Classification of 3-dimensional integrable scalar discrete equations

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Abstract

We classify all integrable 3-dimensional scalar discrete quasilinear equations $Q_3 = 0$ on an elementary cubic cell of the lattice $\mathbb{Z}^3$. An equation $Q_3 = 0$ is called integrable if it may be consistently imposed on all 3-dimensional elementary faces of the lattice $\mathbb{Z}^3$. Under the natural requirement of invariance of the equation under the action of the complete group of symmetries of the cube we prove that the only nontrivial (non-linearizable) integrable equation from this class is the well-known dBKP-system.

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1 Introduction

Although definitively shaped a decade ago, discrete differential geometry (see e.g. [3, 4]) has already provided much insight into structures that are fundamental both to classical differential geometry and to the theory of integrable PDEs. In addition to such purely mathematical fields, results in discrete differential geometry have a great potential in computer graphics and architectural design: it turns out that discrete surfaces parameterized by discrete conjugate lines and discrete curvature lines — the basic structures in discrete differential geometry — have superior approximation properties and other useful features (see [9]).

In this paper we consider the cubic lattice $\mathbb{Z}^n$ with vertices at integer points in the $n$-dimensional space $\mathbb{R}^n = \{(x_1, \ldots, x_n) | x_s \in \mathbb{R}\}$. With each vertex (with integer coordinates $(i_1, \ldots, i_n)$, $i_s \in \mathbb{Z}$) we associate a scalar field variable $f_{i_1 \ldots i_n} \in \mathbb{C}$.

In what follows we need to consider the elementary cubic cell $K_n = \{ (i_1, \ldots, i_n) | i_s \in \{0, 1\} \}$ of the lattice $\mathbb{Z}^n$. The field variables $f_{i_1 \ldots i_n}$ are associated to its $2^n$ vertices. We will use the short notation $\mathbf{f}$ for the set $(f_{00\ldots0}, \ldots, f_{11\ldots1})$ of all these $2^n$ variables.

An $n$-dimensional discrete system of the type considered here is given by an equation of the form

$$Q_n(\mathbf{f}) = 0,$$

on the field variables on the elementary cubic cell $K_n$. For the other elementary cubic cells of $\mathbb{Z}^n$ the equation is the same, after shifting the indices of $\mathbf{f}$ suitably (see Figure 1).

In the last two decades the study of special classes of (1) which are “integrable” (in one sense or another) has become very popular. We give below only a brief account of the current state of this field of research, for a more detailed account cf. [1]–[4] and the references given therein. In fact, discrete integrable systems underlie many classical integrable nonlinear PDEs, like the Krichever-Novikov equation and other examples,
the latter appear as a continuous limit along some of the discrete directions. Other well known classes of integrable geometric objects (with \( n = 3 \)), like minimal surfaces, conjugate nets, constant curvature surfaces, Moutard nets, isothermic surfaces, orthogonal curvilinear coordinates etc., are also obtained as some smooth limits along any two of the three directions of the respective discrete system. The remaining third discrete direction automatically provides us with a transformation known in the classical continuous geometric context as Jonas/Ribaucour/Bäcklund transformation between surfaces of the given class (see [1]–[4] for more details). On the other hand, starting from the classical theorems on non-linear superposition principles and permutability of the aforementioned transformations between smooth surfaces of one of these types we obtain precisely the underlying discrete system. One of the cornerstones of the discrete differential geometry (the idea to look for cubic nonlinear superposition formulas of Bäcklund transformations of nonlinear integrable PDEs) was laid down in [5]. The duality between the smooth objects in any of the geometric classes of integrable smooth surfaces mentioned above and their Bäcklund-type transformations is therefore put into a symmetric form of a single discrete \( n \)-dimensional system and is encoded as the notion of \((n + 1)\)-dimensional consistency [3]:

An \( n \)-dimensional discrete equation (1) is called consistent, if it may be imposed in a consistent way on all \( n \)-dimensional faces of a \((n + 1)\)-dimensional cube.

This can be also understood as the possibility to take \( \mathbb{Z}^{n+1} \) and prescribe the \( n \)-dimensional equation (1) to hold on every \( n \)-dimensional face of every elementary \((n + 1)\)-dimensional cube (of size 1, with edges parallel to the coordinate axes) without side relations to appear. For this reason (1) is often called a “face formula”. A precise definition of consistency, suitable for the class of discrete equations treated in this paper, will be formulated in the next section.

This paper is devoted to application of computer algebra systems \textsc{Reduce} and \textsc{FORM} ([10]), in particular the \textsc{Reduce} package \textsc{Crack} ([11, 12]), to the classification of 3-dimensional integrable discrete systems.

The paper is organized as follows. In Section 2 we give a brief description of the known results on 2-dimensional integrable scalar discrete equations of type (1) and the precise definition of \((n + 1)\)-dimensional consistency condition for such discrete \( n \)-dimensional systems.

Section 3 is devoted to the classification of symmetry types of quasilinear equations (1) for dimensions \( n = 2, 3, 4 \).

In Section 4 we describe the results of our computations (Theorem 2): the only nontrivial (non-linearizable) integrable scalar quasilinear 3d-face equation invariant w.r.t. the complete group of symmetries of the cube is given by the formula (5) below.

Appendices A–E describe the technical details of the computations.
2 The setup

The simplest but very important class of 2-dimensional integrable face formulas was investigated in detail in [1, 2]. They have the form

\[ Q(f_{00}, f_{10}, f_{01}, f_{11}) = 0, \]  

where \( f_{ij} \) are scalar fields attached to the vertices of a square (see Fig. 2) with two main requirements:

1) **Quasilinearity.** \( Q \) is affine linear w.r.t. every \( f_{ij} \), i.e. \( Q \) has degree 1 in any of its four variables: \( Q = c_1(f_{10}, f_{01}, f_{11})f_{00} + c_2(f_{10}, f_{01}, f_{11})f_{10} + c_3(f_{00}, f_{01}, f_{11})f_{11} + \cdots = q_{1111}f_{00}f_{10}f_{01}f_{11} + q_{1110}f_{00}f_{10}f_{01} + q_{1101}f_{00}f_{10}f_{11} + \cdots + q_{0000}. \)

2) **Symmetry.** Equation (2) should be invariant w.r.t. the symmetry group of the square or its suitably chosen subgroup.

A few other requirements were given in [1], in particular the formula (2) involved parameters attached to the edges of the square.

The second requirement of symmetry is obviously very important for the formulation of the condition of 3-dimensional consistency of (2). Namely, suppose we have an elementary cube (3d-cell of \( \mathbb{Z}^3 \), cf. Fig. 3) and impose (2) to hold on three “initial 2d-faces” \( \{ x_1 = 0 \} \): \( Q(f_{000}, f_{010}, f_{001}, f_{011}) = 0 \); \( \{ x_2 = 0 \} \): \( Q(f_{000}, f_{100}, f_{001}, f_{101}) = 0 \); \( \{ x_3 = 0 \} \): \( Q(f_{000}, f_{100}, f_{010}, f_{110}) = 0 \) (these are used to find \( f_{011}, f_{101}, f_{110} \) from \( f_{000}, f_{100}, f_{010}, f_{001} \)). Then we impose (2) to hold on the other three “final 2d-faces” \( \{ x_1 = 1 \} \): \( Q(f_{100}, f_{110}, f_{101}, f_{111}) = 0 \); \( \{ x_2 = 1 \} \): \( Q(f_{010}, f_{110}, f_{011}, f_{111}) = 0 \); \( \{ x_3 = 1 \} \): \( Q(f_{001}, f_{101}, f_{011}, f_{111}) = 0 \); so for the last field variable \( f_{111} \) we can find 3 (apriori) different rational expressions in terms of the “initial data” \( f_{000}, f_{100}, f_{010}, f_{001} \). The 3d-consistency is the requirement that these three expressions of \( f_{111} \) in terms of the initial data should be identically equal. The subtle point of this process consists in the non-uniqueness of the mappings of a given square (Fig. 2) onto the six 2d-faces of the cube. The requirement of symmetry given above guarantees that we can choose any identification of the vertices of the 2-dimensional faces of the 3-dimensional elementary cube (Fig. 3) with the vertices of the “standard” square where (2) is given; certainly this identification should preserve the combinatorial structure of the square (neighbouring vertices remain neighbouring). In [1] a complete classification of 3d-consistent 2d-face formulas (in a slightly different setting) was obtained; in [2] a similar classification was
given for the case when one does not assume that the formula (2) is the same on all the 6 faces of the 3-dimensional cube.

In the next sections we give a symmetry classification of all possible 3d-face formulas defined on some “standard” 3-dimensional cube:

\[ Q(f_{000}, f_{100}, f_{010}, f_{001}, f_{110}, f_{101}, f_{011}, f_{111}) = 0 \]  

(3)

with respect to the complete symmetry group of the cube. Here, as everywhere in the paper, indices of the field variables \( f_{ijk} \) give the coordinates of the corresponding vertices of the standard 3d-cube where our formula (3) is defined.

The requirement of consistency is now formulated similarly to the 2d-case: given a 4d-cube with field values \( f_{ijkl} \), \( i, j, k, l \in \{0, 1\} \), one should impose the formula (3) on every 3d-face of it, by fixing one of the indices \( i, j, k, l \), and making it 0 for the faces which we will call below “initial faces”, or respectively 1 for the faces which we will call “final faces”. One also needs to fix some mapping from the initial “standard” cube (with the vertices labelled \( f_{ijk} \)) onto every one of the eight 3d-faces (for example \( \{ f_{i1kl} \} \) on \( \{ x_2 = 1 \} \)). This can certainly be done using the trivial lexicographic correspondence of the type \( f_{ijk} \mapsto f_{i1jk} \). Geometrically this lexicographic correspondence is less natural since it is not invariant w.r.t. the symmetry group of a 4d-cube. On the other hand there is an important example of such a non-symmetric formula corresponding to the discrete BKP equation ([1], equation (76)). Another possibility to avoid this problem is to impose the requirement of symmetry. More precisely, if one applies any one of the transformations from the group of symmetries of the 3d-cube, (3) shall be transformed into an equation with the left hand side proportional to the original expression \( Q: Q \mapsto \lambda \cdot Q \). Since this symmetry group is generated by reflections, one has \( \lambda^2 = 1 \), so this proportionality multiplier \( \lambda \) may be either (+1) or (−1) for any particular transformation in the complete symmetry group.

From results in [1] we know that there are important 4d-consistent 3d-face formulas which are preserved under a suitable subgroup of the complete symmetry group of the 3d-cube. No classification of 3d-face formulas with such restricted symmetry property has been carried out yet.

### 3 Symmetry classification

Every \( n \)-dimensional face formula \( Q_n = 0 \) which satisfies the requirement of quasilinearity has a left hand side of the form

\[ Q_n = \sum_{\mathcal{D}} q_{\mathcal{D}} \prod_{i=0,1} (f_{i_1...i_n})^{D_{i_1...i_n}} \]  

(4)

with constant coefficients \( q_{\mathcal{D}} \), where the summation is taken over all \( 2^n \) many \( 2^n \)-tuples \( \mathcal{D} = (D_{00...0}, \ldots, D_{11...1}) \), each power \( D_{i_1...i_n} \) of the respective vertex variable \( f_{i_1...i_n} \) being either 0 or 1. In other words: the \( 2^n \) indices of \( q_{\mathcal{D}} \) are the exponents of the \( 2^n \) vertex field variables \( f_{i_1...i_n} \), each exponent \( D_{i_1...i_n} \) being 0 or 1. For example,
$Q_2 = q_{1111}f_{00}f_{10}f_{01}f_{11} + q_{1110}f_{00}f_{10}f_{01} + q_{1101}f_{00}f_{10}f_{11} + \ldots + q_{0000}$ has $2^{2^2} = 16$ terms, $Q_3$ has respectively $2^{2^3} = 256$ terms, and $Q_4$ has already $2^{2^4} = 65536$ terms.

In this Section we classify $n$-dimensional quasi-linear equations $Q_n \equiv 0$ for $n = 2, 3, 4$ that are invariant w.r.t. the complete symmetry group of the respective $n$-dimensional cube. This problem can be reduced to the enumeration of irreducible representations of this group in the space of polynomials of the form \( [4] \). Here this is done in a straightforward way: the group in question is generated by one reflection w.r.t. the plane $x_1 = 1/2$ and $(n-1)$ diagonal reflections w.r.t. the planes $x_1 = x_s$, $s = 2, \ldots, n$ (here $x_k$ denote the coordinates in $\mathbb{R}^n$).

To every reflection $R$ from this generating set we assign $(-)$ or $(+)$ and require the equality $Q(f) = -Q(R(f))$ (respectively $Q(f) = Q(R(f))$) to hold identically in all vertex variables $f$; this gives us a set of equations for the coefficients $q_D$. Running through all possible choices of the signs for the generating reflections we solve the united sets of simple linear equations for the coefficients $q_D$ for every such choice. The main problem consists in the size of the resulting set of equations: for $n = 3$ we have for each combination of $\pm$ for the 3 generating reflections around 770 equation for the 256 coefficients $q_{D_1 \ldots D_4}$; for $n = 4$ every set of equations for the coefficients of $Q_4$ has around 250,000 equations for the 65536 coefficients $q_{D_1 \ldots D_{16}}$. Naturally, not every combination of signs for the generating reflections is possible, most of the resulting sets of equations for $q_{D_1 \ldots D_{(2^n)}}$ allow only trivial solution $q_{D_1 \ldots D_{(2^n)}} = 0$. The results of our computation are given in table form in Theorem \( [4] \) below.

As it turns out, for $n = 2$ three symmetry types of quasi-linear expressions $Q_2$ are possible. In the notation of Table \( [4] \) the first sign refers to the reflection on the line $x_1 = 1/2$, the next sign stands for the reflection on the line $x_1 = x_2$. For example, the expressions of the first type $(+)$ are invariant w.r.t. to the reflection on the line $x_1 = 1/2$, and show a change of sign after the reflection on the line $x_1 = x_2$. The last case $(++)$ consists of expressions which are invariant w.r.t. any element of the complete group of symmetries of the square which corresponds to the choice of the $(+)$ signs for the two generating reflections of the square w.r.t. the lines $x_1 = 1/2$ and $x_1 = x_2$.

For $n = 3$ alongside with the completely symmetric quasi-linear expressions $Q_3$ (the third case $(+++)$ below), there are two other cases $(- - -)$ and $(-- +)$ of nontrivial quasi-linear $Q_3$. In this notation the first sign refers to the reflection on the plane $x_1 = 1/2$, the next signs stand for reflections on the planes $x_1 = x_2$, $x_1 = x_3$ (and $x_1 = x_4$ for $n = 4$).

The resulting numbers of free coefficients $q_D$ in the symmetric face formulas $Q_n$ are given for each of the nontrivial cases in Table \( [1] \). We also give the number of nonzero terms in $Q_n$ for each case.

Especially remarkable is the totally skew-symmetric case $(- - -)$ for $n = 3$: it has

\footnote{To solve the sparse but rather extensive linear systems for the coefficients $q_D$ appearing after the splitting w.r.t. the variables $f_{i_1 \ldots i_n}$, a special linear equation solver had to be written. It can be downloaded together with other material related to this publication (for details see http://lie.math.brocku.ca/twolff/papers/TsWo2007/readme).}

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only one (up to a constant multiple) nontrivial expression

\[ Q_{(- --)} = (f_{100} - f_{001})(f_{010} - f_{111})(f_{101} - f_{110})(f_{011} - f_{001}) - \\
(f_{001} - f_{010})(f_{111} - f_{100})(f_{000} - f_{101})(f_{110} - f_{011}). \] (5)

Precisely this expression gives the so-called discrete Schwarzian bi-Kadomtsev-Petviashvili system (dBKP-system) — an integrable discrete system found in [6, 7] and studied in [1] where the fact of its 4d-consistency was first established. The dBKP-system has many equivalent forms and appears in very different contexts. In addition to the known geometric interpretations and a reformulation as Yang-Baxter system (8), the dBKP-system may be considered as a nonlinear superposition principle for the classical 2-dimensional Moutard transformations (15).

The expression (5) enjoys an extra symmetry property: the equation \( Q_{(- --)} = 0 \) is invariant under the action of the \( SL_2(\mathbb{C}) \) group of fractional-linear transformations

\[ f \mapsto (af + b)/(cf + d) \] (6)

(all group parameters \( a, b, c, d \) are the same for all the vertices of the cube). This is in fact a direct consequence of its uniqueness in this class. Since this \( SL_2(\mathbb{C}) \) action obviously preserves the symmetry type of an expression w.r.t. the group action of the cube symmetry group, it is reasonable to find the subclasses of \( SL_2 \)-invariant face equations in each symmetry class. This is given in the third column of the table.

**Theorem 1** The nonempty symmetry classes of formulas \( Q_n \) for face dimensions \( n = 2, 3, 4 \) are:

| \( n \) | types of symmetry, number of parameters and terms | number of parameters in \( SL_2 \)-invariant subcases |
|---|---|---|
| 2 | 1) \((+-)\): 1 param.; 4 terms | 1) 1 param.; 4 terms |
| | 2) \((-+)\): 3 param.; 10 terms | 2) none |
| | 3) \((++)\): 6 param.; 16 terms | 3) 1 param.; 6 terms |
| 3 | 1) \((--\)\): 1 param.; 24 terms | 1) 1 param.; 24 terms |
| | 2) \((+++)\): 13 param.; 186 terms | 2) none |
| | 3) \((+++\)\): 22 param.; 256 terms | 3) 3 param.; 114 terms |
| 4 | 1) \((---\)\): 94 param.; 29208 terms | 1) 5 param.; 15480 terms |
| | 2) \((+++\)\): 77 param.; 26112 terms | 2) none |
| | 3) \((+++)\): 349 param.; 60666 terms | 3) 3 param.; 15809 terms |
| | 4) \((+++\)\): 402 param.; \( 2^{16} \) terms | 4) 18 param.; 96314 terms |

Table 1: Symmetry classification of the face formulas w.r.t. the complete symmetry group of the cube
In particular, the explicit expression for the (non-$SL_2$-symmetric) $2d$-face formulas are:

\begin{align*}
(+-) : \quad Q &= q_1(f_{11}f_{10} - f_{11}f_{01} - f_{10}f_{00} + f_{01}f_{00}) = q_1(f_{11} - f_{00})(f_{10} - f_{01}), \\
(-+) : \quad Q &= q_1(f_{11} - f_{10} - f_{01} + f_{00}) + q_2(f_{11}f_{00} - f_{10}f_{01}) + \\
&\quad q_3(f_{11}f_{10}f_{01} - f_{11}f_{10}f_{00} - f_{11}f_{01}f_{00} + f_{10}f_{01}f_{00}), \\
(++) : \quad Q &= q_1 + q_2(f_{11} + f_{10} + f_{01} + f_{00}) + q_3(f_{11}f_{00} + f_{10}f_{01}) + \\
&\quad q_4(f_{11}f_{10} + f_{11}f_{01} + f_{10}f_{00} + f_{01}f_{00}) + \\
&\quad q_5(f_{11}f_{10}f_{01} + f_{11}f_{10}f_{00} + f_{11}f_{01}f_{00} + f_{10}f_{01}f_{00}) + \\
&\quad q_6f_{11}f_{10}f_{01}f_{00}.
\end{align*}

The explicit form of the $SL_2$-symmetric face formula for $n = 2$ in the first $++$ case is the same:

$$Q = q_1(f_{11} - f_{00})(f_{10} - f_{01}).$$

For $n = 2$ and the symmetric case $++$ the $SL_2$-symmetric face formula is:

$$Q = q_1(f_{11}f_{10} + f_{11}f_{01} - 2f_{11}f_{00} - 2f_{10}f_{01} + f_{10}f_{00} + f_{01}f_{00}).$$

It should be noted that (8) is 3d-consistent; it can be simplified using $SL_2$ transformations and put into explicitly linear form $Q = f_{11} - f_{10} - f_{01} + f_{00}$ or into $Q = f_{11}f_{00} - f_{10}f_{01}$, so the respective equation $Q = 0$ is equivalent to a linear equation

$$\log f_{11} + \log f_{00} = \log f_{10} + \log f_{01}.$$

\section{Nonexistence of nontrivial integrable face formulas in other symmetry classes for $n = 3$}

In this section we give a computational proof of our main result:

\textbf{Theorem 2} Among the three possible symmetry types of 3-dimensional quasilinear face formulas given in the second column of Table 1, only formula \ref{eq:4} gives a nontrivial 4d-compatible face equation. Any 4d-compatible face formula in the other two symmetry types may be transformed using the action of the group $SL_2(\mathbb{C})$ on the field variables to one of the following linearizable forms:

\begin{equation}
Q^{(1)} = f_{000}f_{001}f_{010}f_{011}f_{100}f_{101}f_{110}f_{111} - \sigma,
\end{equation}

\begin{equation}
Q^{(2)} = f_{001}f_{010}f_{100}f_{111} - \sigma f_{000}f_{011}f_{101}f_{110},
\end{equation}

\begin{equation}
Q^{(3)} = (f_{001} + f_{010} + f_{100} + f_{111}) - \sigma(f_{000} + f_{011} + f_{101} + f_{110}),
\end{equation}

where $\sigma = \pm 1$.

Technically, in order to find 4d-consistent 3d-face formulas among the other two 3-dimensional cases ($--+$) and ($+++$) (as listed in Table 1), one shall run the following algorithmic steps:
Step 1. Take a copy of this \( Q_3 \) formula, map it onto the four initial faces of the 4d-cube (where one of the coordinates \( x_i = 0 \)), solve the mapped equations with respect to \( f_{0111}, f_{1011}, f_{1101}, \) and \( f_{1110} \) leaving the other variables free.

Step 2. Then substitute the obtained rational expressions for \( f_{0111}, f_{1011}, f_{1101}, \) and \( f_{1110} \) into the copies of the face formula mapped onto the four final faces (where one of the coordinates \( x_i = 1 \)), finding respectively four different expressions for the last vertex field \( f_{1111} \).

Step 3. Equate these 4 expressions for \( f_{1111} \) obtaining three rational equations in terms of 11 free variables \( f_{0000}, f_{0010}, f_{0101}, \ldots \) and the parametric coefficients \( q_{D_1 \ldots D_8} \) left free in the chosen symmetry class.

Step 4. Removing the common denominators of the equations and splitting the resulting polynomials w.r.t. the 11 free variables \( f_{ijkl} \) one obtains a polynomial system of equations for the free coefficients \( q_{D_1 \ldots D_8} \).

Step 5. The latter should be solved, resulting in a complete classification of 4d-consistent quasilinear scalar 3d-face formulas.

This approach, applied in a straightforward way, results in extremely huge expressions. Even building the rational expressions in Step 4 in a straightforward way seems to be unrealistic: for a typical 3d-face formula from Table 1 Step 4 should end up (as our test runs allowed for an estimate) in an expression with around \( 10^{14} \) terms, which is beyond the reach of computer algebra systems in the foreseeable future. Even brute force testing of 4d-consistency of the smallest solution (face formula (5) which has no free parametric coefficients \( q_{D_1 \ldots D_8} \)) results in \( 2 \cdot 10^9 \) terms (after substituting the expressions for \( f_{0111}, f_{1011}, f_{1101}, f_{1110} \), collecting the terms over the common denominator and expanding the brackets before the cancellation can start in Step 4). Technically this is explained by the presence of 4 different symbolic denominators of the rational expressions for \( f_{0111}, f_{1011}, f_{1101}, f_{1110} \) and their various products. A careful step-by-step substitution and cancellation of like terms in several stages still can be done even on a modest computer for this \((--)\) case. Using the system FORM (this system was specially designed for large symbolic computations), one can prove that all terms finally cancel out for the case of the integrable 3d-face formula (5) thus giving a computational proof of its 4d-consistency in 3 min CPU time (3 GHz Intel running Linux SUSE 9.3) and less than 200 Mb disk space for temporary data storage.

As the estimates given in Appendix A show, the straightforward approach based on Steps 1–4 is unrealistic for the other two 3-dimensional cases listed in Table 1 even at the stage of generation of the consistency conditions (Step 4).

In order to classify discrete integrable 3d-face formulas \( Q_3 = 0 \) for the case \((--+)\) and the hardest case \((+++\)) we used a totally different randomized “probing” strategy, explained in detail in Appendices B, C, D.

After the computation (cf. Appendices B–E) the list of candidate face formulas \( Q_3 \) for the case \((+++\)) included 5 formulas (before the verification that these formulas,
obtained by our “probing” method, really give 4d-consistent 3d-formulas). For the case
(−+++) the list of candidate face formulas included 3 formulas. All of them include a few free parameters. As one can show, all these formulas can be greatly simplified using
the action of the group $SL_2(\mathbb{C})$ on the field variables $\mathbf{f}$, resulting in the 4d-consistent
3d-face formulas (10), (11), (12). The first two formulas (10), (11) can be linearized
using the logarithmic substitution $\tilde{f}_{ijk} = \log f_{ijk}$.

The expressions for the aforementioned candidates, the FORM procedures and their
logfiles showing the simplification process can be downloaded from
\texttt{http://lie.math.brocku.ca/twolf/papers/TsWo2007/} Here we just give one example of such a $SL_2$-simplification: the $Q$-expression for the case $Q1$ in
\texttt{http://lie.math.brocku.ca/twolf/papers/TsWo2007/SL2-simplification/Case+++/}
is

$$Q = q_{105}(f_{001}f_{010}f_{100}f_{111} + f_{000}f_{011}f_{101}f_{110}) +$$

$$+ q_{107}(f_{001}f_{010}f_{100}f_{110}f_{111} + f_{001}f_{010}f_{100}f_{101}f_{111} + f_{001}f_{010}f_{011}f_{100}f_{111} +$$

$$+ f_{000}f_{011}f_{101}f_{110}f_{111} + f_{000}f_{011}f_{100}f_{101}f_{110} + f_{000}f_{010}f_{011}f_{101}f_{110} +$$

$$+ f_{000}f_{011}f_{101}f_{110} + f_{000}f_{001}f_{010}f_{100}f_{111}) +$$

$$\frac{q_{107}^2}{q_{105}}(f_{001}f_{010}f_{100}f_{101}f_{110}f_{111} + f_{001}f_{010}f_{011}f_{100}f_{110}f_{111} +$$

$$+ f_{001}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{011}f_{100}f_{101}f_{110}f_{111} +$$

$$+ f_{000}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{101}f_{100}f_{101}f_{110}f_{111} +$$

$$+ f_{000}f_{010}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{010}f_{101}f_{100}f_{111} +$$

$$+ f_{000}f_{010}f_{101}f_{100}f_{101}f_{110}f_{111}) +$$

$$\frac{q_{107}^3}{q_{105}^2}(f_{001}f_{010}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{010}f_{011}f_{100}f_{101}f_{110}f_{111} +$$

$$+ f_{000}f_{011}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{010}f_{101}f_{100}f_{110}f_{111} +$$

$$+ f_{000}f_{010}f_{101}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{010}f_{011}f_{101}f_{110}f_{111} +$$

$$+ f_{000}f_{010}f_{101}f_{100}f_{101}f_{110}f_{111} + f_{000}f_{010}f_{101}f_{011}f_{100}f_{101}f_{110}f_{111}) +$$

$$2\frac{q_{107}^4}{q_{105}^3}(f_{000}f_{010}f_{101}f_{100}f_{101}f_{110}f_{111})$$

The simplifying transformation consists in the following steps:

1. $f_{ijk} \mapsto \frac{q_{105}}{q_{107}} f_{ijk}$ and $Q \mapsto \frac{q_{107}}{q_{105}} Q$ (this eliminates the parametric $q_{105}$ and $q_{107}$)

2. $f_{ijk} \mapsto \frac{1}{f_{ijk}}$ (and removing the denominator in $Q$ afterwards)

3. $f_{ijk} \mapsto f_{ijk} - 1$.

This produces the simplified form $Q = f_{001}f_{010}f_{100}f_{111} + f_{000}f_{011}f_{101}f_{110}$. 

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Appendix A: The Size of Consistency Conditions

The following considerations are made under the assumption of generic unknown coefficients $q_D$ in the face formula (11) not satisfying additional symmetry conditions.

Any $(n+1)$-dimensional hypercube built from $2^{n+1}$ vertices $f_{i_1...i_{n+1}}, i_k \in \{0, 1\}$ has $2(n+1)$ faces located in the (logical) planes $x_k = 0$ and $x_k = 1, k = 1, \ldots, (n+1)$. The face relations for the $n+1$ faces that correspond to $x_k = 0$ are

\[
0 = \sum_{D} q_D \prod_{i_s=0,1} (f_{i_1...i_{k-1}0i_{k+1}...i_{n+1}})^{D_{i_1...i_{k-1}i_{k+1}...i_{n+1}}}.
\]  

(13)

They can be used to determine $f_{1..101..1}$ with the 0 being in the $k^{th}$ index position. Each of these face relations involves $2^n$ $f$-variables and thus $2^{2n}$ terms, half of them include $f_{1..101..1}$ as a factor and the other half not. Solving the face relation $x_k = 0$:

\[
0 = A_k f_{1..101..1} + B_k
\]  

(14)

where $A_k, B_k$ are expressions in $q_D, f_\beta$ for $f_{1..101..1}$ and substituting $f_{1..101..1} = -B_k/A_k$ in any expression that involves $f_{1..101..1}$ linearly (like other face relations) and taking the numerator over the common denominator amounts to multiplying all terms that involve $f_{1..101..1}$ by $-B_k$ and all other terms by $A_k$. As $A_k$ and $B_k$ involve each $2^{2n}/2 = 2^{2n-1}$ terms this means that a substitution of $f_{1..101..1}$ increases the number of terms by a factor of $2^{2n-1}$, before cancellations and reductions will be made.

The $2^{nd}$ half of face relations for the $n+1$ faces that correspond to $x_k = 1$ are

\[
0 = \sum_{D} q_D \prod_{i_s=0,1} (f_{i_1...i_{k-1}1i_{k+1}...i_{n+1}})^{D_{i_1...i_{k-1}i_{k+1}...i_{n+1}}}.
\]  

(15)

Each one of them involves $f_{11..1}$ and $n$ of those $f$-variables which have exactly one 0 as index in any one of the $n+1$ index positions apart from the $k^{th}$ position. Replacing each one of these $n$ $f$-variables by using the corresponding $x_i = 0$ face relation increases the number of terms by a factor $2^{2n-1}$ each time, giving in total $2^{2n}(2^{2n-1})^n = 2^{2n(n+1)-n}$ terms. In each substitution the degree of the coefficients $q_D$ increases by one, reaching finally $n+1$.

Solving one of the $n+1$ many $x_k = 1$ face relations

\[
0 = G_k f_{11..1} + H_k
\]  

(16)
for $f_{11.1}$ and substituting $f_{11.1} = -H_k/G_k$ in the other face relations gives $n$ independent consistency conditions

$$G_j H_k = G_k H_j, \quad j = 1, \ldots, k - 1, k + 1, \ldots, n + 1$$

with each $G_i$ and $H_i$ having $2^{2(n+1)-n}/2$ terms, i.e. each consistency condition involving $2\left(2^{n+1}-n-1\right)^2$ terms. The total number of terms of the $n$ consistency conditions is thus $n2^{2(n+1)-n-1}$.

To compute an upper bound of the number of conditions that result from splitting each consistency condition with respect to the independent $f$-variables we note that their highest degree is equal to the total degree of all $q_D$, i.e. it is $2n+2$. The only exception is $f_{00...0}$ which does not occur in the $0 = Q_n|_{1\to k}$ face relations and enters only through substitutions, so its highest degree in the constraints is $2n$. We thus get for an upper bound of the number of different products of different powers of $f_{00...0}$ and the other $2^{n+1}-n-3$ independent $f$-variables the value $2n(2n+2)\left(2^{n+1}-n-3\right)$. With this number and the number of terms of each constraint we get with their quotient an estimate of the average number of terms in each equation (see Table 2).

**Remark.** Although, strictly speaking, these are upper bounds for the size of conditions, one shall keep in mind that any computer algebra system *shall generate all these terms* expanding the brackets and only after generating all of them or a part of them it can search for possible cancellations or reductions. As our test runs with FORM had shown, for the 3-dimensional case $(+++)\) given in Table 11 (with a much smaller number of independent $q_D$ than $2^n$ but the same number of terms in $Q_3$), the total number of terms to be generated for each of the 3 consistency conditions is around $10^{14}$ (compared to $1.4 \cdot 10^{17}$ in Table 2). A typical single equation for the coefficients $q_D$ resulting from splitting a partially formed consistency condition has a few thousand terms of degree 8 in the parametric $q_D$.

### Table 2: Some statistics of faces and consistency conditions

| Dimension of face | $n$ | 2 | 3 | 4 | 5 |
|-------------------|-----|---|---|---|---|
| # of $f$-variables in face formula | $2^n$ | 4 | 8 | 16 | 32 |
| # of terms in face formula | $2^{2n}$ | 16 | 256 | 65536 | $4.3 \times 10^9$ |
| # of all $f$-variables in $(n+1)$-dim. hypercube | $2^{n+1}$ | 8 | 16 | 32 | 64 |
| # of indep. $f$-variables in $(n+1)$-dim. hypercube | $2^{n+1} - n - 2$ | 4 | 11 | 26 | 57 |
| # of $n$-dim. faces in $(n+1)$-dim. hypercube | $2(n+1)$ | 6 | 8 | 10 | 12 |
| # of consistency conditions | $n$ | 2 | 3 | 4 | 5 |
| Upper bound on the # of terms of each condition | $2^{2(n+1)} - 2n - 1$ | $5.2 \times 10^5$ | $1.4 \times 10^{17}$ | $2.8 \times 10^{45}$ | $1.9 \times 10^{112}$ |
| Total degree of the $q_D$ in each condition | $2n + 2$ | 6 | 8 | 10 | 12 |
| Upper bound estimate of the # of equations resulting from splitting each condition | $2n(2n+2)\left(2^{n+1} - n - 3\right)$ | 864 | $6.4 \times 10^9$ | $8.0 \times 10^{25}$ | $2.7 \times 10^{61}$ |
| Estimated average # of terms in each equation | $\frac{2^{2n+1} - 2n - 1}{2n(2n+2)\left(2^{n+1} - n - 3\right)}$ | 606 | $2.2 \times 10^7$ | $3.5 \times 10^{19}$ | $7 \times 10^{50}$ |
Appendix B: The computational problem

As outlined before (cf. Section 4), the task consists of the following steps.

1. Formulate the relations for the \( n + 1 \) faces \( x_k = 0 \).
2. Solve them for \( f_{11\ldots101\ldots1} \).
3. Formulate the relations for the \( n + 1 \) faces \( x_k = 1 \).
4. Perform the substitutions obtained under 2. in the relations of 3.
5. Solve one of the resulting relations for \( f_{11\ldots1} \) and
6. Substitute \( f_{11\ldots1} \) in all other \( n \) face relations of 4.
7. Split these consistency conditions with respect to all the occurring independent \( f \)-variables to obtain an overdetermined system of equations for the unknowns \( q_D \).
8. Find the general solution of this system.
9. Reduce the number of free parameters of the solutions using \( SL_2(\mathbb{C}) \)-transformations (6).

Although the algebraic system for the unknown coefficients \( q_D \) is heavily overdetermined the following difficulties appear.

1. Strictly speaking, in order to formulate even only the smallest subset of conditions one would have to formulate at least one consistency condition (by performing steps 1., 2. fully and 3.- 6. for at least two \( x_k = 1 \) face relations before step 7) i.e. to generate an expression with \( 2^{(2^n+1)(n+1)-2n-1} \) terms.
2. If one found a way around this hurdle then the resulting equations are of high degree \( 2n + 2 \) with on average many terms.
3. Even if one were able to generate 100,000’s of equations and thus find shorter ones which one could solve for some unknowns in terms of others, one would face the problem that many cases and sub-sub-cases have to be investigated due to the high degree of the equations and
4. that one has to generate billions of equations to find some that are independent of the ones generated so far. As we explain below, one can not hope that the first, say, \( 10^6 \) conditions will be equivalent to the full system of equations for \( q_D \), even though we have only very few unknown \( q_D \)'s due to the “triangular” form of this huge system as we explain in Appendix D.

In the computations to be described in this paper we have \( n = 3 \). The full problem (without cubical symmetry) would mean to generate 3 consistency conditions involving each about \( 10^{17} \) terms that split into an estimate of \( 10^{10} \) polynomial equations each being homogeneous of degree 8 for 256 unknowns and involving on average over \( 10^7 \) terms.

To make progress we introduce different solving techniques but also simplify the problem:
1. We restrict our problem to face formulas that obey the full cubical symmetry (cf. Section 3). This reduces the problem to 3 cases:

- case \((- - -)\) with 232 identically vanishing \(q_D\) and 24 other \(q_D\) depending on 1 parameter,
- case \((- + +)\) with 70 identically vanishing \(q_D\) and 186 other \(q_D\) depending on 13 parameters, and
- case \((+ + +)\) with no identically vanishing \(q_D\) and all 256 \(q_D\) depending on 22 parameters.

In the third case \((+ + +)\) – the hardest case – the fact that only 22 of the \(q_D\) are parametric simplifies the problem but the simplification is limited because none of the other 234 \(q_D\) needs to vanish just because of the extra cubical symmetry (see Section 3).

2. We ease the formulation of the large consistency conditions (before splitting them into smaller equations) by replacing \(z\) of the \(v := 2^{n+1} - n - 2\) independent \(f\)-variables by zero, and replacing \(u\) of them by random integer values unequal zero, leaving \(s := v - z - n\) of them in symbolic form. As a consequence, we only get a set of necessary and not sufficient equations for the \(q_D\) but we can repeat this procedure on a gradually increasing level of generality (by increasing \(s\) and lowering \(z\)). A crucial feature is that the preliminary knowledge about the solution is used in the formulation of new necessary systems of equations (see Appendix C).

3. We design a dynamic process that automatically organizes an iteration process that generates and solves/simplifies/makes use of necessary equations automatically. As an important feature, a set of newly generated necessary equations is not read into the ongoing computation at once but gradually on demand (see Appendix D).

4. A recent extension is the parallelization of different case investigations on a computer cluster.

We discuss points 1-3 now in more detail.

**Appendix C: Random probing**

In this Appendix we explain the “probing technique” in more detail which was mentioned briefly in Appendix B under point 2. in the list of techniques. It is based on replacing a number \(z\) of the \(v := 2^{n+1} - n - 2\) independent \(f\)-variables by zero (for \(n = 3, v = 11\)), and replacing a number \(u\) of them by random integer non-zero values, leaving \(s := v - z - n\) of them in symbolic form.

The computation has 3 phases: finding solutions, verifying the obtained solutions probabilistically and again rigorously. The first two phases use the probing technique.
Still, it makes sense to distinguish them because optimal values of \( z, u \) and \( s \) differ in both cases.

In the probing technique the two types of replacements, i.e. replacing \( f_\alpha \) by zero or by a non-zero integer, share the same disadvantage: the number of independent parameters, i.e. of symbolic \( f_\alpha \) which allow to split the consistency condition into many smaller equations, is reduced by one.

The advantage of replacing an \( f_\alpha \) by zero instead of a non-zero integer is that expressions shrink more. Also, replacements by non-zero integers often introduce extra solutions to the generated conditions and it appears to be costly to eliminate these spurious temporary solutions by leading them to a contradiction with more conditions to be generated based on other random replacements. Therefore it is more productive to replace, for example, one \( f_\alpha \) by zero than to replace two \( f_\alpha \) by non-zero integers.

Consequently, in the first phase of finding solutions at most one replacement by a non-zero integer is made (i.e. \( u \leq 1 \)) and after starting with \( z = 9, u = 0, s = 2 \) one increases generality gradually by either changing \( u \) from 0 to 1 and decreasing \( z \) by 1, or by decreasing \( u \) from 1 to 0 and increasing \( s \) by 1 until \( z = 1, u = 0, s = 10 \). We need the occasional substitution by one non-zero integer because we want to increase the generality in as small as possible steps in order to avoid the generation of too many too high degree equations with too many terms. This would happen if we decrease \( z \) by 1, keep \( u = 0 \) and increase \( s \) by 1. A run in full generality \( z = 0, u = 0, s = 11 \) is computationally prohibitive, therefore in a second phase of confirming the found solution probabilistically one starts with \( z = 4, u = 1, s = 6 \) and increases generality by decreasing \( z \), increasing \( u \) and keeping \( s \) constant until \( z = 0, u = 5, s = 6 \). By testing a hypothetical solution with this final setting many times, the correctness of the solution is confirmed with an arbitrarily high probability.

In both phases we want to generate as few and as simple equations as possible in each generation step. So we continue using the same setting of \( z, u, s \) as long as possible (i.e. as long as there are still resulting new conditions after randomly choosing other sets of \( z \) many \( f_\alpha \) to be 0, \( s \) of \( f_\alpha \) to be kept symbolic and randomly assigning integer non-zero values to the other \( u \) parametric \( f_\alpha \)) before generalizing it, i.e. making \( s \) larger and \( z \) smaller. This is regulated by one parameter which specifies the maximum number of consecutive times that a ‘probing’ (generation of conditions) attempt yielded only identities before changing \( z, u, s \).

In a third phase, after all hypothetical solutions have been obtained and been checked probabilistically, they are checked again, now rigorously. This has been done either by a brute force check using the computer algebra system FORM or by using \( SL_2(\mathbb{C}) \)-transformations on the field variables \( f \) to reduce solutions to integrable trivial forms \( (10), (11), (12) \) in Section 4.

A helpful and initially unexpected feature of the probing technique is that the resulting equations appear to be somehow triangularized in the following sense. Each unknown \( q_{i_1i_2..i_m}, m = 2^n, i_j \in \{0, 1\} \) is the coefficient of a product of \( i_1 + i_2 + \ldots + i_m \) many different factors \( f_\alpha \). That means, that at the beginning of the computation when many \( f_\alpha \) are replaced by zero, the \( q_D \) with a high index sum do not occur. Only later on as fewer \( f_\alpha \) are replaced by zero, gradually \( q_D \) with higher index sum appear in
the equations. On one hand this is a good feature, providing a partially triangularized system of equations. On the other hand this means that although we only want to compute a relatively small number of unknowns (for \( n = 3 \) and case (++) these are 22 \( q_D \)) it is not enough to formulate only a comparable number of the huge total set of equations (about \( 6.4 \times 10^9 \) equations for \( n = 3 \)). A set of equations that is equivalent to the complete set of equations is only obtained towards the end of generalizations. For example, \( q_{11...1} \) as the coefficient of the product of all the \( 2^n \) many \( f \) occurring in at least one face formula, at most \( 2^n - 1 \) of them belonging to the independent \( f_\alpha \), can only occur in at least one consistency condition if none of the \( f_\alpha \) occurring in at least one face formula is replaced by zero, i.e. if \( u \) and \( s \) are big enough to satisfy \((u + s) \geq (2^n - 1)\).

That in turn means that millions of the early equations are redundant which implies a large inefficiency in generating equations. This can be avoided by using known relations \( q_D = h_D(q'_D) \), which were derived in the solution process so far, as automatic simplification rules when generating new equations.

Three problems remain to be considered.

1. By replacing \( f_\alpha \) through numbers it may happen that \( A_k \) in any one of the \( x_k = 0 \) face relations (14) becomes zero. Then a new equation generation attempt with different or more general random replacements has to be made.

2. Similarly, it may happen that the coefficient of \( q_{11...1} \) in all \( n + 1 \) many \( x_k = 1 \) face relations (16) is zero. Then a different replacement has to be tried as well.

3. Even if none of the \( A_k \) in (16) becomes zero it may happen that \( A_k \) and \( B_k \) in one face relation are not prime and then a solution for the \( q_D \) which makes the greatest common divisor \( GCD(A_k, B_k) \) to zero is potentially lost when performing substitutions \( f_{11...101,1} = -B_k/A_k \). The same applies to common factors of \( G_k \) and \( H_k \) in the single face relation (16) that is applied to replace \( f_{11...1} \). Therefore each consistency condition has to be multiplied with a product of all common factors of any pair \( A_k, B_k \) and of all common factors of the pair \( G_k, H_k \) which is used to replace \( f_{11...1} \). To lower the computational cost one drops multiplicities of the factors. These factors involve in general \( q_D \) as well as \( f_\alpha \) and therefore the multiplication has to be done before splitting the consistency condition with respect to the independent \( f_\alpha \). Alternatively one can split the consistency condition before multiplication and instead multiply and duplicate the equations in the following way.

Let \( P(q_D, f_\alpha) \) be one of the above mentioned factors and let \( 0 = P \) be split into a system \( 0 = P_i(q_D) \) where redundant equations are dropped\(^4\). Instead of multiplying a constraint \( 0 = C(q_D, f_\alpha) \) with \( P \), splitting the equation \( 0 = PC \) into individual equations and factorizing all of them afterwards, it is equivalent but much more efficient to split \( 0 = C \) into a system of equations \( 0 = \hat{C}_j(q_D) \) and \( 0 = P \) into a system \( 0 = \hat{P}_i(q_D) \) and to consider the equivalent system \( 0 = \hat{P}_i \hat{C}_j, \forall i, j \).

To summarize, in order not to lose solutions for the \( q_D \), the procedure is

\(^4\)The definition of ‘redundant’ depends of the effort one wants to spend at this stage. In the implementation of this algorithm the polynomials \( P_i \) are divided by the coefficients of their leading terms with respect to some ordering of the \( q_D \) and then duplicate \( P_i \) are dropped.
to collect all common factors \( P_r \) of all pairs \( A_k, B_k \) and of the pair \( G_k, H_k \) used to substitute \( f_{11...1} \),

- to drop duplicate factors,

- to split all consistency conditions giving a system \( S \) of equations \( 0 = \hat{C}_j \), and

- to split for each factor \( P_r \) the equation \( 0 = P_r \) into a system \( 0 = \hat{P}_{ri}, \ i = 1 \ldots i_r \) where again equations that are redundant within one such system are dropped.

- If a system \( 0 = \hat{P}_{ri} \) includes a non-vanishing \( \hat{P}_{ri} \) either because \( \hat{P}_{ri} = 1 \) or because \( \hat{P}_{ri}(q_D) \) is known to be non-zero based on the inequalities that are known for some \( q_D \) then this system is ignored because the corresponding \( P_r \) is non-zero. For every other such system \( 0 = \hat{P}_{ri} \), replace the system \( S \) of conditions \( 0 = \hat{C}_j \) by the new system \( \hat{S} \) consisting of the equations \( 0 = \hat{P}_{ri}\hat{C}_j, \ \forall i, j \).

Appendix D: A Succession of Generating and Solving Equations

The system of algebraic equations for the \( q_D \) is investigated by the computer algebra package CRACK that aims at solving polynomially algebraic or differential systems, typically systems that are overdetermined and very large. It offers various degrees of interactivity from fully automatic to fully interactive. The package consists of about 40 modules which perform different steps, like substitutions, factorizations, shortenings, Gröbner basis steps, integrations, separations,... which can be executed in any order. In automatic computations their application is governed by a priority list where highly beneficial, low cost and low risk (of exploding the size of equations) steps come first. Modules are tried from the beginning of this list to its end until an attempt is successful and then execution returns to the start of the list and modules are tried again in that order. This simple principle is refined in a number of ways. For more details see [11], [12].

In order to accommodate the dynamic generation of equations and their successive use, all that had to be done was to add two modules:

- one for generating a new set of necessary conditions using the ‘probing’ technique from Appendix C and writing the generated equations into a buffer file, and

- another module for reading one non-trivial equation from this buffer file (i.e. for continuously reading equations until one is obtained that is not instantly simplified to an identity modulo the known equations or until the end of this file is reached),

and to determine the place of these modules in the priority list. The need for a buffer file arose because the number of equations generated in each ‘probing’ is unpredictable,
especially in view of the large impact that factors $\hat{P}_{ri}$ can have on the number of equations (see the end of Appendix C).  

Some more miscellaneous comments.

As shown in Appendix C the generated equations often take the form of products set equal to zero. This leads to many case distinctions of factors being either zero or non-zero and consequently other factors being zero. The depth of sub-case levels sometimes reaches 20. Because buffer files are only valid for the case in which they were generated (because they make use of case-dependent known substitutions $q_D = h_D(q_D')$ and case-dependent inequalities) and because of these deep levels of sub-cases, the number of buffer files easily reaches 100,000 and more (e.g. too many to be deleted with the simple UNIX command `rm *`). Therefore the case label is encoded in the buffer file name allowing to delete buffer files automatically when the case in which the file was created and all its sub-cases are solved.

There is much room for experimenting with the place of the two new modules within the priority list\(^5\). On one hand one wants to read and create early, so that the ongoing computation has many equations to choose from when looking for the most suitable substitutions, shortenings, ... . The problem is that these equations are all generated with the same limited information on relations between $q_D$, and thus they have a high redundancy. Also, dealing with many long equations does slow down CRACK. On the other hand, giving the branching of the computation into sub-cases a higher priority generates an exponential growth of sub- and sub-sub-cases which drastically increases the number of buffer files to be generated because they are only valid for the case they were generated for or for its sub-cases.

With each investigated case, say $q_5 = 0$, the other case $q_5 \neq 0$ generates inequalities which the package CRACK collects, updates and makes heavily use of to avoid further case distinctions as far as possible, and in this computation also to drop factors of the $\hat{P}_{mi}$ as mentioned in Appendix D.

The individual cases can either be investigated serially or in parallel.

If two solutions of two different cases, for example, the solutions for $q_5 = 0$ and $q_5 \neq 0$ can be merged into one analytic form, if necessary by a re-parametrization, then this is achieved by one of the modules of CRACK ([13]).

Appendix E: The Computation

The computation was not performed in a single run. It fact due to the big workload it extended over several months. Simple subcases were solved initially whereas harder

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\(^5\)This arrangement is similar to the design of CPU chips which have not only access to their own register memory (~ the equations known within CRACK) and access to the hard disk (~ the possibility to call the ‘probing’ module) but which also have access to cache memory (~ the buffer file).

\(^6\)apart from the necessity to give the ‘reading from the buffer’ module a higher priority than the ‘creating of a buffer’ module in order to try reading and emptying a buffer file first (if available and not already read completely) and to create a new buffer file only if none is available or if the available one is already completely read in
ones were completed only after the probing technique from Appendix C and its automatic interplay with the package CRACK were developed. Even then it took some time to fine-tune parameters and put the new modules in the right place within the priority list of procedures in CRACK. Finally, it was nearly possible to do the computation fully automatically, only a few times the proper case distinctions had to be initiated manually at the right time to be able to complete the computation. If one would add up purely the necessary computation time without runs following a poor manual choice of case distinctions leading to computations which generated too large systems and which could not be completed then this would amount to 2 weeks of CPU time on a 3GHz AMD64 PC.

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