Gluon density and $F_2$ functions from BK equation with impact parameter dependence

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February 19, 2008

Abstract

In this note we fix the preliminary results obtained in the study of gluon density function of the paper [33]. The LO BK equation for unintegrated gluon density with impact parameter dependence is considered in order to fix the parameters of the proposed model. In particular the form of initial condition for the equations of proton-proton scattering from [33] is determined, which is similar to the form of fenomenological GBW ansatz. The gluon density function and $F_2$ function are also calculated and compared with the results for the gluon density and $F_2$ functions from the GRV parameterization for different values of $Q^2$. It is shown, that the results for $F_2$ structure function of the considered model are in the good accordance with the results obtained from the GRV parameterization of parton densities.

1 Introduction

The attempts to understand the aspects of high energy scattering of nuclei and hadrons in terms of QCD BFKL pomerons, [1], led in the last time to a number of papers concerning the phenomenological applications of the high energy scattering as well as the pure theoretical properties of the theory, [2, 3, 4, 5, 6, 7, 8, 9]. In present paper we fix the first result obtained in the framework of the model proposed in [2, 3, 5], introducing the impact parameter dependence into the initial conditions and solving the equations of the model for each point in impact parameter space. The found parameters of the model, tuned with the help of DIS, are the first step toward the solution of the problem of proton-proton scattering considered in [33].

The DIS process is well described in the frameworks related with the BK equation, [10, 11], fenomenological models with the saturation properties and CGC models, see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Nevertheless, in rapidity evolution the impact parameter dependence of the amplitude was treated only approximately, see for example [17, 18], whereas the fenomenological models, such as GBW model [12, 13], neglect the evolution of the amplitude with rapidity. In papers [14, 21, 23] the approximate treatment of impact parameter dependence of gluon structure function was accounted together with DGLAP evolution of the function, and

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In papers [24, 25, 26] the factorized from momentum function impact parameter dependence in modified BK equation was considered. But still, the treatment of impact parameter dependence of the simple BK equation compatible with the fenomenological models was missed.

In present calculations of solution of BK equation we include the impact parameter dependence of the amplitude at initial values of rapidity and find the amplitude in each point of impact parameter space, solving the evolution BK equation with the help of the methods developed in [5], see also the papers [24, 25, 26] for similar technics of calculations. In order to simplify the calculations, the solution is obtained in LO approximation and we discuss a possible generalization of the solution till NLO order in the conclusion. Another important question, which we tried to answer on, it is a problem of the form of initial condition function for the BK equation with impact parameter dependence. The form of this function is general in the given framework of interacting BFKL pomerons and, as we mentioned above, the same function could be used in the proton-proton scattering, see [5, 33].

Certainly, the results of calculations must be clarified with the help of well established results for gluon density and/or with the help of DIS data. We perform the check of our calculations comparing calculated gluon density function (integrated gluon density) and $F_2$ function with the results given by the LO and NLO GRV parameterizations for DIS data, [27]. This comparison shows, that in the present framework we achieved the satisfactory description of DIS data. The model based on BK equation with impact parameter dependence shows a good coincidence with GRV parameterization and could be used as a independent parameterization of unintegrated gluon density, [33].

The paper is organizes as follows. In the next section we shortly describe a formalism of calculations. In Sec.3 and Sec.4 we present the results of calculations for $F_2$ structure function and gluon density function correspondingly. Section 5 is a conclusion of the paper.

2 The low-x structure function in the momentum representation

In this section we shortly write the main formulae used in our calculations. The $F_2$ structure function of DIS with impact parameter dependence we define as follows

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2} \int d^2 b \int \frac{d^2 k}{k^4} f(x, k^2, b) \left( \Phi_T(k, m_0^2) + \Phi_L(k, m_0^2) \right)$$

(1)

The unintegrated gluon density function $f(x, k^2, b)$ we find solving the BK equation for each point in impact parameter space:

$$\partial_y f(y, k^2, b) = \frac{N_c \alpha_s}{\pi} k^2 \int \frac{da^2}{a^2} \left[ f(a^2, b) - f(k^2, b) \right] + \frac{f(k^2, b)}{[4a^4 + k^4]^2}$$

$$- 2\pi \alpha_s^2 \left[ k^2 \int \frac{da^2}{a^4} f(a^2, b) \int \frac{dc^2}{c^4} \left( f(c^2, k^2, b) + f(k^2, b) \right) \log \left( \frac{a^2}{k^2} \right) f(a^2, b) \right]$$

(2)

where we introduced the rapidity variable $y = \log(1/x)$. In Eq.2 we assumed that the evolution is local in the transverse plane, i.e. impact parameter dependence of $f(y, k^2, b)$ appear only throw the initial condition for $f(y, k^2, b)$

$$f(y = y_0, k^2, b) = f_{in}(k^2, b)$$

(3)

In order to exclude part of ambiguities in the solution of BK equation arising due the non included NLO corrections, we perform the following substitute in the equation

$$f(y, k^2, b) \rightarrow f(y, k^2, b) \frac{\alpha_s(k^2)}{\alpha_s} = \frac{\hat{f}(y, k^2, b)}{\alpha_s}$$

(4)
is changed

Due to including quark masses in the calculations, the rapidity variable \( y \) modified, see details in \([12, 13]\). For each fixed rapidity fitting of DIS data and they are presented in the next section. The plots of the functions 

Additionally to the \( \alpha_s \) we determine fitting the DIS data for the total cross section and \( F_2 \) structure function, \([12,13]\), with introduced impact parameter dependence 

\[
\Phi_L(k, m_q^2) = 32 \pi \alpha \sum_{q=1}^{3} e_q^2 \int_0^1 d\rho d\eta \frac{k^2 \eta (1 - \eta) \rho^2 (1 - \rho)^2 Q^2}{(Q^2 \rho (1 - \rho) + k^2 \eta (1 - \eta) + m_q^2) (Q^2 \rho (1 - \rho) + m_q^2)}
\]

and 

\[
\Phi_T(k, m_q^2) = 4 \pi \alpha \sum_{q=1}^{3} e_q^2 \int_0^1 d\rho d\eta \frac{k^2 Q^2 (\rho^2 + (1 - \rho)^2) \rho (1 - \rho) (\eta^2 + (1 - \eta)^2) + k^2 (\rho^2 + (1 - \rho)^2) m_q^2 + 4 \rho (1 - \rho) \eta (1 - \eta) m_q^2}{(Q^2 \rho (1 - \rho) + k^2 \eta (1 - \eta) + m_q^2) (Q^2 \rho (1 - \rho) + m_q^2)}
\]

We exclude \( \alpha_s \) from the definition of the impact factors rewriting the \( F_2 \) structure function in the following way 

\[
F_2(x, Q^2) = \frac{Q^2}{4 \pi^2 \alpha} \int d^2 b \int \frac{d^2 k}{k^4} \frac{\tilde{f}(x, k^2, b)}{4 \pi} \left( \Phi_T(k, m_q^2) + \Phi_L(k, m_q^2) \right) = \frac{Q^2}{4 \pi^2 \alpha} \int d^2 b S(x, b, Q^2)
\]

Due the including quark masses in the calculations, the rapidity variable \( y \) (Bjorken \( x \)) in BK equation is also modified, see details in \([12,13]\). For each fixed rapidity \( y \) of the process, the value rapidity taken in BK equation is changed 

\[
y \to y - \ln(1 + \frac{4 m_q^2}{Q^2})
\]

The form of the function \( \tilde{f}(y, k^2, b) \) at initial rapidity, i.e. initial condition for the BK equation Eq[5] has been borrowed from the form of GBW ansatz, \([12,13]\], with introduced impact parameter dependence 

\[
\tilde{f}(y = y_0, k^2, b) = \frac{3}{4 \pi \alpha} k^4 R_0^2 e^{\rho^2/R_0^2} \exp(-k^2 R_0^2 e^{\rho^2/R_0^2})
\]

Additionally to the \( \alpha_s \) from the BK equation Eq[5] there are three more parameters, which are initial rapidity of evolution \( y_0 \), radius of the proton \( R_p^2 \) and ”saturation” radius \( R_0^2 \). These parameters must be found from the fitting of DIS data and they are presented in the next section. The plots of the functions \( S(x, b, Q^2) \) from Eq[8] are given in Fig[11]  

### 3 F2 function and the parameters of the model

The parameters of the model we determine fitting the DIS data for the total cross section and \( F_2 \) structure function, \([29]\). In this note we present the parameters of the model and results of the calculations of \( F_2 \) for only few values of \( Q^2 \) with only three light quarks flavors included, more results, including the application of the model to the ”soft” proton-proton scattering, will be presented in the mentioned above paper \([33]\). The Table[4] shows found values of the parameters of the model and plots Fig[2] present the results of calculations for the \( F_2 \) structure function. It must be mentioned, that instead the the \( \lambda \), \( \lambda_{GBW} \) parameters which determines the
Figure 1: The impact parameter profile of $S(x, b, Q^2)$ as a function of $x$.

| $y_0 (x_0)$ | $R_p^2 (GeV^2)$ | $R_0^2 (GeV^2)$ | $\alpha_s$ | $m_q^2 (GeV^2)$ |
|-------------|------------------|------------------|------------|-----------------|
| 3.1 (0.045) | 7.9              | 2.12             | 0.108      | 0.008           |

Table 1: The parameters of the model.
energy dependence of the saturation radius in the CGC and GBW models, see [12] [13] [22] [23], in our calculations the \( \alpha_s \) is the parameter which determines the energy behavior of the gluon density functions. The smallness of obtained value of \( \alpha_s \) is explained by the LO precision of the calculations and by the variable change Eq.1. In present scheme the value of \( \alpha_s \) determines the evolution length in rapidity space independently on values of \( Q^2 \), i.e. this is some ”averaged” value for \( \alpha_s \) found from the data fitting. The found value of quark mass is also different from the numbers of [12] [13] for example, being nevertheless in the range of possible quark masses of light quark flavors used in [21] [23]. The radius of the proton from Table 1 is close to the experimental value of the proton shape found from the t-distribution of \( J/\psi \) meson of [29], in fact the results of this measurements restrict the possible numerical values of this parameter.

The main difference of our calculations from the results of other models are the results for \( F_2 \) structure function at very small values of \( Q^2 \). The two top plots of Fig 2, calculated at small values of \( Q^2 \), show that our model fails to describe the \( F_2 \) data at \( Q^2 \sim 0.25, GeV^2 \). It means, that saturation effects which provides the description of the low \( Q^2 \) data in ”canonical” saturation model, such as [12] for example , in our framework are not so strong as there. The possible reasons for such a deviations from the ”normal” results obtained at low \( Q^2 \) and small \( x \) we discuss in conclusion. Nevertheless, the comparison of the obtained results with the GRV parameterization results for \( F_2 \) function for \( Q^2 > 1 GeV^2 \) shows a good coincidence, they stay in the limits of differences between the results of GRV parameterization with the results given by other parton density parameterizations, such as [30] for example.

4 Integrated gluon density function

In order to estimate the possible effects of the variables change Eq.1 it is instructive to calculate the values of integrated gluon density. Indeed, as it seems from Fig 2 the LO and NLO GRV parameterization give very close results for the \( F_2 \) structure function. In the same time, the integrated gluon density functions are very different for LO and NLO GRV curves at the same values of \( Q^2 \). We present the obtained plots in Fig 3 for the integrated gluon density functions at different values of \( Q^2 \).

There are two possible resulting plots of the model in Fig 3 which we denotes as \( BK_1 \) and \( BK_2 \) curves. The reason for a existing of two curves is a following. Let’s consider the definition of integrated gluon density function

\[
xG(x, Q^2) = \int d^2 b \int_{Q^2} d\frac{k^2}{k^2} f(x, k^2, b)
\]  

(11)

Due the variable change Eq.4 we obtain a integrated gluon density function in terms of the new function \( \tilde{f} \)

\[
xG(x, Q^2) = \int d^2 b \int_{Q^2} d\frac{k^2}{k^2} \tilde{f}(x, k^2, b) \frac{1}{\alpha_s(k^2)} \to \int d^2 b \int_{Q^2} d\frac{k^2}{k^2} \tilde{f}(x, k^2, b)
\]  

(12)

From the Eq.12 it is clear, that the definition of the \( xG(x, Q^2) \) in terms of \( \tilde{f} \) has an ambiguities in LO scheme calculations due the running coupling \( \alpha_s(k^2) \) under the integral over \( k^2 \). In our LO calculations we need to choose the fixed LO value of the \( \alpha_s(k^2) \) and to extract it from the integration over \( k^2 \) in Eq.12. Therefore, we considered two possibilities for the fixed \( \alpha_s \) value. As the first one we took the value of \( \alpha_s \) from the Table 1 obtained in the fitting of the data. This choice results are denoted as \( BK_1 \) curves in the Fig 3. Another choice for the \( \alpha_s \) is the values of \( \alpha_s \) from the NLO GRV parameterization taken separately for corresponding \( Q^2 \). The results for this value of \( \alpha_s \) is denoted as \( BK_2 \) in the Fig 3.
Figure 2: The $F_2$ data for different values of $Q^2$: the present model results (solid lines), LO GRV parameterization (dashed lines) and NLO GRV (doted lines) as functions of $x$. The GRV results are restricted by $Q^2 > 0.8\,GeV^2$. 
Figure 3: The integrated gluon density function: the present model results (solid lines), LO GRV parameterization (dashed lines) and NLO GRV (dotted lines) as functions of $x$. 

$xG(x,Q^2)$

\[ Q^2 = 2.5 \text{ GeV}^2 \]

\[ Q^2 = 60 \text{ GeV}^2 \]

\[ Q^2 = 12 \text{ GeV}^2 \]

\[ Q^2 = 120 \text{ GeV}^2 \]
The simple conclusions, which could be immediately obtained from the present plots, are the following. First of all, the NLO GRV value of $\alpha_s$ for each $Q^2$ in Eq.12 gives more correct value of $xG(x,Q^2)$ comparing with the common $\alpha_s$ value from the data fit. This fact related with the use of the LO $\tilde{f}$ function in our scheme of calculations and, therefore, each calculation of the $xG(x,Q^2)$ needs the redefinition of the present $\alpha_s$ value. The second conclusion concerns the shape of the found curves. It is easy to see, that both $BK_1$ and $BK_2$ curves have a shapes similar to the NLO GRV curve and pretty different from the LO GRV curve for integrated gluon density. This is a sign, that the simple redefinition of the variables Eq.4 in BK equation allows to include a some part of NLO corrections to the integrated gluon density and $F_2$ functions. It is important to underline again, that obtained integrated gluon density function is similar to the integrated gluon density function obtained with the use of GRV parameterization.

5 Conclusion

We demonstrated, that based on QCD BFKL pomerons BK evolution equation for unintegrated gluon density with impact parameter dependence could be used as a calculation tool for the $F_2$ structure function and integrated gluon density function. In the large range of energies and large range of values of $Q^2$ we obtained a good description of DIS data for $F_2$ structure function. We note, that the obtained results are in good agreement with results obtained with the help of GRV parameterization of parton densities and therefore could be used as independent parameterization of unintegrated gluon density. It is important, because for our framework it means that we not only reproduced the results for DIS using more complex theory than usual evolution equations without impact parameter dependence, but also that we found the initial conditions for the proton-proton scattering in the framework of Braun equations [2, 3, 5]. Therefore, the obtained impact parameter dependent parameterization of proton shape Eq.10 with parameters of Table 1 allow to apply formalism of [7] to the important and more general case of proton-proton scattering, see [33]. Another interesting field of the application of the proposed model, is the description of the processes of exclusive particle production. As it was shown in [4], the account of impact parameter dependence of the proton-proton scattering amplitude is very important for the better understanding and better description of the low momentum region and NLL corrections in the resulting amplitude of the process of exclusive Higgs production.

The unexpected result, obtained in present calculations, it is a bad agreement between the calculations in our approach and results of similar approaches in description of $F_2$ function at small $Q^2 = 0.25 \text{GeV}^2$. Usually, this region of the small values of $Q^2$ at high energies is considered as a region where the saturation effects are large, see [12]. In our case, as it seems, the evolution over rapidity in impact parameter space does not lead to the saturation effects which will generate appropriate slope of $F_2$ function at very small $Q^2$ and small $x$. The reasons for such a distinction from usual saturation models behavior is not clear. The including of NLO corrections into the calculation scheme could, in principal, to improve the situation. At small values of $Q^2$ the effect of NLO corrections must be large, it is clear if we will consider the averaged value of $\alpha_s$ obtained in our fit. This value is very small and in the theory with running coupling constant at small $Q^2$ this value must be changed a lot, giving a more appropriate result for $F_2$ at small $Q^2$.

Another approach to this problem, is that the DIS process at small $Q^2$ physically is very similar to the hadron-hadron scattering, see [31] for example. From this point of view it is not clear why the simple ”fan” structure of BK equation must work well at small values of $Q^2$. More complicated ”net” diagrams of interacting pomerons became to be important in this case, see [2, 3, 7, 8, 32], and absence of these diagrams in BK equation could lead to the wrong results for DIS at small $Q^2$. Interesting to note, that from the formal point of view
these ”net” diagrams are also part of NLO correction to the unintegrated gluon density, which arise from the field theory part of the process and not from the corrections to the BFKL kernel. We plan to investigate this question in our future studies of the gluon density function in the framework with NLO corrections included.

Acknowledgments

I am especially grateful to Y. Shabelsky for the discussion on the subject of the paper and to Leszek Motyka for the help and useful comments. This work was done with the support of the Ministerio de Educacion y Ciencia of Spain under project FPA2005-01963 together with Xunta de Galicia (Conselleria de Educacion).

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