Gosper’s algorithm and Bell numbers

Robert Dougherty-Bliss*

October 26, 2022

Abstract

Computers are good at evaluating finite sums in closed form, but there are finite sums which do not have closed forms. Summands which do not produce a closed form can often be “fixed” by multiplying them by a suitable polynomial. We provide an explicit description of a class of such polynomials for simple hypergeometric summands in terms of the Bell numbers.

Evaluating the partial sums of sequences which are products and quotients of polynomials, binomial coefficients, factorials, and so on is a major theme in combinatorics, discrete probability, and computer science. The main tool in this area is Bill Gosper’s marvelous hypergeometric summation algorithm [3].

A sequence $f(k)$ is hypergeometric (or a hypergeometric term) provided that the consecutive quotient $f(k + 1)/f(k)$ is a rational function in $k$. Gosper’s algorithm completely solves the problem of hypergeometric summation in one variable. It constructively determines when $\sum f(k)$ itself is hypergeometric term. We call such hypergeometric terms “Gosper summable.”

Unfortunately, many hypergeometric terms are not Gosper summable. For example, we cannot fill in the following blank with a hypergeometric term:

$$\sum_{k=0}^{n} \frac{1}{k!} = \underline{\hspace{2cm}}.$$ 

However, an upshot of Gosper’s algorithm is that we can often tweak non-Gosper summable terms to make them summable. For example, while $1/k!$ is not Gosper

*Rutgers University, Department of Mathematics
summable, the term \((k - 1)/k!\) is. In fact, lots of multiples of \(1/k!\) are Gosper summable:

\[
\begin{align*}
\sum_{k=0}^{n} \frac{k - 1}{k!} &= -\frac{1}{n!} \\
\sum_{k=0}^{n} \frac{k^2 - 2}{k!} &= -\frac{n + 2}{n!} \\
\sum_{k=0}^{n} \frac{k^3 - 5}{k!} &= -\frac{n^2 + 3n + 5}{n!} \\
\sum_{k=0}^{n} \frac{k^4 - 15}{k!} &= -\frac{n^3 + 4n^2 + 9n + 15}{n!}.
\end{align*}
\]

This list suggests that there exists a sequence of integers \(b(d)\)—beginning 1, 2, 5, 15—such that \((k^d - b(d))/k!\) is Gosper summable. This turns out to be true. Even better, \(b(d)\) turns out to be the \(d\)th Bell number, the number of partitions of \(d\) elements into any number of nonempty subsets.

Indeed, there is a similar statement for hypergeometric terms of the form \(z^k a^k\) and \(z^k/a^k\) for constant \(z\), where \(a^k = a(a+1)\cdots(a+k-1)\) is the rising factorial. Specifically, there are explicit exponential generating functions \(g_{a,z}(x)\) and \(f_{a,z}(x)\) such that \((k^d - c(d))z^k a^k\) is Gosper summable iff \(c(d)\) is the coefficient on \(x^d/d!\) in \(g_{a,z}(x)\), and the analogous statement for \(z^k/a^k\) and \(f_{a,z}(x)\). These generating functions happen to be related to the famous exponential generating function for the Bell numbers \(B(x) = e^{e^x-1}\). Our goal is to explain and prove these facts.

The remainder of this paper is organized as follows. Section 1 gives a quick overview of Gosper’s algorithm. Section 2 establishes the summability results and gives the explicit generating functions. Section 3 shows how to explicitly evaluate a special case of these sums in terms of well-known integer sequences. Section 4 explains how these generating functions are related to the Bell numbers.

1 Gosper’s algorithm

This section provides a brief overview of Gosper’s algorithm. For more details, see [3] or [6].

A sequence \(f(k)\) is hypergeometric, or a hypergeometric term, provided that \(f(k + 1)/f(k)\) is a rational function in \(k\).
Every rational function $R(k)$ can be decomposed as

$$R(k) = \frac{a(k) c(k+1)}{b(k) c(k)},$$

where $a$, $b$, and $c$ are polynomials in $k$ which satisfy $\gcd(a(k), b(k+i)) = 1$ for all nonnegative integers $i$. This is called the polynomial normal form of $R(k)$. If $f(k)$ is hypergeometric and $f(k+1)/f(k)$ has polynomial normal form

$$\frac{f(k+1)}{f(k)} = \frac{a(k) c(k+1)}{b(k) c(k)},$$

then we call $a/b$ the \textit{kernel} of $f$, and $c$ the \textit{shell} of $f$. Note that

$$f(k) = z c(k) \prod_{j=0}^{k-1} (a(j)/b(j))$$

for some constant $z$. For this reason, we sometimes call $c(k)$ the “polynomial part” of $f(k)$ and the remaining product the “purely hypergeometric part.”

Gosper’s algorithm amounts to the following theorem.

\textbf{Theorem.} The hypergeometric term $f(k)$ with polynomial normal form $(a, b, c)$ is Gosper summable if and only if there is a polynomial solution $x(k)$ to

$$x(k+1)a(k) - x(k)b(k-1) = c(k). \quad (1)$$

In that case,

$$\sum_k f(k) = \left(\frac{x(k)b(k-1)}{c(k)}\right)f(k).$$

For example, the term ratio of $f(k) = 1/k!$ has polynomial normal form

$$\frac{f(k+1)}{f(k)} = \frac{1}{k+1}$$

with $(a, b, c) = (1, k+1, 1)$. Therefore $f(k)$ is summable if and only if $x(k+1) - x(k)k = 1$ has a polynomial solution $x(k)$, which it does not. On the other hand, the term ratio of $g(k) = (k-1)/k!$ has polynomial normal form

$$\frac{g(k+1)}{g(k)} = \frac{1}{k+1} \frac{k}{k-1}$$
with \((a, b, c) = (1, k + 1, k - 1)\). Therefore \(g(k)\) is summable if and only if \(x(k + 1) - x(k)k = k - 1\) has a polynomial solution \(x(k)\), which it does, namely \(x(k) = -1\). In addition,

\[
\sum_k \frac{k - 1}{k!} = -\frac{k}{k!} = -\frac{1}{(k-1)!}.
\]

### 2 Summability

For pedagogical purposes, let us first prove the following proposition.

**Proposition 1.** The term

\[
\frac{k^d - b(d)}{k!}
\]

is Gosper summable if and only if \(b(d)\) is the \(d\)th Bell number.

**Proof.** The consecutive term ratios of \(1/k!\) are \(1/(k + 1)\), which has polynomial normal form \((1, k + 1, 1)\). Setting \(x_d(k) = -k^d\) in (1) shows that \(p_d(k) = k^{d+1} - (k + 1)^d\) is a sequence of polynomials such that \(p_d(k)/k!\) is Gosper summable. Since the degree of \(p_d(k)\) is exactly \(d + 1\), these polynomials are linearly independent, and therefore form a basis for the set of all polynomials \(p(k)\) such that \(p(k)/k!\) is Gosper summable. Our proposition amounts to the claim that \(k^d + b(d)\) is a different basis for this space.

The degrees of the \(p_d(k)\) start at 1 and increase by 1 every step, so by subtracting appropriate multiples of previous terms, we can cancel every power of \(k\) in \(p_d(k)\) except the leading term and the constant. That is, there is a basis of the form \(k + c(1), k^d + c(2), \ldots\), obtained by a linear operation on the \(p_d(k)\). In particular, since

\[
p_d(k) = k^{d+1} - \sum_j \binom{d}{j} k^j,
\]

the correct multiples to subtract are as follows:

\[
k^{d+1} + c(d + 1) = p_d(k) + \sum_{j > 0} \binom{d}{j} (k^j + c(j)).
\]
If we set \( c(0) = -1 \) and look at the constant term of both sides, we obtain

\[
c(d + 1) = \sum_j \binom{d}{j} c(j).
\]

This implies \( c(d) = -b(d) \), where \( b(d) \) is the \( d \)th Bell number, since the Bell numbers satisfy the same recurrence and begin with \( 1 \) rather than \(-1\).

The above outline carries over nearly verbatim to other simple hypergeometric terms. A slight difference is that, most of the time, the sequence \( c(d) \) is not well-known, and we have to settle for an explicit exponential generating function. The following propositions neatly summarize the results.

**Proposition 2.** If \( z \neq 0 \) and \( a \) is not a nonpositive integer, then the hypergeometric term \((k^d - c(d))z^k/a^k\) is Gosper summable if and only if

\[
c(d) = [x^d/d!] \exp(-z - (a - 1)x + ze^x) = [x^d/d!] f_{a,z}(x).
\]

**Proof.** The consecutive term ratios of \( z^k/a^k \) are \( z/(a + k) \), so their polynomial normal form is \((z, a + k, 1)\). It follows that the sequence of polynomials

\[
p_d(k) = k^d(a + k - 1) - z(k + 1)^d
\]

for \( d = 0, 1, 2, \ldots \) form a basis for the set of polynomials \( p(k) \) such that \( p(k)z^k/a^k \) is Gosper summable. It suffices to transform this basis by iteratively eliminating all powers of \( k \) from \( p_d(k) \) except its highest power and its constant term, then to show that the constant terms have the quoted exponential generating function.

Note that

\[
p_d(k) = k^{d+1} - (z + 1 - a) k^d - z \sum_{j<d} \binom{d}{j} k^j.
\]

Therefore,

\[
k^{d+1} + c(d + 1) = p_d(k) + (z + 1 - a)(k^d + c(d)) + z \sum_{0<j<d} \binom{d}{j} (k^j + c(j)).
\]

Comparing constant terms, we see that

\[
c(d + 1) = -z + (z + 1 - a) c(d) + z \sum_{0<j<d} \binom{d}{j} c(j)
\]
If we let \( c(0) = -1 \), then this becomes
\[
c(d + 1) = (1 - a)c(d) + z \sum_j \binom{d}{j} c(j).
\]

If \( C(x) = \sum_{d \geq 0} \frac{c(d)}{d!} x^d \) is the exponential generating function of \( c(d) \), then the previous equation implies
\[
C'(x) = (1 - a)C(x) + ze^x C(x).
\]
Solving this linear differential equation yields
\[
C(x) = -e^{-z-(a-1)x+ze^x}.
\]
Therefore \((k^d - c(d))z^k/a^k\) is Gosper summable if and only if \( c(d) \) is the coefficient on \( x^d/d! \) in \( \exp(-z-(a-1)x+ze^x) \).

**Proposition 3.** If \( z \neq 0 \) and \( a \) is not a nonpositive integer, then the hypergeometric term \((k^d - c(d))z^k/a^k\) is Gosper summable with respect to \( k \) if and only if
\[
c(d) = [x^d/d!] \exp(z^{-1} - ax - z^{-1}e^{-x}) = [x^d/d!] g_{a,z}(x).
\]

**Proof.** The consecutive term ratio of \( z^k a^k \) has polynomial normal form \((z(a + k), 1, 1)\). Therefore, as in the proof of Proposition 2, the sequence of polynomials
\[
p_d(k) = z(k + 1)^d(a + k) - k^d
\]
form a basis for the set of all polynomials \( p(k) \) such that \( p(k)z^k a^k \) is Gosper summable, and our job is to simplify it.

Note that
\[
p_d(k) = zk^{d+1} + (z(a + d) - 1)k^d + z \sum_{j<d} \left( a \binom{d}{j} + \binom{d}{j-1} \right) k^j.
\]
Therefore, having constructed basis elements of the form \( k+c(1), k^2+c(2), \ldots, k^d+c(d) \), we have
\[
z(k^{d+1}+c(d+1)) = p_d(k) - (z(a+d)-1)(k^d+c(d)) - z \sum_{0<j<d} \left( a \binom{d}{j} + \binom{d}{j-1} \right) (k^j+c(j)).
\]
Comparing constant coefficients yields

\[ zc(d + 1) = az - (z(a + d) - 1)c(d) - z \sum_{0 \leq j \leq d} \left( a \binom{d}{j} + \binom{d}{j - 1} \right) c(j). \]

If we let \( c(0) = -1 \), then this becomes

\[ zc(d + 1) = c(d) - z \sum_{0 \leq j \leq d} \left( a \binom{d}{j} + \binom{d}{j - 1} \right) c(j). \]

If we move the sum to the left-hand side, the equation reads

\[ c(d) = z \sum_{j} \left( a \binom{d}{j} + \binom{d}{j - 1} \right) c(j). \]

If \( C(x) \) is the exponential generating function of \( c(d) \), then standard techniques give us

\[ C(x) = z(e^x C(x) + e^x C'(x)), \]

whose unique solution with \( C(0) = -1 \) is \( C(x) = - \exp(z^{-1} - ax - z^{-1} e^{-x}). \)

3 Explicit Formulas and Gould Numbers

In the previous section we proved that

\[ \sum_{k} \frac{k^d - b(d)}{k!} \]

is Gosper summable when \( b(d) \) is the \( d \)th Bell number. In this section we will explicitly evaluate this sum in terms of well-known integer sequences.

Equation (2) is essentially a change of basis equation. It tells us how to express the polynomials \( p_d(k) \) in terms of the polynomials \( k^d - b(d) \). The first basis, \( p_d(k) \), has the benefit that

\[ \sum_{k} \frac{p_d(k)}{k!} = - \frac{k^{d+1}}{k!}. \]

So, if we could invert (2) and express \( k^d - b(d) \) in terms of \( p_d(k) \), \( p_{d-1}(k) \), and so on, we could apply linearity to evaluate \( \sum_{k} \frac{k^d - b(d)}{k!} \).
Equation (2) amounts to the following matrix identity:

\[
\begin{bmatrix}
p_0(k) \\
p_1(k) \\
p_2(k) \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
-1 & 1 & 0 & 0 & \cdots \\
-2 & -1 & 1 & 0 & \cdots \\
-3 & -3 & -1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
k - b(1) \\
k^2 - b(2) \\
k^3 - b(3) \\
\vdots
\end{bmatrix}.
\]  

(3)

The coefficient matrix is invertible. The first few rows of \(A^{-1}\) are as follows:

\[
A^{-1} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & \cdots \\
3 & 1 & 1 & 0 & 0 & \cdots \\
9 & 4 & 1 & 1 & 0 & \cdots \\
31 & 14 & 5 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

The OEIS [5] suggests that the columns are the diagonals of A121207, which is a table of values \(T_{dj}\) defined by

\[
T_{(d+1)j} = \sum_{i=0}^{d-j-1} \binom{r}{i} T_{(d-i)j}.
\]

This table is a special case of a family of tables studied by Gould and Quaintance [4]. The numbers \(T_{r1}\) are called the Gould numbers (see A040027). This suggestion turns out to be correct.

**Proposition 4.** For any positive integer \(d\),

\[
\sum_{k} \frac{k^d - b(d)}{k!} = -\sum_{j\geq1} \frac{B_{dj} k^j}{k!},
\]

where the matrix \(B_{dj}\) is defined by \(B_{dd} = 1\) and

\[
B_{(d+1)j} = \sum_{k=0}^{d-j} \binom{d}{k} B_{(d-k)j} \quad (d \geq j).
\]

In particular, \(B_{dj}\) is the \(d\)th element of the \((j - 1)th diagonal of A121207.

8
Proof. The matrix in (3) is defined by

\[ A_{dj} = [d = j] - \binom{d - 1}{j} [d \neq j]. \]

For \( B \) to be its matrix inverse, we must have

\[ \sum_{k \geq 1} A_{(d+1)k} B_{kj} = [d + 1 = j] \tag{4} \]

for all integers \( d \geq 0 \) and \( j \geq 1 \). If we expand \( A_{(d+1)k} \), then this reads

\[ B_{(d+1)j} = \sum_{k \geq 1} \binom{d}{k} B_{kj} + [d + 1 = j]. \tag{5} \]

Note that this implies \( B_{dj} = 0 \) if \( d < j \). Indeed, \( B_{1j} = \binom{0}{1} B_{1k} = 0 \), and if \( d < j - 1 \), then we can apply induction to every term of the right-hand side of (5) to conclude that \( B_{(d+1)j} = 0 \). Hence we can define \( B_{dj} \) as follows:

\[ B_{jj} = 1 \]
\[ B_{(d+1)j} = \sum_{j \leq k \leq d} \binom{d}{k} B_{kj} \quad (d \geq j). \]

This is A121207 shifted so that \( j \) begins at 1 rather than 0.

Written more concretely, this identity reads

\[ \sum_{k=0}^{n-1} \frac{k^d - b(d)}{k!} = -\sum_{j \geq 1} B_{dj} n^j \frac{n!}{n!}. \]

If we multiply by \( n! \) and rearrange things, we obtain the following equality for the bell numbers, valid for \( n \geq 1 \) and \( d \geq 0 \):

\[ b(d) = \frac{\sum_{k=0}^{n-1} k^d \eta^{n-k} + \sum_{j \geq 1} B_{dj} n^j}{\sum_{k=0}^{n-1} \eta^{n-k}}. \tag{6} \]

It seems plausible that this has a combinatorial proof, but the author does not know one.
4 Connections with Bell numbers

The exponential generating functions from the previous section are

\begin{align*}
    f_{a,z}(x) &= \exp(-z - (a - 1)x + ze^x) \\
    g_{a,z}(x) &= \exp(z^{-1} - ax - z^{-1}e^{-x}).
\end{align*}

These functions, and therefore the underlying sequences, are connected with the Bell numbers. In particular, if we let

\[ B(x) = e^{e^x-1} = \sum_{j \geq 0} \frac{b(d)}{d!} x^d \]

be the exponential generating function for the Bell numbers, then for integral \( z \) we have the following identities:

\begin{align*}
    f_{a,z}(x) &= e^{(1-a)x} B(x)^z \\
    g_{a,1/z}(x) &= e^{-ax} B(-x)^{-z}.
\end{align*}

(7) (8)

If \( z \) is positive, the first equation says that the coefficients of \( f_{1,z}(x) \) are the binomial convolution of \((1 - a)^k\) with the convolution of the Bell numbers with themselves \( z \) times. If \( z \) is negative, the second equation says that the coefficients of \( g_{1,1/z}(x) \) are the binomial convolution of \((-a)^k\) with the convolution of the alternating Bell numbers \((-1)^d b(d)\) with themselves \( z \) times.

Examples for \( z^k/a^k \) Setting \( a = z = 1 \) in (7), we get \( f_{1,1}(x) = B(x) \). If we translate this into the vocabulary of the previous section, this says that

\[ \frac{k^d - b(d)}{k!} \]

is Gosper summable, and no other constants will work. Similarly, \( f_{1,2}(x) = B(x)^2 \), so

\[ \frac{(k^d - c(d))2^k}{k!} \]
is Gosper summable only if \( c(d) = \sum_j \binom{d}{j} b(d) b(d-j) \). This sequence begins 2, 6, 22, 94, corresponding to the following identities:

\[
\sum_{k=0}^{n} \frac{(k-2)2^k}{k!} = -\frac{2^{n+1}}{n!}
\]
\[
\sum_{k=0}^{n} \frac{(k^2 - 6)2^k}{k!} = -\frac{(n+3)2^{n+1}}{n!}
\]
\[
\sum_{k=0}^{n} \frac{(k^3 - 22)2^k}{k!} = -\frac{(n^2 + 4n + 11)2^{n+1}}{n!}
\]
\[
\sum_{k=0}^{n} \frac{(k^4 - 94)2^k}{k!} = -\frac{(n^3 + 5n^2 + 17n + 47)2^{n+1}}{n!}
\]

Setting \( a = 1/2 \) and \( z = 1 \) gives \( g_{1/2,1}(x) = e^{x/2} B(x) \), which says that

\[
\frac{k^d - c(d)}{(1/2)^k} = (k^d - c(d)) \frac{4^k k!}{(2k)!}
\]

is Gosper summable only if \( c(d) = \sum_j \binom{d}{j} b(d) / 2^{d-j} \).

**Examples for** \( z^k a^\frac{k}{k} \)  The connection for \( g_{1,1/z}(x) \) is most convenient when \( z \) is a negative integer. Setting \( z = -1 \) in (8) gives \( g_{1,-1}(x) = e^{-x} B(-x) = B'(-x) \), which says that

\[
(k^d - (-1)^d b(d + 1))(-1)^k k!
\]

is Gosper summable, and no constant except \((-1)^d b(d + 1)\) will work. Similarly, \( g_{1,-1/2}(x) = e^{-x} B(-x)^2 = B'(-x) B(-x) \). Therefore,

\[
(k^d - c(d)) \frac{k!}{(-2)^k}
\]

is Gosper summable only if \( c(d) = (-1)^d \sum_j \binom{d}{j} b(j + 1) b(d-j) \). For example,

\[
\sum_{k=0}^{n} \frac{(k^2 - 11)k!}{(-2)^k} = \frac{(n-3)(n+1)!}{(-2)^n} - 8.
\]

Setting \( a = 1/2 \) and \( z = -1 \) gives \( g_{1/2,-1}(x) = e^{-x/2} B(-x) \), so

\[
(k^d - c(d))(-1)^k (1/2)^x = (k^d - c(d))(-1)^k \frac{4^k k!}{4^k k!}
\]

is Gosper summable only if \( c(d) = (-1)^d \sum_j \binom{d}{j} b(d) / 2^{d-j} \).
5 Conclusion

We have given some explicit conditions for the Gosper summability of hypergeometric terms of the form

\[(k^d - c(d))z^ka^k \quad \text{and} \quad (k^d - c'(d))\frac{z^k}{a^k}.\]

Namely, \(c(d)\) and \(c'(d)\) must be the coefficients of explicit exponential generating functions which are related to the Bell numbers. In the special case of \(1/k!\), we gave an explicit evaluation of these sums in terms of the inverse of a matrix involving binomial coefficients. The Bell numbers probably appear by accident. However, should some combinatorial connection be made, the author would like to hear about it.

We have made use of Gosper’s algorithm for hypergeometric summation, but there is a continuous variant of Gosper’s algorithm for hyperexponential integration [1]. We may be able to make statements about when integrals of the form

\[\int (x^d - b(d))e^{-x^2} \, dx\]

are themselves hyperexponential. However, in contrast to the summation problem, we have a solid understanding of all elementary antiderivatives thanks to Liouville’s theorem [2, ch. 12], not just hyperexponential ones. Thus this could be a less satisfying problem.

Finally, we note that the results here work essentially because the space of polynomials \(p(k)\) such that \(p(k)z^ka^k\) are Gosper summable contains polynomials of every degree greater than or equal to 1. More complicated hypergeometric terms will produce spaces with degrees only 2 or greater, or 3 or greater, and so on. In these cases, the basis could not be simplified down to leading powers and constants, so the results would be about terms of the form

\[(k^d - kc_1(d) - c_0(d))f(k),\]

or

\[(k^d - k^2c_2(d) - kc_1(d) - c_0(d))f(k),\]

and so on. The techniques here would certainly apply to such terms, though the results would be more difficult to state.

12
References

[1] Almkvist, G. and Zeilberger, D., 1990. The method of differentiating under the integral sign. J. Symb. Comput., 10(6), pp.571-592.

[2] Geddes, K.O., Czapor, S.R. and Labahn, G., 1992. Algorithms for computer algebra. Springer Science & Business Media.

[3] Gosper, R., 1978. Decision procedure for Indefinite Hypergeometric Summation. Proceedings of the National Academy of Sciences, 75(1), pp.40-42.

[4] Gould, H.W. and Quaintance, J., 2007. A Linear Binomial Recurrence and the bell Numbers and Polynomials. Applicable Analysis and Discrete Mathematics, pp.371-385.

[5] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, Published electronically at http://oeis.org.

[6] Petkovšek, M., Wilf, H., and Zeilberger, D. (1997). A = B. Wellesley: AK Peters.