The Phenomenology of Inclusive Heavy-to-Light Sum Rules

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Abstract

By calculating the $O(\alpha_s)$ corrections to inclusive heavy-to-light sum rules we find model independent upper and lower bounds on form factors for $B\to\pi l\nu$ and $B\to\rho l\nu$. We use the bounds to rule out model predictions. Some models violate the bounds only for certain ranges of sum rule input parameters $\bar{\Lambda} = m_B - m_b$ and $\lambda_1$, or for certain choices of model parameters, while others obey or violate the bounds irrespective of the inputs. We discuss the reliability and convergence of the bounds, point out their utility for extracting $V_{ub}$, and derive from them a new form factor scaling relation.

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1 Introduction

A precise extraction of the Cabibo-Kobayashi-Maskawa parameters is predicated on our ability to control the theoretical uncertainties involved in the predictions of heavy hadron decays. As such, it is imperative that we perform our calculations within a systematic expansion. Heavy quark effective theory[1] leads to one such approximation scheme. It leads to predictions for the inclusive heavy hadron decay rates and spectra[2, 3] as an expansion in $\alpha_s$ and $\Lambda_{QCD}/m_b$, as well as predictions for heavy to heavy exclusive rates at zero recoil. Thus, with our present theoretical methods we are capable of extracting $V_{cb}$ at the level of a few percent. However, prospects for measuring $V_{ub}$ are much bleaker. The standard method of extraction based on the lepton endpoint spectrum will likely be plagued by large theoretical uncertainties[4], while an extraction from the hadronic spectrum[5] will suffer from the difficulty of the measurement itself. Given that exclusive modes have already been measured, it would be nice if we could get a handle on the exclusive form factors themselves. Unfortunately, our ability to calculate in QCD is limited to inclusive quantities due to the fact that quark-hadron duality is in itself an inclusive concept. If we attempt to completely eliminate the excited states from our calculation we begin to probe the dynamics of confinement, of which we are effectively ignorant. However, if we do not eliminate the excited states, but merely note that their contribution is positive definite, then we can derive bounds on the form factors instead of predictions.

This idea was originally formulated by Bjorken to derive upper bounds[6] on form factors as a function of momentum transfer $q^2$, including a well-known bound on the slope of the Isgur-Wise function at zero recoil. In addition, Voloshin[7] was able to find a lower bound at zero recoil. A few years later, Bigi et al.[8] showed that these bounds can be derived from sum rules similar to those developed for deep inelastic scattering, and are in fact the leading order term in a systematic expansion in $\alpha_s$ and $\Lambda_{QCD}/m_b$. These sum rules were then used to refine bounds on heavy-to-heavy matrix elements at or near zero recoil[8, 9, 10, 11, 12, 13] and were applied as well to heavy-to-light form factors far from zero recoil[14]. In the latter reference, momentum-dependent lower bounds were also developed.

These bounds are useful for several reasons. They may be used to fine-tune and discriminate among models used extensively for CKM extractions and Monte-Carlo simulations of backgrounds. Given the measured $q^2$ dependence of a form factor, the upper and lower
bounds may be used to extract a value for $V_{ub}$ without relying on models. Since model predictions for $B \to \pi l \nu$ and $B \to \rho l \nu$ decay rates vary by more than a factor of four, it is reasonable to ask whether comparing the total rate to the integrated bounds may yield a more accurate extraction of $V_{ub}$. We will see that the $B \to \pi l \nu$ sum rule can place a more restrictive lower bound on $V_{ub}$ than can be obtained by the union of models. An upper bound may be derived as well using recent work\[15, 16\] describing the shape of form factors to handle the problematic zero recoil region.

In this letter we continue the work begun in ref. \[14\] by including the effect of radiative corrections on the form factor bounds. These effects are of leading order for the lower bounds, and in one case dominate the upper bound. Due to the length of the calculations we only present partial results here and leave the more technical aspects for a subsequent, longer publication, where we will present the full one loop radiative corrections to the hadronic tensor $T_{\mu \nu}$ defined by

$$T_{\mu \nu}(v \cdot q, q^2) = -\frac{i}{2M_B} \int d^4x \ e^{-iq \cdot x} \langle B(v)|T\left( J^{\mu \dagger}(x) J^{\nu}(0) \right)|B(v)\rangle$$

$$\equiv -g^{\mu \nu}T_1 + v^{\mu}v^{\nu}T_2 - i\epsilon^{\mu \alpha \beta \gamma}v_{\alpha}q_{\beta}T_3 + q^{\mu}q^{\nu}T_4$$

$$+ (q^{\mu}v^{\nu} + q^{\nu}v^{\mu}) T_5,$$

(1)

where $J$ is a flavor-changing current, $v^\mu$ is the $B$ meson four-velocity, and $q^\mu$ is the dilepton four-momentum. These corrections are useful in their own right since they can be used to calculate physical rates with arbitrary cuts, via a simple numerical integration.

Let us briefly review the derivation of the bounds. We follow the notation used in \[14\] and refer the reader to this reference for details. The idea is to equate the inclusive rate, which is calculable within perturbative QCD using an operator product expansion, with the sum over exclusive states. The sum over exclusive states is then truncated, and the equality is changed to an inequality. More explicitly, equating partonic and hadronic sums over intermediate states in Eq. (1), we have

$$\frac{|\langle H|a \cdot J|B\rangle|^2}{4M_B^2 \epsilon_H \epsilon} + \sum_{X \neq H} \frac{|\langle X|a \cdot J|B\rangle|^2}{2M_B(\epsilon + \epsilon_H - \epsilon_X)}$$

$$- \sum_X (2\pi)^3 \delta^{(3)}(\vec{p}_X - \vec{q}) \frac{|\langle B|a \cdot J|X\rangle|^2}{2M_B(\epsilon + \epsilon_H + \epsilon_X - 2M_B)} = a_{\mu}^{\ast} T_{\mu \nu}^{OP} a_{\nu}.$$  

(2)

Here $a_{\mu}$ is an arbitrary four-vector chosen to pick out the form factor of interest for the $B \to H$ transition (eventually we will take $H$ to be a pion or rho), $\epsilon$ is defined as $\epsilon = \cdot$
\( M_B - E_H - v \cdot q \), and \( E_H = \sqrt{M_H^2 + \vec{q}^2} \). The first two terms represent the local cut in the complex \( v \cdot q \) plane corresponding to the semi-leptonic decay process. The third term arises from the “distant cut,” corresponding to the pair production process. The sum over states contains the usual phase space integration \( \int d^3p/(2E) \) for each particle, while \( \sum'_{X \neq H} \) is shorthand for

\[
\sum'_{X \neq H} = \sum_{X \neq H} (2\pi)^3 \delta^{(3)}(\vec{p}_X + \vec{q}).
\]

To justify the use of the OPE on the right hand side of Eq. (2), we perform a contour integral over \( \epsilon \) with a weighting function as described in [11]. We choose the weighting function \( W_\Delta(\epsilon) \) to be \( W = \theta(\Delta - \epsilon) \). Delta has the effect of cutting off the contribution from excited states, which in turn improves the bound beyond what one would get trivially from insisting that the one exclusive mode be less than the inclusive rate. However, as discussed above it is not possible to choose \( \Delta \) to be too small, since as we eliminate the contributions from excited states we become more sensitive to hadronization effects, and errors due to duality violation become important. Thus, we must choose \( \Delta \) small enough to maximize the utility of our bounds, yet large enough to preserve their validity, \( \Lambda_{QCD} < < \Delta < m_b \), where \( \Lambda_{QCD} \) is a typical hadronic scale. Furthermore, the OPE is an expansion in \( \Lambda_{QCD}/E_H \), so we can expect it to converge only when the hadronic three momentum \( q_3 \equiv |\vec{q}| \) is sufficiently large, \( q_3 >> \Lambda_{QCD} \).

By appropriate choice of contour we may eliminate the contribution from the unphysical cut. The upper bound may then be found by noticing that the excited state contribution is positive definite, leading to

\[
\frac{|\langle H|a \cdot J|B\rangle|^2}{4M_B E_H} \leq \int d\epsilon \ W_\Delta(\epsilon) \ a^{\mu\nu} T_{\mu\nu}^{\text{OPE}} a^{\nu}.
\]

A lower bound can be found by using the fact that

\[
(E_1 - E_H) \sum'_{X \neq H} |\langle X|a \cdot J|B\rangle|^2 W_\Delta(E_X - E_H) \leq \sum'_{X \neq H} (E_X - E_H) |\langle X|a \cdot J|B\rangle|^2 W_\Delta(E_X - E_H),
\]

where \( E_1 \) is the energy of the first excited state more massive than \( H \). The contribution of multi-particle states with energies less than that of the first excited resonance has been neglected, as they are suppressed by both phase space and large-\( N_c \) power counting. We

\footnote{In ref. [11] it was shown that for \( b \to c \) the results depend rather mildly on the precise form of the weight function.}
therefore have both upper and lower bounds,

$$\int d\epsilon \ W_{\Delta}(\epsilon) \ a^{\mu*}T_{\mu\nu}^{\text{OPE}}a^{\nu} \geq \frac{|\langle H|a\cdot J|B\rangle|^2}{4M_B E_H} \geq \int d\epsilon \ W_{\Delta}(\epsilon) \ a^{\mu*}T_{\mu\nu}^{\text{OPE}}a^{\nu}\left[1 - \frac{\epsilon}{E_1 - E_H}\right]. \quad (6)$$

Since Eq. (2) forces $\epsilon = E_X - E_H$, we see that the last term above may be interpreted as the average excitation energy of the higher states contributing to the sum rule, weighted by $W_{\Delta}(\epsilon) \ a^\dagger T a$. As emphasized in [14], the excitation energy begins at $\mathcal{O}(\Lambda_{\text{QCD}})$ and receives radiative corrections of order $\alpha_s \Delta$. Since these two quantities are numerically comparable, the lower bounds are not trustworthy without the radiative corrections, which we now include. Furthermore, given that our bounds are only valid when $q_3 >> \Lambda_{\text{QCD}}$, we see that the lower bound will only be useful if

$$\frac{(C_1\Lambda_{\text{QCD}} + C_2\alpha_s\Delta)}{(M_1^2 - M_H^2)} q_3 \lesssim 1. \quad (7)$$

Thus, we see again there is a trade off between the utility and validity of the lower bound, although we are helped in this case by the fact that typical hadronic masses $\sim 1$GeV are numerically larger than nonperturbative condensates such as $\bar{\Lambda}$. The range in $q_3$ for which the lower bounds are both useful and reliable depends on the nonperturbative corrections $C_1\Lambda_{\text{QCD}}$ presented in [14] and the radiative correction $C_2\alpha_s\Delta$ presented here. We will see that the usefulness of the lower bound depends sensitively on the numerical value of the heavy quark parameter $\bar{\Lambda}$.

The values of the parameters $\bar{\Lambda}$ and $\lambda_1$ have been extracted from the shape of the lepton end-point spectrum in inclusive semileptonic $B \to Xl\bar{\nu}_l$ decay [17, 18]. The values found were $\bar{\Lambda} = 0.39 \pm 0.11 \text{ GeV}$ and $\lambda_1 = -0.19 \pm 0.10 \text{ GeV}^2$ (the uncertainty being the 1$\sigma$ statistical error only). The errors are strongly correlated, and therefore we will in general evaluate our bounds for three values of the pair ($\bar{\Lambda}, \lambda_1$):

Set A $\equiv$ ($\bar{\Lambda} = 0.28 \text{ GeV}$, $\lambda_1 = -0.09 \text{ GeV}$)

Set B $\equiv$ ($\bar{\Lambda} = 0.39 \text{ GeV}$, $\lambda_1 = -0.19 \text{ GeV}$)

Set C $\equiv$ ($\bar{\Lambda} = 0.50 \text{ GeV}$, $\lambda_1 = -0.29 \text{ GeV}$) \quad (8)

The value of $\lambda_2 = 0.12 \text{ GeV}^2$ is determined from the $B - B^*$ hyperfine splitting. Eventually, $\bar{\Lambda}$ may be extracted from measurements of semileptonic $\Lambda_b \to \Lambda_c$ decays, because all form factors relevant to this decay can be expressed in terms of $\bar{\Lambda}$ and one universal function[19], and this function is furthermore determined by a single “shape parameter” [16]. Information on $\bar{\Lambda}$ and $\lambda_1$ may also be extracted from the $B \to X_s\gamma$ spectrum.
2 Heavy-to-Light Sum Rules

Let us begin by considering the bounds on the $B \to \pi l \bar{\nu}$ form factor $f^+$ defined by

$$
\langle \pi(p') | \bar{u} \gamma^\mu b | \bar{B}(p) \rangle = (p + p')^\mu f^+ + (p - p')^\mu f^- .
$$

(9)

The choice $a = (q_3, 0, 0, v \cdot q)$ in Eq. (6) leads to the sum rule

$$
\frac{1}{2}(1 - \bar{\Lambda}/M_B)^2 + X \frac{\alpha_s}{\pi} + \frac{\lambda_1}{M_B} \left( \frac{1}{q_3} - \frac{5}{6M_B} \right) + \frac{\lambda_2}{M_B} \left( \frac{1}{q_3} - \frac{1}{2M_B} \right) \\
\geq \frac{f^2 q_3^2}{E \pi M_B} \geq \frac{1}{E_1 - E_H} \left\{ \frac{1}{2}(1 - \bar{\Lambda}/M_B)^2(E_1 - q_3 - \bar{\Lambda}) + Y \frac{\alpha_s}{\pi} \Delta \\
+ \frac{\lambda_1}{6M_B^2} \left[ 5q_3 - 5E_1 - 7M_B + \frac{M_B}{q_3} (6E_1 + M_B) \right] \\
+ \frac{\lambda_2}{4M_B^2} \left[ 2(q_3 - E_1 + M_B) + \frac{M_B}{q_3} (4E_1 - M_B) \right] \right\} ,
$$

(10)

where $X$ and $Y$ are both functions of $\Delta$, $q_3$, and $\bar{\Lambda}$. Their analytic form is rather lengthy and will be presented in the longer version of this article. We choose $\Delta = 1.5$ GeV and $\alpha_s = 0.32$ for all plots in this section.

Figure 1 shows the upper and lower bounds on the form factor $f^+$ as a function of $q^2$, and demonstrates the importance of various perturbative and nonperturbative contributions. The entire kinematic range has been shown, although the bounds can only be trusted for $q^2 \lesssim 18$ GeV$^2$. The three thin solid lines correspond to the upper bound with $\bar{\Lambda}$ and $\lambda_1$ given by data sets A, B, and C, with set A being the most constraining, and set C the least. It is evident the upper bound is rather insensitive to nonperturbative corrections. Neither is it particularly sensitive to perturbative corrections, which alter the tree level result (not plotted) by no more than $\pm 15\%$ over the entire $q^2$ range. The tree level lower bounds for sets A and B are given by the dashed lines (a) and (b), respectively. As expected, the one-loop contributions are comparable in size to the tree level term: they move the dashed curves (a) and (b) into the thick solid curves (A) and (B), respectively. The lower bound for data set C is not shown because it is unphysical once perturbative corrections are included (that is, it becomes negative).

To assess the convergence of the perturbative and nonperturbative expansions, we look at the upper and lower bounds at $q^2 = 17.4$ GeV$^2$, corresponding to $q_3 = 1$ GeV. For data set A with $\Delta = 1.5$ GeV and $\alpha_s = 0.32$, the perturbative corrections in Eq. (10) take the
Figure 1: Upper and lower bounds on the $B \to \pi$ form factor $f_+$ as a function of momentum transfer $q^2$, in GeV$^2$. Fig. 1a: The three thin solid lines correspond to the upper bound for data sets A, B, and C as described in the text. The lower bounds for data sets A and B, respectively, are shown at both tree (dashed a,b) and order $\alpha_s$ (thick solid A,B). Fig. 1b: Thick solid lines redisplay the upper (labeled “upper”) and lower (labeled “B”) bounds for set B and the lower bound for set A (labeled “A”). The thin lines are model predictions as outlined in the text.
values $X = -0.26$ and $Y = -0.44$. Numerically, the upper bound is

$$\frac{f_2^2 q_3^2}{E_\pi M_B} \leq 0.500 - 0.055 + 0.008 - 0.021 - 0.006$$

$$\mathcal{O}(1) + \mathcal{O}(\bar{\Lambda}) + \mathcal{O}(\Lambda_{QCD}^2) + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s \bar{\Lambda}) , \quad (11)$$

where the lower line indicates the order in the double expansion in $\alpha_s$ and $\Lambda_{QCD}$. Both expansions converge quite well at this value of $q_3$. For the lower bound, we note that quantities like $E_1 - q_3 - \bar{\Lambda}$, which represent a mismatch between perturbative and hadronic endpoints, are $\mathcal{O}(\Lambda_{QCD})$. We evaluate such terms exactly, and for this reason do not show the expansion in $\bar{\Lambda}/q_3$, only $\lambda_1/q_3$ and $\lambda_2/q_3$. The resulting lower bound is

$$\frac{f_2^2 q_3^2}{E_\pi M_B} \geq 0.23 - 0.01 - 0.12$$

$$\mathcal{O}(\frac{\bar{\Lambda}}{(E_1 - E_H)}) + \mathcal{O}(\frac{\Lambda_{QCD}^2}{(E_1 - E_H)}) + \mathcal{O}(\frac{\alpha_s}{(E_1 - E_H)}) , \quad (12)$$

The nonperturbative corrections appear to be under control. To gauge the stability of the lower bound with respect to perturbative corrections, it would be very useful to have an estimate of the two loop contribution (e.g., the $\alpha_s^2 \beta_0$ term). In the meantime, it is encouraging to note that increasing the one-loop contribution by 25% strongly modifies the lower bound (A) only for $q^2 \lesssim 15 \text{ GeV}^2$, by shifting the value of $q^2$ where it drops to zero from $q^2 \approx 12 \text{ GeV}^2$ to $q^2 \approx 14 \text{ GeV}^2$ (curve B is modified even less).

The above analysis suggests that both the upper and lower bounds on $f_+$ are reliable for $q^2 \lesssim 18 \text{ GeV}^2$, once the nonperturbative parameters $\bar{\Lambda}, \lambda_1$ are given. Are they useful as well? In Fig. 1b we again plot the two lower bounds (A) and (B) (thick solid lines), as well as the upper bound for the central data set B (thick solid line labeled “upper”). Superimposed on these bounds are four models representing the spread available in the literature. Three of the models include the contribution of the $B^*$ resonance by using heavy meson chiral perturbation theory[20], and therefore depend on the $B^*\cdot B\cdot \pi$ coupling $g$ and the $B$ meson decay constant $f_B$ as input parameters. We adopt values used in the original papers. The topmost, dashed line is the prediction of Casalbuoni et al.[21] with $g = 0.61$ and $f_B = .2 \text{ GeV}$. With this choice of input parameters, it is clearly ruled out by our upper bound over the entire kinematic region (and the corresponding value of $V_{ub}$ eliminated). The thin solid line is the prediction of the Light-Front quark model by Cheung et al.[22], with $g = .75$ and $f_B = .187 \text{ GeV}$. It violates the upper bound for $q^2 > 13 \text{ GeV}^2$. The dot-dashed
curve, corresponding to the “constrained dispersive model” of Burdman and Kambor with $g = 0.5$ and $f_B = 0.15$ GeV, is consistent with both upper and lower bounds over the kinematic range where the bounds are trustworthy. Finally, the dotted curve is the prediction of the ISGW nonrelativistic quark model. It falls below the lower bound (A) at $q^2 = 15$ GeV$^2$, so it is inconsistent with the values $\bar{\Lambda} = 0.28$ GeV, $\lambda_1 = -0.09$ GeV$^2$. It has been pointed out that the ISGW model fails to include the contribution from the $B^*$ pole at large $q^2$; if modified to include this, it may well rise above our lower bounds.

The models plotted in Fig. 1b are merely a representative range – there are many other models and calculations. For example, the prediction of Bagan et al. is similar to that of Burdman and Kambor for $q^2 \lesssim 18$ GeV$^2$, so it is also consistent with the bounds. Lattice simulations are consistent as well.

While the upper bounds are fairly insensitive to $\Delta$, the lower bounds can be somewhat improved by smearing over a smaller range of excitation energies. Taking $\Delta \rightarrow 1$ GeV lowers the value of $q^2$ where the lower bound vanishes by 1 to 2 GeV$^2$. Conversely, increasing $\Delta \rightarrow 2$ GeV weakens the lower bound, but only about half as severely as when $\Delta$ is decreased. We retain $\Delta = 1.5$ GeV as a reasonable compromise between maximizing utility and reliability.

We turn now to the $\bar{B} \rightarrow \rho \ell \bar{\nu}$ form factors $a_\pm$ defined by

$$\langle \rho(p') | \bar{u} \gamma^\mu \gamma_5 b | \bar{B}(p) \rangle = f_\rho \epsilon^{*\mu} + [(p + p')^\mu a_+ + (p - p')^\mu a_-] p \cdot \epsilon^* .$$  \hspace{1cm} (13)

Using $a = (E_\rho, 0, 0, -q_3)$ in Eq. (2) gives the upper bound

$$\frac{1}{2}(E_\rho - q_3)^2 \geq \lambda_1 \left[ \frac{q_3}{3M_B} - \frac{E_\rho}{3q_3} - \frac{E_\rho^2}{3q_3M_B} \right]$$

$$+ \lambda_2 \left[ \frac{q_3}{M_B} - \frac{E_\rho}{2q_3} - \frac{E_\rho^2}{M_Bq_3} \right] + O(\alpha_s q_3^2) \nonumber$$

$$\geq \frac{M_Bq_3^2M_\rho^2}{E_\rho} \left[ a_+ + \frac{1}{2}(a_+ + a_-) \left( \frac{M_B E_\rho}{M_\rho^2} - 1 \right) \right]^2 ,$$  \hspace{1cm} (14)

which is interesting because the leading order term vanishes as $q_3$ becomes large. As pointed out in [14], this implies the $\alpha_s$ contribution dominates at small $q^2$.

It is worth noting that Eq. (14) implies nontrivial scaling relations for the form factors $a_\pm$. In the limit that $\Lambda_{QCD} \ll q_3, M_B$, they must fall off at least as fast as

$$a_+ \sim \frac{1}{(M_Bq_3)^{1/2}} + \frac{[\alpha_s(q_3)q_3]^{1/2}}{M_B^{1/2} \Lambda_{QCD}}$$

$$a_+ (a_+ + a_-) \sim \frac{\Lambda_{QCD}^2}{(M_Bq_3)^{3/2}} + \frac{\alpha_s(q_3)^{1/2} \Lambda_{QCD}}{M_B^{3/2}q_3^{1/2}} .$$  \hspace{1cm} (15)
The scaling with $M_B$ follows from considering the heavy $b$-mass limit, which leads to $a_- = -a_+ [1 + O(1/m_b)]$. That $a_+$, modulo perturbative contributions, falls off with $q_3$ is no surprise, since even a simple pole goes like $1/q_3$. However, the $q_3$ behavior of $(a_+ + a_-)$ is, to our knowledge, a novel result. At $q^2 = 0$, $q_3 \sim M_B$ and the $b$-mass scaling relation need not hold. Instead, the full $q_3$ scaling relations take over. Practically speaking, the large factor $(M_B E_{\rho}^{max} - M_\rho^2)/(2 M_\rho^2) \approx 12$ in Eq. (14) leads to very tight constraints on the combination $a_+ + a_-$, independent of the validity of the $b$-mass scaling relations.

The upper bounds on the magnitude of the dimensionless form factor combination

$$A_+ \equiv (M_B + M_\rho) \left[ a_+ + \frac{1}{2} (a_+ + a_-) \left( \frac{M_B E_\rho}{M_\rho^2} - 1 \right) \right]$$

are plotted in thick solid lines in Fig. 2. The curves (a), (b), and (c) correspond to data sets A, B, and C, respectively. The parameter $\lambda_1$ plays an important role in this case because the leading order $(E_\rho - q_3)^2$ term suffers from a severe cancellation. In fact, for $q_3 > M_\rho$, this term is formally $O(M_B^4 / q_3^2)$, which is subleading to the contributions from $\lambda_1$ and $\lambda_2$ (note that $\bar{\Lambda}$ does not enter the upper bound at tree level). The thin solid line is the tree level upper bound for $\lambda_1 = -0.19$ GeV$^2$ (data set B). As expected, the perturbative corrections dominate at small $q^2$. Even so, changing the one-loop contribution by 25% alters the upper bounds by no more than $\approx 10\%$ (the square root helps), so they are reasonably stable against perturbative corrections. Again, a two-loop calculation or estimate would be quite welcome here.

Also plotted in Fig. 2 are three predictions for $A_+$ taken from the literature. The dashed line is the relativistic quark model of WSB\[28\], which exceeds the upper bound by roughly a factor of five. The reason it violates the bound so spectacularly is that it does not obey the scaling law $a_- = -a_+$. Indeed, in order to comply with the bounds, the scaling violation at $q^2 = 0$ would have to be reduced by a factor of eight, $(a_+ + a_-)/a_+ \lesssim \frac{1}{8}$. The dotted curve is the prediction of another relativistic quark model due to Melikhov and Nikitin\[29\], using their “Set 1” parameters. It is consistent with the upper bounds for $\lambda_1 \geq -0.19$ GeV$^2$ (their other three sets are consistent for $\lambda_1 \geq -0.09$ GeV$^2$). The nonrelativistic ISGW quark model\[24, 30\] obeys the bounds regardless of the data set. We note that at $q^2 = 0$, the model of Cheng et al.\[31\] gives $A_+ = 0.12$, which is consistent with our bounds.

\footnote{This relation was used in \[14\] to simplify Eq. (14). This is only valid for $b \to c$ decays, since otherwise the smallness of $(a_+ + a_-)$ is compensated by $M_B/M_\rho$.}
Figure 2: Upper bounds on the $B^0 \rightarrow \rho^+ l^- \pi$ form factor $A_+$ for data sets A, B, C are given by the thick solid curves (a), (b), (c), respectively. The tree-level upper bound for set B is given by the thin solid curve. The dashed, dotted, and dot-dashed curves are the model predictions of WSB, MN, and ISGW, respectively, as described in the text.
Figure 2 illustrates the utility of the inclusive sum rules for constraining, or even eliminating, models. The sum rules can also yield bounds on the nonperturbative parameters $\bar{\Lambda}$ and $\lambda_1$. For example, the condition $|A_+|^2 \geq 0$ in Eq. (14) gives a lower bound on $\lambda_1$ comparable to $|A_+|^2$, if we use $\Delta = 1.5$ GeV. The lower bound on $A_+$ also yields a restrictive sum rule, but higher order nonperturbative terms become important (this is why we have not plotted the lower bound). The role of heavy-to-light sum rules in constraining nonperturbative parameters will be explored in greater detail in the long version of this article.

3 Conclusions

We have presented one-loop improved upper and lower bounds on the $B \to \pi l\bar{\nu}$ form factor $f_+$, as well as an upper bound on the combination of $B \to \rho l\bar{\nu}$ form factors $A_+$ defined in Eq. (16). We used the upper bounds on $f_+$, which are exceptionally stable with respect to perturbative and nonperturbative corrections, to rule out certain input parameters in two models. A third model may be consistent with the lower bounds on $f_+$, depending on the values of the measurable quantities $\bar{\Lambda}$ and $\lambda_1$. The lower bounds change little if the one-loop corrections are increased by 25%, but an estimation of the two-loop correction is needed to be certain they are reliable.

The upper bounds on $A_+$ are moderately stable against further perturbative corrections, but an estimate of two-loop effects would again be most welcome. The bounds are particularly interesting because they strongly constrain the allowed size of the scale violating (in $m_b$ and $q_3$) quantity $a_+ + a_-$. Indeed, the model of Wirbel, Stech, and Bauer exceeds the upper bound by a factor of five. A model by Melikhov and Nikitin obeys the bounds, but depending on the physical value of $\lambda_1$, their input parameters may be constrained. The ISGW model, by contrast, obeys the bounds rather handily.

The bounds given here demonstrate how inclusive heavy-to-light sum rules can constrain model parameters, or in some cases even invalidate their predictions. They should be useful not only in discriminating between models, but in constructing, constraining, and fine-tuning them. A variety of bounds on various combinations of form factors can be made. The inclusion of $\alpha_s$ perturbative corrections now allows us to consider, with some reliability, lower bounds on form factors, upper bounds on heavy quark violating form factor combinations,
and upper bounds on nonperturbative condensates. The results presented here will be extended in a forthcoming publication[33], including analytic expressions for the perturbative corrections. Improvements to existing sum rules may be possible by either further restricting the spin-parity of contributing intermediate states, or by using data to include the contributions of higher states to the hadronic sum. They should be of use as well in extracting model-independent limits on the CKM element $V_{ub}$.

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