A note on $L(1)$ of Hecke $L$–series associated to the elliptic curves with CM by $\sqrt{-3}$

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Abstract Consider elliptic curves $E : y^2 = x^3 + D^3$ defined over the quadratic field $\mathbb{Q}(\sqrt{-3})$. Hecke $L$–series attached to $E$ are studied, formulae for their values at $s = 1$, and bound of $3$-adic valuations of these values are given. These results are complementary to those in [Q] and [QZ], and are consistent with the predictions of the conjecture of Birch and Swinnerton-Dyer.

Keywords: Elliptic curve, $L$-function, complex multiplication, Birch and Swinnerton-Dyer conjecture.

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1. Introduction and statement of main results

This note is a complement of [Q] and [QZ]. Let $\tau = (-1 + \sqrt{-3})/2$ be a primitive cubic root of unity and $O_K = \mathbb{Z}[\tau]$ the ring of integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-3})$. In this note, we consider the elliptic curves

$$E = E_{D^3} : y^2 = x^3 + D^3, \text{ with } D = \pi_1 \cdots \pi_n, \quad (1.1)$$

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where \( \pi_k \equiv 1 \pmod{12} \) \( (k = 1, \cdots, n) \) are distinct prime elements in \( O_K \). Obviously, \( E \) has complex multiplication by \( O_K \). Let \( S = \{ \pi_1 \cdots \pi_n \} \). For any subset \( T \) of \( \{1, \cdots, n\} \), denote \( D_T = \prod_{k \in T} \pi_k \), \( \hat{D}_T = D/D_T \) and put \( D_\emptyset = 1 \) when \( T = \emptyset \) (empty set). Let \( \psi_{D_T^3} \) be the Hecke character (i.e., Gr"ossencharacter) of \( K \) attached to the elliptic curve \( E_{D_T^3} : y^2 = x^3 + D_T^3 \), and let \( L_S(\overline{\psi}_{D_T^3}, s) \) be the Hecke \( L \)-series of \( \overline{\psi}_{D_T^3} \) (the complex conjugate of \( \psi_{D_T^3} \)) with the Euler factors omitted at all primes in \( S \) (for the definition of such Hecke \( L \)-series attached to an elliptic curve, see [Sil2]). We have the following result about the special value of \( L_S(\overline{\psi}_{D_T^3}, s) \) at \( s = 1 \).

**Theorem 1.1** Let \( D = \pi_1 \cdots \pi_n \), where \( \pi_k \equiv 1 \pmod{12} \) \( (k = 1, \cdots, n) \). Then, for any factor \( D_T \) of \( D \) and the corresponding Hecke character \( \psi_{D_T^3} \), we have

\[
- \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi}_{D_T^3}, 1) = \frac{\sqrt{3}}{4} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{1}{\wp(\sqrt{-3} c \omega_D)} + \frac{\sqrt{3}}{2} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2,
\]

where \( (\cdot)_2 \) is the quadratic residue symbol in \( K \), \( C \) is any complete set of representatives of the relatively prime residue classes of \( O_K \) modulo \( D \), \( \wp(z) \) is the Weierstrass \( \wp \)-function satisfying \( \wp'(z)^2 = 4\wp(z)^3 - 1 \) with period lattice \( L_\omega = \omega O_K \) (corresponding to the elliptic curve \( y^2 = x^3 - \frac{1}{4} \)) and \( \omega = 3.059908 \cdots \) is an absolute constant.

There is much literature studying the special values \( L(1) \) associated to the CM elliptic curves (see e.g., [BSD], [Z1 \sim 3], [Q], [QZ]). In [Q], a similar result of \( L(1) \) was obtained for the Hecke character attached to some special elliptic curves \( y^2 = x^3 - 2^4 3^3 D^3 \). Now for the elliptic curves (1.1) above, to obtain an explicit formula of
L(1) need to overcome more difficulties, especially in calculating the key values of Weierstrass zeta function \( \zeta(z, L_\omega) \), Weierstrass \( \wp \)-function \( \wp(z) (= \wp(z, L_\omega)) \) and its derivative \( \wp'(z) \) (see the proof of Theorem 1.1 in the following).

Let \( \mathbb{Q}_2 \) be the completion of \( \mathbb{Q} \) at the \( 2 \)-adic valuation, \( \overline{\mathbb{Q}} \) and \( \overline{\mathbb{Q}}_2 \) be the algebraic closures of \( \mathbb{Q} \) and \( \mathbb{Q}_p \) respectively, and let \( v_2 \) be the normalized \( 2 \)-adic additive valuation of \( \overline{\mathbb{Q}}_2 \) (i.e., \( v_2(2) = 1 \)). Fix an isomorphic embedding \( \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_2 \). Then, via this embedding, \( v_2(\alpha) \) is defined for any algebraic number \( \alpha \) in \( \overline{\mathbb{Q}} \). The value \( v_2(\alpha) \) for \( \alpha \in \overline{\mathbb{Q}} \) depends on the choice of the embedding \( \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_2 \), but this does not affect our discussion in this paper.

By Corollary 22 of [CW], we know that \( L(\psi_{D^3}, 1)/\omega \) is an algebraic number, i.e., \( L(\psi_{D^3}, 1)/\omega \in \overline{\mathbb{Q}} \). For its \( 2 \)-adic valuation, we have

**Theorem 1.2.** Let \( D = \pi_1 \cdots \pi_n \), where \( \pi_k \equiv 1 \pmod{12} \) are distinct prime elements of \( \mathbb{Z}[\tau] \) \((k = 1, \cdots, n)\), and let \( \psi_{D^3} \) be the Hecke character of \( \mathbb{Q}(\sqrt{-3}) \) attached to the elliptic curve \( E_{D^3} : y^2 = x^3 + D^3 \). Then, for the \( 2 \)-adic valuation of \( L(\psi_{D^3}, 1)/\omega \) we have

\[
v_2 \left( L(\psi_{D^3}, 1)/\omega \right) \geq n - 1.
\]

**2. Proofs of Theorems**

**Proof of Theorem 1.1.** For the elliptic curve \( E_{D^3_T} : y^2 = x^3 + D_{T_1}^3 \), it has complex multiplication by \( O_K \). Since the class number of \( K \) is 1, the period lattice of \( E_{D^3_T} \) should be \( L_T = \omega_T O_K \) for some \( \omega_T \in \mathbb{C}^\times \). Let \( \omega_T = \alpha_T \omega \), \( \alpha_T \in \mathbb{C}^\times \). By Tate's algorithm [T], it is easy to show that the conductor of \( E_{D^3_T} \) is \( N_{E_{D^3_T}} = 12D_{T_1}^2 \), and the conductor of \( \psi_{D^3_T} \) is \( f_{\psi_{D^3_T}} = (2\sqrt{-3}D_T) \). In Prop.A of [QZ] (for a general form,
see Prop. 5.5 in [GS]), putting \( k = 1, \mathfrak{h} = O_K, \mathfrak{g} = (2\sqrt{-3}D), \rho = \frac{\omega_T}{2\sqrt{-3}D}, \phi = \psi_D^{\mathfrak{h}}, \)

then the ray class field of \( K \) modulo \( \mathfrak{g} \) is \( K((E_D^{\mathfrak{h}})_\mathfrak{g}) \) (see the Lemma 4.7 in [GS]), and then

\[
\frac{\bar{\rho}}{|\rho|^2} L_\mathfrak{g} (\overline{\psi}_D^{3\mathfrak{h}}, s) = \sum_{b \in \mathcal{B}} H_1 \left( \frac{\psi_D^{3\mathfrak{h}}(b)\omega_T}{2\sqrt{-3}D}, 0, s, L_T \right) \quad (\text{Re}(s) > 3/2)
\]

with \( \mathcal{B} = \{(6c + D) : c \in \mathcal{C} \} \), such that \( \{\sigma_b : b \in \mathcal{B}\} = \text{Gal} \left( K((E_D^{3\mathfrak{h}})_\mathfrak{g})/K \right) \cong (O_K/(2\sqrt{-3}D))^{\times}/O_K^{\times} \) (via Artin map), where \( \mathcal{C} \) is as in Theorem 1.1, a set of representatives of \( (O_K/(D))^{\times} \). Then

\[
\frac{\bar{\rho}}{|\rho|^2} L_\mathfrak{g} (\overline{\psi}_D^{3\mathfrak{h}}, s) = \sum_{c \in \mathcal{C}} E^*_1 \left( \frac{\psi_D^{3\mathfrak{h}}(6c + D)\omega_T}{2\sqrt{-3}D}, 0, s, \omega_T O_K \right) \quad (\text{Re}(s) > 3/2).
\]

Note that \( H_1 (z, 0, 1, L) \) could be analytically continued by the Eisenstein \( E^*_1 \)-function

(see [W]): \( H_1 (z, 0, 1, L) = E^*_1 (z, L) = E^*_1 (z, L) \). Hence we get

\[
\frac{2\sqrt{-3}D}{\alpha T \omega} L_\mathfrak{g} (\overline{\psi}_D^{3\mathfrak{h}}, 1) = \sum_{c \in \mathcal{C}} E^*_1 \left( \frac{\psi_D^{3\mathfrak{h}}(6c + D)\omega_T}{2\sqrt{-3}D}, \alpha T \omega O_K \right) \quad (2.1)
\]

Since \( D \equiv 1 \pmod{12} \), we have \( 6c + D \equiv 1 \pmod{6} \) for any \( c \in \mathcal{C} \). In particular,

\[
\left( \frac{2}{6c + D} \right)_3 = 1 \quad \text{(see [IR], P.119).}
\]

So by definition (see[Sil2], p.178),

\[
\psi_D^{3\mathfrak{h}}(6c + D) = \frac{4D_T^3}{6c + D} (6c + D) = \left( \frac{D_T}{6c + D} \right)^2 (6c + D).
\]

Moreover, by the quadratic reciprocity law in \( K \) (see [Le], pp.256~260), we have

\[
\left( \frac{D_T}{6c + D} \right)_2 = \left( \frac{6c + D}{D_T} \right)_2 = \left( \frac{6c}{D_T} \right)_2 = \left( \frac{-2 \cdot (\sqrt{-3})^2 c}{D_T} \right)_2 = \left( \frac{-2c}{D_T} \right)_2 = \left( \frac{2c}{D_T} \right)_2,
\]

the last equality holds because \( \left( \frac{1}{D_T} \right)_2 = 1 \) (see [Le], p.111). Therefore, by (2.1) above, and note that \( L_\mathfrak{g} (\overline{\psi}_D^{3\mathfrak{h}}, 1) = L_S (\overline{\psi}_D^{3\mathfrak{h}}, 1) \), we obtain

\[
\frac{2\sqrt{-3}D}{\alpha T \omega} L_S (\overline{\psi}_D^{3\mathfrak{h}}, 1) = \sum_{c \in \mathcal{C}} E^*_1 \left( \left( \frac{-\sqrt{-3}c \omega}{D} - \sqrt{-3} \omega \right) \alpha T \left( \frac{2c}{D_T} \right)_2, \alpha T \omega O_K \right).
\]

(2.2)
Let $\lambda = -\alpha_T \left( \frac{2c}{D_T} \right)^2$, then $\alpha_T \omega O_K = \lambda \omega O_K = \lambda L_\omega$. By formula $E_1^*(\lambda z, \lambda L) = \lambda^{-1} E_1^*(z, L)$, we obtain

$$E_1^* \left( \left( -\frac{\sqrt{-3} c \omega}{D} - \frac{\sqrt{-3} \omega}{6} \right), \alpha_T \left( \frac{2c}{D_T} \right)_2, \alpha_T \omega O_K \right) = E_1^* \left( \left( \frac{\sqrt{-3} c \omega}{D} + \frac{\sqrt{-3} \omega}{6} \right) \lambda, \lambda \omega \right) = -\alpha_T^{-1} \left( \frac{2c}{D_T} \right)_2 E_1^* \left( \frac{\sqrt{-3} c \omega}{D} + \frac{\sqrt{-3} \omega}{6}, L_\omega \right).$$

So by (2.2) above, we get

$$-\frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi D}_2^3, 1) = \frac{1}{2\sqrt{-3}} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_2 E_1^* \left( \frac{\sqrt{-3} c \omega}{D} + \frac{\sqrt{-3} \omega}{6}, L_\omega \right). \tag{2.3}$$

By [QZ], it is easy to see that

$$E_1^*(z, L_\omega) = \zeta(z, L_\omega) - \frac{2\pi \varpi}{\sqrt{3} \omega^2}, \tag{2.4}$$

where $\zeta(z, L) = \frac{1}{z} + \sum_{\alpha \in L - \{0\}} \left( \frac{1}{z - \alpha} + \frac{1}{\overline{\alpha} - \overline{z}} \right)$ is the Weierstrass Zeta-function, an odd function, i.e., $\zeta(-z, L) = -\zeta(z, L)$ (see [Sil 2]). By the addition formula (see [Law])

$$\zeta(z_1 + z_2, L_\omega) = \zeta(z_1, L_\omega) + \zeta(z_2, L_\omega) + \frac{1}{2} \varphi' \left( \frac{z_1}{\varphi(z_1) - \varphi(z_2)} \right), \quad \text{we obtain}$$

$$\zeta \left( \frac{\sqrt{-3} c \omega}{D} + \frac{\sqrt{-3} \omega}{6}, L_\omega \right)$$

$$= \zeta \left( \frac{\sqrt{-3} c \omega}{D}, L_\omega \right) + \zeta \left( \frac{\sqrt{-3} \omega}{6}, L_\omega \right) + \frac{1}{2} \varphi' \left( \frac{\sqrt{-3} c \omega}{D} \right) \varphi' \left( \frac{\sqrt{-3} \omega}{6} \right). \tag{2.5}$$

Now we compute the values of $\zeta \left( \frac{\sqrt{-3} \omega}{6}, L_\omega \right), \varphi \left( \frac{\sqrt{-3} \omega}{6} \right)$ and $\varphi' \left( \frac{\sqrt{-3} \omega}{6} \right)$. Note that $\sqrt{-3} = 1 + 2\tau$, $\tau O_K = O_K$, $\tau L_\omega = L_\omega \varphi(\tau z, L_\omega) = \tau \varphi(z, L_\omega), \varphi'(\tau z, L_\omega) = \varphi'(z, L_\omega)$ and $\zeta(\tau z, L_\omega) = \tau^2 \zeta(z, L_\omega)$ (see [La], p.16, p.240). Also by [St] and [QZ], we know that

$$\varphi \left( \frac{\omega}{3}, L_\omega \right) = 1, \quad \varphi' \left( \frac{\omega}{3}, L_\omega \right) = -\sqrt{3}, \quad \varphi'' \left( \frac{\omega}{3}, L_\omega \right) = 6, \quad \zeta \left( \frac{\omega}{2}, L_\omega \right) = \frac{\pi}{\sqrt{3} \omega},$$

$$\zeta \left( \frac{\omega}{3}, L_\omega \right) = \frac{2\pi \sqrt{3}}{3\sqrt{3}}, \quad \zeta \left( \frac{2\omega}{3}, L_\omega \right) = \frac{4\pi}{3\sqrt{3}} - \frac{1}{\sqrt{3}}. \tag{2.6}$$
For $O_K = \mathbb{Z}[\tau]$, it is easy to see that the Eisenstein series $G_{2k}(O_K)$ is a real number for each positive integer $k \geq 2$. So by the Laurent series expansion $\varphi(z, O_K) = z^{-2} + \sum_{k=1}^{\infty} (2k + 1) G_{2k+2}(O_K) z^{2k}$ (see [Sil1], p.169), it is easy to see that $\varphi(\frac{1}{2}, O_K) \in \mathbb{R}$, a real number, so $\varphi(\frac{\omega}{2}, L_\omega) = \omega^{-2} \varphi(\frac{1}{2}, O_K) \in \mathbb{R}$. Then, since $(\varphi(\frac{\omega}{2}, L_\omega), \frac{1}{2} \varphi'(\frac{\omega}{2}, L_\omega))$ is a point of order 2 of the elliptic curve $y^2 = x^3 - \frac{1}{4}$ mentioned above, one can easily obtain that

$$\varphi'(\frac{\omega}{2}, L_\omega) = 0, \quad \varphi\left(\frac{2\omega}{3}, L_\omega\right) = 1, \quad \varphi'\left(\frac{2\omega}{3}, L_\omega\right) = \sqrt{3}, \quad \varphi\left(\frac{\omega}{2}, L_\omega\right) = \frac{3\sqrt{2}}{2}. \quad (2.7)$$

So by the addition formula of $\zeta(z, L_\omega)$ above, we get

$$\zeta\left(\frac{5\omega}{6}, L_\omega\right) = \zeta\left(\frac{\omega}{2} + \frac{\omega}{3}, L_\omega\right) = \zeta\left(\frac{\omega}{2}, L_\omega\right) + \zeta\left(\frac{\omega}{3}, L_\omega\right) + \frac{1}{2} \cdot \frac{\varphi'(\frac{\omega}{2}) - \varphi'(\frac{\omega}{3})}{\varphi(\frac{\omega}{2}) - \varphi(\frac{\omega}{3})}$$

$$= \frac{5\pi}{3\sqrt{3}\omega} + \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{2} - 2}. \quad (2.8)$$

Moreover, for any $\alpha \in L_\omega$, we have

$$\zeta(z + \alpha, L_\omega) - \zeta(z, L_\omega) = \eta(\alpha, L_\omega) = \alpha s_2(L_\omega) + \pi A(L_\omega)^{-1} = \frac{2\pi \alpha}{\sqrt{3}\omega^2}$$

because $s_2(L_\omega) = \frac{2\pi}{\omega} \zeta(\frac{\omega}{2}, L_\omega) - \frac{2\pi}{\sqrt{3}\omega^2} = 0$ and $A(L_\omega) = \frac{\sqrt{3} \omega^2}{2\pi}$ (see [QZ]). Putting $z = -\frac{\omega}{6}$ and $\alpha = \omega$, then we obtain

$$\zeta\left(\frac{5\omega}{6}, L_\omega\right) + \zeta\left(\frac{\omega}{6}, L_\omega\right) = \frac{2\pi}{\sqrt{3}\omega}. \quad \text{So by (2.8), we get}$$

$$\zeta\left(\frac{\omega}{6}, L_\omega\right) = \frac{\pi}{3\sqrt{3}\omega} - \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{\sqrt{2} - 2}. \quad (2.9)$$

Also by taking $u = \frac{2\omega}{3}$ and $v = \frac{\omega}{6}$ in the formula (see [Law], p.161)

$$\zeta(u + v, L_\omega) + \zeta(u - v, L_\omega) - 2\zeta(u, L_\omega) = \frac{\varphi'(u)}{\varphi(u) - \varphi(v)}, \quad (2.10)$$

we get

$$\zeta\left(\frac{2\omega}{3} + \frac{\omega}{6}, L_\omega\right) + \zeta\left(\frac{2\omega}{3} - \frac{\omega}{6}, L_\omega\right) - 2\zeta\left(\frac{2\omega}{3}, L_\omega\right) = \frac{\varphi'(\frac{2\omega}{3})}{\varphi(\frac{2\omega}{3}) - \varphi(\frac{\omega}{6})},$$

which implies

$$\varphi\left(\frac{\omega}{6}\right) = 1 + \sqrt{2} + \sqrt{4.} \quad (2.11)$$
Then by taking \( u = \frac{\omega}{6} \) and \( v = \frac{\tau \omega}{3} \) in the formula (2.10) above, we get

\[
\wp' \left( \frac{\omega}{6} \right) = -\sqrt{3} \left( 3 + 2 \cdot \sqrt{2} + 2 \cdot \sqrt{4} \right). \tag{2.12}
\]

Now, by substituting these values into the addition formula of \( \wp(z) \) (see [Law], p.162), we have

\[
\wp \left( \sqrt{-\frac{3\omega}{6}} \right) = \wp \left( \frac{\omega}{6} + \frac{\tau \omega}{3} \right) = \frac{1}{4} \left( \frac{\wp' \left( \frac{\omega}{6} \right) - \wp' \left( \frac{\tau \omega}{3} \right)}{\wp \left( \frac{\omega}{6} \right) - \wp \left( \frac{\tau \omega}{3} \right)} \right)^2 - \wp \left( \frac{\omega}{6} \right) - \tau \wp \left( \frac{\omega}{3} \right) = -\sqrt{3},
\]

that is \( \wp \left( \sqrt{-\frac{3\omega}{6}} \right) = -\sqrt{3} \). \tag{2.13}

Next, by putting \( u = \frac{\omega}{6} \) and \( v = \frac{\tau \omega}{3} \) into the following formula (see [Law], p.183, Exer. 15)

\[
\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = \frac{\wp'(v) + \wp'(u + v)}{\wp(v) - \wp(u + v)}, \quad \text{we obtain} \quad \wp' \left( \sqrt{-\frac{3\omega}{6}} \right) = -3\sqrt{3} \cdot \sqrt{1}. \tag{2.14}
\]

Again by the addition formula of \( \zeta(z, L_\omega) \) above, we get

\[
\zeta \left( \sqrt{-\frac{3\omega}{6}}, L_\omega \right) = \zeta \left( \frac{\omega}{6} + \frac{\tau \omega}{3}, L_\omega \right) = \zeta \left( \frac{\omega}{6}, L_\omega \right) + \zeta \left( \frac{\tau \omega}{3}, L_\omega \right) + \frac{1}{2} \cdot \frac{\wp' \left( \frac{\omega}{6} \right) - \wp' \left( \frac{\tau \omega}{3} \right)}{\wp \left( \frac{\omega}{6} \right) - \wp \left( \frac{\tau \omega}{3} \right)}
\]

\[
= -\frac{\pi \cdot \sqrt{-1}}{3\omega} - \frac{\sqrt{-1}}{2} \cdot \sqrt{4}, \quad \text{and then by (2.5) we obtain}
\]

\[
\zeta \left( \sqrt{-\frac{3\omega}{6}}, L_\omega \right) = \zeta \left( \frac{\sqrt{-3\omega}}{D}, L_\omega \right) - \frac{\pi \cdot \sqrt{-1}}{3\omega} - \frac{\sqrt{-1}}{2} \cdot \sqrt{4} + \frac{1}{2} \cdot \frac{\wp' \left( \sqrt{-\frac{3\omega}{6}} \right)}{\wp \left( \sqrt{-\frac{3\omega}{6}} \right)} + 3\sqrt{-1} \tag{2.16}
\]
Substituting it into (2.4) and (2.3), we obtain
\[- \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\bar{\psi}_{D_T^1}, 1) = \frac{\sqrt{3}}{4} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{1}{\varphi \left( \frac{\sqrt{-3c\omega}}{D} \right)} + \frac{3}{\sqrt{2}} - \frac{\sqrt{4}}{4\sqrt{3}} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{1}{\varphi \left( \frac{\sqrt{-3c\omega}}{D} \right)} + \frac{3}{\sqrt{2}}.
\]

\[+ \frac{1}{2\sqrt{-3}} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \left( \zeta \left( \frac{\sqrt{-3c\omega}}{D}, L_\omega \right) + \frac{1}{2} \cdot \frac{\varphi' \left( \frac{\sqrt{-3c\omega}}{D} \right)}{\varphi \left( \frac{\sqrt{-3c\omega}}{D} \right)} + \frac{3}{\sqrt{2}} + \frac{2\pi \sqrt{-1}}{\omega} \cdot \frac{c}{D} \right).
\]

Since \( D = \pi_1 \cdots \pi_n \) with \( \pi_k \equiv 1 \pmod{12} \), so we may choose the set \( C \) in such a way that \( -c \in C \) when \( c \in C \). Obviously \((-c/D_T)_2 = (c/D_T)_2\). Also since \( \zeta(z, L_\omega) \) and \( \varphi'(z, L_\omega) \) are odd functions, and \( \varphi(z, L_\omega) \) is an even function, so
\[\sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \zeta \left( \frac{\sqrt{-3c\omega}}{D}, L_\omega \right) = \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{\varphi' \left( \frac{\sqrt{-3c\omega}}{D} \right)}{\varphi \left( \frac{\sqrt{-3c\omega}}{D} \right)} + \frac{3}{\sqrt{2}} = \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{c}{D} = 0.
\]

Therefore
\[- \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\bar{\psi}_{D_T^1}, 1) = \frac{\sqrt{3}}{4} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{1}{\varphi \left( \frac{\sqrt{-3c\omega}}{D} \right)} + \frac{3}{\sqrt{2}} - \frac{\sqrt{4}}{4\sqrt{3}} \sum_{c \in C} \left( \frac{c}{D_T} \right)_2 \frac{1}{\varphi \left( \frac{\sqrt{-3c\omega}}{D} \right)} + \frac{3}{\sqrt{2}}.
\]

This proves Theorem 1.1. \( \square \)

**Remark 2.1.** (1) It follows from the above proof that Theorem 1.1 holds for all \( D = \pi_1 \cdots \pi_n \) with \( \pi_k \equiv 1 \pmod{4\sqrt{-3}} \).

(2) In particular, by taking \( D = 1 \) in the formula of Theorem 1.1, it is easy to see that \( L(E_1/Q, 1) = L(\bar{\psi}_1, 1) = \frac{3\sqrt{3}}{4\sqrt{3}} \cdot \omega \) for the elliptic curve \( E_1 : y^2 = x^3 + 1 \).

**Lemma 2.2.** For the Weierstrass \( \varphi \)-function \( \varphi(z, L_\omega) \) in Theorem 1.1 and any \( c \in C \), we have
\[v_2 \left( \varphi \left( \frac{\sqrt{-3c\omega}}{D}, L_\omega \right) + \sqrt{2} \right) = 0.
\]

**Proof.** Taking \( r = 1, \gamma = 1, \Delta = D, \beta = \sqrt{-3c} \) and \( \lambda = \frac{1}{2}(1 - 3^{1-r}) = 0 \) in the lemmas 2 and 1 in [St], by the above (2.14), it then follows that
\[v_2 \left( \varphi \left( \frac{\sqrt{-3c\omega}}{D} \right) \right) = 0, \text{ so } v_2 \left( \varphi \left( \frac{\sqrt{-3c\omega}}{D} \right) + \sqrt{2} \right) = 0.
\]
The proof is completed. □

**Proof of Theorem 1.2.** Add up the two sides of the formula in Theorem 1.1 over all subsets $T$ of $\{1, \ldots, n\}$, we obtain

$$-\sum_T \frac{D}{\omega} \left( \frac{2}{D_T} \right)^2 L_S(\overline{\psi}_{D_T^2}, 1) = \frac{\sqrt{3}}{4} \sum_{c \in \mathcal{C}} \frac{1}{\psi(\sqrt{3/\omega} D_T)} + \sqrt{2} \sum_T \left( \frac{c}{D_T} \right)^2 - \frac{\sqrt{7}}{4\sqrt{3}} \# \mathcal{C}. \tag{2.17}$$

By assumption,

$$v_2 \left( \frac{\sqrt{4}}{4\sqrt{3}} \cdot \# \mathcal{C} \right) = v_2 \left( \frac{\sqrt{4}}{4\sqrt{3}} \cdot \prod_{k=1}^n (\pi_k \pi_k - 1) \right) \geq \frac{2}{3} - 2 + 2n = 2n - \frac{4}{3}. \quad (n \geq 1)$$

Note that by our choice $-c \in \mathcal{C}$ when $c \in \mathcal{C}$, and $\left( \frac{-c}{D_T} \right)^2 = \left( \frac{c}{D_T} \right)^2$, so by Lemma 2.2 we know that the first term in the right side of (2.17) has $2$–adic valuation $\geq -2 + 1 + n = n - 1$. Therefore

$$v_2 \left( \sum_T \frac{D}{\omega} \left( \frac{2}{D_T} \right)^2 L_S(\overline{\psi}_{D_T^2}, 1) \right) \geq n - 1. \tag{2.18}$$

By definition, we know that, if $T = \{1, \ldots, n\}$, then $L_S(\overline{\psi}_{D_T^2}, 1) = L(\overline{\psi}_{D^2}, 1)$; and if $T = \emptyset$, then $L_S(\overline{\psi}_{D_T^2}, 1) = L(\overline{\psi}, 1) = L(\overline{\psi}, 1) \prod_{k=1}^n \left( 1 - \frac{1}{\pi_k} \right) = \frac{\sqrt{4}}{4\sqrt{3}} \cdot \omega \prod_{k=1}^n \left( 1 - \frac{1}{\pi_k} \right)$ (see the above Remark 2.1.(2)). So we have

$$v_2 \left( L_S(\overline{\psi}, 1)/\omega \right) \geq -\frac{4}{3} + 2n \geq n - 1 \quad (\text{Since } v_2(\pi_{k-1} - 1) \geq 2). \tag{2.19}$$

Now we use induction method on $n$ to prove $v_2 \left( L(\overline{\psi}_{D^2}, 1)/\omega \right) \geq n - 1$. When $n = 1$, $D = \pi_1$, $v_2 \left( L(\overline{\psi}, 1)/\omega \right) \geq -\frac{4}{3} + 2 = \frac{2}{3}$. Also by taking $n = 1$ in (2.18),

$$v_2 \left( \frac{\pi_1}{\omega} \left( \frac{2}{D_0} \right)^2 L(\overline{\psi}, 1) + \frac{\pi_1}{\omega} \left( \frac{2}{\pi_1} \right)^2 L(\overline{\psi}, 1) \right) \geq 1 - 1 = 0.$$

So $v_2 \left( L(\overline{\psi}, 1)/\omega \right) = v_2 \left( \frac{\pi_1}{\omega} \left( \frac{2}{\pi_1} \right)^2 L(\overline{\psi}, 1) \right) \geq 0$. Assume our conclusion is true for $1, 2, \ldots, n-1$, and consider the case $n$, $D = \pi_1 \cdots \pi_n$. For any non-trivial subset
$T$ of $\{1, \cdots, n\}$, denote $t = t(T) = \sharp T$, by definition, we have

$$v_2 \left( \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi_{D_T^3}}, 1) \right) = v_2 \left( \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L(\overline{\psi_{D_T^3}}, 1) \prod_{\pi_k|\hat{D}_T} \left( 1 - \left( \frac{D_T}{\pi_k} \right)^2 \frac{1}{\pi_k} \right) \right)$$

$$= v_2 \left( L(\overline{\psi_{D_T^3}}, 1)/\omega \right) + \sum_{\pi_k|\hat{D}_T} v_2 \left( 1 - \left( \frac{D_T}{\pi_k} \right)^2 \frac{1}{\pi_k} \right). \quad (2.20)$$

Note that $0 < t(T) < n$, by induction assumption we have $v_2 \left( L(\overline{\psi_{D_T^3}}, 1)/\omega \right) \geq t(T) - 1$. Also $\left( \frac{D_T}{\pi_k} \right)^2 = 1$ or $-1$ for each $\pi_k|\hat{D}_T$. So by (2.20) above, we get

$$v_2 \left( \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi_{D_T^3}}, 1) \right) \geq t(T) - 1 + n - t(T) = n - 1.$$

Then together with (2.19) of the case $T = \emptyset$, we obtain

$$v_2 \left( L(\overline{\psi_{D_T^3}}, 1)/\omega \right) = v_2 \left( \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi_{D_T^3}}, 1) \right)$$

$$= v_2 \left( \left( \sum_{T \subset \{1, \cdots, n\}} \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi_{D_T^3}}, 1) \right) - \left( \sum_{T \subset \{1, \cdots, n\}} \frac{D}{\omega} \left( \frac{2}{D_T} \right)_2 L_S(\overline{\psi_{D_T^3}}, 1) \right) \right)$$

$$\geq n - 1.$$

This proves our conclusion by induction, and the proof is completed. □

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