The enhancement of the decay $\Upsilon(1D) \rightarrow \eta \Upsilon(1S)$ by the axial anomaly in QCD

M.B. Voloshin
Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455
and
Institute of Theoretical and Experimental Physics, Moscow, 117259

Abstract

It is shown that the rates of the decays $\Upsilon(1^3D_1) \rightarrow \eta \Upsilon(1S)$ and $\Upsilon(1^3D_2) \rightarrow \eta \Upsilon(1S)$ should be comparable to and likely exceed that of the recently discussed in the literature two-pion transition $\Upsilon(1D) \rightarrow \pi \pi \Upsilon(1S)$. The reason for this behavior is that the discussed $\eta$ transitions are enhanced by the contribution of the anomaly in the flavor singlet axial current in QCD.
1 Introduction

The $D$-wave states in the family of the $b\bar{b}$ resonances present a new interesting testing ground for the study of heavy quark dynamics. The $^3D_J$ states with $J = 1, 2, 3$ should form a closely spaced triplet of resonances, of which one (most likely the $^3D_2$) with the mass of about 10.16 GeV has been recently observed in the CLEO experiment through the radiative transitions to and from the $D$ wave state. It is clear however that similarly to other excited $b\bar{b}$ resonances there should also be strong-interaction transitions from the $D$ states to lower resonances with emission of light mesons, i.e. of two pions, $\eta$, and also a weaker isospin violating transition with emission of $\pi^0$. In particular the transitions of the type $\Upsilon(1D) \rightarrow \pi\pi \Upsilon(1S)$ were discussed in the literature in some detail. The interest to the two-pion transitions is explained by that these are well known to be dominant for the more familiar cases of the hadronic transitions between $^3S_1$ states, i.e. from $\psi'$ in charmonium and from $\Upsilon(2S)$, while the rate of similar decays with the $\eta$ emission being considerably smaller, and the related isospin violating transition with emission of a single $\pi^0$ being greatly suppressed further. The amplitudes of these transitions between the $^3S_1$ states of heavy quarkonium and the pattern of the rates were understood long ago within the general method of describing the hadronic transitions in heavy quarkonium using the multipole expansion in QCD for the interaction of the heavy quarkonium with soft gluonic field. In the transitions between the $^3S_1$ heavy resonances the relevant amplitudes for production of the light mesons are determined by the low energy theorems arising from the quantum anomalies in QCD: the emission of an $S$ wave pair of pions is dominated by the anomaly in the trace of the energy-momentum tensor, while the $P$ wave emission of the $\eta$ is regulated by the anomaly in the flavor singlet axial current. The anomalous contribution greatly enhances both rates in agreement with the available data.

The purpose of the present paper is to point out that the relation between the rates of the two-pion and $\eta$ transitions from $\Upsilon(1^3D_{1,2})$ to $\Upsilon(1S)$ should be quite different from the pattern observed in the transitions between the $^3S_1$ states. Namely, in the $1D \rightarrow 1S$ transitions the pions are emitted in the $D$ wave, and the corresponding amplitude for the production of the pions by the relevant gluonic operator decouples from the anomaly. On the other hand, the $P$ wave emission of $\eta$ is still possible for transitions to the $^3S_1$ state from the $1^3D_J$ resonances with $J = 1$ and $J = 2$, and, as will be shown here, is indeed contributed by the axial anomaly in QCD. The resulting enhancement of the amplitude of the $\eta$ emission turns out to be sufficient to compensate for the suppression factors inherent in this decay (the
flavor SU(3) violation, as well as a suppression by the inverse of the $b$ quark mass), so that the rate of the $\eta$ transitions should be comparable to that of the two-pion ones, and in fact is quite likely to be the largest among the hadronic transitions from the $3D_J$ states\(^1\). A more definite quantitative estimate of the ratio of the rates, $\Gamma(1^3D_{1,2} \to \eta 1^3S_1)/\Gamma(1D \to \pi \pi 1S)$, is hindered by the present poor understanding of a parameter governing the non-anomalous amplitude of production of the pion pair by gluonic operators, as will be discussed in Sect.4.

Within the QCD multipole expansion treatment of the hadronic transitions in a heavy quarkonium, outlined below in Sect.2, the evaluation of the amplitude of the $\eta$ emission requires knowledge of the matrix element $\langle \eta | G_{\mu\nu}D_{\rho}G_{\lambda\sigma}|0 \rangle$, where $G_{\mu\nu}^a$ is the gluonic field tensor, and $D_{\rho}$ is the covariant derivative. It will be shown in Sect.3 that this matrix element is completely determined by the well known expression\(^{11}\)

$$\langle \eta | G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a |0 \rangle = 8\pi^2 \sqrt{\frac{2}{3}} f_\eta m_\eta^2 ,$$  

(1)

following from the anomaly in the divergence of the flavor singlet axial current in QCD. In eq.\(^1\) $f_\eta$ is the $\eta$ ‘decay constant’, equal to the pion decay constant $f_\pi \approx 130$ MeV in the limit of exact flavor SU(3) symmetry, and $f_\eta$ is likely to be larger due to effects of the SU(3) violation. Also throughout this paper the normalization of the gluon field tensor includes the QCD coupling $g$ (so that e.g. the Lagrangian for the gluon field reads as $L_g = -G^2/(4g^2)$).

The amplitudes and probabilities of specific decays are calculated in Sect.4. Besides a numerical comparison of the rates for the two-pion and $\eta$ transitions between the $3D_J$ and the $3S_1$ resonances in the $b\bar{b}$ system, also discussed there are the greatly suppressed transitions with emission of a $\pi^0$.

Finally, the Section 5 contains a summary of the discussion in the present paper.

## 2 Transition amplitudes in the multipole expansion

We start with a brief reminder of the leading terms in the multipole expansion in QCD which are relevant to the discussed transitions\(^8\)\(^9\)\(^\dagger\).

\(^\dagger\)This behavior is reminiscent of that expected\(^12\) for transitions between $^1P_1$ and $^3S_1$ states. There the two-pion emission, also decoupled from the anomaly, is additionally kinematically suppressed, so that the isospin violating single $\pi^0$ emission becomes more probable due to the contribution of the axial anomaly (while the $\eta$ emission is impossible kinematically).
The two-pion transition arises in the second order in the $E_1$ interaction with the chromoelectric gluon field $\vec{E}^a$ described by the Hamiltonian

$$H_{E_1} = -\frac{1}{2}\xi^a \vec{r} \cdot \vec{E}^a(0),$$

where $\xi^a = t_1^a - t_2^a$ is the difference of the color generators acting on the quark and antiquark (e.g. $t_1^a = \lambda^a/2$ with $\lambda^a$ being the Gell-Mann matrices), and $\vec{r}$ is the vector for relative position of the quark and the antiquark.

The transitions of the type $^3D_J \to \eta^3S_1$ are induced by the interference of the $E_1$ interaction in eq.(2) with the $M_2$ term containing the chromomagnetic field $\vec{B}^a$ and described by the Hamiltonian

$$H_{M_2} = -(4m_Q)^{-1}\xi^a S^i r_i \left(D_j B_k(0)\right)^a,$$

where $D$ is the QCD covariant derivative, $m_Q$ is the heavy quark mass, and $\vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2)/2$ is the operator of the total spin of the quark-antiquark pair. It should be noted that the $M_1$ term, formally of a lower order in the multipole expansion, is proportional to the spin-flip operator $(\vec{\sigma}_1 - \vec{\sigma}_2)$ and thus does not contribute to transitions between states with the same total spin.

Using the expressions (2) and (3) the transition amplitudes are found in the standard way:

$$A_{\pi\pi} \equiv A(^3D_J \to \pi^3S_1) = \langle \pi\pi|E_i^a E_j^a|0\rangle A_{ij},$$

$$A_{\eta}^{(J)} \equiv A(^3D_J \to \eta^3S_1) = m_Q^{-1} \langle \eta|E_i^a (D_j B_k)^a + (D_j B_k)^a E_i^a|0\rangle A_{ijk}^{(J)},$$

where $A_{ij}$ and $A_{ijk}^{(J)}$ are the heavy quarkonium amplitudes, defined as

$$A_{ij} = \frac{1}{32} \langle 1S|\xi^a r_i \mathcal{G} r_j \xi^a|1D\rangle$$

and

$$A_{ijk}^{(J)} = \frac{1}{64} \langle ^3S_1|\xi^a r_i \mathcal{G} r_j \xi^a S_k| ^3D_J\rangle$$

with $\mathcal{G}$ being the Green’s function of the heavy quark pair in a color octet state, and also in these expressions a use is made of the fact that in matrix elements between color singlet states one can replace $\xi^a \ldots \xi^b$ by $(\delta^{ab}/8)\xi^c \ldots \xi^e$.

The (chromo)electric dipole interaction in eq.(2) does not involve the spin of the heavy quarks. Thus in the leading nonrelativistic limit, assumed throughout this paper, where the spin and coordinate degrees of freedom can be considered as independent, the amplitude $A_{ij}$
in eq. (4) does not depend on the spin variables of the initial $D$ wave state or of the final $S$ wave one. For this reason the rates of transitions from each of the $^3D_J$ states to $\pi\pi\ ^3S_1$ (and also of $^1D_2 \to \pi\pi\ ^1S_0$) are all the same [13, 2] (modulo small differences in the energy release, whose effect is formally beyond the assumed approximation) and can in fact be calculated for spinless quarks.

The amplitude $A_{ijk}^{(J)}$ does depend on the spin-orbital state of the quark pair and is different for different values of the total momentum $J$. For the purpose of the present discussion in the leading approximation of the decoupled spin variables it is convenient to first represent this amplitude not in the basis of states with definite $J$, but rather in a form with explicitly factorized spin and orbital components in the Cartesian coordinates. For this representation we denote as $\zeta_i$ and $\chi_i$ the spin polarization amplitude of respectively the initial $^3D$ state and the final $^3S_1$ state, and as $\psi_{ij}$ the orbital polarization amplitude of the $L = 2$ wave in the initial $D$ states. The tensor $\psi_{ij}$ is symmetric and traceless, as appropriate for an $L = 2$ state, and all these amplitudes are assumed to be normalized in the standard way, so that the sums over the polarization states are defined as

$$
\sum_{\text{pol}} \chi_i^* \chi_j = \sum_{\text{pol}} \zeta_i^* \zeta_j = \delta_{ij}, \quad \sum_{\text{pol}} \psi_{ij}^* \psi_{kl} = \frac{1}{2} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right). \quad (8)
$$

In this notation the amplitude $A_{ij}$ (for spinless quarks) is proportional to $\psi_{ij}$ and can thus be written in terms of a scalar quantity $A_2$ as $A_{ij} = \psi_{ij} A_2$, while the amplitude $A_{ijk}^{(J)}$ is expressed in terms of the same $A_2$ as

$$
A_{ijk}^{(J)} = \frac{i}{2} \epsilon_{klm} \chi_i^* P^{(J)} \psi_{ij} \zeta_m A_2, \quad (9)
$$

where $P^{(J)}$ is the projector on states with definite $J$, acting on the product of the spin and orbital polarization amplitudes $\psi_{ij} \zeta_m$.

The quantity $A_2$ depends on details of dynamics of heavy quarkonium, and at present is highly model-dependent. For this reason a prediction of the absolute rates of the discussed decays involves a considerable uncertainty. Clearly, however, $A_2$ cancels in the considered here ratio of the rates of the two-pion and $\eta$ transitions, which is thus determined by the ratio of the matrix elements entering the equations (4) and (5), describing the production by the gluon operators of the corresponding light meson states.
3 Matrix elements for production of light mesons by gluonic operators

The gluonic matrix element for the two-pion production in eq.(11) multiplies the traceless tensor $\psi_{ij}$ and thus receives no contribution from the (enhanced) trace anomaly in QCD. Rather this matrix element is parameterized in terms of the QCD coupling $\alpha_s$ and the parameter $\rho_G$ introduced in Ref.[6] as ‘the fraction of the pion momentum carried by gluons’. Using this parameterization, one can write

$$A_{\pi^+\pi^-} = \langle \pi^+\pi^-|E_i^a E_j^a|0\rangle \psi_{ij} (\chi^*_k \zeta_k) A_2 = 4\pi \alpha_s \rho_G p^+_i p^-_j \psi_{ij} (\chi^*_k \zeta_k) A_2 ,$$

where the final state with charged pions is assumed for definiteness, and $p^\pm$ stand for the momenta of the pions in the heavy quarkonium rest frame.

The matrix element of the gluonic operators in eq.(11) can be found as soon as the amplitudes of general Lorentz structure $\langle \eta| G_{\mu\nu}^a (D_\rho G_{\lambda\sigma})^a_0|0\rangle$ and $\langle \eta| (D_\rho G_{\mu\nu})^a G_{\lambda\sigma}^a|0\rangle$ are known. These structures can in fact be reduced to (a total derivative of) the amplitude described by eq.(11). The possibility of the reduction of the structures with derivatives to the expression in eq.(11) is determined by the general theory[14]. Here we present the specific implementation of this reduction, which uses the following simple algebraic identity valid for an arbitrary four-vector $p$:

$$p_\rho \epsilon_{\mu\nu\lambda\sigma} = p_\lambda \epsilon_{\mu\nu\rho\sigma} - p_\rho \epsilon_{\mu\nu\lambda\sigma} - p_\mu \epsilon_{\nu\rho\lambda\sigma} + p_\nu \epsilon_{\mu\rho\lambda\sigma} ,$$

where the convention $\epsilon_{0123} = 1$ is assumed. The antisymmetry of the field tensor $G_{\mu\nu}$ then allows one to write the general form of the first of the discussed matrix elements in the linear order in the $\eta$ momentum $p$ in terms of two scalars $X$ and $Y$:

$$i \langle \eta(p)|G_{\mu\nu}^a (D_\rho G_{\lambda\sigma})^a_0|0\rangle = X p_\rho \epsilon_{\mu\nu\lambda\sigma} + Y (p_\lambda \epsilon_{\mu\nu\rho\sigma} - p_\rho \epsilon_{\mu\nu\lambda\sigma} - p_\mu \epsilon_{\nu\rho\lambda\sigma} + p_\nu \epsilon_{\mu\rho\lambda\sigma}) .$$

The third structure, allowed by the symmetry and proportional to $(p_\mu \epsilon_{\nu\rho\lambda\sigma} - p_\nu \epsilon_{\mu\rho\lambda\sigma})$, is reduced to the first two due to the identity (11). Furthermore, applying in eq.(12) the equations of motion (the Jacobi identity): $D_\rho G_{\lambda\sigma} + D_\sigma G_{\rho\lambda} + D_\lambda G_{\sigma\rho} = 0$, one arrives at the relation $X = 2Y$.

Likewise, writing the second of the discussed matrix elements in terms of two scalars $\tilde{X}$ and $\tilde{Y}$ as

$$i \langle \eta(p)|(D_\rho G_{\mu\nu})^a G_{\lambda\sigma}^a_0|0\rangle = \tilde{X} p_\rho \epsilon_{\mu\nu\lambda\sigma} + \tilde{Y} (p_\mu \epsilon_{\lambda\sigma\rho\nu} - p_\nu \epsilon_{\lambda\sigma\rho\mu}) ,$$

where the convention $\tilde{\epsilon}_{0123} = 1$ is assumed. The antisymmetry of the field tensor $G_{\mu\nu}$ then allows one to write the general form of the second of the discussed matrix elements in the linear order in the $\eta$ momentum $p$ in terms of two scalars $\tilde{X}$ and $\tilde{Y}$ as

$$i \langle \eta(p)|(D_\rho G_{\mu\nu})^a G_{\lambda\sigma}^a_0|0\rangle = \tilde{X} p_\rho \epsilon_{\mu\nu\lambda\sigma} + \tilde{Y} (p_\mu \epsilon_{\lambda\sigma\rho\nu} - p_\nu \epsilon_{\lambda\sigma\rho\mu}) .$$
and applying the Jacobi identity, one finds the relation $\tilde{X} = 2\tilde{Y}$.

The sum of the expressions (12) and (13) should combine into a total derivative, i.e. the sum should be proportional to $p_{\rho}$. This is possible due to the identity (11) under the condition that $\tilde{Y} = Y$, so that all the considered scalar form factors are expressed in terms of one of them, e.g. in terms of $X$: \(^2\)

$$\tilde{X} = X, \quad \tilde{Y} = Y = \frac{1}{2}X . \quad (14)$$

Using this relation and contracting the sum of the expressions (12) and (13) with $\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}$ the form factor $X$ is identified from the equation (1) as

$$X = -\frac{1}{30} \langle \eta|G^a\tilde{G}^a|0\rangle = -\frac{4\pi^2}{15} \sqrt{\frac{2}{3}} f_\eta m_\eta^2 . \quad (15)$$

The relations (12) - (14) fully define the gluonic matrix element in eq.(5) in terms of $X$:

$$i \langle \eta|E_i^a (D_jB_k)^a + (D_jB_k)^a E_i^a|0\rangle = -2X(3p_j \delta_{ik} - p_i \delta_{jk}) . \quad (16)$$

One can now use this expression and the form of the amplitude $A_{ijk}^{(J)}$ from eq.(9) to write the full amplitude of the $\eta$ transition as

$$A_\eta^{(J)} = -\frac{2}{m_Q} X \sum_i \epsilon_{jlm} \chi_i^* P^{(J)} \psi_{ij} \zeta_m A_2 , \quad (17)$$

where the symmetry of the tensor amplitude $\psi_{ij}$ is taken into account.

### 4 Relations between the decay rates

It is quite clear from the equation (17) that due to the presence of $\epsilon_{jlm}$ the amplitude of the $\eta$ emission vanishes for the $J = 3$ state, which is totally symmetric in the indices $i, j, l$, when expressed in terms of the product $\psi_{ij}\zeta_m$. This previously mentioned behavior is naturally expected, given that the $\eta$ meson is emitted in the $P$ wave. The projection of the latter product on the state with $J = 1$ is given by

$$P^{(1)} \psi_{ij} \zeta_m = \frac{3}{10} (\delta_{im} \psi_{nj} \zeta_n + \delta_{jm} \psi_{ni} \zeta_n - \frac{2}{3} \delta_{ij} \psi_{mn} \zeta_n) . \quad (18)$$

\(^2\)An alternative derivation of two of these relations, namely $\tilde{X} = X$ and $\tilde{Y} = Y$, would be by arguing that in the particular amplitudes (12) and (13) the operators $G^a$ and $(DG)^a$ can be considered as commuting with each other, so that the expressions (12) and (13) differ only by re-labeling the indices.
The normalization factor here is readily found from eq.(8) and the condition that the sum over all the polarization states should be equal to the number of the polarization states of $^3D_1$: \[ \sum_{i,j,m} |P^{(1)}_{ij}\zeta_m|^2 = 3 \, . \]

Using the explicit expression (18) in the formula (17) for the transition amplitude, it is a straightforward exercise to find the square of the amplitude averaged over polarizations of the initial $^3D_1$ state and summed over the polarizations of the final $^3S_1$ state:

\[ \overline{|A^{(1)}_\eta|^2} = \frac{10}{9} p^{2} \frac{X^2}{m_Q^2} |A_2|^2 , \]  
(19)

where the overline in the l.h.s. denotes the prescribed averaging-summation operation, and $p_\eta = |p|$.

In order to find the similar quantity for the transitions from the $^3D_2$ states, one generally would have to consider the projector $P^{(2)}$ similarly to the previous treatment of the projector $P^{(1)}$. However, at this point it is simpler to use the fact that only the $J = 1$ and $J = 2$ states contribute to the ‘grand total’ sum of the square of the amplitude over all orbital and spin polarization states, and to find the $J = 2$ contribution by using eq.(19) and subtracting the sum over the $J = 1$ states from the ‘grand total’:

\[ \overline{|A^{(2)}_\eta|^2} = \frac{1}{5} \left( \sum_{J,i,j,k} |A^{(J)}_{ijk}|^2 - 3 \overline{|A^{(1)}_\eta|^2} \right) = \frac{1}{5} \left( \frac{40}{3} - \frac{10}{3} \right) \frac{p^{2}_\eta X^2}{m_Q^2} |A_2|^2 = 2 \, \frac{p^{2}_\eta X^2}{m_Q^2} |A_2|^2 . \]  
(20)

A comparison of the latter result with that in eq.(19) immediately leads to the prediction of the ratio of the decay rates:

\[ \Gamma(^3D_1 \to \eta^3S_1) = \frac{5}{9} \Gamma(^3D_2 \to \eta^3S_1) . \]  
(21)

It should be noted that this relation is obtained in the limit where all effects of the spin-dependent interaction in the heavy quarkonium are neglected. The ignored effects include in particular the fine-structure splitting between the masses of the $^3D_2$ and $^3D_1$. However this splitting is expected to be quite small, not larger than about 10 MeV (for a summary of potential model predictions see e.g. Ref.[15]). Thus if the difference in the kinematical factors $p^2_\eta$ in the decay rate is used as a representative measure of the contribution of the unaccounted corrections, one might expect that the accuracy of the relation (21) should be about 10%.

In order to estimate the relative rate of the discussed $\eta$ and $\pi\pi$ transitions we need to compare the corresponding phase space integrals of the corresponding amplitude squared at
the energy release $\Delta = M(1^3D_J) - M(\Upsilon) \approx 700 \text{ MeV}$, i.e. to compare the expression

$$W_{\eta}^{(J)} = \int |A_{\eta}^{(J)}|^2 2\pi \delta(\Delta - \varepsilon_\eta) \frac{d^3p_\eta}{(2\pi)^3 2\varepsilon_\eta} = |A_{\eta}^{(J)}|^2 \frac{p_\eta}{2\pi}$$

(22)

(numerically, $p_\eta \approx 435 \text{ MeV}$) for the $\eta$ emission with the integral

$$W_{\pi\pi} = \int |A_{\pi\pi}|^2 2\pi \delta(\Delta - \varepsilon_1 - \varepsilon_2) \frac{d^3p_1}{(2\pi)^3 2\varepsilon_1} \frac{d^3p_2}{(2\pi)^3 2\varepsilon_2}$$

(23)

for the two-pion emission, where in each of the integrals $\varepsilon$ stands for the energy of the corresponding light meson.

The square of the $\pi^+\pi^-$ transition amplitude from eq.(10), averaged over the initial and summed over the final polarization states, is given by

$$|A_{\pi^+\pi^-}|^2 = \frac{8}{5} \pi^2 (\alpha_s \rho G)^2 \left[p_1^2 p_2^2 + \frac{1}{3} (\vec{p}_1 \cdot \vec{p}_2)^2 \right] |A_2|^2 \rightarrow \frac{16}{9} \pi^2 (\alpha_s \rho G)^2 p_1^2 p_2^2 |A_2|^2,$$

(24)

where $\vec{p}_{1,2}$ are the momenta of the pions, and $p_{1,2} = |\vec{p}_{1,2}|$. Also in the last transition in eq.(24) the averaging over the relative angle between the momenta is performed, as appropriate for the purpose of calculating the integral in eq.(23). A straightforward integration in eq.(23) of the expression (24) yields

$$W_{\pi^+\pi^-} = 0.44 \frac{\Delta^7}{630\pi} (\alpha_s \rho G)^2 |A_2|^2,$$

(25)

where the numerical factor 0.44 accounts for the finite mass of the pions in the phase space integral (i.e. for massless pions this factor would be equal to one). Thus using in eq.(22) the expression for the $\eta$ emission amplitude from eq.(20), the ratio of the decay rates is found as

$$\frac{\Gamma(\Upsilon(1^3D_2) \rightarrow \eta \Upsilon)}{\Gamma(\Upsilon(1^3D_2) \rightarrow \pi^+\pi^- \Upsilon)} = \frac{W_{\eta}^{(2)}}{W_{\pi^+\pi^-}} = \frac{X^2 p_\eta^3}{0.44 (\alpha_s \rho G)^2 \Delta^7 m_b^2} = \frac{448}{15} \frac{\pi^4}{0.44 (\alpha_s \rho G)^2} \frac{f_\eta^2 m_\eta^4 p_\eta^3}{m_b^2 \Delta^7} \approx \left( \frac{0.64}{\alpha_s \rho G} \right)^2.$$

(26)

Here the expression for $X$ given by eq.(15) is used, and also in the final numerical calculation rather conservative values of $f_\eta$ and $m_b$ are assumed: $f_\eta \approx f_\pi \approx 130 \text{ MeV}$, $m_b \approx 5 \text{ GeV}$.

Clearly, the main uncertainty in evaluation of the ratio of the decay rates comes from poor knowledge of the dimensionless parameter $\alpha_s \rho G$ with both factors normalized at a scale $\mu$ set by the characteristic size of the quarkonium. The estimates of the relevant value of this parameter range from $\alpha_s \rho G \approx 0.2$ [2] to $\alpha_s \rho G \approx 0.59$ [3], with a realistic value likely
being close to 0.35. In either case, the numerical result in the equation (26) predicts that the \( \eta \) transition rate should be not smaller, but most plausibly larger, than the rate of the transition with the emission of two pions.

The presented analysis can also be readily applied for an estimate of the rate of the isospin violating transition with emission of a single \( \pi^0 \). In the discussed axial-anomaly-dominated processes the amplitude of the \( \pi^0 \) transition is simply related\[7\] to the amplitude of the \( \eta \) emission by the ratio\[11\] of the corresponding gluonic matrix elements:

\[
\frac{A_\pi}{A_\eta} = \frac{\langle \pi^0 | G^a \tilde{G}^a | 0 \rangle}{\langle \eta | G^a G^a | 0 \rangle} \frac{p_\pi}{p_\eta} \approx \sqrt{\frac{m_d - m_u}{m_d + m_u}} \frac{f_\pi}{f_\eta} \frac{m_\pi^2}{m_\eta^2} \frac{p_\pi}{p_\eta} ,
\]

where \( m_u \) and \( m_d \) are the masses of the up and down quarks, and the ratio of the decay rates is thus estimated as

\[
\frac{\Gamma(\Upsilon(1^{3}D_J) \to \pi^0 \Upsilon)}{\Gamma(\Upsilon(1^{3}D_J) \to \eta \Upsilon)} = 3 \left( \frac{m_d - m_u}{m_d + m_u} \right)^2 \frac{f_\pi^2}{f_\eta^2} \frac{m_\pi^4}{m_\eta^4} \frac{p_\pi^2}{p_\eta^2} .
\]

For \( (m_d - m_u)/(m_d + m_u) \approx 0.3 \)[10] this estimate gives numerically about \( 4 \times 10^{-3} \), so that the rate of the transition with the emission of a single pion should be quite small.

5 Summary

Within the description of hadronic transitions among the levels of a heavy quarkonium, based on the multipole expansion in QCD, the interference of the \( E1 \) interaction in eq.(2) with the \( M2 \) term in eq.(3) gives rise to transitions between the \( ^3D_J \) and \( ^3S_1 \) resonances in the \( b\bar{b} \) system with the emission of the \( \eta \) meson in \( P \) wave. These processes are observable as the decays \( \Upsilon(1^{3}D_J) \to \eta \Upsilon \) of the \( D \) wave resonances with \( J = 1 \) and \( J = 2 \). The dependence on the coordinates of the heavy quarks of the transition amplitude describing these decays coincides with that of the amplitude for the transitions \( \Upsilon(1^{3}D_J) \to \pi\pi \Upsilon \). Thus the highly model-dependent factor related to the dynamics of the heavy quarkonium cancels in the ratio of the amplitudes, which ratio is thus determined by the amplitudes of production of the corresponding light meson states by gluonic operators. The relevant amplitude for the production of \( \eta \) (eqs.[15][19]) is shown to be completely determined by the low energy theorem in eq.(1) directly related to the anomaly in the flavor singlet axial current in QCD. The enhancement of the \( \eta \) emission due to the anomaly contribution, which emission normally would be suppressed by the inverse of the \( b \) quark mass, and the relative smallness of the
flavor SU(3) breaking, makes the rate of this process comparable to, or most likely greater than that of the two-pion emission, as described by the equation (26). This pattern of the relative rates is qualitatively different from that observed in the $2^3S_1 \rightarrow 1^3S_1$ transitions, both in charmonium and in the $b\bar{b}$ system, where the two-pion transitions dominate. The difference is that in the latter transitions the two-pion emission amplitude is also enhanced by an anomaly, in this case the conformal anomaly in QCD, while the amplitudes of the $D \rightarrow \pi\pi S$ decays receive no enhanced anomalous contribution. The presented treatment, based on the multipole expansion, predicts the ratio of the $\eta$ transition rates to $\Upsilon$ from the $^3D_1$ and $^3D_2$ resonances (eq.(21)) with plausibly an accuracy of about 10%, while the numerical value of the ratio of these rates to that of the two-pion emission, given by eq.(26), contains a considerably larger uncertainty associated with the present poor knowledge of the parameter $\alpha_s \rho_G$, describing the emission of two pions (eq.(10)).

6 Acknowledgements

I gratefully acknowledge enlightening conversations with Arkady Vainshtein. This work is supported in part by the DOE grant DE-FG02-94ER40823.

References

[1] S.E. Csorna et. al. [CLEO Collaboration], CLEO Report No. CLEO-CONF-02-06, July 2002; [hep-ex/0207060].
[2] P. Moxhay, Phys. Rev. D 37 (1988) 2557.
[3] P. Ko, Phys. Rev. D 47 (1993) 208.
[4] J.L. Rosner, U. Chicago report No. EFI 03-06, February 2003; [hep-ph/0302212].
[5] M. Voloshin and V. Zakharov, Phys. Rev. Lett. 45 (1980) 688.
[6] V.A. Novikov and M.A. Shifman, Zeit. Phys. C 8 (1981) 43.
[7] B.L. Ioffe and M.A. Shifman, Phys. Lett. 95B (1980) 99.
[8] K. Gottfried, Phys. Rev. Lett. 40 (1978) 598.
[9] M.B. Voloshin, Nucl. Phys. B154 (1979) 365.

[10] R. Crewther, Phys. Rev. Lett. 28 (1972) 1421;
    M. Chaniwitz and J. Ellis, Phys. Lett. 40B (1972) 397;
    J. Collins, L. Duncan, and S. Joglekar, Phys. Rev. D 16 (1977) 438.

[11] D.J. Gross, S.B. Treiman, and F. Wilczek, Phys. Rev. D 19 (1979) 2188;
    V.A. Novikov et.al., Nucl. Phys. B165 (1980) 55.

[12] M.B. Voloshin, Yad.Phys. 43 (1986) 1571, Sov. J. Nucl. Phys. 43 (1986) 1011.

[13] T.-M. Yan, Phys. Rev. D 22 (1980) 1652.

[14] E.V. Shuryak and A.I. Vainshtein, Nucl. Phys. B199 (1982) 451; Nucl. Phys. B201 (1982) 141.

[15] S. Godfrey and J.L. Rosner, Phys.Rev.D 64 (2001) 097501; 66 (2002) 059902(E).

[16] J. Gasser and H. Leutwyler, Phys. Rep. 87C (1982) 77.