Driven-dissipative time crystalline phases in a two-mode bosonic system with Kerr nonlinearity

L.R. Bakker,1,2 M.S. Bahovadinov,2,3 D.V. Kurlov,2 V. Gritsev,1,2 Aleksey K. Fedorov,2,4 and Dmitry O. Krimer5

1Institute for Theoretical Physics, Universiteit van Amsterdam, Science Park 904, Amsterdam, The Netherlands
2Russian Quantum Center, Skolkovo, Moscow 143025, Russia
3Physics Department, National Research University Higher School of Economics, Moscow, 101000, Russia
4National University of Science and Technology “MISIS”, Moscow 119049, Russia
5Institute for Theoretical Physics, Vienna University of Technology (TU Wien), Wiedner Hauptstraße 8-10/136, A–1040 Vienna, Austria

For the driven-dissipative system of two coupled bosonic modes in a nonlinear cavity resonator, we demonstrate a sequence of phase transitions from a trivial steady state to two distinct dissipative time crystalline phases. These effects are already anticipated at the level of the semiclassical analysis of the Lindblad equation using the theory of bifurcations and are further supported by the full quantum (numerical) treatment. The system is predicted to exhibit different dynamical phases characterized by an oscillating non-equilibrium steady state with non-trivial periodicity, which is a hallmark of time crystals. We expect that these phases can be directly probed in various cavity QED experiments.

Introduction. Nonlinear quantum optical effects are of great importance both for fundamental research and various applications, in particular in quantum information technologies [1–6]. Realistic settings of quantum experiments require considering not only sizable nonlinear effects, but also an interplay between external driving and dissipation caused by the fundamentally open nature of such systems. A system of paramount importance is a driven-dissipative model of bosonic modes with the Kerr nonlinearity [7–11]. For example, a qubit encoded in quantum harmonic oscillators [12] can be made stable against environment-induced decay using an interplay between Kerr-type interactions and squeezing [13–15]. On the fundamental side, non-equilibrium bosonic systems with a Kerr nonlinearity may exhibit novel dynamical phases, such as time crystals [16–18].

The time crystal (TC) phase of matter has been predicted theoretically in isolated Floquet driven systems and driven-dissipative systems [19–31] and has recently been observed experimentally [32–37]. Time crystals were originally introduced as the temporal analogue of spatial crystals where the time (rather than spatial) translation symmetry of a system is broken [38]. Crucially, the time crystalline phase of matter would then be resistant to entropy increase [32, 39–41]. This property makes the TC phase of matter an interesting candidate for quantum hardware devices, where entropy growth and spontaneous decay leads to corruption of stored information.

In this work we demonstrate that a system of two driven-dissipative coupled bosonic modes that are trapped in an optical cavity with Markovian dissipation exhibits intriguing dynamical behavior featuring inter alia time-crystalline phases. In the semiclassical regime, the system is shown to undergo a series of sub- and supercritical Hopf bifurcations between different stationary solutions. The Hopf bifurcations are responsible for the periodic dynamics emerging in the form of limit cycles in the phase space of a system [42] - a phenomenon that is absent in a single-mode bosonic system with a Kerr nonlinearity [43, 44]. Most importantly, we find a period doubling behavior suggesting existence of multiple distinct, non-trivial TC phases present in the system. The presence of the limit cycles on the semiclassical level can be considered as an indicator for possible (dissipative) TC phases in the full quantum dynamics of our system. Indeed, in the quantum regime we observe signatures of multiple nonequilibrium phase transitions in the form of the closure of the dissipative gap in the Liouvillian spectrum. We provide evidence that the semiclassical pre-
dictions are in many aspects consistent with the results obtained in the framework of the full quantum mechanical approach.

Our analytical approach is based on a combination of the Lie-algebraic disentanglement technique [45–51] and a semiclassical approximation (see Supplementary material for detailed exposition). The results in the quantum regime are found using exact diagonalization (ED) methods and by performing Monte Carlo simulations for trajectories of observables. The Monte Carlo simulations are not as sensitive to system size scaling as ED computations and therefore allow us to investigate larger system sizes. Using a combination of all aforementioned methods we conclude that different time crystalline phases exist in a broad range of values of the single-photon driving amplitude.

The model. We consider two driven-dissipative coupled modes in a cavity [52] (see Fig. 1) described by the following Hamiltonian ($\hbar = 1$):

$$\hat{H} = \omega_a \hat{a}^\dagger \hat{a} + g \hat{b}^\dagger \hat{a} + g^* \hat{a} \hat{b}^\dagger + \omega_b \hat{b}^\dagger \hat{b} + E_1(t) \hat{b} + E_1^*(t) \hat{b}^\dagger + \frac{U}{2} \hat{b}^\dagger \hat{b} \hat{b} \hat{b},$$

where $\hat{a}$, $\hat{b}$ ($\hat{a}^\dagger$, $\hat{b}^\dagger$) are bosonic annihilation (creation) operators. Parameters $\omega_j > 0$ are the cavity frequencies of the $a$- and $b$-modes, $g$ is the coupling strength between the modes, $E_1(t)$ determines the driving protocol of the $b$ mode and $E$ is the Kerr interaction strength. In nonlinear media $U \sim n_2 \omega_0^2 / (n_0^2 V_{\text{eff}})$, where $n_{0,2}$ are linear and nonlinear refractive indexes, $\omega_0$, $V_{\text{eff}}$ are the mode frequency and effective volume respectively. Both $a$- and $b$-modes are coupled to a zero-temperature Markovian environment. The $a$-mode experiences only the single-photon losses, whereas the $b$-mode is prone to both single- and two-photon losses [53]. The overall time evolution of the system is then governed by the Lindblad equation [54],

$$\dot{\rho} = -i[\hat{H}, \rho] + \frac{\gamma_a}{2} D[\hat{a}] \rho + \frac{\gamma_b}{2} D[\hat{b}] \rho + \frac{\chi_b}{2} D[\hat{b}^\dagger \hat{b}] \rho \equiv \mathcal{L} \rho,$$

where $D[\hat{L}] = 2 \hat{L} \rho \hat{L}^\dagger - \hat{L}^\dagger \hat{L} \rho - \rho \hat{L}^\dagger \hat{L}$ is the dissipator, $\hat{H}$ is given by Eq. (1), and $\mathcal{L}$ is the Liouvillian. Moreover, $\gamma_j > 0$ and $\chi_j > 0$ represent the cavity single and double mode loss rates, correspondingly.

Semiclassical analysis. In the semiclassical approximation, the Lindblad equation is reduced to the master equation (see supplemental information for details)

$$\dot{\xi}(t) = A (|z|^2) \xi(t) + \eta,$$

where the matrix $A$ reads as

$$A (|z|^2) = \begin{pmatrix} \tilde{\kappa}_a & -ig^* & 0 & 0 \\ -ig & \varphi (|z|^2) & 0 & 0 \\ 0 & 0 & \tilde{\kappa}_a & ig \\ 0 & 0 & ig^* & \varphi^* (|z|^2) \end{pmatrix}.$$
Hopf bifurcation). Above this threshold value, there is a stable stationary state between the points H2 and SN1.

In addition, we disclose the bifurcation scenario following an inverse route by starting from the stationary state slightly below the saddle-node bifurcation, SN1, and gradually decreasing the driving amplitude $\mathcal{E}_1$ [green path designated by arrows in Fig. 2(a)]. This will lead to a partially different dynamical scenario associated with the hysteretic behavior shown in this figure. Specifically, when one decreases $\mathcal{E}_1$ below H2, limit cycles are time-dependent steady states in the interval between H2 and PD1 as expected. Subsequently, the limit cycles double their period at PD1. For driving amplitudes below H1, unstable limit cycles are also possible solutions [dashed limit cycle in Fig. 2(b)] that, however, can not be experimentally observed. At the point PD2, the limit cycles half their period. With further decrease of $\mathcal{E}_1$, the stable and unstable limit cycles ultimately annihilate at the Limit Point Cycle (LPC). When decreasing the driving amplitude below the LPC point, the time-dependent steady state will jump down to a stationary state lying on the lower red curve. Thus, on the semiclassical level, the system demonstrates a series of continuous and discontinuous phase transitions with hysteretic behavior.

In the next step, we compare the semiclassical results to the full quantum mechanical dynamics of the system. Using a representation of bosonic creation and annihilation operators in a truncated Fock basis, we can compute the spectrum of our system. The thermodynamic limit is reached when the driving amplitude approaches infinity, $\mathcal{E}_1 \to \infty$, while the product $F\sqrt{U}$ is kept fixed (the so-called ‘weak interaction limit’) [19, 59]. In addition, the product of $F$ should remain constant. We introduce a dimensionless parameter $N$ to keep track of the particle number and quantify the large $N$-limit as follows:

$$\mathcal{E}_1 = \tilde{\mathcal{E}}_1 \sqrt{N}, \quad U = \tilde{U}/N, \quad \chi = \tilde{\chi}/N.$$  \hspace{1cm} (5)

We obtain a qualitative picture of the quantum mechanical solution as a function of $N$ which is summarized in Fig. 3. In general, the quantum mechanical results agree to a large extent with the semiclassical predictions. In the region of optical bistability shown in Fig. 2 ($7 \lesssim \tilde{\mathcal{E}}_1/\gamma_b \lesssim 20$), the dissipative gap, defined as the largest real part of the non-zero eigenvalues, closes rapidly, indicating a presence of a dissipative phase transition within this region. As we increase the driving amplitude to $\tilde{\mathcal{E}}_1/\gamma_b \approx 13$ (starting from $\tilde{\mathcal{E}}_1/\gamma_b \approx 7$), a pair of eigenvalues starts to approach the imaginary axis as a function of $N$ (see Fig. 3(c)). The resulting quantum oscillations are the quantum mechanical analogue of the limit cycles observed in the semiclassical case and indicate the time-crystalline phase. This analogy between semiclassical and quantum oscillations can be derived by comparing the inverse of the imaginary parts of the eigenvalues responsible for quantum oscillations with the periods of limit cycles [see Fig. 2(d), (e)]. When increasing the driving amplitude further to a value of $\tilde{\mathcal{E}}_1/\gamma_b \approx 16$...
In this work we demonstrate that a system of two coupled bosonic modes in a dissipative cavity exhibits rich behavior related to time crystalline phases. Based on the semiclassical approach, we compare semiclassical and quantum approaches: On Fig. 2 we observe the appearance of limit cycles for driving amplitudes between the points $H_1$ and $H_2$ which indicates a broken continuous time translation symmetry of the set of equations (3) within this interval of $\tilde{E}_1$. This is manifested by the discrete peak structure on Fig. 2 (d,e). In the quantum case we observe nearly non-decaying (almost zero real part of the Liouville eigenvalues) oscillating coherences at corresponding frequencies, see Fig. 3 (c,d). Semiclassical peaks on Fig. 2 (d,e) correspond to the points inside the thick red arrows and ovales on Fig. 3 (c,d) respectively.

**Conclusions and discussions.** In this work we demonstrated that a system of two coupled bosonic modes in a dissipative cavity exhibits rich behavior related to time crystalline phases. Based on the semiclassical approach,
we have identified a parameter range in which a time crystalline phase emerges in the form of usual limit cycles or limit cycles featuring a doubled-loop structure associated with period doubling. Results of computations in quantum regime in the identified parameter range qualitatively agree with the global picture sketched by the semiclassical approach: A series of phase transitions is observed where oscillating coherences and period doubling modes emerge. These transitions are accompanied by the closure of the Liouvillean gap in the thermodynamic limit. Computational limitations do not allow us to probe the system at sufficiently large excitation number $N$ to make more precise, quantitative predictions on the phase transitions of the model discussed in this work. At this stage, experimental investigations, like in [7, 8, 10, 11] are the natural next step for a detailed investigation of the predicted non-equilibrium phase transitions.

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SUPPLEMENTAL MATERIAL

Lie algebraic solution of time dependent Lindblad equation

Here, we describe in detail the Lie algebraic method used to solve the time dependent Lindblad equation [S1–S7]. Our starting point is to treat the \( b \)-mode semiclassically by replacing \( b \)-operators with a c-number, \( \hat{b} \rightarrow b \). This simplification allows us to solve the rest of the equation for the \( a \)-mode exactly using the Lie-algebraic approach since the remaining quantum operators form a closed Lie algebra. Using this approach we can keep track of any possible time dependencies exactly. Note that for time independent system parameters the semiclassical solution is in fact the exact solution to this system. For the driving protocol used in this work, time independence of the system parameters can be achieved through a rotating wave transformation. Here, however, we solve the system for any arbitrary choice of time dependent system parameters. For the \( a \)-mode from equation (2) in the main text we have

\[
\dot{\rho} = -i \left[ \omega_a \hat{a}^\dagger \hat{a} + gb^*(t)\hat{a} + g^*b(t)\hat{a}^\dagger, \rho \right] + \frac{\gamma_a}{2} D[\hat{a}]\rho = \mathcal{L}\rho, \tag{S.1}
\]

whereas for the \( b \)-mode in the semiclassical limit we find:

\[
\dot{b} = i\kappa_b b + g(\hat{a}) + iK|b|^2b + E_b(t)^*. \tag{S.2}
\]

where \( K = -\chi_b - iU \) and \( \kappa_b = -\gamma_b/2 - i\omega_b \). Eqs. (S.1) and (S.2) form a coupled system. Together they provide a self-consistent solution to equation (2) in the main text in the absence of quantum fluctuations in the \( b \)-mode. Our quantum mechanical problem is therefore reduced to that of a simple harmonic oscillator with time dependent coherent drive and dissipation. In order to solve the system of Eqs (S.1) and (S.2) for any choice of the time dependent parameters, we can use the Lie-algebraic properties of the operators. The main idea behind this approach is that the time-evolution operator, being a time-ordered exponent, is an element of a Lie group, as long as the operators in the Liouvilian (S.1) form a closed Lie algebra. When this algebraic structure holds, one can make a solution ansatz of the following form:

\[
\rho(t) = \prod_j e^{c_j(t)O_j}\rho(0), \tag{S.3}
\]

where \( c_j(t) \) are time dependent functions that depend on the system parameters, and \( g_j \) are the systems superoperators. Defining the commutator for two superoperators \( O_1 \) and \( O_2 \) as

\[
[O_1, O_2]\rho = O_1(O_2\rho) - O_2(O_1\rho), \tag{S.4}
\]

and identifying the superoperators in Eq. (S.1) (i.e. the set of \( O_j \)'s in (S.3)) as

\[
J\rho = a^\dagger a \rho, \quad B_L\rho = a^\dagger a^\dagger a^\dagger a^\dagger \rho, \quad B_R\rho = \rho a^\dagger a^\dagger a^\dagger a^\dagger, \\
A_L\rho = a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger \rho, \quad A_R\rho = a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger \rho, \tag{S.5}
\]

we find that these superoperators generate the following Lie algebra:

\[
[A_L, A_R^\dagger] = 1, \quad [A_R, A_R^\dagger] = 1, \\
[B_L, A_L] = -A_L, \quad [B_R, A_R] = -A_R, \\
[B_L, A_R^\dagger] = A_L^\dagger, \quad [B_R, A_R^\dagger] = A_R^\dagger, \\
[B_L, J] = -J, \quad [B_R, J] = -J, \\
[J, A_L^\dagger] = A_R, \quad [J, A_R^\dagger] = A_L, \tag{S.6}
\]

with all other commutators being zero. In terms of these superoperators, Liouvilian (S.1) can be written as

\[
\mathcal{L} = \kappa_a B_L + \kappa_a^* B_R + \gamma_a J + ig^*b^* \left( A_R^\dagger - A_L \right) + ig^*b \left( A_R - A_R^\dagger \right), \tag{S.7}
\]

where we introduce the same parameters \( \kappa_a = -i\omega_a - \gamma_a/2 \) as in the main text. Since the Liouvilian is generated by elements forming the closed Lie algebra, the solution to (S.1) is given by a product of ordinary (opposed to the time-ordered in the general case) time dependent exponentials of the form in eq. (S.3) Obviously, the choice for this
we get follows:

\[ \rho(t) = e^{i(t(t))} e^{\hat{b}^\dagger(t) B_L} e^{\hat{b}^\dagger(t) B_R e^{\alpha(t)} A_L^\dagger e^{-\alpha(t)} A_L} e^{\alpha(t) A_L^\dagger} e^{-\alpha(t) A_R} \rho(0). \]  

(S.9)

Here, the relationships between the different time dependent functions \( c_j(t) \) can be derived from the hermiticity of the density matrix \( \rho(t) \). Note the presence of the prefactor \( \exp \{ f(t) \} \). It appears because the Lie algebra in Eq. (S.6) is spanned not only by the superoperators (S.5), but it also contains the identity operator, as can be seen from the commutation relations. The Ansatz in Eq. (S.9) is one of the most natural ones because the action of the group elements on the initial state (S.8) is very simple. Differentiating \( \rho(t) \) in Eq. (S.9) and using various adjoint actions (in order to commute all exponentials to the right), such as

\[
\begin{align*}
    e^{g_J} B_L e^{-g_J} &= B_L + gJ, \\
e^{g_J} A_L^\dagger e^{-g_J} &= A_L^\dagger + gA_R, \\
e^{g_K} J e^{-g_K} &= J - gA_L, \\
e^{g_K} B_L e^{-g_K} &= B_L + gA_L, \\
e^{g_K} A_L e^{-g_K} &= A_L - g, \\
e^{g_K} A_R e^{-g_K} &= e^{-g_K} A_R, \\
e^{g_K} B_L e^{-g_K} &= e^{-g_K} A_L,
\end{align*}
\]

we get

\[
\hat{\rho} = \left\{ \hat{\beta}^\dagger B_R + e^{\hat{\beta}^\dagger} A_R + \left( he^\hat{\beta} - e^{-\hat{\beta}} \right) \hat{\alpha} A_R + \hat{\beta} B_L + e^\hat{\beta} \hat{\alpha} A_L^\dagger \\
+ \left( he^{\hat{\beta}^\dagger} - e^{-\hat{\beta}} \right) \hat{\alpha} A_L + \left[ \hat{h} \hat{h} \left( \hat{\beta} + \hat{\beta}^\dagger \right) \right] J + \hat{f} + \partial_t |\alpha|^2 \right\} \rho(t).
\]  

(S.11)

Matching the expression in the curly brackets with Liouvillian (S.7), we obtain a system of differential equations for \( f(t), h(t), \alpha(t) \), and \( \beta(t) \):

\[
\begin{align*}
    \dot{\beta} &= \kappa_\alpha, \\
    \dot{\alpha} &= -ig^* be^{-\beta}, \\
    \dot{h} - \gamma_\alpha h - \gamma_\alpha &= 0, \\
    \dot{f} + \partial_t |\alpha|^2 &= 0,
\end{align*}
\]  

(S.12)

with zero initial conditions. Note that the above equations are coupled with the semiclassical equation for the \( b \)-mode Eq. (S.2). This particular system of differential equations, Eqs. (S.12), can be solved in terms of quadratures as follows:

\[
\begin{align*}
    \beta(t) &= \int_0^t d\tau \kappa_\alpha(\tau), \\
    \alpha(t) &= -i \int_0^t d\tau e^{-\beta(\tau)} g^*(\tau) b(\tau), \\
    h(t) &= e^{-(\beta(t) + \beta^*(t))} - 1 = e^{\int_0^t d\tau \gamma_\alpha(\tau)} - 1, \\
    f(t) &= -|\alpha(t)|^2,
\end{align*}
\]  

(S.13)

where \( b(t) \) is a solution to Eq. (S.2). Indeed, taking into account the definitions of \( A_{L,R}\rho \) and \( A_{L,R}\rho^\dagger \), along with the expression for the displacement operator in terms of \( a \) and \( a^\dagger \),

\[
|\alpha_0\rangle = D(\alpha_0) |0\rangle = e^{-|\alpha_0|^2/2} e^{\alpha_0 a^\dagger} e^{-\alpha_0^* a} |0\rangle,
\]  

(S.14)

where \( |\alpha_0\rangle \) is a coherent state, the action of the last four exponents in Eq. (S.9) can be written as

\[
e^{[\alpha(t)]^2} D(\alpha(t)) |\alpha_0\rangle |\alpha_0\rangle D(-\alpha(t)) = e^{[\alpha(t)]^2} |\alpha_0 + \alpha(t)\rangle |\alpha_0 + \alpha(t)\rangle.
\]  

(S.15)

Here we took into account that \( |\alpha_0\rangle\langle\alpha_0| = D(\alpha_0)|0\rangle\langle0| D(-\alpha_0) \) and used the following property of the displacement operators: \( D(\alpha) D(\alpha_0) = \exp\{\alpha \hat{\alpha}_0^\dagger - \alpha^* \hat{\alpha} \} D(\alpha_0 + \alpha) \). Note that the exponential prefactor cancels with the factor of
where $|v\rangle$ is a coherent state, we get
\[ e^{\beta^* B_L} e^{\beta B_R} |\alpha_0 + \alpha \rangle \langle \alpha_0 + \alpha| = e^{(e^{2Re(\beta)} - 1)|\alpha_0 + \alpha|^2} |\alpha_0 + \alpha \rangle \langle \alpha_0 + \alpha| e^{\beta^*}. \] (S.17)

Finally, the action of $\exp(hJ)$ is trivially found using the formal power series expansion of the exponent
\[ e^{hJ} |v\rangle \langle v| = \sum_{n=0}^{+\infty} \frac{h^n}{n!} |v\rangle \langle v| (a^\dagger)^n = e^{h|v|^2} |v\rangle \langle v|. \] (S.18)

In our case this yields
\[ e^{hJ} |(\alpha_0 + \alpha)e^{\beta^*}\rangle \langle (\alpha_0 + \alpha)e^{\beta}| = \exp \left\{ h e^{2Re(\beta)} |\alpha_0 + \alpha|^2 \right\} |(\alpha_0 + \alpha)e^{\beta^*}\rangle \langle (\alpha_0 + \alpha)e^{\beta}|. \] (S.19)

Then, taking into account that $h \exp[2Re(\beta)] = 1 - \exp[2Re(\beta)]$, as follows from Eq. (S.13), we combine Eqs. (S.15), (S.17), and (S.19), and observe that all exponential prefactors cancel out. Thus, the density matrix from Eq. (S.9) reduces to
\[ \rho(t) = |(\alpha_0 + \alpha(t)) e^{\beta(t)}\rangle \langle (\alpha_0 + \alpha(t)) e^{\beta(t)}|. \] (S.20)

This exact solution allows us to compute the expectation value of $\hat{a}(t)$ as
\[ \alpha(t) = \langle a(t) \rangle = \text{Tr} [\hat{a} \rho(t)] \] (S.21)
thus effectively reducing our problem to solving the following set of coupled differential equations,
\[ i\dot{a} = g^* e^{-\beta} b, \]
\[ i\dot{b} = i\kappa_b b + g e^{\beta} (\alpha_0 + \alpha) + iK|b|^2 b + E_1(t)^*. \] (S.22)

Our problem can further be reduced to an autonomous system of equations by assuming that $g$, $\kappa_a$, $\kappa_b$, and $U$ are time-independent, and the driving amplitude being an oscillatory function $E_1(t) = \mathcal{E}_1 e^{i\omega_1 t}$. In this case, using an appropriate choice of variables,
\[ y(t) = e^{\beta(t) + i\omega_1 t} (\alpha_0 + \alpha(t)), \]
\[ z(t) = e^{i\omega_1 t} b(t), \] (S.23)

one can reduce Eq. (S.22) to
\[ \dot{y} = \tilde{\kappa}_a y - ig^* z, \]
\[ \dot{z} = -ig y + \tilde{\kappa}_b z + K|z|^2 z - i\mathcal{E}_1^* \] (S.24)
which is used to find the results reported in equation (3) in the main text.

**Weak interaction limit**

The system we consider here is, technically speaking, a zero dimensional system, and as such there is no obvious concept of taking a thermodynamic limit. However, Casteels et al. [S8] made a inspirational argument based on the works of Carmicheal [S9]: they compared the Fourier transforms of the Liouvillian (S.1) and a system of $N$ copies of dissipative coherently driven Bose-Hubbard chains, where $N$ is the number of cavities. This resembles a thermodynamic limit. The suggested equivalence can readily be derived by substituting Fourier transformed bosonic operators into the Hamiltonian for the 1D Bose Hubbard chain. Note that the homogeneous drive corresponds only
to the $k = 0$ mode in the expansion. The only difference between the arguments made in [S8] and this work is that we also added a two-mode dissipation channel. This results in rescaling the parameters $\hat{\mathcal{E}}$, $U$ and the dissipation rate $\chi$. The scaling of $\chi$ turns out to be the same as that of $U$. Omitting the details of the trivial calculation we conclude that we need to rescale the system parameters as follows

$$\hat{\mathcal{E}} = \hat{\mathcal{E}}_1 \sqrt{N}, \quad U = \frac{\tilde{U}}{N}, \quad \chi = \frac{\tilde{\chi}}{N}.$$  

(S.25)

Using this notion of thermodynamic limit we can perform numerical computations to probe the full quantum phase diagram of our system. Fig. S1(a) shows that the quantum mechanical description of the system converges to the semiclassical prediction as we increase $N$, the steady state expectation value of the particle number operator. Note that in the quantum regime there is no bi-stability because of the single-valuedness of the wavefunction. Instead, we observe a sharp transition. The increase in the slope of the transition is expected to grow proportionally to $N$. Convergence was carefully monitored to ensure that these results are trustworthy. An example of the convergence that was monitored is given in Fig S1(b). Here we were looking at the steady state expectation value $\text{Tr}(\hat{b}^\dagger \hat{b} \rho_{ss})$ measured by evaluating $\text{Tr}(1 - [\hat{b}^\dagger, \hat{b}])$ for different values of $N$. The closer this value is to zero, the better the convergence is. Fock space dimensionality cutoff is the same for both the $a$- and $b$-modes. The parameters are the same as in Fig (a). The convergence is measured at the point $\hat{\mathcal{E}}_1 / \gamma_b = 30$ which requires the largest dimensionality of the Hilbert space to converge, since the steady state particle number increases as a function of $\hat{\mathcal{E}}_1$.

FIG. S1. (a) Comparison of semi classical steady state solution as a function of $\hat{\mathcal{E}}_1 / \gamma_b$ (Red curve) and the numerically computed, exact solutions of the steady state using a truncated Fock space representation of the bosonic $a$ and $b$ modes (dashed curves). The parameters are $g = 5, \arg \mathcal{E}_1 = 0, \gamma_a = \chi_b = 1, \Delta_a = U = 10$ and $\Delta_b = -20$ (in units of $\gamma_b$). Cutoff dimensionality for both modes is based on the results from Fig (b) and are 17, 17, 24, 28, 30 for the modes $N = 1 \ldots 5$ respectively. (b) Convergence of steady-state expectation value $\text{Tr}(\hat{b}^\dagger \hat{b} \rho_{ss})$ measured by evaluating $\text{Tr}(1 - [\hat{b}^\dagger, \hat{b}])$ for different values of $N$. The closer this value is to zero, the better the convergence is. Fock space dimensionality cutoff is the same for both the $a$- and $b$-modes. The parameters are the same as in Fig (a). The convergence is measured at the point $\hat{\mathcal{E}}_1 / \gamma_b = 30$ which requires the largest dimensionality of the Hilbert space to converge, since the steady state particle number increases as a function of $\hat{\mathcal{E}}_1$.
FIG. S2. Monte Carlo trajectories of particle number expectation values $y_0 = \text{Tr}[\hat{a} \hat{a}^\dagger \rho(t)]$. Figs. (a), (b) and (c) show the averaged quantum trajectories at a driving amplitude of $\tilde{E}_1/\gamma_b = 13$, the associated Fourier spectrum and error. The orange and red lines show the behavior for $N = 10$ and $N = 25$ respectively with Fock-dimension cutoffs of 100 and 250. The semiclassical prediction is also included (dashed black lines). The initial state of the quantum trajectories is a coherent state, with an expectation value predicted by the semiclassical solutions. The quantum trajectories have been computed for a total time of $t_{\text{total}}/\gamma_b = 15$, with $dt_{\gamma_b} = 0.005$ and averaged over 3000 individual trajectories. Figs. (d), (e) and (f) show the same as the figures in the row above, but now at a driving amplitude of $\tilde{E}_1/\gamma_b = 16$. Note that for this driving amplitude, the system decays faster to the steady state. This is expected from the larger dissipative gap at this parameter choice. The period doubled mode is visible in the semiclassical regime in the Fourier spectrum in figure (e). All parameters in the figure are the same as in the figures of main text.

For the excited modes a trick like the one mentioned above will not work and one has to resort to using sparse LU-decompositions. Using this method we can compute the eigenvalues presented in the main text up to a cutoff of the Hilbert space dimension equal to 31. We note, however, that this is not enough to ensure high convergence for e.g. the period doubling modes for $N = 3$ and 4. Nevertheless, we can qualitatively extrapolate the behavior of this system for higher values of $N$ as we did in the main text. Larger computational power, or more efficient schemes should be used to probe the period doubling behavior in more detail.

Monte Carlo trajectories

In order to overcome the limitations of direct diagonalization we studied the quantum trajectories of the system using the QuTiP package for python [S11]. This procedure is less sensitive to scaling of matrix sizes, and allows for larger system sizes up to $N = 25$ at a cutoff in Hilbert space dimensions (in the number basis) by 250. Results are shown in Fig. S2. We clearly see that the trajectories decay to a steady state expectation value. This is coming from the fact that the dissipative gap is still finite. Increasing the parameter $N$ slows down the decay. From the Fourier spectrum we determine the periodicity of oscillations which closely match our semiclassical predictions. The period doubling mode remains elusive from these figures. This is due to the choice of the initial state. For the trajectories shown in Fig. S2, we have chosen a coherent initial state on a point of the limit cycle predicted by the semiclassical analysis. This state has a negligible overlap with the period doubled mode, which can be deduced using the results of exact diagonalization methods. Therefore, the quantum trajectories are exceedingly unlikely to exhibit period doubling in their Fourier spectrum at the level of precision of the computations in this work.
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