Covariant Perturbation Theory (IV). Third Order in the Curvature

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Abstract

The trace of the heat kernel and the one-loop effective action for the generic differential operator are calculated to third order in the background curvatures: the Riemann curvature, the commutator curvature and the potential. In the case of effective action, this is equivalent to a calculation (in the covariant form) of the one-loop vertices in all models of gravitating fields. The basis of nonlocal invariants of third order in the curvature is built, and constraints arising between these invariants in low-dimensional manifolds are obtained. All third-order form factors in the heat kernel and effective action are calculated, and several integral representations for them are obtained. In the case of effective action, this includes a specially generalized spectral representation used in applications to the expectation-value equations. The results for the heat kernel are checked by deriving all the known coefficients of the Schwinger-DeWitt expansion including $a_3$ and the cubic terms of $a_4$. The results for the effective action are checked by deriving the trace anomaly in two and four dimensions. In four dimensions, this derivation is carried out by several different techniques elucidating the mechanism by which the local anomaly emerges from the nonlocal action. In two dimensions, it is shown by a direct calculation that the series for the effective action terminates at second order in the curvature. The asymptotic behaviours of the form factors are calculated including the late-time behaviour in the heat kernel and the small-$\Box$ behaviour in the effective action. In quantum gravity, the latter behaviour contains the effects of vacuum radiation including the Hawking effect.

⋆This paper appeared in February 1993 as the University of Manitoba report, SPIRES-HEP: PRINT-93-0274 (MANITOBA). The purpose of the present publication is to make it more accessible. As compared to the original text, a minor error in the Appendix is corrected. Mathematica files with the results of this paper are included in the source file of the present submission.
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1. Introduction

The present paper is a sequel of the series of papers where covariant perturbation theory was proposed [1] and used [2,3] for the calculation of the heat kernel, effective action, and expectation-value equations in field theory. We refer to ref. [1] as paper I; papers II and III are refs. [2] and [3] respectively.

In the present paper, the trace of the heat kernel and the one-loop effective action for the general background are calculated to third order in the field strengths (curvatures). The calculation of the heat kernel is carried out for an arbitrary space-time dimension, and the effective action is computed in dimensions two and four. For the form factors in the heat kernel and four-dimensional effective action, several integral representations are obtained including the spectral representation used in the expectation-value equations [1,4]. Also, tables of various asymptotic behaviours of the form factors are presented. The results are checked by comparison with the Schwinger-DeWitt expansion for the heat kernel, and by derivation of the trace anomaly in four dimensions. In two dimensions, it is explicitly checked that the expansion of the effective action terminates at second order in the curvature; terms of third order vanish. The mechanism of this vanishing is revealed and shown to be the same by which, in four dimensions, the local trace anomaly emerges from the nonlocal effective action. The derivation of the trace anomaly from the effective action is carried out in several different ways including the technique of spectral representation.

For the present work one needs a classification of nonlocal curvature invariants, including constrains arising between these invariants in low-dimensional manifolds. Such a classification is given to third order in the curvature. The paper is supplied with an appendix which contains a systematic analysis of identities for nonlocal cubic invariants in four dimensions. As a by-product of this analysis, a mechanism is discovered by which the Gauss-Bonnet invariant becomes topological in four dimensions (eq. (A.39) of Appendix).

The present paper, like the preceding ones, deals only with the version of covariant perturbation theory appropriate for noncompact asymptotically flat manifolds. The effective action is computed for euclidean signature of the metric, which, for certain quantum states, is sufficient for obtaining the expectation-value equations of lorentzian field theory [1].

For the motivation of the present study, discussion of the method, and relevant physical problems we refer to the preceding papers and the recent report [5] where covariant perturbation theory along with some of its applications is reviewed. Here we only remind of the notation.

The subject of calculation is the heat kernel

$$K(s) = \exp sH$$  \hspace{1cm} (1.1)

where $H$ is the generic second-order operator

$$H = g^{\mu\nu} \nabla_\mu \nabla_\nu \hat{1} + (\hat{P} - \frac{1}{6} \hat{R} \hat{1}), \quad g^{\mu\nu} \nabla_\mu \nabla_\nu \equiv \Box$$  \hspace{1cm} (1.2)

acting on small disturbances of an arbitrary set of fields $\varphi^A(x)$. Here $A$ stands for any set of discrete indices, and the hat indicates that the quantity is a matrix acting on the vector of small disturbances $\delta \varphi^A$:

$$\hat{1} = \delta^A_B, \quad \hat{P} = P^A_B, \text{ etc.}$$

(1.3)
The matrix trace will be denoted by tr:

\[\text{tr} \hat{1} = \delta^A_A, \quad \text{tr} \hat{P} = P^A_A, \text{ etc.} \quad (1.4)\]

In (1.2), \(g_{\mu\nu}\) is a positive-definite metric characterized by its Riemann curvature \(* R^\alpha_{\beta\mu\nu}\), \(\nabla_\mu\) is a covariant derivative (with respect to an arbitrary connection) characterized by its commutator curvature

\[(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \delta \varphi^A = \mathcal{R}^A_{B\mu\nu} \delta \varphi^B, \quad \mathcal{R}^A_{B\mu\nu} \equiv \hat{\mathcal{R}}_{\mu\nu}, \quad (1.5)\]

and \(\hat{P}\) is an arbitrary matrix. The redefinition of the potential in (1.2) by inclusion of the term in the Ricci scalar \(R\) is a matter of convenience.

For the set of the field strengths (curvatures)

\[R^\alpha_{\beta\mu\nu}, \quad \hat{\mathcal{R}}_{\mu\nu}, \quad \hat{P} \quad (1.6)\]

classifying the background we use the collective notation \(\mathcal{R}\). The calculations in covariant perturbation theory are carried out with accuracy \(O[\mathcal{R}^n]\), i.e. up to terms of \(n\)th and higher power in the curvatures (1.6). It is worth noting that, since the calculations are covariant, any term in \(g_{\mu\nu}\) is in fact of infinite power in the curvature, and \(O[\mathcal{R}^n]\) means terms containing \(n\) or more curvatures explicitly.

The heat kernel (1.1) governs the covariant diagrammatic technique [5–8] to all loop orders. At one-loop order, the effective action is given by the trace of the heat kernel

\[-W = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} K(s) + \int d^{2\omega} x \delta^{(2\omega)}(x,x)(\ldots) \quad (1.7)\]

where \(\text{Tr}\), as distinct from \(\text{tr}\) in (1.4), denotes the functional trace

\[\text{Tr} K(s) = \int d^{2\omega} x \text{ tr}[K(s) \delta^{(2\omega)}(x,y)] \bigg|_{y=x}. \quad (1.8)\]

Here and below, \(2\omega\) is the space-time dimension, and the term with ellipses (\ldots) stands for the contribution of the local functional measure [9], proportional to the delta-function at coincident points. As shown in [10,11], this contribution always cancels the volume divergences of the loop \(*\) which otherwise would appear in (1.7) in the form of a divergent cosmological term. In the case of a massless operator (1.2), the result of this cancellation can be written down as

\[-W = \frac{1}{2} \int_0^\infty \frac{ds}{s} \left( \text{Tr} K(s) - \text{Tr} K(s) \bigg|_{\mathcal{R}=0} \right) \quad (1.9)\]

where subtracted is the zeroth-order term of the covariant expansion in powers of the curvature. The masslessness of the operator (1.2) means that, like the Riemann and commutator curvatures, the potential \(\hat{P}\) falls off at infinity of the manifold. For the precise conditions of this fall off see [2].

\(*\) We use the conventions \(R^\mu_{\alpha\nu\beta} = \partial_\nu \Gamma^\mu_{\alpha\beta} - \cdots\), \(R^\mu_{\alpha\beta} = R^\mu_{\alpha\mu\beta}\), \(R = g^{\alpha\beta} R_{\alpha\beta}\).

\(*\) Loops with the heat kernels are finite, and the ultraviolet divergences appear as divergences of the integrals over \(s\) at the lower limits.
The paper is organized so that the first twelve sections contain only the presentation of the final results, and the remaining eight sections contain their derivations. The final result for the trace of the heat kernel is given in sec. 2. The late-time and early-time asymptotic behaviours of $\text{Tr} K(s)$ are considered in sects. 3 and 4 where also a comparison with the Schwinger-DeWitt expansion is carried out. Sect. 5 contains the calculation of the effective action in two dimensions. The final result for the effective action in four dimensions is given in sect. 6. Integral representations of the form factors in the effective action are given in sects. 7 – 9: the $\alpha$-representation (sect. 7), the Laplace representation (sect. 8), and the spectral representation (sect. 9). Sects. 10 and 11 contain the tables of asymptotic behaviours of the form factors. The trace anomaly is derived in sect. 12. Sects. 13–16 contain the derivation of the result for the trace of the heat kernel, and in sect. 15 an alternative representation of the form factors in $\text{Tr} K(s)$ is given. Sects. 17–20 contain the derivation of the results for the effective action in four dimensions. Appendix contains the analysis of identities for local and nonlocal cubic invariants in low-dimensional manifolds.

The authors dispose of the results of the present paper in the format of the computer algebra program Mathematica. These files are available from the source file submitted at http://arxiv.org.

2. Final result for the trace of the heat kernel to third order in the curvature

The result is

$$\text{Tr} K(s) = \frac{1}{(4\pi s)^{3/2}} \int dx g^{1/2} \text{tr}\{ \hat{1} + s \hat{P} \}$$

$$+ s^2 \sum_{i=1}^{5} f_i(-s \square_2) \mathcal{R}_1 \mathcal{R}_2(i)$$

$$+ s^3 \sum_{i=1}^{11} F_i(-s \square_1, -s \square_2, -s \square_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)$$

$$+ s^4 \sum_{i=12}^{25} F_i(-s \square_1, -s \square_2, -s \square_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)$$

$$+ s^5 \sum_{i=26}^{28} F_i(-s \square_1, -s \square_2, -s \square_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)$$

$$+ s^6 F_{29}(-s \square_1, -s \square_2, -s \square_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(29)$$

$$+ O[\mathcal{R}^4] \}. \tag{2.1}$$

Here terms of zeroth, first and second order in the curvature reproduce the results of paper II *. There are five quadratic structures

$$\mathcal{R}_1 \mathcal{R}_2(1) = R_{1\mu\nu} R^{\mu\nu}_2 \hat{1}, \tag{2.2}$$

*To second order in the curvature, the result for $\text{Tr} K(s)$ was first published in A.O.Barvinsky and G.A.Vilkovisky, Proceedings of the Fourth Seminar on Quantum Gravity, May 25-29, 1987, Moscow, eds. M.A.Markov, V.A.Berezin and V.P.Frolov (World Scientific, Singapore, 1988) p.217.
\[ R_1 R_2 (2) = R_1 R_2 \hat{1}, \quad (2.3) \]
\[ R_1 R_2 (3) = \hat{P}_1 R_2, \quad (2.4) \]
\[ R_1 R_2 (4) = \hat{P}_1 \hat{P}_2, \quad (2.5) \]
\[ R_1 R_2 (5) = \hat{R}_{1\mu} \hat{R}_{2\mu}, \quad (2.6) \]

whose contributions are of the form

\[ \int dx g^{1/2} f(-s \Box) R_1 R_2 = \int dx g^{1/2} R f(-s \Box) R, \quad (2.7) \]

and the notation on the left-hand side of (2.7) assumes that \( \Box \) acts on \( R_2 \). The form factors \( f_i, i = 1 \) to 5, are functions of the operator

\[ -s \Box = \xi \quad (2.8) \]

(with \( \Box \) defined in (1.2)) and are expressed through the basic second-order form factor

\[ f(\xi) = \int_{\alpha \geq 0} d^2 \alpha \delta(1 - \alpha_1 - \alpha_2) \exp(-\alpha_1 \alpha_2 \xi) = \int_0^1 d\alpha e^{-\alpha(1-\alpha)\xi} \quad (2.9) \]

as follows

\[ f_1(\xi) = \frac{(f(\xi) - 1 + \frac{1}{6} \xi)}{\xi^2}, \quad (2.10) \]
\[ f_2(\xi) = \frac{1}{8} \left[ \frac{1}{36} f(\xi) + \frac{1}{3} \frac{(f(\xi) - 1)}{\xi} - \frac{(f(\xi) - 1 + \frac{1}{6} \xi)}{\xi^2} \right], \quad (2.11) \]
\[ f_3(\xi) = \frac{1}{12} f(\xi) + \frac{1}{2} \frac{(f(\xi) - 1)}{\xi}, \quad (2.12) \]
\[ f_4(\xi) = \frac{1}{2} f(\xi), \quad (2.13) \]
\[ f_5(\xi) = -\frac{1}{2} \frac{(f(\xi) - 1)}{\xi}. \quad (2.14) \]

Terms of third order in the curvature in (2.1) are given by a sum of contributions of twenty nine cubic structures. Eleven of them contain no derivatives

\[ R_1 R_2 R_3 (1) = \hat{P}_1 \hat{P}_2 \hat{P}_3, \quad (2.15) \]
\[ R_1 R_2 R_3 (2) = \hat{R}_{1\alpha} \hat{R}_{2\beta} \hat{R}_{3\mu}, \quad (2.16) \]
\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (3) = \hat{\mathcal{R}}_{1}^{\mu \nu} \hat{\mathcal{R}}_{2 \mu \nu} \hat{P}_3, \) \hspace{1cm} (2.17)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (4) = R_1 R_2 \hat{P}_3, \) \hspace{1cm} (2.18)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (5) = R_1^{\mu \nu} R_2 \mu \nu \hat{P}_3, \) \hspace{1cm} (2.19)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (6) = \hat{P}_1 \hat{P}_2 R_3, \) \hspace{1cm} (2.20)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (7) = R_1 \hat{\mathcal{R}}_{2}^{\mu \nu} \hat{\mathcal{R}}_{3 \mu \nu}, \) \hspace{1cm} (2.21)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (8) = R_1^{\alpha \beta} \hat{\mathcal{R}}_{2 \alpha \mu} \hat{\mathcal{R}}_{3 \beta \mu}, \) \hspace{1cm} (2.22)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (9) = R_1 R_2 R_3 \hat{1}, \) \hspace{1cm} (2.23)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (10) = R_1^{\mu} R_2^{\alpha} R_3^{\beta} \hat{1}, \) \hspace{1cm} (2.24)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (11) = R_1^{\mu \nu} R_2 \mu \nu R_3 \hat{1}, \) \hspace{1cm} (2.25)

fourteen contain two derivatives

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (12) = \hat{\mathcal{R}}_{1}^{\alpha \beta} \nabla^\mu \hat{\mathcal{R}}_{2 \mu \alpha} \nabla^\nu \hat{\mathcal{R}}_{3 \nu \beta}, \) \hspace{1cm} (2.26)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (13) = \hat{\mathcal{R}}_{1}^{\mu \nu} \nabla^\mu \hat{P}_2 \nabla^\nu \hat{P}_3, \) \hspace{1cm} (2.27)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (14) = \nabla^\mu \hat{R}_{1}^{\mu \alpha} \nabla^\nu \hat{R}_{2 \nu \alpha} \hat{P}_3, \) \hspace{1cm} (2.28)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (15) = R_1^{\mu \nu} \nabla^\mu R_2 \nabla^\nu \hat{P}_3, \) \hspace{1cm} (2.29)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (16) = \nabla^\mu R_1^{\mu \alpha} \nabla^\nu R_2 \mu \alpha \hat{P}_3, \) \hspace{1cm} (2.30)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (17) = R_1^{\mu \nu} \nabla^\mu \nabla^\nu \hat{P}_2 \hat{P}_3, \) \hspace{1cm} (2.31)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (18) = R_1^{\alpha \beta} \nabla^\mu \hat{R}_{2}^{\mu \alpha} \nabla^\nu \hat{R}_{3 \nu \beta}, \) \hspace{1cm} (2.32)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (19) = R_1^{\alpha \beta} \nabla^\alpha \hat{R}_{2}^{\mu \alpha} \nabla^\beta \hat{R}_{3 \mu \beta}, \) \hspace{1cm} (2.33)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (20) = R_1 \nabla^\alpha \hat{R}_{2}^{\alpha \mu} \nabla^\beta \hat{R}_{3 \beta \mu}, \) \hspace{1cm} (2.34)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (21) = R_1^{\mu \nu} \nabla^\mu \nabla^\lambda \hat{R}_{2}^{\lambda \alpha} \hat{R}_{3 \alpha \nu}, \) \hspace{1cm} (2.35)

\( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (22) = R_1^{\alpha \beta} \nabla^\alpha R_2 \nabla^\beta R_3 \hat{1}, \) \hspace{1cm} (2.36)
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(23) = \nabla^\mu R_1^{\mu \alpha} \nabla_\nu R_2_{\mu \alpha} R_3 \hat{1}, \quad (2.37) \]
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(24) = R_1^{\mu \nu} \nabla_\mu R_2^{\alpha \beta} \nabla_\nu R_3_{\alpha \beta} \hat{1}, \quad (2.38) \]
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(25) = R_1^{\mu \nu} \nabla_\alpha R_2_{\beta \mu} \nabla^\beta R_3^{\alpha \nu} \hat{1}, \quad (2.39) \]

three contain four derivatives
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(26) = \nabla_\alpha \nabla_\beta R_1^{\mu \nu} \nabla_\mu R_2^{\alpha \beta} \hat{P}_3, \quad (2.40) \]
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(27) = \nabla_\alpha \nabla_\beta R_1^{\mu \nu} \nabla_\mu R_2^{\alpha \beta} R_3 \hat{1}, \quad (2.41) \]
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(28) = \nabla_\mu R_1^{\alpha \lambda} \nabla_\nu R_2^{\beta \mu} \nabla^\alpha R_3^{\lambda \nu} \hat{1}, \quad (2.42) \]

and one contains six derivatives
\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(29) = \nabla_\lambda \nabla_\sigma R_1^{\alpha \beta} \nabla_\alpha R_2^{\mu \nu} \nabla_\mu R_3^{\lambda \sigma} \hat{1}. \quad (2.43) \]

These twenty nine structures form a complete basis of nonlocal invariants of third order in the curvature. (Ten of them are purely gravitational, and with gravity switched off there are six.)

The third-order form factors \( F_i, \, i = 1 \text{ to } 29 \), in the expressions
\[ F(-s \square_1, -s \square_2, -s \square_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \]

are functions of three operator arguments
\[ -s \square_1 = \xi_1, \quad -s \square_2 = \xi_2, \quad -s \square_3 = \xi_3, \quad (2.45) \]

and it is assumed that \( \square_1 \) acts on the curvature with the label 1, \( \square_2 \) acts on the curvature with the label 2, \( \square_3 \) acts on the curvature with the label 3. There is no question about a commutativity of \( \square_m \) with \( \nabla \)'s acting on \( \mathcal{R}_m \) in (2.26)–(2.43) since a contribution of any such commutator is already \( O[\mathcal{R}^4] \).

In expression (2.1), the form factors \( F_i \) automatically acquire symmetries (if any) of their respective curvature structures \( \text{tr} \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) \). These symmetries (under permutations of the labels 1,2,3) follow from the table (2.15)–(2.43) and imply the following symmetrization of the form factors:

\[ F_1^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{3} \left( F_1(\xi_1, \xi_2, \xi_3) + F_1(\xi_3, \xi_1, \xi_2) + F_1(\xi_2, \xi_3, \xi_1) \right), \quad (2.46) \]
\[ F_2^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{3} \left( F_2(\xi_1, \xi_2, \xi_3) + F_2(\xi_3, \xi_1, \xi_2) + F_2(\xi_2, \xi_3, \xi_1) \right), \quad (2.47) \]
\[ F_3^{\text{sym}}(\xi_1, \xi_2, \xi_3) = F_3(\xi_1, \xi_2, \xi_3), \quad (2.48) \]
\[
\begin{align*}
F_4^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_4(\xi_1, \xi_2, \xi_3) + F_4(\xi_2, \xi_1, \xi_3) \right), \\
F_5^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_5(\xi_1, \xi_2, \xi_3) + F_5(\xi_2, \xi_1, \xi_3) \right), \\
F_6^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_6(\xi_1, \xi_2, \xi_3) + F_6(\xi_2, \xi_1, \xi_3) \right), \\
F_7^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_7(\xi_1, \xi_2, \xi_3) + F_7(\xi_1, \xi_3, \xi_2) \right), \\
F_8^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_8(\xi_1, \xi_2, \xi_3) + F_8(\xi_1, \xi_3, \xi_2) \right), \\
F_9^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{6} \left( F_9(\xi_1, \xi_2, \xi_3) + F_9(\xi_3, \xi_1, \xi_2) \\
&\quad + F_9(\xi_2, \xi_3, \xi_1) + F_9(\xi_2, \xi_1, \xi_3) \\
&\quad + F_9(\xi_3, \xi_2, \xi_1) + F_9(\xi_1, \xi_3, \xi_2) \right), \\
F_{10}^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{6} \left( F_{10}(\xi_1, \xi_2, \xi_3) + F_{10}(\xi_3, \xi_1, \xi_2) \\
&\quad + F_{10}(\xi_2, \xi_3, \xi_1) + F_{10}(\xi_2, \xi_1, \xi_3) \\
&\quad + F_{10}(\xi_3, \xi_2, \xi_1) + F_{10}(\xi_1, \xi_3, \xi_2) \right), \\
F_{11}^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_{11}(\xi_1, \xi_2, \xi_3) + F_{11}(\xi_2, \xi_1, \xi_3) \right), \\
F_{12}^\text{sym}(\xi_1, \xi_2, \xi_3) &= F_{12}(\xi_1, \xi_2, \xi_3), \\
F_{13}^\text{sym}(\xi_1, \xi_2, \xi_3) &= F_{13}(\xi_1, \xi_2, \xi_3), \\
F_{14}^\text{sym}(\xi_1, \xi_2, \xi_3) &= F_{14}(\xi_1, \xi_2, \xi_3), \\
F_{15}^\text{sym}(\xi_1, \xi_2, \xi_3) &= F_{15}(\xi_1, \xi_2, \xi_3), \\
F_{16}^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_{16}(\xi_1, \xi_2, \xi_3) + F_{16}(\xi_2, \xi_1, \xi_3) \right), \\
F_{17}^\text{sym}(\xi_1, \xi_2, \xi_3) &= F_{17}(\xi_1, \xi_2, \xi_3), \\
F_{18}^\text{sym}(\xi_1, \xi_2, \xi_3) &= \frac{1}{2} \left( F_{18}(\xi_1, \xi_2, \xi_3) + F_{18}(\xi_1, \xi_3, \xi_2) \right),
\end{align*}
\]
\[ F_{19}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{19}(\xi_1, \xi_2, \xi_3) + F_{19}(\xi_1, \xi_3, \xi_2) \right), \quad (2.64) \]

\[ F_{20}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{20}(\xi_1, \xi_2, \xi_3) + F_{20}(\xi_1, \xi_3, \xi_2) \right), \quad (2.65) \]

\[ F_{21}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = F_{21}(\xi_1, \xi_2, \xi_3), \quad (2.66) \]

\[ F_{22}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{22}(\xi_1, \xi_2, \xi_3) + F_{22}(\xi_1, \xi_3, \xi_2) \right), \quad (2.67) \]

\[ F_{23}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{23}(\xi_1, \xi_2, \xi_3) + F_{23}(\xi_2, \xi_1, \xi_3) \right), \quad (2.68) \]

\[ F_{24}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{24}(\xi_1, \xi_2, \xi_3) + F_{24}(\xi_1, \xi_3, \xi_2) \right), \quad (2.69) \]

\[ F_{25}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{25}(\xi_1, \xi_2, \xi_3) + F_{25}(\xi_1, \xi_3, \xi_2) \right), \quad (2.70) \]

\[ F_{26}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{26}(\xi_1, \xi_2, \xi_3) + F_{26}(\xi_2, \xi_1, \xi_3) \right), \quad (2.71) \]

\[ F_{27}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{27}(\xi_1, \xi_2, \xi_3) + F_{27}(\xi_2, \xi_1, \xi_3) \right), \quad (2.72) \]

\[ F_{28}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( F_{28}(\xi_1, \xi_2, \xi_3) + F_{28}(\xi_2, \xi_1, \xi_3) \right), \quad (2.73) \]

\[ F_{29}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{3} \left( F_{29}(\xi_1, \xi_2, \xi_3) + F_{29}(\xi_3, \xi_1, \xi_2) + F_{29}(\xi_2, \xi_3, \xi_1) \right), \quad (2.74) \]

When taken separately from their curvature structures, the functions \( F_i \) make sense only being explicitly symmetrized as above.

All the twenty nine functions \( F_i(\xi_1, \xi_2, \xi_3) \) are expressed through the basic third-order form factor

\[ F(\xi_1, \xi_2, \xi_3) = \int_{\alpha \geq 0} d^3 \alpha \, \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \times \exp(-\alpha_1 \alpha_2 \xi_3 - \alpha_2 \alpha_3 \xi_1 - \alpha_1 \alpha_3 \xi_2) \quad (2.75) \]

(which is completely symmetric in \( \xi_1, \xi_2, \xi_3 \)) and the basic second order form factor \( f(\xi) \) introduced in (2.9). The coefficients of these expressions are rational functions with a universal denominator

\[ \Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1 \xi_2 - 2\xi_1 \xi_3 - 2\xi_2 \xi_3 \quad (2.76) \]

raised to a certain power. In terms of (2.9), (2.75) and (2.76), the explicit expressions for
the (not symmetrized) third-order form factors are as follows:

\[ F_1(\xi_1, \xi_2, \xi_3) = \frac{1}{3} F(\xi_1, \xi_2, \xi_3), \tag{2.77} \]

\[ F_2(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ \frac{4 \xi_1 \xi_2 \xi_3}{3 \Delta^3} (-3 \xi_1^2 \xi_2 - 3 \xi_1 \xi_2 \xi_3 + 3 \xi_3^3) \right. \]
\[ + \frac{4}{\Delta^2} \left( -\xi_1^2 \xi_2 - \xi_1 \xi_2 \xi_3 + 2 \xi_1 \xi_2 \xi_3 + \xi_3^3 \right) \]
\[ + f(\xi_1) \frac{8 \xi_1 \xi_2 \xi_3}{\Delta^3} (\xi_1^2 - \xi_2^2 + 2 \xi_1 \xi_2 - \xi_3^2) \]
\[ + \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{4 \xi_1}{\Delta^2} (3 \xi_1^2 - 2 \xi_1 \xi_2 - \xi_2^2 - 2 \xi_1 \xi_3 + 2 \xi_2 \xi_3 - \xi_3^2) \]
\[ - 2 \frac{1}{\xi_1 - \xi_2} \left( \frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_2) - 1}{\xi_2} \right), \tag{2.78} \]

\[ F_3(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ \frac{2 \xi_1 \xi_2}{\Delta^2} (\xi_1 - \xi_2 - \xi_3)(-\xi_1 + \xi_2 - \xi_3) \right. \]
\[ - \frac{2}{\Delta} (\xi_1 + \xi_2 - \xi_3) + f(\xi_1) \frac{4 \xi_1 \xi_2}{\Delta^2} (-\xi_1 + \xi_2 - \xi_3) \]
\[ + f(\xi_2) \frac{4 \xi_1 \xi_2}{\Delta^2} (\xi_1 - \xi_2 - \xi_3) \]
\[ + f(\xi_3) \frac{1}{\Delta^2} (\xi_1^3 - \xi_1^2 \xi_2 + \xi_1 \xi_2^2 + \xi_2^3 - 3 \xi_1^2 \xi_3 \]
\[ + 6 \xi_1 \xi_2 \xi_3 - 3 \xi_2^2 \xi_3 + 3 \xi_1^2 \xi_3^2 + 3 \xi_2 \xi_3^2 - \xi_3^3), \tag{2.79} \]

\[ F_4(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ \frac{1}{36 \Delta^4} (-4 \xi_1^8 - 4 \xi_1^7 \xi_2 + 32 \xi_1^6 \xi_2^2 - 28 \xi_1^5 \xi_2^3 \right. \]
\[ + 4 \xi_1^4 \xi_2^4 + 2 \xi_1^7 \xi_3 + 26 \xi_1^6 \xi_2 \xi_3 - 90 \xi_1^5 \xi_2^2 \xi_3 + 62 \xi_1^4 \xi_2^3 \xi_3 \]
\[ + 38 \xi_1^6 \xi_3^2 - 60 \xi_1^5 \xi_2 \xi_3^2 + 42 \xi_1^4 \xi_2^2 \xi_3^2 - 20 \xi_1^3 \xi_2 \xi_3^3 - 82 \xi_1^2 \xi_2^3 \xi_3^3 \]
\[ + 62 \xi_1^4 \xi_2 \xi_3^3 + 20 \xi_1^3 \xi_2^2 \xi_3^3 + 50 \xi_1^4 \xi_2^3 \xi_3^3 - 28 \xi_1^3 \xi_2 \xi_3 + 6 \xi_1^2 \xi_2^2 \xi_3^3 \]
\[ + 14 \xi_1^2 \xi_3^5 + 6 \xi_1^2 \xi_2 \xi_3^5 - 22 \xi_1 \xi_2 \xi_3^6 + 2 \xi_1 \xi_3^7 + \xi_2 \xi_3^7 \]
\[ - \frac{4}{3 \Delta^3} (-3 \xi_1^5 \xi_2^2 + 5 \xi_1^4 \xi_3 - 2 \xi_1^3 \xi_2 \xi_3 - 3 \xi_1^2 \xi_2^2 \xi_3 \]
\[ + \xi_1^3 \xi_2^3 + 2 \xi_1^2 \xi_2 \xi_3^2 - 3 \xi_1^2 \xi_3^3 + \xi_1 \xi_2 \xi_3^3 - 2 \xi_1^2 \xi_3^3 \]
\[ + \xi_1^3 \xi_3^3 - \xi_1^2 \xi_3^4 + \xi_5 \xi_3) \]
\[ - \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{2}{\Delta^2} (4 \xi_1^2 + 2 \xi_1 \xi_2 - 2 \xi_1 \xi_3 - \xi_3^2) \]
\[ - f(\xi_1) \frac{1}{24 \Delta^4} \left( \xi_1^8 - 2 \xi_1^7 \xi_2 + 34 \xi_1^6 \xi_2^2 - 74 \xi_1^5 \xi_2^3 + 52 \xi_1^4 \xi_2^4 \right. \]
\[ - 38 \xi_1^3 \xi_2^5 + 46 \xi_1^2 \xi_2^6 - 14 \xi_1 \xi_2^7 - 5 \xi_2^8 - 8 \xi_1 \xi_3 + 38 \xi_1 \xi_2 \xi_3 \]
\[ - 76 \xi_1^3 \xi_2 \xi_3 + 90 \xi_1^2 \xi_2^2 \xi_3 + 16 \xi_1 \xi_2^3 \xi_3 - 134 \xi_1 \xi_2 \xi_3^2 + 68 \xi_1 \xi_2^2 \xi_3 \]
\[ + 6 \xi_2 \xi_3 + 2 \xi_1 \xi_2 \xi_3^2 - 106 \xi_1 \xi_2 \xi_3^2 + 30 \xi_1 \xi_2 \xi_3^2 + 28 \xi_1 \xi_3^2 \xi_3 \]
\[ + 88 \xi_1 \xi_3^2 \xi_3^2 - 114 \xi_1 \xi_3^2 \xi_3^2 + 46 \xi_2 \xi_3^2 - 56 \xi_1 \xi_3^3 + 78 \xi_1 \xi_2 \xi_3^3 \]
\(-8\xi_1^2\xi_2^2\xi_3^3 - 12\xi_1^2\xi_2^2\xi_3^3 + 48\xi_1^2\xi_2^4\xi_3^3 - 146\xi_2^5\xi_3^3 + 70\xi_1^4\xi_3^4 \\
+ 58\xi_1^3\xi_2\xi_3^4 + 94\xi_1^2\xi_2^2\xi_3^4 + 78\xi_1^2\xi_2^3\xi_3^4 + 180\xi_2^4\xi_3^4 - 56\xi_1^3\xi_3^5 \\
- 110\xi_1^2\xi_2^3\xi_3^5 - 108\xi_1^2\xi_2^3\xi_3^5 - 110\xi_2^5\xi_3^5 + 28\xi_1^2\xi_3^6 + 50\xi_1^2\xi_2\xi_3^6 \\
+ 34\xi_2^3\xi_3^6 - 8\xi_1^3\xi_3^7 - 6\xi_2\xi_3^7 + \xi_3^8) \\
- f(\xi_3) \frac{1}{24\Delta^2}(-\xi_1^8 + \xi_1^7\xi_2 - 14\xi_1^6\xi_2^2 + 46\xi_1^5\xi_2^3 \\
- 32\xi_1^4\xi_2\xi_3 - 8\xi_1^7\xi_3 - 30\xi_1^6\xi_2\xi_3 + 44\xi_1^5\xi_2^2\xi_3 - 34\xi_1^4\xi_2^3\xi_3 \\
+ 12\xi_1^3\xi_2^4\xi_3 - 28\xi_1^6\xi_2^3\xi_3 - 78\xi_1^5\xi_2^4\xi_3 - 18\xi_1^4\xi_2^5\xi_3 - 32\xi_1^3\xi_2^6\xi_3 \\
+ 56\xi_1^2\xi_3^3\xi_3^3 - 22\xi_1^3\xi_2^2\xi_3^3 - 16\xi_1^4\xi_2^3\xi_3^3 - 18\xi_1^3\xi_2^4\xi_3^3 + 70\xi_1^2\xi_3^5 \\
- 128\xi_1^3\xi_2^4\xi_3^4 + 82\xi_1^2\xi_2^5\xi_3^4 + 56\xi_1^3\xi_3^5 + 166\xi_1^2\xi_2^6\xi_3^5 + 78\xi_1^2\xi_2^2\xi_3^5 \\
- 28\xi_1^2\xi_3^6 - 78\xi_1\xi_2\xi_3^6 + 8\xi_1\xi_3^7 + 7\xi_2\xi_3^7 - \xi_3^8) \\
- \frac{\left(f(\xi_1) - 1\right)}{\xi_1} \frac{1}{4\Delta^3\xi_2}(-\xi_1^6 + 24\xi_1^5\xi_2 + 41\xi_1^4\xi_2^2 - 24\xi_1^3\xi_2^3 - 13\xi_1^2\xi_2^4 \\
- 32\xi_1^5\xi_2^5 + 3\xi_2^6 - 61\xi_3^6 - 8\xi_1^4\xi_2\xi_3 + 12\xi_1^3\xi_2^2\xi_3 + 40\xi_1^2\xi_2^3\xi_3 \\
+ 74\xi_1^3\xi_2^4\xi_3 - 16\xi_1^6\xi_2^3\xi_3 + 15\xi_1^5\xi_2^4\xi_3 - 32\xi_1^3\xi_2^5\xi_3 - 26\xi_1^2\xi_2^6\xi_3 \\
- 24\xi_1\xi_3^3\xi_3^3 + 35\xi_1^3\xi_2^3\xi_3 + 20\xi_1^3\xi_2^3\xi_3^3 + 16\xi_1^2\xi_2^4\xi_3 + 52\xi_1^2\xi_2^4\xi_3^3 \\
- 40\xi_1^3\xi_3^3 + 15\xi_1\xi_2\xi_3^3 + 40\xi_1\xi_3^4 + 25\xi_2^3\xi_3^4 - 6\xi_1\xi_3^5 - 8\xi_2\xi_3^5 + \xi_3^6) \\
- \frac{\left(f(\xi_3) - 1\right)}{\xi_3} \frac{1}{4\Delta^3\xi_2}(-\xi_1^6 + \xi_1^5\xi_2 - 15\xi_1^4\xi_2^2 + 9\xi_1^3\xi_2^3 + 6\xi_1^2\xi_2^4 \\
- 42\xi_1^4\xi_2\xi_3 + 18\xi_1^3\xi_2^2\xi_3 + 18\xi_1^2\xi_2^3\xi_3 - 15\xi_1^3\xi_2^3 + 17\xi_1^2\xi_2^3 \\
- 2\xi_1^2\xi_3^3 + 20\xi_1^3\xi_2^3 + 52\xi_1^2\xi_2^3\xi_3 + 8\xi_1\xi_2^2\xi_3 + 15\xi_1\xi_2^3 \\
- 23\xi_1\xi_3^3 + 6\xi_1\xi_3^4 + 5\xi_2\xi_3^5 - \xi_3^6) \\
- \frac{1}{\xi_2 - \xi_3} \left( f(\xi_2) - f(\xi_3) \right) \frac{\xi_2 - \xi_3}{24\xi_1} \\
- \frac{1}{\xi_2 - \xi_3} \left( f(\xi_2) - 1 \right) \frac{\xi_2}{\xi_2} \frac{f(\xi_3) - 1}{\xi_3} \frac{\xi_3 - \xi_2 - \xi_3}{4\xi_1}, (2.80) \\
F(\xi_1, \xi_2, \xi_3) = \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{2}{\xi_1\xi_2} - f(\xi_3) \frac{-2\xi_2 + \xi_3}{8\xi_1\xi_2} \\
- \frac{\left(f(\xi_3) - 1\right)}{\xi_3} (-2\xi_2 + 5\xi_3), (2.81) \\
F_0(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ - \frac{1}{6\Delta^2} (2\xi_1^4 + 4\xi_1^3\xi_2 - 6\xi_1^2\xi_2^2 \\
- 2\xi_1^3\xi_3 + 2\xi_1^2\xi_2\xi_3 - 2\xi_1\xi_2\xi_3^2 - 2\xi_1\xi_3^3 + \xi_3^4) - \frac{2\xi_1}{\Delta} \right] \\
- f(\xi_1) \frac{1}{2\Delta^2} (\xi_3^3 + 5\xi_1^2\xi_2 - 5\xi_1\xi_2^2 - \xi_3^2 + \xi_2^3 + \xi_2^3\xi_3 - \xi_2\xi_3^2 - \xi_3^3) \\
- f(\xi_3) \frac{1}{2\Delta^2} (-2\xi_1^3 + 2\xi_1^2\xi_2 + 2\xi_1^2\xi_3 - 2\xi_1\xi_2\xi_3 - 2\xi_1\xi_3^2 + \xi_3^3), (2.82)
\[F_7(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \frac{2\xi_2 \xi_3}{3\Delta^4} (\xi_1^6 - 5\xi_1^5 \xi_3 + 6\xi_1^4 \xi_2 \xi_3 + \xi_1^4 \xi_3^2
\]
\(- 6\xi_1^3 \xi_2 \xi_3^2 + 8\xi_1^2 \xi_2^2 \xi_3^2 + 6\xi_1^3 \xi_3^3 - 4\xi_1^2 \xi_2 \xi_3^3 - 2\xi_1 \xi_2^2 \xi_3^3
\]
\(+ 8\xi_2^3 \xi_3^3 - 4\xi_2^2 \xi_4^3 + 3\xi_1 \xi_2 \xi_3^4 - 9\xi_2^2 \xi_3^3 - \xi_1 \xi_3^5 + \xi_3^6)
\]
\(+ \frac{2}{3\Delta^3} (\xi_1^5 - 7\xi_1^4 \xi_3 + 24\xi_1^3 \xi_2 \xi_3 + 8\xi_1^3 \xi_3^2 - 42\xi_1^2 \xi_2 \xi_3^2 + 34\xi_1 \xi_2^2 \xi_3^2
\]
\(- 2\xi_1^2 \xi_3^3 - 32\xi_1 \xi_2 \xi_3^3 - 34\xi_2 \xi_3^3 - 2\xi_1 \xi_3^4 + 33\xi_2 \xi_3^4 + \xi_3^5))
\]
\(+ \left(F(\xi_1, \xi_2, \xi_3) \frac{2}{\Delta^2} (\xi_1^2 - 4\xi_1 \xi_2 + 10\xi_2 \xi_3 + 2\xi_2^2)
\right)
\]
\(+ f(\xi_1) \frac{1}{12\Delta^4} (-\xi_1^7 + 14\xi_1^6 \xi_3 - 10\xi_1^5 \xi_2 \xi_3 - 7\xi_1^5 \xi_3^2 + 2\xi_1^4 \xi_2 \xi_3^2
\]
\(+ 62\xi_1^3 \xi_2 \xi_3^2 + 76\xi_1^3 \xi_2 \xi_3^3 + 8\xi_1^3 \xi_2^2 \xi_3^3 + 4\xi_1^2 \xi_2 \xi_3^4 + 116\xi_1 \xi_2 \xi_3^4
\]
\(- 70\xi_1^3 \xi_3^4 - 4\xi_1^2 \xi_2 \xi_3^4 - 178\xi_1^2 \xi_3^4 + 58\xi_2 \xi_3^4 + 42\xi_2 \xi_3^5
\]
\(+ 76\xi_2 \xi_3^5 + 90\xi_2 \xi_3^5 - 14\xi_2 \xi_3^6 - 34\xi_2 \xi_3^6 + 2\xi_3^7)
\]
\(+ f(\xi_1) \frac{4\xi_2 \xi_3}{3\Delta^4} (-\xi_1^5 + 7\xi_1^4 \xi_2 - 8\xi_1^3 \xi_2^2 + 2\xi_1^2 \xi_2^3 + 2\xi_1 \xi_2^4 - \xi_2^5
\]
\(+ 3\xi_1^4 \xi_3 - 8\xi_1^3 \xi_2 \xi_3 + 6\xi_1^2 \xi_2 \xi_3^2 - \xi_2^4 \xi_3 + 3\xi_1^3 \xi_3^2 - 4\xi_2 \xi_2 \xi_3^2
\]
\(- 6\xi_1 \xi_2 \xi_2 \xi_3^2 + 8\xi_2 \xi_3 - 4\xi_2 \xi_3^3 + 4\xi_2 \xi_2 \xi_3^3 - 8\xi_2^2 \xi_3^3 + \xi_3^5)
\]
\(- \left(\frac{f(\xi_1)}{\xi_1} - \frac{1}{2\Delta^3} (-\xi_1^5 - 2\xi_1^4 \xi_3 - 36\xi_1^3 \xi_2 \xi_3 + 20\xi_1^3 \xi_3^2
\]
\(+ 28\xi_1^2 \xi_2 \xi_3^2 - 54\xi_1 \xi_2 \xi_3^2 - 28\xi_1 \xi_3^2 + 40\xi_1 \xi_2 \xi_3^3
\]
\(- 4\xi_2 \xi_3^3 + 14\xi_2 \xi_3^4 + 6\xi_2 \xi_3^4 - 2\xi_2^5)
\]
\(+ \left(\frac{f(\xi_3)}{\xi_3} - \frac{1}{2\Delta^3} \xi_3^6 - 6\xi_1^5 \xi_5 + 15\xi_4^4 \xi_2^2 - 20\xi_1^3 \xi_2^3 + 15\xi_1^2 \xi_2^4
\]
\(- 6\xi_1 \xi_2^5 + \xi_2^6 - 8\xi_1^5 \xi_3 + 28\xi_1^4 \xi_2 \xi_3 - 32\xi_1^3 \xi_2 \xi_3 + 8\xi_1^2 \xi_2 \xi_3^2
\]
\(+ 8\xi_1 \xi_2 \xi_3^2 + 4\xi_1^2 \xi_3^2 - 7\xi_1^2 \xi_2 \xi_3^2 + 118\xi_1 \xi_2 \xi_3^2
\]
\(- 6\xi_1 \xi_2 \xi_3^2 + 5\xi_2 \xi_3^2 - 32\xi_1 \xi_2 \xi_3^2 - 32\xi_1 \xi_2 \xi_3^2 - 13\xi_1 \xi_3^4
\]
\(+ 8\xi_1 \xi_2 \xi_3^4 - 5\xi_2 \xi_3^4 + 8\xi_1 \xi_3^5 + 4\xi_2 \xi_3^6)
\]
\(+ \frac{1}{\xi_2 - \xi_3} \left(\frac{f(\xi_3)}{\xi_2} - \frac{f(\xi_3)}{\xi_3} \right) \xi_2 \xi_3^2
\] \right)
\] (2.83)

\[F_8(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[\frac{4\xi_1 \xi_2 \xi_3}{\Delta^4} (-\xi_1^5 + 6\xi_1^4 \xi_3 - 8\xi_1^3 \xi_2 \xi_3 - 4\xi_1^3 \xi_3^2
\]
\(+ 12\xi_1^2 \xi_2 \xi_3^2 - 6\xi_1 \xi_2 \xi_3^2 + 4\xi_1^2 \xi_3^3 + 4\xi_2 \xi_3 + 6\xi_1 \xi_3^4 - 2\xi_2 \xi_3^4 - 2\xi_3^5)
\]
\(+ \frac{8\xi_2 \xi_3}{\Delta^3} (-7\xi_1^3 + 18\xi_1^2 \xi_3 - 14\xi_1 \xi_2 \xi_3 + 6\xi_1 \xi_3^2 + 10\xi_2 \xi_3^2 - 10\xi_3^3)
\]
\(- \left(F(\xi_1, \xi_2, \xi_3) \frac{8}{\Delta^2 \xi_1} (-\xi_1^3 + 6\xi_1^2 \xi_3 + 10\xi_1 \xi_2 \xi_3 - 6\xi_1 \xi_3^2
\]
\(- 2\xi_2 \xi_3^2 + 2\xi_2^3)
\]
\(- f(\xi_1) \frac{8\xi_1 \xi_2 \xi_3}{\Delta^4} (\xi_1 + 4\xi_1^3 \xi_3 + 4\xi_1 \xi_2 \xi_3 - 4\xi_1 \xi_2 \xi_3^2 + 2\xi_2 \xi_3^2
\]
\[ + 4\xi_1\xi_3^2 - 2\xi_3^4 \]
\[ + f(\xi_2)\frac{16\xi_1\xi_2\xi_3}{\Delta^4}(\xi_1^4 - 2\xi_1^3\xi_2 + 2\xi_1\xi_2^3 - \xi_2^4 - 4\xi_1^3\xi_3 \]
\[ + 6\xi_1^2\xi_2\xi_3 - 2\xi_2^3\xi_3 + 6\xi_1\xi_2^2\xi_3^2 - 6\xi_1\xi_2\xi_3^2 - 4\xi_1^3\xi_3^3 + 2\xi_2\xi_3^3 + \xi_3^4 \]
\[ - \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{32\xi_1\xi_2\xi_3}{\Delta^3}(3\xi_1^2 - 2\xi_1\xi_3 + 4\xi_2\xi_3 - 4\xi_3^2) \]
\[ - \left( \frac{f(\xi_2) - 1}{\xi_2} \right) \frac{2}{\Delta^3}(\xi_1^6 - 2\xi_1^5\xi_2 - 5\xi_1^4\xi_2^2 + 20\xi_1^3\xi_2^3 - 25\xi_1^2\xi_2^4 \]
\[ + 14\xi_1\xi_2^5 - 3\xi_1^6 - 6\xi_1^5\xi_3 + 2\xi_1^4\xi_2\xi_3 - 44\xi_1^3\xi_2^2\xi_3 - 44\xi_1^2\xi_2^3\xi_3 \]
\[ + 82\xi_1\xi_2^4\xi_3 + 10\xi_2^5\xi_3 + 15\xi_1^4\xi_3^2 + 12\xi_1^3\xi_2\xi_3^2 + 114\xi_1^2\xi_2^2\xi_3^2 \]
\[ - 36\xi_1^3\xi_2^3\xi_3^2 - 9\xi_1^4\xi_3^3 - 20\xi_1^3\xi_2^3\xi_3^3 - 76\xi_1\xi_2^2\xi_3^3 - 4\xi_2^3\xi_3^3 \]
\[ + 15\xi_1^2\xi_3^5 + 22\xi_1\xi_2^3\xi_3^4 + 11\xi_2^2\xi_3^4 - 6\xi_1\xi_3^6 - 6\xi_2\xi_3^5 + \xi_3^6), \]  
\[ (2.84) \]

\[ F_9(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ \frac{1}{108\Delta^6}(6\xi_1^{11}\xi_2 - 24\xi_1^{10}\xi_2^2 - 26\xi_1^9\xi_2^3 + 126\xi_1^8\xi_2^4 \]
\[ - 108\xi_1^7\xi_2^5 + 24\xi_1^6\xi_2^6 - 54\xi_1^5\xi_2^8\xi_3 + 150\xi_1^4\xi_2^7\xi_3 - 156\xi_1^3\xi_2^6\xi_3 \]
\[ + 60\xi_1^2\xi_2^5\xi_3^2 - 456\xi_1^1\xi_2^4\xi_3^3 - 60\xi_1^0\xi_2^3\xi_3^4 + 222\xi_1^1\xi_2^4\xi_3^3 - 396\xi_1^1\xi_2^4\xi_3^3 \]
\[ + 162\xi_1^2\xi_2^4\xi_3^3 + 304\xi_1^3\xi_2^4\xi_3^3 + 186\xi_1^3\xi_2^4\xi_3^3 - 3\xi_1^5\xi_2^3 + \xi_2^3 \]
\[ - \frac{1}{36\Delta^5}(130\xi_1^8\xi_2^2 + 400\xi_1^7\xi_2^2 - 560\xi_1^6\xi_2^3 + 172\xi_1^5\xi_2^4 \]
\[ - 8\xi_1^5\xi_2^2\xi_3 + 760\xi_1^5\xi_2^3\xi_3 - 570\xi_1^4\xi_2^4\xi_3 + 400\xi_1^4\xi_2^4\xi_3 \]
\[ - 200\xi_1^3\xi_2^4\xi_3^3 - 396\xi_1^2\xi_2^4\xi_3^3 - 156\xi_1\xi_2^2\xi_3^5 + 71\xi_1^3 \]
\[ - \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{4}{\Delta^4}(\xi_1^6 + 5\xi_1^5\xi_2 - 9\xi_1^4\xi_2^2 + 5\xi_1^3\xi_2^3 + 12\xi_1^2\xi_2^4 \]
\[ + 4\xi_1^2\xi_2^2\xi_3 - \xi_1^3\xi_2^3 - \xi_1^3\xi_2^3\xi_3^2 + 7\xi_1^2\xi_2^3\xi_3^3 - 8\xi_1\xi_2\xi_3^5 - 7\xi_3^7) \]
\[ + \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24} \right) \frac{2}{\Delta^3}(2\xi_1^5\xi_2 - 10\xi_1^4\xi_2^2 \]
\[ + 10\xi_1^3\xi_2^3 + 18\xi_1^3\xi_2^2\xi_3 - 20\xi_1^2\xi_2^2\xi_3^2 + 22\xi_1\xi_2\xi_3^4 + \xi_3^6) \]
\[ + f(\xi_1)\frac{1}{288\Delta^6}(5\xi_1^4 + 50\xi_1^3\xi_3 - 2\xi_2^9\xi_2\xi_3 - 12\xi_1^9\xi_3^2 \]
\[ - 124\xi_1^8\xi_2^2\xi_3^2 + 508\xi_1^7\xi_2^2\xi_3^2 - 114\xi_1^6\xi_2^3\xi_3^3 - 48\xi_1^5\xi_2^6\xi_3^3 \]
\[ - 664\xi_1^6\xi_2^4\xi_3^3 + 1144\xi_1^5\xi_2^5\xi_3^3 + \xi_2^8\xi_3^3^4 + 68\xi_1^6\xi_2^6\xi_3^4 \]
\[ - 1196\xi_1^5\xi_2^6\xi_3^4 - 1100\xi_1^4\xi_2^7\xi_3^4 + 1462\xi_1^3\xi_2^8\xi_3^4 + 180\xi_1^5\xi_2^6\xi_3^5 \]
\[ + 232\xi_1^2\xi_2^9\xi_3^5 + 1299\xi_1^2\xi_2^2\xi_3^5 - 2000\xi_1^3\xi_2^3\xi_3^6 - 268\xi_1^2\xi_2^2\xi_3^5 \]
\[ - 140\xi_1^2\xi_2^2\xi_3^5 - 180\xi_1^2\xi_2^2\xi_3^6 + 36\xi_1^4\xi_2^6\xi_3^6 + 72\xi_1^3\xi_2^3\xi_3^6 \]
\[ + 234\xi_1^2\xi_2^3\xi_3^6 + 476\xi_1\xi_2^4\xi_3^6 + 180\xi_2^5\xi_3^6 - 228\xi_1^4\xi_3^7 \]
\[ - 304\xi_1^3\xi_2^7 - 1200\xi_1^2\xi_2^2\xi_3^7 - 496\xi_1^3\xi_2^3\xi_3^7 - 132\xi_2^4\xi_3^7 \]
\[ + 114\xi_1^2\xi_3^8 + 10\xi_1^2\xi_2^2\xi_3^8 + 86\xi_1\xi_2^2\xi_3^8 - 210\xi_1^2\xi_3^8 + 122\xi_1^2\xi_3^9 \]
\[ + 124\xi_1\xi_2\xi_3^9 + 202\xi_2^2\xi_3^9 - 50\xi_1\xi_3^{10} - 30\xi_2\xi_3^{10} - 10\xi_3^{11}) \]
\[- \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{1}{12\Delta^5} (-47\xi_1^9 - \xi_1^8\xi_3 - 103\xi_1^7\xi_2\xi_3 \\
+ 256\xi_1^7\xi_3^2 + 42\xi_1^6\xi_2\xi_3^2 - 174\xi_1^5\xi_2^2\xi_3 - 124\xi_1^6\xi_3^3 \\
+ 418\xi_1^5\xi_2\xi_3^3 + 294\xi_1^4\xi_2^2\xi_3^3 - 30\xi_1^3\xi_2^3\xi_3^3 - 12\xi_1^5\xi_3^4 \\
- 416\xi_1^4\xi_2\xi_3^4 - 1\xi_1^3\xi_2^2\xi_3^4 + 128\xi_1^2\xi_2^3\xi_3^4 - 190\xi_1\xi_2^4\xi_3^4 \\
+ 122\xi_1^4\xi_3^5 + 278\xi_1^3\xi_2\xi_3^5 - 186\xi_1^2\xi_2^2\xi_3^5 + 330\xi_1\xi_2^3\xi_3^5 \\
+ 16\xi_2^4\xi_3^5 - 240\xi_1^3\xi_2^3\xi_3^5 + 54\xi_1^2\xi_2^2\xi_3^5 - 60\xi_1\xi_2\xi_3^5 - 26\xi_2^3\xi_3^5 \\
+ 4\xi_2^2\xi_3^7 - 170\xi_1\xi_2\xi_3^7 + 10\xi_2^2\xi_3^7 + 90\xi_1\xi_3^8 + \xi_2\xi_3^8 - \xi_3^9) \\
+ \left( \frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} \right) \frac{1}{8\Delta^4\xi_2\xi_3} (-4\xi_1^8 + 116\xi_1^7\xi_3 + 369\xi_1^6\xi_2\xi_3 \\
- 400\xi_1^7\xi_3^2 - 146\xi_1^6\xi_2\xi_3^2 + 174\xi_1^5\xi_2^2\xi_3^2 + 536\xi_1^6\xi_3^3 \\
+ 362\xi_1^5\xi_2\xi_3^3 - 46\xi_1^4\xi_2^2\xi_3^3 - 1070\xi_1^3\xi_2^3\xi_3^3 - 176\xi_1^5\xi_3^4 \\
- 354\xi_1^4\xi_2\xi_3^4 + 690\xi_1^3\xi_2^2\xi_3^4 - 740\xi_1^2\xi_2^3\xi_3^4 - 180\xi_1\xi_2^4\xi_3^4 \\
- 272\xi_1^4\xi_3^5 + 70\xi_1^3\xi_2\xi_3^5 + 1650\xi_1^2\xi_2^2\xi_3^5 + 306\xi_1\xi_2^3\xi_3^5 \\
- 26\xi_2^4\xi_3^5 + 304\xi_1^3\xi_2\xi_3^5 - 790\xi_1^2\xi_2^2\xi_3^5 - 132\xi_1\xi_2^3\xi_3^5 + 58\xi_2^3\xi_3^6 \\
- 120\xi_1^2\xi_3^7 - 18\xi_1\xi_2\xi_3^7 - 50\xi_2^2\xi_3^7 + 24\xi_1\xi_3^8 + 22\xi_2\xi_3^8 - 4\xi_3^9) \\
+ \frac{1}{576\xi_1 - \xi_2} \left( \frac{f(\xi_1) - f(\xi_2)}{\xi_1^2} \right) + \frac{1}{48\xi_1 - \xi_2} \left( \frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_2) - 1}{\xi_2} \right) \\
- \frac{1}{\xi_1 - \xi_2} \left( \frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} - \frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{\xi_2^2} \right) \left( \frac{\xi_1}{2\xi_3} - \frac{3}{16} \right), \quad (2.85) \]

\[F_{10}(\xi_1, \xi_2, \xi_3) = \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24} \right) \frac{8}{3\xi_1\xi_2\xi_3} \]

\[- \left( \frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} \right) \frac{2\xi_1}{\xi_2\xi_3}, \quad (2.86) \]

\[F_{11}(\xi_1, \xi_2, \xi_3) = - \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{2}{3\Delta^2\xi_1\xi_2} (\xi_1^4 + 2\xi_1^3\xi_2 - 3\xi_1^2\xi_2^2 \\
- \xi_1^3\xi_3 + \xi_1^2\xi_2\xi_3 - 3\xi_1\xi_2^2\xi_3^2 + 4\xi_1^2\xi_2\xi_3^2 - 5\xi_1\xi_3^3 - \xi_3^4) \\
+ \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24} \right) \frac{2}{\Delta^2\xi_1\xi_2} (-2\xi_1 + \xi_3) \\
- f(\xi_3) \frac{1}{96\xi_1\xi_2} (-2\xi_1 + \xi_3) - \left( \frac{f(\xi_3) - 1}{\xi_3} \right) \frac{1}{6\xi_1\xi_2} (-\xi_1 + \xi_3) \\
- \left( \frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} \right) \frac{1}{\Delta^2\xi_2} (\xi_1^4 + 6\xi_1^3\xi_2 - 8\xi_1^2\xi_2^2 + 2\xi_1\xi_2^3 \\
- \xi_2^4 + 2\xi_1^3\xi_3 + 2\xi_1\xi_2^2\xi_3 + 4\xi_2^3\xi_3 - 8\xi_1^2\xi_2^2 \\
- 10\xi_1\xi_2\xi_3 - 6\xi_2^2\xi_3^2 + 6\xi_1\xi_3^3 + 4\xi_2\xi_3^3 - \xi_3^4) \\
+ \left( \frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2} \right) \frac{1}{8\Delta^2\xi_1\xi_2} (-2\xi_1^5 + 6\xi_1^4\xi_2 - 4\xi_1^3\xi_2^2 \right) \]
\[F_{12}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ -\frac{2\xi_1}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)^2(-\xi_1 - \xi_2 + \xi_3) \right] \]
\[\times (\xi_1 - \xi_2 + \xi_3) - \frac{8}{\Delta^2}(-2\xi_1^2 + \xi_1\xi_2 + \xi_2^2 + \xi_1\xi_3 - 2\xi_2\xi_3 + \xi_3^2) \]
\[- f(\xi_1) \frac{4\xi_1}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3)(\xi_1 - \xi_2 + \xi_3) \]
\[+ f(\xi_2) \frac{4\xi_2}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)^2(-\xi_1 - \xi_2 + \xi_3) \]
\[- f(\xi_3) \frac{4\xi_3}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)(\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 - \xi_3^2) \]
\[+ \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{2}{\Delta^2}\xi_2\xi_3(-\xi_1^3\xi_2 + 3\xi_1^2\xi_2^2 - 3\xi_1^2\xi_2^3 + \xi_2^4 \]
\[- \xi_1^3\xi_3 + 18\xi_1^2\xi_2\xi_3 + 3\xi_1^2\xi_2^3 + 4\xi_2^3\xi_3 + 3\xi_1^2\xi_3^2 \]
\[+ 3\xi_1\xi_2^3 + 6\xi_1^2\xi_2^2 - 3\xi_1^3\xi_3^3 - 4\xi_2^3\xi_3^3 + \xi_3^4) \]
\[+ \left( \frac{f(\xi_2) - 1}{\xi_2} \right) \frac{2}{\Delta^2}\xi_3(-3\xi_1^2\xi_2 + 3\xi_1\xi_2^2 - \xi_2^3 \]
\[- 3\xi_1^2\xi_3 - 6\xi_1\xi_2\xi_3 - 7\xi_2^2\xi_3 + 3\xi_1\xi_3^2 + 9\xi_2\xi_3^2 - \xi_3^3) \]
\[+ \left( \frac{f(\xi_3) - 1}{\xi_3} \right) \frac{2}{\Delta^2}\xi_2(-3\xi_1^2\xi_2 + 3\xi_1\xi_2^2 - \xi_2^3 \]
\[- 3\xi_1^2\xi_3 - 6\xi_1\xi_2\xi_3 + 9\xi_2^2\xi_3 + 3\xi_1\xi_3^2 - 7\xi_2\xi_3^2 - \xi_3^3) \]
\[+ \frac{1}{\xi_1 - \xi_2} \left( \frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_2) - 1}{\xi_2} \right) \frac{2}{\xi_3} \]
\[+ \frac{1}{\xi_1 - \xi_3} \left( \frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_3) - 1}{\xi_3} \right) \frac{2}{\xi_2} \]
\[\] (2.88)

\[F_{13}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \frac{2}{\Delta}(-\xi_1 + \xi_2 + \xi_3) - f(\xi_1) \frac{4}{\Delta} \]
\[- f(\xi_2) \frac{2}{\Delta\xi_1}(-\xi_1 - \xi_2 + \xi_3) - f(\xi_3) \frac{2}{\Delta\xi_1}(-\xi_1 + \xi_2 - \xi_3) \]
\[- \frac{1}{\xi_2 - \xi_3}(f(\xi_2) - f(\xi_3)) \frac{2}{\xi_1} \]
\[\] (2.89)

\[F_{14}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ -\frac{2}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3) \times \right. \]
\[\times (-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3) + \frac{8}{\Delta} \]
\[\]
\begin{align}
  &+ f(\xi_1) \frac{4}{\Delta^2} (-\xi_1 + \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3) \\
  &- f(\xi_2) \frac{4}{\Delta^2} (-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3) \\
  &- f(\xi_3) \frac{4}{\Delta^2} (\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 - \xi_3^2),
\end{align}

\begin{align}
  F_{15}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3) \left[ -\frac{2\xi_1}{3\Delta^4} (-\xi_1 + \xi_2 + \xi_3)^2 (\xi_1^4 + 2\xi_1^3\xi_2 \\
  &- 6\xi_1^2\xi_2^2 + 2\xi_1\xi_2^3 + \xi_2^4 + 2\xi_1^3\xi_3 + 4\xi_1^2\xi_2\xi_3 - 2\xi_1\xi_2^2\xi_3 - 4\xi_2^3\xi_3 \\
  &- 6\xi_1^2\xi_3^2 - 2\xi_1\xi_2\xi_3^2 + 6\xi_2^2\xi_3^2 + 2\xi_1\xi_3^3 - 4\xi_2\xi_3^3 + \xi_3^4) \\
  &- \frac{4}{3\Delta^2} (19\xi_1^4 - 22\xi_1^3\xi_2 - 12\xi_1^2\xi_2^2 + 14\xi_1\xi_2^3 + \xi_2^4 \\
  &- 22\xi_1^3\xi_3 + 40\xi_1^2\xi_2\xi_3 - 14\xi_1\xi_2^2\xi_3 - 4\xi_2^3\xi_3 - 12\xi_1^2\xi_3^2 \\
  &- 14\xi_1\xi_2\xi_3^2 + 6\xi_2^2\xi_3^2 + 14\xi_1\xi_3^3 - 4\xi_2\xi_3^3 + \xi_3^4) \\
  &- \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{48\xi_1}{\Delta^2} \\
  &+ f(\xi_1) \frac{4\xi_1}{3\Delta^4} (-\xi_1 + \xi_2 + \xi_3)(\xi_1^4 + 2\xi_1^3\xi_2 - 6\xi_1^2\xi_2^2 \\
  &+ 2\xi_1^2\xi_3^2 + \xi_2^4 + 2\xi_1^3\xi_3 + 4\xi_1^2\xi_2\xi_3 - 2\xi_1\xi_2^2\xi_3 - 4\xi_2^3\xi_3 \\
  &- 6\xi_1^2\xi_3^2 - 2\xi_1\xi_2\xi_3^2 + 6\xi_2^2\xi_3^2 + 2\xi_1\xi_3^3 - 4\xi_2\xi_3^3 + \xi_3^4) \\
  &- f(\xi_2) \frac{1}{6\Delta^4\xi_1} (-\xi_1 - \xi_2 + \xi_3)(9\xi_1^6 - 8\xi_1^5\xi_2 - 35\xi_1^4\xi_2^2 \\
  &+ 64\xi_1^3\xi_2^3 - 37\xi_1^2\xi_2^4 + 8\xi_1\xi_2^5 - \xi_2^6 - 8\xi_1^5\xi_3 \\
  &+ 6\xi_1^4\xi_2\xi_3 + 20\xi_1^2\xi_2^3\xi_3 - 24\xi_1\xi_2^4\xi_3 + 6\xi_2^5\xi_3 \\
  &- 35\xi_1\xi_3^3 + 34\xi_1^2\xi_2^2\xi_3^2 + 16\xi_1\xi_2\xi_3^3 - 15\xi_2^4\xi_3^2 \\
  &+ 64\xi_1^3\xi_3^3 + 20\xi_1^2\xi_2\xi_3^3 + 16\xi_1\xi_2^2\xi_3^3 + 20\xi_2^3\xi_3^3 \\
  &- 37\xi_2^2\xi_3^4 - 24\xi_1\xi_2^3\xi_3^4 - 15\xi_2\xi_3^5 + 8\xi_1\xi_3^5 + 6\xi_2\xi_3^5 - \xi_3^6) \\
  &- f(\xi_3) \frac{1}{6\Delta^4\xi_1} (-9\xi_1^7 + 17\xi_1^6\xi_2 + 27\xi_1^5\xi_2^2 - 99\xi_1^4\xi_2^3 \\
  &+ 101\xi_1^3\xi_2^4 - 45\xi_1^2\xi_2^5 + 9\xi_1\xi_2^6 - \xi_2^7 - \xi_1^6\xi_3 - 6\xi_1^5\xi_2\xi_3 \\
  &+ 41\xi_1^4\xi_2^2\xi_3 - 84\xi_1^3\xi_2^3\xi_3 + 81\xi_1^2\xi_2^4\xi_3 - 38\xi_1\xi_2^5\xi_3 \\
  &+ 7\xi_2^6\xi_3 + 43\xi_1^5\xi_3^2 - 41\xi_1^4\xi_2\xi_3^2 - 34\xi_1^3\xi_2^2\xi_3^2 \\
  &- 2\xi_1^4\xi_3^3 + 2\xi_2^4\xi_3^2 + 55\xi_1^2\xi_2^3\xi_3^2 - 21\xi_2^5\xi_3^2 - 29\xi_1^3\xi_2\xi_3^2 \\
  &+ 44\xi_1^3\xi_3^3 - 30\xi_1^2\xi_2^2\xi_3^3 - 20\xi_1\xi_2^3\xi_3^3 + 35\xi_2^4\xi_3^3 \\
  &- 27\xi_1^3\xi_3^4 - 33\xi_1^2\xi_2\xi_3^4 - 25\xi_1\xi_2^2\xi_3^4 - 35\xi_2^3\xi_3^4 \\
  &+ 29\xi_1^2\xi_3^5 + 26\xi_1\xi_2\xi_3^5 + 21\xi_2^2\xi_3^5 - 7\xi_1\xi_3^6 - 7\xi_2\xi_3^6 + \xi_3^7) \\
  &- \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{16\xi_1^2}{\Delta^3} (3\xi_1^2 - \xi_1\xi_2 - 2\xi_2^2 - \xi_1\xi_3 + 4\xi_2\xi_3 - 2\xi_3^2) \\
  &+ \left( \frac{f(\xi_2) - 1}{\xi_2} \right) \frac{1}{\Delta^3\xi_1} (-\xi_1 - \xi_2 + \xi_3)(3\xi_1^4 - 46\xi_1^3\xi_2 + 44\xi_1^2\xi_2^2)}
\end{align}
\[-2\xi_1\xi_2^3 + \xi_2^4 - 10\xi_1^3\xi_3 + 40\xi_1^2\xi_2\xi_3 - 2\xi_1\xi_2^2\xi_3 - 4\xi_2^3\xi_3 \\
+ 12\xi_1^2\xi_2^2 + 10\xi_1\xi_2^3 + 6\xi_2^2\xi_3^2 - 6\xi_1\xi_3^3 - 4\xi_2\xi_3^3 + \xi_3^4) \\
+ \left(\frac{f(\xi_1) - 1}{\xi_3}\right) \frac{1}{\Delta^3\xi_1}(-3\xi_1^5 + 13\xi_1^4\xi_2 - 22\xi_1^3\xi_2^2 + 18\xi_1^2\xi_2^3 \\
- 7\xi_1\xi_2^4 + \xi_2^5 + 43\xi_1^4\xi_3 - 76\xi_1^3\xi_2\xi_3 + 18\xi_1^2\xi_2^2\xi_3 + 20\xi_1\xi_2^3\xi_3 \\
- 5\xi_2^4\xi_3 + 2\xi_1^3\xi_3^2 + 6\xi_1^2\xi_2\xi_3^2 - 18\xi_1\xi_2^2\xi_3^2 + 10\xi_2^3\xi_3^2 - 42\xi_2\xi_3^3 \\
+ 4\xi_1\xi_2\xi_3^3 - 10\xi_2^2\xi_3^3 + \xi_1^3\xi_3^4 + 5\xi_2\xi_3^4 - \xi_3^5) \\
+ \frac{1}{\xi_2 - \xi_3}(f(\xi_2) - f(\xi_3)) \frac{1}{6\xi_1} \\
+ \frac{1}{\xi_2 - \xi_3}(f(\xi_2) - f(\xi_3)) \frac{1}{\xi_3} \frac{1}{\xi_1}, \tag{2.91}
\]

\[F_{16}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \frac{8}{\Delta^2}(-2\xi_1^2 + 2\xi_1\xi_2 + \xi_3^2) \]
\[- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{8}{\Delta\xi_1\xi_2}(2\xi_1 - \xi_3) \\
- f(\xi_3) \frac{1}{2\xi_1\xi_2} + \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{32\xi_1}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3) \\
- \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{1}{\Delta^2\xi_1\xi_2}(2\xi_1^4 - 8\xi_1^3\xi_2 + 6\xi_1^2\xi_2^2 - 16\xi_1^3\xi_3 \\
+ 16\xi_1^2\xi_2\xi_3 + 36\xi_1^2\xi_2^3 - 20\xi_1\xi_2\xi_3^2 - 32\xi_2^3\xi_3^3 + 5\xi_3^4), \tag{2.92}
\]

\[F_{17}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\left[- \frac{2\xi_1}{\Delta^2}(-\xi_1 + \xi_2 + \xi_3)^2 - \frac{4}{\Delta}\right] \\
+ f(\xi_1) \frac{4\xi_1}{\Delta^3}(-\xi_1 + \xi_2 + \xi_3) - f(\xi_2) \frac{1}{\Delta^2\xi_1}(-\xi_1 - \xi_2 + \xi_3) \times \\
\times (3\xi_1^2 - 4\xi_1\xi_2 + \xi_2^2 - 4\xi_1\xi_3 - 2\xi_2\xi_3^2 + \xi_3^3) \\
- f(\xi_3) \frac{1}{\Delta^2\xi_1}(-\xi_1 + \xi_2 - \xi_3)(3\xi_1^2 - 4\xi_1\xi_2 + \xi_2^2 - 4\xi_1\xi_3 - 2\xi_2\xi_3 + \xi_3^2) \\
- \frac{1}{\xi_2 - \xi_3}(f(\xi_2) - f(\xi_3)) \frac{1}{\xi_1}, \tag{2.93}
\]

\[F_{18}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\left[- \frac{2\xi_1}{\Delta^4}(\xi_1^6 - 8\xi_1^5\xi_3 + 14\xi_1^4\xi_2\xi_3 + 10\xi_1^4\xi_2^2) \\
- 32\xi_1^3\xi_2\xi_3^2 + 18\xi_1^2\xi_2^2\xi_3^2 + 8\xi_1^2\xi_2\xi_3^3 - 16\xi_1\xi_2^2\xi_3^3 + 4\xi_2^3\xi_3^3 \\
- 10\xi_1^2\xi_3^4 + 8\xi_1\xi_2\xi_3^4 + 2\xi_2^2\xi_3^4 + 8\xi_1^3\xi_3^5 - 4\xi_2\xi_3^5 - 2\xi_3^6) \\
- \frac{4}{\Delta^3}(7\xi_1^4 - 32\xi_1^3\xi_3 + 32\xi_1^2\xi_2\xi_3 + 12\xi_1^2\xi_3^2 \\
- 32\xi_1\xi_2\xi_3^2 + 10\xi_2^2\xi_3^2 + 16\xi_1\xi_3^3 - 10\xi_3^4) \\
- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{32}{\Delta^2\xi_1}(\xi_1^2 - \xi_1\xi_3 + \xi_2\xi_3 - \xi_3^2)
\]

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\begin{align}
&+ f(\xi) \frac{4\xi_1}{\Delta^4} (-\xi_1^5 + 6\xi_1^4\xi_3 - 8\xi_1^3\xi_2\xi_3 - 4\xi_1^3\xi_3^2 + 12\xi_1^2\xi_2\xi_3^2 \\
&- 6\xi_1^2\xi_2\xi_3^2 - 4\xi_1^2\xi_2^3 + 4\xi_2^2\xi_3^3 + 6\xi_1\xi_3^4 - 2\xi_2\xi_3^4 - 2\xi_3^5) \\
&- f(\xi) \frac{8\xi_1}{\Delta^4} (-\xi_1 + \xi_2 + \xi_3)(\xi_1^4 - 2\xi_1^3\xi_2 + 2\xi_1\xi_2^3 - \xi_2^4 - 4\xi_1^3\xi_3 \\
&+ 6\xi_1^2\xi_2\xi_3 - 2\xi_2^3\xi_3 + 6\xi_1^2\xi_3^2 - 6\xi_1\xi_2\xi_3^2 - 4\xi_1\xi_3^3 + 2\xi_2\xi_3^3 + \xi_3^4) \\
&+ \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{16\xi_1}{\Delta^3} (-3\xi_1^3 + 8\xi_1^2\xi_3 - 6\xi_1\xi_2\xi_3 + 2\xi_1\xi_3^2 + 4\xi_2\xi_3^2 - 4\xi_3^3) \\
&+ \left( \frac{f(\xi_2) - 1}{\xi_2} \right) \frac{4}{\Delta^3} (-\xi_1^5 + 17\xi_1^4\xi_2 - 2\xi_1^3\xi_2^2 - 46\xi_1^2\xi_2^3 + 35\xi_1\xi_2^4 \\
&- 3\xi_2^5 + 5\xi_1^4\xi_3 - 44\xi_1^3\xi_2\xi_3 + 22\xi_1^2\xi_2^2\xi_3 + 4\xi_1\xi_2^3\xi_3 + 13\xi_2^4\xi_3 \\
&- 10\xi_1^3\xi_3^2 + 30\xi_1^2\xi_2\xi_3^2 - 38\xi_1\xi_2^2\xi_3^2 - 22\xi_2^3\xi_3^2 + 10\xi_1^2\xi_3^3 \\
&+ 4\xi_1\xi_2\xi_3^3 + 18\xi_2^2\xi_3^3 - 5\xi_1^4\xi_3 - 7\xi_2\xi_3^4 + \xi_3^5),
\end{align}

(2.94)
\[ + 64\xi_1\xi_2\xi_3^2 - 42\xi_2^2\xi_3^2 - 64\xi_1\xi_3^3 + 8\xi_2\xi_3^3 + 34\xi_3^4 \]
\[ + \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{24}{\Delta^2} (\xi_1 - 2\xi_3) \]
\[ + f(\xi_1) \frac{4}{3\Delta^4} (\xi_1^6 - 5\xi_1^5\xi_3 + 6\xi_1^4\xi_2\xi_3 + \xi_1^4\xi_3^2 \]
\[ - 6\xi_1^2\xi_2\xi_3^2 + 8\xi_1^2\xi_2^2\xi_3^2 + 6\xi_1^3\xi_3^3 - 4\xi_1^2\xi_2\xi_3^3 - 2\xi_1\xi_2^2\xi_3^3 \]
\[ + 8\xi_2^3\xi_3^3 - 4\xi_2^2\xi_3^4 + 3\xi_1\xi_2\xi_3^4 - 9\xi_2^3\xi_3^4 - \xi_1\xi_3^5 + \xi_3^6 \]
\[ - f(\xi_3) \frac{4}{3\Delta^4} (2\xi_1^6 - 9\xi_1^5\xi_2 + 15\xi_1^4\xi_2^2 - 10\xi_1^3\xi_2^3 + 3\xi_1\xi_2^5 \]
\[ - \xi_2^6 - 5\xi_1^5\xi_3 + 18\xi_1^4\xi_2\xi_3 - 22\xi_1^3\xi_2^2\xi_3 + 8\xi_1^2\xi_2^3\xi_3 \]
\[ + 3\xi_1\xi_2^4\xi_3 - 2\xi_2^5\xi_3 + \xi_1^4\xi_3^2 - 2\xi_1^3\xi_2\xi_3^2 + 8\xi_1^2\xi_2^2\xi_3^2 \]
\[ - 14\xi_1\xi_2^3\xi_3^2 + 7\xi_1^2\xi_2^3\xi_3^2 + 6\xi_1^3\xi_3^3 - 12\xi_1^2\xi_2\xi_3^3 + 6\xi_1\xi_2^2\xi_3^3 \]
\[ - 4\xi_2^3\xi_3^4 + 3\xi_1\xi_2\xi_3^4 - 7\xi_2^2\xi_3^4 - \xi_1\xi_3^5 + 2\xi_2\xi_3^5 + \xi_3^6 \]
\[ - \left( \frac{f(\xi_1) - 1}{\xi_1} \right) \frac{2}{\Delta^3} (\xi_1^4 + 24\xi_1^3\xi_3 - 20\xi_1^2\xi_2\xi_3 + 20\xi_1^2\xi_3^2 \]
\[ + 24\xi_1\xi_2\xi_3^2 + 6\xi_2^2\xi_3^2 - 24\xi_1\xi_3^3 - 8\xi_2\xi_3^3 + 2\xi_3^4 \]
\[ - \left( \frac{f(\xi_3) - 1}{\xi_3} \right) \frac{2}{\Delta^3} \xi_3 \]
\[ - \frac{1}{\xi_2 - \xi_3} \left( \frac{f(\xi_2) - 1}{\xi_2} - \frac{f(\xi_3) - 1}{\xi_3} \right) \frac{1}{\xi_1}, \] (2.96)

\[ F_{21}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ - \frac{8\xi_1\xi_3}{\Delta^4} (-\xi_1 + \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3) \times \right. \]
\[ \times (-\xi_1 + \xi_2 + \xi_3)^3 + \frac{16}{\Delta^3} (-\xi_1 + \xi_2 + \xi_3) \times \]
\[ \times (-\xi_1^3 + 3\xi_1^2\xi_2 - 3\xi_1\xi_2^2 + \xi_2^3 - 6\xi_1^2\xi_3 \]
\[ + 4\xi_1\xi_2^2\xi_3 + 2\xi_2^2\xi_3 + 3\xi_1\xi_3^2 - 7\xi_2\xi_3^2 + 4\xi_3^3 \] \]
\[ + \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{32}{\Delta^2} \xi_1 \xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 + 4\xi_1\xi_3 - 2\xi_2\xi_3 + \xi_3^2 \right) \]
\[ + f(\xi_1) \frac{16\xi_3^3}{\Delta^4} (-\xi_1 + \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3)^2 \]
\[ - f(\xi_2) \xi_1 \xi_3 \frac{16\xi_1\xi_3}{\Delta^4} (-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3)^3 \]
\[ - f(\xi_3) \frac{16\xi_1\xi_3}{\Delta^4} (-\xi_1 + \xi_2 + \xi_3)(-\xi_1^3 + 3\xi_1^2\xi_2 - 3\xi_1\xi_2^2 \]
\[ + \xi_2^3 + \xi_1^2\xi_3 - 2\xi_1\xi_2\xi_3 + \xi_2^2\xi_3 + \xi_1\xi_3^2 - 2\xi_2\xi_3^2 - \xi_3^3) \]

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\[
\begin{align*}
- & \left( \frac{f(\xi) - 1}{\xi} \right) \frac{32\xi_1}{\Delta^3} (-\xi_1^3 + 3\xi_1^2\xi_2 - 3\xi_1\xi_2^2 + \xi_2^3) \\
- & 5\xi_1^2\xi_2 + 4\xi_1\xi_2^2 + \xi_2^3 + 3\xi_1\xi_3^2 - 5\xi_2\xi_3^2 + 3\xi_3^3) \\
- & \left( \frac{f(\xi) - 1}{\xi} \right) \frac{8}{\Delta^3\xi_1} (-\xi_1^5 + 9\xi_1^4\xi_2 - 22\xi_1^3\xi_2^2 + 22\xi_1^2\xi_2^3 \\
+ & 9\xi_1^2\xi_2^4 + \xi_2^5 + 4\xi_1^4\xi_3 - 4\xi_1^3\xi_2\xi_3 + 16\xi_1^2\xi_2^2\xi_3 - 12\xi_1\xi_2^3\xi_3 \\
- & 4\xi_2\xi_3^4 - 6\xi_1^3\xi_2^2 - 18\xi_1^2\xi_2\xi_3^2 + 10\xi_1\xi_2^2\xi_3^2 + 6\xi_2^3\xi_3^2 \\
+ & 4\xi_2^2\xi_3^3 + 12\xi_1\xi_2^3\xi_3^3 - 4\xi_2^2\xi_3^3 - \xi_1\xi_3^4 + \xi_2\xi_3^4 \\
- & \left( \frac{f(\xi) - 1}{\xi} \right) \frac{8\xi_3}{\Delta^3\xi_1} (\xi_1^4 - 4\xi_1^3\xi_2 + 6\xi_1^2\xi_2^2 - 4\xi_1\xi_2^3 \\
+ & \xi_2^4 + 20\xi_1^3\xi_3 - 44\xi_1^2\xi_2\xi_3 + 28\xi_1\xi_2^2\xi_3 - 4\xi_2^3\xi_3 - 2\xi_1^2\xi_3^2 \\
- & 4\xi_1\xi_2\xi_3^2 + 6\xi_2^2\xi_3^2 - 20\xi_1\xi_3^3 - 4\xi_2\xi_3^3 + \xi_3^4 \\
- & \frac{1}{\xi_2 - \xi_3} \left( \frac{f(\xi) - 1}{\xi} - \frac{f(\xi) - 1}{\xi_3} \right) \frac{4}{\xi_1} \right) \\
& (2.97)
\end{align*}
\]

\[F_{22}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ \frac{\xi_1}{18\Delta^6} (\xi_1^{10} + 4\xi_1^9\xi_3 + 2\xi_1^8\xi_2\xi_3 - 30\xi_1^6\xi_3^2 \\
- 32\xi_1^7\xi_3^2 + 148\xi_1^6\xi_2\xi_3^2 + 16\xi_1^6\xi_2^2\xi_3^2 - 240\xi_1^5\xi_2\xi_3^3 + 248\xi_1^4\xi_2^2\xi_3^3 \\
+ 156\xi_1^5\xi_2^3 - 8\xi_1^5\xi_2\xi_3^2 + 284\xi_1^4\xi_2^2\xi_3^2 - 192\xi_1^3\xi_2^3\xi_3 \\
+ 230\xi_1^3\xi_2^2\xi_3^2 - 264\xi_1^2\xi_2^3\xi_3^2 + 136\xi_1^2\xi_2^2\xi_3^3 + 416\xi_1\xi_2^3\xi_3^3 \\
- 240\xi_1^3\xi_2^2\xi_3^3 + 16\xi_1^4\xi_2\xi_3^3 \\
- 224\xi_1^4\xi_2^2\xi_3^3 + 72\xi_1^3\xi_2^3\xi_3^3 + 176\xi_1^2\xi_2^3\xi_3^3 + 28\xi_2^3\xi_3^3 \\
+ 112\xi_1^2\xi_2^3\xi_3^3 - 80\xi_1^2\xi_2^2\xi_3^3 + 16\xi_2^3\xi_3^3 - 30\xi_1\xi_2^3\xi_3^3 \\
+ 4\xi_1\xi_2^3\xi_3^3 + 26\xi_2^2\xi_3^3 + 4\xi_1\xi_3^3 - 12\xi_2\xi_3^3 + 2\xi_3^10) \\
+ \frac{1}{9\Delta^5} (73\xi_1^8 - 52\xi_1^7\xi_3 + 148\xi_1^6\xi_2\xi_3 - 844\xi_1^6\xi_2\xi_3 \\
+ 130\xi_1^5\xi_3^2 + 1008\xi_1^4\xi_2\xi_3^2 + 1004\xi_1^4\xi_2\xi_3^2 - 1648\xi_1^4\xi_2\xi_3^3 \\
- 760\xi_1^3\xi_2\xi_3^3 + 952\xi_1^3\xi_2^2\xi_3^3 + 320\xi_1^4\xi_2^3\xi_3^3 + 1556\xi_1^3\xi_2\xi_3^4 \\
- 1116\xi_1^2\xi_2\xi_3^4 - 244\xi_1\xi_2\xi_3^4 + 250\xi_2^4\xi_3^4 + 796\xi_1\xi_3^5 \\
+ 72\xi_1^2\xi_2\xi_3^5 + 612\xi_1\xi_2^2\xi_3^5 + 256\xi_2\xi_3^5 + 92\xi_1\xi_3^6 \\
- 532\xi_1\xi_3^6 - 88\xi_1^2\xi_3^6 + 164\xi_1\xi_3^7 + 128\xi_2\xi_3^7 - 34\xi_3^8) \right] \\
- \left( \frac{F(\xi_1, \xi_2, \xi_3) - 1}{\xi} \right) \frac{2}{\Delta^4\xi_1\xi_2\xi_3} (\xi_1^8 - 16\xi_1^7\xi_3 - 44\xi_1^6\xi_2\xi_3 \\
+ 44\xi_1^6\xi_3^2 + 36\xi_1^5\xi_2\xi_3^2 - 116\xi_1^4\xi_2\xi_3^2 - 52\xi_1^5\xi_3^3 \\
+ 208\xi_1^4\xi_2\xi_3^3 + 64\xi_1^4\xi_2\xi_3^3 + 64\xi_1^2\xi_2\xi_3^3 + 20\xi_1^4\xi_3^4 \\
- 72\xi_1^3\xi_2\xi_3^4 + 16\xi_1^2\xi_2\xi_3^4 + 52\xi_1\xi_2\xi_3^4 - 10\xi_1^4\xi_3^4 \\
+ 84\xi_1^2\xi_3^5 - 96\xi_1\xi_2\xi_3^5 - 84\xi_2\xi_3^5 + 8\xi_3^3\xi_3^5 - 4\xi_1^2\xi_3^6 \\
+ 36\xi_1\xi_3^6 + 8\xi_2^2\xi_3^6 - 4\xi_1\xi_3^7 - 8\xi_2\xi_3^7 + 2\xi_3^8) \right)
\]
\[+
296\xi_1 \varepsilon_2^4 - 292\xi_1 \varepsilon_2^5 + 170\xi_1 \varepsilon_2^6 - 58\xi_1 \varepsilon_2^7 + 11\xi_1 \varepsilon_2^8
\]
\[-\xi_2^9 + 185\xi_1 \varepsilon_2 - 258\xi_1 \varepsilon_2^2 + 696\xi_1 \varepsilon_2^3 + 1586\xi_1 \varepsilon_2^4 \]
\[-690\xi_1 \varepsilon_2^5 - 510\xi_1 \varepsilon_2^6 + 488\xi_1 \varepsilon_2^7 - 114\xi_1 \varepsilon_2^8 \]
\[+ 9\xi_2^8 + 116\xi_1 \varepsilon_2^2 - 90\xi_1 \varepsilon_2^3 - 790\xi_1 \varepsilon_2^4 \]
\[+ 480\xi_1 \varepsilon_2^5 - 696\xi_1 \varepsilon_2^6 - 854\xi_1 \varepsilon_2^7 + 298\xi_1 \varepsilon_2^8 \]
\[-36\xi_2^7 + 892\xi_1 \varepsilon_2^2 - 1014\xi_1 \varepsilon_2^3 + 510\xi_1 \varepsilon_2^4 \]
\[-1148\xi_1 \varepsilon_2^5 - 324\xi_1 \varepsilon_2^6 - 202\xi_1 \varepsilon_2^7 + 84\xi_1 \varepsilon_2^8 \]
\[+ 198\xi_1 \varepsilon_2^4 - 1164\xi_1 \varepsilon_2^5 + 1050\xi_1 \varepsilon_2^6 - 282\xi_1 \varepsilon_2^7 \]
\[-240\xi_1 \varepsilon_2^8 - 1262\xi_1 \varepsilon_2^9 + 846\xi_1 \varepsilon_2^{10} + 90\xi_1 \varepsilon_2^{11} \]
\[-352\xi_1 \varepsilon_2^{12} + 394\xi_1 \varepsilon_2^{13} + 126\xi_1 \varepsilon_2^{14} - 348\xi_1 \varepsilon_2^{15} \]
\[+ 310\xi_1 \varepsilon_2^{16} - 106\xi_1 \varepsilon_2^{17} - 84\xi_1 \varepsilon_2^{18} - 140\xi_1 \varepsilon_2^{19} \]
\[-78\xi_1 \varepsilon_2^{20} + 36\xi_1 \varepsilon_2^{21} + 37\xi_1 \varepsilon_2^{22} - 9\xi_1 \varepsilon_2^{23} + \xi_3^9)\]
\[-\left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right) \frac{2}{\Delta^4 \xi_2 \xi_3} (3\xi_1^8 - 42\xi_1^7 \xi_3 - 72\xi_1^6 \xi_2 \xi_3\]
\[+ 104\xi_1^6 \xi_2^2 - 114\xi_1^5 \xi_2 \xi_3^2 - 128\xi_1^4 \xi_2^2 \xi_3^2 - 126\xi_1^5 \xi_3^3 \]
\[+ 288\xi_1^4 \xi_2 \xi_3^3 - 12\xi_1^3 \xi_2^2 \xi_3^3 - 48\xi_1^2 \xi_2^3 \xi_3^3 + 96\xi_1^4 \xi_3^4 \]
\[+ 82\xi_1^3 \xi_2 \xi_3^4 + 144\xi_1^2 \xi_2^2 \xi_3^4 + 90\xi_1 \xi_2 \xi_3^5 + 70\xi_2 \xi_3^4 \]
\[-70\xi_1 \xi_3^5 - 144\xi_1^2 \xi_2 \xi_5^3 - 112\xi_1 \xi_2 \xi_3^5 \]
\[+ 48\xi_1 \xi_2 \xi_3^6 + 90\xi_1 \xi_2 \xi_3^6 + 56\xi_1 \xi_2 \xi_3^6 - 18\xi_1 \xi_3^7 - 16\xi_2 \xi_3^7 + 2\xi_3^8)\]
\[-\left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right) \frac{1}{2\Delta^4 \xi \xi_2} (8\xi_1^8 - 55\xi_1^7 \xi_2 + 167\xi_1^6 \xi_2^2 \]
\[+ 25\xi_1^5 \xi_2^3 - 8\xi_1^4 \xi_2^4 - 253\xi_1^3 \xi_2^5 + 125\xi_1^2 \xi_2^6 - 37\xi_1 \xi_2^7 + 5\xi_2^8 \]
\[-72\xi_1 \xi_3^7 + 285\xi_1^6 \xi_2 \xi_3 - 414\xi_1^5 \xi_2^2 \xi_3 + 195\xi_1^4 \xi_2^3 \xi_3 + 180\xi_1^3 \xi_2^4 \xi_3 \]
\[-333\xi_1^2 \xi_2^5 \xi_3 - 510\xi_1 \xi_2^6 \xi_3 + 280\xi_1^2 \xi_3^2 \xi_3 - 189\xi_1^1 \xi_2^3 \xi_3^2 \]
\[-859\xi_1 \xi_2 \xi_3^3 - 70\xi_1 \xi_2^2 \xi_3^3 + 714\xi_1 \xi_2 \xi_3^4 \xi_3^3 - 407\xi_1 \xi_2 \xi_3^5 \xi_3^2 \]
\[+ 153\xi_1 \xi_2 \xi_3^6 + 520\xi_1 \xi_2 \xi_3^7 + 641\xi_1 \xi_2 \xi_3^7 + 204\xi_1 \xi_2 \xi_3^8 \]
\[-250\xi_1 \xi_2 \xi_3^9 + 116\xi_1 \xi_2 \xi_3^{10} - 191\xi_1 \xi_2 \xi_3^{11} + 456\xi_1 \xi_2 \xi_3^{12} \]
\[-421\xi_1 \xi_2 \xi_3^{13} + 609\xi_1 \xi_2 \xi_3^{14} + 573\xi_1 \xi_2 \xi_3^{15} + 95\xi_1 \xi_2 \xi_3^{16} \]
\[-152\xi_1 \xi_2 \xi_3^{17} - 873\xi_1 \xi_2 \xi_3^{18} - 718\xi_1 \xi_2 \xi_3^{19} + 25\xi_2 \xi_3^{20} \xi_3^5 + 8\xi_1 \xi_2 \xi_3^6 \]
\[+ 287\xi_1 \xi_2 \xi_3^6 + 51\xi_2 \xi_3^6 - 24\xi_1 \xi_3^7 - 53\xi_2 \xi_3^7 + 16\xi_3^8)\]
\[+ \frac{1}{\xi_2 - \xi_3} \left(\frac{f(\xi_2) - f(\xi_3)}{144\xi_1} + \frac{1}{\xi_2 - \xi_3} \left(\frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{\xi_2^2} - \frac{f(\xi_3) - 1}{\xi_3^2}\right)\right) \frac{1}{12\xi_1}\]
\[-\frac{1}{\xi_1 - \xi_3} \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} - \frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right) \frac{2}{\xi_2}\]
\[-\frac{1}{\xi_2 - \xi_3} \left(\frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{\xi_2^2} - \frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right) \frac{1}{4\xi_1}\]
\[ 
\begin{align*}
&+ 3\xi_1^4\xi_3^3 - 2\xi_1^3\xi_2^2\xi_3 - 4\xi_1^4\xi_3^2 - 4\xi_1^3\xi_2\xi_3^2 + 8\xi_1^2\xi_2^2\xi_3^2 \\
&+ 6\xi_1^3\xi_3^3 - 6\xi_1^2\xi_2\xi_3^3 + \xi_1^2\xi_3^4 + 6\xi_1\xi_2\xi_3^4 - 5\xi_1\xi_3^5 + \xi_3^6) \\
&- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{8}{3\Delta^3\xi_1\xi_2}(-\xi_1^5 - 33\xi_1^4\xi_2 + 34\xi_1^3\xi_2^2 \\
&+ 2\xi_1^4\xi_3 + 32\xi_1^3\xi_2\xi_3 - 34\xi_1^2\xi_2^2\xi_3 + 2\xi_1^3\xi_2^3 + 42\xi_1^2\xi_2^2\xi_3^2 \\
&- 8\xi_1^2\xi_3^3 - 24\xi_1\xi_2\xi_3^3 + 7\xi_2^4\xi_3 - \xi_3^5) \\
&+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right) \frac{8}{\Delta^2\xi_1\xi_2}(2\xi_1^2 + 10\xi_1\xi_2 - 4\xi_1\xi_3 + \xi_3^2) \\
&- f(\xi_3) \frac{1}{2\Delta^3\xi_1\xi_2} - \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{16\xi_1}{3\Delta^3}(-\xi_1 + \xi_2 - \xi_3)(\xi_1^4 + 2\xi_1^3) \\
&- 6\xi_1^2\xi_2^2 + 2\xi_1\xi_3^2 + \xi_2^4 - \xi_1^3\xi_3 + \xi_1^2\xi_2\xi_3 + \xi_1\xi_2^2\xi_3 - \xi_2^3\xi_3 \\
&- 3\xi_2^4\xi_3^2 - 8\xi_2\xi_3^3 - 3\xi_2^2\xi_3^2 + 5\xi_1\xi_3^3 + 5\xi_2\xi_3^3 - 2\xi_3^4) \\
&- \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{2}{3\Delta^3\xi_1\xi_2}(\xi_1^2 - 8\xi_1\xi_2 + 28\xi_1\xi_3^2 - 56\xi_1^5\xi_3^2) \\
&+ 35\xi_1^4\xi_2^4 - 9\xi_1^7\xi_3 + 57\xi_1^6\xi_2\xi_3 - 117\xi_1^5\xi_2^2\xi_3 \\
&+ 69\xi_1^5\xi_2^3\xi_3 + 35\xi_1^6\xi_3^2 - 110\xi_1^5\xi_2\xi_3^2 + 125\xi_1^4\xi_2^2\xi_3^2 \\
&- 50\xi_1^3\xi_2^3\xi_3^2 - 77\xi_1^5\xi_3^3 + 63\xi_1^4\xi_2\xi_3^3 + 14\xi_1^3\xi_2^2\xi_3^3 \\
&+ 105\xi_1^4\xi_3^3 + 36\xi_1^3\xi_2^3\xi_3^3 - 13\xi_1^2\xi_2^2\xi_3^3 - 91\xi_1^3\xi_3^3 \\
&- 73\xi_1^2\xi_2\xi_3^4 + 49\xi_1\xi_2^2\xi_3^4 + 25\xi_1\xi_2\xi_3^5 - 15\xi_1\xi_3^7 + \xi_3^8) \\
&+ \left(\frac{f(\xi_1) - 1 + \frac{1}{5}\xi_1}{\xi_1^2}\right) \frac{4}{\Delta^3\xi_2\xi_3}(-\xi_1^5\xi_2 + 5\xi_1^4\xi_2^2 \\
&- 10\xi_1^3\xi_2^3 + 10\xi_1^2\xi_2^4 - 5\xi_1\xi_2^5 + \xi_2^6 + \xi_1^5\xi_3 \\
&+ 50\xi_1^4\xi_2\xi_3 - 22\xi_1^3\xi_2^2\xi_3 - 36\xi_1^2\xi_2^3\xi_3 + 13\xi_1\xi_2^4\xi_3 - 6\xi_2^5\xi_3 \\
&+ \xi_1^4\xi_2^2 - 22\xi_1^3\xi_2^3\xi_3 + 56\xi_1^2\xi_2^4\xi_3 - 2\xi_1\xi_2^5\xi_3^2 + 15\xi_2^4\xi_3^2 \\
&- 10\xi_1^3\xi_2^3 - 44\xi_1^2\xi_2\xi_3^3 - 22\xi_1\xi_2^2\xi_3^3 - 20\xi_2^3\xi_3^3 + 14\xi_1^2\xi_3^4 \\
&+ 23\xi_1\xi_2\xi_3^4 + 15\xi_2^2\xi_3^4 - 7\xi_1^5\xi_3^3 - 6\xi_2\xi_3^5 + \xi_3^6) \\
&+ \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right) \frac{1}{2\Delta^3\xi_1\xi_2}(2\xi_1^6 - 12\xi_1^5\xi_2 + 30\xi_1^4\xi_2^2 \\
&- 20\xi_1^3\xi_2^3 - 4\xi_1^5\xi_3 + 12\xi_1^4\xi_2\xi_3 - 8\xi_1^3\xi_2^2\xi_3 - 26\xi_1^4\xi_3^2 \\
&- 184\xi_1^3\xi_2\xi_3^2 + 210\xi_1^2\xi_2^2\xi_3^2 + 72\xi_1^3\xi_3^3 - 136\xi_1^2\xi_2\xi_3^3 \\
&- 50\xi_1^2\xi_2^2 + 162\xi_1\xi_2\xi_3^4 - 4\xi_1\xi_3^5 + 5\xi_3^6) \\
&+ \frac{1}{\xi_1 - \xi_2} \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} - \frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{\xi_2^2}\right) \frac{2}{\xi_3}.
\end{align*}
\]

\[F_{24}(\xi_1, \xi_2, \xi_3) = \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{4\xi_1}{\Delta^3\xi_1\xi_2}(\xi_1^2 - 4\xi_1\xi_3 + 2\xi_2\xi_3 + 2\xi_3^2) \\
+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right) \frac{8}{\Delta^2\xi_2\xi_3},\]
\[ F_{26}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left[ \frac{4\xi_1 \xi_2}{\Delta^4} (2\xi_1^4 - 8\xi_1^3 \xi_2 + 6\xi_1^2 \xi_2^2 - 4\xi_1^2 \xi_3^2) + 4\xi_1 \xi_2 \xi_3^2 + \xi_1^4 \right] - \frac{16}{\Delta^3} (-3\xi_1^3 + 3\xi_1^2 \xi_2 + 4\xi_1^2 \xi_3) - 4\xi_1^2 \xi_3 + \xi_1 \xi_2^2 - \xi_3^3) \right] + \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{8}{\Delta^2 \xi_1 \xi_2} (2\xi_1^2 + 4\xi_1 \xi_2 - 4\xi_1 \xi_3 + \xi_3^2) - f(\xi_1) \frac{16\xi_1 \xi_2}{\Delta^4} (-\xi_1 + \xi_2 - \xi_3)^2 (-\xi_1 + \xi_2 + \xi_3) - f(\xi_3) \frac{1}{2\Delta^4 \xi_1 \xi_2} (-2\xi_1^7 + 18\xi_1^6 \xi_2 - 42\xi_1^5 \xi_2^2 + 26\xi_1^4 \xi_2^3 + 14\xi_1^6 \xi_3 - 76\xi_1^5 \xi_2 \xi_3 + 178\xi_1^4 \xi_2^2 \xi_3 - 116\xi_1^3 \xi_2 \xi_3^3 + 42\xi_1^5 \xi_2^2 + 110\xi_1^4 \xi_2 \xi_3^2 - 68\xi_1^3 \xi_2 \xi_3^2 + 70\xi_1^2 \xi_2 \xi_3 - 40\xi_1^3 \xi_2 \xi_3^3 - 30\xi_1^2 \xi_2^2 \xi_3^2 - 70\xi_1^3 \xi_2^3 - 50\xi_1^2 \xi_2 \xi_3^4 + 42\xi_1^2 \xi_3^5 - 26\xi_1 \xi_2 \xi_3^5 - 14\xi_1 \xi_3^6 + \xi_3^7) \]
\begin{align*}
+ & \left( \frac{f(\xi) - 1}{\xi_3} \right) \frac{32\xi}{\Delta^3}(3\xi^1 - \xi_1\xi_2 - 2\xi_2^2 - \xi_1\xi_3 + 4\xi_2\xi_3 - 2\xi_3^2) \\
- & \left( \frac{f(\xi) - 1}{\xi_3} \right) \frac{1}{3\Delta^3\xi_1\xi_2}(-2\xi_1^5 + 6\xi_1^4\xi_2 - 4\xi_1^3\xi_2^2 + 18\xi_1^4\xi_3 \\
+ & 24\xi_1^2\xi_2\xi_3 - 42\xi_1^2\xi_2^2\xi_3 - 52\xi_1^2\xi_3^2 + 52\xi_1^2\xi_2\xi_3^2 \\
+ & 68\xi_1^2\xi_3^3 - 20\xi_1\xi_2\xi_3^3 - 42\xi_1\xi_3^4 + 5\xi_3^5), \tag{2.102}
\end{align*}

\begin{align*}
F_{27}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\left[-\frac{2\xi_1\xi_2}{3\Delta^3}(2\xi_1^8 - 4\xi_1^7\xi_2 - 16\xi_1^6\xi_2^2 + 68\xi_1^5\xi_2^3 \\
- & 50\xi_1^4\xi_2^4 - 2\xi_1^7\xi_3 + 10\xi_1^6\xi_2\xi_3 - 18\xi_1^5\xi_2^2\xi_3 + 10\xi_1^4\xi_2^3\xi_3 - 4\xi_1^6\xi_3^2 \\
- & 4\xi_1^3\xi_2^3\xi_3 + 52\xi_1^4\xi_2^3\xi_3^2 - 44\xi_1^3\xi_2^3\xi_3^2 + 2\xi_1^5\xi_3^3 - 6\xi_1^4\xi_2\xi_3^3 \\
+ & 4\xi_1^3\xi_2^3\xi_3^3 + 4\xi_1^3\xi_2^3\xi_3^4 - 8\xi_1^2\xi_2^3\xi_3^4 + 2\xi_1^3\xi_3^5 \\
- & 2\xi_1^2\xi_2^3\xi_3^5 - 4\xi_1^2\xi_2^3\xi_3^6 + 2\xi_1\xi_2\xi_3^6 - 2\xi_1\xi_3^7 + \xi_3^8)
\right] \\
+ & \frac{8}{3\Delta^3}(-3\xi_1^7 - 30\xi_1^5\xi_2 + 108\xi_1^5\xi_2^2 - 75\xi_1^4\xi_2^3 + \xi_1^6\xi_3 \\
+ & 30\xi_1^5\xi_2\xi_3 - 129\xi_1^4\xi_2^2\xi_3 + 98\xi_1^3\xi_2^3\xi_3 + 15\xi_1^5\xi_3^2 \\
+ & 21\xi_1^4\xi_2^3\xi_3^2 - 36\xi_1^3\xi_2^3\xi_3^2 - 12\xi_1^4\xi_3^3 - 12\xi_1^3\xi_2\xi_3^3 \\
+ & 24\xi_1^2\xi_2\xi_3^3 - 17\xi_1^3\xi_3^4 - 12\xi_1^2\xi_2\xi_3^4 + 21\xi_1^2\xi_3^5 \\
+ & 3\xi_2\xi_3^5 - 3\xi_1\xi_3^6 - \xi_3^7) \\
- & \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{4}{3\Delta^3\xi_1\xi_2\xi_3}(-6\xi_1^7 + 30\xi_1^6\xi_2 - 54\xi_1^5\xi_2^2 \\
+ & 30\xi_1^5\xi_2^3 + 38\xi_1^6\xi_3 + 120\xi_1^5\xi_2\xi_3 + 258\xi_1^4\xi_2^2\xi_3 \\
- & 416\xi_1^4\xi_2^3\xi_3 - 66\xi_1^5\xi_3^2 + 222\xi_1^4\xi_2^2\xi_3 + 288\xi_1^3\xi_2^3\xi_3^2 \\
+ & 6\xi_1^3\xi_3^3 - 12\xi_1^3\xi_2\xi_3^3 - 174\xi_1^2\xi_2^2\xi_3^3 + 86\xi_1^3\xi_3^4 \\
+ & 78\xi_1^2\xi_2^3\xi_3^4 - 78\xi_1^2\xi_3^5 - 6\xi_1\xi_2\xi_3^5 + 18\xi_1\xi_3^6 + \xi_3^7) \\
+ & \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24} \right) \frac{16}{\Delta^3\xi_1\xi_2\xi_3}(\xi_1^4 - 4\xi_1^3\xi_2 + 3\xi_1^2\xi_2^2 \\
- & 7\xi_1\xi_2^3 - 23\xi_2^2\xi_3 + 9\xi_1^2\xi_3^2 + 11\xi_1\xi_2\xi_3^2 - \xi_1\xi_3^3 - \xi_3^4) \\
+ & f(\xi_1) \frac{8\xi_1\xi_2}{3\Delta^6}(-\xi_1 + \xi_2 - \xi_3^2)(-\xi_1 + \xi_2 + \xi_3) \times \\
\times (\xi_1^4 + 2\xi_1^3\xi_2 - 6\xi_1^2\xi_2^2 + 2\xi_1\xi_2^3 + \xi_2^4 - \xi_1^3\xi_3 + \xi_1^2\xi_2\xi_3 \\
+ \xi_1\xi_2^3 - \xi_2^3\xi_3 - 2\xi_1\xi_2\xi_3^2 - \xi_1^3\xi_3^2 - \xi_2\xi_3^3 + \xi_3^4) \\
- & \frac{f(\xi_3)}{24\Delta^6\xi_1\xi_2}(-2\xi_1^{11} + 26\xi_1^{10}\xi_2 - 222\xi_1^9\xi_2^2 + 790\xi_1^8\xi_2^3 \\
- & 1268\xi_1^7\xi_2^4 + 676\xi_1^6\xi_2^5 + 22\xi_1^{10}\xi_3 - 212\xi_1^9\xi_2\xi_3 \\
+ & 894\xi_1^5\xi_3^2\xi_3 - 2224\xi_1^7\xi_2^3\xi_3 + 3692\xi_1^6\xi_2^4\xi_3 - 2172\xi_1^5\xi_2^5\xi_3 \\
- & 110\xi_1^9\xi_3^2 + 738\xi_1^8\xi_2^3 - 1816\xi_1^7\xi_2^2\xi_3 + 2120\xi_1^6\xi_2^3\xi_3^2 \\
- & 932\xi_1^5\xi_3^2\xi_3^2 + 330\xi_1^8\xi_3^3 - 1392\xi_1^7\xi_2\xi_3^3 + 1912\xi_1^6\xi_2^2\xi_3^3 \\
+ & 400\xi_1^5\xi_3^2\xi_3^3 - 450\xi_1^4\xi_2^4\xi_3^3 - 660\xi_1^7\xi_2^3 + 1428\xi_1^6\xi_2^4\xi_3^2 \\
- & 484\xi_1^5\xi_2^2\xi_3^4 - 284\xi_1^4\xi_2^3\xi_3^4 + 924\xi_1^6\xi_2^3\xi_3^5 - 504\xi_1^5\xi_2^3\xi_3^5)
\right.
\end{align*}
\[-380\xi_1^4\xi_2^2\xi_3^5 - 40\xi_1^3\xi_2^3\xi_3^5 - 924\xi_1^5\xi_3^6 - 588\xi_1^4\xi_2\xi_3^6
\- 504\xi_1^3\xi_2^3\xi_3^6 + 660\xi_1^4\xi_3^7 + 912\xi_1^3\xi_2^3\xi_3^7 + 524\xi_1^2\xi_2^2\xi_3^7
\- 330\xi_1^2\xi_3^8 - 558\xi_1^2\xi_2^3\xi_3^8 + 110\xi_1\xi_3^9
\+ 86\xi_1\xi_2\xi_3^9 - 22\xi_1\xi_3^10 + \xi_3^{11})
\- \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{16\xi_1}{3\Delta^3}(3\xi_1^6 - 32\xi_1^5\xi_2 - 76\xi_1^4\xi_2^2 + 8\xi_1^3\xi_2^3 + 67\xi_1^2\xi_2^4
\- 32\xi_1\xi_2^5 - 2\xi_2^6 + 2\xi_1^5\xi_3 + 3\xi_1^4\xi_2\xi_3 + 51\xi_1^3\xi_2^2\xi_3 - 119\xi_1^2\xi_2^3\xi_3
\+ 63\xi_1^2\xi_2^4\xi_3 - 13\xi_1^4\xi_3^2 - 30\xi_1^3\xi_2\xi_3^2 + 53\xi_1^2\xi_2^2\xi_3^2 - 28\xi_1\xi_2\xi_3^3
\+ 18\xi_1\xi_2^2\xi_3^2 - \xi_3^3 \cdot 17\xi_1\xi_2^3\xi_3^2 - 10\xi_1\xi_2^3\xi_3^3 - 32\xi_2\xi_3^3
\+ 16\xi_1^2\xi_2^4 + 12\xi_1\xi_2\xi_3^4 + 18\xi_2^2\xi_3^4 - 5\xi_1\xi_3^5 - 2\xi_3^6)
\- \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{1}{3\Delta^5\xi_1\xi_2}(-2\xi_1^9 + 20\xi_1^8\xi_2 - 70\xi_1^7\xi_2^2
\+ 110\xi_1^6\xi_2^3 - 58\xi_1^5\xi_2^4 + 20\xi_1^4\xi_3 - 194\xi_1^3\xi_2\xi_3 + 44\xi_1^2\xi_2^2\xi_3
\+ 1250\xi_1^5\xi_2^3 - 120\xi_1^4\xi_2\xi_3 - 88\xi_1^7\xi_2^3 + 390\xi_1^6\xi_2^2\xi_3
\- 642\xi_1^5\xi_2^3\xi_3^2 + 340\xi_1^4\xi_2\xi_3^2 + 224\xi_1^3\xi_2\xi_3^2 - 378\xi_1^2\xi_2\xi_3^2
\+ 552\xi_1^2\xi_2\xi_3^2 - 30\xi_2\xi_3^3 + 125\xi_1\xi_2\xi_3^3 - 364\xi_1^4\xi_2^4 + 326\xi_1^4\xi_2^4
\+ 38\xi_1^3\xi_2^4 + 392\xi_1^4\xi_3^5 - 86\xi_1^3\xi_2\xi_3^5 + 102\xi_1^2\xi_2\xi_3^5
\- 28\xi_1\xi_2\xi_3^6 - 254\xi_1\xi_2\xi_3^6 + 128\xi_1\xi_2\xi_3^7 + 105\xi_1\xi_2\xi_3^7 - 34\xi_1\xi_3^8 + 2\xi_3^9)
\+ \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{1}{2\Delta^4\xi_1\xi_2}4\xi_1^6 - 22\xi_1^5\xi_2^2 + 35\xi_1^4\xi_2^2 - 20\xi_1^3\xi_2^3
\- 5\xi_1^2\xi_2^4 + 10\xi_1\xi_2^5 - 3\xi_2^6 - 27\xi_1^5\xi_3 - 97\xi_1^4\xi_2\xi_3
\- 166\xi_1^3\xi_2^2\xi_3 + 222\xi_1^2\xi_2^2\xi_3 + 49\xi_1\xi_2^2\xi_3 + 19\xi_1^2\xi_2\xi_3 + 36\xi_1^4\xi_3^2
\+ 80\xi_1^3\xi_2^2\xi_3^2 - 192\xi_1^2\xi_2^2\xi_3^2 - 120\xi_1\xi_2^2\xi_3^2 - 44\xi_2^2\xi_3^2
\+ 2\xi_1^3\xi_3^3 + 12\xi_1\xi_2^2\xi_3^3 + 62\xi_1\xi_2^2\xi_3^3 + 45\xi_2\xi_3^3
\- 27\xi_1^2\xi_2^4 - 10\xi_1\xi_2^4 - 19\xi_2^2\xi_3^4 + 9\xi_1\xi_3^5 - 5\xi_2\xi_3^5 + 2\xi_3^6)
\- \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{1}{2\Delta^4\xi_1\xi_2}(-6\xi_1^7 + 30\xi_1^6\xi_2 - 54\xi_1^5\xi_2^2
\+ 30\xi_1^4\xi_2^3 + 74\xi_1^6\xi_3 - 204\xi_1^5\xi_2\xi_3 + 150\xi_1^4\xi_2^2\xi_3 - 20\xi_1^3\xi_2^3\xi_3
\- 294\xi_1^5\xi_3^2 - 558\xi_1^4\xi_2\xi_3^2 + 852\xi_1^3\xi_2\xi_3^2 + 378\xi_1^4\xi_3^3
\+ 728\xi_1^3\xi_2^2\xi_3^3 - 34\xi_1^2\xi_2^2\xi_3^4 - 382\xi_1\xi_2\xi_3^5
\- 210\xi_1\xi_2\xi_3^5 + 154\xi_1\xi_2\xi_3^5 + 78\xi_1\xi_3^6 + 7\xi_3^7),
\]

\[F_{28}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \frac{16\xi_3}{\Delta^4}(-2\xi_1^4 + 2\xi_1^2\xi_2^2
\+ 4\xi_1^3\xi_3 - 4\xi_1^2\xi_2\xi_3 + 4\xi_1\xi_2^2\xi_3 - 4\xi_1\xi_3^3 + \xi_3^4)
\+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{16}{\Delta^3\xi_1\xi_2}(-4\xi_1^4 - 4\xi_1^3\xi_2 + 8\xi_1^2\xi_2^2 + 10\xi_1^3\xi_3
\- 18\xi_1^2\xi_2\xi_3 - 6\xi_1^2\xi_2^2 + 12\xi_1\xi_2\xi_3^2 - 2\xi_1\xi_3^3 + \xi_3^4)\]

\[25\]
\[ F_{29}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \left\{ \frac{8 \xi_1 \xi_2 \xi_3}{3 \Delta^6}(6 \xi_1^5 \xi_2 + 3 \xi_1^4 \xi_2^2 - 12 \xi_1^3 \xi_2^3 + 6 \xi_1^5 \xi_3 \\
+ 12 \xi_1^3 \xi_2^2 \xi_3 + 3 \xi_1^4 \xi_2^3 + 12 \xi_1^3 \xi_2 \xi_3^2 - 10 \xi_1^2 \xi_2^2 \xi_3^2 - 18 \xi_1 \xi_2 \xi_3^4 - 3 \xi_3^6) \\
+ \frac{16}{\Delta^5}(3 \xi_1^5 \xi_2 + 6 \xi_1^4 \xi_2^2 - 14 \xi_1^3 \xi_2^3 + 3 \xi_1^5 \xi_3 + 18 \xi_1^3 \xi_2 \xi_3 \\
+ 6 \xi_1^4 \xi_3^2 + 18 \xi_1^3 \xi_2^2 \xi_3^2 - 16 \xi_1^2 \xi_2^2 \xi_3^3 - 21 \xi_1 \xi_2 \xi_3^4 - 2 \xi_3^6) \right\} \\
+ \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \Delta^4 \xi_1 \xi_2 \xi_3 \left[ 9 \xi_1^4 \xi_2^2 - 16 \xi_1^3 \xi_2^3 + 30 \xi_1^3 \xi_2^2 \xi_3 \\
+ 9 \xi_1^4 \xi_3^2 + 30 \xi_1^3 \xi_2 \xi_3^2 - 34 \xi_1^2 \xi_2 \xi_3^3 - 30 \xi_1 \xi_2 \xi_3^4 - 4 \xi_3^6 \right] \\
- \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24} \right) \frac{32}{\Delta^3 \xi_1 \xi_2 \xi_3}(-3 \xi_1^2 \xi_2 - 3 \xi_1 \xi_3) \\
+ 4 \xi_1 \xi_2 \xi_3 + 3 \xi_3^3) \\
+ f(\xi_1) \frac{16 \xi_1 \xi_2 \xi_3}{\Delta^6}(-\xi_1^5 + \xi_1^4 \xi_2 + 2 \xi_1^3 \xi_2^2 - 2 \xi_1^2 \xi_2^3 - \xi_1 \xi_2^4 \\
+ \xi_2^5 + \xi_1 \xi_2^4 - 4 \xi_1^4 \xi_2 \xi_3 + 2 \xi_1^3 \xi_2 \xi_3 + 4 \xi_1 \xi_2^3 \xi_3 - 3 \xi_2^4 \xi_3 \\
+ 2 \xi_1^3 \xi_3^2 + 2 \xi_1 \xi_2 \xi_3^2 - 6 \xi_1 \xi_2^2 \xi_3^2 + 2 \xi_2^3 \xi_3^2 \\
- 2 \xi_1^2 \xi_3^3 + 4 \xi_1 \xi_2 \xi_3^3 + 2 \xi_2^2 \xi_3^3 - \xi_1 \xi_3^4 - 3 \xi_2^3 \xi_3^2 + 3 \xi_3^5) \\
+ \left( f(\xi_1) - \frac{1}{\xi_1} \right) \frac{32 \xi_1}{\Delta^5}(-2 \xi_1^5 + \xi_1^4 \xi_2 + 7 \xi_1^3 \xi_2^2 - 7 \xi_1^2 \xi_2^3 \\
- \xi_1 \xi_2^4 + 2 \xi_2^5 + \xi_1 \xi_2^4 - 18 \xi_1^3 \xi_2 \xi_3 + 7 \xi_1 \xi_2^2 \xi_3 + 16 \xi_1 \xi_2 \xi_3^2 \\
- 6 \xi_2^4 \xi_3 + 7 \xi_1^3 \xi_3^2 + 7 \xi_1 \xi_2 \xi_3^2 - 30 \xi_1 \xi_2 \xi_3^2 - 4 \xi_2^3 \xi_3^2 \\
- 7 \xi_1^2 \xi_3^3 + 16 \xi_1 \xi_2 \xi_3^3 + 4 \xi_2^2 \xi_3^3 - \xi_1 \xi_3^4 - 6 \xi_2 \xi_3^4 + 2 \xi_3^5) \\
+ \left( f(\xi_1) - \frac{1}{\xi_1} \right) \frac{24 \xi_1}{\Delta^4 \xi_2 \xi_3}(-\xi_1^5 - 3 \xi_1^4 \xi_2 + 14 \xi_1^3 \xi_2^2 \\
- 2 \xi_1^2 \xi_2^3 + \xi_1 \xi_2^4 + 2 \xi_2^5 - 2 \xi_1^3 \xi_2^2) \right\} \right. \\
\]
\[ -14\xi_1^2\xi_2^3 + 3\xi_1\xi_2^4 + \xi_2^5 - 3\xi_1^4\xi_3 - 36\xi_1^3\xi_2\xi_3 + 22\xi_1^2\xi_2^2\xi_3 \\
+ 20\xi_1\xi_2^3\xi_3 - 3\xi_2^4\xi_3 + 14\xi_1^3\xi_3^2 + 22\xi_1^2\xi_2\xi_3^2 - 46\xi_1\xi_2^2\xi_3^2 \\
+ 2\xi_2^3\xi_3^2 - 14\xi_1^2\xi_3^3 + 20\xi_1\xi_2\xi_3^3 + 2\xi_2^2\xi_3^3 \\
+ 3\xi_1\xi_3^4 - 3\xi_2\xi_3^4 + \xi_3^5. \]  
\hspace{1cm} (2.105)

For the derivation of these results see sects. 13–16. The differential equations for the basic form factors, and comments on the form of the expressions above will be found in sect. 16. Another representation for the third-order form factors is given in sect. 15.

3. The late-time behaviour of the trace of the heat kernel

Derivation of the late-time behaviour of the form factors in the heat kernel was given in paper II to all orders in the curvature (see also sect. 15 below). For the basic form factors (2.9) and (2.75) this behaviour is

\[ f(-s\Box) = -\frac{1}{s}\frac{2}{6\Box} + O\left(\frac{1}{s^2}\right), \quad s \to \infty \]  
\hspace{1cm} (3.1)

\[ F(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{s^2}\left(\frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3}\right) + O\left(\frac{1}{s^3}\right), \quad s \to \infty. \]  
\hspace{1cm} (3.2)

The late-time behaviour of all second-order and third-order form factors follows then from the explicit expressions above *. With the symmetries (2.46)–(2.74) taken into account, one obtains

\[ f_1(-s\Box) = -\frac{1}{s}\frac{1}{6\Box} + O\left(\frac{1}{s^2}\right), \]  
\hspace{1cm} (3.3)

\[ f_2(-s\Box) = \frac{1}{s}\frac{1}{18\Box} + O\left(\frac{1}{s^2}\right), \]  
\hspace{1cm} (3.4)

\[ f_3(-s\Box) = \frac{1}{s}\frac{1}{3\Box} + O\left(\frac{1}{s^2}\right), \]  
\hspace{1cm} (3.5)

\[ f_4(-s\Box) = -\frac{1}{s}\frac{1}{\Box} + O\left(\frac{1}{s^2}\right), \]  
\hspace{1cm} (3.6)

\[ f_5(-s\Box) = -\frac{1}{s}\frac{1}{2\Box} + O\left(\frac{1}{s^2}\right), \]  
\hspace{1cm} (3.7)

and

\[ F_1^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{s^2}\frac{1}{3}\left(\frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3}\right) + O\left(\frac{1}{s^3}\right), \]  
\hspace{1cm} (3.8)

*Another way is to use the \(\alpha\)-representation of the form factors in sect. 15, and eqs. (15.54), (15.55).
\[ F_{2}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{s^2} \left( \frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3} \right) + O\left(\frac{1}{s^3}\right), \quad (3.9) \]

\[ F_{3}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.10) \]

\[ F_{4}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{s^2} \left( \frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3} \right) + O\left(\frac{1}{s^3}\right), \quad (3.11) \]

\[ F_{5}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.12) \]

\[ F_{6}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{s^2} \left( \frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3} \right) + O\left(\frac{1}{s^3}\right), \quad (3.13) \]

\[ F_{7}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.14) \]

\[ F_{8}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{s^2} \left( \frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} \right) + O\left(\frac{1}{s^3}\right), \quad (3.15) \]

\[ F_{9}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{s^2} \left( \frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3} \right) + O\left(\frac{1}{s^3}\right), \quad (3.16) \]

\[ F_{10}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.17) \]

\[ F_{11}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{s^2} \left( \frac{1}{\Box_1\Box_2} - \frac{1}{\Box_1\Box_3} - \frac{1}{\Box_2\Box_3} \right) + O\left(\frac{1}{s^3}\right), \quad (3.18) \]

\[ sF_{12}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{s^2} \frac{1}{\Box_1\Box_2\Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.19) \]

\[ sF_{13}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{s^2} \frac{1}{\Box_1\Box_2\Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.20) \]

\[ sF_{14}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{s^2} \frac{1}{\Box_1\Box_2\Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.21) \]

\[ sF_{15}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.22) \]
\[ s F_{16}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.23) \]

\[ s F_{17}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.24) \]

\[ s F_{18}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = \frac{1}{s^2} \frac{2}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.25) \]

\[ s F_{19}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = -\frac{1}{s^2} \frac{1}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.26) \]

\[ s F_{20}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = \frac{1}{s^2} \frac{1}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.27) \]

\[ s F_{21}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = \frac{1}{s^2} \frac{1}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.28) \]

\[ s F_{22}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = \frac{1}{s^2} \frac{1}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.29) \]

\[ s F_{23}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = -\frac{1}{s^2} \frac{2}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.30) \]

\[ s F_{24}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = -\frac{1}{s^2} \frac{1}{\Box_1 \Box_2 \Box_3} + O\left(\frac{1}{s^3}\right), \quad (3.31) \]

\[ s F_{25}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.32) \]

\[ s^2 F_{26}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.33) \]

\[ s^2 F_{27}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.34) \]

\[ s^2 F_{28}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right), \quad (3.35) \]

\[ s^3 F_{29}^{\text{sym}}(-s \Box_1, -s \Box_2, -s \Box_3) = O\left(\frac{1}{s^3}\right). \quad (3.36) \]

The result is that the behaviour of the trace of the heat kernel at large \(s\) is \(s^{-\omega+1}\), and the coefficient of this asymptotic behaviour is obtained to third order in the curvature \(\star\). As

\(^*\)As shown in paper II, this behaviour holds at all orders in the curva-
seen from the expressions above, not all basis structures contribute to the leading asymptotic behaviour. The asymptotic form of $\text{Tr}K(s)$ is as follows:

\[
\text{Tr}K(s) = \frac{s}{(4\pi s)^{\omega}} \int dx \, g^{1/2} \text{tr} \left\{ \hat{P} \right\}
\]

\[
- \hat{P} \frac{1}{\hat{P}} - \frac{1}{2} \hat{R}_{\mu\nu} \frac{1}{\hat{R}_{\mu\nu}} + \frac{1}{3} \hat{P} \frac{1}{\hat{P}} R
\]

\[
- \frac{1}{6} R_{\mu\nu} \frac{1}{R_{\mu\nu}} \hat{R}_{\mu\nu} \hat{R}_{\mu\nu} + \frac{1}{18} R \frac{1}{R} \hat{R}
\]

\[
+ \hat{P} \frac{1}{\hat{P}} - 2 \hat{R}^\mu_{\alpha} \frac{1}{\hat{R}^\alpha_{\beta}} \hat{R}^\beta_{\mu}
\]

\[
+ \frac{1}{36} R \frac{1}{R} \hat{P} + \frac{1}{18} R \frac{1}{R} \hat{P}
\]

\[
- \frac{1}{6} \hat{P} \frac{1}{\hat{P}} R - \frac{1}{3} \hat{P} \frac{1}{\hat{P}} R
\]

\[
+ 2 \frac{1}{2} R_{\alpha\beta} \frac{1}{\hat{R}_{\alpha} \hat{R}_{\beta}} - \frac{1}{216} R \frac{1}{R} \hat{R} \hat{R}
\]

\[
+ \frac{1}{12} R^\mu_{\nu} \frac{1}{R^\mu_{\nu}} R \hat{R} - \frac{1}{6} R^\mu_{\nu} \frac{1}{R^\mu_{\nu}} R \hat{R}
\]

\[
+ 2 \frac{1}{2} \hat{R}^\mu_{\nu} \frac{1}{\hat{R}^\mu_{\nu}} \hat{R} \frac{1}{\hat{R}} - 2 \frac{1}{2} \hat{R}^\mu_{\nu} \frac{1}{\hat{R}^\mu_{\nu}} \hat{R} \frac{1}{\hat{R}}
\]

\[
- 2 \frac{1}{2} \hat{R}^\mu_{\nu} \frac{1}{\hat{R}^\mu_{\nu}} \hat{R} \frac{1}{\hat{R}} - 2 \frac{1}{2} \hat{R}^\mu_{\nu} \frac{1}{\hat{R}^\mu_{\nu}} \hat{R} \frac{1}{\hat{R}}
\]

\[
- \frac{1}{6} \frac{1}{R} \frac{1}{R} \hat{R} \hat{R}
\]

\[
= \mathcal{O}(\mathbb{R}^4) + \mathcal{O}(\frac{1}{s^{\omega}}), \quad s \to \infty.
\]
4. The early-time behaviour of the trace of the heat kernel, and comparison with the Schwinger-DeWitt expansion

Derivation of the early-time behaviour of the form factors in the heat kernel presents no problem. One may use either the explicit expressions in sect. 2 or the $\alpha$-representation in sect. 15 and eqs. (15.51), (15.53). For the basic form factors (2.9) and (2.75) one obtains

$$f(-s\Box) = 1 + \frac{1}{6}s\Box + \frac{1}{60}s^2\Box^2 + O\left(s^3\right), \quad s \to 0$$  

$$F(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{2} + \frac{1}{24}s(\Box_1 + \Box_2 + \Box_3) + O\left(s^2\right), \quad s \to 0$$

and the full table of asymptotic behaviours for the second-order and third-order form factors is as follows:

$$f_1(-s\Box) = \frac{1}{60} + \frac{1}{840}s\Box + \frac{1}{15120}s^2\Box^2 + O\left(s^3\right),$$  

$$f_2(-s\Box) = -\frac{1}{180} - \frac{1}{3780}s\Box + O\left(s^3\right),$$  

$$f_3(-s\Box) = \frac{1}{180}s\Box + \frac{1}{1260}s^2\Box^2 + O\left(s^3\right),$$  

$$f_4(-s\Box) = \frac{1}{2} + \frac{1}{132}s\Box + \frac{1}{120}s^2\Box^2 + O\left(s^3\right),$$  

$$f_5(-s\Box) = \frac{1}{12} + \frac{1}{120}s\Box + \frac{1}{1680}s^2\Box^2 + O\left(s^3\right),$$  

$$F_1^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{6} + s\left(\frac{\Box_3}{72} + \frac{\Box_2}{72} + \frac{\Box_1}{72}\right) + O\left(s^2\right),$$  

$$F_2^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{45} + s\left(-\frac{\Box_3}{1890} - \frac{\Box_2}{1890} - \frac{\Box_1}{1890}\right) + O\left(s^2\right),$$  

$$F_3^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{12} + s\left(\frac{\Box_1}{180} + \frac{\Box_2}{180} + \frac{\Box_3}{90}\right) + O\left(s^2\right),$$  

$$F_4^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = s\left(\frac{\Box_2}{15120} - \frac{\Box_3}{15120} + \frac{\Box_1}{15120}\right) + O\left(s^2\right),$$  

$$F_5^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \left(\frac{1}{180} - \frac{\Box_3}{90\Box_1} + \frac{\Box_3^2}{180\Box_1\Box_2} + \frac{\Box_1}{180\Box_2} + \frac{\Box_2}{180\Box_1}ight)$$  

$$- \frac{\Box_3}{90\Box_1^2} + s\left(\frac{\Box_3^3}{1120\Box_1\Box_2} + \frac{\Box_1^2}{3360\Box_2} + \frac{\Box_2^2}{3360\Box_1} + \frac{\Box_1}{3360} + \frac{\Box_2}{3360}ight)$$  

$$- \frac{\Box_3^2}{672\Box_1} + \frac{\Box_2^3}{3360\Box_1} + \frac{\Box_1^3}{3360\Box_2} - \frac{\Box_3^2}{672\Box_2} + \frac{\Box_3}{2520}\right) + O\left(s^2\right),$$
\[ F_6^{\text{sym}}(-s \square_1, -s \square_2, -s \square_3) = s \frac{\square_3}{720} + O\left(s^2\right), \quad (4.13) \]

\[ F_7^{\text{sym}}(-s \square_1, -s \square_2, -s \square_3) = s\left( \frac{13 \square_1}{30240} - \frac{\square_2}{30240} - \frac{\square_3}{30240} \right) + O\left(s^2\right), \quad (4.14) \]

\[ F_8^{\text{sym}}(-s \square_1, -s \square_2, -s \square_3) = \left( \frac{1}{45} + \frac{\square_3}{36 \square_1} + \frac{\square_2}{36 \square_1} \right) + s\left( \frac{\square_1}{840} + \frac{\square_2}{420} + \frac{\square_3}{420} \right) \]
\[ + \frac{\square_2 \square_3}{210 \square_1} + \frac{\square_2^2}{420 \square_1} + \frac{\square_3^2}{420 \square_1} \right) + O\left(s^2\right), \quad (4.15) \]

\[ F_9^{\text{sym}}(-s \square_1, -s \square_2, -s \square_3) = \frac{1}{s} \left( \frac{\square_1}{1080 \square_2 \square_3} + \frac{\square_2}{1080 \square_1 \square_3} + \frac{\square_3}{1080 \square_1 \square_2} \right) \]
\[ + \left( \frac{\square_3^2}{15120 \square_1} - \frac{1}{3240} + \frac{\square_2}{15120 \square_1} + \frac{\square_1}{15120 \square_3} + \frac{\square_2^2}{10080 \square_1 \square_3} \right) \]
\[ + \frac{\square_3^2}{15120 \square_2} + \frac{\square_1}{15120 \square_2} + \frac{\square_2}{15120 \square_3} + \frac{\square_3}{15120 \square_2} + \frac{\square_3^2}{10080 \square_2 \square_3} \]
\[ + \frac{\square_2^2}{15120 \square_1} + \frac{\square_1}{15120 \square_3} + \frac{\square_2}{15120 \square_2} + \frac{\square_3}{15120 \square_2} + \frac{\square_3}{15120 \square_2} \]
\[ + \frac{\square_3^2}{15120 \square_1 \square_2} + \frac{\square_2^2}{15120 \square_3} + \frac{\square_3^2}{15120 \square_1} - \frac{\square_2}{75600} - \frac{\square_3}{75600} \]
\[ - \frac{\square_1}{75600} \right) + O\left(s^2\right), \quad (4.16) \]

\[ F_{10}^{\text{sym}}(-s \square_1, -s \square_2, -s \square_3) = \frac{1}{s} \left( -\frac{1}{135 \square_1} - \frac{1}{135 \square_3} - \frac{1}{135 \square_2} + \frac{\square_3}{270 \square_1} \right) \]
\[ + \frac{\square_1}{270 \square_3} + \frac{\square_2}{270 \square_3} \right) + \left( \frac{\square_1^2}{2520 \square_2 \square_3} - \frac{\square_2}{2520 \square_2} - \frac{\square_2}{2520 \square_1} \right) \]
\[ + \frac{\square_2^2}{2520 \square_1 \square_3} - \frac{\square_1}{2520 \square_1} + \frac{\square_3^2}{2520 \square_3} - \frac{\square_2}{2520 \square_3} - \frac{\square_3}{2520 \square_1} \]
\[ - \frac{\square_2}{2520 \square_2} + \left( \frac{\square_1^3}{37800 \square_2 \square_3} + \frac{\square_3^3}{37800 \square_1 \square_2} + \frac{\square_2^3}{37800 \square_1 \square_3} \right) \]
\[ - \frac{\square_2^2}{56700 \square_1} - \frac{\square_1^2}{56700 \square_2} - \frac{\square_2^2}{56700 \square_3} - \frac{\square_3^2}{56700 \square_2} - \frac{\square_3^2}{56700 \square_1} \]
\[ - \frac{\square_3^2}{56700 \square_3} - \frac{\square_2}{56700 \square_2} - \frac{\square_3}{56700 \square_1} - \frac{\square_2}{56700 \square_3} - \frac{\square_3}{56700 \square_3} \]
\[ - \frac{\square_1}{37800} \right) + O\left(s^2\right), \quad (4.17) \]

\[ F_{11}^{\text{sym}}(-s \square_1, -s \square_2, -s \square_3) = \frac{1}{s} \left( \frac{1}{180 \square_1} - \frac{\square_3}{180 \square_1 \square_2} + \frac{1}{180 \square_2} \right) \]

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\begin{align*}
+ \left( \frac{1}{15120} + \frac{\Box_3}{3780\Box_2} - \frac{\Box_2}{30240\Box_1} - \frac{\Box_1}{30240\Box_2} + \frac{\Box_3}{3780\Box_1} - \frac{\Box_3^2}{4320\Box_1\Box_2} \right) \\
+ s \left( \frac{\Box_3}{25200} - \frac{13\Box_3^2}{30240\Box_1} + \frac{\Box_2\Box_3}{37800\Box_1} + \frac{\Box_1}{30240} + \frac{\Box_2}{30240} + \frac{\Box_3^3}{50400\Box_1\Box_2} \\
- \frac{13\Box_3^2}{30240\Box_2} - \frac{\Box_1^2}{37800\Box_2} + \frac{\Box_3}{302400\Box_1} + \frac{\Box_2^2}{302400\Box_1} + \frac{1}{32} + \frac{1}{32} \right) + O \left( s^2 \right),
\end{align*}
(4.18)

\begin{align*}
s_{F_{12}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= s \frac{1}{252} + O \left( s^2 \right),
\end{align*}
(4.19)

\begin{align*}
s_{F_{13}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= s \frac{1}{60} + O \left( s^2 \right),
\end{align*}
(4.20)

\begin{align*}
s_{F_{14}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= s \frac{1}{180} + O \left( s^2 \right),
\end{align*}
(4.21)

\begin{align*}
s_{F_{15}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= -s \frac{1}{1890} + O \left( s^2 \right),
\end{align*}
(4.22)

\begin{align*}
s_{F_{16}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= \left( \frac{1}{45\Box_1} - \frac{\Box_3}{45\Box_1\Box_2} + \frac{1}{45\Box_2} \right) \\
+ s \left( \frac{\Box_3}{420\Box_1} - \frac{\Box_3^2}{280\Box_1\Box_2} + \frac{\Box_3}{420\Box_2} + \frac{\Box_1}{840\Box_2} + \frac{1}{630} + \frac{\Box_2}{840\Box_1} \right) + O \left( s^2 \right),
\end{align*}
(4.23)

\begin{align*}
s_{F_{17}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= s \frac{1}{180} + O \left( s^2 \right),
\end{align*}
(4.24)

\begin{align*}
s_{F_{18}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= \frac{1}{18\Box_1} + s \left( \frac{1}{1260} + \frac{\Box_2}{210\Box_1} + \frac{\Box_3}{210\Box_1} \right) + O \left( s^2 \right),
\end{align*}
(4.25)

\begin{align*}
s_{F_{19}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= -\frac{1}{36\Box_1} + s \left( -\frac{1}{1260} - \frac{\Box_2}{420\Box_1} - \frac{\Box_3}{420\Box_1} \right) + O \left( s^2 \right),
\end{align*}
(4.26)

\begin{align*}
s_{F_{20}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= -s \frac{1}{7560} + O \left( s^2 \right),
\end{align*}
(4.27)

\begin{align*}
s_{F_{21}}^{\text{sym}} (-s\Box_1, -s\Box_2, -s\Box_3) &= \frac{1}{9\Box_1} + s \left( \frac{\Box_3}{105\Box_1} + \frac{1}{630} + \frac{\Box_2}{105\Box_1} \right) + O \left( s^2 \right),
\end{align*}
(4.28)
\[
\begin{align*}
 s F_{22}^{\text{sym}}(s \Box_1, -s \Box_2, -s \Box_3) &= \frac{1}{s} \left( -\frac{1}{60 \Box_2 \Box_3} - \frac{1}{180 \Box_1 \Box_3} - \frac{1}{180 \Box_1 \Box_2} \right) \\
 &\quad + \left( -\frac{\Box_2}{2520 \Box_1 \Box_3} - \frac{\Box_1}{630 \Box_2 \Box_3} - \frac{1}{2520 \Box_3} - \frac{1}{2520 \Box_2} + \frac{1}{2520 \Box_1} - \frac{1}{2520 \Box_2 \Box_3} \right) \\
 &\quad + s \left( \frac{1}{226800} - \frac{\Box_1}{16800 \Box_2} - \frac{\Box_3}{50400 \Box_2} - \frac{\Box_1^2}{10080 \Box_2 \Box_3} - \frac{\Box_2}{50400 \Box_3} + \frac{1}{50400 \Box_1} - \frac{\Box_2^2}{50400 \Box_1 \Box_3} \right) + O\left(s^2\right), \quad (4.29)
\end{align*}
\]

\[
\begin{align*}
 s F_{23}^{\text{sym}}(s \Box_1, -s \Box_2, -s \Box_3) &= \frac{1}{s} \left( \frac{1}{45 \Box_1 \Box_2} + \left( -\frac{1}{7560 \Box_1} - \frac{1}{7560 \Box_2} + \frac{\Box_3}{1080 \Box_1 \Box_2} \right) \\
 &\quad + s \left( -\frac{\Box_1}{75600 \Box_1} + \frac{\Box_2}{75600 \Box_2} + \frac{1}{10800 \Box_2} + \frac{\Box_3}{10800 \Box_1} - \frac{\Box_1^2}{12600 \Box_1 \Box_2} - \frac{\Box_2}{12600 \Box_1 \Box_2} \right) + O\left(s^2\right), \quad (4.30)
\end{align*}
\]

\[
\begin{align*}
 s F_{24}^{\text{sym}}(s \Box_1, -s \Box_2, -s \Box_3) &= \frac{1}{s} \left( \frac{1}{45 \Box_2 \Box_3} + \left( -\frac{1}{840 \Box_3} - \frac{1}{840 \Box_2} + \frac{\Box_1}{504 \Box_2 \Box_3} \right) \\
 &\quad + s \left( -\frac{1}{9450} - \frac{\Box_3}{25200 \Box_2} + \frac{\Box_1}{25200 \Box_2} \right) + O\left(s^2\right), \quad (4.31)
\end{align*}
\]

\[
\begin{align*}
 s F_{25}^{\text{sym}}(s \Box_1, -s \Box_2, -s \Box_3) &= \frac{1}{s} \left( \frac{1}{45 \Box_1 \Box_2} - \frac{1}{45 \Box_2 \Box_3} + \frac{1}{45 \Box_1 \Box_3} \right) \\
 &\quad + \left( -\frac{\Box_1}{420 \Box_2 \Box_3} - \frac{1}{315 \Box_1} + \frac{\Box_3}{420 \Box_1 \Box_3} + \frac{\Box_2}{420 \Box_1 \Box_3} \right) \\
 &\quad + s \left( \frac{1}{18900 \Box_2} - \frac{1}{18900 \Box_2} - \frac{1}{6300 \Box_1} + \frac{\Box_2}{18900 \Box_3} - \frac{\Box_1}{18900 \Box_3} - \frac{1}{3150} - \frac{1}{6300 \Box_2 \Box_3} + \frac{1}{6300 \Box_1 \Box_3} + \frac{1}{6300 \Box_1 \Box_3} - \frac{1}{6300 \Box_1 \Box_3} \right) + O\left(s^2\right), \quad (4.32)
\end{align*}
\]

\[
\begin{align*}
 s^2 F_{26}^{\text{sym}}(s \Box_1, -s \Box_2, -s \Box_3) &= \frac{1}{45 \Box_1 \Box_2} + s \left( \frac{1}{840 \Box_2} + \frac{\Box_3}{280 \Box_1 \Box_2} + \frac{1}{840 \Box_1} \right) \\
 &\quad + O\left(s^2\right), \quad (4.33)
\end{align*}
\]

\[
\begin{align*}
 s^2 F_{27}^{\text{sym}}(s \Box_1, -s \Box_2, -s \Box_3) &= \frac{1}{s} \left( \frac{1}{504 \Box_1 \Box_2} + \left( -\frac{1}{504 \Box_1 \Box_3} - \frac{1}{504 \Box_2 \Box_3} \right) \\
 &\quad - \frac{1}{756 \Box_1 \Box_2} + s \left( \frac{1}{6300 \Box_3} - \frac{1}{8400 \Box_2 \Box_3} - \frac{1}{7560 \Box_2} - \frac{1}{7560 \Box_1} \right) + O\left(s^2\right), \quad (4.34)
\end{align*}
\]
\[ s^2 F_{28}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = -\frac{1}{45s} \frac{2}{\Box_1\Box_2\Box_3} + \left( -\frac{1}{210s} \right) - \frac{1}{315s} \]
\[ \frac{1}{\Box_3} + s\left( -\frac{1}{315s} \Box_1 - \frac{1}{630s} \Box_2 - \frac{1}{315s} \right) - \frac{2}{4725s} \]
\[ \frac{1}{\Box_3} + O\left(s^2\right), \] (4.35)

\[ s^3 F_{29}^{\text{sym}}(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{1890s} \frac{1}{\Box_1\Box_2\Box_3} + s\left( \frac{1}{1890s} \Box_2 \right) + \frac{1}{1890s} \]
\[ \frac{1}{\Box_3} + O\left(s^2\right). \] (4.36)

It is striking that, in the early-time expansion, the third-order form factors are still nonlocal and, for some of them, the expansion starts with a negative power of \( s \). By making a comparison with the table of tensor structures in (2.15)–(2.43), one can see that such a behaviour is inherent only in the gravitational form factors, and, moreover, the nonlocal operators \( 1/\Box \) in the asymptotic expressions above act only on the gravitational curvatures. As discussed in paper II, these features will persist at all higher orders in \( \Re \), and the cause is that the basis set of curvatures for the heat kernel does not contain the Riemann tensor. In covariant perturbation theory, the Riemann tensor gets automatically excluded via the Bianchi identities [2]. Below we show that restoring the Riemann tensor restores the locality of the early-time expansion.

The early-time expansion of the heat kernel is known as the Schwinger–DeWitt expansion. For \( \text{Tr}K(s) \) it is of the form [6]

\[ \text{Tr}K(s) = \frac{1}{(4\pi s)^2} \sum_{n=0}^{\infty} s^n \int dx g^{1/2} \text{tr} \hat{a}_n(x, x) \] (4.37)

where \( \hat{a}_n(x, x) \) are the DeWitt coefficients with coincident arguments. All \( \hat{a}_n(x, x) \) are local functions of the background fields entering the operator (1.2). There exist independent methods for obtaining these coefficients, and, for \( n = 0, 1, 2, 3, 4 \), the \( \hat{a}_n(x, x) \) have been calculated explicitly [6,7,12–15]. A comparison with these known expressions, carried out below, provides a powerful check of the present results.

By inserting the expansions (4.3)–(4.36) in (2.1), one arrives at eq. (4.37) with the following results for the (integrated) DeWitt coefficients \( a_0 \) to \( a_4 \):

\[ \int dx \, g^{1/2} \text{tr} \hat{a}_0(x, x) = \int dx \, g^{1/2} \text{tr} \hat{1}, \] (4.38)

\[ \int dx \, g^{1/2} \text{tr} \hat{a}_1(x, x) = \int dx \, g^{1/2} \text{tr} \hat{P}, \] (4.39)

\[ \int dx \, g^{1/2} \text{tr} \hat{a}_2(x, x) = \int dx \, g^{1/2} \text{tr} \left\{ \frac{1}{2} \hat{P}_1 \hat{P}_2 + \frac{1}{12} \hat{R}_{1\mu
u} \hat{R}_2^{\mu\nu} \right. \]
\[ + \frac{1}{60} R_{1\mu\nu} R_2^{\mu\nu} \hat{1} - \frac{1}{180} R_1 R_2 \hat{1} + \frac{1}{360s} R_1 R_2 R_3 \hat{1} \]
\[\begin{align*}
&\left( -\frac{1}{45\Box_3} + \frac{\Box_3}{90\Box_1 \Box_2} \right) R_{1\alpha}^{\mu} R_{2\beta}^{\nu} R_{3\mu}^{\beta} \hat{1} + \left( -\frac{1}{90\Box_2} - \frac{\Box_3}{180\Box_1 \Box_2} \right) R_{1\mu}^{\mu} R_{2\mu}^{\nu} R_{3\nu}^{\mu} \\
&\left( -\frac{1}{90\Box_2} - \frac{1}{60\Box_2 \Box_3} \right) R_{1\alpha}^{\mu} \nabla_\alpha R_{2} \nabla_\beta R_{3} \hat{1} + \frac{1}{45\Box_1 \Box_2} \nabla_\mu R_{1\alpha}^{\mu} \nabla_\nu R_{2\mu} R_{3\nu} \hat{1} \\
&\frac{1}{45\Box_3} \nabla_\alpha R_{1\nu} \nabla_\mu R_{2\nu} \nabla_\beta R_{3\beta} \hat{1} + \left( -\frac{2}{45\Box_1} - \frac{1}{45\Box_2 \Box_3} \right) R_{1\mu}^{\mu} \nabla_\alpha R_{2\nu} \nabla_\mu R_{3\nu} \hat{1} \\
&- \frac{1}{45\Box_1 \Box_2} \nabla_\mu R_{1\nu} \nabla_\nu R_{2\lambda} \nabla_\mu \nabla_\beta R_{3\nu}^{\beta} \hat{1} \\
&- \frac{2}{45\Box_1 \Box_2 \Box_3} \nabla_\mu R_{1\lambda} \nabla_\sigma R_{2\lambda} \nabla_\nu \nabla_\beta R_{3\nu} \hat{1} \right) + O[\Box^4],
\end{align*}\] (4.40)

\[\begin{align*}
\int dx \, g^{1/2} \text{tr} \hat{a}_3(x, x) &= \int dx \, g^{1/2} \text{tr} \left\{ \frac{\Box_2}{12} \hat{P}_1 \hat{P}_2 + \frac{\Box_2}{120} \hat{R}_{1\mu\nu} \hat{R}_{2\mu}^{\nu} + \frac{\Box_2}{180} \hat{P}_1 \hat{R}_2 \\
&+ \frac{\Box_2}{840} R_{1\mu\nu} R_{2\nu}^{\mu} \hat{1} - \frac{\Box_2}{3780} R_{1\mu} R_{2\nu} \hat{1} + \frac{1}{6} \hat{P}_1 \hat{P}_2 \hat{P}_3 - \frac{1}{45} \hat{R}_{1\alpha}^{\mu} \hat{R}_{2\beta} \hat{R}_{3\mu} \hat{1} \\
&\frac{1}{12} \hat{R}_{1\mu}^{\mu} \hat{R}_{2\mu} \hat{P}_3 + \left( \frac{1}{180} + \frac{\Box_1}{90\Box_2} - \frac{\Box_3}{45\Box_2} + \frac{\Box_2}{180\Box_1 \Box_2} \right) R_{1\mu}^{\mu} R_{2\nu} \hat{1} \\
&+ \left( \frac{1}{45} + \frac{\Box_3}{18\Box_1} \right) R_{1\alpha}^{\mu} \hat{R}_{2\mu}^{\alpha} \hat{R}_{3\mu} \hat{1} + \left( -\frac{1}{3240} + \frac{\Box_1}{2520\Box_3} + \frac{\Box_2}{3360\Box_1 \Box_2} \right) R_{1\mu} R_{2\nu} R_{3}\hat{1} \\
&+ \left( -\frac{1}{1890} - \frac{\Box_1}{42\Box_3} + \frac{\Box_3^2}{840\Box_1 \Box_2} \right) R_{1\nu}^{\nu} R_{2\beta} R_{3\mu} \hat{1} \\
&\frac{1}{15120} - \frac{1}{15120 \Box_2} + \frac{\Box_3}{1890 \Box_2} - \frac{\Box_3^2}{4320 \Box_1 \Box_2} \right) R_{1\mu}^{\mu} R_{2\nu} R_{3}\hat{1} \\
&\frac{2}{45\Box_2} - \frac{\Box_3}{45\Box_1 \Box_2} \right) \nabla_\mu R_{1\nu}^{\alpha} \nabla_\nu R_{2\mu} \hat{1} + \frac{1}{1801} R_{1\alpha}^{\beta} \nabla_\mu R_{2\nu}^{\alpha \beta} \hat{1} \\
&\left( -\frac{1}{36\Box_1} - \frac{1}{2520 \Box_1} - \frac{\Box_1}{1260 \Box_2} - \frac{\Box_1}{630 \Box_3} - \frac{\Box_3}{1260 \Box_1 \Box_2} \right) R_{1\mu}^{\mu} \nabla_\nu R_{2\nu} \hat{1} \\
&+ \left( -\frac{1}{3780 \Box_2} + \frac{\Box_3}{1080 \Box_1 \Box_2} \right) \nabla_\mu R_{1\nu}^{\alpha} \nabla_\nu R_{2\mu} \hat{1} \\
&\left( -\frac{1}{420 \Box_2} + \frac{\Box_1}{504 \Box_2 \Box_3} \right) R_{1\mu}^{\mu} \nabla_\nu R_{2\nu}^{\alpha \beta} \hat{1} \\
&\left( -\frac{1}{315 \Box_1} - \frac{\Box_1}{420 \Box_2 \Box_3} + \frac{\Box_3}{210 \Box_1 \Box_2} \right) R_{1\mu}^{\mu} \nabla_\nu R_{2\nu} \hat{1} \\
&\frac{1}{45\Box_1 \Box_2} \nabla_\mu R_{1\nu}^{\nu} \nabla_\nu R_{2\beta} \hat{1} \\
&\left( -\frac{1}{756 \Box_1 \Box_2} - \frac{\Box_3}{252 \Box_2 \Box_3} \right) \nabla_\nu R_{1\nu}^{\mu} \nabla_\mu R_{2\nu} \hat{1} \\
&\left( -\frac{1}{315 \Box_1 \Box_2} - \frac{\Box_1}{105 \Box_2 \Box_3} \right) \nabla_\mu R_{1\alpha}^{\lambda} \nabla_\nu R_{2\nu} \hat{1} \\
&\frac{1}{1890 \Box_1 \Box_2 \Box_3} \nabla_\lambda \nabla_\sigma R_{1\alpha}^{\lambda} \nabla_\beta R_{2\nu} \nabla_\mu \nabla_\nu R_{3\nu} \hat{1} \right) + O[\Box^4],
\end{align*}\] (4.41)
\[
\int dx \, g^{1/2} \text{tr} \, \hat{a}_4(x, x) = \int dx \, g^{1/2} \text{tr} \left\{ \frac{\square_2^2}{120} \hat{P}_1 \hat{P}_2 + \frac{\square_2^2}{1260} \hat{P}_1 \hat{R}_2 + \frac{\square_2^2}{1680} \hat{R}_{1\mu
u} \hat{R}_{2\mu
u}^\nu \\
+ \frac{\square_2^3}{15120} R_{1\mu
u} R_{2\mu
u}^\nu \hat{1} + \frac{\square_3^3}{24} \hat{P}_1 \hat{P}_2 \hat{P}_3 - \frac{\square_3^3}{630} \hat{R}_{1\mu\alpha} \hat{R}_{2\beta\beta} \hat{R}_{3\mu} \mu \\
+ \left( \frac{\square_1}{180} + \frac{\square_2}{180} + \frac{\square_3}{90} \right) \hat{R}_{1\mu
u} R_{2\mu\nu}^\nu \hat{P}_3 + \left( \frac{\square_1}{7560} - \frac{\square_3^3}{15120} \right) R_1 \hat{R}_2 \hat{P}_3 \\
+ \left( \frac{\square_1}{1680} + \frac{\square_2^2}{2520} - \frac{\square_3^2}{336} - \frac{\square_3^2}{1120} \square_2 \right) R_{1\mu
u} R_{2\mu\nu}^\nu \hat{P}_3 \\
+ \frac{\square_3^3}{720} \hat{P}_1 \hat{P}_2 \hat{R}_3 + \left( \frac{\square_3^3}{30240} - \frac{\square_3^3}{15120} \right) R_1 \hat{R}_2 \hat{R}_3^\nu \hat{R}_3^\mu \\
+ \left( \frac{\square_3}{840} + \frac{\square_2}{210} + \frac{\square_2 \square_3}{210 \square_1} + \frac{\square_3^2}{210 \square_1} \right) R_{1\alpha\beta} \hat{R}_2 \hat{R}_{3\mu} \hat{R}_{3\beta} \mu \\
+ \left( \frac{\square_3}{25200} \square_3 + \frac{\square_1 \square_2}{50400} - \frac{\square_3}{25200} + \frac{\square_3}{50400} \square_1 \square_2 \right) R_1 \hat{R}_2 \hat{R}_3 \\
+ \left( - \frac{\square_3}{9450} - \frac{\square_3}{18900} - \frac{\square_3}{3} + \frac{\square_3}{2600} \square_1 \square_2 \right) R_{1\mu\alpha} R_{2\beta\beta} R_{3\mu} \hat{1} \\
+ \left( \frac{\square_3}{15120} - \frac{\square_3}{15120} \square_2 \square_3 + \frac{\square_3}{25200} + \frac{\square_3}{18900} \square_2 - \frac{\square_3}{15120} \square_2 \right) R_1 \hat{R}_2 \hat{R}_3 \\
+ \frac{1}{60} \hat{R}_{1\mu
u} \nabla_\mu \hat{P}_2 \nabla_\nu \hat{P}_3 + \frac{1}{180} \nabla_\mu \hat{R}_{1\mu\alpha} \nabla_\nu \hat{R}_{2\nu\alpha} \hat{P}_3 - \frac{1}{1890} R_{1\mu\nu} \nabla_\mu R_2 \nabla_\nu \hat{P}_3 \\
+ \left( \frac{\square_2}{630} + \frac{\square_2}{420} \square_2 + \frac{\square_3}{210 \square_2} - \frac{\square_3}{280} \square_1 \square_2 \right) \nabla_\mu R_{1\mu\alpha} \nabla_\nu R_{2\nu\alpha} \hat{P}_3 \\
+ \frac{1}{180} R_{1\mu\nu} \nabla_\mu \nabla_\nu \hat{P}_2 \hat{P}_3 + \left( \frac{\square_3}{1260} + \frac{\square_3}{105 \square_1} \right) R_{1\alpha\beta} \nabla_\mu \hat{R}_2 \nabla_\nu \hat{R}_3^{\nu\beta} \\
+ \left( - \frac{\square_3}{1260} - \frac{\square_3}{210 \square_1} \right) R_{1\beta\gamma} \nabla_\alpha \hat{R}_2 \nabla_\beta \hat{R}_3^{\mu\nu} - \frac{1}{7560} R_1 \nabla_\alpha \hat{R}_{2\mu\nu} \nabla_\beta \hat{R}_3^{\beta\mu} \\
+ \left( \frac{1}{630} + \frac{\square_2}{105 \square_1} + \frac{\square_3}{105 \square_1} \right) R_{1\mu
u} \nabla_\mu \nabla_\chi \hat{R}_2^{\chi\alpha} \hat{R}_3^{\alpha\nu} + \left( \frac{1}{226800} - \frac{\square_3}{8400} \square_2 \right) \\
- \frac{1}{10800} \square_2 \square_3 + \frac{25200 \square_1}{25200 \square_2} - \frac{25200 \square_1}{25200 \square_2} \right) R_{1\alpha\beta} \nabla_\alpha R_2 \nabla_\beta R_3 \hat{1} \\
+ \left( \frac{\square_3}{37800} \square_2 + \frac{\square_3}{5400 \square_2} - \frac{\square_3}{12600} \square_1 \square_2 \right) \nabla_\mu R_{1\mu\alpha} \nabla_\nu R_{2\nu\alpha} \hat{R}_3 \hat{1} \\
+ \left( \frac{\square_3}{9450} - \frac{\square_3}{12600} \square_2 + \frac{\square_3}{8400} \square_2 \square_3 - \frac{\square_3}{6300} \square_2 \right) R_{1\mu\nu} \nabla_\mu R_{2\beta\beta} \nabla_\nu R_{3\alpha\beta} \hat{1} \\
+ \left( \frac{1}{3150} - \frac{\square_3}{9450} \square_2 - \frac{\square_3}{6300} \square_2 \square_3 - \frac{\square_3}{3150 \square_1} + \frac{\square_3}{3150 \square_2} \right) \times R_{1\mu\nu} \nabla_\alpha R_{2\beta\mu} \nabla_\beta R_3^{\nu\beta} \hat{1} + \left( \frac{1}{4200 \square_2} + \frac{\square_3}{280 \square_1 \square_2} \right) \nabla_\alpha \nabla_\beta R_{1\mu\nu} \nabla_\mu R_{2\nu\beta} \hat{P}_3 \\
+ \left( \frac{1}{37800} - \frac{\square_3}{6300} \square_3 + \frac{25200 \square_1 \square_2}{4200 \square_2 \square_3} \right) \nabla_\alpha \nabla_\beta R_{1\mu\nu} \nabla_\mu \nabla_\nu R_{2\beta\beta} R_3 \hat{1} 
\right\}.
\]
\[
\begin{split}
&+ \left( -\frac{1}{1575}\square_2 - \frac{2}{4725}\square_3 - \frac{\square_1}{1575\square_2\square_3} - \frac{\square_2}{6300\square_1\square_2} \right) \nabla_\mu R^\alpha_1 \nabla_\nu R^\beta_2 \nabla_\alpha \nabla_\beta R^\mu_3 \nabla_\nu R^\beta_3 \\
&+ \frac{1}{6300\square_1\square_2} \nabla_\lambda \nabla_\sigma R^\alpha_1 \nabla_\alpha \nabla_\beta R^\mu_2 \nabla_\mu \nabla_\nu R^\lambda_3 \nabla_\gamma R^\nu_3 \right) + O[\Re^1].
\end{split}
\] (4.42)

The task is now to bring expressions (4.40)–(4.42) to a local form by restoring the Riemann tensor. The expression for the Riemann tensor solving the Bianchi identities to second order in the Ricci tensor is given in Appendix (eq. (A.32)). The procedure that we use is as follows. For each \( a_n \), we first consider a linear combination of all possible local invariants of the appropriate dimension with unknown coefficients. Next, in this combination, we exclude the Riemann tensor, and equate the result to the nonlocal expression above. This gives a set of equations for the coefficients, which, in each case, has a unique solution. In the case of \( a_2 \), there is only one local invariant with explicit participation of the Riemann tensor:

\[
\int dx g^{1/2} R_{\alpha\beta} R_{\mu\nu} R^{\alpha\beta\mu\nu}.
\] (4.43)

In the case of \( a_3 \), there are seven (the integral over space-time is assumed):

\[
\begin{align*}
\text{tr} \hat{P} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, \quad & \quad \text{tr} \hat{R}^{\alpha\beta} \hat{R}^{\mu\nu} R_{\alpha\beta\mu\nu}, \\
R^{\alpha\beta\mu\nu} R_{\mu\nu\alpha\beta}, \quad & \quad R^{\alpha\beta\mu\nu} R^{\rho\sigma}_{\alpha\beta}, \\
R^{\alpha\beta\mu\nu} R^{\beta\mu\nu}_{\alpha\beta}, \quad & \quad RR^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, \\
R^{\alpha\beta\mu\nu} R^{\beta\mu\nu}_{\alpha\beta}, \quad & \quad R^{\alpha\beta\mu\nu} R^{\beta\mu\nu}_{\alpha\beta},
\end{align*}
\] (4.44)

and the coefficient of the sixth turns out to be zero. In the case of \( a_4 \), there are ten (counting only cubic):

\[
\begin{align*}
\text{tr} \Box \hat{P} R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}, \quad & \quad \text{tr} \hat{P} \nabla_\mu \nabla_\alpha \nabla_\beta R^{\mu\nu\alpha\beta}, \\
\text{tr} \hat{R}^{\alpha\beta} \Box \hat{R}^{\mu\nu} R_{\alpha\beta\mu\nu}, \quad & \quad \Box R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}, \\
\Box R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, \quad & \quad R_{\mu\nu} \nabla_\alpha R_{\nu\beta} \nabla_\mu R_{\alpha\beta}, \\
R \nabla_\mu \nabla_\alpha R_{\nu\beta} R^{\mu\nu\alpha\beta}, \quad & \quad \nabla_\mu R_{\nu\alpha} \nabla_\rho R_{\mu\rho \nu}, \\
\nabla_\alpha R_{\beta\lambda} \nabla_\mu R^{\lambda}_\mu R^{\alpha\beta\mu\nu}, \quad & \quad R^{\alpha\beta\mu\nu} \Box R^{\beta\mu\nu} R_{\alpha\beta\mu\nu},
\end{align*}
\] (4.45)

and the last one proves to be absent. The number of invariants with the Riemann tensor does not grow fast owing to the Bianchi identities and, particularly, their corollary

\[
\Box R^{\alpha\beta\mu\nu} \equiv \frac{1}{2} \left( \nabla_\mu \nabla_\alpha R^{\beta\nu} + \nabla_\alpha \nabla_\mu R^{\beta\nu} - \nabla_\nu \nabla_\alpha R^{\beta\mu} - \nabla_\alpha \nabla_\nu R^{\beta\mu} \right)
\]

\[
- \nabla_\mu \nabla_\beta R^{\mu\alpha} - \nabla_\beta \nabla_\mu R^{\mu\alpha} + \nabla_\nu \nabla_\beta R^{\mu\alpha} + \nabla_\beta \nabla_\nu R^{\mu\alpha}
\]

\[
+ R^{[\mu}_{\nu} \lambda^{\beta\alpha} - R^{[\alpha}_{\lambda} \beta^{\mu\nu}} + R^{[\beta}_{\nu} \gamma^{\mu\nu} \lambda^{\nu} - R^{\alpha}_{\sigma \lambda} R^{\mu\nu \sigma \lambda}
\] (4.46)

which excludes \( \Box R_{\alpha\beta\mu\nu} \) in a local way.

*On the bases of local and nonlocal invariants see Appendix.
The final results are as follows. The expressions (4.38) and (4.39) are already in the local form. The expression (4.40) is brought to a local form by using eq. (A.38) of Appendix:

$$
\int dx \, g^{1/2} \text{tr} \hat{a}_2(x, x) = \int dx \, g^{1/2} \text{tr} \left\{ \frac{1}{12} \hat{P} \hat{P} + \frac{1}{120} \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} \\
+ \left[ \frac{1}{180} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} \right] \hat{1} \right\} + O[\mathcal{R}^4],
$$

(4.47)

and the expressions (4.41), (4.42) rewritten in terms of invariants (4.44) and (4.45) take the form

$$
\int dx \, g^{1/2} \text{tr} \hat{a}_3(x, x) = \int dx \, g^{1/2} \text{tr} \left\{ \frac{1}{12} \hat{P} \square \hat{P} + \frac{1}{120} \hat{R}_{\mu\nu} \square \hat{R}^{\mu\nu} \\
+ \frac{1}{180} \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} + \frac{1}{180} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \hat{P} \\
+ \frac{1}{840} R_{\mu\nu} \square R^{\mu\nu} - \frac{1}{3780} R \square R \right\} \hat{1} + \frac{1}{6} \hat{P} \hat{P} \hat{P} - \frac{1}{45} \hat{R}_{\mu} \hat{R}^{\alpha} \hat{R}^{\beta} \hat{R}_{\mu} \\
+ \frac{1}{12} \hat{P} \hat{R}_{\alpha\beta} \hat{R}_{\alpha\beta} + \frac{1}{72} \hat{R}_{\mu\nu} \hat{R}_{\alpha\beta} \hat{R}_{\mu\nu} \\
- \frac{1}{180} R^{\mu\nu\alpha\beta} \hat{R}_{\mu\nu} + \frac{1}{180} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \hat{P} \\
+ \frac{1}{180} R^{\alpha\beta} R_{\alpha\beta} \hat{P} + \left[ -\frac{1}{1620} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} + \frac{17}{45360} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} + \frac{17}{45360} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} \\
+ \frac{1}{45360} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} + \frac{1}{2835} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} + \frac{1}{2835} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} \\
+ \frac{1}{945} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} + \frac{4}{2835} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} + \frac{4}{2835} \hat{R}^{\alpha \beta} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} \hat{R}^{\sigma \rho} \hat{R}_{\alpha \beta} \\
- \frac{1}{180} R^{\mu\nu\alpha\beta} \hat{R}_{\mu\nu\alpha\beta} \right\} \hat{1} + O[\mathcal{R}^4],
$$

(4.48)
\[ + \frac{1}{1120} R_{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha R_{\nu\beta} + \frac{1}{420} R_{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha R_{\nu\beta} \]
\[ + \frac{13}{30240} \nabla_\alpha \hat{R}_{\mu\nu} - \frac{1}{15120} R_{\mu\nu} \nabla_\alpha \hat{R}_{\mu\nu} \]
\[ - \frac{1}{7560} R_{\mu\nu} \hat{R}_{\alpha\beta} \nabla_\mu \nabla_\alpha \hat{R}_{\beta\mu} \]
\[ - \frac{1}{1260} R_{\mu\nu} \hat{R}_{\alpha\beta} \nabla_\mu \nabla_\alpha \hat{R}_{\beta\mu} \]
\[ + \frac{1}{37800} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]
\[ + \left( \frac{1}{50400} R_{\mu\nu\alpha\beta} \nabla_\mu R_{\alpha\beta} \right) + \frac{1}{6300} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]
\[ - \frac{1}{25200} R_{\alpha\beta} \nabla_\mu R_{\mu\nu\alpha\beta} \nabla_\nu R_{\nu\beta} - \frac{1}{37800} R_{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha R_{\nu\beta} \]
\[ - \frac{1}{6300} R_{\mu\nu} \nabla_\alpha R_{\beta\mu} \nabla_\beta R_{\alpha\beta} - \frac{1}{9450} R_{\mu\nu} \nabla_\alpha R_{\beta\mu} \nabla_\beta R_{\alpha\beta} \]
\[ + \frac{1}{18900} \nabla_\mu R_{\mu\nu\alpha\beta} \nabla_\nu R_{\mu\nu} \]
\[ + \frac{1}{37800} R_{\mu\nu} \nabla_\mu R_{\mu\nu\alpha\beta} \nabla_\nu R_{\mu\nu} \]
\[ + \frac{1}{7560} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]
\[ - \frac{1}{100800} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]
\[ - \frac{1}{7560} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]
\[ - \frac{1}{100800} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]
\[ - \frac{1}{100800} \nabla_\mu R_{\alpha\beta} \nabla_\mu R_{\alpha\beta} \]

(4.49)

The expressions (4.38), (4.39) and (4.47) for \( a_0, a_1 \) and \( a_2 \) coincide with the results obtained by other methods \([6,7,12–14]\). It is easy to compare expression (4.48) for \( a_3 \) with the result in \([14,15]\) since they differ only by the substitution:

\[ \int dx \frac{g^{1/2}}{2} \text{tr} \nabla_\alpha \hat{R}^{\mu\nu} \nabla_\beta \hat{R}_{\mu\nu} = \int dx \frac{g^{1/2}}{2} \left( -\frac{1}{2} \hat{R}_{\tau\mu\nu} \hat{R}^{\mu\nu} + 2 \hat{R}^{\mu\nu} \hat{R}_{\alpha\beta} \hat{R}^{\alpha\beta} \right). \]

(4.50)

and it is very difficult to compare expression (4.49) for \( a_4 \) with the result in \([15]\). Some identities used for this purpose will be found in sect. 14. The coincidence does take place with accuracy \( O[\Re^4] \) but expression (4.49) is a result of such drastic simplifications that it should be considered as new. It goes without saying that, although all the equations (4.47)–(4.49) are presently obtained with accuracy \( O[\Re^4] \), the results for \( a_2 \) and \( a_3 \) are exact.

### 5. The effective action in two dimensions

As discussed in paper II, the effective action (1.9)

\[ -W = \frac{1}{2} \int_0^\infty \frac{ds}{s} \left( \text{Tr}K(s) - \text{Tr}K(s)|_{\Re=0} \right) \]

(5.1)

*Which in eq. (8.21) of paper II figures with the cubic terms omitted.*
in two dimensions is generally nonanalytic in the curvature. An exceptional case is the
conformal invariant scalar quantum field for which
\[ \text{tr} \hat{1} = 1, \quad \hat{R}_{\mu \nu} = 0, \quad \hat{P} = \frac{1}{6} \hat{R}, \quad R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} R, \]
\[ g^{1/2} R = \text{a total derivative}, \quad \omega = 1. \] (5.2)

In this case, the effective action is expandable in powers of the curvature because the inte-
gral (5.1) converges at the upper limit at each order of this expansion owing to specific
cancellations in the asymptotic behaviours of the form factors at large \( s \). Furthermore, in
the case (5.2), the expansion of \( W \) in powers of the curvature should terminate at the second
power thereby yielding an exact result; the terms of third and higher powers in the curvature
should vanish order by order. Our present purpose is to check explicitly the vanishing of the
third-order terms.

By using the conditions (5.2) in (2.1), we obtain \( \text{Tr} K(s) \) as an expansion in powers of the
Ricci scalar only:
\[
\text{Tr} K(s) = \frac{1}{4 \pi s} \int dx \, g^{1/2} \left\{ 1 + s^2 \sum_{i=1}^{5} c_i f_i(-s \Box_2) R_1 R_2 + s^3 \sum_{i=1}^{20} C_i F_i(-s \Box_1, -s \Box_2, -s \Box_3) R_1 R_2 R_3 + O[R^4] \right\}, \quad \omega = 1
\] (5.3)
where
\[ c_1 = \frac{1}{2}, \quad c_2 = 1, \quad c_3 = \frac{1}{6}, \quad c_4 = \frac{1}{36}, \quad c_5 = 0, \]
and
\[ C_1 = \frac{1}{216}, \quad C_4 = \frac{1}{6}, \quad C_5 = \frac{1}{12}, \quad C_6 = \frac{1}{36}, \quad C_9 = 1, \quad C_{10} = \frac{1}{4}, \quad C_{11} = \frac{1}{2}, \]
\[ C_{15} = \frac{s}{24}(\Box_1 - \Box_2 - \Box_3), \quad C_{16} = \frac{s}{48}(\Box_3 - \Box_2 - \Box_1), \quad C_{17} = \frac{s}{72} \Box_2, \]
\[ C_{22} = \frac{s}{4}(\Box_1 - \Box_2 - \Box_3), \quad C_{23} = \frac{s}{8}(\Box_3 - \Box_2 - \Box_1), \quad C_{24} = \frac{s}{8}(\Box_1 - \Box_2 - \Box_3), \]
\[ C_{25} = \frac{s}{16}(\Box_1 - \Box_2 - \Box_3), \quad C_{26} = \frac{s^2}{24} \Box_1 \Box_2, \quad C_{27} = \frac{s^2}{4} \Box_1 \Box_2, \]
\[ C_{28} = \frac{s^2}{16} \Box_3(\Box_3 - \Box_2 - \Box_1), \quad C_{29} = \frac{s^3}{8} \Box_1 \Box_2 \Box_3, \]
\[ C_2 = C_3 = C_7 = C_8 = C_{12} = C_{13} = C_{14} = C_{18} = C_{19} = C_{20} = C_{21} = 0. \] (5.4)

After insertion in (5.3) of the coefficients (5.4) and the expressions for the form factors \( f_i \)
and \( F_i \) given in sect. 2, \( \text{Tr} K(s) \) divided by \( s \) takes the form
\[
\frac{1}{s} \text{Tr} K(s) = \frac{1}{4 \pi} \int dx \, g^{1/2} \left\{ \frac{1}{s^2} + \frac{1}{32} f(-s \Box_2) \frac{f(-s \Box_2) - 1}{s \Box_2} + \frac{3}{8} \left( f(-s \Box_2) - 1 - \frac{1}{6} s \Box_2 \right) \right\} R_1 R_2
\]
\[ \begin{align*}
&+ \left[ -sF(-s\Box_1, -s\Box_2, -s\Box_3) \frac{\Box_1^2 \Box_2^2 \Box_3^2}{3D^3} \\
&+ f(-s\Box_1) \frac{1}{32D^3\Box_2} (\Box_1^6 - 4\Box_1^5\Box_2 - 4\Box_1^5\Box_3 + 3\Box_1^4\Box_2\Box_3 \\
&+ 24\Box_1^3\Box_2^2\Box_3 + 5\Box_1^4\Box_3^2 + 24\Box_1^3\Box_2\Box_3^2 - 2\Box_1^2\Box_2^2\Box_3^2 \\
&+ 32\Box_1^3\Box_2^3 - 25\Box_1^2\Box_2\Box_3^3 - 36\Box_1\Box_2^2\Box_3^3 + 5\Box_2^3\Box_3^3 \\
&- 5\Box_1^2\Box_3^4 - 9\Box_2^2\Box_3^4 + 4\Box_1\Box_3^5 + 5\Box_2\Box_3^5 - \Box_3^6) \\
&- \left( f(-s\Box_1) - 1 \right) \frac{1}{s\Box_1} \frac{1}{8D^2\Box_2} (\Box_1^4 - 2\Box_1^3\Box_3 - 12\Box_1^2\Box_2\Box_3 \\
&- 10\Box_1\Box_2^2\Box_3 + 8\Box_1\Box_2\Box_3^2 - 2\Box_2^2\Box_3^2 + 2\Box_1\Box_3^3 \\
&+ 3\Box_2\Box_3^3 - \Box_3^4) \\
&+ \left( f(-s\Box_1) - 1 - \frac{1}{6} s\Box_1 \right) \frac{3}{8D^2\Box_2} (\Box_1^2 + 4\Box_1\Box_2 + \Box_2\Box_3 - \Box_3^2) \\
&- \frac{1}{\Box_2 - \Box_3} \frac{\Box_2}{32\Box_1} (f(-s\Box_2) - f(-s\Box_3)) \\
&+ \frac{1}{\Box_2 - \Box_3} \frac{\Box_2}{8\Box_1} \left( \frac{f(-s\Box_2) - 1}{s\Box_2} - \frac{f(-s\Box_3) - 1}{s\Box_3} \right) \\
&- \frac{1}{\Box_2 - \Box_3} \frac{3\Box_2}{8\Box_1} \left( \frac{f(-s\Box_2) - 1 - \frac{1}{6} s\Box_2}{(s\Box_2)^2} \\
&- \frac{f(-s\Box_3) - 1 - \frac{1}{6} s\Box_3}{(s\Box_3)^2} \right) \right] R_1 R_2 R_3 + O[R^4] \right), \quad \omega = 1 \tag{5.5}
\end{align*} \]

in terms of the basic form factors (2.9) and (2.75), and

\[ D = \Box_1^2 + \Box_2^2 + \Box_3^2 - 2\Box_1\Box_2 - 2\Box_1\Box_3 - 2\Box_2\Box_3. \]

By using the asymptotic behaviours (3.1) and (3.2), one can now check that, at \( s \to \infty \), the leading terms \( 1/s \) in (5.5) cancel at both second order and third order in the curvature so that

\[ \frac{1}{s} \text{Tr}K(s) = O\left( \frac{1}{s^2} \right), \quad s \to \infty. \tag{5.6} \]

As a result, the integral (5.1) converges at the upper limit. The convergence at the lower limit in the curvature-dependent terms holds trivially. Only the term of zeroth order in the curvature is ultraviolet divergent but, in the effective action (5.1), this term gets subtracted [10,11].

For the calculation of the integral (5.1), one may use the differential equations for the basic form factors (eqs. (16.39), (16.40) and (16.42) of sect. 16) to make the following substitutions in (5.5):

\[ \begin{align*}
&-s\frac{\Box_1 \Box_2 \Box_3}{D} F(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{d}{ds} \left( sF(-s\Box_1, -s\Box_2, -s\Box_3) \right) \\
&+ \frac{\Box_1 (\Box_3 + \Box_2 - \Box_1)}{2D} f(-s\Box_1) + \frac{\Box_2 (\Box_1 + \Box_3 - \Box_2)}{2D} f(-s\Box_2) \\
&+ \frac{\Box_3 (\Box_1 + \Box_2 - \Box_3)}{2D} f(-s\Box_3), \tag{5.7}
\end{align*} \]
\[
\frac{f(-s\Box) - 1}{s \Box} = \frac{d}{ds} \left( \frac{-2}{\Box} f(-s\Box) \right) + \frac{1}{2} f(-s\Box), \quad (5.8)
\]
\[
\frac{f(-s\Box) - 1 - \frac{1}{6}s\Box}{(s\Box)^2} = \frac{d}{ds} \left( \frac{-2}{3\Box} f(-s\Box) - \frac{1}{3\Box} f(-s\Box) \right) + \frac{1}{12} f(-s\Box). \quad (5.9)
\]

The result of these substitutions is that the expression (5.5) becomes a total derivative in \(s\):
\[
\frac{1}{s} \text{Tr} K(s) = \frac{1}{4\pi} \int dx g^{1/2} \frac{d}{ds} \left\{ \frac{1}{s} + l(s, \Box) R_1 R_2 \right. \\
+ h(s, \Box_1, \Box_2, \Box_3) R_1 R_2 R_3 + O[R^4]\left\}, \quad \omega = 1 \quad (5.10)
\]

where
\[
l(s, \Box) = \frac{1}{\Box} \left( \frac{1}{8} f(-s\Box) - \frac{1}{4} \frac{f(-s\Box) - 1}{s\Box} \right), \quad (5.11)
\]
\[
h(s, \Box_1, \Box_2, \Box_3) = h_1^{\text{sym}} + h_2^{\text{sym}} + h_3^{\text{sym}}, \quad (5.12)
\]
and \(h_1^{\text{sym}}, h_2^{\text{sym}}, h_3^{\text{sym}}\) are the completely symmetrized in \(\Box_1, \Box_2, \Box_3\) functions
\[
h_1 = s F(-s\Box_1, -s\Box_2, -s\Box_3) \frac{\Box_1 \Box_2 \Box_3}{3D^2}, \quad (5.13)
\]
\[
h_2 = f(-s\Box_1) \frac{1}{8D^2 \Box_1 \Box_2} (\Box_1^4 - 2 \Box_1^3 \Box_3 + 2 \Box_1 \Box_3^3 - \Box_3^4 - 2 \Box_1^3 \Box_2 \\
+ 3 \Box_2 \Box_3^3 - 8 \Box_1 \Box_2 \Box_3 + 8 \Box_1 \Box_2 \Box_3^2 - 10 \Box_1 \Box_2^2 \Box_3 - 2 \Box_2^2 \Box_3^2) \\
- \left( \frac{f(-s\Box_1) - 1}{s\Box_1} \right) \frac{1}{4D \Box_1 \Box_2} (\Box_1^2 + 4 \Box_1 \Box_2 + \Box_2 \Box_3 - \Box_3^2), \quad (5.14)
\]
\[
h_3 = \frac{1}{\Box_2 - \Box_3} \frac{\Box_2}{\Box_1} \left[ - \frac{1}{8} \left( \frac{1}{\Box_2} f(-s\Box_2) - \frac{1}{\Box_3} f(-s\Box_3) \right) \\
+ \frac{1}{4} \left( \frac{1}{\Box_2} \frac{f(-s\Box_2) - 1}{s\Box_2} - \frac{1}{\Box_3} \frac{f(-s\Box_3) - 1}{s\Box_3} \right) \right]. \quad (5.15)
\]

Insertion of (5.10) in (5.1) gives for the effective action:
\[
W = \frac{1}{8\pi} \int dx g^{1/2} \left( l(0, \Box_2) R_1 R_2 \\
+ h(0, \Box_1, \Box_2, \Box_3) R_1 R_2 R_3 + O[R^4]\right), \quad \omega = 1 \quad (5.16)
\]

where use is made of the fact that the functions \(l\) and \(h\) vanish at \(s \to \infty\). With the asymptotic behaviours (4.1) and (4.2), and the explicit expressions above for the functions \(l\) and \(h\), we obtain
\[
h_1^{\text{sym}}|_{s=0} = 0, \quad (5.17)
\]
\[
h_2^{\text{sym}}|_{s=0} = -h_3^{\text{sym}}|_{s=0} = -\frac{1}{36} \frac{\Box_1 + \Box_2 + \Box_3}{\Box_1 \Box_2 \Box_3}. \quad (5.17)
\]
The result is
\[ l(0, \Box) = \frac{1}{12} \Box, \quad h(0, \Box_1, \Box_2, \Box_3) = 0, \] (5.18)
and
\begin{align*}
W &= \frac{1}{96\pi} \int dx \, g^{1/2} \frac{1}{\Box} R + O[R^4], \quad \omega = 1. 
\end{align*}
(5.19)
Here the term of second order in the curvature reproduces the result of paper II (and the results of refs. [16,17] obtained by integrating the trace anomaly).

Thus the third–order contribution in \( W \) really vanishes, and the mechanism of this vanishing is that, under special conditions like (5.2), the third-order contribution in \( s^{-1}\text{Tr}K(s) \) becomes a total derivative of a function vanishing at both \( s = 0 \) and \( s = \infty \). This mechanism underlies all ”miraculous” cancellations of nonlocal terms including the trace anomaly in four dimensions.

6. Final result for the effective action in four dimensions. Explicit representation of the form factors

The result for the one-loop effective action (1.9) to third order in the curvature is of the form (\( \omega = 2 \))
\begin{align*}
-W &= \frac{1}{2(4\pi)^2} \int dx \, g^{1/2} \, \text{tr} \left\{ \sum_{i=1}^{5} \gamma_i(-\Box_2) \mathcal{R}_1 \mathcal{R}_2(i) 
+ \sum_{i=1}^{29} \Gamma_i(-\Box_1, -\Box_2, -\Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) + O[\mathcal{R}^4] \right\}. 
\end{align*}
(6.1)
Here terms of zeroth and first order in the curvature are omitted *. Terms of second order in the curvature are given by five quadratic structures (2.2)--(2.6), and terms of third order by twenty nine cubic structures (2.15)--(2.43). The second-order form factors \( \gamma_i(-\Box), \ i = 1 \) to 5, are of the form
\begin{align*}
\gamma_1(-\Box) &= \frac{1}{60} \left( -\ln \left( -\frac{\Box}{\mu^2} \right) + \frac{16}{15} \right), 
\gamma_2(-\Box) &= \frac{1}{180} \left( \ln \left( -\frac{\Box}{\mu^2} \right) - \frac{37}{30} \right), 
\gamma_3(-\Box) &= -\frac{1}{18},
\end{align*}
(6.2)-(6.4)

*Since these terms are local and at most quadratic in derivatives, they must be removed by renormalization. The zeroth-order term violates the boundary condition of asymptotic flatness but, in the present case of massless quantum fields, it is cancelled by the contribution of the functional measure [10,11].
\[ \gamma_4(-\Box) = -\frac{1}{2} \ln \left( -\frac{\Box}{\mu^2} \right), \]  
\[ \gamma_5(-\Box) = \frac{1}{12} \left( -\ln \left( -\frac{\Box}{\mu^2} \right) + \frac{2}{3} \right) \]  

(6.5)  
(6.6)

where the parameter \( \mu^2 > 0 \) accounts for the ultraviolet arbitrariness. The form factor \( \gamma_3(-\Box) \) is local and independent of this arbitrariness. To second order in the curvature, the expressions above reproduce the results of the paper II.

The third-order form factors

\[ \Gamma_i(-\Box_1, -\Box_2, -\Box_3), \quad i = 1 \text{ to } 29, \]  

(6.7)

contain no arbitrary parameters and are expressed through the basic third-order form factor

\[ \Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_{\alpha \geq 0} d^3 \alpha \, \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \times (-\alpha_1 \alpha_2 \Box_3 - \alpha_1 \alpha_3 \Box_2 - \alpha_2 \alpha_3 \Box_1)^{-1} \]  

(6.8)

and the second-order form factors

a) \( \ln(\Box_n/\Box_m), \)  
b) \( \frac{\ln(\Box_n/\Box_m)}{\Box_n - \Box_m}, \)  
\( n, m = 1, 2, 3. \)  

(6.9)

The coefficients of these expressions are rational functions of the following general form:

\[ \frac{P(\Box)}{D^6 \Box_1^2 \Box_2^2 \Box_3^2} \]  

(6.10)

where \( P(\Box) \) is a polynomial, and

\[ D = \Box_1^2 + \Box_2^2 + \Box_3^2 - 2\Box_1 \Box_2 - 2\Box_1 \Box_3 - 2\Box_2 \Box_3. \]  

(6.11)

There are also purely rational contributions of the form (6.10). In this representation, the explicit expressions for \( \Gamma_i \) are given below. In sects. 7–9 we present also several integral representations of the form factors \( \Gamma_i \). The derivations of these results are given in sects. 17–20.

When taken separately from their curvature structures, the form factors (6.7) should be explicitly symmetrized

\[ \Gamma_i \rightarrow \Gamma_i^{\text{sym}} \]  

(6.12)

according to the same laws (2.46)–(2.74) as the form factors in the heat kernel. In a not symmetrized form, the explicit expressions for the third-order form factors are as follows:

\[ \Gamma_1(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{3} \Gamma(-\Box_1, -\Box_2, -\Box_3), \]  

(6.13)
\begin{align}
\Gamma_2(-\square_1, -\square_2, -\square_3) &= \Gamma(-\square_1, -\square_2, -\square_3) \frac{1}{3D^3} \left(8\square_1^4\square_2\square_3 \\
&- 8\square_1^3\square_2^2\square_3 - 8\square_1^2\square_2^3\square_3 + 8\square_1\square_2^4\square_3 - 8\square_1^3\square_2\square_3^2 \\
&+ 32\square_1\square_2^2\square_3^2 - 8\square_1\square_2^3\square_3^2 - 8\square_1^2\square_2\square_3^3 \\
&- 8\square_1\square_2^2\square_3^3 + 8\square_1\square_2\square_3^4\right) \\
&+ \frac{\ln(\square_1/\square_2)}{9D^3} \left(8\square_1^4\square_2 - 24\square_1^3\square_2^2 + 24\square_1^2\square_2^3 \\
&- 8\square_1\square_2^4 + 4\square_1^4\square_3 + 96\square_1^3\square_2\square_3 - 96\square_1\square_2^3\square_3 \\
&- 4\square_2^4\square_3 - 12\square_1^3\square_3^2 - 108\square_1^2\square_2\square_3^2 + 108\square_1\square_2^2\square_3^2 \\
&+ 12\square_2^3\square_3^3 - 12\square_2^2\square_3^3 - 4\square_1\square_3^4 + 4\square_2\square_3^4\right) \\
&+ \frac{1}{3D^2} \left(- 2\square_1^3 + 2\square_1^2\square_2 + 2\square_1\square_2^2 - 2\square_2^3 + 2\square_1\square_3 \\
&- 28\square_1\square_2\square_3 + 2\square_2^2\square_3 + 2\square_1\square_3^2 + 2\square_2\square_3^2 - 2\square_3^3\right), \tag{6.14}
\end{align}

\begin{align}
\Gamma_3(-\square_1, -\square_2, -\square_3) &= \Gamma(-\square_1, -\square_2, -\square_3) \frac{1}{D^2} \left(- 2\square_1^3\square_2 \\
&+ 4\square_1^2\square_2^2 - 2\square_1\square_2^3 - 2\square_1^2\square_2\square_3 - 2\square_1\square_2^2\square_3 + 4\square_1\square_2\square_3^2\right) \\
&+ \frac{\ln(\square_1/\square_2)}{3D^2} \left(- \square_1^3 + 9\square_1^2\square_2 + 9\square_1\square_2^2 + \square_2^3 \\
&+ 2\square_1\square_3 - 2\square_2^2\square_3 - \square_1\square_3^2 + \square_2\square_3^2\right) \\
&+ \frac{\ln(\square_1/\square_3)}{3D^2} \left(- 2\square_1^3 - 3\square_1^2\square_2 + 6\square_1\square_2^2 - \square_2^3 + 4\square_1\square_3^2 \\
&- 12\square_1\square_2\square_3 + 2\square_2^2\square_3 - 2\square_1\square_3^2 - 2\square_2\square_3^2\right) \\
&+ \frac{\ln(\square_2/\square_3)}{3D^2} \left(- \square_1^3 + 6\square_1^2\square_2 - 3\square_1\square_2^2 - 2\square_2^3 \\
&+ 2\square_1\square_3^2 - 12\square_1\square_2\square_3 + 4\square_2^2\square_3 - \square_1\square_3^2 - 2\square_2\square_3^2\right) \\
&+ \frac{1}{D} \left(\square_1 + \square_2 - \square_3\right), \tag{6.15}
\end{align}

\begin{align}
\Gamma_4(-\square_1, -\square_2, -\square_3) &= \Gamma(-\square_1, -\square_2, -\square_3) \frac{1}{36D^4} \left(- 4\square_1^8 \\
&- 4\square_1^7\square_2 + 32\square_1^6\square_2^2 - 28\square_1^5\square_2^3 + 4\square_1^4\square_2^4 \\
&+ 2\square_1^7\square_3 - 118\square_1^6\square_2\square_3 - 90\square_1^5\square_2^2\square_3 + 206\square_1^4\square_2^3\square_3 \\
&+ 38\square_1^6\square_3^2 + 180\square_1^5\square_2\square_3^2 - 198\square_1^4\square_2^2\square_3^2 - 236\square_1^3\square_2^3\square_3^2 \\
&- 82\square_1^5\square_3^3 + 110\square_1^4\square_2\square_3^3 + 188\square_1^3\square_2^2\square_3^3 + 50\square_1^2\square_3^4\right),
\end{align}
\begin{align*}
&- 172 \Box_1 \Box_2 \Box_3^4 + 90 \Box_1^2 \Box_2^2 \Box_3^4 + 14 \Box_1^3 \Box_3^5 - 90 \Box_1^2 \Box_2 \Box_3^5 \\
&- 22 \Box_1^2 \Box_3^6 + 46 \Box_1 \Box_2 \Box_3^6 + 2 \Box_1 \Box_3^7 + \Box_3^8 \bigg) + \frac{\ln(\Box_1/\Box_2)}{18 D^4} \bigg(- 6 \Box_1^7 - 8 \Box_1^6 \Box_2 + 24 \Box_1^5 \Box_2^2 \\
&+ 10 \Box_1^4 \Box_2^3 + 11 \Box_1^6 \Box_3 - 64 \Box_1^5 \Box_2 \Box_3 - 125 \Box_1^4 \Box_2^2 \Box_3 \\
&+ 10 \Box_1^5 \Box_3^2 + 112 \Box_1^4 \Box_2 \Box_3^2 + 54 \Box_1^3 \Box_2^2 \Box_3^2 - 30 \Box_1^4 \Box_3^3 \\
&+ 12 \Box_1^3 \Box_2 \Box_3^3 + 10 \Box_1^3 \Box_3^4 - 58 \Box_1^2 \Box_2 \Box_3^4 + 11 \Box_1^2 \Box_3^5 - 6 \Box_1 \Box_3^6 \bigg) + \frac{\ln(\Box_1/\Box_3)}{18 D^4} \bigg(- 6 \Box_1^7 + 2 \Box_1^6 \Box_2 + 12 \Box_1^5 \Box_2^2 \\
&+ 2 \Box_1^4 \Box_2^3 - 8 \Box_1^3 \Box_2^4 - 12 \Box_1^2 \Box_2^5 + 10 \Box_1 \Box_2^6 \\
&+ \Box_1^6 \Box_3 - 80 \Box_1^5 \Box_2 \Box_3 - 46 \Box_1^4 \Box_2^2 \Box_3 + 72 \Box_1^3 \Box_2^3 \Box_3 \\
&+ 79 \Box_1^2 \Box_2 \Box_3^4 - 16 \Box_1 \Box_2^5 \Box_3 - 10 \Box_2^6 \Box_3 + 38 \Box_1^5 \Box_3^2 \\
&+ 56 \Box_1^4 \Box_2 \Box_3^2 - 60 \Box_1^3 \Box_2^2 \Box_3^2 - 114 \Box_1^2 \Box_2 \Box_3^3 - 56 \Box_1^2 \Box_3^4 \\
&+ 28 \Box_1 \Box_2^3 \Box_3^3 - 45 \Box_1^4 \Box_3^3 + 72 \Box_1^3 \Box_2 \Box_3^4 + 18 \Box_1^2 \Box_2^2 \Box_3^3 \\
&+ 60 \Box_1 \Box_2 \Box_3^5 - 15 \Box_2^4 \Box_3^2 - 10 \Box_1 \Box_3^6 - 2 \Box_1^2 \Box_3^5 \\
&+ 56 \Box_1 \Box_2 \Box_3^4 - 20 \Box_2^3 \Box_3^4 + 31 \Box_2^2 \Box_3^5 - 48 \Box_1 \Box_2 \Box_3^5 \\
&+ 20 \Box_2^2 \Box_3^5 - 6 \Box_1 \Box_3^6 - 3 \Box_3^7 \bigg) + \frac{1}{24 D^3} \bigg(6 \Box_1^5 + 4 \Box_1^4 \Box_2 - 20 \Box_1^3 \Box_2^2 - 22 \Box_1^4 \Box_3 \\
&+ 44 \Box_1^3 \Box_2 \Box_3 + 50 \Box_1^2 \Box_2^2 \Box_3 - 20 \Box_1^3 \Box_3^2 - 24 \Box_1^2 \Box_2 \Box_3^2 \\
&+ 32 \Box_1 \Box_2 \Box_3^3 - 14 \Box_1 \Box_2 \Box_3^3 + 4 \Box_1 \Box_3^4 - 5 \Box_3^5 \bigg), \tag{6.16}
\end{align*}

\begin{align*}
\Gamma_5(-\Box_1, -\Box_2, -\Box_3) &= \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{\Box_1 \Box_2 \Box_3^2}{D^2} \\
&+ \frac{\ln(\Box_1/\Box_2)}{18 D^2 \Box_2} \bigg(- \Box_1^4 + 3 \Box_1^3 \Box_2 - 4 \Box_1^2 \Box_2^2 + 3 \Box_1 \Box_3^3 \bigg) \\
&+ 4 \Box_1^2 \Box_2 \Box_3 - 3 \Box_1^2 \Box_3^2 - 7 \Box_1 \Box_2 \Box_3^2 + \Box_1 \Box_3^3 \bigg) + \frac{\ln(\Box_1/\Box_3)}{18 D^2 \Box_2} \bigg(- 2 \Box_1^5 + 6 \Box_1^4 \Box_2 - 5 \Box_1^3 \Box_2^2 - \Box_1^2 \Box_3^3 \\
&+ 3 \Box_1 \Box_2^4 - \Box_2^5 + 6 \Box_1^4 \Box_3 + 8 \Box_1^3 \Box_2 \Box_3 - 21 \Box_1^2 \Box_2^2 \Box_3 \\
&+ 4 \Box_1 \Box_2 \Box_3^2 - 3 \Box_2^2 \Box_3^2 - 6 \Box_1^3 \Box_3^2 - 14 \Box_1^2 \Box_2 \Box_3^2 \\
&- 7 \Box_1 \Box_2 \Box_3^3 + \Box_2^2 \Box_3^3 \bigg) + \frac{1}{24 D \Box_1 \Box_2} \bigg(6 \Box_1^3 - 6 \Box_1^2 \Box_2 - 14 \Box_1 \Box_3^2 \\
&+ 14 \Box_1 \Box_2 \Box_3 + 10 \Box_1 \Box_3^2 - \Box_3^3 \bigg), \tag{6.17}
\end{align*}

\begin{align*}
\Gamma_6(-\Box_1, -\Box_2, -\Box_3) &= \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{6 D^2} \bigg(- 2 \Box_1^4
\end{align*}
\[ -4\Box_1^3\Box_2 + 6\Box_1^2\Box_2^2 + 2\Box_1^3\Box_3 - 14\Box_1^2\Box_2\Box_3 \\
+ 2\Box_1\Box_2\Box_3^2 + 2\Box_1\Box_3^3 - \Box_3^4 \]
\[ + \frac{\ln(\Box_1/\Box_2)}{3D^2} \left( -2\Box_1^3 - 6\Box_1^2\Box_2 + \Box_1\Box_3^2 + \Box_1\Box_3^2 \right) \]
\[ + \frac{\ln(\Box_1/\Box_3)}{6D^2} \left( -5\Box_1^3 - 3\Box_1^2\Box_2 + 9\Box_1\Box_2^2 - \Box_2^3 + \Box_1\Box_3^2 \right) \\
- 12\Box_1\Box_2\Box_3 - \Box_2^2\Box_3 + \Box_1\Box_3^2 - \Box_2\Box_3^2 + 3\Box_3^3 \]
\[ + \frac{\Box_1}{D}, \]  \hspace{1cm} (6.18)

\[
\Gamma_7(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3D^4} \left( 4\Box_1^6\Box_2\Box_3 - 24\Box_1^5\Box_2^2\Box_3 \\
+ 18\Box_1^4\Box_2^3\Box_3 + 8\Box_1^3\Box_2^4\Box_3 - 12\Box_1^2\Box_2^5\Box_3 + 2\Box_2^7\Box_3 + 63\Box_1^4\Box_2^2\Box_3^2 \right) \\
- 108\Box_1^3\Box_2^3\Box_3^2 - 66\Box_1^2\Box_2^4\Box_3^2 + 72\Box_1\Box_2^5\Box_3^2 + 114\Box_1\Box_2^3\Box_3^3 \\
- 72\Box_1\Box_2^4\Box_3^3 - 18\Box_2^5\Box_3^3 + 16\Box_2^4\Box_3^4 \right) \\
\[ + \frac{\ln(\Box_1/\Box_2)}{36D^4} \left( 3\Box_1^8 - 12\Box_1^7\Box_2 + 12\Box_1^6\Box_2^2 + 16\Box_1^5\Box_2^3 - 50\Box_1^4\Box_2^4 \right) \\
+ 52\Box_1^3\Box_2^5 - 28\Box_1^2\Box_2^6 + 8\Box_1\Box_2^7 - \Box_2^8 - 18\Box_1^7\Box_3 \\
+ 198\Box_1^6\Box_2\Box_3 - 194\Box_1^5\Box_2^2\Box_3 - 350\Box_1^4\Box_2^3\Box_3 + 498\Box_1^3\Box_2^4\Box_3 - 46\Box_1^2\Box_2^5\Box_3 - 94\Box_1\Box_2^6\Box_3 + 60\Box_1^7\Box_3^2 + 48\Box_1^6\Box_2^3 - 430\Box_1^5\Box_2^2\Box_3^2 \\
+ 1230\Box_1^4\Box_2^3\Box_3^2 + 40\Box_1^3\Box_2^4\Box_3^2 - 960\Box_1^2\Box_2^3\Box_3^2 + 86\Box_1\Box_2^4\Box_3^2 \\
- 14\Box_2^6\Box_3^2 - 76\Box_1^5\Box_2^3 + 218\Box_1^4\Box_2^2\Box_3 - 1096\Box_1^3\Box_2\Box_3^2 - 3\Box_3^2 \right) \\
+ 1224\Box_1^2\Box_2^3\Box_3^3 + 228\Box_1\Box_2^4\Box_3^3 + 14\Box_2^5\Box_3^3 + 80\Box_1^4\Box_3^4 \\
+ 132\Box_1^3\Box_2^4\Box_3 - 960\Box_1^2\Box_2^3\Box_3 + 372\Box_1\Box_2^4\Box_3^2 - 58\Box_1^5\Box_3^5 \\
- 122\Box_1^2\Box_2^3\Box_3 + 130\Box_1\Box_2^2\Box_3^5 - 14\Box_2^3\Box_3^5 + 28\Box_1^2\Box_3^6 \\
+ 22\Box_1\Box_2^3\Box_3^6 + 14\Box_2^2\Box_3^6 - 8\Box_1^2\Box_3^7 - 6\Box_2\Box_3^7 + \Box_3^8 \right) \\
\[ + \frac{\ln(\Box_2/\Box_3)}{18D^4} \left( -3\Box_1^7\Box_2 + 18\Box_1^6\Box_2^2 - 46\Box_1^5\Box_2^3 + 65\Box_1^4\Box_2^4 \right) \\
- 55\Box_1^3\Box_2^5 + 28\Box_1^2\Box_2^6 - 8\Box_1\Box_2^7 + \Box_2^8 - 118\Box_1^5\Box_2^2\Box_3 \\
+ 284\Box_1^4\Box_2^3\Box_3 - 183\Box_1^3\Box_2^4\Box_3 - 38\Box_1^2\Box_2^5\Box_3 + 58\Box_1\Box_2^6\Box_3 \\
- 6\Box_2^7\Box_3 - 568\Box_1^3\Box_2^3\Box_3^2 + 432\Box_1^2\Box_2^4\Box_3^2 + 22\Box_1\Box_2^5\Box_3^2 \\
+ 14\Box_2^6\Box_3^2 - 300\Box_1\Box_2^4\Box_3^3 - 14\Box_2^5\Box_3^3 \right) \\
\[ + \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)} \left( -\Box_2 \right) \\
\[ + \frac{1}{24D^3} \left( -3\Box_1^5 + 26\Box_1^4\Box_2 - 44\Box_1^3\Box_2^2 + 36\Box_1^2\Box_2^3 \right) \\
- 14\Box_1\Box_2^4 + 2\Box_2^5 - 92\Box_1^3\Box_2\Box_3 + 180\Box_1^2\Box_2^2\Box_3 \\
+ 88\Box_1\Box_2^3\Box_3 - 110\Box_2^4\Box_3 - 218\Box_1\Box_2^2\Box_3^2 + 108\Box_2^3\Box_3^2 \right), \]  \hspace{1cm} (6.19)

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\begin{align*}
\Gamma_\delta(\Box_1, \Box_2, \Box_3) &= \Gamma(\Box_1, \Box_2, \Box_3) \frac{1}{D^4} \left( -4 \Box_1^5 \Box_2 \Box_3 ight) \\
&+ 24 \Box_1^5 \Box_2 \Box_3 - 16 \Box_1^4 \Box_2^2 \Box_3 - 16 \Box_1^3 \Box_2^4 \Box_3 + 24 \Box_1^2 \Box_2^5 \Box_3 \\
&- 8 \Box_1 \Box_2 \Box_3 - 8 \Box_1^4 \Box_2^2 \Box_3^2 + 168 \Box_1^3 \Box_2^3 \Box_3^2 + 72 \Box_1^2 \Box_2^4 \Box_3^2 \\
&- 96 \Box_1 \Box_2^5 \Box_3^2 - 176 \Box_1^2 \Box_2 \Box_3^3 + 104 \Box_1 \Box_2^4 \Box_3^3 \\
&+ \frac{\ln(\Box_1/\Box_2)}{9D^4 \Box_1} \left( -3 \Box_1^8 + 16 \Box_1^7 \Box_2 - 32 \Box_1^6 \Box_2^2 + 24 \Box_1^5 \Box_2^3 + 10 \Box_1^4 \Box_2^4 \\
&- 32 \Box_1^3 \Box_2^5 + 24 \Box_1^2 \Box_2^6 - 8 \Box_1 \Box_2^7 + 2 \Box_2^8 + 20 \Box_1^7 \Box_3 - 192 \Box_1^6 \Box_2 \Box_3 \\
&+ 180 \Box_1^5 \Box_2^2 \Box_3 - 392 \Box_1^4 \Box_2^3 \Box_3 - 628 \Box_1^3 \Box_2^4 \Box_3 + 192 \Box_1^2 \Box_2^5 \Box_3 \\
&+ 44 \Box_1 \Box_2^6 \Box_3 - 8 \Box_2^7 \Box_3 - 58 \Box_1^6 \Box_3^2 + 396 \Box_1^5 \Box_2 \Box_3^2 \\
&- 1374 \Box_1^4 \Box_2^2 \Box_3^2 + 80 \Box_1^3 \Box_2^3 \Box_3^2 + 930 \Box_1^2 \Box_2^4 \Box_3^2 + 4 \Box_1 \Box_2^5 \Box_3^2 \\
&+ 22 \Box_2^6 \Box_3^2 + 96 \Box_1^5 \Box_3^3 - 176 \Box_1^4 \Box_2 \Box_3^3 + 1408 \Box_1^3 \Box_2^2 \Box_3^3 \\
&- 1128 \Box_1^2 \Box_2^3 \Box_3^3 - 176 \Box_1 \Box_2^4 \Box_3^3 - 24 \Box_2^5 \Box_3^3 - 100 \Box_1^4 \Box_3^4 \\
&- 176 \Box_1^3 \Box_2 \Box_3^4 - 156 \Box_1^2 \Box_2^2 \Box_3^4 + 176 \Box_1 \Box_2^3 \Box_3^4 + 68 \Box_1^3 \Box_3^5 \\
&+ 168 \Box_1^2 \Box_2 \Box_3^5 - 4 \Box_1 \Box_2^2 \Box_3^5 + 24 \Box_2^3 \Box_3^5 - 30 \Box_1^2 \Box_3^6 \\
&- 44 \Box_2 \Box_3^6 - 22 \Box_2^2 \Box_3^6 + 8 \Box_1 \Box_3^7 + 8 \Box_2 \Box_3^7 - \Box_3^8) \\
&+ \frac{\ln(\Box_2/\Box_3)}{9D^4 \Box_1} \left( 4 \Box_1^7 \Box_2 - 26 \Box_1^6 \Box_2^2 + 72 \Box_1^5 \Box_2^3 - 110 \Box_1^4 \Box_2^4 \\
&+ 100 \Box_1^3 \Box_2^5 - 54 \Box_1^2 \Box_2^6 + 16 \Box_1 \Box_2^7 - 2 \Box_2^8 + 216 \Box_1^5 \Box_2^2 \Box_3 \\
&- 568 \Box_1^4 \Box_2^3 \Box_3 - 45 \Box_1^3 \Box_2^4 \Box_3 - 24 \Box_1^2 \Box_2^5 \Box_3 - 88 \Box_1 \Box_2^6 \Box_3 \\
&+ 16 \Box_2^7 \Box_3 + 132 \Box_1^3 \Box_2^3 \Box_3^2 - 1086 \Box_1^2 \Box_2^4 \Box_3^2 - 8 \Box_1 \Box_2^5 \Box_3^2 \\
&- 44 \Box_2 \Box_3^2 + 352 \Box_1 \Box_2^3 \Box_3 + 48 \Box_2^5 \Box_3^3 \\
&+ \frac{1}{6D^3 \Box_1} \left( \Box_1^6 - 8 \Box_1^5 \Box_2 - 10 \Box_1^4 \Box_2^2 - 10 \Box_1^3 \Box_2^4 - 8 \Box_1 \Box_2^5 - 2 \Box_2^6 \\
&+ 74 \Box_1^4 \Box_2^3 - 192 \Box_1^3 \Box_2^5 \Box_3 - 24 \Box_1^2 \Box_2^6 \Box_3 + 72 \Box_1 \Box_2^7 \Box_3 + 4 \Box_2^8 \Box_3 \\
&+ 274 \Box_1^2 \Box_2^2 \Box_3^2 - 80 \Box_1 \Box_2^3 \Box_3^2 + 2 \Box_2^4 \Box_3^2 - 4 \Box_2 \Box_3^3 \Box_3^2 \right) \\
\end{align*}

\begin{align*}
\Gamma_\delta(\Box_1, \Box_2, \Box_3) &= \Gamma(\Box_1, \Box_2, \Box_3) \frac{1}{324D^6} \left( 3 \Box_1^{12} + 18 \Box_1^{11} \Box_2 \\
&- 72 \Box_1^{10} \Box_2^2 - 78 \Box_1^{9} \Box_2^3 + 378 \Box_1^{8} \Box_2^4 - 324 \Box_1^{7} \Box_2^5 \\
&+ 72 \Box_1^{6} \Box_2^6 + 792 \Box_1^{10} \Box_2 \Box_3 - 2304 \Box_1^{9} \Box_2^2 \Box_3 - 3150 \Box_1^{8} \Box_2^3 \Box_3 \\
&+ 7164 \Box_1^{7} \Box_2^4 \Box_3 - 3312 \Box_1^{6} \Box_2^5 \Box_3 + 9090 \Box_1^{5} \Box_2^6 \Box_3 \\
&- 16200 \Box_1^{4} \Box_2^7 \Box_3 - 1836 \Box_1^{5} \Box_2^4 \Box_3^2 + 2232 \Box_1^{4} \Box_2^5 \Box_3^2 \\
&+ 16320 \Box_1^{3} \Box_2^6 \Box_3^2 - 6732 \Box_1^{2} \Box_2^7 \Box_3^2 - 4246 \Box_1 \Box_2^8 \Box_3^2 \\
&+ \frac{\ln(\Box_1/\Box_2)}{1080D^6 \Box_2 \Box_3} \left( 6 \Box_1^{12} \Box_2^2 - 72 \Box_1^{11} \Box_2^3 + 360 \Box_1^{10} \Box_2^4 - 984 \Box_1^{9} \Box_2^5 \\
&+ 1566 \Box_1^{8} \Box_2^6 - 1296 \Box_1^{7} \Box_2^7 + 3 \Box_1^{12} \Box_3 - 33 \Box_1^{11} \Box_2 \Box_3 \\
&+ 1216 \Box_1^{10} \Box_2^2 \Box_3 - 3944 \Box_1^{9} \Box_2^3 \Box_3 + 3827 \Box_1^{8} \Box_2^4 \Box_3 \\
&+ 1143 \Box_1^{7} \Box_2^5 \Box_3 - 5576 \Box_1^{6} \Box_2^6 \Box_3 - 36 \Box_1^{11} \Box_3^2 + 824 \Box_1^{10} \Box_2 \Box_3^2 \right) \\
\end{align*}
\[
+ 6216 \Box_4^9 \Box_3^2 \Box_3^2 - 1176 \Box_7^8 \Box_2^3 \Box_3^2 - 20260 \Box_{17}^7 \Box_2^4 \Box_3^2 \\
+ 15280 \Box_1^6 \Box_2^5 \Box_3^2 + 180 \Box_1^{10} \Box_3^3 - 2860 \Box_1^9 \Box_2 \Box_3^3 \\
- 13704 \Box_2^8 \Box_2^2 \Box_3^3 + 60960 \Box_7^7 \Box_2^3 \Box_3^3 - 11220 \Box_6^8 \Box_2^4 \Box_3^3 \\
- 18396 \Box_1^5 \Box_2^5 \Box_3^3 - 492 \Box_1^9 \Box_3^4 + 3376 \Box_1^8 \Box_2 \Box_3^4 \\
+ 4276 \Box_1^7 \Box_2^2 \Box_3^3 - 61416 \Box_1^6 \Box_2^2 \Box_3^4 + 14112 \Box_1^5 \Box_2^4 \Box_3^4 \\
+ 783 \Box_1^8 \Box_3^5 - 495 \Box_1^7 \Box_2 \Box_3^5 + 176 \Box_1^6 \Box_2^2 \Box_3^5 \\
+ 1296 \Box_1^5 \Box_2^2 \Box_3^5 + 10506 \Box_1^4 \Box_2^2 \Box_3^5 - 648 \Box_1^7 \Box_3^6 \\
- 2500 \Box_1^6 \Box_2 \Box_3^6 + 4632 \Box_1^5 \Box_2 \Box_3^6 + 31092 \Box_1^4 \Box_2 \Box_3^6 \\
+ 3076 \Box_1^5 \Box_2 \Box_3^7 - 15104 \Box_1^4 \Box_2 \Box_3^7 - 50196 \Box_1^3 \Box_2 \Box_3^7 \\
+ 648 \Box_1^5 \Box_3^8 - 1638 \Box_1^4 \Box_2 \Box_3^8 + 24536 \Box_1^3 \Box_2 \Box_3^8 - 783 \Box_1^4 \Box_3^9 \\
- 451 \Box_1^3 \Box_2 \Box_3^9 - 12528 \Box_1^2 \Box_2 \Box_3^9 + 492 \Box_1^3 \Box_3^{10} + 1084 \Box_1^2 \Box_2 \Box_3^{10} \\
- 180 \Box_1 \Box_2 \Box_3^{11} - 392 \Box_1 \Box_2 \Box_3^{12} + 36 \Box_1 \Box_3^{12} - 3 \Box_3^{13}
\]

\[
\ln(\Box_1/\Box_2) \left(\frac{-6 \Box_1 + \Box_3}{720 \Box_3}\right)
\]

\[
\frac{1}{2160 \Box_5 \Box_2 \Box_3} \left( - \Box_1^{11} + 32 \Box_1^{10} \Box_2 - 140 \Box_1^9 \Box_2^2 + 224 \Box_1^8 \Box_2^3 \\
+ 66 \Box_1^7 \Box_2^4 - 768 \Box_1^6 \Box_2^5 + 588 \Box_1^5 \Box_2^6 - 418 \Box_1^4 \Box_2 \Box_3 \\
+ 120 \Box_1^4 \Box_2 \Box_3 + 4556 \Box_1^6 \Box_2 \Box_3^2 - 6508 \Box_1^5 \Box_2 \Box_3^2 \\
+ 2636 \Box_1^5 \Box_2 \Box_3^3 - 10326 \Box_1^4 \Box_2 \Box_3^3 + 20320 \Box_1^4 \Box_2 \Box_3^3 \\
+ 760 \Box_1^5 \Box_2 ^4 \Box_3^2 - 408 \Box_1^4 \Box_2^2 \Box_3^2 - 27064 \Box_1^5 \Box_2 ^3 \Box_3^3 \\
+ 7428 \Box_1^4 \Box_2 ^4 \Box_3^3 + 5978 \Box_1^3 \Box_2^4 \Box_3^4\right)
\]

(6.21)

\[
\Gamma_1 (-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3 \Box_3^3} \left(6 \Box_1^4 \Box_2 \Box_3 \\
- 12 \Box_1^3 \Box_2 \Box_3 + 8 \Box_1 \Box_2 \Box_3^2\right)
\]

\[
\ln(\Box_1/\Box_2) \left(4 \Box_1^4 \Box_2 - 12 \Box_1^3 \Box_2^2 + 2 \Box_1^4 \Box_3 + 48 \Box_1^3 \Box_2 \Box_3 \\
- 6 \Box_1^3 \Box_3^2 - 54 \Box_1^2 \Box_2 \Box_3^2 + 6 \Box_1^2 \Box_3^3 - 2 \Box_1 \Box_3^4\right)
\]

\[
\frac{1}{540 \Box_2 \Box_3} \left( - \Box_1^5 + 12 \Box_1^4 \Box_2 - 30 \Box_1^3 \Box_2^2 + 20 \Box_1^2 \Box_2^3 \\
- 108 \Box_1 \Box_2 \Box_3 + 204 \Box_1 \Box_2^2 \Box_3 - 418 \Box_1 \Box_2^2 \Box_3^2\right)
\]

(6.22)

\[
\Gamma_1 (-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3 \Box_3^4} \left(5 \Box_1^5 \Box_2 \Box_3^2 \\
- 2 \Box_1^4 \Box_2^2 \Box_3^2 - 3 \Box_1^3 \Box_2^3 \Box_3^2 - 11 \Box_1^4 \Box_2 \Box_3^3 + 17 \Box_1^3 \Box_2^2 \Box_3^3 \\
+ 3 \Box_1^3 \Box_2 \Box_3^4 - 11 \Box_1^2 \Box_2^2 \Box_3^4 + 7 \Box_1 \Box_2 \Box_3^5 - 2 \Box_1 \Box_2 \Box_3^6\right)
\]

\[
\frac{\ln(\Box_1/\Box_2)}{540 \Box_4^2} \left(2 \Box_1^8 - 20 \Box_1^7 \Box_2 + 74 \Box_1^6 \Box_2^2 - 150 \Box_1^5 \Box_2^3\right)
\]
\[ + 202\square_1^4\square_2^4 - 20\square_1^7\square_3 + 144\square_1^6\square_2\square_3 - 208\square_1^5\square_2^2\square_3^3 + 64\square_1^4\square_2^3\square_3 + 81\square_1^6\square_2^3 - 321\square_1^5\square_2\square_3^2 + 1251\square_1^4\square_2^2\square_3^2 + 1173\square_1^3\square_2^3\square_3^2 - 178\square_1^5\square_2\square_3 + 232\square_1^4\square_2\square_3^3 - 2266\square_1^3\square_2^2\square_3^3 + 235\square_1^4\square_3^4 + 99\square_1^3\square_2\square_3^4 + 1208\square_1^2\square_2^3\square_3^4 - 192\square_1^3\square_3^5 - 216\square_1^2\square_2^3\square_3^5 + 95\square_1^2\square_2^6\square_3^6 + 85\square_1\square_2\square_3^6 - 26\square_1\square_3^7 + 3\square_3^8 \]

\[ \frac{\ln(\square_1/\square_2)}{540D^3\square_1\square_2} \left( 4\square_1^9 - 31\square_1^8\square_2 + 97\square_1^7\square_2^2 - 159\square_1^6\square_2^3 + 143\square_1^5\square_2^4 - 59\square_1^4\square_2^5 - 9\square_1^3\square_2^6 + 23\square_1^2\square_2^7 - 11\square_1\square_2^8 + 2\square_2^9 - 40\square_1^8\square_3 + 216\square_1^7\square_2\square_3 - 368\square_1^6\square_2^2\square_3 + 272\square_1^5\square_2^3\square_3 - 180\square_1^4\square_2^4\square_3 + 208\square_1^3\square_2^5\square_3 - 160\square_1^2\square_2^6\square_3 + 72\square_1\square_2^7\square_3 - 20\square_2^8\square_3 + 162\square_1^7\square_3^2 - 390\square_1^6\square_2\square_3^2 + 1395\square_1^5\square_2^2\square_3^2 - 75\square_1^4\square_2^3\square_3^2 - 1248\square_1^3\square_2^4\square_3^2 + 144\square_1^2\square_2^5\square_3^2 - 69\square_1\square_2^6\square_3^2 + 81\square_1^5\square_3^2 - 356\square_1^4\square_2\square_3^3 - 40\square_1^3\square_2^2\square_3^3 - 1262\square_1^2\square_2\square_3^4 - 1597\square_1\square_2^2\square_3^4 + 2184\square_1^4\square_3^3 + 1004\square_1^3\square_2^2\square_3^3 - 272\square_1^2\square_2^3\square_3^3 - 178\square_2\square_3^3 - 4701\square_1\square_3^4 + 882\square_1^2\square_2\square_3^4 - 389\square_1^3\square_2^2\square_3^4 - 1597\square_1^2\square_2^3\square_3^4 + 729\square_1^2\square_2^4\square_3^4 + 235\square_1^5\square_3^4 - 384\square_1^3\square_2^2\square_3^5 - 936\square_1^2\square_2^3\square_3^5 + 216\square_1^2\square_2^4\square_3^5 - 720\square_1\square_2^5\square_3^5 - 192\square_1\square_2^2\square_3^6 + 422\square_1^2\square_2^3\square_3^6 + 337\square_1\square_2^4\square_3^6 + 95\square_2\square_3^6 - 52\square_1\square_2\square_3^7 - 72\square_1\square_2\square_3^7 - 26\square_2^2\square_3^7 + 6\square_1\square_3^8 + 3\square_2\square_3^8 \right) \]

\[ \frac{1}{120 (\square_1 - \square_2)} \]

\[ \frac{1}{1080D^3\square_1\square_2} \left( 19\square_1^7 - 77\square_1^6\square_2 + 117\square_1^5\square_2^2 - 59\square_1^4\square_2^3 - 100\square_1^3\square_2^4 - 48\square_1^2\square_2^5 - 228\square_1\square_2^6 - 80\square_2^7 + 219\square_1^5\square_3^2 + 555\square_1^4\square_2\square_3^2 - 1854\square_1^3\square_2^2\square_3^2 - 260\square_1^2\square_2\square_3^3 + 632\square_1^5\square_3^3 + 696\square_1^4\square_2\square_3^3 + 185\square_1^3\square_2\square_3^4 + 201\square_1^2\square_2^2\square_3^4 - 84\square_1^2\square_3^5 - 12\square_1\square_2\square_3^5 + 25\square_1\square_3^6 - 2\square_3^7 \right), \]

(6.23)
\(\Gamma_{13}(-\square_1,-\square_2,-\square_3) = \Gamma(-\square_1,-\square_2,-\square_3) \frac{1}{D} \left(2\square_1 - 2\square_2 - 2\square_3\right)\)
\[\begin{align*}
&+ \frac{\ln(\square_1/\square_2)}{3D\square_1} \left(6\square_1 + 2\square_2 - 2\square_3\right) \\
&+ \frac{\ln(\square_1/\square_3)}{3D\square_1} \left(6\square_1 - 2\square_2 + 2\square_3\right) \\
&+ \frac{\ln(\square_2/\square_3)}{3D\square_1} \left(-4\square_2 + 4\square_3\right) \\
&+ \frac{\ln(\square_2/\square_3) 2}{(\square_2 - \square_3) \square_1},
\end{align*}\]
(6.25)

\[\Gamma_{14}(-\square_1,-\square_2,-\square_3) = \Gamma(-\square_1,-\square_2,-\square_3) \frac{1}{D^2} \left(-2\square_1^3\right)\]
\[+ 2\square_1^2 \square_2 + 2\square_1 \square_2^2 - 2\square_2^3 + 2\square_1^2 \square_3 - 12\square_1 \square_2 \square_3\]

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\[+ 2\Box_2^2 \Box_3 + 2 \Box_1 \Box_3^2 + 2 \Box_2 \Box_3^2 - 2 \Box_3^3\]
\[+ \frac{\ln(\Box_1/\Box_2)}{D^2} \left(- 4 \Box_1^2 + 4 \Box_2^2 + 4 \Box_1 \Box_3 - 4 \Box_2 \Box_3\right)\]
\[+ \frac{\ln(\Box_1/\Box_3)}{D^2} \left(- 4 \Box_1^2 + 4 \Box_1 \Box_2 - 4 \Box_2 \Box_3 + 4 \Box_3^2\right)\]
\[+ \frac{\ln(\Box_2/\Box_3)}{D^2} \left(4 \Box_1 \Box_2 - 4 \Box_2^2 - 4 \Box_1 \Box_3 + 4 \Box_3^2\right)\]
\[+ \frac{1}{D}, \quad (6.26)\]

\[
\Gamma_{15}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3D^4} \left(2 \Box_1^7 - 18 \Box_1^5 \Box_2^2\right)\]
\[+ 32 \Box_1^4 \Box_2^3 - 18 \Box_1^3 \Box_2^4 + 2 \Box_1 \Box_2^6 + 72 \Box_1^5 \Box_2 \Box_3 - 72 \Box_1^4 \Box_2^2 \Box_3\]
\[- 72 \Box_1^3 \Box_2^3 \Box_3 + 72 \Box_1^2 \Box_2^4 \Box_3 - 18 \Box_1^5 \Box_3^2 - 72 \Box_1^4 \Box_2 \Box_3^2\]
\[+ 252 \Box_1^4 \Box_2 \Box_3^2 - 72 \Box_1^2 \Box_2^3 \Box_3^2 - 180 \Box_1 \Box_2 \Box_3^4 + 32 \Box_1^4 \Box_3^3\]
\[- 72 \Box_1^3 \Box_2 \Box_3^3 - 72 \Box_1^2 \Box_2^2 \Box_3^3 + 32 \Box_1 \Box_2^3 \Box_3^3 - 18 \Box_1^3 \Box_3^4\]
\[+ 72 \Box_1^2 \Box_2 \Box_3^4 - 18 \Box_1 \Box_2^2 \Box_3^4 + 2 \Box_1 \Box_3^6\]
\[+ \frac{\ln(\Box_1/\Box_2)}{9D^4} \left(18 \Box_1^6 + 10 \Box_1^5 \Box_2 - 120 \Box_1^4 \Box_2^2\right)\]
\[+ 120 \Box_1^{3} \Box_2^{3} - 10 \Box_1^{2} \Box_2^{4} - 18 \Box_1 \Box_2^{5} - 22 \Box_1^{5} \Box_3\]
\[+ 264 \Box_1^{3} \Box_2 \Box_3 - 264 \Box_1^{2} \Box_2^{3} \Box_3 + 22 \Box_1 \Box_2^{4} \Box_3 - 42 \Box_1^{4} \Box_3^{2}\]
\[- 306 \Box_1^{3} \Box_2^{2} \Box_3^{2} + 306 \Box_1^{2} \Box_2^{2} \Box_3^{3} + 42 \Box_1 \Box_2^{3} \Box_3^{3} + 78 \Box_1^{3} \Box_3^{3}\]
\[- 78 \Box_1 \Box_2^{2} \Box_3^{3} - 32 \Box_1^{2} \Box_3^{4} + 32 \Box_1 \Box_2 \Box_3^{4}\]
\[+ \frac{\ln(\Box_1/\Box_3)}{9D^4} \left(18 \Box_1^6 - 22 \Box_1^5 \Box_2 + 42 \Box_1^4 \Box_2^2\right)\]
\[+ 78 \Box_1^{3} \Box_2^{3} - 32 \Box_1^{2} \Box_2^{4} + 10 \Box_1^{5} \Box_3 + 264 \Box_1^{4} \Box_2 \Box_3\]
\[- 306 \Box_1^{3} \Box_2 \Box_3^{2} + 32 \Box_1 \Box_2^{4} \Box_3 - 120 \Box_1^{4} \Box_3^{2} + 306 \Box_1^{3} \Box_2^{2} \Box_3^{2}\]
\[- 78 \Box_1 \Box_2^{3} \Box_3^{2} + 120 \Box_1^{3} \Box_3^{3} - 264 \Box_1^{2} \Box_2 \Box_3^{3} + 42 \Box_1 \Box_2^{2} \Box_3^{3}\]
\[- 10 \Box_1^{2} \Box_3^{4} + 22 \Box_1 \Box_2 \Box_3^{4} - 18 \Box_1^{3} \Box_3^{5}\]
\[+ \frac{\ln(\Box_2/\Box_3)}{9D^4} \left(- 32 \Box_1^{5} \Box_2 + 78 \Box_1^{4} \Box_2^{2} - 42 \Box_1^{3} \Box_2^{3}\right)\]
\[- 22 \Box_1^{2} \Box_2^{4} + 18 \Box_1 \Box_2^{5} + 32 \Box_1^{5} \Box_3 - 306 \Box_1^{3} \Box_2^{2} \Box_3\]
\[+ 264 \Box_1^{3} \Box_2^{3} \Box_3 + 10 \Box_1 \Box_2^{4} \Box_3 - 78 \Box_1^{4} \Box_3^{2} + 306 \Box_1^{3} \Box_2 \Box_3^{2}\]
\[- 120 \Box_1 \Box_2^{3} \Box_3^{2} + 42 \Box_1^{3} \Box_3^{3} - 264 \Box_1^{2} \Box_2 \Box_3^{3} + 120 \Box_1 \Box_2^{2} \Box_3^{3}\]
\[+ 22 \Box_1^{2} \Box_3^{4} - 10 \Box_1 \Box_2 \Box_3^{4} - 18 \Box_1^{3} \Box_3^{5}\]
\[+ \frac{1}{3D^3} \left(- 13 \Box_1^{4} + 13 \Box_1^{3} \Box_2 + 13 \Box_1^{2} \Box_2^{2}\right)\]
\[- 13 \Box_1^{2} \Box_3^{2} + 13 \Box_1 \Box_2^{3} - 62 \Box_1^{2} \Box_2 \Box_3 + 13 \Box_1 \Box_2 \Box_3^{2}\]
\[+ 13 \Box_1^{2} \Box_3^{2} + 13 \Box_1 \Box_2 \Box_3^{2} - 13 \Box_1 \Box_3^{3}\), \quad (6.27)\]
\[ \Gamma_{16}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{D^3}(16\Box_1^3\Box_2\Box_3 \\
- 16\Box_1^2\Box_2^2\Box_3 + 8\Box_1^2\Box_2\Box_3^2 - 12\Box_1\Box_2\Box_3^3) \\
+ \frac{\ln(\Box_1/\Box_2)}{9D^3\Box_2} \left( - 2\Box_1^5 + 12\Box_1^4\Box_2 - 18\Box_1^3\Box_2^2 \\
+ 8\Box_1^4\Box_2^3 + 168\Box_1^2\Box_2^2\Box_3 - 12\Box_1^3\Box_3^2 - 36\Box_1^2\Box_2\Box_3^2 \\
+ 8\Box_1^2\Box_3^3 + 24\Box_1\Box_2\Box_3^3 - 2\Box_1\Box_3^4 \right) \\
+ \frac{\ln(\Box_1/\Box_2)}{9D^3\Box_2} \left( - 4\Box_1^6 + 24\Box_1^5\Box_2 - 54\Box_1^4\Box_2^2 \\
+ 60\Box_1^3\Box_2^3 - 36\Box_1^2\Box_2^4 + 12\Box_1\Box_2^5 - 2\Box_2^6 \\
+ 16\Box_1^5\Box_3 + 72\Box_1^4\Box_2\Box_3 - 96\Box_1^2\Box_2^3\Box_3 + 8\Box_2^5\Box_3 \\
- 24\Box_1^4\Box_2^2 - 72\Box_1^3\Box_2^2\Box_3^2 + 216\Box_1^2\Box_2^2\Box_3^2 - 36\Box_1\Box_2^3\Box_3^2 \\
- 12\Box_2^4\Box_3^3 + 16\Box_1^3\Box_3^3 + 48\Box_1^2\Box_2\Box_3^3 + 24\Box_1\Box_2^2\Box_3^3 \\
+ 8\Box_2^3\Box_3^3 - 4\Box_2^2\Box_3^4 - 2\Box_2^2\Box_3^3 \right) \\
+ \frac{1}{6D^2\Box_1\Box_2}(6\Box_1^4 - 32\Box_1^3\Box_2 + 26\Box_1^2\Box_2^2 - 20\Box_1^3\Box_3 - 4\Box_1\Box_2\Box_3 \\
+ 24\Box_1^2\Box_3^2 + 24\Box_1\Box_2\Box_3^2 - 12\Box_1\Box_3^3 + \Box_3^4 \right), \quad (6.28) \]

\[ \Gamma_{17}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{D^2}(2\Box_1^3 \\
- 4\Box_1^2\Box_2 + 2\Box_1\Box_2^2 - 4\Box_1^2\Box_3 + 8\Box_1\Box_2\Box_3 + 2\Box_1\Box_3^2) \\
+ \frac{\ln(\Box_1/\Box_2)}{3D^2\Box_1}(9\Box_1^3 - 5\Box_1^2\Box_2 - 5\Box_1\Box_2^2 + \Box_2^3 \\
- 13\Box_1^2\Box_3 - 3\Box_2^2\Box_3 + 5\Box_1\Box_3^2 + 3\Box_2\Box_3^2 - \Box_3^3) \\
+ \frac{\ln(\Box_1/\Box_3)}{3D^2\Box_1}(9\Box_1^3 - 13\Box_1^2\Box_2 + 5\Box_1\Box_2^2 - \Box_2^3 - 5\Box_1\Box_3^2 \\
+ 3\Box_2^2\Box_3 - 5\Box_1\Box_3^2 - 3\Box_2\Box_3^2 + \Box_3^3) \\
+ \frac{\ln(\Box_2/\Box_3)}{3D^2\Box_1}( - 8\Box_1^2\Box_2 + 10\Box_1\Box_2^2 - 2\Box_2^3 + 8\Box_1^2\Box_3 \\
+ 6\Box_2^2\Box_3 - 10\Box_1\Box_3^2 - 6\Box_2\Box_3^2 + 2\Box_3^3) \\
+ \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)\Box_1} \left(-2\right) \frac{1}{D} \right), \quad (6.29) \]

\[ \Gamma_{18}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{D^4}(2\Box_1^7 \]

54
\[
- 16\Box_1^6\Box_2 + 20\Box_1^5\Box_2^2 - 20\Box_1^3\Box_2^4 + 16\Box_1^2\Box_2^5 \\
- 4\Box_1\Box_2^6 + 56\Box_1^5\Box_2\Box_3 - 192\Box_1^4\Box_2^2\Box_3 \\
+ 64\Box_1^3\Box_2^3\Box_3 + 80\Box_1^2\Box_2^4\Box_3 - 48\Box_1\Box_2^5\Box_3 + 180\Box_1^3\Box_2^2\Box_3^2 \\
- 176\Box_1^2\Box_2^3\Box_3^2 - 12\Box_1\Box_2^4\Box_3^2 + 64\Box_1\Box_2^3\Box_3^3 \\
+ \frac{\ln(\Box_1/\Box_2)}{D^4\Box_1} \left( 78\Box_1^7 - 186\Box_1^6\Box_2 - 42\Box_1^5\Box_2^2 \right) \\
+ 470\Box_1^4\Box_2^3 - 470\Box_1^3\Box_2^4 + 162\Box_1^2\Box_2^5 - 14\Box_1\Box_2^6 \\
- 2\Box_2^7 - 294\Box_1^6\Box_3 + 1056\Box_1^5\Box_2\Box_3 - 582\Box_1^4\Box_2^2\Box_3 \\
- 752\Box_1^3\Box_2^3\Box_3 + 478\Box_1^2\Box_2^4\Box_3 + 112\Box_1\Box_2^5\Box_3 - 18\Box_2^6\Box_3 \\
+ 402\Box_1^5\Box_2^2\Box_3^2 - 1554\Box_1^4\Box_2\Box_3^2 + 1284\Box_1^3\Box_2^2\Box_3^2 - 12\Box_1^2\Box_2^3\Box_3^2 \\
- 182\Box_1\Box_2^4\Box_3^2 + 62\Box_2^5\Box_3^2 - 230\Box_1^4\Box_3^3 + 608\Box_1^3\Box_2\Box_3^3 \\
- 780\Box_1^2\Box_2^3\Box_3^3 - 110\Box_2^4\Box_3^3 + 50\Box_1^3\Box_3^4 + 170\Box_1^2\Box_2\Box_3^4 \\
+ 182\Box_1\Box_2^2\Box_3^4 + 110\Box_2^3\Box_3^4 - 18\Box_1^2\Box_3^5 - 112\Box_1\Box_2\Box_3^5 \\
- 62\Box_2^2\Box_3^5 + 14\Box_1\Box_3^6 + 18\Box_2\Box_3^6 - 2\Box_3^7 \\
+ \frac{\ln(\Box_2/\Box_3)}{D^4\Box_3} \left( - 108\Box_1^6\Box_2 + 444\Box_1^5\Box_2^2 - 706\Box_1^4\Box_2^3 \\
+ 520\Box_1^3\Box_2^4 - 180\Box_1^2\Box_2^5 + 28\Box_1\Box_2^6 - 4\Box_2^7 \\
- 972\Box_1^4\Box_2^2\Box_3 + 1360\Box_1^3\Box_2\Box_3^2 - 308\Box_1^2\Box_2^4\Box_3 \\
- 224\Box_1\Box_2^5\Box_3 + 36\Box_2^6\Box_3 - 768\Box_1^2\Box_2^3\Box_3^2 + 364\Box_1\Box_2^4\Box_3^2 \\
- 124\Box_2^5\Box_3^2 - 220\Box_2^4\Box_3^3 \right) \\
+ \frac{1}{3D^4\Box_1} \left( - 20\Box_1^5 + 106\Box_1^4\Box_2 - 76\Box_1^3\Box_2^2 - 8\Box_1^2\Box_2^3 \\
+ 20\Box_1\Box_2^4 - 2\Box_2^5 - 12\Box_1^3\Box_2\Box_3 + 128\Box_1^2\Box_2^2\Box_3 \\
+ 16\Box_1\Box_2^3\Box_3 + 6\Box_2^4\Box_3 - 36\Box_1\Box_2^2\Box_3^2 - 4\Box_2^3\Box_3^2 \right), \\
(6.30)
\]

\[\Gamma_1 (-\Box_1, -\Box_2, -\Box_3) = \Gamma (-\Box_1, -\Box_2, -\Box_3) \frac{1}{D^4} (-8\Box_1^5\Box_2\Box_3 \\
+ 24\Box_1^4\Box_2^2\Box_3 + 24\Box_1^3\Box_2^3\Box_3 - 56\Box_1^2\Box_2^4\Box_3 + 24\Box_1\Box_2^5\Box_3 \\
- 72\Box_1^3\Box_2^2\Box_3^2 + 96\Box_1^2\Box_2^3\Box_3^2 + 24\Box_1\Box_2^4\Box_3^2 - 48\Box_1\Box_2^3\Box_3^3 \\
+ \frac{\ln(\Box_1/\Box_2)}{D^4\Box_1} \left( - 12\Box_1^6\Box_2 + 48\Box_1^5\Box_2^2 - 72\Box_1^4\Box_2^3 \\
+ 48\Box_1^3\Box_2^4 - 12\Box_1^2\Box_2^5 - 6\Box_3^6\Box_3 - 228\Box_1^5\Box_2\Box_3 \\
+ 78\Box_1^4\Box_2^2\Box_3 + 528\Box_1^3\Box_2^3\Box_3 - 346\Box_1^2\Box_2^4\Box_3 - 28\Box_1\Box_2^5\Box_3 \\
+ 2\Box_2^6\Box_3 + 24\Box_1^5\Box_3^2 + 402\Box_1^4\Box_2\Box_3^2 - 912\Box_1^3\Box_2^2\Box_3^2 \\
- 72\Box_1^2\Box_2^3\Box_3^2 + 56\Box_1\Box_2^4\Box_3^2 - 10\Box_2^5\Box_3^2 - 36\Box_1^4\Box_3^3 \\
- 48\Box_1^3\Box_2\Box_3^3 + 576\Box_1^2\Box_2^2\Box_3^3 + 20\Box_2^4\Box_3^3 + 24\Box_1^3\Box_3^4 \\
- 140\Box_1^2\Box_2^3\Box_3^4 - 56\Box_1\Box_2^2\Box_3^4 - 20\Box_2^3\Box_3^4 - 6\Box_1^2\Box_3^5 \\
+ 28\Box_1\Box_2\Box_3^5 + 10\Box_2^3\Box_3^5 - 2\Box_2^2\Box_3^6 \right) \]
\[
\Gamma_{20}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3D^4} \left( -2\Box_1^7 
\right.
\]
\[
+ 14\Box_1^6\Box_2 - 12\Box_1^5\Box_2^2 - 10\Box_1^4\Box_2^3 + 20\Box_1^3\Box_2^4 
\]
\[
- 6\Box_1^2\Box_2^5 - 4\Box_1\Box_2^6 - 60\Box_1^5\Box_2\Box_3 
\]
\[
+ 198\Box_1^4\Box_2^2\Box_3 - 16\Box_1^3\Box_2^3\Box_3 - 150\Box_1^2\Box_2^4\Box_3 + 72\Box_1\Box_2^5\Box_3 
\]
\[
+ 2\Box_2^6\Box_3 - 216\Box_1^3\Box_2^2\Box_3^2 + 228\Box_1^2\Box_2^3\Box_3^2 + 36\Box_1\Box_2^4\Box_3^2 
\]
\[
- 18\Box_1^5\Box_3^2 - 104\Box_1^3\Box_2^3\Box_3^3 + 14\Box_1^2\Box_3^4 \right)
\]
\[
\frac{\ln(\Box_2/\Box_3)}{9D^4\Box_1} \left( 40\Box_1^6\Box_2 - 152\Box_1^5\Box_2^2 + 210\Box_1^4\Box_2^3 
\right.
\]
\[
- 120\Box_1^3\Box_2^4 + 20\Box_1^2\Box_2^5 + 2\Box_2^7 + 402\Box_1\Box_2^2\Box_3 
\]
\[
- 572\Box_1^3\Box_2^3\Box_3 + 124\Box_1^2\Box_2^4\Box_3 + 100\Box_1\Box_2^5\Box_3 - 14\Box_2^6\Box_3 
\]
\[
+ 408\Box_1^2\Box_2^3\Box_3^2 - 200\Box_1\Box_2^4\Box_3^2 + 42\Box_2^5\Box_3^2 - 70\Box_2^4\Box_3^3 \right)
\]
\[
+ \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)(6\Box_1)} \left( -1 \right)
\]
\[
+ \frac{1}{6D^3} \left( 15\Box_1^4 - 68\Box_1^3\Box_2 + 24\Box_1^2\Box_2^2 + 36\Box_1\Box_2^3 - 22\Box_2^4 
\right.
\]
\[
+ 96\Box_1\Box_2^2\Box_3 - 108\Box_1\Box_2^3\Box_3 - 16\Box_2^3\Box_3^2 + 38\Box_2^2\Box_3^3 \right),
\]

(6.32)
\[ \Gamma_2(\square_1, -\square_2, -\square_3) = \Gamma(-\square_1, -\square_2, -\square_3) \frac{1}{D^4} \left( -8\square_1^6\square_3 \right) \\
+ 8\square_1^5\square_2\square_3 + 48\square_1^4\square_2^2\square_3 - 112\square_1^3\square_2^3\square_3 + 88\square_1^2\square_2^4\square_3 \\
- 24\square_1^5\square_3 + 24\square_1^4\square_2\square_3 - 144\square_1^3\square_2^2\square_3 + 144\square_1^2\square_2^3\square_3 \\
+ 48\square_1^3\square_2^3 - 72\square_1^2\square_2^4\square_3 - 163\square_1^3\square_3^2 + 192\square_1^2\square_2\square_3^3 \\
- 336\square_1^2\square_2^3\square_3^3 - 128\square_1\square_2^3\square_3^3 - 16\square_1^3\square_3^4 + 16\square_1^2\square_2\square_3^4 \\
+ 48\square_1^2\square_2\square_3^3 + 24\square_1^2\square_3^5 - 72\square_1\square_2\square_3^5 - 8\square_1\square_3^6 \right) \\
+ \frac{\ln(\square_1/\square_2)}{9D^4}\left( -8\square_1^7 + 40\square_1^6\square_2 - 80\square_1^5\square_2^2 \\
+ 80\square_1^4\square_2^3 - 40\square_1^3\square_2^4 + 8\square_1^2\square_2^5 - 116\square_1^6\square_3 \\
- 80\square_1^5\square_2\square_3 + 900\square_1^4\square_2^2\square_3 - 1056\square_1^3\square_2^3\square_3 + 308\square_1^2\square_2^4\square_3 \\
+ 48\square_1^3\square_3 - 4\square_1^2\square_3^2 + 340\square_1^5\square_3^3 - 1044\square_1^4\square_2\square_3^2 \\
+ 96\square_1^3\square_2^2\square_3^2 + 672\square_1^2\square_2^3\square_3^2 - 84\square_1\square_2^4\square_3^2 + 20\square_2^5\square_3^2 \\
- 240\square_1^4\square_3^3 + 1248\square_1^3\square_2\square_3^3 - 936\square_1^2\square_2^2\square_3^3 - 32\square_1\square_2^3\square_3^3 \\
- 40\square_2^4\square_3^3 - 40\square_1^3\square_3^4 - 120\square_1^2\square_2\square_3^4 + 120\square_1\square_2^2\square_3^4 \\
+ 40\square_2^3\square_3^4 + 68\square_1^2\square_3^5 - 48\square_1\square_2\square_3^5 - 20\square_2^2\square_3^5 \\
- 4\square_1^3\square_3^6 + 4\square_2^2\square_3^6 \right) \\
+ \frac{\ln(\square_1/\square_3)}{9D^4}\left( -4\square_1^7 + 20\square_1^6\square_2 - 40\square_1^5\square_2^2 \\
+ 40\square_1^4\square_2^3 - 20\square_1^3\square_2^4 + 4\square_1^2\square_2^5 - 160\square_1^6\square_3 \\
+ 320\square_1^5\square_2\square_3 + 36\square_1^4\square_2^2\square_3 - 432\square_1^3\square_2^3\square_3 + 280\square_1^2\square_2^4\square_3 \\
- 48\square_1^3\square_3 - 4\square_1^2\square_3^2 + 284\square_1^5\square_3^3 - 1548\square_1^4\square_2\square_3^2 \\
+ 1824\square_1^3\square_2^2\square_3^2 - 624\square_1^2\square_2^3\square_3^2 + 84\square_1\square_2^4\square_3^2 - 20\square_2^5\square_3^2 \\
+ 144\square_1^4\square_3^3 + 336\square_1^3\square_2\square_3^3 - 696\square_1^2\square_2^2\square_3^3 + 32\square_1\square_2^3\square_3^3 \\
+ 40\square_2^4\square_3^3 - 476\square_1^3\square_3^4 + 828\square_1^2\square_2\square_3^4 - 120\square_1\square_2^2\square_3^4 \\
- 40\square_2^3\square_3^4 + 208\square_1^2\square_3^5 + 48\square_1\square_2\square_3^5 + 20\square_2^2\square_3^5 \\
+ 4\square_1^3\square_3^6 - 4\square_2^2\square_3^6 \right) \\
+ \frac{\ln(\square_2/\square_3)}{9D^4}\left( 4\square_1^7 - 20\square_1^6\square_2 + 40\square_1^5\square_2^2 - 40\square_1^4\square_2^3 \\
+ 20\square_1^3\square_2^4 - 4\square_1^2\square_2^5 - 44\square_1^6\square_3 + 400\square_1^5\square_2\square_3 \\
- 864\square_1^4\square_2^2\square_3 + 624\square_1^3\square_2^3\square_3 - 28\square_1^2\square_2^4\square_3 - 96\square_1\square_2^5\square_3 \\
+ 8\square_2^6\square_3 - 56\square_1^5\square_3^2 - 504\square_1^4\square_2\square_3^2 + 1728\square_1^3\square_2^2\square_3^2 \\
- 1296\square_1^2\square_2^3\square_3^2 + 168\square_1\square_2^4\square_3^2 - 40\square_2^5\square_3^2 + 384\square_1^4\square_3^3 \\
- 912\square_1^3\square_2^3 + 240\square_1^2\square_2^4\square_3^2 + 64\square_1\square_2^5\square_3^2 + 80\square_2^2\square_3^3 \\
- 436\square_1^3\square_3^4 + 948\square_1^2\square_2\square_3^4 - 240\square_1\square_2^2\square_3^4 - 80\square_2^3\square_3^4 \\
+ 140\square_1^2\square_3^5 + 96\square_1\square_2\square_3^5 + 40\square_2^2\square_3^5 + 8\square_1\square_3^6 - 8\square_2\square_3^6 \right) \\
\]
\[
\Gamma_{22}(-\square_1, -\square_2, -\square_3) = \Gamma(-\square_1, -\square_2, -\square_3) \frac{1}{18D^6} (-\square_1^{11} \\
- 4\square_1^{10}\square_2 + 30\square_1^9\square_2^2 - 156\square_1^7\square_2^4 + 264\square_1^6\square_2^5 \\
- 156\square_1^5\square_2^6 + 30\square_1^3\square_2^8 - 4\square_1^2\square_2^9 - 2\square_1\square_2^{10} \\
- 184\square_1^9\square_2^3 + 352\square_1^8\square_2^2\square_3 + 1672\square_1^7\square_2^3\square_3 \\
- 2720\square_1^6\square_2^4\square_3 + 304\square_1^5\square_2^5\square_3 + 1168\square_1^4\square_2^6\square_3 \\
- 152\square_1^3\square_2^7\square_3 - 332\square_1^2\square_2^8\square_3 + 80\square_1\square_2^9\square_3 \\
- 1884\square_1^7\square_2^2\square_3^2 + 1320\square_1^6\square_2^3\square_3^2 + 7036\square_1^5\square_2^4\square_3^2 \\
- 3456\square_1^4\square_2^5\square_3^2 - 3240\square_1^3\square_2^6\square_3^2 + 2008\square_1^2\square_2^7\square_3^2 \\
- 282\square_1\square_2^8\square_3^2 - 4640\square_1^5\square_2^3\square_3^3 + 1136\square_1^4\square_2^4\square_3^3 \\
+ 7512\square_1^3\square_2^5\square_3^3 - 3992\square_1^2\square_2^6\square_3^3 + 192\square_1\square_2^7\square_3^3 \\
- 4150\square_1^9\square_2^2\square_3^4 + 2320\square_1^8\square_2^3\square_3^4 + 540\square_1^7\square_2^4\square_3^4 \\
- 528\square_1^5\square_2^5\square_3^5 ) \\
+ \frac{\ln(\square_1/\square_2)}{270D^6\square_1\square_2^2\square_3} ( - 12\square_1^{12}\square_2 + 144\square_1^6\square_2^2 - 756\square_1^{10}\square_2^3 \\
+ 2316\square_1^9\square_2^4 - 4632\square_1^8\square_2^5 + 6384\square_1^7\square_2^6 \\
- 6216\square_1^6\square_2^7 + 4296\square_1^5\square_2^8 - 2076\square_1^4\square_2^9 \\
+ 672\square_1^3\square_2^{10} - 132\square_1^2\square_2^{11} + 12\square_1\square_2^{12} \\
- 6\square_1^{12}\square_3 + 117\square_1^{11}\square_2\square_3 - 2129\square_1^{10}\square_2^2\square_3 \\
+ 7757\square_1^9\square_2^3\square_3 - 10791\square_1^8\square_2^4\square_3 + 4142\square_1^7\square_2^5\square_3 \\
+ 5386\square_1^6\square_2^6\square_3 - 8706\square_1^5\square_2^7\square_3 + 7208\square_1^4\square_2^8\square_3 \\
- 4259\square_1^3\square_2^9\square_3 + 1479\square_1^2\square_2^{10}\square_3 - 203\square_1\square_2^{11}\square_3 \\
+ 5\square_2^{12}\square_3 + 72\square_1^{11}\square_3^2 - 156\square_1^{10}\square_2\square_3^2 \\
- 2598\square_1^9\square_2^3\square_3^2 - 12647\square_1^8\square_2^4\square_3^2 + 47868\square_1^7\square_2^5\square_3^2 \\
- 18226\square_1^6\square_2^6\square_3^2 - 31616\square_1^5\square_2^7\square_3^2 + 10014\square_1^4\square_2^8\square_3^2 \\
+ 15452\square_1^3\square_2^9\square_3^2 - 7919\square_1^2\square_2^{10}\square_3^2 + 1222\square_1\square_2^{11}\square_3^2 \\
- 55\square_2^{11}\square_3^2 - 378\square_1^{10}\square_3^3 + 6091\square_1^9\square_2\square_3^3 \\
+ 1199\square_1^8\square_2^2\square_3^3 - 49968\square_1^7\square_2^3\square_3^3 - 22068\square_1^6\square_2^4\square_3^3 \\
+ 84806\square_1^5\square_2^5\square_3^3 + 1758\square_1^4\square_2^6\square_3^3 - 36528\square_1^3\square_2^7\square_3^3 )
\]
\[
\Gamma_{23}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3D^5} \left( -8\Box_1^7\Box_2\Box_3 
\right. \\
+ 72\Box_1^5\Box_2^3\Box_3 - 64\Box_1^4\Box_2^4\Box_3 + 28\Box_1^6\Box_2\Box_3^2 - 228\Box_1^5\Box_2^2\Box_3^2 \\
+ 200\Box_1^4\Box_2^3\Box_3^2 - 32\Box_1^5\Box_2\Box_3^3 + 280\Box_1^4\Box_2^2\Box_3^3 \\
- 296\Box_1^3\Box_2^3\Box_3^3 + 8\Box_1^4\Box_2^4\Box_3 + 120\Box_1^3\Box_2^2\Box_3^4 \\
+ 8\Box_1^3\Box_2\Box_3^5 - 84\Box_1^2\Box_2^2\Box_3^5 - 4\Box_1^2\Box_2\Box_3^6 \\
\left. \right) \\
+ \frac{\ln(\Box_2/\Box_3)}{135D^5\Box_2\Box_3} \left( 6\Box_1^9\Box_2 - 54\Box_1^8\Box_2^2 + 216\Box_1^7\Box_2^3 
\right. \\
- 504\Box_1^6\Box_2^4 + 756\Box_1^5\Box_2^5 + 2\Box_1^9\Box_3 - 80\Box_1^8\Box_2\Box_3 \\
+ 414\Box_1^7\Box_2^3\Box_3 - 896\Box_1^6\Box_2^2\Box_3 + 860\Box_1^5\Box_2^4\Box_3 - 22\Box_1^8\Box_3^2 \\
+ 424\Box_1^7\Box_2^3\Box_3^2 - 1978\Box_1^6\Box_2^2\Box_3^2 - 42\Box_1^5\Box_2^3\Box_3^2 \\
+ 7246\Box_1^4\Box_2^4\Box_3^2 + 101\Box_1^7\Box_3^3 - 1110\Box_1^6\Box_2^3\Box_3 \\
+ 3130\Box_1^5\Box_2^3\Box_3^3 - 8758\Box_1^4\Box_2^2\Box_3^3 - 259\Box_1^6\Box_3^4 \\
+ 1555\Box_1^5\Box_2^3\Box_3^4 - 873\Box_1^4\Box_2^2\Box_3^4 + 10225\Box_1^3\Box_2^3\Box_3^4 \\
+ 413\Box_1^5\Box_3^5 - 1124\Box_1^4\Box_2^2\Box_3^5 + 1435\Box_1^3\Box_2^3\Box_3^5 \\
- 427\Box_1^4\Box_2^3\Box_3^6 + 285\Box_1^3\Box_2^2\Box_3^6 + 760\Box_1^2\Box_2^2\Box_3^6 \\
+ 287\Box_1^3\Box_2^3\Box_3^7 + 106\Box_1^2\Box_2^2\Box_3^7 - 121\Box_1^2\Box_2\Box_3^8 \\
- 65\Box_1\Box_2^3\Box_3^8 + 29\Box_1\Box_3^9 - 3\Box_3^{10} \\
\left. \right) \\
+ \frac{\ln(\Box_1/\Box_3)}{135D^5\Box_1\Box_2\Box_3} \left( 3\Box_1^{10}\Box_2 - 27\Box_1^9\Box_2^2 + 108\Box_1^8\Box_2^3 - 252\Box_1^7\Box_2^4 \\
+ 378\Box_1^6\Box_2^5 - 378\Box_1^5\Box_2^6 + 252\Box_1^4\Box_2^7 - 108\Box_1^3\Box_2^8 
\right) \\
\right) 
\]
\[ + 27\, \Box_1^2 \Box_2^9 - 3\, \Box_1 \Box_2^{10} + 4\, \Box_1^{10} \Box_3 - 70\, \Box_1^9 \Box_2 \Box_3 + 342\, \Box_1^8 \Box_2^2 \Box_3 - 808\, \Box_1^7 \Box_2^3 \Box_3 + 1060\, \Box_1^6 \Box_2^4 \Box_3 \\
- 756\, \Box_1^5 \Box_2^5 \Box_3 + 200\, \Box_1^4 \Box_2^6 \Box_3 + 88\, \Box_1^3 \Box_2^7 \Box_3 - 72\, \Box_1^2 \Box_2^8 \Box_3 + 10\, \Box_1 \Box_2^9 \Box_3 + 2\, \Box_2^{10} \Box_3 - 44\, \Box_1^9 \Box_3^2 + 443\, \Box_1^8 \Box_2 \Box_3^2 \\
- 2189\, \Box_1^7 \Box_2^2 \Box_3^2 + 2523\, \Box_1^6 \Box_2^3 \Box_3^2 + 2081\, \Box_1^5 \Box_2^4 \Box_3^2 \\
- 5165\, \Box_1^4 \Box_2^5 \Box_3^2 + 2565\, \Box_1^3 \Box_2^6 \Box_3^2 - 211\, \Box_1^2 \Box_2^7 \Box_3^2 \\
+ 19\, \Box_1 \Box_2^8 \Box_3^2 - 22\, \Box_2^9 \Box_3^2 + 202\, \Box_1^8 \Box_3^3 - 1140\, \Box_1^7 \Box_2 \Box_3^3 \\
+ 2585\, \Box_1^6 \Box_2^2 \Box_3^3 + 14265\, \Box_1^5 \Box_2^4 \Box_3^3 \\
- 3340\, \Box_1^3 \Box_2^5 \Box_3^3 - 545\, \Box_1^2 \Box_2^6 \Box_3^3 - 30\, \Box_1 \Box_2^7 \Box_3^3 + 101\, \Box_2^8 \Box_3^3 \\
- 518\, \Box_1^7 \Box_3^4 + 1220\, \Box_1^6 \Box_2 \Box_3^4 + 1917\, \Box_1^5 \Box_2^2 \Box_3^4 + 1625\, \Box_1^4 \Box_2^3 \Box_3^4 - 8600\, \Box_1^3 \Box_2^4 \Box_3^4 + 2790\, \Box_1^2 \Box_2^5 \Box_3^4 \\
- 335\, \Box_1 \Box_2^6 \Box_3^4 + 259\, \Box_2^7 \Box_3^4 + 826\, \Box_1^6 \Box_3^5 + 20\, \Box_1^5 \Box_2 \Box_3^5 \\
- 3551\, \Box_1^4 \Box_2^2 \Box_3^5 + 8664\, \Box_1^3 \Box_2^3 \Box_3^5 - 2116\, \Box_1^2 \Box_2^4 \Box_3^5 \\
+ 1144\, \Box_1 \Box_2^5 \Box_3^5 + 413\, \Box_2^6 \Box_3^5 - 854\, \Box_1^5 \Box_3^6 \\
- 1320\, \Box_1^4 \Box_2 \Box_3^6 - 319\, \Box_1^3 \Box_2^2 \Box_3^6 + 1079\, \Box_1^2 \Box_2^3 \Box_3^6 \\
- 1605\, \Box_1 \Box_2^4 \Box_3^6 - 427\, \Box_2^5 \Box_3^6 + 574\, \Box_1^4 \Box_3^7 \\
+ 1292\, \Box_1^3 \Box_2 \Box_3^7 + 1683\, \Box_1^2 \Box_2^2 \Box_3^7 + 1186\, \Box_1 \Box_2^3 \Box_3^7 \\
+ 287\, \Box_2^4 \Box_3^7 - 242\, \Box_1^3 \Box_3^8 - 535\, \Box_1^2 \Box_2 \Box_3^8 - 470\, \Box_1 \Box_2^2 \Box_3^8 \\
- 121\, \Box_2^3 \Box_3^8 + 58\, \Box_1^2 \Box_3^9 + 90\, \Box_1 \Box_2 \Box_3^9 + 29\, \Box_2 \Box_3^9 \\
- 6\, \Box_1^3 \Box_3^{10} - 3\, \Box_2 \Box_3^{10}) \\
+ \frac{\ln(\Box_1/\Box_2)}{(\Box_1 - \Box_2)} \left( \frac{-1}{30\Box_3} \right) \\
+ \frac{1}{270D^4\Box_1^2} \left( 19\, \Box_1^8 - 116\, \Box_1^7 \Box_2 + 460\, \Box_1^6 \Box_2^2 \\
- 1100\, \Box_1^5 \Box_2^3 + 737\, \Box_1^4 \Box_2^4 - 119\, \Box_1^3 \Box_2^5 \\
+ 2293\, \Box_1^3 \Box_2^5 \Box_3 + 2979\, \Box_1^2 \Box_2^6 \Box_3 - 3089\, \Box_1 \Box_2^7 \Box_3 + 319\, \Box_2^8 \Box_3^2 \\
+ 264\, \Box_1^5 \Box_2 \Box_3^2 - 4983\, \Box_1^4 \Box_2^2 \Box_3^2 \\
+ 6560\, \Box_1^3 \Box_2^3 \Box_3^2 - 479\, \Box_1^5 \Box_2^3 \Box_3^2 - 1055\, \Box_1^4 \Box_2 \Box_3^3 \\
- 1082\, \Box_1^3 \Box_2^2 \Box_3^3 + 445\, \Box_1^4 \Box_3^4 + 1060\, \Box_1^3 \Box_2 \Box_3^4 \\
+ 1503\, \Box_1^2 \Box_2^2 \Box_3^4 - 269\, \Box_1^3 \Box_3^5 - 489\, \Box_1 \Box_2 \Box_3^5 \\
+ 109\, \Box_2^2 \Box_3^6 + 68\, \Box_1 \Box_2 \Box_3^6 - 29\, \Box_1 \Box_3^7 + 2\, \Box_3^8 \right), \tag{6.35} \]

\[
\Gamma_{24}(\Box_1, \Box_2, \Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{D^4} \left( 2\, \Box_1^5 \Box_2 \Box_3 \\
- 12\, \Box_1^3 \Box_2^3 \Box_3 + 8\, \Box_1^2 \Box_2^4 \Box_3 + 8\, \Box_1^3 \Box_2 \Box_3^2 - 8\, \Box_1^2 \Box_2^3 \Box_3^2 \right) \\
+ \frac{\ln(\Box_1/\Box_2)}{45D^4\Box_1 \Box_2 \Box_3} \left( 2\, \Box_1^8 \Box_2 - 16\, \Box_1^7 \Box_2^2 + 52\, \Box_1^6 \Box_2^3 \\
- 90\, \Box_1^5 \Box_2^4 + 90\, \Box_1^4 \Box_2^5 - 52\, \Box_1^3 \Box_2^6 + 16\, \Box_1^2 \Box_2^7 \right) 
\]
\[-2\Box_1 \Box_2^8 + \Box_1^8 \Box_3 - 24\Box_1^7 \Box_2 \Box_3 + 120\Box_1^6 \Box_2^2 \Box_3 \\
- 184\Box_1^5 \Box_2^3 \Box_3 + 36\Box_1^4 \Box_2^4 \Box_3 + 136\Box_1^3 \Box_2^5 \Box_3 - 108\Box_1^2 \Box_2^6 \Box_3 \\
+ 24\Box_1 \Box_2 \Box_3 - \Box_2^8 \Box_3 - 8\Box_1^7 \Box_3^2 + 102\Box_1^6 \Box_2^2 \Box_3^2 \\
+ 141\Box_1^5 \Box_2^2 \Box_3^2 + 89\Box_1^4 \Box_2^3 \Box_3^2 - 392\Box_1^3 \Box_2^4 \Box_3^2 + 146\Box_1^2 \Box_2^5 \Box_3^2 \\
- 85\Box_1^3 \Box_2^3 \Box_3^2 + 7\Box_2^7 \Box_3^2 + 26\Box_1^6 \Box_3^3 - 176\Box_1^5 \Box_2 \Box_3^3 \\
- 203\Box_1^4 \Box_2^2 \Box_3^3 + 456\Box_1^3 \Box_2^3 \Box_3^3 + 38\Box_1^2 \Box_2^4 \Box_3^3 + 136\Box_1 \Box_2^5 \Box_3^3 \\
- 21\Box_2^6 \Box_3^3 - 45\Box_1^5 \Box_3^4 + 123\Box_1^4 \Box_2 \Box_3^4 - 106\Box_1^3 \Box_2^2 \Box_3^4 \\
- 158\Box_1^2 \Box_2^3 \Box_3^4 - 105\Box_1 \Box_2^4 \Box_3^4 + 35\Box_2^5 \Box_3^4 + 45\Box_1^4 \Box_3^5 \\
- 16\Box_1^3 \Box_2 \Box_3^5 + 70\Box_1^2 \Box_2^2 \Box_3^5 + 32\Box_1 \Box_2^3 \Box_3^5 - 35\Box_2^4 \Box_3^5 \\
- 26\Box_2^3 \Box_3^6 - 12\Box_1^2 \Box_2 \Box_3^6 + \Box_1 \Box_2^2 \Box_3^6 + 21\Box_2^3 \Box_3^6 \\
+ 8\Box_1^2 \Box_3^7 - 7\Box_2^2 \Box_3^7 - \Box_1 \Box_3^8 + \Box_2 \Box_3^8 \right) \\
+ \ln(\Box_2 / \Box_3) \left( -\frac{1}{45D} \Box_1^3 \Box_3 + 45 \Box_1 \Box_2 \Box_3^2 - 45 \Box_1^4 \Box_2^2 + 45 \Box_1^5 \Box_2^3 \\
- 45 \Box_1^4 \Box_2 + 26 \Box_1^3 \Box_2^5 - 8 \Box_1^2 \Box_2^6 + \Box_1 \Box_2^7 \\
- 18 \Box_1 \Box_2 \Box_3 + 8 \Box_1 \Box_2 \Box_3^2 + 8 \Box_1 \Box_2 \Box_3^3 + 152 \Box_1 \Box_2 \Box_3^4 \\
+ 96 \Box_1 \Box_2 \Box_3^2 - 24 \Box_1 \Box_2 \Box_3^3 + 2 \Box_2 \Box_3^4 - 292 \Box_1 \Box_2 \Box_3^5 \\
+ 286 \Box_1 \Box_2 \Box_3^3 - 76 \Box_1 \Box_2 \Box_3^4 + 86 \Box_1 \Box_2 \Box_3^5 - 14 \Box_2 \Box_3^6 \\
- 196 \Box_1 \Box_2 \Box_3^5 - 104 \Box_1 \Box_2 \Box_3^6 + 42 \Box_2 \Box_3^7 - 70 \Box_2 \Box_3^8 \right) \right] \\
+ \frac{\ln(\Box_2 / \Box_3)}{45D} \left( \frac{1}{\Box_2 - \Box_3} \left( -\frac{1}{30} \right) \right) \\
+ \frac{1}{540D \Box_3} \left( \Box_1^6 + 6 \Box_1^5 \Box_2 - 24 \Box_1^4 \Box_2^2 - 4 \Box_1^3 \Box_2^3 \\
+ 66 \Box_1^4 \Box_2^4 - 66 \Box_1^3 \Box_2^5 + 20 \Box_2^6 - 90 \Box_1^4 \Box_2 \Box_3 \\
- 312 \Box_1^3 \Box_2 \Box_3^2 + 1092 \Box_1^2 \Box_2 \Box_3^3 - 558 \Box_1 \Box_2 \Box_3^4 - 48 \Box_2 \Box_3^5 \\
- 1230 \Box_1 \Box_2 \Box_3^6 + 624 \Box_1 \Box_2 \Box_3^7 + 12 \Box_2 \Box_3^8 + 16 \Box_2 \Box_3^9 \right) , \tag{6.36} \]
\[\Gamma_{26}(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{D^4}(8\Box_1^5\Box_2^2 - 32\Box_1^4\Box_2^2 + 24\Box_1^3\Box_2^3 + 48\Box_1^3\Box_2\Box_3^2 + 48\Box_1^3\Box_2^2\Box_3 - 72\Box_1^3\Box_2^2\Box_3^2 + 96\Box_1^2\Box_2^2\Box_3^2 - 32\Box_1^2\Box_2^2\Box_3^3 + 24\Box_1\Box_2\Box_3^4)\]
\[+ \frac{\ln(\Box_1/\Box_2)}{9D^4\Box_1}\left(-2\Box_1^6 + 22\Box_1^5\Box_2 + 118\Box_1^4\Box_2^2\right)\]
\[-450\Box_1^2\Box_3^2 + 10\Box_1^5\Box_3 - 32\Box_1^4\Box_2\Box_3 + 474\Box_1^3\Box_2^2\Box_3 - 20\Box_1^4\Box_3^2 - 36\Box_1^3\Box_2\Box_3^2 - 624\Box_1^2\Box_2^2\Box_3^2 + 20\Box_1^3\Box_3^3 + 80\Box_1^2\Box_2\Box_3^3 - 10\Box_1^2\Box_3^4 - 34\Box_1\Box_2\Box_3^4 + 2\Box_1\Box_3^5\right)\]
\[+ \frac{\ln(\Box_1/\Box_3)}{9D^4\Box_1}\left(-4\Box_1^7 + 44\Box_1^6\Box_2 - 22\Box_1^5\Box_2^2 + 174\Box_1^4\Box_2^3 + 276\Box_1^3\Box_2^4 - 140\Box_1^2\Box_2^5 + 22\Box_1\Box_2^6 - 2\Box_2^7 + 20\Box_1^6\Box_3 - 64\Box_1^5\Box_2\Box_3 + 726\Box_1^4\Box_2^2\Box_3^2 - 912\Box_1^3\Box_2^3\Box_3^2 + 252\Box_1^2\Box_2^4\Box_3^3 - 32\Box_1\Box_2^5\Box_3^3 + 10\Box_2^6\Box_3^3 - 40\Box_1^5\Box_3^2\right)\]
\[-72\,\Box_1^4\Box_2^2\Box_3^2 - 120\,\Box_1^3\Box_2^2\Box_3^1 + 504\,\Box_1^2\Box_2^3\Box_3^2 - 36\,\Box_1^4\Box_2^4\Box_3^2\
+ 20\,\Box_2^5\Box_3^2 + 40\,\Box_1^4\Box_3^3 + 160\,\Box_1^3\Box_2\Box_3^3 - 552\,\Box_1^2\Box_2^2\Box_3^3\
+ 80\,\Box_2^3\Box_3^4 + 20\,\Box_1^4\Box_3^4 - 20\,\Box_1^3\Box_3^5 - 68\,\Box_1^2\Box_3^6\Box_3^1\
- 34\,\Box_1^2\Box_2\Box_3^4 - 10\,\Box_2^3\Box_3^4 + 4\,\Box_1^2\Box_3^5 + 2\Box_2^2\Box_3^5\)\
+ \frac{1}{6D^4\Box_1\Box_2}\left(6\Box_1^5 - 70\Box_1^4\Box_2 + 64\Box_1^3\Box_2^2 - 26\Box_1^4\Box_3\right)\
+ 100\Box_1^3\Box_2\Box_3 - 146\Box_1^2\Box_2^2\Box_3 + 44\Box_1^3\Box_3^2 + 24\Box_1^2\Box_2\Box_3^2\
- 36\Box_1^2\Box_3^3 - 34\,\Box_1^2\Box_3^2 + 14\,\Box_1\Box_3^4 - \Box_3^5), \quad (6.38)\]

\[
\Gamma_27(-\Box_1, -\Box_2, -\Box_3) = \Gamma(-\Box_1, -\Box_2, -\Box_3) \frac{1}{3D^6}\left(-4\Box_1^9\Box_2\right)\
+ 80\Box_1^8\Box_2^2 + 32\Box_1^7\Box_3^2 - 136\Box_1^6\Box_2^4 + 100\Box_1^5\Box_2^5\
- 32\Box_1^6\Box_2\Box_3 - 200\Box_1^5\Box_2^3\Box_3 + 792\Box_1^4\Box_2^3\Box_3 - 560\Box_1^4\Box_2^4\Box_3\
+ 60\Box_1^7\Box_2\Box_3^2 - 352\Box_1^6\Box_2\Box_3^2 - 1292\Box_1^5\Box_2\Box_3^2\
+ 1584\Box_1^4\Box_2\Box_3^2 + 320\Box_1^5\Box_2^2\Box_3 + 960\Box_1^5\Box_2^2\Box_3^2\
- 1232\Box_1^4\Box_2\Box_3^3 + 4\Box_1^5\Box_2\Box_3^4 - 296\Box_1^4\Box_2^2\Box_3^4 + 556\Box_1^3\Box_2^3\Box_3^4\
- 192\Box_1^3\Box_2\Box_3^5 + 136\Box_1^3\Box_2^2\Box_3^5 + 116\Box_1^3\Box_2\Box_3^6 - 160\Box_1^2\Box_2^2\Box_3^6\
+ 64\Box_1^2\Box_2\Box_3^7 - 24\Box_1\Box_2\Box_3^8)\
+ \frac{\ln(\Box_1/\Box_2)}{135D^6\Box_2}\left(-\Box_1^{10} + 9\Box_1^9\Box_2 - 681\Box_1^8\Box_2^2\right)\
+ 1073\Box_1^7\Box_2^3 + 3954\Box_1^6\Box_2^4 - 12530\Box_1^5\Box_2^5\
+ 9\Box_1^9\Box_3 - 84\Box_1^8\Box_2\Box_3 - 2205\Box_1^7\Box_2^2\Box_3\
- 16728\Box_1^6\Box_2^3\Box_3 + 40362\Box_1^5\Box_2^4\Box_3 - 30\Box_1^8\Box_3^2\
+ 114\Box_1^7\Box_2\Box_3^2 + 5112\Box_1^6\Box_2^3\Box_3 - 6432\Box_1^5\Box_2^3\Box_3^2\
- 73320\Box_1^4\Box_2^4\Box_3^2 + 42\Box_1^7\Box_3^3 + 252\Box_1^6\Box_2\Box_3^3\
+ 704\Box_1^5\Box_2^3\Box_3 + 28524\Box_1^4\Box_2^3\Box_3^3 - 660\Box_1^5\Box_2\Box_3^4\
- 180\Box_1^4\Box_2^2\Box_3^4 - 2160\Box_1^3\Box_2^3\Box_3^4 - 84\Box_1^5\Box_2^3\
+ 276\Box_1^4\Box_2\Box_3^5 - 7320\Box_1^3\Box_2^2\Box_3^5 + 126\Box_1^4\Box_2^3\
+ 414\Box_1^3\Box_2\Box_3^3 - 4688\Box_1^2\Box_2^2\Box_3^3 - 90\Box_1^3\Box_2\Box_3^7\
- 444\Box_1^2\Box_2\Box_3^7 + 33\Box_1^2\Box_3^8 + 123\Box_1\Box_2\Box_3^8 - 5\Box_1\Box_3^9)\
+ \frac{\ln(\Box_1/\Box_3)}{135D^6\Box_1\Box_2}\left(-2\Box_1^{11} + 18\Box_1^{10}\Box_2 - 663\Box_1^9\Box_2^2\right)\
+ 1999\Box_1^8\Box_2^3 - 588\Box_1^7\Box_2^4 - 4852\Box_1^6\Box_2^5\
+ 7678\Box_1^5\Box_2^6 - 4542\Box_1^4\Box_2^7 + 926\Box_1^3\Box_2^8\
+ 18\Box_1^2\Box_2^9 + 9\Box_1\Box_2^{10} - \Box_2^{11} + 18\Box_1^{10}\Box_3\
- 168\Box_1^9\Box_2\Box_3 - 3447\Box_1^8\Box_2^2\Box_3 - 8652\Box_1^7\Box_2^3\Box_3\
+ 44112\Box_1^6\Box_2^4\Box_3 - 42372\Box_1^5\Box_2^5\Box_3 + 3750\Box_1^4\Box_2^6\Box_3\
+ 8076\Box_1^3\Box_2^7\Box_3 - 1242\Box_1^2\Box_2^8\Box_3 - 84\Box_1\Box_2^9\Box_3\right]

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\[ + 9 \Box_{2}^{10} \Box_{3} - 60 \Box_{1}^{9} \Box_{3}^{2} + 228 \Box_{1}^{8} \Box_{2} \Box_{3}^{2} + 5166 \Box_{1}^{7} \Box_{2}^{2} \Box_{3}^{2} \\
- 35922 \Box_{1}^{6} \Box_{2}^{3} \Box_{3}^{2} - 6690 \Box_{1}^{5} \Box_{2}^{4} \Box_{3}^{2} + 66630 \Box_{1}^{4} \Box_{2}^{5} \Box_{3}^{2} \\
- 294903 \Box_{1}^{3} \Box_{2}^{6} \Box_{3}^{2} + 54 \Box_{1}^{2} \Box_{2}^{7} \Box_{3}^{2} + 114 \Box_{1} \Box_{2}^{8} \Box_{3}^{2} \\
- 30 \Box_{2}^{9} \Box_{3}^{2} + 84 \Box_{1}^{8} \Box_{2} \Box_{3}^{3} + 504 \Box_{1}^{7} \Box_{2}^{2} \Box_{3}^{3} + 2614 \Box_{1}^{6} \Box_{2}^{3} \Box_{3}^{3} \\
+ 39324 \Box_{1}^{5} \Box_{2}^{4} \Box_{3}^{3} - 66330 \Box_{1}^{4} \Box_{2}^{5} \Box_{3}^{3} + 10800 \Box_{1}^{3} \Box_{2}^{6} \Box_{3}^{3} \\
+ 1910 \Box_{1}^{2} \Box_{2}^{7} \Box_{3}^{3} + 252 \Box_{1} \Box_{2}^{8} \Box_{3}^{3} + 42 \Box_{2}^{9} \Box_{3}^{3} \\
- 1320 \Box_{1}^{6} \Box_{2}^{4} \Box_{3} + 1980 \Box_{1}^{5} \Box_{2}^{2} \Box_{3}^{4} + 2880 \Box_{1}^{4} \Box_{2}^{3} \Box_{3}^{4} \\
+ 5040 \Box_{1}^{3} \Box_{2}^{4} \Box_{3}^{4} + 2160 \Box_{1}^{2} \Box_{2}^{5} \Box_{3}^{4} - 660 \Box_{1} \Box_{2}^{6} \Box_{3}^{4} \\
- 168 \Box_{1}^{6} \Box_{2}^{5} + 552 \Box_{1}^{5} \Box_{2}^{2} \Box_{3}^{5} - 7764 \Box_{1}^{4} \Box_{2}^{3} \Box_{3}^{5} \\
+ 6732 \Box_{1}^{3} \Box_{2}^{4} \Box_{3}^{5} - 444 \Box_{1}^{2} \Box_{2}^{5} \Box_{3}^{5} + 276 \Box_{1} \Box_{2}^{6} \Box_{3}^{5} \\
- 84 \Box_{2}^{6} \Box_{3}^{5} + 252 \Box_{1}^{5} \Box_{2}^{3} \Box_{3}^{6} + 828 \Box_{1}^{4} \Box_{2}^{4} \Box_{3}^{6} \\
- 1262 \Box_{1}^{3} \Box_{2}^{2} \Box_{3}^{6} - 5950 \Box_{1}^{2} \Box_{2}^{3} \Box_{3}^{6} + 414 \Box_{1} \Box_{2}^{4} \Box_{3}^{6} \\
+ 1264 \Box_{1}^{6} \Box_{2}^{3} \Box_{3}^{7} - 180 \Box_{1}^{5} \Box_{2}^{4} \Box_{3}^{7} - 888 \Box_{1}^{4} \Box_{2}^{5} \Box_{3}^{7} \\
+ 3258 \Box_{1}^{3} \Box_{2}^{6} \Box_{3}^{7} - 444 \Box_{1}^{2} \Box_{2}^{7} \Box_{3}^{7} - 90 \Box_{1} \Box_{2}^{8} \Box_{3}^{7} \\
+ 66 \Box_{2}^{6} \Box_{3}^{8} + 246 \Box_{1}^{5} \Box_{2}^{2} \Box_{3}^{8} + 123 \Box_{1}^{4} \Box_{2}^{3} \Box_{3}^{8} + 33 \Box_{1}^{3} \Box_{2}^{4} \Box_{3}^{8} \\
- 10 \Box_{1}^{2} \Box_{2}^{9} - 5 \Box_{2}^{9} \Box_{3}^{9} ) \]

\[ + \frac{1}{540 D^5 \Box_{1}^{2} \Box_{2}^{3}} \left( - 2 \Box_{1}^{10} + 20 \Box_{1}^{9} \Box_{2} - 90 \Box_{1}^{8} \Box_{2}^{2} \\
+ 240 \Box_{1}^{7} \Box_{2}^{3} - 420 \Box_{1}^{6} \Box_{2}^{4} + 252 \Box_{1}^{5} \Box_{2}^{5} + 44 \Box_{1}^{4} \Box_{2}^{6} \\
- 260 \Box_{1}^{3} \Box_{2}^{7} + 6904 \Box_{1}^{2} \Box_{2}^{8} - 19592 \Box_{1}^{1} \Box_{2}^{9} \\
+ 12904 \Box_{1}^{9} \Box_{2}^{10} - 318 \Box_{1}^{8} \Box_{2}^{11} + 2104 \Box_{1}^{7} \Box_{2}^{12} \\
+ 10056 \Box_{1}^{6} \Box_{2}^{13} + 51336 \Box_{1}^{5} \Box_{2}^{14} - 63178 \Box_{1}^{4} \Box_{2}^{15} \\
+ 1092 \Box_{1}^{3} \Box_{2}^{16} - 3956 \Box_{1}^{2} \Box_{2}^{17} + 23604 \Box_{1}^{1} \Box_{2}^{18} \\
+ 48068 \Box_{1}^{10} \Box_{2}^{19} - 2088 \Box_{1}^{9} \Box_{2}^{20} + 488 \Box_{1}^{8} \Box_{2}^{21} \\
+ 4048 \Box_{1}^{7} \Box_{2}^{22} - 16832 \Box_{1}^{6} \Box_{2}^{23} + 2364 \Box_{1}^{5} \Box_{2}^{24} \\
+ 7420 \Box_{1}^{4} \Box_{2}^{25} - 10296 \Box_{1}^{3} \Box_{2}^{26} - 1584 \Box_{1}^{2} \Box_{2}^{27} \\
- 6440 \Box_{1}^{1} \Box_{2}^{28} + 5844 \Box_{1}^{2} \Box_{2}^{29} + 588 \Box_{1}^{3} \Box_{2}^{30} \\
+ 1396 \Box_{1}^{2} \Box_{2}^{31} - 102 \Box_{1}^{1} \Box_{2}^{32} + 98 \Box_{1}^{2} \Box_{2}^{33} \\
+ 8 \Box_{1}^{3} \Box_{2}^{34} - 3 \Box_{1}^{10} \Box_{2}^{35} \right), \quad (6.39) \]
\[+ 2808\square_1^4\square_2^3\square_3 - 24\square_1^6\square_3^2 + 200\square_1^5\square_2\square_3^2\]
\[- 88\square_1^4\square_2^2\square_3^2 - 7800\square_1^3\square_2^3\square_3 + 60\square_1^5\square_3^3\]
\[- 80\square_1^4\square_2^3\square_3 - 4380\square_1^3\square_2\square_3^3 - 80\square_1^4\square_3^4\]
\[- 240\square_1^3\square_2\square_3^4 - 3040\square_1^2\square_2\square_3^4 + 60\square_1^3\square_3^5\]
\[+ 280\square_1^2\square_2\square_3^5 - 24\square_1^2\square_3^6 - 88\square_1\square_2\square_3^6 + 4\square_1\square_3^7\)
\[\frac{\ln(\square_1/\square_3)}{45D^5\square_1^2} \left( - 16\square_1^7\square_2^2 + 80\square_1^6\square_2^3 - 160\square_1^5\square_2^4\right)
\[+ 160\square_1^4\square_2^5 - 80\square_1^3\square_2^6 + 16\square_1^2\square_2^7 + 8\square_1\square_2^8\]
\[- 144\square_1^7\square_3^3 - 844\square_1^6\square_2\square_3^3 + 3048\square_1^5\square_2^3\square_3\]
\[- 2700\square_1^4\square_2^4\square_3 + 240\square_1^3\square_2^5\square_3 + 460\square_1^2\square_2^6\square_3\]
\[- 72\square_1^2\square_2^7\square_3 + 4\square_2^8\square_3 - 84\square_1^7\square_3^2 + 400\square_1^6\square_2\square_3^2\]
\[- 2888\square_1^5\square_2^3\square_3^2 - 1320\square_1^4\square_2^4\square_3^2 + 6480\square_1^3\square_2^5\square_3^2\]
\[- 2800\square_1^2\square_2^6\square_3^2 + 200\square_1^2\square_2^6\square_3^2 - 24\square_2^7\square_3^2\]
\[+ 120\square_1^6\square_3^3 - 160\square_1^5\square_2\square_3^3 + 6060\square_1^4\square_2^3\square_3\]
\[- 9120\square_1^3\square_2^3\square_3^3 + 1680\square_1^2\square_2^4\square_3^3 - 80\square_1\square_2^5\square_3^3\]
\[+ 60\square_2^6\square_3^3 - 160\square_1^5\square_3^4 - 480\square_1^4\square_2\square_3^4 - 80\square_1^3\square_2^2\square_3^4\]
\[+ 2960\square_1^2\square_2^3\square_3^4 - 240\square_1^2\square_2^4\square_3^4 - 80\square_2^5\square_3^4\]
\[+ 120\square_1^4\square_3^5 + 560\square_1^3\square_2\square_3^5 - 2148\square_1^2\square_2^2\square_3^5\]
\[+ 280\square_1\square_2^3\square_3^5 + 60\square_2^4\square_3^5 - 48\square_1^7\square_3^6 - 176\square_1^6\square_2\square_3^6\]
\[- 88\square_1\square_2^2\square_3^6 - 24\square_2^3\square_3^6 + 8\square_1^2\square_3^7 + 4\square_2^2\square_3^7\)
\[+ \frac{1}{135D^4\square_1^2\square_3^3} \left( 2\square_1^8 - 16\square_1^7\square_2 + 56\square_1^6\square_2^2\right)
\[- 112\square_1^5\square_2^3 + 70\square_1^4\square_2^4 - 28\square_1^3\square_2^5 + 212\square_1^2\square_2^6\square_3\]
\[- 468\square_1^5\square_2\square_3 + 284\square_1^4\square_2^3\square_3 + 86\square_1^6\square_2\square_3^2 + 360\square_1^5\square_2^2\square_3^2\]
\[+ 3690\square_1^4\square_2^2\square_3^2 - 4136\square_1^3\square_2^3\square_3^2 - 760\square_1^5\square_3^3 - 1900\square_1^4\square_2\square_3^3\]
\[+ 4136\square_1^3\square_2^3\square_3^3 - 70\square_1^4\square_3^4 + 1616\square_1^3\square_2^3\square_3^4\]
\[+ 690\square_1^2\square_2^3\square_3^4 + 188\square_1^3\square_3^5 + 108\square_1^2\square_2\square_3^5\]
\[- 142\square_1\square_3^6 - 212\square_1\square_2\square_3^6 + 44\square_1\square_3^7 - 2\square_3^8\),
\]
\[\Gamma_{29}(\square_1, - \square_2, - \square_3) = \Gamma(\square_1, - \square_2, - \square_3) \frac{1}{3D^6} \left( 48\square_1^7\square_2\square_3\right)
\[- 64\square_1^6\square_2^2\square_3 - 176\square_1^5\square_2\square_3^3 + 384\square_1^4\square_2^4\square_3 - 176\square_1^3\square_2^5\square_3\]
\[- 64\square_1^2\square_2^6\square_3 + 48\square_1^4\square_2^7\square_3 - 64\square_1^6\square_2\square_3^2 + 64\square_1^5\square_2^2\square_3^2\]
\[- 576\square_1^4\square_2^3\square_3^2 - 576\square_1^3\square_2\square_3^3 + 640\square_1^2\square_2^4\square_3^3 - 64\square_1\square_2^6\square_3^2\]
\[- 176\square_1^5\square_2\square_3^4 - 576\square_1^4\square_2\square_3^3 + 1728\square_1^3\square_2^3\square_3^3\]
\[- 576\square_1^2\square_2^4\square_3^3 - 176\square_1\square_2^5\square_3^3 + 384\square_1^4\square_2\square_3^4 - 576\square_1^3\square_2^2\square_3^4\]
\[- 576\square_1^2\square_2^3\square_3^4 + 384\square_1\square_2^4\square_3^4 - 176\square_1\square_2^3\square_3^5 + 64\square_1^2\square_2^2\square_3^5\]
\[- 176\square_1\square_2^3\square_3^5 - 64\square_1\square_2^2\square_3^6 - 64\square_1\square_2\square_3^6 + 48\square_1\square_2\square_3^7\)
\]
\[66\]
\[
\begin{align*}
&+ \ln(\Box_1/\Box_2) + \frac{1}{45D^6\Box_1\Box_2\Box_3}\left(-8\Box_1^9\Box_2^2 + 56\Box_1^8\Box_2^3 - 168\Box_1^7\Box_2^4 \right. \\
&+ 280\Box_1^6\Box_2^5 - 280\Box_1^5\Box_2^6 + 168\Box_1^4\Box_2^7 - 56\Box_1^3\Box_2^8 \\
&+ 8\Box_1^2\Box_2^9 + 216\Box_1^8\Box_2^2\Box_3 - 84\Box_1^7\Box_2^3\Box_3 + 108\Box_1^6\Box_2^4\Box_3 \\
&- 108\Box_1^5\Box_2^6\Box_3 + 864\Box_1^4\Box_2^7\Box_3 - 216\Box_1^3\Box_2^8\Box_3 - 4\Box_1^9\Box_2^4 \\
&+ 108\Box_1^8\Box_2^3\Box_8^2 + 3600\Box_1^7\Box_2^2\Box_3^2 - 624\Box_1^6\Box_2^3\Box_3^2 \\
&- 16848\Box_1^5\Box_2^4\Box_3^2 + 16848\Box_1^4\Box_2^5\Box_3^2 + 624\Box_1^3\Box_2^6\Box_3^2 \\
&- 3600\Box_1^2\Box_2^7\Box_3^2 - 108\Box_1^8\Box_2^3\Box_8^3 + 4\Box_2^9\Box_3^2 + 28\Box_1^9\Box_3^3 \\
&- 432\Box_1^7\Box_2^3\Box_3^3 - 6624\Box_1^6\Box_2^2\Box_3^3 + 29136\Box_1^5\Box_2^3\Box_3^3 \\
&- 29136\Box_1^3\Box_2^5\Box_3^3 + 6624\Box_1^2\Box_2^6\Box_3^3 + 432\Box_1^7\Box_2^3\Box_3^3 \\
&- 28\Box_2^8\Box_3^3 - 84\Box_1^7\Box_3^3 + 540\Box_1^6\Box_2^2\Box_3^3 \\
&- 4068\Box_1^5\Box_2^3\Box_3^4 - 33300\Box_1^4\Box_2^4\Box_3^4 + 33300\Box_1^3\Box_2^6\Box_3^4 \\
&+ 4068\Box_1^2\Box_2^5\Box_3^4 - 540\Box_1^6\Box_2^6\Box_3^4 + 84\Box_1^7\Box_3^4 + 140\Box_1^6\Box_3^5 \\
&+ 12780\Box_1^5\Box_2^3\Box_3^5 - 12780\Box_1^4\Box_2^4\Box_3^5 - 140\Box_2^6\Box_3^5 \\
&- 140\Box_1^5\Box_3^6 - 540\Box_1^4\Box_2^2\Box_3^6 - 6000\Box_1^3\Box_2^3\Box_3^6 \\
&+ 6000\Box_2^3\Box_3^6 + 540\Box_1^2\Box_2^4\Box_3^6 + 140\Box_1^5\Box_3^6 \\
&+ 84\Box_1^4\Box_3^7 + 432\Box_1^3\Box_2^3\Box_3^7 - 432\Box_1^2\Box_3^3\Box_8^7 - 84\Box_2^4\Box_3^7 \\
&- 28\Box_3^8 + 108\Box_1^2\Box_2^3\Box_3^8 + 108\Box_1^2\Box_2^3\Box_3^8 + 28\Box_3^3\Box_8 \\
&+ 4\Box_1^2\Box_3^9 - 94\Box_2^2\Box_3^9 \right)
\end{align*}
\]

\[
\frac{1}{135D^5\Box_1\Box_2\Box_3}\left(-\Box_1^9 + 13\Box_1^8\Box_2 - 50\Box_1^7\Box_2^2 \right.
\]

\[
\begin{align*}
&+ 82\Box_1^6\Box_2^3 - 44\Box_1^5\Box_2^4 - 44\Box_1^4\Box_2^5 + 82\Box_1^3\Box_2^6 \\
&- 50\Box_1^2\Box_2^7 + 13\Box_1^8\Box_2 - 29\Box_1^8\Box_2^3 \\
&- 154\Box_1^7\Box_2\Box_3 + 64\Box_1^6\Box_2^2\Box_3 + 922\Box_1^5\Box_2\Box_3^3 - 1690\Box_1^4\Box_2^2\Box_3 \\
&+ 922\Box_1^3\Box_2\Box_3^4 + 64\Box_1^2\Box_2^3\Box_3 - 154\Box_1^2\Box_2\Box_3^4 + 13\Box_1^2\Box_3^4 \\
&- 50\Box_1^2\Box_3^3 + 50\Box_1^6\Box_2^2\Box_3 - 50\Box_1^6\Box_2\Box_3 \\
&+ 50\Box_1^5\Box_2^2\Box_3^2 + 5074\Box_1^4\Box_2^2\Box_3^2 - 50\Box_1^6\Box_2^2\Box_3 \\
&+ 64\Box_1^2\Box_2^6\Box_3^2 + 82\Box_1^6\Box_3^3 + 922\Box_1^5\Box_2^2\Box_3 \\
&+ 50\Box_1^5\Box_2^2\Box_3 + 17196\Box_1^3\Box_2^3\Box_3^2 + 5074\Box_1^2\Box_2^4\Box_3^2 \\
&+ 922\Box_1^2\Box_2^5\Box_3^3 + 82\Box_2^6\Box_3^3 - 44\Box_1^5\Box_2^3\Box_3 - 1690\Box_1^4\Box_2^2\Box_3 \\
&+ 50\Box_1^5\Box_2^2\Box_3^4 + 5074\Box_1^4\Box_2^2\Box_3^4 - 1690\Box_1^2\Box_2^5\Box_3 \\
&- 44\Box_1^5\Box_2^3\Box_3 - 44\Box_1^4\Box_2^4\Box_3 + 922\Box_1^3\Box_2^2\Box_3^2 - 50\Box_1^5\Box_2^3\Box_3 \\
&+ 922\Box_1^3\Box_2^5\Box_3^3 + 82\Box_1^3\Box_3^3 + 64\Box_1^2\Box_2^3\Box_3 \\
&+ 64\Box_1^2\Box_2^2\Box_3^5 + 82\Box_2^2\Box_3^5 - 50\Box_1^2\Box_3^7 - 154\Box_1^2\Box_2\Box_3 \\
&- 50\Box_2^2\Box_3^7 + 13\Box_1^2\Box_3^6 + 13\Box_2^2\Box_3^6 - 3\Box_3^9 \bigg).
\end{align*}
\]

(6.41)

It should be noted that, in four dimensions, the basis of twenty nine cubic structures (2.15)–(2.43) is overcomplete (see Appendix). There exists a constraint between the purely gravitational structures (eq. (A.35) of Appendix) which reduces the dimension of the basis by one. The results above are given in the reduced basis obtained by elimination of the
completely symmetric part of the structure $R_1 R_2 R_3(28)$. This is seen from the fact that the
form factor $\Gamma_{28}$ in eq. (6.40) possesses the property
\[ \Gamma_{28}(\Box, \Box, \Box) + \Gamma_{28}(\Box, \Box, \Box) + \Gamma_{28}(\Box, \Box, \Box) \\
+ \Gamma_{28}(\Box, \Box, \Box) + \Gamma_{28}(\Box, \Box, \Box) + \Gamma_{28}(\Box, \Box, \Box) = 0. \tag{6.42} \]

Other properties of the form factors, including their asymptotic behaviours at large and
small arguments, are studied below. The differential equations for the basic form factor
(6.8), and comments on the expressions above will be found in sect. 18.

7. The $\alpha$-representation of the third-order form factors in the effective action

Below we consider separately the third-order form factors (6.7) in the effective action. When
written down explicitly as above, they are very cumbersome. Most compact is their
integral $\alpha$-representation which is in fact the one obtained initially (see sects. 17–19) and
from which all the other representations are derived including the one given above.

In the $\alpha$-representation, the functions (6.7) are given in terms of the integrals
\[ \left\langle \frac{P(\alpha, \Box)}{-\Omega} \right\rangle_3 = \int_{\alpha \geq 0} d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \frac{P(\alpha, \Box)}{-\Omega} \tag{7.1} \]
where
\[ \Omega = \alpha_2 \alpha_3 \alpha_1 + \alpha_1 \alpha_2 \alpha_3 + \alpha_1 \alpha_2 \alpha_3, \]
and $P(\alpha, \Box)$ is a polynomial in $\alpha$’s, boxes and inverse boxes. There are also explicit contributions of two types: purely tree terms and terms proportional to (6.9b) with tree coefficients.

The expressions for the twenty nine form factors (6.7) in the $\alpha$-form are as follows:

\[ \Gamma_1(-\Box, -\Box, -\Box) = \left\langle \frac{1}{-\Omega} \left( \frac{1}{3} \right) \right\rangle_3, \tag{7.2} \]

\[ \Gamma_2(-\Box, -\Box, -\Box) = \left\langle \frac{1}{-\Omega} \left( \frac{4}{3} \alpha_1 \alpha_2 \alpha_3 \right) \right\rangle_3 + \frac{1}{3} \ln \left( \frac{\Box_1}{\Box_2} \right), \tag{7.3} \]

\[ \Gamma_3(-\Box, -\Box, -\Box) = \left\langle \frac{1}{-\Omega} (2\alpha_1 \alpha_2) \right\rangle_3, \tag{7.4} \]

\[ \Gamma_4(-\Box, -\Box, -\Box) = \left\langle \frac{1}{-\Omega} \left( -\frac{1}{9} \alpha_1 + \frac{7}{9} \alpha_2 - \alpha_3 \alpha_2 \alpha_3 + \frac{1}{36} \alpha_3 - \frac{4}{9} \alpha_1 \alpha_3 + \alpha_1 \alpha_3 + \frac{8}{9} \alpha_1 \alpha_2 \alpha_3 - 2 \alpha_1 \alpha_2 \alpha_3 \right) \right\rangle_3, \tag{7.5} \]
\[ \Gamma_5(-\Box_1, -\Box_2, -\Box_3) = \langle \frac{1}{-\Omega} \left( \frac{1}{9} \alpha_1 + \frac{1}{9} \alpha_1 \alpha_2 - \frac{2}{9} \alpha_1 \alpha_3 ight. \\
+ \frac{\Box_1}{\Box_2} \left( \frac{1}{9} \alpha_2 + \frac{1}{9} \alpha_2^2 \right) \rangle_3 \\
+ \frac{1}{4 \Box_2} - \frac{\Box_3}{24 \Box_1 \Box_2}, \]  
\text{(7.6)}

\[ \Gamma_6(-\Box_1, -\Box_2, -\Box_3) = \langle \frac{1}{-\Omega} \left( -\frac{1}{6} + \alpha_1 - \alpha_1^2 \right) \rangle_3, \]  
\text{(7.7)}

\[ \Gamma_7(-\Box_1, -\Box_2, -\Box_3) = \langle \frac{1}{-\Omega} \left( -\frac{1}{12} \alpha_2 + \frac{5}{6} \alpha_2 \alpha_3 \\
- \frac{1}{4} \alpha_1 \alpha_2 \alpha_3 - 2 \alpha_1 \alpha_2^2 \alpha_3 - 2 \alpha_2^3 \alpha_3 \\
+ \frac{\Box_2}{\Box_1} \left( -\frac{1}{12} \alpha_1 \right) \rangle_3 \\
+ \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (-\Box_2) \\
+ \frac{\alpha_1 \alpha_3 + 8 \alpha_1^2 \alpha_2 \alpha_3}{12 \Box_1}, \]  
\text{(7.8)}

\[ \Gamma_8(-\Box_1, -\Box_2, -\Box_3) = \langle \frac{1}{-\Omega} \left( \frac{1}{6} \alpha_2 + \frac{5}{3} \alpha_1 \alpha_2 \\
- 2 \alpha_1^2 \alpha_2 - 4 \alpha_1 \alpha_2 \alpha_3 + 8 \alpha_1 \alpha_2^2 \alpha_3 \\
+ \frac{\Box_2}{\Box_1} \left( -\frac{1}{6} \alpha_1 + \frac{1}{3} \alpha_1 \alpha_2 + 2 \alpha_1^2 \alpha_2 \\
+ \alpha_1 \alpha_3 + 8 \alpha_1^2 \alpha_2 \alpha_3 \right) \rangle_3, \]  
\text{(7.9)}

\[ \Gamma_9(-\Box_1, -\Box_2, -\Box_3) = \langle \frac{1}{-\Omega} \left( -\frac{1}{648} + \frac{1}{216} \alpha_1 - \frac{5}{108} \alpha_1^2 \\
+ \frac{2}{135} \alpha_2 + \frac{1}{36} \alpha_1 \alpha_2 + \frac{35}{216} \alpha_1^2 \alpha_2 - \frac{13}{1080} \alpha_2^2 \\
+ \frac{1}{180} \alpha_1 \alpha_2^2 - \frac{14}{45} \alpha_1^2 \alpha_2^2 + \frac{11}{1080} \alpha_2^3 + \frac{41}{270} \alpha_1 \alpha_2^3 \\
- \frac{4}{9} \alpha_1^2 \alpha_2^3 + \frac{13}{360} \alpha_2 \alpha_3 + \frac{1}{72} \alpha_1 \alpha_2 \alpha_3 - \frac{11}{135} \alpha_1^2 \alpha_2 \alpha_3 \\
- \frac{19}{360} \alpha_2^2 \alpha_3 - \frac{1}{6} \alpha_1 \alpha_2^2 \alpha_3 - \frac{113}{54} \alpha_1^2 \alpha_2^2 \alpha_3 + \frac{47}{540} \alpha_2^3 \alpha_3 \\
+ \frac{7}{108} \alpha_1 \alpha_2^3 \alpha_3 + \alpha_1^2 \alpha_2^3 \alpha_3 - \frac{1}{10} \alpha_2^2 \alpha_3^2 - \frac{1}{12} \alpha_1 \alpha_2 \alpha_3^2 \\
+ \frac{2}{3} \alpha_1^2 \alpha_2 \alpha_3^2 + \frac{1}{9} \alpha_2^3 \alpha_3^2 \right) \rangle_3. \]  
\text{(7.9)}
\[
\begin{align*}
\Gamma_1 & \left( \frac{2}{135} \alpha_1 - \frac{11}{540} \alpha_1^2 - \frac{37}{540} \alpha_1 \alpha_2 \right) \\
\Gamma_2 & \left( \frac{7}{90} \alpha_1 \alpha_2 - \frac{1}{10} \alpha_1^2 \alpha_2 - \frac{1}{15} \alpha_1 \alpha_2^2 \right) \\
\Gamma_3 & \left( \frac{11}{135} \alpha_1 \alpha_3 - \frac{1}{180} \alpha_1 \alpha_3^2 - \frac{1}{180} \alpha_1 \alpha_2 \alpha_3 \right) \\
\Gamma_4 & \left( \frac{19}{36} \alpha_1 \alpha_2 \alpha_3 + \frac{17}{36} \alpha_1 \alpha_2^2 \alpha_3 - \frac{5}{36} \alpha_1 \alpha_2^3 \alpha_3 \right) \\
\Gamma_5 & \left( \frac{7}{60} \alpha_1 \alpha_3^2 + \frac{17}{45} \alpha_1 \alpha_3^2 - \frac{31}{60} \alpha_1 \alpha_2 \alpha_3^2 + \frac{1}{36} \alpha_1 \alpha_2 \alpha_3^3 \right) \\
\Gamma_6 & \left( \frac{29}{36} \alpha_1 \alpha_2 \alpha_3^3 + \alpha_1 \alpha_2 \alpha_3^3 + \alpha_1 \alpha_2 \alpha_3^4 \right) \\
\Gamma_7 & \left( \frac{1}{720} - \frac{\alpha_1}{120 \alpha_3} - \frac{\alpha_1}{2160 \alpha_2 \alpha_3} \right) \\
\end{align*}
\]

\[\Gamma_{10}(-\alpha_1, -\alpha_2, -\alpha_3) = \left\langle -\frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \right\rangle_3 \]

\[\Gamma_{11}(-\alpha_1, -\alpha_2, -\alpha_3) = \left\langle -\frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \right\rangle_3 \]
\[ + \frac{\Box_1}{\Box_2} \left( - \frac{7}{540} \alpha_2 + \frac{1}{540} \alpha_1 \alpha_2 + \frac{2}{135} \alpha_2^2 \right) - \frac{7}{180} \alpha_1 \alpha_2^2 - \frac{1}{60} \alpha_2 \alpha_3 + \frac{7}{90} \alpha_1 \alpha_2 \alpha_3 - \frac{1}{9} \alpha_1^2 \alpha_2 \alpha_3 \\
+ \frac{2}{45} \alpha_2^2 \alpha_3 - \frac{22}{45} \alpha_1 \alpha_2^2 \alpha_3 + \frac{1}{3} \alpha_1^2 \alpha_2^2 \alpha_3 + 2 \frac{\alpha_2 \alpha_3^2}{45} \\
- \frac{4}{9} \alpha_1 \alpha_2 \alpha_3^2 + \frac{2}{9} \alpha_1^2 \alpha_2 \alpha_3^2 - \frac{2}{9} \alpha_2^3 \alpha_3^2 \right) \rangle_3 \\
+ \frac{1}{120 (\Box_1 - \Box_2)} - \frac{\Box_3}{540 \Box_1 \Box_2}, \tag{7.12} \]

\[ \Gamma_{12}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ \left\langle \frac{1}{-\Omega} \left( \frac{8}{3} \alpha_1^2 - 4 \alpha_1^3 \right) \right\rangle_3 + \frac{\ln(\Box_1/\Box_2)}{(\Box_1/\Box_2)} + \frac{\ln(\Box_1/\Box_3)}{(\Box_1/\Box_3)} \right] \tag{7.13} \]

\[ \Gamma_{13}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ \left\langle \frac{1}{-\Omega} \left( 2 \alpha_1 \right) \right\rangle_3 + \frac{2 \ln(\Box_1/\Box_3)}{(\Box_1/\Box_3)} \right], \tag{7.14} \]

\[ \Gamma_{14}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_3} \left\langle \frac{1}{-\Omega} \left( 2 \alpha_3 - 4 \alpha_3^2 \right) \right\rangle_3, \tag{7.15} \]

\[ \Gamma_{15}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left\langle \frac{1}{-\Omega} \left( \frac{2}{3} \alpha_1^2 - 4 \alpha_1^3 + 4 \alpha_1^4 \right) \right\rangle_3, \tag{7.16} \]

\[ \Gamma_{16}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left\langle \frac{1}{-\Omega} \left( \frac{4}{9} \alpha_1 + \frac{4}{3} \alpha_2 \alpha_1^2 - \frac{4}{9} \alpha_1 \alpha_3 - \frac{4}{3} \alpha_1^2 \alpha_3 \right) \right\rangle_3 \\
+ \frac{1}{6 \Box_1 \Box_2}, \tag{7.17} \]

\[ \Gamma_{17}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ \left\langle \frac{1}{-\Omega} \left( 2 \alpha_1 \right) \right\rangle_3 + \frac{\ln(\Box_2/\Box_3)}{(\Box_2/\Box_3)} \right], \tag{7.18} \]

\[ \Gamma_{18}(\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left\langle \frac{1}{-\Omega} \left( 2 \alpha_1^2 - 8 \alpha_1^2 \alpha_2 + 8 \alpha_1^2 \alpha_2 \alpha_3 \right) \right\rangle_3, \tag{7.19} \]
\[
\Gamma_{19}(-\square_1, -\square_2, -\square_3) = \frac{1}{\square_1} \left[ \left( \frac{1}{-\Omega} \left( -4\alpha_1^2\alpha_2\alpha_3 \right) \right) \right]_3
+ \frac{1}{6} \ln(\square_2/\square_3) \right],
\]

\[
\Gamma_{20}(-\square_1, -\square_2, -\square_3) = \frac{1}{\square_1} \left[ \left( \frac{1}{-\Omega} \left( -\frac{2}{3} \alpha_1^2 + \frac{10}{3} \alpha_1^2\alpha_2 
- \frac{2}{3} \alpha_1\alpha_2^2 \right) \right. 
- \frac{11}{3} \alpha_1^2\alpha_2\alpha_3 + \frac{2}{3} \alpha_1\alpha_2^2\alpha_3 \right)_3 
- \frac{1}{6} \ln(\square_2/\square_3) \right],
\]

\[
\Gamma_{21}(-\square_1, -\square_2, -\square_3) = \frac{1}{\square_1} \left[ \left( \frac{1}{-\Omega} \left( 8\alpha_1^2\alpha_3 - 16\alpha_1^2\alpha_3^2 \right) \right)_3 
- \frac{2}{3} \ln(\square_2/\square_3) \right],
\]

\[
\Gamma_{22}(-\square_1, -\square_2, -\square_3) = \frac{1}{\square_1} \left[ \left( \frac{1}{-\Omega} \left( \frac{1}{90} \alpha_1 + \frac{1}{5} \alpha_1^2 - \frac{23}{54} \alpha_1\alpha_3 
- \frac{101}{90} \alpha_1^2\alpha_3 + \frac{29}{90} \alpha_1\alpha_2\alpha_3 + \frac{68}{45} \alpha_1^2\alpha_2\alpha_3 + \frac{1}{6} \alpha_1\alpha_3^2 + \frac{4}{3} \alpha_1^2\alpha_3^2 
+ \frac{4}{45} \alpha_1\alpha_2\alpha_3^2 + \frac{2}{9} \alpha_1\alpha_2^2\alpha_3^2 + \frac{4}{9} \alpha_1\alpha_3^3 - \frac{2}{9} \alpha_1\alpha_3^4 \right) \right)_3 
+ \frac{2}{90} \ln(\square_2/\square_3) \right],
\]

\[
\Gamma_{23}(-\square_1, -\square_2, -\square_3) = \frac{1}{\square_1} \left[ \left( \frac{1}{-\Omega} \left( \frac{2}{45} \alpha_2 + \frac{2}{9} \alpha_1^2\alpha_2 + \frac{2}{45} \alpha_2^2 - \frac{2}{45} \alpha_1\alpha_2^2 \right) \right. 
- \frac{28}{45} \alpha_1^2\alpha_2^2 - \frac{4}{9} \alpha_1^2\alpha_2\alpha_3 - \frac{7}{54} \alpha_2\alpha_3 + \frac{7}{10} \alpha_1\alpha_2\alpha_3 
- \frac{46}{45} \alpha_1^2\alpha_2\alpha_3 + \frac{1}{18} \alpha_2^2\alpha_3 + \frac{4}{45} \alpha_1\alpha_2^2\alpha_3 + \frac{38}{9} \alpha_1^2\alpha_2^2\alpha_3 
- \frac{4}{15} \alpha_1^2\alpha_2\alpha_3^2 - \frac{7}{15} \alpha_2\alpha_3^2 - \frac{17}{9} \alpha_1\alpha_2\alpha_3^2 + \frac{19}{9} \alpha_1^2\alpha_2\alpha_3^2 
- \frac{8}{15} \alpha_2^2\alpha_3^2 + \frac{11}{9} \alpha_1\alpha_2^2\alpha_3^2 - \frac{4}{9} \alpha_1^2\alpha_2^2\alpha_3^2 + \frac{10}{9} \alpha_2\alpha_3^3 
+ \frac{11}{9} \alpha_1\alpha_2\alpha_3^3 + \frac{2}{9} \alpha_2^2\alpha_3^3 - \frac{2}{9} \alpha_2\alpha_3^4 \right)_3 
+ \frac{1}{30} \ln(\square_1/\square_3) \right],
\]

\[
- \frac{1}{270 \square_1 \square_2}.
\]
\begin{align}
\Gamma_{23}(-\Box_1, -\Box_2, -\Box_3) &= \frac{1}{\Box_1} \left[ \left( \frac{1}{\Omega} \right) \left( -\frac{4}{45} \alpha_1 + \frac{2}{135} \alpha_1^2 - \frac{16}{135} \alpha_1 \alpha_2 ight) + \frac{2}{9} \alpha_1^2 \alpha_2 + \frac{4}{15} \alpha_1^3 \alpha_2 - \frac{34}{45} \alpha_1 \alpha_2^2 + \frac{76}{45} \alpha_1^2 \alpha_2^2 ight.
\nonumber
& \quad - \frac{8}{9} \alpha_1^3 \alpha_2^2 + \frac{8}{9} \alpha_1 \alpha_2^4 + \frac{7}{45} \alpha_1 \alpha_3 - \frac{1}{5} \alpha_1^2 \alpha_3
\nonumber
& \quad - \frac{19}{45} \alpha_1 \alpha_2 \alpha_3 - \frac{28}{45} \alpha_1^2 \alpha_2 \alpha_3 + \frac{20}{9} \alpha_1 \alpha_2^2 \alpha_3 - \frac{4}{3} \alpha_1^2 \alpha_2^2 \alpha_3
\nonumber
& \quad + \frac{1}{15} \alpha_1 \alpha_3 \alpha_3 + \frac{8}{45} \alpha_1^2 \alpha_3 \alpha_3 + \frac{32}{45} \alpha_1 \alpha_2 \alpha_3 \alpha_3 - \frac{4}{9} \alpha_1^2 \alpha_2 \alpha_3 \alpha_3
\nonumber
& \quad - \frac{4}{3} \alpha_1 \alpha_2 \alpha_3 \alpha_3 - \frac{8}{45} \alpha_1 \alpha_3 \alpha_3 \alpha_3 - \frac{4}{9} \alpha_1 \alpha_2 \alpha_3 \alpha_3
\nonumber
& \quad + \frac{1}{\Box_3} \left( -\frac{2}{45} \alpha_3 - \frac{4}{45} \alpha_1 \alpha_3 - \frac{14}{45} \alpha_1^2 \alpha_3 + \frac{4}{9} \alpha_1^4 \alpha_3 ight)
\nonumber
& \quad - \frac{4}{45} \alpha_2 \alpha_3 + \frac{4}{45} \alpha_1 \alpha_2 \alpha_3 + \frac{52}{45} \alpha_1^2 \alpha_2 \alpha_3 - \frac{23}{45} \alpha_2^2 \alpha_3
\nonumber
& \quad + \frac{52}{45} \alpha_1 \alpha_2 \alpha_3 - \frac{8}{9} \alpha_1^2 \alpha_2 \alpha_3 + \frac{4}{9} \alpha_2^4 \alpha_3 + \frac{14}{15} \alpha_1^2 \alpha_3 \alpha_3
\nonumber
& \quad - \frac{4}{45} \alpha_1 \alpha_2 \alpha_3 \alpha_3 - \frac{4}{45} \alpha_1 \alpha_2^2 \alpha_3 \alpha_3 + \frac{14}{15} \alpha_2^2 \alpha_3 \alpha_3 - \frac{4}{9} \alpha_1 \alpha_2^2 \alpha_3 \alpha_3
\nonumber
& \quad - \frac{2}{9} \alpha_1^2 \alpha_3 \alpha_3 - \frac{2}{9} \alpha_2^2 \alpha_3 \alpha_3 \right) \rangle_3
\nonumber
& + \frac{\ln(\Box_1/\Box_2)}{(\Box_1 - \Box_2)} \left( \frac{\Box_1}{30 \Box_3} \right)
\nonumber
& + \frac{1}{135 \Box_1 \Box_2},
\end{align}

(7.24)

\begin{align}
\Gamma_{24}(-\Box_1, -\Box_2, -\Box_3) &= \frac{1}{\Box_2} \left[ \left( \frac{1}{\Omega} \right) \left( -\frac{5}{54} \alpha_2 - \frac{23}{270} \alpha_1 \alpha_2 ight) + \frac{2}{5} \alpha_1^2 \alpha_2 + \frac{4}{15} \alpha_1^3 \alpha_2 + \frac{1}{270} \alpha_2^2 
\nonumber
& \quad + \frac{13}{270} \alpha_2 \alpha_3 - \frac{1}{5} \alpha_1 \alpha_2 \alpha_3 + \frac{4}{15} \alpha_1 \alpha_2 \alpha_3 \alpha_3
\nonumber
& \quad + \frac{1}{\Box_2} \left( -\frac{2}{45} \alpha_1 + \frac{1}{45} \alpha_1 \alpha_2 + \frac{1}{45} \alpha_1 \alpha_3 \right) \right) \rangle_3
\nonumber
& + \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3) \cdot 30 \Box_1} \right]
\nonumber
& + \frac{1}{540 \Box_2 \Box_3},
\end{align}

(7.25)

\begin{align}
\Gamma_{25}(-\Box_1, -\Box_2, -\Box_3) &= \frac{1}{\Box_1} \left[ \left( \frac{1}{\Omega} \right) \left( -\frac{13}{135} \alpha_1 - \frac{56}{135} \alpha_1 \alpha_2 ight) 
\nonumber
& \quad + \frac{2}{5} \alpha_1^2 \alpha_2 + \frac{4}{15} \alpha_1^3 \alpha_2 + \frac{1}{270} \alpha_2^2 
\nonumber
& \quad + \frac{13}{270} \alpha_2 \alpha_3 - \frac{1}{5} \alpha_1 \alpha_2 \alpha_3 + \frac{4}{15} \alpha_1 \alpha_2 \alpha_3 \alpha_3
\nonumber
& \quad + \frac{1}{\Box_1} \left( -\frac{2}{45} \alpha_1 + \frac{1}{45} \alpha_1 \alpha_2 + \frac{1}{45} \alpha_1 \alpha_3 \right) \right) \rangle_3
\nonumber
& + \frac{\ln(\Box_1/\Box_2)}{(\Box_1 - \Box_2) \cdot 30 \Box_2} \right]
\nonumber
& + \frac{1}{540 \Box_1 \Box_3},
\end{align}

(7.26)
\[ + \frac{28}{45} \alpha_1^2 + \frac{32}{45} \alpha_1^2 \alpha_2 \alpha_3 + \frac{16}{15} \alpha_1 \alpha_2 \alpha_3 \]
\[ + \frac{\Box_1}{\Box_3} \left( - \frac{8}{45} \alpha_3 - \frac{37}{135} \alpha_1 \alpha_3 + \frac{16}{45} \alpha_1^3 \alpha_3 + \frac{11}{135} \alpha_2 \alpha_3 \right) \]
\[ + \frac{28}{45} \alpha_1 \alpha_2 \alpha_3 - \frac{4}{45} \alpha_2^2 \alpha_3 - \frac{16}{45} \alpha_1 \alpha_2^2 \alpha_3 + \frac{1}{135} \alpha_3^2 \]
\[ + \frac{32}{45} \alpha_1 \alpha_3^2 - \frac{16}{45} \alpha_1 \alpha_2 \alpha_3^2 + \frac{16}{45} \alpha_2 \alpha_3^2 \right) \right) \right) \right) \]
\[ + \frac{\ln(\Box_1/\Box_3)}{(\Box_1 - \Box_2) (\Box_1 \Box_3)} \left( -\frac{2\Box_1}{15\Box_3} \right) \right) \right) \right) \]
\[- \frac{1}{135} \Box_1 \Box_3 + \frac{1}{270} \Box_2 \Box_3, \]  
(7.26)

\[ \Gamma_{26}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2} \left\langle \frac{1}{-\Omega} \left( 4 \alpha_1^2 \alpha_2^2 \right) \right\rangle_3, \quad \]  
(7.27)

\[ \Gamma_{27}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2} \left\langle \frac{1}{-\Omega} \left( - \frac{4}{3} \alpha_1 \alpha_2^2 + \frac{8}{3} \alpha_3 \alpha_2^3 \right. \right. \]
\[ - 3 \alpha_1 \alpha_2^2 \alpha_3^2 \right) \right\rangle_3 \]
\[- \frac{1}{540} \Box_1 \Box_2 \Box_3, \]  
(7.28)

\[ \Gamma_{28}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2} \left\langle \frac{1}{-\Omega} \left( \frac{8}{3} \alpha_1 \alpha_2^2 \alpha_3 \right) \right\rangle_3 \]
\[ + \frac{1}{135} \Box_1 \Box_2 \Box_3, \]  
(7.29)

\[ \Gamma_{29}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2 \Box_3} \left\langle \frac{1}{-\Omega} \left( \frac{8}{3} \alpha_1 \alpha_2^2 \alpha_3 \right) \right\rangle_3. \]  
(7.30)

The \( \alpha \)-representation exists also for the form factors in the heat kernel (sect. 15), and in both cases it suffers from one and the same shortcoming: it is not unique in a sense that the vanishing of an integral like (7.1) does not imply the vanishing of the polynomial \( P(\alpha, \Box) \). The minor cause of this nonuniqueness is the presence in (7.1) of the delta-function which confines \( P(\alpha, \Box) \) to \( \sum \alpha = 1 \), and the major one is the fact that \( P(\alpha, \Box) \) depends not only on \( \alpha \) but also on \( \Box \). In expressions (7.2)–(7.30), this dependence manifests itself in the presence of the factors \( \Box_1 \) and \( 1/\Box_{nm} (n, m = 1, 2, 3) \) in the coefficients of the \( \alpha \)-polynomials. Because the arguments of the functions \( \Gamma_i \) enter not only the kernel \( 1/(-\Omega) \), (7.1) is not a proper integral representation. In consequence of this fact, there exists a hierarchy of nontrivial identities between the averages of the form (7.1). Examples of such identities will be found in sect. 18.
The form factors of the curvature structures with and without derivatives have different dimension. In the form factors $\Gamma_1$ to $\Gamma_{11}$ corresponding to the structures without derivatives, the $\Box$-coefficients of the $\alpha$-polynomials are always of the form $\Box_n / \Box_m$. As the analysis in sects. 19, 20 shows, it is these coefficients that determine the asymptotic behaviours of the form factors. Each factor $\Box_n / \Box_m$ causes the logarithmic growth at large $\Box_n$ and the power growth at small $\Box_m$. This correlation can be traced by a comparison of the exact expressions (7.2)–(7.30) with the tables of asymptotic behaviours in sects. 10, 11. In calculations with the exact form factors, it is the coefficients $\Box_n / \Box_m$ that cause the problem of nonuniqueness.

Another problem with the $\alpha$-representation is the presence of nonanalytic terms that are not in the $\alpha$-form. There are two types of such terms:

\begin{equation}
\begin{align*}
\text{a) } & \ln \left( \frac{\Box_n / \Box_k}{\Box_n - \Box_k} \right), \\
\text{b) } & \ln \left( \frac{\Box_n / \Box_k}{\Box_n - \Box_k} \right) \frac{\Box_n}{\Box_n - \Box_k} \Box_m.
\end{align*}
\end{equation} (7.31)

The terms (7.31a) are really independent but the terms (7.31b) mix up with the $\alpha$-averages in the limits $\Box_n \to -\infty$ and $\Box_m \to 0$. This can also be traced by a comparison of expressions (7.2)–(7.30) with the tables of asymptotic behaviours below.

In eqs. (7.27)–(7.30), the form factor $\Gamma_{29}$ of the structure with six derivatives and the form factors $\Gamma_{26}$ to $\Gamma_{28}$ of the structures with four derivatives contain the overall factors $1 / \Box_1 \Box_2 \Box_3$ and $1 / \Box_1 \Box_2$ respectively which can be attached to the basis structures themselves to give these structures the standard dimension. The respectively redefined form factors are then in no way different from the form factors of the structures without derivatives. A similar redefinition in the case of the form factors $\Gamma_{12}$ to $\Gamma_{25}$ corresponding to the structures with two derivatives encounters a difficulty since it is not immediately clear which of the three inverse boxes $1 / \Box_1, 1 / \Box_2, 1 / \Box_3$ should be attached to the curvature structure. It turns out that at least a partial answer to this question can be given on the basis of the asymptotic behaviours of the form factors. With one exception, there exists a choice (and sometimes more than one) such that the redefined form factor does not acquire a growth at large arguments. In expressions (7.13)–(7.26) above, the $1 / \Box$ satisfying this criterion is written down as an overall factor. The exception is $\Gamma_{22}$. This form factor can only be written as a sum of two each of which satisfying the above criterion. This is fixed in the form of expression (7.23). Strictly speaking, $\Gamma_{25}$ in eq. (7.26) is, in this sense, also a sum of two form factors but one of the summands is a pure tree.

Inspection of expressions (7.2)–(7.30) shows that they obey the following general rule. Each $1 / \Box$ multiplier in the $\alpha$-polynomial appears only in a product with the like $\alpha$, e.g. $\alpha_1 / \Box_1$, $\alpha_1 \alpha_2 / \Box_1 \Box_2$, $\alpha_1 \alpha_2 \alpha_3 / \Box_1 \Box_2 \Box_3$, etc. This "rule of the like $\alpha$" plays an important role both in calculations with the exact form factors and in the forms of their asymptotic behaviours. The work of this rule is discussed in detail in sects. 19, 20.

The nonuniqueness of the $\alpha$-representation makes it unfit for carrying out checks like the check of the trace anomaly. The explicit representation in sect. 6 possesses the advantage of being unique but is cumbersome and unfit for applications to the expectation-value problems because the nonlocal operators are not expressed through the Green function [1,4]. This compels looking for unique integral representations of the form factors. Two such representations are given below.

It should be emphasized that only the symmetrized form factors $\Gamma_i^{\text{sym}}$ make sense (eq. (6.12)). In a not symmetrized form, various expressions for $\Gamma_i$ may differ by terms vanishing after the symmetrization.
8. The Laplace representation of the third-order form factors in the effective action

The Laplace form of the form factors arises naturally when the loop diagrams are calculated in terms of the heat kernels. Therefore, this representation is readily obtained in the present technique.

The Laplace representation proper

\[ f(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.1) \]

\[ \sum u_\Box \equiv u_1\Box_1 + u_2\Box_2 + u_3\Box_3, \quad \Box_1, \Box_2, \Box_3 < 0 \]

exists only for functions \( f \) decreasing at large values of each of the arguments (and is insensitive to a power growth at small values). This representation is, therefore, useful for studying the large-\( \Box \) limit. In the process of giving the form factors the Laplace guise (sect. 19), their nondecreasing terms get detached and take the form of Laplace integrals multiplied by powers of \( \Box \)'s. It is important that the Laplace originals in these terms depend on only two of the three arguments. In consequence of this fact, the nondecreasing terms factorize into functions of one or two variables which are, moreover, elementary.

Another property of the form factors, which is a direct consequence of the "rule of the like \( \alpha \)" (see sect. 7), is that all their Laplace originals are rational.

The expressions for the form factors (6.7) in the Laplace form are as follows:

\[ \Gamma_1(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_1(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.2) \]

\[ \Gamma_2(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_2(u_1, u_2, u_3)e^{\sum u_\Box} \]

\[ + \frac{\Box_1}{3} \int_0^\infty d^3u (u_2 + u_3)^{-1}e^{\sum u_\Box}, \quad (8.3) \]

\[ \Gamma_3(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_3(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.4) \]

\[ \Gamma_4(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_4(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.5) \]

\[ \Gamma_5(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_5(u_1, u_2, u_3)e^{\sum u_\Box} \]

\[ - \frac{\Box_1}{6} \int_0^\infty d^3u (u_1 + u_3)^{-1}e^{\sum u_\Box} \]

\[ - \frac{\Box_1\Box_3}{4} \int_0^\infty d^3u e^{\sum u_\Box} \]

\[ + \frac{\Box_3^2}{24} \int_0^\infty d^3u e^{\sum u_\Box}. \quad (8.6) \]
\[ \Gamma_6(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_6(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.7) \]

\[ \Gamma_7(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_7(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.8) \]

\[ \Gamma_8(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_8(u_1, u_2, u_3)e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_2}{3} \int_0^\infty d^3u (u_2 + u_3)^{-3}(-u_2^2 - 4u_2u_3 - u_3^2)e^{\sum u_\Box}, \quad (8.9) \]

\[ \Gamma_9(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_9(u_1, u_2, u_3)e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_1}{720} \int_0^\infty d^3u \left[(u_2 + u_3)^{-1} \right. \]
\[ \quad \left. + 4(u_1 + u_2)^{-4}(-5u_1^3 + 20u_1^2u_2 - 2u_1u_2^2) \right] e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_1^2}{2160} \int_0^\infty d^3u e^{\sum u_\Box}, \quad (8.10) \]

\[ \Gamma_{10}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_{10}(u_1, u_2, u_3)e^{\sum u_\Box} \]
\[ \quad - \frac{\Box_1^2}{270} \int_0^\infty d^3u e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_1^2}{540} \int_0^\infty d^3u e^{\sum u_\Box}, \quad (8.11) \]

\[ \Gamma_{11}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_{11}(u_1, u_2, u_3)e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_3}{1080} \int_0^\infty d^3u \left[9(u_1 + u_2)^{-1} \right. \]
\[ \quad \left. + (u_1 + u_3)^{-5}(-18u_1^4 - 103u_1^3u_3 - 57u_1^2u_3^2 \right. \]
\[ \quad \left. - 33u_1u_3^3 - 13u_3^4 \right] e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_1}{1080} \int_0^\infty d^3u (u_1 + u_3)^{-5}(-u_1^4 + 153u_1^3u_3 + 165u_1^2u_3^2 \right. \]
\[ \quad \left. + 71u_1u_3^3 + 12u_3^4)e^{\sum u_\Box} \]
\[ \quad + \frac{\Box_3^2}{540} \int_0^\infty d^3u e^{\sum u_\Box}, \quad (8.12) \]

\[ \Gamma_{12}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \rho_{12}(u_1, u_2, u_3)e^{\sum u_\Box}, \quad (8.13) \]
\[\Gamma_{13}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{13}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.14)\]

\[\Gamma_{14}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{14}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.15)\]

\[\Gamma_{15}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{15}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.16)\]

\[\Gamma_{16}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{16}(u_1, u_2, u_3) e^{\sum u^\Box} - \frac{\Box_3}{6} \int_0^\infty d^3 u e^{\sum u^\Box}, \quad (8.17)\]

\[\Gamma_{17}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{17}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.18)\]

\[\Gamma_{18}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{18}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.19)\]

\[\Gamma_{19}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{19}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.20)\]

\[\Gamma_{20}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{20}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.21)\]

\[\Gamma_{21}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{21}(u_1, u_2, u_3) e^{\sum u^\Box}, \quad (8.22)\]

\[\Gamma_{22}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{22}(u_1, u_2, u_3) e^{\sum u^\Box} + \frac{\Box_2}{270} \int_0^\infty d^3 u e^{\sum u^\Box}, \quad (8.23)\]

\[\Gamma_{23}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{23}(u_1, u_2, u_3) e^{\sum u^\Box} - \frac{\Box_3}{135} \int_0^\infty d^3 u e^{\sum u^\Box}, \quad (8.24)\]

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\[
\Gamma_{24}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{24}(u, 2u, 3) e^{\sum_u}
- \frac{\Box_1}{540} \int_0^\infty d^3 u e^{\sum_u}, \quad (8.25)
\]

\[
\Gamma_{25}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{25}(u, u, 3) e^{\sum_u}
- \frac{\Box_1}{270} \int_0^\infty d^3 u e^{\sum_u}
+ \frac{\Box_2}{135} \int_0^\infty d^3 u e^{\sum_u}, \quad (8.26)
\]

\[
\Gamma_{26}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{26}(u, 2u, 3) e^{\sum_u}, \quad (8.27)
\]

\[
\Gamma_{27}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{27}(u, u, 3) e^{\sum_u}, \quad (8.28)
\]

\[
\Gamma_{28}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{28}(u, 2u, 3) e^{\sum_u}, \quad (8.29)
\]

\[
\Gamma_{29}(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3 u \rho_{29}(u, 2u, 3) e^{\sum_u}, \quad (8.30)
\]

where

\[
\rho_1 = \frac{1}{3} [u_1 u_2 + u_1 u_3 + u_2 u_3]^{-1}, \quad (8.31)
\]

\[
\rho_2 = \frac{1}{3} [(u_1 u_2 + u_1 u_3 + u_2 u_3)^4]^{-1} (4u_1^2 u_2^2 u_3^2), \quad (8.32)
\]

\[
\rho_3 = [(u_1 u_2 + u_1 u_3 + u_2 u_3)^3]^{-1} (2u_1 u_2 u_3^2), \quad (8.33)
\]

\[
\rho_4 = \frac{1}{36} [(u_1 u_2 + u_1 u_3 + u_2 u_3)^5]^{-1} (u_1^4 u_2^4 - 10u_1^4 u_2^3 u_3
+ 6u_1^4 u_2^2 u_3^2 + 6u_1^3 u_2^3 u_3^2 + 14u_1^4 u_2 u_3^3 + 6u_1^3 u_2^2 u_3^3
- 4u_1^4 u_3^4 + 20u_1^3 u_2 u_3^4 - 12u_1^2 u_2 u_3^4), \quad (8.34)
\]
\[
\rho_5 = \frac{1}{6} \left[ (u_1 + u_3)^2 (u_2 + u_3)^2 (u_1 u_2 + u_1 u_3 + u_2 u_3)^3 \right]^{-1} \\
\times \left( 3u_1^3 u_2^3 u_3^2 + 10u_1^3 u_2^2 u_3^3 + 5u_1^3 u_2 u_3^4 + 6u_1^2 u_2^2 u_3^4 \\
+ u_1^3 u_3^5 + 3u_1^2 u_2 u_3^5 \right), \\
(8.35)
\]

\[
\rho_6 = \frac{1}{6} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^3 \right]^{-1} \left( -u_1^2 u_2^2 + 2u_1^2 u_2 u_3 \\
- 2u_1^2 u_3^2 + 4u_1 u_2 u_3^2 \right), \\
(8.36)
\]

\[
\rho_7 = \frac{1}{12} \left[ (u_2 + u_3)^2 (u_1 u_2 + u_1 u_3 + u_2 u_3)^5 \right]^{-1} \left( -8u_1^4 u_2^5 u_3 \\
+ 16u_1^4 u_2^4 u_3^2 - 4u_1^3 u_2^5 u_3^2 + 24u_1^4 u_2^3 u_3^3 - 12u_1^3 u_2^4 u_3^3 \\
- 4u_1^2 u_2^5 u_3^3 - 4u_1^2 u_2^4 u_3^4 - 10u_1 u_2^5 u_3^4 - u_2^5 u_3^5 \right), \\
(8.37)
\]

\[
\rho_8 = \left[ (u_2 + u_3)^4 (u_1 u_2 + u_1 u_3 + u_2 u_3)^5 \right]^{-1} \left( -4u_1^3 u_2^7 u_3^2 \\
+ 20u_1^3 u_2^6 u_3^3 - 4u_1^2 u_2^7 u_3^3 + 80u_1^3 u_2^5 u_3^4 + 24u_1^2 u_2^4 u_3^4 \\
+ 28u_1^2 u_2^5 u_3^5 \right), \\
(8.38)
\]

\[
\rho_9 = \frac{1}{540} \left[ (u_1 + u_2)^4 (u_1 + u_3)^4 (u_2 + u_3)^4 (u_1 u_2 + u_1 u_3 + u_2 u_3)^7 \right]^{-1} \\
\times \left( 10u_1^{14} u_2^{10} + 40u_1^{13} u_2^{11} + 3u_1^{12} u_2^{12} - 50u_1^{14} u_2^9 u_3 \\
- 250u_1^{13} u_2^{10} u_3 - 1094u_1^{12} u_2^{11} u_3 - 30u_1^{14} u_2^8 u_3^2 \\
+ 250u_1^{13} u_2^9 u_3^2 - 1700u_1^{12} u_2^{10} u_3^2 - 2088u_1^{11} u_2^{11} u_3^2 \\
+ 240u_1^{14} u_2^7 u_3^3 + 4020u_1^{13} u_2^8 u_3^3 + 7830u_1^{12} u_2^9 u_3^3 \\
+ 2910u_1^{11} u_2^{10} u_3^3 + 180u_1^{14} u_2^6 u_3^4 + 6900u_1^{13} u_2^7 u_3^4 \\
+ 20070u_1^{12} u_2^8 u_3^4 + 11340u_1^{11} u_2^9 u_3^4 - 2172u_1^{10} u_2^{10} u_3^4 \\
- 30u_1^{14} u_2^5 u_3^5 + 5040u_1^{13} u_2^6 u_3^5 + 10176u_1^{12} u_2^7 u_3^5 \\
- 47400u_1^{11} u_2^8 u_3^5 - 160176u_1^{10} u_2^9 u_3^5 - 1464u_1^{12} u_2^6 u_3^6 \\
- 153500u_1^{11} u_2^7 u_3^6 - 523172u_1^{10} u_2^8 u_3^6 \\
- 377568u_1^9 u_2^9 u_3^6 - 371690u_1^{10} u_2^7 u_3^7 \\
- 1452596u_1^9 u_2^8 u_3^7 - 330957u_1^8 u_2^8 u_3^7 \right), \\
(8.39)
\]

\[
\rho_{10} = \frac{1}{3} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \left( u_1^2 u_2^2 u_3^2 \right), \\
(8.40)
\]
\[ \rho_{11} = \frac{1}{90} \left[ (u_1 + u_3)^3 (u_2 + u_3)^3 (u_1 u_2 + u_1 u_3 + u_2 u_3)^5 \right]^{-1} \]
\[ \times \left( -30 u_1^6 u_2^6 u_3^2 - 190 u_1^6 u_2^5 u_3^3 - 215 u_1^6 u_2^4 u_3^4 - 270 u_1^5 u_2^5 u_3^4 \\
- 106 u_1^6 u_2^3 u_3^5 - 505 u_1^5 u_2^4 u_3^5 - 28 u_1^6 u_2^2 u_3^6 - 143 u_1^5 u_2^3 u_3^6 \\
- 170 u_1^4 u_2^4 u_3^6 - 8 u_1^6 u_2 u_3^7 + 21 u_1^5 u_2^2 u_3^7 - u_1^6 u_3^8 \\
+ 11 u_1^5 u_2 u_3^8 + 60 u_1^4 u_2^2 u_3^8 + 30 u_1^3 u_2^3 u_3^8 + 2 u_1^5 u_3^9 \\
+ 10 u_1^4 u_2 u_3^9 + 20 u_1^3 u_2^2 u_3^9 \right), \quad (8.41) \]

\[ \rho_{12} = \frac{1}{3} \left[ (u_1 + u_2) (u_1 + u_3) (u_1 u_2 + u_1 u_3 + u_2 u_3)^3 \right]^{-1} \]
\[ \times \left( u_1^4 u_2^3 + 3 u_1^4 u_2^2 u_3 + 4 u_1^3 u_2^3 u_3 + 3 u_1^4 u_2 u_3^2 \\
+ 12 u_1^3 u_2^2 u_3^2 + 9 u_1^2 u_2^3 u_3^2 + 3 u_1^4 u_3^3 + 4 u_1^3 u_2 u_3^3 \\
+ 9 u_1^2 u_2^2 u_3^3 + 6 u_1 u_2 u_3^3 \right), \quad (8.42) \]

\[ \rho_{13} = \left[ (u_2 + u_3) (u_1 u_2 + u_1 u_3 + u_2 u_3) \right]^{-1} (2 u_2 u_3), \quad (8.43) \]

\[ \rho_{14} = \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^2 \right]^{-1} (2 u_1 u_2 u_3), \quad (8.44) \]

\[ \rho_{15} = \frac{1}{3} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \left( u_1^2 u_2^3 u_3^2 \\
+ u_1^2 u_2^2 u_3^3 - 2 u_1^2 u_2 u_3^3 \right), \quad (8.45) \]

\[ \rho_{16} = \frac{1}{3} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^3 \right]^{-1} \left( -2 u_1^3 u_2 u_3 \\
- 2 u_1^3 u_3^2 - 6 u_1^2 u_2 u_3^2 \right), \quad (8.46) \]

\[ \rho_{17} = \left[ (u_2 + u_3) (u_1 u_2 + u_1 u_3 + u_2 u_3)^2 \right]^{-1} (u_2^2 u_3^2), \quad (8.47) \]

\[ \rho_{18} = \frac{1}{3} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \left( -2 u_1^4 u_2^3 - 10 u_1^4 u_2^2 u_3 \\
- 8 u_1^3 u_2^3 u_3 - 16 u_1^3 u_2^2 u_3^2 - 18 u_1^2 u_2^3 u_3^2 - 6 u_1 u_2^3 u_3^3 \right), \quad (8.48) \]
\[ \rho_{19} = \frac{1}{6} \left( (u_2 + u_3)(u_1u_2 + u_1u_3 + u_2u_3)^4 \right)^{-1} \left( 2u_1^4u_2^4 \right. \\
+ 12u_1^4u_2^3u_3 + 8u_1^3u_2^4u_3 + 16u_1^4u_2^2u_3^2 + 40u_1^3u_2^3u_3^2 \\
+ 12u_1^2u_2^4u_3^2 + 12u_1^2u_2^3u_3^3 + 8u_1u_2^4u_3^3 + u_2^4u_3^4 \left), \right. \]
\[ (8.49) \]

\[ \rho_{20} = \frac{1}{6} \left( (u_2 + u_3)(u_1u_2 + u_1u_3 + u_2u_3)^4 \right)^{-1} \x ( -4u_1^3u_2^4u_3 - 2u_1^2u_2^4u_3^2 - 2u_1^2u_2^3u_3^3 - u_2^4u_3^4 \right), \]
\[ (8.50) \]

\[ \rho_{21} = \frac{1}{3} \left[ (u_2 + u_3)(u_1u_2 + u_1u_3 + u_2u_3)^4 \right]^{-1} \x ( -2u_1^4u_2^4 - 12u_1^4u_2^3u_3 - 8u_1^3u_2^4u_3 - 20u_1^4u_2^2u_3^2 \\
- 40u_1^3u_2^3u_3^2 - 24u_1^2u_2^4u_3^2 - 12u_1^2u_2^3u_3^3 - 40u_1^3u_2^2u_3^3 \\
- 36u_1^2u_2^3u_3^3 - 8u_1u_2^4u_3^3 - 2u_1^2u_3^4 + 2u_1^3u_2^3u_3^4 - 8u_1^2u_2^3u_3^4 - 2u_2^4u_3^4 \right), \]
\[ (8.51) \]

\[ \rho_{22} = \frac{1}{90} \left[ (u_1 + u_2)(u_1 + u_3)(u_2 + u_3)(u_1u_2 + u_1u_3 + u_2u_3)^6 \right]^{-1} \x ( -3u_1^8u_2^6 - 3u_1^7u_2^7 - 3u_1^6u_2^8 - 18u_1^5u_2^5u_3 \\
- 39u_1^7u_2^6u_3 - 39u_1^6u_2^7u_3 - 18u_1^5u_2^8u_3 - 45u_1^4u_2^4u_3^2 \\
- 153u_1^7u_2^5u_3^2 - 218u_1^6u_2^6u_3^2 - 164u_1^5u_2^7u_3^2 - 51u_1^4u_2^8u_3^2 \\
- 30u_1^4u_2^6u_3^3 - 285u_1^7u_2^5u_3^3 - 17u_1^6u_2^7u_3^3 - 644u_1^5u_2^8u_3^3 \\
- 330u_1^4u_2^7u_3^3 - 78u_1^3u_2^8u_3^3 - 435u_1^6u_2^4u_3^4 - 1286u_1^5u_2^5u_3^4 \\
- 996u_1^4u_2^6u_3^4 - 403u_1^5u_2^7u_3^4 - 63u_1^2u_2^8u_3^4 - 717u_1^4u_2^5u_3^5 \\
- 845u_1^3u_2^6u_3^5 - 239u_1^2u_2^7u_3^5 - 24u_1^2u_2^8u_3^5 - 175u_1^2u_2^6u_3^6 \\
- 42u_1u_2^7u_3^6 - 3u_2^8u_3^6 - 2u_2^7u_3^7 \right), \]
\[ (8.52) \]

\[ \rho_{23} = \frac{1}{90} \left[ (u_1 + u_2)(u_1u_2 + u_1u_3 + u_2u_3)^5 \right]^{-1} \x ( 6u_1^6u_2^4 + 3u_1^5u_2^5 + 28u_1^6u_2^3u_3 + 58u_1^5u_2^4u_3 \\
+ 42u_1^6u_2^2u_3^2 + 152u_1^5u_2^3u_3^2 + 110u_1^4u_2^4u_3^2 + 24u_1^6u_2u_3^3 \\
+ 124u_1^5u_2^2u_3^3 + 300u_1^4u_2^3u_3^3 + 4u_1^6u_3^4 + 24u_1^5u_2u_3^4 \\
+ 60u_1^4u_2^2u_3^4 + 40u_1^3u_2^3u_3^4 \right), \]
\[ (8.53) \]
\[ \rho_{24} = \frac{1}{30} \left[ (u_2 + u_3)(u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \]
\[ \times \left( 2u_1^3 u_2^5 + 10u_1^3 u_2^4 u_3 + 8u_1^2 u_2^5 u_3 + 20u_1^3 u_2^3 u_3^2 \right. \]
\[ + 32u_1^2 u_2^4 u_3^2 + 8u_1 u_2^5 u_3^2 + 24u_1^2 u_2^3 u_3^3 + 16u_1 u_2^4 u_3^3 \]
\[ + 2u_2^5 u_3^3 + u_2^4 u_3^4 \right), \quad (8.54) \]

\[ \rho_{25} = \frac{1}{15} \left[ (u_1 + u_2)(u_1 + u_3)(u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \]
\[ \times \left( 2u_1^6 u_2^3 + 10u_1^6 u_2^2 u_3 + 12u_1^5 u_2^3 u_3 + 20u_1^6 u_2^2 u_3^2 \right. \]
\[ + 30u_1^4 u_2^3 u_3^2 + 10u_1^3 u_2^3 u_3^3 \right), \quad (8.55) \]

\[ \rho_{26} = \frac{1}{6} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^3 \right]^{-1} \left( u_1^3 u_2^3 + 6u_1^2 u_2^2 u_3 \right. \]
\[ \left. + 4u_1^3 u_2 u_3^2 + 6u_1^2 u_2^2 u_3^2 \right), \quad (8.56) \]

\[ \rho_{27} = \frac{1}{540} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^5 \right]^{-1} \left( u_1^5 u_2^5 + 10u_1^5 u_2^4 u_3 \right. \]
\[ + 50u_1^5 u_2^3 u_3^2 + 20u_1^5 u_2^4 u_3^2 + 62u_1^5 u_2^2 u_3^3 + 210u_1^4 u_2^3 u_3^3 \]
\[ + 22u_1^5 u_2 u_3^3 + 100u_1^4 u_2^2 u_3^4 + 150u_1^3 u_2^3 u_3^4 + 2u_1^5 u_3^5 \right. \]
\[ \left. + 10u_1^4 u_2^3 u_3^5 + 20u_1^3 u_2^2 u_3^5 \right), \quad (8.57) \]

\[ \rho_{28} = \frac{1}{135} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \left( 2u_1^4 u_2^4 + 16u_1^4 u_2^3 u_3 \right. \]
\[ + 6u_1^4 u_2^2 u_3^2 + 24u_1^3 u_2^3 u_3^2 - 8u_1^4 u_2 u_3^3 - 24u_1^3 u_2^2 u_3^3 \]
\[ - 2u_1^4 u_3^4 - 8u_1^3 u_2 u_3^4 - 6u_1^2 u_2^2 u_3^4 \right), \quad (8.58) \]

\[ \rho_{29} = \frac{1}{45} \left[ (u_1 u_2 + u_1 u_3 + u_2 u_3)^4 \right]^{-1} \left( - u_1^4 u_2^3 u_3 - 4u_1^4 u_2^3 u_3^2 \right. \]
\[ - 4u_1^3 u_2^2 u_3^3 - 4u_1^3 u_2^3 u_3^3 \right). \quad (8.59) \]

Although the nondecreasing form factors are not completely presentable in the Laplace form, the explicit dependence on the ∇’s outside the kernel exp(∑u∇) is purely local.

Apart from the nondecreasing terms which are elementary, the Laplace representation is unique. The calculations with the form factors can, therefore, be carried out in terms of the Laplace originals but if, in these calculations, the form factors are multiplied by new
powers of $\Box$’s, the whole procedure of absorbing these multipliers in the Laplace originals and detaching the nondecreasing terms should be repeated. This situation is encountered in the calculation of the trace anomaly (sect. 12), and it makes the Laplace representation inconvenient for this calculation. In addition, this representation is well adapted only to the euclidean signature of the metric. When going over to the lorentzian signature, setting the retarded boundary conditions [1,4] for the kernel $\exp(\sum u\Box)$ is embarrassing. Only in the spectral representation are all the drawbacks removed.

9. The generalized spectral representation of the third-order form factors in the effective action

Spectral representation of the form factors is of special importance for applications because it allows going over to the lorentzian signature in the expectation-value equations [1,4]. The generalized spectral representation retains this quality and, in addition, makes it possible to overcome difficulties connected with the discontinuous nature of the spectral weight in the third-order form factors [3] (see sect. 20).

In the generalized spectral representation, there is one extra integration over a parameter entering the spectral weight. Namely, for each of the arguments $\Box_1, \Box_2, \Box_3$ in the triple form factors we introduce the following spectral integral

$$S(y, \Box) = \frac{y}{2} \int_0^\infty dm \frac{J_1(ym)}{m^2 - \Box}$$

(9.1)

depending on a parameter $y$, where $J_1$ is the order-1 Bessel function. Next, we introduce the operator

$$C = y^2 \frac{\partial}{\partial y^2}$$

(9.2)

and denote, for short,

$$S_n = S(y_n, \Box_n), \quad C_n = y_n^2 \frac{\partial}{\partial y_n^2}, \quad n = 1, 2, 3.$$  

(9.3)

The third-order form factors $\Gamma_i$ are then expressed through integrals of the form

$$2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P(C_1, C_2, C_3)S_1S_2S_3$$

(9.4)

where $P(C_1, C_2, C_3)$ is a polynomial, and it is understood that $C_1, C_2, C_3$ act on $S_1, S_2, S_3$ respectively with subsequently setting $y_1 = y_2 = y_3 = y$.

The spectral representation (both generalized and ordinary) is sensitive to the behaviour of a function at small arguments. Only the functions that behave in each argument like $\mathcal{O}/\Box, \mathcal{O} \to 0$ at $\Box \to -0$ admit this representation. * In fact, as pointed out in sect. 7, the form factors $\Gamma_1$ to $\Gamma_{11}$ contain the coefficients $\Box_n/\Box_m$ which cause the $1/\Box_m$ behaviour.

*The exception to this rule pointed out in sect. 20 does not concern the triple and double spectral forms used here.
at small $\Box_m$ (and the $\ln(-\Box_n)$ behaviour at large $\Box_n$). In the spectral technique, the terms with these coefficients get detached and take the form

$$\frac{\Box_n}{\Box_m} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 P(C_n, C_k) S_n S_k, \quad k \neq m, \quad k \neq n. \quad (9.5)$$

The double-spectral integral in (9.5) gives the coefficient of the $1/\Box_m$ asymptotic behaviour as a function of $\Box_n$ and $\Box_k$. The total form factor is a sum of the triple-spectral contributions (9.4), double-spectral contributions (9.5), and tree terms.

The form factors $\Gamma_{12}$ to $\Gamma_{29}$ are given below in their redefined versions, with the overall $1/\Box$ factors (see sect. 7). Since, in this version, $\Gamma_{22}$ is a sum of two form factors, it has two contributions of the form (9.4) with the polynomials $P_{22}^{-1}$ and $P_{22}^{-2}$.

The expressions for the form factors (6.7) in the generalized spectral form are as follows:

$$\Gamma_1(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_1(C_1, C_2, C_3) S_1 S_2 S_3, \quad (9.6)$$

$$\Gamma_2(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_2(C_1, C_2, C_3) S_1 S_2 S_3$$
$$+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{1}{3} C_1 C_2 \right) S_1 S_2, \quad (9.7)$$

$$\Gamma_3(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_3(C_1, C_2, C_3) S_1 S_2 S_3, \quad (9.8)$$

$$\Gamma_4(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_4(C_1, C_2, C_3) S_1 S_2 S_3, \quad (9.9)$$

$$\Gamma_5(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_5(C_1, C_2, C_3) S_1 S_2 S_3$$
$$+ \frac{\Box_1}{\Box_2} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( \frac{1}{24} C_1 C_3 \right) S_1 S_3$$
$$+ \frac{1}{4 \Box_2} - \frac{\Box_3}{24 \Box_1 \Box_2}, \quad (9.10)$$

$$\Gamma_6(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_6(C_1, C_2, C_3) S_1 S_2 S_3, \quad (9.11)$$

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\[ \Gamma_7(-\square_1, -\square_2, -\square_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_7(C_1, C_2, C_3)S_1S_2S_3, \quad (9.12) \]

\[ \Gamma_8(-\square_1, -\square_2, -\square_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_8(C_1, C_2, C_3)S_1S_2S_3 \]
\[ + \frac{\Box_2}{\Box_1} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( \frac{1}{3}C_2C_3 - \frac{7}{24}C_2^2C_3 \right. \]
\[ + \frac{1}{24}C_2^3C_3 - \frac{7}{24}C_2C_3^2 + \frac{1}{6}C_2^2C_3^2 + \frac{1}{24}C_2C_3^3 \left) S_2S_3, \quad (9.13) \]

\[ \Gamma_9(-\square_1, -\square_2, -\square_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_9(C_1, C_2, C_3)S_1S_2S_3 \]
\[ + \frac{\Box_1}{\Box_2} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{1}{120}C_1C_3 + \frac{17}{2160}C_1^2C_3 \right. \]
\[ + \frac{1}{2160}C_1^3C_3 + \frac{1}{4320}C_1C_3^2 - \frac{1}{80}C_1^2C_3^2 - \frac{1}{2160}C_1^3C_3^2 \]
\[ + \frac{1}{432}C_1C_3^3 + \frac{1}{216}C_1^2C_3^3 - \frac{1}{864}C_1C_3^4 \right) S_1S_3 \]
\[ + \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{1}{120}C_1C_2 \right) S_1S_2 \]
\[ - \frac{\Box_1}{2160\Box_2\Box_3}, \quad (9.14) \]

\[ \Gamma_{10}(-\square_1, -\square_2, -\square_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{10}(C_1, C_2, C_3)S_1S_2S_3 \]
\[ + \frac{1}{270\Box_3} - \frac{\Box_1}{540\Box_2\Box_3}, \quad (9.15) \]

\[ \Gamma_{11}(-\square_1, -\square_2, -\square_3) = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{11}(C_1, C_2, C_3)S_1S_2S_3 \]
\[ + \frac{\Box_1}{\Box_2} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{7}{320}C_1C_3 + \frac{3289}{103680}C_1^2C_3 \right. \]
\[ - \frac{49}{4320}C_1^3C_3 + \frac{191}{103680}C_1^4C_3 - \frac{1}{8640}C_1^5C_3 + \frac{3049}{103680}C_1C_3^2 \]
\[ - \frac{3949}{103680}C_1^2C_3^2 + \frac{307}{34560}C_1^3C_3^2 - \frac{71}{103680}C_1^4C_3^2 \]
\[- \frac{1213}{103680} C_1 C_3^3 + \frac{157}{11520} C_1^2 C_3^3 - \frac{11}{6912} C_1^3 C_3^3 \]
\[+ \frac{143}{103680} C_1 C_3^4 - \frac{17}{11520} C_1^2 C_3^4 + \frac{1}{103680} C_1 C_3^5 \]  
\[S_1 S_3 \]
\[+ \frac{\Box_3}{\Box_1} \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^2 \left( \frac{149}{8640} C_2 C_3 - \frac{1973}{103680} C_2^2 C_3 \right) \]
\[+ \frac{767}{103680} C_2^3 C_3 - \frac{163}{103680} C_2^4 C_3 + \frac{13}{103680} C_2^5 C_3 \]
\[+ \frac{C_2 C_3^2}{2573} - \frac{1009}{103680} C_2^2 C_3^2 - \frac{41}{11520} C_2^3 C_3^2 \]
\[+ \frac{11}{34560} C_2^4 C_3^2 + \frac{17280}{34560} C_2^3 C_3^3 - \frac{263}{34560} C_2^2 C_3^3 \]
\[+ \frac{19}{34560} C_2^3 C_3^3 - \frac{103}{103680} C_2 C_3^4 \]
\[+ \frac{C_2^2 C_3^4}{103} + \frac{1}{5760} C_2^3 C_3^5 \]  
\[S_2 S_3 \]
\[+ \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^2 \left( - \frac{1}{120} C_1 C_2 \right) S_1 S_2 \]
\[- \frac{\Box_3}{540 \Box_1 \Box_2} \]  
\[= (9.16) \]

\[\Gamma_{12}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} 2 \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^3 \]  
\[P_{12}(C_1, C_2, C_3) S_1 S_2 S_3, \]  
\[= (9.17) \]

\[\Gamma_{13}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ 2 \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^3 \right. \]  
\[P_{13}(C_1, C_2, C_3) S_1 S_2 S_3 \]
\[+ \left. \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^2 \left( -2 C_2 C_3 \right) S_2 S_3 \right], \]  
\[= (9.18) \]

\[\Gamma_{14}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_3} 2 \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^3 \]  
\[P_{14}(C_1, C_2, C_3) S_1 S_2 S_3, \]  
\[= (9.19) \]

\[\Gamma_{15}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} 2 \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^3 \]  
\[P_{15}(C_1, C_2, C_3) S_1 S_2 S_3, \]  
\[= (9.20) \]

\[\Gamma_{16}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} 2 \int_0^{\infty} dy^2 \left( \frac{4}{y^2} \right)^3 \]  
\[P_{16}(C_1, C_2, C_3) S_1 S_2 S_3 \]
\[+ \frac{1}{6 \Box_1 \Box_2}, \]  
\[= (9.21) \]

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\[ \Gamma_{17}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{17}(C_1, C_2, C_3) S_1 S_2 S_3 \\
+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 (-C_2 C_3) S_2 S_3 \right], \quad (9.22) \]

\[ \Gamma_{18}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{18}(C_1, C_2, C_3) S_1 S_2 S_3, \quad (9.23) \]

\[ \Gamma_{19}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{19}(C_1, C_2, C_3) S_1 S_2 S_3 \\
+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{1}{6} C_2 C_3 \right) S_2 S_3 \right], \quad (9.24) \]

\[ \Gamma_{20}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{20}(C_1, C_2, C_3) S_1 S_2 S_3 \\
+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( \frac{1}{6} C_2 C_3 \right) S_2 S_3 \right], \quad (9.25) \]

\[ \Gamma_{21}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{21}(C_1, C_2, C_3) S_1 S_2 S_3 \\
+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( \frac{2}{3} C_2 C_3 \right) S_2 S_3 \right], \quad (9.26) \]

\[ \Gamma_{22}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} \left[ 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{22-1}(C_1, C_2, C_3) S_1 S_2 S_3 \\
+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{1}{90} C_2 C_3 \right) S_2 S_3 \right] \\
+ \frac{1}{\Box_2} \left[ 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{22-2}(C_1, C_2, C_3) S_1 S_2 S_3 \\
+ \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 \left( -\frac{1}{30} C_1 C_2 \right) S_1 S_2 \right] \\
- \frac{1}{270\Box_1\Box_3}, \quad (9.27) \]
\[\Gamma_{23}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{23}(C_1, C_2, C_3) S_1 S_2 S_3\]
\[+ \frac{1}{135 \Box_1 \Box_2}, \quad (9.28)\]
\[\Gamma_{24}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_2} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{24}(C_1, C_2, C_3) S_1 S_2 S_3\]
\[+ \frac{1}{540 \Box_2 \Box_3}, \quad (9.29)\]
\[\Gamma_{25}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{25}(C_1, C_2, C_3) S_1 S_2 S_3\]
\[- \frac{1}{135 \Box_1 \Box_3} + \frac{1}{270 \Box_2 \Box_3}, \quad (9.30)\]
\[\Gamma_{26}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{26}(C_1, C_2, C_3) S_1 S_2 S_3, \quad (9.31)\]
\[\Gamma_{27}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{27}(C_1, C_2, C_3) S_1 S_2 S_3\]
\[- \frac{1}{540 \Box_1 \Box_2 \Box_3}, \quad (9.32)\]
\[\Gamma_{28}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{28}(C_1, C_2, C_3) S_1 S_2 S_3\]
\[+ \frac{1}{135 \Box_1 \Box_2 \Box_3}, \quad (9.33)\]
\[\Gamma_{29}(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1 \Box_2 \Box_3} 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 P_{29}(C_1, C_2, C_3) S_1 S_2 S_3 \quad (9.34)\]

where
\[P_1 = \frac{1}{3} C_1 C_2 C_3, \quad (9.35)\]
\[ P_2 = \frac{2}{9} C_1 C_2 C_3 - \frac{2}{3} C_1^2 C_2 C_3 + \frac{2}{3} C_1^2 C_2^2 C_3 - \frac{2}{9} C_1^2 C_2 C_3^2, \quad (9.36) \]

\[ P_3 = C_1 C_2 C_3 - C_1^2 C_2 C_3 - C_1 C_2^2 C_3 + C_1^2 C_2^2 C_3, \quad (9.37) \]

\[ P_4 = \frac{11}{108} C_1 C_2 C_3 - \frac{5}{36} C_1^2 C_2 C_3 - \frac{1}{108} C_1^3 C_2 C_3 - \frac{19}{216} C_1^2 C_2^2 C_3 + \frac{11}{54} C_1^3 C_2^2 C_3 - \frac{1}{24} C_1^3 C_2^3 C_3 - \frac{13}{108} C_1 C_2 C_3^2 + \frac{17}{108} C_1^2 C_2 C_3^2 - \frac{1}{12} C_1^3 C_2 C_3^2 \]
\[ + \frac{11}{108} C_1^2 C_2^2 C_3^2 - \frac{1}{12} C_1^3 C_2^2 C_3^2, \quad (9.38) \]

\[ P_5 = -\frac{1}{6} C_1 C_3 + \frac{1}{3} C_1^2 C_3 - \frac{1}{6} C_1^3 C_3 \]
\[ + \frac{1}{9} C_1 C_2 C_3 - \frac{2}{9} C_1^2 C_2 C_3 + \frac{1}{18} C_1^3 C_2 C_3 + \frac{1}{18} C_1^2 C_2^2 C_3 \]
\[ + \frac{1}{9} C_1 C_2 C_3^2 - \frac{1}{9} C_1^2 C_2 C_3^2, \quad (9.39) \]

\[ P_6 = -\frac{1}{6} C_1 C_2 C_3 + \frac{1}{2} C_1^2 C_2 C_3 - \frac{1}{2} C_1^3 C_2 C_3, \quad (9.40) \]

\[ P_7 = \frac{1}{12} C_2 C_3 - \frac{3}{8} C_1 C_2 C_3 + \frac{5}{24} C_1^2 C_2 C_3 - \frac{1}{6} C_2^2 C_3 \]
\[ + \frac{7}{6} C_1 C_2^2 C_3 - \frac{1}{2} C_1^2 C_2^2 C_3 + \frac{1}{12} C_2^3 C_3 - \frac{7}{12} C_1 C_2^3 C_3 \]
\[ + \frac{1}{12} C_1^2 C_2 C_3^2 + \frac{1}{12} C_1 C_2^2 C_3^2 - \frac{19}{24} C_1 C_2^2 C_3^2 + \frac{7}{24} C_1^2 C_2^2 C_3^2 \]
\[ + \frac{7}{12} C_1 C_2^3 C_3^2 - \frac{1}{12} C_1^2 C_2^3 C_3^2 - \frac{1}{12} C_1 C_2^4 C_3^2, \quad (9.41) \]

\[ P_8 = -2 C_2 C_3 + \frac{5}{3} C_1 C_2 C_3 + \frac{1}{6} C_1^2 C_2 C_3 - \frac{1}{3} C_1^3 C_2 C_3 \]
\[ + 6 C_2^2 C_3 - \frac{14}{3} C_1 C_2^2 C_3 + \frac{1}{6} C_1^2 C_2^2 C_3 + \frac{1}{3} C_1^3 C_2 C_3 \]
\[ - \frac{11}{3} C_3^2 C_3 + 3 C_1 C_2^3 C_3 - \frac{1}{3} C_1^2 C_2^3 C_3 + \frac{5}{6} C_2^4 C_3 \]
\[ - \frac{2}{3} C_1 C_2^4 C_3 - \frac{8}{3} C_2^2 C_3^2 + 2 C_1 C_2^2 C_3^2 - \frac{1}{3} C_1^2 C_2^2 C_3^2 \]
\[ + \frac{11}{6} C_3^2 C_3^2 - \frac{5}{3} C_1 C_2^3 C_3^2 + \frac{1}{3} C_1^2 C_2^3 C_3^2 - \frac{1}{3} C_2^4 C_3 \]
\[ + \frac{1}{3} C_1 C_2^4 C_3^2, \quad (9.42) \]
\[ P_9 = \frac{13}{72} C_1 C_2 + \frac{1801}{2160} C_1^2 C_2 + \frac{3209}{4320} C_1^3 C_2 \]
\[ + \frac{265}{864} C_1^4 C_2 - \frac{43}{864} C_1^5 C_2 + \frac{1}{4320} C_1^6 C_2 - \frac{1117}{1440} C_1^2 C_2^2 \]
\[ + \frac{4639}{4320} C_1^3 C_2^2 - \frac{31}{96} C_1^4 C_2^2 + \frac{47}{1440} C_1^5 C_2^2 \]
\[ + \frac{1}{864} C_1^6 C_2^2 + \frac{67}{240} C_1^3 C_2^3 + \frac{127}{1080} C_1^4 C_2^3 \]
\[ - \frac{1}{144} C_1^5 C_2^3 + \frac{7}{864} C_1^4 C_2^4 + \frac{77}{720} C_1 C_2 C_3 \]
\[ - \frac{4859}{12960} C_1^2 C_2 C_3 + \frac{5783}{12960} C_1^3 C_2 C_3 - \frac{631}{2592} C_1^4 C_2 C_3 \]
\[ + \frac{43}{864} C_1^5 C_2 C_3 - \frac{1}{240} C_1^6 C_2 C_3 + \frac{7}{108} C_1^2 C_2^2 C_3 \]
\[ - \frac{59}{240} C_1^3 C_2^2 C_3 + \frac{1553}{12960} C_1^4 C_2^2 C_3 - \frac{47}{4320} C_1^5 C_2^2 C_3 \]
\[ + \frac{1}{2592} C_1^6 C_2^2 C_3 - \frac{1}{32} C_1^3 C_2^3 C_3 + \frac{1}{360} C_1^5 C_2^3 C_3 \]
\[ + \frac{1}{720} C_1^4 C_2^3 C_3 - \frac{373}{3240} C_1^2 C_2^2 C_3^2 - \frac{41}{810} C_1^3 C_2^2 C_3^2 \]
\[ - \frac{1}{12960} C_1^4 C_2^2 C_3^2 + \frac{19}{12960} C_1^3 C_2 C_3^2 + \frac{1}{720} C_1^4 C_2 C_3^2 \]
\[ + \frac{1}{1080} C_1^3 C_2^3 C_3^3, \quad (9.43) \]

\[ P_{10} = \frac{1}{18} C_1 C_2 C_3 - \frac{1}{6} C_1^2 C_2 C_3 + \frac{1}{6} C_1^2 C_2^2 C_3 - \frac{1}{18} C_1^2 C_2 C_3^2, \quad (9.44) \]

\[ P_{11} = \frac{59}{540} C_1 C_3 - \frac{53}{216} C_1^2 C_3 + \frac{47}{270} C_1^3 C_3 \]
\[ - \frac{37}{1080} C_1^4 C_3 - \frac{1}{540} C_1^5 C_3 - \frac{29}{3240} C_1 C_2 C_3 \]
\[ + \frac{163}{6480} C_1^2 C_2 C_3 - \frac{449}{12960} C_1^3 C_2 C_3 + \frac{7}{12960} C_1^4 C_2 C_3 \]
\[ + \frac{1}{360} C_1^5 C_2 C_3 + \frac{119}{6480} C_1^2 C_2^2 C_3 - \frac{7}{12960} C_1^3 C_2^2 C_3 \]
\[ + \frac{1}{648} C_1^4 C_2^2 C_3 + \frac{1}{4320} C_1^3 C_2^3 C_3 - \frac{19}{12960} C_1^4 C_2^3 C_3 \]
\[ - \frac{11}{72} C_1 C_3^2 + \frac{319}{1080} C_1^2 C_3^2 - \frac{109}{540} C_1^3 C_3^2 \]
\[ + \frac{17}{540} C_1^4 C_3^2 - \frac{1}{270} C_1^5 C_3^2 + \frac{31}{2160} C_1 C_2 C_3^2 \]
\[ - \frac{373}{6480} C_1^2 C_2 C_3^2 + \frac{83}{1728} C_1^3 C_2 C_3^2 + \frac{47}{5184} C_1^4 C_2 C_3^2 \]

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\[-\frac{1}{360} C_1^5 C_2 C_3^2 + \frac{319}{12960} C_1^2 C_2^2 C_3^2 - \frac{223}{8640} C_1^3 C_2^2 C_3^2 \]
\[+ \frac{5}{5184} C_1^4 C_2^2 C_3^2 + \frac{13}{4320} C_1^3 C_2^3 C_3^2 + \frac{83}{1080} C_1^4 C_3^3 \]
\[-\frac{103}{1080} C_1^2 C_3^3 + \frac{73}{1080} C_1^3 C_3^3 - \frac{1}{360} C_1^4 C_3^3 \]
\[-\frac{1}{540} C_1^5 C_3^3 - \frac{349}{12960} C_1 C_2 C_3^3 + \frac{127}{8640} C_1^2 C_2 C_3^3 \]
\[-\frac{197}{25920} C_1^3 C_2^2 C_3^3 - \frac{1}{216} C_1^4 C_2 C_3^3 - \frac{13}{960} C_1^2 C_2^2 C_3^3 \]
\[+ \frac{193}{25920} C_1^3 C_2 C_3^3 - \frac{1}{60} C_1^4 C_3^3 + \frac{7}{1080} C_1^2 C_3^4 \]
\[-\frac{1}{72} C_1^3 C_3^4 + \frac{1}{60} C_1 C_2 C_3^4 + \frac{1}{180} C_1^2 C_2 C_3^4 \]
\[+ \frac{1}{540} C_1^3 C_3^5 - \frac{1}{540} C_1 C_2 C_3^5 - \frac{1}{540} C_1^2 C_2 C_3^5 , \] (9.45)

\[P_{12} = -\frac{1}{3} C_1 C_2 + \frac{2}{3} C_1^2 C_2 - \frac{1}{3} C_1^3 C_2 \]
\[-\frac{1}{3} C_1 C_3 + \frac{2}{3} C_1^2 C_3 - \frac{1}{3} C_1^3 C_3 - \frac{4}{3} C_1 C_2 C_3 \]
\[+ \frac{10}{3} C_1^2 C_2 C_3 - \frac{8}{3} C_1^3 C_2 C_3 + \frac{2}{3} C_1^4 C_2 C_3 , \] (9.46)

\[P_{13} = 2C_1 C_2 C_3 - 2C_1^2 C_2 C_3 , \] (9.47)

\[P_{14} = -2C_1 C_2 C_3 + 4C_1 C_2 C_3^2 - 2C_1 C_2 C_3^3 , \] (9.48)

\[P_{15} = \frac{2}{3} C_1 C_2 C_3 - 2C_1^2 C_2 C_3 + \frac{13}{6} C_1^3 C_2 C_3 \]
\[- C_1^4 C_2 C_3 + \frac{1}{6} C_1^5 C_2 C_3 , \] (9.49)

\[P_{16} = \frac{2}{9} C_1 C_2 C_3 - \frac{2}{9} C_1^2 C_2 C_3 - \frac{4}{9} C_1 C_2^2 C_3 \]
\[+ \frac{2}{3} C_1 C_2 C_3^2 - \frac{2}{9} C_1^3 C_2 C_3^2 + \frac{2}{3} C_1 C_2 C_3^2 \]
\[- \frac{8}{9} C_1^2 C_2 C_3^2 + \frac{2}{9} C_1^3 C_2 C_3^2 , \] (9.50)
\[ P_{17} = 2C_1C_2C_3 - 3C_1^2C_2C_3 + C_1^3C_2C_3, \quad (9.51) \]

\[ P_{18} = \frac{4}{3} C_1C_2^2C_3 - 2C_1^2C_2^2C_3 + \frac{2}{3} C_1^3C_2^2C_3 + \frac{2}{3} C_1C_2^3C_3 \]

\[ - C_1^2C_2^2C_3^2 + \frac{1}{3} C_1^3C_2^2C_3^2, \quad (9.52) \]

\[ P_{19} = -\frac{1}{3} C_1C_2C_3 + \frac{1}{2} C_1^2C_2C_3 - \frac{1}{6} C_1^3C_2C_3 \]

\[ + \frac{2}{3} C_1C_2^2C_3 - C_1^2C_2^2C_3 + \frac{1}{3} C_1^3C_2C_3 \]

\[ - \frac{1}{3} C_1C_2^2C_3^2 + \frac{1}{2} C_1^2C_2^2C_3^2 - \frac{1}{6} C_1^3C_2^2C_3^2, \quad (9.53) \]

\[ P_{20} = -\frac{1}{36} C_1C_2C_3 - \frac{1}{24} C_1^2C_2C_3 + \frac{5}{72} C_1^3C_2C_3 - \frac{11}{36} C_1C_2^2C_3 \]

\[ + \frac{5}{9} C_1^2C_2^2C_3 - \frac{1}{4} C_1^3C_2^2C_3 - \frac{1}{12} C_1C_2^3C_3 + \frac{1}{12} C_1^2C_2^3C_3 \]

\[ - \frac{2}{9} C_1C_2^2C_3^2 + \frac{3}{8} C_1^2C_2^2C_3^2 - \frac{11}{72} C_1^3C_2^2C_3^2 - \frac{1}{36} C_1C_2^3C_3^2 \]

\[ + \frac{1}{36} C_1^2C_2^3C_3^2, \quad (9.54) \]

\[ P_{21} = \frac{4}{3} C_1C_2C_3^2 - 2C_1^2C_2C_3^2 + \frac{2}{3} C_1^3C_2C_3^2 \]

\[ - \frac{4}{3} C_1C_2C_3^3 + 2C_1^2C_2C_3^3 - \frac{2}{3} C_1^3C_2C_3^3, \quad (9.55) \]

\[ P_{22-1} = \frac{22}{135} C_1C_2C_3 - \frac{1}{4} C_1^2C_2C_3 + \frac{47}{540} C_1^3C_2C_3 \]

\[ - \frac{46}{135} C_1C_2^2C_3 + \frac{241}{540} C_1^2C_2^2C_3 - \frac{19}{180} C_1^3C_2^2C_3 \]

\[ + \frac{53}{270} C_1C_2^3C_3 - \frac{34}{135} C_1^2C_2^3C_3 + \frac{1}{18} C_1^3C_2^3C_3 \]

\[ - \frac{1}{540} C_1C_2^5C_3 + \frac{1}{540} C_1^2C_2^5C_3 + \frac{28}{135} C_1C_2^3C_3^2 \]

\[ - \frac{73}{270} C_1C_2^2C_3^2 + \frac{17}{270} C_1^3C_2^2C_3^2 - \frac{2}{135} C_1C_2^3C_3^2 \]

\[ + \frac{2}{135} C_1^2C_2^3C_3^2 + \frac{1}{540} C_1C_2^3C_3^3 - \frac{1}{540} C_1^3C_2^3C_3^3, \quad (9.56) \]
\[ P_{22-2} = \frac{19}{90} C_1 C_2 C_3 - \frac{1}{36} C_1^2 C_2 C_3 - \frac{4}{135} C_1^3 C_2 C_3 \]
\[ - \frac{17}{45} C_1 C_2^2 C_3 + \frac{71}{360} C_1^2 C_2^2 C_3 - \frac{1}{72} C_1^3 C_2^2 C_3 \]
\[ + \frac{227}{1080} C_1 C_2^3 C_3 - \frac{29}{540} C_1^2 C_2^3 C_3 + \frac{7}{1080} C_1^3 C_2^3 C_3 \]
\[ - \frac{1}{24} C_1 C_2^4 C_3 + \frac{11}{1080} C_1^2 C_2^4 C_3 - \frac{1}{540} C_1^3 C_2^5 C_3 \]
\[ - \frac{59}{270} C_1 C_2 C_3^2 - \frac{1}{10} C_1^2 C_2 C_3^2 + \frac{7}{90} C_1^3 C_2 C_3^2 \]
\[ + \frac{529}{1080} C_1 C_2^2 C_3^2 - \frac{11}{54} C_1^2 C_2^2 C_3^2 + \frac{1}{216} C_1^3 C_2^2 C_3^2 \]
\[ - \frac{131}{540} C_1 C_2^3 C_3^2 + \frac{17}{360} C_1^2 C_2^3 C_3^2 - \frac{1}{1080} C_1^3 C_2^3 C_3^2 \]
\[ + \frac{47}{1080} C_1 C_2^4 C_3^2 - \frac{11}{1080} C_1^2 C_2^4 C_3^2 + \frac{1}{540} C_1^3 C_2^5 C_3^2 \]
\[ - \frac{1}{90} C_1 C_2 C_3^3 + \frac{7}{45} C_1^2 C_2 C_3^3 - \frac{31}{540} C_1^3 C_2 C_3^3 \]
\[ - \frac{109}{1080} C_1 C_2^2 C_3^3 - \frac{11}{1080} C_1^2 C_2^2 C_3^3 + \frac{2}{135} C_1^3 C_2^2 C_3^3 \]
\[ + \frac{7}{216} C_1 C_2^3 C_3^3 + \frac{11}{1080} C_1^2 C_2^3 C_3^3 - \frac{1}{180} C_1^3 C_2^3 C_3^3 \]
\[ - \frac{1}{540} C_1 C_2^4 C_3^3 + \frac{1}{54} C_1 C_2 C_3^4 - \frac{1}{36} C_1^2 C_2 C_3^4 \]
\[ + \frac{1}{108} C_1^3 C_2 C_3^4 - \frac{1}{90} C_1 C_2^2 C_3^4 + \frac{1}{60} C_1^2 C_2^2 C_3^4 \]
\[ - \frac{1}{180} C_1^3 C_2^2 C_3^4, \quad (9.57) \]

\[ P_{23} = - \frac{3}{5} C_1 C_2 + \frac{1021}{540} C_1 C_2^2 - \frac{1219}{540} C_1^3 C_2 \]
\[ + \frac{233}{180} C_1^4 C_2 - \frac{23}{60} C_1^5 C_2 + \frac{8}{135} C_1^6 C_2 \]
\[ - \frac{1}{270} C_1 C_2^2 C_3 - \frac{277}{540} C_1 C_2^2 C_3 - \frac{173}{135} C_1^2 C_2 C_3 \]
\[ + \frac{113}{108} C_1 C_2 C_3^2 - \frac{3}{10} C_1^2 C_2 C_3^2 + \frac{1}{45} C_1^3 C_2 C_3^2 \]
\[ - \frac{17}{108} C_1 C_2^3 C_3 - \frac{29}{90} C_1^2 C_2^3 C_3 - \frac{89}{540} C_1^3 C_2^3 C_3 \]
\[ - \frac{1}{135} C_1^4 C_2 C_3 - \frac{1}{135} C_1^5 C_2 C_3 - \frac{1}{27} C_1^6 C_2 \]
\[ - \frac{2}{27} C_1^2 C_2^4 + \frac{1}{27} C_1^3 C_2^4 - \frac{1}{270} C_1^4 C_2 \]
\[ + \frac{1}{135} C_1^2 C_2^5 - \frac{1}{270} C_1^3 C_2^5 + \frac{43}{135} C_1 C_2 C_3 \]
\[
\begin{align*}
&\frac{101}{135} C_1^2 C_2 C_3 + \frac{16}{27} C_1^3 C_2 C_3 - \frac{26}{135} C_1^4 C_2 C_3 \\
&+ \frac{4}{135} C_1^5 C_2 C_3 - \frac{41}{90} C_1 C_2^2 C_3 + \frac{161}{270} C_1^2 C_2^2 C_3 \\
&- \frac{5}{54} C_1^3 C_2^2 C_3 - \frac{7}{135} C_1^4 C_2 C_3 + \frac{1}{270} C_1^5 C_2 C_3 \\
&+ \frac{41}{135} C_1 C_2^3 C_3 - \frac{47}{135} C_1^2 C_2^3 C_3 + \frac{1}{30} C_1^3 C_2^3 C_3 \\
&+ \frac{1}{90} C_1^4 C_2^3 C_3 - \frac{2}{27} C_1 C_2^4 C_3 + \frac{2}{27} C_1^2 C_2^4 C_3 \\
&+ \frac{1}{135} C_1 C_2^5 C_3 - \frac{1}{135} C_1^2 C_2^5 C_3 + \frac{1}{9} C_1^3 C_2^5 C_3 \\
&- \frac{133}{540} C_1^2 C_2 C_3^2 + \frac{79}{540} C_1^3 C_2 C_3^2 - \frac{7}{540} C_1^4 C_2 C_3^2 \\
&+ \frac{1}{540} C_1^5 C_2 C_3^2 + \frac{1}{105} C_1^2 C_2^3 C_3^2 + \frac{1}{15} C_1^2 C_2^3 C_3^2 \\
&- \frac{41}{540} C_1^3 C_2^2 C_3^2 - \frac{19}{540} C_1 C_2^3 C_3^2 + \frac{1}{45} C_1^2 C_2^3 C_3^2 \\
&+ \frac{7}{540} C_1^3 C_2^3 C_3^2 - \frac{23}{270} C_1 C_2 C_3^3 + \frac{14}{135} C_1 C_2 C_3^3 \\
&- \frac{1}{54} C_1^3 C_2 C_3^3 + \frac{1}{30} C_1^2 C_2 C_3^3 - \frac{1}{27} C_1^2 C_2 C_3^3 \\
&+ \frac{1}{270} C_1^3 C_2^2 C_3^3 - \frac{90}{270} C_1^2 C_2^2 C_3^3 + \frac{1}{90} C_1^2 C_2^2 C_3^3 \\
&+ \frac{1}{54} C_1 C_2 C_3^4 - \frac{1}{45} C_1^2 C_2 C_3^4 + \frac{1}{240} C_1^3 C_2 C_3^4 \\
&- \frac{1}{270} C_1^2 C_2 C_3^4 + \frac{1}{270} C_1^2 C_2 C_3^4,
\end{align*}
\]

\[
P_{24} = \frac{1}{90} C_2 C_3 - \frac{7}{135} C_1 C_2 C_3 - \frac{13}{540} C_1^2 C_2 C_3 \\
+ \frac{1}{90} C_1^4 C_2 C_3 - \frac{1}{90} C_2^2 C_3 + \frac{1}{20} C_1 C_2 C_3^2 + \frac{13}{540} C_1^2 C_2 C_3 \\
- \frac{1}{90} C_1^4 C_2 C_3 - \frac{1}{90} C_2^3 C_3 + \frac{1}{540} C_1 C_2 C_3^3 + \frac{1}{90} C_1^2 C_3 \\
+ \frac{90}{270} C_2 C_3^2 - \frac{13}{540} C_1 C_2 C_3^2 - \frac{1}{45} C_2^2 C_3^2 + \frac{13}{540} C_1 C_2 C_3^2 \\
+ \frac{1}{90} C_2 C_3^2 + \frac{1}{90} C_1 C_2 C_3^3 - \frac{1}{90} C_1^2 C_2 C_3^3 - \frac{1}{90} C_1 C_2 C_3^3 \\
+ \frac{1}{90} C_1 C_2^2 C_3^3,
\]

\[
P_{25} = -\frac{5}{27} C_1 C_2 + \frac{191}{270} C_1^2 C_2 - \frac{44}{45} C_1^3 C_2 + \frac{53}{90} C_1^4 C_2
\]

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\[-\frac{4}{27} C_1^5 C_2 + \frac{2}{135} C_1^6 C_2 + \frac{1}{10} C_1 C_2^2 - \frac{31}{135} C_1^2 C_2^2 \]
\[+ \frac{43}{270} C_1^3 C_2^2 - \frac{4}{135} C_1^4 C_2^2 + \frac{4}{135} C_1 C_2^3 \]
\[+ \frac{2}{27} C_1^2 C_2^3 + \frac{8}{135} C_1^3 C_2^3 - \frac{2}{135} C_1^4 C_2^3 \]
\[+ \frac{7}{30} C_1 C_2 C_3 - \frac{58}{135} C_1^2 C_2 C_3 + \frac{53}{270} C_1^3 C_2 C_3 \]
\[+ \frac{2}{135} C_1^4 C_2 C_3 - \frac{2}{135} C_1^5 C_2 C_3 - \frac{22}{45} C_1 C_2^2 C_3 \]
\[+ \frac{8}{15} C_1^2 C_2^2 C_3 - \frac{2}{135} C_1^3 C_2^2 C_3 - \frac{4}{135} C_1^4 C_2^2 C_3 \]
\[+ \frac{2}{9} C_1 C_2^3 C_3 - \frac{4}{15} C_1^2 C_2^3 C_3 + \frac{2}{45} C_1^3 C_2^3 C_3 \]
\[+ \frac{2}{15} C_1 C_2^2 C_3^2 - \frac{2}{15} C_1^2 C_2^2 C_3^2 - \frac{2}{45} C_1 C_2^3 C_3^2 \]
\[+ \frac{2}{45} C_1^2 C_2^3 C_3^2, \quad (9.60) \]

\[P_{26} = \frac{2}{3} C_1 C_2 C_3 - 2 C_1^2 C_2 C_3 + \frac{2}{3} C_1^3 C_2 C_3 \]
\[+ \frac{3}{2} C_1 C_2^2 C_3 - C_1^3 C_2^2 C_3 + \frac{1}{6} C_1 C_2^3 C_3, \quad (9.61) \]

\[P_{27} = \frac{4}{135} C_1 C_2 C_3 - \frac{17}{135} C_1^2 C_2 C_3 + \frac{7}{90} C_1^3 C_2 C_3 \]
\[+ \frac{1}{135} C_1^4 C_2 C_3 - \frac{1}{135} C_1^5 C_2 C_3 + \frac{37}{270} C_1^2 C_2^2 C_3 \]
\[+ \frac{97}{540} C_1^3 C_2^2 C_3 + \frac{7}{270} C_1^4 C_2^2 C_3 + \frac{1}{180} C_1^5 C_2^2 C_3 \]
\[+ \frac{7}{108} C_1^3 C_2^3 C_3 - \frac{7}{270} C_1^4 C_2^3 C_3 - \frac{1}{540} C_1^5 C_2^3 C_3 \]
\[+ \frac{1}{270} C_1^4 C_2^4 C_3 + \frac{1}{45} C_1 C_2 C_3^2 - \frac{1}{15} C_1^2 C_2^2 C_3^2 \]
\[+ \frac{1}{45} C_1^3 C_2 C_3^2 + \frac{1}{20} C_1^2 C_2^2 C_3^2 - \frac{1}{30} C_1^3 C_2^2 C_3^2 \]
\[+ \frac{1}{180} C_1^3 C_2^3 C_3^2 - \frac{1}{135} C_1 C_2 C_3^3 + \frac{1}{45} C_1^2 C_2 C_3^3 \]
\[+ \frac{1}{135} C_1^3 C_2 C_3^3 - \frac{1}{60} C_1^2 C_2^2 C_3^3 + \frac{1}{90} C_1^3 C_2^2 C_3^3 \]
\[+ \frac{1}{540} C_1^3 C_2^3 C_3^3, \quad (9.62) \]

\[P_{28} = \frac{4}{45} C_1 C_2 C_3 - \frac{4}{15} C_1^2 C_2 C_3 + \frac{4}{45} C_1^3 C_2 C_3 \]
\[\frac{1}{5}C_1^2C_2^2C_3 - \frac{2}{15}C_1^3C_2^2C_3 + \frac{1}{45}C_1^3C_2^3C_3\]
\[-\frac{4}{45}C_1C_2C_3^2 + \frac{4}{15}C_1^2C_2C_3^2 - \frac{4}{45}C_1^3C_2C_3^2\]
\[-\frac{1}{5}C_1^2C_2^2C_3^2 + \frac{2}{15}C_1^3C_2^2C_3^2 - \frac{1}{45}C_1^3C_2^3C_3^2,\] (9.63)

\[P_{29} = \frac{4}{135}C_1C_2C_3 - \frac{2}{15}C_1^2C_2C_3 + \frac{2}{45}C_1^3C_2C_3\]
\[+ \frac{1}{5}C_1^3C_2^2C_3 - \frac{1}{15}C_1^3C_2C_3 - \frac{1}{15}C_1^2C_2^3C_3\]
\[+ \frac{1}{45}C_1^3C_2^3C_3 - \frac{1}{10}C_1^3C_2^2C_3^2 + \frac{1}{10}C_1^3C_2^2C_3^2\]
\[-\frac{1}{30}C_1^3C_2^3C_3^2 + \frac{1}{270}C_1^3C_2^3C_3^3.\] (9.64)

For calculations within the generalized spectral technique, it is important to know the asymptotic behaviours of the integral (9.1). They are as follows:

\[S = -\frac{1}{2\Box} + O\left(e^{-\sqrt{-\Box}}\right), \quad y \to \infty,\] (9.65)

and

\[S = S^M + O\left(y^{2(M+1)}\ln y^2\right), \quad y \to 0,\] (9.66)

\[S^M = \sum_{m=1}^{M} \frac{(y^2)^m(-\Box)^{m-1}}{4^m(m!)^2} \left[ \frac{m}{2} \ln \left(-\frac{1}{4}y^2\Box\right) - m\psi(m+1) + \frac{1}{2} \right], \quad M \geq 1\] (9.67)

where \(\psi(x)\) is the Euler \(\psi\)-function.

The generalized spectral representation can be extended to products of the form factors with any positive powers of \(\Box\)’s. This is achieved by a repeated use of the relation

\[\Box S = -\frac{1}{2} - \frac{4}{y^2}(C - 1)CS\] (9.68)

along with the commutation rule

\[P(C) \left(\frac{1}{y^2}\right)^N = \left(\frac{1}{y^2}\right)^N P(C - N)\] (9.69)

valid for any polynomial \(P(C)\). The result for a product of \(\Box_1^{M_1}\Box_2^{M_2}\Box_3^{M_3}\) with the integrals in (9.4) or (9.5) is then again a sum of integrals of the form

\[\int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^N P(C, \ldots C) S_{\underline{\underline{\cdots}}_K}.\] (9.70)
Owing to this property, the generalized spectral representation solves the notorious problem of nonuniqueness. An example showing how it makes manifest the hidden identities between the form factors will be considered in sect. 12. The only arbitrariness that remains in this representation corresponds to the possibility of integration by parts over $y^2$ and is expressed by the identity

$$C_1 + C_2 + C_3 = 2 \quad (9.71)$$

in the integral (9.4),

$$C_n + C_k = 1 \quad (9.72)$$

in the integral (9.5), and

$$\sum C = N - 1 \quad (9.73)$$

in the integral (9.70). This arbitrariness is removed by excluding everywhere a $C$ with any chosen (but one and the same) index.

The formal use of the identity (9.73) is safe if, in eq. (9.70), $N \leq K$, as is the case in the expressions for the form factors above. However, in the case of products of the form factors with the positive powers of $\Box$’s, the integral (9.70) may appear with $N > K$. The polynomial $P(C, \ldots C)$ will then contain the factors

$$\prod_{p=1}^{M} (C - p)^2 \quad (9.74)$$

improving the asymptotic behaviour of the respective $S$ so that the integral (9.70) converge at the lower limit. The $S$ on which the operator (9.74) acts can be replaced by $(S - S^M)$ with $S^M$ in (9.67). Only after this replacement has been made, can the identity (9.73) be applied to exclude $C$ in the operator (9.74).

The last limitation is connected with the fact that $S$ does not decrease at $y \to \infty$ but the action of at least one $C$ makes it decreasing exponentially. One must, therefore, ensure that at least one $S$ in the integral (9.70) be accompanied by at least one like $C$. In the expressions for the form factors above, this condition is always fulfilled.

For the details and derivations see sect. 20.

10. **The large-$\Box$ asymptotic behaviours of the third-order form factors**

After varying of one of the curvatures in the effective action (6.1), the respective $\Box$ argument of the form factor will become distinguished for it will refer to the observation point of the current. Obviously, each of the $\Box$ arguments will, in its turn, find itself in this role while the other arguments will refer to the points of internal integrations. Therefore, of interest are the asymptotic behaviours of the form factors $\Gamma_i$ in one (each) of the three arguments $\Box_1, \Box_2, \Box_3$ with the two others fixed. Here we present such asymptotic behaviours of $\Gamma_i$ at $\Box_m \to -\infty$, and in the next section at $\Box_m \to -0$ $(m = 1, 2, 3)$.

Although the total dimension of the form factors $\Gamma_1$ to $\Gamma_{11}$ is $\Box^{-1}$, in individual arguments they grow, generally, like $\Box_m^{m+1}$ and $\ln(-\Box_m)$, $\Box_m \to -\infty$. The growth is, however, present only in the gravitational form factors; the nongravitational ones tend to a constant at $\Box_m \to$
The constant in the asymptotic behaviour, as a function of two other □'s, is either a tree or the function (7.31a).

In the asymptotic expressions below, O denotes decreasing terms. The behaviour of the form factors $\Gamma_1$ to $\Gamma_{11}$ is obtained with this accuracy. The behaviour of the form factors $\Gamma_{12}$ to $\Gamma_{29}$ is obtained with a higher accuracy so that for the redefined form factors of the standard dimension the accuracy be O. Such a redefinition is discussed in sect. 7, and the form in which the results for $\Gamma_{12}$ to $\Gamma_{29}$ are presented below corresponds to the form of the exact expressions in sect. 7.

The asymptotic expressions for the form factors (6.7) at large ($-\Box_m$) are as follows:

\[
\Gamma_{1\text{sym}} = O, \quad \Box_1 \to -\infty \quad \text{or} \quad \Box_2 \to -\infty \quad \text{or} \quad \Box_3 \to -\infty \quad (10.1)
\]

\[
\Gamma_{2\text{sym}} = \begin{cases} 
\ln(\Box_2/\Box_3)/9(\Box_2-\Box_3) + O, & \Box_1 \to -\infty \\
\ln(\Box_1/\Box_3)/9(\Box_1-\Box_3) + O, & \Box_2 \to -\infty \\
\ln(\Box_1/\Box_2)/9(\Box_1-\Box_2) + O, & \Box_3 \to -\infty
\end{cases} \quad (10.2)
\]

\[
\Gamma_{3\text{sym}} = O, \quad \Box_1 \to -\infty \quad \text{or} \quad \Box_2 \to -\infty \quad \text{or} \quad \Box_3 \to -\infty \quad (10.3)
\]

\[
\Gamma_{4\text{sym}} = O, \quad \Box_1 \to -\infty \quad \text{or} \quad \Box_2 \to -\infty \quad \text{or} \quad \Box_3 \to -\infty \quad (10.4)
\]

\[
\Gamma_{5\text{sym}} = \begin{cases} 
-\ln(\Box_3/\Box_1)/12\Box_2 + \frac{1}{8\Box_2} + O, & \Box_1 \to -\infty \\
-\ln(\Box_2/\Box_1)/12\Box_3 + \frac{1}{8\Box_1} + O, & \Box_2 \to -\infty \\
-\Box_3/24\Box_1\Box_2 + \frac{1}{8\Box_1} + \frac{1}{8\Box_2} + O, & \Box_3 \to -\infty
\end{cases} \quad (10.5)
\]

\[
\Gamma_{6\text{sym}} = O, \quad \Box_1 \to -\infty \quad \text{or} \quad \Box_2 \to -\infty \quad \text{or} \quad \Box_3 \to -\infty \quad (10.6)
\]

\[
\Gamma_{7\text{sym}} = O, \quad \Box_1 \to -\infty \quad \text{or} \quad \Box_2 \to -\infty \quad \text{or} \quad \Box_3 \to -\infty \quad (10.7)
\]

\[
\Gamma_{8\text{sym}} = \begin{cases} 
O, & \Box_1 \to -\infty \\
-\ln(\Box_2/\Box_3)/6\Box_1 - \frac{1}{6\Box_1} + O, & \Box_2 \to -\infty \\
-\ln(\Box_3/\Box_2)/6\Box_1 - \frac{1}{6\Box_1} + O, & \Box_3 \to -\infty
\end{cases} \quad (10.8)
\]
\[
\Gamma_{\text{sym}}^{9} = \begin{cases} 
-\frac{\Box_1}{6480\Box_2\Box_3} + \frac{\ln(\Box_2/\Box_3)}{2160(\Box_2-\Box_3)} \\
+ \frac{1}{1080\Box_2} + \frac{1}{1080\Box_3} + O, \quad \Box_1 \to -\infty \\
-\frac{\Box_2}{6480\Box_1\Box_3} + \frac{\ln(\Box_1/\Box_3)}{2160(\Box_1-\Box_3)} \\
+ \frac{1}{1080\Box_1} + \frac{1}{1080\Box_3} + O, \quad \Box_2 \to -\infty \\
-\frac{\Box_3}{6480\Box_1\Box_2} + \frac{\ln(\Box_1/\Box_2)}{2160(\Box_1-\Box_2)} \\
+ \frac{1}{1080\Box_1} + \frac{1}{1080\Box_2} + O, \quad \Box_3 \to -\infty 
\end{cases}
\] (10.9)

\[
\Gamma_{\text{sym}}^{10} = \begin{cases} 
-\frac{\Box_1}{1620\Box_2\Box_3} + \frac{1}{810\Box_2} + \frac{1}{810\Box_3} + O, \quad \Box_1 \to -\infty \\
-\frac{\Box_2}{1620\Box_1\Box_3} + \frac{1}{810\Box_1} + \frac{1}{810\Box_3} + O, \quad \Box_2 \to -\infty \\
-\frac{\Box_3}{1620\Box_1\Box_2} + \frac{1}{810\Box_1} + \frac{1}{810\Box_2} + O, \quad \Box_3 \to -\infty 
\end{cases}
\] (10.10)

\[
\Gamma_{\text{sym}}^{11} = \begin{cases} 
\frac{\ln(\Box_1/\Box_3)}{180\Box_2} + \frac{19}{2160\Box_2} + O, \quad \Box_1 \to -\infty \\
\frac{\ln(\Box_2/\Box_3)}{180\Box_1} + \frac{19}{2160\Box_1} + O, \quad \Box_2 \to -\infty \\
-\frac{\Box_3}{540\Box_1\Box_2} - \frac{\ln(\Box_1/\Box_1)}{120\Box_2} - \frac{\ln(\Box_3/\Box_2)}{120\Box_1} \\
+ \frac{\ln(\Box_1/\Box_2)}{120(\Box_1-\Box_2)} + \frac{1}{2160\Box_1} + \frac{1}{2160\Box_2} + O, \\
\Box_3 \to -\infty
\end{cases}
\] (10.11)

\[
\Gamma_{\text{sym}}^{12} = \frac{1}{\Box_1} O, \quad \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty
\] (10.12)

\[
\Gamma_{\text{sym}}^{13} = \frac{1}{\Box_1} \left( 2\frac{\ln(\Box_2/\Box_3)}{(\Box_2-\Box_3)} + O \right), \\
\Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty
\] (10.13)

\[
\Gamma_{\text{sym}}^{14} = \frac{1}{\Box_3} O, \quad \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty
\] (10.14)

\[
\Gamma_{\text{sym}}^{15} = \frac{1}{\Box_1} O, \quad \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty
\] (10.15)
\[ \Gamma_{16}^{\text{sym}} = \frac{1}{\Box_1} \left( \frac{1}{12 \Box_2} + O \right) + (1 \leftrightarrow 2), \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.16)

\[ \Gamma_{17}^{\text{sym}} = \frac{1}{\Box_1} \left( \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)} + O \right), \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.17)

\[ \Gamma_{18}^{\text{sym}} = \frac{1}{\Box_1} O, \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.18)

\[ \Gamma_{19}^{\text{sym}} = \frac{1}{\Box_1} \left( \frac{1}{6} \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)} + O \right), \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.19)

\[ \Gamma_{20}^{\text{sym}} = \frac{1}{\Box_1} \left( -\frac{1}{6} \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)} + O \right), \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.20)

\[ \Gamma_{21}^{\text{sym}} = \frac{1}{\Box_1} \left( -\frac{2}{3} \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)} + O \right), \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.21)

\[ \Gamma_{22}^{\text{sym}} = \frac{1}{\Box_1} \left( \frac{1}{90} \frac{\ln(\Box_2/\Box_3)}{(\Box_2 - \Box_3)} + O \right) + \left[ \frac{1}{\Box_2} \left( \frac{1}{60} \frac{\ln(\Box_1/\Box_3)}{(\Box_1 - \Box_3)} - \frac{1}{540 \Box_1} + O \right) + (2 \leftrightarrow 3) \right], \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.22)

\[ \Gamma_{23}^{\text{sym}} = \frac{1}{\Box_1} \left( \frac{1}{270 \Box_2} + O \right) + (1 \leftrightarrow 2), \quad \Box_1 \rightarrow -\infty \text{ or } \Box_2 \rightarrow -\infty \text{ or } \Box_3 \rightarrow -\infty \] (10.23)
\[ \Gamma^{\text{sym}}_{24} = \frac{1}{\Box_2} \left( \frac{1}{1080 \Box_3} + O \right) + (2 \leftrightarrow 3), \]
\[ \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty \] (10.24)

\[ \Gamma^{\text{sym}}_{25} = \frac{1}{270 \Box_2 \Box_3} + \frac{1}{\Box_1} \left( -\frac{1}{270 \Box_2} - \frac{1}{270 \Box_3} + O \right), \]
\[ \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty \] (10.25)

\[ \Gamma^{\text{sym}}_{26} = \frac{1}{\Box_1 \Box_2} O, \quad \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty \] (10.26)

\[ \Gamma^{\text{sym}}_{27} = \frac{1}{\Box_1 \Box_2} \left( -\frac{1}{540 \Box_3} + O \right), \]
\[ \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty \] (10.27)

\[ \Gamma^{\text{sym}}_{28} = \frac{1}{\Box_1 \Box_2} \left( \frac{1}{135 \Box_3} + O \right), \]
\[ \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty \] (10.28)

\[ \Gamma^{\text{sym}}_{29} = \frac{1}{\Box_1 \Box_2 \Box_3} O, \]
\[ \Box_1 \to -\infty \text{ or } \Box_2 \to -\infty \text{ or } \Box_3 \to -\infty. \] (10.29)

For the derivation of these results see sect. 19. The nondecreasing terms in the asymptotic expressions above are in complete agreement with the ones appearing in the Laplace representation of the form factors (sect. 8).

11. The small-\(\Box\) asymptotic behaviours of the third-order form factors

The small-\(\Box\) asymptotic behaviours of the form factors will be the first among the present results used in the study of the gravitational expectation-value equations [4]. In an asymptotically flat space-time, they should determine the behaviour of the vacuum current at spatial and null infinities. The rate of the energy radiation by the gravitational collapse, as a nonlocal functional of the curvature [17], should, in the first place, follow from these results. Furthermore, to lowest order in the curvature this calculation has been carried out
for the spherically symmetric state, and it was shown that the Hawking stable component of the radiation is contained in the third-order form factors (see [5]).

These applications determine also the accuracy with which the asymptotic behaviours of the form factors should be calculated. Terms $O(1)$ at $\Box \to -0$ give already vanishing contributions at the asymptotically flat infinity. In fact, the gravitational form factors behave, generally, like $1/\Box$ and $\ln(-\Box)$ at $\Box \to -0$. The coefficients of these behaviours are functions of two other $\Box$’s, for which we introduce the following notations. In the form factor $\Gamma_i$, the coefficient of the $1/\Box$ behaviour will be denoted $A_i(\Box)$, and the coefficient of the $\ln(-\Box)$ behaviour $B_i(\Box)$. These functions are of the following general form ($m = 1, 2, 3$):

$$A_i(1) = a_i(1) \left| \frac{\ln(j_2 \Box_2 / j_3 \Box_3)}{(j_2 \Box_2 - j_3 \Box_3)} \right|_{j=1},$$

$$A_i(2) = a_i(2) \left| \frac{\ln(j_1 \Box_1 / j_3 \Box_3)}{(j_1 \Box_1 - j_3 \Box_3)} \right|_{j=1},$$

$$A_i(3) = a_i(3) \left| \frac{\ln(j_1 \Box_1 / j_2 \Box_2)}{(j_1 \Box_1 - j_2 \Box_2)} \right|_{j=1},$$

$$B_i(1) = b_i(1) \left| \frac{\ln(j_2 \Box_2 / j_3 \Box_3)}{(j_2 \Box_2 - j_3 \Box_3)} \right|_{j=1},$$

$$B_i(2) = b_i(2) \left| \frac{\ln(j_1 \Box_1 / j_3 \Box_3)}{(j_1 \Box_1 - j_3 \Box_3)} \right|_{j=1},$$

$$B_i(3) = b_i(3) \left| \frac{\ln(j_1 \Box_1 / j_2 \Box_2)}{(j_1 \Box_1 - j_2 \Box_2)} \right|_{j=1}$$

where $a_i(\Box)$ and $b_i(\Box)$ are polynomials in $\partial/\partial j$. The asymptotic expressions below are supplied with the forms of these polynomials. Some of the functions $A_i(\Box)$ or $B_i(\Box)$ vanish, and in addition to these functions, the coefficients of the leading asymptotic behaviours have tree contributions.

With $O$ denoting terms $O(1)$, the asymptotic expressions for the form factors (6.7) at small $\Box$ are as follows:

$$\Gamma_{\text{sym}}^i = \begin{cases} 
\ln(-\Box_1)B_1(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2)B_1(2) + O, & \Box_2 \to -0 \\
\ln(-\Box_3)B_1(3) + O, & \Box_3 \to -0 
\end{cases}$$

(11.7)
\[ b_1(1) = \frac{1}{3}, \quad (11.8) \]
\[ b_1(2) = \frac{1}{3}, \quad (11.9) \]
\[ b_1(3) = \frac{1}{3}, \quad (11.10) \]
\[ \Gamma_{\text{sym}}^2 = \begin{cases} 
\ln(-\Box_1) \left( - \frac{1}{9g_2} - \frac{1}{9g_3} \right) + O, & \Box_1 \to 0 \\
\ln(-\Box_2) \left( - \frac{1}{9g_1} - \frac{1}{9g_3} \right) + O, & \Box_2 \to 0 \\
\ln(-\Box_3) \left( - \frac{1}{9g_1} - \frac{1}{9g_2} \right) + O, & \Box_3 \to 0 
\end{cases} \quad (11.11) \]
\[ \Gamma_{\text{sym}}^3 = \begin{cases} 
O, & \Box_1 \to 0 \text{ or } \Box_2 \to 0 \\
\ln(-\Box_3) B_3(3) + O, & \Box_3 \to 0 
\end{cases} \quad (11.12) \]
\[ b_3(3) = \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2}, \quad (11.13) \]
\[ \Gamma_{\text{sym}}^4 = \begin{cases} 
\ln(-\Box_1) B_4(1) + O, & \Box_1 \to 0 \\
\ln(-\Box_2) B_4(2) + O, & \Box_2 \to 0 \\
\ln(-\Box_3) B_4(3) + O, & \Box_3 \to 0 
\end{cases} \quad (11.14) \]
\[ b_4(1) = -\frac{1}{36} \frac{\partial^2}{\partial j_2^2} - \frac{1}{36} \frac{\partial}{\partial j_3} \frac{\partial}{\partial j_2} - \frac{1}{9} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} - \frac{1}{12} \frac{\partial^2}{\partial j_2^2} \frac{\partial}{\partial j_3}, \quad (11.15) \]
\[ b_4(2) = -\frac{1}{36} \frac{\partial^2}{\partial j_1^2} - \frac{1}{36} \frac{\partial}{\partial j_3} \frac{\partial}{\partial j_1} - \frac{1}{9} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} - \frac{1}{12} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3}, \quad (11.16) \]
\begin{align}
\Gamma_{5}^{\text{sym}} &= \begin{cases}
\frac{1}{12} \left( \frac{1}{8} - \frac{A_5(1)}{24 \sqrt{2}} \right) + O, & \Box_1 \to -0 \\
\frac{1}{12} \left( \frac{1}{8} - \frac{A_5(2)}{24 \sqrt{2}} \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{1}{12 \sqrt{2}} + \frac{1}{12 \sqrt{2}} \right) + O, & \Box_3 \to -0
\end{cases} \\
&= \begin{cases}
\ln(-\Box_1) B_6(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) B_6(2) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) B_6(3) + O, & \Box_3 \to -0
\end{cases} \\
&= \begin{cases}
\ln(-\Box_1) B_6(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) B_6(2) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) B_6(3) + O, & \Box_3 \to -0
\end{cases} \\
&= \begin{cases}
\ln(-\Box_1) B_7(1) + O, & \Box_1 \to -0 \\
O, & \Box_2 \to -0 \text{ or } \Box_3 \to -0
\end{cases} \\
&= \begin{cases}
\ln(-\Box_1) B_7(1) + O, & \Box_1 \to -0 \\
O, & \Box_2 \to -0 \text{ or } \Box_3 \to -0
\end{cases}
\end{align}
\[ b_7(1) = \frac{1}{24} \frac{\partial}{\partial j_2} + \frac{1}{96} \frac{\partial^2}{\partial j_2^2} + \frac{1}{24} \frac{\partial}{\partial j_3} + \frac{19}{48} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} \]

\[ - \frac{1}{24} \frac{\partial^3}{\partial j_2^3} \frac{\partial}{\partial j_3} + \frac{1}{192} \frac{\partial^2}{\partial j_2^2} - \frac{1}{24} \frac{\partial}{\partial j_2} \frac{\partial^3}{\partial j_3^3} \]

\[ + \Box_3 \left( \frac{1}{96} \frac{\partial^2}{\partial j_2^2} - \frac{1}{96} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} \right) \]

\[ + \Box_2 \left( \frac{1}{96} \frac{\partial^2}{\partial j_2^2} + \frac{1}{96} \frac{\partial}{\partial j_2} \frac{\partial^2}{\partial j_3^2} \right), \quad (11.26) \]

\[ \Gamma_{8}^{\text{sym}} = \begin{cases} \frac{1}{12} A_8(1) + O, & \Box_1 \to -0 \\ \ln(-\Box_2) \left( \frac{1}{60} \right) + O, & \Box_2 \to -0 \\ \ln(-\Box_3) \left( \frac{1}{60} \right) + O, & \Box_3 \to -0 \end{cases} \quad (11.27) \]

\[ a_8(1) = \Box_3 \left( \frac{1}{12} + \frac{1}{4} \frac{\partial}{\partial j_2} + \frac{1}{4} \frac{\partial}{\partial j_3} - \frac{1}{6} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} \right) \]

\[ + \Box_2 \left( \frac{1}{12} + \frac{1}{4} \frac{\partial}{\partial j_2} + \frac{1}{4} \frac{\partial}{\partial j_3} - \frac{1}{6} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} \right), \quad (11.28) \]

\[ \Gamma_{9}^{\text{sym}} = \begin{cases} \frac{1}{12} \left( - \frac{1}{6480 \Box_2} - \frac{1}{6480 \Box_3} + A_9(1) \right) + \ln(-\Box_1) \left( \frac{1}{1296 \Box_2} \right) \\ + \frac{1}{1296 \Box_3} + B_9(1) \right) + O, & \Box_1 \to -0 \\ \frac{1}{12} \left( - \frac{1}{6480 \Box_2} - \frac{1}{6480 \Box_3} + A_9(2) \right) + \ln(-\Box_2) \left( \frac{1}{1296 \Box_2} \right) \\ + \frac{1}{1296 \Box_3} + B_9(2) \right) + O, & \Box_2 \to -0 \\ \frac{1}{12} \left( - \frac{1}{6480 \Box_2} - \frac{1}{6480 \Box_3} + A_9(3) \right) + \ln(-\Box_3) \left( \frac{1}{1296 \Box_2} \right) \\ + \frac{1}{1296 \Box_3} + B_9(3) \right) + O, & \Box_3 \to -0 \end{cases} \quad (11.29) \]

\[ a_9(1) = \Box_3 \left( \frac{1}{1296} \frac{\partial^3}{\partial j_2^3} - \frac{1}{324} \frac{\partial^2}{\partial j_2^2} \frac{\partial}{\partial j_3} + \frac{1}{3240} \frac{\partial}{\partial j_2} \frac{\partial^2}{\partial j_3^2} \right) \]

\[ + \Box_2 \left( \frac{1}{3240} \frac{\partial^2}{\partial j_2^2} \frac{\partial}{\partial j_3} - \frac{1}{324} \frac{\partial}{\partial j_2} \frac{\partial^2}{\partial j_3^2} + \frac{1}{1296} \frac{\partial^3}{\partial j_3^3} \right), \quad (11.30) \]
\[ b_3(1) = -\frac{49}{4665600} \frac{\partial^6}{\partial j_1^6} + \frac{1}{25920} \frac{\partial^5}{\partial j_1^5} \frac{\partial}{\partial j_2} + \frac{1}{20736} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_2^2} \]
\[-\frac{17}{72900} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_2^3} + \frac{1}{20736} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_2^4} + \frac{1}{25920} \frac{\partial}{\partial j_1} \frac{\partial^5}{\partial j_2^5} \]
\[-\frac{49}{4665600} \frac{\partial^6}{\partial j_3^6} \]
\+ \frac{\Box_3}{\Box_2} \left( -\frac{13}{1555200} \frac{\partial^6}{\partial j_2^6} + \frac{29}{518400} \frac{\partial^5}{\partial j_2^5} \frac{\partial}{\partial j_3} - \frac{17}{388800} \frac{\partial^4}{\partial j_2^4} \frac{\partial^2}{\partial j_3^2} \right)
\- \frac{1}{8100} \frac{\partial^3}{\partial j_2^3} \frac{\partial^3}{\partial j_3^3} + \frac{133}{1555200} \frac{\partial^2}{\partial j_2^2} \frac{\partial^4}{\partial j_3^4} - \frac{23}{1555200} \frac{\partial}{\partial j_2} \frac{\partial^5}{\partial j_3^5} \right)
\+ \frac{\Box_2}{\Box_3} \left( -\frac{23}{1555200} \frac{\partial^5}{\partial j_2^5} \frac{\partial}{\partial j_3} + \frac{133}{1555200} \frac{\partial^4}{\partial j_2^4} \frac{\partial^2}{\partial j_3^2} - \frac{1}{8100} \frac{\partial^3}{\partial j_2^3} \frac{\partial^3}{\partial j_3^3} \right)
\- \frac{17}{388800} \frac{\partial^2}{\partial j_2^2} \frac{\partial^4}{\partial j_3^4} + \frac{29}{518400} \frac{\partial}{\partial j_2} \frac{\partial^5}{\partial j_3^5} - \frac{13}{1555200} \frac{\partial^6}{\partial j_3^6} \right), \quad (11.31)\]

\[ a_3(2) = \frac{\Box_3}{\Box_2} \left( \frac{1}{1296} \frac{\partial^3}{\partial j_1^3} - \frac{1}{324} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_2} + \frac{1}{3240} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} \right) \]
\+ \frac{\Box_1}{\Box_2} \left( \frac{1}{324} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_2} - \frac{1}{324} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} + \frac{1}{1296} \frac{\partial^3}{\partial j_2^3} \right), \quad (11.32)\]

\[ b_3(2) = -\frac{49}{4665600} \frac{\partial^6}{\partial j_1^6} + \frac{1}{25920} \frac{\partial^5}{\partial j_1^5} \frac{\partial}{\partial j_2} + \frac{1}{20736} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_2^2} \]
\[-\frac{17}{72900} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_2^3} + \frac{1}{20736} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_2^4} + \frac{1}{25920} \frac{\partial}{\partial j_1} \frac{\partial^5}{\partial j_2^5} \]
\[-\frac{49}{4665600} \frac{\partial^6}{\partial j_3^6} \]
\+ \frac{\Box_3}{\Box_1} \left( -\frac{13}{1555200} \frac{\partial^6}{\partial j_1^6} + \frac{29}{518400} \frac{\partial^5}{\partial j_1^5} \frac{\partial}{\partial j_2} - \frac{17}{388800} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_2^2} \right)
\- \frac{1}{8100} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_2^3} + \frac{133}{1555200} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_2^4} - \frac{23}{1555200} \frac{\partial}{\partial j_1} \frac{\partial^5}{\partial j_2^5} \right)
\- \frac{23}{1555200} \frac{\partial^5}{\partial j_1^5} \frac{\partial}{\partial j_2} + \frac{133}{1555200} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_2^2} - \frac{1}{8100} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_2^3} \right)
\- \frac{17}{388800} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_2^4} + \frac{29}{518400} \frac{\partial}{\partial j_1} \frac{\partial^5}{\partial j_2^5} - \frac{13}{1555200} \frac{\partial^6}{\partial j_2^6} \right), \quad (11.33)\]

\[ a_3(3) = \frac{\Box_2}{\Box_1} \left( \frac{1}{1296} \frac{\partial^3}{\partial j_1^3} - \frac{1}{324} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_2} + \frac{1}{3240} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} \right) \]
\+ \frac{\Box_1}{\Box_2} \left( \frac{1}{324} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_2} - \frac{1}{324} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} + \frac{1}{1296} \frac{\partial^3}{\partial j_2^3} \right), \quad (11.34)\]
\( b_9(3) = -\frac{49}{466560} \frac{\partial^6}{\partial j_1^6} + \frac{1}{25920} \frac{\partial^5 \partial}{\partial j_1^5 \partial j_2} + \frac{1}{20736} \frac{\partial^4 \partial^2}{\partial j_1^4 \partial j_2^2} \\
- \frac{17}{72900} \frac{\partial^3 \partial^3}{\partial j_1^3 \partial j_2^3} + \frac{1}{20736} \frac{\partial^2 \partial^4}{\partial j_1^2 \partial j_2^4} + \frac{1}{25920} \frac{\partial \partial^5}{\partial j_1 \partial j_2^5} \\
- \frac{49}{466560} \frac{\partial^6}{\partial j_2^6} \right) + \square_2 \left( -\frac{13}{1555200} \frac{\partial^6}{\partial j_1^6} \right) + \frac{1}{518400} \frac{\partial^5 \partial}{\partial j_1^5 \partial j_2} - \frac{17}{388800} \frac{\partial^4 \partial^2}{\partial j_1^4 \partial j_2^2} \\
- \frac{1}{8100} \frac{\partial^3 \partial^3}{\partial j_1^3 \partial j_2^3} + \frac{133}{1555200} \frac{\partial^2 \partial^4}{\partial j_1^2 \partial j_2^4} - \frac{23}{1555200} \frac{\partial \partial^5}{\partial j_1 \partial j_2^5} \right)
+ \square_1 \left( -\frac{23}{1555200} \frac{\partial^5 \partial}{\partial j_1^5 \partial j_2} + \frac{133}{1555200} \frac{\partial^4 \partial^2}{\partial j_1^4 \partial j_2^2} - \frac{1}{8100} \frac{\partial^3 \partial^3}{\partial j_1^3 \partial j_2^3} \\
- \frac{17}{388800} \frac{\partial^2 \partial^4}{\partial j_1^2 \partial j_2^4} + \frac{29}{518400} \frac{\partial \partial^5}{\partial j_1 \partial j_2^5} - \frac{13}{1555200} \frac{\partial^6}{\partial j_2^6} \right), (11.35)

\[
\Gamma_{10}^{\text{sym}} = \left\{
\begin{array}{l}
\frac{1}{\mathcal{U}_1} \left( \frac{1}{810} - \frac{\partial_1}{1620 \mathcal{U}_3} - \frac{\partial_3}{1620 \mathcal{U}_2} \right) + O, \quad \square_1 \to -0 \\
\frac{1}{\mathcal{U}_2} \left( \frac{1}{810} - \frac{\partial_2}{1620 \mathcal{U}_3} - \frac{\partial_3}{1620 \mathcal{U}_1} \right) + O, \quad \square_2 \to -0 \\
\frac{1}{\mathcal{U}_3} \left( \frac{1}{810} - \frac{\partial_3}{1620 \mathcal{U}_2} - \frac{\partial_2}{1620 \mathcal{U}_1} \right) + O, \quad \square_3 \to -0
\end{array}\right.
\]

(11.36)

\[
\Gamma_{11}^{\text{sym}} = \left\{
\begin{array}{l}
\frac{1}{\mathcal{U}_1} \left( -\frac{\partial_1}{3400 \mathcal{U}_2} + A_{11}(1) \right) + O, \quad \square_1 \to -0 \\
\frac{1}{\mathcal{U}_2} \left( -\frac{\partial_2}{3400 \mathcal{U}_1} + A_{11}(2) \right) + O, \quad \square_2 \to -0 \\
\ln(-\mathcal{U}_3) \left( \frac{1}{180 \mathcal{U}_2} - \frac{1}{180 \mathcal{U}_1} \right) + O, \quad \square_3 \to -0
\end{array}\right.
\]

(11.37)

\[
a_{11}(1) = \square_3 \left( -\frac{17}{1080} - \frac{17}{2160} \frac{\partial}{\partial j_2} + \frac{11}{2160} \frac{\partial^2}{\partial j_2^2} + \frac{1}{720} \frac{\partial^3}{\partial j_2^3} \\
- \frac{1}{180} \frac{\partial}{\partial j_3} + \frac{1}{540} \frac{\partial}{\partial j_2} - \frac{1}{180} \frac{\partial^2}{\partial j_2^2} + \frac{1}{1080} \frac{\partial^2}{\partial j_2^2} \\
+ \frac{1}{360} \frac{\partial^2}{\partial j_2 \partial j_3^2} + \frac{1}{1080} \frac{\partial^2}{\partial j_2 \partial j_3^2} \right) \\
+ \square_2 \left( \frac{1}{360} - \frac{1}{360} \frac{\partial}{\partial j_2} - \frac{1}{2160} \frac{\partial}{\partial j_2^2} + \frac{1}{270} \frac{\partial}{\partial j_2} - \frac{1}{1080} \frac{\partial^2}{\partial j_2^2} \\
- \frac{1}{540} \frac{\partial^2}{\partial j_2^2} - \frac{1}{1080} \frac{\partial^2}{\partial j_2^2} - \frac{1}{1080} \frac{\partial^2}{\partial j_2^2} \right),
\]

(11.38)
\( a_{11}(2) = \Box_3 \left( -\frac{17}{1080} - \frac{17}{1080} \frac{\partial}{\partial j_1} + \frac{11}{2160} \frac{\partial^2}{\partial j_1^2} + \frac{1}{720} \frac{\partial^3}{\partial j_1^3} \right. \\
- \frac{1}{180} \frac{\partial}{\partial j_3} + \frac{1}{540} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} - \frac{1}{2160} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3} + \frac{1}{1080} \frac{\partial^2}{\partial j_3^2} \\
+ \frac{1}{360} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_3^2} + \frac{1}{1080} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_3^2} \right) \\
+ \Box_1 \left( \frac{1}{360} - \frac{1}{360} \frac{\partial}{\partial j_1} - \frac{1}{2160} \frac{\partial}{\partial j_3} + \frac{1}{270} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} - \frac{1}{1080} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3} \\
- \frac{1}{540} \frac{\partial^2}{\partial j_3^2} - \frac{1}{108} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_3^2} - \frac{1}{1080} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_3^2} \right), \quad (11.39) \)

\[ \Gamma_{12}^{\text{sym}} = \begin{cases} 
\ln(-\Box_1) \left( \frac{-2}{3\sqrt{j_3}} + B_{12}(1) \right) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) \left( \frac{-1}{3\sqrt{j_3}} + B_{12}(2) \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{-1}{3\sqrt{j_3}} + B_{12}(3) \right) + O, & \Box_3 \to -0 
\end{cases} \quad (11.40) \]

\[ b_{12}(1) = \frac{1}{\Box_2} \left( -\frac{1}{3} \frac{\partial}{\partial j_2} \right) + \frac{1}{\Box_3} \left( -\frac{1}{3} \frac{\partial}{\partial j_3} \right), \quad (11.41) \]

\[ b_{12}(2) = \frac{1}{\Box_1} \left( \frac{5}{4} \frac{\partial^2}{\partial j_1^2} + \frac{2}{3} \frac{\partial^3}{\partial j_1^3} + \frac{1}{12} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} \right) \\
+ \frac{1}{\Box_3} \left( -\frac{1}{3} \frac{\partial}{\partial j_3} + \frac{1}{12} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} - \frac{1}{12} \frac{\partial^2}{\partial j_3^2} \right), \quad (11.42) \]

\[ b_{12}(3) = \frac{1}{\Box_1} \left( \frac{5}{4} \frac{\partial^2}{\partial j_1^2} + \frac{2}{3} \frac{\partial^3}{\partial j_1^3} + \frac{1}{12} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} \right) \\
+ \frac{1}{\Box_2} \left( -\frac{1}{3} \frac{\partial}{\partial j_2} + \frac{1}{12} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} - \frac{1}{12} \frac{\partial^2}{\partial j_2^2} \right), \quad (11.43) \]

\[ \Gamma_{13}^{\text{sym}} = \begin{cases} 
\ln(-\Box_1) B_{13}(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) \left( \frac{-2}{\sqrt{j_1 j_3}} + B_{13}(2) \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{-2}{\sqrt{j_1 j_3}} + B_{13}(3) \right) + O, & \Box_3 \to -0 
\end{cases} \quad (11.44) \]
\[ b_{13}(1) = \frac{1}{\Box_2} \left( -\frac{1}{2} \frac{\partial^2}{\partial j_2^2} + \frac{1}{2} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} \right) + \frac{1}{\Box_3} \left( \frac{1}{2} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} - \frac{1}{2} \frac{\partial^2}{\partial j_3^2} \right), \]  
(11.45)

\[ b_{13}(2) = \frac{1}{\Box_1} \left( -2 \frac{\partial}{\partial j_1} \right), \]  
(11.46)

\[ b_{13}(3) = \frac{1}{\Box_1} \left( -2 \frac{\partial}{\partial j_1} \right), \]  
(11.47)

\[ \Gamma_{14}^{\text{sym}} = \begin{cases} 
\ln(-\Box_1)B_{14}(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2)B_{14}(2) + O, & \Box_2 \to -0 \\
\ln(-\Box_3)B_{14}(3) + O, & \Box_3 \to -0 
\end{cases} \]  
(11.48)

\[ b_{14}(1) = \frac{1}{\Box_3} \left( -2 \frac{\partial}{\partial j_3} - 2 \frac{\partial^2}{\partial j_3^2} \right), \]  
(11.49)

\[ b_{14}(2) = \frac{1}{\Box_3} \left( -2 \frac{\partial}{\partial j_3} - 2 \frac{\partial^2}{\partial j_3^2} \right), \]  
(11.50)

\[ b_{14}(3) = \frac{1}{\Box_1} \left( -\frac{1}{2} \frac{\partial^2}{\partial j_1^2} + \frac{1}{2} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} \right) + \frac{1}{\Box_2} \left( \frac{1}{2} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} - \frac{1}{2} \frac{\partial^2}{\partial j_2^2} \right), \]  
(11.51)

\[ \Gamma_{15}^{\text{sym}} = \begin{cases} 
O, & \Box_1 \to -0 \\
\ln(-\Box_2)B_{15}(2) + O, & \Box_2 \to -0 \\
\ln(-\Box_3)B_{15}(3) + O, & \Box_3 \to -0 
\end{cases} \]  
(11.52)

\[ b_{15}(2) = \frac{1}{\Box_1} \left( \frac{1}{3} \frac{\partial^2}{\partial j_1^2} + \frac{2}{3} \frac{\partial^3}{\partial j_1^3} + \frac{1}{6} \frac{\partial^4}{\partial j_1^4} \right), \]  
(11.53)

\[ b_{15}(3) = \frac{1}{\Box_1} \left( \frac{1}{3} \frac{\partial^2}{\partial j_1^2} + \frac{2}{3} \frac{\partial^3}{\partial j_1^3} + \frac{1}{6} \frac{\partial^4}{\partial j_1^4} \right), \]  
(11.54)
\[
\Gamma^{\text{sym}}_{16} = \begin{cases}
\frac{1}{\partial_1} \left( \frac{1}{\partial_2} + A_{16}(1) \right) + O, & \Box_1 \to -0 \\
\frac{1}{\partial_2} \left( \frac{1}{\partial_1} + A_{16}(2) \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{1}{\partial_2 \partial_3} \right) + O, & \Box_3 \to -0
\end{cases}
\]

\(a_{16}(1) = \frac{1}{3} \frac{\partial}{\partial j_2},\) (11.56)

\(a_{16}(2) = \frac{1}{3} \frac{\partial}{\partial j_1},\) (11.57)

\[
\Gamma^{\text{sym}}_{17} = \begin{cases}
O, & \Box_1 \to -0 \\
\ln(-\Box_2) \left( \frac{1}{\partial_1 \partial_3} + B_{17}(2) \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{1}{\partial_1 \partial_2} + B_{17}(3) \right) + O, & \Box_3 \to -0
\end{cases}
\]

\(b_{17}(2) = \frac{1}{\Box_1} \left( \frac{\partial^2}{\partial j_1^2} \right),\) (11.59)

\(b_{17}(3) = \frac{1}{\Box_1} \left( \frac{\partial^2}{\partial j_1^2} \right),\) (11.60)

\[
\Gamma^{\text{sym}}_{18} = \begin{cases}
\frac{1}{\partial_1} A_{18}(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) B_{18}(2) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) B_{18}(3) + O, & \Box_3 \to -0
\end{cases}
\]

\(a_{18}(1) = -1 \left( \frac{2}{3} \frac{\partial}{\partial j_2} + \frac{2}{3} \frac{\partial}{\partial j_3} - \frac{1}{3} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} \right),\) (11.62)

\(b_{18}(2) = \frac{1}{\Box_1} \left( \frac{\partial^2}{\partial j_1^2} + \frac{2}{3} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3} \right),\) (11.63)
\( b_{18}(3) = \frac{1}{\Box_1} \left( \frac{\partial^2}{\partial j_1^2} + \frac{2}{3} \frac{\partial^2}{\partial j_1^2 \partial j_2} \right), \) \hspace{1cm} (11.64)

\[
\Gamma_{19}^{\text{sym}} = \begin{cases} 
\frac{1}{\Box_1} A_{19}(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) \left( \frac{1}{\Box_1 \Box_3} \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{1}{\Box_1 \Box_2} \right) + O, & \Box_3 \to -0 
\end{cases} \hspace{1cm} (11.65)
\]

\( a_{19}(1) = \frac{1}{6} + \frac{1}{6} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3}, \) \hspace{1cm} (11.66)

\[
\Gamma_{20}^{\text{sym}} = \begin{cases} 
\ln(-\Box_1) B_{20}(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) \left( \frac{1}{\Box_1 \Box_3} + B_{20}(2) \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{1}{\Box_1 \Box_2} + B_{20}(3) \right) + O, & \Box_3 \to -0 
\end{cases} \hspace{1cm} (11.67)
\]

\[
b_{20}(1) = \frac{1}{\Box_2} \left( \frac{1}{144} \frac{\partial^4}{\partial j_2^4} - \frac{5}{144} \frac{\partial^3}{\partial j_2^3} \frac{\partial}{\partial j_3} + \frac{5}{144} \frac{\partial^2}{\partial j_2^2} \frac{\partial^2}{\partial j_3^2} - \frac{1}{144} \frac{\partial}{\partial j_2} \frac{\partial^3}{\partial j_3^3} \right) \\
+ \frac{1}{\Box_3} \left( -\frac{1}{144} \frac{\partial^3}{\partial j_3^3} \frac{\partial}{\partial j_2} + \frac{5}{144} \frac{\partial^2}{\partial j_3^2} \frac{\partial^2}{\partial j_2^2} - \frac{5}{144} \frac{\partial}{\partial j_3} \frac{\partial^3}{\partial j_2^3} + \frac{1}{144} \frac{\partial^4}{\partial j_2^4} \right), \hspace{1cm} (11.68)
\]

\[
b_{20}(2) = \frac{1}{\Box_1} \left( -\frac{1}{3} \frac{\partial^2}{\partial j_1^2} - \frac{5}{18} \frac{\partial^2}{\partial j_1^2 \partial j_2} + \frac{1}{18} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} \right), \hspace{1cm} (11.69)
\]

\[
b_{20}(3) = \frac{1}{\Box_1} \left( -\frac{1}{3} \frac{\partial^2}{\partial j_1^2} - \frac{5}{18} \frac{\partial^2}{\partial j_1^2 \partial j_2} + \frac{1}{18} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} \right), \hspace{1cm} (11.70)
\]

\[
\Gamma_{21}^{\text{sym}} = \begin{cases} 
\frac{1}{\Box_1} A_{21}(1) + O, & \Box_1 \to -0 \\
\ln(-\Box_2) \left( \frac{2}{\Box_1 \Box_3} + B_{21}(2) \right) + O, & \Box_2 \to -0 \\
\ln(-\Box_3) \left( \frac{2}{\Box_1 \Box_2} \right) + O, & \Box_3 \to -0 
\end{cases} \hspace{1cm} (11.71)
\]
\[ a_{21}(1) = -\frac{2}{3} + \frac{4}{3} \frac{\partial}{\partial j_3} + \frac{2}{3} \frac{\partial^2}{\partial j_3^2}, \quad (11.72) \]

\[ b_{21}(2) = \frac{1}{\Box_1} \left( -\frac{4}{3} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3} - \frac{2}{3} \frac{\partial^2}{\partial j_1^2} \frac{\partial^2}{\partial j_3^2} \right), \quad (11.73) \]

\[ \Gamma_{22}^{\text{sym}} = \begin{cases} 
\frac{1}{\Box_1} \left( -\frac{1}{1800 \Box_2} - \frac{1}{1800 \Box_3} + A_{22}(1) \right) + O, & \Box_1 \to -0 \\
\frac{1}{\Box_2} \left( -\frac{1}{1800 \Box_1} + A_{22}(2) \right) + \ln(-\Box_2) \left( -\frac{1}{36 \Box_1 \Box_3} + B_{22}(2) \right) + O, & \Box_2 \to -0 \\
\frac{1}{\Box_3} \left( -\frac{1}{1800 \Box_1} + A_{22}(3) \right) + \ln(-\Box_3) \left( -\frac{1}{36 \Box_1 \Box_2} + B_{22}(3) \right) + O, & \Box_3 \to -0 
\end{cases} \quad (11.74) \]

\[ a_{22}(1) = -\frac{1}{60}, \quad (11.75) \]

\[ a_{22}(2) = \frac{1}{360} \frac{\partial^3}{\partial j_1^3} + \frac{1}{72} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3} + \frac{1}{120} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_3^2} + \frac{1}{360} \frac{\partial^3}{\partial j_3^3}, \quad (11.76) \]

\[ b_{22}(2) = \frac{1}{\Box_1} \left( \frac{13}{64800} \frac{\partial^6}{\partial j_1^6} + \frac{89}{194400} \frac{\partial^5}{\partial j_1^5} \frac{\partial}{\partial j_3} - \frac{47}{38880} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_3^2} \right. \\
+ \frac{1}{21600} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_3^3} + \frac{29}{194400} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_3^4} + \frac{1}{38880} \frac{\partial}{\partial j_1} \frac{\partial^5}{\partial j_3^5} \left. \right) \\
+ \frac{1}{\Box_3} \left( \frac{1}{3600} \frac{\partial^6}{\partial j_1^6} \frac{\partial}{\partial j_3} - \frac{31}{194400} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_3^2} + \frac{7}{38880} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_3^3} \right. \\
- \frac{37}{64800} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_3^4} + \frac{11}{48600} \frac{\partial}{\partial j_1} \frac{\partial^5}{\partial j_3^5} + \frac{1}{38880} \frac{\partial^6}{\partial j_3^6} \right), \quad (11.77) \]

\[ a_{22}(3) = \frac{1}{360} \frac{\partial^3}{\partial j_1^3} + \frac{1}{72} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_2} + \frac{1}{120} \frac{\partial}{\partial j_1} \frac{\partial^2}{\partial j_2^2} + \frac{1}{360} \frac{\partial^3}{\partial j_2^3}, \quad (11.78) \]
\[ b_{22}(3) = \frac{1}{\Delta_1} \left( \frac{13}{64800} \frac{\partial^6}{\partial j_1^6} + \frac{89}{194400} \frac{\partial^5}{\partial j_1^5} \frac{\partial}{\partial j_2} - \frac{47}{38880} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_2^2} \right) \\
+ \frac{1}{\Delta_2} \left( \frac{1}{21600} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_2^3} + \frac{29}{194400} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_2^4} + \frac{1}{38880} \frac{\partial}{\partial j_1^1} \frac{\partial^5}{\partial j_2^5} \right) \\
+ \frac{1}{\Delta_3} \left( \frac{1}{3600} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_2} - \frac{31}{194400} \frac{\partial}{\partial j_1^1} \frac{\partial^4}{\partial j_2^4} + \frac{37}{38880} \frac{\partial}{\partial j_1^1} \frac{\partial^2}{\partial j_2^2} \right) \\
- \frac{64800}{64800} \frac{\partial}{\partial j_1^1} \frac{\partial^2}{\partial j_2^2} + \frac{48600}{38880} \frac{\partial}{\partial j_1^1} \frac{\partial^2}{\partial j_2^2} + \frac{1}{38880} \frac{\partial^6}{\partial j_2^6} \right), \] (11.79)

\[
\Gamma_{23}^{\text{sym}} = \begin{cases} 
\frac{1}{\Delta_1} \left( \frac{1}{1350} + A_{23}(1) \right) + O, & \Delta_1 \to -0 \\
\frac{1}{\Delta_2} \left( \frac{1}{1350} + A_{23}(2) \right) + O, & \Delta_2 \to -0 \\
\ln(-\Delta_3) \left( \frac{1}{450} \frac{1}{\Delta_3} \right) + O, & \Delta_3 \to -0 
\end{cases} \] (11.80)

\[
a_{23}(1) = \frac{1}{90} \frac{\partial^2}{\partial j_2^2} + \frac{2}{45} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} + \frac{1}{60} \frac{\partial^2}{\partial j_3^2}, \] (11.81)

\[
a_{23}(2) = \frac{1}{90} \frac{\partial^2}{\partial j_1^2} + \frac{2}{45} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} + \frac{1}{60} \frac{\partial^2}{\partial j_3^2}, \] (11.82)

\[
\Gamma_{24}^{\text{sym}} = \begin{cases} 
\ln(-\Delta_1) \left( \frac{1}{300} \frac{1}{\Delta_1} \right) + O, & \Delta_1 \to -0 \\
\frac{1}{\Delta_2} \left( \frac{1}{540} + A_{24}(2) \right) + O, & \Delta_2 \to -0 \\
\frac{1}{\Delta_3} \left( \frac{1}{540} + A_{24}(3) \right) + O, & \Delta_3 \to -0 
\end{cases} \] (11.83)

\[
a_{24}(2) = \frac{1}{60} \frac{\partial^2}{\partial j_1^2} + \frac{1}{20} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_3} + \frac{1}{60} \frac{\partial^2}{\partial j_3^2}, \] (11.84)

\[
a_{24}(3) = \frac{1}{60} \frac{\partial^2}{\partial j_1^2} + \frac{1}{20} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} + \frac{1}{60} \frac{\partial^2}{\partial j_2^2}, \] (11.85)
\[ \Gamma_{25}^{\text{sym}} = \begin{ \{ } 
\frac{1}{\Box_1} \left( -\frac{1}{270 \Box_2} - \frac{1}{270 \Box_3} + A_{25}(1) \right) \\
\frac{1}{\Box_2} \left( -\frac{1}{270 \Box_1} + \frac{1}{270 \Box_3} \right) + O, \quad \Box_1 \to -0 \\
\frac{1}{\Box_3} \left( -\frac{1}{270 \Box_1} + \frac{1}{270 \Box_2} \right) + O, \quad \Box_3 \to -0 
\end{ \{ } \] (11.86)

\[ a_{25}(1) = \frac{1}{30} \frac{\partial^2}{\partial j_2^2} + \frac{2}{15} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} + \frac{1}{30} \frac{\partial^2}{\partial j_3^2}, \] (11.87)

\[ \Gamma_{26}^{\text{sym}} = \begin{ \{ } 
\frac{1}{\Box_1} A_{26}(1) + O, \quad \Box_1 \to -0 \\
\frac{1}{\Box_2} A_{26}(2) + O, \quad \Box_2 \to -0 \\
\ln(-\Box_3) B_{26}(3) + O, \quad \Box_3 \to -0 
\end{ \{ } \] (11.88)

\[ a_{26}(1) = \frac{1}{\Box_2} \left( -\frac{1}{6} \frac{\partial^2}{\partial j_2^2} \right), \] (11.89)

\[ a_{26}(2) = \frac{1}{\Box_1} \left( -\frac{1}{6} \frac{\partial^2}{\partial j_1^2} \right), \] (11.90)

\[ b_{26}(3) = \frac{1}{\Box_1 \Box_2} \left( \frac{1}{6} \frac{\partial^2}{\partial j_1^2} \frac{\partial^2}{\partial j_2^2} \right), \] (11.91)

\[ \Gamma_{27}^{\text{sym}} = \begin{ \{ } 
\frac{1}{\Box_1} \left( -\frac{1}{540 \Box_2 \Box_3} + A_{27}(1) \right) + O, \quad \Box_1 \to -0 \\
\frac{1}{\Box_2} \left( -\frac{1}{540 \Box_1 \Box_3} + A_{27}(2) \right) + O, \quad \Box_2 \to -0 \\
\frac{1}{\Box_3} \left( -\frac{1}{540 \Box_1 \Box_2} \right) + \ln(-\Box_3) B_{27}(3) + O, \quad \Box_3 \to -0 
\end{ \{ } \] (11.92)

\[ a_{27}(1) = \frac{1}{\Box_2} \left( \frac{1}{180} \frac{\partial^2}{\partial j_2^2} + \frac{1}{135} \frac{\partial^3}{\partial j_2^3} + \frac{1}{1080} \frac{\partial^4}{\partial j_2^4} + \frac{1}{540} \frac{\partial^2}{\partial j_2^2} \frac{\partial^2}{\partial j_3^2} \right), \] (11.93)
\( a_{27}(2) = \frac{1}{\Box_1} \left( \frac{1}{180} \frac{\partial^2}{\partial j_1^2} + \frac{1}{135} \frac{\partial^3}{\partial j_1^3} + \frac{1}{1080} \frac{\partial^4}{\partial j_1^4} + \frac{1}{540} \frac{\partial^2}{\partial j_1^2} \frac{\partial^2}{\partial j_2^2} \right), \quad (11.94) \)

\( b_{27}(3) = \frac{1}{\Box_1 \Box_2} \left( - \frac{1}{1080} \frac{\partial^4}{\partial j_1^4} \frac{\partial^2}{\partial j_1^2} \frac{\partial^2}{\partial j_2^2} + \frac{1}{270} \frac{\partial^3}{\partial j_1^3} \frac{\partial^3}{\partial j_2^3} - \frac{1}{1080} \frac{\partial^2}{\partial j_1^2} \frac{\partial^4}{\partial j_2^2} \right), \quad (11.95) \)

\[
\Gamma_{28}^{\text{sym}} = \begin{cases} 
\frac{1}{\Box_1} \left( \frac{1}{1350} + A_{28}(1) \right) + O, & \Box_1 \rightarrow 0 \\
\frac{1}{\Box_2} \left( \frac{1}{1350} + A_{28}(2) \right) + O, & \Box_2 \rightarrow 0 \\
\frac{1}{\Box_3} \left( \frac{1}{1350} \right) + O, & \Box_3 \rightarrow 0 
\end{cases} \quad (11.96) \]

\[ a_{28}(1) = \frac{1}{\Box_2} \left( \frac{1}{45} \frac{\partial^2}{\partial j_2^2} \frac{\partial}{\partial j_3} \right), \quad (11.97) \]

\[ a_{28}(2) = \frac{1}{\Box_1} \left( \frac{1}{45} \frac{\partial^2}{\partial j_1^2} \frac{\partial}{\partial j_3} \right), \quad (11.98) \]

\[
\Gamma_{29}^{\text{sym}} = \begin{cases} 
\frac{1}{\Box_1} A_{29}(1) + O, & \Box_1 \rightarrow 0 \\
\frac{1}{\Box_2} A_{29}(2) + O, & \Box_2 \rightarrow 0 \\
\frac{1}{\Box_3} A_{29}(3) + O, & \Box_3 \rightarrow 0 
\end{cases} \quad (11.99) \]

\[ a_{29}(1) = \frac{1}{\Box_2 \Box_3} \left( - \frac{1}{270} \frac{\partial^2}{\partial j_2^2} \frac{\partial^2}{\partial j_3^2} \right), \quad (11.100) \]

\[ a_{29}(2) = \frac{1}{\Box_1 \Box_3} \left( - \frac{1}{270} \frac{\partial^2}{\partial j_1^2} \frac{\partial^2}{\partial j_3^2} \right), \quad (11.101) \]

\[ a_{29}(3) = \frac{1}{\Box_1 \Box_2} \left( - \frac{1}{270} \frac{\partial^2}{\partial j_1^2} \frac{\partial^2}{\partial j_2^2} \right). \quad (11.102) \]

For the derivation of these results see sect. 20 where also the spectral form of the functions (11.1)–(11.6) is given.
12. The trace anomaly in four dimensions

A crucial check of the results above is a derivation of the trace anomaly for a conformal invariant quantum field in four dimensions. To have as many curvature structures as possible involved in the check, we choose the following quantum field model ($\omega = 2$):

$$S[\varphi] = \frac{1}{2} \int dx \, g^{1/2} \left( \nabla_\mu \varphi^T \nabla^\mu \varphi + \frac{R}{6} \varphi^T \varphi + \frac{\lambda^2}{4!} (\varphi^T \varphi)^2 \right),$$  \hspace{1cm} (12.1)

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$  \hspace{1cm} (12.2)

$$\nabla_\mu \varphi = \partial_\mu \varphi + A_\mu \hat{G} \varphi, \quad \nabla_\mu \varphi^T = \partial_\mu \varphi^T + A_\mu \varphi^T \hat{G}^T,$$  \hspace{1cm} (12.3)

$$\hat{G} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (12.4)

where (12.1) is the euclidean action of the complex scalar quantum field

$$\varphi = \varphi_1 + i \varphi_2,$$  \hspace{1cm} (12.5)

rewritten in terms of the real components. The electromagnetic and gravitational fields in (12.1) are classical.

The action (12.1) is invariant under the local conformal transformations

$$\delta_\sigma g^{\mu\nu}(x) = \sigma(x) g^{\mu\nu}(x), \quad \delta_\sigma \varphi(x) = \frac{1}{2} \sigma(x) \varphi(x), \quad \delta_\sigma A_\mu(x) = 0$$  \hspace{1cm} (12.6)

with the parameter $\sigma(x)$. The hessian of the action (12.1) has the form (1.2) (times a local matrix) in which the potential is

$$\hat{P} = -\frac{2\lambda^2}{4!} \left( \frac{3\varphi_1^2 + \varphi_2^2}{2\varphi_1 \varphi_2} \quad \frac{2\varphi_1 \varphi_2}{3\varphi_2^2 + \varphi_1^2} \right),$$  \hspace{1cm} (12.7)

and the commutator curvature is

$$\hat{R}_{\mu\nu} = \hat{G}(\partial_\mu A_\nu - \partial_\nu A_\mu)$$  \hspace{1cm} (12.8)

with $\hat{G}$ in (12.4).

From (12.6)–(12.8) we find the conformal transformation laws for the curvatures and $\Box$-operators:

$$\delta_\sigma \hat{P} = \sigma \hat{P}, \quad \delta_\sigma \hat{R}_{\mu\nu} = 0,$$  \hspace{1cm} (12.9)

$$\delta_\sigma R_{\mu\nu} = (\nabla_\mu \nabla_\nu + \frac{1}{2} g_{\mu\nu} \Box) \sigma, \quad \delta_\sigma R = (3 \Box + R) \sigma,$$  \hspace{1cm} (12.10)

$$(\delta_\sigma \Box) \hat{P} = \sigma \Box \hat{P} - \nabla_\alpha \sigma \nabla^\alpha \hat{P}, $$  \hspace{1cm} (12.11)
(\delta \square) \hat{R}_{\mu\nu} = \sigma \square \hat{R}_{\mu\nu} + \hat{R}_{\mu\nu} \square \sigma \\
+ \nabla_\mu \sigma \nabla^\alpha \hat{R}_{\alpha\nu} - \nabla_\nu \sigma \nabla^\alpha \hat{R}_{\alpha\mu}, \tag{12.13}

(\delta \sigma) R_{\mu\nu} = \sigma \square R_{\mu\nu} + R_{\mu\nu} \square \sigma + \nabla_\alpha \sigma \nabla^\alpha R_{\mu\nu} \\
+ \nabla_\mu (\sigma \nabla^\nu) R - 2 \nabla^\alpha \sigma \nabla_{(\mu} R_{\nu)\alpha}, \tag{12.14}

(\delta \sigma) R = \sigma \square R - \nabla_\alpha \sigma \nabla^\alpha R. \tag{12.15}

Having got these laws, one may already forget the particular content of the model, and merely consider the transformation (12.9)–(12.15) in the effective action. For the dimensionally regularized * one-loop effective action (6.1), the result should be exactly

\[- \delta_\sigma W = - \frac{1}{2(4\pi)^2} \int dx \, g^{1/2} \sigma \tr \hat{a}_2(x, x) \tag{12.16}\]

where \(\hat{a}_2(x, x)\) is the second DeWitt coefficient * at coincident points [6,7]:

\[\hat{a}_2(x, x) = \frac{1}{6} \hat{P} + \frac{1}{180} \hat{R} \hat{I} + \frac{1}{180} (R_{\alpha\beta\mu\nu}^2 - R_{\alpha\beta}^2) \hat{I} + \frac{1}{12} \hat{R}_{\mu\nu}^2 + \frac{1}{2} \hat{P}^2. \tag{12.17}\]

Expression (12.16) is the general form of the conformal anomaly in four dimensions [18–22]. For the model above,

\[\delta_\sigma = \int dx \left( \sigma(x) g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} + \frac{1}{2} \sigma(x) \varphi \frac{\delta}{\delta \varphi} \right), \tag{12.18}\]

and

\[g^{-1/2} \left( g^{\mu\nu} \frac{\delta W}{\delta g^{\mu\nu}} + \frac{1}{2} \frac{\delta W}{\delta \varphi} \right) = \frac{1}{2(4\pi)^2} \tr \hat{a}_2(x, x). \tag{12.19}\]

In the present technique, eq. (12.16) can be obtained only with a given accuracy O[\(\Re^n\]) and with the Riemann tensor expressed through the Ricci tensor. To lowest order, one may use the expression for \(R_{\alpha\beta\mu\nu}^2\) given in Appendix A of paper II. After elimination of the Riemann tensor from (12.17), eq. (12.16) takes the form

\[- \delta_\sigma W = \frac{1}{2(4\pi)^2} \int dx \, g^{1/2} \tr \left\{ - \frac{1}{6} (\square \hat{P}) \sigma \right. \\
- \frac{1}{180} (\square R) \sigma \hat{I} - \frac{1}{12} \hat{R}_{\mu\nu}^2 \sigma - \frac{1}{2} \hat{P}^2 \sigma \right\}. \]

*The dimensional regularization was used in paper II for the derivation of the quadratic terms in \(W\). In fact, important is only belonging of the regularization to one of the two alternative classes discussed in [22].

*Since the function \(\sigma(x)\) is arbitrary in any compact domain, the anomaly (12.16) provides a check of \(\tr \hat{a}_2(x, x)\) itself whereas in sect. 4 we dealt only with the integral

\[\int dx \, g^{1/2} \tr \hat{a}_2(x, x).\]

Hence the differences between the expressions (4.47) and (12.17).
\[
- \frac{1}{180} \left( 1 + 2 \frac{\Box_1}{\Box_2} - 4 \frac{\Box_3}{\Box_1} + \frac{\Box_3^2}{\Box_1 \Box_2} \right) R_{1}^{\mu \nu} R_{2 \mu \nu} \sigma_3 \hat{1} \\
- \frac{1}{45} \left( 2 \frac{\Box_3}{\Box_1} - \frac{\Box_3}{\Box_1 \Box_2} \right) \nabla^\mu R_{1}^{\nu \lambda} \nabla_\nu R_{2 \mu \lambda} \sigma_3 \hat{1} \\
- \frac{1}{45} \frac{\Box_1}{\Box_1 \Box_2} \nabla_\alpha \nabla_\beta R_{1 \mu \nu} \nabla^\mu \nabla^\nu R_{2}^{\alpha \beta} \sigma_3 \hat{1} \right\} + O[\Re^3] \quad (12.20)
\]

where the notation in the nonlocal terms is the same as before with \( \sigma \) playing the role of the third curvature. It is the latter equation that will be checked below by a direct calculation with \( W \) in (6.1).

We begin this check with calculating the result of the transformation (12.9)–(12.15) in the quadratic terms of \( W \). For the quadratic terms of (6.1) we have

\[
\delta_\sigma \int dx g^{1/2} \text{tr} \left\{ \sum_{i=1}^{5} \gamma_i (-\Box_2) \Re_1 \Re_2 (i) \right\} = \int dx g^{1/2} \text{tr} \left\{ -\frac{3}{18} \hat{P} \Box \sigma \\
+ \frac{1}{30} R^{\mu \nu} \left[ \gamma (-\Box) + \frac{16}{15} \left( \nabla_\mu \nabla_\nu \sigma + \frac{1}{2} g_{\mu \nu} \sigma \right) \right] \hat{1} \\
- \frac{3}{90} R \left[ \gamma (-\Box) + \frac{37}{30} \Box \sigma \right] \hat{1} \\
+ \frac{1}{12} \hat{\Re}_{\mu \nu} \left( \delta_\sigma \gamma (-\Box) \right) \hat{\Re}_{\mu \nu} + \frac{1}{2} \hat{P} \left( \delta_\sigma \gamma (-\Box) \right) \hat{P} \\
+ \frac{1}{60} R_{\mu \nu} \left( \delta_\sigma \gamma (-\Box) \right) R^{\mu \nu} \hat{1} - \frac{1}{180} R \left( \delta_\sigma \gamma (-\Box) \right) R \hat{1} \right\} \quad (12.21)
\]

where

\[
\gamma (-\Box) = -\ln \left( -\frac{\Box}{\mu^2} \right). \quad (12.22)
\]

In the term linear in \( R^{\mu \nu} \), for being able to use the Bianchi identity, one must commute \( \gamma (-\Box) \) with \( \nabla_\mu \nabla_\nu \), and the commutator cannot be neglected. As a result of this commutation, the linear nonlocal terms cancel, and we obtain

\[
\delta_\sigma \int dx g^{1/2} \text{tr} \left\{ \sum_{i=1}^{5} \gamma_i (-\Box_2) \Re_1 \Re_2 (i) \right\} = \int dx g^{1/2} \text{tr} \left\{ -\frac{1}{6} (\hat{P} \Box) \sigma - \frac{1}{180} (\Box R) \sigma \hat{1} \\
+ \frac{1}{30} R_{\mu \nu} \left[ \gamma (-\Box), \nabla^\mu \nabla^\nu \right] \sigma \hat{1} \\
+ \frac{1}{12} \hat{\Re}_{\mu \nu} \left( \delta_\sigma \gamma (-\Box) \right) \hat{\Re}_{\mu \nu} + \frac{1}{2} \hat{P} \left( \delta_\sigma \gamma (-\Box) \right) \hat{P} \\
+ \frac{1}{60} R_{\mu \nu} \left( \delta_\sigma \gamma (-\Box) \right) R^{\mu \nu} \hat{1} - \frac{1}{180} R \left( \delta_\sigma \gamma (-\Box) \right) R \hat{1} \right\} \quad (12.23)
\]

where the first two terms correctly reproduce the linear contributions to the anomaly, and the remaining terms are already quadratic in the curvature.

For the calculation of the quadratic terms in (12.23) we use the spectral representation

\[
\gamma (-\Box) = \int_0^\infty dm^2 \left( \frac{1}{m^2 - \Box} - \frac{1}{m^2 + \mu^2} \right). \quad (12.24)
\]
and the commutation (variation) rule for the inverse operator

\[
\left[ \frac{1}{m^2 - \Box}, \nabla^\mu \nabla^\nu \right] = - \frac{1}{m^2 - \Box} \left[ - \Box, \nabla^\mu \nabla^\nu \right] \frac{1}{m^2 - \Box}, \tag{12.25}
\]

\[
\delta_\sigma \frac{1}{m^2 - \Box} = - \frac{1}{m^2 - \Box} \delta_\sigma (-\Box) \frac{1}{m^2 - \Box} \tag{12.26}
\]

where, within the required accuracy, the factors on the right-hand sides can already be commuted freely. Doing the spectral-mass integral then gives

\[
\int dx \, g^{1/2} R_{\mu\nu} [\gamma (-\Box), \nabla^\mu \nabla^\nu] \sigma
\]

\[
= - \int dx \, g^{1/2} \ln(\Box / \Box_3) [\Box_3, \nabla_3^\mu \nabla_3^\nu] R_{1\mu\nu} \sigma_3 + O[R^3], \tag{12.27}
\]

and, similarly,

\[
\int dx \, g^{1/2} \text{tr} \mathcal{R}_1 (\delta_\sigma \gamma (-\Box_2)) \mathcal{R}_2
\]

\[
= - \int dx \, g^{1/2} \text{tr} \ln(\Box_1 / \Box_2) (\delta_\sigma \Box_2) \mathcal{R}_1 \mathcal{R}_2 + O[\mathcal{R}^3]. \tag{12.28}
\]

There remain to be used in (12.28) the transformation laws (12.12)–(12.15), and in (12.27) the expression for the commutator

\[
[\Box, \nabla_\mu \nabla_\nu] \sigma = 2 \nabla_\mu (R_{\nu\alpha}) \nabla^\alpha \sigma + 2 R_{\alpha(\mu} \nabla_{\nu)} \nabla^\alpha \sigma
\]

\[
- \nabla_\alpha R_{\mu\nu} \nabla^\alpha \sigma - 2 R_{\alpha\beta\mu} \nabla^\alpha \nabla^\beta \sigma \tag{12.29}
\]

in which the Riemann tensor should be expressed through the Ricci tensor.

The final result for (12.23) is

\[
\delta_\sigma \int dx \, g^{1/2} \text{tr} \left\{ \sum_{i=1}^{5} \gamma_i (-\Box_2) \mathcal{R}_1 \mathcal{R}_2 (i) \right\}
\]

\[
= \int dx \, g^{1/2} \text{tr} \left\{ \frac{1}{6} (\Box \hat{P}) \sigma - \frac{1}{180} (\Box R) \sigma \hat{1}
\]

\[
+ \sum_{i=1}^{10} M_i (\Box_1, \Box_2, \Box_3) \mathcal{R}_1 \mathcal{R}_2 \sigma_3 (i) \right\} + O[\mathcal{R}^3] \tag{12.30}
\]

where \( \mathcal{R}_1 \mathcal{R}_2 \sigma_3 (i) \) are the following ten tensor structures:

\[
\mathcal{R}_1 \mathcal{R}_2 \sigma_3 (1) = R_1 R_2 \sigma_3 \hat{1}, \tag{12.31}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \sigma_3 (2) = R_1^{\mu\nu} R_{2\mu\nu} \sigma_3 \hat{1}, \tag{12.32}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \sigma_3 (3) = R_1^{\mu\nu} \nabla_\nu \nabla_\mu R_2 \sigma_3 \hat{1}, \tag{12.33}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \sigma_3 (4) = \nabla_\mu R_1^{\nu\lambda} \nabla_\nu R_{2\mu\lambda} \sigma_3 \hat{1}, \tag{12.34}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \sigma_3 (5) = \nabla_\alpha \nabla_\beta R_{1\mu\nu} \nabla^\mu \nabla^\nu R_2^{\alpha\beta} \sigma_3 \hat{1}, \tag{12.35}
\]
\( \mathcal{R}_1 \mathcal{R}_2 \sigma_3(6) = \hat{P}_1 R_2 \sigma_3, \)  
(12.36)

\( \mathcal{R}_1 \mathcal{R}_2 \sigma_3(7) = \nabla_\alpha \nabla_\beta \hat{P}_1 R_2^{\alpha \beta} \sigma_3, \)  
(12.37)

\( \mathcal{R}_1 \mathcal{R}_2 \sigma_3(8) = \hat{P}_1 \hat{P}_2 \sigma_3, \)  
(12.38)

\( \mathcal{R}_1 \mathcal{R}_2 \sigma_3(9) = \hat{R}_1^{\mu \nu} \hat{R}_2^{\mu \nu} \sigma_3, \)  
(12.39)

\( \mathcal{R}_1 \mathcal{R}_2 \sigma_3(10) = \nabla_\mu \hat{R}_1^{\mu \alpha} \nabla_\nu \hat{R}_2^{\nu \alpha} \sigma_3, \)  
(12.40)

and for the form factors \( M_i(\Box_1, \Box_2, \Box_3) \) we obtain

\[
M_1 = \frac{(-2 \Box_1 + 5 \Box_3) \ln(\Box_1 / \Box_2)}{720 (\Box_1 - \Box_2)} + \frac{(\Box_1 + \Box_2 - \Box_3) \ln(\Box_1 / \Box_3)}{120 (\Box_1 - \Box_3)},
\]

(12.41)

\[
M_2 = \frac{(-2 \Box_1 - \Box_3) \ln(\Box_1 / \Box_2)}{120 (\Box_1 - \Box_2)} + \frac{(\Box_1 - \Box_2 - \Box_3) \ln(\Box_1 / \Box_3)}{60 \Box_2 (\Box_1 - \Box_3)},
\]

(12.42)

\[
M_3 = -\frac{1}{30} \ln(\Box_1 / \Box_3) - \frac{1}{30} \ln(\Box_2 / \Box_3),
\]

(12.43)

\[
M_4 = -\frac{1}{30} \ln(\Box_1 / \Box_2) + \frac{(\Box_1 - \Box_2 - \Box_3) \ln(\Box_1 / \Box_3)}{15 \Box_2 (\Box_1 - \Box_3)},
\]

(12.44)

\[
M_5 = \frac{1}{15 \Box_2 (\Box_1 - \Box_3)},
\]

(12.45)

\( M_6 = 0, \)

(12.46)

\( M_7 = 0, \)

(12.47)

\[
M_8 = \frac{(-2 \Box_2 - \Box_3) \ln(\Box_1 / \Box_2)}{4 (\Box_1 - \Box_2)},
\]

(12.48)

\[
M_9 = -\frac{\Box_3 \ln(\Box_1 / \Box_2)}{12 (\Box_1 - \Box_2)},
\]

(12.49)

\[
M_{10} = \frac{\ln(\Box_1 / \Box_2)}{6 (\Box_1 - \Box_2)}.
\]

(12.50)
The conformal transformation in the cubic terms of the effective action (6.1) is easier to carry out because, within the required accuracy, only the curvatures in $\mathbb{R}_1 \mathbb{R}_2 \mathbb{R}_3$ need be varied. The result is again a sum of contributions of the ten tensor structures (12.31)–(12.40):

\[
\delta_\sigma \int dx \, g^{1/2} \text{tr} \left\{ \sum_{i=1}^{29} \Gamma_i (-\Box_1, -\Box_2, -\Box_3) \mathbb{R}_1 \mathbb{R}_2 \mathbb{R}_3(i) \right\} = \int dx \, g^{1/2} \text{tr} \left\{ \sum_{i=1}^{10} N_i (\Box_1, \Box_2, \Box_3) \mathbb{R}_1 \mathbb{R}_2 \sigma_3(i) \right\} + O[\Box^3] \tag{12.51}
\]

where the form factors $N_i (\Box_1, \Box_2, \Box_3)$ are the following combinations of the form factors $\Gamma_i (-\Box_1, -\Box_2, -\Box_3)$:

\[
N_1 = (-\Box_1 + 2\Box_3) \Gamma_{11}^{\text{sym}} \mid_{\Box_1 \leftrightarrow \Box_3} \\
+ \frac{1}{4} ((\Box_1 - \Box_2 - \Box_3)(-\Box_1 + \Box_2 - \Box_3) \\
+ \Box_3(-\Box_1 - \Box_2 + \Box_3)) \Gamma_{22}^{\text{sym}} \mid_{\Box_1 \leftrightarrow \Box_3} \\
+ \left( \frac{1}{4} (\Box_1 - \Box_2 - \Box_3)^2 - \frac{1}{2} \Box_1 \Box_2 \Box_3 \right) \\
+ \left( \frac{1}{4} (\Box_1 - \Box_2 - \Box_3)(-\Box_1 - \Box_2 + \Box_3) \right. \\
\left. - \frac{1}{4} \Box_3(-\Box_1 - \Box_2 + \Box_3) \right) \Gamma_{22}^{\text{sym}} \mid_{\Box_1 \leftrightarrow \Box_3} \\
+ \frac{1}{16} (-\Box_1 - \Box_2 + \Box_3)^2 \Gamma_{25}^{\text{sym}} \mid_{\Box_1 \leftrightarrow \Box_3} \\
+ \frac{1}{2} (-\Box_1 + \Box_3) \left( \frac{1}{2} (\Box_1 - \Box_2 - \Box_3)^2 + \Box_2 \Box_3 \right) \Gamma_{27}^{\text{sym}} \mid_{\Box_1 \leftrightarrow \Box_3} \\
\left. + \frac{1}{16} (-\Box_1 + \Box_2 - \Box_3)(-\Box_1 - \Box_2 + \Box_3)^2 \Gamma_{28}^{\text{sym}} \mid_{\Box_1 \leftrightarrow \Box_3} \right. \\
\left. + 9 \Box_3 \Gamma_{9}^{\text{sym}} + \frac{3}{8} (-\Box_1 - \Box_2 + \Box_3) \Gamma_{10}^{\text{sym}} \right. \\
\left. - \frac{3}{2} \Box_3(-\Box_1 - \Box_2 + \Box_3) \Gamma_{22}^{\text{sym}} \right. \\
\left. + \frac{1}{4} (\Box_1 - 2\Box_3)(-\Box_1 - \Box_2 + \Box_3) \Gamma_{24}^{\text{sym}} \right. \\
\left. + \frac{1}{8} (\Box_1 - \Box_2 - \Box_3)(-\Box_1 - \Box_2 + \Box_3) \Gamma_{25}^{\text{sym}} \right. \\
\left. + \frac{1}{32} (-\Box_1 - \Box_2 + \Box_3)( (\Box_1 - \Box_2 - \Box_3)(-\Box_1 + \Box_2 - \Box_3) \right. \\
\left. + \Box_3(-\Box_1 - \Box_2 + \Box_3) \right) \Gamma_{28}^{\text{sym}}, \tag{12.52}
\]

\[
N_2 = \frac{1}{4} ((\Box_1 - \Box_2 - \Box_3)(-\Box_1 + \Box_2 - \Box_3)
\]

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\[
N_3 = -\frac{1}{2} (-\Box_1 - \Box_2 + \Box_3) \Gamma_{24}^{\text{sym}} |_{\Box_1 \rightarrow \Box_2} \\
+ \frac{1}{2} (\Box_1 + \Box_2 - \Box_3) \Gamma_{25}^{\text{sym}} |_{\Box_1 \rightarrow \Box_2} \\
+ \frac{1}{2} (-\Box_1 - \Box_2 + \Box_3) \Gamma_{25}^{\text{sym}} |_{\Box_1 \rightarrow \Box_3} \\
+ \frac{1}{4} (-\Box_1 + \Box_2 - \Box_3)(-\Box_1 - \Box_2 + \Box_3) \Gamma_{28}^{\text{sym}} |_{\Box_1 \rightarrow \Box_3} \\
+ 2 \Gamma_{11}^{\text{sym}} |_{\Box_2 \rightarrow \Box_3} + (-\Box_1 + \Box_2 - \Box_3) \Gamma_{25}^{\text{sym}} |_{\Box_2 \rightarrow \Box_3} \\
+ \left(\frac{1}{2} (-\Box_1 + \Box_2 - \Box_3)^2 + \Box_1 \Box_3\right) \Gamma_{27}^{\text{sym}} |_{\Box_2 \rightarrow \Box_3} \\
+ \frac{1}{4} (\Box_1 - \Box_2 - \Box_3)(-\Box_1 - \Box_2 + \Box_3) \\
- \Box_3(-\Box_1 - \Box_2 + \Box_3) \Gamma_{28}^{\text{sym}} |_{\Box_2 \rightarrow \Box_3} \\
+ 3 \Gamma_{10}^{\text{sym}} - 6 \Box_3 \Gamma_{22}^{\text{sym}} + (\Box_1 - 2 \Box_3) \Gamma_{24}^{\text{sym}} \\
+ \frac{1}{2} (\Box_1 - \Box_2 - 2 \Box_3) \Gamma_{25}^{\text{sym}} \\
+ \frac{1}{4} (\Box_1 - \Box_2 - \Box_3)(-\Box_1 + \Box_2 - \Box_3) \\
+ \Box_3(-\Box_1 - \Box_2 + \Box_3) \Gamma_{28}^{\text{sym}} \\
+ \frac{3}{8} (-\Box_1 + \Box_3)((-\Box_1 + \Box_2 - \Box_3)^2 + 2 \Box_1 \Box_3) \Gamma_{29}^{\text{sym}},
\]

(12.53)

\[
N_4 = \frac{1}{2} \Box_3 \Gamma_{25}^{\text{sym}} |_{\Box_1 \rightarrow \Box_3} + 3 \Gamma_{10}^{\text{sym}} + 3 \Box_3 \Gamma_{23}^{\text{sym}} + (\Box_1 - \Box_2 - 2 \Box_3) \Gamma_{25}^{\text{sym}} \\
+ \frac{1}{4} (\Box_1 - \Box_2 - \Box_3)(-\Box_1 + \Box_2 - \Box_3) \\
+ \Box_3(-\Box_1 - \Box_2 + \Box_3) \Gamma_{28}^{\text{sym}},
\]

(12.54)

\[
N_5 = \Gamma_{25}^{\text{sym}} |_{\Box_1 \rightarrow \Box_3} - (\Box_1 - \Box_2 + 2 \Box_3) \Gamma_{28}^{\text{sym}} |_{\Box_1 \rightarrow \Box_3} \\
- 2 \Gamma_{24}^{\text{sym}} + 3 \Box_3 \Gamma_{27}^{\text{sym}} \\
+ \frac{3}{4} ((-\Box_1 + \Box_2 - \Box_3)^2 + 2 \Box_1 \Box_3) \Gamma_{29}^{\text{sym}},
\]

(12.55)

\[
N_6 = -\frac{3}{4} \Box_3((-\Box_1 - \Box_2 + \Box_3) \Gamma_{15}^{\text{sym}} |_{\Box_1 \rightarrow \Box_2, \Box_2 \rightarrow \Box_3, \Box_3 \rightarrow \Box_1}
\]

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The total result of (12.30) and (12.51) is the following conformal variation of the effective action (6.1):

\[-\delta_\omega W = \frac{1}{2(4\pi)^2} \int dx \, g^{1/2} \, \text{tr} \left\{ -\frac{1}{6}(\square P)\sigma - \frac{1}{180}(\square R)\sigma \hat{I} \\
+ \sum_{i=1}^{10} (M_i + N_i)\mathcal{R}_i R_2 \sigma_3(i) \right\} + O[\mathcal{R}^3] \]
where the quadratic terms are determined by the sum $M_i + N_i$. There remain to be calculated the linear combinations of the third-order form factors $\Gamma_i$ in (12.52)–(12.61). This is the most important part of the calculation because it checks both: our results for the third-order form factors and our capability of working with them. The simplest is to use the explicit forms of $\Gamma_i$ in (6.13)–(6.41). This way of calculation is most straightforward and least instructive. It is, nevertheless, gratifying to observe that all terms with the basic third-order form factor $\Gamma(-\Box_1, -\Box_2, -\Box_3)$ cancel in the combinations $N_i$, all terms with the second-order form factors $\ln(\Box_n/\Box_m)$ cancel in the combinations $N_i + M_i$, and there remain only trees:

$$\frac{1}{2} \left( M_1 + N_1 + M_1|_{\Box_1 \rightarrow \Box_2} + N_1|_{\Box_1 \rightarrow \Box_2} \right) = 0, \quad (12.63)$$

$$\frac{1}{2} \left( M_2 + N_2 + M_2|_{\Box_1 \rightarrow \Box_2} + N_2|_{\Box_1 \rightarrow \Box_2} \right) = -\frac{1}{180} \Box_3 + \frac{1}{180} \Box_2 - \frac{1}{180} \Box_1, \quad (12.64)$$

$$M_3 + N_3 = 0, \quad (12.65)$$

$$\frac{1}{2} \left( M_4 + N_4 + M_4|_{\Box_1 \rightarrow \Box_2} + N_4|_{\Box_1 \rightarrow \Box_2} \right) = -\frac{1}{45} \Box_3 \Box_1 + \frac{1}{45} \Box_3 \Box_2, \quad (12.66)$$

$$\frac{1}{2} \left( M_5 + N_5 + M_5|_{\Box_1 \rightarrow \Box_2} + N_5|_{\Box_1 \rightarrow \Box_2} \right) = -\frac{1}{45} \Box_1 \Box_2, \quad (12.67)$$

$$M_6 + N_6 = 0, \quad (12.68)$$

$$M_7 + N_7 = 0, \quad (12.69)$$

$$\frac{1}{2} \left( M_8 + N_8 + M_8|_{\Box_1 \rightarrow \Box_2} + N_8|_{\Box_1 \rightarrow \Box_2} \right) = -\frac{1}{2}, \quad (12.70)$$

$$\frac{1}{2} \left( M_9 + N_9 + M_9|_{\Box_1 \rightarrow \Box_2} + N_9|_{\Box_1 \rightarrow \Box_2} \right) = -\frac{1}{12}, \quad (12.71)$$

$$\frac{1}{2} \left( M_{10} + N_{10} + M_{10}|_{\Box_1 \rightarrow \Box_2} + N_{10}|_{\Box_1 \rightarrow \Box_2} \right) = 0. \quad (12.72)$$

The symmetrizations on the left-hand sides of these equations correspond to the symmetries of the tensor structures (12.31)–(12.40). With the form factors (12.63)–(12.72), eq. (12.62) takes the final form

$$-\delta W = \frac{1}{2(4\pi)^2} \int dx g^{1/2} \text{tr} \left\{ -\frac{1}{6} (\hat{P}) \sigma - \frac{1}{180} (\Box R) \sigma \hat{1} 
- \frac{1}{180} \left( 1 + \frac{4 \Box_3}{\Box_2} + \frac{\Box_3^2}{\Box_1 \Box_2} \right) \mathcal{R}_1 \mathcal{R}_2 \sigma_3(2) 
- \frac{1}{45} \left( 2 \frac{\Box_3}{\Box_1} - \frac{\Box_3}{\Box_1 \Box_2} \right) \mathcal{R}_1 \mathcal{R}_2 \sigma_3(4) 
- \frac{1}{45} \frac{1}{\Box_1 \Box_2} \mathcal{R}_1 \mathcal{R}_2 \sigma_3(5) - \frac{1}{2} \mathcal{R}_1 \mathcal{R}_2 \sigma_3(8) 
- \frac{1}{12} \mathcal{R}_1 \mathcal{R}_2 \sigma_3(9) \right\} + O[\mathfrak{F}^3] \quad (12.73)$$
which is precisely the trace anomaly (12.20).

The derivation of the anomaly is not, however, an end in itself. It is also important that this and similar derivations be feasible within the working technique used in applications. The generalization of the spectral representation elaborated in sect. 20 and summarized in sect. 9 serves this purpose. The task is again to calculate the linear combinations (12.52)–(12.61) of the form factors $\Gamma_i$ but in terms of integral originals. As noted repeatedly in the present paper, the difficulty that such a calculation encounters is connected with the fact that the coefficients of the linear combinations of the form factors contain $\Box$’s. Because these $\Box$’s appear outside the kernel of the integral representation, there exist nontrivial linear identities between the form factors not expressible in terms of the originals. Examples of such identities are considered in sect. 18. In the calculation of expressions (12.52)–(12.61), this difficulty is present in full measure. The result should be an almost complete cancellation leading to eqs. (12.63)–(12.72), and this cancellation is based entirely on the hidden identities of the said type. Below we illustrate the derivation of the anomaly within the generalized spectral technique by deriving one of these identities.

The example we shall consider is given in eq. (18.11) of sect. 18. In terms of the $\alpha$-integrals, the following linear combination of the third-order form factors:

$$L = (\Box_1 - \Box_2) \left\langle \frac{\alpha_3}{-\Omega} \right\rangle_3 + \Box_3 \left\langle \frac{(\alpha_1 - \alpha_2)}{-\Omega} \right\rangle_3$$

(12.74)

should equal the second-order form factor

$$L = \ln(\Box_2/\Box_1).$$

(12.75)

In terms of the generalized spectral integrals, expression (12.74) is of the form (see sect. 9)

$$L = -(\Box_1 - \Box_2)2 \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^3 (C_3 - 1)C_1C_2C_3S_1S_2S_3$$

$$- \Box_32 \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^3 (C_1 - C_2)C_1C_2C_3S_1S_2S_3. $$

(12.76)

In order to bring it to the final form (12.75), one must first absorb the $\Box$’s by using relations (9.68) and (9.69). The result is

$$L = 2 \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^4 \left[(C_3 - 1)(C_1 - 1)^2 - (C_3 - 1)(C_2 - 1)^2ight.$$

$$+ (C_1 - C_2)(C_3 - 1)^2 \left] \right. (C_1)(C_2)(C_3 - 1)^2$$

$$+ (C_3)(2C_1 - 3)(C_3 - 1)^2 \right) C_1C_2C_3S_1S_2S_3$$

(12.77)

where the convergence at the lower limit holds owing to the presence of the operator $(C-1)^2$ in each term. Next, one must use in (12.77) the relation (9.73)

$$C_1 + C_2 + C_3 = 3$$

(12.78)

to remove the operators $C$ from one of the $S$’s, say $S_1$, but then, in the term with $(C_1 - 1)^2$, the bare $S_1$ will appear with a subtraction:

$$L = 2 \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^4 \left\{(C_3 - 1)(C_2 + C_3 - 2)^2C_1C_2C_3(S_1 - S_1^1)S_2S_3ight.$$

$$- \left[(C_3 - 1)(C_2 - 1)^2 + (2C_2 + C_3 - 3)(C_3 - 1)^2 \right.\right.$$}

$$\times C_1C_2C_3S_1S_2S_3\}$$

(12.79)
where $S^4$ is $S^M$ in (9.67) with $M = 1$. At this stage, the triple-spectral contribution cancels, and we obtain

$$L = -2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^4 \left[ (C_3 - 1)(C_2 - 1)^2 + 2(C_2 - 1)(C_3 - 1)^2 + (C_3 - 1)^3 \right] C_1 C_2 C_3 S_1^1 S_2 S_3. \quad (12.80)$$

Next, one must again use the relation (12.78), to remove the operators $C$ from $S_2$, but, in the term with $(C_2 - 1)^2$, the bare $S_2$ will appear with a subtraction. At this stage, the double-spectral contribution vanishes owing to the fact that

$$(C_1 - 1)^2 S_1^1 = 0, \quad (12.81)$$

and we obtain

$$L = 2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^4 (C_1 - C_2)(C_3 - 1)^2 C_1 C_2 C_3 S_1^1 S_2^1 S_3. \quad (12.82)$$

With $S^4$ in (9.67), this integral takes the form

$$L = \frac{1}{2} \ln(\Box_2/\Box_1) \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 (C_3 - 1)^2 C_3 S_3 \quad (12.83)$$

where the integrand is a total derivative. Hence

$$L = -8 \ln(\Box_2/\Box_1) \left[ \frac{1}{y^2} (C_3 - 1) C_3 S_3 \right]_{y=0} = \ln(\Box_2/\Box_1). \quad (12.84)$$

These are the typical cancellations occurring in expressions (12.52)–(12.61). The amount of calculations with the spectral forms is much smaller than with the explicit forms of sect. 6, and the result is again eqs. (12.63)–(12.72). In this way the anomaly is derived in the generalized spectral technique.

Finally, the third way of deriving the anomaly that we present is making the conformal transformation in the trace of the heat kernel. The expansion of $\text{Tr}K(s)$ in powers of the curvatures is given in eq. (2.1). To enable a comparison with the effective action (6.1), one must subtract from the heat kernel the terms of zeroth and first order in the curvature (see a remark to eq. (6.1)). These are the first two terms of expression (2.1). For $\text{Tr}K(s)$ with these terms subtracted we introduce the notation ($\omega = 2$)

$$\text{Tr}K'(s) = \text{Tr}K(s) - \frac{1}{(4\pi s)^2} \int dx g^{1/2} \text{tr} (\hat{1} + s\hat{P}). \quad (12.85)$$

The second-order terms in (2.1) transform like in eq. (12.23) but, instead of the form factor $\gamma(-\Box)$, one has to deal with the form factors

$$f(-s\Box), \quad \frac{f(-s\Box) - 1}{s\Box}, \quad \frac{f(-s\Box) - 1 - \frac{1}{6}s\Box}{(s\Box)^2} \quad (12.86)$$
where \( f(-s\Box) \) is given in (2.9). The counterparts of eqs. (12.27) and (12.28) are, in this case *

\[
\int dx \, g^{1/2} R_{\mu\nu} \left[ f(-s\Box) , \nabla^\mu \nabla^\nu \right] \sigma \\
= \int dx \, g^{1/2} \frac{f(-s\Box_1) - f(-s\Box_3)}{\Box_1 - \Box_3} \left[ \Box_3, \nabla_3^\mu \nabla_3^\nu \right] R_{1\mu\nu} \sigma_3 + O[R^3..], \\
(12.87)
\]

\[
\int dx \, g^{1/2} \, \text{tr} \, \mathcal{R}_1 \left( \delta_\sigma f(-s\Box_2) \right) \mathcal{R}_2 \\
= \int dx \, g^{1/2} \, \text{tr} \left( \frac{f(-s\Box_1) - f(-s\Box_2)}{\Box_1 - \Box_2} \right) \delta_\sigma \Box_2 \mathcal{R}_1 \mathcal{R}_2 + O[\mathcal{R}^3]. \\
(12.88)
\]

The third-order terms in (2.1) transform in a way completely similar to the above. An important distinction from the previous case is that the linear nonlocal terms do not cancel.

The total result for \( \text{Tr} K'(s) \) divided by \( s \) is of the form

\[
\frac{1}{s} \delta_\sigma \text{Tr} K'(s) = \frac{1}{(4\pi)^2} \int dx \, g^{1/2} \, \text{tr} \left\{ \sigma \Box t_1(s, \Box) \hat{P} + \sigma \Box t_2(s, \Box) \hat{R} \hat{P} \\
+ \sum_{i=1}^{10} T_i(s, \Box_1, \Box_2, \Box_3) \mathcal{R}_1 \mathcal{R}_2 \sigma_3(i) \right\} + O[\mathcal{R}^3] \\
(12.89)
\]

where \( \mathcal{R}_1 \mathcal{R}_2 \sigma_3(i) \) are the tensor structures (12.31)–(12.40), and the functions \( t_1, t_2, T_i \) are obtained as certain combinations of the form factors in the heat kernel. The differential equations for these form factors can next be used the same way as in sect. 5 * to bring the functions \( t_1, t_2, T_i \) to the form of total derivatives:

\[
t_1 = \frac{d}{ds} \bar{t}_1, \quad t_2 = \frac{d}{ds} \bar{t}_2, \quad T_i = \frac{d}{ds} \bar{T}_i. \\
(12.90)
\]

The final result for \( \bar{t}_1, \bar{t}_2, \bar{T}_i \) is worth presenting in full. In terms of the basic form factors in the heat kernel:

\[
f(-s\Box), \quad F(-s\Box_1, -s\Box_2, -s\Box_3) \\
(12.91)
\]

(eqs. (2.9) and (2.75)), and the determinant \( D \) in eq. (6.11), we obtain

\[
\bar{t}_1 = \frac{f(-s\Box) - 1}{s\Box}, \\
(12.92)
\]

\[
\bar{t}_2 = \frac{1}{12} \frac{f(-s\Box) - 1}{s\Box} - \frac{1}{2} \frac{f(-s\Box) - 1 - \frac{1}{6} s\Box}{(s\Box)^2}, \\
(12.93)
\]

\[
\bar{T}_1 = F(-s\Box_1, -s\Box_2, -s\Box_3) \frac{1}{36D^4} (\Box_3^8 - 4\Box_1\Box_3^7 - 16\Box_1^2\Box_3^6) \\
+ 68\Box_1^3\Box_3^5 - 100\Box_1^4\Box_3^4 + 68\Box_1^5\Box_3^3 - 16\Box_1^6\Box_3^2 - 4\Box_1^7\Box_3
\]

*For the derivation see sect. 14, eq. (14.8).

*One makes the substitutions (5.7)–(5.9).
\[ T_2 = \frac{1}{s^2} \left( F(-s \circ_1, -s \circ_2, -s \circ_3) - \frac{1}{2} - \frac{1}{24} s(\circ_1 + \circ_2 + \circ_3) \right) \frac{2}{\circ_1 \circ_2} - \left( \frac{f(-s \circ_3) - 1}{s \circ_3} \right) \frac{\circ_3}{24D^4} \left( \circ_3^7 + 2\circ_3^6 \circ_1 - 38\circ_3^5 \circ_1^2 + 90\circ_3^4 \circ_1^3 \right) \\
- \left( \frac{f(-s \circ_1) - 1 - \frac{4}{3} s \circ_1}{(s \circ_1)^2} \right) \frac{2\circ_1^2}{D^3} \left( \circ_3^4 - 4\circ_2 \circ_3^3 - 6\circ_1^2 \circ_3^2 \right) \\
+ \left( \frac{f(-s \circ_3) - 1 - \frac{4}{3} s \circ_3}{(s \circ_3)^2} \right) \frac{\circ_3}{4D^3} \left( 7\circ_3^5 - 22\circ_1 \circ_3^4 - 20\circ_3^3 \circ_1^2 \right) \\
+ 52\circ_3^2 \circ_1^3 - 26\circ_3 \circ_1^4 + 2\circ_1^5 + 36\circ_3^3 \circ_1 \circ_2 - 52\circ_3^2 \circ_2 \circ_1^2 \]

\[ + 8\circ_3 \circ_1^3 \circ_2 - 6\circ_2 \circ_1^4 + 18\circ_3 \circ_1^2 \circ_2^2 + 4\circ_2^2 \circ_1^3, \]

(12.94)
\[
\hat{T}_3 = -F(-s\Box_1, -s\Box_2, -s\Box_3) \frac{2\Box_1}{3D^4} (\Box_1^6 - 9\Box_1^4\Box_2^2 - 9\Box_3^2\Box_1^4
\]
\[
- 2\Box_1^4\Box_2\Box_3 + 16\Box_1^3\Box_2^3 + 8\Box_1^3\Box_2^2\Box_3 + 8\Box_1^3\Box_2\Box_3^2 + 16\Box_3^3\Box_1^3
\]
\[
+ 10\Box_1^2\Box_2^3 \Box_1^2 - 9\Box_1^2\Box_2^2 - 9\Box_3^4\Box_1^2 - 12\Box_1^2\Box_2\Box_3^2 - 12\Box_1^2\Box_2^3\Box_3
\]
\[
+ 8\Box_1\Box_2^3\Box_3 - 8\Box_1\Box_2^2\Box_3^2 - 8\Box_1\Box_2\Box_3^3 + 8\Box_1\Box_2^3\Box_3 - 2\Box_2^5\Box_3
\]
\[
- \Box_2^3\Box_3^3 - 2\Box_3^5\Box_2 + \Box_3^6 + 4\Box_2^3\Box_3^3 - \Box_2^3\Box_3^3 + \Box_2^4 + \Box_2^3\Box_3
\]
\[
+ \frac{1}{s} \left( F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} \right) \frac{4}{3D^3} (40\Box_1^2\Box_2\Box_3 + 19\Box_1^4
\]
\[
- 22\Box_1^3\Box_2 - 22\Box_1^3\Box_3 - 12\Box_1^2\Box_2^2 - 12\Box_1^2\Box_2^2 - 12\Box_1^2\Box_2^3\Box_3
\]
\[
- 14\Box_1\Box_2^3\Box_3 + 14\Box_1\Box_2^2\Box_3 + 14\Box_1\Box_2^3\Box_3 + \Box_2^4 + 6\Box_2^2\Box_3^2 - 4\Box_2^3\Box_3
\]
\[
- 4\Box_2^3\Box_3^3 + \Box_3^4)
\]
\[
\frac{1}{s^2} \left( F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} - \frac{1}{24} s(\Box_1 + \Box_2 + \Box_3) \right) \frac{48\Box_1}{D^2}
\]
\[
+ \left( \frac{f(-s\Box_1) - 1}{s\Box_1} \right) \frac{4\Box_1^2}{3D^4} (\Box_1^5 + \Box_1^4\Box_2 + \Box_1^3\Box_3 - 8\Box_1^3\Box_3^2
\]
\[
- 8\Box_1^3\Box_2^2 + 8\Box_1^2\Box_2^3 + 8\Box_1^2\Box_3^3 - \Box_1\Box_2^4 - 4\Box_1\Box_2\Box_3 + \Box_2^3\Box_3
\]
\[
- \Box_1\Box_3^4 - 4\Box_2\Box_3^3\Box_1 + 10\Box_1\Box_2^2\Box_3 + 3\Box_2\Box_3^3 - \Box_2^5 + \Box_3^5 - 2\Box_2^3\Box_3^3 - 2\Box_2\Box_3^2 + 3\Box_2\Box_3)
\]
\[
- \left( \frac{f(-s\Box_2) - 1}{s\Box_2} \right) \frac{1}{6\Box_1D^4} (\Box_1^8 + \Box_1^7 + 28\Box_1^6\Box_3^2 - 56\Box_1^5\Box_3^3
\]
\[
- 56\Box_1^3\Box_3^5 + 70\Box_1^4\Box_3^4 + 28\Box_1^3\Box_3^6 + 8\Box_1^3\Box_3^7 - 8\Box_1^7\Box_3 - 8\Box_1^7\Box_3
\]
\[
- 61\Box_1^3\Box_3^7 - 7\Box_2\Box_3^7 + 35\Box_2^4\Box_3^4 - 35\Box_2^3\Box_3^5 + 21\Box_2^2\Box_3^6 + 99\Box_1\Box_2^2
\]
\[
- 99\Box_1^5\Box_2 - 29\Box_3^4\Box_2 + \Box_1^4\Box_2^2 + 29\Box_1^6\Box_2^2 + 29\Box_1^6\Box_2^2
\]
\[
+ \Box_1^7\Box_2 - \Box_1\Box_2^7 - \Box_2^7\Box_3 - 21\Box_2^5\Box_3 + 7\Box_2^6\Box_3^2 + 23\Box_1\Box_2^6\Box_3
\]
\[
- 27\Box_1\Box_2^2\Box_3^5 - 5\Box_1^4\Box_2^2\Box_3^2 + 139\Box_1^4\Box_2^2\Box_3^3 - 99\Box_1^5\Box_2\Box_3^2
\]
\[
+ 31\Box_2^4\Box_3^6 + 100\Box_1^3\Box_2^2\Box_3^2 + 50\Box_1^3\Box_2^3\Box_3^2 + 81\Box_1^4\Box_2^3\Box_3
\]
\[
- 45\Box_1^2\Box_2^3\Box_3^4 + 66\Box_1^2\Box_2^4\Box_3^2 - 39\Box_1^2\Box_2^5\Box_3 - 47\Box_1^2\Box_2^4\Box_3^2
\]
\[
+ 14\Box_1^3\Box_2^3\Box_3^2 + 60\Box_1^2\Box_2^4\Box_3^2 - 41\Box_1^3\Box_2^3\Box_3^2 - 66\Box_1^5\Box_2^2\Box_3
\]
\[
- 15\Box_1\Box_2^3\Box_3^4 - 34\Box_1\Box_2^2\Box_3^4 + 18\Box_1\Box_2^2\Box_3^2\Box_3^3)
\]
\[
- \left( \frac{f(-s\Box_3) - 1}{s\Box_3} \right) \frac{\Box_3}{6\Box_1D^4} (\Box_1^6\Box_3 + 29\Box_1^4\Box_3^3 - \Box_3^7 + 9\Box_1^7 + \Box_2^7
\]
\[
- 7\Box_2^6\Box_3 - 35\Box_2^4\Box_3^3 - 21\Box_2^2\Box_3^5 + 7\Box_2^2\Box_3^6 + 35\Box_2^2\Box_3^4
\]
\[
+ 7\Box_2^6\Box_1 - 43\Box_1^5\Box_3^2 - 27\Box_1^5\Box_2^2 + 27\Box_1^3\Box_3^4 - 29\Box_1^2\Box_3^5
\]
\[
- 17\Box_1^6\Box_2 - 9\Box_1^6\Box_2 + 45\Box_1^2\Box_2^5 - 101\Box_1^3\Box_2^4 + 99\Box_1^4\Box_2^3
\]
\[
+ 21\Box_1^2\Box_3^6 + 6\Box_1^5\Box_2^2\Box_3 + 30\Box_1^2\Box_3^2\Box_3^3
\]
\[
+ 25\Box_1^2\Box_3^4 - 81\Box_1^2\Box_3^4\Box_3 + 84\Box_1^3\Box_3^2\Box_3 + 20\Box_1^2\Box_3^3\Box_3
\]
\[
+ 34\Box_1^3\Box_3^2\Box_3^2 - 55\Box_1^2\Box_3^3\Box_3^2 + 2\Box_1^2\Box_3^3\Box_3^2 - 41\Box_1^4\Box_2^2\Box_3
\]
\]
\[(12.95)\]
\[ + 41\square_1^4\square_2\square_3^2 - 44\square_1^3\square_2^2\square_3^3 + 33\square_1^2\square_2\square_3^4 + 38\square_1\square_2^5\square_3 \\
- \left( \frac{f(-s\square_1) - 1 - \frac{1}{6}s\square_1}{(s\square_1)^2} \right) \frac{16\square_1^3}{D^3} (-2\square_3^2 + 4\square_2\square_3 + 3\square_1^2) \\
- \square_1\square_2 - 2\square_2^2 - \square_3\square_1 \\
+ \left( \frac{f(-s\square_2) - 1 - \frac{1}{6}s\square_2}{(s\square_2)^2} \right) \frac{1}{\square_1 D^3} (\square_1^6 + \square_3^6 + 58\square_1^4\square_2^2 \\
+ 15\square_3^2\square_1^4 - 18\square_1^3\square_2^3 - 20\square_3^3\square_1^3 - 27\square_1^2\square_2^4 + 15\square_3^4\square_1^2 \\
- \square_2^5\square_3 + 10\square_3^2\square_1^4 - 5\square_3^5\square_2 - 10\square_3^2\square_1^3 + 5\square_3^2\square_2^4 \\
+ 31\square_3^4\square_1\square_2 - 34\square_1^3\square_2^2\square_3^2 + 68\square_1^3\square_2^2\square_3 + 6\square_1^2\square_2\square_3^3 \\
+ 12\square_3^2\square_1^2\square_2^2 + 22\square_1\square_2^4\square_3 + 8\square_1\square_2^2\square_3^3 - 6\square_1^2\square_2^3\square_3 \\
- 30\square_1\square_2^3\square_3^2 + 11\square_1\square_2^2\square_3^4 - 6\square_1^5\square_3 - 9\square_1^5\square_2 \\
- 5\square_1\square_2^5 - 6\square_1\square_3^5) \\
- \left( \frac{f(-s\square_3) - 1 - \frac{1}{6}s\square_3}{(s\square_3)^2} \right) \frac{\square_3}{\square_1 D^3} (-\square_2^5 - 10\square_2^3\square_3^2 - 5\square_2\square_3^4 \\
+ 10\square_2^2\square_3^3 + \square_3^5 - \square_1\square_3^4 + 3\square_1^5 - 6\square_1^2\square_2\square_3^2 - 4\square_2^3\square_3\square_1 \\
- 18\square_1^3\square_2^3 - 22\square_1^3\square_2^2 + 18\square_1\square_2^2\square_3^2 + 76\square_1^3\square_2\square_3 \\
- 20\square_1^2\square_2^3\square_3 - 18\square_1^2\square_2^2\square_3 + 5\square_2^4\square_3 - 13\square_1^4\square_2 \\
- 43\square_1^4\square_3^2 - 2\square_3^3\square_3^2 + 42\square_1^2\square_3^3 + 7\square_1\square_2^4) \\
- \frac{1}{\square_2 - \square_3} \left( \frac{f(-s\square_2) - 1 - \frac{1}{6}s\square_2}{(s\square_2)^2} - \frac{f(-s\square_3) - 1 - \frac{1}{6}s\square_3}{(s\square_3)^2} \right) \frac{\square_3}{6\square_1} \\
+ \frac{1}{\square_2 - \square_3} \left( \frac{f(-s\square_2) - 1 - \frac{1}{6}s\square_2}{(s\square_2)^2} - \frac{f(-s\square_3) - 1 - \frac{1}{6}s\square_3}{(s\square_3)^2} \right) \frac{\square_3}{6\square_1}, \quad (12.96) \]

\[ \bar{T}_4 = -\frac{1}{s^2} \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} \right) \frac{8}{D^2} (-\square_3^2 + 2\square_1^2 - 2\square_2\square_1) \\
+ \frac{1}{s^2} \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} - \frac{1}{24}s(\square_1 + \square_2 + \square_3) \right) \\
\times \frac{8(-\square_3^2 + 2\square_1)}{D\square_1\square_2} \\
- \left( \frac{f(-s\square_3) - 1}{s\square_3} \right) \frac{\square_3}{2\square_2\square_1} \\
+ \left( \frac{f(-s\square_1) - 1 - \frac{1}{6}s\square_1}{(s\square_1)^2} \right) \frac{32\square_1^2(\square_3 + \square_1 - \square_2)}{D^2} \\
+ \frac{\left( f(-s\square_3) - 1 - \frac{1}{6}s\square_3 \right) \frac{\square_3}{D^2\square_2\square_1} (5\square_3^4 - 32\square_3^3\square_1 + 36\square_3^2\square_1^2} \\
- 16\square_3\square_1^3 + 2\square_1^4 - 20\square_3^2\square_1\square_2 + 16\square_2\square_3\square_1^2 \\
- 8\square_2\square_1^3 + 6\square_1^2\square_2^2), \quad (12.97) \]

\[ \bar{T}_5 = F(-s\square_1, -s\square_2, -s\square_3) \frac{4\square_1^2\square_2}{D^4} (\square_3^4 - 4\square_1^2\square_3^2 + 2\square_1^4) \]
\[+4\square_1 \square_2 \square_3^2 - 8\square_2 \square_1^3 + 6\square_1^2 \square_2^2\]
\[- \frac{1}{s} \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} \right) \frac{16}{D^3} \left( \square_3^3 - \square_1 \square_3^2 - 4\square_1^2 \square_3 + 3\square_1^3 + 4\square_1 \square_2 \square_3 - 3\square_1^2 \square_2 \right)\]
\[+ \frac{1}{s^2} \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} - \frac{1}{24} s(\square_1 + \square_2 + \square_3) \right) \frac{8}{D^2 \square_1 \square_2} \times (\square_3^2 - 4\square_1 \square_3 + 2\square_1^2 + 4\square_1 \square_2)\]
\[- \left( \frac{f(-s\square_1) - 1}{s \square_1} \right) \frac{16 \square_1^2 \square_2}{D^4} (\square_1^3 + \square_1^2 \square_3 - 3\square_1^2 \square_2 + 3\square_1 \square_2^2)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{2D^4 \square_1 \square_2} (-\square_3^3 + 14\square_1 \square_3^2 - 42\square_1^2 \square_3^5)\]
\[+70 \square_1^2 \square_3^3 - 70\square_1^4 \square_3^3 + 42\square_1^5 \square_3^3 - 14\square_3^5 \square_1^6 + 2\square_1^7\]
\[-26\square_1 \square_2 \square_3^5 + 50\square_1^2 \square_2 \square_3^4 + 40\square_1^3 \square_2 \square_3^3 - 110\square_1^4 \square_2 \square_3^2 + 76\square_1^5 \square_2 \square_3 - 18\square_1^6 \square_2 + 30\square_1^2 \square_2 \square_3^2 + 68\square_1^3 \square_2 \square_3^2\]
\[-178\square_1^4 \square_2 \square_3^3 + 42\square_1^5 \square_2 \square_3 + 116\square_1^3 \square_2 \square_3^3 - 26\square_1^4 \square_2 \square_3^3\]
\[+ \left( \frac{f(-s\square_1) - 1}{s \square_1} \right) \frac{32 \square_1^2}{D^3} (3\square_1^2 - \square_1 \square_3 - \square_1 \square_2)\]
\[+ 4\square_2 \square_3 - 2\square_3^2 - 2\square_2^2)\]
\[+ \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^4 \square_1 \square_2} (-5\square_3^3 + 42\square_1 \square_3^4 - 68\square_1^2 \square_3^3)\]
\[+ 52\square_1^3 \square_3^2 - 18\square_1^4 \square_3 + 2\square_1^5 + 20\square_1 \square_2 \square_3^3 - 52\square_1^2 \square_2 \square_3^2\]
\[- 24\square_1^3 \square_3 \square_3 - 6\square_1^4 \square_2 + 42\square_1^2 \square_2 \square_3^3 + 4\square_1^3 \square_2^2)\]
\[+ 1 \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} \right) \frac{4\square_1}{D}\]
\[+ \left( \frac{f(-s\square_1) - 1}{s \square_1} \right) \frac{4 \square_1^2}{D^2} (\square_1 \square_2 + \square_1 \square_3 - \square_2^2 - \square_3^2 + 2\square_2 \square_3)\]
\[- \left( \frac{f(-s\square_2) - 1}{s \square_2} \right) \frac{\square_2}{D^2} (\square_1^3 + \square_2 \square_1^2 - \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^2} (\square_1^3 - \square_2 \square_1^2 + \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^2} (\square_1^3 + \square_2 \square_1^2 - \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[= \frac{4\square_2}{D^2} (\square_3^2 + 2\square_1 \square_3 - 2\square_2 \square_3 + \square_1^2)\]

\[\tilde{\mathbf{T}}_6 = \frac{1}{3D^2} (2\square_1^2 - 4\square_2 \square_1^3 + 2\square_1 \square_3^3 + 6\square_2 \square_3^3)\]
\[- 6\square_1^2 \square_3^2 - 2\square_1 \square_2 \square_3 + 4\square_1^2 \square_3 \square_3 + 2\square_1^3 \square_2 - 4\square_2 \square_3^3\]
\[- 2\square_1 \square_2 \square_3^2 + \square_3^4 - 6\square_3^2 \square_1^2 + \square_1^4 + 2\square_1 \square_3^3 + 2\square_3 \square_1^3)\]
\[+ \frac{1}{s} \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} \right) \frac{4\square_1}{D}\]
\[+ \left( \frac{f(-s\square_1) - 1}{s \square_1} \right) \frac{4 \square_1^2}{D^2} (\square_1 \square_2 + \square_1 \square_3 - \square_2^2 - \square_3^2 + 2\square_2 \square_3)\]
\[- \left( \frac{f(-s\square_2) - 1}{s \square_2} \right) \frac{\square_2}{D^2} (\square_1^3 + \square_2 \square_1^2 - \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^2} (\square_1^3 - \square_2 \square_1^2 + \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^2} (\square_1^3 + \square_2 \square_1^2 - \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[+ 1 \left( F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} \right) \frac{4\square_1}{D}\]
\[+ \left( \frac{f(-s\square_1) - 1}{s \square_1} \right) \frac{4 \square_1^2}{D^2} (\square_1 \square_2 + \square_1 \square_3 - \square_2^2 - \square_3^2 + 2\square_2 \square_3)\]
\[- \left( \frac{f(-s\square_2) - 1}{s \square_2} \right) \frac{\square_2}{D^2} (\square_1^3 + \square_2 \square_1^2 - \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^2} (\square_1^3 - \square_2 \square_1^2 + \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[- \left( \frac{f(-s\square_3) - 1}{s \square_3} \right) \frac{\square_3}{D^2} (\square_1^3 + \square_2 \square_1^2 - \square_1^2 \square_3 - \square_2^2 \square_1)\]
\[= \frac{4\square_2}{D^2} (\square_3^2 + 2\square_1 \square_3 - 2\square_2 \square_3 + \square_1^2)\]
\[-2\Box_1 \Box_2 + \Box_2^2)\]
\[-\frac{1}{s} \left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} \right) \frac{8}{D} \]
\[+ \left( \frac{f(-s\Box_1) - 1}{s \Box_1} \right) \frac{2}{\Box_2 D^2} (\Box_1^3\Box_2 + \Box_1^3\Box_3 - 7\Box_1^2\Box_2^2 - 3\Box_1^2\Box_3^2 \]
\[-6\Box_1^2\Box_2\Box_3 - 11\Box_1\Box_2\Box_3 + 3\Box_1\Box_3^3 + 7\Box_1\Box_2^3 + \Box_1 \Box_2 \Box_3^2 \]
\[+ 4\Box_2^3 \Box_3 - 6\Box_2^2 \Box_3^2 + 4\Box_2 \Box_3^3 - \Box_2^4 - \Box_3^4) \]
\[+ \left( \frac{f(-s\Box_2) - 1}{s \Box_2} \right) \frac{8 \Box_2^2(\Box_3 + \Box_1 - \Box_2)}{D^2} \]
\[-\left( \frac{f(-s\Box_3) - 1}{s \Box_3} \right) \frac{2 \Box_3}{\Box_2 D^2} (\Box_1^3 - 3\Box_3 \Box_1^2 - 5\Box_2 \Box_1^2 + 2\Box_3 \Box_1 \Box_2 \]
\[+ 7\Box_1^2 \Box_2^2 + 3\Box_1 \Box_3^2 + 3\Box_2 \Box_3^2 + \Box_2^2 \Box_3 - 3\Box_3^3 - 3\Box_2^3) \]
\[-\frac{1}{\Box_1 - \Box_3} \left( \frac{f(-s\Box_1) - 1}{s \Box_1} - \frac{f(-s\Box_3) - 1}{s \Box_3} \right) \frac{2 \Box_3}{\Box_2}, \] (12.100)

\[\tilde{T}_8 = F(-s\Box_1, -s\Box_2, -s\Box_3), \] (12.101)

\[\tilde{T}_9 = -F(-s\Box_1, -s\Box_2, -s\Box_3) \frac{2 \Box_1 \Box_2}{D^2} (-\Box_3^2 + 2\Box_1^2 - 2\Box_1 \Box_2) \]
\[+ \frac{1}{s} \left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} \right) \frac{2(-\Box_3 + 2\Box_1)}{D} \]
\[+ \left( \frac{f(-s\Box_1) - 1}{s \Box_1} \right) \frac{8 \Box_2 \Box_1^2(\Box_3 - \Box_2 + \Box_1)}{D^2} \]
\[-\left( \frac{f(-s\Box_3) - 1}{s \Box_3} \right) \frac{\Box_3}{D^2} (-\Box_3^3 + 6\Box_3^2 \Box_1 - 6\Box_3 \Box_1^2 + 2\Box_1^3 \]
\[+ 6\Box_3 \Box_2 \Box_1 - 2\Box_2 \Box_1^2), \] (12.102)

\[\tilde{T}_{10} = -F(-s\Box_1, -s\Box_2, -s\Box_3) \frac{2}{D^2} (\Box_3^3 - 2\Box_1 \Box_3^2 - 2\Box_1^2 \Box_3 + 2\Box_1^3 \]
\[+ 2\Box_1 \Box_2 \Box_3 - 2\Box_2 \Box_1^2) \]
\[+ \frac{1}{s} \left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} \right) \frac{8}{D} \]
\[-\left( \frac{f(-s\Box_1) - 1}{s \Box_1} \right) \frac{8 \Box_1}{D^2} (\Box_3^2 - 2\Box_2 \Box_3 - \Box_1^2 + \Box_2^2) \]
\[+ \left( \frac{f(-s\Box_3) - 1}{s \Box_3} \right) \frac{4 \Box_3 (\Box_3^2 - 2\Box_1^2 + 2\Box_1 \Box_2)}{D^2}. \] (12.103)

The conformal variation of the effective action (6.1) can now be obtained as

\[-\delta_\sigma W = \frac{1}{2} \int_0^\infty \frac{ds}{s} \delta_\sigma \text{Tr} K''(s). \] (12.104)
From (12.89) and (12.90) we find

\[-\delta_\sigma W = -\frac{1}{2(4\pi)^2} \int dx \, g^{1/2} \text{tr} \left\{ \sigma \Box \bar{t}_1(0, \Box) \hat{P} + \sigma \Box \bar{t}_2(0, \Box) \hat{R} \right\}
\]

\[+ \sum_{i=1}^{10} \bar{T}_i(0, \Box_1, \Box_2, \Box_3) \Re \Re_1 \Re_2 \sigma_3(i) \right\} + O[\Re^3] \quad (12.105)\]

where use is made of the fact that the functions \(\bar{t}_1, \bar{t}_2, \bar{T}_i\) as given in eqs. (12.92)–(12.103) vanish at \(s \to \infty\). The behaviours of these functions at \(s = 0\) follow from the results of sect. 4 :

\[\bar{t}_1(0, \Box) = \frac{1}{6}, \quad (12.106)\]

\[\bar{t}_2(0, \Box) = \frac{1}{180}, \quad (12.107)\]

\[\frac{1}{2} (\bar{T}_1 + \bar{T}_1|\Box_1 \leftrightarrow \Box_2) = 0, \quad s = 0 \quad (12.108)\]

\[\frac{1}{2} (\bar{T}_2 + \bar{T}_2|\Box_1 \leftrightarrow \Box_2) = \frac{1}{180} + \frac{\Box_1}{180 \Box_2} + \frac{\Box_2}{180 \Box_1} - \frac{\Box_3}{90 \Box_1} - \frac{\Box_3}{90 \Box_2} + \frac{\Box_3^2}{180 \Box_1 \Box_2}, \quad s = 0 \quad (12.109)\]

\[\bar{T}_3 = 0, \quad s = 0 \quad (12.110)\]

\[\frac{1}{2} (\bar{T}_4 + \bar{T}_4|\Box_1 \leftrightarrow \Box_2) = \frac{1}{45 \Box_1} + \frac{1}{45 \Box_2} - \frac{\Box_3}{45 \Box_1 \Box_2}, \quad s = 0 \quad (12.111)\]

\[\frac{1}{2} (\bar{T}_5 + \bar{T}_5|\Box_1 \leftrightarrow \Box_2) = \frac{1}{45 \Box_1 \Box_2}, \quad s = 0 \quad (12.112)\]

\[\bar{T}_6 = 0, \quad s = 0 \quad (12.113)\]

\[\bar{T}_7 = 0, \quad s = 0 \quad (12.114)\]

\[\frac{1}{2} (\bar{T}_8 + \bar{T}_8|\Box_1 \leftrightarrow \Box_2) = \frac{1}{2}, \quad s = 0 \quad (12.115)\]

\[\frac{1}{2} (\bar{T}_9 + \bar{T}_9|\Box_1 \leftrightarrow \Box_2) = \frac{1}{12}, \quad s = 0 \quad (12.116)\]

\[\frac{1}{2} (\bar{T}_{10} + \bar{T}_{10}|\Box_1 \leftrightarrow \Box_2) = 0, \quad s = 0. \quad (12.117)\]
With these expressions inserted in (12.105), one arrives at eq. (12.73) which is the correct trace anomaly.

Because the conformal transformation is inhomogeneous in the curvature, the expansion in powers of the curvature does not preserve the exact conformal properties of the effective action. These properties can only be recovered order by order. One can try to remove this shortcoming of covariant perturbation theory by using the ideas of ref. [22] but such an improvement is already beyond the scope of the present paper.

This concludes the presentation of the results concerning third order in the curvature, and the remaining part of the paper is devoted to their derivations. We start in the next section with perturbation theory for the heat kernel, and subsequently outline in succession the techniques used for obtaining the results in sects. 2 and 6–11.

13. Third order of perturbation theory for the trace of the heat kernel

In covariant perturbation theory [1,2], the heat kernel is first expanded in powers of perturbations:

$$K(s) = \sum_{n=0}^{\infty} K_n(s)$$

where $K_n(s)$ is a term of $n$-th power in the perturbations of metric, connection and potential

$$h^{\mu\nu}, \quad \hat{\Gamma}_\mu, \quad \hat{P} - \frac{1}{6} R \hat{1}. \quad (13.2)$$

When calculated by the algorithm of paper II, the trace of $K_n(s)$ is obtained in the form

$$\text{Tr} K_n(s) = \frac{1}{(4\pi s)^{n/2}} \int dx \tilde{g}_i^{1/2}(x) \int_{\alpha_i \geq 0} d^n \alpha \delta(1 - \sum_{i=1}^{n} \alpha_i)$$

$$\times \text{tr} \left\{ \text{exp} \left[ s \Omega_n(\alpha_1, \ldots, \alpha_n | \tilde{\nabla}^i) \right] \right\}$$

$$\times \sum_{l=0}^{n} s^l \tilde{B}_n^l(\alpha_1, \ldots, \alpha_n | x_i) \right|_{x_i = x} \quad (13.3)$$

(eq. (5.46) of paper II), or, with the notation

$$\int_{\alpha_i \geq 0} d^n \alpha \delta(1 - \sum_{i=1}^{n} \alpha_i) f(\alpha_1, \ldots, \alpha_n | x_i) \right|_{x_i = x} = \langle f \rangle_n, \quad (13.4)$$

$$\text{Tr} K_n(s) = \frac{1}{(4\pi s)^{n/2}} \int dx \tilde{g}_i^{1/2} \sum_{l=0}^{n} s^l \text{tr} (e^{s \Omega_n} \tilde{B}_n^l) \right|_{x_i = x} \quad (13.5)$$

Here $\tilde{g}$ and $\tilde{\nabla}$ are auxiliary, flat, metric and covariant derivative, $\Omega_n(\alpha_1, \ldots, \alpha_n | \tilde{\nabla}^i)$ is an operator of second order in $\tilde{\nabla}^i$, and $\tilde{\nabla}^i$ acts on the perturbation number $i$ contained in $\tilde{B}_n^l$. Each term in $\tilde{B}_n^l(\alpha_1, \ldots, \alpha_n | x_i)$ where $i$ ranges from 1 to $n$ is a
product of \( n \) perturbations (13.2) at the points \( x_1, \ldots, x_n \) respectively, and the label \( i \) on a
perturbation means that the perturbation is at the point \( x_i \). For example,

\[
h^{\mu\nu} \hat{\Gamma}_{2\alpha} \hat{P}_3 = h^{\mu\nu}(x_1) \hat{\Gamma}_\alpha(x_2) \hat{P}(x_3).
\]

After the action of \( \tilde{\nabla}^i \), all points \( x_i \) are made coincident with the integration point \( x \) in (13.3)
or (13.5). The possibility of integration over \( x \) by parts is then expressed by the identity

\[
\sum_{i=1}^n \tilde{\nabla}^i = 0
\]

which is used to put the third-order form factors in the form given below.

Since each perturbation \( h^{\mu\nu} \) or \( \hat{\Gamma}_\mu \) is an infinite series in the curvature, the expansion to
a given order in the curvature involves all lower orders in perturbations. Therefore, we begin
with quoting the results of [2] for \( n = 1 \) and \( n = 2 \) in (13.3). For \( n = 1 \),

\[
\Omega_1 = 0, \tag{13.7}
\]

\[
\hat{B}^0_1 = -\frac{1}{2} h \hat{1}, \tag{13.8}
\]

\[
\hat{B}^1_1 = \frac{1}{3} \tilde{\nabla}_\mu \tilde{\nabla}_\nu h^{\mu\nu} \hat{1} - \frac{1}{12} \tilde{\square} h \hat{1} - \tilde{\nabla}_\mu \hat{\Gamma}^{\mu} + \hat{P} - \frac{1}{6} R \hat{1}. \tag{13.9}
\]

Here and below

\[
h = h^{\mu\nu} \bar{g}_{\mu\nu}, \quad \tilde{\square} = \bar{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu, \tag{13.10}
\]

and the indices of \( \tilde{\nabla}_\mu \) and the perturbations are raised and lowered with the flat metric \( \bar{g}_{\mu\nu} \),
extcept for

\[
\hat{\Gamma}_\mu \equiv (\bar{g}^{\mu\nu} + h^{\mu\nu}) \hat{\Gamma}_\nu. \tag{13.11}
\]

For \( n = 2 \),

\[
\Omega_2(\alpha_1, \alpha_2 | \tilde{\nabla}^i) = \alpha_1 \alpha_2 \tilde{\nabla}_2, \tag{13.12}
\]

\[
\hat{B}^0_2(\alpha_1, \alpha_2 | x_i) = \left( \frac{1}{4} h_1 h_2 + \frac{1}{2} h_{1\mu\nu} h^{\mu\nu}_2 \right) \hat{1}, \tag{13.13}
\]

\[
\hat{B}^1_2(\alpha_1, \alpha_2 | x_i) = -\alpha_1^2 (\tilde{\nabla}_\mu \tilde{\nabla}_\nu h^{\mu\nu}_1) h_2 \hat{1} \\
- 2\alpha_1 \alpha_2 (\tilde{\nabla}_\nu h^{\mu\nu}_1) \bar{g}_{\mu\alpha} (\tilde{\nabla}_\beta h^{\beta\alpha}_2) \hat{1} - 2 \hat{\Gamma}_1^{\mu} \bar{g}_{\mu\nu} \hat{\Gamma}_2^{\nu} + 2 \alpha_2 h_1 (\tilde{\nabla}_\mu \hat{\Gamma}^{\mu}_2) \\
+ 4 \alpha_1 (\tilde{\nabla}_\mu h^{\mu\nu}_1) \bar{g}_{\nu\alpha} \hat{\Gamma}^{\alpha}_2 - h_1 (\hat{P}_2 - \frac{1}{6} R_2 \hat{1}), \tag{13.14}
\]

\[
\hat{B}^2_2(\alpha_1, \alpha_2 | x_i) = \hat{1} (\alpha_1 \alpha_2)^2 (\tilde{\nabla}_\mu \tilde{\nabla}_\nu h^{\mu\nu}_1) (\tilde{\nabla}_\alpha \tilde{\nabla}_\beta h^{\alpha\beta}_2) \\
+ 4 \alpha_1 \alpha_2 (\tilde{\nabla}_\mu \hat{\Gamma}^{\mu}_1) (\tilde{\nabla}_\nu \hat{\Gamma}^{\nu}_2) - 4 \alpha_1^2 \alpha_2 (\tilde{\nabla}_\mu \tilde{\nabla}_\nu h^{\mu\nu}_1) (\tilde{\nabla}_\alpha \hat{\Gamma}^{\alpha}_2) \\
+ 2 \alpha_1^2 (\tilde{\nabla}_\mu \tilde{\nabla}_\nu h^{\mu\nu}_1) (\hat{P}_2 - \frac{1}{6} R_2 \hat{1}) \\
- 4 \alpha_2 (\hat{P}_1 - \frac{1}{6} R_1 \hat{1}) (\tilde{\nabla}_\mu \hat{\Gamma}^{\mu}_2) + (\hat{P}_1 - \frac{1}{6} R_1 \hat{1}) (\hat{P}_2 - \frac{1}{6} R_2 \hat{1}). \tag{13.15}
\]
A routine calculation by the algorithm of paper II gives for $n = 3$

$$\Omega_3(\alpha_1, \alpha_2, \alpha_3 | \nabla^i) = \alpha_2 \alpha_3 \tilde{n}_1 + \alpha_1 \alpha_3 \tilde{n}_2 + \alpha_1 \alpha_2 \tilde{n}_3,$$

(13.16)

$$\hat{B}_3^0(\alpha_1, \alpha_2, \alpha_3 | x_i) = -\hat{\lambda} \left( \frac{1}{8} h_1 h_2 h_3 + \frac{3}{4} h_1 h_2^\mu h_3^{\nu\sigma} + h_1^\alpha h_2^\mu h_3^{\nu\sigma} \right),$$

(13.17)

$$\hat{B}_3^1(\alpha_1, \alpha_2, \alpha_3 | x_i) =$$

$$\hat{\lambda} \left[ 3(\bar{g}_{\alpha\beta} \bar{g}_{\mu\nu} + 2\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta}) \left( D^1_\lambda D^2_\sigma + \frac{1}{4} D^2_\lambda D^2_\sigma \right) h_1^{\alpha\beta} h_2^\mu h_3^{\nu\sigma} \right.$$

$$\left. - 3 \left( \bar{g}_{\mu\nu\lambda\sigma} D^1_\mu + \bar{g}_{\mu\nu\alpha\sigma} D^1_\mu + \frac{1}{2} \bar{g}_{\mu\nu\lambda\sigma} D^3_\alpha \right) \hat{\Gamma}_1^\alpha h_2^\mu h_3^{\nu\sigma} \right. + 3(\bar{g}_{\alpha\beta} \hat{\Gamma}_1^\alpha \hat{\Gamma}_2^\beta h_3 + 2\hat{\Gamma}_1^\alpha \hat{\Gamma}_2^\beta h_3) + \frac{3}{4} \bar{g}_{\mu\nu\alpha\beta} \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) h_2^\mu h_3^{\nu\alpha\beta}, \right.$$

(13.18)

$$\hat{B}_3^2(\alpha_1, \alpha_2, \alpha_3 | x_i) =$$

$$-\hat{\lambda} \left( \frac{3}{2} \bar{g}_{\alpha\beta} D^1_\mu D^1_\nu D^2_\lambda D^2_\sigma + 6\bar{g}_{\mu\alpha} D^3_\beta D^1_\nu D^2_\lambda D^2_\sigma \right) h_1^{\alpha\beta} h_2^\mu h_3^{\nu\sigma} + 12 \bar{g}_{\alpha\beta} D^2_\mu \hat{\Gamma}_1^\alpha \hat{\Gamma}_2^\beta h_3$$

$$+ (3\bar{g}_{\mu\nu} D^1_\alpha D^2_\lambda D^2_\sigma + 3\bar{g}_{\lambda\sigma} D^3_\alpha D^1_\mu D^2_\nu D^2_\sigma$$

$$+ 6\bar{g}_{\alpha\lambda} D^1_\mu D^1_\nu D^2_\sigma + 12\bar{g}_{\mu\nu} D^3_\alpha D^1_\mu D^2_\sigma) \hat{\Gamma}_1^\alpha \hat{\Gamma}_2^\beta h_3^{\mu\nu}$$

$$- (6\bar{g}_{\mu\nu} D^3_\alpha D^1_\beta + 12\bar{g}_{\nu\alpha} D^3_\beta D^2_\nu$$

$$+ 12\bar{g}_{\mu\nu} D^3_\alpha D^2_\nu) \hat{\Gamma}_1^\alpha \hat{\Gamma}_2^\beta h_3^{\mu\nu}$$

$$- \frac{3}{2} \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \left( \hat{P}_2 - \frac{1}{6} \hat{R}_2 \hat{1} \right) h_3$$

$$+ (3\bar{g}_{\mu\nu} D^1_\alpha + 6\bar{g}_{\mu\alpha} D^2_\nu - 3\bar{g}_{\nu\mu} D^3_\alpha - 6\bar{g}_{\mu\alpha} D^1_\nu) \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \hat{\Gamma}_2^\alpha h_3^{\mu\nu}$$

$$- 6\bar{g}_{\nu\alpha} \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \hat{\Gamma}_2^\alpha \hat{\Gamma}_3^\beta h_3$$

$$- \frac{3}{2} \left( \bar{g}_{\nu\mu} D^2_\alpha D^2_\sigma + \bar{g}_{\lambda\sigma} D^1_\mu D^1_\nu + 4\bar{g}_{\mu\lambda} D^1_\nu D^2_\sigma \right) \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) h_2^\mu h_3^{\nu\lambda\sigma}, \right.$$ 

(13.19)

$$\hat{B}_3^3(\alpha_1, \alpha_2, \alpha_3 | x_i) = D^3_\alpha D^3_\beta D^1_\mu D^1_\nu D^2_\lambda D^2_\sigma h_1^{\alpha\beta} h_2^\nu h_3^{\mu\lambda\sigma} \hat{1}$$

$$- 8D^2_\alpha D^1_\beta D^1_\mu D^2_\nu D^2_\lambda D^2_\sigma h_1^{\alpha\beta} h_2^\nu h_3^{\mu\lambda\sigma} \hat{1}$$

$$- 12D^3_\beta D^3_\alpha D^1_\mu D^2_\nu D^2_\lambda h_2^\nu h_3^{\mu\lambda\sigma} \hat{1}$$

$$+ \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \left( \hat{P}_2 - \frac{1}{6} \hat{R}_2 \hat{1} \right) \left( \hat{P}_3 - \frac{1}{6} \hat{R}_3 \hat{1} \right)$$

$$+ 3D^2_\alpha D^2_\beta \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \left( \hat{P}_2 - \frac{1}{6} \hat{R}_2 \hat{1} \right) h_3^{\alpha\beta}$$

$$- 6D^1_\mu \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \left( \hat{P}_2 - \frac{1}{6} \hat{R}_2 \hat{1} \right) \hat{\Gamma}_3^\mu$$

$$+ 6 \left( D^3_\beta D^1_\mu D^1_\nu - D^3_\alpha D^2_\nu D^2_\sigma \right) \left( \hat{P}_1 - \frac{1}{6} \hat{R}_1 \hat{1} \right) \hat{\Gamma}_2^\alpha h_3^{\mu\nu},$$
\[ + 12D^1_\mu D^2_\beta (\hat{P}_1 - \frac{1}{6}R_1 \hat{1}) \hat{\Gamma}_2^\beta \]
\[ + 3D^1_\mu D^1_\lambda D^2_\sigma (\hat{P}_1 - \frac{1}{6}R_1 \hat{1}) h^\mu_2 h^\nu_3^\sigma \]

where
\[ \begin{align*}
D^1_\mu & = \alpha_3 \tilde{\nabla}^2_\mu - \alpha_2 \tilde{\nabla}^3_\mu, \\
D^2_\mu & = \alpha_1 \tilde{\nabla}^3_\mu - \alpha_3 \tilde{\nabla}^1_\mu, \\
D^3_\mu & = \alpha_2 \tilde{\nabla}^1_\mu - \alpha_1 \tilde{\nabla}^2_\mu,
\end{align*} \]

and
\[ \tilde{g}^{(2)}_{\mu\nu;\alpha\beta} = \tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta} \tilde{g}_{\nu\alpha} + \tilde{g}_{\mu\nu} \tilde{g}_{\alpha\beta}. \]

14. Expansion of \( \text{Tr} K(s) \) in powers of the curvatures to third order

Next step is replacing the perturbations \( h^{\mu\nu} \) and \( \tilde{\Gamma}_\mu \) by their expressions through the curvatures, and eliminating the auxiliary quantities \( \tilde{g}_{\alpha\beta} \) and \( \tilde{\nabla}_\alpha \). The iterational solutions for \( h^{\mu\nu} \) and \( \tilde{\Gamma}_\mu \) are needed now to third order in the curvature whereas in paper II they were obtained to second order (eqs. (4.28), (4.29) of II). It is, of course, possible to work out these solutions to third order but we shall avoid this calculation by using the following trick. Let us rewrite eqs. (13.1), (13.5) as follows:

\[ \begin{align*}
\text{Tr} K(s) & = \frac{1}{(4\pi s)^2} \int dx \tilde{g}^{1/2} \text{tr} \left[ \hat{1} + \hat{B}_1^0 + \frac{1}{2}(\hat{B}_2^0)_{1/2} + \frac{1}{3}(\hat{B}_3^0)_{1/3} \right] \\
& + \frac{s}{(4\pi s)^2} \int dx \tilde{g}^{1/2} \text{tr} \left[ \hat{B}_1^1 + \frac{1}{2}(\hat{B}_2^1)_{2/2} + \frac{1}{3}(\hat{B}_3^1)_{3/3} \right] \\
& + \frac{s^2}{2(4\pi s)^2} \int dx \tilde{g}^{1/2} \text{tr} \left[ \left\langle \frac{e^{s\Omega_2} - 1 - s\Omega_2}{s^2} \hat{B}_2^0 \right\rangle_{2/2} \\
& + \left\langle \frac{e^{s\Omega_2} - 1}{s} \hat{B}_2^1 \right\rangle_{1/2} + \left\langle e^{s\Omega_2} \hat{B}_2^2 \right\rangle_{2} \right] \\
& + \frac{s^3}{3(4\pi s)^2} \int dx \tilde{g}^{1/2} \text{tr} \left[ \left\langle \frac{e^{s\Omega_3} - 1 - s\Omega_3}{s^3} \hat{B}_3^0 \right\rangle_{3/3} \\
& + \left\langle \frac{e^{s\Omega_3} - 1}{s^2} \hat{B}_3^1 \right\rangle_{2/3} + \left\langle e^{s\Omega_3} \hat{B}_3^2 \right\rangle_{3} \\
& + \left\langle e^{s\Omega_3} \hat{B}_3^3 \right\rangle_{3} + O[3^4] \right].
\end{align*} \]

The purpose of this decomposition is to single out from the outset the terms of zeroth and first order in the curvature which can only be local and coincident with the coefficients
\(a_0(x, x)\) and \(a_1(x, x)\) of the Schwinger-DeWitt expansion. The connection between covariant perturbation theory and the Schwinger-DeWitt technique was discussed in paper II where an expression was obtained for the DeWitt coefficients in terms of perturbation theory (eq. (8.8) of II). In particular,

\[
\int dx \, g^{1/2} \text{tr} \, \hat{a}_0(x, x) = \int dx \, \bar{g}^{1/2} \text{tr} \, \left[ \hat{1} + \hat{B}_1^0 \right. \\
+ \frac{1}{2} \langle \hat{B}_2^0 \rangle_2 + \frac{1}{3} \langle \hat{B}_3^0 \rangle_3 \left. + O[\mathcal{R}^4] \right],
\]

(14.2)

\[
\int dx \, g^{1/2} \text{tr} \, \hat{a}_1(x, x) = \int dx \, \bar{g}^{1/2} \text{tr} \, \left[ \hat{B}_1^1 + \frac{1}{2} \langle \hat{B}_2^0 \hat{B}_2^0 \rangle_2 + \frac{1}{2} \langle \hat{B}_1^1 \rangle_2 \\
+ \frac{1}{2} \langle \Omega_3 \hat{B}_3^0 \rangle_3 + \frac{1}{3} \langle \hat{B}_3^1 \rangle_3 \right] + O[\mathcal{R}^4],
\]

(14.3)

and, therefore, for the first two integrals in (14.1) we can use the known exact results

\[
\int dx \, g^{1/2} \text{tr} \, \hat{a}_0(x, x) = \int dx \, g^{1/2} \text{tr} \, \hat{1},
\]

(14.4)

\[
\int dx \, g^{1/2} \text{tr} \, \hat{a}_1(x, x) = \int dx \, g^{1/2} \text{tr} \, \hat{P}.
\]

(14.5)

To second order in the curvature, it was checked in paper II that these results really follow from eqs. (14.2) and (14.3). To third order in the curvature such a check would require a knowledge of the iterational solutions for \(h^{\mu \nu}\) and \(\hat{\Gamma}_\mu\) to third order. Instead of deriving these solutions, we shall take eqs. (14.4) and (14.5) for granted. Then, after elimination of the first two integrals in expression (14.1), the remaining terms in this expression are already of second and third order in perturbations, and, therefore, the knowledge of \(h^{\mu \nu}\) and \(\hat{\Gamma}_\mu\) to second order in the curvature is sufficient for their calculation.

Consider now the third integral in (14.1) which involves terms of second order in perturbations: \(\hat{B}_2^0, \hat{B}_2^1, \hat{B}_2^2\). It is straightforward to substitute in (13.13)–(13.15) the equations of paper II expressing \(h^{\mu \nu}\) and \(\hat{\Gamma}_\mu\) through \(R^{\mu \nu}\) and \(\hat{R}^{\mu \nu}\) (use is made of eqs. (4.28)–(4.34), (7.3)–(7.5) of II) but, in the second-order form factor itself

\[
e^{\phi \Omega_2} = e^{\phi_0 \alpha_2 \tilde{\phi}},
\]

(14.6)

when expressing \(\tilde{\phi}\) through \(\phi\), the terms linear in the curvature should also be retained. The expression of \(\tilde{\phi}\) through \(\phi\) is generally of the form

\[
\tilde{\phi} = \phi + O(\mathcal{R}, \nabla) + O[\mathcal{R}^2]
\]

(14.7)

where \(O(\mathcal{R}, \nabla)\) is an operator containing the curvature linearly. Its explicit forms are given in paper II for all cases where \(\phi\) acts on a scalar, tensor, matrix, etc (eqs. (4.30)–(4.34) of II). The expansion of the form factor \(e^{\phi \Omega_2}\) is then accomplished as follows:

\[
\int dx \, g^{1/2} \, \mathcal{R}_1 (e^{\alpha \tilde{\phi}} - e^{\alpha \phi}) \mathcal{R}_2
\]

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\[
\begin{align*}
= \int dx \, g^{1/2} \mathcal{R}_1 \int_0^a dt \, e^{(a-t)\Box} \mathcal{O}(\mathcal{R}, \nabla) e^{t\Box} \mathcal{R}_2 + O[\mathcal{R}^4] \\
= \int dx \, g^{1/2} \int_0^a dt \, e^{(a-t)\Box_1 + t\Box_2} \mathcal{O}(\mathcal{R}_3, \nabla_2) \mathcal{R}_1 \mathcal{R}_2 + O[\mathcal{R}^4] \\
= \int dx \, g^{1/2} \frac{e^{\alpha_2} - e^{\alpha_1}}{\Box_2 - \Box_1} \mathcal{O}(\mathcal{R}_3, \nabla_2) \mathcal{R}_1 \mathcal{R}_2 + O[\mathcal{R}^4]
\end{align*}
\] (14.8)

with \(a = s\alpha_1\alpha_2\). Here the numbers on the arguments of \(\mathcal{O}(\mathcal{R}_3, \nabla_2)\) mean that \(\mathcal{O}\) as an operator acts on \(\mathcal{R}_2\), and the curvature that it contains acquires the number 3. Expression (14.8) represents a third-order contribution with the form factor of a new type.

In this way, for the terms with the second-order form factor in (14.1), we obtain the expansions which extend to third order eqs. (7.7), (7.8) and (7.9) of paper II:

\[
\begin{align*}
\int dx \, g^{1/2} \text{tr} \left( \frac{e^{\alpha_2} - 1 - s\Omega_2}{s^2} \hat{B}_2 \right)_2 \\
= \int dx \, g^{1/2} \text{tr} \left( \mathcal{C}_2 (R_1 R_2 + 2R_{1\mu\nu} R_{2\mu\nu}) \hat{1} \right)_2 \\
+ \int dx \, g^{1/2} \text{tr} \left( -\frac{1}{\Box_2} \mathcal{C}_1 + 2\frac{1}{\Box_2} \mathcal{C}_3 - s\mathcal{W}_{12} + \frac{1}{\Box_3} s\mathcal{W}_{12} \right) R_1 R_2 R_3 \hat{1} \\
+ 4 \frac{1}{\Box_2} \mathcal{C}_1 R_{1\mu}^\alpha R_{2\beta}^\nu R_{3\mu\nu} \hat{1} \\
+ \left( 3 \frac{1}{\Box_2} \mathcal{C}_3 - 2 \frac{1}{\Box_1} \mathcal{C}_1 + 2 \frac{1}{\Box_2} \mathcal{C}_3 - 2 \frac{1}{\Box_2} \mathcal{C}_1 - s\mathcal{W}_{12} \right) R_{1\mu}^\alpha R_{2\mu\nu} R_{3\nu} \hat{1} \\
+ \left( 4 \frac{1}{\Box_2} \mathcal{C}_1 - 4 \frac{1}{\Box_3} s\mathcal{W}_{12} + 2 \frac{1}{\Box_1} s\mathcal{W}_{23} \right) \mathcal{R}_1^{\alpha\beta} \nabla_\alpha R_2 \nabla_\beta R_3 \hat{1} \\
+ \left( 8 \frac{1}{\Box_3} \mathcal{C}_2 - 4 \frac{1}{\Box_1} \mathcal{C}_3 + 4 \frac{1}{\Box_3} s\mathcal{W}_{12} \right) \nabla_\mu R_1^{\alpha\nu} \nabla_\nu R_2 \mathcal{R}_3 \hat{1} \\
+ \left( 8 \frac{1}{\Box_1} \mathcal{C}_2 + 4 \frac{1}{\Box_1} \mathcal{C}_3 + 4 \frac{1}{\Box_3} s\mathcal{W}_{12} \right) \mathcal{R}_1^{\mu\nu} \nabla_\mu R_2^{\alpha\beta} \nabla_\nu R_3 \mathcal{R}_3 \hat{1} \\
+ \left( 16 \frac{1}{\Box_1} \mathcal{C}_2 + 8 \frac{1}{\Box_2} \mathcal{C}_3 + 16 \frac{1}{\Box_3} s\mathcal{W}_{12} \right) \mathcal{R}_1^{\mu\nu} \nabla_\alpha R_2 \mathcal{R}_3 \mathcal{R}_3 \mathcal{R}_3 \mathcal{R}_3 \hat{1} \\
+ O[\mathcal{R}^4],
\end{align*}
\] (14.9)

\[
\begin{align*}
\int dx \, \tilde{g}^{1/2} \text{tr} \left( \frac{e^{\alpha_2} - 1}{s} \hat{B}_2 \right)_2 \\
= \int dx \, g^{1/2} \text{tr} \left( \mathcal{B}_2 \left[ \hat{R}_1^{\mu\nu} \hat{R}_2^{\mu\nu} \\
+ (2\alpha_1\alpha_2 - 2\alpha_2^2 + \frac{1}{3}) R_1 R_2 \hat{1} - 2R_1 \hat{P}_2 \right] \right)_2 \\
+ \int dx \, g^{1/2} \text{tr} \left( \left( -\frac{1}{\Box_2} \mathcal{B}_1 + 2\frac{1}{\Box_1} \mathcal{B}_3 - \frac{1}{\Box_2} \mathcal{B}_3 \right) R_1 R_2 \hat{P}_3 \\
+ \frac{1}{\Box_1} \mathcal{B}_2 - \frac{1}{2} s\mathcal{V}_{23} \right) R_1 \hat{R}_2^{\mu\nu} \hat{R}_3^{\mu\nu}
\end{align*}
\]
\[-4 \frac{1}{\Box_1} B_3 R_{1\alpha}^{\alpha\beta} \hat{R}_{2\alpha}^{\beta\mu} \hat{R}_{3\beta\mu} \]

\[+ \left[ \left( 2\alpha_1 \alpha_2 - 2\alpha_1^2 + \frac{1}{6} \right) \Box_1 B_1 + 2(\alpha_1^2 - \alpha_1 \alpha_2) \frac{1}{\Box_2} B_1 \right. \]

\[-\left( \alpha_1 \alpha_2 - \alpha_1^2 + \frac{1}{6} \right) s \nu_3 \right] R_1 R_2 R_3 \hat{1} \]

\[+ \left[ \left( 2\alpha_1 \alpha_2 - 4\alpha_1^2 + \frac{1}{2} \right) \Box_3 B_3 \right. \]

\[+ \left( 4\alpha_1^2 - 4\alpha_1 \alpha_2 - \frac{2}{3} \right) \Box_1 B_3 \]

\[\left. \right] R_1^{\mu\nu} R_2^{\mu\nu} R_3 \hat{1} \]

\[+ \left( -4 \frac{1}{\Box_1} B_3 - 4 \frac{1}{\Box_1} s \nu_3 \right) R_1^{\mu\nu} \nabla_\mu R_2 \nabla_\nu \hat{P}_3 \]

\[+ 4 \frac{1}{\Box_2} B_3^{\mu\nu} R_1^{\mu\alpha} \nabla_\nu R_2 \nabla_\beta \hat{R}_{3\mu
u} \]

\[+ \left[ \left( 4\alpha_1^2 - \frac{2}{3} \right) \Box_1 B_3 \right. \]

\[+ \left( 4\alpha_1 \alpha_2 - 4\alpha_1^2 + \frac{2}{3} \right) \Box_1 \nu_3 \right] R_1^{\alpha\beta} \nabla_\alpha R_2 \nabla_\beta R_3 \hat{1} \]

\[+ \left( 8\alpha_1^2 - 8\alpha_1 \alpha_2 - \frac{2}{3} \right) \Box_1 B_3 \nabla_\mu R_1^{\mu\nu} \nabla_\nu R_2 \nabla_\mu R_3 \hat{1} \]

\[+ 4 \frac{1}{\Box_3} \nu_3 \nabla_\nu R_2 \nabla_\mu R_3 \]

\[+ \left[ -\frac{1}{\Box_3} B_2 R_1^{\alpha\beta} \nabla_\alpha \hat{R}_{2\alpha}^{\beta\nu} \nabla_\nu \hat{R}_{3\mu
u} \right] + O[\mathcal{R}^4], \quad (14.10) \]

\[\int dx \, g^{1/2} \text{tr} \left\langle e^{\alpha_3} B_2^2 \right\rangle \]

\[= \int dx \, g^{1/2} \text{tr} \left\langle A_2 \left[ \hat{P}_1 \hat{P}_2 + (2\alpha_1^2 - \frac{1}{3}) R_1 \hat{P}_2 \right. \right. \]

\[\left. + \left( \alpha_1^2 \alpha_2 - \frac{1}{3} \alpha_1 + \frac{1}{36} \right) R_1 R_2 \hat{1} \right] \left. \right\rangle_2 \]

\[+ \int dx \, g^{1/2} \text{tr} \left[ \left[ \alpha_1^2 \Box_3 A_3 + 2 \frac{1}{\Box_1} \alpha_1^2 A_1 - 2 \alpha_1 \frac{1}{\Box_1} A_3 \right. \right. \]

\[-\frac{1}{3} \Box_2 A_1 + \left( \alpha_1^2 - \frac{1}{6} \right) \left( \Box_3 - \Box_2 - \Box_1 - 1 \right) s \nu_3 \right] R_1 R_2 \hat{P}_3 \]

\[+ \alpha_1^2 \left( \Box_3 A_3 - 2 \frac{1}{\Box_1} A_3 \right) R_1^{\mu\nu} R_2 \nabla_\mu \hat{P}_3 + \frac{1}{\Box_3} A_1 \hat{P}_1 \hat{P}_2 R_3 \]

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+ \left[ (\alpha_1^2 \alpha_2^2 - \frac{1}{6} \alpha_1^2) \frac{\Box_1}{\Box_2 \Box_3} A_1 - (\alpha_1^2 \alpha_2^2 - \frac{1}{36}) \frac{1}{\Box_2} A_1 \right. \\
- \frac{1}{2} \left( (\alpha_1^2 \alpha_2^2 - \frac{1}{3} \alpha_1^2 + \frac{1}{36}) s \mathcal{U}_{23} \right) R_{12} R_{31} \hat{1} \\
+ \left( \alpha_1^2 \alpha_2^2 - \frac{1}{6} \alpha_1^2 \right) \left( \frac{3}{\Box_1 \Box_2} A_3 - 2 \frac{1}{\Box_1} A_3 \right) R_{1}^{\mu \nu} R_{2 \mu \nu} R_{31} \hat{1} \\
+ 4\alpha_2 \frac{1}{\Box_1 \Box_2} A_3 \nabla_\mu \hat{R}_{1}^{\mu \nu} \nabla_\nu \hat{R}_{2 \nu \alpha} \hat{P}_3 \\
+ 4 \left( \alpha_1^2 - \frac{1}{6} \right) \frac{1}{\Box_1} s \mathcal{U}_{23} R_{1}^{\mu \nu} \nabla_\mu R_{2} \nabla_\nu \hat{P}_3 \\
- 4\alpha_1^2 \frac{1}{\Box_1 \Box_2} A_3 \nabla_\mu R_{1}^{\mu \nu} \nabla_\nu R_{2 \mu \alpha} \hat{P}_3 \\
- 2 \frac{1}{\Box_1} s \mathcal{U}_{23} R_{1}^{\mu \nu} \nabla_\mu \nabla_\nu \hat{P}_2 \hat{P}_3 \\
+ 4 \alpha_2 \left( \alpha_1^2 - \frac{1}{6} \right) \frac{1}{\Box_1 \Box_2} A_1 R_1 \nabla_\alpha \hat{R}_{2}^{\alpha \beta} \nabla_\beta \hat{R}_{3 \beta \mu} \\
+ 2 \left( \alpha_1^2 \alpha_2^2 - \frac{1}{3} \alpha_1^2 + \frac{1}{36} \right) \frac{1}{\Box_1} s \mathcal{U}_{23} R_{1}^{\alpha \beta} \nabla_\alpha R_{2} \nabla_\beta R_{3} \hat{1} \\
- 4 \left( \alpha_1^2 \alpha_2^2 - \frac{1}{6} \alpha_1^2 \right) \frac{1}{\Box_1 \Box_2} A_3 \nabla_\mu R_{1}^{\nu \alpha} \nabla_\nu R_{2 \mu \alpha} \hat{R}_3 \hat{1} \\
- 4 \alpha_2 \left( \alpha_1^2 - \frac{1}{6} \right) \frac{1}{\Box_1 \Box_2} A_3 \nabla_\mu \hat{R}_{1 \nu \mu} \nabla_\mu R_{2} R_{3} \\
- 4 \frac{1}{\Box_1} s \mathcal{U}_{23} \hat{R}_{1}^{\mu \nu} \nabla_\mu \hat{P}_2 \nabla_\nu \hat{P}_3 \\
- 4 \alpha_2 \frac{1}{\Box_1 \Box_2} A_3 \nabla_\nu \hat{R}_{1 \nu \mu} \nabla_\mu R_{2} \hat{P}_3 \right) + O[\mathcal{R}^4] \quad (14.11)

where \((m, n = 1, 2, 3; \ m \neq n)\)

\[ A_m = e^{\alpha_1 \alpha_2 \Box_m}, \quad (14.12) \]

\[ B_m = \frac{e^{\alpha_1 \alpha_2 \Box_m} - 1}{s \Box_m}, \quad (14.13) \]

\[ C_m = \frac{e^{\alpha_1 \alpha_2 \Box_m} - 1 - s \alpha_1 \alpha_2 \Box_m}{(s \Box_m)^2}, \quad (14.14) \]

\[ \mathcal{U}_{mn} = \frac{e^{\alpha_1 \alpha_2 \Box_m} - e^{\alpha_1 \alpha_2 \Box_n}}{s(\Box_m - \Box_n)}, \quad (14.15) \]

\[ \mathcal{V}_{mn} = \frac{1}{s(\Box_m - \Box_n)} \left( \frac{e^{\alpha_1 \alpha_2 \Box_m} - 1}{s \Box_m} - \frac{e^{\alpha_1 \alpha_2 \Box_n} - 1}{s \Box_n} \right), \quad (14.16) \]

\[ \mathcal{W}_{mn} = \frac{1}{s(\Box_m - \Box_n)} \left( \frac{e^{\alpha_1 \alpha_2 \Box_m} - 1 - s \alpha_1 \alpha_2 \Box_m}{(s \Box_m)^2} \right) \]
\[ e^{\alpha_1 \alpha_2 \Box_n} - \frac{1 - s \alpha_1 \alpha_2 \Box_n}{(s \Box_n)^2} \]  

(14.17)

(the numbers on the form factors refer to the numbers on the boxes appearing in them), and the averaging \( \langle \rangle_2 \) is defined in (13.4).

Finally, the last integral in (14.1), with \( \hat{B}_0^3, \hat{B}_1^3, \hat{B}_2^3, \hat{B}_3^3 \), is already of third order in perturbations. Therefore, it is sufficient to substitute in eqs. (13.16)–(13.20) the lowest-order expressions

\[ h_{\mu \nu} = 2 \frac{1}{\Box} R_{\mu \nu} + O[\Re^2], \]  

(14.18)

\[ \hat{\Gamma}_\mu = \nabla_\nu \frac{1}{\Box} \hat{R}_{\nu \mu} + O[\Re^2], \]  

(14.19)

\[ \tilde{\nabla}_\mu = \nabla_\mu + O[\Re], \]  

(14.20)

The final result is presented below. As distinct from the form factors (14.12)–(14.17) coming from the second order, the form factor appearing in the last terms of (14.1):

\[ e^{s \Omega_3} = \exp\left[s (\alpha_2 \alpha_3 \Box_1 + \alpha_1 \alpha_3 \Box_2 + \alpha_1 \alpha_2 \Box_3)\right] + O[\Re] \]  

(14.21)

is an irreducible nonlocal function of all the three boxes.

On the whole, there appear initially forty different cubic structures with derivatives that do not contract in the \( \Box \) operators. However, by using eq. (13.6) and its consequences

\[ \nabla_1 + \nabla_2 + \nabla_3 = 0, \]  

(14.22)

\[ 2 \nabla_1 \nabla_2 = \Box_3 - \Box_1 - \Box_2, \]  

(14.23)

the Jacobi and Bianchi identities

\[ \nabla_\lambda \hat{R}_{\beta \mu} + \nabla_\mu \hat{R}_{\lambda \beta} + \nabla_\beta \hat{R}_{\mu \lambda} = 0, \]  

(14.24)

\[ \nabla^\mu R_{\mu \alpha} = \frac{1}{2} \nabla_\alpha R, \]  

(14.25)

and the possibility of discarding terms \( O[\Re] \) when commuting the covariant derivatives (since their contribution is already \( O[\Re^4] \)), we reduce the number of independent cubic structures to thirty three *.

Here are some examples of this reduction.

The identity

\[ \text{tr} \hat{R}_1^{\alpha \beta} \nabla_\alpha \hat{R}_2^{\mu \nu} \nabla_\beta \hat{R}_3^{\mu \nu} = \text{tr} \left( \Box_1 \hat{R}_1^{\mu \nu} \hat{R}_2^{\alpha \beta} \hat{R}_3^{\mu \nu} + \hat{R}_3^{\alpha \beta} \nabla_\nu \hat{R}_2^{\mu \nu} \nabla_\mu \hat{R}_1^{\alpha \beta} \right) + O[\Re^4] + \text{a total derivative} \]  

(14.26)

*Below, in sect. 16, it will be shown that the contributions of four of these structures vanish. In this way the final basis consisting of twenty-nine structures (2.15)–(2.43) is obtained. For a particular space-time dimension, the dimension of the basis of nonlocal invariants can be smaller (see Appendix).
transforms the left-hand side to the structures 2 and 12 of the table (2.15)–(2.43). The identity

\[ \text{tr} \nabla_\mu \nabla_\lambda \hat{R}_{2 \rho}^{\alpha \beta} \nabla_\alpha \hat{R}_{2 \sigma}^{\rho \mu} \hat{R}_{3 \lambda}^{\beta} = - \frac{1}{2} \text{tr} \left( \Box_3 \hat{R}_{2 \alpha \beta}^{\mu \nu} \nabla_\mu \hat{R}_{3 \lambda}^{\beta \nu} \right) \]

transforms the left-hand side to the structures 7, 19, 20 and 21 of the table. The identities

\[ \nabla_\alpha R_1^{\beta \mu} \nabla_\beta \hat{R}_{2 \rho}^{\alpha \beta} \hat{R}_{3 \lambda}^{\beta \nu} = \frac{1}{2} R_2^{\alpha \beta} \nabla_\alpha \hat{R}_{2 \rho}^{\alpha \beta} \nabla_\beta \hat{R}_{3 \lambda}^{\beta \nu} + \text{tr} \left[ \frac{1}{2} R_1^{\mu \nu} \nabla_\mu \nabla_\lambda \hat{R}_{3 \mu}^{\lambda \beta} + O[\mathcal{R}] \right] + \text{total derivative}, \]

transform their left-hand sides to the structures 8, 18 and 21 of the table. The identity

\[ \text{tr} \nabla_\mu R_1^{\beta \alpha} \nabla_\beta \hat{R}_{2 \rho}^{\alpha \beta} \hat{R}_{3 \lambda}^{\beta \nu} \]

transforms the left-hand side to the structure 12 of the table. The identities

\[ \nabla_\alpha R_1^{\beta \mu} \nabla_\beta \hat{R}_{2 \rho}^{\alpha \beta} \hat{R}_{3 \lambda}^{\beta \nu} = \frac{1}{2} R_2^{\alpha \beta} \nabla_\alpha \hat{R}_{2 \rho}^{\alpha \beta} \nabla_\beta \hat{R}_{3 \lambda}^{\beta \nu} + \text{tr} \left[ \frac{1}{2} R_1^{\mu \nu} \nabla_\mu \nabla_\lambda \hat{R}_{3 \mu}^{\lambda \beta} + O[\mathcal{R}] \right] + \text{total derivative} \]

transform their left-hand sides to the structures 7, 19, 20 and 21 of the table. The identity

\[ \text{tr} \nabla_\mu R_1^{\mu \nu} \nabla_\nu \hat{R}_{2 \rho}^{\alpha \beta} \hat{R}_{3 \lambda}^{\beta \nu} \]

transforms its left-hand side to the structures 8, 18 and 21 of the table. The identity (14.26) is obtained by using (14.22) for both \( \nabla\)’s on the left-hand side to write

\[ 2 \nabla^2_{[\alpha} \nabla^3_{\beta]} = \nabla^1_{[\alpha} \nabla^2_{\beta]} - \nabla^1_{[\alpha} \nabla^3_{\beta]} + O[\mathcal{R}], \]

and next applying in succession: the Jacobi identity to \( \hat{R}_2 \) and \( \hat{R}_3 \), (14.22) to \( \nabla \) other than \( \nabla_\alpha \) or \( \nabla_\beta \), the Jacobi identity to \( \hat{R}_1 \), and again (14.22) to \( \nabla \) other than \( \nabla_\alpha \) or \( \nabla_\beta \). The identity (14.27) is obtained by writing

\[ \nabla^\mu \hat{R}_1^{\lambda \nu} = \nabla^\mu \hat{R}_1^{\lambda \nu} + \nabla^{(\mu} \hat{R}_1^{\lambda \nu)}, \]

applying (14.24) to the first term, (14.22) to the second term, next the Jacobi identity to \( \hat{R}_2 \) and \( \hat{R}_3 \), and noting that what is left over forms a total derivative up to \( O[\mathcal{R}] \). The identity (14.28) is obtained by first removing the derivative acting on \( R_1 \), and next using the Jacobi identity for \( \hat{R}_3 \). The identity (14.29) is obtained by replacing the derivative acting on \( R_1 \), so as to use (14.25), and next removing the derivative from the Ricci scalar, applying the
Jacobi identity to $\hat{\mathcal{R}}_2$ and $\hat{\mathcal{R}}_3$ in the unwanted terms, and using (14.23). The identity (14.30) is obtained by applying (14.24) to both $\hat{\mathcal{R}}$'s, using (14.23), and again applying (14.24).

The contribution of one of the structures:

$$\nabla_{\lambda} \hat{\mathcal{R}}_{1}^{\lambda \sigma} \nabla_{\sigma} \nabla_{\alpha} R_{2}^{\mu \nu} \nabla_{\mu} \nabla_{\nu} R_{3}^{\alpha \beta} = - \nabla_{\lambda} \hat{\mathcal{R}}_{1}^{\lambda \sigma} \nabla_{\sigma} \nabla_{\beta} R_{2}^{\mu \nu} \nabla_{\mu} \nabla_{\nu} R_{3}^{\alpha \beta} + O[\Re^4] + \text{a total derivative} \quad (14.31)$$

vanishes because its form factor turns out to be symmetric under a permutation of the labels 2 and 3 whereas the structure itself is antisymmetric under this permutation.

The final result of the calculations above is as follows.

15. The $\alpha$-representation of the form factors in the trace of the heat kernel

We obtain

$$\text{Tr}K(s) = \frac{1}{(4\pi s)^2} \int dx \, g^{1/2} \text{tr} \{ \hat{1} + s \hat{\mathcal{P}} + s^2 \sum_{i=1}^{5} f_i(-s \Box_2) \hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2(i)$$

$$+ \left(s^3 \sum_{i=1}^{11} + s^4 \sum_{i=12}^{25} + s^5 \sum_{i=26}^{28} \right) F_i(-s \Box_1, -s \Box_2, -s \Box_3) \hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(i)$$

$$+ s^6 F_{29}(-s \Box_1, -s \Box_2, -s \Box_3) \hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(29)$$

$$+ \left[ s^4 \sum_{i=30}^{32} F_i(-s \Box_1, -s \Box_2, -s \Box_3) \hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(i)$$

$$+ s^5 F_{33}(-s \Box_1, -s \Box_2, -s \Box_3) \hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(33) \right] + O[\Re^4] \} \quad (15.1)$$

where $\hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2(i)$ with $i = 1$ to 5 are quadratic structures of the table (2.2)–(2.6), $\hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(i)$ with $i = 1$ to 29 are cubic structures of the table (2.15)–(2.43), and there are four additional cubic structures linear in $\hat{\mathcal{R}}_{\mu \nu}$:

$$\hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(30) = \nabla_{\beta} \hat{\mathcal{R}}_1^{\beta \alpha} \nabla_{\alpha} R_2 R_3, \quad (15.2)$$

$$\hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(31) = \nabla_{\mu} \hat{\mathcal{R}}_1^{\mu \alpha} R_2 \alpha \beta \nabla^{\beta} R_3, \quad (15.3)$$

$$\hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(32) = \hat{\mathcal{P}}_1 \nabla_{\beta} \hat{\mathcal{R}}_2^{\beta \alpha} \nabla_{\alpha} R_3, \quad (15.4)$$

$$\hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2 \hat{\mathcal{R}}_3(33) = \nabla_{\alpha} \hat{\mathcal{R}}_1^{\alpha \beta} \nabla_{\beta} R_2 \mu \nu \nabla_{\mu} \nabla_{\nu} R_3. \quad (15.5)$$

The form factors

$$f_i(-s \Box), \quad F_i(-s \Box_1, -s \Box_2, -s \Box_3) \quad (15.6)$$

are obtained as integrals over the parameters

$$\langle \ldots \rangle_2 = \int_{\alpha_2 \geq 0} d\alpha_1 d\alpha_2 \delta(1 - \alpha_1 - \alpha_2) \langle \ldots \rangle, \quad (15.7)$$

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\[(\ldots)_3 = \int_{\alpha \geq 0} d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3)(\ldots), \quad (15.8)\]

and, in this form, are represented by two nonlocal kernels:

\[\exp(s\alpha_1\alpha_2\Box)\]

and

\[\exp(s\Omega), \quad \Omega = \alpha_2\alpha_3\Box_1 + \alpha_1\alpha_3\Box_2 + \alpha_1\alpha_2\Box_3. \quad (15.10)\]

The function (15.9) appears in the combinations

\[\mathcal{A}, \quad \mathcal{B}, \quad \mathcal{C}, \quad \mathcal{U}, \quad \mathcal{V}, \quad \mathcal{W} \quad (15.11)\]

introduced in (14.12)–(14.17), and the function (15.10) appears in the combinations (cf. (14.1))

\[e^{s\Omega}, \quad e^{s\Omega} - 1, \quad e^{s\Omega} - 1 - s\Omega \quad (15.12)\]

which figure explicitly in the expressions below. The coefficients of these functions are polynomials in \(\alpha\)'s, boxes, and inverse boxes.

In this representation, the second-order form factors are of the form

\[f_1 = \langle \mathcal{C} \rangle_2, \quad (15.13)\]

\[f_2 = \left\langle \frac{1}{2} \left( \alpha_1^2\alpha_2^2 - \frac{1}{3}\alpha_1^2 + \frac{1}{36} \right) \mathcal{A} + \left( \alpha_1\alpha_2 - \frac{1}{2}\alpha_1^2 + \frac{1}{6} \right) \mathcal{B} + \frac{1}{2}\mathcal{C} \right\rangle_2, \quad (15.14)\]

\[f_3 = \left\langle \left( \alpha_1 - \frac{1}{6} \right) \mathcal{A} - \mathcal{B} \right\rangle_2, \quad (15.15)\]

\[f_4 = \left\langle \frac{1}{2} \mathcal{A} \right\rangle_2, \quad (15.16)\]

\[f_5 = \left\langle \frac{1}{2} \mathcal{B} \right\rangle_2, \quad (15.17)\]

and the results for the third-order form factors are as follows:

\[F_1 = \left\langle \frac{1}{3} e^{s\Omega} \right\rangle_3, \quad (15.18)\]

\[F_2 = \left\langle \frac{4}{3} \alpha_1\alpha_2\alpha_3 e^{s\Omega} \right\rangle_3 + \left\langle -2\mathcal{V}_{12} \right\rangle_2, \quad (15.19)\]

\[F_3 = \left\langle 2\alpha_1\alpha_2 e^{s\Omega} \right\rangle_3, \quad (15.20)\]
\[ F_4 = \left( \frac{1}{\Box_1 \Box_2} \left( \frac{e^{s \Omega} - 1}{s^2} \right) + \left[ \frac{1}{\Box_1} \left( \frac{1}{3} - \alpha_1^2 - \alpha_2^2 + 3 \alpha_1 \alpha_3 + \alpha_2 \alpha_3 - \alpha_3^2 \right) \right] \right) \]

\[ + \frac{\Box_3}{\Box_1 \Box_2} (3 \alpha_2 \alpha_3) \left( \frac{e^{s \Omega}}{s^2} \right) \left( \left( \frac{1}{36} + \frac{1}{2} \alpha_1^4 - \frac{1}{6} \alpha_1 \alpha_2 - \frac{1}{3} \alpha_2^2 \right) - \frac{1}{2} \alpha_1^2 \alpha_2^2 - \frac{1}{6} \alpha_2 \alpha_3 + \frac{1}{6} \alpha_3^2 + \frac{1}{2} \alpha_1 \alpha_2 \alpha_3 + \frac{3}{2} \alpha_2^2 \alpha_3^2 \right) \]

\[ + \frac{\Box_2}{\Box_1} \left( \frac{1}{2} \alpha_1^2 - \frac{1}{6} \alpha_1 \alpha_2 - \frac{1}{2} \alpha_1 \alpha_2 - \frac{1}{2} \alpha_2^2 \alpha_3 + \frac{1}{2} \alpha_1 \alpha_2^2 \alpha_3 + \frac{1}{2} \alpha_1 \alpha_2^2 \alpha_3 + \frac{3}{2} \alpha_1 \alpha_2 \alpha_3^2 - \frac{1}{3} \alpha_1 \alpha_3 \right) \]

\[ + \frac{\Box_3}{\Box_1} \left( \frac{1}{2} \alpha_1^2 + \frac{1}{6} \alpha_1 \alpha_2 + \alpha_1 \alpha_2 + \frac{3}{2} \alpha_1 \alpha_2 \alpha_3 - \frac{1}{3} \alpha_1 \alpha_3 \right) \]

\[ + \frac{1}{2} \alpha_1^3 \alpha_3 + 2 \alpha_1^2 \alpha_2 \alpha_3 + \alpha_1 \alpha_2^2 \alpha_3 - \frac{1}{2} \alpha_1 \alpha_3^2 - \alpha_1 \alpha_2 \alpha_3 \right) \]

\[ + \frac{\Box_3^2}{\Box_1 \Box_2} \left( \frac{1}{2} \alpha_1^3 \alpha_2 - \frac{1}{2} \alpha_1 \alpha_2^3 - \frac{3}{2} \alpha_1 \alpha_2 \alpha_3 \right) \left( \frac{e^{s \Omega}}{s^2} \right) \]

\[ + \left( \frac{1}{\Box_1} \left( \frac{1}{6} + \alpha_1^2 \right) \frac{A_2}{s} + \left[ \frac{1}{\Box_1} (-\alpha_1^2) + \frac{\Box_3}{\Box_1 \Box_2} \left( \frac{1}{2} \alpha_1^2 \right) \right] \frac{A_3}{s} \right) \]

\[ - \frac{1}{\Box_1} \frac{B_2}{s} + \left( \frac{1}{\Box_1} - \frac{1}{2} \frac{\Box_3}{\Box_1 \Box_2} \right) \frac{B_3}{s} \]

\[ + \frac{\Box_3}{\Box_1} \left( \frac{1}{2} \alpha_1^2 - \frac{1}{12} \right) + \frac{\Box_1}{\Box_1} \left( \frac{1}{2} \alpha_1^2 + \frac{1}{12} \right) \]

\[ + \left( \frac{1}{2} \alpha_1^2 + \frac{1}{12} \right) \left( \frac{U_{23}}{s} + \left( 2 + \frac{1}{2} \frac{\Box_3}{\Box_1 \Box_2} - \frac{1}{2} \frac{\Box_3}{\Box_1 \Box_2} \right) \right) \frac{V_{23}}{s} \right) \right), \quad (15.21) \]

\[ F_5 = \left( \frac{2}{\Box_1 \Box_2} \left( \frac{e^{s \Omega} - 1}{s^2} \right) \right) + \left( \frac{1}{\Box_1} (-\alpha_1^2) + \frac{\Box_3}{\Box_1 \Box_2} \left( \frac{1}{2} \alpha_1^2 \right) \right) \frac{A_3}{s} \]

\[ + \left( \frac{1}{\Box_1} + \frac{3}{\Box_1 \Box_2} \right) \frac{B_3}{s} \right) \right), \quad (15.22) \]

\[ F_6 = \left( \frac{e^{s \Omega}}{s} + \left[ \frac{1}{6} + \alpha_1^2 + \alpha_1 \alpha_3 \right] + \frac{1}{\Box_3} \alpha_1 \alpha_3 - \alpha_2 \alpha_3 \right) \frac{e^{s \Omega}}{s} \]

\[ + \left( \frac{1}{2} \frac{A_1}{s} \right) \right), \quad (15.23) \]

\[ F_7 = \left( \frac{1}{\Box_1} \left( 2 \alpha_1^2 - 2 \alpha_1 \alpha_3 - 4 \alpha_2 \alpha_3 \right) \right) \frac{e^{s \Omega}}{s} \]

\[ + \left[ \left( \frac{1}{3} \alpha_2 \alpha_3 + \alpha_1 \alpha_2 \alpha_3 + 2 \alpha_1 \alpha_2 \alpha_3 + 2 \alpha_1^2 \alpha_3 \right) \right] \frac{1}{\Box_1} \left( 2 \alpha_1 \alpha_2 \alpha_3 \right) \frac{e^{s \Omega}}{s} \]

\[ + \frac{\Box_3}{\Box_1} \left( -2 \alpha_1 \alpha_2 \alpha_3 + 2 \alpha_1 \alpha_2 \alpha_3 \right) \frac{e^{s \Omega}}{s} \]

\[ + \left( \frac{1}{2} \frac{B_2}{s} + \frac{1}{2} \frac{V_{23}}{s} \right) \right), \quad (15.24) \]
\[ F_8 = \left( \frac{1}{2} \frac{(-4\alpha_1^2 + 16\alpha_1 \alpha_2) e^{s\Omega}}{s} + \left[ (-4\alpha_1^2 \alpha_2 \alpha_3) + \frac{3}{\Omega_1} (8\alpha_1^2 \alpha_2 \alpha_3) \right] e^{s\Omega} \right)_3 \\
+ \left( -2 \frac{1}{\Omega_1} \frac{B_3}{s} \right)_2, \] (15.25)

\[ F_9 = \left( -\frac{1}{3} \frac{1}{\Omega_1 \Omega_2 \Omega_3} \left( e^{s\Omega} - 1 - s\Omega \right) + \frac{1}{\Omega_1 \Omega_2} \left( -\frac{1}{6} + \alpha_1^2 - 2\alpha_1 \alpha_2 \right.ight.
\left. -2\alpha_2 \alpha_3 + \frac{3}{2} \alpha_2^2 \left( e^{s\Omega} - 1 \right) \frac{1}{s^2} + \left[ \frac{1}{\Omega_1} \left( -\frac{1}{36} + \frac{1}{6} \alpha_1^2 - \frac{1}{2} \alpha_1^4 \right.ight.ight.
\left. + \frac{1}{3} \alpha_2^2 + 2\alpha_1 \alpha_2^3 - \frac{1}{2} \alpha_2^4 - \frac{1}{2} \alpha_1 \alpha_3 - \frac{1}{6} \alpha_2 \alpha_3 \right.
\left. - \alpha_1^2 \alpha_2 \alpha_3 + 4\alpha_1 \alpha_2^2 \alpha_3 + 2\alpha_2^3 \alpha_3 - \frac{5}{2} \alpha_1^2 \alpha_2^3 + \frac{3}{2} \alpha_2^2 \alpha_3^2 \right)
\left. \left. + \frac{1}{\Omega_2 \Omega_3} \left( -2\alpha_1^3 \alpha_2 + \frac{3}{2} \alpha_1^2 \alpha_2^2 + \frac{1}{2} \alpha_2^4 + \frac{1}{2} \alpha_1 \alpha_3 \right.ight.ight.
\left. - \frac{1}{2} \alpha_1^2 \alpha_2 \alpha_3 - \alpha_1 \alpha_2 \alpha_3 - \frac{1}{2} \alpha_2^3 \alpha_3 - \alpha_1 \alpha_3^3 \right) e^{s\Omega} \right)
\left. + \left[ \left( -\frac{1}{648} + \frac{1}{24} \alpha_1^2 - \frac{1}{12} \alpha_1^4 - \frac{1}{6} \alpha_1^3 \alpha_2 + \frac{1}{36} \alpha_1 \alpha_3 \right.ight.
\left. - \frac{1}{6} \alpha_1^2 \alpha_2 \alpha_3 - \alpha_1^3 \alpha_2^2 \alpha_3 - \frac{1}{6} \alpha_1^2 \alpha_3^2 - \frac{1}{3} \alpha_1 \alpha_2^2 \alpha_3^2 \right)
\left. + \frac{1}{\Omega_2 \Omega_3} \left( \frac{1}{36} \alpha_1 \alpha_2 - \frac{1}{12} \alpha_1^3 \alpha_2 + \frac{1}{6} \alpha_1^2 \alpha_2^2 - \frac{1}{2} \alpha_1^4 \alpha_2^2 - \frac{1}{12} \alpha_1 \alpha_2^3 \\
- \frac{1}{2} \alpha_1^2 \alpha_2^4 - \frac{1}{36} \alpha_2 \alpha_3 + \frac{1}{12} \alpha_1^2 \alpha_2 \alpha_3 - \frac{1}{2} \alpha_1 \alpha_2 \alpha_3 - \frac{1}{4} \alpha_1 \alpha_2 \alpha_3 \right.
\left. + \frac{1}{2} \alpha_1 \alpha_2^4 \alpha_3 - \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \alpha_3^2 + \frac{1}{2} \alpha_1^3 \alpha_2 \alpha_3^2 - \frac{1}{3} \alpha_2 \alpha_3^2 + 2\alpha_1^2 \alpha_2 \alpha_3^2 \right.
\left. + \frac{1}{2} \alpha_1 \alpha_2^3 \alpha_3^2 + \alpha_2 \alpha_3^2 + \alpha_1 \alpha_2 \alpha_3^3 + 3\alpha_1 \alpha_2 \alpha_3 \alpha_3 \alpha_3 \right.
\left. + \alpha_1 \alpha_2 \alpha_3^4 + \alpha_2 \alpha_3^4 \right) + \frac{\Omega_1^2}{\Omega_2 \Omega_3} \left( \frac{1}{4} \alpha_1 \alpha_2 \alpha_3^2 - \alpha_1^3 \alpha_2 \alpha_3 \right.
\left. + \frac{1}{12} \alpha_2 \alpha_3^3 - \alpha_1^2 \alpha_2 \alpha_3^2 + \frac{1}{12} \alpha_1 \alpha_2 \alpha_3^3 \\
- \frac{1}{2} \alpha_1^2 \alpha_2 \alpha_3^3 - \frac{1}{2} \alpha_2 \alpha_3^3 \right) e^{s\Omega} \right)_3 \\
+ \left( \frac{\Omega_1}{\Omega_2 \Omega_3} \left( -\frac{1}{12} \alpha_1^2 + \frac{1}{2} \alpha_2^2 \alpha_3^2 \right) \frac{A_1}{s} + \frac{1}{\Omega_1} \left( \frac{1}{72} - \frac{1}{2} \alpha_1 \alpha_2 \alpha_3 \right) \frac{A_2}{s} \\
+ \frac{\Omega_1}{\Omega_2 \Omega_3} \left( \frac{1}{12} \alpha_1^2 + \alpha_1 \alpha_2 \right) \frac{B_1}{s} + \frac{1}{\Omega_1} \left( \alpha_1 - \alpha_1 \alpha_2 \alpha_2 \right) \frac{B_2}{s} \\
+ \frac{\Omega_1}{\Omega_2 \Omega_3} \left( \frac{1}{2} \alpha_1 \alpha_2 \alpha_2 - \frac{1}{12} \frac{A_1^2}{s} \right) \frac{C_1}{s} + \frac{1}{\Omega_1} \left( \alpha_1^2 - \frac{1}{4} \alpha_1^2 \alpha_2 \alpha_2 - \frac{1}{144} \right) \frac{C_2}{s} \\
+ \left( \frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_1 \alpha_2 - \frac{1}{12} \right) \frac{V_1}{s} + \left( \frac{1}{2} \alpha_1 - \frac{1}{2} \right) \frac{W_1}{s} \right)_2, \] (15.26)
\[
F_{10} = \left\langle -\frac{8}{3} \frac{1}{\Box_1 \Box_2 \Box_3} \frac{(e^{s\Omega} - 1 - s\Omega)}{s^3} \right\rangle_3 + \left\langle \frac{2}{\Box_2 \Box_3} \frac{C_1}{s} \right\rangle_2, \\
\]

\[
F_{11} = \left\langle -\frac{2}{\Box_1 \Box_2 \Box_3} \frac{(e^{s\Omega} - 1 - s\Omega)}{s^3} + \left[ \frac{1}{\Box_1 \Box_2} \left( -\frac{1}{3} + 2\alpha_1^2 + 2\alpha_1 \alpha_3 + \alpha_3^2 \right) \\
+ \frac{1}{\Box_1 \Box_3} \left( -2\alpha_1 \alpha_2 \alpha_3 + 2\alpha_2 \alpha_3 \right) \frac{(e^{s\Omega} - 1)}{s^2} \right]_3 \\
+ \left[ \frac{1}{\Box_1} \left( \frac{1}{6} \alpha_1^2 - \alpha_1 \alpha_2 \alpha_2^2 \right) + \frac{1}{\Box_1 \Box_2} \left( -\frac{1}{12} \alpha_1^2 + \frac{1}{2} \alpha_1^2 \alpha_2^2 \right) \right] \frac{A_3}{s} \\
+ \left[ \frac{1}{\Box_1} \left( -\frac{1}{6} + 2\alpha_1^2 - 2\alpha_1 \alpha_2 \right) \right] \frac{B_3}{s} \\
+ \left( -\frac{1}{\Box_2} + \frac{1}{\Box_2 \Box_3} \right) \frac{C_1}{s} + \left( -\frac{1}{\Box_1} + \frac{3}{2 \Box_1 \Box_2} \right) \frac{C_3}{s} - \frac{1}{2} \mathcal{W}_{12} \right\rangle_2, \\
\]

\[
F_{12} = \left\langle \frac{1}{\Box_1 \Box_3} \frac{-2\alpha_1 + 2\alpha_2 + 2\alpha_3}{s^2} \frac{e^{s\Omega}}{s} + \left[ \frac{1}{\Box_2} \left( 2\alpha_1 \alpha_2 \alpha_3 \right) \right]_3 \\
+ \left[ \frac{1}{\Box_3} \left( 2\alpha_1 \alpha_2 \alpha_3 \right) + \frac{1}{\Box_1 \Box_2} \frac{e^{s\Omega}}{s} \right]_3 \\
+ \left\langle -2 \frac{1}{\Box_2} \frac{B_1}{s^2} - 2 \frac{1}{\Box_3} \frac{V_{12}}{s} - 2 \frac{1}{\Box_2} \frac{V_{13}}{s} \right\rangle_2, \\
\]

\[
F_{13} = \left\langle \frac{1}{\Box_1} \left( 2\alpha_1 \right) \frac{e^{s\Omega}}{s} \right\rangle_3 + \left\langle -2 \frac{1}{\Box_1} \frac{U_{23}}{s} \right\rangle_2, \\
\]

\[
F_{14} = \left\langle -\frac{2}{\Box_1 \Box_2 \Box_3} \frac{s^2}{s^2} + \left[ \frac{1}{\Box_1} \left( 2\alpha_1 \alpha_2 \right) + \frac{1}{\Box_2} \left( 2\alpha_1 \alpha_2 \right) \right]_3 \\
+ \left[ \frac{1}{\Box_1 \Box_2} \left( -2\alpha_1 \alpha_2 \right) \right] \frac{e^{s\Omega}}{s} + \left\langle \frac{1}{\Box_1 \Box_2} \left( 2\alpha_2 \right) \frac{A_3}{s^2} \right\rangle_2, \\
\]

\[
F_{15} = \left\langle \frac{1}{\Box_1 \Box_2} \left( -4\alpha_1 + 12\alpha_1^2 \right) \frac{e^{s\Omega}}{s^2} + \left[ \frac{1}{\Box_1} \left( \frac{2}{5} \alpha_1^2 - 2\alpha_1^4 - 2\alpha_1^3 \alpha_2 \right.ight. \\
\left. - 2\alpha_1^2 \alpha_2 \alpha_3 - 2\alpha_1 \alpha_2 \alpha_3^2 \right) + \frac{1}{\Box_2} \left( -2\alpha_1^3 \alpha_2 + 2\alpha_1^2 \alpha_2^2 + 2\alpha_1^2 \alpha_2 \alpha_3 \right) \right]_3 \\
+ \left[ \frac{1}{\Box_1 \Box_2} \left( 2\alpha_1^3 \alpha_2 - 2\alpha_1^2 \alpha_2^2 - 2\alpha_1^2 \alpha_2 \alpha_3 \right) \right] \frac{e^{s\Omega}}{s} \right\rangle_3 \\
+ \left\langle -2 \frac{1}{\Box_1 \Box_2} \frac{B_3}{s^2} + \frac{1}{\Box_1} \left( 2\alpha_1^2 - \frac{1}{3} \right) \frac{U_{23}}{s} - 2 \frac{1}{\Box_1} \frac{V_{23}}{s} \right\rangle_2, \\
\]

\[
F_{16} = \left\langle \frac{1}{\Box_1 \Box_2} \left( 8\alpha_1 \alpha_2 \right) \frac{e^{s\Omega}}{s^2} \right\rangle_3 + \left\langle \frac{1}{\Box_1 \Box_2} \left( -2\alpha_1^2 \right) \frac{A_3}{s^2} \right. \\
\left. + 2 \frac{1}{\Box_1 \Box_2} \frac{B_3}{s^2} \right\rangle_2, \\
\]

\(149\)
\[ F_{17} = \left\langle \frac{1}{\Box_1} \left( 2\alpha_1^2 \right) \frac{e^{s\Omega}}{s} \right\rangle_3 + \left\langle -\frac{1}{\Box_1} \frac{U_{23}}{s} \right\rangle_2 , \]  

\[ F_{18} = \left\langle 4 \frac{1}{\Box_1 \Box_2 \Box_3} \frac{(e^{s\Omega} - 1)}{s^3} + \left[ \frac{1}{\Box_2 \Box_3} (-8\alpha_1 \alpha_2 + 2\alpha_1^2) \right. \right. + \frac{1}{\Box_1} \left. \right. (4\alpha_1^2 \alpha_2 \alpha_3) \] 
\[ + \left\langle -4 \frac{1}{\Box_1 \Box_3} \frac{B_2}{s^2} \right\rangle_2 , \]  

\[ F_{19} = \left\langle \frac{1}{\Box_1} (-4\alpha_1^2 \alpha_2 \alpha_3) \frac{e^{s\Omega}}{s} \right\rangle_3 + \left\langle -\frac{1}{\Box_1} \frac{V_{23}}{s} \right\rangle_2 , \]  

\[ F_{20} = \left\langle 2 \frac{1}{\Box_1 \Box_2 \Box_3} \frac{(e^{s\Omega} - 1)}{s^3} + \left[ \frac{1}{\Box_1 \Box_2} \left( 2\alpha_2^2 - 4\alpha_1 \alpha_3 - 8\alpha_2 \alpha_3 + 2\alpha_3^2 \right) \right. \right. \] 
\[ + \frac{1}{\Box_2 \Box_3} \left. \left. \right. \left( \frac{1}{3} - \alpha_1^2 - 4\alpha_2^2 + 4\alpha_2 \alpha_3 \right) \right\rangle_3 \] 
\[ \left. \left. + \left[ \frac{1}{\Box_2} \left( -\frac{2}{3} \alpha_2 \alpha_3 + 2\alpha_1^2 \alpha_2 \alpha_3 + 4\alpha_1 \alpha_2^2 \alpha_3 + 2\alpha_2^3 \alpha_3 + 2\alpha_2 \alpha_3^3 \right) \right. \right. \] 
\[ + \frac{1}{\Box_1 \Box_3} \left. \left. \left( \frac{1}{3} \alpha_2 \alpha_3 - \alpha_1^2 \alpha_2 \alpha_3 - 2\alpha_1 \alpha_2 \alpha_3 - 2\alpha_3^3 \right) \right. \right. \] 
\[ + \left. \frac{1}{\Box_3} \left( -2\alpha_1 \alpha_2^2 \alpha_3 + 2\alpha_1 \alpha_2 \alpha_3^2 \right) \right\rangle_3 \] 
\[ + \left\langle \frac{1}{\Box_2 \Box_3} \left( -\frac{1}{3} \alpha_2 + 2\alpha_1 \alpha_2 \right) \frac{A_1}{s^2} + \frac{1}{\Box_1} \frac{V_{23}}{s} \right\rangle_2 , \]  

\[ F_{21} = \left\langle \frac{1}{\Box_1 \Box_2} (-8\alpha_1^2 + 16\alpha_1 \alpha_3) \frac{e^{s\Omega}}{s^2} + \left[ \frac{1}{\Box_1} (8\alpha_1^2 \alpha_2 \alpha_3) \right. \right. \] 
\[ + \frac{1}{\Box_2} (-8\alpha_1 \alpha_2 \alpha_3) + \frac{\Box_3}{\Box_1 \Box_2} (8\alpha_1^2 \alpha_2 \alpha_3) \right\rangle_3 \] 
\[ + \left\langle 4 \frac{1}{\Box_1} \frac{V_{23}}{s} \right\rangle_2 , \]  

\[ F_{22} = \left\langle \frac{1}{\Box_1 \Box_2} \left( -10\alpha_1^2 + 24\alpha_1 \alpha_3 + 4\alpha_2 \alpha_3 \right) \right. \] 
\[ + \left[ \frac{1}{\Box_1 \Box_2} \left( -10\alpha_1^2 + 24\alpha_1 \alpha_3 + 4\alpha_2 \alpha_3 \right) \right. \] 
\[ + \left. \left[ \frac{1}{\Box_1 \Box_2} \left( 2\alpha_1^4 - \frac{4}{3} \alpha_1^2 + \frac{2}{3} \alpha_1 \alpha_2 - 12\alpha_1^3 \alpha_2 + 18\alpha_1^2 \alpha_2^2 + \frac{2}{3} \alpha_1 \alpha_3 \right. \right. \] 
\[ - 4\alpha_1^3 \alpha_3 + 12\alpha_1^2 \alpha_2 \alpha_3 + 4\alpha_1 \alpha_2^2 \alpha_3 - 6\alpha_1^2 \alpha_3^2 - 4\alpha_1 \alpha_2 \alpha_3^2 \right) \] 
\[ \right\rangle_2 , \]
\[ F_{23} = \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} \left( 8 \alpha_1^2 - 16 \alpha_1 \alpha_2 + 8 \alpha_2 \alpha_3 \right) \right\rangle_{s^3} \left( e^{s\Omega} - 1 \right) \]
\[ + \left\langle \frac{1}{\Box_1 \Box_2} \left( 3 \alpha_1 - 8 \alpha_1 \alpha_2 + 4 \alpha_1 \alpha_2 \alpha_3 \right) \right\rangle_{s^3} \]
\[ + \left\langle \frac{1}{\Box_2 \Box_3} \left( 8 \alpha_1^2 \alpha_2 - 8 \alpha_1 \alpha_2 \alpha_3 \right) \right\rangle_{s^3} \]
\[ + \left\langle \frac{1}{\Box_1 \Box_2} \left( 3 \alpha_1 - 2 \alpha_1 \alpha_2 \right) \right\rangle_{s^3} \frac{A_3}{s^2} + \frac{1}{\Box_1 \Box_2} \left( \frac{1}{3} \right) \frac{B_3}{s^2} \]
\[ + 4 \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} \left( 4 \alpha_1^2 \alpha_2 \right) \right\rangle_{s^3} \]
\[ = 151 \]
\[ F_{24} = \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} \left( -4 \alpha_1^2 \right) \right\rangle_{s^3} \]
\[ + \left\langle \frac{2}{\Box_1 \Box_2 \Box_3} \left( \frac{1}{3} \right) \right\rangle_{s^3} \frac{C_1}{s^2} + \frac{1}{\Box_1 \Box_2} \left( \frac{1}{3} \right) \frac{C_3}{s^2} + \frac{1}{\Box_1} \left( \frac{1}{3} \right) \frac{W_{12}}{s} \]
\[ = 154 \]
\[ F_{25} = \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} \left( -16 \alpha_2 \alpha_3 \right) \right\rangle_{s^3} \]
\[ + \left\langle \frac{4}{\Box_2 \Box_3} \left( \frac{1}{3} \right) \right\rangle_{s^3} \frac{C_1}{s^2} + \frac{8}{\Box_1 \Box_3} \left( \frac{1}{3} \right) \frac{C_2}{s^2} + \frac{8}{\Box_3} \left( \frac{1}{3} \right) \frac{W_{12}}{s} \]
\[ = 155 \]
\[ F_{26} = \left\langle \frac{1}{\Box_1 \Box_2} \left( 4 \alpha_1 \alpha_2 \right) \right\rangle_{s^2} \]
\[ F_{27} = \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} \left( 8\alpha_1^3 \alpha_2 - 12\alpha_1^2 \alpha_2^2 - 8\alpha_1^2 \alpha_2 \alpha_3 \right) \frac{e^{s\Omega}}{s^3} \right\rangle_3, \]

\[ F_{28} = \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} (-16\alpha_1 \alpha_2 \alpha_3^2) \frac{e^{s\Omega}}{s^3} \right\rangle_3, \]

\[ F_{29} = \left\langle \frac{1}{\Box_1 \Box_2 \Box_3} \left( \frac{8}{3} \alpha_1^2 \alpha_2^2 \alpha_3^2 \right) \frac{e^{s\Omega}}{s^3} \right\rangle_3, \]

\[ F_{30} = \left\langle \left[ \frac{1}{\Box_2 \Box_3} \left( 2\alpha_3^2 - 4\alpha_1 \alpha_3^2 - 4\alpha_1^2 \alpha_3 \right) \right. \right.
\[ + \frac{1}{\Box_1 \Box_2} \left( -\frac{1}{3} - 2\alpha_1^3 - 4\alpha_1^2 \alpha_2 + 2\alpha_2^2 
\[ + 2\alpha_2^3 + 2\alpha_1^2 \alpha_3 + 4\alpha_1 \alpha_3^2 - 2\alpha_2 \alpha_3^2 \right) \frac{e^{s\Omega}}{s^2} \left. \right\rangle_3 
\[ + \left[ \frac{\Box_3}{\Box_1 \Box_2} \left( -\frac{1}{3} \alpha_1 \alpha_2 + 2\alpha_1 \alpha_2^3 + 2\alpha_1^2 \alpha_2 \alpha_3 \right) \right. \right.
\[ + \frac{\Box_1 \Box_2}{\Box_2 \Box_3} (-2\alpha_1 \alpha_2 \alpha_3^2) + \frac{1}{\Box_1} \left( 2\alpha_1^2 \alpha_3^2 - 2\alpha_1 \alpha_2 \alpha_3^2 - \frac{1}{3} \alpha_1 \alpha_3 \right) \right.
\[ + \frac{1}{\Box_2} \left( \frac{1}{3} \alpha_1 \alpha_2 - 2\alpha_1 \alpha_2 \alpha_3 - 2\alpha_1^2 \alpha_3 + 4\alpha_1 \alpha_2 \alpha_3^2 \right) \frac{e^{s\Omega}}{s} \left. \right\rangle_3 
\[ + \left\langle \frac{1}{\Box_1 \Box_2} \left( \frac{1}{3} \alpha_2 - 2\alpha_1 \alpha_2 \right) \frac{A_3}{s^2} \right\rangle_2, \]

\[ F_{31} = \left\langle -4 \frac{1}{\Box_1 \Box_2 \Box_3} \frac{(e^{s\Omega} - 1)}{s^3} + \left\langle \frac{1}{\Box_1 \Box_2} \left( -4\alpha_1 \alpha_2 + 4\alpha_2^2 \right) \right. \right.
\[ + \frac{1}{\Box_1 \Box_3} (4\alpha_2 \alpha_3) + \frac{1}{\Box_2 \Box_3} (-4\alpha_2 \alpha_3) \frac{e^{s\Omega}}{s^2} \left. \right\rangle_3 
\[ + \left\langle \frac{1}{\Box_2 \Box_3} \left( 4\alpha_1 \right) \frac{B_1}{s^2} \right\rangle_2, \]

\[ F_{32} = \left\langle 2 \frac{1}{\Box_2 \Box_3} \frac{e^{s\Omega}}{s^2} + \left[ \frac{1}{\Box_2} \left( 2\alpha_1 \alpha_2 - 2\alpha_2^2 \right) + \frac{1}{\Box_3} (-2\alpha_2 \alpha_3) \right. \right.
\[ + \frac{\Box_1}{\Box_2 \Box_3} (2\alpha_2 \alpha_3) \frac{e^{s\Omega}}{s} \left. \right\rangle_3 + \left\langle -\frac{1}{\Box_2 \Box_3} \frac{A_1}{s^2} \right\rangle_2, \]
\[ F_{33} = \left( \frac{1}{\Box_1 \Box_2 \Box_3} (8\alpha_1 \alpha_2 - 4\alpha_2^2) e^{s\Omega} \right) \frac{1}{s^3} + \left( \frac{1}{\Box_1 \Box_3} (-4\alpha_1 \alpha_2^2 \alpha_3) \right) \frac{e^{s\Omega}}{s^3} \]  
+ \left( \frac{1}{\Box_2 \Box_3} (4\alpha_1 \alpha_2^2 \alpha_3) + \frac{1}{\Box_1 \Box_2} (4\alpha_1^2 \alpha_2^2 - 4\alpha_1 \alpha_2^3) \right) \frac{e^{s\Omega}}{s^2} \right)_3. \]  

(15.50)

The \( \alpha \)-representations is the starting point for all the further derivations. Therefore, we present here several reference formulae concerning the \( \alpha \)-integrals. One has

\[ \langle \alpha_1^n \alpha_2^m \rangle_2 = \frac{n! m!}{(n + m + 1)!}, \]  

(15.51)

\[ \langle \alpha_1^n \alpha_2^m \ln(\alpha_1 \alpha_2) \rangle_2 = \frac{n! m!}{(n + m + 1)!} \times \left( \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{m} \frac{1}{k} - 2 \sum_{k=1}^{n+m+1} \frac{1}{k} \right), \]  

(15.52)

\[ \langle \alpha_1^n \alpha_2^m \alpha_3^k \rangle_3 = \frac{n! m! k!}{(n + m + k + 2)!}. \]  

(15.53)

These equations are, in particular, useful for obtaining the early-time asymptotic behaviours of the form factors. The late-time behaviour was studied in paper II. The relevant results are

\[ \langle P(\alpha_1, \alpha_2) \exp(s\alpha_1 \alpha_2 \Box) \rangle_2 = -\frac{1}{s} \frac{P(1, 0) + P(0, 1)}{\Box} + O\left( \frac{1}{s^2} \right), \]

\[ s \to \infty \]

(15.54)

\[ \langle P(\alpha_1, \alpha_2, \alpha_3) \exp(s\Omega) \rangle_3 = \frac{1}{s^2} \left[ \frac{P(1, 0, 0)}{\Box_2 \Box_3} + \frac{P(0, 1, 0)}{\Box_1 \Box_3} + \frac{P(0, 0, 1)}{\Box_1 \Box_2} \right] + O\left( \frac{1}{s^3} \right), \]

\[ s \to \infty \]

(15.55)

where \( P \)'s are polynomials in \( \alpha \). It follows, in particular, that the leading asymptotic terms are absent if a monomial in \( \alpha \) contains at least two unlike \( \alpha \)'s. This fact is useful when checking the infrared finiteness of the effective action in two dimensions.

16. Reduction of the form factors in \( \text{Tr}K(s) \) to the basic form factors

The problem with the \( \alpha \)-representation is that the \( s\Box \)-arguments of the form factors appear not only in the kernels (15.9) and (15.10). As seen from the expressions above, they enter also the coefficients of the polynomials in \( \alpha \) onto which the form factors are mapped. For this reason, the \( \alpha \)-representation is not unique in a sense that, even with the delta-function in (15.8) taken into account, the vanishing of an integral like

\[ \langle P(\alpha, \Box)e^{s\Omega} \rangle_3 \]  

(16.1)
does not imply the vanishing of the polynomial $P(\alpha, \Box)$. This nonuniqueness obscures the properties of the form factors and makes difficult various checks like the check of the trace anomaly. In particular, the fact that the contributions of the structures (15.2)–(15.5) vanish (see below) is not seen from (15.47)–(15.50). The origin of the $\Box$'s in the coefficients is the tree formulae which express the perturbations through the curvatures. The problem of nonuniqueness persists in all representations of the form factors in the heat kernel and effective action. Most of the further work with the form factors is aimed at removing this defect.

One way of obtaining a unique representation for the form factors in the heat kernel is elimination of all polynomials in $\alpha$. All form factors will then be explicitly expressed through the basic ones

$$f(\xi) = \langle e^{-\alpha_1 \alpha_2 \xi} \rangle_2, \quad \xi = -s\Box,$$
$$F(\xi_1, \xi_2, \xi_3) = \langle e^{s\Omega} \rangle_3, \quad \xi_m = -s\Box_m. \quad (16.2)$$

The technique of eliminating the polynomials in $\alpha$ is as follows.

After a use of the delta-function in (15.7) and (15.8), there remain to be considered the contributions of the monomials:

$$\langle \alpha_1^n e^{-\alpha_1 \alpha_2 \xi} \rangle_2 = \int_0^1 d\alpha \alpha^n \exp \left[ -\alpha (1 - \alpha)\xi \right], \quad (16.4)$$

$$\langle \alpha_1^n \alpha_2^m e^{s\Omega} \rangle_3 = \int_0^1 d\alpha_2 \int_0^{1-\alpha_2} d\alpha_1 \alpha_1^n \alpha_2^m \times \exp \left[ -\alpha_2 (1 - \alpha_1 - \alpha_2)\xi_1 - \alpha_1 (1 - \alpha_1 - \alpha_2)\xi_2 - \alpha_1 \alpha_2 \xi_3 \right]. \quad (16.5)$$

For the case (16.4), eq. (7.12) of paper II:

$$\int_0^1 d\alpha \frac{d}{d\alpha} \alpha^n \exp \left[ -\alpha (1 - \alpha)\xi \right] = \begin{cases} 0, & n = 0 \\ 1, & n > 0 \end{cases} \quad (16.6)$$

yields the following recurrence relations

$$\langle \alpha_1 e^{-\alpha_1 \alpha_2 \xi} \rangle_2 = \frac{1}{2} \langle e^{-\alpha_1 \alpha_2 \xi} \rangle_2, \quad (16.7)$$

$$\langle \alpha_1^n e^{-\alpha_1 \alpha_2 \xi} \rangle_2 = \frac{1}{2} \langle \alpha_1^{n-1} e^{-\alpha_1 \alpha_2 \xi} \rangle_2 - \frac{1}{2} (n - 1) \langle \alpha_1^{n-2} \left( \frac{e^{-\alpha_1 \alpha_2 \xi} - 1}{\xi} \right) \rangle_2, \quad n \geq 2 \quad (16.8)$$

which make it possible to express all integrals (16.4) through the basic form factor (16.2). Note that this procedure automatically leads to the appearance of the form factors with subtractions (14.13)–(14.14). The recurrence relations for them are similar:

$$\langle \alpha_1 \left( \frac{e^{-\alpha_1 \alpha_2 \xi} - 1}{\xi} \right) \rangle_2 = \frac{1}{2} \langle \left( \frac{e^{-\alpha_1 \alpha_2 \xi} - 1}{\xi} \right) \rangle_2, \quad (16.9)$$
\[
\alpha_1^n \left( \frac{e^{-\alpha_1 \alpha_2 \xi} - 1}{\xi} \right)_2 = \frac{1}{2} \alpha_1^{n-1} \left( \frac{e^{-\alpha_1 \alpha_2 \xi} - 1}{\xi} \right)_2 \\
- \frac{1}{2} (n-1) \left( \alpha_1^{n-2} \left( \frac{e^{-\alpha_1 \alpha_2 \xi} - 1 + \alpha_1 \alpha_2 \xi}{\xi^2} \right)_2 \right), \quad n \geq 2
\]

(16.10)

as one can check with the aid of eq. (15.51). The appearance of the subtractions is explained by analyticity of the integral (16.4) in \(\xi\) at \(\xi = 0\). Since the recurrence relations imply a division by \(\xi\), the appearing subtractions maintain the analyticity. For the form factors with subtractions one has

\[
\left\langle \frac{e^{-\alpha_1 \alpha_2 \xi} - 1}{\xi} \right\rangle_2 = \frac{f(\xi) - 1}{\xi},
\]

(16.11)

\[
\left\langle \frac{e^{-\alpha_1 \alpha_2 \xi} - 1 + \alpha_1 \alpha_2 \xi}{\xi^2} \right\rangle_2 = \frac{f(\xi) - 1 + \frac{1}{6} \xi}{\xi^2}
\]

(16.12)

in terms of (16.2).

Elimination of the polynomials in \(\alpha\) from the third-order form factors is based on integration by parts in (16.5):

\[
\int_0^1 d\alpha_2 \int_0^{1-\alpha_2} \left( \frac{\alpha_1}{\alpha_2} \right)^m \exp \left( s\Omega \right|_{\alpha_3 = 1 - \alpha_1 - \alpha_2} \right) \\
= \left\{ \begin{array}{ll}
\left\langle \alpha_2^m \left( e^{-\alpha_1 \alpha_2 \xi_3} - e^{-\alpha_1 \alpha_2 \xi_1} \right) \right\rangle_2, & n = 0 \\
\left\langle \alpha_1 \alpha_2^m e^{-\alpha_1 \alpha_2 \xi_3} \right\rangle_2, & n > 0
\end{array} \right.
\]

(16.13)

\[
\int_0^1 d\alpha_2 \int_0^{1-\alpha_2} \left( \frac{\alpha_1}{\alpha_2} \right)^m \exp \left( s\Omega \right|_{\alpha_3 = 1 - \alpha_1 - \alpha_2} \right) \\
= \left\{ \begin{array}{ll}
\left\langle \alpha_1^n \left( e^{-\alpha_1 \alpha_2 \xi_3} - e^{-\alpha_1 \alpha_2 \xi_2} \right) \right\rangle_2, & m = 0 \\
\left\langle \alpha_1^m \alpha_2^n e^{-\alpha_1 \alpha_2 \xi_3} \right\rangle_2, & m > 0
\end{array} \right.
\]

(16.14)

where the second-order form factors appearing on the right-hand sides are subject to the recurrence relations above. By performing the differentiations on the left-hand sides of (16.13),(16.14), one obtains two linear algebraic equations for the quantities

\[
\left\langle \alpha_1^{n+1} \alpha_2^m e^{s\Omega} \right\rangle_3, \quad \left\langle \alpha_1^n \alpha_2^{m+1} e^{s\Omega} \right\rangle_3
\]

containing the highest-order monomials. The discriminant of this linear system is

\[
\Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1 \xi_2 - 2\xi_1 \xi_3 - 2\xi_2 \xi_3,
\]

(16.15)

and the recurrence relations obtained in this way are of the form

\[
\left\langle \alpha_1^{n+1} \alpha_2^m e^{s\Omega} \right\rangle_3 = - \frac{\xi_1 (\xi_3 + \xi_2 - \xi_1)}{\Delta} \left\langle \alpha_1^n \alpha_2^m e^{s\Omega} \right\rangle_3 \\
+ 2n \frac{\xi_1}{\Delta} \left\langle \alpha_1^{n-1} \alpha_2^m e^{s\Omega} \right\rangle_3 \\
+ m \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left\langle \alpha_1^n \alpha_2^{m-1} e^{s\Omega} \right\rangle_3 \\
- \frac{(\xi_3 + \xi_1 - \xi_2)}{\Delta} \left\langle \alpha_1^n \alpha_2^m e^{-\alpha_1 \alpha_2 \xi_3} \right\rangle_2 + \beta(n, m),
\]

(16.16)
\[
\beta(n, m) = 0, \quad n > 0, \quad m > 0 \tag{16.17}
\]
\[
\beta(n, 0) = \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left\langle a_1^n e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2, \quad n > 0 \tag{16.18}
\]
\[
\beta(0, m) = 2 \frac{\xi_1}{\Delta} \left\langle a_1^m e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2, \quad m > 0 \tag{16.19}
\]
\[
\beta(0, 0) = 2 \frac{\xi_1}{\Delta} \left\langle e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2 + \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left\langle e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2, \tag{16.20}
\]
\[
\left\langle a_1^{n+1} a_2^m e^{s\Omega} \right\rangle_3 = \frac{-\xi_2 \xi_3 + \xi_1 - \xi_2}{\Delta} \left\langle a_1^n a_2^{m-1} e^{s\Omega} \right\rangle_3 + 2n \frac{\xi_2}{\Delta} \left\langle a_1^n a_2^{m-1} e^{s\Omega} \right\rangle_3 + m \left(\frac{\xi_3 - \xi_2 - \xi_1}{\Delta} \right) \left\langle a_1^{n-1} a_2^{m} e^{s\Omega} \right\rangle_3 - \frac{\xi_3 + \xi_2 - \xi_1}{\Delta} \left\langle a_1^n a_2^m e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2 + \delta(n, m), \tag{16.21}
\]
\[
\delta(n, m) = 0, \quad n > 0, \quad m > 0 \tag{16.22}
\]
\[
\delta(n, 0) = 2 \frac{\xi_2}{\Delta} \left\langle a_1^n e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2, \quad n > 0 \tag{16.23}
\]
\[
\delta(0, m) = \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left\langle a_1^m e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2, \quad m > 0 \tag{16.24}
\]
\[
\delta(0, 0) = 2 \frac{\xi_2}{\Delta} \left\langle e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2 + \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left\langle e^{-\alpha_1 \alpha_2 \xi_2} \right\rangle_2. \tag{16.25}
\]
Together with (16.7)–(16.8) these relations make it possible to express all integrals (16.5) through the basic form factors (16.2) and (16.3). Again one can show that the \( \alpha \)-polynomials do not destroy the combinations (15.11) and (15.12) in which the form factors appear. The recurrence relations with subtractions obtained with the aid of eq. (15.53) are of the form
\[
\left\langle a_1^{n+1} a_2^m (e^{s\Omega} - 1) \right\rangle_3 = \frac{-\xi_2 \xi_3 + \xi_1 - \xi_2}{\Delta} \left\langle a_1^n a_2^{m-1} (e^{s\Omega} - 1) \right\rangle_3 + 2n \frac{\xi_1}{\Delta} \left\langle a_1^n a_2^{m-1} (e^{s\Omega} - 1 - s\Omega) \right\rangle_3 + m \left(\frac{\xi_3 - \xi_2 - \xi_1}{\Delta} \right) \left\langle a_1^{n-1} a_2^{m-1} (e^{s\Omega} - 1 - s\Omega) \right\rangle_3 - \frac{(\xi_3 + \xi_1 - \xi_2)}{\Delta} \left\langle a_1^n a_2^m (e^{-\alpha_1 \alpha_2 \xi_2} - 1 + \alpha_1 \alpha_2 \xi_2) \right\rangle_2 + \gamma(n, m), \tag{16.26}
\]

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\[ \gamma(n, m) = 0, \quad n > 0, \quad m > 0 \]  \hspace{2cm} (16.27)

\[ \gamma(n, 0) = \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left( \alpha_1^n \left( e^{-\alpha_1 \alpha_2 \xi_2} - 1 + \alpha_1 \alpha_2 \xi_2 \right) \right)_2, \quad n > 0 \]  \hspace{2cm} (16.28)

\[ \gamma(0, m) = 2 \frac{\xi_1}{\Delta} \left( \alpha_1^m \left( e^{-\alpha_1 \alpha_2 \xi_1} - 1 + \alpha_1 \alpha_2 \xi_1 \right) \right)_2, \quad m > 0 \]  \hspace{2cm} (16.29)

\[ \gamma(0, 0) = 2 \frac{\xi_1}{\Delta} \left( (e^{-\alpha_1 \alpha_2 \xi_1} - 1 + \alpha_1 \alpha_2 \xi_1) \right)_2 \]  \hspace{2cm} (16.30)

\[ \left( \alpha_1^n \alpha_2^{m+1} (e^{s\Omega} - 1) \right)_3 = -\frac{\xi_2(\xi_3 + \xi_1 - \xi_2)}{\Delta} \left( \alpha_1^n \alpha_2^m (e^{s\Omega} - 1) \right)_3 \]  \hspace{2cm} (16.31)

\[ \sigma(n, m) = 0, \quad n > 0, \quad m > 0 \]  \hspace{2cm} (16.32)

\[ \sigma(n, 0) = 2 \frac{\xi_2}{\Delta} \left( \alpha_1^n \left( e^{-\alpha_1 \alpha_2 \xi_2} - 1 + \alpha_1 \alpha_2 \xi_2 \right) \right)_2, \quad n > 0 \]  \hspace{2cm} (16.33)

\[ \sigma(0, m) = \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta} \left( \alpha_1^n \left( e^{-\alpha_1 \alpha_2 \xi_1} - 1 + \alpha_1 \alpha_2 \xi_1 \right) \right)_2, \quad m > 0 \]  \hspace{2cm} (16.34)

\[ \sigma(0, 0) = 2 \frac{\xi_2}{\Delta} \left( (e^{-\alpha_1 \alpha_2 \xi_2} - 1 + \alpha_1 \alpha_2 \xi_2) \right)_2 \]  \hspace{2cm} (16.35)

and, for the combinations (15.12) themselves, one has

\[ \left( e^{s\Omega} - 1 \right)_3 = F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}, \]  \hspace{2cm} (16.36)

\[ \left( e^{s\Omega} - 1 - s\Omega \right)_3 = F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{1}{24} (\xi_1 + \xi_2 + \xi_3). \]  \hspace{2cm} (16.37)
in terms of \( (16.3) \). However, the analyticity in \( \xi_1, \xi_2, \xi_3 \) is now maintained by a more general mechanism. The analyticity holds only in the sum of the form factors on the right-hand side of \( (16.16) \) (and, similarly, \( (16.21), (16.26), (16.31) \)), and it is a nontrivial fact that, when these form factors are expanded in power series in \( \xi \), the denominator \( \Delta \) gets always cancelled. The mechanism of maintaining analyticity is based on the existence of linear differential equations which the functions \( (16.2) \) and \( (16.3) \) satisfy.

The differential equations for the basic form factors can be derived with the aid of the recurrence relations above. From \( (16.2) \), one has
\[
-\frac{d}{d\xi}f(\xi) = \langle \alpha_1 \alpha_2 e^{-\alpha_1 \alpha_2 \xi} \rangle_2 \tag{16.38}
\]
which, by means of \( (16.8) \), leads to the following equation for the function \( f(\xi) \):
\[
-\frac{d}{d\xi}f(\xi) = \frac{1}{4} f(\xi) + \frac{1}{2} \frac{f(\xi) - 1}{\xi}. \tag{16.39}
\]
The form factor \( (16.12) \) with two subtractions is expressed through the second derivative of \( f(\xi) \):
\[
\frac{d^2}{d\xi^2}f(\xi) = \frac{1}{16} f(\xi) + \frac{1}{4} \frac{f(\xi) - 1}{\xi} + \frac{3}{4} \frac{f(\xi) - 1 + \frac{1}{6} \xi}{\xi^2}. \tag{16.40}
\]
Similarly, one obtains the equation for the form factor \( (16.3) \):
\[
-\frac{\partial}{\partial \xi_1} F(\xi_1, \xi_2, \xi_3) = \frac{1}{\Delta^2} \left[ (\xi_1 - \xi_2 - \xi_3) \Delta + \xi_2 \xi_3 (2\xi_2 \xi_3 - \xi_2^2 - \xi_3^2 + \xi_1^2) \right] F(\xi_1, \xi_2, \xi_3) + \frac{1}{2} \frac{8 \xi_1 \xi_2 \xi_3 + (\xi_2 + \xi_3 - \xi_1) \Delta}{\Delta^2} f(\xi_1) + 2 \frac{\xi_2 \xi_3 (\xi_3 - \xi_2 - \xi_1)}{\Delta^2} f(\xi_2) + 2 \frac{\xi_2 \xi_3 (\xi_2 - \xi_3 - \xi_1)}{\Delta^2} f(\xi_3). \tag{16.41}
\]
The function \( F(\xi_1, \xi_2, \xi_3) \) is completely symmetric in \( \xi_1, \xi_2, \xi_3 \) and, therefore, satisfies two other equations, with \( \partial/\partial \xi_2 \) and \( \partial/\partial \xi_3 \), derivable from \( (16.41) \) by symmetry. Finally, as a consequence of these equations, one can derive an equation for the form factor \( (16.3) \) as a function of \( s \):
\[
-s \frac{\partial}{\partial s} F(-s \square_1, -s \square_2, -s \square_3) = \left( \frac{s \square_1 \square_2 \square_3}{D} + 1 \right) F(-s \square_1, -s \square_2, -s \square_3) + \frac{\square_1 (\square_3 + \square_2 - \square_1)}{2D} f(-s \square_1) + \frac{\square_2 (\square_3 + \square_1 - \square_2)}{2D} f(-s \square_2) + \frac{\square_3 (\square_1 + \square_2 - \square_3)}{2D} f(-s \square_3), \tag{16.42}
\]
\[ D = \Box_1^2 + \Box_2^2 + \Box_3^2 - 2\Box_1\Box_2 - 2\Box_1\Box_3 - 2\Box_2\Box_3. \] (16.43)

By applying the reduction technique above to expressions (15.13)–(15.50), the form factors \( f_i \) with \( i = 1 \) to 5 and \( F_i \) with \( i = 1 \) to 29 are brought to their final forms presented in the tables (2.10)–(2.14) and (2.77)–(2.105). For the form factors \( F_i \) with \( i = 30 \) to 33 the following results are obtained:

\[
\begin{align*}
F_{30} &= F(\xi_1, \xi_2, \xi_3) \left[ \frac{1}{3\xi_1^3} (-\xi_3 - \xi_2 + \xi_1)(\xi_1^6 - 5\xi_1^5\xi_2 + 8\xi_1^3\xi_2^3 \\
- 13\xi_1^2\xi_2^2 + 5\xi_2^3\xi_1 - 6\xi_3^5\xi_2 + 24\xi_3^4\xi_2^2 - 36\xi_3^3\xi_3^3 + 24\xi_2^4\xi_2^2 \\
+ 8\xi_1^3\xi_3^3 + 4\xi_1^4\xi_3^2 - 13\xi_1^2\xi_3^4 - 5\xi_1^5\xi_3 + 4\xi_1^4\xi_2^2 - 12\xi_1^3\xi_3^2\xi_2 \\
+ 32\xi_2^2\xi_3^3\xi_2 - 38\xi_1^2\xi_3^2\xi_2^2 - 6\xi_1^4\xi_3\xi_2 - 12\xi_1^3\xi_3\xi_3^2 + 32\xi_1^2\xi_3^2\xi_3 \\
- 3\xi_3^4\xi_1^2 - 2\xi_1^3\xi_3^2\xi_2^2 - 2\xi_1^2\xi_3^3\xi_2^3 - 3\xi_1^2\xi_3^3\xi_3^2 - 6\xi_5^5\xi_3 + 5\xi_1^2\xi_5^5) \\
- \frac{2}{3\xi_1^3\Delta^3} (84\xi_3^2\xi_1^2 + 17\xi_1^2\xi_3\xi_2^2 - 68\xi_2^3\xi_3\xi_1 - 63\xi_1^3\xi_3^2 + 7\xi_1^2\xi_3^3 \\
+ 26\xi_1\xi_2^4 - 68\xi_3^3\xi_1\xi_2 + 17\xi_3^2\xi_2^3\xi_2 + 26\xi_1^3\xi_3^2\xi_2 + 31\xi_1^4\xi_2 - 12\xi_3^3\xi_2^2 \\
- 12\xi_3^2\xi_3^2 + 18\xi_4^4 - 6\xi_5^5 + 18\xi_3^4\xi_2 + 7\xi_3^3\xi_1^2 - 63\xi_3^2\xi_3^3 + 31\xi_1^4\xi_3 \\
+ 5\xi_1^5 + 26\xi_3^4\xi_1 - 6\xi_5^5))^3 \\
+ \left( F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{4}{\xi_1\xi_2\xi_3\Delta} (\xi_1^2 + 2\xi_2\xi_3) \\
+ f(\xi_1) \frac{2}{3\Delta^4} (\xi_1^6 - 5\xi_1^5\xi_2 + 8\xi_1^3\xi_2^3 - 13\xi_1^2\xi_2^4 + 5\xi_2^5\xi_1 - 6\xi_3^5\xi_2 \\
+ 24\xi_3^4\xi_2^2 - 36\xi_3^2\xi_3^3 + 24\xi_4^4\xi_3^2 + 8\xi_1^3\xi_3^3 + 4\xi_1^4\xi_3^2 - 13\xi_1^2\xi_3^4 \\
- 5\xi_1^5\xi_3 + 4\xi_1^4\xi_2^2 - 12\xi_1^3\xi_3\xi_2 + 32\xi_1^2\xi_3^2\xi_2 - 38\xi_1^2\xi_3^2\xi_3 \\
- 6\xi_1^4\xi_3\xi_2 - 12\xi_1^3\xi_3\xi_3^2 + 32\xi_1^2\xi_3^3\xi_2 - 3\xi_3^4\xi_1\xi_2 - 2\xi_1^2\xi_3^3\xi_2 \\
- 2\xi_1^2\xi_3^3\xi_2 - 3\xi_1\xi_2\xi_3^2 - 6\xi_5^5\xi_3 + 5\xi_1^2\xi_5^5) \\
+ f(\xi_2) \frac{1}{12\xi_2^3\Delta^4} (20\xi_2^3\xi_3^3\xi_1 - 66\xi_2^3\xi_3^2\xi_2^2 + 121\xi_2^4\xi_3^2\xi_1 + 148\xi_2^3\xi_3^3\xi_3 \\
+ 81\xi_2^4\xi_3^2\xi_3 - 74\xi_2^5\xi_3\xi_1 - 151\xi_2^2\xi_3^4\xi_1 + 194\xi_2^2\xi_3^3\xi_1^2 + 34\xi_2^2\xi_3^2\xi_3^3 \\
- 151\xi_2^2\xi_3^3\xi_3 - 35\xi_2^4\xi_3^4 + 35\xi_2^4\xi_3^3 - 21\xi_2^5\xi_3^2 + 29\xi_3^5\xi_1^4 - 69\xi_2^4\xi_3^3 \\
+ 19\xi_2^5\xi_1^2 + 7\xi_2^6\xi_1 - 3\xi_2^2\xi_5^5 + 53\xi_2^2\xi_1^5 - 47\xi_1^6\xi_2 + 91\xi_1^5\xi_3^4 \\
- 189\xi_1^4\xi_3^3 + 165\xi_1^5\xi_3^2 - 9\xi_1\xi_3^6 - 65\xi_1^6\xi_3 - 225\xi_1^2\xi_2\xi_3^4 \\
+ 180\xi_1^2\xi_2\xi_3^3 - 41\xi_1^2\xi_2\xi_3^2 + 54\xi_1^5\xi_2\xi_3 + 86\xi_1\xi_2\xi_3^5 \\
- 3\xi_2^5\xi_1^2 + 7\xi_2^6\xi_2 + 9\xi_1^7 + 7\xi_2^6\xi_1 - 9\xi_7) \\
- f(\xi_3) \frac{1}{12\xi_3^2\Delta^4} (20\xi_2^3\xi_3^3\xi_1 + 194\xi_2^3\xi_3^3\xi_2^2 - 151\xi_2^4\xi_3^3\xi_1 + 180\xi_2^3\xi_3^3\xi_3 \\
- 225\xi_2^2\xi_3^3\xi_2 + 86\xi_2^3\xi_3\xi_3 + 121\xi_2^2\xi_3^4\xi_1 + 66\xi_2^2\xi_3^3\xi_2^2 + 34\xi_2^2\xi_3^2\xi_3^3 \\
- 4\xi_2^4\xi_1^3 + 35\xi_2^4\xi_3 - 35\xi_2^4\xi_3^3 - 21\xi_2^5\xi_3^2 - 189\xi_2^3\xi_1^4 \\
+ 91\xi_2^4\xi_3^3 - 3\xi_2^5\xi_1^2 - 7\xi_2^6\xi_2^3 - 21\xi_2^2\xi_3^5 + 165\xi_1^2\xi_5^5 + 65\xi_1^6\xi_2 \\
- 69\xi_2^5\xi_1^3 + 29\xi_1^4\xi_3^3 + 53\xi_1^5\xi_3^2 + 7\xi_1\xi_3^6 + 47\xi_1^6\xi_3 + 81\xi_2^2\xi_3^4 \\
+ 148\xi_2^3\xi_3^3 - 151\xi_4\xi_2\xi_3^2 + 54\xi_2^5\xi_1^3 - 74\xi_1^2\xi_3^5 \\
+ 19\xi_2^5\xi_3^5 + 7\xi_2^6\xi_2 - 9\xi_7 + 9\xi_2^6\xi_1 + 9\xi_7) \\
+ 159.
\end{align*}
\]
\[ + \left( f(\xi_1) - 1 \right) \frac{1}{\xi_1^2 \xi_2 \Delta^3} (\xi_1^5 \xi_2 + 6 \xi_1^3 \xi_2^3 - 4 \xi_1^2 \xi_2^4 + \xi_3^5 \xi_1 - 6 \xi_3^5 \xi_2 \\
+ 24 \xi_1^4 \xi_2^2 - 36 \xi_1^3 \xi_2^3 + 24 \xi_1^2 \xi_2^4 + 6 \xi_1^3 \xi_2^3 - 4 \xi_1^2 \xi_2^4 - 4 \xi_1 \xi_2^5 + \xi_1^5 \xi_3 \\
- 4 \xi_1^2 \xi_2^4 - 46 \xi_1^3 \xi_2^3 \xi_2 + 60 \xi_1^2 \xi_2^4 \xi_2 - 80 \xi_1^3 \xi_2^3 \xi_2 - 14 \xi_1^4 \xi_3 \xi_2 \\
- 46 \xi_1^3 \xi_2^3 \xi_2 + 60 \xi_1^2 \xi_2^4 \xi_3 + 5 \xi_3^4 \xi_1 \xi_2 - 6 \xi_3 \xi_1^3 \xi_2^2 - 6 \xi_1 \xi_2^3 \xi_2^2 \\
+ 5 \xi_1^2 \xi_3^4 \xi_3 - 6 \xi_2^5 \xi_3 + (\xi_4 \xi_3^5) \\
+ \left( f(\xi_2) - 1 \right) \frac{1}{\xi_2^2 \xi_1 \Delta^3} (-3 \xi_3^6 + 2 \xi_1^5 \xi_2 + 12 \xi_1^3 \xi_2^3 - 8 \xi_1^2 \xi_2^4 + 3 \xi_3^5 \xi_1 \\
- 7 \xi_3^5 \xi_2 + 6 \xi_1^4 \xi_2^2 + 34 \xi_1^3 \xi_2^3 - 53 \xi_1^2 \xi_2^4 - 2 \xi_1^3 \xi_2^3 + 3 \xi_1^4 \xi_2^4 - 2 \xi_1^2 \xi_2^4 \\
- \xi_1^3 \xi_3 + 8 \xi_1^4 \xi_2^3 \xi_2 - 106 \xi_1^2 \xi_2^4 \xi_2 + 66 \xi_1^2 \xi_2^4 \xi_2^2 + 9 \xi_1^4 \xi_3 \xi_2 \\
+ 70 \xi_1^3 \xi_2^2 \xi_3^2 + 14 \xi_1^2 \xi_2^3 \xi_3^3 + 70 \xi_3^4 \xi_1 \xi_2 - 102 \xi_1 \xi_2^3 \xi_2^2 + 112 \xi_1 \xi_2^3 \xi_3^2 \\
- 85 \xi_1 \xi_2^4 \xi_3 + 21 \xi_2^5 \xi_3 + 2 \xi_1 \xi_2^5) \\
- \left( \frac{f(\xi_3) - 1 + \frac{1}{3} \xi_1}{\xi_3^2 \Delta^3} \xi_1^5 \xi_2 + 2 \xi_1^3 \xi_2^3 + 2 \xi_1^2 \xi_2^4 - 2 \xi_3^5 \xi_1 \\
- 21 \xi_3^5 \xi_2 + 53 \xi_1^4 \xi_2^2 - 34 \xi_1^3 \xi_2^3 - 62 \xi_1^2 \xi_2^4 - 12 \xi_1 \xi_2^3 \xi_3 + 8 \xi_1^4 \xi_3^2 \\
+ 8 \xi_1^3 \xi_2^4 - 2 \xi_1^5 \xi_2^3 - 34 \xi_1^4 \xi_2^4 - 70 \xi_1^3 \xi_2^3 \xi_2 + 14 \xi_1^2 \xi_2^3 \xi_2 - 6 \xi_1 \xi_2^4 \xi_2^2 \\
- 9 \xi_1 \xi_2^5 \xi_3 - 32 \xi_3^4 \xi_1 \xi_2^2 + 106 \xi_1^2 \xi_2^4 \xi_3 + 85 \xi_1^3 \xi_2^3 \xi_1 - 11 \xi_1 \xi_2^3 \xi_3^2 \\
+ 102 \xi_1 \xi_2^3 \xi_3^2 - 70 \xi_1 \xi_2 \xi_3^3 + 7 \xi_2^5 \xi_3 - 3 \xi_1 \xi_2^5 + \xi_2^6) \\
+ \left( \frac{f(\xi_1)}{\xi_1^2 \Delta^2} + \frac{1}{3} \xi_1 \right) 4 \xi_1^2 \xi_2 \xi_3 \xi_2^4 (-3 \xi_3^2 + 2 \xi_2 \xi_3 - 2 \xi_1 \xi_2 - \xi_2^2 - 2 \xi_1 \xi_2 + 3 \xi_1^2) \\
- \left( \frac{f(\xi_2)}{\xi_2^2 \Delta^2} + \frac{1}{3} \xi_2 \right) 4 \xi_2 \xi_1 \xi_2 \xi_3 (2 - 2 \xi_1 \xi_3 - 3 \xi_2^2 + \xi_1^2 + 2 \xi_1 \xi_2 + 2 \xi_2 \xi_3) \\
- \left( \frac{f(\xi_3)}{\xi_3^2 \Delta^2} + \frac{1}{3} \xi_3 \right) 4 \xi_3 \xi_2 \xi_3 \xi_3 (-2 \xi_1 \xi_2 + 2 \xi_1 \xi_3 + 2 \xi_2 \xi_3 + \xi_1^2 + \xi_2^2 - 3 \xi_3^2),
\]

(16.44)

\[ F_{31}(\xi_1, \xi_2, \xi_3) = 0, \]  

(16.45)

\[ F_{32}(\xi_1, \xi_2, \xi_3) = 0, \]  

(16.46)

\[ F_{33}(\xi_1, \xi_2, \xi_3) = 0. \]  

(16.47)

The nonvanishing form factor \( F_{30} \) is, however, symmetric under a permutation of the labels 2 and 3:

\[ F_{30}(\xi_1, \xi_2, \xi_3) = F_{30}(\xi_1, \xi_3, \xi_2) \]  

(16.48)

as one can check by a direct inspection of expression (16.44). On the other hand, the structure 30 in eq. (15.2) is antisymmetric under this permutation:

\[ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(30) = [\nabla_\beta \mathcal{R}_1^\beta \mathcal{R}_2 \nabla_\alpha R_2 R_3 \\
\mathcal{R}_1 \mathcal{R}_2 R_3(30) = -[\nabla_\beta \mathcal{R}_1^\beta \mathcal{R}_2 \nabla_\alpha R_3 R_2 + \mathcal{R}^4 + \text{a total derivative}], \]  

(16.49)
and, therefore, the contribution of this structure vanishes

$$\int dx \, \sqrt{g} \, \text{tr} F_{30}(-s \Box_1, -s \Box_2, -s \Box_3) R_1 R_2 R_3(30) = O[\mathcal{R}^4]$$

(16.50)

by the same mechanism as the contribution of the structure (14.31). The difference is only
that neither of the properties (16.45)–(16.48) is seen before the form factors are brought
to a unique representation by eliminating the $\alpha$-polynomials. Thus the contributions of four
extra structures: $R_1 R_2 R_3(i)$ with $i = 30$ to 33 in eq. (15.1) vanish, and there remain only the
contributions of the twenty nine cubic structures presented in the final table (2.15)–(2.43).

Note that all the structures (14.31) and (15.2)–(15.5) whose contributions vanish are
linear in $\hat{\mathcal{R}}_{\mu\nu}$. In the final result for the trace of the heat kernel, there remains only one
cubic structure linear in $\hat{\mathcal{R}}_{\mu\nu}$:

$$R_1 R_2 R_3(13) = \hat{\mathcal{R}}_{\mu\nu} \nabla_\mu \hat{P}_2 \nabla_\nu \hat{P}_3$$

(16.51)

(eq. (2.27) of the final table). Its form factor is symmetric under a permutation of the labels
2 and 3:

$$F_{13}(\xi_1, \xi_2, \xi_3) = F_{13}(\xi_1, \xi_3, \xi_2)$$

(16.52)

(eq. (2.89)) but, because all the three curvatures in (16.51) are matrices, the structure
(16.51) possesses no antisymmetry under this permutation. Its contribution can be written
down as

$$\int dx \, \sqrt{g} \, \text{tr} F_{13}(-s \Box_1, -s \Box_2, -s \Box_3) R_1 R_2 R_3(13)$$

$$= \frac{1}{2} \int dx \, \sqrt{g} \, \text{tr} \hat{\mathcal{R}}_{\mu\nu} \nabla_\mu \hat{P}_2 \nabla_\nu \hat{P}_3,$$

(16.53)

and it does not vanish in the general case, as one can convince oneself by considering simple
examples.

17. Third-order form factors in the effective action ($\omega = 2$). Finiteness

By integrating the form factors

$$F_i(-s \Box_1, -s \Box_2, -s \Box_3)$$

over $s$ with the appropriate weights (following from eqs. (1.9) and (15.1)) we obtain the
third-order form factors in the effective action:

$$G_i(-\Box_1, -\Box_2, -\Box_3) = (4\pi)^2 \int_0^\infty \frac{ds}{s} \frac{s^{p_i}}{(4\pi s)^{2\omega}} F_i(-s \Box_1, -s \Box_2, -s \Box_3),$$

(17.1)

$$p_i = \begin{cases} 
3, & i = 1 \text{ to } 11, \\
4, & i = 12 \text{ to } 25, \\
5, & i = 26 \text{ to } 28, \\
6, & i = 29. 
\end{cases}$$
The normalization \((4\pi)^2\) is intended for the case \(\omega = 2\) which is the only case we shall be interested in.

Inspection of eqs. (15.18)–(15.46) for the form factors \(F_i\) in the \(\alpha\)-representation shows that all \(G_i\) in (17.1) have one and the same structure and can generically be presented in the form

\[
G(-\square_1, -\square_2, -\square_3) = (4\pi)^{2-\omega} \int_0^\infty ds \left\{ \left( a \frac{e^{s\Omega}}{s^{\omega-2}} + b \frac{e^{s\Omega}}{s^{\omega-1}} + c \right) + \frac{e^{s\Omega}}{s^\omega} \sum_{n=1}^3 \left( g_n \frac{e^{s\alpha_2}\square_n}{s^\omega} - h_n \frac{e^{s\alpha_2}\square_n - 1}{s^\omega} \right) + \frac{e^{s\alpha_2}\square_n - 1}{s^\omega} \right\}^3 + \frac{e^{s\alpha_2}\square_n - 1}{s^\omega} \sum_{1 \leq n < m \leq 3} \frac{1}{\square_n - \square_m} \left( \begin{array}{c} h_{mn} \frac{e^{s\alpha_2}\square_m - 1}{\square_n} - h_{mn} \frac{e^{s\alpha_2}\square_m - 1}{\square_m} \\ \end{array} \right) + \frac{e^{s\alpha_2}\square_m - 1}{\square_m} \right\}^2 \right\} \right\}
\]

(17.2)

where

\[
a, b, c, g_n, g_n, h_{nm}, h_{nm}, h_{nm}, \Phi_{nm}, \Phi_{nm}
\]

(17.3)

are functions only of \(\alpha\)'s and \(\square\)'s. If, instead of the \(\alpha\)-representation for the form factors \(F_i\), one starts with their explicit forms in eqs. (2.77)–(2.105), the result for \(G_i\) will anyway have the form (17.2) with the only difference that the coefficients (17.3) will be functions only of \(\square\)'s, and then the \(\alpha\)-averages in (17.2) will be identified directly with the basic form factors \(f(\xi)\) and \(F(\xi_1, \xi_2, \xi_3)\) (cf. eqs. (16.2), (16.3), (16.11), (16.12), (16.36), (16.37)). However, in this case, the functions \(f(\xi)\) and \(F(\xi_1, \xi_2, \xi_3)\) themselves should be put back in the \(\alpha\)-form because the next step in the calculation is commuting the \(\alpha\)- and \(s\)-integrations.

At \(\omega = 2\), the \(a\)-term and all \(h\)-terms in (17.2) are finite whereas all \(b\)-terms and all \(g\)-terms are ultraviolet divergent. On the other hand, it is well known that, in four dimensions, the ultraviolet divergences of the one-loop effective action are limited to terms of zeroth, first and second orders in the curvature [6]; terms of third order should be finite already. They are finite indeed. There are two mechanisms by which the ultraviolet divergences appearing in the third-order form factors (17.2) cancel. In the form factors \(G_i\) of all structures except purely gravitational the divergences appear only because of the division of the heat kernel into \(b\)-terms and \(g\)-terms (and subdivisions into \(b, \bar{b}, b, b, g, \bar{g}, g\) and \(g, \bar{g}, g\)). In the sum of these terms the divergences cancel, and the form factors \(G_i\) themselves are finite. The situation with the purely gravitational structures is different. Their form factors \(G_i\) are actually divergent.

*The reason why these divergences appear at all is the presence of the Riemann tensor in the DeWitt coefficient \(a_2(x, \bar{x})\) which governs the ultraviolet
but the divergences cancel in the sum
\[ \sum_i G_i R_1 R_2 R_3(i) \]
owing to a nonlocal constraint which holds between the purely gravitational structures in four dimensions (see Appendix). Below, these conclusions are confirmed by a direct calculation.

Although the final quantity of interest is finite, for dealing with intermediate divergent quantities, it is convenient to use the method of dimensional regularization. By applying the rules of dimensional regularization to the integrals in (17.2) we obtain (see e.g. [7])

\[
(4\pi)^{2-\omega} \int_0^\infty ds \frac{e^{sE}}{s^{\omega-2}} = \int_0^\infty ds e^{sE} + O(2 - \omega),
\]

(17.4)

\[
(4\pi)^{2-\omega} \int_0^\infty ds \frac{e^{sE}}{s^{\omega-1}} = \left(\frac{1}{2 - \omega} + \ln 4\pi\right) - E \int_0^\infty ds \ln s e^{sE} + O(2 - \omega),
\]

(17.5)

\[
(4\pi)^{2-\omega} \int_0^\infty ds \frac{e^{sE} - 1}{s^\omega} = \left(\frac{1}{2 - \omega} + \ln 4\pi\right)E + E
- E^2 \int_0^\infty ds \ln s e^{sE} + O(2 - \omega),
\]

(17.6)

\[
(4\pi)^{2-\omega} \int_0^\infty ds \frac{e^{sE} - 1 - sE}{s^{\omega+1}} = \left(\frac{1}{2 - \omega} + \ln 4\pi\right)\frac{1}{2} E^2 + \frac{3}{4} E^2
- \frac{1}{2} E^3 \int_0^\infty ds \ln s e^{sE} + O(2 - \omega),
\]

(17.7)

and

\[
\int_0^\infty ds e^{sE} = -\frac{1}{E},
\]

(17.8)

\[
\int_0^\infty ds \ln s e^{sE} = \frac{1}{E} \left( C + \ln(-E) \right)
\]

(17.9)

divergences in four dimensions (see sect. 4). Since covariant perturbation theory expands the Riemann tensor in an infinite series in powers of the Ricci tensor (see paper II and Appendix below), the divergent term with the Riemann tensor brings divergent contributions to the third and all higher orders in the curvature. The problem vanishes, however, if one takes into account the Gauss-Bonnet identity which, in four dimensions, eliminates the Riemann tensor from the (integrated) \( a_2(x,x) \). Automatically eliminated then are also all divergent contributions of higher orders in the curvature. At each order, there exists a nonlocal constraint which ensures this elimination. The hierarchy of these constraints is generated by the expansion of the Gauss-Bonnet invariant (see eq. (A.38) of Appendix).
where
\[ E = \begin{cases} \Omega & , \quad E < 0, \\ \alpha_1 \alpha_2 \square & , \end{cases} \]  
(17.10)

and C is the Euler constant. As a result, the form factor (17.2) takes the form
\[
G(-\square_1, -\square_2, -\square_3) = \left( \frac{1}{2 - \omega} + \ln 4\pi - C \right)
\times \left[ \left\langle b + \overline{b} \Omega + \frac{1}{2} \overline{b} \Omega^2 \right\rangle_3 + \sum_{n=1}^{3} \left\langle g_n + \alpha_1 \alpha_2 \overline{g}_n + \frac{1}{2} (\alpha_1 \alpha_2)^2 \overline{g}_n \right\rangle_2 \right]
\]
\[
- \left[ \left\langle \ln(-\Omega) \left( b + \overline{b} \Omega + \frac{1}{2} \overline{b} \Omega^2 \right) \right\rangle_3 \right.
\]
\[
+ \sum_{n=1}^{3} \ln(-\square_n) \left\langle g_n + \alpha_1 \alpha_2 \overline{g}_n + \frac{1}{2} (\alpha_1 \alpha_2)^2 \overline{g}_n \right\rangle_2 \right]
\]
\[
+ \left\langle a \frac{1}{2 - \Omega} \right\rangle_3 - \sum_{1 \leq n < m \leq 3} \ln(\square_n/\square_m) \left\langle h_{nm} + \alpha_1 \alpha_2 \overline{h}_{nm} + \frac{1}{2} (\alpha_1 \alpha_2)^2 \overline{h}_{nm} \right\rangle_2 \right.
\]
\[
+ r(\square_1, \square_2, \square_3) \right) \]  
(17.11)

where
\[
r(\square_1, \square_2, \square_3) = \left\langle \overline{b} \Omega + \frac{3}{4} \overline{b} \Omega^2 \right\rangle_3 + \sum_{n=1}^{3} \left\langle \overline{g}_n \alpha_1 \alpha_2 + \frac{3}{4} \overline{g}_n (\alpha_1 \alpha_2)^2 \right\rangle_2 \right.
\]
\[
- \sum_{n=1}^{3} \left\langle \ln(\alpha_1 \alpha_2) \left( g_n + \alpha_1 \alpha_2 \overline{g}_n + \frac{1}{2} (\alpha_1 \alpha_2)^2 \overline{g}_n \right) \right\rangle_2 \right.
\]  
(17.12)

is a rational (tree) function of \( \square \)'s which can be calculated explicitly with the aid of (15.51)–(15.53).

The first group of terms in (17.11), with the pole in \( \omega \), represents the logarithmic divergences. The second group contains accompanying them by dimension \( \ln \square \) terms. Since the divergences must cancel, the log's with an arbitrary scaling must cancel as well; only log's of ratios, like \( \ln(\square_n/\square_m) \), may and do survive. To single out the log's with an arbitrary scaling, we denote for short
\[
\overline{b} + \overline{b} \Omega + \frac{1}{2} \overline{b} \Omega^2 = \overline{b}, \]  
(17.13)

\[
g_n + \alpha_1 \alpha_2 \overline{g}_n + \frac{1}{2} (\alpha_1 \alpha_2)^2 \overline{g}_n = \overline{g}_n, \]  
(17.14)

\[
h_{nm} + \alpha_1 \alpha_2 \overline{h}_{nm} + \frac{1}{2} (\alpha_1 \alpha_2)^2 \overline{h}_{nm} = \overline{h}_{nm}, \]  
(17.15)

and write
\[
\left\langle \ln(-\Omega) \overline{b} \right\rangle_3 = \frac{1}{3} \left( \sum_{n=1}^{3} \ln(-\square_n) \right) \left\langle \overline{b} \right\rangle_3 + \left\langle \ln \left( \frac{-\Omega}{(-\square_1 \square_2 \square_3)^{1/3}} \right) \overline{b} \right\rangle_3, \]  
(17.16)
\[ \sum_{n=1}^{3} \ln(-\varpi_n) \langle g_n \rangle_2 = \frac{1}{3} \left( \sum_{n=1}^{3} \ln(-\varpi_n) \right) \sum_{m=1}^{3} \langle g_m \rangle_2 + \frac{1}{3} \sum_{1 \leq n < m \leq 3} \left( \ln(\varpi_n/\varpi_m) \langle g_n - g_m \rangle_2 \right). \quad (17.17) \]

Then we have
\[ G(-\varpi_1, -\varpi_2, -\varpi_3) = G^{\text{div}} + G^{\text{fin}}, \quad (17.18) \]
\[ G^{\text{div}} = \left( \frac{1}{2 - \omega} + \ln 4\pi - C - \frac{1}{3} \sum_{n=1}^{3} \ln(-\varpi_n) \right) \left[ \langle b \rangle_3 + \sum_{m=1}^{3} \langle g_m \rangle_2 \right], \quad (17.19) \]
\[ G^{\text{fin}} = \langle a \frac{1}{-\Omega} \rangle_3 - \langle b (\ln(-\Omega) - \frac{1}{3} \sum_{n=1}^{3} \ln(-\varpi_n)) \rangle_3 - \frac{1}{3} \sum_{1 \leq n < m \leq 3} \left( \langle g_n - g_m \rangle_2 \ln(\varpi_n/\varpi_m) - \sum_{1 \leq n < m \leq 3} \langle h_{nm} \rangle_2 \ln(\varpi_n/\varpi_m) \right) \]
\[ + r(\varpi_1, \varpi_2, \varpi_3). \quad (17.20) \]

The averages in (17.19) are \(\alpha\)-integrals of pure polynomials and are easily calculated for each \(G_i\) with the aid of (15.51), (15.53). The result of this calculation can be presented in the form
\[
\int dx \ g^{1/2} \ tr \sum_{i=1}^{29} G^{\text{div}}_i R_1 R_2 R_3(i) \\
= \int dx \ g^{1/2} \ tr \left( \frac{1}{2 - \omega} + \ln 4\pi - C - \frac{1}{3} \sum_{n=1}^{3} \ln(-\varpi_n) \right) \times \left\{ \begin{array}{l}
\frac{1}{360} R_1 R_2 R_3(9) \\
+ \frac{1}{45} \left( \frac{1}{2} \varpi_1 \varpi_2 \varpi_3 - \frac{1}{\varpi_1} \right) R_1 R_2 R_3(10) \\
+ \frac{1}{90} \left( -\frac{1}{2} \varpi_1 \varpi_2 \varpi_3 + \frac{1}{\varpi_1} \right) R_1 R_2 R_3(11) \\
+ \frac{1}{30} \left( -\frac{1}{2} \varpi_1 \varpi_2 \varpi_3 - \frac{1}{\varpi_1} \right) R_1 R_2 R_3(22) \\
+ \frac{1}{45} \varpi_1 \varpi_2 \varpi_3 R_1 R_2 R_3(23) + \frac{1}{45} \varpi_1 \varpi_2 \varpi_3 R_1 R_2 R_3(24) \\
+ \frac{1}{45} \left( -\frac{1}{\varpi_1 \varpi_2 \varpi_3} + 2 \frac{1}{\varpi_1} \right) R_1 R_2 R_3(25) \\
+ \frac{1}{45} \left( -\frac{1}{\varpi_1 \varpi_2 \varpi_3} \right) R_1 R_2 R_3(27) \\
+ \frac{2}{45} \left( -\frac{1}{\varpi_1 \varpi_2 \varpi_3} \right) R_1 R_2 R_3(28) \end{array} \right\} \quad (17.21) \]
whence it is seen that $G^{\text{div}}$ (properly symmetrized) is nonvanishing only for purely gravitational structures, and, among the latter, the structure with the maximum number of derivatives: $\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(29)$ has $G^{\text{div}} = 0$. The fact that $G^{\text{div}}_{29} = 0$ is explainable from the viewpoint of the analysis carried out in Appendix. If in the identity (A.35) of Appendix one puts

$$F_{\text{sym}}(\square_1, \square_2, \square_3) = -\frac{2}{45} \frac{1}{\square_1 \square_2 \square_3} \left( \frac{1}{2 - \omega} \ln 4\pi - C - \frac{3}{2} \sum_{n=1}^{3} \ln(-\square_n) \right)$$

(17.22)

the result will be precisely the right-hand side of (17.21). Hence

$$\int dx \, g^{1/2} \, \text{tr} \sum_{i=1}^{29} G_{i}^{\text{div}} \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) = 0.$$ 

(17.23)

As discussed in Appendix, the identity (A.35) does in fact mean that, in four dimensions, the basis of nonlocal gravitational invariants can be reduced by one structure. It is in the overcomplete basis that the gravitational form factors contain divergences. Our final results in sects. 6–9 are given in the reduced basis obtained by eliminating the completely symmetric part of the structure 28. In the heat kernel, this reduction of the basis amounts to replacing the form factors $F_i$ by the following modified ones:

$$F_{i}^{\text{mod}} = F_i, \quad i \neq 9, 10, 11, 22, 23, 24, 25, 27, 28,$$

(17.24)

$$F_{9}^{\text{mod}} = F_9 - \frac{1}{3} \frac{1}{\square_1 \square_2 \square_3} \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s} \right)_3,$$

(17.25)

$$F_{10}^{\text{mod}} = F_{10} - \frac{4}{3} \frac{1}{\square_1 \square_2 \square_3} (\square_1^2 + \square_2^2 + \square_3^2)
- 2 \square_1 \square_2 - 2 \square_1 \square_3 - 2 \square_2 \square_3 \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s} \right)_3,$$

(17.26)

$$F_{11}^{\text{mod}} = F_{11} - \frac{2(\square_1 + \square_2 - \square_3)}{\square_1 \square_2} \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s} \right)_3,$$

(17.27)

$$F_{22}^{\text{mod}} = F_{22} + 2 \frac{3(\square_1 + \square_2 + \square_3)}{\square_1 \square_2 \square_3} \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^2} \right)_3,$$

(17.28)

$$F_{23}^{\text{mod}} = F_{23} - 8 \frac{1}{\square_1 \square_2} \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^2} \right)_3,$$

(17.29)

$$F_{24}^{\text{mod}} = F_{24} - 8 \frac{1}{\square_2 \square_3} \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^2} \right)_3,$$

(17.30)

$$F_{25}^{\text{mod}} = F_{25} - 8 \frac{1}{\square_1 \square_2 \square_3} \left( \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^2} \right)_3.$$ 

(17.31)
\[ F_{27}^{\text{mod}} = F_{27} + 8 \frac{1}{\Box_1 \Box_2 \Box_3} \left\langle \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^3} \right\rangle_3, \quad (17.32) \]

\[ F_{28}^{\text{mod}} = F_{28} + 16 \frac{1}{\Box_1 \Box_2 \Box_3} \left\langle \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^3} \right\rangle_3. \quad (17.33) \]

For the respectively modified form factors in the effective action we introduce the notation

\[ \Gamma_i(-\Box_1, -\Box_2, -\Box_3) = (4\pi)^2 \int_0^\infty ds \frac{s^{p_i}}{(4\pi s)^\omega} F_i^{\text{mod}}(-s\Box_1, -s\Box_2, -s\Box_3), \quad (17.34) \]

where the exponents \( p_i \) are the same as in (17.1).

It is easy to make sure that

\[ \int dx g^{1/2} \text{tr} \sum_{i=1}^{29} s^{p_i} F_i^{\text{mod}} \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (i) = \int dx g^{1/2} \text{tr} \sum_{i=1}^{29} s^{p_i} F_i \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (i), \quad (17.35) \]

and, respectively,

\[ \int dx g^{1/2} \text{tr} \sum_{i=1}^{29} \Gamma_i \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (i) = \int dx g^{1/2} \text{tr} \sum_{i=1}^{29} G_i \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (i) \quad (17.36) \]

since (17.35) is a special case of the identity (A.35) corresponding to

\[ \mathcal{F}^{\text{sym}}(\Box_1, \Box_2, \Box_3) = \frac{16}{\Box_1 \Box_2 \Box_3} \left\langle \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^3} \right\rangle_3. \quad (17.37) \]

Thus, the above modification of the form factors has no effect on the trace of the heat kernel and the effective action in four dimensions. On the other hand, with the expression (15.45) for \( F_{28} \), the completely symmetric part of \( F_{28}^{\text{mod}} \) in (17.33) is

\[ -\frac{16}{\Box_1 \Box_2 \Box_3} \left\langle \frac{1}{3} (\alpha_1 \alpha_2 \alpha_3^2 + \alpha_1 \alpha_3 \alpha_2^2 + \alpha_2 \alpha_3 \alpha_1^2) \frac{e^{s\Omega}}{s^3} \right\rangle_3 + \frac{16}{\Box_1 \Box_2 \Box_3} \left\langle \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \frac{e^{s\Omega}}{s^3} \right\rangle_3 = 0 \quad (17.38) \]

by virtue of the delta-function \( \delta(\sum \alpha - 1) \) contained in \( \left\langle \right. \rangle_3 \). Since removal of the symmetric part of \( G_{28} \) removes \( G_{28}^{\text{div}} \), it automatically, via the identity (A.35), removes all \( G_i^{\text{div}} \). As a result, in the reduced basis, the form factors \( \Gamma_i \) themselves are finite.

18. Reduction of the form factors in \( W (\omega = 2) \) to the basic form factors

In terms of the representation (17.2), the transition from \( G_i \) to \( \Gamma_i \) changes only the coefficient \( b \). Since this change turns into zero the coefficients of all \( G_i^{\text{div}} \) in (17.19), the form factors \( \Gamma_i \) have the same form as \( G^{\text{fin}} \) in (17.20) with the modified \( b \). Each

\[ \Gamma_i(-\Box_1, -\Box_2, -\Box_3), \quad i = 1 \text{ to } 29 \quad (18.1) \]
is, therefore, a sum of contributions of the following five types:

\[
\left\langle P(\alpha, \Box) \frac{1}{-\Omega} \right\rangle_3, \tag{18.2}
\]

\[
\left\langle P(\alpha, \Box) \left( \ln(-\Omega) - \frac{1}{3} \sum_{n=1}^{3} \ln(-\Box_n) \right) \right\rangle_3, \tag{18.3}
\]

\[
\sum_{1 \leq m < n \leq 3} \left\langle P_{nm}(\alpha, \Box) \right\rangle_2 \ln(\Box_n/\Box_m), \tag{18.4}
\]

\[
\sum_{1 \leq m < n \leq 3} \left\langle P_{nm}(\alpha, \Box) \right\rangle_2 \frac{\ln(\Box_n/\Box_m)}{\Box_n - \Box_m}, \tag{18.5}
\]

\[
\left\langle P(\alpha, \Box) \right\rangle_3, \quad \left\langle P(\alpha, \Box) \right\rangle_2 \tag{18.6}
\]

where \(P(\alpha, \Box)\) are polynomials in \(\alpha\)'s, boxes and inverse boxes. In (18.4)–(18.6), the \(\alpha\)-averages can be calculated explicitly. Below we summarize a technique by which i) the contributions (18.3), (18.4) and (18.6) can be put in the form (18.2), and ii) the contributions of the form (18.2) can be expressed in either an algebraic or a differential way through elementary functions and the basic third-order form factor

\[
\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty ds F(-s\Box_1, -s\Box_2, -s\Box_3) = \left\langle \frac{1}{-\Omega} \right\rangle_3. \tag{18.7}
\]

The contributions of the type (18.5) are of a special origin (see sect. 14) and remain unaffected by these transformations.

The reduction is mainly based on the formulae derived in paper III. Eqs. (2.17), (2.18), (2.19) and (4.11) of paper III, after a minor modification and adaptation to the present notation, read

\[
\left\langle \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \right\rangle_3 = \frac{1}{(n_1 + n_2 + n_3)!} \times (-\Box_1)^{n_1} \frac{\partial^{n_1}}{\partial \Box_1^{n_1}} \times (-\Box_2)^{n_2} \frac{\partial^{n_2}}{\partial \Box_2^{n_2}} \times (-\Box_3)^{n_3} \frac{\partial^{n_3}}{\partial \Box_3^{n_3}} \Gamma(-\Box_1, -\Box_2, -\Box_3), \tag{18.8}
\]

\[
\left\langle \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \left( \ln(-\Omega) - \frac{1}{3} \sum_{m=1}^{3} \ln(-\Box_m) \right) \right\rangle_3 = \frac{n_1! n_2! n_3!}{(n_1 + n_2 + n_3 + 2)!} \times \left( \frac{4}{3} \sum_{l=1}^{n_1} \frac{1}{l} + \frac{4}{3} \sum_{l=1}^{n_2} \frac{1}{l} + \frac{4}{3} \sum_{l=1}^{n_3} \frac{1}{l} - 2 \sum_{l=3}^{n_1+n_2+n_3+2} \frac{1}{l} \right)
\]

\[
- \frac{2}{(n_1 + n_2 + n_3 + 2)!} (-\Box_1)^{n_1+1} \frac{\partial}{\partial \Box_1^{n_1}} (-\Box_2)^{n_2+1} \frac{\partial}{\partial \Box_2^{n_2}} \times (-\Box_3)^{n_3+1} \frac{\partial}{\partial \Box_3^{n_3}} \left\langle \ln(-\Omega) - \frac{1}{3} \sum_{m=1}^{3} \ln(-\Box_m) \right\rangle_3, \tag{18.9}
\]
\[
\left< \ln(-\Omega) - \frac{1}{3} \sum_{m=1}^{3} \ln(-\Box_m) \right>_3 \\
= -\frac{3}{2} - \frac{1}{6} \sum_{m=1}^{3} \Box_m \left( 1 + \Box_m \frac{\partial}{\partial \Box_m} \right) \Gamma(-\Box_2, -\Box_3), \\
\text{(18.10)}
\]

\[
\ln(\Box_1/\Box_2) = \left[ \Box_1 \Box_3 \left( \frac{\partial}{\partial \Box_1} + \frac{\partial}{\partial \Box_3} \right) - \Box_2 \Box_3 \left( \frac{\partial}{\partial \Box_2} + \frac{\partial}{\partial \Box_3} \right) \right] \Gamma(-\Box_1, -\Box_2, -\Box_3), \\
\text{(18.11)}
\]

and the identity
\[
\frac{1}{2} = \left< \frac{-\Omega}{-\Omega} \right>_3 \\
\text{(18.12)}
\]

adds here one more relation:
\[
-1 = \Box_1 \Box_2 \Box_3 \left( \frac{\partial}{\partial \Box_1} \frac{\partial}{\partial \Box_2} + \frac{\partial}{\partial \Box_2} \frac{\partial}{\partial \Box_3} + \frac{\partial}{\partial \Box_3} \frac{\partial}{\partial \Box_1} \right) \Gamma(-\Box_2, -\Box_3). \\
\text{(18.13)}
\]

These relations make it possible to express all contributions (18.2), (18.3), (18.4), (18.6) through the derivatives of the basic form factor \( \Gamma \), and, on the other hand, relation (18.8) establishes a one-to-one correspondence between derivatives of \( \Gamma \) and averages of the form (18.2). Therefore, all form factors (18.1) can be put in either of the two equivalent forms
\[
\Gamma_i(-\Box_1, -\Box_2, -\Box_3) = \left< \frac{P_i(\alpha, \Box)}{-\Omega} \right>_3 + \text{terms (18.5),} \\
\text{(18.14)}
\]

\[
\Gamma_i(-\Box_1, -\Box_2, -\Box_3) \\
= P_i' \left( \Box, \frac{\partial}{\partial \Box} \right) \Gamma(-\Box_2, -\Box_3) + \text{terms (18.5).} \\
\text{(18.15)}
\]

In this way the results in sect. 7 are obtained. The final expressions for \( \Gamma_i \) in the \( \alpha \)-representation given in sect. 7 differ from (18.14) only in that some tree terms of the type (18.6), those responsible for the power growth of the form factors, are written down explicitly. Singling out of such terms from the general expression (18.14) is discussed in the next section in connection with the asymptotic behaviour of the form factors.

The representations (18.14) and (18.15) are not unique. Differentiation of eq. (18.11) and similar equations with permuted indices gives rise to a hierarchy of identities between the derivatives of \( \Gamma \), or the averages of the form (18.2). For example, by differentiating (18.11) once with respect to \( \Box_3 \), one obtains the identity
\[
\left[ \Box_1 \frac{\partial}{\partial \Box_1} - \Box_2 \frac{\partial}{\partial \Box_2} + (\Box_1 - \Box_2) \frac{\partial}{\partial \Box_3} - \Box_1 \Box_3 \left( \frac{\partial^2}{\partial \Box_1 \partial \Box_3} + \frac{\partial^2}{\partial \Box_2 \partial \Box_3} \right) \right] \Gamma(-\Box_2, -\Box_3) = 0, \\
\text{(18.16)}
\]

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\text{(18.16)}
\]

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\text{(18.16)}
\]
or
\[
\left\langle \frac{1}{\Omega} \left( (\Box_1 - \Box_2)(2\alpha_3^2 - \alpha_3) + \Box_3(\alpha_1 - \alpha_2)(2\alpha_3 - 1) \right) \right\rangle_3 = 0. \tag{18.17}
\]

This arbitrariness, and the constraint \(\sum \alpha = 1\) have been used to bring the form factors to their final forms (7.2)–(7.30) and, in particular, to secure the fulfilment of the "rule of the like \(\alpha\)" observed in sect. 7. The fact that expressions (18.14) can be transformed so that this rule hold is a nontrivial property of the form factors which has important implications.

The cause of nonuniqueness of the representations (18.14) and (18.15) is the existence of linear differential equations satisfied by the function (18.7). For the derivation of these equations one may use the equations for the basic form factor in the heat kernel

\[ F = F(-s\Box_1, -s\Box_2, -s\Box_3). \]

By combining eqs. (16.41) and (16.42), one has
\[
\frac{\partial}{\partial \Box_1} F = \frac{(\Box_2 - \Box_3)^2 - \Box_1^2}{D\Box_1} \frac{\partial}{\partial s}(sF) + \frac{\Box_2 + \Box_3 - \Box_1}{D} F \\
+ \frac{1}{2D} \left( \Box_1 f(-s\Box_1) - \Box_2 f(-s\Box_2) \right) \\
+ \frac{1}{2D} \left( \Box_1 f(-s\Box_1) - \Box_3 f(-s\Box_3) \right) \\
+ \frac{(\Box_2 - \Box_3)}{2D\Box_1} \left( \Box_3 f(-s\Box_3) - \Box_2 f(-s\Box_2) \right). \tag{18.18}
\]

When this equation is integrated over \(s\) from 0 to \(\infty\), the total derivative term with \(F\) vanishes by virtue of the asymptotic behaviours (3.2) and (4.2). Moreover, since the leading asymptotic behaviour (3.1) of the function \(f\) cancels in the appearing differences, the integrals of these differences converge and give
\[
\int_0^\infty ds \left( \Box_1 f(-s\Box_1) - \Box_2 f(-s\Box_2) \right) = -2 \ln(\Box_1/\Box_2). \tag{18.19}
\]

The result is the following equation for the function (18.7):
\[
\frac{\partial}{\partial \Box_1} \Gamma = \frac{\Box_2 + \Box_3 - \Box_1}{D} \Gamma \\
+ \frac{1}{D} \ln(\Box_2/\Box_1) + \frac{1}{D} \ln(\Box_3/\Box_1) + \frac{\Box_2 - \Box_3}{D\Box_1} \ln(\Box_2/\Box_3), \tag{18.20}
\]

and two other equations, with \(\partial/\partial \Box_2\) and \(\partial/\partial \Box_3\), obtained from (18.20) by symmetry.

After the use of these equations in (18.15), all third-order form factors become expressed in an algebraic way through elementary functions and the function \(\Gamma\), and this representation is unique already. In this way the explicit expressions (6.13)–(6.41) are obtained.
19. Derivation of the large-$\Box$ asymptotic behaviours and Laplace originals of the form factors in $W$ ($\omega = 2$)

The Laplace representation for the basic form factor (18.7) is obtained by writing

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty ds \int_{\alpha_i \geq 0} d^3\alpha \, \delta(1 - \sum_{1}^{3} \alpha_i) e^{s\Omega}$$

(19.1)

and making the replacement of variables

$$s, \alpha_1, \alpha_2, \alpha_3 \to u_1, u_2, u_3 :$$

(19.2)

$$\alpha_1 = \frac{u_2u_3}{u_1u_2 + u_2u_3 + u_1u_3},$$

$$\alpha_2 = \frac{u_1u_3}{u_1u_2 + u_2u_3 + u_1u_3},$$

$$\alpha_3 = \frac{u_1u_2}{u_1u_2 + u_2u_3 + u_1u_3},$$

$$s = \frac{(u_1u_2 + u_2u_3 + u_1u_3)^2}{u_1u_2u_3},$$

$$\left(0 \leq s < \infty, \alpha_i \geq 0, \sum_1^{3} \alpha_i = 1 \right) \to (0 \leq u_i < \infty),$$

$$ds d^3\alpha \, \delta(1 - \sum_{1}^{3} \alpha_i) = \frac{d^3u}{u_1u_2 + u_2u_3 + u_1u_3},$$

$$s\Omega = u_1\Box_1 + u_2\Box_2 + u_3\Box_3 \equiv \sum u\Box.$$

The result is

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty d^3u \frac{1}{u_1u_2 + u_2u_3 + u_1u_3} e^{\sum u\Box}. \quad (19.3)$$

The Laplace originals of all form factors (18.14) can then be obtained by introducing the generating function

$$Z(j_1, j_2, j_3) = \Gamma(-j_1\Box_1, -j_2\Box_2, -j_3\Box_3)$$

$$= \int_0^\infty d^3u \frac{e^{\sum u\Box}}{u_1u_2j_3 + u_2u_3j_1 + u_1u_3j_2}$$

(19.4)

and noting that

$$\left\langle \frac{\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}}{-\Omega} \right\rangle_3 = (-1)^{n_1+n_2+n_3} (n_1+n_2+n_3)! \left( \frac{\partial}{\partial j_1} \right)^{n_1} \left( \frac{\partial}{\partial j_2} \right)^{n_2} \left( \frac{\partial}{\partial j_3} \right)^{n_3} Z \bigg|_{j=1}$$

(19.5)
as is obvious from the replacement (19.2) or eq. (18.8). To complete representing the functions (18.14) in the form of Laplace integrals, eq. (19.5) should be supplemented with the inverse Laplace transformation for the terms of the type (18.5). For the further use we present this transformation in the form of the generating equation

\[
\frac{1}{j_1 \Box_1} \ln \left( \frac{j_2 \Box_2}{j_3 \Box_3} \right) = \int_0^\infty d^3u \frac{e^{\sum u \Box}}{j_2 u_3 + j_3 u_2}
\]  

which can next be differentiated with respect to \( j \).

The problem is, however, that again the \( \Box \) arguments of the form factors enter not only the kernel \( e^{\sum u \Box} \) but also the coefficients of the \( \alpha \)-polynomials in (18.14). Originally *, in the form factors of the structures without derivatives, these coefficients are of the form

\[
a) \frac{\Box_k}{\Box_n}, \quad b) \frac{\Box_k^2}{\Box_n \Box_m}; \quad m, n, k = 1, 2, 3
\]

for the structures with two derivatives they are of the form

\[
a) \frac{1}{\Box_n}, \quad b) \frac{\Box_k}{\Box_n \Box_m}, \quad m, n, k = 1, 2, 3,
\]

for the structures with four derivatives of the form

\[
\frac{\Box_m}{\Box_n}; \quad m, n = 1, 2, 3,
\]

and, for the structure 29 with six derivatives, the \( \Box \) coefficient in the form factor is

\[
\frac{1}{\Box_1 \Box_2 \Box_3}.
\]

The task is now to try to absorb these \( \Box \) and \( 1/\Box \) multipliers in the Laplace originals; otherwise the representation will not be unique and all advantages of dealing with the integral originals will be lost. Indeed, with the aid of (19.5), the identities like (18.17) can immediately be translated in the language of Laplace integrals to give relations of the form

\[
\int_0^\infty d^3u \rho(u, \Box) e^{\sum u \Box} = 0
\]

with nonvanishing \( \rho(u, \Box) \).

As will be seen below, for the Laplace representation, the \( 1/\Box \) multipliers in (19.7)–(19.10) present no problem; the problem arises only with the positive powers of \( \Box_k \) in (19.7) and (19.8) because they enhance the behaviour of the form factors at large \( \Box_k \). The point is that the Laplace representation exist only for functions decreasing at large values of their arguments whereas, at small values, any power growth is admissible.

*In the final expressions (7.2)–(7.30), the \( \alpha \)-integrals with the coefficients (19.7b) and (19.8b) are transformed into tree terms by using eqs. (19.21), (19.22) below.
To study the behaviour of the form factors at large negative $\Box_3$, it suffices to consider the generating function (19.4). In the form

$$Z = \int_0^\infty du_3 \, Y(u_3) e^{u_3\Box_3},$$

$$Y(u_3) = \int_0^\infty du_1 \, du_2 \frac{e^{u_1\Box_1 + u_2\Box_2}}{u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2},$$

the behaviour of $Z$ at large $(-\Box_3)$ is determined by the behaviour of $Y(u_3)$ at small $u_3$. For the asymptotic behaviour of $Y(x)$ at small $x$ one obtains

$$Y(x) = \frac{1}{j_3} \ln (-x^{\frac{j_1}{j_3}}) \ln \left(-x^{\frac{j_2}{j_3}}\right)$$
$$- \frac{\Gamma'(1)}{j_3} \ln \left(x^2 \frac{j_1 j_2 \Box_1 \Box_2}{j_3^2}\right) + \frac{2\Gamma''(1)}{j_3} + O(x), \quad x \to 0$$

(19.13)

where $\Gamma'(1), \Gamma''(1)$ are derivatives of the Euler $\Gamma$-function at the point 1. Hence the asymptotic behaviour of $Z$ at large negative $\Box_3$ is

$$Z = -\frac{1}{j_3 \Box_3} \ln^2(-\Box_3) + \frac{1}{j_3 \Box_3} \ln(-\Box_3) \ln \left(\frac{j_1 j_2 \Box_1 \Box_2}{j_3^2}\right)$$
$$- \frac{1}{j_3 \Box_3} \left[ \ln \left(-\frac{j_1}{j_3}\right) \ln \left(-\frac{j_2}{j_3}\right) + 2\zeta(2) \right] + O \left(\frac{1}{\Box_3^2}\right), \quad -\Box_3 \to \infty$$

(19.14)

where $\zeta(2)$ is the Riemann $\zeta$-function at the point 2. By using (19.5), (19.14) and the expressions (7.2)–(7.30) for $\Gamma_i$ in the $\alpha$-representation, one can obtain the asymptotic behaviours of all form factors at large values of each of the three arguments. The final table of asymptotic behaviours is given in sect. 10.

Thus, apart from the multipliers (19.7)–(19.10), the behaviour of the form factors is generally $\ln^2(-\Box)/\Box, \Box \to -\infty$ in each of the arguments. The presence of the multipliers (19.7)–(19.10) changes in this behaviour not only the power of $\Box$ but also the power of $\ln(-\Box)$. The cause is the “rule of the like $\alpha$” mentioned in sect 7. By this rule, each $1/\Box$ multiplier appears only in a product with the like $\alpha$, e.g. $\alpha_1/\Box_1, \alpha_1 \alpha_2/\Box_1 \Box_2$, etc. Since each $\alpha$ is equivalent to a derivative with respect to $j$, the negative powers of $\Box$ in (19.7)–(19.10) appear only in the combinations

$$\frac{1}{\Box_1} \frac{\partial}{\partial j_1} Z = \frac{1}{j_1 j_3 \Box_1 \Box_3} \ln \left(\frac{j_3 \Box_3}{j_2 \Box_2}\right) + O \left(\frac{1}{\Box_3^2}\right), \quad -\Box_3 \to \infty$$

(19.15)

$$\frac{1}{\Box_1 \Box_2} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} Z = -\frac{1}{\Box_1 \Box_2 \Box_3 j_1 j_2 j_3} + O \left(\frac{1}{\Box_3^2}\right), \quad -\Box_3 \to \infty$$

(19.16)

and similar combinations with permuted indices. As seen from (19.7)–(19.10), the terms leading at large $(-\Box_3)$ are always proportional to $1/\Box_1 \Box_2$ and, therefore, by (19.16), have a purely power asymptotic behaviour. We conclude that the large-$\Box$ behaviour of the form factors in individual $\Box$ arguments is generally

$$\Gamma_{1-11} \propto \Box_k, \quad -\Box_k \to \infty$$

(19.17)
for the structures without derivatives,
\[ \Gamma_{12-25} \propto \text{const}, \quad -\Box_k \to \infty \] (19.18)
for the structures with two derivatives,
\[ \Gamma_{26-28} \propto \frac{1}{\Box_k^2}, \quad -\Box_k \to \infty \] (19.19)
for the structures with four derivatives, and
\[ \Gamma_9 \propto \frac{1}{\Box_k^2}, \quad -\Box_k \to \infty \] (19.20)
for the structure with six derivatives.

Since the form factors (19.17) and (19.18) do not decrease at large values of their arguments, they do not admit a Laplace representation. The best one can do is to single out the nondecreasing terms and treat them separately. The first step is to get rid of the coefficients (19.7b) and (19.8b) which cause the strongest growth. This can be done as follows:

\[ \Gamma_{1-11} + O \left( \ln(-\Box_3) \right) = \left\langle \frac{\Box_3^2 \alpha_1 \alpha_2 P(\alpha)}{\Box_1 \Box_2 - \Omega} \right\rangle_3 \]
\[ = \left\langle \frac{\Box_3}{\Box_1 \Box_2} \left( \Omega - \Box_1 \alpha_2 \alpha_3 - \Box_2 \alpha_1 \alpha_3 \right) \frac{P(\alpha)}{-\Omega} \right\rangle_3 \]
\[ = -\frac{\Box_3}{\Box_1 \Box_2} \left\langle P(\alpha) \right\rangle_3 - \left\langle \left( \frac{\Box_3 \alpha_2 \alpha_3}{\Box_2^2} + \frac{\Box_3 \alpha_1 \alpha_3}{\Box_1^2} \right) \frac{P(\alpha)}{-\Omega} \right\rangle_3, \] (19.21)
and, similarly,

\[ \Gamma_{12-25} + O \left( \ln(-\Box_3) \right) = \left\langle \frac{\Box_3 \alpha_1 \alpha_2 P(\alpha)}{\Box_1 \Box_2 - \Omega} \right\rangle_3 \]
\[ = -\frac{1}{\Box_1 \Box_2} \left\langle P(\alpha) \right\rangle_3 - \left\langle \left( \frac{\alpha_2}{\Box_2} + \frac{\alpha_1}{\Box_1} \right) \frac{\alpha_3 P(\alpha)}{-\Omega} \right\rangle_3 \] (19.22)
where the "rule of the like \( \alpha \)" works again. Here the purely tree terms on the right-hand sides are the leading asymptotic terms (19.17) and (19.18) respectively. Their contributions in the Laplace representation can, at best, be written down as

\[ \Gamma_{1-11} + O \left( \ln(-\Box_3) \right) \propto \Box_3^2 \int_0^\infty d^3u e^{\Sigma u}, \] (19.23)
\[ \Gamma_{12-25} + O \left( \ln(-\Box_3) \right) \propto \Box_3 \int_0^\infty d^3u e^{\Sigma u}. \] (19.24)

*Notwithstanding that the total dimension of the form factors is \( \Box^{-1} \) for \( \Gamma_{1-11}, \Box^{-2} \) for \( \Gamma_{12-25}, \Box^{-3} \) for \( \Gamma_{26-28} \), and \( \Box^{-4} \) for \( \Gamma_9 \).
In the remaining terms, the $1/\Box$ multipliers in (19.7)–(19.10) can easily be absorbed in the Laplace originals by integration by parts. Generally, this leads to the appearance of logarithmic originals but, owing to the "rule of the like $\alpha$", we need only

$$
\frac{1}{\Box_1 \partial j_1} \frac{\partial}{\partial j_1} Z = \int_0^\infty d^3 u \frac{u_1}{j_1 u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2} e^{\sum u^\Box} ,
$$

(19.25)

$$
\frac{1}{\Box_1 \Box_2 \partial j_1 \partial j_2} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} Z = \int_0^\infty d^3 u \frac{u_1 u_2}{j_1 j_2 u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2} e^{\sum u^\Box} ,
$$

(19.26)

$$
\frac{1}{\Box_1 \Box_2 \Box_3 \partial j_1 \partial j_2 \partial j_3} \frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} \frac{\partial}{\partial j_3} Z = \int_0^\infty d^3 u \frac{u_1 u_2 u_3}{j_1 j_2 j_3 u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2} e^{\sum u^\Box} ,
$$

(19.27)

and the Laplace originals remain rational.

There remain to be considered the terms with the coefficient (19.7a). Such terms exist only in the form factors of the structures without derivatives, and, as follows from (19.15), they grow like $(\ln(-\Box_k) + \text{const})$, $\Box_k \rightarrow -\infty$. To single out the growing contribution, eq. (19.25) should be rewritten as

$$
\frac{1}{\Box_1} \frac{\partial}{\partial j_1} Z
= -\int_0^\infty d^3 u \frac{u_2 u_3 (u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2)^{-1} (j_2 u_3 + j_3 u_2)^{-1}}{e^{\sum u^\Box}} + \frac{1}{j_1} \int_0^\infty d^3 u \frac{u_2 u_3 + j_3 u_2}{e^{\sum u^\Box}} .
$$

(19.28)

Here the second integral contains the asymptotic term (19.15) whereas the first integral is already $O(1/\Box_3^2)$, $\Box_3 \rightarrow -\infty$. Therefore, upon multiplication by $\Box_3$, the multiplier $\Box_3$ can be absorbed in the first integral of (19.28) by integration by parts. We obtain

$$
\frac{\Box_3}{\Box_1} \frac{\partial}{\partial j_1} Z
= \int_0^\infty d^3 u \frac{u_2 [j_3 u_1 u_2 (u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2)^{-2} (j_2 u_3 + j_3 u_2)^{-1} - j_2 u_3 (u_1 u_2 j_3 + u_2 u_3 j_1 + u_1 u_3 j_2)^{-1} (j_2 u_3 + j_3 u_2)^{\n}]}{e^{\sum u^\Box}} + \frac{\Box_3}{j_1} \int_0^\infty d^3 u \frac{u_2 u_3 + j_3 u_2}{e^{\sum u^\Box}} .
$$

(19.29)

which is a sum of a decreasing Laplace integral and the growing contribution. Note that the latter contribution is precisely of the form (19.6) multiplied by $\Box_3$.

In some cases, as a result of the action of the derivatives $\partial/\partial j$, in the leading asymptotic behaviour $(\ln(-\Box_3) + \text{const})$ of (19.29), the "$\ln(-\Box_3)$" cancels, and the "const" remains:

$$
P_{\text{special}} \left( \frac{\partial}{\partial j} \right) \frac{\Box_3}{\Box_1} \frac{\partial}{\partial j_1} Z = \frac{1}{\Box_1} + O \left( \frac{1}{\Box_3} \right) , \quad \Box_3 \rightarrow -\infty .
$$

(19.30)
The nondecreasing tree term $1/\Box_1$ that appears here can be singled out explicitly already at the level of the $\alpha$-representation by using the identities of sect. 18. In the Laplace representation, it can be put in the form

$$\frac{1}{\Box_1} = -\Box_2 \Box_3 \int_0^\infty d^3 u e^\sum u \Box_1$$ (19.31)

similar to (19.23), (19.24).

There are only two types of nondecreasing contributions: (19.6) multiplied by either $\Box_2$ or $\Box_3$, and the tree terms (19.23), (19.24), (19.31). Both factorize into elementary functions of one or two variables. The nonfactorizable triple form factors are given by proper Laplace integrals with rational originals.

In this way the final results in sect. 8 are obtained.

20. Derivation of the small–$\Box$ asymptotic behaviors, and the generalized spectral technique for the form factors in $W(\omega = 2)$

The spectral representation of the third-order form factors was studied in paper III. For the basic form factor (18.7) it is of the form [3, 23]

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty \frac{dm_1^2 dm_2^2 dm_3^2 \rho(m_1, m_2, m_3)}{(m_1^2 - \Box_1)(m_2^2 - \Box_2)(m_3^2 - \Box_3)}$$ (20.1)

with the discontinuous spectral weight

$$\rho(m_1, m_2, m_3) = \begin{cases} 1/4\pi S &\text{if there exists a triangle with the sides } m_1, m_2, m_3, \\ 0 &\text{otherwise,} \end{cases}$$ (20.2)

where $m_1 = \sqrt{m_1^2}$, $m_2 = \sqrt{m_2^2}$, $m_3 = \sqrt{m_3^2}$, and $S$ is the area of a triangle with the sides $m_1$, $m_2$, $m_3$.

The form factors with $\alpha$-polynomials are then obtained by noting that

$$(-\Box)^n \frac{\partial^n}{\partial \Box^n} \frac{1}{m^2 - \Box} = \left( \frac{\partial}{\partial m^2} \right)^n \frac{(m^2)^n}{m^2 - \Box}$$ (20.3)

and using eq. (18.8):

$$\left\langle \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \right\rangle_3 = \frac{1}{(n_1 + n_2 + n_3)!} \int_0^\infty dm_1^2 dm_2^2 dm_3^2 \rho(m_1, m_2, m_3) \times \left( \frac{\partial}{\partial m_1^2} \right)^{n_1} \left( \frac{\partial}{\partial m_2^2} \right)^{n_2} \left( \frac{\partial}{\partial m_3^2} \right)^{n_3} \frac{(m_1^2)^{n_1} (m_2^2)^{n_2} (m_3^2)^{n_3}}{m_1^2 - \Box_1 m_2^2 - \Box_2 m_3^2 - \Box_3}.$$ (20.4)

*Such tree terms and the tree terms obtained in (19.21), (19.22) figure in the final expressions of sect. 7 for the $\alpha$-representation of the form factors.
Although the problem of expressing the form factors through the Green function $1/(m^2 - \Box)$ is thereby solved, the action of the derivatives $\partial/\partial m^2$ results in the appearance of powers of $1/(m^2 - \Box)$. A natural next step would be to integrate in (20.4) by parts in order that the derivatives $\partial/\partial m^2$ act on $\rho$ but, as seen from the Heron formula for the area of a triangle

$$\frac{1}{4\pi S} = \frac{1}{\pi} \left[ (m_1 + m_2 + m_3)(m_1 + m_2 - m_3) \times (m_1 + m_3 - m_2)(m_3 + m_2 - m_1) \right]^{-\frac{1}{2}},$$

(20.5)

this integration by parts will result in a divergence of the integral at the integration boundary

$$(m_1 = m_2 + m_3) \bigcup (m_2 = m_1 + m_3) \bigcup (m_3 = m_1 + m_2).$$

(20.6)

On the other hand, with the derivatives acting on the Green functions, the representation in eq. (20.4) is different for different $n_1, n_2, n_3$, and, because of this nonuniqueness, unfit for verification of hidden identities between the form factors.

The success comes with the recognition of the following remarkable fact (see, e.g., [24]):

$$\int_0^\infty dy^2 J_0(y n_1) J_0(y n_2) J_0(y n_3) = 4 \rho(m_1, m_2, m_3)$$

(20.7)

where $J_0$ is the order-0 Bessel function, and $\rho$ is given in (20.2). The use of this relation in (20.1) gives

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = 2 \int_0^\infty dy^2 K_0(y \sqrt{-\Box_1}) K_0(y \sqrt{-\Box_2}) K_0(y \sqrt{-\Box_3})$$

(20.8)

where $K_0$ is the order-0 Macdonald function for which we have the spectral representation

$$K_0(y \sqrt{-\Box}) = \frac{1}{2} \int_0^\infty \frac{dm^2 J_0(ym)}{m^2 - \Box}. $$

(20.9)

Eq. (20.8) is a generalized spectral representation in which there is one extra integration over a parameter entering the spectral weight. This integration will always be considered as the last one. Two important advantages of this representation are that i) the boundary (20.6) disappears owing to the properties of the integral (20.7), and ii) in the integral over the parameter, the triple form factor factorizes into functions of one variable.

The generalized spectral representation of the form factors with $\alpha$-polynomials is obtained by noting that

$$(-\Box)^n \frac{\partial^n}{\partial \Box^n} K_0(y \sqrt{-\Box}) = (-y^2)^n \left(\frac{\partial}{\partial y^2}\right)^n K_0(y \sqrt{-\Box})$$

(20.10)

and using eq. (18.8):

$$\left\langle \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \right\rangle_3 = \frac{2}{(n_1 + n_2 + n_3)!} \int_0^\infty dy^2(-y^2)^{n_1+n_2+n_3} \left(\frac{\partial}{\partial y_{1}^2}\right)^{n_1} \left(\frac{\partial}{\partial y_{2}^2}\right)^{n_2} \left(\frac{\partial}{\partial y_{3}^2}\right)^{n_3}

\times K_0(y_1 \sqrt{-\Box_1}) K_0(y_2 \sqrt{-\Box_2}) K_0(y_3 \sqrt{-\Box_3}) \bigg|_{y_1=y_2=y_3=y}. $$

(20.11)

*Eq. (20.8) can be proved independently by using the integral representation for $K_0$. 

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This is equivalent to introducing again the generating function (19.4) for which we have
\[ Z(j_1, j_2, j_3) = 2 \int_0^\infty dy^2 K_0(y j_1^{1/2} \sqrt{-\Box_1}) K_0(y j_2^{1/2} \sqrt{-\Box_2}) K_0(y j_3^{1/2} \sqrt{-\Box_3}). \tag{20.12} \]

In what follows, it will be convenient to express the derivatives in (20.11) in terms of the conformal operator
\[ C \equiv y^2 \frac{d}{dy^2}. \tag{20.13} \]
For the calculations with \( C \), it will suffice to use the commutation rule
\[ P(C) \left( \frac{1}{y^2} \right)^N = \left( \frac{1}{y^2} \right)^N P(C - N) \tag{20.14} \]
where \( P(C) \) is an arbitrary polynomial in \( C \). For example, it is easy to see that the operator in (20.10) expands in powers of \( C \) as follows:
\[ (-y^2)^n \left( \frac{\partial}{\partial y^2} \right)^n = (-1)^n C(C - 1)(C - 2) \ldots (C - n + 1). \tag{20.15} \]

In terms of \( C \), the form factor with the general \( \alpha \)-polynomial takes the form
\[ \left\langle \frac{P(\alpha_1, \alpha_2, \alpha_3)}{-\Omega} \right\rangle_3 = 2 \int_0^\infty dy^2 Q(C_1, C_2, C_3) \times K_0(y_1 \sqrt{-\Box_1}) K_0(y_2 \sqrt{-\Box_2}) K_0(y_3 \sqrt{-\Box_3}) \tag{20.16} \]
where both \( P \) and \( Q \) are polynomials, and the operators \( C_1, C_2, C_3 \) act on \( y_1, y_2, y_3 \) respectively with subsequently setting \( y_1 = y_2 = y_3 = y \). The only arbitrariness of this representation is the one corresponding to the constraint
\[ \alpha_1 + \alpha_2 + \alpha_3 = 1. \tag{20.17} \]
Now it becomes the arbitrariness of integration by parts in \( y^2 \), and the constraint (20.17) turns into the identity
\[ C_1 + C_2 + C_3 = -1. \tag{20.18} \]
The general rule of integration by parts in the expression
\[ \int_0^\infty dy^2 \left( \frac{1}{y^2} \right)^N Q(C_1, \ldots C_n) f(y_1, \ldots y_n) \bigg|_{y_1=\ldots=y} \tag{20.19} \]
is
\[ C_1 + \ldots + C_n = N - 1 \tag{20.20} \]
_provided that the boundary contributions vanish_. In the integral (20.16), the latter condition is always fulfilled because the action of the operators \( C \) cannot deteriorate the behaviours
\[ K_0(y \sqrt{-\Box}) = O(\ln y), \quad y \to 0, \tag{20.21} \]
\[ \mathcal{K}_0(y \sqrt{-\Box}) = O \left( e^{-y \sqrt{-\Box}} \right), \quad y \to \infty. \] (20.22)

In a more general case, the validity of (20.20) may depend on which of the \( C \)'s will be expressed through the others. Such case will be encountered below.

Eq. (20.16) should be supplemented with the spectral form of the second-order form factor in (18.5). In the ordinary spectral representation of this form factor

\[
\frac{\ln(\Box_1/\Box_2)}{(\Box_1 - \Box_2)} = - \int_0^\infty \frac{dm^2}{(m^2 - \Box_1)(m^2 - \Box_2)} \\
= - \int_0^\infty dm_1^2 dm_2^2 \frac{\delta(m_1^2 - m_2^2)}{(m_1^2 - \Box_1)(m_2^2 - \Box_2)},
\]

(20.23)

the double-spectral weight is again discontinuous, while its generalized spectral representation is similar to (20.8):

\[
\frac{\ln(\Box_1/\Box_2)}{(\Box_1 - \Box_2)} = - \int_0^\infty dy^2 \mathcal{K}_0(y \sqrt{-\Box_1})\mathcal{K}_0(y \sqrt{-\Box_2}).
\]

(20.24)

Even a tree can be put in the generalized spectral form:

\[ \frac{1}{\Box} = - \frac{1}{2} \int_0^\infty dy^2 \mathcal{K}_0(y \sqrt{-\Box}) \]

(20.25)

but this is already a signal that the use of this representation needs a reserve.

The reserve concerns the asymptotic behaviours of generalized spectral integrals at small \( \Box \)'s. For the ordinary spectral representation to exist the function should behave like \( O/\Box \) where \( O \to 0 \) at \( \Box \to -0 \). This is also true of the generalized triple and double spectral forms but not of the single one. Indeed, the behaviours of the triple and double forms (20.8) and (20.24) at \( \Box_k \to -0 \) \((k = 1, 2, 3)\) follow the behaviour of the respective \( \mathcal{K}_0 \) because, when this \( \mathcal{K}_0 \) is expanded at small \( y \), the responsibility for the convergence at the upper limit rests with another \( \mathcal{K}_0 \). With a single \( \mathcal{K}_0 \), as in (20.25), this is no longer the case. Hence the imitation spectral weight of a massless Green function.

In the case of the triple form factors, the representation (20.12) readily gives the small-\( \Box \)-asymptotic behaviour of the generating function. By expanding one of the \( \mathcal{K}_0 \)'s, we obtain

\[
Z = - \ln(-j_3 \Box_3) \int_0^\infty dy^2 \mathcal{K}_0(y j_1^{1/2} \sqrt{-\Box_1})\mathcal{K}_0(y j_2^{1/2} \sqrt{-\Box_2}) \\
+ O(1), \quad -\Box_3 \to 0.
\]

(20.26)

Together with (19.5), this gives the asymptotic behaviours of all \( \alpha \)-averages with accuracy \( O(1) \). However, because of the presence of the multipliers \( \Box_1/\Box_k \) in the \( \alpha \)-polynomials, to have this accuracy in the total form factor, the generating function must be known with accuracy \( O(\Box_k) \), \( \Box_k \to -0 \). Owing to the “rule of the like \( \alpha \)”, we need only

\[
\frac{1}{\Box_3} \frac{\partial}{\partial j_3} Z = - \frac{1}{j_3 \Box_3} \int_0^\infty dy^2 \mathcal{K}_0(y j_1^{1/2} \sqrt{-\Box_1})\mathcal{K}_0(y j_2^{1/2} \sqrt{-\Box_2}) \\
+ \frac{1}{4} \ln(-j_3 \Box_3) \int_0^\infty dy^2 y^2 \mathcal{K}_0(y j_1^{1/2} \sqrt{-\Box_1})\mathcal{K}_0(y j_2^{1/2} \sqrt{-\Box_2}) \\
+ O(1), \quad -\Box_3 \to 0
\]

(20.27)
The coefficients of the asymptotic terms in (20.26) and (20.27) can be calculated explicitly: we have eq. (20.24) and

\[
\int_0^\infty dy^2 y^2 K_0(ay)K_0(by) = \frac{4}{(a^2 - b^2)^2} \left[ (a^2 + b^2) \frac{\ln(a^2/b^2)}{a^2 - b^2} - 2 \right],
\]

with \(a > 0, b > 0\).

(20.28)

In this way the final results in sect. 11 are obtained.

In this way the large-\( \Box \) behaviours of the generalized and ordinary spectral integrals differ drastically. The ordinary spectral representation exists only for functions that behave like \( O \to 0 \) at \( \Box \to -\infty \) whereas the generalized one can stand any power growth at large arguments. For our purposes, this is an important advantage because it enables us to absorb in the originals any positive powers of \( \Box \)'s. Indeed, from the Bessel equation for \( K_0 \)

\[
\frac{d}{dz^2} z^2 \frac{d}{dz} z^2 K_0(z) + K_0(z) = 0
\]

(20.29)

with the argument \( z = y\sqrt{\Box} \), we obtain

\[
\Box K_0 = -\frac{4}{y^2} C^2 K_0,
\]

(20.30)

\[
\Box^2 K_0 = \left( -\frac{4}{y^2} \right)^2 (C - 1)^2 C^2 K_0,
\]

(20.31)

and, generally,

\[
\Box^M K_0 = \left( -\frac{4}{y^2} \right)^M \left[ \prod_{p=0}^{M-1} (C - p)^2 \right] K_0.
\]

(20.32)

In combination with (20.16), this gives, for example,

\[
\left\langle \frac{\Box^M P(\alpha_1, \alpha_2, \alpha_3)}{-\Omega} \right\rangle_3 = 2 \int_0^\infty dy^2 \left( -\frac{4}{y^2} \right)^M Q(C_1 - M, C_2, C_3)
\]

\[
\times \left[ \prod_{p=0}^{M-1} (C_1 - p)^2 \right] K_0(y_1\sqrt{-\Box_1})K_0(y_2\sqrt{-\Box_2})K_0(y_3\sqrt{-\Box_3}).
\]

(20.33)

Thus, the situation is opposite to the one in the case of the Laplace representation. Now, the positive powers of \( \Box \) can be easily absorbed in the originals, and the negative powers require a special procedure of detaching the inadmissibly growing terms. The procedure is as follows. Owing to the “rule of the like \( \alpha \)”, a multiplier \( 1/\Box \) may appear only in a product with the operator \( C \). By using eq. (20.9), we find

\[
\frac{C}{\Box} K_0(y\sqrt{-\Box}) = -\frac{1}{2} \int_0^\infty dm^2 \frac{ym}{2} J_1(ym) \frac{1}{(m^2 - \Box)\Box}
\]

\[
= -\frac{1}{2} \int_0^\infty dm^2 \frac{ym}{2m} \frac{1}{m^2 - \Box} - \frac{1}{2\Box}
\]

(20.34)
where $J_1$ is the order-1 Bessel function, and the tree term $1/\Box$ that got detached is the leading asymptotic term at $\Box \to -0$. The remaining spectral integral with $J_1$ is already $O/\Box$. $\Box \to -0$.

Since the multipliers $1/\Box_k$, $k = 1, 2, 3$, do appear in our form factors (and higher powers of $1/\Box_k$ never appear), we introduce

$$S(y, \Box) = \frac{1}{2} \int_0^\infty dm^2 \frac{y J_1(y m)}{2m} \frac{1}{m^2 - \Box}$$

(20.35)

as a basic spectral integral instead of (20.9). Eq. (20.34) expresses $S$ through $K_0$:

$$S = -\frac{C}{\Box} K_0 - \frac{1}{2\Box}$$

(20.36)

but then the Bessel equation (20.30) expresses also $K_0$ through $S$:

$$K_0 = \frac{4}{y^2} CS$$

(20.37)

The latter relation should be used in all expressions like (20.16), (20.24), (20.33), etc. to replace everywhere $K_0$ by $S$. The substitutions to be made are

$$P(C)K_0 = \frac{4}{y^2} P(C - 1)CS,$$

(20.38)

$$\frac{1}{\Box} P(C)CK_0 = -P(C)S - \frac{P(0)}{2\Box},$$

(20.39)

$$\Box^M P(C)K_0 = -\left(-\frac{4}{y^2}\right)^{M+1} P(C - M - 1) \left[ \prod_{p=1}^{M} (C - p)^2 \right] CS$$

(20.40)

with any polynomial $P(C)$. This gives the final form of the representation.

An important point is that $S$, as distinct from $K_0$, does not decrease at $y \to \infty$. As seen from (20.36),

$$S = -\frac{1}{2\Box} + O \left(e^{-y\sqrt{-\Box}}\right), \quad y \to \infty$$

(20.41)

but the action of at least one $C$ makes $S$ decreasing exponentially. Therefore, the generalized spectral integrals must necessarily contain at least one differentiated $S$, which means that more than two $1/\Box$ multipliers cannot be absorbed in the triple form factors. Denoting for short

$$S(y_1, \Box_1) = S_1, \quad S(y_2, \Box_2) = S_2, \quad S(y_3, \Box_3) = S_3$$

(20.42)

and omitting, for simplicity, the $\alpha$-polynomial, we have

$$\left\langle \frac{1}{-\Omega} \right\rangle_3 = 2 \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^3 C_1 C_2 C_3 S_1 S_2 S_3,$$

(20.43)

$$\left\langle \frac{1}{-\Omega} \alpha_1 \right\rangle_3 = 2 \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^2 C_2 C_3 S_1 S_2 S_3 + \frac{1}{\Box_1} \int_0^\infty dy^2 \left(\frac{4}{y^2}\right)^2 C_2 C_3 S_2 S_3,$$

(20.44)
\[
\left\langle \frac{1}{-\Omega} \square_1 \square_2 \right\rangle_3 = \int_0^\infty dy^2 \left( \frac{4}{y^2} \right) C_3 S_1 S_2 S_3 \\
+ \frac{1}{2 \Box_1} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right) C_3 S_2 S_3 \\
+ \frac{1}{2 \Box_2} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right) C_3 S_1 S_3 \\
+ \frac{1}{4 \Box_1 \Box_2} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right) C_3 S_3 
\]
\[(20.45)\]

but one more application of eq. (20.39), to incorporate the factor \((\alpha_1 \alpha_2 \alpha_3)/(\Box_1 \Box_2 \Box_3)\), would result in the appearance of a divergent integral with three bare \(S\).

If the form factors of the structures with derivatives are brought to the standard dimension by the redefinition discussed in sect. 7, then one has to deal only with the ratio \(\Box_m/\Box_n\).

The expression for the triple form factor in this case (with the \(\alpha\)-polynomial omitted) is

\[
\left\langle \frac{1}{-\Omega} \square_1 \right\rangle_3 = -2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^3 (C_2 - 1)^2 C_2 C_3 S_1 S_2 S_3 + \\
+ \frac{\Box_2}{\Box_1} \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^2 C_2 C_3 S_2 S_3. 
\]
\[(20.46)\]

The integrals with one or two \(S\), that get detached in (20.44)–(20.46), reproduce the leading asymptotic terms of the form factors. In this way the results in sect. 9 are obtained.

The last point to be discussed is integration by parts over \(y\) in the case like (20.33) where the total power of \(\Box_1, \Box_2, \Box_3\) in the coefficient of the \(\alpha\)-polynomial is positive. This case is encountered in the calculation of the trace anomaly (sect. 12). It will suffice to consider, as an example, eq. (20.33) with the \(\alpha\)-polynomial omitted. In terms of \(S\),

\[
\left\langle \frac{\Box_1^M}{-\Omega} \right\rangle_3 = -2 \int_0^\infty dy^2 \left( \frac{4}{y^2} \right)^{M+3} \left[ \prod_{p=1}^M (C_1 - p)^2 \right] C_1 C_2 C_3 S_1 S_2 S_3. 
\]
\[(20.47)\]

The convergence of this integral at the lower limit is based on the following property of the operator \(C\):

\[(C - p)^{y+1}(y^2)^p \left( \ln y^2 \right)^q = 0. \]
\[(20.48)\]

Since the asymptotic expansion at small \(y\) of both \(K_0\) and \(S\) is a series of \((y^2)^p\) and \((y^2)^p \ln y^2\) which starts in the case of \(K_0\) with \(p = 0\), and in the case of \(S\) with \(p = 1\), we have at \(y \to 0\)

\[
S = O \left( y^2 \ln y^2 \right), \quad \text{(20.49)}
\]

\[
(C - 1)^2 S = O \left( y^4 \ln y^2 \right), \quad \text{(20.50)}
\]

\[
(C - 2)^2 (C - 1)^2 S = O \left( y^6 \ln y^2 \right), \quad \text{(20.51)}
\]
and, generally, 

$$
\left[ \prod_{p=1}^{M} (C - p)^2 \right] S = O \left( y^{2M+2} \ln y^2 \right). 
$$

The problem is, however, that the integration by parts cannot be applied to the operators 

$$(C_1 - p)^2$$

in (20.47) because the result will be divergent. On the other hand, to remove the arbitrariness connected with the identity (20.20), one may need to have in (20.47) the bare $S_1$. The procedure of integration by parts should then be modified as follows. Let us denote $S^M$ the first $M$ terms of the asymptotic series for $S$ so that

$$(S - S^M) = O \left( y^{2(M+1)} \ln y^2 \right).$$

Since

$$
\left[ \prod_{p=1}^{M} (C - p)^2 \right] S^M = 0,
$$

we may replace in (20.47) $S_1$ by $(S_1 - S^M_1)$. After this replacement, the integration by parts can already be done by the rule (20.20), and the result is

$$
\left\langle \frac{\Box^M}{-\Omega} \right\rangle_3 = 2 \int_0^\infty dy^2 \left( -\frac{4}{y^2} \right)^{M+3} (S_1 - S^M_1) \left[ \prod_{p=1}^{M} (C_2 + C_3 - p - 1)^2 \right] 
\times (C_2 + C_3 - M - 2)C_2C_3S_2S_3.
$$

No boundary terms ever appear but the bare $S$ appears with a subtraction. The goal is, nevertheless, reached: the representation like in (20.55) is unique and can be used for a verification of hidden identities between the form factors. An example of such a verification is given in sect. 12. The only question that may arise is that the subtraction $S^M$ is not in the spectral form but putting it in the spectral form presents no problem since the only nonanalytic function of $\Box$ in $S^M$ is $\ln(-\Box)$.

The generalized spectral representation makes it possible to carry out calculations like the check of the trace anomaly within the working technique used in applications.

**Appendix: Identities for nonlocal cubic invariants**

We begin with the local identities for a tensor possessing the symmetries of the Weyl tensor:

$$C_{\alpha\beta\gamma\delta} = C_{[\alpha\beta][\gamma\delta]} = C_{\alpha\beta[\gamma\delta]},$$

(A.1)

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta},$$

(A.2)

$$C_{\alpha\beta\gamma\delta} + C_{\alpha\delta\beta\gamma} + C_{\alpha\gamma\delta\beta} = 0,$$

(A.3)

$$C_{\alpha\beta\gamma\delta} = 0.$$  

(A.4)
Here the Ricci identity (A.3) can be rewritten as
\[ C_{\alpha[\beta\gamma]} = -\frac{1}{2} C_{\alpha\delta\beta\gamma} \] (A.5)
and, with the use of (A.1), as
\[ C_{\alpha[\beta\gamma\delta]} = 0 \] (A.6)
where the complete antisymmetrization in three indices is meant. Eq. (A.5) is useful when forming contractions because it shows that the contractions of the form
\[ C_{\cdot\cdot\alpha\beta} C_{\cdot\cdot\alpha\beta} \]
express through one another.

In view of applications to the gravitational equations [25], we first list all possible cubic contractions having two free indices. The symmetries above allow only
\[ J_{1\mu} = C_{\mu\beta\gamma\delta} C_{\nu\alpha\sigma} C_{\alpha\sigma\beta\gamma\delta} \] (A.7)
\[ J_{2\mu} = C_{\mu\beta\gamma\delta} C_{\nu\alpha\gamma\sigma} C_{\alpha\sigma\beta\delta} \] (A.8)
\[ J_{3\mu} = C_{\mu\beta\gamma\delta} C_{\nu\alpha\gamma\sigma} C_{\sigma\alpha\beta\delta} \] (A.9)
\[ J_{4\mu} = C_{\mu\gamma\beta\delta} C_{\nu\beta\alpha\sigma} C_{\alpha\sigma\gamma\delta} \] (A.10)
\[ J_{5\mu} = C_{\nu\sigma\mu\kappa} C_{\sigma\alpha\beta\gamma} C_{\alpha\beta\gamma\kappa} \] (A.11)
and, furthermore, by (A.5), one has
\[ J_{4\mu} = \frac{1}{2} J_{1\mu} \] (A.12)
and, by applying the Ricci identity to the last \( C \) in (A.8), one obtains
\[ J_{3\mu} = J_{2\mu} - \frac{1}{2} J_{4\mu} = J_{2\mu} - \frac{1}{4} J_{1\mu} \] (A.13)

Thus, for an arbitrary space-time dimension \( 2\omega \), there are three independent contractions:
\( J_{1\mu}, J_{2\mu}, J_{5\mu} \).

For a particular space-time dimension, the number of independent contractions can be smaller because of the existence of identities obtained by antisymmetrization of \( (2\omega + 1) \) indices. Note that such an antisymmetrization must not involve more than two indices of each \( C \) tensor; otherwise the identity will be satisfied trivially by virtue of (A.6). Hence, for three \( C \) tensors, the number of indices involved in the antisymmetrization should not exceed six, and, therefore, the space-time dimension \( 2\omega \) for which nontrivial identities exist cannot exceed five. For \( 2\omega \leq 5 \) we have
\[ C_{\alpha\beta\gamma\delta} C_{\gamma\delta\nu\mu} C_{\alpha\beta\nu\mu} \equiv 0, \quad 2\omega \leq 5 \] (A.14)
with the complete antisymmetrization of six lower indices. When written down explicitly, this identity takes the form

\[ J_1^{\nu} - 4J_3^{\nu} - 2J_5^{\nu} = 0, \quad 2\omega \leq 5 \]  \hspace{1cm} (A.15)

or, by (A.13), the form

\[ J_2^{\nu} = \frac{1}{2}J_1^{\nu} - \frac{1}{2}J_5^{\nu}, \quad 2\omega \leq 5 \]  \hspace{1cm} (A.16)

and reduces the number of independent contractions down to two: \( J_1^{\nu} \) and \( J_5^{\nu} \).

Finally, for \( 2\omega = 4 \) (the lowest dimension in which a nonvanishing Weyl tensor exists), the identity (A.14) becomes a linear combination of the identities

\[ C_{[\alpha\beta}^{\gamma\delta}C_{\gamma\delta}^{\kappa\mu}C_{\kappa\mu]}^{\alpha\beta} \equiv 0, \quad 2\omega = 4 \]  \hspace{1cm} (A.17)

with the antisymmetrization over only five indices, and there is one more identity, quadratic in \( C \):

\[ C_{[\alpha\beta}^{\gamma\delta}C_{\gamma\delta}^{\alpha\beta}C_{\mu}^{\nu} \equiv 0, \quad 2\omega = 4. \]  \hspace{1cm} (A.18)

Its explicit form is

\[ C^{\alpha\beta\gamma\nu}C_{\alpha\beta\gamma\mu} = \frac{1}{4}\delta_{\mu}^{\nu}C_{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}, \quad 2\omega = 4. \]  \hspace{1cm} (A.19)

When this relation is used in (A.11), the result is

\[ J_5^{\nu} = 0, \quad 2\omega = 4 \]  \hspace{1cm} (A.20)

by (A.4). Thus, in four dimensions, there is only one independent contraction: \( J_1^{\nu} \).

Similar results hold for invariants except that the complete contraction of \( J_5 \) in (A.11) is zero for any space-time dimension. Therefore, initially one has four different \( C^3 \) invariants

\[ I_1 = C_{\mu\beta\gamma\delta}C^{\mu\beta\alpha\sigma}C_{\alpha\sigma\gamma\delta}, \]  \hspace{1cm} (A.21)

\[ I_2 = C_{\mu\beta\gamma\delta}C^{\mu\alpha\gamma\sigma}C_{\alpha\beta\sigma\delta}, \]  \hspace{1cm} (A.22)

\[ I_3 = C_{\mu\beta\gamma\delta}C^{\mu\alpha\gamma\sigma}C_{\sigma\beta\alpha\delta}, \]  \hspace{1cm} (A.23)

\[ I_4 = C_{\mu\gamma\beta\delta}C^{\mu\beta\alpha\sigma}C_{\alpha\sigma\gamma\delta} \]  \hspace{1cm} (A.24)

with the relations

\[ I_3 = I_2 - \frac{1}{4}I_1, \quad I_4 = \frac{1}{2}I_1, \]  \hspace{1cm} (A.25)

and, for \( 2\omega \leq 5 \), the identity (A.14) (contracted in \( \mu, \nu \)) adds one more relation:

\[ I_2 = \frac{1}{2}I_1, \quad 2\omega \leq 5. \]  \hspace{1cm} (A.26)

When going over from \( 2\omega = 5 \) to \( 2\omega = 4 \), the identity (A.19) leads to no further reduction. Thus, the dimension of the basis of local \( C^3 \) invariants is 2 for \( 2\omega > 5 \), and 1 for both \( 2\omega = 5 \) and \( 2\omega = 4 \).
For invariants with the Riemann tensor, the counting is different because, in this case, the quadratic identity (A.19) begins working. For $2\omega \leq 5$, the identity (A.26) with the Weyl tensor expressed through the Riemann tensor reduces the number of independent cubic invariants by one. For $2\omega = 4$, the identity (A.19) contracted with the Ricci tensor:

$$C^{\alpha\beta\gamma\nu} C_{\alpha\beta\gamma\mu} R^\mu_\nu = \frac{1}{4} R C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \quad 2\omega = 4$$  \hspace{1cm} (A.27)

reduces this number by one more. These results agree with the group-theoretic analysis carried out in [26]. According to [26], the dimension of the basis of local cubic invariants with the Riemann tensor (without derivatives) is 8 for $2\omega > 5$, 7 for $2\omega = 5$, and 6 for $2\omega = 4$.

We shall now concentrate on the space-time dimension $2\omega = 4$ and go over to the consideration of nonlocal invariants cubic in the curvature. Owing to their algebraic nature, the identities above admit easily a nonlocal generalization. Indeed, the two cubic identities obtained by antisymmetrizations in four dimensions: eq. (A.17) contracted in $\mu, \nu$, and eq. (A.18) contracted with $R^\mu_\nu$ can in fact be written down for three different tensors and, in particular, for the curvature tensors at three different points. One can then multiply them by arbitrary form factors and next make the points coincident. It will, in addition, be more convenient to deal now with the Riemann tensor rather than the Weyl tensor. The nonlocal identities obtained in this way are of the form

$$\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) R_{1[\alpha\beta} \gamma_\delta R_{2\gamma_\delta}^{\kappa\mu} R_{3\kappa\mu]\alpha\beta} = 0, \quad 2\omega = 4$$ \hspace{1cm} (A.28)

$$\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) R_{1[\alpha\beta} \gamma_\delta R_{2\gamma_\delta}^{\kappa\mu} R_{3\kappa\mu]\alpha\beta} = 0, \quad 2\omega = 4$$ \hspace{1cm} (A.29)

with arbitrary $\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3)$ and $\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3)$. Since nothing is involved here except inexistence of five different indices in four dimensions, these identities are obviously correct in the present nonlocal form as well.

When written down explicitly, the left-hand sides of eqs. (A.28), (A.29) take the form (for arbitrary dimension)

$$\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) R_{1[\alpha\beta} \gamma_\delta R_{2\gamma_\delta}^{\kappa\mu} R_{3\kappa\mu]\alpha\beta} \equiv \frac{1}{15} \tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) \left[ R_{1\alpha\beta\gamma\delta} R_{2}^{\delta\mu\nu} R_{3\mu\nu}^{\alpha\beta} - 2 R_{1\alpha\beta\gamma\delta} R_{2}^{\mu\nu\gamma\delta} R_{3}^{\beta\mu\nu\delta} - 3 R_{1\alpha\beta} R_{2}^{\alpha\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} - 2 R_{1\alpha\beta} R_{2}^{\alpha\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} - R_{3\alpha\beta} R_{2}^{\alpha\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} + \frac{1}{2} R_{1\alpha\beta\gamma\delta} R_{2}^{\gamma\delta\mu\nu} R_{3}^{\beta\mu\nu\delta} + 2 R_{1\alpha\beta\gamma\delta} R_{2}^{\gamma\delta\mu\nu} R_{3}^{\beta\mu\nu\delta} - 2 R_{1\alpha\beta} R_{2}^{\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} - R_{1\alpha\beta} R_{2}^{\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} \right]$$ \hspace{1cm} (A.30)

$$\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) R_{1[\alpha\beta} \gamma_\delta R_{2\gamma_\delta}^{\kappa\mu} R_{3\kappa\mu]\alpha\beta} \equiv - \frac{2}{15} \tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) \times \left[ R_{1\alpha\beta\gamma\mu} R_{2}^{\alpha\beta\gamma\delta} R_{3}^{\mu\nu} - \frac{1}{4} R_{1\alpha\beta\gamma\delta} R_{2}^{\alpha\beta\gamma\delta} R_{3}^{\mu\nu} - 2 R_{1\alpha\beta} R_{2}^{\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} - R_{1\alpha\beta\gamma\delta} R_{2}^{\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} - R_{1\alpha\beta} R_{2}^{\beta\gamma\delta} R_{3}^{\gamma\delta\mu\nu} \right]$$ \hspace{1cm} (A.31)
Since these identities contain arbitrary form factors, they suggest that, in virtue of (A.28)-(A.29), in four dimensions, the basis of nonlocal gravitational invariants may be redundant. However, to convert relations (A.28), (A.29) into constraints between the basis structures (A.29), in four dimensions, the basis of nonlocal gravitational invariants may be redundant. Since these identities contain arbitrary form factors, they suggest that, in virtue of (A.28)-(A.29), in four dimensions, the basis of nonlocal gravitational invariants may be redundant. However, to convert relations (A.28), (A.29) into constraints between the basis structures one must make one more step: eliminate the Riemann tensor.

As discussed in paper II, the Riemann tensor expresses through the Ricci tensor in a nonlocal way once the boundary conditions for the gravitational field are specified. Obtaining this expression amounts to solving iteratively a differentiated Bianchi identity (eq. (A.3) in Appendix A of paper II). For the present case of a positive-signature asymptotically euclidean space, the solution to lowest order in the curvature is given in paper II. This accuracy is sufficient for obtaining relations between cubic invariants via the identities (A.30) and (A.31). However, for the discussion of the Schwinger-DeWitt coefficients in sect. 4 above, the solution for the Riemann tensor is needed with accuracy $O[\mathcal{R}^3]$. By making one more iteration, we obtain the needed expression which extends to second order eq. (A.4) of paper II:

\[
R^{\alpha\beta\mu\nu} = \frac{1}{2} \left( \nabla^\mu \nabla^\alpha R^{\beta\gamma} + \nabla^\alpha \nabla^\mu R^{\beta\gamma} - \nabla^\nu \nabla^\alpha R^{\mu\beta} - \nabla^\alpha \nabla^\nu R^{\mu\beta} 
- \nabla^\nu \nabla^\beta R^{\alpha\mu} - \nabla^\beta \nabla^\nu R^{\alpha\mu} + \nabla^\nu \nabla^\beta R^{\mu\alpha} + \nabla^\beta \nabla^\nu R^{\mu\alpha} \right) 
+ 2R_{\lambda}^{[\mu} \left( \nabla^\lambda \nabla^[\mu R^{\beta\nu]} \right) + 2R_{\lambda}^{[\alpha} \left( \nabla^\lambda \nabla^[\mu R^{\beta\nu]} \right) 
- 2R_{\lambda}^{[\nu} \left( \nabla^\nu \nabla^[\mu R^{\beta\lambda]} \right) - 2R_{\lambda}^{[\alpha} \left( \nabla^\beta \nabla^[\nu R^{\mu\lambda]} \right) 
- 8 \left( \nabla^\lambda \nabla^[\mu R^{\beta\sigma]} \right) \left( \nabla^\lambda \nabla^[\mu R^{\beta\sigma]} \right) - 8 \left( \nabla^\lambda \nabla^[\mu R^{\beta\sigma]} \right) \left( \nabla^\beta \nabla^[\mu R^{\sigma\lambda]} \right) 
- 8 \left( \nabla^\lambda \nabla^[\mu R^{\beta\sigma]} \right) \left( \nabla^\sigma \nabla^[\mu R^{\lambda\sigma]} \right) + 8 \left( \nabla^\mu \nabla^[\sigma R^{\lambda\sigma]} \right) \left( \nabla^\mu \nabla^[\sigma R^{\lambda\sigma]} \right)
- 8 \left( \nabla^\mu \nabla^[\sigma R^{\lambda\sigma]} \right) \left( \nabla^\mu \nabla^[\sigma R^{\lambda\sigma]} \right) 
+ 8 \left( \nabla^\lambda \nabla^[\nu R^{\mu\lambda]} \right) \left( \nabla^\lambda \nabla^[\nu R^{\mu\lambda]} \right) + 8 \left( \nabla^\nu \nabla^[\lambda R^{\mu\lambda]} \right) \left( \nabla^\nu \nabla^[\lambda R^{\mu\lambda]} \right) 
+ 8 \left( \nabla^\lambda \nabla^[\nu R^{\mu\lambda]} \right) \left( \nabla^\lambda \nabla^[\nu R^{\mu\lambda]} \right) - 8 \left( \nabla^\mu \nabla^[\sigma R^{\lambda\sigma]} \right) \left( \nabla^\nu \nabla^[\lambda R^{\sigma\lambda]} \right) 
- 8 \left( \nabla^\mu \nabla^[\sigma R^{\lambda\sigma]} \right) \left( \nabla^\nu \nabla^[\lambda R^{\sigma\lambda]} \right)
\right) 
\right)
+ O[\mathcal{R}^2].
\]

(A.32)

Here the antisymmetrizations on the right-hand side are with respect to $\mu\nu$ and $\alpha\beta$.

Elimination of the Riemann tensor from the identities (A.30) and (A.31) with the use of (A.32) (to lowest order) brings these identities to the following form:

\[
\tilde{F}(\Box_1, \Box_2, \Box_3)R_1^\alpha_{\alpha\beta} \gamma^d R_2^\gamma_{\gamma\mu} R_3^\mu_{\mu\alpha} \equiv - \frac{1}{15} \left( 1 \frac{\Box_1}{\Box_1} + 1 \frac{\Box_2}{\Box_2} + 1 \frac{\Box_3}{\Box_3} \right) R_1 R_2 R_3 
+ \tilde{F}' \left( \frac{\Box_1}{\Box_1} - \frac{\Box_2}{\Box_2} - \frac{\Box_3}{\Box_3} + \frac{1}{2} \frac{\Box_1}{\Box_1} + \frac{1}{2} \frac{\Box_2}{\Box_2} + \frac{1}{2} \frac{\Box_3}{\Box_3} \right) R_1^\mu_{\alpha} R_2^\alpha_{\beta} R_3^\beta_{\mu}
\]

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\[
\begin{align*}
&+ \frac{1}{4} \left( \frac{1}{\Box_1} + \frac{1}{\Box_2} - \frac{3}{\Box_1 \Box_2} \right) \left[ \tilde{\mathcal{F}} + \tilde{\mathcal{F}}|_{1\to3} + \tilde{\mathcal{F}}|_{2\to3} \right] R_{1}^{\mu \nu} R_{2 \mu \nu} R_3 \\
&+ \frac{1}{4} \left( \frac{1}{\Box_1 \Box_2} + \frac{1}{\Box_1 \Box_3} + \frac{3}{\Box_2 \Box_3} \right) \left[ -\tilde{\mathcal{F}}' - \tilde{\mathcal{F}}|_{1\to2} - \tilde{\mathcal{F}}|_{1\to3} \right] R_{1}^{\alpha \beta} \nabla_\alpha R_2 \nabla_\beta R_3 \\
&+ \frac{1}{\Box_1 \Box_2} \left[ \tilde{\mathcal{F}} + \tilde{\mathcal{F}}|_{1\to3} + \tilde{\mathcal{F}}|_{1\to3} \right] \nabla^\mu R_{1}^{\mu \alpha \nu} \nabla_\nu R_{2 \mu \alpha} R_3 \\
&+ \frac{1}{\Box_1 \Box_2} \left[ \tilde{\mathcal{F}}' + \tilde{\mathcal{F}}|_{1\to3} + \tilde{\mathcal{F}}|_{1\to3} \right] R_{1}^{\mu \nu} \nabla_\mu R_{2 \beta} \nabla_\beta R_{3 \alpha} \nabla_\alpha R_3 \\
&+ \left( \frac{1}{\Box_1 \Box_2} + \frac{1}{\Box_1 \Box_3} - \frac{1}{\Box_2 \Box_3} \right) \left[ -\tilde{\mathcal{F}} - \tilde{\mathcal{F}}|_{1\to3} - \tilde{\mathcal{F}}|_{1\to3} \right] \nabla_\alpha R_1 \nabla_\nu R_{2 \lambda} \nabla_\alpha \nabla_\beta R_{3 \mu} \\
&+ \left( \tilde{\mathcal{F}} - \tilde{\mathcal{F}}|_{1\to3} - \tilde{\mathcal{F}}|_{1\to3} \right) \equiv (\Box_1 - \Box_2 - \Box_3) \tilde{\mathcal{F}}, \quad (A.33)
\end{align*}
\]

\[
\begin{align*}
\tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) R_{1 \alpha \beta} \gamma^\delta R_{2 \gamma \delta} \alpha^\beta R_{3 \mu} &\equiv -\frac{2}{15} \left( \frac{1}{8} \tilde{\mathcal{F}} \left( \frac{1}{\Box_1} + \frac{1}{\Box_2} + \frac{3}{\Box_1 \Box_2} \right) R_1 R_2 R_3 \\
&+ \tilde{\mathcal{F}} \left( -1 + \frac{1}{2 \Box_2} - \frac{1}{2 \Box_1} + \frac{3}{\Box_1 \Box_2} + \frac{1}{2 \Box_1 \Box_2} \right) R_{1 \alpha} R_{2 \beta} R_{3 \mu} \\
&+ \frac{1}{4} \left[ -\tilde{\mathcal{F}}(\Box_3) + \tilde{\mathcal{F}}(\Box_1 \Box_2) + \tilde{\mathcal{F}}|_{1\to3} \left( 1 - \frac{3}{\Box_1} + \frac{1}{\Box_2} \right) + \tilde{\mathcal{F}}|_{2\to3} \left( 1 - \frac{3}{\Box_1} + \frac{1}{\Box_2} \right) \right] R_{1 \mu \nu} R_{2 \mu \nu} R_3 \\
&+ \frac{1}{4} \left[ \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_1} + 3 \frac{3}{\Box_2} + \frac{1}{\Box_1 \Box_2} \right) - \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_1} + \frac{3}{\Box_2} + 3 \frac{3}{\Box_1 \Box_2} \right) \right] R_{1 \alpha \beta} \nabla_\alpha R_2 \nabla_\beta R_3 \\
&+ \left( \tilde{\mathcal{F}} \left( \frac{1}{\Box_1} + \tilde{\mathcal{F}}(\Box_1 \Box_2) \right) + \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_1} + \frac{1}{\Box_2} + \frac{3}{\Box_1 \Box_2} \right) \right) \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_1} + \frac{3}{\Box_2} + \frac{3}{\Box_1 \Box_2} \right) \\
&+ \left( \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_2} + \frac{1}{\Box_3} - \frac{1}{\Box_1 \Box_2} \right) + \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_1} + \frac{1}{\Box_2} + \frac{3}{\Box_1 \Box_2} \right) \right) R_{1 \mu \nu} \nabla_\mu R_{2 \alpha \beta} \nabla_\nu R_{3 \alpha} \\
&+ \left( \tilde{\mathcal{F}}(\Box_1 \Box_2) + \tilde{\mathcal{F}}(\Box_1 \Box_2) \right) \left( \frac{1}{\Box_1} + \frac{1}{\Box_2} + \frac{3}{\Box_1 \Box_2} \right) \tilde{\mathcal{F}}(\Box_1 \Box_2) \right) \right) R_{1 \mu \nu} \nabla_\mu R_{2 \beta} \nabla_\beta R_{3 \mu} \\
&+ \left( \tilde{\mathcal{F}} \left( \frac{1}{\Box_1 \Box_2} + \tilde{\mathcal{F}}(\Box_1 \Box_2) \right) + \tilde{\mathcal{F}}|_{1\to3} \left( \frac{1}{\Box_1 \Box_2} + \frac{1}{\Box_1} + \frac{3}{\Box_2} \right) \right) \nabla_\alpha R_{1 \alpha \beta} \nabla_\nu R_{2 \nu \beta} \nabla_\beta R_{3 \mu} \\
&+ \frac{2}{\Box_1 \Box_2} \left( \tilde{\mathcal{F}} \left( \frac{1}{\Box_1 \Box_2} + \tilde{\mathcal{F}}(\Box_1 \Box_2) \right) + \tilde{\mathcal{F}}(\Box_1 \Box_2) \right) \nabla_\mu R_{1 \alpha \beta} \nabla_\nu R_{2 \nu \beta} \nabla_\beta R_{3 \mu} \right) \\
&+ \text{a total derivative + O}[R^4], \quad (A.34)
\end{align*}
\]

where

\[
\tilde{\mathcal{F}}|_{1\to2} \equiv \tilde{\mathcal{F}}(\Box_2, \Box_1, \Box_3), \quad \text{etc.}
\]

It is now seen that, up to total derivatives and terms O[R^4], (A.33) is an equivalent form of (A.34) corresponding to

\[
\tilde{\mathcal{F}} = \frac{1}{2} \frac{1}{\Box_1} \tilde{\mathcal{F}} = \frac{1}{2} \frac{1}{\Box_1} \tilde{\mathcal{F}}. 
\]

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Thus, assuming integration over the space-time which will make irrelevant total derivatives, we conclude that, at third order in the curvature, of the two generally different identities for nonlocal invariants, existing in four dimensions, only one is independent: eq. (A.29). This equation is brought to its final form by putting
\[ \tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) = -\frac{1}{6} \Box_1 \Box_2 \mathcal{F}(\Box_1, \Box_2, \Box_3) \]
where \( \mathcal{F}(\Box_1, \Box_2, \Box_3) \) is a new arbitrary function, and taking into account the symmetries of the tensor structures entering (A.34). The result is the following constraint between the basis invariants listed in the table (2.15)-(2.43):
\[
\int dx \, g^{1/2} \operatorname{tr} \mathcal{F}^{\text{sym}}(\Box_1, \Box_2, \Box_3) \times \left\{ -\frac{1}{48}(\Box_1^2 + \Box_2^2 + \Box_3^2) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(9) \\
- \frac{1}{12}(\Box_1^2 + \Box_2^2 + \Box_3^2 - 2\Box_1 \Box_2 - 2\Box_2 \Box_3 - 2\Box_1 \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(10) \\
- \frac{1}{8} \Box_3(\Box_1 + \Box_2 - \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(11) \\
+ \frac{1}{8}(3\Box_1 + \Box_2 + \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(22) - \frac{1}{2} \Box_3 \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(23) \\
- \frac{1}{2} \Box_1 \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(24) - \frac{1}{2}(\Box_2 + \Box_3 - \Box_1) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(25) \\
+ \frac{1}{2} \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(27) \\
+ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(28) \right\} + O[\mathcal{R}^4] = 0, \quad 2\omega = 4 \tag{A.35} \]
where \( \mathcal{F}^{\text{sym}}(\Box_1, \Box_2, \Box_3) \) is a completely symmetric but otherwise arbitrary function. This constraint, valid in four dimensions, reduces the basis of nonlocal gravitational invariants by one structure. With its aid one can exclude everywhere either the structure 9 or 10 or the completely symmetric (in the labels 1,2,3) part of anyone of the remaining purely gravitational structures except \( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(29) \). The latter structure which is the only one containing six derivatives is absent from the constraint (A.35) and is, therefore, inexcusable. This can be explained by the fact that its local version is the only independent contraction of three Weyl tensors: eq. (A.21) (with eq. (A.32) used).

As seen from eq. (A.35), elimination of any structure except 27 and 28 will result in the appearance of new nonanalytic terms in the \( \alpha \)-representation of the form factors which may complicate obtaining further representations like the Laplace and spectral ones. Therefore, in the text, the constraint (A.35) is used to eliminate the completely symmetric part of the structure 28.

At least apparently, eqs. (A.28) and (A.29) are not the most general nonlocal identities that can be written down by antisymmetrizing five indices. More generally, one can apply this procedure to three tensors with arbitrary indices and arbitrary number of uncontracted derivatives. Therefore, to make sure that there are no more constraints between the basis invariants, an independent check is needed. Since, at third order in the curvature, the maximum number of derivatives that do not contract in the box operators is six, we begin
this check with nonlocal structures having three Ricci tensors and six derivatives. There exist only two such, and only one of them is independent:

\[ \hat{\nabla}_\alpha \nabla_\beta R^{\gamma\delta}_1 \nabla_\gamma \nabla_\delta R_{2 \mu \nu} \nabla_\mu \nabla_\nu R_3^{\alpha\beta} = \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (29), \]  

\[ \hat{\nabla}_\alpha \nabla_\beta R^{\gamma\delta}_1 \nabla_\gamma \nabla_\delta R_{2 \mu \nu} \nabla_\mu \nabla_\nu R_3^{\beta\mu} = -\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (29) + \ldots \]  

(A.36)

(A.37)

where the ellipses \ldots stand for total derivatives and terms with derivatives contracting in the box operators. Eq. (A.37) is obtained by three integrations by parts applied to \( \nabla_\alpha \), \( \nabla_\mu \) and \( \nabla_\delta \). Since not more than one index of each Ricci tensor and not more than one derivative acting on each Ricci tensor may participate in the antisymmetrization (otherwise the result will be either trivial or \( \mathcal{O}[R^4] \)), there are only two possible 5-antisymmetrizations of (A.36):

\[ \nabla_\alpha \nabla_\beta R^{\gamma\delta}_1 \nabla_\gamma \nabla_\delta R_{2 \mu \nu} \nabla_\mu \nabla_\nu R_3^{\alpha\beta} = 0, \]

\[ \nabla_\alpha \nabla_\beta R^{\gamma\delta}_1 \nabla_\gamma \nabla_\delta R_{2 \mu \nu} \nabla_\mu \nabla_\nu R_3^{\beta\mu} = 0. \]

In each of these cases, upon calculation, the terms (A.36) and (A.37) appear in a sum with equal coefficients and, therefore, cancel. This proves that the structure with six derivatives remains unconstrained. Among the invariants with three Ricci tensors and four derivatives, only two are independent: the basis structures 27 and 28, and only the latter admits non-trivial 5-antisymmetrizations. There is, moreover, only one such:

\[ \nabla_\mu R_{1 \lambda}^{\alpha} \nabla^\nu R_{2 \beta}^{\lambda} \nabla_\alpha \nabla^\mu R_{3 \gamma}^{\beta} = 0. \]

Upon calculation and multiplication by an arbitrary form factor, the latter identity gives precisely the constraint (A.35). The invariants with the commutator curvature and four derivatives are reducible (see sect. 14) and, therefore, absent from the basis. Finally, invariants with two derivatives do not admit a nontrivial 5-antisymmetrization since, for that, one needs at least ten indices: five uncontracted to be involved in the antisymmetrization, and five more to make a complete contraction.

Thus, in four dimensions, there is only one constraint between the basis structures, and the dimension of the basis of nonlocal cubic invariants which is generally 29 and in the case of the gravitational invariants 10 becomes respectively 28 and 9.

The nonlocal identity obtained above has a direct relation to the Gauss-Bonnet identity in four dimensions. Indeed, by calculating the square of the Riemann tensor with the aid of eq. (A.32), one finds for arbitrary \( 2\omega \):

\[ \int dx \ g^{1/2} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \]

\[ = \int dx \ g^{1/2} \left[ \frac{1}{2} \square_1 \ R_1 \ R_2 \ R_3 \right. \]

\[ + 2 \left( \frac{1}{\square_2} \ - \frac{1}{\square_1} \right) R_{1 \alpha}^{\alpha} \ R_{2 \beta}^{\beta} \ R_{3 \mu}^{\mu} \]

\[ + \left( \frac{1}{\square_1} \ - \frac{1}{\square_2} \right) R_{1 \mu}^{\mu} \ R_{2 \mu}^{\mu} \ R_3 \]

\[ + \left( -2 \frac{1}{\square_1} \ -3 \frac{1}{\square_2} \right) R_{1 \alpha}^{\alpha} \ R_{2 \beta}^{\beta} \ R_3 \]

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\[ + 4 \frac{1}{\Box_1 \Box_2} \nabla^\mu R_{1}^{\alpha \nu} \nabla_\nu R_{2 \mu \alpha} R_{3} \\
\[ + 4 \frac{1}{\Box_2 \Box_3} R_{1}^{\mu \nu} \nabla_\mu R_{2}^{\alpha \beta} \nabla_\nu R_{3}^{\alpha \beta} \\
\[ + 4 \left( 2 \frac{1}{\Box_1 \Box_2} - \frac{1}{\Box_2 \Box_3} \right) R_{1}^{\mu \nu} \nabla_\alpha R_{2 \beta \mu} \nabla_\beta R_{3}^{\alpha} \\
\[ - 4 \frac{1}{\Box_1 \Box_2 \Box_3} \nabla_\alpha \nabla_\beta R_{1}^{\mu \nu} \nabla_\mu \nabla_\nu R_{2}^{\alpha \beta} R_{3} \\
\[ - 8 \frac{1}{\Box_1 \Box_2 \Box_3} \nabla_\mu R_{1}^{\alpha \lambda} \nabla_\nu R_{2 \lambda \nu} \nabla_\alpha \nabla_\beta R_{3}^{\mu \nu} \] + O[R^4]. \tag{A.38} \]

In agreement with the result of paper II, a contribution of second order in the curvature is absent from this expression for any space-time dimension. The third-order contribution (A.38) does not generally vanish but vanishes in four dimensions because it coincides with the left-hand side of the identity (A.35) if in the latter one puts

\[ J^{\text{sym}}(\Box_1, \Box_2, \Box_3) = -8(\text{tr} \hat{\nabla})^{-1} \frac{1}{\Box_1 \Box_2 \Box_3}, \]

Comparison of eqs. (A.38) and (A.34) gives

\[ \int dx \, g^{1/2} \left( R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} - 4 R_{\mu \nu} R^{\mu \nu} + R^2 \right) \\
\[ = \int dx \, g^{1/2} \left( -10 \frac{1}{\Box_3} \right) R_{1 \alpha \beta \gamma \delta} R_{2 \gamma \delta \alpha \beta} R_{3 \mu} + O[R^4]. \tag{A.39} \]

This relation valid for any number of space-time dimensions elucidates the mechanism by which the Gauss-Bonnet identity arises in four dimensions.

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References

[1] A.O.Barvinsky and G.A.Vilkovisky, Nucl. Phys. B282 (1987) 163
[2] A.O.Barvinsky and G.A.Vilkovisky, Nucl. Phys. B333 (1990) 471
[3] A.O.Barvinsky and G.A.Vilkovisky, Nucl. Phys. B333 (1990) 512
[4] G.A.Vilkovisky, Class. Quantum Grav. 9 (1992) 895

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[5] G.A.Vilkovisky, Preprint CERN-TH.6392/92; Publication de l’Institut de Recherche Mathématique Avancée, R.C.P. 25, vol. 43 (Strasbourg, 1992) p.203

[6] B.S.DeWitt, Dynamical theory of groups and fields (Gordon and Breach, New York, 1965)

[7] A.O.Barvinsky and G.A.Vilkovisky, Phys. Reports 119 (1985) 1

[8] A.O.Barvinsky and G.A.Vilkovisky, Quantum field theory and quantum statistics, vol. 1, eds. I.A.Batalin, C.J.Isham and G.A.Vilkovisky (Hilger, Bristol, 1987) p.245

[9] E.S.Fradkin and G.A.Vilkovisky, Phys. Rev. D8 (1973) 4241

[10] E.S.Fradkin and G.A.Vilkovisky, Lett. Nuovo Cim. 19 (1977) 47; Report of the Institute for Theoretical Physics at Bern (1976)

[11] G.A.Vilkovisky, Report of the Naples University DSF-T-92/19 (1992)

[12] G. ‘t Hooft, Nucl. Phys. B62 (1973) 444

[13] G. ‘t Hooft and M.Veltman, Ann. Inst. Henri Poincaré XX (1974) 69

[14] P.B.Gilkey, J. Diff. Geom. 10 (1975) 601

[15] I.G.Avramidi, Phys. Lett. B238 (1990) 92; Nucl. Phys. B355 (1991) 712

[16] A.M.Polyakov, Phys. Lett. B103 (1981) 207

[17] V.P.Frolov and G.A.Vilkovisky, Proc. second seminar on quantum gravity, 1981, Moscow, eds. M.A.Markov and P.C.West (Plenum, London, 1983) p.267

[18] S.Deser. M.J.Duff and C.J.Isham, Nucl. Phys. B114 (1976) 29

[19] M.J.Duff, Nucl. Phys. B125 (1977) 334

[20] L.S.Brown, Phys. Rev. D15 (1977) 1469

[21] L.S.Brown and J.P.Cassidy, Phys. Rev. D15 (1977) 2810

[22] E.S.Fradkin and G.A.Vilkovisky, Phys. Lett. B73 (1978) 209

[23] N.Nakanishi, Prog. Theor. Phys. 24 (1960) 1275

[24] G.N.Watson, Theory of Bessel functions (Cambridge, 1922)

[25] V.P.Frolov and A.I.Zel’nikov, Phys. Rev. D29 (1984) 1057

[26] S.A.Fulling, R.C.King, B.G. Wybourne and C.J.Cummins, Class. Quantum Grav. 9 (1992) 1151