Solutions of the Quantum-Yang-Baxter-Equation from Symmetric Spaces

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Abstract

We show that for each semi-Riemannian locally symmetric space the curvature tensor gives rise to a rational solution \( r \) of the classical Yang-Baxter equation with spectral parameter. For several Riemannian globally symmetric spaces \( M \) such as real, complex and quaternionic Grassmann manifolds we explicitly compute solutions \( R \) of the quantum Yang Baxter equations (represented in the tangent spaces of \( M \)) which generalize the quantum \( R \)-matrix found by Zamolodchikov and Zamolodchikov in 1979.

1 Introduction and Results

Since their discoveries not only the classical Yang-Baxter equation (CYBE) but also the quantum Yang-Baxter equation (QYBE) are playing an important rôle in several branches of physics and mathematics. Following Sklyanin, the CYBE can be formulated for an element \( r \in \mathfrak{g} \wedge \mathfrak{g} \) on an abstract Lie algebra \( \mathfrak{g} \) and takes the form

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 .
\]

We use the standard notation from the theory of Hopf algebras, where subscripts describe on which places in an \( n \)-fold tensor product the quantities in question act. We refer to solutions of

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Solutions of this equation are commonly called quantum $R$-matrices. For a precise treatment see the important work of Drinfel’d [3] or the standard literature on quantum groups (e.g. [2] and references therein). As it occurs in statistical models for example, these equations possibly depend on additional spectral parameters.

A special case of interest are quantum-$R$-matrices which can be written as a deformation of classical $r$-matrices:

$$R = \text{id} + \hbar r + o(\hbar^2).$$

What we are actually looking for are quantum $R$-matrices fulfilling this relation for some given classical $r$-matrices.

The classical $r$-matrices we have in mind are arising from locally symmetric spaces. It can be shown that curvature tensor of these spaces obey CYBE with spectral parameter (theorem 2.1). This is because there exists a Casimir element of a reductive Lie algebra $\mathfrak{k}$ of the isotropy group $K$ of a symmetric space obeying CYBE. The curvature tensor is exactly this Casimir element being represented in the tangent space of the symmetric space (theorem 2.2). The problem is now how to obtain a deformation in the sense of (2) fulfilling QYBE for a representation of given $\mathfrak{k}$ in the tangent space. It turns out that a heuristic ansatz in quantities one can associate to a symmetric space in a canonical manner yield such solutions (section 3). Using this we explicitly calculate for some of the symmetric spaces which has been classified in [4] quantum $R$-matrices. In particular for spaces of constant sectional curvature, projective spaces and Grassmann manifolds the ansatz turned out to be successful (see section 4).

### 2 Symmetric Spaces and CYBE

Let us briefly recall the geometric background necessary for our considerations (see e.g. [4] and [7] for more details):

A semi-Riemannian differentiable manifold $(M, g)$ is called locally symmetric iff around every point $m$ the geodesic symmetry $(x \mapsto -x$ in an exponential chart) is a local isometry. An equivalent criterion is the statement that the curvature tensor $R$ of its Levi-Civita connection $\nabla$ is covariantly constant, $\nabla R = 0$. Standard examples are the $n$-spheres, but also all Riemann surfaces equipped with the standard constant curvature metric. A semi-Riemannian locally symmetric space is called globally symmetric or symmetric iff the geodesic symmetry around every point extends to a global isometry of $M$. This is for instance no longer the case for Riemann surfaces of genus $\geq 2$. Semi-Riemannian globally symmetric spaces are known to be homogeneous under its Lie group of all isometries whence they admit a group theoretic representation $M = G/K$ where $G$ is a Lie group acting isometrically on $M$ and $K$ is the closed subgroup of all those transformations of $G$ fixing a chosen point $p$. Consider a pair $(G, K)$ of a Lie group $G$ and a closed subgroup $K$ with its associated pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$. $(G, K)$ is called a semi-Riemannian symmetric pair iff i) there is an involutive automorphism $\sigma$ of $G$ such that $K$ is a subgroup of the group $G_\sigma$ of all the fixed points of $\sigma$ and contains the identity component of $G_\sigma$, and ii) there is a nondegenerate symmetric bilinear form $( , )$ on $\mathfrak{m}$, the eigenspace associated to the eigenvalue $-1$ of $\theta := T_p\sigma$, which is invariant under $\theta$ and the restriction of the adjoint representation of $G_\sigma$ to $\mathfrak{m}$. Note that $\mathfrak{m}$ is identified with the tangent space $T_pM$ and that the Lie subalgebra $\mathfrak{k}$ is the eigenspace associated
to the eigenvalue +1 of θ and that θ is an involutive automorphism of the Lie algebra g. There are the well-known commutation relations:

\[ [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}. \quad (3) \]

Such a pair (g, ℓ) will be also called symmetric pair for short. If \( M = G/K \) is a Riemannian globally symmetric space then the pair can be chosen in such a way that K is a compact subgroup of G.

The most known examples of symmetric spaces are spaces of constant sectional curvature (like the sphere \( S^n \) and the hyperbolic space \( H^n \)), the complex and quaternionic projective spaces \( \mathbb{C}P^n \) and \( \mathbb{H}P^n \), and the Grassmann manifolds \( SO(p+q)/SO(p) \times SO(q) \), \( SU(p+q)/SU(p) \times U(q) \) and \( Sp(p+q)/Sp(p) \times Sp(q) \).

We are going to prove in two ways that the curvature tensor of a symmetric space fulfills CYBE. The first one is based on direct differential geometric calculations, the second will make this fact obvious for group theoretical reasons.

Let \((M, g)\) a Riemannian manifold with metric tensor \(g\) and Levi-Civita connection \(\nabla\). For the second covariant derivative (along two vector fields \(X, Y\) on \(M\)) the equation (3) holds

\[ \nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}. \quad (4) \]

is valid. Therefore we obtain for any \((r, s)\)-tensor field \(T \in \mathcal{T}_r^s(M)\) on \(M\) that its antisymmetric part of the second covariant derivative obeys

\[ (\nabla^2_{XY} - \nabla^2_{YX}) T = R(X, Y) \circ T, \quad (5) \]

or in local coordinates (with \(R_{klmn}^{i} \frac{\partial}{\partial x^i} := R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})\frac{\partial}{\partial x^m}\)):

\[ T_{i_1 \ldots i_r}^{i_1 \ldots i_r} - T_{j_1 \ldots j_s}^{j_1 \ldots j_s} = \sum_{\alpha} R_{klmn}^{i} T_{j_1 \ldots j_s}^{i_1 \ldots i_r} - \sum_{\alpha} R_{klmn}^{m} T_{j_1 \ldots j_s}^{i_1 \ldots i_r}. \quad (6) \]

We are now able to formulate the following theorem.

**Theorem 2.1** Let \(M\) be a semi-Riemannian locally symmetric space, \(R_{ijkl}\) the components of the curvature tensor. Then, at each point \(p \in M\),

\[ \hat{r} := \frac{1}{\lambda_1 - \lambda_2} R^{k}_{\ i} E^{j}_{\ i} \otimes E^{l}_{\ k} \in \text{End}(T_p M) \otimes \text{End}(T_p M) \otimes \mathbb{C}(\lambda_1, \lambda_2) \quad (7) \]

is a rational solution of the CYBE with spectral parameters.

**Proof:** The fact that \(R\) is covariant constant is essential for the proof. Inserting \(R\) in equation (3) and rearranging the indices by using the metric, one finds

\[ R^{i}_{\ m} R^{k}_{\ l} R^{n}_{\ r s} - R^{m}_{\ n} R^{i}_{\ k} R^{n}_{\ l s} + R^{i}_{\ m} R^{k}_{\ n} R^{r}_{\ l s} - R^{i}_{\ n} R^{m}_{\ l} R^{r}_{\ k s} = 0. \quad (8) \]

This is exactly the condition for the vanishing of the commutators one has to calculate in CYBE with (6) as classical \(r\)-matrix.

This key observation was the starting point of our considerations. For semi Riemannian globally symmetric spaces this theorem can also be proven by group theoretic methods using representation theory of the Casimir element.

Let \(M\) be a globally symmetric space, \((g, \ell)\) be the associated symmetric pair to \(M \cong G/K\) and \(\rho\) the adjoint representation of \(\ell\) on \(m\). Suppose \(\ell\) is a reductive Lie algebra, i.e. there exists a
nondegenerate, symmetric and invariant bilinear form on \( \mathfrak{k} \) for which we can define the corresponding Casimir element \( t \). Identifying \( \mathfrak{k} \) with its dual via the bilinear form, and introducing an orthonormal basis \( \{e_\alpha\} \), we can write \( t = \sum e_\alpha \otimes e_\alpha \). This element gives rise to rational solutions of CYBE with spectral parameters (cf. [3]). Under these assumptions we can show the following theorem which was also proven in [3].

**Theorem 2.2** The curvature tensor \( R \) of a locally symmetric space is the in \( \mathfrak{m} \cong T_p M \) represented Casimir element \( t \),

\[
R(p) = (\rho \otimes \rho)(t) .
\]

**Proof:** The curvature tensor of a symmetric space at a point \( p \in M \) can be written as

\[
(R(X, Y)Z)(p) = -[[X, Y], Z](p)
\]

(\( \forall X, Y, Z \in \mathfrak{m} \)). Then the statement follows from the fact that there exists an \( \text{Ad}K \)-invariant, non-degenerate scalar product on \( \mathfrak{g} \) and can be easily worked out using an orthonormal basis. \( \square \)

From this point of view theorem 2.1 becomes obvious since the solutions of CYBE obtained there are just because of the special representation we have chosen.

### 3 Quantum-\( \mathcal{R} \)-Matrices associated to Symmetric Pairs

In this section we will shortly explain the main ideas how to obtain solutions of the QYBE from the curvature tensor of a symmetric space. We assume to have the following ingredients, where \( \mathfrak{g} \) denotes a finite dimensional Lie algebra over the field \( \mathbb{K} \):

i.) On \( \mathfrak{g} \) there exists an invariant, symmetric, nondegenerate bilinear form \( \kappa \).

ii.) There exists an involutive automorphism \( \theta : \mathfrak{g} \to \mathfrak{g} \) splitting the Lie Algebra into a direct sum of vector spaces \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) with commutations relations (2).

iii.) We have a faithful representation \( \rho : \mathfrak{g} \to \mathcal{A} \) in some associative algebra \( \mathcal{A} \).

Because of i.) we can identify \( \mathfrak{g} \) with its dual and get the isomorphism (\( \forall X, Y, Z \in \mathfrak{g} \))

\[
\Phi : \mathfrak{g} \otimes \mathfrak{g} \to \text{End}(\mathfrak{g}) , \\
\Phi(X \otimes Y)(Z) = X\kappa(Y, Z) .
\]

Using ii.) and iii.) we define the following objects.

**Definition 3.1** Under above assumptions define the elements:

\[
C := \Phi^{-1}(\text{id}) \in \mathfrak{g} \otimes \mathfrak{g} , \\
G := \Phi^{-1}(\theta) \in \mathfrak{g} \otimes \mathfrak{g}
\]

(10)

and their pendants represented in \( \mathcal{A} \):

\[
\hat{C} := (\rho \otimes \rho)C \in \mathcal{A} \otimes \mathcal{A} , \\
\hat{G} := (\rho \otimes \rho)G \in \mathcal{A} \otimes \mathcal{A}
\]

(11)
Since it is easily verified that $C$ is the so-called quadratic Casimir element of $\mathfrak{g}$ and is therefore $\mathfrak{g}$-invariant, it obeys the CYBE with spectral parameters (see e.g. [3]). Although the sum of $C$ and $G$ is not $\mathfrak{g}$- but $\mathfrak{k}$-invariant it yields a solution of this equation (cf. e.g. [3], where all solutions of CYBE with spectral parameters depending on $C$ and $G$ have been classified.) by setting

$$
r := \frac{C + G}{\lambda_1 - \lambda_2} \in \mathfrak{k} \otimes \mathfrak{k} \otimes \mathbb{C}(\lambda_1, \lambda_2).$$

(12)

The classical $r$-matrix defined above has universal character because it is defined in the abstract Lie algebra $\mathfrak{k}$ and does not depend on any representation. For that reason it is clear that we obtain matrix solutions $\hat{r} = (\rho \otimes \rho)r \in A \otimes A \otimes \mathbb{C}(\lambda_1, \lambda_2)$ for any representation $\rho$.

In order to maintain the aspect that the classical $r$-matrix corresponds the curvature tensor we choose $m$ as representation space for $\mathfrak{g}$, i.e. $A = \text{End}(m)$. We are actually looking for $\mathcal{R}$-matrices obeying the QYBE, which can be written as a deformation of the classical $r$-matrix $\hat{r}$:

$$
\hat{R} = \text{id} + \hbar \hat{r} + \sum_{k=2}^{\infty} h^k \hat{R}^{(k)}, \quad \hat{R}^{(k)} \in \text{End}(m) \otimes \text{End}(m) \otimes \mathbb{C}(\lambda_1, \lambda_2),
$$

(13)

where $\hbar$ is an arbitrary parameter. The aim is to find explicit expressions for the quantities $\hat{R}^{(k)}$, where we make –guided by the classical solution– a heuristic ansatz, that the solutions (13) of QYBE can be written as

$$
\hat{R} = \text{id} + a(\lambda_1 - \lambda_2, \hbar)\hat{G} + b(\lambda_1 - \lambda_2, \hbar)\hat{C},
$$

(14)

where $a, b \in \mathbb{C}(\lambda_1, \lambda_2)[[\hbar]]$ have to be determined. One should keep in mind that this ansatz is by no means unique, but in some cases successful as we show below.

However, in general it is not possible to realize the whole Lie algebra $\mathfrak{g}$ on the vector space $m$, whereas the Lie subalgebra $\mathfrak{k}$ is canonical represented by the adjoint representation. This rather technical problem can be avoided if we introduce a new symmetric pair $(\tilde{\mathfrak{g}}, \mathfrak{k})$ with an involutive automorphism $\tilde{\theta}$ such that there exists a representation of $\tilde{\mathfrak{g}}$ on $m$. Therefore, one has to replace $\theta$ by $\tilde{\theta}$ in the above definition and the resulting $\tilde{C}$ and $\tilde{G}$ constitute the $\mathcal{R}$-matrix for which we have to find the coefficients $a$ and $b$.

4 Examples

We will present examples (cf. [3]) for which we could successfully determine the coefficients for solutions of (14). Instead of proving everything in detail we just sketch the proof. Taking equation (13) and inserting it in QYBE, we obtain restrictions for the coefficients, which can be solved. The other way around, one easily but lengthy checks by direct calculations that the given $\mathcal{R}$-matrices fulfill the QYBE.

In the succeeding examples we use the following notation. By $E_{ij}$ we denote the elementary matrices of given dimension having a 1 in the $i$-th row and $j$-th column and being 0 elsewhere. All vector spaces and their tensor products are considered over the reals. For $\rho$ we choose the usual matrix multiplication.

i.) Spaces of constant curvature. $k$ takes the value 1, 0, −1 depending whether it is a space of
positive, zero or negative sectional curvature.

$\tilde{g} = \mathfrak{gl}(n, \mathbb{R})$

$\mathfrak{t} = \mathfrak{so}(n)$

$m = \mathbb{R}^n$

$\tilde{\theta}(X) = -X^T$

$\kappa(X, Y) = \text{tr } (\rho(X)\rho(Y))$

$\hat{G} = -E_{ij} \otimes E_{ij}$

$\hat{C} = E_{ij} \otimes E_{ji}$

$a = \frac{\hbar}{\lambda_1 - \lambda_2 + n\hbar}$

$b = \frac{\hbar}{\lambda_1 - \lambda_2}$

In order to compute the coefficients $a$ and $b$ we made use of the following identities satisfied by $\hat{C}$ and $\hat{G}$ which are readily verified:

(a) $\hat{G}_{12}\hat{G}_{23} = -\hat{C}_{13}\hat{G}_{23} = -\hat{G}_{12}\hat{C}_{13} = -\hat{G}_{12}\hat{G}_{13}\hat{G}_{23} = 1/n\hat{G}_{12}\hat{C}_{13}\hat{G}_{23} = -\hat{G}_{12}\hat{C}_{13}\hat{C}_{23}$

(b) $\hat{G}_{23}\hat{G}_{12} = -\hat{G}_{23}\hat{C}_{13} = -\hat{G}_{23}\hat{G}_{13}\hat{G}_{12} = 1/n\hat{G}_{23}\hat{C}_{13}\hat{G}_{12} = -\hat{C}_{23}\hat{C}_{13}\hat{C}_{12}$

(c) $\hat{C}_{12}\hat{C}_{13}\hat{G}_{23} = \hat{G}_{23}\hat{G}_{13}\hat{C}_{12} = -\hat{G}_{23}$

(d) $\hat{G}_{12}\hat{G}_{13}\hat{G}_{23} = \hat{C}_{23}\hat{G}_{13}\hat{G}_{12} = -\hat{G}_{12}$

(e) $\hat{C}_{12}\hat{C}_{13}\hat{C}_{12} = \hat{C}_{23}\hat{C}_{13}\hat{C}_{12} = \hat{C}_{13}$

(f) $\hat{G}_{13}\hat{G}_{23} = -\hat{C}_{12}\hat{G}_{23} = -\hat{G}_{13}\hat{C}_{12}$

(g) $\hat{G}_{12}\hat{G}_{13} = -\hat{C}_{23}\hat{G}_{13} = -\hat{C}_{12}\hat{C}_{23}$

(h) $\hat{G}_{13}\hat{G}_{12} = -\hat{C}_{23}\hat{G}_{12} = -\hat{G}_{13}\hat{C}_{23}$

(i) $\hat{G}_{23}\hat{G}_{13} = -\hat{C}_{12}\hat{G}_{13} = -\hat{C}_{23}\hat{C}_{12}$

(j) $\hat{C}_{12}\hat{C}_{13} = \hat{C}_{23}\hat{C}_{12} = \hat{C}_{13}\hat{C}_{23}$

(k) $\hat{C}_{13}\hat{C}_{12} = \hat{C}_{12}\hat{C}_{23} = \hat{C}_{23}\hat{C}_{13}$

The solution for the sphere $S^n$, i.e. $k = 1$, is already presented in $\mathfrak{f}$ and was originally computed by $\mathfrak{f}$.

ii.) The complex projective space $\mathbb{C}P^n$. 

$\tilde{g} = \mathfrak{gl}(n, \mathbb{C})$

$\mathfrak{t} = \mathfrak{u}(n)$

$m = \mathbb{C}^n$

$\tilde{\theta}(X) = -X^\dagger$

$\kappa(X, Y) = \Re \text{tr } (\rho(X)\rho(Y))$

$\hat{G} = -E_{ij} \otimes E_{ij} - iE_{ij} \otimes iE_{ij}$

$\hat{C} = E_{ij} \otimes E_{ji} - iE_{ij} \otimes iE_{ji}$

$a = \frac{\hbar}{\lambda_1 - \lambda_2 + n\hbar}$

$b = \frac{\hbar}{\lambda_1 - \lambda_2}$
Here the identities (c)–(k) of the preceding example remain whereas (a) and (b) are replaced by:

\[(a) \hat{G}_{12} \hat{G}_{23} = -\hat{C}_{13} \hat{G}_{23} = -\hat{G}_{12} \hat{C}_{13} = 1/(2n)\hat{G}_{12} \hat{C}_{13} \hat{G}_{23},
\]

\[(b) \hat{G}_{23} \hat{G}_{12} = -\hat{G}_{23} \hat{C}_{13} = -\hat{C}_{13} \hat{G}_{12} = 1/(2n)\hat{G}_{23} \hat{C}_{13} \hat{G}_{12},
\]

iii.) The quaternionic projective space $\mathbb{HP}^n$.

\[\tilde{g} = \mathfrak{gl}(n, \mathbb{H})\]
\[\tilde{f} = \mathfrak{sp}(n)\]
\[m = \mathbb{H}^n\]
\[\tilde{\theta}(X) = -X^T\]
\[\kappa(X, Y) = \text{Sc} \text{tr} (\rho(X)\rho(Y))\]
\[\hat{G} = -E_{ij} \otimes E_{ij} - iE_{ij} \otimes iE_{ij} - jE_{ij} \otimes jE_{ij} - kE_{ij} \otimes kE_{ij}\]
\[\hat{C} = E_{ij} \otimes E_{ji} - iE_{ij} \otimes iE_{ji} - jE_{ij} \otimes jE_{ji} - kE_{ij} \otimes kE_{ji}\]
\[a = \frac{\lambda_1 - \lambda_2}{(2n+2)\hbar}\]
\[b = \frac{\lambda_1 - \lambda_2}{\hbar}\]

Here again the identities (c)–(k) of the preceding example remain whereas (a) and (b) are replaced by:

\[(a) \hat{G}_{12} \hat{G}_{23} = -\hat{C}_{13} \hat{G}_{23} = -\hat{G}_{12} \hat{C}_{13} = (1/2)\hat{G}_{12} \hat{C}_{13} \hat{G}_{23} = 1/(4n)\hat{G}_{12} \hat{C}_{13} \hat{G}_{23} = (1/2)\hat{G}_{12} \hat{C}_{13} \hat{G}_{23} = (1/2)\hat{G}_{12} \hat{C}_{13} \hat{G}_{23}\]

\[(b) \hat{G}_{23} \hat{G}_{12} = -\hat{G}_{23} \hat{C}_{13} = -\hat{C}_{13} \hat{G}_{12} = (1/2)\hat{G}_{23} \hat{C}_{13} \hat{G}_{12} = 1/(4n)\hat{G}_{23} \hat{C}_{13} \hat{G}_{12} = (1/2)\hat{G}_{23} \hat{C}_{13} \hat{G}_{12} = (1/2)\hat{G}_{23} \hat{C}_{13} \hat{G}_{12}\]

After having introduced solutions for projective spaces the following three examples we encounter unfortunately possess a less geometric meaning. Let $p, q \in \mathbb{N}$ and set $P = \{1, \ldots, p\}$, $Q = \{p + 1, \ldots, p + q\}$. We do not indicate the identities for $\hat{C}$ and $\hat{G}$ (as in the preceding examples); they are computed similarly.

iv.) The symmetric pair $(\mathfrak{gl}(p + q, \mathbb{R}), \mathfrak{gl}(p) \times \mathfrak{gl}(q))$. Defining

\[I_{p,q} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}\]
where \(1_p\) respectively \(1_q\) is the \(p\)-respectively \(q\)-dimensional identity matrix, we find

\[
\tilde{g} = \mathfrak{gl}(p + q, \mathbb{R})
\]
\[
\mathfrak{t} = \mathfrak{gl}(p) \times \mathfrak{gl}(q)
\]
\[
m = \mathbb{R}^{(p+q)}
\]
\[
\tilde{\theta}(X) = I_{p,q}XI_{p,q}
\]
\[
\kappa(X, Y) = \text{tr} (\rho(X)\rho(Y))
\]
\[
\hat{G} = \sum_{i,j \in P} + \sum_{i,j \in Q} - \sum_{j \in P} + \sum_{j \in Q} E_{ij} \otimes E_{ji}
\]
\[
\hat{C} = \sum_{i,j \in P \cup Q} E_{ij} \otimes E_{ji}
\]
\[
a = \frac{b}{\lambda_1 - \lambda_2}
\]
\[
b = \frac{b}{\lambda_1 - \lambda_2}
\]

v.) The symmetric pair \((\mathfrak{gl}(p + q), \mathfrak{so}(p, q))\).

\[
\tilde{\mathfrak{g}} = \mathfrak{gl}(p + q)
\]
\[
\mathfrak{t} = \mathfrak{so}(p, q)
\]
\[
m = \mathbb{R}^{(p+q)}
\]
\[
\tilde{\theta}(X) = -I_{p,q}X^TI_{p,q}
\]
\[
\kappa(X, Y) = \text{tr} (\rho(X)\rho(Y))
\]
\[
\hat{G} = \sum_{i,j \in P} - \sum_{i,j \in Q} + \sum_{i \in P} + \sum_{j \in P} E_{ij} \otimes E_{ij}
\]
\[
\hat{C} = \sum_{i,j \in P \cup Q} E_{ij} \otimes E_{ji}
\]
\[
a = \frac{b}{\lambda_1 - \lambda_2 + \frac{p+q}{2}\hbar}
\]
\[
b = \frac{b}{\lambda_1 - \lambda_2}
\]

vi.) The symmetric pair \((\mathfrak{gl}(2n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{C}))\). Introducing

\[
I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

we get the following solution of QYBE.

\[
\tilde{\mathfrak{g}} = \mathfrak{gl}(2n, \mathbb{R})
\]
\[
\mathfrak{t} = \mathfrak{gl}(n, \mathbb{C})
\]
\[
m = \mathbb{R}^{2n}
\]
\[
\tilde{\theta}(X) = IXI
\]
\[
\kappa(X, Y) = \text{tr} (\rho(X)\rho(Y))
\]
\[
\hat{G} = \sum_{i,j}^{n} E_{i+n,j+n} \otimes E_{j,i} + E_{i,j} \otimes E_{j+n,i+n} - E_{i,j+n} \otimes E_{j,i+n} - E_{i+n,j} \otimes E_{j+n,i}
\]
\[
\hat{C} = \sum_{i,j=1}^{2n} E_{ij} \otimes E_{ji}
\]
\[
a = \frac{b}{\lambda_1 - \lambda_2}
\]
\[
b = \frac{b}{\lambda_1 - \lambda_2}
\]
All examples considered so far have in common that they make use of the trick mentioned in the last section. The $\mathcal{R}$-matrices are unitary up to a factor as readily verified and could been determined by the ansatz (14). Moreover, expanding the coefficients $a$ into geometric series and comparing it with (13) we can read off explicitly the $\hat{\mathcal{R}}(k)$.

The next examples we would like to discuss are Grassmann manifolds. Here our ansatz fails. Nevertheless we immediately see, that the solutions we obtained for projective spaces yield solutions of QYBE for these spaces. We will concentrate on the real Grassmann manifolds of the form $SO(p+q)/S(O(p) \times O(q))$ since the complex and quaternionic cases can be treated in a similar way. The corresponding Lie algebra $\mathfrak{g} = \mathfrak{so}(p+q)$ splits into

$$\mathfrak{k} = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \mid A \in \mathfrak{so}(p), D \in \mathfrak{so}(q) \right\} \cong \mathfrak{so}(p) \times \mathfrak{so}(q),$$

$$\mathfrak{m} = \left\{ \left( \begin{array}{cc} 0 & B \\ -B^T & 0 \end{array} \right) \right\} \mid B \text{ arbitrary } (p \times q)\text{-matrix} \cong \text{End}(\mathbb{R}^q, \mathbb{R}^p).$$

The canonical representation of $\mathfrak{k}$ on $\mathfrak{m}$ is given by

$$\rho : \mathfrak{k} \rightarrow \text{End}(\mathfrak{m}),$$

$$(15) \quad \rho(A, D)(B) := AB - BD$$

Let us recall the solution we found in our first example i.) for different dimensions $p$ and $q$. Namely, we get

$$\mathcal{R}^p = \text{id} + \frac{\hbar}{\lambda_1 - \lambda_2 + \mu^2 \hbar} \hat{G}_p + \frac{\hbar}{\lambda_1 - \lambda_2} \hat{C}_p,$$

$$\mathcal{R}^q = \text{id} + \frac{\hbar}{\lambda_1 - \lambda_2 + \nu^2 \hbar} \hat{G}_q + \frac{\hbar}{\lambda_1 - \lambda_2} \hat{C}_q.$$

We immediately can formulate

**Theorem 4.1** Let $M = SO(p+q)/S(O(p) \times O(q))$ and $\mathcal{R}$ the in $\mathfrak{m} \cong T_p M$ represented Casimir element. Let $v \in \mathfrak{m} \otimes \mathfrak{m}$ be an arbitrary element. Then

$$\hat{\mathcal{R}}(v) := \mathcal{R}^p v \mathcal{R}^q$$

is a solution of the QYBE whose leading terms in the power series expansion in the parameter $\hbar$ take the form

$$\hat{\mathcal{R}} = \text{id} + \frac{\hbar}{\lambda - \mu} \hat{t} + \frac{\hbar^2}{(\lambda - \mu)^2} \left( \frac{1}{2} \hat{t}^2 - \text{id} \right) + \mathcal{O}(\hbar^3).$$

**Proof:** The first assumption is obvious, since $\mathcal{R}^p$ and $\mathcal{R}^q$ are solutions of QYBE themselves. We then obtain

$$\hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{13} \hat{\mathcal{R}}_{23}(v) = \mathcal{R}^p_{12} \mathcal{R}^p_{13} \mathcal{R}^p_{23} v \mathcal{R}^q_{23} \mathcal{R}^q_{13} \mathcal{R}^q_{12}$$

$$= \mathcal{R}^q_{23} \mathcal{R}^p_{13} \mathcal{R}^p_{12} v \mathcal{R}^q_{12} \mathcal{R}^q_{13} \mathcal{R}^q_{23}$$

$$= \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{13} \hat{\mathcal{R}}_{12}(v).$$

The second part can be proven by straightforward calculation. \[\square\]

In the same manner we find the solutions for the remaining Grassmann manifolds there the classical $r$-matrix corresponds to the curvature tensor.
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