ON INVERSION SETS AND THE WEAK ORDER IN COXETER GROUPS

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Abstract. In this article, we investigate the existence of joins in the weak order of an infinite Coxeter group \( W \). We give a geometric characterization of the existence of a join for a subset \( X \) in \( W \) in terms of the inversion sets of its elements and their position relative to the imaginary cone. Finally, we discuss inversion sets of infinite reduced words and the notions of biconvex and biclosed sets of positive roots.

1. Introduction

The Cayley graph of a Coxeter system \((W, S)\) is naturally oriented: we orient an edge \( w \rightarrow ws \) if \( w \in W \) and \( s \in S \) such that \( \ell(ws) > \ell(w) \). Here \( \ell(u) \) denotes the length of a reduced word for \( u \in W \) over the alphabet \( S \). The Cayley graph of \((W, S)\) with this orientation is the Hasse diagram of the (right) weak order \( \leq \), see [4, Chapter 3] for definitions and background.

The weak order encodes a good deal of the combinatorics of reduced words associated to \( W \). For instance for \( u, v \in W \), \( u \leq v \) if and only if a reduced word for \( u \) is a prefix for a reduced word for \( v \). Moreover, Björner [3, Theorem 8] shows that the poset \( (W, \leq) \) is a complete meet semilattice: for any \( A \subseteq W \), there exists a infimum \( \bigwedge A \in W \), also called the meet of \( A \), see also [4, Theorem 3.2.1]. This means, in particular, that any \( u, v \in W \) have a common greatest prefix that is the unique longest \( g = u \wedge v \in W \) for which a reduced word is the prefix of a reduced word for \( u \) and a reduced word for \( v \).

In the case of finite Coxeter systems, the weak order turns out to be a complete ortholattice [3, Theorem 8], thanks to the existence of the unique longest element \( w_0 \) in \( W \): \( u \leq v \) if and only if \( uw_0 \geq vw_0 \) and the supremum of \( A \subseteq W \) exists and is \( \bigvee A = (\bigwedge(Aw_0))w_0 \), also called the join of \( A \). In particular, for any \( u, v \in W \), there exists a unique smallest \( g = u \vee v \in W \) with two reduced words, one with prefix a reduced word for \( u \) and the other with prefix a reduced word for \( v \). The existence of \( w_0 \) and the fact that the weak order is a lattice play important roles in the study of structures related to \((W, S)\) such as Cambrian lattices and cluster combinatorics [30, 31, 32], Garside elements in spherical Artin-braid groups, see for instance [10], or reflection orders, their initial sections and Kazhdan-Lusztig polynomials, see the discussion in [16, §2].
When trying to generalize this technology to infinite Coxeter systems, the absence of \( w \) and of a join in general is crucially missed. For instance the Coxeter sortable elements and Cambrian fans fail to recover the whole cluster combinatorics \([33, \S 1.2]\). Examples suggest that we need to adjoin new elements to the weak order of \((W, S)\) in order to generalize properly its combinatorial usage to infinite Coxeter groups. In other words, we need to define a larger family of reduced words that would have greatest common prefixes (meet) and least common multiples (join).

This bring us to the following question: is there, for each infinite Coxeter system, a suitable complete ortholattice that contains the weak order? A natural candidate would be to consider the set of finite and infinite reduced words over \( S \), but this fails already in the case of infinite dihedral groups since the two infinite words do not have a join in that case. A completion was obtained by Dyer \([17, \text{Corollary 10.8}]\) using the theory of rootoids, but it is not suitable to our combinatorial need. A natural suitable extension of the weak order, containing the set of infinite words, was proposed by M. Dyer and was conjectured to be a complete ortholattice \([16, \text{Conjecture 2.5}]\). This extension uses the notion of bi-closed subsets of positive roots, a suitable generalizations of inversion sets of words, which we recall in \([2]\). This conjecture is still open. The main difficulties are the following:

1. understand bi-closed sets in general. Finite bi-closed sets are the inversion sets of the elements of \( W \), see \([2]\) but what about the other bi-closed sets?
2. understand the possible candidates for a join.

In this article we investigate these two points. After surveying what is known about the notions of bi-closed sets and inversion sets in \([2]\) we give in \([3]\) a geometric characterization criterion for the existence of a join in \((W, \leq)\): the join of a subset \( X \subseteq W \) exists if and only if \( X \) is finite and the cone spanned by the inversion sets of the elements in \( X \) is strictly separated from the imaginary cone – the conic hull of the limit roots –, see Theorem \([3.2]\). As a corollary, we obtain that a subset of positive roots is bi-closed if and only if it is strictly separated from the imaginary cone (Corollary \([3.4]\)). The proof is based on a characterization of the existence of the join obtained by Dyer \([16]\) and on the study of limit roots and imaginary cones started in \([20, 15, 18]\). In \([4]\) we use our geometric criterion on finite bi-closed sets to give a (partially conjectural) characterization of bi-closed sets corresponding to the inversion sets of infinite words, see Corollary \([4.4]\) and Conjecture \([5]\). This characterization extends to arbitrary Coxeter systems a result of P. Cellini and P. Papi \([7]\) valid for affine Coxeter systems (see Remark \([4.7]\)). Finally, in \([5]\) we discuss the similarities and especially the differences between the notions of bi-closed, biconvex and separable subsets of positive roots. The notion of biconvex sets in particular was used in recent works \([2, 27]\) related to the affine cases in which the authors also conjectured an extension of the weak order similar to Dyer’s conjecture.

2. Bi-closed sets, inversion sets and join in the weak order

2.1. Geometric representation of a Coxeter system. Let \((V, B)\) be a quadratic space: \( V \) is a real vector space endowed with a symmetric bilinear form \( B \). The group of linear maps that preserves \( B \) is denoted by \( O_B(V) \). The isotropic cone of \((V, B)\) is \( Q = \{ v \in V \mid B(v, v) = 0 \} \). To any non-isotropic vector \( \alpha \in V \setminus Q \), we associate the \( B \)-reflection \( s_\alpha \in O_B(V) \) defined by \( s_\alpha(v) = v - 2\frac{B(\alpha, v)}{B(\alpha, \alpha)} \alpha \).
A geometric representation of \((W, S)\) is a representation of \(W\) as a subgroup of \(O_B(V)\) such that \(S\) is mapped into a set of \(B\)-reflections associated to a simple system \(\Delta = \{\alpha_s \mid s \in S\}\) (\(s = s_{\alpha_s}\)). Recall that a simple system in \((V, B)\) is a finite subset \(\Delta\) in \(V\) such that:

(i) \(\text{cone}(\Delta) \cap \text{cone}(-\Delta) = \{0\}\) (here \(\text{cone}(X)\) is the set of non-negative linear combinations of vectors in \(X\));

(ii) for all \(\alpha, \beta \in \Delta\) distinct, \(B(\alpha, \beta) \in ]-\infty, -1]\cup \{-\cos\left(\frac{\pi}{k}\right), k \in \mathbb{Z}_{\geq 2}\}\);

(iii) for all \(\alpha \in \Delta\), \(B(\alpha, \alpha) = 1\).

Note that if the order \(m_{st}\) of \(st\) is finite, then \(B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right)\) and that \(B(\alpha_s, \alpha_t) \leq -1\) if and only if the order of \(st\) is infinite. In the case where \(a_{st} = B(\alpha_s, \alpha_t) < -1\), we label the corresponding edge in the Coxeter graph by \(a_{st}\) instead of \(\infty\).

A geometric representation of a Coxeter system is always faithful and gives rise to a root system \(\Phi = W(\Delta)\), which is partitioned into positive roots \(\Phi^+ = \text{cone}(\Delta) \cap \Phi\) and negative roots \(\Phi^- = -\Phi^+\).

The rank of the root system is the cardinality \(|\Delta| = |S|\) of \(\Delta\). The dimension of the geometric representation is the dimension of the linear span of \(\Phi\). The classical geometric representation is obtained by assuming that \(\Delta\) is a basis of \(V\) and that \(B(\alpha_s, \alpha_t) = -1\) if the order of \(st\) in \(W\) is infinite. For more details on geometric representations, we refer the reader to \([20\ §1]\).

### 2.2. Inversion sets and the weak order.

We recall here a useful geometric interpretation of the weak order. First, we review the vocabulary concerning the combinatorics of reduced words. For \(u, v, w \in W\), we say that

- \(w = uv\) is reduced if \(\ell(w) = \ell(u) + \ell(v)\);
- \(u\) is a prefix of \(w\) if a reduced word for \(u\) is a prefix of a reduced word for \(w\);
- \(v\) is a suffix of \(w\) if a reduced word for \(v\) is a suffix of a reduced word for \(w\).

Observe that if \(w = uv\) is reduced then the concatenation of any reduced word for \(u\) with any reduced word for \(v\) is a reduced word for \(w\). Also \(u \leq w\) in the right weak order if and only if \(u\) is a prefix of \(w\). Similarly, \(v \leq_L w\) in the left weak order if and only if \(v\) is a suffix of \(w\). In this article, we consider only the right weak order and so any mention to the weak order denotes the right weak order.

The (left) inversion set of \(w \in W\) is \(N(w) := \Phi^+ \cap w(\Phi^-)\), and its cardinality is \(\ell(w) = |N(w)|\). Left inversion sets have a strong connection to the weak order as stated in the following well-known proposition.

**Proposition 2.1.** Let \(u, v, w \in W\).

(i) If \(w = s_1 \cdots s_k\) is a reduced word for \(w\), then

\[
N(w) = \{\alpha_s, s_1(\alpha_{s_2}), \cdots, s_1 \cdots s_{k-1}(\alpha_{s_k})\}.
\]

(ii) If \(w = uv\) is reduced then \(N(w) = N(u) \cup u(N(v))\).

(iii) The map \(N\) is a poset monomorphism from \((W, \leq)\) to \((P(\Phi^+), \subseteq)\).

**Proof.** The first item follows easily from the geometric interpretation of the length function, see for instance \([21\ §5.6]\). The second item is a direct consequence of the first one by considering a reduced word of \(w\) obtained by the concatenation of a reduced word for \(u\) and a reduced word for \(h\). The fact that \(N\) is a poset
morphism follows from (ii) and the definitions above. Finally, to show that $N$ is injective, consider $u, v$ such that $N(u) = N(v)$. By definition of $N$ we have $\Phi^+ \cap u(\Phi^-) = \Phi^+ \cap v(\Phi^-)$. Then $\Phi^- \cap u(\Phi^+) = \Phi^- \cap v(\Phi^+)$. By taking complements of $N(u) = N(v)$ in $\Phi^+$, we obtain $\Phi^+ \cap u(\Phi^+) = \Phi^+ \cap v(\Phi^+)$. It follows that $u(\Phi^+) = v(\Phi^+)$. In other words, $N(v^{-1}u) = \emptyset$ and therefore $v^{-1}u = e$, which concludes the proof. □

2.3. Biclosed sets and the weak order. We shall now describe the image of the map $N$. A subset $A \subseteq \Phi^+$ is closed if $\text{cone}(\alpha, \beta) \cap \Phi \subseteq \Phi^+$, for all $\alpha, \beta \in A$, and $A$ is coclosed if $A^c = \Phi^+ \setminus A$ is closed. A subset of $\Phi^+$ is biclosed if it is both closed and coclosed. Observe that the intersection of closed sets is closed and that the union of coclosed sets is coclosed. Let $\mathcal{B}(\Phi^+) = \mathcal{B}$ denote the set of biclosed sets and $\mathcal{B}_0(\Phi^+) = \mathcal{B}_0$ the set of finite biclosed sets. The following proposition can be found in [16, Lemma 4.1 (d) and (f)] for instance.

**Proposition 2.2.** The map $N : (W, \leq) \rightarrow (\mathcal{B}_0, \subseteq)$ is a poset isomorphism. In particular, a subset $A \subseteq \Phi^+$ is finite biclosed if and only if there is a (unique) $w \in W$ such that $A = N(w)$. Moreover $u \leq w$ if and only if $u$ is a prefix of $w$ if and only if $N(u) \subseteq N(w)$.

Examples are given in Figure 1 and Figure 2

![Figure 1](image1)

**Figure 1.** The weak order on the Coxeter group of type $A_2$.

**Remark 2.3.** As far as we know, this proposition was stated first in an article by Kostant [24, Proposition 5.10], then as an exercise in Bourbaki [6, p.225, Exercice 16], in the case of finite crystallographic root system. An extended version for a possible infinite non crystallographic root system in the classical geometric representation was stated in [3, Proposition 2 and Proposition 3], see also [12, §2.11]. The proof for an arbitrary root system in an arbitrary geometric representation is precisely the same as the ones given before and can be found in [16, Lemma 4.1].

2.4. Join in the weak order. The weak order is a complete meet semi-lattice [4, Theorem 3.2.1]: any non-empty subset $X \subseteq W$ admits a greater lower bound called the meet of $X$ and denoted by $\bigwedge X$. As explained in the introduction, the weak order turns out to be a complete ortholattice whenever $W$ is finite. So in this case

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1The fact that $N$ is injective is classical, see for instance [6, Exercise 16, p.225] in the case of Weyl groups or [19, Proposition 3.2] in general, we give a proof here for completeness.
X always admits a least upper bound called the join of X, denoted by $\bigvee X$. When $W$ is infinite, however, it is not immediately clear if and when the join of X exists and how to compute it. From the point of view of combinatorics of reduced words, the join of $X$, if it exists, should be the smallest length element in $W$ that has any element in $X$ as a prefix.

**Remark 2.4.** In general, $N(\bigvee X) \neq \bigcap_{x \in X} N(x)$. For instance, let $(W, S)$ be of type $A_2$ and with $S = \{s_1, s_2\}$ as in Figure 1. The inversion set

$$N(s_1s_2 \land s_2s_1) = N(e) = \emptyset$$

is not equal to $N(s_1s_2) \cap N(s_2s_1) = \{\alpha_{s_1} + \alpha_{s_2}\}$, which is not even biclosed.

In [16], Dyer studied the problem of the existence of join in the weak order. We explain now the characterization he obtains.

**Definition 2.5.**

1. We say that $X$ is bounded in $W$ if there is $g \in W$ such that $x \leq g$ for all $x \in X$. Observe that $X$ is necessarily finite since there are only a finite number of prefixes of $g$.
2. Similarly, we say that $A \subseteq \Phi^+$ is bounded in $\Phi^+$ if there is a finite biclosed set $B \in B_0$ containing $A$. Hence $A$ is therefore finite.
3. The 2-closure $\overline{A}$ of $A \subseteq \Phi^+$ is the intersection of all closed subsets of $\Phi^+$ containing $A$. Since the intersection of closed sets is closed, $\overline{A}$ is the smallest closed subset of $\Phi^+$ containing $A$.

**Example 2.6.** Assume that $W$ is the infinite dihedral group generated with Coxeter graph:

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Then $X$ is bounded if and only if $X$ is finite and any $x \in X$ starts with the same letter (either $s$ or $t$), see Figure 2.

By abuse of notation, we write $N(X) := \bigcup_{x \in X} N(x)$.

**Theorem 2.7** (Dyer [16, Theorem 1.5]). Let $X \subseteq W$. Then the following statements are equivalent:

(i) $\bigvee X$ exists;
(ii) $X$ is bounded in $W$;
(iii) $N(X)$ is bounded in $\Phi^+$.

Moreover, in this case, $N(\bigvee X) = \overline{N(X)}$.

We discuss briefly some consequences and questions raised by this theorem.

(a) Theorem 2.7 gives necessary and sufficient conditions for the join to exist and a way to compute it: if $X$ is bounded then $\overline{N(X)}$ is finite and is the inversion set of the join $\bigvee X$. Unfortunately, the computation of the 2-closure of a set of positive roots is not necessarily easy and we do not know of any combinatorial rule to produce $\bigvee X$, not even if $X = \{u, v\}$. However, we explain in the next part of this section how to compute the join geometrically, if it exists.

(b) The theorem states also that if $\bigvee X$ exists, then $\overline{N(X)}$ is a finite biclosed set since it is the inversion set of an element, by Proposition 2.2. Dyer [16, Remarks 1.5] asks if the converse of this last statement is true. This is discussed in [3].
If \( W \) is finite, then \( B = B_s \) and the poset \((B, \subseteq)\) is a complete ortholattice isomorphic to the weak order with maximal element \( N(w_o) = \Phi^+ \); the join is given by the formula given in Theorem 2.7 and the meet is \( \bigwedge A = \overline{A}^c \) for \( A \in B \). Dyer conjectures that this is true even if \( W \) is infinite.

**Conjecture 1** (Dyer [16, Conjecture 2.5]). The poset \((B, \subseteq)\) of biclosed sets ordered by inclusion is a complete ortholattice. The join of a family \( X \subseteq B \), is \( \bigvee X = \bigcup X \), and the ortholattice complement is the set complement in \( \Phi^+ \).

In other words, for this conjecture to be true, we have to show that the closed set \( \bigcup X \) is also coclosed; this is the case for infinite dihedral groups as readily seen in Figure 2. The present work was motivated by this conjecture. We believe that our present investigation of biclosed sets and joins in the weak order will lead to an understanding of the existence of \( \bigvee X \) in the case where the family \( X \) in Conjecture 1 is constituted of finite biclosed sets and biclosed sets corresponding to infinite reduced words. This would be a first step toward answering this conjecture.

![Figure 2. The weak order on biclosed sets for the infinite dihedral group.](image)

**Remark 2.8.** Conjecture 1 is related to the theory of initial sections and reflection orders, developed by Dyer [12, 13] as a tool to study Kazhdan-Lusztig polynomials, see [16, Introduction and §2]. Initial sections and reflection orders appeared also in the works of Cellini-Papi [7] and Ito [22, 23]. Recently, questions about initial sections and the weak order appeared in the work of Baumann, Kamnitzer and Tingley [2, §2.5]. The same year, Lam and Pylyavskyy [27, Theorem 4.10] showed that the set of biclosed sets corresponding to inversion sets of finite and infinite reduced words in an affine Coxeter group is a meet
2.5. **Geometric construction of the join in the weak order.** Here we extract from [16, §11 and Theorem 11.6] that if the join exists, then the 2-closure can be replaced by the conic closure. For \( A \subseteq V \), denote \( \text{cone}(A) \) the set of all non-negative linear combinations of elements of \( A \) and \( \text{cone}_\Phi(A) = \text{cone}(A) \cap \Phi \) the (possibly empty) set of roots contained in \( \text{cone}(A) \). Observe that \( \text{cone}_\Phi(\Delta) = \Phi^+ \). As an immediate consequence, we have that if \( A \subseteq \text{cone}(\Delta) \) then \( \text{cone}_\Phi(A) \subseteq \Phi^+ \).

**Definition 2.9.** Let \( A \subseteq \Phi^+ \).

1. The set \( A \) is convex if \( A = \text{cone}_\Phi(A) \) and coconvex if \( A^c = \Phi^+ \setminus A \) is convex.
2. The set \( A \) is biconvex if \( A \) is convex and coconvex.
3. The set \( A \) is separable if \( \text{cone}(A) \cap \text{cone}(A^c) = \{0\} \).

Note that \( A \) is biconvex (resp. separable) if and only if \( A^c \) is. The next lemma exhibits some relationship between these notions.

**Lemma 2.10.** Let \( A \subseteq \Phi^+ \).

(i) If \( A \) is convex, then \( A \) is closed.
(ii) If \( A \) is separable, then \( A \) is convex and coconvex.
(iii) For any linear hyperplane \( H \), the set of positive roots contained (strictly or not) in one side of the halfspace bounded by \( H \) is separable (hence biconvex).

**Proof.** The first item follows easily from the definitions. The second implication of (ii) follows from (i). Assume that \( A \) is separable. Suppose by contradiction that \( \text{cone}_\Phi(A) \neq A \). Note that \( A \subseteq \text{cone}_\Phi(A) \subseteq \text{cone}(A) \). So there is \( \beta \in A^c \cap \text{cone}_\Phi(A) \). Hence \( \beta \in \text{cone}(A) \cap \text{cone}(A^c) \), contradicting the definition of separable. Thus \( A \) is convex. The proof is similar to show that \( A \) is coconvex. Therefore \( A \) is biconvex.

Now let us prove (iii). Consider a hyperplane \( H = \ker \rho \) (\( \rho \) is a linear form on \( V \)) and \( A := \{ \beta \in \Phi^+ | \rho(\beta) \geq 0 \} \). Write \( H_{\geq 0} = \{ v \in V | \rho(v) \geq 0 \} \) and \( H_{< 0} = \{ v \in V | \rho(v) < 0 \} \). Note that \( A^c \subseteq H_{< 0} \). Therefore we have \( \text{cone}(A) \subseteq H_{\geq 0} \) and \( \text{cone}(A^c) \setminus \{0\} \subseteq H_{< 0} \). So those two cones can only intersect in 0, which implies that \( A \) is separable.

The converse of (i) is false, a counterexample lives in the finite Coxeter group of type \( D_4 \), see [29, p.3192]. If \( A \) is infinite, the converse of any of the implications in (ii) is not true nor is the converse of (iii), as it is shown in [5]. However, those equivalences holds if \( A \) is finite as stated in the following proposition.

**Proposition 2.11.** Let \( A \subseteq \Phi^+ \) be finite. Then the following assertions are equivalent:

(i) \( A = N(w) \) for some \( w \in W \);
(ii) \( A \) is biclosed;
(iii) \( A \) is biconvex;
(iv) \( A \) is separable;
(v) There exists a hyperplane \( H \) such that \( A \) is strictly on one side of \( H \) and \( A^c \) is strictly on the other side of \( H \).

This proposition is a reformulation of [16, Proposition 11.6] as well as the beginning of its proof. We give here a proof for completeness. For further discussions about closed and convex sets on different types of finite root systems, see [29].
Proof. By Proposition 2.2 and Lemma 2.10 we only have to show that (i) implies (v). Take \( w \in W \) and write \( A = N(w) \). Note that \( \Delta \) is finite so \( \text{cone}(\Delta) \) is a simplicial convex cone. Since \( \Delta \) is positively linearly independent, we have \( \text{cone}(\Delta) \cap -\text{cone}(\Delta) = \{0\} \). Observe that \( \text{cone}(\Delta) \) is pointed at 0, so a supporting hyperplane \( H_0 \) of the face 0 separates strictly \( \text{cone}(\Delta) \setminus \{0\} \) and \( -\text{cone}(\Delta) \setminus \{0\} \) (see for instance [35, 34] for further information on cones and polytopes).

Therefore the sets \( \Phi^+ \) and \( \Phi^- \) are strictly separated by \( H_0 \), since \( \Phi^\pm = \pm \text{cone}(\Phi)(\Delta) \).

Therefore the hyperplane \( H = w(H_0) \) strictly separates \( A = N(w) \) and \( A^c \) since \( N(w) \subseteq w(\Phi^-) \) and \( \Phi^+ \setminus N(w) \subseteq w(\Phi^+) \). \( \square \)

Example 2.12. Let \((W, S) = (A_2, \{s_1, s_2\})\). Figure 3 shows the corresponding root system from which the biclosed, biconvex and separable sets can be easily obtained.

![Figure 3. The root system of type A2. The inversion sets \( \emptyset, \{s_1\}, \{s_2\}, \{s_1, s_1 + s_2\}, \{s_2, s_1 + s_2\} \) and \( \{s_1, s_2, s_1 + s_2\} \) are the biclosed, biconvex and separable sets of \( \Phi^+ \).](image)

The following corollary is due to Dyer [16, Proposition 11.6].

Corollary 2.13. Let \( A \subseteq \Phi^+ \) such that \( A \) is a finite biclosed set. Then

\[
\overline{A} = \text{cone}_\Phi(A).
\]

Proof. On one hand \( A' := \text{cone}_\Phi(A) \) is convex by definition, hence closed by Lemma 2.10. So by minimality \( \overline{A} \subseteq A' \). On the other hand, we have \( A \subseteq \overline{A} \). Thus \( \text{cone}(A) \subseteq \text{cone}(\overline{A}) \). Since \( \overline{A} \) is finite biclosed, it is biconvex by Proposition 2.11. So \( \text{cone}_\Phi(\overline{A}) = \overline{A} \) and \( A' \subseteq \overline{A} \). \( \square \)

By abuse of notation, we write \( N(X) := \bigcup_{x \in X} N(x) \).

Proposition 2.14. Let \( X \) be a bounded subset of \( W \). Then the join \( \bigvee X \) exists and

\[
N(\bigvee X) = \overline{N(X)} = \text{cone}_\Phi(N(X)).
\]

Proof. This is an immediate consequence of Theorem 2.7 and Corollary 2.13. \( \square \)

Remark 2.15. (1) Such a formula to compute the join in finite Coxeter groups was originally proven by Björner, Edelman and Ziegler in [5, Theorem 5.5], as a special case of their study of ‘Hyperplane arrangements with a lattice of regions’. It was then extended to infinite Coxeter group by Dyer in [16, Proposition 11.6].
(2) The second author (JPL) shows in his thesis [26, Corollary 2.36] that Conjecture 1 is true for rank 3 Coxeter system (i.e. $|S| = 3$) if we consider biconvex sets instead of biclosed sets. In this case, the join of two biclosed sets is given by the conic closure, as in Proposition 2.14. The authors tried in vain to prove or disprove that, for Coxeter groups of rank 3 or affine Coxeter groups, biclosed sets are all biconvex sets and that the 2-closure can be replaced by the conic closure in Conjecture 1. However, this is no longer true if the Coxeter group is indefinite (not finite nor affine) of rank $\geq 4$ as shown in [26, Section 2.4.2].

3. Existence of the join and the imaginary cone

3.1. Projective representation and normalized roots. In this article we also refer to the projective representation of $W$ and to the associated normalized root system $\hat{\Phi}$. Since the root system is encoded by the set of positive roots $\Phi^+$, we represent the root system by an “affine cut” of $\Phi^+$, see for instance [20, Figures 2 and 3]. Observe that $\hat{\Phi}$ is contained in the polytope $\text{conv}(\hat{\Delta})$. If $W$ is infinite, so is $\hat{\Phi}$, and thus $\hat{\Phi}$ admits a set $E$ of accumulation points that we call the set of limit roots, see [20, Figures 4–7] [26, Appendix A].

Assuming $(W,S)$ to be irreducible, we have $E = \emptyset$ if and only if $W$ is finite; $E$ is a singleton if and only if $(W,S)$ is affine and irreducible. Moreover, limit roots are in the isotropic cone $Q$ of $B$:

$$E \subseteq \hat{Q} = \{x \in V_1 | B(x,x) = 0\}.$$ 

A nice observation is that for any roots $\alpha, \beta \in \Phi^+$, the dihedral reflection subgroup generated by $s_\alpha, s_\beta$ is finite if and only if the line $L(\hat{\alpha}, \hat{\beta}) \cap Q = \emptyset$. Otherwise $L(\hat{\alpha}, \hat{\beta})$ intersects $Q$ in one or two points and contains an infinite number of normalized roots.

In this framework, conic closure is replaced by convex hull and $A \subseteq \Phi^+$ is replaced by $\hat{A} \subseteq \hat{\Phi}$. For instance, this gives for $A \subseteq \Phi^+$:

- The set $A$ is closed if for any $\alpha, \beta \in A$, the normalized roots in the segment $[\hat{\alpha}, \hat{\beta}]$ are all contained in $A$.
- The set $A$ is convex if $\hat{A} = \text{conv}_\Phi(\hat{A}) = \text{conv}(\hat{A}) \cap \hat{\Phi}$.
- The set $A$ is separable if $\text{conv}(\hat{A}) \cap \text{conv}(\hat{A}^c) = \emptyset$.

We write $\hat{N}(w)$ instead of $\widehat{N(w)}$ and we refer to the map $\hat{N} : W \to \hat{\Phi}$. We omit the notation $\hat{\cdot}$ in figures. Within this setting, Proposition 2.14 translates as follows (see Figure 4 for an illustration):

**Proposition 3.1.** Let $X$ be a bounded subset of $W$. Then the join $\bigvee X$ exists and $\hat{N}(\bigvee X) = \text{conv}_\Phi(\hat{N}(X))$. 
3.2. A geometric criterion for the existence of the join in $W$. A very useful feature studied in [15,18] is the imaginary convex body $\conv(E)$ which is shown to be a compact set [18, Theorem 2.2]. This lead us to the following result, which is a geometric generalization of Theorem 2.7. Recall that, by abuse of notation, we write $N(X) := \bigcup_{x \in X} N(x)$.

**Theorem 3.2.** Let $X \subseteq W$. Then the following statements are equivalent:

(i) $\bigvee X$ exists;

(ii) $X$ is bounded;

(iii) $X$ is finite and $\conv(\hat{N}(X)) \cap \conv(E) = \emptyset$.

(iv) $X$ is finite and the convex set $\conv(\hat{N}(X))$ and the imaginary convex body $\conv(E)$ are strictly separated by an hyperplane.

In this case: $N(\bigvee X) = \cone_{\Phi}(N(X))$.

**Proof.** The equivalence between (i) and (ii) is Dyer’s Theorem 2.7.

Assume (i): write $A = \cone_{\Phi}(N(x))$. Since $\bigvee X$ exists, the set $X$ is finite and $N(\bigvee X) = A$ by Proposition 2.14. By Proposition 2.11, there is a hyperplane $H$ separating strictly $A = \cone_{\Phi}(N(x))$ and $A^c$. That means, if $H^+$ (resp. $H^-$) denotes the open half-space with boundary $H$ containing $A$ (resp. $A^c$), then $\Phi^+ \cap H^+ = A$. We now show that $E \cap \hat{H}^+ = \emptyset$. Indeed, otherwise there is $x \in E \cap \hat{H}^+$. Since $\hat{H}^+$ is open, there is a neighbourhood of $x$ in $\hat{H}^+$, which contains an infinite number of normalized roots. This means that there is an infinite number of roots in $H^+ \cap \Phi^+ = A$, contradicting the fact that $A$ is finite. Thus $E$ is contained in the closed half-space $\hat{H} \cup \hat{H}^-$, so is $\conv(E)$. Since $\conv(A) \subseteq \hat{H}^+$, we conclude that $\conv(\hat{N}(X)) \cap \conv(E) = \emptyset$. So this proves that both (iii) and (iv) are true.

Assume (iii) to be true: observe that since $X$ is finite, $\conv(\hat{N}(X))$ is a polytope, hence compact. Since both convex sets $\conv(E)$ and $\conv(\hat{N}(X))$ are compact, there is an hyperplane $H_1$ in $V_1$ that separates strictly $\conv(E)$ and $\conv(\hat{N}(X))$ by the Hahn-Banach Separation Theorem [34, Theorem 2.4.10]. Hence (iv) is true. Now we continue our line of reasoning and prove (i). Let $H$ be the linear hyperplane in $V$ spanned by $H_1$. Set $H^+$ to be the open half space bounded by $H$ and containing $N(X)$. So $H^- = V \setminus (H \cup H^+)$ is the open half space bounded by $H$ and containing $\conv(E)$, hence $E$. The set $A := H^+ \cap \Phi^+$ is therefore bicomplemented by Lemma 2.10.(iii). Assume $A$ is infinite: since $A \subseteq \hat{A}$ is discrete, its accumulation points lie in $(H^+ \cup H) \cap E = \emptyset$ a contradiction. So $A$ is a finite bicomplemented set containing $N(X)$ by definition. So $N(X)$ is bounded in $\Phi^+$ and therefore $\bigvee X$ exists by Theorem 2.7. \hfill $\square$

**Example 3.3.** Consider the affine Coxeter group of type $\hat{C}_2$ given by the following Coxeter graph:

\[
\begin{array}{ccc}
 s_\alpha & 4 & s_\beta \\
 \bigcirc & \bigcirc & \bigcirc \\
 s_\gamma & & \\
\end{array}
\]

The first roots are shown in Figure 4. Here $E = \hat{Q} = \{\delta\}$ is a singleton represented by a red dot in the center of Figure 4. Consider $X = \{s_\alpha, s_\gamma s_\beta\}$. Then

$N(X) = N(s_\alpha) \cup N(s_\gamma s_\beta) = \{\alpha, \gamma, s_\gamma(\beta)\}$.

We see that $\delta \notin \conv_{\Phi}(\hat{N}(g))$, so by Theorem 3.2 the join $g = s_\alpha \vee s_\beta s_\gamma$ exists and its inversion set is equal to $N(g) = \cone_{\Phi}(\alpha, \gamma, s_\gamma(\beta))$ (the blue triangle at the bottom of Figure 4).
of Figure 4. Since $\gamma \in N(g)$, we know that there is $g' \in W$ such that $g = s_\gamma g'$ is reduced. By Proposition 2.1(ii), we have therefore $N(g) = \{\gamma\} \cup s_\gamma(N(g'))$; in Figure 4, $s_\gamma(N(g'))$ is constituted of the roots whose normalization are elements of the segment between $\hat{\alpha}$ and $s_\gamma(\hat{\beta})$. By repeating this process with $g'$ and $\alpha \in N(g')$, we obtain recursively that $g = s_\gamma(s_\alpha s_\beta)^4$.

However, $\delta \in \text{conv}_\Phi(\hat{\beta}, \hat{\alpha}, s_\beta(\gamma))$ (the orange triangle at the top of Figure 4) so the join of $s_\alpha$ and $s_\beta s_\gamma$ does not exist.

Figure 4. The join of $s_\alpha$ and $s_\gamma s_\beta$ exists and is the element $g = s_\gamma(s_\alpha s_\beta)^4$ whose inversion set is constituted of the roots in the blue triangle that represents $\text{cone}(\gamma, s_\gamma(\beta), \alpha)$. However the join of $s_\alpha$ and $s_\beta s_\gamma$ does not exist since the red dot is contained in the red triangle that represents $\text{cone}(\beta, s_\beta(\gamma), \alpha)$.

The end of the proof of Theorem 3.2 gives the following useful characterization of inversion sets, see Figure 5 for an illustration.

**Corollary 3.4.** Let $H$ be a hyperplane that does not intersect the imaginary convex body $\text{conv}(E)$. Then the positive roots on the other side of $\text{conv}(E)$ relatively to $H$ form an inversion set $N(w)$ for a $w \in W$. Moreover, any finite biclosed set is obtained in this way.

3.3. **Toward an even simpler criterion for the existence of a join.** As we stated after Theorem 2.7, Dyer asks the question if we could replace the bounded hypothesis in Theorem 2.7 by “$N(X)$ is finite”. We believe it to be true and state it as a conjecture:

**Conjecture 2** (Dyer [16, Remarks 1.5]). If $N(X)$ is finite, then $\bigvee X$ exists.

In regard of Theorem 3.2, a strategy for answering Conjecture 2 would be to study $\text{conv}(\bar{N}(X)) \cap \text{conv}(E)$ whenever $\bar{N}(X)$ is finite. Indeed, since convex closure implies
In the 2-closure, we have $\overline{N(X)} \subseteq \text{cone}_\Phi(N(X))$; therefore $\text{cone}(N(X)) \subseteq \text{cone}(N(X))$ and by minimality of the conic closure we have $\text{cone}(N(X)) = \text{cone}(N(X))$. So if $\text{conv}(\tilde{N}(X)) \cap \text{conv}(E) = \emptyset$, this would imply that $N(X)$ is bounded and so the join would exist. This discussion leads us naturally to state the following weaker conjecture.

**Conjecture 3.** If $\text{cone}_\Phi(N(X))$ is finite, then join $\bigvee X$ exists.

Note that the converse is true by Proposition 2.14. Note also that if $\text{cone}_\Phi(N(x))$ is finite, so is $N(X)$. Therefore $P = \text{conv}(\tilde{N}(X))$ is a polytope (and so is compact). One strategy to prove Conjecture 3 would be to prove that $P$ and $\text{conv}(E)$ are strictly separated. The difficulty to answer Conjecture 3 lies then in the fact that the set of the limit root does not restrict well to root subsystems, see [20, 18] for a discussion. More precisely, if $A \subseteq \Phi^+$ is finite and if $\text{conv}(E)$ intersects a proper face $F$ of the polytope $\text{conv}(\tilde{A})$, we would need to show that this face contains a limit root $x \in E(\Phi')$ where $\Phi' = \Phi \cap \text{aff}(F)$ (here $\text{aff}(F)$ is the affine subspace generated by $F$), which would imply that $\text{cone}_\Phi(A)$ is infinite. This would require a continuation of the study of the faces of the imaginary convex body $\text{conv}(E)$ initiated in [15] and [18, §3]. However, it is not too difficult to show that if $W$ is of rank $\leq 3$, then Conjecture 3 is true by following the above outlined strategy.

4. **Biclosed sets corresponding to inversion sets of infinite words**

The aim of this section is to discuss a generalization of Corollary 3.4 to inversion sets of infinite reduced words on $S$. 

---

**Figure 5.** Illustration of Corollary 3.4: the set of roots on the side of $H_2$ not containing $\text{conv}(E)$ forms a finite biclosed set, whereas neither side of $H_1$ provides a finite biclosed set.
4.1. Infinite reduced words and limit roots. Let \( \omega = s_1 s_2 s_3 \cdots \in S^* \) be an infinite word on \( S \). We say that \( \omega \) is an infinite reduced word if any prefix \( w_i = s_1 \cdots s_i, \ i \geq 1 \) is reduced in \( W \). We denote by \( W \) the set of infinite reduced words on \( S \). Two infinite reduced words \( w' \) and \( w \) are equivalent, denoted by \( \omega \sim \omega' \), if \( \omega' \) can be obtained from \( \omega \) by a possibly infinite number of braid moves. More precisely, we write \( \omega < \omega' \) if for any prefix \( w_i \) of \( \omega \), there is a prefix \( w'_j \) of \( \omega' \) such that \( w_i \leq w'_j \). Then we write \( \omega \sim \omega' \) if \( \omega \prec \omega' \) and \( \omega' \prec \omega \).

Let \( W^\infty \) be the quotient set of \( W \) by this equivalence relation. We say that \( w \in W \) is a prefix of \( \omega \in W^\infty \) if a reduced expression of \( w \) is a prefix of some infinite reduced word in the equivalence class of \( \omega \), i.e., there is a prefix of \( w'_j \) of \( \omega' \) such that \( w \leq w'_j \).

For \( \omega \in W \), we consider the following sequence of roots:

\[
\beta_1 := \alpha_{s_1} \in \Delta \text{ and } \beta_i := w_{i-1} (\alpha_{s_i}) \in \Phi^+, \quad \text{for } i \geq 2.
\]

The inversion set of \( \omega \) is then

\[
N(\omega) := \bigcup_{i \in \mathbb{N}} N(w_i) = \{ \beta_i \mid i \in \mathbb{N}^* \} \subseteq \Phi^+.
\]

We illustrate this notion in Figure 2 in Example 4.1 and Figure 6.

**Example 4.1.** Consider the affine Coxeter group of type \( \tilde{A}_2 \) given by the following Coxeter graph:

![Coxeter Graph]

Consider \( \omega = (s_\alpha s_\beta s_\gamma)^\infty \) and \( \omega' = s_\beta (s_\alpha s_\beta s_\gamma)^\infty = s_\beta \omega' \): these two infinite reduced words are distinct in \( W^\infty \) since there is no prefix \( w'_i \) of \( \omega' \) such that \( s_\beta \leq w'_i \), but \( \omega \prec \omega' \) since: \( s_\alpha \leq s_\beta s_\alpha s_\beta, s_\alpha s_\beta \leq s_\beta s_\alpha s_\beta; s_\alpha s_\beta s_\gamma \leq s_\beta s_\alpha s_\beta s_\gamma = s_\alpha s_\beta s_\gamma s_\alpha s_\gamma = s_\alpha s_\beta s_\gamma s_\alpha s_\gamma, \) etc., see Figure 6.

**Proposition 4.2.** Let \( \omega, \omega' \in W \).

(i) \( N(\omega) \) is separable (hence biconvex and biclosed).
(ii) \( w \in W \) is a prefix of \( \omega \) if and only \( N(w) \subseteq N(\omega) \).
(iii) If \( \omega \leq \omega' \) then \( N(\omega) \subseteq N(\omega') \).
(iv) \( N(\omega) = N(\omega') \) if and only if \( \omega \sim \omega' \).
(v) \( N : W \cup W^\infty \rightarrow \mathcal{B}(\Phi^+) \) is a monomorphism.

**Remark 4.3.** (a) The notion of inversion set of infinite words appear first, as far as we know, in the work of Cellini and Papi [27] in the context of affine Weyl groups. The equivalence class, as well as the results stated in Proposition 4.2, appears in the works of Ito [22, 23]. It was later used in the work of Lam and Pylyavskyy [27], and Lam and Thomas [26].

(b) Lam and Pylyavskyy define in [27] the limit weak order in \( W^\infty \): \( \omega \preceq \omega' \) in \( W^\infty \) if there is \( \omega_1 \sim \omega \) and \( \omega_1 \sim \omega' \) such that \( \omega_1 \prec \omega'_1 \). Thanks to Proposition 4.2 above, it is clear that the limit weak order is a subposet of the poset \( (\mathcal{B}, \subseteq) \);
recall that the poset $(B, \subseteq)$ is conjectured to be a lattice (see Conjecture 1).

The limit weak order is studied further in Lam and Thomas [28].

Proof. (i) Assume that there is $\beta \in \text{cone}(N(\omega)) \cap \text{cone}(N(\omega)^c)$. So there is a reduced prefix $w_i$ of $\omega$ such that $\beta \in \text{cone}(N(w_i))$. Since $N(w_i) \subseteq N(\omega)$, we have $\beta \in \text{cone}(N(\omega)^c) \subseteq \text{cone}(N(w_i)^c)$. So $\beta \in \text{cone}(N(w_i)) \cap \text{cone}(N(w_i)^c)$, implying $\beta = 0$ since $N(w_i)$ is separable by Proposition 2.11, a contradiction. (ii) is an immediate consequence of the definitions.

(iii) The direct implication follows from definition. Now, let $w \in W$ such that $N(w) \subseteq N(\omega)$. Since $N(w)$ is finite and $N(\omega)$ is the union of increasing finite subsets of $N(\omega)$, there is a prefix $w_i$ of $\omega$ such that $N(w) \subseteq N(w_i)$. Hence $w$ is a prefix of $w_i$ by Proposition 2.2. Therefore $w$ is a prefix of $\omega$.

(iv) The right to left implication follows from (iii) and the definition of the equivalence relation. Now, assume that $N(\omega) = N(\omega')$. Let $w_i$ a finite reduced prefix of $\omega$, so $N(w_i) \subseteq N(\omega) = N(\omega')$. Therefore $w_i$ is a prefix of $\omega'$ by (iii). Therefore $\omega \prec \omega'$. We show similarly that $\omega' \prec \omega$, hence that $\omega \sim \omega'$. Finally (v) follows from (iv). □

We state now a corollary of Theorem 3.2, extending partially Corollary 3.4 to infinite reduced words.

**Corollary 4.4.** Let $\omega \in W^{\infty}$ then $\text{conv}(\tilde{N}(\omega)) \cap \text{conv}(E) = \emptyset$.

Proof. If there is $x \in \text{conv}(\tilde{N}(\omega)) \cap \text{conv}(E)$, then there are roots $\beta_1, \ldots, \beta_k \in N(\omega)$ such that $x = \sum_{i=1}^k a_i \beta_i$, $a_i \geq 0$. By definition of $N(\omega)$, there is a prefix $w_i \in W$ of $\omega$ such that $\beta_1, \ldots, \beta_k \in N(w_i)$. Hence $x \in \text{cone}(N(w_i))$. Therefore $x \in \text{conv}(\tilde{N}(w_i)) \cap \text{conv}(E)$ contradicting Theorem 3.2 □
Remark 4.5 (Separable does not mean separable by an hyperplane). It is not true in general that an infinite subset of $\Phi^+$ is separable if and only if there is an hyperplane that separates it from its complement. Take the universal Coxeter group of rank 3 generated by $s, t, r$ as in Figure 8. Then $A = N(s(tr)^\infty)$ is an infinite separable set and $x = s(\alpha_t + \alpha_t) \in E$ is the accumulation point of $A$. But $A$ is not strictly separated by an hyperplane from its complement. Indeed, assume by contradiction that there is an hyperplane $H$ separating $\hat{A}$ and $\hat{A}^c$. Since in this case $E = \hat{Q}$ (see [18, Theorem 4.4]), the only point of $\hat{Q}$ contained in the hyperplane $H$ separating $\hat{A}$ and $\hat{A}^c$ has to be $\hat{x}$. Therefore, $H$ is tangent to the circle and contains $N(s(rt)^\infty) \subseteq A^c$, a contradiction. Such a counterexample that lies in the affine Coxeter group of type $\tilde{A}_2$ can be found in the second author’s thesis [26, Figure 2.11].

4.2. Toward a geometric characterization of the inversion sets of infinite reduced words. In Corollary 3.4, we give a geometric characterization of biclosed sets that are inversion sets of finite words. We present here a conjectural generalization to infinite reduced words.

Conjecture 4. Let $w$ be an infinite reduced word. Then $\text{Acc}(\hat{N}(w))$ is a singleton.

Remark 4.6. (a) This conjecture is equivalent to showing that the injective sequence $(\hat{\beta}_n)_{n \in \mathbb{N}^*}$ converges, where $\hat{\beta}_n$ is defined in Equation (1). Indeed, the sequence $(\hat{\beta}_n)_{n \in \mathbb{N}^*}$ is bounded, and therefore the equivalence follows from the following general topological fact based on the Bolzano–Weierstraß Theorem: a bounded sequence in a metric space converges if and only if it has a unique accumulation point.

(b) The conjecture is obviously true for irreducible affine types, since $E$ is a singleton.

(c) This conjecture is true for based root systems of Lorentzian type, i.e., the signature of the root system is $(n-1,1)$ as shown by Chen and the second author in [5 Theorem 2.8]. Note that in the proof, the authors use the fact that $Q$ is strictly convex in this case and their approach works also for linearly dependent bases, see [5 Remark 2.4]. Maybe the argument of strict convexity of $Q$ could be replaced in full generality by using [13 Corollary 6.8 (ii)] and a result similar to [5 Proposition 2.1]. In the Lorentzian case, this gives an alternate - but less straightforward - proof of [5 Theorem 2.8].

Conjecture 4 leads us to state the following conjectural geometric characterizations of inversion sets of infinite words, which can be seen as generalization of Cellini and Papi [7 Theorem 3.12] as explained in Remark 4.7.

Conjecture 5. Let $A \in \mathcal{B}$. Then $A = N(\omega)$ for some $\omega \in \mathcal{W}^\infty$ if and only if $A$ is biconvex, $\text{Acc}(\hat{A})$ is a singleton and $\text{conv}(\hat{A}) \cap \text{conv}(E) = \emptyset$.

Remark 4.7. By a result of Cellini and Papi [7 Theorem 3.12], this conjecture is true in affine types as mentioned in [2 Proposition 2.6]. Indeed in the affine type, $E = \{\hat{\delta}\}$ is a singleton, and $\delta$ is called the imaginary root. So if $\text{cone}(A)$ does not contain $\delta$, we apply Cellini and Papi’s result and obtain an infinite reduced word $\omega$ such that $A = N(\omega)$. 
4.3. Limit sets of inversion sets and paths in the imaginary cone. It would be interesting to better understand biclosed sets in relation with their limit roots and subsets of the imaginary cone. Here, we propose an approach to solve Conjecture 4.

The imaginary convex body \( \text{conv}(E) \) has a tiling parameterized by \( W \): let \( K = \{ u \in \text{conv}(\Delta) \mid B(u, \alpha_s) \leq 0 \} \), then \( \text{conv}(E) = W \cdot K \), where \( w \cdot x = \omega(x) \) is well-defined on \( \text{conv}(E) \) for any \( w \in W \), see [15, 18].

Assume from now on that \((\Phi, \Delta)\) is an indefinite based root system (i.e., not finite nor affine). So the relative interior of \( K \) exists. Take \( z \) in the relative interior of \( K \), i.e., \( B(z, \alpha) < 0 \) for all \( \alpha \in \Delta \). Then \( \omega \in W \) can be seen, with the same notations as in §4.1, as a sequence \( z, w_1 \cdot z, w_2 \cdot z, \ldots, w_n \cdot z \) within \( \text{conv}(E) \). Note that \( w_n \cdot z \) is an injective sequence, since \( w_n \cdot z \in w \cdot K \) and \( K \) is a fundamental domain for the action of \( W \) on \( W \cdot K \) (see Figure 7).

Conjecture 6. Let \( \omega \) be an infinite reduced word and \((w_n)_{n \in \mathbb{N}^*}\) the increasing sequence of its prefixes. Let \( z \) in the relative interior of \( K \). Then there is \( x \in E \) such that

\[
\text{Acc}((w_n \cdot z)_{n \in \mathbb{N}^*}) = \text{Acc}(\hat{N}(\omega)) = \{x\}.
\]

In other words both sequences \((w_n \cdot z)_{n \in \mathbb{N}^*}\) and \((\hat{\beta}_n)_{n \in \mathbb{N}^*}\) converge to \( x \in E \).

Remark 4.8. (a) Conjecture 6 has a taste of [18, Theorem 3.11].

(b) We may ask a weaker question: do we have \( \text{Acc}((w_n \cdot z)_{n \in \mathbb{N}^*}) = \text{Acc}(\hat{N}(\omega)) \)?

(c) As stated in [18, Remark 6.16(b)], we do not know if \( \text{Acc}(W \cdot x) = E \) for \( x \in W \cdot K \) in general. We know that \( \text{Acc}(W \cdot x) \supseteq E \), see [18, Corollary 6.15].
So Conjecture [6] slightly improves the statement of [18, Corollary 6.15]: any sequence of the form \( \{w_n \cdot x\}_{n \in \mathbb{N}} \subseteq W \cdot x \) where \( w_n \) are increasing prefixes of an infinite reduced word on \( S \) has an accumulation point in \( E \).

(d) It would be very interesting to fit Lam and Thomas’ results [28] in our framework by interpreting the imaginary convex body as a realization of the Davis complex, see [9, 1]. It would allow the use of tools from CAT(0) spaces to explore Conjecture [6]. Let \( S' \) be the set of spherical subsets of \( S \): \( I \in S' \) if \( W_I = \langle I \rangle \) is finite. A way to start is to take a point \( z \) in the relative interior of \( K \). For any \( I \subset S' \), define

\[ P_z^I := \text{conv}(W_I \cdot z). \]

Then \( P_z^I \) is the permutahedron of the finite Coxeter group \( W_I \) with based point \( x \). Now take

\[ D = W \cdot \left( \bigcup_{I \subseteq S'} P_z^I \right) \subseteq \text{conv}(E). \]

Finally, assume that \( D \) is endowed with the piecewise metric \( \mu \) given by the Euclidean metric on each permutahedron \( w \cdot P_z^I \), for any \( w \in W \). Is \( (D, \mu) \) a geometric realization of the Davis complex? An approach to answer this problem can be found in [25, Appendix B4-B5].

5. Relationships between biclosed and biconvex sets

We know that for a finite set, the notion of separable, biconvex and biclosed are the same, see Proposition [2.11]. In this final section, we discuss the differences between the definition of biclosed, biconvex and separable in the case of infinite sets.

5.1. On infinite biclosed sets and biconvex sets. We establish in the following examples that biclosed sets are not biconvex in general.

**Example 5.1** (A biclosed set that is not biconvex in a geometric representation of dimension 3 of a Coxeter group of rank 4). We consider \((U_3, S)\) to be the universal Coxeter system of rank 3 whose Coxeter graph is:

\[
\begin{array}{ccc}
\infty & s & \infty \\
\downarrow & \downarrow & \downarrow \\
r & \infty & t
\end{array}
\]

Let \( W' \) the reflection subgroup of \( U_3 \) generated by \( S' = \{srs, sts, r, t\} \). It is not difficult to see that \( \Delta' = \{s(\alpha_r), s(\alpha_t), \alpha_r, \alpha_t\} \) is a simple system, with

\[ B(\alpha_r, \alpha_t) = B(s(\alpha_r), s(\alpha_t)) = B(\alpha_r, s(\alpha_t)) = B(\alpha_t, s(\alpha_t)) = -1 \]

and

\[ B(\alpha_r, s(\alpha_t)) = B(s(\alpha_r), \alpha_t) = -3. \]

Note that \( \Delta' \) is positively linearly independent but not linearly independent. It follows that \((W', S')\) is a universal Coxeter system of rank 4 (geometrically represented in dimension 3). Now consider the standard parabolic subgroup \( W_I \) of \((W', S')\) given by \( I = \{sts, r, t\} \subseteq S' \), with associated simple system \( \Delta_I = \{\alpha_t, \alpha_r, s(\alpha_t)\} \).

The associated positive root subsystem \( \Phi_I^+ = W_I(\Delta_I) \) is a biclosed set in \( \Phi^{+} = \)
$W'(\Delta') \cap \Phi^+$. But $\Phi^+_j$ is not biconvex in $\Phi^+$. Indeed $s(\alpha_r) \notin \Phi^+_j$ so $ts(\alpha_r) \notin \Phi^+_j$.

But $ts(\alpha_r) = \alpha_r + 2\alpha_s + 6\alpha_t = \alpha_r + 5\alpha_t + s(\alpha_t) \in \text{cone}_\Phi(\Phi_j^+)$. So $\text{cone}_\Phi(\Phi_j^+) \neq \Phi^+_j$. An illustration is given in Figure 8.

**Figure 8.** Example of a biclosed set that is not biconvex in a geometric representation of dimension 3 of the universal Coxeter group $W' = U_4$ of rank 4; the root system is represented as a root subsystem in the classical geometric representation of the universal group $U_3$ of rank 3.

**Remark 5.2.** However, in the case of an irreducible affine Coxeter group, Dyer [14] informed us that he can prove that biclosed sets are precisely biconvex sets.

**Example 5.3** (A biclosed set that is not biconvex in the classical geometric representation of a Coxeter group of rank 4). We consider $(W, S)$ to be the Coxeter system of rank 4 whose Coxeter graph is:

Using the above example, we show that such a non-biconvex biclosed set lives in the root system of the classical geometric representation of $(W, S)$. Consider $\Delta' = \{r(\alpha_u), r(\alpha_t), s(\alpha_u), s(\alpha_t)\}$ and $S' = \{rur, rtr, sus, sts\}$. Note that $r(\alpha_u) = \alpha_r + \alpha_u$, $r(\alpha_t) = \alpha_r + \alpha_t$, $s(\alpha_u) = \alpha_s + \alpha_u$, $s(\alpha_t) = \alpha_s + \alpha_t$. We have

$$B(r(\alpha_u), r(\alpha_t)) = B(s(\alpha_u), s(\alpha_t)) = B(r(\alpha_u), s(\alpha_t)) = B(s(\alpha_t), r(\alpha_t)) = 1$$
and

\[ B(r(\alpha_u), s(\alpha_t)) = B(r(\alpha_t), s(\alpha_u)) = -3. \]

So \((W', S')\) is a universal Coxeter system with geometric representation of dimension 3 and simple system \(\Delta'\) as in 5.1. Let \(H = \ker \rho\) be the hyperplane generated by \(\Delta'\). We know by a result of Dyer [11, Theorem 4.4.] that \(\Phi'^+ = \Phi^+ \cap H\) is a positive root system associated to a reflection subgroup of \(W\); its associated simple system is the basis of \(\text{cone}(\Phi'^+)\), which turns out to be \(\Delta'\). So \(\Phi'^+\) is the positive root system of \(W'\). Consider now \(I = \{rur, rtr, sus\} \subset S'\), so by Example 5.1, \(\Phi_I\) is biclosed but not biconvex. Then \(A = \Phi_I^+ \cup \{\beta \in \Phi^+ | \rho(\beta) > 0\}\) is biclosed but not biconvex. We illustrate this example in Figure 9.

**Figure 9.** Example of a biclosed set that is not biconvex in the classical geometric representation of a Coxeter group of rank 4.

**Remark 5.4.** We do not know of any example of a biclosed set that is not biconvex in a geometric representation of dimension 3 of a Coxeter system of rank 3.

5.2. **On infinite biconvex and separable sets.** We establish in the following example that biconvex sets are not separable.

We consider \(W\) to be of type \(\tilde{A}_3\) generated by \(\{s_1, s_2, s_3, s_4\}\). We know that \(E = \{\tilde{\delta}\}\) is a singleton (the red dot in the center of the tetrahedron in Figure 10). Let \(X = \{s_2s_1s_3s_2s_1, s_2s_1s_4\}\) and \(A = \text{cone}_4(N(X))\). So \(\hat{A}\) is the union of all the segments \([\beta, \tilde{\delta}] \cap \hat{\Phi}\), with \(\beta \in N(X)\) (in Figure 10, \(\hat{N}(214)\) is the blue triangle on the bottom face and \(\hat{N}(21321)\) is in yellow on the left face). One checks that \(A\) is biconvex. But \(A\) is not separable since \(\delta \in \text{conv}(\hat{A}) \cap \text{conv}(\hat{A}^c)\). One can also check that \(A = \overline{N(X)}\). Therefore this example also shows that the infinite join of \(N(X)\), for \(X = \{s_2s_1s_3s_2s_1, s_2s_1s_4\}\), exists in \((B, \subseteq)\), is not separable and is not the inversion set (nor the complement) of a finite or infinite reduced word.
5.3. Infinite biconvex sets in the Kac-Moody/Lie setting. Biclosed sets are often called biconvex, or compatible, in the literature concerned with root systems associated to a Kac-Moody algebra, see for instance [7, 22, 23, 2, 27]. One has to be very careful with these two notions of biconvexity since they are not the same in general when applied to infinite sets of positive roots. Indeed, in the definition of biconvex in the Kac-Moody setting, the complement is not taken in $\Phi^+$ but in $\Phi^+ \cup \Phi^+_{im}$, where $\Phi^+_{im}$ denotes the positive imaginary roots. So a subset of $\Phi^+ \cup \Phi^+_{im}$ is said to be biconvex in a Kac-Moody root system if $A$ and $(\Phi^+ \cup \Phi^+_{im}) \setminus A$ are convex (we have to consider imaginary roots as well as real roots). In affine types, it is known that $\Phi^+_{im} = \mathbb{Z}_{>0}\delta$, so $\hat{\Phi}^+_{im} = \{\hat{\delta}\}$ is a singleton. Therefore $\delta$ cannot be in $A$ and its complement. Cellini and Papi [7] proved in the affine case that if $A$ is biconvex then either $A = N(\omega)$ for some $\omega \in W^\infty$ if $\delta \notin A$ or $(\Phi^+ \cup \Phi^+_{im}) \setminus A = N(\omega)$ if $\delta \in A$, see also Remark [4,7]. In particular, this means that the real part of any biconvex set in the sense of the affine Kac-Moody setting is separable in our setting by Proposition [4,2]. In Figure 10, we provide a biconvex set $A$ (in the sense of our definition) that is not separable, so it is not a biconvex set in the sense of a Kac-Moody root system.

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![Figure 10. A non separable biclosed set in the affine root system of type $\hat{A}_3$.](image-url)
example given in Figure 10. The second author would like to thank Karim Adiprasito, Hao Chen, Emerson Leon, Vivien Ripoll and Günter M. Ziegler for many important discussions on convex geometry throughout the writing of this article.

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