CONVERGENCE OF APPROXIMATIONS OF MONOTONE GRADIENT SYSTEMS

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Abstract. We consider stochastic differential equations in a Hilbert space, perturbed by the gradient of a convex potential. We investigate the problem of convergence of a sequence of such processes. We propose applications of this method to reflecting O.U. processes in infinite dimension, to stochastic partial differential equations with reflection of Cahn-Hilliard type and to interface models.

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1. Introduction

Consider a separable Hilbert space $H$, which could be finite or infinite dimensional, and a Stochastic Differential Inclusion of the form:

\[
\begin{align*}
\frac{dX_t}{dt} & \in (AX_t - \partial U(X_t)) dt + dW_t, \\
X_0(x) & = x \in H,
\end{align*}
\]

(1.1)

where $A$ is self-adjoint in $H$, $U : H \mapsto [0, +\infty]$ is convex and lower semi-continuous with sub-differential $\partial U$ and $W$ a cylindrical white noise in $H$.

We recall that, if $U$ is Fréchet differentiable on $H$, then $\partial U$ coincides with the gradient $\nabla U$. For general convex $U$, the subdifferential $\partial U(x)$ at $x \in H$ is the set $\{y \in H : U(\xi) \geq U(x) + \langle y, \xi - x \rangle, \forall \xi \in H\}$ and the differential inclusion (1.1) is a formal way of writing which needs to be made precise. Notice that $\partial U(x)$ is a nonempty closed convex set for all $x$ such that $U(x) < \infty$ and there exists an element of minimal norm $\partial_0 U(x)$.

If $\nabla U$ is Lipschitz-continuous, then existence and uniqueness of the SDE (1.1) is well known under suitable conditions on $A$. Moreover, in this case $X$ is reversible with respect to a probability measure $\nu := \frac{1}{2}e^{-2U}d\mu$, where $\mu$ is the Gaussian measure $\mathcal{N}(0, (-2A)^{-1})$ and $X$ can be characterized as the diffusion process in $H$ associated with the Dirichlet form obtained closing in $L^2(H, \nu)$ the symmetric bilinear form:

\[
\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle_H d\nu, \quad \varphi, \psi \in C^1_c(H).
\]

(1.2)

In this paper we consider a sequence of processes $X^n$, solving (1.1) for some choice of $(H, A, U, W) = (H_n, A_n, U_n, W_n)$, and we give conditions under which $X^n$ converges to a solution of (1.1) for some choice of $(H_\infty, A_\infty, U_\infty, W_\infty)$. More precisely, we adopt here the Dirichlet form setting, and consider $X^n$ associated with $\mathcal{E}^n$ for some choice of $(H_n, \nu^n)$ in (1.2) and we give conditions for convergence of $X^n$ to a process $X$ associated with $\mathcal{E}$ for some choice of $(H, \nu)$. Our result includes pointwise convergence of the transition semigroups $P^n$ of $X^n$ to the transition semigroups $P$ of $X$. 


An important example of this situation is the following: consider the Yosida approximations of $U$:

$$U_n(x) := \inf_{y \in H} \left\{ U(y) + n \|x - y\|^2 \right\}.$$ (1.3)

Then it is well known that $U_n$ is smooth and $U_n$, respectively $\nabla U_n$, converge to $U$, resp. $\partial_0 U$. An interesting particular case is:

$$U(x) := \begin{cases} 0 & x \in K \\ +\infty & x \notin K \end{cases} \tag{1.4}$$

where $K \subset H$ is a non-empty closed convex set. However we have several other interesting applications in mind besides Yosida approximations: see the end of this introduction.

There is an extensive literature on equations of the type (1.1). If $H = \mathbb{R}^d$ is finite-dimensional, then well-posedness and existence of strong solutions of (1.1) even with more general drift and diffusion coefficients has been established by Cépa in [11]. If $\dim H = \infty$, then Da Prato and Röckner prove well-posedness of (1.1) in the class of weak solutions: see [8]. However, in the latter paper, $\nabla U$ is assumed to satisfy suitable integrability conditions, and in particular the interesting case (1.4) is not covered.

Notice also that (1.4) is naturally associated with the following second-order elliptic operator:

$$L \varphi := \frac{1}{2} \text{Tr}[D^2 \varphi(x)] + \langle Ax, \nabla \varphi(x) \rangle - \langle \partial_0 U(x), \nabla \varphi(x) \rangle, \quad \forall x \in D(A),$$ (1.5)

where $\varphi$ is a smooth test-function. Da Prato, starting with the paper [6], has investigated the analytical properties of $L$ in a suitable $L^2(H, \mu)$ space, where the probability measure $\mu$ makes $L$ essentially self-adjoint. The same analytical approach is used by Da Prato and Lunardi in [7] in the finite-dimensional case. Again, if $H = \mathbb{R}^d$ then the paper [7] covers essentially the most general situation, while, if $\dim H = \infty$, several interesting situations including (1.4) are not covered by the existing literature.

All papers cited above (and many others) use the Yosida approximations (1.3) and the approximating processes $X^n$, solving (1.1) with $U = U_n$. If $H = \mathbb{R}^d$, Cépa [11] proves that $X^n$ converges almost surely to the solution of the limit differential inclusion; moreover, Da Prato and Lunardi prove that the operator $L$ defined in (1.5), with a Neumann condition at the boundary of $K := \{ U < +\infty \}$, is self-adjoint in $L^2(\mathbb{R}^d, e^{-2U} \, dx)$. In infinite dimension, only weaker results are known under more restrictive assumptions.

Our approach allows to prove that under rather general conditions, the transition semigroup of $X^n$

$$P_t^n \varphi(x) := \mathbb{E} [\varphi(X^n_t(x))] , \quad \varphi \in C_0(H), \ t \geq 0, \ x \in H,$$

converges as $n \to \infty$ to a semigroup $P_t$, and in particular that the finite dimensional distributions of the Markov process $X^n$ converge. We can prove this, for instance, when $(-2A)^{-1}$ is trace-class, for any convex lower semi-continuous $U : H \mapsto ]-\infty, +\infty]$ such that $\mu(U < \infty) > 0$ with $\mu := \mathcal{N}(0, (-2A)^{-1})$, and $U_n$ is the Yosida approximation of $U$. For instance, in the case (1.4) with $\dim H = \infty$ this result seems to be new.

We stress that we do not prove that the limit process $X$, which exists by the Kolmogorov extension theorem, solves (1.1); this can be (and has been) done in several interesting situations, like for equations (1.1) and (3.1) below, but only in particular cases and using additional information about the model. For instance, a detailed description of the right hand side of the integration by parts formula (2.7) below, combined with the Fukushima decomposition, can lead to interesting results: see [17].
We describe now briefly some applications of our general result. First, we discuss the following Stochastic Partial Differential Equation of Cahn-Hilliard type and reflection at 0, that has been considered in [10]:

\[
\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial \theta^2} \left( \frac{\partial^2 u}{\partial \theta^2} + \eta \right) + \frac{\partial}{\partial \theta} \tilde{W}, \quad u \geq 0, \quad d\eta \geq 0, \quad \int u \, d\eta = 0 \quad (1.6)
\]

where \( u \) is a continuous function of \( (t, \theta) \in [0, +\infty) \times [0, 1] \), \( \eta \) is a locally finite positive measure on \( (0, +\infty) \times [0, 1] \) called the reflection measure, preventing \( u \) from becoming negative, and \( \tilde{W} \) is a space-time white noise on \( [0, +\infty) \times [0, 1] \). In this case we have \( H = H^{-1}(0, 1) \), Sobolev space of order \( -1 \), and \( U \) of the form \( C_1 \) with \( K := \{ x \in L^2(0, 1) : x \geq 0 \} \). If we choose:

\[
U_n(x) := \begin{cases} 
  n \int_0^1 [x(\theta) \wedge 0]^2 \, d\theta & x \in L^2(0, 1) \\
  +\infty & x \in H^{-1}(0, 1) \setminus L^2(0, 1)
\end{cases}
\]

then our method applies and we can prove convergence of \( X^n \) as \( n \to \infty \). Notice that \( U_n \) is not smooth in the topology of \( H \) (in fact, \( U_n \) is the Yosida approximation of \( U \) in \( L^2(0, 1) \) rather than in \( H \)). We notice that \( X^n \) is not a monotone sequence in \( n \), and no deterministic method is known to prove convergence of \( X^n \).

Another interesting application is provided by interface models: in [15], Funaki and Olla introduce a finite dimensional SDE whose solution \( \phi_i, i = 1, \ldots, N \) is stationary with respect to a 1-dimensional Gibbs measure. The process \( \phi_i \) is always non-negative and \( \phi \) models the motion of an interface between a gas and a liquid phase. Under a suitable rescaling, the process \( \phi \) converges in law to the solution a SPDE with reflection:

\[
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial \theta^2} + \zeta + \tilde{W}, \quad v \geq 0, \quad d\zeta \geq 0, \quad \int v \, d\zeta = 0.
\]

Also this problems fits in the general scheme we discuss here: a sequence \( \phi^N \) associated with a Dirichlet form like (1.2) with \( \nu^N = \frac{1}{N} e^{-2U_n} \, dx \) and \( U_n \) convex, and a limit process \( v \) associated with a Dirichlet form like (1.2) with \( \nu \) weak limit of \( \nu^N \). Indeed, also the results of [15] can be proven using the techniques of this paper: we refer to [18].

The paper is organized as follows: in section 2 we present the setting and the main results; in section 3 we show some applications; in section 4 we give a tightness lemma and in section 5 we prove the main results.

1.1. **Notation.** For a real separable Hilbert space \( H \), we denote the scalar product by \( \langle \cdot, \cdot \rangle_H \) and the associated norm by \( \| \cdot \|_H \). If \( J \) is Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and associated norm \( \| \cdot \|_J \) such that \( J \) is continuously embedded in \( H \), we denote by \( C_b(J) \), respectively \( C^1_b(J) \), the space of all bounded uniformly continuous functions on \( J \), resp. bounded and uniformly continuous together with the first Fréchet derivative. For all \( \varphi \in C^1_b(J) \) we the directional derivative of \( \varphi \) along \( h \in J \) denote by \( \partial_h \varphi \):

\[
\partial_h \varphi(x) := \lim_{t \to 0} \frac{1}{t} (\varphi(x + th) - \varphi(x)), \quad x \in J.
\]

If \( \nabla_J \varphi : J \to J \) denotes the Fréchet derivative of \( \varphi \), then we have:

\[
\langle \nabla_J \varphi(x), h \rangle_J = \partial_h \varphi(x), \quad \forall \ x \in J, \ h \in J.
\]
If $\Gamma \subset H$ is closed and convex, we denote by $\text{Lip}(\Gamma)$ the set of all bounded $\varphi : \Gamma \mapsto \mathbb{R}$ with:
\[
[\varphi]_{\text{Lip}(\Gamma)} := \sup \left\{ \frac{|\varphi(h) - \varphi(k)|}{\|h - k\|_H}, \; h \neq k, \; h, k \in \Gamma \right\} < \infty.
\]
Finally, if $D \subset H$ is dense, we define $\text{Exp}_D(H) \subset C_0(H)$ as the linear span of $\{\cos((h, \cdot)_H), \sin((h, \cdot)_H) : h \in D\}$.

2. Main result

In this section we describe the general setting and the main result. In the next section we show some concrete example.

All processes we consider take values in the fixed separable Hilbert space $H$, which could be finite or infinite dimensional. In order to cover also convergence of finite dimensional approximations, we consider a sequence $(H_n)_n$ of closed affine subspaces of $H$, each itself a Hilbert space endowed with a scalar product $(\cdot, \cdot)_H$ and associated norm $\| \cdot \|_H$, such that for a fixed constant $c > 0$:
\[
\frac{1}{c} \|h\|_H \leq \|h\|_{H_n} \leq c \|h\|_H, \quad \forall h \in H_n, \; n \in \mathbb{N}.
\]
For all $n \in \mathbb{N}$ we consider a probability measure $\nu^n$ on $H_n$ with topological support $K_n \subseteq H_n$ and we suppose that $K_n$ is convex and closed. Moreover we consider a continuous Markov process $X^n$ in $K_n$ such that:

**Hypothesis 1.** Let $c$ as in (2.1). Then for all $n \in \mathbb{N}$:

1. The transition semigroup $(P^n_t)_{t \geq 0}$ of $X^n$ acts on $\text{Lip}(K_n)$ and for all $\varphi \in \text{Lip}(K_n)$:
\[
|P^n_t \varphi(x) - P^n_t \varphi(y)| \leq c [\varphi]_{\text{Lip}(K_n)} \|x - y\|_H, \quad x, y \in K_n, \; t \geq 0.
\]
2. The following bilinear form is closable:
\[
\mathcal{E}^n(\varphi, \psi) := \frac{1}{2} \int \langle \nabla_{H_n} \varphi, \nabla_{H_n} \psi \rangle_{H_n} \, d\nu^n, \quad \forall \varphi, \psi \in C^1_b(H_n).
\]
and the closure $(\mathcal{E}^n, D(\mathcal{E}^n))$ is a Dirichlet form with associated semigroup $(P^n_t)_{t \geq 0}$.
3. For all $h$ in a dense subset $D_n \subset H_n$ there exists a finite signed measure $\Sigma_h^n$ on $H_n$ such that:
\[
\int \partial_t \varphi \, d\nu^n = -\int \varphi \, d\Sigma_h^n, \quad \forall \varphi \in C^1_b(H_n).
\]
We recall that a *finite signed measure* on $H$ is a map $\Sigma$ from the Borel subsets $\mathcal{B}(H)$ of $H$ to $\mathbb{R}$, such that $\Sigma(\emptyset) = 0$ and for any sequence $(E_h)_h \subset \mathcal{B}(H)$ of pairwise disjoint sets:
\[
\Sigma \left( \bigcup_{h=0}^{\infty} E_h \right) = \sum_{h=0}^{\infty} \Sigma(E_h),
\]
where in the series in the right hand side we have absolute convergence. This notion generalizes the definition of a measure to $\mathbb{R}$-valued set functions. We also recall that we can associate to $\Sigma$ a finite positive measure $|\Sigma|$, called the total variation of $\Sigma$, defined by:
\[
|\Sigma|(E) := \sup \left\{ \sum_{h=0}^{\infty} \Sigma(E_h) : E_h \in \mathcal{B}(H) \text{ pairwise disjoint}, \; E = \bigcup_{h=0}^{\infty} E_h \right\}.
\]
Then $|\Sigma|$ is the smallest positive measure $\gamma$ such that $|\Sigma(E)| \leq \gamma(E)$ for all $E \in \mathcal{B}(H)$. We refer to [1, Chap. I].
Hypothesis 2.

(4) \( \nu^n \) converges weakly on \( H \) to a probability measure \( \nu \), with convex topological support \( K \subseteq H \).

(5) For all \( h \) in a dense subset \( D \subset H \) there is a sequence \( h_n \in D \) with \( h_n \to h \) in \( H \) such that for all \( k_n \in H_n \) with \( k_n \to k \in H \):

\[
\lim_{n \to \infty} \|h_n\|^2_{H_n} = \|h\|^2_H, \quad \lim_{n \to \infty} (h_n, k_n)_{H_n} = (h, k)_H, \tag{2.4}
\]

and moreover there exist a finite signed measure \( \Sigma_h \) on \( H \) and a sequence of compact sets \( (J_m)_m \) in \( H \) such that:

\[
\lim_{n \to \infty} \int \varphi \, d\Sigma^n_{h_n} = \int \varphi \, d\Sigma_h, \quad \forall \varphi \in C_b(H), \tag{2.5}
\]

\[
|\Sigma^n_{h_n}|(H \setminus J_m) \leq \frac{1}{m}, \quad \forall \, n, m \in \mathbb{N}. \tag{2.6}
\]

Notice that (2.6) is a tightness condition for \( |\Sigma^n_{h_n}| \). For a sequence of probability measures, (2.5) and (2.6) are equivalent, but this is not the case for signed measures: consider the example \( H = \mathbb{R}, \Sigma^n = \delta_{n + \frac{1}{n}} - \delta_n \) with \( \delta_n \) Dirac mass at \( a \); then \( \Sigma^n \) converges to the measure identically 0 on \( C_b(\mathbb{R}) \) (recall that this is the space of all bounded uniformly continuous functions on \( \mathbb{R} \)), but the sequence \( |\Sigma^n| = \delta_{n + \frac{1}{n}} + \delta_n \) is not tight. Therefore (2.6) does not follow from (2.5) and has to be proven separately.

By (2.3) and (2.5), we have the following integration by parts formula for \( \nu \):

\[
\int \partial_h \varphi \, d\nu = - \int \varphi \, d\Sigma_h, \quad \forall \, h \in D, \quad \forall \varphi \in C_b(H), \tag{2.7}
\]

We denote the projection from \( H \) to the element in \( K \) with minimal distance by \( \Pi_{K_n} \):

\[
\Pi_{K_n} : H \mapsto K_n, \quad \|\Pi_{K_n}(x) - x\|_H \leq \|k - x\|_H \quad \forall \, k \in K_n.
\]

Under the above assumptions, we have the following Theorem, main result of the paper:

**Theorem 2.1.** Suppose that Hypothesis 1 and 2 hold. Then:

(1) There exists a semigroup \( (P_t)_{t \geq 0} \) of operators acting on \( C_b(H) \) such that for all \( \varphi \in C_b(H), \, x \in K \), setting \( x_n := \Pi_{K_n}(x) \):

\[
\lim_{n \to \infty} P^n_t \varphi(x_n) = P_t \varphi(x), \quad \forall \, t \geq 0.
\]

Moreover \( [P_t \varphi]_{\text{Lip}(H)} \leq c \|\varphi\|_{\text{Lip}(H)} \) for all \( \varphi \in \text{Lip}(H), \, t \geq 0 \).

(2) For all \( x \in K \) there is a Markov process \( (X_t)_{t \geq 0} \), defined on a probability space \( (\Omega, \mathcal{P}_x) \), with state space \( K \) and transition semigroup \( (P_t)_{t \geq 0} \), such that \( P_x(X_0 = x) = 1 \). Moreover \( P_t \varphi(x) \to \varphi(x) \) as \( t \to 0 \), for all \( \varphi \in C_b(H) \).

(3) For all \( \varphi_1, \ldots, \varphi_m \in C_b(H), \, 0 \leq t_1 \leq \ldots \leq t_m \) and \( x \in K \), setting \( x_n := \Pi_{K_n}(x) \)

\[
\lim_{n \to \infty} \mathbb{E}_x [\varphi_1(X^n_{t_1}) \cdots \varphi_m(X^n_{t_m})] = \mathbb{E}_x [\varphi_1(X_{t_1}) \cdots \varphi_m(X_{t_m})].
\]

(4) The following bilinear form is closable:

\[
\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int \langle \nabla_H \varphi, \nabla_H \psi \rangle_H \, d\nu, \quad \forall \, \varphi, \psi \in \text{Exp}_D(H),
\]

and the closure \( (\mathcal{E}, D(\mathcal{E})) \) is a Dirichlet form with associated semigroup \( (P_t)_{t \geq 0} \). Moreover \( \text{Lip}(H) \subset D(\mathcal{E}) \) and \( \mathcal{E}(\varphi, \varphi) \leq [\varphi]_{\text{Lip}(H)}^2 \).
(5) The stationary Markov process $\hat{X}^n$ with transition semigroup $P^n$ and initial distribution $\nu^n$ converges in law to the stationary Markov process $\hat{X}$ with transition semigroup $P$ and initial distribution $\nu$.

3. Applications

Before proving Theorem 2.1, we discuss some interesting situations where the assumptions of Hypothesis 1 and 2 hold and therefore the results of Theorem 2.1 apply.

3.1. Gradient systems with convex potential. Let now $H$ be of infinite dimension. We consider a closed operator $A : D(A) \subset H \to H$ such that:

1. $A$ is self-adjoint in $H$ and $(Ax, x)_H \leq -\omega \|x\|^2$ for all $x \in D(A)$, with $\omega > 0$.
2. $Q := (-2A)^{-1}$ is trace class in $H$.

Let $H_n = H$ for all $n \in \mathbb{N}$ and consider a convex lower semi-continuous function $U : H \to [-\infty, +\infty]$, setting $K := \{U < +\infty\}$. We define the Yosida approximations of $U$:

$$U_n(x) := \inf_{y \in H} \{U(y) + n \|x - y\|^2_H\}.$$  

We recall that $U_n$ is convex, differentiable, and:

$$U_n(x) \leq U_{n+1}(x), \quad \lim_{n \to \infty} U_n(x) = U(x), \quad \forall x \in H. \tag{3.1}$$

Moreover $\nabla U_n$ is Lipschitz-continuous with Lipschitz constant not larger than $2n$: see [1, Prop. 2.6, Prop. 2.11]. By the Hahn-Banach Theorem, see e.g. [5, Proposition I.9], and by the monotonicity of $n \mapsto U_n$, there exist constants $A \in \mathbb{R}$, $B > 0$ such that:

$$U_n(x) \geq A - B \|x\|_H, \quad \forall x \in H, \quad n \in \mathbb{N}. \tag{3.2}$$

Since $\nabla U_n$ is Lipschitz-continuous, there exists a unique strong solution $X^n(x)$ of:

$$\begin{cases}
    dX^n_t = (AX^n_t - \nabla_H U_n(X^n_t)) \, dt + dW_t, \\
    X^n_0(x) = x \in H,
\end{cases}$$

where $W$ is a cylindrical Wiener process in $H$. Moreover:

$$\|X^n_t(x) - X^n_t(y)\| \leq \exp(-\omega t) \|x - y\|, \quad x, y \in H, \quad t \geq 0,$n

which yields (3.2). We also define the probability measures on $H$:

$$\mu(dx) := N(0, Q)(dx), \quad \nu^n(dx) := \frac{1}{Z^n} \exp(-2U_n(x)) \mu(dx),$$

where $Z^n$ is a normalization constant. We also have for all $h \in D(A)$:

$$\Sigma^n_h(dx) := 2 \left(\langle x, Ah \rangle_H - \langle \nabla_H U_n(x), h \rangle_H\right) \nu^n(dx).$$

Proposition 3.1. If $\mu(K) > 0$, then the assumptions of Hypothesis 1 and 2 are fulfilled.

Proof. Set $H_n := H$, $D = D_n := D(A)$. Since $\mu(K) > 0$, $\nu^n$ converges to:

$$\nu := \frac{1}{Z} e^{-2U} \, dx,$$

where $Z > 0$ is a normalization constant. Let now $h = h_n \in D(A)$. For all $\varphi \in C_b^1(H)$:

$$\exists \lim_n \int \varphi d\Sigma^n_h = -\lim_n \int \partial_h \varphi d\nu^n = -\int \partial_h \varphi d\nu =: \int \varphi d\Sigma_h.$$
We do not know yet that $\Sigma_h$ is a signed measure. We set:

$$\left| \Sigma_h \right|(H) := \sup \left\{ \left| \int \varphi \, d\Sigma_h \right| : \varphi \in C^1_b(H), \|\varphi\|_\infty \leq 1 \right\}.$$  

Then we claim that $\Sigma^n_h$ and $\Sigma_h$ are indeed finite signed measures on $H$ and:

$$\sup_n \left| \Sigma^n_h \right|(H) < \infty, \quad \lim_{n \to \infty} \left| \Sigma_h \right|(H) = \left| \Sigma_h \right|(H) < \infty. \quad (3.3)$$

Before proving (3.3) we show that it implies (2.5) and (2.6). Indeed, setting $T_n : C_b(H) \to \mathbb{R}$, $T_n \varphi := \int \varphi \, d\Sigma^n_h$, the family of linear functionals $(T_n)_n$ is equicontinuous and converges on the dense set $C^1_b(H)$ in the sup-norm and therefore converges on $C_b(H)$, i.e. we have (2.5). The proof of the density of $C^1_b(H)$ in $C_b(H)$ in the sup-norm can be found in \cite[Theorem 2.2.1]{9}.

On the other hand, consider a dense sequence $(x_j)_j$ in $H$ and set $A^k_j := \bigcup_{i=1}^j B(x_j, 1/k)$. It is enough to prove that there exists $\Sigma^n_h|_{\partial_i} := \bigcup_{i=1}^j A^k_i$ is a compact set such that $|\Sigma^n_h|\left(H\setminus J_m\right) \leq 1/m$. If we can not find such $\Sigma^n_h$, then for all $i$ there is $n(i)$ such that $|\Sigma^n_h(\partial_i)| \leq |\Sigma^n_h(\partial_i)| - 2^{-k/m}$. By the lower semi-continuity of the total variation on open sets we find: $|\Sigma_h(\partial_i)| \leq |\Sigma_h(\partial_i)| - 2^{-k/m}$ for all $i$ and therefore $|\Sigma_h(\partial_i)| \leq |\Sigma_h(\partial_i)| - 2^{-k/m}$, a contradiction. Therefore (2.6) is proven.

We prove now (3.3). Let $Z$ be a $H$-valued random variable with distribution $\mu = \mathcal{N}(0, Q)$ and $h \in D(A)$. We can suppose that $(-2Ah, h)_H = 1$. Setting $Y := Z + \langle Z, 2Ah \rangle h$, we have that $Z = Y - \langle Z, 2Ah \rangle h$ and $Y$ is independent of $\langle Z, 2Ah \rangle$. Notice that $Y \sim \mathcal{N}(0, Q - h \otimes h)$ and $\langle Z, 2Ah \rangle \sim \mathcal{N}(0, 1)$. Then for all $z \in H$ we can write uniquely $z = y + th$ with $t \in \mathbb{R}$ and $y \in (Ah)^\perp \subset H$ and with this notation:

$$\mu(dz) = \mathcal{N}(0, 1)(dt) \cdot \mathcal{N}(0, Q - h \otimes h)(dy).$$

Let now $V : H \mapsto \mathbb{R}$ be convex and lower semi-continuous. Fix $y \in H$. Then by (3.2) the function $R \ni t \mapsto v(t) := V(ht + y) + \frac{t^2}{2}$ is convex and tends to $+\infty$ as $|t| \to \infty$. Necessarily $t \mapsto e^{-v(t)}$ is non-decreasing on a half-line $(-\infty, t_0]$ and non-increasing on $(t_0, +\infty)$, where $v(t_0) = \min v$. Then $e^{-v(-)}$ has bounded variation on $R$ and:

$$\left| \frac{d}{dt} e^{-v(-)} \right|(dt) = 1_{(t_{\leq t_0})} \frac{d}{dt} e^{-v(-)}(dt) - 1_{(t_{> t_0})} \frac{d}{dt} e^{-v(-)}(dt).$$

We obtain that $\frac{d}{dt} e^{-v}$ is a finite signed measure on $\mathbb{R}$, and:

$$\int_{\mathbb{R}} \left| \frac{d}{dt} e^{-v} \right| (dt) = 2 e^{-v(t_0)} = 2 e^{-\min v}.$$  

Now, by Fubini-Tonelli’s Theorem, for all $\varphi \in C^1_b(H)$:

$$\int \partial_h \varphi e^{-V} \, d\mu = \int \left[ \int \frac{\partial \varphi(ht + y)}{\partial t} e^{-V(ht + y)} \mathcal{N}(0, 1)(dt) \right] \mathcal{N}(0, Q - h \otimes h)(dy)$$

$$= \int \left[ \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} e^{-V(ht + y) - t^2/2}(dt) \right] \varphi(ht + y) \mathcal{N}(0, Q - h \otimes h)(dy).$$

Applying this formula to $V = 2U_n$ and $V = 2U$ and taking the supremum over all $\varphi \in C^1_b(H)$ such that $\|\varphi\|_\infty \leq 1$ we obtain:

$$\left| \Sigma^n_h \right|(H) = \frac{1}{Z^n} \int \sqrt{\frac{2}{\pi}} e^{-\min v} e^{2U_n(ht + y) + \frac{t^2}{2}} \mathcal{N}(0, Q - h \otimes h)(dy),$$
We define $\int_{K}$ as the non-empty interior. If we want to consider a case where $K$ necessarily has non-empty interior, then we have to define $H$ in analogy with the finite dimensional example of the next subsection, if $K \neq H$ then $\Sigma_0$ is expected to contain an infinite dimensional boundary term: this has been explicitly computed in several recent papers, for instance [17], [3], [14], [10]. If $U$ is, as in [14], equal to 0 on $K$ and to $+\infty$ on $H \setminus K$, then Theorem [2.4] gives a construction of the reflecting O.U. process in $K$, in analogy with Fukushima’s approach to reflecting processes in finite dimension: see examples 4.4.2 and 4.5.3 in [13].

For instance, let us consider the case $H = L^2(0,1)$, $A$ the realization of $\partial^2$ with homogeneous Dirichlet condition at $\{0,1\}$ and $U$ of the form (1.4) with $K := \{x \in L^2(0,1) : x \geq -\alpha\}$ with $\alpha \geq 0$. Then the Yosida approximation of $U$ is the distance from $K$ in $H$, the process $X^n$ is the solution of the SPDE:

$$\frac{\partial X^n}{\partial t} = \frac{\partial^2 X^n}{\partial \theta^2} + \frac{\partial^2 W}{\partial t \partial \theta} + 2n(X^n + \alpha)^-, \quad \theta \in [0,1]$$

where $W$ is a Brownian sheet, and the limit $X$ solves the following SPDE with reflection:

$$\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial \theta^2} + \frac{\partial^2 W}{\partial t \partial \theta} + \eta(t, \theta), \quad X \geq -\alpha, \quad d\eta \geq 0, \quad \int (X + \alpha) \, d\eta = 0,$$

where $\eta$ is a reflection term which prevents the continuous solution $X$ from becoming less than $-\alpha$: see [17]. If $\alpha > 0$ then we are in the situation of Proposition [3.1] If $\alpha = 0$ then $\mu(K) = 0$ and Proposition [3.1] does not apply directly. However in this case it is possible to let first $n \to \infty$ with $\alpha > 0$ and afterwards let $\alpha \to 0$: both these limits satisfy the assumptions of Hypothesis [1] and [2].

Another interesting example is the Cahn-Hilliard equation with reflection (1.6) described in the introduction.

### 3.2. Gradient systems with convex potential in finite dimension.

Let $H = H_n = \mathbb{R}^d$ and consider a convex lower semi-continuous $U : \mathbb{R}^d \mapsto ]-\infty, +\infty]$, such that $Z := \int \, e^{-2U} \, dx < \infty$. By the convexity, the latter assumption is equivalent to:

$$\lim_{\|x\| \to \infty} U(x) = +\infty.$$  \quad (3.5)

We define $K := \{U < \infty\}$. Then $K$ is a compact convex set and we assume that $K$ has non-empty interior. If we want to consider a case where $K \subset \mathbb{R}^d$ has empty interior, then we have to define $H'$ as the linear span of $K$ and work in $H'$ rather than in $H$. Notice that $K$ necessarily has non-empty interior in $H'$.

As in the previous subsection, we define the Yosida approximations of $U$ as $U_n(x) := \inf_{y \in \mathbb{R}^d} \{U(y) + n \|x - y\|^2\}$. Then $U_n$ satisfies (3.4) and (3.5). Let $W$ be a $d$-dimensional
Brownian motion. Then by the Lipschitz-continuity of $\nabla U_n$ there exists a unique strong solution $X^n(x)$ of:

$$dX^n_t = -\nabla U_n(X^n_t) \, dt + dW_t, \quad X^n_0(x) = x \in \mathbb{R}^d,$$

(3.6)

We also define the probability measure on $H$:

$$\nu^n(dx) := \frac{1}{Z^n} \exp (-2 U_n(x)) \, dx,$$

where $Z^n$ is a normalization constant. Then, recalling (2.5), we have for all $h \in \mathbb{R}^d$:

$$\Sigma^n_h(dx) := -2 \langle \nabla U_n(x), h \rangle \nu^n(dx).$$

**Proposition 3.2.** In this setting, the assumptions of Hypothesis 1 and 2 are fulfilled.

**Proof.** The proof of Proposition 3.1 can be repeated literally, substituting the Gaussian measure $\mu$ with the Lebesgue measure $dx$ on $\mathbb{R}^d$ and the Gaussian measure $\mathcal{N}(0, Q - h \otimes h)$ with the $(d-1)$-dimensional Lebesgue measure on $h^\perp$. □

In particular, Theorem 2.1 applies to (3.6). We notice that a stronger result is proven in [11], namely almost sure convergence of $X^n$ as $n \to \infty$. However the convergence of $\Sigma^n_h$ to $\Sigma_h$ in the sense of (2.5) and (2.6), is quite general and of independent interest.

Consider for instance the example (1.4) with $K \subset \mathbb{R}^d$ a convex compact set with non-empty interior. In this case $U_n = nd^2_K$, where $d_K(x)$ is the distance of $x$ from $K$. Proposition 3.2 shows that (2.5) and (2.6) hold with:

$$\Sigma^n_h(dx) = -4n d_K(x) \langle \nabla d_K(x), h \rangle \frac{1}{Z^n} e^{-2n d_K^2(x)} \, dx,$$

$$\Sigma_h(dx) := -\frac{1}{|K|} \langle \tilde{n}(x), h \rangle 1_{x \in \partial K} \mathcal{H}^{d-1}(dx)$$

as $n \to \infty$, where $\tilde{n}$ is an outer normal vector to the boundary of $K$, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and $|K| \in (0, \infty)$ is the Lebesgue measure of $K$. When $K$ has smooth boundary, then the normal vector and the boundary measure are classical objects; a general bounded convex $K$ is a Lipschitz set, and the existence of the same objects follows from De Giorgi’s work: see e.g. [11] Theorem 3.36. The Dirichlet form approach to reflecting Brownian motion in Lipschitz domains is developed by Fukushima in [13] and Bass and Hsu in [2].

**3.3. SPDEs of Cahn-Hilliard type and Interface models.** Proposition 3.1 can be extended to other situations where $U_n$ is not the Yosida approximation of $U$ but enjoys similar properties. For instance, this is the case for SPDEs with reflection of Cahn-Hilliard type, considered in [10], and interface models, as in [15] and [18]: see the Introduction of this paper.

**4. Tightness of Stationary Approximations**

Let $\hat{X}^n$ denote the stationary Markov process with transition semigroup $P^n$ and initial distribution $\nu^n$. If $H = \mathbb{R}^d$, then in the following lemma we can set $H^{(1)} = H^{(2)} = \mathbb{R}^d$. If $H$ is infinite dimensional, then we fix Hilbert spaces $H^{(1)}$ and $H^{(2)}$ such that $H$ is embedded in $H^{(1)}$ with Hilbert-Schmidt inclusion, and $H^{(1)}$ is compactly embedded in $H^{(2)}$. We can suppose that $H$ is dense in $H^{(2)}$.

**Lemma 4.1.** For all $T > 0$, the laws of $(\hat{X}^n)_{n \in \mathbb{N}}$ are tight in $C([0, T]; H^{(2)})$. 


We claim that for all \( p > 1 \) there exists \( C_p \in (0, \infty) \), independent of \( n \), such that:

\[
\left( \mathbb{E} \left[ \left\| \hat{X}_t^n - \hat{X}_s^n \right\|_H^p \right] \right)^{\frac{1}{p}} \leq C_p \left| t - s \right|^{\frac{1}{2}}, \quad t, s \geq 0, \quad \forall \ n \in \mathbb{N}. \tag{4.1}
\]

To prove (4.1), we fix \( n \in \mathbb{N} \) and \( T > 0 \) and use the Lyons-Zheng decomposition, see e.g. Th. 5.7.1, to write for \( t \in [0, T] \) and \( h \in H_n \):

\[
\langle h, \hat{X}_t^n - \hat{X}_{t-h}^n \rangle_{H_n} = \frac{1}{2} M_t - \frac{1}{2} (N_T - N_{T-h}),
\]

where \( M, \) respectively \( N \), is a martingale w.r.t. the natural filtration of \( \hat{X}_t^n \), respectively of \( (\hat{X}_t^n_{-t}, \ t \in [0, T]) \). Moreover, the quadratic variations are both equal to: \( \langle M \rangle_t = \langle N \rangle_t = t \cdot \|h\|_{H_n}^2 \). If \( c_p \) is the optimal constant in the Burkholder-Davis-Gundy inequality and \( \kappa \) is the Hilbert-Schmidt norm of the inclusion of \( H \) into \( H^{(1)} \), then (4.1) holds with \( C_p = c_p \kappa \).

Since the law of \( \hat{X}_0^n \) is \( \nu^n \) which converges as \( n \to \infty \) in \( H \) and a fortiori in \( H^{(2)} \), tightness of the laws of \( \hat{X}_n^{(n)\nu} \) in \( C([0, T] ; H^{(2)}) \) follows e.g. by Theorem 7.2 in Chap. 3 of [12].}

\[\square\]

5. CONVERGENCE OF THE SEMIGROUPS

In this section we prove Theorem 2.1. The proof is achieved using the theory of Dirichlet Forms, the uniform Feller property (2.2) of \( X^n \) and the integration by parts formula (2.7).

We define for all \( \varphi \in C_b(K_n) \) the resolvent of \( X^n \):

\[
R^n_\lambda \varphi(x) := \int_0^\infty e^{-\lambda t} P^n_t \varphi(x) \, dt, \quad x \in K_n, \ \lambda > 0.
\]

We prove first convergence of the resolvent \( R^n \) in Proposition 5.1 and then Theorem 2.1.

**Proposition 5.1.**

1. \((\mathcal{E}, \text{Exp}_D(H))\) is closable in \( L^2(\nu) \) and the closure \((\mathcal{E}, D(\mathcal{E}))\) is a symmetric Dirichlet form such that \( \text{Lip}(H) \subset D(\mathcal{E}) \) and \( \mathcal{E}(\varphi, \varphi) \leq \|\varphi\|_{\text{Lip}(H)}^2 \). The associated resolvent operator \((R^n_\lambda)_{\lambda > 0}\) acts on \( C_b(H) \).

2. For all \( \phi \in C_b(H) \) and \( x \in K \): \( R^n_\lambda \phi(\Pi_{K_n}(x)) \to R_\lambda \phi(x) \) as \( n \to \infty \).

We describe the idea of the proof: for \( \varphi \in \text{Lip}(H) \), by the uniform Feller property (2.2) we have that \( R^n_\lambda \phi \) is an equicontinuous and equibounded family. By Arzelà-Ascoli’s Theorem we can extract converging subsequences on compact sets with large mass with respect to \( \nu^n \) and \( \Sigma_n^n \). Now we consider the formula which characterizes \( R^n_\lambda \phi \) for \( \lambda > 0 \):

\[
\lambda \int R^n_\lambda \varphi \psi \, d\nu^n + \mathcal{E}^n(R^n_\lambda \varphi, \psi) = \int \psi \varphi \, d\nu^n, \quad \forall \ \psi \in D(\mathcal{E}^n).
\]

We would like to pass to the limit, but \( \mathcal{E}^n \) contains the gradient of \( R^n_\lambda \varphi \). However, if \( \psi = \exp(i \langle \cdot, h \rangle) \in \text{Exp}_D(H) \), with \( i^2 = -1 \), then we can use the integration by parts formula (2.3) and write:

\[
\mathcal{E}^n(R^n_\lambda \varphi, \psi) = -i \int R^n_\lambda \varphi \psi \, d\Sigma^n_h \to -i \int F \psi \, d\Sigma_h, \quad n \to \infty,
\]

where \( F \) is a pointwise limit of \( (R^n_\lambda \varphi)_n \). Using (2.7), the latter expression is equal to \( \mathcal{E}(F, \psi) \), i.e. we obtain:

\[
\lambda \int F \psi \, d\nu + \mathcal{E}(F, \psi) = \int \psi \varphi \, d\nu, \quad \forall \ \psi \in \text{Exp}_D(H),
\]
which is very close to characterize $F$ as the $\lambda$-resolvent of $E$ in $L^2(\nu)$ applied to $\varphi$. The proof of Proposition 5.1 makes these arguments rigorous.

In the following proofs we use a number of times, often without further mention, the following easily proven fact.

**Lemma 5.2.** Let $E$ be a Polish space, $(M_n : n \in \mathbb{N} \cup \{\infty\})$ a sequence of finite signed measures on $E$ and $(\varphi_n : n \in \mathbb{N} \cup \{\infty\})$ a sequence of functions on $E$, such that:

1. for all $\varphi \in C_b(E)$:
   \[
   \lim_{n \to \infty} \int \varphi \, dM_n = \int \varphi \, dM_{\infty}
   \]

2. there exists a sequence of compact sets $(J_m)_m$ in $E$ such that:
   \[
   \lim_{m \to \infty} \sup_{n \in \mathbb{N}} |M_n|(E \setminus J_m) = 0.
   \]

3. $(\varphi_n : n \in \mathbb{N} \cup \{\infty\})$ is equi-bounded and equi-continuous

4. $\varphi_n$ converges pointwise to $\varphi_{\infty}$ on $\bigcup_{m} J_m$.

Then:
\[
\lim_{n \to \infty} \int_{E} \varphi_n \, dM_n = \int_{E} \varphi_{\infty} \, dM_{\infty}.
\]

**Proof.** We notice that by Arzelà-Ascoli’s Theorem, $\varphi_n$ converges uniformly to $\varphi$ on $J_m$ for all $m \in \mathbb{N}$. Moreover, by the Banach-Steinhaus Theorem the norms of the functionals $C_b(E) \ni \varphi \mapsto \int_{E} \varphi \, dM_n$ are bounded, therefore $|M_n|(E) \leq C < \infty$ for all $n \in \mathbb{N}$. Then:
\[
\left| \int_{E} \varphi_n \, dM_n - \int_{E} \varphi_{\infty} \, dM_{\infty} \right| \leq \int_{E} (\varphi_n - \varphi_{\infty}) \, dM_n + \int_{E} \varphi_{\infty} (dM_n - dM_{\infty})
\]
and the second term in the right hand side tends to 0 by our first assumption. Now:
\[
\int_{E} (\varphi_n - \varphi_{\infty}) \, dM_n \leq \int_{J_m} |\varphi_n - \varphi_{\infty}| \, d|M_n| + \int_{E \setminus J_m} |\varphi_n - \varphi_{\infty}| \, d|M_n|
\]
\[
\leq \sup_{J_m} |\varphi_n - \varphi_{\infty}| \cdot C + \|\varphi_n - \varphi_{\infty}\|_{\infty} |M_n|(E \setminus J_m).
\]
Taking the limsup as $n \to \infty$ and then letting $m \to \infty$ we have the thesis. $\square$

**Proof of Proposition 5.1.** We divide the proof in several steps.

**Step 1.** We recall that $\Pi_{K_n} : H \mapsto K_n$ is 1-Lipschitz in $H$, and therefore, by (2.2):
\[
\|(R^{\lambda}_n \psi) \circ \Pi_{K_n}\|_{\infty} \leq \|\psi\|_{\infty}, \quad [(R^{\lambda}_n \psi) \circ \Pi_{K_n}]_{\text{Lip}(H)} \leq c[\psi]_{\text{Lip}(H)}, \quad \forall \psi \in C^1_b(H). \tag{5.1}
\]

Fix $\psi \in C^1_b(H)$. Let $(m_j)_{j}$ be any sequence in $\mathbb{N}$ and $(x_k)_k$ a countable dense set in $H$. With a diagonal procedure, we can find a subsequence $(m_i)_i$ and a function $F : \{x_k, k \in \mathbb{N}\} \mapsto \mathbb{R}$ such that $R^{\lambda}_n \psi(\Pi_{K_n}(x_k)) \mapsto F(x_k)$ as $n = m_i \to \infty$ for all $k \in \mathbb{N}$. By (5.1), $F$ is Lipschitz on $\{x_k, k \in \mathbb{N}\}$ and therefore can be extended to a function in $\Psi_{\lambda, \psi} \in \text{Lip}(H)$ and:
\[
\Psi_{\lambda, \psi}(x) = \lim_{i \to \infty} R^{m_i}_\lambda \psi(\Pi_{K_{m_i}}(x)) \quad \forall x \in H. \tag{5.2}
\]
Finally, by a diagonal procedure, we can suppose that such limit holds along the same subsequence for all $\lambda \in \mathbb{N}$. Notice that in fact we are going to prove that the limit exists as $n \to \infty$ for all $\lambda > 0$. We define $\Delta := \text{Span}\{\Psi_{\lambda, \psi} : \psi \in C^1_b(H), \lambda \in \mathbb{N}\} \subset \text{Lip}(H)$. 

Step 2. We would like to apply the integration by parts formula (2.7) to $\Psi_{\lambda,\psi}$, which is however not in $C^1_b(H)$ but only in $\text{Lip}(H)$. However, notice that for all $\varphi, \Phi \in C^1_b(H)$:

$$\int \varphi \partial_h \Phi \, d\nu = - \int \Phi \partial_h \varphi \, d\nu - \int \varphi \Phi \, d\Sigma_h, \quad \forall \ h \in D. \tag{5.3}$$

If now $\Phi \in \text{Lip}(H)$, then there exists a sequence $(\Phi_m)_m \subset C^1_b(H)$ such that:

$$\lim_m \Phi_m(x) = \Phi(x), \quad \forall \ x \in H, \quad \|\Phi_m\|_{\infty} + |\Phi_m|_{\text{Lip}(H)} \leq \|\Phi\|_{\infty} + |\Phi|_{\text{Lip}(H)}.$$

By (5.3) we have that $\partial_h \Phi_m$ converges weakly in $L^2(\nu)$ to an element of $L^\infty(\nu)$ that we call $\partial_h \Phi$ and with this definition (2.7) holds for all $\Phi \in \text{Lip}(H)$. Moreover, we obtain in this way that $\nabla_H \Phi \in L^\infty(H, \nu; H)$ is well defined and:

$$\mathcal{E}(\Phi, \Phi) \leq \liminf_{m} \mathcal{E}(\Phi_m, \Phi_m) \leq \liminf_{m} |\Phi_m|^2_{\text{Lip}(H)} \leq |\Phi|^2_{\text{Lip}(H)}.$$

Moreover, for all $\Phi \in \text{Lip}(H)$ it is possible to find a multi-sequence $(\Phi_M)_M \subset \text{Exp}_D(H)$, where $M = (m_1, \ldots, m_5) \in \mathbb{N}^5$, such that $\Phi_M$ converges to $\Phi$ pointwise and:

$$\sup_M \left( \|\Phi_M\|_{\infty} + |\Phi_M|_{\text{Lip}(H)} \right) < \infty, \quad \lim_M \mathcal{E}(\Phi_M, \Psi) = \mathcal{E}(\Phi, \Psi), \quad \forall \ \Psi \in \text{Lip}(H), \tag{5.4}$$

where $\lim_M$ means that we let $m_1 \to \infty$, then $m_2 \to \infty$ and so on until $m_5 \to \infty$ (see [9 Proposition 11.2.10] for similar results).

Step 3. We want to prove now that for all $\lambda \in \mathbb{N}$ and $\Psi_{\lambda,\psi}$ as in the first step:

$$\mathcal{E}_\lambda(\Psi_{\lambda,\psi}, v) := \lambda \int \Psi_{\lambda,\psi} v \, d\nu + \mathcal{E}(\Psi_{\lambda,\psi}, v) = \int v \, d\nu, \quad \forall \ v \in \Delta. \tag{5.5}$$

First we prove (5.5) for $v \in \text{Exp}_D(H)$. Fix $h \in D$ and $h_n \in D_n$ as in Hypothesis 2 and set:

$$\varphi_n(k) := \exp(i \langle h_n, \Pi_{H_n} k \rangle_{H_n}), \quad \varphi(k) := \exp(i \langle h, k \rangle_H), \quad k \in H,$$

where $i \in \mathbb{C}$ with $i^2 = -1$ and $\Pi_{H_n}$ denotes the orthogonal projection from $H$ to $H_n$. By Hypothesis 2 $\|k - \Pi_{H_n} k\|_H \to 0$ for all $k \in H$. Indeed, this is true for all $k \in D$ since there is a sequence $k_n \in H_n$ such that $k_n \to k$ and by density of $D$ in $H$ we conclude, since $\Pi_{H_n}$ is 1-Lipschitz continuous in $H$. Therefore, by (2.4): $\varphi_n(k) \to \varphi(k)$ for all $k \in H$.

Since $R^n_\lambda$ is the resolvent operator associated with $\mathcal{E}^n$:

$$\mathcal{E}^n_\lambda(R^n_\lambda \psi, \varphi_n) := \lambda \int R^n_\lambda \psi \varphi_n \, d\nu^n + \mathcal{E}^n(R^n_\lambda \psi, \varphi_n) = \int \varphi_n \, d\nu^n.$$

Notice that $\nabla_{H_n} \varphi_n = i h_n \varphi_n$. Then, by the integration by parts formula (2.3):

$$2 \mathcal{E}^n(R^n_\lambda \psi, \varphi_n) = \int R^n_\lambda \psi \, d_h \|h_n\|_{H_n}^2 \varphi_n \, d\nu^n - i \int R^n_\lambda \psi \varphi_n \, d\Sigma^n_{h_n}.$$

Since $\nu^n \to \nu$ and $\Sigma^n_{h_n} \to \Sigma$ as $n \to \infty$, by (2.4), (2.5) and Lemma 5.2

$$\lim_{n \to \infty} \int g \, d_h \|h_n\|_{H_n}^2 \varphi_n \, d\nu^n - i \int g \varphi_n \, d\Sigma^n_{h_n} = \int g \, d_h \|h\|_{H}^2 \varphi \, d\nu - i \int g \varphi \, d\Sigma_h, \quad \forall g \in C_b(H).$$

The crucial fact is now the following: by (2.5)-(2.6), (5.1) and Lemma 5.2 we can substitute $g$ with $R^n_\lambda \psi$ in the last formula and prove that:

$$\lim_{i \to \infty} \int R^n_\lambda \psi \varphi_n \, d\Sigma^n_{h_{mi}} = \int \Psi_{\lambda,\psi} \varphi \, d\Sigma_h. \tag{5.6}$$
In particular we obtain:
\[ \int \psi \varphi \, d\nu = \lim_{i \to \infty} \int \psi \varphi_i \, d\nu^m_i = \int \Psi_{\lambda,\psi} \left( \lambda + \frac{1}{2} \|h\|^2 \right) \varphi \, d\nu - \frac{1}{2} \int \Psi_{\lambda,\psi} \varphi \, d \Sigma_h \]
and by the integration by parts formula (2.7) the last expression is equal to \( \mathcal{E}_\lambda(\Psi_{\lambda,\psi}, \varphi) \), i.e. we have proven (5.5) for \( v = \varphi \). By linearity we obtain (5.5) for all \( v \in \text{Exp}_D(H) \). By (5.4) we obtain (5.5) for all \( v \in \Delta \).

**Step 4.** We want to prove now that the bilinear form \((\mathcal{E}, \Delta)\) is closable and the closure is a Dirichlet form. By Lemma I.3.4 in [16], it is enough to prove that if \((u_n)_n \subset \Delta \) and \( u_n \to 0 \) in \( L^2(\nu) \) then \( \mathcal{E}(u_n, v) \to 0 \) for any \( v \in \Delta \). But this is true, since, by (5.5), for all \( u \in \Delta \) there exists some \( \psi_u \in C_b(H) \) such that:
\[ \mathcal{E}(u, v) = \int \psi_u \, v \, d\nu, \quad \forall \, v \in \Delta. \]
We denote the closure of \((\mathcal{E}, \Delta)\) by \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\). We also obtain that \( \Psi_{n,\psi}(x) = \tilde{R}_n \psi(x) \) for all \( x \in K \) and \( n \in \mathbb{N} \), where \((\tilde{R}_\lambda)_{\lambda > 0}\) is the resolvent of \(\tilde{\mathcal{E}}\).

**Step 5.** Finally, we want to show that \((\mathcal{E}, \text{Exp}_D(H))\) is closable and that the closure coincides with \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\) constructed in the previous step. To this aim we show first that \(D(\tilde{\mathcal{E}})\) contains all Lipschitz functions on \( K \) and in particular \(\text{Exp}_D(H)\).

Consider \( \psi \in \text{Lip}(H) \subset D(\mathcal{E}) \): by the general theory of Dirichlet Forms,
\[ \psi \in D(\tilde{\mathcal{E}}) \iff \sup_{\lambda > 0} \int \lambda (\psi - \lambda \tilde{R}_\lambda \psi) \, d\nu < \infty. \]
By (2.2) we have:
\[ \int \lambda (\psi - \lambda R^n_\lambda \psi) \, d\nu^n + \mathcal{E}_n(\lambda R^n_\lambda \psi, \psi) \leq [\psi]^2_{\text{Lip}(H)}, \]
so that letting \( n \to \infty \):
\[ \int \lambda (\psi - \lambda \tilde{R}_\lambda \psi) \, d\nu \leq [\psi]^2_{\text{Lip}(H)}, \]
and therefore \( \text{Lip}(H) \subset D(\tilde{\mathcal{E}}) \). Since by construction \( \Delta \subset \text{Lip}(H) \), then the closure of \((\mathcal{E}, \text{Exp}_D(H))\) is \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\). Now, in order to prove the density of \(\text{Exp}_D(H)\) in \(D(\tilde{\mathcal{E}})\), we remark that the density with respect to the norm-topology is equivalent to the density in the weak topology, which follows from (5.4) and from the density of \( \text{Lip}(H) \) in \( D(\tilde{\mathcal{E}}) \).

Notice that the limit Dirichlet form \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\) does not depend on the subsequences \((n_j)_j\) and \((m_i)_i\) chosen in step 1, since it is the closure of \((\mathcal{E}, \text{Exp}_D(H))\). Then \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}})) = (\mathcal{E}, D(\mathcal{E})) \) is the Dirichlet form we wanted to construct and \( R_\lambda = \tilde{R}_\lambda \) is the associated resolvent operator. In particular the limit in (5.2) does not depend on the subsequence \((m_i)_i\) and
\[ R_\lambda \psi(x) = \lim_{n \to \infty} R^n_\lambda \psi(\Pi_{K_n}(x)) \quad \forall \, x \in K, \, \lambda \in \mathbb{N}. \]
We can now repeat the argument of step 1 and step 3 and obtain that the latter formula holds for all \( \lambda > 0 \). \((\mathcal{E}, D(\mathcal{E}))\) is a Dirichlet Form, because \( R^n_\lambda \) is given by a Markovian kernel, so that \( R_\lambda \) is also Markovian and the result follows from Theorem 4.4 of [16]. The Feller property follows from (2.2). By the density of \( \text{Lip}(H) \) in \( C_b(H) \), \( R^n_\lambda \psi(\Pi_{K_n}(x)) \) converges to \( R_\lambda \psi(x) \) for all \( \psi \in C_b(H) \). □
Proof of Theorem 2.1 We want to prove first that there exists a measurable kernel 
\( r_{\lambda}(x,A) : \lambda > 0, x \in K, A \in \mathcal{B}(H) \), such that for all \( \varphi \in C_b(H) \) and \( x \in K \):
\[
R_{\lambda} \varphi(x) = \int \varphi(y) r_{\lambda}(x,dy).
\]
(5.7)

By Lemma 4.1 there exists a limit \( \tilde{X} \) in distribution of \( \tilde{X}^n \) along a subsequence \((n_i)_i\).
Then for all \( \varphi, \psi \in C_b(H^{(2)}) \subset C_b(H) \):
\[
\int_0^\infty e^{-\lambda t} \mathbb{E}[\psi(\tilde{X}_0) \varphi(\tilde{X}_t)] dt = \lim_{i \to \infty} \int_0^\infty e^{-\lambda t} \mathbb{E}[\psi(\tilde{X}^{n_i}_0) \varphi(\tilde{X}^{n_i}_t)] dt
\]
\[
= \lim_{i \to \infty} \int \psi \, R_{\lambda}^n \varphi \, d\nu^{n_i} = \int \psi \, R_{\lambda} \varphi \, d\nu.
\]
Since this is true for any \( \psi \in C_b(H^{(2)}) \) we obtain:
\[
R_{\lambda} \varphi(x) = \int_0^\infty e^{-\lambda t} \mathbb{E}[\varphi(\tilde{X}_t) | \tilde{X}_0 = x] dt, \quad \nu \text{ a.e. } x.
\]

Therefore for \( \lambda > 0 \) fixed, the measurable kernel \( r_{\lambda}(x,\cdot) \) exists for \( \nu \)-a.e. \( x \), in particular for \( x \) in a set \( K_0 \) dense in \( K \). Let now any \( x \in K \setminus K_0 \) and \( \varphi_n \in C_b(H) \) a monotone sequence converging pointwise to \( \varphi \in C_b(H) \) and let \( x_m \in K \) converging to \( x \). Then \( R_{\lambda} \varphi_n(x_m) \) converges to \( R_{\lambda} \varphi(x_m) \) by Dominated Convergence. Since \( (R_{\lambda} \varphi_n)_n \) is equicontinuous, then necessarily also \( R_{\lambda} \varphi_n(x) \) must converge to \( R_{\lambda} \varphi(x) \), and the existence of \( r_{\lambda}(x,\cdot) \) follows from general measure theory.

We prove now convergence of \( P_t^n \varphi(\Pi K_n(x)) \) to \( P_t \varphi(x) \). Let \( \varphi \in \text{Lip}(H) \): by (2.2), for every \( t > 0 \) and for any sequence \( \varepsilon_n \to 0 \), we have pointwise convergence of \( [P_t^n \varphi] \circ \Pi K_n \) along a subsequence \((m_i)_i\) to \( F \in C_b(H) \); see step 1 of the proof of Proposition 5.1. We want to prove that \( F = P_t \varphi \). Let \( m_{n,\varphi} \) be the spectral measure of the generator of \( \mathcal{E}^n \) associated with \( \varphi \):
\[
\int_{-\infty}^0 \frac{1}{(\lambda-x)^{\ell}} m_{n,\varphi}(dx) = \int \varphi (R_{\lambda}^n)^{\ell} dx, \quad \lambda > 0, \ \ell \in \mathbb{N}.
\]

Analogously, we denote the spectral measure of the generator of \( \mathcal{E} \) associated with \( \varphi \) by \( m^\varphi \). By (5.7) and Lemma 5.2 we obtain that \( (R_{\lambda}^n)^{\ell} \) converges pointwise to \( (R_{\lambda})^{\ell} \), and therefore for all \( \lambda > 0, \ \ell \in \mathbb{N} \), by Proposition 5.1
\[
\int_{-\infty}^0 \frac{1}{(\lambda-x)^{\ell}} m_{n,\varphi}(dx) \xrightarrow{n \to \infty} \int_{-\infty}^0 \frac{1}{(\lambda-x)^{\ell}} m^\varphi(dx).
\]

By the Stone-Weierstrass theorem, the vector space spanned by the set of functions \( \{ x \mapsto (\lambda-x)^{-\ell}, \lambda > 0, \ \ell \in \mathbb{N} \} \) is dense in the set \( C_0((-\infty,0]) \) of continuous functions on \( (-\infty,0] \) vanishing at \( -\infty \). In particular we obtain:
\[
\int \varphi P_t^n \varphi \, d\nu^n = \int_{-\infty}^0 e^{tx} m_{n,\varphi}(dx) \xrightarrow{n \to \infty} \int_{-\infty}^0 e^{tx} m^\varphi(dx) = \int \varphi P_t \varphi \, d\nu.
\]

By polarization we have for all \( \varphi, \psi \in C_b(H) \):
\[
\int \psi P_t^n \varphi \, d\nu^n \xrightarrow{n \to \infty} \int \psi P_t \varphi \, d\nu
\]
and we conclude that \( \lim_{n \to \infty} P_t^n \varphi(\Pi K_n(x)) = P_t \varphi(x) \) for all \( x \in K \). By the density of \( \text{Lip}(H) \) in \( C_b(H) \), \( P_t^n \varphi(\Pi K_n(x)) \) converges to \( P_t \varphi(x) \) for all \( \varphi \in C_b(H) \).
Arguing like for the existence of the kernel $r_\lambda$, we can prove that there exists a measurable kernel $(p_t(x, A) : t \geq 0, x \in K, A \in \mathcal{B}(H))$, such that:

$$P_t \varphi(x) = \int \varphi(y) p_t(x, dy), \quad \forall x \in K, \quad \varphi \in C_b(H).$$

By Lemma 5.2 we can now prove the convergence stated in the third assertion of Theorem 2.1, obtaining also that $(p_t(x, dy))_{t,x}$ satisfies the Chapman-Kolmogorov equation. By the Kolmogorov extension Theorem we have existence of the Markov process $X$. Convergence of $P_t \varphi(x)$ to $\varphi(x)$ follows from the strong continuity of $P_t$ in $L^2(\nu)$, the estimate $|P_t \varphi|_{\text{Lip}(H)} \leq c |\varphi|_{\text{Lip}(H)}$ which follows from (2.2) and the density of $\text{Lip}(H)$ in $C_b(H)$. The closability of $\mathcal{E}$ has been proven in Proposition 5.1. The last assertion of Theorem 2.1 follows from Lemma 4.1 and the third assertion. □

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