DIFFERENCE OF A HAUPTMODUL FOR $\Gamma_0(N)$ AND CERTAIN GROSS-ZAGIER TYPE CM VALUE FORMULAS

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Dedicated to the memory of my grandparents

Abstract. In this work, we show that the difference of a Hauptmodul for a genus zero group $\Gamma_0(N)$ as a Hilbert modular function on $Y_0(N) \times Y_0(N)$ is a Borcherds lift of type $(2,2)$. As applications, we derive Monster denominator formula like product expansions for these Hilbert modular functions and certain Gross-Zagier type CM value formulas.

1. Introduction

In his seminal work \cite{borcherds1,borcherds2}, Borcherds develops a remarkable method to construct meromorphic modular forms $\Psi(z, f)$ on an orthogonal Shimura variety associated to some rational quadratic space of signature $(n,2)$ from some weakly holomorphic modular form $f$ for the Weil representation of $SL_2(\mathbb{Z})$ via regularizing an integral called theta-lift against the Siegel theta function. We now call such meromorphic modular forms Borcherds lifts of type $(n,2)$. Moreover, Borcherds shows that $\Psi(z, f)$ has a beautiful product representation called Borcherds product for its Fourier expansion near a cusp of the orthogonal Shimura variety. One of the most famous Borcherds lifts is the difference $j(\tau_1) - j(\tau_2)$ of the well known Klein’s modular $j$-invariant as a Hilbert modular function for $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, which a Borcherds lift of type $(2,2)$ with $j-744$ as its theta-lift input, namely, $j(\tau_1) - j(\tau_2) = \Psi(z, j - 744)$. If we write $j(\tau) = q^{-1} + \sum_{n \geq 0} c(n) q^n$ with $q := \exp(2\pi i \tau)$, then computing the Borcherds product of $\Psi(z, j - 744)$ near the cusp of the underlying Shimura variety of type $(2,2)$, which is identified with the cusp $(i\infty, i\infty)$ of $Y(1) \times Y(1)$, we can recover the famous Monster denominator formula \cite{monstrous}, namely,

$$j(\tau_1) - j(\tau_2) = (q_1^{-1} - q_2^{-1}) \prod_{m,n > 0} (1 - q_1^m q_2^n)^{c(mn)}.$$ 

Furthermore, another interesting application of the fact that $j(\tau_1) - j(\tau_2)$ is a Borcherds lift $\Psi(z, j - 744)$ is to serve as a key ingredient in the use of the so called big CM value formula \cite{grosszagier} in reproving the interesting and famous Gross-Zagier CM value formula \cite{grosszagier}, namely,

$$\left| \text{Norm} \left( j \left( \frac{d_1 + \sqrt{d_1}}{2} \right) - j \left( \frac{d_2 + \sqrt{d_2}}{2} \right) \right) \right|_{w_1w_2}^{\frac{g}{2}} = \prod_{\substack{x,n,n' \in \mathbb{Z} \\ n,n' > 0 \atop x^2 + 4nn' = d_1d_2}} n^{\epsilon(n')},$$

where $d_1$ and $d_2$ are coprime negative fundamental discriminants, $\epsilon(n')$ is multiplicative, and $\epsilon(p)$ is defined via the local Hilbert symbol at a prime $p$. Such a formula was first discovered and proved by Gross and Zagier \cite{grosszagier}, and it presents a beautiful and remarkable prime factorization formula.
for the rational norm of the algebraic integer \( j \left( \frac{d_1 + \sqrt{d_1}}{2} \right) - j \left( \frac{d_2 + \sqrt{d_2}}{2} \right) \) and reveals the arithmetic information encoded in the exponents of the prime factors via Hilbert symbols. The reproof of the Gross-Zagier CM value formula as mentioned above has been recently worked out by Yang and Yin [29], and it yields an equivalent form of (1.1) as follows.

**Theorem 1.1 (Gross and Zagier).** Let \( E_i = \mathbb{Q}(\sqrt{d_i}) \) be two imaginary quadratic fields of fundamental discriminants \( d_i \) with \( (d_1, d_2) = 1 \). Let \( F = \mathbb{Q}(\sqrt{D}) \) with \( D = d_1d_2 \) and \( E = E_1E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \). Then

\[
\sum_{([a_1],[a_2]) \in \text{Cl}(E_1) \times \text{Cl}(E_2)} \log |j(\tau_{a_1}) - j(\tau_{a_2})|^{1/8} = \sum_{t = 2^{m+D} + \sqrt{D}} \sum_{p \text{ inert in } E/F} \frac{1 + \text{ord}_p(O_E)}{2} \rho(t^{-1}p) \log(N(p)),
\]

where \( \text{Cl}(E_i) \) is the ideal class group of \( E_i \), \( \tau_a = \frac{b + \sqrt{a}}{2a} \) is the unique CM point in the upper half plane \( \mathbb{H} \) given by the integral ideal \( a = [a, \frac{b+\sqrt{a}}{2a}] \), \( w_i \) is the number of roots of unity in \( E_i \), and for an integral ideal \( a \) of \( F \)

\[
\rho(a) := |\{ \mathfrak{B} \subset O_E : N_{E/F}(\mathfrak{B}) = a \}|
\]

which can be computed via calculating its local factors \( \rho_q(a) \) (see (6.4)).

**Remark 1.2.** To see how (1.2) is equivalent to (1.1), we refer the reader to [35] Eq. (7.1) and [29, Remark 4.1].

All of these interesting phenomenon and relation mentioned above that are related to \( j(\tau) \) have greatly motivated us to look into the genus zero Hecke subgroups \( \Gamma_0(N) \) cases. Now recall that for a genus zero congruence subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{R}) \) commensurable with \( \text{SL}_2(\mathbb{Z}) \), the function field on \( X(\Gamma) \) can be generated by a single modular function, and such function is called a Hauptmodul for \( \Gamma \) if it has a unique simple pole of residue 1 at the cusp \( i \infty \), i.e., it has Fourier expansion of the form \( q^{-1/h} + c(0) + c(1)q^{1/h} + \cdots \) with \( q = \exp(2\pi i \tau) \) at the cusp \( i \infty \) where \( h \) is the width of the cusp \( i \infty \). In this work, we first aim to extend the fact that \( j(z_1) - j(z_2) \) is a Borcherds lift of type \((2,2)\) to any genus zero groups \( \Gamma_0(N) \) and show that the difference of a Hauptmodul \( \pi_N(\tau) \) for a genus zero group \( \Gamma_0(N) \) as a Hilbert modular function for \( \Gamma_0(N) \times \Gamma_0(N) \) is a Borcherds lift of type \((2,2)\) by a uniform approach. These extend Scheithauer’s results [22], in which he works only on \( N \) square free using the twisted denominator identity of the Monster Lie algebra. The method we use is different from Scheithauer’s and is more natural from the analytic point of view. Later, employing the big CM value formula [9] together with the fact that \( \pi_N(z_1) - \pi_N(z_2) \) is a Borcherds lift shown in this work, we derive Gross-Zagier type CM value formulas for \( \pi_p(\tau) \) for \( p \in \{3, 5, 7, 13\} \), the only odd primes such that \( \Gamma_0(p) \) are of genus zero.

**Remark 1.3.** The way we compute the Gross-Zagier type CM value formulas in this work was first initiated in [29] in which Yang and Yin prove a Gross-Zagier type CM value formula for \( \pi_2(\tau) \) first conjectured by Yui and Zagier [32]. In general, one can similarly obtain Gross-Zagier type CM value formulas for all \( \pi_N(\tau) \) by carefully computing the constant terms of the corresponding theta-lift input and relevant local Whittaker functions (see Lemma [3.3] and Subsection [7.2]). We refer the reader to [29, 30] for comprehensive descriptions of these computations, and leave the details to the reader.

**Remark 1.4.** Similar formulas for \( \Gamma_0(N)+ \) with \( N \) square free have first been worked out in [7] by Borcherds. In recent work [11], Carnahan obtains similar formulas for completely replicable modular functions.
Now we state the first main result of this work.

**Theorem 1.5.** Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ for $N \geq 2$. Then $\pi_N(z_1) - \pi_N(z_2)$ is a Borcherds lift $\Psi(z, F_N)$ of type $(2, 2)$ for some weakly holomorphic modular function $F_N$ for the Weil representation of $\text{SL}_2(\mathbb{Z})$.

Define, here and throughout the remainder of this work, $\mathcal{C}(\Gamma_1(N))$ to be the set of inequivalent cusps $s = a_s/c_s \in \mathbb{Q}$ of $\Gamma_1(N)$ with $(a_s, c_s) = 1$, $M_s = (a_s \ b_s \ c_s \ d_s)^T \in \text{SL}_2(\mathbb{Z})$, $m_s = (c_s, N)$, and write $h_s = N/m_s$. Note that when $N \neq 4$, $\Gamma_1(N)$ has no irregular cusps, and then $h_s = N/m_s$ is the width of cusp $s \in \mathcal{C}(\Gamma_1(N))$. If $s$ is regular write for $0 \leq t < h_s$

\[(f|M_s)_t = \sum_{n > -\infty} A_s(nh_s + t)q^{(nh_s + t)/h_s}\]

where \[f|M_s = \sum_{n > -\infty} A_s(n)q^{n/h_s}\] and \[f|M_s := f\left(\frac{a_s\tau + b_s}{c_s\tau + d_s}\right)\].

Then we can derive a Monster denominator formula like product expansion for $\pi_N(z_1) - \pi_N(z_2)$ as follows.

**Corollary 1.6.** Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ and define for $d|N$,

\[
\sum_{\ell = -1}^{\infty} A(\ell, d)q^\ell = \frac{2}{\lambda_{2, N}|\Gamma_0(N) : \Gamma_1(N)|} \left\{ \sum_{s \in \mathcal{C}(\Gamma_1(N)) \atop s \ regular} [(\pi_N|M_s)_0 - A_s(0)] + \sum_{s \in \mathcal{C}(\Gamma_1(N)) \atop s \ irregular \ m_s = d} \frac{1}{h_s} [(\pi_N|M_s)_0 - A_s(0)] \right\},
\]

where $\lambda_{2, N} = 2$ or $1$ depending on whether $N = 2$ or not. Then we have

(1.3) \[\pi_N(z_1) - \pi_N(z_2) = (q_1^{-1} - q_2^{-1}) \prod_{m,n > 0 \ mod \ N} \left(1 - (q_1^m q_2^n)^N\right) A(mn, d)\]

where $q_j = \exp(2\pi iz_j)$.

Here are two concrete examples following from Corollary 1.6

**Example 1.7.** Let $\pi_5(\tau)$ be a Hauptmodul for $\Gamma_0(5)$ given by

\[\pi_5(\tau) = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6,\]

where $\eta(\tau)$ is the Dedekind eta function. Then

\[
\left(\frac{\eta(z_1)}{\eta(5z_1)}\right)^6 - \left(\frac{\eta(z_2)}{\eta(5z_2)}\right)^6 = (q_1^{-1} - q_2^{-1}) \prod_{m,n > 0 \ mod \ 5} (1 - q_1^m q_2^n)^A(mn, 5) (1 - q_1^{5m} q_2^{5n})^A(mn, 1),
\]

where

\[
\sum_{\ell = -1}^{\infty} A(\ell, 5)q^\ell = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6 + 6,
\]

\[A(\ell, 1) = a(5\ell) \quad \text{and} \quad \sum_{\ell = 0}^{\infty} a(\ell)q^\ell = 125 \left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6.\]
Example 1.8. Let \( \pi_8(\tau) \) be a Hauptmodul for \( \Gamma_0(8) \) given by
\[
\pi_8(\tau) = \frac{\eta(\tau)^4 \eta(4\tau)^2}{\eta(2\tau)^2 \eta(8\tau)^4}.
\]
Then
\[
\frac{\eta(z_1)^4 \eta(4z_1)^2}{\eta(2z_1)^2 \eta(8z_1)^4} - \frac{\eta(z_2)^4 \eta(4z_2)^2}{\eta(2z_2)^2 \eta(8z_2)^4} = (q_1^{-1} - q_2^{-1}) \prod_{m,n > 0, d|8} \left(1 - (q_1^m q_2^n)^{8}\right)^{A(mn,d)}.
\]
where
\[
\sum_{\ell = 1}^{\infty} A(\ell, 8) q^\ell = \frac{\eta(\tau)^4 \eta(4\tau)^2}{\eta(2\tau)^2 \eta(8\tau)^4} + 4,
\]
\[
\sum_{\ell = 1}^{\infty} A(\ell, 4) q^\ell = 4 - 4 \frac{\eta(\tau)^4 \eta(8\tau)^4 \eta(2\tau)^2}{\eta(4\tau)^{10}},
\]
\[
A(\ell, 2) = a(2\ell) \quad \text{and} \quad \sum_{\ell = 1}^{\infty} a(\ell) q^\ell = 8 - 8e(1/12) \frac{\eta(2\tau)^2 \eta(4\tau)^4}{\eta(8\tau)^2 \eta(\tau + 1/4)^4} = 8 - 8 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 - q^{4n})^4}{(1 - q^{8n})^2 (1 - (iq)^n)^4},
\]
\[
A(\ell, 1) = b(8\ell) \quad \text{and} \quad \sum_{\ell = 1}^{\infty} b(\ell) q^\ell = 32 \frac{\eta(8\tau)^4 \eta(2\tau)^2}{\eta(4\tau)^2 \eta(\tau)^4}.
\]

Following from Corollary 1.6 we can relate the canonical basis elements in \( z_2 \) of the space of weakly holomorphic modular functions for \( \Gamma_0(N) \) with pole supported only at \( i\infty \) to the logarithmic derivative of \( \pi_N(z_1) - \pi_N(z_2) \) with respect to \( z_1 \) as follows.

Corollary 1.9. Let \( \pi_N(\tau) \) be a Hauptmodul for a genus zero \( \Gamma_0(N) \). Then
\[
-\frac{1}{2\pi i} \frac{\pi'_N(z_1)}{\pi_N(z_1) - \pi_N(z_2)} = 1 + \sum_{n=1}^{\infty} P_{N,n}(\pi_N(z_2)) q^n
\]
for some polynomial \( P_{N,n} \) of degree \( n \). Then \( P_{N,n}(\pi_N(z_2)) = a_2^{-n} + O(q_2) \).

The second main results of this work are the Gross-Zagier type CM value formulas in the following.

Theorem 1.10. Let \( E_i = \mathbb{Q}(\sqrt{d_i}) \) be two imaginary quadratic fields of fundamental discriminants \( d_i \) with \( (d_1, d_2) = 1 \). Let \( F = \mathbb{Q}(\sqrt{D}) \) with \( D = d_1 d_2 \) and \( E = E_1 E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \). Let \( \pi_p(\tau) \) be a Hauptmodul of \( \Gamma_0(p) \). Then
\[
\sum_{([a_1],[a_2]) \in S(p,d_1,d_2)} \log |\pi_p(\tau_{\varphi(a_1)}) - \pi_p(\tau_{\varphi(a_2)})|
\]
\[
= -\frac{|S(p,d_1,d_2)|}{32 h(E_1) h(E_2)} \left( \sum_{t = 2m + D \sqrt{D}} a \left( \frac{t}{\sqrt{D}}, \phi_0, 0 \right) + \frac{24}{p-1} \sum_{k=1}^{p-1} a_0(\phi_0, k) \right)
\]
where
\[
S(p,d_1,d_2) := \{([a_1],[a_2]) \in \text{Cl}(E_1, N) \times \text{Cl}(E_2, N) : \exists \text{ fractional ideals } a_i \text{ such that } N(a_1) = N(a_2) \},
\]
Cl($E_{i,p}$) denote the ring class group of conductor $p$ of $E_i$, the map $\varphi$ is $\varphi([a, p^{\frac{b+\sqrt{4}}{2}}]) = [a, \frac{b+\sqrt{4}}{2}]$ for an integral ideal representative $a = [a, p^{\frac{b+\sqrt{4}}{2}}]$ in $\text{Cl}(E_{i,p})$, and $a(\frac{\sqrt{D}}{9}, \phi_{0,0})$ and $a_0(\phi_{0,k})$ are computed and expressed explicitly in Section 6.

In particular, the left hand side of (1.4) can be reformulated in the language of quadratic forms

$$\sum_{([a_1],[a_2]) \in S(p,d_1,d_2)} \log |\pi_p(\tau_{\varphi(a_1)}) - \pi_p(\tau_{\varphi(a_2)})| = \sum_{(Q_1,Q_2) \in S_Q(p,d_1,d_2)} \log |\pi_p(\tau_{Q_1}) - \pi_p(\tau_{Q_2})|,$$

where

$$S_Q(p,d_1,d_2) := \{(Q_1,Q_2) \in Q_d(p) \times Q_{d_2}(p) / \Gamma_0(p), Q_1(1,0) = Q_2(1,0)\}.$$

$Q_d(p)$ denotes the set of primitive and positive definite binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $d$ with $(a,p) = 1$, and $\tau_Q$ is the unique CM point in $\mathbb{H}$ given by $Q(\tau,1) = 0$ for a quadratic form $Q(X,Y)$.

**Example 1.11.** Taking $p = 3$, $d_1 = -8$ and $d_2 = -11$, we can first check that $S(3,-8,-11) = \text{Cl}(E_{1,3}) \times \text{Cl}(E_{2,3})$, and then by computing the corresponding $a(\frac{t}{\sqrt{D}}, \phi_{0,0})$ and $a_0(\phi_{0,k})$ via Section 6, we obtain the following prime factorization for the product of CM values

$$\prod_{([a_1],[a_2]) \in \text{Cl}(E_{1,3}) \times \text{Cl}(E_{2,3})} |\pi_3(\tau_{a_1}) - \pi_3(\tau_{a_2})| = 2^63^37^2.$$

**Remark 1.12.** Although we call (1.4) a Gross-Zagier type CM value formula, one might note that in general, it only tells a formula for the value

$$\frac{1}{|S(p,d_1,d_2)|} \sum_{([a_1],[a_2]) \in S(p,d_1,d_2)} \log |\pi_p(\tau_{\varphi(a_1)}) - \pi_p(\tau_{\varphi(a_2)})|,$$

where the set $S(p,d_1,d_2)$ is not explicit in general. The appearance of $S(p,d_1,d_2)$ follows from an injection of a big CM cycle of a Shimura variety into $\text{Cl}(E_{1,p}) \times \text{Cl}(E_{2,p})$ (see Lemma 5.3). If one had $S(p,d_1,d_2) = \text{Cl}(E_{1,p}) \times \text{Cl}(E_{2,p})$, then (1.4) does give a formula for the product of CM values

$$\prod_{([a_1],[a_2]) \in \text{Cl}(E_{1,p}) \times \text{Cl}(E_{2,p})} |\pi_p(\tau_{a_1}) - \pi_p(\tau_{a_2})|$$

as we have seen in Example 1.11 and it is more “Gross-Zagier-type”. According to a number of computational calculations, it seems to us that such an equality may be true for any $p \in \{3,5,7,13\}$ and any coprime negative fundamental discriminants $d_1$ and $d_2$. However, we were unable to prove or disprove this claim in the present work. Moreover, a natural question may be raised for general integer $N$ instead of merely prime $p$. We leave these interesting observations as a conjecture and an open question below, and these can be the topics for further investigation.

**Conjecture 1.13.** For $p \in \{3,5,7,13\}$ and any coprime negative fundamental discriminants $d_1$ and $d_2$, we have $S(p,d_1,d_2) = \text{Cl}(E_{1,p}) \times \text{Cl}(E_{2,p})$. In the language of quadratic form, it is equivalent to say that for every pair of primitive and positive definite quadratic forms $a_iX^2 + b_iXY + c_iY^2$ of $Q_d(p)$, there always exists a positive integer $n$ such that $a_iX^2 + b_iXY + c_iY^2 = n$ both have integral solutions $(X_i,Y_i)$ with $\gcd(X_i,Y_i) = 1$.

**Question 1.14.** It is known that $S(1,d_1,d_2) = \text{Cl}(E_{1,1}) \times \text{Cl}(E_{2,1}) = \text{Cl}(E_1) \times \text{Cl}(E_2)$. In general, when does $S(N,d_1,d_2) = \text{Cl}(E_{1,N}) \times \text{Cl}(E_{2,N})$ hold?

**Remark 1.15.** Gross-Zagier type CM value formulas for the genus zero Fricke subgroups $\Gamma_0(p)$ have been recently derived in another work of the author [31], using the so called small CM value formula [25]. We refer the reader to [9] for a brief explanation on the distinction between the
concepts related to “big CM” and “small CM” and to [31] for a brief explanation on why the small CM value formula may not work in our case.

This work is divided into two parts and is organized as follows. In the first part, we briefly review the theory of Borcherds lift, realize a family of Shimura varieties as Hilbert modular surfaces, set up several preliminary results and construct the desired weakly holomorphic modular form $F_N$ for the Weil representation. Proofs of Theorem 1.5 Corollaries 1.6 and 1.9 are given in Section 4. In the second part of this work, we will briefly review the concepts of big CM cycles and big CM value formula and show how to employ such formula together with Theorem 1.5 to obtain Theorem 1.10 in Section 5. Computations of $a(\frac{t}{\sqrt{D}}, \phi_{0,0})$ and $a_0(\phi_{0,k})$ mentioned in Theorem 1.10 are carried out in Section 6.

Part 1. Difference of a Hauptmodul for $\Gamma_0(N)$

2. Review of Borcherds Lifts

In this section, we briefly review the theory of Borcherds lift in the adelic setting and relevant concepts such as Shimura variety and special divisor, and we also realize a family of Shimura varieties as Hilbert modular surfaces on which the difference of Hauptmoduls are defined. We rely heavily on [17] (also see [29]).

2.1. Shimura Variety of Type $(n,2)$. Let $V$ be a rational quadratic space with a quadratic form $Q(\cdot)$ of signature $(n,2)$ and the associated bilinear form $(\cdot,\cdot)$. Let $H = \text{GSpin}(V)$ be the general spin group of $V$, then there is an exact sequence

$$1 \to \mathbb{G}_m \to H \to \text{SO}(V) \to 1.$$ 

For a field $F \supset \mathbb{Q}$, we write $V_F$ for $V \otimes_{\mathbb{Q}} F$. A Hermitian symmetric domain for $H(\mathbb{R})$ is the oriented Grassmannian of negative 2-planes of $V_\mathbb{R}$, denoted by $\mathcal{D}$. Denote by $\mathbb{A}_K$ the adele ring over a number field $K$ and by $\mathbb{A}_K,f$ the associated finite adele ring. For a compact open subgroup $K \subset H(\mathbb{A}_Q,f)$, there is an associated Shimura variety $X_K$ over $\mathbb{Q}$ such that

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathcal{D} \times H(\mathbb{A}_Q,f)/K).$$

Let

$$\mathcal{L} = \{ Z \in V_{\mathbb{C}} | (Z, Z) = 0 \text{ and } (Z, \bar{Z}) < 0 \}.$$ 

Then we can see that $\mathcal{D}$ possesses a complex structure via the isomorphism $pr : \mathcal{L}/\mathbb{C}^* \to \mathcal{D}$ sending $Z = X + i Y$ to $\mathbb{R}X + \mathbb{R}(-Y)$. There is another useful realization for $\mathcal{D}$ as follows. Take two isotropic (zero-norm) elements $\ell$ and $\ell'$ of $V$ with $(\ell, \ell') = 1$, and let $V_0 = (Q\ell + Q\ell')^\perp$ be the orthogonal complement of the plane spanned by $\ell$ and $\ell'$. Then we define a so-called tube domain associated to $\ell$ and $\ell'$ by

$$\mathcal{H} = \{ Z = X + i Y \in V_{0,\mathbb{C}} | X, Y \in V_{0,\mathbb{R}} \text{ and } Q(Y) < 0 \}$$

which is isomorphic to $\mathcal{L}/\mathbb{C}^*$ via $w(Z) = \ell' - Q(Z)\ell + Z$. Then the map $w$ induces an action of $\Gamma = K \cap H(\mathbb{Q})^+$ on $\mathcal{H}$, where $H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+$ and $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$, and induces an automorphy factor $j(g, Z)$ characterized by the following identity

$$g \cdot w(Z) = \nu(g) j(g, Z) w(g \cdot Z),$$

where $\nu(g)$ is the spinor norm of $g$. Note that this action preserves the two connected components $\mathcal{H}^\pm$ of $\mathcal{H}$. Fix one of these two connected components, say, $\mathcal{H}^+$. Assuming for simplicity that $H(\mathbb{A}_Q) = H(\mathbb{Q})H(\mathbb{R})^+K$, which is guaranteed by an appropriate choice of $K$ and the strong approximation theorem (see, e.g., [20 Ch. 7]), we have the identification $X_K \cong \Gamma/\mathcal{H}^+$. 

6
2.3. Theta-lift and Borcherds Theorem. and for $X$ and our Gaussian for $V$ is the function 

$$\varphi_\infty(X, Z) = e^{-\pi(X, Z)}.$$ 

For $\tau \in \mathbb{H}$ with $\tau = u + iv$, let 

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix},$$ 

and $g'_\tau = (g_\tau, 1) \in \text{Mp}_2(\mathbb{R})$, the metaplectic group. Let $l = \frac{n}{2} - 1$, $G = \text{SL}_2$ and $\rho$ be the Weil representation of the metaplectic group $G'(\mathbb{A}_Q)$ on $S(V_{A_Q})$. Then for the linear action of $H(A_Q, \mathbb{f})$ we write $\rho(h)\varphi(X) = \rho(h^{-1} \cdot X)$ for $\varphi \in S(V_{A_Q})$. For $Z \in \mathbb{D}$ and $h \in H(A_Q, \mathbb{f})$, we have the linear functional on $S(V_{A_Q})$ given by 

$$\varphi \rightarrow \theta(\tau, Z, h; \varphi) := v^{-\frac{l}{2}} \sum_{X \in V_Q} \rho(g'_{\tau}) (\varphi_\infty(\cdot, Z) \otimes \rho(h)\varphi)(X).$$ 

Let $L$ be a lattice of $V$, and let $L'$ be the dual lattice of $L$ defined by 

$$L' = \{X \in V \mid (X, L) \subset \mathbb{Z} \}.$$ 

Let $S_L$ be the subspace of $S(V_{A_Q})$ consisting of functions with support in $L'$ and constant on cosets of $\hat{L}$, where $\hat{L} = L \otimes_\mathbb{Z} \hat{\mathbb{Z}}$. Then 

$$S_L = \bigoplus_{\eta \in L'/L} \mathbb{C}\phi_\eta, \quad \phi_\eta = \text{Char}(\eta + \hat{L}),$$ 

where $\text{Char}(\cdot)$ denotes the characteristic function. One can check that $S_L$ is $\text{SL}_2(\mathbb{Z})$-invariant under the Weil representation $\rho$. Associated to the lattice $L$, we denote by $\rho_L$, and one has 

$$\rho_L(T)\phi_\mu = e(-Q(\mu))\phi_\mu,$$
\[ \rho_L(S)\phi_\mu = \frac{1}{\sqrt{|L'/L|}} \sum_{\gamma \in L'/L} e((\gamma, \mu))\phi_\gamma \]

where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Let \( \Gamma' = \text{Mp}_2(\mathbb{Z}) \) be the full inverse image of \( \text{SL}_2(\mathbb{Z}) \subset G(\mathbb{R}) \) in \( \text{Mp}_2(\mathbb{R}) \).

**Definition 2.2.** A holomorphic function \( F : \mathbb{H} \to SL \) is a weakly holomorphic modular form of weight \( k \) for the Weil representation \( \rho_L \) if

(i) \( F(\gamma' \tau) = (c\tau + d)^k \rho_L(\gamma')F(\tau) \) for all \( \gamma' \in \Gamma' \),

(ii) \( F(\tau) \) has a Fourier expansion

\[ F(\tau) = \sum_{\eta \in L'/L} \sum_{m \in -Q(\eta) + \mathbb{Z}} c(m, \eta)q^m \phi_\eta \]

where the condition \( m \equiv -Q(\eta) \pmod{\mathbb{Z}} \) follows from the transformation law for \( T' \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Furthermore, denote by \( M_{1,k,\rho_L} \) the space of weakly holomorphic modular forms of weight \( k \) for the Weil representation \( \rho_L \).

For the theta function called Siegel theta function

\[ \theta(\tau, Z, h) = \sum_{\mu \in L'/L} \theta(\tau, Z, h; \phi_\mu), \]

we can pair it with \( F(\tau) \) by the following \( \mathbb{C} \)-bilinear pairing

\[ \langle F(\tau), \theta(\tau, Z, h) \rangle = \sum_{\mu \in L'/L} \sum_{m \in -Q(\mu) + \mathbb{Z}} c(m, \mu)q^m \theta(\tau, Z, h; \phi_\mu). \]

Using this pairing, we define a regularized integral as in [A], called theta-lift,

\[ \Phi(Z, h; F) := \text{CT} \left\{ \lim_{s \to 0} \int_{\mathcal{F}_t} \langle F(\tau), \theta(\tau, Z, h) \rangle v^{-2} dudv \right\} \]

where CT denotes the constant term in the Laurent expansion at \( s = 0 \) of

\[ \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle F(\tau), \theta(\tau, Z, h) \rangle v^{-2} dudv, \]

\( \mathcal{F}_t \) is the truncated fundamental domain defined by

\[ \mathcal{F}_t := \{ \tau \in \mathcal{F} | \text{Im}(\tau) \leq t \} \]

and \( \mathcal{F} \) is the usual fundamental domain for the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \).

Now we are ready to state the celebrated Borcherds Theorem. Under the assumption \( H(\mathbb{A}_\mathbb{Q}) = H(\mathbb{Q})H(\mathbb{R})^+K \), we can simply set \( h = 1 \) and omit it for the simplicity of notation.

**Theorem 2.3** (Borcherds). For a given \( f \in M_{1, -\frac{n}{2}, \rho_L} \) with

\[ f(\tau) = \sum_{\mu \in L'/L} \sum_{n \in -Q(\mu) + \mathbb{Z}} c(n, \mu)q^n \phi_\mu, \]

and \( c(n, \mu) \in \mathbb{Z} \) for \( n < 0 \), and assuming \( \Gamma \subset \text{Aut}(L, f) \) where \( \text{Aut}(L, f) \) is the subgroup of automorphism group of \( L \) that stabilizes \( f \), there is a meromorphic modular form \( \Psi(Z, f) \) called Borcherds lift on \( \mathbb{D} \times H(\mathbb{A}_\mathbb{Q}, f) \) (or say on \( \mathcal{H}^+ \)) of weight \( c(0, 0)/2 \) for \( \Gamma \) such that
(1) the divisor of $\Psi(Z, f)$ on $X_K$ is given by
\[ \text{div}(\Psi(Z, f)^2) = \sum_{\mu \in L'/L} \sum_{n \in \Omega(\mu) + \mathbb{Z}} c(-n, \mu) \delta(\mu) Z(n, \mu), \]

where $Z(n, \mu)$ is the special divisor of index $(n, \mu)$ and $\delta(\mu) = 2$ or $1$ depending on whether $2\mu \in L$ or not, and can be written as
\[ Z(n, \mu) = \{ Z \in \mathbb{D}^+ | (Z, X) = 0 \text{ for some } X \in \mu + L \text{ with } Q(X) = n \}, \]

(2) the following relation
\[ \Phi(Z, f) = -4 \log |\Psi(Z, f)| - c(0, 0) (2 \log |Y| + \Gamma(1) + \log(2\pi)) \]
holds,

(3) near each cusp $\mathbb{Q} \ell$ of $X_K$, the meromorphic function $\Psi(Z, f)$ has a product expansion called Borcherds product of the form
\[ \Psi(Z, f) = Ce((Z, \rho(W_{f,\ell,M}, f)) \prod_{\lambda \in M'_L} \prod_{\mu \in L'/L} \prod_{(\lambda, W, \ell, M) > 0} \prod_{\mu(\mu) = \lambda + M_\ell} [1 - e((\lambda, Z) + (\mu, \ell'))]^{c(-Q(\mu), \mu)} \]

where $C$ is a constant with absolute value
\[ \left| \prod_{\delta \in \mathbb{Z}/N} (1 - e(\delta/N))^{\frac{c(0, \delta)}{2}} \right|. \]

For the definitions of the notation, we refer the reader to [29, Subsection 2.1].

2.4. A Shimura Variety as a Hilbert Modular Surface. In this subsection, we will see how to realize a family of Shimura varieties $X_{K_N}$ of type $(2, 2)$ for some open compact subgroup $K_N$ as Hilbert modular surfaces $Y_0(N) \times Y_0(N)$.

Let $V = M_2(\mathbb{Q})$ be a rational quadratic space with the quadratic form $Q(\cdot) := \det(\cdot)$ of signature $(2, 2)$. Then the general spin group $H = \text{GSpin}(V)$ of $V$ is
\[ H = \{(g_1, g_2) \in \text{GL}_2 \times \text{GL}_2 | \det g_1 = \det g_2\} \]
and acts on $V$ via $(g_1, g_2) \cdot X = g_1 X g_2^{-1}$. Taking $\ell = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, one can check that
\[ \mathcal{H} = \left\{ \begin{pmatrix} z_1 \\ 0 \\ 0 \\ -z_2 \end{pmatrix} \middle| \text{Im}(z_1) \text{Im}(z_2) > 0 \right\}. \]
The following proposition is useful and well known (see, e.g., [29, Proposition 3.1]).

**Proposition 2.4.** Define
\[ \tilde{w} : \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \to \mathcal{L} \]
by $\tilde{w}((z_1, z_2)) = \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix}$. Then the composition $pr \circ \tilde{w}$ gives an isomorphism between $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ and $\mathbb{D}$. One can check that such an isomorphism induces an action of $H(\mathbb{R})$ on $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ via the usual fractional linear transformation, i.e.,
\[ (g_1, g_2) \cdot (z_1, z_2) = (g_1 \cdot z_1, g_2 \cdot z_2), \]
and an automorphy factor $j(g_1, g_2; z_1, z_2) = (c_1 z_1 + d_1)(c_2 z_2 + d_2)$ for $g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$. 
Proof. By the identification given in Subsection 2.4, we have
\begin{equation}
\left\{ g \in \text{GL}_2(\hat{\mathbb{Z}}) \middle| g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}
\end{equation}
and denote \( K = (K_0(N) \times K_0(N)) \cap H(\mathbb{A}_f) \) by \( K_N \). Then \( \Gamma = H(\mathbb{Q})^+ \cap K_N = \Gamma_0(N) \times \Gamma_0(N) \) denoted by \( \Gamma_N \). The strong approximation theorem tells that \( H(\mathbb{A}_f) = H(\mathbb{Q})H(\mathbb{R})^+K_N \), and together with Proposition 2.4 it implies the realization of \( X_{K_N} \cong Y_0(N) \times Y_0(N) \) as a Hilbert modular surface. Moreover, set \( L = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \) and denote it by \( L_N \). Then one can check that \( L_N' = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/N \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \) and the special divisor of index \((n, \mu)\) can be expressed as a divisor on \( Y_0(N) \times Y_0(N) \)
\begin{equation}
Z(n, \mu) = (\Gamma_0(N) \times \Gamma_0(N)) \setminus \left\{ z = (z_1, z_2) \in \mathbb{H}^2 | \bar{w}(z) \perp X \text{ for some } X \in \mu + L_N \right\}.
\end{equation}
Finally, one can check that \( \Gamma_N \subset \text{Aut}(L_N'/L_N) \cap \text{Aut}(L_N) \), where \( \text{Aut}(L_N'/L_N) \) and \( \text{Aut}(L_N) \) denote the automorphism groups of \( L_N'/L_N \) and \( L_N \), respectively.

3. Preliminary Results

In this section, we aim to construct an appropriate Borcherds lift input \( F_N \) for \( \pi_N(z_1) - \pi_N(z_2) \). First, with \( L_N = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \) and \( L_N' = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/N \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \), we note that \( \{\mu\}_{\mu \in L_N'/L_N} = \left\{ \begin{pmatrix} 0 & j/N \\ k & 0 \end{pmatrix} \right\}_{0 \leq j,k \leq N-1} \). Then to simplify our notation, we write \( \mu_{j,k} \) for \( \begin{pmatrix} 0 & j/N \\ k & 0 \end{pmatrix} \), and write \( L_N'/L_N = \{\mu_{j,k}\}_{0 \leq j,k \leq N-1} \). We also write \( \phi_{j,k} \) for \( \phi_{\mu_{j,k}} \in \mathcal{S}_{L_N} \).

Lemma 3.1. Let \( Z(1, \mu_0, 0) \) be the special divisor of index \((1, \mu_0, 0)\). Then
\begin{equation}
Z(1, \mu_0, 0) = \left\{ (\tau, \tau) | \tau \in Y_0(N) \right\}.
\end{equation}

Proof. By the identification given in Subsection 2.4 we have
\begin{equation}
Z(1, \mu_0, 0) = (\Gamma_0(N) \times \Gamma_0(N)) \setminus \left\{ (z_1, z_2) \in \mathbb{H}^2 | \bar{w}(z_1, z_2) \perp X \text{ for some } X \in \mu_0 + L_N \right\}
\end{equation}
\begin{equation}
= (\Gamma_0(N) \times \Gamma_0(N)) \setminus \left\{ (z_1, z_2) \in \mathbb{H}^2 | z_2 = \frac{az_1 + b}{cz_1 + d} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}
\end{equation}
\begin{equation}
= \left\{ (\tau, \tau) | \tau \in Y_0(N) \right\}.
\end{equation}

Note that the divisor of \( \pi_N(z_1) - \pi_N(z_2) \) is \( \left\{ (\tau, \tau) | \tau \in Y_0(N) \right\} \). Then Lemma 3.1 tells us that \( \text{div}(\Psi(z, F_N)) \) must be \( Z(1, \mu_0, 0) \), and thus the \( \phi_{0,0} \)-component function of \( F_N \) must have a simple pole at \( i\infty \) of residue 1, and the other component functions are all holomorphic at \( i\infty \).

Lemma 3.2. For \( N \neq 4 \), the following set
\begin{equation}
\bigcup_{d | N} \left\{ \sum_{k=0}^{d-1} \sum_{j=0}^{d-1} \phi_{jd,k} \right\}
\end{equation}
is a basis for the subspace of \( M^!_{0,\rho_{L_N}} \) consisting of elements with constant component functions. For \( N = 4 \), the following set
\begin{equation}
\{ \phi_{0,0} + \phi_{1,0} + \phi_{2,0} + \phi_{3,0}, \phi_{0,0} + \phi_{0,1} + \phi_{0,2} + \phi_{0,3}, \phi_{2,0} - \phi_{0,1} + \phi_{2,2} - \phi_{0,3} \}
\end{equation}
is a basis for the subspace of \( M^!_{0,\rho_{L_4}} \) consisting of elements with constant component functions.
Proof. Setting up the equations
\[ \rho L_N(T) \left( \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{j,k} \phi_{j,k} \right) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{j,k} \phi_{j,k}, \]
\[ \rho L_N(S) \left( \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{j,k} \phi_{j,k} \right) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{j,k} \phi_{j,k}, \]
expanding the left hand side by the Weil representation, and equating the coefficients by the linear independence of \( \phi_{j,k} \), we can obtain the desired results after some routine calculations. \( \square \)

**Lemma 3.3.** Let \( c_{s,N}^{-1} \) be the inverse of \( c_s \) in \((\mathbb{Z}/N\mathbb{Z})^\times \) when \( m_s = 1 \), and \( c_{s,N}^{-1} = m_s \) otherwise. For \( s \in C(\Gamma_1(N)) \), and integers \( j \) and \( k \), define \( t_{s,j,k} \) by \( t_{s,j,k}/h_s \equiv jk \pmod{1} \). Let \( \pi_N = \pi_N(\tau) \) be a Hauptmodul for a genus zero group \( \Gamma_0(N) \) and \( f_N = f_N(\tau) \) be the \( \Gamma_0(N) \)-induction of \( \pi_N(\tau) \) against \( \phi_{0,0} \) defined by
\[ f_N = \sum_{M \in \Gamma_0(N) \setminus SL_2(\mathbb{Z})} \pi_N(M \cdot \rho L_N(M^{-1}) \phi_{0,0}). \]

Then \( f_N \) is in \( M_{0,\rho L_N}^! \), and
\[ f_N = \frac{2}{|\Gamma_0(N) : \Gamma_1(N)|} \sum_{M \in \Gamma_1(N) \setminus SL_2(\mathbb{Z})} \frac{1}{2} \pi_N(M \cdot \rho(M^{-1}) \phi_{0,0} \right) \]
where
\[ f_N = \frac{2}{\lambda_{2,N}|\Gamma_0(N) : \Gamma_1(N)|} \left[ \sum_{s \in C(\Gamma_1(N))} \pi_N(M_s) \phi_{0,0} + \sum_{s \in C(\Gamma_1(N)) \atop s \text{ regular}} \pi_N(M_s) \left( \sum_{j=1}^{h_s-1} \phi_{0,m_s,j} \right) + \sum_{s \in C(\Gamma_1(N)) \atop s \text{ regular} \atop m_s \neq N} \left( \sum_{k=1}^{h_s-1} \phi_{0,m_s,k} \right) \right], \]
and \( \lambda_{2,N} = 2 \) or 1 depending on whether \( N = 2 \) or not.

Proof. These follow from [3, Theorem 5.4] and [23, Theorem 3.7] by setting \( D = L'_N/L_N \) with quadratic form \( Q(\cdot) = \det(\cdot) \) and \( e^\gamma = \phi_{0,0} \), and realizing that
\[ D_{c_s} = \{ \phi_{jhs,khs} \}_{0 \leq j,k \leq m_s-1} \quad \text{and} \quad D^{c_s} = \{ \phi_{jm_s,kms} \}_{0 \leq j,k \leq h_s-1}. \]
Lemma 3.4. Let \( \pi_N = \pi_N(\tau) \) be a Hauptmodul for genus zero group \( \Gamma_0(N) \), and let
\[
\pi_N \mid M_s = \sum_{n=-1}^{\infty} A_s(n)q^{n/h_s}
\]
where \( h_s = 1 \) if \( N = 4 \) and \( s = 1/2 \), otherwise, \( h_s = h_s \) as defined in Section 7. Let \( f_N = f_N(\tau) \) be the \( \Gamma_0(N) \)-induction as defined in Lemma 3.2, For \( N \neq 4 \), let \( F_N = F_N(\tau) \) be defined by
\[
F_N = f_N - \frac{2}{\lambda_2(N) \Gamma_0(N) : \Gamma_1(N)} \left[ \sum_{s \in \mathcal{C}(\Gamma_1(N))} A_s(0) \sum_{k=0}^{N-1} \phi_{0,k} + \sum_{s \in \mathcal{C}(\Gamma_1(N))} A_s(0) \left( \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \phi_{jms, kh_s} - \sum_{k=0}^{N-1} \phi_{0,k} \right) \right].
\]
For \( N = 4 \), let \( F_4 = F_4(\tau) \) be defined by
\[
F_4 = f_4 - \left( A_{1/4}(0) + A_{0/1}(0) + \frac{1}{2} A_{1/2}(0) \right) \sum_{k=0}^{3} \phi_{0,k} - A_{0/1}(0) \sum_{k=1}^{3} (\phi_{k,0} - \phi_{0,k}) - \frac{1}{2} A_{1/2}(0) (\phi_{2,0} - \phi_{0,1} + \phi_{2,2} - \phi_{0,3}).
\]
Then we have
1. \( F_N(\tau) \) is in \( M_{\rho, L_N}^{\prime} \), and is invariant under \( \text{Aut}(\mathcal{L}_N / \mathcal{L}_N) \),
2. \( c(0, \mu_0, 0) = 0 \) and the \( \phi_{0,0} \)-component of \( F_N \) has Fourier expansion
\[
q^{-1} + \frac{2}{\lambda_2(N) \Gamma_0(N) : \Gamma_1(N)} \left[ \sum_{s \in \mathcal{C}(\Gamma_1(N))} \sum_{n=1}^{\infty} A_s(n)q^n + \sum_{s \in \mathcal{C}(\Gamma_1(N))} \sum_{n=1}^{\infty} \frac{1}{h_s} A_s(n)q^n \right],
\]
3. \( c(0, \mu_j, 0) = 0 \) for \( 0 \leq j \leq N - 1 \),
4. for \( d \mid N \) and \( d \neq N \),
\[
\sum_{k=0}^{N-1} \sum_{j=0}^{d-1} c(0, \mu_j, 0, k, 0, d) = 24.
\]
Proof. Assertion (1) follows from Lemmas 3.2 and 3.3.
By collecting the terms attached to \( \phi_{0,0} \) in (3.2), we obtain that the \( \phi_{0,0} \)-component of \( f_N \) is
\[
q^{-1} + \frac{2}{\lambda_2(N) \Gamma_0(N) : \Gamma_1(N)} \left[ \sum_{s \in \mathcal{C}(\Gamma_1(N))} A_s(0) + \sum_{s \in \mathcal{C}(\Gamma_1(N))} \frac{1}{h_s} A_s(0) \right]
\]
(3.5)
Then Assertion (2) follows from (3.5) and the definition of $F_N$.

By extracting the constant terms attached to $\phi_{j,0}$ for $1 \leq j \leq N - 1$ in (3.2) of $F_N$, we obtain

\begin{equation}
\lambda_{2,N}|\Gamma_0(N) : \Gamma_1(N)| \sum_{s \in \mathbb{C} \cap (\Gamma_1(N))} A_s(0) \sum_{j=1}^{h_s-1} \phi_{jm_s,0} + \sum_{s \in \mathbb{C} \cap (\Gamma_1(N)), s \text{ irregular}} \frac{1}{h_s} \sum_{j=1}^{h_s-1} A_s(0) \phi_{jm_s,0}.
\end{equation}

Then Assertion (3) follows from (3.6) and the definition of $F_N$.

For Assertion (4), by (3.2), (3.3) and (3.4), it can be verified case by case that

\begin{equation}
\sum_{k=0}^{N} \sum_{j=0}^{d-1} c(0, \phi_{j,kd}) = -\frac{2}{\lambda_{2,N}|\Gamma_0(N) : \Gamma_1(N)|} \sum_{s \in \mathbb{C} \cap (\Gamma_1(N))} A_s(0) \left( \frac{d(m_s, N/d)^2 - N}{m_s} \right).
\end{equation}

Let

\[ H_d(\tau) = \frac{N}{d} E_2 \left( \frac{N}{d} \tau \right) - E_2(\tau) \]

where $E_2(\tau)$ is the normalized weight 2 Eisenstein series. It is well known [13, Section 1.2] that $H_d(\tau)$ is a weight 2 modular form for $\Gamma_0(N/d)$, so it is a weight 2 modular form for $\Gamma_1(N)$. Let $B_s(0)$ be the constant term of the Fourier expansion of $H_d(\tau)$ at the cusp $s = a_s/c_s \in \mathbb{C} \cap (\Gamma_1(N))$. It is easy to show that

\[ B_s(0) = \frac{(m_s, N/d)^2}{N/d} - 1 \]

and

\[ H_d(\tau) = \frac{N}{d} - 1 + 24q + O(q^2). \]

Since $\pi_N(\tau)$ is a weight 0 weakly holomorphic modular form for $\Gamma_0(N)$, then it is also on $\Gamma_1(N)$, and by Serre duality [6, Theorem 3.1], we have

\begin{equation}
0 = 24 \times \frac{\lambda_{2,N}|\Gamma_0(N) : \Gamma_1(N)|}{2} + \sum_{s \in \mathbb{C} \cap (\Gamma_1(N))} h_s B_s(0) A_s(0)
\end{equation}

\begin{equation}
= 24 \times \frac{\lambda_{2,N}|\Gamma_0(N) : \Gamma_1(N)|}{2} + \sum_{s \in \mathbb{C} \cap (\Gamma_1(N))} h_s \left( \frac{(m_s, N/d)^2}{N/d} - 1 \right) A_s(0)
\end{equation}

\begin{equation}
= 24 \times \frac{\lambda_{2,N}|\Gamma_0(N) : \Gamma_1(N)|}{2} + \sum_{s \in \mathbb{C} \cap (\Gamma_1(N))} A_s(0) \left( \frac{d(m_s, N/d)^2 - N}{m_s} \right).
\end{equation}

Finally, Assertion (4) follows from (3.7) and (3.8). \qed

4. Proofs of Theorem 1.5 and Corollaries 1.6 and 1.9

4.1. Proof of Theorem 1.5. This subsection is devoted to the proof of our first main result, Theorem 1.5. We rely heavily on Borcherds Theorem, especially the third part of it. To assist in understanding the proof, we first review the definitions of the notation used in Theorem 2.3(3) for
our case. Let $M_\ell = L \cap (\mathbb{Q}\ell + \mathbb{Q}\ell')$ be the Lorentzian lattice of $L$ associated to $\ell$ and $\ell'$. Assume that $(\ell, L) = N_\ell \mathbb{Z}$. Choose a $\xi \in L$ such that $(\xi, \ell) = N_\ell$. Let $L'_\ell$ be a sublattice of $L'$ defined by $L'_\ell = \{x \in L' | (\ell, x) \equiv 0 \pmod{N_\ell}\}$.

Then there is a projection $P : L'_\ell \to M'_\ell$, $p(x) = x_M - \frac{(x, \ell)}{N_\ell} \xi_M$, where $x_M$ and $\xi_M$ are the orthogonal projections of $x$, $\xi \in V$ to $M_{\mathbb{R}}$. So it induces a projection, which is also denoted by $p$, from $L'_\ell / L$ to $M'_\ell / M_\ell$. Next, we define the Weyl chamber $W_{f,\ell_M}$ for

![Insert Diagram Here]

Let $Gr(M_\ell)$ be the Graassmannian of negative lines of $M_{\ell,\mathbb{R}}$, which is a real manifold of dimension 1. For $\lambda \in M'_\ell / M_\ell$ and $n \in \mathbb{Q}$, let $Z_M(n, \lambda) = \{z \in Gr(M_\ell) | z \perp x \text{ for some } x \in \lambda + M_\ell \text{ with } Q(x) = n\}$, which is either empty or a real divisor of $Gr(M_\ell)$. The Weyl chambers $W_f$ associated with $f$ are the connected components of

$Gr(M_\ell) - \bigcup_{\mu \in L'_\ell / L} \bigcup_{n > 0, c(-n, \mu) \neq 0} Z_M(n, p(\mu))$.

Let $\ell_M \in M_\ell$ and $\ell'_M \in M'_\ell$ be isotropic elements with $(\ell_M, \ell'_M) = 1$, whose existence is guaranteed in our case. For general case, we refer the reader to [2, Section 9] and [10, Section 5]. Choose a Weyl chamber such that $\ell_M$ is contained in its closure, and denote such Weyl chamber by $W_{f,\ell_M}$. We now define its Weyl vector $\rho(W_{f,\ell_M}, f)$. Define

$f_M = \sum_{\lambda \in M'_\ell / M_\ell} f_M, \phi_{M,\lambda}$

where $\phi_{M,\lambda} \in \mathbb{C}[M'_\ell / M_\ell]$ and

$f_M, \lambda = \sum_{\mu \in L'_\ell / L, p(\mu) = \lambda + M_\ell} f_\mu$.

Let $P_\ell_M$ be the Lorentzian sublattice of $M_\ell$ associated with $\ell_M$. Clearly, $P_\ell_M = \{0\}$ and $P'_\ell_M / P_\ell_M$ is trivial. Construct $f_P$ from $f_M$ in the same way as $f_M$ constructed from $f$, and obtain

$f_P = \sum_{\lambda \in M'_\ell / M_\ell} \sum_{\mu \in L'_\ell / L, p(\mu) = \lambda + M_\ell} f_\mu$.

Then the Weyl vector $\rho(W_{f,\ell_M}, f)$ is defined by

$\rho(W_{f,\ell_M}, f) = \rho_{\ell_M, \ell_M} + \rho_{\ell'_M} \ell'_M$

where

$\rho_{\ell'_M} = -1 + \frac{c_P(0)}{24} = -1 + \frac{1}{24} \sum_{\lambda \in M'_\ell / M_\ell} \sum_{\mu \in L'_\ell / L, p(\mu) = \lambda + M_\ell} c(0, \mu)$,

$\rho_{\ell_M} = -\frac{1}{4} \sum_{\lambda \in M'_\ell / M_\ell} c_M(0, \lambda) B_2((\lambda, \ell'_M))$.
and \( B_2(x) = x^2 - x + \frac{1}{6} \) is the second Bernoulli polynomial.

**Proof of Theorem 1.5.** Let \( \ell_s = \begin{pmatrix} 0 & a_s \\ 0 & c_s \end{pmatrix} \in L_N \) and \( \ell'_s = \begin{pmatrix} -b_s \\ -d_s \end{pmatrix} \in L'_N \). Then we have

\[
L'_{N, \ell_s} = \left( \frac{Z}{m_s \mathbb{Z}} \right), \quad L'_{N, \ell_s}/L_N = \left\{ \mu_j m_k = \begin{pmatrix} 0 & j/N \\ -m_k & 0 \end{pmatrix} \right\}_{0 \leq j \leq N-1, 0 \leq k \leq h_s-1}
\]

\[
M_s := M_{\ell_s} = \left\{ \begin{pmatrix} a_s h_s x & b_s y \\ c_s h_s x & d_s y \end{pmatrix} \right\}_{x, y \in \mathbb{Z}} \quad \text{and} \quad M'_s := M'_{\ell_s} = \left\{ \begin{pmatrix} a_s x & b_s y \\ c_s x & d_s y \end{pmatrix} \right\}_{x, y \in \mathbb{Z}},
\]

and

\[
z_{\ell_s} = \begin{pmatrix} -a_s z_1 \\ c_s z_1 \end{pmatrix} \quad \text{and} \quad w(z_{\ell_s}) = \begin{pmatrix} -a_s z_1 - b_s \\ -c_s z_1 - d_s \end{pmatrix} + \begin{pmatrix} b_s z_2 + a_s z_1 z_2 \\ d_s z_2 + c_s z_1 z_2 \end{pmatrix}.
\]

Also, we deduce that

\[
\text{Gr}(M_s) = \left\{ \mathbb{R} \begin{pmatrix} -a_s x \\ -c_s x \end{pmatrix} \bigg| x > 0 \right\}
\]

and

\[
Z_{M_s}(1, \mu_{0,0}) = \left\{ Z \in \text{Gr}(M_s) \big| Z \perp X \text{ for some } X \in \mu_{0,0} + M_s \text{ with } Q(X) = 1 \right\}
\]

\[
= \begin{cases} \emptyset & \text{if } m_s \neq N, \\ \{ \mathbb{R} \begin{pmatrix} -a_s \\ -c_s \end{pmatrix} \bigg| x > 0 \} & \text{if } m_s = N.
\end{cases}
\]

And thus

\[
\text{Gr}(M_s) - Z_{M_s}(1, \mu_{0,0}) = \begin{cases} \text{Gr}(M_s) & \text{if } m_s \neq N, \\ \{ \mathbb{R} \begin{pmatrix} -a_s x \\ -c_s x \end{pmatrix} \bigg| 0 < x < 1 \} \cup \{ \mathbb{R} \begin{pmatrix} -a_s x \\ -c_s x \end{pmatrix} \bigg| x > 1 \} & \text{if } m_s = N.
\end{cases}
\]

Let \( \ell_{M_s} = \begin{pmatrix} a_s h_s \\ 0 \\ c_s h_s \end{pmatrix} \) and \( \ell'_{M_s} = \begin{pmatrix} 0 \\ b_s \| h_s \end{pmatrix} \). Then the Weyl chamber whose closure contains \( \mathbb{Q} \ell_{M_s} \) is

\[
W_{\ell_{M_s}} = \left\{ \mathbb{R} \begin{pmatrix} -a_s x \\ -c_s x \end{pmatrix} \bigg| x > 0 \right\}
\]

if \( m_s \neq N \), and

\[
W_{\ell'_{M_s}} = \left\{ \mathbb{R} \begin{pmatrix} -a_s x \\ -c_s x \end{pmatrix} \bigg| x > 1 \right\}
\]

if \( m_s = N \),

and thus for \( X = \begin{pmatrix} a_s n & -b_s m \\ c_s n & d_s m \end{pmatrix} \in M'_s \), we have that

\[
(X, W_{\ell_{M_s}}) > 0 \quad \text{if and only if} \quad \begin{cases} m, n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s \neq N, \\ m \geq 0, m + n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s = N,
\end{cases}
\]

and we can check that \( (X, z_{\ell_s}) = \frac{\mathbb{Q} \ell_{M_s}}{\mathbb{Q} z_{\ell_s}} z_1 m + z_2 n = \frac{1}{h_s} z_1 m + z_2 n \). We also have

\[
M'_{s, \ell_{M_s}} = \left\{ \begin{pmatrix} a_s x \\ c_s x \end{pmatrix} \bigg| x, y \in \mathbb{Z} \right\} \quad \text{and} \quad M'_{s, \ell'_{M_s}}/M_s = \left\{ \begin{pmatrix} a_s k \\ c_s k \end{pmatrix} \bigg| 0 \leq k \leq h_s-1 \right\}.
\]
Let $\tilde{x}, \tilde{y} \in \mathbb{Z}$ such that $c_s \tilde{x} - aN \tilde{y} = m_s$. Then $p : L'_N,\ell'_s/L_N \to M'_s/M_s$ is

$$p \left( \begin{pmatrix} 0 & \frac{j}{N} \\ m_s k & 0 \end{pmatrix} \right) = \begin{pmatrix} a_s k \tilde{x} & -\frac{b_s c_s}{N} j \\ c_s k \tilde{x} & -\frac{d_s c_s}{N} j \end{pmatrix},$$

and thus

$$F_{N,p} = \sum_{\lambda \in M'_s,\ell'_s/M_s} \left( \sum_{\mu \in L'_N,\ell'_s/L_N} F_{N,\phi_{\mu}} \right) = \sum_{k=0}^{h_s-1} \sum_{j=0}^{m_s-1} \sum_{\lambda \in M'_s,\ell'_s/M_s} F_{N,\phi_{j\lambda k m_s}}.$$ 

Thus,

$$c_\ell(0, \phi_{0,0}) = \sum_{k=0}^{h_s-1} \sum_{j=0}^{m_s-1} c(0, \phi_{j\lambda k m_s}) = \begin{cases} 24 & \text{if } m_s \neq N, \\ 0 & \text{if } m_s = N, \end{cases}$$

and

$$\rho_{\ell'_s, M_s} = \begin{cases} 0 & \text{if } m_s \neq N, \\ -1 & \text{if } m_s = N. \end{cases}$$

Therefore, the Weyl vector of $W_{\ell M_s}$ associated to $F_N$ is

$$\rho(W_{\ell M_s}, F_N) = \begin{cases} \rho_{\ell'_s, M_s} & \text{if } m_s \neq N, \\ -\rho_{\ell'_s, M_s} & \text{if } m_s = N, \end{cases}$$

and thus

$$(z_{\ell_s}, \rho(W_{\ell M_s}, F_N)) = \begin{cases} -h_s \rho_{\ell'_s, M_s} z_2 & \text{if } m_s \neq N, \\ -z_1 & \text{if } m_s = N. \end{cases}$$

We can also show that $h_s \rho_{\ell'_s, M_s} = 1$, but we do not need this fact in our proof, so we leave the details of computations to the reader. Similarly, if we let $\ell_s = \begin{pmatrix} c_s & -a_s \\ 0 & 0 \end{pmatrix} \in L_N$ and $\ell'_s = \begin{pmatrix} 0 & 0 \\ d_s & -b_s \end{pmatrix} \in L'_N$, then we have

$$M_s = \left\{ \begin{pmatrix} -d_s y & b_s y \\ c_s h_s x & -a_s h_s x \end{pmatrix} \mid x, y \in \mathbb{Z} \right\} \quad \text{and} \quad M'_s = \left\{ \begin{pmatrix} -d_s y & b_s y \\ c_s x & -a_s x \end{pmatrix} \mid x, y \in \mathbb{Z} \right\},$$

$$\ell_{M_s} = \begin{pmatrix} 0 & 0 \\ c_s h_s & -a_s h_s \end{pmatrix} \quad \text{and} \quad \ell'_{M_s} = \begin{pmatrix} 0 & 0 \\ -d_s h_s & b_s h_s \end{pmatrix},$$

$$z_{\ell_s} = \begin{pmatrix} d_s z_1 & -b_s z_1 \\ c_s z_2 & -a_s z_2 \end{pmatrix} \quad \text{and} \quad w(z_{\ell_s}) = \begin{pmatrix} d_s z_1 + c_s z_1 z_2 -b_s z_1 -a_s z_1 z_2 \\ c_s z_2 + d_s -a_s z_2 -b_s \end{pmatrix},$$

$$W_{\ell M_s} = \left\{ \begin{pmatrix} d_s & -b_s \\ c_s x & -a_s x \end{pmatrix} \mid x > 0 \right\} \quad \text{if } m_s \neq N,$$

$$W_{\ell M_s} = \left\{ \begin{pmatrix} d_s & -b_s \\ c_s x & -a_s x \end{pmatrix} \mid x > 1 \right\} \quad \text{if } m_s = N,$$

$$\left( X, W_{\ell M_s} \right) > 0 \quad \text{if and only if} \quad \begin{cases} m, n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s \neq N, \\ n \geq 0, m + n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s = N, \end{cases}$$
for $X = \begin{pmatrix} -\frac{d_s}{h_s} n & \frac{b_s}{h_s} m \\ -c_s m & a_s m \end{pmatrix} \in M'_s$, and
\[
\rho(W_{\ell M_s}, F_N) = \begin{cases} \rho_{\ell M_s} \ell M_s & \text{if } m_s \neq N, \\ -\rho_{\ell M_s} & \text{if } m_s = N. \end{cases}
\]

Also, we can check $(X, z_{\ell_s}) = z_1 m + \frac{1}{n_s} z_2 n$, and
\[
(z_{\ell_s}, \rho(W_{\ell M_s}, F_N)) = \begin{cases} -h_s \rho_{\ell M_s} z_1 & \text{if } m_s \neq N, \\ -z_2 & \text{if } m_s = N. \end{cases}
\]

Now we are ready for the proof of Theorem 1.5. We aim to show that
\[
\pi_N(z_1) - \pi_N(z_2) = \Psi(z, F_N).
\]

We first note by Borcherds Theorem, Proposition 2.4 and Lemma 3.4(ii) that $\Psi(z, F_N)$ can be viewed as either a meromorphic function on the Shimura variety $X_K$ or a meromorphic function on $Y_0(N) \times Y_0(N)$, and $h_N(z_1, z_2) := \pi_N(z_1) - \pi_N(z_2)$ is a meromorphic function on $Y_0(N) \times Y_0(N)$. Let
\[
g(z_1, z_2) = \frac{\Psi(z, F_N)}{h_N(z_1, z_2)}.
\]

Then $g(z_1, z_2)$ is a meromorphic function on $Y_0(N) \times Y_0(N)$ with no zeros or poles by Lemma 3.1.

Let us fix $z_2 \in \mathbb{H}$. Then $g_1(z_1) := g(z_1, z_2)$ is a meromorphic function in $z_1$ on $Y_0(N)$. Let us investigate the behavior of $g_1(z_1)$ at the cusps of $Y_0(N)$. By Proposition 2.4 we can know that the Fourier expansion of $\Psi(z, F_N)$ at the cusp $(s = a_s/c_s, i\infty)$ is equal to the Borcherds product expansion of $\Psi(z, F_N)$ at the cusp $\mathbb{Q} \ell_s$ where $\ell_s = \begin{pmatrix} 0 & a_s \\ 0 & c_s \end{pmatrix}$. Then by Theorem 2.3(3) together with (4.1) and (4.2), we have
\[
\Psi(z, F_N) = \begin{cases} C_s q_2 \prod_{m,n \geq 0 \atop m^2+n^2>0} (1 - q_1^{m/h_s} q_2^{n/(\mu, \ell_s^0)})^{c(mn, \phi_{\mu})}, & \text{if } m_s \neq N, \\
C_s (q_1^{-1} - q_2^{-1}) \prod_{m \geq 0 \atop m+n \geq 0 \atop m^2+n^2>0} (1 - q_1^{m/h_s} q_2^{n/(\mu, \ell_s^0)})^{c(mn, \phi_{\mu})}, & \text{if } m_s = N, \end{cases}
\]

near the cusp $\mathbb{Q} \ell_s$ for some nonzero constant $C_s$ depending on the choice of cusp $s$, which can be identified by the condition given in Borcherds' Theorem. Then when $z_2 \in \mathbb{H}$ is fixed, as a meromorphic function on $Y_0(N)$, the order of vanishing of $\Psi(z, F_N)$ at the cusp $s = a_s/c_s$ can be computed
\[
\text{ord}_s(\Psi(z, F_N)) = \begin{cases} 0 & \text{if } m_s \neq N, \\ -1 & \text{if } m_s = N. \end{cases}
\]

Also, we can easily compute the Fourier expansion of $h_N(z_1, z_2)$ at the same cusp, and obtain the order of vanishing of $h_N(z_1, z_2)$ at the cusp $s = a_s/c_s$ when $z_2 \in \mathbb{H}$ is fixed, which is
\[
\text{ord}_s(h_N) = \begin{cases} 0 & \text{if } m_s \neq N, \\ -1 & \text{if } m_s = N. \end{cases}
\]
Therefore, as a meromorphic function on $Y_0(N)$, the modular function $g_1(z_1)$ is holomorphic at all of the cusps, and thus $g(z_1, z_2)$ is constant on $Y_0(N) \times \{z_2\}$. Similarly, if we fix $z_1 \in \mathbb{H}$, by Theorem 1.3, 1.4, and 1.5, we can show that $g(z_1, z_2)$ is constant on $\{z_1\} \times Y_0(N)$. Hence, $g(z_1, z_2)$ is constant on $Y_0(N) \times Y_0(N)$, which is 1 by comparing the Fourier expansions of $\Psi(z, F_N)$ and $h_N(z_1, z_2)$ at $(i\infty, i\infty) = (1/N, 1/N)$, and this completes the proof.

4.2. Proofs of Corollaries 1.6 and 1.9. We end this section with the proofs to Corollaries 1.6 and 1.9.

Proof of Corollary 1.6. We first note (3.2)–(3.4) that the $\phi_{j,0}$-component for $0 \leq j \leq N - 1$ of $F_N$ is

\[
\sum_{\ell = -1}^{\infty} c(\ell, \mu_{j,0}) q^\ell
\]

\[
= \frac{2}{\lambda_{2,N} |\Gamma_0(N) : \Gamma_1(N)|} \sum_{s \in \mathcal{C}(\Gamma_1(N))} \left[ (\pi_N | M_s)_0 - A_s(0) \right] + \sum_{s \in \mathcal{C}(\Gamma_1(N))} \frac{1}{h_s} \left[ (\pi_N | M_s) - A_s(0) \right]
\]

\[
= \sum_{d | N} \frac{2}{d \lambda_{2,N} |\Gamma_0(N) : \Gamma_1(N)|} \sum_{s \in \mathcal{C}(\Gamma_1(N))} \left[ (\pi_N | M_s)_0 - A_s(0) \right] + \sum_{s \in \mathcal{C}(\Gamma_1(N))} \frac{1}{h_s} \left[ (\pi_N | M_s) - A_s(0) \right]
\]

\[
= \sum_{d | N} \sum_{d | j} A(\ell, d) q^\ell
\]

\[
= \sum_{\ell = -1}^{\infty} \left( \sum_{d | N} A(\ell, d) \right) q^\ell.
\]

Now from the proof of Theorem 1.5 and 1.5 together with Borcherds Theorem, we deduce that

\[
(\pi_N(z_1) - \pi_N(z_2)) \prod_{\gamma \in \Gamma_1(N) \setminus \Gamma_0(N)} (1 - \pi_N(\gamma z_1))^{-1} \pi_N(\gamma z_2))
\]

\[
= (q_1^{-1} - q_2^{-1}) \prod_{m,n>0} \prod_{j=0}^{N-1} (1 - q_1^m q_2^n e(-j/N)) c(mn, \phi_{j,0})
\]

\[
= (q_1^{-1} - q_2^{-1}) \prod_{m,n>0} \prod_{j=0}^{N-1} (1 - q_1^m q_2^n e(-j/N)) A(mn,d)
\]
\[
(q_1^{-1} - q_2^{-1}) \prod_{m,n>0, d|N} \prod_{j=0}^{\lfloor \frac{N}{d} \rfloor - 1} (1 - q_1^{m} q_2^{n} e^{-j'd/N}) A(mn,d)
\]
\[
= (q_1^{-1} - q_2^{-1}) \prod_{m,n>0, d|N} \left( 1 - (q_1^{m} q_2^{n})^{\frac{N}{d}} \right) A(mn,d).
\]

\[\square\]

**Proof of Corollary 1.9.** Taking the logarithmic derivative of both sides of (1.3), we have that
\[
\frac{1}{2\pi i} \frac{\pi_N(z_1)}{\pi_N(z_1) - \pi_N(z_2)}
\]
\[
= \frac{1}{1 - q_2^{-1} q_1} + \sum_{m,n>0, d|N} \frac{A(mn,d) (q_1^{m} q_2^{n})^{\frac{N}{d}}}{1 - (q_1^{m} q_2^{n})^{\frac{N}{d}}}
\]
\[
= \sum_{n=0}^{\infty} (q_2^{-1} q_1)^n + \sum_{m,n>0, d|N} \sum_{i=1}^{\infty} A(mn,d) (q_1^{m} q_2^{n})^{\frac{N}{d}},
\]
and this proves the corollary.

\[\square\]

**Part 2. Gross-Zagier Type CM Value Formulas**

5. **Big CM Cycles and Big CM Value Formula**

In this section, we briefly review the concepts of big CM cycles and big CM value formula (see [9, Sec. 2–4] for details), based on which we realize a big CM cycle in the Hilbert modular surface \(Y_0(N) \times Y_0(N)\) and prove Theorem 1.10 at the end of Subsection 5.1.

5.1. **Big CM Cycles in \(X_K\).** Let \(F\) be a totally real number field of degree \(d + 1\) and \(W\) be an \(F\)-quadratic space with an \(F\)-quadratic form \(Q_F(\cdot)\) of signature \((2,0),\ldots,(2,0),(0,2)\) with respect to the \(d + 1\) embeddings \(\{\sigma_i\}_{i=1}^{d+1}\) of \(F\) such that \(\text{Res}_{F/Q} W\) is a rational quadratic space of signature \((2d,2)\) with the quadratic form \(Q(\cdot) = \text{tr}_{F/Q} \circ Q_F(\cdot)\) induced from \(Q_F(\cdot)\). Then we have an orthogonal direct sum decomposition
\[
\text{Res}_{F/Q} W = \bigoplus_{i=1}^{d+1} W_{\sigma_i}
\]
where \(W_{\sigma_i} = W \otimes_{F,\sigma_i} \mathbb{R}\). The negative 2-plane \(W_{\sigma_{d+1}}\) gives rise to a two-point (big CM points) subset \(\{z_{\sigma_{d+1}}^\pm\}\) of \(\mathcal{D}\). Let \(T\) be the preimage of \(\text{Res}_{F/Q} \text{SO}(W) \subset \text{SO}(\text{Res}_{F/Q} W)\) in \(H\), the general spin group of \(\text{Res}_{F/Q} W\). Then we have the following commutative diagram.
\[
1 \longrightarrow G_m \longrightarrow T \longrightarrow \text{Res}_{F/Q} \text{SO}(W) \longrightarrow 1
\]
\[
\downarrow \quad \downarrow \quad \downarrow
\]
\[
1 \longrightarrow G_m \longrightarrow H \longrightarrow \text{SO}(V) \longrightarrow 1
\]
and this implies that \(T\) is a maximal torus associated to the CM number field \(E = F(\sqrt{-\det W})\). And we obtain a so called big CM cycle in \(X_K\), the Shimura variety associated to the compact open subgroup \(K\),
\[
Z(W, z_{\sigma_{d+1}}^\pm) = T(\mathbb{Q}) \setminus \left\{ z_{\sigma_{d+1}}^\pm \right\} \times T(\mathcal{A}_Q,f)/K_T
\]
where \(K_T = K \cap T(\mathcal{A}_Q,f)\). The CM cycle \(Z(W, z_{\sigma_{d+1}}^\pm)\) is defined over \(F\), and the formal sum \(Z(W)\) of all of its Galois conjugates is a 0-cycle in \(X_K\) defined over \(\mathbb{Q}\).
5.2. **Big CM Value Formula.** Associated to the $F$-quadratic space $W$ and the additive adelic character $\psi_F = \psi \circ \text{tr}_{F/Q}$ is a Weil representation $\omega = \omega_{\psi_F}$ of $\text{SL}_2(\mathbb{A}_F)$ (and thus $T(\mathbb{A}_F)$) on $\mathcal{S}(W_{\mathbb{A}_F}) = \mathcal{S}(V_{\mathbb{A}_F})$. Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of $E$ associated to $E/F$. Then $\chi = \chi_W$ is also the quadratic Hecke character of $F$ associated to $W$, and there is an $\text{SL}_2(\mathbb{A}_F)$-equivalent map

$$\lambda = \prod_{\nu} \lambda_{\nu} : \mathcal{S}(W_{\mathbb{A}_F}) \to I(0, \chi)$$

via $\lambda(\phi)(g) = \omega(g)\phi(0)$. where $I(s, \chi) = \text{Ind}_{B_{\mathbb{A}_F}}^{\text{SL}_2(\mathbb{A}_F)} \chi \cdot |^s$ is the principal series whose elements are smooth functions $\Phi$ on $\text{SL}_2(\mathbb{A}_F)$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^s\Phi(g, s)$$

for $b \in \mathbb{A}_F$ and $a \in \mathbb{A}_F^\times$ with $n(b)m(a) \in B_F$, where

$$n(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad m(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

and $B_{\mathbb{A}_F}$ the standard Borel subgroup of $\text{SL}_2(\mathbb{A}_F)$. Such an element is called factorizable if $\Phi = \otimes \Phi_{\nu}$ with $\Phi_{\nu} \in I(s, \chi)$. It is called standard if $\Phi_{\text{SL}_2(\mathbb{O}_F)\text{SO}_2(\mathbb{R})^2}$ is independent of $s$. For a standard element $\Phi$, its associated Eisenstein series is defined for $\text{Re}(s) \gg 0$ by

$$E(g, s, \Phi) = \sum_{\gamma \in \mathbb{B}_{\mathbb{A}_F}\backslash\text{SL}_2(\mathbb{A}_F)} \Phi(\gamma g, s).$$

For $\phi \in \mathcal{S}(V_{\mathbb{A}_F}) \subset \mathcal{S}(W_{\mathbb{A}_F})$, let $\Phi_f$ be the standard element associated to $\lambda_f(\phi) \in I(0, \chi)$. For each real embedding $\sigma_i$ of $F$, let $\Phi_{\sigma_i} \in I(s, \chi_{\mathbb{C}/\mathbb{R}}) = I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}})$ be the unique ‘weight one’ eigenvector of $\text{SL}_2(\mathbb{R})$ given by

$$\Phi_{\sigma_i}(n(b)m(a)g_\theta) = \chi_{\mathbb{C}/\mathbb{R}}(a)|a|^s e^{i\theta}$$

for $b \in \mathbb{R}$, $a \in \mathbb{R}^\times$ and $g_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \text{SO}_2(\mathbb{R})$. We define for $\tau = (\tau_1, \ldots, \tau_{d+1}) \in \mathbb{H}^{d+1}$,

$$E(\tau, s, \phi) = \left( \prod_{i=1}^{d+1} v_i \right)^{-\frac{1}{2}} E(g_\tau, s, \Phi_f \otimes \Phi_{\sigma_1} \otimes \Phi_{\sigma_2}),$$

where $\tau_i = u_i + iv_i$ and $g_\tau = (n(u_i)m(\sqrt{v_i}))_{1 \leq i \leq d+1}$. It is a non-holomorphic Hilbert modular form of scalar weight 1 for some congruence subgroup of $\text{SL}_2(\mathbb{O}_F)$. We normalize $E(\tau, s, \phi)$ by

$$E^*(\tau, s, \phi) = \Lambda(s + 1, \chi)E(\tau, s, \phi)$$

where

$$\Lambda(s, \chi) = \left( N_{F/Q}(\mathcal{O}_F) d_{E/F} \right)^{\frac{1}{2}} \left( \pi^{\frac{d+1}{2}} \Gamma \left( \frac{s+1}{2} \right) \right)^{d+1} L(s, \chi).$$

We write the Fourier expansion of $E^*(\tau, s, \phi)$ as

$$E^*(\tau, s, \phi) = E_{0}^*(\tau, s, \phi) + \sum_{t \in \mathbb{F}^\times} E_t^*(\tau, s, \phi).$$

If one assumes that $\phi$ is factorizable, then

$$E_t^*(\tau, s, \phi) = \prod_{p | \infty} W_{t,p}^*(\tau, s, \phi) \prod_{i=1}^{d+1} W_{t,\sigma_i}^*(\tau_i, s, \Phi_{\sigma_i}).$$
Theorem 1.10. We first realize and interpret big CM cycles in the Hilbert modular surfaces for

\[ \text{Proof of Theorem 1.10.} \]

5.3. Let \( \Phi(\tau, s, \phi) = |N_{F/Q}(\partial F d_{E/F})|^{-\frac{s+1}{2}} L_p(1+s, \chi) W^*_{t,\psi}(s, \phi) \) be the theta-lift defined as in Subsection 2.3. Then we have

\[ W^*_{t,\sigma_i}(\tau_i, s, \Phi_{\sigma_i}) = v_i^{-1/2} \pi^{-\frac{s+2}{2}} W^*_{t,\psi}(s, \phi) \]

are the normalized local Whittaker functions, and \( \gamma(W_{\sigma_i}) \) are Weil indices (see, e.g., [18, 28]). Moreover, for \( m > 0 \), we write

\[ a_m(\phi) = \sum_{t \in F_+^*} a(t, \phi), \]

where \( F_+^* \) consists of all totally positive elements in \( F \), the term \( a(t, \phi) \) is the \( t \)-th Fourier coefficient of \( E^{*'}(\tau^\Delta, 0, \phi) \) and \( \tau^\Delta = (\tau, \ldots, \tau) \) the diagonal element, and write the constant term of \( E^{*'}(\tau, s, \phi) \) as

\[ \phi(0)\Lambda(0, \chi) \log \left( \prod_{i=1}^{d+1} v_i \right) + a_0(\phi) \]

for a constant \( a_0(\phi) \) depending on \( \phi \). One can check that \( a(t, \phi) = 0 \) unless \( t - Q_F(\mu) \in \partial F^{-1} \) where \( \mu \) is the element in \( F \) associated with \( \phi \) (see, e.g., [18, 28]).

Now we are ready to state the big CM value formula due to Bruinier, Kudla and Yang [9, Thm. 5.2], which expresses the sum of the values of a theta-lift on a Shimura variety \( X_K \) over a big CM cycle in terms of the coefficients of an incoherent Eisenstein series of weight 1.

**Theorem 5.1** (Bruinier, Kudla, and Yang). For a given \( f \in M_{1-d,\rho_L}^* \) with

\[ f(\tau) = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Z}} c(n, \mu) q^n \phi_\mu, \]

let \( \Phi(Z, f) \) be the theta-lift defined as in Subsection 2.3. Then we have

\[ \sum_{Z \in Z(W)} \Phi(Z, f) = \frac{\deg(Z(W, s^\pm_{\sigma_{d+1}}))}{\Lambda(0, \chi)} \sum_{\mu \in L'/L} c(-m, \mu) a_m(\phi_\mu). \]

5.3. **Proof of Theorem 1.10.** This subsection is devoted to the proof of our second main result, Theorem 1.10. We first realize and interpret big CM cycles in the Hilbert modular surfaces for \( \Gamma_0(N) \) under the identification \( Y_0(N) \times Y_0(N) \cong X_{K_N} \) as we have seen in Subsection 2.4. Then we conclude this section with the proof.

Let \( E_i = \mathbb{Q}(\sqrt{d_i}) \) for \( i = 1, 2 \) with \( (d_1, d_2) = 1 \) be two imaginary quadratic fields of fundamental discriminants \( d_i \), and let \( E = E_1 \otimes E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \). Let \( F = \mathbb{Q}(\sqrt{D}) \) with \( D = d_1 d_2 \) be the maximal totally real subfield of \( E \). We can view \( E \) as a \( F \)-quadratic space \( W \) with \( F \)-quadratic
form $Q_F(z) = \frac{z\bar{z}}{\sqrt{D}}$. Then we can also view it as a rational quadratic space $\text{Res}_{F/Q}W$ with quadratic form $Q(z) = t_{F/Q} \circ Q_F(z)$. Let $\sigma_i$ for $i = 1, 2$ be the two real embeddings of $F$ with $\sigma_i(\sqrt{D}) = (-1)^{i-1}\sqrt{D}$. Then we can see that $W_{\sigma_1}$ has signature $(2, 0)$ at $\sigma_1$ and $W_{\sigma_2}$ has signature $(0, 2)$ at $\sigma_2$. We choose a $\mathbb{Z}$-basis for $\mathcal{O}_F$ as follows

$$e_1 = 1 \otimes 1, \quad e_2 = \frac{-d_1 + \sqrt{d_1}}{2} \otimes 1, \quad e_3 = 1 \otimes \frac{d_2 + \sqrt{d_2}}{2}, \quad e_4 = e_2 e_3,$$

and throughout the remainder of this section, we will drop $\otimes$ when there is no ambiguity. Then we can identify $(\text{Res}_{F/Q}W, Q(\cdot))$ with $(V = M_2(\mathbb{Q}), \det)$ considered in Subsection 2.4 by

$$\sum_{i=1}^{4} x_i e_i \mapsto \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix}.$$

In particular, under this identification, we have

$$L_N \cong \mathbb{Z} + \frac{d_1 + \sqrt{d_1}}{2} + \frac{d_2 + \sqrt{d_2}}{2} + \frac{N(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4},$$

which is of index $N$ in $\mathcal{O}_F$.

In such a case, the maximal torus $T$ over $\mathbb{Q}$ is given by (see [16] or [9, Section 6])

$$T(R) = \{(t_1, t_2) \in (E_1 \otimes \mathbb{Q} R)^{\times} \times (E_2 \otimes \mathbb{Q} R)^{\times} | t_1 \bar{t}_1 = t_2 \bar{t}_2\}$$

for any $\mathbb{Q}$-algebra $R$. Then the map from $T$ to $E$ is given by $(t_1, t_2) \mapsto t_1/\bar{t}_2$. By the theory of complex multiplication [20], there is an embedding

$$\iota_i : E_i \to M_2(\mathbb{Q})$$

such that

$$\iota_i(t) \begin{pmatrix} e_{i+1} \\ e_1 \end{pmatrix} = \begin{pmatrix} te_{i+1} \\ te_1 \end{pmatrix}.$$

Then $\iota = (t_1, t_2)$ gives the embedding from $T$ to $H$, and one has

$$K_{N,T} := K_N \cap T(\mathbb{Q}) = \{(t_1, t_2) \in T(\mathbb{A}_f) | t_i \in \iota_i^{-1}(K_0(N))\}.$$

In the following, we will interpret the big CM cycle

$$Z(W, z_{\sigma_2}^\pm) = T(\mathbb{Q}) \setminus \{\{z_{\sigma_2}^\pm\} \times T(\mathbb{A}_{Q,f})/K_{N,T}\}$$

in $X_{K_N}$ as a 0-cycle in $Y_0(N) \times Y_0(N)$.

**Lemma 5.2.** Under the identification between $\mathbb{D}$ and $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ (see Proposition 2.7), the big CM points $z_{\sigma_2}$ and $z_{\sigma_2}^\pm$ are identified with $(\tau_1, \tau_2) \in \mathbb{H}^2$ and $(-\tau_1, -\tau_2) \in (\mathbb{H}^-)^2$, respectively, where

$$\tau_i = \frac{d_i + \sqrt{d_i}}{2}.$$

**Proof.** See [29] Lemma 3.4. \qed

**Lemma 5.3.** Let $\text{Cl}(E_{i,N})$ be the ring class group of conductor $N$ of $E_i$ for $i = 1, 2$. Then there is an injection

$$j : T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K_{N,T} \to \text{Cl}(E_{1,N}) \times \text{Cl}(E_{2,N})$$

with image

$$S(N, d_1, d_2) := \{(a_1, a_2) \in \text{Cl}(E_{1,N}) \times \text{Cl}(E_{2,N}) : \exists \text{ fractional ideals } a_i \text{ such that } N(a_1) = N(a_2)\} \cong \{(Q_1, Q_2) \in \mathcal{Q}_{d_1}(N)/\Gamma_0(N) \times \mathcal{Q}_{d_2}(N)/\Gamma_0(N) | a_1 = a_2, \text{ i.e., } Q_1(1, 0) = Q_2(1, 0)\}$$
where $Q_d(N)$ denotes the set of primitive and positive definite binary quadratic forms $aX^2 + bXY + cY^2$ of discriminant $d$ with $(a, N) = 1$. The isomorphism is given by

$$[aX^2 + bXY + cY^2] \to \left[ a, N \frac{-b + \sqrt{d}}{2} \right].$$

**Proof.** It is not hard to check that

$$\tau_i \left( x + y \frac{d_i + \sqrt{d}}{2} \right) = \left( x + yd_i, y \frac{d_i - d_2}{4} \right)$$

for $x, y \in \mathbb{Q}$. Thus, one has $\tau_i^{-1}(K_0(N)) = \mathcal{O}_{i,N}$ where $\mathcal{O}_{i,N}$ is the order of conductor $N$ of $E_i$ and $E_i \backslash \mathbb{A}_{E_i,f} / \tau_i^{-1}(K_0(N)) \cong \text{Cl}(E_i,N)$ by [7] Section 4.4. This fact together with [29] Lemma 3.5 implies the first assertion of the lemma. The isomorphism is due to Chen and Yui [12, Thm. 4.4].

**Proposition 5.4.** Let $G_i$ be the associated Galois group of the ring class field of conductor $N$ of $E_i$. Then the points

$$\left[ z_{\sigma_1}^+, (t_1^{-1}, t_2^{-1}) \right] \in T(\mathbb{Q}) \setminus \{ z_{\sigma_2}^+ \} \times T(\mathbb{A}_{q,f})/K,N,T$$

are identified with

$$[\tau_1^{\sigma_1}, \tau_2^{\sigma_2}], [(-\tau_1)^{\sigma_1}, (-\tau_2)^{\sigma_2}] \in Y_0(N) \times Y_0(N),$$

respectively, where $\sigma_a \in G_i$ is the Galois element associated to $[a_i] \in \text{Cl}(E_i,N)$ via Artin map, and $[a_i] \in \text{Cl}(E_i,N)$ is the ideal class associated to the idele class $[t_i] \in T(\mathbb{Q}) \setminus T(\mathbb{A}_{q,f})/K,N,T$.

In particular, by the Shimura reciprocity law, one has $\tau_i^{\sigma_1} = \varphi(a_i)$ where $\tau_a$ is the CM point associated to the ideal $a$, and $\varphi([a, N^{b+\sqrt{d}}]) = [a, b+\sqrt{d}]$.

**Proof.** The former results follow from Lemma 5.3 and [29] Proposition 3.6. The latter result follows from [26, Chapter 6] (or see [27] for a nice summary of Shimura reciprocity law) and [12, Theorem III].

Now we are ready for

**Proof of Theorem 7.10.** By Theorem 2.3(2) and Theorem 1.5, we deduce that

$$\Phi(z, F_N) = -4 \log |\Psi(z, F_N)| = -4 \log |\pi_N(z_1) - \pi_N(z_2)|$$

since $c(0,0) = 0$ by Lemma 3.4. Then together with Proposition 5.4 and Theorem 5.1, equation (5.8) implies that

$$4 \sum_{([a_1],[a_2]) \in S(p,d_1,d_2)} \left( \log |\pi_N(\tau_1^{\sigma_1}) - \pi_N(\tau_2^{\sigma_2})| + \log |\pi_N((-\tau_1)^{\sigma_1}) - \pi_N((-\tau_2)^{\sigma_2})| \ight.$$

$$\left. + \log |\pi_N((\tau_1)^{\sigma_1}) - \pi_N((-\tau_2)^{\sigma_2})| + \log |\pi_N((-\tau_1)^{\sigma_1}) - \pi_N((-\tau_2)^{\sigma_2})| \right)$$

$$= -\frac{2|S(N,d_1,d_2)|}{\Lambda(0,\chi)} \left[ a_1(\phi_{0,0}) + \sum_{\mu \in L'/L} c(0,\mu)a_0(\phi_{\mu}) \right]$$

(5.9)  $$= -\frac{|S(N,d_1,d_2)|w_1w_2}{2h(d_1)h(d_2)} \left[ a_1(\phi_{0,0}) + \sum_{\mu \in L'/L} c(0,\mu)a_0(\phi_{\mu}) \right],$$

where the simplification in the second equality follows from Lemma 3.4 which tells that $c(-m,\mu) = 0$ except $c(-1,\mu_{0,0}) = 1$ for $m > 0$, and the last equality follows from the fact that $\Lambda(0,\chi) =$
\( \Lambda(0, \chi_{E_1/Q}) \Lambda(0, \chi_{E_2/Q}) \). Since \( \pi_N(\tau + 1) = \pi_N(\tau) \) and \( -\tau_i = -d_i + \tau_i, \) then the left hand side of (5.9) is simply
\[
16 \sum_{([a],[b]) \in S(p,d_1,d_2)} \log |\pi_N(\tau_1^\sigma_1) - \pi_N(\tau_2^\sigma_2)|, \]
and thus we have
\[
\sum_{([a],[b]) \in S(p,d_1,d_2)} \log |\pi_N(\tau_1^\sigma_1) - \pi_N(\tau_2^\sigma_2)|
= -\frac{|S(N,d_1,d_2)|w_1w_2}{32\sqrt{d_1(d_2)}} \left( a_1(\phi_0,0) + \sum_{\mu \in \mathbb{L}'/L} c(0,\mu)a_0(\phi_\mu) \right)
= -\frac{|S(N,d_1,d_2)|w_1w_2}{32\sqrt{d_1(d_2)}} \left( \sum_{t \in \mathbb{F}_q^\times} a(t,\phi_0,0) + \sum_{\mu \in \mathbb{L}'/L} c(0,\mu)a_0(\phi_\mu) \right)
= \frac{|S(N,d_1,d_2)|w_1w_2}{32\sqrt{d_1(d_2)}} \left( \sum_{t = 2m+D} a \left( \frac{t}{\sqrt{D}},\phi_0,0 \right) + \sum_{\mu \in \mathbb{L}'/L} c(0,\mu)a_0(\phi_\mu) \right)
\]
where (5.10) follows from the rescaling \( t \rightarrow \frac{t}{\sqrt{D}} \) and the fact that \( a \left( \frac{t}{\sqrt{D}},\phi_0,0 \right) = 0 \) unless \( \frac{t}{\sqrt{D}} \in \partial_F^{-1} = \frac{1}{\sqrt{D}} \mathcal{O}_F. \) Moreover, by Lemma 3.4 we can easily show that for \( p \in \{3,5,7,13\}, \) the constant terms \( c(0,\mu_0,k) = \frac{24}{p-1} \) for \( 1 \leq k \leq p - 1 \) and \( c(0,\mu) = 0 \) for \( \mu \neq \mu_0,k, \) and these simplify the right hand side of (5.10) and yield (1.4) in Theorem 1.10. Finally, the quadratic form interpretation of the left hand side of (1.4) follows from the isomorphism given in Lemma 5.3. \( \square \)

6. Computations of \( a_1 \left( \frac{t}{\sqrt{D}},\phi_0,0 \right) \) and \( a_0(\phi_0,k) \)

In this section, we compute \( a_1 \left( \frac{t}{\sqrt{D}},\phi_0,0 \right) \) and \( a_0(\phi_0,k) \) explicitly. To compute \( a_1 \left( \frac{t}{\sqrt{D}},\phi_0,0 \right), \) one needs to calculate the local Whittaker functions \( W_{\psi_F}^{\psi_F}(s,\phi) \). We note that if one lets \( \psi_F(t) = \psi_F \left( \frac{t}{\sqrt{D}} \right) \) and \( W' = W \) with \( F \)-quadratic form \( Q'_F(z) = z \bar{z}, \) then the Weil representations associated to \( (W,Q_F,\psi_F) \) and \( (W',Q'_F,\psi'_F) \) are the same, and one has by [16] Lemma 4.2.2,\
\[
W_{\psi_F}^{\psi_F}(s,\phi) = |D|_{\mathbb{F}_p}^{-\frac{s}{2}} W_{\psi_F}^{\psi_F}(s,\phi).
\]
Thus
\[
\frac{W_{\psi_F}^{\psi_F}(s,\phi)}{\gamma(W_p)} = |D|_{\mathbb{F}_p}^{-\frac{s}{2}} L_p(s+1,\chi) W_{\psi_F}^{\psi_F}(s,\phi) \gamma(W_p),
\]
and we will compute \( a_1 \left( \frac{t}{\sqrt{D}},\phi_0,0 \right) \) via \( W_{\psi_F}^{\psi_F}(s,\phi) \) whose calculations are tidier due to the normalization (see [30] Sec. 6). In addition, for \( a_0(\phi_0,k) \), by the definition (5.3), one has
\[
a_0(\phi_0,k) = -\bar{W}_{0,f}(0,\phi_0,k)
\]
with

\[ W_{0,f}(s, \phi_\mu) = \prod_{p \neq \infty} \left| D_p \right|^{\frac{1}{2}} \frac{L_p(s + 1, \chi)}{L_p(s, \chi)} \frac{W^{\phi_\mu}_0(s, \phi_\mu)}{\gamma(W_p)} \]

provided \( \phi_\mu \) is factorizable.

We will compute \( a_1 \left( \frac{t}{\sqrt{D}}, \phi_{0,0} \right) \) and \( a_0(\phi_{0,k}) \) case by case according to the ramification of \( p \) in \( F \) and \( E \). Throughout the remainder of this section, for \( 0 \leq k \leq p - 1 \), we write \( \phi_{0,k} \) for \( \text{Char}(\mu_{0,k} + L_p \otimes \hat{\mathbb{Z}}) \). Clearly, by (5.7), one has

\[ \phi_{0,k,p} = \phi_{0,k,p} \prod_{p \neq \infty} \phi_{0,0,p} \]

where \( \phi_{0,k,p} = \text{Char}(\mu_{0,k} + L_p) \), \( L_p = L_p \otimes \mathbb{Z}_p \) and \( \phi_{0,0,p} = \text{Char}(\mathcal{O}_E) \) for \( p \nmid p \). One key step to computing these coefficients via local Whittaker functions is the factorizability of \( \phi_\mu \), so our main strategy is to write \( \phi_{0,k,p} \) as a sum of products factorizable over \( p \mid p \).

Case 1. When \( \left( \frac{D}{p} \right) = \left( \frac{d}{p} \right) = 1 \), then \( p \) is completely split in \( F \) and in \( E \), that is, \( p \mathcal{O}_F = p_1 p_2 \) and \( p \mathcal{O}_E = \mathfrak{B}_1 \mathfrak{B}_1 \mathfrak{B}_2 \mathfrak{B}_2 \). Similar to [29 Section 5], it is not hard to check that

\[ L_p = \prod_{i=0}^{p-1} (M_i \times M_i) \]

where \( M_i = \left\{ (x_1, x_2) \in \mathbb{Z}_p^2 \mid x_1 + x_2 \equiv i \ (\text{mod} \ p) \right\} \), and thus for \( 0 \leq k \leq p - 1 \),

\[ \phi_{0,k} = \sum_{i=0}^{p-1} \phi_{0,k}^{(i)} \]

where

\[ \phi_{0,k}^{(i)} = \phi_{0,k}^{(i)} \prod_{p \neq \infty} \phi_{0,0,p} \]

and \( \phi^{(i)} = \text{Char}(M_i) \). In this case, we aim to compute

\[ a_1(\phi_{0,0}) = \sum_{i=0}^{p-1} a_1(\phi_{0,0}^{(i)}) \]

where

\[ a_1(\phi_{0,0}^{(i)}) = \sum_{t=2m+D \sqrt{t} \sqrt{D}}_{|2m+D| < \sqrt{D}} \frac{a \left( \frac{t}{\sqrt{D}} \phi_{0,0}^{(i)} \right)}{m \in \mathbb{Z}} \]

and for \( 1 \leq k \leq p - 1 \),

\[ a_0(\phi_{0,k}) = \sum_{i=0}^{p-1} a_0(\phi_{0,k}^{(i)}) \]

where

\[ a_0(\phi_{0,k}^{(i)}) = -\tilde{W}_{0,f}^{(i)}(0, \phi_{0,k}^{(i)}). \]

Now we briefly explain how to compute \( a \left( \frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)} \right) \). First denote by Diff\((W, t/\sqrt{D})\) the set of prime ideals \( p \) of \( F \) such that \( W_p \) does not represent \( t/\sqrt{D} \), i.e., \( p \in \text{Diff}(W, t/\sqrt{D}) \) if and only if \( t \neq z \bar{z} \) for any \( z \in E_F^\times \) if and only if \( p \) is inert in \( E/F \) and ord\(_p(t) \) is odd. By [29 Prop. 2.7(1)] and (5.2), we can see that \( a \left( \frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)} \right) = 0 \) when \( |\text{Diff}(W, t/\sqrt{D})| > 1 \)
since \( W^*(\mathbf{p},\phi_{0,0}^{(i)}) = 0 \) for \( \mathbf{p} \in \text{Diff}(W,t/\sqrt{D}) \). Then when \( \text{Diff}(W,t/\sqrt{D}) = \{\mathbf{p}\} \), by [29, Prop. 2.7(2)] or direct calculations from \((6.2)\), one has

\[
(6.3) \quad a\left(\frac{t}{\sqrt{D}},\phi_{0,0}^{(i)}\right) = -4W^*_{\sqrt{p},p}(0,\phi_{0,0}^{(i)}) \prod_{q|p} W^*_q(0,\phi_{0,0}^{(i)}).
\]

Clearly, such a prime ideal \( \mathbf{p} \) must be neither \( \mathbf{p}_1 \) nor \( \mathbf{p}_2 \). In addition, by [29, Prop. 2.7(3), (4)] or direct calculations from \((5.2)\), one has

\[
W^*_{\sqrt{p},p}(0,\phi_{0,0}^{(i)}) = \frac{1 + \text{ord}_p(t)}{2} \log N(\mathbf{p})
\]

and

\[
W^*_q(0,\phi_{0,0}^{(i)}) = \frac{\gamma(W_q)}{\gamma(W)}
\]

for \( q \neq \mathbf{p} \), where

\[
(6.4) \quad \rho_q(a) = \begin{cases} 1 & \text{if } q \text{ is ramified in } E, \\ \frac{1 + (-1)^\text{ord}_q a}{2} & \text{if } q \text{ is inert in } E, \\ 1 + \text{ord}_q a & \text{if } q \text{ is split in } E. \end{cases}
\]

Then together with the facts that \( \prod_{q \neq \mathbf{p}} \rho_q(tp^{-1}) = 1 \), we can rewrite \((6.3)\) as

\[
a\left(\frac{t}{\sqrt{D}},\phi_{0,0}^{(i)}\right) = -4 \frac{1 + \text{ord}_p(t)}{2} \log N(\mathbf{p}) \prod_{q|p} \rho_q(tp^{-1}) \frac{1}{2} \prod_{j=1}^2 W^*_q(0,\phi_{p_j}^{(i)}) \frac{1}{\gamma(W)} \prod_{q|p} \rho_q(tp^{-1}) \frac{1}{2} \prod_{j=1}^2 W^*_q(0,\phi_{p_j}^{(i)}) \frac{1}{\gamma(W)}
\]

where the third equality follows from \((6.2)\). Finally, since this happens for exactly one prime ideal \( \mathbf{p} \) such that \( \text{Diff}(W,t/\sqrt{D}) = \{\mathbf{p}\} \), we can write it in a unified form as

\[
a\left(\frac{t}{\sqrt{D}},\phi_{0,0}^{(i)}\right) = -4 \frac{p^2}{(p - 1)^2} \frac{1 + \text{ord}_p(t)}{2} \log(N(\mathbf{p})) \prod_{q|p} \rho_q(tp^{-1}) \frac{1}{2} \prod_{j=1}^2 W^*_q(0,\phi_{p_j}^{(i)}) \frac{1}{\gamma(W)} \prod_{q|p} \rho_q(tp^{-1}) \frac{1}{2} \prod_{j=1}^2 W^*_q(0,\phi_{p_j}^{(i)}) \frac{1}{\gamma(W)}
\]

The local Whittaker functions \( W^*_{p_j}(0,\phi_{p_j}^{(i)})/\gamma(W_{p_j}) \) are computed explicitly in Corollary \((6.2)\).

For \( a_0(\phi_{0,k}^{(i)}) \), by definition, we first know that

\[
a_0(\phi_{0,k}^{(i)}) = - \left( \frac{L(1 + s, \chi_{p_1}) W^*_{0,p_1}(s, \phi_{p_1}^{(i)})}{L(s, \chi_{p_1}) \gamma(W_{p_1})} \frac{L(1 + s, \chi_{p_2}) W^*_{0,p_2}(s, \phi_{p_2}^{(i+k)})}{L(s, \chi_{p_2}) \gamma(W_{p_2})} \prod_{p|p} W^*_p(0,\phi_{0,0,p}) \right)_{s=0}.
\]
By [30] Prop. 5.7, one can check that
\[
\left. \left( \frac{L(1 + s, \chi_p)W_{0,p}^\psi(s, \phi_{0,0,p})}{L(s, \chi_p)\gamma(W_p)} \right)' \right|_{s=0} = 0
\]
and
\[
\left. \frac{L(1 + s, \chi_p)W_{0,p}^\psi(s, \phi_{0,0,p})}{L(s, \chi_p)\gamma(W_p)} \right|_{s=0} = 1.
\]
Thus, we can deduce that
\[
a_0(\phi^{(i)}_{0,k}) = -\left( \frac{L(1 + s, \chi_{p_1})W_{0,p_1}^\psi(s, \phi^{(i)}_{p_1})}{L(s, \chi_{p_1})\gamma(W_{p_1})} - \frac{L(1 + s, \chi_{p_2})W_{0,p}^\psi(s, \phi^{(i+k)}_{p_2})}{L(s, \chi_p)\gamma(W_p)} \right)' \bigg|_{s=0},
\]
and by Lemma 6.1 we know that for \( i \neq 0 \)
\[
\left. \frac{L(1 + s, \chi_{p_j})W_{0,p_j}^\psi(s, \phi^{(i)}_{p_j})}{L(s, \chi_{p_j})\gamma(W_{p_j})} \right|_{s=0} = 0
\]
and
\[
\left. \frac{L(1 + s, \chi_{p_j})W_{0,p_j}^\psi(s, \phi^{(i)}_{p_j})}{L(s, \chi_{p_j})\gamma(W_{p_j})} \right|_{s=0} = \frac{2\log p}{p-1}.
\]
Also, by Lemma 6.1 one can check that
\[
\left. \frac{L(1 + s, \chi_{p_j})W_{0,p_j}^\psi(s, \phi^{(0)}_{p_j})}{L(s, \chi_{p_j})\gamma(W_{p_j})} \right|_{s=0} = 1.
\]
Therefore, we have
\[
a_0(\phi_{0,k}) = -\frac{4\log p}{p-1}.
\]
Case 2. When \( \left( \frac{d_1}{p} \right) = \left( \frac{d_2}{p} \right) = -1 \), then \( p\mathcal{O}_F = p_1p_2 \) and \( p_i \) are inert in \( E/F \). We have
\[
E_p = E_{p_1} \times E_{p_2}
\]
and the following identification (see, e.g., [21] Prop. 4.31)
\[
(\lambda_{p_1}, \lambda_{p_2}) : E_p \to E_{p_1} \times E_{p_2}
\]
\[
\sqrt{d_1} \to (\sqrt{d_1}, \sqrt{d_1}),
\]
\[
\sqrt{d_2} \to (\sqrt{d_2}, -\sqrt{d_2}).
\]
Since \( p\mathcal{O}_F \subset L_p \), then it is easy to see that
\[
\mathcal{L}_p = \prod_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} \{ (\lambda_{p_1}(\gamma_{hij}), \lambda_{p_2}(\gamma_{hij})) + p\mathcal{O}_{E_{p_1}} \times p\mathcal{O}_{E_{p_2}} \}
\]
where
\[
\gamma_{hij} = h + i\frac{-d_1 + \sqrt{d_1}}{2} + j\frac{d_2 + \sqrt{d_2}}{2}.
\]
Thus, one has
\[
\phi_{0,k,p} = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} \phi^{(k)}_{hij,p_1} \phi^{(k)}_{hij,p_2}
\]
where for \( l = 1, 2, \)
\[
\phi_{hij,p_l}^{(k)} = \text{Char}(\lambda_{p_l}(\gamma_{hij}^{(k)} + p\mathcal{O}_{E_{p_l}}))
\]
and
\[
\gamma_{hij}^{(k)} = h + i \frac{-d_1 + \sqrt{d_1}}{2} + j \frac{d_2 + \sqrt{d_2}}{2} + k \frac{(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4}.
\]

Then we can write
\[
\phi_{0,k} = \sum_{h, i, j \in \mathbb{Z}/p\mathbb{Z}} \Phi_{hij}^{(k)}
\]
where
\[
\Phi_{hij}^{(k)} = \phi_{hij,p_1}^{(k)} \phi_{hij,p_2}^{(k)} \prod_{p \mid p} \phi_{0,0,p}.
\]
and we need to compute
\[
a_1(\phi_{0,0}) = \sum_{h, i, j \in \mathbb{Z}/p\mathbb{Z}} a_1(\Phi_{hij}^{(0)})
\]
where
\[
a_1(\Phi_{hij}^{(0)}) = \sum_{t = 2m + D \sqrt{D} \mid \mathbb{Z}} \sum_{m \in \mathbb{Z}} a\left(\frac{t}{\sqrt{D}} \Phi_{hij}^{(0)}\right),
\]
and for \( 1 \leq k \leq p - 1, \)
\[
a_0(\phi_{0,k}) = \sum_{h, i, j \in \mathbb{Z}/p\mathbb{Z}} a_0(\Phi_{hij}^{(k)})
\]
where
\[
a_0(\Phi_{hij}^{(k)}) = -\tilde{W}_{0,f}'(0, \Phi_{hij}^{(k)}).
\]

Similar to Case 1, it is not hard to deduce that
\[
a\left(\frac{t}{\sqrt{D}} \Phi_{hij}^{(0)}\right) = -4 \begin{cases} 
\frac{p^2}{(p - 1)^2} \frac{1 + \text{ord}_p(t)}{2} \log(N(p)) \prod_{q \mid p} \rho_q(t)^{-1} \prod_{l=1}^2 W_{t,p_l}(0, \phi_{hij,p_l}^{(0)}) \gamma(W_{p_l}) & \text{if } p \nmid p,
\frac{p}{p - 1} \left( \frac{p^{s+1} - 1}{p^{s+1} - 1} \frac{\gamma(W_{p_l})}{\gamma(W_{p_l}')} \right)^{s=0} \prod_{q \mid p} \rho_q(t) \frac{W_{t,p_l}'(0, \phi_{hij,p_l}^{(0)})}{\gamma(W_{p_l}')} & \text{if } p \mid p \text{ and above } t,
\end{cases}
\]
and for \( 1 \leq k \leq p - 1, \)
\[
a_0(\Phi_{hij}^{(k)}) = -p^2 \left( \prod_{l=1}^2 \frac{W_{t,p_l}'(s, \phi_{hij,p_l}^{(k)})}{(p^{s+1} - 1) \gamma(W_{p_l})} \right)^{s=0} = -p^2 \left( \prod_{l=1}^2 \frac{(p^s - 1) \gamma(W_{p_l}')}{p^{s+1} - 1} \gamma(W_{p_l}) \right)^{s=0},
\]
where, by [30, Cor. 5.3],
\[
\frac{W_{t,p_l}'(s, \phi_{hij,p_l}^{(k)})}{\gamma(W_{p_l})}
\]
Case 3. When \( \Phi(\gamma_{hij}) > 0 \) and we have the following identification

\[
\phi_k = \phi_{0,k} \prod_{p \mid p} \phi_{0,0,p}
\]

where \( \phi_{0,k} = \text{Char}(\mu_{0,k} + \mathcal{L}_p) \) and \( \phi_{0,0,p} = \text{Char}(\mathcal{O}_{E,p}) \). Similar to Case 2, we have

\[
\phi_{0,k,p} = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} \phi_{hij}^{(k)}
\]

where

\[
\phi_{hij}^{(k)} = \text{Char}(\lambda(\gamma_{hij}^{(k)} + p\mathcal{O}_{E,p})
\]

and

\[
\gamma_{hij}^{(k)} = h + i \frac{-d_1 + \sqrt{d_1}}{2} + j \frac{d_2 + \sqrt{d_2}}{2} + k \frac{(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4}.
\]

And similarly, we write

\[
\phi_{0,k} = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} \Phi_{hij}^{(k)}
\]

where

\[
\Phi_{hij}^{(k)} = \phi_{hij}^{(k)} \prod_{p \mid p} \phi_{0,0,p}.
\]

and we aim to compute

\[
a_1(\phi_{0,0}) = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} a_1(\Phi_{hij}^{(k)})
\]
where
\[ a_1(\Phi_{hij}^{(0)}) = \sum_{t=2m+D+\sqrt{D}, \quad m \in \mathbb{Z}} a\left(\frac{t}{\sqrt{D}}\Phi_{hij}^{(0)}\right), \]
and for \(1 \leq k \leq p-1,\)
\[ a_0(\phi_{0,k}) = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} a_0(\Phi_{hij}^{(k)}) \]
where
\[ a_0(\Phi_{hij}^{(k)}) = -\tilde{W}^{(l)}_{0,f}(0, \Phi_{hij}^{(k)}). \]
Similar to Case 1, we can easily deduce that
\[ a\left(\frac{t}{\sqrt{D}}\Phi_{hij}^{(0)}\right) = -4\frac{p}{p^2-1} \sum_{p \text{ inert in } E/F} \frac{1 + \text{ord}_p(\ell)}{2} \log(N(p)) \prod_{q \nmid p} \rho_q(tp^{-1}) \frac{W_{t,p}^{\psi_p}(0, \phi_{hij}^{(0)})}{\gamma(W_p^\prime)}. \]
Also, for \(a_0(\Phi_{hij}^{(k)})\), we aim to compute
\[ a_0(\Phi_{hij}^{(k)}) = -\left(\frac{L(1+s, \chi_p)^{W_{0,p}^{\psi_p}(s, \phi_{hij}^{(k)})}}{L(s, \chi_p)\gamma(W_p)} \prod_{p \mid p} \frac{L(1+s, \chi_p)^{W_{0,p}^{\psi_p}(s, \phi_{hij}^{(0)})}}{L(s, \chi_p)\gamma(W_p)}\right)^{l}. \]
Again, similar to Case 1, we can easily deduce that
\[ a_0(\Phi_{hij}^{(k)}) = -\frac{2p^2 \log p}{p^2-1} \frac{W_{0,p}^{\psi_p}(0, \phi_{hij}^{(k)})}{\gamma(W_p)} = -\frac{2p^2 \log p}{p^2-1} \frac{W_{0,p}^{\psi_p}(0, \phi_{hij}^{(0)})}{\gamma(W_p^\prime)}, \]
where the second equality follows from [6,1]. Finally, by [30, Cor. 5.3], we have
\[ W_{t,p}^{\psi_p}(0, \phi_{hij}^{(k)}) = \begin{cases} 0 & \text{if } o(p^2 \eta_{hij}^{(k)} - t) < o_{E_p}(\eta_{hij}^{(k)}) + 2, \\ p^{2o_{E_p}(\eta_{hij}^{(k)}) + 2} & \text{if } o(p^2 \eta_{hij}^{(k)} - t) \geq o_{E_p}(\eta_{hij}^{(k)}) + 2, \end{cases} \]
where \(\eta_{hij}^{(k)} = \frac{1}{p} \gamma_{hij}(k), o(x) = \text{ord}_{F_p}(x), \pi_{F_p} \) is the uniformizer of \(F_p, o_{E_p}(x) = \min\{o(x_1), o(x_2)\}\) for \(x = (x_1, x_2)\) under the given identification \(E_p = F_p \times F_p.\)

Case 4. When \(p \nmid d_1\) and \((\frac{d_2}{p}) = 1,\) then \(pO_F = \mathfrak{p}^2\) and \(\mathfrak{p}\) is split in \(E/F.\) Then we have \(F_p \ncong F_p \times F_p\) and
\[ E_p \cong F_p \times F_p \]
with the following identification
\[ \lambda : E_p \to F_p \times F_p \]
\[ \sqrt{d_1} \to (\sqrt{d_1}, -\sqrt{d_1}), \]
\[ \sqrt{d_2} \to (\sqrt{d_2}, -\sqrt{d_2}). \]
The treatment if essentially the same as Case 3, and in this case, we have
\[ a\left(\frac{t}{\sqrt{D}}\Phi_{hij}^{(0)}\right) = -4\frac{p}{p-1} \sum_{p \text{ inert in } E/F} \frac{1 + \text{ord}_p(\ell)}{2} \log(N(p)) \prod_{q \nmid p} \rho_q(tp^{-1}) \frac{W_{t,p}^{\psi_p}(0, \phi_{hij}^{(0)})}{\gamma(W_p^\prime)}. \]
and

$$a_0(\phi_{hi}^{(k)}) = -\frac{p \log p}{p-1} \frac{W_{0,\tilde{p}}^{\psi_f}(0, \phi_{hi}^{(k)})}{\gamma(W_{\tilde{p}}^f)}$$

where

$$W_{0,\tilde{p}}^{\psi_f}(0, \phi_{hi}^{(k)}) = \begin{cases} 0 & \text{if } o(p^2 \eta_{hi}^{(k)} - t) < o_{E_p}(\eta_{hi}^{(k)}) + 4, \\ p^oE_p(\eta_{hi}^{(k)}) + 2 & \text{if } o(p^2 \eta_{hi}^{(k)} - t) \geq o_{E_p}(\eta_{hi}^{(k)}) + 4, \end{cases}$$

$$\eta_{hi}^{(k)} = \frac{1}{p} \left( h + i \frac{d_1 + \sqrt{d_1}}{2} + j \frac{d_2 + \sqrt{d_2}}{2} + k \frac{(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4} \right),$$

and $\pi_{\tilde{p}}$ is the uniformizer of $F_p$, $o_{E_p}(x) = \min\{o(x_1), o(x_2)\}$ for $x = (x_1, x_2)$ under the given identification $E_p = F_p \times F_{\tilde{p}}$.

Case 5. When $p \nmid d_i$ and $\left( \frac{a_p}{p} \right) = -1$, then $pO_F = \tilde{p}^2$ and $\tilde{p}$ is inert in $E/F$. Similarly, the treatment of this case is essentially the same as the previous two, but one needs to consider an extra subcase, say Diff$(W, t/\sqrt{D}) = \{\tilde{p}\}$. Thus we have

$$a \left( \frac{t}{\sqrt{D}}, \Phi_{hi}^{(0)} \right)$$

$$\left\{ \begin{array}{ll} \frac{p}{p-1} \frac{1 + \text{ord}_p(t) \log(N(p))}{2} \prod_{q \parallel p} \rho_q(t p^{-1}) \frac{W_{t,\tilde{p}}^{\psi_f}(s, \phi_{hi}^{(0)})}{\gamma(W_{\tilde{p}}^f)} & \text{if } \text{Diff}(W, t/\sqrt{D}) = \{p\} \text{ and } p \neq \tilde{p}, \\ \left( \frac{p^{2s+1}}{p^{s+1} - 1} \frac{W_{t,\tilde{p}}^{\psi_f}(s, \phi_{hi}^{(0)})}{\gamma(W_{\tilde{p}}^f)} \right)^t \prod_{q \parallel p} \rho_q(t) & \text{if } \text{Diff}(W, t/\sqrt{D}) = \{\tilde{p}\}, \end{array} \right.$$
Finally, it remains to compute the local Whittaker functions $W_{i_1,p_j}(0, \phi_{i_1}^{(i)})/\gamma(W_{i_1,p_j})$ mentioned in Case 1. The following lemma essentially gives what we need.

**Lemma 6.1.** Let $W = \mathbb{Q}_p^2$ with the quadratic form $Q(x) = x_1 x_2$ and $\phi^{(i)}$ be defined as above. Let $N(k,m) = |\{i \in (\mathbb{Z}/p\mathbb{Z})^\times | i(k-i) \equiv m \pmod{p}\}|$. Then the local Whittaker function $W_i(s, \phi^{(i)})$ for $t \in \mathbb{Z}_p$ is given by

$$\frac{W_i(s, \phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{1}{p} \left| \frac{1}{p^{1+\sigma}} \right| & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left( \frac{k}{p} \right) = 1, \\
\frac{1}{p} \left| \frac{1}{p^{1+\sigma}} \right| & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left( \frac{k}{p} \right) = -1, \\
\frac{1}{p} \left| \frac{1}{p^{1+\sigma}} \right| \sum_{n=2}^{p-1} \frac{1}{p^{1+(1+\sigma)(n)}} & \text{if } t \in p\mathbb{Z}_p,
\end{cases}$$

and for $1 \leq i \leq p-1$,

$$\frac{W_i(s, \phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{1}{p} \left| \frac{1}{p^{1+\sigma}} \right| (N(i,m) - 1) & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times.
\end{cases}$$

Especially, when $s = 0$, we have

$$\frac{W_i(0, \phi^{(i)})}{\gamma(W)} = \begin{cases} 0 & \text{if } t \in j + p\mathbb{Z}_p \text{ and } \left( \frac{j}{p} \right) = -1, \\
\frac{2}{p} & \text{if } t \in j + p\mathbb{Z}_p \text{ and } \left( \frac{j}{p} \right) = 1, \\
\frac{2}{p} \left( \frac{1}{p^{1+(1+\sigma)}} \right) (\text{ord}_p(t) - 1) & \text{if } t \in p\mathbb{Z}_p,
\end{cases}$$

and for $1 \leq i \leq p-1$,

$$\frac{W_i(0, \phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{N(i,m)}{p} & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times, \\
\frac{2}{p} & \text{if } t \in p\mathbb{Z}_p.
\end{cases}$$

**Proof.** By the definition of $W_i(0, \phi)/\gamma(W)$ and unfolding, we have

$$\frac{W_i(0, \phi^{(i)})}{\gamma(W)} = \int_{\mathbb{Q}_p} J_i(b) \psi(-tb) |a(wn(b))|^2 db$$

$$= \int_{\mathbb{Z}_p} J_i(b) \psi(-tb) db + \sum_{n=1}^{\infty} p^n \int_{\mathbb{Z}_p^\times} J_i(p^{-n}b) \psi(-p^{-n}tb) |a(wn(p^{-n}b))|^2 db$$

where

$$J_i(b) = \int_{M_i} \psi(bx_1 x_2) dx_1 dx_2.$$

We first compute the integral $J_i(b)$. For $i = 0$, we can deduce that

$$J_0(b) = \int_{x_1 + x_2 \equiv 0 \pmod{p}} \psi(bx_1 x_2) dx_1 dx_2$$

$$= \frac{1}{p} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \psi(b(py - x_1)x_1) dy dx_1$$

$$= \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2) \left( \int_{\mathbb{Z}_p} \psi(bpy_1) dy \right) dx_1$$

$$= \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2) \text{Char}(\mathbb{Z}_p)(bpz_1) dx_1.$$
Now we compute the last integral case by case.

1. For $b \in \mathbb{Z}_p$,

\[
J_0(b) = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2)\text{Char}(\mathbb{Z}_p)(bp_1)dx_1 \\
= \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \\
= \frac{1}{p}.
\]

2. For $b \in \frac{k}{p} + \mathbb{Z}_p$ with $1 \leq k \leq p - 1$,

\[
J_0(b) = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2)\text{Char}(\mathbb{Z}_p)(bp_1)dx_1 \\
= \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \\
= \frac{1}{p} \left( \int_{\mathbb{Z}_p} 1dx_1 + \int_{\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \right) \\
= \frac{1}{p} \left( \frac{1}{p} + \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \right) \\
= \frac{1}{p} \left( \frac{1}{p} + \frac{1}{p} \sum_{j=1}^{p-1} e^{-\frac{2\pi i j^2 k}{p}} \right),
\]

3. For $b \notin \frac{1}{p} \mathbb{Z}_p$,

\[
J_0(b) = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2)\text{Char}(\mathbb{Z}_p)(bp_1)dx_1 \\
= \frac{1}{p} \int_{\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \\
= \frac{1}{p} \int_{\mathbb{Z}_p} 1dx_1 \\
= |b|^{-1}.
\]

In summary, one has

\[
J_0(b) = \begin{cases} 
\frac{1}{p}, & \text{if } b \in \mathbb{Z}_p; \\
\frac{1}{p} \left( 1 + \sum_{j=1}^{p-1} e^{-\frac{2\pi i j^2 k}{p}} \right), & \text{if } b \in \frac{k}{p} + p\mathbb{Z}_p \subset \frac{1}{p} \mathbb{Z}_p; \\
|b|^{-1}, & \text{if } b \notin \frac{1}{p} \mathbb{Z}_p.
\end{cases}
\]

Now we are ready to compute

\[
\frac{W_t(s, \phi^{(0)})}{\gamma(W)} = \int_{\mathbb{Z}_p} J_0(b)\psi(-tb)db + \sum_{n=1}^{\infty} p^{-ns} \int_{\mathbb{Z}_p^\times} J_0(p^{-n}b)\psi(-p^{-n}tb)db.
\]

33
\[ \frac{1}{p} + p^{1-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-1}b)\psi(-p^{-1}tb)db + \sum_{n=2}^{\infty} p^{n-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-n}b)\psi(-p^{-n}tb)db, \]

where

(1) for \( t \in -k + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times \),

\[ \frac{1}{p} + p^{1-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-1}b)\psi(-p^{-1}tb)db + \sum_{n=2}^{\infty} p^{n-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-n}b)\psi(-p^{-n}tb)db \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{j=1}^{p-1} \left( 1 + \sum_{m=1}^{p-1} e^{-\frac{2\pi im^2}{p}} \right) e^{\frac{2\pi ikb}{p}} \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \left( -1 + \sum_{1 \leq m \leq p-1} (p-1) \left( \frac{k}{b} \right) \sum_{1 \leq m \leq p-1} \left( \frac{k}{b} \right) \right) \]

\[ = \begin{cases} \frac{1}{p} - \frac{1}{p^{1+s}} & \text{if } \left( \frac{k}{b} \right) = -1, \\ \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } \left( \frac{k}{b} \right) = 1, \end{cases} \]

(2) for \( t \in p\mathbb{Z}_p \),

\[ \frac{1}{p} + p^{1-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-1}b)\psi(-p^{-1}tb)db + \sum_{n=2}^{\infty} p^{n-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-n}b)\psi(-p^{-n}tb)db \]

\[ = \frac{1}{p} + \frac{\text{ord}_p(t)+1}{p^{2+s}} \int_{\mathbb{Z}_p^\times} \psi(p^{-n+\text{ord}_p(t)}b)db \]

\[ = \frac{1}{p} + \frac{p-1}{p} \sum_{n=2}^{\text{ord}_p(t)} \frac{1}{p^{n+s}} - \frac{1}{p^{1+(1+\text{ord}_p(t))s}}. \]

Therefore, one has

\[ \frac{W_1(s, \phi(0))}{\gamma(W)} = \begin{cases} \frac{1}{p} - \frac{1}{p^{1+s}} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left( \frac{k}{b} \right) = 1, \\ \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left( \frac{k}{b} \right) = -1, \\ \frac{1}{p} + \frac{p-1}{p} \sum_{n=2}^{\text{ord}_p(t)} \frac{1}{p^{n+s}} - \frac{1}{p^{1+(1+\text{ord}_p(t))s}} & \text{if } t \in p\mathbb{Z}_p, \end{cases} \]

Now for \( 1 \leq i \leq p - 1 \), similarly, we have

\[ J_i(b) = \int_{x_1+x_2 \equiv i \pmod{p}} \psi(bx_1x_2)dx_1dx_2 \]

\[ = \frac{1}{p} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \psi(bx_1(i+py-x_1))dydx_1 \]

\[ = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(bx_1(i-x_1))\text{Char}(\mathbb{Z}_p)(bp_1)x_1dx_1. \]

Then

34
(1) for $b \in \mathbb{Z}_p$,

$$J_i(b) = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(b x_1(i - x_1)) \text{Char}(\mathbb{Z}_p)(bpx_1) dx_1$$

$$= \frac{1}{p} \int_{\mathbb{Z}_p} 1 dx_1$$

$$= \frac{1}{p},$$

(2) for $b \in \frac{k}{p} + \mathbb{Z}_p \subset \frac{1}{p} \mathbb{Z}_p^\times$,

$$J_i(b) = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(b x_1(i - x_1)) \text{Char}(\mathbb{Z}_p)(bpx_1) dx_1$$

$$= \frac{1}{p} \int_{\mathbb{Z}_p} \psi(b x_1(i - x_1)) dx_1$$

$$= \frac{1}{p} \left( \int_{\frac{k}{p} \mathbb{Z}_p} 1 dx_1 + \sum_{j=1}^{p-1} \int_{j+\frac{k}{p} \mathbb{Z}_p} \psi(b x_1(i - x_1)) dx_1 \right)$$

$$= \frac{1}{p} \left( \frac{1}{p} + \frac{1}{p} \sum_{j=1}^{p-1} e^{2\pi i (kj(i-j))} \right),$$

(3) for $b \in \frac{1}{p} \mathbb{Z}_p$,

$$J_i(b) = \frac{1}{p} \int_{\mathbb{Z}_p} \psi(b x_1(i - x_1)) \text{Char}(\mathbb{Z}_p)(bpx_1) dx_1$$

$$= \frac{1}{p} \int_{\frac{1}{p} \mathbb{Z}_p} \psi(b x_1(i - x_1)) dx_1$$

$$= \frac{1}{p} \left( \int_{\frac{1}{p} \mathbb{Z}_p} \psi(b x_1(i - x_1)) dx_1 + \int_{\frac{1}{p} \mathbb{Z}_p^\times} \psi(b x_1(i - x_1)) dx_1 \right)$$

$$= \frac{1}{p} \left( |b|^{-1} + \sum_{j=1}^{p-1} |b|^{-1} e^{2\pi i (oji)} \right)$$

$$= 0.$$

These can be summarized as follows.

$$J_i(b) = \begin{cases} 
\frac{1}{p} & \text{if } b \in \mathbb{Z}_p, \\
\frac{1}{p} \left( 1 + \sum_{j=1}^{p-1} e^{2\pi i (k(j-i))} \right) & \text{if } b \in \frac{k}{p} + \mathbb{Z}_p \subset \frac{1}{p} \mathbb{Z}_p^\times, \\
0 & \text{if } b \not\in \frac{1}{p} \mathbb{Z}_p.
\end{cases}$$

Then for $1 \leq i \leq p - 1$, $W_i(s, \phi^{(i)})/\gamma(W)$ can be computed in a similar way as $i = 0$.

$$\frac{W_i(s, \phi^{(i)})}{\gamma(W)} = \int_{\mathbb{Z}_p} J_i(b) \psi(-tb) db + \sum_{n=1}^{\infty} p^{n-ns} \int_{\mathbb{Z}_p^\times} J_i(p^{-n}b) \psi(-p^{-n}tb) db$$
\[ \frac{1}{p} + \frac{1}{p^{1+s}} \sum_{k=1}^{p-1} \left( 1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj-i)} \right) \int_{-j+pZ_p} \psi(p^{-1}tb) \, db, \]

where

(1) for \( t \in p\mathbb{Z}_p \),

\[ \frac{1}{p} + \frac{1}{p^{1+s}} \sum_{k=1}^{p-1} \left( 1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj-i)} \right) \int_{-j+p\mathbb{Z}_p} \psi(p^{-1}tb) \, db \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{k=1}^{p-1} \left( 1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj-i)} \right) \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \left( p - 1 - \sum_{1 \leq j \leq p-1, j \neq i} 1 + p - 1 \right) \]

\[ = \frac{1}{p} + \frac{1}{p^{1+s}}, \]

(2) for \( t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times \),

\[ \frac{1}{p} + \frac{1}{p^{1+s}} \sum_{k=1}^{p-1} \left( 1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj-i)} \right) \int_{-j+p\mathbb{Z}_p} \psi(p^{-1}tb) \, db \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{k=1}^{p-1} \left( 1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj-i)} \right) e^{-\frac{2\pi ik}{p}} \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{k=1}^{p-1} \left( e^{-\frac{2\pi ik}{p}} + \sum_{j=1}^{p-1} e^{\frac{2\pi i(j-i)k}{p}} \right) \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} \left( -1 + \sum_{1 \leq j \leq p-1} (p-1) - \sum_{1 \leq j \leq p-1} 1 \right) \]

\[ = \frac{1}{p} + \frac{1}{p^{2+s}} (-1 + (p-1)N(i,m) - (p-1) + N(i,m)) \]

\[ = \frac{1}{p} + \frac{1}{p^{1+s}} (N(i,m) - 1). \]

Therefore, we have

\[ W_i(s, \phi(i)) \gamma(W) = \begin{cases} 
\frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } t \in p\mathbb{Z}_p, \\
\frac{1}{p} + \frac{1}{p^{1+s}}(N(i,m) - 1) & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times.
\end{cases} \]
Hence, when $s = 0$,
\[
\frac{W_t(0, \phi^{(0)})}{\gamma(W)} = \begin{cases} 
0 & \text{if } t \in k + p\Z_p \text{ and } \left( \frac{k}{p} \right) = 1, \\
\frac{2}{p} & \text{if } t \in k + p\Z_p \text{ and } \left( \frac{k}{p} \right) = -1, \\
\frac{p-1}{p} (\text{ord}_p(t) - 1) & \text{if } t \in p\Z_p
\end{cases}
\]
and for $1 \leq i \leq p - 1$,
\[
\frac{W_t(0, \phi^{(i)})}{\gamma(W)} = \begin{cases} 
\frac{2}{p} & \text{if } t \in p\Z_p, \\
\frac{N(i, m)}{p} & \text{if } t \in m + p\Z_p \subset \Z_p^\times.
\end{cases}
\]

Following from Lemma 6.2, the local Whittaker functions $W_{t,p_j}^{\psi'}(0, \phi_p^{(i)})/\gamma(W_{p_j})$ can be computed explicitly as follows.

**Corollary 6.2.** Fix $d \in (\Z/p\Z)^\times$ such that $\sqrt{D} \equiv d \pmod{p\Z_p}$. Write $t = \frac{2m + D + \sqrt{D}}{2}$ and $\tilde{d} = \frac{d^2 + d}{2}$. Then for $1 \leq i \leq p - 1$,

1. when $m \equiv -\tilde{d} \pmod{p}$,
   \[
   \frac{W_{t,p_1}^{\psi'}(0, \phi_{p_1}^{(i)})}{\gamma(W_{p_1}')} = \frac{2}{p} \quad \text{and} \quad \frac{W_{t,p_2}^{\psi'}(0, \phi_{p_2}^{(i)})}{\gamma(W_{p_2}')} = \frac{N(i, p - \tilde{d})}{p};
   \]

2. when $m \equiv -\tilde{d} + d \pmod{p}$,
   \[
   \frac{W_{t,p_1}^{\psi'}(0, \phi_{p_1}^{(i)})}{\gamma(W_{p_1}')} = \frac{N(i, d)}{p} \quad \text{and} \quad \frac{W_{t,p_2}^{\psi'}(0, \phi_{p_2}^{(i)})}{\gamma(W_{p_2}')} = \frac{2}{p};
   \]

3. when $m \equiv -\tilde{d} + k \pmod{p}$ with $k \not\equiv 0, d \pmod{p}$,
   \[
   \frac{W_{t,p_1}^{\psi'}(0, \phi_{p_1}^{(i)})}{\gamma(W_{p_1}')} = \frac{N(i, k)}{p} \quad \text{and} \quad \frac{W_{t,p_2}^{\psi'}(0, \phi_{p_2}^{(i)})}{\gamma(W_{p_2}')} = \frac{N(i, k - d)}{p},
   \]

and for $i = 0$,

1. when $m \equiv -\tilde{d} \pmod{p}$ and $\left( \frac{\tilde{d}}{p} \right) = -1$,
   \[
   \frac{W_{t,p_1}^{\psi'}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = \frac{p-1}{p} (\text{ord}_{p_1}(\sigma_1(t)) - 1) \quad \text{and} \quad \frac{W_{t,p_2}^{\psi'}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = 0;
   \]

2. when $m \equiv -\tilde{d} \pmod{p}$ and $\left( \frac{\tilde{d}}{p} \right) = 1$,
   \[
   \frac{W_{t,p_1}^{\psi'}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = \frac{p-1}{p} (\text{ord}_{p_1}(\sigma_1(t)) - 1) \quad \text{and} \quad \frac{W_{t,p_2}^{\psi'}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = \frac{2}{p};
   \]

3. when $m \equiv -\tilde{d} + d \pmod{p}$ and $\left( \frac{-\tilde{d}}{p} \right) = -1$,
   \[
   \frac{W_{t,p_1}^{\psi'}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = 0 \quad \text{and} \quad \frac{W_{t,p_2}^{\psi'}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = \frac{p-1}{p} (\text{ord}_{p_2}(\sigma_2(t)) - 1);
   \]
(4) when \(m \equiv -\tilde{d} + d \pmod{p}\) and \(\left(\frac{-\tilde{d}}{p}\right) = 1\),
\[
\frac{W_{t,p_1}^{\psi_F}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = \frac{2}{p} \quad \text{and} \quad \frac{W_{t,p_2}^{\psi_F}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = \frac{p-1}{p} (\operatorname{ord}_{p_2}(\sigma_2(t)) - 1);
\]

(5) when \(m \equiv -\tilde{d} + k \pmod{p}\) with \(k \neq 0, d \pmod{p}\), \(\left(\frac{-k}{p}\right) = -1\) and \(\left(\frac{d-k}{p}\right) = 1\),
\[
\frac{W_{t,p_1}^{\psi_F}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = 0 \quad \text{and} \quad \frac{W_{t,p_2}^{\psi_F}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = \frac{2}{p} ;
\]

(6) when \(m \equiv -\tilde{d} + k \pmod{p}\) with \(k \neq 0, d \pmod{p}\), \(\left(\frac{-k}{p}\right) = -1\) and \(\left(\frac{d-k}{p}\right) = -1\),
\[
\frac{W_{t,p_1}^{\psi_F}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = 0 \quad \text{and} \quad \frac{W_{t,p_2}^{\psi_F}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = 0 ;
\]

(7) when \(m \equiv -\tilde{d} + k \pmod{p}\) with \(k \neq 0, d \pmod{p}\), \(\left(\frac{-k}{p}\right) = 1\) and \(\left(\frac{d-k}{p}\right) = 1\),
\[
\frac{W_{t,p_1}^{\psi_F}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = \frac{2}{p} \quad \text{and} \quad \frac{W_{t,p_2}^{\psi_F}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = \frac{2}{p} ;
\]

(8) when \(m \equiv -\tilde{d} + k \pmod{p}\) with \(k \neq 0, d \pmod{p}\), \(\left(\frac{-k}{p}\right) = 1\) and \(\left(\frac{d-k}{p}\right) = -1\),
\[
\frac{W_{t,p_1}^{\psi_F}(0, \phi_{p_1}^{(0)})}{\gamma(W_{p_1}')} = \frac{2}{p} \quad \text{and} \quad \frac{W_{t,p_2}^{\psi_F}(0, \phi_{p_2}^{(0)})}{\gamma(W_{p_2}')} = 0 .
\]

Proof. These follow directly from Lemma 6.2. \(\square\)

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