Stability estimate for the relativistic Schrödinger equation with time-dependent vector potentials

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Abstract
We consider the relativistic Schrödinger equation with a time dependent vector and scalar potential on a bounded cylindrical domain. Using a geometric optics ansatz we establish a logarithmic stability estimate for the recovery of vector and scalar potentials.

Keywords: inverse problems, hyperbolic, vector and scalar potentials, relativistic Schrödinger equation, time-dependent, Klein–Gordon

1. Introduction
Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, consider the hyperbolic equation with time dependent coefficients

$$(−i\partial_t + A_0(t, x))^2u - \sum_{j=1}^{n}(−i\partial_{x_j} + A_j(t, x))^2u + V(t, x)u = 0 \quad \text{in } \mathbb{R} \times \Omega,$$

where $A_j(t, x), 0 \leq j \leq n$, and $V(t, x)$ are compactly supported smooth functions.

The vector field $A(t, x) = (A_0(t, x), \ldots, A_n(t, x))$ is called the vector potential, the function $V(t, x)$ is called the scalar potential and equation (1) is often referred to as the relativistic Schrödinger equation or, in the case where the vector potential is zero and the scalar potential is proportional to the mass of a free particle, it is referred to a the Klein–Gordon equation.

We impose the initial and boundary conditions

$${u}(t, x) = \partial_t u(t, x) = 0 \quad \text{for } t \ll 0$$

and

$${u}(t, x) = f(t, x) \quad \text{on } \mathbb{R} \times \partial \Omega,$$
where \( f \) is a compactly supported smooth function on \( \mathbb{R} \times \partial \Omega \). Solutions to (1) satisfying (2) and (3) exist and are unique (Theorem 8.1 in [8]) and we can define the Dirichlet to Neumann operator by

\[
A(f) := (\partial_\nu + i A(t, x) \cdot \nu)u(t, x)|_{\mathbb{R} \times \partial \Omega},
\]

where \( u \) is the solution of (1)–(3), \( \nu \) is the exterior unit normal to \( \partial \Omega \) and we have set \( A(t, x) = (A_1(t, x), \ldots, A_n(t, x)) \). The inverse boundary value problem is the recovery of \( A(t, x) \) and \( V(t, x) \) knowing \( A(f) \) for all \( f \in C^\infty_0(\mathbb{R} \times \partial \Omega) \).

**Definition 1.1.** The pair \( (A(t, x), V(t, x)) \) and \( (A'(t, x), V'(t, x)) \) are said to be gauge equivalent if there exists \( \phi \in C^\infty(\Omega \times \partial \Omega) \) such that \( \phi = 1 \) on \( \mathbb{R} \times \partial \Omega \) and

\[
A'(t, x) = \frac{i}{g(t, x)} V_{t,x} \phi(t, x),
\]

where \( V_{t,x} := (\partial_x, \partial_t) = (\partial_1, \partial_2, \ldots, \partial_{n+1}) \) is the \((n + 1)\)-dimensional gradient. The mapping \( (A, V) \to (A', V') \) is called a gauge transform.

The Dirichlet to Neumann maps \( A \) and \( A' \) are said to be gauge equivalent if for all \( f(t, x) \in C^\infty_0(\mathbb{R} \times \partial \Omega) \),

\[
A'(g(t, x)f(t, x)) = g(t, x)A(f(t, x)).
\]

**Remark:** When \( \Omega \) is simply connected, the gauge \( g \) has the particular form \( g(t, x) = e^{i\varphi(t, x)} \) where \( \varphi(t, x) \in C^\infty(\mathbb{R} \times \Omega) \). Then \( -\frac{i}{g(t, x)} V_{t,x} \varphi(t, x) = V_{t,x} \varphi(t, x) \) and two vector potentials are gauge equivalent if their difference is the gradient of a smooth function.

Inverse problems is a topic in mathematics that has been growing in interest in part, due to its wide range of applications, from medicine to acoustics to electromagnetism (see for instance [8] for some of the latest tools and techniques employed in the solutions of these problems). In the case of the hyperbolic inverse boundary value problem (1)–(4) with time independent coefficients, a powerful tool called the boundary control method, or BC-method for short, was discovered by Belishev (see [2]). It was later developed by Belishev, Kurylev, Lassas, and others ([10, 11]). A new approach to this problem based on the BC-method was developed by Eskin in ([3, 4]). On a similar note, Stefanov and Uhlmann established stability results for the wave equation in anisotropic media (see [18, 19] and [21] for a survey of these results).

In the case where the scalar potential is time-dependent and the vector potential is identically equal to zero \( (A \equiv 0 \text{ in } (1)) \), Stefanov [17] and Ramm and Sjöstrand [13], have shown that the Dirichlet to Neumann map completely determines the scalar potentials. In [5], Eskin considered the case with time-dependent potentials that are analytic in time (this case is more general in terms of the complexity of the PDE but less general with its assumption of analiticity). The analiticity of the time variable is related to the use of a unique continuation theorem established by Tataru in [20]. More recently, the results of [13, 17] were generalized by the author in [14] for the case of vector potentials, where it was shown that the Dirichlet-to-Neumann operator determines the vector and scalar potentials up to a gauge transform.
Regarding the stability in the hyperbolic case, the first results were obtained by Isakov in [6]. Isakov and Sun [7] obtained estimates for two coefficients of a hyperbolic partial differential equation from all measurements on a part of the lateral boundary. In [18] Stefanov and Uhlmann studied the hyperbolic Dirichlet to Neumann map associated to the wave equation in anisotropic media; and in [19], they consider the more general case of determining a Riemannian metric on a Riemannian manifold with boundary from the boundary measurements. More recently in [12], Montalto recovers the metric, a covector field and a potential from the hyperbolic Dirichlet to Neumann map.

However, stability in the case of time-dependent vector has not been considered before. In this paper, which is based on [14, 15], we take advantage of a result by Begmatov [1], where he proves a stability estimate for a time-dependent scalar function when information about its x-ray transforms is known on a cone. In our work we establish stability estimates for vector and scalar potentials when they are compactly supported in space and time. This work is structured as follows. In section 2 we briefly review the construction of the geometric optics ansatz (GO for short) as well as the Green’s formula developed in [14]. This construction is later used to obtain estimates for the x-ray transform along ‘light rays’ of particular combinations of the components of the vector potentials. A logarithmic stability estimate for vector potentials is also established in this section. In section 3 we prove an estimate for the case when both vector and scalar potentials are present.

2. Stability of the vector potentials

The following geometric optics construction as well as the Green’s formula are the same in [14] (see also [16] for specific details regarding this particular equation).

**Lemma 2.1.** For the hyperbolic problem (1)–(3) GO ansatz supported near light rays can be constructed having the form

\[ u(t, x) = \exp \left[ i k (t - \omega \cdot x) - i R_1(t, x; \omega) \left( \chi_1(t', x') + O(k^{-1}) \right) \right], \]

where

\[ R_1(t, x; \omega) = \int_{-\infty}^{(t + \omega \cdot x)/2} \sum_{j=0}^{n} \omega_j A_j \left( t' + s, x' + s \omega \right) ds, \]

\((t', x') = (t, x) - \frac{1}{2} (t + \omega \cdot x) (1, \omega)\) is the projection of \((t, x)\) into \(\Pi_{1, \omega}\), the \(n\)-dimensional linear subspace perpendicular to \((1, \omega)\) (see figure 1 in [14]), and \(\chi_1\) is any real valued function that is constant along the direction given by \((1, \omega)\), and whose support is contained in a neighborhood of the light ray \(\gamma = \{(t', x') + s (1, \omega) | s \in \mathbb{R}\}\). A geometric optics solution for the backwards hyperbolic problem can be obtained in the same fashion, with another real valued function \(\chi_2\) constant along a given light ray.

**Lemma 2.2.** Let \(T_1\) and \(T_2\) be two real numbers with \(T_1 \ll 0 \ll T_2\), and consider the forward and backward hyperbolic equations

\[ L_1 u = 0 \quad \text{in} \quad [T_1, T_2] \times \Omega \]
\[ L_2^* v = 0 \quad \text{in} \quad [T_1, T_2] \times \Omega \]
\[ u = \partial_t u = 0 \quad \text{for} \quad t = T_1 \]
\[ v = \partial_t v = 0 \quad \text{for} \quad t = T_2 \]
\[ u = f \quad \text{on} \quad [T_1, T_2] \times \partial \Omega \]
\[ v = g \quad \text{on} \quad [T_1, T_2] \times \partial \Omega, \]
Denoting by $\langle , \rangle_{\Omega \times \partial \Omega}$ and $\langle , \rangle_{\Omega \times \partial \Omega}$ the $L^2$ inner products in $[T_1, T_2] \times \Omega$, $[T_1, T_2] \times \partial \Omega$; the following Green’s formula holds (see [14, 16] for complete details)

$$\sum_{j=0}^{n} r_j \left( A_j u, -i\partial_t v \right)_{\Omega} + \sum_{j=0}^{n} r_j \left( \left( A_j^{(2)} \right)^2 - \left( A_j^{(1)} \right)^2 u, v \right)_{\Omega} = -\langle Vu, v \rangle_{\Omega},$$

(8)

where $\lambda_0 = t$, $A_j = A_j^{(2)} - A_j^{(1)}$ for $0 \leq j \leq n$, $V = V^{(2)} - V^{(1)}$, $n_0 = -1$, and $r_j = 1$ for $1 \leq j \leq n$.

The remaining proofs in this section closely follow [14]. We assume that the components of the vector potentials $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ as well as the scalar potentials $V^{(1)}$ and $V^{(2)}$ are real valued, smooth and compactly supported in both $t$ and $x$. We write

$$A = A^{(1)} - A^{(2)}$$

where $\mathcal{A}^{(k)} = \left( A_0^{(k)}, \ldots, A_n^{(k)} \right)$, $k = 1, 2$, and as before we denote by $\Pi_{(1, \omega)}$ the $n$-dimensional linear subspace perpendicular to $(1, \omega)$.

In symbols

$$\Pi_{(1, \omega)} = \{ (t, x): t + \omega \cdot x = 0 \}.$$

**Lemma 2.3.** If $A_k$, $k = 1, 2$ represents the Dirichlet to Neumann operator for the hyperbolic equations

$$\left( -i\partial_t + A_0^{(k)}(t, x) \right)^2 - \sum_{j=1}^{n} \left( -i\partial_x + A_j^{(k)}(t, x) \right)^2 + V^{(k)}(t, x) \right) u = 0,$$

(9)

then for all $(t, x) \in \mathbb{R}^{n+1}_+, \omega \in S^{n-1}$, the vectorial ray transform of $A = (A_0^{(2)} - A_0^{(1)}, \ldots, A_n^{(2)} - A_n^{(1)})$ along the light rays

$$\gamma_{(t, x, \omega)} = \{ (t, x) + s(1, \omega): s \in \mathbb{R} \},$$

satisfies

$$\left| \exp \left[ i \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds \right] - 1 \right| \leq C \||A_1 - A_2||,$$

(10)

where $\|| \|$ represents the operator norm between $H^1([T_1, T_2] \times \partial \Omega)$ and $L^2([T_1, T_2] \times \partial \Omega)$.

**Remark:** We point that this result is independent of the presence of scalar potentials.
Proof. Owing to (6) and (7), GO ansatz for the forward and backward hyperbolic equations are given by

\[ u(t, x) = \exp\left[i k(t - \omega \cdot x) - i R_1(t, x; \omega)\right] \left(\chi_1 + O\left(k^{-1}\right)\right), \quad (11) \]

\[ v(t, x) = \exp\left[-ik(t - \omega \cdot x) + i R_2(t, x; \omega)\right] \left(\chi_2 + O\left(k^{-1}\right)\right), \quad (12) \]

where

\[ R_1(t, x; \omega) = \int_{-\infty}^{(t + \omega \cdot x)/2} \sum_{j=0}^n \omega_j A_j^{(1)}(t' + s, x' + s\omega) ds, \quad (13) \]

\[ R_2(t, x; \omega) = \int_{-\infty}^{(t + \omega \cdot x)/2} \sum_{j=0}^n \omega_j A_j^{(2)}(t' + s, x' + s\omega) ds, \quad (14) \]

where \( \chi_1, \chi_2 \) are constant along, and supported on a small neighborhood of the light ray \( \gamma_{(t, x; \omega)} \), and where \((t', x')\) is the projection of \((t, x)\) into \(\Pi_{(t, x; \omega)}\).

For \(0 \leq j \leq n\), differentiation of (11) with respect to \(s_j\) combined with estimate (14) in [16] lead to

\[ \partial_{s_j} u = k \exp\left[i k(t - \omega \cdot x) - i R_1(t, x; \omega)\right] \left(-i r_j \omega_j \chi_1 + O\left(k^{-1}\right)\right), \quad (15) \]

where \(x_0 = t, \omega_0 = 1, r_0 = -1\) and \(r_j = 1\) when \(j \neq 0\). Then by (12)

\[ \left(-i \partial_{s_j} u(t, x)\right)v(t, x) = -k e^{i\left(R_2(t, x; \omega) - R_1(t, x; \omega)\right)} \left(r_j \omega_j \chi_2 + O\left(k^{-1}\right)\right). \quad (16) \]

Similarly, (11) yields

\[ u(t, x)\left(-i \partial_{s_j} v(t, x)\right) = -k e^{i\left(R_2(t, x; \omega) - R_1(t, x; \omega)\right)} \left(r_j \omega_j \chi_2 + O\left(k^{-1}\right)\right). \quad (17) \]

Denoting by \(I_R\) the right-hand side of (8), we obtain via the previous two formulas

\[ I_R = Ck \int_{T_1}^{T_2} \int_B \sum_{j=0}^n \left(A_0 + \sum_{j=1}^n \omega_j A_j\right)(t, x) \chi_1(t, x) \chi_2(t, x) \times \exp\left[i\left(R_2(t, x; \omega) - R_1(t, x; \omega)\right)\right] dx \, dt + \cdots \]

which in turn leads to

\[ I_R = Ck \int_{T_1}^{T_2} \int_B \left(A_0 + \sum_{j=1}^n \omega_j A_j\right)(t, x) \chi_1(t, x) \chi_2(t, x) \times \exp\left[-i\sum_{j=1}^n \omega_j A_j(t' + s, x' + s\omega) ds\right] dx \, dr + \cdots \quad (18) \]

where \(C\) is a constant and ‘\(\cdots\)’ represents terms of order \(O(1)\).

We turn now our attention to the left-hand side of (8). Denoting by \(f\) and \(g\) the restrictions of \(u\) and \(v\) to \([T_1, T_2] \times \partial\Omega\), that is

\[ f = u(t, x)\big|_{[-T_1,T_2] \times \partial\Omega} \quad g = v(t, x)\big|_{[-T_1,T_2] \times \partial\Omega}, \]

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we have by the Cauchy–Schwarz inequality
\[ |I_R| = \left| \left( (A_1 - A_2)(f), g \right)_{\mathcal{T}, \mathcal{E}} \right| \leq \|A_1 - A_2\| \times \|f\|_{L^2(\mathcal{T}, \mathcal{E})} \|g\|_{L^2(\mathcal{T}, \mathcal{E})}. \]

Using (11) the latter norm can be estimated by
\[ \|g\|_{L^2(\mathcal{T}, \mathcal{E})} = \|X_2(t, x)\left( 1 + \mathcal{O}(k^{-1}) \right)\|_{L^2(\mathcal{T}, \mathcal{E})} \leq \|X_2(t, x)\|_{L^2(\mathcal{T}, \mathcal{E})} + \mathcal{O}(k^{-1}), \quad (19) \]
whereas by (15) the middle norm can be estimated by
\[ \|f\|_{L^2(\mathcal{T}, \mathcal{E})} \leq C \left[ \left\| \left| X_1 \right|_{L^2(\mathcal{T}, \mathcal{E})} + \mathcal{O}(1) \right\| \right] = \left[ \left\| \left| X_1 \right|_{L^2(\mathcal{T}, \mathcal{E})} + \mathcal{O}(k^{-1}) \right\| \right]. \quad (20) \]

In addition, since \( \Omega \) is bounded and \( \chi_j, j = 1, 2, \) is localized near a light ray, we have \( \|Z_j\|_{L^2(\mathcal{T}, \mathcal{E})} \leq C \). Therefore, by (19) and (20)
\[ |I_R| \leq C \left[ \left\| \left| A_1 - A_2 \right| + \mathcal{O}(k^{-1}) \right\| \right]. \quad (21) \]

Dividing both sides of Green’s formula (8) by \( k \) (i.e., (18) and (21)) and taking the limit as \( k \to \infty \), we obtain via the triangle inequality and the change of coordinates \((t, x) = \sigma(1, \omega) + Y', Y' \in \mathcal{P}(1, \omega)\)
\[ \left| \int_{\mathcal{P}(1, \omega)} \int_{\mathcal{R}} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right) (Y' + \sigma(1, \omega)) X_1(Y') X_2(Y') \right| \times e^{-\frac{i}{k} \int_{\sigma} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right) (Y' + \sigma(1, \omega)) d\sigma} \leq C \left\| \left| A_1 - A_2 \right| \right\|. \quad (22) \]

If we set
\[ a(Y) := \int_{\mathcal{R}} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right) (Y' + \sigma(1, \omega)) e^{-\frac{i}{k} \int_{\sigma} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right) (Y' + \sigma(1, \omega)) d\sigma}, \]
equation (22) can be rewritten as
\[ \left| \int_{\mathcal{P}(1, \omega)} a(Y) X_1(Y') X_2(Y') dS_Y \right| \leq C \left\| \left| A_1 - A_2 \right| \right\|. \]

The conditions imposed on the support of \( \chi_j, j = 1, 2, \) guarantee that the above estimate holds for any \( \chi_j \) satisfying \( \int_{\mathcal{P}(1, \omega)} \left| \chi_j(Y') \right|^2 dS_Y \leq 1 \), thus \( a \) is a bounded linear functional on \( L^1(\mathcal{P}(1, \omega)) \) and the estimate

\[ |I_R| \leq C \left[ \left\| \left| A_1 - A_2 \right| \right\| \right]. \]

The conditions imposed on the support of \( \chi_j, j = 1, 2, \) guarantee that the above estimate holds for any \( \chi_j \) satisfying \( \int_{\mathcal{P}(1, \omega)} \left| \chi_j(Y') \right|^2 dS_Y \leq 1 \), thus \( a \) is a bounded linear functional on \( L^1(\mathcal{P}(1, \omega)) \) and the estimate

\[ |I_R| \leq C \left[ \left\| \left| A_1 - A_2 \right| \right\| \right]. \]
holds. To finish the proof, we invoke the fundamental theorem of calculus and rewrite the integral in the original coordinate system to obtain

\[ \left| \exp \left[ i \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds \right] - 1 \right| \leq C \| |A_1 - A_2||. \]

\[ \blacksquare \]

**Corollary 2.4.** Let \( A_1, A_2 \), represent the Dirichlet to Neumann operators for the hyperbolic equations (9), and let

\[ \alpha := \sup \left| \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=0}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds \right| \]

where the supremum is taken over \((t, x, \omega) \in [T_1, T_2] \times \Omega \times S^{n-1}\). If \( \alpha < 2\pi \), then for all \((t, x) \in \mathbb{R}_{+}^{n+1}, \omega \in S^{n-1}\).

\[ \left| \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds \right| \leq C \| |A_1 - A_2||, \quad (23) \]

where \( \| \| \) represents the operator norm between \( H^1([T_1, T_2] \times \partial \Omega) \) and \( L^2([T_1, T_2] \times \partial \Omega) \).

**Proof.** Denoting by \( \beta \) the integral \( \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds \), we have

\[ \left| \frac{e^{i\beta} - 1}{|\beta|} \right| = \left| \frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}} \right|. \quad (24) \]

Since \( |\beta| < 2\pi < \alpha \), the right-hand side of (24) is bounded from below. It then follows that

\[ \left| \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds \right| \leq C \left| \frac{1}{|\beta|} \right| \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^{n} \omega_j A_j \right)(t + s, x + s\omega) ds - 1, \]

which in turn leads to (23). \( \blacksquare \)

To deal with the fact that uniqueness of the vector potentials is expected only up to a gauge transform we impose the divergence condition
\[ \text{div } \mathbf{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_x A_j(t, x) = 0. \]  

(25)

By the remark after the definition of gauge equivalent pairs of potentials, we know that the difference of vector potentials is the gradient of a scalar function. The divergence condition then implies that said scalar function must also be harmonic and hence equal to zero by the support conditions imposed on the vector potentials.

Denoting by \( F \) the ray transform of \( A_0 + \sum_{j=1}^n \omega_j A_j \) along light rays \( \gamma(t, x; \omega) \), we can rewrite (23) as

\[ |F(t, x; \omega)| \leq C \| A_1 - A_2 \| \]  

(26)

for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \omega \in S^{n-1} \). Taking the Fourier transform of \( F \) in the variables \( x_1, \ldots, x_n \) yields

\[
(F_{(t, \omega)} F(t, \omega)) (\xi) = \int_{\mathbb{R}^n} e^{-i\xi x} \int_{\mathbb{R}} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) ds dx,
\]

and the change of coordinates \( \tilde{x} = x + s\omega, \tilde{t} = t + s \), with Jacobian \( \frac{\partial(t, \omega)}{\partial(t', \omega')} = 1 \) leads to

\[
(F_{(t, \omega)} F(t, \omega)) (\xi) = e^{-i(x\cdot\xi)} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{-i\tilde{x}\cdot\xi} e^{-i(t\cdot\xi)} d\tilde{x} d\tilde{t},
\]

where the right-hand side of the above equation is the Fourier transform (in all variables) of \( A_0 + \sum_{j=1}^n \omega_j A_j \) at the point \((-\omega \cdot \xi, \xi)\). The above equation can be rewritten as

\[
e^{i t (\omega \cdot \xi)} (F_{(t, \omega)} F(t, \omega)) (\xi) = \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (-\omega \cdot \xi, \xi)
\]

and since the right-hand side is independent of \( t \), so is the left-hand side. In particular when \( t = 0 \) we have

\[
\left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (-\omega \cdot \xi, \xi) = (F_{(0, \omega)} F(0, \omega)) (\xi) = G(\xi; \omega).
\]  

(27)

Since the potentials \( A_j \) are smooth and compactly supported, \( F(0, \omega) : \mathbb{R}_+^n \times S^{n-1} \to \mathbb{R} \) is also smooth and compactly supported because for \( |x| \) big enough, the light rays with direction \((1, \omega)\) emanating from the point \((0, x)\) do not intersect the support of the potentials \( A_j \). Moreover by (26) it is uniformly bounded by \( C \| A_1 - A_2 \| \), and

\[
|G(\xi; \omega)| = \left| \int_{\mathbb{R}^n} e^{-i\omega \cdot \xi} F(0, x; \omega) dx \right| 
\leq \| F(0, \omega) \|_{L^\infty(\mathbb{R}_+^n \times S^{n-1})} \text{Vol}(B_n(0)) 
\leq CR^n \| A_1 - A_2 \|
\]  

(28)

shows that \( G \) is uniformly bounded in \( \mathbb{R}_+^n \times S^{n-1} \).
Lemma 2.5. Let $\Lambda_1, \Lambda_2$ represent the Dirichlet to Neumann operators for the hyperbolic equations (9), and let $\alpha$ be as in corollary 2.4. If $\alpha < 2\pi$ and the divergence condition (25) holds, then

$$\left| \tilde{\Lambda}_j(\tau, \xi) \right| \leq C \left\| \Lambda_1 - \Lambda_2 \right\|, \quad 0 \leq j \leq n, \quad (29)$$

on the set \{$(\tau, \xi) : |\xi| \leq \frac{1}{\tau} \}$.

Proof. Proceeding as in the proof of theorem 3.3 in [14], for $(\tau, \xi)$ fixed with $|\tau| < \frac{1}{\tau} |\xi|$ we can find unit vectors $\omega = \omega(\tau, \xi)$ parametrized by an $(n - 2)$-dimensional sphere with radius $r$, $\frac{\tau}{\sqrt{2}} \leq r \leq 1$, (we denote it by $rS^{n-2}$), satisfying $\tau + \omega(\tau, \xi) \cdot \xi = 0$, as well as $\omega(\theta \tau, \theta \xi) = \omega(\tau, \xi)$ for $\theta > 0$. In other words, we can find $\omega(\tau, \xi)$ homogeneous of degree 0 in $(\tau, \xi)$, such that $(\tau, \xi) \perp (1, \omega(\tau, \xi))$. If $n \geq 3$, we consider a maximal one dimensional sphere contained in $rS^{n-2}$ and choose unit vectors $\omega^{(1)}(\tau, \xi), ..., \omega^{(n)}(\tau, \xi)$ forming the vertices of a regular polygon with $n$ sides. If $n = 2$ we let $\omega^{(1)}(\tau, \xi)$ and $\omega^{(2)}(\tau, \xi)$ be the only two elements of $rS^{0}$. In both cases we then study the set of $n + 1$ equations

$$\begin{cases}
\tilde{\Lambda}_0(\tau, \xi) + \sum_{j=1}^{n} \omega^{(j)}(\tau, \xi) \tilde{\Lambda}_j(\tau, \xi) = G(\xi; \omega^{(k)}(\tau, \xi)), \quad k = 1, ..., n \\
\frac{1}{\sqrt{\tau^2 + |\xi|^2}} \left( \tau \tilde{\Lambda}_0(\tau, \xi) + \sum_{j=1}^{n} \xi_j \tilde{\Lambda}_j(\tau, \xi) \right) = 0,
\end{cases} \quad (30)$$

where the last equation is a simple consequence of the divergence condition (25). The left-hand side of (30) can be expressed as $M(\tau, \xi) \tilde{A}(\tau, \xi)$, where

$$M(\tau, \xi) = \begin{pmatrix}
1 & \omega^{(1)}_1(\tau, \xi) & \ldots & \omega^{(1)}_n(\tau, \xi) \\
1 & \omega^{(2)}_1(\tau, \xi) & \ldots & \omega^{(2)}_n(\tau, \xi) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega^{(n)}_1(\tau, \xi) & \ldots & \omega^{(n)}_n(\tau, \xi)
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & \omega^{(1)}_1(\tau, \xi) & \ldots & \omega^{(1)}_n(\tau, \xi) \\
1 & \omega^{(2)}_1(\tau, \xi) & \ldots & \omega^{(2)}_n(\tau, \xi) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega^{(n)}_1(\tau, \xi) & \ldots & \omega^{(n)}_n(\tau, \xi)
\end{pmatrix}$$

has homogeneous entries of degree 0 in $(\tau, \xi)$. We claim that $M(\tau, \xi)$ is invertible. To prove this statement it suffices to show that the homogeneous system

$$\begin{cases}
\tilde{\Lambda}_0(\tau, \xi) + \sum_{j=1}^{n} \omega^{(j)}(\tau, \xi) \tilde{A}_j(\tau, \xi) = 0, \quad k = 1, ..., n \\
\frac{1}{\sqrt{\tau^2 + |\xi|^2}} \left( \tau \tilde{\Lambda}_0(\tau, \xi) + \sum_{j=1}^{n} \xi_j \tilde{A}_j(\tau, \xi) \right) = 0,
\end{cases} \quad (31)$$

has no non-trivial solution. By theorem 3.4 in [14], potentials satisfying the first $n$ equations are those of the form
\[ \widehat{A}_0(\tau, \xi) = \tau \Phi(\tau, \xi), \]
\[ \widehat{A}_j(\tau, \xi) = \xi_j \Phi(\tau, \xi), \quad 1 \leq j \leq n, \]
for some smooth function \( \Phi \). The last equation in (31) gives \( \Phi(\tau, \xi) \sqrt{\tau^2 + |\xi|^2} = 0 \), which in turn leads to \( \Phi \equiv 0 \), and \( \widehat{A} = 0 \).

Since \( M(\tau, \xi) \) is invertible we can write
\[ \sum_{\tau} \sum_{\xi} A(\tau, \xi) \Lambda A(\tau, \xi) \leq C \left| |A_1| - A_2| \right| \sum_{k=1}^n |c_{k,j}(\tau, \xi)|, \]  
where in the last line of the previous inequality we used the uniform bound (28).

In view of the homogeneity of the functions \( c_{k,j}(\tau, \xi) \) it suffices to work on the compact set \{ \( (\tau, \xi) \) : \( \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2} \) \}. The entries of the inverse matrix of \( M(\tau, \xi) \) have the form
\[ c_{k,j}(\tau, \xi) = \frac{1}{\text{det}(\tau, \xi)} C_{j,k}(\tau, \xi) \]
where \( C_{j,k}(\tau, \xi) \) is the \((j, k)\)-cofactor of \( M(\tau, \xi) \). Since the entries of \( M(\tau, \xi) \) have absolute value less or equal to one, and since \( C_{j,k}(\tau, \xi) \) consists of sums of products of \( n \) such entries, we have
\[ |c_{k,j}(\tau, \xi)| \leq \frac{|C_{j,k}(\tau, \xi)|}{|\text{det}(\tau, \xi)|} \leq \frac{n}{|\text{det}(\tau, \xi)|}. \]
The quantity \( \text{ldet}(\tau, \xi) \) represents the \((n+1)\)-dimensional volume generated by the vectors \( \{(1, \omega^{(1)}(\tau, \xi)), \ldots, (1, \omega^{(n)}(\tau, \xi))\} \). Due to our choice of \( \omega^{(1)}(\tau, \xi), \ldots, \omega^{(n)}(\tau, \xi) \) this volume does not depend on the point \( (\tau, \xi) \). Moreover, \( \text{ldet}(\tau, \xi) = V \times P(\tau, \xi) \) where \( P(\tau, \xi) \) is the projection of \( (\tau, \xi) \) into the linear subspace generated by the set of vectors \( \{(1, \omega^{(1)}(\tau, \xi)), \ldots, (1, \omega^{(n)}(\tau, \xi))\} \) and \( V \) is the \( n \)-dimensional volume generated by these vectors. This projection is given by \( C \sin \varphi \) where \( \varphi \) is the angle between \( (\tau, \xi) \) and said subspace. Since the vectors \( (1, \omega^{(1)}(\tau, \xi)), 1 \leq k \leq n \), are located in the boundary of the light cone \( \{ (\tau, \xi) : |\tau| \geq \frac{|\xi|}{2} \} \), this angle is bounded below by \( \frac{\pi}{8} \).

Therefore the value \( \text{ldet}(\tau, \xi) \) is uniformly bounded from below by \( V \sin \frac{\pi}{8} \) on \( \{ (\tau, \xi) : \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2} \} \). Hence
\[ |c_{k,j}(\tau, \xi)| \leq \frac{n}{V \sin \frac{\pi}{8}}, \]
and by (32) we obtain the uniform estimate
\[ |\widehat{A}_j(\tau, \xi)| \leq C \left| |A_1| - A_2| \right| \]  
on the set \( \{ (\tau, \xi) : |\tau| \leq \frac{|\xi|}{2} \} \). □
The following statement is a result about harmonic measures, its proof can be found in [1].

**Lemma 2.6.** Consider the strip

\[ S = \{ z = z_1 + iz_2 : z_1 \in \mathbb{R}, |z_2| < 2 |\tau_0|, \tau_0 \neq 0 \} \]

and the rays

\[ p_1 = \{ z : -\infty < z_1 \leq -2 |\tau_0|, z_2 = 0 \}, \quad p_2 = \{ z : 2 |\tau_0| \leq z_1 < \infty, z_2 = 0 \} \]

in the complex plane \( \mathbb{C} \).

If \( E = p_1 \cup p_2 \) and \( G = S \setminus E \) is the strip with cuts along the rays \( p_1 \) and \( p_2 \), we have

\[ \frac{2}{3} < \varpi(z, E, G) \leq 1, \tag{33} \]

where \( \varpi(z, E, G) \) is the harmonic measure of \( E \) with respect to \( G \). More precisely

\[ \varpi(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\zeta_2}{(t - \zeta_1)^2 + \zeta_2^2} \, dt, \tag{34} \]

where \( \chi_E(t) \) is the characteristic function of the set

\[ E' = \{ t \in \mathbb{R} : |t| \leq 1 \} \cup \{ t \in \mathbb{R} : |t| > \varepsilon \} \]

We now perform a rotation in \( \xi \) space to make any given vector \( (\tau, \xi) = (\tau, \xi_1, \ldots, \xi_{n-1}, \xi_n) \) have the representation \( (\tau, 0, \ldots, 0, \nu) \). Based on the previous statements we want to `embed' the \( \nu \)-axis into a strip in the complex plane and use the bounds developed in the previous lemma.

**Lemma 2.7.** Let \( A_1, A_2 \) represent the Dirichlet to Neumann operators for the hyperbolic equations (9), and let \( \alpha \) be as in corollary 2.4. If \( \alpha < \frac{\pi}{2} \) and the divergence condition (25) holds, then on the set \( \{ (\tau, \xi) : |\tau| > \frac{|\xi|}{2} \} \) we have

\[ \left| \widehat{A}_j(\tau, \xi) \right| \leq C \frac{e^{\alpha|\xi|}}{|\tau|^\frac{1}{2}} \left\| A_1 - A_2 \right\|_F, \tag{35} \]

where \( a \) is some positive number bigger than the diameter of \( \Omega \).

**Proof.** Since the potentials \( A_j, 0 \leq j \leq n \), are compactly supported, the functions \( \widehat{A}_j(\tau_0, 0, \ldots, 0, \nu) \) admit an analytic extension in \( \nu \) into the complex plane. Letting

\[ \Pi = \{ \nu = (\nu_1, \nu_2) : \nu_1, \nu_2 \in \mathbb{R}, |\nu_2| < 2 |\tau_0|, \tau_0 \neq 0 \} ; \]

\[ q_1 = \{ \nu = (\nu_1, \nu_2) : -\infty < \nu_1 \leq -2 |\tau_0|, \nu_2 = 0 \} ; \]

\[ q_2 = \{ \nu = (\nu_1, \nu_2) : 2 |\tau_0| \leq \nu_1 < \infty, \nu_2 = 0 \} \]
and restricting the potentials to the $\nu$-axis, (33) leads to
\[
\frac{2}{3} < m(\nu, E_1, G_1) \leq 1,
\]
where $E_1 = q_1 \cup q_2$ and $G_1 = \Pi \cap E_1$. Denoting by $v_j(\nu) = \hat{A}_j(2\tau_0, 0,...,0, \nu)$, the above restriction we have by the two-constant theorem (see [9] theorem 9.4.5)
\[
\left| v_j(\nu) \right| \leq m_j^\frac{2}{3} M_j^\frac{1}{3},
\]
(36)
where $m_j$ and $M_j$ are the respective upper bounds of the modulus of $v(\nu)$ on the rays $q_1$ and $q_2$ and on the union of the lines $q_1' = \{ (\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, \nu_2 = -2|\tau_0|\pi \}$ and $q_2' = \{ (\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, \nu_2 = 2|\tau_0|\pi \}$. We point out that the rays $q_1$ and $q_2$ are contained in the set $\{ (r, \xi) : |r| \leq \frac{\xi}{2} \}$ and that (29) provides an estimate for $|v_j(\nu)|$ in that region. To compute $M_j$ we resort to the equality
\[
v_j(\nu) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(\nu' + \nu_2)x_n} W_j(2\tau_0, 0,...,0, x_n) \, dx_n,
\]
where $W_j$ is the Fourier transform of $A_j$ in all variables except $x_n$. These functions are compactly supported in $x_n$ and the above integrand is non-zero only on a bounded subset of the real numbers. Hence on $q_1' \cup q_2'$
\[
\left| v_j(\nu) \right| \leq \frac{1}{2\pi} \sup_{x_n \in (-a(\Omega), a(\Omega))} \left| W_j(2\tau_0, 0,...,0, x_n) \right| \int_{-a(\Omega)}^{a(\Omega)} e^{-|\nu'|x_n} \, dx_n,
\]
where $a$ is a positive number bigger than diam($\Omega$). Integration in $x_n$ then leads to
\[
\left| v_j(\nu) \right| \leq C e^{\frac{|\nu|}{a}} \frac{m_j^\frac{1}{3}}{\tau_0^\frac{1}{6}},
\]
where $|\nu| < q_j' \cup q_2'$ and $a = \hat{a}$. Therefore, when $\nu$ is a real number satisfying $-2|\tau_0| < \nu < 2|\tau_0|$ we have by (36)
\[
\left| v_j(\nu) \right| \leq C e^{\frac{|\nu|}{a}} \frac{m_j^\frac{1}{3}}{\tau_0^\frac{1}{6}}.
\]
The above arguments work for any line contained in the hyperplane $\nu = \tau_0$ that passes through the origin. Hence by (29), for $|\tau| > \frac{|\tau_0|}{2}$ we have
\[
\left| \hat{A}_j(\tau, \xi) \right| \leq C e^\frac{|\nu_1|}{a} \frac{||A_1 - A_2||^\frac{1}{3}}{|\tau|}.\]
\[
\square
\]
We can now establish the desired stability estimate for the vector potentials. The general idea is to use the inequality $\|f\|_{L^2} \leq C \|f\|_{L^2}$ and partition $\mathbb{R}_r \times \mathbb{R}_\xi^2$ in an appropriate way.

**Theorem 2.8.** Suppose that the vector and scalar potentials $A^{(l)} = (A_1^{(l)},...,A_n^{(l)})$, $V^{(l)}$, $l = 1, 2$, are real valued, compactly supported and $C^\infty$ in $t$ and $x$. Let $A = (A_0, A_1,...,A_n)$ where $A_j = A_j^{(1)} - A_j^{(2)}$ and suppose that the following divergence condition holds
\[
\text{div } A = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_x A_j(t, x) = 0,
\]
and that the entries of the vector potential satisfy
\[
\sup \left| \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=0}^{n} \omega_j A_j \right) (t + s, x + s \omega) \, ds \right| < 2\pi,
\]
where the supremum is taken over \((t, x; \omega) \in [T_1, T_2] \times \Omega \times S^{n-1}\).

If \(A_j\) represents the Dirichlet to Neumann operator associated to the hyperbolic problem \((1)-(4)\), then the stability estimate
\[
\max_{0 \leq j \leq n} \left| A_j^{(1)}(t, x) - A_j^{(2)}(t, x) \right| \leq C \left[ \log \frac{1}{|||A_1 - A_2|||} \right]^{-1}
\]
holds for \(A_1, A_2\) satisfying \(|||A_1 - A_2||| \ll 1\).

**Proof.** Let \(\alpha\) be as in corollary 2.4. Since \(\alpha < 2\pi\), from the Fourier inversion formula we have
\[
A_j(t, x) = \frac{1}{(2\pi)^{n+1}} \int \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(t + x \cdot \xi)} \tilde{A}_j(\tau, \xi) \, d\tau d\xi.
\]
Taking absolute values we have for \(\rho > 0\)
\[
|A_j(t, x)| \leq \frac{1}{(2\pi)^{n+1}} \int \int_{B(\rho_1)} \left| \tilde{A}_j(\tau, \xi) \right| \, d\tau d\xi
\]
\[
\leq \frac{1}{(2\pi)^{n+1}} \int \int_{B(\rho)} \left| \tilde{A}_j(\tau, \xi) \right| \, d\tau d\xi
\]
\[
+ \frac{1}{(2\pi)^{n+1}} \int \int_{B(\rho)^C} \left| \tilde{A}_j(\tau, \xi) \right| \, d\tau d\xi
\]
\[
= I_1 + I_2,
\]
where \(B(\rho)\) denotes the \((n + 1)\)-dimensional ball \(B(\rho) = \{(\tau, \xi) : |\tau|^2 + |\xi|^2 \leq \rho^2\}\). Since for \(0 \leq j \leq n\), the potentials \(A_j\), are \(C_0^\infty\) in \(t\) and \(x\), for any \(\beta > 0\), \(\rho_1 > 0\), if \(|\tau|^2 + |\xi|^2 \geq \rho_1^2\) we have
\[
|\tilde{A}_j(\tau, \xi)| \leq \frac{C}{\left( |\tau|^2 + |\xi|^2 \right)^{\beta}},
\]
where \(C\) depends on the derivatives of \(A_j(t, x)\) up to order \(\beta\). When \(\beta > n + 1\), the integral \(I_2\) converges. Moreover, when \(\beta > n + 2\) and \(\rho > 1\), the following estimate holds
\[
I_2 = \int \int_{B(\rho)} \left| \tilde{A}_j(\tau, \xi) \right| \, d\tau d\xi \leq \frac{C}{\rho^{\beta-n+1}} \leq \frac{C}{\rho}.
\]
To estimate \(I_1\) we break up the ball \(B(\rho)\) into two smaller pieces
\[
C_1 = B(\rho) \cap \left\{ (\tau, \xi) : |\tau| < \frac{|\xi|}{2} \right\} \quad \text{and} \quad C_2 = B(\rho) \cap \left\{ (\tau, \xi) : |\tau| \geq \frac{|\xi|}{2} \right\}.
\]
Then
\[ I_1 \leq \int \int \int \int C_1 \left| \Lambda_j(\tau, \xi) \right| \, d\tau \, d\xi + \int \int \int \int C_2 \left| \Lambda_j(\tau, \xi) \right| \, d\tau \, d\xi, \]
and since \( C_1 \) is a subset of \( B(\rho) \) we have
\[ I_1 \leq C_1 \rho^{n+1} \left[ \left| A_1 - A_2 \right| \right] + \int \int \int \int C_2 \left| \Lambda_j(\tau, \xi) \right| \, d\tau \, d\xi. \]

With this decomposition, \( C_2 \) is contained in the set \( \{ (\tau, \xi) : |\tau| > \frac{|\xi|}{2} \} \). Thus by (35)
\[ I_2 \leq C_2 \rho^{n+1} \left[ \left| A_1 - A_2 \right| \right] + C' \rho^{2n+2} \left[ \left| A_1 - A_2 \right| \right]^{\frac{2}{3}} \].

Equations (38)–(40) lead to
\[ \left| A_j(t, x) \right| \leq C \left[ \frac{1}{\rho} + \rho^{n+1} \left[ \left| A_1 - A_2 \right| \right] + \rho^{n+2} \epsilon^{2n+3} \left[ \left| A_1 - A_2 \right| \right]^{\frac{2}{3}} \right]. \]  

The rest of the proof is fairly standard. First we seek to impose a condition on \( \left| A_1 - A_2 \right| \) so that the third term in the right-hand side of (41) dominates the second one. This can be done by simple minimization in \( \rho \) of the function \( \sum_{j=1}^{n} \) over the interval \( [1, +\infty) \). If \( a < \frac{1}{2} \) we want \( \left| A_1 - A_2 \right| < 2a \) and if \( a \geq \frac{1}{2} \) then \( \left| A_1 - A_2 \right| < e^{2a} \). In both cases, if \( \left| A_1 - A_2 \right| \ll 1 \) then
\[ \left| A_j(t, x) \right| \leq C \left[ \frac{1}{\rho} + \rho^{n+2} \epsilon^{2n+3} \left[ \left| A_1 - A_2 \right| \right]^{\frac{2}{3}} \right]. \]

The next step is to choose \( \rho \) so that the two terms in the right-hand side of (42) are comparable. In other words we want \( \rho \) to satisfy the identity
\[ \frac{C}{\rho} = \rho^{n+2} \epsilon^{2n+3} \left[ \left| A_1 - A_2 \right| \right]^{\frac{2}{3}} \]
for some constant \( C \). Taking logarithms on both sides of the previous equation yields the following equivalent identity
\[ 2 \log \left( \frac{C}{\left| A_1 - A_2 \right|} \right) = (3n + 5) \log \rho + 2a \rho, \]
where the right-hand side of (43) is one to one when \( \rho > 0 \) and hence it admits a unique solution. On the other hand, the inequality \( \log \rho \leq \rho \) for positive \( \rho \) as well as (43) lead to
\[ 2 \log \left( \frac{C}{\left| A_1 - A_2 \right|} \right) \leq (3n + 5 + 2a) \rho, \]
or
\[ \frac{1}{\rho} \leq \frac{3n + 5 + 2a}{2} \left[ \log \left( \frac{C}{\left| A_1 - A_2 \right|} \right) \right]^{-1}, \]
and (42) becomes
\[ \left| A_j(t, x) \right| \leq C \left[ \log \left( \frac{C'}{\left| A_1 - A_2 \right|} \right) \right]^{-1} \leq C \left[ \log \left( \frac{1}{\left| A_1 - A_2 \right|} \right) \right]^{-1}, \]
where \( C \) depends on \( n, \Omega \) and derivatives of \( A_j(t, x) \) for \( 0 \leq j \leq n \).
3. Stability of the scalar potentials

In this section we establish a log-log type estimate for the scalar potentials. We point out that the estimate from theorem 2.8 is independent of the scalar potentials. This is because the term involving the difference of said potentials is not the leading term in the asymptotics (18) and it does not survive the process of dividing by \( k \) and taking the limit as \( k \to +\infty \). In the following lines, we reuse the techniques developed in the previous sections while following closely the ideas from Isakov and Sun in [7].

**Theorem 3.1.** Suppose that the vector and scalar potentials \( \mathcal{A}^{(l)} = (A_0^{(l)}, ..., A_n^{(l)}) \), \( l = 1, 2 \), are real valued, compactly supported and \( C^\infty \) in \( t \) and \( x \). Let \( V = V^{(1)} - V^{(2)} \), \( \mathcal{A} = (A_0, A_1, ..., A_n) \), where \( A_j = A_j^{(1)} - A_j^{(2)} \) and suppose that the following divergence condition holds

\[
\text{div} \, \mathcal{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_x A_j(t, x) = 0,
\]

and that the entries of the vector potential satisfy

\[
\sup \left| \int_{-\infty}^\infty (A_0 + \sum_{j=0}^n \omega_j A_j)(t + s, x + s\omega)ds \right| < 2\pi,
\]

where the supremum is taken over \((t, x; \omega) \in [T_1, T_2] \times \Omega \times S^{n-1}\).

If \( \Lambda \) represents the Dirichlet to Neumann operator associated to the hyperbolic problem (1)-(4), then for \( \Lambda_1, \Lambda_2 \) satisfying \( |||\Lambda_1 - \Lambda_2||| \ll 1 \), the following stability estimates hold

\[
||| \mathcal{A} \|||_0 \leq C \left( \log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right)^{-1},
\]

\[
|| V ||_{L^\infty(R_1 \times R^n)} \leq C \left( \log \left( \log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right) \right)^{-1},
\]

where

\[
||| \mathcal{A} \|||_0 = |||\mathcal{A}^{(1)} - \mathcal{A}^{(2)}|||_0 := \max_{0 \leq j \leq n} || A_j^{(1)}(t, x) - A_j^{(2)}(t, x) ||_{L^\infty(R_1 \times R^n)}.
\]

**Proof.** In view of our previous results, it is enough to obtain a uniform estimate for the x-ray transform along light rays of the difference of the scalar potentials. By theorem 2.8, for arbitrary smooth compactly supported scalar potentials \( V^{(1)} \neq V^{(2)} \), we have

\[
||| \mathcal{A} \|||_0 \leq C \left( \log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right)^{-1}.
\]

Green’s formula (8) with \( u \) and \( v \) solutions of the forward and backward hyperbolic problem respectively, as well as the triangle inequality give
When \( u \) and \( v \) are given by the GO ansatz developed in section 2, the discussion of the asymptotics of the derivatives \( \partial_{\alpha\beta} u, \partial_{\alpha\beta} v, 0 \leq j \leq n \), give the estimates

\[
\left\| \left( A_{j} - A_{2} \right) u, v \right\|_{[\Omega]} \leq C k \left\| A_{1} - A_{2} \right\| + O \left(k^{-1}\right),
\]

\[
\left\| \left( A_{j}^{(1)} - A_{j}^{(2)} \right) u, \left( -i \partial_{\alpha\beta} v \right) \right\|_{[\Omega]} \leq C k \left\| A \right\|_{0} + O \left(k^{-1}\right),
\]

\[
\left\| \left( A_{j}^{(1)} - A_{j}^{(2)} \right) \left( -i \partial_{\alpha\beta} u \right), v \right\|_{[\Omega]} \leq C k \left\| A \right\|_{0} + O \left(k^{-1}\right),
\]

\[
\left\| \left( A_{j}^{(2)} \right)^{2} - \left( A_{j}^{(1)} \right)^{2} \right\|_{[\Omega]} \leq C \left\| A \right\|_{0},
\]

where the last inequality follows from the fact that \( |A_{j}^{(1)}(t, x) - A_{j}^{(2)}(t, x)| \leq C \) for all \( (t, x) \in \mathbb{R} \times \mathbb{R}_{x}^{n} \).

On the other hand, the l.h.s of (45) gives

\[
\int_{\Omega} \int_{\Omega} V(t, x)u(t, x)v(t, x) \, dx \, dr = \int_{\Omega} \int_{\Omega} V(t, x)\chi_{j}(t, x)\chi_{2}(t, x) \, dx \, dr + \cdots
\]

where \((t', x')\) is the projection of \((t, x)\) onto \( \Pi_{(1, \omega)} \) and \( \cdots \) represents terms of order \( O(k^{-1}) \).

Also, a simple analysis of \( e^{ix} \) for small \(|x|\) gives

\[
e^{-i \int_{-\omega}^{\omega} \left( A_{0} + \sum_{j=1}^{n} a_{j} \right) (t + r, x + \lambda) dr} = 1 + O\left(\left\| A \right\|_{0}\right)
\]
and thus (45) leads to
\[
\left| \int_{I_2} \int_{\Omega} V(t, x) \chi_1(t, x) \chi_2(t, x) \, dx \, dt \right| \\
\leq C_1 k \left( \| A \|_0 + \| A_1 - A_2 \| \right) + C_2 \| A \|_0 + \mathcal{O} \left( k^{-1} \right)
\]
As in previous cases, the fact that the functions $\chi_j$ are supported near light rays shows that for $k > 0$ the following estimate holds
\[
\left| \int_{-\infty}^{\infty} V(t + s, x + s \omega) \, ds \right| \\
\leq C_1 k \left( \| A \|_0 + \| A_1 - A_2 \| \right) + C_2 \| A \|_0 + \frac{C_3}{k}.
\] (47)

Next we choose $k$ so that the first and last terms in the previous equation are comparable in size. To this end, let
\[ k = \left( \| A_1 - A_2 \| + \| A \|_0 \right)^{-\frac{1}{2}}, \]
then
\[ \left( \| A_1 - A_2 \| + \| A \|_0 \right) k = \frac{1}{k} \leq C \left( \| A_1 - A_2 \| + \| A \|_0 \right)^{\frac{1}{2}}, \]
and (47) gives
\[
\left| \int_{-\infty}^{\infty} V(t + s, x + s \omega) \, ds \right| \\
\leq C_1 \left( \| A_1 - A_2 \| + \| A \|_0 \right)^{\frac{1}{2}} + C_2 \| A \|_0
\]
\[
\leq C \left( \| A \|_0^{\frac{1}{2}} + \| A_1 - A_2 \|^{\frac{1}{2}} \right),
\]
where the last inequality holds when both $\| A_1 - A_2 \|, \| A \|_0 < 1$ (recall that if $0 < \epsilon < 1$, then $\epsilon < \sqrt{\epsilon} < 1$). Estimate (44) then gives
\[
\left| \int_{-\infty}^{\infty} V(t + s, x + s \omega) \, ds \right| \leq C \left( \log \frac{1}{\| A_1 - A_2 \|} \right)^{\frac{1}{2}} + \| A_1 - A_2 \|^{\frac{1}{2}}
\]
\[
\leq C \left( \log \frac{1}{\| A_1 - A_2 \|} \right)^{\frac{1}{2}}.
\] (48)

As in section 2 we get from (48)
\[
\| V \|_{\mathcal{L}^{1}(\mathbb{R}, \mathcal{M}^\omega_x)} \leq C \left( \log \frac{1}{\| A_1 - A_2 \|} \right)^{-1}
\]
\[
\leq C \left( \log \log \frac{1}{\| A_1 - A_2 \|} \right)^{-1/2}.
\] □
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