COMPLEX HESSIAN EQUATION ON KÄHLER MANIFOLD

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1. Introduction

Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$. In 1978, Yau [Yau78] proved the famous Calabi-Yau conjecture by solving following complex Monge-Ampère equation on $M$

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = f \omega^n$$

with positive function $f$. Later on, using tools from pluri-potential theory, Kołodziej [Kol98] studied the same equation with weaker smoothness assumption on $f$.

In this paper, we consider following complex Hessian equation on $(M, \omega)$:

$$\omega^k \wedge \omega^{n-k} = (\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^k \wedge \omega^{n-k} = f \omega^n,$$

where $k$ is a fixed integer between 2 and $n - 1$, and $f$ is a non-negative function on $M$ satisfying the compatibility condition:

$$\int_M f \omega^n = \int_M \omega^n.$$

Noticed that if $k = n$, equation (1) is just the complex Monge-Ampère equation, while if $k = 1$, equation (1) becomes the Laplacian equation. So equation (1) is a generalization of both complex Monge-Ampère equation and Laplacian equation over compact Kähler manifold.

Similar nonlinear equations have been studied extensively by many authors, see [BT76, CNS84, CKNS85, CNS85, Li90, Tru95, GL96, TW99, Li04, B105] and the reference there.

The Main result of this paper is following

**Theorem 1.1.** Let $(M, \omega)$ be a compact Kähler manifold with non-negative holomorphic bisectional curvature, and $f$ is a strictly positive smooth function, then equation (1) has smooth solution unique up to a constant.

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Our approach to Theorem 1.1 is similar to Yau’s approach to the complex Monge-Ampère equation, i.e. continuity method and a priori estimate. By the standard theory of Krylov and Evans, it suffices to prove a priori $C^2$ estimate for equation (1). More precisely, we have

**Proposition 1.2.** If $\varphi$ solves equation (1) and $\sup_M \varphi = 0$, then $\forall q > 2n$,

$$\|\varphi\|_{C^0} \leq C_q, \quad (3)$$

where $C_q$ is a constant depends on $(M, \omega)$, $q$ and $\|f - 1\|_{L^q}$.

**Proposition 1.3.** If $(M, \omega)$ has non-negative holomorphic bisectional curvature and $\varphi$ solves equation (1), then

$$\|\nabla \varphi\|_{C^0(\omega)} \leq C_1, \quad (4)$$

where $C_1$ is a constant depends on $(M, \omega)$, $\|f^{1/k}\|_{C^1(\omega)}$ and $\text{osc} \varphi$.

**Proposition 1.4.** If $(M, \omega)$ has non-negative holomorphic bisectional curvature and $\varphi$ solves equation (1), then

$$\|\partial \bar{\partial} \varphi\|_{C^0(\omega)} \leq C_2, \quad (5)$$

where $C_2$ is a constant depends on $(M, \omega)$, $\sup f$, $\inf \Delta_\omega(f^{1/k})$ and $\text{osc} \varphi$.

Both Proposition 1.3 and Proposition 1.4 require non-negative holomorphic bisectional curvature for the underline Kähler manifold. However Yau’s result requires no curvature condition, even though complex Monge-Ampère equation is worse than complex Hessian equation in certain sense, because Monge-Ampère equation is more nonlinear. One possible explanation is that the convex cone of positive real $(1, 1)$ form is independent of the Kähler metric, while the convex cone of $k$-positive real $(1, 1)$ form does depend on the Kähler metric. Besides, $\omega_{\varphi}$ being positive is a much stronger condition than $\omega_{\varphi}$ being $k$-positive. In fact, Yau used the positivity of $\omega_{\varphi}$ to control some third order terms in his proof of a priori $C^2$ estimate. Also noticed that Li [Li90] studied some nonlinear equations with certain structure conditions over compact Riemannian manifold which include the real Hessian equation as a special case. In Li’s treatment, the non-negativity of sectional curvature is needed. Right now, We don’t know whether the non-negativity of holomorphic bisectional curvature is an essential condition for the solvability of equation (1) or it is just a technical requirement.

\[1\text{In a recent paper [HMW08], we removed the curvature assumption imposed in this paper.}\]
We don’t require the function $f$ to be strictly positive in both Proposition 1.3 and Proposition 1.4, i.e. equation (1) can be degenerate. However, in order to use the theory of Krylov and Evans to get higher regularity, we need the equation to be uniformly elliptic. Hence in Theorem 1.1 we require $f$ to be strictly positive.

Bobcki [Blo05] studied the weak solution of complex Hessian equation over bounded domain in $\mathbb{C}^n$. The concept of weak solution and complex Hessian measure can be extended to the study of complex Hessian equation over compact Kähler manifold, and one can also study the corresponding potential theory.

We organize the rest of the paper as follows: in Section 2 we provide some necessary results on convex cones related to elementary symmetric functions; in Section 3 we study the uniqueness of solution and the $C^0$ estimate; in Section 4 and Section 5 we derive the a priori $C^1$ and $C^2$ estimate respectively.

2. Preliminary

Let $S_k$ be the normalized $k$-th elementary symmetric function defined on $\mathbb{R}^n$ and let

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid S_j(\lambda) > 0, j = 1, \cdots, k \}.$$ 

It is well known that $\Gamma_k$ is an open convex cone in $\mathbb{R}^n$. We call $\Gamma_k$ the $k$-positive cone in $\mathbb{R}^n$.

**Proposition 2.1.** For the $k$-positive cone $\Gamma_k$ in $\mathbb{R}^n$,

- $\Gamma_n = \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \mid \lambda_j > 0, j = 1, \cdots, n \}$. We call $\Gamma_n$ the positive cone in $\mathbb{R}^n$.
- $\Gamma_k$ is the connected component of $\{ \lambda \in \mathbb{R}^n \mid S_k(\lambda) > 0 \}$ containing $\Gamma_n$.
- $\forall \lambda \in \Gamma_k, \forall j = 1, \cdots, n$,

  $$\frac{\partial S_k}{\partial \lambda_j}(\lambda) > 0.$$ (6)

- (Newton inequalities) $\forall \lambda \in \mathbb{R}^n, \forall j = 1, \cdots, n-1$,

  $$S_{j-1}(\lambda)S_{j+1}(\lambda) \leq S_j(\lambda)^2.$$  

- (Maclaurin inequalities) For $\lambda \in \Gamma_k$,

  $$0 < S_k^{1/k}(\lambda) \leq \cdots \leq S_2^{1/2}(\lambda) \leq S_1(\lambda).$$ (7)

- (Generalized Newton-Maclaurin inequalities) For $\lambda \in \Gamma_k$

  $$\left( \frac{S_k(\lambda)}{S_1(\lambda)} \right)^{\frac{1}{k-1}} \leq \left( \frac{S_r(\lambda)}{S_2(\lambda)} \right)^{\frac{1}{r-2}},$$
where $0 \leq s < r \leq k$ and $0 \leq l < k$.

- (Garding inequality) Let $P_k$ be the complete polarization of $S_k$, then for $\Lambda^{(1)}, \ldots, \Lambda^{(k)} \in \Gamma_k$,
  \[ P_k(\Lambda^{(1)}, \ldots, \Lambda^{(k)}) \geq S_k(\Lambda^{(1)})^{1/k} \cdots S_k(\Lambda^{(k)})^{1/k}. \quad (8) \]

- $S_k^{1/k}$ is concave on $\Gamma_k$.
- If $0 \leq l < k$, then $(S_k/S_l)^{1/(k-l)}$ is concave on $\Gamma_k$.

The properties listed above are well known, for proof see [CNS85, Gar59]. In this paper, we also need following result.

**Lemma 2.2.** Suppose $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$ with

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \]

then

\[ \frac{\partial S_k}{\partial \lambda_1} \leq \frac{\partial S_k}{\partial \lambda_2} \leq \cdots \leq \frac{\partial S_k}{\partial \lambda_n}. \quad (9) \]

Moreover

\[ \lambda_1 \frac{\partial S_k}{\partial \lambda_1} \geq \frac{k}{n} S_k. \quad (10) \]

**Proof.** Equation (9) is obvious, we will only prove equation (10). Let $\sigma_k$ be the $k$-th elementary symmetric function, and $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$, then

\[ n\lambda_1 \frac{\partial \sigma_k}{\partial \lambda_1} - k\sigma_k = (n - k)\lambda_1 \sigma_{k-1}(\Lambda \setminus \lambda_1) - k\sigma_k(\Lambda \setminus \lambda_1). \]

By Newton’s Inequality for $\Lambda \setminus \lambda_1$,

\[ \lambda_1 \geq S_1(\Lambda \setminus \lambda_1) \geq \frac{S_k(\Lambda \setminus \lambda_1)}{S_{k-1}(\Lambda \setminus \lambda_1)} = \frac{k\sigma_k(\Lambda \setminus \lambda_1)}{(n - k)\sigma_{k-1}(\Lambda \setminus \lambda_1)}. \]

So\n
\[ n\lambda_1 \frac{\partial S_k}{\partial \lambda_1} \geq kS_k. \]

\[ \square \]

Let $\mathcal{H}(n)$ be the set of $n \times n$ Hermitian matrices. We extend the definition of $S_k$ on $\mathbb{R}^n$ to $\mathcal{H}(n)$ by

\[ S_k(A) = S_k(\lambda(A)) \]

where $\lambda(A)$ is the eigenvalues of Hermitian matrix $A$, then $S_k$ is a homogeneous polynomial of degree $k$ on $\mathcal{H}(n)$ and

\[ \det(A + tI_n) = \sum_{j=0}^{n} \binom{n}{j} S_j(A) t^{n-j}. \quad (11) \]
We define the $k$-positive cone in $\mathcal{H}(n)$ by
\[
\Gamma_k(\mathcal{H}(n)) = \{ A \in \mathcal{H}(n) \mid S_j(A) > 0, j = 1, \cdots, k \}.
\]
Because of equation (11), it is easy to see that $S_k$ is invariant under the adjoint action of $U(n)$, hence $\Gamma_k(\mathcal{H}(n))$ is also $U(n)$-invariant. Besides, all the the properties listed in Proposition 2.1 are also true for $S_k$ defined on $\mathcal{H}(n)$. Especially, for any $A = (A_{\alpha\bar{\beta}}) \in \Gamma_k(\mathcal{H}(n))$, the matrix with entries given by
\[
F^{\alpha\bar{\beta}} = \frac{\partial \log S_k(A)}{\partial A_{\alpha\bar{\beta}}}
\] (12)
is a positive definite Hermitian matrix.

Now consider a complex vector space $V$ of complex dimension $n$ with a fixed Hermitian metric $g$. Let $\omega$ be the Hermitian form of $g$. After fixing an unitary basis $\{\theta^1, \cdots, \theta^n\}$ for $V^*$, any real $(1,1)$ form $\chi$ can be written as
\[
\chi = \sqrt{-1} \chi_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}},
\]
where $A_{\chi} = (\chi_{\alpha\bar{\beta}})$ is a Hermitian matrix. We define the $k$-th Hermitian $S_k(\chi)$ of $\chi$ with respect to $\omega$ as
\[
S_k(\chi) = S_k(A_{\chi}) = S_k((\chi_{\alpha\bar{\beta}})).
\]
The definition of $S_k$ is independent of the choice of unitary basis, in fact $S_k(\chi)$ can be defined without the use of unitary basis by
\[
\chi^k \wedge \omega^{n-k} = S_k(\chi) \omega^n.
\]
Let $\Lambda^{1,1}_R V^*$ be the space of real $(1,1)$ form. We define the $k$-positive cone in $\Lambda^{1,1}_R V^*$ by
\[
\Gamma_k(V^{1,1}) = \{ \chi \in \Lambda^{1,1}_R V^* \mid S_j(\chi) > 0, j = 1, \cdots, k \}.
\]
All the properties listed in Proposition 2.1 continue to be true for the $k$-th Hessian of real $(1,1)$ forms. Especially, for any $\chi_1, \cdots, \chi_k \in \Gamma_k(V^{1,1}),$
\[
\chi_1 \wedge \cdots \wedge \chi_k \wedge \omega^{n-k} > 0.
\] (13)

Let $(M, \omega)$ be a Kähler manifold with Kähler form $\omega$. The tangent space of $M$ at every point is a complex vector space with Hermitian metric, so the construction of $\Gamma_k(V^{1,1})$ can be carried out pointwise on $M$, hence we get a distribution of open convex cones in the space of real $(1,1)$ form on $M$. Since the parallel transportation keeps the Kähler metric, so this distribution of convex cones is also invariant under the parallel transportation. For simplicity, we still use $\Gamma_k$ to denote these convex cones.
Definition 2.3. A real \((1,1)\) form \(\chi \in \Omega^{1,1}(M, \mathbb{R})\) is \(k\)-positive with respect to \(\omega\), if \(\chi \in \Gamma_k\).

Let \(C^\infty(M, \mathbb{R})\) be the set of real valued smooth functions on \(M\). Denote
\[
P_k(M, \omega) = \{ \varphi \in C^\infty(M, \mathbb{R}) \mid \omega \varphi = \omega + \sqrt{-1} \bar{\partial} \partial \varphi \text{ is } k\text{-positive} \}.
\]

Proposition 2.4. If \(\varphi \in C^\infty(M, \mathbb{R})\) solves (1), then \(\varphi \in P_k(M, \omega)\).

Proof. Since \(\omega \varphi\) is a positive form at the point where \(\varphi\) achieves minimum, so \(\omega \varphi \in \Gamma_n \subset \Gamma_k\) at the minimum point. This together with the facts that \(\Gamma_k \subset \mathcal{H}(n)\) is the connected component of \(\{ S_k > 0 \}\) containing \(\Gamma_n\), and the distribution of these cones is invariant under parallel transportation shows that \(\omega \varphi \in \Gamma_k\) at every point, hence \(\varphi \in P_k(M, \omega)\).

Proposition 2.5. If \(\varphi \in P_k(M, \omega)\), then the operator
\[
\varphi \mapsto S_k(\omega \varphi) = \frac{\omega^k \wedge \omega^{n-k}}{\omega^n}
\]
is elliptic at \(\varphi\).

3. Uniqueness

Suppose both \(\varphi\) and \(\psi\) solve equation (1), then
\[
0 = \int_M (\psi - \varphi) (\omega^k \wedge \omega^{n-k} - \omega^k \wedge \omega^{n-k})
= \sum_{l=0}^{k-1} \int_M (\psi - \varphi) \sqrt{-1} \bar{\partial} \partial (\psi - \varphi) \wedge \omega^k \wedge \omega^{k-1-l} \wedge \omega^{n-k}
= - \sum_{l=0}^{k-1} \int_M \sqrt{-1} \bar{\partial} (\psi - \varphi) \wedge \partial (\psi - \varphi) \wedge \omega^l \wedge \omega^{k-1-l} \wedge \omega^{n-k}
\leq 0
\]
The last inequality is true because
\[
\sqrt{-1} \bar{\partial} (\psi - \varphi) \wedge \partial (\psi - \varphi) \in \Gamma_n, \quad \omega \varphi \in \Gamma_k \quad \text{and} \quad \omega \psi \in \Gamma_k,
\]
hence by equation (13), for \(l = 0, \ldots, k - 1\),
\[
\sqrt{-1} \bar{\partial} (\psi - \varphi) \wedge \partial (\psi - \varphi) \wedge \omega^l \wedge \omega^{k-1-l} \wedge \omega^{n-k} \geq 0.
\]
So \(\partial (\psi - \varphi) = 0\), and
\[
\omega \varphi = \omega \psi.
\]

Same positivity argument can be used to prove Proposition 1.2 by Yau’s Moser iteration, see [Siu87] or [Tia00] for details.
4. $C^1$ ESTIMATE

We will follow Blocki’s approach in [Blo07] to get the a priori $C^1$ estimate. But unlike Blocki, we will use the covariant derivative with respect to $\omega$ throughout this paper.

First let’s fix some notation. Let $\{e_1, \cdots, e_n\}$ be a fixed unitary frame for $(M, \omega)$, and $\{\theta^1, \cdots, \theta^n\}$ be the duel frame. For $\varphi \in \mathcal{P}_k(M, \omega)$ with
\[ \sqrt{-1}\partial\bar{\partial}\varphi = \sqrt{-1}\varphi_{\alpha\bar{\beta}} \theta^\alpha \wedge \bar{\theta}^\beta, \]
let
\[ G^{\alpha\bar{\beta}} = F^{\alpha\bar{\beta}}(I_n + (\varphi_{\gamma\delta})) \]
where $(F^{\alpha\bar{\beta}})$ is the matrix-valued function defined on $\Gamma_k(\mathcal{H}(n))$ by equation (12). Let $(G_{\alpha\bar{\beta}})$ be the inverse matrix of $(G^{\alpha\bar{\beta}})$, then
\[ G = \sqrt{-1}G_{\alpha\bar{\beta}} \theta^\alpha \wedge \bar{\theta}^\beta \]
induces a Hermitian metric $G$ on $M$, and this metric is independent of the choice of unitary frame.

Suppose $\varphi$ solves equation (1) with $\inf \varphi = 0$ and $\sup \varphi = \text{osc} \varphi = C_0$. Let
\[ B = \|\nabla \varphi\|^2_\omega = \varphi_{,\gamma}\varphi^{,\gamma} \quad \text{and} \quad A = \log B - h(\varphi), \]
with $h(t) = \frac{1}{2} \log(2t + 1)$. Let $\Delta' = G^{\alpha\bar{\beta}}\nabla_{e_\alpha} \nabla_{e_{\bar{\beta}}}$, then
\[ \Delta'A = \frac{\Delta'B}{B} - \frac{1}{B^2}G^{\alpha\bar{\beta}}B_{,\alpha}B_{,\bar{\beta}} - h'(\varphi)\Delta'\varphi - h''(\varphi)G^{\alpha\bar{\beta}}\varphi_{,\alpha}\varphi_{,\bar{\beta}} \]
and
\[ \Delta'B = G^{\alpha\bar{\beta}}(\varphi^{,\gamma}_{,\alpha\bar{\beta}}\varphi_{,\gamma} + \varphi^{,\bar{\gamma}}_{,\alpha\bar{\beta}}\varphi_{,\gamma} + \varphi^{,\gamma}_{,\alpha}\varphi_{,\bar{\beta}} + \varphi^{,\bar{\gamma}}_{,\alpha}\varphi_{,\bar{\beta}}) \geq G^{\alpha\bar{\beta}}(\varphi^{,\gamma}_{,\alpha\bar{\beta}}\varphi_{,\gamma}). \]
By Ricci identity
\[ \varphi^{,\gamma}_{,\alpha\bar{\beta}} = \varphi_{,\alpha\bar{\beta}}^{,\gamma} \quad \text{and} \quad \varphi_{,\gamma\alpha\bar{\beta}} = \varphi_{,\alpha\bar{\beta}}^{,\gamma} + \varphi_{,\eta}R^\eta_{\alpha\gamma\bar{\beta}}. \]
Since $\varphi$ solves equation (1), so
\[ G^{\alpha\bar{\beta}}\varphi_{,\alpha\bar{\beta}} = (\log f)^,\gamma \quad \text{and} \quad G^{\alpha\bar{\beta}}\varphi_{,\alpha\bar{\beta}} = (\log f)^,\gamma. \]
Hence
\[ \Delta'B \geq 2\text{Re}(\nabla(\log f), \nabla \varphi)_\omega + G^{\alpha\bar{\beta}}\varphi_{,\eta}\varphi^{,\gamma}R^\eta_{\alpha\gamma\bar{\beta}}. \]
If $(M, \omega)$ has non-negative holomorphic bisectional curvature, then
\[ G^{\alpha\bar{\beta}}\varphi_{,\eta}\varphi^{,\gamma}R^\eta_{\alpha\gamma\bar{\beta}} \geq 0 \]
therefore
\[ \Delta'B \geq 2\text{Re}(\nabla(\log f), \nabla \varphi)_\omega. \]
Noticed that $S_k$ is homogeneous of polynomial of degree $k$, so we have
\[ \Delta' \varphi = G^{\alpha \bar{\beta}} \varphi_{,\alpha \bar{\beta}} = k - \text{tr}_{\omega} G. \]

If $A$ achieves maximum at point $p$, then at point $p$, $\nabla A = 0$, i.e.
\[ \frac{B_{,\alpha}}{B} = h'(\varphi) \varphi_{,\alpha} \quad \text{and} \quad \frac{B_{,\bar{\beta}}}{B} = h'(\varphi) \varphi_{,\bar{\beta}}. \]

Hence at the maximum point $p$,
\[ \Delta' A \geq \frac{2}{B} \text{Re} \langle \nabla (\log f), \nabla \varphi \rangle - (h'' + h'^2) G^{\alpha \bar{\beta}} \varphi_{,\alpha} \varphi_{,\bar{\beta}} - k h' + h' \text{tr}_{\omega} G. \]

Since $\varphi$ takes value between 0 and $C_0$, so
\[ \frac{1}{2C_0 + 1} \geq h' \geq 1 \quad \text{and} \quad -h'' - h'^2 \geq \frac{1}{(2C_0 + 1)^2}. \]

Therefore
\[ \Delta' A \geq \frac{2}{B} \text{Re} \langle \nabla (\log f), \nabla \varphi \rangle + \frac{G^{\alpha \bar{\beta}} \varphi_{,\alpha} \varphi_{,\bar{\beta}}}{(2C_0 + 1)^2} \cdot k + \frac{\text{tr}_{\omega} G}{2C_0 + 1}. \quad (19) \]

If the eigenvalue of $\omega + \sqrt{-1} \partial \bar{\partial} \varphi$ with respect to $\omega$ is $(\lambda_1, \cdots \lambda_n)$, then
\[ \text{tr}_{\omega} G = k \frac{S_{k-1}(\lambda)}{S_k(\lambda)} \quad \text{and} \quad f = S_k(\lambda). \]

Noticed
\[ |\nabla \log f| = \frac{k |\nabla (f^{1/k})|}{f^{1/k}} \leq |\nabla (f^{1/k})| \text{tr}_{\omega} G, \]

here we’ve used the Maclaurin Inequality
\[ \frac{1}{f^{1/k}} = \frac{1}{S_k^{1/k}(\lambda)} \leq \frac{S_{k-1}(\lambda)}{S_k(\lambda)} = \frac{1}{k} \text{tr}_{\omega} G. \]

So at the maximum point $p$,
\[ 0 \geq \left( \frac{1}{2C_0 + 1} - \frac{2|\nabla (f^{1/k})|}{|\nabla \varphi|} \right) \text{tr}_{\omega} G + \frac{1}{(2C_0 + 1)^2} G^{\alpha \bar{\beta}} \varphi_{,\alpha} \varphi_{,\bar{\beta}} - k \quad (20) \]

We may also assume that
\[ ||\nabla \varphi|| \leq \frac{1}{2(2C_0 + 1)} ||\nabla (f^{1/k})||, \]

therefore at $p$,
\[ k \geq \frac{1}{(2C_0 + 1)^2} G^{\alpha \bar{\beta}} \varphi_{,\alpha} \varphi_{,\bar{\beta}} + \frac{1}{2(2C_0 + 1)} \text{tr}_{\omega} G. \]

So at the maximum point $p$,
\[ \text{tr}_{\omega} G \leq C, \quad \text{and} \quad G^{\alpha \bar{\beta}} \varphi_{,\alpha} \varphi_{,\bar{\beta}} \leq C \quad (21) \]
where $C$ is a constant depends on $C_0$ only. By the Generalized Newton-Maclaurin Inequality,
\[
S_1(\lambda) \leq S_k(\lambda) \left( \frac{S_{k-1}(\lambda)}{S_k(\lambda)} \right)^{k-1} \leq f \left( \frac{\text{tr}_\omega G}{k} \right)^{k-1},
\]
(22)
$S_1(\lambda)$ is bounded by constant depends on $C_0$ and $\sup f$ at point $p$. Noticed that $\forall \lambda \in \mathbb{R}^n$,
\[
\sum_j \lambda_j^2 = (\sum_j \lambda_j)^2 - \sum_{i \neq j} \lambda_i \lambda_j = (nS_1(\lambda))^2 - n(n-1)S_2(\lambda).
\]
Since $\lambda \in \Gamma_k$ with $k \geq 2$, so $S_2(\lambda) \geq 0$ and
\[
\sup_j |\lambda_j| \leq nS_1(\lambda).
\]
(23)
Therefor at the maximum point $p$, the eigenvalue of $\omega_\phi$ with respect to $\omega$ is bounded. If we further assume that $\lambda_1 \geq \cdots \geq \lambda_n$, then the smallest eigenvalue of matrix $(G^{\alpha\bar{\beta}})$ is
\[
1, \frac{\partial S_k}{\partial \lambda_1} \geq \frac{k}{n\lambda_1},
\]
(24)
Combine equation (21) and (22), $\nabla \phi$ is bounded at the maximum point of $A$, therefore $A$ is bounded everywhere, hence $\nabla \phi$ is also bounded everywhere, and this finishes the proof of Proposition 1.3.

5. $C^2$ Estimate

Same as in Section 4, we will use maximum principle to get some a priori estimate in this section. We can keep using the Hermitian metric introduced by equation (14). But in order to get better regularity result, we will introduce a new Hermitian metric. For $A = (A_{\alpha\bar{\beta}}) \in \Gamma_k(H(n))$, denote
\[
\tilde{F}^{\alpha\bar{\beta}}(A) = \frac{\partial S_k^{1/k}}{\partial A_{\alpha\bar{\beta}}},
\]
then $(\tilde{F}^{\alpha\bar{\beta}}(A))$ is positive definite Hermitian matrix. Using the same frame $(e_1, \cdots, e_n)$ and co-frame $(\theta^1, \cdots, \theta^n)$ as in Section 4 for $\varphi \in \mathcal{P}_k(M, \omega)$ with
\[
\bar{\partial} \bar{\partial} \varphi = \varphi,_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}},
\]
let
\[
H^{\alpha\bar{\beta}} = \tilde{F}^{\alpha\bar{\beta}}(I_n + (\varphi,_{\alpha\bar{\beta}})),
\]
and $(H_{\alpha\bar{\beta}}) = (H^{\alpha\bar{\beta}})^{-1}$, then
\[
H = \sqrt{-1}H_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta
\]
(25)
induces a Hermitian metric on $M$.

Let $\tilde{\Delta} = H^{\alpha\bar{\beta}} \nabla_{e\bar{\beta}} \nabla_{e\alpha}$, then

$$\tilde{\Delta}(n + \Delta \varphi) = H^{\alpha\bar{\beta}} \varphi_{\gamma \gamma \alpha \bar{\beta}}.$$  

From the equation

$$S_k^{1/k}(I_n + (\varphi_{\alpha \bar{\beta}})) = f^{1/k},$$

one get

$$H^{\alpha\bar{\beta}} \varphi_{\alpha \bar{\beta} \gamma \gamma} \geq \Delta(f^{1/k}).$$

Here we’ve used the concavity of $S_k^{1/k}$. By Ricci identity,

$$\varphi_{\alpha \bar{\beta} \gamma \gamma} = \varphi_{\gamma \gamma \alpha \bar{\beta}} + \varphi_{\xi \bar{\beta}} R^\xi_{\gamma \alpha \bar{\beta}} - \varphi_{\alpha \bar{\beta}} R_{\bar{\alpha} \alpha \gamma \gamma}$$

So

$$\tilde{\Delta}(n + \Delta \varphi) \geq \Delta f + H^{\alpha\bar{\beta}} \varphi_{\gamma \gamma \alpha \bar{\beta}} R^\xi_{\gamma \alpha \bar{\beta}} - H^{\alpha\bar{\beta}} \varphi_{\xi \bar{\beta}} R^\xi_{\alpha \gamma \gamma}$$

Choose a unitary frame so that $\varphi_{\alpha \bar{\beta}}$ is diagonal matrix, then

$$H^{\alpha\bar{\beta}} \varphi_{\xi \bar{\beta}} R^\xi_{\gamma \alpha \bar{\beta}} - H^{\alpha\bar{\beta}} \varphi_{\xi \bar{\beta}} R^\xi_{\alpha \gamma \gamma}$$

$$= \sum_{\alpha, \gamma} (H^{\alpha\bar{\alpha}} \varphi_{\gamma \gamma \alpha \bar{\alpha}} - H^{\alpha\bar{\alpha}} \varphi_{\alpha \bar{\alpha} R_{\bar{\alpha} \alpha \gamma \gamma}})$$

$$= \sum_{\alpha, \gamma} H^{\alpha\bar{\alpha}} R_{\gamma \alpha \bar{\alpha}} (\varphi_{\gamma \gamma} - \varphi_{\alpha \bar{\alpha}})$$

$$= \sum_{\alpha < \gamma} R_{\gamma \alpha \bar{\alpha}} (\varphi_{\gamma \gamma} - \varphi_{\alpha \bar{\alpha}}) (H^{\alpha \bar{\alpha}} - H^{\gamma \gamma})$$

If $\omega$ is diagonalized as $(\lambda_1, \cdots, \lambda_n)$ with respect to $\omega$, then

$$\varphi_{\alpha \bar{\alpha}} = \lambda_\alpha - 1 \quad \text{and} \quad H^{\alpha \bar{\alpha}} = \frac{1}{k} S_k^{\alpha \bar{\alpha}} \frac{\partial S_k}{\partial \lambda_\alpha}.$$  

Hence

$$\varphi_{\alpha \bar{\alpha}} = \lambda_\alpha - 1 \quad \text{and} \quad H^{\alpha \bar{\alpha}} = \frac{1}{k} S_k^{\alpha \bar{\alpha}} \frac{\partial S_k}{\partial \lambda_\alpha}.$$  

If $(M, \omega)$ has non-negative holomorphic bisectional curvature, i.e.

$$R_{\gamma \alpha \bar{\alpha}} \geq 0,$$

then

$$H^{\alpha\bar{\beta}} \varphi_{\xi \bar{\beta}} R^\xi_{\gamma \alpha \bar{\beta}} - H^{\alpha\bar{\beta}} \varphi_{\xi \bar{\beta}} R^\xi_{\alpha \gamma \gamma} \geq 0$$

and

$$\tilde{\Delta}(n + \Delta \varphi) \geq \Delta(f^{1/k}).$$ (28)
Consider $n + \Delta \varphi - \varphi$. At the point where $n + \Delta \varphi + \varphi$ achieves the maximum,

$$0 \geq \tilde{\Delta}(n + \Delta \varphi - \varphi) \geq \Delta(f^{1/k}) - f^{1/k} + \text{tr}_\omega H.$$

Hence at the maximum point

$$\text{tr}_\omega H \leq f^{1/k} - \Delta(f^{1/k}) \leq C$$

where $C$ is a constant depends on $\sup f$ and $\inf \Delta(f^{1/k})$. Noticed that

$$\text{tr}_\omega H = kS_k^{\frac{1}{2}} \frac{S_{k-1}}{S_k},$$

and

$$n + \Delta \varphi = S_1(\lambda) \leq S_k \left( \frac{S_{k-1}}{S_k} \right)^{k-1} = S_k \left( \frac{S_{k-1}}{S_k} \right)^{k-1},$$

so by bounding $\text{tr}_\omega H$, we also bound $n + \Delta \varphi$ by constant depends on $\sup f$ and $\Delta(f^{1/k})$ at the maximum point of $n + \Delta \varphi - \varphi$. Therefore get a global bound for $n + \Delta \varphi$. Then by equation (23), we also get global bound for the eigenvalues of $\omega_\varphi$ with respect to $\omega$, i.e.

$$\|\partial \bar{\partial} \varphi\|_{C_0(\omega)} \leq C_2$$

with $C_2$ depends on $\sup f$, $\inf \Delta(f^{1/k})$ and $\text{osc} \varphi$.

References

[Blo05] Zbigniew Błocki. Weak solutions to the complex Hessian equation. *Ann. Inst. Fourier (Grenoble)*, 55(5):1735–1756, 2005.

[Blo07] Zbigniew Błocki. A gradient estimate in the calabi-yau theorem. *preprint*, 2007.

[BT76] Eric Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37(1):1–44, 1976.

[CKNS85] L. Caffarelli, J. J. Kohn, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations. *Comm. Pure Appl. Math.*, 38(2):209–252, 1985.

[CNS84] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.*, 37(3):369–402, 1984.

[CNS85] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.*, 155(3-4):261–301, 1985.

[Går59] Lars Gårding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, 8:957–965, 1959.

[GL96] Bo Guan and Yan Yan Li. Monge-Ampère equations on Riemannian manifolds. *J. Differential Equations*, 132(1):126–139, 1996.

[HMW08] Zuoliang Hou, Xi-Nan Ma, and Damin Wu. Complex hessian equations on a compact kähler manifold. *preprint*, 2008.
[Kol98] Slawomir Kołodziej. The complex Monge-Ampère equation. *Acta Math.*, 180(1):69–117, 1998.

[Li90] Yan Yan Li. Some existence results for fully nonlinear elliptic equations of Monge-Ampère type. *Comm. Pure Appl. Math.*, 43(2):233–271, 1990.

[Li04] Song-Ying Li. On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian. *Asian J. Math.*, 8(1):87–106, 2004.

[Siu87] Yum Tong Siu. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, volume 8 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1987.

[Tia00] Gang Tian. *Canonical metrics in Kähler geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld.

[Tru95] Neil S. Trudinger. On the Dirichlet problem for Hessian equations. *Acta Math.*, 175(2):151–164, 1995.

[TW99] Neil S. Trudinger and Xu-Jia Wang. Hessian measures. II. *Ann. of Math. (2)*, 150(2):579–604, 1999.

[Yau78] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

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