A Decomposition Approach to Multiagent Systems With Bernoulli Packet Loss

Christian Hespe, Hamideh Saadabadi, Adwait Datar, and Herbert Werner, Member, IEEE, and Yang Tang

Abstract—In this article, we extend the decomposable systems’ framework to multiagent systems with Bernoulli-distributed packet loss with a uniform probability. The proposed sufficient analysis conditions for mean-square stability (MSS) and $H_\infty$-performance—which are expressed in the form of linear matrix inequalities—scale linearly with an increased network size and, thus, allow us to analyze even very large scale multiagent systems. A numerical example demonstrates the potential of the approach by application to a first-order consensus problem.

Index Terms—Graph theory, linear matrix inequalities (LMIs), multiagent systems (MASs), networked control systems, packet loss, stochastic systems.

I. INTRODUCTION

CONTROLLING large-scale networks of dynamic systems is a challenging problem that has attracted a lot of research interest. Due to their vast size, systematic centralized controller synthesis or system analysis quickly becomes infeasible due to computational demands. For that reason, decentralized or distributed approaches have become the methods of choice for this class of system [1].

One particular type of such a large-scale system is the multiagent system (MAS), in which groups of simple systems—called agents—collectively solve tasks by applying agent-level rules. Examples of such tasks include formation control, distributed estimation, or source seeking [2]. In order to analyze these MASs, the flexible and powerful framework of decomposable systems introduced by Massioni and Verhaegen in [1] can be employed. It is built upon the idea of decoupling the MASs into smaller modal subsystems and analyzing these subsystems independently, a technique that was originally introduced in [3] for stability analysis only. By decoupling the analysis, the framework improves the scalability in terms of computational complexity from quadratic to linear, in some instances, even constant in the number of agents [1]. Originally proposed for linear-time invariant (LTI) systems, the framework has been extended to linear parameter-varying systems [4] and analysis using integral quadratic constraints [5].

An important aspect of MAS is how the exchange of information is implemented. Depending on the requirements, relative measurements or a communication network is preferable. In this article, we will be focusing on the latter and consider the case where the communication network is subject to stochastic uncertainty in the form of lost information. More specifically, we investigate how to analyze the effect of packet loss described by independent Bernoulli-distributed random variables with uniform probability on stability and performance in a scalable manner. As noted by Ma et al. [6], most existing work on networked MASs with stochastic packet loss assumes identical loss, i.e., that all communication links fail at the same time, an assumption very few systems satisfy in practice. Among others, this scenario is studied in [7], [8], and [9] for Bernoulli and Markov packet loss models, respectively. On the other hand, there are approaches that consider not identical loss but uniform loss probability, e.g., [2], [10], [11], [12]. All four assume symmetric loss, i.e., that link failure is identical in both directions. Finally, Bernoulli packet loss with nonuniform probabilities and independent links is considered in [13] for directed tree graphs using only a lower bound on the transmission probabilities and in [6] for general graphs with known probability for each link.

Of the aforementioned papers, only the papers in [2], [10], and [12] consider system performance in addition to stability, the first two in terms of the convergence rate, and the third using the $L_2$ system norm. Another important performance measure for MAS is the $H_2$-norm, see [1], [12], [14], and [15] among others. A stochastic generalization of this norm for Markov jump linear systems (MJLS) was introduced for optimal control in [16] and used for optimal filtering in [17]. An existing approach for analyzing large MASs with MJLS can be found in [18]. However, while the conditions scale linearly with the number of agents, they scale exponentially with the maximum vertex degree and are thus intractable for many MASs.

Modeling packet loss with identically Bernoulli-distributed random variables is invalid in many real-world scenarios.
Nonetheless, this article provides a first step toward scalable analysis of MASs with more realistic modeling networks.

A. Contributions

The main contribution of this article is the sufficient analysis condition for mean-square stability (MSS) and $H_2$-performance of MASs presented in Theorems 6 and 7 that scale linearly with the number of agents in the presence of nonidentical Bernoulli-distributed packet loss with uniform probability. The conditions are based on extending the decomposable systems’ framework to stochastic jump linear systems and the analytic calculation of the expected Laplacian matrices in Lemma 4. Similar analytic calculations have been presented before in [11]; however, in contrast to previous works and at the cost of losing necessity, this article does not rely on having symmetric packet loss and brings out the inherent structure of the expected Laplacian matrices allowing for decomposition, which is exploited in Lemma 5.

Two further smaller contributions are necessary conditions in Theorem 10 supporting the sufficient conditions and an analysis approach for uncertain transmission probabilities and communication topologies that are based on convexity arguments.

B. Outline

The rest of this article is organized as follows. Section II proceeds with defining notation, setting up the problem and extending the decomposable systems’ framework. Section III contains the calculation of the expected Laplacians. The main results are presented in Section IV, followed by a numerical example in Section V. Finally, Section VI concludes this article.

II. PROBLEM STATEMENT

A. Notation and Definitions

We let $I_N$ denote the $N \times N$ identity matrix and $1_N$ the vector in $\mathbb{R}^N$ with all entries equal to 1. $M \geq (\geq 0)$ or $M < (\leq 0)$ mean that $M$ is positive or negative (semi-) definite. $M_1 \otimes M_2$ is the Kronecker product, which has the mixed product property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. $I_S$ denotes the set-membership indicator function defined as

$$\mathbb{1}_S(x) := \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{else} \end{cases}$$

Depending on the context, we use $\|z\|$ either for the Euclidean vector norm, the induced matrix 2-norm, or the 2-norm for (stochastic) signals defined by $\|z\|^2 := \sum_{k=0}^{\infty} \mathbb{E}[z^T(k)z(k)]$.

The interconnections between agents are modeled using graphs $G := (\mathcal{V}, \mathcal{E})$, which are composed of the vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where an edge $e_{ij} := (v_j, v_i)$ is read as pointing from $v_j$ to $v_i$, and $e_{ij} \notin \mathcal{E}$ if $e_{ij} \notin \mathcal{E} \Rightarrow e_{ji} \in \mathcal{E}$. The set $\mathcal{N}_i^- := \{v_j \in \mathcal{V} : e_{ij} \in \mathcal{E}\}$ is called the in-neighborhood of $v_i$ and its cardinality $d_i^- := |\mathcal{N}_i^-|$ is the in-degree of $v_i$. Equivalently, define the out-neighborhood $\mathcal{N}_i^+ := \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$ and out-degree $d_i^+ := |\mathcal{N}_i^+|$. If for every vertex in $\mathcal{V}$ the in- and out-degree are identical, $G$ is said to be balanced. A sequence of vertices is called a directed path on $G$ if $e_{ij} \in \mathcal{E}$ for all pairs of consecutive vertices $(v_i, v_j)$. If there exists a directed path from all $v_i \in \mathcal{V}$ to all other $v_i \in \mathcal{V} \setminus v_r$, $G$ is said to be strongly connected.

The transpose $G^T$ is defined as the graph in which the direction of every edge is inverted, i.e., $G^T := (\mathcal{V}, \mathcal{E}^T)$ with $e_{ij} \notin \mathcal{E} \iff e_{ji} \in \mathcal{E}^T$.

For a graph $G$, define elementwise the Laplacian matrix $L(G) := [l_{ij}(G)]$, where

$$l_{ij}(G) := \begin{cases} -1, & \text{if } i \neq j \text{ and } v_j \in \mathcal{N}_i^- \\ 0, & \text{if } i \neq j \text{ and } v_j \notin \mathcal{N}_i^- \\ d_i^+, & \text{if } i = j \end{cases}$$

$L(G)$ is symmetric if and only if $G$ is undirected. We will drop the argument from the notation if the corresponding graph can be determined from the context.

B. Jump Linear Systems for Modeling Packet Loss

The focus of this article is MAS, which is subject to stochastic packet loss. This kind of system cannot be modeled in a time-invariant manner, since loss of packets means that connections between individual agents break momentarily and, thus, the interconnection topology between agents is time-varying. For this reason, we will use a special case of MJLS to model the MAS.

An MJLS is a discrete-time, switched linear system, whose switching is controlled by a corresponding Markov chain. At every time instance, the MJLS is in exactly one of $m$ possible modes, where each mode can have a different dynamic behavior. It is described by the state-space system

$$G : \begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}w(k) \\ z(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}w(k) \end{cases}$$

where $x(k) \in \mathbb{R}^{Nn_x}$ is the dynamic state, $\sigma(k) \in K := \{i \in \mathbb{N} : 1 \leq i \leq m\}$ is the state of the Markov chain, and $w(k) \in \mathbb{R}^{Nn_w}$ and $z(k) \in \mathbb{R}^{Nn_z}$ are the performance input and output, respectively. $N$ denotes the number of agents in the system, the initial state of the system is $x(0) = x_0$, and the Markov chain is initially distributed according to $\sigma(0) = \sigma_0$. For each mode $i \in K$, the dynamics of the system are governed by the matrices $A_i, B_i, C_i$, and $D_i$. Note that system (3) does not have a control input or measured output since this article is only concerned with system analysis in contrast to controller synthesis. Equation (3) should thus be considered as a closed-loop model, containing an agent model and potentially a controller.

In this article, we only consider the case where the switching probability of the Markov chain is independent of the chain’s state; thus, the distribution of $\{\sigma(k)\}$ is stationary and described by

$$\Pr(\sigma(k) = i) = t_i$$

for all $k \geq 0$.

There is a variety of definitions for stability in the context of MJLS. Among them are stability in expectation, almost sure stability, and MSS. Here, we will focus on the latter. In comparison, MSS has the advantage that it is easy to test for and implies stability as in the other two definitions [19].

Definition 1 (Mean-Square Stability [19]): The MJLS (3) is mean-square stable if

$$\lim_{k \to \infty} \mathbb{E}[\|x(k)\|^2] = 0$$

and

$$\lim_{k \to \infty} \mathbb{E}[\|x(k)x^T(k)\|] = 0$$

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
for all initial conditions $x_0$ and initial distributions $\sigma_0$.

In the following, we will often refer to an MJLS as stable if it is MSS. As shown in [19], the stability of the individual modes of an MJLS is neither necessary nor sufficient for MSS. Instead, we will make use of the following linear matrix inequality (LMI)-based stability test:

**Theorem 1 (LMI Condition for MSS [20]):** The MJLS (3) is mean-square stable if and only if there exists a $Q > 0$ such that

$$\sum_{i \in K} t_i A_i^TQA_i - Q < 0.$$  \hspace{1cm} (5)

**Remark:** Note that we can express the above LMI using an unconditional expectation. Thus, (5) is equivalent to

$$E[A_i^TQA_i] - Q < 0$$

where the expectation is taken with respect to $\sigma$.

This theorem is a specialization of the general stability test from [20] to MJLS with state-independent switching probabilities as given in (4). Compared to the general case, this theorem results in a sizeable reduction in computational complexity, since the stability test contains only a single-matrix variable $Q$ and a single LMI constraint, instead of having one of both for each mode. Still, it is necessary to enumerate all modes in (5), which renders the analysis of systems with numerous agents intractable. For the specific system structure that is introduced in the next section, we will develop an approach that eliminates the need for mode enumeration.

In addition to MSS, we consider system performance in terms of the $H_2$-norm from input $w$ to output $z$. For the special case of mode-independent transition probabilities in the Markov chain, the norm is defined as follows.

**Definition 2 ($H_2$-norm for MJLS [21]):** The $H_2$-norm of the stable MJLS (3) is defined as

$$\|G\|_{H_2}^2 := \sum_{i \in K} \sum_{s=1}^{n_s} t_i \|z^{s,i}\|^2$$

where $z^{s,i}$ is the response of $G$ to a discrete impulse applied into the $s$th input with $x_0 = 0$ and $\sigma_0 = i$.

Similar to Theorem 1 for MSS, we can exploit the stationarity of the transition probabilities to obtain an analysis condition in terms of LMI constraints. For general MJLS, the corresponding condition requires two LMIs and variables in two variables and LMI constraints. For general MJLS, the need for mode enumeration.

**Theorem 2 ($H_2$-norm for MJLS [21]):** Given the stable MJLS (3), $\|G\|_{H_2}^2 < \gamma$ if and only if there exists a $Q > 0$ and a symmetric $Z$ with $\text{tr}(Z) < \gamma$ such that

$$\sum_{i \in K} t_i \left(A_i^TQA_i + C_i^TQC_i\right) - Q < 0,$$

where $A_i$ and $C_i$ are matrices associated with mode $i$.

$$\sum_{i \in K} t_i \left(B_i^TQB_i + D_i^TD_i\right) - Z < 0.$$  \hspace{1cm} (7b)

**Remark:** Theorem 2 reformulates the result from the work in [21] by introducing $Z$. To see that both are equivalent, use $Y - X < 0 \Rightarrow \text{tr}(X) > \text{tr}(Y)$ with symmetric $X, Y$ for the first direction and chose $Z = \varepsilon I + \sum_{i \in K} t_i (B_i^TQB_i + D_i^TD_i)$ with sufficiently small $\varepsilon > 0$ for the other.

For the same reason as for Theorem 1, direct application of the above result to large MAS would quickly lead to numerically intractable problems. We introduce a subset of jump systems for which the computational complexity can be vastly reduced next.

C. Decomposable Jump Linear Systems

Coming from the general MJLS in Section II-B, this article considers systems with a specific structure in their state-space matrices, which allows us to utilize the decomposable systems’ framework introduced by Massioni and Verhaegen in [1]. According to their definition, a matrix $M$ is said to be decomposable if it can be split up into a decoupled component $M^d$ and a coupled component $M^c$ as $M = I_N \otimes M^d + P \otimes M^c$, where $P$ is called the pattern matrix. Moreover, an LTI system is called decomposable if all matrices of its state-space representation are decomposable with respect to the same pattern matrix.

Applying this concept to the MJLS (3) means that $A_i, B_i, C_i,$ and $D_i$ can be decomposable. We do, however, not insist on having the same pattern matrix for all modes $i \in K$. On the contrary, we will assume the pattern matrix is the only part of the system that changes between the modes. This choice is motivated by the fact that—in the context of networked MASs—the pattern matrix is given by the graph Laplacian and that the communication graph is a stochastic process due to packet loss. We then introduce the nominal graph $G^0 = (V, E^0)$ and its corresponding Laplacian $L^0 := L(G^0)$. Altogether, this leads to the decomposable MJLS

$$G^0 : \begin{cases} x(k+1) = (I_N \otimes A^d + L(G^0) \otimes A^c) x(k) \\
+ (I_N \otimes B^d + L(G^0) \otimes B^c) w(k) \\
z(k) = (I_N \otimes C^d + L(G^0) \otimes C^c) x(k) \\
+ (I_N \otimes D^d + L(G^0) \otimes D^c) w(k) \end{cases} \hspace{1cm} (8)$$

where $G_i := (V, E_i)$ and $E_i \subseteq E^0$ is the subset of edges that successfully transmit a packet in mode $i$ of the MJLS. Analogous to $L^0$, define $L_i := L(G_i)$ as shorthand notation.

More specifically, consider a stationary stochastic process $\{\alpha_{ij}(k)\}$ for each $e_{ij} \in E^0$, where $\alpha_{ij}(k) \in \{0,1\}$. Here, $\alpha_{ij}(k) = 1$ means the edge $e_{ij}$ is active or, equivalently, that the packet is transmitted while $\alpha_{ij}(k) = 0$ means $e_{ij}$ is inactive and the packet is lost. The edges might fail asymmetrically, i.e., it might happen that $\alpha_{ij}(k) \neq \alpha_{ji}(k)$. In the following, we assume that the stochastic processes are Bernoulli distributed and independent in time. Furthermore, at any given time instant, the packet loss between two different pairs of vertices is assumed to be independent. This is formalized in the following assumption.

**Assumption 1:** The stochastic processes $\{\alpha_{ij}(k)\}$ are partially independent and identically Bernoulli distributed such that, for all $k, k' \geq 0$, $e_{ij}, e_{rs} \in E^0$, we have

$$\text{Pr}(\alpha_{ij}(k) = 1) = p \quad \text{Pr}(\alpha_{ij}(k) = 0) = 1 - p$$

with $p \in [0,1]$ and $\alpha_{ij}(k)$ and $\alpha_{rs}(k')$ are independent random variables whenever $k \neq k'$ or $(r,s) \neq (i,j)$ and $(r,s) \neq (j,i)$.

**Remark:** For many real-world scenarios, modeling packet loss as independent Bernoulli-distributed random variables with
uniform probability is an idealization. Similar to the work in [23], we proceed in this way for reasons of mathematical tractability. Note that compared to assuming identical or symmetric loss as, e.g., in [7], [9], and [11], Assumption 1 is closer to reality due to allowing opposing links to be correlated or not.

To map from the stochastic processes \( \{ \alpha_{ij}(k) \} \) to the MJLS (8), define a function \( \nu : E^0 \to \{1, \ldots, |E^0|\} \) that assigns each \( \alpha_{ij} \in E^0 \) a unique integer. Then, we have

\[
\sigma(k) = 1 + \sum_{\alpha_{ij} \in E^0} \alpha_{ij}(k)2^{\nu(\alpha_{ij})-1}
\]

and accordingly \( m = 2^{|E^0|} \). The map from \( \alpha_{ij}(k) \) to \( \sigma(k) \) is bijective, such that we can equivalently represent the Bernoulli packet loss model in the form of the MJLS. Thus, our communication model has two parameters: 1) the graph \( G^0 \) and 2) the probability of successful transmission \( p \).

To reap maximum benefit from introducing the decomposable system framework, we will impose that the matrix variable \( Q \) has a block repeated structure. While this may be a conservative choice, it allows for generating stability and performance tests that are particularly easy to check.

**Corollary 3 (MSS for Decomposable Jump Systems):** The decomposable jump system (8) is mean square stable if there exists a \( Q > 0 \) such that

\[
I_N \otimes (A^{dT}QA^d - Q) + \mathbb{E}[L_s] \otimes (A^{dT}QA^d) + \mathbb{E}[L_s] \otimes (A^{dT}QA^d) < 0.
\]

**Proof:** Take the LMI condition (6) and insert \( Q = I_N \otimes \bar{Q} \) as well as the decomposable MJLS from (8) for \( A_s \), resulting in

\[
\mathbb{E}[(I_N \otimes A^d + L_s \otimes A^c)^T(I_N \otimes \bar{Q})] - I_N \otimes \bar{Q} < 0.
\]

Using the mixed product rule and the commutation property \((M_1 \otimes I)(I \otimes M_2) = (I \otimes M_2)(M_1 \otimes I)\), we obtain

\[
I_N \otimes (A^{dT}QA^d - Q) + \mathbb{E}[L_s] \otimes (A^{dT}QA^d) + L_s^T \otimes (A^{dT}QA^d) < 0
\]

from where we get to (11) by linearity of the expectation, \( \mathbb{E}[X \otimes M] = \mathbb{E}[X] \otimes M \), which holds if \( X \) is a random matrix and \( M \) a constant, and renaming \( \bar{Q} \to Q \).

**Remark** The only source of conservatism in Corollary 3 is the assumption on \( Q \) to have a block repeated structure. A similar result can be obtained without this assumption; however, isolating the expectation of \( L_s^TL_s \) would not be possible, since the commutation property cannot be used. Instead, one would have to consider a weighted squared expectation of the form \( \mathbb{E}[(L_s^T \otimes I)Q(L_s \otimes I)] \), similar to [11, Lemma 1] but with additional Kronecker products. In that case, the analysis conditions cannot be decomposed using the approach proposed in Section IV-A.

A similar corollary can be derived for the \( H_2 \)-performance conditions in Theorem 2. However, since the derivation would be analogous to the proof of Corollary 3, we skip this intermediate result.

### III. Expected Laplacian Matrices

From Corollary 3, we have seen how the expectation of the Laplacian is essential in determining if a decomposable MJLS is MSS or not. We thus derive an analytic calculation of the expectation in terms of \( G^0 \) and \( p \) in the following.

As preparation, notice how the elements of the Laplacian change compared to (2) when packet loss is introduced. Using the elementwise notation \( L_{ij}(k) = [l_{ij}(k)] \), the stochastic Laplacian is given by

\[
l'_ij(k) = \begin{cases} 
-\alpha_{ij}(k), & \text{if } i \neq j \quad \text{and } v_j \in \mathcal{N}_i^- \\
0, & \text{if } i \neq j \quad \text{and } v_j \notin \mathcal{N}_i^- \\
\sum_{v_m \in \mathcal{N}_i} \alpha_{im}(k), & \text{if } i = j.
\end{cases}
\]

As noted earlier, \( \{L_{ij}(k)\} \) is a stochastic process due to packet loss. Since \( \{\alpha_{ij}(k)\} \) and \( \{L_{ij}(k)\} \) are stationary by Assumption 1, we will drop the index \( k \) in the remainder of the article when referring to an instance of these processes.

**Lemma 4 (Expected Laplacian Matrices):** Given the nominal graph \( G^0 \) and packet loss according to Assumption 1, we have

\[
\mathbb{E}[L_s] = pL(G^0) \quad \mathbb{E}[L_s^T L_s] = p^2 L(G^0)^T L(G^0) + p(1-p) \left( L(G^0) + L(G^0)^T \right).
\]

**Proof:** See Appendix A.

**Lemma 4** enables us to calculate the expected Laplacians analytically from the two parameters \( G^0 \) and \( p \), which allows applying Corollary 3 and Theorem 2 effectively without expensive numerical calculation of the expectations by enumeration of all modes. Note that \( L(G^0)^T = L(G^0) \) if and only if \( G \) is balanced. For certain graphs \( G^0 \), we can further exploit the following diagonalizability property.

**Lemma 5 (Simultaneous Diagonalizability):** Given the nominal graph \( G^0 \) and packet loss according to Assumption 1, there exists a similarity transformation \( U \) that diagonalizes \( \mathbb{E}[L_s] \) and \( \mathbb{E}[L_s^T L_s] \), and \( \mathbb{E}[L_s^T L_s] \) if and only if \( L(G^0) \) is normal, i.e., \( L(G^0)L(G^0)^T = L(G^0)^T L(G^0) \).

**Proof:** According to [24, p. 62], there exists a similarity transformation that diagonalizes two simultaneous matrices at the same time if and only if the matrices commute. As shown in [25], \( L(G^0) \) being normal and having zero row sum implies that it has zero column sum as well; therefore, \( G^0 \) is balanced and \( L(G^0)^T = L(G^0) \). By Lemma 4, we thus need to show that \( L(G^0)^T \) and \( L(G^0)^T L \) commute. From the definition of normality, this is trivial for the first pair and easy to verify for \( L(G^0)^T \) and \( L(G^0)^T L \). Conversely, if there exists a transformation that diagonalizes both \( \mathbb{E}[L_s] \) and \( \mathbb{E}[L_s^T L_s] \), then \( L(G^0)^T \) and \( L(G^0)^T L \) commute, implying that \( L(G^0) \) is normal.

For some scenarios in the context of MAS control and distributed consensus, the normality of the Laplacian is too restrictive for Lemma 5 to be applicable. In particular, leader–follower schemes cannot be handled, since they require unbalanced communication graphs.
IV. SCALABLE ANALYSIS WITH PACKET LOSS

A. Decomposed Analysis LMIs

With the results from Section III, we can now formulate our final MSS and $H_2$-performance analysis conditions for decomposable MJLS. The motivation for defining a decomposable system like in [1] is that we can decouple the system as long as we can diagonalize the pattern matrix. Assuming there exists a transformation $U$ such that $U P U^{-1}$ is diagonal—with $P$ being the pattern matrix, then $(U \otimes I)$ decouples the system matrices. In particular, if the underlying graph $G^0$ is undirected, such a transformation is guaranteed to exist with $U U^T = I$. We will thus make the following assumption in the remainder of the article.

**Assumption 2:** The communication graph $G^0$ is undirected.

Assumption 2 restricts the classes of MAS the following results can be applied to. Note, however, that it is different from assuming the packet loss is symmetric, which would be equivalent to assuming all $G_i$ are undirected, in contrast to just $G^0$. Applied to Corollary 3, this gives rise to the following stability test consisting of a set of decoupled LMIs:

**Theorem 6 (Decomposed MSS Test):** Given the MJLS (8) with nominal communication graph $G^0$ and packet loss satisfying Assumptions 1 and 2, the MJLS is mean-square stable if there exists a $Q > 0$ such that

$$
(A^d + p \lambda_i A^c) Q (A^d + p \lambda_i A^c) - Q + 2p(1-p) \lambda_i A^c Q A^c < 0
$$

for all $i \in \{1, \ldots, N\}$, where $\lambda_i$ are the eigenvalues of $L^0$.

**Proof:** Since $G^0$ is undirected, $L^0$ is symmetric, and we know from Lemma 5 that $E[\sigma], E[L_\sigma^T L_\sigma], \text{ and } E[L_\sigma^T L_\sigma]$ can be diagonalized using an orthogonal matrix $U$. Apply a congruence transformation to (11) from Corollary 3 by multiplying with $(U \otimes I_n)$ and $(U^T \otimes I_n)$ from the left and right, respectively. Using the mixed product rule and commutation property of the Kronecker product, this results in

$$
I_N \otimes (A^T Q A^d - Q) + (U E[L_\sigma] U^T) \otimes (A^c Q A^c) + (U E[L_\sigma] U^T) \otimes (A^c Q A^c) < 0
$$

By Lemma 4, we have $U E[L_\sigma] U^T = U E[L_\sigma^T L_\sigma] U^T = p \Lambda$ and $U E[L_\sigma^T L_\sigma] U^T = p^2 \Lambda^2 + 2p(1-p) \Lambda$, where $\Lambda$ is a diagonal matrix containing the eigenvalues of $L^0$. After the transformation, we have

$$
I_N \otimes (A^T Q A^d - Q) + (U E[L_\sigma] U^T) \otimes (A^c Q A^c) + (p^2 \Lambda^2 + 2p(1-p) \Lambda) \otimes (A^c Q A^c) < 0
$$

which is a block-diagonal matrix inequality. Finally, (13) can be obtained by algebraic matrix manipulations and considering the blocks independently. □

**Theorem 6** has multiple advantages in terms of computational complexity compared to the original stability test in Theorem 1. The first and most impactful is replacing the mode enumeration of the MJLS by the formula given in Lemma 4. Since the number of modes scales at least with $2^N$ for strongly-connected graphs—there exists at least one edge per agent—the original formulation has exponential complexity while the analytic calculation scales quadratically. The second improvement comes from decomposing the single large constraint on the whole network into multiple smaller ones with the size of a single agent. In analogy to the modal subsystems from [1], we may term these as modal constraints. Instead of scaling the number of variables and constraints quadratically with the agent count, the decoupled formulation is of constant complexity in the variables and linear complexity in the constraints. Analogous steps can be applied to the $H_2$-performance analysis LMIs from Theorem 2.

**Theorem 7 (Decomposed $H_2$-Performance):** Given the MJLS (8) with nominal communication graph $G^0$ and packet loss satisfying Assumptions 1 and 2, $\hat{G}$ is mean-square stable and $\|G\|_{H_2} < \gamma$ if there exists a $Q > 0$ and symmetric $Z_i$ with $\sum_{i=1}^N \text{tr}(Z_i) < \gamma^2$ such that

$$
A_i^T Q A_i + C_i^T D_i - Q + 2p(1-p) \lambda_i (A_i^T Q A^c + C_i^T C^c) < 0
$$

$$
B_i^T Q B_i + D_i^T D_i - Z_i + 2p(1-p) \lambda_i (B_i^T Q B^c + D_i^T D^c) < 0
$$

for all $i \in \{1, \ldots, N\}$, where $\lambda_i$ are the eigenvalues of $L^0$, $A_i$ denotes $A^d + p \lambda_i A^c$ and equivalently for $B_i$, $C_i$, and $D_i$.

**Proof:** See Appendix B.

The computational performance improvements achieved by Theorems 6 and 7 come at the cost of some conservatism due to imposing that $Q$ is identical for all modal constraints. As noted in the remark to Corollary 3, this restriction is inherently required to utilize the commutation property of the Kronecker product and, thus, to apply Lemmas 4 and 5 for the calculation of the expected Laplacians. The calculation of weighted expected Laplacians and whether their structure allows for a decomposition of the analysis is subject to further research. To evaluate how much conservatism is introduced by the restriction to a single $Q$, we will present a numerical example that demonstrates the tradeoff between computational speed and overestimation of the $H_2$-norm in Section V.

B. Handling Uncertain Loss Probabilities

Theorems 6 and 7 consider the case where the transmission probability $p$ is known exactly. In practice that is often not the case and only a lower bound $p \geq 0$ on the transmission probability is known. If an upper bound $\bar{p} \leq 1$ is provided in the same vein, whether the MJLS (8) is stable or has $H_2$-norm less than $\gamma$ for a constant but uncertain transmission probability can be answered by applying the theorems for all $p$ in $[\bar{p}, \overline{p}]$. However, since $[\bar{p}, \overline{p}]$ is a real interval, numerical evaluation of the LMI constraints for all such $p$ is intractable. Instead, we can make use of the fact that all three LMIs are convex in $p$ under conditions specified in the following lemma.

**Lemma 8 (Convexity in $p$):** With fixed $Q > 0$ and $Z_i$, define the quadratic form $V_i(p, y) := y^T M_i(p)y$, where $M_i(p)$ is the left-hand side of either (11), (14a), or (14b). $V_i(p, y)$ is convex
in \( p \) for all \( i \in \{1, \ldots, N\} \) if and only if either the relevant matrices from \( \{A^c, B^c, C^c, D^c\} \) are zero matrices or all nonzero eigenvalues of \( L^0 \) satisfy \( \lambda_i \geq 2 \).

**Proof:** We prove the lemma for (14a) as representative of all three inequalities. \( V_i(p, y) \) is convex in \( p \) if and only if if \([24]\)

\[
\frac{\partial^2 V_i(p, y)}{\partial p} \geq 0 \quad \forall y \Leftrightarrow (\lambda_i^2 - 2\lambda_i) \left(A^{cT}QA^c + C^{cT}C^c\right) \geq 0.
\]

From \( Q > 0 \), it follows that \( A^{cT}QA^c \geq 0 \) and \( C^{cT}C^c \geq 0 \) for all \( A^c, C^c \). We then distinguish two cases: If \( \lambda_i^2 \geq 2\lambda_i \), we are done. This condition is satisfied for \( \lambda_i = 0 \) and otherwise equivalent to \( \lambda_i \geq 2 \) since all eigenvalues of \( L^0 \) are nonnegative \([2]\). On the other hand, if \( \lambda_i^2 < 2\lambda_i \), we must have \( A^{cT}QA^c + C^{cT}C^c \leq 0 \), and thus, \( A^{cT}QA^c + C^{cT}C^c = 0 \), which, in turn, implies \( A^c = 0 \) and \( C^c = 0 \). The proof for (11) and (14b) follows along the same lines, replacing \( (A^c, C^c) \) by \( (B^c, D^c) \) for (14b) and considering just \( A^c \) for (11).

For a convex function \( V(x) \), its sublevel set \( \{x : V(x) < 0\} \) is convex as well. On the interval \([p, \bar{p}]\), this ensures that checking the condition on the boundary is sufficient to verify it is satisfied throughout. Thus, assuming that the conditions of Lemma 8 are fulfilled, the problem is reduced to applying Theorems 6 or 7 at \( p \) and \( \bar{p} \) with shared \( Q \) and \( Z_i \).

To give some meaning to the conditions from the lemma, the zero-matrix condition implies that there exists no coupling between the agents and is, thus, irrelevant to the analysis of MAS in practice. The remaining condition on the eigenvalues of \( L^0 \) can be seen as a lower bound on the connectivity of the underlying communication graph \( G^0 \). In particular, if \( G^0 \) is undirected and connected, it is a lower bound on the Fiedler eigenvalue \( \lambda_2 \), the smallest nonzero eigenvalue of \( L^0 \) \([2]\).

### C. Handling Uncertain Nominal Communication Graphs

In the form stated earlier, Theorems 6 and 7 require complete knowledge of the spectrum of \( L^0 \) and, thus, centralized information. However, it is possible to utilize another convexity property of the LMIs to relax this restriction.

**Lemma 9 (Convexity in \( \lambda_i \)):** With fixed \( Q > 0 \) and \( Z_i \), define the quadratic form \( V_p(\lambda_i, y) := y^T M_p(\lambda_i) y \), where \( M_p(\lambda_i) \) is the left-hand side of either (11), (14a), or (14b). \( V_p(\lambda_i, y) \) is convex in \( \lambda_i \) for all \( p \).

**Proof:** The proof is analogous to the proof of Lemma 8, outlined exemplarily for (14a). \( Q > 0 \) guarantees that

\[
\frac{\partial^2 V_p(\lambda_i, y)}{\partial \lambda_i} \geq 0 \quad \forall y \Leftrightarrow p^2 \left(A^{cT}QA^c + C^{cT}C^c\right) \geq 0
\]

is always satisfied regardless of \( p \). \( \square \)

Lemma 9 implies that knowledge of the boundary of the spectrum of \( L^0 \) is sufficient to evaluate Theorems 6 and 7. An upper bound on \( \lambda_N \) can, for example, be obtained from the maximum node degree and Cheeger’s inequality could be used to bound \( \lambda_2 \) \([2\text{ Sec. II-D2}]\). For Theorem 6, this adaptation comes without additional conservatism, giving sufficient stability conditions independent of network size. For Theorem 7 on the other hand, one needs to further restrict \( Z_i = \tilde{Z}_i \) for all \( i, j \in \{1, \ldots, N\} \), making the upper bound on the \( H_2 \)-norm possibly more conservative.

### D. Necessary Conditions for the Analysis of MJLS

To evaluate the conservatism introduced by restricting Theorems 6 and 7 to a single \( Q \) for all modal constraints, we can compare their results to those obtained from the lossless theorems from Section II-B. However, this comparison is only tractable for MAS with few agents because of the exponential scaling of the lossless theorems. Thus, we propose necessary conditions for Theorems 1 and 2 that can be checked with the same (linear) complexity as the sufficient conditions from Section IV-A, which enables us to estimate the conservatism for large MAS.

For the analysis, we introduce the *mean system* \( \bar{G} \), which is the LTI system whose system matrices are given by the mean of the MJLS matrices. For the MJLS \( G \), this results in the LTI state-space model

\[
\begin{align}
\bar{x}(k+1) &= \left(I_N \otimes A^d + pL^0 \otimes A^c\right) \bar{x}(k) \\
&\quad + \left(I_N \otimes B^d + pL^0 \otimes B^c\right) \bar{w}(k) \\
\bar{z}(k) &= \left(I_N \otimes C^d + pL^0 \otimes C^c\right) \bar{x}(k) \\
&\quad + \left(I_N \otimes D^d + pL^0 \otimes D^c\right) \bar{w}(k).
\end{align}
\]

The mean system can be seen as advancing the ensemble average state in time, in contrast to the MJLS, which advances one specific realization. Based on the mean system, we can then state the following result.

**Theorem 10 (LTI Necessary Conditions):** Given the decomposable MJLS \( G \), its mean \( \bar{G} \), and any \( \gamma > 0 \), the following implications hold:

1. \( \bar{G} \) is MSS \( \implies \bar{G} \) is stable.
2. \( \|\bar{G}\|_H^2 < \gamma \implies \|\bar{G}\|_H^2 < \gamma \).

**Proof:** See Appendix C. \( \square \)

Theorem 10 implies that stability of \( \bar{G} \) is necessary for MSS of \( G \) and that \( \|\bar{G}\|_H^2 \) is a lower bound for \( \|G\|_H^2 \). A similar result can be obtained for general MJLS in that stability in the second moment, i.e., MSS, implies stability in the first moment \([19, \text{Proposition 3.6}]\). Since \( \bar{G} \) is LTI, we may apply the analysis based on modal subsystems proposed by Massioni and Verhaegen in \([1]\), resulting in LMI conditions that scale linearly in the number of agents.

### V. Example: First-Order Consensus

#### A. Setting up the Problem

To demonstrate the scalability of and judge the amount of conservatism in the analysis conditions from Section IV, let us now finally apply the approach to a numerical example. The example we chose is the *discrete-time first-order consensus problem*, which can be described as the problem of reaching an agreement in a network of linear first-order integrators while each agent is only communicating to a subset of the remaining agents. The communication between agents is modeled using the graph \( G^0 \), with agent \( i \) receiving information according to its in-neighborhood \( N_i^c \). For each individual agent, the dynamics are then described by (16) with \( x_i(k) \in \mathbb{R} \) while a solution to the
An issue with analyzing the system that describes the solution to the consensus problem is its inherent marginal stability. In the decomposable systems’ framework without packet loss from [1], a convenient approach to resolve this issue is to neglect the modal subsystem that corresponds to the 0 eigenvalue of the Laplacian [3]. For the calculation of the $H_2$-norm, this means that $(\Pi \otimes I_n)G$ is analyzed instead of $G$, where $\Pi := I_N - \frac{1}{\sqrt{n}} N_1 N^T$ is the orthogonal projection onto the disagreement space. The same approach can be applied to the decoupled analysis conditions from Theorems 6 and 7 as well as the necessary LTI conditions in Theorem 10 and—in adapted form—the coupled LMIs in Theorems 1 and 2.

Consider again a transformation $U$ with $U^T U = I_N$ such that $U^T L^0 U$ is diagonal, which does exist under Assumption 2. Note that because $L^0$ has zero row and column sum, $U$ can be chosen as $U = [1_N / \sqrt{N}]$ with $\Pi U = [0 \ U]$. We can then apply $U$ as state and signal transformation to $(\Pi \otimes I_n)G$, giving $\tilde{x}(k) := (U^T \otimes I_n) x(k)$, $\tilde{w}(k) := (U^T \otimes I_n w(k)$ and $\tilde{z}(k) := ((U^T \Pi) \otimes I_n) z(k)$, resulting in

$$
\tilde{x}(k + 1) = \begin{bmatrix}
A^d & \hat{I}_i \otimes A^c & \hat{I}_i \otimes A^c \\
0 & I_{N-1} \otimes A^d & \hat{I}_i \otimes A^c \\
B^d & \hat{I}_i \otimes B^c & \hat{I}_i \otimes B^c
\end{bmatrix}
\tilde{x}(k) + 
\begin{bmatrix}
B^d & \hat{I}_i \otimes B^c & \hat{I}_i \otimes B^c
\end{bmatrix}
\tilde{w}(k) \quad (19a)
$$

$$
\tilde{z}(k) = 
\begin{bmatrix}
0 & 0 & 0 \\
0 & I_{N-1} \otimes C^d + \hat{I}_i \otimes C^c & 0 \\
0 & 0 & I_{N-1} \otimes D^d + \hat{I}_i \otimes D^c
\end{bmatrix}
\tilde{z}(k) + 
\begin{bmatrix}
0 & 0 & 0
\end{bmatrix}
\tilde{w}(k) \quad (19b)
$$

where $\hat{I}_i := \frac{1}{\sqrt{N}} L_i U / \sqrt{N}$ and $\hat{L}_i := U^T L_i U$. It was shown in [1] that the $H_2$-norm is invariant under this kind of orthogonal transformation of input and output.

In the transformed system (19), it is apparent that the center of gravity, which $\tilde{x}(k)$ contains in its first $n_x$ entries, does not affect the remaining states, since the bottom left block of every system matrix is 0. The converse does, however, only hold if all $\mathcal{G}_i$ are balanced, since this implies that $\hat{I}_i = 0$ for all $i \in K$. For stability analysis using Theorem 1, this one-way coupling may be ignored, as stability of the remaining system would imply the center of gravity stays finite as long as the decoupled part $A^d$ is at least marginally stable. On the other hand, when calculating the $H_2$-norm of the system, we can take advantage of the fact that the full first columns are zero for the $C$ and $D$ matrices of the transformed system. This implies that even though the center of gravity is affected by the remaining system, this is not apparent in the output $\tilde{z}(k)$ and thus does not increase the $H_2$-norm. The desired $H_2$-norm can, therefore, be obtained by only considering the bottom right block of the transformed system.

### B. Numerical Results

To analyze the scalability and conservatism of the approaches described in this article, we implemented the LMI conditions from Theorems 2, 7, and 10 in MATLAB using the YALMIP [27]
toolbox. All three conditions are affine in $\gamma^2$ such that we can directly minimize $\gamma^2$—and thus $\gamma$—subject to either of the LMIs. The minimum $\gamma$ obtainable by each of the conditions will be plotted ahead as the respective $H_2$-performance. All source code is available in [28].

Let us first evaluate how the $H_2$-performance changes with the transmission probability $p$ for each of the three conditions. For two test graphs, $G_3^\triangle$ and $G_{50}^\triangle$, we perform a sweep over $p$, which is shown in Fig. 2. In the figure, the “decomposed” graph refers to the best upper bound on the $H_2$-norm that can be obtained from Theorem 7, “mean” is the lower bound based on Theorem 10, and “enumerated” corresponds to the original analysis condition in Theorem 2 and, thus, shows the true $H_2$-norm of the system. Theorem 2 is only applied to the small MAS in Fig. 2(a), since $G_{50}^\triangle$ has $m = 2^{|E_{50}^\triangle|} = 2^{7350}$ modes, which are intractable to enumerate.

As expected, the performance figures obtained from the decomposed analysis results in Theorems 7 and 10 do not match the $H_2$-norm of the system but over- and underestimate it, respectively. Furthermore, the gap between the upper and lower bound is significantly increased for the larger MAS. However, while the mean system recovers the exact norm for $p = 1$ because it coincides with the MJLS, the upper bound is conservative for all transmission probabilities.

In the second step, we compare how the analysis conditions from Theorems 2 and 7 scale in terms of computational speed and conservatism of the calculated $H_2$-norm. We start by analyzing the MAS with the circular graphs $G_N^\circ$ for $N$ between 2 and 12. Because the number of edges is relatively small for these graphs, we can apply all three conditions. For each $N$, we calculate the $H_2$-performance with $p = 0.5$, which was chosen since it is the transmission probability with the largest variance. The results are shown in Fig. 3.

As observed before, the conservatism of Theorem 7 grows with an increasing agent count. The lower bound obtained from the mean system is close to the exact norm regardless of the agent count. Concerning the computational speed, it is apparent that the analysis conditions from Theorem 2 show an exponential growth in complexity such that the problem will quickly become intractable even for networks of moderate size. On the other hand, the conditions from Theorem 7 show no substantial increase in computation time.
returns when the number of agents is decreased below that threshold.

VI. CONCLUSION AND FUTURE WORK

This article proposes an extension of the decomposable systems’ framework to stochastic jump linear systems in order to analyze the effect of Bernoulli-distributed packet loss with uniform packet loss probability on MASs. Based on analytic expressions for the expected Laplacians, sufficient analysis conditions for MSS and bounds on the $H_2$-norm that scale linearly with the number of agents were derived. Finally, it was demonstrated that the proposed conditions are applicable to very large networks but that their conservatism increases with the size of the network.

In future work, it will be investigated if the restriction to identical matrix variables in the presented analysis conditions can be removed without losing sufficiency, possibly leading to lossless complexity reduction similar to the LTI case. An instrumental step would be to extend the result on simultaneous diagonalizability to the more general weighted expectation. Furthermore, current research is aiming at how the restrictive assumption of Bernoulli-distributed loss with uniform probability can be relaxed.

APPENDIX A

PROOF OF LEMMA 4

Proof: The expectations are calculated elementwise. Thus, for $\mathbb{E}[L_0]$, we get $\mathbb{E}[t^0_{ij}] = -p$ if $v_j \in \mathcal{N}^-_i$ and $\mathbb{E}[t^0_{ij}] = 0$, otherwise for the off-diagonal entries. On the diagonal, we have $\mathbb{E}[t^0_{ii}] = pd^0_i$. Together, this is equal to $pL(G^0)$.

On the other hand, for the expectation of $L^T_2L_\sigma$, calculate the entries of $(G^0)^T L(G^0)$ first. We get

\begin{align}
I_i^T I_0 & = (d_i^-)^2 + d_i^+ \\
I_i^T I_j & = I_i^T \mathbb{I}_{\mathcal{N}^+} - d_i^- \mathbb{I}_{\mathcal{N}^-} (v_j) - d_j^- \mathbb{I}_{\mathcal{N}^-} (v_i)
\end{align}

for the diagonal and off-diagonal entries respectively, where $I_i^0$ and $I_j^0$ are, respectively, the $i$th and $j$th columns of $L^0$. Notice that $v_i \in \mathcal{N}^-_i \Leftrightarrow v_i \in \mathcal{N}^+_i$ and $\{v_i, v_j\} \subseteq \mathcal{N}^-_i \Leftrightarrow v_i \in \mathcal{N}^+_i \cap \mathcal{N}^+_j$. Then, define $\beta_{ij}$ as the elements of $\mathbb{E}[L^T_2 L_\sigma]$ and see that

\begin{equation}
\beta_{ij} = \mathbb{E} [t^T_j t^T_i] = \sum_{s=1}^{N} \mathbb{E}[t^T_{ij}s] = \sum_{s=1}^{N} \beta_{s,ij}.
\end{equation}

To calculate their values, recall the definition of $l^T_{ij}$ in (12) and distinguish the following five cases:

\begin{align}
\beta^s_{ij} & = \begin{cases} 
\quad p^\perp_{\mathcal{N}^+} (v_s), & \text{if } i = j \neq s \\
\quad (d_i^-)^2 p^2 + d_i^- p (1 - p), & \text{if } i = j = s \\
\quad p^2 \mathbb{I}_{\mathcal{N}^+ \cap \mathcal{N}^+_j} (v_s), & \text{if } s \neq i \neq j \\
\quad - (d_i^- p^2 + p (1 - p)) \mathbb{I}_{\mathcal{N}^+} (v_j), & \text{if } s = i \neq j \\
\quad - (d_j^- p^2 + p (1 - p)) \mathbb{I}_{\mathcal{N}^+} (v_i), & \text{if } i \neq j = s.
\end{cases}
\end{align}

With all five cases covered, sum up the results according to (22) to obtain $\beta_{ij}$. On the main diagonal, we have

\begin{equation}
\beta_{ii} = p^2 \left( (d_i^-)^2 + d_i^+ \right) + p (1 - p) \left( d_i^- + d_i^+ \right)
\end{equation}
while for the off-diagonal entries use \( I_{N_i} (v_i) = I_{N_j} (v_j) \) to arrive at
\[
\beta_{ij} = p^2 \left( \left| N_i \cap N_j \right| - d_i I_{N_i} (v_i) - d_j I_{N_j} (v_j) \right) - p(1 - p) \left( I_{N_i} (v_i) + I_{N_j} (v_j) \right).
\]
(24)

Notice that (23) and (24) contain \( p^2 \) multiplied by (20) and (21), respectively, resulting in the first term in the lemma. Finally, the fact that, in the transposed graph \( G^0^T \), the in- and neighborhood terms are exchanged compared to the original graph \( G^0 \) to see that the remaining terms correspond to the second part of the equation.

Remark: It is possible to exclude opposing links from the independence clause in Assumption 1 because there are no products between \( \alpha_{ij} \) and \( \alpha_{ji} \) for any pair \((i, j)\) in the calculations leading up to \( \beta_{ij} \).

APPENDIX B

PROOF OF THEOREM 7

Proof: First, notice that (7a) is equivalent to
\[
E \left[ A_0^T Q A_0 + C_0^T C_0 \right] - Q \prec 0.
\]
(25)

Then, imposing \( Q = I_N \otimes \tilde{Q} \), apply the same steps as in the proof to Corollary 3 to arrive at
\[
I_N \otimes \left( A_0^T \tilde{Q} A_0 + C_0^T C_0 \right) - \tilde{Q} + 2 \sum_{i \in K} T^i Q A_i C_i + C_i^T C_i + 2 \tilde{Q} \leq E \left[ \tilde{A}_0^T \tilde{A}_0 \right] + E \left[ C_0^T C_0 \right] - Q \prec 0,
\]
(26)

Following the proof of Theorem 6, we utilize Lemmas 4 and 5 to apply a congruence transformation, resulting in:
\[
I_N \otimes \left( A_0^T \tilde{Q} A_0 + C_0^T C_0 \right) - \tilde{Q} + \left( 2 \tilde{Q} \right) = \sum_{i \in K} T^i Q A_i C_i + C_i^T C_i + 2 \tilde{Q} \leq E \left[ \tilde{A}_0^T \tilde{A}_0 \right] + E \left[ C_0^T C_0 \right] - Q \prec 0.
\]
Since every component is block diagonal, this is equivalent to (14a). Moreover, apply the same steps to (7b) without imposing additional constraints on \( Z \), leading to
\[
I_N \otimes \left( B_0^T \tilde{Q} B_0 + D_0^T D_0 \right) - \tilde{Z} + \left( 2 \tilde{Q} \right) = \sum_{i \in K} T^i Q A_i C_i + C_i^T C_i + 2 \tilde{Q} \leq E \left[ \tilde{A}_0^T \tilde{A}_0 \right] + E \left[ C_0^T C_0 \right] - Q \prec 0.
\]

REFERENCES

[1] P. Massioni and M. Verhaegen, “Distributed control for identical dynamically coupled systems: A decomposition approach,” IEEE Trans. Autom. Control, vol. 54, no. 1, pp. 124–135, Jan. 2009.
[2] C. Hoffmann, A. Eichler, and H. Werner, “Distributed control of linear parameter-varying decomposable systems,” in Proc. IEEE Amer. Control Conf., 2013, pp. 2380–2385.
[3] A. Eichler, C. Hoffmann, and H. Werner, “Robust stability analysis of interconnected systems with uncertain time-varying time delays via IQCS,” in Proc. IEEE 52nd Conf. Decis. Control, 2013, pp. 2799–2804.
[4] E. J. A. Fax and R. M. Murray, “Information flow and cooperative control of vehicle formations,” IEEE Trans. Autom. Control, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.
[5] A. Eichler, C. Hoffmann, and H. Werner, “Distributed control of linear parameter-varying decomposable systems,” in Proc. IEEE Amer. Control Conf., 2013, pp. 2380–2385.
[6] C. Hoffmann, A. Eichler, and H. Werner, “Distributed consensus of linear multi-agent systems with nonidentical random packet loss,” in Proc. 59th IEEE Conf. Decis. Control, 2020, pp. 4374–4379.
[7] W. Zhang, X. Wang, J. Kurths, and J. Kurths, “Sampled-data consensus of linear multi-agent systems with packet losses,” IEEE Trans. Neural Netw. Learn. Syst., vol. 28, no. 11, pp. 2516–2527, Nov. 2017.
[8] X. Wang, H. Wang, J. Huang, and J. Kurths, “Sampled-data consensus of multi-agent system in the presence of packet losses,” IEEE Trans. Neural Netw. Learn. Syst., vol. 6, pp. 54844–54853, 2018.
L. Xu, Y. Mo, and L. Xie, “Distributed consensus over Markovian packet loss channels,” IEEE Trans. Autom. Control, vol. 65, no. 1, pp. 279–286, Jan. 2020.

S. Patterson and B. Bamieh, “Convergence rates of consensus algorithms in stochastic networks,” in Proc. IEEE 49th Conf. Decis. Control, 2010, pp. 6608–6613.

J. Wu and Y. Shi, “Average consensus in multi-agent systems with time-varying delays and packet losses,” in Proc. IEEE Amer. Control Conf., 2012, pp. 1579–1584.

R. Ghadami, “Distributed control of multi-agent systems with switching topology, delay, and link failure,” Ph.D. dissertation, Northeastern Univ., Boston, MA, USA, Aug. 2012.

Y. Zhang and Y.-P. Tian, “Maximum allowable loss probability for consensus of multi-agent systems over random weighted lossy networks,” IEEE Trans. Autom. Control, vol. 57, no. 8, pp. 2127–2132, Aug. 2012.

A. A. Stoorvogel, A. Saberi, Z. Liu, and D. N. Jovanovski, “H₂ and H∞ almost output synchronization of heterogeneous continuous-time multi-agent systems with passive agents and partial-state coupling via static protocol,” Int. J. Robust Nonlinear Control, vol. 29, no. 17, pp. 6244–6255, Aug. 2019.

A. Raza, M. Iqbal, J. Moon, and S.-I. Azuma, “Performance measure of hierarchical structures for multi-agent systems,” Int. J. Control Autom. Syst., vol. 20, no. 3, pp. 780–788, Mar. 2022.

O. L. d. V. Costa, J. B. R. D. Val, and J. C. Geromel, “A convex programming approach to H₂ control of discrete-time Markovian jump linear systems,” Int. J. Control, vol. 66, no. 4, pp. 557–580, Jan. 1997.

A. R. Fioravanti, A. P. C. Gonçalves, and J. C. Geromel, “H₂ filtering of discrete-time Markov jump linear systems through linear matrix inequalities,” Int. J. Control, vol. 81, pp. 1221–1231, Jun. 2008.

K. Lee and R. Bhatnacharya, “Stability analysis of large-scale distributed networked control systems with random communication delays: A switched system approach,” Syst. Control Lett., vol. 85, pp. 77–83, Nov. 2015.

O. L. D. V. Costa, R. P. Marques, and M. D. Fragoso, Discrete-Time Markov Jump Linear Systems. Berlin, Germany: Springer, 2005.

O. L. D. V. Costa and M. D. Fragoso, “Stability results for discrete-time linear systems with Markovian jumping parameters,” J. Math. Anal. Appl., vol. 179, no. 1, pp. 154–178, Oct. 1993.

A. R. Fioravanti, A. P. C. Gonçalves, and J. C. Geromel, “Optimal and mode-independent filters for generalised Bernoulli jump systems,” Int. J. Syst. Sci., vol. 46, no. 3, pp. 405–417, Jun. 2013.

A. R. Fioravanti, A. P. C. Gonçalves, G. S. Deaceto, and J. C. Geromel, “Equivalent LMI constraints: Applications to discrete-time MJLS and switched systems,” in Proc. IEEE 51st Conf. Decis. Control, 2012, pp. 1313–1318.

L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, “Foundations of control and estimation over lossy networks,” Proc. IEEE, vol. 95, no. 1, pp. 163–187, Jan. 2007.

R. A. Horn and C. R. Johnson, Matrix Analysis. New York, NY, USA: Cambridge Univ. Press, 2012.

C. W. Wu and L. Chua, “Synchronization in an array of linearly coupled dynamical systems,” IEEE Trans. Circuits Syst., vol. 42, no. 8, pp. 430–447, Aug. 1995.

R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus in networks of multi-agent systems,” Proc. IEEE, vol. 95, no. 1, pp. 215–233, Jan. 2007.

J. Löfberg, “YALMIP: A toolbox for modeling and optimization in MATLAB,” in Proc. IEEE Int. Conf. Rob. Automat., 2004, pp. 284–289.

C. Hespe, H. Saadabadi, A. Datar, H. Werner, and Y. Tang, “Code for a network of agents with ideal and nonideal communication,” ArXiv, 2021, arXiv:1903.08599.

Christian Hespe received the B.Sc. degree in electrical and information engineering from the Hamburg University of Applied Sciences, Germany, in 2018, and the M.Sc. degree in electrical engineering in 2020 from the Hamburg University of Technology, Germany, where he is currently working toward the Ph.D. degree in control systems.

His current research interests include cooperative multiagent systems, their behavior under unreliable communication, and model predictive control.

Christian Hespe received the B.Sc. degree in electrical and information engineering from the Hamburg University of Applied Sciences, Germany, in 2018, and the M.Sc. degree in electrical engineering in 2020 from the Hamburg University of Technology, Germany, where he is currently working toward the Ph.D. degree in control systems.

His current research interests include cooperative multiagent systems, their behavior under unreliable communication, and model predictive control.

Hamidreza Saadabadi received the B.Sc. degree in electrical engineering from Ahvaz University, Ahvaz, Iran, in 2011, and the M.Sc. degree in control systems from Shiraz University, Shiraz, Iran, in 2014.

Since 2018, she has been a Research Assistant at the Hamburg University of Technology, Hamburg, Germany. Her current research interest includes distributed event-triggered control for a network of agents with ideal and nonideal communication.

Adwait Datar received the B.E. degree in mechanical engineering from the University of Pune, Pune, India, in 2012, the M.Sc. degree in mathematical engineering from the University of L’Aquila, L’Aquilla, Italy, in 2016, and the Ph.D. degree in control theory from the School of Electrical Engineering, Computer Science and Mathematics, Hamburg University of Technology, Hamburg, Germany, in 2022.

His research interests include multiagent systems, robust control with integral quadratic constraints, and data-driven techniques in control.

Herbert Werner (Member, IEEE) received a Dipl.-Ing. degree in electrical engineering from the Ruhr University Bochum, Germany, in 1989, an M.Phil. degree in control systems from the University of Strathclyde, Glasgow, UK, in 1991, and the Ph.D. degree in control systems from the Tokyo Institute of Technology, Tokyo, Japan, in 1995.

From 1995 to 1998, he was at Control Engineering Laboratory, Ruhr University Bochum, Bochum, Germany and from 1999 to 2002, at the Control Systems Centre, University of Manchester Institute of Science and Technology, Manchester, U.K. Since 2002, he has been the Head of the Institute of Control Systems, Hamburg University of Technology, Hamburg, Germany. His research interests include linear systems theory, robust and model-predictive control systems, networked control systems, and modeling of uncertain, nonlinear, and time-varying systems.

Yang Tang received the B.S. and Ph.D. degrees in electrical engineering from Donghua University, Shanghai, China, in 2006 and 2010, respectively.

From 2008 to 2010, he was a Research Associate with the Hong Kong Polytechnic University, Hong Kong. From 2011 to 2015, he was a Postdoctoral Researcher with the Humboldt University of Berlin, Berlin, Germany, and with the Potsdam Institute for Climate Impact Research, Potsdam, Germany. Since 2015, he has been a Professor with the East China University of Science and Technology, Shanghai. His current research interests include distributed estimation and control, optimization, cyber-physical systems, hybrid dynamical systems, computer vision, reinforcement learning, and their applications.

Dr. Tang was a recipient of the Alexander von Humboldt Fellowship and has been the ISI Highly Cited Researchers Award by Clarivate Analytics since 2017. He is a Senior Board Member of Scientific Reports, an Associate Editor for IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS, IEEE TRANSACTIONS ON CYBERNETICS, IEEE TRANSACTIONS ON INDUSTRIAL INFORMATICS, IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: REGULAR PAPERS, IEEE TRANSACTIONS ON EMERGING TOPICS IN COMPUTATIONAL INTELLIGENCE, IEEE SYSTEMS JOURNAL AND ENGINEERING APPLICATIONS OF ARTIFICIAL INTELLIGENCE, IFAC Journal, and others. He is a leading Guest Editor for special issues in IEEE TRANSACTIONS ON EMERGING TOPICS IN COMPUTATIONAL INTELLIGENCE and IEEE TRANSACTIONS ON COGNITIVE AND DEVELOPMENTAL SYSTEMS.