Lagrangian torus fibrations and homological mirror symmetry for the conifold

Kwokwai Chan, Daniel Pomerleano, Kazushi Ueda

Abstract

We discuss homological mirror symmetry for the conifold from the point of view of the Strominger-Yau-Zaslow conjecture.

1 Introduction

The behavior of strings and branes near the tip of a cone has been studied extensively in string theory. The case when the cone is a Gorenstein affine toric 3-fold is of particular importance, not only from the point of view of mirror symmetry, but also for application to geometric engineering of Seiberg-Witten theory [KKV97] and the AdS/CFT correspondence [AGM00].

Let \( Z \) be a Gorenstein affine toric 3-fold and \( \varphi : X \to Z \) be a crepant resolution. The convex hull \( \Delta \) of primitive generators of one-dimensional cones of the fan describing \( Z \) as a toric variety is a lattice polygon, which lies on the plane

\[
\mathcal{N} = \{ n = (n_1, n_2, n_3) \in \mathbb{N} \mid n_3 = 1 \}
\]

under a suitable choice of a coordinate \( N \cong \mathbb{Z}^3 \) on the lattice of one-parameter subgroups of the dense torus.

If \( \Delta \) contains an interior lattice point, then \( X \) is derived-equivalent to the total space \( \mathcal{K}_X \) of the canonical bundle of a 2-dimensional toric Fano stack \( \mathcal{X} \), and homological mirror symmetry for \( X \) is related to homological mirror symmetry for \( \mathcal{X} \) by suspension [Sei10]. The case when \( \Delta \) does not contain any interior lattice point is more elusive, and we discuss such a case in this paper.

Let \( Z \) be the conifold, which is a synonym for a 3-dimensional ordinary double point;

\[
Z = \{ (u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1 v_1 = u_2 v_2 \}.
\]

The lattice polygon \( \Delta \) for \( Z \) is the unit lattice square, which does not contain any interior lattice points. The smoothing

\[
Y = \{ (u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1 v_1 = u_2 v_2 - \epsilon \}
\]

of the conifold is expected to be mirror to the small resolution \( \varphi : X \to Z \) (cf. e.g. [ST01, Gro01]).

In this paper, we discuss homological mirror symmetry for the conifold from the point of view of the Strominger-Yau-Zaslow conjecture [SYZ96]. To do this, it is convenient to
consider an open subvariety $Y^0$ of $Y$, which is the complete intersection in $\mathbb{C}^\times \times \mathbb{C}^4 = \text{Spec} \mathbb{C}[z, z^{-1}, u_1, u_2, v_1, v_2]$ defined by
\[
\begin{cases}
u_1v_1 = z - a, \\
u_2v_2 = z - b.
\end{cases}
\tag{1.1}
\]
Here $a$ and $b$ are distinct non-zero complex numbers, which we assume to be negative real numbers for simplicity in this section. We equip $Y^0$ with the restriction $\omega$ of the symplectic form on $\mathbb{C}^\times \times \mathbb{C}^4$ obtained as the sum of the cylindrical Kähler form on $\mathbb{C}^\times$ and the Euclidean Kähler form on $\mathbb{C}^4$. The map
\[
\rho: \quad Y^0 \quad \rightarrow \quad \mathbb{R}^3 \quad \psi \\
(z, u_1, v_1, u_2, v_2) \quad \mapsto \quad (\log |z|, \frac{1}{2} (|u_1|^2 - |v_1|^2), \frac{1}{2} (|u_2|^2 - |v_2|^2))
\]
is a Lagrangian torus fibration, whose discriminant loci is given by the disjoint union of two skew lines
\[
\Gamma = \{(|a|, 0, \lambda_2) \in B \mid \lambda_2 \in \mathbb{R}\} \cup \{(|b|, \lambda_1, 0) \in B \mid \lambda_1 \in \mathbb{R}\}
\]
as shown in Figure 1.1.

![Figure 1.1: The base of the SYZ fibration](image)

The regular fibers of $\rho$ are special with respect to the holomorphic volume form
\[
\Omega = d\log z \wedge d\log u_1 \wedge d\log u_2,
\]
and we will refer to $\rho$ as the SYZ fibration. The mirror $X^0$ of $Y^0$ is identified in [AAK Theorem 11.1] as the complement of a divisor in the resolved conifold;
\[
X^0 = X \setminus D, \\
X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).
\]
Here, the divisor $D$ is the pull-back of the divisor $\{w_1w_2 = 0\}$ on the conifold
\[
Z = \{(u, v, w_1, w_2) \in \mathbb{C}^4 \mid uv = (1 + w_1)(1 + w_2)\}
\]
along the crepant resolution $\varphi : X \to Z$. The natural projection and the inclusion of the zero-section will be denoted by $\pi : X^0 \to \mathbb{P}^1$ and $\iota : \mathbb{P}^1 \to X^0$ respectively. Let $E \subset X^0$ be the image of $\iota$, which is the exceptional locus of the resolution. We write $\mathcal{O}_{X^0}(i) := \pi^*\mathcal{O}_{\mathbb{P}^1}(i)$ and $\mathcal{O}_E(i) := \iota_*\mathcal{O}_{\mathbb{P}^1}(i)$ for short.

To a strongly admissible path $\gamma$, the definition of which we defer to Section 3, one can associate an exact noncompact Lagrangian submanifold $L_\gamma \subset Y^0$, which is a section of the SYZ fibration $\rho : Y^0 \to \mathbb{R}^3$. The SYZ transform $[\text{AP01}, \text{LYZ00}]$ of a Lagrangian section of an SYZ fibration is a holomorphic line bundle on the mirror, obtained as a kind of Fourier transform.

**Theorem 1.1.** The SYZ transform $L_\gamma$ of the Lagrangian section $L_\gamma$ associated with a strongly admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ is the line bundle $\mathcal{O}_{X^0}(-w(\gamma))$ on $X^0$.

Here $w(\gamma)$ denotes the winding number defined in Section 3. Let $\gamma_0$ and $\gamma_1$ be admissible paths shown in Figure 1.2. The associated Lagrangian submanifolds of $Y^0$ will be denoted by $L_0 := L_{\gamma_0}$ and $L_1 := L_{\gamma_1}$, whose winding numbers are 0 and $-1$ respectively. Let $\mathcal{W}$ be the wrapped Fukaya category of $Y^0$ consisting of $L_0$ and $L_1$.

![Figure 1.2: Non-compact Lagrangians](image1.png)

![Figure 1.3: Compact Lagrangians](image2.png)

**Theorem 1.2.** There is an equivalence

$$D^b\mathcal{W} \cong D^b\text{coh}X^0$$

(1.2)

of triangulated categories sending $L_i$ to $\mathcal{O}_{X^0}(i)$ for $i = 0, 1$.

There is a natural choice of a pair $(S_0, S_1)$ of Lagrangian 3-spheres in $Y^0$ which are dual to $(L_0, L_1)$; they are $T^2$-fibrations over the paths shown in Figure 1.3.

**Theorem 1.3.** The SYZ transforms of the Lagrangian 3-spheres $S_0$ and $S_1$ are the line bundles $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ on the exceptional locus $E$ respectively.

Let $\mathcal{F}_0$ be the Fukaya category of $Y^0$ consisting of $S_0$ and $S_1$, and $\text{coh}_0X^0$ be the abelian category of coherent sheaves supported on the exceptional locus of the resolution $\varphi : X \to Z$.

**Theorem 1.4.** There is an equivalence

$$D^b\mathcal{F}_0 \cong D^b\text{coh}_0X^0$$

(1.3)

of triangulated categories sending $S_0$ and $S_1$ to $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ respectively.
This paper is organized as follows: We recall the construction of the SYZ mirror for the conifold from [AAK] in Section 2. In Section 3, we discuss the construction of Lagrangian submanifolds in $Y^0$ from paths on the $z$-plane. In Section 4, we recall the definition of the SYZ transform from [AP01, LYZ00] and prove Theorems 1.1 and 1.3. In Section 5, we give an explicit description of the derived category of coherent sheaves on the resolved conifold. In Section 6, we study the wrapped Fukaya category of $Y^0$ and prove Theorem 1.2. In Section 7, we study $A_\infty$-operations on vanishing cycles in $Y^0$ and prove Theorem 1.4. In Section 8, we discuss extension of the main results of this paper to more general small toric Calabi-Yau 3-folds.

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2 The construction of the SYZ mirror

Recall that $Y^0$ is given by the complete intersection

\begin{align*}
u_1v_1 &= z - a, \\
u_2v_2 &= z - b,
\end{align*}

in $\mathbb{C}^\times \times \mathbb{C}^4$, where $a$ and $b$ are distinct negative real numbers. Without loss of generality, we assume that $a < b$. To construct the mirror of $Y^0$, it is also convenient to regard $Y^0$ as the complement of the anticanonical divisor

$$H = \{(z, u_1, v_1, u_2, v_2) \in Y \mid z = 0\}$$

in

$$Y = \{(z, u_1, v_1, u_2, v_2) \in \mathbb{C}^5 \mid u_1v_1 = z - a, \ u_2v_2 = z - b\}.$$In the following, we shall briefly review the construction of the mirror for $Y^0$ (or $Y$ with respect to the divisor $H$) following the SYZ approach in [Aur07, Aur09]; note that our example is a special case of a much more general construction in [AAK, Section 11].

First of all, there is a Hamiltonian $T^2$-action on $(Y^0, \omega)$:

$$(e^{is}, e^{it}) \cdot (z, u_1, v_1, u_2, v_2) = (z, e^{is}u_1, e^{-is}v_1, e^{it}u_2, e^{-it}v_2)$$

whose moment map is given by

$$\phi(z, u_1, v_1, u_2, v_2) = \left(\frac{1}{2} (|u_1|^2 - |v_1|^2), \frac{1}{2} (|u_2|^2 - |v_2|^2)\right).$$

This action extends to $Y$ and preserves the anticanonical divisor $H$. The SYZ fibration is given by

$$\rho : Y^0 \sslash \omega \rightarrow B := \mathbb{R}_{>0} \times \mathbb{R}^2,$$

$$(z, u_1, v_1, u_2, v_2) \mapsto (|z|, \phi(z, u_1, v_1, u_2, v_2)).$$
Note that we use $|z|$ here instead of $\log |z|$ and the base is $\mathbb{R}_{>0} \times \mathbb{R}^2$ instead of $\mathbb{R}^3$. This harmless change is more convenient for us because we would like to extend this map to $\rho : Y \to \bar{B} := \mathbb{R}_{>0} \times \mathbb{R}^2$ so that the preimage of the boundary $\{0\} \times \mathbb{R}^2$ is precisely given by the hypersurface $H$.

Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$. We denote by

$$L_{r, \lambda} = \{(z, u_1, v_1, u_2, v_2) \in Y \mid |z| = r, \phi(z, u_1, v_1, u_2, v_2) = \lambda\}$$

the fiber of $\rho$ over $(r, \lambda) \in B = \mathbb{R}_{>0} \times \mathbb{R}^2$. Consider the double conic fibration $f : Y \to \mathbb{C}$ given by projection to the $z$-coordinate. Then $L_{r, \lambda}$ can be viewed as a fibration, via $f$, over the circle $C_r = \{z \in \mathbb{C}^\times \mid |z| = r\}$ with generic fiber $T^2$. The fiber $L_{r, \lambda}$ is singular precisely when

(i) $r = |a|$ and $\lambda = (0, \lambda_2)$; or

(ii) $r = |b|$ and $\lambda = (\lambda_1, 0)$;

so the discriminant loci of $\rho$ is the disjoint union of two lines

$$\Gamma = \{(|a|, 0, \lambda_2) \in B \mid \lambda_2 \in \mathbb{R}\} \cup \{(|b|, \lambda_1, 0) \in B \mid \lambda_1 \in \mathbb{R}\}.$$

We denote by $B^{\text{sm}} := B \setminus \Gamma$ the smooth loci of the base of the SYZ fibration.

When $L_{r, \lambda}$ is smooth, it is a special Lagrangian torus in $Y^0$ with respect to the symplectic form $\omega$ and the holomorphic volume form

$$\Omega = d \log z \wedge d \log u_1 \wedge d \log u_2.$$

Let

$$L'_{r, \lambda_1} = \{(u_1, v_1) \in \mathbb{C}^2 \mid |u_1v_1 + a| = r, \ |u_1|^2 - |v_1|^2 = 2\lambda_1\},$$

and let

$$L''_{r, \lambda_2} = \{(u_2, v_2) \in \mathbb{C}^2 \mid |u_2v_2 + b| = r, \ |u_2|^2 - |v_2|^2 = 2\lambda_2\}.$$

Via the map $f' : \mathbb{C}^2 \to \mathbb{C}$ given by $(u_1, v_1) \mapsto u_1v_1 + a$, we can think of $L'_{r, \lambda_1}$ as an $S^1$-bundle over the circle $C_r$. Similarly, via the map $f'' : \mathbb{C}^2 \to \mathbb{C}$ given by $(u_2, v_2) \mapsto u_2v_2 + b$, $L''_{r, \lambda_2}$ can be thought of as an $S^1$-bundle over the same circle. Then $L_{r, \lambda}$ is nothing but the fibred product

$$\begin{array}{ccc}
L_{r, \lambda} & = & L'_{r, \lambda_1} \times_\mathbb{C} L''_{r, \lambda_2} \\
\downarrow & \searrow f' & \downarrow \\
L'_{r, \lambda_1} & \to & \mathbb{C}.
\end{array}$$

To construct the SYZ mirror, we compute the superpotential [CO06, FOOO09, FOOO10, AAK] which counts Maslov index two holomorphic discs in $Y$ (caution: not $Y^0$!) with boundary on the Lagrangian torus fibers of $\rho$.

Omitting subscripts for convenience, for $r$ large, the Lagrangian $L' \times_\mathbb{C} L''$ is Hamiltonian isotopic to a Lagrangian of the form

$$(S^1(r_1) \times S^1(r_2)) \times_\mathbb{C} (S^1(r_3) \times S^1(r_4)).$$
Following the same argument as in [Aur07, Aur09], the $S^1(r_1) \times S^1(r_2)$ component bounds two families of Maslov index two discs, which we will denote by $\beta'_1$ and $\beta'_2$, and the $S^1(r_3) \times S^1(r_4)$ component also bounds two families of Maslov index two discs, which we will denote by $\beta''_1$ and $\beta''_2$. Then clearly $L' \times_C L''$ bounds four families of Maslov index two discs which we will denote by $(\beta'_i, \beta''_j)$ for $i, j = 1, 2$. Let $z_1, z_2, z_3, z_4$ be the weights corresponding respectively to

$$(\beta'_1, \beta''_1), (\beta'_1, \beta''_2), (\beta'_2, \beta''_1), (\beta'_2, \beta''_2).$$

Since

$$(\beta'_1, \beta''_2) + (\beta'_2, \beta''_1) - (\beta'_1, \beta''_1) = (\beta'_2, \beta''_2),$$

we have the relation

$$z_2z_3/z_1 = z_4.$$

It follows that the superpotential for large $r$ is given by

$$W = z_1 + z_2 + z_3 + \frac{z_2z_3}{z_1}.$$

**Remark 2.1.** Note that this is exactly the Hori-Vafa superpotential corresponding to the singular toric variety

$$Z = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1v_1 - u_2v_2 = 0\}.$$

In a sense, we can think of $r$ large as corresponding to some ‘toric limit’.

Using the description of $L = L_{r, \lambda}$ as a fibred product, it is easy to see that

**Proposition 2.2.** A Lagrangian torus fiber $L_{r, \lambda}$ bounds a nontrivial Maslov index zero holomorphic disc in $Y$ if and only if $r = |a|$ or $r = |b|$. In other words, there are exactly two walls.

Recall that we have $a < b < 0$ so that $|b| < |a|$. Let $\alpha'$ be the Maslov index zero disc bounded by the $L'$ factor and $\alpha''$ the one bounded by the $L''$ factor. Also let $w_1$ and $w_2$ be the corresponding weights. When $r$ is small, the Lagrangian torus $L$ is a fibred product of Chekanov tori $L' \times_C L''$, with each factor bounding one family of disc $\beta'_0$ and $\beta''_0$ respectively. So $L$ bounds one family of disc with relative homotopy class $(\beta'_0, \beta''_0)$. Let $u$ be the weight corresponding to $(\beta'_0, \beta''_0)$. Then for small $r$, the superpotential is just

$$W = u.$$

To analyze the wall-crossing for counting of Maslov index two discs, we first assume that $\lambda_1 > 0$. As $r$ increases and passes through the first wall $r = |b|$, the class $(\beta'_0, \beta''_0)$ deforms naturally to

$$(\beta'_1, \beta''_0)$$

but it may also pick up the Maslov index zero disc $\alpha'$ and deform into $(\beta'_1 + \alpha', \beta''_0) = (\beta'_2, \beta''_0)$. Similarly, assuming $\lambda_2 > 0$, as $r$ passes through the second wall $r = |a|$, $(\beta'_1, \beta''_0)$ naturally deforms to $(\beta'_1, \beta''_0)$ but it may also deform to $(\beta'_1, \beta''_0 + \alpha'') = (\beta'_1, \beta''_2)$. 

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Hence, the wall-crossing formula for the first wall reads
\[ u \mapsto \hat{z}_1(1 + w_1), \]  
where \( w_1 = \hat{z}_3/\hat{z}_1 \), and the the wall-crossing formulas for the second wall are given by
\[ \hat{z}_1 \mapsto z_1(1 + w_2), \hat{z}_3 \mapsto z_3(1 + w_2), \]  
where \( w_2 = z_2/z_1 \). Composing these formulas gives
\[ u \mapsto z_1 + z_3 + z_2 + \frac{z_2 z_3}{z_1}, \]
so the wall-crossing formulas do make the superpotential for \( r \) small agree with that for \( r \) large.

Although we have assumed that \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), the above calculations also work for other cases when \( \lambda_1 < 0 \) or \( \lambda_2 < 0 \). Letting \( v = z_1^{-1} \), we conclude that the (uncompleted) SYZ mirror of \( Y \) with respect to the anticanonical divisor \( H \) is the Landau-Ginzburg model \((\tilde{Y}_0, W)\) with total space
\[ \tilde{Y}_0 = \{(u, v, \hat{z}_1, w_1, w_2) \in \mathbb{C}^3 \times (\mathbb{C}^\times)^2 \mid u = \hat{z}_1(1 + w_1), \ v\hat{z}_1 = 1 + w_2\} \]
and superpotential
\[ W = u. \]

We will now show that there is a natural completion of this Landau-Ginzburg model.
\[ Z = \{(u, v, w_1, w_2) \in \mathbb{C}^4 \mid uv = (1 + w_1)(1 + w_2)\} \]
and its crepant resolution
\[ X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \]

Let \( X^0 = X \setminus D \), where \( D \) is the pull-back of the divisor \( \{w_1 w_2 = 0\} \) on \( Z \) along the crepant resolution \( \varphi : X \to Z \). Then we can write
\[ X = \{(u, v, w_1, w_2, [x_1 : x_2]) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 \times \mathbb{P}^1 \mid ux_2 = (1 + w_2)x_1, \ (1 + w_1)x_2 = vx_1\}. \]

Note that \( \tilde{Y}_0 \) is an open affine subvariety of \( X^0 \), and the superpotential \( W \) naturally extends to \( X^0 \). We conclude that

**Proposition 2.3** ([AAK, Section 11]). *The Landau Ginzburg model \((X^0, W)\) is the completed, corrected SYZ mirror to \( Y \) with respect to the anticanonical divisor \( H \). In particular, \( X^0 \) is the completed, corrected SYZ mirror to \( Y^0 = Y \setminus H \).*

### 3 Lagrangian submanifolds fibred over paths

We introduce a class of Lagrangian submanifolds in \( Y^0 \) which are fibred over paths in the \( z \)-plane. Let us start with noncompact Lagrangian submanifolds.
Definition 3.1. A smooth path $\gamma : \mathbb{R} \to \mathbb{C}^\times$ on the $z$-plane such that $\lim_{t \to -\infty} |\gamma(t)| = 0$ and $\lim_{t \to \infty} |\gamma(t)| = 0$ is said to be admissible if it intersects the interval $\epsilon := [a, b]$ transversally and does not intersect the discriminant $\Delta = \{a, b\}$ of the double conic fibration $f : Y^0 \to \mathbb{C}^\times$. The winding number $w(\gamma)$ of an admissible path $\gamma$ is defined as its intersection number with $\epsilon$. We choose the orientation so that a path intersecting $\epsilon$ transversally once and in the counterclockwise direction contributes $+1$ to the intersection number.

Let $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ be an admissible path. The symplectic fibration $f : Y^0 \to \mathbb{C}^\times$ induces a natural horizontal distribution given by symplectic orthogonal to the fiber. Parallel transport with respect to this horizontal distribution gives symplectomorphisms between the smooth fibers of $f$. A 3-dimensional submanifold $L \subset f^{-1}(\gamma)$ is Lagrangian if and only if it is swept by the parallel transport of a Lagrangian cycle in a fiber along $\gamma$ (cf. [Aur07, Section 5.1]). Therefore, by fixing $t_0 \in I$ and choosing a Lagrangian cycle $A_0$ in the double conic fiber

$$f^{-1}(\gamma(t_0)) = (f')^{-1}(\gamma(t_0)) \times (f'')^{-1}(\gamma(t_0)),$$

one can construct a Lagrangian submanifold $L_{\gamma,A_0} \subset Y$ as the submanifold in $f^{-1}(\gamma)$ swept out by the parallel transport of $A_0$ along $\gamma$.

Notice that the winding number $w(\gamma)$ and the Hamiltonian isotopy class of the Lagrangian submanifold $L_{\gamma}$ are invariant when we deform $\gamma$ in a fixed isotopy class relative to the boundary conditions. In particular, we can always deform $\gamma$ so that $\gamma(t)$ lies on the positive real axis for $t < -T$ for some fixed $T > 0$. Then we consider the direct product

$$A_t := \{ (\gamma(t), u_1, v_1, u_2, v_2) \in f^{-1}(\gamma(t)) \mid u_1, v_1, u_2, v_2 \in \mathbb{R} \},$$

of the real loci (see Figure 5.2) in the factors $(f')^{-1}(\gamma(t))$ and $(f'')^{-1}(\gamma(t))$ of the double conic fiber $f^{-1}(\gamma(t))$ for each $t < -T$. The Lagrangian cycle $A_t$ is invariant under symplectic parallel transport for $t < -T$. We then set

$$L_{\gamma} := L_{\gamma,A_t},$$

i.e. the submanifold in $Y^0$ swept out by parallel transport of $A_0$ (for some fixed $t_0 < -T$). This defines a Lagrangian submanifold in $(Y^0, \omega)$ homeomorphic to $\mathbb{R}^3$.

Definition 3.2. An admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ is said to be strongly admissible if $|\gamma| : \mathbb{R} \to \mathbb{R}_{>0}$ is a strictly increasing function.

Proposition 3.3. Let $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ be a strongly admissible path. Then the Lagrangian submanifold $L_{\gamma}$ we define above is a section of the SYZ fibration $\rho : Y^0 \to B$.

Proof. The proof is essentially the same as that of [CU, Proposition 3.4]. The restriction of the moment map $\phi$ to $A_t$ (for $t$ sufficiently small), which is just the direct product of the real loci (Figure 5.2), is injective. Since $\mathbb{T}^2$ acts fiberwise and it acts by symplectomorphisms on $Y^0$, the symplectic parallel transport induces $\mathbb{T}^2$-equivariant symplectomorphisms between fibers of $f$. So the restriction of $\phi$ to a parallel transport of $A_t$ remains injective. Together with the condition that $|\gamma(t)|$ is strictly increasing, we see that $L_{\gamma}$ is intersecting each fiber of the SYZ fibration $\rho : Y^0 \to B$ at one point.  

Remark 3.4. Given a strongly admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$, we can as well choose any Lagrangian cycle $A_0 \subset f^{-1}(\gamma(t_0))$ such that $\phi|_{A_0}$ is an injective map, then the resulting Lagrangian submanifold $L_{\gamma, A_0}$ is also a section of the SYZ fibration.

An example is given by the path

$$\gamma_0 : \mathbb{R} \to \mathbb{C}^\times, \ t \mapsto e^t,$$

which runs through the whole positive real axis, which is obviously strongly admissible. The corresponding Lagrangian submanifold $L_0 := L_{\gamma_0}$ is simply the real locus in $Y$ which we choose as the zero-section of the SYZ fibration.

To construct compact Lagrangian submanifolds in $(Y^0, \omega)$, we consider bounded paths, which are smooth paths $\sigma : [0, 1] \to \mathbb{C}^\times$ starting from the critical value $a$ of one conic fibration and ending at the critical value $b$ of the other conic fibration. The fiber product of the Lefschetz thimbles of each conic fibrations along a bounded path $\sigma$ gives a Lagrangian submanifold $L_{\sigma}$ of $Y^0$, which is a $T^2$ fibration over the bounded path. One $S^1$-factor collapses to a point on one end and the other $S^1$-factor collapses to a point on the other end, so that the total space $L_{\sigma}$ is homeomorphic to $S^3$.

Definition 3.5. We call a bounded path $\sigma : [0, 1] \to \mathbb{C}^\times$ going from $b$ to $a$ strongly admissible if $|\sigma| : [0, 1] \to \mathbb{R}_{>0}$ is a strictly increasing function and $\sigma$ intersects the interval $\epsilon^- := [-b, -a]$ transversally. As in [Cha], in order to define the SYZ transform, we need to choose a reference path $\sigma_0$, relative to which we measure the winding numbers. Since we have chosen the Lagrangian $L_0$ associated to the positive real axis $\gamma_0$ as the zero-section of the SYZ fibration, and we would like the Floer cohomology between the Lagrangians fibered over the two reference paths $\gamma_0$ and $\sigma_0$ to have the correct dimension, we shall impose the condition that the reference paths $\gamma_0$ and $\sigma_0$ intersect transversally at one point (with the correct orientation). For this reason, we choose $\sigma_0$ to be the path corresponding to the Lagrangian 3-sphere $S_0$ as shown in Figure 1.3.

Definition 3.6. The winding number $w(\sigma)$ of a strongly admissible bounded path $\sigma : [0, 1] \to \mathbb{C}^\times$ going from $b$ to $a$ is defined to be the winding number of the concatenation of paths $\sigma_0 \circ \sigma$ with respect to the counterclockwise isomorphism $\pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$, where $\sigma_0$ denotes the path $\sigma_0$ with reversed orientation.

With this definition, the bounded path $\sigma_1$, which corresponds to the Lagrangian 3-sphere $S_1$ in Figure 1.3 has winding number 1.

It is easy to see that the Lagrangian 3-sphere $L_{\sigma}$ associated with a strongly admissible bounded path $\sigma : [0, 1] \to \mathbb{C}^\times$ is fibred by $T^2$ over the line segment (the red line in Figure 1.1)

$$\ell_0 := ([|b|, |a|] \times \{0\}$$

in the base $B$ of the SYZ fibration, and the $T^2$ fiber degenerates to an $S^1$ at both ends ($|a|, 0)$ and $|b|, 0$).
4 SYZ transforms

Let \( x_1 = -\lambda_1, \ x_2 = -\lambda_2 \) and \( x_3 \) be affine coordinates (action coordinates) on the smooth locus \( B^{sm} \) of the SYZ fibration; note that \( x_1 = -\lambda_1 \) and \( x_2 = -\lambda_2 \) are globally defined coordinates. We denote by \( \Lambda^\vee \subset T^*B^{sm} \) the family of lattices locally generated by \( dx_1, dx_2, dx_3 \), and let

\[
\omega_0 := dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 + dx_3 \wedge d\xi_3
\]

be the standard symplectic structure on the quotient \( T^*B^{sm}/\Lambda^\vee \) of the cotangent bundle \( T^*B^{sm} \) by \( \Lambda^\vee \), where \((\xi_1, \xi_2, \xi_3)\) denote the fiber coordinates on \( T^*B^{sm} \). Since we have a global Lagrangian section \( L_0 \) (the zero-section) of the SYZ fibration \( \rho : Y \to B \), there exists a fiber-preserving symplectomorphism \([\text{Dui}80]\)

\[
\Theta : (T^*B^{sm}/\Lambda^\vee, \omega_0) \xrightarrow{\sim} (\rho^{-1}(B^{sm}), \omega)
\]

so that \( L_0 \) corresponds to the zero section of \( T^*B^{sm}/\Lambda^\vee \).

We then take an open cover \( \{U_i\} \) of \( B^{sm} \) such that each \( U_i \) is contractible. As we have seen in Section 2, the SYZ mirror \( X^0 \) is constructed by gluing the open pieces \( TU_i/TU_i \cap \Lambda \) together according to the wall-crossing formulas (2.2), (2.3) (and then extending by analytic continuation). Let \( y_1, y_2, y_3 \) be the coordinates on \( TB^{sm} \) which are dual to the angle coordinates \( \xi_1, \xi_2, \xi_3 \) on \( T^*B^{sm}/\Lambda^\vee \). The local complex coordinates on \( X^0 \) are then given by \( w_1 = \exp 2\pi (x_1 + \sqrt{-1} y_1), \ w_2 = \exp 2\pi (x_2 + \sqrt{-1} y_2) \) and \( \exp 2\pi (x_3 + \sqrt{-1} y_3) \).

Let \( L \subset Y^0 \) be a Lagrangian cycle, given as the quotient of a translate of the co normal bundle \( N^*S \) of an integral affine linear subspace \( S \subset B \) by the lattice \( N^*S \cap \Lambda^\vee \), and equipped with a flat \( U(1) \)-connection \( \nabla \). The SYZ transform of \((L, Y^0)\) is given by a pair \((C, \nabla)\) consisting of the complex submanifold \( C \), which is given by gluing the open pieces

\[
T(S \cap U_i)/T(S \cap U_i) \cap \Lambda
\]

described according to the wall-crossing formulas (2.2), (2.3), and a \( U(1) \)-connection \( \nabla \), the \( (0, 2) \)-part of the curvature two form of which is trivial and hence defines a holomorphic line bundle \( \mathcal{L} \) over \( C \subset X^0 \).

**Definition 4.1.** We define the SYZ transform of the Lagrangian submanifold \( L \) equipped with the flat \( U(1) \)-connection \( \nabla \) to be the holomorphic line bundle \( \mathcal{L} \) over the complex submanifold \( C \subset X^0 \).

We refer the reader to the original papers \([\text{LYZ}00, \text{AP}01]\) for more details and the precise formulas; see also \([\text{Cha}]\).

By Proposition 3.3, the noncompact Lagrangian submanifold \( L_{\gamma} \) associated with a strongly admissible path \( \gamma : \mathbb{R} \to \mathbb{C}^\times \) is a section of the SYZ fibration \( \rho : Y^0 \to B \), so its SYZ transform should produce a holomorphic line bundle over \( X^0 \). Via the symplectomorphism \( \Theta \), we can write \( L_{\gamma} \) as a section of \( T^*B^{sm}/\Lambda^\vee \).

\[
L_{\gamma} = \{(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \in T^*B^{sm}/\Lambda^\vee \mid \xi_j = \xi_j(\gamma) \text{ for } j = 1, 2, 3\},
\]

where \( \xi_j = \xi_j(x_1, x_2, x_3) \) \( (j = 1, 2, 3) \) are smooth functions on \( B^{sm} \). The condition that \( L_{\gamma} \) being Lagrangian is then equivalent to saying that the functions \( \xi_1, \xi_2, \xi_3 \) satisfy the relations

\[
\frac{\partial \xi_j}{\partial x_i} = \frac{\partial \xi_i}{\partial x_j}.
\]
for $j, l = 1, 2, 3$.

The restriction of the Lagrangian section $L_γ$ to an open set $U_i \subset B^{sm}$ is transformed to a family of connections $\{\nabla_{ξ(x)} \mid x \in U_i\}$ which patch together to give a $U(1)$-connection over $U_i$ that can locally be written as

$$\nabla_{U_i} = d + 2\pi \sqrt{-1}(ξ_1dy_1 + ξ_2dy_2 + ξ_3dy_3)$$

over the open piece $TU_i/TU_i \cap \Lambda \subset X^0$. Since the $(0, 2)$-part of the curvature two form for each connection vanishes and the wall-crossing formulas are holomorphic, these connections glue together to give globally a holomorphic line bundle $\mathcal{L}_γ$ over $X^0$.

Notice that the isomorphism class of $\mathcal{L}_γ$ is unchanged when we deform $L_γ$ in a fixed Hamiltonian isotopy class (or deforming $γ$ in a fixed homotopy class relative to the boundary conditions $\lim_{t \to -∞} |γ(t)| = 0$ and $\lim_{t \to +∞} |γ(t)| = +∞$). Therefore, we will regard this as defining the SYZ transform of the Hamiltonian isotopy class of the Lagrangian submanifold $L_γ$ as an isomorphism class of holomorphic line bundle over $X^0$.

As an immediate example, the SYZ transformation of the zero section $L_0$ gives the structure sheaf $\mathcal{O}_{X^0}$ over $X^0$.

To compute (the isomorphism class of) the line bundle $\mathcal{L}_γ$, note that the degree of its restriction to the exceptional curve $E \cong \mathbb{P}^1$ in $X^0$ is given by

$$\deg \mathcal{L}_γ|_E = \int_E \frac{\sqrt{-1}}{2\pi} F_γ = -\int_E dξ_3 ∧ dy_3 = -(ξ_3(|b|, 0) - ξ_3(|a|, 0)).$$

We have the second equality because $y_1, y_2$ are constant (and $x_i = λ_i = 0$ for $i = 1, 2$) on $E$. Hence the isomorphism class of the line bundle $\mathcal{L}_γ$ is completely determined by the increment of the angle coordinate $ξ_3$ on the Lagrangian section $L_γ$ from $(0, 0, |b|)$ to $(0, 0, |a|)$ (which is measured with reference to the path $γ_0$).

**Proof of Theorem 1.1**. Arguing as in the proof of [CU] Theorem 1.1, we first deform $γ$ so that $γ(\log |b|) = -b$ and $γ(\log |a|) = -a$ and $γ(t) \in \mathbb{R}_{>0}$ for $t \neq (\log |b|, \log |a|)$ (up to a re-parametrization if necessary). We then further deform $γ|_{(\log |b|, \log |a|)}$ to the concatenation of $γ_0|_{(\log |b|, \log |a|)}$ (the positive real axis) with a loop winding around the circle $C_{|b|} = \{z \in \mathbb{C}^× \mid |z| = |b|\}$ for $w(γ)$ times. Along $γ_0$, the angle coordinate $ξ_3$ is constantly zero, and $ξ_3$ increases by one when we wind around $C_{|a|}$ once in the counterclockwise direction. Hence, the increment $ξ_3(|b|, 0) - ξ_3(|a|, 0)$ is precisely given by the winding number $w(γ)$. This completes the proof of Theorem 1.1.

Let $γ_0$ and $γ_1$ be admissible paths shown in Figure 1.2 which have winding numbers 0 and −1 respectively. Their associated Lagrangian submanifolds are denoted by $L_0 := L_{γ_0}$ and $L_1 := L_{γ_1}$ respectively. By Theorem 1.1 the SYZ transform of $L_i$ is precisely given by the line bundle $\mathcal{O}_{X^0}(i)$ for $i = 0, 1$.

Next we consider a strongly admissible bounded path $σ : [0, 1] \to \mathbb{C}^×$ going from $b$ to $a$. Recall that the corresponding compact Lagrangian 3-sphere $L_σ$ is a $T^2$-fibration over the line segment $ℓ_0 = ([b], [a]) × \{0\}$ in the base $B = \mathbb{R}_{>0} \times \mathbb{R}^2$ of the SYZ fibration $ρ : Y^0 \to B$ such that the $T^2$-fiber degenerates to an $S^1$ over the endpoints $([a], 0)$ and $([b], 0)$ of $ℓ_0$.

Let $L_σ^0 = L_σ \cap ρ^{-1}(ℓ_0)$, i.e. $L_σ$ with the two $S^1$’s over the end points of $ℓ_0$ removed. Then $L_σ^0$ is (the quotient by a lattice of) a translate of the conormal bundle of $ℓ_0$. Recall
that the coordinates \(w_1, w_2\) on \(X^0\) are given by 
\[w_1 = \exp 2\pi(x_1 + \sqrt{-1}y_1)\]
and 
\[w_2 = \exp 2\pi(x_2 + \sqrt{-1}y_2)\]. We equip \(L_\sigma\) with the flat \(U(1)\)-connection
\[\nabla_0 = d - \pi\sqrt{-1}(d\xi_1 + d\xi_2)\].

Then the SYZ transform of \((L_\sigma, \nabla_0)\) produces the complex submanifold in \(X^0\) defined by 
\[x_1 = x_2 = 0, y_1 = y_2 = 1/2\] or simply 
\[w_1 = w_2 = -1\], which is precisely the exceptional locus \(E \cong \mathbb{P}^1 \subset X^0\) (cf. [Cha, Section 2]).

We also get the \(U(1)\)-connection
\[\tilde{\nabla} = d + 2\pi\sqrt{-1}\xi_3(x, \vec{0})dy_3\]
on \(E\) which defines a holomorphic line bundle over \(E\) whose degree can be computed as
\[\int_E \frac{\sqrt{-1}}{2\pi} F_{\tilde{\nabla}} = -\int_E d\xi_3 \wedge dy_3 = -(\xi_3(|a|, \vec{0}) - \xi_3(|b|, \vec{0})) = -w(\sigma)\].

We have the last equality because the increment of the angle coordinate \(\xi_3\) is measured relative to the reference path \(\sigma_0\) which is computed by the winding number of the loop \(\sigma_0 \circ \sigma\) and this is by definition \(w(\sigma)\). This proves the following:

**Theorem 4.2.** The SYZ transform of the compact Lagrangian 3-sphere \(L_\sigma\) associated to a strongly admissible bounded path \(\sigma : [0, 1] \to \mathbb{C}^\times\) is given by the line bundle \(O_E(-w(\sigma))\) over the exceptional locus \(E \subset X^0\).

Theorem 4.3 is an immediate consequence of Theorem 4.2 since the bounded paths defining \(S_0\) and \(S_1\) have winding numbers 0 and 1 respectively.

### 5 Coherent sheaves on the resolved conifold

Let \(\mathbb{C}^\times\) acts on \(\mathbb{C}^4 = \text{Spec} \mathbb{C}[x, y, t_1, t_2]\) in such a way that \(\alpha \in \mathbb{C}^\times\) maps \((x, y, t_1, t_2)\) to \((\alpha x, \alpha y, \alpha^{-1}t_1, \alpha^{-1}t_2)\). It is convenient to realize the resolved conifold as the quotient
\[X = (\mathbb{C}^4 \setminus \Sigma)/\mathbb{C}^\times\]
where \(\Sigma := \{(x, y, t_1, t_2) \in \mathbb{C}^4 \mid x = y = 0\}\). In these coordinates, the morphism
\[\varphi : X \to Z = \{(u, v, w_1, w_2) \in \mathbb{C}^4 \mid uv = (1 + w_1)(1 + w_2)\}\]
to the conifold is given by
\[u = xt_1, v = yt_2, w_1 = xt_2 - 1, w_2 = yt_1 - 1.\]

**Definition 5.1.** An object \(\mathcal{E}\) in a triangulated category \(\mathcal{T}\) is a **tilting object** if

1. \(\mathcal{E}\) is **acyclic** in the sense that \(\text{Ext}^k(\mathcal{E}, \mathcal{E}) = 0\) for any \(k \neq 0\), and
2. \(\mathcal{E}\) is a **classical generator**, in the sense that the smallest, thick, triangulated subcategory generated by \(\mathcal{E}\) is all of \(\mathcal{T}\).
Note that any classical generator $E$ generates $T$ in the sense that $\text{Hom}^k(E, A) = 0$ for some $A \in T$ and all $k \in \mathbb{Z}$ implies $A \cong 0$ (cf. e.g. [BvdB03, Section 2.1]). The proof of the following theorem can be found in [TU10, Lemma 3.3], and goes back at least to [Ric89, Bon89].

**Theorem 5.2.** Let $E$ be a tilting object in the derived category $D^b\text{coh} X$ of coherent sheaves on a smooth quasi-projective variety $X$. Then $D^b\text{coh} X$ is equivalent to the bounded derived category of finitely-generated right modules over $\text{Hom}(E, E)$.

The following is well-known (cf. e.g. [VdB04]):

**Theorem 5.3.** The direct sum $O_X \oplus O_X(1)$ is a tilting object in $D^b\text{coh} X$, whose endomorphism algebra is described by the quiver

$$
\begin{array}{ccc}
O & \xrightarrow{y} & O(1) \\
\circ & t_1 & \circ \\
\circ & t_2 & \circ \\
\end{array}
$$

with relations

$$\mathcal{I} = (xt_1y - yt_1x, xt_2y - yt_2x, t_1xt_2 - t_2xt_1, t_1yt_2 - t_2yt_1).$$

Let $\{P_{a,i_1,i_2}\}_{(a,i_1,i_2) \in \mathbb{Z} \times \mathbb{N}^2}$ be the basis of $\text{Hom}(O_X, O_X) \cong \text{Hom}(O_X(1), O_X(1)) \cong \Gamma(O_X)$ defined by

$$P_{a,i_1,i_2} = \begin{cases} 
  u^{-a}w_1^{i_1}w_2^{i_2} & a < 0, \\
  v^aw_1^{i_1}w_2^{i_2} & a \geq 0.
\end{cases}$$

Similarly, we define the bases $\{Q_{a,i_1,i_2}\}_{(a,i_1,i_2) \in (\mathbb{Z}+\frac{1}{2}) \times \mathbb{N}^2}$ and $\{R_{a,i_1,i_2}\}_{(a,i_1,i_2) \in (\mathbb{Z}+\frac{1}{2}) \times \mathbb{N}^2}$ of $\text{Hom}(O_X, O_X(1))$ and $\text{Hom}(O_X(1), O_X)$ as

$$Q_{a,i_1,i_2} = \begin{cases} 
  xu^{-a-1/2}w_1^{i_1}w_2^{i_2} & a < 0, \\
  yv^{-a-1/2}w_1^{i_1}w_2^{i_2} & a \geq 0.
\end{cases}$$

and

$$R_{a,i_1,i_2} = \begin{cases} 
  t_1u^{-a-1/2}w_1^{i_1}w_2^{i_2} & a < 0, \\
  t_2v^{-a-1/2}w_1^{i_1}w_2^{i_2} & a \geq 0.
\end{cases}$$

respectively.

We have the following elementary algebra calculation.

**Proposition 5.4** (cf. [Pasa, Proposition 4.5]). The composition of $P_{a,i_1,i_2}$ is given by

$$P_{b,j_1,j_2} \cdot P_{a,i_1,i_2} = \sum_{s_1, s_2 = 0}^{k} \binom{k}{s_1} \binom{k}{s_2} P_{a+b,i_1+s_1,i_2+j_2+s_2}$$

(5.3)
where

\[ k = \begin{cases} 
\min\{|a|, |b|\} & \text{a and b have different signs,} \\
0 & \text{otherwise.}
\end{cases} \]

The composition of \( P_{a,i_1,i_2} \) and \( Q_{b,j_1,j_2} \) is given by

\[
Q_{b,j_1,j_2} \cdot P_{a,i_1,i_2} = \sum_{s_1, s_2 = 0}^{k} \binom{k}{s_1} \binom{k}{s_2} Q_{a+b, i_1+j_1+s_1,j_2+s_2} \quad (5.4)
\]

where

\[ k = \begin{cases} 
\min\{|a|, |b| - 1/2\} & \text{a and b have different signs,} \\
0 & \text{otherwise,}
\end{cases} \]

and similarly for the composition of \( P_{a,i_1,i_2} \) and \( R_{b,j_1,j_2} \). The composition of \( Q_{a,i_1,i_2} \) and \( R_{b,j_1,j_2} \) is given by

\[
R_{b,j_1,j_2} \cdot Q_{a,i_1,i_2} = \sum_{s_1}^{k_1} \sum_{s_2}^{k_2} \binom{k_1}{s_1} \binom{k_2}{s_2} P_{a+b, i_1+j_1+s_1,j_2+s_2} \quad (5.5)
\]

where

\[
\begin{align*}
    k_1 &= \begin{cases} 
    \min\{|a| - 1/2, |b| - 1/2\} + 1 & \text{a < 0 and b > 0,} \\
    \min\{|a| - 1/2, |b| - 1/2\} & \text{a > 0 and b < 0,} \\
    0 & \text{otherwise,}
    \end{cases} \\
    k_2 &= \begin{cases} 
    \min\{|a| - 1/2, |b| - 1/2\} & \text{a < 0 and b > 0,} \\
    \min\{|a| - 1/2, |b| - 1/2\} + 1 & \text{a > 0 and b < 0,} \\
    0 & \text{otherwise.}
    \end{cases}
\end{align*}
\]

The mirror \( X^0 \) is the complement \( X \setminus D \) of the divisor \( D = \{w_1w_2 = 0\} \) on \( X \).

**Corollary 5.5.** The direct sum \( \mathcal{O}_{X^0} \oplus \mathcal{O}_{X^0}(1) \) is a tilting object in \( D^b \text{coh}\ X^0 \).

**Proof.** The fact that \( \mathcal{O}_{X^0} \oplus \mathcal{O}_{X^0}(1) \) is a classical generator follows immediately from the fact that \( \mathcal{O}_X \oplus \mathcal{O}_X(1) \) is a classical generator and the equivalence

\[
D^b \text{coh}\ X/D^b \text{coh}\ D X \xrightarrow{\sim} D^b \text{coh}\ X^0
\]

of triangulated categories [Orl11, Lemma 2.2]. The acyclicity of \( \mathcal{O}_{X^0} \oplus \mathcal{O}_{X^0}(1) \) follows from the acyclicity of \( \mathcal{O}_X \oplus \mathcal{O}_X(1) \) and the description

\[
H^k(\mathcal{O}_{X^0}(i)) = \lim_{\rightarrow} \left( H^k(\mathcal{O}_X(i)) \xrightarrow{w_1w_2} H^k(\mathcal{O}_X(i)) \xrightarrow{w_1w_2} H^k(\mathcal{O}_X(i)) \xrightarrow{w_1w_2} \cdots \right)
\]

of the cohomology as a direct limit [Sei08, (1.13)]. \( \square \)
The derived category $D^{b}\text{coh}_{0}X$ of coherent sheaves on $X$ supported on the exceptional locus $E$ of the resolution $\varphi : X \to Z$ is generated by $\mathcal{O}_{E}$ and $\mathcal{O}_{E}(-1)[1]$, which are Koszul dual to $\mathcal{O}_{X}$ and $\mathcal{O}_{X}(1)$ in the sense that
\[
\text{Hom}^{0}(\mathcal{O}_{X}, \mathcal{O}_{E}) = \mathbb{C}, \quad \text{Hom}^{0}(\mathcal{O}_{X}, \mathcal{O}_{E}(-1)[-1]) = 0, \quad \text{Hom}^{0}(\mathcal{O}_{X}(1), \mathcal{O}_{E}) = 0, \quad \text{Hom}^{0}(\mathcal{O}_{X}(1), \mathcal{O}_{E}(-1)[1]) = \mathbb{C}.
\]

The endomorphism $A_{\infty}$-algebra of $\mathcal{O}_{E} \oplus \mathcal{O}_{E}(-1)$ is Koszul dual to the endomorphism algebra of $\mathcal{O}_{X} \oplus \mathcal{O}_{X}(1)$. A convenient way to describe it is given by the dimer model shown in Figure 5.1.

\[\begin{align*}
\text{Figure 5.1: The dimer model} & \quad \text{Figure 5.2: The corresponding quiver}
\end{align*}\]

It is a graph $G$ drawn on the real 2-torus consisting of two nodes and four edges. One node is painted in black, and the other is painted in white. The dual graph of $G$ is combinatorially identical to $G$, and we turn each edge of the dual graph into an arrow by giving the orientation such that the white node is on the right of the arrow. This makes the dual graph of $G$ into the quiver $Q = (V, A)$ shown in Figure 5.2 with two vertices $V = \{0, 1\}$ and four arrows $A = \{x, y, t_{1}, t_{2}\}$. For each arrow $a$ in the quiver, there are two paths $p_{+}(a)$ and $p_{-}(a)$ from the target of $a$ to the source of $a$; the former goes around the white node, and the latter goes around the black node. Then we can equip the quiver with the relation such that $p_{+}(a)$ is equivalent to $p_{-}(a)$ for all arrows; $I = (p_{+}(a) - p_{-}(a))_{a \in A}$. One can easily see that this relation is identical to the one in (5.2).

Now the endomorphism $A_{\infty}$-algebra of $\mathcal{O}_{E} \oplus \mathcal{O}_{E}(-1)[1]$ is described as follows [FU10, Definition 2.1 and Proposition 2.2]:

- The vertices 0 and 1 of $Q$ correspond to objects $\mathcal{O}_{E}$ and $\mathcal{O}_{E}(-1)[1]$ respectively.
- For a pair $v$ and $w$ of vertices, the space of morphisms is given by
  \[
  \text{Hom}^{i}(v, w) = \begin{cases} 
  \mathbb{C} \cdot \text{id}_{v} & \text{if } i = 0 \text{ and } v = w, \\
  \text{span}\{a \mid a : w \to v\} & \text{if } i = 1, \\
  \text{span}\{a^{\lor} \mid a : v \to w\} & \text{if } i = 2, \\
  \mathbb{C} \cdot \text{id}_{w}^{i} & \text{if } i = 3 \text{ and } v = w, \\
  0 & \text{otherwise}.
  \end{cases}
  \]
- Non-zero $A_{\infty}$-operations are
  \[
  m_{2}(x, \text{id}_{w}) = m_{2}(\text{id}_{w}, x) = x
  \]
for any \( x \in \text{Hom}(v, w) \),
\[
m_2(a, a^\vee) = \text{id}_v^\vee
\]
and
\[
m_2(a^\vee, a) = \text{id}_w^\vee
\]
for any arrow \( a \) from \( v \) to \( w \),
\[
m_k(a_1, \ldots, a_k) = a_0.
\]
for any cycle \( (a_0, \ldots, a_k) \) of the quiver going around a white node, and
\[
m_k(a_1, \ldots, a_k) = -a_0.
\]
for any cycle \( (a_0, \ldots, a_k) \) of the quiver going around a black node.

• The pairing
\[
\langle \bullet, \bullet \rangle : \text{Hom}(w, v) \otimes \text{Hom}(v, w) \to \mathbb{C}[3]
\]
defined by
\[
\langle a^\vee, a \rangle = \langle \text{id}_v^\vee, \text{id}_w \rangle = 1
\]
and zero otherwise makes the endomorphism \( A_\infty \)-algebra into a cyclic \( A_\infty \)-algebra of dimension three.

To be more explicit, one has

\[
\begin{align*}
\text{Hom}^i(\mathcal{O}_E, \mathcal{O}_E) &= \begin{cases} 
\mathbb{C} \cdot \text{id}_{\mathcal{O}_E} & i = 0, \\
\mathbb{C} \cdot \text{id}_{\mathcal{O}_E}^\vee & i = 3, \\
0 & \text{otherwise,}\n\end{cases} \\
\text{Hom}^i(\mathcal{O}_E(-1)[1], \mathcal{O}_E(-1)[1]) &= \begin{cases} 
\mathbb{C} \cdot \text{id}_{\mathcal{O}_E(-1)[1]} & i = 0, \\
\mathbb{C} \cdot \text{id}_{\mathcal{O}_E(-1)[1]}^\vee & i = 3, \\
0 & \text{otherwise,}\n\end{cases} \\
\text{Hom}^i(\mathcal{O}_E(-1)[1], \mathcal{O}_E) &= \begin{cases} 
\mathbb{C} \cdot t_1^\vee \oplus \mathbb{C} \cdot y^\vee & i = 1, \\
\mathbb{C} \cdot t_1 \oplus \mathbb{C} \cdot t_2 & i = 2, \\
0 & \text{otherwise,}\n\end{cases} \\
\text{Hom}^i(\mathcal{O}_E, \mathcal{O}_E(-1)[1]) &= \begin{cases} 
\mathbb{C} \cdot t_1 \oplus \mathbb{C} \cdot t_2 & i = 1, \\
\mathbb{C} \cdot x^\vee \oplus \mathbb{C} \cdot y^\vee & i = 2, \\
0 & \text{otherwise,}\n\end{cases}
\end{align*}
\]

with \( A_\infty \)-operations
\[
\begin{align*}
m_3(y, t_1, x) &= -t_2^\vee, & m_3(t_2, y, t) &= -x^\vee, & m_3(x, t_2, y) &= -t_1^\vee, & m_3(t_1, x, t_2) &= -y^\vee, \\
m_3(y, t_2, x) &= t_1^\vee, & m_3(t_1, y, t_2) &= x^\vee, & m_3(x, t_1, y) &= t_2^\vee, & m_3(t_2, x, t_1) &= y^\vee,
\end{align*}
\]
and
\[
\begin{align*}
m_2(x, x^\vee) &= \text{id}_{\mathcal{O}_E}^\vee, & m_2(y, y^\vee) &= \text{id}_{\mathcal{O}_E}^\vee, & m_2(s^\vee, s) &= \text{id}_{\mathcal{O}_E}^\vee, & m_2(t_1^\vee, t_1) &= \text{id}_{\mathcal{O}_E}^\vee, \\
m_2(t_2, t_2^\vee) &= \text{id}_{\mathcal{O}_E(-1)[1]}^\vee, & m_2(t_1, t_1^\vee) &= \text{id}_{\mathcal{O}_E(-1)[1]}^\vee, & m_2(x^\vee, x) &= \text{id}_{\mathcal{O}_E(-1)[1]}^\vee, & m_2(y^\vee, y) &= \text{id}_{\mathcal{O}_E(-1)[1]}^\vee.
\end{align*}
\]
All the other non-zero \( A_\infty \)-operations just say that \( \text{id}_{\mathcal{O}_E} \) and \( \text{id}_{\mathcal{O}_E(-1)[1]} \) are the identity elements for \( m_2 \).
6 Wrapped Fukaya category

We prove Theorem 1.2 in this section. We set $a = \sqrt{-1}$ and $b = -\sqrt{-1}$ as in Figure 6.1 for convenience in this section. We take wrapping Hamiltonians of the forms

$$H_i = H_b + H_{f_1,i} + H_{f_2,i},$$

where the base Hamiltonian $H_b$ wraps the $z$-plane as shown in Figure 6.3 and the fiber Hamiltonians $H_{f_i,i}$ wrap the fiber either as in Figure 6.4 or Figure 6.5. Let $\phi_t : Y^0 \to Y^0$ be the time $t$ flow by the wrapping Hamiltonian $H_i$. The wrapped Floer cohomology is defined as

$$\text{Hom}_{W_i}(L_j, L_k) = \lim_{t \to \infty} \text{Hom}_F(\phi_t L_j, L_k)$$

where $\text{Hom}_F(\phi_t L_j, L_k)$ is the ordinary Floer cohomology.

**Remark 6.1.** Our choice of Hamiltonian is slightly different from that in [AS10], but is very suitable for analyzing fibrations. In the appendix, we provide a detailed comparison of the two approaches. It is also important to note that, while we don’t construct $A_\infty$-operations on our wrapped Floer cohomology, all of our wrapped Floer groups are concentrated in degree zero and thus any such enhancement would actually be quasi-isomorphic to its cohomology algebra.

**Proposition 6.2.** There is a ring isomorphism

$$\bigoplus_{i,j=0} \text{Hom}_{W_i}(L_i, L_j) \xrightarrow{\sim} \bigoplus_{i,j=0} \text{Hom}(O_X(i), O_X(j)).$$

**Proof.** Let us first consider the composition

$$\text{Hom}(\phi_n(L_0), L_0) \otimes \text{Hom}(\phi_{m+n}(L_0), \phi_n(L_0)) \to \text{Hom}(\phi_{m+n}(L_0), L_0).$$

The intersection points in $\phi_n(L_0) \cap L_0$ can be labeled as $p_{a_{i_1,i_2}}$ as in Figure 6.6. We view the $z$-plane as a cylinder, which is obtained by identifying the horizontal edges of the
rectangle in Figure 6.6. We choose a coordinate on the rectangle in such a way that the top right and the bottom left corners have coordinates $(1, 1)$ and $(-1, -1)$ respectively.

Intersections between the Lagrangians $\phi_n(L_0)$ and $L_0$ are parameterized by triplets of integers $(a, i_1, i_2)$. The integer $a \in [-n + 1, n - 1]$ parametrizes the intersection point of the $z$-projections $\sigma_\gamma(\gamma_0)$ and $\gamma_0$ of the Lagrangians $\phi_n(L_0)$ and $L_0$. The integers $i_1$ and $i_2$ in $[0, [(n - |a|)/2]]$ parametrize the intersection points on the fiber just as in [Pasa, Section 3.3.4].

Our arguments will be based upon the following adaptation of Pascaleff’s theorem [Pasa, Proposition 4.4] to this setting. Its proof follows mutatis-mutandis from Pascaleff’s paper.

**Lemma 6.3.** Let $L$, $L’$, and $L''$ be Lagrangian submanifolds of $Y^0$ fibered over paths $\gamma$, $\gamma’$ and $\gamma''$ in $\mathbb{C}^\times$. Assume that a holomorphic triangle $u : D^2 \to \mathbb{C}^\times$ bounded by $\gamma$, $\gamma’$ and $\gamma''$ with vertices $o \in \gamma \cap \gamma’$, $o’ \in \gamma’ \cap \gamma''$ and $o'' \in \gamma \cap \gamma''$ intersects the discriminants $a$ and $b$ in $\mathbb{C}^\times$ exactly $d_1$ and $d_2$ times respectively. Then holomorphic sections over $u$ contributes to the triangle product $\text{Hom}(L’, L'') \otimes \text{Hom}(L, L’) \to \text{Hom}(L, L’’)$ as

$$m_2(o’_{j_1,j_2}, o_{i_1,i_2}) = \sum_{s_1 = 0}^{d_1} \sum_{s_2 = 0}^{d_2} \binom{d_1}{s_1} \binom{d_2}{s_2} o''_{i_1+j_1+s_1,j_2+s_2}.$$
where $o_{i_1,i_2} \in L \cap L'$ is the intersection point above $o \in \gamma \cap \gamma'$, which is the $i_1$-th one from the bottom in the $u_1v_1$-direction and the $i_2$-th one from the bottom in the $u_2v_2$-direction.

The universal cover of the cylinder in Figure 6.6 is an infinite strip $\{ (s, t) \in \mathbb{R}^2 \mid -1 \leq s \leq 1 \}$. A lift of the $z$-projection $\gamma_{0,n}$ of the wrapped Lagrangian $\phi_n(L_0)$ to the universal cover is given by a line with slope $n$, passing through $(0, k)$ with $k \in \mathbb{Z}$. The discriminants of the conic fibrations are given by $(0, -1/4 + \mathbb{Z})$ and $(0, 1/4 + \mathbb{Z})$ respectively. The projection of the intersection point $p_{b,j_1,j_2} \in \text{Hom}(\phi_n(L_0), L_0)$ has the $s$-coordinate $b/n$, and we choose the lift to the universal cover to be $(b/n, 0)$.

Consider the lift of $\gamma_{0,n}$ passing through $(b/n, 0)$. The induced lift of the intersection point corresponding to $p_{a,i_1,i_2} \in \text{Hom}(\phi_{m+n}(L_0), \phi_n(L_0))$ will have coordinate $(a/m, na/m - b)$. If we then take the lift of $\gamma_{0,m+n}$ passing through this point, it intersects with the lift of $\gamma_0$ at $((a + b)/(m + n), 0)$ as shown in Figure 6.7 or Figure 6.8 depending on the order of $a$ and $b$.

In either case, one can see from Figure 6.9 or Figure 6.10 that the triangle hits both of the discriminants $(0, -1/4 + \mathbb{Z})$ and $(0, 1/4 + \mathbb{Z})$ $k$ times, where $k$ is min{$|a|, |b|$} if $a$ and $b$ has different signs and 0 otherwise. Then one has

$$m_2(p_{b,j_1,j_2}, p_{a,i_1,i_2}) = \sum_{s_1, s_2=0}^{k} \binom{k}{s_1} \binom{k}{s_2} p_{a+b,i_1+j_1+s_1,i_2+j_2+s_2} \tag{6.4}$$

by Pascaleff’s formula, in agreement with (5.3).
Next we consider the composition
\[ \text{Hom}(\phi_n(L_1), L_0) \otimes \text{Hom}(\phi_{m+n}(L_0), \phi_n(L_1)) \to \text{Hom}(\phi_{m+n}(L_0), L_0). \]

A lift of the \( z \)-projection \( \gamma_{1,n} \) of the wrapped Lagrangian \( \phi_n(L_1) \) to the universal cover is given by a line with slope \( n \) passing through \( (0, k + 1/2) \) with \( k \in \mathbb{Z} \). The intersections of the curves on the \( z \)-planes are as in Figure 6.7 or Figure 6.8 again, with \( \gamma_{0,n} \) replaced with \( \gamma_{1,n} \) and \( a \) and \( b \) being half-integers. One can see from Figure 6.9 and Figure 6.10 that the triangle hits the discriminants at \((0, -1/4 + k)\) and \((0, 1/4 + k)\) for \( k \) times, where \( k = \min \{|a| - 1/2, |b| - 1/2\} \) if \( a \) and \( b \) have different signs, and \( k = 0 \) otherwise. Then one has
\[
m_2(r_{b,j_1,j_2}, q_{a,i_1,i_2}) = \sum_{s_1=0}^{k_1} \sum_{s_2=0}^{k_2} \binom{k}{s_1} \binom{k}{s_2} p_{a+b-1,i_1+j_1+s_1,i_2+j_2+s_2}. \tag{6.5}
\]
This is in complete agreement with (5.5). Other compositions can be calculated similarly, and Proposition 6.2 is proved.

The choice of partial wrapping function \( H_{f,1} \) corresponds to the fact that the mirror of the resolved conifold \( X \) is in fact the Landau Ginzburg model \((Y^0, u_1 + u_2)\). See [AAK] for more discussion. Since wrapping by \( H_{f,2} \) corresponds to multiplication by \( w_i \), one obtains the following:

**Corollary 6.4.** There is a ring isomorphism
\[
\bigoplus_{i,j=0} \text{Hom}(W_2(L_i, L_j)) \xrightarrow{\sim} \bigoplus_{i,j=0} \text{Hom}(\mathcal{O}_{X^0}(i), \mathcal{O}_{X^0}(j)). \tag{6.6}
\]

Theorem 1.2 is an immediate consequence of Corollary 5.5, Corollary 6.4, and Theorem A.3.

### 7 Mirror symmetry for vanishing cycles

We prove Theorem 7.1 in this section. Theorem 1.4 follows immediately since \( D^b \text{coh}_0 X^0 \) is generated by \( \mathcal{O}_E \) and \( \mathcal{O}_E(-1) \) as a triangulated category.

**Theorem 7.1.** Let \( \mathcal{F}_0 \) be the Fukaya category of \( Y^0 \) consisting of \( S_0 \) and \( S_1 \). Then \( \mathcal{F}_0 \) is quasi-equivalent to the full subcategory of (the dg enhancement of) \( D^b \text{coh} X^0 \) consisting of \( \mathcal{O}_E \) and \( \mathcal{O}_E(-1) \).

**Proof.** We use the chain model for the Fukaya category of the plumbing by Abouzaid [Abo11, Appendix A]. Let \( Q_1 \) and \( Q_2 \) be a pair of graded exact Lagrangian submanifolds in an exact symplectic manifold equipped with a trivialization of the canonical bundle. Assume that \( Q_1 \) and \( Q_2 \) intersect cleanly along a submanifold \( B \), which consists of \( r \) connected components \( B^1, \ldots, B^r \):
\[
B = Q_1 \cap Q_2, \quad B = B^1 \cup \cdots \cup B^r.
\]

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Since $Q_1$ and $Q_2$ are Lagrangian submanifolds intersecting cleanly along $B$, the normal bundles $N_{Q_1}B$ and $N_{Q_2}B$ are isomorphic as real vector bundles. Choose closed tubular neighborhoods $N_i$ of $Q_i$ and triangulations $Q_i$ of $Q_i$ such that $Q_i$ induce triangulations $N_i$ of $N_i$ and the isomorphism $N_{Q_i}B \cong N_{Q_i}B$ induces a cellular homeomorphism $N_i \cong N_2$. Let $\mathcal{N} = N_1 \sqcup \cdots \sqcup N_r$ be the decomposition of the abstract simplicial complex $\mathcal{N} = N_1 \cong N_2$ into connected components. Then the chain model for the Fukaya category consisting of $Q_i$ is given by

\[
\text{Hom}(Q_i, Q_i) = C^*(Q_i),
\]

\[
\text{Hom}(Q_1, Q_2) = \bigoplus_{k=1}^{r} C^*(\mathcal{N}^k)[m_k],
\]

\[
\text{Hom}(Q_2, Q_1) = \bigoplus_{k=1}^{r} C^*(\mathcal{N}^k, \partial \mathcal{N}^k)[-m_k].
\]

Here integers $m_k$ comes from the gradings on the Lagrangian submanifolds.

Now we apply this construction to the case when $Q_1 = S_0 \cong S^1$, $Q_2 = S_1 \cong S^3$, $B = B^1 \sqcup B^2 = S^1 \sqcup S^1$ and $\mathcal{N}^k \cong \mathbb{D}^2 \times S^1$.

\[
\text{Hom}(S_i, S_i) \cong C^*(S^3),
\]

\[
\text{Hom}(S_1, S_0) \cong C^*(\mathbb{D}^2 \times S^1)[-1] \oplus C^*(\mathbb{D}^2 \times S^1)[-1],
\]

\[
\text{Hom}(S_0, S_1) \cong C^*(\mathbb{D}^2 \times S^1, \partial \mathbb{D}^2 \times S^1)[1] \oplus C^*(\mathbb{D}^2 \times S^1, \partial \mathbb{D}^2 \times S^1)[1].
\]

In this formula, cochains denote simplicial cochains with respect to a suitable triangulation. We have chosen the gradings on $S_0$ and $S_1$ in such a way that $m_1 = m_2 = -1$. We view each copy of $S^3$ via its Hopf decomposition

\[
S^3 = \mathbb{D}^2 \times S^1 \cup \tau_2 \mathbb{D}^2 \times S^1.
\]

In our example, we can work with the smaller cellular model described below, which can be easily seen to be that we get a quasi-isomorphic dg-category if we choose to view each $\mathbb{D}^2$ as a two simplex $\Delta_2$ and $S^1$ as the union of three one simplices $\Delta_1$ in the usual way.

We have the cochain models

\[
C^*(\mathcal{N}^1) = \text{span}_\mathbb{C} \left\{ 1 \right\},
\]

\[
C^*(\mathcal{N}^2) = \text{span}_\mathbb{C} \left\{ x, \ y \to \ xy \to \ yz \right\},
\]

\[
C^*(\mathcal{N}^k) = C^*(\mathbb{D}^2 \times S^1). \text{ Arrows show the differential in such a way that } d(x) = z \text{ and similarly for other arrows. The cohomological degrees are given by } \deg(x) = \deg(y) = 1 \text{ and } \deg(w) = 2. \text{ The elements } w \text{ and } z \text{ are the cellular cochains which are dual to the disc } \mathbb{D}^2 \text{ as shown in Figure 7.2.} \text{ We use the same symbols } x \text{ and } y \text{ for those generators which will be identified under the Hopf gluing. In the first copy of } \mathbb{D}^2 \times S^1, x \text{ is the cochain dual to the boundary of } \mathbb{D}^2 \text{ and } y \text{ is the cellular cochain dual to the } S^1\text{.}
factor. The roles of these cochains are reversed in the second copy of \( \mathbb{D}^2 \times S^1 \). The chain model for one copy of \( C^*(\partial \mathbb{D}^2 \times S^1) \) is given by

\[
C^*(\partial \mathbb{D}^2 \times S^1) = \text{span}_\mathbb{C} \left\{ 1, x, xy, y \right\},
\]

and similarly for the other copy. Accordingly, we have the chain model

\[
C^*(S^3) = \text{span}_\mathbb{C} \left\{ 1, x, xy, yz, xw, w \right\},
\]

for \( C^*(S^3) \) where \( d(xy) = yz + xw \). Using these basic models, we construct the chain level model for the category as follows: For \( C^*(S^3) \), we take the above cochain algebra. For the other groups, we preserve the letters corresponding to the above models to make clear the geometric origins of the generators and use \( \overrightarrow{m} \) to denote a morphism in \( \text{Hom}(S_1, S_0) \) and \( \overleftarrow{m} \) to denote a morphism in \( \text{Hom}(S_0, S_1) \). For \( \text{Hom}(S_0, S_1) \) we take as a basis:

\[
\overleftarrow{z}, \overrightarrow{yz}, \overleftarrow{w}, \overrightarrow{xw}, \quad d = 0.
\]

We have that \( \text{Hom}(S_1, S_0) \) is the sum of two complexes,

\[
\overrightarrow{u}_1 \quad \overrightarrow{x}_1 \quad \overrightarrow{y}_1 \quad \overrightarrow{z}_1 \quad \overrightarrow{z}_1 \quad \overrightarrow{y}_1 \quad \overrightarrow{x}_2 \quad \overrightarrow{z}_1 \quad \overrightarrow{z}_1 \quad \overrightarrow{x}_2 \quad \overrightarrow{z}_2.
\]

The differential is as in the model for \( C^*(\mathbb{D}^2 \times S^1) \). Compositions are the natural ones described in [Abo11 Section 2.1].

**Lemma 7.2.** We have

\[
\begin{align*}
\text{m}_3(\overrightarrow{u}_1, \overrightarrow{z}, \overrightarrow{u}_2) &= \overrightarrow{x}_2, \\
\text{m}_3(\overrightarrow{w}, \overrightarrow{u}_1, \overrightarrow{z}) &= \overrightarrow{yz}, \\
\text{m}_3(\overrightarrow{u}_2, \overrightarrow{w}, \overrightarrow{u}_1) &= \overrightarrow{y}_1, \\
\text{m}_3(\overrightarrow{z}, \overrightarrow{u}_2, \overrightarrow{w}) &= \overrightarrow{xw}.
\end{align*}
\]

The other \( \text{m}_3 \)'s are determined by the natural cyclic Calabi-Yau structure.
Proof. Given a dga \((V, d)\), we choose

- an injection \(i : \mathbb{H}^*(V) \to V\),
- a projection operator \(P : V \to i(\mathbb{H}^*(V))\) such that \(P|_{i(\mathbb{H}^*(V))} = i|_{i(\mathbb{H}^*(V))}\), and
- a chain homotopy \(Q\) such that \(\text{id} - [d, Q] = P\).

Then we define a series of linear maps

\[\lambda_n : V^\otimes n \to V\]

by setting

\[\lambda_2(v_1, v_2) = v_1 \cdot v_2\]

and inductively define

\[\lambda_n(v_1, \cdots, v_n) := (-1)^{n-1}[Q\lambda_{n-1}(v_1, \cdots, v_{n-1})]v_n - (-1)^{\text{deg}(v_1)}v_1[Q_{n-1}(v_2, \cdots, v_n)]
- \sum_{k,l \geq 2} (-1)^{k+l-1}(\text{deg}(v_1) + \cdots + \text{deg}(v_k))Q_k(v_1, \cdots, v_k)[Q_l(v_{k+1}, \cdots, v_n)].\]

for \(n \geq 3\). Now the operators

\[m_n : \mathbb{H}^*(V)^\otimes n \to \mathbb{H}^*(V)\]

are defined by \(m_n = P \circ \lambda_n\).

Theorem 7.3 ([Mer99]). The operators \(m_n\) define the structure of an \(A_{\infty}\)-algebra on \(\mathbb{H}^*(V)\) quasi-isomorphic to \((V, d)\).

We now compute \(Q\) in our setting. Since the differential vanishes on our model for \(\text{Hom}(S_0, S_1)\), the operator \(Q\) also vanishes on \(\text{Hom}(S_0, S_1)\). On \(\text{Hom}(S_0, S_0)\) and \(\text{Hom}(S_1, S_1)\), we can set

\[Q(z) = x, \quad Q(w) = y, \quad Q(yz) = \frac{xy}{2}, \quad Q(xw) = \frac{xy}{2}\]

and everything else to be zero. In the first summand of \(\text{Hom}(S_1, S_0)\), the operator \(Q\) is given by

\[Q(\zeta^1_1) = \xi^1, \quad Q(y\zeta^1_1) = \xi^1 y^1.\]

A similar formula holds in the second summand.

To compute \(m_3(\xi^1_1, \zeta^2, \xi^1_2)\), we notice that \(\xi^1 \cdot \xi^1_2 = 0\), so that

\[m_3(\xi^1_1, \zeta^2, \xi^1_2) = (1 - [d, Q])(Q(\xi^1_1 \cdot \zeta^2) \cdot \xi^1_2)
= (1 - [d, Q])(Q(\zeta^2) \cdot \xi^1_2)
= (1 - [d, Q])(x \cdot \xi^1_2)
= (1 - [d, Q])(\xi^1_2)
= \xi^1_2.\]

The other formulas can be calculated similarly, and Lemma 7.2 is proved. \(\square\)
We also have the following result:

**Lemma 7.4.** All $A_\infty$-operations $m_n$ for $n \geq 4$ vanish.

**Proof.** We argue using Merkulov’s formula by showing that a higher product cannot have a non-trivial component in any of the cohomology classes. First we notice that no cohomology class can be written as a product of two cochains which are in the image of $Q$. To avoid repeated arguments, we will demonstrate why we cannot have

$$m_n(x_1, \cdots, x_n) = \frac{y_z}{w}.$$  

All other cases can be addressed using the same type of arguments.

The only way to write $\frac{y_z}{w}$ as a non-trivial product is as $y \cdot \frac{z}{w}$. Using Merkulov’s formula and the above observation, we can assume without loss of generality that $Q\lambda_{n-1}(x_1, \cdots, x_{n-1})$ has non-trivial coefficient in the basis vector $y$ and that $x_n$ has a non-trivial component in the basis vector $\frac{z}{w}$. This would in turn imply that $\lambda_{n-1}(x_1, \cdots, x_{n-1})$ has non-trivial coefficient in $w$, which is not possible unless $n = 3$ because $w$ cannot be written as the product of cochains, $s_1s_2$, where either $s_1$ or $s_2$ is in the image of $Q$. $\Box$

Lemma 7.2 and Lemma 7.4 show that the $A_\infty$-operations on $F_0$ is identical to those for the endomorphism algebra of $O_E \oplus O_E(-1)$ described in Section 5 and Theorem 7.1 is proved. $\Box$

For the remainder of this section, we offer an alternative approach to Theorem 7.1, which stands on a conjecture that we were not able to verify, but hope is not outside the reach of current technology. Let $\hat{W}$ be the wrapped Fukaya category of $Y^0$ consisting of $L_0$, $L_1$, $S_0$, and $S_1$.

**Conjecture 7.5.** For any pair $(M_1, M_2)$ of objects in $\text{D}^b \hat{W}$, one has $\text{Hom}(M_1, M_2) = 0$ if $\text{Hom}(M_2, M_1) = 0$.

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. We will always assume that triangulated categories have enhancements in terms of dg categories [BK90] or $A_\infty$-categories (cf. e.g. [Kel01]). The right orthogonal to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of objects $M$ satisfying $\text{Hom}(\mathcal{N}, M) = 0$ for any $\mathcal{N} \in \mathcal{N}$. The left orthogonal $\mathcal{N}^\perp$ is defined similarly. The subcategory $\mathcal{N}$ is said to be right admissible if the embedding $I : \mathcal{N} \hookrightarrow \mathcal{D}$ has a right adjoint functor $Q : \mathcal{D} \rightarrow \mathcal{N}$. Left admissibility is defined similarly as the existence of a left adjoint functor, and $\mathcal{N}$ is said to be admissible if it is both right and left admissible.

A subcategory $\mathcal{N}$ is right admissible if and only if for any $X \in \mathcal{D}$, there exists a distinguished triangle $\mathcal{N} \rightarrow X \rightarrow M \rightarrow \mathcal{N}[1]$ with $\mathcal{N} \in \mathcal{N}$ and $M \in \mathcal{N}^\perp$. Such a triangle is unique up to isomorphism, and one has $Q(X) = \mathcal{N}$ in this case. If $\mathcal{N}$ is right admissible, then the quotient category $\mathcal{D}/\mathcal{N}$ is equivalent to $\mathcal{N}^\perp$. Analogous statements also hold for left admissible subcategories. A sequence $(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ of admissible subcategories in a triangulated category $\mathcal{D}$ is called a semiorthogonal decomposition $\mathcal{N}_j \subset \mathcal{N}_i^\perp$ for any $1 \leq j < i \leq n$, and $\mathcal{N}_1, \ldots, \mathcal{N}_n$ generates $\mathcal{D}$ as a triangulated category. A semiorthogonal decomposition will be denoted by

$$\mathcal{D} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle.$$
An object $E$ of $\mathcal{D}$ is *almost exceptional* if $\text{Ext}^i(E, E) = 0$ for $i \neq 0$ and the algebra $A := \text{Hom}(E, E)$ has finite homological dimension \cite[Definition 2.1]{BK04}. Let $\mathcal{E}$ be the smallest full subcategory of $\mathcal{D}$ containing $E$ and closed under cones and direct summands. Then one has a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{E}, \mathcal{E}^\perp \rangle$$

as in \cite[Theorem 3.2]{Bon89}; the object $N \in \mathcal{E}$ in the decomposition $N \to X \to M \to N[1]$ of an object $X \in \mathcal{D}$ is given by

$$N = \text{hom}^\bullet(E, X) \overset{L}{\otimes}_A E,$$

and $M \in \mathcal{E}^\perp$ is the mapping cone

$$M = \text{Cone} \left( \text{hom}^\bullet(E, X) \overset{L}{\otimes}_A E \xrightarrow{\cong} X \right)$$

of the evaluation morphism.

*An alternative proof of Theorem 7.1 assuming Conjecture 7.5.* Corollary 6.4 and Corollary 5.5 show that $L_0 \oplus L_1$ is an almost exceptional object in $\mathcal{D}^b\tilde{W}$, so that one has a semiorthogonal decomposition

$$\mathcal{D}^b\tilde{W} = \langle \mathcal{D}^bW^\perp, \mathcal{D}^bW \rangle. \quad (7.1)$$

Conjecture 7.5 implies that (7.1) is an orthogonal decomposition;

$$\mathcal{D}^b\tilde{W} = \mathcal{D}^bW^\perp \oplus \mathcal{D}^bW. \quad (7.2)$$

Since $\text{End}(S_0) \cong H^0(S^3) \cong \mathbb{C}$, the objects $S_0$ is indecomposable and belongs to either $\mathcal{D}^bW$ or $\mathcal{D}^bW^\perp$. The latter is impossible since $\text{Hom}(L_0, S_0) = \mathbb{C}$. This implies that $S_0 \in \mathcal{D}^bW$, and similarly for $S_1$. The fact

$$\text{Hom}^i(L_j, S_k) = \begin{cases} \mathbb{C} & i = 0 \text{ and } j = k, \\ 0 & \text{otherwise} \end{cases}$$

shows that $S_i$ goes to $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ under the derived equivalence

$$\mathcal{W} \cong \mathcal{D}^b\text{coh} X^0,$$

and Theorem 7.1 is proved.

\[\square\]

8 Small toric Calabi-Yau 3-folds

Let $Y^0$ be the complete intersection in $\mathbb{C}^5 \times \mathbb{C}^4 = \text{Spec} \mathbb{C}[z, z^{-1}, u_1, u_2, v_1, v_2]$ defined by

$$u_1v_1 = (z - a_1) \cdots (z - a_k), \quad (8.1)$$

$$u_2v_2 = (z - b_1) \cdots (z - b_l). \quad (8.2)$$
The SYZ mirror for $Y^0$ is the complement

$$X^0 = X \setminus D$$

of a divisor $D$ in a crepant resolution $X$ of the toric variety whose fan polytope is shown in Figure 8.1.

Here, the fan polytope of a toric variety is the convex hull of the primitive generators of one-dimensional cones of the fan. The fan structure induces a polyhedral decomposition of the fan polytope, and the fan polytope equipped with this polyhedral decomposition is called a \textit{toric diagram}.

The construction of the SYZ mirror of a complete intersection in [AAK, Section 11] shows that primitive generators of one-dimensional cones of the fan for $X$ are given by $(0, 1, 0), (1, 1, 0), \ldots, (k, 1, 0), (0, 0, 1), (1, 0, 1), \ldots, (l, 0, 1)$. The first $k + 1$ points comes from monomials in (8.1), and the next $l + 1$ points comes from monomials in (8.2).

One can map these points by the automorphism of $\mathbb{N} \approx \mathbb{Z}^3$ sending $(n_1, n_2, n_3)$ to $(n_1, n_2, n_2 + n_3)$, so that the fan polytope is the quadrangle on the hyperplane \{$(n_1, n_2, n_3) \in \mathbb{N}_\mathbb{R} \mid n_3 = 1$\} shown in Figure 8.1. The toric Calabi-Yau 3-fold $X$ is \textit{small} in the sense that the resolution $X \to Z = \text{Spec } \mathbb{C}[X]$ does not have 2-dimensional fibers (in other words, the toric variety $X$ has no compact toric divisors).

It is sometimes convenient to consider a stacky resolution $\mathcal{X}$ of $Z$, whose toric diagram is obtained by the triangulation of the fan polytope. Let consider the case when the fan for $\mathcal{X}$ has two 3-dimensional cones, one of which is generated by

$$v_1 = (0, 0, 1), \; v_3 = (0, 1, 0), \; \text{and} \; v_4 = (l, 0, 1),$$

and the other is generated by

$$v_2 = (k, 1, 0), \; v_3, \; \text{and} \; v_4.$$

Let $\varphi : \mathbb{Z}^4 \to N \approx \mathbb{Z}^3$ be the homomorphism sending the $i$-th standard basis $e_i \in \mathbb{Z}^4$ to $v_i \in N$ for $i = 1, \ldots, 4$. Then the toric stack $\mathcal{X}$ is the quotient

$$\mathcal{X} := [((\mathbb{C}^4 \setminus \Sigma)/K],$$

where $\Sigma := \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1 = x_2 = 0\}$ is the Stanley-Reisner locus and

$$K = \text{Ker } (\varphi \otimes \mathbb{C}^\times : (\mathbb{C}^\times)^4 \to N_{\mathbb{C}^\times} \cong (\mathbb{C}^\times)^3)$$

$$= \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{C}^\times)^4 \mid \alpha_2^k \alpha_4^l = \alpha_2 \alpha_3 = \alpha_1 \alpha_4 = 1\}$$

$$\cong K_1 \times K_2.$$
Here $K_1$ and $K_2$ are the subgroups of $K$ given by
\[K_1 := \{ (\alpha^k, \alpha^l, \alpha^{-k}) \in (\mathbb{C}^\times)^3 \mid \alpha \in \mathbb{C}^\times \} \cong \mathbb{C}^\times, \quad (8.3)\]
\[K_2 := \{ (\alpha, 1, 1, \alpha^{-1}) \in (\mathbb{C}^\times)^4 \mid \alpha \in \mathbb{C}^\times, \ \alpha^g = 1 \} \cong \mathbb{Z}/g\mathbb{Z} \quad (8.4)\]
where $g = \gcd(k, l)$. This shows that the toric stack $\mathcal{X}$ is the total space of the direct sum of two line bundles on the quotient stack $X := X_{k,l} := \mathbb{P}(k', l')/(\mathbb{Z}/g\mathbb{Z})$ of the weighted projective line $\mathbb{P}(k', l')$ for $k = gk'$ and $l = gl'$.

The toric stack $\mathcal{X}$ has the following description due to Geigle and Lenzing [GL87]: Let $S = \mathbb{C}[x_1, x_2]$ be the polynomial ring in two variables, graded by the abelian group $\alpha(\pi) = \{x_1, \ldots, x_n, 1\}$ of characters of $K$ and one has
\[\mathcal{X} \cong [(\text{Spec } S \setminus 0)/K].\]

The Picard group of $\mathcal{X}$ can be identified with the group $\text{Hom}(\mathcal{X}, \mathbb{C}^\times)$ of characters of $\mathcal{X}$ and one has
\[\mathcal{X} \cong \mathcal{O}_\mathcal{X}(-\vec{x}_1) \oplus \mathcal{O}_\mathcal{X}(-\vec{x}_2)\]
of line bundles.

Choose $a_i$ and $b_j$ in such a way that all of them are on the unit circle and mutually distinct. Let $(\gamma_i)_{i=1}^{k+l-1}$ be a collection of strongly admissible paths, such that for any connected component of $S^1 \setminus \Delta$ for $S^1 = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}$ and $\Delta = \{a_1, \ldots, a_k, b_1, \ldots, b_l\}$, there is a unique $i$ such that $\gamma_i$ intersects it. Let $\mathcal{W}$ be the wrapped Fukaya category of $Y$ consisting of $L_i := L_{\gamma_i}$ for $i = 0, \ldots, k + l - 1$. Define the collection $(\mathcal{L}_i)_{i=0}^{k+l-1}$ of line bundles on $\mathcal{X}$ inductively by $\mathcal{L}_0 = \mathcal{O}_\mathcal{X}$ and
\[\mathcal{L}_i = \begin{cases} \mathcal{L}_i \otimes \pi^*(\mathcal{O}(\vec{x}_1)) & a_j \text{ for some } j \text{ lies between } \gamma_{i-1} \text{ and } \gamma_i, \\ \mathcal{L}_i \otimes \pi^*(\mathcal{O}(\vec{x}_2)) & b_j \text{ for some } j \text{ lies between } \gamma_{i-1} \text{ and } \gamma_i, \end{cases}\]
where $\pi : \mathcal{X} \to \mathcal{Y}$ is the natural projection.

Then the proof of Theorem 1.2 can be easily adapted to prove the following:

**Theorem 8.1.** There is an equivalence
\[D^b\mathcal{W} \cong D^b \text{coh} \mathcal{X}^0 \quad (8.5)\]
of triangulated categories sending $L_i$ to $\mathcal{L}_i$ for $i = 0, \ldots, k + l - 1$.

## A Lefschetz wrapped Floer cohomology

### A.1 Liouville domains and wrapped Floer cohomology

An exact symplectic manifold with contact type boundary, or a *Liouville domain* for short, is a pair $(M, \theta)$ of a compact manifold $M$ with boundary and a one-form $\theta$ on $M$ called the *Liouville one-form* such that
• the two-form $\omega := d\theta$ is a symplectic form on $M$, and
• the Liouville vector field $Z$ defined by $\iota_Z\omega = \theta$ points strictly outward along the boundary $\partial M$.

The restriction $\alpha := \theta|_{\partial M}$ of the Liouville one-form is a contact one-form on $\partial M$. The Reeb vector field $R$ on $\partial M$ is defined by $R \in \ker \alpha$ and $\alpha(R) = 1$. The symplectic completion $\hat{M}$ of $M$ is obtained by gluing the positive part
\[
(\partial M \times [1, \infty), d(r\alpha))
\]
of the symplectization of $\partial M$ onto $M$;
\[
\hat{M} := M \cup_{\partial M} (\partial M \times [1, \infty)).
\]

Let $L$ be a compact Lagrangian submanifold $L$ of $M$ such that
• $\theta|_L$ is exact; $\theta|_L = df$,
• $L$ intersects $\partial M$ transversally, and
• $\theta|_L$ vanishes to infinite order along the boundary $\partial L := L \cap \partial M$.

In this setting, the completion
\[
\hat{L} := L \cup_{\partial L} (\partial L \times [1, \infty)),
\]
of $L$ is a Lagrangian submanifold of $\hat{M}$.

A Hamiltonian function $H \in C^\infty(\hat{M})$ is admissible if there are constants $K > 0$, $a > 0$, and $b$ such that
\[
H(x, r) = ar + b, \quad \forall (x, r) \in \partial M \times [K, \infty) \subset \hat{M}.
\]
(A.1)
The constant $a$ is called the slope of $H$. An almost complex structure $J$ on $\hat{M}$ is admissible if it is of contact type outside a compact set;
\[
dr = \theta \circ J, \quad \forall (x, r) \in \partial M \times [K, \infty) \subset \hat{M}.
\]
(A.2)

A Reeb chord of length $w$ is a trajectory $x : [0, w] \to \partial M$ of the flow along $R$ such that $x(0) \in L$ and $x(w) \in L$. An integer Reeb chord is a Reeb chord of integer length. A Hamiltonian chord is defined similarly as the trajectory of the Hamiltonian vector field starting and ending at $L$. If we write the time $t$ Hamiltonian flow as $\varphi_t : \hat{M} \to \hat{M}$, then a Hamiltonian chord of length $w$ corresponds to an intersection point $p \in L \cap \varphi_w(L)$. A Hamiltonian chord is non-degenerate if the corresponding intersection is transversal.

Fix an admissible Hamiltonian $H$ of unit slope. If $\dim M \geq 4$, then by perturbing $L$ by an exact symplectic isotopy if necessary, we may assume [AS10] Lemmas 8.1 and 8.2] that
• there are no integer Reeb chords,
• all integer Hamiltonian chords are non-degenerate, and
• no point of $L$ is both a starting point of an integer Hamiltonian chord and an endpoint of an integer Hamiltonian chord, which may or may not be the same chord.

For an integer $w$, the set of Hamiltonian chords of length $w$ will be denoted by $\mathcal{X}_w$. The set $\mathcal{X}_w$ is finite since all the integer Hamiltonian chords are non-degenerate. The action of $x \in \mathcal{X}_w$ is defined by

$$A_{wH}(x) = \int_0^1 (x^*\theta - wH(x(t))dt) + h(x(1)) - h(x(0)).$$

The Floer complex is defined as the direct sum

$$CF^*(\hat{L}; wH) := \bigoplus_{x \in \mathcal{X}_w} \mathbb{C}[x],$$

equipped with the grading coming from the Maslov index. The differential $\delta$ on $CF(\hat{L}; wH)$ is given by counting solutions to Floer’s equation

$$\left\{ \begin{array}{l}
  u : \mathbb{R} \times [0, 1] \to \bar{M}, \\
  u(\mathbb{R} \times \{0, 1\}) \subset \bar{L}, \\
  \lim_{s \to \pm \infty} u(s, \cdot) = x_{\pm}(\cdot), \\
  \partial_s u + J_x(\partial_t u - wX) = 0
\end{array} \right.$$
The continuation map is defined more generally for a family $H_s$ of admissible Hamiltonians with monotonically decreasing slope, and satisfies the transitive law; if one divides a family $\{H_s\}_s$ from $H_{-\infty}$ to $H_\infty$ smoothly into two, then the diagram

$$
\begin{array}{ccc}
HF(\hat{L}; H_\infty) & \xrightarrow{\varphi} & HF(\hat{L}; H_{-\infty}) \\
\varphi_+ & \downarrow & \varphi_- \\
HF(\hat{L}; H_0)
\end{array}
$$

consisting of continuation maps commutes. We say that a family $\{H_p\}_p$ of admissible Hamiltonians is cofinal if the slope of $H_p$ goes to infinity as $p$ goes to infinity. The wrapped Floer cohomology can also be defined as the limit of Floer cohomologies with respect to any cofinal family of non-degenerate admissible Hamiltonians.

The triangle product on wrapped Floer cohomology is defined by counting solutions of the inhomogeneous Cauchy-Riemann equation

$$
\begin{cases}
  u : S \to \hat{M}, \\
  u(\partial S) \subset \hat{L}, \\
  \lim_{s \to \pm \infty} u(\epsilon(s, \cdot)) = x^k(\cdot), \quad k = 0, 1, 2, \\
  (du_z - X_{u(z)} \otimes \gamma_z) \circ j + J_{z, u(z)} \circ (du_z - X_{u(z)} \otimes \gamma_z) = 0,
\end{cases}
$$

where

- $u^k \in \mathbb{N}$ and $x^k \in \mathcal{X}_{w^k}$ for $k = 0, 1, 2$,
- $S = D^2 \setminus \{\zeta^0, \zeta^1, \zeta^2\}$ is a disk with three points on the boundary removed,
- $\epsilon^0 : (-\infty, 0] \times [0, 1] \to S$ and $\epsilon^{1, 2} : [0, \infty) \times [0, 1] \to S$ are strip-like ends,
- $j$ is the complex structure on $S$,
- $\{J_z\}_{z \in S}$ is a family of almost complex structures on $\hat{M}$,
- $\gamma$ is a one-form on $S$ satisfying
  - $\gamma|_{\partial S} = 0$,
  - $d\gamma < 0$ on $S$,
  - $d\gamma = 0$ in a neighborhood of $\partial S$,
  - $(\epsilon^k)^* \gamma = w^k dt$ on the strip-like ends, and
- $X \otimes \gamma \in \text{Hom}(TS, u^* T\hat{M})$ is obtained by composing $\gamma \in C^\infty(T^* S)$ with $u^* X \in C^\infty(u^* TM)$.

A more careful discussion on the wrapped Floer cohomology can be found in [Rit]. To define higher $A_\infty$-operations, one takes the homotopy colimit of the Floer cochain complex instead of the colimit of the cohomology, and use moduli spaces of stable popsicle maps [AS10].
A.2 Lefschetz fibrations and wrapped Floer cohomology

This section follows McLean [McL09] closely. An exact Lefschetz fibration is a proper map \( \pi : E \to S \) from a compact manifold \( E \) with corners to a compact surface \( S \) with boundary satisfying the following:

- \( \partial E \) consists of the vertical boundary \( \partial_v E := \pi^{-1}(\partial S) \) and the horizontal boundary \( \partial_h E := \partial E \setminus \partial_v E \) meeting in a codimension 2 corner.

- \( \pi \) is a \( C^\infty \)-map with finitely many critical points \( E^\text{crit} \subset E \) with critical values \( S^\text{crit} \subset S \). Every critical point is non-degenerate in the sense that the Hessian is non-degenerate. Different critical points have distinct critical values.

- \( E \) is equipped with a one-form \( \Theta \) such that \( \Omega = d\Theta \) is a symplectic form on \( E \setminus E^\text{crit} \) for every \( s \in S \), where \( E_s := \pi^{-1}(s) \) is the fiber of \( \pi \).

- \( \Omega \) is a Kahler form for \( J_0 \) near \( E^\text{crit} \).

There is a natural connection for \( \pi \) given by the horizontal distribution defined as the \( \Omega \)-orthogonal to the tangent space to the fiber. Parallel transport with respect to this connection gives exact symplectomorphisms between smooth fibers of \( \pi \). We write a smooth fiber of \( \pi \) considered as an abstract exact symplectic manifold as \( F \).

We say that \( E \) is a compact convex Lefschetz fibration if \( (F, \Theta|_F) \) is a Liouville domain. Choose a Louville one-form \( \theta_S \) on the base \( S \). Then there is a constant \( K > 0 \) such that for all \( k \geq K \), one has

- \( \omega := \Omega + k\pi^*\omega_S \) is a symplectic form

- the \( \omega \)-dual \( \lambda \) of \( \theta := \Theta + k\pi^*\theta_S \) is transverse to \( \partial E \) and pointing outward by [McL09 Theorem 2.15]. One can complete a compact convex Lefschetz fibration to a complete convex Lefschetz fibration \( \hat{\pi} : \hat{E} \to \hat{S} \) in a natural way, whose base is the completion \( \hat{S} \) of the base \( S \) and whose fiber is a completion \( \hat{F} \) of the fiber \( F \). The completion \( \hat{E} \) can be partitioned into

- \( E \subset \hat{E} \)

- \( A := F_e \times \hat{S} \) where \( F_e \) is the cylindrical end of \( \hat{F} \), and

- \( B := \hat{E} \setminus (A \cup E) \)
as in [McL09, Figure 1].

The completion \( \hat{E} \) is isomorphic to the completion \( \hat{M} \) of a Liouville domain \( M \), obtained by smoothing out the corner of \( E \). We write the radial coordinates for cylindrical
ends of $E$, $S$ and $F$ as $r_s$, $r_S$ and $r_F$. There exists a positive constant $\varpi$ such that $r_s \leq \varpi r$ and $r_F \leq \varpi r$ by [McL09, Lemma 5.7].

A map $H : \hat{E} \to \mathbb{R}$ is a Lefschetz admissible Hamiltonian if $H|_{A} = \pi^* H_S + \pi^* H_F$ outside some large compact set [McL09, Definition 2.21]. Here $H_S$ and $H_F$ are admissible Hamiltonians on $\hat{S}$ and $\hat{F}$ such that $H_S = 0$ on $S \subset \hat{S}$ and $H_F = 0$ on $F \subset \hat{F}$ respectively [McL09, Page 1905], and $\pi_1 : A = F_e \times \hat{S} \to F_e$ is the first projection.

Let $\gamma : [0, 1] \to S$ be a path on the base such that $\gamma((0, 1)) \subset S \setminus S^{\text{crit}}$. Recall that a Lagrangian submanifold fibered over $\gamma$ is a Lagrangian submanifold $L$ of $E$ obtained as the trajectory of the parallel transport along $\gamma$ of a Lagrangian submanifold $L_s$ in a fiber $E_s = \pi^{-1}(s)$. We assume that $L$ is exact, $L$ intersects $\partial E$ transversally, and $\theta|_L$ vanishes to infinite order along $\partial L$. If an endpoint of $\gamma$ is in the interior of $S$, then it must be a critical value of $\pi$. If exactly one endpoint of $\gamma$ is in the interior of $S$, then $L$ is a Lefschetz thimble. If both endpoints of $\gamma$ are in the interior of $S$, then $L$ is a Lagrangian sphere. The Lagrangian $L \subset M$ can be completed to a Lagrangian $\hat{L} \subset \hat{M}$ by first taking the completion $L_s := L_s \cup_{\partial L_s} ([1, \infty) \times \partial L_s) \subset E_s$ in the fiber direction and then taking its parallel transport along $\hat{\gamma} = \gamma \cup_{\partial \gamma} ([1, \infty) \times \partial \gamma) \subset \hat{S}$. Since $\hat{L} \cap A$ is the product $(\hat{L}_s \setminus L_s) \times \hat{\gamma}$ and $\hat{L} \cap B$ is the product $L_s \times (\hat{\gamma} \setminus \gamma)$, one has a maximum principle which applies to Lefschetz admissible $H$:

**Lemma A.1.** For any Floer trajectory $u : D \to \hat{E}$, the functions $r_S \circ u$ and $r_F \circ u$ do not admit local maxima for large $r_S$ and $r_F$.

This allows one to define the Floer differential and the continuation map, which gives the Lefschetz wrapped Floer cohomology

$$ HW^*_\gamma(\hat{L}) := \lim_{w \to \gamma} HF^*(\hat{L}; wH). $$

The Lefschetz wrapped Floer cohomology $HW^*_\gamma(\hat{L})$ does not depend on the choice of a Lefschetz admissible Hamiltonian just as in the case of the ordinary wrapped Floer cohomology.

**Theorem A.2.** One has an isomorphism

$$ HW^*(\hat{L}) \cong HW^*_\gamma(\hat{L}) $$

of graded rings.

The isomorphism (A.3) is obtained by

$$ HW^*(\hat{L}) \cong \lim_{p} HF^*(-\epsilon, \infty)(\hat{L}; \theta_p) $$

(A.4)

$$ \cong \lim_{p} HF^*_{[-\epsilon, \infty]}(\hat{L}; \theta_p) $$

(A.5)

$$ \cong \lim_{p} HF^*_{[-\epsilon, \infty]}(\hat{L}; K_p) $$

(A.6)

$$ \cong \lim_{p} HF^*_{[-\epsilon, \infty]}(\hat{L}; G_p) $$

(A.7)

$$ \cong \lim_{p} HF^*(\hat{L}; G_p) $$

(A.8)

$$ \cong HW^*_\gamma(\hat{L}), $$

(A.9)
which is an adaptation of the proof of [McL09, Theorem 2.22]. Here \( \varrho_p \) is a Hamiltonian function on \( \hat{M} \) satisfying

(i) \( \varrho_p < 0 \) on \( M \),

(ii) \( \varrho_p \) goes to zero in the \( C^2 \) norm on \( M \) as \( p \) goes to infinity,

(iii) \( \varrho_p \) depends only on the radial coordinate on the cylindrical end;

\[ \varrho_p(x, r) = h_p(r), \quad \forall (x, r) \in \partial M \times [1, \infty) \subset \hat{M}. \]

(iv) \( h'_p(r) \geq 0 \) and \( h''_p(r) \geq 0 \) for all \( r \in [1, \infty) \),

(v) \( h'_p(r) = p \) for \( r \in [2, \infty) \), and

(vi) for any \( K > 0 \) and any \( r \in (1, \infty) \), there is an integer \( N \) such that

\[ rh'_p(r) - h_p(r) > K, \quad \forall p > N. \]

The sequence \( \{ \varrho_p \}_p \) is a cofinal family of admissible Hamiltonians, so that the isomorphism (A.4) comes from the definition of the wrapped Floer cohomology.

The condition (iii) implies that for any \( \epsilon > 0 \), the action of any Hamiltonian chord of \( \varrho_p \) in \( M \) is greater than \( -\epsilon \) for sufficiently large \( p \). The condition (iii) implies that the Hamiltonian vector field of \( \varrho_p \) in the cylindrical end is \( h'_p(r) \) times the Reeb vector field on \( \partial M \). It follows that Hamiltonian chords of length one are in one-to-one correspondence with Reeb chords of length \( h'_p(r) \), and the action of a Hamiltonian chord \( (x, r) : [0, 1] \to M \times [1, \infty) \) is given by

\[ A_{\varrho_p}(x, r) = \int_0^1 (x^* \theta - \varrho(x, r)) dt \]

\[ = rh'_p(r) - h_p(r) + f(x(1)) - f(x(0)). \]

The condition \( \theta|_{\partial \hat{L}} = 0 \) implies \( \theta|_{\hat{L} \setminus L} = 0 \), so that the primitive function \( f(x) = \int_0^x \theta \) is constant on each connected component of \( \hat{L} \setminus L \) and hence bounded. Then the condition (vi) shows that the actions of Hamiltonian chords on the cylindrical end are positive for sufficiently large \( p \). As a result, one obtains the isomorphism (A.5), where \( HF^\ast_{(-\epsilon, \infty)}(\hat{L}; \varrho_p) \) is the subgroup of \( HF^\ast(\hat{L}; \varrho_p) \) generated by chords of action greater than or equal to \( \epsilon \).

The construction of \( K_p \) from \( \varrho_p \) proceeds in two steps [Her00, McL09]: First one modifies \( \varrho_p \) to a Hamiltonian \( \varsigma_p \) which is constant outside a large compact set \( \kappa \) while only adding chords of action less than \( -\epsilon \). Then one adds to \( \varsigma_p \) a Lefschetz admissible Hamiltonian \( L_p \), which is zero in the region \( \kappa \) but has action bounded above, so that Hamiltonian chords of \( K_p := L_p + \varsigma_p \) outside \( \kappa \) have action less than \( -\epsilon \). For a suitable choice of a family of admissible almost complex structures,

- there is a bijection between Hamiltonian chords of \( K_p \) of action greater than \( -\epsilon \) and Hamiltonian chords of \( \varrho_p \), and

- this bijection inducing an isomorphism of the moduli spaces of Floer trajectories.
This gives the isomorphism (A.6). The sequence \( \{G_p\}_p \) is a cofinal family of Lefschetz admissible Hamiltonians such that

- there are sequences \( p_i \) and \( q_i \) of positive integers such that
  \[
  K_{p_i} \leq G_{q_i} \leq K_{p_{i+1}}
  \]
  for all \( i \), and
- all Hamiltonian chords of \( G_p \) have action greater than \(-\epsilon\).

This induces the isomorphisms (A.7) and (A.8). The isomorphism (A.9) comes from the cofinality of \( \{G_p\}_p \) and Theorem A.2 is proved.

### A.3 Fiber products of Lefschetz fibrations

Let \( \pi_1 : E_1 \to S \) and \( \pi_2 : E_2 \to S \) be exact Lefschetz fibrations and consider the fiber product

\[
E := E_1 \times_S E_2
\]

By smoothing corners, we obtain a Liouville domain \( M \) with a Liouville one-form \( \theta = \tau_1^* \Theta_1 + \tau_2^* \Theta_2 + k\pi^* \theta_S \) for sufficiently large \( k \) whose completion \( \hat{M} \) is symplectomorphic to \( \hat{E} := \hat{E}_1 \times_S \hat{E}_2 \). We have fiberwise cylindrical coordinates \( r_{F_i} \), \( i = 1, 2 \) and a cylindrical coordinate \( r_S \) on the base. We say that a Hamiltonian \( H : E \to \mathbb{R} \) is fibered admissible if

\[
H = \pi^* H_S + \tau_1^* H_{F_1} + \tau_2^* H_{F_2}
\]

where

- \( H_S \) is an admissible Hamiltonian on \( \hat{S} \), and
- \( H_{F_i} \) is a Hamiltonian on \( \hat{E}_i \) which is
  - zero on \( E_i \cup B_i \), and
  - a pull-back of an admissible Hamiltonian of \( (F_i)_e \) on \( A_i := (F_i)_e \times \hat{S} \).

Lagrangian submanifolds \( L_i \) of \( E_i \) fibered over a common path \( \gamma : [0, 1] \to S \) gives a Lagrangian submanifold \( L := L_1 \times_\gamma L_2 \) of \( E \), which can be completed to a Lagrangian submanifold \( \hat{L} \) of \( \hat{E} \). Although \( \pi : E \to S \) is not a Lefschetz fibration but a Bott-Morse analog of a Lefschetz fibration, the proof of Theorem A.2 can be adapted in a straightforward way to prove the following:
Theorem A.3. One has an isomorphism

\[ HW^*(\hat{L}) \cong \lim_{\to} HF^*(\hat{L}; pH) \quad (A.10) \]

of rings.

The right hand side does not depend on the choice of a fibered admissible Hamiltonian \( H \), or a cofinal family \( \{H_p\}_p \) of fibered admissible Hamiltonians in general. One starts with a cofinal family \( \{\varrho_p\}_p \) of admissible Hamiltonians, truncate outside a large compact set \( \kappa \) to obtain \( \varsigma_p \), then adds a fibered admissible Hamiltonian \( L_p \) supported outside of \( \kappa \) to obtain \( K_p = \varsigma_p + L_p \). This process can be performed without changing chords with actions greater than \( -\epsilon \), and one obtains the isomorphism (A.10).

A.4 Symplectic cohomology and the bulk-boundary map

In view of McLean’s work, it is also natural to discuss the implication of the calculations in this paper for symplectic cohomology. Our treatment here is less detailed because the discussion which follows is complementary to our main topic.

Theorem A.4. Let \( L \) be an admissible Lagrangian which is also a section of the SYZ fibration for the conifold. Then we have an isomorphism of rings \( SH^0(\hat{M}) \to WF(L) \).

Proof. Using the Lefschetz admissible Hamiltonians \( H \) considered in the main part of this paper, we observe that the orbits of the Hamiltonian vector field occur in \( T^3 \) submanifolds. While these orbits are not isolated, Morse-Bott theory allows one to find a perturbation which has exactly 8 orbits corresponding to generators of \( H^*(T^3) \) for each submanifold of Hamiltonian orbits. Next, we observe that Abouzaid has defined a closed-open ring morphism

\[ CO : SH^*(\hat{M}) \to WF(L). \]

This map is again defined by counting solutions to a perturbed \( J \)-holomorphic curve equation with one interior puncture which is required to be asymptotic to our Hamiltonian orbit. In our setting this map can be completely calculated. More precisely the map sends the class in \( H^0(T^3) \) to the unique Reeb chord of \( L \) in each submanifold worth of orbits.

The non-trivial component of our map corresponds to a low-energy curve, which in the Morse-Bott limit corresponds to the classical intersection \( T^3 \cap L \). In our setting, there are no other curves which contribute to the morphism \( CO \) because such a curve would necessarily preserve the homotopy class of the projection of the chord to \( \mathbb{C}^* \). Thus there can be no non-trivial curves connecting different Morse-Bott submanifolds.

In particular, this map induces an isomorphism

\[ SH^0(\hat{M}) \cong WF(L). \]

Thus, the computations in this paper allow us to compute the \( SH^0 \) piece of symplectic cohomology as well.

Corollary A.5. \( SH^0(\hat{M}) \cong \mathbb{C}[u,v,w_1^{-1},w_2^{-1}]/(uv = (1+w_1)(1+w_2)) \)
Remark A.6. While finishing this paper, we noticed that Pascaleff [Pasb] has very recently proven a similar theorem in his study of log Calabi-Yau surfaces. While our notion of Lagrangian section comes from an SYZ fibration, Pascaleff considers Lagrangian sections of an SYZ fibration defined only in a neighborhood of the compactifying divisor for log Calabi-Yau surfaces. It would be interesting to study the relationship between these approaches in more detail.

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Kwokwai Chan
Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

e-mail address: kwchan@math.cuhk.edu.hk
Daniel Pomerleano  
Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo,  
5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.  
\textit{e-mail address} : daniel.pomerleano@gmail.com

Kazushi Ueda  
Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama  
1-1, Toyonaka, Osaka, 560-0043, Japan.  
\textit{e-mail address} : kazushi@math.sci.osaka-u.ac.jp