POSITIVE GROUND STATE SOLUTIONS FOR FRACTIONAL LAPLACIAN SYSTEM WITH ONE CRITICAL EXPONENT AND ONE SUBCRITICAL EXPONENT

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ABSTRACT. In this paper, we consider the following fractional Laplacian system with one critical exponent and one subcritical exponent

\[
\begin{aligned}
(\Delta)^s u + \mu u &= |u|^{p-1}u + \lambda v, \quad x \in \mathbb{R}^N, \\
(\Delta)^s v + \nu v &= |v|^{2^*-2}v + \lambda u, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

where \((\Delta)^s\) is the fractional Laplacian, \(0 < s < 1\), \(N > 2s\), \(\lambda < \sqrt{\mu \nu}\), \(1 < p < 2^* - 1\) and \(2^* = \frac{2N}{N-2s}\) is the Sobolev critical exponent. By using the Nehari manifold, we show that there exists a \(\mu_0 \in (0, 1)\), such that when \(0 < \mu \leq \mu_0\), the system has a positive ground state solution. When \(\mu > \mu_0\), there exists a \(\lambda_{\mu, \nu} \in [\sqrt{(\mu - \mu_0)\nu}, \sqrt{\nu}]\) such that if \(\lambda > \lambda_{\mu, \nu}\), the system has a positive ground state solution, if \(\lambda < \lambda_{\mu, \nu}\), the system has no ground state solution.

1. Introduction. In the past decades, the Laplacian equation or system has been widely investigated and there are many results about ground state solutions, multiple positive solutions, sign-changing solutions, etc (see \[9, 10, 11, 12, 13, 24, 25\] and references therein).

Compared to the Laplacian problem, the fractional Laplacian problem is non-local and more challenging. Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications (see \[3, 6, 7, 14, 20, 22, 27, 29\] and references therein). This type of operator arises in a quite natural way in many different contexts, such as, the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and a great deal of others (see \[1, 16, 23, 30\] and references therein).

For the case of fractional Laplacian equation, the existence and nonexistence of solutions has been studied by a lot of researchers. For example

\[
(\Delta)^s u + u = f(u) \text{ in } \mathbb{R}^N,
\]
has been studied by numerous authors under various hypotheses on the nonlinearity f. Such as, when f is subcritical, Wang and Zhou [31] obtained the existence of a radial sign-changing solution for equation (1) by using variational method and Brouwer degree theory. When the nonlinearity f satisfies the general hypotheses introduced by Berestycki and Lions [4], Chang and Wang [8] proved the existence of a radially symmetric ground state solution with the help of the Pohožaev identity for (1). However, in all these works, they only consider the existence and nonexistence solutions, but there are few results about the uniqueness of solution for fractional Laplacian equation. In [17, 18], for the subcritical case, when f(u) = |u|^{p-2}u, p \in (2, 2^*), R.L. Frank and E. Lenzmann [17] showed the uniqueness of non-linear ground states solutions to the equation (1) for one dimension case and R.L. Frank, E. Lenzmann and L. Silvestre [18] showed the general unique ground state solution to the equation (1) for dimension greater than one.

It is also nature to study the coupled system. For the following fractional Laplacian system,

\[
\begin{aligned}
(-\Delta)^s u &= F(u, v), \quad x \in \mathbb{R}^N, \\
(-\Delta)^s v &= G(u, v), \quad x \in \mathbb{R}^N,
\end{aligned}
\]

has been investigated by many authors under various hypotheses on the nonlinearity F(u, v) and G(u, v). For example, when F(u, v) = f(u) + \lambda v - u, G(u, v) = g(u) + \lambda u - v, D.F. Lü and S.J. Peng [21] showed the vector ground state solution under suitable condition of f, g and \lambda. When F(u, v) = \mu_1 |u|^{2^*-2}u + \frac{\alpha^2}{2^2} |u|^{\alpha-2}u|v|^\beta, G(u, v) = \mu_2 |v|^{2^*-2}v + \frac{\beta^2}{2^2} |u|^{\alpha}|v|^{\beta-2}v, M. D. Zhen, J. C. He and H. Y. Xu [33] showed that the existence and nonexistence of ground state solutions under suitable condition of \alpha, \beta, \gamma, s, N and Z. Guo, S. Luo and W. Zou [20] showed that under suitable condition of \alpha, \beta, s, N the system has a positive ground state solution for all \gamma > 0. When F(u, v) = (|u|^{2p} + b|u|^{p-1}|v|^{p+1})u - u, G(u, v) = (|v|^{2p} + b|v|^{p-1}|u|^{p+1})v - \omega^{2\alpha}v, Q. Guo and X. He [19] proved the existence of a least energy solution via Nehari manifold method and showed that if b is large enough, it has a positive least energy solution. Note that in all these works, they only consider subcritical case or critical case. As far as we know, there are few results for the fractional Laplacian system with one critical exponent and one subcritical exponent on \mathbb{R}^N. In the case of Laplacian system, the problem has been investigated by Z. Chen, W. Zou in [10].

The system we consider is the following

\[
\begin{aligned}
(-\Delta)^s u + \mu u &= |u|^{p-1}u + \lambda v, \quad x \in \mathbb{R}^N, \\
(-\Delta)^s v + \nu v &= |v|^{2^*-2}v + \lambda u, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

where \((-\Delta)^s\) is the fractional Laplacian, \(0 < s < 1\), \(N > 2s\), \(\lambda < \sqrt{\mu\nu}\), \(1 < p < 2^* - 1\), \(2^* = \frac{2N}{N-2s}\) is the Sobolev critical exponent. The fractional Laplacian \((-\Delta)^s\) is defined by

\[-(-\Delta)^s u(x) = \frac{C(N, s)}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N\]

with

\[C(N, s) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos(x)}{|x|^{N+2s}} dx \right)^{-1} = 2^{2s} \Gamma(s) \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} s(1-s).\]
Let \( D^s(\mathbb{R}^N) \) be the Hilbert space defined as the completion of \( C_0^\infty(\mathbb{R}^N) \) with the scalar product
\[
\langle u, v \rangle_{D^s(\mathbb{R}^N)} = \frac{C(N, s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy
\]
and norm
\[
\|u\|_{D^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx = \frac{C(N, s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 \, dx \, dy.
\]

Let \( H^s(\mathbb{R}^N) \) be the Hilbert space of function in \( \mathbb{R}^N \) endowed with the standard scalar product and norm
\[
\langle u, v \rangle = \int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v + uv) \, dx, \quad \|u\|_{H^s(\mathbb{R}^N)} = \langle u, u \rangle.
\]

Let \( C_{p+1} \) be the sharp constant of the Sobolev embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N) \),
\[
C_{p+1} = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{H^s(\mathbb{R}^N)}^p}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}},
\]
and let \( S_s \) be the sharp imbedding constant of \( D^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \),
\[
S_s = \inf_{u \in D^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{D^s(\mathbb{R}^N)}^2}{\|u\|_{L^2(\mathbb{R}^N)}^2}.
\]

From [15] we have \( S_s \) is attained in \( \mathbb{R}^N \) by \( \tilde{u}(x) = \kappa (\varepsilon^2 + |x - x_0|^2)^{-\frac{N-2s}{2}} \), where \( \kappa \neq 0 \in \mathbb{R}, \varepsilon > 0 \) are fixed constants and \( x_0 \in \mathbb{R}^N \).

Denote \( \mathcal{H} = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \) and \( \mathcal{D} = D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N) \), with the norm given by
\[
\| (u, v) \|_{\mathcal{H}} = \|u\|_{H^s(\mathbb{R}^N)}^2 + \|v\|_{H^s(\mathbb{R}^N)}^2 = \|u\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_{L^2(\mathbb{R}^N)}^2,
\]
where \( \| (u, v) \|_{\mathcal{D}}^2 = \|u\|_{D^s(\mathbb{R}^N)}^2 + \|v\|_{D^s(\mathbb{R}^N)}^2. \)

The energy functional associated with (2) is given by
\[
E_{\mu, \nu, \lambda}(u, v) = \frac{1}{2} \| (u, v) \|_{\mathcal{D}}^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\mu u^2 + \nu v^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 \, dx - \lambda \int_{\mathbb{R}^N} uv \, dx.
\]

Define the Nehari manifold
\[
\mathcal{M} := \left\{ (u, v) \in \mathcal{H} \setminus \{(0, 0)\}, \|u\|_{\mathcal{D}}^2 + \int_{\mathbb{R}^N} (\mu u^2 + \nu v^2) \, dx = \int_{\mathbb{R}^N} (|u|^{p+1} + |v|^2) \, dx + 2\lambda \int_{\mathbb{R}^N} uv \, dx \right\}.
\]

We say that \((u, v)\) is a nontrivial solution of (2) if \( u \neq 0, v \neq 0 \) and \((u, v)\) solves (2). From (2) it is easy to see \((u, 0)\) and \((0, v)\) cannot be solution of (2). If \((u, v)\) is a nontrivial solution of (2), then \((u, v)\) is in \(\mathcal{M}\). We know \(\mathcal{M} \neq \emptyset\), since if we take \(\varphi, \psi \in C_0^\infty(\mathbb{R}^N)\) with \(\varphi, \psi \neq 0\) and \(\text{supp}(\varphi) \cap \text{supp}(\psi) = \emptyset\), then there exist \(t_1, t_2 > 0\) such that \((t_1 \varphi, t_2 \psi) \in \mathcal{M}\).

Let
\[
\mu_0 = \left[ \frac{2s(p+1)}{N(p-1)} S_s^{\frac{p+1}{2}} C_{p+1} \right] \left( \frac{p+1}{2} - \frac{s}{N} \right)^{-1}.
\]
Our main result is:

**Theorem 1.1.** Assume $N > 2s$, $1 < p < 2^* - 1$ and $\mu, \nu > 0$, $0 < \lambda < \sqrt{\mu \nu}$. Let $\mu_0$ be in (6).

1. If $0 < \mu \leq \mu_0$, then the system (2) has a positive ground state solution.
2. If $\mu > \mu_0$, then there exists $\lambda_{\mu, \nu} \in [\sqrt{\mu - \mu_0 \nu}, \sqrt{\mu \nu})$ such that,
   (i) if $\lambda < \lambda_{\mu, \nu}$, the system (2) has no ground state solution,
   (ii) if $\lambda > \lambda_{\mu, \nu}$, the system (2) has a positive ground state solution.

We sketch our idea of the proof. It is well known that the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ are not compact for $2 \leq p \leq 2^*$. Hence, the associated functional of problem (2) does not satisfy the Palais-Smale condition. In order to overcome the lack of compactness, we first set our work space in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, where

$$H^s(\mathbb{R}^N) = \{ \varphi \in H^s(\mathbb{R}^N) : \varphi \text{ is radial} \}$$

and $H^s(\mathbb{R}^N)$ is endowed with the $H^s(\mathbb{R}^N)$ topology: $||\varphi||_{H^s(\mathbb{R}^N)} = ||\varphi||_{H^s(\mathbb{R}^N)}$. Let

$$D^s(\mathbb{R}^N) = \{ \varphi \in D^s(\mathbb{R}^N) : \varphi \text{ is radial} \}.$$

By the properties of symmetric radial decreasing rearrangement, we know $C_{p+1}$ is achieved by radial functions in $H^s(\mathbb{R}^N)$ and $S_s$ is achieved by radial functions in $D^s(\mathbb{R}^N)$. By principle of symmetric criticality (Theorem 1.28 in [32]), the solutions for (2) in function space $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ are also those in function space $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

Second, we show that if the critical value $c$ of the functional (5) is strictly less than $\frac{s}{N} S_s^\frac{N}{s}$, then the corresponding critical sequence will satisfy $(PS)_c$ condition.

Finally, we prove that the mountain pass value for (5) is less than $\frac{s}{N} S_s^\frac{N}{s}$ under some proper conditions.

**Remark 1.** We own our result to [10]. We apply the technique used in [10] to show that the mountain pass critical value associated with (5) is below $\frac{s}{N} S_s^\frac{N}{s}$ under the similar conditions given on $\mu, \nu$ and $\lambda$.

To show the convergence of corresponding critical sequence, in [10], the authors use a limiting argument to deal with the problem in Laplacian case and the $C^2$ regularity of the solution solutions are needed. Since for the fractional Laplacian equation, regularity of the solutions is still a big issue. We mainly use the variational argument to prove the convergence of critical sequence.

The paper is organized as follows. In section 2, we introduce some preliminaries that will be used to prove Theorem 1.1. In section 3, we prove Theorem 1.1.

2. Some preliminaries. As mentioned earlier, we will only work in the radial function space. Set $\mathcal{H}_r$ and $\mathcal{D}_r$ as the following

$$\mathcal{H}_r = \{(u, v) \in \mathcal{H} : u, v \text{ are radial} \},$$
$$\mathcal{D}_r = \{(u, v) \in \mathcal{D} : u, v \text{ are radial} \},$$

with norm deduced from $\mathcal{H}$ and $\mathcal{D}$ respectively. Define the Nehari manifold in $\mathcal{H}_r$ as

$$\mathcal{M}_r = \{(u, v) \in \mathcal{M} : u, v \text{ are radial} \}.$$

Theorem 1.1 is proved by the mountain pass theorem [2], we show that $E_{\mu, \nu, \lambda}$ has a $(PS)_c$ sequence in $\mathcal{H}_r$. Choose $\varphi, \psi \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{H}_r(\mathbb{R}^N)$ with $\varphi, \psi \neq 0$ and
supp(\varphi) \cap \text{supp}(\psi) = \emptyset$, then there exists $t_0 > 0$ such that $E_{\mu, \nu, \lambda}(t_0 \varphi, t_0 \psi) < 0$ for all $t \geq t_0$. Take $(u_0, v_0) = (t \varphi, t \psi)$ with $t \geq t_0$ large enough. Let

$$\Gamma = \{ \gamma \in C([0, 1], \mathcal{H}_r) : \gamma(0) = (0, 0), \quad \gamma(1) = (u_0, v_0) \}.$$ 

Define

$$\tilde{A}_{\mu, \nu, \lambda} := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E_{\mu, \nu, \lambda}(\gamma(t)).$$

**Lemma 2.1.** Under the condition $0 < \lambda < \sqrt{\mu\nu}$, there exists a Palais-Smale sequence $\{(u_n, v_n)\} \subset \mathcal{H}_r$ such that

$$E_{\mu, \nu, \lambda}(u_n, v_n) \to \tilde{A}_{\mu, \nu, \lambda} \quad \text{and} \quad E'_{\mu, \nu, \lambda}(u_n, v_n) \to 0 \quad \text{as} \quad n \to +\infty. \quad (7)$$

**Proof.** We first claim that $E_{\mu, \nu, \lambda}$ possesses a mountain pass geometry around $(0, 0)$:

1. there exist $\alpha, \rho > 0$, such that $E_{\mu, \nu, \lambda}(u, v) > \alpha$ for all $||(u, v)||_{\mathcal{H}_r} > \rho$;
2. there exist $(u_0, v_0) \in \mathcal{H}_r$ such that $||(u_0, v_0)||_{\mathcal{H}_r} > \rho$ and $E_{\mu, \nu, \lambda}(u_0, v_0) < 0$.

To claim (1), since $\lambda < \sqrt{\mu\nu}$, we can take a small $\tau > 0$ such that $\lambda^2 = (\mu - \tau)(\nu - \tau)$, then by Sobolev imbedding,

$$E_{\mu, \nu, \lambda}(u, v) = \frac{1}{2}|||u, v|||^2_{\mathcal{H}_r} + \frac{1}{2} \int_{\mathbb{R}^N} (\mu u^2 + \nu v^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx$$

$$- \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx - \lambda \int_{\mathbb{R}^N} uv dx$$

$$\geq \frac{1}{2}|||u, v|||^2_{\mathcal{H}_r} + \frac{\tau}{2} \int_{\mathbb{R}^N} (u^2 + v^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx$$

$$\geq \min\left(\frac{1}{2}, \frac{\tau}{2}\right)|||u, v|||^2_{\mathcal{H}_r} - C_1|||u, v|||_{\mathcal{H}_r}^{p+1} - C_2|||u, v|||_{\mathcal{H}_r}^{2^*}.$$

Choose $\rho > 0$ sufficiently small, if $||(u, v)||^2_{\mathcal{H}_r} = \rho$, then

$$E_{\mu, \nu, \lambda}(u, v) \geq \min\left(\frac{1}{2}, \frac{\tau}{2}\right)|||u, v|||^2_{\mathcal{H}_r} - C_1|||u, v|||_{\mathcal{H}_r}^{p+1} - C_2|||u, v|||_{\mathcal{H}_r}^{2^*} > \frac{1}{4} C_3 \rho^2 > 0.$$

(2) is obvious if we choose $t$ large such that $||(u_0, v_0)||_{\mathcal{H}_r} = t|||(\varphi, \psi)|||_{\mathcal{H}_r} > \rho$.

By the mountain pass theorem, for the constant

$$\tilde{A}_{\mu, \nu, \lambda} := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E_{\mu, \nu, \lambda}(\gamma(t)) > 0,$$

there exist a $(PS)_{\tilde{A}_{\mu, \nu, \lambda}}$ sequence $\{(u_n, v_n)\} \subset \mathcal{H}_r$, that is

$$E_{\mu, \nu, \lambda}(u_n, v_n) \to \tilde{A}_{\mu, \nu, \lambda} \quad \text{and} \quad E'_{\mu, \nu, \lambda}(u_n, v_n) \to 0 \quad \text{as} \quad n \to +\infty,$$

where

$$\Gamma = \{ \gamma \in C([0, 1], \mathcal{H}_r) : \gamma(0) = (0, 0), \gamma(1) = (u_0, v_0) \}.$$ 

Define

$$\mathcal{F}_{\mu, \nu, \lambda} := \inf_{\mathcal{H}_r \backslash \{(0, 0)\}} \max_{t > 0} E_{\mu, \nu, \lambda}(tu, tv)$$

and

$$A_{\mu, \nu, \lambda} := \inf_{(u, v) \in \mathcal{M}_r} E_{\mu, \nu, \lambda}(u, v)$$

$$= \inf_{(u, v) \in \mathcal{M}_r} \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} |u|^{p+1} dx + \frac{8}{N} \int_{\mathbb{R}^N} |v|^{2^*} dx.$$
Lemma 2.2. Under the condition $0 < \lambda < \sqrt{pN}$, $\hat{A}_{\mu,\nu,\lambda} = \overline{A}_{\mu,\nu,\lambda} = A_{\mu,\nu,\lambda}$.

Proof. We first claim $\overline{A}_{\mu,\nu,\lambda} = A_{\mu,\nu,\lambda}$. For any $(u, v) \in \mathcal{H}_r$ with $(u, v) \neq (0, 0)$, there exists a unique $t_{\lambda,u,v} > 0$ such that

$$\max_{t > 0} E_{\mu,\nu,\lambda}(tu,tv) = E_{\mu,\nu,\lambda}(t_{\lambda,u,v}u,t_{\lambda,u,v}v) \quad (8)$$

where $t_{\lambda,u,v} > 0$ satisfies $\varphi(\lambda, u, v, t_{\lambda,u,v}) = 0$ and

$$\varphi(\lambda, u, v, t) = ||(u, v)||^2_\mathcal{H} + \int_{\mathbb{R}^N} (\mu u^2 + \nu v^2)dx - t^{p-1} \int_{\mathbb{R}^N} |u|^{p+1}dx - t^{2^* - 2} \int_{\mathbb{R}^N} |v|^{2^*}dx - 2\lambda \int_{\mathbb{R}^N} uv dx,$$

which implies that $(t_{\lambda,u,v}u, t_{\lambda,u,v}v) \in \mathbb{M}_r$. From (8), we get for any $(u, v) \in \mathcal{H}_r$, $(u, v) \neq (0, 0)$,

$$\max_{t > 0} E_{\mu,\nu,\lambda}(tu,tv) = E_{\mu,\nu,\lambda}(t_{\lambda,u,v}u,t_{\lambda,u,v}v) \geq A_{\mu,\nu,\lambda},$$

therefore $\overline{A}_{\mu,\nu,\lambda} \geq A_{\mu,\nu,\lambda}$. Similarly, we can show that $\overline{A}_{\mu,\nu,\lambda} \leq A_{\mu,\nu,\lambda}$, thus $\overline{A}_{\mu,\nu,\lambda} = A_{\mu,\nu,\lambda}$.

Next, we claim $\hat{A}_{\mu,\nu,\lambda} = \overline{A}_{\mu,\nu,\lambda}$.

For any $\epsilon > 0$, we can take a $(u, v) \neq (0, 0)$ such that

$$E_{\mu,\nu,\lambda}(t_{\lambda,u,v}u,t_{\lambda,u,v}v) > \max_{t > 0} E_{\mu,\nu,\lambda}(tu,tv) + \epsilon \quad (9)$$

We take a two dimensional space $\mathcal{S}$ in $\mathcal{H}_r$ contain $(u, v)$ and $(u_0, v_0)$ and we choose a large $R > \max(||(t_{\lambda,u,v}u,t_{\lambda,u,v}v)||,||u_0,v_0)||) > 0$, such that $E_{\mu,\nu,\lambda}(u, v) < 0$ for all $(u, v) \in \mathcal{S}$ with $||(u, v)|| = R$. Now, we define a path $\Gamma$ connecting $(0, 0)$ and $(u_0, v_0)$ as follows. If $t \in [0, \frac{1}{2}]$, let $(u_t, v_t) = (\frac{2Rt}{R^2 + 2R\mu} \frac{4Rt}{R^2 + 2R\nu})$ is the segment connecting $(0, 0)$ and $(\frac{R_0}{\|u_0\|}, \frac{R_0}{\|v_0\|})$. If $t \in [\frac{1}{2}, 1]$ define $(u_t, v_t) = (\frac{4(1-t)R_0}{\|u_0\|} + (4t-3)u_0, \frac{4(1-t)R_0}{\|v_0\|} + (4t-3)v_0)$ which is a segment connecting $(\frac{R_0}{\|u_0\|}, \frac{R_0}{\|v_0\|})$ and $(u_0, v_0)$. Let $\gamma(t) = (u_t(\cdot), v_t(\cdot)), t \in [0, 1]$, then it is easy to check that

$$\overline{A}_{\mu,\nu,\lambda} + \epsilon > \max_{k > 0} E_{\mu,\nu,\lambda}(ku,kv) = \max_{t \in [0, \frac{1}{2}]} E_{\mu,\nu,\lambda}(u_t,v_t) = \max_{t \in [0,1]} E_{\mu,\nu,\lambda}(\gamma(t)) \geq \hat{A}_{\mu,\nu,\lambda},$$

which implies that $\overline{A}_{\mu,\nu,\lambda} \geq \hat{A}_{\mu,\nu,\lambda}$. Thus $\hat{A}_{\mu,\nu,\lambda} \leq \overline{A}_{\mu,\nu,\lambda} = A_{\mu,\nu,\lambda}$.

Let $\{(u_n, v_n)\}$ is a $(PS)_{\hat{A}_{\mu,\nu,\lambda}}$ sequence, then

$$E_{\mu,\nu,\lambda}(u_n,v_n) \to \hat{A}_{\mu,\nu,\lambda} \text{ and } (E_{\mu,\nu,\lambda})'(u_n,v_n) \to 0, \text{ as } n \to +\infty.$$

We claim $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}_r$. For $n$ large enough, since $\lambda < \sqrt{pN}$, we can take a small $\tau > 0$ such that $\lambda^2 = (\mu - \tau)(\nu - \tau)$, then by Sobolev imbedding

$$\hat{A}_{\mu,\nu,\lambda} + o(1)(||(u_n, v_n)||) = E_{\mu,\nu,\lambda}(u_n,v_n) - \frac{1}{p+1}(E'_{\mu,\nu,\lambda}(u_n,v_n),(u_n,v_n)) \quad (10)$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\int_{\mathbb{R}^N} |(u,v)|^2_\mathcal{H} + \int_{\mathbb{R}^N} (\mu u^2 + \nu v^2)dx - 2\lambda \int_{\mathbb{R}^N} uvdx\right)$$
Lemma 2.4. For any \( f \) and denote \( f \) \( \hat{t} \). Moreover, there exist \( t \) \( \hat{t} \) easy to show that the value \( \hat{t} \).

Remark 2. Consequently, \( \{u_n, v_n\} \) is bounded in \( \mathcal{H}_r \). Since \( \hat{A}_{\mu, \nu, \lambda} > 0 \), \( \{u_n, v_n\} \) is not \( (0, 0) \) for \( n \) large.

Let \( (t_n u_n, t_n v_n) \in \mathcal{M}_r \), from

\[
\langle (E_{\mu, \nu, \lambda})' (u_n, v_n), (u_n, v_n) \rangle \to 0 \quad \text{as} \quad n \to +\infty,
\]

it is easy to see that \( t_n \to 1 \) as \( n \to \infty \). So

\[
\hat{A}_{\mu, \nu, \lambda} = \lim_{n \to \infty} E_{\mu, \nu, \lambda}(u_n, v_n) = \lim_{n \to \infty} E_{\mu, \nu, \lambda}(t_n u_n, t_n v_n)
\]

\[
\geq \liminf_{n \to \infty} E_{\mu, \nu, \lambda}(t_n u_n, t_n v_n) \geq A_{\mu, \nu, \lambda} = \hat{A}_{\mu, \nu, \lambda}.
\]

Consequently, \( \hat{A}_{\mu, \nu, \lambda} = \overline{A}_{\mu, \nu, \lambda} = A_{\mu, \nu, \lambda} \). This completes the proof of Lemma 2.2.

Theorem 1.1, we need the following lemma.

Lemma 2.3. ([5] Brezis-Lieb Lemma) Let \( \{u_n\} \subset L^{p+1}(\mathbb{R}^N), 0 < p < \infty \). If \( \{u_n\} \) is bounded in \( L^{p+1}(\mathbb{R}^N) \) and \( u_n \to u \) a.e on \( \mathbb{R}^N \), then

\[
\int_{\mathbb{R}^N} |u_n|^{p+1} dx = \int_{\mathbb{R}^N} |u|^{p+1} dx + \int_{\mathbb{R}^N} |u_n - u|^{p+1} dx + o(1).
\]

Next, we borrow some ideas from [10] to show that \( A_{\mu, \nu, \lambda} \leq \frac{\sqrt{N}}{N} S_{\alpha}^N \). Define

\[
f_{\beta, \gamma}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx + \frac{\beta}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx + \frac{\gamma}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx,
\]

(11)

\[
g(v) = \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |v|^{p+1} \, dx,
\]

(12)

and denote \( f_{\beta} = f_{\beta, 1} \). Then we have following Lemma:

Lemma 2.4. For any \((u, v) \in \mathcal{H}_r \) with \( u \neq 0 \) and \( v \neq 0 \), there holds

\[
\max_{t > 0} E_{\mu, \nu, \lambda}(tu, tv) > \min \{ \max_{t > 0} f_{\mu - \frac{\lambda^2}{\nu}}(tu), \max_{t > 0} g(tv) \}.
\]

Proof. Since \( 2\lambda uv \leq \frac{\lambda^2}{\nu} u^2 + \nu v^2 \), we have

\[
E_{\mu, \nu, \lambda}(tu, tv) \geq f_{\mu - \frac{\lambda^2}{\nu}}(tu) + g(tv).
\]

Moreover, there exist \( t_1, t_2 > 0 \) such that

\[
\max_{t > 0} f_{\mu - \frac{\lambda^2}{\nu}}(tu) = f_{\mu - \frac{\lambda^2}{\nu}}(t_1 u), \quad \max_{t > 0} g(tv) = g(t_2 v).
\]

Since \( f_{\mu - \frac{\lambda^2}{\nu}}(0) = 0 \), \( g(0) = 0 \) and \( f_{\mu - \frac{\lambda^2}{\nu}}(tu) \) is increasing in \([0, t_1] \), decreasing in \([t_1, +\infty) \), \( g(tv) \) is increasing in \([0, t_2] \), decreasing in \([t_2, +\infty) \). Thus, if \( t_1 < t_2 \),
By \[17, 18\], there exists a unique positive radial ground state solution \( w \) to the equation \((-\Delta)^s u + u = u^p \) in \( H^s(\mathbb{R}^N) \). By (3),

\[
 f_1(w) = \left( \frac{1}{2} - \frac{1}{p + 1} \right) C_{p+1}^{\frac{p+1}{p+1}},
\]

where \( f_1 \) is defined in (11). Let \( w_{\beta,\gamma}(x) := \beta x^\gamma \cdot \gamma^{-\frac{1}{p-1}} w(\beta x) \), then \( w_{\beta,\gamma}(x) \) is the unique positive radial solution of \((-\Delta)^s u + \beta u = \gamma u^p \), \( u \in H^s(\mathbb{R}^N) \) with the energy

\[
 f_{\beta,\gamma}(w_{\beta,\gamma}) = \gamma^{-\frac{2}{p-2}} \beta^{\frac{p+1}{p-1} - \frac{N}{2}} f_1(w) = \left( \frac{1}{2} - \frac{1}{p + 1} \right) \gamma^{-\frac{2}{p-2}} \beta^{\frac{p+1}{p-1} - \frac{N}{2}} C_{p+1}^{\frac{p+1}{p+1}}. \tag{13}
\]

For convenience, we denote \( w_{\beta} = w_{\beta,1} \). Define \( \alpha = N\left(\frac{1}{p+1} - \frac{1}{2^*}\right) \in (0, s) \), then

\[
 \frac{s}{p + 1} = \frac{\alpha}{2} + \frac{s - \alpha}{2^*},
\]

Let \( \bar{\nu}_0 = \frac{s}{2^*} \left( \frac{s - \alpha}{s} \right)^{\frac{N - 2s}{2s} \left( \frac{p+1}{p+1} - \frac{N}{2} \right)^{-1} } \), we have the following Lemma:

**Lemma 2.5.** For \( \mu_0 > 0 \) defined in (6), we have \( \mu_0 < \bar{\nu}_0 \) and

\[
 f_{\mu}(w_{\mu}) = \begin{cases} 
 > \frac{s}{N} S_{\mu}^N, & \text{if } \mu > \mu_0, \\
 = \frac{s}{N} S_{\mu}^N, & \text{if } \mu = \mu_0, \\
 < \frac{s}{N} S_{\mu}^N, & \text{if } \mu < \mu_0.
\end{cases} \tag{14}
\]

**Proof.** By (6) and (13), we have \( f_{\mu_0}(w_{\mu_0}) = \frac{s}{N} S_{\mu_0}^N \). Since \( p < 2^* - 1 \), we have

\[
 \frac{p + 1}{p - 1} - \frac{N}{2s} > 0.
\]

From (13) it is easy to obtain (14). In order to prove \( \bar{\nu}_0 > \mu_0 \), by (14) we need to show \( f_{\mu_0}(w_{\mu_0}) > \frac{s}{N} S_{\mu_0}^N \). By Hölder inequality and Young inequality, we have

\[
 \left( \int_{\mathbb{R}^N} |u|^{p+1} dx \right)^{\frac{1}{p+1}} \leq \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{2^*(s-\alpha)}{2s}}
\]

\[
 \leq \frac{\alpha}{s} \epsilon \int_{\mathbb{R}^N} |u|^2 dx + \frac{s - \alpha}{s} \epsilon^{-\frac{s}{s-\alpha}} \left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{2^*}{s}}.
\]

If we choose \( C_0 > 0, \epsilon_0 > 0 \) such that

\[
 C_0 \frac{\alpha}{s} \epsilon_0^\frac{s}{s-\alpha} = 1, \quad C_0 \frac{s - \alpha}{s} \epsilon_0^{-\frac{s}{s-\alpha}} = S_\alpha,
\]

then, we have

\[
 C_0 = S_\alpha \frac{s-\alpha}{s} \left( \frac{s}{s-\alpha} \right)^{\frac{\alpha}{s}} \left( \frac{s}{\alpha} \right)^{\frac{s}{s-\alpha}},
\]

where \( S_\alpha = \frac{\alpha}{s} \left( \frac{s}{s-\alpha} \right)^{\frac{s}{s-\alpha}} \left( \frac{s}{\alpha} \right)^{\frac{s}{s-\alpha}} \).
and
\[
\|u\|_{H^s(R^N)}^2 + \int_{\mathbb{R}^N} |u|^2 \, dx > S_s \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{\frac{2}{2^*}} + \int_{\mathbb{R}^N} |u|^2 \, dx \\
\geq C_0 \left( \int_{\mathbb{R}^N} |u|^{p+1} \, dx \right)^{\frac{p+1}{p+1}}.
\]

This implies that \(C_{p+1} > C_0\). Combining this with (13), we obtain
\[
f_\mu(w_\mu) > \left( \frac{1}{2} - \frac{1}{p+1} \right) \mu^{\frac{p+1}{2^*}} \left[ \frac{s-a}{s-a} \left( \frac{s-a}{s} \right)^{\frac{a-s}{s}} \right]^{\frac{p+1}{2^*}}.
\]

If we choose
\[
\mu_0 = \frac{\alpha}{s} \left( \frac{s-a}{s} \right)^{\frac{a-s}{s}} \left( \frac{s-a}{s} \right)^{\frac{p+1}{2^*}}
\]

we have \(f_{\mu_0}(w_{\mu_0}) > \frac{s}{N} S_s^\frac{N}{p} \). Thus, \(\mu_0 > \mu_0\). This completes the proof. \(\square\)

Remark 3. From the proof of the above lemma, though the exact values of \(C_{p+1}\) and \(\mu_0\) are unknown, we have a lower bound estimate for \(C_{p+1}\) and an upper bound estimate for \(\mu_0\):
\[
C_{p+1} > C_0 = S_s \left( \frac{s-a}{s-a} \right)^{\frac{a-s}{s}} \left( \frac{s-a}{s} \right)^{\frac{p+1}{2^*}}
\]

\[
\mu_0 < \mu_0 = \frac{\alpha}{s} \left( \frac{s-a}{s} \right)^{\frac{a-s}{s}} \left( \frac{s-a}{s} \right)^{\frac{p+1}{2^*}}
\]

Since \(\alpha \in (0, s)\), it is easy to see that \(\mu_0 < 1\).

For any \(\mu > \mu_0\), \(\nu > 0\), we define a \(C^1\) function \(h_{\mu, \nu} : (0, +\infty) \to \mathbb{R}\) by
\[
h_{\mu, \nu}(a) = \frac{\mu + \nu a^2}{2a} - \frac{\mu_0}{2a} (1 + a^2)^{-\frac{N}{p+1}} \left( \frac{p+1}{p+1} - \frac{N}{2^*} \right).
\]

Then,
\[
h_{\mu, \nu}(a) > \frac{\mu - \mu_0 + \nu a^2}{2a} \geq \sqrt{(\mu - \mu_0)\nu}.
\]

Thus, \(h_{\mu, \nu}(a) \to +\infty\) as \(a \to 0_+\) and \(h_{\mu, \nu}(a)\) is increasing in \([\sqrt{\nu}, +\infty)\). Therefore, there exists \(a_{\mu, \nu} \in (0, \sqrt{\nu})\) such that
\[
\tilde{\lambda}_{\mu, \nu} := h_{\mu, \nu}(a_{\mu, \nu}) = \min_{a \in (0, +\infty)} h_{\mu, \nu}(a).
\]

Since \(h_{\mu, \nu}(\sqrt{\nu}) < \sqrt{\nu}\), we have
\[
\sqrt{(\mu - \mu_0)\nu} < \tilde{\lambda}_{\mu, \nu} < \sqrt{\nu}.
\]

Lemma 2.6. (1) If \(0 < \mu \leq \mu_0\), then \(A_{\mu, \nu, \lambda} < \frac{s}{N} S_s^\frac{N}{p}\).

(2) If \(\mu > \mu_0\), then there exists a \(\lambda_{\mu, \nu} \in [\sqrt{(\mu - \mu_0)\nu}, \tilde{\lambda}_{\mu, \nu})\) such that

(i) if \(0 < \lambda \leq \lambda_{\mu, \nu}\), then \(A_{\mu, \nu, \lambda} = \frac{s}{N} S_s^\frac{N}{p}\),

(ii) if \(\lambda_{\mu, \nu} < \lambda < \sqrt{\nu}\), then \(A_{\mu, \nu, \lambda} < \frac{s}{N} S_s^\frac{N}{p}\),

where \(\tilde{\lambda}_{\mu, \nu}\) is from (16).
Proof. (1) If \( \mu \in (0, \mu_0) \), by Lemma 2.5, we have
\[
\max_{t>0} E_{\mu, \nu, \lambda}(tw_0, 0) = \max_{t>0} f_\mu(tw_0) = f_\mu(w_\mu) < \frac{s}{N} S_s^N.
\]
Thus \( A_{\mu, \nu, \lambda} < \frac{s}{N} S_s^N \).

When \( \mu = \mu_0 \), then \( A_{\mu_0, \nu, \lambda} \leq f_{\mu_0}(w_{\mu_0}) = \frac{s}{N} S_s^N \). Assume by contradiction that
\[
A_{\mu_0, \nu, \lambda} = \frac{s}{N} S_s^N,
\]
then
\[
E_{\mu_0, \nu, \lambda}(w_{\mu_0}, 0) = A_{\mu_0, \nu, \lambda}, \quad (w_{\mu_0}, 0) \in M_r.
\]
Thus, \((w_{\mu_0}, 0)\) is a ground state solution of (2). Since \( \lambda > 0 \), if \((w_{\mu_0}, 0)\) is a ground state solution of (2), then \( w_{\mu_0} \equiv 0 \). This contradicts with \( w_{\mu_0} \neq 0 \), so
\[
A_{\mu_0, \nu, \lambda} < \frac{s}{N} S_s^N.
\]

(2) In order to prove second part of Lemma 2.6, we divide the proof into two steps.

**Step 1.** We first show that for any fixed \( \mu > 0 \), \( \nu > 0 \) and \( \lambda > 0 \), it holds
\[
A_{\mu, \nu, \lambda} \leq \frac{s}{N} S_s^N. \tag{18}
\]

From [15], we know that \( S_s \) is attained in \( \mathbb{R}^N \) by \( u(x) = \kappa(\varepsilon^2 + |x|^2)^{-\frac{N+2s}{2}} \), where \( \kappa \neq 0 \in \mathbb{R}, \varepsilon > 0 \) are fixed constants. Let
\[
U_\varepsilon(x) = C(N, s) \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2s}{2}},
\]
with \( C(N, s) \) chosen so that
\[
\|U_\varepsilon(x)\|_{2^*}^2 = \int_{\mathbb{R}^N} |U_\varepsilon(x)|^{2^*} dx = S_s^N.
\]
Take \( \eta(|x|) \in C_0^\infty(\mathbb{R}^N, [0, 1]) \) be a radial cut-off function such that \( 0 \leq \eta \leq 1, \eta = 1 \) on \( B(0, 1) \) and \( \eta = 1 \) on \( \mathbb{R}^N \setminus B(0, 2) \). Let \( v_\varepsilon = \eta(|x|)U_\varepsilon(x) \). Then, the following estimates holds true (Proposition 21 in [28])
\[
\int_{\mathbb{R}^{2N}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x - y|^{N+2s}} dxdy \leq S_s^N + o(\varepsilon^{N-2s}),
\]
\[
\int_{\mathbb{R}^N} |v_\varepsilon|^2 dx = \begin{cases} C\varepsilon^{2s} + o(\varepsilon^{N-2s}) & \text{if } N > 4s \\ C\varepsilon^{2s} \log(\frac{1}{\varepsilon}) + o(\varepsilon^{2s}) & \text{if } N = 4s \\ C\varepsilon^{-2s} + o(\varepsilon^{2s}) & \text{if } N < 4s, \end{cases}
\]
\[
\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} dx = S_s^N + o(\varepsilon^N).
\]
We claim that
\[
\inf_{v \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|v\|_{2^*}^2 + \int_{\mathbb{R}^N} \nu|v|^2 dx}{(\int_{\mathbb{R}^N} |v|^{2^*} dx)^{\frac{2}{2^*}}} = S_s.
\]
By the above estimates, we can deduce that
\[
\inf_{v \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|v\|_{2^*}^2 + \int_{\mathbb{R}^N} \nu|v|^2 dx}{(\int_{\mathbb{R}^N} |v|^{2^*} dx)^{\frac{2}{2^*}}} \leq \liminf_{\varepsilon \to 0^+} \frac{\|v_\varepsilon\|_{2^*}^2 + \int_{\mathbb{R}^N} \nu|v_\varepsilon|^2 dx}{(\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} dx)^{\frac{2}{2^*}}} \leq S_s.
\]
Since \( \nu > 0 \), it is clear that
\[
\inf_{v \in H^s_0(\mathbb{R}^N) \setminus \{0\}} \frac{\|v\|^2_{D^s_2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \nu |v|^2 dx}{(\int_{\mathbb{R}^N} |v|^2 dx)^{\frac{p}{2}}} \geq S_s.
\]
Thus,
\[
\inf_{v \in H^s_0(\mathbb{R}^N) \setminus \{0\}} \frac{\|v\|^2_{D^s_2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \nu |v|^2 dx}{(\int_{\mathbb{R}^N} |v|^2 dx)^{\frac{p}{2}}} = S_s.
\]
Take a minimization sequence \( v_n \in H^s_0(\mathbb{R}^N) \) such that
\[
\frac{\|v_n\|^2_{D^s_2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \nu |v_n|^2 dx}{(\int_{\mathbb{R}^N} |v_n|^2 dx)^{\frac{p}{2}}} \leq S_s + \frac{1}{n},
\]
can deduce that
\[
A_{\mu,\nu,\lambda} \leq \liminf_{n \to \infty} \max_{t > 0} E_{\mu,\nu,\lambda}(0, tv_n) = \frac{s}{N} S_s^{\frac{N}{2}}.
\]
Next, assume \( \mu > \mu_0 \) and \( 0 < \lambda \leq \sqrt{(\mu - \mu_0)\nu} \), then \( \mu - \frac{N}{\nu} \geq \mu_0 \). By the same arguments as in Lemma 2.2, we have
\[
f_{\mu}(w_{\mu}) = \inf_{u \in H^s_0(\mathbb{R}^N) \setminus \{0\}} \max_{t > 0} f_{\mu}(tu).
\]
By (4), we have
\[
\inf_{u \in H^s_0(\mathbb{R}^N) \setminus \{0\}} \max_{t > 0} g(tu) = \frac{s}{N} S_s^{\frac{N}{2}}.
\]
For any \( (u, v) \in H_\nu \setminus \{(0, 0)\} \), if \( v = 0 \), then \( \max_{t > 0} E_{\mu,\nu,\lambda}(tu, 0) = \max_{t > 0} f_{\mu}(tu) \geq \frac{s}{N} S_s^{\frac{N}{2}} \). If \( u = 0 \), then \( \max_{t > 0} E_{\mu,\nu,\lambda}(0, tv) = \max_{t > 0} g(tv) \geq \frac{s}{N} S_s^{\frac{N}{2}} \). \( (u, v) \neq (0, 0) \), then by Lemma 2.4 and Lemma 2.5, we have
\[
\max_{t > 0} E_{\mu,\nu,\lambda}(tu, tv) > \min \{ \max_{t > 0} f_{\mu - \frac{N}{\nu}}(tu), \max_{t > 0} g(tv) \} \geq \frac{s}{N} S_s^{\frac{N}{2}}.
\]
Together with (18), when \( \mu > \mu_0 \) and \( 0 < \lambda \leq \sqrt{(\mu - \mu_0)\nu} \), it holds
\[
A_{\mu,\nu,\lambda} = \frac{s}{N} S_s^{\frac{N}{2}}.
\]
**Step 2.** We prove (i) - (iii) in (2).
For \( \mu > \mu_0 \) and \( 0 < \lambda < \sqrt{\mu \nu} \), we define
\[
\beta := \frac{\mu + \nu a_{\mu,\nu}^2 - 2\lambda a_{\mu,\nu}}{1 + a_{\mu,\nu}^2}, \quad \gamma := \frac{1}{1 + a_{\mu,\nu}^2},
\]
where \( a_{\mu,\nu} \) is from (16). Since \( 0 < \lambda < \sqrt{\mu \nu} \), direct computation shows \( \beta > 0 \).
Then, it follows from (13),
\[
A_{\mu,\nu,\lambda} \leq \max_{t > 0} E_{\mu,\nu,\lambda}(tw_{\beta,\gamma}, t(a_{\mu,\nu}w_{\beta,\gamma}))
\]
\[
< \max_{t > 0} \left( \frac{1 + a_{\mu,\nu}^2}{2} \|tw_{\beta,\gamma}\|^2_{D^s_2(\mathbb{R}^N)} + \frac{(1 + a_{\mu,\nu}^2)\beta}{2} \int_{\mathbb{R}^N} (tw_{\beta,\gamma})^2 \right.
\]
\[
- \frac{1}{p + 1} \int_{\mathbb{R}^N} (tw_{\beta,\gamma})^{p+1}
\]
\[
= (1 + a_{\mu,\nu}^2) \max_{t > 0} f_{\beta,\gamma}(tw_{\beta,\gamma}) + (1 + a_{\mu,\nu}^2) f_{\beta,\gamma}(w_{\beta,\gamma})
\]
\[
= (1 + a_{\mu,\nu}^2)^{\frac{N}{2}} (\mu + \nu a_{\mu,\nu}^2 - 2\lambda a_{\mu,\nu})^{\frac{p+1}{2}} \left( \frac{1}{2} - \frac{1}{p+1} \right) C_{p+1}^{\frac{p+1}{2}} = A_0.
\]
By (6), to show $A_0 \leq \frac{s}{N} S_s^{\frac{N}{2}}$, we just need to show that

$$(1 + a_{\mu, \nu}^2)^{\frac{\mu}{\nu}} (\mu + \nu a_{\mu, \nu}^2 - 2\lambda a_{\mu, \nu}) \leq \frac{s}{N} S_s^{\frac{N}{2}}.$$  

By (15) and (16), we can deduce that the above inequality is equivalent to $\lambda \geq \tilde{\lambda}_{\mu, \nu}$.

Combining with (17) and (20), for any $\lambda \in [\tilde{\lambda}_{\mu, \nu}, \sqrt{\mu \nu}]$, we have $A_{\mu, \nu, \lambda} < \frac{s}{N} S_s^{\frac{N}{2}}$.

Define

$$\lambda_{\mu, \nu} := \inf\{\lambda < \sqrt{\mu \nu} : A_{\mu, \nu, \lambda} < \frac{s}{N} S_s^{\frac{N}{2}}\}, \forall \lambda \in [\sqrt{\mu \nu}, \tilde{\lambda}_{\mu, \nu}].$$

Then, by (19), we know $\lambda_{\mu, \nu} \in [\sqrt{\mu \nu}, \tilde{\lambda}_{\mu, \nu}]$ and for any $\lambda \in (\lambda_{\mu, \nu}, \sqrt{\mu \nu})$, there holds $A_{\mu, \nu, \lambda} < \frac{s}{N} S_s^{\frac{N}{2}}$. This completes the proof of (ii).

We show that $A_{\mu, \nu, \lambda_{\mu, \nu}} = \frac{s}{N} S_s^{\frac{N}{2}}$, which implies $\lambda_{\mu, \nu} < \tilde{\lambda}_{\mu, \nu}$ immediately.

By (18), we have $A_{\mu, \nu, \lambda_{\mu, \nu}} \leq \frac{s}{N} S_s^{\frac{N}{2}}$. By the definition of $\lambda_{\mu, \nu}$, there exists $\lambda_n < \lambda_{\mu, \nu}$, $n \geq 1$ such that

$$\lim_{n \to +\infty} \lambda_n = \lambda_{\mu, \nu}, \quad A_n := A_{\mu, \nu, \lambda_n} \geq \frac{s}{N} S_s^{\frac{N}{2}}, \quad \forall \ n \geq 1.$$  

For any $(u, v) \in \mathcal{H}_r \setminus \{(0, 0)\}$, there exists $t_n > 0$ such that $\max_{t > 0} E_{\mu, \nu, \lambda_n}(tu, tv) = E_{\mu, \nu, \lambda_n}(t_n u, t_n v)$. Since $\lambda_n \to \lambda_{\mu, \nu}$, we have $t_n \to t_0$ as $n \to +\infty$, where $t_0$ satisfies $\max_{t > 0} E_{\mu, \nu, \lambda_{\mu, \nu}}(tu, tv) = E_{\mu, \nu, \lambda_{\mu, \nu}}(t_0 u, t_0 v)$. Then,

$$\limsup_{n \to +\infty} A_n \leq \limsup_{n \to +\infty} E_{\mu, \nu, \lambda_n}(t_n u, t_n v) = E_{\mu, \nu, \lambda_{\mu, \nu}}(t_0 u, t_0 v).$$

This implies

$$\frac{s}{N} S_s^{\frac{N}{2}} \leq \limsup_{n \to +\infty} A_n \leq A_{\mu, \nu, \lambda_{\mu, \nu}}.$$  

Thus, $A_{\mu, \nu, \lambda_{\mu, \nu}} = \frac{s}{N} S_s^{\frac{N}{2}}$.

Next, we claim $A_{\mu, \nu, \lambda}$ is non-increasing with respect to $\lambda > 0$.

Since

$$\max_{t > 0} E_{\mu, \nu, \lambda}(tu, tv) \geq \max_{t > 0} E_{\mu, \nu, \lambda}(t|u|, |v|),$$  

then we have

$$A_{\mu, \nu, \lambda} = \inf_{\mathcal{H}_r \setminus \{(0, 0)\}} \max_{t > 0} E_{\mu, \nu, \lambda}(t|u|, |v|).$$  

Let $\lambda_1 < \lambda_2$. Then for any $(u, v) \in \mathcal{H}_r \setminus \{(0, 0)\}$ and $t > 0$, we obtain

$$E_{\mu, \nu, \lambda_1}(t|u|, |v|) \geq E_{\mu, \nu, \lambda_2}(t|u|, |v|).$$

Thus, $A_{\mu, \nu, \lambda_1} \geq A_{\mu, \nu, \lambda_2}$. Consequently, $A_{\mu, \nu, \lambda}$ is non-increasing with respect to $\lambda > 0$.

Combining this fact with (19), we see that (i) holds. This completes the proof.
3. Proof of Theorem 1.1.

Proof of Theorem 1.1. We prove Theorem 1.1 by two steps. First, we prove the existence of ground state solutions for system (2), then we claim there exists a positive ground state solution.

Step 1. Prove the existence of ground state solutions for system (2).

By (7) and the proof of Lemma 2.2, there exists a bounded sequence \( \{u_n, v_n\} \subset \mathcal{H}_r \), such that

\[
E_{\mu, \nu, \lambda}(u_n, v_n) \rightarrow \hat{A}_{\mu, \nu, \lambda} \quad \text{and} \quad E_{\mu, \nu, \lambda}'(u_n, v_n) \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.
\]

Thus, by Sobolev Imbedding Theorem, there exists \((u, v) \in \mathcal{H}_r\) such that

\[
\begin{align*}
(u_n, v_n) & \rightarrow (u, v), \quad \text{weakly in} \quad \mathcal{H}_r, \\
(u_n, v_n) & \rightarrow (u, v), \quad \text{strongly in} \quad L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \quad \text{for} \quad 2 < p < 2^*, \quad \text{a.e.} \quad \mathbb{R}^N.
\end{align*}
\]

Then, we have

\[
E_{\mu, \nu, \lambda}'(u, v) = 0.
\]

Let \( w_n = u_n - u \) and \( \sigma_n = v_n - v \). We claim

\[
E_{\mu, \nu, \lambda}(w_n, \sigma_n) = E_{\mu, \nu, \lambda}(u_n, v_n) - E_{\mu, \nu, \lambda}(u, v) + o(1). \tag{25}
\]

By Lemma 2.3, there holds

\[
\begin{align*}
\|u_n\|_2^2 & = \|u\|_2^2 + \|w_n\|_2^2 + o_n(1), \\
\|v_n\|_2^2 & = \|v\|_2^2 + \|\sigma_n\|_2^2 + o_n(1), \\
\|u_n\|_{p+1} & = \|u\|_{p+1} + \|w_n\|_{p+1} + o_n(1) \\
& = \|u\|_{p+1} + o_n(1) \quad \text{as} \quad \|w_n\|_{p+1} \rightarrow 0, \\
\|v_n\|_{2^*}^2 & = \|v\|_{2^*}^2 + \|\sigma_n\|_{2^*}^2 + o_n(1).
\end{align*}
\]

Since

\[
\begin{align*}
\|u_n\|_{D^2(\mathbb{R}^N)}^2 & = \|u\|_{D^2(\mathbb{R}^N)}^2 - \|w_n\|_{D^2(\mathbb{R}^N)}^2 + o_n(1), \\
\|\sigma_n\|_{D^2(\mathbb{R}^N)}^2 & = \|v_n\|_{D^2(\mathbb{R}^N)}^2 - \|\sigma_n\|_{D^2(\mathbb{R}^N)}^2 + o_n(1)
\end{align*}
\]

and

\[
\int_{\mathbb{R}^N} w_n \sigma_n dx = \int_{\mathbb{R}^N} u_n v_n dx - \int_{\mathbb{R}^N} u v dx + o(1).
\]

From above equalities, we obtain (25).

Since, for any \((\phi, \varphi) \in \mathcal{H}_r\), we have

\[
o(\|\phi, \varphi\|) = \langle E_{\mu, \nu, \lambda}'(u_n, v_n), (\phi, \varphi) \rangle
\]

\[
= \langle E_{\mu, \nu, \lambda}'(w_n, \sigma_n), (\phi, \varphi) \rangle + \langle E_{\mu, \nu, \lambda}'(u, v), (\phi, \varphi) \rangle
\]

\[
- \int_{\mathbb{R}^N} \left[ |v_n|^{2^* - 2} v_n - |\sigma_n|^{2^* - 2} \sigma_n - |v|^{2^* - 2} v \right] \varphi dx
\]

\[
- \int_{\mathbb{R}^N} \left[ |u_n|^{p-1} u_n - |w_n|^{p-1} w_n - |u|^{p-1} u \right] \phi dx.
\]

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By Vatali’s Theorem [26], we have
\[
\int_{\mathbb{R}^N} \left[ |v_n|^{2^* - 2}v_n - |\sigma_n|^{2^* - 2}\sigma_n - |v|^{2^* - 2}v \right] \varphi dx = \int_{\mathbb{R}^N} \int_0^1 \frac{d}{dt} \left[ (tv_n + (1 - t)\sigma_n)^{2^* - 1} - (tv)^{2^* - 1} \right] \varphi dx
\]
\[
= \int_{\mathbb{R}^N} \int_0^1 (2^* - 1) [(tv_n + (1 - t)\sigma_n)^{2^* - 2} - (tv)^{2^* - 2}] \varphi dt dx \to 0 \text{ as } n \to +\infty.
\]
Similarly, we have
\[
\int_{\mathbb{R}^N} \left[ |u_n|^{p-1}u_n - |w_n|^{p-1}w_n - |u|^{p-1}u \right] \phi dx \to 0 \text{ as } n \to +\infty.
\]
(26) can be derived from above equalities.
Thus, by (24),(25) and (26), we deduce
\[
o(1) + \tilde{ sol}_{\mu,\nu,\lambda}(u_n, v_n) = E_{\mu,\nu,\lambda}(w_n, \sigma_n) + E_{\mu,\nu,\lambda}(u, v) + o(1).
\]
We show \((u, v)\) is a nontrivial solution. Since (2) has no solution of form \((u, 0)\) or \((0, v)\), if \((u, v) \equiv (0, 0)\), we have
\[
E_{\mu,\nu,\lambda}(u_n, v_n) = \frac{1}{2} \|(w_n, \sigma_n)\|_{D^r}^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\mu w_n^2 + \nu \sigma_n^2) dx - \lambda \int_{\mathbb{R}^N} w_n \sigma_n dx - \frac{1}{2\lambda} \int_{\mathbb{R}^N} |\sigma_n|^{2^*} dx + o_n(1).
\]
By (26),
\[
\|(w_n, \sigma_n)\|_{D^r}^2 + \int_{\mathbb{R}^N} (\mu w_n^2 + \nu \sigma_n^2) dx - \int_{\mathbb{R}^N} |\sigma_n|^{2^*} dx - 2\lambda \int_{\mathbb{R}^N} w_n \sigma_n dx = o(1).
\]
Thus, we obtain
\[
E_{\mu,\nu,\lambda}(u_n, v_n) = \frac{s}{N} \|(w_n, \sigma_n)\|_{D^r}^2
\]
\[
+ \frac{s}{N} \left[ \int_{\mathbb{R}^N} (\mu w_n^2 + \nu \sigma_n^2) dx - 2\lambda \int_{\mathbb{R}^N} w_n \sigma_n dx \right] + o_n(1)
\]
\[
\geq \frac{s}{N} \|(w_n, \sigma_n)\|_{D^r}^2 + o_n(1).
\]
Without loss of generality, we assume
\[
\lim_{n \to +\infty} \|(w_n, \sigma_n)\|_{D^r}^2 = l > 0.
\]
By (28) and \(\lambda < \sqrt{\mu \nu}\), we have
\[
\|(w_n, \sigma_n)\|_{D^r}^2 \leq \int_{\mathbb{R}^N} |\sigma_n|^{2^*} dx + o_n(1).
\]
By Sobolev imbedding \(D^r_s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)\), we have
\[
|\sigma_n|_{D^r_s(\mathbb{R}^N)}^2 \geq S_s \left( \int_{\mathbb{R}^N} |\sigma_n|^{2^*} dx \right)^{\frac{2^*}{2^*}}.
\]
Combining (30) with (31), we can deduce that
\[
l \geq S_s^{\frac{2^*}{2^*}}.
\]
Let \( n \to +\infty \) in (29), we obtain
\[
A_{\mu,\nu,\lambda} \geq \frac{s}{N} S_{s}^{\frac{N}{2}}.
\]
This contradict with Lemma 2.6. Therefore \((u, v)\) is nontrivial solution.

Next, we show \((u, v)\) is a nontrivial ground state solution of system (2). Since \( E_{\mu,\nu,\lambda}(u, v) \geq \hat{A}_{\mu,\nu,\lambda} \), by (27) and Lemma 2.2, we have
\[
E_{\mu,\nu,\lambda}(w_{n}, \sigma_{n}) \leq o(1).
\]
Since \( \lambda < \sqrt{\mu\nu} \), we can take a small \( \tau > 0 \) such that \( \lambda^2 = (\mu - \tau)(\nu - \tau) \), by Sobolev imbedding, (32) and (28), we find
\[
o(1) \geq \frac{s}{N} \left[ ||(w_{n}, \sigma_{n})||_{L_{\infty}}^{2} + \int_{\mathbb{R}^{n}} (\mu w_{n}^{2} + \nu \sigma_{n}^{2})dx - 2\lambda \int_{\mathbb{R}^{n}} w_{n} \sigma_{n}dx \right]
\geq \frac{s}{N} \left[ ||(w_{n}, \sigma_{n})||_{L_{\infty}}^{2} + \tau \int_{\mathbb{R}^{n}} (w_{n}^{2} + \sigma_{n}^{2}) \right]
\geq \frac{s}{N} \min(1, \tau)||{(w_{n}, \sigma_{n})}||_{H_{r}}^{2},
\]
which implies that
\[
(u_{n}, v_{n}) \to (u, v) \text{ strongly in } H_{r}.
\]
As a result, \((u, v)\) is a critical point of \( E_{\mu,\nu,\lambda} \) and by Lemma 2.2, we have
\[
E_{\mu,\nu,\lambda}(u, v) = A_{\mu,\nu,\lambda} \text{ and } E'_{\mu,\nu,\lambda}(u, v) = 0.
\]
Thus, \((u, v)\) is a ground state solution.

**Step 2.** We claim that there exist a positive ground state solution.

Since
\[
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{N+2s}} dxdy - \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{||u(x)|| - |u(y)||}{|x - y|^{N+2s}} dxdy = 2 \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{||u(x)||u(y) - u(x)u(y)||}{|x - y|^{N+2s}} dxdy \geq 0,
\]
we have
\[
||u||_{D^{s}_{2}(\mathbb{R}^{n})} \leq ||u||_{D^{s}_{2}(\mathbb{R}^{n})}.
\]
Then, for the minimizing sequence \((u_{n}, v_{n}) \in \mathbb{M}_{r}\), we have
\[
||u_{n}||_{D^{s}_{2}(\mathbb{R}^{n})}^{2} + ||v_{n}||_{D^{s}_{2}(\mathbb{R}^{n})}^{2} + \int_{\mathbb{R}^{n}} (\mu |u_{n}|^{2} + \nu |v_{n}|^{2})dx \geq \int_{\mathbb{R}^{n}} (|u_{n}|^{p+1} + |v_{n}|^{2})dx + \int_{\mathbb{R}^{n}} 2\lambda |u_{n}||v_{n}|dx,
\]
this implies that there exists \( t_{n} \in (0, 1) \) such that \((t_{n}|u_{n}|, t_{n}|v_{n}|) \in \mathbb{M}_{r}\). Hence, we can choose a minimizing sequence \((\pi_{n}, \tau_{n}) = (t_{n}|u_{n}|, t_{n}|v_{n}|)\) and the weak limit \((\pi, \tau)\) is nonnegative. By Strong maximum principle for fractional Laplacian( see, Proposition 2.17 in [30]), we have \( \pi \) and \( \tau \) are both positive.

Next, we claim if \( \mu > \mu_{0} \) and \( 0 < \lambda < \lambda_{\mu,\nu} \), then system (2) has no ground state solution.

Assume by contradiction that there exist \( \lambda \in (0, \lambda_{\mu,\nu}) \) such that system (2) has a ground state solution \((u_{\lambda}, v_{\lambda}) \neq (0, 0)\). Then \( E_{\mu,\nu,\lambda}(u_{\lambda}, v_{\lambda}) = A_{\mu,\nu,\lambda} = \frac{s}{N} S_{s}^{\frac{N}{2}} \). By (21) and (22) we may assume that \( u_{\lambda} \geq 0, v_{\lambda} \geq 0, \) by Strong maximum principle...
for fractional Laplacian (see Proposition 2.17 in [30]), we have \( u_{\lambda} > 0, v_{\lambda} > 0 \). If we take \( \lambda_1 \in (\lambda, \lambda_{\mu,\nu}) \), then by Lemma 2.6 and (8), we have

\[
\frac{s}{N} S_{\frac{N}{s}}^\frac{2}{s} = \lambda_1 \leq \max_{t > 0} E_{\mu,\nu,\lambda_1}(tu_{\lambda_1}, tv_{\lambda_1}) = E_{\mu,\nu,\lambda_1}(t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}} u_{\lambda_1}, t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}} v_{\lambda_1})
\]

\[
= E_{\mu,\nu,\lambda_1}(t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}}, t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}}) - (\lambda_1 - \lambda) t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}} \int_{\mathbb{R}^N} u_{\lambda_1} v_{\lambda_1} dx
\]

\[
< E_{\mu,\nu,\lambda_1}(t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}}, t_{\lambda_1, u_{\lambda_1}, v_{\lambda_1}}) \leq E_{\mu,\nu,\lambda_1}(u_{\lambda_1}, v_{\lambda_1}) = \frac{s}{N} S_{\frac{N}{s}}^\frac{2}{s},
\]

a contradiction. This completes the proof. \( \square \)

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