A general approach to systems with randomly pinned particles: Unfolding and clarifying the Random Pinning Glass Transition

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Abstract – Pinning a fraction of particles from an equilibrium configuration in supercooled liquids has recently proposed as a way to induce a new kind of glass transition, the Random Pinning Glass Transition (RPGT). The RPGT has been predicted to share some features of standard thermodynamic glass transitions and usual first-order ones. Thanks to its special nature, the approach and the study of the RPGT appears to be a fairly reachable task compared to the daunting problem of inspecting standard glass transitions. In this letter we generalize the pinning particle procedure. We study a mean-field system where the pinned configuration is extracted from the equilibrium distribution at temperature $T'$ and the thermodynamics of the non-pinned particles is observed at a lower temperature $T$. A more complicated physics emerges from this generalization eventually clarifying the origin and the peculiar characteristics of the RPGT.

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Introduction. – Experimental observations [1–4] on super-cooled liquids and plastic materials show a quite universal growth of relaxation time, which spans many orders of magnitude when temperature decreases in a rather restricted temperature window. Soon enough the relaxation time is so large that it is not possible to experimentally equilibrate these systems. This equilibrium/non-equilibrium crossover occurs at a temperature $T_g$, the temperature of the so-called glass transition.

Despite a long-lasting debate on static and dynamic approaches to the problem of glass transition [5,6], the nature and the origin of the dynamical slowing-down in super-cooled liquids remain an open problem. According to one of the possible (static) explanations, the glass transition is the result of a particularly severe critical slowing-down in proximity of a new thermodynamic transition, i.e. the ideal glass transition [7,8], at a temperature $T_K < T_g$. The corresponding critical properties are characterized by an exponential relation between correlation time and cooperative length-scales. In agreement with this picture, it has been observed in numerical simulations [9–14] that typical static length-scales cover only few inter-particle distances when the relaxation time exceeds the observation time scale. In other words the putative universal behavior does not clearly emerge in the temperature range where it is possible to equilibrate the system and a proper analysis of the critical properties cannot be achieved. Moreover, the whole low-temperature phase is out of reach of any possible equilibrium observation. As a consequence, the mere existence of the glass transition and its position on the temperature axis, at a zero or finite temperature $T_K$, are a matter of debate. All these issues hamper in practice any stringent experimental test of this particular theory of the glass transition and, in general, the comparison between predictions coming from static and dynamic views.

Important results of the thermodynamic approach to glass transition have been recently obtained via the procedure of pinning a number of particles in the system [9,15–22]. This is a new tool [23] specifically designed to study the properties of disordered systems, where the lack of a simple order parameter is a major issue. The idea consists in pinning a fraction of degrees of freedom from an equilibrium configuration at temperature $T$ and observing the thermodynamics of the remaining free particles. This procedure has been successfully used to determine the existence of static cooperative length-scales in numerical models of molecular liquids [9,12]. Moreover, recently it has been suggested [22,24,25] that for a particular realization of this procedure, i.e. when the $cN$ pinned particles are randomly chosen among the
$N$ particles of a system, a genuine thermodynamic transition can be induced in three dimensions [12,16] at temperature $T_K(c) > T_K$. The interest in this second result is manifold. First, this set of predictions can be only obtained within a specific thermodynamic theory of the glass transition called Random First Order Transition (RFOT) theory [7,8]. Within other static theories and dynamic approaches only the occurrence of a simple crossover instead of a transition is invariantly envisioned [22] and obtained [26]. Hence, testing this prediction represents a new way to make distinction among different theories of glass transition. Second, the nature of the new thermodynamic transition, called Random Pinning Glass Transition (RPGT), is not exactly the same as the usual ideal glass transition called Random First Order Transition (RFOT) glass transition. Second, the nature of the new thermodynamic transition called Random Pinning Glass Transition (RPGT), is not exactly the same as the usual ideal glass transition instead of a transition is invariantly envisioned [22] and obtained [26]. Hence, testing this prediction represents a new way to make distinction among different theories of glass transition. Second, the nature of the new thermodynamic transition called Random Pinning Glass Transition (RPGT), is not exactly the same as the usual ideal glass transition called Random First Order Transition (RFOT) glass transition. Second, the nature of the new thermodynamic transition called Random Pinning Glass Transition (RPGT), is not exactly the same as the usual ideal glass transition instead of a transition is invariantly envisioned [22] and obtained [26]. Hence, testing this prediction represents a new way to make distinction among different theories of glass transition.

The purpose of this letter is to shed light on the origin of the RPTG and of its particular features that could ultimately enable the inspection of an ideal glass transition in real systems. To this aim, we propose a generalization of the pinning particle procedure. We analyze systems where pinned particles belong to equilibrium configurations at temperature $T'$ and the thermodynamics of the remaining free particles is studied for temperatures $T < T'$. In particular, we will focus on the case $T' = T/\alpha$ with $0 < \alpha < 1$ (see footnote\(^1\)). This generalization also provides new important predictions able to clarify recent results [21] or promote new numerical tests and experiments.

In practice, we present the results of replica computations in the spherical $p$-spin model, the ubiquitous spin model which provides a reliable mean-field description of supercooled liquids in proximity of the glass transition [27–30]. In this case a fraction $c$ of spins is blocked in the positions that they have in an equilibrium configuration at temperature $T'$. The thermodynamics of the remaining spins is studied at $T < T'$. Before presenting the results of this computation, we briefly review the outcomes previously obtained in the same model using other methods to bias the system and induce thermodynamic transitions.

**The $p$-spin model and early bias-induced thermodynamic transitions.** – The spherical $p$-spin model [29] is a lattice spin model defined by the Hamiltonian

$$H = - \sum_{(i,j,...,k)} J_{ij,...,k} s_i s_j ... s_k,$$  \hspace{1cm} (1)

where the sum is performed over all the possible groups of $p$ spins in a system of $N$ spins, $J_{ij,...,k}$ are i.i.d. Gaussian random variables with zero mean and variance $\sigma_f = p! / 2N^{p-1}$, and the spin $s_i$ are continuous variables bounded by the following spherical constraint: $N = \sum_{i=1}^{N} s_i^2$. At a temperature $T_K$, a singularity in the equilibrium free energy identifies the ideal glass transition between a paramagnetic phase and the so-called 1RSB low-temperature phase. This phase is characterized by a non-trivial bimodal distribution of the order parameter $Q$, the similarity between two equilibrium configurations of the system. The other characteristic temperature in the model is the dynamic transition temperature $T_d$. It marks the presence of a dynamical transition akin to the one found in the Mode Coupling Theory.

As pointed out in [22], the emergence of the particular nature of the RPGT and of its trivial low-temperature phase is directly due to the special choice of the configuration used to pin spins (an equilibrium configuration from the same temperature $T$) and to the pinning protocol itself. Let us consider two different procedures: the $\epsilon$-coupling approach by Franz and Parisi [31] and the procedure of pinning spins form a totally random configuration [24]. In the first case, a glassy system at temperature $T$ is coupled with strength $\epsilon$ to a reference configuration, a typical equilibrium configuration of the same system at the same temperature. At variance with the pinning procedure, the coupling is introduced by adding an explicit term in the Hamiltonian that alters the equilibrium energy of the system. For a small strength $\epsilon$, a purely first-order transition [31] between a paramagnetic (the equivalent of the liquid phase) and a RS amorphous phase is observed in the system at temperature $T_\epsilon(\epsilon) > T_K$.

In the second case, a number $cN$ of spins among $N$ spins are frozen in a totally random configuration. A more complicated phase diagram emerges [24]. At small $c$ a usual glass transition is induced at $T_K(c) > T_K$ towards a 1RSB phase. At large $c$ a continuous transition between the 1RSB phase and a RS amorphous phase also takes place. The case of a system with particles pinned from an equilibrium configuration is strikingly different from the two previous examples. A detailed analysis of the differences between the pinning procedure and the $\epsilon$-coupling approach has been recently presented in [24]. The comparison between two pinning procedures, using equilibrium or

\(^1\)The imposed condition on $T'$ is such that the interesting case $T' = T$ where the RPGT appears is recovered for $\alpha = 1$. 56001-p2
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random configurations, is even more interesting because it sheds light on the intrinsic hidden differences between low-temperature equilibrium configurations and random configurations. The best way to approach this problem is to study the interpolating case of a system where particles are pinned from equilibrium configurations at temperature $T' = T/\alpha$, intermediate between the $T' = T$ ($\alpha = 1$) case and the random choice $T' = \infty$ ($\alpha = 0$).

The $p$-spin model with a finite fraction of randomly pinned particles: the general case. The free energy of a $p$-spin model with a finite fraction of spins blocked from an equilibrium configuration is obtained as a result of a double average: first the average over the equilibrium configurations at temperature $T'$ of the pinned spins, then the average over different realizations of the couplings $\{J_{i,j,\ldots,k}\}$. The free energy hence reads

$$\mathcal{F}(c, T; T') = -\frac{T}{N} \sum_{C} P^J(C', T') \log Z^J_c(c, T) ,$$

where $P^J(C', T') = \exp(-\beta J^J(C'))/Z^J(T')$ is the equilibrium probability of configurations $C'$ of free systems at temperature $T' = 1/\beta'$, $Z^J_c(c, T)$ is the partition function of the system at $T$ where $cN$ spins at random are constrained to be in the same configuration as in the reference configuration $C$, and $Z^J(T') = \sum_c \exp(-\beta J^J(C))$ is the partition function for the free system at $T'$. Introducing replicas in accordance with the procedure exposed in [24], performing the double average and summing over all the configurations, we obtain the following expression for the non-trivial part of the free energy of the system:

$$\mathcal{F}(c, T; T') = \lim_{N \to \infty} \frac{-T}{N n(m-1)} \log \int d\mathcal{Q} \exp[-NS(\mathcal{Q})] ,$$

where

$$S(\mathcal{Q}) = -\frac{1}{\alpha} \sum_{a,b} \left[ \frac{\beta^2}{4} \sum_{a \neq b} (c + (1-c)Q_{a,b})^p + \frac{\beta}{2} \sum_a (c + (1-c)Q_{a,1})^p \right] - \frac{(1-c)}{2} \log \det(\mathcal{Q})$$

and the $\mathcal{Q}$ matrix with elements $Q_{a,b}$ is an $nm \times nm$ diagonal block matrix with $n \times n$ blocks each one of $m \times m$ elements. Each block has a special replica, say replica 1, equilibrated at temperature $T'$ and $m-1$ replicates at temperature $T$. The off-diagonal blocks have all zero components. The diagonal blocks have 1 on the diagonal elements, the first row and column is characterized by the parameter $q$, and the remaining $(m-1) \times (m-1)$ elements are shaped in a 1RSB structure with parameter $q_1$, $q_0$, and $\mu$. This structure of the overlap matrix brings into the problem all the relevant order parameters: $q$ is the overlap between the reference configuration $C$ and the biased equilibrium configurations $C$, $q_1(q_0)$ is the overlap between two biased configurations belonging to the same (different) minimum and $\mu$ is the probability of finding biased configurations in different minima. According to this ansatz the action $S(Q, c, T; T')$ reads as follows:

$$S(q, q_1, q_0, \mu) = n(m-1) \left\{ \frac{\beta^2}{4} (1-\mu) (c + (1-c)q_1)^p + \frac{\beta^2}{4} (\mu - (m-1))(c + (1-c)q_0)^p - \frac{\beta}{2} (c + (1-c)q)^p \right\}$$

$$- \frac{1-c}{2} \left[ \left( \frac{1}{1-\mu} \right) \log(q_1) + \frac{1}{\mu} \log(1-q_1 + \mu(q_1 - q_0)) + \frac{1}{1-\mu} \log(1-q_1 + \mu(q_1 - q_0)) \right] \right\} .$$

Equilibrium free energy and physical values of the parameters $q, q_1, q_0$ and $\mu$ come from saddle points of $S$. Finding them amounts to solving the following set of equations:

$$\frac{\beta^2}{2} p(c + (1-c)q)^{p-1} = \frac{q}{1-q_1 + \mu(q_1 - q_0)} ,$$

$$\frac{\beta^2}{2} p(c + (1-c)q_0)^{p-1} = \frac{q_0 - q^2}{(1-q_1 + \mu(q_1 - q_0))^2} ,$$

$$\frac{\beta^2}{2} p(c + (1-c)q_1)^{p-1} = \frac{1}{\mu} \left[ \frac{1}{1-q_1} - \frac{1}{1-q_1 + \mu(q_1 - q_0)} + \frac{q_1 - q_0^2}{(1-q_1 + \mu(q_1 - q_0))^2} \right] ,$$

where $\beta'$ has been replaced by $\alpha \beta$. These equations have two different classes of solutions. The first one is the replica symmetric (RS) solution with $\mu = 1$, and $q_0^{\text{RS}} = q_1^{\text{RS}}$, meaning that the overlap between two randomly chosen biased configurations $C$ is always $q_1$ while the overlap between one of them and the reference configuration $C'$ is $q$. The second one is the 1RSB solution with $\mu^{\text{1RSB}} < 1$, $q_0^{\text{1RSB}} < q_1^{\text{1RSB}}$, and $q_1^{\text{1RSB}} < q_1^{\text{RS}}$, implying an overlap $q$ between $C$ and $C'$ and two possible overlap values between two $C$'s: $q_0$ with probability $\mu$ and $q_1$ with probability $1-\mu$. The appearance of the 1RSB solution with $\mu < 1$ defines the dynamic transition line $T_d(c)$ [32,33]. The

\footnote{This choice for $\mathcal{Q}$ properly restitutes a non-trivial 1RSB phase for the biased system when $T' \geq T_K$. For $T' < T_K$ a more complicated 1RSB structure instead of the diagonal block structure has to be considered for the $nm \times nm$ matrix reflecting the 1RSB structure of reference configurations.}
glass transition is found at the temperature where the free energies of the RS and the 1RSB solutions become equal: $\mathcal{F}_{\text{RS}} = \mathcal{F}_{1\text{RSB}}$ or when eqs. (6) are satisfied by the $\mu = 1$ condition. This defines a glass transition line $T_K(c)$ in the $c, T$ phase diagram.

Generalizing the procedure in [30], we also found a solution to eqs. (6) in the limit $q_1 - q_0 \to 0$:

$$\frac{\beta^2}{2} p(c + (1 - c) q) p^{-1} = \frac{q}{1 - q_0},$$  \hspace{1cm} (7a)

$$\frac{\beta^2}{2} p(c + (1 - c) q_0) p^{-1} = \frac{q_0 - q^2}{(1 - q_0)^2},$$  \hspace{1cm} (7b)

![](https://example.com/image1.png)

![Fig. 1: (Colour on-line) Comparison of two phase diagrams for different values of the parameter $\alpha \in (0, 1)$. A simple first-order transition (long-dashed line) appears for small $\alpha$'s, cutting the continuous transition line (continuous line). Eventually the first-order transition line and the discontinuous glass transition line (short-dashed line) superimpose giving rise to the RPGT line in the case $T' = T$. The dynamical transition line is also reported (dotted line).](https://example.com/image2.png)

$$\frac{\beta^2}{2} p(p - 1)(1 - c)(c + (1 - c) q_0) p^{-2} = \frac{1}{(1 - q_0)^2},$$  \hspace{1cm} (7c)

$$\mu = \frac{(p - 2)(1 - c)(1 - q_0)}{2(c + (1 - c) q_0)}.$$  \hspace{1cm} (7d)

The set of equations (7) defines the continuous transition line $T_d(c)$ in the physically meaningful range $0 < \mu < 1$ corresponding to $\frac{1}{2} q_0^2 \leq q_0 \leq 1$. Actually, the continuous transition line spans an even smaller region. Indeed, we found that the 1RSB solution, that gives rise to the continuous transition, is metastable in certain regions of the phase diagram. The reason is that a third kind of thermodynamic transition occurs in the general case $0 < \alpha < 1$. It is a first-order transition between a low-$q$ 1RSB phase and a high-$q$ RS phase or, in some cases, between two 1RSB phases with low and high $q$.

The three lines of thermodynamic transition arrange in a quite elaborate phase diagram for intermediate $\alpha$ and disappear or merge in the extreme cases $\alpha = 0$ and $\alpha = 1$. In the following section we present a detailed analysis of the phase diagrams for $\alpha = 0.5, 0.9$. We refer to [24] for the $\alpha = 0, 1$ phase diagrams. The dynamic transition line $T_d(c, \alpha)$, indicating where the metastable 1RSB solution with $\mu = 1$ appears, will also be discussed.

The phase diagrams. – In this section we show how the phase diagram changes when $\alpha$ varies from 0 to 1. As we will show, the 1RSB phase shrinks and the continuous transition line contracts to eventually give rise to the peculiar $\alpha = 1$ phase diagram presented in [22,24], where the replica symmetric solution is everywhere stable and a new kind of discontinuous glass transition emerges.

The $\alpha = 0$ phase diagram [24] is qualitatively similar to the one obtained for the $p$-spin model in a random field [30,32]. In fact, a strong local random field has the same effect as pinning. As usual in this case, a 1RSB phase covers the low-temperature region of the phase diagram. Referring to the parameters introduced here to describe the most general case, the 1RSB phase is characterized by $q_1 \geq q_0$, $q = 0$, and $0 < \mu < 1$. This phase is closed off by a discontinuous glass transition line $T_K(c)$, in every respect similar to the ideal glass transition at $T_K(c = 0)$ where $\mu = 1$ and $q_1 > q_0$, and by a continuous glass transition line $T_d(c)$ where $\mu < 1$ and $q_1 - q_0 \to 0^+$. At the sides of this region two RS phases lie: a paramagnetic phase (the analogous of the liquid phase for particle systems) with low $q_1 = q_0$ in the leftmost part, and a frozen amorphous phase selected by pinned particles for high concentrations $c$ with high $q_1 = q_0$. In both cases we have $q = 0$.

For a non-zero $\alpha$, e.g. in the $\alpha = 0.5$ case shown in fig. 1(a), the 1RSB phase shrinks in favor of the frozen amorphous RS phase as a consequence of a folding of the $T_d(c)$ continuous transition line, while the $T_K$ and $T_d$ lines are slightly shifted. On top of this, two 1RSB phases appear in the 1RSB region. They are separated by a first-order transition line. The same line at lower temperature divides the 1RSB region by the frozen RS phase. Two critical points are found: $c_{\alpha}, T_\alpha$, between the discontinuous and the continuous glass transition lines, and $c_2, T_2$, a second-order transition point at the end of the first-order transition line. Physically, right above
$T = \alpha T_{K} \quad (\text{such that } T' = T_{K})$ each equilibrated reference configuration $c_{\alpha}$ belongs to a well-shaped metastable minimum among few others. An infinitesimal bias in this direction, i.e., a small $c$, naturally selects it giving immediately rise to a RS phase. At higher temperature the reference minima become non-trivial hence the first-order transition occurs between two 1RSB phases, through a discontinuous jump from a $q_{1} > q \geq q_{0}$ phase at low $c$ to a $q_{1} > q_{0} > q$ phase at high $c$. Then the $q_{1} > q_{0} > q$ phase undergoes the continuous transition defined by (7). Interestingly this continuous transition is thermodynamically relevant only where the 1RSB solution with $q_{1} - q_{0} \to 0^{+}$ is stable, i.e., on the right of the $T_{1ST}$ line or above the critical point $c_{x}, T_{x}$. Otherwise, it pertains to the free-energy metastable branch, hence it does not emerge in the equilibrium behavior of the system.\(^3\)

As the parameter $\alpha$ increases towards one, see the $\alpha = 0.9$ case in fig. 1(b), the phase diagram does not change qualitatively but it shows how the $\alpha = 1$ phase diagram can be recovered. The 1RSB region shrinks, the continuous glass transition line shortens, and first-order transitions approach discontinuous glass transitions. The first-order transition line ends on the continuous glass transition.

In the $\alpha = 1$ limiting case the two discontinuous transitions and the two critical points superimpose and the continuous transition line disappears.\(^4\) As a result, the 1RSB region vanishes and only one line of glass transitions remains in the phase diagram. This line represents a new kind of glass transition, which has been previously observed in [22,34] and discussed in the context of glasses in [23]. For every $\alpha < 1$, the $T_{K}(c)$ line represents the usual glass transition between a paramagnetic phase, with a single low-overlap value between biased configurations, and a glass phase. This second phase is characterized by a double-peak probability distribution function of the order parameter, where the weight of the high-overlap peak continuously grows from 0 as the transition is left behind. Only beyond the continuous or the first-order transition line a RS phase with a single value of the overlap between the biased configurations can be recovered. On the contrary, when the region of the 1RSB phase vanishes and the first-order transition line merges with the usual glass transition line, the overlap distribution function is never double peaked and the resulting transition is associated to a genuine jump of the overlap from a single low value to a high one. For $\alpha = 1$ the $T_{K}(c)$ line becomes a line of strongly discontinuous glass transitions. As a result the low-temperature/high-concentration phase is constituted by a simple minimum selected by the reference configuration of pinned particles. It is already well formed when it becomes stable, as it usually happens in first-order phase transitions. On the other hand, the physics in the paramagnetic part of the phase diagram is the one that emerges in proximity of the usual glass transition: it is characterized by an exponential number of metastable states and a well-defined configurational entropy that decreases when temperature decreases and vanishes at the transition [24].

Finally, we note that: first, for $T = T'$ we obtain the interruption in a critical point of the RPGT line already claimed in [22]. In our study the critical point is originated by the merging of $c_{0}, T_{0}$ and $c_{x}, T_{x}$. Second, every pair of points in the $T = T'$ phase diagram can be naturally connected by a continuous path that does not cross any thermodynamic singularity [22] because the disconnected 1RSB phase has been squeezed on the RPGT line.

**Conclusion.** – The study of a $p$-spin system at temperature $T$ where a fraction $c$ of spins are blocked in an equilibrium configuration at temperature $T'$ provides intermediate results between the case of random pinning of spins from an equilibrium configuration [22] and from a completely random configuration [24]. A first-order transition line appears in this general case. It always interrupts the continuous glass transition line already present in the completely random configuration case [24,30] and eventually it merges, for $\alpha \to 1$, with the usual glass transition line (between a RS and a 1RSB phase) giving rise to the RPGT line when $T = T'$.

In this paper we imposed $T < T'$. The opposite choice, $T' < T$, or $\alpha > 1$, gives rise to a much simpler phase diagram: a first-order transition between the paramagnetic/liquid phase and the frozen amorphous phase intervenes before the glass transition has been attained, similarly to what happens in the $c$-coupling approach. The present RS ansatz for the reference configurations gives valuable results in the temperature $T$ range such that $T' > T_{K}$. Despite the little practical relevance, it could be at least theoretically interesting to extend the results and the picture we have obtained in this letter to the region $T' < T_{K}$. This can be done introducing a replica symmetry-breaking scheme for the reference configurations used to pin particles in the system.

As usual in $p$-spin glass models, mean-field results can be used to obtain predictions for real liquids within the framework of a landscape-controlled picture of glass transition. We discuss this issue in the following paragraphs. In real systems the dynamic transition becomes a crossover, we expect the same for the dynamic transition line for every value of $\alpha$. On the contrary, the other thermodynamic transition lines should be present also in finite dimensions $D$ (at least for $D \geq 3$), though they will be characterized by different critical properties. Both the usual glass transition line and the RPGT line are induced in the system by the configurational entropy vanishing [22,24]. Hence, according to the RFOT [7,8], activated events involving a larger and larger number of particles should lead to an exponential growth of the relaxation time and of the equilibrium correlation time. At variance, in proximity of the first-order transition line,
the equilibrium correlation time remains almost constant, while the time scale required to relax from non-equilibrium configurations diverges and the phenomenology associated to metastability and nucleation emerges. Finally, the continuous glass transition line could be associated to a critical power-law divergence of relaxation and correlation time when it is approached from the frozen amorphous phase in the rightmost part of the phase diagram. In the non-trivial glass phase (1RSB) of the intermediate region, equilibrium configurations can only be obtained as a result of a formidable optimization task for medium-size systems and are impossible to be determined in the large-size limit. Correlation time and relaxation time would be infinite in this region as it is expected for the glass phase of non-pinned glass formers below $T_K$. In experiments in 2D or 3D colloidal systems, thanks to the technology of the optical tweezers, or in molecular liquids, the simplest observables to probe are correlation and relaxation time, and average overlap, hence this set of predictions can be directly tested. Note that, in the $α=1$ case the RPGT is a true transition only for $D \geq 3$. In $D = 2$ it becomes a cross-over due to the intrinsic disorder of the system. A particularly interesting question is what happens in 2D in the $α < 1$ phase diagram where also continuous transitions are present.

Equilibrium is not always attainable in the thermodynamic limit, but finite-size scaling allows for equilibrium results in every region of the phase diagram. In particular, among the possible static observables, it is worth looking at the probability distribution function of the overlap between equilibrium configurations. This has already been done in a recent numerical work [21] that we can naturally ascribe to the $α \sim 0$ case and in [36] on the $α = 1$ case. Supporting evidences of both the continuous glass transition and the RPGT emerge from these works. Further numerical studies on the intermediate $α$ cases would be suitable to utterly validate the predictive skill of this thermodynamic approach to the glass transition.

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