Symmetry transformations for magnetohydrodynamics and Chew–Goldberger–Low equilibria revisited

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Abstract
We generalize the symmetry transformations for magnetohydrodynamic (MHD) equilibria with isotropic pressure and incompressible flow parallel to the magnetic field introduced by Bogoyavlenskij in the case of the respective Chew–Goldberger–Low (CGL) equilibria with anisotropic pressure. We find that the geometrical symmetry of the field-aligned equilibria can be broken by those transformations only when the magnetic field is purely poloidal. In this situation we derive three-dimensional CGL equilibria from given axisymmetric ones. Also, we examine the generic symmetry transformations for MHD and CGL equilibria with incompressible flow of an arbitrary direction, introduced in a number of papers, and find that they cannot break the geometrical symmetries of the original equilibria, unless the velocity and magnetic field are collinear and purely poloidal.

Keywords: symmetry transformations, MHD equilibria, breaking of geometrical symmetry, plasma flow

(Some figures may appear in colour only in the online journal)

1. Introduction

Two of the most important models widely applied to describe plasma equilibria are the isotropic ideal magnetohydrodynamic (MHD) and the anisotropic CGL models. In references [2–9] methods for constructing new continuous families of equilibria in the framework of the above mentioned models, once a given equilibrium is known, are introduced. More specifically, in references [2–5] three sets of equilibrium transformations under the framework of the MHD model were presented. The first set is applied to a given equilibrium with an incompressible flow of arbitrary direction, while the second one is applied to both static equilibrium and stationary equilibria with field-aligned incompressible flow. The third set of transformations concerns plasma equilibria with compressible flow. In addition, in reference [6] symmetry transformations that produce an infinite class of anisotropic CGL equilibria, on the basis of the prescribed CGL ones with flow of arbitrary direction are introduced; also in references [6–9] symmetry transformations mapping static or stationary MHD equilibria into CGL ones are presented. All these symmetry transformations depend on a number of scalar functions which have to be constant on the magnetic field lines. This implies that the new equilibria resulting from the transformations depend on the structure of the magnetic fields of the original ones, and thus, the topology of the original equilibria is essential for these transformations.

A magnetic field line before closing to itself may either cover a surface, if such a surface exists, or fill a volume. Any surface that is traced out by a number of magnetic field lines is called a magnetic surface. In plasmas of fusion devices, however, the name is usually reserved for nested toroidal surfaces. The generic structure of a magnetic field can be either ‘open’, in the sense that it closes to itself through infinity, as for example in a magnetic mirror, screw pinch, and the Earth’s magnetosphere, or closed if it remains in a spatially finite region, as for example in the central region of a tokamak and stellarator. It can be proved that if the magnetic
field lines lie on some closed surfaces contained in a bounded region and do not have any singularities, then they must be toroids (topological tori) [10–12]. Hamiltonian theory guarantees the existence of magnetic surfaces in systems with three kinds of continuous geometrical symmetry: axisymmetry, as in an ideal tokamak, helical symmetry which can approximately describe a ‘straight stellarator’ without toroidal curvature, and translational symmetry in which the system is unbounded along the symmetry direction. However, the latter category can represent a ‘straight tokamak’ when the magnetic field is periodic along the direction of symmetry and therefore can be considered as a toroidal field, since a single period of such a field is topologically equivalent to a torus. For such symmetric systems the magnetic surfaces are well-defined by the level sets of a function $\psi(q^1, q^2)$, with the third coordinate $q^3$ being ignorable. In nonsymmetric devices, on the other hand, magnetic surfaces do not rigorously exist everywhere because the magnetic field may cover regions of finite volume. Open-ended systems, such as magnetic mirrors, do not possess magnetic surfaces that are traced out by one single line. This leads to a considerable degree of arbitrariness.

The lines of force lying on nested toroidal magnetic surfaces encircle the magnetic axis. This encirclement is arbitrary. By one single line. This leads to a considerable degree of arbitrariness.

In the present work we make an extensive revision of the transformations presented previously in references [2–9] concerning equilibria with incompressible flows. In section 2 we introduce a set of transformations that can be applied to any known anisotropic CGL equilibria with field-aligned incompressible flows (or static equilibria) and an anisotropy function constant on the magnetic field lines, and produce an infinite family of anisotropic equilibria with collinear velocity and magnetic fields, but with density and anisotropy functions that may remain arbitrary. These transformations consist of a generalization of the ones introduced in reference [2] for field-aligned MHD equilibria. We also prove that all transformations presented in references [2–9] can break the geometrical symmetries of a known given equilibria, static or with field-aligned flow, if and only if its magnetic field is purely poloidal. In section 3 we construct three-dimensional (3D) equilibria by applying the introduced transformations to known axisymmetric equilibria with field-aligned incompressible flow, pressure anisotropy, and a purely poloidal magnetic field, related to the symmetry breaking. In section 4 we examine the aforementioned symmetry transformations for flow of arbitrary direction and check their validity in connection with the structure of the magnetic fields of the original equilibria and the existence of magnetic surfaces. Finally, section 5 summarizes our conclusions.

2. Symmetry transformations for field-aligned equilibria

2.1. Review of the transformations for MHD equilibria

In section 4 of reference [2] transformations between MHD equilibria with parallel flows are presented. Specifically, it is stated therein that if $\{B, v, p, \phi\}$ is a solution of the ideal MHD equilibrium system of equations with field-aligned incompressible flow:

$$\rho (v \cdot \nabla)v = J \times B - \nabla p, \quad \nabla \cdot B = 0,$$

$$\nabla \cdot (\rho v) = 0, \quad \nabla \times B = \mu_0 J,$$  \hspace{1cm} (1)

then $\{B_1, v_1, p_1, \phi_1\}$ is defined by the following symmetry transformations, that depend on the arbitrary functions $a(r), b(r), c(r)$, and $\lambda (r)$, and consist of a new solution to the MHD equilibrium set of equations with field-aligned flows:

$$B_1 = b(r)B, \quad v_1 = \frac{c(r)}{a(r)} \frac{B}{\sqrt{\mu_0 \phi}},$$

$$\rho_1(r) = a^2(r)\rho, \quad p_1(r) = C \left( p + \frac{B^2}{2\mu_0} \right) - \frac{B^2}{2\mu_0},$$

$$C = \frac{b^2(r) - c^2(r)}{1 - \lambda^2(r)} \equiv \text{const.} \approx 0. \hspace{1cm} (2)$$

The above special transformations are defined only when the velocity and magnetic field of the original equilibria are related through $v = (\lambda/\sqrt{\mu_0 \phi})B$, and are also valid in the static limit, $v = 0$. Their reductive form for constants $a, b, c$, and $\lambda$ was first derived in reference [4] from the given axisymmetric equilibria found in reference [19]. According to
reference [2] the functions \(a(r), b(r), c(r)\), depending on the topology of the original equilibria may either (i) be constant on magnetic surfaces, or (ii) in the case of symmetry involving two-dimensional (2D) dependence, depend on two transversal variables (i.e. variables not depending explicitly on the ignorable coordinate), or (iii) are constants in the case of force-free equilibria. Also it is claimed therein that transformations (2) can break the geometrical symmetry of the original equilibria (1) with general field-aligned incompressible flow.

2.2. Generalized symmetry transformations for anisotropic pressure

In this subsection we first generalize the transformations (2) introduced in reference [2] for CGL anisotropic equilibria with field-aligned incompressible flow and show that the only situation in which the symmetry of the original equilibria can be broken is that for purely poloidal magnetic fields. These considerations are summarized in the following theorem:

**Theorem 1.** Let \([\mathbf{B}, \mathbf{v}, \varphi, p_\perp, p_\parallel]\) be a known solution to the CGL equilibrium system of equations with field-aligned incompressible flows,

\[
\mathbf{v} = \frac{\lambda(r)}{\sqrt{\mu_0 \varphi}} \mathbf{B},
\]

and pressure anisotropy function,

\[
\sigma_d = \mu_0 \frac{p_\perp - p_\parallel}{\mathbf{B}^2},
\]

being constant on the magnetic field lines, \(\mathbf{B} \cdot \nabla \sigma_d = 0\). Then \([\mathbf{B}_1, \mathbf{v}_1, \varphi_1, p_{\perp 1}, p_{\parallel 1}]\) are given by the following transformations:

\[
\begin{align*}
\mathbf{B}_1 &= \frac{b(r)}{n(r)} \mathbf{B}, \\
\mathbf{v}_1 &= \frac{c(r) \sqrt{1 - \sigma_d}}{a(r) \sqrt{\mu_0 \varphi}} \mathbf{B}, \\
\varphi_1(r) &= a^2(r) \varphi, \\
p_{\perp 1} &= C \left( p_\perp + \frac{\mathbf{B}^2}{2 \mu_0} \right) - \frac{\mathbf{B}_1^2}{2 \mu_0}, \\
p_{\parallel 1} &= C \left( \frac{p_\parallel + \mathbf{B}^2}{2 \mu_0} \right) + \left[ 1 - 2n^2(r)(1 - \sigma_d) \right] \frac{\mathbf{B}_1^2}{2 \mu_0}
\end{align*}
\]

where \(a(r) = 0, b(r), c(r), \) and \(n(r) \neq 0\) are arbitrary functions, define a solution to the CGL set of equilibrium equations with field-aligned flows, if and only if the functions

\[
g(r) = \frac{b(r)}{n(r)}, \quad f(r) = a(r)c(r),
\]

are constant on the magnetic field lines of the original equilibria.

**Proof.** The original equilibria \([\mathbf{B}, \mathbf{v}, \varphi, p_\perp, p_\parallel]\) satisfy the CGL equilibrium equations with field-aligned flows (3):

\[
\begin{align*}
\varphi (\mathbf{v} \cdot \nabla) \mathbf{v} &= \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}, \\
\nabla \cdot (\varphi \mathbf{v}) &= 0, \\
\nabla \cdot \mathbf{B} &= \mu_0 \mathbf{J},
\end{align*}
\]

where the CGL pressure tensor is defined as

\[
P = p_\perp \mathbf{I} + \frac{\sigma_d}{\mu_0} \mathbf{B} \mathbf{B},
\]

with the function \(\sigma_d\) measuring the pressure anisotropy. It is assumed that the flow is incompressible, \(\mathbf{v} \cdot \nabla \varphi = 0\), which by the continuity equation implies that the mass density is constant on streamlines, \(\mathbf{v} \cdot \nabla \varphi = 0\); it is also assumed that the anisotropy function is constant on the magnetic field lines, \(\mathbf{B} \cdot \nabla \sigma_d = 0\). When the equilibria possess some geometrical symmetry, the latter hypothesis for the function \(\sigma_d\) in conjunction with incompressibility, lead to the derivation of a single Grad–Shafranov (GS) equation that governs them [20–22]; also, according to reference [23] this assumption on \(\sigma_d\) may be the only suitable one for satisfying the boundary conditions on a fixed, perfectly conducting wall. It may be noted that for the given field-aligned equilibria, the vectors \(\mathbf{v}\) and \(\mathbf{B}\) are collinear (parallel) and therefore the magnetic field lines are the same as the velocity streamlines. It follows that the function \(\lambda(r)\) must be constant on the magnetic field lines, \(\mathbf{B} \cdot \nabla \lambda(r) = 0\). Also, the force balance equation of the set (7) can be cast into the useful form

\[
(1 - \sigma_d - \frac{1}{\mu_0}) \mathbf{B} \times (\nabla \times \mathbf{B}) + (\sigma_d + \frac{1}{\mu_0}) \nabla \mathbf{B} = 0.
\]

In order for the new solution (5) to be valid it must satisfy the following set of CGL equilibrium equations:

\[
\begin{align*}
\varphi (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 &= \mathbf{J}_1 \times \mathbf{B}_1 - \nabla \cdot \mathbf{P}_1, \\
\nabla \cdot (\varphi \mathbf{v}_1) &= 0, \\
\nabla \cdot \mathbf{B}_1 &= \mu_0 \mathbf{J}_1,
\end{align*}
\]

where

\[
\mathbf{p}_1 = p_{\perp 1} \mathbf{I} + \frac{\sigma_d}{\mu_0} \mathbf{B}_1 \mathbf{B}_1, \quad \sigma_d = \mu_0 \frac{p_{\perp 1} - p_{\parallel 1}}{\mathbf{B}_1^2},
\]

\[
= 1 - n^2(r)(1 - \sigma_d).
\]

Note that systems (7) and (10) are reductions of the generic CGL equilibrium equations since for field-aligned flows it holds that \(\mathbf{v} \times \mathbf{B} = \mathbf{v}_1 \times \mathbf{B}_1 = 0\), and therefore the electric field vanishes by Ohm’s law.

Substituting (5) into (10) yields

\[
\mathbf{B} \cdot \nabla \left( \frac{b(r)}{n(r)} \right) = 0,
\]

\[
\mathbf{B} \cdot \nabla (a(r)c(r)) = 0,
\]
With the use of equation (9) and assuming that \( \sigma_d = 1 \) (in which case \( v_1 = 0 \), \( C = 0 \) and the transformations (5) are not invertible), equation (14) takes the form

\[
B \cdot \left( b^2 \frac{\nabla n}{n} + c^2 \frac{\nabla a}{a} \right) = 0.
\]

(15)

Now with the aid of (6), equations (12), (13), and (15) assume the forms

\[
B \cdot \nabla g(r) = 0,
\]

(16)

\[
B \cdot \nabla f(r) = 0,
\]

(17)

\[
\frac{C}{2} B \cdot \nabla \left( \frac{1 - \sigma_d - \lambda}{1 - \sigma_d} \right) + c^2(r) \frac{f(r)}{g(r)} B \cdot \nabla f(r)
\]

\[
- \frac{b^2(r)}{g(r)} B \cdot \nabla g(r) = 0.
\]

(18)

Since the first term on the left-hand side of equation (18) vanishes, it is apparent that if equations (16) and (17) are valid, then equation (18) is trivially satisfied. Thus, we conclude that in order for transformations (5) to be valid, equations (16) and (17) must be satisfied, or equivalently, both functions \( g(r) \) and \( f(r) \) have to be constant on the magnetic field lines of the original equilibria; quod erat demonstrandum.

Therefore, the anisotropic CGL equilibrium systems (7) and (8) possess a family of intrinsic symmetries since the constituent differential equations are invariant under the transformed variables (5).

**Remark 1.** The symmetry transformations (5) presented herein are defined only when the velocity and the magnetic field of the original equilibria are related through equation (3), and transformations (2) introduced in reference [2] for field-aligned MHD equilibria consisting of a special case of (5) for \( \sigma_d = 0 \) and \( n(r) = 1 \). While both sets of transformations (5) and (2) can change the magnitude of the physical equilibrium quantities or create stationary configurations from static ones, they also preserve the topology of the magnetic surfaces, \( \psi = \text{const.} \) (if such surfaces exist), since it holds that \( B \cdot \nabla \psi = B_\psi \cdot \nabla \psi = 0 \).

Let us now examine the structure of the arbitrary scalar functions in connection with the magnetic field by first noting that the magnetic and velocity fields of the original equilibria (3) lie on the surfaces \( \lambda(r) = \text{const.} \), \( B \cdot \nabla \lambda(r) = v \cdot \nabla \lambda(r) = 0 \), whenever such surfaces exist.

If the magnetic field lines are spatially bounded closed curves, labeled by \( l \), then the surfaces \( \lambda = \text{const.} \) are defined in the neighborhood of \( l \), with \( l \) being itself both a magnetic field line and a streamline. In this situation, the function \( \lambda \) depends on two transversal variables which must define a plane normal to every point of \( l \). If the magnetic field lines are ‘open’, i.e., they approach infinity in one direction, defined by a variable \( q^3 \), then the magnetic field should be finite as \( q^3 \to \infty \). The function \( \lambda(r) \) should depend on two transversal variables when the third one goes to infinity, \( \lambda(q^1, q^2, q^3 \to \infty) = \lambda(q^1, q^2) \), and thus, must depend on these two variables in the whole plasma domain for magnetic surfaces \( \lambda(q^1, q^2) = \text{const.} \). In both of the above kinds of magnetic field lines (bounded and ‘open’), the functions \( g(r), f(r) \) have to be functions of two transversal variables, i.e. \( q^1, q^2 \). One could suggest that this does not restrict the functions \( a(r), b(r), c(r), n(r) \) to have the same 2D dependency (i.e., \( a(r) = A(q^1, q^2)d(q^3) \) and \( c(r) = K(q^1, q^2)/d(q^3) \), such that \( f(r) = A(q^1, q^2)K(q^1, q^2) \)). However, the equation \( \nabla \cdot (\rho v_1) = 0 \) yields \( B \cdot \nabla a(r) = 0 \), which means that \( a = a(q^1, q^2) \) and consequently \( c = c(q^1, q^2) \). Then from the definition of the constant \( C \) it follows that \( b = b(q^1, q^2) \), and as a result \( n = n(q^1, q^2) \). Thus, if the magnetic field lines are finite closed loops or go to infinity in some direction, all functions of transformations must, in general, depend on two variables transversal to this direction.

If the magnetic field lines cover densely everywhere (ergodically) closed magnetic surfaces, \( \lambda(r) = \text{const.} \) (which are toroids), then the functions \( g(r), f(r) \) must be constant on them, and so must be all four functions of the transformations. In this situation, if the field possesses some geometrical symmetry, with ignorable variable \( q^3 \), the surfaces \( \lambda(r) = \text{const.} \) are nested, with \( \lambda = \lambda(q^1, q^2) \). Then all functions \( a(r), b(r), c(r), n(r) \) have the same symmetry (i.e., are functions only of \( q^1, q^2 \)). However, there exists an exception; the one when the original equilibrium has some known geometrical symmetry with a purely poloidal magnetic field to be examined as follows.

**Axial symmetry:** Consider the case that the original equilibria are axially symmetric with field-aligned incompressible flows and an anisotropy function constant on magnetic surfaces [21]. Employing cylindrical coordinates \( (\rho, \psi, \phi) \) we have

\[
B = \frac{I}{\rho} \hat{\phi} + \frac{\hat{\phi}}{\rho} \times \nabla \psi(\rho, \psi, \phi) = \frac{M_\phi B}{\sqrt{\mu_0 B^2}}, \quad \mathbf{v} = \frac{M_\phi B}{\sqrt{\mu_0 B^2}}
\]

(19)

where the function \( I \) relates to the toroidal magnetic field and \( \psi(\rho, \phi) = \text{const.} \), labels the magnetic surfaces. Thus, \( \lambda(r) = \lambda(\psi) = M_\psi(\psi) \), where \( M_\psi = (\sqrt{\mu_0 B^2} V_{\text{pol}})/|B_{\text{pol}}| \) is the poloidal Alfvén Mach function, which for parallel flows equals to the total Mach function \( (M = \sqrt{|V_{\text{tot}}|^2}/|B|) \). To examine whether transformations (5) can break axisymmetry we permit the transformation functions to depend, in addition to \( \psi \), explicitly on \( \phi \), i.e. \( f = f(\psi, \phi), g = g(\psi, \phi) \). Then
equations (16) and (17) yield
\[
\frac{I}{\rho^2} \left( \frac{\partial g}{\partial \rho} \right) = 0,
\]
\[
\frac{I}{\rho^2} \left( \frac{\partial f}{\partial \rho} \right) = 0.
\quad (20)
\]
Set (20) is satisfied either if functions \( g, f \) are constant on the magnetic surfaces, or \( I = 0 \). The latter case implies that transformations (5) can break the axial symmetry of field-aligned equilibria with a purely poloidal magnetic field. The same statement holds for translationally symmetric equilibria with field-aligned flows [24], while the more generic case of helical symmetry will be studied separately below.

**Helical symmetry:** Consider now that the original equilibria are helically symmetric with field-aligned incompressible flows and an anisotropy function constant on magnetic surfaces for which the following relations hold [22]:
\[
B = I/h + h \times \nabla \psi(r, u), \quad \mathbf{v} = \frac{M_p(\psi)}{\sqrt{\mu_0 U^2}} \mathbf{B}.
\quad (21)
\]
Here \((r, u, \xi)\) are helical coordinates defined through the usual cylindrical ones \((\rho, \phi, z)\) as \( r = \rho, u = m\phi - kz, \xi = z \); \( I \) relates to the helicoidal magnetic field and \( \psi(r, u) = \) const. labels the magnetic surfaces; the vector \( \mathbf{h} = (m/(k^2 r^2 + m^2))g_{\xi} \) points along the helical direction, where \((k, m)\) are integers, and the covariant helical basis vectors, \( g_i, i = r, u, \xi \), are defined through the respective cylindrical unit vectors as \( g_r = \hat{\rho}, g_u = (r/m)\hat{\phi}, g_\xi = (rk/m)\hat{\phi} + \hat{z} \). For the adopted non-orthogonal helical coordinates, the \( u-\) and \( \xi-\)covariant and contravariant components of a given vector \( \mathbf{A} \) differ from each other, \( A_i = A'_i (i = u, \xi) \). The magnetic field written in contravariant components is
\[
B = B^i g^i + B'\mathbf{g}_r + B''\mathbf{g}_u.
\quad (22)
\]
The definition of a usual flux function \( \psi (r, u) \) so as equation \( \nabla \cdot \mathbf{B} = 0 \) to be satisfied reduces equation (22) into [see also [25]]:
\[
B = B^\xi g_\xi + \frac{1}{m} g^\xi \times \nabla \psi,
\quad (23)
\]
where \( g^\xi = \xi \) is the contravariant basis vector. Observe that
\[
g_\xi \cdot (g^\xi \times \nabla \psi) = \frac{kr}{m} \frac{\partial \psi}{\partial r} = 0.
\quad (24)
\]
This in fact dictated us to define the helical vector \( \mathbf{h} \) that points into the symmetry direction. Then the magnetic field is written in the form (21) with
\[
I \equiv \frac{B^\xi}{mq} + \frac{kr}{m} \frac{\partial \psi}{\partial r}.
\quad (25)
\]
If we define the poloidal magnetic field as
\[
B_{\text{pol}} = \frac{1}{m} g^\xi \times \nabla \psi,
\quad (26)
\]
then the field on the plane normal to \( \mathbf{h} \) is expressed as
\[
\mathbf{h} \times \nabla \psi = B_{\text{pol}} - kqr \frac{\partial \psi}{\partial r} g_\xi.
\quad (27)
\]
Now let \( f = f(\psi, \zeta), g = g(\psi, \zeta) \). Then satisfaction of equations (16) and (17) requires
\[
B^i \left( \frac{\partial g}{\partial \zeta} \right) = 0,
\]
\[
B^i \left( \frac{\partial f}{\partial \zeta} \right) = 0.
\quad (28)
\]
Similar to the case of equation (20), equation (28) leads to
\[
B^i = 0 \Rightarrow I = \frac{kr}{m} \frac{\partial \psi}{\partial r} \Rightarrow \mathbf{B} = \mathbf{B}_{\text{pol}}.
\quad (29)
\]
Thus, transformations (5) can also break the helical symmetry of the original equilibrium with a field-aligned incompressible flow and pressure anisotropy, if and only if the magnetic field is purely poloidal.

**Remark 2.** The flow of the transformed 3D equilibria, obtained from the application of transformations (5) to geometrically symmetric equilibria with purely poloidal and field-aligned incompressible flow, is indeed incompressible, \( \nabla \cdot \mathbf{v}_1 = 0 \). However, the transformed mass density, \( \rho_1 \), may vary on the magnetic surfaces since the continuity equation of the set (10) is trivially satisfied for purely poloidal velocity, \( \mathbf{v}_1 \).

Finally, it may happen that \( \lambda = \) const. and consequently, \( \nabla \cdot \mathbf{B} = 0 \) in the whole plasma domain. In this situation the force balance equation (9) is written in the form
\[
(1 - \sigma_d - \lambda^2) \mathbf{J} \times \mathbf{B} = \nabla \left( \rho + \chi \frac{B^2}{2\mu_0} \right) - \frac{B^2}{2\mu_0} \nabla \sigma_d.
\quad (30)
\]
where \( \bar{p} = (p_\rho + p_\parallel)/2 \) is defined as an effective isotropic pressure. In this case a family of magnetic surfaces \( w(\mathbf{r}) = \) const., where \( w \equiv \bar{p} + \lambda^2 \frac{B^2}{2\mu_0} \), can be defined, in which both magnetic field lines and velocity streamlines lie on, \( \mathbf{B} \cdot \nabla \mathbf{w}(\mathbf{r}) = 0 \). Analogous considerations can be made on the structure of these surfaces.

Now it may happen \( w = \) const. with \( \nabla \mathbf{w} = 0 \) if \( \mathbf{J} = \mathbf{y}(\mathbf{r}) \mathbf{B} \), that is the current density is parallel to the magnetic field, or equivalently \( \nabla \times \mathbf{v} = \mathbf{r}(\nabla \mathbf{w}) \), that is the velocity is parallel to the vorticity. This is the case of force-free or Beltrami equilibria. Then magnetic surfaces \( y(\mathbf{r}) = \) const. can be yet defined, \( \mathbf{B} \cdot \nabla y(\mathbf{r}) = 0 \). But in the particular case \( y \equiv \) const. (everywhere) and therefore \( \nabla y = 0 \) (and then as well \( t \equiv \) const., with \( \nabla t = 0 \)), we finally escape the topological constraint that magnetic field lines lie on surfaces. The lines of force may be chaotic (space-filling) in this case, and all functions \( a(\mathbf{r}), b(\mathbf{r}), c(\mathbf{r}), n(\mathbf{r}) \) have to be constant.

The above conclusions lead us to formulate the following corollary:

**Corollary 1.** Transformations (5) can break the geometrical symmetry, either axial, translational, or helical, of the original field-aligned equilibria with incompressible flow and anisotropy function constant on magnetic surfaces, if and only if its magnetic field is purely poloidal. Otherwise, the transformed equilibrium retains the original symmetry.
All conclusions derived herein concerning the validity of the transformations, the structure of the arbitrary functions, and the symmetry breaking, also hold for the respective transformations (2) with isotropic pressure (cf. remark 1). In this respect, the symmetry breaking of the static helically symmetric equilibria related to astrophysical jets, examined in section VIII of reference [2], should be revised, since the magnetic field of the original equilibria (cf. equation (8.1) therein) is not purely poloidal unless the constant \( \alpha \) is equal to zero.

3. Construction of 3D CGL equilibria with field-aligned flows

As mentioned in section 2.2 every equilibrium state either static or stationary with incompressible flows, having some continuous geometrical symmetry with a pressure anisotropy function uniform on magnetic surfaces, is governed by a GS equation for the flux function \( \psi \) [20–22, 24–26]. Such an equation contains a quadratic term as \(|\nabla \psi|^2\). For this reason we have introduced the integral transformation [21, 22]

\[
U(\psi) = \int_0^\psi \left[ 1 - \sigma_d(\chi) - M_p^2(\chi) \right] d\chi
\]  
(31)

under which the respective transformed GS equation can be solved by analytical techniques in the \( U \)-space, since equation (31) eliminates a quadratic term as \(|\nabla U|^2\). Transformation (31) does not affect the magnetic surfaces, it just relabels them by the flux function \( U \), and it is a generalization of that in reference [26] for isotropic equilibria with incompressible flow (\( \sigma_d = 0 \)) and that in reference [20] for static anisotropic equilibrium (\( M_p^2 = 0 \)).

Consider axisymmetric equilibria [21] with field-aligned incompressible flows, pressure anisotropy, and a purely poloidal magnetic field. In this case the equilibrium quantities are expressed as

\[
\mathbf{B} = (1 - \sigma_d - M_p^2)^{-1/2} \frac{\hat{\phi}}{\rho} \times \nabla U,
\]

\[
\mu_0 \mathbf{J} = \frac{1}{\rho} \left[ (1 - \sigma_d - M_p^2)^{-1/2} \Delta^8 U \right. \\
+ \frac{1}{2} \frac{d(\sigma_d + M_p^2)}{dU} (1 - \sigma_d - M_p^2)^{-3/2} \left| \nabla U \right|^2 \] \frac{\hat{\phi}}{\rho} \times \nabla U,
\]

\[
\mathbf{v} = \frac{M_p}{\sqrt{\mu_0 \rho}} \mathbf{B}, \quad \tilde{\rho} = \tilde{\rho}_s(U) = M_p^2 \frac{\mathbf{B}^2}{2 \mu_0},
\]

(32)

and the steady states obey the following generalized GS equation:

\[
\Delta^8 U + \mu_0 \rho^2 \frac{d\tilde{\rho}}{dU} = 0, \quad \sigma_d + M_p^2 < 1.
\]

(33)

Here the elliptic operator is defined as \( \Delta^8 = \rho^2 \nabla \cdot (\nabla / \rho^2) \); \( \tilde{\rho} \) is the effective pressure in the absence of flow, and the functions \( \sigma_d, \sigma_d, M_p \) are uniform on magnetic surfaces, \( U(\rho, z) = \text{const.} \). Assigning the surface function to be linear in \( U \), \( \tilde{\rho}_s(U) = (\tilde{\rho}_s(U)/\mu_0 \) with \( \tilde{\rho}_s \) being a free parameter, equation (33) takes the linearized form

\[
\frac{\partial^2 U}{\partial \rho^2} + \frac{\partial^2 U}{\partial z^2} - \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \tilde{\rho}_s \rho^2 = 0.
\]

(34)

The partial differential equation (34), consisting of a reduction of the respective one found in reference [27], admits a generalized Solovev analytical solution of the form

\[
U = \rho \left( \sum_j (w_j J_i(j \rho)e^{k} + w_2 J_i(j \rho)e^{-k}) \right.
\]

\[
+ w_3 Y_i(j \rho)e^{k} + w_4 Y_i(j \rho)e^{-k} \right] - \frac{\tilde{\rho}_s \rho^2}{8}.
\]

(35)

where \( J_i \) and \( Y_i \) are first order Bessel functions of the first and second kind, and \( j = 1, 2, 3, \ldots \), while the coefficients \( w_i, i = 1, 2, 3, 4 \) can be specified by appropriate boundary conditions. To completely specify the equilibrium we chose the following peaked on-axis profiles for the following free surface functions:

\[
\sigma_d = \sigma_d (\frac{U}{U_a})^2, \quad M_p^2 = M_p^2 (\frac{U}{U_a})^2, \quad \rho = \rho_a (\frac{U}{U_a})^{1/2},
\]

(36)

where the parameters \( \sigma_d, M_p^2, \rho_a, \) and \( U_a \) denote the values of the respective functions on the magnetic axis. Imposing the condition that the plasma extends up to the magnetic surface \( U = 0 \), such that both the pressure and the flow vanish thereon, we construct the up-down symmetric with respect to the plane \( z = 0 \) configuration of the triangular magnetic surfaces shown in figure 1, the magnetic axis of which is located at the position \( (\rho_a = 2.16 \text{ m}, z_a = -6 \times 10^{-4} \text{ m}) \). The cross sections of the magnetic surfaces of the above axisymmetric equilibria remain invariant along the toroidal direction, and the streamlines of its purely poloidal \( \mathbf{B} \) and \( \mathbf{v} \) fields lying on those surfaces are presented in figure 2.
Furthermore, every physical quantity associated with equilibria (32) is \(\phi\)-independent. Figure 3 shows how anisotropy and density uniformly change along the radial direction for every angle \(\phi\) on the plane \(z = z_a\), while the respective variations of the magnitude of the magnetic field on the same plane are presented in figure 4.

Applying the symmetry transformations (5), with \(\lambda = M_0(U)\), to equation (32) we find the following expressions for the physical quantities of the transformed equilibria:

\[
B_1 = \frac{b}{n} (1 - \sigma_d - M_p^2)^{-1/2} \frac{\phi}{\rho} \times \nabla U,
\]

\[
v_1 = \frac{c}{a} \sqrt{\frac{1 - \sigma_d}{n}} (1 - \sigma_d - M_p^2)^{-1/2} \frac{\phi}{\rho} \times \nabla U,
\]

\[
\mu_0 J_1 = \frac{b}{n} \left[ \frac{d(\sigma_d + M_p^2)}{dU} \left( 1 - \sigma_d - M_p^2 \right)^{-3/2} |\nabla U| \right] \frac{\phi}{\rho}
\]

\[
+ \frac{1}{2} \left( 1 - \sigma_d - M_p^2 \right)^{-1/2} \frac{\partial(b/n)}{\partial U} |\nabla U|^2 \frac{\phi}{\rho}
\]

\[
- \frac{\partial(b/n)}{\partial \phi} (1 - \sigma_d - M_p^2)^{-1/2} \nabla U \frac{\phi}{\rho^2},
\]

\[
\rho_1 = a^2 \rho, \quad J_1 = C p_y(U) - c^2 (1 - \sigma_d) \frac{B^2}{2 \mu_0},
\]

\[
\sigma_d = 1 - n^2 (1 - \sigma_d), \tag{37}
\]

where the functions \(a, b, c,\) and \(n\) may depend, in addition to \(U\), on the toroidal angle \(\phi\). However, if either of the functions \(n(r)\) or \(a(r)\) remains constant on magnetic surfaces, the breaking of the geometrical symmetry of the original equilibria remains unaffected. Note that the transformed current density \(J_1\) has a component perpendicular to the magnetic surfaces which is undesirable for confinement but this component vanishes when the function \(g = b/n\) is \(\phi\)-independent. This choice, however, yields special equilibria with purely poloidal magnetic field, \(\mathbf{B}_1 = \kappa(U) \mathbf{B}\) and permits only 3D variations of velocity and pressure.

To construct a specific equilibrium let us make the following choice for the arbitrary functions:

\[
c(U, \phi) = \sinh(\cos(\phi)) \left( \frac{1 - \sigma_d - M_p^2}{1 - \sigma_d} \right)^{1/2},
\]

\[
b(U, \phi) = \cosh(\cos(\phi)) \left( \frac{1 - \sigma_d - M_p^2}{1 - \sigma_d} \right)^{1/2},
\]

\[
a(U, \phi) = \cos(\cos(\phi))(1 - \sigma_d)^{-1/2},
\]

\[
n(U, \phi) = [\cosh(\cos(\phi))]^{1/2} (1 - \sigma_d)^{-1/2}. \tag{38}
\]

It is apparent that equation (37) together with (38) define exact 3D equilibria with a purely poloidal magnetic field and field-aligned flow by breaking the axisymmetry of the original equilibria (32). We note that the above equilibria preserve the topology of the original magnetic field lines and streamlines, \(\mathbf{B}_1 \cdot \nabla U = v_1 \cdot \nabla U = \mathbf{B} \cdot \nabla U = v \cdot \nabla U = 0\), and do not obey a GS-like equation analogous to equation (33). Although the topology of the original field lines is preserved, the strength of the transformed magnetic field in now dependent on the angle \(\phi\). The magnitude of \(\mathbf{B}_1\) is not constant along a specific field line along the toroidal direction, as shown in figure 5; it also differs from the magnitude of the original \(\mathbf{B}\), as shown in figure 6.

The flow of the transformed equilibria remains incompressible (cf remark 2). However, on account of the profile of the function \(a(U, \phi)\) the mass density \(\rho_1\) is not a surface function, and its variation along the radial direction is not uniform for every toroidal angle \(\phi\), in contrast to the original density \(\rho\), as shown in figure 7. In addition, the anisotropy

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Topology of the purely poloidal magnetic field lines (left) and velocity streamlines (right) for the constructed axisymmetric equilibria (32).
function of the transformed equilibria, $\sigma_d$, does not remain constant on the magnetic surfaces, depending on the toroidal angle $\phi$, in connection with the scalar function $n(U, \phi)$. In contrast to $\sigma_d$ which is positive and peaked on the axis, meaning that $p_1 > p_\perp$ within the plasma region for every $\phi$, the function $\sigma_d$ can take negative values along the toroidal direction; for specific values of $\phi$ it becomes zero, and thus, $p_0$, and $p_\perp$ equilibrate there. These features can be seen in figure 8.

Finally, on the basis of the choices for the arbitrary scalar functions (38) the function $g = b/n$ depends in addition to $U$, on the toroidal angle $\phi$. Owing to this dependence, the transformed purely poloidal magnetic field is no longer axisymmetric but varies periodically within $\phi$; therefore, the transformed current density in equation (37) has an additional component perpendicular to the magnetic surfaces. Note that the direction of the transformed poloidal magnetic field follows the behavior of the $\phi$-derivative of the function $g(U, \phi)$, which varying sinusoidally changes sign periodically at a period of $\pi/2$ as shown in figure 9. As a result, as $\phi$ varies the direction of the radial-current density (perpendicular to the magnetic surfaces) reverses from outwards to inwards of the magnetic surfaces, while it vanishes when $\phi = \delta \pi$, $\delta = 0, 1, 2, \ldots$, at which angles the transformed current density becomes purely toroidal (but not axisymmetric). The topology of the radial-current density component for different toroidal angles is shown in figure 10.

Though it is well known that toroidal plasma confinement is not possible with a purely poloidal magnetic field, it is interesting that in that case transformations (5) can break the geometrical symmetry and yield 3D equilibria. These equilibria may be of astrophysical interest.

4. Transformations for flow of arbitrary direction

4.1. Review of transformations between MHD into MHD and CGL into CGL equilibria

In references [2, 3, 6] symmetry transformations that produce an infinite family of MHD (CGL) equilibria with arbitrary incompressible flow once a respective MHD (CGL) equilibrium with incompressible flow is given, were introduced as follows.

**MHD into MHD:** In the case of isotropic pressure, suppose that $\{B, v, p, \varphi\}$ is a known solution of the MHD equilibrium system with flow of arbitrary direction

$$\varphi (v \cdot \nabla)v = J \times B - \nabla p, \quad \nabla \cdot (\varphi v) = 0,$$

$$\nabla \times B = \mu_0 J, \quad \nabla \cdot B = 0, \quad v \times B = \nabla \Phi,$$  \hspace{1cm} (39)

where $\Phi$ is the electrostatic potential. The flow is assumed to be incompressible, $\varphi = \varphi(\psi)$, and the function $\psi$ labels the common magnetic and velocity surfaces, if such surfaces exist. Note that these two sets of surfaces should coincide for flows of arbitrary direction because of the Faraday and Ohm’s laws. Then according to references [2, 3], $\{B_1, v_1, p_1, \varphi_1\}$ is defined by the following symmetry transformations
The magnitude of the transformed magnetic field, $B_1$, does change periodically along the toroidal direction (left). Variation of $B_1$ along the radial direction on the plane $z = z_0$ for different values of the toroidal angle $\phi$ (right).

While the original $B$ at a poloidal point (chosen as the point $(\rho = 1.8 \text{ m}, z = z_0$ in the figure) is $\phi$-independent, the respective transformed $B_1$ varies along the toroidal direction. The magnitude of the transformed field is in general higher than the respective original one except from some specific narrow toroidal regions.

Figure 5.

Figure 6.

Consists of a new family of solutions to the MHD equilibrium system.

CGL into CGL: For anisotropic pressure let $\{B, v, \rho, p_\perp, p_\parallel\}$ be a given solution of the CGL equilibrium system of equations:

\[
\rho (v \cdot \nabla) v = J \times B - \nabla \cdot P, \quad \nabla \cdot (\rho v) = 0, \\
\nabla \times B = \mu_0 J, \quad \nabla \cdot B = 0, \quad \nabla \times B = \nabla \Phi,
\]

with arbitrary incompressible flow implying $\rho = \rho(\psi)$, and the anisotropy function constant on magnetic surfaces, $\sigma_d = \sigma_d(\psi)$. Then, according to reference [6], $\{B_1, v_1, \rho_1, p_\perp, p_\parallel\}$ is defined by the following symmetry transformations:

\[
B_1 = \frac{b(r)}{n(r)} B + \frac{c(r)}{n(r) \sqrt{1 - \sigma_d}} \mathbf{v}, \\
v_1 = \frac{c(r)}{a(r) \sqrt{\mu_0 \rho}} B + \frac{b(r)}{a(r)} \mathbf{v}, \\
\rho_1(r) = a^2(r) \rho, \quad p_{\perp 1} = C \left( p_\perp + \frac{B_1^2}{2\mu_0} \right) - \frac{B_1^2}{2\mu_0}, \\
C \equiv b^2(r) - a^2(r) = \text{const.} \neq 0,
\]

is also a solution. Note that transformations (42) depend on the arbitrary functions $a(r), b(r), c(r), n(r)$, and reduce to the respective ones for isotropic pressure given by the set (40) when $\sigma_d = 0$ and $n(r) = 1$.

As stated in reference [6] the functions $a(r), b(r), c(r), n(r)$ have to be constant on the magnetic surfaces. Below we examine the validity of these transformations and whether they can break the geometrical symmetry of the original equilibria.

4.1.1. Validation of equilibrium equations for the transformed fields. In order for the new solution (42) to be valid it must satisfy the following set of CGL equilibrium equations:

\[
\nabla \cdot (\varrho_1 \mathbf{v}_1) = 0, \\
\nabla \cdot (\varrho_1 \mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 = \mathbf{J}_1 \times \mathbf{B}_1 - \nabla \cdot \mathbf{P}_1, \\
\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{J}_1, \\
\n\nabla \times \mathbf{E}_1 = 0 \implies \mathbf{E}_1 = -\nabla \Phi_1, \\
\n\nabla \cdot \mathbf{B}_1 = 0, \\
\n\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_1 = 0,
\]

where $\mathbf{P}_1$ and $\sigma_d$ are given by equation (11).

As expressed in equation (43) the transformed fields in terms of the original ones by means of equation (42) leads to
the following system of equations:

\[
\mathbf{B} \cdot \nabla b + \Lambda \mathbf{v} \cdot \nabla c - (b \mathbf{B} + c \Lambda \mathbf{v}) \cdot \frac{\nabla n}{n} = 0, \tag{44}
\]

It is apparent that if all four functions appearing in the symmetry transformations are constant on the magnetic surfaces, equations (44)-(49) are trivially satisfied; otherwise the above system of six equations for the four functions \(a(r), b(r), c(r), n(r)\) is in general overdetermined. However, if the

\[
\Lambda \mathbf{v} \cdot \nabla b + \mathbf{B} \cdot \nabla c + (\Lambda \mathbf{v} + c \mathbf{B}) \cdot \frac{\nabla a}{a} = 0, \tag{45}
\]

\[
\mathbf{B} \cdot \left( \frac{\nabla a}{a} + \frac{\nabla n}{n} \right) = 0, \tag{46}
\]

\[
\mathbf{v} \cdot \left( \frac{\nabla a}{a} + \frac{\nabla n}{n} \right) = 0, \tag{47}
\]

\[
-\mathbf{B} \cdot \left( b^2 \frac{\nabla n}{n} + c^2 \frac{\nabla a}{a} \right) - bc \Lambda \mathbf{v} \cdot \left( \frac{\nabla n}{n} + \frac{\nabla a}{a} \right)
+ \Lambda \mathbf{v} \cdot (b \nabla c - c \nabla b) = 0, \tag{48}
\]

\[
\Lambda \mathbf{v} \cdot \left( c^2 \frac{\nabla n}{n} + b^2 \frac{\nabla a}{a} \right) + bc \mathbf{B} \cdot \left( \frac{\nabla n}{n} + \frac{\nabla a}{a} \right)
+ \mathbf{B} \cdot (b \nabla c - c \nabla b) = 0, \tag{49}
\]

where \(\Lambda = \sqrt{\mu_0} \beta / \sqrt{1 - \sigma_d}\).

Figure 7. The transformed density \(q_1(U, \phi)\) on the plane \(z = z_0\) does not remain uniform on the magnetic surfaces as the toroidal angle \(\phi\) varies (left). The value of \(q_1\) on the magnetic axis varies along \(\phi\) and is higher than the respective value of the original \(\varphi\), which is constant and maximum thereon (right).

Figure 8. The variation of the anisotropy function, \(\sigma_{d1}\), of the transformed equilibria is not uniform along the toroidal direction (left). The perpendicular pressure of the transformed equilibria is higher than the respective parallel one, except from the toroidal angles \(\pi/2\) and \(3\pi/2\) for which \(p_{th} = p_{th'}\) in contrast to the original \(\sigma_d\) which is always peaked on-axis and positive for every \(\phi\) (right).

Figure 9. The \(\phi\)-derivative of the function \(g\) changes sign periodically with respect to \(\phi\) at a period \(\pi/2\), thus vanishing for \(\phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\). This results in a similar variation of the radial-current density component (perpendicular to the magnetic surfaces) from negative to positive values.
functions $a(r), b(r), c(r), n(r)$ are chosen so that

\[- \nabla a = \nabla n = \nabla (b + c) \quad (b + c), \]

being satisfied when

\[a = \frac{1}{n}, \quad n = b + c, \quad (51)\]

then equations (46) and (47) are trivially satisfied, while equations (44), (45), (48), and (49) reduce to the single relationship:

\[(B - \Lambda v) \cdot (b \nabla c - c \nabla b) = 0. \quad (52)\]

Since $b \neq \pm c$ for the transformation to be invertible, equation (52) is satisfied only for parallel flows:

\[v = \frac{\sqrt{1 - \sigma_d}}{\sqrt{\mu_0 \theta}} B. \quad (53)\]

Note that the field-aligned equilibria (53) and (3) differ from each other, and thus, transformations (42) introduced in reference [6] for the flow of an arbitrary direction are not reducible into the respective transformations (5) for parallel flows presented herein (cf section 2). This result also holds for the respective isotropic transformations (40) and (2) derived in reference [2]. For the special equilibria with field-aligned flows satisfying equation (53) and $a(r), b(r), c(r), n(r)$ generally not constant on magnetic surfaces, transformations (42) reduce to

\[B_1 = B, \quad v_1 = (b + c)^2 v, \quad \varphi_1 = \varphi / (b + c)^2, \quad (42)\]

\[p_{\parallel} = C p_{\parallel} + (C + 1 - 2(b + c)^2(1 - \sigma_d)) \frac{B^2}{2\mu_0}, \quad (54)\]

\[p_{\perp} = C p_{\parallel} + (C - 1) \frac{B^2}{2\mu_0}, \quad \sigma_d = 1 - (b + c)^2(1 - \sigma_d), \quad C = b^2 - c^2 = \text{const.} \neq 0. \quad (55)\]

For equilibria with isotropic pressure satisfying equation (40), being recovered from equation (42) for $\sigma_d = 0$ and $n = a = b + c = 1$, the choice of equation (51) leads to

\[B_1 = B, \quad v_1 = v, \quad \varphi_1 = \varphi, \quad (42)\]

\[p_{\parallel} = C p_{\parallel} + (C - 1) \frac{B^2}{2\mu_0}, \quad (54)\]

\[C = b - c = \text{const.} \neq 0. \quad (55)\]

With the aid of equations (54) and (55) we observe that in the presence of pressure anisotropy the transformed velocity and mass density differ from the respective original ones.

Now suppose that the original equilibrium is helically symmetric [22]. Then the following relations hold:

\[B = I h + h \times \nabla \varphi, \quad (56)\]

\[v = \frac{\Theta}{\theta} h + \frac{M_p}{\sqrt{\mu_0 \theta}} h \times \nabla \varphi, \quad (57)\]

\[\frac{1}{q} \frac{d\Phi}{d\psi} = \frac{IM_p}{\sqrt{\mu_0 \theta}} - \frac{\Theta}{\theta}. \quad (58)\]

where the function $\Theta$ relates to the helical velocity field and $\Phi = \Phi(\psi)$. Equation (51) implies $t = (\sqrt{1 - \sigma_d / \sqrt{\mu_0 \theta}} \Theta / \varphi)$ and $d\Phi / d\psi = 0$. These relations lead to the following one:
\[ \frac{I}{\sqrt{\mu_0 \varrho}} (M_p - \sqrt{1 - \sigma_d}) = 0, \]  

(59)

which implies either \( I = 0 \) or \( M_p^2 + \sigma_d = 1 \). It turns out again that symmetry breaking is possible only for purely poloidal parallel flows \( (I = 0) \). The relation \( M_p^2 + \sigma_d = 1 \) is connected to the Alfvén singularity. The same conclusion holds for axially and translationally symmetric original equilibria with or without pressure anisotropy.

### 4.1.2. Arbitrary functions constant on magnetic surfaces

In the above subsection we found that the symmetry transformations (42) (and the respective transformations (40) for isotropic pressure) are valid when the arbitrary functions are constant on magnetic surfaces, since equations (44)–(49) are trivially satisfied. Here we examine the equilibria derived from a given geometrically symmetric one of this kind.

Let the original CGL equilibria (41) possess magnetic surfaces \( \psi = \text{const} \), which both \( B \) and \( v \) lie on. Also, suppose that the respective surfaces \( \psi_i = \text{const} \) are defined for the transformed equilibria (42) which \( B_1 \) and \( v_1 \) lie on. It holds that

\[ B_1 \times v_1 = \frac{C}{n(\psi) a(\psi)} B \times v, \]

(60)

and thus, the magnetic surfaces through the transformation are preserved:

\[ \psi_1 = F(\psi). \]

(61)

This means that all vectors \( B, B_1, v, v_1 \) lie on the surfaces \( \psi = \text{const} \). As a result, if the original equilibria has some known geometrical symmetry, the transformed equilibria will have the same symmetry, too.

Consider now helically symmetric equilibria with incompressible flow of arbitrary direction and an anisotropy function constant on magnetic surfaces [22]:

\[ B = l h + h \times \nabla \psi(r, u), \]

\[ v = -h + \frac{M_p}{\sqrt{\mu_0 \varrho}} h \times \nabla \psi(r, u), \]

\[ \mu_0 J = (L \psi(r, u) + 2 k m q I(\psi, r)) h - h \times \nabla I(\psi, r), \]

\[ \bar{\rho} = \bar{\rho}_1(\psi) = \left[ \frac{\sqrt{\mu_0 \varrho}}{2} \left( \frac{1 - \sigma_d}{q(1 - \sigma_d - M_p^2)} \right)^2 \right] \]

(62)

where the elliptic operator is defined as \( L = (\nabla \cdot (q \nabla)) / q \). Note that the current density lies on well-defined helicoidal surfaces \( I = \text{const.} \), while the effective pressure is uniform on the surfaces defined by \( \bar{\rho} = \text{const.} \), both sets of surfaces not coinciding with the magnetic surfaces. By applying the symmetry transformations (42) with \( a = a(\psi), \ b = b(\psi), \ c = c(\psi), \ n = n(\psi) \), we obtain the following class of equilibria:

\[ B_1 = \frac{1}{n} \left[ b l + c \frac{\sqrt{\mu_0 \varrho}}{\sqrt{1 - \sigma_d}} \Theta_1 \right] h \]

\[ + \frac{1}{n} \left[ b + c \frac{M_p}{\sqrt{1 - \sigma_d}} \right] h \times \nabla \psi, \]

(63)

\[ v_1 = \frac{1}{a} \left[ c \frac{\sqrt{1 - \sigma_d}}{\sqrt{\mu_0 \varrho}} l + b \Theta \right] h \]

\[ + \frac{1}{a} \left[ c \frac{1 - \sigma_d}{\sqrt{\mu_0 \varrho}} + b \frac{M_p}{\sqrt{1 - \sigma_d}} \right] h \times \nabla \psi, \]

(64)

\[ \mu_0 J_1 = \left( G \mathcal{L} \psi + \frac{dG}{d\psi} |\nabla \psi|^2 + 2 k m q I_1 \right) h - h \times \nabla I_1, \]

(65)

\[ \bar{p}_1 = C \bar{p} + (1 - \sigma_d) (C B^2 - n^2 B_1^2) \]

\[ \times \frac{2}{\mu_0 \varrho}, \]

(66)

Note that although the magnetic surfaces are preserved, neither the transformed current density nor the transformed effective pressure remains on the surfaces of the respective original quantities; \( J_1 \cdot \nabla I = 0, \ \bar{p}_1 = \bar{p} \).

Now since \( \psi_1 = F(\psi) \) and the original equilibria are helically symmetric the transformed ones should retain that symmetry. This means that the transformed fields can also be written in a form similar to equation (62); in particular for the transformed velocity we have

\[ v_1 = \frac{\Theta_1}{\varrho_1} h + \frac{M_{p1}}{\sqrt{\mu_0 \varrho_1}} h \times \nabla \psi_1(r, u), \]

(67)

where

\[ M_{p1}^2 = \frac{v_{pol1}^2}{B_{pol1}^2 / \mu_0 \varrho_1} = (n \sqrt{1 - \sigma_d}) \frac{c \sqrt{1 - \sigma_d} + b M_p}{b \sqrt{1 - \sigma_d} + c M_p}. \]

(68)

Equality of the poloidal velocity components in equations (64) and (67) yields

\[ \frac{d \psi_1}{d \psi} = \frac{b \sqrt{1 - \sigma_d} + c M_p}{n \sqrt{1 - \sigma_d}} \Rightarrow \psi_1(\psi) \]

\[ = \int_0^\psi \frac{b(\chi) \sqrt{1 - \sigma_d(\chi)}}{n(\chi) \sqrt{1 - \sigma_d(\chi)}} d\chi. \]

(69)

Adopting equation (31) both for the original and the transformed helically symmetric equilibria, equation (69) yields

\[ \frac{d U_1(U)}{d U} = C_1^{1/2} \Rightarrow U_1 = C_1^{1/2} U. \]

(70)

Therefore the transformed equilibria differ from the starting ones only by a constant factor \( C_1^{1/2} \), in agreement with the conclusions drawn in the previous sections; the geometrical symmetry of the original equilibria can break only for purely
poloidal magnetic field, otherwise the transformed equilibria retain the original symmetry.

4.2. Transformations between MHD and CGL equilibria

In references [6–8] transformations that produce CGL anisotropic equilibria from given isotropic MHD ones are introduced as follows: if \( \{ \mathbf{B}, \mathbf{v}, p, \varphi \} \) is a known solution of the MHD equilibrium system (39), then the following symmetry transformations

\[
\mathbf{B}_1 = f_1(\mathbf{r})\mathbf{B}, \quad \mathbf{v}_1 = g_1(\mathbf{r})\mathbf{v}, \quad \varphi_1 = \frac{C_0\mu_0}{\delta_1^2(\mathbf{r})}\varphi,
\]

\[
p_{1z} = C_0\mu_0p + C_1 + \left( C_0\mu_0 - f_1^2(\mathbf{r}) \right)\frac{\mathbf{B}^2}{2\mu_0},
\]

\[
p_{1\theta} = C_0\mu_0p + C_1 - \left( C_0\mu_0 - f_1^2(\mathbf{r}) \right)\frac{\mathbf{B}^2}{2\mu_0},
\]

(71)

where \( C_0 \) and \( C_1 \) are arbitrary constants, produce an infinite family of CGL equilibria satisfying equation (43). Transformations (71) are also valid in the static limit, \( \mathbf{v} = 0 \). Let us examine their validity.

Substituting equation (71) into equation (43) we obtain

\[
\mathbf{B} \cdot \nabla f_1(\mathbf{r}) = 0,
\]

\[
\mathbf{v} \cdot \nabla g_1(\mathbf{r}) = 0.
\]

(72)

Thus, in order for transformations (71) to be valid, the functions \( f_1(\mathbf{r}) \) and \( g_1(\mathbf{r}) \) must be constant on the magnetic field lines and velocity streamlines of the original equilibria, and respective considerations on their structure can be made as those in section 2. Therefore it turns out again that the only way that the geometrical symmetry of the original isotropic equilibrium can be broken is if and only if the magnetic and velocity fields are collinear and purely poloidal. In this context, conclusions for the geometrical symmetry of astrophysical jets with magnetic field lines going to infinity in connection with the coordinate \( z \), examined in references [6, 8] in the static limit, should be reconsidered.

5. Conclusions

In the present work we extensively revised the symmetry transformations previously introduced in a series of papers in references [2–9], which once applied to known MHD and/or CGL equilibria produce an infinite new continuous families of respective equilibria. These transformations contain some arbitrary scalar functions, the structure of which depends on the topology of the given equilibria. We examined both transformations that map MHD into MHD, CGL into CGL, and MHD into CGL equilibria, either with field-aligned or arbitrary incompressible flows, particularly as concerns their validity and applicability. All these symmetry transformations can change the magnitude of the equilibrium quantities or create stationary configurations from static ones without changing the solutions’ topology. In addition, we examined whether these transformations can break the geometrical symmetry of the original equilibrium.

In section 2 we presented a new set of symmetry transformations that can be applied to any known CGL equilibria with special field-aligned incompressible flow satisfying equation (3) and pressure anisotropy function, \( \sigma_d \), constant on the magnetic field lines, to produce an infinite class of equilibria with collinear \( \mathbf{v}_1 \) and \( \mathbf{B}_1 \) fields, and with \( \varphi_1 \) and \( \sigma_d \) functions that in general may be arbitrary. These transformations consist of a generalization of the ones introduced in reference [2] for the same kind of field-aligned incompressible flow and isotropic pressure, and can also be applied to static anisotropic equilibria.

In addition, we examined the structure of the arbitrary scalar functions included in the symmetry transformations in relation to the topology of the magnetic field of the original equilibrium and the existence of magnetic surfaces, and proved that if the original equilibrium possesses some known geometrical symmetry, this can be broken by the transformations if and only if the magnetic field is purely poloidal. In this respect, in section 3 we applied the aforementioned symmetry transformations to specifically prescribed axisymmetric CGL equilibria with collinear and purely poloidal \( \mathbf{v} \) and \( \mathbf{B} \) fields, incompressible flow, and \( \sigma_d \) uniform on the magnetic surfaces. We constructed 3D equilibria with collinear and purely poloidal \( \mathbf{v}_1 \) and \( \mathbf{B}_1 \), incompressible flow, but with mass density and anisotropy function varying on the magnetic surfaces, all physical quantities of which depending on all three spatial variables not being invariant along the toroidal direction \( \phi \).

In section 4 we examined the transformations introduced in references [2–9] applied to given equilibria with incompressible flow non-collinear to the magnetic field. We showed that these transformations are valid if the arbitrary functions included therein are either constant on the magnetic surfaces, if such surfaces exist, or if they are related by a special relationship; in the latter case it turns out that the fields \( \mathbf{v} \) and \( \mathbf{B} \) of the original equilibria are restricted to be collinear. If the original equilibria have certain geometrical symmetry, in the former case they differ from the transformed ones only by a constant factor, while in the latter case this symmetry can be broken only for purely poloidal magnetic fields.

Summarizing, the overall conclusion of this study is that both transformations introduced in references [2–9] and the ones presented herein (cf section 2) can break the geometrical symmetry of the original equilibria, both static and/or with field-aligned incompressible flow and both isotropic and/or anisotropic with the function \( \sigma_d \), constant on the magnetic field lines, if and only if the magnetic field is purely poloidal. Otherwise the transformed equilibria retain the geometrical symmetry of the original ones.

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