LAURENT SERIES OF HOLOMORPHIC FUNCTIONS SMOOTH UP TO THE BOUNDARY

ANIRBAN DAWN

Abstract. It is shown that the Laurent series of a holomorphic function smooth up to the boundary on a Reinhardt domain in $\mathbb{C}^n$ converges unconditionally to the function in the Fréchet topology of the space of functions smooth up to the boundary.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a domain i.e. $\Omega$ is an open and connected subset of $\mathbb{C}^n$. Denote by $\mathcal{A}^\infty(\Omega)$ the space of holomorphic functions smooth up to the boundary of $\Omega$, i.e. the space of holomorphic functions whose derivatives of all orders can be extended continuously up to the boundary. For a sequence of functions $\{f_j\} \subset \mathcal{A}^\infty(\Omega)$, $f_j \to f$ in $\mathcal{A}^\infty(\Omega)$ means that for every compact subset $K \subset \overline{\Omega}$, $f_j \to f$ uniformly on $K$ along with all partial derivatives. In particular, if $\Omega$ is bounded, then $f_j \to f$ in $\mathcal{A}^\infty(\Omega)$ means that $f_j \to f$ uniformly on $\overline{\Omega}$ along with all partial derivatives.

Recall that a domain $\Omega \subset \mathbb{C}^n$ is called a Reinhardt domain if for $z = (z_1, \cdots, z_n) \in \Omega$, one has $(\lambda_1 z_1, \cdots, \lambda_n z_n) \in \Omega$, where $|\lambda_j| = 1$ for $j = 1, 2, \cdots, n$. For a detailed expository of Reinhardt domains, see [6]. Let $\Omega$ be a Reinhardt domain and $f \in \mathcal{O}(\Omega)$, the space of holomorphic functions on $\Omega$. It is well known that $f$ admits a unique Laurent series expansion which converges absolutely and uniformly on compact subsets of $\Omega$ to the function $f$, i.e. the Laurent series of $f$ converges to $f$ in the Fréchet topology of $\mathcal{O}(\Omega)$ (cf. [9, p. 46]). The focus of this paper is to prove a result similar to this, for the space $\mathcal{A}^\infty(\Omega)$.

Theorem 1.1. Let $\Omega$ be a Reinhardt domain in $\mathbb{C}^n$ and $f \in \mathcal{A}^\infty(\Omega)$. Then the Laurent series of $f$ converges unconditionally to the function $f$ in the topology of $\mathcal{A}^\infty(\Omega)$.

We say a formal series $\sum_{\alpha \in \Gamma} x_\alpha$, where $\Gamma$ is a countable index set, in a locally convex topological vector space (LCTVS) $X$ is unconditionally convergent if for every bijection $\sigma : \mathbb{N} := \{0, 1, 2, \cdots \} \to \Gamma$, the series $\sum_{j=0}^{\infty} x_{\sigma(j)}$ converges in the topology of $X$ (cf. [11, p. 9]).

Convergence results similar to Theorem 1.1 for other classical function spaces have been proved. For $1 < p < \infty$, it is well known that the partial sums of the Taylor series of $f$ in $H^p(\mathbb{D})$, the Hardy space on the unit disc in $\mathbb{C}$, converges to $f$ in the $H^p(\mathbb{D})$ norm (cf. [4, p. 104-110]). For the same range of $p$, convergence of partial sums of the Taylor series of $f$ in $A^p(\mathbb{D})$, the space of holomorphic $L^p$ functions on the unit disc, has been proved in [11]. For $p = 1$, the sequence of partial sums does not converge in $H^1(\mathbb{D})$ and $A^1(\mathbb{D})$ norms. However, it has been shown in [8] that the sequence of partial sums of $f \in H^1(\mathbb{D})$ is norm convergent in the weaker norm $A^1(\mathbb{D})$. A more general result can be found in [8] where it is proved that for a bounded Reinhardt domain $\mathcal{R}$ in $\mathbb{C}^n$, the “square partial sums” of the Laurent series of $f$ in $A^p(\mathcal{R})$ converges to the function $f$ in the $A^p(\mathcal{R})$ norm. Notice that for a general Reinhardt domain $\Omega$, the convergence of the Laurent series in $\mathcal{A}^\infty(\Omega)$ and
The topology of \( \mathcal{A}^\infty(\Omega) \) is interesting, which is not the case in \( H^p(D) \) or \( A^p(\mathbb{R}) \).

Theorem \[1.1\] is interesting because of the intrinsic importance of the space \( \mathcal{A}^\infty(\Omega) \) in complex analysis. For example, it is known that each smoothly bounded pseudoconvex domain \( \Omega \) is a so called \( \mathcal{A}^\infty(\Omega) \)-domain of holomorphy (see [1] and [5] for details). However, it is also known that pseudoconvex domains with non-smooth boundaries may not be \( \mathcal{A}^\infty(\Omega) \)-domain of holomorphy. This was first noticed for Hartogs triangle \( \{ z = (z_1, z_2) : |z_1| < |z_2| < 1 \} \subset \mathbb{C}^2 \) by Sibony (cf. [10]) and generalised recently by Chakrabarti to Reinhardt domains in \( \mathbb{C}^n \) with 0 as a boundary point (cf. [2]).

The paper is organised as follows. In Section 2.2 we define the notion of absolute convergence of a series in \( \mathcal{A}^\infty(\Omega) \) and we prove that absolute convergence of a series in \( \mathcal{A}^\infty(\Omega) \) implies unconditional convergence. In addition, we show that absolutely convergent series in \( \mathcal{A}^\infty(\Omega) \) converges in the net of partial sums (see Section 2.2 for more details). At the end we prove Theorem \[1.1\] in Section 3 by showing that the Laurent series of \( f \in \mathcal{A}^\infty(\Omega) \) converges absolutely (and therefore unconditionally) in \( \mathcal{A}^\infty(\Omega) \) to the function \( f \).

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2. Preliminaries

2.1. **The topology of \( \mathcal{A}^\infty(\Omega) \).** Let \( \Omega \subset \mathbb{C}^n \) be a domain. We now describe the topology of the space \( \mathcal{A}^\infty(\Omega) \), the space of holomorphic functions smooth up to the boundary of \( \Omega \). First, assume \( \Omega \) is bounded. Then \( \mathcal{A}^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} \mathcal{A}^k(\Omega) \), where for every \( k \in \mathbb{N} \), \( \mathcal{A}^k(\Omega) := \mathcal{C}^k(\overline{\Omega}) \cap \mathcal{O}(\Omega) \) and \( \mathcal{C}^k(\overline{\Omega}) \) denotes the space of \( k \)-times continuously differentiable \( \mathbb{C} \)-valued functions whose derivatives up to order \( k \) can be extended continuously up to the boundary of \( \Omega \). The space \( \mathcal{A}^\infty(\Omega) \) is a Fréchet space and its Fréchet topology is generated by the \( \mathcal{C}^k \)-seminorms given by,

\[
\|f\|_{k,\Omega} := \sup \left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha| \leq k \right\}
\]  
(2.1)

where \( k \) ranges over \( \mathbb{N} \), \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \) is a multi-index with length \( |\alpha| = \sum_{j=1}^{n} \alpha_j \), and

\[
D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.
\]  
(2.2)

Recall that a collection of seminorms \( \{p_k : k \in \Lambda\} \), where \( \Lambda \) is an index set, generates the topology of an LCTVS \( X \) if for every continuous seminorm \( p \) of \( X \), there exists a finite subset \( \{k_1, k_2, \cdots, k_\ell\} \) of \( \Lambda \) and a constant \( C > 0 \) such that

\[
p(x) \leq C \cdot \sup_{1 \leq j \leq \ell} p_{k_j}(x) \quad \text{for all } x \in X.
\]  
(2.3)

Now, assume \( \Omega \) is unbounded. For \( m \in \mathbb{N} \), let \( \Omega_m = \Omega \cap P_m \) where \( P_m = \{ z : |z_j| < m \text{ for all } j \} \) is the polydisc of radius \( m \). Then \( \Omega_m \) is bounded for each \( m \) and we write \( \Omega = \bigcup_{m=0}^{\infty} \Omega_m \). The Fréchet topology of \( \mathcal{A}^\infty(\Omega) \) is generated by the collection of seminorms \( \{\|f\|_{k,\Omega_m} : k, m \in \mathbb{N}\} \), where

\[
\|f\|_{k,\Omega_m} := \sup \left\{ |D^\alpha f(z)| : z \in \Omega_m, |\alpha| \leq k \right\}.
\]  
(2.4)
Note that for a sequence of functions \( \{f_N\} \subset \mathcal{A}_m(\Omega) \), \( f_N \to f \) in \( \mathcal{A}_\infty(\Omega) \) as \( N \to \infty \) if and only if \( f_N \to f \) in \( \mathcal{A}_m(\Omega_m) \) for every \( m \in \mathbb{N} \), as \( N \to \infty \).

Now we describe another collection of seminorms that generates the same locally convex topology of \( \mathcal{A}_\infty(\Omega) \), where \( \Omega \subset \mathbb{C}^n \) is bounded. For \( \alpha \in \mathbb{Z}^n \), we define
\[
|\alpha|_\infty := \max \{ |\alpha_j| : 1 \leq j \leq n \}. \tag{2.5}
\]

For \( k \in \mathbb{N} \), define
\[
\tilde{\mathcal{A}}^k(\Omega) := \left\{ f \in \mathcal{A}^k(\Omega) : D^\alpha(f) \in \mathcal{A}^0(\Omega) \text{ where } |\alpha|_\infty \leq k \right\}, \tag{2.6}
\]
where \( \alpha = (\alpha_1, \cdots, \alpha_n) \) is a multi-index in \( \mathbb{N}^n \) and \( D^\alpha \) is defined in (2.2). Note that \( \tilde{\mathcal{A}}^k(\Omega) \) is a Banach space with the norm,
\[
\|f\|_{k,\Omega} = \sup \left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha|_\infty \leq k \right\}. \tag{2.7}
\]
When \( n = 1 \), \( \tilde{\mathcal{A}}^k(\Omega) \) coincides with \( \mathcal{A}^k(\Omega) \). Observe that for \( n \geq 2 \), \( \mathcal{A}^{nk}(\Omega) \subsetneq \tilde{\mathcal{A}}^k(\Omega) \subsetneq \mathcal{A}^k(\Omega) \). Moreover, for each \( k \in \mathbb{N} \), the inclusion maps \( \mathcal{A}^{nk}(\Omega) \hookrightarrow \tilde{\mathcal{A}}^k(\Omega) \hookrightarrow \mathcal{A}^k(\Omega) \) are bounded with norm 1. The next result is now immediate.

**Lemma 2.1.** For a bounded \( \Omega \subset \mathbb{C}^n \), the collection of seminorms \( \{\|\cdot\|_{k,\Omega} : k \in \mathbb{N}\} \) generates the same Fréchet topology of \( \mathcal{A}_\infty(\Omega) \) as the collection \( \{\|\cdot\|_{k,\Omega} : k \in \mathbb{N}\} \), the \( C^k \)-seminorms of \( \Omega \).

**Proof.** Let \( k \in \mathbb{N} \). Note that for every \( f \in \mathcal{A}_\infty(\Omega) \),
\[
\|f\|_{k,\Omega} = \sup \left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha|_\infty \leq k \right\}
= \sup \left\{ |D^\alpha f(z)| : z \in \Omega, \alpha_j \leq k \text{ for all } j = 1, 2, \cdots, n \right\}
\leq \sup \left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha| \leq nk \right\} = \|f\|_{nk,\Omega}.
\]
Also observe that for every \( f \in \mathcal{A}_\infty(\Omega) \),
\[
\|f\|_{k,\Omega} = \sup \left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha| \leq k \right\}
\leq \sup \left\{ |D^\alpha f(z)| : z \in \Omega, \alpha_j \leq k \text{ for all } j = 1, 2, \cdots, n \right\}
= \sup \left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha|_\infty \leq k \right\} = \|f\|_{k,\Omega}.
\]

\[\square\]

### 2.2. Absolute and unconditional convergence

A formal series \( \sum_{\alpha \in \Gamma} x_\alpha \) in an LCTVS \( X \), where \( \Gamma \) is a countable index set, is said to be **absolutely convergent** if there exists a bijection \( \sigma : \mathbb{N} \to \Gamma \) such that for every continuous seminorm \( p \) of \( X \), \( \sum_{j=0}^\infty p(x_{\sigma(j)}) \) is a convergent series of non-negative real numbers. Let \( \mathcal{P} \) be a collection of continuous seminorms that generates the topology of \( X \). Then, to prove absolute convergence of the series \( \sum_{\alpha \in \Gamma} x_\alpha \), it is sufficient to show that there exists a bijection \( \sigma : \mathbb{N} \to \Gamma \) such that for every \( p \in \mathcal{P} \), the series \( \sum_{j=0}^\infty p(x_{\sigma(j)}) < \infty \).
Let \( (\Gamma, \geq) \) be a directed set. A net \((s_\alpha)\) in \(X\) is said to be a \textit{Cauchy net} if for every \(\epsilon > 0\) and every continuous seminorm \(p\) of \(X\), there exists \(\gamma \in \Gamma\) such that whenever \(\alpha, \beta \geq \gamma\), we have \(p(s_\alpha - s_\beta) < \epsilon\). The net \((s_\alpha)\) converges to an element \(s \in X\) if for every \(\epsilon > 0\) and every continuous seminorm \(p\) of \(X\), there exists \(\gamma \in \Gamma\) such that whenever \(\alpha \geq \gamma\), we have \(p(x_\alpha - x) < \epsilon\). The space \(X\) is said to be \textit{complete} if every Cauchy net of \(X\) converges. Note that replacing continuous seminorms by continuous generating seminorms one can give equivalent definitions of Cauchy net and convergence in an LCTVS. The next result shows that in a complete LCTVS, an absolutely convergent series is unconditionally convergent.

\textbf{Lemma 2.2.} Let \(\sum_{\alpha \in \Gamma} x_\alpha\) be an absolutely convergent series in a complete LCTVS \(X\), where \(\Gamma\) is a countable index set. Then the series converges unconditionally.

\textit{Proof.} Let \(P\) be a collection of generating seminorms of \(X\) and \(p \in P\). Since \(\sum_{\alpha \in \Gamma} x_\alpha\) is absolutely convergent, there exists a bijection \(\sigma : \mathbb{N} \to \Gamma\) such that the series \(\sum_{j=0}^{\infty} p(x_{\sigma(j)})\) converges. Let \(y_j = x_{\sigma(j)}\) and \(s_k = \sum_{j=0}^{k} y_j\). Since \(\sum_{j=0}^{\infty} p(y_j)\) converges, for \(\epsilon > 0\) there exists \(N_0 \in \mathbb{N}\) such that whenever \(m, \ell \in \mathbb{N}\) with \(m \geq \ell \geq N_0\), \(\sum_{j=\ell+1}^{m} p(y_j) < \epsilon\). Therefore for \(m \geq \ell \geq N_0\),

\[ p(s_m - s_\ell) = p\left( \sum_{j=\ell+1}^{m} y_j \right) \leq \sum_{j=\ell+1}^{m} p(y_j) < \epsilon. \]  

(2.8)

It follows from (2.8) that the net \(\{s_k\}\) is Cauchy in a complete LCTVS \(X\), with directed set \((\mathbb{N}, \geq)\), and therefore converges. Let \(s_k \to s\) as \(k \to \infty\). In order to complete the proof, it suffices to show that for every bijection \(\tau : \mathbb{N} \to \mathbb{N}\), the series \(\sum_{j=0}^{\infty} \tau(y_j)\) converges to the same limit \(s\). Let \(s_k^\tau = \sum_{j=0}^{k} \tau(y_j)\). We show \(s_k^\tau \to s\) as \(k \to \infty\). Choose \(u \in \mathbb{N}\) such that the set of integers \(\{0, 1, 2, \cdots, N_0\}\) is contained in the set \(\{\tau(0), \tau(1), \cdots, \tau(u)\}\). Then, if \(k > u\), the elements \(y_1, \cdots, y_{N_0}\) get cancelled in the difference \(s_k - s_k^\tau\) and we have \(p(s_k - s_k^\tau) < \epsilon\) by (2.8). This proves that the sequence \(\{s_k\}\) and \(\{s_k^\tau\}\) converges to the same sum. So, \(s_k^\tau \to s\) as \(k \to \infty\). \(\square\)

\textbf{2.3. Convergence in the net of partial sums.} Let \(\Gamma\) be a countable index set. Let \((\mathfrak{F}(\Gamma), \subset)\) be the directed set of all finite subsets of \(\Gamma\) with inclusion as its order. The net \(\{\{\sum_{\alpha \in J} x_\alpha\}, J \in \mathfrak{F}(\Gamma)\}\) is said to be the \textit{net of partial sums} of the formal series \(\sum_{\alpha \in \Gamma} x_\alpha\) in an LCTVS \(X\). Let \(P\) be a collection of continuous seminorms that generates the topology of \(X\). The net of partial sums of the series is said to be convergent if for every \(\epsilon > 0\) and every \(p \in P\), there exists \(I \in \mathfrak{F}(\Gamma)\), such that for all \(J \in \mathfrak{F}(\Gamma)\) with \(I \subset J\), \(p(\sum_{\alpha \in J} x_\alpha) < \epsilon\).

The net of partial sums of the series is \textit{Cauchy} if for every \(\epsilon > 0\) and \(p \in P\) there exists \(I \in \mathfrak{F}(\Gamma)\) such that for all \(J, K \in \mathfrak{F}(\Gamma)\) with \(I \subset J\) and \(I \subset K\),

\[ p\left( \sum_{\alpha \in J} x_\alpha - \sum_{\alpha \in K} x_\alpha \right) < \epsilon. \]

The next result shows that if a series is absolutely convergent in a complete LCTVS, then the net of partial sums of the series is Cauchy, hence converges.

\textbf{Lemma 2.3.} Let \(\sum_{\alpha \in \Gamma} x_\alpha\) be an absolutely convergent series in a complete LCTVS \(X\), where \(\Gamma\) is a countable index set. Then the net of partial sums of the series converges.

\textit{Proof.} Let \(P\) be a collection of continuous seminorms that generates the topology of \(X\) and \(p \in P\). Since the series \(\sum_{\alpha \in \Gamma} x_\alpha\) is absolutely convergent, there exists a bijection \(\sigma : \mathbb{N} \to \Gamma\)
such that the series of non-negative reals \( \sum_{j=0}^{\infty} p(x_{\sigma(j)}) \) converges. Let \( \epsilon > 0 \). There exists \( N \in \mathbb{N} \) such that for all \( m, k \in \mathbb{N} \) with \( m \geq k > N \),

\[
\sum_{j=k}^{m} p(x_{\sigma(j)}) < \epsilon/2. \tag{2.9}
\]

Let \( (\mathcal{F}(\Gamma), \subset) \) be the directed set of all finite subsets of \( \Gamma \) with inclusion as its order. Let \( I = \{ \sigma(0), \sigma(1), \ldots, \sigma(N) \} \), then \( I \in \mathcal{F}(\Gamma) \). Now, whenever \( J, K \in \mathcal{F}(\Gamma) \) with \( I \subset J \) and \( I \subset K \),

\[
p \left( \sum_{\alpha \in J} x_{\alpha} - \sum_{\alpha \in K} x_{\alpha} \right) \leq p \left( \sum_{\alpha \in J \setminus I} x_{\alpha} - \sum_{\alpha \in K \setminus I} x_{\alpha} \right) \leq p \left( \sum_{\alpha \in J \setminus I} x_{\alpha} \right) + p \left( \sum_{\alpha \in K \setminus I} x_{\alpha} \right) \leq \sum_{\alpha \in J \setminus I} p(x_{\alpha}) + \sum_{\alpha \in K \setminus I} p(x_{\alpha}) \leq \epsilon/2 + \epsilon/2 = \epsilon, \text{ from (2.9).}
\]

This shows that the net of partial sums of the series \( \sum_{\alpha \in \Gamma} x_{\alpha} \) is Cauchy. Since \( \Gamma \) is complete, it is convergent. \( \square \)

2.4. Covering a Reinhardt domain by polyannuli. Let \( \Omega \subset \mathbb{C}^n \) be a Reinhardt domain. For \( 1 \leq j \leq n \), let \( A_j \subset \mathbb{C}^1 \) be either an annulus \( \{ r < |z| < R \} \) or a disc \( \{ |z| < R \} \), where \( 0 < r < R < \infty \). Let \( P = \prod_{j=1}^{n} A_j \). The set \( P \) is said to be a polyannulus in \( \Omega \).

**Lemma 2.4.** Each Reinhardt domain in \( \mathbb{C}^n \) is a union of polyannuli.

**Proof.** Let \( \Omega \subset \mathbb{C}^n \) be Reinhardt and let \( P \) be a polyannulus contained in \( \Omega \). Then \( \bigcup_{P \subset \Omega} P \subset \Omega \). On the other hand let \( a \in \Omega \). Since \( \Omega \) is open, there exists \( \rho > 0 \) such that \( B_{\rho}(a) \), the ball centered at \( a \) with radius \( \rho \) is contained in \( \Omega \). Denote by \( |\Omega| \) the Reinhardt shadow of \( \Omega \), that is \( |\Omega| := \{ (|z_1|, \ldots, |z_n|) : z \in \Omega \} \). Let \( \tau : \Omega \to |\Omega| \) be the map defined by

\[
\tau(z) = (|z_1|, \ldots, |z_n|) \quad \text{for } z \in \Omega. \tag{2.10}
\]

Note that \( \tau \) is an open map. Therefore \( \tau(B_{\rho}(a)) \) is an open subset of \( |\Omega| \). So, there exists \( \delta > 0 \) such that \( \tau(a) \in \Delta_\delta(a) \subset \tau(B_{\rho}(a)) \), where \( \Delta_\delta(a) \) is the (open) \( n \)-cube in \( |\Omega| \) with sides \( \delta \) centered at \( \tau(a) \). Therefore \( a \in \tau^{-1}(\Delta_\delta(a)) \). Since \( \Omega \) is Reinhardt, \( \tau^{-1}(\Delta_\delta(a)) \) is a polyannulus in \( \Omega \). Consequently \( \bigcup_{P \subset \Omega} P \supset \Omega \). \( \square \)

3. Absolute convergence of Laurent series for a bounded domain

Let \( f = \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha} e_{\alpha} \) be the Laurent series expansion of \( f \in \mathcal{A}^\infty(\Omega) \), where \( e_{\alpha} \) denotes the Laurent monomial of exponent \( \alpha : e_{\alpha}(z) = z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \). In this section our goal is to prove the following result.

**Theorem 3.1.** Let \( \Omega \) be a bounded Reinhardt domain in \( \mathbb{C}^n \) and \( f \in \mathcal{A}^\infty(\Omega) \). Then the Laurent series of \( f \) converges absolutely in the topology of \( \mathcal{A}^\infty(\Omega) \).

Recall that if \( \Omega \) is bounded, the collection of seminorms \( \{ ||\cdot||_{k,\Omega} : k \in \mathbb{N} \} \) generates the Fréchet topology of \( \mathcal{A}^\infty(\Omega) \), where for each \( k \), the seminorm \( ||\cdot||_{k,\Omega} \) is as in (2.7). To prove Theorem 3.1, we need to show that there exists a bijection \( \sigma : \mathbb{N} \to \mathbb{Z}^n \) such that for every \( k \in \mathbb{N} \), \( \sum_{j=0}^{\infty} ||c_{\sigma(j)}e_{\sigma(j)}||_{k,\Omega} < \infty \).
The following proposition is the key to prove Theorem 3.1.

**Proposition 3.1.** Let $\mathcal{P}$ be a bounded polyannulus in $\mathbb{C}^n$, that is $\mathcal{P} = \prod_{j=1}^{n} A_j$ and for each $1 \leq j \leq n$, $A_j \subset \mathbb{C}^1$ is either an annulus $\{ z \in \mathbb{C}^1 : r_j < |z| < R_j \}$ or a disc $\{ z \in \mathbb{C}^1 : |z| < R_j \}$ where $0 < r_j < R_j < \infty$. For an integer $\ell$, consider

$$\mu_{\ell} = \begin{cases} \frac{1}{\ell(\ell - 1)} & \text{if } \ell \neq 0, 1, \\ 1 & \text{if } \ell = 0, 1. \end{cases} \quad (3.1)$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, $k \in \mathbb{N}$ and $M_{\alpha,k} = \prod_{j=1}^{n} \mu_{\alpha_j - k}$. Suppose $f = \sum_{\gamma \in \mathbb{Z}^n} c_{\gamma} e_{\gamma}$ is the Laurent series expansion of $f \in A^\infty(\mathcal{P})$, where $e_{\gamma}$ is the monomial function of exponent $\gamma$. Then

$$\|c_{\alpha} e_{\alpha}\|_{k,p} \leq M_{\alpha,k} \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \| f \|_{k+2,p}. \quad (3.2)$$

where $R_j$’s are as above and $\| \cdot \|_{k,p}$ is as in (2.7).

The following simple result is used in the proof of Proposition 3.1.

**Lemma 3.1.** Let $\Omega \subset \mathbb{C}$ be a domain and $g \in A^\infty(\Omega)$. Then

$$\frac{\partial^2 g}{\partial \theta^2} = - \left( z \frac{\partial g}{\partial z} + z^2 \frac{\partial^2 g}{\partial z^2} \right) \quad (3.3)$$

on $\overline{\Omega} \setminus \{ 0 \}$; where $z = re^{i\theta}$ is the natural coordinate on $\mathbb{C}$.

**Proof.** Since $g \in A^\infty(\Omega)$, we need to prove (3.3) at a point $z \in \Omega \setminus \{ 0 \}$. Since $g$ is holomorphic at $z$, the derivative $g'(z) = \frac{\partial g}{\partial z}(z)$ can be computed as a directional derivative in the direction perpendicular to the ray from 0 to $z$. Therefore with $z = re^{i\theta}$,

$$\frac{\partial g}{\partial z}(z) = \lim_{\phi \to \theta} \frac{g(re^{i\phi}) - g(re^{i\theta})}{re^{i\phi} - re^{i\theta}} = \frac{1}{r} \lim_{\phi \to \theta} \frac{\phi - \theta}{e^{i\phi} - e^{i\theta}} = \frac{1}{r} \frac{\partial g}{\partial \theta}(z) \cdot \frac{1}{i} \frac{1}{z} \frac{\partial g}{\partial \theta}(z). \quad (3.4)$$

This shows $\frac{\partial g}{\partial \theta} = iz \frac{\partial g}{\partial z}$. Differentiating once more with respect to $\theta$,

$$\frac{\partial^2 g}{\partial \theta^2} = i^2 re^{i\theta} \frac{\partial g}{\partial z} + iz \frac{\partial^2 g}{\partial z^2} ire^{i\theta} = - \left( z \frac{\partial g}{\partial z} + z^2 \frac{\partial^2 g}{\partial z^2} \right). \quad (3.5)$$

\[\square\]

**Proof of Proposition 3.1.** Let $Z = \{ z \in \mathbb{C}^n : z_j = 0 \text{ for some } j \}$. If $z \in Z \cap \mathcal{P}$, the result is trivial since the left hand side of (3.2) is identically 0. If $z \in \mathcal{P} \setminus Z$, one can write the coefficient $c_{\alpha} = c_{\alpha}(f) \in \mathbb{C}$ of the Laurent series of $f$ using Cauchy formula:

$$c_{\alpha} = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=|z_1|} \cdots \int_{|\zeta_n|=|z_n|} \frac{f(\zeta)}{\zeta^\alpha} \frac{d\zeta_n}{\zeta_n} \cdots \frac{d\zeta_1}{\zeta_1} \quad (3.6)$$
Note that if \( A_j \) is a disc for some \( j \), then \( c_\alpha = 0 \) whenever \( \alpha_j < 0 \). Fix some multi-index notations: \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) and \( z \cdot e^{i\theta} = (z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) \). Parametrize the contours in (3.6) by \( \zeta_j = z_j e^{i\theta_j} \) for every \( j \) to get

\[
c_\alpha = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \frac{f(z \cdot e^{i\theta})}{z^\alpha \exp(i(\alpha_1 \theta_1 + \cdots + \alpha_n \theta_n))} \ d\theta_n \cdots d\theta_1. \tag{3.7}
\]

We want to bound \( c_\alpha z^\alpha \) in the \( \| \|_{k, \mathbb{P}} \) seminorms. First consider the case \( k = 0 \). Introduce multi-index \( \beta \in \mathbb{Z}^n \) (depending on \( \alpha \)) as

\[
\beta_j = \begin{cases} 2 & \text{if } \alpha_j \neq 0, 1 \\ 0 & \text{if } \alpha_j = 0, 1. \end{cases}
\]

We claim that

\[
c_\alpha z^\alpha = \frac{z^\beta}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \left( \prod_{j=1}^{n} U(\alpha_j, \theta_j) \right) \frac{\partial^{\beta_1} f}{\partial \theta_1^{\beta_1} \cdots \partial \theta_n^{\beta_n}} (z \cdot e^{i\theta}) \ d\theta_n \cdots d\theta_1, \tag{3.8}
\]

where, for each \( 1 \leq j \leq n \)

\[
U(\alpha_j, \theta_j) = \begin{cases} e^{-i(\alpha_j-1)\theta_j} & \text{if } \alpha_j \neq 0, 1 \\ \frac{\alpha_j}{\alpha_j-1} & \text{if } \alpha_j = 0, 1. \end{cases}
\]

To prove (3.8), let us write (3.7) as

\[
c_\alpha z^\alpha = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \left( \int_{\theta_n=0}^{2\pi} \frac{f(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n})}{\exp(i\alpha_n \theta_n)} \ d\theta_n \right) \frac{d\theta_{n-1} \cdots d\theta_1}{\exp(i(\alpha_1 \theta_1 + \cdots + \alpha_{n-1} \theta_{n-1}))}. \tag{3.9}
\]

Let \( I_n \) be the integral inside the parentheses in (3.9). If \( \alpha_n = 0, 1 \), we set \( \beta_n = 0 \) and one can write an expression for \( I_n \) (in terms of \( \beta_n \)) directly from (3.9) as

\[
I_n = z_n^{\beta_n} \int_{\theta_n=0}^{2\pi} e^{-i\alpha_n \theta_n} \frac{\partial^{\beta_n} f}{\partial \theta_n^{\beta_n}} (z \cdot e^{i\theta}) \ d\theta_n \tag{3.10}
\]

If \( \alpha_n \neq 0, 1 \), we integrate \( I_n \) by parts with respect to \( \theta_n \) as follows: take \( u = u(\theta_n) = f(z \cdot e^{i\theta}) \) and \( dv = e^{-i\alpha_n \theta_n} \ d\theta_n \) in the formula \( \int_0^{2\pi} u dv = [uv]_0^{2\pi} - \int_0^{2\pi} v du \) and note that the first term vanishes due to periodicity. We finally get

\[
I_n = z_n^{\beta_n} \int_{\theta_n=0}^{2\pi} e^{-i(\alpha_n-1)\theta_n} \frac{\partial f}{\partial \theta_n} (z \cdot e^{i\theta}) \ d\theta_n
\]

Using integration by parts again in the same way we get

\[
I_n = z_n^{\beta_n} \int_{\theta_n=0}^{2\pi} e^{-i(\alpha_n-2)\theta_n} \frac{\partial^2 f}{\partial \theta_n^2} (z \cdot e^{i\theta}) \ d\theta_n \tag{3.11}
\]

Recall we set \( \beta_n = 2 \) for \( \alpha_n \neq 0, 1 \). Therefore (3.11) can be rewritten as

\[
I_n = z_n^{\beta_n} \int_{\theta_n=0}^{2\pi} e^{-i(\alpha_n-2)\theta_n} \frac{\partial^{\beta_n} f}{\partial \theta_n^{\beta_n}} (z \cdot e^{i\theta}) \ d\theta_n. \tag{3.12}
\]
We substitute the expressions (3.10) or (3.12) for $I_n$ in (3.9) (depending on the values of $\alpha_n$, and therefore $\beta_n$). Rearranging the terms and the integrals we write (3.9) as

\[
|c_\alpha z^\alpha| = \frac{z_\alpha^{\beta_n}}{(2\pi)^n} \int U(\alpha_n, \theta_n) \int \cdots \int \left( \prod_{j=1}^{n} U(\alpha_j, \theta_j) \right) \left| \partial^{\beta_1} f \partial^{\beta_2} \cdots \partial^{\beta_n} f (z \cdot e^{i\theta}) \right| d\theta_n \cdots d\theta_1
\]

(3.13)

Let $I_{n-1}$ be the inner integral in (3.13). Depending on the values of $\alpha_{n-1}$ we repeat the earlier procedure. If $\alpha_{n-1} = 0, 1$, we write the expression of $I_{n-1}$ in terms of $\beta_{n-1}$ directly from (3.13), otherwise we integrate $I_{n-1}$ by parts with respect to the variable $\theta_{n-1}$ as previous. We substitute the expressions for $I_{n-1}$ in (3.13) and so on. To prove our claim that (3.8) is true, we repeat the same procedure $(n-2)$ more times.

We now take absolute values on both sides in (3.8) to get,

\[
|c_\alpha z^\alpha| = \left| \frac{z_\alpha^{\beta_n}}{(2\pi)^n} \int U(\alpha_n, \theta_n) \int \cdots \int \left( \prod_{j=1}^{n} U(\alpha_j, \theta_j) \right) \left| \partial^{\beta_1} f \partial^{\beta_2} \cdots \partial^{\beta_n} f (z \cdot e^{i\theta}) \right| d\theta_n \cdots d\theta_1 \right|
\]

\[
\leq R^\delta \left( \prod_{j=1}^{n} |U(\alpha_j, \theta_j)| \right) \left| \partial^{\beta_1} f \partial^{\beta_2} \cdots \partial^{\beta_n} f (z \cdot e^{i\theta}) \right| d\theta_n \cdots d\theta_1
\]

(3.14)

where $R^\delta = R^{\beta_1} \cdots R^{\beta_n}$, $T = \{ |\zeta_j| = |z_j| : 1 \leq j \leq n \}$ is a torus contained in $\Omega$ and $\| \cdot \|_T$ is the sup norm on $T$. If we apply Lemma 3.1 with respect to each of the variable $\zeta_1, \cdots, \zeta_n$ of the function $f$ one after another, we get

\[
\left| \partial^{\beta_1} f \partial^{\beta_2} \cdots \partial^{\beta_n} f (\zeta) \right| = (-1)^{\frac{|\beta|}{2}} \sum_{1 \leq j \leq \beta_j} z^\delta D^\delta f (\zeta)
\]

(3.15)

where the notation $D^\delta$ is defined in (2.2). Therefore

\[
\left| \partial^{\beta_1} f \partial^{\beta_2} \cdots \partial^{\beta_n} f (\zeta) \right| = (-1)^{\frac{|\beta|}{2}} \sum_{1 \leq j \leq \beta_j} z^\delta D^\delta f (\zeta) \leq \sum_{1 \leq j \leq \beta_j} R^\delta \left| D^\delta f (\zeta) \right|
\]

(3.16)

Observe that,

\[
\sum_{1 \leq j \leq \beta_j} R^\delta \left| D^\delta f (\zeta) \right| \leq \sum_{1 \leq j \leq \beta_j} R^\delta \left| D^\delta f (\zeta) \right| \leq \| f \|_{2, \mathcal{P}} \sum_{1 \leq j \leq \beta_j} R^\delta \leq \| f \|_{2, \mathcal{P}} \prod_{j=1}^{n} (1 + R_j)^2
\]

(3.17)

where we recall $|\beta| := \max \{ |\beta_j|, 1 \leq j \leq n \}$. We write $\| f \|_2$ instead of $\| f \|_{2, \mathcal{P}}$ since the domain $\mathcal{P}$ is clear from the context. From (3.16) and (3.17) we get,

\[
\left\| \partial^{\beta_1} f \partial^{\beta_2} \cdots \partial^{\beta_n} f \right\|_T \leq \| f \|_2 \prod_{j=1}^{n} (1 + R_j)^2
\]

(3.18)

So, it follows from (3.14), (3.15) and (3.17) that,

\[
|c_\alpha z^\alpha| \leq R^\delta \left( \prod_{j=1}^{n} |U(\alpha_j, \theta_j)| \right) \left( \prod_{j=1}^{n} (1 + R_j)^2 \right) \| f \|_2 = \left( \prod_{j=1}^{n} |U(\alpha_j, \theta_j)| \right) \left( \prod_{j=1}^{n} R^{\beta_j} (1 + R_j)^2 \right) \| f \|_2
\]

(3.19)
Note that \( \prod_{j=1}^{n} R_j^2 (1 + R_j)^2 \leq \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \). Therefore,

\[
|c_{\alpha} z^\alpha| \leq \left( \prod_{j=1}^{n} |U(\alpha_j, \theta_j)| \right) \cdot \left( \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \right) \|f\|_2 = M_{\alpha,0} \cdot \left( \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \right) \|f\|_2,
\]

where we note from (3.1) that for each \( 1 \leq j \leq n \), \( |U(\alpha_j, \theta_j)| = \mu_{\alpha_j} \) and therefore \( M_{\alpha,0} = \prod_{j=1}^{n} \mu_{\alpha_j} = \prod_{j=1}^{n} |U(\alpha_j, \theta_j)| \). This proves the result for the case \( k = 0 \).

Let \( k \geq 1 \) and \( \gamma \in \mathbb{N}^n \). Use “vector-like” notation: \( \langle \alpha, \theta \rangle = \alpha_1 \theta_1 + \cdots + \alpha_n \theta_n \). We multiply by \( z^\alpha \) and apply \( D^\gamma \) in both sides of (3.7) to get,

\[
D^\gamma (c_{\alpha}(f) z^\alpha) = D^\gamma \left( \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \frac{f(z \cdot e^{i\theta})}{\exp(i \langle \alpha, \theta \rangle)} d\theta_1 \cdots d\theta_n \right) 
\]

\[
= \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} D^\gamma (f(z \cdot e^{i\theta})) e^{-i \langle \alpha, \theta \rangle} d\theta_1 \cdots d\theta_n 
\]

\[
= \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} (D^\gamma f)(z \cdot e^{i\theta}) e^{i \langle \gamma, \theta \rangle} e^{-i \langle \alpha, \theta \rangle} d\theta_1 \cdots d\theta_n 
\]

\[
= c_{\alpha-\gamma} (D^\gamma f) z^{\alpha-\gamma}, \quad \text{by (3.7).} \tag{3.21}
\]

So, using the result in (3.20) and the fact that \( D^\gamma f \in A^\infty(\Omega) \) we get,

\[
|D^\gamma (c_{\alpha}(f) z^\alpha)| = |c_{\alpha-\gamma} (D^\gamma f) z^{\alpha-\gamma}| \leq M_{\alpha-\gamma,0} \left( \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \right) \|D^\gamma f\|_2 \tag{3.22}
\]

Let \( \gamma \) be such that \( |\gamma|_\infty \leq k \). Since \( \gamma_j \leq k \) for every \( 1 \leq j \leq n \), we have \( \frac{1}{\alpha_j-\gamma_j} \leq \frac{1}{\alpha_j-k} \).

So, \( \mu_{\alpha_j-\gamma_j} \leq \mu_{\alpha_j-k} \) and therefore it follows that

\[
M_{\alpha-\gamma,0} = \prod_{j=1}^{n} \mu_{\alpha_j-\gamma_j} \leq \prod_{j=1}^{n} \mu_{\alpha_j-k} = M_{\alpha,k}. \tag{3.23}
\]

Moreover,

\[
\|D^\gamma f\|_2 = \sup \left\{ |D^\alpha (D^\gamma f)(z)| : z \in \Omega, |\alpha|_\infty \leq 2 \right\} = \sup \left\{ \left| D^\beta f(z) \right| : z \in \Omega, |\beta|_\infty \leq |\gamma_j| + 2 \right\} 
\]

\[
\leq \sup \left\{ \left| D^\beta f(z) \right| : z \in \Omega, |\beta|_\infty \leq |\gamma|_\infty + 2 \right\} 
\]

\[
= \|f\|_{|\gamma|_\infty+2} \leq \|f\|_{k+2}, \quad \text{since } |\gamma|_\infty \leq k. \tag{3.24}
\]

Therefore from (3.22), (3.23) and (3.24) we get

\[
\sup_{z \in \mathcal{P}} |D^\gamma (c_{\alpha}(f) z^\alpha)| \leq M_{\alpha,k} \left( \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \right) \|f\|_{k+2} \tag{3.25}
\]

Taking supremum in the left over all \( \gamma \) such that \( |\gamma|_\infty \leq k \) we get the result. \( \square \)
Proof of Theorem 3.1. Let \( z \in \Omega \). It follows from Lemma 2.4 that every Reinhardt domain is a union of polyannuli, so there exists a polyannulus \( \mathcal{P} \subset \Omega \) such that \( z \in \mathcal{P} \), where \( \mathcal{P} = \prod_{j=1}^{n} A_j \) and for each \( j \), \( A_j \subset \mathbb{C}^1 \) is either an annulus \( \{ z \in \mathbb{C} : r_j < |z| < R_j \} \) or a disc \( \{ z : |z| < R_j \} \) where \( 0 < r_j < R_j < \infty \). Let \( f = \sum_{\gamma \in \mathbb{Z}^n} c_{\gamma} e_{\gamma} \) be the Laurent series expansion of a function \( f \in \mathcal{A}^\infty(\Omega) \subset \mathcal{A}^\infty(\mathcal{P}) \). Let \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{Z}^n \). Let \( M_{\alpha,k} \) and \( \mu_{\alpha,j-k} \) be as in the statement of Proposition 3.1. Therefore it follows from Proposition 3.1 that
\[
\|c_{\alpha} e_{\alpha}\|_{k,\mathcal{P}} \leq C_{\alpha,\mathcal{P}} \|f\|_{k+2,\mathcal{P}}. \tag{3.26}
\]
where \( C_{\alpha,\mathcal{P}} = M_{\alpha,k} \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \). Since \( \mathcal{P} \subset \Omega \), we get
\[
\|c_{\alpha} e_{\alpha}\|_{k,\mathcal{P}} \leq C_{\alpha,\mathcal{P}} \|f\|_{k+2,\Omega}. \tag{3.27}
\]
Note that the constant on the right depends on \( \mathcal{P} \). Let \( R' = \prod_{j=1}^{n} (1 + R_j + R_j^2)^2 \). Since \( \Omega \) is bounded, \( R' \) is finite. Let \( C_{\alpha} = M_{\alpha,k} \cdot R' \), then \( C_{\alpha} \) is independent of \( \mathcal{P} \) and it follows that
\[
\|c_{\alpha} e_{\alpha}\|_{k,\mathcal{P}} \leq C_{\alpha} \|f\|_{k+2,\Omega}. \tag{3.28}
\]
Since the constant on the right does not depend on \( \mathcal{P} \), we take supremum in the left of (3.28) over all \( \mathcal{P} \)'s contained in \( \Omega \) to get
\[
\|c_{\alpha} e_{\alpha}\|_{k,\Omega} \leq C_{\alpha} \|f\|_{k+2,\Omega}. \tag{3.29}
\]
Recall \( |\alpha|_{\infty} := \max\{ |\alpha_j|, 1 \leq j \leq n \} \), where \( \alpha \in \mathbb{Z}^n \). It is easy to see that for every \( N \in \mathbb{N} \),
\[
\sum_{|\alpha|_{\infty} \leq N} C_{\alpha} = R' \cdot \prod_{j=1}^{n} \left( \sum_{\alpha_j = -N}^{N} \mu_{\alpha,j-k} \right) \tag{3.30}
\]
Observe that for every \( j \), \( \lim_{N \to \infty} \sum_{\alpha_j = -N}^{N} \mu_{\alpha,j-k} = \lim_{N \to \infty} \sum_{\alpha_j = -N}^{1} \mu_{\alpha,j-k} + \lim_{N \to \infty} \sum_{\alpha_j = 1}^{N} \mu_{\alpha,j-k} \) and both the terms are finite (using the limit comparison test with the series \( \sum_{j=2}^{\infty} \frac{1}{j(j-1)} \)). So,
\[
\lim_{N \to \infty} \sum_{\alpha_j = -N}^{N} \mu_{\alpha,j-k} < \infty \text{ and therefore}
\]
\[
\lim_{N \to \infty} \sum_{|\alpha|_{\infty} \leq N} C_{\alpha} = R' \cdot \lim_{N \to \infty} \prod_{j=1}^{n} \left( \sum_{\alpha_j = -N}^{N} \mu_{\alpha,j-k} \right) < \infty. \tag{3.31}
\]
Choose a bijection \( \sigma : \mathbb{N} \to \mathbb{Z}^n \) such that for every \( N \in \mathbb{N} \),
\[
\sigma^{-1}\{ \alpha \in \mathbb{Z}^n : |\alpha|_{\infty} \leq N \} \subset \{0, 1, 2, \cdots , (2N + 1)^n\} \subset \sigma^{-1}\{ \alpha \in \mathbb{Z}^n : |\alpha|_{\infty} \leq N + 1 \}.
\]
So, for every \( M \in \mathbb{N} \) there exists \( N_1 = N_1(M) \in \mathbb{N} \) such that
\[
\sum_{|\alpha|_{\infty} \leq N_1} \|c_{\alpha} e_{\alpha}\|_{k,\Omega} < \sum_{j=0}^{M} \|c_{\sigma(j)} e_{\sigma(j)}\|_{k,\Omega} < \sum_{|\alpha|_{\infty} \leq N_1 + 1} \|c_{\alpha} e_{\alpha}\|_{k,\Omega}. \tag{3.32}
\]
where we can explicitly calculate \( N_1 \) by the choice of \( \sigma \) above,
\[
N_1 = \begin{cases} \left\lfloor \frac{\sqrt{M} - 1}{2} \right\rfloor & \text{if } M \geq 1 \\ 0 & \text{if } M = 0, \end{cases} \tag{3.33}
\]
and $|\cdot|$ is the floor function. Observe that if $M \to \infty$, then $N_1 \to \infty$ as well. It follows from (3.29) and (3.31) that,

$$\lim_{N_1 \to \infty} \sum_{|\alpha| \leq N_1} \|c_\alpha e_\alpha\|_{k,\Omega} < \infty. \quad (3.34)$$

Therefore from (3.32),

$$\sum_{j=0}^\infty \|c_{\sigma(j)} e_{\sigma(j)}\|_{k,\Omega} = \lim_{M \to \infty} \sum_{j=0}^M \|c_{\sigma(j)} e_{\sigma(j)}\|_{k,\Omega} < \infty.$$

The absolute convergence follows from here.

\[ \square \]

4. Proof of main theorem

In this section we prove Theorem 1.1. The following result is useful.

**Proposition 4.1.** To show that the Laurent series of a function $f \in \mathcal{A}^\infty(\Omega)$ converges absolutely in the topology of $\mathcal{A}^\infty(\Omega)$, it is sufficient to take $\Omega$ to be bounded.

**Proof.** If $\Omega$ is unbounded, recall that one can write $\Omega = \bigcup_{m \in \mathbb{N}} \Omega_m$, where $\Omega_m$ is bounded for each $m \in \mathbb{N}$. We also recall that the collection of seminorms given in (2.4) generates the topology of $\mathcal{A}^\infty(\Omega)$. To prove absolute convergence one needs to show that there exists a bijection $\sigma : \mathbb{N} \to \mathbb{Z}^n$ such that for every $k, m \in \mathbb{N}$, $\sum_{j=0}^\infty \|c_{\sigma(j)} e_{\sigma(j)}\|_{k,\Omega_m} < \infty$.

Therefore it suffices to take $\Omega$ to be bounded and by Lemma 2.1 it is enough to show that there exists a bijection $\sigma : \mathbb{N} \to \mathbb{Z}^n$ such that for every $k \in \mathbb{N}$, $\sum_{j=0}^\infty \|c_{\sigma(j)} e_{\sigma(j)}\|_{k,\Omega} < \infty$, where $\|\cdot\|_{k,\Omega}$ is defined in (2.2). \[ \square \]

**Proof of Theorem 1.1.** It is sufficient to consider $\Omega$ to be bounded (cf. Proposition 4.1). Let $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha e_\alpha$ be the Laurent series expansion of $f \in \mathcal{A}^\infty(\Omega)$. It follows from Theorem 3.1 that the Laurent series of $f$ converges absolutely in $\mathcal{A}^\infty(\Omega)$, therefore unconditionally in $\mathcal{A}^\infty(\Omega)$ (cf. Lemma 2.2). Let $\sigma : \mathbb{N} \to \mathbb{Z}^n$ be a bijection and $g = \lim_{N \to \infty} \sum_{j=0}^N c_{\sigma(j)} e_{\sigma(j)}$ in $\mathcal{A}^\infty(\Omega)$. Therefore, $g = \lim_{N \to \infty} \sum_{j=0}^N c_{\sigma(j)} e_{\sigma(j)}$ also in $\mathcal{O}(\Omega)$. But, by a classical result in ([9], p. 46) the above limit is $f$. Therefore $f = g$. \[ \square \]

**References**

[1] D. Catlin. Boundary behavior of holomorphic functions on pseudoconvex domains. *J. Differential Geom.*, 15(4):605–625, 1980.

[2] D. Chakrabarti. On an observation of sibony. *Proceedings of the American Mathematical Society*, 147(8):3451–3454, 2019.

[3] D. Chakrabarti, L. Edholm, and J. McNeal. Duality and approximation of bergman spaces. *Advances in Mathematics*, 341:616–656, 2019.

[4] J. B. Garnett. *Bounded Analytic Functions*, volume 96. Academic Press, 1981.

[5] M. Hakim and N. Sibony. Spectre de $\mathcal{A}(\Omega)$ pour des domaines bornes faiblement pseudoconvexes réguliers. *J. Funct. Anal.*, 37:127–135, 1980.

[6] M. Jarnicki and P. Pflug. *First steps in several complex variables: Reinhardt domains*, volume 7. European Mathematical Society, 2008.

[7] V. Kadets and M. Kadets. *Series in Banach spaces: conditional and unconditional convergence*, volume 94. Birkhäuser, 1997.

[8] J. McNeal and J. Xiong. Norm convergence of partial sums of h 1 functions. *International Journal of Mathematics*, 29(10):1850065, 2018.
[9] R. M. Range. *Holomorphic functions and integral representations in several complex variables*, volume 108. Springer, 1986.

[10] N. Sibony. Prolongement des fonctions holomorphes bornées et métrique de carathéodory. *Inventiones mathematicae*, 29(3):205–230, 1975.

[11] K. H. Zhu et al. Duality of bloch spaces and norm convergence of taylor series. *The Michigan Mathematical Journal*, 38(1):89–101, 1991.

Central Michigan University, 1200 S Franklin St, Mt Pleasant, MI 48859, USA
E-mail address: dawn1a@cmich.edu