GAUGE FIELDS WITH QUAZINILPOTENT GAUGE GROUP.
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Abstract
We investigate non-linear generalization of Maxwell theory of electromagnetic field keeping the gauge invariance of Lagrangian. New theory, which is standard Yang-Mills theory, is based on Harmonic Oscillator $HO(N, R)$ gauge group. It’s a solvable Lie group with nilpotent normal subgroup of codimension 1. We wright down the Yang-Mills equation and point out their peculiarities and connection with standard Maxwell theory.

Field equations and interpretation.
We shall operate with so-called Harmonic Oscillator Lie algebra $ho(4, R)$, which is quazinilpotent and admitts non-degenerate invariant bilinear form:
$b : L \times L \rightarrow R$, such that $\forall x, y, z \in L$ the following condition:

$$b(ad_z(x), y) + b(x, ad_z(y)) = 0$$

is hold true.

Consider a phase space of harmonic oscillator with generalized variables: $p, q$ and Hamiltonian: $H = \frac{1}{2}(p^2 + q^2)$. It’s known that a set $< 1, p, q >$ forms a Lie algebra in respect to Poisson brackets:

$$\begin{align*}
[1, p] &= [1, q] = 0, \\
[p, q] &= 1,
\end{align*}$$

which is a nilpotent Lie algebra called Heisenberg algebra.
If we join them a Hamiltonian the new set \( <1, p, q, H > \) will form new algebra:
\[
\begin{align*}
[1, p] &= [1, q] = 0, \\
[p, q] &= 1, \\
[H, p] &= q, \\
[H, q] &= -p,
\end{align*}
\]
which is a quazinilpotent Lie algebra and contains Heisenberg algebra as an ideal. Indeed it’s a semi-simple sum of 3-dimensional Heisenberg algebra \( <1, p, q > \) and one-dimensional algebra \( <H> \). This algebra was generalized in papers [1](See also [2]). The main property of this algebra is that it admits non-degenerate invariant bilinear form:
\[
<u, v> = u^sv^s + u^1v^1 + u^4v^4 + zu^4v^4
\]
Here \( s, r, t, ... \) = \((2,3)\), \( z - \) any real constant and \( u^1, u^s, u^4 - \) are the components of the element \( u \) of Lie algebra in the basis \( <1, p, q, H> \):
\[
u = u^11 + u^2p + u^3q + u^4H.
\]
Due to the existence of this form we can construct for the gauge fields \( \hat{A}_\alpha \) a Lagrangian \( L = \frac{1}{4} <\hat{F}_{\alpha\beta}, \hat{F}^{\alpha\beta}> \) which extremals are the Yang-Mills equations exactly!
Here hat \( \hat{A}_\alpha \) means that it belongs to the matrix representation of gauge Lie algebra, and \( \alpha, \beta, \gamma, ... \) are the indices on the bundle (Minkowski) manifold.
\( \hat{F}_{\alpha\beta} = \partial_\alpha \hat{A}_\beta - \partial_\beta \hat{A}_\alpha + [\hat{A}_\alpha, \hat{A}_\beta] - \) curvature tensor of gauge field \( \hat{A}_\alpha \),

Yang-Mills equations are:
\[
\begin{align*}
\partial^\alpha F^{1}_{\alpha\beta} + \omega_{st}A^{s\alpha} F_{t\alpha\beta} &= 0, \\
\partial^\alpha F^{s}_{\alpha\beta} + 2\omega_{st}A^{\alpha[t} F^{t\alpha\beta} &= 0, \\
\partial^\alpha F^{4}_{\alpha\beta} &= 0,
\end{align*}
\]
coupled with system
\[
\begin{align*}
F^{1}_{\alpha\beta} &= 2\partial_{[\alpha} A^{1}_{\beta]} + \omega_{st}A^{s}_{\alpha} A^{t}_{\beta}, \\
F^{s}_{\alpha\beta} &= 2\partial_{[\alpha} A^{s}_{\beta]} + 2\omega_{st}A^{4}_{\alpha} A^{t}_{\beta}, \\
F^{4}_{\alpha\beta} &= 2\partial_{[\alpha} A^{4}_{\beta]}.
\end{align*}
\]
Here \( \omega_{st} = -\omega_{ts} \). The system is semi-splitted in three parts. We see that \( A^4 \) component is a pure Maxwell field. We substitute this field to the
second part and find the $A^s$ components. The first part can be rewritten as a Maxwell equations with sources:

\[
\begin{align*}
\varphi_{\alpha\beta} &= 2\partial_\alpha A^1_\beta, \\
\partial^\alpha \varphi_{\alpha\beta} &= J_\beta.
\end{align*}
\]

where $J_\beta = -\omega_{st} \partial^\alpha (A^s_{\alpha} A^t_\beta) - \omega_{st} A^{s0}_t F^t_{\alpha\beta} - "sources"$. Note that center of $ho(N, R)$ is $<1>$ and from physical point of view it has no useful information, because the element 1 was coupled to the set $p, q$ to complete it to the Lie algebra, what is simply a mathematical trick. It brings to the idea that component $A^1$ don’t represent the real physical field, and putting it to be trivial we can regard the first equation like a constrain on sources, more precisely on physical fields $A^2, A^3$.

To write down the gauge transformations for this theory we have first to construct a Lie group with $ho(4, R)$ Lie algebra. We call this group $HO(4, R)$ a harmonic oscillator Lie group [4]. There are two types of gauge transformations (connected with semi-simple splitting of algebra into two subalgebras):

First type:

\[
\begin{align*}
A^1 &\to A^1, \\
A^s &\to (e^{-\omega \lambda^4})^s_t A^t, \\
A^4 &\to A^1 + \partial \lambda^4.
\end{align*}
\]

Transformation for $A^4$ is usual gauge transformation for Maxwell field (generally for any abelian gauge field). And transformation for $A^s$ are simply rotation in the $p, q$ plane at angle $\lambda^4$:

\[
\begin{align*}
A^2 &\to \cos(\lambda^4) A^2 + \sin(\lambda^4) A^3, \\
A^3 &\to -\sin(\lambda^4) A^2 + \cos(\lambda^4) A^3.
\end{align*}
\]

Second type:

\[
\begin{align*}
A^1 &\to A^1 + \partial \lambda^1 + \omega_{st} \partial \lambda^s \lambda^t + (\lambda^s \lambda^s) A^4, \\
A^s &\to A^s + \partial \lambda^s + \omega_{st} \lambda^t A^4, \\
A^4 &\to A^4.
\end{align*}
\]

**Conclusion.** We see that putting $p, q$—components of gauge field to zero we come to standard Maxwell theory for $A^4$—component. Then there are two aspects of this theory.

The local is connected with the physical meaning of $p, q$—components which satisfy linear equations and non-linear constraints. These components are not independent because of the gauge transformations and we can well put one of them (say $A^3$) to zero, providing the gauge for
the usual Maxwell field $A^4$ being fixed. In this case the constraints will be satisfied automatically and we come to the ordinary two-component Maxwell theory with linear constraints on one of the component.

Another aspect is global. We know that in the Maxwell theory all regular in $R^3$ monopole solutions are trivial. It turns out that the same is hold true for $HO(4, R)$-theory, but it’s much more difficult to prove this [3]. In Maxwell theory we are forced to regard singular monopole solutions (like Dirac magnetic monopole), but it’s energy is infinite. Does it valid for the $HO(4, R)$ – theory? We just can claim the same thing in the case when Maxwell sector is trivial ($A^4 = 0$) [3], but it’s not clear in the general case.

And the last question: is it possible for non-trivial $A^8, A^4$ field to give a zero energy? In this case we can speak about energyless electrodynamics, which can lead to the interesting physical effects. to

References

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