The Quantum-Statistical Condensate of One-Dimensional Anyons

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We develop an exact many-body formalism for one-dimensional anyons, the hybrid particles between bosons and fermions. Besides providing characteristic observables, we reveal the quantum-statistical condensate. This genuine many-body condensate is created purely by quantum-statistical attraction. It is potentially more stable than a Bose-Einstein condensate and carries a rich structure of degenerate internal excitations.

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Introduction – Modern physics has made it possible to construct nanoscopic systems in which electronic excitations are effectively confined to a lower-dimensional world. An unexpected consequence of a reduced spatial dimension is the occurrence of particles that neither obey Fermi nor Bose statistics: anyons [1–4]. While anyons in two dimensions have been theoretically extensively studied [2–6] and indicated to exist in several experimental systems [10–16], they have been comparably neglected in one dimension. This may be because the spatial exchange of two-dimensional anyons results in fascinating physics [17], while in one dimension, anyons only collide, remaining their order. However, exchangeability is installable considering ringlike systems or T-structures [18, 19]. Sparked by this idea, the interest in lower-than-two-dimensional anyons has recently risen, especially in conjunction with the possible detection of Majorana bound states in quantum wires [20–24]. Those are expected to be non-abelian anyons with potential application to topological quantum computing [19, 25].

There exist different theories for one-dimensional particles of intermediate quantum statistics that are all referred to as anyons [5–32]; see [42] for a brief summary. Within this work, we employ the concept of anyons introduced in the seminal work by Leinaas and Myrheim [1]. Their approach advantageously bases only on one fundamental idea: to set up the proper classical theory of indiscernible particles and then quantize it. In two dimensions, this results in “standard” anyons, imaginable as bosons with an attached flux acquiring an Aharonov-Bohm phase when physically exchanged [6, 9]. Applied to one dimension, Leinaas’ and Myrheim’s idea leads to a description of either intrinsically one-dimensional anyons or quasi-one-dimensional anyons that are created from two-dimensional ones due to confinement by an external potential [33]. Concretely, we might imagine a fractional quantum Hall insulator [11], where anyonic bulk excitations are confined to one dimension by an electric potential. Despite its fundamentality, Leinaas’ and Myrheim’s approach to one dimensional anyons has rarely been applied [8, 33] and has remained, to the best of our knowledge, in its elementary state until now.

In this manuscript, we develop the quantum many-body formalism for one-dimensional anyons à la Leinaas and Myrheim. We construct the space of admissible wave functions, develop the second quantization formalism and calculate observables [20, 21, 43, 45] for confined anyons: the energy spectrum, momentum density, and finite size density oscillations. Our results are readily applicable to quasi-particle excitations in quasi one-dimensional systems, like interacting cold atom/ion chains and edge liquids of topological insulators, that potentially carry anyonic excitations [20, 21, 41, 46, 47]. While developing the formalism, we take particular care to include complex momenta, which leads to peculiar statistical bound states. Although the two-particle precursor of these states has already been discovered at the origins of anyons [1], they have been rather neglected in subsequent works [8, 33]. Their full importance unfolds in the exact many-body solutions, where they generalize to the quantum-statistical condensate, a remarkably stable quantum phase exhibiting a complex structure of degenerate internal excitations. Our work shows that one-dimensional anyons offer rich and elegant physics even in the absence of exchangeability.

Model – Let us concisely recapitulate and slightly extend the theory of Leinaas and Myrheim [1]. Consider n classical, indiscernible particles in a region M of real space. The spatial configurations of a system of discernible particles would be described by tuples of positions \( \mathbf{x} = (x_1, \ldots, x_n) \), where \( x_j \) lies in \( M \). Because the particles are indiscernible, however, using tuples is prodigious: for \( n = 2 \), \((x_1, x_2) \) and \((x_2, x_1) \) label the same configuration. Instead, we employ the sets \( \{x_1, \ldots, x_n\} \) of \( n \) distinct positions. The family of all these sets is called configuration space \( \mathcal{C} \) and inherits various properties by local equivalence to \( M^n \). For \( n \) indiscernible particles on a line, the real variables \( x_1 < \cdots < x_n \) parametrize \( \mathcal{C} \), where \( x_1 \) denotes the leftmost and \( x_n \) the rightmost particle. To obtain the quantum mechanical theory, space and momentum variables get promoted to the usual operators acting on the wave functions \( \Psi : \mathcal{C} \to \mathbb{C} \). Additionally, the Hamiltonian \( \mathcal{H} \) must be hermitian. For concreteness,
we consider
\[ \mathcal{H} = -\frac{\hbar^2}{2m} \sum_{j=1}^{n} \partial_{x_j}^2, \]
where \( m \) denotes the mass of the particles. Electromagnetic potentials and particle interactions can be added without changing the formalism. Hermiticity is granted by fulfilling the Robin boundary conditions
\[ (\partial_{x_{j+1}} - \partial_{x_j}) \Psi(x) \big|_{x_j \rightarrow x_{j+1}} = \eta \Psi(x) \big|_{x_j \rightarrow x_{j+1}}, \]
for each \( j \) between 1 and \( n-1 \). These boundary conditions classify the scattering behavior of indiscernible particles. As such, it is not surprising that they depend on the relative momentum \( i\hbar(\partial_{x_{j}} - \partial_{x_{j+1}}) \) of adjacent anyons [48].

The statistical parameter \( \eta \) is a real momentum that characterizes the anyonic species interpolating between bosons and fermions. Coherently, Eq. (1) reduces to the Neumann and Dirichlet boundary conditions of bosons (at \( \eta = 0 \)) and fermions (\( \eta = \pm \infty \)) [49].

Construction of the wave functions - We next construct all wave functions that fulfill Eq. (2) by deriving a proper basis. To this end, we write a general wave function as an integral over momentum space by
\[ \Psi(x) = \int_{\mathbb{C}^{n}} d^{n}k \alpha(k) e^{i\mathbf{k} \cdot \mathbf{x}}. \]

Complex momenta are explicitly included. These are needed to describe anyonic bound states that may potentially form for a negative statistical parameter. To understand why anyons can bind, note that a scattering event of two anyons with a negative statistical parameter results in a negative time shift of the corresponding matter wave [33]. Thus, anyons effectively scatter backwards in time. They then have the chance to scatter again and form a bound state by infinite repetition [50]. In momentum space, the boundary conditions translate to
\[ \alpha(k) = e^{-i\phi_{\eta}(k_{j+1} - k_j)} \alpha(\sigma_j \mathbf{k}) \quad \text{if} \quad k_{j+1} - k_j \neq i\eta, \]
\[ \alpha(k) = 0 \quad \text{if} \quad k_{j+1} - k_j = i\eta. \]

Here, \( \sigma_j \) denotes the elementary permutation which permutes the \( j^{th} \) and \((j+1)^{th}\) element of a tuple and
\[ \phi_{\eta}(k_{j+1} - k_j) = 2 \arctan \left[ \frac{\eta}{(k_{j+1} - k_j)} \right] \]
is the statistical phase. By iteration, these conditions connect coefficients of relatively permuted momenta \( \alpha(P\mathbf{k}) = e^{i\phi_{\eta}(P\mathbf{k})} \alpha(P\mathbf{k}) \). Here, if \( P = \sigma_j \ldots \sigma_{j_3} \) is a general permutation written with an \( r \) as small as possible, then
\[ \phi_{\eta}(P\mathbf{k}) = \sum_{r=1}^{r} \phi_{\eta} \left[ (\sigma_{j_1} \ldots \sigma_{j_r} k_{j_1} \ldots k_{j_r}) \right]. \]
The basis functions are therefore of the form \( \Psi_k \propto \sum_{P \in S_n} e^{i\phi_{\eta}(P\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}} \). Divergent elements of this set are excluded by only permitting \( \mathbf{k} \) that are the concatenation [51] of disjoint tuples which are rigorously protected against permutation by Eq. (2).

We call these tuples clusters motivated by their physical interpretation, which is discussed in a moment. Technically, a cluster is a tuple of complex momenta \( \mathbf{\mu} = (\mu_1, \ldots, \mu_n) \) ordered by imaginary part, where the sum of all momenta is real and the difference of adjacent momenta \( \mu_{j+1} - \mu_j \) is either zero or \(-i\eta \). Thereby, a cluster defines an integer partition \( \mathbf{\nu} \) of \( n \) by counting the numbers of equal momenta starting from the left. Examples of clusters and their integer partitions are sketched in Fig. 2. We call a cluster irreducible if its elements cannot be reordered to form two (nonempty) clusters. Physically, irreducible clusters with more than one element represent composite anyons whose constituents are localized within a characteristic length scale of \( 1/\eta \) from each other and move collectively. They should be conceived as individual particle themselves.

Each possible cluster structure, as defined by the integer partition, represents a different composite anyon species. We elaborate on this below. For positive \( \eta \), irreducible clusters only consist of single particles, which describes free, unbound, anyons. In order to uniquely label a basis function, we introduce the cluster ordering \( \mathcal{O} \). To apply \( \mathcal{O} \) to a tuple \( \mathcal{D} \) of irreducible clusters, first merge clusters that are not disjoint to larger clusters by taking their union and reordering appropriately. Then, sort the resulting clusters by real part and lexicographically by their integer partitions. Finally, concatenate the clusters in this order to obtain the tuple \( \mathcal{O}(\mathcal{D}) \).

In conclusion, the basis wave functions describe composite and free anyons in momentum space. Given an ordered tuple \( \mathcal{O}(\mathcal{D}) \) of irreducible clusters, the correspond-
ing basis function obtains the form
\[ \Psi_{(k=\mathcal{O}(D))}(x) = N_k \sum_{p \in S_n} e^{i\varphi_p(k)} e^{i(pk)x}, \]
where \( N_k \) is the normalization \(^{52}\) and \( \varphi_p(k) \) plays the role of a generalized Slater determinant.

**Second quantization** – Given the basis wave functions of Eq. \(^{49}\), second quantization amounts to defining creation operators to construct all basis states from a vacuum state \(^{53}\). For an irreducible cluster \( \mu \), we define its creation operator by
\[ a_{\mu}^{\dagger} \Psi_{\mathcal{O}(D)} = \sqrt{n_{\mu}} + 1 \ e^{i\varphi_{n\mu}(\mu)} \Psi_{\mathcal{O}(\{\mu\};JD)} \]
and linear continuation to all states. Here, \( n_{\mu} \) is the number of irreducible clusters \( \mu \) in \( \mathcal{D} \). The phase \( \Phi_{\mu}(M) = \sum_{\mu \in \mathcal{O}} \phi_{\mu}(\mu) \), is composed of the cluster-cluster exchange phases \( \Phi_{\mu}(\mu) = \sum_{j=1}^{n_{\mu}} \phi_{\mu}(\mu_j - \mu_1) \). Employing the latter, the algebra of the cluster creation operators is
\[ a_{\mu_1}^{\dagger} a_{\mu_2}^{\dagger} = e^{i\varphi_{\mu_1\mu_2}} a_{\mu_2}^{\dagger} a_{\mu_1}^{\dagger}. \]
To be concrete, we consider the case of unbound anyons, described by clusters with exactly one element. Here,
\[ a_{\mu_1}^{\dagger} a_{\mu_2} = e^{-i\varphi_{\mu_1}(p-q)} a_{\mu_2}^{\dagger} a_{\mu_1}^{\dagger}, \]
where the annihilation operator \( a_p \) is the hermitian conjugate of \( a_p^{\dagger} \). The real space algebra for
\[ \Psi^{\dagger}(x) = \int_{-\infty}^{\infty} dp \ e^{i\varphi_p(x)} a_p^{\dagger}, \]
is readily obtained \(^{42}\) to be
\[ \{ \Psi^{\dagger}(x), \Psi^{\dagger}(y) \} = \int_{0}^{\infty} \frac{dz \ e^{-\eta z}}{|\eta|} \Psi^{\dagger}(y-z) \Psi^{\dagger}(x+z), \]
which yields a smeared anyonic Pauli principle for \( x = y \).

Finally, it is well known that the concept of statistics in one dimension is fuzzy as there exist several ways to transform between different statistics \(^{21,54}\). Likewise here, there is a generalized Jordan-Wigner transformation from anyons to bosons, which we derive in \(^{42}\).

**Systems of finite size** – When anyons are confined to the length \( L \), one would expect the Dirichlet boundary conditions \( \Psi(0,x_2,\ldots,x_n) = \Psi(x_1,\ldots,x_{n-1},L) = 0 \) to quantize the allowed momenta, similar to the particle in a box problem. In fact, the conditions translate to
\[ \alpha(-k_1,\ldots,k_n) = -\alpha(k), \]
\[ \alpha(k_1,\ldots,-k_n) = -e^{2i\pi k_n} L \alpha(k). \]
These constraints are only consistent with Eqs. \(^{4} \) and \(^ {41}\) if the system of transcendental equations
\[ L k_j + \sum_{1 \leq (i \neq j) \leq n} [\phi_{\eta}(k_i - k_j) - \phi_{\eta}(k_i + k_j)]/2 = \pi z_j \]
is fulfilled for \( j \) between 1 and \( n \). Here, the \( z_j \) are positive integers. The momenta that solve Eq. \(^{12}\) are discrete and readily numerically obtainable.

**Application** – Equipped with the developed formalism, we now consider observables of experimental interest. First, we calculate the spectrum of two confined anyons numerically by solving Eq. \(^{12}\) as depicted in Fig. \(^{2}\). The anyonic spectra interpolate between the familiar bosonic and fermionic particle-in-a-box spectra for positive \( \eta \). For instance, in units of \( \hbar^2 \pi^2/(2mL^2) \), the bosonic level at energy 2 continuously evolves to the fermionic level at energy 5. At negative \( \eta \), the anyonic levels form two-anyon bound states with an energy proportional to \(-\eta^2\) in the infinite-size limit \( L \rightarrow \infty \). Energetically higher anyonic bound states correspond to kinematic excitations of the composite anyon in analogy to the behavior of a single particle in a box. Some anyonic levels refuse to form bound states and instead converge to fermionic energies as \( \eta \rightarrow -\infty \). These levels ensure that the finite-size spectrum coherently converges to the infinite-size spectrum. Energy spectra could be a viable observable in systems with few anyons, like, potentially, interacting cold-atom chains \(^{20,21,54}\), and detectable by spectroscopic techniques. Turning to systems containing many anyons, as possibly the case in solid state systems, unavoidable level broadening renders an accurate measurement of the discrete spectrum unfeasible; yet the momentum distribution could uncover the character of the anyons \(^{20}\). We depict the momentum density \( n_k \) at zero temperature in Fig. \(^{2}\). This function gives the number of anyons with momentum between \( k_1 \) and \( k_2 \) by \( \int_{k_1}^{k_2} dk n_k \). For bosons and fermions, it is proportional to the Bose-Einstein and Fermi-Dirac distribution, respectively. Anyons with a positive statistical parameter transform these distributions into each other, still remaining a sharply defined chemical potential reflected by a discontinuity in \( n_k \). If the spectral properties of a system are inaccessible, the statistics is still inferable via local properties, e.g., the finite size density fluctuations \(^{21,45}\). While bosons condense to the middle of the system, fermions distribute equally spaced (by Pauli repulsion), resulting in oscillations of the particle density. Figure \(^{2}\) depicts the scenario for four anyons in the ground state. Unbound anyons suppress the fermionic peaks and broaden the bosonic one, which is characteristic to intermediate statistics \(^{21,45}\).

The quantum-statistical condensate – For \( \eta < 0 \), the anyonic ground state is a maximally bound cluster of the form \( \mu_j = j(\eta - (n - 1)/2) \) as depicted in Fig. \(^{4}\). We call this state the quantum-statistical condensate. Being an irreducible cluster it can be conceived as a single composite anyon. Therefore, its local density is the same as for the Bose-Einstein condensate, as depicted in Fig. \(^{2}\). In fact, the Bose-Einstein condensate can be interpreted as the limit of the quantum-statistical condensate as \( \eta \rightarrow 0^+ \). Besides this, both condensates dif-
fer profoundly: bosons condense into their single-particle ground state, but anyons into an inseparable many-body ground state. Let us derive further characteristics of the quantum-statistical condensate. First, we obtain its ground state energy

$$\epsilon_{GS} = -\frac{\hbar^2}{24m} \eta^2(n-1)n(n+1)$$

by Eq. (13). The proportionality to $n^3$ reveals an exceptional stability of the condensate. Let us, for a moment, regard charged anyons exhibiting Hubbard repulsion, which is proportional to $n^2$. Then, providing a sufficiently large number of anyons, the negative statistical energy outperforms the positive one created by charge repulsion. The quantum-statistical condensate is hence stable against the introduction of charge. We conjecture that this property could lead to remarkable anyon superconductivity [55, 57]. Next, we consider the excitations of the statistical condensate, i.e., the anyonic clusters themselves. A cluster behaves as an individual anyon, the energy of which separates into a kinetic and an internal part. Additionally, by Eq. (13), clusters acquire different statistical phases than their constituents. For instance, clusters of two anyons behave like anyons with the statistical phase $2\phi_\eta + \phi_{2\eta}$ [42]. In the vocabulary of topological field theories for two-dimensional anyons [3, 17, 58], the formation of clusters is linked to anyon fusion [59]. Interestingly, the composition process is not unique for clusters containing more than two anyons. This property corresponds to the existence of different fusion channels, leading to non-abelian anyons by the condensation of abelian ones [58, 60]. Non-abelian anyons should be reflected by a systematic degeneracy of the spectrum. To see this degeneracy, note that inverting and negating a cluster leaves its energy unchanged. We call this operation cluster conjugation. Examples for conjugate clusters are $-\eta\mu (-1, -1, 2)$ and $-\mu (-2, 1, 1)$, depicted in Fig. [42]. Given the aforementioned results, the anyons treated in this manuscript neither behave like the more familiar Ising anyons (or Majorana zero modes [24]) nor like Fibonacci anyons [61] but generally in a more complex way.

Conclusions – On the basis of the general assumptions of [1], we derive an exact quantum many-body formalism for one-dimensional anyons including the exact wave functions, the second quantization, and the momentum discretizing equations for anyons in a box. Employing this formalism, we numerically calculate characteristic observables, namely, the energy spectrum, the momentum statistics, and the finite-size density fluctuations. For a negative statistical parameter, anyons attract each other with a force purely induced by their quantum statistics and form the quantum-statistical condensate. This genuine many-body phase possesses a rich structure of degenerate internal excitations and is more stable than the Bose-Einstein condensate. In particular, the statistical condensate is stable for charged anyons in the presence of Coulomb repulsion. Because of these properties, we deem the statistical condensate a promising candidate for exhibiting anyonic superconductivity and carrying non-abelian anyons. Our work shows that one-dimensional anyons exhibit original and interesting physics even in the absence of exchangeability.

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The concatenation of tuples means to write their elements in a row to form a combined tuple, i.e., $k = (\mu_1, \mu_2, \ldots, \mu_{m_n}, \ldots, \mu_{m_1})$. If $\mu_1, \ldots, \mu_{m_n}$ are tuples of lengths $n_1, \ldots, n_m$ and $k$ is their concatenation.

For real vectors $k$, we have $N_k = \sqrt{1/(2\pi^n n!)}$.

Technically, we define the vacuum state as $\Psi_\mu = 1$.

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We relate to the physically motivated concept of anyon fusion. A rigorous mathematical mapping of our theory to a topological field theory is not straightforwardly given.

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SUPPLEMENTAL MATERIAL

With this supplemental material, we support the main manuscript by giving an overview about intermediate statistics in one dimension, elaborate on the interpretation of anyon clusters as individual anyons, provide the real space algebra for anyonic creation and annihilation operators, and derive the generalized Jordan-Wigner transformation to convert from an anyonic second quantized algebra to a bosonic one.

Notions of intermediate statistics in one spatial dimension

There exists a variety of formalisms describing particles of intermediate statistics in one dimension, which are expected to be applicable to different physical situations. Albeit they differ in their phenomenology, these particles are all occasionally called anyons. For clarity, we shortly discuss the most broadly known theories that are applicable to one spatial dimension.

If the occupation number of a single particle quantum state is restricted to maximally assume a given integer, the particles can be described as parafermions, which are closely related to Potts and clock models and Gentile statistics. Such particles are, amongst others, expected to exist as magnetic excitations separately in the case of a Tomonaga-Luttinger liquid. It is known that these particles (considering each channel the Haldane-Shastry chain, and the fractional systems linked to the Calogero-Sutherland model, the Haldane-Shastry chain, and the fractional excitations in Tomonaga-Luttinger liquids. It is known that these particles (considering each channel separately in the case of a Tomonaga-Luttinger liquid) break time reversal symmetry on the fundamental level of their quantum brackets, reflected by an asymmetric momentum distribution.

Interpretation of anyon clusters as individual anyons

We want to show how the exchange phase of clusters can be interpreted as the statistical phase of a composite species of anyons reaching further than the interpretation supported by Eq. (S1). To this end, we consider two clusters of anyons \( \mu_1 = (K_1 + i\eta/2, K_1 - i\eta/2) \) and \( \mu_2 = (K_2 + i\eta/2, K_2 - i\eta/2) \), the cluster structures of which are depicted in Fig. S1. We introduce the center of mass coordinates \( X_1 = (x_1 + x_2) / 2 \) and \( X_2 = (x_3 + x_4) / 2 \), as well as the separation coordinates \( Z_1 = (x_2 - x_1) / 2 \) and \( Z_2 = (x_4 - x_3) / 2 \). Under the assumption that the two clusters are sufficiently far away from each other, i.e., \( X_2 - X_1 \to \infty \) and \( Z_1, Z_2 \) finite, we obtain

\[
\Psi_{(\mu_1, \mu_2)}(X_1, X_2, Z_1, Z_2) \propto \left[ e^{i2i(K_1 X_1 + K_2 X_2)} + e^{i2\eta_1 \mu_2} e^{i2i(K_2 X_1 + K_1 X_2)} \right] e^{\eta(Z_1 + Z_2)}. \tag{S1}
\]

This wave function obtains the form of a wave function of two composite anyons with an altered statistical phase of \( \nu_1, \mu_2 \), especially if we recall that \( Z_1 \) and \( Z_2 \) are of the order of \( 1/\eta \). This can be physically interpreted as the fusion of anyons to clusters which themselves behave as a composite anyon species. Interestingly, the new statistical phase is

\[
\nu^{\mu_1, \mu_2} = 2\phi_\eta(K_2 - K_1) + \phi_\eta(K_2 - K_1) \tag{S2}
\]

where \( \phi_\eta \) is the statistical phase defined in Eq. (5) of the main manuscript. This has an appealing geometric interpretation, which we depict in Fig. S1.
Real space operator algebra for single anyons

Given the momentum space operator algebra of Eq. (9), we can ask about the algebra of the real space operators \( \Psi \dagger(x) = \int_{-\infty}^{\infty} dp \frac{e^{-i p x}}{\sqrt{2\pi a}} a_p \). We obtain

\[
\{ \Psi \dagger(x), \Psi(y) \} = \delta(x-y) + \int_{0}^{\infty} \frac{2e^{-z/|\eta|}}{|\eta|} \Psi \dagger(y-z) \Psi(x-z),
\]

\[
\{ \Psi \dagger(x), \Psi \dagger(y) \} = \int_{0}^{\infty} \frac{2e^{-z/|\eta|}}{|\eta|} \Psi \dagger(y-z) \Psi \dagger(x+z),
\]

where \( \{\ldots, \ldots\} \) is the anticommutator. Here, \( \lim_{\eta \to 0} \int_{0}^{\infty} 1/|\eta| e^{-z/|\eta|} f(z) = f(0) \) yields the bosonic commutation algebra, while the fermionic anticommutation relations for \( \eta \to \infty \) are trivially contained. If we set \( x = y \), we obtain a generalized, smeared Pauli principle in real space represented by

\[
(\Psi \dagger(x))^2 = \int_{0}^{\infty} dz \frac{1}{|\eta|} e^{-z/|\eta|} \Psi \dagger(x-z) \Psi \dagger(x+z). \quad \text{(S4)}
\]

Generalized Jordan Wigner transformation: anyons to bosons

It is possible to describe the anyonic creation and annihilation operators in terms of bosonic ones, as inspired by the Jordan-Wigner transformation of Ref. [20]. Ultimately, this reflects the fact that the Fock space of anyons is isomorphic to the one of bosons (if \( \eta \neq \pm \infty \)). To this end, consider the bosonic operators \( b \) with the algebra \( [b_k, b_l^\dagger] = \delta(k-l) \) and \( [b_k, b_l^\dagger] = 0 \) with \( k, l \in \mathbb{R} \). For \( \eta \neq \pm \infty \), we define the generalized Jordan-Wigner transformation

\[
\tilde{a}(j) = \lim_{\epsilon \to 0^+} e^{i \int_{j-\epsilon}^{j} \Psi \dagger(k) b_k \phi_\eta(k-j) b(k)}. \quad \text{(S5)}
\]

Calculating the algebra of \( \tilde{a} \), we find \( \tilde{a}_j \tilde{a}_k = \tilde{a}_k \tilde{a}_j e^{i \phi_\eta(j-k)} \) and \( \tilde{a}_j \tilde{a}_k^\dagger = \tilde{a}_k^\dagger \tilde{a}_j e^{-i \phi_\eta(j-k)} + \delta(j-k) \), which is exactly the anyonic algebra described in Eq. (9). Interestingly, we have \( \tilde{a}_k^\dagger \tilde{a}_k = b_k^\dagger b_k \), which results in the same free Hamiltonian using either the bosonic or the anyonic description.

\[
H = \int dk \epsilon(k) b_k^\dagger b_k = \int dk \epsilon(k) \tilde{a}_k^\dagger \tilde{a}_k. \quad \text{(S6)}
\]

Hence, Hamiltonians that are diagonal in momentum space acquire the same form when represented either in anyonic or bosonic degrees of freedom. In order to distinguish between anyons or bosons on a line, it is therefore important to regard scattering processes of anyons, or non-momentum-conserving interactions, or single-particle observables that are not diagonal in momentum space. For finite systems, the difference between anyons and bosons is more obvious by the momentum quantization described in the main manuscript.