An improved Trudinger-Moser inequality involving

$N$–Finsler–Laplacian and $L^p$ norm

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Abstract: Suppose $F : \mathbb{R}^N \to [0, +\infty)$ be a convex function of class $C^2(\mathbb{R}^N \setminus \{0\})$ which is even and positively homogeneous of degree 1. We denote $\gamma_1 = \inf_{u \in W^{1,N}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} F^N(\nabla u) \, dx}{\|u\|_p^N}$. Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a smooth bounded domain. Then for $p > 1$ and $0 \leq \gamma < \gamma_1$, we have

$$\sup_{u \in W^{1,N}_0(\Omega), \|u\|_{N,F,\gamma,p} \leq 1} \int_{\Omega} e^{\lambda|u|_{N,F,\gamma,p}^N} \, dx < +\infty,$$

where $0 < \lambda \leq \lambda_N = \frac{N^N}{N^N - 1} \kappa_N^{\frac{1}{N^N - 1}}$ and $\kappa_N$ is the volume of a unit Wulff ball. Moreover, by using blow-up analysis and capacity technique, we prove that the supremum can be attained for any $0 \leq \gamma < \gamma_1$.

Keywords: N–Finsler–Laplacian; Trudinger-Moser inequality; Extremal function; Blow-up analysis; Elliptic regularity theory

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1 Introduction and main results

Suppose $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded smooth domain. When $1 < p < N$, the Sobolev embedding theorem implies that $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $1 \leq q \leq \frac{Np}{N-p}$. In particular, $W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < \infty$, but the...
embedding $W^{1,N}_0(\Omega) \not\to L^\infty(\Omega)$. A counterexample is given by the function $u(x) = (-\ln|\ln|x||)_+$ as $\Omega$ is the unit ball. It was proposed independently by Yudovich [55], Pohozaev [38], Peetre [39] and Trudinger [46] that $W^{1,N}_0(\Omega)$ is embedded in the Orlicz space $L_{\varphi_\alpha}(\Omega)$ determined by the Young function $\varphi_\alpha(t) = e^{\alpha|t|^{N/(N-1)}} - 1$ for some positive number $\alpha$, it was sharpened by Moser [36] who found the best exponent and proved the following result:

**Theorem A** There exists a sharp constant $\alpha_N := N \frac{N}{N-1} \omega_N^{1/N}$ such that

$$
\sup_{u \in W^{1,N}_0(\Omega), \|\nabla u\|_N \leq 1} \int_{\Omega} e^{\alpha|u|^{N/(N-1)}} \, dx < +\infty, \forall \alpha \leq \alpha_N,
$$

where $\omega_N$ is the volume of unit ball in $\mathbb{R}^N$. Moreover, the supremum in (1.1) is $+\infty$ if $\alpha > \alpha_N$. Related inequalities for unbounded domains were proposed by D. M. Cao [7] in dimension two and J. M. do Ó [15], Adachi-Tanaka [1] in high dimension, however they just considered the subcritical Trudinger-Moser inequality. Ruf [40] (for the case $N = 2$), Li and Ruf [26] (for the general case $N \geq 2$) obtained the Trudinger-Moser inequality in the critical case by replacing the Dirichlet norm with the standard Sobolev norm in $W^{1,N}(\mathbb{R}^N)$. Subsequently, Masmoudi and Sani [34] derived Trudinger-Moser inequalities with the exact growth condition in $\mathbb{R}^N$, These inequality plays an important role in geometric analysis and partial differential equations, we refer to [10, 12, 4, 13, 23, 16, 34] and references therein. In [24], Lam, Lu and Zhang provide a precise relationship between subcritical and critical Trudinger-Moser inequality. The similar result in Lorentz-Sobolev norms was also proved by Lu and Tang [31]. Trudinger-Moser inequality for first order derivatives was extended to high order derivatives by D. Adams [2] for bounded domains when dimension $N \geq 2$. B. Ruf and F. Sani [41] studied the Adams type inequality with higher derivatives of even orders for unbounded domains in $\mathbb{R}^N$. In [22], Lam and Lu applied a rearrangement-free argument to prove sharp Adams’ inequality in general case.

One important problem on Trudinger-Moser inequalities is whether or not extremal functions exist. Existence of extremal functions for the Trudinger-Moser inequality was first obtained by Carleson-Chang [8] when $\Omega$ is the unit ball, by M. Struwe [42] when $\Omega$ is close to the ball in the sense of measure, then by M. Flucher
and K. Lin when Ω is a general bounded smooth domain. Recently based on the work by Malchiodi and Martinazzi in [33], Mancini and Martinazzi [35] reproved the Carleson and Chang’s result by using a new method based on the Dirichlet energy, also allowing for perturbations of the functional. In the entire Euclidean space, existence of extremal functions was proved by Ruf [40] (for the case \( N = 2 \)) and Li and Ruf [26] (for the general case \( N \geq 2 \)). For extremal functions of singular version, Csató and Roy [11] proved that extremal functions exist in bounded domain of 2 dimension. Li and Yang [27] proved that extremal functions exist in the entire Euclidean space.

Moreover, there are some extensions of the Trudinger-Moser inequality. Let \( \alpha_1(\Omega) \) be the first eigenvalue of the Laplacian, Adimurthi and O. Druet [3] proved that

\[
\sup_{u \in W^{1,2}_0(\Omega), |\nabla u|^2 \leq 1} \int_\Omega e^{4\pi u^2(1+\alpha \|u\|^2_2)} dx < +\infty
\]

for \( 0 \leq \alpha < \alpha_1(\Omega) \), the supremum is infinity for any \( \alpha \geq \alpha_1(\Omega) \). This result was generalized by Yang [51, 52] to the cases of high dimension and a compact Riemannian surface. Lu-Yang [32] and J. Zhu [58] considered the case involving the \( L^p \) norm for any \( p > 1 \). For existence of extremal functions of Adimurthi-Druet type inequalities, they proved in [52, 32] that suprema (\( N = 2 \)) are attained for sufficiently small \( \alpha \geq 0 \), and that the supremum (\( N \geq 3 \))[51] is attained for all \( \alpha, 0 \leq \alpha < \alpha_1(\Omega) \). Subsequently, J.M. do Ó and M. de Souza generalized the similar result in whole Euclidean space [14] and high dimension case [17], and the existence of extremal functions was also obtained. A stronger version was established by Tintarev [45], namely,

\[
\sup_{u \in W^{1,2}_0(\Omega), |\nabla u|^2 \leq 1} \int_\Omega e^{4\pi u^2} dx < +\infty, \quad 0 \leq \alpha < \alpha_1(\Omega), \quad (1.2)
\]

Yang [53] obtained extremal functions for (1.2), which was also extended to singular version (see [54]). In [37], the author extends the result of Tintarev to the higher dimension as the following result:

**Theorem B.** Let \( \Omega \subset \mathbb{R}^N(\geq 2) \) be a smooth bounded domain and define

\[
\alpha(\Omega) = \inf_{u \in W^{1,N}_0(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^N dx}{\|u\|^N_N}.
\]
Then for any $0 \leq \alpha < \alpha(\Omega)$,
\[
\sup_{u \in W^{1,N}_0(\Omega), \int_\Omega |\nabla u|^N dx - \alpha \|u\|^N_N \leq 1} \int_{\Omega} e^{\alpha_N|u|^N} dx < +\infty,  \tag{1.3}
\]
where $\alpha_N := N^{\frac{N}{N-1}} \frac{1}{\omega_N}$ and $\omega_N$ is the volume of unit ball in $\mathbb{R}^N$.

Another interesting research is that Trudinger-Moser inequality has been generalized to the case of anisotropic norm. In this paper, denote that $F \in C^2(\mathbb{R}^N \setminus 0)$ is a positive, convex and homogeneous function, $F_\xi = \frac{\partial F}{\partial \xi}$ and its polar $F^o(x)$ represents a Finsler metric on $\mathbb{R}^N$. We will replace the isotropic Dirichlet norm $\|u\|_{W^{1,N}_0(\Omega)} = \left( \int_{\Omega} |\nabla u|^N dx \right)^{\frac{1}{N}}$ by the anisotropic Dirichlet norm $\left( \int_{\Omega} F^N(\nabla u) dx \right)^{\frac{1}{N}}$ in $W^{1,N}_0(\Omega)$. In [49], Wang and Xia proved the following result:

**Theorem C (Anisotropic Trudinger-Moser Inequality).** Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a smooth bounded domain. Let $u \in W^{1,N}_0(\Omega)$ and $\left( \int_{\Omega} F^N(\nabla u) dx \right)^{\frac{1}{N}} \leq 1$. Then
\[
\sup_{u \in W^{1,N}_0(\Omega), \int_{\Omega} F^N(\nabla u) dx \leq 1} \int_{\Omega} e^{\lambda_N u^N} dx < +\infty,  \tag{1.4}
\]
where $\lambda_N = N^{\frac{N}{N-1}} \frac{1}{\kappa_N}$ and $\kappa_N = \{ x \in \mathbb{R}^N : F^o(x) \leq 1 \}$. $\lambda_N$ is sharp in the sense that if $\lambda > \lambda_N$ then there exists a sequence $(u_n)$ such that $\int_{\Omega} e^{\lambda u_n^N} dx$ diverges.

The above inequality is related with $N$-Finsler-Laplacian operator $Q_N$ which is defined by
\[
Q_N u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (F^{N-1}(\nabla u) F_\xi (\nabla u)),
\]
when $N = 2$ and $F(\xi) = |\xi|$, $Q_2$ is just the ordinary Laplacian. This operator is closely related to a smooth, convex hypersurface in $\mathbb{R}^N$, which is called the Wulff shape (or equilibrium crystal shape) of $F$. This operator $Q_N$ was studied in some literatures, see [5, 6, 18] and the references therein. In [56], they obtained the existence of extremal functions for the sharp geometric inequality (1.4).

Our aim is to establish and find extremal functions for Trudinger-Moser inequality involving $N$–Finsler–Laplacian and $L^p$ norm. For $p > 1$, we denote
\[
\gamma_1 = \inf_{u \in W^{1,N}_0(\Omega), u \neq 0} \frac{\int_{\Omega} F^N(\nabla u) dx}{\|u\|^N_p},
\]
and
\[
\left\|u\right\|_{N, F, \gamma, p} = \left( \int_{\Omega} F^N(\nabla u) dx - \gamma \|u\|^N_p \right)^{\frac{1}{N}}.
\]
Theorem 1.1. Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a smooth bounded domain. Then for any \( 0 \leq \gamma < \gamma_1 \),
\[
\Lambda_\gamma = \sup_{u \in W^{1,N}_0(\Omega), \|u\|_{N,F,\gamma,p} \leq 1} \int_{\Omega} e^{\lambda_N |u|^N_{N,F,\gamma}} dx < +\infty, \tag{1.5}
\]
where \( \lambda_N = N \frac{N}{N-1} \kappa_N^{1/N} \) and \( \kappa_N \) is the volume of a unit Wulff ball.

**Remark 1.2.** From Theorem 1.1, for \( 0 \leq \gamma < \gamma_1 \), we can derive the following weak version:
\[
\sup_{u \in W^{1,N}_0(\Omega), \|u\|_{N,F,\gamma,p} \leq 1} \int_{\Omega} e^{\lambda_N (1+\gamma\|u\|_p^N) |u|^N_{N,F,\gamma}} dx < +\infty, \tag{1.6}
\]
for the special case \( p = N \) in (1.6), we refer reader to [57]. Next, let’s show that we have got better result. Indeed, if \( 0 \leq \gamma < \gamma_1 \), \( u \in W^{1,N}_0(\Omega) \) and \( \int_{\Omega} F^N(\nabla u) dx \leq 1 \). Set \( v = (1+\gamma\|u\|_p^N)^{1/p} u \in W^{1,N}_0(\Omega) \), note that \( F \) is a positively homogeneous function of degree 1. Thus
\[
\|v\|_{N,F,\gamma,p}^N = (1 + \gamma\|u\|_p^N) \int_{\Omega} F^N(\nabla u) dx - \gamma\|u\|_p^N - \gamma^2\|u\|_p^{2N} \\
\leq \int_{\Omega} F^N(\nabla u) dx \leq 1.
\]
Applying (1.5) to the function \( v \), we obtain (1.6). This implies that (1.5) is a stronger inequality.

Theorem 1.2. Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a smooth bounded domain. Then the supremum
\[
\Lambda_\gamma = \sup_{u \in W^{1,N}_0(\Omega), \|u\|_{N,F,\gamma,p} \leq 1} \int_{\Omega} e^{\lambda_N |u|^N_{N,F,\gamma}} dx \tag{1.7}
\]
can be attained by \( u_0 \in W^{1,N}_0(\Omega) \cap C^1(\overline{\Omega}) \) with \( \|u_0\|_{N,F,\gamma,p} = 1 \).

This paper is organized as follows: In Section 2, we give some preliminaries, meanwhile, under anisotropic Dirichlet norm and \( L^p \) norm, we establish the Lions type concentration-compactness principle of Trudinger-Moser Inequalities. In Section 3, we give the existence of subcritical maximizers. In Section 4, we analyze the convergence of maximizing sequence and its blow-up behavior, an upper bound is established by capacity estimates. In Section 5, we provide the proof of Theorems 1.1 and 1.2 by contradiction arguments and the construction of test function.
2 Preliminaries

In this section, we will give some preliminaries for our use later. Let \( F : \mathbb{R}^N \to [0, +\infty) \) be a convex function of class \( C^2(\mathbb{R}^N \setminus \{0\}) \) which is even and positively homogeneous of degree 1, so that
\[
F(t \xi) = |t| F(\xi) \quad \text{for any } t \in \mathbb{R}, \; \xi \in \mathbb{R}^N.
\] (2.1)

We also assume that \( F(\xi) > 0 \) for any \( \xi \neq 0 \) and \( Hess(F^2) \) is positive definite in \( \mathbb{R}^N \setminus \{0\} \). Then by Xie and Gong \([50]\), \( Hess(F^N) \) is also positive definite in \( \mathbb{R}^N \setminus \{0\} \).

A typical example is \( F(\xi) = (\sum_i |\xi_i|^q)^{\frac{1}{q}} \) for \( q \in [1, \infty) \). Let \( F^o \) be the support function of \( K := \{ x \in \mathbb{R}^N : F(x) \leq 1 \} \), which is defined by
\[
F^o(x) := \sup_{\xi \in K} \langle x, \xi \rangle,
\]
so \( F^o : \mathbb{R}^N \to [0, +\infty) \) is also a convex, homogeneous function of class \( C^2(\mathbb{R}^N \setminus \{0\}) \).

From \([5]\), \( F^o \) is dual to \( F \) in the sense that
\[
F^o(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^o(\xi)}.
\]

Consider the map \( \phi : S^{N-1} \to \mathbb{R}^N, \phi(\xi) = F_\xi(\xi) \). Its image \( \phi(S^{N-1}) \) is smooth, convex hypersurface in \( \mathbb{R}^N \), which is called the Wulff shape (or equilibrium crystal shape) of \( F \). Then \( \phi(S^{N-1}) = \{ x \in \mathbb{R}^N : F^o(x) = 1 \} \) (see \([48]\), Proposition 2.1).

Denote \( W_r(x_0) = \{ x \in \mathbb{R}^N : F^o(x - x_0) \leq r \} \), we call \( W_r(0) \) a Wulff ball of radius \( r \) with center at 0. We will use the convex symmetrization, which is defined in \([5]\). The convex symmetrization generalizes the Schwarz symmetrization (see \([44]\)).

Let us consider a measured function \( u \) on \( \Omega \subset \mathbb{R}^N \), one dimensional decreasing rearrangement of \( u \) is
\[
u^\diamond(t) = \sup\{ s \geq 0 : |\{ x \in \Omega : u(x) \geq s \}| > t \} \quad \text{for } t \in \mathbb{R}.
\] (2.2)

The convex symmetrization of \( u \) with respect to \( F \) is defined as
\[
u^*(x) = \nu^\diamond(\kappa_N F^o(x)^N) \quad \text{for } x \in \Omega^*.
\] (2.3)

Here \( \kappa_N F^o(x)^N \) is just the Lebesgue measure of a homothetic Wulff ball with radius \( F^o(x) \) and \( \Omega^* \) is the homothetic Wulff ball centered at the origin having the same
measure as $\Omega$. In [5], the authors proved a Pólya-Szegö principle and a comparison result for solutions of the Dirichlet problem for elliptic equations for the convex symmetrization, which generalizes the classical results for Schwarz symmetrization due to Talenti [44].

Now, we give the definition of anisotropic perimeter of a set with respect to $F$, a co-area formula and an isoperimetric inequality. Precisely, for a a subset $E \subset \Omega$ and a function of bounded variation $u \in BV(\Omega)$, anisotropic bounded variation of $u$ with respect to $F$ is

$$\int_{\Omega} |\nabla u|_F = \sup \left\{ \int_{\Omega} u \cdot \text{div} \sigma dx : \sigma \in C^1_0(\Omega; \mathbb{R}^N), F^1(\sigma) \leq 1 \right\}.$$  

Define the anisotropic perimeter of $E$ with respect to $F$ as

$$P_F(E) := \int_{\Omega} |\nabla \chi_E|_F,$$

where $\chi_E$ is the characteristic function of the set $E$. From the reference [20], we have the co-area formula

$$\int_{\Omega} |\nabla u|_F = \int_0^\infty P_F(|u| > t) dt$$

and the isoperimetric inequality

$$P_F(E) \geq N \kappa_1^{1/N} |E|^{1 - \frac{1}{N}}.$$  

We will establish the Lions type concentration-compactness principle [30] for Trudinger-Moser Inequalities under anisotropic Dirichlet norm and $L^p$ norm, which is the extention of Theorem 1.1 in [9] and Lemma 2.3 in [56].

**Lemma 2.1.** Suppose $0 \leq \gamma < \gamma_1$. Let $\{u_n\} \subset W^{1,N}_0(\Omega)$ be a sequence such that $\|u\|_{N,F,\gamma,p} = 1$, $u_n \rightharpoonup u \neq 0$ weakly in $W^{1,N}_0(\Omega)$. Then for any

$$0 < q < q_N(u) := \left(1 - \|u\|_{N,F,\gamma,p}^N\right)^{-\frac{1}{N-1}},$$

we have

$$\int_{\Omega} e^{\lambda_N q|u_n|^N} dx < +\infty$$

where $\lambda_N = N \kappa_1^{N-1} \kappa_N$ and $\kappa_N$ is the volume of a unit Wulff ball. Moreover, this conclusion fails if $p \geq p_N(u)$.  

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Proof. Since \( \|u\|_{N,F,\gamma,p} = 1 \), we have

\[
\lim_{n \to \infty} \int_{\Omega} F^N(\nabla u_n) = \lim_{n \to \infty} (1 + \gamma \|u_n\|_p^N) = 1 + \gamma \|u\|^N_p.
\]

Let

\[
v_n = \frac{u_n}{(\int_{\Omega} F^N(\nabla u_n)dx)^{\frac{1}{N}}},
\]

we have \( \int_{\Omega} F^N(\nabla v_n)dx = 1 \) and \( v_n \rightharpoonup v := u/(1 + \gamma \|u\|_p^N)^{\frac{1}{N}} \) weakly in \( W^{1,N}_0(\Omega) \).

By Lemma 2.3 of [56], it holds

\[
\lim \sup_{n \to \infty} \int_{\Omega} e^{\lambda N}r|v_n|^\frac{N}{N-1} dx < \infty \quad (2.7)
\]

for any \( 0 < r < (1 - \int_{\Omega} F^N(\nabla u_n)dx)^{-\frac{1}{N-1}} \). Since \( q < q_N(u) = (1 - \|u\|_{N,F,\gamma,p}^N)^{-\frac{1}{N-1}} \), we have

\[
\lim_{n \to \infty} q \int_{\Omega} F^N(\nabla u_n)dx)^{\frac{1}{N-1}} = q(1 + \gamma \|u\|_p^N)^{\frac{1}{N-1}}
\]

\[
< \left( \frac{1 + \gamma \|u\|_p^N}{1 - (\int_{\Omega} F^N(\nabla u)dx + \gamma \|u\|_p^N)} \right)^{\frac{1}{N-1}}
\]

\[
= (1 - \int_{\Omega} F^N(\nabla v_n)dx)^{-\frac{1}{N-1}} \quad (2.8)
\]

Take \( r < (1 - \int_{\Omega} F^N(\nabla u_n)dx)^{-\frac{1}{N-1}} \) such that \( q \int_{\Omega} F^N(\nabla u_n)dx)^{\frac{1}{N-1}} < r \) for \( n \) large enough. Thus

\[
\int_{\Omega} e^{\lambda N}q|v_n|^\frac{N}{N-1} dx = \int_{\Omega} e^{\lambda N}q(\int_{\Omega} F^N(\nabla u_n)dx)^{\frac{1}{N-1}}|v_n|^\frac{N}{N-1} dx < \int_{\Omega} e^{\lambda N}r|v_n|^\frac{N}{N-1} dx \quad (2.9)
\]

for \( n \) large enough. The result is followed from (2.7).

Now, as the similar process of Theorem 2.2 in [43], we give the following estimate involving N-Finsler-Laplacian and \( L^p \) norm.

**Lemma 2.2.** Assume that \( p > 1 \) and \( 0 \leq \gamma < \gamma_1 \). Let \( f \in L^1(\Omega) \) and \( u \in C^1(\Omega) \cap W^{1,N}_0(\Omega) \) satisfies

\[
- Q_N u = f + \gamma \|u\|_{p,N-1}^N|u|^{p-2}u \quad \text{in} \quad \Omega, \quad (2.10)
\]

where \( Q_N u = \text{div}(F^{N-1}(\nabla u)F_{\xi_i}(\nabla u)) \). Then for any \( 1 < q < N, u \in W^{1,q}_0(\Omega) \) and

\[
\|u\|_{W^{1,q}_0(\Omega)} \leq C(q,N,\gamma,\gamma_1)\|f\|_{L^1(\Omega)}.
\]
**Proof.** Fix $t > 0$. Testing (2.10) by $u^t := \min\{u, t\} \in W_0^{1,N}(\Omega)$ and integrating by parts, we have

$$
\int_{\Omega} F^N(\nabla u^t) dx \leq \int_{\Omega} f u^t dx + \gamma \int_{\Omega} \|u^t\|^{N-p}_{\rho}|u^t|^{p} dx
$$

$$
\leq t\|f\|_{L^1(\Omega)} + \frac{\gamma}{\gamma_1} \int_{\Omega} F^N(\nabla u^t) dx. \quad (2.11)
$$

Thus

$$
\int_{\Omega} F^N(\nabla u^t) dx \leq \frac{\gamma_1}{\gamma_1 - \gamma} t\|f\|_{L^1(\Omega)}. \quad (2.12)
$$

Denote $W_r(0) = \{x \in \mathbb{R}^N : F^0(x) \leq r\}$ be a Wulff ball of the same measure as $\Omega$. Let $v^*$ be the convex symmetrization of $u^t$ with respect to $F$ and $|W_\rho(0)| = |x \in W_r(0) : v^* \geq t|$. In [5], the authors proved the Pólya-Szego principle

$$
\int_{W_r(0)} F^N(\nabla v^*) dx \leq \int_{\Omega} F^N(\nabla u^t) dx.
$$

Thus

$$
\inf_{\phi \in W_0^{1,N}(W_r(0)), \phi|_{W_\rho(0)}=t} \int_{W_r(0)} F^N(\nabla \phi) dx \leq \int_{W_r(0)} F^N(\nabla v^*) dx
$$

$$
\leq \frac{\gamma_1}{\gamma_1 - \gamma} t\|f\|_{L^1(\Omega)}. \quad (2.13)
$$

On the other hand, the above infimum can be achieved by

$$
\phi_0(x) = \begin{cases} 
\frac{t \log r}{F^0(x)} / \log \frac{r}{\rho} & \text{in } W_r(0) \setminus W_\rho(0), \\
t & \text{in } W_\rho(0). 
\end{cases} \quad (2.14)
$$

Since $F(\nabla F^0(x)) = 1$, through the direct computation, we have

$$
\int_{W_r(0)} F^N(\nabla \phi_0) dx = \int_{\rho} F^N \left( \frac{t}{\log \frac{r}{\rho}} \frac{\nabla F^0(x)}{-s} \right) \int_{\partial \omega_s} \frac{1}{|\nabla F^0(x)|} d\sigma ds
$$

$$
= \int_{\rho} \frac{t^N}{(\log \frac{r}{\rho})^N} \frac{1}{s^N} \int_{\partial \omega_s} \frac{1}{|\nabla F^0(x)|} d\sigma ds
$$

$$
= \int_{\rho} \frac{t^N}{(\log \frac{r}{\rho})^N} \frac{1}{s^N} \kappa_N s^{N-1} ds
$$

$$
= \frac{\kappa_N t^N}{(\log \frac{r}{\rho})^N-1}. \quad (2.15)
$$
Hence \( \frac{N_{K_N}t^N}{(\log \frac{1}{t})^{N-1}} \leq \frac{\gamma_1}{\gamma_1 - \gamma}t\|f\|_{L^1(\Omega)} \), it holds
\[
|\{x \in \Omega : u(x) \geq t\}| = |W_\rho(0)| = \kappa N^\rho \leq \kappa N^\rho \exp(-N(N_{K_N})^\frac{\gamma_1}{\gamma_1 - \gamma}t(\frac{\gamma_1}{\gamma_1 - \gamma}t)^{\frac{1}{N-1}})
\leq |\Omega| \exp(-N(N_{K_N})^\frac{\gamma_1}{\gamma_1 - \gamma}t(\frac{\gamma_1}{\gamma_1 - \gamma}t)^{\frac{1}{N-1}})
= : |\Omega| e^{-NC_1 t}.
\]
(2.16)

For every \( 0 < b < NC_1 \),
\[
\int_\Omega e^{bu} dx \leq \int_{\{x: u(x) \leq 1\}} e^{bu} dx + \int_{\{x: u(x) \geq 1\}} e^{bu} dx
\leq e^b|\Omega| + \sum_{k=1}^\infty e^{b(k+1)}|\{x \in \Omega : k \leq u \leq k + 1\}|
\leq e^b|\Omega| + e^b|\Omega| \sum_{k=1}^\infty e^{(b-NC_1)k} \leq C(b)|\Omega|.
\]
(2.17)

From (2.12), we have
\[
\int_{\{x: u \leq t\}} F^N(\nabla u) dx \leq \frac{\gamma_1}{\gamma_1 - \gamma}t\|f\|_{L^1(\Omega)}.
\]
(2.18)

Hence,
\[
\int_\Omega \frac{F^N(\nabla u)}{1+u^2} dx = \int_{\{x: u(x) \leq 1\}} \frac{F^N(\nabla u)}{1+u^2} dx + \int_{\{x: u(x) \geq 1\}} \frac{F^N(\nabla u)}{1+u^2} dx
\leq \int_{\{x: u(x) \leq 1\}} \frac{F^N(\nabla u)}{u^2} dx + \sum_{m=0}^\infty \int_{\{x: 2^m \leq u(x) \leq 2^{m+1}\}} \frac{F^N(\nabla u)}{u^2} dx
\leq \int_{\{x: u(x) \leq 1\}} \frac{\gamma_1}{\gamma_1 - \gamma}\|f\|_{L^1(\Omega)} + \sum_{m=0}^\infty \frac{1}{2^m} \frac{2\gamma_1}{\gamma_1 - \gamma}\|f\|_{L^1(\Omega)}
= \frac{5\gamma_1}{\gamma_1 - \gamma}\|f\|_{L^1(\Omega)},
\]
(2.19)

where we have used the estimate (2.18) for \( t = 1 \) and \( t = 2^{m+1} \) in last inequality.

For \( 1 < q < N, \frac{q}{N} + \frac{N-q}{N} = 1 \), by Young’s inequality, we have
\[
\int_\Omega F^q(\nabla u) dx = \int_\Omega \frac{F^q(\nabla u)}{(1+u^2)^{q/N}}(1+u^2)^{q/N} dx
\leq \int_\Omega \frac{F^N(\nabla u)}{1+u^2} dx + \int_\Omega (1+u^2)^{q/(N-q)} dx
\]
(2.20)

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The desired bound now follows from (2.17) and (2.19).

3 Maximizer of the subcritical case

In this section, we will show the existence of the maximizers for Trudinger-Moser in the subcritical case.

Lemma 3.1. Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a smooth bounded domain. Then for any $\epsilon \in (0, \lambda_N)$, the supremum

$$\Lambda_{\gamma, \epsilon} = \sup_{u \in W^{1,N}_0(\Omega), \|u\|_{N,F,\gamma,p} \leq 1} \int_{\Omega} e^{(\lambda_N - \epsilon)|u|^{N-\gamma}} dx$$

(3.1)

can be attained by $u_\epsilon \in W^{1,N}_0(\Omega) \cap C^1(\overline{\Omega})$ with $\|u_{\epsilon,n}\|_{N,F,\gamma,p} = 1$. In the distributional sense, $u_\epsilon$ satisfies the following equation

$$\begin{cases}
  -Q_N u_\epsilon - \gamma \|u_\epsilon\|^{N-1}_{p} u_\epsilon^{p-1} = \frac{1}{\lambda_\epsilon^N} e^{(\lambda_N - \epsilon)|u_\epsilon|^{N-1}} \\  u_\epsilon = 0 \text{ on } \partial\Omega, \\  \lambda_\epsilon = \int_{\Omega} u_\epsilon^{N-1} e^{(\lambda_N - \epsilon)|u_\epsilon|^{N-1}} dx.
\end{cases}$$

(3.2)

Moreover,

$$\liminf_{\epsilon \to 0} \lambda_\epsilon > 0$$

and

$$\liminf_{\epsilon \to 0} \Lambda_{\gamma, \epsilon} = \Lambda_{\gamma}.$$  (3.3)

Proof. Let $u_{\epsilon,n}$ be a maximizing sequence for $\Lambda_{\gamma, \epsilon}$, i.e., $u_{\epsilon,n} \in W^{1,N}_0(\Omega)$, $\|u_{\epsilon,n}\|_{N,F,\gamma,p} \leq 1$ and

$$\int_{\Omega} e^{(\lambda_N - \epsilon)|u_{\epsilon,n}|^{N-1}} dx \to \Lambda_{\gamma, \epsilon}$$

(3.4)

as $n \to \infty$. Since $\gamma < \gamma_1$, thus

$$(1 - \frac{\gamma}{\gamma_1}) \int_{\Omega} F^N(\nabla u_{\epsilon,n}) dx \leq \int_{\Omega} F^N(\nabla u_{\epsilon,n}) dx - \gamma \|u_{\epsilon,n}\|_{p}^{N} \leq 1,$$

which lead to $u_{\epsilon,n}$ is bounded in $W^{1,N}_0(\Omega)$, so there exists some $u_\epsilon \in W^{1,N}_0(\Omega)$ such that up to a subsequence, $u_{\epsilon,n} \rightharpoonup u_\epsilon$ weakly in $W^{1,N}_0(\Omega)$, $u_{\epsilon,n} \to u_\epsilon$ strongly in $L^q(\Omega)$ for any $q \geq 1$, and $u_{\epsilon,n} \to u_\epsilon$ a.e. in $\Omega$. We claim that $u_\epsilon \neq 0$. If otherwise, then
\[
\limsup_{n \to \infty} \int_\Omega F^N(\nabla u_{\epsilon,n}) \, dx \leq 1.
\]
The anisotropic Trudinger-Moser inequality implies \(e^{(\lambda_N - \epsilon) \left\| u_{\epsilon,n} \right\|^{\frac{N}{N-1}}}\) is uniformly bounded in \(L^s(\Omega)\) for any \(1 < s < \frac{\lambda_N}{\lambda_N - \epsilon}\). Thus
\[
\Lambda_{\gamma,\epsilon} = \lim_{n \to \infty} \int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon,n} \right\|^{\frac{N}{N-1}}} \, dx = |\Omega|, \quad (3.5)
\]
which is impossible. So \(u_\epsilon \neq 0\) and Lemma 2.1 implies \(\int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon,n} \right\|^{\frac{N}{N-1}}} \, dx \) in \(L^s(\Omega)\) for some \(s > 1\). Consequently, we have
\[
\Lambda_{\gamma,\epsilon} = \lim_{n \to \infty} \int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon,n} \right\|^{\frac{N}{N-1}}} \, dx = \int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon} \right\|^{\frac{N}{N-1}}} \, dx, \quad (3.6)
\]
Then \(u_\epsilon\) attains the supremum. We claim \(\| u_\epsilon \|_{N,F,\gamma,p} = 1\). In fact, if \(\| u_\epsilon \|_{N,F,\gamma,p} < 1\), then
\[
\Lambda_{\gamma,\epsilon} = \int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon} \right\|^{\frac{N}{N-1}}} \, dx < \int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon} \right\|_{N,F,\gamma,p}^{\frac{N}{N-1}}} \, dx = \Lambda_{\gamma,\epsilon}, \quad (3.7)
\]
which is a contradiction. Furthermore, we know that \(u_\epsilon\) satisfies the Euler–Lagrange equation in distributional sense. By regularity theory obtained in [28], we have \(u_\epsilon \in C^1(\Omega)\).

Since \(e^t \leq 1 + te^t\) for \(t \geq 0\), we have
\[
\int_\Omega e^{(\lambda_N - \epsilon) \left\| u_\epsilon \right\|^{\frac{N}{N-1}}} \, dx \\
\leq |\Omega| + (\lambda_N - \epsilon) \int_\Omega \left\| u_\epsilon \right\|_{N,F,\gamma,p}^{\frac{N}{N-1}} e^{(\lambda_N - \epsilon) \left\| u_\epsilon \right\|^{\frac{N}{N-1}}} \, dx \\
= |\Omega| + (\lambda_N - \epsilon) \lambda_\epsilon.
\]
This leads to \(\liminf_{\epsilon \to 0} \lambda_\epsilon > 0\).

Obviously, \(\limsup_{\epsilon \to 0} \Lambda_{\gamma,\epsilon} \leq \Lambda_{\gamma}\). On the other hand, for any \(u \in W^{1,N}_0(\Omega)\) with \(\| u_\epsilon \|_{N,F,\gamma,p} \leq 1\), by Fatou's lemma, we have
\[
\int_\Omega e^{\lambda_N \left\| u \right\|^{\frac{N}{N-1}}} \, dx \leq \liminf_{\epsilon \to 0} \int_\Omega e^{(\lambda_N - \epsilon) \left\| u_{\epsilon} \right\|^{\frac{N}{N-1}}} \, dx \leq \liminf_{\epsilon \to 0} \Lambda_{\gamma,\epsilon},
\]
which implies that \(\liminf_{\epsilon \to 0} \Lambda_{\gamma,\epsilon} \geq \Lambda_{\gamma}\). Thus \(\liminf_{\epsilon \to 0} \Lambda_{\gamma,\epsilon} = \Lambda_{\gamma}\).

\section{Maximizers of the critical case}

In this section, by using blow-up analysis, we analyze the behavior of the maximizers \(u_\epsilon\) in section 3. Since \(u_\epsilon\) is bounded in \(W^{1,N}_0(\Omega)\), up to a subsequence, we can assume \(u_\epsilon \rightharpoonup u_0\) weakly in \(W^{1,N}_0(\Omega)\), \(u_\epsilon \to u_0\) strongly in \(L^q(\Omega)\) for any \(q \geq 1\), and \(u_\epsilon \to u_0\) a.e. in \(\Omega\) as \(\epsilon \to 0\).
4.1 Blow-up analysis

Let $c_\epsilon = \max u_\epsilon = u_\epsilon(x_\epsilon)$. If $c_\epsilon$ is bounded, then for any $u \in W_{0}^{1,N}(\Omega)$ with $\|u\|_{N,F,\gamma,p} \leq 1$, by Lebesgue dominated convergence theorem, we have

$$
\int_{\Omega} e^{\lambda N |u|^{N-1}} dx = \lim_{\epsilon \to 0} \int_{\Omega} e^{(\lambda N - \epsilon) |u|^{N-1}} dx
\leq \lim_{\epsilon \to 0} \int_{\Omega} e^{(\lambda N - \epsilon) |u_\epsilon|^{N-1}} dx
= \int_{\Omega} e^{\lambda N |u_0|^{N-1}} dx
$$

(4.1)

Therefore $u_0$ is the desired maximizer. Moreover, $u_\epsilon \to u_0$ and $u_0 \in C^1(\overline{\Omega})$ by standard elliptic regularity theory. In the following, we consider another case, we assume $c_\epsilon \to +\infty$ and $x_\epsilon \to x_0$ as $\epsilon \to 0$. We assume $x_0 \in \Omega$, at the end of this section, we shall exclude the case $x_0 \in \partial \Omega$.

**Lemma 4.1.** $u_0 \equiv 0$ and $F^N(\nabla u_\epsilon) dx \rightharpoonup \delta_{x_0}$ weakly in the sense of measure as $\epsilon \to 0$, where $\delta_{x_0}$ is the Dirac measure at $x_0$.

**Proof.** Suppose $u_0 \not\equiv 0$. Notice that $\liminf_{\epsilon \to 0} \lambda_\epsilon > 0$, by Lemma 2.1 and Hölder inequality, we have $\frac{1}{\lambda_\epsilon} u_\epsilon |u_\epsilon|^{\frac{2N}{N-1}} e^{(\lambda N - \epsilon)|u_\epsilon|^{\frac{N}{N-1}}}$ is uniformly bounded in $L^q(\Omega)$ for some $q > 1$. Then by Lemma 2.2 in [56], $u_\epsilon$ is uniformly bounded in $\Omega$, which contradicts $c_\epsilon \to +\infty$ as $\epsilon \to 0$. Hence $u_0 \equiv 0$.

Notice that $\int_{\Omega} F^N(\nabla u_\epsilon) dx = 1 + \gamma \|u_\epsilon\|_p^N \to 1$. If $F^N(\nabla u_\epsilon) dx \rightharpoonup \mu \not= \delta_{x_0}$ in the sense of measure as $\epsilon \to 0$, then there exists $\theta < 1$ and $r > 0$ small enough such that

$$
\lim_{\epsilon \to 0} \int_{\mathcal{W}_r(x_0)} F^N(\nabla u_\epsilon) dx < \theta,
$$

Consider the cut-off function $\phi \in C^\infty_0(\Omega)$, which is supported in $\mathcal{W}_r(x_0)$ for some $r > 0$, $0 \leq \phi \leq 1$ and $\phi = 1$ in $\mathcal{W}_{\frac{r}{2}}(x_0)$. Since $u_\epsilon \to 0$ in $L^q(\Omega)$ for any $q > 1$, we have

$$
\limsup_{\epsilon \to 0} \int_{\mathcal{W}_r(x_0)} F^N(\nabla \phi u_\epsilon) dx \leq \lim_{\epsilon \to 0} \int_{\mathcal{W}_r(x_0)} F^N(\nabla u_\epsilon) dx < \theta.
$$

For sufficiently small $\epsilon > 0$, using anisotropic Moser-Trudinger inequality to $\phi u_\epsilon$, we have $e^{\lambda N u_\epsilon^{N-1}} dx$ is uniformly bounded in $L^s(\mathcal{W}_{\frac{r}{2}}(x_0))$ for some $s > 1$. From Lemma 2.2 in [56], $u_\epsilon$ is uniformly bounded in $\mathcal{W}_{\frac{r}{2}}(x_0)$, which contradicts to $c_\epsilon \to +\infty$. Hence $F^N(\nabla u_\epsilon) dx \rightharpoonup \delta_0$ as $\epsilon \to 0$. \qed
Let
\[ r_\epsilon^N = \lambda_\epsilon c_\epsilon^{-N/(N-1)} e^{-(\lambda_\epsilon - \epsilon)c_\epsilon^{N/(N-1)}}. \] (4.2)

Denote \( \Omega_\epsilon = \{ x \in \mathbb{R}^N : x_\epsilon + r_\epsilon x \in \Omega \} \). Define
\[ \psi_\epsilon(x) = c_\epsilon^{-1} u_\epsilon(x_\epsilon + r_\epsilon x) \] (4.3)
and
\[ \varphi_\epsilon(x) = c_\epsilon^{N/(N-1)}(\psi_\epsilon(x) - 1) = c_\epsilon^{1/(N-1)}(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon). \] (4.4)

We have the following:

**Lemma 4.2.** Let \( r_\epsilon, \psi_\epsilon(x) \) and \( \varphi_\epsilon \) be defined as in (4.2)-(5.3). Then
(i) there hold \( r_\epsilon \to 0 \) as \( \epsilon \to 0 \);
(ii) \( \psi_\epsilon(x) \to 1 \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \);
(iii) \( \varphi_\epsilon(x) \to \varphi(x) \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \), where
\[ \varphi(x) = -\frac{N-1}{\lambda_N} \log \left( 1 + \kappa_\epsilon^{N-1} F^0(x) \frac{N}{N-1} \right). \] (4.5)

Moreover,
\[ \int_{\mathbb{R}^N} e^{\frac{N}{N-1}\lambda_N \varphi} dx = 1. \] (4.6)

**Proof.** (i) From (4.2), we have
\[ r_\epsilon^N c_\epsilon^{N/(N-1)} = \lambda_\epsilon e^{-(\lambda_N - \epsilon)c_\epsilon^{N/(N-1)}} \]
\[ = \int_\Omega |u_\epsilon|^{N-1} e^{(\lambda_N - \epsilon)|u_\epsilon|^{\frac{N}{N-1}}} dx \cdot e^{-(\lambda_N - \epsilon)c_\epsilon^{N/(N-1)}} \]
\[ \leq c_\epsilon^{N/(N-1)} e^{\frac{\lambda_N}{\lambda} - \epsilon} c_\epsilon^{N/(N-1)} \int_\Omega e^{\frac{\lambda_N}{\lambda} |u_\epsilon|^{\frac{N}{N-1}}} dx \cdot e^{-(\lambda_N - \epsilon)c_\epsilon^{N/(N-1)}} \]
\[ \leq C c_\epsilon^{N/(N-1)} e^{-\frac{\lambda_N}{\lambda} c_\epsilon^{N/(N-1)}} \to 0, \]
as \( \epsilon \to 0 \). Thus \( \lim_{\epsilon \to 0} r_\epsilon = 0 \).

(ii) By a direction calculation, \( 0 \leq \psi_\epsilon \leq 1 \) and \( \psi_\epsilon \) is a weak solution to
\[ - \text{div}(F^{N-1}(\nabla \psi_\epsilon) F_\epsilon(\nabla \psi_\epsilon)) \]
\[ = c_\epsilon^{-N} e^{-(\lambda_N - \epsilon)c_\epsilon^{N/(N-1)}} (\psi_\epsilon^{\frac{N}{N-1}} - 1) \psi_\epsilon^{\frac{1}{N-1}} + \gamma c_\epsilon^{p-N} r_\epsilon^{N} |u_\epsilon|^{p-N} \psi_\epsilon^{-p+1} \quad \text{in} \quad \Omega_\epsilon. \]
Since \( c_\epsilon^{-N}e^{(\lambda N-\epsilon)c_\epsilon^{N/((N-1)(N-1))} \psi_\epsilon^{-1}} \leq c_\epsilon^{-N} \to 0 \) as \( \epsilon \to 0 \), and
\[
\left( \int_{B_{r_\epsilon}^\eta} (c_\epsilon^{p-N} r_\epsilon^N \|u_\epsilon\|_p^{N-p} \psi_\epsilon^{-p/(p-1)}dx) \right)^{\frac{p-1}{p}}
\]
\[
= c_\epsilon^{1-N} r_\epsilon^N \|u_\epsilon\|_p^{N-1} \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
Thus we obtain that \( \text{div}(F^{N-1}(\nabla \psi_\epsilon) F_\xi(\nabla \psi_\epsilon)) \) is bounded in \( L^{p/(p-1)}(B_{r_\epsilon}^\eta) \). Using elliptic regularity theory (see [47]), we have \( \psi_\epsilon \to \psi \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \), where \( \psi \) is a weak solution to the equation
\[
-\text{div}(F^{N-1}(\nabla \psi) F_\xi(\nabla \psi)) = 0 \quad \text{in} \quad \mathbb{R}^N.
\]
The Liouville theorem (see [21]) implies \( \psi \equiv 1 \).

(iii) We have
\[
-\text{div}(F^{N-1}(\nabla \varphi_\epsilon) F_\xi(\nabla \varphi_\epsilon))
\]
\[
= e^{(\lambda N-\epsilon)c_\epsilon^{N/((N-1)(N-1))} \psi_\epsilon^{-1}} + \gamma c_\epsilon^p r_\epsilon^N \|u_\epsilon\|_p^{N-p} \psi_\epsilon^{-1} \quad \text{in} \quad \Omega_\epsilon.
\]
When \( p > N \), for \( R > 0 \) and sufficiently small \( \epsilon \),
\[
\|u_\epsilon\|_p^{N-p} \leq \left( \int_{B_{R\epsilon}} u_\epsilon^p dx \right)^{N/P-1}
\]
\[
= \epsilon_\epsilon^{N-p} \|u_\epsilon\|_p^N \left( \int_{B_R} \psi_\epsilon^p dx \right)^{N/P-1}
\]
Since \( 0 \leq \psi_\epsilon \leq 1 \), \( \psi_\epsilon \to 1 \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \), we have \( \int_{B_R} \psi_\epsilon^p dx > 0 \). Thus
\[
c_\epsilon^p r_\epsilon^N \|u_\epsilon\|_p^{N-p} \psi_\epsilon^{-1} \leq 2 \left( \int_{B_R} \psi_\epsilon^p dx \right)^{N/P-1} c_\epsilon^N r_\epsilon^{N^2/p} \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
When \( 1 < p \leq N \), obviously, \( c_\epsilon^p r_\epsilon^N \|u_\epsilon\|_p^{N-p} \psi_\epsilon^{-1} \to 0 \) as \( \epsilon \to 0 \). From regularity theory in [47], up to a subsequence, there exists \( \varphi \in C^1(\mathbb{R}^N) \) such that \( \varphi_\epsilon \to \varphi \) in
From the definition of $\psi_\epsilon$ and $\varphi_\epsilon$, we have
\[
u_\epsilon(x_\epsilon + r_\epsilon x) = c_\epsilon^{N-1} \left( \psi_\epsilon(x)^{\frac{N}{N-1}} - 1 \right) = c_\epsilon^{\frac{N}{N-1}} \left( 1 + \psi_\epsilon(x) - 1 \right)^{\frac{N}{N-1}} - 1
= \frac{N}{N-1} \varphi_\epsilon(x) + c_\epsilon^{\frac{N}{N-1}} O((\psi_\epsilon(x) - 1)^2)
= \frac{N}{N-1} \varphi_\epsilon(x) + O(\psi_\epsilon(x) - 1).
\]

By $\epsilon \to 0$, $\varphi$ satisfies
\[
\begin{aligned}
& \begin{cases}
- \text{div}(F^{N-1}(\nabla \varphi)F_\epsilon(\nabla \varphi)) = e^{\frac{N}{N-1} \lambda N \varphi} & \text{in } \mathbb{R}^N, \\
\varphi(x) \leq \varphi(0) = 0.
\end{cases}
\end{aligned}
\]

Combing (4.2) and (4.3), for any $R > 0$, we have
\[
\int_{W_R(0)} e^{\frac{N}{N-1} \lambda N \varphi} dx = \lim_{\epsilon \to 0} \int_{W_R(0)} e^{(\lambda N - \epsilon)u_\epsilon(x_\epsilon + r_\epsilon x)} c_\epsilon^{\frac{N}{N-1}} dx
= \lim_{\epsilon \to 0} \lambda^{N-1} \epsilon \int_{W_R r_\epsilon(x_\epsilon)} e^{(\lambda N - \epsilon)u_\epsilon(y)} dy
= \lim_{\epsilon \to 0} \lambda^{N-1} \epsilon \int_{W_R r_\epsilon(x_\epsilon)} u_\epsilon(y)^{N/(N-1)} e^{(\lambda N - \epsilon)u_\epsilon(y)}^{\frac{N}{N-1}} dy
\leq 1.
\]

This leads to
\[
\int_{\mathbb{R}^N} e^{\frac{N}{N-1} \lambda N \varphi} dx \leq 1.
\]

From co-area formula (2.4) and isoperimetric inequality (2.5), through a simple computation, it follows from Hölder inequality, we have $\int_{\mathbb{R}^N} e^{\frac{N}{N-1} \lambda N \varphi} dx \geq 1$. Thus
\[
\int_{\mathbb{R}^N} e^{\frac{N}{N-1} \lambda N \varphi} dx = 1, \tag{4.7}
\]
which implies $\varphi$ is symmetric with respect to $F^\alpha$, i.e., $\varphi(x) = \varphi(F^\alpha(x))$ (see [48], Prop. 6.1). Thus we get
\[
\varphi(x) = -\frac{N-1}{\lambda} \log \left( 1 + \kappa^{\frac{1}{N-1}} F^\alpha(x)^{\frac{N}{N-1}} \right). \tag{4.8}
\]
Denote $W_R(x_\epsilon)$ be a Wulff ball of radius $R$ with center at $x_\epsilon$. For any $0 < a < 1$, use the notation

$$u_{\epsilon,a} = \min\{u_\epsilon, ac_\epsilon\}.$$ 

Then we have the following:

**Lemma 4.3.** For any $0 < a < 1$, there holds

$$\lim_{\epsilon \to 0} \int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx = a.$$

**Proof.** Testing the equation (3.3), we have

$$\int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx = \int_{\Omega} F^{N-1}(\nabla u_\epsilon) \cdot F_\xi(\nabla u_\epsilon) dx$$

$$= - \int_{\Omega} \text{div}(F^{N-1}(\nabla u_\epsilon)) F_\xi(\nabla u_\epsilon) u_{\epsilon,a} dx$$

$$= \int \frac{1}{4} u_\epsilon^{N-1} e^{(\lambda_N - \epsilon)u_\epsilon^N} u_\epsilon^{N-1} u_{\epsilon,a} dx + \gamma \int_{\Omega} ||F_{p}^{N-1} u_\epsilon^{p-1} u_{\epsilon,a} dx.$$ 

For any $R > 0$, we have $W_R(x_\epsilon) \subset \{u_\epsilon > ac_\epsilon\}$ for $\epsilon > 0$ small enough. Thus

$$\int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx = \int_{\{u_\epsilon \leq ac_\epsilon\}} \frac{1}{1 - \epsilon} u_\epsilon^{N-1} e^{(\lambda_N - \epsilon)u_\epsilon^N} u_{\epsilon,a} dx$$

$$+ \int_{\{u_\epsilon > ac_\epsilon\}} \frac{1}{1 - \epsilon} u_\epsilon^{N-1} e^{(\lambda_N - \epsilon)u_\epsilon^N} u_{\epsilon,a} dx + \gamma \int_{\Omega} ||F_{p}^{N-1} u_\epsilon^{p-1} u_{\epsilon,a} dx > ac_\epsilon \int_{W_R(x_\epsilon)} \frac{1}{1 - \epsilon} u_\epsilon^{N-1} e^{(\lambda_N - \epsilon)u_\epsilon^N} u_{\epsilon,a} dx + o_\epsilon(1).$$

Set $x = x_\epsilon + r_\epsilon y$, we have

$$ac_\epsilon \int_{W_R(x_\epsilon)} \frac{1}{1 - \epsilon} u_\epsilon^{N-1} e^{(\lambda_N - \epsilon)u_\epsilon^N} u_{\epsilon,a} dx$$

$$= a \int_{W_R(0)} \psi_\epsilon(y)^{\frac{1}{N-1}} e^{(\lambda_N - \epsilon)\psi_\epsilon^N(y)^{\frac{N}{N-1}-1}} dy \to \int_{W_R(0)} e^{\frac{N}{N-1}\lambda_N \psi_\epsilon^N dy}.$$ 

Letting $R \to \infty$, we derive that

$$\liminf_{\epsilon \to 0} \int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx \geq a.$$ 

Similarly, we choose $(u_\epsilon - u_{\epsilon,a})$ as a test function of (3.3), we obtain $\liminf_{\epsilon \to 0} \int_{\Omega} F^N(\nabla (u_\epsilon - u_{\epsilon,a})) \geq 1 - a$. Notice that

$$\int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx = \int_{\Omega} F^N(\nabla u_\epsilon) dx - \int_{\Omega} F^N(\nabla (u_\epsilon - u_{\epsilon,a})) dx$$

$$= 1 + \gamma ||u_\epsilon||^N_p - \int_{\Omega} F^N(\nabla (u_\epsilon - u_{\epsilon,a})) dx.$$
which implies
\[ \limsup_{\epsilon \to 0} \int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx \leq a. \]

We have finished the proof of the lemma.

**Lemma 4.4.** There holds
\[ \lim_{\epsilon \to 0} \int_{\Omega} e^{(\lambda_N-\epsilon)|u_{\epsilon}|^{N \over N-1}} dx \leq |\Omega| + \limsup_{\epsilon \to 0} \frac{\lambda_{\epsilon}}{c_{\epsilon}^{N/(N-1)}}. \]

**Proof.** For any \(0 < a < 1\), by Lemma 4.3, we have \(\lim_{\epsilon \to 0} \int_{\Omega} F^N(\nabla u_{\epsilon,a}) dx = a < 1\). From anisotropic Trudinger-Moser inequality, \(e^{\lambda_N u_{\epsilon,a}}\) is bounded in \(L^q(\Omega)\) for some \(q > 1\). Notice that \(u_{\epsilon,a} \to 0\) a.e. in \(\Omega\), which implies \(\lim_{\epsilon \to 0} \int_{\Omega} e^{(\lambda_N-\epsilon)|u_{\epsilon}|^{N \over N-1}} dx = |\Omega|\). Hence
\[
e^{(\lambda_N-\epsilon)u_{\epsilon}} = \int_{\{u_{\epsilon} \leq ac_{\epsilon}\}} e^{(\lambda_N-\epsilon)u_{\epsilon}} dx + \int_{\{u_{\epsilon} > ac_{\epsilon}\}} e^{(\lambda_N-\epsilon)u_{\epsilon}} dx \\
\leq \int_{\{u_{\epsilon} \leq ac_{\epsilon}\}} e^{(\lambda_N-\epsilon)u_{\epsilon}} dx + \frac{1}{(ac_{\epsilon})^{N-1}} \int_{\{u_{\epsilon} > ac_{\epsilon}\}} e^{(\lambda_N-\epsilon)u_{\epsilon}} dx \\
\leq \int_{\Omega} e^{(\lambda_N-\epsilon)u_{\epsilon}}^{N \over N-1} dx + \frac{\lambda_{\epsilon}}{(ac_{\epsilon})^{N-1}}.
\]

Letting \(\epsilon \to 0\) and \(a \to 1\), we conclude our proof.

As a consequence of Lemma 4.4, for any \(\theta < \frac{N}{N-1}\), there holds
\[ \lim_{\epsilon \to 0} \frac{\lambda_{\epsilon}}{c_{\epsilon}^{\theta}} = +\infty. \quad (4.9) \]

If otherwise, \(\frac{\lambda_{\epsilon}}{c_{\epsilon}^{\theta/(N-1)}} \to 0\) as \(\epsilon \to 0\), by Lemma 4.4, we have \(\lim_{\epsilon \to 0} \int_{\Omega} e^{(\lambda_N-\epsilon)|u_{\epsilon}|^{N \over N-1}} dx \leq |\Omega|\), which is impossible.

**Lemma 4.5.** For any \(1 < q < N\), \(c_{\epsilon}^{1 \over q} u_{\epsilon} \rightharpoonup G\) in \(W^{1,q}_0(\Omega)\), where \(G\) is a distributional solution to
\[
\begin{cases} 
-\text{div}(F^{N-1}(\nabla G)F_{\xi}(\nabla G)) = \delta_{x_0} + \gamma \|G\|_p^{N-p}G^{p-1} & \text{in } \Omega, \\
G = 0 & \text{on } \partial \Omega.
\end{cases} \quad (4.10)
\]

Furthermore, \(c_{\epsilon}^{1 \over q} u_{\epsilon} \rightharpoonup G\) in \(C^{1}_{\text{loc}}(\overline{\Omega}\setminus\{x_0\})\), and \(G\) has the form
\[ G(x) = -\frac{1}{(N\kappa_N)^{\frac{1}{N-1}}} \log F^0(x - x_0) + A x_0 + \xi(x), \quad (4.11) \]
where $A_{x_0}$ is a constant depending only on $x_0$, $\xi \in C^0(\Omega) \cap C^1_{loc}(\Omega \setminus \{x_0\})$ and $\xi(x) = O(f_o(x - x_0))$ as $x \to x_0$.

**Proof.** Firstly, we claim for any function $\phi \in C(\Omega)$, it holds

$$
\lim_{\epsilon \to 0} \int_{\Omega} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx = \phi(x_0).
$$

In fact, for any $0 < a < 1$ and $R > 0$, we have $W_{R\epsilon}(x_\epsilon) \subset \{u_\epsilon > a\}$ as $\epsilon > 0$ small enough, we denote

$$
\int \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx
= \int_{\{u_\epsilon \leq a\}} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx + \int_{W_{R\epsilon}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx
+ \int_{\{u_\epsilon > a\}} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx
:= I + II + III.
$$

By Lemma 4.3 and (4.10), we have

$$
I = \frac{c_\epsilon}{\lambda_\epsilon} \int_{\{u_\epsilon \leq a\}} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx
\leq \frac{c_\epsilon}{\lambda_\epsilon} \int_{\Omega} u_\epsilon^{-\frac{1}{N}} e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}} \phi dx
= o(1)O(a).
$$

Making the change in variable $x = x_\epsilon + r_\epsilon y$, since $\phi(x_\epsilon + r_\epsilon y) \to \phi(x_0)$ uniformly in $W_R(0)$. Note $r_\epsilon^N = \lambda_\epsilon c_\epsilon^{-N/(N-1)} e^{-(\lambda N - \epsilon)c_\epsilon^{N/(N-1)}}$, together with (4.7), we have

$$
II = \int_{W_{R\epsilon}(x_\epsilon)} \frac{c_\epsilon r_\epsilon^{-N} e^{N/(N-1)}}{\lambda_\epsilon} \frac{1}{e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}}} \phi dx
= \int_{W_{R\epsilon}(x_\epsilon)} \frac{c_\epsilon r_\epsilon^{-N} e^{N/(N-1)}}{\lambda_\epsilon} \frac{1}{e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}}} \phi dx
= \int_{W_R(0)} \frac{c_\epsilon r_\epsilon^{-N} e^{N/(N-1)}}{\lambda_\epsilon} \frac{1}{e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}}} \phi(x_\epsilon + r_\epsilon y) r_\epsilon^N dy
= \int_{W_R(0)} \frac{1}{e^{(\lambda N - \epsilon)u_\epsilon^{\frac{N}{N-1}}}} \phi(x_\epsilon + r_\epsilon y) dy
\to \phi(x_0)
$$

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as $\epsilon \to 0$ and $R \to \infty$.

$$III = \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} \phi dx$$

$$\leq \|\phi\|_\infty \left( \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} dx - \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} dx \right)$$

$$\leq \|\phi\|_\infty \left( \frac{1}{a} \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} u_\epsilon^{-\frac{N}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} dx - \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} dx \right)$$

$$\leq \|\phi\|_\infty \left( \frac{1}{a} - \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} dx \right).$$

Thus

$$\lim_{a \to 1} \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_{\{u_\epsilon > a_\epsilon\} \cap W_{Rr}(x_\epsilon)} \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} \phi dx = 0.$$ 

Combing the above discussion, we have proved the claim. From (3.3), we have

$$-Q_N(c_\epsilon^{-\frac{1}{N-1}} u_\epsilon) = \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} + \gamma \|c_\epsilon^{-\frac{1}{N-1}} u_\epsilon\|_{p}^{p} (c_\epsilon^{-\frac{1}{N-1}} u_\epsilon)^{p-1}. \quad (4.13)$$

It follows from (4.12) that $c_\epsilon^{-\frac{1}{N-1}} u_\epsilon$ is bounded in $L^1(\Omega)$. From Lemma 2.2, we know that $c_\epsilon^{-\frac{1}{N-1}} u_\epsilon$ is bounded in $W^{1,q}(\Omega)$ for any $1 < q < N$. Hence there exists some $G \in W^{1,q}_0(\Omega)$ such that $c_\epsilon^{-\frac{1}{N-1}} u_\epsilon \to G$ in $W^{1,q}_0(\Omega)$ for any $1 < q < N$.

Multiplying (4.13) with $\phi \in C_0^\infty(\Omega)$, we get

$$-\int_{\Omega} \phi Q_N(c_\epsilon^{-\frac{1}{N-1}} u_\epsilon) dx = \int_{\Omega} \phi \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon^{-\frac{1}{N-1}} e^{-(\lambda N - \epsilon)u_\epsilon \frac{N}{N-1}} dx + \gamma c_\epsilon^{-\frac{1}{N-1}} u_\epsilon \|_{p}^{p} \int_{\Omega} \phi (c_\epsilon^{-\frac{1}{N-1}} u_\epsilon)^{p-1} dx.$$ 

Let $\epsilon \to 0$, using again (4.12), we have

$$\int_{\Omega} \nabla \phi F^{N-1}(\nabla G) F_\xi(\nabla G) dx = \phi(x_0) + \gamma \|G\|_{p}^{N-p} \int_{\Omega} \phi G^{p-1} dx.$$ 

Therefore, in the distributional sense,

$$-\text{div}(F^{N-1}(\nabla G) F_\xi(\nabla G)) = \delta_{x_0} + \gamma \|G\|_{p}^{N-p} G^{p-1} \quad \text{in} \quad \Omega.$$ 

Applying with the standard elliptic regularity theory in [47], we get $c_\epsilon^{-\frac{1}{N-1}} u_\epsilon \to G$ in $C^1_{loc}(\Omega \setminus \{x_0\})$. Finally, replacing the right hand term in equation (4.11), we use
the similar discussion as Lemma 4.7 in [57], the asymptotic representation of Green function can immediately derived. This complete the proof of Lemma 4.5.

Next, we will exclude the case \( x_0 \in \partial \Omega \). Denote \( d_\epsilon = \text{dist}(x_\epsilon, \partial \Omega) \) and \( r_\epsilon \) be defined by (4.2). We obtain

**Lemma 4.6.** If \( 0 \leq \gamma < \gamma_1 \) and \( x_0 \in \partial \Omega \), then \( \frac{r_\epsilon}{d_\epsilon} \to 0 \) as \( \epsilon \to 0 \). Moreover, \( c_\epsilon^{\frac{1}{p - 1}} u_\epsilon \to 0 \) weakly in \( W_0^{1,q}(\Omega)(1 < q < N) \) and \( c_\epsilon^{\frac{1}{p - 1}} u_\epsilon \to 0 \) strongly in \( C^1(\overline{\Omega}\setminus\{x_0\}) \).

**Proof.** Firstly, we prove that \( \frac{r_\epsilon}{d_\epsilon} \to 0 \). If not, there exist some constant \( \delta \) such that \( \frac{r_\epsilon}{d_\epsilon} \geq \delta \) and \( y_\epsilon \in \partial \Omega, d_\epsilon = |x_\epsilon - y_\epsilon| \). Let

\[
\tilde{\psi}_\epsilon = \frac{u_\epsilon(y_\epsilon + r_\epsilon x)}{c_\epsilon}.
\]

As the similar procedure of interior case, we have

\[
\tilde{\psi}_\epsilon \to 1 \quad \text{in} \quad C^1(B_R^+) \quad \text{for} \quad \|\tilde{\psi}_\epsilon\|_{L^\infty(B_R^+)} = 1.
\]

This is impossible because of \( \tilde{\psi}_\epsilon(0) = 0 \). Secondly, let \( \Omega_\epsilon = \{x \in \mathbb{R}^N : x_\epsilon + r_\epsilon \epsilon \in \Omega \} \), we have know that \( \frac{r_\epsilon}{d_\epsilon} \to 0 \) by the first step, then \( \Omega_\epsilon \to \mathbb{R}^N \). Let \( \varphi_\epsilon \) and \( \varphi \), the same argument as the proof of Lemma 4.2, we get \( \varphi_\epsilon \to \varphi \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \). By the similar process as interior case, we have \( c_\epsilon^{\frac{1}{p - 1}} u_\epsilon \to \tilde{G} \) in weakly in \( W_0^{1,q}(\Omega)(1 < q < N) \) and in \( C^1(\overline{\Omega}\setminus\{x_0\}) \) with \( \tilde{G} \) satisfying \( -Q_N \tilde{G} = \gamma \|\tilde{G}\|_p^{N-p} \tilde{G}^{p-1} \) in \( \Omega \) and \( \tilde{G} = 0 \) on \( \partial \Omega \). By the standard elliptic regularity theory, we have \( \tilde{G} \in C^1(\overline{\Omega}) \). Since \( \gamma \leq \gamma_1 \), test the equation with function \( \tilde{G} \), we get \( \tilde{G} \equiv 0 \). Thus we have \( c_\epsilon^{\frac{1}{p - 1}} u_\epsilon \to 0 \) weakly in \( W_0^{1,q}(\Omega)(1 < q < N) \) and \( c_\epsilon^{\frac{1}{p - 1}} u_\epsilon \to 0 \) strongly in \( C^1(\overline{\Omega}\setminus\{x_0\}) \). \( \Box \)

**Lemma 4.7.** If \( 0 < \gamma < \gamma_1 \), the blow-up point \( x_0 \notin \partial \Omega \).

**Proof.** Suppose \( x_0 \in \partial \Omega \). Then \( \|u_\epsilon\|_N^N \to 0 \) implies

\[
(1 + \gamma\|u_\epsilon\|_p^N)^{-N-1} = 1 - \frac{\gamma}{N-1} \|u_\epsilon\|_p^N + O(\|u_\epsilon\|_p^{2N}).
\]
Let $w_\epsilon = \frac{u_\epsilon}{\int_\Omega F^N(\nabla u_\epsilon)}$, since $\int_\Omega F^N(\nabla u_\epsilon) = 1 + \gamma \|u_\epsilon\|_p^N$, we have

$$\Lambda_{\gamma,\epsilon} = \int_\Omega e^{(\lambda_N - \epsilon)|u_\epsilon| \frac{N}{N-1}} \, dx = \int_\Omega e^{(\lambda_N - \epsilon)(1 + \gamma \|u_\epsilon\|_p^N)|u_\epsilon| \frac{N}{N-1}} \, dx$$

$$= \int_\Omega e^{(\lambda_N - \epsilon)(1 + \gamma \|u_\epsilon\|_p^N)|u_\epsilon| \frac{N}{N-1}} \, dx \leq \int_\Omega e^{\lambda_N(1 - (1 + \gamma \|u_\epsilon\|_p^N)|u_\epsilon| \frac{N}{N-1}} \, dx$$

$$\leq e^{\lambda_N \left( \frac{\|u_\epsilon\|_p^N}{\|u_\epsilon\|_p^N} - \frac{\|u_\epsilon\|_p^N}{\|u_\epsilon\|_p^N} \right) \|u_\epsilon\|_p^N \frac{N}{N-1}} \Lambda_0.$$

From Lemma 4.6, we have $\|c_\epsilon \|_p^N u_\epsilon \|_p^N \to 0$. Thus letting $\epsilon \to 0$ and using (3.3), we have $\Lambda_\gamma \leq \Lambda_0$.

On the other hand, according to the anisotropic Trudinger-Moser inequality, $\Lambda_0$ is attained by a function $u \in W_0^{1,N}(\Omega)$ with $\int_\Omega F^N(\nabla u) = 1$. Define $v = u/(1 - \gamma \int_\Omega F^N(\nabla u))$. Thus

$$\|v\|_{N,F,\gamma,p} = \left( \int_\Omega F^N(\nabla v) dx - \gamma \|v\|_p^N \right)^{\frac{1}{N}} = 1$$

Since $u \neq 0$ and $\gamma > 0$, we get $|v| \leq |u|$ and $|u| \neq |v|$. Thus

$$\Lambda_\gamma \geq \int_\Omega e^{\lambda_N |v| \frac{N}{N-1}} \, dx > \int_\Omega e^{\lambda_N |u| \frac{N}{N-1}} \, dx = \Lambda_0,$$

which is a contradiction with $\Lambda_\gamma \leq \Lambda_0$.  

4.2 The upper bound estimate

We will use the capacity technique to give an upper bound estimate, which was used by Y. Li [25] and Yang-Zhu [54], our main result of this subsection is an upper bound estimate.

Lemma 4.8. $\Lambda_\gamma \leq |\Omega| + \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N-1} \frac{1}{\epsilon}}$.

Proof. Notice that $x_0 \in \Omega$. Take $\delta > 0$ such that $W_\delta(x_0) \subset \Omega$. For any $R > 0$, we assume that $\epsilon$ is so small that $\delta > Rr_\epsilon$. We denote by $a_\epsilon(1)$ ($o_\delta(1)$; $o_R(1)$) the terms
which tend to 0 as $\epsilon \to 0$ ($\delta \to 0$; $R \to \infty$). From Lemma 4.5, we have

\[
\int_{\Omega \setminus W_{\delta}(x_\epsilon)} F^N(\nabla u_\epsilon)\, dx = \frac{1}{c^N_\epsilon} \left( \int_{\Omega \setminus W_{\delta}(x_\epsilon)} F^N(\nabla G)\, dx + o_\epsilon(1) \right)
\]

\[
\quad = \frac{1}{c^N_\epsilon} \left( \int_{\Omega \setminus W_{\delta}(x_\epsilon)} -\text{div}(F^{N-1}(\nabla G)F_\xi(\nabla G))G\, dx \right.
\]

\[
\quad \left. + \int_{\partial(\Omega \setminus W_{\delta}(x_\epsilon))} GF^{N-1}(\nabla G)(F_\xi(\nabla G), \nu)\, dx + o_\epsilon(1) \right)
\]

\[
\quad = \frac{1}{c^N_\epsilon} (\gamma \int_{\Omega \setminus W_{\delta}(x_\epsilon)} \|G\|_p^{-p} \, G^p\, dx
\]

\[
\quad + \int_{\partial(\Omega \setminus W_{\delta}(x_\epsilon))} GF^{N-1}(\nabla G)(F_\xi(\nabla G), \nu)\, dx + o_\epsilon(1))
\]

\[
= \frac{1}{c^N_\epsilon} (\gamma \|G\|_p^{N} - \int_{\partial W_{\delta}(x_\epsilon)} GF^{N-1}(\nabla G)(F_\xi(\nabla G), \nu)\, dx + o_\delta(1) + o_\epsilon(1))
\]

\[
= \frac{1}{c^N_\epsilon} (\gamma \|G\|_p^{N} - \frac{1}{(N\kappa_N)^{N-1}} \log \delta + A_{x_0} + o_\delta(1) + o_\epsilon(1))
\]

\[
= \frac{1}{c^N_\epsilon} (\gamma \|G\|_p^{N} - \frac{N}{\lambda_N} \log \delta + A_{x_0} + o_\delta(1) + o_\epsilon(1)).
\]

From (5.3), we have on $W_{Rr_\epsilon}(x_\epsilon)$ that $u_\epsilon(x) = c_\epsilon^{-\frac{1}{N}} \varphi_\epsilon(\frac{x - x_\epsilon}{r_\epsilon}) + c_\epsilon$. Thus

\[
\int_{W_{Rr_\epsilon}(x_\epsilon)} F^N(\nabla u_\epsilon)\, dx = c_\epsilon^{-\frac{N}{N-1}} \int_{W_{R}(0)} F^N(\nabla \varphi_\epsilon)\, dx = \frac{1}{c^N_\epsilon} \left( \int_{W_{R}(x_0)} F^N(\nabla \varphi)\, dx + o_\epsilon(1) \right).
\]

By a straightforward computation, we have

\[
\int_{W_{R}(x_0)} F^N(\nabla \varphi)\, dx = \frac{N}{\lambda_N} \log R + \frac{1}{\lambda_N} \log \kappa_N - \frac{N - 1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + o_R(1).
\]

Let $i_\epsilon = \inf_{\partial W_{Rr_\epsilon}(x_\epsilon)} u_\epsilon$, $s_\epsilon = \sup_{\partial W_{\delta}(x_\epsilon)} u_\epsilon$. Since $\psi_\epsilon(x) \to 1$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, together with Lemma 4.5, we have $i_\epsilon > s_\epsilon$ for $\epsilon > 0$ small enough. Define a function space

\[
S(i_\epsilon, s_\epsilon) = \{ u \in W_\delta(x_\epsilon) \setminus W_{Rr_\epsilon}(x_\epsilon) : u|_{\partial W_{Rr_\epsilon}} = i_\epsilon, u|_{\partial W_{\delta}(x_\epsilon)} = s_\epsilon \},
\]

and \( \inf_{u \in S(i_\epsilon, s_\epsilon)} \int_{B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)} F^N(\nabla u)\, dx \) is attained by $h(x)$ satisfying

\[
\begin{cases}
-Q Nh = 0 & \text{in } W_\delta(x_\epsilon) \setminus W_{Rr_\epsilon}(x_\epsilon), \\
h|_{\partial W_{Rr_\epsilon}} = i_\epsilon, \\
h|_{\partial W_{\delta}(x_\epsilon)} = s_\epsilon.
\end{cases}
\]

(4.17)
The unique solution is
\[ h(x) = \frac{s_\varepsilon (\log F^0(x-x_\varepsilon) - \log (R r_\varepsilon)) + i_\varepsilon (-\log \delta - \log F^0(x-x_\varepsilon))}{\log \delta - \log (R r_\varepsilon)}, \] (4.18)
and hence
\[ \int_{W_\delta(x_\varepsilon) \setminus W_{R r_\varepsilon}(x_\varepsilon)} F^N(\nabla h) dx = N \kappa_N \frac{(i_\varepsilon - s_\varepsilon)^N}{(\log \delta - \log (R r_\varepsilon))^{N-1}}. \] (4.19)
Define \( \tilde{u}_\varepsilon = \max\{s_\varepsilon, \min\{u_\varepsilon, i_\varepsilon\}\} \). Then \( \tilde{u}_\varepsilon \in S(i_\varepsilon, s_\varepsilon) \) and \( F(\tilde{u}_\varepsilon) \leq F(u_\varepsilon) \) for \( \varepsilon > 0 \) small enough. Therefore
\[ \int_{W_\delta(x_\varepsilon) \setminus W_{R r_\varepsilon}(x_\varepsilon)} F^N(\nabla h) dx = \int_{W_\delta(x_\varepsilon) \setminus W_{R r_\varepsilon}(x_\varepsilon)} F^N(\nabla \tilde{u}_\varepsilon) dx \leq \int_{W_\delta(x_\varepsilon) \setminus W_{R r_\varepsilon}(x_\varepsilon)} F^N(\nabla u_\varepsilon) dx \]
\[ = 1 + \gamma \| u_\varepsilon \|^N_p - \int_{W_{R r_\varepsilon}(x_\varepsilon)} F^N(\nabla u_\varepsilon) dx - \int_{\Omega \setminus W_\delta(x_\varepsilon)} F^N(\nabla u_\varepsilon) dx. \] (4.20)
Since \( \gamma \| u_\varepsilon \|^N_p = \gamma c_\varepsilon^{-N^{-1}}(\| G \|^N_p + o_\varepsilon(1)) \), combing (4.14)-(4.16) and (4.19)-(4.20), we obtain
\[ \frac{\lambda_N (i_\varepsilon - s_\varepsilon)^N}{N \log \frac{R}{R} - \log r_\varepsilon} \leq \left( 1 + \frac{N}{\lambda_N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N + \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} - A_{x_0} + o_\delta(1) + o_\varepsilon(1) + o_\varepsilon(1) + o_R(1) \right) \frac{1}{c_\varepsilon^{N^{-1}}} \]
\[ \leq 1 + \frac{N}{\lambda_N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N + \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} - A_{x_0} + o_\delta(1) + o_\varepsilon(1) + o_\varepsilon(1) + o_R(1) \]
\[ \leq (N-1) c_\varepsilon^{N^{-1}} \] (4.21)
here we use the inequality \((1 + t)^{N^{-1}} \leq 1 + \frac{1}{N-1} t^N\) for any \(-1 \leq t \leq 0\) with
\[ -1 \leq \frac{N}{\lambda_N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N + \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} - A_{x_0} + o_\delta(1) + o_\varepsilon(1) + o_\varepsilon(1) + o_R(1) \leq 0. \]
On the other hand, using (4.8) and Lemma 4.5, we have
\[ (i_\varepsilon - s_\varepsilon)^N \leq i_\varepsilon^{N^{-1}} \left( 1 + \frac{N}{\lambda_N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N - A_{x_0} + o_\delta(1) + o_\varepsilon(1) + o_R(1) \right) \]
\[ \leq c_\varepsilon^{N^{-1}} \left( N \frac{1}{\lambda_N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N - A_{x_0} + o_\delta(1) + o_\varepsilon(1) + o_R(1) \right)^{N^{-1}}, \] (4.22)
here we use the inequality \((1 + t)^{\frac{N}{N-1}} \geq 1 + \frac{N}{N-1}t\) for any \(-1 \leq t \leq 0\) with
\[-1 \leq \frac{\frac{N}{N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N - A_{x_0} + o_{\delta}(1) + o_{\epsilon}(1) + o_R(1)}{c_{\epsilon}\frac{N}{N-1}} \leq 0.\]

Since \(\log \frac{\delta}{R} - \log r_\epsilon = \log \frac{\delta}{R} + \frac{\lambda_N - \epsilon}{N} c_{\epsilon}\frac{N}{N-1} - \frac{1}{N} \log \frac{\lambda_N}{c_{\epsilon}\frac{N}{N-1}}\), combing (4.21) and (4.22), we have
\[
\frac{\lambda_N}{N} \left[ c_{\epsilon}\frac{N}{N-1} + \frac{N}{N-1} \left( \frac{N}{\lambda_N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N - A_{x_0} + o_{\delta}(1) + o_{\epsilon}(1) + o_R(1) \right) \right] \\
\leq \left( \log \frac{\delta}{R} + \frac{\lambda_N - \epsilon}{N} c_{\epsilon}\frac{N}{N-1} - \frac{1}{N} \log \frac{\lambda_N}{c_{\epsilon}\frac{N}{N-1}} \right) \\
\times \left[ 1 + \frac{\frac{N}{N} \log \frac{\delta}{R} - \frac{1}{\lambda_N} \log \kappa_N + \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} - A_{x_0} + o(1) }{N(N-1)c_{\epsilon}\frac{N}{N-1}} \right] \\
\leq \frac{\lambda_N - \epsilon}{N} c_{\epsilon}\frac{N}{N-1} + \frac{N}{N-1} \log \frac{\delta}{R} - \frac{1 + o(1)}{N} \log \frac{\lambda_N}{c_{\epsilon}\frac{N}{N-1}} - \frac{1}{N(N-1)} \log \kappa_N \\
+ \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{k} - \frac{\lambda_N}{N(N-1)} A_{x_0} + o(1).
\]

Thus
\[
\frac{1 + o(1)}{N} \log \frac{\lambda_N}{c_{\epsilon}\frac{N}{N-1}} \leq \frac{1}{N} \log \kappa_N + \sum_{k=1}^{N-1} \frac{1}{k} + \frac{\lambda_N}{N} A_{x_0} + o_{\delta}(1) + o_{\epsilon}(1) + o_R(1),
\]

which lead to
\[
\limsup_{\epsilon \to 0} \frac{\lambda_N}{c_{\epsilon}\frac{N}{N-1}} \leq \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N-1} \frac{1}{k}}.
\]

Recall Lemma 4.4, we have finished the proof. \(\Box\)

5 Proof of main Theorems

Proof of Theorem 1.1. Let \(0 \leq \gamma < \gamma_1\). If \(c_\epsilon\) is bounded, the inequality (4.1) implies that the Theorem holds. If \(c_\epsilon \to +\infty\) is bounded, we know that the blow-up point \(x_0 \in \Omega\) by Lemma 4.7 and the result is followed from Lemma 4.8. \(\Box\)

Proof of Theorem 1.2. Let \(0 \leq \gamma < \gamma_1\). We prove that the blow-up phenomena do not occur. In fact, if \(c_\epsilon \to +\infty\). In Lemma 4.8, we have got the upper bound of \(\Lambda_\gamma\), that is to say
\[
\Lambda_\gamma \leq |\Omega| + \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N-1} \frac{1}{k}},
\]

(5.1)
where $x_0$ is the blow-up point. If we can construct a sequence $\phi_\epsilon \in W^{1,N}(\Omega)$ with
\[ \|\phi_\epsilon\|_{N,F,\gamma,p} = \int_{\Omega} F^N(\nabla \phi_\epsilon) dx - \gamma \|\phi_\epsilon\|^N_p = 1, \]
but
\[ \int_{\Omega} e^{\lambda N|\phi_\epsilon|^{N-1}} dx > |\Omega| + \kappa_N e^{\lambda N Ax_0 + \sum_{k=1}^{N-1} \frac{1}{k} + 1}. \] (5.2)

This is the contradiction with (5.1), which implies that $c_\epsilon$ must be bounded and can be attained by the discussion at the beginning of subsection 4.1. Thus it suffice to construct the sequence $\phi_\epsilon$ such that (5.2) holds when the blow-up phenomena occur.

From Lemma 4.5,
\[ G(x) = \frac{-1}{(N\kappa_N)^{\frac{1}{N-1}}} \log F^0(x - x_0) + A x_0 + \xi(x). \]

Define a sequence of functions
\[
\phi_\epsilon(x) = \begin{cases}
C + C^{-\frac{1}{N-1}}(-\frac{N-1}{\lambda_N} \log(1 + \kappa_N^{\frac{1}{N-1}}(F^0(x - x_0)e^{-1})^{\frac{N}{N-1}} + B), & x \in \bar{W}_R(x_0), \\
C^{-\frac{1}{N-1}}(G - \eta \xi), & x \in \Omega \setminus \bar{W}_R(x_0), \\
C^{-\frac{1}{N-1}}G, & x \in \bar{W}_2(x_0) \setminus \bar{W}_R(x_0),
\end{cases}
\] (5.3)

where $B$ and $C$ are constants depending only on $\epsilon$, which will be determined later.

The cutoff function $\eta \in C^1_0(\mathcal{W}_2(x_0))$, $0 \leq \eta \leq 1$ and $\eta = 1$ in $\mathcal{W}_R(x_0)$. To ensure $\phi_\epsilon \in W^{1,N}(\Omega)$, we require for all $x \in \partial \mathcal{W}_R(x_0)$, there holds
\[ C + C^{-\frac{1}{N-1}}(-\frac{N-1}{\lambda_N} \log(1 + \kappa_N^{\frac{1}{N-1}} R^{\frac{N}{N-1}} + B) = C^{-\frac{1}{N-1}}(-\frac{1}{(N\kappa_N)^{\frac{1}{N-1}}} \log(R\epsilon) + A x_0), \]

which implies
\[ B = -C^{\frac{N}{N-1}} + \frac{(N-1)}{\lambda_N} \log(1 + \kappa_N^{\frac{1}{N-1}} R^{\frac{N}{N-1}}) - \frac{N}{\lambda_N} \log(R\epsilon) + A x_0. \]

On one hand, since
\[ F^N(\nabla \phi_\epsilon) = C^{-\frac{N}{N-1}} F^N(\nabla \phi_\epsilon) (1 + O(R\epsilon)) \]
uniformly in $W_2R(x_0) \setminus W_R(x_0)$ as $\epsilon \to 0$. Thus, by Lemma 4.5, we have

$$
\int_{\Omega \setminus W_R(x_0)} F^N(\nabla \phi_\epsilon) \, dx = \int_{\Omega \setminus W_{2R}(x_0)} F^N(\nabla \phi_\epsilon) \, dx + \int_{W_{2R}(x_0) \setminus W_R(x_0)} F^N(\nabla \phi_\epsilon) \, dx \\
= C^{-N \frac{1}{N-1}} \int_{\Omega \setminus W_{2R}(x_0)} F^N(\nabla G) \, dx + \int_{W_{2R}(x_0) \setminus W_R(x_0)} F^N(\nabla G)(1 + O(Re)) \, dx \\
= C^{-N \frac{1}{N-1}} \int_{\Omega \setminus W_R(x_0)} F^N(\nabla G) \, dx + \int_{W_R(x_0) \setminus W_R(x_0)} F^N(\nabla G)O(Re) \, dx \\
= C^{-N \frac{1}{N-1}} \left( \int_{\Omega \setminus W_R(x_0)} F^N(\nabla G) \, dx + O(-Re \log(Re)) \right) \\
= C^{-N \frac{1}{N-1}} \left( \gamma \|G\|_p^N - \frac{N}{\lambda_N} \log(1 + \kappa_N^{1-N}) - \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + o_R(1) \right).
$$

On the other hand, through the direct calculation, we have

$$
\int_{W_R(x_0)} F^N(\nabla \phi_\epsilon) \, dx = C^{-N \frac{1}{N-1}} \left( \frac{N-1}{\lambda_N} \log(1 + \kappa_N^{1-N} R^{-\frac{N}{N-1}}) - \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + o_R(1) \right).
$$

Thus

$$
\int_{\Omega} F^N(\nabla \phi_\epsilon) \, dx = \int_{\Omega \setminus W_R(x_0)} F^N(\nabla \phi_\epsilon) \, dx + \int_{W_R(x_0)} F^N(\nabla \phi_\epsilon) \, dx \\
= C^{-N \frac{1}{N-1}} \left( \gamma \|G\|_p^N - \frac{N}{\lambda_N} \log(1 + \kappa_N^{1-N}) - \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + O(-Re \log(Re)) \right),
$$

we also have

$$
\|\phi_\epsilon\|_p^N = C^{-N \frac{1}{N-1}} \left( \|G\|_p^N + O((Re)^N (- \log(Re))^N) \right).
$$

Take $R = -\log \epsilon$, we get

$$
\|\phi_\epsilon\|_{N,F,\gamma,p}^N = \int_{\Omega} F^N(\nabla \phi_\epsilon) \, dx - \gamma \|\phi_\epsilon\|_p^N \\
= C^{-N \frac{1}{N-1}} \left( \frac{N}{\lambda_N} R + A_{x_0} + \frac{1}{\lambda_N} \log(1 + \kappa_N^{1-N}) - \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + O(R^{-N \frac{N}{N-1}}) \right)
$$

Choosing

$$
C^{-N \frac{1}{N-1}} = -\frac{N}{\lambda_N} \log \epsilon + A_{x_0} + \frac{1}{\lambda_N} \log(1 + \kappa_N^{1-N}) - \frac{N-1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + O(R^{-N \frac{N}{N-1}}),
$$

(5.5)
which implies
\[
B = \frac{N - 1}{\lambda_N} \sum_{k=1}^{N-1} \frac{1}{k} + O(R^{-\frac{N}{n-1}}) + o_R(1). \quad (5.6)
\]

Then we can get \( \| \phi_\epsilon \|^N_{N,F,\gamma,p} = \int_\Omega F^N(\nabla \phi_\epsilon)dx - \gamma \| \phi_\epsilon \|^N_p = 1. \)

Now we estimate \( \int_\Omega e^{\lambda_N|\phi|_{L^N}}. \) Let \( F^0(x-x_0) = e_y, \) we get
\[
\int_{W_R(x_0)} e^{\lambda_N|\phi|_{L^N}} \geq \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N} \frac{1}{k} + O(R^{-\frac{N}{n-1}})} \int_{W_R(0)} (1 + \kappa_N^{\frac{1}{N}} |y|^{-\frac{N}{n-1}})^{-N} dy
\]
\[
= \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N} \frac{1}{k} + O(R^{-\frac{N}{n-1}})} (1 + O(R^{-\frac{N}{n-1}}))
\]
\[
= \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N} \frac{1}{k} + O(R^{-\frac{N}{n-1}})}. \quad (5.7)
\]

From the inequality \( e^t \geq 1 + \frac{t^{N-1}}{(N-1)!}, \) we have
\[
\int_{\Omega \setminus W_{R_\epsilon}(x_0)} e^{\lambda_N|\phi|_{L^N}} dx \geq \int_{\Omega \setminus W_{2R_\epsilon}(x_0)} \left( 1 + \frac{\lambda_N^{N-1} |\phi|_{L^N}}{(N-1)!} \right) dx
\]
\[
= |\Omega| - |W_{2R_\epsilon}(x_0)| + C^{-\frac{N}{n-1}} \frac{\lambda_N^{N-1}}{(N-1)!} \| G \|_N^N + O((R\epsilon)^N (-\log(R\epsilon))^N)
\]
\[
= |\Omega| + C^{-\frac{N}{n-1}} \frac{\lambda_N^{N-1}}{(N-1)!} \| G \|_N^N + O((-\log \epsilon)^{-\frac{N}{n-1}}). \quad (5.8)
\]

Thus
\[
\int_\Omega e^{\phi_\epsilon} dx = \int_{W_{R_\epsilon}(x_0)} e^{\lambda_N|\phi|_{L^N}} dx + \int_{\Omega \setminus W_{R_\epsilon}(x_0)} e^{\lambda_N|\phi|_{L^N}} dx
\]
\[
\geq |\Omega| + \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N} \frac{1}{k} + C^{-\frac{N}{n-1}} \frac{\lambda_N^{N-1}}{(N-1)!} \| G \|_N^N + O((-\log \epsilon)^{-\frac{N}{n-1}})}, \quad (5.9)
\]

By choosing a small \( \epsilon > 0, \) we conclude \( \int_\Omega e^{\phi_\epsilon} dx \geq |\Omega| + \kappa_N e^{\lambda_N A_{x_0} + \sum_{k=1}^{N} \frac{1}{k}}. \) Hence we finish the proof of Theorem 1.2. \( \square \)

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