1 Introduction

Quantum groups provide an interesting example of non-commutative geometry \([1]\) (for a review with theoretical physics in mind see, e.g., \([2]\)). They may be looked at in various ways \([3]\). From a mathematical point of view, they may be introduced by making emphasis on their \(q\)-deformed enveloping algebra aspects \([4, 5]\), which leads to the quantized universal enveloping algebras, or by making emphasis in the \(R\)-matrix formalism that describes the deformed group algebra \([6]\). A useful point of view for possible physical applications is to look at quantum groups as a generalization (deformation) of symmetry groups acting on generalized representation spaces or quantum spaces \([7]\) (see also \([6, 8, 9]\)). Quantum groups are mathematically well defined in the framework of Hopf algebras \([10]\); their algebraic properties depend on one deformation parameter \(q\) (or more) in such a way that for \(q=1\) the deformed structures become the standard (non-deformed, Lie) ones. Thus, the essential feature in the field of quantum groups (we shall not discuss their dual quantum algebra aspect above) is in some sense similar to the relation between classical and quantum mechanics, where the commutative algebra of functions on phase space (the algebra of observables) becomes non-commutative after quantization. In the case of Lie groups, the commutative algebra of functions on the group manifold is replaced by a non-commutative algebra after quantization (or \(q\)-deformation); in particular, the matrix elements generating the algebra become non-commutative. This analogy justifies the ‘quantum’ name given to these structures. It must be said, nevertheless, that in physics ‘quantum’ refers to the appearance of the Planck constant and that its relation to \(q\) is, at best, unclear. Even the analysis of the ‘quasiclassical limit’, usually introduced by assuming that \(q = e^\hbar\), requires writing \(q = e^{\gamma \hbar}\) if \(\hbar\) is the Planck constant, \(i.e.,\) it introduces a new dimensional constant \(\gamma\) in classical physics.

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2 Quantum groups as $q$-symmetries

2.1 $GL_q$ and quantum matrices

Let us start by writing down some simple algebraic quantum group aspects relevant for our discussion. An element of $GL(2, \mathbb{C})$ is a regular $2 \times 2$ matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(2.1)

with $a, b, c, d \in \mathbb{C}$. The quantum group $GL_q(2)$ is the associative algebra (quantum groups are not really group manifolds) generated by the entries $a, b, c, d$ of a matrix $T$ satisfying the homogeneous quadratic relations ($\lambda \equiv q - q^{-1}$, $q \in \mathbb{C}$, $q \neq 0$)

$$ab = qba, \quad bd = qdb, \quad bc = cb,$$

$$ac = qca, \quad cd = qdc, \quad [a, d] = \lambda bc.$$  

(2.2)

By this reason, the notation $Fun(GL_q(2))$ is also used. The relations (2.2) are not arbitrary; they are the result of certain requirements. The most important properties of the matrices $T$ satisfying (2.2), or quantum matrices, are (see [11, 9]):

1) The matrix (co)multiplication preserves (2.2). If we take a second matrix $T'$ with entries satisfying (2.2) and commuting with those of $T$, the entries $T'' = TT'$ satisfy (2.2) again. In contrast, the entries of $T^n$ satisfy (2.2) after replacing $q$ by $q^n$ (quantum matrices do not form a group); this product should not be confused with the comultiplication $T'' = TT'$ above.

2) The relations (2.2) are consistent, i.e., they do not generate higher order relations. For example, let us take the same relations as in (2.2) except for $bc = qcb$. If we put $acd$ into the order $dca$ by exchanging $ac$ first or by moving first $cd$ and compare a new, third order relation (if $q \neq 1$) $bc^2 = 0$ appears. The form of the equations in (2.2) guarantees that no such higher order relations arise; they lead to a finitely generated quadratic algebra.

To go from $GL_q(2)$ (eqs. (2.2)) to $SL_q(2)$ we have to remove one generator. The element (see (A.3))

$$ad - qbc = da - q^{-1}bc := det_q T,$$  

(2.3)

is a central (commuting) element of the algebra which defines the $q$-determinant of the matrix $T$; the addition of the constraint $det_q T = 1$ to eqs. (2.2) consistently reduces the number of generators to three. In the ('classical') limit $q=1$, (2.2) just expresses that the algebra generated by the elements of $SL(2, \mathbb{C})$ is commutative, and (2.3) is the usual determinant.

The above discussion does not make apparent why eqs. (2.2) plus $det_q T=1$ define $SL_q(2)$, nor how to generalize them to the $SL_q(n)$ case. The $q$-group structure becomes clearer by using the $R$-matrix formalism originally developed in the framework of the quantum inverse scattering method (see [8] and references therein). The
non-commutativity of the entries of $T$ may be expressed by saying that
\[
T_1T_2 = \begin{bmatrix}
aa & ab & ba & bb \\
ac & ad & bc & bd \\
ca & cb & da & db \\
cc & cd & dc & dd \\
\end{bmatrix} \neq \begin{bmatrix}
aa & ba & ab & bb \\
ca & da & cb & db \\
ac & bc & ad & bd \\
cc & cd & dc & dd \\
\end{bmatrix} = T_2T_1 \tag{2.4}
\]
where $T_1 = T \otimes I$, $T_2 = I \otimes T$ (see Appendix for notation). Then, eqs. (2.2) may be rewritten as ‘RTT’ (or ‘FRT’ \[6\]) relations,
\[
R_{12}T_1T_2 = T_2T_1R_{12} \quad , \quad (R_{ij,ab}T_{ak}T_{bd} = T_{jc}T_{id}R_{dc,kl}) \quad , \tag{2.5}
\]
where $R_{12}$ is the $4 \times 4$ numerical matrix given in (A.30). In this form, they may be generalized to any dimension; all is needed is the appropriate $n^2 \times n^2$ $R$-matrix which for $GL_q(n)$ is \[3\]
\[
R_{ij,kl} = \delta_{ik}\delta_{jl}(1 + \delta_{ij}(q - 1)) + \lambda\delta_{il}\delta_{jk}\theta(i - j) \quad i, j, ... = 1...n \tag{2.6}
\]
\[
\theta(i - j) = \begin{cases}
0 & i \leq j \\
1 & i > j
\end{cases}
\]
The central $GL_q(n)$ $q$-determinant of $T = (t_{ij})$ $(i, j = 1, ..., n)$ is given \[3\] by
\[
det_q T = \sum_{s \in S_n} (-q)^{l(s)} t_{1s(1)}...t_{ns(n)} \tag{2.7}
\]
where $l(s)$ is the ‘parity’ of the permutation $s$; the condition $\det_q T = 1$ defines $SL_q(n)$. Moreover, there exist matrices $R$ which define the $q$-deformation of all $A_l$, $B_l$, $C_l$, $D_l$-type simple groups and their corresponding real forms as well as for the exceptional groups and supergroups; we refer to \[3\] for details.

One could insert other matrix as $R$ in (2.3) in order to reproduce (2.2). However, the natural ones (as (A.30)) satisfy the Yang-Baxter equation (YBE)
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.8}
\]
which ensures the consistency of (2.3) (for other aspects of the YBE see \[12, 3\] and references therein). This means that no further relations for the generators higher than the quadratic ones (2.2) may be derived from (2.7) and the requirement of associativity for the algebra, which is postulated from the very beginning and is independent of (2.3). This equation is sometimes introduced by reordering $T_1T_2T_3$ to $T_3T_2T_1$ by two different paths using the RTT relation and the associativity property of the algebra. In this way one is lead to $(R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12})T_1T_2T_3 = T_3T_2T_1(R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12})$. Thus, eq. (2.8) is consistent with eq. (2.5), but it is not implied by it. To see this explicitly, consider eq. (2.3) rewritten in the form $\hat{R}T_1T_2 = T_1T_2\hat{R}$ using (A.23) $(\hat{R} = \mathcal{P}R, \hat{R}_{ij,kl} = R_{ij,kl})$. Then, due to (A.34) we get $(q^2 + 1 \neq 0)$
\[
P_+T_1T_2P_- = 0 \quad , \quad P_-T_1T_2P_+ = 0 \tag{2.9}
\]
The first equation implies $ab - qba = 0$, $cd - qdc = 0$ and $[a, d] = qbc - q^{-1}cb$, while the second gives $ac - qca = 0$, $bd - qdb = 0$ and $[a, d] = qcb - q^{-1}bc$. In all, these equations
reproduce (2.2). These equations also follow from $P\pm T_1 T_2 = T_1 T_2 P\pm$, i.e., from an ‘RTT’ relation with $P\pm$ as an $\hat{R}$-matrix, although $P P\pm$ are not invertible and does not satisfy the YBE (2.3).

Since quantum groups are very close to the algebra of functions on a Lie group, we may expect them to have other characteristics pertaining to the group multiplication rule, inverse (antipode) and unit elements, etc. In fact, they may be characterized as Hopf algebras [4, 5, 6] (see [13] for reviews), but this aspect will not be considered here.

2.2 The quantum plane

Let us now introduce a deformed ‘representation space’ for $GL_q(2)$ (and hence for $SL_q(2)$). This is the quantum plane $C_q^2$, or associative algebra (a $q$-plane is not a manifold) generated by two elements $(x, y) = X$ (a ‘$q$-two-vector’) subjected to the commutation property [7]

$$xy = qyx \quad (2.10)$$

The commutation relation (2.10) can also be expressed by using the $q$-symplectic metric $e^q$ [11, 12]

$$e^q = \left( \begin{array}{cc} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{array} \right), \quad (e^q)^2 = -I \quad (2.11)$$

by the equation

$$X' e^q X = 0 \quad , \quad e^q_{ij} X_i X_j = 0 \quad (2.12)$$

which reflects that the $q$-symplectic norm of a $q$-two-vector vanishes. It is also possible to introduce a pair of (odd) variables $(\xi, \eta) = \Omega$ (an odd $q$-two-vector) satisfying

$$\xi \eta = -\frac{1}{q} \eta \xi \quad , \quad \xi^2 = 0 = \eta^2 \quad ; \quad (2.13)$$

for $q=1$, $(x, y)$ commute and $(\xi, \eta)$ anticommute (in a non-commutative differential calculus this second set of variables may be identified [8, 14] with the differentials of $(x, y)$). When it is required that after the transformation (coaction) $X' = T X$, $\Omega' = T \Omega$ (the entries of $T$ commute with those of $X$ and $\Omega$) the new entities $(x', y')$, $(\xi', \eta')$ satisfy also (2.10), (2.13), the commutation properties of the elements of $T$ are completely determined and (2.2) is obtained. This allows us to consider the quantum plane (2.10) as the ‘representation’ space of the $GL_q(2)$ quantum group (2.2). Since the non-commuting properties of the quantum group are encoded in the $R$-matrix by (2.5), it is natural to define the non-commuting properties of the $q$-plane analogously.

Indeed, eqs. (2.10) and (2.13) may be expressed as

$$R_{12} X_1 X_2 = q X_2 X_1 \quad \iff \quad R_{21}^{-1} X_1 X_2 = q^{-1} X_2 X_1 \quad , \quad (2.14)$$

$$R_{12} \Omega_1 \Omega_2 = -q^{-1} \Omega_2 \Omega_1 \quad \iff \quad R_{21}^{-1} \Omega_1 \Omega_2 = -q \Omega_2 \Omega_1 \quad , \quad (2.15)$$

where $X_1 X_2$ and $X_2 X_1$ are, respectively, the four-vectors $(xx, xy, yx, yy)$ and $(x, y, x, y)$ (analogously for $\Omega_1 \Omega_2$ and $\Omega_2 \Omega_1$). In components, (2.14) reads $R_{i,j,ki} X_k X_i =$

\footnote{Specifically, the mapping (coaction) $\varphi : C_q^2 \to GL_q(2) \otimes C_q^2$, $\varphi(X_i) = T_{ij} \otimes X_j$ is an algebra homomorphism, and $C_q^2$ is a left $GL_q(2)$-comodule, see below.}
\( qX_j X_i, \ R_{ij,kl} X_k X_l = qX_i X_j \) similar expressions are obtained for eq. (2.15). Both relations in (2.14) (and in (2.13)) are equivalent since \((\mathcal{P}\mathcal{R}\mathcal{P})_{ij,kl} = R_{ji,lk} \) (i.e., \(\mathcal{P}R_{12}\mathcal{P} = R_{21}\)) and \((\mathcal{P}X_1 X_2)_{ij} = (X_1 X_2)_{ji}\) i.e., \(\mathcal{P}X_1 X_2 = X_2 X_1\). Eqs. (2.14) and (2.15) are preserved by the \(q\)-transformations \(X' = TX\) and \(\Omega' = T\Omega\) since the components of \(X\) and \(\Omega\) are assumed to commute with the entries of \(T\). For instance,

\[
R_{12}X'_i X'_j = R_{12}(T_1 X_1)(T_2 X_2) = R_{12}T_1 T_2 X_1 X_2
\]

\[
= T_2 T_1 R_{12} X_1 X_2 = qT_2 T_1 X_2 X_1 = qX'_2 X'_1
\]

(2.16)

using (2.3): the invariance of the commutation properties (2.14) under a ‘\(q\)-symmetry’ transformation requires (2.3). Although the indices in all previous expressions take the values 1, 2, the \(R\)-matrix form of (2.14) and (2.3) makes it clear how to generalize them to \(GL_q(n)\); all that is needed is the appropriate \(n^2 \times n^2\) \(R\)-matrix given by eq. (2.9). With it, the relations defining the ‘quantum hyperplane’

\[
X = (x_1, ..., x_n), \quad x_i x_j = qx_j x_i \quad (i < j) \quad i, j = 1...n
\]

(2.17)

are again expressed by (2.14) and preserved under \(GL_q(n)\) because of (2.5).

All this is well known. Let us now show how to extend these \(q\)-vector constructions (see also [15] and references therein). We shall consider here the simplest example of \(q\)-twistors constructed from \(q\)-two-vectors (spinors) [(2.10), (2.12), (2.14)] and the application to \(q\)-Minkowski space algebras.

\section{Beyond \(q\)-vectors}

\subsection{Second rank \(q\)-tensors. \(q\)-twistors}

Since a two-dimensional object transforming under \(SL(2, \mathbb{C})\) is a spinor, an element of the \(q\)-plane should be called a \(q\)-spinor rather than \(q\)-vector. Since the complex conjugation of \(A \in SL(2, \mathbb{C})\) defines an inequivalent representation, spinors come in two varieties, dotted and undotted. The classical construction of a (Minkowski) real four-vector uses both

\[
K_{\alpha\beta} = (\sigma_i x^i)_{\alpha\beta} \quad \alpha, \beta = 1, 2, \quad K = \sigma_0 x^0 + \sigma_i x^i = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix},
\]

(3.1)

and \(K'_{\alpha\beta} = A_{\alpha} \cdot \gamma K_{\gamma\delta}(\bar{A}^{-1})^{\delta\beta}, \) where \(A\) and \(\bar{A} \equiv (A^{-1})^\dagger\) are the two fundamental representations of \(SL(2, \mathbb{C}), D_{\pm}^{\pm} 0\) and \(D_{0}^{0}; \sigma^0 = I_2, \sigma^i (i = 1, 2, 3)\) are the Pauli matrices and \(detK = (x^0)^2 - \bar{x}^2 = detK'\) is the square of Minkowski length. Thus, a \(q\)-deformation of the Lorentz group, \(L_q,\) may be obtained \([10]-[13]\) by replacing the classical \((q = 1)\) \(A\) and \(\bar{A}\) by two copies \(T\) and \(\bar{T}\) of \(SL_q(2)\) plus the reality condition (*-structure) \(T^\dagger = \bar{T}^{-1}.\) The commutation relations in this general situation may be expressed in terms of four \(R\)-matrices \(R^{(i)}, i = 1, ..., 4,\)

\[
R^{(1)} T_1 T_2 = T_2 T_1 R^{(1)} \quad , \quad T_1^\dagger R^{(2)} T_2 = T_2 R^{(2)} T_1^\dagger \quad , \quad R^{(4)} T_1 T_2^\dagger = T_2^\dagger T_1^\dagger R^{(4)} \quad , \quad T_2^\dagger R^{(3)} T_1 = T_1 R^{(3)} T_2^\dagger \quad ,
\]

(3.2)
where $R^{(2)} = R^{(3)\dagger}$, $R^{(1)}$ is the $SL_q(2)$ $R$-matrix ($R^{(1)}$ may be taken as $R_{12}$ or $R_{21}^{-1}$) and, since $\tilde{T} = (T^\dagger)^{-1}$ is another copy of $SL_q(2)$, consistency requires $R^{(4)} = \mathcal{P}R^{(1)\dagger}\mathcal{P} = R^{(1)\dagger}$ and $q$ real (from now on, we shall take $q \in \mathbb{R}$). The matrix $R^{(2)}$ (and hence $R^{(3)}$) defines how the elements of both quantum groups $T$ and $\tilde{T}$ commute ($T$ and $\tilde{T}$ are independent) and it is not a priori fixed; in fact, $L_q$ is not uniquely defined (see [20] and [21]).

Consider two $q$-spinors $X$ and $Z$, and their hermitian conjugates $X^\dagger$ and $Z^\dagger$, transforming under the coaction of $SL_q(2)$ with the matrices $T$ and $T^\dagger$ respectively by

\[
X' = TX \quad , \quad X'^\dagger = X^\dagger T^\dagger \quad , \quad Z' = TZ \quad , \quad Z'^\dagger = Z^\dagger T^\dagger .
\]

The form of the eqs. (3.2) is the result of the equations which express the commutation relations among the components of the vectors $X$ and $Z$. Since in principle $T$ and $T^\dagger$ do not commute, we have to allow for possibly non-trivial commutation relations among the components of $X$ and $Z^\dagger$. Thus, the set of commutation relations left invariant is given by

\[
R^{(1)} X_1 X_2 = q X_2 X_1 \quad , \quad Z^\dagger R^{(2)} X_2 = X_2 Z^\dagger \quad ,
q Z^\dagger Z^\dagger = Z^\dagger Z^\dagger R^{(4)} \quad , \quad Z^\dagger R^{(3)} X_1 = X_1 Z^\dagger ,
\]

where $q$ is the deformation parameter of $R^{(1)}$. The invariance of the first and third equations is proven as in Sec.2 and the others similarly. In the second equation, for example, we can check that

\[
Z^\dagger R^{(2)} X_2 = (Z^\dagger T^\dagger)R^{(2)} (T_2 X_2) = Z^\dagger T_2 R^{(2)} T^\dagger X_2
= T_2 Z^\dagger R^{(2)} X_2 T^\dagger = T_2 X_2 Z^\dagger T^\dagger = X_2' Z^\dagger
\]

using the second equations in (3.2) and (3.4), respectively, in the second and fourth equalities. In particular, if $R^{(2)} = I = R^{(3)}$, $T$ and $T^\dagger$ commute, which is reflected in the fact that the components of $X$ and $Z^\dagger$ commute.

Let us use the above construction to introduce another covariant object which generalizes (with some restrictions) the concept of twistor to the $q$-deformed case. Let $X$ and $Z^\dagger$ be $q$-two-vectors ($q$-spinors) of $SL_q(2)$. Tensoring them we introduce the object

\[
K \equiv XZ^\dagger \quad (K_{ij} = X_i Z_j^\dagger) .
\]

Then, the transformation of $K$ induced by (3.3) (the $q$-Lorentz coaction) is

\[
\varphi : K \longmapsto K' = TKT^\dagger \quad (K'_{ij} = T_{im} K_{mn} T_{nj}^\dagger) .
\]

The entries of $K$ are, of course, non-commuting. We shall see that these commutation relations can be expressed by a closed and simple equation which permits to extract the algebra generated by the entries of $K$ without considering its explicit realization in terms of the components of $X$ and $Z^\dagger$. Using the above relations we may now derive the equation describing the commutation relations which define the algebra generated by the entries of $K$. With $K_1 = X_1 Z_1^\dagger$ ($K_1 \equiv (K \otimes 1)_{ij,kl} = X_i Z_k^\dagger \delta_{jl}$) and $K_2 = X_2 Z_2^\dagger$ ($K_2 \equiv (1 \otimes K)_{ij,kl} = \delta_{ik} X_j Z_l^\dagger$), we find using (3.4) that
\[ R^{(1)} K_1 R^{(2)} K_2 = R^{(1)} X_1 Z_1^\dagger R^{(2)} X_2 Z_2^\dagger = R^{(1)} X_1 X_2 Z_1^\dagger Z_2^\dagger = X_2 X_1 Z_2^\dagger Z_1^\dagger R^{(4)} = X_2 Z_2^\dagger R^{(3)} X_1 Z_1^\dagger R^{(4)} \] (3.8)

Hence, the commuting properties of the quantum twistor are given by
\[ R^{(1)} K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R^{(4)} \], (3.9)

which is nothing else (see [22, 13]) than the reflection equation (RE) with no spectral parameter dependence (see [23, 24] and references therein). Eq. (3.9) reflects, with (3.2), the invariance of the commuting properties of the entries of \( K \) by the coaction (3.7). As shown here, eq. (3.9) also follows from interpreting \( K \) as an object made out of two \( q \)-’vectors’ with commuting properties defined by (3.4).

Let \( X, Z \) be two \( q \)-two-vectors (spinors). We may construct the following hermitian objects (quantum twistors)
\[ K = X X^\dagger \quad \text{or} \quad K = X Z^\dagger + Z X^\dagger \], (3.10)

and find that (3.7) preserves the hermiticity property of \( K \). As we shall see, the quantum determinant \( (\text{det}_q K) \) of \( K = X X^\dagger \) (null \( q \)-twistor) is necessarily zero (as it would be as well for \( X Z^\dagger \)). In contrast, the \( q \)-twistor \( K = X Z^\dagger + Z X^\dagger \) has \( \text{det}_q K \neq 0 \).

To compute the commutation properties of these hermitian \( K \) matrices, the complete set of relations among \( X, Z, X^\dagger \) and \( Z^\dagger \) are required. Thus, besides (3.4), we require the following set of covariant relations
\[ R^{(1)} X_1 Z_2 = Z_2 X_1 \quad , \quad Z_2^\dagger R^{(2)} Z_2 = Z_2 Z_1^\dagger \quad , \quad X_2^\dagger Z_2^\dagger = Z_2^\dagger X_1^\dagger R^{(4)} \quad , \quad X_2^\dagger R^{(3)} X_1 = X_1 X_2^\dagger \], (3.11)

the structure of which is again dictated from (3.2) by covariance. It is easily seen, using eqs. (3.4), (3.11) and Hecke’s condition for the matrix \( R^{(1)} = R^{(4)} \), that the commutation properties of the \( q \)-twistors \( K \) are also governed by eq. (3.9).

Notice that \( K = X Z^\dagger + Z X^\dagger \) in (3.10) is constructed from two parts, each one of them satisfying the same algebra relations (3.9):
\[ K = K^{(1)} + K^{(2)} \quad , \quad K^{(1)} \equiv X Z^\dagger \quad , \quad K^{(2)} \equiv Z X^\dagger \]. (3.12)

These two pieces have specific commutation properties among themselves. Indeed, the (mixed) commutation relations (3.11) lead to the following non-commuting property between the matrices \( K^{(1)} \) and \( K^{(2)} \) (non-symmetric under the interchange of \( K^{(1)} \) and \( K^{(2)} \))
\[ R^{(1)} K^{(1)} R^{(2)} K^{(2)} = K^{(2)} R^{(2)} K^{(1)} (\mathcal{P} R^{(4)} \mathcal{P})^{-1} \]. (3.13)

Setting \( R^{(1)} = R_{12} \) or \( R_{21}^{-1} \) produces two different equations which transform into each other by the exchange \( K^{(1)} \leftrightarrow K^{(2)} \). Both equations had to be possible since \( K = K^{(1)} + K^{(2)} \) is symmetric under this exchange. Eq. (3.13), which allows the sum (3.12) of two objects to satisfy the same commutation properties is an example of ‘additive braiding’ equation [27] here obtained from the commutation relations (3.11).

\footnote{Equations of this type were independently used in [25] in the context of braided algebras.}

\footnote{For a discussion of braided geometry and the role of \( q \) see [26].}
are central, the centrality of $q$-spinors. When \( \text{(3.11)} \) are the \textit{braiding relations for $q$-spinors}. 

Supposing that $\hat{R}^{(1)} = PR^{(1)}$ has a spectral decomposition like $\text{(A.34)}$ with a rank three projector $P_+$ and a rank one projector $P_-$, and that $det_qT$ $\text{(A.37)}$ and $det_qT^\dagger$ are central, the $q$-determinant of the $2 \times 2$ matrix $K$ is given by the expression 

\[
(det_qK)P_\dagger = (-q)P_\dagger K_1 \hat{R}^{(3)} K_1 P_-
\]  

(3.14)

When $(det_qT)(det_qT^\dagger) = 1$, $det_qK$ is invariant under the coaction $\text{(3.7)}$. Using the last eq. in $\text{(3.2)}$ and $\text{(A.37)}$

\[
det_q(TKT^\dagger) = P_\dagger (T_1K_1T_1^\dagger) \hat{R}^{(3)} (T_1K_1T_1^\dagger) P_-
\]

\[
= P_\dagger T_1T_2K_1 \hat{R}^{(3)} K_1 T_2T_1^\dagger P_\dagger = (det_qT)(det_qK)(det_qT^\dagger)
\]

(3.15)

Thus, since $(det_qT) = (det_qT^\dagger) = 1$ we obtain that $det_q(TKT^\dagger) = det_qK$. The centrality of $det_qK$ requires some YBE-like conditions on the $R^{(i)}$ $(i = 1, 2, 3, 4)$ matrices in $\text{(3.3)}$.

Using the definition $\text{(3.14)}$ and the $R$-matrix property $\hat{R}^{(3)}_{ab,cd} = R^{(3)}_{ba,cd}$, we can compute explicitly the $q$-determinant of $K$ in the following realizations

1. For the matrix $K = XZ^\dagger$ (and hence for the $q$-twistor $K = XX^\dagger$)

\[
(det_qK)P_\dagger_{ijkl} = P_\dagger_{ijkl}K_{ac} \hat{R}^{(3)}_{ab,mm} K_{mp} P_\dagger_{pn,kl} \propto \epsilon^{eq}_{ij} \epsilon^{eq}_{ab} X_a Z^\dagger_c p^{eq}_{bc,mm} X_m Z^\dagger_p e^{eq}_{pq} e^{eq}_{kl} 
\]

\[
= \epsilon^{eq}_{ij} \epsilon^{eq}_{ab} X_a X_b Z^\dagger_c Z^\dagger_p e^{eq}_{pq} e^{eq}_{kl} = \epsilon^{eq}_{ij} (X^\dagger e^q X) (Z^\dagger e^q Z) e^{eq}_{kl} = 0
\]

(3.16)

since $(X^\dagger e^q X) = 0 = (Z^\dagger e^q Z)$. This reflects the well-known fact in non deformed twistor theory that twistors constructed out of two spinors determine null length vectors.

2. For the $q$-twistor $K = XZ^\dagger + ZX^\dagger$, a similar calculus to the previous one gives

\[
(det_qK)P_\dagger_{ijkl} \propto \epsilon^{eq}_{ij} \epsilon^{eq}_{ab} (X_a Z^\dagger_c + Z_a X^\dagger_c) \hat{R}^{(3)}_{bc,mm} (X_m Z^\dagger_p + Z_m X^\dagger_p) e^{eq}_{pq} e^{eq}_{kl} 
\]

\[
= \epsilon^{eq}_{ij} [(X^\dagger e^q Z)(X^\dagger e^q Z)^\dagger + (Z^\dagger e^q X)(Z^\dagger e^q X)^\dagger] e^{eq}_{kl} \neq 0
\]

(3.17)

Thus, to get twistors with non-null $q$-determinant we need four spinors in the definition of $K$ (notice that $X, Z, X^\dagger$ and $Z^\dagger$ are all algebraically independent). If the scalar products $(X^\dagger e^q Z)$ and $(Z^\dagger e^q X)$ are central in the algebra generated by $X, Z, X^\dagger$ and $Z^\dagger$ the $q$-determinant of $K$ is also central.

### 3.2 An example of $q$-Minkowski space

Since, by assumption, $T$ and $\tilde{T} = (T^\dagger)^{-1}$ are $SL_q(2)$ matrices, i.e.,

\[
R_{12}T_1T_2 = T_2T_1R_{12} \quad , \quad R_{21}T_1T_2^\dagger = T_2^\dagger T_1^\dagger R_{21}, \quad (3.18)
\]

the first and third equations of $\text{(3.2)}$ are fulfilled if $R^{(4)} = R_{12}$ and $R^{(4)} = R_{21}$. Assuming, for instance, the specific non-trivial commutation relations between $T$ and $T^\dagger$ given by

\[
T^\dagger R_{12}T_1 = T_1 R_{12}T_2^\dagger
\]

(3.19)
we find that $R^{(2)} = R_{21}$. Then, eq. (3.19) leads to

$$R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21}$$

(3.20)

In terms of the entries of $K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, eq. (3.20) leads to the algebra (see also [23, 24])

$$\begin{align*}
\alpha \beta &= q^{-2}\beta \alpha \\
\alpha \gamma &= q^2\gamma \alpha \\
\alpha \delta &= \delta \alpha \\
[\beta, \delta] &= -q(\lambda/2)^2 \alpha \beta ,
\end{align*}$$

(3.21)

These commutation properties are preserved by (3.7) and define the quantum Minkowski algebra $M_q$ of [17, 18, 19]. Its linear central term (the $q$-trace of $K$ ([A.39])) is identified with time,

$$x^0 \sim tr_q K = q^{-1}\alpha + q \delta ,$$

(3.22)

and the $q$-determinant, defined by (3.14) with $\hat{R}^{(3)} = \hat{R}$,

$$(det_q K)P_- = (-q)P_- K_1 \hat{R}K_1P_- = (q \beta)P_- ,$$

(3.23)

gives the invariant quadratic central element which is identified with the $q$-Minkowski length. For other examples of $q$-Minkowski algebras and further discussions, see [20].

4 Non-commutative differential calculus

The development of a non-commutative differential calculus on the quantum groups (see, e.g., [28, 29, 30] and references therein and [31] for reviews) as well as on the quantum spaces (see, e.g., [8, 14, 32] and references therein) requires including, in a first stage, derivatives and differentials. A Cartan calculus involving the inner derivation and the Lie derivative (see [33, 34] and references therein) and the $q$-Hodge star operator [35] may also be introduced but will not be discussed here.

4.1 Differential calculus à la Wess-Zumino

The covariant differential calculus on the quantum planes was discussed by J. Wess and B. Zumino [8] for the $A_l$-type plane in which the corresponding $\hat{R}$-matrix has two different eigenvalues. The method developed there was generalized to the quantum orthogonal plane ($B_l$, $D_l$-type) [37, 38, 39], to the $q$-symplectic quantum plane ($C_l$-type) [38], and, in general, to a quantum plane given in terms of an $\hat{R}$-matrix with any number of eigenvalues [10]. This approach is also useful to discuss the differential calculus on the $q$-Minkowski space [19, 21].

4.1.1 The formalism

Consider the associative algebra (quantum space) generated by variables (coordinates) $x^i$, $i = 1, 2, ..., n$, with relations (cf. (2.14))

$$x^i x^j - B_{ij,kl} x^k x^l = 0$$

(4.1)
which may be rewritten \( X = (x^1, ..., x^n) \)

\[
(I - B_{12})X_1X_2 = 0 . \tag{4.2}
\]

Following [8] (see also [14, 30]), we first introduce the exterior derivative \( d \). \( d : x^i \mapsto dx^i \) is nilpotent, \( d^2 = 0 \), and satisfies Leibniz’s rule (for a modified version see [36]). In terms of the derivatives,

\[
d = dx^i \partial_i , \quad i = 1, 2, ..., n , \quad \partial_i \equiv \frac{\partial}{\partial x^i} . \tag{4.3}
\]

The differential calculus on the quantum space is defined by a set of quadratic algebraic relations among all the fundamental objects: coordinates \( x^i \), \( q \)-differentials \( dx^i \) and \( q \)-derivatives \( \partial_i \). These commutation relations must be introduced according to two essential requirements:

1. Covariance under the transformation (coaction) generated by a quantum group matrix \( T \)

\[
X \mapsto X' = TX , \quad dX \mapsto dX' = TdX , \quad \partial \mapsto \partial' = (T^t)^{-1}\partial , \tag{4.4}
\]

in this way, \( d \) is invariant under this transformation, \( d = dX^t\partial = dX'^t\partial' \).

2. Consistency, which means that the quadratic algebra generated by \( x^i \), \( dx^i \) and \( \partial_i \) \( (i = 1, 2, ..., n) \) is associative and there are not independent higher order relations.

In this way, the cross commutation relations among the differential quantities are given in matrix form by [8]

\[
(I - B_{12})X_1X_2 = 0 , \quad \partial_2\partial_1(I - F_{12}) = 0 , \quad \partial_1X_1 = I + X_2C_{12}^t\partial_2 , \quad (I + C_{12})dX_1dX_2 = 0 , \quad \partial_1dX_1 = dX_2D_{12}^t\partial_2 , \quad X_1dX_2 = C_{12}dX_1X_2 , \tag{4.5}
\]

where \( B, C, D \) and \( F \) are numerical matrices to be determined. The second requirement leads to a set consistency conditions which can be summarized in the following relations

\[
(I - B_{12})(I + C_{12}) = 0 , \quad (I + C_{12})(I - F_{12}) = 0 , \quad (I_2 - B_{12})C_{23}C_{12} = C_{23}C_{12}(I_{23} - B_{23}) , \quad C_{12}C_{23}(I_2 - F_{12}) = (I_{23} - F_{23})C_{12}C_{23} , \quad C_{12}C_{23}D_{12} = D_{23}C_{12}C_{23} , \quad D = C^{-1} . \tag{4.6}
\]

Now, for a given \( \hat{R} \)-matrix, the solution of these consistency conditions may be obtained by writing, in an appropriate way, the matrices \( (I - B), (I - F) \) and \( (I + C) \) in terms of the projectors obtained from the spectral decomposition of \( \hat{R} \).

### 4.1.2 The two-dimensional quantum plane case

The \( GL_q(2) \) \( \hat{R} \)-matrix has the spectral decomposition [A.34] where the projector \( P_+ \) \( (P_-) \) is the deformed version of the symmetrizer (antisymmetrizer) in two dimensions. In view of (2.9), it is not surprising that these projectors also allow us to define quantum planes. For instance, as for \( q = 1 \), the commutation relations for
the two-dimensional quantum space are obtained by requiring the vanishing of their $q$-antisymmetric products

$$P_\pm X_1X_2 = 0 \quad .$$

(4.7)

This equation is obtained from eq. (2.14) since, using there the spectral decomposition (A.34) of $\hat{R}$ one gets $(qP_+ - q^{-1}P_-)X_1X_2 = q(P_+ + P_-)X_1X_2$. Analogously, substituting $dX$ for $\Omega$ in (2.15) it follows that the $q$-symmetric products of $q$-differentials must vanish,

$$P_+dX_1dX_2 = 0$$

(4.8)

since, using (A.34), eq. (2.15) may be rewritten as $(qP_+ - q^{-1}P_-)dX_1dX_2 = -q^{-1}(P_+ + P_-)dX_1dX_2$. Obviously, all these relations involving projectors are preserved by the $GL_q(2)$ coaction. The consistency equations (4.6) are fulfilled by taking $\hat{R}$

$$I - B = I - F = P_- \quad \rightarrow \quad B = F = q^{-1}\hat{R} \quad ,$$

$$I + C = P_+ \quad \rightarrow \quad C = q\hat{R} \quad ,$$

(4.9)

and the covariant differential calculus for $q$-two-vectors (or for $q$-spinors) is defined by (compare with the matrix notation used in (1.5))

$$x^i dx^j = q\hat{R}_{ijkl}dx^k dx^l \quad , \quad dx^i dx^j = -q\hat{R}_{ijkl}dx^k dx^l \quad ,$$

$$\partial_\mu x^i = \delta_\mu^i + q\hat{R}_{ijkl}\partial_\mu x^j \partial_\nu \partial_\mu \partial_\nu = q^{-1}\hat{R}_{ijkl}\partial_\mu \partial_\nu \partial_\mu \partial_\nu \quad .$$

(4.10)

In the limit $q=1$, $\hat{R}_{ijkl} = \delta_{ik}\delta_{jk}$ reproduces the usual relations. All these relations are preserved under the quantum group transformations (4.4). For instance, multiplying the inhomogeneous eq. $\partial_1X_1 = I + qX_2\hat{R}_{12}\partial_2$ by $(T_1^1)^{-1}$ from the left and by $T_1^2$ from the right and using the RTT relation in the form $(T_1^1)^{-1}\hat{R}_{12}T_1^2 = T_2^1\hat{R}_{12}(T_2^2)^{-1}$ (cf. (A.32)), it follows that $\partial_1^iX_1^i = I + qX_2^i\hat{R}_{12}^i\partial_2^i$.

4.1.3 Differential calculus on $q$-Minkowski space

The previous formalism was used in (37) (see also (38, 39)) to develop the $SO_q(n)$-covariant differential calculus on $n$-dimensional $q$-Euclidean spaces. In this case, the $\hat{R}$-matrix of $SO_q(n)$, has three eigenvalues (3). In contrast, the situation for the $q$-Lorentz group and the $q$-Minkowski space $M_q$ is slightly more complicated. The commutation relations among the components of the $q$-Minkowski vector are computed by using the commutation relations among $q$-spinors (see (17, 18, 19) for explicit calculations and formulas). This leads to two different $\hat{R}$-matrices, $\hat{R}_I$ and $\hat{R}_{II}$, both $16\times16$ matrices satisfying the Yang-Baxter equation and with eigenvalues $q^2$, $q^{-2}$ and $-1$; $\hat{R}_I$ and $\hat{R}_{II}$ give rise to three projectors each. In each case, one of the subspaces can be further decomposed so that there are four independent projectors in all. These projectors allow us to write down commutation relations for the $q$-symmetric variables (coordinates) and for the $q$-antisymmetric ones ($q$-differentials). Thus, as it was shown in (13), the commutation relations among coordinates, differentials and derivatives satisfying the requirements of covariance and consistency, which define the differential calculus on the $q$-Minkowski space, can be expressed in terms of $\hat{R}_I$ and $\hat{R}_{II}$. 
4.2 RE formalism and $q$-Minkowski space calculus

We shall now look at the $q$-Minkowski space differential calculus by expressing the different commutation relations in terms of appropriate RE, so that the $q$-derivatives ($D_q$) and the $q$-forms ($\Lambda_q$) algebras will also be defined by RE \cite{22,20}. Consider first an object $Y$ transforming covariantly, i.e.,

$$Y \mapsto Y' = (T^\dagger)^{-1}YT^{-1}$$

(cf. (3.7), which will be taken as contravariant). It is easily seen, using (3.2), that the invariance of the commutation properties of the matrix elements of $Y$ gives

$$R^{(1)}Y_1R^{(3)}^{-1}Y_2 = Y_2R^{(2)}^{-1}Y_1R^{(4)}$$

A quadratic and $L_q$-invariant element ($q$-determinant) is defined through

$$(det_q Y)P_- = (-q^{-1})P_- Y_1\hat{R}^{(3)}^{-1}Y_1P_- ;$$

$$\Box_q \equiv det_q Y$$ becomes the $q$-D’Alembertian once the components of $Y$ are associated with the $q$-derivatives. As the $K$ matrix entries were associated with the generators of $M_q$, we shall consider the elements of $Y$ as generating the algebra $D_q$ of the $q$-Minkowski derivatives. For the example of Sec.3.2 the commutation properties are given by

$$R_{12}Y_1R_{12}^{-1}Y_2 = Y_2R_{21}^{-1}Y_1R_{21}$$

and the $q$-determinant

$$det_q Y)P_- = (-q^{-1})P_- Y_1\hat{R}^{(3)}^{-1}Y_1P_- = (uz - q^{-2}vw)P_-$$

is central, $[\Box_q , Y] = 0$. We now need to establish the commutation properties among coordinates and derivatives extending the classical relation $\partial_\mu x^\nu = x^\nu \partial_\mu + \delta^\nu_\mu, \partial^\dagger = -\partial$ to the non-commutative case, in a $L_q$-invariant manner. This is achieved by an inhomogeneous RE \cite{22} of the form

$$Y_2R^{(1)}K_1R^{(2)} = R^{(3)}K_1R^{(1)}^{-1}Y_2 + \eta J,$$

where $\eta$ is a constant, $\eta J \rightarrow I_4$ in the $q \rightarrow 1$ limit, and $J$ is invariant,

$$J \mapsto (T_2^\dagger)^{-1}T_1JT_1^\dagger T_2^{-1} = J, \quad T_1JT_1^\dagger = T_2^\dagger JT_2.$$  

As for $J$, setting $J \equiv J'P$ in eq. (4.17) gives $T_1J'T_2^\dagger = T_2^\dagger J'T_1$, hence $J = R^{(3)}P$ (the same result follows if we set $J = P'J'$). In the previous example $J = R^{(3)}P = R_{12}P$, and in order to have the inhomogeneous term in the simplest form (the analogue of the $\delta_{\mu}^\nu$ of the $q = 1$ case) it is convenient to take $\eta = q^2$. Then, the commutation relations of the entries of $K$ (generators of the algebra $(M_q$ of coordinates) and those of $Y$ (generators of the algebra ($D_q$ of derivatives) are given by

$$Y_2R_{12}K_1R_{21} = R_{12}K_1R_{12}^{-1}Y_2 + q^2R_{12}P.$$  

12
This equation is not invariant under hermitian conjugation. In fact, it is not possible to have simultaneously coordinates and derivatives with the usual hermiticity properties \((K = K^\dagger\) and \(Y = -Y^\dagger\)) [19] (see also [12]).

The determination of the commutation relations for the \(q\)-De Rham complex requires incorporating the exterior derivative \(d\). To the four generators of \(M_q\) and of \(D_q\) we now add the four elements of \(dK\) (\(q\)-one-forms), which generate the de Rham complex algebra \(\Lambda_q\) (the degree of a form is defined as in the classical case). As in (4.4), \(d\) commutes with the \(q\)-Lorentz coaction (3.7), so that

\[
dK' = TdKT^\dagger \quad .
\] (4.19)

Applying \(d\) to (3.20) we obtain

\[
R_{12}dK_1R_{21}K_2 + R_{12}K_1R_{21}dK_2 = dK_2R_{12}K_1R_{21} + K_2R_{12}dK_1R_{21} \quad .
\] (4.20)

We now use that \(R_{12} = R_{21}^{-1} + \lambda P\) (and the same for \(1\leftrightarrow 2\)) to replace one \(R\) in each term in such a way that the terms in \(P K_1R_{21}dK_2\) and in \(PdK_1R_{21}K_2\) may cancel. In this way we obtain two solutions of (4.20). Since the relations obtained are not invariant under hermitian conjugation, we may use one of them for \(dK\) and the other for the hermitian conjugate \(dK^\dagger \equiv (dK)^\dagger\)

\[
R_{12}K_1R_{21}dK_2 = dK_2R_{12}K_1R_{12}^{-1} \quad , \quad R_{12}dK_1^\dagger R_{21}K_2 = K_2R_{12}dK_1^\dagger R_{12}^{-1} \quad ,
\] (4.21)

from which follows that

\[
R_{12}dK_1R_{21}dK_2 = -dK_2R_{12}dK_1R_{12}^{-1} \quad , \quad R_{12}dK_1^\dagger R_{21}dK_2 = -dK_2R_{12}dK_1^\dagger R_{12}^{-1} \quad .
\] (4.22)

The exterior derivative \(d\), as its invariance suggests, has the form

\[
d = tr_q(dKY) \quad .
\] (4.23)

For an explicit comparison with the formalism of Sec.4.1.3 [19] see [20, 43].

What about the physical applications of non-commutative geometry to physics? One of the reasons for introducing \(q\) was to see whether the infinities in quantum field theory could be made milder. For the moment, however, there is no ‘\(q\)-special relativity theory’ or ‘\(q\)-deformed quantum field theory’. We shall conclude by mentioning just one interesting application to particle physics, the Connes-Lott version of the standard model [14]; very recently, it has been used to give an indication of the Higgs mass [15].

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A Appendix. Notation and useful expressions

We list here some expressions and conventions that are useful in the main text. ‘\(RTT\)’ relations as those in (2.7), (3.2) follow the usual conventions i.e., the \(4\times4\)
matrices $T_1, T_2$ are the tensor products
\[ T_1 = T \otimes I, \quad T_2 = I \otimes T. \] (A.24)

The tensor product of two matrices, $C = A \otimes B$, reads in components
\[ C_{ij,kl} = A_{ik}B_{jl}, \] (A.25)
so that the comma separates the row and column indices of the two matrices. Thus,
\[ (A_1)_{ij,kl} = A_{ik}\delta_{jl}; \quad (A_2)_{ij,kl} = A_{jl}\delta_{ik}. \]
The transposition in the first and second spaces is given by
\[ C^t_{ij,kl} = C_{kj,il}, \quad C^t_{ij,kl} = C_{il,kj}, \] (A.26)
i.e., $C^t_1 = A^t \otimes B$ (resp. $C^t_2 = A \otimes B^t$) is given by a matrix in which the blocks 12 and 21 are interchanged (each of the four blocks is replaced by its transpose). Of course, $C^t_{ij,kl} \equiv C^t_{kl,ij}$ is the ordinary transposition. Similarly, the traces in the first and second spaces are given by
\[ (\text{tr}_{(1)} C)_{jl} = C_{ij,il}, \quad (\text{tr}_{(2)} C)_{ik} = C_{ij,kj}. \] (A.27)
They correspond, respectively, to replacing the $4 \times 4$ matrix $C$ by the $2 \times 2$ matrix resulting from adding its two diagonal boxes or by the $2 \times 2$ matrix obtained by taking the trace of each of its four boxes. If $C = A \otimes B$, $\text{tr}_{(1)} C = (\text{tr} A)B$ and $\text{tr}_{(2)} C = A(\text{tr} B)$.

The action of the permutation matrix $P_{12} \equiv P$ is defined by $(PCP)_{ij,kl} = C_{ji,lk}$ ($P(A \otimes B)P = B \otimes A$ if the entries of $A$ and $B$ commute); thus
\[ (PA_1P)_{ij,kl} = (A_1)_{ji,lk} = A_{jl}\delta_{ik} = (A_2)_{ij,kl} \quad ; \] (A.28)
\[ (PC)_{ij,kl} = C_{ji,kl}, \quad (CP)_{ij,kl} = C_{ij,lk}. \quad \text{Explicitly, } P=P^{-1} \text{ is given by} \]
\[ P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_{ij,kl} = \delta_{il}\delta_{jk} \quad ; \] (A.29)
acting from the left (right) it interchanges the second and third rows (columns).

For $GL_q(2)$ (and $SL_q(2)$), the $R_{12}(q) \equiv R_{12} \equiv R$ and $\mathcal{P}R_{12} \equiv \hat{R}_{12} \equiv \hat{R}$ matrices are given by
\[
R = \begin{bmatrix} q & 0 \\ 1 & \lambda \\ \lambda & 1 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} q & \lambda & 1 \\ 1 & 1 & 0 \end{bmatrix} = \hat{R}^t, \] (A.30)
\[
R_{12}(q^{-1}) = R_{12}^{-1}(q), \quad \hat{R}_{12}^{-1}(q) = \hat{R}_{21}(q^{-1}); \] (A.31)

*Most formulae in this Appendix are also valid for the general $GL_q(n)$ case by setting $i, j, k, ... = 1, 2, ..., n.$
where \( \lambda \equiv q - q^{-1} \); \( \hat{R}_{21} = \mathcal{P} \hat{R}_{12} \mathcal{P} \). In terms of \( \hat{R} \), the RTT equation reads

\[
\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12} , \quad (\hat{R}_{ij,ab} T_{ak} T_{bl} = T_{ic} T_{jd} \hat{R}_{cd,kl}) . \tag{A.32}
\]

Similarly, \( \mathcal{P} R_{12} \mathcal{P} = R_{21} = R_{12}^t \), but the last equality is due to the specific form of \( R_{12} \). \( \hat{R} \) satisfies Hecke’s condition

\[
\hat{R}^2 - \lambda \hat{R} - I = 0 , \quad (\hat{R} - q)(\hat{R} + q^{-1}) = 0 \tag{A.33}
\]

and

\[
\hat{R} = q P_+ - q^{-1} P_- , \quad \hat{R}^{-1} = q^{-1} P_+ - q P_- , \quad [\hat{R}, P_\pm] = 0 , \quad P_\pm \hat{R} P\pm = 0 , \tag{A.34}
\]

where the projectors \( P_{\pm 12} \equiv P_\pm \) (q-(anti)symmetrizer) are given by

\[
P_+ = \frac{1}{[2]} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & [2] \end{bmatrix}, \quad P_- = \frac{1}{[2]} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.35}
\]

with \([2] \equiv (q + q^{-1}) \). It is often convenient to express the 4×4 matrix \( P_- \) in the form

\[
(P_-)_{ij,kl} = \frac{1}{[2]} \epsilon^q_{ij} \epsilon^{-q}_{kl} , \quad ([x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}) \ , \tag{A.36}
\]

where \( \epsilon^q = -(\epsilon^{-q})^{-1} \neq (\epsilon^q)^q \) is given in \([2.14]\). The determinant of an ordinary 2×2 matrix may be defined as the proportionality coefficient in \( (\det T) P_- = P_- T_1 T_2 \) where \( P_- \) is obtained from \([A.33]\) setting \( q=1 \). The analogous definition in the \( q \neq 1 \) case

\[
(\det_q T) P_- := P_- T_1 T_2 , \quad (\det_q T) P_- = T_2^T T_1^T P_- \ , \tag{A.37}
\]

\((\det_q T) = (\det_q T)^\dagger\) leads to the expression for \( \det_q T \) given in \([2.3]\). For the \( K \) matrix, the definition of \( \det_q K \) is given by \([3,14]\).

The \( q \)-trace of a matrix \( B \) is defined by (see \([3,30]\))

\[
tr_q(B) = tr(DB) , \quad D = q^2 tr_{(2)}(\mathcal{P}((R^{(1)})^{t_1})^{-1}) \quad , \tag{A.38}
\]

where the superscript \( t_1 \) means transposition in the first space, the trace \( tr_{(2)} \) is taken in the second space and \( q \) is the deformation parameter in \( R^{(1)} \) (see \([3.2]\) and \([3.9]\)). This \( q \)-trace is invariant under the quantum group coaction \( B \mapsto TBT^{-1} \) (as well as under the coaction \( C \mapsto (T^\dagger)^{-1} CT^\dagger \) since \( R^{(1)} = R^{(4)} t_1 = \mathcal{P} R^{(4)} \mathcal{P} \)). In particular, if \( R^{(1)} = R_{12} \) given in \([A.30]\) the \( q \)-trace of \( B \) is \([3.22]\),

\[
tr_q B = tr(DB) , \quad D = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \quad . \tag{A.39}
\]
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