A study on resistance matrix of graphs

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Abstract

In this article we consider resistance matrix of a connected graph. For un-
weighted graph we study some necessary and sufficient conditions for resistance
regular graphs. Also we find some relationship between Laplace matrix and resis-
tance matrix in case of weighted graphs where all edge weights are positive definite
matrices of given order.

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rose inverse, (1)-inverse.

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1 Introduction

Throughout the article all our graphs are finite, undirected, connected and simple. By
order of a graph we mean the number of vertices in the graph. If $i, j \in V(G)$ in a graph
$G$ then we write $i \sim j$ or $ij \in E(G)$ to mean that $i$ is adjacent to $j$ in $G$. If $i$ is a vertex of
a graph $G$ then by $N(i)$ we denote the set of all vertices in $G$ adjacent to $i$ and we call it
the neighbourhood of $i$. The degree of a vertex $i \in V(G)$ is the cardinality of $N(i)$ and is
denoted by $d_i$. A graph $G$ is said to be regular or degree regular if all the vertex degrees
are equal. In that case if all the vertex degrees are equal to $d$ then we call the graph as
d-$regular$ graph. If $V_1 \subseteq V(G)$ and $E_1 \subseteq E(G)$, then by $G - V_1$ and $G - E_1$ we mean
the graphs obtained from $G$ by deleting the vertices in $V_1$ and the edges $E_1$ respectively.
In particular case when $V_1 = \{u\}$ or $E_1 = \{e\}$, we simply write $G - V_1$ by $G - u$ and
$G - E_1$ by $G - e$ respectively. Tree is a connected graph having no cycle and a $k$-forest
is a union of $k$ disjoint trees.

By $M_{m,n}$ we denote the class of all matrices of size $m \times n$. Also by $M_n$ we denote the
class of all square matrices of order $n$. For $M \in M_n$ we write $m_{ij}$ or $M_{ij}$ to denote the
$ij-i^{th}$ element of $M$. We set $[n] = \{1, 2, \ldots, n\}$. For $S \subset [n]$, we write $M(S)$ to denote the
submatrix of $M$ obtained by removing rows and columns corresponding to $S$. By eigenpair
$(\lambda, x)$ of a square matrix $M$ we mean that $\lambda$ is an eigenvalue of $M$ with corresponding
eigenvector $x$. By an eigenvector we always mean the normalized eigenvector and when we
talk about all the eigenvectors of a matrix we consider them as normalized and mutually orthogonal. By \( \mathbb{J} \) and \( \mathbb{I} \) we mean the matrix of all one’s and vector of all one’s respectively of suitable order. We will mention its order wherever its necessary. Adjoint of a matrix \( M \in \mathbb{M}_n \) will be denoted by \( \text{adj}(M) \).

**Definition 1.1** The adjacency matrix of a graph \( G \) denoted by \( A = A(G) \) is defined as

\[
A_{ij} = \begin{cases} 
1 & \text{if } (i, j) \in E \\
0 & \text{otherwise.}
\end{cases}
\]

**Definition 1.2** The degree matrix of a graph \( G \) denoted by \( D = D(G) \) is a diagonal matrix such that \( D_{ii} = d_i \).

**Definition 1.3** The Laplacian matrix of a graph \( G \) denoted by \( L = L(G) \) is defined as \( L = D - A \).

**Definition 1.4** Moore-Penrose inverse of \( M \in \mathbb{M}_{m,n} \) is defined as a matrix \( M^+ \in \mathbb{M}_{n,m} \) satisfying the following four conditions

(i) \( MM^+M = M \),  
(ii) \( M^+MM^+ = M^+ \),  
(iii) \( (MM^+)^* = MM^+ \) and  
(iv) \( (M^+M)^* = M^+M \).

For \( A \in \mathbb{M}_{m,n} \) if the matrix \( C \) satisfies (i) of definition (1.4), then we say \( C \) to be a \((1)\) - inverse of \( A \) and we write it by \( C = A^{(1)} \).

**Definition 1.5** The resistance distance denoted by \( r_{ij} \) between vertices \( v_i \) and \( v_j \) of a graph is defined as the effective resistance between the two vertices computed with Ohm’s law, when the graph is viewed as an electrical network with each edge carrying unit resistance and attaching a battery at vertex \( v_i \) and \( v_j \).

**Definition 1.6** If \( S \) be a set then a function \( \rho : S \times S \rightarrow \mathbb{R} \) is said to be distance function if \( \rho \) satisfies the following conditions for any \( x, y, z \in S \)

(i) \( \rho(x, y) \geq 0 \)  
(ii) \( \rho(x, y) = 0 \iff x = y \)  
(iii) \( \rho(x, y) = \rho(y, x) \)  
(iv) \( \rho(x, y) + \rho(y, z) \geq \rho(x, z) \).
The resistance matrix of a connected graph $G$ of order $n$ is defined to be $R(G) = (r_{ij})$, where $r_{ij}$ is the resistance distance between the vertices $i$ and $j$ in $G$. We call a connected graph $G$ to be resistance regular if all the row sums of $R(G)$ are equal.

We now outline the contents of this article. In section 2 we establish some necessary and sufficient conditions for a connected graph to be resistance regular. Also we discuss some properties of resistance regular graph in that section. In section 3 we study weighted connected graphs where all the weights are positive definite matrices of same order. We consider Laplacian matrix and resistance matrix and establish some relationships between them for matrix weighted connected graphs. Also we show that a matrix weighted tree is completely determined from its resistance matrix.

## 2 Resistance regular graphs

**Lemma 2.1** [5] Let $M$ be a square matrix of order $n$ and $(\lambda_k, c_k); k = 1, \ldots, n$ are its eigenpairs, then

$$M^+_ij = \sum_{k=1}^{n} g(\lambda_k)c_{ik}c_{jk},$$

where

$$g(\lambda_k) = \begin{cases} \frac{1}{\lambda_k} & \text{if } \lambda_k \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.2** [5] For any connected graph $G$

$$r_{ij} = \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k} (c_{ki} - c_{kj})^2$$

where $(\lambda_k, c_k)$ is an eigenpair of $L(G)$.

**Definition 2.3** The Kirchhoff index $Kf(G)$ of a connected graph $G$ is defined by

$$Kf(G) = \frac{1}{2} \sum_{i,j \in V(G)} r_{ij}.$$

**Lemma 2.4** [11] Let $G$ be a connected graph of order $n$ with $\lambda_k$ as eigenvalue of $L(G)$ for $k = 1, \ldots, n$. Then

$$Kf(G) = n \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k}.$$

**Theorem 2.5** For a connected graph $G$ if $R_i$ denote the $i$–th row sum of $R(G)$ and $L^+_ii$ denote the $i$–th diagonal entry of $L^+(G)$, then $R_i = \frac{Kf(G)}{n} + nL^+_ii$ for each $i \in V(G)$.

**Proof.** If $(\lambda_k, c_k)$ is an eigenpair of $L(G)$, then we have from Lemma \((2.2)\)

$$r_{ij} = \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k} (c_{ki} - c_{kj})^2.$$
Therefore we get

\[
R_i = \sum_{j \in V(G)} \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k} (c_{ki} - c_{kj})^2
\]

\[
= \sum_{\lambda_k \neq 0} \left( \sum_{j \in V(G)} \frac{1}{\lambda_k} (c_{ki} - c_{kj})^2 \right)
\]

\[
= \sum_{\lambda_k \neq 0} \left[ \frac{1}{\lambda_k} \left( nc_{ki}^2 + \sum_{j \in V(G)} c_{kj}^2 - 2c_{ki} \sum_{j \in V(G)} c_{kj} \right) \right]
\]

\[
= \sum_{\lambda_k \neq 0} \left[ \frac{1}{\lambda_k} \left( nc_{ki}^2 + 1 - 2c_{ki} \times 0 \right) \right]
\]

\[
= \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k} + n \sum_{k \in V(G)} \frac{c_{ki}^2}{\lambda_k}
\]

\[
= \frac{Kf(G)}{n} + nL^+_{ii}, \quad \text{using Lemmas (2.4) and (2.1)}.
\]

As a Corollary to the above Theorem (2.5) we get the following result.

**Corollary 2.6** [10] A graph \( G \) of order \( n \) is resistance regular if and only if \( L^+_{ii} = \ldots = L^+_{nn} \).

**Lemma 2.7** (Matrix-Tree Theorem) [2] Let \( G \) be a graph and \( W \subset V(G) \), then \( \det(L(W)) \) equals the number of spanning forests of \( G \) with \( |W| \) components in which each component contains one vertex of \( W \).

**Lemma 2.8** [2] For a connected graph \( G \) with \( L = L(G) \),

\[
r_{ij} = L^+_{ii} + L^+_{jj} - 2L^+_{ij} = \frac{\det(L(i,j))}{\det(i)}.
\]

**Theorem 2.9** For any connected graph \( G \) of order \( n \) if \( L^+_{ii} \) denote the \( i - \)th diagonal entry of \( L(G) \) and \( t \) denote the number of spanning trees in \( G \) having two components with \( 'i' \) in one the components is \( t \left( \frac{Kf(G)}{n} + nL^+_{ii} \right) \).

**Proof.** If \( R_i \) denote the \( i - \)th row sum of \( R(G) \) then from Lemma (2.8) we have

\[
R_i = \sum_{j \in V(G)} r_{ij} = \frac{1}{t} \sum_{j \in V(G)} L(i,j).
\]

Now by Matrix-Tree Theorem \( \det(L(i,j)) \) represents the number of spanning forests in \( G \) with two components with \( 'i' \) in one of them and \( 'j' \) on the other. Hence on using Theorem (2.5), we are done.

**Theorem 2.10** If \( t \) be the number of spanning trees in a connected graph \( G \) then the number of spanning forests in \( G \) with two components is equal to \( t(Kf(G)) \).
Proof. By Matrix-Tree Theorem $\det L(i,j)$ represents the number of spanning forests in $G$ with two components with 'i' in one of them and 'j' on the other. Therefore the required result follows from the facts that $r_{ij} = \frac{\det L(i,j)}{t}$ and $Kf(G) = \frac{1}{2} \sum_{i \in V(G)} \sum_{j \in V(G)} r_{ij}$. □

**Lemma 2.11** [2] In a connected graph $G$ resistance distance is a distance function on the set $V(G)$.

**Lemma 2.12** [3] Let $G$ be a connected graph with a cut vertex $k$. If vertices $i$ and $j$ lie in different component of $G - k$ then $r_{ij} = r_{ik} + r_{kj}$.

**Theorem 2.13** Resistance regular graphs cannot have cut vertices.

Proof. If possible suppose $G$ is a resistance regular graph with a cut vertex $v$. Let $H_1$ is a component of $G - v$ with minimum number of vertices and $H_2 = G - H_1 - v$. We consider $(v \sim) u \in H_1$.

Since $v$ is a cut vertex we have from Lemmas (2.12) and (2.11)

$$r_{ux} = r_{vx} + r_{uv} \quad \forall x \in H_2 \quad (2.1)$$

$$r_{xu} + r_{uv} \geq r_{xv} \quad \forall x \in H_1 \quad (2.2)$$

Now

$$R_u = \sum_{x \in G} r_{xu}$$

$$= r_{uv} + \sum_{x \in H_1 \cup H_2} r_{xu}$$

$$\geq r_{uv} + \sum_{x \in H_1} (r_{xv} - r_{uv}) + \sum_{x \in H_2} (r_{xv} + r_{uv}) \quad \text{using (2.1) and (2.2)}$$

$$= r_{uv} + \sum_{x \in H_1 \cup H_2} r_{xu} + r_{uv} \left(|V(H_2)| - |V(H_1)|\right)$$

$$\geq r_{uv} + R_v \quad \text{as} \quad |V(H_2)| \geq |V(H_1)|$$

$$> R_v, \text{ a contradiction.}$$

Hence the result holds. □

**Theorem 2.14** If $R = (r_{ij})$ be the resistance matrix of a connected graph $G$ with $R_i$ as the $i$-th row sum of $R$, then $L_{ij}^+(G) = \frac{R_i + R_j}{2n} - \frac{Kf(G)}{n^2} - \frac{r_{ij}}{2}$.

Proof. From Theorem (2.5) we have

$$L_{ii}^+ = \frac{R_i}{n} - \frac{Kf(G)}{n^2}$$

Now

$$r_{ij} = L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+$$

$$= \frac{R_i + R_j}{n} - \frac{2Kf(G)}{n^2} - 2L_{ij}^+$$

$$\Rightarrow L_{ij}^+ = \frac{R_i + R_j}{2n} - \frac{Kf(G)}{n^2} - \frac{r_{ij}}{2}.$$
Corollary 2.15 If \( G \) be a resistance regular graph, then \( L^+_{ij}(G) = \frac{Kf(G)}{n^2} - \frac{r_{ij}}{2} \).

**Proof.** In Theorem (2.14), putting \( R_i = R_j = \frac{2Kf(G)}{n} \) we get

\[
L^+_{ij} = \frac{4Kf(G)}{n \times 2n} - \frac{Kf(G)}{n^2} - \frac{r_{ij}}{2}
\]

\[
= \frac{Kf(G)}{n^2} - \frac{r_{ij}}{2}
\]

In Theorem (2.14) taking summation over the neighbourhood of any given vertex \( i \in V(G) \) of a graph \( G \) and using \( \sum_{i \in V(G)} R_i = 2Kf(G) \) we get the following Corollary.

**Corollary 2.16** If \( R_i \) be the \( i - \text{th} \) row sum of \( R(G) \) for a connected graph \( G \), then

\[
\sum_{j \in N(i)} L^+_{ij} = \frac{R_i}{n} \left[ \frac{R_i}{2} - \frac{Kf(G)}{n} \right] + \frac{1}{2} \left[ \frac{1}{n} \sum_{j \in N(i)} R_j - \sum_{j \in N(i)} r_{ij} \right]
\]

where \( N(i) \) is the set of all vertices in \( G \) adjacent to \( i \).

**Lemma 2.17** [2] If \( G \) be a graph of order \( n \) and the eigenvalues of \( L(G) \) be \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} > \lambda_n = 0 \), then for any \( a \in \mathbb{R} \) the eigenvalues of \( L + aJ \) are \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1} \) and \( na \).

**Lemma 2.18** [2] If \( G \) be a graph of order \( n \) and the eigenvalues of \( L(G) \) be \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} > \lambda_n = 0 \), then the number of spanning trees of \( G \) equals \( \frac{\lambda_1 \lambda_2 \ldots \lambda_{n-1}}{n} \).

**Lemma 2.19** [2] If \( G \) is a connected graph with \( L = L(G) \) then

\[
L^+ = \left( L + \frac{J}{n} \right)^{-1} - \frac{J}{n}
\]

**Corollary 2.20** If \( R_i \) be the \( i - \text{th} \) row sum of \( R(G) \) for a connected graph \( G \), then the \( ij - \text{th} \) cofactor of \( (L + \frac{J}{n}) \) equals

\[
t \left[ 1 + \frac{R_i + R_j}{2} - \frac{Kf(G)}{n} - \frac{n r_{ij}}{2} \right],
\]

where \( t \) is the number of spanning trees in \( G \).

**Proof.** From Lemma (2.19) we have

\[
(L + \frac{J}{n})^{-1} = L^+ + \frac{J}{n}
\]

(2.3)

Again from Lemmas (2.17) and (2.18), we get

\[
det \left( L + \frac{J}{n} \right) = tn
\]

(2.4)
where $t$ is the number of spanning trees of $G$.

Now using (2.4) in (2.3) we get

$$\frac{\text{adj} (L + \frac{3}{n})}{tn} = L_{ij}^+ + \frac{1}{n}$$  \hspace{1cm} (2.5)

Applying Theorem (2.14) in (2.5),

$$\text{det} \left(L + \frac{3}{n}\right)(i, j) = (-1)^{i+j}t \left[1 + \frac{R_i + R_j}{2} - \frac{Kf(G)}{n} - \frac{n r_{ij}}{2}\right].$$

Hence the result follows. \hfill \blacksquare

**Theorem 2.21** If $G$ is $k$ resistance regular graph of order $n$, then $L_{ij}^+(G) = \frac{k}{n^2}$ and therefore from Corollary (2.15), we get

$$L_{ii}^+ = \frac{Kf(G)}{n^2} = \frac{k}{2n} \quad \forall \, i \in [n]$$  \hspace{1cm} (2.6)

Now by Lemma (2.7),

$$s_{ij} = \text{det} L(i, j).$$  \hspace{1cm} (2.7)

Also from Lemmas (2.8) and (2.7) for any connected graph we have $r_{ij} = \frac{\text{det} L(i, j)}{t}$ and from Corollary (2.6) for resistance regular graph $r_{ij} = 2 \left(L_{ii}^+ - L_{ij}^+\right)$. Comparing these we get

$$L_{ij}^+ = L_{ii}^+ - \frac{\text{det} L(i, j)}{2t} = \frac{k}{2n} - \frac{s_{ij}}{2t}, \quad \text{using (2.6) and (2.7).}$$

Hence the result follows. \hfill \blacksquare

**Definition 2.22** A connected graph $G$ is said to be equiarboreal if the number of spanning trees containing a given edge in $G$ is independent of the choice of edge.

**Lemma 2.23** A connected graph $G$ is equiarboreal if and only if $r_{ij} = r_{uv}$ for any $ij, uv \in E(G)$.

**Lemma 2.24** A connected graph $G$ of order $n$ with $R = R(G)$ is resistance regular if and only if

$$\sum_{j \sim i} r_{ij} = 2 - \frac{2}{n} \quad \forall \, i \in V(G).$$

**Theorem 2.25** An equiarboreal graph is resistance regular if and only if it is degree regular.
Proof. If $G$ be an equiarboreal graph, then let $k = r_{ij}$ for $ij \in E(G)$.

\[ \therefore \sum_{j=1}^{\infty} r_{ij} = d_i k \]

Thus if $R_i$ denote the $i$-th row sum of $R(G)$, then using Lemma (2.24) we get

\[ R_i = R_l \]

\[ \iff \sum_{j=1}^{\infty} r_{ij} = \sum_{j=1}^{\infty} r_{lj} \]

\[ \iff d_i = d_l. \]

Hence the result. \hfill \Box

Lemma 2.26 [6] For $M \in M_n$, the sum of the principal minors of $M$ of order $s \in [n]$ equals the sum of the products of the eigenvalues of $M$ taken $s$ at a time.

Lemma 2.27 [2] Let $G$ be a connected graph of order $n$ and $L = L(G)$. If $M$ is any proper principal submatrix of $L$, then $M^{-1}$ exist and is entrywise nonnegative matrix.

Lemma 2.28 [5] If $G$ be a connected graph of order $n$ and $L = L(G)$ then

\[ L^+ J = 0 = J L^+, \quad LL^+ = I - \frac{J}{n} = L^+ L. \]

Definition 2.29 By bottleneck matrix at vertex $v$ of a graph $G$ with $L = L(G)$ we mean the matrix $(L(v))^{-1}$ i.e. the inverse of the principal submatrix of $L$ obtained by removing row and column corresponding to $v$.

Theorem 2.30 For a connected graph $G$ of order $n$ if $L = L(G)$ and $(\lambda_k, c_k)$ are eigenpairs of $L$ then the following are all equivalent.

(i) $G$ is resistance regular.

(ii) $L^+_i(G) = \frac{Kf(G)}{n^2}$ for all $i \in V(G)$.

(iii) $r_{ij} + 2L^+_i(G)$ is constant for all $i, j \in V(G)$.

(iv) $R = 2(L^+_i - L^+)$.  

(v) $\|B_i\|_1$ is constant for all $i \in V(G)$, where $B_i$ is the bottleneck matrix of $G$ at vertex $i$ and $\|M\|_1$ is the sum of all entries of $M$.

(vi) $\sum_{\lambda_k \neq 0} c_k^2 \lambda_k$ is constant for all $i \in V(G)$, where $\lambda_k$ is an eigenvalue of $L(G)$ with corresponding eigenvector $(c_{1k}, \ldots, c_{nk})^T$.

(vii) All principal minors of $L + \frac{J}{n}$ of order $n-1$ are equal to $t(1 + \frac{Kf(G)}{n})$, where $t$ is the number of spanning trees in $G$.  

(viii) \[ \sum_{j \in N(i)} L^+_{ij}(G) = \frac{d_i Kf(G)}{n^2} + \frac{1}{n} - 1 \] for all \( i \in V(G) \).

**Proof.** \((i) \Leftrightarrow (ii)\) From Corollary (2.6) we have \((ii) \Rightarrow (i)\). Again from the same Corollary (2.6), \((i)\) implies

\[ L^+_{11} = \ldots = L^+_{nn} = l \text{ (say)}. \]

Then

\[ n l = \sum_{i \in V(G)} L^+_{ii} = \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k} \text{ as eigenvalues of } L^+ \text{ are } 0 \text{ and } \frac{1}{\lambda_k}, \text{ for } \lambda_k \neq 0 \]

\[ = \frac{Kf(G)}{n} \text{ using Lemma (2.4)} \]

\[ \Rightarrow l = \frac{Kf(G)}{n^2} \]

Thus \((i) \Rightarrow (ii)\).

\((i) \Leftrightarrow (iii)\) First \((iii)\) implies

\[ r_{ij} + 2L^+_{ij}(G) = c \text{ (say)} \quad \forall i, j \in V(G) \]

\[ \Rightarrow \sum_{j \in V(G)} r_{ij} + 2 \sum_{j \in V(G)} L^+_{ij} = \sum_{j \in V(G)} c \]

\[ \Rightarrow R_i + 0 = cn \quad \text{as } L^+ \mathbb{J} = 0 \text{ by Lemma (2.28)} \]

\[ \Rightarrow (i) \]

Again using Corollary (2.6), \((i)\) implies

\[ L^+_{11} = \ldots = L^+_{nn} = l \text{ (say)}. \]

\[ \therefore r_{ij} = L^+_{ii} + L^+_{jj} - 2L^+_{ij} \]

\[ = 2(l - L^+_{ij}) \]

\[ \Rightarrow r_{ij} + 2L^+_{ij} = 2l, \text{ a constant } \forall i, j \in V(G). \]

\((iii) \Leftrightarrow (iv)\) is obvious from the proof of \((i) \Leftrightarrow (iii)\).

\((iv) \Leftrightarrow (v)\) From [4] we have

\[ L^+ = (L + \mathbb{J})^{-1} - \frac{1}{n^2} \mathbb{J} \quad (2.8) \]

Again for a non singular matrix \(M\), we have from [2]

\[ \det (M + \mathbb{J}) = \det M(1 + \mathbb{I}^T M^{-1} \mathbb{I}) \quad (2.9) \]

Therefore using (2.8) in Corollary (2.6) we get that \(G\) is resistance regular if and only if all principal minors of order \(n - 1\) of \(L + \mathbb{J}\) are equal.

\[ \Leftrightarrow (L + \mathbb{J})(i, i) \text{ is constant } \forall i \in V(G) \]

\[ \Leftrightarrow t(1 + ||B_i||_l_i) \text{ is constant } \forall i \in V(G) \quad \text{by (2.9) and Lemma (2.27)} \]

\[ \Leftrightarrow ||B_i||_l_i \text{ is constant } \forall i \in V(G). \]
(i) ⇔ (vi) Immediately follows from Corollary (2.6) and Lemma (2.1).

(i) ⇔ (vii) If \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} > \lambda_n = 0 \) are the eigenvalues of \( L \) then from Lemma (2.17) we see that eigenvalues of \( L + \frac{J}{n} \) are \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, 1 \). Again from Lemma (2.19) we have

\[
L^+ = (L + \frac{J}{n})^{-1} - \frac{J}{n} \tag{2.10}
\]

Now using (2.10) in Corollary (2.6) we observe that a graph is resistance regular if and only if all principal minors of \( L + \frac{J}{n} \) of order \( n - 1 \) are equal (\( k, \) say). Then using Lemma (2.20) we get

\[
k n = \sum_{i=1}^{n-1} \frac{\prod_{i=1}^{n-1} \lambda_i}{\lambda_i} + t n
\]

\[
= \sum_{i=1}^{n-1} \frac{t n}{\lambda_i} + t n \quad \text{by Lemma (2.18)}
\]

\[
\Rightarrow k = t \left(1 + \sum_{i=1}^{n-1} \frac{1}{\lambda_i}\right)
\]

\[
= t \left(1 + \frac{Kf(G)}{n}\right).
\]

(ii) ⇔ (viii) From Lemma (2.28) we have for any graph

\[
LL^+ = I - \frac{J}{n}
\]

\[
\Rightarrow d_i L^+_i - \sum_{j \in N(i)} L^+_{ij} = 1 - \frac{1}{n}
\]

Therefore the result holds.

From theorem (2.30) we immediately get the following results.

**Corollary 2.31** If \( G \) is a resistance regular graph, then

\[
L^+_i > L^+_j \quad \forall i, j, k \in V(G) \text{ with } j \neq k
\]

**Corollary 2.32** A \( d \)-regular graph \( G \) of order \( n \) is resistance regular if and only if

\[
\sum_{j \in N(i)} L^+_{ij} = \frac{d}{n^2} Kf(G) + \frac{1}{n} - 1 \quad \forall i \in V(G)
\]

where \( N(i) \) is the set of vertices adjacent to \( i \).

**Corollary 2.33** A resistance regular graph \( G \) is regular if and only if \( \sum_{j \in N(i)} L^+_{ij} \) is constant for all \( i \in V(G) \) where \( N(i) \) is the set of vertices adjacent to \( i \).
Theorem 2.34 Let $u$ and $v$ be any two vertices of a graph $G$ of order $n$ and suppose that

$$L(G) + \frac{J}{n} = \begin{pmatrix}
    d_u + 1 & k & x^T \\
    k & d_u + 1 & y^T \\
    x & y & P
\end{pmatrix}$$

If $P$ is invertible then $G$ is resistance regular if and only if $d_v - d_u = y^T P^{-1} y - x^T P^{-1} x$.

Proof. From Lemma (3.11) and Corollary (2.6) we have

$G$ is resistance regular $\iff$ all the principal minors of $L + \frac{J}{n}$ are equal.

$\iff (d_v + 1) \det P - y^T (\text{adj} P) y = (d_u + 1) \det P - x^T (\text{adj} P) x$

$\iff (d_v - d_u) \det P = y^T (\text{adj} P) - x^T (\text{adj} P) x$

$\iff d_v - d_u = y^T P^{-1} y - x^T P^{-1} x$.

\[\blacksquare\]

3 Resistance matrix for matrix weighted graph

In this section we consider connected graphs with every edge associated with positive definite matrices of given order. We call such a graph to be matrix weighted graph. If $G$ be a connected graph of order $n$ and $W_{ij}$ is the weight (positive definite matrix of order $k$) of the edge connecting vertices $i$ and $j$ in $G$, then the Laplacian matrix of $G$ is defined as the $n k \times n k$ matrix (block matrix) whose $ij$ – th block is given by

$$L_{ij} = \begin{cases}
    \sum_{j \in V(G)} W_{ij}^{-1} & \text{if } i = j \\
    -W_{ij}^{-1} & \text{if } i \neq j \text{ and } i \sim j \\
    0 & \text{if } i \neq j \text{ and } i \not\sim j
\end{cases}$$

Also if $L^{(1)}$ is any $(1)$ – inverse of $L(G)$ then the $ij$ – th block of the resistance matrix $R(G)$ of the graph $G$ is defined as $R_{ij} = L_{ii}^{(1)} + L_{jj}^{(1)} - L_{ij}^{(1)} - L_{ji}^{(1)}$.

If $M$ and $N$ are two block matrices of same order and same block partition then by $M \otimes N$ we mean the block matrix whose $ij$ – th block is given by the matrix product $M_{ij} N_{ij}$, where $M_{ij}$ and $N_{ij}$ are $ij$ – th block of the respective matrices. Again for any two matrices $M$ and $N$ by $M \otimes N$ we mean the block matrix whose $ij$ – th block is $m_{ij} N$, where $m_{ij}$ is the $ij$ – th entry of the matrix $M$. If $M$ is a block matrix with all blocks of same order then by $||M||$ we mean the sum of all blocks of $M$, i.e. $||M|| = \sum M_{ij}$, where the summation runs over all blocks of $M$. By $M_B^T$ we mean the block transpose of $M$, i.e. the $ij$ – th block of $M_B^T$ is the $ji$ – th block of $M$. Also we write $\text{trace}_B(M)$ to denote the sum of all diagonal blocks of $M$.

Let $X = \left( L + \frac{1}{n} \mathbb{J}_n \otimes I_k \right)^{-1}$ and $\tilde{X}$ be a block matrix with

$$\tilde{X}_{ij} = \begin{cases}
    X_{ii} & \text{if } i = j \\
    0, & \text{otherwise}
\end{cases}$$
Then it can be easily verified that $R_{ij} = X_{ii} + X_{jj} - X_{ij} - X_{ji}$ and $R = \bar{X}(J_n \otimes I_k) + (I_k \otimes J_n)\bar{X} - X - X_B^T$.

**Theorem 3.1** *For matrix weighted connected graph G of order n with $L = L(G)$*

$$L^+ = X - \frac{1}{n}J_n \otimes I_k$$

where each weight is a positive definite matrix of order $k$.

**Proof.** Since $X = \left( L + \frac{1}{n}J_n \otimes I_k \right)^{-1}$ we get

$$X \left( L + \frac{1}{n}J_n \otimes I_k \right) = I_{nk} = \left( L + \frac{1}{n}J_n \otimes I_k \right)X$$

$$\Rightarrow LX = I_{nk} - \frac{1}{n}(J_n \otimes I_k)X \text{ and } XL = I_{nk} - \frac{1}{n}X(J_n \otimes I_k).$$

Now

$$L\left( X - \frac{1}{n}J_n \otimes I_k \right)L = \left( LX - \frac{1}{n}L(J_n \otimes I_k) \right)L$$

$$= \left( I_{nk} - \frac{1}{n}(J_n \otimes I_k)X - 0 \right)L$$

$$= L - \frac{1}{n}(J_n \otimes I_k)XL$$

$$= L - \frac{1}{n}(J_n \otimes I_k)\left(I_{nk} - \frac{1}{n}(J_n \otimes I_k)\right)$$

$$= L - \frac{1}{n}J_n \otimes I_k + \frac{1}{n^2}(J_n \otimes I_k)^2$$

$$= L - \frac{1}{n}J_n \otimes I_k + \frac{1}{n^2}n(J_n \otimes I_k)$$

$$= L$$

Similarly the other three conditions of Moore-Penrose inverse can also be easily established.

**Theorem 3.2** *For any connected matrix weighted graph G with $L = L(G)$*

$$LL^+ = I_{nk} - \frac{1}{n}J_n \otimes I_k = L^+ L = L X$$

where each weight is a positive definite matrix of order $k$.

**Proof.** From definition of the matrix $X$, and Theorem (3.1) we have

$$LL^+ = L\left( X - \frac{1}{n}J_n \otimes I_k \right)$$

$$= LX - \frac{1}{n}L(J_n \otimes I_k)$$

$$= LX - 0$$

$$= I_{nk} - \frac{1}{n}(J_n \otimes I_k)X$$

$$= I_{nk} - \frac{1}{n}J_n \otimes I_k \text{ as block column sums of } X \text{ are all equal to } I_k.$$
Definition 3.3 The Kirchhoff index $Kf(G)$ of a connected matrix weighted graph $G$ is defined by $Kf(G) = \frac{1}{2} \sum_{i \in V(G)} \sum_{j \in V(G)} R_{ij}$, where $R_{ij}$ is the $ij$-th block of $R(G)$.

Theorem 3.4 For matrix weighted connected graph $G$ of order $n$ with $L = L(G)$

$$Kf(G) = n \sum_{i \in V(G)} L_{ii}^+$$

Proof. By definition we have

$$2Kf(G) = \sum_{i,j \in V(G)} R_{ij}$$

$$= \sum_{i,j \in V(G)} (L_{ii}^+ + L_{jj}^+ - L_{ij}^+ - L_{ji}^+)$$

$$= 2(n-1) \sum_{i \in V(G)} L_{ii}^+ - 2 \sum_{i \neq j} L_{ij}^+$$

$$= 2(n-1) \sum_{i \in V(G)} L_{ii}^+ + 2 \sum_{i \in V(G)} L_{ii}^+$$

$$= 2n \sum_{i \in V(G)} L_{ii}^+$$

Hence the result follows. \qed

Theorem 3.5 For matrix weighted connected graph $G$ of order $n$ with $L = L(G)$ and $R = R(G)$

$$||L \otimes R|| = -2(n-1)I_k$$

where each weight is a positive definite matrix of order $k$.

Proof. We have

$$\left( L + \frac{1}{n} J_n \otimes I_k \right) X = I_{nk}$$

$$\Rightarrow \sum_{j=1}^{n} L_{ij} X_{ji} + \frac{1}{n} \sum_{j=1}^{n} j = 1^n X_{ji} = I_k \quad \forall i \in V(G)$$

$$\Rightarrow \sum_{j=1}^{n} L_{ij} X_{ji} + \frac{1}{n} I_k = I_k \quad \forall i \in V(G)$$

$$\Rightarrow \sum_{j=1}^{n} L_{ij} X_{ji} = \left( 1 - \frac{1}{n} \right) I_k \quad \forall i \in V(G)$$

$$\therefore \quad ||L \otimes X|| = n \left( 1 - \frac{1}{n} \right) I_k \quad (3.11)$$
Now

\[ R = \tilde{X}(J_n \otimes I_k) + (J \otimes I_k) \tilde{X} - X - X_B^T \]

\[ \Rightarrow LR = \tilde{X}(J_n \otimes I_k) - (I_n \otimes I_k - \frac{1}{n} J_n \otimes I - k) - LX_B^T \] using Theorem (3.2) and \( L(J \otimes I_k) = 0 \)

Again

\[ ||L \otimes R|| = \sum_{i,j \in V(G)} L_{ij} R_{ij} \]

\[ = \text{trace}_B(LR) \]

\[ = ||\tilde{L} \tilde{X}|| + 0 - \text{trace}_B(I_n \otimes I_k - \frac{1}{n} J_n \otimes I_k) - ||L \otimes X|| \]

\[ = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} L_{ij} \right) X_{jj} - n \left( 1 - \frac{1}{n} \right) I_k - n \left( 1 - \frac{1}{n} \right) I_k \] using (3.11)

\[ = 0 - 2(n - 1)I_k \]

Hence the result.

For matrix weighted connected graph \( G \) of order \( n \) with \( L = L(G) \) and \( R = R(G) \) let us define

\[ \tau_i = 2I_k + \sum_{j \in V(G)} L_{ij} R_{ij} \text{ and } \tau = (\tau_1, \tau_2, \ldots, \tau_n)^T. \]

**Theorem 3.6** If \( G \) is a matrix weighted connected graph of order \( n \) with \( L = L(G) \) and \( R = R(G) \), then

\[ (I_n^T \otimes I_k) \tau = 2I_k, \] where each weight matrix is of order \( k \) and \( \tau \) is as defined above.

**Proof.** We have

\[ \tau_i = 2I_k + \sum_{j \in V(G)} L_{ij} R_{ij} \]

\[ \Rightarrow \sum_{i=1}^{n} \tau_i = 2nI_k + ||L \otimes R|| \]

\[ \Rightarrow (I_n^T \otimes I_k) \tau = 2nI_k - 2(n - 1)I_k \] using Theorem (3.5)

\[ = 2I_k. \]

**Lemma 3.7** [1] If \( T \) be a matrix weighted tree on \( n \) vertices such that each weight matrix is of order \( k \), then rank of \( L(T) \) is \( (n - 1)k \).

Similar to ( [10] Theorem (9) ) we get the following for matrix weighted graph.

**Theorem 3.8** If \( G \) be a matrix weighted tree of order \( n \), then \( \begin{pmatrix} (L(u))^{-1} & 0 \\ 0 & 0 \end{pmatrix} \) is a \((1)\) inverse of \( L \), where \( u \) is the vertex corresponding to the last row(column) of \( L \).
Proof. Let \( L = \begin{pmatrix} L(u)^{-1} & x \\ x^T & L_{uu} \end{pmatrix} \), where \( L_{uu} = L_{nn} \). By Schur complement formula we have

\[
\text{rank}(L) = \text{rank}(L(u)) + \text{rank}(L_{uu} - x^T(L(u))^{-1}x).
\]

But from Lemma (3.7) we see that \( \text{rank}(L) = (n-1)k = \text{rank}(L(u)) \) and therefore

\[
L_{uu} = x^T L(u)^{-1}x.
\]  \hfill (3.12)

Now

\[
L \begin{pmatrix} L(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{(n-1)k} & 0 \\ x^T L(u)^{-1} & 0 \end{pmatrix}
\]

\[
\Rightarrow L \begin{pmatrix} L(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} L = \begin{pmatrix} L(u) & x \\ x^T & x^T L(u)^{-1}x \end{pmatrix}
\]

\[
= L. \quad \text{using (3.12)}
\]

Hence the result. \hfill \blacksquare

**Theorem 3.9** A matrix weighted tree is completely determined from its resistance matrix.

**Proof.** If \( G \) is a matrix weighted tree then from Theorem (3.8) we have \( \begin{pmatrix} L(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \) as a \((1)^{-}\text{inverse}\) of \( L(G) \).

Now if \( R \) is known then from the relation \( R_{ij} = L_{ii}^{(1)} + L_{jj}^{(1)} - L_{ij}^{(1)} - L_{ji}^{(1)}, L(u)^{-1} \) is known. Thus \( L(u) \) is determined by \( R \).

Again as \( L_{nk} = 0 = \mathbb{J}_{nk}L \), the structure of \( G \) and edge weights of \( G \) are completely known from \( R \) upto isomorphism. \hfill \blacksquare

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