ON LANGLANDS FUNCTIONALITY- REDUCTION TO THE SEMISTABLE CASE

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Abstract. We generalize a beautiful method of Blasius and Ramakrishnan, that in order to exhibit particular instances of the Langlands functorial correspondence, it is enough to show that the correspondence holds in the semistable case, provided the arithmetic data we are considering is closed with respect to cyclic base change and descent.

1. Introduction

Let $F$ be a global field, and let $H$, $G$ be reductive algebraic groups defined over $F$. Suppose $G$ is quasi-split, and we have a morphism

$$\phi : \mathcal{L}H \to \mathcal{L}G$$

of the Langlands dual groups $\mathcal{L}H$ and $\mathcal{L}G$ defined over the complex numbers. The conjectures of Langlands roughly predicts the existence of a functorial transfer from the collection of automorphic representations $\mathcal{A}(H)$ on $H$ to the collection of automorphic representations $\mathcal{A}(G)$ on $G$.

An example of functoriality is obtained upon specialising $H$ to be the group consisting of a single element, and $G$ to be $GL_n$ over $F$. Let $G_F$ denote the Galois group of a separable closure $\bar{F}$ of $F$ over $F$, and $\mathbb{A}_F$ denote the adele ring of $F$. The required functoriality, known as the strong Artin conjecture, is the problem of attaching an automorphic representation of $\pi(\rho)$ of $GL_n(\mathbb{A}_F)$ to a linear representation $\rho$ of $G_F$ to $GL_n(\mathbb{C})$. When $n = 1$, this correspondence is given by the Artin reciprocity law. The compatibility with the local correspondences at all places translates into

$$L(s, \rho) = L(s, \pi(\rho)),$$

where the $L$-function on the left is the $L$-function defined by Artin for a Galois representation, and the $L$-function on the right is the
one attached to an automorphic representation of $GL_n(\mathbb{A}_F)$ by the method of Tate-Godement-Jacquet. In particular, the strong Artin conjecture implies the Artin conjecture that the functions $L(s,\rho)$ are entire, provided $\rho$ does not contain the trivial representation.

Another instance of functoriality is obtained by considering a Galois extension $K/F$. Let $H$ be split over $F$, and let $G = R_{K/F}(H)$, the Weil restriction of scalars of $H$ considered over $K$ to $F$. At the level of $L$-groups, there is a diagonal embedding of $^LH$ to $^LG$. The corresponding lifting is known as base change. When $K/F$ is a cyclic extension of number fields of prime degree and $H = GL(n)$, the existence and properties of base change, in particular the descent criterion that invariant cuspidal automorphic representations lie in the image of base change, has been shown by Langlands [L] for $GL(2)$, and by Arthur and Clozel for $GL(n)$ in general [AC].

One of the properties required of the global correspondence is that it be compatible with the local correspondences at the unramified places as follows: given an automorphic representation $\pi$ of $H(\mathbb{A}_F)$, let $\Pi$ be the corresponding lift to an automorphic representation of $G(\mathbb{A}_F)$. Let $v$ be a place of $F$ at which the local components $\pi_v$ and $\Pi_v$ respectively of $\pi$ and $\Pi$ at $v$ are unramified. The required compatibility is that

$$\phi(S(\pi_v)) = S(\Pi_v),$$

where for an unramified representation $\eta$ of a unramified, reductive group $H$ over a local field $F$, $S(\eta)$ denotes the Satake parameter lying in the dual group $^LH$.

More generally for a local field $F$, let $W_F$ denote the Weil group of $F$, and if $F$ is non-archimedean, let $W'_F = W_F \times SU_2(\mathbb{R})$ be the Weil-Deligne group attached to $F$ (for $F$ archimedean, let $W'_F = W_F$). Let $\Phi(G)$ denote the set of equivalence classes of $L$-parameters,

$$\phi : W'_F \to ^L\hat{G},$$

satisfying the following properties:

- composition with the projection $^L\hat{G} \to G_F$ is the natural map occurring in the definition of Weil groups.
- $\phi(w)$ is semisimple for every $w \in W_F$.

$^L\hat{G}$ is the semidirect product of the dual group $\hat{G}$ defined over the complex numbers and $G_F$, and the equivalence relation is taken with respect to the conjugacy action of $\hat{G}$ on $^L\hat{G}$. Let $\Pi(G)$ denote the collection of irreducible, admissible, representations of $G(F)$. Then the local Langlands conjecture predicts a natural bijection between $\Phi(G)$, and $L$-packets of representations in $\Pi(G)$. The global correspondence
is then required to be compatible with the local correspondence at all places.

To enunciate the main principle of the paper, we need a concept of when the local component of an automorphic representation is ‘nice’.

**Definition 1.1.** Let $F$ be a non-archimedean local field, and $G$ be a reductive group defined over $F$.

1. $G$ is said to be unramified, if $G$ is quasi-split and splits over an unramified extension of $F$.
2. A $L$-parameter $\phi : W'_F \to {}^iG$ is semistable, if $G$ is unramified and if the image of the inertia group of $F$ with respect to $\phi$ acts trivially on $\hat{G}$.

In other words, a parameter is semistable, if restricted to $W_F$ it is an unramified parameter in the traditional sense. Since unramified representations are semistable, for almost all places the local components of an automorphic representation are semistable. At the representation theoretic level, a related notion for a representation of a split, reductive $p$-adic group to be semistable, is that the representation has Iwahori fixed vectors. For $GL(n)$ over local fields, this is compatible with the definition by parametrizations of the Weil-Deligne group to the $L$-group $LGL(n)^0 = GL(n)$, given by Bernstein and Zelevinskii [BZ]. In general, we have the results of Kazhdan and Lusztig [KL], relating the different notions of semistability given by the Deligne-Langlands conjecture.

It was shown by Grothendieck that any $l$-adic representation of $G_F$ is potentially semistable (see next section). Thus the existence of the local Langlands correspondence or a suitable theory of local base change, in particular should imply that given any representation $\pi \in \Pi(G(F))$ as above, there exists an extension $K$ of $F$, such that the restriction of the parameter to the Weil group of $K$ is semistable.

The main principle we want to illustrate here is the following:

**Principle.** Suppose $F$ is a global field and there is a morphism $\phi_F : {}^iH \to {}^iG$ of $L$-groups. Assume that the following conditions are satisfied:

1. for any extension $K$ of $F$, and a semistable automorphic representation $\pi$ of $H(\mathbb{A}_K)$, the automorphic lift $\phi(\pi)$ exists.
2. given $\pi$ an automorphic representation of $H(\mathbb{A}_K)$, by a succession of cyclic base changes of $\pi$, the base changed automorphic representation is semistable.
3. the collection of cuspidal automorphic representations $\pi$ of $G(\mathbb{A}_L)$, satisfy the descent properties of base change for cyclic extensions $L/K$ of prime degree, and $K$ containing $F$, i.e., if $\pi$ is
any $\text{Gal}(L/K)$-invariant, cuspidal, automorphic representation of $G(\mathbb{A}_L)$, then $\pi$ lies in the image of base change. Further any two representations of $G(\mathbb{A}_K)$ which base change to $\pi$, differ upto twisting by an idèle class character corresponding to the extension $L/K$.

Then the Langlands transfer corresponding to $\phi$, can be defined for all cuspidal, automorphic representations of $H(\mathbb{A}_F)$.

The principle indicates a more central role for cyclic base change in the context of establishing reciprocity laws. The underlying reasons for the validity of the principle, lies in the fact that the local Galois groups are solvable, and that local extensions can be approximated by global extensions. The main ingredient that goes into the proof is the theory of cyclic base change, and that locally the class of data considered are potentially semistable. Using the Grunwald-Wang theorem, it is possible to produce an infinite family of cyclic extensions, over which the extent of ‘ramified behavior’ of the given representation comes down, and an induction argument combined with the properties of base change, allows us to conclude the theorem.

The method can be considered as a refinement of the Chebotarev-Artin trick, and appears in Blasius and Ramakrishnan [BR]. It has been used in many different contexts, by Ramakrishnan in showing the automorphy of Rankin-Selberg $L$-functions for $GL(2) \times GL(2)$ [R], and in proving some cases of Artin’s conjecture for Galois representations with solvable image in $GO(4)$ [R]. Harris and Taylor have used it to extend Clozel’s results attaching Galois representations to regular, self-dual automorphic forms on $GL(n)$ [HT]. It is our hope that by placing the method in a general context, the usefulness and the fundamental nature of the principle outlined here will become more apparent and known.

The principal motivation for this paper was to extend the geometric correspondence of attaching a $l$-adic representation to cusp forms on $GL(n)$ over a function field of a curve over a finite field, provided the cuspidal representation is unramified at all places. Such a geometric correspondence has been established in the case $n = 2$ by Drinfeld, and by Gaitsgory, Frenkel and Vilonen for $n \geq 3$ [Lau1]. The results outlined here indicate that it is indeed possible to obtain the correspondence in general using these methods, provided the methods of Gaitsgory, Frenkel and Vilonen can be extended to cover the semistable case, and if the required properties of base change are established for $GL(n)$ over function fields. This will then provide a different and geometric way of obtaining one half of the Langlands correspondence, than
the proof using converse theory and the principle of recurrence given by Deligne and Piatetskii-Shapiro [Lau], which reduces the proof to Lafforgue’s theorem attaching automorphic representations to $l$-adic representations.

Even in the proof given by Lafforgue, the principle of reduction to the semistable case can be applied, and the proof reduced to the case when the automorphic representation is semistable, provided cyclic base change for $GL(n)$ over function fields is available. The advantage is that the compactification of the space of shtukas with multiplicity free (reduced) level structure enjoys nice properties [Laf, Theorem III.17], which are not available in general.

In Section 2, we illustrate this principle in the situation of attaching automorphic representations to $l$-adic representations of $G_F$, $F$ a number field. In the last section we discuss the transfer at the ramified places, and present a method to go from weak to strong lifting in the context of the methods used in this paper. Here our attempt is to clarify the use of different triple product L-functions that appear in the proof of automorphicity of the Rankin-Selberg convolution for $GL(2) \times GL(2)$, given by Ramakrishnan [R].

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Dedication. It is a pleasure to dedicate this paper to M. S. Raghunathan on his sixtieth birthday. His infectious enthusiasm, dynamism, attitude towards mathematics and openness have been of great inspiration to me. We wish him and his family the best in the years to come.

2. Reciprocity: reduction to the semistable case

Let $F$ be a number field. Denote by $\mathcal{A}(n, F)$ the set of irreducible cuspidal, automorphic representations of $GL(n, A_F)$, where $A_F$ is the adele ring of $F$. Let $l$ be a prime distinct from $p$, and let $S$ be a finite set of places of $F$ containing the primes of $F$ lying above $l$. Denote by $\mathcal{G}(n, F)$ the set of irreducible $\lambda$-adic Galois representations
\[
\rho : G_F \rightarrow GL_n(\overline{\mathbb{Q}}_l),
\]

of the absolute Galois group $G_F$ of an algebraic closure $\overline{F}$ of $F$ over $F$, unramified outside $S$.

For a place $v$ of $F$, let $\rho_v : G_v \rightarrow GL_n(\overline{\mathbb{Q}}_l)$ be the local representation at $v$ associated to $\rho$, where $G_v$ is the local Galois group.

**Definition 2.1.** Let $v$ be a finite place of $F$, with residue characteristic coprime to $l$. $\rho_v$ is **semistable** at a place $v$ of $F$, if $\rho_v|_{I_v}$ is unipotent. $\rho_v$
is potentially semistable at a place $v$ of $F$, if $\rho_v|_{I_v}$ is quasi-unipotent, i.e., there is a subgroup $I_v'$ of $I_v$ of finite index, such that $\rho_v|_{I_v'}$ is unipotent.

$\rho$ is (potentially) semistable, if at all finite places $v$ of $F$, $\rho_v$ is (potentially) semistable. (Additionally we can require that $F$ be a totally CM field).

Remark 1. At the finite places of $F$ lying over $l$, the corresponding notions of a $l$-adic representation to be semistable have been defined by Fontaine.

It was proved by Grothendieck that any $\lambda$-adic representation $\rho_v$ is potentially semistable. From the correspondence between $\lambda$-adic representations of the local Galois group and representations of the Weil-Deligne group of the local field as given in [1, Theorem 4.2.1], we see that the two definitions of semistability given in the previous section and the foregoing one are equivalent.

Fix an isomorphism of $\overline{\mathbb{Q}}_l$ with $\mathbb{C}$. Let $v$ be a finite place of $F$ not in $S$. Denote by $\rho(\sigma_v)$ the corresponding Frobenius conjugacy class in $GL(n, \mathbb{C})$.

**Definition 2.2.** An automorphic representation $\pi$ of $GL_n(\mathbb{A}_F)$ is a (weak) lifting of $\rho$, if at any unramified finite place $v$ of $\rho$ and $\pi$ and $v$ not dividing $l$, we have

$$\rho(\sigma_v) = S(\pi_v),$$

where $S(\pi_v)$ is the conjugacy class in $GL(n, \mathbb{C})$ defined by the Satake parameter $S(\pi_v)$.

We illustrate the principle stated above by the following theorem. If $K$ is a finite extension of $F$, let $\rho^K$ denote the restriction of $\rho$ to $G_K$.

**Theorem 1.** Let $\rho : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ be an irreducible $\lambda$-adic representation of $G_F$. Suppose that for any finite extension $K$ of $F$, such that the restriction $\rho^K$ is semistable, there exists a cuspidal, automorphic representation $\pi(\rho^K)$ of $GL(n, \mathbb{A}_K)$ which is a weak lift of $\rho^K$.

Then there exists a cuspidal, automorphic representation $\pi(\rho)$ of $GL(n, \mathbb{A}_F)$, which is a weak lifting of $\rho$.

**Proof.** The proof has three main ingredients. The first main ingredient involved in the proof is the structure of the absolute Galois groups of local fields. The results that we need from the structure theory of local Galois groups, are

- for any place $v$ of a number field $K$, the local Galois group $G_{K_v}$ is solvable.
- Grothendieck’s theorem. Any $\lambda$-adic representation is potentially semistable.
Consider the collection of pairs \( S \) of the form \((G, p)\), where \( G \) is a solvable group, \( p \) is a prime number, and there exists a surjective homomorphism \( G \to \mathbb{Z}/p\mathbb{Z} \). Define the modulus of such a pair to be \((|G|, p)\), and the prime modulus to be \( p \). Order these pairs by:

\[(G, p) \leq (G', p') \text{ iff } |G| \leq |G'| \]

or \(|G| = |G'| \) and \( p \leq p' \).

Given a \( \lambda \)-adic representation \( \rho : G_F \to GL_n(\overline{\mathbb{Q}}_l) \), let \( S' \) denote the set of finite places \( v \) of \( F \), at which the local component \( \rho_v \) is not semi-stable. For each place \( v \in S' \), choose a solvable extension \( L_v \) of minimal degree over \( F_v \) and with Galois group \( H_v \), such that the restriction \( \rho_v|_{G_{L_v}} \) is semistable. Consider the collection of pairs \( S_\rho \) given by \((H_v, p)\), satisfying that there exists a surjective morphism from \( H_v \to \mathbb{Z}/p\mathbb{Z} \). Define the ramification index \( R(\rho) \) of \( \rho \) to be the modulus of a maximal element in \( S_\rho \) (depends on the choice of \( L_v \)), and let \( T \) be the set of places at which the maximum is attained.

The proof of the theorem is by induction on \( R(\rho) \), and we assume that given a representation \( \rho \), we have proved the theorem over all extensions \( K \) of \( F \), such that \( R(\rho_K) < R(\rho) \).

The second main ingredient involved in the proof is the Grunwald-Wang theorem [AT, Chapter XV]: given a finite set of characters \( \eta_v : G_v \to \mathbb{C}^* \), of order \( n_v \) at a finite set of places \( v \in T' \) of \( F \), there exists an idele class character \( \chi \) of \( F \), such that we have an equality of local components for all \( v \in T' \):

\[ \chi_v = \eta_v. \]

Let \( m \) be least common multiple of the orders \( n_v \). Let \( s \) be the least integer \( \geq 2 \), such that \( \zeta_{2^s} + \zeta_{2^s}^{-1} \in F \), but \( \zeta_{2^{s+1}} + \zeta_{2^{s+1}}^{-1} \notin F \), and let

\[ a_0 = (1 + \zeta_{2^s})^m. \]

In general, \( \chi \) can be chosen to have period \( 2m \), but the period can be made equal to \( m \), provided when \( F \) is special, the following auxiliary condition (see [AT, Theorem 5, Chapter 10]) holds:

\[ \prod_{v \in T'} \eta_v(a_0) = 1. \]

In our situation, if we assume further that either \( \eta_v \) is trivial, or it is of order \( p \) for a fixed prime \( p \) at all the places \( v \in T' \), then we can arrange for \( \chi \) to have order \( p \), by adding an extra fixed finite place \( v_0 \) to the collection \( T' \), and choosing an appropriate character at \( v_0 \), such that
in the special case appearing in the hypothesis of the Grunwald-Wang theorem, the above extra condition needed for \( \chi \) to have order \( p \) is satisfied (this is required only when \( F \) is a number field).

We need a preliminary lemma before applying the Grunwald-Wang theorem:

**Lemma 1.** Let \( \rho : G_F \to GL_n(\overline{\mathbb{Q}}_l) \) be an irreducible \( \lambda \)-adic representation of \( G_F \). Then there exists only finitely many characters \( \chi : G_F \to \overline{\mathbb{Q}}^*_l \), such that

\[
\rho \simeq \rho \otimes \chi.
\]

**Proof.** The ramification of \( \chi \) is bounded by the ramification of \( \rho \), and on comparing determinants, we find that \( \chi \) is of order dividing \( n \). The lemma follows from the theorem of Hermite-Minkowski. \( \square \)

We now apply the Grunwald-Wang theorem, and construct characters depending on an auxiliary prime \( w \) not belonging to \( T \). Let \( T \) be the set of places of \( F \), where the ramification index of \( \rho \) attains its maximum. For each \( v \in T \), choose a character \( \eta_v \) of order \( p \), where \( p \) is the prime modulus and such that the kernel of \( \eta_v \) is a Galois extension of \( F_v \) contained in the chosen extension \( L_v \). Choose, and fix a place \( v_0 \) of \( F \) not in \( T \) (if needed), and a character \( \eta_{v_0} \), such that in the special case the extra condition needed in the Grunwald-Wang theorem for the global character to have order exactly the least common multiple of the local orders, is satisfied (see [AT, Theorem 5, Chapter 10]). Finally, let \( K' \) be the compositum of the finitely many cyclic extensions of \( F \) defined by the kernels of the characters satisfying the hypothesis of the above lemma. Choose a prime \( w_0 \neq v_0 \) of \( F \) which is inert in \( K' \), and let

\[
\eta_{w_0} = 1.
\]

This has the consequence that the fields we construct will be disjoint from \( K' \). Let \( w \) be a prime of \( F \) distinct from the primes belonging to \( T \) and from \( v_0 \) and \( w_0 \), and let \( \eta_w = 1 \). Let \( T(w) = T \cup \{w_0\} \cup \{w\} \cup \{v_0\} \). By the Grunwald-Wang theorem, we obtain an idele class character \( \chi^w \), such that the local components at \( v \in T(w) \) satisfy,

\[
\chi^w_v = \eta_v.
\]

Denote by \( K^w \) the corresponding cyclic extension of \( F \) defined by the kernel of \( \chi^w \). By construction, \( K^w \) satisfies the following properties:

- \( K^w \) is of order \( p \) over \( F \), where \( p \) is a prime.
- \( K^w \) is completely split over \( F \) at the primes \( w_0 \) and \( w \). In particular, \( K^w \) is disjoint from \( K' \).
• the ramification index $R(\rho^{K_w})$ is less than $R(\rho)$.

The third main ingredient involved in the proof is the theory of base change, due to Langlands for $n = 2$, and by Arthur and Clozel for general $n$ ([L], [AC]). Let $K/F$ be a cyclic extension of prime degree with Galois group $G(K/F)$. An automorphic representation $\Pi$ of $GL(n, \mathbb{A}_K)$ is invariant with respect to $G(K/F)$, if $\Pi^\sigma \simeq \Pi$, for any $\sigma \in G(K/F)$. Let $\Pi$ be a cuspidal, automorphic invariant representation of $GL(n, \mathbb{A}_F)$. Then there exists a cuspidal, automorphic representation $\pi$ of $GL(n, \mathbb{A}_F)$, which base changes to $\Pi$. Further if $\pi_1$ and $\pi_2$ are two such representations which base change to $\Pi$, then

$$\pi_1 \simeq \pi_2 \otimes \psi,$$

where $\psi$ is an idele class character of $F$, corresponding via class field theory to a character of $G_F \to \mathbb{C}^*$, whose kernel contains $G_K$. As a consequence of the descent criterion, we have

**Lemma 2.** Suppose $K_1, \ldots, K_m$ are cyclic, linearly disjoint extensions of $F$ of prime degree. For $i = 1, \ldots, m$, let $\pi_i$ be a cuspidal, automorphic representations of $GL(n, \mathbb{A}_{K_i})$, invariant with respect to the actions of $G(K_i/F)$. Assume further that for any pair of integers $1 \leq i, j \leq m$, the base change $\pi^{ij}$ of $\pi_i$ and $\pi_j$ to the compositum $GL(n, \mathbb{A}_{K_i K_j})$ are isomorphic. Then there exists a unique, cuspidal automorphic representation $\pi$ of $GL(n, \mathbb{A}_F)$, which base changes to $\pi_i$ for $1 \leq i \leq m$.

**Proof.** We first prove the lemma when $m = 2$. By hypothesis, let $\pi'_1, \ldots, \pi'_l$ be the cuspidal automorphic representations of $GL(n, \mathbb{A}_F)$, which base changes to $\pi_1$, where $l = [K_1 : F]$. Now the various possible descents differ from one another by an idele class character on $F$, corresponding to the extension $K_1/F$. Since $K_2$ is linearly disjoint with $K_1$ over $F$, these characters remain distinct upon base changing to $K_2$, and hence the base change $\Pi'_i$ to $K_2$ of the representations $\pi'_i$ remain distinct. By hypothesis, the base change of $\Pi'_i$ and $\pi_2$ to the compositum $K_1K_2$ are isomorphic. Since there are exactly $[K_1K_2 : K_2] = l$ such representations of $GL(n, \mathbb{A}_{K_2})$, there is precisely one index $j$ such that $\Pi'_j \simeq \pi_2$, and that gives us the lemma when $m = 2$.

Now we consider the general case. The descent $\pi_{12}$ living on $F$, upon base changing to a field $K_i$ different from $K_1$ and $K_2$, satisfies the property that upon further base changing to the fields $K_1K_1$ and $K_2K_1$ becomes isomorphic to $\pi^{12}$ and $\pi^{21}$ respectively. Hence by the uniqueness proved above for two components, the base change of $\pi_{12}$ to $K_i$ is isomorphic to $\pi_i$, and we have the lemma. \qed
Now we proceed to the proof of the Theorem. The restriction \( \rho^w := \rho_{K^w} \) is a \( \lambda \)-adic representation of \( G_{K^w} \), with ramification index less than that of \( \rho \). By induction hypothesis, we have a cuspidal, automorphic representations \( \pi^w \) of \( GL(n, \mathbb{A}_{K^w}) \), which is a (weak) lifting of \( \rho^w \). By strong multiplicity one, \( \pi^w \) is invariant with respect to the action of \( G(K^w/F) \), since \( \rho^w \) is invariant.

Hence by the lemma proved above, we have a unique cuspidal, automorphic representation \( \pi \) of \( GL(n, \mathbb{A}_F) \), which base changes to \( \pi^w \) for each \( w \). By the inductive hypothesis, the local components of \( \pi^w \) and \( \rho^w \) at an unramified prime of \( K^w \) lying above \( w \) agree. But \( w \) splits completely in \( K^w \), and thus these local components are isomorphic respectively to the local components of \( \rho \) and \( \pi \) at the place \( w \). Hence at any place \( w \) of \( F \), not belonging to \( T \cup \{v_0\} \) and where \( \rho_w \) and \( \pi_w \) are unramified, we obtain that the Frobenius conjugacy class defined by \( \rho_w \) is equal to the Satake parameter associated to \( \pi_w \). By choosing a different \( v_0 \), and appealing to strong multiplicity one, we obtain the compatibility of the lift at \( v_0 \) too, and that proves the theorem. \( \square \)

Remark 2. The above theorem can be used in the reverse direction too, from knowing how to attach Galois representations to ‘semistable, cuspidal, automorphic representations’ to attaching Galois representation to any cuspidal, automorphic representation. For this, one has to know that locally the representations are potentially semistable, either via the local Langlands correspondence or using the notion that a representation being semistable amounts to the representation having non-zero Iwahori fixed vectors. For \( GL(n) \) over local fields, this is compatible with the definition by parametrizations of the Weil-Deligne group to the \( L \)-group \([EZ]\), and in general we have the results of Kazhdan and Lusztig \([KL]\) establishing the Deligne-Langlands conjecture.

2.1. Function fields. Let \( F \) be a global field of positive characteristic. The Langlands correspondence for \( GL(2) \) was established by Drinfeld. In \([La]\), Lafforgue has established the Langlands conjecture, giving a canonical bijection between the collection of cuspidal, automorphic representations of \( GL_n (\mathbb{A}_F) \), and the collection of irreducible \( \lambda \)-adic representations of \( G_F \to GL_n (\bar{\mathbb{Q}}_l) \) unramified outside a finite set of places of \( F \), compatible with the local Langlands correspondence at all places \( v \) of \( F \). The method of Lafforgue is to attach a Galois representation \( \rho(\pi) \) to any cuspidal automorphic representation \( \pi \in \mathcal{A}(n, F) \). The reverse map \( \rho \mapsto \pi(\rho) \), is the ‘principle of recurrence’, due to Deligne, Laumon and Piatetskii-Shapiro. This is given by appealing to the converse theorems proved by Cogdell and Piatetskii-Shapiro, and
the results of Grothendieck that the $L$-functions attached to Galois representations have nice analytic properties.

Using geometric methods, Gaitsgory, Frenkel and Vilonen have established the correspondence $\rho \mapsto \pi(\rho)$, attaching a cuspidal automorphic representation $\pi(\rho)$ to any unramified Galois representation $\rho$ of $G_F$, and compatible with the local correspondence (see [Lau1]). The main motivation for this paper is to show that this correspondence can be extended to cover all representations $\rho \in \mathcal{G}(n, F)$, provided we know the correspondence for all ‘semistable representations’, and we have the Langlands-Arthur-Clozel theory of cyclic base change for extensions of prime degree.

Although the results that we require about base change follow from Lafforgue’s results, it would be desirable to have a direct proof of base change, and to characterize the local lifts in terms of character identities, since it reduces the proof of Lafforgue’s theorem to the case when the automorphic representation is semistable. Granting the existence of local base change asserting that the local components are potentially semistable, by the application of the principle given above, we only need to attach Galois representations to semistable automorphic representations. This amounts to considering the space of shtukas with reduced level structure and their compactifications, which enjoys nice properties [Laf, Theorem III.17], and are not available in general.

3. Weak to strong lifting: Matching of $L$ and $\epsilon$-factors

Our main aim in this section is to indicate how to go from a weak lifting to a strong lifting, where we obtain the matching of $L$ and $\epsilon$-factors match at all places, assuming that in the case of lifting of semistable data, the local $L$ and $\epsilon$-factors match at all places. The applications we have in mind here are to clarify the use of different $L$-functions associated to triple products that appears in the proof of automorphy of the Rankin-Selberg convolution for $GL(2) \times GL(2)$ established by Ramakrishnan [R], and to provide a slightly different perspective of Laumon’s theorem proving the factorisation of the global epsilon factor occurring in Grothendieck’s functional equation for $L(s, \rho)$, in terms of the local constants constructed by Deligne and Langlands.

Again the main ingredient is the theory of base change, combined together with Grunwald-Wang theorem, which allows us to concentrate our attention at only one place. The additional assumption that we require, is the existence of Jacquet-Shalika type bounds at the semistable places and the global functional equation. For the case of automorphic representations on $GL(n)$, Jacquet-Shalika type bounds are available
from the existence and properties of Rankin-Selberg convolution proved by Jacquet, Piatetskii-Shapiro, Shalika and Shahidi ([JS], [JPSh], [Sh]).

For the matching of epsilon factors, we assume the factorisation of the global \( \varepsilon \)-factor, and then use the defining properties of the epsilon factors, together with the functional equation. We illustrate the matching at all places of the local \( L \) and \( \varepsilon \)-factors, in the context of the principle.

We first abstract a notion of an ‘arithmetic \( L \)-data’ \( \mathcal{A} \), which is stable with respect to base change.

**Definition 3.1.** An *arithmetic \( L \)-data*, is a collection \( \mathcal{A} \) satisfying the following:

1. to any element \( \pi \in \mathcal{A} \), we have an associated global field \( K \). We will say that \( \pi \) is defined over \( K \). Further there is an \( L \)-function, 
   
   \[
   L(s, \pi) = \prod_{v \in \Sigma_K} L_v(s, \pi),
   \]

   indexed over the places \( v \in \Sigma_K \) of \( K \). It is required that \( L_v(s, \pi) \) are meromorphic and non-vanishing on the entire plane. At the archimedean places, \( L_v(s, \pi) \) can be expressed in terms of \( \Gamma \)-functions. At the finite places, we require that
   
   \[
   L_v(s, \pi) = \prod_{i=1}^{d_v} (1 - \alpha_v,i q_v^{-s}),
   \]

   where \( q_v \) is the cardinality of the residue field, and this will be referred to as the local \( L \)-factors of \( \pi \). Further it is required that the product defining \( L(s, \pi) \) be absolutely convergent in some right half plane.

2. **Functional equation.** \( L(s, \pi) \) has a meromorphic analytic continuation to the entire plane, and satisfies a functional equation in the form,
   
   \[
   L(1 - s, \tilde{\pi}) = \epsilon(s, \pi) L(s, \pi),
   \]

   where \( \tilde{\pi} \in \mathcal{A} \) is the ‘contragredient’ of \( \pi \), and is defined over \( K \). We have that \( \tilde{\pi} = \pi \), i.e., the contragredient operation is an involution on \( \mathcal{A} \). \( \epsilon(s, \pi) \), called the global \( \epsilon \)-factor associated to \( \pi \), is an entire, non-vanishing function of \( s \). Define the archimedean component \( L_\infty(s, \pi) \) of \( L(s, \pi) \) by
   
   \[
   L_\infty(s, \pi) = \prod_{v \in \Sigma_\infty} L_v(s, \pi),
   \]

   the product taken over the collection of archimedean places \( \Sigma_\infty \) of \( K \).
(3) **Convolution with Dirichlet characters.** If \( \pi \) is defined over \( K \), and \( \chi \) is a Dirichlet character of \( K \), i.e., an idele class character of finite order, then there is an element \( \pi \otimes \chi \in \mathcal{A} \). With the above notation for \( L_v(s, \pi) \), at a finite place \( v \) of \( K \) where \( \chi \) is unramified, the local \( L \)-factor is given by

\[
L_v(s, \pi \otimes \chi) = \prod_{i=1}^{d_v(\pi)} (1 - \chi(\omega_v)\alpha_{v,i}(\pi)q_v^{-s})^{-1},
\]

where \( \omega_v \) is an uniformising parameter for \( K_v \). The contragredient of \( \pi \otimes \chi \) is \( \tilde{\pi} \otimes \chi^{-1} \).

(4) **Base change.** For any cyclic extension \( L/K \) with Galois group \( G(L/K) \) and \( \pi \) defined over \( K \), we have a base change element \( B_{L/K}(\pi) \in \mathcal{A} \) defined over \( L \), with the local components of the \( L \)-function at any place \( w \) of \( L \) dividing a place \( v \) of \( K \) given by,

\[
L_w(s, B_{L/K}(\pi)) = \prod_{\chi \in \hat{G}(L/K)} L_v(s, \pi \otimes \chi),
\]

where \( \hat{G}(L/K) \) denotes the set of characters of \( G(L/K) \). Here we are using abelian class field theory in thinking of \( \chi \) also as an idele class character, with abuse of notation. If \( v \) is a place of \( K \), and \( w \) a place of \( L \) of degree one over \( K \), then

\[
L_v(s, \pi) = L_v(s, B_{L/K}(\pi)).
\]

(5) **Potential semistability.** There is a concept of \( \pi \) being semistable at any place of \( K \), satisfying:

- \( \pi \) is semistable at all but finitely many places of \( K \).
- if \( \pi \) is semistable at \( v \), and \( L/K \) is a cyclic extension, then \( B_{L/K}(\pi) \) is semistable at all places of \( L \) lying over \( v \).
- given a place \( v \) of \( K \), there is sequence of cyclic extensions \( L = K_n \supset K_{n-1} \supset \cdots \supset K_0 = K \), with \( K_{i+1}/K_i \) cyclic such that \( v \) splits completely in \( L \), and the successive base changes of \( \pi \) defined with respect to this filtration exist, with the property that

\[
B_{L/K}(\pi) := B_{L/K_{n-1}} \circ \cdots \circ B_{K_1/K}(\pi),
\]

is semistable at all places of \( L \) not dividing \( v \).

Remark 3. It is possible to relax the condition of \( \pi \) being potentially semistable at all places of \( K \), and instead require that the above conditions be valid for a particular class of places of the global fields (for example the class of totally real fields), closed with respect to divisibility.
The last condition in the notion of potential semistability, is motivated by the expectation that locally the components of automorphic representations are potentially semistable, and the use of Grunwald-Wang theorem as in the proof of Theorem 1. We have stated it in this form, rather than put in a condition of local base change and declaring that the local components are potentially semistable.

**Definition 3.2.** Let $A_1, A_2$ be two classes of arithmetical $L$-data as above. Given $\pi_1 \in A_1$ and $\pi_2 \in A_2$, we say that $\pi_1$ and $\pi_2$ are *weakly compatible* (or weakly match, or is a weak lift of one.), if at all places $v$ of $K$, where $\pi_1$ and $\pi_2$ are semistable, we have

$$L_v(s, \pi_1) = L_v(s, \pi_2).$$

$\pi_1$ and $\pi_2$ are *strongly compatible* if at all places $v$ of $K$, we have $L_v(s, \pi_1) = L_v(s, \pi_2)$.

If $\pi_1$ and $\pi_2$ are strongly compatible, we have as a consequence that $\epsilon(s, \pi_1) = \epsilon(s, \pi_2)$.

Let $R : A_1 \rightarrow A_2$ be a map. We say that $R$ is a *weak* (resp. strong) transfer (or weak (resp. strong) lift, or weakly (resp. strongly) compatible), if for every $\pi_1 \in A_1$, $\pi_1$ and $R(\pi_1)$ are weakly (resp. strongly) compatible.

**Definition 3.3.** $\pi \in A$ satisfies the Jacquet-Shalika (JS) condition at a place $v$ of $K$, if $L_v(s, \pi)$ is holomorphic in the half plane $\text{Re}(s) \geq 1/2$.

**Remark 4.** As remarked above, this condition is satisfied for isobaric, automorphic representations of $GL(n, \mathbb{A}_F)$, by the results of Jacquet and Shalika [JS].

**Proposition 2.** Suppose we are given two sets of arithmetic data $A_1$ and $A_2$ as above, and a weak transfer $R : A_1 \rightarrow A_2$. Assume that either $A_1$ or $A_2$ has the property, that for any element $\pi$ in one of them, and any place $v$ of $K$ at which $\pi$ is semistable, $L(s, \pi)$ satisfies the Jacquet-Shalika property at $v$.

Then $R$ is a strong transfer.

**Proof.** Given $\pi \in A_1$ and a finite place $v$ of $K$, we can find a sequence of cyclic extensions $L = K_n \supset K_{n-1} \supset \cdots \supset K_0 = K$, with $K_{i+1}/K_i$ cyclic, such that $v$ splits completely in $L$, and

$$B_{L/K}(\pi), B_{L/K}(\tilde{\pi}), B_{L/K}(R(\pi)), B_{L/K}(\tilde{R(\pi)})$$

are semistable at all places of $L$ not lying over $v$. Since the local $L$-factors do not change for primes splitting completely in $L$, it is enough now to prove the equality of the $L$-factors at a place $w$ of $L$ lying above $v$. 
Let \( w' \) be a place of \( L \) not lying over \( v \). Again we can assume that there is a sequence cyclic extensions \( M = L_m \supset L_{m-1} \supset \cdots \supset L_0 = L \), with \( L_{j+1}/L_j \) cyclic, such that

\[
B_{M/K}(\pi), \ B_{M/K}(\hat{\pi}), \ B_{M/K}(R(\pi)), \ B_{M/K}(\hat{R}(\pi))
\]

are semistable at all places of \( M \).

We argue by induction on \( j \) ranging from 1 to \( m \). Suppose \( M_1/M_2 \) is a cyclic extension, and we have \( \pi \) defined over \( M_2 \). Assume further that \( B_{M_1/M_2}(\pi) \) exist and satisfies the Jacquet-Shalika condition at a place \( w \) of \( M_1 \) lying over a place \( v \) of \( M_2 \). Then from the equality,

\[
L_w(s, B_{M_1/M_2}(\pi)) = \prod_{\chi \in \hat{G}(M_1/M_2)} L_v(s, \pi \otimes \chi),
\]

we obtain that \( L_v(s, \pi) \) also satisfied the JS-condition, since we have the local \( L \)-factors are non-vanishing everywhere. Hence, we obtain by induction that the local components of \( B_{L/K}(\pi) \) and \( B_{L/K}(R(\pi)) \) also satisfy the Jacquet-Shalika condition at any place of \( L \).

We further have the functional equations,

\[
L(1-s, \hat{\pi}) = \epsilon(s, \pi)L(s, \pi)
\]

and \( L(1-s, \hat{R}(\pi)) = \epsilon(s, \pi)L(s, R(\pi)) \).

Dividing these two functional equations, and using the fact that the local components are same at all the places not lying over \( v \) by the hypothesis and semistability condition, we obtain

\[
\frac{L_w(1-s, \hat{\pi})}{L_w(s, \hat{R}(\pi))} = \frac{\epsilon(s, \pi)}{\epsilon(s, R(\pi))} \frac{L_w(s, \pi)}{L_w(s, R(\pi))}.
\]

From the Jacquet-Shalika condition we have that the right hand side is holomorphic and non-vanishing in the half plane \( \text{Re}(s) \geq 1/2 \). From the above functional equation and the assumptions about \( \epsilon \)-factors,, using the fact that contragedient is an involution, we obtain that

\[
\frac{L_w(s, \pi)}{L_w(s, R(\pi))} = \prod_{i=1}^{d_w(\pi)} (1 - \alpha_{w,i}(R(\pi))q_v^{-s}) \frac{1}{\prod_{i=1}^{d_w(\pi)} (1 - \alpha_{v,i}(\pi)q_v^{-s})},
\]

is an entire, non-vanishing function of \( s \). But then the right hand side has to be a constant.

The proof of matcheing of \( L \)-factors at the archimedean places, is similar, where we observe that a ratio of products of Gamma functions cannot be entire unless it is a constant.

\[\square\]

Remark 5. In the context of the applicability of converse theorems, a similar statement is proved in [BR, Proposition 4.1]. Instead of cyclic
base change, what is required in \[BR\], are converse theorems and strong multiplicity one. Indeed by the converse theorems we have attached two possible (cuspidal) automorphic representation, which agree at all places except the given ones, and by strong multiplicity one, we have the matching at all places.

3.1. Automorphy of Rankin-Selberg. The principle outlined in this paper, has been derived from the method of proof of the automorphy of the Rankin-Selberg convolution for \(GL(2) \times GL(2)\) proved by Ramakrishnan in \[R\]. Here we clarify the use of different triple product \(L\)-functions appearing in Ramakrishnan’s proof, and show that it is enough to consider the triple product \(L\)-function constructed by Shahidi.

By the principle outlined in the first part, to prove the automorphy of the Rankin-Selberg convolution, it is enough to prove it in the case when the automorphic representations involved have semistable reduction at all places. To prove this for \(GL(2) \times GL(2)\), we have to show the following: if \(\pi_1\) and \(\pi_2\) are semistable, cuspidal, automorphic representations of \(GL(2, \mathbb{A}_K)\) for a number field \(K\), and if \(\eta\) is an arbitrary cuspidal, automorphic representation of \(GL(2, \mathbb{A}_K)\) with the set of places of \(K\) at which \(\eta\) is ramified disjoint from the set of places at which either \(\pi_1\) or \(\pi_2\) is ramified, then the triple product \(L\)-function \(L(s, \pi_1 \times \pi_2 \times \eta)\) is ‘nice’ in the sense of converse theory, i.e., it has an analytic continuation to the entire plane, is entire and satisfies a suitable functional equation, and is bounded in vertical strips.

One knows by the work of Shahidi \[Sh2\], and Kim and Shahidi \[KS\], that the triple product \(L\)-function can be analytically continued as an entire function, and that it satisfies a suitable functional equation. Further by Ramakrishnan \[R\], and more generally by Gelbart-Shahidi \[GS\], these \(L\)-functions are bounded in vertical strips. The required matching at the semistable places of the \(L\) and \(c\)-factors is given by Part i) of Theorem 3.5 in \[Sh3\] at the non-archimedean places, and for the archimedean places it is given by the archimedean Langlands conjecture proved by Langlands.

This gives us the weak lifting for semistable, cuspidal, automorphic representations. It can be seen along the lines of the proof of Theorem 1, that the principle is valid in this setting, as the necessary hypothesis are satisfied, and hence we obtain a weak lifting in general. By Proposition 2, we obtain a strong lifting too.

Remark 6. In view of the results on lifting from classical groups to \(GL_n\), it seems possible to establish the lifting for semistable representations
(those with Iwahori fixed vectors) from classical groups to $GL_n$, by extending Proposition 3.1 of [CKPS] to the general case.

3.2. Function fields and Laumon’s factorisation theorem. We now consider the situation over function fields. For the rest of the discussion, we are assuming the existence and properties of cyclic base change for $GL(n)$ as outlined above. In order to attach an automorphic representation to a Galois representation, it is enough now to apply the ‘principle of recurrence’ of Deligne and Piatetski-Shapiro and obtain a weak automorphic lifting of a semistable, irreducible Galois representation $\rho$ of degree $n$. The analytic properties of the $L$-function attached to a $\lambda$-adic representation has been proved by Grothendieck. To apply the converse theorem, it remains to provide a factorisation of the epsilon factor $\epsilon(s, \rho \otimes \rho(\eta))$ occurring in Grothendieck’s functional equation, in terms of the local constants constructed by Deligne and Langlands, as given by the theorem of Laumon [Lau, Section 3]. Here $\rho(\eta)$ is the Galois representation of degree $m$ (less than $n$) attached to a cuspidal, automorphic representation of $GL(m, \mathbb{A}_K)$ by Lafforgue [Laf]. The factorisation of the $\epsilon$-constant for $\rho(\eta)$ follows from Lafforgue’s theorem, or from Lafforgue’s theorem in the semistable case, and applying Theorem 1 and Proposition 2. We can further assume that the ramified primes of $\eta$ are disjoint from the ramified primes of $\rho$. Thus we require Laumon’s theorem in the special case when $\rho$ is semistable, and assuming the factorisation for $\rho$ and $\rho(\eta)$ (with disjoint ramification loci), to prove the factorisation for the tensor product $\rho \otimes \rho(\eta)$. The converse theorem machinery proves the existence of a weak automorphic lift for $\rho$. As a consequence, we also obtain Laumon’s theorem proving the factorisation of the global epsilon factor occurring in Grothendieck’s functional equation for $L(s, \rho)$, in terms of the local constants constructed by Deligne and Langlands.

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