Iterated integrals and Borwein-Chen-Dilcher polynomials

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Abstract

We study the zero location and the asymptotic behavior of iterated integrals of polynomials. Borwein-Chen-Dilcher’s polynomials play an important role in this issue. For these polynomials we find their strong asymptotics and give the limit measure of their zero distribution. We apply these results to describe the zero asymptotic distribution of iterated integrals of ultraspherical polynomials with parameters $(2\alpha + 1)/2$, $\alpha \in \mathbb{Z}_+$. 

1 Introduction and main results

Several problems require the location of the zeros of a polynomial in areas such as numerical analysis, approximation theory, differential equations, and complex dynamics. The zeros of a polynomial can represent equilibrium points in a certain force field, geometric points of certain curves, critical points and so on (see [12,13,14,15], and the references therein). The objective of this paper is the study of some algebraic and asymptotic properties of the zeros of iterated integrals of polynomials.

Given a monic polynomial $p_n$ of degree $n$, $\lambda \in \mathbb{C}$, and $m \in \mathbb{Z}_+$, its $m$−fold integral

$$ I_{m,\lambda}(p_n)(z) := \frac{(n+m)!}{n!} \int_\lambda^z \int_\lambda^{t_{m-1}} \cdots \int_\lambda^{t_1} p_n(t_0) \, dt_0 \cdots dt_{m-2} \, dt_{m-1} $$

(1)

defines a monic polynomial of degree $n + m$ for which the derivatives of order $j$, $0 \leq j \leq m - 1$, at $\lambda$ are zero and

$$ (I_{m,\lambda}(p_n))^{(m)}(z) = \frac{(n+m)!}{n!} p_n(z), $$

(2)

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where \( m \in \mathbb{Z}_{+} \) and \( \lambda \in \mathbb{C} \). Of course, \( I_{0,\lambda}(p_{n}) := p_{n} \). When \( \lambda = 0 \), for simplicity of notation, let \( I_{m}(p_{n}) := I_{m,0}(p_{n}) \). The interchange of the order of integration or integration by parts yields

\[
I_{m,\lambda}(p_{n})(z) = \frac{(n+m)!}{(m-1)!n!} \int_{\lambda}^{z} (z-s)^{m-1}p_{n}(s) \, ds
\]

\[
= \frac{(n+m)!}{(m-1)!n!} \int_{\lambda/z}^{1} (1-t)^{m-1}p_{n}(zt) \, dt.
\]

Let \( Q_{n,m} \) be the polynomials of degree \( n \) given by

\[
Q_{n,m}(z) := \frac{(n+m)!}{n!} \sum_{k=0}^{n} \frac{n!}{k!(m+k)!} z^{k} = \frac{n!}{m+1} \sum_{k=0}^{m} \frac{n+(m+k)}{m+k} z^{k},
\]

\[
\text{(3)}
\]

\[
= \frac{(n+m)!}{(m-1)!n!} \int_{1}^{1} (1-t)^{m-1}(1+zt)^{n} \, dt = \frac{(n+m)!}{n!z^{m}} I_{m}(1+z)^{n}.
\]

\[
\text{(4)}
\]

Borwein, Chen and Dilcher (4, Th. 1) prove that the zeros of \( Q_{n,n+1} \) are dense in the curve

\[
\Gamma := \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^{2}}{4z} \right| = 1 \text{ and } |z| \geq 1 \right\},
\]

and these are the only limit points of the zeros. Using this result, they give estimations for the radius of a disc containing the zeros of \( I_{n}(p_{n}) \) for every \( n \). Moreover, they also obtain the curve to which the zeros of the \( n \)-fold integral of the \( n \)-th Legendre polynomial converge, as \( n \) goes to infinity. We call \( Q_{n,m} \) Borwein-Chen-Dilcher polynomials.

We give strong asymptotics for \( Q_{n,n} \). This allows us to characterize the measure which describes their zero distribution. The steepest descent method is used to obtain this estimation. For this description, let us consider two regions induced by \( \Gamma \):

\[
\mathcal{E}_{1} := \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^{2}}{4z} \right| > 1, |z| > 1 \right\},
\]

\[
\mathcal{E}_{2} := \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^{2}}{4z} \right| < 1 \right\} \bigcup \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^{2}}{4z} \right| > 1, |z| < 1 \right\}.
\]

**Theorem 1.** We have

\[
Q_{n,n}(z) = \begin{cases} 
\frac{(z+1)^{2n}}{z^{n}} (1 + o(1)) & \text{if } z \in \mathcal{E}_{1}, \\
\frac{2^{2n}}{\pi n(1-z)} (1 + o(1)) & \text{if } z \in \mathcal{E}_{2}, 
\end{cases}
\]

uniformly, as \( n \to \infty \), on compact subsets of each stated domain\(^1\)

A consequence of the above result is the zero distribution of \( Q_{n,n} \). Define

\[
\nu[Q_{n,n}] := \frac{1}{n} \sum_{Q_{n,n}(z) = 0} \delta_{z},
\]

\(^{1}\text{Here we state results for } m = n \text{ but analogous statements hold for } m = n + j \text{ with } j \text{ a fixed integer.}\)
its weak-* limit is given in terms of the equilibrium measure of $\Gamma$. For each Borel set $B \subset \mathbb{C}$, $\mu_\Gamma(B) = m(\varphi(B \cap \Gamma))$, where $dm = d\theta / (2\pi)$, the normalized arc-length on $\partial D_1$, and $\varphi(z) = \frac{(z+1)^2}{4\pi}$. 

**Corollary 1.** It holds $\lim_{n \to \infty} m \left[ Q_{n,n} \right] = \mu_\Gamma$, where $\mu_\Gamma$ is the equilibrium measure on $\Gamma$. The support of $\mu_\Gamma$ is $\Gamma$. The measure $\mu_\Gamma$ is the pre-image of the normalized arc-length on $\partial D_1$ under the mapping $\varphi(z) = \frac{(z+1)^2}{4\pi}$ from $\Gamma$ to $\partial D_1$. We have $\text{cap} (\Gamma) = 4$. Moreover, for $n$ large enough and all $m \in \mathbb{Z}_+$ the zeros of $Q_{n,m}$ are inside $\Gamma$.

Another consequence of Theorem 1 is the strong asymptotics and zeros distribution of iterated integrals of ultraspherical polynomials. Let $\alpha \in \mathbb{Z}_+$ be a positive integer and let $\hat{P}_n^{(\alpha+1/2)}$ be monic ultraspherical polynomials with parameter $\alpha + 1/2$. The polynomials $\hat{P}_n^{(1/2)}$ are monic Legendre polynomials.

**Theorem 2.** Let $\alpha \in \mathbb{Z}_+$ be a natural number. It holds

$$
I_{2n} \left( \hat{P}_n^{(\alpha+1/2)} \right) (z) = \begin{cases} 
(\zeta^2 - 1)^{2n-\alpha}\zeta^{2\alpha}(1+o(1)) & \text{if } z \in \mathbb{F}_1, \\
(-1)^\alpha\zeta^{2n}\zeta^{\alpha} + (1+o(1)) & \text{if } z \in \mathbb{F}_2, 
\end{cases}
$$

$$
I_{2n+1} \left( \hat{P}_n^{(\alpha+1/2)} \right) (z) = \begin{cases} 
(\zeta^2 - 1)^{2n+1-\alpha}\zeta^{2\alpha}(1+o(1)) & \text{if } z \in \mathbb{F}_1, \\
(-1)^\alpha\zeta^{2n+1}\zeta^{\alpha} + (1+o(1)) & \text{if } z \in \mathbb{F}_2, 
\end{cases}
$$

as $n \to \infty$, where $\mathbb{F}_j$ are the pre-image of $\mathcal{D}_j$ under the transformation $\varphi_1(z) = -\zeta^2$. Moreover, its zero distribution $\beta$ is supported on $\varphi_1(\Gamma)$ and for each Borel set $E$ this measure satisfies

$$
\beta(E) = \mu_\Gamma(\varphi_1(E)).
$$

Ultimately, the behavior of $Q_{n,n}$ is the same as that obtained by integrating polynomials a number of times which does not change with the integrand (see Section 2). In order to set our result in this context, let us fix some notations. Let $A$ be a Jordan rectifiable arc in $\mathbb{C}$ and $\Omega := \mathbb{C} \setminus A$. Given $r \in (0, \infty)$, $D_r := \{z \in \mathbb{C} : |z| < r\}$, $\partial D_r := \{z \in \mathbb{C} : |z| = r\}$, $\partial D_r^* := \{z \in \mathbb{C} : |z| > r\}$, and $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$. Let $\tau$ denote the conformal mapping of $\Omega$ onto $D_1$ such that $\tau'(\infty) := \text{cap}(A) := \lim_{z \to \infty} \frac{\tau(z)}{z}$ and $\text{lim}_{z \to \zeta} |\tau(z)| = 1, \zeta \in \Lambda$. If $\hat{\lambda} \in \Omega$, let $\Lambda_{\hat{\lambda}} := \{z \in \Omega : |\tau(z)| < |\tau(\hat{\lambda})|\}$, $\partial \Lambda_{\hat{\lambda}} := \{z \in \Omega : |\tau(z)| = |\tau(\hat{\lambda})|\}$, $\bar{\Lambda}_{\hat{\lambda}} := \{z \in \Omega : |\tau(z)| > |\tau(\hat{\lambda})|\}$, when $\hat{\lambda} \in A$, we consider $\partial \Lambda_{\hat{\lambda}} := A$ and $\bar{\Lambda}_{\hat{\lambda}} := \Omega$. Let $(\phi_n)$ denote a sequence of monic polynomials such that $\deg \phi_n = n$ for all $n$ and

$$
\lim_{n \to \infty} \frac{\phi_n(z)}{\text{cap}(A)^{1/n} \tau^n(z)} = \mathfrak{F}(z),
$$

uniformly on compact subsets of $\Omega$, in which the analytic function $\mathfrak{F}$ has no zeros. There are several sequences of monic polynomials which satisfy condition (6) such as extremal polynomials with

\[\text{The zeros of } I_n(\hat{P}_n) \text{ different from 0.}\]
respect to a measure whose weight satisfies the Szegő condition (see \cite{17}). If \((\phi_n)\) is a sequence of monic orthogonal polynomials with respect to a measure \(\mu\), \((I_{m,\lambda}(\phi_n))_{n \in \mathbb{Z}^+}\) is a sequence of polynomials orthogonal in a non-standard sense, i.e., they satisfy Sobolev-type orthogonality with respect to the inner product
\[
\langle f, g \rangle := \sum_{k=0}^{m} \int f^{(k)}(k) g^{(k)}(k) \, d\mu_k,
\]
where \(\mu_k = \delta_{\lambda}, 0 \leq k \leq m - 1\), and \(\mu_m = \mu\). There are several papers on this issue (e.g., \cite{1, 2, 8, 9, 12, 14}). These polynomials are useful in Fourier analysis (\cite{5, 14}), numerical analysis (\cite{6}), and so on (see \cite{9} and references therein).

**Theorem 3.** Let \((\phi_n)\) be a sequence of monic polynomials which satisfies \((6)\).  

(i) If \(\lambda \in A\), then
\[
\lim_{n \to \infty} \frac{\phi_n(z)}{I_{m,\lambda}(\phi_n)(z)} = \psi^m(z),
\]
uniformly on compact subsets of \(\Omega\), where the function \(\psi(z) := \frac{\tau'(z)}{\tau(z)}\). Also, we have
\[
\text{w-lim}_{n \to \infty} \nu^*[I_{m,\lambda}(\phi_n)] = \mu_A,
\]
where \(\mu_A\) is the equilibrium measure on the arc \(A\) and
\[
\nu^*[I_{m,\lambda}(\phi_n)] = \sum_{I_{m,\lambda}(\phi_n)(\xi) = 0, \xi \neq 0} \delta_{\xi}.
\]

(ii) If \(\lambda \in \Omega\), then
\[
\lim_{n \to \infty} \frac{\phi_n(z)}{I_{m,\lambda}(\phi_n)(z)} = -\frac{(m - 1)! \psi^m(z)}{(z - \lambda)^{m-1}}
\]
uniformly on compact subsets of \(\Lambda_{\lambda}\) and
\[
\lim_{n \to \infty} \frac{\phi_n(z)}{I_{m,\lambda}(\phi_n)(z)} = \psi^m(z),
\]
uniformly on compact subsets in \(\mathcal{N}_{\lambda}\). Moreover, \(\text{w-lim}_{n \to \infty} \nu^*[I_{m,\lambda}(\phi_n)] = \mu_{\Lambda_{\lambda}}\), where \(\mu_{\Lambda_{\lambda}}\) is the equilibrium measure on the arc \(\partial \Lambda_{\lambda}\).

Section 5 is devoted to the proof of the above theorem. Theorem 2 is proved in Section 3. In Section 4 we state some results about the location of the zeros of iterated integrals of polynomials and the next section includes the proof of Theorem 1 as well as some of its consequences.
2 Proof of Theorem 1 and some consequences

Let \( P_{n,m}(z) := \int_0^1 (1-t)^{m-1}(1+zt)^n \, dt \). The relations in (5) are equivalent to prove that

\[
P_{n,n}(z) = \begin{cases} \frac{\sqrt{n}}{2n} \frac{(z+1)^{2n}}{z^n}(1+o(1)) & \text{if } z \in \partial_1, \\ \frac{1}{n(1-z)}(1+o(1)) & \text{if } z \in \partial_2, \end{cases}
\]

uniformly, as \( n \to \infty \), on compact subsets of each stated domain.

Changing \( t = \frac{1}{2}(1 - \frac{1}{z}) + \frac{1}{2}(1 + \frac{1}{z})x \), a straightforward computation gives us

\[
P_{n,n}(z) := \int_0^1 (1-t)^{n-1}(1+zt)^n \, dt = \frac{(z+1)^{2n}}{2^{2n}n^2} \int_w^1 (1-x^2)^{n-1}(1+x) \, dx,
\]

where \( w = \frac{1-z}{1+z} \). Observe that the bi-linear function \( w = \frac{1-z}{1+z} \) transforms \( \overline{D}_1 \) onto \( \mathcal{R}(w) < 0 \) and \( 1-w^2 = \frac{4z}{(1+z)^2} \). Define

\[
G_n(w) := \int_w^1 [(1-x^2)^n \, dx.
\]

We have \( \int_w^1 (1-x^2)^{n-1}(1+x) \, dx = G_{n-1}(w) + \frac{(1-w^2)^n}{2n} \), so, the theorem is equivalent to check:

(i) Uniformly on compact subsets of \( \mathcal{G}_2 := \{ w \in \mathbb{C} : |1-w^2| > 1 \} \cup \{ w \in \mathbb{C} : |1-w^2| < 1 \text{ and } \mathcal{R}(w) > 0 \} \),

\[
G_n(w) = \frac{(1-w^2)^{n+1}}{2n} (1+o(1)).
\]

(ii) Uniformly on compact subsets of \( \mathcal{G}_1 := \{ w \in \mathbb{C} : |1-w^2| < 1 \text{ and } \mathcal{R}(w) < 0 \} \),

\[
G_n(w) = \frac{\sqrt{\pi}}{\sqrt{n}} (1+o(1)).
\]

The integral (8) does not depend on the curve of integration. Thus, we deform the integration interval \([w,1]\) as we need in each case.

First, we obtain (9). Let \( K \subset (\mathcal{G}_2 \cap \{ w \in \mathbb{C} : \mathcal{R}(w) \geq 0 \}) \) be a compact set and \( w \in K \). From the change of variable \( t \frac{2}{n} = x^2 - w^2 \) and the dominated converge theorem, we get

\[
\int_w^1 (1-x^2)^n \, dx = \frac{(1-w^2)^{n+1}}{n} \int_0^{\sqrt{n}} \frac{(t - \frac{1}{n})^n}{\sqrt{(1-t^2)^2 + w^2}} t \, dt = \frac{(1-w^2)^{n+1}}{2nw} (1+o(1)).
\]

If \( K \subset (\mathcal{G}_2 \cap \{ w \in \mathbb{C} : \mathcal{R}(w) \leq 0 \}) \) is a compact set and \( w \in K \), the proof is reduced to the former case because \( G_n(w) = -G_n(-w) + 2G_n(0) \), and

\[
G_n(0) = \frac{\Gamma(n+1)\Gamma(1/2)}{2\Gamma(n+3/2)} = \frac{\sqrt{\pi}}{2\sqrt{n}} (1+o(1)).
\]
Second, we get (10). Let $K \subset \mathcal{G}_1$ and $w \in K$; we can choose $w_1 \in (-1,0)$ such that $K \subset \{ u : \Re(u) < w_1 \}$. Then

$$G_n(w) = \int_w^{w_1} (1-x^2)^n \, dx + \int_0^0 (1-x^2)^n \, dx + G_n(0) = 2G_n(0)(1+o(1)).$$

□

The following result is a consequence of Theorem 1.

**Corollary 2.** We have

$$\lim_{n \to \infty} |Q_{n,n}(z)|^{1/n} = \left| \frac{(z+1)^2}{2z} \right|, \quad z \in \mathcal{E}_1,$$

$$\lim_{n \to \infty} |Q_{n,n}(z)|^{1/n} = 4, \quad z \in \mathcal{E}_2,$$

uniformly on compact subsets of each mentioned region. Moreover,

$$\lim_{n \to \infty} \left( \max_{z \in \Gamma} |Q_{n,n}(z)| \right)^{1/n} = 4. \quad (12)$$

**Proof.** We will only check (12). From (7), it is equivalent to

$$\lim_{n \to \infty} \left( \max_{w \in \partial \mathcal{G}_1} |G_n(w)| \right)^{1/n} = 1.$$

By the maximum principle, $G_n(0) \leq \max_{w \in \partial \mathcal{G}_1} |G_n(w)| \leq 1$. From (11) the proof is concluded straightforward.

□

**Remark 1.** If $z \in \mathcal{E}_1 := \{ z \in \mathbb{C} : \left| \frac{(z+1)^2}{4z} \right| \geq 1, |z| \geq 1 \}$, then there exists $K_1(z) > 0$ ($K_1(z)$ only depends on $z$) such that

$$\frac{K_1(z)}{\sqrt{n}} \leq |Q_{n,n}(z)|,$$

for $n$ large enough. In fact, from (3) and (7), the above relation is equivalent to

$$\frac{K_2(w)}{\sqrt{n}} \leq |G_n(w)|$$

for $w \in \mathcal{G}_1 := \{ w \in \mathbb{C} : |1-w^2| \leq 1 \text{ and } \Re(w) \leq 0 \}$, where $K_2(z)$ is a positive function. The case $w = 0$ is straightforward. If $w \neq 0$, $G_n(w) = 2G_n(0) - \int_{-w}^1 (1-x^2)^n \, dx$ and there exists $C(w) > 0$ such that

$$\left| \int_{-w}^1 (1-x^2)^n \, dx \right| \leq \frac{C(w)}{n}.$$

Actually, if $z$ is outside a disc with center at 1 ($w$ is outside of a disc with center at 0) with small radius, then we can take the positive constant $C(w)$ independent of $w$.

With the same arguments we obtain

$$Q_{n,n+1}(z) = \begin{cases} \frac{(z+1)^2}{2z} \frac{1}{\sqrt{n}} (1+o(1)) & \text{if } z \in \mathcal{E}_1, \\ \frac{2^{n+1}}{\sqrt{\pi n (1-z^2)}} (1+o(1)) & \text{if } z \in \mathcal{E}_2, \end{cases} \quad (13)$$

as $n \to \infty$ uniformly on compact subsets of the mentioned regions.
2.1 Proof of Corollary 1

Given that \( w = \frac{(z+1)^2}{z} \) is a conformal transform from \( \mathcal{D}_1 \) onto \( \overline{D}_4 \), and \( \text{cap}(D_4) = 4 \) (see [11 Th. 5.2.3]), we have \( \text{cap}(\Gamma) = 4 \). Moreover, by the subordination principle ([11 Th. 4.3.8], for each Borel set \( B \subset C \), \( \mu_r(B) = m(\phi(B \cap \Gamma)) \), where \( dm = d\theta/(2\pi) \) on \( \partial D_1 \).

Let \( \mu \) be a weak-* limit of \( v[Q_{n,n}] := \frac{1}{n} \Sigma_{z \in Q_{n,n}(z) = 0} \delta_z \), i.e. there exists a subsequence \((v[Q_{n_k,n_k}])\) such that \( \lim_{k \to \infty} \int f(\zeta) dV(Q_{n_k,n_k})(\zeta) = \int f(\zeta) d\mu(\zeta) \) for all continuous function \( f \) in \( C \) with compact support. To simplify notation, we write \( n \) instead of \( n_k \). By Theorem 1, \( \text{supp}(\mu) \subset \Gamma \). It is well known that

\[
\lim_{n \to \infty} V(v[Q_{n,n}], z) = V(\mu, z)
\]

uniformly on compact subsets of \( C \setminus \Gamma \). Then, by (3), and Corollary 2, we obtain

\[
V(v[Q_{n,n}], z) = \frac{1}{n} \log \left( \frac{(n-1)\Gamma n!}{(2n)!} \right) + V(v[P_n], z)
\]

and

\[
V(\mu, z) = \log \left( \frac{1}{\pi} \right) + \begin{cases} 
0 & \text{if } z \in \mathcal{D}_2, \\
\log \left| \frac{4z}{(z+1)^2} \right| & \text{if } z \in \mathcal{D}_1.
\end{cases}
\]

Then, from the identity principle for harmonic functions (see [11 Th. 1.1.7]) we get \( \text{supp}(\mu_\Gamma) = \Gamma \) and from the principle of descent, if \( z \in \Gamma \) and \( (z_m) \subset \mathcal{D}_2 \) such that \( \lim_{m \to \infty} z_m = z \),

\[
V(\mu, z) \leq \liminf_{n \to \infty} V(v[Q_{n,n}], z_n) = \log \left( \frac{1}{\pi} \right).
\]

Since \( \mu \) is a probability measure, \( I(\mu) = \int V(\mu, z) d\mu(z) \leq \log(\frac{1}{\pi}) \), but \( \mu_\Gamma \) is the unique measure which minimizes the energy between the probability measures with support on \( \Gamma \), and \( I(\mu_\Gamma) = \log \frac{1}{\text{cap}(\Gamma)} = \log(\frac{1}{\pi}) \). Therefore, \( \mu = \mu_\Gamma \), w-lim \( v[Q_{n,n}] = \mu_\Gamma \). \( \square \)

In Figures 1 we can see the zeros of \( Q_{n,m} \) for several values of \( n \) and \( m \).

3 Asymptotic analysis for the integral of ultraspherical polynomials

Proof of Theorem 2

We shall do induction on the parameter \( \alpha \). For \( \alpha = 0 \), we have Legendre polynomials. By Rodrigues’ formula (see [16 (4.3.1)]) the monic Legendre polynomials \( \tilde{P}_{n}^{(1/2)} \) are given by

\[
\tilde{P}_{n}^{(1/2)}(z) := (-1)^n \frac{n!}{(2n)!} \partial_z^n \left( (1 - z^2)^n \right).
\]

\(^3\)Hereafter, if \( \nu \) is a positive Borel measure with compact support in the complex plane, its logarithm potential and energy are respectively \( V(\nu, z) := \int \frac{1}{|z-x|} d\nu(x) \) and \( I(\nu) := \int V(\nu, z) d\nu(z) \).
Figure 1: a) Zeros of $Q_{n,n}$ for $n = 20, \ldots, 70$. b) Zeros of $Q_{100,100}$. c) Zeros of $Q_{n,m}$ for $n = 70$ and $m = 20, \ldots, 70$. d) Zeros of $Q_{n,m}$ for $n = 100$ and $m = 20, \ldots, 70$. In all cases the curve is $\left| \frac{(z+1)^2}{4z} \right| = 1$.

So, $I_{2n}(\hat{P}_{2n}^{(1/2)})(z) = (-z^2)^n Q_{n,n}(-z^2)$ and $I_{2n+1}(\hat{P}_{2n+1}^{(1/2)})(z) = -(-z^2)^{n+1} Q_{n,n+1}(-z^2)$.

From (13), Theorem 1 and Corollary 1, we obtain

$$I_{2n}(\hat{P}_{2n}^{(1/2)})(z) = \begin{cases} (z^2 - 1)^{2n}(1 + o(1)) & \text{if } z \in \mathcal{F}_1; \\ \frac{(-1)^n z^{2n}}{\sqrt{4n}} \frac{z}{1+z^2}(1 + o(1)) & \text{if } z \in \mathcal{F}_2, \end{cases}$$

where

$$\mathcal{F}_1 := \{z \in \mathbb{C} : \frac{z^2 - 1}{2z^2} > 1, |z| > 1\}, \quad \mathcal{F}_2 := \{z \in \mathbb{C} : \frac{z^2 - 1}{2z^2} < 1\} \cup \{z \in \mathbb{C} : \frac{z^2 - 1}{2z^2} > 1, |z| < 1\}.$$  

Observe that the coefficient of $z^{2n}$ in $I_{2n}(\hat{P}_{2n}^{(1/2)})$ is $\frac{p_{2n}^{(1/2)(0)}}{\kappa_{2n}} \sim \frac{(-1)^n z^{2n}}{\sqrt{4n}}$ as $n \to \infty$, where $\kappa_{2n}$ is the leading coefficient of $P_{2n}^{(1/2)}$. 

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uniformly on compact subsets of $F$.

Moreover, it holds
\[
\text{as } h \to \infty, \text{ where } F_j \text{ are the pre-image of } D_j \text{ under the transformation } q_1(z) = -z^2. \text{ Moreover, its zero distribution } \nu \text{ is supported on } q_1(\Gamma) \text{ and for each Borel set } E \text{ this measure satisfies } \nu(E) = \mu_{\Gamma}(q_1(E)).
\]

Next, we assume that the statement holds for $\alpha$. According to [16, (4.21.7)], we have
\[
\partial_z \left( N_n^{(k+1/2)}(z) \right) = \frac{1}{2} (n + 2\alpha + 1) P_{n-1}^{(k+3/2)}(z),
\]
so
\[
I_{n-1} \left( P_{n-1}^{(k+3/2)}(z) \right) = I_{n-1} \left( P_{n-1}^{(k+1/2)}(z) \right) - \frac{1}{2n(2n-1)} \partial_z^2 \left( I_n \left( P_n^{(k+1/2)}(z) \right) \right).
\]

Moreover, it holds
\[
I_n \left( P_n^{(k+1/2)}(z) \right) = \frac{1}{2n(2n-1)} \partial_z^2 \left( I_n \left( P_n^{(k+1/2)}(z) \right) \right).
\]

Next, we consider $n = 2k + 2$ even. By the Cauchy integral formula for the derivative and induction hypothesis, we get
\[
I_{2k} \left( P_{2k}^{(k+1/2)}(z) \right) = \frac{1}{(4k+2)(4k+1)} \partial_z^2 \left( I_{2k} \left( P_{2k}^{(k+1/2)}(z) \right) \right)
\]
uniformly on compact subsets of $F_1$. Observe that on $F_1 \left| \frac{z^2 - 1}{2z} \right| > 1$, and by [16, (4.7.31)]
\[
P_{2k}^{(k+1/2)}(0) = \frac{(-1)^{k+1} \Gamma(k + \alpha + 3/2) \Gamma(2k + 3)}{2^{2k+2} \Gamma(k+1)}
\]
Moreover, by Stirling’s formula
\[
\frac{\Gamma(k + \alpha + 3/2)}{\Gamma(k + 2)} = k^{\alpha-1/2}(1 + o(1)), \quad \frac{(4k+2)!}{(2k)!} = \frac{2^{4k+2}}{\sqrt{2\pi}k}(1 + o(1)).
\]
Thus,
\[
I_{2k} \left( P_{2k}^{(k+1/2)}(z) \right) - \frac{\Gamma(k + \alpha + 3/2) \Gamma(2k + 3)}{2^{2k+2} \Gamma(k+1)} = I_{2k} \left( P_{2k}^{(k+1/2)}(z) \right) (1 + o(1))
\]
\[
= (z^2 - 1)^{2k+2-\alpha} z^2 \alpha (1 + o(1)).
\]

\footnote{The zeros of $I_n \left( P_n^{(k+1/2)}(z) \right)$ different from 0.}
\footnote{Remember that $I_n(z)$ are monic polynomials and $\partial_z^j (I_n(z)) = 0$, $j = 0, 1, \ldots, m - 1$. The factor $\frac{(2n-2)}{(n-2)m!}$ is to guarantee that $\partial_z^{m-2} \left( I_{n-1} \left( P_{n-1}^{(k+3/2)}(z) \right) \right) = 0.$}
uniformly on compact subsets of $\mathcal{F}_1$. Therefore,
\[
I_{2k+1}\left(\hat{P}_{2k+1}^{(\alpha+3/2)}(z)\right) = (z^2 - 1)^{2k+1-(\alpha+1)}z^{2(\alpha+1)}(1 + o(1)),
\]
uniformly on compact subsets of $\mathcal{F}_1$.
Consider $z \in \mathcal{F}_2$, we have
\[
\frac{1}{(4k+4)(4k+3)}\partial_z^2(I_{2k+2}\left(\hat{P}_{2k+2}^{(\alpha+1/2)}(z)\right)) = \frac{(-1)^{k+1}2^{2k+2-\alpha}z^{2k}}{2^2\sqrt{\pi k}(1+z^2)}(1 + o(1)).
\]
In fact, it is immediately checked that
\[
\partial_z^2\left(\frac{(-1)^{k+1}2^{2k+2-\alpha}z^{2k}}{\sqrt{\pi k}(1+z^2)}\right) = \frac{(-1)^{k+1}2^{2k+2-\alpha}}{\sqrt{\pi k}}\partial_z^2\left(\frac{z^{2k+2}}{1+z^2}\right) = \frac{(-1)^{k+1}2^{2k+2-\alpha}}{\sqrt{\pi k}} \cdot \frac{2z^{2n}(1 - 3z^2 + n(1 + z^2)(3 + 2n + (-1 + 2n)z^2))}{(1 + z^2)^3},
\]
and $\frac{z^{2k}}{1+z^2} - z^{2k} = \frac{-z^{2k+2}}{1+z^2}$. Therefore,
\[
I_{2k+1}\left(\hat{P}_{2k+1}^{(\alpha+3/2)}(z)\right) = \frac{(-1)^{k+1}2^{2k+1-(\alpha+1)}}{\sqrt{\pi k}} \cdot \frac{z^{2k+2}}{1+z^2}(1 + o(1))
\]
uniformly on compact subsets of $\mathcal{F}_2$.
If $n$ is odd, then $\hat{P}_n^{(\alpha+1/2)}(0) = 0$,
\[
I_{2n}\left(\hat{P}_{2n}^{(\alpha+3/2)}(z)\right) = \frac{1}{(4n+2)(4n+1)}\partial_z^2\left(I_{2n+1}\left(\hat{P}_{2n+1}^{(\alpha+1/2)}(z)\right)\right),
\]
and the proof is concluded as the former case.

The conclusion about the limiting distribution of the zero of $I_n\left(\hat{P}_n^{(\alpha+1/2)}(z)\right)$ follows straightforward from their asymptotic behavior and the unicity theorem for potentials (see $[13, Theorem 2.1, p. 97]$).

In Figures 2 and 3 we can see the zeros of iterated integral of ultraspherical polynomials, in particular, of Legendre polynomials, for different values of $n$.

## 4 Zeros of polynomials with critical points on a disc

It is clear that if we know the location of the critical points of a polynomial and one of its zeros, the remaining zeros are uniquely determined. Nonetheless, there are only a few general results about the zero locations of polynomials in terms of their critical points and a given zero, most of them contained in $[10, \S 4.5]$ and $[4]$.
Figure 2: Zeros of the iterated integral of Legendre polynomials $I_n(P_n^{(1/2)})$: to the left, for $n$ even $n = 10$ to $n = 80$. To the right, for $n$ odd $n = 11$ to $n = 81$. In both the curve $\left|\frac{1-z^2}{2z}\right| = 1$.

Since

$$Q_{n,m}(z) = \frac{(n+m)!}{m!} \frac{1}{z^m} I_m[(1+z)^n] = \frac{1}{z^m} \left( (1 + z)^{n+m} - \sum_{k=0}^{m-1} \binom{n+m}{k} z^k \right),$$

we have, Lemma 1.

$$Q_{n,m}(z) = \begin{cases} \frac{(z+1)^{n+m}}{z^{m-1}} (1 + o(1)) & \text{if } |z+1| > 1, \\ -\frac{1}{z^{m-1}} (1 + o(1)) & \text{if } |z+1| < 1. \end{cases}$$

as $n \to \infty$. Moreover, w-limit $\nu[Q_{n,m}] = \mu_{D_1(-1)}$, where $\mu_{D_1(-1)}$ is the equilibrium measure on $D_1(-1)$. So it is given by the normalized arc length on $D_1(-1)$.

If $p_n$ is a polynomial of degree $n$, $p_n(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$, then $I_{m,0}(p_n)(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$, where

$$c_k = a_k b_k,$$

then $\sum_{k=0}^{n} \binom{n}{k} b_k z^k = Q_{n,m}(z)$.

This means that, as Borwein, Chen and Dilcher (4) observed, $I_{m,0}$ is a Hadamard product of $p_n$ and $Q_{n,m}$. Then, by a theorem of Szegő and Schur (4), for $n$ large enough, we have:

**Corollary 3.** If the zeros of $p_n$ lie in the disc $D_r$, then the zeros of $I_{m,0}(p_n)$ lie in $D_{(2+\varepsilon(n))r}$, where $(\varepsilon(n))$ is a decreasing sequence of positive number with $\lim_{n \to \infty} \varepsilon_n = 0$. 

Figure 3: Zeros of the integral of ultraspherical polynomials $I_n(P_n^{(13/2)})$: to the left, for $n$ even, $n = 20, \ldots, 160$. To the right, for $n = 120$. In both the curve $\left| \frac{1-z^2}{2z} \right| = 1$.

The next result also helps to locate the zeros of the iterated integral of a polynomial. This is an extension of [15, Th. 5.7.8] and the proof is carried out with analogous arguments.

Lemma 2. Let $P$ be a polynomial of degree $n \geq 2$ with all its critical points in the closed disc $D_r$, where $r \in \mathbb{R}_+$ is fixed. If $P(\lambda) = P(z) = 0$, with $\lambda, z \in \mathbb{C}$, then

(i) there exists $w \in D_1$ such that

$$z = F_r(w) := 2rw - \overline{\lambda}w^2.$$  \hfill (14)

(ii) $||z| - |\lambda|| \leq 2r$.

(iii) $F_r$ is univalent on $D_1$ if and only if $|\lambda| \leq r$.

Proof. As $z$ and $\lambda$ are zeros of $P$, from the bisector lemma (see [10, Th. 4.3.1]), if we draw a straight line $\ell$ which cuts perpendicularly the segment joining the two zeros at its middle point, then $P'$ has at least one zero in each of the closed half planes in which $\ell$ divides the complex plane. But, we have assumed that all the zeros of $P'$ lie in $\overline{D}_r$ and therefore $\ell$ must intersect $\partial D_r$. Hence, there exists $u \in \partial D_1$ such that $|z - ru| = |ru - \lambda| = |r - \overline{\lambda}u|$. It follows that there exists $v \in \partial D_1$ such that $z - ru = v(r - \overline{\lambda}u)$, where we have $z = r(u + v) - \overline{\lambda}uv$. This expresses $z$ as a value of a symmetric linear form in the variables $u$ and $v$ taking their values on $\partial D_1$, and therefore in $\overline{D}_1$. It follows from Walsh’s coincidence lemma [10, Th. 3.4.1b] that $z$ is a value of the polynomial obtained by putting $w = u = v$ with $w \in \overline{D}_1$, which establishes (14) and the inequality in statement (ii) as an immediate consequence.

If $\lambda = 0$ then obviously $F_r$ is univalent. Assume that $\lambda \neq 0$, if there exist $w_1, w_2 \in \overline{D}_1$ such that $w_1 \neq w_2$ and $F_r(w_1) = F_r(w_2)$, we get that $w_1 + w_2 = 2(r/\lambda)$. Therefore, $F_r$ is univalent on $D_1$ if and only if $|\lambda| \leq r$ and we get the third statement of the theorem. \hfill \Box

Remark 2. Under the above assumptions, as a consequence of Lemma 2, the possible region of zeros of $P$ is the set $F_r(D_1)$ and if $|\lambda| \leq r$ then $F_r$ maps $\partial D_1$ onto a Jordan curve (for $r = 1$ see Figure 4).
Corollary 4. Given two integers \( n, m > 0 \) and \( \lambda \in \mathbb{C} \), let \( \rho = 2^m(\lambda + r) - |\lambda| \). If all the zeros of \( \phi_n \) are in \( \bar{D}_r \), then all the zeros of the polynomials \( I_{m, \lambda}(\phi_n) \) lie in the closed disc \( \bar{D}_\rho \).

Proof. For \( m = 1 \), as all the zeros of \( \phi_n \) lie in \( D_r \) (i.e. the critical points of \( I_{1, \lambda}(\phi_n) \)), the assertion follows from Lemma 2. The rest of the proof runs by induction.

5 Proof of Theorem 3

The proof of Theorem 3 is divided into three subsections: first, two auxiliary lemmas, second, when \( \lambda \in A \), and third, when \( \lambda \notin A \).

5.1 Auxiliary lemmas

The following result plays a main role in obtaining the strong asymptotic behavior of the \( m \)th iterated integrals \( I_{m, \lambda}(\phi_n) \).
Lemma 3. Set $\lambda \in \Omega$, and $K_1$, $K_2$ two compact sets with $K_1 \subset (\Lambda_\lambda \cup A)$ and $K_2 \subset \overline{K_\lambda}^c$. Let $z_1 \in K_1$, $z_2 \in K_2$ and $\{\phi_n\}_{n \in \mathbb{Z}_+}$ be a sequence of polynomials which satisfies (9). Then

$$\lim_{n \to \infty} \left| \int_{z_1}^{z_2} \phi_n(s) \, ds \right|^{1/n} = \text{cap}(A) |\tau(z_2)| \quad \text{uniformly on } K_1 \text{ and } K_2. \quad (15)$$

Proof. For $z_1 \in K_1$ and $z_2 \in K_2$, denote $J_n := \int_{z_1}^{z_2} \phi_n(s) \, ds$. This integral is independent of the contour of integration from $z_1$ to $z_2$. Then, by the maximum principle for holomorphic functions and (6) we have

$$\limsup_{n \to \infty} |J_n|^{1/n} \leq \left( \int_{z_1}^{z_2} |ds| \max_{\{z \in \mathbb{C} : \tau(z) = \tau(z_2)\}} |\phi_n(z)| \right)^{1/n} = \text{cap}(A) |\tau(z_2)|.$$

Since the convergence in (9) is uniform on compact subsets of $\Omega$, the above relation also holds uniformly on $K_1$ and $K_2$.

The proof of the inequality

$$\liminf_{n \to \infty} |J_n|^{1/n} \geq \text{cap}(A) |\tau(z_2)| \quad (16)$$

requires a more detailed analysis. We chose $\zeta_0$ near to $z_2$ such that $[\zeta_0, z_2] \cap A = \emptyset$,

$$|\tau(\zeta_0)| > |\tau(\lambda)| \quad \text{and} \quad \text{Arg}(\tau(\zeta_0)) = \text{Arg}(\tau(z_2)). \quad (17)$$

Let $\Delta_1$ be a Jordan rectifiable arc from $z_1$ to $\zeta_0$ in $\{z \in \mathbb{C} : |\tau(z_1)| \leq |\tau(z)| \leq |\tau(\zeta_0)|\}$ and let $\Delta_2$ be the arc given by $r(t) = \tau^{-1}(t \tau(z_2) + (1 - t) \tau(\zeta_0))$, $t \in [0, 1]$, which satisfies

$$\text{Arg}(\tau(r(t))) \quad \text{is constant and} \quad r'(t) = \frac{\tau(z_2) - \tau(\zeta_0)}{\tau'(t \tau(z_2) + (1 - t) \tau(\zeta_0))} \neq 0, \quad t \in [0, 1].$$

We take as integration path in $J_n$ the curve $\Delta := \Delta_1 + \Delta_2$. Then,

$$|J_n| = \left| \int_{\Delta_1} \phi_n(s) \, ds \right| \geq \left| \int_{\Delta_2} \phi_n(s) \, ds \right| - \left| \int_{\Delta_1} \phi_n(s) \, ds \right| |ds|.$$

By the maximum modulus principle and (6), it follows

$$\limsup_{n} \left( \int_{\Delta_1} |\phi_n(s)| \, ds \right)^{1/n} \leq \text{cap}(A) |\tau(\zeta_0)|. \quad (18)$$

Since $\Re(r(1)) \frac{\tau'(r(1))}{\tau'(r(1))} \neq 0$, we have that its real or its imaginary part is different from zero. We assume that $\Re\left( \frac{\phi_n(s) \tau'(s)}{\text{cap}(A)^n \, \tau'(s)^2} \right) \neq 0$. Other case is reduced to this one by multiplying $\phi_n$ by $i$. So, by (6), we can chose $\Delta_2^1 \subset \Delta_2$ a piece of arc of $\Delta_2$ containing $z_2$ such that

$$\lim_{n \to \infty} \Re\left( \frac{\phi_n(s) \tau'(s)}{\text{cap}(A)^n \, \tau'(s)^2} \right) = \Re\left( \frac{\phi_n(s) \tau'(s)}{\tau'(s)^2} \right) \neq 0.$$
uniformly on $\Delta_2^1$ and $H_n(s) := \Re \left( \frac{\phi_n(s)\tau'(s)}{\text{cap}(A)n\tau'(s)^2} \right)$ has constant sign for $s \in \Delta_2^1$ and $n$ large enough. From (17), we have

$$
\left| \int_{\Delta_2} \phi_n(s) ds \right| \geq \int_{\Delta_2^1} \frac{\phi_n(s)}{\text{cap}(A)n}\tau''(s)|\tau'(s)| ds - \int_{\Delta_2 \setminus \Delta_2^1} |\phi_n(s)| |ds|
$$

(19)

$$
\geq \int_J H_n(r(t))(\text{cap}(A))n|\tau''(r(t))| dt - \int_{\Delta_2 \setminus \Delta_2^1} |\phi_n(s)| |ds|
$$

where $J$ denote the parameter interval for the $\Delta_2^1$ path. By the choice of $\Gamma_2^1$, it follows

$$
\limsup_{n \to \infty} \left( \int_{\Delta_2 \setminus \Delta_2^1} |\phi_n(s)| |ds| \right)^{1/n} < |\text{cap}(A)\tau(z_2)|.
$$

Now, we deduce

$$
\lim_{n \to \infty} \left| \int_J H_n(r(t))(\text{cap}(A))n|\tau''(r(t))| dt \right|^{1/n} = |\text{cap}(A)\tau(z_2)|.
$$

(20)

In fact, since $|\tau(r(t))| \leq |\tau(r(1))| = |\ell(z_2)|$ for all $t \in J$, we have

$$
\limsup_{n \to \infty} \left| \int_J H_n(r(t))(\text{cap}(A))n|\tau''(r(t))| dt \right|^{1/n} \leq |\text{cap}(A)\tau(z_2)|.
$$

On the other hand, given $\varepsilon \in (0,|\tau(z_2)|)$, we chose $\delta > 0$ such that $[1-\delta,1] \subset J$ and $|\tau(r(t))| > |\tau(z_2)| - \varepsilon$, $\forall t \in J$. Then,

$$
\left| \int_J H_n(r(t))(\text{cap}(A))n|\tau''(r(t))| dt \right| \geq \int_{[1-\delta,1]} H_n(r(t)) dt (\text{cap}(A))^n (|\tau(z_2)| - \varepsilon)^n.
$$

and we get

$$
\liminf_{n \to \infty} \left| \int_J H_n(r(t))(\text{cap}(A))n|\tau''(r(t))| dt \right|^{1/n} \geq |\text{cap}(A)\tau(z_2)|.
$$

Once we have obtained (20), from (17)–(18) and (19)–(20) follow readily (15) and (16). Observe that the limit in (16) is uniform on $z_1$ and $z_2$. By the continuity of $\frac{\delta(s)\tau'(s)}{|\tau'(s)|^2}$, we can consider that the relations (17)–(20) hold for all $z'_1$ and $z'_2$ in a closed disc around $z_1$ and $z_2$. \hfill \square

The following lemma is a well-known result, but we include a proof for an easy reading.

**Lemma 4.** We have

$$
\psi(z) = \int \frac{d\mu_A(w)}{z - w}, \quad z \in \Omega, \quad \text{where, as it was defined in Theorem 3,} \quad \psi(z) = \frac{\tau'(z)}{\tau(z)}.
$$

**Proof.** Let $\{\phi_n\}$ be a sequence of monic polynomials, which satisfy (9). So, theirs zeros tend to $A$ and $w$-lim $\nu[\phi_n] = \mu_A$. (See arguments to check $w$-lim $\nu[I_{1,\lambda}(\phi_n)] = \mu_A$ in the proof of statement (i) of Theorem 3 in the next section.) Let $(g_n)$ be the sequence of analytic functions

$$
g_n(z) := (\text{cap}(A))^n\tilde{\gamma}(z)(\tau(z))^n, \quad z \in \Omega, \quad n \in \mathbb{N}.
$$
Then, \( \lim_{n \to \infty} \frac{g_n'(z)}{n g_n(z)} = \psi(z) \), uniformly on compact subsets of \( \Omega \). Since
\[
\left( \frac{\phi_n(z)}{g_n(z)} \right)' = \frac{\phi_n(z)}{g_n(z)} \left( \frac{\phi_n'(z)}{\phi_n(z)} - \frac{g_n'(z)}{g_n(z)} \right),
\]
we get \( \lim_{n \to \infty} \frac{\phi_n'(z)}{n \phi_n(z)} = \lim_{n \to \infty} \frac{g_n'(z)}{n g_n(z)} = \psi(z) \), uniformly on compact subsets of \( \Omega \) and
\[
\psi(z) = \lim_{n \to \infty} \frac{\phi_n'(z)}{n \phi_n(z)} = \lim_{n \to \infty} \int \frac{d\nu[\phi_n](w)}{z - w} = \int \frac{d\mu_A(w)}{z - w}, \quad z \in \Omega.
\]
\[\square\]

5.2 Proof of Theorem 3(ii): \( \lambda \not\in A \)

We only consider the case \( m = 1 \) because another step of induction on \( m \) follows with the same argument. It is well known that, if \( P_n \) is a monic polynomial of degree \( n \geq 1 \), we have \( \|P_n\|_A \geq (\text{cap}(E))^n \). (See [11, Th. 5.5.4].) Combining this relation with the definition of \( I_{1,\lambda}(\phi_n) \) and the condition \( (5) \) on \( \phi_n \), it plainly follows that \( \lim_{n \to \infty} \|I_{1,\lambda}(\phi_n)\|_A^{1/n} = \text{cap}(A) \). Thus, since \( A \) has empty interior and a connected complement (see \( 3 \)), we have
\[
\lim_{n \to \infty} v[I_{1,\lambda}(\phi_n)] = \mu_A.
\]
As \( \tau \) has no zeros in \( \Omega \), by Lemma 3 we know that the zeros of \( (I_{1,\lambda}(\phi_n)) \) converge to \( A \). So, we obtain
\[
\frac{\phi_n(z)}{I_{1,\lambda}(\phi_n)(z)} = \frac{(I_{1,\lambda}(\phi_n))'(z)}{(n + 1)I_{1,\lambda}(\phi_n)(z)} = \int \frac{d\nu[I_{1,\lambda}(\phi_n)](w)}{z - w},
\]
and by \( (21) \),
\[
\lim_{n \to \infty} \frac{\phi_n(z)}{I_{1,\lambda}(\phi_n)(z)} = \int \frac{d\mu_A(w)}{z - w} = \psi(z), \quad z \in \Omega.
\]
Therefore, the statement (i) of Theorem 3 follows immediately from the hypothesis \( (6) \) on \( \phi_n \).
\[\square\]

5.3 Proof of Theorem 3(iii): \( \lambda \not\in A \)

We chose \( \lambda_0 \in A \) and \( \lambda \in \Omega \). From \( 1 \), \( I_{m,\lambda}(\phi_n) \) can be written alternatively as
\[
I_{m,\lambda}(\phi_n)(z) = I_{m,\lambda_0}(\phi_n)(z) - P_{m-1}(z),
\]
where \( P_{m-1}(z) \) is the \( m - 1 \)-th Taylor polynomial of \( I_{m,\lambda_0}(\phi_n) \) in powers of \( (z - \lambda) \). From \( 2 \), we have that \( (I_{m,\lambda}(\phi_n))^{(k)}(\lambda) k! = \binom{n}{k} I_{m-k,\lambda_0}(\phi_n)(\lambda) \), \( 0 \leq k \leq m-1 \). Therefore,
\[
P_{m-1}(z) = \sum_{k=0}^{m-1} \binom{n+m}{k} I_{m-k,\lambda_0}(\phi_n)(\lambda)(z - \lambda)^k.
\]
The proof is completed using (i) of Theorem 3 for \( I_{m-k,\lambda_0}(\phi_n) \).
\[\square\]

By way of Theorem 3’s illustration, Figures 5 and 6 include several examples of the curve \( \partial \Lambda_\lambda \), the zeros of the \( \phi_n \), and the zeros of their iterated integrals for different values of \( \lambda, m \).
Figure 5: Here we illustrate Theorem 3 for ultraspherical polynomials on $[-1, 1]$: $I_6(P_n^{(3/2)})$.

Figure 6: We illustrate Theorem 3 for orthogonal polynomials on the arc \( \{ e^{it} : \theta \leq t \leq 2\pi - \theta \} \) of the unit circle. In each case we take the Chebyshev measure on the arc. The zeros of $\phi_n$ are marked by bullets, the zeros of $I_{1, \lambda}(\phi_n)$ by square, and the zeros of $I_{2, \lambda}(\phi_n)$ by diamonds. In each case, the dashes circle is $\partial D_1$ and the gray path is the curve $\partial \Lambda_\lambda$ for the corresponding values of $\lambda$.

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