by the sensors. $\square$

5 Constraints on agents’ speed

We proposed a control law to guide the agents toward the critical points, that is, to their respective centroid, and observed that the closed loop system for agents modeled as first order dynamical system, is globally asymptotically stable. Here we impose some constraints on the agent speeds and analyze the dynamics of closed loop system.

5.1 Maximum speed constraint

Let the agents have a constraint on maximum speed of $U_{max_i}$, for $i = 1, \ldots, n$. Now consider the control law

$$u_i = \begin{cases} -k_{prop}(p_i - \tilde{C}_{V_i}), & \text{If } |k_{prop}(p_i - \tilde{C}_{V_i})| \leq U_{max_i} \\ -U_{max_i} \frac{(p_i - \tilde{C}_{V_i})}{\|p_i - \tilde{C}_{V_i}\|} & \text{Otherwise} \end{cases}$$

(21)

The control law (21) makes the agents move toward their respective centroids with saturation on speed.

It is easy to see, that the control law (21) is spatially distributed in the generalized Delaunay graph.
Theorem 5 The trajectories of the agents governed by the control law (21), starting from any initial condition \( P(0) \in Q^N \), will asymptotically converge to the critical points of \( H \).

Proof. Consider \( V(P) = -H \), where \( P = \{ p_1, p_2, \ldots, p_N \} \) represents the configuration of \( N \) agents.

\[
\dot{V}(P) = -\frac{dH}{dt} = -\sum_{i \in \{1,2,\ldots,N\}} \frac{\partial H^n}{\partial p_i} \dot{p}_i \tag{22}
\]

\[
= \begin{cases} 
-2\alpha \sum_{i \in \{1,2,\ldots,N\}} \tilde{M}_V((p_i - \tilde{C}_V)k_{prop} \times (p_i - \tilde{C}_V), \quad \text{if } |k_{prop}(p_i - \tilde{C}_V)| \leq U_{max} \\
-2\alpha \sum_{i \in \{1,2,\ldots,N\}} \tilde{M}_V((p_i - \tilde{C}_V)U_{max}(\|p_i - \tilde{C}_V\|), \quad \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
-2\alpha k_{prop} \sum_{i \in \{1,2,\ldots,N\}} \tilde{M}_V(||p_i - \tilde{C}_V||^2, \quad \text{if } |k_{prop}(p_i - \tilde{C}_V)| \leq U_{max} \\
-2\alpha \sum_{i \in \{1,2,\ldots,N\}} U_{max} \tilde{M}_V(||p_i - \tilde{C}_V||^2), \quad \text{otherwise}
\end{cases}
\]

We observe that \( V : Q \rightarrow \mathbb{R} \) is continuously differentiable in \( Q \) as \( \{ V_i \} \) depends at least continuously on \( P \) (Theorem 1), and \( \dot{V} \) is continuous as \( u \) is continuous; \( M = Q \) is a compact invariant set; \( \dot{V} \) is negative definite in \( M \); \( E = \dot{V}^{-1}(0) = \{ \tilde{C}_V \} \), which itself is the largest invariant subset of \( E \) by the control law (21). Thus, by LaSalle’s invariance principle, the trajectories of the agents governed by control law (21), starting from any initial configuration \( P(0) \in Q^N \), will asymptotically converge to the set \( E \), the critical points of \( H \), that is, the generalized centroidal Voronoi configuration with respect to the density function as perceived by the sensors. \( \square \)

5.2 Constant speed control

The agents may have a constraint of moving with a constant speed \( U_i \). But we let the agents slow down as they approach the critical points. For \( i = 1, \ldots, n \), consider the control law

\[
u_i = \begin{cases} 
-U_i \frac{(p_i - \tilde{C}_V)}{\|p_i - \tilde{C}_V\|}, & \text{if } \|p_i - \tilde{C}_V\| \geq \delta \\
-U_i \frac{(p_i - \tilde{C}_V)}{\delta}, & \text{otherwise}
\end{cases} \tag{23}
\]

where, \( \delta > 0 \), predefined value, such that the control law (23) makes the agents move toward their respective centroids with a constant speed of \( U_i \) when they at a distance greater than \( \delta \) from the corresponding centroids and slow down as they approach them.

It is easy to see, that the control law (23) is spatially distributed in the generalized Delaunay graph.

Theorem 6 The trajectories of the agents governed by the control law (23), starting from any initial condition \( P(0) \in Q^N \), will asymptotically converge to the critical points of \( H \).

Proof. Consider \( V(P) = -H \), where \( P = \{ p_1, p_2, \ldots, p_N \} \) represents the configuration of \( N \) agents.

\[
\dot{V}(P) = -\frac{dH}{dt} = -\sum_{i \in \{1,2,\ldots,N\}} \frac{\partial H^n}{\partial p_i} \dot{p}_i \tag{24}
\]

\[
= \begin{cases} 
-2\alpha \sum_{i \in \{1,2,\ldots,N\}} U_i \tilde{M}_V(||p_i - \tilde{C}_V||^2, \quad \text{if } \|p_i - \tilde{C}_V\| \geq \delta \\
-2\alpha \sum_{i \in \{1,2,\ldots,N\}} U_i \tilde{M}_V(p_i - \tilde{C}_V)/\delta, \quad \text{otherwise}
\end{cases}
\]

We observe that \( V : Q \rightarrow \mathbb{R} \) is continuously differentiable in \( Q \) as \( \{ V_i \} \) depends at least continuously on \( P \) (Theorem 1), and \( \dot{V} \) is continuous as \( u \) is continuous; \( M = Q \) is a compact invariant set; \( \dot{V} \) is negative definite in \( M \); \( E = \dot{V}^{-1}(0) = \{ \tilde{C}_V \} \), which itself is the largest invariant subset of \( E \) by the control law (23). Thus, by LaSalle’s invariance principle, the trajectories of the agents governed by control law (23), starting from any initial configuration \( P(0) \in Q^N \), will asymptotically converge to the set \( E \), the critical points of \( H \), that is, the generalized centroidal Voronoi configuration with respect to the density function as perceived by the sensors. \( \square \)

6 Limited range sensors

In reality the sensors will have limited range. In this section we formulate a spatially distributed limited range locational optimization problem.

Let \( R_i \) be the limit on range of the sensors and \( \tilde{B}(p_i, R_i) \) be a closed ball centered at \( p_i \) with a radius of \( R_i \). The \( i \)-th sensor has access to information only from points in the set \( V_i \cap \tilde{B}(p_i, R_i) \). Let us also assume that \( f_i(R_i) = f_j(R_j), \forall i, j \in \{1, \ldots, N\} \), that is, we assume that the cutoff range for all sensors is the same. Consider the objective function to be maximized,

\[
\tilde{H}(P) = \sum_i \int_{(V_i \cap \tilde{B}(p_i, R_i))} \phi_n(q) \tilde{f}_i(||p_i - q||)dQ \tag{25}
\]

where, \( \tilde{f}_i(r) = \begin{cases} 
f_i(r) & \text{if } r \leq R_i \\
f_i(R_i) & \text{otherwise}
\end{cases} \)
pa similar result is proved. But the objective function is made up of sums of the contributions from sets \( V_i \cap B(p_i, R_i) \), enabling the sensors to solve the optimization problem in a spatially distributed manner.

In reality for range limited sensors the effectiveness should be zero beyond the range limit. Consider \( \tilde{f}_i(.) = \tilde{f}_i(.) - f_i(R_i) \) (Figure 1). It can be shown that the objective function (25) has the same critical points if \( \tilde{f}_i \) is replaced with \( f_i \), as the difference in two objective functions will be a constant term \( f_i(R_i) \) (Note that we have assumed \( f_i(R_i) = f_j(R_j) \forall i, j \in \{1, \ldots, N\} \).

The gradient of (25) with respect to \( p_i \) can be determined as

\[
\frac{\partial \mathcal{H}}{\partial p_i}(P) = 2\mathcal{M}(V_i \cap B(p_i, R_i)) (\tilde{C}_{(V_i \cap B(p_i, R))} - p_i) \tag{26}
\]

We use the control law

\[
u_i = -k_{\text{prop}}(p_i - \tilde{C}_{(V_i \cap B(p_i, R_i))}) \tag{27}\]

It is easy to show, that the gradient (26) and the control law (27) are spatially distributed in the \( r \)-limited Delaunay graph \( G_{LD} \), the Delaunay graph incorporating the sensor range limits.

**Theorem 7** The trajectories of the sensors governed by the control law (27), starting from any initial condition \( P(0) \), will asymptotically converge to the critical points of \( \mathcal{H} \).

**Proof.** The proof is similar to that of the Theorem 4 with \( V = -\tilde{\mathcal{H}}(P) \). It is It can be shown that \( V \) is continuously differentiable based on Theorem 2.2 in [12], where a similar result is proved.

Fig. 1. Illustration of \( \tilde{f}_i \) and \( f_i \) in the presence of a limit on sensor range. The solid curve represent the sensor effective function \( \tilde{f}_i \) and dotted curve is the actual sensor effective function \( f_i(r) = \tilde{f}_i(r) + (1 - f_i(R)) \) with \( R = 6 \).

The objective function (25) has the same critical points if \( \tilde{f}_i \) is replaced with \( f_i \), as the difference in two objective functions will be a constant term \( f_i(R_i) \) (Note that we have assumed \( f_i(R_i) = f_j(R_j) \forall i, j \in \{1, \ldots, N\} \).

The gradient of (25) with respect to \( p_i \) can be determined as

\[
\frac{\partial \mathcal{H}}{\partial p_i}(P) = 2\mathcal{M}(V_i \cap B(p_i, R_i)) (\tilde{C}_{(V_i \cap B(p_i, R))} - p_i) \tag{26}
\]

We use the control law

\[
u_i = -k_{\text{prop}}(p_i - \tilde{C}_{(V_i \cap B(p_i, R_i))}) \tag{27}\]

It is easy to show, that the gradient (26) and the control law (27) are spatially distributed in the \( r \)-limited Delaunay graph \( G_{LD} \), the Delaunay graph incorporating the sensor range limits.

**Theorem 7** The trajectories of the sensors governed by the control law (27), starting from any initial condition \( P(0) \), will asymptotically converge to the critical points of \( \mathcal{H} \).

**Proof.** The proof is similar to that of the Theorem 4 with \( V = -\tilde{\mathcal{H}}(P) \). It is It can be shown that \( V \) is continuously differentiable based on Theorem 2.2 in [12], where a similar result is proved.

Fig. 2. Trajectories of sensors along with final Voronoi cells, with (a) an uniform uncertainty density, and (b) \( \phi(x, y) = 0.9e^{-0.04((x-10)^2+(y-10)^2)} \). ‘+’ indicate initial positions of the sensors, and ‘o’ indicate the centroid of Voronoi cells corresponding to agent locations at the end of simulation.

**7 Illustrative example**

Here we provide simulation results as an illustration of the optimal deployment using the proposed heterogeneous locational optimization. In this example we consider 10 mobile sensors without any sensor range limits in a square area of size 10 \times 10. We used \( f_i = -\alpha_i r_i^2 \) as node functions, with \( \{\alpha_1, \ldots, \alpha_{10}\} = \{1, 1.25, 1.5, 0.75, 0.8, 1.3, 0.9, 1.1, 1.4, 1.2\} \). Figure 2 (a) shows the trajectories of the sensors with a uniform uncertainty density and Figure 2 (b) shows the trajectories with \( \phi(x, y) = 0.9e^{-0.04((x-10)^2+(y-10)^2)} \). The sensors reach the optimal configuration which is the centroids of multiplicatively weighted Voronoi cells in this case. In both figures, ‘o’ represents the centroids of the generalized Voronoi cell corresponding to the final agent configuration, and it can be seen that the each agent is sufficiently close to corresponding centroid. The simulations were terminated when \( \max \|p_i - (C)_{V_i}\| < 0.5 \). The generalized Voronoi partition corresponding to final agent configurations are also shown in Figure 2.

Figures 3 (a) shows the plot of the normalized objective function with time steps. It can be observed that the control law successfully makes the agents/sensors move toward the optimal deployment maximizing the objective function. As noted earlier, the strategy does not guarantee global optimal configuration. Plot of an error measure of \( \sum_{i=1}^{N} \|p_i - (C)_{V_i}\|^2 \) with time steps is shown in Figure 3 (b). It can be observed that the control law successfully drives the agents toward the respective centroids.

**8 Conclusions**

A generalization of Voronoi partition has been proposed and the standard Voronoi decomposition and its
variations are shown as special cases. A heterogeneous spatially distributed locational optimization has been formulated and solved using the proposed partitioning scheme and the a variation of centroidal Voronoi configuration is shown to be the optimal deployment of the limited range sensors with heterogeneous capabilities. Illustrative simulation result was provided to support the theoretical results.

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