Ferrers functions of arbitrary degree and order
and related functions

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1 Introduction

Ferrers functions $P_{-\mu}^-\nu(x)$ (associated Legendre functions of the first kind on the cut) are of utmost importance in theoretical, applied and computational mathematics. They are solutions of the Legendre differential equation

$$\frac{d}{dx} \left( (1-x^2) \frac{dy}{dx} \right) + \left( \nu (\nu + 1) - \frac{\mu^2}{1-x^2} \right) y = 0$$

on the interval $-1 < x < 1$ and have found wide applications in analysis, numerical methods, classical and quantum physics, mechanics and engineering. In particular, $P_{-\mu}^-\nu(x)$ arise as a result of the separation of variables in various physical problems, in expansions of functions in series and integrals of Ferrers functions as well as in analytical and numerical approximations based on using orthogonal polynomials or functions. There are numerous publications in this classical field, and recent articles by Bakaleinikov and Silbergleit [4], Bielski [6], Cohl and Costas-Santos [12, 11], Cohl, Dang and Dunster [10], Durand [14, 15], Maier [21, 22], Nemes and Olde Daalhuis [25], Szmytkowski [31, 32], Wang and Qiao [36], and Zhou [38] should be mentioned in this connection. In this article, we obtain for Ferrers functions novel integral representations, which are used, together with analytical continuation, for the systematic derivation of numerous series representations, integral and series connection formulas, asymptotic and differentiation formulas, generating functions and additional theorems for $P_{-\mu}^-\nu(\tanh (\alpha + \beta))$. Various formulas for Gegenbauer polynomials and associated Legendre functions (associated Legendre polynomials) are obtained as special cases. Surprisingly, most results have not been given in the literature before. New derivations, refinements or new forms are suggested for a number of known relations as well. Because results of this paper are addressed also to
physicists and engineers, the author tried his utmost to attain an accessible, simple and detailed style of the article and proofs.

In this paper, traditional definitions and notations for special functions are accepted, \( \mathbb{R} \) is the set of real numbers, \( \mathbb{C} \) is the set of complex numbers, \( \mathbb{C}^2 = \mathbb{C} \times \mathbb{C} \), \( \mathbb{Z} \) is the set of integers, \( \mathbb{N} \) is the set of natural numbers and \( \mathbb{N}_0 \) is the set of whole numbers.

Ferrers functions of the first kind are defined by means of the Gauss hypergeometric function \([10],[19]\), namely

\[
P_{\nu}^{-\mu}(x) = P_{-1-\nu}^{-\mu}(x) = \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} \frac{F((-\nu, \nu + 1; 1 + \mu; \frac{x-1}{x+1})}{\Gamma(1 + \mu)}, \quad |x| < 1, \quad (1)
\]
or on using the Pfaff transformation for Gauss functions

\[
P_{\nu}^{-\mu}(x) = \frac{2^{-\nu} (1-x)^{\frac{\mu}{2}}}{\Gamma(1 + \mu) (1+x)^{\frac{\mu}{2} - \nu}} F\left(-\nu, \mu - \nu; 1 + \mu; \frac{x-1}{x+1}\right). \quad (2)
\]

Ferrers functions and their derivatives are entire functions of each of the parameters \( \nu \) and \( \mu \). This analyticity allows us to use analytic continuation when a certain relation is valid in some range of the parameters values. In many cases, to make use of analytic continuation, we will exploit the following simple proposition arising from the uniform convergence of the series.

**Proposition 1** Suppose that \( \mathfrak{S} \subset \mathbb{C}^2 \) is a bounded domain and functions \( Q_n(\nu, \mu) \), \( n \in \mathbb{N}_0 \), are analytic functions of \( \nu \) and \( \mu \) on \( \mathfrak{S} \) such that for certain \( a, b, c \in \mathbb{R} \), \( \left|Q_n(\nu, \mu) n^{a \mu + b \nu + c}\right| = O(1) \) as \( n \to \infty \). Then \( \sum_{n=0}^{\infty} Q_n(\nu, \mu) x^n \) is an analytic function of each of the variables \( \nu \) and \( \mu \) if \( |x| = 1 \) while \( \Re \mu + b \Re \nu + c > 1 \) for all \( (\nu, \mu) \in \mathfrak{S} \) or if \( |x| < 1 \).

One can see from \([2]\) that \( P_{\nu}^{m}(x) \equiv 0 \) as \( m, k \in \mathbb{N}_0 \) and \( m > k \). If \( \mu = \nu - n \) or \( \nu = n, n \in \mathbb{N}_0 \), then the right side of \([2]\) is proportional to Jacobi polynomials \([1],[26]\):

\[
P_{\nu}^{\nu-n}(x) = \frac{n! (1-x^2)^{\frac{\nu-n}{2}}}{2^{\nu-n} \Gamma(\nu+1)} P_{n}^{(\nu-n, \nu-n)}(x), \quad (3)
\]

\[
\lim_{\eta \to k} \Gamma(-\eta) P_{\eta}^{n+\eta+1}(x) = 2^{k+n+1} P_{n}^{(-k-n-1, -k-n-1)}(x),
\]

and

\[
P_{n}^{-\mu}(x) = \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} \frac{n! P_{n}^{(\mu, -\mu)}(x)}{\Gamma(n+\mu+1)}. \quad (4)
\]

Since \( P_{n}^{(\alpha, \beta)}(x) = (-1)^n P_{n}^{(\beta, \alpha)}(-x) \), the above relations entails

\[
P_{\nu}^{-\nu}(x) = (-1)^n P_{\nu}^{-\nu}(-x), \quad P_{-1-\nu}^{-\mu}(x) = (-1)^k \frac{\Gamma(k-\mu+1)}{\Gamma(k+\mu+1)} P_{k}^{\mu}(-x). \quad (5)
\]
Also, \( \{ P_{n+\mu+}^{\nu,\mu} (x) \}, \) \( \Re \mu > -1, \) and \( \{ P_{n+\mu}^{\nu} (x) \}, \) \( |\Re \mu| < 1, \) constitute complete systems of orthogonal functions as \( n \in \mathbb{N}_0. \) In fact, \( P_{n+\mu+}^{\nu} (x) \) are related to Gegenbauer polynomials \( C_n^{(\tau)} (x) = (-1)^{n} C_n^{(\tau)} (-x), \) \( \tau \in \mathbb{C}, \) defined on \((-1,1)\) by the relation

\[
C_n^{(\tau)} (x) = \frac{(2\pi)^\frac{\tau}{2}}{\Gamma (\frac{\tau + 1}{2})} P_n^{\frac{\tau - \frac{\tau}{2} \tau}{} (1 - x^2)^{\frac{\tau}{2}}} (x) = \frac{(2\pi)^\frac{\tau}{2}}{\Gamma (\frac{\tau + 1}{2})} \frac{P_n^{\frac{\tau - \frac{\tau}{2} \tau}{} (1 - x^2)^{\frac{\tau}{2}}} (x)}{2^{\frac{\tau}{2} n! (1 - x^2)^{\frac{\tau}{2}}}}, \tag{6}
\]

Gegenbauer polynomials are entire functions of the parameter \( \tau \) and have the generating function \( [3, 16] \)

\[
(\tau^2 + 2sx + 1)^{-\tau} = \sum_{n=0}^{\infty} (-1)^n s^n C_n^{(\tau)} (x), \ |s| < 1. \tag{7}
\]

As \( n \to \infty, \) there is an asymptotic estimate \( [10] \)

\[
\left| C_n^{(\tau)} (x) \right| = O \left( n^{\Re \tau - 1} \right), \ -1 < x < 1. \tag{8}
\]

## 2 Integral and series representations for Ferrers functions

Our starting point is employing Euler’s integral representation of the Gauss hypergeometric function (Erdelyi at al. (1955) \( [16] \)) to write

\[
P_{\nu}^{\mu} (x) = \frac{2^{-\nu} (1 + x)^{\nu - \frac{\mu}{2}} (1 - x)^{\frac{\mu}{2}}}{\Gamma (\mu - \nu) \Gamma (1 + \nu)} \int_0^1 s^{\nu - 1} (1 - s)^{\nu} ds \frac{1}{\left( 1 - \frac{s}{x} \right)^{\nu}}, \tag{9}
\]

\( \Re \mu > \Re \nu > -1, \ -1 < x < 1. \)

The changes \( x = \tanh \alpha \) and \( s = \exp (-t) \) yield the representation in the form of a Laplace integral, namely

\[
P_{\nu}^{\mu} (\tanh \alpha) = \frac{2^\nu e^{-\nu} \cos \nu \cdot \alpha}{\Gamma (\mu - \nu) \Gamma (1 + \nu)} \int_0^\infty \frac{e^{-\nu \cdot t} dt}{\sinh \frac{t}{2} \cos \left( \frac{\nu}{2} + \alpha \right)} \tag{10}
\]

\( \Re \mu > \Re \nu > -1. \)

Another integral representation follows in the same way from the relations \( F (a, b, c, z) = F (b, a, c, z), \) \( [2] \) and Euler’s integral representations as \( \mu = \sigma + \gamma, \nu = -\gamma, \Re \gamma > 0, \) and \( \Re \sigma > -1, \)

\[
P_{-\gamma}^{\sigma - \gamma} (\tanh \alpha) = \frac{P_{-\gamma - 1}^{\sigma - 1} (\tanh \alpha)}{\cosh \gamma \cdot \alpha} = \frac{2^\gamma e^{-(\sigma + 2\gamma)\alpha}}{\Gamma (\gamma) \Gamma (\sigma + 1)} \int_0^1 \frac{s^{\gamma - 1} (1 - s)^{\sigma} ds}{(1 + e^{-2\alpha s})^{\sigma + 2\gamma}}
\]

\[
= \frac{2^{-\gamma}}{\Gamma (\gamma) \Gamma (\sigma + 1)} \int_0^\infty \frac{\sinh \frac{t}{2} dt}{\cosh^{\gamma + 2\gamma} \left( \frac{t}{2} + \alpha \right)}. \tag{11}
\]
By exploiting the binomial series
\[
\cosh^\gamma \alpha \sinh^\sigma \frac{t}{2} \cosh^{\sigma+\gamma} \left( \frac{t}{2} + \alpha \right) = \frac{\tanh^\sigma \frac{t}{2}}{n\cosh^\sigma \alpha \cosh^\gamma \frac{t}{2}} \left( 1 + \tanh \alpha \tanh \frac{t}{2} \right)^{-\gamma - 2\sigma}
\]
and integrating term-by-term, we obtain from (11), as \( \tanh \alpha = x \), the representation in terms of a generalized hypergeometric function \(_2R_1(a; b; c; \gamma; z)\)  

\[
P_{\gamma-1}^{-\sigma-\gamma} (x) = \frac{2^{-\gamma} (1 - x^2)^{\frac{\sigma+\gamma}{2}}}{\Gamma(\sigma + 1)} \sum_{n=0}^{\infty} \frac{(\sigma + 2\gamma)_n \Gamma \left( \frac{n + \sigma + 1}{2} \right)}{n!} (-x)^n
\]

\[
= \frac{2^{-\sigma-\gamma} \exp \left( 1 - x^2 \right)^{\frac{\sigma+\gamma}{2}}}{\Gamma(\sigma + 2\gamma) \Gamma(\sigma + 1)} \sqrt{\pi} \Psi_0 \left( \frac{\sigma + \gamma + 1}{2} ; \frac{\sigma + 1}{2} + \gamma; \frac{1}{2}; -x \right)
\]

and hence, using Legendre’s duplication formula for gamma functions, the representation in terms of a Fox-Wright hypergeometric function \(_2\Psi_0(z)\)  

\[
P_{\gamma-1}^{-\sigma-\gamma} (x) = \frac{2^{\sigma+\gamma-1}}{\sqrt{\pi}} \Gamma(\sigma + 2\gamma) \Gamma(\sigma + 1) \sqrt{\pi} \Psi_0 \left( \frac{\sigma + \gamma + 1}{2} ; \frac{\sigma + 1}{2} + \gamma; \frac{1}{2}; -x \right)
\]

By making use of the asymptotic expansion for gamma functions 11

\[
\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta} + O \left( \frac{1}{z} \right) \text{ as } z \to \infty,
\]

one can see that terms of the above power series can be written in the form \( Q_n(\sigma, \gamma)n^{\sigma+\gamma+c}x^n \), \( |Q_n(\sigma, \gamma)| \leq Q \), i.e., the power series converge absolutely and, according to proposition mentioned in section 1, are analytic functions of \( \sigma \) and \( \gamma \). This means that the formulas obtained are valid for all values of \( \sigma, \gamma \in \mathbb{C} \) by virtue of analytic continuation. Note that the even and odd parts of (12) are proportional to Gauss hypergeometric functions and hence \( P_{\gamma-1}^{-\sigma-\gamma}(x) \) can be written in the form of the well-known combination of Gauss functions 16. As \( -\sigma - 1 = l = n_0 \) or \( -\sigma - 2\gamma = l = n_0 \), the series turns into finite sums. On denoting \( \gamma = -\lambda \),

\[
P_{\lambda-\gamma}^{\gamma}(x) = \frac{2^l \Gamma(2\lambda - l + 1)}{\Gamma(2\lambda - l + 1)} \sum_{n=0}^{l} \frac{(-1)^l \Gamma(l + \frac{\lambda}{2}) x^n}{n! (l - n)!} \sum_{j=0}^{l} \frac{(-1)^j \Gamma(l - j + \frac{\lambda}{2}) x^{l-2j}}{2^{2j}j! (l - 2j)!}.
\]
Also, we obtain from (12) the relation
\[
\left( \frac{d^n}{dx^n} P_{\nu}^{-\mu} (x) \right)_{x=0} = (-1)^n 2^{n+\mu-1} \frac{\Gamma \left( \frac{n+\mu+\nu+1}{2} \right)}{\sqrt{\pi} \Gamma (\mu - \nu) \Gamma (\mu + \nu + 1)}.
\] (15)

New integrals follow from (10) and (11) by changing \( t = 4z \) and using the identity
\[
\cosh (\alpha + 2z) = 1 + 2 \tanh \alpha \tanh z + \tanh^2 z.
\]

Then, as \( x = \tanh \alpha \), the series in (7) converges uniformly on the interval \( s \in [0,1] \) by virtue of the asymptotic estimate (8). Inserting the above-mentioned,
\[
P_{\nu}^{-\mu} (x) = \frac{\Gamma^{-1} (1 + \nu)}{\Gamma (\mu - \nu)} \left( \frac{1 - x}{1 + x} \right)^{\frac{\gamma}{2}} \int_0^1 \frac{4^\nu s^\nu (1 - s)^{2\mu-2\nu-1} ds}{(1 + s)^{2\mu+2\nu+1} (s^2 + 2sx + 1)},
\] (16)

and on denoting \( \sigma = \tau + 2\nu \) and \( \gamma = -\nu \) (11) turns into
\[
P_{\nu}^{-\nu-\sigma} (x) = \frac{2^{\tau+3\nu+2} (1 - x^2)^{\frac{\mu-\nu}{2}}}{\Gamma (-\nu) \Gamma (\tau + 2\nu + 1)} \int_0^1 \frac{(1 - s^2)^{-2\nu-1} s^\tau + 2\nu ds}{(s^2 + 2sx + 1)},
\] (17)

\( \text{Re} \mu > \text{Re} \nu > -1, -1 < x < 1, \)

### Theorem 2

Let \( \nu, \mu, \tau \in \mathbb{C} \). There are the expansions of Ferrers functions into series of Gegenbauer polynomials:
\[
\frac{P_{\nu}^{-\nu-\nu} (x)}{(1 - x^2)^{\frac{\mu-\nu}{2}}} = \frac{2^{\tau+\nu} \Gamma \left( \frac{1}{2} - \nu \right)}{\sqrt{\pi} \Gamma (\tau + 2\nu + 1)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma \left( \frac{n+\tau+2\nu}{2} \right)}{\Gamma \left( \frac{n+\tau-2\nu}{2} \right)} C_n^{(\tau)} (x),
\] (18)

\( \text{Re} (\tau + 2\nu) < 0, \nu - \frac{1}{2} \notin \mathbb{N}_0, \)

and
\[
P_{\nu}^{-\mu} (x) = \left( \frac{1 - x}{1 + x} \right)^{\frac{\mu-\nu}{2}} \frac{\Gamma (\mu - \nu + \frac{1}{2})}{4^\nu \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n c_n C_n^{(-\nu)} (x)},
\] (20)

\( \nu - \mu - \frac{1}{2} \notin \mathbb{N}_0, \text{Re} (2\mu - \nu) > 0, \)

where
\[
c_n = \frac{(\nu + 1)_n}{\Gamma (n + 2\mu - \nu + 1)} F \left( 2\mu + 2\nu + 1, n + \nu + 1; n + 2\mu - \nu + 1; -1 \right)
\]
\[
= \frac{(\nu + 1)_n}{\Gamma (n + 2\mu - \nu + 1)} F \left( 2\mu + 2\nu + 1, 2\mu - 2\nu; 2\mu + n - \nu + 1; \frac{1}{2} \right).
\]

**Proof.** First we will examine (16) as \(-1 < \text{Re} \nu < 0\) and \(\text{Re} (2\mu - 2\nu) \geq 1\). In this case, as \(\tau = -\nu\), the series in (7) converges uniformly on the interval \(s \in [0,1]\) by virtue of the asymptotic estimate (8). Inserting the above-mentioned
series, one is allowed to integrate term-by-term. On evaluating arising integrals with Euler’s integral representation of the Gauss hypergeometric function we arrive at (20) in this initial case. Now, asymptotic expansions (8) and (13) manifest that series in (20) is absolutely convergent under conditions (21), i.e., it is an analytic function of $\mu$ and $\nu$ (see section 1), and therefore (20) is valid due to analytic continuation. The expansion (18) can be proved in the same manner by examination (17) where initially is taken $\Re \nu < 0$, $\Re \tau < 0$, and $\Re (\tau + 2\nu) > -1$.

Since $C_n^{(\tau)}(x)$ are orthogonal for $\Re \tau > -1/2$, coefficients of the series might be expressed as certain new integrals of products of Gegenbauer polynomials and Ferrers functions while Parseval’s equation enables us to evaluate new integrals of products of Ferrers functions.

3 The generalized Mehler-Dirichlet integral

Let $I_{\mu}(z)$ be a modified Bessel function and $\chi_{\kappa,\sigma}(\xi, u)$ be a combination of Bessel functions of the first and second kind, namely

$$\chi_{\kappa,\sigma}(\xi, u) = J_{\kappa} \left( \xi e^{-iu/2} \right) Y_{\sigma} \left( \xi e^{iu/2} \right) - Y_{\kappa} \left( \xi e^{-iu/2} \right) J_{\sigma} \left( \xi e^{iu/2} \right).$$

**Theorem 3** The integral representation

$$\frac{P_{\nu-\frac{1}{4}}(\cos \theta)}{\sqrt{2} \sin \mu \theta} = \frac{\mu^\frac{\mu+\frac{1}{2}}{2}}{2^\mu \pi} \int_0^\theta \hat{\chi}_\nu \left( \xi, u \right) \frac{J_{-\mu-\frac{1}{4}} \left( \xi \sqrt{2 (\cos u - \cos \theta)} \right)}{(\cos u - \cos \theta)^{\frac{\mu+\frac{1}{2}}{2}}} du \quad (22)$$

$$= \frac{\mu^\frac{\mu+\frac{1}{2}}{2}}{2^\mu \pi} \int_0^\theta \hat{\chi}_\nu \left( \xi, u \right) \frac{J_{-\mu-\frac{1}{4}} \left( \xi \sqrt{2 (\cos u - \cos \theta)} \right)}{e^{iu/2} (\cos u - \cos \theta)^{\frac{\mu+\frac{1}{2}}{2}}} du \quad (23)$$

$$\hat{\chi}_\nu \left( \xi, u \right) = e^{-iu/2} \chi_{\nu+1,\nu} \left( \xi, u \right) + e^{iu/2} \chi_{\nu+1,\nu} \left( \xi, -u \right),$$

is valid on $\theta \in (0, \pi)$ as $\xi, \nu, \mu \in \mathbb{C}$ and $\Re \mu < \frac{1}{2}$.

**Proof.** We are basing on the integral

$$\chi(\xi, \theta) = \int_0^\theta \frac{P_{\nu-\frac{1}{4}}(\cos u) I_{-\mu-\frac{1}{4}} \left( \xi \sqrt{2 (\cos u - \cos \theta)} \right)}{\left( \sqrt{2 (\cos u - \cos \theta)} \right)^{\frac{\mu+\frac{1}{2}}{2}}} \frac{1}{\sin^{\mu-1} \theta} du,$$

$$\chi(\xi, \theta) = e^{-i\pi} \sqrt{\frac{\pi}{2}} \xi^{\frac{\mu-\frac{1}{2}}{2}} \chi_{\nu,\nu}(\xi, \theta),$$

$$-\frac{1}{2} < \Re (\mu) < 1, \theta \in [0, \pi), \xi \in \mathbb{C}, \nu \in \mathbb{C},$$
which follows from the integral (16) of the article [23]. The changes $2x = 1 - \cos \theta$ and $2t = 1 - \cos s$ convert the above integral into a convolution, and then
\[
\int_0^x f(t) \frac{I_{\mu-\frac{1}{2}} (2\xi \sqrt{x-t})}{(x-t)^{1/4}} dt = g(\xi, x), \quad 0 \leq x < 1, \quad (24)
\]
\[
f(t) = \frac{P_{\nu-\frac{1}{2}} (1-2t)}{t^{\frac{1}{2}} (1-t)^{1/2}}, \quad g(\xi, x) = 2^{\frac{\mu+1}{2}} \chi (\xi, \arccos (1-2x)).
\]
By taking $f(t) = 0$ as $t \geq 1$ and defining $g(\xi, x)$ for $x \geq 1$ as the left side of (24), we redefine (24) as an equation on the interval $0 \leq x < \infty$. Now, by applying Laplace integral transform $F(s) = \mathcal{L} (f(x))$, one can readily find
\[
F(s) = \xi^{\frac{\mu+1}{2}} e^{-\frac{\xi^2}{4}} G(s). \quad (25)
\]
The inverse Laplace transform of (25) can be written as a convolution
\[
f(x) \xi^{\frac{\mu-1}{2}} = \mathcal{L}^{-1} \left( s^{\mu+\frac{1}{2}} e^{-\frac{\xi^2}{4}} G(s) \right) = \mathcal{L}^{-1} (sG(s)) * \mathcal{L}^{-1} \left( s^{\mu-\frac{1}{2}} e^{-\frac{s^2}{4}} \right).
\]
By taking into account that $g(\xi, 0) = 0$, the above relation yields as $0 < x < 1$,
\[
\frac{P_{\nu-\frac{1}{2}} (1-2x)}{\pi e^{\frac{\xi^2}{4}} \sin \theta} = \int_0^\theta \frac{J_{\mu-\frac{1}{2}} (\xi \sqrt{2 (\cos u - \cos \theta)}) d (\chi_{\nu, \nu} (\xi, u))}{2^{\frac{\mu+1}{2}} \xi^{\mu-\frac{1}{2}} (\cos \theta - \cos \theta)^{1/4}}, \quad (26)
\]
where by virtue of analytic continuation $\Re \mu < \frac{1}{2}$. Finally, by exploiting formulas for derivatives of Bessel functions, we obtain (22), and then (23). □

Make the changes $\theta = \pi - \vartheta$ and $u = \pi - s$ in (22). Then, by exploiting [5] and analytic continuations of cylindrical functions [1], one can obtain as $\nu = n - 1/2, n \in \mathbb{N}$, and $\xi = i\eta$:

**Corollary 4** As $n \in \mathbb{N}$, $\mu, \eta \in \mathbb{C}$, $\Re \mu < \frac{1}{2}$, and $0 < \vartheta < \pi$,
\[
\frac{P_{n-\frac{1}{2}} (\cos \theta)}{\sqrt{\pi} \sin \theta} = \eta^{\mu+\frac{1}{2}} \Gamma (n - \mu) \int_0^\pi W_n (\eta, s) \frac{I_{\mu-\frac{1}{2}} (\eta \sqrt{2 (\cos s - \cos \vartheta)} )}{(\cos \vartheta - \cos s)^{1/4}} ds,
\]
\[
W_n (\eta, s) = \frac{e^{-is/2} \bar{\chi}_n (\eta, s) - e^{is/2} \bar{\chi}_n (\eta, -s)}{2i},
\]
\[
\bar{\chi}_n (\eta, u) = J_{n+\frac{1}{2}} (\eta e^{is/2}) Y_{n-\frac{1}{2}} (\eta e^{-is/2}) + Y_{n+\frac{1}{2}} (\eta e^{is/2}) J_{n-\frac{1}{2}} (\eta e^{-is/2}).
\]
As $\xi = 0$ and $\nu = \gamma + 1/2$, the integral representation turns into the famous Mehler-Dirichlet integral:

**Corollary 5** As $\gamma, \mu \in \mathbb{C}$, $\Re \mu < 1/2$, and $0 < \theta < \pi$,

\[ P_{\gamma}^{\mu} (\cos \theta) = \sqrt{\frac{2}{\pi}} \frac{\sin^\mu \theta}{\Gamma \left( \frac{1}{2} - \mu \right)} \int_0^\theta \frac{\cos \left( \gamma + \frac{1}{2} \right) u}{(\cos u - \cos \theta)^{\frac{1}{2} + \mu}} du. \] (28)

The above relation can be viewed as a fractional type integral operator transforming trigonometric functions into Ferrers functions of the first kind. Another similar operator can be obtained from (27) as $\eta = 0$,

\[ P_{n-1}^{-\mu} (\cos \vartheta) = \sqrt{\frac{2}{\pi}} \frac{\Gamma (n - \mu) \sin^\mu \vartheta}{\Gamma \left( \frac{1}{2} - \mu \right) \Gamma (n + \mu)} \int_0^\pi \frac{\sin \left( n - \frac{1}{2} \right) s}{(\cos \vartheta - \cos s)^{\frac{1}{2} + \mu}} ds, \] (29)

$\Re \mu < 1/2$, $n \in \mathbb{N}$, $0 < \vartheta < \pi$.

Certain other fractional type integral representations will be derived in the next section.

### 4 Representations Ferrers functions in the form of fractional type integrals

As a results of the changes $t = 2u - 2\alpha$ and $t = u - \alpha$, (10) becomes

\[ P_{n}^{-\mu} (\tanh \alpha) = \frac{2^{1+\nu} e^{\alpha \mu} \cosh^{-\nu} \alpha}{\Gamma (\mu - \nu) \Gamma (1 + \nu)} \int_{-\alpha}^{\alpha} e^{-2\mu u} du \] (30)

\[ = \frac{2^{1+\nu} e^{\alpha \mu}}{\Gamma (\mu - \nu) \Gamma (1 + \nu)} \int_{-\alpha}^{\alpha} e^{-2\mu u} \cosh^{2\nu} u du \] (31)

and

\[ P_{\nu}^{-\mu} (\tanh \alpha) = \frac{\cosh^{-\nu} \alpha}{\Gamma (\mu - \nu) \Gamma (1 + \nu)} \int_{-\alpha}^{\alpha} e^{-\mu u} \frac{du}{(\sinh u - \sinh \alpha)^{-\nu}}. \] (32)

The above formulas hold as $\Re \mu > \Re \nu > -1$.

Now, additional integral representations of Ferrers functions are obtained by changing variables and parameters: as $\Re \mu < - \Re \nu < 1$,

\[ P_{\nu}^{\mu} (-\tanh \alpha) = \frac{2^{\nu+1} e^{\alpha \mu}}{\Gamma (-\mu - \nu) \Gamma (1 + \nu)} \int_{-\infty}^{\alpha} e^{-2\mu u} \cosh^{2\nu} u du, \] (33)

\[ P_{\nu}^{\mu} (-\tanh \alpha) = \frac{\cosh^{-\nu} \alpha}{\Gamma (-\mu - \nu) \Gamma (1 + \nu)} \int_{-\infty}^{\alpha} e^{-\mu u} \frac{du}{(\sinh \alpha - \sinh u)^{-\nu}}. \] (34)
The integral representation (32) is a source of an asymptotic expansion of $P^{-\mu}_\lambda (x)$ as $\mu \to \infty$ in the right half-plane. We write

$$
\left( \frac{\sinh u - \sinh \alpha}{\cosh \alpha} \right)^\nu = (u - \alpha)^\nu T(u),
$$

where $T(u)$ is an infinitely differentiable function and can be represented in the vicinity of the point $u = \alpha$ by the Taylor series, that is,

$$
\left( \frac{\sinh u - \sinh \alpha}{\cosh \alpha} \right)^\nu = \sum_{k=0}^{\infty} \frac{t_{k,\nu} (\alpha)}{k!} (u - \alpha)^{k+\nu},
$$

with bounded for $\alpha \in \mathbb{R}$ functions $t_{k,\nu} (\alpha)$ defined by the relation

$$
t_{0,\nu} (\alpha) = 1, \quad t_{1,\nu} (\alpha) = \frac{\nu}{2} \tanh \alpha, \quad t_{2,\nu} (\alpha) = \frac{\nu (\nu - 1)}{4} \tanh^2 \alpha + \frac{\nu}{3}.
$$

Now, Watson’s lemma gives

**Theorem 6** Let the lying in the half-plane $\text{Re}\mu \geq \varepsilon > 0$ curve $\mathcal{L}$ be going to infinity and, as $\mu \to \infty$ along $\mathcal{L}$, $\nu = \nu (\mu)$ be bounded and $-1 < \text{Re} \nu < \text{Re} \mu$. If $K$ is an arbitrary fixed positive integer, then as $\mu \to \infty$ along $\mathcal{L}$,

$$
e^{\alpha \mu} P^{-\mu}_\nu (\tanh \alpha) = \frac{1}{\Gamma (\mu - \nu)} \left( \frac{\nu + 1}{\mu^{k+\nu+1} k!} \sum_{k=0}^{K-1} t_{k,\nu} (\alpha) + O \left( \frac{1}{\mu^{K+\nu+1}} \right) \right), \quad (35)
$$

uniformly with respect to $\alpha \in \mathbb{R}$.

As $\mu \to \infty$, another form of the asymptotic expansion is given by the terms of the hypergeometric series (1). Various other asymptotic expansions for $P^{-\mu}_\nu (\tanh \alpha)$ of large order are presented in [13], [26], [25], [18].

The integrals in (31), (32), (33) and (34) can be viewed as fractional type integral operators transforming exponential functions into Ferrers functions, i.e., operators connecting index integral transforms whose kernels contain Ferrers functions (see [5], [20], [24]) with Fourier and Laplace transforms or as operators connecting series of exponents (Fourier series of periodic functions in particular) with series of Ferrers functions.

Fractional type integral representations connecting Ferrers functions with power functions can be obtained by changing $t = 2u - 2\alpha$ in (11):

$$
P^{-\sigma-\gamma}_\gamma (\tanh \alpha) = \frac{2^{1-\gamma} \cosh \gamma \alpha}{\Gamma (\gamma) \Gamma (\sigma + 1)} \int_\alpha^{\infty} \frac{\sinh \sigma (u - \alpha)}{\cosh^{2\gamma+\sigma} u} du + O \left( \frac{1}{\mu^{K+\nu+1}} \right), (36)
$$

$$
= \frac{2^{1-\gamma} \cosh \sigma \gamma \alpha}{\Gamma (\gamma) \Gamma (\sigma + 1)} \int_\alpha^{\infty} \frac{(\tanh u - \tanh \alpha)^\sigma}{\cosh^{2\gamma} u} du, (37)
$$

$\text{Re} \gamma > 0, \text{Re} \sigma > -1,$
and hence, as \( \text{Re} \gamma > 0 \) and \( \text{Re} \sigma > -1 \),

\[
P_{\gamma-1}^{-\sigma-\gamma}(-\tanh \alpha) = \frac{2^{1-\gamma} \cosh^{\sigma+\gamma} \alpha}{\Gamma(\gamma) \Gamma(\sigma + 1)} \int_{-\infty}^{\alpha} \frac{\cosh^{-2\gamma} \alpha u \, du}{(\tanh \alpha - \tanh u)^{-\sigma}}. \tag{38}
\]

The integral representations obtained in this section lead to fractional type integral operators relating Ferrers functions.

**Theorem 7** There are fractional type integral operators relating Ferrers functions of different degrees

\[
P_{\lambda+\rho+1}^{-\mu}(\tanh \alpha) = \frac{\Gamma(\mu - \rho) \Gamma^{-1} (1 + \lambda)}{\Gamma(\mu - \lambda - \rho - 1)} \int_{\alpha}^{\infty} P_{\rho}^{-\mu} (\tanh s) \cosh^{\rho+1} s \frac{\cosh^{-\rho} s \, ds}{(\sinh s - \sinh \alpha)^{-\lambda - \rho}}, \tag{39}
\]

\( \text{Re} (\mu - \rho) > \text{Re} \lambda + 1 > 0 \),

or Ferrers functions of different orders

\[
P_{\nu}^{-\lambda-1}(\tanh \alpha) = \frac{1}{\Gamma(\lambda + 1)} \int_{\alpha}^{\infty} \frac{\cosh^{-\lambda-2} s P_{\nu}^{-\mu} (\tanh s)}{(\tanh s - \tanh \alpha)^{-\lambda}} \, ds, \tag{40}
\]

\( \text{Re} \mu > -1, \text{Re} \lambda > -1 \).

**Proof.** Equations (39) is obtained by exploiting the integral

\[
(\eta(u) - \eta(\alpha))^{\lambda+\rho+1} = \frac{1}{B(1 + \lambda, 1 + \rho)} \int_{\alpha}^{u} \frac{(\eta(s) - \eta(\alpha))^{\lambda}}{(\eta(u) - \eta(s))^{-\rho}} \, d\eta(s), \tag{41}
\]

\( \text{Re} \lambda > -1, \text{Re} \rho > -1, u \geq \alpha \),

where \( \eta(s) \) is a continuous monotonically increasing function on the interval \( s \in [u, \alpha] \). Insert (41) with \( \eta(s) = \sinh s \) and \( \rho = \nu - \lambda - 1 \) into (32). Interchanging the order of integration and computing arising integrals by exploiting (32) yield the proof of (39). Equation (40) is derived in the same way from (37) as \( \eta(s) = \tanh s, \rho + \gamma = \mu, \) and \( \gamma - 1 = \nu \).

On making the changes \( \sigma = \lambda + 1, x = \tanh \alpha, \) and \( t = \tanh s, \) (40) turns into the integral derived by Collins [9], although his proof is somewhat more complicated. Collins noted that the Mehler-Dirichlet integral (28) can be obtained from the above-mentioned relation as \( \mu = -1/2 \). The connection relation (39) is new. A curious shift operator for exponents is derived from (39) by setting \( \mu = \pm 1/2, \rho = -\nu - 1, \lambda = \eta - 1/2, \tanh \alpha = \cos \varphi, \) and \( \tanh s = \cos \theta \) and using expressions of Ferrers functions of order \( \pm 1/2 \) in terms of trigonometric functions [1].

**Corollary 8**

\[
e^{i(\nu-\eta)\varphi} = \frac{\Gamma(\nu + \frac{1}{2}) \sin^{\frac{1}{2} - \nu} \varphi}{\Gamma(\nu + \frac{1}{2}) \Gamma(\nu - \eta)} \int_{0}^{\pi} e^{i(\nu+\frac{1}{2})\theta} \sin^{\nu-\eta-1} \theta \frac{\sin^{\frac{1}{2} - \eta} (\varphi - \theta)}{\sin^{\frac{1}{2} - \eta} (\varphi - \theta)} \, d\theta,
\]

\( 0 < \varphi, \theta < \pi, \text{Re} \nu > \text{Re} \eta > -\frac{1}{2} \).
A novel fractional type integral representation of Ferrers functions, can be derived from (39) on setting \( \rho = -k - \mu - 1, \ k \in \mathbb{N}_0, \) and making use of (30):

**Corollary 9** Let \( 2 \Re \mu + k > \Re \lambda > -1. \) Then

\[
P_{\lambda - k - \mu} (\tanh \alpha) = \frac{2^\mu k! \Gamma (\mu + \frac{1}{2}) \cosh^{k+\mu-\lambda} \alpha}{\sqrt{\pi} \Gamma (k + 2\mu - \lambda) \Gamma (1 + \lambda)} \int_\alpha^{\infty} \frac{C_k^{(\mu+\frac{1}{2})} (\tanh s)}{(\sinh s - \sinh \alpha)^{\lambda-2}} \cosh^{k+2\mu} s. \tag{42}\]

In particular, as \( 2 \Re \mu > \Re \lambda > -1, \)

\[
P_{\lambda - \mu} (\tanh \alpha) = \frac{2^\mu \Gamma (\mu + \frac{1}{2}) \cosh^{\mu-\lambda} \alpha}{\sqrt{\pi} \Gamma (2\mu - \lambda) \Gamma (1 + \lambda)} \int_\alpha^{\infty} \frac{\cosh^{-2\mu} s ds}{(\sinh s - \sinh \alpha)^{\lambda-2}}. \tag{43}\]

Setting into (39) \( \rho = -k-1, \ k \in \mathbb{N}_0, \) and employing (41), we arrive at another fractional type integral representation of Ferrers functions generalizing (32):

**Corollary 10** Let \( \Re \mu + k > \Re \lambda > -1. \) Then

\[
P_{\lambda - k} (\tanh \alpha) = \frac{k! \cosh^{k-\lambda} \alpha}{\Gamma (k + \mu - \lambda) \Gamma (1 + \lambda)} \int_\alpha^{\infty} \frac{e^{-\mu s} P_k^{(\mu, -\mu)} (\tanh s)}{(\sinh s - \sinh \alpha)^{\lambda-2}} ds. \tag{44}\]

Note that integral representations obtained in this section, together with analytic continuation, give rise to the following simple differentiation formulas (which can be also obtained by using the differentiation formulas for Gauss hypergeometric functions): as \( n \in \mathbb{N}_0, \)

\[
\frac{d^{n+1}}{dx^{n+1}} P_{\gamma - 1}^{\gamma - \gamma} (x) = \frac{(-1)^{n+1} 2^{1-\gamma}}{\Gamma (\gamma) (1 - x^2)^{1-\gamma}},
\]

\[
\frac{d^{n+1}}{dx^{n+1}} \left( P_{n-\mu} (x) \left( \frac{1 - x}{1 + x} \right)^{\frac{\mu}{2}} \right) = \frac{2^{n+1} (1 - x)^{\mu-n-1}}{\Gamma (\mu-n) (1 + x)^{\mu+n-1}},
\]

\[
\left( (1 - x^2)^{\frac{\mu}{2}} \frac{d}{dx} \right)^{n+1} P_{n-\mu} (x) = \frac{(-1)^{n} (1 - x)^{\frac{\mu+n}{2}}}{\Gamma (\mu-n) (1 + x)^{\frac{\mu-n}{2}}},
\]

\[
\left( (1 - x^2)^{\frac{\mu}{2}} \frac{d}{dx} \right)^{n+1} P_{n-\mu} (x) = \frac{2^n \Gamma (\mu + \frac{1}{2}) (1 - x)^{\mu}}{\sqrt{\pi} \Gamma (2\mu - n)}.
\]

A representation of Ferrers functions in the form of a fractional type integral containing a Gauss hypergeometric function can be derived by exploiting the integral (41). As \( \eta (s) = \tanh s, \) on inserting (41) into (41) and interchanging the order of integration

\[
P_{\nu - \mu} (\tanh \alpha) = \frac{2^{1+\nu} e^{\alpha \mu}}{\Gamma (\mu - \nu) \Gamma (\lambda + 1) \Gamma (\nu - \lambda)} \int_\alpha^{\infty} \frac{J (u, s) du}{\cosh^{2} s (\tanh s - \tanh \alpha)^{\lambda}},
\]

\[
J (u, s) = \int_{s}^{\infty} \frac{e^{-2\mu} \cosh^{2\nu} u}{(\tanh u - \tanh s)^{\lambda+1-\nu}} du.
\]
The integral \( J(u, s) \) can be readily evaluated by making the change \( e^{-2u} = e^{-2s}z \) and using Euler’s integral representation of the Gauss hypergeometric function. Finally,

\[
\frac{2^\nu e^{-\alpha \mu} P_{\nu-\mu} (\tanh \alpha)}{\Gamma^{-1} (\mu - \lambda) \Gamma^{-1} (1 + \lambda)} = \int_\alpha^\infty \frac{F \left( -\nu - \lambda - 1, \mu - \nu; \mu - \lambda; -e^{-2s} \right) ds}{(\tanh s - \tanh \alpha)^{-\lambda} e^{(2\nu - \nu - \lambda - 1)s} \cosh^{\nu - \lambda + 1} s},
\]

\( \Re \mu > \Re \lambda > -1 \).

5 Series and integral relations connecting Ferrers functions of different degrees and orders

We commence with

**Theorem 11** Let \( \Re (\mu + \nu) > -1 \). Then on \(-1 < x < 1\),

\[
\frac{P_{\nu}^{-\mu} (x)}{\Gamma (\nu - \mu + 1)} = \frac{2^{-\mu} (1 - x)^\mu}{\Gamma (\nu + \mu + 1)} \sum_{n=0}^\infty \frac{(-2\mu)_n (-\nu)_n}{n!} \left( \frac{1 + x}{1 - x} \right)^\frac{\mu}{\nu} P_{\nu-\mu}^{-n} (x)
\]

(45)

**Proof.** Suppose first that \((\Re \gamma, \Re \sigma)\) is an interior point of the triangle \( \Delta \) in the \((\Re \gamma, \Re \sigma)\)-plane with \( \Delta \) described by the inequalities \( \Re \gamma > 0, \Re \sigma > -1, \Re (\sigma + \gamma) > 0, \) and \( \Re (\sigma + 2\gamma) < 1 \). By using the absolutely convergent binomial series

\[
2^\kappa \sinh^{\frac{\kappa}{2}} t = \sum_{n=0}^\infty \frac{(-\kappa)_n e^{(\frac{\kappa}{2} - n)t}}{n!}, \quad \kappa > 0, \quad t \geq 0,
\]

we rewrite (11) in the form

\[
P_{-\gamma}^{-\sigma} (\tanh \alpha) = \frac{2^{-2\gamma - 3\gamma} \cosh \alpha}{\Gamma (\gamma) \Gamma (\sigma + 1)} \lim_{\varepsilon \to +0} \int_{\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \sum_{n=0}^\infty \frac{(-2\gamma - 2\gamma)_n e^{(\sigma + \gamma - n)t}}{n! \cosh \left( \frac{\varepsilon}{2} + \alpha \right) [\sinh \frac{\varepsilon}{2} \cosh \left( \frac{\varepsilon}{2} + \alpha \right)]^{\sigma + 2\gamma}} dt,
\]

Since the above series converge absolutely and uniformly on \( 0 < \varepsilon \leq t \leq 1/\varepsilon \), one is allowed to integrate term-by-term. By exploiting the integral representation (10) and denoting \( \gamma = -\nu \) and \( \sigma + \gamma = \mu \), we obtain the expression

\[
\frac{\Gamma (\nu + \mu + 1) P_{\nu}^{-\mu} (\tanh \alpha)}{2^{-\nu} \Gamma (\nu - \mu + 1) \cosh^{-\nu} \alpha} = \sum_{n=0}^\infty \frac{(-2\mu)_n (-\nu)_n}{n! e^{(\mu - n)\alpha}} P_{\nu-\mu}^{-n} (\tanh \alpha) - \lim_{\varepsilon \to +0} P_{\varepsilon},
\]

(46)
where

\[
\text{Re}(\mu - \nu) < 1, \text{ Re}(\mu + \nu) > -1, \text{ Re}\nu < 0, \text{ Re}\mu > 0,
\]

\[
\frac{P_{\zeta}}{P(\alpha)} = \sum_{n=0}^{\infty} \frac{(-2\mu)_n}{n!} V_n(\alpha, \varepsilon),\quad P(\alpha) = \frac{2^{\nu-\mu} \cosh^{-\nu-\mu} \alpha}{\Gamma(\nu - \mu + 1) \Gamma(-\nu)},
\]

\[
V_n(\alpha, \varepsilon) = \left( \int_{0}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\infty} \right) \frac{e^{(\mu-n)\varepsilon t} dt}{\sinh \left( \frac{t}{2} \cosh \left( \frac{\mu}{2} + \alpha \right) \right)^{\nu - \varepsilon}},
\]

\[
|V_n(\alpha, \varepsilon)| \leq V(\varepsilon) = \left( \int_{0}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\infty} \right) \frac{e^{\text{Re} \mu \varepsilon t} dt}{\sinh \left( \frac{t}{2} \cosh \left( \frac{\mu}{2} + \alpha \right) \right)^{\text{Re}(-\nu)}}.
\]

The inequality \(|P_{\zeta}(\alpha)| \leq V(\varepsilon)|P(\alpha)| \sum_{n=0}^{\infty} (-2\mu)_n / n!\) manifests that the second term in (45) vanishes because the integrand in \(V(\varepsilon)\) is an integrable function. Now, after the change \(\tanh \alpha = x\), we obtain (45) which is valid for \((\text{Re}\mu, \text{Re}\nu)\) belonging to the triangle \(\text{(47)}\). By taking into account an asymptotic formula

\[
\Gamma(\lambda - \rho) P_{\tau}^{-\lambda}(\tanh \alpha) = \frac{\Gamma(\lambda - \rho)}{\Gamma(\lambda + 1)} e^{-\lambda \alpha} \left( 1 + O \left( \frac{1}{\lambda} \right) \right)
\]

\[
= \frac{e^{-\lambda \alpha}}{\lambda^{-\rho}} \left( 1 + O \left( \frac{1}{\lambda} \right) \right), \quad (48)
\]

which follows on from \(1\) and \(13\) as \(|\lambda| \to \infty\) in the sector \(|\arg \lambda| \leq \delta < \pi\), one can infer that the series in (45) converges absolutely and uniformly with respect to \(\mu\) and \(\nu\) on any bounded region of \(\mathbb{C}^2\) in which \(\text{Re}(\mu + \nu) > -1\). Then the right side of (45) is an analytic function \(\mu\) and \(\nu\) on such a region, and the theorem is proven by virtue of analytic continuation. ■

**Corollary 12** As \(x \in (-1, 1), \mu, \nu \in \mathbb{C},\) and \(l, k \in \mathbb{N}_0\), there are the connection relations

\[
\frac{P_{\kappa}^{-\mu}(x)}{\Gamma(k - \mu + 1)} = \frac{-2^{-\mu} (1 - x)^{\mu} k!}{\Gamma(k + \mu + 1)} \sum_{n=0}^{k} \frac{(-1)^n (-2\mu)_n}{n! (k - n)!} \left( \frac{1 + x}{1 - x} \right)^{\frac{\mu}{2} - n} P_{k - \mu}^{\mu-n}(x); (49)
\]

\[
\frac{P_{\nu}^{-\frac{\mu}{2}}(x)}{\Gamma(\nu - \frac{\mu}{2} + 1)} = \frac{-2^{-\frac{\mu}{2}} (1 - x)^{\frac{\mu}{2}} l!}{\Gamma(\nu + \frac{\mu}{2} + 1)} \sum_{n=0}^{l} \frac{(-1)^n (-\nu)_n}{n! (l - n)!} \left( \frac{1 + x}{1 - x} \right)^{\frac{\mu}{2} - n} P_{\nu - \frac{\mu}{2}}^{\frac{\mu}{2}-n}(x); (50)
\]

The restriction on \(\mu\) and \(\nu\) in the above corollary was again discarded due to analytic continuation.

Another connection relation can be derived by employing the absolutely and uniformly convergent on \(y \geq 0\) series

\[
e^{-\mu y} = \left( \frac{1 + \sqrt{1 - \cosh^2 \frac{\mu}{2} y}}{\cosh \frac{2\mu y}{2}} \right)^{-2\mu} = \sum_{n=0}^{\infty} \frac{A_{\mu}^{(n)}(\mu)}{\cosh^{2n+2\mu} \frac{y}{2}}, \quad (51)
\]

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where $\mu \in \mathbb{C}$ and
\[ A^{(\mu)}_n = \mu \frac{(2\mu + n + 1)_{n-1}}{2^{2n+2\mu-1}n!} \]
are Taylor-Maclaurin coefficients of the function $(1 + \sqrt{1 - x})^{-2\mu}$ found with Lagrange’s expansion by noting that $y(x) = 1 + \sqrt{1 - x}$ is a solution of the equation $y = a - x/y$ as $a = 2$.

**Theorem 13** Let $\mu, \nu \in \mathbb{C}$. Then on $0 < x < 1$,
\[
\frac{P^{-\mu}_\nu(x)}{(1 + x)^\mu} = \mu \sum_{n=0}^{\infty} \frac{(2\mu + n + 1)_{n-1}}{2^{2\mu+3\mu-1}n!} (\mu - \nu) P^{-\mu-n}_{n+\mu-\nu-1}(x),
\]
\[
\frac{P^{-2\mu}_\nu(x)}{(1 - x^2)^{\frac{\nu}{2}}} = \frac{\mu}{2^\nu} \sum_{n=0}^{\infty} \frac{\Gamma(n + \mu)(2\mu - \nu)_{2n}}{2^n \Gamma(n + 2\mu + 1)n!} P^{-\mu-n}_{n+\mu-\nu-1}(x).
\]

**Proof.** When $\alpha > 0$ and $\text{Re} \mu > \text{Re} \nu > 0$, inserting the above series into (51) with $y = t + 2\alpha$ into (10) and integrating term-by-term lead to a series of integrals. On evaluating arising integrals with (11) one obtains
\[
P^{-\mu}_\nu(\tanh \alpha) = e^{\mu \alpha} \sum_{n=0}^{\infty} A^{(\mu)}_n \frac{\Gamma(n + \mu - \nu)}{\Gamma(\mu - \nu) \cosh^{n+\mu} \alpha}.
\]

Now, note that as $\mu = \lambda + n$ and $\nu = \gamma + n$, $n \in \mathbb{N}_0$, the asymptotic formula for Gauss hypergeometric functions with large parameters [26], [27] shows the hypergeometric function in the definition (2) to be a bounded quantity when $\alpha > 0$, $\lambda$, and $\gamma$ are fixed. Then,
\[
|\Gamma(\lambda + n + 1) P^{-\lambda-n}_{\gamma+n}(\tanh \alpha)| \leq B(\alpha, \lambda, \gamma) \frac{2^n}{\cosh^n \alpha}, \quad 0 < \alpha < \infty.
\]

The above estimate manifests that series in (54) converges absolutely and uniformly with respect to $(\mu, \nu)$ belonging to any bounded region of $\mathbb{C}^2$, that is, restrictions on the parameters $\mu$ and $\nu$ can be discarded due to analytic continuation. Finally, on changing $\tanh \alpha = x$ the proof of (52) is completed. The connection formula (53) is proved in the same manner by inserting (51) into (52) and evaluating arising integrals with (43). $\blacksquare$

By employing (5), we have

**Corollary 14** Let $\mu \in \mathbb{C}$ and $k \in \mathbb{N}_0$. Then on $0 \leq x < 1$,
\[
\frac{P^{\mu}_{k+\mu}(x)}{(1 + x)^\mu} = \frac{\mu k!}{2^{3\mu-1}} \sum_{n=0}^{k} (-1)^n (2\mu + n + 1)_{n-1} P^{-n-\mu}_{k-n}(x),
\]
and on $-1 < x < 1$,
\[
\frac{P^{-2\mu}_{k+2\mu}(x)}{(1 - x^2)^{\frac{k}{2}}} = \frac{\mu k!}{2^\mu} \sum_{n=0}^{[k]} \frac{\Gamma(n + \mu)(1 - x^2)^{\frac{k}{2}}}{2^n \Gamma(n + 2\mu + 1)(k - 2n)!} P^{-n-\mu}_{k-n}(x).
\]

One can obtain additional similar relations as follows.
Theorem 15 Let $\sigma, \gamma, \nu, \mu \in \mathbb{C}$. Then on $0 < x < 1$,

$$P_{\gamma-1}^{\sigma-\gamma} (x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! (\gamma - 1)^n}{2^{n-2} (n!)^2 (1-x^2)^{n-2}} P_{n+\gamma-2}^{\sigma-\gamma-n} (x); \quad (58)$$

$$P_{\lambda-\mu}^{-\mu} (x) = \frac{\Gamma (\mu + \frac{3}{2})}{\Gamma (\mu + 1)} \sum_{n=0}^{\infty} \frac{(2n)! (2\mu - \lambda)_{2n} P_{n-\lambda+\mu+\frac{1}{2}}^{-n-\mu+\frac{1}{2}} (x)}{(-1)^n 2^{3n-\frac{1}{2}} n! (\mu + 1) (1-x^2)^{n-2}}, \quad (59)$$

$$-\mu - \frac{3}{2} \notin \mathbb{N}_0.$$

Proof. Let $\alpha > 0$, $2 \text{Re} \mu > \text{Re} \lambda > 1$, $\text{Re} \gamma > 1$, and $\text{Re} \sigma > -1$. Then integrating the integral representations (37) and (38) by parts, we get

$$P_{\gamma-1}^{\sigma-\gamma} (\tanh \alpha) = \frac{2^{2-\gamma} \cosh^{\sigma+\gamma} \alpha}{\Gamma (\gamma - 1) \Gamma (\sigma + 2)} \int_{\alpha}^{\infty} \frac{(\tanh u - \tanh \alpha)^{\sigma+1} \sinh u}{\cosh^{2\gamma-1} u} du,$$

$$P_{\lambda-\mu}^{-\mu} (\tanh \alpha) = \frac{2^{\mu+1} \Gamma (\mu + \frac{3}{2}) \cosh^{\mu-\lambda} \alpha}{\sqrt{\pi} \Gamma (2\mu - \lambda) \Gamma (\lambda + 2)} \int_{\alpha}^{\infty} \frac{\sinh^{2n-2} u \sinh u du}{(\sinh u - \sinh \alpha)^{\lambda-1}}.$$ 

Substituting the binomial series

$$\sinh u = \cosh u \sqrt{1 - \frac{1}{\cosh^2 u}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n+1} n! \cosh^{2n+1} u}, \quad u > 0,$$

and integrating term-by-term lead to the expansions (58) and (59) where $x = \tanh \alpha$. Finally, due to (55) one can ascertain that the theorem holds. ■

Corollary 16 Let $\mu, \sigma \in \mathbb{C}$ and $k \in \mathbb{N}_0$. Then on $0 \leq x < 1$,

$$P_{k-1}^{k-\sigma} (x) = k! \sum_{n=0}^{k} \frac{(2n)!}{2^{n-2} (k-n)! (n!)^2} P_{k-n-\sigma-1}^{k-n} (x), \quad (60)$$

If $-\mu - \frac{3}{2} \notin \mathbb{N}_0$, then on $-1 < x < 1$,

$$P_{k+\mu}^{-\mu} (x) = \frac{\Gamma (\mu + \frac{3}{2}) k!}{\Gamma (\mu + 1)} \sum_{n=0}^{\infty} \frac{(2n)! (1-x^2)^{\frac{2n+1}{2}}}{(-1)^n 2^{3n-\frac{1}{2}} (k-2n)! n! (\mu + 1)_n} P_{k+\mu-n+\frac{1}{2}}^{-n-\mu+\frac{1}{2}} (x). \quad (61)$$

As $\text{Re} \sigma > -1$ and $\text{Re} \gamma > 0$, a new integral connection between Ferrers functions can be obtained by rewriting (37) in the form

$$P_{\gamma-1}^{\sigma-\gamma} (\tanh \alpha) = \frac{2^{1-\gamma} \cosh^{\sigma+\gamma} \alpha}{\Gamma (\gamma) \Gamma (\sigma + 1)} \int_{-\infty}^{\infty} f (u, \alpha) \frac{e^{2\alpha u}}{\cosh^{2(\gamma+\sigma)} u} du, \quad (62)$$

$$f (u, \alpha) = H (u - \alpha) e^{-2\alpha u} (\tanh u - \tanh \alpha)^{\sigma} \cosh^{2\sigma} u,$$
where \( \text{Re} \sigma < \text{Re} \epsilon < \text{Re} (\gamma + \sigma) \) and \( H(u) \) is the Heaviside unit function. Employ Parseval’s equation for Fourier transforms \( [33] \),

\[
\int_{-\infty}^{\infty} g_1(u) g_2(u) \, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_1(\omega) G_2(-\omega) \, d\omega, \quad (63)
\]

\[
G_m(\omega) = \int_{-\infty}^{\infty} g_m(u) e^{i\omega u} \, du,
\]

\[
g_1(u) \in L^p(-\infty, \infty), \ G_2(\omega) \in L^p(-\infty, \infty), \ 1 \leq p \leq 2,
\]
to transform \( (62) \). On exploiting \( (31) \) and

\[
\int_{-\infty}^{\infty} \frac{e^{pz}}{\cosh^q u} \, du = 2^{q-1} \frac{\Gamma \left( \frac{q+1}{2} \right) \Gamma \left( \frac{q-p}{2} \right)}{\Gamma (q)}, \ \text{Re} (q \pm p) > 0, \quad (64)
\]

it results after the changes \( z = \epsilon - i \frac{\omega^2}{2}, \ \sigma + \gamma = \mu, \) and \( \gamma = -\nu \) in the connection formula

\[
P_{-\mu}^{-\nu} (\tanh \alpha) = - \frac{2\mu}{2 \pi i} \int_{\text{Re} z = \epsilon} \frac{P_{\mu+\nu}^{-z} (\tanh \alpha) \Gamma (\mu - z) e^{-z\alpha}}{\Gamma^{-1} (\mu + z) \Gamma^{-1} (z - \mu - \nu)} \, dz, \quad (65)
\]

where \( \text{Re} \mu > 0, \ \text{Re} \nu < 0, \) and \( \max (-\text{Re} \mu, \text{Re} (\mu + \nu)) < \text{Re} \epsilon < \text{Re} \mu \).

The integral in \( (65) \) can be evaluated as \( \alpha > 0 \) and \( \alpha < 0 \) in the form of certain series by applying the residue theorem. This evaluation and analytic continuation yield as \( \text{tanh} \alpha = x, \)

**Theorem 17** As \( \nu, \mu \in \mathbb{C} \) and \( 0 < x < 1, \)

\[
P_{\nu}^{-\mu} (x) = 2^\mu (1 + x)^{-\mu} \sum_{n=0}^{\infty} \frac{(-1)^n (2\mu)_n (-\nu)_n}{n!} n \left( \frac{1 - x}{1 + x} \right)^{\frac{n}{2}} P_{\mu+\nu-n}^n \left( x \right), \]

If \( 2\mu + \nu \notin \mathbb{Z} \) when \( \nu \notin \mathbb{Z} \) and \( -1 < x < 0, \)

\[
\frac{2^{-\mu} P_{\nu}^{-\mu}(x)}{(1 - x^2)^{-\frac{\mu}{2}}} = \sum_{n=0}^{\infty} \frac{(2\mu)_n \Gamma (-\nu - 2\mu - n)}{(-1)^n \Gamma (-\nu) n!} \frac{1 + x}{1 - x} \frac{n + \mu}{2} P_{\nu+\mu+n}^{\mu+n} \left( x \right)
\]

\[
+ \sum_{n=0}^{\infty} \frac{(-\nu)_n \Gamma (2\mu + \nu - n)}{(-1)^n n! \Gamma (2\mu)} \frac{1 + x}{1 - x} \frac{n - \mu}{2} P_{\nu+\mu-n}^{\nu-n} \left( x \right),
\]

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Corollary 18 As $x \in (-1, 1)$, $k, l \in \mathbb{N}_0$, and $\mu, \nu \in \mathbb{C}$, 

\[
\frac{2^{-\mu} P^{-\mu}_k (x)}{k! (1 + x)^{-\mu}} = \sum_{j=0}^{k} \frac{(2\mu)_j}{(k-j)!j!} \left( \frac{1-x}{1+x} \right)^j P^{-j-\mu}_{k+j} (x), \tag{66}
\]

\[
\frac{2^{\nu} P^\nu_l (x)}{\nu! (1 + x)^{\nu}} = \sum_{j=0}^{l} \frac{(-\nu)_j}{(l-j)!j!} \left( \frac{1-x}{1+x} \right)^j P^{-j+\nu}_{l-j} (x). \tag{67}
\]

The integral connection formula

\[
P^\mu_{\lambda-\mu} \left( \frac{\tanh \alpha}{\cosh \mu} \right) = \frac{\mathcal{A}_0}{2\pi i} \int_{\text{Re} z = \epsilon} \frac{\Gamma (z - \lambda) P^{-z}_{\lambda-\mu} \left( \frac{\tanh \alpha}{\cosh \mu} \right)}{\Gamma^{-1} \left( \mu - \frac{1}{2} \right) \Gamma^{-1} \left( \mu + \frac{1}{2} \right)} dz,
\]

\[
\mathcal{A}_0 = \frac{\Gamma \left( 2\mu - \lambda \right) \Gamma (\mu)}{\Gamma (2\mu - \lambda) \Gamma (\mu)}, \quad \text{Re} \lambda > -1, \text{Re} \mu > 0,
\]

is derived in the same manner from \[63\] by exploiting \[32\] and \[62\]. Evaluating the integral by the residue theorem and using analytic continuation, we have

Theorem 19 As $\lambda, \mu \in \mathbb{C}$ and $x > 0$,

\[
P^{-\mu}_{\lambda-\mu} (x) = \frac{2^{3\mu} \Gamma \left( \mu + \frac{1}{2} \right)}{\sqrt{\pi} (1-x^2)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2\mu - \lambda)_{2n} (2\mu)_n}{(-1)^n n!} P^{-2\mu-2n}_{\lambda-2n} (x).
\]

Corollary 20 Let $\mu \in \mathbb{C}$ and $k, \lambda \in \mathbb{N}_0$. Then as $-1 < x < 1$,

\[
P^{-\mu}_{k+\mu} (x) = \frac{2^{3\mu} k! \Gamma \left( \mu + \frac{1}{2} \right)}{\sqrt{\pi} (1-x^2)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2\mu - \lambda)_{2n} (2\mu + k)_{2n}}{(-1)^n n! (k-2n)!} P^{-2\mu-2n}_{\lambda} (x) \tag{68}
\]

and

\[
P^k_{\lambda+k} (x) = (-2)^{k} k! (1-x^2)^{\frac{1}{2}} \sum_{n=0}^{2k} \frac{(-2k-\lambda)_{2n}}{n! (2k-n)!} P^{2k-2n}_{\lambda} (x). \tag{69}
\]

If $\lambda \in \mathbb{C}\setminus\mathbb{N}_0$, then \[69\] holds as $0 \leq x < 1$.

The formula connecting Ferrers functions with products of Ferrers functions, namely

\[
P^{-\mu}_\nu (x) = B \int_{-\infty}^{\infty} \frac{P^{-\mu_{1}-i\omega}_{\nu_{1}} (x) P^{-\mu_{2}+i\omega}_{\nu_{2}} (x)}{\Gamma^{-1} (\mu_{1} - \nu_{1} + i\omega) \Gamma^{-1} (\mu_{2} - \nu_{2} - i\omega)} d\omega, \tag{70}
\]

\[
B = \frac{\Gamma (1 + \nu_{1}) \Gamma (1 + \nu_{2})}{2\pi \Gamma (1 + \nu) \Gamma (\mu - \nu)},
\]

where

\[
\mu = \mu_{1} + \mu_{2}, \nu = \nu_{1} + \nu_{2}, \nu > -1, -\nu_{k} \notin \mathbb{N}, \text{Re} (\nu_{k} - \mu_{k}) \notin \mathbb{N}_0, \tag{71}
\]

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is obtained from (10) by using Parseval’s equation for Fourier integral transforms, making the change \( \tanh \alpha = x \), and taking such \( \mu_k \) and \( \nu_k \) that \( \text{Re} \mu_k > \text{Re} \nu_k > -1 \). One can see from (48) that for any \( \mu_k \) the integrand in (70) is \( O(\omega^{2-\nu}) \) as \( |\omega| \to \infty \). If \( \text{Re}(\nu_k - \mu_k) \notin \mathbb{N}_0 \), the integrand is an analytic function of parameters \( \nu_k \) and \( \mu_k \), and then integral (70) holds under conditions (71) due to analytic continuation.

On evaluating integral (70) by the residue theorem and denoting \( \nu_1 = \lambda \) we obtain after analytic continuation

\[
P_{\nu}^{-\mu}(x) = \Gamma(1+\lambda) \frac{\Gamma(1+\nu-\lambda)}{\Gamma(1+\nu)} \sum_{n=0}^{\infty} (-1)^n \frac{(\mu-\nu)_n}{n!} P_{\lambda}^{n-\lambda}(x) P_{\nu-\lambda}^{n-\mu+\lambda}(x). \tag{72}
\]

The above expression can be transformed by some manipulations to the form established by Campos [7] whose proof is quite different.

Employing Parseval’s equations for cosine and sine Fourier integral transforms again leads from (10) to similar integral relations. In particular, as \( \nu, \mu \in \mathbb{R}, \mu_1 = \mu_2 = \mu/2, \nu_1 = \nu_2 = \nu/2, \) and \( \mu > \nu > -1, \)

\[
P_{\nu}^{-\mu}(tanh \alpha) = \frac{2\Gamma^2(1+\nu/2)}{\pi \Gamma(\mu-\nu) \Gamma(1+\nu)} \int_0^\infty \left( \frac{\text{Re} e^{-i\omega \alpha} P_{\frac{\mu}{2}}^{-\frac{\mu}{2}+i\omega}(tanh \alpha)}{\Gamma^{-1}\left(\frac{\mu-\nu}{2} - i\omega\right)} \right)^2 d\omega
\]

\[
= \frac{2\Gamma^2(1+\nu/2)}{\pi \Gamma(\mu-\nu) \Gamma(1+\nu)} \int_0^\infty \left( \frac{\text{Im} e^{-i\omega \alpha} P_{\frac{\mu}{2}}^{-\frac{\mu}{2}+i\omega}(tanh \alpha)}{\Gamma^{-1}\left(\frac{\mu-\nu}{2} - i\omega\right)} \right)^2 d\omega
\]

\[
= \frac{\Gamma^2(1+\nu/2)}{\pi \Gamma(\mu-\nu) \Gamma(1+\nu)} \int_0^\infty \left| \frac{P_{\frac{\mu}{2}}^{-\frac{\mu}{2}+i\omega}(tanh \alpha)}{\Gamma^{-1}\left(\frac{\mu-\nu}{2} - i\omega\right)} \right|^2 d\omega.
\]

### 6 Differential relations between Ferrers functions of different degrees and orders

By denoting \( \rho = \nu - n \) and \( \lambda = n - 1, n \in \mathbb{N}, \) (89) is written as

\[
\frac{P_{\nu}^{-\mu}(tanh \alpha)}{\cosh^{-\nu} \alpha} = \frac{(\mu - \nu)_n}{(n-1)!} \int_\alpha^\infty \frac{P_{\nu-n}^{-\mu}(tanh s) \cosh^{n+1} s}{(\sinh s - \sinh \alpha)^{n-1}} ds \tag{73}
\]

\[
\text{Re}(\mu - \nu) > -1.
\]

Therefore,

\[
\frac{(\mu - \nu)_n P_{\nu-n}^{-\mu}(tanh \alpha)}{(-1)^n \cosh^{n-\nu} \alpha} = \left( \frac{1}{\cosh \alpha} \frac{d}{dx} \right)^n P_{\nu}^{-\mu}(tanh \alpha) \cosh^{-\nu} \alpha. \tag{74}
\]
Rewriting (73) in the form
\[
\frac{\cosh^\nu \alpha}{\omega^{1-n}(\alpha)} P^\nu_{\nu}(\tanh \alpha) = \frac{(\mu - \nu)_n}{(n-1)!} \int_\omega^\infty \frac{P^\nu_{\nu-n}(\tanh s) \cosh^{\nu-n+1}s}{\omega^{n-1}(s) (\omega(s) - \omega(\alpha))^{1-n}} ds,
\]
we obtain
\[
\frac{(-1)^n (\mu - \nu)_n}{(z + \sinh \alpha)^{n-1} \cosh^{n-\nu} \alpha} P^\nu_{n-n-\nu}(\tanh \alpha) = \left( \frac{(z + \sinh \alpha)^2}{\cosh \alpha} \right)^n \frac{d^n}{dx^n} \left( \frac{\cosh^\nu \alpha P^\nu_{\nu}(\tanh \alpha)}{(z + \sinh \alpha)^{n-1}} \right). \quad (75)
\]

Now, if \(-1 < x < 1\) and \(x + z \sqrt{1 - x^2} \neq 0, z \in \mathbb{C}\), then
\[
P^\nu_{n-n-\nu}(x) = \frac{(-1)^n}{(1 - x^2)^{\frac{n+1}{2}}} \frac{d^n}{dx^n} \left( \frac{P^\nu_{\nu}(x)}{(1 - x^2)^{\frac{n}{2}}} \right). \quad (76)
\]
\[
q^{n+1}(x, z) P^\nu_{n-n-\nu}(x) = \frac{(-1)^n}{(1 - x^2)^{\frac{n+1}{2}}} \frac{d^n}{dx^n} \left( \frac{q^{1-n}(x, z) P^\nu_{\nu}(x)}{(1 - x^2)^{\frac{n-1}{2}}} \right). \quad (77)
\]
\[
q(x, z) = x + z \sqrt{1 - x^2}.
\]

In the same way, as \(\mu = \sigma - n\) and \(\lambda = n - 1\), it follows from (40) that if \(-1 < x < 1\) and \(x + z \neq 0\), then
\[
\frac{(-1)^n P^\nu_{\nu-\sigma}(x)}{P^\nu_{\nu}(x)} = \frac{d^n}{dx^n} \left( \frac{P^\nu_{\nu-\sigma}(x)}{(1 - x^2)^{\frac{n-\nu}{2}}} \right). \quad (78)
\]
\[
\frac{(x + z)^{n+1} P^\nu_{\nu-\sigma}(x)}{(1 - x^2)^{\frac{n+1}{2}}} = \left( \frac{(x + z)^2}{(1 - x^2)^{\frac{n+1}{2}}} \right) \frac{d^n}{dx^n} \left( \frac{P^\nu_{\nu}(x)}{(x + z)^{\frac{n-1}{2}}} \right). \quad (79)
\]

The well-known (usually as \(\sigma\) is an integer) elementary differential relation (75) for \(n = 1\) also follows from the formula of the derivative of Ferrers functions [20], [28] and then by induction for any \(n\). The differential relation (76) can be obtained in the same manner as well (though surprisingly there is no source of reference known to the author in the general case \(n > 1\)). Readers can find several similar differential recurrences (for \(n = 1\)), including (76) and (78), in papers by Celeghini and del Olmo [8] when \(\nu, \mu\) are integers, and by Maier [21] when \(\nu, \mu\) are arbitrary numbers. The differential relations (77) and (79) are not so evident and probably are new.

Combining or setting specific values of parameters, one can obtain from (76), (77), and (79) various relations. For example, as \(z = \tan \varphi\) and \(x = \cos (\varphi + \theta), 0 < \varphi + \theta < \pi\), (77) results in
\[
\frac{(\mu - \nu)_n}{\cos^{n-1} \theta \sin^{n+1}(\theta + \varphi)} \frac{P^\nu_{\nu-n-\nu}(\cos (\theta + \varphi))}{P^\nu_{\nu}(\cos (\theta + \varphi))} = \left( \frac{\cos^2 \theta d}{d\theta} \right)^n \frac{P^\nu_{\nu}(\cos (\theta + \varphi))}{\cos^{\nu+1-n}(\theta + \varphi)}. \quad (77)
\]

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Another example is the relation obtained from (78) by making use of (14), namely

\[
P_{\nu}^{n-\nu} (x) = \frac{(-1)^n (1 - x^2)^{\frac{\mu - \nu}{2}}}{2^n \Gamma (\nu + 1)} \frac{d^n}{dx^n} (1 - x^2)^{\nu},
\]

that due to (3) is the classical Rodrigues formula for Jacobi polynomials \( P_n^{(\nu-n,\nu-n)} (x) \).

7 Generating functions

Ferrers functions \( P_{\nu}^{-\mu} (x), x \in (1,1) \), can be analytically continued (by means of (2) or (3)) to the complex \( z \)-plane with cuts connecting branch points \( z = \pm 1 \) and the point at infinity. We denote such analytic functions as \( \hat{P}_{\nu}^{-\mu} (z) \) to avoid some confusion between associated Legendre functions \( P_{\nu}^{-\mu} (z) \) and continuations of Ferrers functions.

**Theorem 21** Let \( \mu, \nu, s \in \mathbb{C} \). Then, for \( x \in (1,1) \) there are the generating functions:

\[
\sum_{n=0}^{\infty} \left( \mu - \nu \right)_n P_{n-\nu-1}^{\nu-\mu} (x) \frac{s^n}{n!} = \frac{\hat{P}_{\nu}^{-\mu} \left( \sqrt{1 - 2xs + x^2} \right)}{(1 - 2sx + s^2)^{\frac{s}{4}}},
\]

where \( s \in \mathbb{C} \) if \( \nu - \mu \in \mathbb{N}_0 \) and \( |s| < 1 \) for \( \nu - \mu \notin \mathbb{N}_0 \). If \( |s| = 1 \) and \( \nu - \mu \notin \mathbb{N}_0 \), then (87) is valid either as \( \text{Re} \nu > 1/2 \) or as \(-1/2 < \text{Re} \nu \leq 1/2 \) while \( \text{arg} s \neq \pm \arccos x \);

\[
\sum_{n=0}^{\infty} \frac{P_{\nu}^{n-\mu} (x) s^n}{(1 - x^2)^{\frac{n-\mu}{2}}} \frac{n!}{n} = \frac{\hat{P}_{\nu}^{-\mu} (x - s)}{1 - (x - s)^2}^{-\frac{1}{4}},
\]

where \( s \in \mathbb{C} \) if \( \nu, \mu \in \mathbb{Z} \); if \( \nu, \mu \notin \mathbb{Z} \), then either \( |s| \leq 1 - |x| \) as \( \text{Re} \mu > 0 \) or \( |s| < 1 - |x| \) as \( \text{Re} \mu \leq 0 \); if \( \nu \in \mathbb{Z} \) and \( \mu \notin \mathbb{Z} \), then either \( |s| \leq 1 + x \) as \( \text{Re} \mu > 0 \) or \( |s| < 1 + x \) as \( \text{Re} \mu \leq 0 \); if \( \nu \in \mathbb{Z} \) and \( \mu \notin \mathbb{Z} \), then either \( |s| \leq 1 - x \) as \( \text{Re} \mu > 0 \) or \( |s| < 1 - x \) as \( \text{Re} \mu \leq 0 \);

\[
\sum_{n=0}^{\infty} P_{\nu}^{n-\mu} (x) \frac{s^n}{n!} = \frac{\hat{P}_{\nu}^{-\mu} (x - s \sqrt{1 - x^2})}{(1 + 2s \frac{x}{\sqrt{1 - x^2}} - s^2)^{\frac{1}{4}}},
\]

where \( s \in \mathbb{C} \) if \( \nu, \mu \in \mathbb{Z} \), if \( \nu, \mu \notin \mathbb{Z} \), then either \( |s| \leq \sqrt{1 - |x|}/\sqrt{1 + |x|} \) as \( \mu > 0 \) or \( |s| < \sqrt{1 - |x|}/\sqrt{1 + |x|} \) as \( \text{Re} \mu \leq 0 \); if \( \nu \in \mathbb{Z} \) and \( \mu \notin \mathbb{Z} \), then either \( |s| \leq \sqrt{1 + x}/\sqrt{1 - x} \) as \( \text{Re} \mu > 0 \) or \( |s| < \sqrt{1 + x}/\sqrt{1 - x} \) as \( \text{Re} \mu \leq 0 \); if \( \mu \in \mathbb{Z} \) and \( \nu \notin \mathbb{Z} \), then either \( |s| \leq \sqrt{1 - x}/\sqrt{1 + x} \) as \( \mu > 0 \) or \( |s| < \sqrt{1 - x}/\sqrt{1 + x} \) as \( \mu \leq 0 \).

**Proof.** On changing \( \sinh \alpha = t, -\infty < t < \infty \), (74) leads to the relation

\[
\frac{(-1)^n (\mu - \nu)_n}{(1 + t^2)^{\frac{\nu}{2}}} P_{n-\nu-1}^{\nu-\mu} \left( \frac{t}{\sqrt{1 + t^2}} \right) = \frac{d^n P_{\nu}^{\nu-\mu} \left( \frac{t}{\sqrt{1 + t^2}} \right)}{dt^n} \left( \frac{1 + t^2}{1 + t^2} \right)^{-\frac{1}{4}},
\]

(84)
which holds for $n \in \mathbb{N}$. Write the Taylor series

$$
\sum_{n=0}^{\infty} \frac{(-1)^n (\mu - \nu)^n}{(1 + u^2)^{n+\nu}} P_{n-\nu-1}^{-\nu} \left( \frac{u}{\sqrt{1+u^2}} \right) \frac{(t-u)^n}{n!} = \frac{P_{\nu}^{-\mu} \left( \frac{t}{\sqrt{1+t^2}} \right)}{(1+t^2)^{-\virg{\frac{\nu}{2}}}}.
$$

As $P_{\nu}^{-\mu} \left( t/\sqrt{1+t^2} \right)$ is replaced by $\hat{P}_{\nu}^{-\mu} \left( t/\sqrt{1+t^2} \right)$, (85) is valid by virtue of analytic continuation for complex $t$ lying within the disk of convergence. Now, by changing

$$
\frac{u}{\sqrt{1+u^2}} = x, \quad \frac{u-t}{\sqrt{1+u^2}} = s,
$$

we obtain (81). If $\nu - \mu \notin \mathbb{N}_0$, then the series in (81) becomes a finite sum and $s \in \mathbb{C}$. If $\nu - \mu \notin \mathbb{N}_0$, then the well-known asymptotics of Ferrers functions [13, 1]: for $\lambda \to \infty$ and $-1 + \varepsilon < x < 1 - \varepsilon$, $\varepsilon > 0$,

$$
P_{\lambda}^{-\mu} (x) = \frac{\Gamma (\lambda - \mu + 1)}{\Gamma (\lambda + \frac{\nu}{2})} \left( x + O \left( \frac{1}{\lambda} \right) \right),
$$

$$
p (\lambda, x) = \sqrt{\frac{2}{\pi}} \cos \left( \frac{\lambda + \frac{\nu}{4}}{2} \right) \arccos x - \frac{\pi}{16} - \frac{\pi}{2} x \sin \pi \nu
$$

together with the asymptotics of gamma functions, leads to the estimate: as $n \to \infty$

$$
(\mu - \nu) s^n P_{n-\nu-1}^{-\mu} (x) \frac{s^n}{n!} = \frac{s^n}{n^{\nu+\frac{1}{2}}} \left( \frac{p (n-\nu-1, x)}{\Gamma (\mu - \nu)} + O \left( \frac{1}{n} \right) \right),
$$

which yields the restrictions imposed on $|s|$.

The generating functions (82) is derived in the same way as (81) by employing (83). If $\nu, \mu \in \mathbb{Z}$, then the series in (82) is truncated and $s \in \mathbb{C}$. In the case when at least one of the numbers $\nu$ and $\mu$ is not an integer, we suppose by virtue of (1) that $\text{Re} \nu > -1$. Invoking the asymptotic formula (33) and using the relation

$$
\sin \pi (\nu - \sigma) \frac{\Gamma (\nu + \sigma + 1)}{\Gamma (\nu - \sigma + 1)} P_{\nu}^{-\sigma} (x) = \frac{P_{\nu}^{-\sigma} (x) \sin \pi \nu}{\Gamma (\nu - \sigma + 1)} - \frac{\sin \pi \sigma P_{\nu}^{-\sigma} (-x)}{\Gamma (\nu - \sigma + 1)}
$$

manifest that as $n \to \infty$,

$$
P_{n}^{-\mu} \left( x \right) = \frac{\Gamma (n-\mu+\nu+1)}{n^{\nu+1}} \left( \frac{1-x}{1+x} \right)^{\nu} C_n (x) \sin \pi \mu + \frac{\Gamma (n+\nu+1)}{n^{\nu+1}} \left( \frac{1+x}{1-x} \right)^{\nu} \alpha (x),
$$

$$
C_n (x) = \alpha \left( x \right) + O \left( \frac{1}{n} \right), \quad \alpha \left( x \right) = \frac{1-x}{1+x} \frac{\Gamma (n+\nu+1)}{n^{\nu+1}} \left( 1-x^2 \right)^{\nu},
$$

hence

$$
P_{n}^{-\mu} \left( x \right) \frac{s^n}{n! (1-x^2)^{\nu}} = \frac{s^n}{n^{\nu+1}} \left( \frac{\alpha \left( x \right) + O \left( \frac{1}{n} \right)}{(1+x)^n} \sin \pi \mu + \frac{\alpha (-x) + O \left( \frac{1}{n} \right)}{(1-x)^n} \sin \pi \nu \right).
$$
Analyzing the above relations, we arrive to the restrictions imposed on the parameter $s$.

The generating function (83) follows on from (82) by a change of a parameter. The proof is completed.

The generating functions (81) and (83) were obtained by Truesdell [34] but restrictions on the parameter $s$ were not indicated, and besides, analytic continuation of Ferrers functions is required as $s \in \mathbb{C}$. Truesdell also noted that the generating function for Gegenbauer polynomials (7) is a special case of (81).

Setting $x = 0$ and $s = -t/\sqrt{1-t^2}$ into (81) and using (15) yield new representations for Ferrers functions.

**Corollary 22** Ferrers functions are represented in the forms

$$P_{\nu}^{\mu} (t) = \frac{\sqrt{\pi}}{2\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (\mu - \nu)_n t^n (1-t^2)^{\frac{\nu-n}{2}}}{\Gamma \left( \frac{n+\mu+\mu+1}{2} \right) \Gamma \left( \frac{\mu+\nu-n+2}{2} \right) n!}$$

(86)

$$|t| < \frac{\sqrt{2}}{2} \text{ if } \nu - \mu / \in \mathbb{N}_0.$$

If $\nu - \mu = l \in \mathbb{N}_0$,

$$P_{l+\mu}^{\nu} (t) = \frac{\sqrt{\pi l!}}{2\mu} \sum_{j=0}^{l} \frac{t^n (1-t^2)^{\frac{l+\mu+n}{2}}}{\Gamma \left( \frac{n-\nu+1}{2} \right) \Gamma \left( \frac{2\mu+l-n+2}{2} \right) n!}$$

$$= \frac{l!}{2^{\nu} \mu} \sum_{j=0}^{l} \frac{(-1)^j t^l 2j (1-t^2)^{j+\frac{l}{2}}}{(l-2j)! \Gamma \left( j+\mu+1 \right)}.$$ 

(87)

Setting $x = 0$ and $s = -t$ into (83) and using (15) lead to

**Corollary 23** Ferrers functions are represented on $-1 < t < 1$ in the form

$$\frac{P_{\nu}^{\mu} (t)}{(1-t^2)^{\frac{\mu}{2}}} = \frac{\sqrt{\pi}}{2\nu} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{\Gamma \left( \frac{\mu-n-\nu+1}{2} \right) \Gamma \left( \frac{\mu+\nu-n+2}{2} \right) n!}$$

(88)

$$= -\frac{\sin \pi \nu}{2\nu+1\pi} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{\Gamma \left( \frac{\nu+1}{2} \right) \Gamma \left( \frac{\mu-n-\nu+1}{2} \right) \Gamma \left( \frac{\mu+\nu-n+2}{2} \right) n!}$$

$$= -\frac{\sin \pi \nu}{2\nu+1\pi} \Psi_0 \left( \frac{\nu+1}{2}, \frac{\mu-n-\nu+1}{2}; -t \right)$$

$$-\frac{\sin \pi \nu}{2\nu+1\pi} \Psi_0 \left( \frac{\nu+1}{2}, \frac{\mu-n-\nu+1}{2}; t \right).$$

Note that even and odd parts of (86) and (88) can be written in terms of Gauss hypergeometric functions and then Ferrers functions are expressed in the form similar to that known for associated Legendre functions.

By making in (81) the changes $x = \cos \beta$ and $s = (\sin \alpha)^{-1} \sin (\alpha - \beta)$, we arrive at
Corollary 24 If $\beta \in (-\pi, \pi)$ and $\alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$, $\alpha/\pi \notin \mathbb{Z}$, then

\[
P_{\nu}^{-\mu}(\cos \alpha) = \frac{1}{\sin \beta} \sum_{n=0}^{\infty} \frac{(\mu - \nu)_n \sin^n (\alpha - \beta)}{n! \sin^{n-\nu} \alpha} P_{n-\nu-1}^{-\mu}(\cos \beta).
\] (89)

When $\nu - \mu \notin \mathbb{N}$, there are the additional restrictions: $\sin (2\alpha_1 - \beta) < 0$ and

\[
\alpha = \frac{\beta}{2}
\] also holds as $\sin (2\alpha_1 - \beta) = 0$ if either $\text{Re} \nu > 1/2$ or $-1/2 < \text{Re} \nu \leq 1/2$ while $\arg (\sin (\alpha - \beta) / \sin \alpha) \neq \pm \beta$.

The following formulas originate from (82) on changing variables.

Corollary 25 Let $x, y \in \mathbb{R}$. If $x^2 + y^2 < 2$, then

\[
P_{\nu}^{-\mu}(xy) = \frac{4^{-\mu}}{(1 - x^2 y^2)^{\frac{\mu}{2}}} \sum_{n=0}^{\infty} \frac{P_{\nu}^{-\mu}((x+y)^2)}{4^n} (x - y)^{2n}. \] (90)

If $-1 < x, y < 1$, then

\[
P_{\nu}^{-\mu}(x) = \frac{2^{-\mu}}{(1 - x^2)^{\frac{\mu}{2}}} \sum_{n=0}^{\infty} \frac{P_{\nu}^{-\mu}(\frac{x+y}{2})}{4^n} (y - x)^n. \] (91)

If $0 < \theta + \varphi < \pi$, then

\[
\frac{P_{\nu}^{-\mu}(\cos (\theta + \varphi))}{\sin^{-\mu}(\theta + \varphi)} = \sum_{n=0}^{\infty} \frac{\sin^n \theta \sin^n \varphi}{n!} \frac{P_{\nu}^{-\mu}(\cos \theta \cos \varphi)}{(1 - \cos^2 \theta \cos^2 \varphi)^{\frac{n-\mu}{2}}}. \] (92)

\[
\frac{2^\mu P_{\nu}^{-\mu}(\cos \theta \cos \varphi)}{(1 - \cos^2 \theta \cos^2 \varphi)^{\frac{\mu}{2}}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cos^n (\theta - \varphi)}{[4 - \cos^2 (\theta + \varphi)]^{\frac{n-\mu}{2}}} \frac{P_{\nu}^{-\mu}(\cos(\theta - \varphi))}{n!}. \] (93)

A new generating function for Ferrers functions $P_{\nu}^{\mu-k}(x)$ arises as a partial case of the next result.

Theorem 26 Let $z > -1$, $\lambda, t \in \mathbb{C}$. If $|t| < z^{-2} - 1$ as $-1 < z < -2^{-1/2}$ while $-\lambda \notin \mathbb{N}$, $|t| < 1$ as $z \geq -2^{-1/2}$ or $-\lambda \in \mathbb{N}$, then

\[
f_{\lambda}(t, z) = (1 + z\sqrt{1 - t})^{-\lambda} = \sum_{n=0}^{\infty} \mathfrak{A}_{\lambda, n}(z) \frac{t^n}{n!}, \] (94)

where

\[
\mathfrak{A}_{\lambda, n}(z) = \frac{(\lambda)_n z^n}{(z + 1)^{\lambda+n}} \frac{(1 - n, n; 1 - n - \lambda; 1 + z/2z)}{2^n n!}. \] (95)

and also if $-\lambda \notin \mathbb{N}$ or $|z| < 1$,

\[
\mathfrak{A}_{\lambda, n}(z) = 2^n \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \Gamma (\frac{j+1}{2} - n + 1)}{\Gamma (\frac{j}{2} - n + 1) j!}. \] (96)
The functions $\mathcal{A}_n^{(\lambda)}(z)$ are analytic functions of the parameter $\lambda$ obeying the recurrence relations

$$
\mathcal{A}_{n+1}^{(\lambda)}(z) = 2n\mathcal{A}_n^{(\lambda)} - z\frac{d\mathcal{A}_n^{(\lambda)}}{dz},
$$

(97)

$$
\mathcal{A}_0^{(\lambda)} = \frac{1}{(1 + z)^\lambda}, \quad \mathcal{A}_1^{(\lambda)} = \frac{\lambda z}{(1 + z)^{\lambda+1}},
$$

and

$$(z^2 - 1) \mathcal{A}_{n+2}^{(\lambda)}(z) + \alpha(z, \lambda, n) \mathcal{A}_{n+1}^{(\lambda)}(z) + \beta(\lambda, n) z^2 \mathcal{A}_n^{(\lambda)}(z) = 0,
$$

(98)

where

$$
\alpha(z, \lambda, n) = 2n + 1 - (4n + 2\lambda + 3) z^2,
$$

$$
\beta(\lambda, n) = 4n^2 + 2n(2\lambda + 1) + \lambda(\lambda + 1).
$$

**Proof.** To obtain the explicit expression (95), we note that the function $y = 1 + z\sqrt{1 - t}$ is a solution of the equation $y = a - tz^2/y + z - 1$ as $a = z + 1$. Then Lagrange’s expansion in powers of $t$ for the function $y^{-\lambda} = (1 + z\sqrt{1 - t})^{-\lambda}$ manifests that as $n \geq 1$ and $z > 1$,

$$
\mathcal{A}_n^{(\lambda)}(z) = 2^n \left[ \frac{d^{n-1}}{da^{n-1}} \left( \frac{(-z^2)^n}{(a + z - 1)^n} \right) \right]_{a = z+1}
$$

$$
= \lambda \left(2z^2\right)^n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-j-1)!(a + z - 1)^{2n-j-1}}
$$

$$
= \frac{(z + 1)^{-\lambda}}{\Gamma(\lambda)} \sum_{j=0}^{n-1} \frac{\Gamma(j + \lambda + 1)\Gamma(2n - j)}{j!(n-j)!2^{n-j}}.
$$

On making the change $j = n - 1 - k$ and employing Euler’s reflection formula for gamma functions the above sum becomes

$$
\mathcal{A}_n^{(\lambda)}(z) = \frac{z^n\Gamma(1 - \lambda)}{(z + 1)^{\lambda+n}} \sum_{k=0}^{n-1} \frac{(-1)^{n-k}\Gamma(n + k)(z + 1)^k}{\Gamma(k + 1 - n - \lambda)(n - k - 1)!k!}
$$

Then, by taking into account that $(-m)_k = (-1)^k m!/ (m - k)!$ as $m \geq k$, $m, k \in \mathbb{N}_0$,

$$
\mathcal{A}_n^{(\lambda)}(z) = \frac{z^n\Gamma(1 - \lambda)}{(z + 1)^{\lambda+n}} \sum_{k=0}^{n-1} \frac{(n)_k (1 - n)_k(z + 1)^k}{\Gamma(1 - n - \lambda)_k k!}
$$

$$
= \frac{z^n\Gamma(n + \lambda)}{(z + 1)^{\lambda+n}} \sum_{k=0}^{n-1} \frac{(n)_k (1 - n)_k(z + 1)^k}{\Gamma(1 - n - \lambda)_k k!},
$$

24
and we arrive at (95), which holds for \( n = 0 \) also. The conditions imposed on the radius of convergence of the power series (94) arise from the locations of the singular points of \( f_{\lambda}(t, z) \).

Expanding \((1 + zu)^{-\lambda}\) with \( u = \sqrt{1 - t} \), into a binomial series and substituting a binomial series in powers of \( t \) for \( u^n \), on interchanging the order of summation one readily obtains the expansions (94) with \( A_{\lambda}^n(z) \) given by the series (96) which is absolutely convergent as \(-1 < z < 1\) or truncated as \(-\lambda \notin \mathbb{N}_0\).

To derive recurrence relations, note that
\[
\frac{\partial f_{\lambda}}{\partial t} = \frac{\lambda z}{2\sqrt{1 - t}} f_{\lambda + 1},
\]
\[
z \frac{\partial f_{\lambda}}{\partial z} = -\lambda z \sqrt{1 - t} f_{\lambda + 1} = -2 \lambda (1 - t) \frac{\partial f_{\lambda}}{\partial t},
\]
and hence
\[
f_{\lambda + 1} = f_{\lambda} - z \sqrt{1 - t} f_{\lambda + 1} = f_{\lambda} - \frac{2}{\lambda} (1 - t) \frac{\partial f_{\lambda}}{\partial t},
\]
(99)

Note that
\[
\frac{\lambda (\lambda + 1) z^2}{4} f_{\lambda + 2} = \sqrt{1 - t} \frac{\partial}{\partial t} \sqrt{1 - t} \frac{\partial f_{\lambda}}{\partial t} = (1 - t) \frac{\partial^2 f_{\lambda}}{\partial t^2} - \frac{1}{2} \frac{\partial f_{\lambda}}{\partial t},
\]
Then, excluding \( f_{\lambda + 2} \) from (100) yields the differential equation
\[
\frac{4 (z^2 (1 - t) - 1)}{\lambda (\lambda + 1) z^2} \left[ (1 - t) \frac{\partial^2 f_{\lambda}}{\partial t^2} - \frac{1}{2} \frac{\partial f_{\lambda}}{\partial t} \right] - \frac{4}{\lambda} (1 - t) \frac{\partial f_{\lambda}}{\partial t} + f_{\lambda} = 0.
\]
(101)
Now, (98) is obtained by inserting the series (94) into (101) and equating coefficients of like powers. In the same way, inserting the series (94) into (99) leads to (97). The proof is completed. ■

The series representation (96) gives rise to

**Corollary 27** If \( r \in \mathbb{N}_0 \), then the functions \( \mathfrak{A}_n^{(r)}(z) \) are polynomials

\[
\mathfrak{A}_n^{(r)}(z) = (-1)^n 2^n r! \sum_{j=0}^r \frac{\Gamma \left( \frac{r}{2} + 1 \right)}{(r-j)! \Gamma \left( \frac{r}{2} + n + 1 \right)} \frac{z^j}{j!}.
\]
(102)

If \( z > 1 \), then the functions \( \mathfrak{A}_n^{(\lambda)}(z) \) by virtue of (11) and (13) are expressed in terms of Ferrers functions

\[
\mathfrak{A}_n^{(\lambda)}(z) = \frac{(-z)^n \Gamma (1 - \lambda) p_n^{n+\lambda}}{(z^2 - 1)^{\lambda/2 + n}} \left( -\frac{1}{z} \right) = \frac{\lambda z^n \Gamma (2n + \lambda)}{(z^2 - 1)^{\lambda/2 + n}} p_{n-1}^{-n-\lambda} \left( \frac{1}{z} \right).
\]
Making the changes $z = \sec \theta, t = s \sin \theta, \lambda = -\mu - 1$ and $n = k + 1$ in (94), we obtain by differentiation with respect to the parameter $s$

**Corollary 28** The generating function

$$\sum_{k=0}^{\infty} \Gamma (2k + 1 - \mu) P_k^\mu (\cos \theta) \frac{s^k}{2^k k!} = \left(\cos \theta + \sqrt{1 - s \sin \theta}\right)^\mu \sqrt{1 - s \sin \sin \theta}$$

is valid as $\mu, s \in \mathbb{C}, \theta \in (0, \pi/2)$, and $|s| \sin \theta < 1$.

**8 Addition theorems**

In this section, analogs of addition theorems expressing $P_{\nu}^\mu (\tanh (\alpha + \beta))$ as series of products of the form $(1 + \tanh \beta \tanh \alpha)^{-\nu} V_n (\beta) V_n (\alpha) P_{\nu - \alpha n}^\alpha (\tanh \alpha)$ will be obtained. Such theorems are interesting because $P_{\nu}^\mu (\tanh \alpha)$ arise in various problems of mathematical physics described by a Schrödinger operator with a regular Pöschl-Teller potential on the line. The simplest addition theorem expressing $P_{\nu}^\mu (x + y)$ in terms of $P_{\nu}^\mu (x)$ is given by (82) as $x, y \in \mathbb{R}$ and $|x| + |y| < 1$. If $x = (1 + \tanh \alpha \tanh \beta)^{-1} \tanh \alpha$ and $y = (1 + \tanh \alpha \tanh \beta)^{-1} \tanh \beta$, then $P_{\nu}^\mu (x + y) = P_{\nu}^\mu (\tanh (\alpha + \beta))$.

We commence with

**Theorem 29** Let $\gamma, \mu \in \mathbb{C}$, Then the addition formula

$$\frac{P_{\nu - 1}^\mu (\tanh (\alpha + \beta))}{(1 + \tanh \beta \tanh \alpha)^{-\gamma}} = \sum_{n=0}^{\infty} \frac{(\gamma)_n 2^{n(\gamma + \mu)} (\tanh \beta) P_{\gamma + n - 1}^\mu (\tanh \alpha)}{n! \cosh^\mu (\beta) \cosh^\nu \alpha}$$

(103)

is valid as $-\gamma - \mu \in \mathbb{N}_0$ or $\alpha > 0$ and $\cosh \alpha > \sinh \beta$.

**Proof.** According to (36), for $\text{Re} \, \gamma > 0$ and $\mu - \gamma = \sigma, \text{Re} \, \sigma \geq 0$,

$$\frac{P_{\gamma - 1}^{\sigma - \gamma} (\tanh (\alpha + \beta))}{\cosh^\gamma (\alpha + \beta)} = \frac{2^{1-\gamma}}{\Gamma (\gamma) \Gamma (\sigma + 1)} \int_{\alpha + \beta}^{\infty} \sinh^\sigma (u - \alpha - \beta) \frac{1}{\cosh^{2\gamma + \sigma} u} du,$$

or by making the change $u = t + \beta$,

$$\frac{P_{\gamma - 1}^{\sigma - \gamma} (\tanh (\alpha + \beta))}{\cosh^\gamma (\alpha + \beta)} = \mathcal{W} (\beta) \int_0^{\infty} \sinh^\sigma (t - \alpha) \frac{1}{\cosh^{2\gamma + \sigma} t} f_{2\gamma + \sigma} \left(\frac{1}{\cosh^2 t}, \tanh \beta\right) dt,$$

$$\mathcal{W} (\beta) = \frac{2^{1-\gamma}}{\Gamma (\gamma) \Gamma (\sigma + 1) \cosh^{2\gamma + \sigma} \beta}.$$
estimate \( \text{55} \) and the asymptotic formula \( \text{15} \), one can see that absolute values of terms of the series in \( \text{103} \) are less than

\[
A(\mu, \gamma, \alpha) = \sum_{n=0}^{\infty} \left| \mathfrak{A}_n^{(k)}(\tanh \beta) \right| \frac{n^{\gamma-\mu-1}}{2^n n! \cosh^{2n} \alpha}.
\]

Then, we infer by employing the preceding theorem that the series in \( \text{103} \) converges absolutely and uniformly with respect to \((\mu, \gamma)\) belonging to any bounded region of \( \mathbb{C}^2 \), and therefore it is an analytic function of parameters \( \mu \) and \( \gamma \). The proof of the theorem is completed by analytic continuation. \( \blacksquare \)

**Theorem 30** The addition formula

\[
\frac{P_{\nu}^{-\mu}(\tanh (\alpha + \beta))}{(1 + \tanh \beta \tanh \alpha)^{-\nu}} = \sum_{n=0}^{\infty} \frac{(\mu - \nu)_n (\nu)_n}{2^n (-1)^n n! \cosh^{2n} \alpha} \left( \frac{P_{\nu}^{-n}(\tanh \alpha)}{2^n n! \cosh^{2n} \alpha} \right), \quad (104)
\]

\[
\mathcal{F}_n(\beta) = \frac{e^{\beta(\nu-\mu)}}{\cosh^\nu \beta} F(-n, -\nu; 1 - \nu - n; e^{-2\beta}), \quad \alpha > 0,
\]

is valid as \( \alpha + \beta > 0 \) or as \( \beta \in \mathbb{R} \) and \( \nu \in \mathbb{N}_0 \).

**Proof.** According to \( \text{30} \), as \( \mu > \nu > 0 \),

\[
\frac{P_{\nu}^{-\mu}(\tanh (\alpha + \beta))}{\cosh^{-\nu}(\alpha + \beta)} = \frac{2^{1+\nu} e^{(\alpha+\beta)\mu}}{\Gamma(\mu - \nu) \Gamma(1 + \nu)} \int_{\alpha+\beta}^{\infty} \frac{e^{-2u\mu} \cosh^\nu u du}{\sinh^{-\nu}(u - \alpha - \beta)},
\]

or on making the change \( u = s + \beta \) and using the identity \( \cosh^\nu (s + \beta) = g_{\nu}(e^{-2s}, e^{-2\beta}) e^{\nu \cosh^\nu s} \) with \( g_{\nu}(t, z) = (1 + zt)^\nu (1 + t)^{-\nu} \),

\[
P_{\nu}^{-\mu}(\tanh (\alpha + \beta)) = \mathcal{P}(\alpha, \beta) \int_{\alpha}^{\infty} \frac{e^{-2\mu s} g_{\nu}(e^{-2s}, e^{-2\beta}) ds}{[\sinh(s - \alpha) \cosh s]^{-\nu}}, \quad (105)
\]

\[
\mathcal{P}(\alpha, \beta) = \frac{2^{1+\nu} e^{\alpha\mu} e^{(\nu-\mu)\beta}}{\Gamma(\mu - \nu) \Gamma(1 + \nu)} \cosh^{-\nu}(\alpha + \beta).
\]

Expand \( g_{\nu}(t, z) \) into a series in powers of \( t \). On multiplying binomial series for \((1 + t)^{-\nu} \) and \((1 + zt)^{-\nu} \) we have as \(|zt|, |t| < 1 \) or as \(|t| < 1 \) and \( \nu \in \mathbb{N}_0 \),

\[
g_{\nu}(t, z) = \left( \sum_{k=0}^{\infty} \frac{\Gamma(1 - \nu) t^k}{(1 - \nu - k)!} \right) \left( \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k - \nu) t^k z^k}{(\nu) k!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1 - \nu)(k - \nu) z^k}{k!} \right) t^n.
\]

Now, \( g_{\nu}(t, z) \) is seen to be a generating function for hypergeometric polynomials

\[
g_{\nu}(t, z) = \sum_{n=0}^{\infty} \frac{\Gamma(1 - \nu) n!}{n! \Gamma(1 - \nu - n)} \left( \sum_{k=0}^{n} \frac{(-n)_k (-\nu)_k z^k}{(1 - \nu - n) k!} \right) t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n (\nu)_n}{n!} F(-n, -\nu; 1 - \nu - n; z) t^n, \quad (106)
\]
and therefore \( g_\nu(e^{-2s}, e^{-2\beta}) \) in (105) can be written as an absolutely and uniformly convergent power series in powers of \( e^{-2s} \) on any interval \( s \geq \alpha_0 > 0 \). Now, (104) follows for \( \mu > \nu > 0 \) as the result of term-by-term integration of the series (by virtue of the uniform convergence) and making use of (30). Note that the series in (104) converges absolutely and uniformly on any bounded region of \( \mu \) and \( \nu \) for \( \alpha \geq \alpha_0 > 0 \) due to the estimate

\[
\left| \sum_{n=0}^{\infty} \frac{(\mu - \nu)_n}{(1 + \mu - \nu)_n} n! e^{\mu n} P_{\nu}^{-\mu-n} (\tanh \alpha) \right| \leq \mathcal{D}_0 \sum_{n=0}^{\infty} \frac{|(\nu)_n F_n(\beta)|}{(1 - \nu)_n n!} e^{\nu n} \leq \mathcal{D} \sum_{n=0}^{\infty} \frac{|(\nu)_n F_n(\beta)|}{n!} \leq e^{\nu n} \alpha_n
\]

involving absolute convergence of the power series (106), and therefore it is an analytic function of parameters \( \mu \) and \( \gamma \). Then analytic continuation brings in the proof of the theorem. ■

**Theorem 31** The addition formula

\[
\frac{P_{\nu}^\mu (\tanh (\alpha + \beta))}{(1 + \tanh \beta \tanh \alpha)^{-\nu}} = \sum_{n=0}^{\infty} \frac{(-\nu)_n}{(2\mu)_n} \frac{F_n(\beta)}{\Gamma(\gamma)} P_{\mu+\nu}^{-\mu-n} (\tanh \alpha), \tag{107}
\]

is valid for \( \mu, \nu \in \mathbb{C} \) if \( \alpha > 0 \) and \( \alpha + \beta \geq 0 \).

**Proof.** Initially, according to (36), as \( \Re \gamma > 0 \) and \( \Re \sigma \geq 0 \),

\[
P_{-\sigma-\gamma} (\tanh (\alpha + \beta)) = \frac{2^{1-\gamma}}{\Gamma(\gamma) \Gamma(\sigma + 1)} \int_{\alpha + \beta}^{\infty} \frac{\sinh^\gamma (u - \alpha - \beta)}{\cosh^{2\gamma + \sigma} u} \, du
\]

\[
= \frac{2^{\gamma+2\sigma+1}}{\Gamma(\gamma) \Gamma(\sigma + 1)} \int_{\alpha}^{\infty} \frac{e^{-2(\gamma + \sigma)} q_{\sigma,\gamma} (e^{-2u}, e^{-2\beta})}{(\sinh (u - \alpha) \cosh u)^{-\sigma}} \, du,
\]

where the function

\[
q_{\sigma,\gamma} (t, z) = (1 + t)^{-\sigma} (1 + zt)^{-2\gamma - \sigma} \text{is a generating function for hypergeometric polynomials} \tag{108}
\]

\[
q_{\sigma,\gamma} (t, z) = \sum_{n=0}^{\infty} \frac{(2\sigma + 2\gamma)_n}{(-1)^n n!} F(-n, 2\gamma + \sigma; 2\sigma + 2\gamma; 1 - z) t^n,
\]

\(|zt| < 1, \ |t| < 1 \).

By inserting (108) and integrating term-by-term with exploiting (36), one obtains (107) in the considered initial case on denoting \( \mu = \sigma + \gamma \) and \( \nu = -\gamma \). Now, the proof of the theorem for any \( \mu, \nu \in \mathbb{C} \) is accomplished by analytic continuation based on the uniform convergence of the series (107). The mentioned uniform convergence can be readily shown by using the absolute convergence of (108) and the asymptotic formula (59). ■
9 Gegenbauer polynomials

By setting $\nu = n + \tau - \frac{1}{2}$ in (80) and employing (6), we obtain the classical Rodrigues formula for Gegenbauer polynomials. The differential relations (76), (77), (79) and (84), together with (6), are sources of various other Rodrigues-like formulas. In particular, one gets by setting $\mu = \tau - \frac{1}{2}$ and $\nu = -\tau - \frac{1}{2}$ and using (14): from (84) as $-\infty < t < \infty$,

$$C_n^{(\tau)} \left( \frac{t}{\sqrt{1 + t^2}} \right) = \frac{(-1)^n}{n!} (1 + t^2)^{\frac{n+2\tau}{2}} \frac{d^n}{dt^n} \left( 1 + t^2 \right)^{-\tau};$$

from (75) as $\sinh \alpha = 1/t - z, z \in \mathbb{R},$ and $0 < t \leq \infty$,

$$C_n^{(\tau)} \left( \frac{1 - tz}{\sqrt{t^2 + (1 - tz)^2}} \right) = \left( \frac{t^2 + (1 - t^2)^2}{n!t^{2\tau - 1}} \right)^{\frac{n+2\tau}{2}} \frac{d^n}{dt^n} \left( \frac{t^{n+2\tau-1}}{(t^2 + (1 - t^2)^2)} \right);$$

and from (77) as $z = i$ and $x = \cos \theta, 0 < \theta < \pi$,

$$C_n^{(\tau)} (\cos \theta) = \frac{e^{-i(n+1)\theta}}{n!\sin^{2\tau - 1} \theta} \left( e^{2i\theta} \frac{d}{d\theta} \right)^n e^{-i(n-1)\theta} \sin^{n+2\tau-1} \theta.$$

The first of the above Rodrigues-like formulas is akin to the Tricomi formula derived in a more complicated manner [16].

Writing in (80) $P_n^{\nu-\nu}(x)$ instead of $P_n^{\nu-\nu}(x)$ and setting $\nu = \frac{1}{2} - m, n - \nu = k + \frac{1}{2}$ yield as $k, m \in \mathbb{N}_0$ and $k \geq m$,

$$C_{m+k}^{(-k)}(x) = \frac{(-1)^{k+m} 2^{m+k+1} k! (2m)!}{(k+m)! (k-m)! m!} \frac{d^{k-m}}{dx^{k-m}} \left( 1 - x^2 \right)^{-\frac{1}{2} - m}.$$

The well-known explicit representation for Gegenbauer polynomials [30] follows from (14) as $\lambda = \tau - \frac{1}{2} + l, l \in \mathbb{N}$.

Another simple explicit representation originates from (67), namely

$$C_l^{(\tau)} (t) = (2\tau)_l \sum_{j=0}^{l} \left[ \frac{l}{2} \right] \frac{1}{j!} \frac{t^{l-2j}}{l^{l-2j+1}} \left( 1 - t^2 \right)^{\tau}.$$
A relation connecting different Gegenbauer polynomials can be derived from (18) as \( \nu = -l - \mu - 1/2, \tau = l + 2\mu, \) and \( l \in \mathbb{N}, \)

\[
C_l^{(\mu)}(x) = \frac{\sqrt{\pi} \Gamma(l + 2\mu) \Gamma(l + \mu + 1)}{2^{-l-1} \Gamma(\mu)} \sum_{n=0}^{l} \frac{\Gamma^{-1} \left( \frac{n-l+1}{2} \right) \Gamma^{(l+2\mu)}(x)}{2^n \Gamma \left( \frac{n+3l+4\mu+2}{2} \right) (l-n)!} C_n^{(l+2\mu)}(x)
\]

\[
= \frac{2 \Gamma(l + 2\mu) \Gamma(l + \mu + 1)}{\Gamma(\mu)} \left[ \frac{l}{2} \right] \sum_{j=0}^{l/2} (l+1)^j C_{l-2j}^{(l+2\mu)}(x).
\]

It is an elegant special case of the known general relation established by basing on properties of Jacobi polynomials [3, Theorem 7.1.4].

Three other connection relations can be established from (57), (68) and (61):

\[
C_{k-2n}^{(\mu+\frac{1}{2})}(x) = \frac{4\mu + 1}{k + 2\mu + 1} \sum_{n=0}^{[\frac{k}{2}]} \frac{(2\mu)_{2n} (1 - x^2)^n C_{k-2n}^{(\mu+\frac{1}{2})}(x)}{2^{2n} n! (2\mu + 1)_n},
\]

\[
C_{k-2n}^{(\nu)}(x) = \frac{\Gamma(k + 2\nu + 1)}{2^{-4\nu} \sqrt{\pi}} \sum_{n=0}^{[\frac{k}{2}]} \frac{4^n (2\nu)_n \Gamma(2n + 2\nu + \frac{1}{2}) C_{k-2n}^{(2\nu+2\nu+\frac{1}{2})}(x)}{(-1)^n n! \Gamma(2n + k + 4\nu + 1) (1 - x^2)^{-n}},
\]

\[
C_{k-2n}^{(\nu+\frac{1}{2})}(x) = \frac{2\mu + 1}{k + 2\mu + 1} \sum_{n=0}^{[\frac{k}{2}]} \frac{(2\mu)_n (1 - x^2)^n C_{k-2n}^{(\nu+\frac{1}{2})}(x)}{(-1)^n n! 2^{2n} n!} C_{k-2n}^{(\nu+\frac{1}{2})}(x).
\]

A relation expressing an Gegenbauer polynomial as a sum of products of Gegenbauer polynomials follows from (12) as \( \nu = l + \mu, \) \( l \in \mathbb{N}_0, \) and \( \mu = \tau - \frac{1}{2}, \)

\[
C_l^{(\tau)}(x) = a(\tau, \lambda, l) \sum_{n=0}^{l} b_n(\tau, \lambda, l) C_n^{(\lambda - n + \frac{1}{2})}(x) C_{l-n}^{(\tau - \lambda + n)}(x),
\]

\[
a(\tau, \lambda, l) = \frac{\Gamma(l + 2\tau) \Gamma(l + \tau - \lambda + \frac{1}{2}) \Gamma(1 + \lambda)}{\sqrt{\pi} \Gamma(\tau) \Gamma(l + \tau + \frac{1}{2})},
\]

\[
b_n(\tau, \lambda, l) = \frac{\Gamma(\lambda - n + \frac{1}{2}) \Gamma(n + \tau - \lambda)}{\Gamma(2\lambda - n + 1) \Gamma(n + l + 2\tau - 2\lambda)}.
\]

\[\lambda - \tau - l + \frac{1}{2} \notin \mathbb{N}, -\lambda \notin \mathbb{N}, \lambda - \tau \notin \mathbb{N}_0, l - \lambda - \frac{1}{2} \notin \mathbb{N}_0.\]

Results obtained in section 7 and section 8 give a number of additional formulas. Two of them yield finite sums as \( \mu = \tau - \frac{1}{2} \) and \( \nu = l + \tau - \frac{1}{2}, \)
\( l \in \mathbb{N}_0. \) Then, the addition theorem [104] turns into an addition theorem for Gegenbauer polynomials:

\[
C_l^{(\tau)} \left( \frac{x + y}{1 + xy} \right) = \frac{1}{(1 + xy)^{l}} \sum_{n=0}^{l} \frac{(\tau)_n (l + \tau - \frac{1}{2})_n \Omega_n(y) C_{l-n}^{(\tau+n)}(x)}{2^{-n} n! (l + 2\tau)_n (1 - x)^{-n}},
\]

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where \(0 < x < 1, 0 < x + y < 1,\) and \(\Omega_n(y)\) are polynomials, namely
\[
\Omega_n(y) = (1 + y)^l F\left(-n, \frac{1}{2} - l; \frac{3}{2} - l - n; \frac{1 - y}{1 + y}\right).
\]

A finite sum case of the generating function (81)
\[
C_l^{(\tau)}\left(\frac{x - s}{\sqrt{1 - 2sx + s^2}}\right) = (1 - 2sx + s^2)^{-\frac{\tau}{2}} \sum_{j=0}^l (2\tau)_j (-1)^{l+j} (2\tau)_j (l - j)! C_j^{(\tau)}(x) \frac{s^j}{(1 + y)^j},
\]
\[x \in (-1, 1), \ s \in \mathbb{C},\]
is known [28].

The above expression involves helpful identities: as \(s = 2x,\)
\[
C_l^{(\tau)}(x) = \sum_{j=0}^l \frac{(2\tau)_j (-1)^j}{(2\tau)_j (l - j)!} C_j^{(\tau)}(x) (2x)^{l-j};
\]
as \(s = \pm 1,\)
\[
C_l^{(\tau)}\left(\sqrt{\frac{1 \pm x}{2}}\right) = \frac{\Gamma (l + 2\tau)}{2^{\frac{\tau}{2}} (1 \pm x)^{\frac{\tau}{2}}} \sum_{j=0}^l \frac{(-1)^j x^j (2\tau)_j (l - j)!}{\Gamma (j + 2\tau) (l - j)!} C_j^{(\tau)}(x);
\]
as \(s = 1/x,\)
\[
C_l^{(\tau)}\left(\sqrt{1 - x^2}\right) = (1 - x^2)^{-\frac{\tau}{2}} \sum_{j=0}^l \frac{(2\tau)_j (-1)^j x^j}{(2\tau)_j (l - j)!} C_j^{(\tau)}(x).
\]

In this case, (29) turns into the relation
\[
C_l^{(\tau)}(\cos \alpha) = \left(\frac{\sin \alpha}{\sin \beta}\right)^l \sum_{j=0}^l \frac{(2\tau)_j (-1)^j}{(2\tau)_j (l - j)!} \left(\frac{\sin (\beta - \alpha)}{\sin \alpha}\right)^{l-j} C_j^{(\tau)}(\cos \beta),
\]
\[\beta \in (0, \pi), \ \alpha \in \mathbb{C}, \ \alpha/\pi \notin \mathbb{Z},\]
which was derived in [29] by inventive manipulations in a rather tedious way.

In conclusion, we note that a number of integral and series representations for Gegenbauer polynomials, series connecting or containing Gegenbauer polynomials and integrals of Gegenbauer polynomials can be written by basing on results obtained in this article. For example, new fractional type integral operators relating Gegenbauer polynomials and Jacobi polynomials \(P_k^{(\mu,-\mu)}(x)\) follows from (42) and (43) by setting \(\lambda = \mu, \ k = n + 1,\) and \(\lambda = -\mu, \ k = n + 1,\) respectively:
\[
P_n^{(\mu,-\mu)}(\tanh \alpha) = \frac{(n + 1) \Gamma (\mu + \frac{1}{2})}{\sqrt{\pi 2^{\mu + n}} \Gamma (1 + \mu)} \int_0^\infty \frac{C_{n+\frac{1}{2}}^{(\mu+\frac{1}{2})}(\tanh s) ds}{(\sinh s - \sinh \alpha)^{-\mu} \cosh^{n+2\mu+1} s},
\]
\[\Re \mu > -1, \ n \in \mathbb{N}_0,\]
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relations

Ferrers associated Legendre functions (associated Legendre polynomials) are defined as

\[ P_{\mu}^m(x) = \frac{n!}{(\mu + \frac{1}{2}) (n + \mu + 1)!} \left( \frac{\sinh \alpha}{\cosh n + 2\mu + 1} \right) \int_{-n+1/2}^{n+1/2} e^{-\mu s} P_{\mu}^{n} \left( \frac{\sinh s}{\sinh \alpha} \right) \cos n + \mu + 1 s \, ds, \]

where \( \mu \) and \( \nu \) are integers, \( \nu > 0 \), and \( x = \cos \theta \).

10 Ferrers associated Legendre functions

Ferrers associated Legendre functions (associated Legendre polynomials) are defined as \( P_{\mu}^m(x) \) with \( k, m \in \mathbb{N}_0 \) and \( m \leq k \).

As \( \nu = k \), \( n = k - m \), and \( \mu = m \), (110) takes the form of the well-known relations

\[ P_{k}^m(-x) = (-1)^{k+m} P_{k}^m(x), \]

(110)

\[ P_{k}^m'(x) = (-1)^m \frac{(k-m)!}{(k+m)!} P_{k}^{m}(x). \]

(111)

If an order \( m \) is fixed and \( k - m \in \mathbb{N}_0 \),

\[ P_{k}^m(x) = (-1)^m \frac{(k+m)}{(k-m)!} P_{k-m}^m(x) = \frac{(2m)!}{(-2)^m m!} C_{m}^{(m+\frac{1}{2})}(x), \]

and therefore it is a complete orthogonal system on \((-1, 1)\) (see also [2] on orthogonal properties of Gegenbauer polynomials (and then associated Legendre functions) on the interior of an ellipse in the complex plane).

The classical Rodrigues formula for Ferrers associated Legendre functions is given by (30) where \( \nu = k \) and \( n = m + k \). Various Rodrigues-like relations can be derived from the differential relations (76), (77), (79), and (84). In particular, by setting \( \mu = m \), \( \nu = -m - 1 \), and \( n = k - m \) into (84) and (77) and by using (111), one obtains

\[ P_{k}^m \left( \frac{t}{\sqrt{1 + t^2}} \right) = \frac{2^{-m}}{(k-m)!} \frac{(1+t^2)^{\frac{1}{2}k-m}}{(-1)^k m! (k-m)!} \frac{d^k}{dt^k} \left( 1 + t^2 \right)^{\frac{2}{2}m+1} C_{k-m}^{m} \left( 1 + t^2 \right), \]

\[ P_{k}^m(\cos \theta) = \frac{(2m)! e^{-i(k-m+1)\theta}}{(-2)^m m! (k-m)! \sin^m \theta} \left( e^{2i\theta} \frac{d}{d\theta} \right)^{k-m} \frac{\sin^{m+k} \theta}{e^{i(k-m+1)\theta}}. \]

In another case, as \( \sigma = \nu = k \), \( n = k + m \), and \( z = \pm 1 \), (49) becomes

\[ P_{k}^m(x) = \frac{(-1)^{k+m} (1-x^2)^m}{2k! (1 \pm x)^{k-m+1}} \left( (1 \pm x)^2 \frac{d}{dx} \right)^{k+m} \frac{(1-x^2)^k}{(1 \pm x)^{k-m+1}}. \]

\[ P_{k}^m(x) = \frac{(-1)^k (k+m)! (1-x^2)^m}{2k! (k-m)! (1 \pm x)^{k-m+1}} \left( (1 \pm x)^2 \frac{d}{dx} \right)^{k-m} \frac{(1-x^2)^k}{(1 \pm x)^{k-m+1}}. \]
The known explicit representations for Ferrers associated Legendre functions

\[
P_k^m (x) = \frac{(1 - x^2)^{\frac{m}{2}}}{2^k} \sum_{j=0}^{[\frac{k-m}{2}]} \frac{(-1)^j (2k - 2j)! x^{k-m-2j}}{j! (k-m-2j)! (k-j)!},
\]

\[
P_k^m (x) = \frac{(k+m)!}{2^k (k-m)!} \sum_{j=0}^{[\frac{k+m}{2}]} \frac{(-1)^j (2k - 2j)! x^{k+m-2j}}{j! (k+m-2j)! (k-j)!},
\]

\[
P_k^m (x) = \frac{(k+m)!}{2m} \sum_{j=0}^{\frac{k-m}{2}} \frac{(-1)^j x^{k-m-2j} (1-x^2)^{\frac{m}{2}}}{2^{2j} j! (k-m-2j)! (j+m)!}.
\]

follow from (14) and (87). As \( x = 1 \) and \( m \geq 1 \), the above representations imply the combinatorial identity

\[
\sum_{j=0}^{[\frac{k}{2}]} \frac{(-1)^j (2k - 2j)!}{j! (n-2j)! (k-j)!} = 0, \; k, n \in \mathbb{N} \text{ and } k < n \leq 2k.
\]

Connection relations for associated Legendre functions are derived from (66) as \( \mu = \pm m \) and from (67) as \( l = 2m \) and \( \nu = -k - 1 \). On employing (111) we obtain:

\[
\frac{P_k^m (x)}{(1 + x)^m} = \frac{2^m (k+m)!k!}{(k-m)!} \sum_{n=0}^{\min(2m,k)} \binom{2m}{n} \frac{(-1)^n (2m)_n}{n! (k+2m+n)!} P_{k+n}^{m+n} (x),
\]

\[
\frac{P_k^m (x)}{(1 + x)^m} = \frac{(2m)!l!}{2^m} \sum_{n=\max(2m-k,0)}^{\min(2m,k)} \binom{2m}{n} \frac{(1-x)^{\frac{m}{2} + \frac{m}{2}}}{1+x} \frac{(-1)^n P_{k-n}^{m-n} (x)}{(k-n)! (2m-n)! n!},
\]

\[
\frac{P_k^m (x)}{(1 + x)^m} = \frac{(2m)!2m!}{2^m k!} \sum_{n=0}^{\min(2m,k)} \binom{2m}{n} \frac{(1-x)^{\frac{m}{2} + \frac{m}{2}}}{1+x} \frac{(k+n)! P_{k+m}^{m-n} (x)}{n! (2m-n)!}.
\]

Note that the above relations can be derived from (49) and (50) as well.

Another connection relation can be found from (57) as \( \mu = m \) and \( k = l-m \):

\[
P_{l+m}^{2m} (x) = \frac{(l+3m)!m}{(l+m)!} \sum_{n=0}^{\min(l,m)} \binom{m+n-1}{m-n} \frac{(1-x^2)^{\frac{m-n}{2}}}{(-2)^{n+m} (2m+n)! n!} P_{l+m-n}^{m+n} (x).
\]

A reciprocal formula

\[
P_l^m (x) = \frac{2^{m+1} (l+m)!}{(m-1)! (1-x^2)^{\frac{m}{2}}} \sum_{n=0}^{\min(l,m)} \frac{(-1)^{n+m} (2m+n-1)! P_{l+m-n}^{m+n} (x)}{(2n+l+3m)! n!}.
\]
and an additional connection relation

\[ P_l^m(x) = \frac{(-2)^{-m}(l+m)!m!}{(1-x^2)^{\frac{m}{2}}} \sum_{n=\max\left(\frac{1+\nu}{2},1\right)}^{\min\left(\frac{1+\nu}{2},2m\right)} \frac{P_{l-m}^{2m-2n}(x)}{(2m-n)!(l+m-2n)!n!} \]

are derived from (18) and (20) as polynomials, and an additional connection relation

\[ P_l^m(x) = \frac{r!(k-r)!(k+m)!}{(k-r)^{l-1}} \sum_{n=\max(0,2r-m-k)}^{\min(k-m,2r)} \frac{P_r^{n-r}(x)}{(k-m-n)!n!} P_{k-r}^{r-m-n}(x), \]

where \( r \in \mathbb{N}_0, 0 < r < k; \)

\[ P_k^m(x) = \frac{r!(k-r)!(k+m)!}{(k-r)^{l-1}} \sum_{n=\max(0,2r+m-k)}^{\min(k+m,2r)} \frac{P_r^{n-r}(x)}{(k+m-n)!n!} P_{k-r}^{r+m-n}(x), \]

where \( r \in \mathbb{N}_0, 0 < r < k. \)

The following expansions of associated Legendre functions into sums of Gegenbauer polynomials are found from (18) and (20) as \( \nu = -k - 1: \)

\[ P_k^m(x) = \frac{\Gamma\left(k + \frac{3}{2}\right)(m+k)!}{2^{m-1}\sqrt{\pi}(1-x^2)^{\frac{m}{2}}} \sum_{j=0}^{\min\left(k-m+1\right)\left(k+m+1\right)} (-1)^j C_{k+m-2j}\left(k-1\right)\frac{\left(k-j\right)!\left(k-m-j\right)!}{\left(1-x^2\right)^{\frac{m-j}{2}}}, \]

\[ \left(1 + \frac{x}{1-x}\right)\frac{m}{\sqrt{\pi}} P_k^{-m}(x) = \frac{(2k+2m+1)!}{22^k\left(k+1\right)\left(k+m\right)} \sum_{n=0}^{k} \alpha_n C_n^{(k+1)}(x), \]

where coefficients \( \alpha_n \) are given by

\[ \alpha_n = \frac{\Gamma\left(n-k,2m-2k-1,n+k+2m+2,-1\right)}{(n+k+2m+1)!(k-n)!}. \]

Another expression of associated Legendre functions in terms of Gegenbauer polynomials,

\[ P_{k+m}^m(x) = \frac{(2m+1)!}{(k+2m+1)m!} \sum_{n=0}^{k} \frac{(2n)!}{(-1)^n+m!} \frac{2^{2m}C_{k-2m}^{(n+m+1)}(x)}{2^{n+m}n!}, \]

is obtained from (61).

As \( \nu = k + m, \mu = m, \) and \( l = n = j, \) we get from (61) as \( s \in \mathbb{C}, \)

\[ \sum_{j=0}^{k} \frac{(-1)^{k+j}k^{-j}P_{j+m}^{m}(x)}{(l+2m)!\left(j-k\right)!} = \frac{P_{k+m}^{m}\left(\sqrt{1-2sx+s^2}\right)}{(k+2m)!(1-2sx+s^2)^{-\frac{m}{2}}}. \]

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and hence

\[
P_k^m(x) = (k + m)! \sum_{j=0}^{k-m} \frac{(-1)^j (2x)^{k-m-j} P_{j+m}^m(x)}{(j + 2m)! (k - m - j)!},
\]

\[
P_k^m \left( \sqrt{1 \pm \frac{x}{2}} \right) = \frac{(k + m)!}{2^\frac{k}{2} (1 \pm x)^{\frac{k}{2}}} \sum_{j=0}^{k-m} \frac{(-1)^j x^j P_{j+m}^m(x)}{(j + 2m)! (k - m - j)!},
\]

\[
P_k^m \left( \sqrt{1 - x^2} \right) = \frac{(k + m)!}{(1 - x^2)^{\frac{k}{2}}} \sum_{j=0}^{k-m} \frac{(-1)^j x^j P_{j+m}^m(x)}{(j + 2m)! (k - m - j)!},
\]

\[
\frac{\hat{P}_k^m(\cos \alpha)}{(k + 2m)!} = \sum_{j=0}^{k} \frac{\sin^{k-j} (\beta - \alpha) \sin^{m+j} \alpha}{(j + 2m)! (k - j)! \sin^{k+m} \beta} P_{j+m}^m(\cos \beta).
\]

As \( \nu = k \) and \( \mu = \pm m \), (82) leads to

\[
\sum_{n=0}^{k \pm m} \frac{P_{n+m}^m(x)}{(1 - x^2)^{\frac{n}{2} + \frac{m}{2}}} s^n n! = \frac{\hat{P}_k^m(x - s)}{(1 - (x - s)^2)^{\frac{n}{2} + \frac{m}{2}}},
\]

Now, (100), (101), (102), and (103) turn into finite sums and give us another sort of relations for associated Legendre functions. In particular, as \( \theta + \varphi \in (0, \pi) \),

\[
\frac{P_k^m(\cos (\theta + \varphi))}{\sin^{m+1} (\theta + \varphi)} = \sum_{n=0}^{k \pm m} \frac{\sin^n \theta \sin^n \varphi \cos^n \theta \cos^n \varphi}{n!} \frac{P_{n+m}^m(\cos \theta \cos \varphi)}{(1 - \cos^2 \theta \cos^2 \varphi)^{\frac{n}{2} + \frac{m}{2}}},
\]

\[
2^{\pm m} P_k^m(\cos \theta \cos \varphi) \left[ 1 - \cos^2 \theta \cos^2 \varphi \right]^{\frac{n}{2} + \frac{m}{2}} = \sum_{n=0}^{k \pm m} \frac{(-1)^n \cos^n (\theta - \varphi) P_{n+m}^m(\cos (\theta + \varphi))}{[4 - \cos^2 (\theta + \varphi)]^{\frac{n}{2} + \frac{m}{2}} n!}.
\]

The generation function (83) takes the form

\[
\sum_{n=0}^{k \pm m} P_{n+m}^m(x) \frac{s^n}{n!} = \frac{\hat{P}_k^m(x - s \sqrt{1 - x^2})}{(1 + \frac{2sx}{\sqrt{1 - x^2}} - s^2)^{\frac{n}{2} + \frac{m}{2}}},
\]

and then as \( s = i \) and \( x = \cos \theta \), \( \theta \in (0, \pi) \),

\[
\hat{P}_k^m(e^{\pm i\theta}) = 2^{\pm \frac{m}{2}} e^{\frac{im\pi}{2}} e^{-\frac{imx}{2}} \sin^{\frac{m}{2}} \theta \sum_{n=0}^{k \pm m} \frac{\sin^{\frac{m}{2}} \theta}{n!} P_{n+m}^m(\cos \theta).
\]

The addition theorems (104) and (107) can be rewritten in the forms of addition theorems for associated Legendre functions: as \( 0 < x < 1 \) and \( 0 < x + y < 1 \),
\[
P_k^m \left( \frac{x+y}{1+xy} \right) = \frac{(k+m)!}{(k-1)!} \sum_{n=0}^{k+m} \frac{(k+n-m-1) \Phi_n,k,\pm m (y) \left( \frac{1-x}{1+x} \right)^{\frac{m}{2}}}{(k+1+n)!} P_n^{\pm m} (x),
\]

\[
\Phi_n,\pm m (y) = (-1)^{n+\frac{m}{2}} (1+y)^{k+\frac{m}{2}} F \left( -n, -k; 1-k-n; \frac{1-y}{1+y} \right);
\]

\[
P_k^m \left( \frac{x+y}{1+xy} \right) = \frac{(k+m)!k!}{(k-m)! (2m-1)!} \sum_{n=0}^{k} \frac{(-1)^n \Lambda_n,k,m (y) P_n^{m+n} (x)}{n! (1-x)^{\frac{n+m}{2}} (1+x)^{\frac{n-m}{2}}};
\]

\[
\Lambda_n,k,m (y) = \frac{(2m+n-1)! (1-y)^{m-k}}{(k+2m+n)!} F \left( -n-m-k; 2m; \frac{2y}{1+y} \right);
\]

\[
P_k^m \left( \frac{x+y}{1+xy} \right) = \frac{(2m)!}{k!} \sum_{n=0}^{2m} \frac{(k+n)! \tilde{\Lambda}_n,k,m (y) (1-x)^{\frac{n-m}{2}}}{n! (2m-n)! (1+x)^{\frac{n-m}{2}}} P_n^{m-n} (x);
\]

\[
\tilde{\Lambda}_n,k,m (y) = \frac{(1-y)^{k-m+1}}{1+y} F \left( -n,k-m+1; -2m; \frac{2y}{1+y} \right);
\]

\[
\Theta_n,k,m (y) = \frac{(1+y)^{m+k}}{1-y} F \left( -n,-k-m; -2m; \frac{2y}{1+y} \right).
\]

By setting \( \gamma = -k, \mu = \pm m, \) and using (111), (5) and (6), the addition theorem (103) turns into the expansion

\[
P_k^m \left( \frac{x+y}{1+xy} \right) = \Psi (x,y) \sum_{n=0}^{\min(k,2m)} \frac{(-1)^n \Theta_n,k,m (y) (1-x)^{\frac{n-m}{2}} P_n^{m-n} (x)}{(k-n)!(2m-n)! (1+x)^{\frac{n-m}{2}}},
\]

\[
\Psi (x,y) = \frac{(-1)^m k!}{\sqrt{\pi} 2^{\pm m} (1+xy)^{\frac{m}{2}}} \left( \frac{1-y^2}{1-x^2} \right)^{\frac{m}{2}}, \quad -1 < x, y < 1,
\]

where the polynomials \( \Phi_n^{(\pm m-k)} (z) \) are defined in (102).

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