HOLOMORPHIC MAPPINGS BETWEEN HYPERQUADRICS WITH SMALL SIGNATURE DIFFERENCE

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Abstract. In this paper, we study holomorphic mappings sending a hyperquadric of signature \( \ell \) in \( \mathbb{C}^n \) into a hyperquadric of signature \( \ell' \) in \( \mathbb{C}^N \). We show (Theorem 1.1) that if the signature difference \( \ell' - \ell \) is not too large, then the mapping can be normalized by automorphisms of the target hyperquadric to a particularly simple form and, in particular, the image of the mapping is contained in a complex plane of a dimension that depends only on \( \ell \) and \( \ell' \), and not on the target dimension \( N \). We also prove a Hopf Lemma type result (Theorem 1.2) for such mappings.

1. Introduction

In a recent paper of the first and the third authors [BH], it was shown that a holomorphic mapping \( U \rightarrow \mathbb{C}^N \), where \( U \) is an open connected subset of \( \mathbb{C}^n \) and \( N \geq n \), sending a piece of a real hyperquadric with positive signature into a real hyperquadric with the same signature either possess the super-rigidity and the CR transversality (or Hopf lemma) properties or sends the whole open neighborhood \( U \) into the target hyperquadric. The super-rigidity phenomenon obtained in [BH] contrasts with the rigidity of holomorphic mappings between Heisenberg hypersurfaces (i.e. hyperquadrics with 0-signature) in complex spaces of different dimensions, which holds only when the difference in dimension is small. (See [HJ] for a survey on this matter.) The result obtained in [BH] is more along the lines of the behavior of holomorphic mappings between bounded symmetric domains of rank at least two (see [Mo]). In this paper, we study holomorphic mappings between hyperquadrics with different signatures. We will show that both a suitably reinterpreted rigidity phenomenon and a weaker notion of the Hopf lemma property hold when the difference between the signatures is not too large; see Theorems 1.1 and 1.2 for the precise formulation.

To state our main result, we first recall some notation and definitions. For \( 0 \leq \ell \leq n - 1 \), we define the generalized Siegel upper-half space

\[
S^n_\ell := \{ (z, w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n : \ w = u + iv, \ v > - \sum_{j=1}^\ell |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2 \},
\]

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where the first sum is understood to be 0 if $\ell = 0$. The boundary of $S^n_\ell$ is the standard hyperquadric

\begin{equation}
H^n_\ell := \{(z, w) = (z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \ w = u + iv, v = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\}.
\end{equation}

If $0 < \ell < n - 1$, it is well known that any CR function defined over a connected open piece $M$ of $H^n_\ell$ extends to a holomorphic function in a neighborhood of $M$ in $\mathbb{C}^n$ (see e.g. [1]). We denote by $\text{Aut}_0(H^n_\ell)$ the stability group of $H^n_\ell$ at 0, i.e. the group of local biholomorphisms of $\mathbb{C}^n$ sending 0 to itself and a piece of $H^n_\ell$ near the origin into $H^n_\ell$. We point out that if $0 \leq \ell \leq (n-1)/2$, then $\ell$ is the signature of the hyperquadric $H^n_\ell$. (By the signature of a connected Levi nondegenerate hypersurface, we mean the minimum of the number of positive and negative eigenvalues of a representative of the Levi form at any point.) In what follows, we shall mainly consider this case.

Our main result in this paper concerns holomorphic mappings $F$ defined in an open connected neighborhood $U$ of 0 in $\mathbb{C}^n$, valued in $\mathbb{C}^N$, and sending $H^n_\ell \cap U$ into $H^N_{\ell'}$. Using the coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and $(z^*, w^*) \in \mathbb{C}^{N-1} \times \mathbb{C}$, we shall write the components of $F$ in the form $z^* = f(z, w)$, $w^* = g(z, w)$. We remark that it is easy to see that the derivative $\partial g/\partial w(0)$ is a real number. Using the notation above, we can now state our first main result as follows.

**Theorem 1.1.** Let $F$ be a holomorphic map from an open connected neighborhood $U$ of 0 in $\mathbb{C}^n$ into $\mathbb{C}^N$ with $1 < n < N$ and $F(0) = 0$. Assume that $F$ maps $H^n_\ell \cap U$ into $H^N_{\ell'}$, with $\ell \leq (n-1)/2$ and $\ell' \leq (N-1)/2$. Then, the following hold:

(a) If $\partial g/\partial w(0) > 0$, then $\ell \leq \ell'$ and $n - \ell \leq N - \ell'$. Moreover, if $\ell' < 2\ell$, then there is $\gamma \in \text{Aut}_0(H^N_{\ell'})$ such that

\begin{equation}
\gamma \circ F(z, w) = (z_1, \ldots, z_\ell, \psi(z, w), z_{\ell+1}, \ldots, z_{n-1}, \psi(z, w), 0, \ldots, 0, w),
\end{equation}

where $\psi = (\psi_1, \ldots, \psi_{\ell'-\ell})$ is holomorphic near 0 (with the understanding that this term is not present when $\ell' = \ell$).

(b) If $\partial g/\partial w(0) < 0$, then $\ell' \geq n - 1 - \ell$ and $N - 1 - \ell' \geq \ell$. Moreover, if $\ell' < n - 1$, then there is $\gamma \in \text{Aut}_0(H^N_{\ell'})$ such that

\begin{equation}
\gamma \circ F(z, w) = (z_{\ell+1}, \ldots, z_{n-1}, \psi(z, w), z_1, \ldots, z_\ell, \psi(z, w), 0, \ldots, 0, -w),
\end{equation}

where $\psi = (\psi_1, \ldots, \psi_{\ell'-\ell(n-1-\ell)})$ is holomorphic near 0 (with the understanding that this term is not present when $\ell' = n - 1 - \ell$).

We would like to make a few remarks.

**Remark 1.1.** In the notation of Theorem 1.1, if $\partial g/\partial w(0) \neq 0$ and $\ell' < n - 1 - \ell$, then by part (b) of the theorem, necessarily $\partial g/\partial w(0) > 0$. Thus, if $\partial g/\partial w(0) \neq 0$, $\ell' < n - 1 - \ell$, and $\ell' < 2\ell$, then the conclusion in Theorem 1.1 (a) holds.

Recall that a holomorphic mapping $F : U \subset \mathbb{C}^n \to \mathbb{C}^N$ sending a real hypersurface $M \subset \mathbb{C}^n$ into a real hypersurface $M' \subset \mathbb{C}^N$ is said to be CR transversal to $M'$ at $p \in M$ if

\[ T_{F(p)}^{1,0} M' + dF(T_{p}^{1,0} M) = T_{F(p)}^{1,0} \mathbb{C}^N. \]
It is well known and not difficult to see that a holomorphic mapping $F$ as in Theorem 1.1 is CR transversal to $\mathbb{H}^N_n$ at 0 if and only if $\partial g/\partial w(0) \neq 0$. Moreover, if $\partial g/\partial w(0) \neq 0$, then $\partial g/\partial w(0) > 0$ if and only if there is a small open neighborhood $V$ of 0 in $\mathbb{C}^n$ such that $F(S^\ell_n \cap V) \subset S^\ell_N$. Similarly, if $\partial g/\partial w(0) \neq 0$, then $\partial g/\partial w(0) < 0$ if and only if there is a small open neighborhood $V$ of 0 in $\mathbb{C}^n$ such that $F(S^\ell_n \cap V) \subset \mathbb{C}^N \setminus S^\ell_N$.

Also, recall that if $M'$ is a real-analytic hypersurface in $\mathbb{C}^N$, defined near a point $p' \in M'$ by a real-analytic defining equation $\rho'(Z, Z) = 0$, then for $q'$ near $p'$ the Segre variety $Q'_q$ of $M'$ at $q'$ is the complex manifold defined by the holomorphic equation $\rho'(Z, \bar{q'}) = 0$. In particular, the Segre variety of $\mathbb{H}^N_n$ at 0 is given by $Q'_0 = \{(z^*, w^*) : w^* = 0\}$. Hence, in the notation of Theorem 1.1, $F$ sends $U$ into $Q'_0$ if and only if $g \equiv 0$. We now state our second main result of this paper.

**Theorem 1.2.** Let $F$ be a holomorphic map from an open neighborhood $U$ of 0 in $\mathbb{C}^n$ into $\mathbb{C}^N$ with $1 < n < N$ and $F(0) = 0$. Assume that $\mathbb{H}^n_\ell \cap U$ is connected and $F$ maps $\mathbb{H}^n_\ell \cap U$ into $\mathbb{H}^n_\ell'$, with $\ell \leq (n-1)/2$ and $\ell' \leq (N-1)/2$. Then, the following hold:

(a) If there is a point $p \in \mathbb{H}^n_\ell \cap U$ and an open neighborhood $V$ of $p$ in $U$ such that $F(S^\ell_n \cap V) \subset \mathbb{S}^\ell_N$ and $\ell' < 2\ell$, then $g \equiv 0$.

(b) If there is a point $p \in \mathbb{H}^n_\ell \cap U$ and an open neighborhood $V$ of $p$ in $U$ such that $F(S^\ell_n \cap V) \subset \mathbb{C}^N \setminus \mathbb{S}^\ell_N$ and $\ell' < n-1$, then $g \equiv 0$.

**Remark 1.2.** By Lemma 4.1 of [BH] and the above discussion, under the assumption that $\ell' < n-1$ and $F(U) \not\subset \mathbb{H}^n_\ell$, there must be a point $p \in \mathbb{H}^n_\ell \cap U$ and an open neighborhood $V$ of $p$ in $\mathbb{C}^n$ such that either $F(S^\ell_n \cap V) \subset \mathbb{S}^\ell_N$ or $F(S^\ell_n \cap V) \subset \mathbb{C}^N \setminus \mathbb{S}^\ell_N$. Also, when $F(U) \subset \mathbb{H}^n_\ell$, we have $g \equiv 0$ (see Lemma 4.1 in [BH] or p. 605 in [BER2]). We remark that in [BH] it was proved that if $\ell' = \ell$ and $\partial g/\partial w(0) = 0$, then the stronger conclusion $F(U) \subset \mathbb{H}^n_\ell$ holds. The latter conclusion does not always hold when $\ell' > \ell$ as is illustrated by Example 5.1 below.

**Remark 1.3.** The hypotheses on the signatures $\ell$ and $\ell'$ in Theorems 1.1 and 1.2 are sharp in the sense that the conclusions fail when the strict inequalities are replaced by equalities. This is illustrated by Example 5.2 below.

As in [BH], Theorem 1.1 has immediate applications to the study of proper holomorphic mappings between classical domains in complex projective spaces. For $0 \leq \ell < n$, denote by $\mathbb{B}^n_\ell$ the domain in $\mathbb{CP}^n$ given by

$$\mathbb{B}^n_\ell := \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : |z_0|^2 + \cdots + |z_\ell|^2 > |z_{\ell+1}|^2 + \cdots + |z_n|^2\}.$$

Then it is well known that the Cayley transformation $\Psi_n : \mathbb{C}^n \to \mathbb{CP}^n$ given by

$$\Psi_n(z, w) := [i + w, 2z, i - w]$$

biholomorphically maps the generalized Siegel upper-half space $S^\ell_n$ and its boundary $\mathbb{H}^n_\ell$ into $\mathbb{B}^n_\ell \setminus \{[z_0, \ldots, z_n] : z_0 + z_n = 0\}$ and $\partial \mathbb{B}^n_\ell \setminus \{[z_0, \ldots, z_n] : z_0 + z_n = 0\}$, respectively.

For $0 \leq k \leq m$, let $E_{(k,m)}$ denote the $m \times m$ diagonal matrix with its first $k$ diagonal elements $-1$ and the rest $+1$, and define

$$U(n + 1, \ell + 1) = \{A \in GL(n + 1, \mathbb{C}) : AE_{(\ell+1,n+1)}A^T = E_{(\ell+1,n+1)}\}.$$
In what follows, we will regard \( U(n + 1, \ell + 1) \) as a subgroup of the automorphism group of \( \mathbb{CP}^n \) by identifying an element \( A \) in \( U(n + 1, \ell + 1) \) with the holomorphic linear map \( \sigma \in \text{Aut}(\mathbb{CP}^n) \) defined by \( \sigma([z_0, \cdots, z_n]) = [z_0, \cdots, z_n]A \). Then, it is well known that, with this identification, we have \( U(n + 1, \ell + 1) = \text{Aut}(\mathbb{B}^n_\ell) \), transitively acting on \( \mathbb{B}^n_\ell \) (see e.g. [CM, §1]).

By repeating the arguments in the beginning of the proof of Theorem 1.1 in [BH], the following corollary can easily be deduced from Theorem 1.1 above.

**Corollary 1.3.** Let \( \ell \leq (n - 1)/2 \) and \( \ell' \leq (N - 1)/2 \). Let \( F \) be a holomorphic map from \( \mathbb{B}^n_\ell \) into \( \mathbb{CP}^N \) with \( 1 < n \leq N \). Then, the following hold:

(a) If \( \ell' < 2\ell \) and \( F \) is proper from \( \mathbb{B}^n_\ell \) into \( \mathbb{B}_N^{\ell'} \), then \( F(\mathbb{B}^n_\ell) \) is contained in a linear projective subspace of \( \mathbb{CP}^N \) of dimension \( n + \ell' - \ell \).

(b) If \( \ell' < n - 1 \) and \( F \) is proper from \( \mathbb{B}^n_\ell \) into \( \mathbb{B}^{N-1}_{N-\ell'} \), then \( F(\mathbb{B}^n_\ell) \) is contained in a linear projective subspace of \( \mathbb{CP}^N \) of dimension \( \ell' + \ell + 1 \).

2. Notation, Definitions, and Two Basic Lemmas

We shall use the notation and definitions introduced in [BH], which we recall here for the reader’s convenience. Let \( \mathbb{H}^n_{\ell} \subset \mathbb{C}^n \) and \( \mathbb{H}_N^{\ell'} \subset \mathbb{C}^N \) be the standard hyperquadrics defined by (1.2):

\[
\mathbb{H}^n_{\ell} := \{(z, w = u + iv) \in \mathbb{C}^n, \ v = \text{Im} \ w = \sum_{j=1}^{n-1} \delta_{j,\ell}|z_j|^2\};
\]

\[
\mathbb{H}_N^{\ell'} := \{(z^*, w^* = u^* + iv^*) \in \mathbb{C}^N, \ v^* = \sum_{j=1}^{N-1} \delta_{j,\ell'}|z^*_j|^2\}.
\]

Here and in what follows, we denote by \( \delta_{j,\ell} \) the symbol which takes value \(-1\) when \( 1 \leq j \leq \ell \) and \( 1 \) otherwise. For \( \ell' \geq \ell \) and \( N \geq n > \ell - 1 \), we define

\[
\mathbb{H}_N^{\ell'}|_{\ell',n} := \{(z^*, w^*) \in \mathbb{C}^N, \ \text{Im} \ w^* = \sum_{j=1}^{N-1} \delta_{j,\ell',n}|z^*_j|^2\}.
\]

with \( \delta_{j,\ell',n} = -1 \) for \( j \leq \ell \) or \( n \leq j \leq n + \ell' - \ell - 1 \), and \( \delta_{j,\ell',n} = 1 \) otherwise. When \( \ell' > \ell \), \( \mathbb{H}_N^{\ell'} \) is biholomorphically equivalent to \( \mathbb{H}_N^{\ell'}|_{\ell',n} \) by the linear map

\[
\sigma_{\ell',n}(z^*, w^*) := (z_1^*, \cdots, z^*_\ell, z^*_{\ell+1}, \cdots, z^*_{n-1}, z^*_{\ell+1}, \cdots, z^*_n, z^*, \cdots, z^*_n) = (z, w).
\]

Write \( L_j = 2i\delta_{j,\ell}z_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j} \) for \( j = 1, \cdots, n-1 \) and \( T = \frac{\partial}{\partial u} \). Then \( L_1, \cdots, L_{n-1} \) form a global basis for the complex tangent bundle \( T^{(1,0)}\mathbb{H}^n_\ell \) and \( T \) is a tangent vector field of \( \mathbb{H}^n_\ell \) transversal to \( T^{(1,0)}\mathbb{H}^n_\ell \oplus T^{(0,1)}\mathbb{H}^n_\ell \). Parameterize \( \mathbb{H}^n_\ell \) by the map \( (z, \overline{z}, u) \mapsto (z, u + \sum_{j=1}^{n-1} \delta_{j,\ell}|z_j|^2) \). In what follows, we will assign the weights of \( z \) and \( u \) to be 1 and 2, respectively. For a non-negative integer \( m \), a function \( h(z, \overline{z}, u) \) defined in a small neighborhood \( M \) of 0 in \( \mathbb{H}^n_\ell \) is said to be \( o_w(m) \), if \( h(tz, t\overline{z}, t^2u)/|t|^m \to 0 \) uniformly for \( (z, u) \) on any compact subset of \( M \) for
t \in \mathbb{R}, t \to 0$. (In this case, we write \( h = o_{\text{wt}}(m) \)). By convention, we write \( h = o_{\text{wt}}(0) \) if \( h(z, \overline{z}, u) \to 0 \) as \((z, \overline{z}, u) \to 0 \). For a smooth function \( h(z, \overline{z}, u) \) defined in \( M \), we denote by \( h^{(k)}(z, \overline{z}, u) \) the sum of terms of weighted degree \( k \) in the Taylor expansion of \( h \) at \( 0 \). We also denote by \( h^{(k)}(z, \overline{z}, u) \) a weighted homogeneous polynomial of weighted degree \( k \), (even if there is no specified function \( h \)).

For a sufficiently smooth function \( h = h(x_1, \ldots, x_m) \) defined in an open subset of \( \mathbb{C}^m \) and any multiple index \( \alpha = (\alpha_1, \ldots, \alpha_m) \), we write

\[
D^\alpha_x h = \frac{\partial^{\alpha_1 + \cdots + \alpha_m} h}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}.
\]

For two \( m \)-tuples \( x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \) of complex numbers, we write

\[
\langle x, y \rangle_\ell = \sum_{j=1}^m \delta_{j,\ell} x_j y_j, \quad \text{and} \quad |x|_{\ell}^2 = \langle x, x \rangle_\ell.
\]

For \( \ell' \geq \ell \) and \( \ell - 1 \leq n \leq m \), we write \( \langle x, y \rangle_{\ell', n} = \sum_{j=1}^m \delta_{j,\ell'} x_j y_j \).

For the proof of Theorems 1.1 and 1.2 it will be more convenient to assume that the map \( F \) sends \( \mathbb{H}^n_\ell \cap U \) into \( \mathbb{H}^N_{\ell', \ell'} \cap U \), with the conclusions modified accordingly. The proof of the theorem is based on an induction argument, in the spirit of the Chern-Moser theory, using the weighted expansion of the basic equations describing the inclusion \( F(U \cap \mathbb{H}^n_\ell) \subset \mathbb{H}^N_{\ell', \ell'} \subset U \). The method used here is largely motivated by the work in Huang [Hu1], Ebenfelt-Huang-Zaitsev [EHZ2], and Baouendi-Huang [BH].

The following two basic lemmas, also stated and used in [BEH], will be crucial in the proof of Theorem 1.1.

**Lemma 2.1.** Let \( k, \ell, n \) be nonnegative integers such \( 1 \leq k \leq n - 2 \). Assume that \( a_1, \ldots, a_k, b_1, \ldots, b_k \) are germs at \( 0 \in \mathbb{C}^{n-1} \) of holomorphic functions such that

\[
\sum_{i=1}^k a_i(z) \overline{b_i}(\xi) = A(z, \xi) \langle z, \overline{\xi} \rangle_\ell, \tag{2.1}
\]

where \( A(z, \overline{\xi}) \) is a germ at \( 0 \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \) of a holomorphic function in \((z, \xi)\). Then \( A(z, \overline{\xi}) \equiv 0 \).

Lemma 2.1 was proved in [Hu1] (see Lemma 3.2, [Hu1]). In that paper, the lemma is stated only for \( \ell = 0 \), but the proof for \( \ell > 0 \) is identical (see also Lemma 2.1 in [EHZ2]). Lemma 2.1 was also a crucial tool in the papers [Hu1], [EHZ1], [EHZ2]. We shall also need the following corollary, which follows from repeated use of Lemma 2.1.

**Corollary 2.2.** Let \( \ell, n, k_0, k_1, \ldots, k_r \) be nonnegative integers such \( 1 \leq k_j \leq n - 2 \) for \( j = 0, \ldots, r \). Assume that \( a_i^j, b_i^j \) for \( j = 0, \ldots, r \) and \( i = 1, \ldots, k_j \) are germs at \( 0 \in \mathbb{C}^{n-1} \) of
holomorphic functions such that

\[
\sum_{j=0}^{r} \left( \sum_{i=1}^{k_j} a_j^i(z) \overline{b_j^i(\xi)} \right) \langle z, \overline{\xi} \rangle_\ell = A(z, \overline{\xi}) \langle z, \overline{\xi} \rangle_{r+1}^\ell,
\]

where \( A(z, \overline{\xi}) \) is a germ at \( 0 \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \) of a holomorphic function in \((z, \overline{\xi})\). Then

\[
\sum_{j=1}^{k} a_j^i(z) b_j^i(\xi) = 0, \quad j = 0, \ldots, r,
\]

and \( A(z, \overline{\xi}) \equiv 0 \).

The second lemma that we shall need is the following.

**Lemma 2.3.** Let \( k, \ell, m, n \) be nonnegative integers such that \( k < l \leq (n - 1)/2 \) and \( k \leq m \). Assume that \( a_1, \ldots, a_k, b_1, \ldots, b_m \) are germs at \( 0 \in \mathbb{C}^{n-1} \) of holomorphic functions such that

\[
- \sum_{i=1}^{k} |a_i(z)|^2 + \sum_{j=1}^{m} |b_j(z)|^2 = A(z, \overline{z}) |z|_{\ell}^2,
\]

where \( A(z, \overline{\xi}) \) is a germ at \( 0 \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \) of a holomorphic function of \((z, \overline{\xi})\). Then \( A(z, \overline{\xi}) \equiv 0 \) and \((b_1, \ldots, b_m) = (a_1, \ldots, a_k) \cdot U\), where \( U \) is a constant \((k \times m)\)-matrix such that \( U \cdot \overline{U} = \text{Id}_{k \times k} \).

The proof of the first part of Lemma 2.3, \( A(z, \overline{\xi}) \equiv 0 \), can be obtained from Lemma 4.1 of [BH] (with \( \ell' = \ell \) and after a direct application of Lemma 2.1 of [BH]). This part of the lemma also follows in a straightforward way from Theorem 5.7 in the subsequent work [BER2]. The last conclusion follows directly from a lemma by D’Angelo [DA].

### 3. Proof of Theorem 1.1

We shall first prove part (a) of Theorem 1.1 and hence, in addition to the main hypotheses in the theorem, we assume also that \( \partial g/\partial w(0) > 0 \). We use the notation introduced in §2 together with the basic set-up in [BH]. First, we note that the conclusions \( \ell \leq \ell' \) and \( n - \ell \leq N - \ell' \) follow immediately by counting the number of negative and positive eigenvalues on both sides in equation (2.6) in [BH] (as in Lemma 2.1 (a) in [BH]).

In what follows, we identify \( \mathbb{H}_\ell^N \) with \( \mathbb{H}_{\ell, \ell', n}^N \) as explained in Section 2. The fact that the mapping \( F \) sends \( \mathbb{H}_\ell^N \) into \( \mathbb{H}_{\ell, \ell', n}^N \) means that the following basic equation for \( F = (f, \varphi, g) = (f_1, \ldots, f_{n-1}, \varphi_1, \ldots, \varphi_{N-n}, g) \) holds:

\[
\text{Im } g = \langle f, \overline{f} \rangle_\ell + \langle \varphi, \overline{\varphi} \rangle_\tau \quad \text{when Im } w = \langle z, \overline{z} \rangle_\ell.
\]

Here and in what follows, we use the notation \( \tau := \ell' - \ell \). As in the introduction, we shall also use the notation \( \overline{f} := (f, \varphi) \). By the first part of Lemma 2.2 in [BH], we can assume, without
loss of generality, that $F$ has the following normalization $F = (f, \varphi, g)$:

\begin{align*}
f(z, w) &= z + \frac{i}{2}a^{(1)}(z)w + o_{wt}(3), \\
\varphi(z, w) &= \varphi^{(2)}(z) + o_{wt}(2), \\
g(z, w) &= w + o_{wt}(4)
\end{align*}

(3.2)

with

\begin{equation}
\langle a^{(1)}(z), \tau \rangle_{\ell} \equiv \langle \varphi^{(2)}(z), \varphi^{(2)}(z) \rangle_{\tau}.
\end{equation}

(3.3)

We remark that to achieve the normalization (3.2) above, it suffices to compose the original map $F$ from the left with $\tilde{\sigma} \in Aut_{0}(\mathbb{H}^{N}_{\ell, \ell})$ (see Lemma 2.2 in [BH]). Since $\tau = \ell' - \ell < \ell$ by the assumption in part (a) of the theorem, it follows from (3.3) and Lemma 2.3 that $\langle a^{(1)}(z), \tau \rangle_{\ell} \equiv 0$. Thus, $a^{(1)}(z) \equiv 0$ (and hence $f^{(3)} \equiv 0$) and

\begin{equation}
\langle \varphi^{(2)}(z), \varphi^{(2)}(z) \rangle_{\tau} \equiv 0
\end{equation}

(3.4)

Assume that we have shown $g^{(t)} \equiv 0$, $f^{(t-1)} \equiv 0$ for $3 \leq t < s$. Observe that we have shown this for $s = 5$. Then collecting terms of weighted degree $s$ in (3.1), we obtain

\begin{equation}
\text{Im} \ \{g^{(s)}(z, w) - 2i\langle \tau, f^{(s-1)}(z, w) \rangle_{\ell} \} = \\
\sum_{s_1 + s_2 = s} \langle \varphi^{(s_1)}(z, w), \varphi^{(s_2)}(z, w) \rangle_{\tau}, \text{ when } w = u + i(z, \bar{z})_{\ell}.
\end{equation}

(3.5)

For convenience, we shall use the notation

\begin{equation}
\mathcal{L}(p, q)(z, \bar{z}, u) := \text{Im} \ \{q(z, w) - 2i\langle \tau, p(z, w) \rangle_{\ell} \} \bigg|_{w = u + i(z, \bar{z})_{\ell}},
\end{equation}

(3.6)

where $p(z, w) = (p_1(z, w), \ldots, p_{n-1}(z, w))$ and $q(z, w)$ are holomorphic polynomials. Thus, the equation (3.5) can be written

\begin{equation}
\mathcal{L}(f^{(s-1)}, g^{(s)})(z, \bar{z}, u) = \\
\sum_{s_1 + s_2 = s} \langle \varphi^{(s_1)}(z, w), \varphi^{(s_2)}(z, w) \rangle_{\tau} \bigg|_{w = u + i(z, \bar{z})_{\ell}}
\end{equation}

(3.7)

The main step in the proof of Theorem 1.1 is an induction based on the following result.

**Proposition 3.1.** Let $F = (f, \varphi, g)$ be any normalized map as in (3.2) sending an open piece of $\mathbb{H}^{n}_{\ell}$ near the origin into $\mathbb{H}_{\ell, \ell}$. Assume that for all $3 \leq t \leq 2(s^* - 1)$

\begin{equation}
f^{(t-1)} \equiv 0, \quad g^{(t)} \equiv 0, \quad \langle \varphi^{(s_1)}, \varphi^{(s_2)} \rangle_{\tau} \equiv 0, \quad \forall(s_1, s_2) : s_1 + s_2 = t.
\end{equation}

(3.8)

Then (3.8) holds also for $t = 2(s^* - 1) + 1$ and $t = 2s^*$ for any such an $F$.

We point out that we have already proved that (3.8) holds for all $3 \leq t \leq 4$. Hence, once the proposition has been proved, we conclude by induction that (3.8) holds for all $t \geq 3$. For the proof of Proposition 3.1 we shall need some notation and results from [EHZ2].
Given a real-valued power series \( A(z, \bar{z}, w, \bar{w}) \), we shall use the expansion
\[
A(z, \bar{z}, w, \bar{w}) = \sum_{\mu, \nu, \gamma, \delta} A_{\mu \nu \gamma \delta}(z, \bar{z}) w^\gamma \bar{w}^\delta, \quad (z, w) \in \mathbb{C}^{n-1} \times \mathbb{C},
\]
where \( A_{\mu \nu \gamma \delta}(z, \bar{z}) \) is a bihomogeneous polynomial in \((z, \bar{z})\) of bidegree \((\mu, \nu)\) for every \( (\mu, \nu, \gamma, \delta) \in \mathbb{Z}_+^4 \). We recall from [EHZ2] that \( A(z, \bar{z}, w, \bar{w}) \) is said to belong to the class \( \mathcal{S}_k \), where \( k \) is a positive integer, if \( A \) vanishes at least to order 2 at 0 and for every \( (\mu, \nu, \gamma, \delta) \in \mathbb{Z}_+^4 \) we have
\[
A_{\mu \nu \gamma \delta}(z, \bar{z}) w^\gamma \bar{w}^\delta = \sum_{j=1}^{k} p_j(z, w) q_j(z, w),
\]
where the \( p_j \) and \( q_j \) are homogeneous holomorphic polynomials of the appropriate degrees. We shall use the following result, which is direct consequence of Theorem 2.2 and Lemma 4.2 in [EHZ2].

**Theorem 3.2.** Let \( A(z, \bar{z}, w, \bar{w}) \) be a real-valued weighted homogeneous polynomial of degree \( s \geq 5 \), and assume that \( A \in \mathcal{S}_{n-2} \). If \( p(z, w) = (p_1(z, w), \ldots, p_{n-1}(z, w)) \) and \( q(z, w) \) are weighted homogeneous holomorphic polynomials of degree \( s - 1 \) and \( s \), respectively, such that
\[
\mathcal{L}(p, q)(z, \bar{z}, u) = A(z, \bar{z}, w, \bar{w}) \big|_{w = u + i(z, \bar{z})},
\]
then \( p \equiv 0 \), \( q \equiv 0 \), and \( A \equiv 0 \).

We shall now give the proof of Proposition 3.1.

**Proof of Proposition 3.1.** We shall first prove that (3.8) holds for \( t = 2(s^* - 1) + 1 = 2s^* - 1 \) with \( s^* \geq 3 \). As mentioned above, the hypotheses in the proposition imply that (3.7) holds with \( s = t \). We note that the hypotheses also imply that \( \langle \varphi^{(s_1)}, \overline{\varphi^{(s_1)}} \rangle \equiv 0 \) for \( 2 \leq s_1 \leq s^* - 1 \). By a lemma of D’Angelo [DA], we conclude that there are constants \( a_j^{s_k} \) such that
\[
\varphi_j^{(s_1)} = \sum_{k=1}^{N-n} a_j^k \varphi_k^{(s_1)} \quad \text{for} \quad j = \tau + 1, \ldots, N - n, \quad 2 \leq s_1 \leq s^* - 1.
\]
Now, if \( s_1 + s_2 = 2s^* - 1 \), then \( \min(s_1, s_2) \leq s^* - 1 \). If, say, \( s_1 = \min(s_1, s_2) \leq s^* - 1 \) (the case where \( s_2 = \min(s_1, s_2) \) is completely analogous and left to the reader), then it follows from the identities in (3.12) that
\[
\langle \varphi^{(s_1)}, \varphi^{(s_2)} \rangle \equiv \sum_{j=1}^{N-n} \delta_{j \tau} \varphi_j^{(s_1)} \overline{\varphi_j^{(s_2)}}
\]
\[
\equiv \sum_{j=1}^{\tau} \varphi_j^{(s_1)} q_j^{(s_2)},
\]
where
\[
q_j^{(s_2)}(z, w) := \delta_{j \tau} \varphi_j^{(s_2)}(z, w) + \sum_{k=\tau+1}^{N-n} \delta_{k \tau} a_j^{s_1} \varphi_k^{(s_2)}(z, w).
\]
Since $\ell' < 2\ell \leq n - 1$ and $\tau = \ell' - \ell$, it follows that $\tau \leq n - 2$. We conclude that

$$A(z, \bar{z}, w, \bar{w}) := \sum_{s_1 + s_2 = 2s^* - 1} \langle \varphi^{(s_1)}(z, w), \varphi^{(s_2)}(z, w) \rangle_\tau$$

belongs to $\tilde{S}_{n-2}$. It follows from (3.7) and Theorem 3.2 that $f^{(s-1)}(z, w) \equiv 0$ with $s = 2s^* - 1$, and $A \equiv 0$. By the definition of $A$, we conclude that $\langle \varphi^{(s_1)}, \varphi^{(s_2)} \rangle_\tau \equiv 0$ for $s_1 + s_2 = 2s^* - 1$. This completes the proof of Proposition 3.1 for $t = 2(s^* - 1) + 1 = 2s^* - 1$.

It remains to prove (3.8) for $t = 2s^* \geq 6$. Complexifying (3.5) with $s = 2s^*$, we get

$$g^{(2s^*)}(z, w) = 2i \sum_k \langle \varphi^{(k)}(z, w), \varphi^{(2s^*-k)}(\xi, \eta) \rangle_\tau,$$

when $w = \overline{\eta} + 2i \langle z, \xi \rangle_\ell$.

Let us denote by $L_j = \frac{\partial}{\partial z_j} + 2i \delta_j \overline{\xi_j} \frac{\partial}{\partial w}$ for $j = 1, \ldots, n - 1$. Observe that $L_j$ is tangent to the complex hypersurface defined by $w = \overline{\eta} + 2i \langle z, \xi \rangle_\ell$. As a first step towards finishing the induction step for $s = 2s^*$ in the proof of Proposition 3.1, we shall prove that

$$f^{(2s^*-2)}(z, w) \equiv 0, \quad g^{(2s^*)}(z, w) \equiv 0.$$

We begin by establishing the following preliminary claim.

**Claim 3.3.** We have $f^{(2s^*-1)}(z, w) = a^{(1)}_j(z)w^{s^*-1}$, $g^{(2s^*)}(z, w) = d^{(0)}w^{s^*}$, and

$$a^{(1)}_j(z) = d^{(0)}z_j,$$

for $j = 1, \ldots, n - 1$.

**Proof of Claim 3.3.** Applying $L_j$ to (3.15), we obtain for $w = \overline{\eta} + 2i \langle z, \xi \rangle_\ell$

$$L_j \left( g^{(2s^*)}(z, w) \right) - 2i \langle \xi, L_j(f^{(2s^*-1)}(z, w)) \rangle_\ell - 2i f_j(\xi, \eta) \delta_{j\ell}$$

$$= 2i \sum_k \left( L_j \varphi^{(k)}(z, w), \varphi^{(2s^*-k)}(\xi, \eta) \right)_\tau.$$

Let us write

$$f^{(s-1)}(z, w) = \sum a^{(r_j)}_j(z)w^{r_j}, \quad \varphi^{(k)}(z, w) = \sum b^{(\mu_k)}_k(z)w^{\mu_k}, \quad g^{(s)}(z, w) = \sum d^{(j)}_j(z)w^{n_j},$$

where the sums run over all indices such that

$$\tau_j + 2\tau^*_j = s - 1, \quad \mu_k + 2\mu^*_k = k, \quad j + 2n^*_j = s.$$
Recall that $s = 2s^*$. Letting $w = 0$, $\eta = 2i\langle \tau, \xi \rangle$, and collecting terms of degree $K > 2$ in $\xi$ and degree $P$ in $z$ in \[(3.18)\], we obtain

\[
\begin{align*}
-2ia_j^{(K-P)}(\xi)\eta^P\delta_{j\ell} &= 2i\sum_{k=2}^{s-2} \left\langle \varphi_{z_j}^{(k)}(z,0), \sum_{s-k-2}^{b_{s-k}}(\xi)\eta^{\mu_{s-k}} \right\rangle
- 4\sum_{k'=3}^{s-2} \delta_{j\ell} \xi_j \left\langle \varphi_{w}^{(k')}^{(k')} (z,0), \sum_{s-k'-2}^{b_{s-k'}}(\xi)\eta^{\mu_{s-k'}} \right\rangle,
\end{align*}
\]

where the sums inside $\langle \cdot, \cdot \rangle_{\tau}$ run over the indices

\[\mu_{s-k} + \mu_{s-k}^* = K, \quad k - 1 + \mu_{s-k}^* = P\]
and

\[\mu_{s-k'} + \mu_{s-k'}^* + 1 = K, \quad k' - 2 + \mu_{s-k'}^* = P\]

Taking into account also the fact that $\mu_{s-k} + 2\mu_{s-k}^* = s - k$ and $\mu_{s-k'} + 2\mu_{s-k'}^* = s - k'$, we see that \[(3.22)\] and \[(3.23)\] have solutions only when

\[K + P + 1 = s.\]

We shall let $K = s^* + p$ and $P = s^* - p - 1$. We then get

\[\mu_{s-k} = k + 2p, \quad \mu_{s-k}^* = s^* - p - k\]

and

\[\mu_{s-k'} = k' + 2p - 2, \quad \mu_{s-k'}^* = s^* - p - k' + 1.\]

Since both $\mu_{s-k}^*$ and $\mu_{s-k'}^*$ must be non-negative, we can rewrite \[(3.21)\] as follows

\[
\begin{align*}
-2ia_j^{(2p+1)}(\xi)\eta^{s^* - p - 1}\delta_{j\ell} &= 2i\sum_{k=2}^{s^* - p} \left\langle \varphi_{z_j}^{(k)}(z,0), \sum_{s-k-2}^{b_{s-k}^{(k+2p)}}(\xi)\eta^{s^* - p - k} \right\rangle
- 4\sum_{k'+3}^{s^* + p - 1} \delta_{j\ell} \xi_j \left\langle \varphi_{w}^{(k')}^{(k')} (z,0), \sum_{s-k'-2}^{b_{s-k'}}(\xi)\eta^{s^* - p - k} \right\rangle,
\end{align*}
\]

which can be rewritten as

\[
\begin{align*}
-2ia_j^{(2p+1)}(\xi)\eta^{s^* - p - 1}\delta_{j\ell} =
\sum_{k=2}^{s^* - p} \left(2i \left\langle \varphi_{z_j}^{(k)}(z,0), \sum_{s-k-2}^{b_{s-k}^{(k+2p)}}(\xi)\right\rangle - 4\delta_{j\ell} \xi_j \left\langle \varphi_{w}^{(k+1)}^{(k+1)} (z,0), \sum_{s-k-2}^{b_{s-k}^{(k+2p-1)}}(\xi)\right\rangle \right) \eta^{s^* - p - k},
\end{align*}
\]
This equation is valid for all \( p = 0, 1, \ldots, \sigma^* - 1 \) and the sum on the right hand side of (3.28) is void when \( p = \sigma^* - 1 \). When \( q \leq \sigma^* - 1 \) we can use (3.12) to rewrite

\[
\left< \phi_{z_j}^q(z, 0), \frac{b^q_{s-q}}{b^q_{s-q}}(\xi) \right> = \sum_{i=1}^{N-n} \delta_{ir} \phi_{i,z_j}^q(z, 0) \frac{b_{i,s-q}^{(q+2p)}}{b_{i,s-q}^{(q+2p)}}(\xi)
\]

(3.29)

\[
\sum_{i=1}^{N-n} \frac{\phi_{i,z_j}^q(z, 0)c_i(\xi)},
\]

where

\[
c_i(\xi) := \delta_{ir} \frac{b_{i,s-q}^{(q+2p)}}{b_{i,s-q}^{(q+2p)}}(\xi) + \sum_{m=r+1}^{N-n} \delta_{mr} \frac{a_{m_i}^{(q+2p)}}{b_{m,s-q}^{(q+2p)}}(\xi).
\]

(3.30)

We can make a similar substitution in \( \langle \phi_{w}^q(z, 0), \frac{b^q_{s-q}}{b^q_{s-q}}(\xi) \rangle \), again for \( q \leq \sigma^* - 1 \). Hence, if \( 2 \leq p \leq \sigma^* - 1 \), then we conclude by Corollary 2.2, since \( 2^{r} \leq n - 2 \), that

\[
a_j^{(2p+1)}(z) \equiv 0, \quad j = 1, \ldots, n - 1, \quad p = 2, \ldots \sigma^* - 1.
\]

(3.31)

When \( p = 1 \), the equation (3.28) can be written as

\[
-2ia_j^{(3)}(\xi)\eta^s \delta_{j} = \sum_{k=2}^{s^*-2} \left( 2i \left< \phi_{z_j}^k(z, 0), \frac{b_{s-k}^{(k+2)}}{b_{s-k}^{(k+2)}}(\xi) \right> - 4\delta_{j} \bar{\xi} \left< \phi_{w}^{(k+1)}(z, 0), \frac{b_{s-k}^{(k+1)}}{b_{s-k}^{(k+1)}}(\xi) \right> \right) \eta^{s-k} + 2i \left< \phi_{z_j}^{(s^*-1)}(z, 0), \frac{b_{s^*+1}^{(s^*-1)}}{b_{s^*+1}^{(s^*-1)}}(\xi) \right> - 4\delta_{j} \bar{\xi} \left< \phi_{w}^{(s^*)}(z, 0), \frac{b_{s^*}^{(s^*)}}{b_{s^*}^{(s^*)}}(\xi) \right> \tau
\]

(3.32)

We now turn to the equation (3.15) in which we set \( w = 0 \) and \( \eta = -2i(z, \xi) \). Collecting terms of degree \( s^* \) in \( z \) and \( s^* \) in \( \xi \), we obtain

\[
-\overline{d^{(0)}} \eta^{s^*} - 2i \left< \phi_{z_j}^{(s^*)}(z, 0), \frac{b_{s-k}^{(s^*)}}{b_{s-k}^{(s^*)}}(\xi) \right> \tau \eta^{s^*-k} = 2i \sum_{k=2}^{s^*} \left< \phi_{z_j}^{(k)}(z, 0), \frac{b_{s-k}^{(k)}}{b_{s-k}^{(k)}}(\xi) \right> \tau \eta^{s^*-k}.
\]

(3.33)

This can be rewritten as

\[
-\overline{d^{(0)}} \eta^{s^*} - 2i \left< \phi_{z_j}^{(s^*)}(z, 0), \frac{b_{s-k}^{(s^*)}}{b_{s-k}^{(s^*)}}(\xi) \right> \tau \eta^{s^*-k} = 2i \left< \phi_{z_j}^{(s^*)}(z, 0), \frac{b_{s-k}^{(s^*)}}{b_{s-k}^{(s^*)}}(\xi) \right> \tau
\]

(3.34)

Since the left hand side is divisible by \( \eta \) and \( \eta < s^* \), it follows from Lemma 2.3 that

\[
\left< \phi_{z_j}^{(s^*)}(z, 0), \frac{b_{s-k}^{(s^*)}}{b_{s-k}^{(s^*)}}(\xi) \right> \equiv 0
\]

(3.35)
and there are constants $c_{jk}^{(s^*)}$ such that
\begin{equation}
(3.36) \quad b_{js^*}^{(s^*)}(z) = \sum_{k=1}^{\tau} c_{jk}^{(s^*)} b_{ks^*}^{(s^*)}(z), \quad j = \tau + 1, \ldots, N - n,
\end{equation}
where $b_{js^*}^{(s^*)}(z)$ is the $j$th component of the vector valued function $b_{s^*}^{(s^*)}(z)$. By using Lemma \ref{lem:2.1} repeatedly (as in the proof of Corollary \ref{cor:2.2}), we also conclude that
\begin{equation}
(3.37) \quad \langle b_k^{(s^*)}(z), b_{s-k}^{(s^*)}(\xi) \rangle_{\tau} = 0, \quad k = 2, \ldots, s^* - 1,
\end{equation}
and
\begin{equation}
(3.38) \quad d^{(0)} \eta = 2i \left\langle z, a^{(1)}(\xi) \right\rangle_{\ell},
\end{equation}
Equation \eqref{eq:3.38} implies \eqref{eq:3.17}. If we use \eqref{eq:3.36} to substitute for $b_{js^*}^{(s^*)}(z)$ in \eqref{eq:3.32} as above, then we conclude, again by Corollary \ref{cor:2.2} that
\begin{equation}
(3.39) \quad a_j^{(3)} \equiv 0, \quad j = 1, \ldots, n - 1.
\end{equation}
Thus, the equations \eqref{eq:3.31} and \eqref{eq:3.39} together imply that $f^{2s^*-1}(z, w) = a^{(1)}(z)w^{s^*-1}$.

To show that $g^{(s^*)}(z, w) = d^{(0)} w^{s^*}$, we go back to \eqref{eq:3.15} in which we again set $w = 0$ and $\eta = -2i(z, \xi)_{\ell}$. Note that the degree of $f^{2s^*-1}(\xi, \eta)$ in $\xi$ is $s^*$ by what we have already proved. Thus, if we collect terms of degree $K = s^* + p$ in $\xi$ with $p \geq 1$, then we obtain
\begin{equation}
(3.40) \quad - \langle d^{(2p)}(\xi) \eta^{s^*-p} = 2i \sum_{k=2}^{s^*-2} \left\langle \phi^{(k)}(z, 0), \sum_{\mu=-k}^{s^*-k} b_{s-k}^{(\mu)}(\xi) \eta^{\mu} \right\rangle_{\tau},
\end{equation}
where the sum inside $\langle \cdot, \cdot \rangle_{\tau}$ runs over the indices
\begin{equation}
\mu_s - k + \mu_{s-k} = s^* + p, \quad \mu_{s-k} + 2\mu_{s-k} = s - k.
\end{equation}
As above, equation \eqref{eq:3.40} can be rewritten as
\begin{equation}
(3.41) \quad - \langle d^{(2p)}(\xi) \eta^{s^*-p} = 2i \sum_{k=2}^{s^*-p} \left\langle \phi^{(k)}(z, 0), b_{s-k}^{(k+2p)}(\xi) \right\rangle_{\tau} \eta^{s^*-p-k},
\end{equation}
As above, letting $p = 1, \ldots, s^*$ (understanding the sum on the right to be void when $p \geq s^* - 2$) and substituting for $\phi^{(k)}(z, 0)$ using \eqref{eq:3.12}, we conclude that
\begin{equation}
(3.42) \quad d^{(2p)}(\xi) \equiv 0, \quad p = 1, \ldots, s^*.
\end{equation}
This completes the proof of Claim \ref{claim:3.3}. \hfill \Box

Remark 3.1. For future reference, we note that we may combine equations \eqref{eq:3.35} and \eqref{eq:3.37} to conclude
\begin{equation}
(3.43) \quad \langle b_k^{(s^*)}(z), b_{s-k}^{(s^*)}(\xi) \rangle_{\tau} = 0, \quad k = 2, \ldots, s^*.
\end{equation}
By complex conjugating and switching the roles of $z$ and $\xi$ in \eqref{eq:3.43}, we also obtain
\begin{equation}
(3.44) \quad \langle b_k^{(s^*-k)}(z), b_{s-k}^{(s^*-k)}(\xi) \rangle_{\tau} = 0, \quad k = s^*, \ldots, s - 2.
\end{equation}
To complete the proof of (3.16), we need to employ the moving point trick first introduced in [Hu1], and also crucially used in the later works [Hu2] and [BH]. We first recall some notation and definitions from [Hu2] and [BH].

Let \( F_p = \tau_p^{*} \circ F \circ \sigma_p^{0} \) be as in (3.1) in [BH] and let \( F_p^{**} = (f_p^{**}, \varphi_p^{**}, g_p^{**}) \) be its second normalization at the base point \( p \in M \) (see p. 390 in [BH]). In particular, we have

\[
(3.45) \quad (f_p^{**})_j = \frac{(f_p^{*})_j - a_j(p)g_p^{*}}{1 + 2i \left\langle \tilde{f}_p^{*}, a(p) \right\rangle_{t,t',n} + (r(p) - i|a(p)|^2_{t,t',n})g_p^{*}}, \quad j = 1, \ldots, n - 1.
\]

Here, \( a(p) = T(\tilde{f}_p^{*})(0) \), which can be written

\[
a_j(p) = \frac{1}{\lambda(p)} \left\langle T(\tilde{f}), L_j(f) \right\rangle_{t,t',n} \bigg|_p \quad \text{for} \quad j \leq n - 1.
\]

and

\[
a_j(p) = \frac{1}{\lambda(p)} \left\langle T(\tilde{f}), D_j(p) \right\rangle_{t,t',n} \bigg|_p \quad \text{for} \quad j \geq n,
\]

where

\[
L_j = \frac{\partial}{\partial z_j} + 2i\delta_{jl} \frac{\partial}{\partial w}, \quad j = 1, \ldots, n - 1, \quad T = \frac{\partial}{\partial w},
\]

the vectors \( D_j(p) \) and the scalar \( \lambda(p) \) depend on first order derivatives of the mapping \( F \) at \( p \).

The reader is referred to Sections 2 and 3 in [BH], and also [Hu1], and also crucially used in the later works [Hu2] and [BH]. We first recall some notation and definitions from [Hu2] and [BH].

We state here some facts that can be found in [BH]. As functions of \( p \in \mathbb{H}^n_t \), \( \text{deg}_{wt}(a_j(p)) \geq 2 \) for \( j \leq n - 1 \) and \( \text{deg}_{wt}a_j(p) \geq 1 \) for \( j \geq n \). To emphasize that the weighted degree is as a function of \( p \), we shall write \( a_j(p) = O_{wt,p}(1) \), \( j \geq n \), and so on in what follows. Also, we have

\[
(3.46) \quad |a(p)|^2_{t,t',n} = \frac{1}{\lambda(p)} |T(\tilde{f})|^2_{t,t',n} = O_{wt,p}(2), \quad (r + i|a(p)|^2_{t,t',n}) = O_{wt,p}(2), \quad \lambda(p) = 1 + O_{wt,p}(2),
\]

\[
g_p^{*} = \frac{1}{\lambda(p)} \left( g - 2i \left\langle \tilde{f}, \tilde{f}(p) \right\rangle_{t,t',n} \right) \circ \sigma_0^p,
\]

\[
\tilde{f}_p^{*} = (\tilde{f} - \tilde{f}(p)) \circ \sigma_0^p \circ \tilde{A}^{-1},
\]

\[
(f_p^{**})_j = \delta_{j,t} \left\langle \frac{1}{\lambda}(\tilde{f} - \tilde{f}(p)) \circ \sigma_0^p, L_j(\tilde{f}) \right\rangle_{t,t',n}.
\]

We observe further that \( F_p^{**} \) is a normalized map sending the origin to the origin and an open piece of \( \mathbb{H}^n_{t,t'} \) into \( \mathbb{H}^n_{t,t'} \). Hence, the conclusion of Claim 3.3 holds for the components \( f_p^{**} \) and \( g_p^{**} \) for every \( p \in \mathbb{H}^n_t \) near 0. We shall prove the following lemma, which then completes the proof of (3.16).

**Lemma 3.4.** Suppose that the hypotheses in Proposition 3.1 hold. Assume further that for any \( p \in M \), we have

\[
(3.47) \quad f_p^{**}(z,w) = z + a_p^{(1)}(z)w^{s-1} + O_{wt}(s), \quad g_p^{**}(z,w) = w + d_p^{(0)}w^s + O_{wt}(s+1),
\]
where $s = 2s^*$. Then, $a_p^{(1)}(z) \equiv 0$, $a_p^{(0)} = 0$, and hence

$$f(z, w) = z + O_{wt}(s), \quad g(z, w) = w + O_{wt}(s + 1).$$

We remark that by the hypothesis in Proposition 3.1 and what we did above, the assumption in (3.47) always holds.

**Proof of Lemma 3.4.** Let us identify the coefficients, as functions of $p \in \mathbb{H}^n_\ell$, in front of $w^{s^* - 1}$ in the Taylor expansion on both sides of (3.45) for a fixed $1 \leq j \leq n - 1$. We note, in view of the assumption in the lemma, that the coefficient on the left is 0. To find the coefficient on the right, we first consider the coefficient of $(f_p^*)^j$, which in view of (3.46) equals

$$\frac{1}{\lambda(p)} \left< T^{s^* - 1} \hat{f}(p), L_j \hat{f}(p) \right>_{\ell, \ell', n}.$$

If we use the following expansion of $\hat{f} = (f, \varphi)$,

$$f_j(z, w) = z + a_j^{(1)}(z) w^{s^* - 1} + O_{wt}(s), \quad \varphi(z, w) = \varphi^{(2)}(z) + \varphi^{(3)}(z, w) + \cdots + \varphi^{(s^* - 2)}(z, w) + \cdots$$

with

$$\varphi^{(s^* - 2)}(z, w) = b^{(0)}_{s-2} w^{s^* - 2} + \sum_{\mu_{s-2} = 0}^{s-3} b^{(s^* - 2 - \mu_{s-2})}_{s-2}(z) w^{\mu_{s-2}},$$

then, with $p = (z_p, w_p)$, we obtain

$$\frac{1}{\lambda(p)} \left< T^{s^* - 1} \hat{f}(p), L_j \hat{f}(p) \right>_{\ell, \ell', n} = (s^* - 1)! \left( \delta_j \delta^{(1)}_j(z_p) + \left< b^{(0)}_{s-2}, \varphi^{(2)}_{\ell}(z_p) \right> \right) + O_{wt, p}(2).$$

Since $a_j(p) = O_{wt, p}(2)$, for $j = 1, \ldots, n - 1$, and the denominator of (3.45) is $1 + O_{wt, p}(1)$, we conclude that the coefficient of $w^{s^* - 1}$ on the right hand side of (3.45) equals (3.51) modulo $O_{wt, p}(2)$. The conclusion is that

$$\delta_j \delta^{(1)}_j(z_p) = \left< b^{(0)}_{s-2}, \varphi^{(2)}_{\ell}(z_p) \right> = 0.$$

Since this holds for all $j = 1, \ldots, n - 1$ and all $p \in \mathbb{H}^n_\ell$, and hence for all $z_p$ in a neighborhood of 0, we conclude that

$$a_j^{(1)}(z) \equiv 0, \quad j = 1, \ldots, n - 1.$$

This also implies that $g(z, w) = w + O_{wt}(s + 1)$ in view of (3.17). This completes the proof of Lemma 3.4.

To complete the induction step in the proof of Proposition 3.1 we must show that

$$\left< \varphi^{(k)}(z, w), \varphi^{(s^*-k)}(\xi, \eta) \right> = 0, \quad k = 2, \ldots, s - 2,$$

for $s = 2s^*$. Using the expansion of $\varphi^{(k)}(z, w)$ in (3.19), we observe that it suffices to show that

$$\left< b^{(k)}_k(z), b^{(s^*-k)}_{s-k}(\xi) \right> = 0.$$
for all indices such that

\[(3.54) \quad k = 2, \ldots, s - 2, \quad \mu_k + 2\mu^*_k = k, \quad \mu_{s-k} + 2\mu^*_{s-k} = s - k.\]

For this purpose, let us consider equation \((3.15)\) with \(w = \bar{\eta} + 2i\langle z, \bar{\xi} \rangle\) and collect terms of degree \(p\) in \(\eta\), \(q\) in \(z\), and \(r\) in \(\xi\). Since the left hand side of \((3.15)\) vanishes by \((3.16)\), we obtain

\[(3.55) \quad \sum_{k=2}^{s-2} \sum_{j=0}^{\mu^*_k} (2i)^{\mu^*_k-j} \langle \mu_k \rangle \bar{\eta}^{\mu^*_k-j} \langle z, \bar{\xi} \rangle^{\mu^*_k-j} \left\langle b_k^\mu(z), b_{s-k}^{(\mu^*_{s-k})}(\xi) \right\rangle = 0,
\]

where the notation introduced in \((3.19)\) has been used and as above \(s = 2s^*\). The middle sum in \((3.55)\) ranges over those indices \((\mu_k, \mu^*_k, \mu_{s-k}, \mu^*_{s-k})\) that satisfy the equations

\[(3.56) \quad \mu_k + \mu^*_k - j = q
\]

\[\mu_k + 2\mu^*_k = k
\]

\[\mu^*_k + \mu_{s-k} - j = r
\]

\[\mu_{s-k} + 2\mu^*_{s-k} = s - k
\]

\[\mu^*_{s-k} + j = p.
\]

For fixed \(2 \leq k \leq s - 2\), we may view the two last sums in \((3.55)\) as ranging over those \(\mu_k, \mu^*_k, \mu_{s-k}, \mu^*_{s-k}\) and \(j\) that satisfy the following system of equations:

\[(3.57) \quad \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mu_k \\ \mu^*_k \\ \mu_{s-k} \\ \mu^*_{s-k} \\ j \end{pmatrix} = \begin{pmatrix} q \\ k \\ r \\ s - k \\ p \end{pmatrix}
\]

subject to the constraints

\[(3.58) \quad \mu_k \geq 0, \quad \mu^*_k \geq 0, \quad \mu_{s-k} \geq 0, \quad \mu^*_{s-k} \geq 0, \quad 0 \leq j \leq \mu^*_k.
\]

A straightforward row reduction shows that \((3.57)\) is equivalent to the system

\[(3.59) \quad \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mu_k \\ \mu^*_k \\ \mu_{s-k} \\ \mu^*_{s-k} \\ j \end{pmatrix} = \begin{pmatrix} q \\ k - q \\ r + q - k \\ s - r - q \\ p \end{pmatrix}
\]

We conclude that the system is solvable when

\[(3.60) \quad 2p + q + r = s,
\]
in which case we may solve for \( \mu_k, \mu^*_k, \mu_{s-k}, \) and \( \mu^*_{s-k} \) in terms of \( j \) to obtain

\[
\begin{align*}
\mu_k &= 2q + 2j - k \\
\mu^*_k &= k - q - j \\
\mu_{s-k} &= r + q + 2j - k \\
\mu^*_{s-k} &= p - j.
\end{align*}
\]

(3.61)

Let us begin by choosing \( p = 0 \) and, for integral \( m \) with \(-\langle s^*-2 \rangle \leq m \leq s^*-2\), choose \( q = s^*+m \) and \( r = s^*-m \). The constraints in (3.58) imply that \( j = 0 \) and \( s^* + m \leq k \leq s + 2m \). Note that in this induction step of the proof of Proposition 3.1 only \( b^{(\mu_k)}(z) \) with \( 2 \leq k \leq s - 2 \) are involved. For simplicity of notation, with \( s = 2s^* \) fixed, let us introduce \( b^{(\mu_k)}(z) \) for any integer \( k \) by defining

\[
(3.62) \quad b^{(\mu_k)}(z) \equiv 0, \quad \text{if } k \leq 1 \text{ or } k \geq s - 1.
\]

We then conclude from (3.15) and the above that

\[
(3.63) \quad \sum_{k=s^*+m}^{s+2m} (2i)^{k-s^*-m} \langle z, \xi \rangle^{k-s^*-m}_\ell \left\langle b^{(s+2m-k)}(z), \overline{b^{(s-k)}(\xi)} \right\rangle_\tau = 0.
\]

Note that with the convention (3.62), the sum really ranges over \( k \) from \( \max(2, s^* + m) \) and \( \min(s - 2, s + 2m) \). If \( m \geq 1 \) or \( m < -s^*/2 \), then no term of the form \( b^{(\mu_\tau)}(z) \) (or its complex conjugate) appears in the sum in (3.63) and it follows from Corollary 2.2 and (3.12) (as in the proof of Claim 3.3 above) that

\[
(3.64) \quad \left\langle b^{(s+2m-k)}(z), \overline{b^{(s-k)}(\xi)} \right\rangle_\tau = 0, \quad k = s^* + m, \ldots, s + 2m.
\]

With \(-s^*/2 \leq m \leq 0 \), we can rewrite (3.63) as follows

\[
(3.65) \quad \sum_{k=s^*+m}^{s^*-1} (2i)^{k-s^*-m} \langle z, \xi \rangle^{k-s^*-m}_\ell \left\langle b^{(s+2m-k)}(z), \overline{b^{(s-k)}(\xi)} \right\rangle_\tau +
\]

\[
(2i)^{-m} \langle z, \xi \rangle^{-m}_\ell \left\langle b^{(s^*+2m)}(z), \overline{b^{(s^*)}(\xi)} \right\rangle_\tau 
\]

\[
+ \sum_{k=s^*+1}^{s+2m} (2i)^{k-s^*-m} \langle z, \xi \rangle^{k-s^*-m}_\ell \left\langle b^{(s+2m-k)}(z), \overline{b^{(s-k)}(\xi)} \right\rangle_\tau = 0,
\]

where we understand any sum over an index set such that the upper limit is strictly less than the lower limit to not be present. By using (3.36) to rewrite (as in (3.29))

\[
\left\langle b^{(s^*+2m)}(z), \overline{b^{(s^*)}(\xi)} \right\rangle_\tau
\]

appearing in the middle term in (3.65), and (3.12) to rewrite the remaining terms, we conclude from Corollary 2.2 that also in this case

\[
(3.66) \quad \left\langle b^{(s-2-k)}(z), \overline{b^{(s-k)}(\xi)} \right\rangle_\tau = 0, \quad k = s^* + m, \ldots s + 2m.
\]
We can summarize the above, using also the convention (3.62), as follows
\begin{equation}
\left\langle b_k^{s+2m-k}(z), b_{s-k}^{s-k}(\xi) \right\rangle_\tau = 0, \quad s^* + m \leq k \leq s + 2m, \quad -s^* \leq m \leq s^*.
\end{equation}

Now, for general \( 0 \leq p \leq s^* \), \( q = s^* + m - p \) and \( r = s^* - m - p \), the equations (3.61) can be rewritten
\begin{align*}
\mu_k &= s + 2m - 2p + 2j - k \\
\mu_k^* &= k - s^* - m + p - j \\
\mu_{s-k} &= s - 2p + 2j - k \\
\mu_{s-k}^* &= p - j.
\end{align*}

The constraints in (3.58) amount to \( 0 \leq j \leq p \) and
\begin{equation}
s^* + m - p + 2j \leq k \leq \min(s - 2p + 2j, s + 2m - 2p + 2j).
\end{equation}

We can rewrite (3.55) as follows
\begin{equation}
\sum_{j=0}^{p} \sum_{k=s^*+m-p+2j}^{s+(2m)-2p+2j} (2i)^{k-s^*-m+p-2j} \left( \binom{k-s^*-m+p-j}{j} \right) \bar{\eta}_p \times
\langle z, \xi \rangle_{\ell}^{k-s^*-m+p-2j} \left\langle b_k^{s+2m-2p+2j-k}(z), b_{s-k}^{s-2p+2j-k}(\xi) \right\rangle_\tau = 0,
\end{equation}
where the parenthetical \((+2m)\) in the upper limit in the sum over \( k \) is only present when \( m < 0 \).

**Claim 3.5.** We have, for every \( 0 \leq p \leq s^* \),
\begin{equation}
\left\langle b_k^{s+2m-2p-k}(z), b_{s-k}^{s-2p-k}(\xi) \right\rangle_\tau = 0,
\end{equation}
for all
\begin{equation}
s^* + m - p \leq k \leq s + 2m - 2p, \quad \text{when } -(s^* - p) \leq m < 0
\end{equation}
and
\begin{equation}
s^* + m - p \leq k \leq s^* - p, \quad \text{when } 0 \leq m \leq s - 2p,
\end{equation}
where we understand \( b_k^{\mu_k}(z) \equiv 0 \) if \( k \leq 1 \) or \( k \geq s - 1 \). In addition, there are constants \( c_{jk}^{s^* - 2l} \) such that
\begin{equation}
b_{js^{*}-2l}(z) = \sum_{k=1}^{\tau} c_{jk}^{s^* - 2l} b_{ks^{*}-2l}(z), \quad j = \tau + 1, \ldots, N - n, \quad l = 0, 1, \ldots, \min(p, [s^*/2]),
\end{equation}
where \( b_{js^{*}-2l}(z) \) is the \( j \)th component of the vector valued function \( b_s^{s^* - 2l}(z) \) and \([s^*/2]\) denotes the largest integer \( \leq s^*/2 \).

**Proof of Claim 3.5.** We shall prove Claim 3.5 by induction on \( p \). We observe that for \( p = 0 \), the equations in (3.71) reduce to (3.67). (Note that (3.71) is automatic for \( k \leq 1 \) and \( k \geq s - 1 \).)
by the convention introduced above.) Also, (3.74) is just (3.36). Let $p^* < s^*$ and assume that (3.71), and (3.74) hold for all $p \leq p^*$. Consider the term

$$
(3.75) \quad \left\langle b_k^{(s+2m-2p+2j-k)}(z), b_{s-k}^{(s-2p+2j-k)}(\xi) \right\rangle \tau
$$

in the sum (3.70) with $p = p^* + 1$ and $m = m^*$, i.e.

$$
(3.76) \quad \left\langle b_k^{(s+2m^*-2p^*-2+2j-k)}(z), b_{s-k}^{(s-2p^*-2+2j-k)}(\xi) \right\rangle \tau,
$$

where $0 \leq j \leq p^* + 1$ and

$$
(3.77) \quad s^* + m^* - p^* - 1 + 2j \leq k \leq s(2m^*) - 2p^* - 2 + 2j
$$

where the parenthetical term (+2$m^*$) is only present when $m^* < 0$. By the induction hypothesis, this term vanishes if there are $-(s^* - p) \leq m \leq s^* - p$ and $0 \leq p \leq p^*$ such that

$$
(3.78) \quad \begin{align*}
2m - 2p &= 2m^* - 2p^* - 2 + 2j \\
2p &= 2p^* + 2 - 2j
\end{align*}
$$

satisfying

$$
(3.79) \quad s^* + m - p \leq k \leq s(2m) - 2p,
$$

where the parenthetical term (+2$m$) is only present when $m < 0$. (In what follows, we shall continue to use this convention.) Solving for $m$ and $p$ in (3.78), we obtain

$$
(3.80) \quad m = m^*, \quad p = p^* + 1 - j.
$$

The constraint $p \leq p^*$ is satisfied if $j \geq 1$. We note that

$$
(3.81) \quad s^* + m - p = s^* + m^* - p^* - 1 + j \leq s^* + m^* - p^* - 1 + 2j \leq k.
$$

We also have $s(+2m) - 2p = s(+2m^*) - 2p^* - 2 + 2j \geq k$. We conclude that the constraint (3.79) is satisfied. Hence, the terms (3.76) that appear in (3.70) (with $p = p^* + 1$ and $m^*$) vanish for all $j \geq 1$. Consequently, it follows from (3.70), with $p = p^* + 1$ and $m = m^*$, that

$$
(3.82) \quad \sum_{k=s^*+m-p^*-1}^{s(+2m)-2p^*-2} (2i)^{k-s^*-m+p^*+1} \left\langle z, \xi \right\rangle \tau^{k-s^*-m+p^*+1} \left\langle b_k^{(s+2m-2p^*-2-k)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right\rangle \tau = 0.
$$

Assume first that $p^* \leq s^*/2 - 1$. Then if $m \geq p^* + 2$ or $m \leq -s^*/2 + p^* + 1/2$, no term involving $b_k^{(s^*/2)}(z)$ appears in the sum and equation (3.81) implies, by using Corollary 2.2 and (3.12) as above, that

$$
(3.83) \quad \left\langle b_k^{(s+2m-2p^*-2-k)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right\rangle \tau = 0, \quad k = s^* + m - p^* - 1, \ldots, s(+2m) - 2p^* - 2.
$$
When \(-s^*/2 + p^* + 1/2 < m \leq p^* + 1\), we can rewrite (3.81) as follows

\[
(3.83) \quad \sum_{k = s^* + m - p^* - 1}^{s^* - 1} (2i)^{k-s^*-m+p^*+1} \langle z, \bar{\xi} \rangle_{\ell}^{k-s^*-m+p^*+1} \left< b_k^{(s+2m-2p^*-2-k)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right>_{\tau} +
\]

\[
(2i)^{-m+p^*+1} \left< z, \bar{\xi} \right>_{\ell}^{-m+p^*+1} \left< b_{s^*}^{(s+2m-2p^*-2)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right>_{\tau} +
\]

\[
\sum_{k = s^* + 1}^{s^* + (-2m-2p^* - 2)} (2i)^{k-s^*-m+p^*+1} \langle z, \bar{\xi} \rangle_{\ell}^{k-s^*-m+p^*+1} \left< b_k^{(s+2m-2p^*-2-k)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right>_{\tau} = 0,
\]

where we understand any sum over an index set such that the upper limit is strictly less than the lower limit to be present. Let us first consider the case \(1 \leq m \leq p^* + 1\). Note that in this case \(s^* + 2m - 2p^* - 2\) can be written as \(s^* - 2l\) with \(l = p^* + 1 - m\) and then \(l \leq p^*\), and hence we can use the induction hypothesis (3.74) to rewrite the middle term

\[
\left< b_{s^*}^{(s+2m-2p^*-2)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right>_{\tau}
\]

as a sum involving only \(\tau\) terms (as in (3.29)). All other terms can be similarly rewritten using (3.12). We conclude by Corollary 2.2 that (3.82), which was shown above to hold for \(m \geq p^* + 2\) and \(m < -s^*/2 + p^* + 1\), holds also for \(1 \leq m \leq p^* + 1\).

Next, we consider \(m = 0\). In this case, we first use Lemma 2.1 repeatedly (as in the proof of Corollary 2.2) to deduce that each term in the first sum in (3.83) vanishes, i.e.

\[
(3.84) \quad \left< b_{k}^{(s-2p^*-2-k)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right>_{\tau} = 0, \quad k = s^* - p^* - 1, \ldots, s^* - 1.
\]

By canceling a factor \(2i\langle z, \bar{\xi} \rangle^{p^* + 1}\), we are left with

\[
(3.85) \quad \left< b_{s^*}^{(s^*-2p^*-2)}(z), b_{s-k}^{(s^*-2p^*-2)}(\xi) \right>_{\tau} + \sum_{k = s^* + 1}^{s^* - 2p^* - 2} (2i)^{k-s^*+m-p^*-1} \langle z, \bar{\xi} \rangle_{\ell}^{k-s^*+m-p^*-1} \left< b_k^{(s-2p^*-2-k)}(z), b_{s-k}^{(s-2p^*-2-k)}(\xi) \right>_{\tau} = 0.
\]

It follows from Lemma 2.3 that

\[
\left< b_{s^*}^{(s^*-2p^*-2)}(z), b_{s^*}^{(s^*-2p^*-2)}(\xi) \right>_{\tau} = 0
\]

and

\[
(3.86) \quad b_{j}^{(s^*-2p^*-2)}(z) = \sum_{k=1}^{\tau} c_{jk}^{s^*-2p^*-2} b_{k}^{(s^*-2p^*-2)}(z), \quad j = \tau + 1, \ldots, N - n
\]

for some constants \(c_{jk}^{s^*-2p^*-2}\). Note that (3.86) and the induction hypothesis yield (3.74) with \(p = p^* + 1\). Now, applying Corollary 2.2 to the remaining equation, we conclude that (3.82) holds for \(m = 0\).

Finally, we consider the remaining cases \(-s^*/2 + p^* + 1 \leq m \leq -1\). We can rewrite the terms in the first and last sum using (3.12) and the middle term using (3.86). We conclude, by
Corollary \textbf{2.2} that (3.82) holds for these $m$ as well. Consequently, we have proved (3.71), for the ranges of $k$ and $m$ given by (3.5) and (3.73) and $l = p^* + 1$, when $p^* \leq s^*/2 - 1$.

We now assume $p^* > s^*/2 - 1$. In this case the upper limit in the sum over $k$ in (3.81) satisfies $s(+2m) - 2p^* - 2 < s^*$ for all $m$, since the term $(+2m)$ is only present when $m < 0$. Hence, no term $b^{(m,s^*)}_l(z)$ will appear in the equation. The conclusion (3.71), for the ranges of $k$ and $m$ given by (3.5) and (3.73) and $p = p^* + 1$, then follows also in this case. Note that when $p^* > s^*/2 - 1$, then $\min(p^* + 1, [s^*/2]) = [s^*/2] \leq p^*$ and, hence, (3.71) trivially holds also for $p = p^* + 1$ by the induction hypothesis. Claim 3.5 now follows by induction. \hfill \Box

To prove (3.53) for all indices such that (3.54) holds, it suffices in view of Claim 3.5 to show that for any integer $2 \leq k \leq s - 2$ and any integers $0 \leq \mu_k \leq k$, $0 \leq \mu_{s-k} \leq s - k$ of the same parity as $k$, we can find integers $m$ and $p$ satisfying
\begin{equation}
(3.87) \quad s + 2m - 2p - k = \mu_k, \quad s - 2p - k = \mu_{s-k}
\end{equation}
such that the constraints
\begin{equation}
(3.88) \quad 0 \leq p \leq s^*, \quad -(s^* - p) \leq m \leq s^* - p, \quad s^* + m - p \leq k \leq s(+2m) - 2p
\end{equation}
hold, where again the parenthetical term $(+2m)$ is only present when $m < 0$. We shall treat here the case of even $k = 2l$, with $1 \leq l \leq s^* - 1$. The case of odd $k$ is similar and left to the reader. We can write $\mu_k = 2x$ and $\mu_{s-k} = 2y$, where $0 \leq x \leq l$ and $0 \leq y \leq s^* - l$. The unique solution to (3.87) is then
\begin{equation}
(3.89) \quad p = s^* - y - l, \quad m = x - y.
\end{equation}
Clearly, we have $0 \leq p \leq s^*$. We find that $s^* - p = y + l$. It follows that $m \leq x \leq l \leq y + l = s^* - p$ and $m \geq -y \geq -(y + l) = -(s^* - p)$, and hence $m$ satisfies the middle constraints in (3.88). Moreover, we have $s^* + m - p = x + l \leq 2l = k$. When $m \geq 0$, the upper limit for $k$ in (3.88) equals $s(+2m) - 2p = s - 2p = 2(s^* - p) = 2y + 2l \geq 2l = k$. When $m < 0$, the upper limit for $k$ equals $s(+2m) - 2p = 2(s^* - p + m) = 2x + 2l \geq 2l = k$. We conclude that all constraints in (3.88) are satisfied. Thus, we have proved (3.53) for all indices such that the constraints in (3.54) hold. This completes the induction step for $s = 2s^*$ and, hence, the proof of Proposition 3.1 is complete. \hfill \Box

Proof of Theorem 3.1 (a). As explained in the beginning of this section, we may assume that the mapping $F$ satisfies the normalization (3.2). By induction and Proposition 3.1, we conclude that
\begin{equation}
f(z, w) \equiv z, \quad g(z, w) \equiv w, \quad \left< \varphi(z, w), \overline{\varphi(\xi, \eta)} \right> \equiv 0,
\end{equation}
where $\tau = l' - l$ as above. By a lemma of D’Angelo, there is a constant $(N - n - \tau) \times (N - n - \tau)$ matrix $U$ such that
\begin{equation}
U \overline{U} = I_{(N - n - \tau) \times (N - n - \tau)}
\end{equation}
and
\begin{equation}
(\varphi_{\tau+1}, \ldots, \varphi_{N-n})U = (\varphi_1, \ldots, \varphi_\tau, 0, \ldots, 0).
\end{equation}
If we let $\gamma$ be the automorphism of $\mathbb{H}^N_{l'}$ given by
\begin{equation}
\gamma(z, w) := (z', z'', z'''U, w),
\end{equation}
where \( z = (z', z'', z''') \) with
\[
z' = (z_1, \ldots, z_{n-1}), \quad z'' = (z_n, \ldots, z_{n+r-1}), \quad z''' = (z_{n+r}, \ldots, z_{N-1})
\]
then \( \gamma \circ F \) satisfies the conclusion of Theorem 1.1 (a). The proof of Theorem 1.1 (a) is complete.

**Proof of Theorem 1.1** (b). We now assume that \( \frac{\partial g}{\partial w}(0) = \lambda < 0 \). By counting the number of negative and positive eigenvalues in \([2.6], BH\), we similarly see that \((n-1-\ell) \leq \ell' \) and \( N-1-\ell' \geq \ell \).

Assume that \( \ell' < n-1 \). Define \( \ell^* = n-1-\ell \) and \( \tau^* = \ell' - (n-1-\ell) \). Then we have \( \tau^* < \ell \leq \frac{n-1}{2} \). Notice that Lemma 2.3 still holds when \( \ell, \tau \) are replaced by \( \ell^* \) and \( \tau^* \), respectively. Indeed, to see this, we need only to observe that \( A(z, \tau)|z|_{\ell^*} = -A(z, \tau)(-|z_{\ell^*+1}|^2 - \cdots - |z_{n-1}|^2 + |z_1|^2 + \cdots + |z_{\tau^*}|^2) \) with \( \frac{n-1}{2} \geq n-1-\ell^* > \tau^* \). Certainly Lemma 2.1 holds when \( \ell \) is replaced by \( \ell^* \).

Let \( \sigma^*(z, w) = (z_{\ell^*+1}, \ldots, z_{n-1}, z_1, \ldots, z_{\tau^*}, -w) \) and consider \( F^* = F \circ \sigma^* \). Then \( F^* \) maps a small piece of \( \mathbb{H}^n_p \) near the origin into \( \mathbb{H}^N_{\ell^*, \tau^*} \). Now, although \( \ell^* \geq (n-1)/2 \), the same argument as in [BH] still shows that we can still normalize \( F^* \) by composing a certain linear fraction map from \( \mathbb{H}^N_{\ell^*, \tau^*} \) to \( \mathbb{H}^N_{\ell^*, \tau^*} \) from the left to get the same normalization for \( F^* \) as in (3.2) (with \( \ell, \tau \) being replaced by \( \ell^* \) and \( \tau^* \), respectively). Still write \( F^* \) for the normalized \( F^* \). Now, the argument in the proof of Theorem 1.1 goes through without any change when we replace \( \ell, \tau \) by \( \ell^* \) and \( \tau^* \), respectively. (Details are left to the reader.) Hence, one easily see that in this setting, we have the statement in Theorem 1.1 (b). This completes the proof of Theorem 1.1

\[ \square \]

### 4. PROOF OF THEOREM 1.2

We present in this section the proof of Theorem 1.2. Our argument here is partially motivated by the work of Zhang [Zha].

**Proof of Theorem 1.2.** As in Section 3 above, for \( p \in \mathbb{H}^n_\ell \) we let \( F_p = (f_p, \phi_p, q_p) := \tau^F_p \circ F \circ \sigma^p_0 \) as in (3.1) in [BH], where \( \sigma^p_0 \) and \( \tau^F_p \) are linear automorphisms of the source and target hyperquadrics such that \( F_p(0) = 0 \). We shall denote by \( M \) a sufficiently small open neighborhood of 0 in \( \mathbb{H}^n_\ell \). Recall that \( F \) is CR transversal to \( \mathbb{H}^N_{\ell, \tau} \) at \( p \in M \) if and only if \( \partial g_p/\partial w(0) \neq 0 \). Let \( E = \{ p \in M : F \text{ is not transversal to } \mathbb{H}^N_{\ell, \tau} \text{ at } p \} \). The set \( E \) is a real-analytic subvariety near 0 of \( M \). If \( E \) contains an open neighborhood of 0, then it follows from Lemma 4.1 in [BH] (see also Theorem 1.1 in [BER2]) that \( F(U) \subset \mathbb{H}^N_{\ell, \tau} \), which as mentioned in Remark 1.2 implies \( g \equiv 0 \). Thus, we may assume that the complement \( C := M \setminus E \) is open and dense. For any \( p \in C \), in view of the discussion preceding Theorem 1.2 we can apply Theorem 1.1 (a) or (b), depending on the sign of \( \partial g_p/\partial w(0) \), to conclude that there is a \( \tau_p \in Aut_0(\mathbb{H}^N_{\ell, \tau}) \) such that

\[
(\tau_p \circ F_p)(z, w) = \begin{cases} 
(z_1, \ldots, z_{\ell}, \psi_p(z, w), z_{\ell+1}, \ldots, z_{n-1}, \psi_p(z, w), 0, \ldots, 0, w), & \text{if } \frac{\partial g_p}{\partial w}(0) > 0 \\
(z_{\ell+1}, \ldots, z_{n-1}, \psi_p(z, w), z_1, \ldots, z_{\ell}, \psi_p(z, w), 0, \ldots, 0, w), & \text{if } \frac{\partial g_p}{\partial w}(0) < 0.
\end{cases}
\]
Since $\tau_p \in \text{Auto}_0(\mathbb{H}^N_{\ell'})$, it is of the form
\begin{equation}
\tau_p(z', w') = \left( \frac{\lambda_p(z' - a_pw')U_p}{\Delta_p(z', w')}, \frac{\epsilon_p \lambda_p^2 w'}{\Delta_p(z', w')} \right),
\end{equation}
where $\lambda_p > 0$, $\epsilon_p = \pm 1$, $U_pE_{(\ell', N-1)} F'_p = \epsilon_p E_{(\ell', N-1)}$, and $\Delta_p(z', w')$ is a linear polynomial in $(z', w')$. Note that both mappings on the right hand side of (4.1) are CR transversal (to $\mathbb{H}^N_{\ell'}$) at every $p \in \mathbb{H}^n_{\ell'}$. Thus, if $q \in M$ is a point such that $F_p(q)$ is not on the polar variety of $\tau_p$, i.e. $\Delta_p(F_p(q)) \neq 0$, then $F_p$ is CR transversal at $q$. It follows that $F$ is CR transversal at $q^* = \sigma_0^p(q)$ and, hence, $q^* \in C$. We conclude that for any $q^* \in E$, the point $q = (\sigma_0^p)^{-1}(q^*)$ belongs to the polar variety of $\tau_p$. On the other hand, the last component of $\tau_p \circ F_p$ is holomorphic near $q$ and, hence, the numerator in the last component of $\tau_p \circ F_p$ must also vanish at $q$, i.e.
\[ g_p(q) = 0. \]

Now, we have
\[ g_p(q) = (g \circ \sigma_0^p)(q) - g(p) - 2i\langle (\tilde{f} \circ \sigma_0^p)(q), \bar{f}(p) \rangle = 0, \]
or, equivalently,
\[ g(q^*) - g(p) - 2i\langle \tilde{f}(q^*), \bar{f}(p) \rangle = 0. \]
or, by complex conjugating the latter,
\begin{equation}
\begin{aligned}
g(p) - g(q^*) - 2i\langle \bar{f}(p), \bar{f}(q^*) \rangle &= 0. \\
\end{aligned}
\end{equation}
Recall that (4.3) holds for all $q^* \in E$ and all $p \in C$. Since $C$ is open and dense in $M$, it follows that for each fixed $q^* \in E$ there is an open neighborhood $U$ of $0$ in $\mathbb{C}^n$ such that (4.3) holds for $p \in U$. If we use the notation $Q'_{q'}$, for the Segre variety of $\mathbb{H}^N_{\ell'}$ at $q' = (z_q', w_q')$, i.e.
\[ Q'_{q'} := \{(z, w): w = \bar{w}_q - 2i\langle \bar{z}', \bar{z}_q \rangle \}, \]
then (4.3) shows that $F(U) \subset Q'_{F(q^*)}$. In particular, if $0 \in E$, then $g(z, w) \equiv 0$. This completes the proof of Theorem 1.2.

Remark 4.1. Note that in the proof of Theorem 1.2 above, we actually proved that
\begin{equation}
E = \{q^* \in M: F(U) \subset Q'_{F(q^*)}, \text{ for some open neighborhood } U \text{ of } 0 \text{ in } \mathbb{C}^n \}. 
\end{equation}

5. Examples

We end this paper with a couple of examples. Our first example shows that a holomorphic mapping $F$ as in Theorem 1.2 can be non-transversal to $\mathbb{H}^N_{\ell'}$ at 0, i.e. $\partial g/\partial w(0) = 0$, without sending a full neighborhood $U$ of 0 in $\mathbb{C}^n$ into $\mathbb{H}^N_{\ell'}$, in contrast with the case $\ell' = \ell$ treated in [BH]. Moreover, the example shows that there are mappings that are CR transversal to $\mathbb{H}^N_{\ell'}$ at all points outside a proper real-analytic subvariety of $\mathbb{H}^n_{\ell'}$ without being CR transversal at all points.
Example 5.1. Consider the following polynomial mapping $F: \mathbb{C}^5 \to \mathbb{C}^7$, 
\begin{equation}
F(z_1, z_2, z_3, z_4, w) := (4z_1z_2, 4z_2^2, 2z_2(i + w), 2z_2(i - w), 4z_2z_3, 4z_2z_4, 0).
\end{equation}
It clearly sends 0 to 0. We claim that it also sends $\mathbb{H}_5^2$ into $\mathbb{H}_7^3$. Let us write
$$
\rho := \text{Im } w - (-|z_1|^2 - |z_2|^2 + |z_3|^2 + |z_4|^2)
$$
and
$$
\rho' := \text{Im } w' - (-|z_1'|^2 - |z_2'|^2 - |z_3'|^2 + |z_4'|^2 + |z_5'|^2 + |z_6'|^2),
$$
so that $\mathbb{H}_5^2$ and $\mathbb{H}_3^3$ are defined by $\rho = 0$ and $\rho' = 0$, respectively. A straightforward computation, left to the reader, shows that
$$
\rho' \circ F = 4|z_2|^2 \rho.
$$
It follows that $F$ does not send a full neighborhood $U$ of 0 in $\mathbb{C}^5$ into $\mathbb{H}_7^3$ and $F$ is CR transversal to $\mathbb{H}_3^3$ precisely at those $p = (z, w) \in \mathbb{H}_5^2$ for which $z_2 \neq 0$ (see e.g. Remark 1.2 in [BER2]). Note that $\ell' = 3 < 4 = 2\ell = n - 1$. Consequently, both Theorem 1.1 and 1.2 apply. The conclusion of Theorem 1.2 is obviously true. Moreover, at any point where the mapping $F$ is CR transversal to $\mathbb{H}_3^3$, it follows from Theorem 1.1 that $F$ can be renormalized by composing on the left with an automorphism of $\mathbb{H}_7^3$ so as to be of the form (1.3) or (1.4). (In this particular case, $F$ can be normalized to satisfy either of (1.3) or (1.4) since the signature of $\mathbb{H}_3^3$ is half its CR dimension.)

We conclude this paper by giving an example that shows that Theorems 1.1 and 1.2 are sharp in the sense that the conclusions fail when the hypotheses on the signatures $\ell$ and $\ell'$ are not satisfied.

Example 5.2. Consider the polynomial mapping $F: \mathbb{C}^3 \to \mathbb{C}^5$ given by:
$$
F(z_1, z_2, w) := \left( z_1 + \frac{z_2^2}{2} - \frac{i}{4} w, z_2 - \frac{z_1 z_2}{2}, z_1 - \frac{z_1^2}{2} + \frac{i}{4} w, z_2 + \frac{z_1 z_2}{2}z_1 w \right).
$$
It sends 0 to 0, and we claim that it sends $\mathbb{H}_3^2$ to $\mathbb{H}_5^2$. With the notation
$$
\rho := \text{Im } w - (-|z_1|^2 + |z_2|^2), \quad \rho' := \text{Im } w' - (-|z_1'|^2 - |z_2'|^2 + |z_3'|^2 + |z_4'|^2)
$$
for the defining equations of $\mathbb{H}_3^2$ and $\mathbb{H}_5^2$, respectively, we compute that
$$
\rho' \circ F = (z_1 + \bar{z}_1) \rho.
$$
We conclude that $F$ sends $\mathbb{H}_3^2$ to $\mathbb{H}_5^2$, as claimed, and that $F$ is CR transversal to $\mathbb{H}_5^2$ at $p \in \mathbb{H}_3^2$ except on the subvariety of $\mathbb{H}_3^2$ given by the intersection with $z_1 + \bar{z}_1 = 0$. (The transversality at most points is predicted by Theorem 1.1 of [BER2].) We note that $\ell' = 2 = 2\ell = n - 1$. Also, note the following:

1. The conclusions of Theorem 1.1 (a) and (b) fail at points where $F$ is CR transversal to $\mathbb{H}_5^2$. Indeed, if $F$ could be renormalized to satisfy either of (1.3) or (1.4), then the image $F(\mathbb{C}^3)$ would be contained in a 4-dimensional subspace of $\mathbb{C}^5$. It can readily be checked that this is not the case.

2. For any point $p \in \mathbb{H}_3^2$ at which transversality fails, there is no open neighborhood $U$ of $p$ in $\mathbb{C}^3$ such that $F(U) \subset Q'_{F(p)}$; here, $Q'_{p'}$ denotes the Segre variety of $\mathbb{H}_5^2$ at $p'$. Indeed, each
Segre variety $Q'_p$ is a hyperplane in $\mathbb{C}^5$ and, as above, it can be checked that the image of $F$ is not contained in any hyperplane. Consequently, the conclusion of Theorem 1.2 also fails.

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