Obstructions for partitioning into forests

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Abstract

For a class $\mathcal{C}$ of graphs, we define $\mathcal{C}$-edge-brittleness of a graph $G$ as the minimum $\ell$ such that the vertex set of $G$ can be partitioned into sets inducing a subgraph in $\mathcal{C}$ and there are $\ell$ edges having ends in distinct parts. We characterize classes of graphs having bounded $\mathcal{C}$-edge-brittleness for a class $\mathcal{C}$ of forests or a class $\mathcal{C}$ of graphs with no $K_4 \setminus e$ topological minors in terms of forbidden obstructions. We also define $\mathcal{C}$-vertex-brittleness of a graph $G$ as the minimum $\ell$ such that the edge set of $G$ can be partitioned into sets inducing a subgraph in $\mathcal{C}$ and there are $\ell$ vertices incident with edges in distinct parts. We characterize classes of graphs having bounded $\mathcal{C}$-edge-brittleness for a class $\mathcal{C}$ of forests in terms of forbidden obstructions.

1 Introduction

For a graph $G$, a subdivision of $G$ is a graph obtained from $G$ by replacing each edge of $G$ with an internally disjoint path of length at least 1. For graphs $G$ and $H$, we say $H$ is a topological minor of $G$ if $G$ has a subgraph that is a subdivision of $H$. We say $G$ is $H$-free, if no topological minor of $G$ is isomorphic to $H$. For this paper, we call a class $\mathcal{C}$ of graphs hereditary if $\mathcal{C}$ is closed under taking topological minors.

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Let $C$ be a hereditary class of graphs. For a graph $G$, the $C$-edge-brittleness of $G$, denoted by $\eta_C(G)$, is the minimum integer $\ell$ such that there is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that $G[V_i] \in C$ for all $i$ and the number of edges having ends in distinct $V_i$’s is $\ell$. It follows easily that $\eta_C(G)$ is the minimum number of edges of $G$ whose deletion makes each component belong to $C$.

The aim of this paper is to study structures of graphs with large $\eta_C$. As will be shown later (Proposition 1.6), taking topological minors does not increase $C$-edge-brittleness. So it is a natural question to characterize hereditary classes of graphs having bounded $\eta_C$ for various hereditary classes $C$ of graphs.

For this question, we give affirmative answers when $C$ is the class of forests or the class of diamond-free graphs. The acyclic edge-brittleness, denoted by $\eta_a$, is defined as $\eta_C$ for the class $C$ of forests. A diamond, denoted by $D$, is the graph obtained from $K_4$ by removing one edge, see Figure 1. The diamond-free edge-brittleness, denoted by $\eta_d$, is defined as $\eta_C$ for the class $C'$ of diamond-free graphs.

We write $K_n$ and $K_{m,n}$ for the complete graph on $n$ vertices and the complete bipartite graph on $m + n$ vertices partitioned into sets of $m$ and $n$ vertices. We write $P_n$ to denote the path graph on $n$ vertices. For graphs $G$ and $H$, and a positive integer $n$, let $G + H$ be the disjoint union of $G$ and $H$, and $nG$ be the graph obtained by taking disjoint union of $n$ copies of $G$. We denote by $\hat{n}D$ the graph obtained from $nD$ by selecting one degree-2 vertex from each component and identifying all of them into one vertex, see Figure 3.

Our first theorem characterizes hereditary classes of graphs with bounded acyclic edge-brittleness.

**Theorem 1.1.** Let $\mathcal{G}$ be a hereditary class of graphs. Then, $\mathcal{G}$ has bounded $\eta_a$ if and only if $\{K_3, 2K_3, 3K_3, \ldots\} \not\subseteq \mathcal{G}$, $\{K_1 + K_2, K_1 + 2K_2, K_1 + 3K_2, \ldots\} \not\subseteq \mathcal{G}$, and $\{K_{2,1}, K_{2,2}, K_{2,3}, \ldots\} \not\subseteq \mathcal{G}$.
Our second theorem characterizes hereditary classes of graphs with bounded diamond-free edge-brittleness.

**Theorem 1.2.** Let \( \mathcal{G} \) be a hereditary class of graphs. Then, \( \mathcal{G} \) has bounded \( \eta_d \) if and only if \( \{D, 2D, 3D, \ldots\} \not\subseteq \mathcal{G} \), \( \{K_1 + P_3, K_1 + 2P_3, K_1 + 3P_3, \ldots\} \not\subseteq \mathcal{G} \), \( \{K_2, 2K_2, 3K_2, \ldots\} \not\subseteq \mathcal{G} \) and \( \{\hat{D}, 2\hat{D}, 3\hat{D}, \ldots\} \not\subseteq \mathcal{G} \).

As corollaries of these theorems, we obtain the following Ramsey-type results as well.

**Corollary 1.3.** Let \( n \) be a positive integer.

- If a graph has sufficiently large acyclic edge-brittleness, then it contains a topological minor isomorphic to \( nK_3 \), \( K_1 + nK_2 \) or \( K_2, n \).

- If a graph has sufficiently large diamond-free edge-brittleness, then it contains a topological minor isomorphic to \( nD \), \( K_1 + nP_3 \), \( K_2, n \) or \( \hat{nD} \).

For the second part of this paper, we present an analogue of edge-brittleness for the partition of edges. We define the \( C \)-vertex-brittleness of \( G \), denoted by \( \kappa_C(G) \), as the minimum integer \( \ell \) such that there is a partition \( (E_1, E_2, \ldots, E_n) \) of \( E(G) \) such that the subgraph of \( G \) induced by the edges in \( E_i \) belongs to \( C \) for each \( i \) and the number of vertices incident with edges in distinct \( E_i \)'s is \( \ell \).

The acyclic vertex-brittleness, denoted by \( \kappa_a \), is defined as the \( C \)-vertex-brittleness for the class \( C \) of forests. As the third theorem, we characterize all hereditary classes of graphs with bounded acyclic vertex-brittleness as follows.

**Theorem 1.4.** Let \( \mathcal{G} \) be a hereditary class of graphs. Then, \( \mathcal{G} \) has bounded \( \kappa_a \) if and only if \( \{K_3, 2K_3, 3K_3, \ldots\} \not\subseteq \mathcal{G} \) and \( \{K_1 + K_2, K_1 + 2K_2, K_1 + 3K_2, \ldots\} \not\subseteq \mathcal{G} \).

This also gives a Ramsey-type result for acyclic vertex-brittleness.

**Corollary 1.5.** Let \( n \) be a positive integer. If a graph has sufficiently large acyclic vertex-brittleness, then it has a topological minor isomorphic to \( nK_3 \) or \( K_1 + nK_2 \).
We finish this section with the proof that taking topological minors does not increase $\mathcal{C}$-edge-brittleness and $\mathcal{C}$-vertex-brittleness.

**Proposition 1.6.** Let $\mathcal{C}$ be a hereditary class of graphs. If $G'$ is a topological minor of a graph $G$, then

$$\eta_{\mathcal{C}}(G') \leq \eta_{\mathcal{C}}(G) \quad \text{and} \quad \kappa_{\mathcal{C}}(G') \leq \kappa_{\mathcal{C}}(G).$$

**Proof.** It is trivial if $G'$ is a subgraph of $G$. So it is enough to prove this when $G$ is a subdivision of $G'$ obtained by subdividing one edge $e = uv$ of $G'$ into a path $uxv$ of length 2.

Let $(V_1, V_2, \ldots, V_n)$ be a partition of $V(G)$ certifying $\eta_{\mathcal{C}}(G)$. Then $G[V_i] \in \mathcal{C}$. We may assume $x \in V_1$. Let $V'_1 = V_1 \setminus \{x\}$. If $u, v \in V_1$, then $G[V_1]$ is a subdivision of $G'[V_1']$ and so $G'[V_1'] \in \mathcal{C}$ as $\mathcal{C}$ is hereditary. If $u \notin V_1$ or $v \notin V_1$, then $G'[V_1'] = G[V_1] \setminus x \in \mathcal{C}$. It is now straightforward to check in both cases that the partition $(V'_1, V_2, \ldots, V_n)$ of $V(G')$ certifies that $\eta_{\mathcal{C}}(G') \leq \eta_{\mathcal{C}}(G)$.

For the second inequality, let $(E_1, E_2, \ldots, E_n)$ be a partition of $E(G)$ certifying $\kappa_{\mathcal{C}}(G)$. Then the subgraph of $G$ induced by $E_i$ is in $\mathcal{C}$ for each $i$. We may assume that $ux \in E_1$. If $xv \in E_1$, then let $E'_1 = (E_1 \setminus \{ux, xv\}) \cup \{e\}$ and then the subgraph of $G'$ induced by $E'_1$ is still in $\mathcal{C}$ because it is a topological minor of the subgraph of $G$ induced by $E_1$. In this case, it is easy to see that the partition $(E'_1, E_2, \ldots, E_n)$ certifies that $\kappa_{\mathcal{C}}(G') \leq \kappa_{\mathcal{C}}(G)$.

Thus we may assume that $xv \in E_2$. If $u$ is not incident with edges in $E_2$, then let $E'_1 = (E_1 \setminus \{ux\})$, $E'_2 = (E_2 \setminus \{xv\}) \cup \{e\}$. Then clearly the subgraph of $G'$ induced by each of $E'_1$ and $E'_2$ is in $\mathcal{C}$ and therefore we can deduce easily that the partition $(E'_1, E'_2, \ldots, E_n)$ certifies that $\kappa_{\mathcal{C}}(G') \leq \kappa_{\mathcal{C}}(G)$. Thus we may assume that $u$ is incident with an edge in $E_2$ and by symmetry, $v$ is incident with an edge in $E_1$. Then both $u$ and $v$ have ends in distinct $E_i$'s. Then $(E_1 \setminus \{ux\}, E_2 \setminus \{xv\}, E_3, \ldots, E_n, \{e\})$ certifies that $\kappa_{\mathcal{C}}(G') \leq \kappa_{\mathcal{C}}(G)$.

Proposition 1.6 does not necessarily hold if $G'$ is a minor of $G$. For example, let $G$ be the graph given in Figure 4 and let $G' = G/e$ for the edge $e$ shown in the figure. Then $\eta_a(G) = 3$ witnessed by a partition $\{(w_1, w_2, w_3, v_1, v_2), \{v_2\}\}$ but $\eta_a(G/e) = 4$. It is also easy to see that $\kappa_a(G) = 2$ and $\kappa_a(G/e) = 3$.

This paper is organized as follows. Section 2 discusses edge-brittleness and proves Theorems 1.1 and 1.2. Section 3 discusses acyclic vertex-brittleness, proving Theorem 1.4.
2 Bounded edge-brittleness

In this section, we prove Theorem 1.1 and Theorem 1.2. We will use the following theorem of Erdős and Pósa [1].

**Theorem 2.1** (Erdős and Pósa [1]). For a positive integer \(k\), there exists a function \(f_a(k) = O(k \log k)\) such that every graph \(G\) contains \(k\) vertex-disjoint cycles or \(G\) has a vertex set \(X\) of at most \(f_a(k)\) vertices such that \(G \setminus X\) has no cycles.

This has been generalized to minors by Robertson and Seymour [2]. We say a graph \(H\) has the Erdős-Pósa property if there is a function \(f : \mathbb{N} \rightarrow \mathbb{N}\) such that for every graph \(G\) and \(k \in \mathbb{N}\), \(G\) contains either \(k\) disjoint \(H\)-minors or a set \(X\) of vertices with \(|X| \leq f(k)\) such that \(G \setminus X\) has no minor isomorphic to \(H\).

**Theorem 2.2** (Robertson and Seymour [2]). A graph \(H\) has the Erdős-Pósa property if and only if \(H\) is planar.

Since the diamond graph \(D\) is planar and has maximum degree 3, we deduce the following corollary.

**Corollary 2.3.** For a positive integer \(k\), there exists a function \(f_d(k)\) such that every graph \(G\) contains \(k\) vertex-disjoint subgraphs, each of which is isomorphic to a subdivision of the diamond graph \(D\), or has a vertex set \(X\) of at most \(f_d(k)\) vertices such that \(G \setminus X\) has no subdivision of \(D\) as a subgraph.

We will use these functions \(f_a\) and \(f_d\) later in the proofs.

Let \(G\) be a graph and let \(X\) be a set of vertices of \(G\). A vertex \(v\) is a neighbor of \(X\), if \(v\) has a neighbor in \(X\). A star \(S\) is a graph isomorphic to \(K_{1,n}\) for some \(n \geq 1\). If \(n \geq 2\), then the center of \(S\) is defined as the unique vertex of degree at least two. If \(n = 1\), then we fix one vertex of \(S\) as its center, so that every star contains exactly one center. A leaf of a tree is a vertex of degree 1. We remark that a one-edge graph is a star with one
center and two leaves. A subdivided $X$-star is a subgraph $S'$ of $G$ isomorphic to a subdivision of a star $S$ where every leaf belongs to $X$. We denote by $c(S')$ the center of $S$, and by $L(S')$ the set of leaves of $S$.

**Lemma 2.4.** Let $n \geq 2$, $k$ and $m$ be positive integers. Let $T$ be a tree and $X$ be a subset of $V(T)$ with $|X| \geq k + (m - 1)(n - 2)(k - 1) + (m - 1)$. Then, $T$ contains either

- a subdivided $X$-star with at least $n$ leaves, or
- $m$ vertex-disjoint subtrees of $T$ each containing at least $k$ vertices in $X$.

**Proof.** We use induction on $m + |V(T)|$. If $m = 1$, then the statement trivially follows since $T$ itself contains at least $k$ vertices in $X$.

Let $m > 1$. If $k = 1$, then $|X| \geq m$ and each vertex in $X$ forms a subtree of $T$ containing one vertex in $X$, so the second statement trivially holds. Now, we assume that $k \geq 2$. If $T$ contains a leaf $\ell$ which is not in $X$, then we are done by applying the induction hypothesis to $T \setminus \ell$. So we may assume that every leaf of $T$ belongs to $X$. If $T$ contains a vertex $v$ with degree at least $n$, then $T$ contains a subdivided $X$-star with center $v$ and at least $n$ leaves, so we may further assume that $T$ has maximum degree less than $n$.

Let $T'$ be the subgraph of $T$ induced by the set of edges $e = uv$ where each component of $T \setminus e$ contains at least $k$ vertices in $X$. Clearly, $T'$ is a proper subgraph of $T$ because every edge incident with a leaf in $T$ is not contained in $T'$. Furthermore, $T'$ has at least one edge. If not, then every edge $uv$ can be oriented so that $uv$ is oriented towards $v$ if the component of $T \setminus uv$ containing $u$ has less than $k$ vertices in $X$. Since $|E(T)| < |V(T)|$, $T$ has a vertex $w$ that is a sink in this orientation. Now the degree of $w$ is at most $n - 1$ and therefore

\[
|X| \leq (n - 1)(k - 1) + 1
\]

\[
< k + (n - 2)(k - 1) + 1 \leq k + (m - 1)(n - 2)(k - 1) + (m - 1),
\]

contradicting the assumption on $|X|$.

Thus, there exists a leaf $v$ of $T'$. Let $u$ be the neighbor of $v$ in $T'$, and $C$ be the component of $T \setminus uv$ containing $v$. By the definition of $T'$, $C$ contains at least $k$ vertices in $X$.

We know that each component of $C \setminus v$ contains less than $k$ vertices in $X$, since every edge in $C$ incident with $v$ does not belong to $E(T')$. As $T$ has maximum degree at most $n - 1$, it follows that $C$ contains at
most \((n - 2)(k - 1) + 1\) vertices in \(X\). Hence, \(T \setminus V(C)\) contains at least \(k + (m - 2)(n - 2)(k - 1) + (m - 2)\) vertices in \(X\), and by the induction hypothesis, either there is a subdivided \(X\)-star in \(T \setminus V(C)\) with at least \(n\) leaves, or \((m - 1)\) vertex-disjoint subtrees of \(T \setminus V(C)\) each containing at least \(k\) vertices in \(X\). If the second case happens, then since \(C\) is vertex-disjoint from \(T \setminus V(C)\) and contains at least \(k\) vertices in \(X\), we obtain desired \(m\) vertex-disjoint subtrees of \(T\). This completes the proof.

2.1 Bounded acyclic edge-brittleness

First let us see some examples of graphs having unbounded acyclic edge-brittleness.

Lemma 2.5. Let \(n \geq 2\) be an integer.

(i) \(\eta_a(nK_3) = 2n\).

(ii) \(\eta_a(K_1 + nK_2) = 2n\).

(iii) \(\eta_a(K_{2,n}) = n\).

Proof. (i): It is easy to see that \(\eta_a(nK_3) \leq 2n\). For the lower bound, suppose that \((V_1, V_2, \ldots, V_k)\) is a partition of \(V(nK_3)\). Each \(K_3\) has at least two edges having ends in distinct \(V_i\)'s and therefore there are at least \(2n\) edges having ends in distinct \(V_i\)'s. Therefore \(\eta_a(nK_3) \geq 2n\).

(ii): Suppose \((V_1, V_2, \ldots, V_k)\) is a partition of \(V(K_1 + nK_2)\), each part inducing a forest. We may assume that the vertex \(v\) from \(K_1\) is in \(V_1\). Then each triangle containing \(v\) must have exactly two edges having one end in \(V_1\) and another end not in \(V_1\). So, the number of edges of \(K_1 + nK_2\) between \(V_1\) and \(\bigcup_{i=2}^{k} V_i\) is at least \(2n\). Therefore, \(\eta_a(K_1 + nK_2) \geq 2n\). It is easy to see that \(\eta_a(K_1 + nK_2) \leq 2n\).

(iii): Let \((A, B)\) be the bipartition of \(K_{2,n}\) where \(A = \{u, v\}\) and \(|B| = n\). Suppose \((V_1, V_2, \ldots, V_k)\) is a partition of \(V(K_{2,n})\). We may assume that \(u\) is in \(V_1\). We claim that the number of edges of \(K_{2,n}\) between \(V_1\) and \(\bigcup_{i=2}^{k} V_i\) is at least \(n\), which implies \(\eta_a(K_{2,n}) \geq n\). If \(v\) is in \(V_1\), then there is at most one vertex in \(B\) contained in \(V_1\), and there are at least \(2(n - 1) (\geq n)\) edges having one end in \(V_1\) and the other end not in \(V_1\). So, we may assume \(v \notin V_1\), and without loss of generality, let \(v \in V_2\). Then for each vertex \(w\) in \(B\), either \(uw\) or \(vw\) has exactly one end in \(V_1\). So, \(\eta_a(K_{2,n}) \geq n\). It is easy to see that \(\eta_a(K_{2,n}) \leq n\).
Theorem 1.1. Let $G$ be a hereditary class of graphs. Then, $G$ has bounded $\eta_a$ if and only if $\{K_3, 2K_3, 3K_3, \ldots \} \not\subset G$, $\{K_1 + K_2, K_1 + 2K_2, K_1 + 3K_2, \ldots \} \not\subset G$, and $\{K_{2,1}, K_{2,2}, K_{2,3}, \ldots \} \not\subset G$.

Proof. By Lemma 2.5, for $n \geq 2$, $nK_3$, $K_1 + nK_2$, and $K_{2,n}$ have $\eta_a$ at least $n$. Thus, the forward implication holds.

For the backward implication, we claim that for every integer $n \geq 3$, if a graph $G$ has no topological minor isomorphic to $nK_3$, $K_1 + nK_2$, or $K_{2,n}$, then $\eta_a(G)$ is at most $(p + 1)p^2n^3/2$ where $p = f_a(n)$ for the function $f_a$ in Theorem 2.1.

We may assume that no component $C$ of $G$ is a tree because otherwise $\eta_a(G \setminus V(C)) = \eta_a(G)$ and so we can simply delete $C$ from $G$. Since $G$ has no $n$ vertex-disjoint cycles, there exists a set $U$ of vertices of $G$ with $|U| \leq p$ such that $G \setminus U$ has no cycles by Theorem 2.1.

$(1)$ For every component $C$ of $G \setminus U$, each vertex $u$ in $U$ has at most $(n-1)^2+1$ neighbors in $C$.

Suppose not. We prove that $G$ contains $K_{2,n}$ or $K_1 + nK_2$ as a topological minor, which leads a contradiction. Let $X$ be the set of all neighbors of $u$ in $V(C)$. Since $C$ is a tree and $|X| \geq (n-1)^2+2$, we can apply Lemma 2.4 to $C$ and $X$ with $m = n$ and $k = 2$. Then, we obtain either a subdivided $X$-star $S$ with at least $n$ leaves, or $n$ vertex-disjoint subtrees, each containing at least 2 vertices in $X$. In the first case, $V(S) \cup \{u\}$ induces a subgraph of $G$ containing a $K_{2,n}$-topological minor. In the second case, in each subtree, we take a path with both ends in $X$ and no interior vertex in $X$. (It is possible since each subtree contains at least 2 vertices in $X$.) Then, these $n$ paths and $u$ induce a subgraph of $G$ containing a $K_1 + nK_2$-topological minor. This proves $(1)$.

We divide components of $G \setminus U$ into three sets $C_1, C_2, C_3$. For each component $C$ of $G \setminus U$,

- $C \in C_1$ if and only if $V(C)$ has at least two neighbors in $U$,
- $C \in C_2$ if and only if $V(C)$ has exactly one neighbor in $U$, say $u$, and $u$ has at least two neighbors in $V(C)$, and
- $C \in C_3$ if and only if $V(C)$ has exactly one neighbor in $U$, say $u$, and $u$ has exactly one neighbor in $V(C)$.

Since no component of $G$ is a tree, every component of $G \setminus U$ has a neighbor in $U$. So, this clearly partitions the set of components of $G \setminus U$. Let $q_i = |C_i|$ for $i = 1, 2, 3$. 

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(2) $q_1 \leq (n - 1)\binom{p}{2}$.

Suppose $q_1 > (n - 1)\binom{p}{2}$. Since each component in $C_1$ has at least two neighbors in $U$, and $|U| \leq p$, the pigeonhole principle implies that there exist $C_1, C_2, \ldots, C_n \in C_1$ and $u_1, u_2 \in U$ such that both $u_1$ and $u_2$ are neighbors of $V(C_i)$ for $i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$, let $P_i$ be a path of length at least 2 in $G$ with ends $u_1$ and $u_2$ and all interior vertices in $V(C_i)$. By the definitions, we know the existence of such a path $P_i$. Since these $n$ paths are internally disjoint, it follows that the union of them is a subdivision of $K_{2,n}$, which leads a contradiction. So, $q_1 \leq (n - 1)\binom{p}{2}$.

(3) $q_2 \leq (n - 1)p$.

If $q_2 > (n - 1)p$, then the pigeonhole principle implies that there exists a vertex $u$ in $U$ such that there are $n$ components in $C_2$ having at least two neighbors of $u$. That would mean that $G$ has a topological minor isomorphic to $K_1 + nK_2$, leading a contradiction. This prove (3).

For each $u \in U$, let $V_u = \{u\} \cup \left(\bigcup_{C \in C_3} \text{a neighbor of } u \text{ in } V(C)\right)$. By the definition of $C_3$, the sets $V_u$ for $u \in U$ are disjoint. Now we consider the partition $\mathcal{P} = \{V(C) \mid C \in C \cup C_2\} \cup \{V_u \mid u \in U\}$. We claim that this partition gives $\eta_\mathcal{P}(G) \leq (p + 1)p^2n^3/2$.

Let $e$ be an edge of $G$ joining two distinct parts of $\mathcal{P}$. Then, $e$ joins two vertices in $U$, or a vertex in $U$ and some component in $C \cup C_2$. Since $G$ is simple, there are at most $\binom{p}{2}$ edges with both ends in $U$. Furthermore, by (1), (2) and (3), we know that the number of edges between $U$ and $\bigcup_{C \in C_3} V(C)$ is at most $p(n - 1)\binom{p}{2}(\frac{n(n - 1)^2}{2}) + 1$, and the number of edges joining $U$ and $\bigcup_{C \in C_2} V(C)$ is at most $p(n - 1)p((n - 1)^2 + 1)$. Therefore, the number of edges of $G$ joining two distinct parts of $\mathcal{P}$ is at most

$$\binom{p}{2} + p(n - 1)\binom{p}{2}(\frac{n(n - 1)^2}{2}) + p(n - 1)p((n - 1)^2 + 1) \leq \frac{(p + 1)p^2n^3}{2}.$$ 

This completes the proof. □

2.2 Bounded diamond-free edge-brittleness

In this subsection, we prove Theorem 1.2.

**Theorem 1.2.** Let $\mathcal{G}$ be a hereditary class of graphs. Then, $\mathcal{G}$ has bounded $\eta_d$ if and only if $\{D, 2D, 3D, \ldots\} \nsubseteq \mathcal{G}$, $\{K_{1+P_3}, K_{1+2P_3}, K_{1+3P_3}, \ldots\} \nsubseteq \mathcal{G}$, $\{K_{2,1}, K_{2,2}, K_{2,3}, \ldots\} \nsubseteq \mathcal{G}$ and $\{\hat{D}, 2\hat{D}, 3\hat{D}, \ldots\} \nsubseteq \mathcal{G}$. 

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The proof is similar to that of Theorem 1.1.

**Proof.** Similar to Lemma 2.5, we can show that for every positive integer $n$, the graphs $nD$, $K_1 + nP_3$, and $nD$ have diamond-free edge-brittleness at least $2n$, and for every integer $n \geq 4$, $K_{2,n}$ has diamond-free edge-brittleness at least $n$. So, the forward implication follows.

We prove the backward implication. Let $f_d$ be the function defined in Corollary 2.3. We claim that for every integer $n \geq 3$, if a graph $G$ has no topological minor isomorphic to $nD$, $K_1 + nP_3$, $K_{2,n}$ or $\hat{n}D$, then $\eta_d(G)$ is at most $n^3m^2(m + 3)$ where $m = f_d(n)$.

We may assume that every component of $G$ contains a subdivision of $D$. By the definition of $f_d$, there exists $U \subseteq V(G)$ with $|U| \leq m$ such that $G \setminus U$ is $D$-free.

Let $C$ be the set of components of $G \setminus U$.

(1) For each vertex $u \in U$, every component $C \in C$ contains at most $(2n - 3)(n - 1) + 2$ neighbors of $u$.

Suppose $u$ has more than $(2n - 3)(n - 1) + 2$ neighbors in $V(C)$. Let $X$ be the set of neighbors of $u$ in $V(C)$. Let $S$ be a spanning tree of $C$. By Lemma 2.4 with $m = n$ and $k = 3$, there exist a subdivided $X$-star with at least $n$ leaves, or $n$ vertex-disjoint subtrees $S_1, S_2, \ldots, S_n$ of $S$ each containing at least 3 vertices in $X$. In the first case, $G$ contains a subdivision of $K_{2,n}$. In the second case, for $i = 1, 2, \ldots, n$, let $v_{i,1}, v_{i,2}, v_{i,3}$ be three neighbors of $u$ in $V(S_i)$. Since $S_i$ is connected, there exists a vertex $v_i \in V(S_i)$ such that three paths from $v_i$ to $v_{i,1}, v_{i,2}, v_{i,3}$ are mutually edge-disjoint. (We regard a one-vertex graph as a path of length zero.) This implies that the three paths together with $\{uv_{i,1}, uv_{i,2}, uv_{i,3}\}$ form a subdivision of $D$, where $u$ corresponds to a vertex of degree three in the subdivision of $D$. Hence, $G$ contains a $K_1 + nP_3$-topological minor, which is a contradiction. This proves (1).

We partition $C$ into four sets $C_1, C_2, C_3$ and $C_4$ as follows: for each $C \in C$,

- $C \in C_1$ if and only if $V(C)$ has at least two neighbors in $U$,
- $C \in C_2$ if and only if $V(C)$ has exactly one neighbor in $U$, say $u$, and $u$ has at least three neighbors in $V(C)$,
- $C \in C_3$ if and only if $V(C)$ has exactly one neighbor in $U$, say $u$, $u$ has at most two neighbors in $V(C)$ and $\{u\} \cup V(C)$ induces a subgraph of $G$ containing a subdivision of $D$, and
- $C \in C_4$ if and only if $V(C)$ has no neighbors in $U$.
- $C \in \mathcal{C}_4$ otherwise.

Clearly, this partitions $\mathcal{C}$.

(2) $|\mathcal{C}_1| \leq (n-1)\binom{m}{2}$.

Suppose $|\mathcal{C}_1| > (n-1)\binom{m}{2}$. Then, by the pigeonhole principle, there exist $u_1, u_2 \in U$ and $C_1, C_2, \ldots, C_n \in \mathcal{C}_1$ such that both $u_1$ and $u_2$ have neighbors in $V(C_i)$ for $i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, let $P_i$ be a path of $G$ joining $u_1$ and $u_2$ of length at least two where every internal vertex belongs to $V(C_i)$. Then, the union $\bigcup_{i=1,\ldots,n} P_i$ forms a subdivision of $K_{2,n}$, a contradiction. This proves (2).

(3) $|\mathcal{C}_2| \leq (n-1)m$.

Suppose not. Then, there exist $u \in U$ and $C_1, C_2, \ldots, C_n \in \mathcal{C}_2$ such that $u$ has at least three neighbors in $V(C_i)$ for each $i = 1, 2, \ldots, n$. Then, by the same argument of the proof of (1), $G$ contains a subdivision of $K_1 + nP_3$, a contradiction.

(4) $|\mathcal{C}_3| \leq (n-1)m$.

Suppose not. Then, there exist $u \in U$ and $C_1, C_2, \ldots, C_n \in \mathcal{C}_3$ such that for each $i = 1, 2, \ldots, n$, adding $u$ and all edges of $G$ between $u$ and $V(C_i)$ to $C_i$ produces a subgraph $D_i$ that is a subdivision of $D$. Since $C_i$ is $D$-free, $D_i$ must contain $u$, and $u$ must have degree 2 in $D_i$ by the assumption of $\mathcal{C}_3$. This implies that the union of $D_1, D_2, \ldots, D_n$ forms a subdivision of $\hat{nD}$ in $G$, a contradiction.

Note that for each $C \in \mathcal{C}_4$, $V(C)$ has exactly one neighbor in $U$, say $u$, and $u$ has at most two neighbors in $V(C)$, and furthermore, $\{u\} \cup V(C)$ induces a $D$-free subgraph of $G$. For each $u \in U$, let $\mathcal{J}_u$ be the set of components $C$ in $\mathcal{C}_4$ such that $u$ has a neighbor in $V(C)$. Clearly, $\{u\} \cup \bigcup_{C \in \mathcal{J}_u} V(C)$ induces a $D$-free subgraph in $G$. Now we consider the partition $\mathcal{P} = \{\{u\} \cup \bigcup_{C \in \mathcal{J}_u} V(C) \mid u \in U\} \cup \{V(C) \mid C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3\}$. By the definition, each component of $G \setminus U$ is $D$-free, and as we mentioned above, $\{u\} \cup \bigcup_{C \in \mathcal{J}_u} V(C)$ induces a $D$-free subgraph. Hence, each part of $\mathcal{P}$ induces a $D$-free subgraph of $G$.

We claim that the number of edges joining two distinct parts of $\mathcal{P}$ is at most $n^3m^2(m + 3)$. 

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We count edges $e$ of $G$ joining two distinct parts of $\mathcal{P}$. There are two cases to consider:

- $e$ joins two vertices in $U$.
- $e$ joins $U$ and some component in $C_1 \cup C_2 \cup C_3$.

In the first case, there are at most $\binom{m}{2}$ edges of $G$ joining two vertices in $U$. In the second case, by (2), (3) and (4), there are at most $(n - 1)\binom{m}{2} + (n - 1)m + (n - 1)m = \frac{(n-1)m(m+3)}{2}$ components in $C_1 \cup C_2 \cup C_3$. So by (1), there are at most $(2n - 3)(n - 1) + 2 \cdot m \cdot \frac{(n-1)m(m+3)}{2} \leq n^2(n - 1)m^2(m + 3)$ edges of $G$ joining $U$ and some component in $C_1 \cup C_2 \cup C_3$. Therefore, the number of edges of $G$ joining two distinct parts of $\mathcal{P}$ is at most

$$\binom{m}{2} + n^2(n - 1)m^2(m + 3) \leq n^3m^2(m + 3),$$

which implies that $\eta_d(G) \leq n^3m^2(m + 3)$. This proves Theorem 1.2. □

3 Bounded acyclic vertex-brittleness

We will now discuss acyclic vertex-brittleness. First let us state a useful lemma.

**Lemma 3.1.** Let $T$ be a tree and $X$ be a subset of $V(T)$ with $|X| \geq 2$. Then, there are vertex-disjoint subdivided $X$-stars $S_1, S_2, \ldots, S_m$ in $T$ such that every vertex in $X$ is a center or a leaf of $S_i$ for some $i$.

**Proof.** We proceed by induction on $|X| + |E(T)|$. If $|X| = 2$ or $|X| = 3$, then there is a subdivided $X$-star $S$ with $X \subseteq \{c(S)\} \cup L(S)$, so we are done.

Suppose $|X| > 3$. If $T$ has a leaf not in $X$, say $\ell$, then we can apply the induction hypothesis to $T \setminus \ell$. So, we may assume that every leaf of $T$ belongs to $X$. We may further assume that for every edge $e \in E(T)$, there is a component of $T \setminus e$ containing at most one vertex in $X$, since otherwise, we can apply the induction hypothesis to each component. So, $T$ is not a path because $|X| \geq 4$. This means that there exists a vertex of $T$ with degree at least three. Indeed, such a vertex uniquely exists, since otherwise, each of the two components of $T \setminus e$, where $e$ is in a path of $T$ connecting two vertices of degree at least three, contains at least two vertices in $X$, a contradiction. Let $c$ be the vertex of $T$ with degree at least three. Let $u$ be a neighbor of $c$. Because the degree of $c$ is at least three, the component of $T \setminus cu$ containing $c$ has at least two vertices in $X$. Hence, no vertex of degree
two belongs to $X$. This completes the proof because $T$ itself is a subdivided $X$-star such that $X \subseteq \{c(T)\} \cup L(T)$.

By properly extending $S_1, S_2, \ldots, S_m$ in Lemma 3.1, we can cover all edges of $T$ by edge-disjoint subtrees $T_1, T_2, \ldots, T_m$ where $T_i$ contains $S_i$ for $i = 1, 2, \ldots, m$. Then, we obtain the following lemma.

**Lemma 3.2.** Let $T$ be a tree, $X$ be a subset of $V(T)$ with $|X| \geq 2$. Then, there are edge-disjoint subtrees $T_1, T_2, \ldots, T_m$ covering all edges of $T$, a partition $(X_1, X_2, \ldots, X_m)$ of $X$, and vertex-disjoint subdivided $X$-stars $S_1, S_2, \ldots, S_m$ such that for $i = 1, 2, \ldots, m$,

- $X_i \subseteq V(S_i) \subseteq V(T_i)$ and $|X_i| \geq 2$, and
- every vertex in $X_i$ is a center or a leaf of $S_i$.

Before proving Theorem 1.4, we first consider a simple case that $V(G)$ can be partitioned into $U$ and $V$ where $U$ is independent and $V$ induces a tree.

**Lemma 3.3.** Let $G$ be a graph with $V(G) = U \cup V$ where $U$ is an independent set, and $G[V]$ is a tree. Suppose $|U| = m$ and $G$ is $K_1 + nK_2$-free. Then, $\kappa_a(G) \leq 2m(n - 1)$.

**Proof.** We may assume that $G$ has no isolated vertices. We may also assume that no vertex in $U$ has degree one, since otherwise, we move such a vertex to $V$. So, every vertex in $U$ has at least two neighbors in $V$. If $U = \emptyset$, then $G$ is a tree, and $\kappa_a(G) = 0 \leq 2m(n - 1)$. So, we assume $U \neq \emptyset$.

Let $T = G[V]$. For $u \in U$, let $X_u$ be the set of neighbors of $u$ in $V$. Since $|X_u| \geq 2$, Lemma 3.2 implies that there exist edge-disjoint subtrees $T_{u,1}, T_{u,2}, \ldots, T_{u,k_u}$ of $T$ covering all edges of $T$, a partition $(X_{u,1}, X_{u,2}, \ldots, X_{u,k_u})$ of $X_u$ and vertex-disjoint subdivided $X_u$-stars $S_{u,1}, \ldots, S_{u,k_u}$ in $T$ satisfying the conditions in Lemma 3.2.

For $i = 1, 2, \ldots, k_u$, since $|X_{u,i}| \geq 2$, it follows that $V(S_{u,i}) \cup \{u\}$ induces a subgraph of $G$ containing a cycle. This means that $\{u\} \cup \bigcup_{i=1,2,\ldots, k_u} V(S_{u,i})$ induces a subgraph of $G$ containing a $K_1 + k_uK_2$-topological minor because $S_{u,1}, \ldots, S_{u,k_u}$ are vertex-disjoint. By the assumption that $G$ is $K_1 + nK_2$-free, we know that $k_u < n$. Let $I_u$ be the subset of $V(T)$ consisting of

- all centers of the subdivided $X_u$-stars $S_{u,1}, \ldots, S_{u,k_u}$, and
- all vertices of $T$ contained in at least two subtrees of $T_{u,1}, T_{u,2}, \ldots, T_{u,k_u}$.
Since \( k_u \leq n - 1 \), we have \(|I_u| \leq k_u + (k_u - 1) \leq 2n - 3 \).

Let \( I = \bigcup_{u \in U} I_u \). We remark that \(|I| \leq m(2n - 3) \). Let \( T_1, T_2, \ldots, T_s \) be the collection of all maximal subtrees of \( T \) having no internal vertex in \( I \).

We claim that for each \( u \in U \), it has at most one neighbor in \( V(T_i) \setminus I \).

Suppose \( u \) is adjacent to two vertices \( v_1, v_2 \in V(T_i) \setminus I \). Let \( P \) be the path between \( v_1 \) and \( v_2 \). Since \( v_1, v_2 \notin I \) and every interior vertex of \( P \) is an interior vertex of \( T_i \), every vertex of \( P \) is not contained in \( I \), in particular, not contained in \( I_u \). This means that \( P \) is a path of some subtree \( T_{u,j} \), \( v_1, v_2 \in X_{u,j} \) and \( T_{u,j} \) contains a subdivided \( X_{u,j} \)-star with center \( c_{u,j} \). Because \( v_1, v_2 \in X_{u,j} \), \( c_{u,j} \) is contained in \( P \), that is, \( P \) contains a vertex in \( I \), which leads a contradiction. Hence, \( u \) has at most one neighbor in \( V(T_i) \setminus I \).

For \( i = 1, 2, \ldots, s \), let \( E_i \) be the union of \( E(T_i) \) and the edges between \( U \) and \( V(T_i) \setminus I \). By the above claim, we know that \( E_i \) induces a tree in \( G \). For each \( v \in I \), let \( E_v \) be the set of edges joining \( v \) and \( U \). We consider the partition \( \mathcal{P} = \{E_j \mid j = 1, 2, \ldots, s\} \cup \{E_v \mid v \in I\} \). Clearly, this partitions \( E(G) \), and each part induces a tree. If \( v \in V(G) \) is incident with edges in distinct parts of \( \mathcal{P} \), then \( v \) is contained in \( U \) or \( I \). Because \(|U| + |I| \leq m + m(2n - 3) = 2m(n - 1) \), it follows that \( \kappa_a(G) \leq 2m(n - 1) \). This completes the proof.

Now we prove Theorem 1.4.

**Theorem 1.4.** Let \( \mathcal{G} \) be a hereditary class of graphs. Then, \( \mathcal{G} \) has bounded \( \kappa_a \) if and only if \( \{K_3, 2K_3, 3K_3, \ldots\} \not\subseteq \mathcal{G} \) and \( \{K_1 + K_2, K_1 + 2K_2, K_1 + 3K_2, \ldots\} \not\subseteq \mathcal{G} \).

**Proof.** Similar to Lemma 2.5, we can show that \( \kappa_a(nK_3) = 2n \) and \( \kappa_a(K_1 + nK_2) = n + 1 \). So the forward implication follows.

For the backward implication, we claim that for every positive integer \( n \), every graph \( G \) with no topological minor isomorphic to \( nK_3 \) or \( K_1 + nK_2 \) has \( \kappa_a \) at most \( 2m^2n^2 \) where \( m = f_a(n) \) for the function \( f_a \) in Theorem 2.1.

Since \( G \) has no topological minor isomorphic to \( nK_3 \), \( G \) has no \( n \) vertex-disjoint cycles and therefore by Theorem 2.1 there exists a set \( U \) of vertices of \( G \) with \(|U| \leq m \) such that \( G \setminus U \) is a forest.

We may assume that no component \( C \) of \( G \) is a tree because otherwise \( \kappa_a(G \setminus V(C)) = \kappa_a(G) \) and so we can simply delete \( C \) from \( G \).

Let \( C_1, C_2, \ldots, C_p \) be the components of \( G \setminus U \) each having at least two neighbors of some vertex in \( U \), and \( D_1, D_2, \ldots, D_q \) be the other components of \( G \setminus U \). By the pigeonhole principle, if \( p > m(n - 1) \), then \( G \) contains a subdivision of \( K_1 + nK_2 \), a contradiction. So, \( p \leq m(n - 1) \).
For $1 \leq i \leq p$, let $G_i$ be the subgraph of $G$ induced by the set of edges with at least one end in $V(C_i)$. Since $V(G_i) \setminus V(C_i)$ is an independent set of size at most $m$ in $G_i$, Lemma 3.3 implies that there is a partition $P_i$ of $E(G_i)$, each part inducing a tree, which gives $\kappa_a(G_i) \leq 2m(n-1)$. For $i = 1, 2, \ldots, q$, let $E_i$ be the set of all edges with at least one end in $V(D_i)$.

We consider the partition $P = \bigcup_{i=1,2,\ldots,p} P_i \cup \{E_1, E_2, \ldots, E_q\} \cup \{\{e\} | e \in E(G[U])\}$. Obviously, each part of $P$ induces a tree. We claim that $P$ gives $\kappa_a(G) \leq 2m^2n^2$. Note that every vertex in $V(D_i)$ is incident with only edges in $E_i$. For $i = 1, 2, \ldots, q$, if a vertex in $V(C_i)$ meets at least two parts of $P$, then it meets at least two parts of $P_i$. So by the construction of $P_i$, there are at most $2m(n-1)$ such vertices in $V(C_i)$. Therefore, the number of vertices meeting at least two parts of $P$ is at most $|U| + 2m(n-1)p \leq m + 2m^2(n-1)^2 \leq 2m^2n^2$. This completes the proof.

References

[1] P. Erdős and L. Pósa. On independent circuits contained in a graph. Canad. J. Math., 17:347–352, 1965.

[2] N. Robertson and P. Seymour. Graph minors. V. Excluding a planar graph. J. Combin. Theory Ser. B, 41(1):92–114, 1986.