Affine bundles are affine spaces over modules

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Abstract. We show that the category of affine bundles over a smooth manifold \(M\) is equivalent to the category of affine spaces modelled on projective finitely generated \(C^\infty(M)\)-modules. Using this equivalence of categories, we are able to give an alternate proof of the main result of [12], showing that the characterization of vector bundles by means of their Lie algebras of homogeneous differential operators also holds for vector bundles of rank 1 and over any base manifolds.

Key words: affine bundle; vector bundle; affine space; module

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1 Introduction

Affine bundles appear in differential geometry mainly from the geometry of a frame-independent formulation of classical mechanics (see for instance [6, 7, 5, 8, 21, 14, 13, 2, 1, 17, 16]) and in the study of jet bundles (see [18, 9]).

As affine spaces are modelled on vector spaces, each affine bundle has an underlying vector bundle. This being said, remember (see [20, 15]) that the functor of global sections provides an equivalence of categories between vector bundles over a smooth manifold \(M\) and projective finitely generated \(C^\infty(M)\)-modules.

In this text, we first emphasize the similarities between affine bundles and affine spaces over modules using the language of torsors (see [3, 4]). Then we show that the category of affine bundles over a smooth manifold \(M\) is equivalent to the category of affine spaces modelled on projective finitely generated \(C^\infty(M)\)-modules.

Finally, by means of this equivalence of categories and of the concept of smooth envelope of an \(\mathbb{R}\)-algebra (see [19]), we are able to give an alternate proof of the main result of [12], showing that the characterization of vector bundles by means of their Lie algebras of homogeneous differential operators (resp. by means of their \(\mathbb{R}\)-algebras of fiberwise polynomial functions) also holds for vector bundles of rank 1 and without any topological assumption about the base manifolds.

2 Affine spaces

First, we translate the definitions of affine spaces modelled on vector spaces and affine maps between them into the language of torsors. This allows us to extend in a natural way the definition to affine spaces over arbitrary modules.

2.1 Affine spaces modelled on vector spaces

Let \(V\) be a vector space. Remember that an affine space modelled on \(V\) is a (nonempty) set \(A\) on which \(V\) acts freely and transitively, i.e. together with a map

\[ t : V \times A \to A : (v, a) \mapsto t_v(a) \]

such that for any two \(a_0, a \in A\), there is a unique \(v \in V\) such that \(a = t(v, a_0)\). We often write \(a + v\) instead of \(t_v(a)\) and \(a - a_0\) for the unique \(v\) “moving” \(a_0\) to \(a\).
Note that the above definition of an affine space makes no use of scalar multiplication in the vector space. What is required is a free and transitive group action: an affine space is nothing but a torsor under the action of a special kind of group (a vector space). We are thus dealing with torsors under the action of groups with an additional structure: scalar multiplication. Morphisms of affine spaces are morphisms of torsors that are compatible with this richer structure: if $V$ and $V'$ are vector spaces and if $A$ and $A'$ be affine spaces respectively modelled on $V$ and $V'$, a map $f : A \to A'$ is called an affine map if there is a linear map $\tilde{f} : V \to V'$ such that the induced action of $V$ on $A'$,

$$V \times A' \to A'(v, a') \mapsto \tilde{f}(v) \cdot a',$$

turns $f : A \to A'$ into a $V$-morphism of torsors, i.e., $f(a_0 + v) = f(a_0) + \tilde{f}(v)$ for all $a_0 \in A$ and all $v \in V$.

### 2.2 Affine spaces modelled on modules

In view of what precedes, nothing prevents us from defining affine spaces modelled on modules over arbitrary commutative rings.

**Definition 1.** Let $R$ be a commutative ring and $P$ be an $R$-module. An $R$-affine space modelled on $P$ is a $P$-torsor, i.e., a set $A$ together with a free and transitive group action

$$t : P \times A \to A : (p, a) \mapsto t_p(a).$$

We also write $a + p$ instead of $t_p(a)$ and $a - a_0$ for the unique $p$ “moving” $a_0$ to $a$.

**Definition 2.** Let $R$ be a commutative ring, $P$ and $P'$ be $R$-modules. If $A$ and $A'$ are affine spaces respectively modelled on $P$ and $P'$, a map $f : A \to A'$ is called an $R$-affine map if there is an $R$-linear map $\tilde{f} : P \to P'$ (the $R$-linear part of $f$) such that the induced action of $P$ on $A'$,

$$P \times A' \to A' : (p, a') \mapsto \tilde{f}(p) \cdot a',$$

turns $f : A \to A'$ into a $P$-morphism of torsors.

The composition of $R$-affine maps is obviously an $R$-affine map. Also the identity maps $\text{id}_A : A \to A$ are $R$-affine maps. We denote by $\text{AS}(\text{Mod } R)$ the category of $R$-affine spaces and $R$-affine maps between them.

**Remark 1.** The category $\text{AS}(\text{Mod } R)$ is not a subcategory of the category of torsors because $R$-affine maps are not necessarily equivariant maps. Roughly speaking, equivariance is “shifted” by the $R$-linear map $\tilde{f}$.

### 3 Vector bundles

**Definition 3.** Let $M$ be a connected $m$-dimensional smooth manifold. A fiber bundle $\eta : E \to M$ is a vector bundle if

1. there is vector space $V$ ((the typical fiber) such that for any $x \in M$, the fiber $\eta_x$ is a vector space isomorphic to $V$;

\[^1\text{Since we consider modules over a commutative ring, we don’t distinguish between left and right modules.}\]
2. the base manifold $M$ is entirely covered with the domains $U$ of fiberwise linear local trivializations $\Phi_U : \eta^{-1}(U) \xrightarrow{\sim} U \times V$:

$$
\begin{array}{ccc}
\eta^{-1}(U) & \xrightarrow{\Phi_U} & U \times V \\
\eta & \downarrow & \downarrow \text{pr}_1 \\
U & \rightarrow & 
\end{array}
$$

Any vector bundle $\eta : E \to M$ admits a canonical global section: the zero section. The latter associates to any point $x \in M$ the origin $0_x$ of the vector space $\eta_x$.

**Remark 2.** Any point in the total space of a vector bundle $\eta : E \to M$ can be reached by a global section. Indeed, any point in the total space of a fiber bundle can be reached by a local section (see [11]) while multiplication by a “bump” function on $M$ allows use to globalize such a local section.

**Proposition 1.** Any vector bundle is a group object in the category $\text{FB}(M)$ of fiber bundles over $M$.

**Proof.** Define a triple $(m, \text{inv}, \epsilon)$ as follows:

$$
m : E \times_M E \to E : (e, e') \mapsto e + e' \in \eta_x ;
$$

$$
\epsilon : M \to E : x \mapsto 0_x \in \eta_x ;
$$

$$
\text{inv} : E \to E : e \mapsto -e \in \eta_x .
$$

These maps are smooth and fiber-preserving. Moreover, they define a group structure on each fiber since they are built fiberwise from vector space structures. It follows that $\eta : E \to M$ is indeed a (particular kind of) group object in $\text{FB}(M)$.

Vector bundles are fiber bundles with a richer structure: fibers are vector spaces. A morphism of vector bundles is not any morphism of fiber bundles: it must be compatible with the vector structure.

**Definition 4.** An $M$-morphism of vector bundles from $\eta : E \to M$ to $\eta' : E' \to M$ is a fiberwise linear $M$-morphism of fiber bundles, i.e. an $M$-morphism such that all restrictions $f|_{\eta_x} : \eta_x \to \eta'_x$ are linear maps.

Vector bundles over $M$ and $M$-morphisms between them form a category, denoted by $\text{VB}(M)$. This category is well-known to be equivalent to a certain category of modules:

**Proposition 2.** The functor of global sections $\Gamma$, from the category $\text{VB}(M)$ of vector bundles over $M$ to the category $\text{Mod}_{\text{fg}} C^\infty(M)$ of projective finitely generated $C^\infty(M)$-modules is an equivalence of categories.

**Proof.** See [20, 15].

**Remark 3.** It is not useful here to enter into details about the notion of projective finitely generated modules. The interested reader is invited to consult [20, 15].
4 Affine bundles

**Definition 5.** Let \( M \) be a connected \( m \)-dimensional smooth manifold and \( \eta : E \to M \) be a vector bundle. A fiber bundle \( \pi : Z \to M \) is an **affine bundle** modelled on \( \eta \) if

1. there is an affine space \( A \) (the **typical fiber**, modelled on the typical fiber \( V \) of \( \eta \)) such that for any \( x \in M \), the fiber \( \pi_x \) is an affine space modelled on \( \eta_x \), isomorphic to \( A \);

2. the base manifold \( M \) is entirely covered with the domains \( U \) of fiberwise affine local trivializations \( \Phi_U : \pi^{-1}(U) \simto U \times A \):

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times A \\
\downarrow & & \downarrow \text{pr}_1 \\
U & & \\
\end{array}
\]

**Remark 4.** Since fibers in an affine bundle are affine spaces, they don’t come with a preferred origin. Consequently, no canonical global section shows up. However, since the fibers are diffeomorphic to a euclidean space, it follows from [19] that any affine bundle admits global sections.

**Proposition 3.** Any affine bundle \( \pi : Z \to M \) modelled on \( \eta : E \to M \) is a torsor under the action of the group object \( \eta \) in the category \( \text{FB}(M) \).

**Proof.** Fiberwise translations in all fibers define a map

\[
t : E \times_M Z \to Z \\
(e, z) \mapsto z + e \in \pi_x(z)
\]

This map is smooth. Indeed, there exists local adapted coordinates in which \( t \) reads

\[
t(x^i; e^\beta; z^\alpha) = (x^i; z^\alpha + e^\alpha)
\]

It makes \( \pi : Z \to M \) a torsor under the group object \( \eta : E \to M \) in the category \( \text{FB}(M) \) since by construction, the restriction of \( t \) on fibers is build from free and transitive actions. \( \blacksquare \)

**Proposition 4.** Conversely, any torsor \( \pi : Z \to M \) under the action of a vector bundle \( \eta : E \to M \) is an affine bundle modelled on \( \eta \).

**Proof.** The action making \( \pi : Z \to M \) a torsor under \( \eta : E \to M \) defines on each fiber \( \pi_x \) the structure of an affine space over the corresponding vector space \( \eta_x \). Take \( A \) to be any affine space modelled on the typical fiber \( V \) of \( \eta \) and build fiberwise affine trivializations from the choice of any global section \( s_0 : M \to Z \) of \( \pi \) and from fiberwise linear trivializations of \( \eta \). \( \blacksquare \)

**Definition 6.** Let \( \pi : Z \to M \) and \( \pi' : Z' \to M \) be affine bundles modelled on \( \eta : E \to M \). An \( \eta \)-**morphism of affine bundles** from \( \pi : Z \to M \) to \( \pi' : Z' \to M \) is an \( \eta \)-morphism of torsors.

**Proposition 5.** A smooth map \( f : Z \to Z' \) is an \( \eta \)-morphism of affine bundles if and only if it is an \( M \)-morphism of fiber bundles such that all restrictions \( f|_{\pi_x} : \pi_x \to \pi'_x \) are \( \eta_x \)-affine maps.

**Proof.** The equivariance of \( f \) under the action of the vector bundle translates fiberwise into equivariance of the restrictions \( f_x : \pi_x \to \pi'_x \). \( \blacksquare \)

Affine bundles are fiber bundles with some extra structure: fibers are affine spaces. A morphism of affine bundles is not any morphism of fiber bundles: it must be compatible with the affine structure.
Definition 7. Let $\pi : Z \to M$ and $\pi' : Z' \to M$ be affine spaces modelled on $\eta : E \to M$ and $\eta' : E' \to M$. An $M$-morphism of affine bundles from $\pi : Z \to M$ to $\pi' : Z' \to M$ is fiberwise affine $M$-morphism of fiber bundles, i.e., an $M$-morphism such that all restrictions $f|_{\pi_x} : \pi_x \to \pi'_x$ are affine maps.

The following proposition shows that the above definition corresponds in some sense to comparing torsors under the action of two different vector bundles over $M$: morphisms of affine bundles have to be compatible with the structure of the underlying vector bundles.

Proposition 6. An $M$-morphism of fiber bundles $f : Z \to Z'$ is an $M$-morphism of affine bundles if and only if there is an $M$-morphism of vector bundles $\tilde{f}$ (the linear part of $f$) from $\eta : E \to M$ and $\eta' : E' \to M$ such that for any $x \in M$, the restriction $f|_{\pi_x} : \pi_x \to \pi'_x$ is an affine map with linear part $f|_{\eta_x} : \eta_x \to \eta'_x$.

Proof. We define $\tilde{f} : E \to E'$ on each fiber $\eta_x$ as the linear part of $f|_{\pi_x} : \pi_x \to \pi'_x$ and we check easily that it is indeed an $M$-morphism of vector bundles. ■

Affine bundles over $M$ and $M$-morphisms between them form a category, denoted by $\text{AB}(M)$.

Remark 5. The category $\text{AB}(M)$ and affine maps is not a subcategory of the category of torsors under the action of vector bundles since morphisms are not necessarily equivariant maps. Here, equivariance is “shifted” by an $M$-morphism of vector bundles.

5 An equivalence of categories

Both affine bundles and affine spaces over projective finitely generated $C^\infty(M)$-modules were shown above to be torsors under the action of particular group objects. Also morphisms in both cases were required to preserve the same kind of “linear property” of these group objects. Finally, these group objects live in two equivalent categories [2, 15]. Let us show that the two affine concepts inherit this equivalence of categories.

Lemma 1. Let $M$ be a connected $m$-dimensional smooth manifold and $\pi : Z \to M$ be an affine bundle modelled on a vector bundle $\eta : E \to M$. The space of global sections $\Gamma(\pi)$ is in a canonical way an affine space over the $C^\infty(M)$-module $\Gamma(\eta)$.

Proof. Fiberwise translations in all fibers define an action

$$t : \Gamma(\eta) \times \Gamma(\pi) \to \Gamma(\pi) : (u, s) \mapsto s + u$$

where the value of the section $s + u$ at a point $x \in M$ is given by

$$(s + u)(x) := s(x) + u(x) \in \pi_x$$

This action free and transitive since it is build fiberwise from affine spaces structures. ■

Lemma 2. Any point $z$ in the total space of an affine bundle $\pi : Z \to M$ can be reached by a global section.

Proof. Having fixed a global section $s_0 : M \to Z$, it suffices to find a global section $u$ of the vector bundle $\eta : E \to M$ passing through $z - s_0(\pi(z)) \in \eta_{\pi(z)}$ because then $s_0 + u$ does the job. Since Remark 2 ensures the existence of such a section $u$, the Lemma follows. ■

Theorem 1. The functor of global sections $\Gamma$, from the category $\text{AB}(M)$ of affine bundles over $M$ to the category $\text{AS}(\text{Mod}_{pf} C^\infty(M))$ of affine spaces over projective finitely generated $C^\infty(M)$-modules is an equivalence of categories.
Proof. Remember that a functor is an equivalence of categories if and only if it is essentially surjective and fully faithful.

First, the functor \( \Gamma \) is essentially surjective. Let \( A \) be a \( C^\infty(M) \)-affine space modelled on a projective finitely generated module \( P \). From the equivalence of categories between vector bundles and projective finitely generated \( C^\infty(M) \)-modules, there is a vector bundle \( \eta : E \to M \) and a \( C^\infty(M) \)-isomorphism \( T : P \xrightarrow{\sim} \Gamma(\eta) \). The \( C^\infty(M) \)-module \( \Gamma(\eta) \) is also a \( C^\infty(M) \)-affine space modelled on itself. Picking any point \( a_0 \in A \), we define an isomorphism of \( C^\infty(M) \)-affine spaces \( T : A \xrightarrow{\sim} \Gamma(\eta) \) by setting \( T(a_0 + p) := T(p) \).

The functor \( \Gamma \) is also faithful. Let \( \alpha \) and \( \beta \) be \( M \)-morphisms of affine bundles from \( \pi : Z \to M \) to \( \pi' : Z' \to M \), such that \( \Gamma(\alpha) = \Gamma(\beta) \). This equality means that \( \alpha \circ s = \beta \circ s \) for every \( s \in \Gamma(\pi) \). It follows that \( \alpha = \beta \) since Lemma 2 ensures that any point in the total space of an affine bundle can be reached by a global section.

Finally, the functor \( \Gamma \) is full. Let \( \pi : Z \to M \) and \( \pi' : Z' \to M \) be affine bundles modelled on \( \eta : E \to M \) and \( \eta' : E' \to M \), respectively. Let \( T : \Gamma(\pi) \to \Gamma(\pi') \) be a \( C^\infty(M) \)-affine map. The linear part \( \bar{T} : \Gamma(\eta) \to \Gamma(\eta') \) is a \( C^\infty(M) \)-homomorphism. From the equivalence of categories between vector bundles and projective finitely generated \( C^\infty(M) \)-modules, there exists an \( M \)-morphism of vector bundles \( f \) from \( \eta : E \to M \) to \( \eta' : E' \to M \), such that \( \bar{T} = \Gamma(f) \). Picking any global section \( s_0 \in \Gamma(\pi) \), we define an \( M \)-morphism of affine bundles \( \alpha : Z \to Z' \) by setting \( \alpha(z) := T(s_0)(\pi(z)) + f(z - s_0(\pi(z))) \). Obviously, \( \Gamma(\alpha) = T \).

6 An algebraic characterization for vector bundles

6.1 Fiberwise polynomial functions on vector bundles

Let \( \pi : E_\pi \to M \) be a vector bundle over \( M \). The pullback map \( \pi^* \) determines an isomorphism of \( \mathbb{R} \)-algebras \( C^\infty(M) \to Pol^0(\pi) \). This way, multiplication by elements of \( Pol^0(\pi) \) endows every \( Pol^k(\pi) \) (\( k \in \mathbb{N} \)) with a natural \( C^\infty(M) \)-module structure: for all \( g \in C^\infty(M) \), \( f \in Pol^k(\pi) \),

\[
gf := \pi^*(g)f \in Pol^k(\pi) \,.
\]

In particular when \( k = 1 \), this \( C^\infty(M) \)-module is isomorphic to the dual module \( \Gamma(\pi)^\vee \) of the module of (global) sections of \( \pi \). For a function \( f \in Pol^1(\pi) \), the corresponding element \( \xi_f \in \Gamma(\pi)^\vee \) is the onesuch that for any \( s \in \Gamma(\pi) \) and any \( x \in M \), we have

\[
\xi_f(s)(x) = f(s(x)) \,.
\]

Proposition 7 \((12)\). Let \( \pi : E_\pi \to M \) and \( \eta : E_\eta \to N \) be two vector bundles. Every isomorphism of \( \mathbb{R} \)-algebras \( \Psi : Pol(E_\pi) \to Pol(E_\eta) \) is filtered with respect to the filtrations of \( Pol(E_\pi) \) and \( Pol(E_\eta) \) associated with their gradings.

6.2 Smooth envelopes and lifting of isomorphisms

Let \( A \) be a \( \mathbb{R} \)-algebra, i.e. an associative algebra over \( \mathbb{R} \) which is commutative and has a unit element \( 1_A \). Let \( \text{Spec}_\mathbb{R} A \) denote the set of all \( \mathbb{R} \)-algebra homomorphisms \( A \to \mathbb{R} \). The set \( \text{Spec}_\mathbb{R} A \) is by definition the \( \mathbb{R} \)-spectrum, while the elements of \( \text{Spec}_\mathbb{R} A \) will sometimes be called \( \mathbb{R} \)-points of the algebra \( A \). Elements of \( A \) can be viewed as functions on \( \text{Spec}_\mathbb{R} A \). More precisely, we can associate a function with each \( a \in A \), namely the function

\[
f_a : \text{Spec}_\mathbb{R} A \to \mathbb{R} : h \mapsto f_a(h) := h(a) \,.
\]

This procedure supply thus any abstract algebra \( A \) with a geometrization as an algebra of functions on the space \( \text{Spec}_\mathbb{R} A \).
**Example 1.** In the case where $A$ is the algebra of smooth functions on a smooth manifold, the $\mathbb{R}$-spectrum identifies with the manifold itself (see [15 paragraph 7.2]).

Consider the subalgebra $A_{\text{geom}} := \{ f_a : a \in A \}$ of the $\mathbb{R}$-algebra $\mathcal{F}(A)$ of all (real-valued) functions on $\text{Spec}_\mathbb{R} A$. The map $A \to A_{\text{geom}}, a \mapsto f_a$ is always a surjective $\mathbb{R}$-homomorphism. When it is injective, we say that the algebra $A$ is geometric. A geometric algebra is thus an algebra which is canonically isomorphic to an algebra of functions through the above homomorphism.

**Lemma 3.** An $\mathbb{R}$-algebra $A$ is geometric if and only if the ideal $I_A := \bigcap_{h \in \text{Spec}_\mathbb{R} A} \ker h$ is trivial. In particular, for any vector bundle $\pi : E_\pi \to M$, $A = C^\infty(E_\pi)$ and $A = \text{Pol}(\pi)$ are geometric.

**Proof.** The first statement follows from the fact that we have

$$f_a = 0 \iff \forall h \in \text{Spec}_\mathbb{R} A, f_a(h) = h(a) = 0 \iff a \in \bigcap_{h \in \text{Spec}_\mathbb{R} A} \ker h .$$

In the case $A = C^\infty(E_\pi)$ or $A = \text{Pol}(\pi)$, the ideal $I_A$ is a subset of $\bigcap_{x \in M} \ker e_v x = 0$ and $I_A$ is thus trivial because only the zero function vanishes at all points of $E_\pi$.

**Definition 8 ([15]).** A geometric $\mathbb{R}$-algebra $A$ (viewed as an algebra of functions) is $C^\infty$-closed if for any for any finite collection of its elements $f_1, \ldots, f_k$ and any function $g \in C^\infty(\mathbb{R}^k)$, there exists an element $f \in A$ such that

$$g(f_1(x), \ldots, f_k(x)) = f(x) \quad \text{for all } x \in \text{Spec}_\mathbb{R} A .$$

**Example 2.** For any vector bundle $\pi : E_\pi \to M$, the algebra $A = C^\infty(E_\pi)$ is $C^\infty$-closed. Indeed, for any finite collection of elements $f_1, \ldots, f_k \in C^\infty(E_\pi)$ and any function $g \in C^\infty(\mathbb{R}^k)$, the composition of the smooth maps $(f_1, \ldots, f_k) : E_\pi \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}$ belongs to $C^\infty(E_\pi)$.

**Definition 9 ([15 paragraph 3.32]).** Let $A$ be a geometric $\mathbb{R}$-algebra. A $C^\infty$-closed $\mathbb{R}$-algebra $\bar{A}$ together with a homomorphism $i : A \to \bar{A}$ is a smooth envelope of $A$ if for any homomorphism $\alpha : A \to B$ from $A$ into a $C^\infty$-closed $\mathbb{R}$-algebra $B$, there exists a unique homomorphism $\bar{\alpha} : \bar{A} \to B$ extending $\alpha$. In other words, under the above assumptions, the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow \scriptstyle{i} & & \downarrow \scriptstyle{\bar{\alpha}} \\
\bar{A} & \xrightarrow{\bar{\alpha}} & B
\end{array}$$

can always be uniquely completed (by the dotted arrow $\bar{\alpha}$) to a commutative one.

**Proposition 8.** For any vector bundle $\pi : E_\pi \to M$, the algebra $C^\infty(E_\pi)$ (together with the inclusion $i_\pi : \text{Pol}(\pi) \to C^\infty(E_\pi)$) is a smooth envelope of $\text{Pol}(\pi)$.

**Proof.** See [15 paragraph 11.58].

**Proposition 9.** Let $\pi : E_\pi \to M$ and $\eta : E_\eta \to M$ be two vector bundles over $M$. For any $\mathbb{R}$-isomorphism $\Psi : \text{Pol}(E_\pi) \to \text{Pol}(E_\eta)$, there is a unique $\mathbb{R}$-homomorphism $\Psi : C^\infty(E_\pi) \to C^\infty(E_\eta)$ extending $\Psi$. In other words, under the above assumptions, the diagram

$$\begin{array}{ccc}
\text{Pol}(\pi) & \xrightarrow{\Psi} & \text{Pol}(\eta) \\
\downarrow \scriptstyle{i_\pi} & & \downarrow \scriptstyle{i_\eta} \\
C^\infty(E_\pi) & \xrightarrow{\Psi} & C^\infty(E_\eta)
\end{array}$$

can always be uniquely completed (by the dotted arrow $\Psi$) to a commutative one. Moreover, $\Psi$ is an $\mathbb{R}$-isomorphism.
In this case, Ψ is an isomorphism, we shall find an inverse homomorphism. Let Ψ be the homomorphism $C^\infty(E_\eta) \to C^\infty(E_\pi)$ obtained from the universal property of $Pol(\eta)$ with $\alpha = i_\eta \circ \Psi : Pol(\pi) \to C^\infty(E_\eta)$.

To show that Ψ is an isomorphism of $\mathbb{R}$-algebras, then their total spaces are diffeomorphic.

**Corollary 1.** If two vector bundles have their algebras of fiberwise polynomial functions isomorphic (as $\mathbb{R}$-algebras), then their total spaces are diffeomorphic.

### 6.3 An algebraic characterization

**Theorem 2.** Let $\pi : E_\pi \to M$ and $\eta : E_\eta \to M$ be two vector bundles over $M$. A map $\Psi : Pol(\pi) \to Pol(\eta)$ satisfying

$$\Psi \circ \pi^* = \eta^*$$

is an isomorphism of $\mathbb{R}$-algebras if and only if there exists an isomorphism of affine bundles $\alpha : \pi \to \eta$ over id$_M$ such that

$$\Psi = \alpha^{-1}|_{Pol(\pi)}$$

In this case, Ψ is filtered and the induced graded isomorphism $\Psi^\text{gr} : Pol(\pi) \to Pol(\eta)$ is related to $\alpha$ via the formula

$$\Psi^\text{gr} = \bar{\alpha}^{-1}|_{Pol(\pi)}$$

where the linear part $\bar{\alpha} : \pi \to \eta$ of $\alpha$ is an isomorphism of vector bundles over $M$.

**Proof.** The existence and uniqueness of $\Psi$ is a direct consequence of Proposition 3 use the universal property of the smooth envelope of $Pol(\pi)$ with $\alpha = i_\eta \circ \Psi : Pol(\pi) \to C^\infty(E_\eta)$.

Starting from id$_{Pol(\pi)}$, we know that Ψ is necessarily filtered. It follows that its restriction $\Psi^{\leq 1} : Pol^{\leq 1}(\pi) \to Pol^{\leq 1}(\eta)$ is a bijection, while hypothesis 3 ensures that this map is $C^\infty(M)$-linear: for any $g \in C^\infty(M), f \in Pol^{\leq 1}(\pi)$, we have

$$\Psi^{\leq 1}(g.f) = \Psi(\pi^*(g))\Psi(f) = \eta^*(g)\Psi(f) = g.\Psi^{\leq 1}(f)$$

Now, the restriction $\Psi|_{Pol^{\leq 1}(\pi)}$ induces a $C^\infty(M)$-homomorphism

$$\Gamma(\pi)^\vee \cong Pol^{\leq 1}(\pi) \hookrightarrow Pol^{\leq 1}(\eta)$$

whose dual map yields a $C^\infty(M)$-homomorphism $Pol^{\leq 1}(\eta)^\vee \to \Gamma(\pi)$ since the module $\Gamma(\pi)$ coincide with its bidual.\(^2\) Composing with the $C^\infty(M)$-affine map $\Gamma(\eta) \to \Gamma(\pi)$.

\(^2\) Indeed, the dual of a module of sections can be seen as the module of sections of the dual bundle, where the duality is that of finite-dimensional vector spaces.

\(^3\) This affine map appears in the study of vector hulls of affine spaces and affine bundles [10, 6].
In view of Theorem 1 there is a unique morphism of affine bundles \( \beta : \eta \to \pi \) over \( \text{id}_M \) for which \( C^\infty(M) \)-affine morphism \( T : \Gamma(\eta) \to \Gamma(\pi) \) reads \( T(t) = \beta \circ t \) for all \( t \in \Gamma(\eta) \). Let us show that we have \( \Psi = \beta^*|_{\text{Pol}(\pi)} \) and \( \Psi^{gr} = \tilde{\beta}^*|_{\text{Pol}(\pi)} \).

Since every point of the total space \( E_\eta \) can be reached by a global section of \( \eta \), it is enough to show \( f \circ \beta(t(x)) = \Psi(f)(t(x)) \) and \( f \circ \tilde{\beta}(t(x)) = \Psi^{gr}(f)(t(x)) \) for all \( f \in \text{Pol}^1(\pi), t \in \Gamma(\eta), x \in M \).

On the one hand, we compute successively

\[
\begin{align*}
\Psi(f)(t(x)) &= \xi_f(\beta \circ t)(x) \\
&= \xi_f(T(t))(x) \\
&= i_{\Gamma(\eta)}(t)(F(\xi_f))(x) \\
&= F(\xi_f) \circ t(x) \\
&\equiv \Psi(f)(t(x)) .
\end{align*}
\]

On the other hand, we have also

\[
\begin{align*}
\Psi^{gr}(f)(t(x)) &= \xi_f(\tilde{\beta} \circ t)(x) \\
&= \xi_f(\tilde{T}(t))(x) \\
&= i_{\Gamma(\eta)}(t)(F(\xi_f))(x) \\
&= \text{pr}^1(F(\xi_f)) \circ t(x) \\
&\equiv \Psi^{gr}(f)(t(x)) .
\end{align*}
\]

Finally, since \( C^\infty(E_\pi) \) (resp. \( C^\infty(E_\eta) \)) is the smooth envelope of \( \text{Pol}(\pi) \) (resp. \( \text{Pol}(\eta) \)), Proposition 9 ensures that \( \Psi : \text{Pol}(\pi) \to \text{Pol}(\eta) \) extends in a unique way to an \( \mathbb{R} \)-homomorphism \( C^\infty(E_\pi) \to C^\infty(E_\eta) \), this unique extension being an isomorphism because \( \Psi \) is. Since \( \beta^* : f \mapsto f \circ \beta \) is such an extension, it is an isomorphism and \( \beta \) is thus a diffeomorphism. Setting \( \alpha := \beta^{-1} \), we get the announced isomorphism of affine bundles.

**Corollary 2.** Two vector bundles \( \pi : E_\pi \to M \) and \( \eta : E_\eta \to N \) are isomorphic if and only if their algebras of fiberwise polynomial functions are isomorphic (as \( \mathbb{R} \)-algebras).

**Proof.** Let \( \Psi : \text{Pol}(\pi) \to \text{Pol}(\eta) \) be an isomorphism of \( \mathbb{R} \)-algebras. If \( M = N \) and if (3) holds, the result is an immediate consequence of Theorem 2.

In the general case, it follows from [12, Lemma1] that \( \Psi \) induces a diffeomorphism \( g : M \to N \). The pullback bundle \( g^*\eta : E_{g^*\eta} \to M \) comes with an isomorphism of vector bundles \( \tilde{g} : g^*\eta \to \eta \) while the map \( \tilde{g}^* \circ \Psi : \text{Pol}(\pi) \to \text{Pol}(g^*\eta) \) is an \( \mathbb{R} \)-isomorphism satisfying (3). We can thus invoke Theorem 2 to show that the bundles \( \pi \) and \( g^*\eta \) are isomorphic too, hence the conclusion.

**Corollary 3.** Two vector bundles are isomorphic if and only if their Lie algebras of homogeneous differential operators are isomorphic (as Lie algebras).

**Proof.** It follows immediately from [12, Proposition 4] and Corollary 2.

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**A Torsors**

**A.1 Torsors under the action of groups**

**Definition 10.** Let \( G \) be a group. A torsor under the action of \( G \) is a (nonempty) set \( X \) on which \( G \) acts freely and transitively, i.e. together with a map

\[
\gamma : G \times X \to X : (g, x) \mapsto g \cdot x
\]

such that for any two \( x, y \in X \), there is a unique \( g \in G \) such that \( y = g \cdot x \).
Asking for the action to be free and transitive amounts to require that the map
\[
\gamma \times \text{id}_X : G \times X \to X \times X : (g, x) \mapsto (g \cdot x, x)
\]
is a bijection (i.e. an isomorphism in the category of sets).

Torsors being spaces endowed with a group action, there are natural candidates for morphisms: equivariant maps.

**Definition 11.** Let \(X\) and \(X'\) be two torsors under the action of \(G\). A map \(f : X \to X'\) is a \(G\)-morphism of torsors if it is \(G\)-equivariant, i.e. for any \(x \in X\) and any \(g \in G\),
\[
f(g \cdot x) = g \cdot f(x).
\]
In other words, we ask for the following diagram to be commutative:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\gamma} & X \\
\downarrow{\text{id}_G \times f} & & \downarrow{f} \\
G \times X & \xrightarrow{\gamma'} & X'
\end{array}
\]

More generally, we can define morphisms of torsors under the actions of two different groups.

**Definition 12.** Let \((X, \gamma)\) and \((X', \gamma')\) be torsors under the actions of \(G\) and \(G'\), respectively. A map \(f : X \to X'\) is a morphism of torsors if there is a group homomorphism \(\bar{f} : G \to G'\) such that the induced action of \(G\) on \(X'\),
\[
G \times X' \to X' : (g, x') \mapsto \bar{f}(g) \cdot x'
\]
turns \(f : X \to X'\) into a \(G\)-morphism of torsors.

### A.2 Group objects in a category

Let \(\mathcal{C}\) be a category with finite products (i.e. \(\mathcal{C}\) has a terminal object \(1\) and any two objects of \(\mathcal{C}\) have a product). A group object in \(\mathcal{C}\) is an object \(G\) in \(\mathcal{C}\) together with three morphisms:

- the group multiplication \(m : G \times G \to G\);
- the inclusion of the unit element \(\epsilon : 1 \to G\);
- the inversion operation \(\text{inv} : G \to G\).

These morphisms must fulfill the following requirements:

- multiplication is associative, i.e. \(m \circ (m \times \text{id}_G) = m \circ (\text{id}_G \times m)\) as morphisms \(G \times G \times G \to G\);
- inclusion of the unit is a two-sided unit of \(m\), i.e. \(m \circ (\text{id}_G \times \epsilon) = \text{pr}_1\), where \(\text{pr}_1 : G \times 1 \to G\) is the canonical projection, and \(m \circ (\epsilon \times \text{id}_G) = \text{pr}_2\), where \(\text{pr}_2 : 1 \times G \to G\) is the canonical projection;
- inversion operation is a two-sided inverse for \(m\), i.e. if \(d : G \to G \times G\) is the diagonal map, and \(\epsilon_G : G \to G\) is the composition of the unique morphism \(G \to 1\) (also called the counit) with \(\epsilon\), then \(m \circ (\text{id}_G \times \text{inv}) \circ d = \epsilon_G\) and \(m \circ (\text{inv} \times \text{id}_G) \circ d = \epsilon_G\).

\[^4\text{We identify } G \times (G \times G) \text{ with } (G \times G) \times G \text{ in a canonical manner.}\]
A.3 Torsors under the action of group objects

Let $G$ be a group object in $\mathcal{C}$. Let $X$ be an object in $\mathcal{C}$. A group action of $G$ on $X$ is a morphism $\gamma : G \times X \to X$ in $\mathcal{C}$ such that $\gamma \circ (\epsilon \times \text{id}_X) = \text{id}_X$ and $\gamma \circ (m \times \text{id}_X) = \gamma \circ (\text{id}_G \times \gamma)$ as morphisms $G \times G \times X \to G$.

Definition 13. A torsor under the action of $G$ is a nonempty object $X$ on which $G$ acts freely and transitively, i.e. the induced map $\gamma \times \text{id}_X : G \times X \to X \times X$ is an isomorphism in $\mathcal{C}$.

Definition 14. Let $X$ and $X'$ be torsors under the action of a group object $G$. A morphism $f \in \text{Hom}_\mathcal{C}(X, X')$ is a $G$-morphism of torsors if $\gamma' \circ (\text{id}_G \times f) = f \circ \gamma$ as morphisms $G \times X \to X'$, i.e., if the following diagram commutes:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\gamma} & X \\
\downarrow{\text{id}_G \times f} & & \downarrow{f} \\
G \times X' & \xrightarrow{\gamma'} & X'
\end{array}
\]

B Fiber bundles

Affine spaces are particular torsors in the category of sets. Affine bundles will be particular torsors in the category of fiber bundles. Let us recall some basic notion about this category.

B.1 The category of fiber bundles over a (fixed) manifold

Let $M$ be a connected $m$-dimensional smooth manifold. Remember that a fiber bundle over $M$ is a surjective smooth submersion $\pi : E \to M$ such that

1. there is a smooth manifold $F$ (the typical fiber) such that for any $x \in M$, $\pi_x := \pi^{-1}(x)$ is diffeomorphic to $F$;

2. any point $x \in M$ has an open neighborhood $U$ such that there is a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times F$ making commutative the diagram below:

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times F \\
\downarrow{\pi} & & \downarrow{\text{pr}_1} \\
U & &
\end{array}
\]

An $M$-morphism of fiber bundles from $\pi : E \to M$ to $\pi' : E' \to M$ is a smooth map $f : E \to E'$ that preserves fibers, i.e., $f(\pi_x) \subset \pi'_{f(x)}$ for all $x \in M$. Fiber bundles over $M$ and $M$-morphisms between them form a category, denoted by $\text{FB}(M)$.

B.2 Group objects and torsors in $\text{FB}(M)$

A group object in $\text{FB}(M)$ is nothing but a fiber bundle $\eta : E \to M$ endowed with

- a fiber-preserving smooth map $m : E \times_M E \to E$,
- a fiber-preserving smooth map $\text{inv} : E \to E$,
- a global section $\epsilon : M \to E$.

\[\text{Again we identify } G \times (G \times X) \text{ with } (G \times G) \times X \text{ in a canonical manner}\]
so that on each fiber \( \eta_x (x \in M) \), the induced triple \((m_x, \text{inv}_x, \epsilon_x)\) defines a group structure.\(^6\)

**Remark 6.** Note that group objects in FB\((M)\) are not Lie groups since only elements in the same fiber can be multiplied with each other.

A group action of a group object \( \eta : E \rightarrow M \) on an arbitrary fiber bundle \( \pi : Z \rightarrow M \) is then an \( M \)-morphism

\[
\gamma : E \times_M Z \rightarrow Z : (e, z) \mapsto e \cdot z
\]

whose restriction to any fiber defines a group action. Finally, a fiber bundle \( \pi : Z \rightarrow M \) is a torsor under \( \eta : E \rightarrow M \) if the induced map

\[
\gamma \times \text{id}_Z : E \times_M Z \rightarrow Z \times_M Z
\]

is an \( M \)-isomorphism of fiber bundles (at the level of fibers, this corresponds to asking for all the actions to be free and transitive).

**Example 3** (Principal bundles). Let \( G \) be a Lie group. A principal \( G \)-bundle is a fiber bundle \( \pi : P \rightarrow M \) with typical fiber \( G \), together with a smooth right action \( P \times G \rightarrow P \) that is fiber-preserving, free and transitive. Nontrivial principal \( G \)-bundles are not group objects in FB\((M)\), but rather torsors under the (right) action of the trivial fiber bundle \( \text{pr}_1 : M \times G \rightarrow M \). In particular, any fiber of a \( G \)-principal bundle is a torsor (in the category of smooth manifolds) under the action of the Lie group \( G \).

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\(^6\)Note that the fibered product \( \times_M \) of fiber bundles over \( M \) defines product objects while the trivial bundle \( \text{id}_M : M \rightarrow M \) is a terminal object in the category FB\((M)\), so that we are in the conditions of Paragraph A.2.
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