A CONJECTURE ON SOME ESTIMATES FOR INTEGRALS

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Abstract. A conjecture concerning some pairs of interfering estimates for some integrals is formulated in three equivalent versions. Its importance for the Paley problem for plurisubharmonic functions and for certain classes of extremal problems for entire functions of several variables is declared.

A «positive function» below means a function whose values are greater than or equal to zero. A real valued function $\phi$ of a real argument is called an «increasing function» if $x_1 \leq x_2$ implies the non-strict inequality $\phi(x_1) \leq \phi(x_2)$.

The following conjecture associated with solving the Paley problem for plurisubharmonic functions and with some extremal problems in the theory of entire functions of several variables was suggested in [1]. It is an open problem since approximately 1992.

Conjecture 1. Let $S$ be a positive increasing function on the ray $[0, +\infty)$ such that $S(0) = 0$ and, moreover, let $S$ be logarithmically convex, i.e. the function $x \mapsto S(e^x)$ is convex on the interval $[-\infty, +\infty)$. Let $\lambda \geq 1/2$, $n \geq 2$, $n \in \mathbb{N}$. Under these assumptions if

$$\int_0^1 S(tx) (1 - x^2)^{n-2} x \, dx \leq t^\lambda \text{ for all } 0 \leq t < +\infty,$$

then

$$\int_0^{+\infty} S(t) \frac{t^{2\lambda-1}}{(1 + t^2)^\lambda} \, dt \leq \frac{\pi (n-1)}{2 \lambda} \prod_{k=1}^{n-1} \left(1 + \frac{\lambda}{2k}\right).$$

Note that the product in the right hand side of (2) is associated with Euler’s Beta and Gamma functions usually demoted through $B$ and $\Gamma$:

$$B(\lambda/2, n) = \int_0^1 x^{\lambda/2-1} (1 - x)^{n-1} \, dx = \frac{\Gamma(\lambda/2) \Gamma(n)}{\Gamma(\lambda/2 + n)} = \frac{\Gamma(\lambda/2) (n-1)!}{\Gamma(\lambda/2) (\lambda/2) (\lambda/2 + 1) \ldots (\lambda/2 + (n-1))} = \frac{2}{\lambda \prod_{k=1}^{n-1} (1 + \frac{\lambda}{2k})}.$$
i.e. the right hand side of (2) can be written as
\[
\frac{\pi(n-1)}{2\lambda} \prod_{k=1}^{n-1} \left(1 + \frac{\lambda}{2k}\right) = \frac{\pi(n-1)}{2\lambda^2} \cdot \frac{1}{B(\lambda/2, n)}.
\]

If we choose
\[
S(t) = 2(n-1) \prod_{k=1}^{n-1} \left(1 + \frac{\lambda}{2k}\right) t^\lambda, \quad t \geq 0, \quad \lambda \geq \frac{1}{2},
\]
then the inequalities (1) and (2) turn to the equalities.

The conjecture 1 is valid for \(\lambda \leq 1\) even without conditions like convexity for \(S\).

It is possible to show, though it is difficult, that for \(\lambda > 1\) one inevitably should impose some conditions of this sort for \(S\).

Beyond these facts, for example, it is even unknown if the conjecture 1 is valid in the case where \(n = 2\) for at least one value of \(\lambda > 1\).

Below we consider the case \(\lambda > 1\) only.

**Lemma 1** (was formulated as the Proposition 5.1 in [3]). A real valued function \(S = S(x)\) on the ray \([0, +\infty)\) such that \(S(0) = 0\) is an increasing logarithmically convex function if and only if there is an increasing function \(s = s(t)\) on the ray \([0, +\infty)\) such that \(S\) is presented as
\[
S(x) = \int_0^x \frac{s(t)}{t} \, dt.
\]

Using this lemma, we can write the integral from (1) as
\[
\int_0^1 S(t) \frac{t^{2\lambda-1}}{(1+t^{2\lambda})^2} \, dt = -\frac{1}{2(n-1)} \int_0^1 \left( \int_0^{tx} \frac{s(t)}{\tau} \, d\tau \right) d(1-x^2)^{n-1}.
\]

Integrating this equality by parts, we get
\[
\int_0^1 S(t) \frac{t^{2\lambda-1}}{(1+t^{2\lambda})^2} \, dt = -\frac{1}{2(n-1)} \int_0^1 \frac{s(tx)}{x} (1-x^2)^{n-1} \, dx.
\]

Similarly for the integral (2), we have
\[
\int_0^{+\infty} S(t) \frac{t^{2\lambda-1}}{(1+t^{2\lambda})^2} \, dt = \frac{1}{2\lambda} \int_0^{+\infty} \frac{s(t)}{t} \, dt \frac{1}{1+t^{2\lambda}}.
\]

Thus the inequalities (1) and (2) are transformed to the following inequalities
\[
\frac{1}{2(n-1)} \int_0^1 \frac{s(tx)}{x} (1-x^2)^{n-1} \, dx \leq t^\lambda \text{ for all } 0 \leq t < +\infty,
\]
\[
\frac{1}{2\lambda} \int_0^{+\infty} \frac{s(t)}{t} \, dt \frac{1}{1+t^{2\lambda}} \leq \frac{\pi(n-1)}{2\lambda} \prod_{k=1}^{n-1} \left(1 + \frac{\lambda}{2k}\right) = \frac{\pi(n-1)}{2\lambda^2} \cdot \frac{1}{B(\lambda/2, n)}.
\]
where \( s \geq 0 \) is an increasing function and the identity (3) is used in deriving the above relationships. Note that without loss of generality we can replace \( s \) by an increasing function \( h \geq 0 \) defined through the formula

\[
h(x) = \frac{1}{4(n-1)} s(x) \quad \text{for } x \geq 0.
\]

As a result the above two relationships are transformed to a condition for the function \( h \) and to an inequality for this function:

\[
2 \int_0^1 \frac{h(t^2 x^2)}{x} (1 - x^2)^{n-1} dx \leq (t^2)^{\lambda/2} \quad \text{for all } 0 \leq t < +\infty,
\]

\[
2 \int_0^{+\infty} \frac{h(t^2)}{t} \frac{dt}{1 + t^{2\lambda}} \leq \frac{\pi}{2} \prod_{k=1}^{n-1} \left( 1 + \frac{\lambda}{2k} \right) = \frac{\pi}{\lambda} \cdot \frac{1}{B\left(\frac{\lambda}{2}, n\right)}.
\]

Now, upon the following changes of variables

\[
x^2 = x', \quad t^2 = t' \quad \lambda/2 = \alpha > 1/2
\]

and redesignating the variables \( x' \) and \( t' \) back through \( x \) and \( t \) we get

\[
\int_0^1 \frac{h(tx)}{x} (1 - x')^{n-1} dx' \leq t^\alpha \quad \text{for all } 0 \leq t < +\infty, \quad (6a)
\]

\[
\int_0^{+\infty} \frac{h(t)}{t} \frac{dt}{1 + t^{2\alpha}} \leq \frac{\pi}{2} \prod_{k=1}^{n-1} \left( 1 + \frac{\alpha}{k} \right) = \frac{\pi}{2\alpha} \cdot \frac{1}{B(\alpha, n)}. \quad (6b)
\]

Thus the conjecture 1 with \( \lambda > 1 \) is equivalent to the following more simple conjecture.

**Conjecture 2.** Let \( \alpha > 1/2 \). Then for any increasing function \( h \geq 0 \) on the ray \([0, +\infty)\) the condition (6a) implies the inequality (6b).

There is one more version of the conjectures 1 and 2. First of all note that for some rather evident reasons it is sufficient to prove these conjectures for smooth functions. Therefore we can assume that the function \( h \) in the conjecture 2 has a continuous derivative, i.e. \( q = h' \geq 0 \) on the interval \((0, +\infty)\). Then, integrating by parts, we get a conjecture equivalent to the previous conjectures 1 and 2.

**Conjecture 3.** Let \( \alpha > 1/2 \). If \( q \) is a positive continuous function on the ray \([0, +\infty)\), then the condition

\[
\int_0^1 \left( \int_x^1 (1 - y)^{n-1} \frac{dy}{y} \right) q(tx) dx \leq t^{\alpha-1}
\]

implies the estimate

\[
\int_0^{+\infty} q(t) \ln \left( 1 + \frac{1}{t^{2\alpha}} \right) dt \leq \pi \alpha \prod_{k=1}^{n-1} \left( 1 + \frac{\alpha}{k} \right). \quad (7)
\]
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