Dark matter from Modified Friedmann Dynamics

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Abstract

The contemporary cosmic expansion is considered in the context of Modified Friedmann Dynamics (MOFD). We discuss some relativistic model exploring analogy to MOND modification of Newtonian dynamics. We argue that MOFD cosmologies can explain fraction of dark matter in the accelerating Universe. We discuss some observational constraints on possible evolitional MOFD scenarios of cosmological models coming from SN Ia distant supernovae. We show that Modified Newtonian Dynamics can be obtained as a Newtonian limit of more general relativistic models, with polytropic component of Equation of State. They constitute a special subclass of generalized Cardassian models basing on generalization of the Raychaudhuri equation rather than on generalization of the Friedmann first integral. We demonstrate that MOND cosmologies are compatible with observed accelerated phase of expansion of current universe only for high value of cosmological constant. The Bayesian framework of model selection favored this model over ΛCDM model if Ω_{m,o} is fixed but this evidence is not significant. Moreover obtained from statistical analysis value of the MOND characteristic β parameter is far from value required for explanation of the flat rotation curves of spiral galaxies.

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I. INTRODUCTION

The idea that dark matter manifestation in flat rotation curves of spiral galaxies is a consequence of Modified Newtonian Dynamics (MOND) seems to be both intriguing and controversial how claims Lue and Starkman. In principle there is a simple way to test this theory by consideration observational consequences coming from cosmological models basing on these modifications of gravitational interactions at the late time. Of course the MOND theory is Newtonian but it should be obtained as a limit of more general and fundamental theory (Modified General Relativity – MOGR). If we consider homogeneous and isotropic cosmology in this theory then the modified Friedmann equations will describe the evolution of the Universe.

Recently, different modifications of Friedmann equation were proposed. Motivation of this model is to explain the current acceleration of the Universe without references to the unknown form of dark energy. In this scenario (called the Cardassian or polytropic expansion) there is no dark energy component but the universe is matter dominated. It is accelerating due to the adding a certain additional term to Friedmann first integral of Einstein equations (with the Robertson-Walker symmetry). An important question is whether there is any connections between the MOND driven cosmology and the Cardassian one.

In the paper by Lue and Starkman it is presented interesting idea of derivation modified theory of gravity from constraint coming from the fundamental Birkhoff law. In this approach authors explain how the cosmic acceleration is generated through these modifications.

In this paper we incorporate MOND for the late time cosmological scenario, while early stages of evolution are dominated by usual matter described in a standard way by general relativity. For simplicity (without losing the generality) we assume that our universe is flat. In derivations of basic dynamical equations both for Newtonian and relativistic model we use particle-like description. In this approach the evolution of the universe is represented by a motion of a unit mass particle under the action of a one-dimensional potential \( V(a) \) which can be simply obtained from the MOND gravitational acceleration postulate. The position variable \( a \) is a scale factor of the universe and all dynamics is determined by the potential function through the analog of Newtonian equations.

The Cardassian models base on the generalization of the Friedmann first integral by
adding in r.h.s. a term which is called the Cardassian term, i.e.,

\[ H^2 = \frac{\rho}{3} + B\rho^n - \frac{k}{a^2}, \]  

where \( \rho \) is the energy density. If the source of gravity is a perfect fluid with pressure \( p = \gamma \rho \) \( (\gamma = \text{const}) \) then \( \rho = \rho_0 (a/a_0)^{-3(1+\gamma)} \), \( a_0 \) is a present value of the scale factor, \( k = 0, \pm 1 \) is the curvature constant, \( H = (\ln a)' \) is the Hubble function. Note that equation (1) is the first integral of the generalized Einstein equation for the Robertson-Walker symmetry. The basic equations constitute the system

\[ \dot{H} = -H^2 - \frac{\rho}{6} (1 + 3\gamma) + \frac{B\rho^n}{2}, \]  

\[ \dot{\rho} = -3H\rho(1 + \gamma) \]  

with the first integral in the form

\[ \left( \frac{H}{H_0} \right)^2 = \Omega_{\gamma,0} \left( \frac{a}{a_0} \right)^{-3(1+\gamma)} + \Omega_{\text{Card},0} \left( \frac{a}{a_0} \right)^{-3n(1+\gamma)} + \Omega_{k,0} \left( \frac{a}{a_0} \right)^{-2}. \]

Equation (2a) is called the Raychaudhuri equation, while equation (2b) is the conservation equation. It seems to be more natural to generalize the Raychaudhuri equation instead of its first integral. In the last case we obtain a more general theory containing MOND cosmologies as a special case. If we consider the standard Cardassian models, then in the right-hand side of equation 3) only power low terms can of type \( a^\beta \) can appears while in the MOND cosmologies some part of potential is logarithmic type.

Our basic idea is to explain the fraction of dark matter in the Universe in analogy to the Milgrom [3] explanation of flat rotation curves of spiral galaxies, i.e. in terms of the MOND conception rather than mysterious dark energy.

The organization of our paper is as follows. In section 2 we provide a brief summary of the features of the Cardassian models and generalized Cardassian models that are relevant for our further discussion. The particle-like description of MOFD cosmologies and constraining model parameters in the light of SNIa data (based on the Riess sample) are presented in section 3. Finally in section 4 some concluding remarks and perspectives for analysis of cosmology in the new MOFD (or MOGR) paradigm are formulated.
II. GENERALIZED CARDASSIAN MODELS AS A NATURAL GENERALIZATION OF FRW MODELS.

By the generalized FRW Cardassian models we understand models which dynamics is governed by the generalized Raychaudhuri equation and conservation condition

\[
\frac{\ddot{a}}{a} = -\frac{\rho(a)}{2} \left( \frac{1}{3} + \gamma(a) \right) - \frac{f(\rho(a))}{6}, \tag{4a}
\]

\[
\dot{\rho} = -3 \left( \frac{\dot{a}}{a} \right) (1 + \gamma(a)) \rho(a), \tag{4b}
\]

where a dot denotes differentiation with respect to the cosmological time \( t \), \( f(\rho(a)) \) defines the type of modification of the standard Raychaudhuri equation for FRW cosmology which holds for \( f = 0 \).

System (4) has a first integral in the form

\[
\rho_{\text{eff}} - 3 \frac{\dot{a}^2}{a^2} = 3 \frac{k}{a^2}, \tag{5}
\]

where \( \rho_{\text{eff}}(a) \) plays the role of effective energy density (see Appendix). Equation (5) is independent (directly) on the special form of matter the Universe is filled with. In the generic case if we put into (4a) \( f(\rho) \propto \rho^n \) and \( \gamma(a) \) like for a mixture of noninteracting matter and radiation then the usual class of the Cardassian models is recovered. However let us note that (5) with \( \rho_{\text{eff}}(a) = \rho(a) + 3B\rho^n \) does not play the role of the first integral in the special case when \( f(\rho)a \propto a^{-1} \). It is just the case of the MOND cosmologies. To illustrate this let us consider the simplest case of single fluid with energy density \( \rho \) and \( \gamma = \text{const} \). Then from equation (4b) we obtain

\[
\rho = \rho_0 \left( \frac{a}{a_0} \right)^{-3(1+\gamma)} \tag{6}
\]

or in term of density parameter \( \Omega_{i,0} \equiv \rho_{i,0}/3H_0^2 \)

\[
\Omega_i = \Omega_{i,0} \left( \frac{a}{a_0} \right)^{-3(1+\gamma)} \tag{7}
\]

Let us substitute \( f(\rho) = 3B\rho^n \). Hence (4a) assumes the form

\[
\frac{\ddot{a}}{a} = -\frac{\rho_0}{2} \left( \frac{1}{3} + \gamma \right) \left( \frac{a}{a_0} \right)^{-3(1+\gamma)} - \frac{\rho_0^2 B}{2} \left( \frac{a}{a_0} \right)^{-3n(1+\gamma)} \tag{8}
\]

It would be useful to consider in (8) two cases

\[
n \neq \frac{2}{3(1+\gamma)} \text{ or } n = \frac{2}{3(1+\gamma)}.
\]
In the first case we obtain a class of cosmologies called the Cardassian models. They can be treated as standard cosmological models where the universe is filled with a mixture of non-interacting perfect fluids with the equation of state \( p = \gamma \rho \) for the first and \( p = (n(1+\gamma) - 1)\rho = w\rho \) for the second one. Therefore the Cardassian models with a single fluid have dynamics in the form of a two-dimensional dynamical system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -\frac{1}{2} \left\{ \Omega_{\gamma,0} x^{-2-3\gamma} (1+3\gamma) + \Omega_{\text{Card},0} x^{-2-3w} (1+3w) \right\} \equiv -\partial V / \partial x,
\end{align*}
\]

where \( \Omega_{\gamma,0} = \rho_{\gamma,0}/3H_0^2, \Omega_{\text{Card},0} = B\rho_{\gamma,0}^{\text{eff}}/H_0^2(1+3w) \), are density parameters of matter and the fictitious Cardassian fluid respectively, \( x = a/a_0 \) is a dimensionless scale factor in the units of its present value \( a_0 \). A dot here denotes differentiation with respect to re-parametrized time variable \( \tau \) defined as \( t \rightarrow \tau : dt|_{H_0} = d\tau \). Of course \( \Omega_{\gamma,0} + \Omega_{\text{Card},0} + \Omega_{k,0} = 1 \) is satisfied. Note also that the potential function \( V \) is determined modulo to any additive constant.

Because \( n \neq \frac{2}{3(1+\gamma)} \) system (10) possesses the first integral in the form

\[
\frac{y^2}{2} + V(x) \equiv 0,
\]

where \( V(x) = -\frac{1}{2} \left\{ \Omega_{\gamma,0} x^{-1-3\gamma} + \Omega_{\text{Card},0} x^{-1-3w} + \Omega_{k,0} \right\} \) and the constant in \( V \) should be chosen such that \( \sum_i \Omega_{i,0} = 1 \). The constraint condition \( \sum_i \Omega_{i,0} = 1 \) in the general relativity reveals the fact that both matter and geometrical term are related. The fact that \( \Omega_{\text{Card},0} \) does not contribute in this relation is a reflection of the fact that we are beyond the standard cosmology. Note that it can be estimated only from the observations.

Let us comment now the second case of \( n = 2/3(1+\gamma) \). Then the potential function assumes very special form with logarithmic component:

\[
V(x) = -\frac{1}{2} \left\{ \Omega_{\gamma,0} x^{-1-3\gamma} + \Omega_{C,0} \ln x + \Omega_{k,0} + (1 - \Omega_{\gamma,0}) \right\},
\]

where \( \Omega_{k,0}^{\text{eff}} = \Omega_{k,0} + (1 - \Omega_{\gamma,0}) \). Then if we substitute this form into the (11) we can obtain the form of the first integral for this case (see Appendix). Note that both last two terms in (12) of the same type can be defined in one term which we called effective curvature density parameter. Usually the form of first integral (11) is treated as a starting point to further analysis of the generalized Friedmann equation. In our opinion the generalization of FRW equations on the level of the Raychaudhuri equation seems to be methodologically more
correct procedure than generalization of its first integral. Moreover is more general because one additional case is included.

Finally the generalized Cardassian models in our terminology constitute larger class of models and both cases for which both \( n \neq \frac{2}{3(1+\gamma)} \) and \( n = \frac{2}{3(1+\gamma)} \) belongs to this class. There are two parameters characterizing models of this class \((n, \Omega_{\gamma,0})\) if \(\Omega_{k,0} = 0\).

III. PARTICLE-LIKE DESCRIPTION OF MOND AND MOFD COSMOLOGIES.

In MOND the gravitational acceleration \( g \) exerted by a body of mass \( M \) at the radial distance \( a \) obeys the relationship

\[
g \propto \begin{cases} 
-a^{-2} & \text{for } |g| > g_0 \\
-a^{-1} & \text{for } |g| < g_0
\end{cases}
\]  

(13)

where \( g_0 \) is a critical value of acceleration. Hence the potential of the gravitational field can be simply calculated from the formula

\[
V(a) = -\frac{1}{M} \int_0^a g(a) da \propto \begin{cases} 
-a^{-1} & \text{for } |g| > g_0 \\
\ln a & \text{for } |g| < g_0
\end{cases}
\]  

(14)

At first we can build the Newtonian (13) cosmological models basing on the particle-like description of quintessential cosmology developed by us earlier [7]. Following this approach the dynamics of Newtonian cosmological models can be represented by a motion of the particle-universe under the action of a one-dimensional potential \( V = V(a) \), where \( a \) is a scale factor of the universe plays the role of positional variable.

The heuristic method of obtaining Newtonian modified potential is basing on consideration Schwarzschild solution of relativistic model. We start from Newtonian model potential and then derive relativistic model. In the FRW cosmology the evolution of the universe can be derived from the Hamiltonian which in terms of dimensionless variable takes the form

\[
\mathcal{H} = \frac{1}{2} y^2 + V(x) \equiv 0, 
\]  

(15)

where \( x = a/a_0, y = \dot{x} \) and \( V(x) \) is in the form

\[
V(x) = -\frac{1}{2} \left\{ \Omega_{\gamma,0} x^{-1-3\gamma} + \Omega_{\text{MOND},0} \ln x + (1 - \Omega_{\gamma,0}) \right\}.
\]  

(16)
The last term \((1 - \Omega_{\gamma,0})\) in (16) plays only the role of negative curvature term \(\Omega_{k,0}^{\text{eff}} = 1 - \Omega_{\gamma,0}\).

The Hamiltonian is defined on zero energy level \(H = E = 0\). The motion in the configuration space is defined in the domain admissible for motion:

\[
D_0 = \{ x : V(x) \leq 0 \}
\]  

(17)

From (15) and (16) we obtain the counterpart of the Friedmann equation in our theory. Of course if we substitute \(H = H_0\) and \(x = 1\) then we recover \(\sum_i \Omega_{i,0} = 1\) as a constraint on density parameters from relation

\[
H^2(x) = H_0^2 \left\{ \Omega_{\gamma,0} x^{-3(1+\gamma)} + \Omega_{\text{MOND},0} x^{-2} \ln x + \Omega_{k,0}^{\text{eff}} x^{-2} \right\}
\]  

(18)

or in the terms of redshift

\[
H^2(z) = H_0^2 \left\{ \Omega_{\gamma,0} (1 + z)^{3(1+\gamma)} - \Omega_{\text{MOND},0} (1 + z)^2 \ln (1 + z) + \Omega_{k,0}^{\text{eff}} (1 + z)^2 \right\}.
\]  

(19)

By comparing (18) with (12) we find strictly correspondence between a special second class of the Cardassian models with \(n = \frac{2}{3(1+\gamma)}\) and MOND cosmologies.

From the Newtonian analogue of the equation of motion \(\ddot{x} = -\partial V / \partial x\) we find that

\[
\ddot{x} = \frac{1}{2} \left\{ \Omega_{\gamma,0} (-1 - 3\gamma) x^{-2 - 3\gamma} + \Omega_{\text{MOND},0} x^{-1} \right\}
\]  

(20)

The universe is accelerating at the present epoch \((x=1)\) if only

\[
\Omega_{\text{MOND},0} > (1 + 3\gamma) \Omega_{\gamma,0}
\]  

(21)

Therefore for \(\gamma = 0\) (dust) \(\Omega_{\text{MOND},0} > \Omega_{m,0}\) is required if \(\Omega_{k,0}^{\text{eff}} = 0\).

The values of model parameter \((\Omega_{m,0}, \Omega_{\text{MOND},0})\) can be obtained from the fitting procedure to SNIa data. The luminosity distance as a function of redshift is given in the form

\[
d_L(z) = \frac{1+z}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1 + z')^3 - \Omega_{\text{MOND},0}(1 + z')^2 \ln (1 + z') + (1 - \Omega_{m,0})(1 + z')^2}}
\]  

(22)

In the mentioned before paper by Lue and Starkman [1, 2] we find very interesting idea of the MOND law of gravitational interacting derived from the general relativity. The authors assuming the validity of the Birkhoff theorem and derive the basic cosmological model equation in the form

\[
\frac{H^2}{H_0^2} = \frac{\dot{x}^2}{x^2} = g \left( \frac{\rho}{\rho_{\text{crit}}} \right) \equiv \begin{cases} 
\Omega_m + C_1 \Omega_m^{2/3} & \text{for the Einstein regime } \Omega_m > \Omega_c \\
\beta \Omega_m^{2/3} \ln \Omega_m + C_2 \Omega_m^{2/3} & \text{for the MOND regime } \Omega_m < \Omega_c
\end{cases}
\]  

(23)
In the Lue and Starkman \cite{1,2} model the evolution of the universe consists of two phases, the first one dominated by gravity following the general relativity and the second one by its modification (MOGR). Our idea is little different because we assume that both effects are acting as different regimal effects but general relativity dominates at early stages of evolution while MOGR describes the late time evolution. The dependence of the Hubble function describes following formula

\[ H^2(x) = H_0^2 \left\{ \Omega_{m,0} x^{-3} + \Omega_{\text{MOND},0} x^{-2} \ln x + \Omega_{k,0} x^{-2} \right\}, \tag{24} \]

where \( \Omega_{\text{eff}} \) is the effective curvature such that

\[ \Omega_{\text{eff},0} = \beta \Omega_{m,0}^{2/3} \ln \Omega_{m,0} + \Omega_{k,0}, \tag{25} \]

\[ \Omega_{k,0} = 1 - \Omega_{m,0}, \tag{26} \]

\[ \Omega_{\text{MOND},0} = -3 \beta \Omega_{m,0}^{2/3}. \tag{27} \]

Finally we obtain the same governing equation as \cite{16} from the general relativistic considerations.

The dynamics of this model can be represented in the form of the autonomous dynamical system

\[ \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{\partial V}{\partial x} \end{cases} \tag{28} \]

where

\[ V(x) = -\frac{1}{2} \left\{ \Omega_{m,0} x^{-1} + \Omega_{\text{MOND},0} \ln x + \Omega_{k,0}^{\text{eff}} \right\}. \tag{29} \]

The system \cite{28} has the first integral in the form

\[ \mathcal{H} = \frac{1}{2} y^2 + V(x) = 0 \]

From the first integral we obtain \( d_L(z) \) relation

\[ d_L(z) = \frac{1 + z}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1 + z')^3 - \Omega_{\text{MOND},0}(1 + z')^2 \ln (1 + z') + \Omega_{k,0}(1 + z')^2}}. \tag{30} \]

The parameter \( \beta \) can be expressed as a function of \( \Omega_{m,0} \). For example for dust matter \( \gamma = 0 \) we obtain constraint on \( \beta \) parameter

\[ \beta = -\frac{1 - \Omega_{m,0}}{3 \Omega_{m,0}^{2/3} \ln \Omega_{m,0}}. \tag{31} \]
TABLE I: Results of the statistical analysis of the MOND model obtained from the best fit with minimum $\chi^2$. F denotes fixed value of parameter.

| sample | $\Omega_{k,0}$ | $\Omega_{m,0}$ | $\Omega_{MOND,0}$ | $\Omega_{\Lambda,0}$ | $\mathcal{M}$ | $\chi^2$ | $\beta$ |
|--------|----------------|----------------|------------------|----------------------|---------------|---------|--------|
| Gold   | -0.88          | 0.00           | -2.00            | 1.88                 | 15.935        | 173.1   | $\infty$ |
|        | $\rightarrow$ | 0.00           | -0.81            | 1.00                 | 15.955        | 175.2   | $\infty$ |
|        | -0.90          | 0.01F          | -2.00            | 1.89                 | 15.935        | 173.1   | 14.36  |
|        | $\rightarrow$ | 0.01F          | -0.78            | 0.99                 | 15.955        | 175.2   | 5.60   |
|        | -0.94          | 0.05F          | -1.95            | 1.89                 | 15.935        | 173.1   | 4.78   |
|        | $\rightarrow$ | 0.05F          | -0.68            | 0.95                 | 15.955        | 175.3   | 1.67   |
|        | -0.83          | 0.30F          | -1.19            | 1.53                 | 15.955        | 173.8   | 0.88   |
|        | $\rightarrow$ | 0.30F          | -0.02            | 0.70                 | 15.955        | 175.8   | 0.01   |
|        | 0.00           | 1.00           | 1.80             | $\rightarrow$ 15.965 | 177.6        | -0.60   |
|        | $\rightarrow$ | 1.00           | 1.80             | $\rightarrow$ 15.965 | 177.6        | -0.60   |

If we define $z_{eq}$ as a moment in the evolution of the universe at which both material and MOND terms are equal we obtain

$$\frac{\Omega_{\text{MOND},0}}{\Omega_{m,0}} = \frac{1 + z_{eq}}{\ln(1 + z_{eq})},$$

(32)

where $\Omega_{\text{MOND},0}/\Omega_{m,0} = -3\beta\Omega_{m,0}^{-1/3}$.

The results of our analysis are based on the Gold Riess Riess et al. [8]) supernovae Ia sample and there are presented in the table. One can see that considered models well fitted SNIa data. However MOND model required value of $\beta \simeq 15$ for possibility of explanation of flat rotation curve. We obtained such value of $\beta$ only for the model with low $\Omega_{m,0} = 0.01$ and $\Omega_{k,0}^{\text{eff}} = -0.9$ (i.e $\Omega_{k,0} = -3.97$. This value of $\Omega_{m,0}$ and $\Omega_{k,0}$ are in disagreement with both results of CMBR and primordial nucleosynthesis.

IV. MOFD MODEL VERSUS $\Lambda$CDM MODEL IN THE LIGHT OF BAYESIAN INFORMATION CRITERION.

In this section we extended previous model by adding dark energy in the form cosmological constant or phantoms We show that the MOFD cosmologies can be obtained as a Newtonian
limit of class of Phantom models which base on a simple modification of the FRW equation. The physical status of both MOFD and Phantom models is similar because they offer the possibility of alternative explanation of dark matter and dark energy, respectively. We investigate some observational constraints on the FRW cosmological models with baryonic matter and MOFD phase squeezed in the evolutionary scenario between the epoch of matter domination and the dark energy epoch. We compare such a model with the concordance ΛCDM model and argue that while both models are indistinguishable (close value of $\chi^2$) the Akaike and Bayesian informative criterions favors MOFD model with baryonic dark matter.

We consider two possible model with the exit on Λ epoch or on the phantom (Cardassian) epoch. For both cases the relations $H(z)$ are (respectively):

$$H = H_0 \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 - \Omega_{MOND,0}(1+z)^2ln(1+z) + \Omega_\Lambda}$$ (33)

and

$$H = H_0 \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 - \Omega_{MOND,0}(1+z)^2ln(1+z) + \Omega_{Ph,0}(1+z)^3n}$$ (34)

where $p = w\rho$, $w < -1$ for phantoms, $n = 1 + w$ for Cardassian.

To compare considered models, how they fitted the data the informative criteria can be useful [9]. The problem of classification of the cosmological models on the light of information criteria on the base of the astronomical data was discussed in our previous papers [10, 11, 12, 13].

The Akaike information criterion (AIC) is defined in the following way

$$AIC = -2\ln L + 2d$$ (35)

where $L$ is the maximum likelihood and $d$ is the number of the model parameters. The best model with a parameter set providing the preferred fit to the data is that minimizes the AIC.

The Bayesian information criterion (BIC) introduced by Schwarz is defined as

$$BIC = -2\ln L + d\ln N$$ (36)

where $N$ is the number of data points used in the fit.

This criterion gives a simple objective criterion for the inclusion of new parameters into the standard ΛCDM model. From the results presented in the Tables [IIIIV] we can draw the following conclusion. The $\Omega_{MOND}$ is needed as a parameter and hence it is more likely that observations were generated in MOFD.

| case | name of model              | $H(z)$                                                                 | free parameters       | $d$  |
|------|----------------------------|-----------------------------------------------------------------------|-----------------------|------|
| 0    | Einstein-de Sitter         | $H = H_0 \sqrt{\Omega_{m,0}(1 + z)^3 + \Omega_{k,0}(1 + z)^2}$       | $H_0, \Omega_{m,0}$  | 2    |
| 1    | ΛCDM                       | $H = H_0 \sqrt{\Omega_{m,0}(1 + z)^3 + \Omega_{k,0}(1 + z)^2 + \Omega_{\Lambda}}$ | $H_0, \Omega_{m,0}, \Omega_{\Lambda}$ | 3    |
| 2a   | MOND, $\Omega_{m,0}$ - fitted | $H = H_0 \sqrt{\Omega_{m,0}(1 + z)^3 + \Omega_{k,0}(1 + z)^2 - \Omega_{\text{MOND},0}(1 + z)^2 \ln(1 + z) + \Omega_{\Lambda}}$ | $H_0, \Omega_{m,0}, \Omega_{\text{MOND},0}, \Omega_{\Lambda}$ | 4    |
| 2b   | MOND, $\Omega_{m,0} = 0.05$ | $H = \Omega_{\text{MOND},0}, \Omega_{\Lambda}$                      | $H_0, \Omega_{\text{MOND},0}, \Omega_{\Lambda}$ | 3    |

**TABLE II:** The Hubble function versus redshift for analyzed scenarios.
| case | AIC (1-Ω_{m,0} - Ω_{Λ,0} = 0) | AIC (1-Ω_{m,0} - Ω_{Λ,0} ≠ 0) | BIC (1-Ω_{m,0} - Ω_{Λ,0} = 0) | BIC (1-Ω_{m,0} - Ω_{Λ,0} ≠ 0) |
|------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 0    | 325.5                         | 194.4                         | 328.6                         | 200.5                         |
| 1    | 179.9                         | 179.9                         | 186.0                         | 189.0                         |
| 2a   | 181.2                         | 181.1                         | 190.3                         | 193.4                         |
| 2b   | 179.3                         | 179.1                         | 185.4                         | 188.3                         |

TABLE III: The values of AIC and BIC for distinguished models (Table II).
V. CONCLUSIONS

Our general conclusion is that MOND cosmology should be treated as a potential alternative to the ΛCDM model in the context of explanation of dark matter. To clarify the status of these model let us consider two sets of best-fitted (gold sample of SNIa, Riess et al. [8]) model parameters (see table I).

Our point of view is following - because the ΛCDM model fits SNIa data as well as the MOND alternative and additionally the second model explain dark matter content in term of $\Omega_{\text{MOND},0}$ the model under consideration should be treated as a possible candidate to explain dark matter in the Universe. However in particular case for the flat universe with $\Omega_{m,0} = 0.3, \Omega_{\Lambda,0} = 0.7$ we obtain $\Omega_{\text{MOND},0} = -0.02$, i.e. $\beta = 0.01$ while for the flat universe with $\Omega_{m,0} = 0.05, \Omega_{\Lambda,0} = 0.95$ we obtain $\Omega_{\text{MOND},0} = -0.68$ i.e. $\beta = 1.67$. The first case is corresponding to the ΛCDM model while the second should be treated as an alternative description of acceleration driven by cosmological constant and dumping by baryonic matter ($\Omega_{m,0} = 0.05$). Both models are indistinguishable—close values of $\chi^2$ (see table II) and as result overlapping Hubble diagrams.

On can see that however considered models well fitted SNIa data MOND model required value of $\beta \simeq 15$ for possibility of explanation of flat rotation curve. We obtained such value of $\beta$ only for the model with $\Omega_{m,0} = 0.01$ and $\Omega_{k,0} = -0.9$ (i.e $\Omega_{k,0} = -3.97$). This value of $\Omega_{m,0}$ and $\Omega_{k,0}$ are in disagreement both with result of CMBR and early nucleosynthesis. Finally we conclude that, the MOND conception explain only separately flat rotation curves of spiral galaxies or the fraction of dark matter in the Universe but it is not able to explain these both facts together.

In this paper we also demonstrate that classical MOND conception can be derived from more fundamental relativistic theory, namely from the generalized Cardassian model.

The main aim of the paper was to show that the existence of the MOND phase during the evolution of the Universe, before the epoch of domination of dark energy can explain the presence of dark matter in the Universe. In other words there are two indistinguishable scenario from the point of view of explanation of the SNIa data. On the other hand if we á priori assume that $\Omega_{m,0} = 0.3$ the observations exclude the cosmological model with the squeezing MOND phase in the cosmological scenario ($\Omega_{\text{MOND},0} \simeq 0$). If we assume flat universe with the value of $\Omega_{m,0} \simeq 0.3$ as it is suggested by extragalactic observations than we
obtain that $\Omega_{\text{MOND},0}$ should be small, but not necessary equal to zero. In our approach we check whether the MOND phase frozen in the cosmological scenario according to Starkman’s idea can give us understanding of the fraction of nonbarionic matter in $\Omega_{m,0}$. We find that such a model well fit supernovae data but value of $\beta$ is far from Lue and Starkman value $\beta = 15$.

The second topic of this paper is the construction of the new class of cosmological models with frozen the MOND phase into evolutional scenario with exit to the Cardassian models. As it is well known the Cardassian models are an alternative to the cosmological models with dark energy in the explanation of present acceleration of the current Universe. In these models instead of dark energy violating the strong energy condition is postulated a simple modification of the Friedmann first integral. In this paper the model is fitted to observations of distant SNIa using the Riess sample. We obtain analogous results as in the case with the exit to the dark energy epoch. The advantage of the model with frozen MOND phase and exit to the Cardassian models is twofold. First, it can explain the acceleration of the Universe. Second, it can explain the fraction of the dark matter.

The other results can be summarized as follows. We propose the theoretical description of cosmology MOFD based on the modified gravity. We find the connection of such models with recently discussed Cardassian models [20]. The parameter $\beta$ characterizing the MOND phase is estimated. We also estimated this parameter for the model with exit to the Cardassian model. In this case we obtain the value of characteristic parameter $\beta$ which is far to the value assumed by Starkman ($\beta = 15$). The value of Cardassian exponent in the term $\rho^n$ in the modified Friedmann equation is close to zero. This situation is very close the model with the cosmological constant but nevertheless $n$ is negative and nonzero.

In this paper we pay attention to the flat cosmological models. It would useful to make some remarks on the non-flat cosmological models. In the models with exit to $\Lambda$ epoch we estimate the curvature type term $\Omega_{k,0}^{\text{eff}}$. From the $\chi^2$ analysis we obtain that non-flat case is more preferable than its flat counterpart. The similar dependence of $d_L(z)$ on the Hubble diagram is obtained for fitting the model with the exit to the Cardassian domination epoch with the SNIa data (without any prior on $n$).

However MOFD cosmologies are compatible with observed late-time accelerated expansion of contemporary universe. The popular method of apriorical generalization of Friedmann equation is adding polytropic component of r.h.s. of $H^2$ relation i.e. generalization
Friedmann first equation. Our proposal is generalization Raychaudhuri equation rather then Friedmann. Then we obtain previous generalization plus one exceptional case which is strictly related with main subject of the paper.

However we still share the opinion expressed by Sahni that there is the fundamental difficulty of MOND gravity because this theory is not embedded within a more comprehensive and fundamental theory of gravitation. We also do not know the Lagrangian for the Cardassian modification of gravity but these models can be treated as a simple modification of the cosmological models with FRW symmetry.

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VII. APPENDIX

In this section we demonstrate how the presence of additional term in the Raychaudhuri equation can be modeled by some noninteracting fictitious fluid \( X \) with energy density \( \rho_X(a) \) and pressure \( p_X(a) \). We start from the basic equations

\[
\frac{\ddot{a}}{a} = -\frac{1}{6} (\rho + 3p) + \frac{B}{6} a^m, \quad (37a)
\]

\[
\dot{\rho} = -3H (\rho + p). \quad (37b)
\]

If we postulate that

\[
-\frac{1}{6} (\rho_X + 3p_X) = \frac{B}{6} a^m \quad (38)
\]

then (37) can be rewritten to the form

\[
\frac{\ddot{a}}{a} = -\frac{1}{6} \sum_{i,X} (\rho_k + 3p_k), \quad (39)
\]

where the summation should be performed over all components of fluid. For any \( i \) fluid conservation equation is satisfied

\[
\dot{\rho}_i = -3H (\rho_i + p_i). \quad (40)
\]

Of course analogical condition should be satisfied by the fluid \( X \), i.e.

\[
\frac{d\rho_X}{da} = -\frac{3}{a} (\rho_X + p_X). \quad (41)
\]
From (38) we calculate $p_X$ and then we substitute this expression into (41). Hence we obtain

$$p_X = -\frac{1}{3}\rho_X - \frac{B}{3} a^m$$

(42)

and

$$\frac{d\rho_X(a)}{da} = -2 a \rho_X(a) + \frac{B}{a} a^m.$$  

(43)

As a solution of (43) we obtain

$$\rho_X(a) = \begin{cases} \frac{C}{a^2} + \frac{B}{m+2} a^m & \text{for } m \neq -2 \\ \frac{C}{a^2} + \frac{B}{m} \ln a & \text{for } m = -2 \end{cases}$$

(44)

$$p_X(a) = \begin{cases} -\frac{C}{3a^2} - \frac{B}{m+3} a^m & \text{for } m \neq -2 \\ -\frac{C}{3a^2} - \frac{B}{3m^2} \ln a & \text{for } m = -2 \end{cases}$$

(45)

Of course system (37) has the first integral in the form

$$\rho_{\text{eff}} - 3 \frac{\dot{a}^2}{a^2} = 3 \frac{k}{a^2} = \sum_i (\rho_i + \rho_X) - 3 \frac{\dot{a}^2}{a^2},$$

(46)

where

$$\dot{\rho}_i = -3H(\rho_i + p_i)$$

(47)

for any $i$-fluid, $\sum_i \rho_i = \rho$ and also $\rho_{\text{eff}} = -3H(\rho_{\text{eff}} + p_{\text{eff}})$. The first integral (46) has different form for both distinguished cases

$$\rho + \frac{C}{a^2} + \frac{B}{m+2} a^m - 3 \frac{\dot{a}^2}{a^2} = 3 \frac{k}{a^2}, \quad \text{for } m \neq -2$$

(48a)

$$\rho + \frac{C}{a^2} + \frac{B}{a^2} \ln a - 3 \frac{\dot{a}^2}{a^2} = 3 \frac{k}{a^2}, \quad \text{for } m = -2$$

(48b)

We require the correspondence with standard FRW model for the case $B = 0$. Hence we obtain $C = 0$. Finally the potential functions for both cases takes the following form

$$V(a) = \begin{cases} -\frac{1}{6}(\rho + \frac{B}{m+2} a^m) a^2 & \text{for } m \neq -2 \\ -\frac{1}{6}(\rho + \frac{B}{a^2} \ln a) a^2 & \text{for } m = -2 \end{cases}$$

(49)

Of course the Hamiltonian system is still determined on the energy level $E = -k/2$ ($\mathcal{H} = \dot{a}^2/2 + V(a) \equiv E$).

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