Counting Steps: A New Approach to Objective Probability in Physics

Amit Hagar and Giuseppe Sergioli*

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Abstract

We propose a new interpretation of objective deterministic chances in statistical physics based on physical computational complexity. This notion applies to a single physical system (be it an experimental set–up in the lab, or a subsystem of the universe), and quantifies (1) the difficulty to realize a physical state given another, (2) the ‘distance’ (in terms of physical resources) of a physical state from another, and (3) the size of the set of time–complexity functions that are compatible with the physical resources required to reach a physical state from another.

1 Introduction

Probabilistic statements in a deterministic dynamical setting are commonly understood as epistemic. Since in such a setting a complete specification of the state of the system at one time – together with the dynamics – uniquely determines the state at later times, the inability to predict an outcome exactly (with probability 1) is predicated on the notion of ignorance, or incomplete knowledge. Such a subjective interpretation is natural in the context of classical statistical mechanics (SM), where a physical state is represented as a point on phase space, and the dynamics is a trajectory in that space, or in the context of Bohmian mechanics, where the phase space is replaced

*Department of HPS and Center for Spacetime Symmetries, College of Arts and Sciences, Indiana University, Bloomington, IN and Università di Cagliari, Sardegna, Italy. emails: hagara@indiana.edu and giuseppe.sergioli@gmail.com

1Lewis 1986 is the locus classicus for such a view.
with a configuration space and the ontology is augmented with the quan-
tum potential, but recently it has been suggested as a viable option also in
the context of orthodox non–relativistic quantum mechanics (QM), where
the state is represented as a vector in the Hilbert space, and the dynamics
is a unitary transformation, i.e., a rotation, in that space (e.g., Caves, Fuchs,
& Schack 2002). In all three cases the dynamics is strictly deterministic,
and the only difference – apart from the representation of the state – is in
the character of the probabilities: in classical SM or Bohmian mechanics
they are subsets of phase space (or configuration space) obeying a Boolean
structure; in QM they are angles between subspaces in the Hilbert space
obeying a non–Boolean structure, whence the famous non–locality, context-
tuality, and the violation of Bell’s inequalities.

Such an epistemic notion of probability in statistical physics appears to
many inappropriate. The problem is not how lack of knowledge can bring
about physical phenomena; it can’t. Neither is it a problem about ontolog-
ical vagueness. Rather, the problem is that an epistemic interpretation of
probability in statistical physics, be it classical SM or QM, turns these the-
ories into a type of statistical inference: while applied to physical systems,
these theories become theories about epistemic judgments in the light of
incomplete knowledge, and the probabilities therein do not represent or
influence the physical situation, but only represent our state of mind (Frigg
2007; Uffink 2011).

In recent years an alternative, objective view of probability has been
defended, both in the foundations of classical SM and in the context of
Bohmian mechanics, based on the notion of typicality (e.g., Maudlin 2007).
Typicality claims tell us what most physical states are, or which dynamical
evolution are overwhelmingly more likely, by assigning measure 1 to the set
of such states or the set of such dynamical evolutions. Such claims make
an analytical connection between a deterministic dynamics and a charac-
terization of certain empirical distributions, hence can be interpreted as ob-
jective, having nothing to do with one’s credence or state of knowledge.
With this notion, or so the story goes, one can treat probabilistic statements

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2Note that the ontological picture in this case is not as clear as in SM or Bohmian me-
chanics.
3See Albert (2000, 64) for such a complaint in the context of classical SM.
4The epistemic view of QM, for example, under specifies what the probabilities in QM
probabilities for, and regards the state as a state of knowledge. See Hagar (2003) for such a
complaint.
5Examples are “most quantum states are mixed, i.e., entangled with their environment”,
or “most systems relax to thermodynamic equilibrium if left to themselves”.
in a deterministic physical setting as arising from an objective state of affairs, and the theories that give rise to these statements as theories about the physical world, rather than theories about our state of knowledge.

This objective view of probability, however, is not problem–free. First, as its proponents admit (Goldstein et al. 2010), the notion of typicality is too weak: a theorem saying that a condition is true of the vast majority of systems does not prove anything about a concrete given system. Next, the notion lacks logical closure: a pair of typical states is not necessarily a typical pair of states, which means that “being typical” is not an intrinsic property of an initial condition, not even for a single system, but depends on the relation between the state and other possible initial conditions (Pitowsky 2011). Possible ways around these difficulties have been suggested, but while these problems may be circumvented, there exists a deeper lacuna underlying the notion of typicality which threatens the entire project.

The point is that typicality claims depend on a specific choice of measure, usually the Lebesgue measure or any other measure absolutely continuous with it. But what justifies this choice of measure when an infinite number of possible measures are equally plausible (Hemmo & Shenker 2011a)? Moreover, even if we have established somehow that, relative to a preferred choice of measure, a certain set of states $T$ is typical, i.e., its members are overwhelmingly more probable with respect to all possible states, what justifies the claim that we are likely to observe, or “pick–up”, members of that set $T$ more often than members of the complement set $\bar{T}$? After all, the measure we have imposed on the space of all possible potential states $(T \cup \bar{T})$ need not dictate the measure we impose on the space of our actual observations. Indeed, while under the choice of the Haar measure “most” quantum states are mixed, hence entangled with their environment, we can still realize (hence observe) pure states in the lab, at least to a certain extent. In what sense, then, are these states “rare”?

The twofold problem of justification of the measure is, on final account, a manifestation of the problem of induction (Pitowsky 1985, 234–238). On a strictly empiricist view, the effort to justify typicality claims is just another (futile) attempt to give demonstrations to matters of fact, or to derive contingent conclusions from necessary truths. The point here is that there is no surrogate for experience in the empirical sciences, and that inductive reasoning is the best one can do in one’s attempts to understand the world.

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6Maudlin (2007, 287), for example, rejects the requirement to assign probabilities to single systems, and Pitowsky (2011) proposes to retain most of the advantages of typicality but to retreat to a full–fledged Lebesgue measure, with its combinatorial interpretation.
We tend to agree with the above criticism, but we also believe the alternative (or rather the lack thereof) it leaves us with is equally unsatisfactory. While typicality arguments do seem to achieve too much, the above pessimism seems to leave us with too little: not only are we unable to make sense of objective deterministic chances on this empiricist account, we are also unable to justify the standard statistical methods in the scientific practice. These methods are commonly understood within the context of the frequentist approach to probability, and yet the latter approach is based on typicality arguments (these appear, e.g., implicitly in the weak law of large numbers, or explicitly in the definition of a random sequence), and so the criticism raised against typicality equally undercuts the attempts to apply frequentist methods in the empirical sciences (Hemmo & Shenker 2011b).

Since we believe there is more to statistical physics than statistical inference, we here propose a new non–frequentist interpretation of physical probability as an alternative to typicality. Our notion is objective, dynamical, applies to individual states or systems, and its definition requires no convergence theorems. This notion completely escapes the problem of justification of the measure imposed on the space of all possible states, while addressing directly the problem of justification of the measure imposed on the space of all actual states. As such it is consistent with the above criticism marshaled against typicality and frequentism, and can serve as a viable alternative to the current epistemic view of probability in statistical physics, turning the latter once again into a physical theory about the natural world.

The paper is structured as follows. In section 2 we spell out the basic assumptions behind our proposal. Warming up, in section 3 we show how by equating “probable” with “easy” (in terms of computational complexity), one can assign measure 1 (0) to a set of states whose realization requires a dynamical evolution with polynomial (exponential) time–complexity. In section 4 we move to a full–fledged definition of objective probability that quantifies how hard it is to realize a physical state, and measures (in terms of physical resources) the ‘distance’ between any such pair of states. In section 5 we discuss how subjective probability can be made objective in situations that include an observer. In doing so we introduce a new interpretation to the probabilities of QM. In section 6 we defend the plausibility of our view in the general “observer–free” context, where the system at hand is the universe as a whole or any subsystem thereof. Section 7 concludes.
2 Assumptions

We start by spelling out the five basic assumptions that underlie our models. They are Determinism, $P \subseteq \text{EXPTIME}$, Boundedness, Discreteness, and Locality. These assumptions are working hypotheses in the framework from which our interpretation of probability stems, namely, physical computational complexity. They help us delineate the two probability spaces in our models: the space of physically allowable states, and the space of physically allowable dynamical evolutions.

A. Determinism

Our models rest on the assumption of strict determinism. This assumption follows from the physical Church–Turing thesis, according to which dynamical evolutions of physical systems can be regarded as computations. Agreed, there exists a vast literature on the physical possibility of super-tasks and “hypercomputation”, that aims to show that Turing–computability is not a natural property, and need not apply a priori in the physical world. The counterexamples for the physical Church–Turing thesis, however, are rather contrived and, as far as we know, are not realizable in the actual universe. Be that as it may, in what follows we disregard naked singularities, closed timelike curves, non–globally hyperbolic spacetime models, ill–posed problems, divergences, and the like, adhering to the idea that every dynamical evolution takes a physical state to one and only one physical state.

B. $P \subseteq \text{EXPTIME}$

The fact that each computation requires physical resources (energy and time) that increase with the size (the number of degrees of freedom) of

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7In this framework, the performance of physical (dynamical) systems is analyzed with notions and concepts that originate in computational complexity theory. See, e.g., Geroch & Hartle 1986, Pitowsky 1990, and Pitowsky 1996.

8The physical Church–Turing thesis is logically independent of the original Church–Turing thesis. See e.g., Pitowsky & Shagrir 2003.

9So far there are two such counterexamples: Pour el & Richards’ (1989) wave equation in 3 dimensions and Pitowsky’s (1990) spacetime model that allows finite–time execution of an infinite number of computational steps. See also Hogarth 1994 for an elaboration on the latter, and Earman & Norton 1993 for further discussion.

10Note that from a strictly dynamical perspective, quantum mechanics is fully deterministic: Schrödinger’s equations takes any quantum state to one and only one quantum state.
the system allows us to classify different dynamical evolutions as either “easy” (i.e., having polynomial time–complexity such as $O(n^c)$) or “hard” (i.e., having exponential time–complexity such as $O(c^n)$). That there exists a meaningful difference between these two classes (and between different degrees of time–complexity within each class) is the consequence of the Time Hierarchy Theorem (Hartmanis and Stearns 1965).

C. Boundedness

Assumption (A) allows us to apply the machinery of complexity theory to dynamical evolutions, by treating them as computations. Assumption (B) allows us to classify states (and the dynamical evolutions that realize them) as “easy” or “hard”. Assumption (C) allows us to impose upper and lower bounds on the set of all possible dynamical evolutions in the actual universe, based on the following two facts:

- The physical resources (time, energy, and number of particles) in the universe are bounded from above; beyond a certain degree of an exponential or a polynomial time evolution, the next computational step would require resources that would supersede this bound.

- The minimum number of computational steps is 1, and so for a given $n$ (the size of the system) there always exists a lower bound on the set of possible dynamical evolutions below which the number of steps required for this input size $n$ is smaller than 1.

D. Discreteness

Assumption (D) allows us to discretize the set of the physically allowable dynamical evolutions. Two facts warrant the elimination of real coefficients in our classification of dynamical evolutions into time–complexity classes. First, each dynamical evolution is governed by a Hamiltonian (the total energy function). Second, the time–energy uncertainty relation limits the ability to resolve arbitrary energy differences between any two Hamiltonians (Childs, Preskill & Renes 2000, Aharonov, Massar, & Popescu 2002). This means that we cannot distinguish between two unknown Hamiltonians with infinite precision, hence the space of possible Hamiltonians is discrete.

11Here $n$ is the input size – in our case the dimension of the system at hand, and $c$ is a (rational, as we shall assume below) coefficient.
E. Locality

Finally, and consistent with the current state of affairs in physics, in physically realizing the Hamiltonians that govern the dynamical evolutions, we allow only local interactions.

Models

The above assumptions allow us to propose two possible models for objective physical probability.

Our first model is constructed on the space of all possible physical states of a given physical system with a given number of degrees of freedom \( n \) in a given moment in time \( t \), confined to a given energy shell \( E \). Each such state, given assumption (A), is a result of a certain dynamical evolution, which, in turn, is generated by a certain Hamiltonian. If we assume further that all dynamical evolutions start from a common, “mother” state, say, the initial state of the universe, we can then assign (using assumptions (B)–(E)) a non-uniform probability distribution on the set of all possible states according to the time–complexity of the dynamical evolution that realizes each state. As we shall see below, for a large number of degrees of freedom, this assignment has interesting consequences.

Our second model is constructed on the space of all possible dynamical evolutions. Here, again, to precisely define the notion of objective physical probability we require the above couple, i.e., the number of degrees of freedom \( n \), time \( t \), and energy \( E \). Given such a triplet, we construct a probability space out of a functional that relates the power \( (E/t) \) of a computation – seen as a dynamical evolution from one state to another – with the relative size of the set of the possible dynamical evolutions that are compatible with it. Our probability function is thus a distance measure on the above functional, that quantifies how hard it is to realize a state, or how far a given system is from that state, in terms of the physical resources available to it relative to the required resources.

3 Warming Up: Not All States Are Born Equal

The standard story about typicality (in classical SM, or in Bohmian mechanics) requires the notion of equiprobability, or a uniform measure, relative to

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12We do not claim that these models are unique or optimal. Our purpose is only to demonstrate the possibility of defining a finite notion of objective probability in physics on the basis of considerations from physical computational complexity.
which a set is declared *typical*. The foundations of SM are saturated with failed attempts to justify this choice of measure\textsuperscript{13} the most famous of which is Boltzmann’s ergodic hypothesis\textsuperscript{14} Our first stab at the notion of physical probability based on computational complexity suggests a possible way out of this problem.

At the crux of the matter lie the primitive notions of *number* and *counting*. The choice of the Lebesgue measure is deemed “natural” when one extends the standard notion of counting from the finite case to the infinite one (Pitowsky 2011). But why treat each state as equal (in number) to another even in the finite case? Agreed, there are physical situations involving symmetries, such as the case of a fair die, that warrant such a treatment (Strevens 1998), but *in general* there is no *a priori* reason to count this way.\textsuperscript{15} Consequently, in what follows we shall treat physical states in an utterly politically incorrect manner, assigning states with non–equal weights inverse proportionally to the degree of time–complexity of the dynamical evolution they are associated with. As it turns out, few empirical facts allow us to get as close to equiprobability as one can get in a finite setting, *without* relying on any *a priori* notion of equiprobability or uniform probability distribution from the outset.

Our probability triplet thus consists of the following elements:

* Ω is the state space of all physical states of a system with *n* degrees of freedom on a given energy shell in a given moment in time, obeying the current laws of physics, i.e., realized by a certain physical process whose time–complexity is either polynomial or exponential (“easy” or “hard”). Note that Ω is a bounded and discrete set of polynomial functions $\in O(n^c)$ and exponential functions $\in O(e^n)$.

* F is the $\sigma$–algebra of Ω, i.e., a non–empty class of subsets of Ω, containing Ω itself, the empty set, and closed under the formation of complements, finite unions, and finite intersections (i.e., F is a discrete, bounded subset of the power set of Ω). The elements of F are pos-

\textsuperscript{13}See, e.g., Sklar 1993, 156–195 for a summary.

\textsuperscript{14}That the ergodic hypothesis falls short of justifying the assumption of equiprobability follows from three facts: (1) many thermodynamic systems are not ergodic, (2) ergodicity holds only at infinite time scales, and (3) such a justification is plainly circular, as it is valid only for a set of “normal” states whose measure 1 is fixed, again, relative to the choice of measure we are trying to justify from the outset. The last two facts are, essentially, equivalent to the recent criticism against typicality.

\textsuperscript{15}See Irvine 2011 on Frege’s failed attempt to justify such an *a priori* conception of counting.
possible physical states, realized by physical processes with a combined
time–complexity, either exponential or polynomial.

- \( P \) is the probability measure that maps \( F \) onto \([0, 1]\), where \( P(\emptyset) = 0 \),
  \( P(F) = 1 \), such that \( P \) is additive.

We now proceed to the assignment of measures on sets of states. Let’s
denote with \( \text{Exp} \) and \( \text{Poly} \) a partition of \( F \) into two subsets with some prior
measures \( \mu_e \) and \( \mu_p = 1 - \mu_e \).

Next, for every function \( f \in (\text{Poly} \cup \text{Exp}) \) we define the weight \( \xi \) of \( f \) at
an arbitrary (natural) point \( n \) as:

\[
\xi_{f(n)} = \mu_e - \frac{\arctan f'(n)}{\Pi/2} \mu_e \quad \text{when } f(n) \in \text{Exp}. 
\]

(1)

and

\[
\xi_{f(n)} = (1 - \mu_e) + [(1 - \mu_e) - \frac{\arctan f'(n)}{\Pi/2} (1 - \mu_e)] \quad \text{when } f(n) \in \text{Poly}. 
\]

(2)

For every function \( f \in (\text{Poly} \cup \text{Exp}) \) we define the probability of \( f \) in an
arbitrary (natural) point \( n \) as

\[
P(f(n)) = \alpha \xi_{f(n)}. 
\]

(3)

where \( \alpha \) is a normalization parameter given by:

\[
\alpha = \frac{1}{\sum_i \xi_{f_i(n)}}. 
\]

(4)

It is easy to show that \( P(f(n)) \in [0, 1] \), that \( P(\emptyset) = 0 \), and that by construc-
tion, the total probability is equal to 1. Furthermore, if we take into account
the complexity degree of each element of the bounded set \((\text{Poly} \cup \text{Exp})\), we
can obtain a partition of \((\text{Poly} \cup \text{Exp})\) where every element of the partition
(indicate by \( I_1, ..., I_n \)) corresponds to a different degree of complexity. We
now define \( P(I_1) = \sum_{f_i \in I_1} P(f_i) \). It follows that:

\[
\forall i, j, P(I_i \cup I_j) = \sum_{f_i \in I_i} P(f_i) + \sum_{f_j \in I_j} P(f_j). 
\]

(5)

\[16\]Metaphorically, dynamical evolutions in \( \text{Exp} \) “lose weight” in direct proportion to their
degree of time–complexity. This total lost weight is now (non–uniformly) distributed on
\( \text{Poly} \) in such a way that the dynamical evolutions in \( \text{Poly} \) gain relative weights inverse
proportionally to their degree of time–complexity.
Thus $P$ satisfies Kolmogorov’s axioms, and as such it is an admissible probability function.

Note that $\xi$ is inverse proportional to the first derivative of the time–complexity function of the dynamical evolution that realizes the state. Consequently, for a large input size (i.e., a large number of degrees of freedom), the following results hold (see appendix A for details):

I. The set of states whose time–complexity is exponential gets assigned a measure close to zero, while the set of states whose time–complexity is polynomial gets assigned a measure close to one.

II. The polynomial states are (almost) uniformly distributed, i.e., their distribution is (almost) a resolution of the identity.

We would like to clarify the following two points:

• Since the $\sigma$–algebra $F$, as we’ve argued, is a bounded and discrete, $\alpha$ is always finite (albeit very small for a large $n$). For this reason, in a finite model such as ours, the assignment of actual measure 0 (or 1 for that matter), as well as the assignment of equiprobability (as a resolution of the identity on Po{\text{oly}}) are fictions, as the set $\text{Exp}$ remains with a finite (albeit very small) measure, and the partition on the set Po{\text{oly}} is never strictly uniform. For a macroscopic system, however, ‘very small’ is an understatement, and the above partition is very close to uniform. When the state at hand is of the universe as a whole (where $n \approx 10^{80}$), the understatement is literally a cosmic one, and equiprobability of all polynomial time–complexity functions is a practical certainty.

• The normalization factor, $\alpha$, is inverse proportional to the number of possible time–complexity functions, and yet for combinatorial reasons, this number, $i$, is directly related to $n$, the number of degrees of freedom, hence, in effect, $\alpha$ is inverse proportional to $n$. One can object here that we implicitly “sneak in” an assumption about equal weights by treating each degree of freedom on equal footing\cite{17}. In response, we stress that the this result rests not on an a priori notion of counting, but rather on our assumption (E) above and on the empirical fact that a concatenation of nearest–neighbor interactions is not

\footnote{\textit{Recall that each time–complexity function represents a specific interaction Hamiltonian that generates it. To get $\alpha \propto \n^{-1}$ we must assume that a two–body system possesses less possible interactions (hence less possible dynamical evolutions) than a many–body system.}}
physically equivalent to one nonlocal interaction[18]. In other words, it is because of this (contingent) nature of physical interactions (and the resources they require) that results (I) and (II) above hold.

Are these results sufficient to support the common lore, according to which ‘most’ states are thermodynamic normal, i.e., *typical*, hence more likely to be observed? The answer is clearly negative, but the reasons for the insufficiency are quite subtle.

First, note that we have deliberately distributed weights on different time–complexity functions in such a way that favors members of Poly and disfavors members of Exp (see equations (1) & (2) above), but nothing justifies such a distribution – we could just as well have constructed a symmetric model in which Exp turned out to be assigned a measure close to 1 and Poly a measure close to 0. This is exactly the problem typicality arguments face (Hemmo & Shenker 2011a), and in this respect, our model fares no better. What are model *does* show, however, is that for a large dimension, and given several plausible assumptions such as ours, equiprobability (or some distribution close to it) holds among members of *one of the two sets*. Moreover, we believe that from the finitist perspective we espouse, a justification for equations (1) & (2) is also uncalled for: the above model is a probability space of *possible* states; what matters to a finitist is the probability space of the *actual* states. The latter space is presented in the next section.

Disregarding this problem of justification, and in order to highlight another interesting feature of our model, let’s assume that (I) and (II) above hold. One could still argue that in order to support the common lore, it is necessary to demonstrate that the time–complexity of the dynamical evolutions associated with those normal states is polynomial. According to our model this would endow such states with high probability. The problem is that the common lore also associates thermodynamically normal behavior with non–integrable systems, yet our probability model is discrete. As such, it harbors only integrable, or periodic dynamical systems. One could still observe chaotic behavior in this context[19], but this would require redefining notions such as “sensitivity to initial conditions” or “dynamical instability” to fit the discrete background, and would also require a careful analysis of time scales[20].

[18] For an analysis of the complexity costs involved in simulating a nonlocal operator with local ones see, e.g., Vidal & Cirac 2002.

[19] There is no compelling reason to associate chaos only with the cardinality of the reals. See Winnie 1992 on the idea of “computable chaos”.

[20] At this point, and for the record, let us acknowledge the discrepancy between our dis-
But such a demonstration would still fall short of supporting the common lore. The standard notion of probability concerns a sequence of events, and in particular, a random choice of such sequences. We could, of course, translate this notion to fit our new definition by exchanging events with physical states, yet nothing constrains us to treat the above probability space of all possible states as isomorphic to the probability space that contains our actual observations. In particular, even if on phase space thermodynamic normal states were members of a set of measure 1 and thermodynamic abnormal states were members of a set of measure 0, the measure imposed on the space of our actual observations could be different: we could, for example, chose between thermodynamic normal or abnormal states by tossing a fair coin, thus endowing them with equiprobability!

In fact, we know from experience that thermodynamic abnormal states can be realized in the lab. The most famous examples for these are the spin echo experiment (Hahn 1950) and the Fermi–Pasta–Ulam (1955) discovery of a violation of the equipartition theorem. If we call such ‘anomalies’ “rare”, we must explain how is it that we can repeat such “rare” events ever so often.

In what follows we shall demonstrate how our new view on physical probability can meet these challenges.

4 One Step at a Time

We suggest to interpret objective probability as a physical magnitude that quantifies how hard it is to realize a physical state, given a triplet of physical resources (energy, time, space). Equivalently, this magnitude quantifies how ‘far’ a given physical system is from a certain state in terms of the physical resources available to it, relative to those required for that state’s realization.

- Take any physical system with dimension \( n \) in a given energy state \( E \) and in a given moment in time \( t \), and let \( \Omega \) be the bounded and discrete set of possible dynamical evolutions obeying the current laws of physics, whose time–complexity is either polynomial or exponential (“easy” or “hard”), that may govern the system’s behavior. The set

\[ \text{crete model and the continuous nature of the time evolutions. The former is used in computer science; the latter in physics. Both are consistent, and the question whether the former approximates the latter or vice versa, i.e., which is more fundamental, seems to us, at least at the current stage of physics, purely metaphysical.} \]
\( \Omega \) contains all possible *dynamical evolutions* that can realize a single actual state.

- Given a certain couple \((n, \frac{E}{t})\), where \(n\) is the dimension of the state, \(E\) is the total possible energy, and \(t\) is the total possible time, we consider the set
  \[
  S_{\bar{n}} = \{ g_{\bar{n}} \in \Omega | O(g_{\bar{n}}) \leq \# \left( \frac{E}{t} \right) \}
  \]
  \(6\)

  where \(g_{\bar{n}}\) is a dynamical evolution that for a given \(n\) "consumes" at most the resources \(E\) in time \(t\) (we denote the power allowed for the computation as \(P_w = E/t\))^21

- \(F\) is the \(\sigma\)–algebra of \(S\), i.e., a non–empty class of subsets of \(S\), containing \(S\) itself, the empty set, and closed under the formation of complements, finite unions, and finite intersections. The elements of \(F\) are dynamical evolutions with a combined time–complexity, either exponential or polynomial. \(F\) is thus a subset of the power set of \(S\), and is bounded and discrete.

Our probability measure \(P\) is given by the mapping^22

\[
\forall A \in F : P_{\{n, \frac{E}{t}\}}(A) = \frac{|A|}{|S|}
\]

^21By "consumes" we mean the following. Take an arbitrary computation. Each computational step "costs" the same amount of time; but if, as in our case, the total time allowed for the computation is fixed, the difference in time–complexity is cashed out in terms of the difference in the frequency of the computation, i.e., the time–difference between any two computational steps. Thus, for a given \(n\) and for a given \(t\), the higher the degree of time–complexity of the function, the higher the frequency of the computation. Since higher frequency means higher power, by setting a bound on \(P_w\), one immediately sets a bound on the number of computational steps allowed for the computation (\(\# \)), and subsequently, a bound on the number of time–complexity functions that can realize the computation.

^22Mathematically speaking, the mapping between \(\#\) and \(P_w\) is discontinuous. We can still define \(P\) with an integral, however, using an approximation, by embedding this mapping into the continuous function \(f\). This embedding has one advantage, namely, it allows us to constrain our model: the curve \(\# = f_{\bar{n}}(P_w)\) relates time–complexity functions (indirectly via the number of steps they require for the computation) with the power of the computation, and is of the general concave form \((n^\alpha P_w)^{\beta}\) – since all time–complexity functions are monotonic, have a common origin, and are otherwise non–intersecting, \(\forall P_{w_i}, P_{w_j}, x\) such that \(P_{w_i} < P_{w_j}\), \(x \in \mathbb{N} : f_{\bar{i}}(P_{w_{i+x}}) - f_{\bar{i}}(P_{w_i}) > f_{\bar{j}}(P_{w_{j+x}}) - f_{\bar{j}}(P_{w_j})\) – where \(\alpha\) and \(\beta\) are free parameters, constrained by the theorems of probability theory (e.g., independence, conditional probability). See appendix B–E.
Where \( \left( \frac{E}{t} \right)_A \) is the available power. To calculate this magnitude we embed it in a continuous function of the general concave form \(# = (n^{\alpha}P_w)^{1/\beta} \), where \( \alpha \) and \( \beta \) are free parameters.

\[
P = \frac{|A|}{|S|} = \frac{(P_w)_A \cdot \#_A - \int_0^{(P_w)_A} \beta n^{\alpha}P_w \, dP_w}{(P_w)_S \cdot \#_S - \int_0^{(P_w)_S} \beta n^{\alpha}P_w \, dP_w}
\]

By construction \( P(A) \in [0, 1] \), \( P(\emptyset) = 0 \), \( P(S) = 1 \), and \( P \) is additive: \( \forall A, B \in F \) such that \( A \cap B = \emptyset \), \( P(A \cup B) = P(A) + P(B) \).

From a computational complexity perspective, the meaning of \( P \) is straightforward:

- \( P = 0 \) means that the desired state is non–Turing–computable, i.e., its realization requires a non–algorithmic process, such as a measurement with infinite precision.

- \( P = 1 \) means that we are ‘at’ the desired state hence its realization requires constant resources. In complexity theory, such a process would be assigned complexity \( O(1) \).

- \( 0 < P < 1 \) means that the given system is ‘\( P \)–distant’ from the desired final state. In other words, \( P \) denotes the transition probability.
between the intermediate states, captured by the size of the set of possible time–complexity functions that are compatible with the resources required for that transition.

5 Ignorance of what?

Consistent with our goal to turn epistemic probability in statistical physics into an objective one, the notion of physical probability here proposed has nothing to do with one’s credence or degrees of belief. It measures, as we have seen, three equivalent physical properties that each pair of physical states objectively possess:

- The difficulty (in terms of physical resources) to realize the transition from one state to another. The more probable a state, the easier it is to reach it from a given state with a given amount of resources.
- The distance (in terms of physical resources) between one state and another. The more probable a state, the shorter is its distance (in terms of physical resources) from the initial state.
- The relative size of the set of time–complexity functions that are compatible with the above two properties. The more probable a state, the larger is this size.

To see how this notion of probability exorcizes “ignorance” from statistical physics, we propose the following intuition.

Start with classical mechanics. Here the common lore traces probabilistic statements to dynamical sensitivity of initial conditions. Omniscient beings such as Laplace’s demon, or so the story goes, could predict with certainty any possible outcome of a dynamical evolution. Short of this a power, finite creatures such as ourselves are constrained to introduce error into their predictions. What we suggest here is that this error has physical meaning, captured by our notion of probability.

Agreed, the intuition that “information is physical” is almost a century old (e.g., Szilard [1929] 2002; Landauer 1996), but so far the attempts to

\footnote{Given that such evolutions are restricted to Turing-computable ones. See Pitowsky 1996.}

\footnote{On another interesting relation between error and complexity costs see Traub & Werschultz 1999.}
make it precise have resulted in much confusion. Our new proposal makes clear exactly where and how “information” becomes “physical”: suppose we would like to predict a certain outcome of an experiment in the lab using classical mechanics. Unless we already possess the initial state of our experiment, we need to realize it. The preparation of the initial state, starting from a specific known state we do possess, requires physical resources (in this case, precision). If these are limited, then our probability $P$ quantifies how far we are from the ideal initial state, or equivalently, what is the error, $\epsilon$, in our preparation, where $P = 1 - \epsilon$. And since resources are always limited, we never start an experiment in the ideal initial state, and so we have an error in the final state, whence objective uncertainty.

One might argue that the uncertainty principle already turns the classical notion of ignorance into an objective one, as it constrains the precision of state–preparation by imposing limitations on the simultaneous measurements of non commuting observables such as position and momentum. There is something to this intuition, but our new notion of probability helps to make it more precise, revealing its fundamental origins. The idea is the following: precision requires error–correction, which is itself a computation with a certain frequency. In other words, precision is a composite physical resource that can be broken down further into more primitive resources such as energy and time. Thus position–momentum is not the relevant uncertainty here – time–energy is – and this means that the inability to predict with probability 1 the state of a physical system governed by deterministic dynamics originates much deeper.

Recall that in quantum mechanics the evolution of a physical system is given by the propagator $U_\beta = e^{-iHt}$ where $H$ is the Hamiltonian of the system. Now, our attempt to estimate this propagator introduces an error and results in an approximate propagator, $U_\alpha$, where the error is given by

$$\epsilon = d(U_\beta, U_\alpha) = \|U_\beta - U_\alpha\|_{op} = \sup_{\|\psi\|=1} \| (U_\beta - U_\alpha) |\psi\rangle \|_{C^2}. \hspace{1cm} (8)$$

where $U_\alpha$ and $U_\beta$ are two different rotation operators along, say, the $y$ axis, and $0 < \alpha < \beta < \frac{\pi}{2}$. Here, again, the error $\epsilon$, and therefore our notion of probability, $P = 1 - \epsilon$, quantify the distance (in terms of energy and

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25 The idea that “information is physical” in the sense that erasure of information increases entropy (“Landauer’s principle”) was conceived in the context of the long–standing attempts to exorcize Maxwell’s demon. See Earman & Norton 1999 for a devastating criticism of these attempts and the above principle.

26 Mathematically, the estimation is done using a Taylor series, truncating the exponential and approximating it with a sum of polynomials.
time) between two physical states – in this case, the distance, relative to a common initial energy state, between the “ideal” energy state and the energy state we can prepare with the physical resources that are available to us. One can prove that \( d(U_\beta, U_\alpha) = 2(1 - \cos(\beta - \alpha)) \) so, by denoting the error \( \epsilon = d(U_\beta, U_\alpha)/2 \), it follows that our probability is related to the quantum probability, namely, \( P_{\text{complexity}} = \sqrt{P_{QM}} \).

What is the physical meaning of “preparation” and “estimation”? Physically, these are measurements – in our case, measurements of energy. But a proper energy measurement probes the time evolution, and therefore cannot be done with arbitrary time, unless one knows in advance the Hamiltonian of the system at hand. Rather, the time required to perform the measurement and the precision of the measurement are constrained by the time–energy uncertainty relation. For this reason, unless we know in advance the Hamiltonian that governs the dynamics of a physical system, we can never predict its future behavior with probability 1. To know the Hamiltonian, however, we need to measure the system’s energy, whence objective uncertainty.

6 Deterministic Chances without an Observer

We have just shown how our new definition of physical probability can turn notions such as “ignorance” or “lack of knowledge” into an objective feature of the world. Since the real meaning of energy in physics is that of governing the time evolution of a system, the true nature of unpredictability in statistical physics resides in the inability to estimate precisely the (deterministic) dynamics of a physical system.

One could argue, however, that the foregoing analysis is adequate only for situations that involve predictions, hence an agent, or an observer. Inside one’s lab, for example, the available physical resources are always limited, and so it makes sense to say that one is “ignorant” of the dynamical evolution (in the above sense of estimating the Hamiltonian by measuring its energy state). But when the physical state at hand is a state outside the lab, say, a hypersurface of the universe or any subsystem thereof, and there

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Aharonov & Bohm (1961), for example, proposed a clever way to bypass the time–energy uncertainty principle by measuring a spin-half particle in a magnetic field. Later it was shown that this ability to bypass the time–energy uncertainty relation rests on the prior knowledge of the Hamiltonian: if one knows the Hamiltonian, one need not spend any time (hence computational – qua physical – resources) in estimating it, but whenever the Hamiltonian of a system is unknown, determining it to precision \( \Delta H \) requires a time \( \Delta t \) given by \( \Delta t \Delta H \geq 1 \). See Aharonov, Massar, & Popescu 2002.
are no observers, then it makes no sense to say that Nature “estimates” its own Hamiltonian, or is “unaware” of it. And so, if our notion of probability requires an observer, then it is not objective at all. In other words, outside the lab, where there are no observers, there is also no place for probability.

This argument, phrased in terms of the above conditional, is clearly valid, and there are two strategies to approach it.

The first strategy, call it OBSERVER, is to agree that the argument is also sound, and to accept that our notion of probability does require an observer. On this view, we have only translated the observer’s ignorance into physical terms. We do not believe that this is the case here, but we use this opportunity to note that even if the above argument were sound, our “translation” has merit as it clarifies some open questions in the foundations of QM.

The second strategy, call it DYNAMICS, is to deny the soundness of the above argument, and to insist that our notion of probability requires no observer to be defined. We believe this view is the correct one, and to this end we shall demonstrate that while our notion may be regarded as a “translation” of the observer’s ignorance into physical terms, in effect it has nothing to do with observers, knowledgable or otherwise. Rather, the important feature of our objective notion of probability is that it is a dynamical notion. From this standpoint, the state inside the lab is completely equivalent to any physical state outside it, and our objective probability equally applies in both cases.

We start with OBSERVER. Here we would like to point out that the ability to use our model in one’s lab and to translate the observer’s “ignorance” into physics using measurement errors – quantified with complexity terms – is already a significant step forward. In effect, since the characterization of macroscopic experiments – even the most mundane ones such as a toss of a fair coin – is quite daunting from the perspective of physical computational complexity, the best way to test our model is to compare its predictions to the predictions of quantum mechanics in specific microscopic experiments where the initial and final states are completely characterized energetically.

From a philosophical perspective, however, note that our “translation” offers a new interpretation to quantum probabilities, and thus adds a possible alternative to the long–standing feud in the foundations of QM.

Recall that on the subjective view of QM, the quantum state is treated

\[\text{[28]Indeed, we would encourage physicists to interact more with complexity theorists in order to quantify the physical resources required for certain experiments, and classify physical states according to their respective physical computational complexity. On such an attempt see Aaronson & Arkhipov 2010.}\]
as a state of knowledge, and quantum probabilities (calculated by the Born rule) are interpreted as “gambling bets” of agents on results of experiments, a-la Ramsey–De Finetti (e.g., Fuchs 2010). In contrast, in Bohmian mechanics, yet another epistemic approach in the foundations of QM, the probabilities are probabilities for particles to have certain positions. The above insistence on “observers” – considered by proponents of the subjective view as their claim to fame (e.g., Fuchs & Peres 2000) – is mocked by Bohmians as ontologically vague; the whole point behind arguments about typicality is to free the discussion from such notions. Our new idea about probability simply avoids this debate altogether by supplying an alternative: yes, one may interpret the quantum state as a state of incomplete knowledge if one really insists on doing so, but this need not entail a choice between two evils (hidden variables vs. instrumentalism in the guise of agnosticism). On our view, what quantum probabilities are probabilities for is neither the positions of particles, nor the gambling bets of learned observers. Rather, quantum probabilities simply quantify how hard it is to realize a physical state; they measure the ‘distance’ between the current state of a physical system and any other state thereof, given the resources (energy/time) that are available to that system at that moment.

Now let’s move to DYNAMICS. As we have stressed, we believe that our notion of probability has nothing to do with observers, knowledge, or ignorance. Indeed, if objective probability is a physical feature of the world, then it must apply even in observer–free scenarios.

In order to show that it does, we remind the reader that our new notion is dynamical, and so the real question is not about observers (inside the lab) or lack thereof (outside it), but about dynamics vs. statics: in the lab we let an experiment run; we start in one state and we end in another, and so we interact with a system that is always changing. Outside the lab the universe as a whole runs its course without anything interacting with it, which means that its Hamiltonian is stationary. Nothing happens to the universe as a whole; its dynamics is completely frozen. A stationary Hamiltonian means a static universe; a static universe means no change in the energy state. Whence our true problem: how can we apply our dynamical probability model to the universe as a whole?

There are two ways out of this conundrum. The first, more radical, is to argue that the Hamiltonian of the universe is actually time–dependent and not stationary. In such a scenario, there is an energy difference between

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29 The issue relates to the famous problem of time in quantum gravity. See, e.g., Butterfield and Isham 2001.
the current state of the universe and any consecutive state thereof, hence it makes sense to say that, while the dynamics of the universe as a whole is deterministic, in any given moment there is an objective notion of transition probability that measures the distance (in terms of energy and time) between the current energy state of the universe and any future possible energy state thereof, given the physical resources that are available to it at any given moment.

Luckily, such a radical solution, in which the universe is regarded as an open system and time–translation–invariance (and with it the conservation of energy) fail globally, is not the only way out. Another, less radical, way to solve this problem is to invoke the empiricist perspective our model stems from. On this view, one may resist the temptation to regard the universe as a whole as a meaningful physical system, as the only systems that are available to physical investigations are subsystems of the universe. If we restrict ourselves to these, and, further, acknowledge that subsystems are always interacting with the rest of the universe, then we have, again, a way to resurrect objective probability in a deterministic dynamical setting.

While the total Hamiltonian of the universe, $H_T$, is indeed stationary,

$$H_T = H_S + H_E + H_I \quad \text{and} \quad \frac{dH_T}{dt} = 0.$$  

(9)

the Hamiltonians of any subsystem $H_S$, its respective environment $H_E$, and the interaction between the two $H_I$, need not be stationary as long as they balance each other. But if the latter three do change locally, we have again the means to erect our model and measure, on the functional $f$ above, how far a given subsystem in a given energy state is from any other possible energy state thereof, given the physical resources that are available to it at any given moment.

7 Conclusion

We often hear locutions such as “30% chance of snow in NYC tomorrow”. What could such a sentence possibly mean?

As we have argued, it cannot possibly mean that we have run 1000 computer simulations and 300 of them resulted in snow around NYC: even if this were the relative frequency of computer simulations that predicted

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30For a way to define local dynamics within a globally static universe see, e.g., Page & Wootters 1983.
snow in NYC, extrapolating from this finite sample would require typicality assumptions and these are unjustifiable. Are the chances in the above locution, then, merely our degrees of belief about the possible future? We (and arguably weather scientists with us) would like to believe that weather science – or any physical science – is aimed at describing the natural world, and not at describing our beliefs about that world, and so the chances in the above locution had better be chances in the world, and not in our mind.

Our new model for objective probability can give sense to such locutions and bring weather science – indeed statistical physics in general – from the realm of statistical inference back to its original cradle. On our view, an exact meaning to this locution can be given only when one specifies the couple \( \{n, E, t\} \), i.e., the dimension of the system at hand, the energy, and the time available. Given this couple, our model can (in principle) yield a precise number for the probability of the physical state “Snow in NYC”, given the current state at NYC.

To sum up, the following is what the above locution should mean, if the chances it refers to are to be considered objective deterministic chances:

- The difficulty (in terms of physical resources) to realize in one day the transition from the current state in NYC to the state “Snow in NYC”. The higher the probability for the state “Snow in NYC”, the easier (in terms of computational complexity) it is to reach it from the current state in NYC.

- The distance (in terms of the physical resources to be spent in one day) between the current state in NYC and the state “Snow in NYC”. The higher the probability, the shorter is this distance (in terms of physical resources).

- The relative size of the set of time–complexity functions that are compatible with the given resources required for “Snow in NYC” to occur in one day. The higher the probability, the larger is this size.

The amount of physical resources that separate two physical states is an objective feature of the world. Computational complexity theory simply allows us to map this feature onto \([0, 1]\), and this mapping, as we have shown, has all the characteristics of a discrete probability function.
Appendix

7.1 Measures

Let us consider an arbitrary exponential function $c^n$ and its associated probability:

$$P(c^n) = [\mu - \frac{\arctan(c^n \ln c)}{\pi/2}]\alpha. \quad (10)$$

where

$$\alpha = \frac{1}{\sum_{f_i|f_i \in \text{Poly}} \xi_{f_i} + \sum_{f_i|f_i \in \text{Exp}} \xi_{f_i}}. \quad (11)$$

It is straightforward to show that for a large dimension $n$

$$\forall c : P(c^n) \approx (\mu - \mu)\alpha = 0. \quad (12)$$

In contrast, since for a large dimension $\frac{\arctan(c^n \ln c)}{\pi/2} \approx 1$, the unnormalized measure $\xi_p$ on $\text{Poly}$ doesn’t change and is $1 - \mu$, but when we normalize, we get a resolution of the identity

$$P(f(n)) = \alpha \xi_p = \frac{1 - \mu}{i(1 - \mu)} = i^{-1} \quad (13)$$

where $i$ is the number of polynomial functions in $\text{Poly}$.

7.2 Joint Probability

To calculate joint probability in case of independence one needs to realize a new probability space $S_3$ from two given probability spaces $S_1$ and $S_2$, where the new couple $\{n_3, (E_t)_3\}$ is given by the respective sums, i.e., $n_3 = n_1 + n_2$ and $(E_t)_3 = (E_t)_1 + (E_t)_2$.

- $S_3$ is defined accordingly as:

$$S_{n_3} = \{g \in \Omega | O(g_{n_3}) \leq \#(\left(\frac{E}{t}\right)_3)\} \quad (14)$$

- The probability measure $P$ is given, again, by the mapping

$$\forall A \in F : P_{\{n_A, (E_t)_A\}}(A) = \frac{|A|}{|S_3|} \quad (15)$$
• One can calculate the joint probability for a combined state \( C = A \cap B \) by taking into considerations the way the physical resources are distributed between the two components of the combined system. Since we are constructing a new probability space, additional constraints must be satisfied to maintain the appropriate relations between this new space and the earlier, atomic ones.

1. We define equivalent processes to have the same power and the same dimension:
   - \( (E_T)_1 = (E_T)_2 \).
   - \( n_{A \cap B} = n_1 + n_2 = 2n_1 = 2n_2 \).

   So in this case, since all interactions are local, the total available power is \( (E_T)_{A \cap B} = (E_T)_1 \frac{n_1}{n_1 + n_2} + (E_T)_2 \frac{n_2}{n_1 + n_2} \).

2. For such processes, since \( n \) increases and the power remains the same, it follows that \( P_3(A \cap B) < P_1(A); P_3(A \cap B) < P_2(B) \) (where the \( P \)s are calculated for each case separately).
3. If $B$ depends on $A$, $P_3(A \cap B) > P_1(A)P_2(B)$. This case is accommodated in our model by noticing that the computation time for $B$ already includes the computation time for $A$, hence the total computation time is shorter than the computation time in the case of independence, hence the power of the computation is higher than the case of independence, as required.

7.3 Conditional Probability

Conditional probability $P(A|B)$ is calculated by rescaling $S_1$ to fit $S_2$ (in terms of $n$), and by calculating the ratio $P(A \cap B)/P(B)$.

7.4 Constraints

These requirements can be used to constrain $f$: when we blow up the dimension by a factor of $\gamma$ and keep the power fixed, the derivative goes down by a factor of $\gamma^\alpha$; when we blow up the dimension and keep the derivative fixed, the power goes up by a factor of $\gamma^{\frac{\alpha}{\beta - 1}}$. For non-equivalent
processes, where \((\frac{E}{t})_A = \delta (\frac{E}{t})_B\) \((0 < \delta < 1)\), we can show that
\[
(\frac{E}{t})_A < (\frac{E}{t})_{A \cap B} < (\frac{E}{t})_B. \tag{16}
\]
If we require further that the power of \(A\) grows less than the “blow–up effect” (that is, that as \(n\) increases the Power axis increases less rapidly than the number–of–steps axis)
\[
(\frac{E}{t})_{A \cap B} < \left( 1 + \frac{1 - \delta}{\delta \gamma} \right). \tag{17}
\]
we get a constraint on \(\alpha\) and \(\beta\):
\[
\frac{\alpha}{\beta - 1} > \log_{\gamma} \left( 1 + \frac{1 - \delta}{\delta \gamma} \right). \tag{18}
\]

7.5 Proofs

For the sake of simplicity, and without loss of generality instead, of taking the function \# = \((n^\alpha P_w)^{1/\beta}\), we concentrate on the inverse function \(P_w = n^{-\alpha} \#^{\beta}\), while recalling that for any continuous, derivable, and monotonic function \(g\), \((g^{-1})'(x) = \frac{1}{g'(y)}\). Let \((\frac{E}{t})_A = P_w A, (\frac{E}{t})_B = P_w B,\) and \(n_A = \Delta n_B\), where \(\Delta \in \mathbb{R}, \Delta > 1\), and let 0 < \(\frac{P_w_A}{P_w_B} = \delta \leq 1\). We prove that: \(P_w A \leq P_w \cap B \leq P_w B\) (equality holds only when \(P_w A = P_w B\)):
\[
P_w A \cap B = P_w A \frac{\Delta n_B}{(\Delta + 1)n_B} + P_w B \frac{n_B}{(\Delta + 1)n_B} = \tag{19}
\]
\[
= \delta P_w B \frac{\Delta}{\Delta + 1} + P_w B \frac{1}{(\Delta + 1)n_B} = \tag{20}
\]
\[
= \frac{\Delta \delta + 1}{\Delta + 1} P_w B = \frac{\Delta \delta + 1}{\Delta \delta + \delta} P_w A. \tag{21}
\]
But,
\[
\frac{\Delta \delta + 1}{\Delta + 1} \leq 1; \quad \frac{\Delta \delta + 1}{\Delta \delta + \delta} \geq 1. \tag{22}
\]
If we blow up \(n\) by a real factor \(\gamma > 1\) and we keep the power fixed:
\[
f_n(\#) = \frac{1}{n^\alpha} \#^\beta; \quad f'_n(\#) = \frac{\beta}{n^\alpha} \#^{\beta - 1}. \tag{23}
\]
Analogously:
\[
f'_{\gamma n}(\#) = \frac{\beta}{(\gamma n)^\alpha} \#^{\beta - 1}. \tag{24}
\]
So,

\[
\frac{f'_{\gamma n}(\#)}{f'_{\bar{n}}(\#)} = \gamma^{-\alpha}.
\]  
(25)

If we blow up \( n \) by a real factor \( \gamma > 1 \) and we keep the derivative fixed:

\[
P_\gamma = f_{\bar{n}}(\#) = \frac{1}{n^\alpha} #^\beta.
\]  
(26)

The derivative for a number of steps \( \# \) will be

\[
f'_{\bar{n}}(\#) = \frac{\beta}{n^\alpha} #^{\beta-1}.
\]  
(27)

Analogously, for the new (blown–up) power \( \tilde{P}_\gamma \)

\[
\tilde{P}_\gamma = f'_{\tilde{\gamma} n}(\tilde{\#}) = \frac{\beta}{(\gamma n)^\alpha} \tilde{\#}^{\beta-1}.
\]  
(28)

In order to maintain the same derivative, we find the value of \( \tilde{\#} \) such that:

\[
\frac{\beta}{(\gamma n)^\alpha} \tilde{\#}^{\beta-1} = \frac{\beta}{n^\alpha} #^{\beta-1}.
\]  
(29)

We obtain:

\[
\tilde{\#} = \gamma^{\frac{\alpha}{\beta-1}} \#.
\]  

So,

\[
f_{\gamma n}(\gamma^{\frac{\alpha}{\beta-1}} \#) = \frac{1}{(\gamma n)^\alpha} (\gamma^{\frac{\alpha}{\beta-1}} \#)^\beta = \gamma^{\frac{\alpha}{\beta}} f_{\bar{n}}(\#).
\]  
(30)

Hence, \( \tilde{P}_\gamma = \gamma^{\frac{\alpha}{\beta-1}} \). Finally, we know that \( \frac{P_\gamma}{P_{\delta A}} = \frac{\Delta \delta + 1}{(\Delta + 1)\delta} \) and that \( \frac{P_{\delta A}}{P_{\delta A}} = (\Delta + 1)^{\frac{\alpha}{\beta-1}} \). Let us recall that \( \gamma = \Delta + 1 \) and that \( 0 < \delta \leq 1 \). We have to prove that there is some constraint on \( \alpha \) and \( \beta \) such that \( \frac{P_\gamma}{P_{\delta A}} < \frac{P_{\delta A}}{P_{\delta A}} \). From

\[
\frac{a \delta + 1}{(\Delta + 1)\delta} < (\Delta + 1)^{\frac{\alpha}{\beta-1}}.
\]  
(31)

we obtain that

\[
\frac{\alpha}{\beta-1} > \log_{\gamma} \frac{\Delta \delta + 1}{\Delta \delta + \delta}.
\]  
(32)

But

\[
\frac{\Delta \delta + 1}{\Delta \delta + \delta} = \frac{\Delta \delta + 1 + \delta - \delta}{\delta(\Delta + 1)} = \frac{\delta(\Delta + 1) + 1 - \delta}{\delta(\Delta + 1)} = 1 + \frac{1 - \delta}{\delta \gamma}.
\]  
(33)

so our constraint is

\[
\frac{\alpha}{\beta-1} > \log_{\gamma} \left( 1 + \frac{1 - \delta}{\delta \gamma} \right).
\]  
(34)
Reference

Aaronson, Scott, and Alex Arkhipov (2010), “The Computational Complexity of Linear Optics”, http://arxiv.org/abs/1011.3245

Aharonov, Yakir, and David Bohm (1961), “Time in the Quantum Theory and the Uncertainty Relation for Time and Energy”, Physical Review 122(5): 1649–1658.

Aharonov, Yakir, Serge Massar, and Sandu Popescu (2002), “Measuring Energy, Estimating Hamiltonians, and the Time–Energy Uncertainty Relation”, Physical Review A 66: 5107.

Albert, David (2000), Time and Chance, Harvard: Harvard University Press.

Butterfield, Jeremy, and Chris J. Isham (2001), “Space-time and the Philosophical Challenges of Quantum Gravity”, in C. Callender and N. Hugget (eds.) Physics Meets Philosophy at the Planck Scale, Cambridge: Cambridge University Press, 33–89.

Caves, Carlton, Chris Fuchs, and Rudiger Schack (2002), “Quantum Probabilities as Bayesian probabilities”, Physical Review A 65: 022305.

Childs, Andrew, John Preskill, and Joseph Renes (2000), “Quantum Information and Precision Measurement”, Journal of Modern Optics 47(213): 155–176.

Earman, John, and John Norton (1993), “Forever is a Day: Supertasks in Pitowsky and Malament–Hogarth Spacetimes”, Philosophy of Science 60: 22–42.

______________ (1999), “EXORCIST XIV: The Wrath of Maxwells Demon. Part II. From Szilard to Landauer and Beyond”, Studies in History and Philosophy of Modern Physics 30: 1–40.

Fermi, Enrico, J. Pasta, S. Ulam, and M. Tsingou (1955), “Studies of Nonlinear Problems I”, Los Alamos preprint LA-1940.

Frigg, Roman (2007), “Probability in Bolzmannian Statistical Mechanics”, in G. Ernst and A. H Utemann (eds.) Time, Chance and Reduction, Philosophical Aspects of Statistical Mechanics, Cambridge: Cambridge University Press.

Fuchs, Chris (2010), “QBism, the Perimeter of Quantum Bayesianism”, http://arxiv.org/abs/1003.5209

Fuchs, Chris, and Asher Peres (2000), “Quantum Theory Needs No Interpretation”, Physics Today 53: 70–71.

Geroch, Robert, and James Hartle (1986), “Computability and Physical Theories”, Foundations of Physics 16(6): 533–550.

Goldstein, Sheldon, Joel, L. Lebowitz, Roderich Tumulka, and Nino Zanghi (2010), “Long-Time Behavior of Macroscopic Quantum Systems: Commentary Accompanying the English Translation of John von Neumann’s 1929 Article on the Quantum Ergodic Theorem”, http://arxiv.org/abs/1003.2129v1
Hagar, Amit, (2003), “A Philosopher Looks at Quantum Information Theory”, Philosophy of Science 70: 752–775.

Hahn, Erwin (1950), “Spin Echoes”, Physical Review 80: 580–594.

Hartmanis, J., and R.E. Stearns (1965) “On the Computational Complexity of Algorithms”, Transactions of the American Mathematical Society 117: 285–306.

Hemmo, Meir, and Orly Shenker (2011a), “Probability and Typicality in Physics”, forthcoming in Probability in Physics, Essays in Memory of Itamar Pitowsky, The Frontiers Collection, Berlin: Springer.

Hemmo, Meir, and Orly Shenker (2011b), “A New Problem in the Frequentist Approach to Probability”, forthcoming in Mind.

Hogarth, Mark, “Non–Turing Computers and Non–Turing Computability”, in D. Hull, M. Forbes, and R. M. Burian (eds.), PSA 1994, Vol. 1. East Lansing: Philosophy of Science Association, 126–138.

Irvine, Andrew (2011), “Frege on Number Properties”, Studia Logica 96: 239–260.

Landauer, Rolf, (1996), “The Physical Nature of Information”, Physics Letters A 217: 188–193.

Lewis, David (1986), Philosophical Papers (Vol. 2). Oxford: Oxford University Press.

Maudlin, Tim (2007), ‘What could be objective about probabilities?’, Studies in History and Philosophy of Modern Physics 38: 275–291.

Page, Don, and William K. Wootters (1983), “Evolution Without Evolution: Dynamics Described by Stationary Observables”, Physical Review D 27: 2885–2892.

Pitowsky, Itamar (1985), “On the Status of Statistical Inferences”, Synthese 63(2): 233–247.

Pitowsky, Itamar (1990), “The Physical Church Thesis and Physical Computational Complexity”, Iyyun 39: 87–99.

Pitowsky, Itamar (1996), “Laplace’s Demon Consults an Oracle: The Computational Complexity of Prediction”, Studies in History and Philosophy of Modern Physics 17: 161–180.

Pitowsky, Itamar (2011), “Typicality and the Role of the Lebesgue Measure in Statistical Mechanics”, forthcoming in Probability in Physics, Essays in Memory of Itamar Pitowsky, The Frontiers Collection, Berlin: Springer.

Pitowsky, Itamar, and Oron Shagrir (2003), “Physical Hypercomputation and the Church–Turing Thesis”, Minds & Machines, 13: 87–101.

Pour–el, Marian and Ian Richards (1989), Computability in Analysis and Physics, Berlin: Springer.
Sklar, Lawrence (1993), *Physics and Chance*, Cambridge: Cambridge University Press.

Strevens, Michael (1998), “Inferring Physical Probabilities from Symmetries”, *Nous* 32: 231–246.

Szilard, Leo ([1929] 2002), “On the Decrease in Entropy in a Thermodynamic System by the Intervention of Intelligent Beings”, *Zeitschrift für Physik* 53: 840–856. English translation by Anatol Rapoport and Mechthilde Knoller in Harvey Leff and Andrew Rex (eds.) *Maxwell’s Demon* 2, 110–119.

Traub, Josef, and Arthur Werschultz (1999), *Complexity and Information*, Cambridge: Cambridge University Press.

Uffink, Jos (2011), “Subjective Probability and Statistical Physics”, in C. Beisbart and S. Hartmann (eds.) *Probabilities in Physics*, Oxford: Oxford University Press.

Vidal, Guifre, and Ignacio Cirac (2002), “Optimal Simulation of Nonlocal Hamiltonians Using Local Operations and Classical Communication”, *Physical Review A* 66, 022315.

Winnie, John A. (1992), “Computable Chaos”, *Philosophy of Science* 59(2): 263–275.