Extended Gini Index

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Abstract

We propose an extended version of Gini index defined on the set of infinite utility streams, $X = Y^\mathbb{N}$ where $Y \subset \mathbb{R}$. For $Y$ containing at most finitely many elements, the index satisfies the generalized Pigou-Dalton transfer principles in addition to the anonymity axiom. However, in the general case of $Y$ containing infinitely many elements, real valued representation satisfying the generalized Pigou-Dalton transfer principle and anonymity is not possible.

Keywords: Anonymity, Extended Gini Index, Generalized Pigou-Dalton transfer principle, Social welfare function.

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1 Introduction

The Gini index (also referred to as Gini Coefficient) is a widely used measure of inequality in income or wealth distribution in the society. It is a measure of dispersion of income distribution for a given population and is sensitive to redistribution of income from rich to poor. In this paper, we propose an extended version of the Gini index as a real valued representation of the infinite utility streams. The new index satisfies generalized version of Pigou-Dalton transfer principle and the anonymity axiom.

A brief review of the two equity axioms is as follows. Anonymity axiom which is an example of procedural equity applies to the situations where the change involved in the infinite utility streams does not tweak the distribution of utilities in the infinite utility stream. The idea of anonymity was introduced in a classic contribution, Ramsey, who observed that discounting one generation’s utility relative to another’s is “ethically indefensible”, and something that “arises merely from the weakness of the imagination”. Diamond formalized the concept of “equal treatment” of all generations (present and future) in the form of an anonymity axiom on social preferences. The anonymity axiom requires that society should be indifferent between two streams of well-being, if one is obtained from the other by interchanging the well-being of any pair of generations.

The second equity concept is an example of the consequentialist equity notions. The particular version considered in this paper is the well-known Pigou-Dalton transfer principle. This axiom applies to the infinite utility streams where modify the distribution of individual utilities are modified. Pigou-Dalton transfer principle compares two infinite utility streams (x and y) in which all generations except two have the same utility levels in both utility streams; regarding the two remaining generations (say, i and j), if \( y_i < x_i < x_j < y_j \) and \( y_i + y_j = x_i + x_j \), then utility stream x is socially preferred to y. This means that if we reduce well-being inequality among generation i and j, leaving all other generations intact, then we obtain a socially preferred welfare distribution. This equity principle ranks utility sequence x superior to y if x is obtained from y by carrying out a non-leaky and non-rank switching transfer of welfare between a pair of generation.

It is easy to infer that this definition would also help us rank utility sequence x superior to y if x is obtained from y by carrying out non-leaky and non-rank switching transfer of welfare among any finitely many pairs of generations. To enable us to compare sequence x and y when x is obtained from y by carrying out arbitrarily many pairs of rich and poor generations, we have introduced a generalized version of the Pigou-Dalton transfer principle in Dubey and Laguzzi. It has been shown that the generalized equity principle admits numerical representation in case Y does not contain more than seven distinct elements.

In this short note we consider weaker version of the generalized equity principle and show (in Proposition 1) that an index defined along the lines of Gini index (which we call extended Gini index) satisfies the equity principle in addition to anonymity axiom when Y contains finitely many distinct elements. Second result (Proposition 2) demonstrates impossibility of representation when Y contains infinitely many distinct elements.

Rest of the paper is organized as follows. Section 2 contains the relevant definitions and in section...
we discuss the generalized equity principle which is the focus of our paper. Section contains the two results with detailed proofs. We conclude in section.

2 Preliminaries

2.1 Notation
Let \( \mathbb{R} \) and \( \mathbb{N} \) be the sets of real numbers and natural numbers respectively. For all \( y, z \in \mathbb{R}^\mathbb{N} \), we write \( y \geq z \) if \( y_n \geq z_n \), for all \( n \in \mathbb{N} \); we write \( y > z \) if \( y \geq z \) and \( y \neq z \); and we write \( y \gg z \) if \( y_n > z_n \) for all \( n \in \mathbb{N} \).

2.2 Definitions

2.2.1 Pairing function
A partial function \( f : X \to Y \) is a function from a subset \( S \) of \( X \) to \( Y \). If \( S \) equals \( X \), the partial function is said to be total. Domain and range of function \( f \) are denoted by \( \text{dom}(f) \) and \( \text{ran}(f) \) respectively. We say that \( \alpha : \mathbb{N} \to \mathbb{N} \) is a pairing function if and only if \( \alpha \) is a partial function satisfying \( \forall n \in \text{dom}(\alpha), \alpha(\alpha(n)) = n \). Note that for every pairing function \( \alpha \) we have \( \text{dom}(\alpha) = \text{ran}(\alpha) \). We denote the set of all pairing functions by \( \Pi \).

2.2.2 Density for subsets of natural numbers
Let \( |S| \) denote the cardinality of the finite set \( S \subset \mathbb{N} \). The lower asymptotic density of \( S \subset \mathbb{N} \) is defined as:

\[
d_L(S) = \liminf_{n \to \infty} \frac{|S \cap \{1, 2, \ldots, n\}|}{n}.
\]

Likewise, the upper asymptotic density of \( S \subset \mathbb{N} \) is defined as:

\[
d_U(S) = \limsup_{n \to \infty} \frac{|S \cap \{1, 2, \ldots, n\}|}{n}.
\]

If the two coincide for a set \( S \subset \mathbb{N} \), it is called the asymptotic density of set \( S \), \( d(S) \).

2.2.3 Social Welfare Order
Let \( Y \), a non-empty subset of \( \mathbb{R} \), be the set of all possible utilities that any generation can achieve. Then \( X \equiv Y^\mathbb{N} \) is the set of all possible utility streams. If \( \langle x_n \rangle \in X \), then \( \langle x_n \rangle = (x_1, x_2, \ldots) \), where, for all \( n \in \mathbb{N} \), \( x_n \in Y \) represents the amount of utility that the generation of period \( n \) earns. We consider binary relations on \( X \), denoted by \( \succeq \), with symmetric and asymmetric parts denoted by \( \sim \) and \( \succ \) respectively, defined in the usual way. A social welfare order (SWO) is a complete and transitive binary relation. A social welfare function (SWF) is a mapping \( W : X \to \mathbb{R} \). Given a SWO \( \succeq \) on \( X \), we say that \( \succeq \) can be represented by a real-valued function if there is a mapping \( W : X \to \mathbb{R} \) such that for all \( x, y \in X \), we have \( x \succeq y \) if and only if \( W(x) \geq W(y) \).
2.2.4 Procedural Equity

The following axioms on social welfare orders are used in the analysis. The procedural equity criterion that we will use is anonymity (also sometimes known as finite anonymity).

**Definition.** (Anonymity - AN): If \( x, y \in X \), and if there exist \( i, j \in \mathbb{N} \) such that \( x_i = y_j \) and \( x_j = y_i \), and for every \( k \in \mathbb{N} \setminus \{i, j\} \), \( x_k = y_k \), then \( x \sim y \).

We set \( \mathcal{P} := \{ \pi : \mathbb{N} \to \mathbb{N} : \text{finite permutation} \} \), and we define, for \( x \in X \), \( f_\pi(x) := \langle x(\pi(n)) : n \in \mathbb{N} \rangle \).

2.2.5 Consequentialist equity principles

The consequentialist equity criterion extensively studied in the literature are:

**Definition.** (Pigou-Dalton transfer principle - PD): If \( x, y \in X \), and there exist \( i, j \in \mathbb{N} \) and \( \varepsilon > 0 \), such that \( x_j = y_j + \varepsilon > y_i = x_i - \varepsilon \), while \( y_k = x_k \) for all \( k \in \mathbb{N} \setminus \{i, j\} \), then \( y \succ x \).

**Definition.** (Strong equity - SE): If \( x, y \in X \), and there exist \( i, j \in \mathbb{N} \), such that \( x_j > y_j > y_i > x_i \) while \( y_k = x_k \) for all \( k \in \mathbb{N} \setminus \{i, j\} \), then \( y \succ x \).

3 Generalized Pigou-Dalton and equity principles

The key observation and reason of the present paper for studying some extended versions of these consequentialist equity principles is motivated by the following observation. Given a set of utilities \( Y := \{a, b, c, d\} \subset \mathbb{R} \) ordered by \( a < b < c < d \), consider the two infinite streams

\[
x := \langle b, c, b, c, b, \cdots \rangle, \quad \text{and} \quad y := \langle a, d, a, d, a, d, \cdots \rangle.
\]

Following the expected interpretation of a distributive equity principle, we should be able to always rank \( y \prec x \). In the finite case, PD (and so SE) together with transitivity is sufficient to secure such a ranking, but in the infinite case transitivity cannot be extended to infinite chains. Hence PD and SE even with transitivity are not sufficient conditions to secure the desired ranking \( y \prec x \), and so an extension is necessary.

**Definition.** (Generalized Pigou-Dalton, GPD) Given \( x, y \in X \) if there is \( \alpha \in \Pi \) such that for every \( j \notin \text{dom}(\alpha) \), \( x_j = y_j \), and for every \( i \in \text{dom}(\alpha) \) there is \( \varepsilon_i \) such that

\[
either y_i + \varepsilon_i = x_i < x_{\alpha(i)} = y_{\alpha(i)} - \varepsilon_i \text{ or } y_{\alpha(i)} + \varepsilon_i < x_{\alpha(i)} < y_i < x_i \sim y_i - \varepsilon_i, \text{ then } y \prec x.
\]

**Definition.** (Generalized Equity, GE) Given \( x, y \in X \) if there is \( \alpha \in \Pi \) such that for every \( j \notin \text{dom}(\alpha) \), \( x_j = y_j \), and for every \( i \in \text{dom}(\alpha) \) one has

\[
either y_i < x_i < x_{\alpha(i)} < y_{\alpha(i)} \text{ or } y_{\alpha(i)} < x_{\alpha(i)} < x_i < y_i, \text{ then } y \prec x.
\]

In [Dubey and Laguzzi](#) we have investigated the existence and representation of these generalized equity principles. The results show that when we do not put any further restriction to the combinatorial characteristics of the pairing function, representation of SWRs satisfying those principles is
rather demanding and hard to obtain. This lead us to investigate more restricted and weaker forms of generalized equity and Pigou-Dalton transfer.  

A first line of weaker variants of GPD and GE is given by imposing some combinatorial restrictions on the pairing functions as explained in what follows. Consider two streams \(x, y \in Y^\mathbb{N}\), with \(Y = \{a, b, c, d\}\) \((a < b < c < d)\):

\[
x(n) = \begin{cases} 
  a & \exists k \in \mathbb{N} (n = 10^k) \\ 
  d & \text{else}
\end{cases} 
\]

\[
y(n) = \begin{cases} 
  b & \text{if } x(n) = d \\
  c & \text{else}
\end{cases}.
\]

By GE we get \(x \prec y\) even if the welfare improving re-distributions, from \((a, d)\) in \(x\) to \((b, c)\) in \(y\) occurs on a very sparse subset of \(\mathbb{N}\). This specifically happens as the pairing function \(\alpha\) is such that for every \(N \in \mathbb{N}\) there exists \(n \in \text{dom}(\alpha)\) such that \(|n - \alpha(n)| > N\). In other words the distances between the generations linked by the pairing function \(\alpha\) grows in an unbounded manner. To avoid this feature, we can require the pairing function \(\alpha\) having some limitations in term of being bounded, in a similar fashion as anonymity could require some restrictions on the family of permutations. The following definitions capture this idea.

**Definition.** We introduce the following collections of pairing functions.

- (Bounded pairing function) We say that \(\alpha \in \Pi\) is a bounded pairing function \((\alpha \in \Pi^b)\) iff there exists \(\lim_{n \to \infty} \frac{\lfloor n - \alpha(n) \rfloor}{n} = 0\).

- (Fixed-step pairing function) We say that \(\alpha \in \Pi\) is a fixed-step pairing function \((\alpha \in \Pi^f)\) iff there exists \(h \in \mathbb{N}\) such that \(\forall n \in \mathbb{N} \forall k \in (n \cdot (h - 1), n \cdot h]\), one has \(\alpha(k) \in n \cdot ((h - 1), n \cdot h]\).

Note that a fixed-step pairing function is also bounded. One can then introduce the following weakening of GE.

**Definition.**  

- (Bounded Generalized Equity, b-GE). Given \(x, y \in X\) if there is \(\alpha \in \Pi^b\) such that for every \(j \notin \text{dom}(\alpha)\), \(x_j = y_j\), and for every \(i \in \text{dom}(\alpha)\) one has

\[
either y(i) < x(i) < x(\alpha(i)) < y(\alpha(i)) or y(\alpha(i)) < x(\alpha(i)) < x(i) < y(i),
\]

then \(y \prec x\).

- (Fixed-step Generalized Equity, s-GE). Given \(x, y \in X\) if there is \(\alpha \in \Pi^s\) such that for every \(j \notin \text{dom}(\alpha)\), \(x_j = y_j\), and for every \(i \in \text{dom}(\alpha)\) one has

\[
either y(i) < x(i) < x(\alpha(i)) < y(\alpha(i)) or y(\alpha(i)) < x(\alpha(i)) < x(i) < y(i),
\]

then \(y \prec x\).

Moreover, one can similarly introduce b-GPD (s-GPD, resp.) and so on, by requiring \(\alpha \in \Pi^b\) (\(\alpha \in \Pi^s\), resp.). Note that GE \(\Rightarrow\) b-GE \(\Rightarrow\) s-GE.

A second line works as follows. Since we are considering an infinite time horizon, the ranking induced by \(\prec\) should be sensitive not only to few changes, but to a number of changes as large as possible. For instance, in an infinite setting one could require that the number of individuals/generations linked via the pairing function (where we can appreciate a reduction of inequality) should be at least an infinite set, and possibly with some non-zero density. This is in line with the weaker forms of Pareto principles extensively studied in the literature, such as infinite Pareto, asymptotic Pareto and weak Pareto. The following definitions capture this relevant idea for our study.
Definition. • (Infinite Equity, IE). Given \( x, y \in X \) if there is \( \alpha \in \Pi \) such that \( d(\text{dom}(\alpha)) \) is infinite, for every \( j \notin \text{dom}(\alpha), x_j = y_j, \) and for every \( i \in \text{dom}(\alpha) \) one has

\[
either y_i < x_i < x_{\alpha(i)} < y_{\alpha(i)} \text{ or } y_{\alpha(i)} < x_{\alpha(i)} < x_i < y_i, \text{ then } y \prec x.
\]

• (Asymptotic Equity, AE). Given \( x, y \in X \) if there is \( \alpha \in \Pi \) such that \( d(\text{dom}(\alpha)) > 0, \) for every \( j \notin \text{dom}(\alpha), x_j = y_j, \) and for every \( i \in \text{dom}(\alpha) \) one has

\[
either y_i < x_i < x_{\alpha(i)} < y_{\alpha(i)} \text{ or } y_{\alpha(i)} < x_{\alpha(i)} < x_i < y_i, \text{ then } y \prec x.
\]

(Weak Equity, WE). Given \( x, y \in X \) if there is \( \alpha \in \Pi \) such that \( d(\text{dom}(\alpha)) = \mathbb{N}, \) for every \( j \notin \text{dom}(\alpha), x_j = y_j, \) and for every \( i \in \text{dom}(\alpha) \) one has

\[
either y_i < x_i < x_{\alpha(i)} < y_{\alpha(i)} \text{ or } y_{\alpha(i)} < x_{\alpha(i)} < x_i < y_i, \text{ then } y \prec x.
\]

Note that \( \text{GE} \Rightarrow \text{IE} \Rightarrow \text{AE} \Rightarrow \text{WE}. \) Moreover, given a certain type of generalized equity principle, the related generalized Pigou-Dalton version is weaker, which means for instance \( \text{GE} \Rightarrow \text{GPD} \) and so on.

Remark 1. Note that we can then easily combine the two types of weaker variants, and obtain principles like s-AE, where we require both that the pairing function \( \alpha \in \Pi^s \) and that \( d(\text{dom}(\alpha)) > 0. \)

In Lemma 1 and 3 of [Dubey and Laguzzi], we have proven that if the utility domain rules out certain types of order-structure one can always prove the existence of SWR satisfying GE (also combined with AN and M). Since all the variants we are considering in this context are weaker versions of GE, those results suffice to obtain the existence of corresponding SWRs. In the next section we study more deeply exactly the various cases and understand when combinations of these generalized equity principles with AN and M are representable and when, on the contrary, they lead to impossibility of representation by social welfare functions. We present a positive and a negative result, and discuss further developments we would tackle in future research.

4 Representation of asymptotic equitable social welfare relations

4.1 Fixed-step asymptotic equity

In [Dubey and Laguzzi] we have proven that any SWO satisfying IE is not representable when the utility domain \( Y \) has at least eight elements. A careful scrutiny of the proof shows that actually non-representability persists even if we weaken IE to AE, since the pairing functions defined in that proof actually has domain with strictly positive density. But what is crucial regarding those pairing functions is that they do not satisfy any particular characteristics in line with Definition 3. The following result shows that there is no way to modify that proof-construction in order to obtain similar result, and indeed if we require fixed-step asymptotic equity, then we obtain an elegant social welfare function, which recalls an extended infinite version of the well-known Gini index.
Proposition 1. Let $X = Y^N$ where $|Y| < \infty$ and $Y \subseteq [0, 1]$. Then there exists a social welfare function $W : X \to \mathbb{R}$ satisfying AN and s-AE.

Proof. We present the result for the variants involving the generalized Pigou-Dalton transfer principles and $Y' = \{1, 2, \ldots, N\}$ and get $W' : Y^N \to \mathbb{R}$ satisfying s-APD and AN. Then by simply using a monotone function $f : Y' \to Y \subseteq [0, 1]$ we can extend the result in order to obtain $W : Y^N \to \mathbb{R}$ for s-AE and AN. Define

$$W'_N(x) := \frac{\sum_{k=1}^{N} \sum_{j=1}^{N} |x_k - x_j|}{N^2},$$

and

$$W'(x) := -\lim_{N \to \infty} W'_N(x),$$

We claim $W$ satisfies AN and p-APD. The former is trivial, so we only have to prove $W$ satisfies s-APD. So pick $x, y \in Y^N$ such that there exists $\alpha \in \Pi'$ so that:

- for all $k \in \text{dom}(\alpha)$, either $x_k < y_k < y_{\alpha(k)} < x_{\alpha(k)}$ or $x_{\alpha(k)} < y_k < x_k$;
- for all $k \notin \text{dom}(\alpha)$, $x_k = y_k$.

Let $\{I_n : n \in \mathbb{N}\}$ be the interval partition associated with $\alpha$ and $H_n := \sup I_n$. In order to show $W'(x) < W'(y)$, we first need to compare

$$\sum_{k=1}^{H_N} \sum_{j=1}^{H_N} |x_k - x_j| \quad \text{and} \quad \sum_{k=1}^{H_N} \sum_{j=1}^{H_N} |y_k - y_j|.$$ 

Some combinatorics and the choice of $x$ and $y$ reveals that, for every $n \in \mathbb{N}$,

$$\sum_{k \in I_n} \sum_{j \in I_n} |x(k) - x(j)| \geq \sum_{k \in I_n} \sum_{j \in I_n} |y(k) - y(j)| + \frac{2}{5} |\text{dom}(\alpha) \cap I_n| \cdot \frac{2}{5} |\text{dom}(\alpha) \cap I_n|.$$ 

To show that we can proceed by an inductive argument on all pairs $(k, j) \in H_N \times H_N$, by computing the values $|x_k - x_j|'$s compared to $|y_k - y_j|'$s. Note that once we have fixed $k \in H_N$ we have four possible cases: 1) $x_k = y_k$ and $x_j = y_j$; 2) $x_j = y_j$ and $y_k < x_k$; 3) $y_k < x_k$ and $x_j < y_j$; 4) $y_k < x_k$ and $y_j < x_j$.

Case 1): $x_k = y_k$ and $x_j = y_j$. We simply notice that obviously $|x_k - y_k| = 0 = |x_j - y_j|$.

Case 2): $x_j = y_j$ and $y_k < x_k$. We compute the values given by $\{j, k\}$ and $\{j, \alpha(k)\}$ and notice that $|x_k - x_j| + |x_j - x_{\alpha(k)}| = |y_k - y_j| + |y_j - y_{\alpha(k)}| + \epsilon_k - \epsilon_k = |y_k - y_j| + |y_k - y_{\alpha(k)}|$.

Case 3): $y_k < x_k$ and $x_j < y_j$. We compute the values given by $\{k, j\}$, $\{k, \alpha(j)\}$, $\{j, \alpha(k)\}$, $\{j, \alpha(j)\}$, $\{\alpha(j), \alpha(k)\}$. Some combinatorial computation provides the following:

- $|x_k - x_j| \geq |y_k - y_j| + \mu_0 \epsilon_k + \mu_0 \epsilon_j$, where $\mu_0$ takes a values in $[-1, 1]$ depending on $x_k, y_k, x_j, y_j$ using the following criteria: $\mu_0 = 1$ if $x_j > x_k$, $\mu_0 = -1$ if $y_j < y_k$, and $\mu_0 \in (-1, 1)$ is described by a monotone increasing function when $y_j \geq y_k$ or $x_j \leq x_k$.

- $|x_k - x_{\alpha(j)}| \geq |y_k - y_{\alpha(j)}| + \mu_1 \epsilon_k - \mu_1 \epsilon_j$, where $\mu_1$ takes a values in $[-1, 1]$ depending on $x_k, y_k, x_{\alpha(j)}, y_{\alpha(j)}$ using the following criteria: $\mu_1 = 1$ if $x_{\alpha(j)} < y_k$, $\mu_1 = -1$ if $x_k < y_{\alpha(j)}$, and $\mu_1 \in (-1, 1)$ is described by a monotone increasing function when $y_{\alpha(j)} \leq x_k$ or $y_k \leq x_{\alpha(j)}$. 


These two observations together, we obtain:

\[ |x_j - x_{\alpha(k)}| \geq |y_j - y_{\alpha(k)}| - \mu_2 \epsilon_k + \mu_2 \epsilon_j, \]

where \( \mu_2 \) takes a values in \([-1, 1]\) depending on \( x_j, y_j, x_{\alpha(k)}, y_{\alpha(k)} \) using the following criteria: \( \mu_2 = 1 \) if \( x_{\alpha(k)} > y_j \), \( \mu_1 = -1 \) if \( x_j > y_{\alpha(k)} \), and \( \mu_1 \in (-1, 1) \) is described by a monotone increasing function when \( y_{\alpha(k)} \geq x_j \) or \( y_j \geq x_{\alpha(k)} \).

\[ |x_{\alpha(k)} - x_{\alpha(j)}| \geq |y_{\alpha(k)} - y_{\alpha(j)}| - \mu_3 \epsilon_k - \mu_3 \epsilon_j, \]

where \( \mu_3 \) takes a values in \([-1, 1]\) depending on \( x_{\alpha(k)}, y_{\alpha(k)}, x_{\alpha(j)}, y_{\alpha(j)} \) using the following criteria: \( \mu_3 = 1 \) if \( x_{\alpha(k)} > x_{\alpha(j)} \), \( \mu_3 = -1 \) if \( y_{\alpha(k)} < y_{\alpha(j)} \), and \( \mu_3 \in (-1, 1) \) is described by a monotone increasing function when \( y_{\alpha(k)} \geq y_{\alpha(j)} \) or \( x_{\alpha(k)} \leq x_{\alpha(j)} \).

\[ |x_j - x_{\alpha(j)}| = |y_j - y_{\alpha(j)}| + 2\epsilon_j. \]

All together we obtain:

\[
|y_k - y_j| + |x_k - x_{\alpha(j)}| + |x_j - x_{\alpha(k)}| + |x_{\alpha(k)} - x_{\alpha(j)}| + |x_j - x_{\alpha(j)}| \\
\geq |y_k - y_j| + |x_k - y_{\alpha(j)}| + |y_j - y_{\alpha(k)}| + |y_{\alpha(k)} - y_{\alpha(j)}| + |x_j - y_{\alpha(j)}| + \\
\epsilon_k (\mu_0 + \mu_1 - \mu_2 - \mu_3) + \epsilon_j (\mu_0 - \mu_1 + \mu_2 - \mu_3) + 2\epsilon_j.
\]

By construction we have \( \mu_1 \leq \mu_0, \mu_3 \leq \mu_2, \mu_2 \leq \mu_0 \) and \( \mu_3 \leq \mu_1 \). Hence the last line is \( \geq 2\epsilon_j \).

Case 4): \( y_k < x_k \) and \( y_j < x_j \). It is analog to the previous case, and the details are left to the reader.

Proceeding inductively and comparing all pairs through the induction on \( j \) and \( k \) following the four cases, we can therefore observe two facts: firstly, whenever at least one of the pairs involved does not belong to \( \text{dom}(\alpha) \), the sum of the absolute values of symmetric differences of the considered combinations of stream \( x \) and stream \( y \) coincide; secondly, whenever the pairs both belong to \( \text{dom}(\alpha) \), then the five combinations considered always reveal that the sum involving the values of stream \( x \) is always larger than the ones referring to \( y \) by \( 2\epsilon \), where \( \epsilon := \min\{\epsilon_j, \epsilon_k\} \). Hence, putting these two observations together, we obtain:

\[
\sum_{k \in I_N} \sum_{j \in I_N} |x_k - x_j| \leq \sum_{k \in I_N} \sum_{j \in I_N} |y_k - y_j| + \frac{2}{5} \epsilon \cdot |\text{dom}(\alpha) \cap I_N| \cdot \frac{2}{5} \epsilon \cdot |\text{dom}(\alpha) \cap I_N|,
\]

and therefore we obtain the desired property, as by the characteristics of \( Y \) it holds \( \epsilon \geq 1 \). Hence we get:

\[
-W'(x) := \liminf_{N \to \infty} \frac{1}{H_N} \sum_{k=1}^{H_N} \sum_{j=1}^{H_N} |x(k) - x(j)| \geq \\
\geq \liminf_{N \to \infty} \frac{1}{H_N^2} \left( \sum_{k=1}^{H_N} \sum_{j=1}^{H_N} |y(k) - y(j)| + \frac{2}{5} |\text{dom}(\alpha) \cap [1, H_N]| \cdot \frac{2}{5} |\text{dom}(\alpha) \cap [1, H_N]| \right) \geq \\
\geq \liminf_{N \to \infty} \frac{1}{H_N^2} \left( \sum_{k=1}^{H_N} \sum_{j=1}^{H_N} |y(k) - y(j)| \right) + \frac{4}{25} \liminf_{N \to \infty} \frac{1}{H_N^2} \left( |\text{dom}(\alpha) \cap [1, H_N]| \right)^2 \geq \\
= -W'(y) + \frac{4}{25} d^2(\text{dom}(\alpha)).
\]

Since by assumption \( d(\text{dom}(\alpha)) > 0 \), we therefore get \( W'(x) < W'(y) \) as desired. \( \square \)
Note that in the inequalities we have used the property \( \lim \inf (a + b) \geq \lim \inf a + \lim \inf b \), \( \lim \inf (a \cdot b) \geq \lim \inf a \cdot \lim \inf b \) and \( \text{dom}(\alpha) \) is strictly positive; therefore the proof cannot be adopted to work for s-IE as well, and this is perfectly coherent with [Dubey and Laguzzi, Proposition 3] where it is shown that the combination of s-IE and AN is not representable for every non-trivial utility domain.

We now want to show that the previous positive result cannot hold when the utility domain gets too complicated. Specifically, next result shows a limitation when \( Y \subseteq [0, 1] \) contains some infinite subsets with particular order type. The set \( Y \) contains a pair of infinite sets one increasing and the other decreasing with well-defined minimum or maximum elements for each subset of \( Y \).

**Proposition 2.** Let \( Y \subseteq [0, 1] \) contain as a subset \( \{ \frac{1}{k} + \frac{1}{k+1} : k \in \mathbb{N} \} \cup \{ \frac{1}{k} - \frac{1}{k+1} : k \in \mathbb{N} \} \). Then any SWO defined on \( X = Y^\mathbb{N} \) satisfying s-AE and AN is not representable.

**Proof.** We establish the claim by contradiction. Let \( W : X \to \mathbb{R} \) be a SWF satisfying s-AE and AN. Let \( q_1, q_2, \ldots \) be any arbitrary enumeration of rational numbers in \((0, 1)\). We keep this enumeration fixed throughout the proof. Pick \( r \in (0, 1) \) and let \( u_1(r) := \min \{n \in \mathbb{N} : q_n \in [r, 1)\} \). Having defined \( u_k(r) \), for every \( k \geq 1 \) we set

\[
 u_{k+1}(r) := \min \{n \in \mathbb{N} \setminus \{u_1(r), u_2(r), \ldots, u_k(r)\} : q_n \in [r, 1)\}.
\]

Let \( U(r) := \{u_k(r) ! : k \in \mathbb{N}\} = \{U_t(r) : t \in \mathbb{N}\} \). Let \( L(r) = \mathbb{N} \setminus U(r) \) with \( l_1(r) < l_2(r) < \cdots < l_k(r) < \cdots \). We define a pair of sequences \( x(r) \) and \( z(r) \) for each \( r \in (0, 1) \) relying on the following infinite pairs of elements of set \( Y \).

\[
y_k = (y_k^1, y_k^2) := \left( \frac{1}{2} - \frac{1}{k+1}, \frac{1}{2} + \frac{1}{k+1} \right), \quad \{y_k : k \in \mathbb{N}\} \subseteq Y.
\]

For \( k < l \),

\[
\frac{1}{2} - \frac{1}{k+1} < \frac{1}{2} - \frac{1}{l+1} < \frac{1}{2} + \frac{1}{l+1} < \frac{1}{2} + \frac{1}{k+1}.
\]

Hence \( y_l > y_k \) by SE. Each element of \( L(r) \) and \( U(r) \) describes a pair of elements in the utility streams \( \langle x(r) \rangle \) and \( \langle z(r) \rangle \). The utility stream \( x(r) \) is:

\[
x'(r) = (x'_{2t-1}(r), x'_2(r)) = \begin{cases} y_1 & \text{if } t \in U(r), \\ y_{m+1} & \text{if } t = l_m(r), \quad m \in \mathbb{N}. \end{cases}
\]

Observe that the sequence \( \langle x(r) \rangle \) takes values \( y_2, y_3, \ldots, y_k \) in increasing order for \( k \in L(r) \) at pair of coordinates \((2k-1, 2k)\) and takes constant value of \( y_1 \) for \( t \in U(r) \) at pair of coordinates \((2t-1, 2t)\). Next, we define utility stream \( \langle z(r) \rangle \) in an identical fashion by taking \( U(r) = U(r) \setminus \{U_1(r)\} \), and \( L(r) = L(r) \cup \{U_1(r)\} \):

\[
z'(r) = (z'_{2t-1}(r), z'_2(r)) = \begin{cases} y_1 & \text{if } t \in U(r), \\ y_{m+1} & \text{if } t = l_m(r), \quad m \in \mathbb{N}. \end{cases}
\]

Hence,

\[\text{For } U = \{1!, 2!, 3!, \ldots \} = \{n! : n \in \mathbb{N}\}, \quad L = \mathbb{N} \setminus U, \quad \text{the first (1!), second (2!), sixth (3!) etc pair of elements are assigned values based on the rule for set } U, \quad \text{and remaining pairs of elements are assigned values based on the rule for set } L. \quad \text{It is easy to see that } d(U) = 0 \text{ and therefore, } d(L) = 1 \text{ and also since for any } r \in (0, 1), \quad U(r) \subseteq U, \quad d(U(r)) = 0, \quad \text{and } d(L(r)) = 1.\]
\[ z'(r) = y_1 = x'(r) \text{ for all } t \in U(r), \]
\[ z'(r) = y_k = x'(r), \quad k > 1 \text{ for all } t < U_1(r), \quad t \in L(r) \]
\[ z^{U_1(r)}(r) = y_k \succ y_1 = x^{U_1(r)}(r) \text{ since } k > 1, \]
\[ z'(r) = y_k \succ y_{k-1} = x'(r) \text{ for all } t > U_1(r), \quad t \in L(r). \]

Note \( d(L(r)) = 1 \) and therefore, \( d(L(r) \setminus \{1, 2, \ldots, U_1(r) - 1\}) = 1 \). Hence, by s-AE we get \( x(r) \prec z(r) \), and
\[ W(x(r)) < W(z(r)). \quad (5) \]

Next, we pick \( s \in (r, 1) \) for which \( \langle x(s) \rangle \) and \( \langle z(s) \rangle \) are defined using the same construction as above. Observe that \( U(s) \subset U(r) \) and let \( U(rs) := U(r) \setminus U(s) \). Note that there are infinitely many pairs of natural numbers in \( U(rs) \) and \( U(s) \) since there are infinitely many rational numbers \( q_n \in [r,s] \) and \( q_n \in [s,1) \) respectively. We list the elements of \( U(rs) \) in increasing order, i.e.,
\[ U(rs) = \{U_k(rs), k \in \mathbb{N} : U_k(rs) < U_{k+1}(rs), \forall k \in \mathbb{N}\}. \]

The sets \( U(s) \) and \( U(rs) \) form a partition of \( U(r) \), i.e., \( U(s) \cap U(rs) = \emptyset \) and \( U(s) \cup U(rs) = U(r) \). By construction, we only have two possible cases: \( U_1(r) = U_1(rs) \) or \( U_1(r) = U_1(s) \). Therefore, there are two cases to consider.

1. \( U_1(r) = U_1(rs) \). In this case,
\[ z'(r) = y_1 = x'(s) \text{ for all } t \in U(s), \]
\[ z'(r) = y_k = x'(s), \quad k > 1 \text{ for all } t < U_2(rs), \quad t \in L(s), \]
\[ z^{U_2(rs)}(r) = y_1 \prec y_k = x^{U_2(rs)}(s) \text{ since } k > 1, \]
\[ z'(r) = y_k \prec y_l = x'(s) \text{ for all } t > U_2(rs), \quad t \in L(s) \text{ since } k < l. \]

Since \( d(L(s)) = 1 \), by s-AE, \( z(r) \prec x(s) \).

2. \( U_1(r) = U_1(s) \). In this case,
\[ z'(r) = y_1 = x'(s) \text{ for all } t > U_1(s), \quad t \in U(s), \]
\[ z'(r) = y_k = x'(s), \quad k > 1 \text{ for all } t < U_1(s), \quad t \in L(s), \]
\[ z^{U_1(s)}(r) = y_k \succ y_1 = x^{U_1(s)}(s) \text{ since } k > 1, \]
\[ z^{U_1(rs)}(r) = y_1 \prec y_k = x^{U_1(rs)}(s) \text{ since } k > 1, \]
\[ z'(r) = y_k \succ y_{k-1} = x'(s) \text{ for all } t \in (U_1(s), U_1(rs)) \cap \mathbb{N}, \]
\[ z'(r) = y_k = x'(s) \text{ for all } t \in (U_1(rs), U_2(rs)) \cap \mathbb{N}, \]
\[ z'(r) = y_k \prec y_l = x'(s) \text{ for all } t > U_2(rs), \quad t \in L(s) \text{ since } k < l. \]

We permute pairs of terms \( z'(r) \) for \( t \in \{U_1(r), \ldots, U_1(rs) - 1\} \) with equally many pairs \( t' \in U(rs) \setminus \{U_1(rs)\} \) to obtain \( \langle z' \rangle := \langle z_{\pi(t)} : t \in \mathbb{N} \rangle \). Then \( z' \) satisfies the following properties:
\[ z''(r) = y_k = x'(s), \quad k > 1 \text{ for all } t < U_1(s), \]
\[ z''(r) = y_1 = x'(s), \text{ for all } t \in U(s), \]
\[ z''(r) = y_1 \prec y_k = x'(s), \quad k > 1 \text{ for all } t \in [U_1(s), U_1(rs)] \cap \mathbb{N}, \]
\[ z''(r) = y_k = x'(s), \quad k \geq 1 \text{ for all } t \in (U_1(rs), U_2(rs)) \cap \mathbb{N}, \]
\[ z''(r) = y_k < y_l = x'(s) \text{ for all } t > U_2(rs), \quad t \in L(s) \text{ since } k < l. \]

Hence, by AN, we have \( z' \sim z(r) \), and \( W(z') = W(z(r)) \). Since \( d(L(s)) = 1 \), by s-AE, \( z(r) \prec x(s) \). Hence
\[
W(z') < W(x(s)). \tag{6}
\]

Combining \( z(r) \sim z' \) and \( z' \prec x(s) \), we get \( z(r) \prec x(s) \), hence
\[
W(z(r)) < W(x(s)). \tag{7}
\]

Consequently, in both cases, we obtain
\[
W(z(r)) < W(x(s)). \tag{8}
\]

Therefore, (5) and (8) imply that \( (W(x(r)), W(z(r))) \) and \( (W(x(s)), W(z(s))) \) are non-empty and disjoint open intervals. Hence, because \( r \) and \( s \), with \( r < s \), were arbitrary, by density of \( \mathbb{Q} \) in \( \mathbb{R} \) we conclude that we have found a one-to-one mapping from \((0, 1)\) to \( \mathbb{Q} \), which is impossible as the latter set is countable.

\[ \square \]

5 Conclusions

In this paper, we have proposed a new version of Gini Coefficient. This index represents social welfare orders satisfying generalized Pigou-Dalton transfer principle and anonymity on the space of infinite utility streams when individual agents’ utility is assigned values from a finite set \( Y \subset \mathbb{R} \).

Since an explicit formula for the index is described, it is useful for policy formulation. We also show that when we consider more general set \( Y \) (i.e., \( Y \) having infinitely many elements of the type considered in Proposition 2), numerical representation is impossible. It is an open question for us to explore in future if social welfare function exists in case \( Y(<) \) is a well-ordered infinite subset of real numbers.
References

H. Dalton. The measurement of the inequality of incomes. *The Economic Journal*, 30(119):348–361, 1920.

P. A. Diamond. The evaluation of infinite utility streams. *Econometrica*, 33(1):170–177, 1965.

R. S. Dubey and G. Laguzzi. Equitable preference relations on infinite utility streams. *Working paper*.

C. Gini. Concentration and dependency ratios, in Italian (1909). English translation. *Rivista di Politica. Economica*, 87:769–789, 1997.

C. Hara, T. Shinotsuka, K. Suzumura, and Y. S. Xu. Continuity and egalitarianism in the evaluation of infinite utility streams. *Social Choice and Welfare*, 31(2):179–191, 2008.

A. C. Pigou. *Wealth and welfare*. Macmillan and Co., Ltd. London, 1912.

F. P. Ramsey. A mathematical theory of saving. *The Economic Journal*, 38:543–59, 1928.