The optimal reordering of measurements for photonic quantum tomography

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Quantum tomography is an essential method of the photonic technology toolbox and is routinely used for evaluation of experimentally prepared states of light and characterization of devices transforming such states. The tomography procedure consists of many different sequentially performed measurements. We present considerable tomography speedup by optimally arranging the individual constituent measurements, which is equivalent to solving an instance of the traveling salesman problem. As an example, we obtain solutions for photonic systems of up to five qubits, and conclude that already for systems of three or more qubits, the total duration of the tomography procedure can be halved. The reported speedup has been verified experimentally for quantum state tomography and also for full quantum process characterization up to six qubits, without resorting to any complexity reduction or simplification of the system of interest. Our approach is versatile, and reduces the time of an input-output characterization of optical devices and various scattering processes as well.

I. INTRODUCTION

Quantum photonic technologies have the potential to revolutionize information and communication systems, employing non-classical states of light to encode, manipulate, and transmit information [1, 2]. There has been a significant progress in secure communications, photonic simulations, and other recently emerging fields, especially with the advent of novel sources of entangled states [3, 4] and quantum integrated circuits [5–7]. The complexity of the quantum states, and their transformations, grows exponentially with the number of quantum bits (qubits). Quantum information processing exploits this scaling to realize quantum algorithms and circuits, showing a potential advantage over their classical counterparts. In order to evaluate the performance of the quantum state sources and the quantum circuits, it is necessary to perform their characterization. Quantum state tomography [8–12], developed to estimate quantum state based on measurement outputs, is an essential part of the quantum information processing toolbox and is routinely used for characterization of experimentally prepared quantum states of light [13, 14]. Also, quantum process tomography [15] is a crucial tool for full characterization of a device that transforms quantum states, like an on-chip interferometric network or a quantum circuit processing qubits in a quantum register, see Figs. 1(b) and 1(c). Despite being a much more complicated procedure, it was shown that quantum process tomography is equivalent to quantum state tomography performed on a larger parameter space [16, 17].

A photonic dual-rail qubit encodes information in two modes, such as horizontal and vertical polarization or lower and upper path of an interferometer, which serve as computational basis states [2, 19]. Estimating an unknown state of a qubit system by means of quantum state tomography requires projecting the state onto quorum states, often chosen as tensor products of single-qubit Pauli eigenstates [20], a minimum set of projections [21], or randomly selected [22, 23]. Despite the choice of set of tomographic measurements, one has to sequentially adjust the individual projections using programmable analyzers, often realized using birefringent wave plates in rotation mounts, voltage controlled liquid crystal modulators, or various phase shifters, see Figs. 1(b) and 1(c). The analyzers need to be readjusted when switching from one measurement to another, which can be time-consuming. It is also generally the case that the transition time depends on the particular measurement sequence. Consequently, it is reasonable to ask whether it is possible to optimize the order of tomographic measurements with respect to the total time spent on readjusting the analyzers [20]. We are then faced with an instance of the traveling salesman problem (TSP) [25].

For quantum process tomography, the optimization can be performed at the measurement stage as well as the preparation stage. We will then further discuss the optimization of the measurement sequence, knowing that the results also hold for state preparation. Indeed, the preparation sequence can always be the subject of optimization even when the measurement is implemented without the necessity of analyzer readjusting [26, 27]. The reduction of quantum tomography duration is highly desirable as it also reduces the overall setup drift and other systematic errors, which typically yields improved quality of the quantum operation or state characterized [28, 29].

In this work, we solve the measurement order optimization for the case of polarization tomography with wave plates in rotation mounts. The optimization for other physical realizations of analyzers or preparation stages can be performed analogously. We illustrate our approach on a one-qubit system, and then find the optimal strategy for systems of increasing numbers of qubits, going up to five. We quantify the time saved on reorienting the wave plates in terms of the speedup factor and find an increasing trend with each added qubit. We also experimentally demonstrate the feasibility of the speedup in systems of up to six qubits, where we achieve a speedup close to the predicted one. Furthermore, we discuss possible optimization for path-encoded qubits on an optical...
FIG. 1. (a) Complete characterization of a quantum circuit based on probing (P) the input qubits by all possible combinations of quantum states selected from a quorum and performing full analysis (A) of the output qubits. The analysis of the output qubit consists of projections of the qubit state onto quorum states. (b) For photonic circuits the analysis (A) is often polarization-encoded and formed by a sequence of wave plates (half-wave, HWP; quarter-wave, QWP), a polarization beam splitter (PBS), and a single-photon detector (SPD). The wave plates are rotated to perform the projections to particular polarization states. (c) When both outputs of the PBS are detected by the SPDs, two orthogonal projections are measured at the same time, reducing the duration of the measurement at the expense of a higher number of SPDs employed. The preparation stage (P) is constructed in a similar way using a proper single-photon source, a polarizer, and a sequence of wave plates.

II. ONE-QUBIT CASE

We start with a simple example of one-qubit state tomography. We consider a tomography implementation with a half-wave plate, a quarter-wave plate, and a polarizing beam splitter with a detector in one of its output ports (further called the six-state scheme) shown in Fig. 1(b), where tomography is realized by projecting on six polarization states in three mutually unbiased bases. These polarization states are namely the horizontal (H), vertical (V), diagonal (D), anti-diagonal (A), right-handed circular (R), and left-handed circular (L). The wave plates are in motorized rotation mounts, and projections on each of these states can be measured after orienting the wave plates so that their axes of birefringence are at specific angles with the plane of horizontal polarization of the measured light. Conventionally, the projections are measured in the order in which the polarization states H through L were presented here unless a Gray-code-like ordering is used where just one wave plate is allowed to move during each readjustment [20].

The wave plate angles for all the projections are shown in Table I. During transitions between individual measurements, it is necessary to rotate the wave plates by different angles, which in turn takes different amounts of time for the motorized mounts to realize. As more than one wave plate will generally have to be rotated, we compare all of the individual wave plate rotation times and take the maximum as the transition time.

Also note that while it is the total time spent on transitions that we ultimately seek to minimize, we will continue to specify the problem in terms of the wave plate rotation angles (absolute valued), and, where convenient, still refer to time instead of angle. This is so that the problem specification and solution are not dependent on the particular rotation mounts used. The two formulations, temporal and angular, are indeed equivalent if the rotation time of the mount scales linearly with the angle traveled, which is practical the case. The temporal formulation has to be used for different polarization analyzers employed, i.e. liquid crystal modules or piezo-based fiber polarization controllers, where the angle is not defined and the transition time does not scale linearly with control voltages.

Now, with a set of states to project onto, and the corresponding wave plate angles, we can proceed to the optimization. In our case, the number of projections to be measured during tomography is \( p = 6 \) and the number of all the possible permutations of their order is \( p! = 720 \). It is then feasible to find the measurement order of the least total transition time using a brute-force method. This approach, however, does not scale well with the number of qubits in the quantum system of interest. For example, in the case of a two-qubit system, the number of measurements is \( p^2 = 36 \) and the number of their possible permutations is \( 36! \approx 10^{41} \). Our only option then is to solve the TSP using the state-of-the-art branch and bound algorithms [25].

The TSP is commonly specified using the adjacency matrix. Its element \( c_{ij} \) is in our case given as the maximal angle traveled by any wave plate during a transition.

\[
\begin{array}{cccccc}
H & V & D & A & R & L \\
H & 0 & 22.5 & 22.5 & 22.5 & 22.5 \\
V & 22.5 & 0 & 22.5 & 67.5 & 45 \\
D & 22.5 & 22.5 & 0 & 45 & 45 \\
A & 22.5 & 67.5 & 45 & 0 & 45 \\
R & 45 & 45 & 45 & 45 & 0 \\
L & 45 & 45 & 45 & 45 & 90 \\
\end{array}
\]

TABLE I. One-qubit polarization tomography (the six-state scheme). The H, V, D, A, R, and L projections are shown with their corresponding wave plate angles in degrees.

TABLE II. The TSP specification using the adjacency matrix consists of the maximal angles rotated by any wave plate during transitions between projections.
TABLE III. The optimal projection measurement sequence for one-qubit tomography.

H L A R V D

from measurement $i$ to measurement $j$. The diagonal elements are zero, and the matrix is symmetrical. The adjacency matrix for the one-qubit, six-state scheme case is shown in Table II. This matrix can then be given as an input to any TSP solver compliant with the standard defined by TSPLIB. We made use of the Concorde solver. The solver finds the shortest cycle of measurements, meaning that the optimization takes into consideration also the transition from the final measurement back to the initial one (the preparation for the next run of tomography). The resulting optimal measurement sequence is shown in Table III.

To quantify the reduction in the total transition time $\tau_{\text{TSP}}$ of the TSP-optimized order of measurements compared to the conventional order, where the time spent on transitions is $\tau_{\text{conv}}$, we use the speedup factor $s = \tau_{\text{conv}}/\tau_{\text{TSP}}$. The time $\tau_{\text{conv}}$ can be readily computed from the adjacency matrix as $\tau_{\text{conv}} = \left(\sum_{i=1}^{5} c_{i,i+1}\right) + c_{6,1}$, the last term being the transition from the final to the initial measurement, completing the cycle. For the one-qubit case, we found that the total angular duration of the cycle of transitions with the conventional order of measurements was 292.5°, whereas for the TSP-optimized order of measurements it was 225°. The same result is obtained using the brute-force method. The speedup factor for single-qubit tomography is therefore $s = 1.3$. The same speedup can also be exploited in classical polarimetry with a strong optical signal.

III. MORE QUBITS

For the tomography of a multi-qubit system it is not enough to perform the one-qubit procedure for each individual qubit. It is necessary to measure the projection onto every combination of the single-qubit polarization states. This requirement leads to a $p^n$ scaling of the number of measurements, $p$ being the number of measurements for a single-qubit state, and $n$ being the number of qubits in the quantum system. This naturally affects the duration of the entire tomography measurement, and gives an even greater incentive to reduce it.

As the number of possible permutations of the order of the tomographic measurements is given by a factorial, we were forced to abandon the brute force method and rely on a TSP solver, where we were able to obtain the optimal measurement orders for up to five-qubit systems in a similar fashion as in the previous single-qubit example. The optimal sequence for two-qubit tomography is shown in Table IV. Measurement sequences for systems of higher numbers of qubits are too long to show in print. They are instead stored online. The size of the adjacency matrix scales as $p^n \times p^n$. For the six-state scheme and a five-qubit system, this means that a 7776 × 7776 matrix was required for the specification of the TSP. Matrices of such size were generated automatically by a computer program.

Additionally, apart from the six-state scheme, another configuration has also been considered. Using a detector in each of the output beam splitter ports as shown in Fig. 1(c), we can measure the projections onto both of the orthogonal polarization states in a given basis simultaneously. This allows for one-qubit tomography to consists of three, and not six, measurements. Not the individual states, but the bases, are readjusted leading to simultaneous measurements of the H and V, D and A, and R and L states, with the wave plate angles the same as for the H, D, and R state, respectively. We will further refer to this configuration as the three-base scheme. It is important to note that employing the three-base scheme is limited to quantum state tomography and cannot be utilized for probe state preparation in quantum process tomography or any input-output system characterization.
The total angular (and thus, temporal) duration of tomographic tasks increases exponentially with the size of the quantum system. However, the TSP optimization of the order of measurements yields reduction of the total tomography duration, shown in Fig. 2, which also scales exponentially. The only exception is the one-qubit, three-base tomography, where the conventional order is optimal already. The speedup factor, plotted in Fig. 3, is generally bigger for the six-state scheme compared to the three-base scheme, but in both cases it follows the same monotonously rising trend, reaching a value of 2 already for three-qubit tomography relying on the six-state scheme. The limitation on the number of qubits for which we were able to find the optimal measurement orders seems to originate from the size of the problem, namely the number of the measurements required for complete tomography. In the six-qubit case, this is 46,656, which is out of reach for the state-of-the-art TSP solvers. Additional concerns come from the memory limitations of the consumer-grade computers we used the TSP solver on. However, even for larger systems, where the complete optimization cannot be performed, it is still possible to utilize the results for a smaller subsystem and achieve the corresponding speedup. For example, six-qubit tomography can be partially optimized by setting the measurements on the five-qubit subsystem in the optimal order and changing the sixth qubit measurement in the conventional order. This approach would still yield the speedup factor corresponding to a five-qubit system.

To demonstrate the speedup under real conditions, we have used the TSP solver to optimize the order of preparations and measurements in existing experimental setups. We performed quantum state tomography of a three-qubit entangled Greenberger–Horne–Zeilinger state prepared by a Toffoli gate (controlled-controlled-NOT gate) \[33\]. Furthermore, quantum process tomography was used to fully characterize a two-qubit SWAP gate \[34\]. The computer program responsible for manipulating the rotation mounts was modified to record timestamps just before and after each transition. We were thus able to compute the speedup factor, and arrived at results close to the predicted values (see Fig. 3). The slight discrepancy between the predicted and the measured values of the speedup is caused by time overhead of the communication with the motorized rotation mounts. Also, small deviation from linear dependence of mounts’ rotation time and the angle travelled contributes to this discrepancy.

An important fact to note is that we have compared the total times spent on reorienting the wave plates, not the total time of tomography. When comparing the latter, we may arrive at a smaller speedup factor. This is due to the fact that the tomography consists of other operations, apart from the plate readjustment, such as data acquisition and hardware communication overhead. Using the optimal order of measurements and a low-latency multi-channel counter \[35\], we have been able to reduce the overall time of full quantum process tomography of a three-qubit quantum controlled-controlled-phase gate \[36\] from approximately 23 hours to less than 11 hours. Consequently, the impact of setup instabilities has been effectively reduced.

### IV. DIFFERENT TOMOGRAPHY SCHEMES AND INFORMATION ENCODINGS

To show the wide applicability of the optimization strategy we also demonstrate tomography speedup for path-encoded quantum circuitry on an optical chip. The path-encoded qubit is formed by two paths of a Mach-Zehnder interferometer with different probabilities of having a photon in the upper (0) and lower (1) arms, and a relative phase between them. The key element here is a phase shifter, often implemented using a resistive heating element on the chip surface, enabling reconfigurability of the waveguide circuit \[3, 37–40\]. The preparation of input states and the projection measurements requires setting the phase and changing the splitting ratio of a waveguide coupler. The former can be performed with the help of the heating element but the latter cannot be achieved easily on the chip. Instead, a Mach-Zehnder interferometer with the heater in one arm is employed to emulate the functionality of a variable-ratio coupler, see Fig. 4. The required phase settings for six-state tomography with a single detector per qubit are shown in Table 4. The phase induced by the heater, and also the required settling time are proportional to the dissipated power, with approximately 0.5 W inflicting the phase change of \(2\pi\) within 1 s \[39\]. Two different optimization goals can be pursued, leading to minimization of either the total time of phase readjusting or the total heat transferred to the chip from all heaters. The time minimization problem
section schemes (the 0, 1, +, −, i, and –i projections are shown with the corresponding phase settings.

| T:R | θ     | ϕ     |
|-----|-------|-------|
| 0   | 100:0 | 0     |
| 1   | 100:0 | π/2   |
| +   | 50:50 | π/2   |
| −   | 50:50 | −π/2  |
| i   | 50:50 | π/2   |
| −i  | 50:50 | −π/2  |

is formulated in phase units (rad) in a similar way as for polarization tomography, with the adjacency matrix elements given by the maximum phase change across all the involved heaters. The achievable speedup reaches 1.43 for 1 qubit and practically saturates at the value of 1.80 for two and more qubits, see Fig. 5(a). The thermal load minimization problem depends severely on the exact temporal response of the heater. For simplicity we assume that the temperature (and the inflicted phase change) increases linearly with time after the heating power is switched on and before the target temperature (phase) is reached. Consequently, the total heat transferred is proportional to the product of the power being dissipated by all the heaters and the settling time of the heater that is responsible for the largest phase change. Even such simplified description requires the use of nontrivial asymmetric adjacency matrices. The optimization yields heat reduction of 0.3 J (reduction factor of 1.59) for a single qubit, which goes up to 343 J (reduction factor of 1.79) for four qubits, see Fig. 5(b). Upon comparing the total duration of the heat-optimized measurement sequences with the time-optimized ones, we found that they were not significantly different. This leads us to the conclusion that both duration and heat reduction can be achieved at once.

The tomography schemes studied so far consist of probing by and projecting to tensor products of the eigenstates of Pauli operators. However, square-root measurements [41], random sampling [42], and compressed sensing techniques [22–24] benefit from random (or at least randomly selected) projection measurements. The optimization of all the schemes in the plethora of measurement frameworks utilizing some form of randomness in state preparation or measurement is beyond the scope of this work. To simply demonstrate the applicability of the proposed TSP optimization strategy, we show the average speedup reached by the optimal reordering of polarization projection measurements with randomly generated rotation angles of the half-wave and quarter-wave plates. The total number of individual measurements were chosen to be the same as for the previous tomographic scheme based on tensor products of single-qubit Pauli eigenstates to facilitate the comparison with the previous results. The random-measurement scheme speedup, averaged from ten different sets of randomly selected wave plate angles, reaches 2.18 for two qubits already.

| T:R | θ     | ϕ     |
|-----|-------|-------|
| 0   | 100:0 | 0     |
| 1   | 100:0 | π     |
| +   | 50:50 | π/2   |
| −   | 50:50 | −π/2  |
| i   | 50:50 | π/2   |
| −i  | 50:50 | −π/2  |

V. CONCLUSION

We have shown that the total duration of quantum tomography can be considerably reduced by an optimal reordering the constituent measurements. We have presented the solutions and experimental verifications of the devised optimization strategy for photonic quantum tomography of up to six polarization encoded qubits. The total time spent on reorienting the polarization-changing elements required for the tomography procedure is minimized, which yields nearly twofold increase in speed already for three-qubit state tomography. The speedup has been analyzed also for advanced input-output process characterization with the optimized sequence of probe state preparations and output measurements. As an example, the duration of full tomographic characterization of a complex quantum logic gate, consisting of 47 thousands constituent measurements, was reduced to less than 50%, compared to the unoptimized conventional strategy.

The optimization strategy presented here for polarization encoding with wave plates can also be used with other analyzers, encodings, and even different optimization goals in mind. For example, on-chip path-encoding utilizing heating elements to change the phase between various paths can be optimized not only to decrease the total measurement duration but also to diminish the overall thermal load. Furthermore, the reported tomography optimization is applicable to continuous-variable [43] or high-dimensional systems [44]. As with the full quantum tomography, also other measurement schemes like direct fidelity estimation [22] and matrix-product-state tomography [45–48] can benefit from the presented optimization approach. The achievable speedup increases with the number of parties and preparation/measurement settings.
FIG. 5. The speedup factor (a) for temporal optimization of the path-encoded state tomography using heater elements practically saturates at the value of 1.8 for more than two qubits. The total heat reduction (b) for heat-optimized tomography, assuming that power of 0.5 W applied over 1 second is required for a $2\pi$ phase change.

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