WHEN A STOCHASTIC EXPONENTIAL IS A TRUE MARTINGALE.
EXTENSION OF A METHOD OF BENEŠ.

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Abstract. Let $\mathfrak{z}$ be a stochastic exponential, i.e., $\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \, dM_s$, of a local martingale $M$ with jumps $\Delta M_t > -1$. Then $\mathfrak{z}$ is a nonnegative local martingale with $E_{\mathfrak{z}} \leq 1$. If $E_{\mathfrak{z}T} = 1$, then $\mathfrak{z}$ is a martingale on the time interval $[0, T]$. Martingale property plays an important role in many applications. It is therefore of interest to give natural and easy verifiable conditions for the martingale property. In this paper, the property $E_{\mathfrak{z}T} = 1$ is verified with the so-called linear growth conditions involved in the definition of parameters of $M$, proposed by Girsanov [10]. These conditions generalize the Beneš idea, [3], and avoid the technology of piece-wise approximation. These conditions are applicable even if Novikov, [30], and Kazamaki, [18], conditions fail. They are effective for Markov processes that explode, Markov processes with jumps and also non Markov processes. Our approach is different to recently published papers [5] and [29].

1. Introduction

Let $M = (M_t)_{t \in [0, T]}$ be a martingale (local martingale) with paths from Skorokhod’s space $D$. So $M = M^c + M^d$, where $M^c$ and $M^d$ are continuous and purely discontinuous martingales respectively. Denote by $\Delta M_t := M_t - M_{t-}$ the jump process of the martingale $M$ and by $\langle M^c \rangle_t$ the predictable quadratic variation of continuous martingale $M^c$. If $\Delta M_t > -1$, $t > 0$, then the Itô equation

$$3_t = 1 + \int_0^t 3_s \, dM_s$$

obeys the unique nonnegative solution (eg. [9])

$$3_t = \exp \left( M_t - \frac{1}{2} \langle M^c \rangle_t \right) \prod_{s=0}^t (1 + \Delta M_s) e^{-\Delta M_s}, \quad t \geq 0,$$

known as Doleans-Dade exponential. It is well known that $\mathfrak{z}$ is a nonnegative local martingale and, therefore, it is a supermartingale with $E_{\mathfrak{z}t} \leq 1$ for any $t \geq 0$. If

$$E_{\mathfrak{z}T} = 1, \quad \exists \ T > 0,$$

then $\mathfrak{z} = (\mathfrak{z}_t)_{t \in [0, T]}$ is a martingale, i.e., $E_{\mathfrak{z}t} \equiv 1, \ t \in [0, T]$. The property (1.3) is used in different applications, where $\mathfrak{z}_T$ plays a role of the Radon-Nikodym derivative of one

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probability measure w.r.t. another one supported on the Skorokhod space. The random variable \( z_T \) is one of important objects involved in Statistics of Random Processes (Liptser-Shiryaev, [26]), in Financial Mathematics (Shiryaev, [35], Sin, Carlos A. [36], etc.), in the proof of existence of weak solutions of Itô’s equations (Rydberg [33]) and many others important applications.

Below we give a short survey of known conditions implying \( \mathbb{E} z_T = 1 \) provided that \( M \equiv M^c_c \), in which case
\[
\mathbb{E} \exp \left( \frac{1}{2} \langle M^c \rangle_T - \frac{1}{2} T \right) = 1.
\]

Girsanov in his classical paper [10] used the condition
\[
\langle M^c \rangle_T \leq \text{const.}
\]

This condition was weakened in many variants and accomplished by Novikov condition, [30],
\[
\mathbb{E} e^{\frac{1}{2} \langle M^c \rangle_T} < \infty,
\]
by Kazamaki condition [18]
\[
\sup_{t \in [0, T]} \mathbb{E} e^{\frac{1}{2} M^c_t} < \infty
\]
and, finally, by Krylov condition [20]:
\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \exp \left( [1 - \varepsilon] \frac{1}{2} \langle M^c \rangle_T \right) < \infty, \quad \lim_{\varepsilon \to 0} \varepsilon \log \sup_{t \in [0, T]} \mathbb{E} \exp \left( [1 - \varepsilon] \frac{1}{2} M^c_t \right) < \infty.
\]

It would be noted that for any \( \varepsilon \in (0, 1/2) \) there exists a martingale \( M^c \) such that \( \mathbb{E} e^{\frac{1}{2} (1 - \varepsilon) \langle M^c \rangle_T} < \infty \), while (1.3) fails (see. [26], p. 224).

Consider a simple example. Let \( B_t \) be Brownian motion and let \( M^c_t = 2 \int_0^t B_s dB_s \).

Then Kazamaki’s condition fails for \( T \geq 1 \), because \( \mathbb{E} \exp \left( \frac{1}{2} B_1^2 - \frac{1}{2} T \right) = \infty \), that is,
\[
\mathbb{E} \exp \left( \int_0^T B_s dB_s \right) = \mathbb{E} \exp \left( \frac{1}{2} B_T^2 - \frac{1}{2} T \right) = \infty.
\]

Since Novikov’s condition implies Kazamaki’s condition Novikov’s condition fails too.

All aforementioned conditions, guaranteeing (1.3), are formulated in terms of \( M^c \) and \( \langle M^c \rangle \). Verifications of these conditions require often complicated and even non-achievable calculations. A natural question is how to check (1.3) in the setting of the aforementioned example for \( T \geq 1 \). It is surprising that it is possible to do with the help of Beneš’s condition, [3].

The main aim of this paper consists in showing that conditions referred to later as “Beneš conditions” provide (1.3) for martingales \( M \) of sufficiently general structure (1.2).

Together with Beneš’ condition we shall use a uniform integrability condition, a test which was proposed by Hitsuda in [11].

Note also that the proposed approach of verification of \( \mathbb{E} z_T = 1 \) fits naturally with the method of establishing \( \mathbb{E} z_\infty = 1 \) in [14].

Approaches used in Markov setting that do not use Beneš’ conditions can be found in recent paper of Cheridito, Filipović, Yor, [5].

In order to formulate our result assume \( M \) in (1.2) is a part of some semimartingale \( X \) involved in typical models met in applications:

1) a non-explosive Markov process
2) a non Markovian semimartingale
3) *a possibly explosive Markov process.*

The general case is analyzed in Section 3. In Section 2, we consider the case of continuous martingales, which is the simplest from technical point of few. In Section 2 we show how to replace the classical Beneš proof by a new one serving most models.

Applicability of the main result is shown on many examples. An auxiliary technical result (generalized Girsanov theorem) is given in Appendix A.

**Remark 1.** The question whether \( z \) a martingale arises often in view of the following problem. Let \( \mu^X \) and \( \mu^Y \) be probability measures supported on Skorokhod space. These measures are distributions of semimartingales \((X, Y) = (X_t, Y_t)_{t \in [0, T]}\) respectively. Then the question becomes under which conditions \( \mu^X \ll \mu^Y \) with the Radon-Nikodym derivative \( \frac{d\mu^Y}{d\mu^X}(X) = z \)? These result can be found in [10] (Girsanov), [7] (Dawson), [12] (Itô - Watanabe), [15] (Kadota - Shepp), [21] (Kunita), [23] (Lépingle - Mémin), [26, Ch. 7] (Liptser - Shiryaev), [14] (Kabanov - Liptser - Shiryaev) and more new papers [33] (Rydberg), [31] (Palmowski - Rolski), [37] (Wong - Heyde ), [5] (Cheridito - Filipović - Yor), [29] (Mijitovic - Urusov), [2] (Baudoin - Nualart), etc.

## 2. Beneš conditions. New proofs

In this Section we show the example where conditions of Kazamaki and Novikov fail, but Beneš’ conditions don’t.

We consider two types of continuous martingales:

\[
M'_t = \int_0^t \sigma(B_s)dB_s \quad \text{and} \quad M''_t = \int_0^t \sigma_s(B)dB_s,
\]

where functions \( \sigma(y) \) and \( \sigma_s(y) \) of arguments \( y \in \mathbb{R} \) and \( (s, y_{[0,s]}) \in \mathbb{R}_+ \times \mathbb{C}_{[0,\infty)} \) are measurable w.r.t. corresponding \( \sigma \)-algebras and satisfy the linear grows conditions:

\[
\sigma^2(y) \leq r [1 + y^2], \quad (2.1)
\]

\[
\sigma^2(s, y_{[0,s]}) \leq r \left[ 1 + \sup_{s' \leq s} y_{s'}^2 \right]. \quad (2.2)
\]

It is well known from the classical Beneš paper [3] (see also Karatzas, Shreve, [17]), that \((1.3)\) holds with any \( T > 0 \). We show that this result can be easily obtained avoiding piece-wise technique approximation, used by Beneš.

Theorems 2.1 and 2.2 were formulated by Beneš, but the proofs below are new.

**Theorem 2.1.** Let \( M_t = M'_t \) and \( 2.1 \) hold true.

Then \((1.3)\) is valid for any \( T > 0 \).

**Proof.** Choose \( \tau_n = \inf \{ t : (\Delta t \vee B^2_t) \geq n \} \geq n \geq 1 \). Write

\[
\Delta t \land \tau_n = 1 + \int_0^t I_{\{s \leq \tau_n\}} \Delta s \sigma(B_s)dB_s.
\]

\[\text{inf}\{\emptyset\} = \infty\]
The definition of $\tau_n$ and condition (2.1) imply that the integrand in the Itô’s integral $\int_0^t I_{s \leq \tau_n} \delta \sigma(B_s) dB_s$ is bounded. Therefore the process $\delta_{t \wedge \tau_n}$ is a square integrable martingale, that is, $\mathbb{E}_{\delta_{t \wedge \tau_n}} = 1$. If a family $(\delta_{t \wedge \tau_n})_{n \geq 1}$ is uniformly integrable, then $\mathbb{E}_{\delta_{t \wedge \tau_n}} \xrightarrow{n \to \infty} 1$ and $\lim_{n \to \infty} \mathbb{E}_{\delta_{t \wedge \tau_n}} = \mathbb{E}_T = 1$. Thus, it is left to check the uniform integrability. Following Hitsuda we apply Vallée de Poussin’s theorem with function $x \log x$, $x \geq 0$ and show that

$$\sup_n \mathbb{E}_{\delta_{t \wedge \tau_n}} \log(\delta_{t \wedge \tau_n}) < \infty.$$ 

Since $\mathbb{E}_{\delta_{T \wedge \tau_n}} = 1$, change the probability measure $\mathbb{Q}^n \ll \mathbb{P}$ with $d\mathbb{Q}^n = \delta_{T \wedge \tau_n} d\mathbb{P}$ and obtain (here $\mathbb{E}^n$ denotes an expectation relative to $\mathbb{Q}^n$)

$$\sup_n \mathbb{E}_{\delta_{T \wedge \tau_n}} \log(\delta_{T \wedge \tau_n}) = \sup_n \mathbb{E}^n log(\delta_{T \wedge \tau_n}).$$

Next, $\delta_t = \int_0^t \sigma(B_s) dB_s - \frac{1}{2} \int_0^t \sigma^2(B_s) ds \leq \int_0^t \sigma(B_s) dB_s$ implies

$$\sup_n \mathbb{E}_{\delta_{T \wedge \tau_n}} \log(\delta_{T \wedge \tau_n}) \leq \sup_n \mathbb{E}^n \int_0^T I_{[s \leq \tau_n]} \sigma(B_s) dB_s$$

which gives us a hint to use a representation of $B_{t \wedge \tau_n}$ as a $\mathbb{Q}^n$ - semimartingale in the formula $\mathbb{E}^n \int_0^T I_{[s \leq \tau_n]} \sigma(B_s) dB_s$. By the classical Girsanov theorem

$$B_{t \wedge \tau_n} = \int_0^t I_{[s \leq \tau_n]} \sigma(B_s) ds + \tilde{B}_t^n$$

with $\mathbb{Q}^n$ - Brownian motion $\tilde{B}_t^n$ stopped at time $\tau_n$ and having the predictable quadratic variation $(\tilde{B})_t = t \wedge \tau_n$. Hence

$$\sup_n \mathbb{E}^n \log(\delta_{T \wedge \tau_n}) \leq \sup_n \mathbb{E}^n \int_0^T I_{[s \leq \tau_n]} \sigma^2(B_s) ds$$

and the proof is reduced to verification of $\mathbb{E}^n \int_0^T \sigma^2(B_s \wedge \tau_n) ds \leq x$ with a constant $x$ independent of $n$. Beneš condition (2.1) is the key point of the required verification. It enables to replace $\mathbb{E}^n \int_0^T \sigma^2(B_s \wedge \tau_n) ds$ by $\mathbb{E}^n \int_0^T (B^n_{s \wedge \tau_n}) ds$ and verify only the validity of

$$\mathbb{E}^n \int_0^T (B^n_{s \wedge \tau_n}) ds \leq \mathbb{E}^n \int_0^T (B^2_{s \wedge \tau_n}) ds \leq x$$

with $x$ independent of $n$. To this end, by applying the Itô formula to $B^2_{t \wedge \tau_n}$, we obtain

$$B^2_{t \wedge \tau_n} = 2 \int_0^t I_{[s \leq \tau_n]} B_s \sigma(B_s) ds + 2 \int_0^t I_{[s \leq \tau_n]} B_s d\tilde{B}_s^n + \langle \tilde{B}^n \rangle_t$$

$$\mathbb{E}^n B^2_{t \wedge \tau_n} = 2 \int_0^t \mathbb{E}^n I_{[s \leq \tau_n]} B_s \sigma(B_s) ds + \mathbb{E}^n \langle \tilde{B}^n \rangle_t.$$ 

Since $|B_s \sigma(B_s)| ds \leq c[1 + B^2_s]$, the following upper bound holds for $V^n_t = \mathbb{E}^n B^2_{t \wedge \tau_n}$:

$$\mathbb{E}^n B^2_{t \wedge \tau_n} \leq 2 \int_0^t \mathbb{E}^n I_{[s \leq \tau_n]} |B_s \sigma(B_s)| ds + \mathbb{E}^n (t \wedge \tau_n).$$

Therefore $V^n_t$ satisfies the Gronwall-Bellman inequality $V^n_t \leq x [1 + \int_0^t V^n_s ds]$ with appropriated positive constant $x$. 


Theorem 2.2. Let $M_t = M_t''$ and condition (2.2) holds.
Then $\mathbb{E}_{3_T} = 1$ for any $T > 0$.

Proof. Choose $\tau_n = \inf \{ t : (\tilde{\mathcal{L}}_t \vee \sup_{s \leq t} B_s^2) \geq n \}$ and obtain $\mathbb{E}_{3_{\tau(n)}} = 1$. Now, the proof of the theorem is reduced to verification of the inequality $\sup_n \mathbb{E}^n_0 \log \left( \mathbb{E}^n_{\tau(n)} \right) < \infty$, where $\mathbb{E}^n_0$ denotes the expectation relative to the probability measure $Q^n$, $Q^n \ll P$, $\frac{dQ^n}{dP} = \tilde{\zeta}_{\tau(n)}$. By Girsanov’s theorem the process $B_{t \wedge \tau_n}$ is a semimartingale w.r.t. the measure $Q^n$:

\[
B_{t \wedge \tau_n} = \int_0^t I_{(s \leq \tau_n)} \sigma_s(B) ds + \tilde{B}_t^n \tag{2.3}
\]

with $Q^n$ Brownian motion $\tilde{B}_t^n$ stopped at time $\tau_n$, having the predictable variation $\langle \tilde{B} \rangle_t = t \wedge \tau_n$. Consequently $\mathbb{E}^n_0 \int_0^t I_{(s \leq \tau_n)} \sigma_s^2(B) d\tilde{B}_s^n = \mathbb{E}^n_0 \int_0^t I_{(s \leq \tau_n)} \sigma_s(B) ds$ and, in view of condition (2.2),

\[
\mathbb{E}^n_0 \int_0^t I_{(s \leq \tau_n)} \sigma_s^2(B) ds \leq T \left[ 1 + \mathbb{E}^n_0 \int_0^t I_{(s \leq \tau_n)} \sup_{s' \leq s} B_{s'}^2 ds \right].
\]

So, it suffices to prove that

\[
\sup_n \mathbb{E}^n_0 \int_0^T \sup_{s' \leq s} B_{s' \wedge \tau_n}^2 ds < \infty. \tag{2.4}
\]

Condition (2.3) and Cauchy-Schwartz inequality enable us to use the following inequality

\[
\sup_{t' \leq t} B_{t' \wedge \tau_n}^2 \leq 2 \left( \int_0^t I_{(s \leq \tau_n)} |\sigma_s(B)| ds \right)^2 + 2 \sup_{t' \leq t} \tilde{B}_{t' \wedge \tau_n}^2 \leq 2t \int_0^t I_{(s \leq \tau_n)} \sigma_s^2(B) ds + 2 \sup_{t' \leq t} \tilde{B}_{t' \wedge \tau_n}^2 \leq 2t \int_0^t I_{(s \leq \tau_n)} \left[ 1 + \sup_{s' \leq s} \sigma_{s'}^2(B) \right] ds + 2 \sup_{t' \leq t} \tilde{B}_{t' \wedge \tau_n}^2.
\]

Moreover, the Doob maximal inequality for square integrable martingales yields

$\mathbb{E}^n_0 \sup_{t' \leq t} |\tilde{B}_{t' \wedge \tau_n}^n|^2 \leq 4(t \wedge \tau_n)$. These inequalities allow us to use Gronwall-Bellman inequality $V^n_t \leq r[1 + \int_0^t V^n_s ds]$, $t \in [0, T]$, where $V^n_t := \mathbb{E}^n_0 \sup_{t' \leq t} B_{t' \wedge \tau_n}^2$, and obtain (2.4). □

3. General model

Assume a martingale $M = M^c + M^d$ is a part of some semimartingale $X$. While it may not be ‘the most general case’, it covers, however, most examples met in applications. Other generalizations in the spirit of our arguments are possible but we decided not pursue them here, as the reader will see that the paper is already technical enough.

Introduce the following notations and assumptions.

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ is a stochastic basis satisfying the general conditions.
4.1. Main result. Examples. Since $X$ is a Markov process, we refine its description, that is, we replace (3.1) by the Itô equation:

$$X_t = X_0 + \int_0^t a_s(X)ds + \int_0^t b_s(X)dB_s + \int_0^t h_s(X, z)[\mu(ds, dz) - dK(dz)].$$

The martingale $M = M^c + M^d$ driven by $X$ is given by

$$M_t = \int_0^t \sigma_s(X)dB_s + \int_0^t \varphi_s(X, z)[\mu(ds, dz) - dK(dz)].$$

4. $X$ is a non-explosive Markov process

The semimartingale $X$ is assumed to be a unique weak solution of the Itô equation:

$$X_t = X_0 + \int_0^t a_s(X)ds + \int_0^t b_s(X)dB_s + \int_0^t h_s(X, z)[\mu(ds, dz) - dK(dz)]. \quad (3.1)$$

The martingale $M = M^c + M^d$ driven by $X$ is given by

$$M_t = \int_0^t \sigma_s(X)dB_s + \int_0^t \varphi_s(X, z)[\mu(ds, dz) - dK(dz)]. \quad (3.2)$$
Remark 2. Looking ahead let us clarify a role of the operator \( L \) is a local martingale. Hence
\[
X_t = 1 + \int_0^t \mathbb{E}_s (X_{s-}) dB_s + \int_0^t \varphi_s (X_{s-}, z) \left[ \mu (ds, dz) - ds \right].
\]

Introduce the following operators (depending on \( s \)), acting on \( (x_s)_{s \in [0,T]} \in \mathbb{D} \)
\[
L_s (x_{s-}) := 2x_{s-} a_s (x_{s-}) + b^2_s (x_{s-}) + \int_R h^2_s (x_{s-}, z)\varphi_s (x_{s-}, z) K (dz);
\]

\[ \mathcal{L}_s (x_{s-}) := 2x_{s-} \left[ a_s (x_{s-}) + b_s (x_{s-}) \sigma_s (x_{s-}) + \int_R h_s (x_{s-}, z) \varphi_s (x_{s-}, z) K (dz) \right] + b^2_s (x_{s-}) + \int_R h^2_s (x_{s-}, z) K (dz) + \int_R h^2_s (x_{s-}, z) \varphi_s (x_{s-}, z) K (dz). \] (4.1)

The role of the operator \( L_s (x_{s-}) \) is clarified as follows.

**Proposition 4.1.** Let \( X^2_0 \leq r \) and \( L_s (x_{s-}) \leq r [1 + x^2_{s-}] \). Then
\[
\sup_{t \in [0,T]} \mathbb{E} X^2_t < \infty;
\]
that is, the process \( X \) does not explode on any finite time interval \([0, T]\).

**Proof.** Applying the Itô formula to \( X^2_t \) we obtain \( X^2_t = X^2_0 + \int_0^t L_s (X_{s-}) ds + \mathcal{M}_t \), where \( \mathcal{M}_t \) is a local martingale.

Note also that the definition of \( L_s (x_{s-}) \) and \( \zeta_n = \inf \{ t : X^2_t \geq n \} \) provide \( \mathbb{E} X^2_{t \wedge \zeta_n} < \infty \).

So, \( X^2_{t \wedge \zeta_n} = X^2_0 + \int_0^t L_s (X_{s-}) ds + M_{t \wedge \zeta_n} \), implies \( \mathbb{E} M^2_{t \wedge \zeta_n} < \infty \) and \( \mathbb{E} M_{t \wedge \zeta_n} = 0 \). Also
\[ V^2_t := \mathbb{E} X^2_{t \wedge \zeta_n} \text{ solves Gronwall-Bellman’s inequality} \]
\[ EV^2_t \leq r [1 + \int_0^t V^2_s ds]. \]

Hence \( \mathbb{E} X^2_{t \wedge \zeta_n} \leq r e^{rt} \) and by Fatou theorem \( \mathbb{E} X^2_t \leq r e^{rt} \).

So, \( \sup_{t \in [0,T]} \mathbb{E} X^2_t \leq r e^{rt} \). \( \square \)

**Remark 2.** Looking ahead let us clarify a role of the operator \( \mathcal{L}_s (x_{s-}) \). Assume \( \mathbb{E} = \mathbb{E}_r = 1 \).

So \( Q \ll P \) with \( dP \mid \mathcal{F} = dQ \) is the probability measure. Then the operator \( \mathcal{L}_s (x_{s-}) \) plays the role of the operator \( L_s (x_{s-}) \) for the process \( X_t \) under the new measure \( Q \).

**Theorem 4.2.** \(|X_0| \leq r\)

1. \( \sigma^2_s (x_{s-}) + \int_R \varphi^2_s (x_{s-}, z) K (dz) \leq r [1 + x^2_{s-}] \)
2. \( L_s (x_{s-}) \leq r [1 + x^2_{s-}] \)
3. \( \mathcal{L}_s (x_{s-}) \leq r [1 + x^2_{s-}] \)

Then \((\mathfrak{Z}_t)_{t \in [0, T]} \) is the martingale for any \( T > 0 \).
The proof of this theorem is given in Section 6.2. Now, we illustrate its applications in various models.

To avoid repetitions we omit “... the conditions of the theorem are fulfilled ...” and “...for any \( T > 0,...)” and use a shorthand “\( \vartheta = \mathcal{E}(M) \)” instead, wherever it does not cause misunderstanding.

**Example 4.1** ([28]). \( X \) is purely discontinuous martingale with independent increments. Let

\[
X_t = \int_0^t \int_{\mathbb{R}} z[\mu(ds, dz) - K(dz)ds] \quad \text{and} \quad M_t = \int_0^t \int_{\mathbb{R}} |X_{s-}|[\mu(ds, dz) - K(dz)ds]
\]

Since

- \( a_s(x_{s-}) = b_s(x_{s-}) = 0 \) \( h_s(x_{s-}, z) = z \),
- \( \sigma_s(x_{s-}) = 0 \) \( \varphi_s(x_{s-}, z) = |x_{s-}z| \),
- \( \int_{\mathbb{R}} z^2 K(dz) < \infty \),

then together with additional condition \( \int_{\mathbb{R}} |z|^3 K(dz) < \infty \) we have

\[
\begin{align*}
\sigma_s^2(x_{s-}) + \int_{\mathbb{R}} \varphi_s^2(x_{s-}, z)K(dz) &= x_{s-}^2 \int_{\mathbb{R}} z^2 K(dz) \\
L_s(x_{s-}) &= \int_{\mathbb{R}} z^2 K(dz) \\
\mathcal{L}_s(x_{s-}) &= 2x_{s-} |x_{s-}| \int_{\mathbb{R}} z K(dz) + \int_{\mathbb{R}} z^2 K(dz) + |x_{s-}| \int_{\mathbb{R}} |z|^3 K(dz)
\end{align*}
\]

\[
\leq r[1 + x_{s-}^2].
\]

Thus, \( \vartheta = \mathcal{E}(M) \).

**Example 4.2.** Constant Elasticity of Variance, [1], [2], [3].

Let

\[
X_t = 1 + \int_0^t X_s ds + \int_0^t \sqrt{X_s^+} dB_s \quad \text{and} \quad M_t = \int_0^t \sqrt{X_s^+} dB_s.
\]

Then,

- \( X_0 = 1 \)
- \( a_s(x_{s-}) = x_{s-}, b_s(x_{s-}) = \sqrt{x_{s-}^+} \) \( h_s(x_{s-}, z) = 0 \),
- \( \sigma_s(x_{s-}) = \sqrt{x_{s-}^+} \) \( \varphi_s(x_{s-}, z) = 0 \).

Consequently

\[
\begin{align*}
\sigma_s^2(x_{s-}) + \int_{\mathbb{R}} \varphi_s^2(x_{s-}, z)K(dz) &= x_{s-}^+ \\
L_s(x_{s-}) &= 2x_{s-} + x_{s-}^+ \\
\mathcal{L}_s(x_{s-}) &= 2x_{s-} |x_{s-} + x_{s-}^+| + x_{s-}^+
\end{align*}
\]

\[
\leq r[1 + x_{s-}^2].
\]

It should be noted that \( \vartheta = \inf\{t : X_t = 0\} \) is the time to absorption at zero of the process \( X_t \) (time to ruin), \( \vartheta < \infty \) with a positive probability, and \( X_t = X_{t\wedge \vartheta} \). Thus \( \vartheta, (\cdot_{t\wedge \vartheta})_{t \leq T} \) is a martingale for any \( T > 0 \).
Example 4.3. Cubic stabilizing drift. Let
\[ X_t = 1 - \int_0^t X_s^3 ds + \int_0^t X_s dB_t \] and \[ M_t = \int_0^t X_s dB_s. \]
Then,
- \( X_0 = 1, \)
- \( a_s(x_{s-}) = -x_{s-}^3, \quad b_s(x_{s-}) = x_{s-} \quad h_s(x_{s-}, z) = 0, \)
- \( \sigma_s(x_{s-}) = x_{s-}^+ \quad \varphi_s(x_{s-}, z) = 0. \)
So,
\[ \sigma_s^2(x_{s-}) = \left( x_{s-}^+ \right)^2 \quad L_s(x_{s-}) = -2x_{s-}^4 + x_{s-}^2 \quad \xi_s(x_{s-}) = -2x_{s-}^4 + x_{s-}^2 + 2|x_{s-}^3| \]
\[ \leq r[1 + x_{s-}^2]. \]
Thus, \( \mathfrak{z} = \mathcal{E}(M). \)

Example 4.4. Brownian Bridge. Zero mean Gaussian process \( X_t \) defined on the time interval is said Brownian bridge if its correlation function
\[ R(t', t) = (t' \wedge t)[1 - (t' \vee t)]. \]
It is also well known that \( X_t \) is the unique solution of Itô’s equation
\[ X_t = -\int_0^t \frac{X_s}{1-s} ds + B_t, \quad t \in [0, 1), \quad \lim_{t \uparrow 1} X_t = 0. \]
Let \( M_t = \int_0^t X_s dB_s, \quad t \leq 1. \)
Here \( T = 1 \) and
- \( X_0 = 0, \)
- \( a_s(x_{s-}) = -x_{s-} (s < 1), \quad b_s(x_{s-}) = x_{s-} \quad h_s(x_{s-}, z) = 0, \)
- \( \sigma_s(x_{s-}) = x_{s-}^+ \quad \varphi_s(x_{s-}, z) = 0. \)
Therefore
\[ \sigma_s^2(x_{s-}) = x_{s-}^2 \quad L_s(x_{s-}) = -2x_{s-}^4 + x_{s-}^2 \quad \xi_s(x_{s-}) = -2x_{s-}^4 + x_{s-}^2 + 2|x_{s-}^3| \]
\[ \leq r[1 + x_{s-}^2] \]
and by Theorem 4.2 \( \mathfrak{z}_1 = 1. \)

Example 4.5. One extension of Mijitovic and Urusov example (see [29])
Let \( \alpha \in (-1, 0] \) and
\[ X_t = 1 + \int_0^t |X_s|^\alpha ds + B_t \] and \( M_t = \int_0^t X_s dB_s. \)
In [29], it is shown , \( \mathfrak{z} = \mathcal{E}(M). \) Theorem 4.2 enables to show that \( \mathfrak{z} = \mathcal{E}_T(M) \) even if \( \alpha = -1. \) In this case \( X_t \) is the Bessel process (see e.g. Exercise 2.25 p. 197 [32]), that is,
\[ X_t = 1 + \int_0^t \frac{ds}{X_s} + B_t. \]
Then
- \( X_0 = 1, \)
- \( a_s(x_{s-}) = \frac{1}{x_{s-}^\alpha 0}, \quad b_s(x_{s-}) = 1 \quad h_s(x_{s-}, z) = 0, \)
\* \( \sigma_s(x_{s-}) = x_{s-} \), \( \varphi_s(x_{s-}, z) = 0 \)

and

\[
\begin{align*}
\sigma_s^2(x_{s-}) &= x_{s-}^2 \\
L_s(x_{s-}) &= 3 \\
\mathcal{L}_s(x_{s-}) &= 3 + 2x_{s-}
\end{align*}
\]

\[ \leq r[1 + x_{s-}^2]. \]

Therefore \( z = \mathcal{E}(M) \).

5. \( X \) is a past-dependent semimartingale

5.1. Main result. Examples. Recall representations (3.1) and (3.2). Operators \( L_s(x) \) and \( \mathcal{L}_s(x) \) are changed as follows

\[
\begin{align*}
L_s(x) := a_s^2(x) + b_s^2(x) + \int_{\mathbb{R}} h_s^2(x, z) K(dz), \\
\mathcal{L}_s(x) := a_s^2(x) + b_s^2(x) + \int_{\mathbb{R}} h_s^2(x, z) K(dz) + b_s^2(x) \sigma_s^2(x) \\
&+ \int_{\mathbb{R}} h_s^2(x, z) K(dz) \int_{\mathbb{R}} \varphi_s^2(x, z) K(dz) + \int_{\mathbb{R}} h_s^2(x, z) \varphi_s(x, z) K(dz).
\end{align*}
\]

(5.1)

Theorem 5.1. If \(|X_0| \leq r \)

1) \( \sigma_s^2(x) + \int_{\mathbb{R}} \varphi_s^2(x, z) K(dz) \leq r\left[1 + \sup_{s' \leq s} x_{s'}^2 \right] \)

2) \( L_s(x) \leq r\left[1 + \sup_{s' \leq s} x_{s'}^2 \right] \)

3) \( \mathcal{L}_s(x) \leq r\left[1 + \sup_{s' \leq s} x_{s'}^2 \right] \),

then \( z = \mathcal{E}(M) \).

Note that the process \( X \) does not explode due to assumption 2).

The proof of this theorem is given in Section 6.3.

Example 5.1. Weak existence for a past-dependent SDE with unit diffusion.

We show that a stochastic differential equation

\[ X_t = \int_0^t a_s(X) ds + B_t. \]

has a weak solution on any time interval \([0, T]\) if

\[ a_s^2(y) \leq r\left[1 + \sup_{s' \leq s} y_{s'}^2 \right], \quad (y_s)_{s \in [0, T]} \in \mathbb{C} \]

(cf assumptions of Theorem 7.2, Ch. 7, §7.2 [26]).

Set \( M_t = \int_0^t \sigma_s(B) dB_t \) with \( \sigma_s(y) \equiv a_s(y) \). Then by Theorem 5.1, \( E_{\mathbb{F}} = 1 \). So, there exists a probability measure \( Q \ll P \), \( \frac{dQ}{dP} = \mathcal{E}_{\mathbb{F}} \). Hence, by Girsanov theorem, the process \( (B_t, Q)_{t \in [0, T]} \) is nothing but a weak solution of Itô’s equation \( B_t = \int_0^t a_s(B) ds + \dot{B}_t \) with \( Q \)-Brownian motion \( \dot{B}_t \).

Note that weak uniqueness of this equation also holds (see Theorem 4.12 in [26, Ch. 4]).
Example 5.2. A past-dependent SDE with a singular diffusion. Assume the Itô equation $X_t = X_0 + \int_0^t b_s(X)dB_s$ with $b_s^2(x) \geq 0$ obeys a weak solution. Then equation with drift

$$X_t = X_0 + \int_0^t a_s(X)ds + \int_0^t b_s(X)dB_s,$$

also has a weak solution. Suppose

- $X_0^2 \leq x$;
- $a_s(x) = a_s(x)1_{\{b_s^2(x) > 0\}}$;
- $b_s^2(x) a_s(x) \leq \left\lfloor \frac{1}{\sup_{s' \leq s} x_s^2} \right\rfloor \leq r \left( 1 + \sup_{s' \leq s} x_s^2 \right)$, $s \in [0, T]$, $(x_s)_{s \in [0, T]} \in \mathbb{C}$.

(\text{cf Ch. 7, §7.6, Theorem 7.19 in [26]). To this end, we choose

$$\sigma_s(x) = \frac{a_s(x)}{b_s^2(x)}1_{\{b_s^2(x) > 0\}} \quad \text{and} \quad M_t = \int_0^t \sigma_s(B)dB_s$$

and then apply Theorem \textbf{5.1} with $\mathfrak{A}_t = 1 + \int_0^t \mathfrak{A}_s \sigma_s(x)dB_s$. Since $\mathbb{E}_{\mathfrak{A}_T} = 1$ there exists the probability measure $Q \ll P$ with density $\frac{dQ}{dP} = \mathfrak{A}_T$.

Then, by Girsanov theorem

$$X_t = X_0 + \int_0^t b_s(X)\sigma_s(X) ds + \int_0^t b_s(X)d\tilde{B}_s,$$

with a Q-Brownian motion $\tilde{B}_t$.

Example 5.3. SDE with delay. Theorem \textbf{5.1} is applicable to semimartingale with past-dependent characteristic in a form of delay. This stochastic model is used often in modern stochastic control.

Let $\vartheta > 0$ denote fixed delay parameter in Itô’s equation

$$X_t = 1_{\{X_s \in [-\vartheta, 0]\}} + \int_0^t a_s(X_{s-\vartheta})ds + \int_0^t b_s(X_{s-\vartheta})dB_s.$$

Let $M_t = \int_0^t \sigma_s(X_{s-\vartheta})dB_s$ and so $\mathfrak{A}_t = 1 + \int_0^t \mathfrak{A}_s \sigma_s(X_{s-\vartheta})dB_s$.

Then, by Theorem \textbf{5.1} $\mathfrak{A}_t = \mathbb{E}(M)$ if the following conditions are satisfied:

$$\begin{aligned}
\sigma_s^2(x_{s-\vartheta}^2) + b_s^2(x_{s-\vartheta}^2) \\
\sigma_s^2(x_{s-\vartheta}^2) a_s^2(x_{s-\vartheta}) b_s^2(x_{s-\vartheta}) \\
\end{aligned} \leq x \left( 1 + \sup_{s' \leq s} x_s^2 \right),$n = [0, T]$, $(x_s)_{s \in [0, T]} \in \mathbb{C}$.

6. Proofs

6.1. Auxiliary result. Proofs of Theorem \textbf{2.1} and Theorem \textbf{5.1} follow the same idea, and rely on an auxiliary result that allows to check uniform integrability in terms of

$$\sup_{n} \mathbb{E}_{\mathfrak{A}_{T \wedge \tau n}} \log(\mathfrak{A}_{T \wedge \tau n}) < \infty.$$
Let $X$ and $M$ be defined by (3.1) and (3.2), and

$$
3_t = 1 + \int_0^t 3_s - \sigma_s(X) dB_s + \int_0^t \int_0^t 3_s - \varphi_s(X, z)[\mu(ds, dz) - dsK(dz)].
$$

Set the localizing sequences:

$$
\tau_n = \inf\{t : (3_t \lor X_t^2) \geq n\} \quad \text{in the proof of Theorem 2.1}
$$

and

$$
\tau_n = \inf\{t : (3_t \lor \sup_{s \leq t} X_s^2) \geq n\} \quad \text{in the proof of Theorem 5.1}
$$

and notice that $3_{(s \land \tau_n)^-}$ and $X_{(s \land \tau_n)^-}$ are bounded processes. Since Dooleans-Dade’s formula (1.2), with martingales $M_t^\circ$ and $M_t^\circ$ defined in (3.2), is the unique solution of SDE

$$
3_t = 1 + \int_0^t 3_s - \sigma_s(X) dB_s + \int_0^t \int_0^t 3_s - \varphi_s(X, z)[\mu(ds, dz) - dsK(dz)],
$$

we have that

$$
\begin{align*}
3_{t \land \tau_n} &= 1 + \int_0^t I_{\{s \leq \tau_n\}} 3_{(s \land \tau_n)^-} \sigma_s(X) dB_s \\
&\quad + \int_0^t \int_0^t I_{\{s \leq \tau_n\}} 3_{(s \land \tau_n)^-} \varphi_s(X, z)[\mu(ds, dz) - dsK(dz)].
\end{align*}
$$

Hence

$$
E(3_{T \land \tau_n} - 1)^2 = E \int_0^T I_{\{s \leq \tau_n\}} 3_{(s \land \tau_n)^-}^2 \sigma_s^2(X) + \int_0^T \varphi_s^2(X, z) K(dz) ds.
$$

By one of the (BC) conditions: for any $s \in [0, T]$, $(x_s)_{s \leq T} \in \mathbb{D}$

$$
\sigma_s^2(x) + \int_\mathbb{R} \varphi_s^2(x, z) K(dz) \leq x \left[ 1 + x_{s-}^2 \right] \quad \text{if } X \text{- (BC-Markov)}
$$

\[ \leq x \left[ 1 + \sup_{s' < s} x_{s'}^2 \right] \quad \text{if } X \text{- (BC-Past Dependent)} \]

It now follows from (6.1) that $3_{t \land \tau_n}$ is a square integrable martingale with $E3_{t \land \tau_n} = 1$.

**Lemma 6.1.** The family \( \{3_{T \land \tau_n}\}_{n \geq 1} \) is uniformly integrable if

$$
\begin{align*}
\sup_n E_n \int_0^T X_{s \land \tau_n}^2 ds, \quad &\text{in Theorem 4.2} \\
\sup_n E_n \int_0^T \sup_{s' < s} X_{s' \land \tau_n}^2 ds, \quad &\text{in Theorem 5.1}
\end{align*}
$$

Proof. The existence of probability measure $Q^n \ll P$ with the density $\frac{dQ^n}{dP} = 3_{T \land \tau_n}$ is obvious. Henceforth $E^n$ is the expectation of $Q^n$ measure. The uniform integrability of the family \( \{3_{T \land \tau_n}\}_{n \rightarrow \infty} \) is verified by the Vallée de Poussin theorem with the function $x \log(x), x \geq 0$. The formula $\sup_n E^n 3_{T \land \tau_n} \log (3_{T \land \tau_n}) < \infty$ is convenient since

$$
3_{t \land \tau_n} = \exp \left( M_{t \land \tau_n} - A_{t \land \tau_n} \right),
$$
where $M_{t \wedge \tau_n}$ is a square integrable martingale and $A_{t \wedge \tau_n}$ is an increasing positive process:

\[
M_{t \wedge \tau_n} = \int_0^t I_{\{s \leq \tau_n\}} \sigma_s(X) dB_s + \int_0^T I_{\{s \leq \tau_n\}} \varphi_s(X, z) [\mu(ds, dz) - dsK(dz)]
\]

\[
A_{t \wedge \tau_n} = \frac{1}{2} \int_0^t I_{\{s \leq \tau_n\}} \sigma_s^2(X) ds + \int_0^T I_{\{s \leq \tau_n\}} \{ \varphi_s(X, z) - \log [1 + \varphi_s(X, z)] \} \mu(ds, dz).
\]

The condition $\varphi_s(X, z) > -1$ implies $\varphi_s(X, z) - \log [1 + \varphi_s(X, z)] \geq 0$. This inequality, jointly with $\sigma_s^2(X) \geq 0$, implies $A_{t \wedge \tau_n} \geq 0$. Therefore $\log (\delta_{t \wedge \tau_n}) \leq M_{t \wedge \tau_n}$ and, so,

\[
\delta_{T \wedge \tau_n} \log (\delta_{T \wedge \tau_n}) \leq \delta_{T \wedge \tau_n} M_{T \wedge \tau_n}.
\]

Both processes $\delta_{t \wedge \tau_n}$ and $M_{t \wedge \tau_n}$ are square integrable martingales having continuous and purely discontinuous components: $\delta_{t \wedge \tau_n} = \delta_{t \wedge \tau_n}^c + \delta_{t \wedge \tau_n}^d$, $M_{t \wedge \tau_n} = M_{t \wedge \tau_n}^c + M_{t \wedge \tau_n}^d$, where

\[
\delta_{T \wedge \tau_n}^c = \int_0^T I_{\{s \leq \tau_n\}} \delta_s \sigma_s(X) dB_s,
\]

\[
M_{T \wedge \tau_n}^c = \int_0^T I_{\{s \leq \tau_n\}} \sigma_s(X) dB_s,
\]

\[
\delta_{T \wedge \tau_n}^d = 1 + \int_0^T \int_\mathbb{R} I_{\{s \leq \tau_n\}} \delta_s \varphi_s(X, z) [\mu(ds, dz) - dsK(dz)],
\]

\[
M_{T \wedge \tau_n}^d = \int_0^T \int_\mathbb{R} I_{\{s \leq \tau_n\}} \varphi_s(X, z) [\mu(ds, dz) - dsK(dz)].
\]

Hence $\mathbb{E}M_{T \wedge \tau_n} \delta_{T \wedge \tau_n} = \bar{\mathbb{E}} M_{T \wedge \tau_n}$. Also

\[
\mathbb{E} M_{T \wedge \tau_n}^c \delta_{T \wedge \tau_n} = \mathbb{E} \int_0^T I_{\{s \leq \tau_n\}} \delta_s \sigma_s^2(X) ds = \mathbb{E} \int_0^T \delta_s X_s^2 ds = \bar{\mathbb{E}} \int_0^T I_{\{s \leq \tau_n\}} \sigma_s^2(X) ds
\]

and

\[
\mathbb{E} M_{T \wedge \tau_n}^d \delta_{T \wedge \tau_n} = \mathbb{E} \int_0^T I_{\{s \leq \tau_n\}} \delta_s \varphi_s^2(X, z) K(dz) ds = \bar{\mathbb{E}} \int_0^T I_{\{s \leq \tau_n\}} \varphi_s^2(X, z) K(dz) ds.
\]

So

\[
\bar{\mathbb{E}} M_{T \wedge \tau_n} = \bar{\mathbb{E}} \int_0^T I_{\{s \leq \tau_n\}} \left( \sigma_s^2(X) + \int_\mathbb{R} \varphi_s^2(X, z) K(dz) \right) ds.
\]

Now, (6.3) enables to finish the proof:

\[
\sup_n \mathbb{E}_{T \wedge \tau_n} \log (\delta_{T \wedge \tau_n}) \leq \mathbb{E} \sup_n \int_0^T X_{s \wedge \tau_n} ds, \quad \text{for theorem 4.2}
\]

\[
\sup_n \mathbb{E}_{T \wedge \tau_n} \log (\delta_{T \wedge \tau_n}) \leq \mathbb{E} \sup_n \int_0^T X_{s \wedge \tau_n} ds \quad \text{for theorem 5.1}
\]

if conditions of the lemma fulfilled.
6.2. Proof of Theorem [4.2] By Lemma [6.1] it suffices to verify that
\[
\sup_n \mathbb{E}^n \int_0^T X_{s \wedge \tau_n}^2 \, ds < \infty
\]
with \( \tau_n \) defined in (6.1) and \( \mathbb{E}^n \) the expectation under \( \mathbb{Q}^n : d\mathbb{Q}^n = \delta_{\tau_n} \, d\mathbb{P} \). Thus we need to know how \( X \) looks like under \( \mathbb{Q}^n \). This is given by a well-known result on semimartingales under a change of measure, Theorem A.1 (Appendix A). It states that \((X_t, \mathbb{Q}^n)_{t \in [0,T]}\) is a semimartingale with decomposition

\[
X_{t \wedge \tau_n} = X_0 + \int_0^t I_{\{s \leq \tau_n\}} \left[ a_s(X_{s-}) + b_s(X_{s-})\sigma_s(X_{s-}) \right. \\
+ \left. \int_{\mathbb{R}} h_s(X_{s-}, z) \varphi_s(X_{s-}, z) K(dz) \right] ds + \tilde{M}^c_{t \wedge \tau_n} + \tilde{M}^d_{t \wedge \tau_n},
\]
with continuous \( \tilde{M}^c_{t \wedge \tau_n} \) and purely discontinuous \( \tilde{M}^d_{t \wedge \tau_n} \) square integrable martingales having predictable quadratic variations:

\[
\langle \tilde{M}^c_{t \wedge \tau_n} \rangle_t = \int_0^t I_{\{s \leq \tau_n\}} b_s^2(X_{s-}) \, ds \\
\langle \tilde{M}^d_{t \wedge \tau_n} \rangle_t = \int_0^t I_{\{s \leq \tau_n\}} h_s^2(X_{s-}, z) [1 + \varphi_s(X_{s-}, z)] K(dz) \, ds.
\]

By Itô’s formula we obtain

\[
X_{t \wedge \tau_n}^2 = X_0^2 + \int_0^t I_{\{s \leq \tau_n\}} 2X_{s-} \left[ a_s(X_{s-}) + b_s(X_{s-})\sigma_s(X_{s-}) \right. \\
+ \left. \int_{\mathbb{R}} h_s(X_{s-}, z) \varphi_s(X_{s-}, z) K(dz) \right] ds + \langle \tilde{M}^c_{t \wedge \tau_n} \rangle_t + \langle \tilde{M}^d_{t \wedge \tau_n} \rangle_t \\
+ \int_0^t I_{\{s \leq \tau_n\}} 2X_{s-} \tilde{M}^c_{s \wedge \tau_n} + \int_0^t I_{\{s \leq \tau_n\}} 2X_{s-} \tilde{M}^d_{s \wedge \tau_n},
\]
where \( [\tilde{M}^c_{t \wedge \tau_n}, \tilde{M}^d_{t \wedge \tau_n}]_t \) is the quadratic variation of \( \tilde{M}^d_{t \wedge \tau_n} \). Therefore

\[
\mathbb{E}^n X_{t \wedge \tau_n}^2 = \mathbb{E}^n X_0^2 + \int_0^t \mathbb{E}^n I_{\{s \leq \tau_n\}} 2X_{s-} \left[ a_s(X_{s-}) + b_s(X_{s-})\sigma_s(X_{s-}) \right. \\
+ \left. \int_{\mathbb{R}} h_s(X_{s-}, z) \varphi_s(X_{s-}, z) K(dz) \right] ds + \mathbb{E}^n \langle \tilde{M}^c_{t \wedge \tau_n} \rangle_t + \mathbb{E}^n \langle \tilde{M}^d_{t \wedge \tau_n} \rangle_t,
\]
where \( [\tilde{M}^d_{t \wedge \tau_n}, \tilde{M}^d_{t \wedge \tau_n}]_t \) is a quadratic variation of the martingale \( \tilde{M}^d_{t \wedge \tau_n} \). In view of

\[
\mathbb{E}^n \langle \tilde{M}^c_{t \wedge \tau_n} \rangle_t = \mathbb{E}^n \int_0^t I_{\{s \leq \tau_n\}} b_s^2(X_{s-}) \, ds \\
\mathbb{E}^n \langle \tilde{M}^d_{t \wedge \tau_n} \rangle_t = \mathbb{E}^n \int_0^t \int_{\mathbb{R}} I_{\{s \leq \tau_n\}} h_s^2(X_{s-}, z) [1 + \varphi_s(X_{s-}, z)] K(dz) \, ds
\]
(6.4) can be presented as

\[
\mathbb{E}^n X_{t \wedge \tau_n}^2 = \mathbb{E}^n X_0^2 + \mathbb{E}^n \int_0^t I_{\{s \leq \tau_n\}} \Lambda_s(X_{s-}) \, ds,
\]
with \( \Lambda_n(X_{s-}) \) defined in (4.1). Thus, we obtain the Gronwall-Bellman inequality: 
\[ \tilde{E}^n X_{t \wedge \tau_n}^2 \leq r \int_0^t \left[ 1 + \tilde{E}^n X_{s \wedge \tau_n}^2 \right] ds \]
which implies the desired estimate
\[ \sup_n \int_0^T \tilde{E}^n X_{s \wedge \tau_n}^2 ds \leq e^{rT} - 1. \]

6.3. **Proof of Theorem 5.1.** Let stopping time \( \tau_n \) (see (6.1)) is adapted to Theorem 5.1. Then, by Lemma 6.1 it suffices to verify the uniform integrability of family of random variables \( \tilde{z}_{T \wedge \tau_n}, n \geq 1 \):
\[
\sup_n \tilde{E}^n \int_0^T \sup_{s' \leq s} X_{s' \wedge \tau_n}^2 ds < \infty.
\]
In view of \( E(\tilde{z}_{T \wedge \tau_n}) = 1 \), the probability measure \( Q^n \) is well defined: \( dQ^n = \tilde{z}_{T \wedge \tau_n} dP \). By Theorem A.1 the process \( (X_{t \wedge \tau_n}, Q^n)_{t \in [0,T]} \) is the semimartingale:
\[
X_{t \wedge \tau_n} = X_0 + \int_0^t I_{\{s \leq \tau_n\}} \left[ a_s(X) + b_s(X)\sigma_s(X) + \int_{\mathbb{R}} h_s(X, z) \varphi_s(X, z) K(dz) \right] ds + \tilde{M}_t^{c,n} + \tilde{M}_t^{d,n},
\]
where \( \tilde{M}_t^{c,n} \) and \( \tilde{M}_t^{d,n} \) are continuous and purely discontinuous square integrable martingales with
\[
\langle \tilde{M}^{c,n} \rangle_t = \int_0^t I_{\{s \leq \tau_n\}} b_s^2(X) ds
\]
\[
\langle \tilde{M}^{d,n} \rangle_t = \int_0^t \int_{\mathbb{R}} I_{\{s \leq \tau_n\}} h_s^2(X, z) \varphi_s(X, z) K(dz) ds.
\]
This semimartingale enables to obtain the following estimate:
\[
\tilde{E}^n \sup_{t' \leq t} |X_{t' \wedge \tau_n}|^2 \leq 4 \left[ \tilde{E}^n X_0^2 + \tilde{E}^n \sup_{t' \leq t} |M_t^{c,n}|^2 + \tilde{E}^n \sup_{t' \leq t} |M_t^{d,n}|^2 \right]
+ \tilde{E}^n \left( \int_0^t I_{\{s \leq \tau_n\}} \left| a_s(X) + b_s(X)\sigma_s(X) + \int_{\mathbb{R}} h_s(X, z) \varphi_s(X, z) K(dz) \right| ds \right)^2.
\]
To this end, we evaluate each term in the right hand side of aforementioned inequality.

\[ 4\tilde{E}^n X_0^2 \leq r \]

by the assumption of theorem;

\[ 4\tilde{E}^n \sup_{t' \leq t} |M_{t'}^{c,n}|^2 \leq 16\tilde{E}^n (M^{c,n})_t = 16\tilde{E}^n \int_0^t I_{\{s \leq \tau_n\}} b_s^2(X) \, ds \]

the maximal Doob inequality;

\[ 4\tilde{E}^n \sup_{t' \leq t} |M_{t'}^{d,n}|^2 \leq 16\tilde{E}^n (M^{d,n})_t \]

\[ = 16\tilde{E}^n \int_0^t \int_{\mathbb{R}} I_{\{s \leq \tau_n\}} h_s^2(X, z)[1 + \varphi_s(X, z)]K(dz) \, ds \]

4J_t \leq 4t\tilde{E}^n \int_0^t I_{\{s \leq \tau_n\}} |a_s(X) + b_s(X)\sigma_s(X) + \int_{\mathbb{R}} h_s(X, z)\varphi_s(X, z)K(dz)|^2 \, ds

\leq 12t\tilde{E}^n \int_0^t I_{\{s \leq \tau_n\}} \left[ a_s^2(X) + b_s^2(X)\sigma_s^2(X) + \left( \int_{\mathbb{R}} h_s(X, z)\varphi_s(X, z)K(dz) \right)^2 \right] \, ds

\leq 12t\tilde{E}^n \int_0^t I_{\{s \leq \tau_n\}} \left[ a_s^2(X) + b_s^2(X)\sigma_s^2(X) + \int_{\mathbb{R}} h_s^2(X, z)K(dz) \int_{\mathbb{R}} \varphi_s(X, z)K(dz) \right] \, ds

the Cauchy-Schwarz inequality.

These upper bounds imply the following inequality

\[ \tilde{E}^n \sup_{t' \leq t} |X_{t' \wedge \tau_n}|^2 \leq r + \int_0^t \tilde{E}^n \sup_{t' \leq t} I_{\{s \leq \tau_n\}} \mathfrak{L}_s(X) \, ds \]

where the operator \( \mathfrak{L}_s(X) \) is defined in (5.1), that is, \( \tilde{E}^n \mathfrak{L}_s(X) \leq r \left[ 1 + \tilde{E}^n \sup_{s' \leq s} |X_{s' \wedge \tau_n}|^2 \right] \).

Thus, we arrive at the Gronwall-Bellman inequality:

\[ \tilde{E}^n \sup_{t' \leq t} |X_{t' \wedge \tau_n}|^2 \leq r \left[ 1 + \int_0^t \tilde{E}^n \sup_{s' \leq s} |X_{s' \wedge \tau_n}|^2 \, ds \right] . \]

So, the desired estimate \( \sup_n \int_0^T \tilde{E}^n \sup_{s' \leq s} |X_{s' \wedge \tau_n}|^2 \, ds \leq e^{rT} - 1 \) holds true.

7. Example when \( X_t \) is a possibly explosive Markov process

In this Section we consider a concrete model (for a different example see Andersen and Piterbarg [1]).

Let

\[ X_t = X_0 + \int_0^t a_s(X_{s-}) \, ds + \int_0^t b_s(X_{s-}) \, dB_s + \int_0^t \int_{\mathbb{R}} h_s(X_{s-}, z) [\mu(ds, dz) - dsK(dz)] \]

\[ \mathcal{L}_t = 1 + \int_0^t \mathcal{L}_{s-}[\sigma_s(X_{s-}) \, dB_s + \int_0^t \int_{\mathbb{R}} \varphi_s(X_{s-}, z) [\mu(ds, dz) - dsK(dz)]]. \]

This example contains purely discontinuous martingales and so, it generalizes a model of Mijitovic-Urusov (see [29]).

Let the following conditions hold.

1. \( a_s(x_{s-}) \geq |x_{s-}|^{\alpha}, \alpha > 3 \)

2. \( b_s^2(x_{s-}) \leq \begin{cases} r, & \alpha \in (3, 4) \\ r|1 + x_{s-}^2|, & \alpha > 4 \end{cases} \)
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3. \( h_s(x_{s-}, z) \equiv z \)

4. \( \sigma^2_s(x_{s-}) \leq r [1 + x^2_{s-}] \)

5. \( \varphi_s(x_{s-}, z) \equiv |z| \)

6. \( \int \mathbb{R} [z^2 + |z|^3] K(dz) < \infty \)

7. \( 0 < X_0 \leq r. \)

We assume condition

\[ \sigma^2_s(x_{s-}) + \int \mathbb{R} \varphi^2_s(x_{s-}, z) K(dz) \leq [1 + x^2_{s-}] \text{ s} \in [0, T] \]

However, conditions related to operators \( L_s(x_{s-}) \) and \( \mathcal{L}_s(x_{s-}) \) fail. Therefore an explosion of the process \( X_t \) towards to \( + \infty \) is possible. In a case of explosion \( \tau_n = \inf \{ t : (3t \vee X^2_t) \geq n \} \), \( n \geq 1 \) obeys a limit \( \tau \leq \infty \) as \( n \to \infty \) with \( \mathbb{P}(\tau < \infty) > 0 \). So, in the case of explosion only \( \mathbb{E}_{\tau \wedge \tau} = 1 \) might be expected. The chance of explosion does not contradict the statement of Lemma 6.1

\[ \sup_{n} \mathbb{E}^n \int_0^T X^2_{s \wedge \tau_n} ds < \infty. \]

However, it is hard to check. Therefore we shall use

\[ \sup_{n} \mathbb{E}^n \int_0^T \frac{|X_{s \wedge \tau_n}|^\alpha}{1 + |X_{s \wedge \tau_n}|^{\alpha-2}} ds < \infty \quad (7.1) \]

instead since for \( \alpha > 3 \)

\[ \sup_{n} \mathbb{E}^n \int_0^T \frac{|X_{s \wedge \tau_n}|^\alpha}{1 + |X_{s \wedge \tau_n}|^{\alpha-2}} ds < \infty \Rightarrow \sup_{n} \mathbb{E}^n \int_0^T X^2_{s \wedge \tau_n} ds < \infty. \]

We choose the function

\[ g_{\alpha}(u) = \int_0^u \frac{1}{y^{\alpha-2}} dy, \quad u \in \mathbb{R} \]

with derivatives

\[ g'_{\alpha}(u) = \frac{1}{1 + |u|^{\alpha-2}}, \quad g''_{\alpha}(u) = -\frac{(\alpha - 2)|u|^{\alpha-3} \text{sign}(u)}{(1 + |u|^{\alpha-2})^2}. \]

Moreover \( a_s(x_{s-}) \geq |x_{s-}| \alpha \) \( |g''_{\alpha}(x_{s-})| \leq r. \)

\[ g'_{\alpha}(x_{s-})a_{s}(x_{s-}) \geq \frac{|x_{s-}|^{\alpha}}{1 + |x_{s-}|^{\alpha-2}}, \]

where the right hand side of this inequality is the integrand in (7.1). Since the process \( X_{t \wedge \tau_n} \), relative to the new measure \( \mathbb{Q}^n \), does not explode for any fixed \( n \), then by Theorem A.1 we have

\[ X_{t \wedge \tau_n} = X_0 + \int_0^t I_{\{s \leq \tau_n\}} \left[ a_{s}(X_{s-}) + b_{s}(X_{s-}) \sigma_s(X_{s-}) \right. \]

\[ + \int \mathbb{R} z|z| K(dz) \big] ds + \tilde{M}^{s,n}_t + \tilde{M}^{d,n}_t, \]
where $\tilde{M}^{c,n}_t$ and $\tilde{M}^{d,n}_t$ are continuous and purely discontinuous square integrable martingales with the predictable quadratic variations

\[
\langle \tilde{M}^{c,n} \rangle_t = \int_0^t I_{\{s \leq \tau_n \}} b_s^2(X_{s-}) ds
\]
\[
\langle \tilde{M}^{d,n} \rangle_t = \int_0^t \int_{\mathbb{R}} I_{\{s \leq \tau_n \}} [z^2 + |z|^3] K(dz) ds.
\]

Now, by applying the Itô formula to $g_\alpha(X_{t \wedge \tau_n})$ we obtain

\[
g_\alpha(X_{T \wedge \tau_n}) = g_\alpha(X_0) + \int_0^T I_{\{s \leq \tau_n \}} g'_\alpha(X_{s-}) \left[ a_s(X_{s-}) + \sigma_s(X_{s-}) b_s(X_s) + \int_\mathbb{R} |z| K(dz) \right] ds
\]
\[
+ \int_0^T I_{\{s \leq \tau_n \}} g'_\alpha(X_{s-}) d\tilde{M}^{c,n}_s + \int_0^T I_{\{s \leq \tau_n \}} g'_\alpha(X_{s-}) d\tilde{M}^{d,n}_s
\]
\[
+ \frac{1}{2} \int_0^T I_{\{s \leq \tau_n \}} g''_\alpha(X_{s-}) b_s^2(X_s) ds
\]
\[
+ \int_0^T \int_{\mathbb{R}} I_{\{s \leq \tau_n \}} \left[ g_\alpha(X_{s-} + z) - g_\alpha(X_{s-}) - g'_\alpha(X_{s-}) z \right] \nu^n(ds, dz).
\]

Hence

\[
\tilde{E}^n g_\alpha(X_{T \wedge \tau_n}) = \tilde{E}^n g_\alpha(X_0)
\]
\[
+ \tilde{E}^n \int_0^T I_{\{s \leq \tau_n \}} g'_\alpha(X_{s-}) a_s(X_s) ds
\]
\[
+ \tilde{E}^n \int_0^T I_{\{s \leq \tau_n \}} g'_\alpha(X_{s-}) \left[ \sigma_s(X_{s-}) b_s(X_s) + \int_\mathbb{R} |z| K(dz) \right] ds
\]
\[
+ \frac{1}{2} \tilde{E}^n \int_0^T I_{\{s \leq \tau_n \}} g''_\alpha(X_{s-}) b_s^2(X_s) ds
\]
\[
+ \tilde{E}^n \int_0^T \int_{\mathbb{R}} I_{\{s \leq \tau_n \}} \left[ g(X_{s-} + z) - g(X_{s-}) - g'_\alpha(X_{s-}) z \right] \nu^n(ds, dz),
\]

where $\nu^n(ds, dz) = I_{\{s \leq \tau_n \}} [1 + |z|] ds K(dz)$ is $\tilde{Q}^n$-compensator of integer-value random measure $I_{\{s \leq \tau_n \}} \mu(ds, dz)$ (see Theorem A.1). So, in view of (7.2), one can derive the next upper bound:
\[ \bar{E}^n \int_0^T I_{\{s \leq \tau_n\}} \left| \frac{X_s - \alpha}{1 + |X_s - \alpha| - 2} \right| ds \leq \bar{E}^n \int_0^T I_{\{s \leq \tau_n\}} g'_\alpha(X_s -) a_s(X_s -) ds \]

\[ \leq \bar{E}^n g_\alpha(X_T \wedge \tau_n) + \bar{E}^n \int_0^T I_{\{s \leq \tau_n\}} g'_\alpha(X_s -) \sigma_s(X) b_s(X -) ds \]

\[ + \bar{E}^n \int_0^T \int_\mathbb{R} I_{\{s \leq \tau_n\}} g''(X_s -) z^2 K(dz) ds \]

\[ + \frac{1}{2} \bar{E}^n \int_0^T \int_\mathbb{R} I_{\{s \leq \tau_n\}} |g'''(X_s -)| b_s^2(X_s -) ds \]

\[ + \bar{E}^n \int_0^T \int_\mathbb{R} I_{\{s \leq \tau_n\}} |g_\alpha(X_s - + z) - g_\alpha(X_s -) - g'_\alpha(X_s -) z| |1 + |z|| K(dz) ds. \]

The absolute value of each summand above is bounded by a constant independent of \( n \):

- \( g_\alpha(X_T \wedge \tau_n) \leq \text{const.}, \quad \alpha > 3 \)
- \( g'_\alpha(X_{(t \wedge \tau_n)-}) \left| \sigma_t(X_{(t \wedge \tau_n)-}) b_t(X_{(t \wedge \tau_n)-}) \right| \]
  \[ \leq \left\{ \begin{array}{l}
\frac{\sqrt{1+X_{(t \wedge \tau_n)-}^2}}{1+|X_{(t \wedge \tau_n)-}|} \leq \text{const.}, \quad \alpha \in (3, 4) \\
\frac{1}{1+|X_{(t \wedge \tau_n)-}|^{\alpha-3}} \leq \text{const.}, \quad \alpha \geq 4
\end{array} \right. \]

- \( | \int_\mathbb{R} g'_\alpha(X_{(t \wedge \tau_n)-}) \int_\mathbb{R} z^2 K(dz) | \leq \text{const.} \)

- \( \int_0^T |g''_\alpha(X_{(s \wedge \tau_n)-})| b_s^2(X_{(s \wedge \tau_n)-}) ds \)
  \[ \leq \int_0^T \frac{(\alpha-2)|X_{(s \wedge \tau_n)-}|^{\alpha-3}}{1+|X_{(s \wedge \tau_n)-}|^{\alpha-2} \left[ 1 + X_{(s \wedge \tau_n)-}^2 \right]} \int_\mathbb{R} z^2 K(dz) ds \leq \text{const.} \]

- \( \int_0^T \int_\mathbb{R} |g_\alpha(X_{s \wedge \tau_n} + z) - g_\alpha(X_{s \wedge \tau_n}) - g'_\alpha(X_{(s \wedge \tau_n)-}) z| \left[ 1 + |z| \right] K(dz) ds \)
  \[ \leq rT \int_\mathbb{R} \left[ z^2 + |z|^3 \right] K(dz). \]

Thus, (7.1) holds.

8. Extensions

8.1. Under \( \mathbb{E}_{T_B} < 1 \) the Beneš condition may fail. Let \( X_t \) be Bessel process, \( X_t = 1 + \int_0^t dB_s + B_t \). By the Itô

\[ \log(X_t) = -\int_0^t \frac{1}{2X_s^2} ds + \int_0^t \frac{dB_s}{X_s}. \]
Consequently one may choose \( M_t = - \int_0^t \frac{d\sigma_s}{X_s} \) and create Doleans-Dade process

\[
\tilde{\sigma}_t = 1 - \int_0^t \frac{dB_s}{X_s}.
\]

Assume there exists a positive time value \( T \) such that \( E_{\tilde{\sigma}} = 1 \). Then also there exists a probability measure \( Q \ll P \) with density \( \frac{dQ}{dP} = \tilde{\sigma} \). So the Girsanov theorem enables present the process \( X_t \) (w.r.t. \( Q \)) is: \( X_t = 1 + \tilde{B}_t \) with \( Q \)-Brownian motion \( \tilde{B}_t \). Hence \( X_t \) is Gaussian process which cannot to be positive on \([0, T]\), a.s. So, \( E_{\tilde{\sigma}} \neq 1 \), i.e., \( E_{\tilde{\sigma}} < 1 \). On the other hand, \( \sigma^2(y) = \frac{1}{1+y^2} \leq r[1+y^2] \).

8.2. **X - Vector diffusion.** In view of diffusion setting we shall use notations: \( x_s, X_s \) instead of \( x_{s-}, X_{s-} \). Let

\[
X_t = X_0 + \int_0^t a_s(X_s)ds + \int_0^t b_s(X_s)dB_s,
\]

where \( a_s(x_s) \) is matrix-valued function, \( b_s(x_s) \) are vector-valued function, \( B_t \) is Brownian vector (column) motion process with independent component (standard Brownian motions). The Bene\'s condition is naturally compatible with vector case. The norm in \( \mathbb{L}^2 \) and the inner product denote by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) respectively, the transposition symbol denote by \( ^* \). Let \( \sigma_s(x_s) \) be vector function (row), such that, the process

\[
\tilde{\sigma}_t = 1 + \int_0^t \tilde{\sigma}_s \langle \sigma_s(x_s), dB_s \rangle
\]

is well defined.

**Theorem 8.1.** Let \( \|X\|_0^2 \leq r \quad (x_s)_{s \in \mathbb{R}_+} \in \mathbb{C} \). Let for any \( (x_s)_{s\in[0,T]} \in \mathbb{C} \) the following property hold:

1) \( \|\sigma_s(x_s)\|^2 \)
2) \( L_s(x_s) = 2\langle x_s, a_s(x) \rangle + \text{trace}[b^*_s(x_s)b_s(x_s)] \)
3) \( L_s(x_s) = 2\langle x_s, a_s(x) \rangle + \text{trace}[b^*_s(x_s)b_s(x_s)] + \text{trace}[b^*_s(x_s)b_s(x_s)] \).

If for any \( s \leq T \)

\[
\|\sigma_s(x_s)\|^2 + L_s(x_s) + L_s(x_s) \leq r[1 + \|x_s\|^2],
\]

then \( E_{\tilde{\sigma}} = 1 \), \( \forall \ T > 0 \).

The proof of this theorem is similar to the proof in scalar setting, so, it is omitted.

**Example 8.1.** Let \( X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t B(X_s)dB_s \) and

\[
\tilde{\sigma}_t = 1 + \int_0^t \tilde{\sigma}_s \left[ \theta(X_s)dW_s + \sigma(X_s)dB_s \right],
\]

where \( W_t \) is Wiener process independent Brownian motion \( B_t \). In spit of \( X_t \) is scalar process, it is convenient to verify \( E_{\tilde{\sigma}} = 1 \) by applying Theorem 8.1. Write

\[
\begin{pmatrix} X_t \\ 0 \end{pmatrix} = \begin{pmatrix} X_0 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} a(X_s) \\ 0 \end{pmatrix} ds + \int_0^t \begin{pmatrix} b(X_s) \\ 0 \end{pmatrix} \begin{pmatrix} dW_s \\ dB_s \end{pmatrix}
\]

\[
\tilde{\sigma}_t = 1 + \int_0^t \tilde{\sigma}_s \left[ \theta(X_s)dW_s + \sigma(X_s)dB_s \right].
\]
If
1) \( X_2^2 \leq r \)
2) \( \sigma^2(x) + \theta^2(x) \leq x[1 + x^2] \)
3) \( 2xa(x) + b^2(x) \leq x[1 + x^2] \)
4) \( 2x[a(x) + b(x)\sigma(x)] + b^2(x) \leq x[1 + x^2] \),
then \( E_{3_T} = 1, \forall T > 0. \)

Assume, \( \sigma^2(x) \equiv 0. \) In this setting, \( X_t \) and \( W_t \) are independent processes, so that, a well-known result holds: \( E_{3_T} = E \exp \left( \int_0^T \theta(X_s)dW_s - \frac{1}{2} \int_0^T \theta^2(X_s)ds \right) = 1. \)

8.3. **Nonlinear version of Hitsuda’s type model.** Let
\[
X_t = \int_0^t \int_0^s l(s, u)dB_u ds + B_t \quad \text{and} \quad \delta_t = 1 + \int_0^t \delta_s \sigma(X_s)dB_s,
\]
where \( l(t, s) \) is Volterra kernel \( \int_0^t \int_0^s l^2(s, u)duds < \infty, \) and where \( \sigma(x) \) is nonlinear function. In the case \( \sigma(x) = x \) the condition \( \int \int l^2(t, s)dtds < \infty \) provides \( E_{3_T} = 1 \) (see [11]). If \( \sigma(x) \) in nonlinear function (possible discontinuous), then, combining Hitsuda’s approach with Beneš condition, it is possible to obtain

**Theorem 8.2.** If
1) \( \sigma^2(x) \leq r[1 + x^2], \) \( s \in [0, T] \)
2) \( \int \int l^2(t, s)dtds < \infty, \)
then \( E_{3_T} = 1. \)

**Proof.** Formally, this model is not compatible with conditions of Theorem 4.2 and Theorem 5.1. Nevertheless the choice of \( \tau_n = \inf \{ t : (\delta_t \vee X_t^2) \geq n \} \) and the condition
\[
\sup_n \mathbb{E}^n \int_0^T X_{t \wedge \tau_n}^2 dt < \infty, \quad (8.1)
\]
(see Lemma 6.1) provide \( E_{3_T} = 1. \) Condition 1) guaranties \( E_{3_T \wedge \tau_n} = 1, \) that is, existence of probability measure \( Q^n. \) Then, by Girsanov theorem, a random process \((B_{t \wedge \tau_n}, Q^n)\) can be presented as:
\[
B_{t \wedge \tau_n} = \int_0^t I_{\{s \leq \tau_n\}} \sigma(X_s)ds + \tilde{B}_n^t,
\]
with \( Q^n \) Brownian motion \( \tilde{B}_n^t \) stopped \( \tau_n. \) Now, a random process \((X_{t \wedge \tau_n}, Q^n)_{t \in [0, T]}\) is semimartingale:
\[
X_{t \wedge \tau_n} = \int_0^t I_{\{s \leq \tau_n\}} \int_0^s l(s, u)dB_u^nds
+ \int_0^t I_{\{s \leq \tau_n\}} \int_0^s l(s, u)\sigma(X_u)duds
+ \int_0^t I_{\{s \leq \tau_n\}} \sigma_s(X)ds + \tilde{B}_n^t. \quad (8.2)
\]
This semimartingale will be applied in the proof of \( S.1 \). First of all we estimate \( \tilde{\mathbb{E}}^n X^2_{t \wedge \tau_n} \). In order to do that we evaluate the expectation (\( \tilde{\mathbb{E}}^n \)) of each term square in the right hand side in \( S.2 \). Applying the Cauchy-Schwarz inequality, and the maximal Doob inequality for square integrable martingale, and condition 1) we obtain,

**First term:**

\[
\tilde{\mathbb{E}}^n \left( \int_0^t I_{\{s \leq \tau_n\}} \int_0^s l(s, u) dB^a_u \, ds \right)^2 \leq \tilde{\mathbb{E}}^n \left( \int_0^t \sup_{s \leq \tau_n} \int_0^s l(s, u) dB^a_u \, ds \right)^2 \\
\leq t \tilde{\mathbb{E}}^n \left( \int_0^t \sup_{s \leq \tau_n} \int_0^s l(s, u) dB^a_u \, ds \right)^2 \leq 4t^2 \int_0^T \int l^2(s, u) \, ds \, ds.
\]

**Second term:**

\[
\tilde{\mathbb{E}}^n \left( \int_0^t I_{\{s \leq \tau_n\}} \int_0^s l(s, u) \sigma(X_u) \, du \, ds \right)^2 \leq t \int_0^t \tilde{\mathbb{E}}^n \left( \int_0^s l(s, u) \sigma(X_u) \, du \right)^2 \, ds \\
\leq t \int_0^t \int_0^t l^2(s, u) \, du \int_0^s \sigma^2(X_v) \, dvds \\
\leq t \int_0^t \int_0^t l^2(s, u) \, du \int_0^s \left[ 1 + \tilde{\mathbb{E}}^n X^2_{v \wedge \tau_n} \right] \, dvds.
\]

**Third term:**

\[
\tilde{\mathbb{E}}^n \left( \int_0^t I_{\{s \leq \tau_n\}} \sigma_s(X) \, ds \right)^2 \leq t \int_0^t \left[ 1 + \tilde{\mathbb{E}}^n X^2_{v \wedge \tau_n} \right] \, dvds.
\]

**Fourth term:**

\[
\left( \int_0^t \tilde{\mathbb{E}}^n B^a_t \right)^2 \leq \mathbb{E}^n (t \wedge \tau_n) \leq t.
\]

Obtained estimates enable arrive at the Gronwall-Bellman inequality:

\[
\tilde{\mathbb{E}}^n X^2_{t \wedge \tau_n} \leq t \int_0^t \int l^2(t, s) \, dt \, ds \left[ 1 + \int_0^t \tilde{\mathbb{E}}^n X^2_{s \wedge \tau_n} \, ds \right]
\]

and, jointly with condition 2) of the theorem, to verify a validity \( S.1 \). □

**APPENDIX A. Generalized Girsanov theorem**

Let \( X \) solves equation \( (3.1) \) while stopping time \( \tau_n \) is defined in \( (6.1) \). Then

\[
X_{t \wedge \tau_n} = X_0 + \int_0^t I_{\{s \leq \tau_n\}} a_s(X) \, ds \\
+ \int_0^t I_{\{s \leq \tau_n\}} b_s(X) \, dB_s + \int_0^t \int_s^t I_{\{s \leq \tau_n\}} E_{R} h_s(X, z)[\mu(ds, dz) - dsK(dz)].
\]

Let the random variable \( 3_{T \wedge \tau_n} \) is defined in \( (6.2) \). Recall that \( \mathbb{E}_3 3_{T \wedge \tau_n} = 1 \) and set a probability measure \( Q^n \) having a density \( \frac{dQ^n}{d\mathbb{P}} = 3_{T \wedge \tau_n} \).
Theorem A.1. (1) \( \nu^n(ds, dz) = I_{\{s \leq \tau_n\}}[1 + \varphi(s, X_{s-}, z)]dsK(dz) \) is \( Q^n \) - compensator of the integer-valued random measure \( I_{\{s \leq \tau_n\}} \mu(ds, dz) \).

(2) \( (X_{t \wedge \tau_n}, \mathcal{F}_t, \tilde{Q}^n)_{t \in [0, T]} \) is a semimartingale with decomposition:

\[
X_{t \wedge \tau_n} = X_0 + \int_0^t I_{\{s \leq \tau_n\}} \left[ a_s(X) + b_s(X)\sigma_s(X) + \int_R h_s(X, z)\varphi_s(X, z)K(dz) \right] ds + \tilde{M}^{c,n}_t + \tilde{M}^{d,n}_t
\]

in which \( (\tilde{M}^{c,n}_t, \tilde{M}^{d,n}_t; \mathcal{F}_t, \tilde{Q}^n)_{t \in [0, T]} \) are continuous and purely discontinuous square integrable martingales. Their predictable quadratic variations is defined below:

\[
\langle \tilde{M}^{c,n} \rangle_t = \int_0^t I_{\{s \leq \tau_n\}} b_s^2(X)ds
\]

\[
\langle \tilde{M}^{d,n} \rangle_t = \int_0^t \int_R I_{\{s \leq \tau_n\}} h_s^2(X, z)[1 + \varphi_s(X, z)]K(dz)ds
\]

Proof. (1) Let \( \theta_n \) be stopping time, \( \theta_n \leq \tau_n \). Let \( \Gamma \in \mathbb{R} \setminus \{0\} \) is a measurable set. Then

\[
\tilde{E}^n \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} \mu(ds, dz) = \tilde{E}^n \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} \nu^n(ds, dz).
\]

On the other hand

\[
\tilde{E}^n \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} \mu(ds, dz) = \mathbb{E}^{\tilde{Q}^n}_{\tilde{F}^{T \wedge \theta_n}} \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} \mu^n(ds, dz)
\]

\[
= \mathbb{E} \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} \tilde{\beta}_{s \wedge \theta_n} \mu^n(ds, dz)
\]

\[
= \mathbb{E} \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} \tilde{\beta}_{s \wedge \theta_n} [1 + \varphi_s(X, z)]dsK(dz)
\]

\[
= \mathbb{E}^{\tilde{Q}^n}_{\tilde{F}^{T \wedge \theta_n}} \int_0^T \int_R I_{\{s \leq \theta_n\}} I_{\{z \in \Gamma\}} [1 + \varphi(s, X_{s-}, z)]dsK(dz)
\]

Now, in view of an arbitrariness \( \theta_n \) and \( \Gamma \), statement (1) holds (about additional details see [25], §5, Ch.4 ) and §3.a - §3.d in [13).

(2) To simplify notations denote

\[
\tilde{M}^{c,n}_t = \int_0^t I_{\{s \leq \tau_n\}} b_s(X)dB_s,
\]

\[
\tilde{M}^{d,n}_t = \int_0^t \int_R I_{\{s \leq \tau_n\}} h_s(X, z)[\mu(ds, dz) - dsK(dz)].
\]
Taking into account that $\tilde{\mathcal{M}}^{c,n}_t$ and $\mathcal{M}^{d,n}_t$, $\mathcal{M}^{c,n}_t$ are square integrable martingales their quadratic characteristics are defined as:

$$\langle 3 \wedge \tau_n, \mathcal{M}^{c,n} \rangle_t = \int_0^t I_{\{s \leq \tau_n\}} 3s - \sigma_s(X)b_s(X)ds$$

$$[3 \wedge \tau_n, \mathcal{M}^{d,n}]_t = \int_0^t \int_\mathbb{R} I_{\{s \leq \tau_n\}} \phi_s(X, z)h_s(X, z)q(ds, dz)$$

Moreover $\langle 3 \wedge \tau_n, \mathcal{M}^{d,n} \rangle_t$, being the compensator of $[3 \wedge \tau_n, \mathcal{M}^{d,n}]_t$, obeys the following presentation

$$\langle 3 \wedge \tau_n, \mathcal{M}^{d,n} \rangle_s = \int_0^t \int_\mathbb{R} I_{\{s \leq \tau_n\}} \phi_s(X, z)h_s(X, z)K(dz)ds.$$

Next, by Theorem 2, [25, §5, Ch.4],

$$\tilde{\mathcal{M}}^{c,n}_t = \mathcal{M}^{c,n}_t - \int_0^t 3^{-1}_{(s \wedge \tau_n)} - d\langle 3 \wedge \tau_n, \mathcal{M}^{c,n} \rangle_s$$

$$\tilde{\mathcal{M}}^{d,n}_t = \mathcal{M}^{d,n}_t - \int_0^t 3^{-1}_{(s \wedge \tau_n)} - d\langle 3 \wedge \tau_n, \mathcal{M}^{d,n} \rangle_s$$

are continuous and purely discontinuous $Q^n$ - martingales. In other words, $(Q^n : \mathcal{M}^{c,n}_t ; \mathcal{M}^{d,n}_t)$ processes obeys the presentations:

$$\mathcal{M}^{c,n}_t = \int_0^t I_{\{s \leq \tau_n\}} \sigma_s(X)b_s(X)ds + \tilde{\mathcal{M}}^{c,n}_t$$

$$\mathcal{M}^{d,n}_t = \int_0^t \int_\mathbb{R} I_{\{s \leq \tau_n\}} \phi_s(X, z)h_s(X, z)ds, + \tilde{\mathcal{M}}^{d,n}_t,$$

where $(Q^n, \tilde{\mathcal{M}}^{c,n}_t)$ is a continuous martingale and $(Q^n, \tilde{\mathcal{M}}^{d,n}_t)$ is purely discontinuous martingale with $(\mathcal{M}^{c,n}_t)$ and $(\mathcal{M}^{d,n}_t)$ given \cite{A.1}. \hfill $\square$

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