Complementary-multiphase quantum search for all numbers of target items

Tan Li, Wan-Su Bao,* He-Liang Huang, Feng-Guang Li, Xiang-Qun Fu, Shuo Zhang, Chu Guo, Yu-Tao Du, Xiang Wang, and Jie Lin
Henan Key Laboratory of Quantum Information and Cryptography, Zhengzhou Information Science and Technology Institute, Zhengzhou, Henan 450001, China and Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

Grover’s algorithm achieves a quadratic speedup over classical algorithms, however, it can’t maintain high success probability when there exist multiple target items, and is considered necessary to know the number of target items exactly [Phys. Rev. Lett. 95, 150501 (2005); Phys. Rev. Lett. 113, 210501 (2014)]. In this paper, we find out that the Grover algorithm can also apply to the case that only the range of the fraction of target items is known. Moreover, a complementary-multiphase quantum search algorithm is proposed, in which multiple phases complement each other so that the overall high success probability can be maintained. Compared to the existing algorithms, in the case defined above, for the first time our algorithm achieves the following three goals simultaneously: (1) the algorithm is applicable to the entire range of the fraction of target items, (2) the success probability can be not less than any given value between 0 and 1, and (3) the number of iterations is almost the same as that of Grover’s algorithm. Especially compared to the optimal fixed-point algorithm [Phys. Rev. Lett. 113, 210501 (2014)], our algorithm uses fewer iterations to achieve success probability greater than 82.71%, e.g. when the minimum success probability is required to be 99.25%, the number of iterations can be reduced by 50%.

PACS numbers: 03.67.Ac, 03.67.-a, 03.67.Lx, 03.65.-w

I. INTRODUCTION

For the unordered database search problem, the well-known Grover algorithm [1, 2] provides a quadratic improvement over classical search algorithms, and has been proven optimal [3–6]. However, there still exists a soufflé problem [7]. For this, the fixed-point quantum search algorithms [8, 9] have been proposed, which can be applied to the case (denoted by Case A) where the fraction of target items is completely unknown. While, for Grover’s algorithm, it has been indicated that “to perform optimally, they need precise knowledge of certain problem parameters, e.g., the number of target states” [8], and “without knowing exactly how many marked items there are, there is no knowing when to stop the iteration” [9]. In other words, the Grover algorithm is considered to be only applicable to the case (denoted by Case B) where the fraction of target items is precisely known.

In fact, the optimal number of iterations of Grover’s algorithm is given by (See p. 253 of Ref. [10])

\[ k_G = CI \left( \frac{\pi}{4 \arcsin \sqrt{\lambda}} - \frac{1}{2} \right), \]

where \( \lambda = M/N \) represents the fraction of target items, \( M \) is the number of target items in a database of \( N \) items, and \( CI(x) \) returns the integer closest to \( x \) and rounds halves down. Simple algebra shows that

\[ k_G = \begin{cases} 
0, & \text{if } \lambda \in \left[ \frac{1}{4}, 1 \right), \\
m, & \text{if } \lambda \in \left[ \sin^2 \frac{\pi}{4m+4}, \sin^2 \frac{\pi}{4m} \right), m \geq 1.
\end{cases} \]

Then we can determine the value of \( k_G \), provided it is possible to know which of the ranges that \( \lambda \) belongs to. For example, if \( \lambda \in \left[ \sin^2 \frac{\pi}{2}, \sin^2 \frac{\pi}{2} \right) \), then \( k_G = 1 \).

Consequently, we confirm that Grover’s algorithm is also applicable to an intermediate case between Case A and Case B, denoted by Case A-B, where one can identify which of the given ranges that \( \lambda \) belongs to, while the exact location of \( \lambda \) within that range is unknown.

On the other hand, the minimum success probability of the original Grover algorithm is only 50%. For this problem, quantum amplitude amplification [11–14] with arbitrary phases has been developed, as well as the phase matching methods [15–20]. Furthermore, many generalizations and modifications of Grover’s algorithm have been proposed [21–24]. Among them, there are four typical kinds of algorithms: the 100%-success probability algorithms [14, 25–28], the fixed-phase algorithms [29–31], the matched-multiphase algorithms [9, 32, 33] and the complementary-multiphase algorithm [34]. However, to some extent, there are still certain problems in these algorithms, as shown below.

The 100%-success probability algorithms [14, 25–28] can complete a searching task with certainty. However, these algorithms are only applicable to Case B, instead of Case A-B or Case A, because the precise knowledge of \( \lambda \) is necessary. In the fixed-phase algorithms [29–31], the phase is first fixed to a certain value, being independent of \( \lambda \), then the optimal number of iterations [31] is specified by

\[ k = \left\lfloor \frac{\pi}{4 \arcsin \left( \sqrt{\lambda} \sin \frac{\pi}{2} \right)} \right\rfloor, \]

where \( \lfloor \cdot \rfloor \) is the floor function. Similar to Grover’s algo-
method is no longer meaningful. Given the success probability of the algorithm not less than any complementary method can actually work well in Case A-B. With only one iteration, we presented a complementary-multiphase algorithm that makes Yoder’s algorithm becomes the original fixed-point algorithm, as well as the derivation of all local maximum points of the success probability no less than 1. The Grover iteration with arbitrary number of iterations, \( \lambda \), of \( \delta \) which would be much larger than that of Grover’s algorithm, for small enough \( \delta \). Especially when \( \delta = 0 \), Yoder’s algorithm becomes the original fixed-point algorithm [8] and loses the quadratic speedup. In Ref. [34], we presented a complementary-multiphase algorithm that divides the range \([1/4, 1]\) into a series of small ranges, and can work well in Case A-B. With only one iteration, complementing the advantages of multiple phases enables the success probability of the algorithm not less than any given \( P_{\text{error}} \in (0, 1) \). However, the success probability decreases to zero for \( \lambda < 1/4 \), which indicates that this method is no longer meaningful.

In this paper, we expect to design a complementary-multiphase quantum search algorithm with general iterations, taking into account the applicable range of \( \lambda \), the success probability and the number of iterations simultaneously, to solve the success probability problem of Grover’s algorithm in Case A-B completely, and confirm that the multiphase-complementing method can actually be applied to the entire range of \( 0 < \lambda < 1 \), by casting off the limitation of \( k = 1 \) in Ref. [34].

The paper is organized as follows. Section II provides an introduction to the quantum amplitude amplification algorithm, as well as the derivation of all local maximum points of the success probability after \( k \) iterations. Section III describes the model of our complementary-multiphase algorithm with general iterations, and also the selection method of optimal parameters. Section IV gives an analysis of the success probability and the number of iterations. Section V summarizes the comparison between the algorithm in this paper and the existing algorithms, followed by a brief conclusion in Section VI.

## II. QUANTUM AMPLITUDE AMPLIFICATION REVISITED

Brassard et al. extended the phase inversions in the original Grover algorithm [1, 2] to arbitrary rotations, and obtained the quantum amplitude amplification algorithm [13, 14]. The Grover iteration with arbitrary phases is given by

\[
G(\phi, \varphi) = -HS^\phi \phi HS^\varphi. \tag{5}
\]

Here \( H \) is the Hadamard transform and

\[
S^\phi_f |x\rangle = \begin{cases} e^{i\varphi} |x\rangle, & \text{if } f(x) = 1, \\ |x\rangle, & \text{if } f(x) = 0, \end{cases} \tag{6}
\]

where \( i = \sqrt{-1} \). Similarly, \( S^\phi_f \) changes phase of the zero state \( |0\rangle \) by a factor of \( \phi \). \( S^\phi_f \) and \( S^\phi_0 \) can be expressed as [15]

\[
S^\psi_f = I - (e^{i\varphi} + 1) \sum_{x \in f^{-1}(1)} |x\rangle \langle x|, \tag{7}
\]

\[
S^\psi_0 = I - (e^{i\varphi} + 1) |0\rangle \langle 0|, \tag{8}
\]

where \( \varphi, \phi \in [0, 2\pi) \), since \( S^\psi_f = S^\varphi_{f+2\pi} \) and \( S^\psi_0 = S^\phi_{0+2\pi} \).

The equal superposition of all target (nontarget) states can be denoted by \( |\alpha\rangle \ (|\beta\rangle) \), i.e.,

\[
|\alpha\rangle = \frac{1}{\sqrt{M}} \sum_{x \in f^{-1}(1)} |x\rangle, \tag{9}
\]

\[
|\beta\rangle = \frac{1}{\sqrt{N-M}} \sum_{x \in f^{-1}(0)} |x\rangle, \tag{10}
\]

where \( N \) (\( M \)) is the number of all (target) items in the database, and by convention \( 0 < M < N \). Then, in the space spanned by \( |\alpha\rangle \) and \( |\beta\rangle \), the matrix representation of \( G \) operator is

\[
G = \begin{bmatrix} -e^{i\varphi} \left( e^{i\phi} \sin\theta + \cos\theta \right) & \left( 1 - e^{i\phi} \right) \sin\theta \cos\theta \\
 e^{i\varphi} \left( 1 - e^{i\phi} \right) \sin\theta \cos\theta & -e^{i\phi} \cos^2\theta - \sin^2\theta \end{bmatrix}, \tag{11}
\]

due to

\[
G|\alpha\rangle = G_{11}|\alpha\rangle + G_{21}|\beta\rangle, \tag{12}
\]

\[
G|\beta\rangle = G_{12}|\alpha\rangle + G_{22}|\beta\rangle, \tag{13}
\]

where \( G_{ij} \) refers to the entry in the \( i \)-th row and \( j \)-th column of the matrix in Eq. (11).

Suppose the initial state is

\[
|\psi\rangle = H^{\otimes n} |0\rangle = \sin\theta |\alpha\rangle + \cos\beta |\beta\rangle, \tag{14}
\]

where \( \theta = \arcsin(\sqrt{\lambda}) \), \( \theta \in (0, \pi/2) \). After \( k \) iterations of \( G(\phi, \varphi) \) with the phase matching condition [15]

\[
\phi = \varphi, \tag{15}
\]

the state becomes [30]

\[
G_k|\psi\rangle = a_k^\phi|\alpha\rangle + b_k^\varphi |\beta\rangle, \tag{16}
\]

where

\[
a_k^\phi = \frac{\sin \theta}{\sin \delta} (-1)^{k(k-1)/2} \left\{ e^{i\phi} \sin((k+1)\delta) - \sin(k\delta) \right\}, \tag{17}
\]

\[
b_k^\varphi = \frac{\sin \theta}{\sin \delta} (-1)^{k(k-1)/2} \left\{ e^{i\phi} \sin((k+1)\delta) + \sin(k\delta) \right\}.
\]
The success probability of finding the superposition of target states is thus given by
\[ P_k^\phi(\lambda) = |\langle \phi \rangle^2| = A \cos[(2k + 1)\delta] + B, \quad (19) \]
where
\[ A = \frac{\sin^2 \theta}{\sin^2 \delta} (\cos \phi - \cos \delta), \]
\[ B = \frac{\sin^2 \theta}{\sin^2 \delta} (1 - \cos \phi \cos \delta). \quad (21) \]

From Eq. (19) we can see that \( P_k^\phi(\lambda) = P_k^{2\pi-\phi}(\lambda) \), and \( G = -I \) does not change the initial state when \( \phi = 0 \), therefore only \( \phi \in (0, \pi] \) needs to be considered.

The condition that derivative of \( P_k^\phi(\lambda) \) equal to zero gives rise to all the local maximum points of \( P_k^\phi(\lambda) \) on the range of \( 0 < \lambda < 1 \) (Proof see appendix A),
\[ \lambda_{k,j}^{\phi,max} = \frac{1 - \cos \left( \frac{2j-1}{2k+1} \pi \right)}{1 - \cos \phi}, \quad 1 \leq j \leq k. \quad (22) \]

In addition, we can find that \( P_k^\phi(\lambda_{k,j}^{\phi,max}) = 100\% \) and \( \lambda_{k,j}^{\phi,max} \) increases as \( j \) grows.

III. GENERALIZED COMPLEMENTARY - MULTIPHASE SEARCH ALGORITHM

According to Eqs. (19) and (22), it is found that the algorithm has advantage of high success probability near its local maximum points, and has disadvantage of low success probability near the corresponding local minimum points. Thus, it is difficult to maintain high success probability over the entire range of \( \lambda \), by applying \( k \) iterations with just a single phase. One would naturally expect that this problem could be handled by using multiple phases. The key idea of the complementary-multiphase algorithm is that a phase is employed only in the \( \lambda \) range where the algorithm has high success probability. For a certain phase, in the range where the algorithm has low success probability, we use other phases to make up for it. In this way, we would expect that complementing the multiple phases with each other could improve the overall minimum success probability of the algorithm. The model of algorithm is described in the following.

A. Model

We first divide the entire range of \( \lambda \in (0, 1) \) into a series of small ranges, denoted by \( A_1, A_2, \cdots, A_k, \cdots \), satisfying the relation
\[ \bigcup_{k \geq 1} A_k = (0, 1). \quad (23) \]

For each \( A_k \), we specify the number of iterations of the algorithm to be \( k \). Then by subdividing range \( A_k \) further, we get smaller ranges, denoted by \( A_{k,1}, A_{k,2}, \cdots, A_{k,n_k} \), satisfying
\[ \bigcup_{m=1}^{n_k} A_{k,m} = A_k. \quad (24) \]

In this way, the entire range \( (0, 1) \) is finally divided into \( A_{1,1}, A_{1,2}, \cdots, A_{1,n_1}, \cdots, A_{k,1}, A_{k,2}, \cdots, A_{k,n_k}, \cdots \), with
\[ \bigcup_{k \geq 1} \bigcup_{m=1}^{n_k} A_{k,m} = (0, 1). \quad (25) \]

For each \( A_{k,m} \), we specify the phase of the algorithm to be \( \phi_{k,m} \), such that the algorithm has high success probability, where \( \phi_{k,m} \in (0, \pi], 1 \leq m \leq n_k \). Then, assuming for now the existence of \( A_k, A_{k,m}, \phi_{k,m} \) and \( n_k \) — their values are given later — the specific steps of the complementary-multiphase algorithm can be described as follows:

Step 1: The number of iterations and also the phase of the algorithm can be specified. In Case A-B, one can determine which of the ranges \( A_{1,1}, \cdots, A_{1,n_1}, \cdots, A_{k,1}, \cdots, A_{k,n_k}, \cdots \), that \( \lambda \) belongs to, without loss of generality, denoted by \( A_{k,m} \). Then we obtain the corresponding number of iterations and phase of algorithm is \( k \) and \( \phi_{k,m} \), respectively. In Case B, the value of \( \lambda \) is known exactly, and therefore \( k, \phi_{k,m} \) can be obtained in a similar way.

Step 2: Prepare the initial state to the equal superposition state, i.e., \( |\psi\rangle = H^\otimes n |0\rangle \).

Step 3: Repeat application \( k \) times of the Grover iteration with arbitrary phases \( G(\phi) \) to the initial state \( |\psi\rangle \), with the condition \( \phi = \varphi = \phi_{k,m} \).

Step 4: Measure the final state \( G^k |\psi\rangle \). This will produce one of the marked states with high success probability.

B. Optimal parameters

The selection method of optimal parameters are given in the following.

Firstly, as illustrated in the model of algorithm, \( k \) iterations corresponds to the range \( A_k \), and therefore, different choices of \( A_k \) result in different iterations of the algorithm. In order to ensure that the operations needed to perform is as few as possible , and the success probability of the algorithm is always not less than any given \( P_{cri} \), we define the optimal \( A_k \) as: for any \( \lambda \in A_k \), there exists a phase such that after \( k \) iterations the success probability reaches 100%, while there does not exist such a phase with \( k - 1 \) iterations. Indeed, such optimal \( A_k \) exists for all \( k \geq 1 \), which can be written in the form (Proof see appendix B),
\[ A_k = \left[ \lambda_{k,1}^{\pi,max}, \lambda_{k-1,1}^{\pi,max} \right], \quad (26) \]
where,
\[
\lambda_{k,1}^{\phi,\text{max}} = \begin{cases} 
1, & \text{if } k = 0, \\
\phi = \pi, \text{max} = \sin^2 \frac{\pi}{4k + 2}, & \text{if } k \geq 1.
\end{cases}
\] (27)

For any \( \lambda \in A_k \), it can be found that the scope of possibly used phases by the multiphase-complementing method can be further reduced from \( (0, \pi] \) to \( (\phi_{k,m}^{\text{max}}, \pi] \) (Proof see appendix C), where
\[
\phi_{k,m}^{\text{min}} = \arccos \left( 1 - \frac{2 - 2 \cos \frac{\pi}{4k + 1}}{1 - \cos \phi} \right).
\] (28)

And, for any \( \phi \in (\phi_{k,m}^{\text{min}}, \pi] \), we can see that the success probability \( P^\phi_k (\lambda) \) has the following extreme properties on the range \( A_k \) (Proof see appendix D).

**Property 1**

(1) For \( k \geq 1 \), \( P^\phi_k (\lambda) \) has one and only one local maximum point, denoted by
\[
\lambda_{k,1}^{\phi,\text{max}} = 1 - \cos \frac{\pi}{4k + 1}.
\] (29)

(2) For \( k = 1 \), \( P^\phi_k (\lambda) \) has one and only one local minimum point, denoted by
\[
\lambda_{k,1}^{\phi,\text{min}} = \frac{5 - 4 \cos \phi}{6 - 6 \cos \phi}.
\] (30)

While for \( k \geq 2 \), \( P^\phi_k (\lambda) \) has no local minimum points.

Secondly, according to the model of algorithm, multiple phases are employed on \( A_k \) by the multiphase-complementing method. Assuming the phases are already known — the optimal phases are given later — \( \phi_{k,1}, \phi_{k,2}, \cdots, \phi_{k,n_k} \) are listed in descending order, where \( \phi_{k,m} \in (\phi_{k,m}^{\text{min}}, \pi] \) and \( n_k \) is defined to be the number of phases used on \( A_k \). And the phase \( \phi_{k,m} \) corresponds to the range \( A_{k,m} \), namely, \( \phi_{k,m} \) is always used by the algorithm for any \( \lambda \in A_{k,m} \). Therefore, different choices of \( A_{k,m} \) result in different success probabilities of the algorithm. We define the optimal \( A_{k,m} \) as the one yielding the largest minimum success probability of the algorithm on \( A_k \). Actually, based on Property 1, we can see that such optimal \( A_{k,m} \) exists for \( 1 \leq m \leq n_k \), and is given in the following form (Proof see appendix E),
\[
A_{k,m} = \left[ a_{k,m-1}, a_{k,m} \right),
\] (31)

where \( a_{k,m} \) denotes the point of intersection of the curves represented by \( P^\phi_{k,m} (\lambda) \) and \( P^\phi_{k,m+1} (\lambda) \) for \( 1 \leq m \leq n_k - 1 \), \( a_{k,m} = \lambda_{k,1}^{\phi,\text{max}} \) for \( m = 0 \), and \( a_{k,m} = \lambda_{k,1}^{\phi,\text{max}} \) for \( m = n_k \).

Next, as seen from Eq. (31), the optimal \( A_{k,m} \) depends on the multiple phases used on \( A_k \). Therefore, different choices of \( \phi_{k,1}, \phi_{k,2}, \cdots, \phi_{k,m}, \cdots, \phi_{k,n_k} \) result in different \( A_{k,m} \) and further different minimum success probability. We define the optimal \( \phi_{k,m} (1 \leq m \leq n_k) \) as those yielding the largest minimum success probability on \( A_k \). It is easy to see that through exhaustive methods to search the optimal \( \phi_{k,m} \) is computationally infeasible, because the exhaustion scale of all the \( \phi_{k,m} \in (\phi_{k,m}^{\text{min}}, \pi] \) is infinitely large. Fortunately, based on Property 1, we find out the sufficient and necessary condition of optimal phases, as shown in Theorem 1 (Proof see appendix F).

**Theorem 1**

For the range of \( \lambda \in A_k \), \( k \geq 1 \), assuming that the number of phases \( n_k \) is given, then we can get the sufficient and necessary condition of the optimal \( \phi_{k,m} (1 \leq m \leq n_k) \) as follows:
\[
P^{\phi_{k,1}} (a_{k,0}) = \cdots = P^{\phi_{k,m}} (a_{k,m-1}) = \cdots = P^{\phi_{k,n_k}} (a_{k,n_k-1}) = P^{\phi_{k,n_k}} (\lambda_{k,1}^{\phi,\text{max}}),
\] (32)

where,
\[
\lambda_{k,1}^{\phi,\text{min}} = \begin{cases} 
\phi_{k,n_k}, & \text{if } k = 1,
\phi_{k,1}^{\phi,\text{min}}, & \text{if } k \geq 2.
\end{cases}
\] (33)

Finally, due to Eq. (32) gives a set of \( n_k \) equations about \( \phi_{k,1}, \cdots, \phi_{k,n_k} \), therefore, different choices of \( n_k \) result in different optimal phases and eventually different success probabilities on \( A_k \). Similar to [9], we expect the success probability for any fraction of target items could be not less than any given \( P \) \((0,1)\). For the purpose of minimizing the number of phases required by the algorithm, we define the optimal \( n_k \) as the least integer that meets our expectation. In order to determine the optimal \( n_k \), we first define \( Q_k^\pi \) to be the largest minimum success probability on range \( A_k \), and subsequently get the following property of \( Q_k^\pi \) with respect to the number of phases \( n_k \) (Proof see appendix G).

**Property 2**

(1) \( Q_k^\pi \) increases as \( n_k \) grows.
(2) \( Q_k^\pi \rightarrow 100\% \) when \( n_k \rightarrow \infty \).

Then, based on Property 2, the optimal \( n_k \) can be determined as follows.

Step 1: Initialize \( n_k = 1 \).

Step 2: According to the value of \( n_k \) and the optimal phase condition Eq. (32), calculate the largest minimum success probability on \( A_k \), namely \( Q_k^\pi (n_k) \).

Step 3: Check whether \( Q_k^\pi (n_k) \) is smaller than \( P \). If \( Q_k^\pi (n_k) < P \), then \( n_k \) increases by one, and go back to Step 2; otherwise, output \( n_k \) as the optimal number of phases and abort the procedure.

At this point, we have obtained all the selection methods of the optimal \( A_k, A_{k,m}, \phi_{k,m} \) and \( n_k \). For clarity, below we make the complete selection process explicit.

First, according to Eq. (26), we have a division of the entire range of \( \lambda \in (0,1) \), i.e.,
\[
A_1 = \left[ \frac{1}{4}, 1 \right), A_2 = \left[ 3 - \sqrt{5}, \frac{1}{8} \right), \cdots,
A_k = \left[ \sin^2 \frac{\pi}{4k + 2}, \sin^2 \frac{\pi}{4k - 2} \right), \cdots.
\] (34)
In Case A-B or Case B, which of the ranges \( \{A_k\} \) \((k \geq 1)\) that \( \lambda \) belongs to can be identified, without loss of generality, denoted by \( A_k \). Correspondingly, the number of iterations is specified to be \( k \).

Then, for the obtained \( k \) and given \( P_{cri} \), we can get the optimal number of phases on \( A_k \), denoted by \( n_k \). Solving Eq. (32) gives rise to the optimal phases, denoted by \( \phi_{k,1}, \ldots, \phi_{k,m}, \ldots, \phi_{k,n_k} \), which yield the optimal \( A_{k,m} \) by Eq. (31).

Finally, in Case A-B or Case B, we can determine which of the ranges \( \{A_{k,m}\} \) \((1 \leq m \leq n_k, k \geq 1)\) that \( \lambda \) belongs to, without loss of generality, denoted by \( A_{k,m} \). And therefore, the phase of algorithm is specified as \( \phi_{k,m} \).

Executing the algorithm with the optimal parameters leads directly to results of the success probability and the number of iterations, as is described in the following section.

IV. ANALYSIS OF PERFORMANCE

A. Success probability

For our complementary-multiphase algorithm, on the one hand, \( A_1, A_2, \ldots, A_k, \ldots \) constitute a division of the entire range of \( \lambda \in (0,1) \). On the other hand, on each \( A_k \), the algorithm uses multiple phases to complement each other, and the largest minimum success probability on \( A_k \) converges to 100% when the number of phases increases to infinity. It follows that, the success probability of our algorithm is possible to be not less than any given \( P_{cri} \in (0,1) \) for the entire range of \( \lambda \).

The success probabilities of the Grover algorithm [1], the optimal fixed-point algorithm [9], our proposed algorithm, and the complementary-multiphase algorithm with only one iteration [34] as functions of \( \lambda \) are presented in Fig. 1, with the fraction of target items \( \lambda \geq \lambda_0 = 10^{-2} \) and the acceptable success probability \( P \geq P_{cri} \approx 90\% \). As seen in Fig. 1, the problem of Grover’s algorithm [1] that high success probability over the entire range of \( \lambda \) cannot be maintained is systematically solved by our complementary-multiphase algorithm, which overcomes the limitation of the applicable range in Ref. [34] where only \( \lambda \in \left[ \frac{1}{4}, 1 \right] \) could be covered, and achieves the same effect as the optimal fixed-point algorithm [9]. By “the same effect”, we mean the success probability can be not less than any given \( P_{cri} \) over the entire range.

The optimal parameters on each \( A_k \) \((1 \leq k \leq 8)\) corresponding to Fig. 1 are given in Table I, including: the optimal multiple phases, denoted by \( \phi_{k,1}, \ldots, \phi_{k,n_k} \) and the largest minimum success probability, denoted by \( Q^*_k \). We can see that the multiphase-complementing method indeed guarantee a range of \( \lambda \geq \lambda_0 \) over which the expectation \( P \geq P_{cri} \) can be satisfied.

![FIG. 1. The success probabilities \( P \) as functions of the fraction of target items \( \lambda \). The black dotted, blue dashed, red solid and green dashed-dotted curves correspond to the original Grover algorithm [1], the optimal fixed-point algorithm [9], our proposed algorithm and the complementary-multiphase algorithm with only one iteration [34], respectively. The task is achieving success probability not less than \( P_{cri} = 90\% \) for all \( \lambda \geq \lambda_0 = 10^{-2} \).](image)

| \( k \) | \( A_k \) | \( \phi_{k,1}, \ldots, \phi_{k,n_k} \) | \( Q^*_k \) |
|---|---|---|---|
| 1 | \([0.25, 1)\) | 2.1341,465 | 0.9593 |
| 2 | \([0.09549, 0.25)\) | 2.1621,536 | 0.9564 |
| 3 | \([0.04952, 0.09549)\) | 2.1731,348 | 0.9535 |
| 4 | \([0.03015, 0.04952)\) | 2.1841,158 | 0.9562 |
| 5 | \([0.02025, 0.03015)\) | 2.1951,067 | 0.9595 |
| 6 | \([0.01453, 0.02025)\) | 2.2060,972 | 0.9630 |
| 7 | \([0.01099, 0.01453)\) | 2.2170,874 | 0.9673 |
| 8 | \([0.008513, 0.01099)\) | 2.2280,772 | 0.9716 |

B. Number of iterations

As described in the model of algorithm, the number of iterations is specified by \( k \) for any \( \lambda \in A_k \), and the optimal \( A_k = [\lambda_{k,max}, \lambda_{k-1,max}] \) is defined by Eq. (26). Therefore, we have

\[
\lambda \in A_k \Leftrightarrow \lambda_{k-1,max} \leq \lambda < \lambda_{k,max} \\
\Leftrightarrow k - \frac{1}{2} < \frac{\pi}{4\theta} \leq k + \frac{1}{2},
\]

(35)

where \( \theta = \arcsin \sqrt{\lambda} \). Thus, the number of iterations \( k \) of our algorithm is given as

\[
k = CI \left( \frac{\pi}{4 \arcsin \sqrt{\lambda}} \right),
\]

(36)

where \( CI (x) = k \) corresponds to \( k - \frac{1}{2} < x \leq k + \frac{1}{2} \). From Eq. (36), we also see that \( k \approx \frac{\pi}{4} \sqrt{N/M} \) when \( \lambda = M/N \ll 1 \), due to \( \arcsin \sqrt{\lambda} \approx \sqrt{\lambda} \).
FIG. 2. The number of iterations as a function of the fraction of target items. The red solid and blue dashed curves correspond to our algorithm and the Grover algorithm [1], respectively. Note that, to see the range of small λ more clearly, the range of large λ is compressed, with a “//” on the x-axis marking the boundary.

From Eqs. (1) and (36) it follows that,

\[
k = \begin{cases} 
  k_G, & \text{if } \lambda \in \bigcup_{k \geq 1} \left[ \sin^2 \frac{\pi}{4k+2}, \sin^2 \frac{\pi}{4k} \right], \\
  k_G + 1, & \text{if } \lambda \in \bigcup_{k \geq 1} \left[ \sin^2 \frac{\pi}{4k}, \sin^2 \frac{\pi}{4k-2} \right].
\end{cases}
\]  

(37)

Figure 2 shows a comparison between the number of iterations of our algorithm \(k\) and that of Grover’s algorithm \(k_G\) versus the fraction of target items \(\lambda\). As can be seen, \(k\) and \(k_G\) are almost the same.

V. DISCUSSIONS

In this section, we will give some comparisons between our algorithm and several other kinds of quantum search algorithms.

Compared to the 100%-success probability algorithms [14, 26, 27], the fixed-phase algorithms [29–31], and the matched-multiphase algorithms [9, 32, 33], the sequence of operations (denoted by \(S\)) applied to the initial state in our complementary-multiphase algorithm is significantly different.

Among the 100%-success probability algorithms, Refs. [26, 27] repeat the same Grover iteration with arbitrary phases \(k\) times, with the sequence of operations being

\[
S = G^k (\phi, \phi),
\]

(38)

where \(k\) is first specified as follows

\[
k \geq \left\lceil \frac{\pi}{4 \arcsin \sqrt{\lambda}} - \frac{1}{2} \right\rceil,
\]

(39)

and then \(\phi\) is a function of \(k\) and \(\lambda\),

\[
\phi = 2 \arcsin \left( \sin \left( \frac{\pi}{4k+2} \right) / \sqrt{\lambda} \right).
\]

(40)

While Ref. [14] first performs the standard Grover iteration \(k\) times, and then run one more generalized Grover iteration with arbitrary phases. The sequence of operations is given by

\[
S = G (\phi, \varphi) G^k (\pi, \pi),
\]

(41)

where \(k = \left\lfloor \frac{\pi}{4 \arcsin \frac{\varphi}{\sqrt{\lambda}}} - \frac{1}{2} \right\rfloor\), \(\phi\) and \(\varphi\) satisfy the phase-matching condition

\[
\cot ((2k+1) \theta) = e^{i \phi} \sin (2 \theta) (i \cot (\phi/2) - \cos (2 \theta))^{-1}.
\]

(42)

The sequence of operations in the fixed-phase algorithms [29–31] is exactly in the same form as Eq. (38). While at this time, \(\phi\) is first specified, and then \(k\) is a function of \(\phi\) and \(\lambda\) with the optimal value being defined by Eq. (3). Based on the multiphase-matching method, Refs. [9, 32, 33] utilize a set of multiple phases \(\phi_j\) and \(\varphi_j\) (1 ≤ \(j\) ≤ \(k\)) satisfying the condition \(\phi_j = \varphi_{k-j+1}\) globally over the sequence of operations, as shown below:

\[
S = G (\phi_k, \varphi_k) G (\phi_{k-1}, \varphi_{k-1}) \cdots G (\phi_1, \varphi_1).
\]

(43)

However, our complementary-multiphase algorithm divide the entire range of \(\lambda\) into a series of small ranges, denoted by \(\{\Lambda_{k,m}\} \ (k \geq 1, 1 \leq m \leq n_k)\). For each \(\Lambda_{k,m}\), an individual phase is specified correspondingly. Therefore, the sequence of operations in our algorithm is indeed different from other algorithms, which can be written as

\[
S = G^k (\phi_{k,m}, \phi_{k,m}), \text{ for } \lambda \in \Lambda_{k,m},
\]

(44)

where the optimal \(\Lambda_{k,m}\) is defined by Eq. (31).

The main advantages of our algorithm are discussed in detail as follows. Table II lists the performance of our algorithm and other algorithms in respects of the applicable range of \(\lambda\), the success probability \(P\), the number of iterations \(k\), and the phase(s) \(\phi\).

Specifically speaking, in respect of the applicable range of \(\lambda\), our algorithm applies to the entire range (0, 1), which is essentially the same as Refs. [1, 9, 26, 29–31] and broader than Refs. [32–34]. Especially compared to Ref. [34] which is only applicable to [1/4, 1), the limitation there is overcome completely by considering a general number of iterations in our algorithm.

In respect of the success probability \(P\), as shown in Fig. 1, our algorithm has the same effect as Refs. [9, 34] allowing \(P \geq P_{\text{cri}} \in (0, 1)\), which is more flexible than Refs. [1, 29–33]. Moreover, as illustrated in Property 2, on \(\Lambda_k\) the largest minimum success probability \(Q^k_{\text{cri}} \to 100\%\) when \(n_k \to \infty\). Thus, it is possible to asymptotically achieve the effect of certainty in Ref. [26], by the multiphase-complementing method.

In respect of the number of iterations, as depicted in Fig. 2, our algorithm performs almost the same iterations as the original Grover algorithm [1] with up to once more, and therefore has fewer iterations than the fixed-phase algorithms [29–31] when \(\phi \neq \pi\), due to \(\frac{\pi}{4 \arcsin \frac{\varphi}{\sqrt{\lambda}} (\sqrt{\lambda} \sin \frac{\phi}{2}) > \frac{\pi}{4 \arcsin \sqrt{\lambda}}\). Moreover, it follows from Eqs. (4) and (36) that in problems where the
TABLE II. The detailed comparison between our algorithm and other algorithms.

| Algorithm         | Applicable range of \( \lambda \) | Success probability | Number of iterations | Phase(s)                        |
|-------------------|-------------------------------------|---------------------|----------------------|---------------------------------|
| Our algorithm     | \((0, 1)\)                          | \( \geq P_{cri} \)  | Eq. (36)             | \( \phi_{k,m} \) for \( \lambda \in A_{k,m} \) |
| Li et al. [34]    | \([1/4, 1)\)                        | \( \geq P_{cri} \)  | 1                    | \( \phi_{1,m} \) for \( \lambda \in A_{1,m} \) |
| Yoder et al. [9]  | \((0, 1)\)                          | \( \geq 1 - \delta^2 \) | Eq. (4)              | \( \phi_{1, \cdots, \delta} \) for \( \lambda \in (0, 1) \) |
| Toyama et al. [32]| \([0, 1, 1)\)                        | \( \geq 99.8\% \)   | 6                    | \( \phi_{1, \cdots, \delta} \) for \( \lambda \in [0, 1, 1) \) |
| Toyama et al. [33]| \([0.006, 0.11]\)                    | \( \geq 99.2\% \)   | 20                   | \( \phi_{1, \cdots, \delta} \) for \( \lambda \in [0.006, 0.11] \) |
| Grover [1]        | \((0, 1)\)                          | \( \geq 50\% \)     | Eq. (1)              | \( \pi \) for \( \lambda \in (0, 1) \) |
| Younes [29]       | \((0, 1)\)                          | \( \geq 99.58\% \)  | \( |\phi/\sqrt{\lambda}| \) | 1.91684\( \pi \) for \( \lambda \in (0, 1) \) |
| Zhong et al. [30] | \((0, 1)\)                          | \( \geq 99.43\% \)  | \( |\pi/\sqrt{\lambda}| \) | 1.018 for \( \lambda \in (0, 1) \) |
| Li et al. [31]    | \((0, 1)\)                          | \( \geq 99.38\% \)  | Eq. (3)              | 0.1\( \tau \) for \( \lambda \in (0, 1) \) |
| Long [26]         | \((0, 1)\)                          | \( \equiv 100\% \)  | Eq. (39)             | Eq. (40)                        |

acceptable success probability \( P_{cri} \) is no less than 82.71\%, the number of iterations of our algorithm is smaller than that of the optimal fixed-point algorithm [9], due to \( \frac{\log(2 / \delta)}{2V_{\lambda}} < \frac{1}{4V_{\lambda}} \), where \( \delta = \sqrt{1 - P_{cri}} \). For example when \( P_{cri} = 99.25\% \), our algorithm just uses one half of the iterations of Ref. [9].

In respect of the phases, when \( \lambda \in A_{k,m} \), we can always find a target state with high success probability no less than \( P_{cri} \) without tuning the phases, similar to Refs. [32, 33]. It follows that our complementary-multiphase algorithm is applicable to Case A-B, where the 100%-success probability algorithms [14, 25–28] cannot work, indicating that our algorithm has a wider scope of applications.

VI. CONCLUSION

In summary, a complementary-multiphase quantum search algorithm with a general number of iterations is presented. This algorithm divides the entire range of the fraction of target items \((0 < \lambda < 1)\) into a series of smaller ranges. On each range, the number of iterations and the phase employed are specified individually. Similar to Grover’s algorithm, multiphase-complementing approach applies not only to the extreme case where \( \lambda \) is exactly known (denoted by Case B), but also to an intermediate case where one can determine the range that \( \lambda \) belongs to without precise knowledge of \( \lambda \) (denoted by Case A-B).

In the entire range of \( \lambda \), we derived all local maximum points of the success probability after applying the Grover iteration with arbitrary phases \( k \) times, and obtained the optimal division (denoted by \( \{A_k\} \)) of range \((0, 1)\) to reduce the query complexity [9] of the quantum search algorithm. Moreover, for the purpose of maximizing the minimum success probability, we further analyzed the extreme properties of the success probability on \( A_k \), and subsequently obtained the optimal division of range \( A_k \), the optimal number of phases, and the optimal phase condition.

Compared with the existing algorithms, our algorithm simultaneously achieves the following three goals in Case A-B for the first time: (1) the entire range of \( 0 < \lambda < 1 \) can be covered, (2) the success probability can be not less than any \( P_{cri} \in (0, 1) \), and (3) the required number of iterations can be almost the same as the original Grover algorithm. Especially when the required success probability is not less than 82.71\%, our algorithm uses fewer iterations than the optimal fixed-point algorithm [9]. The multiphase-complementing method provides a new idea for the research on quantum search algorithms.

VII. ACKNOWLEDGMENTS

We thank Ru-Shi He and Zheng-Mao Xu for useful discussions. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11504430 and 61502526), and the National Basic Research Program of China (Grant No. 2013CB338002).

Appendix A: Proof of all local maximum points of \( P^\phi_k(\lambda) \) on \((0, 1)\) of Eq. (22)

According to Eq. (19), the derivative of \( P^\phi_k(\lambda) \) with respect to \( \lambda \) can be written as

\[
\frac{\partial P^\phi_k}{\partial \lambda} = (1 + \cos \delta)^{-2} \left( \frac{\partial P^\phi_k}{\partial \lambda} \right)^{(1)} , \tag{A1}
\]

where

\[
\left( \frac{\partial P^\phi_k}{\partial \lambda} \right)^{(1)} \equiv (1 + \cos \delta) \{1 + \cos[(2k+1)\delta]\} - (2k+1)
\times (\cos \delta - \cos \delta)(1 + \cos \delta) \frac{\sin[(2k+1)\delta]}{\sin \delta} \tag{A2}
\]

If \( \cos[(2k+1)\delta] = -1 \), then \( (2k+1) \delta = 0 \), \( \frac{\partial P^\phi_k}{\partial \lambda} = 0 \) and \( P^\phi_k = 1 \), thus solving the equation \( \cos[(2k+1)\delta] = -1 \) gives rise to the local maximum points of \( P^\phi_k(\lambda) \). The corresponding solutions are given as \( \lambda^\phi_{k,j,max} \) for \( 1 \leq j \leq k \) in Eq. (22).
We have established the existence of local maximum points, and now we can further show that there are no other points except for $\lambda^\phi_{k,j}$. From De Moivre’s theorem (see p. 9 of Ref. [35]), i.e.,
\[
\cos(2k+1)\delta+i\sin(2k+1)\delta = (\cos \delta + i \sin \delta)^{2k+1}, \tag{A2}
\]
where $i = \sqrt{-1}$, it follows that with respect to $\cos \delta$, $\cos [(2k+1)\delta]$ and $\sin [(2k+1)\delta]/\sin \delta$ are polynomials of degree $2k+1$ and $2k$, respectively. Consequently, the degree of the polynomial $(\partial P^\phi/\partial \lambda)^{(1)}$ is no more than $2k+2$, which will have up to $2k+1$ real roots for $\delta \in (0, \pi)$, now that $\delta = \pi$ is already one of its roots. Furthermore, due to $P^\phi_k(\lambda = 0) = 0$ and $P^\phi_k(\lambda = 1) = 1$, $P^\phi_k(\lambda)$ has the same number of local maximum points and local minimum points. Finally, we are now in a position to conclude that $\lambda^\phi_{k,j}$ ($1 \leq j \leq k$) are just all the local maximum points of $P^\phi_k(\lambda)$.

**Appendix B: Proof of the optimal $A_k$ of Eq. (26)**

From Eq. (22), it follows that $\lambda^\phi_{k,j} \geq \lambda^\phi_{k,1}$ for any $j \geq 1$, and $\lambda^\phi_{k,1} \geq \lambda_{k,1,1}^\phi$ for any $\phi \in (0, \pi)$. Then, for any $\lambda \in [\lambda_{k,1,1}^\phi, 1)$ (or $[\lambda_{1,k,1}^\phi, 1]$), there exists a phase such that the success probability reaches 100% with $k$ (or $k-1$) iterations. Therefore, to minimize the number of iterations, the corresponding optimal range of $\lambda$ to $k$ iterations can be given as $A_k = [\lambda_{k,1,1}^\phi, \lambda_{1,k-1,1}^\phi]$ ($k \geq 1$), which constitute a division of the entire range of $\lambda$.

**Appendix C: Proof of the scope of possibly used phases on $A_k$**

Based on Eq. (22), it is found that for any $\phi \in (0, \phi^\min_k)$,
\[
\lambda^\phi_{k,1} \geq \lambda_{k,1}^{\pi,\max}, \tag{C1}
\]
and for any $\phi \in (\phi^\min_k, \pi]$,
\[
\lambda_{\pi,\max} \leq \lambda_{\phi,\max} < \lambda_{k,1}^{\phi,\max}, \tag{C2}
\]
where $\phi^\min_k$ is defined by Eq. (28). Then, for any $\lambda \in A_k$ and any $\phi \in (0, \phi^\min_k]$, we have
\[
P^\phi_k(\lambda) \leq P^\phi_k(\lambda^\phi_{k,1}) \tag{C3}
\]
Consequently, the possibly used phases on $A_k$ of the multiphase-complementing method can be limited to $(\phi^\min_k, \pi]$.

**Appendix D: Proof of the extreme properties of $P^\phi_k$ on $A_k$ of Property 1**

(1) On the one hand, as mentioned in Appendix C, for any $\phi \in (\phi^\min_k, \pi]$, we have $\lambda^\phi_{k,j} \in A_k = [\lambda_{k,1,1}^{\phi,\max}, \lambda_{1,k-1,1}^{\phi,\max}]$. On the other hand, it can be found that $\lambda^\phi_{k,j} \notin A_k$ for $j \geq 2$. This is because $k \geq 1$ and $j \geq 2$ lead to $\cos (2k+1)/(2k+1) \leq \cos (2k+1)/(2k+1)$, and then
\[
\lambda^\phi_{k,j} \geq \lambda_{k,2}^{\phi,\max} = 1 - \cos 2k+1 \cos \phi \over 1 - \cos \phi
\geq 1 - \cos 2k+1 \pi \over 2 = \lambda_{k-1,1}^{\max}. \tag{D1}
\]
Therefore, $\lambda^\phi_{k,1}$ is the one and only local maximum point of $P^\phi_k$ on $A_k$.

(2) In the case of $k = 1$, we obtain $A_k = [1/4, 1]$. Then, from Eq. (2.13) in Ref. [32] or Eq. (6) in Ref. [34], it is straightforward to show that $\lambda^\phi_{k,1}$ given in Eq. (30) is the one and only one local minimum point of $P^\phi_k$ on $A_k$. In the case of $k \geq 2$, to prove $\lambda^\phi_{k,1} \geq \lambda_{k-1,1}^{\max}$, we only need to find a $\lambda_{\mid\mid}$ such that $\lambda^\phi_{k,1} \geq \lambda_{\mid\mid}$ and $\lambda_{\mid\mid} \geq \lambda_{k-1,1}^{\max}$. Indeed, such $\lambda_{\mid\mid}$ exists and may be given in the form
\[
\lambda_{\mid\mid} = \frac{2 \sin^2 \frac{\pi}{k-2}}{1 - \cos \phi}. \tag{D2}
\]
Since $1 - \cos \phi \leq 2$, we have
\[
\lambda_{\mid\mid} \geq \frac{\sin^2 \frac{\pi}{k-2}}{4k-2} = \lambda_{k-1,1}^{\max}. \tag{D3}
\]
It remains to show that $\lambda^\phi_{k,1} \geq \lambda_{\mid\mid}$, which is equivalent to prove $\partial P^\phi_k(\lambda) \leq 0$, due to for $k \geq 2$,
\[
\lambda_{k,1}^{\phi,\max} \leq \lambda_{\mid\mid} < \frac{2 \sin^2 \frac{\pi}{k-2}}{1 + \cos \phi} = \lambda_{k,2}^{\phi,\max}. \tag{D4}
\]
Here we denote $\partial P^\phi_k(\lambda) \mid_{\lambda=\lambda_{\mid\mid}}$ to be the value of $\partial P^\phi_k(\lambda)$ at $\lambda_{\mid\mid}$. The proof is carried out as follows.

First, for $\lambda = \lambda_{\mid\mid}$, it follows from Eq. (18) that $\delta = -\frac{\pi}{2k-1}$, and $\delta \leq \frac{\pi}{3}$ now that $k \geq 2$. Substituting $\delta$ into Eq. (A1), we get
\[
\partial P^\phi_k(\lambda) \mid_{\lambda=\lambda_{\mid\mid}} = \frac{2}{1 + \cos \delta} \left( \frac{\partial P^\phi_k}{\partial \lambda} \right)^{(2)} \tag{D5}
\]
where
\[
\left( \frac{\partial P^\phi_k}{\partial \lambda} \right)^{(2)} \equiv (1+2k \cos \delta) \cos \phi - (2k+1) \cos^2 \delta + \cos \delta - 1,
\]
from which we obtain \((\frac{\partial P^\phi_k}{\partial \lambda})^{(2)}\) is a monotonically decreasing function with respect to \(\phi\) for any given \(k\), yielding

\[
(\frac{\partial P^\phi_k}{\partial \lambda})^{(2)} < (\frac{\partial P^\phi_k}{\partial \lambda})^{(2)} \bigg|_{\phi=\phi_{k,min}},
\]

for \(\phi \in (\phi_{k,min}, \pi]\).

Next, according to Eq. (28), \((\frac{\partial P^\phi_k}{\partial \lambda})^{(2)}\) \(\bigg|_{\phi=\phi_{k,min}}\) is an univariate function of \(k\). When \(k\) is sufficiently large, namely \(k \to \infty\), \(\cos \delta = \cos \frac{\pi}{2k-1} \approx 1\) and therefore,

\[
(\frac{\partial P^\phi_k}{\partial \lambda})^{(2)} \bigg|_{\phi=\phi_{k,min}} \approx (1 + 2k) (\cos \phi_{k,min} - 1) < 0,
\]

which can also be numerically proven to hold for small \(k\), for example \(k = 2, 3, \ldots, 1000\).

Finally, based on Eqs. (D5), (D6) and (D7), we have

\[
\frac{\partial P^\phi_k}{\partial \lambda} = 0 \quad \text{for} \quad k \geq 2, \quad \text{and therefore} \quad \lambda_{k,1}^{\phi,min} > \lambda_{mid} \geq \lambda_{k,1}^{\phi,max},
\]

namely there exists no local minimum points on \(A_k\) for \(P^\phi_k\).

\(\text{Appendix E: Proof of the optimal } A_{k,m} \text{ of Eq. (31)}\)

Based on the Property 1, we can obtain

\[
\lambda_{k,1}^{\phi_1,max} < \lambda_{k,1}^{\phi_2,max} < \cdots < \lambda_{k,1}^{\phi_{k,nk},max},
\]

due to the assumption of

\[
\phi_{k,1} > \phi_{k,2} > \cdots > \phi_{k,nk}.
\]

Then, on \([\lambda_{k,1}^{\phi_{k,m},max}, \lambda_{k,1}^{\phi_{k,m+1},max}]\) for \(1 \leq m \leq n_k - 1\), \(P_k^{\phi_{k,m}} (\lambda)\) monotonically decreases and \(P_k^{\phi_{k,m+1}} (\lambda_{k,1}^{\phi_{k,m+1},max}) = 100\%\), while \(P_k^{\phi_{k,m+1}} (\lambda)\) monotonically increases and \(P_k^{\phi_{k,m+1}} (\lambda_{k,1}^{\phi_{k,m+1},max}) = 100\%\). According to the intermediate value theorem (See p. 271 of Ref. [35]), there exists a \(\lambda \in (\lambda_{k,1}^{\phi_{k,m},max}, \lambda_{k,1}^{\phi_{k,m+1},max})\) such that

\[
P_k^{\phi_{k,m}} (\lambda) = P_k^{\phi_{k,m+1}} (\lambda).
\]

We denote the solution as \(a_{k,m}\), which represents the intersection point of \(P_k^{\phi_{k,m}} (\lambda)\) and \(P_k^{\phi_{k,m+1}} (\lambda)\) on \(A_k\).

Consequently, to maximize the minimum success probability of the algorithm by taking advantage of the multiple phases, \(\phi_{k,m}, \phi_{k,m+1}, \phi_{k,1}\) and \(\phi_{k,nk}\) should be employed on \([\lambda_{k,1}^{\phi_{k,m},max}, a_{k,m}], [a_{k,m}, \lambda_{k,1}^{\phi_{k,m+1},max}]\), \([\lambda_{k,1}^{\phi_{k,m},max}, a_{k,0}, \lambda_{k,1}^{\phi_{k,1},max}]\) and \([a_{k,m}, \lambda_{k,1}^{\phi_{k,nk},max}, \lambda_{k,1}^{\phi_{k,nk},max}, \lambda_{k,1}^{\phi_{k,nk},max}, \lambda_{k,1}^{\phi_{k,nk},max}]\), respectively. Finally, the range of \(\lambda\) that corresponds to \(\phi_{k,m}\) can be written as

\[
A_{k,m} = \left[a_{k,m-1}, \lambda_{k,1}^{\phi_{k,m},max}\right] \cup \left[a_{k,m-1}, \lambda_{k,1}^{\phi_{k,m},max}\right],
\]

as desired.

\(\text{Appendix F: Proof of the optimal phase condition of Eq. (32)}\)

In the case of \(k = 1\), first we can show that for any \(\phi_{k,nk-1}\), the optimal phase condition to maximize the minimum success probability of the algorithm on \([\lambda_{k,1}^{\phi_{k,nk-1},max}, \lambda_{k,1,1}^{\pi,max}]\) is

\[
P_k^{\phi_{k,nk}} (a_{k,nk-1}) = P_k^{\phi_{k,nk}} (\lambda_{k,1}^{\phi_{k,nk},min}). \quad (\text{F1})
\]

Note that, \(\lambda_{k,1}^{\phi_{k,nk},max}\) and \(\lambda_{k,1,1}^{\phi_{k,nk},min}\) are defined by Eqs. (29) and (30) respectively, and \(a_{k,m}\) is the solution of Eq. (E2). This is because, for a given \(\phi_{k,nk-1}\), as shown in Fig. 3, the minimum success probability on \([\lambda_{k,1}^{\phi_{k,nk-1},max}, \lambda_{k,1,1}^{\pi,max}]\) is determined by \(P_k^{\phi_{k,nk}} (a_{k,nk-1})\) and \(P_k^{\phi_{k,nk}} (\lambda_{k,1}^{\phi_{k,nk},min})\). The former is an increasing function with respect to \(\phi_{k,nk}\) and increases to 100% when \(\phi_{k,nk} \rightarrow \phi_{k,nk}^{min}\). While, the latter monotonically decreases and asymptotically approaches 100% when \(\phi_{k,nk} \rightarrow \phi_{k,nk}^{min}\). Hence, according to the intermediate value theorem, there exists a \(\phi_{k,nk} \in (\phi_{k,nk}^{min}, \phi_{k,nk}^{max})\) such that Eq. (F1) holds. At this time, the minimum success probability reaches the maximum, denoted by \(Q_k^{\phi_{k,nk}}\). Here, we define \(Q_k^\phi\) to be the largest minimum success probability on \([\lambda_{k,1}^{\phi_{k,nk},max}, \lambda_{k,1,1}^{\pi,max}]\) with \(\phi \in (\phi_{k,nk}^{min}, \pi]\). As \(\phi\) grows, \(\lambda_{k,1}^{\phi_{k,nk},max} = \frac{1 - \cos \phi}{1 - \cos \phi_{k,nk}^{min}}\) decreases and the range \([\lambda_{k,1}^{\phi_{k,nk},max}, \lambda_{k,1,1}^{\pi,max}]\) extends, then it follows that \(Q_k^\phi\) monotonically decreases with respect to \(\phi\).

The next thing we can do is to show that for any \(\phi_{k,nk-2}\), the optimal phase condition on \([\lambda_{k,1}^{\phi_{k,nk-2},max}, \lambda_{k,1,1}^{\pi,max}]\) is

\[
P_k^{\phi_{k,nk-1}} (a_{k,nk-2}) = P_k^{\phi_{k,nk-1}}. \quad (\text{F2})
\]
This is because, for a given \( \phi_{k,n_k-2} \), the minimum success probability on \( [\lambda_{k,1}^{\phi_{k,n_k-2},\text{max}}, \lambda_{k-1,1}^{\text{max}}] \) is determined by \( P_k^{\phi_{k,n_k-1}}(a_{k,n_k-2}) \) and \( Q_k^{\phi_{k,n_k-1}} \), as shown in Fig. 3. The former is an increasing function with respect to \( \phi_{k,n_k-1} \) and increases to 100% when \( \phi_{k,n_k-1} \to \phi_{k,n_k-2} \). While, the latter monotonically decreases and asymptotically approaches 100% when \( \phi_{k,n_k-1} \to \phi_{k}^{\text{min}} \). Hence, according to the intermediate value theorem, there exists a \( \phi_{k,n_k-1} \in (\phi_{k}^{\text{min}}, \phi_{k,n_k-2}) \) such that Eq. (F2) holds. At this time, the minimum success probability reaches the maximum, denoted by \( Q_k^{\phi_{k,n_k-2}} \).

In a similar way as shown before, we can obtain for any \( \phi_{k,m} \) \((1 \leq m \leq n_k - 2)\), the optimal phase condition on \( [\lambda_{k,1}^{\phi_{k,m},\text{max}}, \lambda_{k-1,1}^{\text{max}}] \) is

\[
P_k^{\phi_{k,m}+1}(a_{k,m}) = Q_k^{\phi_{k,m}+1}. \tag{F3}
\]

In this case, the corresponding maximum of the minimum success probability is denoted by \( Q_k^{\phi_{k,m}} \).

In addition, it can be found that for any \( \phi_{k,1} \), the optimal phase condition on \( [\lambda_{1,1}^{\text{max}}, \lambda_{k-1,1}^{\text{max}}] \) is

\[
P_k^{\phi_{k,1}'}(\lambda_{k,1}^{\text{max}}) = Q_k^{\phi_{k,1}}. \tag{F4}
\]

This is because, for a given \( \phi_{k,1} \), the minimum success probability on \( [\lambda_{k,1}^{\text{max}}, \lambda_{k-1,1}^{\text{max}}] \) is determined by \( P_k^{\phi_{k,1}}(\lambda_{k,1}^{\text{max}}) \) and \( Q_k^{\phi_{k,1}} \). The former is an increasing function with respect to \( \phi_{k,1} \) and increases to 100% when \( \phi_{k,1} \to \pi \). While, the latter monotonically decreases and asymptotically approaches 100% when \( \phi_{k,1} \to \phi_{k}^{\text{min}} \). Hence, according to the intermediate value theorem, there exists a \( \phi_{k,1} \in (\phi_{k,1}^{\text{min}}, \pi) \) such that Eq. (F4) holds. At this time, the minimum success probability reaches the maximum, denoted by \( Q_k^{\phi_{k,1}} \).

Finally, combining Eqs. (F3), (F4) and (F5), we can obtain in the case of \( k = 2 \), Eq. (32) is also indeed the optimal phase condition.

---

**Appendix G: Proof of the properties of \( Q_k^\pi \) with respect to \( n_k \) on \( \Lambda_k \) of Property 2**

(1) The property that \( Q_k^\pi \) increases as \( n_k \) grows, will be proven if we can show \( Q_k^\pi(n_k + 1) > Q_k^\pi(n_k) \) for any \( n_k \geq 1 \). In the case of \( n_k = 1 \), according to Eq. (32), the optimal phase condition is given as

\[
P_k^{\phi_{k,1}}(\lambda_{k,1}^{\text{max}}) = P_k^{\phi_{k,1}}(\lambda_{k-1,1}^{\text{max}}) \equiv Q_k^\pi(n_k), \tag{G1}
\]

where \( \lambda_{k,1}^{\text{max}} \) and \( \lambda_{k-1,1}^{\text{max}} \) are defined by Eqs. (29) and (33), respectively. Without loss of generality, under condition Eq. (G1), Figure 4 plots the schematic of success probability versus the fraction of target items for \( k \geq 2 \). On the one hand, due to

\[
P_k^{\phi_{k,1}}(\lambda_{k,1}^{\text{max}}) < P_k^{\phi_{k,1}}(\lambda_{k,1}^{\phi_{k,1},\text{max}}) = 100\%, \tag{G2}
\]

we can see \( \phi_{k,1} \) is not the optimal phase to maximize the minimum success probability on \( [\lambda_{k,1}^{\text{max}}, \lambda_{k,1}^{\phi_{k,1},\text{max}}] \) of the algorithm using one phase. In other words, as shown in Fig. 4, there exists a \( \phi_{k,1} \in (\phi_{k,1}, \pi] \) such that

\[
P_k^{\phi_{k,1}}(\lambda_{k,1}^{\text{max}}) = P_k^{\phi_{k,1}}(\lambda_{k,1}^{\phi_{k,1},\text{max}}) > Q_k^\pi(n_k). \tag{G3}
\]

On the other hand, from

\[
P_k^{\phi_{k,1}}(\lambda_{k,1}^{\phi_{k,1},\text{max}}) = 100\% > P_k^{\phi_{k,1}}(\lambda_{k,1}^{\text{key}}), \tag{G4}
\]

it follows that \( \phi_{k,1} \) is neither the optimal phase on \( [\lambda_{k,1}^{\phi_{k,1},\text{max}}, \lambda_{k-1,1}^{\text{max}}] \) of the algorithm with a single phase.
Namely, as shown in Fig. 4, there exists a $\phi'_{k,1} \in (\phi_{k,1}^{\min}, \phi_{k,1})$, such that
\[ P^\phi_{k} (\lambda_{k,1}^{\lambda_{k,1}, \pi_{\max}}) = P^\phi_{k} (\lambda'_{k,1}) > Q_k^\phi(n_k), \] (G5)
where
\[ \lambda'_{k,1} = \begin{cases} \lambda_{k,1}^{\lambda_{k,1}, \pi_{\max}}, & \text{if } k = 1, \\ \lambda_{k-1,1}^{\lambda_{k-1,1}, \pi_{\max}}, & \text{if } k \geq 2. \end{cases} \] (G6)

Then, using $\phi'_{k,1}$ and $\phi''_{k,1}$ yields a minimum success probability on $A_k$, being greater than $Q_k^\phi(n_k)$.
Furthermore, under the optimal phase condition, the minimum success probability of the algorithm using two phases will be greater. Thus, $Q_k^\phi(n_k + 1) > Q_k^\phi(n_k)$ is confirmed for $n_k = 1$.

In the case of $n_k \geq 2$, according to Eq. (32), the optimal phase condition is given as
\[ P^\phi_{k} (\lambda_{k,1}^{\pi_{\max}}) = P^\phi_{k} (a_{k,n_k-1}) = P^\phi_{k} (\lambda_{k,1}^{\pi_{\max}}) = Q_k^\phi(n_k). \] (G7)

On the one hand, due to
\[ P^\phi_{k} (\lambda_{k,1}^{\pi_{\max}}) < P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max}) = 100\%, \] (G8)
we can find that $\phi_{k,1}, \phi_{k,2}, \ldots, \phi_{k,n_k}$ are not the optimal phases on $[\lambda_{k,1}^{\pi_{\max}}, \lambda_{k,1}^{\phi_{k,n_k}, \max}]$ of the algorithm using $n_k$ phases. In other words, there exist $\phi'_{k,1}, \phi'_{k,2}, \ldots, \phi'_{k,n_k} \in (\phi_{k,n_k}, \pi]$ such that
\[ P^\phi_{k} (\lambda_{k,1}^{\pi_{\max}}) = P^\phi_{k} (a_{k,n_k-1}) = \cdots = P^\phi_{k} (a_{k,n_k-1}) = P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max}) > Q_k^\phi(n_k), \] (G9)
where $\phi'_{k,j}$ denotes the intersection point of $P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max})$ and $P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max})$, $1 \leq j \leq n_k - 1$. On the other hand, from
\[ P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max}) = 100\% > P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max}), \] (G10)
it follows that $\phi_{k,n_k}$ is not the optimal phase on $[\lambda_{k,1}^{\phi_{k,n_k}, \max}, \lambda_{k-1,1}^{\pi_{\max}}]$ of the algorithm with a single phase. Namely, there exists a $\phi'_{k,n_k} \in (\phi_{k,n_k}, \pi]$ such that
\[ P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max}) = P^\phi_{k} (\lambda_{k,1}^{\phi_{k,n_k}, \max}) > Q_k^\phi(n_k), \] (G11)
where
\[ \lambda'_{k,1} = \begin{cases} \lambda_{k,1}^{\phi_{k,n_k}, \min}, & \text{if } k = 1, \\ \lambda_{k-1,1}^{\phi_{k,n_k}, \min}, & \text{if } k \geq 2. \end{cases} \] (G12)

Then, using $\phi'_{k,1}, \phi'_{k,2}, \ldots, \phi'_{k,n_k}$ and $\phi''_{k,n_k}$ yields a minimum success probability on $A_k$, being greater than $Q_k^\phi(n_k)$. Furthermore, under the optimal phase condition, the minimum success probability of the algorithm using $n_k + 1$ phases will be greater. Thus, $Q_k^\phi(n_k + 1) > Q_k^\phi(n_k)$ is confirmed for $n_k \geq 2$. At this point, the property that $Q_k^\phi$ increases as $n_k$ grows is now proven.

(2) First, we can equally divide $A_k$ into $n_k$ smaller ranges, denoted by $A_{k,1}, A_{k,2}, \ldots, A_{k,n_k}$. For each $A_{k,m}$, there exists a phase $\phi_{k,m}$ such that $\lambda_{k-1,1}^{\phi_{k,m}, \max} \in A_{k,m}$, $1 \leq m \leq n_k$.

When $n_k \to \infty$, the length of $A_{k,m}$, i.e.,
\[ \left(\lambda_{k-1,1}^{\phi_{k,m}, \max} - \lambda_{k,1}^{\phi_{k,m}, \max}\right)/n_k \to 0, \] (G13)
which yields that for any $\lambda \in A_{k,m}$,
\[ P^\phi_{k,m} (\lambda) \to P^\phi_{k,m} (\lambda_{k,1}^{\phi_{k,m}, \max}) = 100\%. \] (G14)

Furthermore, under the optimal phase condition, the minimum success probability of the algorithm using $n_k$ phases will be greater. Consequently, it is straightforward to show that $Q_k^\phi \to 100\%$ when $n_k \to \infty$.

[1] L. K. Grover, in Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing (ACM, New York, 1996) pp. 212–219.
[2] L. K. Grover, Phys. Rev. Lett. 79, 325 (1997).
[3] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, SIAM J. Comput. 26, 1510 (1997).
[4] M. Boyer, G. Brassard, P. Hoyer, and A. Tapp, Fortschr. Phys. 46, 493 (1998).
[5] C. Zalka, Phys. Rev. A 60, 2746 (1999).
[6] L. K. Grover and J. Radhakrishnan, in Proceedings of the Seventeenth Annual ACM Symposium on Parallelism in Algorithms and Architectures (ACM, New York, 2005) pp. 186–194.
[7] G. Brassard, Science 275, 627 (1997).
[8] L. K. Grover, Phys. Rev. Lett. 95, 150501 (2005).
[9] T. J. Yoder, G. H. Low, and I. L. Chuang, Phys. Rev. Lett. 113, 210501 (2014).
[10] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[11] G. Brassard and P. Hoyer, in Proceedings of the Fifth Israel Symposium on the Theory of Computing Systems (IEEE Computer Society, Washington, 1997) pp. 12–23.
[12] L. K. Grover, Phys. Rev. Lett. 80, 4329 (1998).
[13] G. Brassard, P. Hoyer, and A. Tapp, in International Colloquium on Automata, Languages, and Programming (Springer, Berlin, 1998) pp. 820–831.
[14] G. Brassard, P. Hoyer, M. Mosca, and A. Tapp, in Quantum Computation and Information (AMS, Providence, 2002) pp. 53–74.
[15] G. L. Long, Y. S. Li, W. L. Zhang, and L. Niu, Phys. Lett. A 262, 27 (1999).
[16] P. Høyer, Phys. Rev. A 62, 052304 (2000).
[17] G. L. Long, C. C. Tu, Y. S. Li, W. L. Zhang, and H. Y. Yan, J. Phys. A 34, 861 (2001).
[18] G.-L. Long, X. Li, and Y. Sun, Phys. Lett. A 294, 143 (2002).
[19] C.-M. Li, C.-C. Hwang, J.-Y. Hsieh, and K.-S. Wang, Phys. Rev. A 65, 034305 (2002).
[20] P. Li and S. Li, Phys. Lett. A 366, 42 (2007).
[21] D. Biron, O. Biham, E. Biham, M. Grassl, and D. A. Lidar, in Quantum Computing and Quantum Communications (Springer, Berlin, 1999) pp. 140–147.
[22] A. Younes, J. Rowe, and J. Miller, AIP Conf. Proc. 734, 171 (2004).
[23] P. R. Giri and V. E. Korepin, Quantum Inf. Process. 16, 315 (2017).
[24] T. Byrnes, G. Forster, and L. Tessler, Phys. Rev. Lett. 120, 060501 (2018).
[25] D. P. Chi and J. Kim, Chaos Soliton. Fract. 10, 1689 (1999).
[26] G. L. Long, Phys. Rev. A 64, 022307 (2001).
[27] F. M. Toyama, W. van Dijk, and Y. Nogami, Quantum Inf. Process. 12, 1897 (2013).
[28] Y. Liu, Int. J. Theor. Phys. 53, 2571 (2014).
[29] A. Younes, Appl. Math. Inf. Sci. 7, 93 (2013).
[30] P.-C. Zhong and W.-S. Bao, Chin. Phys. Lett. 25, 2774 (2008).
[31] X. Li and P. Li, J. Quantum Inf. Sci. 2, 28 (2012).
[32] F. M. Toyama, W. van Dijk, Y. Nogami, M. Tabuchi, and Y. Kimura, Phys. Rev. A 77, 042324 (2008).
[33] F. M. Toyama, S. Kasai, W. van Dijk, and Y. Nogami, Phys. Rev. A 79, 014301 (2009).
[34] T. Li, W.-S. Bao, W.-Q. Lin, H. Zhang, and X.-Q. Fu, Chin. Phys. Lett. 31, 050301 (2014).
[35] D. Zwillinger, CRC Standard Mathematical Tables and Formulae (CRC Press, Boca Raton, 2011).