Malcev Yang-Baxter equation, weighted $\Theta$-operators on Malcev algebras and post-Malcev algebras

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Abstract

The purpose of this paper is to study the $\Theta$-operators on Malcev algebras and discuss the solutions of Malcev Yang-Baxter equation by $\Theta$-operators. Furthermore we introduce the notion of weighted $\Theta$-operators on Malcev algebras, which can be characterized by graphs of the semi-direct product Malcev algebra. Then we introduce a new algebraic structure called post-Malcev algebras. Therefore, post-Malcev algebras can be viewed as the underlying algebraic structures of weighted $\Theta$-operators on Malcev algebras. A post-Malcev algebra also gives rise to a new Malcev algebra. Post-Malcev algebras are analogues for Malcev algebras of post-Lie algebras and fit into a bigger framework with a close relationship with post-alternative algebras.

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1. Introduction

Malcev algebras play an important role in the geometry of smooth loops. Just as the tangent algebra of a Lie group is a Lie algebra, the tangent algebra of a locally analytic Moufang loop is a Malcev algebra [18, 21, 27, 28]. A Malcev algebra is a non-associative algebra $A$ with an anti-symmetric multiplication $[\cdot, \cdot]$ that satisfies the Sagle’s identity

$$[[x, z], [y, t]] = [[[x, y], z], t] + [[[y, z], t], x] + [[[z, t], x], y] + [[[t, x], y], z], \forall x, y, z, t \in A.$$ 

Pre-Malcev algebras have been studied extensively since [26] which are the generalization of pre-Lie algebras, in the sense that any pre-Lie algebra is a pre-Malcev algebra but the converse is not true. Studying pre-Malcev algebras independently is significant not only to its own further development, but also to develop the areas closely connected with such
algebras. A pre-Malcev algebra is a vector space $A$ endowed with a bilinear product $\triangleright$ satisfying the following identity for $x, y, z, t \in A$,

$$[y, z] \triangleright (x \triangleright t) + [[x, y], z] \triangleright t + y \triangleright ((x, z] \triangleright t) - x \triangleright (y \triangleright (z \triangleright t)) + z \triangleright (x \triangleright (y \triangleright t)) = 0, \quad (1.1)$$

where $[x, y] = x \triangleright y - y \triangleright x$. The existence of subadjacent Malcev algebras and compatible pre-Malcev algebras was given in [26, Proposition 5]. For a given pre-Malcev algebra $(A, \triangleright)$, there is a Malcev algebra $A^C$ defined by the commutator $[x, y] = x \triangleright y - y \triangleright x$, and the left multiplication operator in $A$ induces a representation of Malcev algebra $A^C$. 

Rota-Baxter operators were introduced by G. Baxter [7] in 1960 in the study of fluctuation theory in Probability. These operators were then further investigated, by G.-C. Rota [30], Atkinson [1], Cartier [9] and others. In the 1980s, the notion of Rota-Baxter operator of weight 0 was introduced in terms of the classical Yang-Baxter equation for Lie algebras (see [4, 5, 13–15, 17, 23] for more details). Later on, B. A. Kupershmidt [19] introduced the notion of $\mathcal{O}$-operator as generalized Rota-Baxter operators to understand classical Yang-Baxter equations and related integrable systems. In fact, a skew-symmetric solution of the analogue of CYBE on Malcev algebras motivated by the point of Kupershmidt and Bai. 

The notion of post-algebras goes back to Rosenbloom in [29] (1942). An equivalent formulation of the class of post-algebras was given by Rousseau in [31] (1969, 1970) which became a starting point for deep research. Post-Lie algebras have been introduced by Vallette in 2007 [33] in connection with the homology of partition posets and the study of Koszul operads. However, J. L. Loday studied pre-Lie algebras and post-Lie algebras within the context of algebraic operad triples, see for more details [24, 25]. In the last decade, many works [8, 10, 34] intrested in post-Lie algebra structures, motivated by the importance of pre-Lie algebras in geometry and in connection with generalized Lie algebra derivations.

Recently, Post-Lie algebras which are non-associative algebras played an important role in different areas of pure and applied mathematics. They consist of a vector space $A$ equipped with a Lie bracket $[\cdot, \cdot]$ and a binary operation $\triangleright$ satisfying the following axioms

$$x \triangleright [y, z] = [x \triangleright y, z] + [y, x \triangleright z], \quad (1.2)$$

$$[x, y] \triangleright z = as_{\triangleright}(x, y, z) - as_{\triangleright}(y, x, z). \quad (1.3)$$

If the bracket $[\cdot, \cdot]$ is zero, we have exactly a pre-Lie structure. It is worth to note that, in spite the post-Lie product does not yield a Lie bracket by antisymmetrization, the bilinear product $\{\cdot, \cdot\} : A \otimes A \to A$, defined for all $x, y \in A$ by

$$\{x, y\} = x \triangleright y - y \triangleright x + [x, y]. \quad (1.4)$$

defines on $A$ another Lie algebra structure. The varieties of pre- and post-Lie algebras play a crucial role in the definition of any pre and post-algebra through black Manin operads product, see details in [3, 12]. Whereas pre-Lie algebras are intimately associated with euclidean geometry, post-Lie algebras occur naturally in the differential geometry of homogeneous spaces, and are also closely related to Cartan’s method of moving frames. Ebrahimi-Fard, Lundervold and Munthe-Kaas [10] studied universal enveloping algebras of post-Lie algebras and the free post-Lie algebra.

In this paper we use weighted $\mathcal{O}$-operators to split operations, although a generalization exists in the alternative setting in terms of bimodules. Diagram (1.5) summarizes the
By (1.5) we investigate the notion of a weighted $\mathcal{O}$-operators structure associated to any post-alternative algebra. The multiplication is given by

$$x \star y = x < y + y > x + x \cdot y.$$ 

In addition, in Section 4 we investigate the notion of a weighted $\mathcal{O}$-operator to construct a post-alternative algebra structure on the $A$-bimodule $\mathbb{K}$-algebra of an alternative algebra $(A, \cdot)$. Section 4 is devoted to introduce the notion of post-Malcev algebra and we show that weighted $\mathcal{O}$-operators can be used to construct post-Malcev algebras. We also reveal a relation between post-Malcev algebras and post-alternative algebras by the commutative diagram (1.5).

Throughout this paper, all algebras are finite-dimensional and over a field $\mathbb{K}$ of characteristic 0.

2. $\mathcal{O}$-operators and Malcev Yang-Baxter equation

In this section, we recall the classical result that a skew-symmetric solution of CYBE in a Malcev algebra gives an $\mathcal{O}$-operator through a duality between tensor product and linear maps. Not every $\mathcal{O}$-operator on a Malcev algebra comes from a solution of CYBE in this way. However, any $\mathcal{O}$-operator can be recovered from a solution of CYBE in a larger Malcev algebra.

We first recall the concept of a representation (see [20]) and construct the dual representation.

**Definition 2.1** ([20]). A representation (or a module) of a Malcev algebra $(A, [\cdot, \cdot])$ on a vector space $V$ is a linear map $\rho : A \rightarrow \text{End}(V)$ such that, for all $x, y, z \in A$,

$$\rho([x, y], z) = \rho(x)\rho(y)\rho(z) - \rho(z)\rho(x)\rho(y) + \rho(y)\rho([x, z]) - \rho([y, z])\rho(x). \quad (2.1)$$

We denote this representation by $(V, \rho)$.

For all $x, y \in A$, define the map $ad : A \rightarrow \text{End}(A)$ by $ad_x(y) = [x, y]$. Then $ad$ is a representation of the Malcev algebra $(A, [\cdot, \cdot])$ on $A$, which is called the adjoint representation.

Let $(A, [\cdot, \cdot])$ be a Malcev algebra and $(V, \rho)$ is a representation on $A$. Consider the dual space $V^*$ of $V$ and $\text{End}(V^*)$. Define the linear map $\rho^* : A \rightarrow \text{End}(V^*)$ by

$$\langle \rho^*(x)a^*, b \rangle = -\langle a^*, \rho(x)b \rangle, \quad \forall x \in A, b \in V, a^* \in V^*, \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between $V^*$ and $V$.

**Proposition 2.1.** With the above notations, $(V^*, \rho^*)$ is a representation of $A$ which is called the dual representation of $(V, \rho)$.

**Proof.** By (2.1), we have, for $x, y, z \in A$,

$$\rho([[y, x], z]) = -\rho([[x, y], z]) = \rho(z)\rho(y)\rho(x) - \rho(y)\rho(x)\rho(z) - \rho(x)\rho([z, y]) + \rho([x, z])\rho(y).$$
So, for any \( x, y, z \in A \), \( a^* \in V^* \), \( b \in V \), we have

\[
\langle \rho^*([[[x, y], z])a^*, b \rangle = -\langle a^*, \rho([[x, y], z])b \rangle = -\langle a^*, -\rho([[x, y], z])b \rangle
\]

\[
= -\langle a^*, (\rho(y)\rho(x)\rho(z) - \rho(z)\rho(y)\rho(x) + \rho(x)\rho(y)\rho(z)) - \rho([x, z])\rho(y))b \rangle
\]

\[
= -\langle ( - \rho^*(z))\rho^*(x)\rho^*(y) + \rho^*(x)\rho^*(y)\rho^*(z) + \rho^*([[z, y]])\rho^*(x) - \rho^*(y)\rho^*(([x, z]))a^*, b \rangle.
\]

Hence, since \( \langle \cdot, \cdot \rangle \) is nondegenerate, we obtain

\[
\rho^*([[x, y], z]) = \rho^*(x)\rho^*(y)\rho^*(z) - \rho^*(z)\rho^*(x)\rho^*(y) + \rho^*(y)\rho^*([z, x]) - \rho^*([[y, z]])\rho^*(x).
\]

**Definition 2.2.** Let \( (A, [, ,]) \) be a Malcev algebra and \( r = \sum_i x_i \otimes y_i \in A \otimes A \). \( r \) is called a solution of Malcev Yang-Baxter equation in \( A \) if \( r \) satisfies

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,
\]

where

\[
r_{12} = \sum_i x_i \otimes y_i \otimes 1, \quad r_{13} = \sum_i x_i \otimes 1 \otimes y_i, \quad r_{23} = \sum_i 1 \otimes x_i \otimes y_i,
\]

and

\[
[r_{12}, r_{13}] = \sum_{i,j} [x_i, x_j] \otimes y_i \otimes y_j, \quad [r_{13}, r_{23}] = \sum_{i,j} x_i \otimes x_j \otimes [y_i, y_j],
\]

\[
[r_{12}, r_{23}] = \sum_{i,j} x_i \otimes [y_i, x_j] \otimes y_j.
\]

Let \( V \) be a vector space. The **twisting operator** \( \sigma : V^{\otimes 2} \rightarrow V^{\otimes 2} \) is defined for all \( a, b \in V \) by

\[
\sigma(a \otimes b) = b \otimes a.
\]

We call \( r = \sum_i a_i \otimes b_i \in V^{\otimes 2} \) **skew-symmetric** (resp. **symmetric**) if \( r = -\sigma(r) \) (resp. \( r = \sigma(r) \)). Furthermore, \( r \) can be regarded as a linear map from \( V^* \) to \( V \) in the following way

\[
\langle a^*, \sigma(b^*) \rangle = \langle a^* \otimes b^*, r \rangle, \quad \forall a^*, b^* \in V^*.
\]

Equation (2.3) gives the tensor form of Malcev Yang-Baxter equation. What we will do next is to replace the tensor form by a linear operator satisfying some conditions.

**Theorem 2.1.** Let \( (A, [, ,]) \) be a Malcev algebra and \( r \in A \otimes A \). Then \( r \) is a skew-symmetric solution of Malcev Yang-Baxter equation in \( A \) if and only if \( r \) satisfies for all \( x, y \in A^* \),

\[
[r(x^*), r(y^*)] = (ad^*r(x^*))(y^*) - (ad^*r(y^*))(x^*).
\]

**Proof.** Let \( \{e_1, ..., e_n\} \) be a basis of \( A \) and \( \{e_1^*, ..., e_n^*\} \) be its dual basis. Suppose that

\[
[e_i, e_j] = \sum_p c_{ij}^p e_p \quad \text{and} \quad r = \sum_{i,j} a_{ij} e_i \otimes e_j.
\]

Hence \( a_{ij} = -a_{ji} \). Now, we have

\[
[r_{12}, r_{13}] = \sum_{i,j} a_{ij} e_i \otimes e_j \otimes 1 \sum_{k,l} a_{kl} e_k \otimes 1 \otimes e_l = \sum_{i,j,k,l,p} a_{ij} a_{kl} c_{ik}^p e_p \otimes e_j \otimes e_l,
\]

\[
[r_{13}, r_{23}] = \sum_{i,j} a_{ij} e_i \otimes 1 \otimes e_j \sum_{k,l} a_{kl} e_k \otimes e_l \otimes e_l = \sum_{i,j,k,l,p} a_{ij} a_{kl} c_{ji}^p e_i \otimes e_k \otimes e_p,
\]

\[
[r_{12}, r_{23}] = \sum_{i,j} a_{ij} e_i \otimes e_j \otimes 1 \sum_{k,l} a_{kl} e_k \otimes e_l \otimes e_l = \sum_{i,j,k,l,p} a_{ij} a_{kl} c_{jk}^p e_i \otimes e_p \otimes e_l.
\]

Then \( r \) is a solution of the Malcev Yang-Baxter equation in \( (A, [, ,]) \) if and only if (for any \( j, p, l \))

\[
\sum_{i,k} \left( a_{ij} a_{kl} c_{ik}^p + a_{kp} a_{ij} c_{ik}^j + a_{pa} a_{kl} c_{jk}^i \right) e_p \otimes e_j \otimes e_l = 0.
\]
On the other hand, by (2.5), we get \( r(e_j^*) = \sum_i a_{ij} e_i \). Then, if we set \( x^* = e_j^* \) and \( y^* = e_i^* \), by (2.6),
\[
\sum_{i,k} \left( a_{ij} a_{kl} c_{ik}^j + a_{kp} a_{ij} c_{ki}^j + a_{pi} a_{kl} c_{ik}^j \right) e_p = 0.
\]
Therefore, it is easy to see that \( r \) is a solution of the Malcev Yang-Baxter equation in \( A \) if and only if \( r \) satisfies (2.6).
\[\square\]

**Definition 2.3.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra. A symmetric bilinear form \( B \) on \( A \) is called *invariant* if, for all \( x, y, z \in A \),
\[
B([x, y], z) = B(x, [y, z]).
\]

**Definition 2.4.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra. A Rota-Baxter operator of weight 0 on \( A \) is a linear map \( \mathcal{R} : A \to A \) satisfying for all \( x, y, z \in A \),
\[
[\mathcal{R}(x), \mathcal{R}(y)] = \mathcal{R}([\mathcal{R}(x), y] + [x, \mathcal{R}(y)]).
\]

**Corollary 2.1.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra and \( r \in A \otimes A \). Assume \( r \) is skew-symmetric and there exists a nondegenerate symmetric invariant bilinear form \( B \) on \( A \). Define a linear map \( \varphi : A \to A^* \) by \( \langle \varphi(x), y \rangle = B(x, y) \) for any \( x, y \in A \). Then \( r \) is a solution of the Malcev Yang-Baxter equation in \( A \) if and only if \( \mathcal{R} = r \varphi : A \to A \) is a Rota-Baxter operator.

**Proof.** For any \( x, y, z \in A \), we have
\[
\langle \varphi(ad(x)y), z \rangle = B([x, y], z) = B(z, [x, y]) = -B(y, [x, z]) = \langle ad^*(x)\varphi(y), z \rangle.
\]
Hence \( \varphi(ad(x)y) = ad^*(x)\varphi(y) \) for any \( x, y \in A \). Let \( x^* = \varphi(x), y^* = \varphi(y) \), then by Theorem 2.1, \( r \) is a solution of the Malcev Yang-Baxter equation in \( A \) if and only if
\[
[r\varphi(x), r\varphi(y)] = [r(x^*), r(y^*)] = r(ad^*r(x^*)(y^*) - ad^*r(y^*)(x^*)) = r\varphi([r\varphi(x), y] + [x, r\varphi(y)]).
\]
Therefore the conclusion holds. \[\square\]

Now, we introduce the notion of \( \mathcal{O} \)-operator of a Malcev algebra.

**Definition 2.5.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra and let \((V, \rho)\) be a representation of \( A \). A linear map \( T : V \to A \) is called an \( \mathcal{O} \)-operator associated to \( \rho \) if for all \( a, b \in V \),
\[
[T(a), T(b)] = T(\rho(T(a))b - \rho(T(b))a).
\]

**Example 2.1.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra. Then a Rota-Baxter operator (of weight zero) is an \( \mathcal{O} \)-operator of \( A \) associated to the adjoint representation \((A, ad)\) and a skew-symmetric solution of Malcev Yang-Baxter equation in \( A \) is an \( \mathcal{O} \)-operator of \( A \) associated to the representation \((A^*, ad^*)\).

Let \((A, [\cdot, \cdot])\) be a Malcev algebra. Let \( \rho^* : A \to gl(V^*) \) be the dual representation of the representation \( \rho : A \to gl(V) \) of the Malcev algebra \( A \). A linear map \( T : V \to A \) can be identified as an element in \( A \otimes V^* \subset (A \otimes \rho^*, V^*) \otimes (A \otimes \rho^*, V) \) as follows. Let \( \{e_1, \cdots, e_n\} \) be a basis of \( L \). Let \( \{v_1, \cdots, v_m\} \) be the dual basis, that is \( v_i^*(v_j) = \delta_{ij} \). Set \( T(v_i) = \sum_{j=1}^n a_{ij} e_j, i = 1, \cdots, m \). Since as vector spaces, \( \text{Hom}(V, A) \cong A \otimes V^* \), we have
\[
T = \sum_{i=1}^m T(v_i) \otimes v_i^* = \sum_{i=1}^m \sum_{j=1}^n a_{ij} e_j \otimes v_i^* 
\in A \otimes V^* \subset (A \otimes \rho^*, V^*) \otimes (A \otimes \rho^*, V^*).
\]

**Theorem 2.2.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra. Then \( T \) is an \( \mathcal{O} \)-operator of \( A \) associated to \((V, \rho)\) if and only if \( r = T - \sigma(T) \) is a skew-symmetric solution of the Malcev Yang-Baxter equation in \( A \otimes \rho^*, V^* \).
Proof. From (2.9), we have \( r = T - \sigma(T) = \sum_i T(v_i) \otimes v_i^* - v_i^* \otimes T(v_i) \). Thus,

\[
[r_{12}, r_{13}] = \sum_{i,k=1}^m \left( [T(v_i), T(v_k)] \otimes v_i^* \otimes v_k^* - \rho^*(T(v_i))v_k^* \otimes v_i^* \otimes T(v_k) \right. \\
&\quad + \rho^*(T(v_k))v_i^* \otimes T(v_i) \otimes v_k^* \right), \\
[r_{12}, r_{23}] = \sum_{i,k=1}^m \left( -v_i^* \otimes [T(v_i), T(v_k)] \otimes v_k^* - T(v_i) \otimes \rho^*(T(v_k)) v_i^* \otimes v_k^* \right. \\
&\quad \left. + v_i^* \otimes \rho^*(T(v_i))v_k^* \otimes T(v_k) \right), \\
[r_{13}, r_{23}] = \sum_{i,k=1}^m \left( v_i^* \otimes v_k^* \otimes [T(v_i), T(v_k)] + T(v_i) \otimes v_k^* \otimes \rho^*(T(v_k)) v_i^* \right. \\
&\quad \left. - v_i^* \otimes T(v_k) \otimes \rho^*(T(v_i)) v_i^* \right).
\]

By the definition of dual representation, we know \( \rho^*(T(v_k))v_i^* = -\sum_{j=1}^m v_i^*(\rho(T(v_k))v_j) \). Thus,

\[
- \sum_{i,k=1}^m T(v_i) \otimes \rho^*(T(v_k))v_i^* \otimes v_k^* = - \sum_{i,k=1}^m T(v_i) \otimes \left[ \sum_{j=1}^m -v_i^*(\rho(T(v_k))v_j) \right] \otimes v_k^* \\
= \sum_{i,k=1}^m \sum_{j=1}^m v_j^*(\rho(T(v_k))v_i)T(v_j) \otimes v_i^* \otimes v_k^* = \sum_{i,k=1}^m T(\sum_{j=1}^m v_j^*(\rho(T(v_k))v_i)v_j) \otimes v_i^* \otimes v_k^* \\
= \sum_{i,k=1}^m T(\rho(T(v_k))v_i) \otimes v_i^* \otimes v_k^*.
\]

Therefore,

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \\
= \sum_{i,k=1}^m \left( [T(v_i), T(v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k) \right) \otimes v_i^* \otimes v_k^* \\
= -v_i^* \otimes [T(v_i), T(v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k) \otimes v_k^* \\
+ v_i^* \otimes v_k^* \otimes [T(v_i), T(v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k) \right).
\]

So \( r \) is a classical \( r \)-matrix in \( \mathcal{A} \ltimes_{\rho^*} V^* \) if and only if \( T \) is an \( \mathcal{O} \)-operator. \( \square \)

In fact, Theorem 2.2 gives a relation between \( \mathcal{O} \)-operator and Malcev Yang-Baxter equation. Then, we get a direct conclusion from Theorems 2.1 and 2.2.

**Corollary 2.2.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra. Let \( \rho : A \to gl(V) \) be a representation of \( A \). Set \( \mathcal{A} = A \ltimes_{\rho^*} V^* \). Let \( T : V \to A \) be a linear map. Then the following three conditions are equivalent:

(i) \( T \) is an \( \mathcal{O} \)-operator of \( A \) associated to \( \rho \);  
(ii) \( T - \sigma(T) \) is a skew-symmetric solution of the Malcev Yang-Baxter equation in \( \mathcal{A} \);  
(iii) \( T - \sigma(T) \) is an \( \mathcal{O} \)-operator of the Malcev algebra \( \mathcal{A} \) associated to \( ad^* \).

3. Alternative and post-alternative algebras

In this section, we recall some basic definitions about alternative and pre-alternative algebras studied in [6, 22].
3.1. Some basic results on alternative algebras

**Definition 3.1.** An alternative algebra $(A, \cdot)$ is a vector space $A$ equipped with a bilinear operation $(x, y) \mapsto x \cdot y$ satisfying, for all $x, y, z \in A$,

$$as_A(x, x, y) = as_A(y, x, x) = 0,$$

where $as_A(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the **associator**.

**Remark 3.1.** If the characteristic of the field is not 2, then an alternative algebra $(A, \cdot)$ also satisfies the stronger axioms, for all $x, y, z \in A$,

$$as_A(x, y, z) + as_A(y, x, z) = 0, \quad (3.2)$$

$$as_A(z, x, y) + as_A(y, z, x) = 0. \quad (3.3)$$

Now, recall that an algebra $(A, \cdot)$ is called admissible Malcev algebra if $(A, [\cdot, \cdot])$ is a Malcev algebra, where $[x, y] = x \cdot y - y \cdot x$.

**Example 3.1.** Any alternative algebra is Malcev admissible. That is if $(A, \cdot)$ be an alternative algebra, then $(A, [\cdot, \cdot])$ is a Malcev algebra, where $[x, y] = x \cdot y - y \cdot x$, for all $x, y \in A$.

**Definition 3.2.** Let $(A, \cdot)$ be an alternative algebra and $V$ be a vector space. Let $l, r : A \to \text{End}(V)$ be two linear maps. Then, $(V, l, r)$ is called a representation or a bimodule of $A$ if, for any $x, y \in A$,

$$\tau(x)\tau(y) + \tau(y)\tau(x) - \tau(x \cdot y) - \tau(y \cdot x) = 0, \quad (3.4)$$

$$l(x \cdot y) + l(y \cdot x) - l(x)l(y) - l(y)l(x) = 0, \quad (3.5)$$

$$l(x \cdot y) + \tau(y)l(x) - l(x)l(y) - l(x)\tau(y) = 0, \quad (3.6)$$

$$\tau(y)l(x) + \tau(y)\tau(x) - l(x)\tau(y) - \tau(x \cdot y) = 0. \quad (3.7)$$

**Definition 3.3.** A pre-alternative algebra is a triple $(A, \prec, \succ)$, where $A$ is a vector space, $\prec, \succ : A \otimes A \to A$ are bilinear maps satisfying for all $x, y, z \in A$ and $x \cdot y = x \prec y + x \succ y$,

$$x \succ y \prec z \prec x \succ (y \prec z) + (y \prec x) \succ z - y \prec (x \cdot z) = 0, \quad (3.8)$$

$$x \succ y \prec z \prec x \succ (y \prec z) + (z \cdot x) \succ y - y \succ (x \succ y) = 0, \quad (3.9)$$

$$x \cdot y \succ z \prec x \succ (y \succ z) + (y \cdot x) \succ z - y \succ (x \succ z) = 0, \quad (3.10)$$

$$z \prec x \succ y - z \prec (x \cdot y) + (z \prec y) \prec x - z \prec (y \cdot x) = 0. \quad (3.11)$$

**Proposition 3.1.** Let $(A, \prec, \succ)$ be a pre-alternative algebra. Then the product $x \cdot y = x \prec y + x \succ y$ defines an alternative algebra $A$. Furthermore, $(A, L_{\succ}, R_{\prec})$, where $L_{\succ}(x)y = x \succ y$ and $R_{\prec}(x)y = y \prec x$, gives a representation of the associated alternative algebra $(A, \cdot)$ on $A$.

**Proposition 3.2.** Let $(A, \prec, \succ)$ be a pre-alternative algebra. Then the product $x \succ y = x \succ y - y \prec x$ defines a pre-Malcev structure in $A$.

3.2. A-bimodule alternative algebras, weighted O-operators and post-alternative algebras

**Definition 3.4.** Let $(A, \cdot)$ be an alternative algebra. Let $(V, \cdot_V)$ be an alternative algebra and $l, r : A \to \text{End}(V)$ be two linear maps. We say that $(V, \cdot_V, l, r)$ is an **A-bimodule alternative algebra** if $(V, l, r)$ is a representation of $(A, \cdot)$ such that the following compatibility conditions hold (for all $x \in A$, $a, b \in V$)

$$\tau(x)(a \cdot_V b) - a \cdot_V (\tau(x)b) + \tau(x)(b \cdot_V a) - b \cdot_V (\tau(x)a) = 0, \quad (3.12)$$

$$(l(x)a) \cdot_V b - l(x)(a \cdot_V b) + (l(x)b) \cdot_V a - l(x)(b \cdot_V a) = 0. \quad (3.13)$$
(l(x)a) \cdot_V b - a \cdot_V (l(x)b) + (\tau(x)a) \cdot_V b - l(x)(a \cdot_V b) = 0, \quad (3.14)
(r(x)a) \cdot_V b - a \cdot_V (r(x)b) + r(x)(a \cdot_V b) - a \cdot_V (r(x)b) = 0. \quad (3.15)

**Proposition 3.3.** Let \((A, \cdot)\) and \((V, \cdot_V)\) be two alternative algebras and \(l, \tau : A \to End(V)\) be two linear maps. Then \((V, \cdot_V, l, \tau)\) is an \(A\)-bimodule alternative algebra if and only if the direct sum \(A \oplus V\) of vector spaces is an alternative algebra (the semi-direct sum) with the product on \(A \oplus V\) defined for all \(x, y \in A, a, b \in V\) by

\[(x + a) \ast (y + b) = x \cdot y + l(x)b + \tau(y)a + a \cdot_V b. \quad (3.16)\]

We denote this algebra by \(A \ltimes_L V\) or simply \(A \ltimes V\). Further, if \((A, \cdot)\) is an alternative algebra, then it is easy to see that \((A, \cdot_L, R)\) is an \(A\)-bimodule alternative algebra, where \(L\) and \(R\) are the left and right multiplication operators corresponding to the multiplication \(\cdot\).

**Proof.** For any \(x, y, z \in A, a, b, c \in V\)

\[
as_{A \otimes V}(x + a, y + b, z + c) + as_{A \otimes V}(y + b, x + a, z + c)
= ((x + a) \ast (y + b)) \ast (z + c) - (x + a) \ast ((y + b) \ast (z + c)) + ((y + b) \ast (x + a)) \ast (z + c)
- (y + b) \ast ((x + a) \ast (z + c))
= (x \cdot y + l(x)b + \tau(y)a + a \cdot_V b) \ast (z + c) - (x + a) \ast (y \cdot z + l(y)c) + \tau(z)b + b \cdot_V c +
+ (y \cdot x + l(y)a + \tau(x)b) + b \cdot_V a) \ast (z + c) - (y + b) \ast (x \cdot z + l(x)c + \tau(z)a + a \cdot_V c)
= (x \cdot y \cdot z + l(x \cdot y)c + \tau(z)(l(x)b + \tau(y)a + a \cdot_V b) + (l(x)b + \tau(y)a + a \cdot_V b) \cdot_V c - x \cdot (y \cdot z) - l(x)(l(y)c + \tau(z)b + b \cdot_V c) - \tau(y \cdot z)a - a \cdot_V (l(y)c + \tau(z)b + b \cdot_V c) +
+ (y \cdot x) \cdot z + l(y \cdot x)c + \tau(z)(l(y)a + \tau(x)b + b \cdot_V a) + (l(y)a + \tau(x)b + b \cdot_V a) \cdot_V c - y \cdot (x \cdot z) - l(y)(l(x)c + \tau(z)a + a \cdot_V c) - \tau(x \cdot z) - b \cdot_V (l(x)c + \tau(z)a + a \cdot_V c).
\]

Hence, \(as_{A \otimes V}(x + a, y + b, z + c) + as_{A \otimes V}(y + b, x + a, z + c) = 0\) if and only if \((3.2), (3.12)\) and \((3.14)\) hold.

Analogously, \(as_{A \otimes V}(z + c, x + a, y + b) + as_{A \otimes V}(z + c, y + b, x + a) = 0\) if and only if \((3.3), (3.13)\) and \((3.15)\) hold. \(\square\)

**Definition 3.5** ([3]). A post-alternative algebra \((A, \prec, \succ, \cdot)\) is a vector space \(A\) equipped with bilinear operations \(\prec, \succ, \cdot : A \otimes A \to A\) obeying the following equations for \(* = \prec + \succ + \cdot\) and all \(x, y, z \in A\),

\[
(x \cdot y) \cdot z - x \cdot (y \cdot z) + (y \cdot x) \cdot z - y \cdot (x \cdot z) = 0, \quad (3.17)
(z \cdot x) \cdot y - z \cdot (x \cdot y) + (z \cdot y) \cdot x - z \cdot (y \cdot x) = 0, \quad (3.18)
(x \cdot y) \prec z - x \cdot (y \prec z) + (y \cdot x) \prec z - y \cdot (x \prec z) = 0, \quad (3.19)
(x \succ y) \cdot z - x \succ (y \cdot z) + (x \cdot y) \succ z - x \succ (y \succ z) = 0, \quad (3.20)
(y \succ x) \cdot z - x \succ (y \succ z) + (x \prec y) \cdot z - x \prec (y \succ z) = 0, \quad (3.21)
(z \prec x) \cdot y - z \cdot (x \prec y) + (z \cdot y) \prec x - z \cdot (y \prec x) = 0, \quad (3.22)
(x \succ y) \prec z - x \succ (y \prec z) + (y \prec x) \prec z - y \prec (x \succ z) = 0, \quad (3.23)
(x \prec y) \prec z - x \prec (y \prec z) + (z \cdot x) \prec y - z \prec (x \prec y) = 0, \quad (3.24)
(x \succ y) \succ z - x \succ (y \succ z) + (y \cdot x) \succ z - y \succ (x \succ z) = 0, \quad (3.25)
(z \prec x) \prec y - z \prec (x \succ y) + (z \cdot y) \prec x - z \prec (y \cdot x) = 0. \quad (3.26)
\]

**Remark 3.2.** Let \((A, \prec, \succ, \cdot)\) be a post-alternative algebra. If the operation \(\cdot\) is trivial, then it is a pre-alternative algebra.

Let \((A, \prec, \succ, \cdot)\) be a post-alternative algebra, it is obvious that \((A, \cdot)\) is an alternative algebra. On the other hand, it is straightforward to get the following conclusion:
Theorem 3.1. If \((A,\prec,\succ,\cdot)\) is a post-alternative algebra, then with a new bilinear operation \(\ast : A \times A \to A\) on \(A\) defined for all \(x,y \in A\) by
\[
x \ast y = x \prec y + x \succ y + x \cdot y,
\] (3.27)
\((A,\ast)\) becomes an alternative algebra. It is called the associated alternative algebra of \((A,\prec,\succ,\cdot)\).

Proof. In fact, for any \(x,y,z \in A\), we have
\[
\begin{align*}
as_A(x,y,z) + as_A(y,x,z) &= (x \ast y) \ast z - x \ast (y \ast z) + (y \ast x) \ast z - y \ast (x \ast z) \\
&= (x \prec y) \prec z + (x \prec y) \succ z + (x \succ y) \prec z - z \prec (x \ast y) + (y \prec x) \ast z - (y \ast x) \prec z \\
&+ (y \prec x) \ast z - (y \prec x \prec y) \succ z - (y \prec x) \succ z \succ z - (y \prec x \succ y) \prec z \\
&+ (y \prec x \prec z - x \prec (y \prec z) - x \succ (y \prec z) - (y \prec x) \prec z \prec (y \prec z) \\
&- (y \prec x \succ z - x \prec (y \prec z) - x \succ (y \prec z) - (y \prec x) \succ z \succ (y \prec z) \\
&+ (y \prec x \succ z - y \prec (x \ast z) - y \succ (x \ast z) - y \prec (x \ast z) - y \succ (x \ast z)
\end{align*}
\]
and then replacing \((x,y,z)\) in this computation by \((z,x,y)\) yields \(as_A(z,x,y) + as_A(z,y,x) = 0\), which completes the proof according to Definition 3.1 and Remark 3.1. □

The following terminology is motivated by the notion of \(\lambda\)-weighted \(O\)-operator as a generalization of (the operator form of) the classical Yang-Baxter equation in [2,19].

Definition 3.6. Let \((A,\cdot)\) be an alternative algebra and \((V,\cdot_V,l,r)\) be an \(A\)-bimodule alternative algebra. A linear map \(T : V \to A\) is called a \(\lambda\)-weighted \(O\)-operator associated to \((V,\cdot_V,l,r)\) if \(T\) satisfies, for all \(a,b \in V\),
\[
T(a) \cdot T(b) = T(l(T(a))b + r(T(b))a) + \lambda a \cdot_V b.
\] (3.28)

When \((V,\cdot_V,l,r) = (A,\cdot,L,R)\), the condition (3.28) becomes
\[
\mathcal{R}(x) \cdot \mathcal{R}(y) = \mathcal{R}(\mathcal{R}(x) \cdot y + x \cdot \mathcal{R}(y) + \lambda x \cdot y).
\] (3.29)
The property (3.29) implies that \(\mathcal{R} : A \to A\) is a \(\lambda\)-weighted Rota-Baxter operator on the alternative algebra \((A,\cdot)\).

Theorem 3.2. Let \((A,\cdot)\) be an alternative algebra and \((V,\cdot_V,l,r)\) be an \(A\)-bimodule alternative algebra. Let \(T : V \to A\) be a \(\lambda\)-weighted \(O\)-operator associated to \((V,\cdot_V,l,r)\). Define three new bilinear operations \(\prec,\succ,\circ : V \otimes V \to V\) on \(V\) as follows:
\[
a \succ b = l(T(a))b, \quad a \prec b = r(T(b))a, \quad a \circ b = \lambda a \cdot_V b.
\] (3.30)
Then \((V,\prec,\succ,\circ)\) becomes a post-alternative algebra and \(T\) is a homomorphism of alternative algebras.

Proof. Since \(A\) is an alternative algebra, (3.17) and (3.18) obviously hold. Furthermore, for any \(a,b,c \in V\), we have
\[
(a \circ b) \prec c - a \circ (b \prec c) + (b \circ a) \prec c - b \circ (a \prec c) = (\lambda a \cdot_V b) \prec c - a \circ (\tau(T(c))b) + (\lambda b \cdot_V a) \prec c - b \circ (\tau(T(c))a) \\
= \lambda (\tau(T(c))a \cdot_V b - a \cdot_V (\tau(T(c))b) + \tau(T(c))(b \cdot_V a) - b \cdot_V (\tau(T(c))a) = 0.
\]
So, (3.19) holds. Moreover, (3.20) holds. Indeed,
\[
\begin{align*}
(a \succ b) \circ c - a \circ (b \succ c) + (a \succ c) \circ b - a \succ (c \circ b) &= ((l(T(a))b) \circ c - a \circ (\lambda b \cdot_V c) + (l(T(a))c) \circ b - a \succ (\lambda c \cdot_V b) \\
&= \lambda ((l(T(a))b) \cdot_V c - l(T(a))(b \cdot_V c) + (l(T(a))c) \cdot_V b - l(T(a))(c \cdot_V b) = 0.
\end{align*}
\]
To prove identity (3.21), we compute as follows

\[
(b \succ a) \circ c - a \circ (b \succ c) + (a \prec b) \circ c - b \succ (a \circ c) = \lambda ((T(b))a) \circ c - a \circ ((T(b))c) + (\tau(T(b))a) \circ c - b \succ (\lambda a \cdot V c) = \lambda ((T(b))a) \cdot V c - a \cdot V ((T(b))c) + (\tau(T(b))a) \cdot V c - ((T(b))(a \cdot V c)) = 0.
\]

The other identities can be shown similarly. \(\square\)

**Corollary 3.1.** Let \((A, \cdot)\) be an alternative algebra and \(\mathcal{R} : A \rightarrow A\) be a \(\lambda\)-weighted Rota-Baxter operator for \(A\). Then \((A, \prec, \succ, \circ)\) is a post-alternative algebra with the operations

\[
x \prec y = x \cdot \mathcal{R}(y), \quad x \succ y = \mathcal{R}(x) \cdot y, \quad x \circ y = \lambda x \cdot y.
\]

## 4. Weighted \(\mathcal{O}\)-operators and post-Malcev algebras

We start this section by introducing the notion of post-Malcev algebra together with some of its basic properties. We will also briefly discuss the post-Malcev algebra structure underneath the \(\lambda\)-weighted \(\mathcal{O}\)-operators. We then show that there is a close relationship between post-Malcev algebras and post-alternative algebras in parallel to the relationship between pre-Malcev and pre-alternative algebras.

### 4.1. \(A\)-module Malcev algebras and weighted \(\mathcal{O}\)-operators

Now, we extend the concept of a module to that of an \(A\)-module algebra by replacing the \(\mathbb{K}\)-module \(V\) by a Malcev algebra. Next, we introduce \(\lambda\)-weighted \(\mathcal{O}\)-operators on Malcev algebras and study some basic properties.

**Definition 4.1.** Let \((A, [\cdot, \cdot])\) and \((V, [\cdot, \cdot]_V)\) be two Malcev algebras. Let \(\rho : A \rightarrow \text{End}(V)\) be a linear map such that \((V, \rho)\) is a representation of \((A, [\cdot, \cdot])\) and the following compatibility conditions hold for all \(x, y, z, a, b, c \in V:\)

\[
\rho([x, y])[a, b]_V = \rho(x)[\rho(y)a, b]_V - [\rho(y)\rho(x)a, b]_V - [\rho(x)\rho(y)b, a]_V + \rho(y)[\rho(x)b, a]_V, \\
\quad (4.1)
\]

\[
[\rho(x)a, \rho(y)b]_V = [\rho([x, y])a, b]_V - [\rho(x)\rho(y)a, b]_V + [\rho(y)\rho(x)a, b]_V + [\rho(y)\rho(x)b, a]_V, \\
\quad (4.2)
\]

\[
[\rho(x)a, [b, c]_V]_V = [[\rho(x)b, a]_V, c]_V - \rho(x)[[b, a]_V, c]_V - [\rho(x)[a, c]_V, b]_V - [[\rho(x)c, b]_V, a]_V. \\
\quad (4.3)
\]

Then \((V, [\cdot, \cdot]_V, \rho)\) is called an \(A\)-module Malcev algebra.

In the sequel, an \(A\)-module Malcev algebra is denoted by \((V, [\cdot, \cdot]_V, \rho)\). It is straightforward to get the following:

**Proposition 4.1.** Let \((A, [\cdot, \cdot])\) and \((V, [\cdot, \cdot]_V)\) be two Malcev algebras and \((V, [\cdot, \cdot]_V, \rho)\) be an \(A\)-module Malcev algebra. Then \((A \oplus V, [\cdot, \cdot]_\rho)\) carries a new Malcev algebra structure with bracket

\[
[x + a, y + b]_\rho = [x, y] + \rho(x)b - \rho(y)a + [a, b]_V, \quad \forall x, y, z \in A, \quad a, b, c \in V.
\]

This is called the semi-direct product, often denoted by \(A \ltimes \rho V\) or simply \(A \ltimes V\).

**Proof.** For \(x, y, z, t \in A\) and \(a, b, c, d \in V\),

\[
[[x + a, z + c]_\rho, [y + b, t + d]_\rho] = [[x, z], [y, t]] + \rho([x, z])\rho(y)d - \rho([x, z])\rho(t)b \\
+ \rho([x, z])[b, d]_V - \rho([y, t])\rho(x)c + \rho([y, t])\rho(z)a - \rho([y, t])[a, c]_V + [\rho(x)c, \rho(y)d]_V \\
- [\rho(x)c, \rho(t)b]_V + [\rho(x)c, [b, d]_V]_V - [\rho(z)a, \rho(y)d]_V + [\rho(z)a, \rho(t)b]_V - [\rho(z)a, [b, d]_V]_V \\
+ [[a, c]_V, \rho(y)d]_V - [[a, c]_V, \rho(t)b]_V + [[a, c]_V, [b, d]_V]_V, \\
[[[x + a, y + b]_\rho, z + c]_\rho, t + d]_\rho = [[[x, y], z], t] + \rho([[x, y], z])d - \rho(t)\rho([x, y])c.
\]
More generally, if we define a

\[ T : V \to A, x, y \in V \to \rho(T(x)) + \rho(T(y)) = \rho(x) + \rho(y) \]

By Proposition 4.2. According to (4.1), we have

\[ [x, y] = \rho(x)y - \rho(y)x + \lambda(x,y) \]

Obviously, a \(\lambda\)-weighted \(\mathcal{O}\)-operator associated to \((V, \rho)\) is a special \(\mathcal{O}\)-operator associated to \((A, [\cdot, \cdot])\), where \(\lambda(x,y)\) is a \(\lambda\)-weight.

**Remark 4.1.** More generally, if we define a \(\lambda\)-semi-direct product denoted by \(A \ltimes_{\lambda} V\) as follow

\[ [x + a, y + b]_{\rho} = [x, y] + \rho(x)b - \rho(y)a + \lambda(a, b)_V, \quad \forall x, y \in A, \quad a, b \in V \quad (4.5) \]

we obtain the same characterization given in the above Proposition.

**Example 4.1.** It is known that \((A, ad)\) is a representation of \(A\) called the adjoint representation. Then \((A, [\cdot, \cdot], ad)\) is an \(A\)-module Malcev algebra.

**Proposition 4.2.** Let \((A, \cdot)\) be an alternative algebra. Then the triplet \((V; [\cdot, \cdot], I - \tau)\) defines an \(A\)-module Malcev admissible algebra of \((A, [\cdot, \cdot])\).

**Proof.** By Proposition 3.3, \(A \ltimes_{\lambda} V\) is an alternative algebra. For its associated Malcev algebra \((A \oplus V, [\cdot, \cdot])\), we have

\[ [x + a, y + b] = (x + a) \cdot (y + b) - (y + b) \cdot (x + a) \]

\[ = x \cdot y + I(x)b + \tau(y)a + a \cdot V - y \cdot x - I(y)a - \tau(x)b - b \cdot V \]

\[ = [x, y] + (I - \tau)(x)b - (I - \tau)(y)a + [a, b]_V. \]

According to (4.4), we deduce that \((V; [\cdot, \cdot], I - \tau)\) is an \(A\)-module Malcev admissible algebra of \((A, [\cdot, \cdot])\).

**Definition 4.2.** Let \((A, [\cdot, \cdot])\) be a Malcev algebra and \((V; [\cdot, \cdot], \rho)\) be an \(A\)-module Malcev algebra. A linear map \(T : V \to A\) is said to be a \(\lambda\)-**weighted \(\mathcal{O}\)-operator** associated to \((V; [\cdot, \cdot], \rho)\) if for all \(a, b \in V\),

\[ T(a), T(b) = T(\rho(T(a))b - \rho(T(b))a + \lambda(a, b)_V). \quad (4.6) \]

Obviously, a \(\lambda\)-weighted \(\mathcal{O}\)-operator associated to \((A, [\cdot, \cdot], ad)\) is just a \(\lambda\)-weighted Rota-Baxter operator on \(A\). A \(\lambda\)-weighted \(\mathcal{O}\)-operator can be viewed as the relative version of a Rota-Baxter operator in the sense that the domain and range of an \(\mathcal{O}\)-operator might be different.
Example 4.2. \(\) (i) A Rota-Baxter operator on \(A\) is simply a \(0\)-weighted \(O\)-operator.
(ii) The identity map \(id : A \to A\) is a \((-1)\)-weighted \(O\)-operator.
(iii) If \(f : A \to A\) is a Malcev algebra homomorphism and \(f^2 = f\) (idempotent condition), then \(f\) is a \((-1)\)-weighted \(O\)-operator.
(iv) If \(T\) is a \(\lambda\)-weighted \(O\)-operator, then for any \(\nu \in \mathbb{K}\), the map \(\nu T\) is a \((\nu \lambda)\)-weighted \(O\)-operator.
(v) If \(T\) is a \(\lambda\)-weighted \(O\)-operator, then \(-\lambda id - T\) is a \(\lambda\)-weighted \(O\)-operator.

In the following, we characterize \(\lambda\)-weighted \(O\)-operators in terms of their graph.

Proposition 4.3. Let \((V; [\cdot, \cdot]_V, \rho)\) be an \(A\)-module Malcev algebra. Then a linear map \(T : V \to A\) is a \(\lambda\)-weighted \(O\)-operator associated to \((V; [\cdot, \cdot]_V, \rho)\) if and only if the graph \(Gr(T) = \{T(a) + a\mid a \in V\}\) of the map \(T\) is a subalgebra of the \(\lambda\)-semi-direct product \(A \ltimes^\lambda V\).

Proof. Let \(T : V \to A\) be a linear map. For all \(a, b \in V\), we have
\[
[T(a) + a, T(b) + b]_\lambda^A = [T(a), T(b)] + \rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V,
\]
which implies that the graph \(Gr(T) = \{T(a) + a\mid a \in V\}\) is a subalgebra of the Malcev algebra \(A \ltimes^\lambda V\) if and only if \(T\) satisfies
\[
[T(a), T(b)] = T(\rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V),
\]
which means that \(T\) is a \(\lambda\)-weighted \(O\)-operator. 

As a consequence of the above proposition, we get the following.

Corollary 4.1. Let \(T : V \to A\) be a \(\lambda\)-weighted \(O\)-operator. Since \(Gr(T)\) is isomorphism to \(V\) as a vector space, we get that \(V\) inherits a new Malcev algebra structure with the bracket
\[
[a, b]_T := \rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V, \quad \text{for } a, b \in V.
\]
In other words, \((V, [\cdot, \cdot]_T)\) is a Malcev algebra, denoted by \(V_T\) (called the induced Malcev algebra). Moreover, \(T : V_T \to A\) is a homomorphism of Malcev algebras.

Let \(T, T' : (A, [\cdot, \cdot]) \to (V, [\cdot, \cdot]_V)\) be two \(\lambda\)-weighted \(O\)-operators. A homomorphism from \(T\) to \(T'\) consists of Malcev algebra homomorphisms \(\phi : A \to A\) and \(\psi : V \to V\) such that
\[
\phi \circ T = T' \circ \psi, \quad \psi(\rho(x)a) = \rho(\phi(x))(\psi(a)), \quad \forall x \in A, a \in V.
\]
In particular, if both \(\phi\) and \(\psi\) are invertible, \((\phi, \psi)\) is called an isomorphism from \(T\) to \(T'\).

Proposition 4.4. Let \((\phi, \psi)\) be a homomorphism of \(\lambda\)-weighted \(O\)-operators from \(T\) to \(T'\). Then \(\psi : V \to V\) is a homomorphism of induced Malcev algebras from \((V, [\cdot, \cdot]_T)\) to \((V, [\cdot, \cdot]_{T'})\).

Proof. For any \(a, b \in V\), we have
\[
\psi([a, b]_T) = \psi(\rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V)
\]
In the sequel, we characterize \( \lambda \)-weighted \( \mathcal{O} \)-operators associated to \((V;[\cdot,\cdot]_V,\rho)\) in terms of the Nijenhuis operators. Recall that a Nijenhuis operator on a Malcev algebra \((A, [\cdot, \cdot])\) is a linear map \(N : A \to A\) satisfying, for all \(x, y \in A\),

\[
[N(x), N(y)] = N([N(x), y] - [N(y), x] - N([x, y])).
\]

**Proposition 4.5.** Let \((V;[\cdot,\cdot]_V,\rho)\) be an \(A\)-module Malcev algebra. Then a linear map \(T : V \to A\) is a \(\lambda\)-weighted \(\mathcal{O}\)-operator associated to \((V;[\cdot,\cdot]_V,\rho)\) if and only if

\[
N_T = \begin{bmatrix}
\lambda id & -T \\
0 & 0
\end{bmatrix}
: A \oplus V \to A \oplus V
\]

is a Nijenhuis operator on the semi-direct product Malcev algebra \(A \ltimes V\).

**Proof.** For all \(x, y \in A, a, b \in V\), on the one hand, we have

\[
[N_T(x + a), N_T(y + b)] = [\lambda x - T(a), \lambda y - T(b)]_\rho
= \lambda^2[x, y] - \lambda[x, T(b)] - \lambda[T(a), y] + [T(a), T(b)].
\]

On the other hand, since \(N_T^2 = N_T\), we have

\[
N_T([N_T(x + a), y + b], x + a)_\rho = \lambda x - T(a) - \lambda y - T(b)]_\rho = \lambda^2[x, y] - \lambda[x, T(b)] - \lambda[T(a), y] + [T(a), T(b)].
\]

Therefore, \(N_T\) is a Nijenhuis operator on the semi-direct product Malcev algebra \(A \ltimes V\) if and only if \((4.6)\) is satisfied. \(\square\)

**Corollary 4.2.** A linear map \(T : V \to A\) is a \(\lambda\)-weighted \(\mathcal{O}\)-operator associated to \((V;[\cdot,\cdot]_V,\rho)\) if and only if the operator

\[
N_T = \begin{bmatrix}
id & -T \\
0 & 0
\end{bmatrix}
: A \oplus V \to A \oplus V
\]

is a Nijenhuis operator on the \(\lambda\)-semi-direct product Malcev algebra \((A \oplus V, [\cdot, \cdot]_\rho^A)\).

### 4.2. Definition and constructions of post-Malcev algebras

In this section, we introduce the notion of post-Malcev algebras. We show that post-Malcev algebras arise naturally from a \(\lambda\)-weighted \(\mathcal{O}\)-operators. Therefore, post-Malcev algebras can be viewed as the underlying algebraic structures of \(\lambda\)-weighted \(\mathcal{O}\)-operators on Malcev algebras. Finally, we study some properties of post-Malcev algebras.

**Definition 4.3.** A post-Malcev algebra \((A, [\cdot, \cdot], \rhd)\) is a Malcev algebra \((A, [\cdot, \cdot])\) together with a bilinear map \(\rhd : A \otimes A \to A\) such that for all \(x, y, z \in A\), and \(\{x, y\} = x \rhd y - y \rhd x + [x, y]\),

\[
\{x, z\} \rhd [y, t] = x \rhd [z \rhd (y \rhd t)] - [z \rhd (x \rhd y) \rhd t] - [x \rhd (z \rhd (y \rhd t))] + [x \rhd (z \rhd y) \rhd t],
\]

\[
[x \rhd z, y \rhd t] = [[x \rhd y, z] \rhd t] - x \rhd ([y \rhd z] \rhd t) + y \rhd (x \rhd [z, t]) + [y \rhd (x \rhd t), z],
\]

\[
[x \rhd z, [y, t]] = [[x \rhd y, z] \rhd t] - x \rhd ([y \rhd z, t]) - [x \rhd [z, t], y] - [[x \rhd t, y], z],
\]

\[
\{x, z\} \rhd t = x \rhd (y \rhd (z \rhd t)) + y \rhd (x \rhd (z \rhd t)) - x \rhd [y, z] \rhd t - [y, z] \rhd (x \rhd t) - [y, z] \rhd t - (x \rhd t).
\]

**Example 4.3.**

1. A pre-Malcev algebra is a post-Malcev algebra with an abelian Malcev algebra \((A, [\cdot, \cdot] = 0, \rhd)\). (See [16, 26] for more details.)
2. Post-Malcev algebras generalize post-Lie algebras.
3. If \((A, [\cdot, \cdot])\) is a Malcev algebra, then \((A, [\cdot, \cdot], \rhd)\) is a post-Malcev algebra, where \(x \rhd y = [y, x]\) for all \(x, y \in A\).
Let \((A, [\cdot, \cdot], \triangleright)\) and \((A’, [\cdot, \cdot]’, \triangleright’)\) be two post-Malcev algebras. A homomorphism of post-Malcev algebras is a linear map \(f : A \rightarrow A’\) such that \(f([x,y]) = [f(x), f(y)]’\) and \(f(x \triangleright y) = f(x) \triangleright’ f(y)\).

**Proposition 4.6.** Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Malcev algebra. Then the bracket
\[
\{x, y\} = x \triangleright y - y \triangleright x + [x, y]
\] (4.13)
defines a Malcev algebra structure on \(A\). We denote this algebra by \(A^C\) and call it the sub-adjacent Malcev algebra of \(A\).

**Proof.** The skew symmetry is obvious. For all \(x, y, z, t \in A\), we have
\[
\begin{align*}
\{\{x, z\}, \{y, t\}\} & = \{x, z\} \triangleright \{y, t\} - \{y, t\} \triangleright \{x, z\} + \{\{x, z\}, \{y, t\}\} \\
= \{x, z\} \triangleright (y \triangleright t) - \{x, z\} \triangleright \{y, t\} + \{x, z\} \triangleright \{y, t\} - \{y, t\} \triangleright \{x, z\} \\
& \quad + \{y, t\} \triangleright (z \triangleright x) - \{x, z\} \triangleright \{y, t\} + \{x, z\} \triangleright \{y, t\} - \{y, t\} \triangleright \{x, z\} \\
& \quad + \{x \triangleright z, \{y, t\}\} - \{z \triangleright x, y \triangleright t\} + \{z \triangleright x, y \triangleright t\} - \{z \triangleright x, y \triangleright t\} \\
& \quad + \{[x, z], y \triangleright t\} - \{z \triangleright x, y \triangleright t\} + \{[x, z], y \triangleright t\}.
\end{align*}
\]
\[
\begin{align*}
\{\{y, z\}, \{t, x\}\} & = \{y, z\} \triangleright x - x \triangleright \{y, z\} + \{\{y, z\}, \{t, x\}\} \\
& = \{y, z\} \triangleright (t \triangleright x) - \{t \triangleright x\} \triangleright \{y, z\} + \{t \triangleright x\} \triangleright \{y, z\} - \{y, z\} \triangleright \{t \triangleright x\} \\
& \quad + \{x \triangleright y, \{t, x\}\} - \{x \triangleright y, \{t, x\}\} + \{x \triangleright y, \{t, x\}\} - \{x \triangleright y, \{t, x\}\} \\
& \quad + \{\{y, z\}, t \triangleright x\} - \{t \triangleright y, \{t, x\}\} + \{t \triangleright y, \{t, x\}\} - \{t \triangleright y, \{t, x\}\} \\
& \quad + \{[y, z], t \triangleright x\} - \{[[y, z], t \triangleright x]\} + \{[[y, z], t \triangleright x]\} - \{[[y, z], t \triangleright x]\} \\
& = \{\{y, z\}, \{t, x\}\} = 0.
\end{align*}
\]
By the identity of Malcev algebra and (4.9)-(4.12), we have
\[
\{\{x, z\}, \{y, t\}\} - \{\{x, z\}, \{y, t\}\} = 0.
\]

**Remark 4.2.** Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Malcev algebra. If \(\triangleright\) is commutative, \(x \triangleright y = y \triangleright x\), then the two Malcev brackets \([\cdot, \cdot]\) and \(\{\cdot, \cdot\}\) coincide.

**Corollary 4.3.** If \((A, [\cdot, \cdot], \triangleright)\) be a post-Malcev algebra, then \((A, \odot)\) is an admissible Malcev algebra, with the product \(\odot\) defined for all \(x, y \in A\) by
\[
x \odot y = x \triangleright y + \frac{1}{2} [x, y].
\] (4.14)
Proposition 4.7. Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Malcev algebra. Define \(L_\triangleright : A \to A\) by 
\(L_\triangleright(x)y = x \triangleright y\) for any \(x, y \in A\). Then \((A; [\cdot, \cdot], L_\triangleright)\) is an \(A\)-module Malcev algebra of \((A^C, \{\cdot, \cdot\})\).

**Proof.** By (4.12), \(L_\triangleright\) is a representation of \((A^C, \{\cdot, \cdot\})\). Indeed, for \(x, y, z, t \in A\), 
\[
L_\triangleright(\{\{x, y\}, z\})t = \{\{x, y\}, z\} \triangleright t
\]
\[
= x \triangleright (y \triangleright (z \triangleright t)) - z \triangleright (x \triangleright (y \triangleright t)) - y \triangleright ((x, z) \triangleright t) - (y, z) \triangleright (x \triangleright t)
\]
\[
= L_\triangleright(x)L_\triangleright(y)L_\triangleright(z)t - L_\triangleright(z)L_\triangleright(x)L_\triangleright(y)t - L_\triangleright(y)L_\triangleright((x, z) \triangleright t)
\]
\[
- L_\triangleright(\{y, z\})L_\triangleright(x)t.
\]
To prove (4.1), according to (4.9) we compute
\[
L_\triangleright(\{x, z\})\{y, t\}
\]
\[
= \{x, z\} \triangleright [y, t] = x \triangleright [z \triangleright y, t] - [z \triangleright (x \triangleright y), t] - [x \triangleright (z \triangleright y), t] + z \triangleright [x \triangleright t, y]
\]
\[
= L_\triangleright(x)L_\triangleright(y)L_\triangleright(z)t - L_\triangleright(z)L_\triangleright(x)L_\triangleright(y)t - L_\triangleright(y)L_\triangleright((x, z) \triangleright t) + L_\triangleright(z)[L_\triangleright(x)t, y].
\]
Similarly, by (4.10) and (4.11), we have
\[
[L_\triangleright(x), L_\triangleright(y)]t
\]
\[
= [x \triangleright z, y \triangleright t] = [\{x, y\} \triangleright z, t] - x \triangleright [y \triangleright z, t] + y \triangleright (x \triangleright [z, t]) + [y \triangleright (x \triangleright t), z]
\]
\[
= L_\triangleright((x, y), z)\triangleright t - L_\triangleright(x)\{L_\triangleright(y), z\} \triangleright t + L_\triangleright(y)L_\triangleright((x, z) \triangleright t) + [L_\triangleright(y), L_\triangleright(x, t), z],
\]
\[
[L_\triangleright(x), [y, t]]
\]
\[
= [x \triangleright z, [y, t]] = x \triangleright [z \triangleright [y, t]] - [z \triangleright (x \triangleright [y, t]), t] - [x \triangleright [z, t], y] - [[x \triangleright t, y], z]
\]
\[
= [[L_\triangleright(x), y], z] \triangleright t - L_\triangleright(x)\{[y, z], t\} - [L_\triangleright(x), [z, t], y] - [[L_\triangleright(x), t], y, z].
\]
Therefore \((A; [\cdot, \cdot], L_\triangleright)\) is an \(A\)-module Malcev algebra of \((A^C, \{\cdot, \cdot\})\). \(\square\)

**Proposition 4.8.** If \((A; [\cdot, \cdot], \triangleright)\) is a post-Malcev algebra, then \((A, -[\cdot, \cdot], \triangleright)\) is also a post-Malcev algebra, where for all \(x, y \in A\),
\[
x \triangleright y = x \triangleright y + [x, y]. \quad (4.15)
\]
Moreover, \((A, [\cdot, \cdot], \triangleright)\) and \((A, -[\cdot, \cdot], \triangleright)\) have the same sub-adjacent Malcev algebra \(A^C\).

**Proof.** We check only that \((A, -[\cdot, \cdot], \triangleright)\) verifies the first post-Malcev identity. The other identities can be verified similarly. In fact, for all \(x, y, z, t \in A\),
\[
- \{x, z\} \triangleright [y, t] + x \triangleright [z \triangleright y, t] - [z \triangleright (x \triangleright y), t] - [x \triangleright (z \triangleright t), y] + z \triangleright [x \triangleright t, y]
\]
\[
= -\{x, z\} \triangleright [y, t] - \{\{x, z\}, [y, t]\} + x \triangleright [z \triangleright y, t] + x \triangleright [\{y, z\}, t] + x \triangleright [z, t \triangleleft y, t]
\]
\[
+ [x, [[y, z], t]] - [z \triangleright (x \triangleright y), t] - [z \triangleright [x, y], t] - [[z, x \triangleright y], t] - [[z, [x, y]], t]
\]
\[
- [x \triangleright (z \triangleright t), y] - [x \triangleright [z, t], y] - [[x, z \triangleright t], y] - [[x, [z, t]], y] + z \triangleright [x \triangleright t, y]
\]
\[
+ z \triangleright [[x, t], y] + [z, [x \triangleright t], y] + [[x, [x, t]], y] = 0.
\]
\(\square\)

**Theorem 4.1.** If \((A; [\cdot, \cdot], \triangleright)\) is a post-Malcev algebra, then 
\((A \times A, [\cdot, \cdot])\) is a Malcev algebra, with the double bracket product \([\cdot, \cdot]\) on \(A \times A\) for all \(a, b, x, y \in A\) by
\[
[[a, x], (b, y)] = (a \triangleright b - b \triangleright a + [a, b], \quad a \triangleright y - b \triangleright x + [x, y]). \quad (4.16)
\]

**Proof.** Let \(x, y, z, t, a, b, c, d \in A\). It is obvious that \([[a, x], (b, y)] = -[[b, y], (a, x)]\). On the other hand,
\[
[[[a, x], (b, y)], (d, t)] =
\]
\[
\{\{a, c\}, \{b, d\}\}, (a \triangleright c) \triangleright (b \triangleright t) - (a \triangleright c) \triangleright (d \triangleright y) - (c \triangleright a) \triangleright (b \triangleright t)
\]
\[
+ (c \triangleright a) \triangleright (d \triangleright y) + [a, c] \triangleright (b \triangleright t) - [a, c] \triangleright (d \triangleright y) + \{a, c\} \triangleright [y, t]
\]
\[
- (b \triangleright d) \triangleright (a \triangleright z) + (b \triangleright d) \triangleright (c \triangleright x) + (d \triangleright b) \triangleright (a \triangleright z) - (d \triangleright b) \triangleright (c \triangleright x)
\]
- \([b, d] \triangleright (a \triangleright z) + [b, d] \triangleright (c \triangleright x) - \{b, d\} \triangleright [x, z] + [a \triangleright z, b \triangleright t]
- \{a \triangleright z, d \triangleright y\} - [c \triangleright x, b \triangleright t] + \{a \triangleright x, d \triangleright y\} - \{a \triangleright z, [y, t]\} + \{c \triangleright x, [y, t]\}
+ [[x, z], b \triangleright t] - [[x, z], d \triangleright y] - [[x, z], [y, t]],

\[\{\{a, b\}, c\}, d\}, (c \triangleright (d \triangleright c)) \triangleright x - (d \triangleright (c \triangleright b)) \triangleright x + (d \triangleright (c \triangleright b)) \triangleright x + [b, c, d] \triangleright x + \{b, c, d\} \triangleright x - a \triangleright ((c \triangleright b) \triangleright t) + a \triangleright ((c \triangleright b) \triangleright t) - a \triangleright ((c \triangleright b) \triangleright t) + a \triangleright ((c \triangleright b) \triangleright t)
- a \triangleright (d \triangleright (c \triangleright y)) + a \triangleright (d \triangleright [y, z]) - a \triangleright [b \triangleright z, t] + a \triangleright [c \triangleright y, t] - a \triangleright [[y, z], t] + \{[a, b] \triangleright t, x\} - [d \triangleright (b \triangleright z), x] + \{d \triangleright (c \triangleright y), x\} - [d \triangleright [y, z], x] + [[b, d] \triangleright z, t] - [c \triangleright [y, t], t] + [c \triangleright [y, t], t] + \{[[y, z], t], t\},

\[\{\{b, c\}, (c, z)\}, (d, t\}, (a, x)\}, (b, y)\}, (c, z)\} =

\[\{\{d, a\}, b\}, (b \triangleright (d \triangleright z)) \triangleright x - (d \triangleright (b \triangleright d)) \triangleright x + (d \triangleright (b \triangleright d)) \triangleright x + [a \triangleright (d \triangleright z), x] + \{a \triangleright (d \triangleright z), x\} + \{[a \triangleright (d \triangleright z), x], x\},

\[\{\{d, t\}, (a, x)\}, (b, y)\}, (c, z)\} =

\[\{\{a, x\}, (c, z)\}, [[b, y], (d, t)]\} - \{\{a, x\}, (b, y)\}, (c, z)\}, (d, t)]\}
- \{\{b, y\}, (c, z)\}, (d, t\}, (a, x\} - \{\{c, z\}, (d, t\}, (a, x\), (b, y)\}]
- \{\{d, t\}, (a, x)\}, (b, y)\}, (c, z)\} = (0, 0).

The following results illustrate that a \(\lambda\)-weighted \(O\)-operator induces a post-Malcev algebra structure.

**Theorem 4.2.** Let \((A, [\cdot, \cdot], \lambda)\) be a Malcev algebra and \((V; [\cdot, \cdot], \rho)\) an \(A\)-module Malcev algebra. Let \(T : V \rightarrow A\) be a \(\lambda\)-weighted \(O\)-operator associated to \((V; [\cdot, \cdot], \rho)\).
Define two new bilinear operations $\cdot : V \times V \to V$ as follows, for all $a, b \in V$,
\[ [a, b] = \lambda[a, b]_V, \quad a \triangleright b = \rho(T(a))b. \] (4.17)

Then $(V, [\cdot, \cdot], \triangleright)$ is a post-Malcev algebra.

(ii) $T$ is a Malcev algebra homomorphism from the sub-adjacent Malcev algebra $(V, \{\cdot, \cdot\})$ given in Proposition 4.6 to $(A, [\cdot, \cdot], T)$.

**Proof.** (i) We use (4.1)-(4.3) of representation of Malcev algebras on $\mathbb{K}$-algebra.

\[
\{a, c\} \triangleright [b, d] - a \triangleright [c \triangleright b, d] + [c \triangleright (a \triangleright b), d] + [a \triangleright (c \triangleright d), b] - c \triangleright [a \triangleright d, b] = (\rho(T(a))c - \rho(T(c))a + \lambda[a, c]_V) \triangleright \lambda[b, d]_V - \rho(T(a))[\rho(T(c))b, d] + [\rho(T(c))\rho(T(a))b, d] + [\rho(T(a))\rho(T(c))d, b] - \rho(T(c))[\rho(T(a))d, b] = \lambda\left(\rho(T(\rho(T(a)))c - T(\rho(T(c))a) + T(\lambda[a, c]_V))\right)[b, d]_V - \rho(T(a))[\rho(T(c))b, d]_V + [\rho(T(c))\rho(T(a))b, d]_V + [\rho(T(a))\rho(T(c))d, b]_V - \rho(T(c))[\rho(T(a))d, b]_V = 0.
\]

Using the condition (2.1) of Definition 2.1, we check

\[
\lambda^2\left([\rho(T(a))c, [b, d]_V]_V - [\rho(T(a))b, [c, d]_V]_V + [\rho(T(a))b, [c, d]_V]_V + [\rho(T(a))c, [d, b]_V]_V + [\rho(T(a))d, [b, c]_V]_V\right) = 0.
\]

(ii) The Malcev bracket $\{\cdot, \cdot\}$ is defined for all $a, b \in V$ by

\[
\{a, b\} = a \triangleright b - b \triangleright a + [a, b] = \rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V.
\]

Then the sub-adjacent Malcev algebra of the above post-Malcev algebra $(V, [\cdot, \cdot], \triangleright)$ is exactly the Malcev algebra $(V, [\cdot, \cdot], T)$ given in Corollary 4.1 Then the result follows. \(\square\)

**Proposition 4.9.** Let $T, T' : (V, [\cdot, \cdot]_V) \to (A, [\cdot, \cdot])$ be two $\mathcal{O}$-operators with respect to an $A$-module Malcev algebra $(V, [\cdot, \cdot]_V, \rho)$. Let $(V, \{\cdot, \cdot\}, \triangleright)$ and $(V, \{\cdot, \cdot\}', \triangleright')$ be the post-Malcev algebras given in Theorem 4.2 and $(\phi, \psi)$ be a homomorphism from $T'$ to $T$. Then $\psi$ is a homomorphism from the post-Malcev algebra $(V, \{\cdot, \cdot\}, \triangleright)$ to the post-Malcev algebra $(V, \{\cdot, \cdot\}', \triangleright')$. 


For all $e,e_0$ let
\begin{align*}
\psi(a,b) &= \psi(a,b)_V = \lambda(\psi(a),\psi(b))_V = \{\psi(a),\psi(b)\}', \\
\psi(a \triangleright b) &= \psi(\rho(T(a))b) = \rho(\psi(T(a)))(\psi(b)) = \rho(T'(\psi(a)))(\psi(b)) = \psi(a) \triangleright' \psi(b),
\end{align*}
which implies that $\psi$ is a homomorphism between the post-Malcev algebras in Theorem 4.2. \hfill $\square$

Given a Malcev algebra, the following result gives a necessary and sufficient condition to have a compatible post-Malcev algebra structure.

**Proposition 4.10.** Let $(A,[\cdot,\cdot])$ be a Malcev algebra. Then there exists a compatible post-Malcev algebra structure on $A$ if and only if there exists an $A$-module Malcev algebra $(V;[\cdot,\cdot]_V,\rho)$ and an invertible $O$-operator $T : V \to A$.

**Proof.** Let $(A,[\cdot,\cdot],\triangleright)$ be a post-Malcev algebra and $(A,[\cdot,\cdot])$ be the associated Malcev algebra. Then the identity map $id : A \to A$ is an invertible $1$-weighted $O$-operator on $(A,[\cdot,\cdot])$ associated to $(A,[\cdot,\cdot],ad)$.

Conversely, suppose that there exists an invertible $1$-weighted $O$-operator of $(A,[\cdot,\cdot])$ associated to an $A$-module Malcev algebra $(V;[\cdot,\cdot]_V,\rho)$ . Then, using Theorem 4.2, there is a post-Malcev algebra structure on $T(V) = A$ given by
\begin{align*}
\{x, y\} &= \lambda T([T^{-1}(x),T^{-1}(y)]_V), \quad x \triangleright y = T(\rho(x)T^{-1}(y)).
\end{align*}
This is compatible post-Malcev algebra structure on $(A,[\cdot,\cdot])$. Indeed,
\begin{align*}
x \triangleright y - y \triangleright x + \{x, y\} &= T(\rho(x)T^{-1}(y) - \rho(y)T^{-1}(x) + [T^{-1}(x),T^{-1}(y)]_V) \\
&= [TT^{-1}(x),TT^{-1}(y)] = [x,y]. \quad \square
\end{align*}

An obvious consequence of Theorem 4.2 is the following construction of a post-Malcev algebra in terms of $\lambda$-weighted Rota-Baxter operator on a Malcev algebra.

**Corollary 4.4.** Let $(A,[\cdot,\cdot])$ be a Malcev algebra and the linear map $\mathcal{R} : A \to A$ is a $\lambda$-weighted Rota-Baxter operator. Then, there exists a post-Malcev structure on $A$ given, for all $x,y \in A$, by
\begin{align*}
\{x, y\} &= \lambda[x,y], \quad x \triangleright y = [\mathcal{R}(x),y].
\end{align*}
If in addition, $\mathcal{R}$ is invertible, then there is a compatible post-Malcev algebra structure on $A$ given, for all $x,y \in A$, by
\begin{align*}
\{x, y\} &= \mathcal{R}([\mathcal{R}^{-1}(x),\mathcal{R}^{-1}(y)]), \quad x \triangleright y = \mathcal{R}([x,\mathcal{R}^{-1}(y)]).
\end{align*}

**Example 4.4.** In this example, we calculate $(-1)$-weighted Rota-Baxter operators on the Malcev algebra $A$ and we give the corresponding post-Malcev algebras. Let $A$ be the simple Malcev algebra over the field of complex numbers $\mathbb{C}$ [11, Example 3]. In this case $A$ has a basis $\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$ with the following table of multiplication:
\[
\begin{array}{cccccccc}
| & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
 e_1 & 0 & 2e_2 & -2e_3 & 2e_4 & -2e_5 & 2e_6 & -2e_7 \\
 e_2 & -2e_2 & 0 & e_1 & 2e_7 & 0 & -2e_5 & 0 \\
 e_3 & 2e_3 & -e_1 & 0 & 0 & -2e_6 & 0 & 2e_4 \\
e_4 & -2e_4 & -2e_7 & 0 & 0 & e_1 & 2e_3 & 0 \\
e_5 & 2e_5 & 0 & 2e_6 & -e_1 & 0 & 0 & -2e_2 \\
e_6 & -2e_6 & 2e_5 & 0 & -2e_3 & 0 & 0 & e_1 \\
e_7 & 2e_7 & 0 & -2e_4 & 0 & 2e_2 & -e_1 & 0
\end{array}
\]
Now, define the linear map $\mathcal{R} : A \to A$ by
\[
\mathcal{R}(e_1) = \frac{1}{2}e_1 + 2\alpha e_2 + 2\beta e_5 + 2\gamma e_6, \quad \mathcal{R}(e_2) = 0, \quad \mathcal{R}(e_3) = e_3 - \alpha e_1 + \delta e_5 - 2\beta e_6,
\]
\[ \mathcal{R}(e_4) = e_4 - \beta e_1 - \delta e_2 + \mu e_6, \quad \mathcal{R}(e_5) = \mathcal{R}(e_6) = 0, \quad \mathcal{R}(e_7) = e_7 - \gamma e_1 + 2\beta e_2 - \mu e_5. \]

Then \( \mathcal{R} \) is a \((-1)\)-weighted Rota-Baxter operator on \( A \). Using Corollary 4.4, we can construct a post-Malcev algebra on \( A \) given by

\[
\{\cdot, \cdot\} \quad \begin{array}{cccccccc}
\cdot & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & 0 & 2\lambda e_2 & -2\lambda e_3 & 2\lambda e_4 & -2\lambda e_5 & 2\lambda e_6 & -2\lambda e_7 \\
e_2 & -2\lambda e_2 & 0 & \lambda e_1 & 2\lambda e_7 & 0 & -2\lambda e_5 & 0 \\
e_3 & 2\lambda e_3 & -\lambda e_1 & 0 & 0 & -2\lambda e_6 & 0 & 2\lambda e_4 \\
e_4 & -2\lambda e_4 & -2\lambda e_7 & 0 & 0 & \lambda e_1 & 0 & 2\lambda e_3 \\
e_5 & 2\lambda e_5 & 0 & 2\lambda e_6 & -\lambda e_1 & 0 & 0 & -2\lambda e_2 \\
e_6 & -2\lambda e_6 & 2\lambda e_5 & 0 & -2\lambda e_3 & 0 & 0 & \lambda e_1 \\
e_7 & 2\lambda e_7 & 0 & -2\lambda e_4 & 0 & 2\lambda e_2 & -\lambda e_1 & 0 \\
\end{array}
\]

The following result establishes a close relation between a post-alternative algebra and a post-Malcev algebra.

**Theorem 4.3.** Let \( T : V \to A \) be a \( \lambda \)-weighted \( \mathcal{O} \)-operator of alternative algebra \( (A, \cdot) \) with respect to \((V, \cdot, V, I, \tau)\) and \((V, \circ, <, >)\) be the associated post-alternative algebra given in Theorem 3.2. Then \( T \) is a \( \lambda \)-weighted \( \mathcal{O} \)-operator on the Malcev admissible algebra \((A, [\cdot, \cdot])\) with respect to an \( A \)-module Malcev algebra \((V; [\cdot, \cdot], V, I, \tau)\).

Moreover, if \((V, \{\cdot, \cdot\}, \triangleright)\) be a post-Malcev algebra associated to the Malcev admissible algebra \((A, [\cdot, \cdot])\) on \((V; [\cdot, \cdot], V, I, \tau)\). Then, the products \((\{\cdot, \cdot\}, \triangleright)\) are related with \((\circ, <, >)\) as follow, for all \( a, b \in V \),

\[
\{a, b\} = a \circ b - b \circ a, \quad a \triangleright b = a > b < a. \tag{4.18}
\]

**Proof.** Using the condition of \( \lambda \)-weighted \( \mathcal{O} \)-operator in (3.28) and Proposition 4.2, for \( a, b \in A \),

\[
[T(a), T(b)] - T(\rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V) = T(a) \cdot T(b) - T(b) \cdot T(a) - T((I - \tau)(T(a))b - (1 - \tau)(T(b))a + \lambda(a \cdot V b - b \cdot V a) = 0.
\]

Then \( T \) is a \( \lambda \)-weighted \( \mathcal{O} \)-operator on the Malcev admissible algebra \((A, [\cdot, \cdot])\) with respect to an \( A \)-module Malcev algebra \((V; [\cdot, \cdot], V, I, \tau)\).

On the other hand, from (3.30) of Theorem 3.2 and (4.17) of Theorem 4.2 that

\[
\{a, b\} = \lambda[a, b]_V = \lambda a \cdot V b - \lambda b \cdot V a = a \circ b - b \circ a, \\
\] 
\[a \triangleright b = (I - \tau)(T(a))b = l(T(a))b - \tau(T(a))b = a > b < a. \]

\[\square\]
Corollary 4.5. Let \((A, \circ, \prec, \succ)\) be a post-alternative algebra given in Corollary 3.1, \((A, \{\cdot, \cdot\}, \triangleright)\) be a post-Malcev algebra associated to the Malcev algebras \((A, [\cdot, \cdot])\) and let \(R\) be a \(\lambda\)-weighted Rota-Baxter operator of \((A, \cdot)\). Then, the operations
\[
\{x, y\} = x \circ y - y \circ x, \quad x \triangleright y = x - y - y \prec x, \quad (4.19)
\]
define a post-Malcev structure in \(A\).

It is easy to see that (4.13) and (4.19) fit into the commutative diagram
\[
\text{Post-alternative alg.} \quad \xymatrix{ x \prec y \ar[r] & x \succ y \ar[r] & x \cdot y \ar[r] & \text{alternative alg.} } \]
\[
\{x, y\} = x \circ y - y \circ x \quad x \triangleright y = x - y - y \prec x \quad x \cdot y - y \cdot x \quad (4.20)
\]

When the operation \(\cdot\) of the post-alternative algebra and the bracket \([\cdot, \cdot]\) of the post-Malcev algebra are both trivial, we obtain the following commutative diagram.

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