The isoperimetric problem for Hölderian curves

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Abstract

We prove a necessary stationary condition for non-differentiable isoperimet-
ric variational problems with scale derivatives, defined on the class of Höl-
der continuous functions.

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Key words: scale calculus, isoperimetric problem, non differentiability.

1 Introduction

An analogue of differentiable calculus for Hölder continuous functions has
been recently developed by J. Cresson, by substituting the classical notion
of derivative by a new complex operator, called the scale derivative [2]. A
Leibniz rule similar to the classical one is proved, and with it a general-
ed Euler-Lagrange equation, valid for nonsmooth curves, is obtained [2]. The
new calculus of variations find applications in scale-relativity theory, and
some applications are given to Hamilton’s principle of least action and to
nonlinear Schrödinger equations [1, 2, 3].

In this note we introduce the isoperimetric problem for Hölder con-
 tinuous curves in Cresson’s setting. Section 2 reviews the quantum cal-
 culus of J. Cresson, fixing some typos found in [2]. Main results are given in
Section 3 where the non differentiable isoperimetric problem is formulated and
respective stationary condition proved (see Theorem 4). We end with Sec-
tion 4 illustrating the applicability of our Theorem 4 to a simple example
that has an Hölder continuous extremal, which is not differentiable in the
classical sense.
2 Preliminaries

In this section we review the quantum calculus [1, 2], which extends the classical differential calculus to nonsmooth continuous curves. As usual, we denote by $C^0$ the set of continuous real valued functions defined on $\mathbb{R}$.

**Definition 1.** ([2]) Let $f \in C^0$ and $\epsilon > 0$. The $\epsilon-$left and $\epsilon-$right quantum derivatives are defined by

$$
\Delta^-\epsilon f(x) = \frac{-f(x-\epsilon) - f(x)}{\epsilon} \quad \text{and} \quad \Delta^+\epsilon f(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon},
$$

respectively. In short, we write $\Delta^\sigma\epsilon f(x)$, $\sigma = \pm$.

Next concept generalizes the derivative for continuous functions, not necessarily smooth.

**Definition 2.** (cf. [2]) Let $f \in C^0$ and $\epsilon > 0$. The $\epsilon$ scale derivative of $f$ at $x$ is defined by

$$
\square\epsilon f(x) = \frac{1}{2}(\Delta^+\epsilon f(x) + \Delta^-\epsilon f(x)) - i\frac{1}{2}(\Delta^+\epsilon f(x) - \Delta^-\epsilon f(x)), \quad i^2 = -1. \quad (1)
$$

If $f$ is a $C^1$ function, and if we take the limit as $\epsilon \to 0$ in (1), we obtain $f'(x)$. To simplify, when there is no danger of confusion, we will write $\square\epsilon f$ instead of $\square\epsilon f/\square x$. For complex valued functions, we define

$$
\square\epsilon f = \frac{\square{\text{Re}(f)}(x) + i\square{\text{Im}(f)}(x)}{\square x}(x).
$$

We now collect the results needed to this work. First the Leibniz rule for quantum calculus:

**Theorem 1.** (cf. [2]) Given $f, g \in C^0$ and $\epsilon > 0$, one has

$$
\square\epsilon (f \cdot g) = \square\epsilon f \cdot g + f \cdot \square\epsilon g + i\frac{\epsilon}{2}(\square\epsilon f \square\epsilon g - \square\epsilon f \square\epsilon g - \square\epsilon f \square\epsilon g), \quad (2)
$$

where $\square\epsilon f$ is the complex conjugate of $\square\epsilon f$.

If $f$ and $g$ are both differentiable, we obtain the Leibniz rule $(f \cdot g)' = f' \cdot g + f \cdot g'$ from [2], taking the limit as $\epsilon \to 0$.

**Definition 3.** Let $f \in C^0$, and $\alpha \in (0,1)$ be a real number. We say that $f$ is H"olderian of H"older exponent $\alpha$ if there exists a constant $c$ such that, for all $\epsilon > 0$, and all $x, x' \in \mathbb{R}$ such that $|x - x'| \leq \epsilon$,

$$
|f(x) - f(x')| \leq c\epsilon^\alpha.
$$

We denote by $H^\alpha$ the set of H"olderian functions with H"older exponent $\alpha$. 

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From now on, we assume that $\alpha \in (0,1)$ is fixed, and $\epsilon$ is a sufficiently small parameter, $0 < \epsilon \ll 1$. Let

$$C^\alpha_\epsilon(a,b) = \{ y : [a-\epsilon, b+\epsilon] \to \mathbb{R} \mid y \in H^\alpha \}.$$ 

A functional is a function $\Phi : C^\alpha_\epsilon(a,b) \to \mathbb{C}$. We study the class of functionals $\Phi$ of the form

$$\Phi(y) = \int_a^b f(x, y(x), \Box_\epsilon y(x)) \, dx,$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a $C^1$ function, called the Lagrangian. We assume that the Lagrangian satisfies

$$\|Df(x, y(x), \Box_\epsilon y(x))\| \leq C,$$

where $C$ is a positive constant, $D$ denotes the differential, and $\| \cdot \|$ is a norm for matrices.

If we consider the class of differentiable functions $y \in C^1$, we obtain the classical functional

$$\Phi(y) = \int_a^b f(x, y(x), \dot{y}(x)) \, dx$$

of the calculus of variations when $\epsilon$ goes to zero.

The methods to solve problems of the calculus of variations admit a common variational approach: we consider a class of functions $\eta(x)$ such that $\eta(a) = 0 = \eta(b)$; and admissible functions $\overline{y} = y + \epsilon_1 \eta$ on the neighborhood of $y$. For $\epsilon_1$ sufficiently small, $\overline{y}$ is infinitely near $y$ and satisfies given boundary conditions $\overline{y}(a) = y(a)$ and $\overline{y}(b) = y(b)$. For our purposes, we need another assumption about functions $\eta$.

**Definition 4.** (2) Let $y \in C^\alpha_\epsilon(a,b)$. A variation $\overline{y}$ of $y$ is a curve of the form $\overline{y} = y + h$, where $h \in C^\beta_\epsilon(a,b)$, $\beta \geq \alpha 1_{[1/2,1]} + (1 - \alpha)1_{[0,1/2]}$, and $h(a) = 0 = h(b)$.

The minimal condition on $\beta$ is to ensure that the variation curve $\overline{y}$ is still on $C^\alpha_\epsilon(a,b)$.

**Definition 5.** (2) A functional $\Phi$ is called differentiable on $C^\alpha_\epsilon(a,b)$ if for all variations $\overline{y} = y + h$, $h \in C^\beta_\epsilon(a,b)$,

$$\Phi(y + h) - \Phi(y) = F_y(h) + R_y(h),$$

where $F_y$ is a linear operator and $R_y(h) = O(h^2)$. 

Theorem 2. (cf. [2]) For all $\epsilon > 0$, the functional $\Phi$ defined by (3) is differentiable, and its derivative is

$$F_y(h) = \int_a^b \left[ \frac{\partial f}{\partial y}(x, y(x), \square_\epsilon y(x)) - \square_\epsilon \left( \frac{\partial f}{\partial \square_\epsilon y}(x, y(x), \square_\epsilon y(x)) \right) \right] h(x) \, dx$$

$$+ \int_a^b \frac{\partial f}{\partial \square_\epsilon y}(x, y(x)) \frac{\partial f}{\partial \square_\epsilon y}(x, y(x)) h(x) \, dx + iR_y(h)$$

with

$$R_y(h) = -\frac{\epsilon}{2} \int_a^b \left[ \square_\epsilon f_\epsilon(x) \square_\epsilon h(x) - \square_\epsilon f_\epsilon(x) \square_\epsilon h(x) - \square_\epsilon f_\epsilon(x) \square_\epsilon h(x) \right] dx$$

where

$$f_\epsilon(x) = \frac{\partial f}{\partial \square_\epsilon y}(x, y(x), \square_\epsilon y(x)).$$

Definition 6. (2) Let $a_p(\epsilon)$ be a real or complex valued function, with parameter $p$. We denote by $[\cdot]_\epsilon$ the (unique) linear operator defined by

$$a_p(\epsilon) - [a_p(\epsilon)]_\epsilon \to_{\epsilon \to 0} 0 \quad \text{and} \quad [a_p(\epsilon)]_\epsilon = 0 \quad \text{if} \quad \lim_{\epsilon \to 0} a_p(\epsilon) = 0.$$

Definition 7. (2) We say that $y$ is an extremal curve for the functional (3) on $C^\beta_\epsilon(a, b)$, if $[F_y(h)]_\epsilon = 0$ for all $\epsilon > 0$ and $h \in C^\beta_\epsilon(a, b)$.

The main result of [2] is a version of the Euler-Lagrange equation for nonsmooth curves:

Theorem 3. (2) The curve $y$ is an extremal for the functional (3) on $C^\beta_\epsilon(a, b)$ if and only if

$$\left[ \frac{\partial f}{\partial y}(x, y(x), \square_\epsilon y(x)) - \square_\epsilon \left( \frac{\partial f}{\partial \square_\epsilon y}(x, y(x), \square_\epsilon y(x)) \right) \right]_\epsilon = 0$$

for every $\epsilon > 0$.

## 3 Main results

The isoperimetric problem is one of the most ancient optimization problems. One seeks to find a continuously differentiable curve $y = y(x)$, satisfying
given boundary condition \( y(a) = a_0 \) and \( y(b) = b_0 \), which minimizes or maximizes a given functional

\[
I(y) = \int_a^b f(x, y(x), \dot{y}(x)) \, dx,
\]

for which a second given functional

\[
G(y) = \int_a^b g(x, y(x), \dot{y}(x)) \, dx
\]

possesses a given prescribed value \( K \). The classical method to solve this problem involves a Lagrange multiplier \( \lambda \) and consider the problem of extremizing the functional

\[
\int_a^b (f - \lambda g) \, dx
\]

using the respective Euler-Lagrange equation. In scale calculus we have an additional problem, because functionals \( I \) and \( G \) take complex values and so the Lagrange multiplier method must be adapted. We will assume that \( \|Dg(\cdot)\| \) is finite.

For our main theorem, we need the following lemma.

**Lemma 1.** If \( \lim_{\epsilon \to 0} (a_p(\epsilon)) \) and \( \lim_{\epsilon \to 0} (b_p(\epsilon)) \) are both finite, then

\[
[a_p(\epsilon) \cdot b_p(\epsilon)]_{\epsilon} = [a_p(\epsilon)]_{\epsilon} \cdot [b_p(\epsilon)]_{\epsilon}.
\]

**Proof.** Since \( \lim_{\epsilon \to 0} (a_p(\epsilon)) \) and \( \lim_{\epsilon \to 0} (b_p(\epsilon)) \) are finite, then \( \lim_{\epsilon \to 0} [a_p(\epsilon)]_{\epsilon} \) and \( \lim_{\epsilon \to 0} [b_p(\epsilon)]_{\epsilon} \) are also finite. Moreover,

1. \[
\lim_{\epsilon \to 0} (a_p(\epsilon) \cdot b_p(\epsilon) - [a_p(\epsilon)]_{\epsilon} \cdot [b_p(\epsilon)]_{\epsilon})
\]
   \[=\lim_{\epsilon \to 0} ((a_p(\epsilon) - [a_p(\epsilon)]_{\epsilon}) \cdot b_p(\epsilon) + [a_p(\epsilon)]_{\epsilon} \cdot (b_p(\epsilon) - [b_p(\epsilon)]_{\epsilon})) = 0.\]

2. If \( \lim_{\epsilon \to 0} (a_p(\epsilon) \cdot b_p(\epsilon)) = 0 \), then \( \lim_{\epsilon \to 0} (a_p(\epsilon)) = 0 \) or \( \lim_{\epsilon \to 0} (b_p(\epsilon)) = 0 \). Therefore, \( [a_p(\epsilon)]_{\epsilon} = 0 \) or \( [b_p(\epsilon)]_{\epsilon} = 0 \) and so \( [a_p(\epsilon)]_{\epsilon} \cdot [b_p(\epsilon)]_{\epsilon} = 0 \).

\[\square\]

**Definition 8.** Given a constraint functional \( G(y) = K \) and a curve \( \overline{y} \), we say that \( \overline{y} \) is an extremal curve for the functional \( I(y) = \int_a^b f(x, y(x), \square_{\epsilon} y(x)) \, dx \)
subject to the constraint $G(y) = K$, if whenever $\hat{y} = \bar{y} + \sum_k h_k$, $h_k \in C^\beta_c(a,b)$, is a variation satisfying the constraint $G(\hat{y}) = K$, then

$$[F_\bar{y}(h_k)]_\epsilon = \int_a^b \left[ \frac{\partial f}{\partial y}(x, \bar{y}(x), \Box_\epsilon \bar{y}(x)) - \Box_\epsilon \left( \frac{\partial f}{\partial \Box_\epsilon y}(x, \bar{y}(x), \Box_\epsilon \bar{y}(x)) \right) \right]_\epsilon h_k(x) \, dx = 0$$

for all $\epsilon > 0$ and for all $k$.

**Theorem 4.** Let $\bar{y} \in C^\alpha_c(a,b)$. Suppose that $\bar{y}$ is an extremal for the functional

$$\begin{align*}
I : C^\alpha_c(a,b) & \to \mathbb{C} \\
y & \mapsto \int_a^b f(x, y(x), \Box_\epsilon y(x)) \, dx
\end{align*}$$

on $C^\beta_c(a,b)$, subject to the boundary conditions $y(a) = a_0$, $y(b) = b_0$ and the integral constraint

$$G(y) = \int_a^b g(x, y(x), \Box_\epsilon y(x)) \, dx = K,$$

where $K \in \mathbb{C}$ is a given constant. If

1. $\bar{y}$ is not an extremal for $G$;

2. and

$$\lim_{\epsilon \to 0} \max_{x \in [a,b]} \left| \left( \frac{\partial f}{\partial y} - \Box_\epsilon \left( \frac{\partial f}{\partial \Box_\epsilon y} \right) \right) \right|_{(x, \bar{y}(x), \Box_\epsilon \bar{y}(x))}$$

and

$$\lim_{\epsilon \to 0} \max_{x \in [a,b]} \left| \left( \frac{\partial g}{\partial y} - \Box_\epsilon \left( \frac{\partial g}{\partial \Box_\epsilon y} \right) \right) \right|_{(x, \bar{y}(x), \Box_\epsilon \bar{y}(x))}$$

are both finite;

then there exists $\lambda \in \mathbb{R}$ such that

$$\left[ \left( \frac{\partial L}{\partial y} - \Box_\epsilon \left( \frac{\partial L}{\partial \Box_\epsilon y} \right) \right) \right]_{(x, \bar{y}(x), \Box_\epsilon \bar{y}(x))} = 0,$$

where $L = f - \lambda g$. In other words, $\bar{y}$ is an extremal for $L$.

**Remark 1.** Hypothesis 2 of Theorem 4 is trivially satisfied in the case where the admissible curves are smooth.

**Proof.** To short, let $u = (x, \bar{y}(x), \Box_\epsilon \bar{y}(x))$. Consider the two-parameter family of variations

$$\hat{y} = \bar{y} + \epsilon_1 \eta_1 + \epsilon_2 \eta_2,$$
such that \( \eta_1, \eta_2 \in C^\beta_c(a, b) \), \( \beta \geq \alpha_1[1/2,1] + (1 - \alpha)1_{0,1/2} \), \( \eta_1(a) = 0 = \eta_1(b) \), \( \eta_2(a) = 0 = \eta_2(b) \), and \( \epsilon_1, \epsilon_2 \in B_r(0) \), with \( r \) sufficiently small. Then, \( \hat{y}(a) = a_0 \) and \( \hat{y}(b) = b_0 \), as prescribed, for all values of the parameters \( \epsilon_1 \) and \( \epsilon_2 \). It is easy to see that \( \hat{y} \in C^\alpha(a, b) \).

1. If we fix two curves \( \eta_1 \) and \( \eta_2 \), we can consider the functions \( \mathcal{T} \) and \( \mathcal{G} \) with two variables \( \epsilon_1 \) and \( \epsilon_2 \), defined by

\[
\mathcal{T}(\epsilon_1, \epsilon_2) = \int_a^b f(x, \overline{y}(x)) + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \quad \mathcal{G}(\epsilon_1, \epsilon_2) = \int_a^b g(x, \overline{y}(x)) + \epsilon_1 \eta_1 + \epsilon_2 \eta_2 \ dx.
\]

Let \( \mathcal{G} = \mathcal{G} - K \).

2. We have \( \nabla \mathcal{G}(0, 0) \neq 0 \). Indeed, since \( g \) is a smooth function, \( \mathcal{G} \) is also smooth and

\[
\frac{\partial \mathcal{G}}{\partial \epsilon_1} \bigg|_{(0,0)} = \int_a^b \left( \eta_1 \frac{\partial g}{\partial y}|_u + \Box \eta_1 \frac{\partial g}{\partial \Box y}|_u \right) dx
\]

\[
= \int_a^b \frac{\partial g}{\partial y}|_u \left( \frac{\partial g}{\partial \Box y}|_u \right) \eta_1 dx + \int_a^b \Box \frac{\partial g}{\partial \Box y}|_u \left( \frac{\partial g}{\partial \Box y}|_u \right) \eta_1 dx - \frac{1}{2} \int_a^b \left[ \Box g \Box \eta_1 - \Box g \Box \eta_1 - \Box g \Box \eta_1 + \Box g \Box \eta_1 \right] dx,
\]

where

\[
g_\epsilon = \frac{\partial g}{\partial \Box y}|_u.
\]

Since

\[
\lim_{\epsilon \to 0} \int_a^b \Box \left( \frac{\partial g}{\partial \Box y}|_u \right) \eta_1(x) dx = 0
\]

and

\[
\lim_{\epsilon \to 0} \epsilon \int_a^b (Op_\epsilon g_\epsilon Op_\epsilon' \eta_1) dx = 0,
\]

where \( Op_\epsilon \) and \( Op_\epsilon' \) is equal to \( \Box \epsilon \) and \( \Box \epsilon \) (cf. [2, Lemma 3.2]), it follows that

\[
\left[ \frac{\partial \mathcal{G}}{\partial \epsilon_1} \right]_{(0,0)} = \int_a^b \left[ \frac{\partial g}{\partial y}|_u - \Box \left( \frac{\partial g}{\partial \Box y}|_u \right) \right] \eta_1(x) dx.
\]
Since \( \overline{y} \) is not an extremal of \( G \), there exists a curve \( \eta_1 \) such that

\[
\left[ \frac{\partial G}{\partial \epsilon_1} \right]_{(0,0)} \neq 0.
\]

Therefore, by the definition of \( \left[ \cdot \right]_\epsilon \), we conclude that

\[
\frac{\partial G}{\partial \epsilon_1} \bigg|_{(0,0)} \neq 0.
\]

3. We can choose \( \epsilon_2 \eta_2 \) in order to satisfy the isoperimetric condition. Since \( \nabla G(0,0) \neq 0 \) and \( G(0,0) = 0 \), by the implicit function theorem, there exists a function \( \epsilon_1 := \epsilon_1(\epsilon_2) \) defined on a neighbourhood of zero such that

\[
G(\epsilon_1(\epsilon_2), \epsilon_2) = 0.
\]

4. We now adapt the Lagrange multiplier method. Since \( \overline{G}(\epsilon_1(\epsilon_2), \epsilon_2) = 0 \), for any \( \epsilon_2 \), then

\[
0 = \frac{d}{d\epsilon_2} \overline{G}(\epsilon_1(\epsilon_2), \epsilon_2) = \frac{d\epsilon_1}{d\epsilon_2} \cdot \frac{\partial G}{\partial \epsilon_1} + \frac{\partial G}{\partial \epsilon_2}
\]

and so, as \( \epsilon \) goes to zero,

\[
\frac{d\epsilon_1}{d\epsilon_2} \bigg|_0 = -\int_a^b \left( \frac{\partial g}{\partial y} \bigg|_u - \frac{\Box x}{\Box y} \left( \frac{\partial g}{\partial \Box y} \bigg|_u \right) \right) \eta_2(x) \, dx + \int_a^b \frac{\partial g}{\partial y} \bigg|_u \cdot \eta_2 \, dx - \ldots
\]

is finite. Observe that

\[
\lim_{\epsilon \to 0} \frac{\partial \overline{T}}{\partial \epsilon_1} \bigg|_u = \lim_{\epsilon \to 0} \int_a^b \left( \frac{\partial f}{\partial y} \bigg|_u - \frac{\Box x}{\Box y} \left( \frac{\partial f}{\partial \Box y} \bigg|_u \right) \right) \eta_1(x) \, dx
\]

and

\[
\lim_{\epsilon \to 0} \frac{\partial \overline{G}}{\partial \epsilon_1} \bigg|_u = \lim_{\epsilon \to 0} \int_a^b \left( \frac{\partial g}{\partial y} \bigg|_u - \frac{\Box x}{\Box y} \left( \frac{\partial g}{\partial \Box y} \bigg|_u \right) \right) \eta_1(x) \, dx
\]

are also finite. Let us prove that

\[
\frac{d}{d\epsilon_2} \left[ T(\epsilon_1(\epsilon_2), \epsilon_2) \right]_{(0,0)} = 0.
\]

(4)
A direct calculation shows that

\[
\frac{d}{d\epsilon_2} \left[ T(\epsilon_1(\epsilon_2), \epsilon_2) \right]_{\epsilon_0} = \left[ \frac{d\epsilon_1}{d\epsilon_2} \frac{\partial T}{\partial \epsilon_1} + \frac{\partial T}{\partial \epsilon_2} \right]_{\epsilon_0} \\
= \left[ \frac{d\epsilon_1}{d\epsilon_2} \right]_{\epsilon_0} \int_a^b \left[ \frac{\partial f}{\partial y} - \frac{x}{y} \left( \frac{\partial f}{\partial \epsilon_2} \right) \right]_{\epsilon_0} \eta_1 \, dx \\
+ \int_a^b \left[ \frac{\partial f}{\partial y} - \frac{x}{y} \left( \frac{\partial f}{\partial \epsilon_2} \right) \right]_{\epsilon_0} \eta_2 \, dx \\
= \left[ \frac{d\epsilon_1}{d\epsilon_2} \right]_{\epsilon_0} \left[ F(\eta_1) \right]_{\epsilon_0} + \left[ F(\eta_2) \right]_{\epsilon_0} = 0
\]

since \( \eta \) is an extremal of \( I \) subject to the constraint \( G = K \). On the other hand, for any \( \epsilon_2 \), we also have

\[
\left[ G(\epsilon_1(\epsilon_2), \epsilon_2) \right]_{\epsilon_0} = 0.
\]

Therefore,

\[
0 = \frac{d}{d\epsilon_2} \left[ \frac{d\epsilon_1}{d\epsilon_2} \frac{\partial G}{\partial \epsilon_1} + \frac{\partial G}{\partial \epsilon_2} \right]_{\epsilon_0} \\
and so
\[
\left[ \frac{d\epsilon_1}{d\epsilon_2} \right]_{\epsilon_0} = -\left[ \frac{\partial G}{\partial \epsilon_2} \right]_{\epsilon_0}.
\]

Using condition \((4)\), we have

\[
\left| \begin{bmatrix} \frac{\partial T}{\partial \epsilon_1} & \frac{\partial G}{\partial \epsilon_1} \\ \frac{\partial T}{\partial \epsilon_2} & \frac{\partial G}{\partial \epsilon_2} \end{bmatrix} \right| = 0.
\]

Since \( \left[ \frac{\partial G}{\partial \epsilon_1} \right]_{\epsilon_0} \neq 0 \), we conclude that there exists some real \( \lambda \) such that

\[
\left( \left[ \frac{\partial T}{\partial \epsilon_1} \right]_{\epsilon_0}, \left[ \frac{\partial T}{\partial \epsilon_2} \right]_{\epsilon_0} \right) = \lambda \left( \left[ \frac{\partial G}{\partial \epsilon_1} \right]_{\epsilon_0}, \left[ \frac{\partial G}{\partial \epsilon_2} \right]_{\epsilon_0} \right).
\]
5. In conclusion, since 

\[ 0 = \left[ \frac{\partial}{\partial \epsilon} \left( I - \lambda G \right) \right]_{(0,0)}^{\epsilon} \]

\[ = \int_a^b \left[ \eta_2 \frac{\partial f}{\partial y} + \epsilon \eta_2 \frac{\partial f}{\partial \epsilon y} \right] - \lambda \left( \eta_2 \frac{\partial g}{\partial y} + \epsilon \eta_2 \frac{\partial g}{\partial \epsilon y} \right) dx \]

\[ = \int_a^b \left[ \frac{\partial L}{\partial y} - \epsilon \frac{\partial \Delta}{\partial \epsilon y} \right] \eta_2 dx \]

and \( \eta_2 \) is any curve, we obtain

\[ \left[ \frac{\partial L}{\partial y} - \epsilon \frac{\partial \Delta}{\partial \epsilon y} \right] = 0. \]

\[ \square \]

4. An example

Let \( f(x, y, v) = (v - \frac{\Box}{\Box x} |x|)^2 \). With simple calculations, one proves that

\[ \frac{\Box}{\Box x} |x| = \begin{cases} 
1 & \text{if } x \geq \epsilon \\
\frac{x}{\epsilon} - i(\epsilon - x)/\epsilon & \text{if } 0 \leq x < \epsilon \\
\frac{x}{\epsilon} - i(\epsilon + x)/\epsilon & \text{if } -\epsilon < x < 0 \\
-1 & \text{if } x \leq -\epsilon 
\end{cases} \]

Suppose we want to find the extremals for the functional

\[ \int_{-1}^1 f(x, y(x), \Box y(x)) \, dx \] (5)

subject to the integral constraint

\[ \int_{-1}^1 g(x, y(x), \Box y(x)) \, dx = \frac{2}{3}, \]

where \( g(x, y, v) = x + y^2 \), and to the boundary conditions \( y(-1) = 1 = y(1) \).

The (nonsmooth) curve \( y = |x| \) satisfies the constraint integral, and the following conditions:
1. \[ \left[ \frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial |x| y} \right) \right]_{\epsilon} = 0; \]
\[ \frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial |x| y} \right) = -\frac{\partial}{\partial x} \left( 2 \left( \frac{\partial}{\partial x} |x| - \frac{\partial}{\partial x} |x| \right) \right) = 0. \]

2. \[ \left[ \frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial g}{\partial |x| y} \right) \right]_{\epsilon} \neq 0; \]
\[ \frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial g}{\partial |x| y} \right) = 2|x|. \]

3. \[ \lim_{\epsilon \to 0} \max_{x \in [-1,1]} \left| \frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial |x| y} \right) \right| = \lim_{\epsilon \to 0} 0 = 0. \]

4. \[ \lim_{\epsilon \to 0} \max_{x \in [-1,1]} \left| \frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial g}{\partial |x| y} \right) \right| = \lim_{\epsilon \to 0} 2 = 2. \]

Observe that, since \( y = |x| \) is actually an extremal of (5), we may take \( \lambda = 0 \).

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