A TRACE FORMULA FOR RIGID VARIETIES, AND MOTIVIC WEIL GENERATING SERIES FOR FORMAL SCHEMES

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Abstract. We establish a trace formula for rigid varieties $X$ over a complete discretely valued field, which relates the set of unramified points on $X$ to the Galois action on its étale cohomology. We develop a theory of motivic integration for formal schemes of pseudo-finite type over a complete discrete valuation ring $R$, and we introduce the Weil generating series of a regular formal $R$-scheme $\mathfrak{X}$ of pseudo-finite type, via the construction of a Gelfand-Leray form on its generic fiber. Our trace formula yields a cohomological interpretation of this Weil generating series.

When $X$ is the formal completion of a morphism $f$ from a smooth irreducible variety to the affine line, then its Weil generating series coincides (modulo normalization) with the motivic zeta function of $f$. When $X$ is the formal completion of $f$ at a closed point $x$ of the special fiber $f^{-1}(0)$, we obtain the local motivic zeta function of $f$ at $x$. In the latter case, the generic fiber of $\mathfrak{X}$ is the so-called analytic Milnor fiber of $f$ at $x$; we show that it completely determines the formal germ of $f$ at $x$.

1. Introduction

Let $R$ be a complete discrete valuation ring, and denote by $k$ and $K$ its residue field, resp. its field of fractions. We assume that $k$ is perfect. We fix a uniformizing parameter $\pi$ in $R$ and a separable closure $K^s$ of $K$, and we denote by $K^t$ and $K^{sh}$ the tame closure, resp. strict henselization of $K$ in $K^s$.

The main goal of the present paper is to establish a broad generalization of the trace formula in [31, 5.4], to the class of special formal $R$-schemes (i.e. separated Noetherian formal $R$-schemes $\mathfrak{X}$ such that $\mathfrak{X}/J$ is of finite type over $R$ for each ideal of definition $J$; these are also called formal $R$-schemes of pseudo-finite type). For any special formal $R$-scheme $\mathfrak{X}$, Berthelot constructed in [8, 0.2.6] its generic fiber $\mathfrak{X}_\eta$, which is a rigid variety over $K$, not quasi-compact in general. If $\mathfrak{X}_\eta$ is smooth over $K$, our trace formula relates the set of unramified points on $\mathfrak{X}_\eta$ to the Galois action on the étale cohomology of $\mathfrak{X}_\eta$, under a suitable tameness condition.

The set of unramified points on $\mathfrak{X}_\eta$ is infinite, in general, but it can be measured by means of the motivic Serre invariant, first introduced in [29] and further refined in [32]. A priori, this motivic Serre invariant is only defined if $\mathfrak{X}_\eta$ is quasi-compact. However, using dilatations, we show that there exists an open quasi-compact rigid subvariety $X$ of $\mathfrak{X}_\eta$ such that $X(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$. Moreover, the motivic Serre invariant of $X$ depends only on $\mathfrak{X}_\eta$, and can be used to define the motivic Serre invariant of $\mathfrak{X}_\eta$. If we denote by $\mathfrak{X}_0$ the underlying $k$-variety of $\mathfrak{X}$ (i.e. the closed subscheme of $\mathfrak{X}$ defined by the largest ideal of definition), then this motivic Serre invariant takes values in a certain quotient of the Grothendieck ring.

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etale cohomology were studied in [31]. In particular, if $k$ is a characteristic zero, we associate to any generically smooth special formal series coincides with Denef and Loeser’s motivic zeta function associated to special formal $R$-scheme and we extend the constructions and result in [31] to this setting. In particular, if $f: X \to \text{Spec } K$ is a regular special formal $R$-scheme of pure relative dimension $m$, we associate to any continuous differential form $\omega$ in $\Omega^m_{X/k}(X)$ its Gelfand-Leray form $\omega/d\pi$, which is a section of $\Omega^m_{\hat{X}/\hat{K}}(\hat{X}_x)$. This construction allows us to define for any regular special formal $R$-scheme $X$ its Weil generating series, a formal power series over the localized Grothendieck ring $M_{\hat{X}_x}$, whose degree $d$ coefficient measures the set of unramified points on $\hat{X}_x \times_K K(d)$. Here $K(d)$ denotes the totally ramified extension $K((\sqrt[d]{\pi}))$ of $K$; the Weil generating series depends on the choice of $\pi$ if $k$ is not algebraically closed.

If $X$ is a smooth irreducible $k$-variety, endowed with a dominant morphism $f: X \to \text{Spec } K$, then we denote by $\hat{X}$ its $\pi$-adic completion. It is a regular special formal $R$-scheme. We show that, modulo normalization, its Weil generating series coincides with Denef and Loeser’s motivic zeta function associated to $f$, and its motivic volume coincides with the motivic nearby cycles [21].

Finally, we study the analytic Milnor fiber $\mathcal{F}_x$ of $f$ at a closed point $x$ of the special fiber $X_s = f^{-1}(0)$. This object was introduced in [30], and its points and étale cohomology were studied in [31]. In particular, if $k = \mathbb{C}$, the étale cohomology of $\mathcal{F}_x$ coincides with the singular cohomology of the topological Milnor fiber of $f$ at $x$, and the Galois action corresponds to the monodromy. We will show that $\mathcal{F}_x$ completely determines the formal germ of $f$ at $x$: it determines the completed local ring $\hat{O}_{X,x}$ with its $R$-algebra structure induced by the morphism $f$. The formal spectrum $\mathcal{X}_x := \text{Spf } \hat{O}_{X,x}$ is a regular special formal $R$-scheme, and its generic fiber is precisely $\mathcal{F}_x$. The Weil generating series of $\mathcal{X}_x$ coincides (modulo normalization) with the local motivic zeta function of $f$ at $x$, and its motivic volume with Denef and Loeser’s motivic Milnor fiber.

To conclude the introduction, we give a survey of the structure of the paper. In Section 2 we study the basic properties and constructions for special formal $R$-schemes: generic fibers, formal blow-ups, dilatations, and resolution of singularities. We also encounter an important technical complication w.r.t. the finite type-case: if $\mathcal{X}$ is a special formal $R$-scheme and $\omega$ is a differential form on $\mathcal{X}_x$, there does not necessarily exist an integer $a > 0$ such that $\pi^a \omega$ is defined on $\mathcal{X}$. We call forms which (locally on $\mathcal{X}$) have this property, $\mathcal{X}$-bounded differential forms; this notion is important for what follows.

In Section 3 we compute the $\ell$-adic tame nearby cycles on a regular special formal $R$-scheme $\mathcal{X}$ whose special fiber is a tame strict normal crossings divisor (Proposition 3.2). We prove that any such formal scheme $\mathcal{X}$ admits an algebraizable étale cover (Proposition 3.3), and then we use Grothendieck’s description of the nearby cycles in the algebraic case [1 Exp. I].
We generalize the theory of motivic integration to the class of special formal $R$-schemes in Section 4. In fact, using dilatations, we construct appropriate models which are topologically of finite type over $R$, and for which the theory of motivic integration was developed in [35] and [29]. Of course, we have to show that the result does not depend on the chosen model. In particular, we associate a motivic Serre invariant to any generically smooth special formal $R$-scheme $X$ (Definition 4.5) and we show that it can be computed on a Néron smoothening (Corollary 4.14).

In Section 5, we construct weak Néron models for tame ramifications of regular special formal $R$-schemes whose special fiber is a strict normal crossings divisor (Theorem 5.1) and we obtain a formula for the motivic Serre invariants of these ramifications. We define the order of a bounded gauge form $\omega$ on the generic fiber $X_\eta$ of a smooth special formal $R$-scheme $X$ along the connected components of $X_0$, and we deduce an expression for the motivic integral of $\omega$ on $X$ (Proposition 5.14).

The trace formula is stated and proven in Section 6.4 (Theorem 6.4). It uses the computation of the motivic Serre invariants of tame ramifications in Section 5, the computation of the tame nearby cycles in Section 3, and Laumon’s result on equality of $\ell$-adic Euler characteristics with and without proper support [28].

In Section 7, we consider regular special formal $R$-schemes $X$ whose special fibers $X_s$ are strict normal crossings divisors, where $\text{char}(k) = 0$. We define the order of a bounded gauge form $\omega$ on the generic fiber $X_\eta$ along the components of $X_s$, and we use this notion to compute the motivic integral of $\omega$ on all the totally ramified extensions of $X_\eta$ (Definition 7.35). In Section 7.3 we introduce the Gelfand-Leray form $\omega/d\pi$ associated to a top differential form $\omega$ on $X$ over $k$ (Definition 7.21), using the fact that the wedge product with $d\pi$ defines an isomorphism between $\Omega^m_{X_\eta/K}$ and $(\Omega^{m+1}_{X/k})_{\text{rig}}$ if $X$ is a generically smooth special formal $R$-scheme of pure relative dimension $m$ (Proposition 7.19). If $X$ is regular and $\omega$ is a gauge form, then $\omega/d\pi$ is a bounded gauge form on $X_\eta$ and the volume Poincaré series of $(X, \omega/d\pi)$ depends only on $X$, and not on $\omega$: we get an explicit expression in terms of any resolution of singularities (Proposition 7.30). We call this series the Weil generating series associated to $X$.

Now let $X$ be a smooth, irreducible variety over $k$, endowed with a dominant morphism $X \to \text{Spec} k[t]$, and denote by $\hat{X}$ its $t$-adic completion. We show in Section 8 that the analytic Milnor fiber of $f$ at a closed point $x$ of the special fiber $X_s = f^{-1}(0)$ completely determines the formal germ of $f$ at $x$ (Proposition 8.7). Finally, in Section 9, we prove that the Weil generating series of $\hat{X}$ and $F_x$ coincide, modulo normalization, with the motivic zeta function of $f$, resp. the local motivic zeta function of $f$ at $x$ (Theorem 9.5 and Corollary 9.6), and we show that the motivic volumes of $\hat{X}$ and $F_x$ correspond to Denef and Loeser’s motivic nearby cycles and motivic Milnor fiber (Theorem 9.7). This refines the comparison results in [31].

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Notation and conventions. Throughout this article, $R$ denotes a complete discrete valuation ring, with residue field $k$ and quotient field $K$, and we fix a uniformizing parameter $\pi$. Some of the constructions require that $k$ is perfect or that $k$ has characteristic zero; this will be indicated at the beginning of the section. For any field $F$, we denote by $F^s$ a separable closure. We denote by $K^{sh}$ the strict henselization of $K$, by $R^{sh}$ the normalization of $R$ in $K^{sh}$, and by $\hat{K}$ the tame closure of $K$ in $K^s$. We denote by $\hat{R}$ and $\hat{\mathcal{X}}$ the completions of $K^{\text{sh}}$ and $K^s$. We denote by $p$ the characteristic exponent of $k$, and we fix a prime $\ell$ invertible in $k$. We say that $R'$ is a finite extension of $R$ if $R'$ is the normalization of $R$ in a finite field extension $K'$ of $K$.

For any integer $d > 0$ prime to the characteristic exponent of $k$, we put $K(d) := K[T]/(T^d - \pi)$. This is a totally ramified extension of degree $d$ of $K$. We denote by $R(d)$ the normalization $R[T]/(T^d - \pi)$ of $R$ in $K(d)$. For any formal $R$-scheme $\mathcal{X}$ and any rigid $K$-variety $X$, we put $\mathcal{X}(d) := \mathcal{X} \times_R R(d)$ and $X(d) := X \times_K K(d)$. Moreover, we put $\mathcal{X} := X \hat{\times}_K \hat{\mathcal{X}}$.

If $S$ is a scheme, we denote by $S_{\text{red}}$ the underlying reduced scheme. A $S$-variety is a separated reduced $S$-scheme of finite type.

If $F$ is a field of characteristic exponent $p$, $\ell$ is a prime invertible in $F$, and $S$ is a variety over $F$, then we say an étale covering $T \to S$ is tame if, for each connected component $S_i$ of $S$, the degree of the étale covering $T \times_S S_i \to S_i$ is prime to the characteristic exponent $p$ of $F$. We call a $\mathbb{Q}_\ell$-adic sheaf $\mathcal{F}$ on $S$ tamely lisse if its restriction to each connected component $S_i$ of $S$ corresponds to a finite dimensional continuous representation of the prime-to-$p$ quotient $\pi_1(S_i, s)^{p'}$, where $s$ is a geometric point of $S_i$. We call a $\mathbb{Q}_\ell$-adic sheaf $\mathcal{F}$ on $S$ tamely constructible if there exists a finite stratification of $S$ into locally closed subsets $S_i$, such that the restriction of $\mathcal{F}$ to each $S_i$ is tamely lisse. If $M$ is a torsion ring with torsion orders prime to $p$, then tamely lisse and tamely constructible sheaves of $M$-modules on $S$ are defined in the same way.

If $\mathcal{X}$ is a Noetherian adic formal scheme and $Z$ is a closed subscheme (defined by an open coherent ideal sheaf on $\mathcal{X}$) we denote by $\mathcal{X}/Z$ the formal completion of $\mathcal{X}$ along $Z$. If $\mathcal{N}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module, we denote by $\mathcal{N}/Z$ the induced coherent $\mathcal{O}_{\mathcal{X}/Z}$-module. We embed the category of Noetherian schemes into the category of Noetherian adic formal schemes by endowing their structure sheaves with the discrete (i.e. $(0)$-adic) topology. If $\mathcal{X}$ is a Noetherian adic formal scheme and $\mathcal{J}$ a coherent ideal sheaf on $\mathcal{X}$, we’ll write $V(\mathcal{J})$ for the closed formal subscheme of $\mathcal{X}$ defined by $\mathcal{J}$.

For the theory of stft formal $R$-schemes (stft=separated and topologically of finite type) and the definition of the Grothendieck ring of varieties $K_0(\text{Var}_Z)$ over a separated scheme $Z$ of finite type over $k$, we refer to [31]. Let us only recall that $\mathbb{L}$ denotes the class of the affine line $\mathbb{A}^1_Z$ in $K_0(\text{Var}_Z)$, and that $\mathcal{M}_Z$ denotes the localized Grothendieck ring $K_0(\text{Var}_Z)[L^{-1}]$. The topological Euler characteristic

$$\chi_{\text{top}}(X) := \sum_{i \geq 0} (-1)^i \dim H^i(X \times_k k^s, \mathbb{Q}_{\ell})$$

induces a group morphism $\chi_{\text{top}} : \mathcal{M}_Z \to \mathbb{Z}$. The definition of the completed localized Grothendieck ring $\hat{\mathcal{M}}_Z$ is recalled in [32 §4.1].

If $V = \oplus_{i \in \mathbb{Z}} V_i$ is a graded vector space over a field $F$, such that $V_i = 0$ for all but a finite number of $i \in \mathbb{Z}$ and such that $V_i$ is finite dimensional over $F$ for all $i$,
and if $M$ is a graded endomorphism of $V$, then we define its trace and zeta function by

$$\text{Tr}(M|V) := \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(M|V_i) \in F$$

$$\zeta(M|V; T) := \prod_{i \in \mathbb{Z}} (\det(Id - TM|V_i))^{(-1)^{i+1}} \in F(T)$$

Finally, if $X$ is a rigid $K$-variety, we put $H(X) := \bigoplus_{i \geq 0} H^i(X, \mathbb{Q}_\ell)$ where the cohomology on the right is Berkovich’ $\ell$-adic étale cohomology [5].

2. Special formal schemes

We recall the following definition: if $A$ is an adic topological ring with ideal of definition $J$, then the algebra of convergent power series over $A$ in the variables $(x_1, \ldots, x_n)$ is given by

$$A\{x_1, \ldots, x_n\} := \lim_{n \to \infty} (A/J^n)[[x_1, \ldots, x_n]]$$

**Definition 2.1.** Let $X$ be a Noetherian adic formal scheme, and let $J$ be its largest ideal of definition. The closed subscheme of $X$ defined by $J$ is denoted by $X_0$, and is called the reduction of $X$. It is a reduced Noetherian scheme.

This construction defines a functor $(\cdot)_0$ from the category of Noetherian adic formal schemes to the category of reduced Noetherian schemes. Note that the natural closed immersion $X_0 \to X$ is a homeomorphism.

**Definition 2.2** (Special formal schemes [7], §1). A topological $R$-algebra $A$ is called special, if $A$ is a Noetherian adic ring and, for some ideal of definition $J$, the $R$-algebra $A/J$ is finitely generated.

A special formal $R$-scheme is a separated Noetherian adic formal scheme $X$ endowed with a structural morphism $X \to \text{Spf} R$, such that $X$ is a finite union of open formal subschemes which are formal spectra of special $R$-algebras. In particular, $X_0$ is a separated scheme of finite type over $k$.

We denote by $X_s$ the special fiber $X \times_R k$ of $X$. It is a formal scheme over $\text{Spec } k$. If $X$ is stft over $R$, then $X_s$ is a separated $k$-scheme of finite type, and $X_0 = (X_s)_{\text{red}}$.

Note that our terminology is slightly different from the one in [7, §1], since we impose the additional quasi-compactness condition.

Special formal schemes are called formal schemes of pseudo-finite type over $R$ in [4]. We adopt their definitions of étale, adic étale, smooth, and adic smooth morphisms [4, 2.6]. If $X$ is a special formal $R$-scheme, we denote by $\text{Sm}(X)$ the open formal subscheme where the structural morphism $X \to \text{Spf } R$ is smooth.

Berkovich shows in [7, 1.2] that a topological $R$-algebra $A$ is special, iff $A$ is topologically $R$-isomorphic to a quotient of the special $R$-algebra

$$R\{T_1, \ldots, T_m\}[[S_1, \ldots, S_n]] = R[[S_1, \ldots, S_n]][T_1, \ldots, T_m]$$

It follows from [37, 38] that special $R$-algebras are excellent, as is observed in [15, p.476].

Any stft formal $R$-scheme is special. Note that a special formal $R$-scheme is stft over $R$ iff it is $R$-adic. If $X$ is a special formal $R$-scheme, and $Z$ is a closed subscheme of $X_0$, then the formal completion $\mathcal{X}/\mathcal{Z}$ of $X$ along $Z$ is special.
We say that a special formal $R$-scheme $\mathfrak{X}$ is algebraizable, if $\mathfrak{X}$ is isomorphic to the formal completion of a separated $R$-scheme $X$ of finite type along a closed subscheme $Z$ of its special fiber $X_s$. In this case, we say that $X/Z$ is an algebraic model for $\mathfrak{X}$. If $\mathfrak{X}$ is stft over $R$ and $Z = X_s$ (i.e. $\mathfrak{X}$ is isomorphic to the $\pi$-adic completion $\hat{X}$ of $X$), then we simply say that $X$ is an algebraic model for $\mathfrak{X}$. Finally, if $\hat{Z}$ is the $\pi$-adic completion of a coherent $\mathcal{O}_X$-module $\mathcal{I}$, we say that $(X, \mathcal{I})$ is an algebraic model for $(\mathfrak{X}, \hat{Z})$.

If $\mathfrak{X}$ is a special formal $R$-scheme, $R'$ is a finite extension of $R$, and $\psi$ is a section in $\mathfrak{X}(R')$, then we denote by $\psi(0)$ the image of $\text{Spf} R'$ in $\mathfrak{X}_0$.

2.1. The generic fiber of a special formal scheme. Berthelot explains in [8 0.2.6] how to associate a generic fiber $\mathfrak{X}_g$ to a special formal $R$-scheme $\mathfrak{X}$ (see also [16 §7]). This generic fiber $\mathfrak{X}_g$ is a separated rigid variety over $K$, not quasi-compact in general, and is endowed with a canonical morphism of ringed sites $sp : \mathfrak{X}_g \to \mathfrak{X}$ (the specialization map). This construction yields a functor $(\cdot)_g$ from the category of special formal $R$-schemes, to the category of separated rigid $K$-varieties. We say that $\mathfrak{X}$ is generically smooth if $\mathfrak{X}_g$ is smooth over $K$.

If $Z$ is a closed subscheme of $\mathfrak{X}_0$, then $sp^{-1}(Z)$ is an open rigid subvariety of $\mathfrak{X}_g$, canonically isomorphic to $(\mathfrak{X}/Z)_g$ by [8 0.2.7]. We call it the tube of $Z$ in $\mathfrak{X}$, and denote it by $\lceil Z \rceil$.

We recall the construction of $\mathfrak{X}_g$ in the case where $\mathfrak{X} = \text{Spf} A$ is affine, with $A$ a special $R$-algebra, following [16 7.1]. The notation introduced here will be used throughout the article.

Let $J$ be the largest ideal of definition of $A$. For each integer $n > 0$, we denote by $A[J^n/\pi]$ the subalgebra of $A \otimes_R K$ generated by $A$ and the elements $j/\pi$ with $j \in J^n$. We denote by $B_n$ the $J$-adic completion of $A[J^n/\pi]$ (this is also the $\pi$-adic completion), and we put $C_n = B_n \otimes_R K$. Then $C_n$ is an affinoid algebra, the natural map $C_{n+1} \to C_n$ induces an open embedding of affinoid spaces $\text{Sp} C_n \to \text{Sp} C_{n+1}$, and by construction, $\mathfrak{X}_g = \cup_{n > 0} \text{Sp} C_n$.

For each $n > 0$, there is a natural ring morphism $A \otimes_R K \to C_n$ which is flat by [16 7.1.2]. These morphisms induce a natural ring morphism

$$i : A \otimes_R K \to \mathcal{O}_{\mathfrak{X}_g}(\mathfrak{X}_g) = \cap_{n > 0} C_n$$

Definition 2.3 ([27],2.3). A rigid variety $X$ over $K$ is called a quasi-Stein space if there exists an admissible covering of $X$ by affinoid opens $X_1 \subset X_2 \subset \ldots$ such that $\mathcal{O}_X(X_{n+1}) \to \mathcal{O}_X(X_n)$ has dense image for all $n \geq 1$.

A crucial feature of a quasi-Stein space $X$ is that $H^i(X, \mathcal{F})$ vanishes for $i > 0$ if $\mathcal{F}$ is a coherent sheaf on $X$, i.e. the global section functor is exact on coherent modules on $X$. This is Kiehl’s “Theorem B” for rigid quasi-Stein spaces [27 2.4].

Proposition 2.4. If $\mathfrak{X} = \text{Spf} A$ is an affine special formal $R$-scheme, then $\mathfrak{X}_g$ is a quasi-Stein space.

Proof. Let $J$ be the largest ideal of definition in $A$. Put $X_n = \text{Sp} C_n$; then

$$X_1 \subset X_2 \subset \ldots$$

is an affinoid cover of $\mathfrak{X}_g$. Fix an integer $n > 0$, and let \{ $g_1, \ldots, g_s$ \} be a set of generators of the ideal $J^n$ in $A$. Then by construction, $X_n$ consists exactly of the
points $x$ in $X_n$ such that $|(g_j/\pi)(x)| \leq 1$ for $j = 1, \ldots, s$, by the isomorphism [16, 7.1.2]

$$C_n \cong C_{n+1}(\{T_1, \ldots, T_s\}/(g_1 - \pi T_1, \ldots, g_s - \pi T_s))$$

As Kiehl observes right after Definition 2.3 in [27], this implies that $X_n$ is quasi-Stein. □

Let $A$ be a special $R$-algebra, and $X = \text{Spf} A$. Whenever $M$ is a finite $A$-module, we can define the induced coherent sheaf $M_{\text{rig}}$ on $X_n$ by

$$M_{\text{rig}}|_{\eta (C_n)} := (M \otimes_A C_n)_{\text{rig}}$$

Here $(M \otimes_A C_n)_{\text{rig}}$ denotes the coherent $O_{\eta (C_n)}$-module associated to the finite $C_n$-module $M \otimes_A C_n$.

If $X = \text{Spf} A$ is topologically of finite type over $R$, then $X_n$ is simply $\text{Spf} (A \otimes_R K)$, and $M_{\text{rig}}$ corresponds to the $(A \otimes_R K)$-module $M \otimes_R K$.

**Lemma 2.5.** If $A$ is a special $R$-algebra and $X = \text{Spf} A$, the functor

$$(\cdot)_{\text{rig}} : (\text{Coh}_X) \to (\text{Coh}_{X_n}) : M \mapsto M_{\text{rig}}$$

from the category $(\text{Coh}_X)$ of coherent $O_X$-modules to the category $(\text{Coh}_{X_n})$ of coherent $O_{X_n}$-modules, is exact.

**Proof.** This follows from the fact that the natural ring morphism $A \to C_n$ is flat for each $n > 0$, by [16, 7.1.2]. □

**Proposition 2.6.** For any special formal $R$-scheme $X$, there exists a unique functor

$$(\cdot)_{\text{rig}} : (\text{Coh}_X) \to (\text{Coh}_{X_n}) : M \mapsto M_{\text{rig}}$$

such that

$$M_{\text{rig}}|_{U_n} = (M|_U)_{\text{rig}}$$

for any open affine formal subscheme $U$ of $X$.

The functor $(\cdot)_{\text{rig}}$ is exact. For any morphism of special formal $R$-schemes $h : \mathcal{Y} \to X$, and any coherent $O_X$-module $M$, there is a canonical isomorphism $(h^* M)_{\text{rig}} \cong (h_n)^* M_{\text{rig}}$. Moreover, if $h$ is a finite adic morphism, and $N$ is a coherent $O_{\mathcal{Y}}$-module, then there is a canonical isomorphism $(h^* N)_{\text{rig}} \cong (h_n)^* (N_{\text{rig}})$.

**Proof.** Exactness follows immediately from Lemma 2.5. It is clear that $(\cdot)_{\text{rig}}$ commutes with pull-back, so let $h : \mathcal{Y} \to X$ be a finite adic morphism of special formal $R$-schemes, and let $N$ be a coherent $O_{\mathcal{Y}}$-module. We may suppose that $X = \text{Spf} A$ is affine; then $\mathcal{Y} = \text{Spf} D$ with $D$ finite and adic over $A$, and $h_* N$ is simply $N$ viewed as an $A$-module. By [16, 7.2.2], the inverse image of $\text{Sp} C_n \subset X_n$ in $\mathcal{Y}_n$ is the affinoid space $\text{Sp} (D \otimes_A C_n)$, so both $(h_* N)_{\text{rig}}|_{\text{Sp} C_n}$ and $(h_n)^* (N_{\text{rig}})|_{\text{Sp} C_n}$ are associated to the coherent $C_n$-module $N \otimes_A C_n$. □

**Example 2.7.** The assumption that $h$ is finite and adic is crucial in the last part of Proposition 2.6. Consider, for example, the special formal $R$-scheme $X = \text{Spf} R[[x]]$, and denote by $h : X \to \text{Spf} R$ the structural morphism. Choose a series $(a_n)$ in $K$ such that $|a_n| \to \infty$ as $n \to \infty$, but with $|a_n| \leq \log n$. Then the power series $f = \sum_{n \geq 0} a_n x^n$ in $K[[x]]$ defines an element of $O_{X_n}(X_n) = (f_n)(O_X)_{\text{rig}}$ since it converges on every closed disc $D(0, \rho)$ with $\rho < 1$, but it does not belong to $R[[x]] \otimes_R K = (h_*(O_X))_{\text{rig}}$. 

Lemma 2.8. If $\mathfrak{X}$ is a special formal $R$-scheme and $M$ is a coherent $O_{\mathfrak{X}}$-module, then the functor $(\cdot)_{\text{rig}}$ induces a natural map of $K$-modules

$$i : M(\mathfrak{X}) \otimes_R K \rightarrow M_{\text{rig}}(\mathfrak{X}_n)$$

and this map is injective. If $M = O_{\mathfrak{X}}$ then $i$ is a map of $K$-algebras.

Proof. The map $i$ is constructed in the obvious way: if $\mathfrak{X} = \text{Spf} A$ is affine and $m$ is an element of $M(\mathfrak{X})$, then the restriction of $i(m)$ to $\text{Sp} C_n$ is simply the element $m \otimes 1$ of $M_{\text{rig}}(\text{Sp} C_n) = M \otimes_A C_n$. The general construction is obtained by gluing.

To prove that $i$ is injective, we may suppose that $\mathfrak{X} = \text{Spf} A$ is affine; we'll simply write $M$ instead of $M(\mathfrak{X})$. Let $m$ be an element of $M$ and suppose that $i(m) = 0$. Suppose that $m$ is non-zero in $M \otimes_R K$, and let $\mathfrak{M}$ be a maximal ideal in $A \otimes_R K$ such that $m$ is non-zero in the stalk $M_{\mathfrak{M}}$. By [16, 7.1.9], $\mathfrak{M}$ corresponds canonically to a point $x$ of $\mathfrak{X}_n$ and there is a natural local homomorphism $(A \otimes_R K)_{\mathfrak{M}} \rightarrow O_{\mathfrak{X}_n,x}$ which induces an isomorphism on the completions, so $i(m) = 0$ implies that $m$ vanishes in the $\mathfrak{M}$-adic completion of $M_{\mathfrak{M}}$. This implies at its turn that $m$ vanishes in $M_{\mathfrak{M}}$ since $M_{\mathfrak{M}}$ is separated for the $\mathfrak{M}$-adic topology [22, 7.3.5]; this contradicts our assumption. \hfill \square

Corollary 2.9. If $\mathfrak{X}$ is a special formal $R$-scheme and $M$ is a coherent $O_{\mathfrak{X}}$-module, then $M_{\text{rig}} = 0$ iff $M$ is annihilated by a power of $\pi$.

Proof. We may assume that $\mathfrak{X}$ is affine, say $\mathfrak{X} = \text{Spf} A$. By Lemma 2.8, $M(\mathfrak{X}) \otimes_R K = 0$ iff $M$ is annihilated by a power of $\pi$. \hfill \square

Lemma 2.10. Let $\mathfrak{X} = \text{Spf} A$ be an affine special formal $R$-scheme, and let $f : M \rightarrow N$ be a morphism of coherent $O_{\mathfrak{X}}$-modules such that the induced morphism of coherent $O_{\mathfrak{X}_n}$-modules $f_{\text{rig}} : M_{\text{rig}} \rightarrow N_{\text{rig}}$ is an isomorphism. Then the natural map

$$f : M(\mathfrak{X}) \otimes_R K \rightarrow N(\mathfrak{X}) \otimes_R K$$

is an isomorphism, and fits in the commutative diagram

$$\begin{array}{ccc}
M(\mathfrak{X}) \otimes_R K & \longrightarrow & N(\mathfrak{X}) \otimes_R K \\
\downarrow & & \downarrow \\
M_{\text{rig}}(\mathfrak{X}_n) & \longrightarrow & N_{\text{rig}}(\mathfrak{X}_n)
\end{array}$$

where the vertical arrows are injections and the horizontal arrows are isomorphisms.

Proof. We extend the morphism $f$ to an exact sequence of coherent $O_{\mathfrak{X}}$-modules

$$0 \longrightarrow \text{ker}(f) \longrightarrow M \longrightarrow^f N \longrightarrow \text{coker}(f) \longrightarrow 0$$

Since $(\cdot)_{\text{rig}}$ is an exact functor by Lemma 2.8 and $(f)_{\text{rig}}$ is an isomorphism by assumption, $\text{ker}(f)_{\text{rig}}$ and $\text{coker}(f)_{\text{rig}}$ vanish, and hence, $\text{ker}(f)$ and $\text{coker}(f)$ are $\pi$-torsion modules, by Corollary 2.8. Since $\mathfrak{X}$ is affine, the above exact sequence gives rise to an exact sequence of $A$-modules

$$0 \longrightarrow \text{ker}(f)(\mathfrak{X}) \longrightarrow M(\mathfrak{X}) \longrightarrow^f N(\mathfrak{X}) \longrightarrow \text{coker}(f)(\mathfrak{X}) \longrightarrow 0$$

and by tensoring with $K$, we obtain the required isomorphism. The remainder of the statement follows from Lemma 2.8. \hfill \square
By [16] 7.1.12, there is a canonical isomorphism of $\mathcal{O}_{X_\eta}$-modules

$$\Omega^i_{X_\eta/K} \cong \left( \Omega^i_{X/R} \right)_{\text{rig}}$$

for any special formal $R$-scheme $X$ and each $i \geq 0$.

2.2. Bounded differential forms.

**Definition 2.11.** Let $X$ be a special formal $R$-scheme. For any $i \geq 0$, we call an element $\omega$ of $\Omega^i_{X_\eta/K}(X_\eta)$ an $X$-bounded $i$-form on $X_\eta$, if there exists a finite cover of $X$ by affine open formal subschemes $\{U^{(j)}\}_{j \in I}$ such that for each $j \in I$, $\omega|_{U^{(j)}}$ belongs to the image of the natural map

$$\Omega^i_{X/R}(U^{(j)}) \otimes_R K \to \Omega^i_{X_\eta/K}(U^{(j)}_\eta)$$

By Lemma 2.11, this definition is equivalent to saying that $\omega$ belongs to the image of the natural map

$$(\Omega^i_{X/R} \otimes_R K)(X) \to \Omega^i_{X_\eta/K}(X_\eta)$$

where $\Omega^i_{X/R} \otimes_R K$ is a tensor product of sheaves on $X$.

If $X$ is a finite cover over $R$ then any differential form on $X_\eta$ is $X$-bounded, by quasi-compactness of $X_\eta$. This is false in general: see Example 2.7 for an example of an unbounded 0-form.

**Lemma 2.12.** If $X = \text{Spf} \, A$ is an affine special formal $R$-scheme, and $i \geq 0$ is an integer, then an element $\omega$ of $\Omega^i_{X_\eta/K}(X_\eta)$ is $X$-bounded iff it belongs to the image of the natural map

$$\Omega^i_{X/R}(X) \otimes_R K \to \Omega^i_{X_\eta/K}(X_\eta)$$

**Proof.** Since $X$ is affine,

$$(\Omega^i_{X/R} \otimes_R K)(X) = \Omega^i_{X/R}(X) \otimes_R K$$

\[\square\]

**Corollary 2.13.** Let $X$ be a special formal $R$-scheme, and let $i \geq 0$ be an integer. If $\omega$ is a $X$-bounded $i$-form on $X_\eta$, then for any finite cover of $X$ by affine open formal subschemes $\{U^{(j)}\}_{j \in I}$, and for each $j \in I$, $\omega|_{U^{(j)}}$ belongs to the image of the natural map

$$\Omega^i_{X/R}(U^{(j)}) \otimes_R K \to \Omega^i_{X_\eta/K}(U^{(j)}_\eta)$$

**Lemma 2.14.** Let $X$ be a special formal $R$-scheme, such that $X_\eta$ is reduced. An element $f$ of $\mathcal{O}_{X_\eta}(X_\eta)$ is $X$-bounded iff it is bounded, i.e. iff there exists an integer $M$ such that $|f(x)| \leq M$ for each point $x$ of $X_\eta$.

**Proof.** Since an element $f$ of $\mathcal{O}_X(X)$ satisfies $|f(x)| \leq 1$ for each point $x$ of $X_\eta$ by [16] 7.1.8.2, it is clear that an $X$-bounded analytic function on $X_\eta$ is bounded. Assume, conversely, that $f$ is a bounded analytic function on $X_\eta$. To show that $f$ is $X$-bounded, we may suppose that $X = \text{Spf} \, A$ is affine and flat. Since the natural map $A \otimes_R K \to \mathcal{O}_{X_\eta}(X_\eta)$ is injective, $A \otimes_R K$ is reduced; since $A$ is $R$-flat, $A$ is reduced. If $A$ is integrally closed in $A \otimes R K$, then the image of the natural map $A \otimes_R K \to \mathcal{O}_{X_\eta}(X_\eta)$ coincides with the set of bounded functions on $X_\eta$, by [16] 7.4.1-2. So it suffices to note that the natural map $A \otimes_R K \to B \otimes_R K$ is bijective, where $B$ is the normalization of $A$ in $A \otimes_R K$ ($B$ is a special $R$-algebra since it is finite over $A$, by excellence of $A$; see [15]).

\[\square\]
2.3. Admissible blow-ups and dilatations. Let $X$ be a Noetherian adic formal scheme, let $I$ be an ideal of definition, and let $C$ be any coherent ideal sheaf on $X$. Following the tft-case in [10, §2], we state the following definition.

**Definition 2.15 (Formal blow-up).** The formal blow-up of $X$ with center $I$ is the morphism of formal schemes

$$X' := \lim_{\leftarrow n \geq 1} \text{Proj} \left( \bigoplus_{d \geq 0} I^d \otimes_{O_X} (O_X/I^n) \right) \to X$$

**Proposition 2.16.** Let $X$ be a Noetherian adic formal scheme with ideal of definition $J$, let $I$ be a coherent ideal sheaf on $X$, and consider the formal blow-up $h : X' \to X$ of $X$ at $I$.

1. If $U = \text{Spf} A$ is an affine open formal subscheme of $X$, then the restriction of $h$ over $U$ is the $J(U)$-adic completion of the scheme-theoretic blow-up of $\text{Spec} A$ at the ideal $I(\text{Spec} A)$.
2. The blow-up morphism $X' \to X$ is adic and topologically of finite type. In particular, $X'$ is a Noetherian adic formal scheme.
3. (Universal property) The ideal $IO_{X'}$ is invertible on $X'$, and each morphism of adic formal schemes $g : Y \to X$ such that $IO_Y$ is invertible, factors uniquely through a morphism of formal schemes $Y \to X'$.
4. The formal blow-up commutes with flat base change: if $f : Y \to X$ is a flat morphism of Noetherian adic formal schemes, then

$$X' \times_X Y \to Y$$

is the formal blow-up of $Y$ at $IO_Y$.
5. If $K$ is an open coherent ideal sheaf on $X$, defining a closed subscheme $Z$ of $X$, then the formal blow-up of $X/Z$ at $I/Z$ is the formal completion along $Z$ of the formal blow-up of $X$ at $I$.

**Proof.** Point (1) follows immediately from the definition, and (2) follows from (1). In (3) and (4) we may assume that $X$ and $Y$ are affine; then the result follows from the corresponding properties for schemes, using (1). Point (5) is a special case of (4). □

**Corollary 2.17.** Let $X$ be a special formal $R$-scheme with ideal of definition $J$, let $I$ be a coherent ideal sheaf of $X$, and consider the admissible blow-up $h : X' \to X$ of $X$ at $I$.

1. The blow-up $X'$ is a special formal $R$-scheme.
2. If $X$ is flat over $R$, then $X'$ is flat over $R$.

**Proof.** Point (1) follows immediately from Proposition 2.16(2). To prove (2), we may assume that $X = \text{Spf} A$ is affine; then flatness of $X'$ follows from Proposition 2.16(1), the fact that the scheme-theoretic blow-up of $\text{Spec} A$ at $I(\text{Spec} A)$ is flat over $R$, and flatness of the completion morphism. □

Let $X$ be a special formal $R$-scheme, and let $J$ be an ideal of definition. Let $I$ be a coherent ideal sheaf on $X$, open w.r.t. the $\pi$-adic topology (i.e. $I$ contains a power of $\pi$). We will call such an ideal sheaf $\pi$-open. We do not demand that $I$ is
open w.r.t. the $J$-adic topology on $\mathfrak{X}$. If $I$ is $\pi$-open, we call the blow-up $\mathfrak{X} \to \mathfrak{X}$ with center $I$ admissible.\footnote{Contrary to the terminology used in [10] for the stft case, our definition of admissible blow-up does not assume any flatness conditions on $\mathfrak{X}$.}

We can give an explicit description of admissible blow-ups in the affine case, generalizing [10, 2.2].

**Lemma 2.18.** Let $A$ be a special $R$-algebra, with ideal of definition $J$, and let $I = (f_1, \ldots, f_p)$ be a $\pi$-open ideal in $A$. Put $\mathfrak{X} = \text{Spf} A$. Let $h : \mathfrak{X} \to \mathfrak{X}$ be the admissible blow-up of $\mathfrak{X}$ at $I$. The scheme-theoretic blow-up of $\text{Spec} A$ at $I$ is covered by open charts $\text{Spec} A_i$, $i = 1, \ldots, p$, where

\[
A'_i = A_i^{\xi_1, \ldots, \xi_p} / (f_i \xi_j - \xi_j f_i)_{j=1,\ldots,p}
\]

\[
A_i = A_i^\prime / (f_i - \text{torsion})
\]

(here the $\xi_i / \xi_j$ serve as variables, except when $i = j$). We write $\widehat{A}_i$ and $\widehat{A}_i^\prime$ for the $J$-adic completions of $A_i$, resp. $A_i^\prime$. Then

\[
\widehat{A}_i^\prime = A_i^{\xi_1, \ldots, \xi_p} / (f_i \xi_j - \xi_j f_i)_{j=1,\ldots,p}
\]

\[
\widehat{A}_i = \widehat{A}_i^\prime / (f_i - \text{torsion})
\]

and $\text{Spf} \widehat{A}_i^\prime$ is the open formal subscheme of $\mathfrak{X}$ where $f_i$ generates $I O_{\mathfrak{X}}$. In particular, $\{\text{Spf} \widehat{A}_i^\prime\}_{i=1,\ldots,p}$ is an open cover of $\mathfrak{X}$.

If, moreover, $A$ is flat over $R$, then

\[
\widehat{A}_i = \widehat{A}_i^\prime / (f_i - \text{torsion}) = \widehat{A}_i^\prime / (\pi - \text{torsion})
\]

**Proof.** The proof is similar to the stft-case [10, 2.2]. First, we show that $\widehat{A}_i = \widehat{A}_i^\prime / (f_i - \text{torsion})$. Since $A_i$ is a finite $A_i^\prime$-module, $\widehat{A}_i = A_i \otimes_{A_i^\prime} \widehat{A}_i^\prime$. Since $A_i^\prime$ is Noetherian, $\widehat{A}_i^\prime$ is flat over $A_i^\prime$, so

\[
A'_i / (f_i - \text{torsion}) \otimes_{A_i^\prime} \widehat{A}_i^\prime = \widehat{A}_i^\prime / (f_i - \text{torsion})
\]

Now, we show that $\pi$-torsion and $f_i$-torsion coincide in $\widehat{A}_i^\prime$ if $A$ is $R$-flat. Since $I$ is $\pi$-open in $A$, it contains a power of $\pi$. Since $f_i$ generates $I \widehat{A}_i^\prime$, the $f_i$-torsion is contained in the $\pi$-torsion. But $A_i$ is $R$-flat since $A$ is $R$-flat, so $\widehat{A}_i = \widehat{A}_i^\prime / (f_i - \text{torsion})$ is $R$-flat, i.e., has no $\pi$-torsion.

The remainder of the statement is clear. \hfill $\Box$

**Proposition 2.19.** Let $\mathfrak{X}$ be a special formal $R$-scheme, and let $h : \mathfrak{X} \to \mathfrak{X}$ be an admissible blow-up with center $I$. The induced morphism of rigid varieties $h_n : \mathfrak{X}_n \to \mathfrak{X}_n$ is an isomorphism.

**Proof.** We may assume that $\mathfrak{X}$ is affine, say $\mathfrak{X} = \text{Spf} A$, with $J$ as largest ideal of definition. Let $I = (f_1, \ldots, f_p)$ be a $\pi$-open ideal in $A$, and let $h : \mathfrak{X} \to \mathfrak{X}$ be the blow-up of $\mathfrak{X}$ at $I$. We define $B_n$ and $C_n$ as in Section 2.24 for $n > 0$.

We adopt the notation from Lemma 2.18. Since the admissible blow-up $h$ is adic, and the induced morphism $V(J O_{\mathfrak{X}}) \to V(J)$ is of finite type, it follows from [10, 7.2.2] that the restriction of $h_n : (\text{Spf} \widehat{A}_i)^n \to \mathfrak{X}_n$ over $\text{Sp} C_n$ is given by

\[
h_{n,i} : \text{Sp} C_n \otimes_{\widehat{A}_i} \text{Sp} C_n
\]
for each $i$ and each $n$. However, the natural map

$$C_n \otimes_A \hat{A}_i' \to C_n \otimes_A \hat{A}_i$$

is an isomorphism since the $f_i$-torsion in $\hat{A}_i'$ is killed if we invert $\pi$ (because $f_i$ divides $\pi$ in $A_i'$, as $f_i$ generates the open ideal $IA_i'$). Hence, $\{h_{n,1}, \ldots, h_{1,p}\}$ is nothing but the rational cover of $\text{Sp} C_n$, associated to the tuple $(f_1, \ldots, f_p)$ (see [9, 8.2.2]); note that these elements generate the unit ideal in $C_n$, since $I$ contains a power of $\pi$, which is a unit in $C_n$. □

**Definition 2.20** (Dilatation). Let $\mathcal{X}$ be a flat special formal $R$-scheme, and let $\mathcal{I}$ be a coherent ideal sheaf on $\mathcal{X}$ containing $\pi$. Consider the admissible blow-up $h : \mathcal{X}' \to \mathcal{X}$ with center $\mathcal{I}$. If $\mathcal{U}$ is the open formal subscheme of $\mathcal{X}'$ where $\mathcal{I}\mathcal{O}_{\mathcal{X}'}$ is generated by $\pi$, we call $\mathcal{U} \to \mathcal{X}$ the dilatation of $\mathcal{X}$ with center $\mathcal{I}$.

**Proposition 2.21.** Let $\mathcal{X}$ be a flat special formal $R$-scheme, let $\mathcal{I}$ be a coherent ideal sheaf on $\mathcal{X}$ containing $\pi$, and let $Z$ be a closed subscheme of $\mathcal{X}_0$. The dilatation of $\mathcal{X}/Z$ with center $\mathcal{I}/Z$ is the formal completion along $Z$ of the dilatation of $\mathcal{X}$ with center $\mathcal{I}$.

If, moreover, $\mathcal{X}$ is stft over $R$ and $(\mathcal{X}, \mathcal{I})$ has an algebraic model $(X, I)$, then the dilatation of $X/Z$ at $\mathcal{I}/Z$ is the formal completion along $Z$ of the dilatation of $X$ at $I$ (as defined in [11, 3.2/1]).

**Proof.** This is clear from the definition. □

**Proposition 2.22.** Let $\mathcal{X}$ be a flat special formal $R$-scheme, and let $\mathcal{I}$ be a coherent ideal sheaf on $\mathcal{X}$ containing $\pi$. Let $h : \mathcal{U} \to \mathcal{X}$ be the dilatation with center $\mathcal{I}$, and denote by $\mathfrak{Z}$ the closed formal subscheme of $\mathcal{X}_s$ defined by $\mathcal{I}$. The dilatation $\mathcal{U}$ is a flat special formal $R$-scheme, and $h_s : \mathcal{U}_s \to \mathfrak{Z}_s$ factors through $\mathfrak{Z}$. The induced morphism $h_n : \mathcal{U}_n \to \mathfrak{Z}_n$ is an open immersion.

If $g : \mathfrak{V} \to \mathcal{X}$ is any morphism of flat special formal $R$-schemes such that $g_s : \mathfrak{V}_s \to \mathfrak{Z}_s$ factors through $\mathfrak{Z}$, then there is a unique morphism of formal $R$-schemes $i : \mathfrak{V} \to \mathcal{U}$ such that $g = h \circ i$.

If $\mathcal{I}$ is open, then $\mathcal{U}$ is stft over $R$.

**Proof.** It is clear that $h_s$ factors through $\mathfrak{Z}$. The morphism $\mathcal{U}_n \to \mathfrak{Z}_n$ is an open embedding because $\mathcal{U}$ is an open formal subscheme of the blow-up $\mathcal{X}'$ of $\mathcal{X}$ at $\mathcal{I}$, and $\mathcal{X}'_n \to \mathfrak{Z}_n$ is an isomorphism by Proposition 2.19.

Since $g_s$ factors through $\mathfrak{Z}$, we have $\mathcal{I}\mathcal{O}_\mathfrak{V} = (\pi)$. In particular, by flatness of $\mathfrak{V}$, the ideal $\mathcal{I}\mathcal{O}_\mathfrak{V}$ is invertible, and by the universal property of the blow-up, $g$ factors uniquely through a morphism $i : \mathfrak{V} \to \mathcal{X}'$ to the blow-up $\mathcal{X}' \to \mathcal{X}$ at $\mathcal{I}$. The image of $\mathfrak{V}$ in $\mathcal{X}'$ is necessarily contained in $\mathcal{U}$ since $\pi$ generates $\mathcal{I}\mathcal{O}_\mathfrak{V}$: if $v$ were a closed point of $\mathfrak{V}$ mapping to a point $x$ in $\mathcal{X}' \setminus \mathcal{U}$, then we could write $\pi = a \cdot f$ in $\mathcal{O}_{\mathcal{X}'_x}$ with $f \in \mathcal{I}\mathcal{O}_{\mathcal{X}'}$ and $a$ not a unit. Thus yields $\pi = i^*a \cdot i^*f$ in $\mathcal{O}_{\mathcal{X}_v}$, but since $i^*f$ belongs to $\mathcal{I}\mathcal{O}_\mathfrak{V}$, we have also $i^*f = b \cdot \pi$ in $\mathcal{O}_{\mathcal{X}_v}$, so $\pi = c \cdot \pi$ with $c$ not a unit, and $0 = (1-c) \cdot \pi$. Since $1-c$ is invertible in $\mathcal{O}_{\mathcal{X}_v}$, this would mean that $\pi = 0$ in $\mathcal{O}_{\mathcal{X}_v}$, which contradicts flatness of $\mathfrak{V}$ over $R$.

Finally, assume that $\mathcal{I}$ is open. Then the ideal $\mathcal{I}\mathcal{O}_\mathfrak{U}$ is open, and by definition of the dilatation, it is generated by $\pi$. This implies that $\mathcal{I}\mathcal{O}_\mathfrak{U}$ is an ideal of definition, and that $\mathfrak{U}$ is stft. □
Proposition 2.23. Let \( \mathfrak{X} \) be a flat special formal \( R \)-scheme, let \( U \) be a reduced closed subscheme of \( \mathfrak{X}_0 \), and denote by \( \Omega \to \mathfrak{X} \) the completion map of \( \mathfrak{X} \) along \( U \). If we denote by \( \Omega' \to \Omega \) and \( \mathfrak{X}' \to \mathfrak{X} \) the dilatations with center \( \mathfrak{X}_0 = U \), resp. \( \mathfrak{X}_0 \), then there exists a unique morphism of formal \( R \)-schemes \( \Omega' \to \mathfrak{X}' \) such that the square

\[
\begin{array}{ccc}
\Omega' & \longrightarrow & \Omega \\
\downarrow & & \downarrow \\
\mathfrak{X}' & \longrightarrow & \mathfrak{X}
\end{array}
\]

commutes, and this morphism \( \Omega' \to \mathfrak{X}' \) is the dilatation of \( \mathfrak{X}' \) with center \( \mathfrak{X}'_s \times_{\mathfrak{X}_0} U \).

Proof. The existence of such a unique morphism \( \Omega' \to \mathfrak{X}' \) follows immediately from Proposition 2.22 to show that this is the dilatation with center \( \mathfrak{X}'_s \times_{\mathfrak{X}_0} U \), it suffices to check that \( \Omega' \to \mathfrak{X}' \) satisfies the universal property in Proposition 2.22. Let \( h : \mathfrak{X} \to \mathfrak{X}' \) be a morphism of flat special formal \( R \)-schemes such that \( h_s : \mathfrak{X}_s \to \mathfrak{X}'_s \) factors through \( \mathfrak{X}'_s \times_{\mathfrak{X}_0} U \). This means that the composed morphism \( \mathfrak{X}_s \to \mathfrak{X}_0 \) factors through \( U \), and hence, \( \mathfrak{X} \to \mathfrak{X}' \) factors through a morphism \( g : \mathfrak{X} \to \Omega' \). Moreover, again by Proposition 2.22 \( g \) factors uniquely through a morphism \( f : \mathfrak{X} \to \Omega' \).

On the other hand, let \( f' : \mathfrak{X} \to \Omega' \) be another morphism of formal \( R \)-schemes such that \( h \) is the composition of \( f' \) with \( \Omega' \to \mathfrak{X}' \). Then the compositions of \( f \) and \( f' \) with \( \Omega' \to \Omega \) coincide, so \( f = f' \) by the uniqueness property in Proposition 2.22 for the dilatation \( \Omega' \to \Omega \).

Berthelot’s construction of the generic fiber of a special formal \( R \)-scheme can be restated in terms of dilatations. Let \( \mathcal{J} \) be an ideal of definition of \( \mathfrak{X} \), and for any integer \( e > 0 \), consider the dilatation

\[ h^{(e)} : \Omega^{(e)} \to \mathfrak{X} \]

with center \((\pi, \mathcal{J}^e)\). The formal \( R \)-scheme \( \Omega^{(e)} \) is stft over \( R \), and \( h^{(e)} \) is an open immersion. By the universal property of the dilatation, \( h^{(e)} \) can be decomposed uniquely as

\[ \Omega^{(e)} \overset{h^{(e,e')}}{\longrightarrow} \Omega^{(e')} \overset{h^{(e')}}{\longrightarrow} \mathfrak{X} \]

for any pair of integers \( e' \geq e \geq 0 \). Moreover, \( h^{(e,e')} \) induces an open immersion \( h^{(e,e')} : \Omega^{(e)}_{\eta} \to \Omega^{(e')}_{\eta} \).

Lemma 2.24. The image of the open immersion

\[ h^{(e)}_{\eta} : \Omega^{(e)}_{\eta} \to \mathfrak{X}_{\eta} \]

consists of the points \( x \in \mathfrak{X}_{\eta} \) such that \( |f(x)| \leq |\pi| \) for each element \( f \) of the stalk \((\mathcal{J}^e)_{\text{sp}(x)}\).

Proof. Let \( R' \) be a finite extension of \( R \), and let \( \psi \) be a section in \( \mathfrak{X}(R') \). Then by the universal property in Proposition 2.22 \( \psi \) lifts to a section in \( \Omega^{(e)}(R') \) iff \((\mathcal{J}^e, \pi)\mathcal{O}_{\text{sp}(x)} \) (the pull-back through \( \psi \)) is generated by \( \pi \). If we denote by \( x \) the image of the morphism \( \psi_{\eta} \) in \( \mathfrak{X}_{\eta} \), this is equivalent to saying that \( |f(x)| \leq |\pi| \) for all \( f \) in \((\mathcal{J}^e)_{\text{sp}(x)}\).

Proposition 2.25. The set \( \{\Omega^{(e)}_{\eta} | e > 0\} \) is an admissible cover of \( \mathfrak{X}_{\eta} \).
Proof. Let \( Y = \text{Sp} \ B \) be any affinoid variety over \( K \), endowed with a morphism of rigid \( K \)-varieties \( \varphi : Y \to \mathfrak{X}_\eta \). We have to show that the image of \( \varphi \) is contained in \( \mathcal{U}^{(e)}_\eta \), for \( e \) sufficiently large. We may assume that \( \mathfrak{X} \) is affine, say \( \mathfrak{X} = \text{Spf} \ A \). Since \( \mathcal{J} \) is an ideal of definition on \( \mathfrak{X} \), we have \( |f(x)| < 1 \) for any \( f \in \mathcal{J}(\mathfrak{X}) \) and any \( x \in \mathfrak{X}_\eta \). Since \( \mathcal{J} \) is finitely generated, and by the Maximum Modulus Principle \([9, 6.2.1.4]\), there exists a value \( e > 0 \) such that for each element \( f \in \mathcal{J}(\mathfrak{X})^e \) and each point \( y \) of \( Y \), \( |f(\varphi(y))| < |\pi| \). By Lemma \([2.24]\) this implies that the image of \( \varphi \) is contained in \( \mathcal{U}^{(e)}_\eta \). \( \square \)

Hence, we could have defined \( \mathfrak{X}_\eta \) as the limit of the direct system \((\mathcal{U}^{(e)}_\eta, h^{(e,e')}_{\eta})\) in the category of rigid \( K \)-varieties.

Remark. If we assume that \( \mathfrak{X} = \text{Spf} \ A \) is affine and that \( \mathcal{J} \) is the largest ideal of definition on \( \mathfrak{X} \), then the dilatation \( \mathcal{U}^{(e)} \to \mathfrak{X} \) is precisely the morphism \( \text{Spf} \ B_n \to \mathfrak{X} \) introduced at the beginning of Section \([2.1]\). This can be seen by using the explicit description in Lemma \([2.18]\) \( \square \)

2.4. Irreducible components of special formal schemes. If \( \mathfrak{X} \) is a special formal \( R \)-scheme, the underlying topological space \(|\mathfrak{X}| = |\mathfrak{X}_0|\), and even the scheme \( \mathfrak{X}_0 \), reflect rather poorly the geometric properties of \( \mathfrak{X} \). For instance, if \( A \) is a special \( R \)-algebra, \(|\text{Spf} \ A|\) can be irreducible even when \( \text{Spec} \ A \) is reducible (e.g. for \( A = R[[x,y]]/(xy) \), where \( (\text{Spf} \ A)_0 = \text{Spec} k \)). Conversely, \(|\text{Spf} \ A|\) can be irreducible even when \( A \) is integral (e.g. for \( A = R\{x,y\}/(\pi - xy) \)).

Therefore, a more subtle definition of the irreducible components of \( \mathfrak{X} \) is needed. We will use the normalization map \( \tilde{\mathfrak{X}} \to \mathfrak{X} \) constructed in \([13]\) (there normalization was already used to define the irreducible components of a rigid variety). The normalization map is a finite morphism of special formal \( R \)-schemes.

Definition 2.26. Let \( \mathfrak{X} \) be a special formal \( R \)-scheme, and let \( \tilde{\mathfrak{X}} \to \mathfrak{X} \) be a normalization map. We say that \( \mathfrak{X} \) is irreducible if \(|\tilde{\mathfrak{X}}|\) is connected.

Definition 2.27. Let \( \mathfrak{X} \) be a special formal \( R \)-scheme, and let \( h : \tilde{\mathfrak{X}} \to \mathfrak{X} \) be a normalization map. We denote by \( \tilde{\mathfrak{X}}_i, i = 1, \ldots, r \) the connected components of \( \tilde{\mathfrak{X}} \) (defined topologically), and by \( h_i \) the restriction of \( h \) to \( \tilde{\mathfrak{X}}_i \).

For each \( i \), we denote by \( \mathfrak{X}_i \) the reduced closed subscheme of \( \mathfrak{X} \) defined by the kernel of the natural map \( \psi_i : \mathcal{O}_{\tilde{\mathfrak{X}}} \to (h_i)_* \mathcal{O}_{\tilde{\mathfrak{X}}_i} \). We call \( \mathfrak{X}_i, i = 1, \ldots, r \) the irreducible components of \( \mathfrak{X} \).

Note that \((h_i)_* \mathcal{O}_{\tilde{\mathfrak{X}}_i} \) is coherent, by finiteness of \( h \), so the kernel of \( \psi_i \) is a coherent ideal sheaf on \( \mathfrak{X} \) and \( \mathfrak{X}_i \) is well-defined.

In the affine case, the irreducible components of \( \mathfrak{X} \) correspond to the minimal prime ideals of the ring of global sections, as one would expect:

Lemma 2.28. Let \( \mathfrak{X} = \text{Spf} \ A \) be an affine special formal \( R \)-scheme, and denote by \( P_i, i = 1, \ldots, r \) the minimal prime ideals of \( A \). Then the irreducible components of \( \mathfrak{X} \) are given by \( \text{Spf} \ A/P_i, i = 1, \ldots, r \).

Proof. If \( A \to \tilde{A} \) is a normalization map, then \( \tilde{A} = \prod_{i=1}^r A/P_i \). Hence, \( \tilde{\mathfrak{X}}_i = \text{Spf} (\tilde{A}/P_i) \) are the connected components of \( \tilde{\mathfrak{X}} \), and \( \mathfrak{X}_i = \text{Spf} A/P_i \) for each \( i \). \( \square \)

Lemma 2.29. With the notations of Definition \([2.27]\), the morphism \( h_i : \tilde{\mathfrak{X}}_i \to \mathfrak{X}_i \) induced by \( h \) is a normalization map for each \( i \). Hence, \( \mathfrak{X}_i \) is irreducible for each \( i \).
Proof. Fix an index \( i \in \{1, \ldots, r\} \), let \( \mathcal{U} = \text{Spf} \ A \) be an open affine formal subscheme of \( \mathfrak{X} \), and denote by \( P_j, j = 1, \ldots, q \) the minimal prime ideals in \( A \). Since normalization commutes with open immersions \([15, 1.2.3]\), \( \hat{\mathcal{X}} \cap h^{-1}(\mathcal{U}) \) is a union of connected components of \( \hat{\mathcal{U}} \), i.e. it is of the form \( \text{Spf} \prod_{j \in J} \hat{A}/P_j \) for some subset \( J \) of \( \{1, \ldots, q\} \). Then by definition, \( \hat{\mathcal{X}}_i \cap \mathcal{U} \) is the closed formal subscheme defined by the ideal \( \pi \cap \mathcal{X}_i \cup \mathcal{U} \cap \mathcal{X}_i \) is a normalization map. \( \square \)

There is an important pathology in comparison to the scheme case: a non-empty open formal subscheme of an irreducible special formal R-scheme is not necessarily irreducible, as is shown by the following example. Put \( A = R\{x, y\}/(xy - \pi) \), and \( \mathfrak{X} = \text{Spf} \ A \). Then \( \mathfrak{X} \) is irreducible since \( A \) is a domain. However, if we denote by \( O \) the point of \( \mathfrak{X} \) defined by the open ideal \((\pi, x, y)\), then \( \mathfrak{X} \setminus \{O\} \) is disconnected.

**Proposition 2.30.** Let \( Y \) be either an excellent Noetherian scheme over \( R \) or a special formal \( R \)-scheme. In the first case, let \( \mathcal{I} \) be a coherent ideal sheaf on \( Y \) such that the \( \mathcal{I} \)-adic completion \( \mathcal{Y} \) of \( Y \) is a special formal \( R \)-scheme; in the second, let \( \mathcal{I} \) be any open coherent ideal sheaf on \( Y \), and denote again by \( \mathcal{Y} \) the \( \mathcal{I} \)-adic completion of \( Y \).

If \( h : \hat{Y} \to Y \) is a normalization map, then its \( \mathcal{I} \)-adic completion \( \hat{h} : \hat{\mathcal{Y}} \to \mathcal{Y} \) is also a normalization map.

**Proof.** We may assume that \( Y \) is affine, say \( Y = \text{Spec} \ A \) (resp. \( \text{Spf} \ A \)), so that \( \mathcal{I} \) is defined by an ideal \( I \) of \( A \). Moreover, we may assume that \( A \) is reduced. Denote by \( \hat{A} \) the \( I \)-adic completion of \( A \). Since \( \hat{A} \) is finite over \( A \), \( B := \hat{A} \otimes_A \hat{A} \) is the \( I \)-adic completion of \( \hat{A} \). Hence, by \([15, 1.2.2]\), it suffices to show that \( B \) is normal, and that \( B/P \) is reduced for all minimal prime ideals \( P \) of \( A \).

Normality of \( B \) follows from excellence of \( \hat{A} \). Let \( P \) be any minimal prime ideal of \( A \). Then \( \hat{A}/P \) is reduced, and hence, so is \( B/P \), again by excellence of \( \hat{A} \) (see \([24, 7.8.3(v)]\)). \( \square \)

**Corollary 2.31.** We keep the notation of Lemma \( 2.30 \), and we denote by \( Z_1, \ldots, Z_q \) the connected components of the closed subscheme \( Z \) of \( Y \) defined by \( \mathcal{I} \). If \( Y_1, \ldots, Y_r \) are the irreducible components of \( \mathcal{Y} \), then the irreducible components of \( \mathcal{Y} \) are given by the irreducible components of \( \hat{Y}_i/Z_j \) for \( i = 1, \ldots, r \) and \( j = 1, \ldots, q \) (where \( \hat{Y}_i/Z_j \) may be empty for some \( i, j \)).

**Proof.** Denote by \( h_i : \hat{Y}_i \to Y \) the restriction of \( h \) to \( \hat{Y}_i \), for each \( i \), and by \( \hat{h}_i : \mathcal{U}_i \to \hat{Y}_i/Z \) its \( \mathcal{I} \)-adic completion. By exactness of the completion functor, the kernel of

\[
\mathcal{O}_{\hat{Y}_i/Z} \to (\hat{h}_i)_* \mathcal{O}_{\mathcal{U}_i}
\]

is the defining ideal sheaf of the completion \( \hat{Y}_i/Z \), for each \( i \).

\( \square \)

2.5. **Strict normal crossings and resolution of singularities.**

**Definition 2.32.** A special formal \( R \)-scheme \( \mathfrak{X} \) is regular if, for any point \( x \) of \( \mathfrak{X} \), the local ring \( \mathcal{O}_{\mathfrak{X},x} \) is regular (it suffices to check this at closed points).

Let \( \mathfrak{X} \) be a regular special formal \( R \)-scheme. We say that a coherent ideal sheaf \( \mathcal{I} \) on \( \mathfrak{X} \) is a strict normal crossings ideal, if the following conditions hold:
(1) there exists at each point \( x \) of \( \mathfrak{X} \) a regular system of local parameters \((x_0, \ldots, x_m)\) in \( \mathcal{O}_{\mathfrak{X}, x} \) such that \( \mathcal{I}_x \) is generated by \( \prod_{j=0}^m (x_j)^{M_j} \) for some tuple \( M \) in \( \mathbb{N}^{(0, \ldots, m)} \).

(2) if \( \mathfrak{E} \) is the closed formal subscheme of \( \mathfrak{X} \) defined by \( \mathcal{I} \), then the irreducible components of \( \mathfrak{E} \) are regular.

**Lemma 2.33.** If (1) holds, then condition (2) is equivalent to the condition that \( \mathcal{O}_{\mathfrak{E}_i, x} \) is a domain for each irreducible component \( \mathfrak{E}_i \) of \( \mathfrak{E} \) and each point \( x \) of \( \mathfrak{E}_i \).

**Proof.** If \( \mathfrak{E}_i \) is regular, then \( \mathcal{O}_{\mathfrak{E}_i, x} \) is regular and hence a domain. Conversely, assume that \( \mathcal{O}_{\mathfrak{E}_i, x} \) is a domain. Using the notations in condition (1) of Definition 2.32 we choose an open affine neighbourhood \( \mathfrak{U} = \text{Spf} \ A \) of \( x \) in \( \mathfrak{X} \) such that \( x_0, \ldots, x_m \) are defined on \( \mathfrak{U} \) and such that \( \mathcal{I} \) is generated by \( \prod_{j=0}^m (x_j)^{M_j} \) on \( \mathfrak{U} \).

Denote by \( \mathfrak{M} \) the maximal ideal of \( A \) defining \( x \).

Since \( \mathcal{O}_{\mathfrak{X}, x}/(x_j) \) is regular for each \( j \), so is \( (A/(x_j))_{\mathfrak{M}} \) (these Noetherian local rings have isomorphic \( \mathfrak{M}\)-adic completions by the proof of [15, 1.2.1]). Hence, shrinking \( \mathfrak{U} \), we may assume that \( A/(x_j) \) is a domain for each \( j \). Then the irreducible components of \( \mathfrak{E} \cap \mathfrak{U} \) are defined by \( x_j = 0 \) for \( j = 0, \ldots, m, M_j \neq 0 \), and since normalization commutes with open immersions [15, 1.2.3], \( \mathfrak{E}_i \cap \mathfrak{U} \) is a union of such components. However, since \( \mathcal{O}_{\mathfrak{E}_i, x} \) is a domain, we see that it is of the form \( \mathcal{O}_{\mathfrak{X}, x}/(x_j) \) for some \( j \). In particular, \( \mathcal{O}_{\mathfrak{E}_i, x} \) is regular. \( \square \)

Hence, if \( \mathfrak{E} \) is a scheme, condition (2) follows from condition (1) (since any local ring of an irreducible scheme is a domain).

We say that a closed formal subscheme \( \mathfrak{E} \) of a regular special formal \( R \)-scheme \( \mathfrak{X} \) is a strict normal crossings divisor if its defining ideal sheaf is a strict normal crossings ideal. We say that a special formal \( R \)-scheme \( \mathfrak{Y} \) has strict normal crossings if \( \mathfrak{Y} \) is regular and the special fiber \( \mathfrak{Y}_x \) is a strict normal crossings divisor, i.e. if the ideal sheaf \( \pi_{\mathfrak{Y}_x} \) is a strict normal crossings ideal.

Now let \( \mathfrak{X} \) be a regular special formal \( R \)-scheme, and let \( \mathcal{I} \) be a strict normal crossings ideal on \( \mathfrak{X} \), defining a closed formal subscheme \( \mathfrak{E} \) of \( \mathfrak{X} \). We can associate to each irreducible component \( \mathfrak{E}_i \) of \( \mathfrak{E} \) a multiplicity \( m(\mathfrak{E}_i) \) as follows. Choose any point \( x \) on \( \mathfrak{E}_i \), denote by \( \mathfrak{P}_i \) the defining ideal sheaf of \( \mathfrak{E}_i \), and by \( \mathfrak{P}_{i, x} \) its stalk at \( x \).

**Lemma 2.34.** The ring \( (\mathcal{O}_{\mathfrak{X}, x})_{\mathfrak{P}_{i, x}} \) is a DVR.

**Proof.** The ring \( (\mathcal{O}_{\mathfrak{X}, x})_{\mathfrak{P}_{i, x}} \) is regular since \( \mathfrak{X} \) is regular, so it suffices to show that \( \mathfrak{P}_{i, x} \) is principal. However, if \((x_0, \ldots, x_m)\) is a regular system of local parameters in \( \mathcal{O}_{\mathfrak{X}, x} \) such that \( \mathcal{I}_x \) is generated by \( \prod_{i=0}^m (x_i)^{M_i} \), then we’ve seen in the proof of Lemma 2.33 that \( \mathfrak{P}_{i, x} \) is generated by \( x_j \) at \( x \), for some index \( j \). \( \square \)

We define the multiplicity \( m(\mathfrak{E}_i, x) \) of \( \mathfrak{E}_i \) at \( x \) as the length of the \( (\mathcal{O}_{\mathfrak{X}, x})_{\mathfrak{P}_{i, x}} \)-module \( (\mathcal{O}_{\mathfrak{E}, x})_{\mathfrak{P}_{i, x}} \).

**Lemma-Definition 2.35.** The multiplicity \( m(\mathfrak{E}_i, x) \) does not depend on the point \( x \). Therefore, we denote it by \( m(\mathfrak{E}_i) \), and we call it the multiplicity of \( \mathfrak{E}_i \) in \( \mathfrak{E} \).

If \( x \) is any point of \( \mathfrak{E}_i \), and if \( \mathcal{I}_x \) is generated by \( \prod_{j=0}^m x_j^{M_j} \) in \( \mathcal{O}_{\mathfrak{X}, x} \), with \((x_0, \ldots, x_m) \) a regular system of local parameters in \( \mathcal{O}_{\mathfrak{X}, x} \), then \( \mathfrak{P}_{i, x} \) is generated by \( x_j \) for some index \( j \), and \( m(\mathfrak{E}_i) = M_j \).
Proof. We’ve seen in the proof of Lemma 2.34 that \( \mathcal{P}_{i,x} \) is generated by \( x_j \) for some index \( j \), and that \( (O_X)_x \mathcal{P}_{i,x} \) is a DVR with uniformizing parameter \( x_j \), so clearly \( m(\mathcal{E}_i,x) = M_j \).

Moreover, there exists an open neighbourhood \( U \) of \( x \) in \( X \) such that \( x_0, \ldots, x_m \) are defined on \( U \), and such that \( \mathcal{P}_j \) is generated by \( x_j \) on \( U \) and \( I \) by \( \prod_{j=0}^{m} x_j^M_j \). This shows that \( m(\mathcal{E}_i,y) = M_j \) for each point \( y \) in a sufficiently small neighbourhood of \( x \). Hence, \( m(\mathcal{E}_i,y) \) is locally constant on \( \mathcal{E}_i \), and therefore constant since \( \mathcal{E}_i \) is connected.

If \( \mathcal{X} \) is a regular special formal \( R \)-scheme and \( \mathcal{E} \) is strict normal crossings divisor, then we write \( \mathcal{E} = \sum_{i \in I} N_i \mathcal{E}_i \) to indicate that \( \mathcal{E}_i, i \in I \), are the irreducible components of \( \mathcal{E} \), and that \( N_i = m(\mathcal{E}_i) \) for each \( i \). We say that \( \mathcal{E} \) is a tame strict normal crossings divisor if the multiplicities \( N_i \) are prime to the characteristic exponent of the residue field \( k \) of \( R \). We say that \( \mathcal{X} \) has tame strict normal crossings if \( \mathcal{X} \) is regular, and \( \mathcal{X}_s \) is a tame strict normal crossings divisor.

If \( Z \) is a separated \( R \)-scheme of finite type, \( Z \) is regular, and its special fiber \( Z_s \) is a (tame) strict normal crossings divisor (in the classical sense), then we say that \( Z \) has (tame) strict normal crossings.

For any non-empty subset \( J \) of \( I \), we define \( \mathcal{E}_J := \cap_{i \in J} \mathcal{E}_i \) (i.e. \( \mathcal{E}_J \) is defined by the sum of the defining ideal sheaves of \( \mathcal{E}_i, i \in J \)), \( E_J := \cap_{i \in J}(\mathcal{E}_i)_0 \) and \( E_J^0 := E_J \setminus (\cup_{i \notin J}(\mathcal{E}_i)_0) \). Moreover, we put

\[
m_J := \gcd\{N_i \mid i \in J\} \]

If \( i \in I \), we write \( E_i \) instead of \( E_{\{i\}} = (\mathcal{E}_i)_0 \). Note that \( \mathcal{E}_i \) is regular and \( E_J = (\mathcal{E}_J)_0 \) for each non-empty subset \( J \) of \( I \).

Example 2.36. Consider the special formal \( R \)-scheme

\[
\mathcal{X} = \text{Spf} R[[x,y]]/(\pi - x^{N_1}y^{N_2})
\]

Then \( \mathcal{X}_s = \text{Spf} k[[x,y]]/((x^{N_1}y^{N_2}) \), and we get \( \mathcal{X}_s = N_1 \mathcal{E}_1 + N_2 \mathcal{E}_2 \) with \( \mathcal{E}_1 = \text{Spf} k[[y]] \) and \( \mathcal{E}_2 = \text{Spf} k[[x]] \). Note that \( E_1^0 = E_2^0 = \emptyset \), while \( E_{\{1,2\}} \) is a point (the maximal ideal \( (\pi,x,y) \).

The varieties \( E_i \) are not necessarily irreducible. Consider, for instance, the smooth special formal \( R \)-scheme

\[
\mathcal{X} = \text{Spf} R[[x,z]]/(x - yz)
\]

Then \( \mathcal{X}_s \) is the formal \( k \)-scheme \( \text{Spf} k[[x]][y,z]/(x - yz) \) which is irreducible, since \( k[[x]][y,z]/(x - yz) \) has no zero-divisors. However, \( \mathcal{X}_0 = \text{Spec} k[y,z]/(yz) \) is reducible.

If \( m_J \) is prime to the characteristic exponent of \( k \), we construct an étale cover \( \mathcal{E}_J \) of \( E_J \) as follows: we can cover \( E_J \) by affine open formal subschemes \( \mathcal{U} = \text{Spf} V \) of \( \mathcal{X} \) such that \( \pi = uv^{m_j} \) with \( u,v \in V \) and \( u \) a unit. We put

\[
\mathcal{U} = \text{Spf} V[T]/(uT^{m_j} - 1)
\]

The restrictions of \( \mathcal{U} \) over \( E_J \) glue together to an étale cover \( \mathcal{E}_J \) of \( E_J \).

If \( \mathcal{X} \) is an \( stfl \) formal \( R \)-scheme, then all these definitions coincide with the usual ones (see [21]). In particular, \( \mathcal{E}_i = E_i \) is a regular \( k \)-variety, and \( N_i \) is the length of the local ring of the \( k \)-scheme \( \mathcal{X}_s \) at the generic point of \( E_i \).
Definition 2.37. Let $\mathfrak{X}$ be a regular special formal $R$-scheme with strict normal crossings, with $\mathfrak{X}_s = \sum_{i \in I} N_i \mathfrak{E}_i$, and let $J$ be a non-empty subset of $I$. We say that an integer $d > 0$ is $J$-linear if there exist integers $\alpha_j > 0$, $j \in J$, with $d = \sum_{j \in J} \alpha_j N_j$. We say that $d$ is $\mathfrak{X}_s$-linear if $d$ is $J$-linear for some non-empty subset $J$ of $I$ with $|J| > 1$ and $E_j^J \neq \emptyset$.

Lemma 2.38. Let $\mathfrak{X}$ be regular special formal $R$-scheme with strict normal crossings. There exists a sequence of admissible blow-ups

$$\pi^{(j)} : \mathfrak{X}^{(j+1)} \to \mathfrak{X}^{(j)}, \quad j = 0, \ldots, r - 1$$

such that

- $\mathfrak{X}^{(0)} = \mathfrak{X}$,
- the special fiber of $\mathfrak{X}^{(j)}$ is a strict normal crossings divisor
  $$\mathfrak{X}^{(j)}_s = \sum_{i \in I^{(j)}} N_i^{(j)} \mathfrak{E}_i^{(j)}$$
- $\pi^{(j)}$ is the formal blow-up with center $\mathfrak{E}_j^{(j)}$, for some subset $J^{(j)}$ of $I^{(j)}$, with $|J^{(j)}| > 1$,
- $d$ is not $\mathfrak{X}^{(r)}$-linear.

Proof. The proof of [32, 5.17] carries over verbatim to this setting.

Definition 2.39. A resolution of singularities of a generically smooth flat $R$-variety $X$ (resp. a generically smooth, flat special formal $R$-scheme), is a proper morphism of flat $R$-varieties (resp. a morphism of flat special formal $R$-schemes) $h : X' \to X$, such that $h$ induces an isomorphism on the generic fibers, and such that $X'$ is regular, with as special fiber a strict normal crossings divisor $X'_s$. We say that the resolution $h$ is tame if $X'_s$ is a tame strict normal crossings divisor.

Lemma 2.40. Let $A$ be a special $R$-algebra, let $X$ be a stft formal $R$-scheme, and let $Z$ be a closed subscheme of $X_s$. Finally, let $U$ be a Noetherian scheme, and let $V$ be a closed subscheme of $U$.

If $\mathfrak{M}$ is an open prime ideal of $A$, defining a point $x$ of $\text{Spf } A$, then the local morphism $A_{\mathfrak{M}} \to \mathcal{O}_{\text{Spf } A, x}$ induces an isomorphism on the completions (w.r.t. the respective maximal ideals)

$$\hat{A}_{\mathfrak{M}} \cong \hat{\mathcal{O}}_{\text{Spf } A, x}$$

If $x$ is a point of $Z$, then the local morphism $\mathcal{O}_{X, x} \to \mathcal{O}_{X/\mathfrak{J}, x}$ induces an isomorphism on the completions

$$\hat{\mathcal{O}}_{X, x} \cong \hat{\mathcal{O}}_{X/\mathfrak{J}, x}$$

If $x$ is a point of $V$, then the local morphism $\mathcal{O}_{U, x} \to \mathcal{O}_{U/\mathfrak{V}, x}$ induces a canonical isomorphism on the completions

$$\hat{\mathcal{O}}_{U, x} \cong \hat{\mathcal{O}}_{U/\mathfrak{V}, x}$$

Proof. The first point is shown in the proof of [15, 1.2.1]. As for the second, if $J$ is the defining ideal sheaf of $Z$ on $X$, then

$$\frac{(\mathcal{O}_{X, x})/(J^n)}{\cong \mathcal{O}_{X/\mathfrak{J}, x} / (J^n)} \cong \frac{(\mathcal{O}_{X/\mathfrak{J}, x})/(J^n)}{\cong \mathcal{O}_{X/\mathfrak{J}, x} / (J^n)}$$

for each $n \geq 1$. The proof of the third point is analogous.
Lemma 2.41. Let $A$ be a special $R$-algebra, let $X$ be a separated scheme of finite type over $R$, or a stft formal $R$-scheme, and let $Z$ be a closed subscheme of $X$. Let $U$ be a Noetherian $R$-scheme, and let $V$ be a closed subscheme of $U$.

(1) $\text{Spec } A$ is regular iff $\text{Spf } A$ is regular. Moreover, $X$ is regular at the points of $Z$ iff $X/Z$ is regular, and $U$ is regular at the points of $V$ iff $U/V$ is regular.

(2) If $(\text{Spec } A)_s$ is a strict normal crossings divisor, then $(\text{Spf } A)_s$ is a strict normal crossings divisor. Moreover, if $X_s$ is a strict normal crossings divisor at the points of $Z$, then $(\widehat{X/Z})_s$ is a strict normal crossings divisor.

(3) $\text{Spf } A$ is generically smooth, iff $(A \otimes_R K)_M$ is geometrically regular over $K$, for each maximal ideal $M$ of $A \otimes_R K$. Moreover, if $X$ is generically smooth, then $\widehat{X/Z}$ is generically smooth.

(4) If $K$ is perfect, any regular special formal $R$-scheme $\mathfrak{x}$ is generically smooth.

Proof. Regularity of a local Noetherian ring is equivalent to regularity of its completion [23 17.1.5], so (1) follows from Lemma 2.40 the fact that regularity can be checked at maximal ideals, and the fact that any maximal ideal of an adic topological ring is open. Point (2) follows from the fact that, for any local Noetherian ring $S$, a tuple $(x_0, \ldots, x_m)$ in $S$ is a regular system of local parameters for $S$ iff it is a regular system of local parameters for $\widehat{S}$. The only delicate point is that we have to check if condition (2) in Definition 2.32 holds for $(\widehat{X/Z})_s$ and $(\widehat{U/V})_s$.

This, however, follows from Corollary 2.31.

Now we prove (3). By [12 2.8], smoothness of $X_\eta$ is equivalent to geometric regularity of $O_{X_\eta,x}$ over $K$, for each point $x$ of $X_\eta$. By [16 7.1.9], $x$ corresponds canonically to a maximal ideal $M$ of $A \otimes_R K$, and the completions of $O_{X_\eta,x}$ and $(A \otimes_R K)_M$ are isomorphic. We can conclude by using the same arguments as in [31 2.4(3)].

To prove the second part of (3), we may assume that $X$ is a stft formal $R$-scheme: if $X$ is a generically smooth separated scheme of finite type over $R$, then its $\pi$-adic completion $\widehat{X}$ is generically smooth by [31 2.4(3)], and we have $\widehat{X/Z} = \widehat{X}/Z$. If $X$ is a stft formal $R$-scheme, (3) follows from the fact that $(\widehat{X/Z})_\eta$ is canonically isomorphic to the tube $|Z|$, which is an open rigid subvariety of $X_\eta$.

For point (4), we may assume that $\mathfrak{x} = \text{Spf } A$ is affine. It suffices to show that $(A \otimes_R K)_M$ is geometrically regular over $K$ for each maximal ideal $M$, by point (3). But $A$ is regular by (1), and since $K$ is perfect, $(A \otimes_R K)_M$ is geometrically regular over $K$. □

Proposition 2.42. If $k$ has characteristic zero, any affine generically smooth flat special formal $R$-scheme $\mathfrak{x} = \text{Spf } A$ admits a resolution of singularities by means of admissible blow-ups.

Proof. By Temkin’s resolution of singularities for quasi-excellent schemes of characteristic zero [36], Spec $A$ admits a resolution of singularities $Y \to \text{Spec } A$ by means

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2The converse implication is false, as is seen by taking a regular flat formal curve $X$ over $R$ whose special fiber $X_s$ is an irreducible curve with a node $x$, and putting $Z = \{x\}$. 

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of blow-ups whose centers contain a power of $\pi$. Completing w.r.t. an ideal of definition $I$ of $A$ yields a resolution $h : \mathcal{Y} \to \text{Spf } A$ by means of admissible blow-ups, by Lemma 2.41(2).

2.6. Étale morphisms of special formal schemes. We define étale and adic étale morphisms of formal $R$-schemes as in [3, 2.6]. A local homomorphism of local rings $(A, \mathfrak{M}) \to (B, \mathfrak{N})$ is called unramified if $\mathfrak{N} = \mathfrak{M}B$ and $B/\mathfrak{N}$ is separable over $A/\mathfrak{M}$. We recall the following criterion.

Lemma 2.43. Let $h : \mathcal{Y} \to \mathcal{X}$ be a morphism of pseudo-finite type of Noetherian adic formal schemes, and let $y$ be a point of $\mathcal{Y}$. Then the following properties are equivalent:

1. The local homomorphism $h^* : \mathcal{O}_{\mathcal{X}, h(y)} \to \mathcal{O}_{\mathcal{Y}, y}$ is flat and unramified.
2. The local homomorphism $\widehat{h}^* : \widehat{\mathcal{O}}_{\mathcal{X}, h(y)} \to \widehat{\mathcal{O}}_{\mathcal{Y}, y}$ is flat and unramified.
3. $h$ is étale at $y$.

Proof. Use [3], (3.1), (4.5), (6.5).

In (2), the completions can be taken either w.r.t. the adic topologies on $\mathcal{X}$ and $\mathcal{Y}$, or w.r.t. the topologies defined by the respective maximal ideals.

Lemma 2.44. Let $\mathcal{X}$ be a regular special formal $R$-scheme (or a regular $R$-variety), such that $\mathcal{X}_s$ is a strict normal crossings divisor. Let $h : \mathcal{Y} \to \mathcal{X}$ be an étale morphism of adic formal schemes. Then $\mathcal{Y}$ is regular, and $\mathcal{Y}_s$ is a strict normal crossings divisor. It is tame if $\mathcal{X}_s$ is tame.

Proof. Let $y$ be a closed point on $\mathcal{Y}_s$, and put $x = h(y)$. Take a regular system of local parameters $(x_0, \ldots, x_m)$ in $\mathcal{O}_{\mathcal{X}, x}$, such that $\pi = u \prod_{i=0}^m x_i^{N_i}$, with $u$ a unit.

By Lemma 2.43, $h^* : \mathcal{O}_{\mathcal{X}, x} \to \mathcal{O}_{\mathcal{Y}, y}$ is flat and unramified. In particular, $(h^* x_0, \ldots, h^* x_m)$ is a regular system of local parameters in $\mathcal{O}_{\mathcal{Y}, y}$. It satisfies

$$\pi = h^* u \cdot \prod_{i=0}^m (h^* x_i)^{N_i}$$

Finally, it is clear that the irreducible components of $\mathcal{Y}_s$ are the connected components of the regular closed formal subschemes $\mathcal{Y}_s \times_{\mathcal{X}} \mathcal{E}_j$, with $\mathcal{E}_j, j \in J$ the irreducible components of $\mathcal{X}_s$.

3. Computation of nearby cycles on formal schemes

3.1. Algebraic covers.

Definition 3.1. Let $\mathcal{X}$ be a flat special formal $R$-scheme. A nice algebraizable cover for $\mathcal{X}$ at a closed point $x$ of $\mathfrak{X}_0$ is a surjective finite adic étale morphism of special formal $R$-schemes $\mathfrak{Y} \to \mathfrak{U}$, with $\mathfrak{U}$ a Zariski-open neighbourhood of $\mathfrak{X}$, such that $\mathfrak{Y}_0/\mathfrak{U}_0$ is a tame étale covering, and such that each point of $\mathfrak{Y}$ has a Zariski-open neighbourhood which is isomorphic to the formal completion of a regular $R$-variety $Z$ with tame strict normal crossings, along a closed subscheme of $Z_s$.

Proposition 3.2. We assume that $k$ is perfect. Let $\mathcal{X}$ be a regular special formal $R$-scheme with tame strict normal crossings. Then $\mathcal{X}$ admits a nice algebraizable cover at any closed point $x$ of $\mathfrak{X}_0$. 

Proof. We may assume that $\mathfrak{X}$ is affine, say $\mathfrak{X} = \text{Spf} A$, and that there exist elements $x_0, \ldots, x_m$ in $A$ with $\pi = u \prod_{i=0}^m x_i^{N_i}$, with $u$ a unit and $N_i \in \mathbb{N}$, and such that $(x_0, \ldots, x_m)$ is a regular system of local parameters on $\mathfrak{X}$ at $x$. Put $d := \text{gcd}(N_0, \ldots, N_m)$, and consider the finite étale morphism
\[
g : \mathfrak{Y} := \text{Spf} A[T]/(uT^d - 1) \to \mathfrak{X}
\]
Since $\mathfrak{X}_s$ is tame, $d$ is prime to the characteristic exponent $p$ of $k$, and $\mathfrak{Y}_0$ is a tame étale covering of $\mathfrak{X}_0$.

By Bezout’s Lemma, we can find integers $a_j, j = 0, \ldots, m$, such that $d = \sum_{j=0}^m a_j N_j$. On $\mathfrak{Y}$, we have $\pi = \prod_{j=0}^m (T^{-a_j} x_j)^{N_j}$. The sections $z_j := T^{-a_j} x_j$ on $\mathfrak{Y}$ define a morphism of formal schemes over $\text{Spec} R$
\[
h : \mathfrak{Y} \to Y := \text{Spec} R[y_0, \ldots, y_m]/(\pi - \prod_{j=0}^m y_j^{N_j})
\]
given by $h^*(y_i) = z_i$, mapping the fiber over $x$ to the origin. Moreover, $(z_0, \ldots, z_m)$ is a regular system of local parameters at any point $z$ of $\mathfrak{Y}$ lying over $x$, and by Lemma 2.43 $h$ is étale at $z$. Hence, shrinking $\mathfrak{X}$, we may assume that $h$ is étale on $\mathfrak{Y}$.

By [3, 7.12], if $z$ is any point of $\mathfrak{Y}$ lying over $x$, there exists a Zariski-open neighbourhood of $z$ in $\mathfrak{Y}$ which is a formal completion of an adic étale $Y$-scheme $Z$ along a closed subscheme of $Z_z$; $Z$ is automatically a regular $R$-variety with tame strict normal crossings, by Lemma 2.44.

In particular, every regular special formal $R$-scheme with tame strict normal crossings is algebraizable locally w.r.t. the étale topology. Combining this with Proposition 2.42, we see that, if $k$ has characteristic zero and $\mathfrak{X}$ is a generically smooth affine special formal $R$-scheme, then there exist a morphism of special formal $R$-schemes $h : \mathfrak{X}' \to \mathfrak{X}$ such that $h_\eta$ is an isomorphism, and an étale cover $\{U_i\}$ of $\mathfrak{X}'$ by algebraizable special formal $R$-schemes $U_i$.

3.2. Computation of the nearby cycles on tame strict normal crossings.
We can use the constructions in the preceding section to generalize Grothendieck’s computation of the tame nearby cycles on a tame strict normal crossings divisor [1 Exp. I], to the case of special formal $R$-schemes [7]. Let $\mathfrak{X}$ be a regular special formal $R$-scheme, such that the special fiber $\mathfrak{X}_s$ is a tame strict normal crossings divisor $\sum_{i \in I} N_i \mathfrak{E}_i$. For any non-empty subset $J$ of $I$, we denote by $M_J$ the kernel of the linear map
\[
\mathbb{Z}^J \to \mathbb{Z} : (z_j) \mapsto \sum_{j \in J} N_j z_j.
\]

Proposition 3.3. Suppose that $k$ is algebraically closed. Let $\mathfrak{X}$ be a regular special formal $R$-scheme, such that $\mathfrak{X}_s$ is a tame strict normal crossings divisor $\sum_{i \in I} N_i \mathfrak{E}_i$. Let $M$ be a torsion ring, with torsion orders prime to the characteristic exponent of $k$. For each non-empty subset $J \subset I$, and each $i \geq 0$, the $i$-th cohomology sheaf of tame nearby cycles $R^i \psi_\eta^J(M)$ associated to $\mathfrak{X}$, is tamely lisse on $E_0$. Moreover, for each $i > 0$, and each point $x$ on $E_J^0$, there are canonical isomorphisms
\[
R^i \psi_\eta^J(M)_x = R^0 \psi_\eta^J(M)_x \otimes \bigwedge M_J^i
\]
and \( R^0 \psi^t_0(M)_x \cong M^{F_J} \), where \( F_J \) is a set of cardinality \( m_J \), on which \( G(K^t/K) \) acts transitively.

**Proof.** By [7], Cor. 2.3, whenever \( h : \frak{V} \to \frak{X} \) is an adic étale morphism of special formal \( R \)-schemes, we have

\[
R^t \psi^t_0(M|_{\frak{V}_e}) \cong h_0^* R^t \psi^t_0(M|_{\frak{X}_e})
\]

Hence, by Proposition 3.2 we may suppose that \( \frak{X} \) is the formal completion of a regular \( R \)-variety \( X \) with tame strict normal crossings, along a closed subscheme of \( X_s \). By Berkovich’ Comparison Theorem [6, 5.1], it suffices to prove the corresponding statements for \( X \) instead of \( \frak{X} \). The computation of the fibers was done in [1, Exp. I]. To see that \( R^t \psi^t_0(M) \) is tamely lissee on \( E^t_J \), one can argue as follows: by the arguments in the proof of Proposition 3.2 one can reduce to the case

\[
X = \text{Spec } R[x_0, \ldots, x_m]/(\pi - x_0^{N_0} \cdots x_q^{N_q})
\]

with \( q \leq m \), \( N_j > 0 \) for each \( j \), and with \( E^t_J \) defined by \( x_0 = \ldots = x_q = 0 \). By smooth base change, one reduces to the case where \( m = q \), and \( E^t_J \) is the origin. Now the statement is trivial. \( \Box \)

**Corollary 3.4.** Suppose that \( k \) is algebraically closed. Let \( \frak{X} \) be a regular special formal scheme over \( R \), such that the special fiber \( \frak{X}_o \) is a tame strict normal crossings divisor \( \sum_{i \in I} N_i \frak{E}_i \). Let \( x \) be a closed point on \( E^t_J \), for some non-empty subset \( J \subset I \), and let \( e > 0 \) be an integer. Let \( \varphi \) be a topological generator of the tame geometric monodromy group \( G(K^t/K) \).

- \( Tr(\varphi^e | R^j \psi^t_0(\mathbb{Q}_\ell)_x) = 0 \), if \( |J| > 1 \), or if \( J \) is a singleton \( \{i\} \) and \( N_i \mid e \),
- \( Tr(\varphi^e | R^j \psi^t_0(\mathbb{Q}_\ell)_x) = N_i \), if \( J = \{i\} \) and \( N_i \mid e \).

**Corollary 3.5.** If \( k \) is algebraically closed, of characteristic zero, then \( R^j \psi^t_0(\mathbb{Q}_\ell) \) is constructible on \( \frak{X}_o \), for any generically smooth special formal \( R \)-scheme \( \frak{X} \), and the action of \( G(K^t/K) \) is continuous.

**Proof.** This follows from Proposition 2.42, Proposition 3.3, and Corollary 7, 2.3. \( \Box \)

### 4. Motivic Integration on Special Formal Schemes

Throughout this section, we assume that \( k \) is perfect. A possible approach to define motivic integration on special formal \( R \)-schemes \( \frak{X} \), would be to introduce the Greenberg scheme \( Gr^R(\frak{X}) \) of \( \frak{X} \) (making use of the fact that \( V(J) \) is of finite type over \( R \) if \( J \) is an ideal of definition on \( \frak{X} \)) and to generalize the constructions in [35] and [29] to this setting. We will take a shortcut, instead, making use of appropriate stft models for special formal \( R \)-schemes. The theory of motivic integration on stft formal \( R \)-schemes was developed in [35], and this theory was used to define motivic integrals of differential forms of maximal degree on smooth quasi-compact rigid varieties in [29]. The constructions were refined to a relative setting in [32], and extended to so-called “bounded” rigid varieties in [34].

**Definition 4.1.** Let \( \frak{X} \) be a special formal \( R \)-scheme. A Néron smoothening for \( \frak{X} \) is a morphism of special formal \( R \)-schemes \( \frak{Y} \to \frak{X} \) such that \( \frak{Y} \) is adic smooth over \( R \), and such that \( \frak{Y}_\eta \to \frak{X}_\eta \) is an open embedding satisfying \( \frak{Y}_\eta(K^{sh}) = \frak{X}_\eta(K^{sh}) \).
Remark. If $\mathfrak{X}$ is stft over $R$, we called this a weak Néron smoothening in [32], to make a distinction with a stronger variant of the definition. Since this distinction is irrelevant for our purposes, we omit the adjective “weak” from our terminology. □

Any generically smooth stft formal $R$-scheme $\mathfrak{Y}$ admits a Néron smoothening $\mathfrak{Y} \to \mathfrak{X}$, by [13 3.1]. Moreover, by [32 6.1], the class $[\mathfrak{Y}]_s$ of $\mathfrak{Y}$ in $\mathcal{M}_{X_0}/(L - [X_0])$ does not depend on $\mathfrak{Y}$. Following [29], we called this class the motivic Serre invariant of $\mathfrak{X}$, and denoted it by $S(\mathfrak{X})$. We will generalize this result to special formal $R$-schemes.

Lemma 4.2. If $\mathfrak{X}$ is a flat special formal $R$-scheme, and $h : \mathfrak{Y} \to \mathfrak{X}$ is the dilatation with center $X_0$, then $\mathfrak{Y}$ is stft over $R$, and $h$ induces an open embedding $\mathfrak{Y} \to \mathfrak{X}$ with $\mathfrak{Y}(K^h) = \mathfrak{X}(K^h)$.

Proof. By the universal property in Proposition 2.22 any $R^h$-section of $\mathfrak{X}$ lifts uniquely to $\mathfrak{Y}$. □

Proposition 4.3. Any generically smooth special formal $R$-scheme $\mathfrak{X}$ admits a Néron smoothening.

Proof. We may assume that $\mathfrak{X}$ is flat over $R$. Take $\mathfrak{Y}$ as in Lemma 4.2, it is generically smooth, since $\mathfrak{Y}_0$ is open in $\mathfrak{X}_0$. If $\mathfrak{Y} \to \mathfrak{X}$ is a Néron smoothening of $\mathfrak{Y}$, then the composed morphism $\mathfrak{Y} \to \mathfrak{X}$ is a Néron smoothening for $\mathfrak{X}$. □

Lemma 4.4. Let $\mathfrak{X}$ be a flat, generically smooth stft formal $R$-scheme, and let $U$ be a closed subscheme of $\mathfrak{X}_s$. If we denote by $\mathfrak{Y} \to \mathfrak{X}$ the dilatation with center $U$, then the image of $S(\mathfrak{Y})$ under the forgetful morphism

$$\mathcal{M}_{\mathfrak{Y}_0}/(L - [\mathfrak{Y}_0]) \to \mathcal{M}_U/(L - [U])$$

coincides with the image of $S(\mathfrak{X})$ under the base change morphism

$$\mathcal{M}_{\mathfrak{X}_0}/(L - [\mathfrak{X}_0]) \to \mathcal{M}_U/(L - [U])$$

Likewise, if $\omega$ is a differential form of maximal degree (resp. a gauge form) on $\mathfrak{X}_0$, then the image of $\int_{\mathfrak{Y}} |\omega|$ coincides with the image of $\int_{\mathfrak{X}} |\omega|$ in $\mathcal{M}_U$, resp. $\mathcal{M}_U$.

Proof. If we denote by $h : \mathfrak{Y} \to \mathfrak{X}$ the blow-up of $\mathfrak{X}$ with center $U$, then $\mathfrak{Y}$ is, by definition, an open formal subscheme of $\mathfrak{Y}^*$. If $g : \mathfrak{Y}^* \to \mathfrak{Y}$ is a Néron smoothening, and if we put $\mathfrak{Y}^* = g^{-1}(\mathfrak{Y})$, then by the universal property of the dilatation in Proposition 2.22, the induced open embedding of Greenberg schemes $Gr^R(\mathfrak{X}) \to Gr^R(\mathfrak{Y}^*)$ is an isomorphism onto the cylinder $(h_s \circ g_s \circ \theta_0)^{-1}(U)$ (here $\theta_0$ denotes the truncation morphism $Gr^R(\mathfrak{Y}^*) \to \mathfrak{Y}^*_s$).

We recall that the motivic integral $\int_{\mathfrak{X}} |\omega|$, with $\mathfrak{X}$ generically smooth and stft over $R$, was defined in [32 §6], refining the construction in [29 4.1.2].

Definition 4.5. Let $\mathfrak{X}$ be a generically smooth, flat special formal $R$-scheme, and let $h : \mathfrak{X}^* \to \mathfrak{X}$ be the dilatation with center $X_0$. We define the motivic Serre invariant $S(\mathfrak{X})$ of $\mathfrak{X}$ by

$$S(\mathfrak{X}) := S(\mathfrak{X}^*) \text{ in } \mathcal{M}_{\mathfrak{X}_0}/(L - [\mathfrak{X}_0])$$

If $\omega$ is a differential form of maximal degree (resp. a gauge form) on $\mathfrak{X}_0$, then we put

$$\int_{\mathfrak{X}} |\omega| := \int_{\mathfrak{X}^*} |\omega|$$
If $X$ is a generically smooth special formal $R$-scheme, we denote by $X^{\text{flat}}$ its maximal flat closed subscheme (obtained by killing $\pi$-torsion) and we put $S(X) = S(X^{\text{flat}})$ and

$$\int_X |\omega| := \int_{X^{\text{flat}}} |\omega|$$

It follows from Lemma 4.4 that this definition coincides with the usual one if $X$ is stft over $R$. Note that even in this case, the dilatation $X' \to X$ is not necessarily an isomorphism, since $X_s$ might not be reduced. In fact, $X'$ might be empty, for instance if $X$ has strict normal crossings with all multiplicities > 1.

**Proposition 4.6.** Let $X$ be a generically smooth special formal $R$-scheme. If $Y \to X$ is a Néron smoothening for $X$, then

$$S(X) = [Y_0] \in \mathcal{M}_{X_0}/(\mathbb{L} - [X_0])$$

For any differential form of maximal degree (resp. gauge form) $\omega$ on $X_\eta$, we have

$$\int_X |\omega| = \int_Y |\omega|$$

in $\hat{\mathcal{M}}_{X_0}$, resp. $\mathcal{M}_{X_0}$.

**Proof.** We may assume that $X$ is flat over $R$; let $X' \to X$ be the dilatation with center $X_0$. By the universal property of the dilatation in Proposition 2.22 and the fact that $Y_0$ is reduced, any Néron smoothening $Y \to X$ factors through a morphism of stft formal $R$-schemes $Y \to X'$, and by Lemma 4.2 this is again a Néron smoothening. So the result follows from (the proof of) [32, 6.11] (see also [34] for an addendum on the mixed dimension case).

We showed in [34] that the generic fiber $X_\eta$ of a generically smooth special formal $R$-scheme $X$ is a so-called “bounded” rigid variety over $K$ (this also follows immediately from Lemma 1.2), and we defined the motivic Serre invariant $S(X_\eta)$ of $X_\eta$, as well as motivic integrals $\int_{X_\eta} |\omega|$ of differential forms $\omega$ on $X_\eta$ of maximal degree.

**Proposition 4.7.** If $X$ is a generically smooth special formal $R$-scheme, then $S(X_\eta)$ is the image of $S(X)$ under the forgetful morphism

$$\mathcal{M}_{X_0}/(\mathbb{L} - [X_0]) \to \mathcal{M}_k/(\mathbb{L} - 1)$$

If $\omega$ is a gauge form on $X_\eta$, then $S(X)$ is the image of $\int_X |\omega|$ in $\mathcal{M}_{X_0}/(\mathbb{L} - [X_0])$. If $\omega$ is a differential form of maximal degree (resp. a gauge form) on $X_\eta$, then $\int_{X_\eta} |\omega|$ is the image of $\int_X |\omega|$ under the forgetful morphism $\hat{\mathcal{M}}_{X_0} \to \hat{\mathcal{M}}_k$, resp. $\mathcal{M}_{X_0} \to \mathcal{M}_k$.

**Proof.** This is clear from the definitions, and the corresponding properties for stft formal $R$-schemes [32, 6.4].

**Definition 4.8.** Suppose that $k$ has characteristic zero. Let $X$ be a generically smooth special formal $R$-scheme. Let $\omega$ be a gauge form on $X_\eta$. We define the volume Poincaré series of $(X, \omega)$ by

$$S(X, \omega; T) := \sum_{d > 0} \left( \int_{X(d)} |\omega(d)| \right) T^d \in \mathcal{M}_{X_0}[[T]]$$
Remark. The motivic Serre invariant $S(X)$ and the motivic integral $\int_X |\omega'|$ (for any differential form $\omega'$ of maximal degree on $X_0$) are independent of the choice of uniformizer $\pi$. The volume Poincaré series, however, depends on the choice of $\pi$, or more precisely, on the $K$-fields $K(d)$. If $k$ is algebraically closed, then $K(d)$ is the unique extension of degree $d$ of $K$, up to $K$-isomorphism, and $S(X, \omega; T)$ is independent of the choice of $\pi$. See also the remark following Definition 7.37. □

Proposition 4.9. Let $X$ be a generically smooth special formal $R$-scheme, and let $U$ be a locally closed subscheme of $X_0$. Denote by $\mathcal{U}$ the formal completion of $X$ along $U$. Then $S(\mathcal{U})$ is the image of $S(X)$ under the base change morphism

$$\mathcal{M}_{X_0}/(L - [X_0]) \to \mathcal{M}_U/(L - [U])$$

If $\omega$ is a gauge form on $X_0$, then $\int_{\mathcal{U}} |\omega|$ is the image of $\int_X |\omega|$ under the base change morphism

$$\mathcal{M}_{X_0} \to \mathcal{M}_U$$

(the analogous statement holds if $\omega$ is merely a differential form of maximal degree).

If, moreover, $k$ has characteristic zero, then $S(\mathcal{U}, \omega; T)$ is the image of $S(X, \omega; T)$ under the base-change morphism

$$\mathcal{M}_{X_0}[[T]] \to \mathcal{M}_U[[T]]$$

In particular, if $X$ is stft over $R$, then $S(\mathcal{U})$ and $S(\mathcal{U}, \omega; T)$ coincide with the invariants with support $S_U(X)$ and $S_U(X, \omega; T)$ defined in [31].

Proof. We may assume that $X$ is flat. If $U$ is open in $X_0$, then these results are clear from the definitions, since dilatations commute with flat base change; so we may suppose that $U$ is a reduced closed subscheme of $X_0$. Let $h : X' \to X$ be the dilatation with center $X_0$, and let $\mathcal{U}' \to \mathcal{U}$ be the dilatation with center $\mathcal{U}_0 = U$. By Proposition 2.23, there exists a unique morphism of stft formal $R$-schemes $\mathcal{U}' \to X'$ such that the square

$$\begin{array}{ccc}
\mathcal{U}' & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X
\end{array}$$

commutes, and $\mathcal{U}' \to X'$ is the dilatation with center $X'_0 \times_{X_0} U$. Now we can conclude by Lemma 1.4. □

Corollary 4.10. If $\{U_i, i \in I\}$ is a finite stratification of $X_0$ into locally closed subsets, and $\mathcal{U}_i$ is the formal completion of $X$ along $U_i$, then

$$S(X) = \sum_{i \in I} S(\mathcal{U}_i)$$

$$\int_X |\omega| = \sum_{i \in I} \int_{\mathcal{U}_i} |\omega|$$

$$S(X, \omega; T) = \sum_{i \in I} S(\mathcal{U}_i, \omega; T)$$

(we applied the forgetful morphisms $\mathcal{M}_{U_i} \to \mathcal{M}_{X_0}$ to the right-hand sides).
Proposition 4.12. If \( X \) is a generically smooth flat special formal \( R \)-scheme, then there exists a composition of admissible blow-ups \( Y \to X \) such that \( Sm(Y) \to X \) is a special Néron smoothing for \( X \).

Proof. Let \( I \) be the largest ideal of definition for \( X \), and let \( h : X' \to X \) be the admissible blow-up with center \( I \). We denote by \( U \) the open formal subscheme of \( X' \) where \( I \mathcal{O}_{X'} \) is generated by \( \pi \), i.e. \( U \to X \) is the dilatation with center \( X_0 \), and \( \mathcal{U} \) is stft over \( R \).

By [13, 3.1] and [10, 2.5], there exists an admissible blow-up \( g : Y' \to U \) such that \( Sm(Y') \to U \) is a Néron smoothing. We will show that this blow-up extends to an admissible blow-up \( Y \to X \). Choose an integer \( j > 0 \) such that \( \pi^j \) is contained in the center \( \mathcal{J} \) of \( g \). By [22, 9.4.7], the pull-back of \( \mathcal{J} \) to \( V(I' \mathcal{O}_U) \) extends to a coherent ideal sheaf \( \mathcal{J}' \) on \( V(I' \mathcal{O}_X') \); we'll denote again by \( \mathcal{J}' \) the corresponding coherent ideal sheaf on \( X' \). Since formal blow-ups commute with open embeddings, the admissible blow-up \( Y \to X' \) with center \( \mathcal{J}' \) extends \( g \).

Finally, let us show that the composed morphism \( Sm(Y) \to X \) is a special Néron smoothing. It suffices to show that the natural map \( Sm(Y)_{\eta}(K^{sh}) \to X_{\eta}(K^{sh}) \) is surjective. By Lemma [32, Lemma 4.2], \( X_{\eta}(K^{sh}) \to \mathcal{U}_\eta(K^{sh}) \) is surjective, and since \( Sm(Y') \to U \) is a Néron smoothing, \( Sm(Y')_{\eta}(K^{sh}) \to \mathcal{U}_\eta(K^{sh}) \) is surjective; but \( Y' \) is an open formal subscheme of \( Y \), so the result follows.

□

Proposition 4.13. If \( X \) is a smooth special formal \( R \)-scheme, then

\[
S(X) = [X_0]
\]

in \( \mathcal{M}_{X_0}/(L - [X_0]) \).

Proof. Stratifying \( X_0 \) in regular pieces, we might as well assume that \( X_0 \) is regular from the start, by Corollary [4.10]. Moreover, we may suppose that \( X = \text{Spf} \mathcal{A} \) is affine, and that \( X_0 \) is defined by a regular sequence \( (\pi,x_1,\ldots,x_q) \) in \( \mathcal{A} \). The dilatation of \( X \) with center \( X_0 \) is given by

\[
Y := \text{Spf} \mathcal{A}[T_1,\ldots,T_q]/(x_i - \pi T_i)_{i=1,\ldots,q} \to X
\]

Now \( Y \) is flat, and

\[
Y_0 = \text{Spec} (\mathcal{A}/(\pi,x_1,\ldots,x_q))[T_1,\ldots,T_q] \cong X_0 \times_k \mathbb{A}^q_k
\]

Since \( k \) is perfect and \( X_0 \) is regular, \( Y_0 \) is smooth over \( k \), and hence \( Y \) is smooth over \( R \), and

\[
S(X) = S(Y) = [Y_0] = [X_0]
\]

in \( \mathcal{M}_{X_0}/(L - [X_0]) \).

□

Corollary 4.14. If \( h : Y \to X \) is a special Néron smoothing, then

\[
S(X) = [Y_0] \in \mathcal{M}_{X_0}/(L - [X_0])
\]
5. Computation of Serre invariants and motivic integrals

Throughout this section, we assume that $k$ is perfect.

5.1. Serre invariants of the ramifications. If $\tilde{X}$ is a special formal $R$-scheme, we denote by $\tilde{X} \to X$ the normalization morphism [15].

**Theorem 5.1.** Let $X$ be a regular special formal $R$-scheme, such that $X_s$ is a tame strict normal crossings divisor $\sum_{i \in I} N_i E_i$, and let $d > 0$ be an integer, prime to the characteristic exponent of $k$. If $d$ is not $X_s$-linear, then

$$h : Sm(\tilde{X}(d)) \to X(d)$$

is a special Néron smoothening. Moreover, if we put

$$\tilde{E}(d)_i = (\tilde{X}(d))_0 \times_{X_0} E_i$$

for each $i$ in $I$, then

$$Sm(\tilde{X}(d))_0 = \bigsqcup_{N_i | d} \tilde{E}(d)_i$$

and the $E_i$-variety $\tilde{E}(d)_i$ is canonically isomorphic to $\tilde{E}_i$ (defined in Section 2.5).

**Proof.** The fact that $Sm(\tilde{X}(d))_0 = \bigsqcup_{N_i | d} \tilde{E}(d)_i$ and that the $E_i$-variety $\tilde{E}(d)_i$ is canonically isomorphic to $\tilde{E}_i$, can be proven exactly as in [31, 4.4].

Since $X(d)_0$ is smooth and, a fortiori, normal, and normalization commutes with taking generic fibers [14 2.1.3], $h_\eta$ is an open embedding. Since $d$ is not $X_s$-linear, the obvious generalization of [32, 5.15] implies that

$$\left( Sm(\tilde{X}(d))_\eta \right) (K(d)^{sh}) = X(d)_\eta (K(d)^{sh})$$

i.e. $h$ is a special Néron smoothening. □

**Corollary 5.2.** Let $X$ be a regular special formal $R$-scheme, such that $X_s$ is a strict normal crossings divisor $\sum_{i \in I} N_i E_i$, and let $d > 0$ be an integer, prime to the characteristic exponent of $k$.

If $d$ is not $X_s$-linear, then

$$S(X(d)) = \sum_{i \in I, N_i | d} \lceil \tilde{E}_i \rceil$$

in $\mathcal{M}_{X_s}/(\mathbb{L} - [X_0])$.

**Proof.** If $X'$ is the completion of $X$ along $\sqcup_{N_i | d} E_i$, then $S(X(d)) = S(X'(d))$, since $d$ is not $X_s$-linear, by a straightforward generalization of [32, 5.15]. Hence, we may as well assume that $X' = \tilde{X}$. In this case, since $d$ is prime to the characteristic exponent of $k$, $X_s$ is tame, so we can use Theorem 5.1 and Corollary 4.14 to conclude. □
5.2. Order of a top form at a section. This subsection is a straightforward generalization of [31 §6.2]. Let $\mathfrak{X}$ be a generically smooth special formal $R$-scheme, of pure relative dimension $m$. Let $R'$ be a finite extension of $R$, of ramification index $e$, and denote by $K'$ its quotient field.

**Definition 5.3.** For any element $\psi$ of $\mathfrak{X}(R')$, and any ideal sheaf $\mathcal{I}$ on $\mathfrak{X}$, we define $\text{ord}(\mathcal{I})(\psi)$ as the length of the $R'$-module $R'/\psi^*\mathcal{I}$.

We recall that the length of the zero module is 0, and the length of $R'$ is $\infty$.

For any element $\psi$ of $\mathfrak{X}(R')$, the $R'$-module $M := (\psi^*\Omega^m_{\mathfrak{X}/R})/(\text{torsion})$ is a free rank 1 submodule of pure relative dimension $m$.

**Definition 5.4.** For any global section $\omega$ of $\Omega^m_{\mathfrak{X}/R}$ and any section $\psi$ in $\mathfrak{X}(R')$, we define the order $\text{ord}(\omega)(\psi)$ of $\omega$ at $\psi$ as follows: we choose an integer $a \geq 0$ such that $\omega' := \pi^a\psi^*(\omega)$ belongs to the submodule $M$ of $(\psi^*\Omega^m_{\mathfrak{X}/R})/(\text{torsion})$, and we put

$$\text{ord}(\omega)(\psi) = \text{length}_{R'}(M/R'\omega') - e.a$$

This definition does not depend on $a$.

If $e = 1$, this definition coincides with the one given in [29 4.1]. It only depends on the completion of $\mathfrak{X}$ at $\psi(0) \in \mathfrak{X}_0$. If $\omega$ is a gauge form on $\mathfrak{X}_n$, $\text{ord}(\omega)(\psi)$ is finite.

Now let $h : \mathfrak{Z} \to \mathfrak{X}$ be a morphism of generically smooth special formal $R$-schemes, both of pure relative dimension $m$. Let $R'$ be a finite extension of $R$, and fix a section $\psi$ in $\mathfrak{Z}(R')$. The canonical morphism

$$h^*\Omega^m_{\mathfrak{X}/R} \to \Omega^m_{\mathfrak{Z}/R}$$

induces a morphism of free rank 1 $R'$-modules

$$(\psi^*h^*\Omega^m_{\mathfrak{X}/R})/(\text{torsion}) \to (\psi^*\Omega^m_{\mathfrak{Z}/R})/(\text{torsion})$$

We define $\text{ord}(\mathcal{J}ac_h)(\psi)$ as the length of its cokernel.

If $\Omega^m_{\mathfrak{Z}/R}/(\text{torsion})$ is a locally free rank 1 module over $O_\mathfrak{Z}$, we define the Jacobian ideal sheaf $\mathcal{J}ac_h$ of $h$ as the annihilator of the cokernel of the morphism

$$h^*\Omega^m_{\mathfrak{X}/R} \to \Omega^m_{\mathfrak{Z}/R}/(\text{torsion})$$

and we have $\text{ord}(\mathcal{J}ac_h)(\psi) = \text{ord}(\mathcal{J}ac_h)(\psi)$. The following lemmas are proved as their counterparts [31], Lemma 6.4-5.

**Lemma 5.5.** Let $h : \mathfrak{Z} \to \mathfrak{X}$ be a morphism of generically smooth special formal $R$-schemes, both of pure relative dimension $m$. Let $R'$ be a finite extension of $R$. For any global section $\omega$ of $\Omega^m_{\mathfrak{X}/K}$, and any $\psi' \in \mathfrak{Z}(R')$,

$$\text{ord}(h^*\omega)(\psi) = \text{ord}(\omega)(h(\psi)) + \text{ord}(\mathcal{J}ac_h)(\psi)$$

**Lemma 5.6.** Let $e \in \mathbb{N}^*$ be prime to the characteristic exponent of $k$. Let $\mathfrak{X}$ be a regular special formal $R$-scheme with tame strict normal crossings. Let $\omega$ be a global section of $\Omega^m_{\mathfrak{X}/K}$. We denote by $\omega(e)$ the pullback of $\omega$ to the generic fiber of $\mathfrak{X}(e)$.

Let $R'$ be a finite extension of $R(e)$, and let $\psi(e)$ be a section in $\text{Sm}(\mathfrak{X}(e))(R')$. If we denote by $\psi$ its image in $\mathfrak{X}(R')$, then

$$\text{ord}(\omega)(\psi) = \text{ord}(\omega(e))(\psi(e))$$
5.3. Order of a gauge form on a smooth formal $R$-scheme. Let $X$ be a smooth special formal $R$-scheme of pure relative dimension $m$, and let $\omega$ be a $X$-bounded gauge form on $X_\eta$ (see Definition 2.11). We denote by $\text{Irr}(X_0)$ the set of irreducible components of $X_0$ (note that $X_0$ is not always smooth over $k$; see Example 2.30). Let $C$ be an irreducible component of $X_0$, and let $\xi$ be its generic point. The local ring $\mathcal{O}_{X,\xi}$ is a UFD, since $X$ is regular. Since $X$ is smooth over $R$, $(\Omega^n_{X/R})_\xi$ is a free rank 1 module over $\mathcal{O}_{X,\xi}$ by [4, 4.8] and [3, 5.10], and $\pi$ is irreducible in $\mathcal{O}_{X,\xi}$. 

**Definition 5.7.** Let $A$ be a UFD, and let $a$ be an irreducible element of $A$. Let $N$ be a free $A$-module of rank one, and let $n$ be an element of $N$. We choose an isomorphism of $A$-modules $A \cong N$, and we define $\text{ord}_an$ as follows: if $n \neq 0$, $\text{ord}_an$ is the largest $q \in \mathbb{N}$ such that $a^q|n$ in $A$. If $n = 0$, we put $\text{ord}_an = \infty$. This definition does not depend on the choice of isomorphism $A \cong N$.

**Definition 5.8.** Let $X$ be a smooth special formal $R$-scheme of pure relative dimension $m$, let $C$ be an irreducible component of $X_0$, and denote by $\xi$ its generic point. If $\omega$ is a $X$-bounded $m$-form on $X_\eta$, we can choose $b \in \mathbb{N}$ such that $\pi^b\omega$ extends to a section $\omega'$ of $(\Omega^n_{X/R})_\xi$, and we put

$$\text{ord}_C\omega := \text{ord}_\pi\omega' - b$$

This definition does not depend on $b$.

**Lemma 5.9.** Let $X$ be a smooth connected special formal $R$-scheme, and let $f$ be an element of $\mathcal{O}_X(\xi)$. If $f$ is a unit on $X_\eta$, and $f$ is not identically zero on $X_0$, then $f$ is a unit on $X$.

**Proof.** We may assume that $X = \text{Spf}A$ is affine. By the correspondence between maximal ideals of $A \otimes_R K$ and points of $X_\eta$ explained in [10, 7.1.9], we see that $f$ is a unit in $A \otimes_R K$, so there exists an element $q \in A$ and an integer $i \geq 0$ such that $f^iq = \pi^i$. We may assume that either $i = 0$ (in which case $f$ is a unit), or $q$ is not divisible by $\pi$ in $A$. Since $X$ is smooth, $\pi$ is a prime in $A$, so $\pi$ divides $f$ if $f$ is not a unit; this contradicts the hypothesis that $f$ does not vanish identically on $X_0$. \hfill \Box

**Lemma 5.10.** Let $X$ be a smooth special formal $R$-scheme, and let $\omega$ be a $X$-bounded gauge form on $X_\eta$. Let $R'$ be a finite unramified extension of $R$, and consider a section $\psi \in \mathcal{O}_X(R')$. If $C$ is an irreducible component of $X_0$ containing $\psi(0)$, then

$$\text{ord}_C(\omega) = \text{ord}(\omega)(\psi)$$

**Proof.** We denote by $\xi$ the generic point of $C$. Multiplying with powers of $\pi$, we may assume that $\omega$ is defined on $X$. Moreover, we may assume that there exists a section $\omega_0$ in $\Omega^n_{X/R}(X/R)$ which generates $\Omega^n_{X/R}$ at each point of $X$, and we write $\omega = f\omega_0$ with $f \in \mathcal{O}_X(\xi)$. Dividing by an appropriate power of $\pi$, we may assume that $\pi^i \not| f$ in $\mathcal{O}_{X,\xi}$ and $\text{ord}_C(\omega) = 0$. Since $\omega$ is gauge on $X_\eta$, $f$ is a unit on $X_\eta$, so by Lemma 5.9 $f$ is a unit on $X$. Hence,

$$\text{ord}(\omega)(\psi) = \text{ord}_\pi\psi^*(f) = 0$$

\hfill \Box

**Corollary 5.11.** If $C_1$ and $C_2$ are irreducible components of the same connected component $C$ of $X_0$, then $\text{ord}_{C_1}(\omega) = \text{ord}_{C_2}(\omega)$, and we denote this value by $\text{ord}_C(\omega)$. 

Proof. We can always find a section $\psi$ in $\mathfrak{X}(R')$ with $R'/R$ finite and unramified, and with $\psi(0) \in C_1 \cap C_2$. □

**Corollary 5.12.** If $\mathfrak{X}_0$ is connected, and $Z$ is a locally closed subset of $\mathfrak{X}_0$, then
$$\text{ord}_C(\omega) = \text{ord}_{\mathfrak{X}_0}(\omega)$$
for any connected component $C$ of $Z$, where the left hand side is computed on the completion $\mathfrak{X}/Z$.

**Lemma 5.13.** Let $\mathfrak{X}$ be a smooth connected special formal $R$-scheme, and let $Z$ be a regular closed subscheme of $\mathfrak{X}_0$. If we denote by $h : Z \rightarrow \mathfrak{X}$ the dilatation with center $Z$, and by $c$ the codimension of $Z$ in $\mathfrak{X}$, then for any $\mathfrak{X}$-bounded gauge form $\omega$ on $\mathfrak{X}$,
$$\text{ord}_{\mathfrak{X}_0}(\omega) = \text{ord}_Z(\omega) + c$$
Proof. By Corollary 5.12 and Proposition 2.21, we may assume that $\mathfrak{X}_0 = Z$. Now the result follows from Lemma 5.10 and Lemma 5.5, since $J_{\text{ac}} = (\pi^{c-1})$. □

**Proposition 5.14.** Let $\mathfrak{X}$ be a smooth special formal $R$-scheme, of pure relative dimension $m$, and denote by $C(\mathfrak{X}_0)$ the set of connected components of $\mathfrak{X}_0$. For any $\mathfrak{X}$-bounded gauge form $\omega$ on $\mathfrak{X}$,
$$\int_{\mathfrak{X}} |\omega| = \mathbb{L}^{-m} \sum_{C \in C(\mathfrak{X}_0)} |C| \mathbb{L}^{-\text{ord}_C(\omega)}$$
in $\mathcal{M}_{\mathfrak{X}_0}$.

Proof. By Corollaries 4.10 and 5.12 we may assume that $\mathfrak{X}_0$ is regular and connected. By definition,
$$\int_{\mathfrak{X}} |\omega| = \int_{Z} |\omega|$$
where $Z \rightarrow \mathfrak{X}$ is the dilatation with center $\mathfrak{X}_0$. In the proof of Proposition 4.13 we saw that $Z$ is smooth, and $\mathfrak{Y}_0 = [\mathfrak{X}_0]\mathbb{L}^{c-1}$, with $c$ the codimension of $\mathfrak{X}_0$ in $\mathfrak{X}$. Hence, we can conclude by Lemma 5.13. □

**Corollary 5.15.** Let $\mathfrak{X}$ be a generically smooth special formal $R$-scheme of pure relative dimension $m$, and let $\omega$ be a gauge form on $\mathfrak{X}$, and $Z$ a variety over $F$. If $\mathfrak{Y} \rightarrow \mathfrak{X}$ is a special Néron smoothening, then
$$\int_{\mathfrak{X}} |\omega| = \mathbb{L}^{-m} \sum_{C \in C(\mathfrak{Y}_0)} |C| \mathbb{L}^{-\text{ord}_C(\omega)}$$
in $\mathcal{M}_{\mathfrak{X}_0}$.

6. The trace formula

Let $F$ be any field, let $Z$ be a variety over $F$, and let $A$ be an abelian group. The abelian group $C(Z, A)$ of constructible $A$-functions on $Z$ is the subgroup of the abelian group of functions of sets $Z \rightarrow A$, consisting of mappings of the form
$$\varphi = \sum_{S \in \mathcal{S}} a_S 1_S$$
where $\mathcal{S}$ is a finite stratification of $Z$ into constructible subsets, and $a_S \in A$ for $S \in \mathcal{S}$, and where $1_S$ denotes the characteristic function of $S$. Note that a constructible function on $Z$ is completely determined by its values on the set of closed points $Z^o$.
of $Z$. If $A$ is a ring, $C(Z,A)$ carries a natural ring structure. If $A = Z$, we call $C(Z,A)$ the ring of constructible functions on $Z$, and we denote it by $C(Z)$.

For any constructible $A$-function

$$\varphi = \sum_{S \in \mathcal{S}} a_S \mathbf{1}_S$$

on $Z$ as above, we can define its integral w.r.t. the Euler characteristic as follows:

$$\int_Z \varphi d\chi := \sum_{S \in \mathcal{S}} a_S \chi_{\text{top}}(S)$$

If the group operation on $A$ is written multiplicatively, we write $\int_X \times Z$ instead of $\int_Z$.

The calculus of integration with respect to the Euler characteristic was, to our knowledge, first introduced in [39].

**Lemma 6.1.** Suppose that $F$ is algebraically closed, and let $Z$ be a variety over $F$. Let $\ell$ be a prime number, invertible in $F$, and let $\mathcal{L}$ be a tamely constructible $\mathbb{Q}_\ell$-adic sheaf on $Z$. Suppose that a finite cyclic group $G$ with generator $g$ acts on $\mathcal{L}$. We denote by $Z^o$ the set of closed points on $Z$.

1. The mapping

$$\text{Tr}(g | \mathcal{L}^e) : Z^o \to \mathbb{Q}_\ell^e : x \mapsto \text{Tr}(g | \mathcal{L}_x)$$

defines a constructible $\mathbb{Q}_\ell^e$-function on $Z$, and

$$\text{Tr}(g | \bigoplus_{i \geq 0} H^i_c(Z, \mathcal{L})) = \int_Z \text{Tr}(g | \mathcal{L}) d\chi$$

2. The mapping

$$\zeta(g | \mathcal{L}^e; T) : Z^o \to \mathbb{Q}_\ell[[T]]^\times : x \mapsto \zeta(g | \mathcal{L}_x; T)$$

defines a constructible $\mathbb{Q}_\ell[[T]]^\times$-function on $Z$, and

$$\zeta(g | \bigoplus_{i \geq 0} H^i_c(Z, \mathcal{L}); T) = \int_Z \zeta(g | \mathcal{L}^e; T) d\chi$$

**Proof.** First, we prove (1). By additivity of $H^i_c(\cdot)$, we may suppose that $Z$ is normal and $\mathcal{L}$ is tamely lisse on $Z$. In this case, $\text{Tr}(g | \mathcal{L})$ is constant on $Z$, and the result follows from [31, 5.1]. Now (2) follows from the identity [17, 1.5.3]

$$\det(\text{Id} - T.M | V)^{-1} = \exp(\sum_{d > 0} \text{Tr}(M^d | V) \frac{T^d}{d})$$

for any endomorphism $M$ on a finite dimensional vector space $V$ over a field of characteristic zero. □

**Lemma 6.2.** Let $G$ be a finite group, let $F$ be an algebraically closed field, and fix a prime $\ell$, invertible in $F$. If $f : Y \to X$ is a morphism of separated $F$-schemes of finite type, and $\mathcal{L}$ is a constructible $E[G]$-sheaf on $Y$, for some finite extension $E$ of $\mathbb{Q}_\ell$, then

$$f_*[\mathcal{L}] = f_! [\mathcal{L}] \text{ in } K_0(X; E[G])$$

In particular, if $X = \text{Spec } F$, then

$$\text{Tr}(g | \bigoplus_{i \geq 0} H^i_c(Y, \mathcal{L})) = \text{Tr}(g | \bigoplus_{i \geq 0} H^i_c(Y, \mathcal{L}))$$

for each element $g$ of $G$. 


Proof. If $G$ is the trivial group, then this is a well-known theorem of Laumon’s. His proof carries over verbatim to the case where $G$ is any finite group. □

Corollary 6.3. Let $G = \langle g \rangle$ be a finite cyclic group. Let $F$ be an algebraically closed field, let $U$ be a variety over $F$, and let $L$ be a tamely constructible $\mathbb{Q}_\ell[G]$-sheaf on $U$, for any prime $\ell$ invertible in $F$. Then

$$\text{Tr}(g \mid \oplus_{i \geq 0} H^i(U, L)) = \text{Tr}(g \mid \oplus_{i \geq 0} H^i_\ell(U, L)) = \int_U \text{Tr}(g \mid L_\ast d\chi)$$

Proof. The first equality follows from Lemma 6.2, while the equality between the second and the third expression follows from Lemma 6.1. □

The following theorem is a broad generalization of [31, 5.4]. It implies, in particular, that the assumptions that $X$ is algebraic and $Z$ is proper, are superfluous in the statement of [31, 5.4].

Theorem 6.4 (Trace formula). Assume that $k$ is perfect. Let $\varphi$ be a topological generator of the tame geometric monodromy group $G(K \times K^h)$. Let $\mathfrak{X}$ be a generically smooth special formal $R$-scheme, and suppose that $\mathfrak{X}$ admits a tame resolution of singularities $h : \mathfrak{Y} \to \mathfrak{X}$, with $\mathfrak{Y}_s = \sum_{i \in I} N_i \mathfrak{E}_i$. For any integer $d > 0$, prime to the characteristic exponent of $k$, we have

$$\chi_{\text{top}}(S(\mathfrak{X}_\eta(d))) = \text{Tr}(\varphi^d \mid H(\mathfrak{X}_\eta)) = \sum_{N_i \mid d} N_i \chi_{\text{top}}(E_i^o)$$

Proof. We may assume that $\mathfrak{Y} = \mathfrak{X}$, and that $k$ is algebraically closed (since the motivic Serre invariant is clearly compatible with unramified extensions of the base $R$). The equality

$$\chi_{\text{top}}(S(\mathfrak{X}_\eta(d))) = \sum_{N_i \mid d} N_i \chi_{\text{top}}(E_i^o)$$

can be proven as in [31, 5.4]: the expression holds if $d$ is not $\mathfrak{X}_\ast$-linear, by Corollary 5.2 and the fact that $E_i^o$ is a degree $N_i$ finite étale cover of $E_i$. Moreover, the right hand side of the equality does not change under blow-ups with center $\mathfrak{E}_J$ with $\emptyset \neq J \subset I$, by the obvious generalization of [31, 5.2], so the expression holds in general by Lemma 2.38.

So it suffices to prove that

$$\text{Tr}(\varphi^d \mid H(\mathfrak{X}_\eta)) = \sum_{N_i \mid d} N_i \chi_{\text{top}}(E_i^o)$$

By [7, 2.3(ii)], there is for each $i \geq 0$ a canonical isomorphism

$$H^i(\mathfrak{X}_0, R\psi^i_\eta(\mathbb{Q}_\ell|_{\mathfrak{X}_\eta})) \cong H^i(\mathfrak{X}_\eta, \mathbb{Q}_\ell)$$

and hence,

$$(6.1) \quad \text{Tr}(\varphi^d \mid H(\mathfrak{X}_\eta)) = \text{Tr}(\varphi^d \mid H(\mathfrak{X}_0, R\psi^i_\eta(\mathbb{Q}_\ell|_{\mathfrak{X}_\eta})))$$

By our local computation in Proposition 5.3 we can filter $R^i\psi^i_\eta(\mathbb{Q}_\ell|_{\mathfrak{X}_\eta})$ by constructible subsheaves which are stable under the monodromy action and such that the action of $\varphi$ on successive quotients has finite order. Hence, we can apply Lemma 6.3. Combined with the computation in Corollary 3.4 we obtain the required equality. □
Corollary 6.5. If $k$ has characteristic zero, and $X$ is a generically smooth special formal $R$-scheme, then
\[ \chi_{\text{top}}(S(\mathcal{X}_\eta(d))) = Tr(\varphi^d | H(\mathcal{X}_\eta)) \]
for any integer $d > 0$. In particular,
\[ \chi_{\text{top}}(S(X(d))) = Tr(\varphi^d | H(X)) \]
for any smooth quasi-compact rigid variety $X$ over $K$.

Proof. Since generically smooth affine special formal $R$-schemes admit a resolution of singularities if $k$ has characteristic zero, by Proposition 2.42 we can cover $X$ by a finite family of open affine formal subschemes, such that
\[ \chi_{\text{top}}(S(\mathcal{X}(d))) = Tr(\varphi^d | H(\mathcal{X})) \]
whenever $\mathcal{X}$ is an intersection of members of this cover. But both sides of this equality are additive w.r.t. $\mathcal{X}$ (for the right hand side, use equation (6.1)) and Lemma 6.3, which yields the result for $\mathcal{X} = \mathcal{X}$. □

Corollary 6.6. If $k$ is an algebraically closed field of characteristic zero, and $X_{K,\eta}$ is the analytification of a proper smooth variety $X_K$ over $K$, then
\[ \chi_{\text{top}}(S(X_{K,\eta}(d))) = Tr(\varphi^d | \oplus_{i\geq 0} H^i(X_K \times_K K^s, \mathbb{Q}_\ell)) \]

Proof. If $X$ is a flat proper $R$-model for $X_K$, then $X_{K,\eta}$ is canonically isomorphic to the generic fiber $X_\eta$ of the $\pi$-adic completion $\hat{X}$, by [8] 0.3.5). Moreover, by [8] 7.5.4], there is a canonical isomorphism
\[ H^i(X_K \times_K K^s, \mathbb{Q}_\ell) \cong H^i(X_\eta) \]
for each $i \geq 0$. □

As we observed in [31] §5], some tameness condition is necessary in the statement of the trace formula: if $R$ is the ring of Witt vectors $W(F_{\ell}^p)$, and $X$ is $\text{Spf} R[x]/(x^p - \pi)$, then $S(X) = 0$ while the trace of $\varphi$ on the cohomology of $X$ is 1. It would be interesting to find an intrinsic tameness condition on $X_\eta$ under which the trace formula holds.

It would also be interesting to find a proof of the trace formula which does not rely on explicit computation, and which does not use resolution of singularities. One could use the following strategy. There’s no harm in assuming that $k$ is algebraically closed. After admissible blow-up, we may suppose that the $R$-smooth part $Sm(X)$ of $X$ is a weak Néron model for $X$, by Proposition 4.12. On $Sm(X)_0$, the tame nearby cycles are trivial, so the trace of $\varphi$ on the cohomology of $Sm(X)_0$ yields $\chi_{\text{top}}(S(X))$. Hence, it suffices to prove that the trace of $\varphi$ on the cohomology of $R\psi^t_\eta(\mathbb{Q}_\ell|_{X_\eta})|_Y$ vanishes, where $Y$ denotes the complement of $Sm(X)_0$ in $X$. We’ll assume the following result: for each $i \geq 0$, there is a canonical $G(K^t/K)$-equivariant isomorphism
\[ H^i(Y, R\psi^t_\eta(\mathbb{Q}_\ell|_{X_\eta})|_Y) \cong H^i(\overline{Y}, \mathbb{Q}_\ell) \]
This is known if $X$ is algebraizable, by Berkovich’ comparison theorem [7] 3.1. If $K$ has characteristic zero and $X$ is $\text{stft}$ over $R$, then it can be proven in the same way as Huber’s result [20] 3.15] (which is the corresponding result for Berkovich’ functor $R\Theta_K$ from $[7]$ §2]).
Assuming \( \text{(6.2)} \), everything reduces to the following assertion: if \( \mathcal{X}_n \) is the smooth generic fiber of a special formal \( R \)-scheme and satisfies \( \mathcal{X}_n(K) = \emptyset \) and an appropriate tameness condition (in particular if \( k \) has characteristic zero), then

\[
\text{Tr}(\varphi \mid H(\mathcal{X}_n)) = 0
\]

At this point, I don’t know how to prove this without making an explicit computation on a resolution of singularities.

7. The volume Poincaré series, and the motivic volume

Throughout this section, we assume that \( k \) has characteristic zero.

7.1. Order of a gauge form along strict normal crossings. Throughout this subsection, \( \mathcal{X} \) denotes a regular special formal \( R \)-scheme of pure relative dimension \( m \), whose special fiber is a strict normal crossings divisor \( \mathcal{X}_s = \sum_{i \in I} N_i \mathcal{E}_i \).

**Definition 7.1.** For each \( i \in I \), and each point \( x \) of \( E_i \), we denote by \( \mathcal{P}_{i,x} \) the (not necessarily open) prime ideal in \( \mathcal{O}_{\mathcal{X},x} \) corresponding to \( \mathcal{E}_i \), and we define \( \mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} \) as the localization

\[
\mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} = (\mathcal{O}_{\mathcal{X},x})_{\mathcal{P}_{i,x}}
\]

Moreover, we introduce the \( \mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} \)-module

\[
\Omega_{\mathcal{X},\mathcal{E}_i,x} := (\Omega^m_{\mathcal{X}/R,x})_{\mathcal{P}_{i,x}} / (\mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} - \text{torsion})
\]

By Lemma 2.34, \( \mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} \) is a discrete valuation ring. Note that the valuation of \( \pi \) in \( \mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} \) equals \( N_i \).

If \( \mathcal{X} \) is \( \text{stft} \) over \( R \), then \( E_i = \mathcal{E}_i \), and if we denote by \( \xi_i \) the generic point of \( E_i \), then \( \mathcal{P}_{i,\xi_i} \) is the maximal ideal of \( \mathcal{O}_{\mathcal{X},\xi_i} \). So \( \mathcal{O}_{\mathcal{X},\mathcal{E}_i,\xi_i} = \mathcal{O}_{\mathcal{X},\xi_i} \), and \( \Omega_{\mathcal{X},\mathcal{E}_i,\xi_i} \) is the module \( \Omega_i \) considered in [31, 6.7]. Note that in the general case, \( E_i \) is not necessarily irreducible (see Example 2.36).

**Lemma 7.2.** If \( h : \mathcal{V} \to \mathcal{X} \) is an étale morphism, and \( \mathcal{C} \) is a connected component of \( h^{-1}(\mathcal{E}_i) \), then the natural map

\[
h^* \Omega^m_{\mathcal{X}/R,x} \to \Omega^m_{\mathcal{V}/R,y}
\]

induces an isomorphism \( \Omega_{\mathcal{X},\mathcal{E}_i,x} \otimes_{\mathcal{O}_{\mathcal{X},\mathcal{E}_i,x}} \mathcal{O}_{\mathcal{V},\mathcal{C},y} \cong \Omega_{\mathcal{V},\mathcal{C},y} \) for each point \( y \) on \( C = \mathcal{E}_0 \) and with \( x = h(y) \).

**Proof.** Since \( h \) is étale,

\[
h^* \Omega^m_{\mathcal{X}/R,x} \to \Omega^m_{\mathcal{V}/R,y}
\]

is an isomorphism by [4] 4.10. Let \( \mathcal{P}_{y} \) be the prime ideal in \( \mathcal{O}_{\mathcal{V},y} \) defining \( \mathcal{C} \). Since \( h \) is étale, the local morphism \( h^* : \mathcal{O}_{\mathcal{X},x} \to \mathcal{O}_{\mathcal{V},y} \) is a flat, unramified monomorphism by Lemma 2.43 and by localization, so is

\[
\mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} \to \mathcal{O}_{\mathcal{V},\mathcal{C},y}
\]

The isomorphism

\[
\Omega^m_{\mathcal{X}/R,x} \otimes_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{V},y} \cong \Omega^m_{\mathcal{V}/R,y}
\]

localizes to an isomorphism

\[
(\Omega^m_{\mathcal{X}/R,x})_{\mathcal{P}_{i,x}} \otimes_{\mathcal{O}_{\mathcal{X},\mathcal{E}_i,x}} \mathcal{O}_{\mathcal{V},\mathcal{C},y} \cong (\Omega^m_{\mathcal{V}/R,y})_{\mathcal{P}_{y}}
\]

which, by flatness of \( \mathcal{O}_{\mathcal{X},\mathcal{E}_i,x} \to \mathcal{O}_{\mathcal{V},\mathcal{C},y} \), induces an isomorphism

\[
\Omega_{\mathcal{X},\mathcal{E}_i,x} \otimes_{\mathcal{O}_{\mathcal{X},\mathcal{E}_i,x}} \mathcal{O}_{\mathcal{V},\mathcal{C},y} \cong \Omega_{\mathcal{V},\mathcal{C},y}
\]

\( \square \)
**Lemma 7.3.** For each $i \in I$ and each point $x$ of $E_i$, the $\mathcal{O}_{X, x}$-module $\Omega_{X, x}$ is free of rank one.

**Proof.** Since $\Omega_{X, x}$ is finite over $\mathcal{O}_{X, x}$ and torsion-free, and $\mathcal{O}_{X, x}$ is a PID, the module $\Omega_{X, x}$ is free over $\mathcal{O}_{X, x}$. Let us show that its rank equals 1. By Lemma 7.2 we may pass to an étale cover and assume that there exists a regular system of local parameters $(x_0, \ldots, x_n)$ in $\mathcal{O}_{X, x}$ with $\mathfrak{p}_{i, x} = (x_i)$ and $\mathfrak{p} = \prod_{i=0}^{n} x_i^{N_i}$. Deriving this expression, we see that $\Omega_{X, x}$ is generated by $dx_0 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_n$. □

Note that the natural map $\Omega_{X/R}^{m}(\mathfrak{x}) \to \Omega_{X, x}$ factors through

$$\Omega_{X/R}^{m}(\mathfrak{x})/(\pi - \text{torsion})$$

since $\Omega_{X, x}$ has no torsion.

**Definition 7.4.** Fix $i \in I$ and let $x$ be a point of $E_i$. For any

$$\omega \in \Omega_{X/R}^{m}(\mathfrak{x})/(\pi - \text{torsion})$$

we define the order of $\omega$ along $\mathfrak{e}_i$ at $x$ as the length of the $\mathcal{O}_{X, x}$-module

$$\Omega_{X, x}/(\mathcal{O}_{X, x} \cdot \omega)$$

and we denote it by $\text{ord}_{\mathfrak{e}_i, x} \omega$.

If $\omega$ is a $\mathfrak{X}$-bounded $m$-form on $\mathfrak{X}_n$, there exists an integer $a \geq 0$ and an affine open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}$ containing $x$, such that $\pi^a \omega$ belongs to

$$\Omega_{\mathfrak{X}/R}(\mathfrak{U})/(\pi - \text{torsion}) \subset \Omega_{\mathfrak{X}/R}(\mathfrak{U}) \otimes_R K$$

We define the order of $\omega$ along $\mathfrak{e}_i$ at $x$ as

$$\text{ord}_{\mathfrak{e}_i, x} \omega := \text{ord}_{\mathfrak{e}_i, x}(\pi^a \omega) - aN_i$$

This definition does not depend on $a$. If $\mathfrak{X}$ is smooth, it coincides with the one given in Section 5.3 in the following sense: if $\mathfrak{X}$ is connected and $x$ is any point of $\mathfrak{X}_0$, then $\text{ord}_{\mathfrak{X}_0, x} \omega = \text{ord}_{\mathfrak{X}_0, x} \omega$.

**Lemma 7.5.** Fix $i \in I$, and let $x$ and $y$ be points of $E_i$ such that $y$ belongs to the Zariski-closure of $\{x\}$. For any $\mathfrak{X}$-bounded $m$-form $\omega$ on $\mathfrak{X}_n$,

$$\text{ord}_{\mathfrak{e}_i, x} \omega = \text{ord}_{\mathfrak{e}_i, y} \omega$$

**Proof.** We may suppose that $\omega \in \Omega_{\mathfrak{X}/R}^{m}(\mathfrak{x})$. The natural localization map $\mathcal{O}_{X, y} \to \mathcal{O}_{X, x}$ induces a flat, unramified local homomorphism $\mathcal{O}_{X, y} \to \mathcal{O}_{X, x}$, and

$$\Omega_{X, x} \cong \Omega_{X, y} \otimes_{\mathcal{O}_{X, y}} \mathcal{O}_{X, x}$$

Hence, we can conclude by the following algebraic property: if $g : A \to A'$ is a flat, unramified morphism of discrete valuation rings, if $M$ is a free $A$-module of rank 1, and $m$ is an element of $M$, then the length of the $A$-module $M/(Am)$ equals the length of the $A'$-module $(M \otimes A')/(A'm)$. Indeed: fixing an isomorphism of $A$-modules $A \cong M$, the length of $A/(Am)$ is equal to the valuation of $m$ in $A$. □

**Corollary 7.6.** Fix $i \in I$. If $\omega$ is a $\mathfrak{X}$-bounded $m$-form on $\mathfrak{X}_n$, then the function

$$E_i \to \mathbb{Z} : x \mapsto \text{ord}_{\mathfrak{e}_i, x} \omega$$

is constant on $E_i$. 
Definition 7.7. For any $i \in I$ and any $\mathcal{X}$-bounded $m$-form on $\mathcal{X}_n$, we define the order of $\omega$ along $\mathcal{E}_i$ by

$$\text{ord}_{\mathcal{E}_i, \omega} := \text{ord}_{\mathcal{E}_i, x, \omega}$$

where $x$ is any point on $E_i$.

By Corollary 7.6, this definition does not depend on the choice of $x$.

Lemma 7.8. Let $h : \mathfrak{Q} \to \mathcal{X}$ be an étale morphism of special formal $R$-schemes. Let $\mathcal{E}$ be a connected component of $h^{-1}(\mathcal{E}_i)$. For any $\mathcal{X}$-bounded $m$-form $\omega$ on $\mathcal{X}_n$,

$$\text{ord}_{\mathcal{E}, \omega} = \text{ord}_{\mathcal{E}} h^* \omega$$

Proof. This follows immediately from Lemma 7.2 and the algebraic argument in the proof of Lemma 7.6.

For each $i \in I$, we denote by $\mathcal{I}_{\mathcal{E}_i}$ the defining ideal sheaf of $\mathcal{E}_i$ in $\mathcal{X}$. For any finite extension $R'$ of $R$ and any $\psi \in \mathcal{X}(R')$, we denote $\text{ord}(\mathcal{I}_{\mathcal{E}_i})(\psi)$ by $\text{ord}_{\mathcal{E}_i}(\psi)$ (see Definition 5.3). If $R'$ has ramification degree $e$ over $R$, and the closed point $\psi(0)$ of the section $\psi$ is contained in $E^0_i$, then the equality $x_i^N \cdot (\text{unit})$ in $\mathcal{O}_{\mathcal{X}, \psi(0)}$ implies that $\text{ord}_{\mathcal{E}_i}(\psi) = e/N_i$.

The following results are proven exactly as their counterparts [31, 6.11-13].

Lemma 7.9. Fix a non-empty subset $J$ of $I$. Let $R'$ be a finite extension of $R$, and let $\psi$ be an element of $\mathcal{X}(R')$, such that its closed point $\psi(0)$ lies on $E^0_J$. For any $\mathcal{X}$-bounded gauge form $\omega$ on $\mathcal{X}_n$,

$$\text{ord}(\omega)(\psi) = \sum_{i \in J} \text{ord}_{\mathcal{E}_i}(\psi)(\text{ord}_{\mathcal{E}_i, \omega} - 1) + \max_{i \in J}\{\text{ord}_{\mathcal{E}_i}(\psi)\}$$

Proposition 7.10. Let $\omega$ be a $\mathcal{X}$-bounded gauge form on $\mathcal{X}_n$. Take a subset $J$ of $I$, with $|J| > 1$, and $E^0_J \neq \emptyset$. Let $h : \mathcal{X}' \to \mathcal{X}$ be the formal blow-up with center $\mathcal{E}_J$, and denote by $\mathcal{E}_J'$ its exceptional component. We have

$$\text{ord}_{\mathcal{E}_J', \omega} = \sum_{i \in J} \text{ord}_{\mathcal{E}_i, \omega}$$

Proposition 7.11. Let $\omega$ be a $\mathcal{X}$-bounded gauge form on $\mathcal{X}_n$. Fix an integer $e > 0$. Denote by $\omega(e)$ the pull-back of $\omega$ to the generic fiber of $\mathcal{X}(e)$. For each $i \in I$, with $N_i e$, and each connected component $C$ of $\text{Sm}(\mathcal{X}(e)) \times_{\mathcal{X}_0} E_i$, we have

$$\text{ord}_C(\omega(e)) = (e/N_i) \cdot \text{ord}_{\mathcal{E}_i, \omega}$$

where the left hand side is computed on the smooth special formal $R$-scheme $\text{Sm}(\mathcal{X}(e))$.

7.2. Volume Poincaré series.

Theorem 7.12. Let $\mathcal{X}$ be a regular special formal $R$-scheme of pure relative dimension $m$, whose special fiber is a strict normal crossings divisor $\mathcal{X}_s = \sum_{i \in I} N_i \mathcal{E}_i$. Let $\omega$ be a $\mathcal{X}$-bounded gauge form on $\mathcal{X}_n$, and put $\mu_i = \text{ord}_{\mathcal{E}_i, \omega}$ for each $i \in I$. Then for any integer $d > 0$,

$$\int_{\mathcal{X}(d)} |\omega(d)| = \mathbb{L}^{-m} \sum_{\emptyset \neq J \subseteq I} (L - 1)^{|J| - 1} |E^0_J| \sum_{k_i \geq 1, i \in J} \mathbb{L}^{-\sum_{i \in J} k_i \mu_i} \text{ in } \mathcal{M}_{\mathcal{X}_0}$$
Proof. We’ll show how the proof of the corresponding statement in [31, 7.6] can be generalized.

First, suppose that \( d \) is not \( X_0 \)-linear. Then it follows from Theorem 5.1 Corollary 5.13 and Proposition 7.11 that

\[
\int \omega(d) = L^{-m} \sum_{N_i \mid d} \left[ \tilde{E}_i \right] L^{-d\mu_i / N_i},
\]

\[
= L^{-m} \sum_{\emptyset \neq J \subseteq I} (L - 1)^{|J| - 1} \left[ \tilde{E}_J \right] (\sum_{k_i \geq 1, \sum_{i \in J} k_i N_i = d} L^{-\sum_{i \in J} k_i \mu_i}), \quad (\ast)
\]

By Lemma 2.32, it suffices to show that the expression \((\ast)\) is invariant under formal blow-ups with center \( E_J \), \( |J| > 1 \). This can be done as in [31, 7.6], using an immediate generalization of the local computation in [31, 7.5]. \( \square \)

Corollary 7.13. Let \( X \) be a generically smooth special formal \( R \)-scheme, of pure relative dimension \( m \). Suppose that \( X \) admits a resolution of singularities \( X' \to X \), with special fiber \( X'_s = \sum_{i \in I} N_i E_i \). Let \( \omega \) be a \( X \)-bounded gauge form on \( X_\eta \).

The volume Poincaré series \( S(X, \omega; T) \) is rational over \( M_{X_0} \). In fact, if we put \( \mu_i := \text{ord}_{E_i} \omega \), then the series is given explicitly by

\[
S(X, \omega; T) = L^{-m} \sum_{\emptyset \neq J \subseteq I} (L - 1)^{|J| - 1} \left[ \tilde{E}_J \right] \prod_{i \in J} \frac{L^{-\mu_i} T^{N_i}}{1 - L^{-\mu_i} T^{N_i}}, \quad \text{in} \ M_{X_0}[[T]].
\]

By Proposition 2.32, any affine generically smooth special formal \( R \)-scheme admits a resolution of singularities. By the additivity of the motivic integral, we obtain an expression for the volume Poincaré series in terms of a finite atlas of local resolutions. In particular, we obtain the following result.

Corollary 7.14. Let \( X \) be a generically smooth special formal \( R \)-scheme, of pure relative dimension \( m \). Let \( \omega \) be a \( X \)-bounded gauge form on \( X_\eta \). The volume Poincaré series \( S(X, \omega; T) \) is rational over \( M_{X_0} \). More precisely, there exists a finite subset \( S \) of \( \mathbb{Z} \times \mathbb{N}^* \) such that \( S(X, \omega; T) \) belongs to the subring

\[
M_{X_0} \left[ \frac{L^a T^b}{1 - L^a T^b} \right]_{(a, b) \in S}
\]

of \( M_{X_0}[[T]] \).

7.3. The Gelfand-Leray form and the local singular series. Let \( X \) be a special formal \( R \)-scheme, of pure relative dimension \( m \). Then \( X \) is also a formal scheme of pseudo-finite type over \( \text{Spec} \, k \), in the terminology of [4], and the sheaves of continuous differential forms \( \Omega^m_{X/k} \) are coherent, by [4, 3.3].

Consider the morphism of coherent \( \mathcal{O}_X \)-modules

\[
i : \Omega^m_{X/k} \xrightarrow{d \pi \wedge} \Omega^{m+1}_{X/k} : \omega \mapsto d \pi \wedge \omega
\]

Since \( d \pi \wedge \Omega^{m-1}_{X/k} \) is contained in its kernel, and

\[
\Omega^m_{X/k} / \left( d \pi \wedge \Omega^{m-1}_{X/k} \right) \cong \Omega^m_{X/R}
\]

by [4, 3.10], \( i \) descends to a morphism of coherent \( \mathcal{O}_X \)-modules

\[
i : \Omega^m_{X/R} \xrightarrow{d \pi \wedge} \Omega^{m+1}_{X/k}
\]
We’ve seen in Section 2.1 that any coherent module $F$ on $X$ induces a coherent module $F_{\text{rig}}$ on $X_\eta$, and this correspondence is functorial. Hence, $i$ induces a morphism of coherent $O_{X_\eta}$-modules

$$i : \Omega^m_{X_\eta/K} \xrightarrow{d\pi \wedge} (\Omega^{m+1}_{X/k})_{\text{rig}}$$

by [16, 7.1.12].

**Definition 7.15.** If $X$ is a special formal $R$-scheme, the Koszul complex of $X$ is by definition the complex of coherent $O_{X_\eta}$-modules

$$0 \longrightarrow O_{X_\eta} \xrightarrow{d\pi \wedge} (\Omega^1_{X/k})_{\text{rig}} \xrightarrow{d\pi \wedge} (\Omega^2_{X/k})_{\text{rig}} \xrightarrow{d\pi \wedge} \cdots$$

**Lemma 7.16.** If $X$ is a special formal $R$-scheme, and $i > 0$ is an integer, then there exists a canonical exact sequence of $O_{X_\eta}$-modules

$$(\Omega^{i-1}_{X/k})_{\text{rig}} \xrightarrow{d\pi \wedge} (\Omega^i_{X/k})_{\text{rig}} \longrightarrow (\Omega^i_{X/k}/d\pi \wedge \Omega^{i-1}_{X/k})_{\text{rig}} \longrightarrow 0$$

**Proof.** Since the functor $(\cdot)_{\text{rig}}$ is exact by Proposition 2.6, we get a canonical exact sequence of $O_{X_\eta}$-modules

$$(\Omega^{i-1}_{X/k})_{\text{rig}} \xrightarrow{d\pi \wedge} (\Omega^i_{X/k})_{\text{rig}} \longrightarrow (\Omega^i_{X/k}/d\pi \wedge \Omega^{i-1}_{X/k})_{\text{rig}} \longrightarrow 0$$

But $\Omega^i_{X/k}/d\pi \wedge \Omega^{i-1}_{X/k} \cong \Omega^i_{X/R}$, so we can conclude by [16, 7.1.12].

**Lemma 7.17.** Let $X$ be a variety over $k$, and consider a morphism $f : X \to \mathbb{A}^1_k = \text{Spec } k[\pi]$ which is smooth of pure relative dimension $m$ over the torus $\mathbb{G}_m = \text{Spec } k[\pi, \pi^{-1}]$. If we denote by $\mathfrak{X}$ the $\pi$-adic completion of $f$, then the Koszul complex of $\mathfrak{X}$ is exact, and

$$i : \Omega^m_{X_\eta/K} \xrightarrow{d\pi \wedge} (\Omega^{m+1}_{X/k})_{\text{rig}}$$

is an isomorphism.

**Proof.** Put $X' = X \times \mathbb{A}^1_k \mathbb{G}_m$. Since $f$ is smooth over $\mathbb{G}_m$,

$$0 \longrightarrow O_{X'} \xrightarrow{d\pi \wedge} \Omega^1_{X'/k} \xrightarrow{d\pi \wedge} \Omega^2_{X'/k} \xrightarrow{d\pi \wedge} \cdots$$

is exact, and hence the cokernels of the inclusion maps

$$d\pi \wedge \Omega^{i-1}_{X/k} \to \ker (d\pi \wedge : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k})$$

are $\pi$-torsion modules. Taking $\pi$-adic completions and using [4, 1.9], we see that the cokernels of the maps

$$d\pi \wedge \Omega^{i-1}_{X/k} \to \ker (d\pi \wedge : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k})$$

are $\pi$-torsion modules, so they vanish by passing to the generic fiber. We can conclude by exactness of $(\cdot)_{\text{rig}}$ (Proposition 2.6) that the Koszul complex of $\mathfrak{X}$ is exact. Moreover, $\Omega^{m+2}_{X/k}$ is the $\pi$-adic completion of $\Omega^{m+2}_{X/k}$; hence, it is $\pi$-torsion, and $(\Omega^{m+2}_{X/k})_{\text{rig}} = 0$. By Lemma 7.16, this implies that

$$i : \Omega^m_{X_\eta/K} \xrightarrow{d\pi \wedge} (\Omega^{m+1}_{X/k})_{\text{rig}}$$

is an isomorphism.
Lemma 7.18. Let \( h : \mathcal{X} \to \mathcal{Y} \) be a morphism of special formal \( R \)-schemes, such that \( h_\eta : \mathcal{X}_\eta \to \mathcal{Y}_\eta \) is étale. If the Koszul complex of \( \mathcal{X} \) is exact, then the natural map

\[
\varphi : h_\eta^!(\Omega^i_{\mathcal{Y}/k})_{\text{rig}} = (h^*\Omega^i_{\mathcal{Y}/k})_{\text{rig}} \to (\Omega^i_{\mathcal{X}/k})_{\text{rig}}
\]

is an isomorphism of coherent \( \mathcal{O}_{\mathcal{X}_\eta} \)-modules, for each \( i \geq 0 \). If, moreover, \( h_\eta \) is surjective, then the Koszul complex of \( \mathcal{Y} \) is exact.

Proof. We proceed by induction on \( i \). For \( i = 0 \), the statement is clear, so assume \( i > 0 \). We put \( \Omega^{-1}_{\mathcal{X}/k} = 0 \) and \( \Omega^{-1}_{\mathcal{Y}/k} = 0 \). Now consider the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
(h^*\Omega^i_{\mathcal{Y}/k})_{\text{rig}} & \xrightarrow{d\pi \wedge} & (h^*\Omega^{i-1}_{\mathcal{Y}/k})_{\text{rig}} \\
\Omega^i_{\mathcal{X}/k} & \xrightarrow{d\pi \wedge} & (\Omega^i_{\mathcal{X}/k})_{\text{rig}}
\end{array}
\end{array}
\]

The bottom row is exact by exactness of the Koszul complex on \( \mathcal{X} \) and by Lemma 7.16. The upper row is exact except maybe at \( (h^*\Omega^{i-1}_{\mathcal{Y}/k})_{\text{rig}} \), by Lemma 7.16 (applied to \( \mathcal{Y} \)) and flatness of \( h_\eta \). The first and second vertical arrows are isomorphisms by the induction hypothesis, and the fourth one is an isomorphism since \( h_\eta \) is étale [12 2.6]. Now a diagram chase shows that the third vertical arrow is an isomorphism as well. If \( h_\eta \) is also surjective, then by faithful flatness the Koszul complex of \( \mathcal{Y} \) is exact since the complex

\[
0 \longrightarrow h_\eta^!\Omega^0_{\mathcal{Y}/k} \xrightarrow{d\pi \wedge} h_\eta^!\Omega^1_{\mathcal{Y}/k} \xrightarrow{d\pi \wedge} h_\eta^!\Omega^2_{\mathcal{Y}/k} \xrightarrow{d\pi \wedge} \ldots
\]

is isomorphic to the Koszul complex of \( \mathcal{X} \) and hence exact. \( \square \)

Proposition 7.19. If \( \mathcal{X} \) is a generically smooth special formal \( R \)-scheme of pure relative dimension \( m \), then the Koszul complex of \( \mathcal{X} \) is exact, and

\[
i : \Omega^m_{\mathcal{X}/k} \xrightarrow{d\pi \wedge} (\Omega^{m+1}_{\mathcal{X}/k})_{\text{rig}}
\]

is an isomorphism.

Proof. We may assume that \( \mathcal{X} \) is affine, say \( \mathcal{X} = \text{Spf} \, A \). We use the notation of Section 2.1. The morphism of special formal \( R \)-schemes

\[
\mathcal{Y} := \text{Spf} \, B_n \to \mathcal{X}
\]

induces an open embedding on the generic fibers. By Lemma 7.16 and Lemma 7.18 it suffices to show that the Koszul complex of \( \mathcal{Y} \) is exact and that \( (\Omega^{m+2}_{\mathcal{Y}/k})_{\text{rig}} = 0 \). Hence, we may as well assume that \( A \) is topologically of finite type over \( R \). By resolution of singularities (Proposition 2.12) and the proof of Proposition 3.2, and again applying Lemma 7.18 we may assume that \( \mathcal{X} = \text{Spf} \, A \) is endowed with an étale morphism of formal \( R \)-schemes

\[
f : \mathcal{Z} \to \text{Spf} \, R\{x_0, \ldots, x_m\}/(\pi - \prod_{i=0}^{m} x_i^{N_i})
\]

with \( N_i \in \mathbb{N} \). Since \( f \) is étale, it is enough to prove the result for

\[
\mathcal{X} = \text{Spf} \, R\{x_0, \ldots, x_m\}/(\pi - \prod_{i=0}^{m} x_i^{N_i})
\]

Now we can conclude by Lemma 7.17. \( \square \)
Corollary 7.20. If, moreover, $\mathfrak{X}$ is affine, then the natural map

$$\Omega^m_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K \xrightarrow{d\pi\wedge} \Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X}) \otimes_R K$$

is an isomorphism, and it fits in a commutative diagram

\[
\begin{array}{ccc}
\Omega^m_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K & \xrightarrow{d\pi\wedge} & \Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X}) \otimes_R K \\
\downarrow & & \downarrow \\
\Omega^m_{\mathfrak{X}_{\eta}/K}(\mathfrak{X}_{\eta}) & \xrightarrow{d\pi\wedge} & (\Omega^{m+1}_{\mathfrak{X}/k})_{rig}(\mathfrak{X}_{\eta})
\end{array}
\]

where the vertical arrows are injections and the horizontal arrows are isomorphisms.

Proof. This follows immediately from Lemma 2.10.

Definition 7.21 (Gelfand-Leray form). If $\mathfrak{X}$ is a generically smooth special formal $R$-scheme of pure relative dimension $m$, and if $\omega$ is an element of $\Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X})$, then we denote by $\omega/d\pi$ the inverse image of $\omega$ under the isomorphism

\[i : \Omega^m_{\mathfrak{X}_{\eta}/K}(\mathfrak{X}_{\eta}) \xrightarrow{d\pi\wedge} (\Omega^{m+1}_{\mathfrak{X}/k})_{rig}(\mathfrak{X}_{\eta})\]

and we call it the Gelfand-Leray form associated to $\omega$.

Remark. Let us compare this definition with the construction made in [31, §9.2]. Let $X$ be a smooth irreducible variety over $k$, of dimension $m+1$, and let $f : X \to \mathbb{A}^1_k = \text{Spec } k[t]$ be a morphism of $k$-varieties, smooth over the torus $\text{Spec } k[t, t^{-1}]$. Let $\omega$ be a gauge form on $X$, and denote by $V$ the complement in $X$ of the special fiber $X_s$ of $f$. In [31] [§9.2], we constructed a relative form $\omega/d\pi$ in $\Omega^m_{V/k}(V)$ and we defined the Gelfand-Leray form as the element of $\Omega^m_{\mathfrak{X}_{\eta}/K}(\mathfrak{X}_{\eta})$ induced by $\omega/d\pi$. It is obvious from the constructions that this form coincides with our Gelfand-Leray form associated to the element of $\Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X})$ obtained from $\omega$ by completion.

Corollary 7.22. If $\mathfrak{X}$ is a regular flat special formal $R$-scheme of pure relative dimension $m$, and if $\omega$ is an element of $\Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X})$, then $\omega/d\pi$ is $\mathfrak{X}$-bounded. If, moreover, $\omega$ is a gauge form on $\mathfrak{X}$ (i.e. a nowhere vanishing section of $\Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X})$), then $\omega/d\pi$ is a bounded gauge form on $\mathfrak{X}_{\eta}$.

Proof. Boundedness follows from Corollary 7.20. Now suppose that $\omega$ is a gauge form on $\mathfrak{X}$. We may assume that $\mathfrak{X} = \text{Spf } A$ is affine. The fact that $\omega$ is gauge means that $\omega \notin \mathfrak{M}\Omega^{m+1}_{\mathfrak{X}/k}(\mathfrak{X})$ for each prime ideal $\mathfrak{M}$ of $A$; using [16] 7.1.9, we see that this implies that the image of $\omega$ in $(\Omega^{m+1}_{\mathfrak{X}/k})_{rig}(\mathfrak{X}_{\eta})$ does not vanish at any point $x$ of $\mathfrak{X}_{\eta}$. Hence, since the map $i$ of Proposition 7.19 is an isomorphism of coherent $\mathcal{O}_{\mathfrak{X}_{\eta}}$-modules, $\omega/d\pi$ is gauge.

Lemma 7.23. If $h : \mathfrak{Y} \to \mathfrak{X}$ is a morphism of generically smooth special formal $R$-schemes, both of pure relative dimension $m$, and if $\omega$ is a global section of $\Omega^{m+1}_{\mathfrak{X}/k}$, then

\[(h^*\omega)/d\pi = (h_\eta)^*(\omega/d\pi)\]

in $\Omega^m_{\mathfrak{Y}_{\eta}/K}(\mathfrak{Y}_{\eta})$. 
Proof. It suffices to show that
\[ d\pi \wedge ((h_\eta)^* \alpha) = (h_\eta)^* (d\pi \wedge \alpha) \]
for any \( m \)-form \( \alpha \) on \( X_\eta \); substituting \( \alpha \) by \( \omega / d\pi \) yields the result. For any \( i \geq 0 \), the square
\[
\begin{array}{ccc}
  h^* \Omega^i_{X/k} & \longrightarrow & \Omega^i_{Y/k} \\
  d\pi \wedge & & \downarrow \quad d\pi \wedge \\
  h^* \Omega^{i+1}_{X/k} & \longrightarrow & \Omega^{i+1}_{Y/k}
\end{array}
\]
commutes, and therefore
\[
\begin{array}{ccc}
  h^* \Omega^m_{X/R} & \longrightarrow & \Omega^m_{Y/R} \\
  d\pi \wedge & & \downarrow \quad d\pi \wedge \\
  h^* \Omega^{m+1}_{X/k} & \longrightarrow & \Omega^{m+1}_{Y/k}
\end{array}
\]
commutes. We can conclude by passing to the generic fiber. \( \square \)

Lemma 7.24. If \( X \) is a separated formal scheme of pseudo-finite type over \( \text{Spec} \, F \), with \( F \) a perfect field, and \( X \) is regular, then \( X \) is smooth over \( F \).

Proof. Let \( x \) be a closed point of \( X_0 \), and let \((x_0, \ldots, x_m)\) be a regular system of local parameters on \( X \) at \( x \). These define a morphism of formal schemes of pseudo-finite type over \( \text{Spec} \, F \)
\[ h : \mathcal{U} \to k^{m+1}_F \]
on some open neighbourhood \( \mathcal{U} \) of \( x \) in \( X \). Since \( k^{m+1}_F \) is smooth over \( F \), it suffices to show that \( h \) is étale at \( x \). This follows immediately from Lemma 2.43 and the fact that \( F \) is perfect. \( \square \)

Lemma 7.25. Let \( X \) be a flat regular special formal \( R \)-scheme, of pure relative dimension \( m \). Then \( X \) is smooth over \( k \), of pure dimension \( m + 1 \), and \( \Omega^{m+1}_{X/k} \) is a locally free sheaf of rank 1 on \( X \).

Proof. The fact that \( X \) is smooth over \( k \) follows from Lemma 7.24 since \( k \) has characteristic zero. The fact that it has pure dimension \( m + 1 \) follows from the flatness of \( X \) over \( R \).

By [3, 4.8], the sheaf of continuous differential forms \( \Omega^1_{X/k} \) is locally free; by [3, 5.10], it has rank \( m + 1 \). \( \square \)

Corollary 7.26. If \( X \) is a regular special formal \( R \)-scheme, then we can cover \( X \) by open formal subschemes \( \mathcal{U} \) such that \( \mathcal{U}_0 \) admits a \( \mathcal{U} \)-bounded gauge form.

Proof. By Lemma 7.26, we can cover \( X \) by open formal subschemes \( \mathcal{U} \) such that \( \mathcal{U}^{m+1}_0 \cong \mathcal{O}_X \). By Corollary 7.22 each \( \mathcal{U}_0 \) admits a \( \mathcal{U} \)-bounded gauge form. \( \square \)

If \( h : X' \to X \) is a morphism of smooth formal \( k \)-schemes of pseudo-finite type of pure dimension \( m + 1 \), we define the Jacobian ideal sheaf of \( h \) as the annihilator of the cokernel of the natural map of locally free rank one \( \mathcal{O}_{X'} \)-modules
\[ \psi : h^* \Omega^{m+1}_{X/k} \to \Omega^{m+1}_{X'/k} \]
and we denote this ideal sheaf by \( \mathcal{J} \text{ac}_{h/k} \) to distinguish it from the Jacobian ideal sheaf \( \mathcal{J} \text{ac}_h \) from Section 5.2.
Lemma 7.27. Let \( h : \mathcal{X}' \to \mathcal{X} \) be a morphism of regular flat special formal \( R \)-schemes, both of pure relative dimension \( m \), such that \( h_\eta \) is étale. Then the Jacobian ideal sheaf \( \text{Jac}_{h/k} \) is invertible, and contains a power of \( \pi \).

Proof. By Lemma 7.18 and exactness of the functor \( (\_)_{\text{rig}} \), we see that \( \text{coker}(\psi)_{\text{rig}} = 0 \). This means that \( \text{Jac}_{h/k} \) contains a power of \( \pi \), by Corollary 2.9.

Both \( h^*\Omega^m_{X/k} \) and \( \Omega^m_{\mathcal{X}/k} \) are line bundles on \( \mathcal{X}' \), by Lemma 7.25. Covering \( \mathcal{X}' \) by sufficiently small affine open formal subschemes \( U = \text{Spf} A \), we may assume that they are trivial; let \( \omega \) and \( \omega' \) be generators for \( h^*\Omega^m_{X/k} \) resp. \( \Omega^m_{\mathcal{X}/k} \). Then we can write \( \psi(\omega) = f\omega' \) on \( U \) with \( f \) in \( A \), and \( f \) generates the ideal sheaf \( \text{Jac}_{h/k} \) on \( U \). \( \square \)

Let \( \mathcal{X} \) be a regular special formal \( R \)-scheme, whose special fiber is a strict normal crossings divisor \( \sum_{i \in I} N_i E_i \). Let \( J \) be an invertible ideal sheaf on \( \mathcal{X} \), and fix \( i \in I \). One can show as in Lemma 7.5 that the length of the \( \mathcal{O}_{\mathcal{X}, E_i, x} \)-module \( \mathcal{O}_{\mathcal{X}, E_i, x}/J \mathcal{O}_{\mathcal{X}, E_i, x} \) is independent of the point \( x \) of \( E_i \). We call this value the multiplicity of \( J \) along \( E_i \).

Definition 7.28. Let \( h : \mathcal{X}' \to \mathcal{X} \) be a morphism of regular flat special formal \( R \)-schemes, both of pure relative dimension \( m \), such that \( h_\eta \) is étale. If \( \mathcal{X}' \) is a strict normal crossings divisor \( \sum_{i \in I} N_i E_i \), then we denote by \( \nu_i - 1 \) the multiplicity of \( J \) along \( E_i \), and we write \( K_{\mathcal{X}'/\mathcal{X}} = \sum_{i \in I} (\nu_i - 1) E_i \).

Lemma 7.29. Let \( h : \mathcal{X}' \to \mathcal{X} \) be a morphism of regular flat special formal \( R \)-schemes, both of pure relative dimension \( m \), such that \( h_\eta \) is étale. If \( \mathcal{X}' \) is a strict normal crossings divisor \( \sum_{i \in I} N_i E_i \), and if

\[
K_{\mathcal{X}'/\mathcal{X}} = \sum_{i \in I} (\nu_i - 1) E_i
\]

then for any gauge form \( \omega \) on \( \mathcal{X} \) and any \( i \in I \),

\[
\text{ord}_{E_i}(h^*_\eta(\omega/d\pi)) = \nu_i - N_i
\]

Proof. First of all, note that \( h^*_\eta(\omega/d\pi) = (h^*\omega)/d\pi \) by Lemma 7.23. Choose an index \( i \) in \( I \) and a point \( x' \) on \( E_i \), and put \( x = h(x') \). Shrinking \( \mathcal{X} \) to an open formal neighbourhood of \( x \), we may suppose that there exists an integer \( a \) such that \( \phi := \pi^a(\omega/d\pi) \) belongs to

\[
\Omega^m_{\mathcal{X}/R}/(\pi - \text{torsion}) \subset \Omega^m_{\mathcal{X}_n/K}(\mathcal{X}_n)
\]

since \( \omega/d\pi \) is \( \mathcal{X} \)-bounded by Lemma 7.22.

Consider the commutative diagram

\[
\begin{array}{ccc}
\Omega^m_{\mathcal{X}/R} & \xrightarrow{d\pi\wedge} & \Omega^m_{\mathcal{X}'/k} \\
\downarrow & & \downarrow \\
h^*\Omega^m_{\mathcal{X}/R} & \xrightarrow{d\pi\wedge} & h^*\Omega^m_{\mathcal{X}'/k}
\end{array}
\]
Since $\Omega_{X/k}^{m+1}$ and $\Omega_{X/k}^{m+1}$ are locally free, we get a commutative diagram (using the notation in Section 7.1).

$$
\begin{array}{ccc}
h^*\Omega_{X/R}^m \otimes \mathcal{O}_{X',i,x'} & \xrightarrow{d\pi \wedge} & h^*\Omega_{X/k}^{m+1} \otimes \mathcal{O}_{X',i,x'} \\
\varphi \downarrow & & \downarrow \psi \\
\Omega_{X',i,x'} & \xrightarrow{d\pi \wedge} & \Omega_{X/k}^{m+1} \otimes \mathcal{O}_{X',i,x'}
\end{array}
$$

By definition,

$$ord_{\omega_i}(h^*_i(\omega/d\pi)) = \text{length } (\Omega_{X',i,x'}/(\varphi(\phi) \cdot \mathcal{O}_{X',i,x'})) - aN_i$$

Since $d\pi \wedge \phi = \pi^a \omega$ and $\omega$ generates $\Omega_{X/k}^{m+1}$, we see that the $\mathcal{O}_{X',i,x'}$-module

$$(\Omega_{X/k}^{m+1} \otimes \mathcal{O}_{X',i,x'})/(\psi(d\pi \wedge \phi) \mathcal{O}_{X',i,x'})$$

has length $N_i - 1$. Since this value does not change if we pass to an étale cover of $X'$ whose image contains $x'$ (by the algebraic argument used in the proof of Lemma 7.5), we may assume that $\pi = \prod_{j=0}^m x_i^{N_j}$ on $X'$, with $(x_0, \ldots, x_m)$ a regular sequence. Hence, taking differentials, we see that $\omega_0 := dx_0 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_m$

generates $\Omega_{X',i,x'}$, and it is clear that $ord_{\omega_i}(d\pi \wedge \omega_0) = N_i - 1$ in $\Omega_{X/k}^{m+1} \otimes \mathcal{O}_{X',i,x'}$ (see Definition 7.7 for the notation $ord_{\omega_i}(\cdot)$).

**Proposition 7.30.** Let $X$ be a regular flat special formal $R$-scheme of pure relative dimension $m$, and let $\omega$ be a gauge form in $\Omega_{X/k}^{m+1}(X)$. The volume Poincaré series $S(X, \omega/d\pi; T)$ only depends on $X$, and not on $\omega$. In fact, if $X' \to X$ is any resolution of singularities, with $X'_i = \sum_{i \in I} N_i X_i$ and $K_{X'}/X = \sum_{i \in I} (\nu_i - 1) X_i$, then $S(X, \omega/d\pi; T)$ is given explicitly by

$$S(X, \omega/d\pi; T) = \mathbb{L}^{-m} \sum_{0 \neq J \subseteq I} (L - 1)^{|J| - 1} \prod_{i \in J} \left[ e_i^{N_i - \nu_i T N_i} \right] \prod_{i \in J} \left[ 1 - \mathbb{L}^{-1} \sum_{i \in J} \nu_i T N_i \right] \text{ in } \mathcal{M}_{X_0}[[T]]$$

**Proof.** By additivity of the motivic integral, we may assume that $X$ is affine. Then $X$ admits a resolution of singularities by Proposition 2.42 and the expression for $S(X, \omega/d\pi; T)$ follows from Corollary 7.13 and Lemma 7.29. This expression is clearly independent of $\omega$.

**Remark.** The fact that $S(X, \omega/d\pi; T)$ does not depend on $\omega$ follows already from the fact that $\omega/d\pi$ is independent of $\omega$ up to multiplication with a unit on $X$. Hence, for each $d > 0$, $\int_{X_0} |(\omega/d\pi)(d)|$ does not depend on $\omega$. Beware that $S(X, \omega/d\pi; T)$ depends on the choice of $\pi$ if $k$ is not algebraically closed; see the remark following Definition 4.8.
Definition 7.31. If $\mathfrak{X}$ is a regular special formal $R$-scheme, and $\mathfrak{X}$ admits a gauge form $\omega$, then we define the local singular series of $\mathfrak{X}$ by

$$F(\mathfrak{X}; *): N^* \to M_{\mathfrak{X}_0}: d \mapsto \int_{\mathfrak{X}(d)} |\frac{\omega}{d\pi}(d)|$$

This definition only depends on $\mathfrak{X}$, and not on $\omega$, by Proposition 7.30.

If $\mathfrak{X}$ is any regular special formal $R$-scheme, we choose a finite cover $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ by open formal subschemes such that each $\mathfrak{U}_i$ admits a gauge form $\omega_i$, and we define the local singular series of $\mathfrak{X}$ by

$$F(\mathfrak{X}; *): N^* \to M_{\mathfrak{X}_0}: d \mapsto \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} F(\bigcap_{j \in J} \mathfrak{U}_j; d)$$

This definition does not depend on the chosen cover, by additivity of the motivic integral. We define the Weil generating series of $\mathfrak{X}$ by

$$S(\mathfrak{X}; T) = \sum_{d>0} F(\mathfrak{X}; d)T^d \in M_{\mathfrak{X}_0}[[T]]$$

In the terminology of [14, 4.4], the Weil generating series is the Mellin transform of the local singular series. If $\mathfrak{X}$ admits a gauge form $\omega$, then by definition, $S(\mathfrak{X}; T) = S(\mathfrak{X}, \omega/d\pi; T)$.

The term “Weil generating series” is justified by the fact that $F(\mathfrak{X}; d)$ can be seen as a measure for the set $\bigcup_{K'} \mathfrak{X}_\eta(K')$ where $K'$ varies over the unramified extensions of $K(d)$. Moreover, we have the following immediate corollary of the trace formula in Theorem 6.4.

Proposition 7.32. Let $\varphi$ be a topological generator of $G(K^s/K^s_{\text{sh}})$. If $\mathfrak{X}$ is a regular special formal $R$-scheme, then for any integer $d > 0$, $\chi_{\text{top}}(F(\mathfrak{X}; d)) = Tr(\varphi^d | H(\mathfrak{X}_\eta))$.

7.4. The motivic volume. Let $\mathfrak{X}$ be a generically smooth, special formal $R$-scheme, of pure relative dimension $m$, and let $\omega$ be a $\mathfrak{X}$-bounded gauge form on $\mathfrak{X}_\eta$. It is not possible to associate a motivic Serre invariant to $\mathfrak{X}$ in a direct way, since the normalization $R^*$ of $R$ in $K^s$ is not a discrete valuation ring. We will define a motivic object by taking a limit of motivic integrals over finite ramifications of $\mathfrak{X}$, instead.

Definition 7.33 ([25], (2.8)). There is a unique $M_{\mathfrak{X}_0}$-linear morphism

$$\lim_{T \to \infty} : M_{\mathfrak{X}_0} \left[ \frac{\prod_{(a,b) \in \mathbb{Z} \times N^*} \mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathbb{Z} \times N^*} \longrightarrow M_{\mathfrak{X}_0}$$

mapping

$$\prod_{(a,b) \in I} \frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b}$$

to $(-1)^{|I|} = (-1)^{|I|} [\mathfrak{X}_0]$, for each finite subset $I$ of $\mathbb{Z} \times N^*$. We call the image of an element its limit for $T \to \infty$.

Proposition 7.34. Let $\mathfrak{X}$ be a generically smooth, special formal $R$-scheme, of pure relative dimension $m$, and let $\omega$ be a $\mathfrak{X}$-bounded gauge form on $\mathfrak{X}_\eta$. The limit of $-S(\mathfrak{X}, \omega; T)$ for $T \to \infty$ is well-defined, and does not depend on $\omega$. If $\mathfrak{X}' \to \mathfrak{X}$
is any resolution of singularities, with \( X'_s = \sum_{i \in I} N_i \xi_i \), then this limit is given explicitly by

\[
\mathbb{L}^{-m} \sum_{\emptyset \neq J \subset I} (1 - \mathbb{L})^{\left| J \right| - 1} [E_J^n]
\]

in \( M_{X_0} \).

Proof. This follows immediately from the computation in Corollary 7.13 and the observation preceding Corollary 7.14.

\( \Box \)

Definition 7.35. Let \( X \) be a generically smooth special formal \( R \)-scheme of pure relative dimension, and assume that \( X_\eta \) admits a \( X \)-bounded gauge form. The motivic volume

\[
S(X; \hat{K}^s) \in M_{X_0}
\]

is by definition the limit of \(-S(X, \omega; T)\) for \( T \to \infty \), where \( \omega \) is any \( X \)-bounded gauge form on \( X_\eta \).

If \( h : \mathfrak{Y} \to X \) is a morphism of generically smooth special formal \( R \)-schemes such that \( h_\eta \) is an isomorphism, and if \( X_\eta \) admits a \( X \)-bounded gauge form, then it is clear from the definition that \( S(X; \hat{K}^s) = S(\mathfrak{Y}; \hat{K}^s) \) in \( M_{X_0} \).

In definition 7.35, the condition that \( X_\eta \) admits a gauge form can be avoided as follows.

Proposition-Definition 7.36. If \( X \) is a generically smooth special formal \( R \)-scheme which admits a resolution of singularities, then there exists a morphism of special formal \( R \)-schemes \( h : \mathfrak{Y} \to X \) such that \( h_\eta \) is an isomorphism, and such that \( \mathfrak{Y} \) has a finite open cover \( \{ U_i \}_{i \in I} \) such that \( U_i \) has pure relative dimension and \( (U_i)_\eta \) admits a \( U_i \)-bounded gauge form for each \( i \). Moreover, the value

\[
S(X; \hat{K}^s) = \sum_{\emptyset \neq J \subset I} (-1)^{\left| J \right| + 1} S(\cap_{i \in J} U_i; \hat{K}^s) \in M_{X_0}
\]

only depends on \( X \).

Proof. Since \( X \) admits a resolution of singularities, we may assume that \( X \) is regular and flat; now it suffices to put \( \mathfrak{Y} = X \) and to apply Lemma 7.26.

The fact that the expression for \( S(X; \hat{K}^s) \) only depends on \( X \), follows from the additivity of the motivic integral, and the fact that we can dominate any two such morphisms \( h \) by a third by taking the fibered product.

\( \Box \)

Definition 7.37. Let \( X \) be a generically smooth special formal \( R \)-scheme, and take a finite cover \( \{ U_i \}_{i \in I} \) of \( X \) by affine open formal subschemes. Then we can define the motivic volume \( S(X; \hat{K}^s) \) by

\[
S(X; \hat{K}^s) = \sum_{\emptyset \neq J \subset I} (-1)^{\left| J \right| + 1} S(\cap_{i \in J} U_i; \hat{K}^s)
\]

This definition only depends on \( X \).

Note that the terms \( S(\cap_{i \in J} U_i; \hat{K}^s) \) are well-defined, since each \( U_i \) admits a resolution of singularities by Proposition 2.42 and hence, Proposition-Definition 7.36 applies.

Remark. Beware that the motivic volume \( S(X; \hat{K}^s) \) depends on the choice of uniformizer \( \pi \) in \( R \) (more precisely, on the fields \( K(d) \)), if \( k \) is not algebraically
closed. For instance, if $k = \mathbb{Q}$ and $\mathfrak{X} = \text{Spf} R[x]/(x^2 - 2\pi)$ then $S(\mathfrak{X}; \widehat{K}^s) = [\text{Spec} \mathbb{Q}(\sqrt{2})] \in \mathcal{M}_k$, while for $\mathfrak{X} = \text{Spf} R[x]/(x^2 - \pi)$ we find $S(\mathfrak{X}; \widehat{K}^s) = 2$ (to see that these are distinct elements of $\mathcal{M}_k$, look at their étale realizations in the Grothendieck ring of $\ell$-adic representations of $G(\overline{\mathbb{Q}}/\mathbb{Q})$).

If $k$ is algebraically closed, the motivic volume is independent of the choice of uniformizer, since $K(d)$ is the unique extension of degree $d$ of $K$ in $K^s$; see the remark following Definition 3.3.

It is not hard to see that for any unramified extension $R'$ of $R$, and any generically smooth special formal $R$-scheme $\mathfrak{X}$, the motivic volume of $\mathfrak{X}' = \mathfrak{X} \times_R R'$ is the image of the motivic volume of $\mathfrak{X}$ under the natural base change morphism $\mathcal{M}_{\mathfrak{X}_0} \to \mathcal{M}_{\mathfrak{X}_0'}$. □

Now we define the motivic volume of a smooth rigid $K$-variety $\mathfrak{X}_0$ that can be realized as the generic fiber of a special formal $R$-scheme $\mathfrak{X}$. If $\mathfrak{X}_0$ is quasi-compact, this definition was given in [31, 8.3]: the image of $S(\mathfrak{X}; \widehat{K}^s)$ under the forgetful morphism $\mathcal{M}_{\mathfrak{X}_0} \to \mathcal{M}_k$ only depends on $\mathfrak{X}_0$, and it was called the motivic volume $S(\mathfrak{X}_0; \widehat{K}^s)$ of $\mathfrak{X}_0$. I do not know if this still holds if $\mathfrak{X}_0$ is not quasi-compact; the problem is that it is not clear if any two formal $R$-models of $\mathfrak{X}_0$ can be dominated by a third. Therefore, we need an additional technical condition (which might be superfluous).

**Definition 7.38.** Let $\mathfrak{X}$ be a special formal $R$-scheme, and suppose that $\mathfrak{X}_0$ is reduced. For any $i \geq 0$, a section of $\Omega^i_{\mathfrak{X}_0}(\mathfrak{X}_0)$ is called a universally bounded $i$-form on $\mathfrak{X}_0$, if it is $\mathfrak{Y}$-bounded for each formal $R$-model $\mathfrak{Y}$ of $\mathfrak{X}_0$.

I don’t know an example of a bounded $i$-form which is not universally bounded. If $\mathfrak{X}_0$ is reduced, then an analytic function on $\mathfrak{X}_0$ is bounded if it is universally bounded, by Lemma 2.14. If $\mathfrak{X}_0$ is quasi-compact, then any differential form on $\mathfrak{X}_0$ is universally bounded.

**Lemma 7.39.** If $\mathfrak{X}$ is an affine special formal $R$-scheme, and $\mathfrak{X}_0$ is reduced, then any $\mathfrak{X}$-bounded $i$-form $\omega$ on $\mathfrak{X}_0$ is universally bounded.

**Proof.** Since it suffices to prove that $\pi^a \omega$ is universally bounded, for some integer $a$, we may suppose that $\omega$ belongs to the image of the natural map

$$\Omega^i_{\mathfrak{X}/R}(\mathfrak{X}) \to \Omega^i_{\mathfrak{X}_0/K}(\mathfrak{X}_0)$$

by Lemma 2.12. This means that we can write $\omega$ as a sum of terms of the form $a_0(da_1 \wedge \cdots \wedge da_i)$ with $a_0, \ldots, a_i$ regular functions on $\mathfrak{X}$, and hence $a_0, \ldots, a_i$ are bounded by 1 on $\mathfrak{X}_0$. If $\mathfrak{Y}$ is any formal $R$-model for $\mathfrak{X}_0$, then by Lemma 2.14, the functions $a_0, \ldots, a_i$ on $\mathfrak{X}_0$ are $\mathfrak{Y}$-bounded; so $\omega$ is $\mathfrak{Y}$-bounded. □

**Proposition-Definition 7.40.** Let $\mathfrak{X}$ be any generically smooth special formal $R$-scheme, and assume that $\mathfrak{X}_0$ admits a universally bounded gauge form $\omega$. The image of $S(\mathfrak{X}; \widehat{K}^s)$ under the forgetful morphism $\mathcal{M}_{\mathfrak{X}_0} \to \mathcal{M}_k$ only depends on $\mathfrak{X}_0$; we call it the motivic volume of $\mathfrak{X}_0$, and denote it by $S(\mathfrak{X}_0; \widehat{K}^s)$.

**Proof.** If $\mathfrak{X}$ is any formal $R$-model for $\mathfrak{X}_0$, then $\omega$ is $\mathfrak{X}$-bounded, and the image of $S(\mathfrak{X}; \widehat{K}^s)$ in $\mathcal{M}_k$ coincides with

$$- \lim_{T \to \infty} \sum_{d > 0} \left( \int_{\mathfrak{X}(d)} |\omega(d)| \right) T^d$$
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by Proposition 4.7. Hence, it does not depend on \( X \) (and neither on \( \omega \)). □

In particular, this definition applies to the generic fiber of an affine regular special formal \( R \)-scheme \( \mathfrak{X} \) that admits a gauge form \( \omega \in \Omega^\text{max}_{X/k}(\mathfrak{X}) \), by Corollary 7.22 and Lemma 7.39. Hence, for any generically smooth special formal \( R \)-scheme \( \mathfrak{X} \), we can cover \( \mathfrak{X}_\eta \) by a finite number of open rigid subvarieties \( U_i, i \in I \) such that \( S(\cap_{i \in I} U_i; \widehat{K}^s) \) is defined for each non-empty subset \( J \) of \( I \), by Proposition 2.42.

However, it is not clear if the value \( \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} S(\cap_{i \in J} U_i; \widehat{K}^s) \) is independent of the chosen cover: if \( V_\ell, \ell \in L \) is another such cover, I do not know if \( V_\ell \cap U_i \) admits a universally bounded gauge form for all \( i \) and \( \ell \).

If \( \mathfrak{X} \) is stft, we recover the definitions from \[31\].

Proposition 7.41. Let \( \mathfrak{X} \) be a generically smooth special formal \( R \)-scheme, and \( V \) a locally closed subset of \( \mathfrak{X}_0 \). If we denote by \( \mathfrak{V} \) the formal completion of \( \mathfrak{X} \) along \( V \), then \( S(\mathfrak{V}; \widehat{K}^s) \) coincides with the image of \( S(\mathfrak{X}; \widehat{K}^s) \) under the base-change morphism \( M_{\mathfrak{X}_0} \to M_V \).

Proof. This follows immediately from Proposition 4.9. □

If \( \mathfrak{X} \) is stft over \( R \), then we called in \[31\] the image of \( S(\mathfrak{X}; \widehat{K}^s) \) in \( M_V \) the motivic volume of \( \mathfrak{X} \) with support in \( V \), and we denoted it by \( S_V(\mathfrak{X}; \widehat{K}^s) \). The above proposition shows that

\[ S_V(\mathfrak{X}; \widehat{K}^s) = S(\mathfrak{V}; \widehat{K}^s) \]

in \( M_V \). In particular, it only depends on \( \mathfrak{V} \), and not on the embedding in \( \mathfrak{X} \).

Proposition 7.42. If \( \mathfrak{X} \) is a generically smooth special formal \( R \)-scheme, then

\[ \chi_{\text{top}}(S(\mathfrak{X}; \widehat{K}^s)) = \chi_{\text{ét}}(\overline{\mathfrak{X}}_\eta) \]

where \( \chi_{\text{ét}} \) is the Euler characteristic associated to Berkovich étale \( \ell \)-adic cohomology for non-archimedean analytic spaces.

In particular, if \( \mathfrak{X}_\eta \) admits a universally bounded gauge form, then

\[ \chi_{\text{top}}(S(\mathfrak{X}_\eta; \widehat{K}^s)) = \chi_{\text{ét}}(\overline{\mathfrak{X}}_\eta) \]

Proof. Let \( \varphi \) be a topological generator of the geometric monodromy group \( G(K^s/K^{s,h}) \).

By definition,

\[ \chi_{\text{top}}(S(\mathfrak{X}; \widehat{K}^s)) = - \lim_{T \to \infty} \left( \sum_{d>0} \chi_{\text{top}}(S(\mathfrak{X}(d)_\eta))T^d \right) \]

Hence, by our Trace Formula in Theorem 6.4,

\[ \chi_{\text{top}}(S(\mathfrak{X}; \widehat{K}^s)) = - \lim_{T \to \infty} \left( \sum_{d>0} Tr(\varphi^d | H(\overline{\mathfrak{X}}_\eta))T^d \right) \]

Recall the identity \[17\ 1.5.3\]

\[ \sum_{d>0} Tr(F^d, V)T^d = T \frac{d}{dT} \log(det(1 - TF, V)^{-1}) \]

\[ = - T \frac{d}{dT} \frac{det(1 - TF, V)}{det(1 - TF, V)} \]
for any endomorphism $F$ on a finite dimensional vector space $V$ over a field of characteristic zero. Taking limits, we get

$$\lim_{T \to \infty} \sum_{d>0} Tr(F^d, V)T^d = \dim(V)$$

Applying this to $F = \varphi$ and $V = H(\mathcal{X}_\eta)$ yields the result. \hfill \Box

8. The analytic Milnor fiber

In this section, we prove that the analytic Milnor fiber introduced in [31] determines a singularity up to formal equivalence. We do not impose any restriction on the residue field $k$.

8.1. Branches of formal schemes.

**Definition 8.1.** Let $\mathcal{X}$ be a flat special formal $R$-scheme, and let $x$ be a closed point of $\mathcal{X}_0$. Consider the normalization $\tilde{\mathcal{X}} \to \mathcal{X}$, and let $x_1, \ldots, x_m$ be the points on $\tilde{\mathcal{X}}_0$ lying over $x$. We call the special formal $R$-schemes $\text{Spf} \hat{\mathcal{O}}_{\tilde{\mathcal{X}}, x_1}, \ldots, \text{Spf} \hat{\mathcal{O}}_{\tilde{\mathcal{X}}, x_m}$ the branches of $\mathcal{X}$ at $x$.

**Remark.** This notion should not be confused with the branches of the special fiber $\mathcal{X}_s$ at $x$. For instance, if $\mathcal{X} = \text{Spf} R\{x,y\}/(xy - \pi)$, then $\mathcal{X}_s = \text{Spec} k[x,y]/(xy)$ has two branches at the origin, while $\mathcal{X}$ is normal. \hfill \Box

**Proposition 8.2.** Let $\mathcal{X}$ be a flat special formal $R$-scheme, and let $x$ be a closed point on $\mathcal{X}_0$. Suppose that $\mathcal{X}|_x$ is normal, and let $x_1, \ldots, x_m$ be the points lying over $x$ in the normalization $h : \tilde{\mathcal{X}} \to \mathcal{X}$. There is a canonical isomorphism

$$\mathcal{O}|_x([x]) = \prod_{i=1}^m \mathcal{O}|_{x_i}([x_i])$$

Moreover, $\prod_{i=1}^m \hat{\mathcal{O}}_{\tilde{\mathcal{X}}, x_i}$ is canonically isomorphic to the subring of $\mathcal{O}|_x([x])$ consisting of the analytic functions $f$ on $[x]$ with supremum norm $|f|_{\sup} \leq 1$.

**Proof.** The map $h_\eta : \tilde{\mathcal{X}}_\eta \to \mathcal{X}_\eta$ is a normalization map by [15 2.1.3], and so is its restriction over $[x]$, by [15 1.2.3]. Hence, $h_\eta$ is an isomorphism over $[x]$, so it is clear that $[x] \cong \bigcup_{i=1}^m [x_i]$. Therefore, we may as well assume that $\mathcal{X}$ is normal. In this case, the result follows from [16 7.4.1]. \hfill \Box

**Corollary 8.3.** Let $\mathcal{X}$ and $\mathcal{Y}$ be flat formal $R$-schemes, and let $x$ and $y$ be closed points of $\mathcal{X}_0$, resp. $\mathcal{Y}_0$. Then $[x]$ and $[y]$ are isomorphic over $K$, if the disjoint unions of the branches of $(\mathcal{X}, x)$, resp. $(\mathcal{Y}, y)$ are isomorphic over $R$. In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are normal at $x$, resp. $y$, then the $R$-algebras $\hat{\mathcal{O}}_{\mathcal{X}, x}$ and $\hat{\mathcal{O}}_{\mathcal{Y}, y}$ are isomorphic iff $[x]$ and $[y]$ are isomorphic over $K$.

**Lemma 8.4.** Let $\mathcal{X}$ be a smooth special formal $R$-scheme, let $R'$ be any finite unramified extension of $R$, and denote by $k'$ its residue field. The natural map

$$\mathcal{X}(R') \to \mathcal{X}_0(k')$$

is surjective.
Proof. By formal smoothness, the map $\mathfrak{X}(R'/\pi^{n+1}) \to \mathfrak{X}(R'/\pi^n)$ is surjective for each $n \geq 0$, so since $R'$ is complete, we see that $\mathfrak{X}(R') \to \mathfrak{X}_0(k')$ is surjective. □

**Corollary 8.5.** Consider a flat special formal $R$-scheme $\mathfrak{X}$ and a point $x$ of $\mathfrak{X}_0(k)$. Then $\mathfrak{X}$ is smooth at $x$, iff $|x|$ is isomorphic to an open unit polydisc

$$\mathbb{B}_K^n = (\text{Spf } R[[x_1, \ldots, x_m]])_\eta$$

for some $m \geq 0$.

Proof. Replacing $\mathfrak{X}$ by its formal completion at $x$, we may as well assume that $\mathfrak{X}_0 = \{x\}$. If $\mathfrak{X}$ is smooth at $x$, then $\mathfrak{X}(R)$ is non-empty by Lemma 8.4 and hence $\mathfrak{X} \cong \text{Spf } R[[x_1, \ldots, x_m]]$ by [11] 3.1/2. Hence, $|x|$ is isomorphic to the open unit polydisc $\mathbb{B}_K^n$. For the converse implication, assume that $\mathfrak{X}_y \cong \mathbb{B}_K^n$. Then $\mathfrak{X}$ is normal since $\mathbb{B}_K^n$ is normal and connected; so we can apply Corollary 8.3 to $(\mathfrak{X}, x)$ and (Spf $R[[x_1, \ldots, x_m]]$, 0). □

### 8.2. The analytic Milnor fiber

In this section, we put $R = k[[\pi]]$ and $K = k((\pi))$. Let $f : X \to \text{Spec } k[\pi]$ be a morphism from a $k$-variety $X$ to the affine line, let $x$ be a closed point on the special fiber $X_s$ of $f$, and assume that $f$ is flat at $x$. Denote by $\hat{X}$ the $\pi$-adic completion of $f$; it is a strict formal $R$-scheme. The tube $\mathcal{F}_x := |x|$ of $x$ in $\hat{X}$ is canonically isomorphic to the generic fiber of the flat special formal $R$-scheme $\text{Spf } \hat{O}_{X,x}$ (the $R$-structure being given by $f$), by [8] 0.2.7. In [31], we called $\mathcal{F}_x$ the analytic Milnor fiber of $f$ at $x$, based on a topological intuition explained in [33] 4.1 and a cohomological comparison result: if $k = \mathbb{C}$ and $X$ is smooth at $x$, then the étale $\ell$-adic cohomology of $\mathcal{F}_x$ corresponds to the singular cohomology of the classical topological Milnor fiber of $f$ at $x$, by [31] 9.2. If $f$ has smooth generic fiber (e.g. when $X - X_s$ is smooth and $k$ has characteristic zero), then the analytic Milnor fiber $\mathcal{F}_x$ of $f$ at $x$ is a smooth rigid variety over $K$.

The arithmetic and geometric properties of $\mathcal{F}_x$ reflect the nature of the singularity of $f$ at $x$ (see for instance Proposition 8.7). We will see in Proposition 8.7 that $\mathcal{F}_x$ is even a complete invariant of the formal germ of the singularity $(f, x)$, if $X$ is normal at $x$.

**Definition 8.6.** Let $X$ and $Y$ be $k$-varieties, endowed with $k$-morphisms $f : X \to \text{Spec } k[\pi]$ and $g : Y \to \text{Spec } k[\pi]$. We say that $(f, x)$ and $(g, y)$ are formally equivalent if $\hat{O}_{X,x}$ and $\hat{O}_{Y,y}$ are isomorphic as $R$-algebras (the $R$-algebra structures being given by $f$, resp. $g$).

**Proposition 8.7.** Let $X$ and $Y$ be irreducible $k$-varieties, and let $f : X \to \text{Spec } k[\pi]$ and $g : Y \to \text{Spec } k[\pi]$ be dominant morphisms. Let $x$ and $y$ be closed points on the special fibers $X_s$, resp. $Y_s$, and assume that $X$ and $Y$ are normal at $x$, resp. $y$. The analytic Milnor fibers $\mathcal{F}_x$ and $\mathcal{F}_y$ of $f$ at $x$, resp. $g$ at $y$, are isomorphic over $K$, iff $(f, x)$ and $(g, y)$ are formally equivalent.

More precisely, the completed local ring $\hat{O}_{X,x}$ is recovered, as a $R$-algebra, as the algebra of analytic functions $f$ on $\mathcal{F}_x$ with $|f|_{\text{sup}} \leq 1$.

Proof. By Proposition 8.2 it suffices to show that $\hat{O}_{X,x}$ and $\hat{O}_{Y,y}$ are normal and flat. Normality follows from normality of $\hat{O}_{X,x}$ and $\hat{O}_{Y,y}$, by excellence, and flatness follows from the fact that $f$ and $g$ are flat.

**Proposition 8.8.** Let $X$ be any $k$-variety, let $f : X \to \text{Spec } k[\pi]$ be a morphism of $k$-varieties, let $x$ be a $k$-rational point on the special fiber $X_s$ of $f$, and assume that
f is flat at x. Then f is smooth at x iff $\mathcal{F}_x$ is isomorphic to an open unit polydisc $B^n_K$ for some $m \geq 0$.

Proof. This follows from Corollary 8.5 (smoothness of f at x is equivalent to smoothness of Spf $\hat{\mathcal{O}}_{X,x}$ over $R$. □

Proposition 8.9. Let X be a smooth irreducible $k$-variety, let $f : X \to \text{Spec } k[[\pi]]$ be a dominant morphism, and let x be a closed point of $X_s$ whose residue field $k_x$ is separable over $k$. The following are equivalent:

1. the morphism f is smooth at x,
2. the analytic Milnor fiber $\mathcal{F}_x$ of f at x contains a $K'$-rational point for some finite unramified extension $K'$ of $K$.

If $k$ is perfect, then each of the above statements is also equivalent to

3. the analytic Milnor fiber $\mathcal{F}_x$ of f at x is smooth over $K$ and satisfies $S(\mathcal{F}_x) \neq 0$.

If $k$ has characteristic zero, and if we denote by $\varphi$ a topological generator of the geometric monodromy group $G(K^s/K^{sh})$, then each of the above statements is also equivalent to

4. the analytic Milnor fiber $\mathcal{F}_x$ of f at x satisfies
$$\text{Tr}(\varphi | H(\mathcal{F}_x \times_K K^s, \mathbb{Q}_\ell)) \neq 0$$

If $k = \mathbb{C}$, and if we denote by $F_x$ the classical topological Milnor fiber of f at x and by M the monodromy transformation on the graded singular cohomology $H^i_{\text{sing}}(F_x, \mathbb{C}) = \oplus_{i \geq 0} H^i_{\text{sing}}(F_x, \mathbb{C})$

then each of the above statements is also equivalent to

5. the topological Milnor fiber $F_x$ of f at x satisfies
$$\text{Tr}(M | H_{\text{sing}}(F_x, \mathbb{C})) \neq 0$$

Proof. The implication (1) $\Rightarrow$ (2) (with $K' = k_x((\pi))$) follows from Lemma 8.4.

The implication (2) $\Rightarrow$ (1) follows from [11, 3.1/2]: denote by $R'$ the normalization of $R$ in $K'$. Since $X_R := X \times_{k[[\pi]]} R$ is regular and $R$-flat, the existence of a $R'$-section through x on the $R$-scheme $X_R$ implies smoothness of $X_R$ at x. But the set $\mathcal{F}_x(K')$ is canonically bijective to the set of $R'$-sections on $X_R$ through x.

The implication (4) $\Rightarrow$ (3) follows from the trace formula (Theorem 6.4), (3) $\Rightarrow$ (2) is obvious, and (1) $\Rightarrow$ (4) follows from [31, 3.5] and the triviality of the $\ell$-adic nearby cycles of f at x. Finally, the equivalence (4) $\Leftrightarrow$ (5) follows from the comparison theorem [31, 9.2]. □

The equivalence of (1) and (5) (for $k = \mathbb{C}$) is a classical result by A’Campo [2].

9. Comparison to the motivic zeta function

We suppose that $k$ has characteristic zero, and we put $R = k[[\pi]]$. Let X be a smooth, irreducible $k$-variety, of dimension $m$, and consider a dominant morphism $f : X \to \text{Spec } k[[\pi]]$. The formal $\pi$-adic completion of f is a generically smooth, flat stft formal $R$-scheme $\mathcal{X}$. We denote by $X_0$ its generic fiber.

Definition 9.1. We call $X_0$ the rigid nearby fiber of the morphism f. It is a separated, smooth, quasi-compact rigid variety over $K = k((\pi))$. 


9.1. The monodromy zeta function.

**Definition 9.2.** Suppose that \( k \) is algebraically closed. For any locally closed subset \( V \) of \( X_s \), we define the monodromy zeta function of \( f \) at \( V \) by

\[
\zeta_{f,V}(T) = \prod_{i \geq 0} \det(1 - T\varphi | H^i(V[\hat{\mathbb{K}}\otimes_{\mathbb{K}}\mathbb{Q}_\ell])^{(-1)+i} \in \mathbb{Q}_\ell[[T]]
\]

where \( \varphi \) is a topological generator of the Galois group \( G(K^s/K) \).

**Lemma 9.3.** If \( k = \mathbb{C} \), and \( x \) is a closed point of \( X_s \), then \( \zeta_{f,x}(T) \) coincides with

\[
\zeta(M | \oplus_{i \geq 0} H^i_{\text{sing}}(F_x, \mathbb{Q}; T)
\]

where \( F_x \) denotes the topological Milnor fiber of \( f \) at \( x \), and \( M \) is the monodromy transformation on the graded singular cohomology space \( \oplus_{i \geq 0} H^i_{\text{sing}}(F_x, \mathbb{Q}) \).

**Proof.** This follows from the comparison result in [31, 9.2]. \( \square \)

For \( k = \mathbb{C} \), the function \( \zeta_{f,x}(T) \) is known as the monodromy zeta function of \( f \) at \( x \).

9.2. Denef and Loeser’s motivic zeta functions. As in [20, p.1], we denote, for any integer \( d > 0 \), by \( \mathcal{L}_d(X) \) the \( k \)-scheme representing the functor

\[
(k - \text{algebras}) \rightarrow (\text{Sets}) : A \mapsto X(A[t]/(t^{d+1}))
\]

Following [21, 3.2], we denote by \( \mathcal{X}_d \) and \( \mathcal{X}_{d,1} \) the \( X_s \)-varieties

\[
\mathcal{X}_d := \{ \psi \in \mathcal{L}_d(X) | \text{ord}_tf(\psi(t)) = d \}
\]

\[
\mathcal{X}_{d,1} := \{ \psi \in \mathcal{L}_d(X) | f(\psi(t)) = t^d \mod t^{d+1} \}
\]

where the structural morphisms to \( X_s \) are given by reduction modulo \( t \).

In [21, 3.2.1], the motivic zeta function \( Z(f; T) \) of \( f \) is defined as

\[
Z(f; T) = \sum_{d=1}^{\infty} [\mathcal{X}_{d,1}]L^{-md}T^d \in \mathcal{M}_{X_s}[[T]]
\]

and the naïve motivic zeta function \( Z^{\text{naive}}(f; T) \) is defined as

\[
Z^{\text{naive}}(f; T) = \sum_{d=1}^{\infty} [\mathcal{X}_d]L^{-md}T^d \in \mathcal{M}_{X_s}[[T]]
\]

If \( U \) is a locally closed subscheme of \( X_s \), the local (naïve) motivic zeta function \( Z_U(f; T) \) (resp. \( Z_U^{\text{naive}}(f; T) \)) with support in \( U \) is obtained by applying the base change morphism \( \mathcal{M}_{X_s}[[T]] \rightarrow \mathcal{M}_U[[T]] \).

Let \( h : X' \rightarrow X \) be an embedded resolution for \( f \), with \( X'_s = \sum_{i \in I} N_iE_i \), and with Jacobian divisor \( K_{X'\mid X} = \sum_{i \in I} (\nu_i - 1)E_i \).

By [21, Theorem 3.3.1], we have

\[
Z(f; T) = \sum_{\emptyset \neq J \subseteq I} (L - 1)^{|J|-1}[E^J_i] \prod_{i \in J} \frac{L^{-\nu_i}T^{N_i}}{1 - L^{-\nu_i}T^{N_i}} \text{ in } \mathcal{M}_{X_s}[[T]]
\]

\[
Z^{\text{naive}}(f; T) = \sum_{\emptyset \neq J \subseteq I} (L - 1)^{|J|}[E^J_i] \prod_{i \in J} \frac{L^{-\nu_i}T^{N_i}}{1 - L^{-\nu_i}T^{N_i}} \text{ in } \mathcal{M}_{X_s}[[T]]
\]
Inspired by the $p$-adic case [18], Denef and Loeser defined the motivic nearby cycles $S_f$ by taking formally the limit of $-Z(f; T)$ for $T \to \infty$, i.e.
\[ S_f = \sum_{\emptyset \neq J \subseteq I} (1 - L_i)^{|J|-1}[\tilde{E}_j] \in \mathcal{M}_X. \]

For each closed point $x$ of $X$, they denote by $S_{f,x}$ the image of $S_f$ under the base change morphism $\mathcal{M}_{X_s} \to \mathcal{M}_x$, and they called $S_{f,x}$ the motivic Milnor fiber of $f$ at $x$. This terminology is justified by the fact that, when $k = \mathbb{C}$, the mixed Hodge structure of $S_{f,x} \in \mathcal{M}_\mathbb{C}$ coincides with the mixed Hodge structure on the cohomology of the topological Milnor fiber of $f$ at $x$ (in an appropriate Grothendieck group of mixed Hodge structures); see [10, 4.2].

**Theorem 9.4.** Let $x$ be a closed point of $X$. The local zeta functions $Z_x(f; T)$ and $Z_{\text{naive}}(f; T)$, and the motivic Milnor fiber $S_{f,x}$, depend only on the rigid $K$-variety $\mathcal{F}_x$, the analytic Milnor fiber of $f$ at $x$.

**Proof.** This follows from Proposition 8.7, since all these invariants can be computed on the $R$-algebra $\hat{O}_{X,x}$, i.e. they are invariant under formal equivalence. To see this, note that any arc $\psi : \text{Spec} k[[t]] \to X$ with origin $x$ factors through a morphism of $k$-algebras $\hat{O}_{X,x} \to k'[[[t]]$, and that $f(\psi) \in k'[[[t]]$ is simply the image of $\pi$ under the composition
\[ k[[\pi]] \xrightarrow{f} \hat{O}_{X,x} \xrightarrow{\psi} k'[[[t]] \]

\[ \Box \]

**Remark.** In fact, the local zeta function $Z_x(f; T)$ carries additional structure, coming from a $\hat{\mu}(k)$-action on the spaces $\mathcal{X}_{d,1}$ (see [21, 3.2.1]); the resulting $\hat{\mu}(k)$-action on the motivic Milnor fiber $S_{f,x}$ captures the semi-simple part of the monodromy action on the cohomology of the topological Milnor fiber, by [21, 3.5.5]. The same argument as above shows that $\mathcal{F}_x$ completely determines the zeta function $Z_x(f; T) \in \mathcal{M}_x^{\hat{\mu}(k)[[T]]}$ and the motivic Milnor fiber $S_{f,x} \in \mathcal{M}_x^{\hat{\mu}(k)}$, where $\mathcal{M}_x^{\hat{\mu}(k)}$ is the localized Grothendieck ring of varieties over $x$ with good $\hat{\mu}(k)$-action [21, 2.4].

In Corollary 9.6 and Theorem 9.7, we’ll realize $Z_x(f; T)$ and $S_{f,x}$ explicitly in terms of the analytic Milnor fiber $\mathcal{F}_x$.

9.3. **Comparison to the motivic zeta function.** We define the local singular series associated to $f$ by
\[ F(f; d) = F(\tilde{X}; d) \in \mathcal{M}_{X_s}. \]
(see Definition 7.31). We define the motivic Weil generating series associated to $f$ by
\[ S(f; T) := S(\tilde{X}; T) = \sum_{d>0} F(f; d)T^d \in \mathcal{M}_{X_s}[[T]] \]

For any locally closed subscheme $U$ of $X$, we define the motivic Weil generating series $S_U(f; T)$ with support in $U$ as the image of $S(f; T)$ under the base-change morphism
\[ \mathcal{M}_{X_s}[[T]] \to \mathcal{M}_U[[T]] \]

If we denote by $\mathcal{U}$ the formal completion of $\tilde{X}$ along $U$, $S_U(f; T)$ coincides with $S(\mathcal{U}; T)$ by Proposition-Definition 4.9 in particular, it depends only on $\mathcal{U}$.
We recall the following result [31, 9.10].

**Theorem 9.5.** We have

\[ S(f; T) = L^{-(m-1)} Z(f; L T) \in \mathcal{M}_X[[T]] \]

**Corollary 9.6.** For any closed point \( x \) on \( X_s \),

\[ L^{-(m-1)} Z_x(f; T) = S(\text{Spf} \hat{O}_{X,s}; T) = \sum_{d>0} \left( \int_{\mathcal{F}_x(d)} |\omega/d\pi(d)| \right) T^d \in \mathcal{M}_x[[T]] \]

where \( \omega \) is any gauge form on \( \text{Spf} \hat{O}_{X,x} \) and where we view \( \mathcal{F}_x \) as a rigid variety over \( k_x((\pi)) \), with \( k_x \) the residue field of \( x \).

Hence, modulo a normalization by powers of \( L \), we recover the motivic zeta function and the local motivic zeta function at \( x \) as the Weil generating series of \( \hat{X} \), resp. \( \text{Spf} \hat{O}_{X,x} \).

**9.4. Comparison to the motivic nearby cycles.**

**Theorem 9.7.** We have

\[ S(\hat{X}; \hat{K}^s) = L^{-(m-1)} S_f \in \mathcal{M}_X \]

For any closed point \( x \) on \( X_s \), we have

\[ S(\mathcal{F}_x; \hat{K}^s) = L^{-(m-1)} S_{f,x} \in \mathcal{M}_x \]

**Proof.** The result is obtained by taking a limit \( T \to \infty \) of the equality in Theorem 9.5 and applying Proposition-Definition 4.9. Note that \( S(\mathcal{F}_x; \hat{K}^s) \) is well-defined, since \( \mathcal{F}_x \) admits a universally bounded gauge form by Corollary 7.22 and Lemma 7.39.

**Proposition 9.8.** Assume that \( k \) is algebraically closed, and let \( \varphi \) be a topological generator of the absolute Galois group \( G(K^s/K) \). For any integer \( d > 0 \),

\[ \chi_{\text{top}} S(\mathcal{F}_x(d)) = \text{Tr}(\varphi^d | H(\mathcal{F}_x \hat{\times} \hat{K}^s, \mathbb{Q}_\ell)) \]

**Proof.** This is a special case of the trace formula in Theorem 6.4.

**Corollary 9.9.** Suppose \( k = \mathbb{C} \), denote by \( F_x \) the topological Milnor fiber of \( f \) at \( x \), and by \( M \) the monodromy transformation on the graded singular cohomology space \( H(F_x, \mathbb{C}) \). For any integer \( d > 0 \),

\[ \chi_{\text{top}} S(\mathcal{F}_x(d)) = \text{Tr}(M^d | H(F_x, \mathbb{C})) \]

**Proof.** This follows from the cohomological comparison in [31, 9.2].

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