Efficient tests for experimental quantum gates

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Abstract
Realistic quantum gates operate at non-vanishing noise levels. Therefore, it is necessary to evaluate the performance of each device according to some experimentally observable criteria of device performance. In this presentation, the characteristic properties of quantum operations are discussed and efficient measurement strategies are proposed.

1 Introduction
The technological foundation of quantum information is the control of physical systems at the quantum level. Within the standard theoretical framework, this level of control is represented by pure states and by unitary transformations. However, it is a rather peculiar feature of quantum theory that these elements of the theory cannot be identified directly with the effects actually observed in the laboratory. Instead, experimental observations are connected to the theoretical formalism using the statistical interpretation given by the measurement postulate. This indirect connection between experimental evidence and theoretical interpretation makes it impossible to define the success or failure of a quantum operation in conventional terms. Specifically, the result of an individual measurement cannot be used to identify the quantum state, so it is never possible to tell what the output state of a quantum process "really" was and how the process actually changed the input state. The evaluation of noisy quantum operations is therefore a highly non-trivial task requiring a more detailed analysis of the connection between the quantum formalism and the observable measurement statistics [1, 2, 3, 4].

From the theoretical side, this problem has been addressed by defining mathematical measures of the distances between states and operations in their respective vector spaces. In particular, the effective overlap between two processes can then be given in terms of the process fidelity $F$, obtained from the trace of the process matrix products $|U|^\dagger |U$). However, it is not immediately clear how the process fidelity relates to individual tests of quantum gate performance in the laboratory.

In this presentation, the problem is therefore approached from the opposite viewpoint: first, the quantum operation is defined in terms of observable effects, then the classical fidelities of these properties are related to the overall process fidelity. In particular, it is pointed out that only two classical operations can already provide a good measure of how well a quantum gate works [5]. Possible selections of classical fidelities are considered for the case of the quantum controlled-NOT, and the problem of verifying entanglement generation by local measurements is addressed.

2 Characteristic observable operations and their classical fidelities
In general, the effect of an ideal deterministic operation on any quantum state $|\psi\rangle$ is described by the unitary operator $\hat{U}_0$. However, this unitary operation is actually a very compact summary of many possible operations that can be performed by a quantum device depending on the choice of input states. In order to verify the actual properties of an experimental quantum process, it is therefore necessary to select a representative set of observable operations by choosing appropriate input states and output measurements.

For a set of $N$ distinguishable (= orthogonal) quantum states $\{|n\rangle\}$ of an $N$-level system, the effect of the unitary operation $\hat{U}_0$ is given by

$$\hat{U}_0 |n\rangle = |f_n\rangle.$$  

(1)
This transition from the set of distinguishable states \(| n \rangle\) in the input to the corresponding distinguishable output states \(| f_n \rangle\) is a characteristic observable operation of the quantum device. It can be verified by performing an appropriate von Neumann measurement of the output, with a fidelity of \(F_{n \rightarrow f_n}\) obtained by averaging the probability of obtaining the correct output result \(f_n\) over all inputs \(n\). 

\[
F_{n \rightarrow f_n} = \frac{1}{N} \sum_n p(f_n|n). \tag{2}
\]

Since the operation transforming \(n\) into \(f_n\) can be defined in classical terms, that is, without any reference to the unobserved effects of quantum coherence, it will be referred to as a classical fidelity in the following, to distinguish it from the quantum mechanical concept of fidelity defined as measures in Hilbert space.

In the case of a quantum controlled-NOT, one of the characteristic observable operations is the classical-controlled-NOT operation observed in the computational basis,

\[
\hat{U}\text{CNOT} | 0_z;0_z \rangle = | 0_z;0_z \rangle \\
\hat{U}\text{CNOT} | 0_z;1_z \rangle = | 0_z;1_z \rangle \\
\hat{U}\text{CNOT} | 1_z;0_z \rangle = | 1_z;1_z \rangle \\
\hat{U}\text{CNOT} | 1_z;1_z \rangle = | 1_z;0_z \rangle. \tag{3}
\]

The classical fidelity of this operation, \(F_{zz \rightarrow zz}\), is obtained by averaging the probability of obtaining the correct measurement outcome in the \(Z\)-basis over the four \(Z\)-basis input states,

\[
F_{zz \rightarrow zz} = \frac{1}{4} (p_{zz|zz}(00|00) + p_{zz|zz}(01|01) + p_{zz|zz}(11|10) + p_{zz|zz}(10|11)).\tag{4}
\]

However, the quantum coherence of the gate implies that a characteristic observable operation exists for any choice of input basis. In the case of a two qubit gate such as the quantum controlled-NOT, it is therefore interesting to consider input states taken from the \(X\) and \(Y\) bases as well. It is then possible to define a set of nine characteristic observable operations of the two qubit gate.

An overview over the characteristic operations for the quantum controlled-NOT is shown in table 1. Like all unitary operations, the quantum controlled-NOT has a set of eigenstates for which the characteristic operation is the identity operation ("identity" in the table). As can be seen from the table, this set of eigenstates is the \(ZX\) basis. The regular controlled-NOT operation ("CNOT" in the table) is observable in both the \(ZZ\) and the \(ZY\) basis, since the conditional spin flip of the target qubit corresponds to a rotation around the \(X\) axis. If the target qubit input is an eigenstate of \(X\), its eigenvalue is preserved and the effect on the control bit is a conditional rotation around the \(Z\) axis. Thus, the gate performs a controlled-NOT operations with reversed roles of the target and the control ("reverse CNOT" in the table), e.g. for the \(XX\) basis,

\[
\hat{U}\text{CNOT} | 0_x;0_x \rangle = | 0_x;0_x \rangle \\
\hat{U}\text{CNOT} | 0_x;1_x \rangle = | 1_x;1_x \rangle \\
\hat{U}\text{CNOT} | 1_x;0_x \rangle = | 1_x;0_x \rangle \\
\hat{U}\text{CNOT} | 1_x;1_x \rangle = | 0_x;1_x \rangle. \tag{5}
\]

The same observable operation is obtained for the \(YX\) basis. Finally, the entanglement generating functions of the gate are described by the observable operations on input states of the \(XY\), \(XZ\), \(YX\) and \(YZ\) bases ("entangle" in the table). In these four cases, the output states form a complete Bell basis of four orthogonal maximally entangled states each, e.g. for the \(XZ\) basis,

\[
\hat{U}\text{CNOT} | 0_x;0_x \rangle = \frac{1}{\sqrt{2}} (| 0_x;0_x \rangle + | 1_x;1_x \rangle) \\
\hat{U}\text{CNOT} | 0_x;1_x \rangle = \frac{1}{\sqrt{2}} (| 0_x;1_x \rangle + | 1_x;0_x \rangle) \\
\hat{U}\text{CNOT} | 1_x;0_x \rangle = \frac{1}{\sqrt{2}} (| 0_x;0_x \rangle - | 1_x;1_x \rangle) \\
\hat{U}\text{CNOT} | 1_x;1_x \rangle = \frac{1}{\sqrt{2}} (| 0_x;1_x \rangle - | 1_x;0_x \rangle). \tag{6}
\]

It should be noted that the direct experimental verification of this operation requires a non-local Bell measurement of the two qubits. Thus, characteristic observable operations are not necessarily local, and it may be useful to distinguish classical local fidelities such as the identity or the controlled-NOT operations from the classical non-local fidelities of entanglement generation. Obviously, the use of the term "classical" in this context is merely based on the absence of coherent superpositions due to the restriction of the operation to well-defined basis sets in the input and the output, not on any problems of separability.

In order to obtain a classical non-local fidelity by local measurements, it is necessary to verify the correlations of the output spin components \(X\), \(Y\), and \(Z\) in three separate measurements. Each of these measurements corresponds to a classical local fidelity. However, since only the correlation is verified, there are two correct outcomes for each input, e.g. in the
Table 1: Characteristic operations of the quantum controlled-NOT for input states taken from the X, Y, and Z bases. See the text for a more detailed explanation of the four types of operations indicated above.

| Target Input | Control Input |
|--------------|---------------|
| X-basis      | reverse CNOT  |
| Y-basis      | reverse CNOT  |
| Z-basis      | identity      |

For the case given by equation (6) above,

\[
F_{xz\rightarrow zz} = \frac{1}{4} (p_{xz|zz}(00|00) + p_{xz|zz}(00|11) + p_{xz|zz}(01|01) + p_{xz|zz}(01|10) + p_{xz|zz}(10|00) + p_{xz|zz}(10|11) + p_{xz|zz}(11|01) + p_{xz|zz}(11|10)) \tag{7}
\]

Using the classical local fidelities for XZ to ZZ, XZ to XX, and XZ to YY, the total non-local fidelity \( F_{xz\rightarrow ent} \) for the characteristic observable operation given in equation (6) can be determined by

\[
F_{xz\rightarrow ent} = \frac{1}{2} (F_{xz\rightarrow xx} + F_{xz\rightarrow yy} + F_{xz\rightarrow zz} - 1) \tag{8}
\]

Entanglement generation can thus be verified by a set of three local fidelities.

### 3 Complementary operations and process fidelity estimates

As the discussion above illustrates, it is possible to characterize the complete quantum coherent operation of a gate in terms of a set of classical operations defined by different input state selections. The effects of quantum coherence are then expressed entirely in terms of directly observable classical input-output relations. In fact, it is possible to uniquely identify a specific unitary transformation using only two characteristic observable operations. The condition for selecting these two operations is that each input state of operation A must overlap with each input state of operation B. The precise output states of operation B then depend on the phase relations between the output states of operation A, allowing a complete test of the quantum coherence described by \( \hat{U}_0 \).

Optimal sensitivity to quantum coherent effects is obtained if the squared overlap of the input states \(| n \rangle \) and the input states \(| k \rangle \) is equal to 1/N for any combination of \( n \) and \( k \), e.g.,

\[
| k \rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp[-i \frac{2\pi}{N} kn] | n \rangle. \tag{9}
\]

The output states of the characteristic operation defined by the input basis \(| k \rangle \) are then given by

\[
| g_k \rangle = \hat{U}_0 | k \rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp[-i \frac{2\pi}{N} kn] | f_n \rangle. \tag{10}
\]

Any phase error in the output state components \(| f_n \rangle \) will reduce the fidelity of this output. With respect to the output components \(| f_n \rangle \), the observable operation \( n \rightarrow f_n \) thus tests the output amplitudes, while the observable operation \( k \rightarrow g_k \) tests the phases. Since the unitary operation \( \hat{U}_0 \) is completely defined by equation (6), a verification of both the amplitudes and the phases of the output components \(| f_n \rangle \) constitutes unambiguous proof that the operation performed is actually \( \hat{U}_0 \).

As has been discussed in more detail elsewhere [5], it is possible to obtain a reliable estimate of the overall process fidelity \( F_{process} \), defined as the overlap of the ideal process matrix with the process matrix of the experimental realization [3,4], using only the two complementary classical fidelities \( F_n \) and \( F_k \),

\[
F_{process} \leq \text{Min}\{F_n, F_k\} \quad F_{process} \geq F_n + F_k - 1. \tag{11}
\]

This result can be explained quite intuitively if one assumes that the process fidelity corresponds to the probability of performing the correct quantum operation. Since performance of the correct quantum operation automatically implies that every characteristic operation is correctly performed, all classical fidelities must be equal to or larger than the process fidelity. However, the classical fidelities also include the probabilities of error processes. Such processes
can reliably perform one of the two complementary processes, thus contributing to either $F_n$ or $F_k$. However, they cannot perform both operations at once, so the maximal average fidelity for error processes is $(F_n + F_k)/2 = 1/2$. The minimal process fidelity is thus obtained by assuming that all error processes have this maximal average fidelity of 1/2.

It is now possible to apply this evaluation to the characteristic operations of the quantum controlled-NOT gate shown in table I. Complementary pairs are obtained by choosing any pair of operations that has a different input input basis for both the control and the target (that is, any pair that is neither in the same line nor in the same column of the table). Experimentally, it may be most convenient to choose a pair of complementary operations that can be verified by local measurements. This condition is fulfilled by pairs consisting of a controlled-NOT and a reverse controlled-NOT operation, i.e. the operation in the computational basis $ZZ$ and the operation in the $XX$ basis. For this example, the estimate of the process fidelity is

$$F_{\text{process}} \leq \min \{F_{zz \rightarrow zz}, F_{xx \rightarrow xx}\}$$

$$F_{\text{process}} \geq F_{zz \rightarrow zz} + F_{xx \rightarrow xx} - 1.$$ (12)

It is thus possible to evaluate the performance of a quantum controlled-NOT by obtaining the fidelities of two classical controlled-NOT operations performed by the gate.

Since it is usually not too difficult to change the local settings for input and output states, the fidelity estimate can be optimized by measuring the classical fidelities of all four controlled-NOT and reverse controlled-NOT operations. The best estimate is then obtained by using the lowest fidelity for the lower bound and the highest pair of fidelities for the upper bound,

$$F_{\text{process}} \leq \min \{F_{zz \rightarrow zz}, F_{xx \rightarrow xx}, F_{zy \rightarrow yz}, F_{yx \rightarrow yx}\}$$

$$F_{\text{process}} \geq \max \{F_{zz \rightarrow zz}, F_{zy \rightarrow yz}\} + \max \{F_{xx \rightarrow xx}, F_{yx \rightarrow yx}\} - 1.$$ (13)

It is thus possible to obtain a fairly reliable evaluation of the quantum controlled-NOT from local measurements of its four classical controlled-NOT operations.

## 4 Classical fidelities and entanglement capabilities

One of the applications of the process fidelity is to provide a lower limit for any classical fidelity of the quantum device. A lower bound of the process fidelity is therefore also a lower bound for the fidelities of all characteristic operation of the device. The bound obtained for a pair of complementary characteristic operations $n$ and $k$ can thus be applied directly to predict the minimal fidelity of any other characteristic operation $l$. According to equation (14), this lower bound for all classical fidelities $F_l$ then reads

$$F_l \geq F_n + F_k - 1.$$ (14)

The measurement of only two complementary fidelities thus provides an estimate for the fidelities of all other characteristic operations that the device could perform.

In the case of a quantum controlled-NOT, this means that an estimate for the fidelities of entanglement generation can be obtained by measuring only the local controlled-NOT and reversed controlled-NOT operations. The entanglement capability of the gate can thus be verified without actually generating entanglement. For example, measurements of the controlled-NOT operation in the computational basis $ZZ$ and of the reverse controlled-NOT operation in the $XX$ basis can be used to estimate the potential for generating entanglement from inputs in the $YY$ basis, since

$$F_{yy \rightarrow \text{ent.}} \geq F_{zz \rightarrow zz} + F_{xx \rightarrow xx} - 1.$$ (15)

Of course the same estimate applies to the fidelities of entanglement generation from $XZ$, $YZ$, and $XY$ inputs. A more precise estimate can be obtained from the four classical local fidelities of the quantum controlled-NOT according to equation (16). This estimate reads

$$F_{ij \rightarrow \text{ent.}} \geq \max \{F_{zz \rightarrow zz}, F_{zy \rightarrow yz}\} + \max \{F_{xx \rightarrow xx}, F_{yx \rightarrow yx}\} - 1,$$ (16)

where the indices $ij$ indicate the four entanglement generating input state selections $XZ$, $YZ$, $XY$, and $YY$.

Since the fidelities $F_{ij \rightarrow \text{ent.}}$ determine the minimal probability of successfully generating a maximally entangled state, it is possible to estimate the minimal amount of entanglement that this fidelity can generate. This estimate is based on the observation that the concurrence $C$ measuring the entanglement of a mixed state $\hat{\rho}$ can be estimated from the fidelity of a maximally entangled state $|E_{\text{max}}\rangle$ using

$$C \geq 2(E_{\text{max}} | \hat{\rho} | E_{\text{max}}) - 1.$$ (17)

Since the minimal fidelity of obtaining a maximally entangled state in the output of each entanglement
generating operation $ij$ is equal to the classical non-local fidelity $F_{ij\rightarrow \text{ent.}}$ of the operation, the corresponding estimate for the concurrence $C_{\text{gate}}$ defining the entanglement capability of the gate reads

$$C_{\text{gate}} \geq 2F_{ij\rightarrow \text{ent.}} - 1. \quad (18)$$

With this estimate, the lower bound for the fidelity of entanglement generation given by equation (16) translates into a lower bound for the entanglement capability given by

$$C_{\text{gate}} \geq 2\max\{F_{zz\rightarrow zz}, F_{zy\rightarrow zy}\} + 2\max\{F_{xz\rightarrow zx}, F_{yx\rightarrow yx}\} - 3,$$  

(19)

that is, the quantum gate can definitely generate entanglement if the average of the maximal controlled-NOT fidelity and the maximal reverse controlled-NOT fidelity is greater than $3/4$. In other words, it is completely impossible that an experimental device performs the classical local controlled-NOT and reverse controlled-NOT operations at fidelities greater than 75% without also generating corresponding amounts of entanglement when operated in any of the four entanglement generating operations.

### 5 Conclusions

In order to evaluate the performance of experimental quantum gate operations, the essential properties of quantum gates have to be defined in experimentally accessible terms. This goal can be achieved by selecting representative sets of characteristic observable operations. Each characteristic observable operation is given by a set of distinguishable input states and their expected distinguishable outputs. It is then possible to measure the fidelity of each characteristic operation by comparing the actual outputs with the expected ones, in close analogy to tests conventionally performed on classical devices.

For the quantum controlled-NOT, a representative set of characteristic operations is shown in table 1. These characteristic operations include the identity operation, four classical controlled-NOT operations, and four entanglement generating operations. The latter operations cannot be verified in a single local measurement, so it may be necessary to define their fidelities in terms of three separate local fidelities for the generation of correlations in $X$, $Y$, and $Z$.

The classical fidelities obtained for characteristic observable operations allow estimates of the general quantum properties of the gate. In particular, a lower bound for both the process fidelity and the entanglement capability of an experimental gate can already be obtained from only two complementary classical fidelities. In the case of experimental quantum controlled-NOT gates, it is therefore possible to predict the amount of entanglement that a gate can generate from the fidelities of two entirely local classical controlled-NOT operations.

In conclusion, representative sets of characteristic observable operations provide experimentally accessible evidence for the successful implementation of quantum coherent gate operations. Reliable estimates of the overall process fidelity and the entanglement capability of the gate can be obtained efficiently by combining the fidelities of two or more characteristic operations. It is thus possible to perform efficient tests of quantum gates using only a small number of well-defined measurements.

### References

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