Loop Quantum Cosmological Perturbations

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Abstract. In the context of loop quantum cosmology, we provide a complete quantization of an inhomogeneous inflationary model consisting of a perturbed Friedmann-Lemaître-Robertson-Walker universe filled with a minimally coupled massive scalar field. We focus on the case of flat, compact spatial sections. After fixing the local gauge freedom, the kinematical Hilbert space is constructed by combining the representation used in loop quantum cosmology for the homogeneous sector with a preferred Fock quantization of the inhomogeneities. We characterize the physical states annihilated by the quantum Hamiltonian constraint and discuss the evolution of the inhomogeneities in the presence of an emergent relational time.

1. Introduction

Loop Quantum Cosmology (LQC) [1] employs the polymer quantization of Loop Quantum Gravity (LQG) [2] in symmetry-reduced systems. Its successful application to the Friedmann-Lemaître-Robertson-Walker (FLRW) model, in which the big-bang is replaced with a bounce, led to the study of more complicated situations, such as the Bianchi or Gowdy models. In the latter, the combination of a polymer quantization of the gravitational homogeneous degrees of freedom and the standard Fock quantization of the inhomogeneities allowed one to deal with systems with an infinite number of degrees of freedom [3]. In this approximation, usually called hybrid quantization, the effects of quantum geometry are included only in the zero modes. Nevertheless, it provides a rigorous way of constructing a kinematical Hilbert space in which the constraints can be represented.

The inclusion of inhomogeneities in LQC is an inevitable step in order to make contact with cosmological observations. In this work, we study the case of a (scalarly) perturbed FLRW model with a massive scalar field as matter content. As it is well known, this simple model can nonetheless undergo inflation, thus providing a natural arena for the emergence of primordial inhomogeneities which is compatible with observations [4].

2. Classical parametrization

The parametrization of the classical system that we use is essentially that of Halliwell and Hawking [5]. The Arnowitt-Deser-Misner variables are expanded in the eigenbasis of the Laplace-Beltrami operator of a reference metric (the three-torus in the case of a flat FLRW model [6] or the three-sphere for a closed scenario [7]). We only analyze scalar harmonics, although the generalization to vector and tensor ones is straightforward. The zero modes are then identified with the homogeneous variables of an FLRW model, while the inhomogeneities are considered perturbations of them. Hence, the action is truncated at quadratic order in the inhomogeneities.
A Legendre transform leads to a Hamiltonian that is a linear combination of constraints: a global Hamiltonian constraint with quadratic corrections, which appears multiplied by the homogeneous lapse, and two local constraints linear in the perturbations, whose corresponding Lagrange multipliers are the modes of the perturbations of the lapse and shift. These linear constraints are removed with a suitable gauge fixing. In particular, we have considered the so-called longitudinal gauge and the gauge in which the spatial sections have constant curvature [6, 7]. The reduced system is parametrized by the homogeneous variables, the modes of the matter-field perturbation and their canonically conjugate momenta. However, these variables may not have canonical Dirac brackets (this is the case e.g. in the longitudinal gauge). In the truncation of the action at quadratic order in perturbations, a new set of canonical variables can be introduced if the homogeneous variables are corrected with quadratic terms [6, 7].

The Fock quantization of a field theory in a (classical) curved spacetime suffers from an infinite ambiguity in the choice of representation for the canonical commutation relations. It has been shown that in many cosmological scenarios this ambiguity can be eluded by imposing (a) the invariance of the vacuum state under the spatial symmetries of the field equation and (b) the unitary implementation of the field dynamics in the quantum theory. Furthermore, these two requirements select a preferred scaling of the field and a particular canonically conjugate momentum, apart from picking out a unique class of unitarily equivalent Fock representations for them [8]. A representative of this class can be constructed by choosing the annihilation- and creation-like variables that would be natural if the field were massless.

We incorporate these criteria in our quantization so as to recover the privileged quantum field theory in the regimes in which the background behaves classically. In the case under discussion, the matter field perturbation must be scaled with the scale factor of the background metric [9]. With this scaling, the field equation becomes a Klein-Gordon equation with background-dependent mass and subdominant ultraviolet corrections. The privileged momentum coincides—up to subdominant ultraviolet corrections—with the time-derivative of the field. To extend this change of variables to a canonical transformation in the complete phase space (at the considered perturbative order), the homogeneous variables need to be corrected with terms quadratic in the perturbations [9].

It is worth mentioning that an alternative quantization with the same requirements of symmetric vacuum and unitary dynamics can be performed in terms of a gauge-invariant canonical pair [6, 7]. Since the canonical transformation that relates these variables with the scaled field perturbation and its privileged momentum is non-local, while the uniqueness results on the Fock representation commented above apply only to local transformations, the two quantizations might—in principle—be inequivalent. Nevertheless, we have proven that the symplectomorphism is unitarily implementable in the classical background [6, 7], and thus the two quantizations are equivalent in that regime.

3. Quantization of the model

In LQG, the fundamental variables are the holonomies of the Ashtekar-Barbero connection and the fluxes of a densitized triad. In a homogeneous and isotropic model, it suffices to consider their counterparts \( N_\mu = \exp(i \mu c/2) \) (\( \mu \) is proportional to the length of the holonomy) and \( p \). Following the so-called improved dynamics scheme [10], we consider the \( p \)-dependent length \( \bar{\mu} = \sqrt{\Delta p} \), where \( \Delta \) is the minimum non-zero eigenvalue of the area operator in LQG. These variables are represented in the space of square-integrable functions in the Bohr compactification of the real line, \( \mathbb{R}_B \), with respect to its Haar measure \( d\mu_B \). Alternatively, we can choose an orthonormal basis \( \{|v\rangle\}_{v \in \mathbb{R}} \) such that

\[
\hat{p}|v\rangle = \text{sgn}(v)(2\pi\gamma G\hbar \sqrt{\Delta}|v\rangle)^{2/3}|v\rangle, \quad \hat{N}_\mu|v\rangle = |v + 1\rangle.
\]

The constants \( \gamma, G, \) and \( \hbar \) are the Immirzi, Newton, and reduced Planck constants, respectively.
The homogeneous part of the scalar field, $\phi$, is quantized à la Schrödinger, whereas for its perturbation, $\delta\phi$, we adopt the preferred Fock representation described above. We denote the corresponding Fock space by $\mathcal{F}$. As anticipated, the total kinematical Hilbert space is just the product of the three pieces, $L^2(\mathbb{R}_B, d\mu_B) \otimes L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}$. This is the space where the Hamiltonian constraint is to be represented. The structure of the constraint is $C = C_0 + \sum C_2(n)$, where $n$ labels the modes of the Laplace-Beltrami operator. We identify the zeroth-order part of the constraint, $C_0$, with that of the unperturbed system. In the case of a flat universe, the prescription we adopt is

$$\dot{C}_0 = \left[\frac{1}{\sqrt{|p|}}\right]^{3/2} \left[ -\frac{6}{\gamma^2} \hat{\Omega}^2 + 8\pi G (\pi_\phi^2 + m^2 |\hat{p}|^2 \phi^2) \right] \left[\frac{1}{\sqrt{|p|}}\right]^{3/2},$$

where we have introduced the regularized operators

$$\left[\frac{1}{\sqrt{|p|}}\right] = \frac{3}{4\pi\gamma G\hbar\sqrt{\Delta}} \text{sign}(p) \sqrt{|\hat{p}|} (\hat{N}_{-\mu} \sqrt{|\hat{p}|} \hat{N}_\mu - \hat{N}_\mu \sqrt{|\hat{p}|} \hat{N}_{-\mu}),$$

$$\hat{\Omega} = \frac{1}{4i\sqrt{\Delta}} \hat{v}^{1/2} [\text{sign}(v) (\hat{N}_{2\mu} - \hat{N}_{-2\mu}) + (\hat{N}_{2\mu} - \hat{N}_{-2\mu}) \text{sign}(v)] \hat{v}^{1/2}.$$ 

In the case of a closed universe, the constraint can be led to a similar form by means of a unitary transformation [7]. In both cases, the homogeneous constraint is a second-order difference operator and the states with support on the semilattices $\mathcal{L}_\varepsilon = \{|v| = \pm (\varepsilon + 4n) | \varepsilon \in (0, 4], n \in \mathbb{N}\}$ form superselection sectors.

The quadratic part of the constraint contains both homogeneous and inhomogeneous variables. Although at present we do not have at our disposal a systematic method to regularize it, we adopt a prescription that respects the superselection sectors of the unperturbed theory [6, 7].

Once the constraint is represented in the kinematical Hilbert space, one has to find the (generalized) states that it annihilates. At this point, the use of a homogeneous scalar field as a relational time is frequently invoked in LQC. The solutions of the Hamiltonian constraint satisfy an equation of the form

$$-\hbar^2 \partial^2_\phi \Psi = (\hat{\mathcal{H}}_0^2 + (0)\hat{\Theta}_2 + i\hbar (1)\hat{\Theta}_2 \partial_\phi) \Psi,$$

where $\hat{\mathcal{H}}_0$ is a zeroth-order operator (perturbatively), while $(0)\hat{\Theta}_2$ and $(1)\hat{\Theta}_2$ are operators quadratic in the perturbations and that contain no derivatives with respect to $\phi$. If $\phi$ can be used as a relational time, this is an evolution equation that characterizes the physical states.

We can adopt the ansatz

$$\Psi(\phi, v, \delta\phi) = \frac{\chi_0(\phi, v) \psi(\phi, \delta\phi)}{\langle \mathcal{H}_0(\phi) \rangle_{\chi_0}},$$

where $\chi_0$ is a positive-frequency background state (such that $-i\hbar \partial_\phi \chi_0 = \mathcal{H}_0 \chi_0$) and $\langle \bullet \rangle_{\chi_0}$ denotes the expectation value at $\chi_0$ with respect to the inner product in the unperturbed theory. Then, if we neglect the contributions that mix different background states and assume that the characteristic ‘time’ scale of the inhomogeneities is much smaller than that of the unperturbed background, we arrive at the first-order differential equation

$$-i\hbar \partial_\phi \psi = \frac{1}{2} \frac{((0)\hat{\Theta}_2 + (1)\hat{\Theta}_2 \mathcal{H}_0) \chi_0 \psi}{\langle \mathcal{H}_0 \rangle_{\chi_0}}.$$
This equation can be interpreted as corresponding to a quantum field theory in an effective curved spacetime, given by the background state.

Actually, there is in fact no need of adopting a relational time to characterize the physical states. Since the constraint is a difference operator that does not mix the semilattices $L^\pm_\varepsilon$, its solutions can be characterized in terms of their initial data on the minimum volume section of each superselection sector, $|v| = \varepsilon$. Assuming a perturbative expansion for the physical states, $\Psi = \Psi_0 + \Psi_2 + \ldots$, and imposing the constraint, one can see that $\Psi_0$ must be a solution of the unperturbed constraint, whereas $\Psi_2$ must satisfy a similar equation with a source term arising from the action of the inhomogeneities on $\Psi_0$. If we endow the space of initial data with a suitable inner product, we obtain the physical Hilbert space $L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}$.

4. Summary and outlook
We have sketched the complete quantization of an FLRW model with scalar perturbations. The gauge is fixed up to (global) time reparametrizations at the classical level, although a description in terms of gauge-invariants is available and leads to consistent results. The parametrization of the inhomogeneous sector is chosen so that unitary dynamics can be recovered in the limit of quantum field theory in a curved spacetime. In order to maintain canonical Dirac brackets, the homogeneous variables need suitable corrections that are quadratic in the perturbations.

The kinematical Hilbert space has been obtained as the product of the LQC one and a Fock space, corresponding to the homogeneous background and the matter-field perturbation, respectively. However, the constraint operator mixes both spaces in a non-trivial way. Nevertheless, we have proposed two different characterization of the physical states. The first one employs the homogeneous scalar field as a relational time, which allows us to interpret the constraint equation as an evolution equation. The other one determines the states in terms of their initial value on the minimum volume section of each superselection sector.

Of course, the goal is now to confront the predictions of this model with observations. A first approach to do this is provided by the effective dynamics of semiclassical states [1]. In the case of the hybrid quantization of the Gowdy models, the analysis of the effective dynamics showed very interesting features, namely the persistence of the bounce and the statistical amplification of the inhomogeneities through it [11]. These results hold qualitatively in the perturbed FLRW model, and will be the object of a forthcoming publication.

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