Symplectic packing in dimension 4

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Abstract
We discuss closed symplectic 4-manifolds which admit full symplectic packings by \( N \) equal balls for large \( N \)’s. We give a homological criterion for recognizing such manifolds. As a corollary we prove that \( \mathbb{C}P^2 \) can be fully packed by \( N \) equal balls for every \( N \geq 9 \).

1 Introduction

Let \( (M^4, \Omega) \) be a closed symplectic 4-manifold. We say that \( (M, \Omega) \) admits a symplectic packing by \( N \) balls of radii \( \lambda_1, \ldots, \lambda_N \) if there exists a symplectic embedding of the disjoint union \( \coprod_{q=1}^{N}(B(\lambda_q), \omega_{std}) \) into \( (M, \Omega) \), where \( (B(\lambda_q), \omega_{std}) \) denotes the standard closed 4-ball of radius \( \lambda_q \), endowed with the standard symplectic form of \( \mathbb{R}^4, \omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \).

We say that \( (M, \Omega) \) admits a full packing by \( N \) equal balls if the supremum of volumes which can be filled by symplectic embeddings of \( N \) disjoint equal balls equals to the volume of \( (M, \Omega) \). Otherwise we say that there is a packing obstruction. Finally, we denote by \( v_N(M, \Omega) \) the ratio between the supremum of the fillable volume by packings with \( N \) equal balls, and the volume of \( (M, \Omega) \).

Symplectic packings were studied for the first time by Gromov in [Gr], and later by McDuff and Polterovich in [M-P]. McDuff and Polterovich discovered that for certain manifolds there are packing obstructions. Moreover, they were able to compute \( v_N \) of certain manifolds for many values of \( N \). In particular they computed \( v_N(\mathbb{C}P^2) \) for any \( N \) which is square and for \( 1 \leq N \leq 8 \). It turned out that for every \( N \) which is a square there exists full packing, while for every \( 1 \leq N \leq 8 \) which is not a square there are packing obstructions. We refer the reader to [M-P] for more details about the symplectic packing problem.

In this paper we continue the above discussion, concentrating on manifolds which admit full packings by \( N \) equal balls for large enough \( N \)’s. We give a

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homological condition for recognizing such manifolds, and a direct method for computing values of $N_0$, such that for all $N \geq N_0$ there exists full packings by $N$ equal balls. We then work out several examples including $CP^2$, which turns out to admit full packings for every $N \geq 9$.

The methods we use are based on the inflation procedure of Lalonde and McDuff and on Taubes theory of Gromov invariants via pseudo-holomorphic curves.

2 Main results

Our main results are concerned with closed symplectic 4-manifolds of the following types:

- manifolds with $b_2^+ = 1, b_1 = 0$.
- ruled manifolds and their blow-ups.

Manifolds of the above types belong to a wider class known as *manifolds which do not have SW-simple type*. We shall denote this class by $\mathcal{C}$, remarking that our main results remain true for manifolds in this class.

We refer the reader to [TAU 1, TAU 2], and to [McD 3, McD 2] for the precise definition of that class. Finally, note that the class $\mathcal{C}$ is closed under the operation of blowing-up.

Given a symplectic manifold $(M, \Omega)$, we shall denote by $c_1 = c_1(TM, J)$ the first Chern class of the complex vector bundle $(TM, J)$, where $J$ is any almost complex structure tamed by $\Omega$.

**Definition 2.A** Let $(M, \Omega)$ be a closed symplectic 4-manifold. Consider the following set

$$D_\Omega = \{ B \in H_2(M; \mathbb{Z}) \mid \Omega(B) > 0, c_1(B) \geq 2, B \cdot B \geq 0 \}.$$ 

Define

$$d_\Omega = \inf_{B \in \mathcal{D}_\Omega} \frac{\Omega(B)}{c_1(B)} \in [0, \infty].$$ 

Here we use the convention that $\inf \emptyset = \infty$.

Before stating our main theorem we mention that given a symplectic manifold $(M, \Omega)$, its volume is defined to be $Vol(M, \Omega) = \int_M \frac{1}{2} \Omega \wedge \Omega$.

**Theorem 2.B** Let $(M, \Omega)$ be a closed symplectic 4-manifold in the class $\mathcal{C}$. Suppose that $0 < d_\Omega \leq \infty$. Then

$$v_N(M, \Omega) \geq \min \{ 1, \frac{N d_\Omega^2}{2 Vol(M, \Omega)} \}.$$
In particular, there exists an integer $N_\Omega$ such that for every $N \geq N_\Omega$, $(M, \Omega)$ admits a full packing by $N$ equal balls. In fact, $N_\Omega$ can be taken to be any integer which satisfies

$$N_\Omega \geq \frac{2\text{Vol}(M, \Omega)}{d_\Omega^2}.$$  

The proof is given in section 4, where a slightly sharper result is stated and proved. The above theorem shows that it makes sense to define the following invariant.

**Definition 2.C** Let $(M, \Omega)$ be a closed symplectic 4-manifold. Define $P(M, \Omega)$, the packing number of $(M, \Omega)$, as follows:

$$P(M, \Omega) = 1 + \max\{N \in \mathbb{N} | \text{there does not exist a full packing by } N \text{ equal balls}\}.$$  

Here we use the convention that $\max\emptyset = 0$, while $\max$ of an unbounded set is $\infty$. When there is no risk of confusion we shall denote the packing number of $(M, \Omega)$ by $P_\Omega$.

As corollary to theorem 2.B we prove:

**Corollary 2.D**

1) $P(\mathbb{C}P^2, \sigma_{\text{std}}) = 9$, where $\sigma_{\text{std}}$ is the standard Kähler form of $\mathbb{C}P^2$.

2) $2\beta^2_0 \leq P(S^2 \times S^2, \sigma \oplus \beta \sigma) \leq 8\beta^2_0$, where $\sigma$ is the standard symplectic form of $S^2$ and $0 < \alpha \leq \beta$.

3) Let $R$ be an orientable surface of genus $g \geq 1$. Let $\Omega = \beta \sigma_R \oplus \alpha \sigma_{S^2}$ be a symplectic form of $R \times S^2$, where $\sigma_R$, $\sigma_{S^2}$ are area forms of $R$, $S^2$ respectively such that $\int_R \sigma_R = \int_{S^2} \sigma_{S^2} = 1$. Then $P_\Omega = \lceil \frac{2\beta^2_0}{\alpha} \rceil$. (Here, $\lceil x \rceil$ denotes the minimal integer which is greater or equal to $x$).

The proof of this corollary is given in section 5.

**Remark 2.E**

1) Notice that in part 3 of the above corollary, unlike in part 2, we do not assume that $\alpha \leq \beta$.

2) From parts 2 and 3 of the above corollary it follows that $P_\Omega$ is not a deformation invariant, since $P_\Omega \to \infty$ as $\alpha \to 0$.

**Theorem 2.F** Let $(M, \Omega)$ be a closed symplectic minimal 4-manifold in the class $C$. Suppose that $(M, \Omega)$ is not rational or ruled, then $D_\Omega = \emptyset$. In particular $P(M, \Omega) = 1$.

**Proof.** Denote by $K = -c_1(TM, J)$ the canonical class of $(TM, J)$, where $J$ is any almost complex structure tamed by $\Omega$. Since $(M, \Omega)$ is minimal but not rational or ruled, then by a theorem due to Liu [Liu] we must have $K^2 \geq 0$ and $K \cdot [\Omega] \geq 0$. Hence $K$ belongs to the closure of the positive light cone

$$\overline{P^+} = \{ a \in H^2(M; \mathbb{R}) | a^2 \geq 0, [\Omega] \cdot a \geq 0 \}.$$  

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Assume that \( D_\Omega \) is not empty, and let \( B \in D_\Omega \). Clearly \( D_\Omega \subseteq P^+ \). Since manifolds in the class \( C \) have \( b_2^+ = 1 \), it follows from the light cone lemma (see [McD 3]) that \( K \cdot B \geq 0 \). But this cannot hold by the definition of the set \( D_\Omega \).

Examples of manifolds satisfying the conditions of the above theorem are: Barlow surfaces, Dolgachev surfaces, hyper-elliptic surfaces and Enriques surface. As explained in [M-S 1] and [McD 2] the above manifolds belong to the class \( C \). Since they are minimal and not rational or ruled they satisfy the conditions of theorem 2.F.

We conclude this section by mentioning that all symplectic packings of manifolds in the class \( C \) are unique in the sense that given \( \lambda_1, \ldots, \lambda_N \), any two symplectic embeddings of the disjoint union of balls of radii \( \lambda_1, \ldots, \lambda_N \) are symplectically isotopic. See [McD 1] for the case of \( CP^2 \) with \( N \leq 2 \), [Bi 1] for the case of \( CP^2 \) with \( N \leq 6 \), and [McD 2] for the general case.

3 Inflation and Gromov invariants

In order to prove theorem 2.B we need a few preliminary lemmas. For completeness we also state some relevant theorems from Taubes theory of Gromov invariants.

Let \((M, \Omega)\) be a closed symplectic 4-manifold. An \( \Omega \)-symplectic exceptional sphere is by definition a symplectically embedded sphere with self intersection number \(-1\). We shall denote by \( E \) the set of all 2-integral homology classes which can be represented by \( \Omega \)-symplectic exceptional spheres. In what follows we shall use the notation \( PD \) for the Poincaré duality.

We first need the following theorem of McDuff which belongs to the framework of Taubes theory of Gromov invariants. For the theory of Gromov invariants see [TAU 1, TAU 2], [McD 3].

**Theorem 3.A** (McDuff [McD 2] lemma 2.2) Let \((M, \Omega)\) be a closed symplectic 4-manifold in the class \( C \). Let \( A \in H_2(M; \mathbb{Q}) \) satisfy \( \Omega(A) > 0 \) and \( A \cdot A > 0 \). Then for sufficiently large \( n \), the class \( nA \) can be represented by a (possibly disconnected) \( J \)-holomorphic curve for generic \( J \). If in addition, for every \( E \in E \) \( A \cdot E \geq 0 \) then there exists an embedded and connected pseudo-holomorphic curve representing the class \( nA \).

Combining this theorem with the inflation procedure of Lalonde and McDuff one obtains the following theorem:

**Theorem 3.B** (McDuff [McD 2]) Let \((M, \Omega)\) be a closed symplectic 4-manifold in the class \( C \). Let \( A \in H_2(M; \mathbb{Q}) \) satisfy \( \Omega(A) > 0 \) and \( A \cdot A > 0 \). Assume that for every \( E \in E \) \( A \cdot E \geq 0 \). Then there exists a closed 2-form \( \rho \), representing \( PD(A) \) and such that \( \Omega + y\rho \) is symplectic for all \( y \geq 0 \).
For more details about this and about the inflation procedure see [McD 2] and [L-M].

**Remark 3.C** An obvious conclusion of the above theorem is:
Under the assumptions of the above theorem, arbitrarily close to \( PD(A) \) there exist cohomology classes which represent symplectic forms in the same deformation class as \( \Omega \).

We shall also need the following well known lemma (see [McD 4], [M-P]):

**Lemma 3.D** Let \((M, \Omega)\) be a closed symplectic 4-manifold. Denote by \( \mathcal{E} \) the set of all homology classes which can be represented by \( \Omega \)-symplectic exceptional spheres. Then:
1) \( \mathcal{E} \) depends only on the deformation class of \( \Omega \).
2) Let \( \mathcal{J}(\Omega) \) be the space of all \( \Omega \)-tamed smooth almost complex structures of \( M \). Then there exists a dense (actually, even residual) subset \( \mathcal{J}_E \subseteq \mathcal{J}(\Omega) \) such that for every \( J \in \mathcal{J}_E \) all classes in \( \mathcal{E} \) admit \( J \)-holomorphic representatives which are connected, embedded and of genus 0.
3) If \( E', E'' \) are distinct classes in \( \mathcal{E} \) then \( E' \cdot E'' \geq 0 \).

Essential to the symplectic packing problem is the symplectic blow-up operation (see [M-P], [M-S 2]). We shall work in the following setting. Let \((M, J)\) be a 4-dimensional almost complex manifold with \( J \) integrable near \( x_1, \ldots, x_N \in M \). Let \((\mathcal{M}, \mathcal{J}) \rightarrow (M, J)\) be the complex blow-up of \((M, J)\) at \( x_1, \ldots, x_N \). Denote by \( \Sigma_q = \Theta^{-1}(x_q) \) \( q = 1, \ldots, N \) the exceptional divisors, and by \( E_q \in H_2(\mathcal{M}; \mathbb{Z}) \) their homology classes. Finally, we set \( e_q = PD(E_q) \).

Recall that a symplectic form taming an almost complex structure \( J \) is said to be \( J \)-standard near \( x \in M \) if the pair \((\Omega, J)\) is diffeomorphic to the standard pair \((\omega_{std}, i)\) of \( \mathbb{R}^4 \), near \( x \).

The following lemma is an obvious generalization of proposition 2.1.C from [M-P].

**Lemma 3.E** Let \((M, \Omega)\) be a closed symplectic 4-manifold. Let \( J \) be an almost complex structure tamed by \( \Omega \), which is integrable near \( x_1, \ldots, x_N \in M \) and suppose that \( \Omega \) is \( J \)-standard near \( x_1, \ldots, x_N \). Let \( \mu_1(0), \ldots, \mu_N(0) \) be positive numbers and let

\[
\varphi = \prod_{q=1}^{N} \varphi_q : \prod_{q=1}^{N} (B(\mu_q(0)), \omega_{std}) \rightarrow (M, \Omega)
\]

be a symplectic embedding which is \((i, J)\)-holomorphic. Denote by \((\mathcal{M}, \mathcal{O})\) the symplectic blow-up of \( \Omega \) with respect to \( \varphi \). Suppose we have a symplectic deformation \( \{\mathcal{O}_t\}_{0 \leq t \leq 1} \) starting with \( \mathcal{O}_0 \), and lying in the cohomology class

\[
[\mathcal{O}_t] = [\Theta^*\Omega] - \pi \sum_{q=1}^{N} \mu_q(t)^2 e_q.
\]
Then $(M, \Omega)$ admits a symplectic packing by $N$ balls of radii $\mu_1(1), \ldots, \mu_N(1)$.

The proof of this lemma goes along the same lines as those of proposition 2.1.C from [M-P], only that here one has to adjust the symplectic forms $\Omega_t$ on the exceptional divisors so that they become standard. This can be done using the symplectic neighborhood theorem.

## 4 Proof of the main theorem

We shall prove a slightly more general version of theorem 2.B, namely:

**Theorem 4.A** Let $(M, \Omega)$ be a closed symplectic 4-manifold in the class $C$. Suppose that $0 < d_\Omega \leq \infty$ and let $\lambda_1, \ldots, \lambda_N < \sqrt{d_\Omega}$ be positive numbers which satisfy

$$\sum_{q=1}^{N} \lambda_q^4 < 2Vol(M, \Omega).$$

Denote by $\overline{M} \xrightarrow{\Theta} M$ a complex blow-up of $M$ at $N$ distinct points. Then the following holds:

1) The cohomology class $[\Theta^* \Omega] - \sum_{q=1}^{N} \lambda_q^2 e_q$ admits a symplectic representative.

2) The manifold $(M, \pi_\Omega)$ admits a symplectic packing by $N$ balls of radii $\lambda_1, \ldots, \lambda_N$.

3) In particular, if

$$N \geq \frac{2Vol(M, \Omega)}{d_\Omega^2}$$

then there exists a full packing of $(M, \Omega)$ by $N$ equal balls.

**Proof.** The idea of the proof goes along the following lines. First, endow $\overline{M}$ with some auxiliary symplectic form $\overline{\Omega}$, obtained from blowing-up $(M, \Omega)$ with respect to some symplectic embedding of very small balls. Next, consider the homology class $A = PD([\Theta^* \Omega]) - \sum_{q=1}^{N} \lambda_q^2 E_q$. The idea is to show that for large enough $n$ the class $nA$ represents an embedded, reduced and connected pseudoholomorphic curve in $\overline{M}$. Once this is proved we can use the inflation procedure to obtain a closed 2-form $\rho$ which lies in the cohomology class $[\Theta^* \Omega] - \sum_{q=1}^{N} \lambda_q^2 e_q$ such that for every $y \geq 0$ the form $\frac{1}{y^2} \overline{\Omega} + y\rho$ is symplectic. Dividing by $y$ we see that for every $y > 0$ the form $\frac{1}{y^2} \overline{\Omega} + \rho$ is symplectic. By taking $y$ to be very large we obtain a symplectic form lying in a cohomology class which is very close to the desired class: $[\Theta^* \Omega] - \sum_{q=1}^{N} \lambda_q^2 e_q$. It turns out that this approximation is enough for our purposes. We now give the precise details of the proof.
Let $J$ be an almost complex structure tamed by $\Omega$ which is integrable near $x_1, \ldots, x_N \in M$. Let $(\overline{M}, J) \xrightarrow{\Theta} (M, J)$ be the complex blow-up of $M$ at $x_1, \ldots, x_N$. Denote by $\Sigma_q$ the exceptional divisors, by $E_q$ their homology classes, and by $e_q$ the Poincaré dual of $E_q$. Finally, denote by $c_1$ the first Chern class of $(TM, J)$ and by $\overline{c}_1$ the first Chern class of $(\overline{T M}, J)$. Notice that $\overline{c}_1 = c_1 - \sum_{q=1}^{N} e_q$, under the natural decomposition $H^2(M; \mathbb{Z}) = H^2(\overline{M}; \mathbb{Z}) \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_N$.

Without loss of generality we may assume that $\Omega$ is $J$-standard near the points $x_1, \ldots, x_N$ since $\Omega$ is isotopic to such a form (see [M-P] proposition 2.1.A).

The proof is divided into two steps. In the first we assume that the cohomology class of $\Omega$ is rational. The second step is a reduction to the first one.

**Step 1:** Assume that $[\Omega]$ is a rational class.

Let $N \in \mathbb{N}$, and let $\lambda_q$ be as in the assumptions of the theorem. Choose $\tilde{\lambda}_q \in \mathbb{Q}$ such that $\lambda_q < \tilde{\lambda}_q < \sqrt{d_\Omega}$ and such that $\sum_{q=1}^{N} \tilde{\lambda}_q^4 < 2Vol(M, \Omega)$. Set

$$a = [\Theta^*\Omega] - \sum_{q=1}^{N} \tilde{\lambda}_q^2 e_q \in H^2(\overline{M}; \mathbb{Q}).$$

Denote by $A$ the Poincaré dual of $a$. Clearly $A \cdot A > 0$.

Let $\overline{\Omega}_\epsilon$ be a symplectic form on $\overline{M}$ obtained from symplectic blowing-up with respect to a holomorphic and symplectic embedding of $N$ balls of very small radii. Hence

$$[\overline{\Omega}_\epsilon] = [\Theta^*\Omega] - \epsilon \sum_{q=1}^{N} e_q.$$

By taking $\epsilon$ be small enough we may assume that $\overline{\Omega}_\epsilon(A) > 0$. Denote by $\mathcal{E}$ the set of homology classes which can be represented by $\overline{\Omega}_\epsilon$-symplectic exceptional spheres in $\overline{M}$.

We claim that $A \cdot E > 0$ for any $E \in \mathcal{E}$. For this purpose, write

$$E = B - \sum_{q=1}^{N} m_q E_q, \text{ where } B \in H_2(M; \mathbb{Z}).$$

Notice first that $\Omega(B) \geq 0$. Indeed, the form $\overline{\Omega}_\epsilon$ can be included in a smooth family of symplectic forms $\{\overline{\Omega}_t\}_{0 < t \leq \epsilon}$ such that

$$[\overline{\Omega}_t] = [\Theta^*\Omega] - t \sum_{q=1}^{N} e_q \text{ for all } 0 < t \leq \epsilon.$$

Since the set $\mathcal{E}$ depends only on the deformation class of $\overline{\Omega}_\epsilon$ we must have $\overline{\Omega}_t(E) > 0$ for all $0 < t \leq \epsilon$ and it easily follows that $\Omega(B) \geq 0$.
Moreover, one can show that:
(i) If $\Omega(B) = 0$ then $B = 0$ and $E = E_q$ for some $1 \leq q \leq N$.
(ii) If $B \neq 0$ then $m_q \geq 0$ for all $1 \leq q \leq N$.

Indeed, suppose that $\Omega(B) = 0$ but $B \neq 0$. Therefore $E \neq E_q$ for all $q$, so by lemma 3.D we have $E \cdot E_q \geq 0$, hence $m_q \geq 0$ for all $q$. It is not possible that all the $m_q$’s are zero, since then $E = B$ and we get $\Omega(E) = \Omega(B) = 0$. So there must be at least one positive $m_q$. But then $\Omega(E) = 0 - \epsilon \sum_{q=1}^{N} m_q < 0$, which is impossible. This proves that $B = 0$, and it easily follows that $E = E_q$ for some $q$. Part (ii) follows immediately from lemma 3.D.

Now we are ready to show that $A \cdot E > 0$. If $\Omega(B) = 0$ then $E = E_q$ for some $q$, hence $A \cdot E = \lambda^2_q > 0$. So assume $\Omega(B) > 0$, from which it follows that $m_q \geq 0$ for all $q$. If all the $m_q$ are zero then clearly $A \cdot E = \Omega(B) = \Omega_\epsilon(B) > 0$ and we are done. So assume that at least one of the $m_q$’s is positive, hence

$$\sum_{q=1}^{N} m_q \geq 1.$$ 

Since $\check{c}_1(E) = 1$ we must have

$$c_1(B) = 1 + \sum_{q=1}^{N} m_q \geq 2.$$ 

Furthermore, since $E \cdot E = -1$ we must have

$$B \cdot B - \sum_{q=1}^{N} m^2_q = -1$$

hence $B \cdot B \geq 0$. This shows that $B \in D_\Omega$. Set $\check{\lambda} = \max\{\check{\lambda}_1, \ldots, \check{\lambda}_N\}$. We have

$$A \cdot E = \Omega(B) - \sum_{q=1}^{N} \check{\lambda}^2_q m_q \geq \Omega(B) - \check{\lambda}^2 (c_1(B) - 1) \geq \Omega(B) - d_\Omega c_1(B) + \check{\lambda}^2 > 0.$$ 

Notice that the case $D_\Omega = \emptyset$ (which corresponds to $d_\Omega = \infty$), has already been treated since when $D_\Omega = \emptyset$, the proof of our claim ends at an earlier stage.

By theorem 3.B there exists a closed 2-form $\rho$, representing the class $a = PD(A)$ such that $\Omega_\epsilon + y\rho$ is symplectic for all $y \geq 0$. As mentioned before the idea now is to consider the symplectic forms $\frac{1}{s} \Omega_\epsilon + \rho$. If we take $y$ large enough we obtain in this way a symplectic form which lies in a cohomology class very close to the desired one.

More precisely, consider the symplectic forms

$$(1) \quad \Omega_s = \frac{1}{1 + \epsilon} \Omega_\epsilon + (s - \epsilon) \rho]$$, \quad \text{where} \quad s \geq \epsilon.$
They lie in the cohomology class
\[
[\Omega_s] = [\Theta^* \Omega] - \frac{1}{1 + s - \epsilon} \sum_{q=1}^N (\epsilon + (s - \epsilon) \lambda_q^2) e_q.
\]

Choose \( s_0 > \epsilon \) so large that
\[
\lambda_q^2 < \frac{\epsilon + (s_0 - \epsilon) \lambda_q^2}{1 + s_0 - \epsilon} < \bar{\lambda}_q^2 \quad (\text{we may assume} \, \epsilon < \bar{\lambda}_q^2 \, \text{for all} \, q).
\]

Notice that equation 1 provides a symplectic deformation \( \{\Omega_s\}_{\epsilon \leq s \leq s_0} \) starting at \( \Omega_\epsilon \) and ending with \( \Omega_{s_0} \), and this deformation satisfies the conditions of lemma 3.E. It follows by this lemma that \((M, \pi\Omega)\) admits a symplectic packing by \( N \) balls of radii \( \sqrt{\frac{\epsilon + (s_0 - \epsilon) \lambda_q^2}{1 + s_0 - \epsilon}} > \lambda_q^2 \) \( (q = 1, \ldots, N) \). In particular all the conclusions of the theorem hold for \( \lambda_1, \ldots, \lambda_N \).

Step 2: Consider the general case.

First notice that the proof of step 1 still holds if we assume that the cohomology class of \( \Omega \) is a real multiple of a rational class. We need now the following general and simple observation:

Let \( a \in \mathbb{R}^n \). Then arbitrarily close to \( a \) there exist \( a_0, \ldots, a_n \in \mathbb{Q}^n \subset \mathbb{R}^n \) and nonnegative real numbers \( \alpha_0, \ldots, \alpha_n \geq 0 \) such that \( \sum_{i=0}^n \alpha_i = 1 \) and \( \sum_{i=0}^n \alpha_i a_i = a \).

Using this, we decompose \( a = [\Theta^* \Omega] - \sum_{q=1}^N \bar{\lambda}_q^2 e_q \) into a sum of the form \( \sum_{i=0}^n \alpha_i a_i \) where \( a_i \in H^2(M; \mathbb{Q}) \), and \( n = b_2(M) + N \). Since the classes \( a_i \) can be taken to be arbitrarily close to \( a \) we can proceed as we did in step 1 of the proof. The only difference is that now we have to do \( n + 1 \) inflations, thus obtaining \( n + 1 \) closed 2-forms \( \rho_0, \ldots, \rho_n \) representing the classes \( a_0, \ldots, a_n \), and such that \( \sum_{i=0}^n \rho_i \) are symplectic for all \( \rho_i \geq 0 \). The proof now proceeds as in step 1 by taking \( \rho_i = s \alpha_i \), with \( s \) very large.

Note that inflation using more than one curve is slightly more complicated than just with one curve. We refer the reader to [McD 2] for more details on this.

Finally, the third statement of the theorem follows easily from the second.

5 Examples

In this section we work out some examples of computing the packing number.

Minimal rational and ruled manifolds

Consider first minimal rational manifolds, that is \( \mathbb{C}P^2 \) and \( S^2 \times S^2 \). We start with \((\mathbb{C}P^2, \sigma_{std})\), where \( \sigma_{std} \) is the standard Kähler form normalized such that
∫_{\mathbb{C}P^2} \sigma_{\text{std}} = 1. Denote by \( L \) the homology class of a projective line in \( \mathbb{C}P^2 \) and by \( l \) its Poincaré dual. We have \( c_1 = 3l, \) \([\sigma_{\text{std}}] = l\), hence \( d_{\sigma_{\text{std}}} = \frac{1}{3} \). Using Theorem 4.A we see that given \( \lambda_1, \ldots, \lambda_N < \frac{1}{\sqrt[4]{3}} \) which satisfy \( \sum_{q=1}^{N} \lambda_q^4 < 1 \), there exists a symplectic packing of \((\mathbb{C}P^2, \pi_{\sigma_{\text{std}}})\) by \( N \) balls of radii \( \lambda_1, \ldots, \lambda_N \).

In particular, if \( N \geq 9 \), \( \mathbb{C}P^2 \) admits a full packing by \( N \) equal balls. Note however, that for 8 equal balls there is a packing obstruction (see [M-P]). Finally, recall that by a theorem of Taubes any two cohomologous symplectic structures on \( \mathbb{C}P^2 \) are diffeomorphic. All the above prove:

**Corollary 5.A** For any symplectic form \( \sigma \) on \( \mathbb{C}P^2 \), \( P_\sigma = 9 \).

Let us consider now \( S^2 \times S^2 \). Again, it is enough to consider the standard split forms, since by theorems of McDuff and Li and Liu (see [L-M]), any two cohomologous symplectic forms of \( S^2 \times S^2 \) are diffeomorphic.

Consider the symplectic form \( \Omega = \alpha \sigma \oplus \beta \sigma \), where \( \int_{S^2} \sigma = 1 \) and \( 0 < \alpha \leq \beta \). Using Gromov’s non-squeezing theorem (see [Gr]), it is not hard to see that if \( B(\lambda) \) embeds symplectically into \((S^2 \times S^2, \Omega)\) then \( \pi \lambda^2 < \alpha \), hence we must have

\[
P_\Omega \geq 2 \frac{\beta}{\alpha}.
\]

Using theorem 4.A we compute an upper bound for \( P_\Omega \) as follows:

Set \( A_1 = [S^2 \times pt] \), \( A_2 = [pt \times S^2] \), and \( a_i = PD(A_i) \) \( i = 1, 2 \). We have \( c_1 = 2(a_1 + a_2) \) and \( \text{Vol}(S^2 \times S^2, \Omega) = \alpha \beta \). Let \( B \in D_\Omega \), say \( B = n_1 A_1 + n_2 A_2 \). It is easy to see that \( n_1, n_2 \) are non-negative. Hence

\[
\frac{\Omega(B)}{c_1(B)} = \frac{\alpha n_1 + \beta n_2}{2n_1 + 2n_2} \geq \frac{\alpha}{2} > 0.
\]

Therefore \( d_\Omega \geq \frac{\alpha}{2} \), hence from theorem 4.A we obtain:

**Corollary 5.B** Let \( 0 < \alpha \leq \beta \) and let \( \sigma \) be any area form of \( S^2 \). Then \( 2 \frac{\beta}{\alpha} \leq P_\Omega \leq 8 \frac{\beta}{\alpha} \).

Note that in some cases it is possible to give sharper bounds for \( P_\Omega \), and even to compute its precise value. It is well known that there exists a diffeomorphism between the blow-up of \( S^2 \times S^2 \) at \( N \) points and the blow-up of \( \mathbb{C}P^2 \) at \( N + 1 \) points. Furthermore it is not hard to compute that this diffeomorphism can be chosen to induce the following correspondence:

packing \((S^2 \times S^2, \pi \Omega)\) by \( N \) balls of radii \( \lambda_1, \ldots, \lambda_N \) correspond to packing \((\mathbb{C}P^2, \pi(\alpha + \beta - \lambda_1^2)\sigma_{\text{std}})\) by \( N + 1 \) balls of radii \( \sqrt{\alpha - \lambda_1^2}, \sqrt{\beta - \lambda_1^2}, \lambda_2, \ldots, \lambda_N \).

Restricting to \( \alpha = \beta \), a straightforward computation shows that there is packing obstruction for 7 balls (one uses the above correspondence and the packing inequalities from [M-P]). Hence, if \( \alpha = \beta \) then \( P_\Omega = 8 \).

Let us consider now irrational ruled surfaces.
Corollary 5.C Let $R$ be an orientable surface of genus $g \geq 1$. Let $\sigma_R$, $\sigma_{S^2}$ be area forms on $R, S^2$ respectively, such that $\int_R \sigma_R = \int_{S^2} \sigma_{S^2} = 1$. Let $\alpha, \beta$ be positive numbers, then

$$P_{(R \times S^2, \beta \sigma_R \oplus \alpha \sigma_{S^2})} = \lceil 2 \frac{\beta}{\alpha} \rceil$$

Proof. Set $\Omega = \beta \sigma_R \oplus \alpha \sigma_{S^2}$. Let $J_s$ be a split complex structure on $R \times S^2$. Denote by $\Omega$ a blow-up of $\Omega$. We claim that the set $E$ of $\Omega$-symplectic exceptional spheres is:

$$E = \{ E_1, \ldots, E_N, S - E_1, \ldots, S - E_N \},$$

where $S = [pt \times S^2]$.

Indeed, let $C$ be an $\Omega$-symplectic exceptional sphere in the class $E$. Choose a generic almost complex structure $J_t$ tamed by $\Omega$ for which $C$ is $J_t$-holomorphic. Consider a generic path $\{ J_t \}_{0 \leq t \leq 1}$ of $\Omega$-tamed almost complex structures with $J_0 = J$ and $J_1 = J_s$. Since $J$ can be chosen to be arbitrarily close to $J_s$, we may assume that for all $0 \leq t < 1$ there exist $J_t$-holomorphic $E$-spheres. Using Gromov’s compactness theorem we obtain a (possibly cusp) $J_s$-holomorphic $E$-curve, $\tilde{C} = C_1 \cup \ldots \cup C_n$, with $\text{genus}(C_i) = 0$.

Denote by $pr_R : R \times S^2 \rightarrow R$ the projection and by $\tilde{pr}_R$ its lifting to $(R \times S^2)_N$. Clearly $\tilde{pr}_R$ is $J_s$-holomorphic. Since $\text{genus}(R) \geq 1$ it follows that for every $j$, $\tilde{pr}_R(C_j)$ must be a point, say $p_j$. As $\tilde{C}$ is connected we see that all the $p_j$’s are equal. Thus $\tilde{pr}_R(\tilde{C})$ is a point, and it immediately follows that $E \in \{ E_1, \ldots, E_N, S - E_1, \ldots, S - E_N \}$. Conversely, it is obvious that $E_q$ and $S - E_q$ are $\Omega$-symplectic exceptional classes.

Since the exceptional spheres provide all the packing obstructions, it follows that $(R \times S^2, \pi \Omega)$, admits a symplectic packing by $N$ balls of radii $\lambda_1, \ldots, \lambda_N$ iff $\lambda_2^4 < \lambda_1^2 < \alpha$ and $\sum_{q=1}^N \lambda_q^4 < 2\alpha \beta$. Restricting to equal balls the result easily follows.

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