WEAK COMPACTNESS OF ALMOST LIMITED OPERATORS

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Abstract. The paper is devoted to the relationship between almost limited operators and weakly compacts operators. We show that if $F$ is a $\sigma$-Dedekind complete Banach lattice then, every almost limited operator $T : E \rightarrow F$ is weakly compact if and only if $E$ is reflexive or the norm of $F$ is order continuous. Also, we show that if $E$ is a $\sigma$-Dedekind complete Banach lattice then the square of every positive almost limited operator $T : E \rightarrow E$ is weakly compact if and only if the norm of $E$ is order continuous.

1. Introduction

Throughout this paper $X, Y$ will denote real Banach spaces, and $E, F$ will denote real Banach lattices. $B_X$ is the closed unit ball of $X$ and $B_X^+ := B_E \cap E^+$ is the positive part of $B_E$. We will use the term operator $T : X \rightarrow Y$ between two Banach spaces to mean a bounded linear mapping. We refer to [1, 5] for unexplained terminology of the Banach lattice theory and positive operators.

Let us recall that a norm bounded set $A$ in a Banach space $X$ is called limited, if every weak$^*$ null sequence $(f_n)$ in $X^*$ converges uniformly to zero on $A$, that is, $\sup_{x \in A}|f_n(x)| \rightarrow 0$. An operator $T : X \rightarrow Y$ is said to be limited whenever $T(B_X)$ is a limited set in $Y$, equivalently, whenever $||T^*(f_n)|| \rightarrow 0$ for every weak$^*$ null sequence $(f_n) \subset Y^*$.

Recently, the authors of [2] considered the disjoint version of limited sets by introducing the class of almost limited sets in Banach lattices. From [2] a norm bounded subset $A$ of a Banach lattice $E$ is said to be almost limited, if every disjoint weak$^*$ null sequence $(f_n)$ in $E^*$ converges uniformly to zero on $A$.

From [4], an operator $T : X \rightarrow E$ is called almost limited if $T(B_X)$ is an almost limited set in $E$, equivalently, $||T^*(f_n)|| \rightarrow 0$ for every disjoint weak$^*$ null sequence $(f_n) \subset E^*$. Note that an almost limited operator need not be weakly compact. In fact, the identity operator of the Banach lattice $\ell^\infty$ is almost limited but it is not weakly compact.

In this paper, we characterize pairs of Banach lattices $E, F$ for which every almost limited operator $T : E \rightarrow F$ is weakly compact. More precisely, we will prove that if $F$ is a $\sigma$-Dedekind complete Banach lattice then, every almost limited operator $T : E \rightarrow F$ is weakly compact if and only if $E$ is reflexive or the norm of $F$ is order continuous (Theorem 2.5). Next, we will prove that if $E$ is a $\sigma$-Dedekind complete Banach lattice then the square of every positive almost limited operator $T : E \rightarrow E$ is weakly compact if and only if the norm of $E$ is order continuous (Theorem 2.9). As consequences, we will give some interesting results.

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2. Main results

Let us recall that a Banach lattice $E$ is said to have the dual positive Schur property if $\|f_n\| \to 0$ for every weak* null sequence $(f_n) \subset (E^*)^\tau$, equivalently, $\|f_n\| \to 0$ for every weak* null sequence $(f_n) \subset (E^*)^\tau$ consisting of pairwise disjoint terms (Proposition 2.3 of [8]). A Banach lattice $E$ has the property (d) whenever $|f_n| \wedge |f_n| = 0$ and $f_n \overset{w^*}{\to} 0$ in $E^*$. It should be noted, by Proposition 1.4 of [8], that every $\sigma$-Dedekind complete Banach lattice has the property (d) but the converse is not true in general. In fact, the Banach lattice $\ell^\infty/c_0$ has the property (d) but it is not $\sigma$-Dedekind complete [8, Remark 1.5].

Our first result shows that we can restrict sequences appearing in the definition of almost limited operator $T : X \to E$ to positive disjoint sequences if the Banach lattice $E$ has the property (d).

**Proposition 2.1.** An operator $T : X \to E$ from a Banach space $X$ into a Banach lattice $E$ with the property (d), is almost limited if and only if $\|T^*(f_n)\| \to 0$ for every weak* null sequence $(f_n)$ in $E^*$ consisting of positive and pairwise disjoint elements.

**Proof.** The “only if” part is trivial. For the “if” part, let $(f_n) \subset E^*$ be a disjoint weak* null sequence. As $E$ has the property (d), $|f_n| \overset{w^*}{\to} 0$. Using the inequalities $0 \leq f_n^+ \leq |f_n|$ and $0 \leq f_n^- \leq |f_n|$, we see that $(f_n^+)$ and $(f_n^-)$ are disjoint weak* null sequences of $E^*$. So, from our hypothesis we see that $\|T^*(f_n^+))\| \to 0$ and $\|T^*(f_n^-)\| \to 0$. This implies that $\|T^*(f_n)\| \to 0$, and hence $T$ is almost limited. \hfill $\square$

The next result follows immediately from Proposition 2.3 of [8] combined with Proposition 2.1.

**Corollary 2.2.** A Banach lattice $E$ with the property (d) has the dual positive Schur property if and only if the identity operator on $E$ is almost limited.

The following result shows that if a positive almost limited operator $T : E \to F$ has its range in a Banach lattice with the property (d), then every positive operator $S : E \to F$ that dominates (i.e., $0 \leq S \leq T$) is also almost limited.

**Proposition 2.3.** Let $E$ and $F$ be two Banach lattices such that $F$ has the property (d). If a positive operator $S : E \to F$ is dominated by an almost limited operator, then $S$ itself is almost limited.

**Proof.** Let $S, T : E \to F$ be two operators such that $0 \leq S \leq T$ and $T$ is almost limited. Let $(f_n)$ be a disjoint sequence in $(F^*)^\tau$ such that $f_n \overset{w^*}{\to} 0$. As $T$ is almost limited, $\|T^*(f_n)\| \to 0$. Using the inequalities $0 \leq S^*(f_n) \leq T^*(f_n)$, we see that $\|S^*(f_n)\| \leq \|T^*(f_n)\|$ for all $n$, from which we get $\|S^*(f_n)\| \to 0$. Now, by Proposition 2.1 $S$ is well almost limited. \hfill $\square$

The next remark will be useful in further considerations.

**Remark 2.4.**

1. Consider the scheme of operators $X \overset{R}{\to} Y \overset{S}{\to} F$. It is easy to see that if $S$ is an almost limited operator, then $S \circ R$ is likewise almost limited.

2. Consider the scheme of operators $X \overset{R}{\to} E \overset{S}{\to} F$. 


(a) If $R$ is an almost limited operator, then $S \circ R$ is not necessarily almost limited. In fact, by a result in [6], there exists a non regular operator $S : \ell^\infty \to c_0$, which is certainly not compact. So by Proposition 4.3 of [4], $S$ is not almost limited. If $R : \ell^\infty \to \ell^\infty$ is the identity operator on $\ell^\infty$ then $R$ is almost limited but $S \circ R = S$ is not almost limited.

(b) However, if $E$ has the dual positive Schur property (for example, $E = \ell^\infty$) and $F$ has the property (d), and $S$ is positive, then $T = S \circ R$ is an almost limited operator. In fact, according to Proposition 2.1, let $(f_n) \subseteq F^*$ be a positive disjoint weak$^*$ null sequence. Clearly $0 \leq S^* f_n \overset{w^*}{\to} 0$ holds in $E^*$. Since $E$ has the dual positive Schur property then $\|S^* f_n\| \to 0$, and hence $\|T^* f_n\| = \|R^* (S^* f_n)\| \to 0$, as desired.

Our next major result characterizes pairs of Banach lattices $E$, $F$ for which every positive almost limited operator $T : E \to F$ is weakly compact.

**Theorem 2.5.** Let $E$ and $F$ be two Banach lattices such that $F$ is $\sigma$-Dedekind complete. Then the following assertions are equivalent:

1. Every almost limited operator $T : E \to F$ is weakly compact.
2. Every positive almost limited operator $T : E \to F$ is weakly compact.
3. One of the following statements is valid:
   a. $E$ is reflexive.
   b. The norm of $F$ is order continuous.

**Proof.** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Assume by way of contradiction that $E$ is not reflexive and the norm of $F$ is not order continuous. We have to construct a positive almost limited operator $T : E \to F$ which is not weakly compact.

Indeed, since the norm of $F$ is not order continuous, then by Corollary 2.4.3 of [5] we may assume that $\ell^\infty$ is a closed sublattice of $F$. As $E$ is not reflexive then $E^*$ is not reflexive, and hence the closed unit ball $B_{E^*}$ of $E^*$ is not weakly compact. So, from $B_{E^*} \subset B_{E^*}^c - B_{E^*}^c$, we see that $B_{E^*}^c$ is not weakly compact. Then, by the Eberlein-Šmulian theorem one can find a sequence $(f_n)$ in $B_{E^*}^c$ which does not have any weakly convergent subsequence. Consider the positive operator $T : E \to \ell^\infty \subseteq F$ defined by

$$T(x) = (f_n(x))_{n=1}^\infty$$

for all $x \in E$. By Remark 2.4(2b) $T$ is an almost limited operator. But $T$ is not weakly compact. In fact, if $T$ were weakly compact then $T^* : (\ell^\infty)^* \to E^*$ would weakly compact. Note that $T^*((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n f_n$ for every $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)^*$. So, if $e_n$ is the usual basis element in $\ell^1$ then $T^* (\lambda_n) = f_n$ so that $(f_n)$ would have a weakly convergent subsequence. This contradicts the choice of $(f_n)$. Therefore, $T$ is not weakly compact, as desired.

(a) $\Rightarrow$ (1) In this case, every operator from $E$ into $F$ is weakly compact.

(b) $\Rightarrow$ (1) By Theorem 4.2 of [4] we see that $T$ is L-weakly compact, and by Theorem 5.61 of [1] $T$ is well weakly compact.

By a similar proof as the previous theorem, we obtain the following result.

**Theorem 2.6.** Let $X$ a Banach space and $F$ a $\sigma$-Dedekind complete Banach lattice. Then the following assertions are equivalent:
(1) Every almost limited operator $T : X \to F$ is weakly compact.

(2) One of the following statements is valid:
   (a) $X$ is reflexive.
   (b) The norm of $F$ is order continuous.

As a consequence of Theorem 2.5, we obtain an operator characterization of order continuity of the norm of a $\sigma$-Dedekind complete Banach lattice.

**Corollary 2.7.** Let $E$ be a $\sigma$-Dedekind complete Banach lattice. Then the following statements are equivalent:

1. Every almost limited operator $T$ from $E$ into $E$ is weakly compact.
2. Every positive almost limited operator $T$ from $E$ into $E$ is weakly compact.
3. The norm of $E$ is order continuous.

Another consequence of Theorem 2.5 is the following result.

**Corollary 2.8.** For a Banach lattice $E$, the following statements are equivalent:

1. Every positive operator $T : E \to F$ from $E$ to an arbitrary infinite dimensional AM-space is weakly compact.
2. Every positive operator $T : E \to \ell_\infty$ is weakly compact.
3. $E$ is reflexive.

**Proof.** (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious. (2) $\Rightarrow$ (3) Follows from Theorem 2.5. □

The following result characterize Banach lattice $E$ for which every positive almost limited operator $T : E \to E$ has a weakly compact square.

**Theorem 2.9.** Let $E$ be a $\sigma$-Dedekind complete Banach lattice. Then the following statements are equivalent:

1. Every positive almost limited operator $T$ from $E$ into $E$ is weakly compact.
2. For every positive almost limited operator $T$ from $E$ into $E$, the operator $T^2$ is weakly compact.
3. The norm of $E$ is order continuous.

**Proof.** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Assume by way of contradiction that the norm of $E$ is not order continuous. So, by Theorem 4.14 of [1] there exists a disjoint sequence $(u_n) \subset E^+$ satisfying $\|u_n\| = 1$ and $0 \leq u_n \leq u$ for all $n$ and for some $u \in E^+$. We can now proceed analogously to the proof of Proposition 0.5.5 of [7]. Let $g_n \in E^+$ be of norm one and such that $g_n(u_n) = \|u_n\| = 1$ and let $P_n$ be the band projection onto $\{u_n\}^{dd}$, where $\{u_n\}^{dd}$ is the band generated by $\{u_n\}$. If $f_n = g_n \circ P_n$, then $f_n \wedge f_m = 0$ for $n \neq m$, $\sup_n \|f_n\| \leq 1$ and $f_n(u_m) = \delta_{nm}$. Hence the operator $S : \ell^\infty \to E$ defined by

$$S((t_n)_{n=1}^\infty) = (\omega \sum_{n=1}^\infty t_n u_n)$$

is a lattice isomorphism from $\ell^\infty$ into $E$, where $(\omega \sum_{n=1}^\infty t_n u_n)$ denotes the order limit of the sequence of the partial sums $\sum_{n=1}^m t_n u_n$ for each $(t_n)_{n=1}^\infty \in \ell^\infty$. Also, let $R : E \to \ell^\infty$ be the positive operator defined by

$$R(x) = (f_n(x))_{n=1}^\infty.$$
So, by Remark 2.4(2b), the positive operator \( T = S \circ R : E \to F \) defined by
\[
T(x) = (o) \sum_{n=1}^{\infty} f_n(x)u_n
\]
is almost limited. But \( T \) is not weakly compact. In fact, let \( x_n = \sum_{k=1}^{n} u_k \) for each \( n \), and note that \( 0 \leq x_n \uparrow u \). Clearly \( T(u_n) = u_n \), and hence \( T(x_n) = x_n \) for all \( n \). If \( x \) is a weak limit of a subsequence of \( (x_n) \), then it is easy to see that \( x_n \uparrow x \) and \( x_n \overset{\omega}{\to} x \) must hold. By Theorem 3.52 of [1] we have \( \|x_n - x\| \to 0 \), and hence \( \|x_{n+1} - x_n\| \to 0 \). But this contradicts \( \|x_{n+1} - x_n\| = \|u_{n+1}\| = 1 \) for all \( n \). Thus \( (x_n) \) has no weakly convergent subsequence, and hence \( T \) is not weakly compact, as desired.

\((3) \Rightarrow (1)\) Follows from Theorem 2.5. \(\square\)

Finally, note that a weakly compact operator \( T : X \to F \) need not be almost limited. In fact, the identity operator of the Banach lattice \( \ell^2 \) is weakly compact but it is not almost limited. However, if \( F \) has the positive Schur property, then the two class coincide. The details follow.

**Proposition 2.10.** An operator \( T : X \to F \) from a Banach space \( X \) to a Banach lattice \( F \) with the positive Schur property is weakly compact if and only if it is almost limited.

**Proof.** The “if” part follows from Theorem 2.6. For the “only if” part, assume that \( T : X \to F \) is weakly compact. It follows from Theorem 3.4 of [3] that \( T \) is L-weakly compact, and hence \( T \) is almost limited [4, Theorem 4.2]. \(\square\)

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