Projective and Non-Projective Varieties of Topological Decomposition of Groups with Embeddings

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Abstract: In general, the group decompositions are formulated by employing automorphisms and semidirect products to determine continuity and compactification properties. This paper proposes a set of constructions of novel topological decompositions of groups and analyzes the behaviour of group actions under the topological decompositions. The proposed topological decompositions arise in two varieties, such as decomposition based on topological fibers without projections and decomposition in the presence of translated projections in topological spaces. The first variety of decomposition introduces the concepts of topological fibers, locality of group operation and the partitioned local homeomorphism resulting in formulation of transitions and symmetric surjection within the topologically decomposed groups. The reformation of kernel under decomposed homeomorphism and the stability of group action with the existence of a fixed point are analyzed. The first variety of decomposition does not require commutativity maintaining generality. The second variety of projective topological decomposition is formulated considering commutative as well as noncommutative projections in spaces. The effects of finite translations of topologically decomposed groups under projections are analyzed. Moreover, the embedding of a decomposed group in normal topological spaces is formulated in this paper. It is shown that Schoenflies homeomorphic embeddings preserve group homeomorphism in the decomposed embeddings within normal topological spaces. This paper illustrates that decomposed group embedding in normal topological spaces is separable. The applications aspects as well as parametric comparison of group decompositions based on topology, direct product and semidirect product are included in the paper.

Keywords: topology; symmetry; group decomposition; Schoenflies embeddings; projection; homeomorphism

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1. Introduction

The applications of group theory are found in the identification of symmetries and in designing cryptographic protocols [1,2]. The topologization of algebras was first proposed by Markov and the topologizability of structures was further studied by extending the Kotov model [3]. Thus, topology and group theory exhibit interplay if the underlying space is considered to be a topological space. Topological groups are special groups with topological properties in underlying spaces. In general, the separable topological spaces are considered while analyzing topological groups, as well as their subgroups, with closure properties [4]. A topological group is compactly generated if the group contains a compact subspace. Moreover, as a consequence, every closed subgroup of a group becomes separable if the group is a compact separable topological group.

The topological spaces can be further equipped with projection varieties. The constructions of various types of projection in topological spaces can be formulated with algebraic and geometric
properties. The geometric variety of projective spaces with distribution of singular points is proposed in [5]. The factoriality of geometric hypersurface under projection is constructed up to $d$-degree containing singular points. There are several interesting applications of concepts of finite projective spaces. For example, the Galois projective spaces (finite variety) have applications in coding theory, group theory and cryptology [6].

The overview of existing theories related to compactification, projections and decomposition of topological groups and associated topological dynamics are presented below, in brief. The motivation for the formulation of topological decomposition of groups is described.

1.1. Compactification and Projections

Topological groups can be formulated with compactification by employing a set of constructions, which can be categorized as [7,8], (a) maximal ideal spaces of function algebras and, (b) Samuel compactification in uniform group. The examples of natural compactifications of topological groups are: greatest ambit compactification (GAC), Roelcke compactification (RC) and, weakly almost periodic compactification (WAPC). The GAC and RC constructions are based on uniform continuity of function algebras, whereas the WAPC construction considers a compact semigroup structure with a semitopological nature. Moreover, the Pontrjagin and Tannaka dualities exist in locally compact Abelian groups and also in compact groups [8,9]. The compactification of semigroups of a locally compact group can be formulated from the corresponding compactification of a normal subgroup [10]. In general, the topological symmetry group preserves its orientation and this is a result of induced homeomorphism. However, it is often challenging to construct a set of decomposition matrices of any symmetry group where the group is finite [11].

The projections in topological spaces generate projective topological spaces, where the space is generally considered to be compact and Hausdorff. The admissibility of continuous maps as well as their composition in projective spaces are formulated considering that the underlying topological space is compact and Hausdorff [12]. In such spaces, the disconnection is defined based on the generation of open sets from the closure of it (i.e., closure of a set not necessarily forming a closed set).

1.2. Decomposition and Soft Sets

A group can be extended by an Abelian group resulting in the formation of a group of equivalence classes. The condition for such extension is that all the groups should be locally compact and Hausdorff, as well as second countable in nature [13]. In the group decomposition and extension sequence, a homeomorphism is a locally trivial fiber space defined over Abelian extending group [13]. In general, the decompositions of finite groups are performed by incorporating concepts of direct product and semidirect product. The decomposability of a $p$-group is analyzed given the quotient space and the associated indirect product [14]. Given any group and its Abelian normal subgroup, the $R$-decomposability of the base group over its Abelian normal subgroup can be formulated [15]. The $R$-decomposition indicates the possibility of embedding the base group in a semidirect product involving an $R$-module. A generalized polar decomposition of Lie groups can be formulated incorporating involutive automorphisms [16]. The polar decompositions exist in close proximity of group identity and can be extended to larger subsets of the group. Furthermore, the Iwasawa type of decomposition of classical Lie groups was proposed by using QR-decomposition as well as symplectic matrix [17–19]. A complete Jordan decomposition of a group of the multiplicative type was constructed in [20]. In this case, the unipotent elements exist in the group with commutativity.

The concepts of soft sets and soft topological spaces were presented in [21]. The continuity of general topological spaces becomes decomposable if the underlying space is a soft topological space [22]. Furthermore, the $\gamma$-operation and decomposition of soft continuity in soft topological spaces are formulated [23]. It categorizes the open sets into several different classes. The decomposition of continuity in ideal topological spaces was constructed in [24]. The decomposition of ideal topological spaces introduces two different notions of continuity, namely $w$-I-contituinity and $w^*$-I-continuity.
The reducibility properties in hypergroups in crisp sets and its extension to fuzzy cases were presented in [25]. The partitioning of a hypergroup is accomplished based on equivalent classes generated in case of crisp set, and it is further extended to hypergroups incorporating fuzzy cases.

1.3. Motivation

The topological decomposition of an algebraic structure is an interesting topic considering that the underlying space is topological in nature having structural embeddings. It is motivating to investigate a different category of group decomposition without involving direct or semidirect products. Furthermore, the study of projective varieties in such decomposed structures needs attention. The question is: how do we construct the group decompositions from the topological perspectives and are the decompositions embeddable? Moreover, what are the topological relations between multiple decomposed groups in the topological spaces in view of Schoenflies embeddings preserving group homeomorphism? This paper proposes the formulation of the topological decomposition of a group structure and investigates the associated properties. The dynamics of group actions as well as transition behaviours in a topologically decomposed group are presented. The topological decomposition of a group considers the generation of multiple topological components as subspaces within the underlying topological space, while preserving the group’s algebraic structure. This paper introduces the concepts of topological fibers, locality of group operation and partitioned local homeomorphism in a topologically decomposed group in axiomatic forms. The proposed nonprojective construction does not presume the Abelian property of the group to maintain generality. Next, the projective variety of topological decomposition is proposed. The projection in topological space considers formation of disjoint open sets in projective subspace, where the underlying topological space is Hausdorff in nature. The formations of projections are based on two varieties: commutative and noncommutative. The noncommutative variety generates disconnected open sets in topological spaces under projection. Thus, from the topological point of view, the translated components of a decomposed group become separable under projection. The separability of decomposed embedded groups in normal topological spaces and the preservation of group homeomorphism in embedded decomposed groups are analyzed in this paper.

The rest of the paper is organized as follows. Section 2 presents preliminary concepts in the domain. Section 3 presents a set of new definitions in axiomatic forms for nonprojective and projective varieties of decompositions. Section 4 presents analytical properties of topological group decomposition for nonprojective as well as projective categories. Section 5 presents application aspects as well as a detailed comparative analysis between topological decomposition and other general decomposition structures of groups (such as, direct and semidirect products). Finally, Section 6 concludes the paper.

2. Preliminary Concepts

The concept of a group is based on the defined binary operation on a set having a closure property with respect to the operation. On the other hand, the topology of the set constructs a set theoretic structure on the underlying set irrespective of any operation. In this section, the preliminary concepts are presented in brief. In this paper, symbol $A \leq B$ denotes that $A$ is a subgroup of $B$ and $\overline{A}$ denotes the closure of corresponding set $A$. The normal subgroup $A$ of group $B$ is specifically denoted as $A \triangleleft B$ and the complement of any arbitrary set $X$ is denoted as $X^c$.

Let $X$ and $S$ be two point sets and $P(X)$ be power set of $X$, where $X \cap S = \phi$. The subset $I \subset \mathbb{Z}_0^+$ is employed as an index set throughout this paper. Let $X$ be equipped with a binary operation, $*: X^2 \rightarrow X$, such that, $\forall x, y \in X, (x * y) \in X, (y * x) \in X$. Let $G = (X, *)$ be equipped with all the axioms of group structure and, thus, $G = (X, *)$ is called a group [1,4]. A group $G = (X, *)$ can be equipped with a topological structure. The topology on $G = (X, *)$ is denoted by $\tau_G \subseteq P(X)$, where it maintains the axioms of topology. A group action is given by, $\beta: G \times S \rightarrow S$ while maintaining certain properties. The properties of group action are (I) $\beta(g, s) = g * s$ and (II) $\beta(g, \beta(h, s)) = \beta((g * h), s)$. The second property asserts associativity of $G = (X, *)$ in the presence of
The decomposability, locality of binary operation and, homeomorphism in decomposed structure are
the decomposition is performed in one-dimensional (1D) space (i.e., reduced dimension of topological
spaces) for simplicity. This is a basic type of topological decomposition. In the second case, the dimension
of topological spaces is considered. The projection segregates the structure of topologically
decomposed group within the topological spaces.

In this paper, two topological decomposition varieties are proposed. In the first case, the topological
decomposition is constructed without involving any projection within the topological spaces.

3. Decomposition Varieties

The decomposition of a group in the topological spaces can be performed in several ways.
In this paper, two topological decomposition varieties are proposed. In the first case, the topological
decomposition is constructed without involving any projection within the topological spaces.

The decomposition is performed in one-dimensional (1D) space (i.e., reduced dimension of topological
spaces) for simplicity. This is a basic type of topological decomposition. In the second case, the dimension
of topological spaces is enhanced to arbitrary nD and the algebraic projection during decomposition in
the topological spaces is considered. The projection segregates the structure of topologically
decomposed group within the topological spaces.

In this section, a set of definitions is presented in two sections. The first section of definitions is
related to the non-projective variety of topological decomposition. The second section of definitions is
concerned with projective variety of topological decomposition in nD space.

3.1. Non-Projective Decomposition: Definitions

The definitions are formulated based on group structure embedded within topological spaces
denoted by, \( T = (G, \tau_G) \), where \( G = (X, \ast) \) is a group. The embedded group action on \( T \) is given by,
\( \Psi = (T, \beta) \). This section presents a set of existing basic definitions and the newly formulated ones.

The decomposability, locality of binary operation and, homeomorphism in decomposed structure are
presented as a set of axioms.

3.1.1. Topological Space Partition

Let \( (X, \tau_X) \) be a topological space. The space partition \( \Pi_X = \{ A_i \subset X : i \in I \} \) is a family of subspaces
such that, \( \forall A_i, A_j \in \Pi_X, A_i \cap A_j = \emptyset \) and, \( \bigcup_{i \in I} A_i = X \). The partitioned topological spaces may be
employed to obtain a homogeneous set, which is topologically relevant in nature [27].

3.1.2. Group Homeomorphisms

Let \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \) be two groups. The group homeomorphism \( h_H : G_1 \rightarrow G_2 \)
is a mapping under the condition given by, \( \forall x, y \in G_1, h_H(x \ast_1 y) = h_H(x) \ast_2 h_H(y) \). The group
homeomorphisms can be strengthened to form isotropy between group structures [28].

3.1.3. \( G \)-Partition

Let \( D_G \subset \tau_G \) be in \( T \) such that if \( i, k \in I, i \neq k \), then \( D_G = \{ A_i \subset X : A_i \in \tau_G \land (A_i \cap A_k = \emptyset) \} \).
The \( G \)-partition in \( T \) is given by \( D_G \).
3.1.4. G–Decomposition

A G–decomposition in T is denoted by \( \Pi_G \subseteq D_G \) if the following axioms are satisfied considering \( A \in \tau_G, B \in \tau_G \) and \( A \cap B = \phi \),

\[
\begin{align*}
X \setminus \{e\} &= A \cup B, \\
\forall a \in A, \exists b \in B : a \ast b = b \ast a = e \in G, \\
\Pi_G &= \{A, B, X \setminus A \cup B\}
\end{align*}
\]

3.1.5. \( \Pi_G \)-Fiber

Let \( f : A \rightarrow B \) be a bijection in the \( \Pi_G \). The function \( f(.) \) is a \( \Pi_G \)-fiber if the following axiom is satisfied,

\[
\forall a \in A, \exists b \in B : a \ast f(a) = b \ast f^{-1}(b) = e
\]

3.1.6. \( \ast \)-Locality of \( \Pi_G \)

A G–decomposition of \( \Pi_G \) of a group \( G = (X, \ast) \) is called \( \ast \)-local if \( \forall a \in A, \forall b \in B, a \ast b \neq e \) and the following axioms are satisfied,

\[
\begin{align*}
a \ast b &\in f(A) \setminus \{b\}, \\
b \ast a &\in f^{-1}(B) \setminus \{a\}, \\
\forall a, c \in A, [a \ast c, c \ast a] \subseteq A, \\
\forall b, d \in B, [b \ast d, d \ast b] \subseteq B
\end{align*}
\]

3.1.7. \( \Pi_G \)-Homeomorphism

Let \( \Pi_{G1} \) and \( \Pi_{G2} \) be two G–decompositions of two groups \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \), respectively. A bijection \( h : \Pi_{G1} \rightarrow \Pi_{G2} \) is a \( \Pi_G \)-homeomorphism between \( G_1 \) and \( G_2 \) if the following axiomatic properties are satisfied,

\[
\begin{align*}
\{A_1, B_1\} &\subseteq \Pi_{G1}, \{A_2, B_2\} \subseteq \Pi_{G2}, \\
h(A_1) &= f^{-1}(B_2), \\
h(B_1) &= f(A_2), \\
(e_1 \in X_1) &= h^{-1}(e_2 \in X_2)
\end{align*}
\]

3.1.8. Homeomorphic \( \ast \)-Locality

Let \( \Pi_{G1} \) and \( \Pi_{G2} \) be two G–decompositions. The homeomorphic \( \ast \)-locality is given by,

\[
\begin{align*}
Y &\subseteq h(X_1 \setminus \{e_1\}), \\
\forall y_1, y_2 \in Y, [y_1 \ast y_2, y_2 \ast y_1] \subseteq Y
\end{align*}
\]

In the above definition, the group operation \( \ast \) is assumed to be in the codomain of \( h(.) \).

3.1.9. Schoenflies Homeomorphic Embeddings

Let \( (X, \tau_X) \) and \( (Y, \tau_Y) \) be two topological spaces and \( F : (A \subset X) \rightarrow Y \) be an embedding. There exists a homeomorphism \( h_S : X \rightarrow Y \), such that the restriction maintains, \( h_S|_{A \subset X} = F \).

This is a generalized version of Schoenflies embeddings theorem, which is applied in this paper. In specific cases, the Schoenflies homeomorphism may generate separable components in topological spaces if \( A \) is homeomorphic to \( S^1 \) [29].
3.1.10. Decomposed Group Embedding

Let \((X, \tau_X)\) be a normal topological space and \(E \in \tau_X\) be a subspace. If \(G = (Y, \ast)\) is a group, then the decomposed group embedding in \(E\) is given by,

\[
\begin{align*}
i, k &\in \mathbb{Z}^+, i, k \in [1, 3], \\
\exists H_i &\subset E, D \in \Pi_G, \\
e_i &: D \rightarrow H_i, \\
[i \neq k] &\Rightarrow [H_i \cap H_k = \phi]
\end{align*}
\]

(6)

Note that embedding \(e_i : D \rightarrow H_i\) is injective in nature within normal topological subspace. It further concludes that \( \bigcup_{i \in [1, 3]} H_i \subset E\) in the topological space \((X, \tau_X)\). The decomposed group embedding is denoted by \((E, G)\).

3.2. Topological Decomposition with Projection: Definitions

In this section, a set of definitions is presented considering \(n\)-dimensional space with embedded topologically decomposed groups. The multidimensional topological decomposition of groups in a more general way allows the inclusion of the concept of projection and its different forms. Let \(X^n\) be an \(n\)-dimensional Hausdorff space equipped with the associative \(n\)-ary closed group operation \(\ast_n : X^{2n} \rightarrow X^n\) characterized by

\[
\forall (x_1)_i^{n}, (y_1)_i^{n} \in X^n, \\
(x_1)_i^{n} \ast_n (y_1)_i^{n} = \left( (x_i \ast y_i)_{i=1}^{n} \in X^n \right), \\
\left( (x_1)_1^{n} \ast_n (y_1)_1^{n} \right)^{n} \ast_n (z_1)_1^{n} = \left( (z_1)_1^{n} \ast_n \left( (y_1)_1^{n} \ast_n (z_1)_1^{n} \right) \right)
\]

(7)

Note that the standard closed group operation in this case is \(\ast : X^2 \rightarrow X\) (component by component) and \((x_1)_i^{n}\) represents a point in \(X^n\) Hausdorff space. Furthermore, it is considered by following the standard axioms of topology that \(\tau_x \subseteq P(X^n)\), where \(P(.)\) is a power set (i.e., decomposition and projection would consider a product topological space). First, a set of definitions are presented in this setting before proceeding to the construction of relevant theorems representing the salient properties.

3.2.1. Gravity of Decomposition

The topological subspace \(A \in \tau_x\) is called gravity of topological decomposition if \(\forall (x_1)_i^{n}, (y_1)_i^{n} \in A, (x_1)_i^{n} \ast_n (y_1)_i^{n} \in A\).

3.2.2. Monotone Class of Decomposition

A monotone class \([B, A \subset B] \subset \tau_x\) is defined as topologically decomposed if the following axioms are satisfied by \(\ast_n : B^n \rightarrow X^n\):

\[
\forall (a_1)_i^{n} \in A, \forall (b_1)_i^{n} \in B|A, (a_1)_i^{n} \ast_n (b_1)_i^{n} \in B|A, \\
\forall (a_1)_i^{n}, (b_1)_i^{n} \in B,A, (a_1)_i^{n} \ast_n (b_1)_i^{n} \in B\bar{c}
\]

(8)

3.2.3. Projection of Decomposition

The decomposed space has algebraic projection if it maintains a set of axioms as mentioned below:

\[
\forall D \in \tau_x : D \subset B^c, \\
\forall (a_1)_i^{n}, (b_1)_i^{n} \in B|A, (a_1)_i^{n} \ast_n (b_1)_i^{n} = \left( (b_1)_i^{n} \ast_n (a_1)_i^{n} \right), \\
(a_1)_i^{n} \ast_n (b_1)_i^{n} \in D
\]

(9)

This above definition portrays that the projection is commutative in nature, preserving symmetry. The next definition presents the noncommutative extension of algebraic projection.
3.2.4. Noncommutative Projection

The associative space has noncommutative projection of decomposition if the following axioms are satisfied considering \( \exists \{ F_1, F_2 \} \subset \mathbb{B}^c, \)

\[
\begin{align*}
F_1 \cup F_2 & \subset B^c, \\
\left\{(x_i \ast y_i)_{i=1}^n \in F_1 \right\} & \Leftrightarrow \left\{(y_i \ast x_i)_{i=1}^n \in F_2 \right\}
\end{align*}
\]  \hspace{1cm} (10)

One can view the noncommutative projection of decomposition as a generalization of projection along with additional requirements. The finiteness of associative space and projection under translation is given in the next definition.

It is important to note that the above mentioned two categories of projections (commutative and noncommutative) may not simultaneously exist in a topologically decomposed group. The reason for this is that they are structurally non-equivalent and are fundamentally related to the Abelian-ness of the original group.

3.2.5. Finiteness of Translated Projection

The projection of decomposition in associative space under functional translation \( f : X^n \to R \) is strictly finite if

\[
\forall (x_i)_{i=1}^n, (y_i)_{i=1}^n \in X^n, \\
\left\{ f\left(\langle x_i \ast y_i \rangle_{i=1}^n \right) \right\} \subset R\setminus \{-\infty, +\infty\}
\]  \hspace{1cm} (11)

Evidently, the finite translation of projection of decomposition does not consider commutativity. There can be additional properties related to finite translation depending on the type of translation of decomposed projection in the associative space. The inflationary nature of bounded translation is given in the next definition.

3.2.6. Inflationary Bounded Translation

A finite noncommutative projection is called inflationary if the following axioms are satisfied.

\[
\forall (x_i)_{i=1}^n \in F_1, \forall (y_i)_{i=1}^n \in F_2, \\
f\left(\langle x_i \rangle_{i=1}^n \right) < f\left(\langle y_i \rangle_{i=1}^n \right)
\]  \hspace{1cm} (12)

The above definition induces the notion of direction or position with respect to algebraic operation within the associative space in a much weaker sense. From the topological point of view, if \((X^n, \tau_x)\) and \((Y^n, \tau_y)\) are two commutative topological spaces having decomposed embedded groups \(G_x = (X^n, \ast^n_x)\) and \(G_y = (Y^n, \ast^n_y)\) respectively, then \(\{B_X, A_X \subset B_X, D_X\} \subset \tau_x\) and, \(\{B_Y, A_Y \subset B_Y, D_Y\} \subset \tau_y\). One can notice that the structure of generalized topological decomposition of groups in the multidimensional space is different as compared to the decomposition in one dimensional space. In case of one-dimensional decomposition, the concepts of monotone class and associated projections are excluded, whereas this is not the case for multidimensional decomposition.

4. Analytical Properties of Decomposition Varieties

The analytical properties of topological decompositions of groups are presented in two sections respective to two decomposition varieties. In the first section, the properties of non-projective variety in reduced dimension are presented. The second section presents the projective variety of decomposition in higher dimensional spaces.
4.1. Properties of Non-Projective Variety

The topological decomposition of multiple groups can preserve the shared identity, if such an element exists. We show that the identity map preserves the invariance of the common identity of groups, even when their union is not a group.

**Theorem 1.** If \( \Pi_G \cap \Pi_G' \neq \emptyset \) then \( \{id(\Pi_G \cap \Pi_G')\} \subset (\Pi_G \cup \Pi_G') \).

**Proof.** Let \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \) be two groups such that, \( X_1 \neq X_2 \). If \( \Pi_G \cap \Pi_G' \neq \emptyset \), then \( e_1 = e_2 \) because \( \forall A \subset X_1 \setminus \{e_1\}, \forall B \subset X_2 \setminus \{e_2\}, A \cap B = \emptyset \). If \( h : \Pi_G \rightarrow \Pi_G' \) is a \( \Pi_G\)–homeomorphism, then the only invariant element is determined by, \( h(\Pi_G \cap \Pi_G') = \{id(e)\} \), where \( e_1 = e_2 = e \). Hence, \( \{id(\Pi_G \cap \Pi_G')\} \subset (\Pi_G \cup \Pi_G') \) in \( G_1, G_2 \). \( \square \)

**Remark 1.** A numerical example is presented here to illustrate the basic concept of preservation of the identity element. Let \( G_1 = (Z^2, \ast_1) \) and \( G_2 = (C, \ast_2) \) be two groups, where \( Z, C \) represent sets of integers and complex numbers, respectively. It is clear that \( Z^2 \neq C \). If one considers that \( \ast_1 = \ast_2 = + \) such that, \( (a,b) + (c,d) = (a+c, (b+d)) \), then in both the cases, \( e_1 = e_2 = (0,0) \). This is a specific form of numerical construction in comparison to the more generalized abstract algebraic construction above.

However, in a subgroup of the topologically decomposed group, the locality of original group operation can be maintained depending upon its original structure. This property is presented in the next theorem.

**Theorem 2.** If \( H \triangleleft G \) is a subgroup of a group \( G \) having \( \ast \)-locality, then the induced decomposition in subgroup \( \Pi_H \) is also \( \ast \)-local.

**Proof.** Let \( G = (X, \ast) \) be a group and, \( H \triangleleft G \). Let \( \Pi_G \) be \( \ast \)-local and \( \Pi_G = \{A \subset X, B \subset X, [e]\} \). As \( H \triangleleft G \), thus \( \Pi_H = \{A_H \subset A, B_H \subset B, [e]\} \). However, \( \forall a \in A, \forall b \in B, a \ast b \in B \) if \( a \ast b \neq e \). So, \( \exists a \in A_H, \) such that \( a \ast b \in B \setminus B_H \). Hence, this is a contradiction, because \( H \triangleleft G \). Hence, \( a \in A_H \) in \( H \triangleleft G \) indicates \( a \ast b \in B_H \). On the other hand, if \( b \in B_H \) and \( H \triangleleft G \), then \( b \ast a \in A_H \). Moreover, \( \forall a_1, a_2 \in A_H, \{(a_1 \ast a_2), (a_2 \ast a_1)\} \subset A_H \) because \( H \triangleleft G \). Similarly, in the decomposed set \( \forall b_1, b_2 \in B_H, \{(b_1 \ast b_2), (b_2 \ast b_1)\} \subset B_H \). Hence, \( \Pi_H = \ast \)-local. \( \square \)

This indicates that the locality of group operation in topologically decomposed group is preserved so that a subgroup maintains locality of group operation if the corresponding locality was present in the original group structure. However, one can construct a surjective map between multiple groups. The surjective mapping between two groups helps in maintaining homeomorphisms in two topologically decomposed groups as presented below.

**Theorem 3.** If \( g : X_1 \rightarrow X_2 \) is a surjection, then \( g(.) \) is a \( \Pi_G \)-homeomorphism if \( (g \circ f) : (A_1 \in \Pi_G) \rightarrow (A_2 \in \Pi_G) \) is also a symmetric surjection.

**Proof.** Let \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \) be two groups, where \( g : X_1 \rightarrow X_2 \) is a surjection, such that \( g(e_1) = e_2 \). Let \( \exists \{a_{11}, a_{12}, a_{13}\} \subset X_1 \) and \( A_1 \in \Pi_G, A_2 \in \Pi_G, \) such that \( \{a_{11}, a_{12}\} \subset A_1 \) and, \( g(a_{11}) = g(a_{12}) = (a_{21} \in A_2) \). Consider \( a_{13} \in A_1 \) and, \( a_{11} \ast_1 a_{12} = a_{13} \) in \( G_1 = (X_1, \ast_1) \). If \( g(.) \) is \( \Pi_G\)-homeomorphic, then \( g(A_1) = f^{-1}(B_2) \), where \( B_2 \in \Pi_G \). It follows that \( g(a_{13}) \in A_2 \) and \( \exists A_2 \in A_2 \), such that \( a_{21} \ast_2 a_{22} = g(a_{13}) \) in \( G_2 = (X_2, \ast_2) \). This indicates that \( \exists A_1 \in A_1 \), such that \( g(a_{14}) \in A_2 \) and \( \{a_{21}, g(a_{13}), g(a_{14})\} \in A_2 \). However, in \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \), \( f(A_1) \in \Pi_G \) and \( A_2 = f^{-1}(B_2) \). Moreover, if \( g(.) \) is \( \Pi_G\)-homeomorphism, then by symmetry, \( (g \circ f)A_1 = f(A_2) \) is also a surjection. Hence, \( (g \circ f) : (A_1 \in \Pi_G) \rightarrow (A_2 \in \Pi_G) \) is a surjection. \( \square \)
It follows that a surjective map can be constructed between two topologically decomposed groups, where the function composition between surjection and corresponding topological fiber maintains symmetric transitions between two decomposed groups.

On the other hand, the structure of the kernel of group homeomorphism may get reshaped due to topological decomposition. This is not the case only if the kernel is trivial in nature. This property is illustrated in next theorem.

**Theorem 4.** If \( h : \Pi_{G_1} \to \Pi_{G_2} \) is a \( \Pi_G \)-homeomorphism between \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \), then the trivial kernel is \( \ker(h^{-1}(X_2)) = X_k \ast_1 f(X_k) \), where \( X_k \subseteq A_1 \).

**Proof.** Let \( h : \Pi_{G_1} \to \Pi_{G_2} \) be a \( \Pi_G \)-homeomorphism between \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \), where \( X_k \subseteq (A_1 \in \Pi_{G_1}) \). However, \( \forall x \in X_k, (x \ast_1 f(x)) = (f(x) \ast_1 x) = e_1 \). Furthermore, \( h(x \ast_1 f(x)) = e_2 \) in \( G_2 = (X_2, \ast_2) \). Hence, the trivial kernel in \( G_2 = (X_2, \ast_2) \) under bijective \( h(.) \) is given by \( \ker(h^{-1}(X_2)) = X_k \ast_1 f(X_k) \). \( \Box \)

This indicates that, if the kernel of a group homeomorphism is trivial, then a transition map between kernel and another topologically decomposed group can be formed through topological fiber considering appropriate subsets generating the trivial kernel.

Moreover, the recovery of the kernel in a decomposed group can be performed through topological mapping in decomposed sets. This is presented in following lemma.

**Lemma 1.** If \( h_H : G_1 \to G_2 \) is a group homeomorphism, then \( \ker(h_H) \subset h^{-1}(X_2) \).

**Proof.** Let \( h_H : G_1 \to G_2 \) be a group homeomorphism. It follows that, \( \ker(h^{-1}(X_2)) \subset \ker(h_H) \). Moreover, \( \exists E \subset (A_1 \cup B_1) \) such that, \( \ker(h_H) = E \) in \( G_1 = (X_1, \ast_1) \), where \( E \subset h^{-1}(X_2) \). As, \( \ker(h^{-1}(X_2)) \subset h^{-1}(X_2) \), hence \( \ker(h_H) \subset h^{-1}(X_2) \). \( \Box \)

Thus, the kernel of a group homeomorphism can be represented through the \( \Pi_G \)-homeomorphism between two topologically decomposed groups. However, the complexity increases if multiple groups are involved. The maintenance of locality of group operations in two topologically decomposed groups allow the recovery of kernels of group homeomorphisms, as presented in next theorem.

**Theorem 5.** If \( h : \Pi_{G_1} \to \Pi_{G_2} \) is a bijective \( \Pi_G \)-homeomorphism, then \( \ker(h(X_a)) = \ker(h^{-1}(Y_a)) \), where \( X_a \subset (A_1 \cup B_1) \) and \( Y_a \) is a \( \ast \)-locality in \( G_2 = (X_2, \ast_2) \).

**Proof.** Let \( h : \Pi_{G_1} \to \Pi_{G_2} \) be a bijective \( \Pi_G \)-homeomorphism and \( X_a \subset (A_1 \cup B_1) \) in \( G_1 = (X_1, \ast_1) \). If \( Y_a \subset X_2 \) of \( G_2 = (X_2, \ast_2) \), such that \( Y_a = h(X_a) \) and, \( Y_a = A_{2a} \cup B_{2a} \), then \( \exists A_{1a} \subset X_1, \exists B_{1a} \subset X_1 \) such that, \( A_{1a} = h^{-1}(A_{2a}) \) and \( B_{1a} = h^{-1}(B_{2a}) \). However, \( Y_a \) is a \( \ast \)-locality in \( G_2 = (X_2, \ast_2) \) and non-trivial kernel is \( \ker(h^{-1}(Y_a)) = A_{ak} \cup B_{ak} \), where \( A_{ak} \subset A_{1a} \) and, \( B_{ak} \subset B_{1a} \) such that, \( h(A_{ak} \ast_1 B_{ak}) = e_2 \). Moreover, as \( h(.) \) is homeomorphic and invertible in topologically decomposed groups, so \( h(A_{ak} \ast_2 B_{ak}) = h(A_{ak} \ast_1 B_{ak}) \). Hence, \( \ker(h(X_a)) = \ker(h^{-1}(Y_a)) \) in \( G_1 = (X_1, \ast_1) \) and \( G_2 = (X_2, \ast_2) \). \( \Box \)

Hence, the determination of a kernel of a group homeomorphism can be done symmetrically (both ways) between two topologically decomposed groups if \( \Pi_G \)-homeomorphism is a bijection and the group operations maintain locality in decomposed groups. The locality of group operation in a topologically decomposed group affects the dynamics of group actions. This property is illustrated in the next theorem.

**Theorem 6.** If \( S \) is arbitrary set and \( G = (X, \ast) \) has \( \ast \)-locality in \( \Pi_G \) and \( \beta : G \times S \to S \) is a group action, then \( \beta(A, S) \cap \beta(B, S) = \phi \), where \( \{A, B\} \subset \Pi_G \).
**Theorem 7.** In $\Pi_G$ of group $G = (X, *)$, the group action $\beta(a, \beta(f(a), s))$ is stable.

**Proof.** Let $G = (X, *)$ be a group and a group action is $\beta : G \times S \to S$ on arbitrary set $S$ generating orbit of $s \in S$ as $O_s \subset S$. Let $\{A, B\} \subset \Pi_G$ and $\leftrightarrow$-locality exists in $\Pi_G$. If $a_1, a_2 \in A$ such that $\beta(c, s \in S) = \beta(a_1, \beta(a_2, s))$, then $c = a_1 * a_2$. Moreover, $c \in A$, because $\leftrightarrow$-locality exists in $\Pi_G$. Similarly, $\beta(d, s \in S) = \beta(b_1 \in B, \beta(b_2 \in B, s))$ indicates that $d \in B \in \Pi_G$. Furthermore, as $\forall s, t \in S, O_s \cap O_t = \phi$ and $A \cap B = \phi$, hence $\beta(A, S) \cap \beta(B, S) = \phi$. $\Box$

This is an extension of the property of group action generating partitions in the context of the topologically decomposed group. It follows that the equivalence relation and resulting set partitioning property is preserved in group action involving the topologically decomposed group. If a group is topologically decomposed, then the stability of corresponding group action needs attention. The main question is: does the decomposition affect stability or is it invariant in any case? The property of stability of group action in a topologically decomposed group is analyzed in the following theorem.

**Theorem 2.** In $\Pi_G$ of group $G = (X, *)$, the group action $\beta(b_2, \beta(f^{-1}(b_2), s))$ is stable, where $b_2 \in B \in \Pi_G$.

**Proof.** Let $G = (X, *)$ be a group. In $A \in \Pi_G$ of the group, $\forall a \in A, a * f(a) = e \in X$. However, according to the property of group action, $\beta(a, \beta(f(a), s)) = e * s = s \in S$. Hence, the group action $\beta(a, \beta(b, s))$ is always stable if $b = f(a)$. $\Box$

This indicates that the stability of group action is maintained in the topologically decomposed group. One reason for this is that the standard property of group action, such as forming equivalence relation, is preserved by topologically decomposed group.

The next lemma presents construction of stable group action in topologically decomposed group.

**Lemma 2.** In $\Pi_G$ of group $G = (X, *)$, the group action $\beta(b_2, \beta(f^{-1}(b_2), s))$ is stable, where $b_2 \in B \in \Pi_G$.

**Proof.** Let $\{A, B\} \subset \Pi_G$ be in $G = (X, *)$ having $\leftrightarrow$-locality. Let $\exists a_1 \in A, \exists b_1 \in b$ be such that, $a_1 * b_1 = b_2 \in B$ due to $\leftrightarrow$-locality. However, $\forall b \in B, \exists a \in A$ such that $a = f^{-1}(b)$. Thus, the group action $\beta(b_2, \beta(f^{-1}(b_2), s))$ results in $(b_2 * a_2) * s$. Moreover, in the topologically decomposed group, $b_2 * a_2 = e = a_2 * b_2$. Hence, it is stable. $\Box$

This extends the stability property to the topological fiber in the decomposed group. The group action in a topologically decomposed group maintains stability in the presence of transitions through topological fiber. The symmetric or identical group actions can be found in two topologically decomposed groups under $\Pi_G$-homeomorphism. This property is presented in the following theorem.

**Theorem 8.** If $h : \Pi_G \to \Pi_G$ is a $\Pi_G$-homeomorphism in $G_1 = (X_1, *)$ and $G_2 = (X_2, *)$, then $\beta_{G_1} : G_1 \times S \to S$ and $\beta_{G_2} : G_2 \times S \to S$ are identical if $\forall g_1 \in X_1, \forall g_2 \in X_2, \beta_{G_1}(g_1, s) \cap \beta_{G_2}(h(g_1), s) \neq \phi$.

**Proof.** Let two groups be denoted as, $G_1 = (X_1, *)$ and $G_2 = (X_2, *)$. Let $\beta_{G_1} : G_1 \times S \to S$ and $\beta_{G_2} : G_2 \times S \to S$ be two group actions, such that $\beta_{G_1}(X_1 \times S) \cap \beta_{G_2}(X_2 \times S) = \phi$. If $h : \Pi_G \to \Pi_G$ is a $\Pi_G$-homeomorphism, then $\beta_{G_1}(h(A_1) \times S) \subset \beta_{G_2}(X_2 \times S)$. This indicates that $\beta_{G_2}(h(A_2) \times S) \cap \beta_{G_2}(h(h(A_1) \times S) = \phi$. According to symmetry of $\Pi_G$, $\beta_{G_2}(B_2 \times S) \cap \beta_{G_2}(h(B_1) \times S) \neq \phi$. Moreover, $\beta_{G_1}(h^{-1}(e_2)) \times S = \beta_{G_2}(h^{-1}(e_2)) \times S$ in $\Pi_G$. Hence, $\forall g_1 \in X_1, \forall g_2 \in X_2, \beta_{G_1}(g_1, s) \cap \beta_{G_2}(h(g_1), s) \neq \phi$. $\Box$

This is another way of relooking into the property of group action in the presence of two different topologically decomposed groups. It asserts that, in any case, the equivalence relation forming partitions in a set under group action is maintained, even if the corresponding group action involves more than one topologically decomposed group. However, the natural question arises about the interplay between existence of any fixed-point and commutativity of the group, if any.
The following theorem presents such interplay and property between two concepts.

**Theorem 9.** In \( G = (X, * ) \), if the group action in \( \Pi _G \setminus \{ e \} \) has no fixed-point then \( G = (X, * ) \) is non-Abelian and group action is asymmetric.

**Proof.** In any group \( G = (X, * ) \), if \( \beta : G \times S \rightarrow S \) is a group action, then \( \exists \{ s_l, s_r \} \subset S \), such that \( s_l = \beta(a, \beta(b, s)) \) and \( s_r = \beta(b, \beta(a, s)) \). Let \( O_G \triangleleft G \) be an orbit stabilizer, where \( O_G = \{ e \} \). However, in \( \Pi _G \setminus \{ e \} \) of topologically decomposed \( G = (X, * ) \), due to *-locality \( a \ast b \in B \) and \( b \ast a \in A \). Moreover, \( \forall a \in A \cup B, \beta(a \ast s) \neq s \) and \( s_r = s_l = s \) if \( b = f(a) \). But \( A \cap B = \emptyset \) in \( \Pi _G \) and thus, \( s_l \neq s_r \neq s \) if \( b \neq f(a) \). This indicates that \( (a \in A) \ast (b \in B) \neq (b \in B) \ast (a \in A) \), where \( b \neq f(a) \). Hence, \( G = (X, * ) \) is non-Abelian and group action is asymmetric. \( \square \)

This property exposes the interplay between a stationary point and group action in the presence of a topologically decomposed group. Due to the locality of group operation, a decomposed group is considered to be non-Abelian. Hence, the stationary or fixed point may not exist under asymmetric group action in topologically decomposed group. The exception is the identity element.

The property of minimal form of orbit stabilizer in a topologically decomposed group under group action is presented in the following theorem.

**Theorem 10.** In \( G = (X, * ) \), if group action in \( \Pi _G \) has orbit stabilizer \( O_G = \{ e \} \in \Pi _G \), then \( \forall s \in S, O_s = E_01 \cup E_02 \), where \( E_01 \cap E_02 = \emptyset \).

**Proof.** Let \( G = (X, * ) \) be a non-Abelian group and \( O_G \triangleleft G \) be the orbit stabilizer for \( \beta : G \times S \rightarrow S \), such that \( O_G = \Pi _G \setminus \{ A, B \} \). As \( A \cap B = \emptyset \) and \( a \ast b \neq b \ast a \), hence \( \beta(a, (b, s)) \neq \beta(b, (a, s)) \), where \( a \in A, b \in B \) and \( b \neq f(a) \). Moreover, \( \forall s \in S, O_s = \{ \beta(a, s) : a \in X \} \) and, \( \forall a, b \in A \cup B, (a \ast s) \neq (b \ast s) \neq s \), where \( O_s \subset S \) and \( a \neq b \). Let, be \( E_01 = \beta(A \times S) \) and \( V = \beta(B \times S) \), where \( \{ A, B \} \in \Pi _G \). Hence, \( O_s = E_01 \cup E_02 \), where \( E_02 = V \cup \{ s \in S \} \) and \( E_01 \cap E_02 = \emptyset \). \( \square \)

This is another way of looking into the formation of set partitioning in the presence of a topologically decomposed group. This indicates that if the identity element is the only orbit stabilizer, then a set would generate more than one partition through group action in the presence of a topologically decomposed group. This property is already present in cases of group action and it is preserved even in the topologically decomposed group.

The next theorem illustrates that the space of embedded decomposed group is separable in the normal topological space.

**Theorem 11.** The space of decomposed group embedding \( (E, G) \) is separable, where \( E \in \tau _X \) and \( G = (Y, * ) \) is a group.

**Proof.** Let \( (X, \tau _X) \) be a normal topological space and \( E \in \tau _X \) is also normal and open. Let \( H_i \subset E \) be such that \( 1 \leq i \leq 3 \) and \( i \in \mathbb{Z}^+ \), where \( \bigcup _{i \in [1, 3]} H_i \subset E \). If \( G = (Y, * ) \) is a group such that \( D \in \Pi _G \) and, \( e_i : D \rightarrow H_i \) then the corresponding decomposed group embedding will be given by \( e_i(D) \subset E \). However, in the decomposed group \( \Pi _G, A \cap B = \emptyset \) and \( e_i(A) \subset H_i, e_k(B) \subset H_k \subset E \), such that \( k \in \mathbb{Z}^+, k \in [1, 3] \) and, \( i \neq k \). Thus, \( e_i(A) \cap e_k(B) = \emptyset \) in \( E \in \tau _X \) and, \( (e_i(A) \cup e_k(B)) \subset E \in (X, \tau _X) \). Similarly, \( \exists j \in \mathbb{Z}^+, j \in [1, 3] \), such that \( i \neq j \neq k \) and \( e_j(Y \cup \overline{A \cup B}) \subset H_j \subset E \), where \( \bigcup _{j \in [1, 3]} H_j \subset E \) in the normal topological space.

Furthermore, as \( H_i \cap H_k = \emptyset \) if \( i \neq k \) and \( (X, \tau _X) \) is a normal topological space, so \( \overline{H_i} \cap \overline{H_k} = \emptyset \), where \( \overline{H_i}, \overline{H_k} \subset E \). Thus, if \( \bigcup _{i \in [1, 3]} \overline{H_i} = \overline{E} \), then \( (E, G) \) consists of countable dense sets. Hence, \( (E, G) \) is separable in normal \( (X, \tau _X) \) containing an embedded decomposed group. \( \square \)
It is well known that Schoenflies homeomorphism is defined for topological spaces with embeddings. The interesting question is whether such homeomorphism preserves group homeomorphism between two embeddings of decomposed groups into spaces. The answer to this question is presented in the next theorem.

**Theorem 12.** Let \((E_1, G_1)\) and \((E_2, G_2)\) be two disjoint decomposed group embeddings in normal topological space \((X, \tau_X)\). If \(h_I : G_1 \rightarrow G_2\) is a group homeomorphism, then there is a Schoenflies homeomorphism \(h_S : E_1 \rightarrow E_2\) preserving group homeomorphism in decomposed embeddings.

**Proof.** Let \((X, \tau_X)\) be a normal topological space and \((E_1, G_1), (E_2, G_2)\) be two disjoint decomposed group embeddings for \(G_1 = (X_1, \tau_1)\) and \(G_2 = (X_2, \tau_2)\). Let \(h_I : G_1 \rightarrow G_2\) be a group homeomorphism. Thus, \(\forall x, y \in A_1 \in \Pi G_1\) if \(x \tau_1 y \in A_1\) then \(h_I(x) \tau_2 h_I(y) \in A_2\), where \(h_I(x) \tau_1(y) \in A_2\). Again, if \(e_1 : (D_1 \in \Pi G_1) \rightarrow (H_1 \subset E_1)\) and \(e_2 : (D_2 \in \Pi G_2) \rightarrow (H_2 \subset E_2)\) are two decomposed group embeddings, then \(e_1(x \tau_1 y) \in H_1\) and \(e_2(h_I(x) \tau_2 h_I(y)) = (e_2 \circ h_I)(x \tau_1 y)\), where \(e_2(h_I(x) \tau_2 h_I(y)) \in H_2\). Thus, if \(h_S : E_1 \rightarrow E_2\) is a bijection in \((X, \tau_X)\), then \((h_S \circ e_1)(x \tau_1 y) \in H_2\). Moreover, \(h_S(e_1(x \tau_1 y) \tau_1 e_1(y)) = e_2(h_I(x) \tau_2 h_I(y))\) in \(E_1 \cup E_2\) is maintained, preserving group homeomorphism in decomposed group embeddings. □

Hence, the Schoenflies homeomorphism preserves group homeomorphism in decomposed group embeddings in disjoint normal topological spaces.

### 4.2. Properties of Projective Variety of Decomposition

This section presents the analytical properties of projective variety of topologically decomposed groups. It is considered that underlying topological space is Hausdorff in nature. The maintenance of subgroup structure in the algebraic projection is formulated and the effect of finite translation on the topological decomposition is investigated. The first interesting property is the noncompactness of gravity subspace of decomposition, where the gravity space embeds a subgroup structure.

**Theorem 13.** The gravity subspace of topological decomposition \(A \in \tau_x\) is a noncompact subgroup \(H = (A, \ast_n)\) in \((X^n, \tau_{x^n})\) if \(\forall (x_i)_{i=1}^{n} \in A, (x_i^{-1})_{i=1}^{n} \in A\).

**Proof.** Let \((X^n, \tau_x)\) be a topological space embedding a topologically decomposed group and, \(A \in \tau_x\). Let \(\ast_n : X^{2n} \rightarrow X^n\) be an \(n\)-dimensional group operation, such that \((X^n, \ast_n)\) is a group. However, as \(A \subset X^n\) is a gravity space so \(\forall (x_i)_{i=1}^{n}, (y_i)_{i=1}^{n} \in A, (x_i)_{i=1}^{n} \ast_n (y_i)_{i=1}^{n} \in A\). Thus, if \(\forall (x_i)_{i=1}^{n} \in A, 3(x_i^{-1})_{i=1}^{n} \in A\) then, \((x_i)_{i=1}^{n} \ast_n (x_i^{-1})_{i=1}^{n} = (e_i)_{i=1}^{n}\) and, \(\forall i \in [1, n], e_i = e_0\). Moreover, as \((X^n, \ast_n)\) is a group, hence it is true that, \(\forall (x_i)_{i=1}^{n}, (y_i)_{i=1}^{n}, (z_i)_{i=1}^{n} \in A, (x_i)_{i=1}^{n} \ast_n (y_i)_{i=1}^{n} \ast_n (z_i)_{i=1}^{n} = (x_i)_{i=1}^{n} (y_i)_{i=1}^{n} (z_i)_{i=1}^{n}\). However, as \(A \in \tau_x\) so \(\partial A \subset X^n \setminus A\) indicating \(H = (A, \ast_n)\) is a noncompact subgroup. □

The structural property of the gravity subspace of a topologically decomposed group prohibits the inclusion of a boundary. It appears that the gravity subspace needs to be open to be an element of topology and also to maintain the subgroup structure. This leads to the assertion of condition of noncompactness of gravity subspace, as given in next theorem.

**Theorem 14.** In \((X^n, \tau_x)\) if gravity subspace is given by \(A \in \tau_x\), then \((\overline{A}, \ast_n)\) is not a subgroup.

**Proof.** Let \((X^n, \tau_x)\) be a topological space embedding a topologically decomposed group \(G = (X^n, \ast_n)\) and, \(A \in \tau_x\) be a gravity subspace in the topologically decomposed group. Let \(\exists B \subset X^n\) be such that \(A \subset B\) and, \(B \in \tau_x\). However, as \(\partial A \subset A^c\), thus it indicates that \(\forall (x_i)_{i=1}^{n} \in \partial A \Rightarrow (x_i)_{i=1}^{n} \in A^c\) and,
if \( \{x_i^m\}_{i=1}^n, \{y_i^m\}_{i=1}^n \subseteq \partial A \), then \( \{(x_i^m)^n\}_{i=1}^n \ast_n \{y_i^m\}_{i=1}^n \in D \), where \( D \in \tau_A \) and \( D \subset B^c \). Hence, \((\overline{A}, \ast_n)\) is not closed under algebraic operation and it is not a subgroup. \( \square \)

Thus, the closure of gravity subspace of a topologically decomposed group is no longer a subgroup. However, the properly chosen topological elements of a decomposed group can form a noncompact subgroup. This property is explained in the next theorem.

**Theorem 15.** If \( [B, D] \subset \tau_A \), such that \( D \subset B^c \) and \( A \subset B \), then \( B \cup D \in \tau_A \) is a noncompact subgroup if \( (D \ast_n D) \subset (B \cup D) \).

**Proof.** Let \((X^n, \tau_x)\) be a topological space containing a topologically decomposed group \( G = (X^n, \ast_n) \). It is already shown that if \( A \subset X^n \) is a gravity subspace then \((A, \tau_n)\) is a noncompact subgroup. However, if \( \{B, D\} \in \tau_A \), such that \( A \subset B \) and \( D \subset B^c \), then \( \forall \{x_i^n\}_{i=1}^n, \{y_i^n\}_{i=1}^n \in B \setminus A \) the corresponding algebraic operation would lead to, \( \{(x_i^n)^n \ast_n (y_i^n)^n\}_{i=1}^n \in D \). Note that \((B, \tau_n)\) and \((D, \ast_n)\) are not subgroups in the topological space. Thus, the group operation \( \ast_n \) is closed in \( B \cup D \) if \( (D \ast_n D) \subset (B \cup D) \). Moreover, if \( \forall \{x_i^n\}_{i=1}^n \in (B \cup D), \exists \{x_i^{-1^n}\}_{i=1}^n \in (B \cup D), \) then \( \{(x_i^n)^n \ast_n (x_i^{-1^n})_{i=1}^n\}_{i=1}^n \in (A \subset B) \). Hence, \((B \cup D, \ast_n)\) is a noncompact subgroup if \((D \ast_n D) \subset (B \cup D)\), where \((B \cup D) \in \tau_A \). \( \square \)

Thus, appropriately selected topological elements from the topologically decomposed groups can form a subgroup under a certain condition. The condition is that an element and its inverse should be within the topological subspaces, such that their union can prepare a subgroup. The algebraic projection of such subgroups maintains the integrity of subgroup structure as illustrated in the corollary given below.

**Corollary 1.** In noncommutative projective space \((X^n, \tau_x)\), the \((E \in \tau_x, \ast_n)\) is a noncommutative and noncompact subgroup, where \( E = B \cup F_1 \cup F_2 \) if \( (E \ast_n E) \subset E \) and \( E \subset X^n \).

**Proof.** Let \((X^n, \tau_x)\) be a noncommutative projective topological space with a topologically decomposed group \((X^n, \ast_n)\), such that \( \{B, F_1, F_2\} \subset \tau_x \). Thus, if \( E = B \cup F_1 \cup F_2 \), then \( E \in \tau_x \) is open. However, in \((X^n, \ast_n)\) the algebraic operation maintains \((B \ast_n F_1) \ast_n F_2 = B \ast_n (F_1 \ast_n F_2)\) and, in \((X^n, \tau_x)\) topological space, \( F_1 \cap F_2 = \phi \). If \( H \in E \), such that \((F_1 \ast_n F_2) \cup (F_1 \ast_n F_1) \cup (F_2 \ast_n F_1) \cup (F_2 \ast_n F_2) \) \( \subseteq H \), then \( E \in \tau_x, \ast_n \) is closed with respect to \( \ast_n \). Moreover, if the topological decomposition is such that \( \forall \{x_i^n\}_{i=1}^n \in E, \exists \{x_i^{-1^n}\}_{i=1}^n \in E \), then \( \{(x_i^n)^n \ast_n (x_i^{-1^n})_{i=1}^n\}_{i=1}^n \in A \), where \( A \subset E \) is a gravity subspace. Hence, \((E \in \tau_x, \ast_n)\) is a noncommutative noncompact subgroup if the corresponding closed topological space maintains the property that, \( X^n \setminus E \neq \phi \). \( \square \)

Thus, the topological decomposition of a group can be performed in a way so that a subgroup structure can be formed containing the gravity subspace. However, the interesting question is: what would be the property of projection after decomposition? This question is addressed in the next theorem.

**Theorem 16.** In \((X^n, \tau_x)\) topological space containing topologically decomposed group \( G = (X^n, \ast_n) \) under translated projection, if \( \{x_i^n\}_{i=1}^n, \{y_i^n\}_{i=1}^n \subset E, E \in \tau_x \), then \( \exists \lambda \in \mathbb{R} (\text{real}) \) such that \( f((x_i^n)^n), f((y_i^n)^n) = \lambda f((x_i^n)^n \ast_n (y_i^n)^n \in D \), where \( E = B \setminus A \) and \( \{(x_i^n)^n \ast_n (y_i^n)^n\}_{i=1}^n \in D \).

**Proof.** Let \((X^n, \tau_x)\) be a topological space containing topologically decomposed group \( G = (X^n, \ast_n) \) under translated projection, where \( E = B \setminus A \). So, \( \forall \{x_i^n\}_{i=1}^n, \{y_i^n\}_{i=1}^n \in E \), the commutative projection holds (by definition), \( \{(x_i^n)^n \ast_n (y_i^n)^n\}_{i=1}^n = \{(y_i^n)^n \ast_n (x_i^n)^n\}_{i=1}^n \) and \( \{(x_i^n)^n \ast_n (y_i^n)^n\}_{i=1}^n \in D \). This indicates that translation of projection maintains \( f((x_i^n)^n \ast_n (y_i^n)^n \in D, \{(y_i^n)^n \ast_n (x_i^n)^n\}_{i=1}^n \) under commutativity, where the finiteness of translation maintains the condition that, \( \forall \{z_i^n\}_{i=1}^n \in D, f((z_i^n)^n) \in (-\infty, +\infty) \).
Thus, in the translated topological projective space of the decomposed group, \( \forall (x_i^{n})_{i=1}^{n}, (y_i^{n})_{i=1}^{n} \in E, \exists r_x, r_y \in R \) such that \( f((x_i^{n})_{i=1}^{n}), f((y_i^{n})_{i=1}^{n}) = r_x r_y \). Moreover, if \( f((x_i^{n})_{i=1}^{n} \ast n (y_i^{n})_{i=1}^{n}) \) then, \( f((x_i^{n})_{i=1}^{n}) = r_x \) and \( \exists \in R \) such that \( f_x r_y \). Hence, in the translated projection of topologically decomposed group, \( f((x_i^{n})_{i=1}^{n}), f((y_i^{n})_{i=1}^{n}) = f((x_i^{n})_{i=1}^{n} \ast n (y_i^{n})_{i=1}^{n}) \). □

However, the translated topologically decomposed group exhibits different behaviour if the decomposed monotone class is involved while performing group operation with the elements residing outside of gravity subspace. This observation is presented in the next theorem in two equally valid forms depending on the nature of space: discrete or continuous.

**Theorem 17.** In \((X^n, \tau_x)\), the translated monotone class of topologically decomposed group maintains strict finiteness condition \( f(A \ast n E) < f(E) \) if \( \forall (x_i^{n})_{i=1}^{n} \in B \in \tau_x, f((x_i^{n})_{i=1}^{n}) \in (0, +\infty) \), where \( A \) is a gravity subspace and, \( E = B \setminus A \).

**Proof.** Let \((X^n, \tau_x)\) be a topological space with a topologically decomposed group \( G = (X^n, \ast n) \) and \( f(.) \) is a monotone class translation of topologically decomposed group. First, let us consider that the space is discrete in nature and Hausdorff. Let the gravity subspace be, \( A \in \tau_x \) and \( E = B \setminus A \) such that \( |B, A \subset B| \subset \tau_x \). If \( H \subset B \) then \((A \ast n H) \subset B \) following the group closure property, which follows that, \( \bigcup \{ (x_i^{n})_{i=1}^{n} \ast n E \in E, \forall (x_i^{n})_{i=1}^{n} \in X^n, f((x_i^{n})_{i=1}^{n}) \in (-\infty, +\infty) \} \). However, if \( \forall (x_i^{n})_{i=1}^{n} \in B \in \tau_x, f((x_i^{n})_{i=1}^{n}) \in (-\infty, +\infty) \), then it may be possible that, \( \sum_{(x_i^{n})_{i=1}^{n} \in A} f((x_i^{n})_{i=1}^{n} \ast n H) = 0 \) if the translation is appropriately balanced around the origin including it. Moreover, due to balancing, if \( \forall (x_i^{n})_{i=1}^{n} \in A, f((x_i^{n})_{i=1}^{n}) \in (0, +\infty) \) and \( \forall (x_i^{n})_{i=1}^{n} \in E, f((x_i^{n})_{i=1}^{n}) \in (-\infty, 0] \), then

\[
\sum_{(x_i^{n})_{i=1}^{n} \in A} f((x_i^{n})_{i=1}^{n}) > \sum_{(x_i^{n})_{i=1}^{n} \in E} f((x_i^{n})_{i=1}^{n}).
\]

Hence, as \((A \ast n E) \subset E \), so \( f(A \ast n E) < f(E) \) if \( \forall (x_i^{n})_{i=1}^{n} \in B, f((x_i^{n})_{i=1}^{n}) \in (0, +\infty) \). The similar proof can be extended for a Hausdorff space under continuity. □

As a consequence, the translation of the gravity subspace of a topologically decomposed group can appear to be invariant under the algebraic group operation. This observation is explained in the next corollary.

**Corollary 2.** In \((X^n, \tau_x)\) containing topologically decomposed group \( G = (X^n, \ast n) \), the gravity translation is invariant, as \( f(A \ast n A) = f(A) \).

**Proof.** Let the topologically decomposed group \( G = (X^n, \ast n) \) be embedded within the topological space \((X^n, \tau_x)\), where the gravity subspace is \( A \in \tau_x \). It is shown earlier that gravity subspace forms a subgroup, as \((A \ast n A) = A \).

As \( \forall (x_i^{n})_{i=1}^{n} \in A, f((x_i^{n})_{i=1}^{n}) \in (-\infty, +\infty) \), hence, \( f(A \ast n A) = f(A) \). □

A similar type of invariance of translation can be observed in monotone class considering the commutativity of projection in topological space. In other words, commutative projection preserves translation invariance within monotone class. This observation is presented as a theorem below.

**Theorem 18.** In the topological space \((X^n, \tau_x)\) with a decomposed group \( G = (X^n, \ast n) \), the commutative projection within monotone class is translation invariant as, \( f(E_1 \ast n E_2) = f(E_2 \ast n E_1) \), where \( A, B, E_1, E_2 \subset \tau_x \) and, \( (E_1, E_2) \subset B \setminus A \).

**Proof.** Let \((X^n, \tau_x)\) be a topological space containing a topologically decomposed group \( G = (X^n, \ast n) \). Let in the space be \( A, B, E_1, E_2 \subset \tau_x \) such that \( A \) is the gravity subspace of decomposition and, \((E_1 \cup E_2) \subset B \setminus A \) where, \( E_1 \cap E_2 = \emptyset \). However, in the monotone class of decomposition, if the projection is commutative then \( E_1 \ast n E_2 = E_2 \ast n E_1 \). Moreover, the projection maintains topological
property that, \((E_1 \ast_n E_2) \subset D\), where \(D \in \tau_x\) and \(D \cap B = \emptyset\). Thus, the translation maintains the invariance under commutative projection as \(f(E_1 \ast_n E_2) = f(E_2 \ast_n E_1)\). □

The translation function can exhibit various properties depending on commutativity as well as the nature of projection of a topologically decomposed group. The translation function can be fixed (constant) or variable, and it determines whether the resulting space is Hausdorff or not. This interplay between commutativity, projection and the nature of translation is presented in the following theorem.

**Theorem 20.** In noncommutative and inflationary projection of topologically decomposed group, the translated topological subspace \(f(F_1 \cup F_2)\) is Hausdorff if translation is not a constant anywhere.

**Proof.** Let \((X^n, \tau_x)\) be a topological space and \(G = (X^n, \ast_n)\) be a topologically decomposed group under noncommutative projection along with translation \(f(.)\). If \(f(.)\) is inflationary and, \([A, B, F_1, F_2] \subset \tau_x\) such that, \(B \cap F_1 = B \cap F_2 = \emptyset\), then \(F_1 \cap F_2 = \emptyset\) in the topological space. Moreover, \(\forall(x_i^n)_{i=1}^n, (y_i^n)_{i=1}^n \in B\backslash A\) the projection maintains the property that \((x_i^n)_{i=1}^n \ast_n (y_i^n)_{i=1}^n \in F_1 \cup F_2\). Furthermore, the translation is finite and strictly ordered, such that \((f(F_1) \cup f(F_2)) \subset (-\infty, +\infty)\) and, \(f(F_1) < f(F_2)\). If the translation is locally constant in \(W \subset (F_1 \cup F_2)\) and \([M_1, M_2] \subset B\backslash A\), then \(f(M_1 \ast_n M_2) = f(M_2 \ast_n M_1)\) where \(M_1 \cap M_2 = \emptyset\) and, \(W = (M_1 \ast_n M_2) \cup (M_2 \ast_n M_1)\). This indicates that, \(\forall(x_i^n)_{i=1}^n, (y_i^n)_{i=1}^n \in W\), the following property is satisfied, \(B_x(f((x_i^n)_{i=1}^n), \varepsilon > 0) \cap B_y(f((y_i^n)_{i=1}^n), \varepsilon > 0) \neq \emptyset\), where \(\varepsilon \in \mathbb{R}^+\). Thus, the resulting translated subspace is not Hausdorff and, as a result, the entire noncommutative projection is not Hausdorff under locally constant translation. Hence, the translation must not be constant anywhere in the entire domain of noncommutative projection of topologically decomposed group in order to form a translated Hausdorff space. □

The noncommutative projection of a topologically decomposed group opens the possibility of characterization of completeness of the resulting space. The analysis of completeness is performed under the translation operation while applying the noncommutative projection of decomposition. As a natural requirement, the translated space is required to be Hausdorff. This interplay is presented in the following theorem.

**Theorem 21.** If \(S_X = \{(x_i^n)_{i=1}^n\}_{k=1}^{+\infty}\) and \(S_Y = \{(y_i^n)_{i=1}^n\}_{k=1}^{+\infty}\) such that, \(S_X \subset F_1, S_Y \subset F_2\) and, \([F_1, F_2] \subset \tau_x\) in a topologically decomposed group \(G = (X^n, \ast_n)\) having noncommutative projection, then \(d(f(a_k \in S_X), f(b_k \in S_Y))_{k=1}^{+\infty}\) is a convergent Cauchy sequence generated by monotonic inflationary translation if the translated space is Hausdorff.

**Proof.** Let \(G = (X^n, \ast_n)\) be a topologically decomposed group in \((X^n, \tau_x)\). Let the translation \(f(.)\) be inflationary, and \([A, B, F_1, F_2] \subset \tau_x\). If the translation \(f(X^n) \subset (-\infty, +\infty)\) generates a Hausdorff space, then the resulting translated space is metrizable and finite. Let \(S_X = \{(x_i^n)_{i=1}^n\}_{k=1}^{+\infty}\) and \(S_Y = \{(y_i^n)_{i=1}^n\}_{k=1}^{+\infty}\) be two sequences, where \(S_X \subset F_1, S_Y \subset F_2\). However, in this case, the translation maintains the following condition, \(f(F_1) < f(F_2)\). Thus, two real intervals \(I_X \subset R, I_Y \subset R\) exist, such that \(f(S_X) \subset I_X\) and \(f(S_Y) \subset I_Y\), where \(\forall a \in I_X, \forall b \in I_Y\), it is true that \(a \prec b\) if \(\text{Sup}(I_X) < \text{Inf}(I_Y)\). Furthermore, in all cases, the translation maintains the property that \(\text{Sup}(I_Y) \in (-\infty, +\infty)\). Thus, each pair of elements taken from \(I_X \times I_Y\) creates a monotone relation \(\forall(a, b) \in I_X \times I_Y, (a, b) \in I_X\), where the translated space is finite. On the other hand, if \(I_X \cap I_Y \neq \emptyset\), then \(\forall(x_i^n)_{i=1}^n \in S_X\) and, \(\forall(y_i^n)_{i=1}^n \in S_Y\) the translation maintains the following condition, \(f((x_i^n)_{i=1}^n) < f((y_i^n)_{i=1}^n)\), where \((a_i \ast_n b_i)_{i=1}^n = (x_i)_{i=1}^n, (b_i \ast_n a_i)_{i=1}^n = (y_i)_{i=1}^n\) and, \((a_i)_{i=1}^n, (b_i)_{i=1}^n \subset B(A)\). Now, if \(S_{XY} = \{(w_k)_{k=1}^{+\infty}\}\) is a sequence under the distance metric \(d : f \times f \rightarrow R^n\) in the metrizable translated space, such that \(w_m = d(f((x_m^n)_{m=1}^n), f((y_m^n)_{m=1}^n))\), then \(S_{XY} = \{(w_k)_{k=1}^{+\infty}\}\) is a Cauchy if it is a monotone decrease, \(w_m > w_{m+1}\). Hence, inflationary finite translation in Hausdorff space generates the convergent Cauchy sequence under monotonicity. □
Remark 2. The immediate consequence of the above theorem is that the monotonic inflationary translation of the topologically decomposed group under noncommutative projection prepares a translated space, which is sequentially complete.

The topological homeomorphism between two decomposed groups can be established under projections. The commutative projections of two topologically decomposed groups can preserve local homeomorphism in topological spaces. This property is presented in the following theorem.

**Theorem 21.** If \((X^n, \tau_x)\) and \((Y^n, \tau_y)\) are two topological spaces containing, respectively, two topologically decomposed groups, \(G_x = (X^n, *^n_x)\) and \(G_y = (Y^n, *^n_y)\) with commutative projections, then \((E_y *^n_y (g(H)) *^n_Y E_y \subseteq g(D_x \in \tau_x)\) under topological homeomorphism \(g : X^n \to Y^n\), where \(E_y \subseteq B_y \setminus A_y\) and \(H = \ker(g)\).

**Proof.** Let \((X^n, \tau_x)\) and \((Y^n, \tau_y)\) be two topological spaces containing, respectively, two topologically decomposed groups, \(G_x = (X^n, *^n_x)\) and \(G_y = (Y^n, *^n_y)\). Let \(g : X^n \to Y^n\) be a topological homeomorphism and the decomposed groups have commutative projections, such that \(\{A_x, B_x, D_x \subset X^n \setminus B_x\} \subset \tau_x\), \(A_y, B_y, D_y \subset Y^n \setminus B_y\) \(\subset \tau_y\), and, \(g((x_i)_{i=1}^n *^n_x (z_i)_{i=1}^n) = (g((x_i)_{i=1}^n) *^n_y g((z_i)_{i=1}^n)) \in Y^n\), \((x_i)_{i=1}^n \in X^n, (z_i)_{i=1}^n \in X^n\). Let \(N(x_i) \subset X^n\) be a neighbourhood of \((x_i)_{i=1}^n\). Hence, due to homeomorphism \(g((x_i)_{i=1}^n) *^n_y g((z_i)_{i=1}^n) \in N(y_i)\), \(\Delta N(y_i) \subset Y^n\) such that, \(g((x_i)_{i=1}^n *^n_x (z_i)_{i=1}^n) \in N(y_i)\), where \(g(N(x_i)) \subset N(y_i)\). However, if \(H = \ker(g)\), then \(H \subset A_x\) and, \(g(H) \subset A_y\). Moreover, if \(E_x \subset B_x \setminus A_x\) and \(E_y \subset B_y \setminus A_y\), then \(g(E_x) *^n_y g(E_y) = g(E_x) *^n_y g(E_y)\) preserving homeomorphism of group structures. Furthermore, due to commutative projections under topological homeomorphism, \((E_x *^n_x E_x) \subset D_x\) in the topological space \((X^n, \tau_x)\) and, \(D_x = g^{-1}(D_y)\). As, \((g(E_x) *^n_y g(E_y)) \subset D_y\) in \((Y^n, \tau_y)\), hence \((E_y *^n_y g(H)) *^n_Y E_y \subseteq g(D_x)\). □

The homeomorphism between two topologically decomposed groups can lead to a lifting property under projections if the underlying topological space is continuous. The lifting property joins the kernel of homeomorphism from one decomposed group to the element of another decomposed group under projection. In another view, the topological lifting property can be incorporated into decomposed groups under projections by constructing functional compositions involving homeomorphism. This construction and lifting property is illustrated in the following theorem.

**Theorem 22.** If \(g : X^n \to Y^n\) is a homeomorphism between \((X^n, \tau_x)\) and \((Y^n, \tau_y)\) containing topologically decomposed groups \(G_x = (X^n, *^n_x)\) and \(G_y = (Y^n, *^n_y)\) under projection, then \(g \circ f_2 = f_1\) is a lifting, where \(f_1 : [0, 1] \to Y^n, f_2 : [0, 1] \to X^n\) and, \(f_1(0) = e_y \in Y^n, f_2(0) = \ker(g)\).

**Proof.** Let \(g : X^n \to Y^n\) be a homeomorphism between \((X^n, \tau_x)\) and \((Y^n, \tau_y)\) containing topologically decomposed groups \(G_x = (X^n, *^n_x)\) and \(G_y = (Y^n, *^n_y)\) under projection, where the underlying topological spaces are continuous. Let \(f_1 : [0, 1] \to Y^n\) and, \(f_2 : [0, 1] \to X^n\) be continuous, such that \(f_1(0) = e_y \in Y^n, f_2(0) = \ker(g)\). Now, if the lifting is constructed as \(g \circ f_2 : [0, 1] \to Y^n\), such that \(g \circ f_2 = f_1\), then \(g \circ f_2(0) = e_y\). Moreover, as \(f_1(0) = \ker(g)\), so \(f_2(0) = \ker(g)\) if \(g \circ f_2 = f_1\) is a lifting. □

The construction of lifting directly translates into the determination of invertibility of path in half open interval \(h : (0, 1] \to X^n\) under the condition that the composed lifting is a proper subset of the path in the closed interval to codomain \(Y^n\). This property is presented in next lemma.

**Lemma 3.** If \(g \circ f_2 = f_1\) is a lifting then \(h : (0, 1] \to X^n\) is invertible, where \(g \circ h \subset f_1\).

**Proof.** Let \((X^n, \tau_x)\) and \((Y^n, \tau_y)\) be two Hausdorff topological spaces containing topologically decomposed groups \(G_x = (X^n, *^n_x)\) and \(G_y = (Y^n, *^n_y)\) under projection. Let \(g : X^n \to Y^n\) be a homeomorphism. If composition \(g \circ f_2 = f_1\) is a lifting, then \(f_2(0) = \ker(g)\). Again, if \(\ker(g) = A_x \in \tau_x,\)
then \( f_2(0) \) is not invertible, as \(|A_2| > 1\). Let \( h : (0,1] \to X^n \) be such that \((g \circ h)l = (g \circ f_2)l\), where \( I \subseteq (0,1] \). Thus, \((g \circ h)I \subseteq \bigcup_{x \in [0,1]} f_1(x)\). Hence, \( \forall (y_i)^n_{i=1} \in f_1(.) \), the following condition is satisfied, \((h^{-1} \circ g^{-1})(y_i)^n_{i=1} \in (0,1] \). This leads to the conclusion that, \( h : (0,1] \to X^n \) is invertible. \( \square \)

The invertibility property incorporates the one-to-one correspondence between two topologically decomposed groups under projection through the composed lifting.

5. Application Aspects and Comparison

The applications of group decompositions are found in solid state physics and also in computer algebra [30]. The concept of topological decomposition of groups can be applied to characterize spaces and multidimensional surfaces. Traditionally, the group decompositions are performed by using direct and semidirect products. If \( G = (X, \ast) \) is a decomposable group, then \( \exists H_1, H_2 \subset X, G = H_1 \otimes H_2 \), where \( H_1 \otimes H_2 \) signifies semidirect product operation between two subgroups if \( G = (X, \ast) \) is a semidirect decomposable type. The restriction in case of semidirect decomposition is that, \( H_1 \triangleleft_N G \) and \( H_2 \triangleleft G \). On the other hand, if \( G = (X, \ast) \) is directly decomposable as a Cartesian product between \( H_1 \times H_2 \), then \( H_1 \triangleleft_N G \) and \( H_2 \triangleleft_N G \). Note that semidirect product-based decomposition is not unique, but the direct product-based decomposition is unique in nature. However, in both types of decomposition, one has to construct suitable subgroups as well as the corresponding normal subgroups depending on the types of decomposition.

In cases of topological decomposition of groups, the condition of availability of such subgroups is further relaxed, because the topological decomposition of groups is based on the formation of topological partitions (i.e., separation) generating components. The comparison of various decomposition structures and properties is presented in Table 1, where \( A \) and \( B \) represent decomposed topological components according to an application context. The similarity between topological decomposition of groups and the semidirect product-based decomposition is that both of these types of decomposition preserve the shared identity element. However, the semidirect product requires a strict condition that the decomposed partitions should be independent subgroups. The overall partitioning structures of topological decomposition and direct product-based decomposition of groups are similar in nature; however, there is no shared identity in the case of direct product-based decomposition. The direct product-based decomposition requires a stricter condition that both the decomposed subgroups should be normal. This condition is further relaxed in topological decomposition of groups. The topological decomposition of groups can be constructed in two varieties, such as with projection or without projection in spaces. The non-projective variety employs topological fibers. However, the other two types of group decomposition incorporate projections component by component.

Table 1. Comparison between various group decomposition structures.

| Decomposition Varieties | Partitioning Structure | Normality Property of Subgroup | Identity Placement | Projection/Fiber |
|-------------------------|------------------------|-------------------------------|--------------------|-----------------|
| Semidirect product      | \( H_1 \otimes H_2 \)  | \( H_1 \) normal             | \( H_1 \cap H_2 = \{e\} \) | Projection      |
| Direct product          | \( H_1 \times H_2 \)   | \( H_1, H_2 \) both normal   | \( e_{H1} \neq e_{H2} \) | Projection      |
| Topological decomposition | \( \left\{A,B,X\backslash(\overline{A} \cup \overline{B})\right\} \) | Not applicable            | \( A \cap B = \phi, e \in X \) | Topological fiber and projection |

6. Conclusions

The topological decomposition of groups is an interesting topic and can be performed without employing semidirect products. The topological decomposition of groups, as well as group actions, requires the construction of underlying topological spaces with group embeddings. There are two different varieties of topological decomposition of groups. In the case of the non-projective variety of topological decomposition, the concepts of topological fibers, locality of group operation
and homeomorphism are required. However, such decomposition does not specifically need any requirement of commutativity of the embedded group within topological spaces. The symmetry of surjection can be maintained in the topologically decomposed group due to function composition and absence of projection. If the group action has no fixed-point in non-projective decomposed group, then the group is non-Abelian in nature within the topological spaces. On the other hand, in the projective variety of topologically decomposed groups, the gravity subspace may be noncompact. The gravity subspace does not automatically form a subgroup in the projective decomposition. In the case of non-commutative projective decomposition, the noncompact and noncommutative subgroups can be formed. However, the finite translations of decomposed group in topological spaces and monotone class have interplay. The gravity subspace is translation invariant in the projective variety of topologically decomposed group. The sequential completeness property can be evaluated in a topologically decomposed group with projection under specific condition. Moreover, the lifting in the topological decomposition of groups with homeomorphism can be constructed in the projective variety. The generalized Schoenflies embeddings of a decomposed group can preserve group homeomorphism if the topological space is normal.

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