OPTIMAL INVESTMENT AND REINSURANCE OF INSURERS WITH LOGNORMAL STOCHASTIC FACTOR MODEL

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Abstract. We propose the stochastic factor model of optimal investment and reinsurance of insurers where the wealth processes are described by a bank account and a risk asset for investment and a Cramér-Lundberg process for reinsurance. The optimization is obtained through maximizing the exponential utility. Owing to the claims driven by a Poisson process, the proposed optimization problem is naturally treated as a jump-diffusion control problem. Applying the dynamic programming, we have the Hamilton-Jacobi-Bellman (HJB) equations and the corresponding explicit solution for the corresponding HJB. Hence, the optimal values and optimal strategies can be obtained. Finally, in numerical analysis, we illustrate the performance of the proposed optimization according to the results of the corresponding value function. In addition, compared to the wealth process without investment, the efficiency of the proposed optimization is discussed in terms of ruin probabilities.

1. Introduction. The optimization problems for insurers have been studied by many researchers. Previous works mainly use the dynamic programming approach. Adapting this approach, Hamilton–Jacobi–Bellman (HJB) equations can be derived. Optimal strategies are constructed by the solutions of HJB equations. In the literature on portfolio optimization, Badaoui et al. [3, 7, 12, 22, 25, 26] studied optimal investment problems for maximizing expected exponential utilities. The optimization problems based on Black-Scholes models are employed by [7, 22, 26]. In particular, since institutional investors such as insurance companies use economic factors such as dividend yields, the rate of inflation, unemployment rates, etc. in order to forecast the return of risky assets, we realize that stochastic factor models seem to be more realistic than Black-Scholes models. Hence, stochastic factor models are also adopted in [3, 4, 12].

In addition, regarding to optimal reinsurance problems, Bai et al. [1, 10, 13, 14, 18, 21, 23, 25, 27, 28] treated optimal investment-reinsurance problems through...
maximizing exponential utility criterions. Guan and Liang [9] considered the similar problem using a power utility criterion. Li et. al [15, 16, 17, 20, 29] studied mean-variance insurer’s optimal investment-reinsurance problems. Moreover, Bo and Wang [5] discussed the stochastic factor model, and adopted a power utility as a counterpart of [1, 13, 14, 18, 21, 23, 25, 27, 28]. Some other works are studied using the martingale method, which is a different effective method from the dynamic programming approach. This method is based on equivalent martingale measures and martingale representation theorems. For instance, optimal investment problems for insurers were solved in [24, 31], and optimal investment and risk control problems was settled in [30].

In particular, we state Badaoui-Fernández [3], Badaoui et al.[4], Fernández et al. [7], Hata and Yasuda [12], Liang et al. [18] and Xu et al. [25]. Fernández et al. [7] studied maximization of the expected exponential utility with Black-Scholes model. Indeed, Badaoui et al. [3, 4] considered a stochastic volatility model as counterparts of [7] where Badaoui and Fernández [3] assumed that the risky assets and factor processes are not correlated and Badaoui et al. [4] allowed that the risky assets and factor processes are correlated. Hata et al. [12] discussed a linear Gaussian stochastic factor model, and constructed the explicit optimal strategy. Note that this model does not belong to a class considered in [4] since the risk-premium process of [12] is unbounded. See Assumption 2 of [4]. Liang et. al [18] treated the optimal investment and reinsurance problem with an Ornstein-Uhlenbeck model. Moreover, Xu et al. [25] studied the optimal investment and reinsurance problem counterpart of [3].

Our objective is to consider the optimal investment and reinsurance problem with the lognormal stochastic factor model. Indeed, we employ a geometric Brownian motion as a stochastic factor. For example, since the unemployment rate is positive in general, it is reasonable to model the unemployment rate as a lognormal factor. Hence, in this paper, we may further assume an economic factor that takes a positive value such as the unemployment rate affects the mean return of a stock price. The model and the corresponding HJB equation are proposed in Section 2.

In Section 3 we study the case of the incomplete market. The optimal value function and the corresponding optimal investment and reinsurance are obtained. In order to have the optimal value function and the strategy, in Theorem 3.1, we solve the HJB equation proposed in Section 2 through (16) which turns out to be a linear partial differential equation (26) by using a suitable transformation. Applying Feynman-Kac’s formula, (26) has a stochastic representation (36). Moreover, (36) can be explicitly solved by using the self-similarity of Brownian motions and Yor’s formula, which is the distribution of the Brownian exponential functional given by [32]. We also prove the uniqueness of the proposed solution through proving the uniqueness of (16) by showing the martingale property of (43). See Lemma B.2. Based on the solution (25), we prove the verification theorem that the maximizer in the proposed HJB equation (14) is indeed optimal. In particular, according to the argument of Lemma B.2 again, we verify the optimal strategy belongs to the set $\mathcal{A}_{t,T}$ of admissible strategies using the martingale property. In Section 4 we consider the complete market scenario. Following the similar way of the case of $|p| < 1$, the explicit solution of the HJB and the uniqueness can be also obtained.

Section 5 illustrates the proposed optimization portfolio through numerical studies. We compare the value function with the varied initial surplus $x$ and the initial
factor $y$ based on two different risk aversion parameters $\alpha$. In addition, the corresponding ruin probabilities of the wealth process with investment and the one without investment are also shown based on the varied $\alpha$, the retention $k$, the payment parameter of claims $\theta$, and the initial surplus $x$. Surprisingly, in some extreme cases, the ruin probability of the case without investment is smaller than the one of the case with investment. However, as we expected, the investment leads to the more stable owing to the moderate slope of ruin probabilities and smaller ruin probabilities in the most cases which is important for the portfolio optimization and risk management.

The features and contributions are mentioned are as follows.

- In this paper, we propose the stochastic factor model which is relatively tractable. The explicit optimal value and the corresponding optimal investment strategy of (P) can be obtained using the dynamic programming. To best of our knowledge, the optimization by using our lognormal stochastic factor model will rarely exist.
- Using an analytical approach without the gradient estimate directly, we show the martingale properties of (43), (45), which prove the uniqueness of (16) and verification theorem of (P) respectively. In general, it is important to treat the gradient of the solution for (16). For example, in the case of linear Gaussian models, since the explicit solutions of HJB equations have quadratic growth, it is known that their gradients have linear growth. Hence, it is not so difficult to prove the martingale properties. See [12]. However, since our solutions (25) and (49) are very complicated, it is hard to get the gradient by a direct calculation. We created a way not to evaluate the gradient directly. See Lemma B.2 and Appendices B.2 and B.3.
- Through the numerical studies, we conclude that the investment implies the stability for the insurance company in the common situation in terms of the smaller ruin probability. In addition, it is also important to note that the investment not only helps the risk management for individuals but also the stability of the financial system for insurers since in the case of the insurers without investment, the ruin probability is too large to create the system crash. Hence, the investment leads to the system stability owing to the moderate slope of ruin probability in the initial surplus or the rate of claims.

2. Investment and reinsurance model. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ be an underlying filtered probability space endowed with a 2-dimensional standard Brownian motion $(w(t))_{t \geq 0}$, where $w := (w_1, w_2)^\top$ ((·)\top denotes the transpose of a vector or a matrix). Consider a market consisting of a bank account and a risky stock. Suppose that $S^0$ and $S$ are described as

$$
\begin{align*}
    dS^0_t &= S^0_t rd_t, \\
    dS_t &= S_t \left((\mu(Y_t) + r)dt + edw(t)\right), \\
    dY_t &= b(Y_t)dt + a(Y_t)dw_2(t),
\end{align*}
$$

(1)

with the initial values $S^0_0 > 0, S_0 > 0, Y_0 = y > 0$ and $r \geq 0$. And,

$$
e := \left(\sqrt{1 - \rho^2}, \rho\right) \in [0, 1] \times [-1, 1]
$$

(2)
is a unit vector with the constant instantaneous correlation:
\[ \rho = \frac{d(S, Y)_t}{d(S)_t d(Y)_t}. \]

Here, if \(|\rho| < 1\), then the market is incomplete. On the other hand, if \(|\rho| = 1\), then the market is complete. Moreover, for \(y > 0\)
\[ \mu(y) := \mu_0 + \mu_1 y, \quad b(y) := by, \quad \text{and} \quad a(y) := ay, \quad (3) \]
with \(b, \mu_0, \mu_1 \in \mathbb{R}\), and \(a > 0\).

Assume that an insurer invests in the risky assets and the bank account, and takes reinsurance. It is well known that the classical Cramér–Lundberg process is defined by
\[ R_t := x + ct - J_t, \]
where \(x\) is the initial surplus of the insurer, and \(c > 0\) is the premium rate. And, the total payment process \(J_t\) is defined by
\[ J_t := \sum_{i=1}^{p_t} Z_i, \]
where \((p_t)_{t \geq 0}\) is a Poisson process with a constant intensity \(\lambda > 0\), and \((Z_i)_{i \geq 1}\) is a sequence of independent non-negative random variables with the exponential distribution \(\nu:\)
\[ \nu(z) := 1 - e^{-\theta z}, \quad \theta > 0. \quad (4) \]
In other words, \(p_t\) is the total number of claims up to time \(t\), and \(Z_i\) is the size of the \(i\)-th claim.

In this paper, we consider the setting that the insurer takes reinsurance. We denote the retention level for reinsurance by \(h_t \in [0, 1]\). This means that for each claim \(Z\) arriving at time \(t\) the insurer and the reinsurer pay \(h_t Z\) and \((1 - h_t) Z\) respectively. In addition, the insurer pays the premium rate \(\tilde{c}(h_t)\) for this reinsurance where \(\tilde{c}(h)\) is defined by
\[ \tilde{c}(h) := k c(1 - h), \quad k \geq 1. \quad (5) \]
The classical Cramér–Lundberg process with the reinsurance function \(\tilde{c}(h)\) is given by
\[ R_t^h := x + \int_0^t \{c - \tilde{c}(h_s)\} ds - \int_0^t h_s dJ_s. \]
Assume that the insurer invests at time \(t\) the amount \(\pi_t\) of wealth \(X_{\pi}^{t,h}\) in the risky asset with price \(S_t\) and the remaining reserve \(X_{\pi}^{t,h} - \pi_t\) is invested in the risk-free asset. Now, for each \(t > 0\) the filtration \((\mathcal{F}_t)_{t \geq 0}\) is defined by
\[ \mathcal{F}_t := \sigma\{w(s), p_s, Z_j 1_{j \leq p_s}; s \leq t, j \geq 1\}. \]
Furthermore, we assume that \((w(t))_{t \geq 0}, (p_t)_{t \geq 0}\) and \((Z_i)_{i \geq 1}\) are mutually independent.
We consider a strategy \((\pi_t, h_t)\) in the space \(L_{2,t,T}\) of investment-reinsurance strategies:
\[
L_{2,t,T} := \{ (\pi_s, h_s) \in \mathbb{R} \times [0,1] : (\pi_s, h_s) \text{ is a } \mathcal{F}_s\text{-progressively measurable, and } \int_t^T |\pi_s|^2 ds < \infty, \ P - a.s. \}.
\]
Then, the insurer’s wealth has the dynamics:
\[
X_{t}^{\pi,h} = R_t^h + \int_0^t \left\{ \pi_s \frac{dS_s}{S_s} + \left( X_s^{\pi,h} - \pi_s \right) \frac{dS^0_s}{S^0_s} \right\} \, ds
= x + \int_0^t \left\{ c - \tilde{c}(h_s) + \pi_s \mu(Y_s) + r X_s^{\pi,h} \right\} \, ds
+ \int_0^t \pi_s \, dw(s) - \int_0^t h_s \, dJ_s.
\]
Here, we note that \(\int_{z>0} z \nu(dz) < \infty\),
and we write Poisson random measure of \(J\) on \([0, \infty) \times [0, \infty)\) as \(N(dt, dz)\):
\[
N([0,t] \times U) := \sum_{0 \leq s \leq t} \mathbf{1}_U(\Delta J_s)
\]
for a Borel set \(U \subset [0, \infty)\), where \(\Delta J_s := J_s - J_{s-}\). And, we can also define the compensated Poisson random measure
\[
\tilde{N}(dt, dz) := N(dt, dz) - \lambda \nu(dz) dt.
\]
In this paper, define an expected exponential type utility of the terminal wealth:
for a given constant \(T > 0\):
\[
J(t, x, y; \pi, h) := E_{t,x,y} \left[ -e^{-\alpha X_T^{\pi,h}} \right], \ \alpha > 0.
\] (6)
Then, we are interested in the following problem:
\[
(P) \quad V(t, x, y) := \sup_{(\pi, h) \in A_{t,T}} J(t, x, y; \pi, h),
\]
where \(A_{t,T} \subset \mathcal{L}_{2,t,T}\) is the set of admissible strategies to be described later.
Now, applying the dynamic programming approach, we shall formally derive the Hamilton-Jacobi-Bellman (HJB) equation for \((P)\). Here, we assume
\[
\theta > \alpha e^{rT}.\] (7)
Under \((7)\) we observe
\[
\int_{z>0} e^{\alpha h_z} e^{(T-t)} \nu(dz) \leq \int_{z>0} e^{\alpha z e^{rT}} \nu(dz) = \frac{\theta}{\theta - \alpha e^{rT}} < \infty.
\]
Then, recalling that
\[
dX_s^{\pi,h} = \left\{ c - \tilde{c}(h_s) + \pi_s \mu(Y_s) + r X_s^{\pi,h} \right\} ds + \pi_s \, dw(s)
- \int_{z>0} h_s \, zN(ds, dz), \quad X_t^{\pi,h} = x,
\] (8)
or equivalently,
\[ X_{T,T}^π,h := e^{r(T-t)} x + \int_t^T e^{r(T-s)} \{ c - \hat{c}(h_s) + \pi_s \mu(Y_s) \} ds \\
+ \int_t^T e^{r(T-s)} \pi_s e dw(s) - \int_t^T e^{r(T-s)} \int_{z>0} h_s z N(ds, dz), \]
we have
\[ e^{-\alpha X_{T,T}^π,h} = e^{-\alpha x e^{r(T-t)} - \alpha \int_t^T e^{r(T-s)} ds + \int_t^T \ell(s, Y_s, \pi_s) - \phi(s,h_s) ds} \cdot e^{\mathcal{E}^0_{t,T}(\pi, h)}, \] (9)
where \( \ell(t, y, \pi), \phi(t, h) \) and \( \mathcal{E}^0_{t,T}(\pi, h) \) are defined as follows:
\[ \ell(t, y, \pi) := \frac{\alpha^2}{2} e^{2r(T-t)} \pi^2 - \alpha e^{r(T-t)} \pi \mu(y), \]
\[ \phi(t, h) = - \left\{ \alpha e^{r(T-t)} \hat{c}(h_s) + \lambda \int_{z>0} \left( e^{\alpha h z e^{r(T-t)}} - 1 \right) \nu(dz) \right\}, \]
\[ \mathcal{E}^0_{t,T}(\pi, h) := e^{-\alpha \int_t^T e^{r(T-s)} \pi_s e dw(s) - \frac{\alpha^2}{2} \int_t^T e^{r(T-s)} \pi^2_s ds} \]
\[ \cdot e^{\int_{z>0} \alpha h z e^{r(T-s)} N(ds, dz) + \lambda \int_{t,s} \left( 1 - e^{\alpha h z e^{r(T-s)}} \right) \nu(dz) ds}. \]
Now, following a standard argument ([8] : Chapter IV), and using the dynamics (1) and (8), we formally derive the Hamilton-Jacobi-Bellman (HJB) equation for \( \mathbf{P} \):
\[ \sup_{(\pi, h) \in \mathbb{R} \times [0,1]} \left[ \frac{\partial V}{\partial t} + \frac{a(y)^2}{2} V_{yy} + b(y) V_y + \frac{\pi^2}{2} V_{xx} + \{ c - \hat{c}(h) + \pi \mu(y) + \rho x \} V_x \right. \\
\left. + \pi \rho a(y) V_{xy} + \lambda \int_{z>0} \{ V(t, x - h z, y) - V(t, x, y) \} \nu(dz) \right] = 0, \]
\[ V(T, x, y) = -e^{-\alpha x}. \]
If we make an ansatz written as
\[ V(t, x, y) := -e^{-\alpha x e^{r(T-t)} - \alpha \int_t^T e^{r(T-s)} ds + \ell(t, y)}, \] (10)
then \( u \) must solve the following partial differential equation:
\[ \sup_{\pi \in \mathbb{R}} \mathcal{L}_\pi u(t, y) + \sup_{h \in [0,1]} \phi(t, h) = 0, \ u(T, y) = 0, \] (11)
where \( \mathcal{L}_\pi u(t, y) \) is defined by
\[ \mathcal{L}_\pi u(t, y) := -\frac{\partial u}{\partial t} - \frac{a(y)^2}{2} D^2 u - \frac{a(y)^2}{2} (Du)^2 - b(y) Du \]
\[ -\frac{\alpha^2}{2} e^{2r(T-t)} \pi^2 + \alpha e^{r(T-t)} \pi \{ \mu(y) + a(y) \rho Du \}. \] (12)
Here, \( Du := \frac{\partial u}{\partial y} \) and \( D^2 u := \frac{\partial^2 u}{\partial y^2} \). Using (5), and noting that
\[-\lambda \int_{z>0} \left( e^{\alpha h z e^{r(T-t)}} - 1 \right) \nu(dz) \]
is strictly concave with respect to \( h \in [0,1] \), we see that there is a unique maximizer \( \hat{h}_t \) of \( \phi(t, h) \):
\[ \hat{h}_t := \arg \sup_{h \in [0,1]} \phi(t, h). \] (13)
The explicit form of \( \hat{h}_t \) will be given in Corollary 1.
If we further assume
\[ v(t, y) := u(t, y) + \int_t^T \phi(s, \tilde{h}_s) ds, \]
then \( v \) solves the following partial differential equation:
\[ \sup_{\pi \in \mathbb{R}} \mathcal{L}_\pi v(t, y) = 0, \quad v(T, y) = 0. \] (14)

Recalling that the maximizer is
\[ \tilde{\pi}(t, y) := \frac{1}{\alpha} e^{-r(T-t)} \{ \mu(y) + \rho a(y) Dv(t, y) \}, \] (15)
we rewrite (14) as
\[ -\partial_tv = \frac{a(y)^2}{2} D^2v + \left\{ b(y) - \rho a(y) \mu(y) \right\} Dv \\
+ \frac{(1 - \rho^2)a(y)^2}{2} (Dv)^2 - \frac{\mu(y)^2}{2}, \] (16)
\[ v(T, y) = 0. \]

We then discuss the solutions and the corresponding verification theorems in the case of the incomplete market with \(|\rho| < 1\) and in the case of the complete market with \(|\rho| = 1\) respectively.

3. The incomplete market case: \(|\rho| < 1\). In this section, we consider (P) in the case of \(|\rho| < 1\). This case means that the financial market is incomplete. In Subsection 3.1, we obtain the explicit solution of (16). In Subsection 3.2, we show the uniqueness for solution of (16). In Subsection 3.3, we prove the verification theorem.

3.1. The explicit solution of (16). Define
\[ \beta_1 := -|\mu_1| < 0, \] (17)
\[ \beta_0 := \frac{1}{\beta_1} \left\{ \mu_0 \mu_1 - \frac{b}{a} (\beta_1 + \rho \mu_1) \right\}, \] (18)
\[ \chi := \frac{1}{1 - \rho^2} \left\{ \frac{\beta_0^2 - \mu_0^2}{2} + \left( \frac{b}{a} - \frac{a}{2} \right) (\beta_0 + \rho \mu_0) \right\}, \] (19)
\[ \xi(y) := \frac{1}{(1 - \rho^2)a} \{(\beta_1 + \rho \mu_1) y + (\beta_0 + \rho \mu_0) \log y\}. \] (20)

Further, define \(G(t, y)\) as
\[ G(t, y) := \frac{1}{1 - \rho^2} \left\{ -\frac{\delta^2}{2} a^2 t - \beta_0 + \rho \mu_0 \log y \right\} \] (21)
\[ + \frac{1}{1 - \rho^2} \log \int_R dz \int_0^\infty du \exp \left[ -\frac{\beta_1 + \rho \mu_1}{a} - \frac{ye^z}{1 - \frac{\beta_1}{a} yu} \right] \left( \frac{\beta_1 + \rho \mu_1}{a} \right) \eta_{\alpha \xi}(u, z), \]
\[ \eta_t(u, z) := \frac{1}{2u} \exp \left( -\frac{2(1 + c^2)}{u} \right) \theta_{4z/2u}(t/4), \]  
(22) 

\[ \theta_r(t) := \frac{r}{\sqrt{2\pi} t} \exp \left( -\frac{\varepsilon_r^2}{2r^2} \right) \int_0^\infty e^{-\frac{\varepsilon_r^2}{2r^2} e^{-r \cosh \xi} \sinh \xi \sin \left( \frac{\pi \xi}{t} \right)} \, d\xi, \]  
and 
\[ \delta := \frac{1}{a^2} \left( b + a\beta_0 - \frac{a^2}{2} \right). \]  
(23) 

Then, we have the following.

**Theorem 3.1.** Assume (1), (2), (3) and 
\[ \rho \mu_1 > 0. \]  
(24) 

Then, (16) has a negative solution \( \hat{\psi}(t, y) \), where \( \hat{\psi}(t, y) \) is defined by 
\[ \hat{\psi}(t, y) = \chi \cdot (T - t) + \xi(y) + G(T - t, y). \]  
(25) 

First, we assume that \( v(t, y) \) is a solution to (16). Then, we see that 
\[ \psi(t, y) := e^{(1 - \rho^2)v(t, y)} \]  
satisfies 
\[ -\partial_t \psi = \frac{a(y)^2}{2} D^2 \psi + \{ b(y) - \rho a(y) \mu(y) \} D \psi - \frac{1 - \rho^2}{2} \mu(y)^2 \psi, \]  
(26) 

and \( \psi(T, y) = 1 \). Define 
\[ F_t := e^{-\rho \int_0^t \mu(Y_s) ds} - \frac{a(y)^2}{2} \int_0^t \mu(Y_s)^2 ds - \rho \int_0^t \mu(Y_s)^2 ds, \]  
(27) 

Then, by Feynman-Kac’s formula, we deduce that 
\[ \hat{\psi}(t, y) := E_{t, y}[F_T/F_t] = E[F_{T-t}] \]  
solves (26), if \( \hat{\psi} \in C^{1,2}([0, T] \times (0, \infty)) \) and \( F_T \in L^1(P) \). And, \( \hat{\psi} \) defined by 
\[ \hat{\psi}(t, y) := \frac{1}{1 - \rho^2} \log E_{t, y} [F_t, T] = \frac{1}{1 - \rho^2} \log E [F_{T-t}] \]  
(28) 

solves (16). Now, we try to check \( \hat{\psi} \in C^{1,2}([0, T] \times (0, \infty)) \). We recall 
\[ \hat{\psi}(t, y) = \hat{E} \left[ e^{-\frac{a(y)^2}{2} \int_0^T \mu(Y_s)^2 ds} \right], \]  
(29) 

where \( \hat{E} [\cdot] \) denotes the expectation with respect to the probability measure \( \hat{P} \) on \( (\Omega, \mathcal{F}) \) defined by 
\[ \left. \frac{d\hat{P}}{dP} \right|_{\mathcal{F}_t} = \mathcal{E}_t := e^{-\rho \int_0^t \mu(Y_s) ds} - \frac{a(y)^2}{2} \int_0^t \mu(Y_s)^2 ds \]  
(30) 

\[ = e^{\int_0^t (-\rho \mu_1 Y_s - \mu_0) ds} - \frac{a(y)^2}{2} \int_0^t (-\rho \mu_1 Y_s - \mu_0)^2 ds. \]  

From (24) and Lemma A.1 below \( \hat{P} \) is actually well-defined. Then, under the measure \( \hat{P} \), 
\[ \hat{w}_2(t) := w_2(t) + \rho \int_0^t \mu(Y_s) ds \]  
is a Brownian motion, and \( Y_t \) solves 
\[ dY_t = \{ b(Y_t) - \rho a(Y_t) \mu(Y_t) \} dt + a(Y_t) d\hat{w}_2(t) \]  
(31) 

with \( Y_0 = y \). Here, \( \tilde{\beta}_1 \) and \( \tilde{\beta}_0 \) are defined as follows: 
\[ \tilde{\beta}_1 := -\rho \mu_1 < 0, \ \text{and} \ \tilde{\beta}_0 := -\rho \mu_0. \]  
(32)
Then, we obtain the following.

**Lemma 3.2.** Assume (1), (2), (3) and (24). The equation (31) has a unique solution given by

\[ Y_t = \frac{ye^{a\bar{w}_2(t)} + a^2\bar{\delta}t}{1 - a\bar{\beta}_1y} > 0, \]

where \( \bar{\delta} \) is defined by

\[ \bar{\delta} := \frac{1}{a^2} \left( b + a\bar{\beta}_0 - a^2 \right). \]

**Proof.** The stochastic differential equation for \( \nu_t := \log Y_t \) is described as

\[ d\nu_t = \left( a^2\bar{\delta} + a\bar{\beta}_1 e^{\nu_t} \right) dt + ad\bar{w}_2(t), \quad \nu_0 = \log y. \]

From Itô’s formula, if \( \bar{\beta}_1 < 0 \), (35) has a strong solution:

\[ \nu_t = \nu_0 + a\bar{w}_2(t) + a^2\bar{\delta}t - \log \left\{ 1 - a\bar{\beta}_1 e^{\nu_0} \int_0^t e^{a\bar{w}_2(s)} + a^2\bar{\delta}s \, ds \right\}. \]

Hence, we obtain (33). Since \( (b + a\bar{\beta}_0)y + a\bar{\beta}_1 y^2 \) is locally Lipschitz continuous, (31) has a unique strong solution (33). From \( a\bar{\beta}_1 < 0 \) \( Y_t \) does not blow up in finite time. \( \Box \)

Then, we prove the smoothness of \( \tilde{v} \).

**Lemma 3.3.** Assume (1), (2), (3) and (24). Then, we have \( \tilde{v} \in C^{1,2}([0, T] \times (0, \infty)) \).

The proof is given in Appendix B. And, we observe the following.

**Lemma 3.4.** Assume (1), (2), (3) and (24). Then, it holds that

\[ \tilde{v}(t, y) = \chi \cdot (T - t) + \xi(y) + \frac{1}{1 - \rho^2} \log E \left[ e^{-(1 - \rho^2)(\xi(Y_{T-t})} \right]. \]

Here \( E[\cdot] \) is the expectation with respect to the probability measure \( \tilde{P} \) on \( (\Omega, \mathcal{F}) \) defined by

\[ \frac{d\tilde{P}}{dP} \bigg|_{\mathcal{F}_t} := e^{f_o'(\beta_0 + \beta_1 Y_s) d\bar{w}_2(s) - \frac{1}{2} f_o''(\beta_0 + \beta_1 Y_s)^2 ds}. \]

**Proof.** Using Itô’s formula, we have

\[ \log F_t - \log \tilde{E}_t = (1 - \rho^2) \{ \chi \cdot t + \xi(y) - \xi(Y_t) \} \]

\[ + \frac{\beta_1^2 - \mu_1^2}{2} \int_0^t Y_s^2 ds + \left\{ \beta_0\beta_1 - \mu_0\mu_1 + \frac{b}{a} (\beta_1 + \rho\mu_1) \right\} \int_0^t Y_s ds. \]

From (17), (18) and (38) we have

\[ F_t = e^{(1 - \rho^2)(\chi \cdot t + \xi(y) - \xi(Y_t))} \tilde{E}_t, \]

from which we have

\[ \tilde{v}(t, y) = \chi \cdot (T - t) + \xi(y) + \frac{1}{1 - \rho^2} \log E \left[ \tilde{E}_{T-t} e^{-(1 - \rho^2)(\xi(Y_{T-t})} \right]. \]

From \( \beta_1 < 0 \) and Lemma A.1 below \( \tilde{P} \) is actually well-defined. Hence, we have (36). \( \Box \)
Under the measure $\hat{P}$,
\[
\hat{w}_2(t) := w_2(t) - \int_0^t (\beta_0 + \beta_1 Y_s) ds
\]
is a Brownian motion, and $Y_t$ solves
\[
dY_t = \{(b + a\beta_0)Y_t + a\beta_1 Y_t^2\} dt + aY_t d\hat{w}_2(t), \quad Y_0 = y. \tag{39}
\]
Then, by the same calculation of Lemma 3.2, we see that the equation (39) has a unique solution given by
\[
Y_t = \frac{ye^{a\hat{w}_2(t)} + a^2 \delta t}{1 - a\beta_1 y \int_0^t e^{a\hat{w}_2(s)} + a^2 \delta s ds}, \tag{40}
\]
where $\delta$ is given by (23).

**Lemma 3.5.** Assume (1), (2), (3) and (24). Then, we have
\[
\hat{E}[e^{-(1-\rho^2)\xi(Y_t)}] = e^{(1-\rho^2)G(t,y)}, \tag{41}
\]
where $G(t,y)$ is given by (21).

**Proof.** First, we shall check that there is $K_{T,y} > 0$ such that
\[
\hat{E}[e^{-(1-\rho^2)\xi(Y_t)}] \leq K_{T,y}. \tag{42}
\]
Using (39), we have
\[
d\xi(Y_t) = \left[\frac{a^2}{2} Y_t^2 \xi''(Y_t) + \{(b + a\beta_0)Y_t + a\beta_1 Y_t^2\} \xi'(Y_t)\right] dt + \xi'(Y_t) aY_t d\hat{w}_2(t),
\]
and
\[
de^{-\rho^2\xi(Y_t)} = e^{-\rho^2\xi(Y_t)} q(Y_t) dt - (1 - \rho^2)e^{-\rho^2\xi(Y_t)} \xi'(Y_t) aY_t d\hat{w}_2(t),
\]
where $q(y)$ is defined by
\[
q(y) := -(1-\rho^2)\left[\frac{a^2}{2} y^2 \xi''(y) + \{(b + a\beta_0)y + a\beta_1 y^2\} \xi'(y) - \frac{(1-\rho^2) a^2}{2} y^2 (\xi'(y))^2\right].
\]
Now, we rewrite $q(y)$ as
\[
q(y) = -\left(1-\rho^2\right)\left[\frac{a^2}{2} y^2 \xi''(y) + \{(b + a\beta_0)y + a\beta_1 y^2\} \xi'(y) - \frac{(1-\rho^2) a^2}{2} y^2 (\xi'(y))^2\right]
\]
\[
= - (\beta_1 \beta_0 - \beta^2 \mu_1 \mu_0 + \frac{b}{a} (\beta_1 + \rho \mu_1)) y
\]
\[
- (\beta_0 + \rho \mu_0) \left(\frac{b}{a} - \frac{\beta_0 - \rho \mu_0}{2}\right).
\]
Hence, we have
\[
\hat{E}[e^{-(1-\rho^2)\xi(Y_t \wedge \tau_n)}] \leq e^{-(1-\rho^2)\xi(y)} + \hat{C} \int_0^t \hat{E}[e^{-(1-\rho^2)\xi(Y_s \wedge \tau_n)}] ds,
\]
where $\tau_n := \inf\{t > 0; Y_t < 1/n \text{ or } n < Y_t\}, n \geq 1$, and $\hat{C} := \sup_{y > 0} q(y)$. From Gronwall’s inequality we have
\[
\hat{E}[e^{-(1-\rho^2)\xi(Y_t \wedge \tau_n)}] \leq e^{-(1-\rho^2)\xi(y) + \hat{C}T}.
\]
As \( n \to \infty \), we have (42) with \( K_{T,y} := e^{-(1-\rho^2)\xi(y)+CT} \). Next, we compute \( \hat{E}[e^{-(1-\rho^2)\xi(Y_t)}] \). Using (20), (40) and the self-similarity of Brownian motions, we have

\[
\hat{E}[e^{-(1-\rho^2)\xi(Y_t)}] = \hat{E} \left[ e^{-\beta_1 + \rho \mu_1 \frac{a}{a^2} \left( \hat{w}_2(t) + \delta t \right)} \hat{w}_2(t) + \delta t \right] \]

\[
= \hat{E} \left[ \exp \left\{ -\frac{\beta_1 + \rho \mu_1}{a} \frac{ye^{\hat{w}_2(t) + \delta t}}{1 - \frac{\beta_1}{a} y \int_0^t e^{\hat{w}_2(s) + \delta s} ds} \right\} \right] \]

\[
= y^{\frac{\beta_0 + \rho \mu_0}{a}} \hat{E} \left[ \exp \left\{ -\frac{\beta_1 + \rho \mu_1}{a} \frac{ye^{\hat{w}_2(t) + \delta t}}{1 - \frac{\beta_1}{a} y \int_0^t e^{\hat{w}_2(s) + \delta s} ds} \right\} \right] \]

where \( \delta \) is given by (23). Now, define a probability measure \( P^* \) as

\[
\frac{dP^*}{dP} \bigg|_{\hat{F}_t} := e^{-\delta \hat{w}_2(t) - \frac{\delta^2}{2} \hat{w}_2(t)}. 
\]

Then \( w^*_t = \hat{w}_2(t) + \delta t \) is a Brownian motion under \( P^* \). So, we obtain

\[
\hat{E}[e^{-(1-\rho^2)\xi(Y_t)}] = y^{\frac{\beta_0 + \rho \mu_0}{a}} E^* \left[ \exp \left\{ -\frac{\beta_1 + \rho \mu_1}{a} \frac{ye^{w^*_t}}{1 - \frac{\beta_1}{a} yA_{a^2t}} \right\} \right] 
\]

\[
= e^{-\frac{\beta_0 + \rho \mu_0}{a} w^*_t} E^* \left[ \exp \left\{ -\frac{\beta_1 + \rho \mu_1}{a} \frac{ye^{w^*_t}}{1 - \frac{\beta_1}{a} yA_{a^2t}} \right\} \right] 
\]

\[
= e^{-\frac{\beta_0 + \rho \mu_0}{a} w^*_t} E^* \left[ \exp \left\{ -\frac{\beta_1 + \rho \mu_1}{a} \frac{ye^{w^*_t}}{1 - \frac{\beta_1}{a} yA_{a^2t}} \right\} \right] 
\]

\[
= e^{-\frac{\beta_0 + \rho \mu_0}{a} w^*_t} E^* \left[ \exp \left\{ -\frac{\beta_1 + \rho \mu_1}{a} \frac{ye^{w^*_t}}{1 - \frac{\beta_1}{a} yA_{a^2t}} \right\} \right] 
\]

where \( E^*[: \] is the expectation with respect to \( P^* \) and

\[
A_t = \int_0^t e^{w^*_u} du. 
\]

From Yor’s formula (see [32]) and the self-similarity of Brownian motions, we obtain

\[
P^*(A_t \in du, w^*_t \in dz) = \eta_t(u, z)dudz, 
\]
where $\eta_t(u, z)$ is given by (22). Hence, we have
\[
\hat{E}\left[e^{-(1-\rho^2)\xi(Y_t)}\right] = e^{(1-\rho^2)G(t,y)},
\]
where $G(t, y)$ is given by (21).

**Proof of Theorem 3.1.** From Lemmas 3.4 and 3.5, (25) solves (16). And, from (28), $\hat{v}(t,y) < 0$ holds.

### 3.2. The uniqueness of the solution for (16)

**Theorem 3.6.** Assume (1), (2), (3) and (24). Then, if $v \in C^{1,2}([0,T] \times (0,\infty))$ is a negative solution of (16), then $v \equiv \hat{v}$ holds.

**Proof.** Assume that $v \in C^{1,2}([0,T] \times (0,\infty))$ is a negative solution of (16). Then, $\psi(t,y) := e^{(1-\rho^2)v(t,y)}$ solves (26). And, we have
\[
d\psi(t, Y_t) = \left\{ \partial_t \psi(t, Y_t) + \frac{a(Y_t)^2}{2}D^2\psi(t, Y_t) + b(Y_t)D\psi(t, Y_t) \right\} dt + D\psi(t, Y_t)a(Y_t)dw_2(t)
\]
\[
= \left\{ \rho a(Y_t)\mu(Y_t)D\psi(t, Y_t) + \frac{1}{2} \rho^2 \mu(Y_t)^2 \psi(t, Y_t) \right\} dt + D\psi(t, Y_t)a(Y_t)dw_2(t).
\]

Further, we recall (27) to obtain
\[
dF_t = F_t \left( -\rho \mu(Y_t)dw_2(t) - \frac{1}{2} \rho^2 \mu(Y_t)^2 dt \right).
\]
Hence, we have
\[
d\{\psi(t, Y_t)F_t\} = \psi(t, Y_t)dF_t + F_t d\psi(t, Y_t) + d\psi(t, Y_t) \cdot dF_t
\]
\[
= \psi(t, Y_t)F_t \left\{ -\rho a(Y_t) + \frac{\partial Y_t}{\psi(t, Y_t)} a(Y_t) \right\} dw_2(t)
\]
\[
= \psi(t, Y_t)F_t \left\{ -\rho a(Y_t) + (1 - \rho^2)Dv(t, Y_t)a(Y_t) \right\} dw_2(t).
\]
Namely, we have
\[
\psi(t, Y_t)F_t = \psi(0, y) \hat{\xi}_t^v,
\]
where $\hat{\xi}_t^v$ is defined by
\[
\hat{\xi}_t^v := \int_0^t \left\{ -\rho \mu(Y_s) + (1 - \rho^2)Dv(s, Y_s)a(Y_s) \right\} dw_2(s) - \frac{1}{2} \int_0^t \left\{ -\rho \mu(Y_s) + (1 - \rho^2)Dv(s, Y_s)a(Y_s) \right\}^2 ds.
\]
(43)

From Lemma B.2 below $\hat{\xi}_t^v$ is a martingale. So, we have
\[
\psi(t, y) = E[F_t F_t^{-1} | Y_t = y] = E[F_t^{-1} | Y_0 = y] = e^{(1-\rho^2)\hat{v}(t,y)},
\]
that is $v \equiv \hat{v}$. □
3.3. The verification theorem. From (4), (5) and Example of [25] we recall that
\[
\phi(t, h_t) = kc(h_t - 1)e^{c(T-t)} + \frac{\lambda\theta}{\alpha h_t e^{c(T-t)} - \theta} + \lambda.
\]
Then, the following result of the optimal reinsurance strategy is known in [25].

**Corollary 1.** ([25] : Example) Assume (4) and (5). Then, \(\hat{h}_t = \arg\sup_{h_t \in [0, 1]} \phi(t, h_t)\) is the optimal reinsurance strategy and is given as follows.

- If \(\theta \leq \frac{\lambda}{kc}\), then \(\hat{h}_t = 0\) for all \(t \in [0, T]\).
- If \(\theta > \frac{\lambda}{kc}\) and \(T > \frac{1}{r} \log \frac{\theta - \frac{\lambda}{kc}}{\alpha}\), then
  \[
  \hat{h}_t = \begin{cases} 
  \frac{\theta - \sqrt{\frac{\lambda\theta}{kc}}}{\alpha e^{c(T-t)}} & \text{for } t \in \left[0, T - \frac{1}{r} \log \frac{\theta - \frac{\lambda}{kc}}{\alpha}\right), \\
  1 & \text{for } t \in \left[T - \frac{1}{r} \log \frac{\theta - \frac{\lambda}{kc}}{\alpha}, T\right].
  \end{cases}
  \]
- If \(\theta > \frac{\lambda}{kc}\) and \(T \leq \frac{1}{r} \log \frac{\theta - \frac{\lambda}{kc}}{\alpha}\), then \(\hat{h}_t = 1\) for \(t \in [0, T]\).

Define \(A_{t,T}\) as
\[
A_{t,T} := \left\{ (\pi_s, h_s)_{s \in [t, T]} \in L_{2,t,T}; E_{t,y}[E_{t,T}(\pi, h)] = 1 \right\},
\]
where \(E_{t,T}(\pi, h) := E_T(\pi, h)/E_t(\pi, h)\) and \(E_t(\pi, h)\) is defined by
\[
E_t(\pi, h) := E_t^c(\pi)E_t^N(h)
\]
\[
E_t^c(\pi) := e^{\int_0^t (-ae^{c(T-s)} \pi_s e + Dc(s, Y_s)f(Y_s))ds} - e^{\frac{1}{2} \int_0^t a e^{c(T-s)} \pi_s e + Dc(s, Y_s)f(Y_s)^2}ds,
\]
\[
E_t^N(h) := e^{\int_0^t \int_{s > 0} ah_s z e^{c(T-s)}N(ds, dz) + \lambda \int_0^t \int_{s > 0} \left(1 - e^{ah_s z e^{c(T-s)}}\right)e(z)dz}ds.
\]
(45)

Here, \(f(y) := (0, a(y))\). Then, we have the following.

**Theorem 3.7.** Consider (P). Assume (1), (2), (3), (4), (5), (7) and (24). Assume also
\[
\theta > 2ae^{rT}.
\]
Then, the strategy \((\hat{\pi}, \hat{h})\) in \(A_{t,T}\) is optimal. Here, \(\hat{\pi}\) is defined by
\[
\hat{\pi}_t := \hat{\pi}(t, Y_t),
\]
\[
\hat{\pi}(t, y) := \frac{1}{\alpha} e^{-r(T-t)} \{\mu(y) + \rho a(y) D\hat{v}(t, y)\},
\]
(47)
where \(\hat{v}\) is defined in (28). Furthermore,
\[
\sup_{(\pi, h) \in A_{t,T}} J(t, x, y; \pi, h) = J(t, x, y; \hat{\pi}, \hat{h}) = \hat{V}(t, x, y),
\]
where \(\hat{V}(t, x, y)\) is defined by
\[
\hat{V}(t, x, y) := -e^{\hat{v}(t,y)-\alpha e^{c(T_t)}-\omega} \int_t^T e^{c(T-s)}ds - \int_t^T \phi(s, \hat{h}_s)ds.
\]
Proof. For any \((\pi, h) \in \mathcal{A}_{t,T}\) we recall, from (9)
\[
J(t,x,y;\pi,h) = -e^{-\alpha x e^r(T-t)}-\cos \int_t^T e^{r(T-s)}ds \cdot E_{t,y} \left[ e^{\int_t^T (\ell(s,Y,s)-\phi(s,h_s))ds} \mathcal{E}_{t,T}^0(\pi,h) \right] = -e^{-\alpha x e^r(T-t)}-\cos \int_t^T e^{r(T-s)}ds \cdot E_{t,y} \left[ e^{\int_t^T (\ell(s,Y,s)-\phi(s,h_s))ds} \mathcal{E}_{t,T}^N(\pi,h) \right] 
\]
\[
\leq -e^{-\alpha x e^r(T-t)}-\cos \int_t^T e^{r(T-s)}ds - \int_t^T \phi(s,h_s)ds 
\]
\[
= \mathcal{E}_{t,T}^N(\pi) = 1 
\]

where \(\mathcal{E}_{t,T}^N(\pi) := \mathcal{E}_{t,T}^N(h)/\mathcal{E}_{t,T}^N(\pi)\). Here, the third inequality follows from \(\hat{h}_s = \arg\sup_{h \in [0,1]} \phi(s,h_s)\). Moreover, we observe
\[
\int_t^T \ell(s,Y,s)ds - \alpha \int_t^T e^{r(T-s)}\pi_sedw(s) - \alpha^2 \int_t^T e^{2r(T-s)}\pi_s^2ds 
\]
\[
= \nu(T,Y_T) + \int_t^T \ell(s,Y,s)ds - \alpha \int_t^T e^{r(T-s)}\pi_sedw(s) - \alpha^2 \int_t^T e^{2r(T-s)}\pi_s^2ds 
\]
\[
= \nu(t,y) + \int_t^T \left\{ \frac{\partial \nu}{\partial s}(s,Y_s) + \frac{a(Y_s)^2}{2}D^2\nu(s,Y_s) + b(Y_s)D\nu(s,Y_s) \right\} ds 
\]
\[
+ \int_t^T D\nu(s,Y_s)a(Y_s)dw_2(s) - \alpha \int_t^T e^{r(T-s)}\pi_sedw(s) 
\]
\[
- \frac{\alpha^2}{2} \int_t^T e^{2r(T-s)}\pi_s^2ds - \int_t^T \ell(s,Y,s)ds 
\]
\[
= \nu(t,y) - \int_t^T \left\{ \frac{\partial \nu}{\partial s}(s,Y_s) - \frac{a(Y_s)^2}{2}D^2\nu(s,Y_s) - b(Y_s)D\nu(s,Y_s) - \ell(s,Y,s) \right\} ds 
\]
\[
+ \frac{\alpha^2}{2} e^{2r(T-s)}\pi_s^2 - \frac{1}{2} \left| -\alpha e^{r(T-s)}\pi_s e + D\nu(s,Y_s)f(Y_s) \right| \right\} ds + \log \mathcal{E}_{t,T}^N(\pi), 
\]
where \(\mathcal{E}_{t,T}^N(\pi) := \mathcal{E}_{t,T}^N(h)/\mathcal{E}_{t,T}^N(\pi)\). So, we have
\[
e^{\int_t^T \ell(s,Y,s,\pi)ds}e^{-\alpha \int_0^T e^{r(T-s)}\pi_sedw(s)} = \mathcal{E}_{t,T}^N(h) 
\]
\[
= \mathcal{E}_{t,T}^N(h) 
\]

Hence, we have
\[
J(t,x,y;\pi,h) \leq -e^{-\alpha x e^r(T-t)}-\cos \int_t^T e^{r(T-s)}ds - \int_t^T \phi(s,h_s)ds + \nu(t,y) 
\]
\[
\leq \hat{V}(t,x,y) \mathcal{E}_{t,T}(\pi,h) 
\]
\[
= \hat{V}(t,x,y), 
\]
where the first inequality comes from the fact that \(\mathcal{L}^\pi \nu(s,Y_s) \leq 0 \text{ a.e.} (s,\omega) \in [t,T] \times \Omega\). See (14). And, in the second inequality holds using \(E_{t,y}[\mathcal{E}_{t,T}(\pi,h)] = 1\) holds for \((\pi, h) \in \mathcal{A}_{t,T}\)
Next, we take \((\widehat{\pi}, \widehat{h})\). By Appendix B.2 below, we see that
\[
E_{t,y} \left[ \mathcal{E}_{t,T}(\widehat{\pi}, \widehat{h}) \right] = E \left[ \mathcal{E}_{T-t}(\widehat{\pi}, \widehat{h}) \right] = 1. \tag{48}
\]
Hence, \((\widehat{\pi}, \widehat{h}) \in A_{t,T}\).

In a similar way above, noting that \(\mathcal{L}^x\pi(s, Y_s) = 0\) a.e.\((s, \omega) \in [t, T] \times \Omega\), and that \(\hat{h}_s = \arg \sup_{h_s \in [0,1]} \phi(s, h_s)\), we have
\[
\begin{align*}
J(t, x, y; \widehat{\pi}, \widehat{h}) &= \widehat{V}(t, x, y)E_{t,y} \left[ e^{-\int_t^T \mathcal{L}^x\pi(s, Y_s) ds} \mathcal{E}_{t,T}(\widehat{\pi}, \widehat{h}) \right] \\
&= \widehat{V}(t, x, y)E_{t,y} \left[ \mathcal{E}_{t,T}(\widehat{\pi}, \widehat{h}) \right] \\
&= \widehat{V}(t, x, y)E \left[ \mathcal{E}_{T-t}(\widehat{\pi}, \widehat{h}) \right] \\
&= \widehat{V}(t, x, y).
\end{align*}
\]
Hence, we conclude this theorem. \(\square\)

4. **The complete market case:** \(|\rho| = 1\). In this section, we consider \((P)\) in the case of \(|\rho| = 1\). This case means that the financial market is complete. In Subsection 4.1, we obtain the explicit solution of (16) by using the similar argument of Subsection 3.1. In Subsection 4.2, we show the uniqueness for solution of (16), and observe a situation different from the case of \(|\rho| = 1\). In Subsection 4.3, the verification theorem is given by following the argument of Subsection 3.3.

4.1. **The explicit solution of (16).**

**Theorem 4.1.** Assume (1), (2), (3), and (24). Then, (16) has a negative solution
\[
\hat{v}(t, y) = H(t, y), \tag{49}
\]
where \(H(t, y)\) is defined by
\[
H(t, y) := -\frac{1}{2} \int_0^T ds e^{-s^2 t} \int_{-\infty}^{+\infty} dz \int_0^\infty du \left\{ \mu_0 + \frac{\mu_1 y e^z}{1 - \frac{\beta_0}{a} y u} \right\}^2 e^{\tilde{\delta} z} \eta_{a^2 t}(u, z). \tag{50}
\]
Here, \(\eta_{a^2 t}(u, z)\) is given in (22) with \(t\) replaced by \(a^2 t\), and \(\tilde{\delta}\) is given in (34). The parameters \(\tilde{\beta}_0\) and \(\tilde{\beta}_1\) are given in (32). Moreover, \(\hat{v}\) satisfies
\[
0 > \hat{v}(t, y) > -C(1 + y). \tag{51}
\]

**Proof.** If \(\rho^2 = 1\), then (16) becomes the following linear partial differential equation:
\[
-\partial_t v = \frac{a(y)^2}{2} D^2 v + \{ b(y) - \rho a(y) \mu(y) \} D v - \frac{\mu(y)^2}{2}, \quad v(T, y) = 0. \tag{52}
\]
Define
\[
\hat{v}(t, y) := -\frac{1}{2} \tilde{E}_{t,y} \left[ \int_t^T \mu(Y_s)^2 ds \right] = -\frac{1}{2} \tilde{E} \left[ \int_0^{T-t} \mu(Y_s)^2 ds \right] \tag{53},
\]
where \(Y_t\) is given by (31). Note that
\[
(b + a \tilde{\beta}_0) Y_t + a \tilde{\beta}_1 Y_t^2 \leq \frac{a \tilde{\beta}_1}{2} Y_t^2 + \tilde{C}.
\]
Based on (31) and (33), we observe
\[
\tilde{E}_{t,y} \left[ \int_t^{T \wedge \tau_n} Y_s^2 ds \right] \leq -\frac{2}{a\beta_1} (y + \tilde{C}T),
\]
where \( \tau_n := \inf\{ s > t; Y_s < 1/n, n < Y_s \} \). Here, \( \tilde{C} \) is independent of \( n \). Using Fatou’s lemma and letting \( n \) to \( \infty \), we have
\[
\tilde{E}_{t,y} \left[ \int_t^T Y_s^2 ds \right] \leq -\frac{2}{a\beta_1} (y + \tilde{C}T).
\] (54)
Hence, we have \( |\tilde{v}(t,y)| < \infty \). Following the argument of Lemma 3.3, we see \( \tilde{v} \in \mathcal{C}^{1,2}(\mathbb{R}^2) \). So, by Feynman-Kac formula, (52) has a negative solution \( \tilde{v} \).

From (33) and the self-similarity of Brownian motions, we have
\[
\tilde{E}[\mu(Y_t)^2] = \tilde{E}\left[ \left( \mu_0 + \frac{\mu_1 y e^{\tilde{w}_2(t)} + \alpha^2 \delta t}{1 - \alpha\beta_1 y \int_0^t e^{\tilde{w}_2(s) + \alpha^2 \delta s} ds} \right)^2 \right]
= \tilde{E}\left[ \left( \mu_0 + \frac{\mu_1 y e^{\tilde{w}_2(t)} + \alpha^2 \delta t}{1 - \beta_1 y \int_0^t e^{\tilde{w}_2(s) + \alpha^2 \delta s} ds} \right)^2 \right].
\]
Now, let \( \tilde{P}^* \) be a probability measure
\[
\frac{d\tilde{P}^*}{d\tilde{P}} \bigg|_{F_t} := e^{-\delta \tilde{w}_2(t) - \frac{\alpha^2}{2} t}.
\]
Then, \( \tilde{w}_t^* = \tilde{w}_2(t) + \delta t \) is a Brownian motion under \( \tilde{P}^* \). So, we have
\[
\tilde{E}^*[\mu(Y_t)^2] = \tilde{E}^*\left[ \left( \mu_0 + \frac{\mu_1 y e^{\tilde{w}_2(t)} + \alpha^2 \delta t}{1 - \beta_1 y \tilde{A}_t e^{\tilde{w}_2(t)}} \right)^2 e^{\delta \tilde{w}_2(t) - \frac{\alpha^2}{2} t} \right]
= e^{-\frac{\alpha^2}{2} t} \tilde{E}^*\left[ \left( \mu_0 + \frac{\mu_1 y e^{\tilde{w}_2(t)} + \alpha^2 \delta t}{1 - \beta_1 y \tilde{A}_t e^{\tilde{w}_2(t)}} \right)^2 e^{\delta \tilde{w}_2(t)} \right],
\]
where \( \tilde{E}^*[\cdot] \) is the expectation with respect to \( \tilde{P}^* \) and
\[
\tilde{A}_t = \int_0^t e^{\tilde{w}_u} du.
\]
From Yor’s formula (see [32]) and the self-similarity of Brownian motions, we have
\[
\tilde{P}^*(\tilde{A}_t \in du, \tilde{w}_t^* \in dz) = \eta_t(u,z)du dz,
\]
where \( \eta_t(u,z) \) is given by (22). Hence, from (53), we have \( \tilde{v}(t,y) = H(t,y) \). Moreover, using (54) and (53), we obtain (51) at once. \( \square \)
4.2. The uniqueness of the solution for (16).

Theorem 4.2. Assume (1), (2), (3), and (24). Let \( v \in \mathcal{C}^{1,2}([0, T] \times (0, \infty)) \) be a solution of (16). If \( v \) satisfies (51) then \( v \equiv \hat{v} \) holds.

Proof. Assume that \( v \in \mathcal{C}^{1,2}([0, T] \times (0, \infty)) \) is a negative solution of (16). Then, we have

\[
-\partial_t(v - \hat{v}) = \frac{a(y)^2}{2}D^2(v - \hat{v}) + \{b(y) - \rho a(y)\mu(y)\}D(v - \hat{v}),
\]

(55)

Using this, (31) and \( \tau_n := \inf\{s > t; Y_s < 1/n \text{ or } Y_s > n\} \), we have

\[
\hat{E}_{t,y}[v(T \wedge \tau_n, Y_{T \wedge \tau_n}) - \hat{v}(T \wedge \tau_n, Y_{T \wedge \tau_n})] = v(t, y) - \hat{v}(t, y).
\]

(56)

Now, using Lemma B.1, we have, for \( \delta > 0 \)

\[
\hat{E}_{t,y}\left[|v(T \wedge \tau_n, Y_{T \wedge \tau_n}) - \hat{v}(T \wedge \tau_n, Y_{T \wedge \tau_n})|^{1+\delta}\right] \leq K_T \left(1 + \hat{E}_{t,y}\left[Y_{T \wedge \tau_n}^{1+\delta}\right]\right) \]

\[
\leq K_T (1 + C_{1+\delta,T,y}).
\]

Since \( K_T (1 + C_{1+\delta,T,y}) \) is independent of \( n \), we verify the uniform integrability of random variables:

\[
\lim_{n \to \infty} \hat{E}_{t,y}[v(T \wedge \tau_n, Y_{T \wedge \tau_n}) - \hat{v}(T \wedge \tau_n, Y_{T \wedge \tau_n})] = 0,
\]

which leads to \( v \equiv \hat{v} \) holds. \( \square \)

4.3. The verification theorem.

Theorem 4.3. Consider (P). Assume (1), (2), (3), (4), (5), (7), (24), and (46). Then, the strategy \((\hat{\pi}, \hat{h}) \in \mathcal{A}_{t,T}\) defined by

\[
\hat{\pi}_t := \hat{\pi}(t, Y_t),
\]

\[
\hat{\pi}(t, y) := \frac{1}{\alpha}e^{-r(T-t)}\left\{\mu(y) + \rho a(y)D\hat{v}(t, y)\right\}
\]

(57)

is optimal. Here, \( \hat{h} \) is given in Corollary 1 and \( \mathcal{A}_{t,T} \) is given in (44). Moreover,

\[
\sup_{(\pi, h) \in \mathcal{A}_{t,T}} J(t, x, y; \pi, h) = J(t, x, y; \hat{\pi}, \hat{h}) = \hat{V}(t, x, y),
\]

where \( \hat{V}(t, x, y) \) is defined by

\[
\hat{V}(t, x, y) := -e^{\hat{v}(t, y) - \alpha xe^{\hat{v}(T - s) - \alpha \int_t^T e^{\hat{v}(T - \tau)ds} - \int_t^T \phi(s, \hat{h}_s)ds}}.
\]

Proof. Following the same argument of Theorem 3.7, we have

\[
J(t, x, y; \pi, h) \leq \hat{V}(t, x, y), \text{ for any } (\pi, h) \in \mathcal{A}_{t,T}.
\]

Next, we take \((\hat{\pi}, \hat{h})\). By Appendix B.3 below, we see that

\[
\hat{E}_{t,y}\left[\hat{E}_{t,T}(\hat{\pi}, \hat{h})\right] = \hat{E}\left[\hat{E}_{T-t}(\hat{\pi}, \hat{h})\right] = 1.
\]

(58)

Hence, \((\hat{\pi}, \hat{h}) \in \mathcal{A}_{t,T}\). According to the same argument of Theorem 3.7 again, we have

\[
J(t, x, y; \hat{\pi}, \hat{h}) = \hat{V}(t, x, y).
\]

So, we conclude this theorem. \( \square \)
5. **Numerical results.** This section is devoted to analysis of the value function and the ruin probability through numerical analysis.

5.1. **The analysis of the value function \( \hat{V}(0, x, y) \).** In Figure 1 and Figure 2, we observe that for the fixed initial surplus \( x \), the value functions \( \hat{V}(0, x, y) \) monotonically decreases in the initial factor \( y \). In addition, as the initial factor \( y \) is large enough, \( \hat{V}(0, x, y) \) is treated as a constant in \( y \). Interestingly, unlike the behavior of \( y \) for the value function \( \hat{V}(0, x, y) \), for the fixed initial factor, if \( y \) is small, \( \hat{V}(0, x, y) \) is obviously convex in the initial surplus \( x \). However, as \( y \) becomes large, \( \hat{V}(0, x, y) \) can be seen as a constant in \( x \).

In particular, in the case of the measure of risk aversion \( \alpha \) being large, for given \( y \), the shape of the convex function for \( \hat{V}(0, x, y) \) is decreasing and increasing sharply in \( x \). Similarly, for given \( x \), \( \hat{V}(0, x, y) \) increases strictly in \( y \).

\[ \text{Figure 1. The value function } \hat{V}(0, x, y) \text{ with } a = b = 1, \beta_0 = \beta_1 = 1, r = 0.05, \rho = 0.2, \mu_0 = \mu_1 = 0.5, \lambda = 10, \theta = 2, c = 1, k = 12, \text{ the varied parameter } \alpha = 0.2. \]

5.2. **Ruin probability.** In the analysis of the ruin probability, we study the wealth processes with investment and without investment. According to the definition in [12], the ruin probability where the wealth process is driven by the optimal investment \( \hat{\pi} \) given by (47) is defined by

\[ P \left( \inf_{0 \leq t \leq T} X_t^{\hat{\pi}, \hat{h}} \leq 0 \right), \]

and the ruin probability of the wealth process without investment is defined by

\[ P \left( \inf_{0 \leq t \leq T} X_t^{0, \hat{h}} \leq 0 \right) = P \left( \inf_{0 \leq t \leq T} R_t^h \leq 0 \right). \]
In order to compare the ruin probabilities using the different sets of parameters, we use the Monte Carlo method with 1000 trajectories based the Euler scheme.

- **Relation between measure of risk aversion and ruin probability:** In Figure 3, as we expected, the ruin probability of the case with no investment is constant in the varied $\alpha$ and the performance of the wealth process with investment is better in terms of the smaller ruin probability. In addition, the ruin probability decreases in the risk aversion parameter $\alpha$ from 0.1 to 1. Based on the solution of $\hat{\pi}$ given by (15), the amount of investing risky assets is decreasing in $\alpha$. However, interestingly, the ruin probability behaves as a convex function in $\alpha$ and the smallest ruin probability is located around 0.4. In particular, Figure 4 shows the small risk aversion parameter $\alpha$ gives much higher proportion of risky assets where the risk governed by risk assets leads to the larger ruin probability comparing to the wealth process without risky assets.

- **Relation between retention and ruin probability:** Figure 5 shows that the retention level $k$ does not affect the ruin probabilities significantly in both cases owing to the ruin probability staying flat with respect to varied $k$. In this particular scenario, the ruin probability of allowing investment is half of the one of no investment.

- **Relation between intensity of claim and ruin probability:** In Figure 6, it is natural to obtain the ruin probabilities both increase in the intensity of claims with or without investment. In addition, in this particular scenario, when $\lambda \leq 7$, the ruin probability with investment is smaller than the one without.
Figure 3. The ruin probability with \( a = b = 0.1, \beta_0 = \beta_1 = 1, r = 0.05, \rho = 0.2, \mu_0 = \mu_1 = 0.5, \lambda = 10, \theta = 1, c = 1, k = 15, x = 10, \) and the varied \( \alpha. \)

Figure 4. The ruin probability with \( a = b = 0.1, \beta_0 = \beta_1 = 1, r = 0.05, \rho = 0.2, \mu_0 = \mu_1 = 0.5, \lambda = 10, \theta = 1, c = 1, k = 15, x = 10, \) and the relative varied small \( \alpha. \)
investment in the sense that the risk of claim is smaller than the risk of investment. Nevertheless, as the intensity of claim increases, the investment is necessary in terms of the smaller ruin probability. Basically, the slope of ruin probability with investment is much flatter than the one without investment leading to stability of wealth.

- **Relation between payment parameter and ruin probability**: It is also natural to observe that in Figure 7, the ruin probabilities in both cases decrease in the claim parameter $\theta$ due to the large payment with the expected payment of claims $1/\theta$. Surprisingly, in this particular case, the wealth without investment gives the smaller ruin probability than the one with investment as $\theta \geq 1.2$. Namely, if the risk given by payment of claims is small, the risk coming from the risky assets dominates the risk given by claims leading to the larger ruin probability for the wealth with investment. However, if the expected payment is larger with the relative smaller $\theta$, the wealth with investment outperforms in terms of smaller ruin probability since the decreasing slope of ruin probability of the wealth with investment is much smaller than the one of the wealth without investment.

- **Relation between initial surplus and ruin probability**: We first simply observe that in Figure 8 the ruin probabilities decrease in the surplus $x$. In addition, as the discussion of the relation between the intensity of claim distribution and the ruin probability, the slope of ruin probability decreases sharply in the case of the wealth without investment. Specifically, as the initial surplus $x$ is greater than 14, the ruin probability of the case of no investment is much
Figure 6. The ruin probability with $\alpha = 0.2 \ a = b = 0.1, \ \beta_0 = \beta_1 = 1, \ r = 0.05, \ \rho = 0.2, \ \mu_0 = \mu_1 = 0.5, \ \theta = 1, \ c = 1, \ k = 15, \ x = 10$, and the varied $\lambda$.

Figure 7. The ruin probability with $\alpha = 0.2 \ a = b = 0.1, \ \beta_0 = \beta_1 = 1, \ r = 0.05, \ \rho = 0.2, \ \mu_0 = \mu_1 = 0.5, \ \lambda = 10, \ c = 1, \ k = 15, \ x = 10$, and the varied $\theta$. 
closer to the one with investment implying that the large initial surplus can cover the risk driven by the claims. The risk created by investment may lead to the worse case in terms of larger ruin probability. However, decreasing trend of the case of no investment is much sharper similar to the case of the varied intensity.

![Figure 8. The ruin probability with $\alpha = 0.2$, $a = b = 0.1$, $\beta_0 = \beta_1 = 1$, $r = 0.05$, $\rho = 0.2$, $\mu_0 = \mu_1 = 0.5$, $\lambda = 10$, $\theta = 1$, $c = 1$, $k = 15$, and the varied surplus $x$.](image)

According to the previous discussion, it is important to note that, the insurers should consider investment for their wealth since the ruin probability becomes much smaller in the most cases. Based on the relation between the exponential utility function and the ruin probability, the insurers can minimize the ruin probability through investment by choosing the proper risk aversion parameter $\alpha$. However, note that if insurers intend to be more risk seeking investors with the relatively small $\alpha$ around $0.03$, the investment will hurt the wealth process due to the larger ruin probability shown in Figure 4.

In addition, based on the analysis of the varied intensity of claim $\lambda$, payment parameter $\theta$ and initial surplus $x$, it is natural to observe that if the intensity of claims and the expected payment are small or the initial surplus is large enough, the investment is not appropriate since the corresponding ruin probability becomes larger due to the risk driven by risky assets. Nevertheless, in general, investment is very helpful in terms of the slighter possibility of bankruptcy based on the small ruin probability.

In particular, from the economic view, it is very important to keep the stability of financial system consisted of insurers. In Figure 7 and Figure 8, when $\theta = 1.5$ and $x = 6$, the ruin probabilities are both around 0.7 leading to possible systemic risk for the insurers. Hence, in order to achieve this goal, we suggest that the stability
of the system can be created by investment owing to the ruin probability with a moderate slope in the varied intensity of claims \( \theta \) or initial surplus \( x \) (solid line). Hence, we conclude that investment is necessary for portfolio optimization and risk management for individuals. Moreover, investment improves the stability of the financial system in advance.

Through the previous results and discussion of the ruin probabilities for varied parameters, from the economical justification point of view, it is crucial to consider the asset allocation such as investment and reinsurance in order to create stability by reducing the risk of individual default in general. However, note that in the case of investment driven by the larger risk or the loss from the claim being smaller, investment may not be a good strategy to prevent the possible default. In addition, investment over aggressively and confidently with smaller risk aversion parameters may also increase the possibility of ruin. Hence, owing to the risk coming from the market, the proper investment is necessary for stability.

6. Conclusions. In this paper, we propose a stochastic factor model for optimal investment and reinsurance of insurers where the wealth processes are described based on a bank account, a risky stock, and the classical Cramér–Lundberg process with the reinsurance function. In addition, the economic factor process is formulated as a geometric Brownian motion which affects the return of the risky stock. Applying the dynamic programming principle to the proposed problem, we have the corresponding HJB equation. Through the explicit solution of the value function, the optimal investment and reinsurance are both obtained. The verification theorem is also discussed. Finally, we analyze the obtained optimal investment and reinsurance through numerical analysis according to the corresponding value function and ruin probability.

The importance of investment for insurers is also illustrated in the financial implication. In general, compared to the strategy without investment, the strategy with investment leads to stability in terms of smaller ruin probability which the probability of the wealth less than or equal to zero. However, note that in some particular cases such as the case the much smaller risk aversion, the smaller intensity of claims, or the smaller payment of each claim, the investment may not be efficient due to the larger ruin probabilities. Namely, in this scenario, the risk driven by the stock dominates the risk coming from the insurer feature and the claims. From the economical justification point of view, asset allocation is crucial for the avoidance of default. However, ruin probabilities may increase due to aggressive investment. Hence, the proper asset allocation through investment is necessary for stability.

The proposed problem can be extended in some directions. First, instead of maximizing the exponential utility, insurers may consider to minimize the ruin probability though their own strategies. The comparison of the proposed two strategies would be very interesting. Second, the factor models can be relaxed from the log-normal case. These above extensions are ongoing research projects.

Appendix A. Proof of Lemmas.

Lemma A.1. Assume (1), (2), and (3). If \( k_1 < 0 \) and \( k_0 \in \mathbb{R} \), then we have
\[
E \left[ \hat{\xi}_1 \right] = 1, \quad t \in [0, T],
\]
where $\hat{\xi}_t$ is defined by

$$
\hat{\xi}_t := e^{\int_0^t (k_0 + k_1 Y_s)dw_2(s)} - \frac{1}{2} \int_0^t (k_0 + k_1 Y_s)^2 ds.
$$

Proof. We apply the idea of Lemma 4.1.1 in [2]. Recall that

$$
d\hat{\xi}_t = \hat{\xi}_t (k_0 + k_1 Y_t) dw_2(t).
$$

Let $\epsilon > 0$ be arbitrary. We apply Itô’s formula to $\frac{\hat{\xi}_t}{1 + \epsilon \hat{\xi}_t}$ and have

$$
d \left( \frac{\hat{\xi}_t}{1 + \epsilon \hat{\xi}_t} \right) = d\hat{\xi}_t - \hat{\xi}_t d\hat{\xi}_t,
$$

where $\hat{M}_t$ and $\hat{A}_t$ are defined by

$$
\hat{M}_t := \int_0^t \frac{\hat{\xi}_s (k_0 + k_1 Y_s)}{1 + \epsilon \hat{\xi}_s}^2 dw_2(s), \quad \text{and} \quad \hat{A}_t := \frac{\epsilon \hat{\xi}_t^2 (k_0 + k_1 Y_t)^2}{(1 + \epsilon \hat{\xi}_t)^3}.
$$

If we can check that there is $K_{1,T,Y} > 0$ such that

$$
E \left[ \int_0^t \hat{\xi}_s (1 + Y_s^2) ds \right] \leq K_{1,T,Y},
$$

then we see that

$$
E \left[ \frac{\hat{\xi}_t}{1 + \epsilon \hat{\xi}_t} \right] = \frac{1}{1 + \epsilon} - E \left[ \int_0^t \hat{\xi}_s ds \right].
$$

Here, we observe the following:

- $\hat{A}_s \to 0$ a.e. a.s. as $\epsilon \to 0$.
- $\hat{A}_s \leq K_2 \hat{\xi}_s (1 + Y_s^2)$, $\exists K_2 > 0$.

Hence, from (62) and the dominated convergence theorem, we have

$$
E \left[ \int_0^t \hat{\xi}_s ds \right] \to 0 \text{ as } \epsilon \to 0.
$$

Meanwhile, since $E \left[ \hat{\xi}_t \right] \leq 1$,

$$
E \left[ \frac{\hat{\xi}_t}{1 + \epsilon \hat{\xi}_t} \right] \to E \left[ \hat{\xi}_t \right] \text{ as } \epsilon \to 0.
$$

Hence, letting $\epsilon \to 0$ in (63), we obtain $E \left[ \hat{\xi}_t \right] = 1$.

Finally, we prove (62). Applying Itô’s formula to $Y_t^2$, we have

$$
dY_t^2 = (2b + a^2)Y_t^2 dt + 2aY_t^2 dw_2(t).
$$

Using (60) and (64), we have

$$
d \{ \hat{\xi}_t Y_t^2 \} = \hat{\xi}_t dY_t^2 + Y_t^2 d\hat{\xi}_t + d\hat{\xi}_t \cdot dY_t^2
$$

$$
= \hat{\xi}_t Y_t^2 (2ak_1 Y_t + 2ak_0 + 2b + a^2) dt + \hat{\xi}_t Y_t^2 (k_1 Y_t + k_0 + 2a) dw_2(t).
$$
Setting $\tau_n := \inf\{t > 0; Y_t < 1/n \text{ or } Y_t > n\}$, we have
\[
E \left[ \hat{\xi}_{T \wedge \tau_n} Y_{s \wedge \tau_n}^2 \right] - y^2 = E \left[ \int_0^{T \wedge \tau_n} \hat{\xi}_s Y_s^2 (2ak_1 Y_s + 2ak_0 + 2b + a^2) ds \right].
\]
Observing that
\[
\hat{\xi}_s Y_s^2 (2ak_1 Y_s + 2ak_0 + 2b + a^2) = \hat{\xi}_s Y_s^2 \{2ak_1 Y_s + (2ak_0 + 2b + a^2 + 1) - 1\} \leq K_3 \hat{\xi}_s - \hat{\xi}_s Y_s^2, \quad \exists K_3 > 0,
\]
we have
\[
E \left[ \int_0^{T \wedge \tau_n} \hat{\xi}_s Y_s^2 ds \right] \leq y^2 + K_3 T.
\]
Using Lemma A.2 below, and tending $n$ to $\infty$, we obtain (62) with $K_{1,T,y} = y^2 + K_3 T$.

**Lemma A.2.** Assume (1), (2), and (3). Then, we have
\[
P(\tau_n \leq T) \to 0, \quad \text{as } n \to \infty.
\]

**Proof.** First, we have
\[
P(\tau_n \leq T) = P \left( \inf_{t \in [0,T]} Y_t \leq \frac{1}{n}, \text{ or } n \leq \sup_{t \in [0,T]} Y_t \right)
\leq P \left( \inf_{t \in [0,T]} Y_t \leq \frac{1}{n} \right) + P \left( n \leq \sup_{t \in [0,T]} Y_t \right).
\]
Next, we have
\[
P \left( n \leq \sup_{t \in [0,T]} Y_t \right) = P \left( n \leq \sup_{t \in [0,T]} ye^{(b - \frac{a^2}{2})t + aw_2(t)} \right)
= P \left( \log n - \log y \leq \sup_{t \in [0,T]} \{(b - \frac{a^2}{2})t + aw_2(t)\} \right)
\leq P \left( \log n - \log y \leq \left\{ \left( b - \frac{a^2}{2} \right) T \lor 0 \right\} + a \sup_{t \in [0,T]} w_2(t) \right)
\leq P \left( \log n - \log y \leq \left\{ \left( b - \frac{a^2}{2} \right) T \lor 0 \right\} + a \sup_{t \in [0,T]} |w_2(t)| \right)
= P \left( \frac{1}{a} \left[ \log n - \log y - \left\{ \left( b - \frac{a^2}{2} \right) T \lor 0 \right\} \right] \leq \sup_{t \in [0,T]} |w_2(t)| \right)
where the last inequality follows from the Burkholder-Davis-Gundy inequality. Moreover, we have

\[
P \left( \inf_{t \in [0,T]} Y_t \leq \frac{1}{n} \right) = P \left( n < \left\{ \inf_{t \in [0,T]} y_t^{(b - \frac{a^2}{2}) t + aw_2(t)} \right\}^{-1} \right)
= P \left( ny < e^{-\inf_{t \in [0,T]} \left( (b - \frac{a^2}{2}) t + aw_2(t) \right)} \right)
= P \left( \log n + \log y < - \inf_{t \in [0,T]} \left\{ \left( b - \frac{a^2}{2} \right) t + aw_2(t) \right\} \right)
= P \left( \log n + \log y < \sup_{t \in [0,T]} \left\{ - \left( b - \frac{a^2}{2} \right) t - aw_2(t) \right\} \right)
\leq P \left( \log n + \log y < \left\{ - \left( b - \frac{a^2}{2} \right) T \vee 0 \right\} + a \sup_{t \in [0,T]} \{-w_2(t)\} \right)
\leq \frac{1}{a} \left[ \log n + \log y - \left\{ - \left( b - \frac{a^2}{2} \right) T \vee 0 \right\} \right] \leq \sup_{t \in [0,T]} |w_2(t)|
\leq \frac{aE \left[ \sup_{t \in [0,T]} |w_2(t)| \right]}{\log n + \log y - \left\{ - \left( b - \frac{a^2}{2} \right) T \vee 0 \right\}}
\leq \frac{\left( \sup_{t \in [0,T]} |w_2(t)| \right)^2}{\log n + \log y - \left\{ - \left( b - \frac{a^2}{2} \right) T \vee 0 \right\}}^{1/2}
\leq \frac{2a \sqrt{T}}{\log n + \log y - \left\{ - \left( b - \frac{a^2}{2} \right) T \vee 0 \right\}},
\]

where (67) follows from (66).

Using (66), (67) and (68), we can obtain (65).
Lemma B.1. Assume (1), (2), (3), and (24). Use (31). Then, for \( p > 1 \) there is \( C_{p,T,y} > 0 \) such that

\[
\tilde{E}[Y_t^p] \leq C_{p,T,y}. \tag{69}
\]

Proof. Now, using (31) we see that, for \( p > 1 \)

\[
dY_t^p = pY_t^{p-1}dY_t + \frac{p(p-1)}{2}Y_t^{p-2}d\langle Y \rangle_t
\]

\[
= p \left[ a\tilde{\beta}_1 Y_t^{p+1} + \left( b + a\tilde{\beta}_0 + \frac{a^2(p-1)}{2} \right) Y_t^p \right] dt + paY_t^p d\tilde{w}_2(t). \tag{70}
\]

Define

\[
\tilde{M}_p := \sup_{y>0} \left[ p \left( a\tilde{\beta}_1 y^{p+1} + \left( b + a\tilde{\beta}_0 + \frac{a^2(p-1)}{2} \right) y^p \right) \right]. \tag{71}
\]

Then, we have

\[
\tilde{E} \left[ Y_{t\wedge \tau_n}^p \right] \leq y^p + \tilde{M}_p T,
\]

where \( \tau_n := \inf \{ t > 0; Y_t < 1/n, n < Y_t \} \). As \( n \to \infty \), we have

\[
\tilde{E} \left[ Y_t^p \right] \leq y^p + \tilde{M}_p T.
\]

Hence, we have (69) with \( C_{p,T,y} = y^p + \tilde{M}_p T \).

\[\square\]

Proof of Lemma 3.6

To show the smoothness of \( \tilde{v} \) we show the smoothness of \( \phi \):

\[
\phi(t, y) := \tilde{E} \left[ e^{\int_0^t \kappa(Y_s)ds} \right], \tag{72}
\]

where \( \kappa(y) \) is defined by

\[
\kappa(y) := -\frac{1 - \rho^2}{2} \mu(y)^2.
\]

We try to prove that \( \phi(t, y) \) is differentiable with respect to \( t \). We observe

\[
\frac{\phi(t+h, y) - \phi(t, y)}{h} = \tilde{E} \left[ \frac{1}{h} \left( e^{\int_0^h \kappa(Y_s)ds} - e^{\int_0^t \kappa(Y_s)ds} \right) \right]
\]

\[
= \tilde{E} \left[ \frac{1}{h} \int_0^h \frac{\partial}{\partial \epsilon} \left( e^{\int_0^{t+\epsilon} \kappa(Y_s)ds} \right) d\epsilon \right]. \tag{73}
\]

We also observe

\[
\left| \frac{1}{h} \int_0^h \frac{\partial}{\partial \epsilon} \left( e^{\int_0^{t+\epsilon} \kappa(Y_s)ds} \right) d\epsilon \right| \leq \frac{1}{h} \int_0^h |\kappa(Y_{t+\epsilon})| d\epsilon
\]

\[
\leq K \left( \sup_{t \in [0,T]} |Y_t|^2 + 1 \right). \tag{74}
\]

Using (70) with \( p = 2 \), we have

\[
dY_t^2 = 2 \left[ a\tilde{\beta}_1 Y_t^3 + \left( b + a\tilde{\beta}_0 + \frac{a^2}{2} \right) Y_t^2 \right] dt + 2aY_t^2 d\tilde{w}_2(t).
\]
Then, we have

\[
\hat{E}\left[ \sup_{t \in [0,T]} Y_t^2 \right] \leq y^2 + \tilde{M}_2T + 2a\hat{E}\left[ \sup_{t \in [0,T]} \int_0^t Y_s^2 d\tilde{w}_2(s) \right] \leq y^2 + \tilde{M}_2T + 2a \hat{E}\left[ \left( \sup_{t \in [0,T]} \int_0^t Y_s^2 d\tilde{w}_2(s) \right)^2 \right]^{1/2} \\
\leq y^2 + \tilde{M}_2T + 8a\hat{E}\left[ \left( \int_0^T Y_s^2 (\tilde{w}_2(s))_T \right)^{1/2} \right] \\
\leq y^2 + \tilde{M}_2T + 8a \left( \int_0^T \hat{E} \left[ Y_t^4 \right] dt \right)^{1/2}.
\]

(75)

In the third inequality we use the Burkholder-Davis-Gundy inequality. And, from Lemma B.1 we have

\[
\hat{E}\left[ \sup_{t \in [0,T]} Y_t^2 \right] \leq y^2 + \tilde{M}_2T + 8a(y^4 + \tilde{M}_4T). 
\]

Hence, using (74) and the dominated convergence theorem, we have

\[
\frac{\partial \phi}{\partial t}(t,y) = \lim_{h \to 0} \frac{1}{h} \hat{E}\left[ \int_0^h \frac{\partial}{\partial \epsilon} \left( e^{\int_0^t \kappa(Y_s + \epsilon) ds} \right) d\epsilon \right] \\
= \hat{E}\left[ \kappa(Y_t) e^{\int_0^t \kappa(Y_s) ds} \right].
\]

Next, we try to show that \( \phi(t,y) \) is differentiable with respect to \( y \). We observe

\[
\frac{\phi(t,y+h) - \phi(t,y)}{h} = \hat{E}\left[ \frac{1}{h} \left( e^{\int_0^t \kappa(Y_{s+h}) ds} - e^{\int_0^t \kappa(Y_s) ds} \right) \right] \\
= \hat{E}\left[ \frac{1}{h} \int_0^h \frac{\partial}{\partial \epsilon} \left( e^{\int_0^t \kappa(Y_{s+h}) ds} \right) d\epsilon \right].
\]

(76)

Here, \( Y_s^\gamma \) solves (31) with \( Y_0^\gamma = z \). We also observe

\[
\left| \frac{1}{h} \int_0^h \frac{\partial}{\partial \epsilon} \left( e^{\int_0^t \kappa(Y_{s+h}) ds} \right) d\epsilon \right| \leq \frac{1}{h} \int_0^h \left| \frac{\partial}{\partial \epsilon} \left( e^{\int_0^t \kappa(Y_{s+h}) ds} \right) \right| d\epsilon.
\]

(77)

We also have

\[
\frac{\partial}{\partial \epsilon} \left( e^{\int_0^t \kappa(Y_{s+h}) ds} \right) \leq \int_0^t \mu(Y_{s+h}) \frac{\partial Y_{s+h}}{\partial \epsilon} ds \leq K_T \int_0^t (1 + Y_{s+h}) \left| \frac{\partial Y_{s+h}}{\partial \epsilon} \right| ds.
\]

(78)
Now, we observe the following:

\[
Y^\varepsilon_t = (y + \varepsilon) e^{a_2(t) + a^2 \delta t} - a_1(y + \varepsilon) \int_0^t e^{a_2(s) + a^2 \delta s} ds + a_2 \delta t.
\]

Hence, we have

\[
0 < Y^\varepsilon_t < (y + 1) e^{a_2(t) + a^2 \delta t}, \text{ and } 0 < \frac{\partial Y^\varepsilon_t}{\partial \varepsilon} < e^{a_2(t) + a^2 \delta t}.
\] (79)

From (78) and (79), we have

\[
\sup_{\varepsilon \in (0,1)} \left| \frac{\partial}{\partial \varepsilon} \left( e^{f_0 \kappa(Y_\varepsilon^+)} ds \right) \right| \leq K_T (y + 1) \int_0^t \left( e^{2a_2(s) + 2a^2 \delta s} + 1 \right) ds.
\] (80)

Using (77) and (80) we have

\[
\left| \frac{1}{h} \int_0^h \frac{\partial}{\partial \varepsilon} \left( e^{f_0 \kappa(Y_\varepsilon^+)} ds \right) \right| \leq K_T (y + 1) \int_0^t \left( e^{2a_2(s) + 2a^2 \delta s} + 1 \right) ds.
\] (81)

And, we have

\[
\tilde{E} \left[ \int_0^t e^{2a_2(s) + 2a^2 \delta s} ds \right] = \int_0^t e^{2a^2 \delta s} ds < \infty.
\] (82)

Using (76), (81), (82) and the dominated convergence theorem, we have

\[
\frac{\partial \phi}{\partial y}(t, y) = \lim_{h \to 0} \tilde{E} \left[ \frac{1}{h} \int_0^h \frac{\partial}{\partial \varepsilon} \left( e^{f_0 \kappa(Y_\varepsilon^+)} ds \right) \right] = \tilde{E} \left[ \frac{\partial}{\partial y} \left( e^{f_0 \kappa(Y_\varepsilon^+)} ds \right) \right].
\]

Finally, we show that \( \phi(t, y) \) is twice differentiable with respect to \( y \). We have

\[
\frac{\partial_y \phi(t, y + h) - \partial_y \phi(t, y)}{h} = \tilde{E} \left[ \frac{1}{h} \int_0^h \frac{\partial^2}{\partial \varepsilon \partial y} \left( e^{f_0 \kappa(Y_\varepsilon^+)} ds \right) \right].
\] (83)

A straightforward calculation shows that

\[
\frac{\partial}{\partial \varepsilon} \left\{ \frac{\partial}{\partial y} \left( e^{f_0 \kappa(Y_\varepsilon^+)} ds \right) \right\} = \int_0^t \frac{\partial^2}{\partial \varepsilon \partial y} \kappa(Y_\varepsilon^+) ds e^{f_0 \kappa(Y_\varepsilon^+)} ds + \left( \int_0^t \frac{\partial}{\partial y} \kappa(Y_\varepsilon^+) ds \right) \left( \int_0^t \frac{\partial}{\partial \varepsilon} \kappa(Y_\varepsilon^+) ds \right) e^{f_0 \kappa(Y_\varepsilon^+)} ds.
\] (84)
Further, we have
\[
\frac{\partial Y_t^{y+\epsilon}}{\partial y} = \frac{e^{a\tilde{w}_2(t) + a^2\delta t}}{\left\{1 - a\tilde{\beta}_1(y + \epsilon) \int_0^t e^{a\tilde{w}_2(s) + a^2\delta s} ds\right\}^3},
\]
and
\[
\frac{\partial^2 Y_t^{y+\epsilon}}{\partial y \partial \epsilon} = \frac{2a\tilde{\beta}_1 \int_0^t e^{a\tilde{w}_2(s) + a^2\delta s} ds \cdot e^{a\tilde{w}_2(t) + a^2\delta t}}{\left\{1 - a\tilde{\beta}_1(y + \epsilon) \int_0^t e^{a\tilde{w}_2(s) + a^2\delta s} ds\right\}^3}.
\]

Then, we observe
\[
0 < \frac{\partial Y_t^{y+\epsilon}}{\partial y} < e^{a\tilde{w}_2(t) + a^2\delta t},
\]
and
\[
\left| \frac{\partial^2 Y_t^{y+\epsilon}}{\partial y \partial \epsilon} \right| \leq \frac{2e^{a\tilde{w}_2(t) + a^2\delta t}}{y \left\{1 - a\tilde{\beta}_1(y + \epsilon) \int_0^t e^{a\tilde{w}_2(s) + a^2\delta s} ds\right\}^2}.
\]

Using (79), (84) and (85), we have
\[
\sup_{\epsilon \in (0,1)} \left| \frac{\partial^2 }{\partial y \partial \epsilon} \left( e^{f_0' \kappa(Y_{y+\epsilon})} ds \right) \right| \leq K_T \left(1 + \frac{1}{y}\right) \int_0^t \left( e^{4a\tilde{w}_2(s) + 4a^2\delta s} + 1 \right) ds. \tag{86}
\]

From (77) and (80) we have
\[
\left| \frac{1}{h} \int_0^h \frac{\partial^2 }{\partial \epsilon \partial y} \left( e^{f_0' \kappa(Y_{y+\epsilon})} ds \right) d\epsilon \right| \leq K_T \left(1 + \frac{1}{y}\right) \int_0^t \left( e^{4a\tilde{w}_2(s) + 4a^2\delta s} + 1 \right) ds. \tag{87}
\]

Further, we observe
\[
\tilde{E} \left[ \int_0^T e^{4a\tilde{w}_2(s) + 4a^2\delta s} ds \right] = \int_0^T e^{8a^2s + 4a^2\delta s} ds < \infty. \tag{88}
\]

Using (83), (87), (88) and the dominated convergence theorem, we have
\[
\frac{\partial^2 \phi}{\partial y^2}(t, y) = \lim_{h \to 0} \tilde{E} \left[ \frac{1}{h} \int_0^h \frac{\partial^2 }{\partial \epsilon \partial y} \left( e^{f_0' \kappa(Y_{y+\epsilon})} ds \right) d\epsilon \right] = \tilde{E} \left[ \frac{\partial^2 }{\partial y^2} \left( e^{f_0' \kappa(Y_{y})} ds \right) \right].
\]

**Lemma B.2.** Assume (1), (2), (3), and (24). Let \( v \) be a negative solution of (16). Then, we have
\[
E \left[ \tilde{c}_t^v \right] = 1, \quad t \in [0, T].
\]

**Proof.** We follow the arguments of Lemma A.1 basically. Recall that
\[
d\tilde{c}_t^v = \tilde{c}_t^v \left\{ -\mu Y_t + (1 - \rho^2) Dv(t, Y_t) a(Y_t) \right\} dw_2(t).
\]

Let \( \epsilon > 0 \) be arbitrary. Then, we have
\[
d \left( \frac{\tilde{c}_t^v}{1 + \epsilon \tilde{c}_t^v} \right) = d\tilde{M}_t - \tilde{A}_t dt,
\]
Here, we check the following:

Here, we use (16). Using (89) and (92), we have

where \( \tilde{M}_t \) and \( \tilde{A}_t \) are defined by

\[
\tilde{M}_t := \int_0^t \frac{\tilde{\xi}_s^v}{(1 + \epsilon \tilde{\xi}_s^v)} \left\{ -\rho \mu(Y_s) + (1 - \rho^2)Dv(s, Y_s) a(Y_s) \right\} dw_2(s),
\]

\[
\tilde{A}_t := \frac{\epsilon (\tilde{\xi}_t^v)^2}{(1 + \epsilon \tilde{\xi}_t^v)^3} \left\{ -\rho \mu(Y_t) + (1 - \rho^2)Dv(t, Y_t) a(Y_t) \right\}^2.
\]

If we can check that there is \( K_{4,T,y} > 0 \) such that

\[
E \left[ \int_0^t \tilde{\xi}_s^v \left\{ \mu(Y_s)^2 + (Dv(s, Y_s))^2 a(Y_s)^2 \right\} ds \right] \leq K_{4,T,y},
\]

then we see that

\[
E \left[ \tilde{M}_t \right] \leq \tilde{C}_v E \left[ \int_0^t \tilde{\xi}_s^v \left\{ \mu(Y_s)^2 + (Dv(s, Y_s))^2 a(Y_s)^2 \right\} ds \right]
\]

\[
\leq \tilde{C}_v K_{4,T,y},
\]

and that \( \tilde{M}_t \) is a square-integrate martingale. Hence, we have

\[
E \left[ \frac{\tilde{\xi}_t^v}{1 + \epsilon \tilde{\xi}_t^v} \right] = \frac{1}{1 + \epsilon} - E \left[ \int_0^t \tilde{A}_s ds \right].
\]

Here, we check the following:

- \( E \left[ \tilde{\xi}_t^v \right] \leq 1. \)
- \( \tilde{A}_t \to 0 \) a.e. a.s. as \( \epsilon \to 0. \)
- \( \tilde{A}_s \leq K_5 \tilde{\xi}_s^v \left\{ \mu(Y_s)^2 + (Dv(s, Y_s))^2 a(Y_s)^2 \right\}, \) \( \exists K_5 > 0. \)

Using (90) and letting \( \epsilon \) to 0 in (91), we have \( E \left[ \tilde{\xi}_t^v \right] = 1. \)

Finally, we prove (90). Applying Itô’s formula to \( v(t, Y_t) \), we have

\[
dv(t, Y_t) = \left\{ \partial_t v(t, Y_t) + \frac{a(Y_t)^2}{2} D^2 v(t, Y_t) + b(Y_t)Dv(t, Y_t) \right\} dt
\]

\[
+ Dv(t, Y_t) a(Y_t) dw_2(t)
\]

\[
= \left[ \rho a(Y_t) \mu(Y_t) Dv(t, Y_t) - \frac{1 - \rho^2}{2} (Dv(t, Y_t))^2 a(Y_t)^2 + \frac{\mu(Y_t)^2}{2} \right] dt
\]

\[
+ Dv(t, Y_t) a(Y_t) dw_2(t).
\]

Here, we use (16). Using (89) and (92), we have

\[
d \left\{ \tilde{\xi}_t^v v(t, Y_t) \right\} = \tilde{\xi}_t^v dv(t, Y_t) + v(t, Y_t) d\tilde{\xi}_t^v + d\tilde{\xi}_t^v \cdot dv(t, Y_t)
\]

\[
= \tilde{\xi}_t^v \left[ 1 - \frac{\rho^2}{2} (Dv(t, Y_t))^2 a(Y_t)^2 + \frac{\mu(Y_t)^2}{2} \right] dt
\]

\[
+ \tilde{\xi}_t^v \left[ Dv(t, Y_t) a(Y_t) + v(t, Y_t) \left\{ -\rho \mu(Y_t) + (1 - \rho^2)Dv(t, Y_t) a(Y_t) \right\} \right] dw_2(t).
\]

Setting \( \tau_n := \inf \{ t > 0; Y_t < 1/n, n < Y_t \} \), we have

\[
2E \left[ \tilde{\xi}_T^v v(T \wedge \tau_n, Y_{T \wedge \tau_n}) \right] - 2v(0, y)
\]

\[
= E \left[ \int_0^{T \wedge \tau_n} \tilde{\xi}_s^v \left\{ (1 - \rho^2)(Dv(s, Y_s))^2 a(Y_s)^2 + \mu(Y_s)^2 \right\} ds \right].
\]
Moreover, using \( v < 0 \), we have
\[
E \left[ \int_0^{T \wedge \tau_n} \bar{\eta}^v_s \left\{ (1 - \rho^2)(Y_t(\omega))^2 \right\} ds \right] \leq -2v(0, y).
\]
Hence, as \( n \to \infty \), we have
\[
E \left[ \int_0^T \bar{\eta}^v_s \left\{ (1 - \rho^2)(Y_t(\omega))^2 \right\} ds \right] \leq -2v(0, y).
\]
We conclude this lemma.

**B.1. Preliminary.** In this section, we introduce the following result, which will be used several times in the proofs of our theorems.

**Lemma B.3.** ([11], Lemma 3.1) For \( f := (f_1, f_2) : (0, \infty) \to \mathbb{R}^2 \), denote \( f(Y) := (f(Y_t))_{t \in [0, T]} \). Suppose \( f(Y) \) is progressively measurable such that \( \int_0^T |f(Y_t)|^2 dt < \infty \) a.e.. Then the martingale property of \( \mathcal{E} \left( \int f(Y)^T dw \right) \) is equivalent to that of \( \mathcal{E} \left( \int f_2(Y) dw_2 \right) \). Here, \( \mathcal{E} \left( \int f(Y)^T dw \right) \) and \( \mathcal{E} \left( \int f_2(Y) dw_2 \right) \) are defined as follows:
\[
\mathcal{E} \left( \int f(Y)^T dw \right)_t := e^{f_0^t f(Y_s)dw(s) - \frac{1}{2} f_0^t |f(Y)|^2 ds},
\]
\[
\mathcal{E} \left( \int f_2(Y) dw_2 \right)_t := e^{f_0^t f_2(Y_s)dw_2(s) - \frac{1}{2} f_0^t |f_2(Y)|^2 ds}.
\]

**Lemma B.4.** Under assumptions (4) and (46) we have
\[
E \left[ \mathcal{E}^N_t (\hat{h}) \right] = 1,
\]
where \( \hat{h} \) is given in Corollary 1.

**Proof.** Recall that
\[
d\mathcal{E}^N_t (\hat{h}) = \mathcal{E}^N_t (\hat{h}) \int_{z > 0} \left( e^{\alpha \hat{h} z e^{(T - t)}} - 1 \right) \hat{N}(dt, dz).
\]
Let \( \epsilon > 0 \) be arbitrary. Using Itô’s formula, we have
\[
d \left( \frac{\mathcal{E}^N_t (\hat{h})}{1 + \epsilon \mathcal{E}^N_t (\hat{h})} \right) = d\bar{M}_t - \bar{A}_t dt,
\]
where \( \bar{M}_t \) and \( \bar{A}_t \) are defined by
\[
\bar{M}_t := \int_0^t \int_{z > 0} \left\{ \frac{\mathcal{E}^N_{u^-} (\hat{h}) e^{\alpha \hat{h} z e^{(T - u)}}}{1 + \epsilon \mathcal{E}^N_{u^-} (\hat{h})} - \frac{\mathcal{E}^N_{u^-} (\hat{h})}{1 + \epsilon \mathcal{E}^N_{u^-} (\hat{h})} \right\} \hat{N}(du, dz),
\]
\[
\bar{A}_t := \epsilon \lambda \int_{z > 0} \frac{\mathcal{E}^N_t (\hat{h})^2}{(1 + \epsilon \mathcal{E}^N_t (\hat{h})^2)(1 + \epsilon \mathcal{E}^N_t (\hat{h}) e^{\alpha \hat{h} z e^{(T - u)}})} \left( e^{\alpha \hat{h} z e^{(T - u)}} - 1 \right)^2 \nu(dz).
\]
Then, we observe that \( (\bar{M}_t)_{t \in [0, T]} \) is a square-integrable martingale. Further, we observe the following:

- \( E \left[ \mathcal{E}^N_t (\hat{h}) \right] \leq 1 \),
- \( \bar{A}_t \to 0 \) a.e. a.s. as \( \epsilon \to 0 \),
- \( \bar{A}_t \leq K_0 \mathcal{E}^N_t (\hat{h}) \left( \int_{z > 0} e^{2\alpha \hat{h} z e^{(T - u)}} \nu(dz) + 1 \right) \leq K_0 \mathcal{E}^N_t (\hat{h}) \left( \frac{\theta}{\theta - 2\alpha e^{(T - u)}} + 1 \right) \)
  \( < \infty \), \( \exists K_0 > 0 \). Here, we use (46).
Integrating (95) on [0, t] and taking expectation for both sides, we have
\[ E \left[ \frac{\mathcal{E}_i^N(\tilde{h})}{1 + e^{\mathcal{E}_i^N(\tilde{h})}} \right] = \frac{1}{1 + \epsilon} - E \left[ \int_0^t \tilde{A}_s ds \right]. \] (96)

Further, using the dominated convergence theorem in (96), we have (93). \( \square \)

B.2. The proof of (48). It suffices to show that \( E \left[ \mathcal{E}_i(\tilde{\pi}, \tilde{h}) \right] = 1, t \in [0, T] \). Noting that \( \tilde{h} \) is the deterministic function and using Lemma B.4, we have
\[ E \left[ \mathcal{E}_i(\tilde{\pi}, \tilde{h}) \right] = E \left[ \mathcal{E}_i(\tilde{\pi}) \right] E \left[ \mathcal{E}_i^N(\tilde{h}) \right] = E \left[ \mathcal{E}_i(\tilde{\pi}) \right]. \]

Observing that
\[-\alpha \rho e^{(T - s)} \tilde{\pi}_s + D\tilde{v}(s, Y_s) a(Y_s) = -\rho \mu(Y_s) + (1 - \rho^2) D\tilde{v}(s, Y_s) a(Y_s),\]
we see that the martingale property of \( \tilde{\mathcal{E}}_i(\tilde{\pi}) \) is equivalent to that of \( \tilde{\mathcal{E}}_i^\tilde{v} \) from Lemma B.3. Here, \( \tilde{\mathcal{E}}_i^\tilde{v} \) is given by (43) when \( v \) is replaced by \( \tilde{v} \). Following the argument of Lemma B.2, we have \( E \left[ \tilde{\mathcal{E}}_i^\tilde{v} \right] = 1 \), which verifies the martingale property of \( \tilde{\mathcal{E}}_i^\tilde{v} \).

Hence, we have \( E \left[ \mathcal{E}_i(\tilde{\pi}) \right] = 1 \), which leads to \( E \left[ \mathcal{E}_i(\tilde{\pi}, \tilde{h}) \right] = 1 \).

B.3. The proof of (58). It suffices to show that \( E \left[ \mathcal{E}_i(\tilde{\pi}, \tilde{h}) \right] = 1, t \in [0, T] \). Noting that \( \tilde{h} \) is the deterministic function and using Lemma B.4, we obtain
\[ E \left[ \mathcal{E}_i(\tilde{\pi}, \tilde{h}) \right] = E \left[ \mathcal{E}_i(\tilde{\pi}) \right] E \left[ \mathcal{E}_i^N(\tilde{h}) \right] = E \left[ \mathcal{E}_i(\tilde{\pi}) \right]. \]

From \( \rho^2 = 1 \) we observe that
\[-\alpha \rho e^{(T - s)} \tilde{\pi}_s + D\tilde{v}(s, Y_s) a(Y_s) = -\rho \mu(Y_s) = -\rho Y_s - \rho_0.\]
And, we see that the martingale property of \( \tilde{\mathcal{E}}_i^\tilde{v}(\tilde{\pi}) \) is equivalent to that of \( \tilde{\mathcal{E}}_i \) from Lemma B.3. Here, \( \tilde{\mathcal{E}}_i \) is given by (30). Using (24) and Lemma A.1, we have \( E \left[ \tilde{\mathcal{E}}_i \right] = 1 \), which verifies the martingale property of \( \left( \tilde{\mathcal{E}}_i \right)_{t \in [0, T]} \). So, we have \( E \left[ \mathcal{E}_i(\tilde{\pi}) \right] = 1 \), which leads to \( E \left[ \mathcal{E}_i(\tilde{\pi}, \tilde{h}) \right] = 1 \).

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