Duals of nonabelian gauge theories in $D$ dimensions

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Abstract

The dual of an arbitrary $D$-dimensional nonabelian lattice gauge theory, obtained after character expansion and integration over the gauge group, is shown to be a local lattice theory in the eigenspace of the Casimir operators. For $D \leq 4$ we also provide the explicit form of the action as a product of character expansion coefficients and Racah coefficients. The representation can be used to facilitate strong coupling expansions. Furthermore, the possibility of simulations, at weak coupling, in the dual representation, is also discussed.

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Ever since their invention by Wilson [1], lattice gauge theories have inspired research in a multitude of directions. One of the most interesting of these has been the investigation of duality properties. The study of duality properties of various spin models has a long history, but more recently duals of compact \( U(1) \) lattice gauge theories were also found [2] [3].

The first step of finding duals of lattice gauge theories is always the character expansion of the Wilson-action. The second step is integration over the gauge degrees of freedom. In the case of \( U(1) \) theories these steps lead to integer valued fields satisfying constraints. The constraints have the form of linear equations for the eigenvalues of Casimir operators. For \( U(1) \) theories these are additive quantum numbers.

Suppose now that the dual picture is to be used for simulations. Then either the additive quantum numbers involved in the constraints have to be summed over, or the constraints (Bianchi identities) have to be resolved in terms of fields with no constraints. The first of these routes is not feasible for \( U(1) \) gauge theories: it would be tantamount to the exact solution of the theory. However, the second route has been successful [2] [3].

When a similar procedure is applied to nonabelian lattice gauge theories the above two steps also lead to local discrete valued models. The partition function of such a model is formed from a sum over products of the character expansion coefficients, depending on the eigenvalue of the quadratic Casimir operator and of Clebsch-Gordan coefficients depending on two kinds of discrete valued fields. Fields belonging to the first group take values that are the eigenvalues of diagonal elements of the Lie algebra, labeling states in a given representation. We will denote the collection of them for a particular group element by \( m, n, M, N, \mu, \nu \). These are additive quantum numbers and they all satisfy constraints in the form of linear equations. Either these constraints have to be resolved, or the summation over the corresponding fields has to be performed. The second group of discrete valued fields take values that are the eigenvalues of Casimir operators. They will be labeled by letters \( j, J, \) and \( l \). They satisfy triangular and tetrahedral inequalities. As will be pointed out later, such constraints would allow simulations respecting detailed balance. For short we will refer to these fields as magnetic quantum number and angular momentum, respectively, though these names are truly appropriate only if the gauge group is \( SU(2) \).

The purpose of this letter is to investigate the possibility of performing summations over magnetic quantum numbers. Of course, this can be done in a formal manner, but the crucial question is whether the resulting effective theory of angular momentum valued fields is local or not. We will answer this question in the affirmative for arbitrary compact gauge group and arbitrary space-time dimension \( D \). In fact, the statement is fairly obvious for the more or less trivial case \( D = 2 \), and it has recently been proved for \( D = 3 \) by several authors [4] [5].

A lattice gauge theory in the fundamental representation (a generalization to other representations is possible) has the form

\[
Z = \prod_{r} \left[ \prod_{i=1}^{D} \int dA_i(r) \prod_{i>k \geq 1} \exp\{ \beta \chi_f([U_{ik}(r)]) \} \right],
\]

where \( \chi_f \) denotes the character of the group in the fundamental representation depending on the plaquette element, to be described later, and \( \int dA_i(r) \) represents integration over the
Haar measure. The product index $i$ represents the axes of the lattice, while the $(ik)$ pair, in conjunction with $r$, represents a particular plaquette (2 dimensional simplex) of the lattice.

The exponentials can be expanded in a series of characters to give

$$\exp\{\beta\chi_f[U_{ik}(r)]\} = \sum_j d_j \chi_j[U_{ik}(r)] c_j(\beta)$$

(2)

where $d_j$ is the dimension of the representation labeled by $j$.

Now the characters are traces of the representation matrix of the plaquette element. The plaquette element can be written as the product of the link elements at the boundary of the plaquette. The link element $U_i(r)$ is associated with a link originating at $r$ and extending one lattice unit into the direction of the $i$th axis. The element $U_i(r)$ is also associated with the positive direction, while its adjoint with the negative direction. Setting $r = 0$, for simplicity, we have

$$\chi_j[U_{ik}(0)] = \text{Tr}_j[U_i^\dagger(0)U_k(0)U_iU_k^\dagger(\hat{e}_i)]$$

(3)

where $\hat{e}_k$ is a unit lattice vector in the positive direction along the $k$th axis.

The dual of the lattice consists of $D$ dimensional hypercubes centered at the vertices of the original lattice. The boundary of a hypercube consists of $2D$ codimension 1 cubes. The dual of a link of the original lattice is such a cube, on which the link variables $U_i$ live. The duals of plaquettes are codimension 2 simplexes (for $D=4$, plaquettes themselves). They are the boundaries of the $D-1$ dimensional cubes that themselves form the boundary of the hypercube. Duals of plaquettes attached to a single link of the original lattice form the boundary of such a $D-1$ dimensional simplex. There are $2D(D-1)$ such simplexes attached to each hypercube.

Our discussion starts with a simple observation:

1) Every pair of adjacent link variables of a plaquette define a unique hypercube.

The four possible pairs one can form on a loop define four adjacent hypercubes.

The proof is simple. Two adjacent links of the original lattice intersect in a lattice point. Let us associate this point with the pair of adjacent link variables. A point of the original lattice is, however, dual to a hypercube (centered at the point). The corners of a plaquette define four separate, but adjacent hypercubes. Their mutual boundaries are the duals of the link, codimension 1 simplexes.

Now saturate each product of link matrices on the plaquette by a complete system of states. Each complete system can be labeled by $(j, m)$, where $m$ represents the collection of quantum numbers labeling states in representation $j$. Then it follows that

2) Every magnetic quantum number, $m$ is associated with a unique hypercube.

This is true because every $m$ is associated with a pair of link matrices, but according to 1) every pair of adjacent link matrices is associated with a hypercube.

The plaquette contribution can then be written as

$$\chi_j[U_{ik}(0)] = D_{m_1m_2}^{ij}(U_i)D_{m_2m_3}^{ij}(U_k)D_{m_3m_4}^{ij}(U_i)D_{m_4m_1}^{ij}(U_k),$$

(4)
where the arguments of the link operators were dropped for simplicity. They are the same as in (3). Each of the four indices \( m \) is associated with an adjacent hypercube, according to 1) and 2).

Fig. 1. shows a plaquette in the \( ik \) plane. It also shows the magnetic quantum numbers labeling systems saturating adjacent products of \( U \)-operators. The magnetic quantum numbers are associated with dual hypercubes centered at corners of the plaquette.

It is worth making a comment at this point. Lemma 1 fails for globally symmetric nonabelian spin models. There is no natural way of associating the two hypercubes, connected by a link, with one or the other of the set of variables \( m_1 \) and \( m_2 \) of the product \( D_{m_1 m_2}^j(U^\dagger)D_{m_2 m_1}^j(U') \). Indeed, their duals are not local in the space of Casimir operators. The results of this paper would fail for nonabelian spin models.

Furthermore, it is easy to see that

3) Every link matrix \( U_i \) is associated with the same hypercube on its left (and also on its right) in every plaquette it enters.

The proof follows from the definition of link variables. Naturally, the association is reversed for the matrix \( U_i^\dagger \). Then we obtain

4) For every link variable, \( U \), the left subscript (right subscript) of each rotation function \( D_{m_i m_j}^j(U) \) it enters is associated with the same hypercube.

4) is a simple consequence of 2) and 3).

Let us prove now the most important result of this letter:

5) Every Clebsch-Gordan coefficient, obtained after combining rotation functions and integrating over the gauge variables, depends on magnetic quantum numbers associated with a single dual hypercube only. The angular momentum variables of the Clebsch-Gordan coefficients are shared by neighboring hypercubes only.

The proof starts with the use of the addition theorem for representation functions to reduce the number of representation functions \((2D - 2)\) depending on the particular link \( U \).

\[
D_{m_1 n_1}^{j_1}(U)D_{m_2 n_2}^{j_2}(U) = \sum_J(2J + 1)(-1)\alpha^{2(j_1 + j_2 - j_1)}\begin{pmatrix} j_1 & j_2 & M \\ m_1 & m_2 & J \end{pmatrix} D_{M N}^J(U) \begin{pmatrix} j_1 & j_2 & N \\ n_1 & n_2 & J \end{pmatrix},
\]

(5)

where the Wigner’s three-\( j \) symbols have been used rather than Clebsch-Gordan coefficients.

When a rotation function with argument \( U \) is combined with another one with argument \( U^\dagger \), then the relation

\[
D_{m n}^j(U^\dagger) = (-1)^{m-n} D_{-n-m}^j(U)
\]

should be first used.

Two important observations concerning (6) should be made at this point. First, the dependence on indices \( m \) and \( n \), associated with different hypercubes, factorizes in the two three-\( j \) symbols. Second, the new rotation function appearing in the addition theorem also satisfies 4), because \( M = m_1 + m_2 \) and \( N = n_1 + n_2 \).
Subsequent applications of the addition theorem reduce the product of $2(D - 1)$ rotation functions to a single rotation function $D^J_{\mu\nu}(U)$ and a product of $2(2D - 3)$ three-$j$ symbols, $2D - 3$ of which depend on $2(D - 1)$ the quantum numbers $m_i$ only, which are associated with one of the hypercube and the same number of them depend on quantum numbers $n_i$ only, which are associated with the other hypercube. Finally, integration over the group implies that $J = \mu = \nu = 0$ in the last rotation function. This, in turn, implies that the last two three-$j$ symbols that contain the quantum numbers $(J, \mu)$ and $(J, \nu)$, respectively, turn into Kronecker deltas for two pairs of angular momentum vectors. That leaves us with the product of $2(D - 2)$ three-$j$ symbols dependent on quantum numbers $m_i$ only and the same number of symbols dependent on the quantum numbers $n_i$ only. The dependence on additive quantum numbers labeling states is completely factorized.

Notice now that a similar construction can be performed on every one of the $2D$ links emanating from a given lattice point. Altogether, we will have $4D(D - 2)$ three-$j$ symbols dependent on $m_i$ quantum numbers associated with a given hypercube only. Furthermore, the angular momenta in these three-$j$ symbols are shared by the two hypercubes joining in the $D - 1$ dimensional simplex, dual to the appropriate link variable, only. This completes the proof of the theorem.

Now the final form of the main result of the paper can be spelled out as

6) Summation over magnetic quantum numbers results in a discrete local field theory in the eigenvalues of Casimir operators.

Summation over the magnetic quantum numbers results in factors associated with hypercubes. The factor depends on angular momenta corresponding to the $2D$ codimension 1 simplexes of the hypercube and additional $2D(D - 2)$ angular momenta obtained at repeated applications of addition theorem (5), also associated with the same hypercube. Thus, the range of the interactions is the size of a single hypercube.

Let us investigate now the simplest cases $D = 2, 3$, and 4. Although $D = 2$ lattice gauge theories are trivial, still the application of our results to them is instructive. Also the algebra is simple enough so that it can be written out in detail. There are $D = 2$ link variables $U_1$ and $U_2$ running out of an arbitrary point of the original lattice and $D = 2$ variables $U_1$ and $U_2$ running into it. The $2D(D - 1) = 4$ plaquettes containing these variables are

$$\text{Tr}(U_2 U_1 ...), \quad \text{Tr}(U_1^\dagger U_2 ...), \quad \text{Tr}(U_2^\dagger U_1^\dagger ...), \quad \text{Tr}(U_1 U_2^\dagger ...),$$

where link operators connecting other dual plaquettes (hypercubes) have been omitted from the traces.

After inserting complete systems of states one obtains the following product of rotation functions:

$$\sum_{m_{12}m_{122}m_{122}} D^{j_{12}}_{m_{12}m_{122}} (U_2) D^{j_{12}}_{m_{122}m_{12}} (U_1) D^{j_{12}}_{m_{12}m_{122}} (U_1^\dagger) D^{j_{12}}_{m_{122}m_{12}} (U_2^\dagger) \times D^{j_{12}}_{m_{12}m_{122}} (U_2^\dagger) D^{j_{12}}_{m_{122}m_{12}} (U_1^\dagger) D^{j_{12}}_{m_{12}m_{122}} (U_1^\dagger) D^{j_{12}}_{m_{122}m_{12}} (U_2^\dagger),$$

where indices associated by other dual plaquettes have been omitted.
Integration over the rotation functions is simple, because there are only two functions for each rotation, each giving a Kronecker delta for the angular momenta and magnetic quantum numbers as well. Thus, the summation over magnetic quantum numbers results in a factor \(2j + 1\) only, where \(j = j_{12} = \ldots = j_{12}\). The factor corresponding to a dual plaquette is proportional to a Kronecker delta for all the angular momenta involved and a trivial \(j\)-dependent multiplier. Since the angular momenta are shared between neighboring dual plaquettes, the partition function becomes diagonal in angular momentum. Thus, the partition function is \(\sum_j (2j + 1)[c_j(\beta)]V\), where \(V\) is the volume of the lattice.

The \(D = 3\) case \(^{2}\) is more complicated. There are \(2D = 6\) link (dual plaquette) variables, and \(2D(D - 1) = 12\) plaquettes (dual links) involved. Each link variable appears in four different plaquettes. Each appearance contributes to the expression by a rotation function, as in \((8)\). The four rotation functions can be combined pairwise using \((9)\) and \((10)\) giving two three-\(j\) symbols for the dual plaquette in question and two rotation functions. Integration over the group space results in the identification of the new angular momenta and magnetic quantum numbers in the three-\(j\) symbols. Before we write down the appropriate expressions obtained after this procedure we introduce a concise notation, due to Wigner \(\cite{6}\). Magnetic quantum numbers will be omitted from three-\(j\) symbols, with the understanding that repeated angular momenta in products imply summation over the corresponding magnetic quantum numbers. Since we restrict our discussion to three-\(j\) symbols associated with a single hypercube, every angular momentum will be uniquely associated with a magnetic quantum number.\(^{2}\)

Labeling angular momenta as in \((8)\) we obtain the following product of three-\(j\) symbols:

\[
(j_{13}j_{12}J_1)(j_{12}j_{13}J_1)(j_{12}j_{23}J_2)(j_{12}j_{23}J_2)(j_{13}j_{23}J_3)(j_{12}j_{23}J_3)
\times
(j_{12}j_{13}J_1)(j_{13}j_{12}J_1)(j_{12}j_{23}J_2)(j_{13}j_{12}J_2)(j_{13}j_{23}J_3)(j_{12}j_{23}J_3),
\]

(9) reduces to

\[
(j_{12}J_2)(j_{23}J_3)(j_{31}J_1)(j_{12}J_2)
= \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\} (j_1J_2J_3)
\]

(10)

and

\[
(j_{12}J_2)(j_{23}J_3)(j_{31}J_1)(J_1J_2J_3)
= \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\} (j_1J_2J_3)
\]

(11)

\(^{2}\) As Wigner points it out \(^{3}\) contravariant and covariant components should be used in the three-\(j\) symbols, which differ in sign. Contravariant component are always contracted with covariant ones. Our final result is not affected by this complication, since we intend to sum over all magnetic numbers. Thus, for the purpose of simplifying notations we will not distinguish contravariant and covariant components.
\[(12)\] has a simple geometrical interpretation if the gauge group is \(SU(2)\). Since angular momenta are attached to plaquettes on the original lattice, and links (vectors) on the dual lattice, each six-\(j\) symbol represents a tetrahedron. The five tetrahedrons corresponding to the five six-\(j\) symbols form a cube-like object with triangular faces. The angular momenta \(J_i\) and \(J_f\) correspond to face diagonals of the cube. All angular momenta are shared by neighboring dual cubes. The angular momenta \(j_{ik}\) are shared by three other, neighboring cubes: the ones displaced by \(\hat{e}_i, \hat{e}_k\), and by \(\hat{e}_i + \hat{e}_k\). Barred indices correspond to negative directions. Each of the angular momenta \(J_i\) is shared only by one other cube, the one displaced by vector \(\hat{e}_i\). The collection of tetrahedra cannot always be embedded in flat three dimensional space. In fact, if the coefficients \(c_j(\beta)\) of \((12)\) are omitted then the partition function reduces to the discrete model of Regge for three dimensional gravity.

Let us turn now to the physically most interesting \(D = 4\) case. Integration over the gauge degrees of freedom leads to the following invariant sum over products of three-\(j\) symbols:

\[
\sum_{J_1 \ldots J_7} \prod_{i=1}^{7} (2J_i + 1) \left\{ \begin{array}{ccc} J_{14} & J_{23} & J_{12} \\ J_{14} & J_{24} & J_{12} \\ J_{23} & J_{24} & J_{12} \end{array} \right\} \left\{ \begin{array}{ccc} J_{23} & J_{24} & J_{23} \\ J_{23} & J_{24} & J_{23} \\ J_{23} & J_{24} & J_{23} \end{array} \right\} \left\{ \begin{array}{ccc} J_{14} & J_{24} & J_{23} \\ J_{14} & J_{24} & J_{23} \\ J_{14} & J_{24} & J_{23} \end{array} \right\} \left\{ \begin{array}{ccc} J_{12} & J_{14} & J_{24} \\ J_{12} & J_{14} & J_{24} \\ J_{12} & J_{14} & J_{24} \end{array} \right\} = (j_1, j_2) = (-1)^{2j_3} \sum_{j'} (2j' + 1) \left\{ \begin{array}{ccc} \hat{j}_1 & \hat{j}_4 & j' \\ \hat{j}_3 & \hat{j}_2 & j \end{array} \right\} (j_1, j_2) (j_3, j_4) (j_5, j_6) (j_7, j_8) \]

(13)

Again the symbols \(j\) correspond to angular momenta carried by plaquettes, while symbols \(J\) represent angular momenta obtained when the \(j\)-s are added. Both of these angular momenta are shared by neighboring hypercubes, exactly the same manner as for \(D = 3\). Note that the subscript of symbols \(J\) indicates the direction of the neighboring hypercube sharing the corresponding angular momentum.

Using relation \((10)\), \((11)\), and \((12)\)

\[
\sum_{J_1 \ldots J_7} \prod_{i=1}^{7} (2J_i + 1) \left\{ \begin{array}{ccc} J_{14} & J_{23} & J_{12} \\ J_{14} & J_{24} & J_{12} \\ J_{23} & J_{24} & J_{12} \end{array} \right\} \left\{ \begin{array}{ccc} J_{23} & J_{24} & J_{23} \\ J_{23} & J_{24} & J_{23} \\ J_{23} & J_{24} & J_{23} \end{array} \right\} \left\{ \begin{array}{ccc} J_{14} & J_{24} & J_{23} \\ J_{14} & J_{24} & J_{23} \\ J_{14} & J_{24} & J_{23} \end{array} \right\} \left\{ \begin{array}{ccc} J_{12} & J_{14} & J_{24} \\ J_{12} & J_{14} & J_{24} \\ J_{12} & J_{14} & J_{24} \end{array} \right\} \left\{ \begin{array}{ccc} J_{12} & J_{14} & J_{24} \\ J_{12} & J_{14} & J_{24} \\ J_{12} & J_{14} & J_{24} \end{array} \right\} = (j_1, j_2) = (-1)^{2j_3} \sum_{j'} (2j' + 1) \left\{ \begin{array}{ccc} \hat{j}_1 & \hat{j}_4 & j' \\ \hat{j}_3 & \hat{j}_2 & j \end{array} \right\} (j_1, j_2) (j_3, j_4) (j_5, j_6) (j_7, j_8) \]

one can express product \((13)\) by means of invariant, six-\(j\) symbols. We obtain
\[
\times \left\{ \begin{array}{ccc}
J_7 & J_{13} & J_5 \\
J_{12} & J_6 & J_{23}
\end{array} \right\} \times \left\{ \begin{array}{ccc}
J_7 & J_3 & J_{34} \\
J_{14} & J_{13} & J_5
\end{array} \right\} \times \left\{ \begin{array}{ccc}
J_7 & J_3 & J_{34} \\
J_{24} & J_{23} & J_{34}
\end{array} \right\},
\]
(15)
where the subscripts of the \(J_{i}^{ab}\) angular momenta, appearing in (13) have been omitted. The angular momenta \(J_1, \ldots, J_7\) appear in the process of reduction of (13). They are not shared by neighboring hypercubes.

Finally, we examine the possibility of simulations, using the dual representation. In principle, such a possibility is exciting, because simulating integer valued theories without the need of tedious matrix multiplications makes computations faster. There are several issues, however, which have to be resolved before simulations can be attempted.

The first issue is detailed balance. Suppose an angular momentum \(j\) is updated. Suppose \(\Delta\) is the range of the allowed random change, i.e. the randomly proposed new value of \(j\) satisfies \(j - \Delta \leq j' \leq j + \Delta\) that is independent of the inequalities that \(j\) satisfies. Then there are two possibilities. Either \(j'\) also satisfies all the required inequalities, or it does not. In the former case the probability of the \(j \rightarrow j'\) transition is the same as that of the transition \(j' \rightarrow j\). In the latter case the two probabilities are still equal, namely they are zero. Thus for such an update procedure the detailed balance condition is satisfied.

The second issue is the complicated form of six-\(j\) symbols. Note, however, that the aim is to perform simulations in the weak coupling regime, relevant in the continuum limit. At weak coupling the average angular momentum is large. In fact, \(j \sim \sqrt{\beta}\). For large values of the angular momentum the semiclassical limit of six-\(j\) symbols can be used. Wigner \(\dagger\) has shown that in that limit six-\(j\) symbols are given in an average sense by
\[
\left\{ \begin{array}{ccc}
\hat{j}_1 & \hat{\hat{j}}_2 & \hat{\hat{j}}_3 \\
\hat{j}_1 & \hat{\hat{j}}_2 & \hat{\hat{j}}_3
\end{array} \right\}^2 \simeq \frac{1}{4\pi |(\hat{j}_1 \times \hat{j}_2) \cdot \hat{j}_3|},
\]
(16)
i.e by the inverse of \(24\pi\) times the volume, \(V\), of the tetrahedron. The volume can be expressed by the edges of the tetrahedron using Cayley’s formula
\[
288V^2 = \begin{vmatrix}
0 & J_1^2 & J_2^2 & J_3^2 & 1 \\
J_1^2 & 0 & J_2^2 & J_3^2 & 1 \\
J_2^2 & J_3^2 & 0 & J_1^2 & 1 \\
J_3^2 & J_1^2 & J_2^2 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{vmatrix}.
\]
(17)
The volume has square root type zeros at the edge of the allowed region, thus, the six-\(j\) symbols have mild, integrable singularities as the tetrahedron becomes degenerate.

One may also consider using continuous angular momentum variables. This would be approximately equivalent to using a noncompact gauge group. It should not alter results in the weak coupling limit.

One very important issue we have not yet resolved is the positivity of the integrand of the functional integral in \(j\)-representation. Indeed, (16) gives an expression for the square of six-\(j\) symbols only. The six-\(j\) symbols themselves oscillate. The phase of oscillation was found by Ponzano and Regge \(\ddagger\). In the semiclassical limit, summations over angular momenta are dominated by values which make all phases of oscillations stationary. The form of the resulting field theory is complicated (at least in the \(D = 4\) case) and will be dealt with in a future publication.
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Figure Captions

Fig.1. A plaquette in the $ik$ plane. $m_i$ represent magnetic quantum numbers associated with dual hypercubes.
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