Parametrised Complexity of Model Checking and Satisfiability in Propositional Dependence Logic

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Abstract

In this paper, we initiate a systematic study of the parametrised complexity in the field of Dependence Logics which finds its origin in the Dependence Logic of Väänänen from 2007. We study a propositional variant of this logic (PDL) and investigate a variety of parametrisations with respect to the central decision problems. The model checking problem (MC) of PDL is NP-complete. The subject of this research is to identify a list of parametrisations (formula-size, treewidth, tree-depth, team-size, number of variables) under which MC becomes fixed-parameter tractable. Furthermore, we show that the number of disjunctions or the arity of dependence atoms (dep-arity) as a parameter both yield a paraNP-completeness result. Then, we consider the satisfiability problem (SAT) showing a different picture: under team-size, or dep-arity SAT is paraNP-complete whereas under all other mentioned parameters the problem is in FPT. Finally, we introduce a variant of the satisfiability problem, asking for teams of a given size, and show for this problem an almost complete picture.

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1 Introduction

The logics of dependence and independence are a recent innovation studying such central formalisms occurring in several areas of research: computer science, logic, statistics, game theory, linguistics, philosophy, biology, physics, and social choice theory [14]. Jouko Väänänen [27] initiated this subfield of research in 2007, and nowadays, it is a vibrant area of study [1]. Its focus widened from initially first-order dependence logic further to modal logic [28], temporal logics [20, 19], probabilistic logics [8], logics for independence [18], inclusion logics [12, 17], multi-team semantics [7], and poly-team semantics [15].

In this paper, we study a sub-logic of the modal variant which is called propositional dependence logic (PDL) [30, 16]. The main concept also in this logic, the dependence atom $\text{dep}(P; Q)$, intuitively states that the variables $p \in P$ functionally determine the values of the variables $q \in Q$. As functional dependence only makes sense on sets of assignments, which Väänänen called teams, team-semantics are the heart of the satisfaction relation $|=\text{ in this logic}$. Formally, a team $T$ is a set of classical propositional assignments $t: \text{Vars} \rightarrow \{0,1\}$, and $T |= \text{dep}(P; Q)$ if and only if for all $t, t' \in T$, we have that $t$ and $t'$ agree on the values of $P$ implies $t$ and $t'$ agree on the values of $Q$.

The model checking question (MC), given a team $T$ and a PDL-formula $\varphi$, asks if $T |= \varphi$ is true. The satisfiability problem (SAT), given a PDL-formula $\varphi$, asks for the existence of a team $T$ such that $T |= \varphi$. It is known that MC as well as SAT are NP-complete by Ebbing and Lohmann [9], respectively, by Lohmann and Vollmer [22]. These authors classify
the complexity landscape of even operator-fragments of PDL yielding a deep understanding of these problems from a classical complexity point of view. For an overview of how other atoms (e.g., inclusion, or independence) influence the complexity of these problems consider the tables in the work Hella et al. [17].

Often, when a problem is shown to be intrinsic hard, a possible way to further unravel the true reasons for the intractability is the framework of parametrised complexity theory [6]. Here, one aims for a more fine-grained complexity analysis involving the study of parametrisations and how they pin causes for intractability substantially. One distinguishes two runtimes of a different quality: \( f(k) \cdot p(|x|) \) versus \( p(|x|)^{f(k)} \), where \( f \) is an arbitrary computable function, \( p \) is a polynomial, \( |x| \) the input length and \( k \) the value of the parameter. Clearly, both runtimes are polynomial but the first one is much better as the polynomial degree is independent of the parameter’s value. Problems that can be solved with algorithms running in a time of the first kind are said to be fixed-parameter tractable (or in FPT). Whereas, problems of category two are in the complexity class XP. It is known that FPT \( \subseteq \) XP [11].

This area of research is tremendously growing and often provides new insights into the inherent difficulty of the studied problems [5]. However, the area of dependence logic is rather blank with respect to this direction of research, only Meier and Reinbold [24] investigated the parametrised enumeration complexity of a fragment of PDL recently. As a subject of this research, we want to initiate and to further push a study of the parametrised complexity of problems in these logics.

Contributions In Table 1 we give an overview of our results. We study eight different parametrisations for each of the model checking problem, the satisfiability problem, as well as a variant of satisfiability (this problem asks for a team of a given size). Thereby, we prove dichotomies for MC and SAT: depending on the parameter the problem is either fixed-parameter tractable or paraNP-complete. Only the satisfiability variant and the parameters treewidth and dep-arity resist a complete classification and are left for further research.

Related work The technique of Courcelle’s theorem [4] (see Prop. 6) has been used in different contexts: temporal logic [23], knowledge representation [13], and nonmonotonic logic [25]. Elberfeld et al. [10] enriched Courcelle’s theorem to also yield results for the complexity class XL. This improvement applies to our results utilising this theorem as well and affects Theorem 12, 16, and 25.

Organisation of the article At first, we introduce some required notions and definitions in (parametrised) complexity theory, dependence logic, and first-order logic. Then we study the parametrised complexity of the model checking problem. Proceed with the satisfiability problem and study a variant of it. Finally, we conclude and discuss open questions. Proofs of results marked with a (⋆) can be found in the appendix.

2 Preliminaries

In this paper, we assume familiarity with standard notions in complexity theory [26] such as the classes NP and P. We will recapitulate some relevant notion of parametrised complexity theory, now. For a broader introduction consider the textbook of Downey and Fellows [6], or Flum and Grohe [11]. A parametrised problem (PP) \( \Pi \subseteq \Sigma^* \times \mathbb{N} \) consists of tuples \((x, k)\), where \( x \) is called the instance and \( k \) the (value of the) parameter.
Table 1 Complexity classification overview with pointers to the theorems. All results are completeness results. The question mark symbol means that the precise complexity is unknown.

| Parameter         | MC   | SAT  | m-SAT |
|-------------------|------|------|-------|
| treewidth         | FPT  | FPT  | ?     |
| team-size         | FPT  | paraNP | paraNP |
| formula-size      | FPT  | FPT  | FPT  |
| treedepth         | FPT  | FPT  | FPT  |
| #variables        | FPT  | FPT  | FPT  |
| #splits           | paraNP | FPT  | ?     |
| dep-arity + #splits | paraNP | FPT  | FPT  |

Definition 1 (Fixed-parameter tractable and paraNP). Let \( \Pi \) be a PP over \( \Sigma^* \times \mathbb{N} \). We say that \( \Pi \) is fixed-parameter tractable (or is in the class FPT) if there exists a deterministic algorithm \( A \) deciding \( \Pi \) in time \( f(k) \cdot |x|^{O(1)} \) for every input \((x,k) \in \Sigma^*\), where \( f \) is a computable function. If \( A \) is a nondeterministic algorithm instead, then \( \Pi \) belongs to the class paraNP.

Propositional dependence logic Let \( \text{VAR} \) be a countably finite set of variables. The syntax of propositional dependence logic (PDL) is defined via the following EBNF:

\[
\varphi :: \top | \bot | x | \neg x | \varphi \lor \varphi | \varphi \land \varphi | \text{dep}(X;Y),
\]

where \( \top \) is verum, \( \bot \) is falsum, \( x \in \text{VAR} \) is a variable, \( X,Y \subseteq \text{VAR} \) are finite sets of variables, \( \text{dep}(\cdot;\cdot) \) is called the dependence atom, and the disjunction \( \lor \) is also called split-junction. Observe that we only consider atomic negation. We let \( \mathcal{P} \) be defined as the PDL-formulas without \( \text{dep}(\cdot;\cdot) \).

Definition 2 (Team semantics). Let \( \varphi,\psi \) be PDL-formulas and \( P,Q \subseteq \text{VAR} \) be two finite sets of variables. A team \( T \) is a set of assignments \( t: \text{VAR} \rightarrow \{0,1\} \). Furthermore, we define the satisfaction relation \( \models \) as follows, where \( T \models \top \) is always true, \( T \models \bot \) is never true, and \( T \models \neg \text{dep}(P;Q) \) iff \( T = \emptyset \):

\[
\begin{align*}
T \models x & \iff \forall t \in T : t(x) \\
T \models \neg x & \iff \forall t \in T : \neg t(x) \\
T \models \varphi \land \psi & \iff T \models \varphi \land T \models \psi \\
T \models \varphi \lor \psi & \iff \exists T_1 \cup T_2 = T : T_1 \models \varphi \text{ and } T_2 \models \psi \\
T \models \text{dep}(P;Q) & \iff \forall t,t' : \bigwedge_{p \in P} t(p) = t'(p) \text{ implies } \bigwedge_{q \in Q} t(q) = t'(q)
\end{align*}
\]

Note that in literature there exist two semantics for the split-junction operator: lax and strict semantics (e.g., Hella et al. [17]). Strict semantics requires the “splitting of the team” to be a partition whereas lax semantics allow an “overlapping” of the team. We use lax semantics here. Also further note that allowing a real negation operator dramatically increases the complexity of SAT in this logic to ATIME-ALT(exp, poly) (alternating exponential time with polynomially many alternations) as shown by Hannula et al. [16]. That is one reason why we stick to atomic negation.
Parametrised Complexity of MC and SAT in Propositional Dependence Logic

In the following, we define three well-known formula properties which are relevant to results in the paper. A formula \( \phi \) is flat if, given any team \( T \), we have that \( T \models \phi \iff \{ s \} \models \phi \) for every \( s \in T \). A logic \( \mathcal{L} \) is downwards closed if for every \( \mathcal{L} \)-formula \( \phi \) and team \( T \), if \( T \models \phi \) then for every \( P \subseteq T \) we have that \( P \models \phi \). A formula \( \phi \) is 2-coherent if for every team \( T \), we have that \( T \models \phi \iff \{ s_i, s_j \} \models \phi \) for every \( s_i, s_j \in T \). The classical \( \mathcal{P} \mathcal{L} \)-formulas are flat. Moreover, \( \mathcal{P} \mathcal{D} \mathcal{L} \) is downwards closed and every dependence atom is 2-coherent.

Monadic second-order logic and the representation of inputs

We consider the syntax (or formula) tree structure underlying a \( \mathcal{P} \mathcal{D} \mathcal{L} \)-formula as part of the representation. The leaves of this tree are atomic subformulas, that is, either propositional variables or the subformulas of the form \( \text{dep}(X; Y) \) for some sets \( X \) and \( Y \) of variables. Every internal node with two child nodes is either a conjunction or a disjunction. Notice that in \( \mathcal{P} \mathcal{D} \mathcal{L} \)-formulas the negation is only allowed in front of atomic formulas wherefore negation nodes can only appear the layer above the leaf nodes.

Given an instance \( \langle T, \Phi \rangle \) of the model checking problem, where \( \Phi \) is a \( \mathcal{P} \mathcal{D} \mathcal{L} \)-formula with propositional variables \( \{ x_1, \ldots, x_n \} \subseteq \text{VAR} \) and \( T = \{ s_1, \ldots, s_m \} \) is a team of assignments \( s_i : \text{VAR} \to \{ 0, 1 \} \). Then we define a graph-structure \( \mathcal{A}_{T, \Phi} \) as follows (we will often write \( \mathcal{A} \) instead when it is clear that \( \mathcal{A}_{T, \Phi} \) corresponds to input \( \langle T, \Phi \rangle \)). We consider the following signature and represent the formula by its syntax tree.

\[
\tau := \{ \text{SF}^1, \text{VAR}^1, \text{NEG}^2, \text{CONJ}^3, \text{DISJ}^3, \text{DEP}^3, \equiv^2, \psi^1, \ldots, \psi^1_m, 0, 1, r, c_1, \ldots, c_m \}
\]

Here, \( \equiv \) is a binary relation, \( r, 0, 1, c_1, \ldots, c_m \) are constant symbols and \( \psi^1, \ldots, \psi^1_m \) are unary function symbols. We will denote the set \( \{ c_1, \ldots, c_m \} \) by \( C \). Note that the team-size is not fixed and accordingly not part of the parameter. Intuitively, we use the symbols in \( C \) to encode the team \( T \). This is not an issue, because for every team-size we then have a reduction to \( \mathcal{M} \mathcal{S} \mathcal{O} \)-MC computable in \( \text{fpt} \)-time. Moreover, \( \text{SF} \) and \( \text{VAR} \) are unary relations representing ‘is a subformula of \( \Phi \)’ and ‘is a variable in \( \Phi \)’ respectively. The details of the \( \tau_{T, \Phi} \)-structure \( \mathcal{A} \) are as follows.

Universe \( A \): the set \( \text{SF}(\Phi) \cup \{ c_1^A, \ldots, c_m^A \} \cup \{ 0^A, 1^A \} \). We will often use 0, 1 for both, the \( \tau \)-symbols as well as the elements of \( A \) which are the interpretations of these symbols. Moreover, we will denote the set \( \{ c_1^A, \ldots, c_m^A \} \) by \( C^A \).

Subformula relation \( \equiv^A \): the binary relation on \( \text{SF}(\Phi) \) such that \( \phi \equiv^A \psi \) iff \( \psi \) is the immediate subformula of \( \phi \).

Root constant \( r \): the element interpreting \( \Phi \) and representing the root of the syntax tree.

Note that \( \Phi \) is a subformula of itself so \( \Phi \in \text{SF}(\Phi) \).

Variable relation \( \text{VAR}(x) \): is true if and only if \( x \) is a variable that appears in \( \Phi \).

Team constants \( C \): Each \( c_i \in C \) is interpreted as \( c_i^A \in C^A \). Where each \( c_i \) corresponds to the index of an assignment \( s_i \in T \) for \( i \leq m \) using the functions defined below.

Team elements functions \( \{ f_1, \ldots, f_m \} \): For each \( i \leq m, f_i^A : \text{VAR}(\Phi) \cup C^A \to \{ 0, 1 \} \) is a unary function. In order to simulate the team \( T \), the function \( f_i \) extends the assignment \( s_i \in T \) for each \( i \leq m \) via

\[
f_i^A(a) := \begin{cases} s_i(a) & \text{if } \text{VAR}(a) \\ 1 & \text{if } a = c_i^A \\ 0 & \text{otherwise.} \end{cases}
\]

That is, for each \( i, f_i \) extends the corresponding assignment function \( s_i \) in such a way that \( c_i^A \mapsto 1 \) and for \( j \neq i, c_j^A \mapsto 0 \). The idea is that each function \( f_i \) simulates the assignment
\[ s_i \in T \text{ and each such } f_i \text{ maps the constant corresponding to its index (that is, } c^A_i \text{) to 1 and to all other indices } (c^A_j, j \neq i) \text{ to 0. We use the index } i \text{ in } c^A_i \text{ as the call for the function } f_i \text{ (or } s_i). \]

**Dependence atom** \( \text{DEP}(x, Y_1, Y_2) \) is true if and only if \( Y_1 \) and \( Y_2 \) are disjoint sets of variables and \( x \) represents the dependence atom \( \text{dep}(Y_1; Y_2) \).

**Negation** \( \text{NEG}(x, y) \) is true if and only if \( x \) represents \( \lnot y \) where \( y \) is a variable.

**Conjunction** \( \text{CONJ}(x, y_1, y_2) \) is true if and only if \( x \) represents \( y_1 \land y_2 \) where \( y_1 \) and \( y_2 \) are subformulas.

**Disjunction** \( \text{DISJ}(x, y_1, y_2) \) is true if and only if \( x \) represents \( y_1 \lor y_2 \) where \( y_1 \) and \( y_2 \) are subformulas.

The reason for using constants is that in \( \mathcal{MSO} \) we cannot quantify over sub-teams (being the \( n \)-ary relations). On the other hand, this is needed because at every occurrence of a split-junction the team must split into two sub-teams. However, by relating these constants with the corresponding indices of the assignments, we can achieve this purpose by quantifying over subsets of constants (or indices). Furthermore, we assume an arbitrary but fixed ordering on the team \( T \).

**Definition 3 (Gaifman graph).** Given a team \( T \) and a \( \mathcal{PDL} \)-formula \( \Phi \), the Gaifman graph \( G_{T,\Phi} = (A, E) \) of the \( \tau_{T,\Phi} \)-structure \( A \) is defined as

\[
E := \{ \{ u, v \} \mid u, v \in A, u \text{ and } v \text{ share a tuple in a relation in } A \}
\]

**Lemma 4 (⋆).** There is a 1-1-correspondence between the collection \( \{ P_i \mid P_i \subseteq T \} \) of sub-teams and \( \{ C_i \mid C_i \subseteq C \} \) of subsets of indices.

We will often write \( C_P \) for a subset of \( C \) that corresponds to the sub-team \( P \). Moreover, when saying ‘a sub-team’ in the context of the structure \( A \) we mean some subset \( C_P \subseteq C^A \).

In this paper, we will consider treewidth \([2]\) of structures as a possible parameter. We will use it in the context of Courcelle’s theorem \([3]\) as a parameter of the occurring first-order structures which are interpreted as Gaifman graphs.

**Definition 5 (Treewidth).** The tree-decomposition of a given graph \( G = (V, E) \) is a tree \( T = (B, E_T) \), where the vertex set \( B \) is called bags and \( E_T \) is the edge relation such that the following is true: (i) \( \bigcup_{b \in B} \neq V \) (ii) for every \( \{ u, v \} \in E \) there is a \( b \in B \) with \( u, v \in b \), and (iii) for all \( v \in V \) the restriction of \( T \) to \( v \) is connected. The width of a given tree-decomposition \( T = (B, E_T) \) is the size of the largest bag minus one: \( \max_{b \in B} |b| - 1 \). The treewidth of a given graph \( G \) is the minimum over all widths of tree-decompositions of \( G \).

Observe that if \( G \) is a tree then the treewidth of \( G \) is one. Intuitively, one can say that treewidth accordingly is a measure of tree-likeness of a given graph. The next result is used in the context of the occurring parameter treewidth to achieve an \( \mathsf{FPT} \) result. We apply this result in Theorem \([12, 16, 25]\).

**Proposition 6 (Courcelle’s Theorem, \([4\) Theorem 6.3 (1)])].** \( \mathcal{MSO} \)-MC is in \( \mathsf{FPT} \) when parametrised by the treewidth of the input structure and the length of the input formula.

### 3 Parametrised Complexity of Model Checking in PDL

In this section, we study the MC question under various parametrisations. Table \([1]\) contains a complete list of results with all the parameters.
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3.1 Parameter: treewidth

Here, we consider the treewidth of the Gaifman graph of $A$ as a parametrisation. We aim to define an $\mathcal{MSO}$-sentence which expresses the model checking problem of $\mathcal{PDL}$-formulas. In the next lemmas, we approach this goal step-wisely.

- **Lemma 7** (*). There is a sentence $\Gamma_{\text{structure}}$ such that $A \models \Gamma_{\text{structure}}$ if and only if $A$ contains the formula-tree of $\Phi$.

  Note, that we require the tree-relation $\succeq^A$ to be not transitive. We also need that the functions $\{ f_i \mid i \leq k \}$ map the variables and the constants to $\{0, 1\}$ such that each function maps exactly one constant to 1.

- **Lemma 8** (*). There exists a sentence $\Gamma_{\text{index}}$ such that $A \models \Gamma_{\text{index}}$ if and only if the functions $A$ maps exactly one constant to 1.

  The following lemma show that there is no relation between treewidth and team-size.

- **Lemma 9** (*). Given a $\mathcal{PDL}$-formula $\Phi$ and a team $T$. Then treewidth of the Gaifman graph corresponding to $A$ is independent of team-size.

Now, we simulate the truth of a subformula under a sub-team in the structure $A$.

- **Lemma 10.** Given a subformula $\phi$ of $\Phi$ and a sub-team $P \subseteq T$ then there exists an $\mathcal{MSO}$-formula $\Gamma_{\text{true}}(Z, x)$ such that the following is true: $P \models \phi$ if and only if $A \models \Gamma_{\text{true}}(C_P, \phi)$.

  **Proof.** First, we define the following formula to simplify our notation:

  \[
  \theta_{\text{index}}(x) := \neg \text{VAR}(x) \land \left( \bigvee_{j \leq m} f_j(x) = 1 \right)
  \]

  Clearly, $A \models \theta_{\text{index}}(a)$ if and only if $a \in C^A$. That is, $f_i^A(a) = 1$ for some $i \leq m$. We use this formula to refer to the function $f_i$ directly. That is, we associate each assignment $s_i \in T$ with a constant element $c_i \in C$ such that the role played by $s_i$ can be simulated by $c_i$ through the relation $A \models \theta_{\text{index}}(c_i)$. Informally, we can quantify over sub-teams of the team by this construction.

  We will also work with set variables and most of the times these will be used for subsets of $C$. Recall that $C_P = \{ c \in C \mid s_i \in P, f_i(c) = 1 \}$. Consequently, $C_P \subseteq C$ and we define $\theta_{\text{index set}}(C_P) := \forall x(C_P(x) \rightarrow \theta_{\text{index}}(x))$. Now, we let

  \[
  \theta_{\text{maps}}(y, x, d) := \theta_{\text{index}}(y) \land \text{VAR}(x) \land \left( \bigwedge_{i \leq m} (f_i(y) = 1 \rightarrow f_i(x) = d) \right),
  \]

  where $d \in \{0, 1\}$. The formula $\theta_{\text{maps}}(c, x, 1)$ ensures that the element $c$ is an index in $C$. Accordingly, for some $i \leq m$, we have that $c = c_i$ and $f_i(c_i) = 1$, where by definition $f_i$ corresponds to the assignment $s_i$. Then, $\theta_{\text{maps}}(c, x, 1)$ intuitively says that the assignment corresponding to $c$ evaluates the variable $x$ to 1.

  Since the truth value of a formula is determined by its subformulas we need the implications giving sufficient conditions for a particular subformula to be true under a given team. Let us start by defining the following formulas that simulate the truth of an atom $\alpha$ under a sub-team $P$.

  For $\text{VAR}(x)$, we denote $\theta_{\text{true var}}(C_P, x)$ if and only if every assignment with its corresponding index in $C_P$ evaluates $x$ to 1. That is, $\theta_{\text{true var}}(C_P, x) := \forall c(C_P(c) \rightarrow \theta_{\text{maps}}(c, x, 1))$, and
similarly, for evaluating to 0, let \( \theta_{\text{false}}(C_P, y) := \forall c (C_P(c) \rightarrow \theta_{\text{maps}}(c, y, 0)) \). For \( Y_1, Y_2 \) and \( x \) such that \( \text{DEP}(x, Y_1, Y_2) \), we set \( \theta_{\text{fix}}(c_1, c_2, y) := \theta_{\text{maps}}(c_1, y, 1) \leftrightarrow \theta_{\text{maps}}(c_2, y, 1) \), and then

\[
\theta_{\text{true}}(C_P, x, Y_1, Y_2) := \forall c_1 \forall c_2 \exists y \left[ (C_P(c_1) \land C_P(c_2)) \rightarrow \left( (\forall y (Y_1(y) \land \theta_{\text{fix}}(c_1, c_2, y))) \rightarrow (\forall y (Y_2(y) \lor \theta_{\text{fix}}(c_1, c_2, y))) \right) \right].
\]

The formula intuitively says that for every pair \( c_1, c_2 \) fixing every variable from \( Y_1 \) also fixes every variable from \( Y_2 \). This is the necessary condition for a dependence atom to be true. For \( \text{NEG}(x, y) \) define \( \theta_{\text{neg}}(C_P, x) := \exists y (\varphi_{\text{VAR}}(y) \land \text{NEG}(x, y) \land \theta_{\text{false}}(C_P, y)) \).

This completes the truth simulation for the base case at the atomic subformulas level. For the remaining cases of internal nodes notice that the truth value of a node can be determined once its immediate nodes have been evaluated. That is, if we know the truth value of \( \phi \) and \( \theta \) and there is a subformula \( \phi \) such that \( \phi = \alpha_1 \land \alpha_2 \) then for any sub-team \( P \subset T \) we have \( P \models \phi \iff P \models \alpha_1 \) and \( P \models \alpha_2 \). Similarly, for disjunction (\( \phi = \alpha_1 \lor \alpha_2 \)) we have that \( P \models \phi \iff \exists P_1 \exists P_2 (P_1 \cup P_2 = P) \) such that \( P_1 \models \alpha_1 \) and \( P_2 \models \alpha_2 \). In view of this, for the general case, we let

\[
\Gamma_{\text{true}}(C_P, \phi) := \exists \alpha_1 \exists \alpha_2 \exists C_{p_1} \exists C_{p_2} \left[ \Gamma_{\text{true}}(C_{p_1}, \alpha_1) \land \Gamma_{\text{true}}(C_{p_2}, \alpha_2) \land \text{CONJ}(\phi, \alpha_1, \alpha_2) \rightarrow \forall c (C_{p_1}(c) \leftrightarrow C_{p_2}(c)) \right] \land \text{DISJ}(\phi, \alpha_1, \alpha_2) \rightarrow \forall c (C_{p_1}(c) \lor C_{p_2}(c)) ,
\]

where the occurrence of \( \Gamma_{\text{true}}(C_{p_i}, \alpha_i) \) for \( i = 1, 2 \) indicates that for \( \phi \models \alpha_i \), the truth value of \( \phi \) under \( P \) is evaluated provided that the truth value for \( \alpha_i \) is known under \( P_i \).

The fact that the negation symbol only appears at the front of a variable implies that the subformulas with negation symbol can only appear at depth \( 1 \). We consider this case separately. This simplifies our construction since for the internal nodes we only have either conjunctions or disjunctions. Combining base and general case we get the following

\[
\Gamma_{\text{true}}(C_P, \phi) := \forall \varphi (\varphi \rightarrow \theta_{\text{true}}(C_P, \varphi)) \land \exists Y_1 \exists Y_2 (\text{DEP}(\phi, Y_1, Y_2) \rightarrow \theta_{\text{true}}(C, \phi, Y_1, Y_2)) \land \exists \alpha (\varphi_{\text{VAR}}(\alpha) \land \text{NEG}(\phi, \alpha) \rightarrow \theta_{\text{false}}(C_P, \alpha)) \land \exists \alpha_1 \exists \alpha_2 (\text{CONJ}(\phi, \alpha_1, \alpha_2) \rightarrow (\Gamma_{\text{true}}(C_P, \alpha_1) \land \Gamma_{\text{true}}(C_P, \alpha_2))) \land \exists \alpha_1 \exists \alpha_2 (\text{DISJ}(\phi, \alpha_1, \alpha_2) \rightarrow \exists C_1 \exists C_2 \forall c (C_{p_1}(c) \leftrightarrow (C_1(c) \lor C_2(c))) \land \Gamma_{\text{true}}(C_1, \alpha_1) \land \Gamma_{\text{true}}(C_2, \alpha_2)) .
\]

Finally, the following claim completes the proof to our lemma. Moreover, this will help us switching between the truth evaluation \( P \models \alpha \) and \( A \models \Gamma_{\text{true}}(C_P, \alpha) \) whenever required.

\begin{itemize}
\item \textbf{Claim 11 (\(*\)}. Let \( A \) be a \( \tau_{T, \Phi} \)-structure such that \( A \models \Gamma_{\text{structure}} \land \Gamma_{\text{index}} \). Then, for every subformula \( \alpha \in \text{SF}(\Phi) \) and every sub-team \( P \subset T \), we have \( P \models \alpha \iff A \models \Gamma_{\text{true}}(C_P, \alpha) \), where \( C_P \) is the index set corresponding to the team \( P \).
\end{itemize}

\begin{itemize}
\item \textbf{Theorem 12 (\(*\)}. \textit{MC parametrised by treewidth of } \( G_{T, \Phi} \) \textit{and treedepth of } \( \Phi \) \textit{is in } \textit{FPT}. \end{itemize}
3.1.1 Layer-wise construction to avoid treedepth

Now, we explain how to modify the formulas in the proof of Lemma 10 such that internal nodes are satisfied under the corresponding sub-teams without using the treedepth requirement. We use the idea of layers of the syntax tree to overcome this hurdle. There are three types of nodes in the syntax tree ($A$, $\Rightarrow$, $\Phi$). Namely the leaf nodes with no child, negation nodes with exactly one child and internal nodes with two child nodes. Accordingly, by quantifying universally over layers, we can avoid relying on treedepth as a parameter.

The tuple $(\alpha, y_1, y_2)$ is a layer if $\alpha$ is a subformula of $\Phi$ and $y_1, y_2$ are immediate subformulas of $\alpha$. We sometimes specify a layer by the parent node $\alpha$ alone. Moreover, in this subsection the indices of the index sets $C_\alpha$ refer to the corresponding subformula $\alpha$.

Note, that the formula $\Gamma_{\text{treedepth}}(C_\varphi, \varphi)$ in the proof of Lemma 10 gives a sufficient condition for $\varphi$ to be true when its immediate subformulas have been evaluated already. We formalise this by the existence of a function that maps each variable $X$ to a corresponding team $P_x$ such that $P_x \models X$, associates every dependence atom with a sub-team satisfying this atom, and maps $y$ (s.t. $y = \neg X$) to a sub-team $P_y \models \neg X$. We rely on the following proposition of Yang.

► Proposition 13 ($[29]$). The truth value of a formula $\Phi$ under a team $T$ is determined by the truth value of leaves in the syntax tree of $\Phi$ under the sub-teams of $T$.

Intuitively speaking, the truth of a non-atomic subformula is evaluated under a sub-team $P \subseteq T$ given the truth evaluation of its atomic subformulas under sub-teams of $P$.

There could be many teams in case of a dependence atom (for example every singleton). For this reason we quantify over teams in such a way that at the root level (the formula $\Phi$) the full team is reached.

► Lemma 14. Let $A$ be a $\tau_T, \varphi$-structure such that $A \models \Gamma_{\text{structure}} \land \Gamma_{\text{index}}$. Then there is an $\mathcal{MSO}$-sentence $\Gamma_{\text{layer}}$ such that $A \models \Gamma_{\text{layer}} \iff T \models \Phi$.

Proof. Let us start to define the layer construction. The following formula maintains the split-junctions and conjunctions accordingly and ensures that the root-team complies with $T$.

$$
\begin{align*}
\Gamma_{\text{induced}}(\alpha, y_1, y_2, C_\alpha, C_{y_1}, C_{y_2}) := & \Gamma_{\text{treedepth}}(C_{y_1}, y_1) \land \Gamma_{\text{treedepth}}(C_{y_2}, y_2) \\
& \land (r(\alpha) \rightarrow \forall c [C_\alpha(c) \leftrightarrow C(c)]) \\
& \land [(\text{CONJ}(\alpha, y_1, y_2) \rightarrow \forall c [C_\alpha(c) \leftrightarrow C_{y_1}(c) \leftrightarrow C_{y_2}(c)])] \\
& \land (\text{DISJ}(\alpha, y_1, y_2) \rightarrow \forall c [C_\alpha(c) \leftrightarrow (C_{y_1}(c) \lor C_{y_2}(c))]),
\end{align*}
$$

where $y_1, y_2$ are immediate subformulas of $\alpha$ as specified by the relations $\text{CONJ}(\alpha, y_1, y_2)$ and $\text{DISJ}(\alpha, y_1, y_2)$. The next formula distinguishes between the leaf and inner cases.

$$
\begin{align*}
\Gamma_{\text{node}}(C_x, x) := & \exists y_1 \exists y_2 \exists C_{y_1} \exists C_{y_2} \\
& [r(x) \rightarrow \forall c [C(c) \rightarrow C_x(c)]] \\
& \land [\theta_{\text{leaf}}(x) \rightarrow \theta_{\text{true}}(C_x, x)] \\
& \land [(y_1 \neq y_2 \land \text{NEG}(x, y_1)) \rightarrow \theta_{\text{false}}(C_x, y_1)] \\
& \land [(y_1 \neq y_2 \land x \equiv y_1 \land x \equiv y_2) \rightarrow \Gamma_{\text{induced}}(x, y_1, y_2, C_x, C_{y_1}, C_{y_2})],
\end{align*}
$$

where $\theta_{\text{leaf}}(x)$ is a formula saying that $x$ is a leaf in the formula tree and

$$
\theta_{\text{true}}^{\text{true}}(C_x, x) := [\text{VAR}(x) \land \theta_{\text{true}}^{\text{var}}(C_x, x)] \lor [\exists Y_1 \exists Y_2 (\text{DEP}(x, Y_1, Y_2) \land \theta_{\text{true}}^{\text{dep}}(C_x, \theta, Y_1, Y_2))].
$$
When quantified universally over each layer $x$, this provides the necessary condition for existence of a truth function that associates a satisfying sub-team $C_x$ to each $x$ such that $A \models \Gamma_{true}(C_x, x)$. That is, $\Gamma_{layer} := \forall x \exists C_x(SF(x) \rightarrow (\theta_{index\ set}(C_x) \land \Gamma_{node}(C_x, x)))$. Notice the existential quantification for subformulas $y_1$ and $y_2$ as well as for their corresponding teams. This highlights the fact that the truth value of every parent node $\alpha$ under any index set $C_x$ is determined by the evaluation of the two children nodes $y_1, y_2$ under the index sets which are specified by the parent node. Now, it suffices to prove the following statement.

\begin{itemize}
\item[$\triangleright$] Claim 15 ($\ast$). $A \models \Gamma_{layer} \iff T \models \Phi$.
\end{itemize}

The following theorem is is a consequence of Lemma 14 in combination with Proposition 6.

\begin{itemize}
\item[$\triangleright$] Theorem 16. MC parametrised by treewidth of $G_{T, \Phi}$ is in FPT.
\end{itemize}

3.2 Parameter: team-size

The main source of difficulty in the model checking problem seems to be the split-junction operator. For a team of size $k$ and a formula with only one split-junction there are $2^k$ many candidates for the correct split and each can be verified in polynomial time. As a result, an exponential runtime in the input length seems necessary. However, if $k$ is considered as a parameter then the problem can be solved in polynomial time with respect to the input and exponentially in the parameter.

\begin{itemize}
\item[$\triangleright$] Theorem 17. MC parametrised by team-size is in FPT.
\end{itemize}

\textbf{Proof.} We claim that Procedure $\text{check}$ solves the task in fpt-time. The correctness follows from the fact that the procedure is simply a recursive definition of truth evaluation of $\mathcal{PDL}$-formulas. The total time for this algorithm can be bounded by $O(2^k) + O(d \cdot 2^k) \cdot |x|^{O(1)}$ for any instance $x = (T, \Phi)$, where $\#\text{splits} = d$. Finally, since $|T, \Phi| \geq d$, we achieve $O(2^k \cdot p(|x|))$ for some polynomial $p$.

3.3 Remaining Cases

Intuitively, under the parametrisation number of occurring split-junctions ($\#\text{splits}$) the problem cannot be in $\text{W}[\mathcal{P}]$ as this would suggest, we can limit the number of non-deterministic steps to the parameter alone. We show $\text{paraNP}$-completeness by reducing from the 3-colourability problem (3COL) and applying the following result.
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![Figure 1](image)

**Figure 1** A graph $G = \langle \{ v_i, v_j, v_k \}, \{ e_l, e_m \} \rangle$ and a corresponding team.

**Proposition 18** ([11, Theorem 2.14]). A nontrivial parameterised problem in paraNP with at least one NP-complete slice is paraNP-complete.

The idea of the reduction from 3COL is to construct a team as shown in Figure 1 in combination with the disjunction of three times the formula $\bigwedge_{x_i=(v_i,v_j)} \text{dep}(y_k;x_i)$. Intuitively, vertices of the graph correspond to assignments in the team and the three splits then map to the three colours.

**Theorem 19** (⋆). MC parametrised by $\#\text{splits}$ is paraNP-complete.

In the reduction from 3COL mentioned above, the arity is already fixed.

**Theorem 20** (⋆). MC parametrised by dep-arity is paraNP-complete.

Notice, that a rather simple PDL-formula was used in the proof of Theorem 19. This gives the following corollary in conjunction with Theorem 20.

**Corollary 21.** MC parametrised by $\#\text{splits} + \text{dep-arity}$ is paraNP-complete.

The remaining cases then can be easily deduced.

**Theorem 22** (⋆). MC for the parameters formula-size, $\#\text{variables}$, or treedepth is in FPT.

### 4 Satisfiability

In this section, we study SAT under various parameterisations, so the question of whether there exists a team $T$ for a given formula $\Phi$ such that $T \models \Phi$. First note, that in the absence of the parameter team-size the question is equivalent to finding a singleton team. As a result, team semantics coincides with the usual Tarskian semantics. This facilitates, for example, determining the truth value of disjunctions in the classical way. The reason for that is, if there is a satisfying team, then a singleton team satisfy the formula witnessed by PDL being downwards closed.

**Theorem 23** (⋆). SAT parametrised by team-size, or dep-arity is paraNP-complete.

Now, we turn towards the FPT-cases of SAT.

#### 4.1 Parameter: treewidth

We again apply Courcelle’s theorem (Prop. 6) to obtain an FPT results for the satisfiability question. However, the construction is different from the one used in Subsection 3.1. Accordingly, we modify the tree structure as the team is now not any longer part of it.

We use the vocabulary $\tau_\Phi = \{ \text{SF}^1, \neg^2, \text{LIT}^4, \text{VAR}^3, \text{NEG}^2, \text{CONJ}^3, \text{DISJ}^3, \text{DEP}^3, r, z_{\text{dep}} \}$.

Different than before, we wish to consider the negated variables (literals) also as leaves. The idea is that with each subformula $\alpha$ we associate a subset of literals that must be mapped true
in order for the formula to be satisfiable. The construction also encompasses the necessary conditions under which the truth value of a parent node is determined by its child nodes. We assure that a label for any leaf node contains a literal at that leaf node whereas a label under any assignment. Whereas, at a conjunction node, when an assignment contains the constant $z_{dep}$ if and only if the node is a dependence atom. This corresponds to our intuition that a dependence atom is always true under any singleton team.

Let $A = SF(\Phi) \cup \{ z_{dep} \}$. The relations are interpreted as before except that now the leaves of $(SF(\Phi), \succ, \Phi)$ are literals and dependence atoms. As a consequence, all internal nodes have two children.

We give an example to elaborate the algorithmic idea with the formula $\Phi = (x_1 \land \neg x_2) \land (((x_1 \land \neg x_2) \land \text{dep}(P; Q)) \lor ((x_1 \lor \neg x_1) \land x_0))$. We label each leaf-literal with itself. Then, taking the union at each conjunction node and keeping each choice separately at disjunction nodes, we get three labels for $\Phi$, namely $L_1 = \{ x_4, \neg x_2, z_{dep}, x_1 \}$, $L_2 = \{ x_4, \neg x_2, x_3, x_6 \}$ and $L_3 = \{ x_4, \neg x_2, \neg x_1, x_6 \}$. Each of these labels is consistent and arises to a satisfying assignment $s_i$ (or the team $\{ s_i \}$). Conversely, each satisfying assignment gives such a label. For example, $s_2(x_4) = s_2(x_3) = s_2(x_6) = 1$ and $s_2(x_2) = 0$, then $\{ s_2 \} \models \Phi$. Now, starting at the root we can label each node. Notice, that this assignment corresponds to $L_2$.

**Lemma 24** (*). Let $\Phi \in \mathcal{PDL}$ and $A_\Phi$ be the corresponding $\tau_\Phi$-structure such that $A_\Phi \models \Gamma_{\text{structure}}$. Then there exists an MSO-sentence $\Gamma_{\text{sat}}$ such that

$$A_\Phi \models \Gamma_{\text{sat}} \iff \text{There is an assignment } s: \text{VAR}(\Phi) \rightarrow \{0, 1\} \text{ s.t. } \{ s \} \models \Phi$$

Let $G_\Phi$ be the Gaifman graph for the structure $A_\Phi$. Then combine Prop. 6 with Lemma 24 to achieve the following theorem.

**Theorem 25.** SAT parametrised by treewidth of $G_\Phi$ is in FPT.

### 4.2 Parameter: \#splits

We present a procedure that constructs a satisfying assignment $s$ such that $\{ s \} \models \Phi$ if there is one, otherwise it answers no. The idea is that this procedure needs to remember the positions where a modification in the assignment is possible. We show that the number of these positions is bounded by the parameter \#splits.

Consider the syntax-tree $(SF(\Phi), \succ, \Phi)$ of $\Phi$ where, as before, multiple occurrences of subformulas are allowed. The procedure starts at the leaf level with satisfying singleton team candidates (for convenience, we will talk only about assignments now). Reaching the root it confirms whether it is possible to have a combined assignment or not. We assume that the leaves of the tree consist of literals or dependence atoms. Accordingly, the internal nodes of the tree are only conjunction and disjunction nodes. The procedure sets all the dependence atoms to be trivially true (as we satisfy them via singletons). Additionally, it sets all the literals satisfied by their respective assignment. Ascending the tree, it checks the relative conditions for conjunction and disjunction by joining the assignments and thereby giving rise to conflicts. A conflict arises (only at a conjunction node) when two assignments are joined with contradicting values for some variable. At this point, it sets this variable $x$ to a conflict state $c$. At disjunction nodes the assignment stores that it has two options and keeps the assignments separately.

Joining a true-value from a dependence atom affects the assignment only at disjunction nodes. This corresponds to the intuition that a formula of the form $\text{dep}(P; Q) \lor \psi$ is true under any assignment. Whereas, at a conjunction node, when an assignment $s$ joins with a true, the procedure returns the assignment $s$. Since at a split the procedure returns both
assignments, for \( k \) splits there could be \( \leq 2^k \)-many assignment choices. At the root node if at least one assignment is consistent then we have a satisfying assignment. Otherwise, if all the choices contain conflicts over some variables then there is no such satisfying singleton team.

▶ **Theorem 26** (⋆). SAT parametrised by \( \# \text{splits} \) is in \( \text{FPT} \). Moreover, there is an algorithm that solves the problem in \( O(2^{\# \text{splits}(\Phi)} \cdot |\Phi|^{O(1)}) \) for any \( \Phi \in \mathcal{PDL} \).

The remaining cases follow from the previous results.

▶ **Theorem 27** (⋆). SAT parametrised by \( \# \text{variables} \), formula-size, or treedepth is in \( \text{FPT} \).

### 4.3 A satisfiability variant

The shown results suggest that it might be interesting to study the following variant of SAT, in which we impose an additional input \( 1^m \) (unary encoding) with \( m \geq 2 \) and ask for a satisfying team of size \( m \). Let us call the problem \( m \text{-SAT} \). We wish to emphasise that \( m \text{-SAT} \) is not the same as the SAT parametrised by team-size.

▶ **Theorem 28** (⋆). \( m \text{-SAT} \) parametrised by \( \# \text{variables} \), formula-size, or treedepth is in \( \text{FPT} \).

One can extend the algorithm from Theorem 26 however, the arity of dependence atoms has to be part of the parameter as well.

▶ **Theorem 29** (⋆). Given a \( \mathcal{PDL} \)-formula \( \Phi \) with \( q \) as the maximum arity of its dependence atoms and \( k \) many split-junctions. Then there is an algorithm that enumerates all the satisfying teams of \( \Phi \) in time \( O(2^{2^q} \cdot 2^k \cdot p(|\Phi|)) \) for some polynomial \( p(x) \).

Neither the arity of the dependence atoms nor the team-size alone are fruitful parameters.

▶ **Theorem 30** (⋆). \( m \text{-SAT} \) parametrised by team-size, or dep-arity is paraNP-complete.

### 5 Conclusion

In this paper, we started a systematic study of the parametrised complexity of model checking and satisfiability in propositional dependence logic. For both problems, we exhibited a complexity dichotomy (see Table 1): depending on the parameter, the problem either is in \( \text{FPT} \) or paraNP-complete. Interestingly, there exist parameters for which MC is easy, but SAT is hard (team-size) and vice versa (\( \# \text{splits} \)).

Furthermore, we introduced a satisfiability question which also asks for a team of a given size (\( m \text{-SAT} \)). Here, we leave the cases for treewidth and \( \# \text{splits} \) open for further research. The parameter treewidth alone reduces to assuring team labels in our construction for SAT. It turns out that a modification of it to include non-trivial labels for each dependence atom does not work. The reason is that at the root level we might want to quantify over sets of labels which seems to be futile in MSO.

The parameter \( \# \text{splits} + \text{dep-arity} \) plays a special role, as dep-arity alone is always hard for all three problems, whereas adding \( \# \text{splits} \) allows for \( m \text{-SAT} \) to reach \( \text{FPT} \) (while the precise complexity for \( \# \text{splits} \) alone is unknown—the proposed algorithm for SAT, in that case, allows for no immediate generalisation).

Another important question for future research is to consider the validity and implication problem for \( \mathcal{PDL} \). Finally, we aim, besides answering the open cases, to study further operators such as independence and inclusion atoms.
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Proof. We specify how the structure $\Gamma$ corresponds to a single assignment $s_i$ through the function $f_i$. For each sub-team $P \subseteq T$ we set $C_P = \{ c \in C \mid s_i(c) = 1 \}$. Similarly for each $C_r \subseteq C$ we set $P_r = \{ s_i \in T \mid f_i(c) = 1 \}$. □

Lemma 7. There is a sentence $\Gamma_{\text{structure}}$ such that $A \models \Gamma_{\text{structure}}$ if and only if $A$ contains the formula-tree of $\Phi$.

Proof. We specify how the structure $A$ contains the correct syntax tree of $\Phi$. For this we need to assure the following.

**Universe** Every element of the universe is either a propositional variable or a constant in $C^4$ or one of either 0 or 1.

$$\psi_0(x) := \text{SF}(x) \lor C(x) \lor x = 0 \lor x = 1$$

**Tree** The root has no $\succ$-predecessor.

$$\psi_1(x) := x = r \iff \neg \exists y (x \succ y)$$

**Leaf** The leaves are atomic subformulas of $\Phi$. That is, either a propositional variable or a dependence atom

$$\theta_{\text{leaf}}(x) := \text{VAR}(x) \lor \exists Y_1 \exists Y_2 (\text{DEP}(x, Y_1, Y_2)) \land \neg \exists y (x \succ y)$$

**Inner node** Every formula that is not a leaf has at most two $\succ$-successors. If it has one successor then the formula is the negation of this successor. If it has two successors the formula is either a conjunction or a disjunction of these successors.

$$\psi_2(x) := \neg \theta_{\text{leaf}}(x) \lor (\exists y [x \succ y \rightarrow \text{NEG}(x, y)] \land \exists Y_1 \exists Y_2 [(x \succ y_1 \land x \succ y_2) \rightarrow \text{CONJ}(x, y_1, y_2) \lor \text{DISJ}(x, y_1, y_2)])$$

|   | $x_1$ | $\ldots$ | $x_n$ | $c_1$ | $c_2$ | $\ldots$ | $c_i$ | $\ldots$ | $c_m$ |
|---|---|---|---|---|---|---|---|---|---|
| $f_1$ | $s_1(x_1)$ | $\ldots$ | $s_1(x_n)$ | 1 | 0 | $\ldots$ | 0 | $\ldots$ | 0 |
| $f_2$ | $s_2(x_1)$ | $\ldots$ | $s_2(x_n)$ | 0 | 1 | $\ldots$ | 0 | $\ldots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $f_i$ | $s_i(x_1)$ | $\ldots$ | $s_i(x_n)$ | 0 | 0 | $\ldots$ | 1 | $\ldots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $f_m$ | $s_m(x_1)$ | $\ldots$ | $s_m(x_n)$ | 0 | 0 | $\ldots$ | 0 | $\ldots$ | 1 |

Table 2 From a team $T = \{ s_1, \ldots, s_m \}$ over $\{ x_1, \ldots, x_n \}$ to $C$ through functions relating them.
The sentence
\[
\Gamma_{\text{structure}} := \forall x (\psi_0(x) \land [\text{SF}(x) \rightarrow (\psi_1(x) \land \psi_2(x) \land \theta_{\text{leaf}}(x))])
\]
encodes the desired tree structure of \( \Phi \).

\textbf{Lemma 8.} There exists a sentence \( \Gamma_{\text{index}} \) such that \( A \models \Gamma_{\text{index}} \) if and only if the functions \( A \) maps exactly one constant to \( 1 \).

\textbf{Proof.} This can be assured by the following formulas
\[
\psi_3(x) := (\text{VAR}(x) \lor C(x)) \rightarrow \bigwedge_{i \leq m} (f_i(x) = 0 \lor f_i(x) = 1)
\]
\[
\psi_4(c, c') := (C(c) \land C(c')) \rightarrow \bigwedge_{i \leq m} [(f_i(c) = 1 \land f_i(c') = 1) \rightarrow c = c']
\]
and letting
\[
\Gamma_{\text{index}} := \forall x \forall c \forall c' (\psi_3(x) \land \psi_4(c, c')).
\]
Then \( A \models \Gamma_{\text{index}} \) implies that constants (indices) and functions are related to each other as claimed.

\textbf{Lemma 9.} Given a \( \text{PDL} \)-formula \( \Phi \) and a team \( T \). Then treewidth of the Gaifman graph corresponding to \( A \) is independent of team-size.

\textbf{Proof.} We consider the treewidth corresponding to only those participating relations from \( \tau_{T,\Phi} \) that simulate the team. From our construction, we have following relations and accordingly their corresponding bags in the tree-decomposition.

\textbf{Function symbols} The propositional variables appear as arguments to functions \( f_1^1, \ldots, f_m^1 \).
Moreover, each function (seen as relation) only relates variables to \( 0, 1 \) and thereby the corresponding bags must be of form \( \{ X_i, 0, 1 \} \) for each variable \( X_i \). Clearly, the team-size (number of functions) does not affect the treewidth since the bag size is constant.

\textbf{Constant symbols} Constants \( c_1, \ldots, c_m \) appear as arguments to functions and only appear in a relation as \( 0, 1 \). The corresponding bags can be taken \( \{ c_i, 0, 1 \} \) for each \( i \leq m \), again the number \( m \) does not change the treewidth.

Note that the vocabulary for Gaifman graph only allows relation symbols but we can circumvent this by interpreting the functions and constants as relation symbols and thereby having a purely relational vocabulary. This does not affect the treewidth of \( A \). Putting it all together, we get that treewidth is independent of team-size as claimed.

\textbf{Claim 11 (⋆).} Let \( A \) be a \( \tau_{T,\Phi} \)-structure such that \( A \models \Gamma_{\text{structure}} \land \Gamma_{\text{index}} \). Then for every subformula \( \alpha \in \text{SF}(\Phi) \) and every sub-team \( P \subseteq T \) the following is true
\[
P \models \alpha \iff A \models \Gamma_{\text{true}}(C_P, \alpha),
\]
where \( C_P \) is the index set corresponding to the team \( P \).

\textbf{Proof.} We prove this claim by an induction on the structure of \( \alpha \). We start with the base cases.
\( \alpha = X \) is a propositional variable Since \( A \models \text{VAR}(X) \) it follows that
\[
A \models \Gamma_{\text{true}}(C_P, X) \iff A \models \theta_{\text{true}}(C_P, X)
\]
\[
\iff A \models \forall c' (C_P(c') \rightarrow \theta_{\text{maps}}(c', X, 1))
\]
and
\[ A \models \theta_{\text{map}}(c', X, 1) \iff A \models \theta_{\text{index}}(c') \land \bigwedge_{i \leq m} (f_i^A(c') = 1 \rightarrow f_i^A(X) = 1) \]

But \( f_i^A(c') = 1 \iff c' = c_i \in C_P \) and consequently \( f_i^A(X) = s_i(X) \). That is,
\[ A \models f_i^A(X) = 1 \iff s_i(X) = 1 \iff \{ s_i \} \models \alpha \]

since this holds for every \( c' \in C_P \) (equivalently for every \( s_i \in P \)) and \( \alpha \) is a propositional variable. We have, by flatness:
\[ A \models f_i^A(X) = 1 \iff s_i(X) = 1 \iff \{ s_i \} \models \alpha \]

Finally yielding \( A \models \Gamma_{\text{true}}(C_P, X) \iff P \models \alpha. \)

\( \alpha = \text{dep}(Y; Z) \) is a dependence atom. Start with
\[ A \models \exists Y_1 \exists Y_2(\text{DEP}(\alpha, Y_1, Y_2)). \]

First note that \( P \models \alpha \) if \( \forall \exists' \forall \exists'' \in P \)
\[ \bigwedge_{y_u \in Y} (s'(y_u) = s''(y_u)) \implies \bigwedge_{z_v \in Z} (s'(z_v) = s''(z_v)) \]

As a result, for every pair \( c', c'' \in C_P \) with \( c' = c_i, c'' = c_j \) and variable \( X \) we have the following equivalences
1. \( A \models \theta_{\text{fix}}(c_i, c_j, X) \)
2. \( A \models f_i^A(X) = f_j^A(X) \)
3. \( s_i(X) = s_j(X) \)

Moreover, the assumption
\[ A \models \forall c_1 \forall c_2 (C_P(c_1) \land C_P(c_2)) \land \bigwedge_{y_i \in Y_1} \theta_{\text{fix}}(c_1, c_2, y_i) \]

implies the following equivalence
\[ A \models \forall c_1 \forall c_2 (C_P(c_1) \land C_P(c_2)) \land \bigwedge_{y_i \in Y_2} \theta_{\text{fix}}(c_1, c_2, y_i) \iff P \models \text{dep}(Y; Z) \]

\( \alpha = \neg X \) is the negation of a variable. Since,
\[ A \models \exists x (\theta_{\text{leaf}}(x) \land \text{NEG}(\alpha, x)) \]

implies
\[ A \models \Gamma_{\text{true}}(C_P, \alpha) \iff A \models \forall c(C_P(c) \rightarrow \theta_{\text{map}}(c, x, 0)) \]

We have
\[ P \models \alpha \iff A \models \Gamma_{\text{true}}(C_P, \alpha) \]

because of the following equivalences and the flatness property.
1. \( A \models \theta_{\text{map}}(c, X, 0) \)
2. \( c = c_i \) for some \( i \leq m \) and \( A \models f_i^A(X) = 0 \)
3. \( s_i(X) = 0 \), where \( s_i \in P \)
4. \( \{ s_i \} \models \neg X \)
\( \alpha = \neg \text{dep}(Y_1; Y_2) \) is the negation of a dependence atom We need to ensure that \( \mathcal{A} \models \alpha \iff C_P = \emptyset \). But this follows from \( P \models \alpha \iff P = \emptyset \) and our construction implies that \( C_P = \emptyset \) in this case.

Now we turn to the induction step. There are two cases which have to be examined. 

\( \alpha = \phi_1 \lor \phi_2 \) is a split-junction Accordingly we have that 
\[
\mathcal{A} \models \exists y_1 \exists y_2 (\text{DISJ}(\alpha, y_1, y_2)).
\]

As a consequence, the following two conditions are equivalent: 
\[
\mathcal{A} \models \Gamma_{\text{true}}(C_P, \alpha) \\
\iff \mathcal{A} \models \exists C_1 \exists C_2 (\forall c [C_P(c) \leftrightarrow (C_1(c) \lor C_2(c))] \land \Gamma_{\text{true}}(C_1, \phi_1) \land \Gamma_{\text{true}}(C_2, \phi_2))
\]

Now \( \mathcal{A} \models \Gamma_{\text{true}}(C_P, \alpha) \) guarantees the split of \( C_P \) into two sets of indices \( C_1, C_2 \) such that for each \( i = 1, 2 \) we have that \( \mathcal{A} \models \Gamma_{\text{true}}(C_i, \phi_i) \). Let us denote the corresponding sub-teams of \( P \) by \( T_1 \) and \( T_2 \), that is, \( T_r = \{ s_i \in T \mid c_i \in C_r \} \) for \( r = 1, 2 \). Then we claim that \( T_1 \cup T_2 = P \) and \( T_r \models \phi_r \) for \( r = 1, 2 \). Note that 
\[
\mathcal{A} \models \Gamma_{\text{true}}(C_r, \phi_r) \iff T_r \models \phi_r
\]

is true by induction hypothesis. Moreover, every assignment in \( T_r \) corresponds to its index in \( C_r \) which is the subset of \( C_P \) for \( r = 1, 2 \). On the other hand every assignment in \( P \) is in \( C_P \) and hence in \( C_1 \cup C_2 \). This shows that \( P = T_1 \cup T_2 \) is true. Conversely let us assume that \( P \models \alpha \) and let \( T_1 \cup T_2 = P \) be the splitting. Let with respect to \( r \in \{1, 2\} \)

\[ C_r = \{ c \in C_P \mid \text{for } s_i \in T_r, f_i(c) = 1 \} \]

By induction hypothesis and the fact that \( T_r \models \phi_r \) for \( r = 1, 2 \), we have that 
\[
\mathcal{A} \models \Gamma_{\text{true}}(C_1, \phi_1) \land \Gamma_{\text{true}}(C_2, \phi_2)
\]

and 
\[
\mathcal{A} \models \forall c [C_P(c) \leftrightarrow (C_1(c) \lor C_2(c))]
\]

are true, implying \( \mathcal{A} \models \Gamma_{\text{true}}(C_P, \alpha) \).

\( \alpha = \phi_1 \land \phi_2 \) is a conjunction Then 
\[
\mathcal{A} \models \exists y_1 \exists y_2 (\text{CONJ}(\alpha, y_1, y_2))
\]
is true and, accordingly, it is true that 
\[
\mathcal{A} \models \Gamma_{\text{true}}(C_P, \alpha) \iff \mathcal{A} \models \Gamma_{\text{true}}(C_P, \phi_1) \land \Gamma_{\text{true}}(C_P, \phi_2),
\]

which, by similar arguments as in the split-junction case, proves the required equivalence. 

\( \triangleright \)

**Theorem** \([12]\) MC parametrised by treewidth of \( G_{T, \Phi} \) and treedepth \( \Phi \) is in FPT.

**Proof.** We reformulate the formula \( \Gamma_{\text{true}}(C, \Phi) \) in the proof of Lemma \([10]\) as \( \Gamma_{\text{true}}(C, \Phi) \). This intuitively says that the root \( \Phi \) at depth \( n \) is satisfied by recursively stating that every successor node is satisfied. Then using Lemmas \([7, 10]\) in combination with Proposition \([9]\) yields the desired result. 

\( \triangleleft \)
Claim 15. \( \mathcal{A} \models \Gamma_{layer} \iff T \models \Phi \).

Proof. We will prove the claim via a characterisation through truth functions. At first, we need to define what a truth function is and need to state a result from Yang [29].

Definition 31. A truth function for a PDL-formula \( \Phi \) and a team \( T \) is defined as a function \( \mathcal{F} : SF(\Phi) \rightarrow 2^T \) such that

- for all \( \theta \in SF(\Phi) : \mathcal{F}(\theta) \models \theta \),
- for \( \theta = \alpha_1 \land \alpha_2 \) then \( \mathcal{F}(\theta) = \mathcal{F}(\alpha_1) = \mathcal{F}(\alpha_2) \),
- for \( \theta = \alpha_1 \lor \alpha_2 \) then \( \mathcal{F}(\theta) = \mathcal{F}(\alpha_1) \cup \mathcal{F}(\alpha_2) \), and
- \( \mathcal{F}(\Phi) = T \).

We use the following characterisation of formula satisfaction under a team in terms of a truth function.

Proposition 32 ([29]). Given a PDL-formula \( \Phi \) and a team \( T = \{ s_1, \ldots, s_m \} \) then \( T \models \Phi \) iff it has a truth function \( \mathcal{F} : SF(\Phi) \rightarrow 2^T \).

Now, it suffices to prove the following equivalence:

\[ \mathcal{A} \models \Gamma_{layer} \iff \text{There is a truth function } \mathcal{F} : SF(\Phi) \rightarrow 2^T. \]

We switch between \( T_\alpha \) and \( C_\alpha \) and use the equivalence \( T_\alpha \models \alpha \iff \mathcal{A} \models \Gamma_{true}(C_\alpha, \alpha) \) by using observations from Claim 11 when necessary.

“\( \Leftarrow \)”: If there is a truth function then for every \( \alpha \) we can take \( T_\alpha = \mathcal{F}(\alpha) \) and \( C_\alpha = \{ c_r \mid s_r \in T_\alpha \} \). Then \( \mathcal{A} \models \Gamma_{layer} \) is immediate from definition of the truth function \( \mathcal{F} \) as illustrated below:

- If \( \alpha \) is a variable or a dependence atom then \( \mathcal{F}(\alpha) \models \alpha \) implies \( \mathcal{A} \models \theta_{true}\{ C_\alpha, \alpha \} \)
- If \( \alpha = \neg \gamma \) then \( \mathcal{F}(\alpha) \models \neg \gamma \Rightarrow \mathcal{A} \models \theta_{false}\{ C_\alpha, \gamma \} \)
- If \( \alpha \) is a conjunction or a disjunction of \( y_1, y_2 \), then it has two child nodes such that \( \mathcal{F}(y_i) = y_i \) for \( i = 1, 2 \). Moreover, \( \mathcal{F}(\alpha) \models \alpha \) is true, where \( \mathcal{F}(\alpha) = \mathcal{F}(y_1) \cup \mathcal{F}(y_2) \) if \( \alpha = y_1 \lor y_2 \) and \( \mathcal{F}(\alpha) = \mathcal{F}(y_1) \cap \mathcal{F}(y_2) \) if \( \alpha = y_1 \land y_2 \). In both cases we claim that \( \mathcal{A} \models \Gamma_{node}(C_\alpha, \alpha) \) since \( \mathcal{A} \models \Gamma_{induced}(\alpha, y_1, y_2, C_\alpha, C_{y_1}, C_{y_2}) \), where \( \mathcal{A} \models \Gamma_{true}(C_{y_i}, y_i) \) is true by induction hypothesis.

As a result, for every \( x \in SF(\Phi) \), there exists a \( C_x \subseteq C \) such that \( \mathcal{A} \models \Gamma_{node}(C_x, x) \) which completes the proof in this direction.

“\( \Rightarrow \)”: We prove that if \( \mathcal{A} \models \Gamma_{layer} \) then there is a truth function with required properties. We first prove the following claim.

Claim 33. If \( \mathcal{A} \models \Gamma_{layer} \) then for every subformula \( \alpha \) there is a sub-team \( C_\alpha \) such that \( \mathcal{A} \models \Gamma_{true}(C_\alpha, \alpha) \).

Proof. The case for leaf subformulas is obvious from the fact that \( \mathcal{A} \models \forall x \exists C_x(\theta_{leaf}(x) \rightarrow \theta_{leaf}(C_x, x)) \). Note that \( \Gamma_{node}(C_\alpha, \alpha) \) also takes care of the nodes \( \alpha \)'s such that \( \text{NEG}(\alpha, x) \). Because \( \mathcal{A} \models \text{NEG}(\alpha, x) \) implies \( \mathcal{A} \models \theta_{false}(C_x, x) \) and by this \( \mathcal{A} \models \theta_{true}(C_\alpha, \alpha) \) is true via taking \( C_\alpha = C_x \).

Finally let \( \psi_1, \psi_2 \) be two subformulas such that there are \( C_1, C_2 \) such that \( \mathcal{A} \models \Gamma_{true}(C_i, \psi_i) \) for \( i = 1, 2 \). Then by taking either the union \( C_1 \cup C_2 \) or intersection \( C_1 \cap C_2 \) gives the desired sub-team for the subformula \( \psi_1 \lor \psi_2 \) and \( \psi_1 \land \psi_2 \) respectively. This completes the claim proof. \( \Box \)
We use the property of dependence logic being downwards closed while showing that $C_1 \cap C_2$ satisfies the conjunction. Now, Claim 33 implies that for every subformula $\alpha$ there is $C_\alpha$ such that $A \models I_\text{true}(C_\alpha, \alpha)$ which, by arguments from Claim 11 implies that $T_\alpha = \{ s_\alpha \mid c_\alpha \in C_\alpha \}$. Accordingly, by taking $F(\alpha) = T_\alpha$, we have $F(\alpha) \models \alpha$. Now let $\alpha = \psi_1 \lor \psi_2$ then $C_\alpha = C_{\psi_1} \cup C_{\psi_2}$ as specified in the layer condition $I_{\text{induced}}(\alpha, \psi_1, \psi_2, C_\alpha, C_{\psi_1}, C_{\psi_2})$. But this implies $F(\alpha) = F(\psi_1) \cup F(\psi_2)$ as required. A similar argument works for the case $\alpha = \psi_1 \land \psi_2$. Finally, when $x = \Phi$ (that is, at the root level) we have an additional condition that $A \models r(\Phi) \to \forall c[C(c) \to C_\alpha(c)]$ is true which assures that $F(\Phi) = T$.

\textbf{Theorem 19.} MC parametrised by \#splits is \textsc{paraNP}-complete.

\textbf{Proof.} We show a reduction from the question of whether a given graph is 3-colourable. Given an instance $\langle G \rangle$ where $G = (V, E)$ is a graph. We map this input to an instance $\langle (T, \Phi), 2 \rangle$ where $T$ is a team, and $\Phi$ is a $\mathcal{PDL}^t$-formula with 2 split-junctions. Let $V = \{ v_1, \ldots, v_m \}$ be the vertex set and $E = \{ e_1, \ldots, e_m \}$ the given set of edges. Then we define

$$\text{VAR}(\Phi) = \{ x_1, \ldots, x_n \} \cup \{ y_1,1, \ldots, y_1,m, \ldots, y_m,1, \ldots, y_m,n \}.$$ 

That is, we have (1) a variable $x_i$ corresponding to each node $v_i$ and (2) a variable $y_{j,k}$ corresponding to each edge $e_j$ and each node $v_k$. For convenience, we will sometimes write $y_{j}$ instead of $(y_{j,1}, \ldots, y_{j,n})$ when it is clear that we are talking about the tuple of variables corresponding to the edge $e_j$. Consequently, we have an $n$-tuple of variables $y_j$ for each edge $e_j$, where $1 \leq j \leq m$. The idea of the team that we construct is that there is an assignment $s_i$ corresponding to each variable $v_i$ that encodes the neighbourhood of $v_i$. The assignment $s_i$ also encodes all edges that $v_i$ participates in. This is achieved by mapping each variable $y_{j,k}$ in tuple $y_{\ell}$ to $1$ if $v_j \in e_\ell$ whereas $y_{j,k} = 0$ if $v_j \notin e_\ell$ and for every $j \neq i$, $y_{\ell,j} = 1$. Figure 1 illustrates an example to get an intuition on this construction.

Formally, we define the team as follows.

1. If $G$ has an edge $e_\ell = \{ v_i, v_j \}$ then we set $s_i(x_j) = 1$ and $s_j(x_i) = 1$, and let $s_l(y_{\ell,1}) = \ldots = s_l(y_{\ell,n}) = 1$ as well as $s_j(y_{\ell,1}) = \ldots = s_j(y_{\ell,n}) = 1$
2. For the case $v_j \notin e_\ell$, we set $s_j(y_{\ell,j}) = 0$ and for the remaining indices $s_j(y_{\ell,i}) = 1$.
3. Since, we can assume w.l.o.g. the graph has no loops (self-edges) we always have $s_i(x_i) = 0$ for all $1 \leq i \leq n$.

Then two assignments $s_i, s_j$ agree on $y_{k}$ if the corresponding edge $e_k$ is the edge between $v_i$ and $v_j$, and we have $s_j(y_k) = 1 = s_j(y_k)$.

Now let $\Phi$ be the following formula

$$\Phi := \bigwedge_{e_k = \{ v_i, v_j \}} \text{dep}(y_k; x_i) \lor \bigwedge_{e_k = \{ v_i, v_j \}} \text{dep}(y_k; x_i) \lor \bigwedge_{e_k = \{ v_i, v_j \}} \text{dep}(y_k; x_i).$$

The choice of $x_i$ or $x_j$ to appear in the formula is irrelevant. The idea is that if there is an edge $e_k$ between two nodes $v_i, v_j$ and accordingly $s_i(y_k) = 1 = s_j(y_k)$ then the two nodes cannot be in the same split of team. This is always true because in that case the assignments $s_i, s_j$ cannot agree on any of $x_i$ or $x_j$. Since, by (3), we have $s_i(x_i) = 0$ but there is an edge to $v_j$ and we have $s_j(x_i) = 1$. The desired result is achieved by the following claim.

\textbf{Claim 34.} $G$ is 3-colourable iff $T \models \Phi$

\textbf{Proof.} $\Rightarrow$: Let $V_1, V_2, V_3$ be the distribution of $V$ into three colours. Consequently, for every $v \in V$ we have an $r \leq 3$ such that $v \in V_r$. Moreover, for every $v_i, v_j \in V_r$ there is no $e_\ell$ s.t. $e_\ell = \{ v_i, v_j \}$. Let $T_r = \{ s_i \mid v_i \in V_r \}$ for each $r \leq 3$, then we show that $\bigcup_{r \leq 3} T_r = T$
$T_r \models \phi$. This will prove that $T \models \Phi$ because we can split $T$ into three sub-teams such that each satisfies the disjunct.

Since for each $s_i, s_j \in T_r$, there is no edge $e_\ell = \{ v_i, v_j \}$ this implies that for the tuple $y_\ell$, we have $s_i(y_\ell) \neq s_j(y_\ell)$ and thereby making the dependence atom trivially true. Moreover, by 2-coherency it is enough to check only for pairs $s_i, s_j$ and since the condition holds for every edge, we have $T_r \models \phi$. Since our assumption is that $V$ can be split into three such sets, we have the split of $T$ into three sub-teams. This gives $T \models \Phi$.

$\Leftarrow$: Conversely, assume that $T$ can be split into three sub-teams each satisfying $\phi$. Then we show that $V_1, V_2, V_3$ is the partition of $V$ into three colours. Let $V_r = \{ v_i \mid s_i \in T_r \}$ then $\bigcup_{r \leq 3} V = V$ and for any $v_i, v_j \in V_r$ there is no edge between $v_i, v_j$. Suppose to the contrary that there is an edge $e_\ell = \{ v_i, v_j \}$. Then we must have $s_i, s_j \in T_\ell$ such that $s_i(y_\ell) = 1 = s_j(y_\ell)$.

That is, $s_i(y_{\ell,1}) = 1 = s_j(y_{\ell,1})$, $s_i(y_{\ell,n}) = 1 = s_j(y_{\ell,n})$. Since we have that $s_i(x_i) = 0$ whereas $s_j(x_i) = 1$, this implies $\{ s_i, s_j \} \not\models \phi$ which is a contradiction. 

This concludes the full proof.

\textbf{Theorem 20} MC parametrised by dep-arity is paraNP-complete.

\textbf{Proof.} The problem is in paraNP since the classical MC is in NP. For the lower bound, notice that the MC problem for the bounded arity fragment of PDL is still NP-complete [21 Thm 4.13]. This gives paraNP-completeness in conjunction with Proposition 18.

\textbf{Theorem 22} MC for the parameters formula-size, \#variables, or treedepth is in FPT.

\textbf{Proof.} We will use the following folklore result from parameterised complexity to get results for the above cases.

\textbf{Proposition 35.} Let $Q$ be a problem such that $(Q,k)$ is in FPT and let $\ell$ be another parameter such that $k \leq f(\ell)$ for some computable function $f$, then $(Q,\ell)$ is also in FPT.

Since team-size $\leq 2^{\text{formula-size}}$ and for team-size the problem MC is in FPT (Thm. 17), we get the same result for formula-size as the parameter.

Now, if a PDL-formula $\Phi$ has $m$ variables then there are $2^m$ many assignments and accordingly, the maximum size for a team is $2^m$. As a result, we have team-size $\leq 2^{\#\text{variables}}$ and reach, as before, FPT for parameter \#variables. Moreover, if MC w.r.t. team-size can be solved in time $O^*(2^d)$ then MC w.r.t. \#variables can be solved in time $O^*(2^d)$.

If a formula $\Phi$ has treedepth $= d$ then there are $\leq 2^d$ leaves and $\leq 2^d$ internal nodes. Consequently, we have formula-size $\leq 2^d$ and thereby MC $\in$ FPT parametrised by treedepth.

\textbf{Theorem 23} SAT parametrised by team-size, or dep-arity is paraNP-complete.

\textbf{Proof.} Regarding team-size, first note, that having this parameter does not impose anything on the problem which asks for some team (in particular, this does not ask for a team of fixed size, now). Then the paraNP lower bound follows from the 1-slice of the problem being NP-complete (use Proposition 15). For the membership result, observe, that we just guess a singleton team and then verify classically.

Under dep-arity, the problem is in paraNP because the classical version is in NP. For the lower bound, notice that the 0-slice of the problem (so no dependence atoms) is equivalent to propositional SAT which is NP-complete (use Proposition 18).
Lemma 24. Let \( \Phi \in PDL \) and \( A_\Phi \) be the corresponding \( r_0 \)-structure. Then there exists an \( MSO \)-sentence \( \Gamma_{sat} \) such that
\[
A_\Phi \models \Gamma_{sat} \iff \text{There is an assignment } s : \text{VAR}(\Phi) \to \{0,1\} \text{ s.t. } s \models \Phi
\]

Proof. Let us define the sentence \( \Gamma_{sat} \) inductively. The base case must assert that
1. the label of a literal (leaf node) \( \ell \) is either empty or contains \( \ell \), and
2. the label of a dependence atom \( \alpha = \text{dep}(P; Q) \) contains \( z_{dep} \).

This is achieved by the following formula.
\[
\Gamma_{leaf}(L_\alpha, \alpha) := [\text{LIT}(\alpha) \to \forall y(L_\alpha(y) \leftrightarrow y = \alpha)] \\
\land [\exists x \exists y (\text{DEP}(\alpha, x, y) \to \forall [L_\alpha(y) \leftrightarrow y = z_{dep}])]
\]

Next we define a formula that specifies how the labels for \( \theta_1 \) and \( \theta_2 \) induce a label for \( \theta_1 \land \theta_2 \) and \( \theta_1 \lor \theta_2 \). We let
\[
\Gamma_{ind}(L_\alpha, \alpha, L_{\theta_1}, \theta_1, L_{\theta_2}, \theta_2) := \text{CONJ}(\alpha, \theta_1, \theta_2) \to [\forall y[L_\alpha(y) \leftrightarrow (L_{\theta_1}(y) \lor L_{\theta_2}(y))]]) \\
\land [\text{DISJ}(\alpha, \theta_1, \theta_2) \to [(\forall y[L_\alpha(y) \leftrightarrow (L_{\theta_1}(y))]) \lor \forall y[L_\alpha(y) \leftrightarrow (L_{\theta_2}(y))]]]
\]

That is, the label for conjunction node is the union of the two labels from child nodes and for disjunction node it is either label from the two nodes. Note that in the construction, we always force splits such that one side is empty because this more likely satisfies the formula. Joining everything together we get the following formula.
\[
\Gamma_{label}(L_x, x) := \exists y_1 \exists y_2 \exists L_{y_1} \exists L_{y_2} [\theta_{leaf}(x) \to \Gamma_{leaf}(L_x, x)] \\
\land [(x \not\supset y_1 \land x \not\supset y_2) \to (\Gamma_{label}(L_{y_1}, y_1) \land \Gamma_{label}(L_{y_2}, y_2) \land \Gamma_{ind}(L_x, x, L_{y_1}, y_1, L_{y_2}, y_2))]
\]

Finally, consider the following sentence which gives the desired conditions to be satisfied by each node.
\[
\Gamma_{sat} := \forall x \exists L_x ([\text{SF}(x) \to \Gamma_{label}(L_x, x)] \land \\
(r(x) \to [\forall y \forall z (\text{VAR}(y) \land \text{NEG}(z, y)) \to (\neg L_x(y) \lor \neg L_x(z)) \land \exists y L_x(y)])]
\]

Intuitively, the above sentence guarantees that for each subformula there is a label (subset of literals) and that the root contains a consistent and non-empty label. Moreover, note that the labels for subformulas other than root can be both empty and even inconsistent as we will see for the case when a subformula appears as a disjunct. We modify the sentence \( \Gamma_{structure} \) from the proof of Lemma 7 such that negated variables also appear as leaves.

Claim 36. \( A_\Phi \models \Gamma_{sat} \iff \text{There is an assignment } s : \text{VAR}(\Phi) \to \{0,1\} \text{ s.t. } s \models \Phi \)

Proof. “\( \Rightarrow \)” Let \( s \) be a satisfying assignment. Consider \( L_s \subseteq \text{LIT}(\Phi) \) such that for \( s(x) = 1 \) we let \( x \in L_s \) and for \( s(x) = 0 \) we let \( \neg x \in L_s \). As a result, \( L_s \) contains all the literals mapped to 1 by \( s \). Then we let \( L_\Phi = L_s \), clearly \( L_\Phi \neq \emptyset \) and \( L_\Phi \) is consistent. We define labels for each subformula \( \psi \in \text{SF}(\Phi) \) in a top-down fashion as below.

Case \( \Phi \): \( L_\Phi = L_s \).

Case \( \alpha = \theta_1 \land \theta_2 \): Given \( L_\alpha \), we let \( L_{\theta_1} = L_\alpha \cap \text{LIT}(\theta_1) \).

Case \( \alpha = \theta_1 \lor \theta_2 \): Given \( L_\alpha \), we let \( L_{\theta_2} = L_\alpha \cap \text{LIT}(\theta_1) \) if \( s \models \theta_1 \) and \( L_{\theta_2} = \emptyset \). Otherwise \( L_{\theta_2} = L_\alpha \cap \text{LIT}(\theta_2) \) and \( L_{\theta_2} = \emptyset \) if \( s \models \theta_2 \).
Finally, at the leaf levels, the labels are either empty (corresponding to leaves that end up in subformulas that form disjuncts) or contain literals (for the subformulas that contribute in the truth definition of $\Phi$ under $s$.) Furthermore, the way we define labels here implies that leaf-labels can only contain literals actually appearing in those leaves as required. This proves the desired result since every subformula gets a label with required conditions met.

"⇒": First note that moving from leaves to root every inconsistent choice arrives at some disjunct. This is because the label for the root is consistent. This argument shows that for every $\psi$ the label $L_\psi$ can be taken to be consistent since if this is inconsistent then take $L_\psi = \emptyset$. Note also that the label for the root node is non-empty. In fact, this label is minimal in the sense that it contains all the leaf-nodes except possibly those that get dropped at a disjunction node. This forces an assignment to map literals that appear in the root-label to $\{0, 1\}$ accordingly thereby satisfying those atoms. Whereas for the remaining variables not in the label the assignment can map them to any value 0 or 1.

Now, we claim that for every $\psi \in SF(\Phi)$ such that $L_\psi$ is consistent, there exists an assignment $s_\psi$ with $s_\psi(x) = 1$ for $x \in L_\psi$ and $s_\psi(x) = 0$ for $\neg x \in L_\psi$ such that $s_\psi \models \psi$.

We prove this by induction. First note that if $L_\psi$ is empty, then the claim is trivially true. If $\psi = x$ then $L_\psi = \{x\}$ and as a result $s_\psi \models \psi$ similarly, for $\psi = \neg x$ we have that $L_\psi = \{\neg x\}$ and for $\psi = \text{dep}(P; Q)$ we have that $L_\psi = \{\text{dep}\}$. For $\psi = \alpha_1 \land \alpha_2$ such that $s_\alpha_1 \models \alpha_1$ and $s_\alpha_2 \models \alpha_2$. Then $s_\psi$ defined from $L_\psi = L_{\alpha_1} \cup L_{\alpha_2}$ has the desired property that $s_\psi \models \psi$ since $L_\psi$ is consistent. Similarly, for $\psi = \alpha_1 \lor \alpha_2$ such that either $s_{\alpha_1} \models \alpha_1$ or $s_{\alpha_2} \models \alpha_2$ then by taking $s_\psi = s_{\alpha_i}$ suffices for $i \in \{1, 2\}$ such that $s_{\alpha_i} \models \alpha_i$. We conclude that the label $L_\Phi$ for the root node can be extended to an assignment $s$ such that $s$ is non-trivial, consistent and $\{s\} \models \Phi$. □

This completes the proof.

\begin{theorem}
\textsc{SAT} parametrised by $\#\text{ splits}$ is in $\mathsf{FPT}$. Moreover, there is an algorithm that solves the problem in $O(2^{\#\text{ splits}(\Phi)} \cdot |\Phi|^{O(1)})$ for any $\Phi \in \mathsf{PDL}$.
\end{theorem}

\textbf{Proof.} We consider partial mappings of the kind $t: \text{VAR} \to \{0, 1, c\}$. Intuitively, these mappings are used to find a satisfying singleton team in the process of the presented algorithm.

If $t, t'$ are two (partial) mappings then $t \oplus t'$ is the assignment such that

$$(t \oplus t')(x) := \begin{cases} 
  c & \text{if both are defined and } t(x) \neq t'(x) \\
  t(x) & \text{if } t(x) \text{ is defined} \\
  t'(x) & \text{if } t'(x) \text{ is defined} \\
  \text{undefined} & \text{otherwise.}
\end{cases}$$

We prove the following claim.

\begin{claim}
The formula $\Phi$ is satisfiable if and only if Algorithm 1 returns a consistent (partial) assignment $s$ that can be extended to a satisfying assignment for $\Phi$ over $\text{VAR}(\Phi)$.
\end{claim}

\textbf{Proof.} We prove using induction on the structure of $\Phi$.

\textbf{Base case} Start with $\Phi = X$ is a variable. Then $\Phi$ is satisfiable and $\{s\} \models \Phi$ such that $s(x) = 1$. Moreover, such an assignment is returned by the procedure as depicted by line 3 of the algorithm. Similarly, the case $\Phi = \neg X$ follows by line 4. The case $\Phi = \text{dep}(P; Q)$ or $\Phi = \top$ is a special case of a $\mathsf{PDL}$-formula since this is true under any assignment. Line 6 in our procedure returns such an assignment that can be extended to any consistent assignment. Finally, for $\Phi = \bot$, the assignment contains a conflict and can not be extended to a consistent assignment as the algorithm returns "$\Phi$ is not satisfiable".
Algorithm 1: SAT-algorithm for \#splits which tries to find a satisfying singleton team.

Input: $\mathcal{PDL}$-formula $\Phi$ represented by a syntax tree

Output: A set of partial assignments $S$ such that $S \models \Phi$ or “$\Phi$ is not satisfiable”

begin
foreach Leaf $\ell$ of the syntax-tree do
  if $\ell = X$ is a variable then $S_\ell \leftarrow \{\{x \mapsto 1\}\}$;
  else if $\ell = \neg X$ is a negated variable then $S_\ell \leftarrow \{\{x \mapsto 0\}\}$;
  else if $\ell = \bot$ then pick an arbitrary $x \in \text{VAR}$ and set $S_\ell \leftarrow \{\{x \mapsto c\}\}$;
  else $S_\ell \leftarrow \{1\}$; // case \& for split-junction

foreach Inner node $\ell$ of the syntax-tree in bottom-up order do
  if $1 \in S_\ell$ then
  \hspace{1em} if $\ell$ is a conjunction then $S_\ell \leftarrow S_{\ell_1} \cup S_{\ell_2}$;
  \hspace{1em} else $S_\ell \leftarrow \{1\}$; // empty split for split-junction

  else if $\ell$ is a conjunction then
    foreach $s_0 \in S_0$ and $s_1 \in S_1$ do $S_\ell \leftarrow S_\ell \cup \{\{s_0 \oplus s_1\}\}$;
  else
    foreach $s_0 \in S_0$ and $s_1 \in S_1$ do $S_\ell \leftarrow S_\ell \cup \{\{s_0\},\{s_1\}\}$;
  \hspace{1em} if there exists a non-conflicting assignment $s$ in root node then return $s$;

else return “$\Phi$ is not satisfiable”;

end

Induction Step Notice first that if either of the two operands is $\top$ then this is a special case and triggers lines 9–11 of the algorithm thereby giving the satisfying assignment.

Suppose now that $\Phi = \psi_1 \land \psi_2$ and that the claim is true for $\psi_1$ and $\psi_2$. As a result, both $\psi_1$ and $\psi_2$ are satisfiable if and only if the algorithm returns a satisfying assignment for each. Let $S_i$ for $i = 0, 1$ be such that every consistent $t_i \in S_i$ can be extended to a satisfying assignment for $\psi_i$. Our claim is that $S_\Phi$ returned by the procedure (line 13) is non-empty and contains a consistent assignment for $\Phi$ if and only if $\Phi$ is satisfiable. First note that, by induction hypothesis, $S_i$ contains all the possible partial assignments that satisfy $\psi_i$ for $i = 0, 1$. Consequently, $S_\Phi$ contains all the possible $\oplus$-joins of such assignments that can satisfy $\Phi$. Let $\psi_0$ be satisfied by an assignment $s_0$ and $\psi_1$ be satisfied by an assignment $s_1$. Moreover, let $s' \in S_\Phi$ be an assignment such that $s' = s_0 \oplus s_1$. If $s'$ is consistent then $s'$ can be extended to a satisfying assignment $s$ for $\Phi$ since $\{s'\} \models \psi_i$ for $i = 0, 1$. On the other hand if every $s = s_0 \oplus s_1$ is conflicting then there is no assignment over $\text{VAR}(\Phi) = \text{VAR}(\psi_1) \cup \text{VAR}(\psi_2)$ that satisfies $\Phi$. Accordingly, $\Phi$ is not satisfiable.

The case for split-junction is simpler. Suppose that $\Phi = \psi_0 \lor \psi_1$ and that the claim is true for $\psi_1$ and $\psi_2$. Then $\Phi$ is satisfiable if and only if either $\psi_0$ or $\psi_1$ is satisfiable. Since the label $S_\Phi$ for $\Phi$ is the union of all the labels from $\psi_0$ and $\psi_1$, it is enough to check that either the label of $\psi_0$ ($S_0$) or the label of $\psi_1$ ($S_1$) contains a consistent partial assignment. By induction hypothesis, this is equivalent to checking whether $\psi_0$ or $\psi_1$ is satisfiable. This completes the case for split-junction and the proof to our claim. $\triangle$

Finally, notice that the label size multiplies at the occurrence of a split-junction. That is, we keep all the assignment candidate separately and each such candidate is present in the label for split-junction node. In contrast, at conjunction nodes, we ‘join’ the assignments and for
Figure 2 (left) syntax tree of example formula, and (right) computation of Algorithm 1. Notation: $x/\neg x/x^c$ means a variable is set to true/false/conflict. Clearly, $\{ x_4, x_1, \neg x_2, x_3 \}$ satisfies the formula.

this reason, the label size is the sum of the two labels. This implies that the maximum size for any label is bounded from above by $2^\#\text{split}$. As a consequence, the above algorithm runs in polynomial time in the input and exponential in the parameter.

Figure 2 presents an example of using the above algorithm.

> **Theorem 27.** SAT parametrised by $\#\text{variables}$, formula-size, or treedepth is in $\text{FPT}$.  

**Proof.** Regarding $\#\text{variables}$, the question $\text{PDL-SAT}$ boils down to $\text{PC-SAT}$ of finding an assignment for a given propositional logic formula. The latter problem, when parameterised by the number of variables in the input formula, is in $\text{FPT}$ which implies that the former problem is also in $\text{FPT}$.

Note that $\text{formula-size} = |\Phi|$ and any PP $II$ is in $\text{FPT}$ for the parametrisation input-length. Consequently, SAT parametrised by $\text{formula-size}$ is in $\text{FPT}$.

If a formula $\Phi$ has treedepth = $d$ then there are $\leq 2^d$ leaves and $\leq 2^d$ internal nodes accordingly we have $\text{formula-size} \leq 2^{2d}$ which shows membership in $\text{FPT}$ parametrised by treedepth.

> **Theorem 28.** $m$-$\text{SAT}$ parametrised by $\#\text{variables}$, formula-size, or treedepth is in $\text{FPT}$.  

**Proof.** In the case of $\#\text{variables}$, the team-size is bounded by $2^\#\text{variables}$. Moreover, there are $2^\#\text{variables}$ such teams and we can find all the satisfying teams in fpt-time and then check whether any such team has size $m$.

For formula-size note that $\text{formula-size} \leq |x|$ and $II \in \text{FPT}$ for any problem PP $II$ with parameter input-size. As a result, we have that $m$-$\text{SAT}$ parametrised by formula-size is in $\text{FPT}$.

For treedepth notice that $\text{formula-size} \leq 2^{2\text{treedepth}}$ and thereby the problem is in $\text{FPT}$ under this parametrisation.

> **Theorem 29.** Given a $\text{PDL}$-formula $\Phi$ with $q$ as the maximum arity of its dependence atoms and $k$ many split-junctions. Then there is an algorithm that enumerates all the satisfying teams of $\Phi$ in time $O(2^{2q} \cdot 2^k \cdot p(|\Phi|))$ for some polynomial $p(x)$.  

**Proof.** We present the idea of how Algorithm 1 can be extended to find satisfying teams of the formula $\Phi$. For bounded arity, the procedure labels all the satisfying teams for an atom
(team-labels) rather than assignments (node-labels) as in Algorithm 1. Accordingly, a single label is not a set of literals now but a set of sets of such literals. The changes in the above procedure include dealing with a set of labels rather than labels themselves. In principle this is possible but the label size for each node having dependence atom as a subformula may explode. The reason is that the above procedure brute-forces at the atomic level for finding a satisfying assignment. Now for a dependence atom, there could be many assignments satisfying the atom and consequently many-many team-labels. Consequently, our procedure labels every node with all the satisfying teams and removes those team labels that contain conflicting assignments.

At conjunction, every assignment-label from a team label of one node joins every assignment label in the team label of the other node using the same rules as above. Care needs to be taken, however, because conflicts arise only when individual assignments from different team-labels are joined and not between the same team labels. That is, a team label can be \( \{ \{ x, y, z \}, \{ \neg x, y, z \} \} \) even though both \( x \) and \( \neg x \) appear together inside one label (\( x \) and \( \neg x \) appear in one team label and not in one assignment label. If they appear in one assignment label then it is a conflict).

As a consequence, we argue that the extension of Algorithm 1 is the desired algorithm that solves the \( m \)-SAT problem. Moreover, the size of each label depends on the maximum arity of any dependence atom as well as the number of splits. This complete the proof.

Example 38. Consider a very simple formula \( x \land \text{dep}(x; y) \). Then team-label for \( x \) constitutes \( \{ \{ x \} \} \) whereas the label for \( \text{dep}(x; y) \) consists of 8 such sets of literals corresponding to each satisfying team. We write all of these to get an idea of how large a label may get. We have the following team-labels for \( \text{dep}(x; y) \):

\[
\{ x, y \}, \{ x, \neg y \}, \{ \neg x, y \}, \{ \neg x, y \}, \\
\{ x, y \}, \{ \neg x, \neg y \}, \{ x, y \}, \{ \neg x, y \}, \{ x, \neg y \}, \{ \neg x, y \} \}
\]

Now, at the next conjunction node we are left with only two consistent team-labels as \( \{ \{ x, y \} \} \) and \( \{ \{ x, \neg y \} \} \) because other labels containing \( \neg x \) conflicts with label \( \{ x \} \) for \( x \). In view of this, we have satisfying teams for \( x \land \text{dep}(x; y) \) as \( T_1 = \{ s_1 \} \) and \( T_2 = \{ s_2 \} \) where \( s_1(x) = s(y) = 1 \) and \( s_2(x) = 1, s_2(y) = 0 \). On the other hand all the teams corresponding to labels of \( \text{dep}(x; y) \) satisfy the formula \( x \lor \text{dep}(x; y) \).

Theorem 30. \( m \)-SAT parametrised by team-size, or dep-arity is paraNP-complete.

Proof. Regarding team-size, the lower bound follows from the 1-slice and propositional SAT (Prop. 18). For membership in paraNP, we simply guess a team of constant size and all linear many splits where one split has exponentially in the parameter many possibilities.

Now turn to dep-arity as the parameter. For the lower bound notice, that SAT parametrised by dep-arity is paraNP-complete. Moreover, the problem SAT provides a lower bound for the problem \( m \)-SAT. Accordingly, \( m \)-SAT parametrised by dep-arity is paraNP-hard. For the paraNP-upper bound, we simply guess a team of size \( m \) which then is linear in the input length.