Partial Differential Equations

Self-similar solutions with fat tails for a coagulation equation with diagonal kernel

Solutions auto-similaires avec queues épaisses d’une équation de coagulation à noyau diagonal

Barbara Niethammer\(^a\), Juan J.L. Velázquez\(^b\)

\(^a\) Oxford Centre for Nonlinear PDE, Mathematical Institute, University of Oxford, 24-29 St. Giles’, Oxford, OX1 3LB, United Kingdom
\(^b\) ICMAT, C/ Nicolas Cabrera 15, Cantoblanco, 28049 Madrid, Spain

1. Introduction

Smoluchowski’s coagulation equation provides a mean-field description of binary coalescence of clusters. If \(\xi\) denotes the size of a cluster and \(f(\xi,t)\) the corresponding number density at time \(t\) then the equation is

\[
\frac{\partial}{\partial t} f(\xi,t) = \frac{1}{2} \int_0^\infty d\eta K(\xi - \eta, \eta) f(\eta,t) f(\xi - \eta,t) - f(\xi,t) \int_0^\infty d\eta K(\xi, \eta) f(\eta,t),
\]

(1)

where \(K(\xi, \eta)\) is a kernel that describes the rate of the coalescence process.

Here we consider a specific diagonal kernel of homogeneity \(\gamma < 1\), given by \(K(\xi, \eta) = \delta(\xi - \eta)\xi^{1+\gamma}\), that reduces (1) to

\[
\frac{\partial}{\partial t} f(\xi,t) = \frac{1}{4} \left(\frac{\xi}{2}\right)^{1+\gamma} f^2\left(\frac{\xi}{2},t\right) - \xi^{1+\gamma} f^2(\xi,t).
\]

(2)
In the following we study self-similar solutions of (2). Such solutions are of the form
\[ f(x, t) = t^{-(1 + (1 + \gamma)/\beta)} g\left(\frac{x}{t^\gamma}\right) \]  
for some positive \( \beta \), where \( g \) satisfies, with \( x = \xi/t^\beta \), that
\[ -\left(1 + (1 + \gamma)/\beta\right) g - \beta x g'(x) = \frac{1}{4} \left(\frac{x}{2}\right)^{1 + \gamma} g^2\left(\frac{x}{2}\right) - x^{1 + \gamma} g^2(x). \]  
(4)

If one looks for solutions with conserved mass, then \( \beta \) is uniquely determined by \( \beta = \beta_\ast := 1/(1-\gamma) \). For further reference we also note that we can integrate the equation in (4) to obtain
\[ \beta x^2 g(x) = \int_{x/2}^{x} s^{2 + \gamma} g^2(s) \, ds + (1-\gamma)(\beta - \beta_\ast) \int_{0}^{x} s g(s) \, ds. \]  
(5)

Here we assumed implicitly that \( xg(x) \) and \( x^{2+\gamma}g^2(x) \) are integrable at zero and that \( \lim_{x \to 0} x^2 g(x) = 0 \). As we will see below (cf. (9)), these properties will be satisfied by the solutions we are going to consider. Notice also that we have the well-known power-law solution
\[ g = x^{-(1+\gamma)} \frac{1}{1-\theta} \text{ with } \theta := 2^{\gamma-1} < 1. \]  
(6)

In [1] a mass-preserving solution of (5), that is a solution for \( \beta = \beta_\ast \), is constructed that is decaying exponentially fast and satisfies
\[ g(x) = x^{-(1+\gamma)} \left(1 - cx^{\mu/(1-\gamma)} + o(x^{\mu/(1-\gamma)})\right) \text{ as } x \to 0, \]  
(7)

where \( \mu > 0 \) satisfies a certain transcendental equation. The constant \( c > 0 \) is not determined due to an invariance of (4) under the rescaling \( g(x) \to a^{1+\gamma} g(ax) \) for any \( a > 0 \). In the case of mass-preserving solutions the constant can be fixed by normalizing the mass of the solution. As is pointed out in [1], the solution is unique in the class of functions satisfying (7), but uniqueness in general is not known.

In [1] the question is raised whether solutions with algebraic decay, others from the one in (6), exist in analogy to the ones that have been found in [2] for the constant and additive kernel. More precisely, for example for the constant kernel, it is established in [2] that there exists a family of self-similar solutions with infinite mass and the decay behavior \( x^{-(1+\rho)} \) for all \( \rho \in (0, 1) \). Furthermore, it is shown that a solution of the coagulation equation converges to the self-similar solution with decay behavior \( x^{-(1+\rho)} \) if and only if the mass-distribution of the initial data is regularly varying with exponent \( 1-\rho \). In this note we prove for the diagonal kernel the existence of a corresponding family of self-similar solutions with infinite mass and asymptotic behavior \( x^{-(1+\rho)} \) as \( x \to \infty \) with \( \rho \in (\gamma, 1) \). Notice, that this gives solutions that are increasing as \( x \to \infty \) if \( \gamma < -1 \). Our proof is simple and exploits strong monotonicity properties of a suitably rescaled version of the equation for the self-similar solution. We presently do not know, however, how to characterize the domains of attraction of these self-similar solutions. The analysis in [2] relies on the fact that the Laplace transform of the equation satisfies a simple ODE, a method that is not applicable in the present situation.

Our main result is the following:

**Theorem 1.** Let \( \gamma < 1 \) and \( \mu \) be the unique positive solution of
\[ \frac{1 + \beta \mu}{2} = \frac{1 - 2^{\gamma-1-\mu}}{1 - 2^{\gamma-1}}. \]  
(8)

Then there exists for any \( \beta > \beta_\ast \) a solution \( g \) of (5) such that
\[ g(x) = x^{-(1+\gamma)} \left(\frac{1}{1-\theta} - cx^{\mu/(1-\gamma)} + o(x^{\mu/(1-\gamma)})\right) \]  
(9)
as \( x \to 0 \) with a positive constant \( c \). Furthermore, \( x^{-(1+\gamma)} g(x) \) is monotonically decreasing and satisfies
\[ g(x) \sim \frac{d}{x^{1+\gamma+1/\beta}} \text{ as } x \to \infty \]  
(10)
for some positive constant \( d \).

As explained above, the constants \( c \) and \( d \) in Theorem 1 are not determined due to the invariance of the equation under appropriate rescaling.
2. Proof

Our proof proceeds similarly to the one in [1] for the mass-conserving solutions. First, to scale out the singular behavior as \( x \to 0 \), we introduce \( h(x) = g(x)x^{1+\gamma} \) such that \( h \) solves

\[
-\beta x^{1+\gamma} h(x) - h(x) = \theta h^2 \left( \frac{x}{2} \right) - h^2(x)
\]

(11)
or, due to \((5)\),

\[
\beta x^{1-\gamma} h(x) = \int_{x/2}^{x} s^{-\gamma} h^2(s) \, ds + (1-\gamma)(\beta - \beta_*) \int_{0}^{x} s^{-\gamma} h(s) \, ds.
\]

(12)

Notice, that the power-law solution \((6)\) corresponds to the constant solution \( h \equiv 1/(1-\theta) \). It is also clear that any solution of \((11)\) for which \( \lim_{x \to 0} h(x) \) exists, that this limit must equal \( 1/(1-\theta) \). We are now looking for solutions that bifurcate from this constant at \( x \to 0 \).

In order to identify the next order behavior, we make the ansatz \( h(x) = 1/(1-\theta) + x^\mu + o(x^\mu) \) as \( x \to 0 \). Plugging this into \((12)\), recalling that \( \beta_* = 1/(1-\gamma) \) and rearranging we find that \( \mu \) must indeed satisfy \((8)\). If we denote by \( F(\mu) = (1 - 2^{1-\mu} - \mu)/(1-\theta) \) we see that \( F(0) = 1 > 1/2 \). On the other hand, \( F \) is increasing and \( \lim_{\mu \to \infty} F(\mu) = 1/(1-\theta) \). Hence, there must be a unique positive solution of \((8)\).

Next, we introduce the function \( j(x) \) via

\[
h(x) = \frac{1}{1-\theta} + x^\mu (-c + j(x)),
\]

(13)

where \( c \in \mathbb{R} \) is a constant. Using Eqs. \((8)\) and \((12)\) we obtain that \( j \) satisfies

\[
j(x) = \frac{1}{\beta} x^{-(1-\gamma+\mu)} \left( \int_{x/2}^{x} s^{-\gamma+\mu} \frac{2}{1-\theta} j(s) \, ds + \int_{x/2}^{x} s^{-\gamma+2\mu} (-c + j(s))^2 \, ds \right.
\]

\[+ (1-\gamma)(\beta - \beta_*) \int_{0}^{x} s^{-\gamma+\mu} j(s) \, ds \right) =: T[j].
\]

(14)

In order to prove that a local solution of \((14)\) exists, we can proceed analogously to [1]. We only indicate the main steps here.

We define for some \( \varepsilon \in (0, \mu) \) and \( z > 0 \) the space

\[
C_\varepsilon(z) := \left\{ f \in C[0, z]; f(0) = 0; \| f \| := \sup_{x \in [0, z]} x^{-\varepsilon}|f(x)| < \infty \right\}.
\]

It is clear that the operator \( T \) maps \( C_\varepsilon(z) \) into itself. Next, we are going to show that \( T \) maps a ball in \( C_\varepsilon(z) \) of a sufficiently small radius \( R \) into itself if \( z \) is sufficiently small. This follows from

\[
\| T[j] \| \leq \frac{1}{\beta} \| j \| \left( \frac{2}{1-\theta} \frac{1}{1-\gamma+\mu+\varepsilon} (1 - 2^{1-\mu-\varepsilon}) + \| j \| \frac{2\mu}{1-\gamma + 2\mu + 2\varepsilon} (1 - 2^{1-\mu-2\varepsilon}) \right.
\]

\[+ c^2 \frac{2\mu}{1-\gamma + 2\mu + 2\varepsilon} \right)
\]

that implies

\[
\| T[j] \| \leq \| j \| \left( \frac{1}{\beta(1-\gamma+\mu+\varepsilon)} (2F(\mu + \varepsilon) + (1-\gamma)(\beta - 1)) + C\mu (\| j \|^2 + 1) \right).
\]

Now we know by the definition of \( \mu \) that \( 2F(\mu + \varepsilon) < 1 + \beta(\mu + \varepsilon) \) and hence

\[
\frac{1}{\beta(1-\gamma+\mu+\varepsilon)} (2F(\mu + \varepsilon) + (1-\gamma)(\beta - 1)) \leq \frac{1}{1-\gamma + \mu + \varepsilon} (\mu + \varepsilon + 1-\gamma) = 1.
\]

Thus, there exists a constant \( k = k(\varepsilon) < 1 \) such that if \( \| j \| \leq R \) we find \( \| T[j] \| \leq kR + C\mu (R^2 + 1) \). For sufficiently small \( z \) and an appropriately small \( R \) the right-hand side is bounded by \( R \). Similarly one can show that \( T \) is a contraction, we omit the details here. Hence, a local solution to \((14)\) exists, and thus also to \((11)\). Next, we choose \( c > 0 \), and claim that \( h \) is decreasing in a neighborhood of zero. To see this, notice that it follows from \((14)\) that \( j'(x) \) exists for \( x > 0 \) and that we have the estimate \( |j'(x)| \leq C \frac{|j(x)|}{x} + Cx^{\mu-1} \) for \( x \in (0, z) \). This in turn implies that
If $z$ is sufficiently small, we find that $h'(x) < 0$ for $x \in (0, z)$. We are going to show that as long as $h$ exists and is positive this property is conserved. Indeed, assume that there exists $x_0 > 0$ such that $h'(x_0) = 0$. Then (11) and the fact that $h$ is decreasing for $x < x_0$ imply that

$$0 = h(x_0)^2 - h(x_0) - \theta h^2 \left( \frac{x_0}{2} \right)^2 < (1 - \theta) h^2(x_0) - h(x_0) = h(x_0) \left( (1 - \theta) h(x_0) - 1 \right).$$

As long as $h$ is positive, the right-hand side is strictly negative, since $h(x_0) < 1/(1 - \theta)$ and we obtain the desired contradiction. Moreover, Eq. (12) implies for $\beta \geq \beta_*$ that $h$ is positive whenever it exists. Hence, using standard results on ordinary differential equations, we obtain global existence of a solution $h$ to (11) which is strictly decreasing. Since $h(x_0) = 0$ it follows that $h(x) \to 0$ as $x \to \infty$.

It remains to show that $h(x) \sim x^{-1/\beta}$ as $x \to \infty$ from which (10) follows. First, due to the invariance of Eq. (11) under the transformation $x \to ax$ for $a > 0$, we can assume without loss of generality that $h(1) = 1/2$. Since $h$ satisfies $\beta x h'(x) + h(x) \leq h^2(x)$ we have by simple comparison that

$$h(x) \leq \frac{1}{1 + x^{1/\beta}} \text{ for } x \geq 1.$$  \hfill (15)

We now introduce $p(x) = x^{1/\beta} h(x)$ that solves

$$\beta p'(x) = x^{-(1+1/\beta)} \left( p^2(x) - \theta 2^2/\beta p^2 \left( \frac{x}{2} \right) \right).$$  \hfill (16)

The estimate (15) in particular implies that $p(x) \leq 1$ for all $x \geq 1$ and thus (16) implies that $\beta |p'(x)| \leq 2x^{-(1+1/\beta)}$ for all $x \geq 2$. Hence $|p(x) - p(x_0)| \leq 2x_0^{1/\beta}$ for any $x_0 \geq 2$ which implies that $\lim_{x \to \infty} p(x)$ exists. In order to complete the proof of Theorem 1 it remains to establish that this limit is strictly positive. To this end we note that (12) implies

$$\beta x^{1-\gamma} h(x) > (1 - \gamma) (\beta - \beta_*) \int_0^x s^{-\gamma} h(s) \, ds.$$  \hfill (17)

If we define $\Phi(x) := \int_0^x s^{-\gamma} h(s) \, ds$ then (17) implies that $\beta x \Phi'(x) - (1 - \gamma)(\beta - \beta_*) \Phi(x) > 0$. Integrating this last inequality we obtain

$$x^{-(1-\gamma)(\beta - \beta_*)} \Phi(x) > \Phi(1) = \int_0^1 s^{-\gamma} h(s) \, dx =: c_0 > 0$$

for all $x \geq 1$. Thus

$$\Phi(x) \geq c_0 x^{-(1-\gamma)(\beta - \beta_*)} = c_0 x^{1-\gamma} x^{-1/\beta}$$

for $x \geq 1$ and plugging this into (17) we find $h(x) \geq \frac{c_0 x^{1-\gamma}}{x^{1/\beta}}$ for all $x \geq 1$, that finishes the proof.

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