LOCO: Distributing Ridge Regression with Random Projections

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Abstract

We propose LOCO, a distributed algorithm which solves large-scale ridge regression. LOCO randomly assigns variables to different processing units which do not communicate. Important dependencies between variables are preserved using random projections which are cheap to compute. We show that LOCO has bounded approximation error compared to the exact ridge regression solution in the fixed design setting. Experimentally, in addition to obtaining significant speedups LOCO achieves good predictive accuracy on a variety of large-scale regression problems. Notably LOCO is able to solve a regression problem with 5 billion non-zeros distributed across 128 workers in 25 seconds.

1 Introduction

In the last few years there has been great interest in solving large-scale optimization and estimation problems. Since multi-core architectures and powerful computing clusters have become commonplace, parallelization has naturally emerged as a technique to leverage these resources to enable increasingly large problems to be solved quickly.

Two obvious questions arise: (1) how should the data and processing tasks be distributed among processing units (workers) and (2) how and what should each worker communicate. The choice of learning algorithm affects both of these points. Stochastic gradient descent (SGD) methods are suited to parallelization over the rows (observations) of the data [24]. However, synchronization of results to ensure each worker is updating the current gradient becomes expensive. This has motivated recent asynchronous approaches to parallel SGD [8][15]. Parallelizing over the columns (features) of the data has been proposed for coordinate descent optimization which is commonly used to solve \( \ell_1 \) penalized problems [7]. To ensure convergence, the inherent dependence between features is typically assumed to be small between blocks which are operated on in parallel. Fewer methods have been proposed for distributed optimization on a cluster of machines where communication costs must be considered [17].

Specifically, we limit our focus to \( \ell_2 \) penalized linear regression for large-scale estimation tasks when the size of the data is such that a single multi-core processor is insufficiently fast. We therefore wish to distribute the problem in a way which allows computation to be shared across many machines which do not share memory. Ideally this is done in such a way that synchronization and...
communication between workers are kept to a minimum. Another setting which motivates distributed estimation is one of privacy preservation. In this framework, no single worker may see all of the features and so even when the data size is not massive, sharing memory and data between workers is not permitted.

Randomized dimensionality reduction based on the Johnson-Lindenstrauss lemma has emerged as a way to quickly obtain provably good approximations to a variety of learning tasks [3]. Notably, structured random projections have been used to speed up approximate kernel expansions [12] and linear regression. For the latter, it can be shown that the least squares solution computed on a random projection of either the row [14] or column [11, 13] space of the data matrix results in a solution which is close to optimal. An obvious downside to dimensionality reduction is that the solution obtained is no longer in the original space. Therefore, the estimated coefficients are difficult to interpret with respect to the observed features – a task often as important as prediction accuracy. Furthermore, in order to compute the projection, a single machine is assumed to have access to the entire dataset.

In this work we propose and analyze LOCO a simple, low-communication distributed algorithm to approximately solve \( \ell_2 \) penalized least squares which crucially requires no synchronization between workers. LOCO assigns features to workers by randomly partitioning the data into \( K \) blocks (alternatively, this may be part of the problem specification). In each block, a small number of cheaply computed random projections are used to approximate the contribution from the remaining columns of the data. Each worker then simply optimizes the objective independently on this compressed dataset, the size of which is proportional to the size of the random projection and the total number of workers. The solution vector returned by LOCO is constructed by collecting the estimates for the respective unprojected “raw” features from each worker such that it lies in the original space.

Outline and Contribution. In §2 we formally describe our estimation problem and the distributed setting which we consider. We also give a brief introduction to random projections, in particular the Subsampled Randomized Hadamard Transform (SRHT). In §3 we describe LOCO, our algorithm for distributed ridge regression. In §4 we show in the fixed design setting that the error between the coefficients estimated by LOCO and the optimal ridge regression coefficients is bounded, under natural assumptions about the problem setting – that some proportion of the signal lies in the top principal components. Importantly, unlike other approaches to parallelizing or distributing optimization, we make no assumptions on sparsity in the data. In §5 we place our contribution in context of recently proposed related approaches to distributed optimization. In §6 we provide implementation details and empirical evaluation of our algorithm on large-scale simulated and real datasets. LOCO typically exhibits near-linear speedups with the number of workers with little loss in prediction accuracy.

2 Problem Setting

Given a matrix of features \( X \in \mathbb{R}^{n \times p} \) and a corresponding vector of responses, \( y \in \mathbb{R}^n \) where the dimensionality \( p \) and sample size \( n \) are very large. We consider solving problems of the form

\[
\min_{\beta \in \mathbb{R}^p} L(\beta) := \min_{\beta \in \mathbb{R}^p} f(\beta) + \mathcal{R}(\beta)
\]

Here, \( f \) is a smooth convex loss and \( \mathcal{R} \) is a convex regularizer which is separable, i.e. \( \mathcal{R}(\beta) = \sum_{j=1}^p r_j(\beta) \). In this work we will concentrate on the case of ridge regression, where \( f(\beta) = \frac{1}{2n}||y - X\beta||^2 \) is the squared error loss and \( \mathcal{R}(\beta) = \lambda||\beta||^2 \) is the ridge penalty.\footnote{Throughout, ||·|| refers to the Euclidean norm for vectors and the spectral norm for matrices.} Ridge regression has a closed-form solution \( \beta^* = (X^\top X + n\lambda I)^{-1} X^\top y \), but clearly when the dimensionality of the data is large, constructing and inverting the covariance matrix is prohibitively expensive. When the number of samples is very large, ridge regression is usually solved using stochastic gradient descent (SGD) or coordinate descent [19].

Distributing ridge regression. We now consider the case where we distribute the features across \( K \) different workers. Formally, let \( P \) be the set of indices \( 1, \ldots, p \). We partition this into \( K \) non-overlapping subsets \( P_1, \ldots, P_K \) of equal size, \( \tau = p/K \) so \( P = \bigcup_{k=1}^K P_k \) and \( |P_k| = |P_1| = \ldots = |P_K| = \tau \). This is for simplicity of notation only, in general the partitions can be of different sizes.
A naive attempt at parallelizing \( (1) \) would simply be solving the minimization problem on each subset of features \( \mathcal{P}_k \) independently. However, without sparsity in the dataset to guide the partitioning process, important dependencies between features in different blocks would not be preserved.

We can rewrite \( (1) \) making explicit the contribution from block \( k \). Letting \( \mathbf{X}_k \in \mathbb{R}^{n \times \tau} \) be the sub-matrix whose columns correspond to the coordinates in \( \mathcal{P}_k \) (the “raw” features of block \( k \)) and \( \mathbf{X}_{(-k)} \in \mathbb{R}^{n \times (p - \tau)} \) be the remaining columns of \( \mathbf{X} \), we have

\[
L(\mathbf{\beta}) = n^{-1} \| \mathbf{y} - \mathbf{X}_k \mathbf{\beta}_{\text{raw}} - \mathbf{X}_{(-k)} \mathbf{\beta}_{(-k)} \|^2 + \lambda \| \mathbf{\beta}_{\text{raw}} \|^2 + \lambda \| \mathbf{\beta}_{(-k)} \|^2.
\]  

(2)

The idea behind our approach is to replace \( \mathbf{X}_{(-k)} \) in each block with a low-dimensional approximation. Since the regularizer is separable across blocks, we only require that the contribution from \( \mathbf{X}_{(-k)} \) to \( f(\mathbf{\beta}) \) is preserved.

Let \( \tilde{\mathbf{X}}_k \in \mathbb{R}^{n \times (K - 1) \tau_{\text{subs}}} \) be the matrix whose columns are a low-dimensional approximation to \( \mathbf{X}_{(-k)} \), i.e., to the columns of \( \mathbf{X} \) not in \( \mathbf{X}_k \), and \( \tau_{\text{subs}} \ll \tau \). Note that each of the other \( K - 1 \) blocks provides an approximation of its respective \( \tau \) raw features of size \( \tau_{\text{subs}} \). We shall call the columns in \( \tilde{\mathbf{X}}_k \) the “random” features of block \( k \). Defining the sub-problem that worker \( k \) solves as

\[
L_k(\mathbf{\beta}_k) = n^{-1} \| \mathbf{y} - \mathbf{X}_k \mathbf{\beta}_{\text{raw}} - \mathbf{X}_{k, \text{rp}} \mathbf{\beta}_{k, \text{rp}} \|^2 + \lambda \| \mathbf{\beta}_{\text{raw}} \|^2 + \lambda \| \mathbf{\beta}_{k, \text{rp}} \|^2,
\]

(3)

we require the approximation \( \tilde{\mathbf{X}}_k \) to be such that the risk of the estimator which minimizes eq. \( (3) \) is similar to the risk of the minimizer of eq. \( (2) \) (we formalize this in \( \ref{eq:randomiser} \)). In order to achieve this we construct the approximation using random projections which we briefly describe below.

**Johnson-Lindenstrauss projections.** Johnson-Lindenstrauss (J-L) projections are low-dimensional embeddings \( \mathbf{\Pi} : \mathbb{R}^\tau \rightarrow \mathbb{R}^{\tau_{\text{subs}}} \) which preserve up to a small distortion — pairwise \( \ell_2 \) distances between vectors according to the J-L lemma (see e.g. \( \ref{eq:JL} \)). Typically, \( \mathbf{\Pi} \in \mathbb{R}^{\tau \times \tau_{\text{subs}}} \) is constructed to be a nearly-orthogonal matrix with entries drawn at random from a sub-gaussian distribution \( \tilde{1} \).

We concentrate on the class of structured random projections, among which the Subsampled Randomized Hadamard Transform (SRHT) has received particular recent attention \( \cite{H.-Lin, W.-W.} \). The SHRT consists of a preconditioning step after which \( \tau_{\text{subs}} \) columns of the new matrix are subsampled uniformly at random. In more detail, it consists of a projection matrix, \( \mathbf{\Pi} = \sqrt{\tau/\tau_{\text{subs}}} \mathbf{DHS} \) \( \cite{H.-Lin, W.-W.} \) with the definitions:

- \( \mathbf{S} \in \mathbb{R}^{\tau \times \tau_{\text{subs}}} \) is a subsampling matrix.
- \( \mathbf{D} \in \mathbb{R}^{\tau \times \tau} \) is a diagonal matrix whose entries are drawn independently from \( \{-1, 1\} \).
- \( \mathbf{H} \in \mathbb{R}^{\tau \times \tau} \) is a normalized Walsh-Hadamard matrix\( \footnote{For the Hadamard transform, \( \tau \) must be a power of two but other transforms exist (e.g. DCT, DFT) with similar theoretical guarantees and no restriction on \( \tau \).} \) which is defined recursively as

\[
\mathbf{H}_\tau = \begin{bmatrix} \mathbf{H}_{\tau/2} & \mathbf{H}_{\tau/2} \\ \mathbf{H}_{\tau/2} & -\mathbf{H}_{\tau/2} \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}.
\]

We set \( \mathbf{H} = \frac{1}{\sqrt{\tau}} \mathbf{H}_\tau \) so it has orthonormal columns.

The SRHT has similar \( \ell_2 \) distance preserving properties as sub-gaussian random projections but has the added benefit of a fast \( \mathcal{O}(\tau \log \tau) \) matrix-vector product due to its recursive definition. Note that the SRHT can also be combined with an i.i.d. Gaussian random vector to obtain a Gaussian matrix with approximately independent entries in the same computational time \( \cite{H.} \).

For moderately sized problems, random projections have been used to reduce the dimensionality of the data prior to performing OLS \( \cite{H.-Lin} \) and ridge regression \( \cite{W.-W.} \). However after projection, the solution vector is in the compressed space and so interpretability of coefficients is lost. Furthermore, for large problems the running time of the SRHT presents a large constant overhead.
3 Algorithm

Our procedure, LOCO for distributed ridge regression is presented in Algorithm 1. We describe the steps in more detail below.

Algorithm 1 LOCO

**Input:** Data: \( \mathbf{X}, \mathbf{y} \), Number of blocks: \( K \), Parameters: \( \tau_{\text{subs}}, \lambda \)

1. Partition \( \mathcal{P} = \{1, \ldots, p\} \) into \( K \) subsets \( \mathcal{P}_1, \ldots, \mathcal{P}_K \) of equal size, \( \tau \).
2. **for each** worker \( k \in \{1, \ldots, K\} \) **in parallel do**
   3. Compute and send random projection \( \bar{X}_k = X_k \Pi_k \).
   4. Construct \( X_k \)
   5. \( \hat{\beta}_k \leftarrow \text{SolveRidge}(\bar{X}_k, \mathbf{y}, \lambda) \)
   6. \( \hat{\beta}_k = [\hat{\beta}_k]_{1: \tau} \)
3. **end for**

**Output:** Solution vector: \( \hat{\beta} = [\hat{\beta}_1, \ldots, \hat{\beta}_K] \)

**Input.** As well as the usual regularization parameter \( \lambda \), LOCO requires the specification of the number of workers \( K \) and the random projection dimension \( \tau_{\text{subs}} \).

**Steps 1 & 3.** We first randomly partition the coordinates into \( K \) subsets. Then each worker computes a random projection, via the SRHT, of its respective block which we denote by \( \bar{X}_k = X_k \Pi_k \in \mathbb{R}^{n \times \tau_{\text{subs}}} \).

**Step 4.** Each worker \( k \) constructs the matrix
\[
\bar{X}_k \in \mathbb{R}^{n \times (\tau + (K-1)\tau_{\text{subs}})} = \left[ X_k, [\bar{X}_k']_{k' \neq k} \right]
\]
which is the column-wise concatenation of the raw feature matrix \( X_k \) and the random approximations from all other blocks.\(^3\) Without loss of generality the raw features will always occupy the first \( \tau \) columns of \( X_k \). The last \( (K-1)\tau_{\text{subs}} \) columns of \( X_k \) are a good approximation of the \( (K-1) \) blocks of the full data matrix not in \( X_k \) and so solving (\ref{eq:ridge}) using \( \bar{X}_k \) obtains a solution which is close to the optimal solution using \( X \). We make this explicit in (\ref{eq:ridge}).

**Steps 5 & 6.** The function \( \text{SolveRidge}(\bar{X}_k, \mathbf{y}, \lambda) \) returns a vector
\[
\hat{\beta}_k \in \mathbb{R}^{\tau + (K-1)\tau_{\text{subs}}} = \arg \min_{\beta_k} n^{-1} \| \mathbf{y} - \bar{X}_k \beta_k \|^2 + \lambda \| \beta_k \|^2
\]
In practice, any fast algorithm which returns an accurate solution to eq. (\ref{eq:ridge}) can be used here. The final solution vector \( \hat{\beta} \) is the concatenation of the first \( \tau \) coordinates of each \( \hat{\beta}_k \) and so lives in the same space as the original data.

**Computational, memory and communication costs.** The cost of computing a fast random projection of the \( \tau \) features in each block is \( O(\tau \log \tau_{\text{subs}}) \). Assuming a solver which scales linearly with the problem dimension (i.e. stochastic gradient descent) is used in \( \text{SolveRidge}(\bar{X}_k, \mathbf{y}, \lambda) \), the part of the computational cost which is dependent on the dimension scales with \( O(\tau \log \tau_{\text{subs}} + \tau + (K-1)\tau_{\text{subs}}) \).

Each machine only needs to store a copy of its block of raw features and a random projection of the remaining features which is \( O(\tau + (K-1)\tau_{\text{subs}}) \). This is substantially smaller than the original dimensionality \( p \). Each worker must communicate its random projection once to all other workers (or to a shared location where the other workers can read it). Aside from this there is no further communication between workers. The small size of the projection ensures that for appropriately sized problems, each worker is able to store its relevant features in local memory.

\(^3\) Alternatively, when \( \Pi \) is defined explicitly, summing the \( \tau \rightarrow \tau_{\text{subs}} \) -dimensional random projections of \( K-1 \) blocks is equivalent to computing the \( (p-\tau) \rightarrow \tau_{\text{subs}} \) -dimensional random projection in one go which allows for a much smaller dimensional but also less accurate representation.
4 Analysis

In this section we will show that in the fixed design setting the coefficients estimated by Loco are close to the full ridge regression solution. The results here are developed for the case where the random features in $\tilde{X}_k$ result from concatenating the SRHT projections of all other blocks and throughout we shall assume that the columns of $X$ and $\tilde{X}_k$ are standardized.

Consider the model

$$y = X\beta^* + \varepsilon,$$  \hspace{1cm} (5)

with fixed $X \in \mathbb{R}^{n \times p}$ and true parameter vector $\beta^* \in \mathbb{R}^p$. Assumption 1 below will formalize our assumptions on the noise, $\varepsilon$. Let $\hat{\beta}''$ denote the ridge estimate for $\beta^*$, so $\hat{\beta}''$ is the solution which results from solving the ridge regression problem in the original space, stated in eq. (2).

In order to formulate our result, we require the following risk function.

**Definition 1 (Risk).** Let $\hat{b}$ be an estimator for $\beta^*$ and define the risk of $\hat{b}$ with fitted values $\hat{y} = X\hat{b} \in \mathbb{R}^n$ as

$$R(X\hat{b}) = n^{-1} E_{\varepsilon} \|X\beta^* - X\hat{b}\|^2.$$  

In the derivation of Theorem 1 we make use of the fact that we can rewrite the regularized optimization problems in eq. (2) and (3) as constrained optimization problems with a monotonic relationship between the regularization parameter $\lambda$ and the constraint $t$ which upper-bounds the $\ell_2$ norm of the solution. In the original space we have

$$\min_{\|\beta\|^2 \leq t} n^{-1}\|y - X\beta\|^2$$  \hspace{1cm} (6)

while each worker solves

$$\min_{\|\beta_k\|^2 \leq t} n^{-1}\|y - X_k\beta_k\|^2$$  \hspace{1cm} (7)

in a compressed space. Recall that $\hat{\beta}''$ minimizes eq. (6) and $\bar{\beta}_k$ minimizes eq. (7).

Before we state our main theorem, we make the natural assumption that the main contribution to the $\ell_2$ norm of the true parameter vector – i.e. most of the important signal – lies in the direction of the first $J$ principal components of $X$. This merely formalizes the conditions under which ridge regression yields good results. Since ridge regression applies more shrinkage in directions associated with smaller eigenvalues [10], if Assumption 1 does not hold we might expect a different estimator to be more appropriate.

**Assumption 1.** Let $w^*$ be the true parameter vector after rotating $X$ to the PCA coordinate system. There exists $1 \leq J \leq \min\{n, p\}$ and $c \in (0, 1/2)$ such that

(A1) the $J$-th largest eigenvalue of the covariance matrix $\lambda_J > 0$,

(A2) the ridge constraint is active: $t \leq (1 - c) \sum_{j=1}^{J} (w_j^*)^2$,

(A3) the errors $\varepsilon_i, i = 1, \ldots, n$ have zero mean, are independent and their variances are bounded by $\sigma^2 > 0$.

To shed some light onto Assumption (A2), consider the noiseless case where the entire signal lies in the first $J$ principal components. Then $c = 0$ implies no shrinkage, while increasing $c$ means that the amount of regularization becomes larger.

If Assumptions (A1) and (A2) do not hold, then ridge regression may not be a suitable estimator for $\beta^*$ in Eq. (5), independent of how we choose the size of the constraint. If, on the other hand, (A1) and (A2) do hold, the amount of required regularization can differ. In problem settings where the signal-to-noise ratio is low, $p > n$ or where the covariance matrix of $X$ is otherwise close to singular, the ridge constraint is active and (A2) covers the relevant section of the regularization parameter. The ridge estimator will then shrink less along directions associated with large variance. If the data are full rank and the noise value is very low, shrinkage may be unnecessary and the ordinary least squares estimator may be more appropriate. This issue is discussed in [SI.2.1] and we derive a similar bound for OLS in [SI.2.2] which is the relevant bound if the ridge constraint is not active and (A2) does not apply.
We now present Theorem 1 which states that the expected difference between the coefficients \( \hat{\beta} \) returned by LOCO and the full ridge regression solution is bounded.

**Theorem 1.** Under Assumption 1, \( \exists n_0(\xi) \) for all \( \xi > K(\delta + (p - \tau)/e^r) \) such that for all \( n \geq n_0 \) with probability at least \( 1 - \xi \)

\[
E_{\epsilon}(\|\hat{\beta}^r - \hat{\beta}\|^2) \leq \frac{5K}{c\lambda_J} \left( \frac{1}{(1 - \rho)^2} - 1 \right) R(X^\hat{\beta}^r)
\]

where \( \rho = C \sqrt{\frac{r \log(2r/\delta)}{(K-1)\tau_{subs}}} \), \( r = \text{rank}(X) \), \( \lambda_J \) denotes the \( J \)th largest non-zero eigenvalue of the covariance matrix and \( R(X^\hat{\beta}^r) \) is the risk of the ridge estimator (see e.g. [13]). The expectation is conditional on the random projection as the uncertainty coming from the SRHT is captured in the probability with which the statement holds.

The exact value of \( n_0 \) depends on \( \xi \) and the exact form is given in the proof of Theorem 1. The proof of Theorem 1 is given in [SI.2.1].

The bound above scales with the number of workers, \( K \) and inversely with \( (1 - \rho)^2 \), which measures the quality of the random feature representation. The latter is improved (for a fixed \( \tau_{subs} \)) by increasing \( K \) although this has the additional effect of increasing the computational overhead per worker which scales as \( O((K - 1)\tau_{subs}) \).

### 5 Related work

Recently, several methods have been proposed for parallelizing convex optimization. Among these, HOGWILD! [15], ASYNCGDA and ASYNCGADGRAD [8] have shown that large speedups are possible with asynchronous gradient updates when data is sparse. These methods rely on the idea that if the number of non-zero coordinates in each stochastic gradient evaluation is small compared to \( p \), workers updating the same solution vector in parallel will rarely propose conflicting updates. As such each worker is allowed to update the solution asynchronously without the need for locking, provided the delay of any processor is not too great.

Similar requirements on sparsity have been used to parallelize coordinate descent [17]. Other approaches rely on alternative, but related conditions which require the spectral norm of the data – which captures the size of dependencies between features – to be small [7][18].

Whilst sparsity is a natural and common feature of large datasets, in some fields the data collected is dense with many correlated features. Furthermore, in the setting where \( n \ll p \), SGD can converge slowly. Under these conditions we might expect the performance of the above mentioned approaches to suffer. Notably, LOCO makes no assumptions about sparsity since each block sees a representation of the remaining features such that updates to the individual solution vectors are not independent of the rest of the dataset. LOCO does not require synchronization between workers since each worker may only update its own part of the solution vector. Although a bound on the maximum delay is not required or assumed, the total running time of LOCO will naturally be dictated by the slowest worker.

Most of the above mentioned approaches implement parallelism on a multi-core architecture with shared memory. For distributed optimization, communicating results between workers introduces overhead. Several communication strategies for distributed coordinate descent were discussed in [17]. In contrast, LOCO requires computing random projections for each block and communicating them once and so no additional synchronization or communication are required until prediction time.

Parallel estimation has also been considered in the case of kernel ridge regression [23]. It has been shown that randomly splitting and distributing the samples among workers and averaging their estimates achieves a superlinear speedup whilst retaining optimality up to a number of workers which is problem dependent. Conceptually, LOCO is perhaps most similar to this approach, although clearly the usual i.i.d. assumption on the observations does not hold for partitioning the features.
6 Experimental Results

Implementation details. We implemented LOCO in Python making use of fast packages for random projections and ridge regression. For the random projection we used the DCT implemented in FFTW. The ridge regression solver is implemented in scikit-learn which outperforms SGD for \( p \gg n \). We ran LOCO on the BRUTUS cluster, where each worker “communicates” by writing its random projections to a Lustre file system to which each worker is connected via InfiniBand network and as such enables very fast simultaneous reads and writes. We consider each machine to be a worker and since the cluster is heterogeneous we do not exploit any further local parallelization. For comparison, we modified the loss and gradient computations of HOGWILD! to perform ridge regression and ran it on a single Xeon E3-1275 V with 32GB of RAM (4 cores, comparable but slightly slower than the test machine of [15]). The difference between setups makes a comparison of absolute timing inaccurate but relative speedup for each method is comparable.

Simulated data. We consider two large-scale simulated problems. The data is generated from a Gaussian distribution with mean zero and a block-wise covariance matrix such that the features are not independent and the block structure is not known to the algorithm a priori. The data simulation method is fully described in [SI.1]. The matrix of simulated data is fully dense, so there is no sparsity that could be exploited for speedups which directly handicaps HOGWILD!.

The first scenario we consider is \( n = 1000, p = 131,072 \) (200M non-zeros, testing set is the same size as the training set) and has rank \( r = 20 \). The purpose of this experiment is to compare LOCO against HOGWILD! in a setting where memory is not a limiting factor and so HOGWILD! can comfortably parallelize the problem and run on a single machine. We aim to compare the speedup of both methods when the data is dense. The behaviour of HOGWILD! should also be similar to the behaviour of ASYNCDA [8].

According to Theorem 1, increasing \( \tau_{subs} \) will improve the prediction error but also increase the computational time. To give a meaningful comparison between the methods we vary \( \tau_{subs} \) and report the time spent training the model and the prediction error achieved. Since for different number of workers, \( \tau = p/K \) is different, the random projection dimension, \( \tau_{subs} \) is chosen relative to \( \tau \), i.e. \( 0 \leq \tau_{subs} \leq 0.1 \tau \). As a baseline we show the results for a standard ridge regression solver. Prediction error is normalized by this result to obtain relative prediction error.

Figure 1(a) shows the average results over 5 trials for \( K = \{2, 4, 8, 16\} \). Comparable test error to full ridge regression (horizontal dotted line at 1) is obtained with very low-dimensional random projections, even for \( K = 16 \). The largest error for each setting occurs when \( \tau_{subs} = 0 \), i.e. the dataset is partitioned but no random features are added, and increases as \( K \) increases. This highlights

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(a) relative prediction error against time and (b) relative speedup comparison between LOCO and HOGWILD! for \( p = 131,072 \).

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[1] http://www.fftw.org

[2] http://en.wikipedia.org/wiki/Brutus_cluster

[3] Code available from: http://hazy.cs.wisc.edu/hazy/victor/Hogwild/
Figure 2: (a) Prediction error against time, (b) relative speedup and (c) relative additional dimensions for \( p = 1.05M \).

the importance of accounting for dependencies between blocks of features and justifies our random projection approach to distributing features across workers.

Figure 1(b) compares the relative speedup for increasing \( K \) for LOCO and HOGWILD!. HOGWILD! exhibits linear speedup for up to 4 threads but no speedup when more are added. LOCO exhibits an almost linear speedup with the number of workers – with \( K = 16 \) we obtain a \( 6 \times \) speedup over \( K = 2 \). The absolute running time of LOCO was faster and the timings for HOGWILD! ignore a large constant overhead for file loading.

We must stress that this experiment is designed so that the theoretical assumptions underpinning HOGWILD! are violated. Clearly if the data were very sparse, HOGWILD! would exhibit large speedups as investigated in \cite{8, 15}. However, for very large datasets it would still be limited by the number of cores and amount of memory available to a single machine. In such scenarios it may be advantageous to combine LOCO and HOGWILD!.

The second scenario we consider is \( n = 5000, p = 1.05M \) and \( r = 2000 \). Since the data is fully dense, there are over 5 billion non-zero\footnote{Comparable in number with the experiment size of \cite{17} and \cite{16} with the key difference that we do not impose block sparsity in the data like \cite{17} and we do not simply sample \( X \) from \( N(0, 1) \) like \cite{16}.}. Now the size of the data starts to become impractical for a single machine (training data is \( \geq 40 \text{GB} \)) and the distributed nature of LOCO is advantageous. We just report the performance of LOCO for \( K = \{64, 128, 256\} \).

Fig. 2(a) shows the normalized mean-squared prediction error against time. LOCO takes just 25 seconds to achieve a prediction error of 0.11 for \( K = 128 \). Fig. 2(b) shows that we obtain an almost \( 2 \times \) speedup when increasing from 64 to 256 workers. The reason for the small speedup can be observed in Fig. 2(c). The x-axis plots the additional overhead in terms of the original dataset size required to obtain a particular error. Since the size of the random feature representation for each worker scales as \((K-1)\tau_{subs}\), for \( K = 256 \), we require almost \( 20p \) features to match the performance of \( K = 64 \). This illustrates the natural tradeoff between parallelism and additional computational cost from adding enough random features to ensure a good representation as discussed in \cite{4}.

7 Discussion

In this work we have presented LOCO, a simple algorithm for distributed ridge regression – requiring minimal communication and no synchronization – based on random projections. We have shown theoretically and empirically that LOCO achieves small additional error compared with the optimal ridge regression solution. It obtains near linear speedups with the number of workers without making any additional assumptions about sparsity in the data. Furthermore, in principle any fast ridge regression solver can be used in conjunction with LOCO to further speed up learning in each individual worker according to whichever computing architecture or data assumptions apply. It should be noted that LOCO is a complementary rather than competing approach to methods such as HOGWILD!. In particular, a large but sparse problem could be solved on a cluster of multi-core machines extremely quickly by combining LOCO with HOGWILD!.

Further Work. Although currently our results are specific for ridge regression, we expect that the same principles can be generalized to other convex optimization problems.
As mentioned in the introduction, distributed optimization – where no single worker sees all of the data – is a natural paradigm when preserving privacy is required. Additionally, the class of J-L projections that we use have been shown to preserve differential privacy [5]. We aim to explore the connection between LOCO and privacy aware learning.

Finally, [22] established bounds on the minimum amount of communication necessary for a distributed estimation task to achieve minimax optimal risk. It would be interesting investigate how LOCO fits into this framework.

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Supplementary Information for LOCO: Distributing Ridge Regression with Random Projections

Here we collect supplementary technical details, empirical results and discussion which support the results presented in the main text.

SI.1 Data generation

Typically parallel optimization methods are evaluated on extremely sparse datasets [15] [8] or uncorrelated simulated data [17] [2] [16] which fulfills the types of assumptions on sparsity or low-correlations between features necessary to obtain theoretical results. Since we do not make these assumptions, we aim to show that LOCO is robust to correlations between features which can be accounted for using random projections.

We generated each observation in the following way. First, an \( r \) -dimensional standard Gaussian vector is generated \( z \in \mathbb{R}^r \sim \mathcal{N}(0, I_r) \).

We then sample a diagonal matrix of eigenvalues, \( \Lambda \in \mathbb{R}^{r \times r} = \text{diag}(\lambda_1, \ldots, \lambda_r) \) where we can choose each \( \lambda_i \) so that it has a flat (i.e. \( \lambda_1 = \ldots = \lambda_r \)) or decaying (i.e. \( \lambda_1 > \ldots > \lambda_r \)) spectrum, along with an orthonormal matrix of eigenvectors \( Q \in \mathbb{R}^{p \times r} \) with columns 

\[
Q = [q_1, \ldots, q_r]
\]

which have the additional structure that only one entry in each row of \( Q \) is non-zero. \( Q \) can easily be generated from the QR decomposition of a \( p \times r \) random matrix. For example, in MATLAB this is achieved by

\[
[Q, \sim] = \text{qr}(\text{randn}(p, r), 0);
\]

Now, we construct \( x \in \mathbb{R}^p = \Lambda^{1/2} z \) so that \( E[zz^\top] = QAQ^\top \). This ensures that features are correlated according to the non-zero entries in each column of \( Q \). In other words, the features indexed by the non-zero entries of each \( q \) are correlated with each other. A feature \( j \) indexed by \( q_r \) and a feature \( j' \) indexed by \( q_r' \) are uncorrelated. If we arrange the non-zero entries in each \( q \) such that they form contiguous regions, \( E[zz^\top] \) would be a block-diagonal covariance matrix.

We initialize the true regression vector as \( \beta^* = 0 \). We then choose \( r \) indices, each corresponding to a randomly chosen coordinate indexed by each of the columns of \( Q \). Denoting these indices by the subscript, \( \beta^*_q \), we have \( \beta^*_q \sim \text{Unif}[-2, 2] \). The remaining \( p - r \) coordinates are drawn from \( \mathcal{N}(0, 0.2) \). This ensures that only a single coordinate from each of the blocks of correlated features contributes a large amount to the response. Finally the response is, \( y = x^\top \beta^* + \sigma^2 \).

Using this routine, we can quickly generate very high dimensional and dense vectors \( x \) which have an interesting, low-dimensional structure and dependencies between features.

SI.2 Proofs of main results

Proof roadmap. Since the modified ridge regression problem (4) that each worker solves is convex, each worker will obtain a global minimizer to its own problem. In order to ensure a good solution to the global problem, we quantify the approximation error each worker incurs for the raw features with respect to the global solution since only these estimates are ultimately used in the solution \( \hat{\beta} \) LOCO returns. This is achieved by first bounding the difference between the risk of ridge regression in the original space \( R(X\hat{\beta}^r) \) and the risk of ridge regression in the compressed space of worker \( k \ R(X_k\beta_k) \). Using this bound and the convexity of \( f \), we will derive the final bound for the expected difference between the estimates.

Before presenting the proof of theorem 1 we shall introduce a few necessary lemmata. Subsequently, we derive a similar bound for OLS regression. An illustration of the bound in Theorem 1 can be found in Remark 1 in Section SI.2.1.
SI.2.1 Ridge Regression

Recall that \( \hat{\beta}_{rr} \) denotes the solution which results from solving the ridge regression problem using the original data and let \( \hat{\beta}^r_k = \hat{\beta}_{rr}^k \) be the solution to solving ridge regression in the compressed domain of worker \( k \). Each of these parameter vectors can be partitioned into two parts - one containing the components corresponding to the raw features of interest while the other part contains either the remaining raw features not in \( P_k \) or the random features. This is clarified in the following definition.

**Definition 2.** Assuming without loss of generality that the problem is permuted so that the raw features of worker \( k \)’s problem are the first \( \tau \) columns of \( X \) and \( X_k \) we have

(a.) the ridge estimate in the original space, \( \hat{\beta}_{rr} = \begin{bmatrix} \hat{\beta}_{rr}^r_k & \hat{\beta}_{rr}^r_{(-k)} \end{bmatrix}, \) and

(b.) the ridge estimate returned by worker \( k, \hat{\beta}^r_k = \begin{bmatrix} \hat{\beta}_{rr}^r_k; \hat{\beta}_{rr}^r_{(-k)} \end{bmatrix}. \)

In the following we will show that \( \hat{\beta}_{rr}^r_{k, raw} \) is approximately equal to the corresponding coordinates of the coefficient vector, (a.) in the original space, i.e.

\[ \hat{\beta}_{rr}^r_k \approx \hat{\beta}_{rr}^r. \]

For each worker we need to bound difference between these components as LOCOS eventually constructs its estimate \( \hat{\beta} \) by concatenating the estimates for the raw features of each worker.

The following lemma establishes a basic inequality containing the estimates worker \( k \) returns for its raw features and the global ridge regression solution.

**Lemma 2.** Setting \( \hat{\beta}_{rr}^r_{k, raw} \) as the first \( \tau \) elements (i.e. those corresponding to the raw features) of the vector which minimizes the ridge problem of worker \( k \)

\[
\min_{\|\beta_k\|^2 \leq t} n^{-1} \|y - \bar{X}_k \beta_k\|^2 = \min_{\|\beta_{raw}\|^2 + \|\beta_{k, rp}\|^2 \leq t} n^{-1} \|y - X_k \beta_{raw} - \bar{X}_k \beta_{k, rp}\|^2
\]

in a compressed space, we have

\[
\min_{\|\hat{\beta}_{rr}^r_{k, raw}\|^2 + \|\hat{\beta}_{rr}^r_{(-k)}\|^2 \leq t} n^{-1} \|y - X_k \hat{\beta}_{rr}^r_{k, raw} - X_{(-k)} \hat{\beta}_{rr}^r_{(-k)}\|^2 \leq \min_{\|\beta\|^2 \leq t} n^{-1} \|y - X \beta\|^2 + \Delta
\]

where \( \hat{\beta}_{rr}^r_{k, raw} \) is now fixed and the design matrices are standardized. \( \Delta \) is the difference between the global objective and the objective of worker \( k \) at their respective minimal values, i.e.

\[
\Delta = n^{-1} \|y - \bar{X}_k \hat{\beta}_{rr}^r_k\|^2 - n^{-1} \|y - X \hat{\beta}_{rr}^r\|^2.
\]

**Proof.** For the same value of \( t \) we have

\[
\min_{\|\beta\|^2 \leq t} n^{-1} \|y - X \beta\|^2 \leq \min_{\|\beta_{raw}\|^2 + \|\beta_{k, rp}\|^2 \leq t} n^{-1} \|y - X_k \beta_{raw} - \bar{X}_k \beta_{k, rp}\|^2
\]

where the inequality follows from the fact that the original space is larger such that \( y \) can be approximated better. In particular, we have more features in \( X_{(-k)} \) than in \( \bar{X}_k \) and the latter lies in the column space of the former.

Replacing \( X_k \beta_{raw} \) with \( X_k \hat{\beta}_{rr}^r_{k, raw} \) on the left hand side in inequality (8) where \( \hat{\beta}_{rr}^r \) is part of the solution to the right hand side we obtain

\[
\min_{\|\hat{\beta}_{rr}^r_{k, raw}\|^2 + \|\hat{\beta}_{rr}^r_{(-k)}\|^2 \leq t} n^{-1} \|y - X_k \hat{\beta}_{rr}^r_{k, raw} - X_{(-k)} \hat{\beta}_{rr}^r_{(-k)}\|^2 \leq \min_{\|\beta_{raw}\|^2 + \|\beta_{k, rp}\|^2 \leq t} n^{-1} \|y - X_k \beta_{raw} - \bar{X}_k \beta_{k, rp}\|^2
\]

which holds due to the same argument as above.
Lastly, we rewrite the right hand side in terms of the original objective
\[
\min_{\|\theta_{k,x}\|^2 \leq \delta} n^{-1} \|y - X_k \beta_{k,x} - X(-k) \beta(-k)\|^2 \leq \min_{\|\theta_{k,x}\|^2 \leq \delta} n^{-1} \|y - X_k \beta_{k,x} - X(-k) \beta(-k)\|^2 + \Delta
\]
where \(\Delta\) accounts for the difference, i.e.
\[
\Delta = \min_{\|\theta_{k,x}\|^2 \leq \delta} n^{-1} \|y - \bar{X}_k \beta_k\|^2 - \min_{\|\theta\|^2 \leq \delta} n^{-1} \|y - X \beta\|^2. \quad (9)
\]

Recall that in step 4, of LOCO we construct the design matrix of worker \(k\) by concatenating the random features from the remaining blocks. The following lemmas use ideas from [13] to quantify the effect of the random projections. First, Lemma 3 establishes a bound on the spectral norm between the design matrix of worker \(k\) and the global design matrix. We use this fact in Lemma 4 to upper bound the largest eigenvalue and lower bound the \(r^{th}\) largest eigenvalue of the covariance matrix of the projected data in terms of the respective eigenvalues of the data covariance matrix. Recall that \(r\) is the rank of \(X\).

**Lemma 3 (Concatenating random features).** Consider the singular value decomposition \(X = UDV^T\) where \(U \in \mathbb{R}^{n \times r}\) and \(V \in \mathbb{R}^{r \times r}\) have orthonormal columns and \(D\) is diagonal; \(r = \text{rank}(X)\). In addition to the raw features, let \(\tilde{X}_k \in \mathbb{R}^{n \times (\tau + (K-1)\tau_{subs})}\) contain random features which result from concatenating the \(K-1\) random projections from the other workers. Furthermore, assume without loss of generality that the problem is permuted so that the raw features of worker \(k\)’s problem are the first \(\tau\) columns of \(X\) and \(\tilde{X}_k\). Finally, let
\[
\Theta_C = \begin{bmatrix} I_\tau & 0 & \ldots & 0 \\ 0 & \Pi_1 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \Pi_{K-1} \end{bmatrix} \in \mathbb{R}^{p \times (\tau + (K-1) \tau_{subs})}
\]
such that \(\tilde{X}_k = X \Theta_C\).

With probability at least \(1 - (\delta + \frac{p - \tau}{\tau})\)
\[
\|V^T \Theta_C \Theta_C^T V - V^T V\| \leq \sqrt{\frac{c \log(2r/\delta)r}{(K-1)\tau_{subs}}}
\]

**Proof.** Let \(V_k\) contain the first \(\tau\) rows of \(V\) and let \(\tilde{V}_k\) contain the rows of \(V\) which are multiplied by the rows of \(\Theta_C\) containing \(\Pi_{k'}\). Decompose the matrix products as follows
\[
V^T V = V_k^T V_k + \sum_{k' \neq k} V_{k'}^T V_{k'} \quad \text{and} \quad V^T \Theta_C \Theta_C^T V = V_k^T V_k + \sum_{k' \neq k} \tilde{V}_k^T \tilde{V}_{k'}.
\]
With \(\tilde{V}_{k'} = V_{k'}^T \Pi_{k'}\) we have
\[
\|V^T \Theta_C \Theta_C^T V - V^T V\| = \|\sum_{k' \neq k} (V_{k'}^T \Pi_{k'} \Pi_{k'}^T V_{k'} - V_{k'}^T \tilde{V}_{k'}\|)
\]
Since \(\Theta_C\) is an orthogonal matrix, from Lemma 3.3 in [20] and Lemma 19 concatenating \((K-1)\) independent SRHTs from \(\tau\) to \(\tau_{subs}\) is equivalent to applying a single SRHT from \(p - \tau\) to \((K-1)\tau_{subs}\). Therefore we can simply apply Lemma 16 to the above to obtain the result.

**Lemma 4.** Let \(\tilde{X}_k \in \mathbb{R}^{n \times (\tau + (K-1)\tau_{subs})}\) be as defined above. For \(X\) with \(r = \text{rank}(X)\) and with probability at least \(1 - (\delta + \frac{p - \tau}{\tau})\)
\[
(1 - \rho)XX^T \preceq \tilde{X}_k \tilde{X}_k^T \preceq (1 + \rho)XX^T
\]
where \(\rho = C \sqrt{\frac{\tau \log(2r/\delta)}{(K-1)\tau_{subs}}}\). Here, \(A \preceq B\) means that \((B-A)\) is a positive semi-definite matrix.
Proof. The proof is analogous to the proof of Corollary 1 in [13]. Using the singular value decomposition $X = UD\Sigma V^\top$, we have

$$X_k X_k^\top = UD\Sigma_C \Theta_C \Theta_C^\top VDV^\top$$

where $\Theta_C$ is as defined in Lemma 5. Lemma 3 implies that

$$(1 - \rho) V^\top V \preceq \Theta_C \Theta_C^\top V \preceq (1 + \rho) V^\top V$$

with probability at least $1 - (\delta + \frac{\rho - 1}{\epsilon^2})$ and $\rho = C \sqrt{\frac{r \log(2r/\delta)}{(K-1)\tau_{\text{subs}}}}$. The result follows by multiplying with $UD$ and $DU^\top$.

Lemma 5. Let $R(\hat{\beta}_k^\tau)$ denote the risk of the global ridge estimate and let $R(\tilde{X}_k \hat{\beta}_k^\tau)$ denote the risk of the ridge estimate worker $k$ returns. For $\tilde{X}_k \in \mathbb{R}^{n \times (\tau + (K-1)\tau_{\text{subs}})}$ with probability at least $1 - (\delta + \frac{\rho - 1}{\epsilon^2})$, the expected difference in the risk is bounded

$$R(\tilde{X}_k \hat{\beta}_k^\tau) - R(\hat{\beta}_k^\tau) = E_\varepsilon \Delta \leq \left( \frac{1}{(1 - \rho)^2} - 1 \right) R(\hat{\beta}_k^\tau)$$

where $\rho = C \sqrt{\frac{r \log(2r/\delta)}{(K-1)\tau_{\text{subs}}}}$ and $\Delta$ is the difference between the objectives (6) and (7) at their minimal values.

Proof. As Lemma 4 holds for $\tilde{X}_k \in \mathbb{R}^{n \times (\tau + (K-1)\tau_{\text{subs}})}$, Lemma 17 applies with probability at least $1 - (\delta + \frac{\rho - 1}{\epsilon^2})$. Thus, we have

$$R(\tilde{X}_k \hat{\beta}_k^\tau) \leq (1 - \rho)^{-2} R(\hat{\beta}_k^\tau) = R(\hat{\beta}_k^\tau) + \left( \frac{1}{(1 - \rho)^2} - 1 \right) R(\hat{\beta}_k^\tau)$$

where $\rho = C \sqrt{\frac{r \log(2r/\delta)}{(K-1)\tau_{\text{subs}}}}$. $\Delta$ was introduced in eq. (9) in Lemma 2 as the difference between the objectives (6) and (7) at their minimal values. Taking the expectation of this difference w.r.t. $\varepsilon$ coincides with the difference between the risk functions. Therefore, we have

$$R(\tilde{X}_k \hat{\beta}_k^\tau) - R(\hat{\beta}_k^\tau) = E_\varepsilon \Delta.$$ 

Lemma 6. Let $\nabla f(\hat{\beta}^\tau)$ denote the gradient of $f$ at $\hat{\beta}^\tau$ and let $\lambda$ be the regularization parameter in the penalized formulation of the objective. Then we have that

$$\nabla f(\hat{\beta}^\tau) \geq -2\lambda \hat{\beta}^\tau.$$ (10)

Proof. Without loss of generality rotate $X$ to the PCA coordinate system such that

$$\Sigma_R = \frac{1}{n} X_R^\top X_R = \text{diag}(\lambda_1, \ldots, \lambda_{\min(n,p)}, 0, \ldots, 0)$$

with $\lambda_1, \ldots, \lambda_{\min(n,p)}$ being the non-zero eigenvalues of $\Sigma_R$ in decreasing order. The subscript indicates that $X_R$ is the rotated design matrix.

Consider the estimator

$$\hat{w}^0 = \lim_{\lambda \to 0} \hat{w}^\tau$$

where $\hat{w}^\tau$ contains the ridge regression estimates in the rotated space. Since the OLS estimator is not uniquely defined if there are zero eigenvalues, $\hat{w}^0$ is the least squares solution with minimal $\ell_2$ norm.

In this orthogonal setting, the ridge estimate for the $j$-th coefficient $\hat{w}_j^\tau$ has a simple relation to the corresponding estimate $\hat{w}_j^0$

$$\hat{w}_j^\tau = \frac{\lambda_j}{\lambda_j + \lambda} \hat{w}_j^0.$$
Furthermore, the loss can be expressed as
\[
f(\hat{\mathbf{w}}^\pi) = \sum_{j=1}^{p} \lambda_j \left(1 - \frac{\lambda_j}{\lambda_j + \lambda}\right)^2 (\hat{\mathbf{w}}_j^0)^2 + c' = \sum_{j=1}^{p} \lambda_j (\hat{\mathbf{w}}_j^0 - \hat{\mathbf{w}}_j^\pi)^2 + c'
\]
where \(c'\) is a constant and the gradient follows as
\[
\nabla f(\hat{\mathbf{w}}^\pi) = -\sum_{j=1}^{p} 2\lambda_j (\hat{\mathbf{w}}_j^0 - \hat{\mathbf{w}}_j^\pi) = -\sum_{j=1}^{p} 2\lambda\lambda_j (\hat{\mathbf{w}}_j^0 - \hat{\mathbf{w}}_j^\pi) = -2\lambda\hat{\mathbf{w}}^\pi.
\]
As \(f\) was not changed by rotating \(\mathbf{X}\), rotating \(\hat{\mathbf{w}}^\pi\) back to retrieve \(\hat{\beta}^\pi\) does not change the gradient. Therefore, we have
\[
\nabla f(\hat{\beta}^\pi) = -2\lambda\hat{\beta}^\pi.
\]

**Lemma 7.** Let \(\lambda\) be the regularization parameter in the penalized formulation of the objective and let \(\lambda_J\) denote the \(J\)-th largest eigenvalue of the covariance matrix. Then under assumptions (A1) through (A3) there exists a \(n_0\) such that for all \(n \geq n_0 = n_0(\gamma, \sigma^2, J, \lambda_J, L_J, c^2)\), with probability at least \(1 - \gamma\),
\[
\lambda \geq \frac{c}{5}\lambda_J.
\]

**Proof.** Let \(\mathbf{X}_R\) denote the design matrix after rotating \(\mathbf{X}\) to the PCA coordinate system and let \(\hat{\mathbf{w}}^\pi\) denote the ridge estimate in this space. Then we can express \(\hat{\mathbf{w}}^\pi_j\) as
\[
\hat{\mathbf{w}}^\pi_j = \frac{\lambda_j}{\lambda_j + \lambda} (\mathbf{w}_j^* + \mathbf{z}_j)
\]
where \(\mathbf{w}_j^*\) is the true parameter vector in the PCA coordinate system and
\[
\mathbf{z}_j = \frac{(\mathbf{X}_R)_{j}^\top \varepsilon}{(\mathbf{X}_R)_{j}^\top (\mathbf{X}_R)_{j}} = \frac{(\mathbf{X}_R)_{j}^\top \varepsilon}{n\lambda_j}.
\]
Define
\[
L_J := \sum_{j=1}^{J} (\mathbf{w}_j^*)^2.
\]
As \(\hat{\beta}^\pi\) is the solution to the constrained optimization problem in eq. (6), we have that \(\|\hat{\mathbf{w}}^\pi\|^2 \leq t\).

As \(J < p\) and using the relation between \(\mathbf{w}^*\) and \(\hat{\mathbf{w}}^\pi\) yields
\[
\sum_{j=1}^{J} \left(\frac{\lambda_j}{\lambda_j + \lambda}\right)^2 (\mathbf{w}_j^* + \mathbf{z}_j)^2 \leq t.
\]
Hence
\[
\left(\sum_{j=1}^{J} \left(\frac{\lambda_j}{\lambda_j + \lambda}\right)^2 (\mathbf{w}_j^*)^2\right)^{1/2} \leq (t)^{1/2} + \left(\sum_{j=1}^{J} \left(\frac{\lambda_j}{\lambda_j + \lambda}\right)^2 \mathbf{z}_j^2\right)^{1/2}.
\]
Using monotonicity of the eigenvalues, the left hand side is bounded by
\[
\sum_{j=1}^{J} \left(\frac{\lambda_j}{\lambda_j + \lambda}\right)^2 (\mathbf{w}_j^*)^2 \geq \left(\frac{\lambda_J}{\lambda_J + \lambda}\right)^2 \sum_{j=1}^{J} (\mathbf{w}_j^*)^2.
\]
Using assumption (A2), we thus have from (11)
\[
\left(\left(\frac{\lambda_J}{\lambda_J + \lambda}\right)^2 L_J\right)^{1/2} \leq ((1 - c)L_J)^{1/2} + \left(\sum_{j=1}^{J} \left(\frac{\lambda_j}{\lambda_j + \lambda}\right)^2 \mathbf{z}_j^2\right)^{1/2}.
\]

(12)
If we can show that there exists some \( n_0 \) such that for all \( n \geq n_0 \) with probability at least \( 1 - \gamma \),

\[
\sum_{j=1}^{J} \left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 z_j^2 \leq \frac{c^2}{16} L_J,
\]

(13)

then, from (12), with probability at least \( 1 - \gamma \),

\[
\left( \left( \frac{\lambda_j}{\lambda_j + \lambda} \right) L_j \right)^{1/2} \leq (1 - c) L_j^{1/2} + (c^2/16) L_j^{1/2},
\]

and hence

\[
\left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 \leq (1 - c) + 2\sqrt{(1 - c) c^2/16 + c^2/16}
\]

\[
\leq (1 - c) + c/2 + c^2/16 \leq 1 - \frac{7c}{16}
\]

And, if (13) is true, there thus exists a \( n_0 \) such that with probability at least \( 1 - \gamma \),

\[
\lambda \geq \left( \frac{1}{\sqrt{1 - \frac{7c}{16}}} - 1 \right) \lambda_J
\]

\[
\geq \frac{c}{5} \lambda_J.
\]

It thus remains to show (13). The mean of

\[
\frac{\lambda_j}{\lambda_j + \lambda} z_j
\]

vanishes and the variance is given, for all \( j = 1, \ldots, J \) by

\[
\frac{\sigma^2 \lambda_j}{n(\lambda + \lambda_j)^2} \leq \frac{\sigma^2}{n \lambda_j}.
\]

Hence, for all \( j = 1, \ldots, J \) and \( a > 0 \),

\[
P \left\{ \left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 z_j^2 \geq a^2 \right\} \leq \frac{1}{a^2} \frac{\sigma^2}{n \lambda_j}.
\]

Thus, using a Bonferroni bound over \( j = 1, \ldots, J \),

\[
P \left\{ \sum_{j=1}^{J} \left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 z_j^2 \geq a^2 \right\} \leq \sum_{j=1}^{J} P \left\{ \max_{j=1, \ldots, J} \left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 z_j^2 \geq a^2 / J \right\}
\]

\[
\leq \frac{J^2}{a^2} \frac{\sigma^2}{n \lambda_j}.
\]

Hence, with probability at least \( 1 - \gamma \),

\[
\sum_{j=1}^{J} \left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 z_j^2 \leq \frac{J^2}{\gamma} \frac{\sigma^2}{n \lambda_j},
\]

(14)

If choosing

\[
n_0 := \frac{16 J^2 \sigma^2}{\gamma \lambda_J L J c^2},
\]

then, with probability at least \( 1 - \gamma \),

\[
\sum_{j=1}^{J} \left( \frac{\lambda_j}{\lambda_j + \lambda} \right)^2 z_j^2 \leq \frac{c^2}{16} L_J,
\]

(15)

which shows (13) and thus completes the proof. \( \square \)

We are now ready to present the proof of our main theorem.
Proof of Theorem 1. Letting $\hat{\beta}_{k,raw}^n$ denote the first $\tau$ elements of $\hat{\beta}_k^n$, we set

$$\tilde{\beta}' = \arg \min_{\|\hat{\beta}_{k,raw}^n\|^2 + \|\beta_{(-k)}\|^2 \leq t} n^{-1}\|y - X_k\hat{\beta}_{k,raw}^n - X_{(-k)}\beta_{(-k)}\|^2,$$

where $\hat{\beta}_{k,raw}^n$ is fixed and the minimization is over $\beta_{(-k)}$ i.e. $\tilde{\beta}' = [\hat{\beta}_{k,raw}^n; \tilde{\beta}'_{(-k)}].$

From Lemma 2, it follows that $f(\tilde{\beta}') - f(\hat{\beta}_{k,raw}^n) \leq \Delta$ and as $f(\beta)$ is convex, we also have that

$$f(\tilde{\beta}') \geq f(\hat{\beta}_{k,raw}^n) + \nabla f(\hat{\beta}_{k,raw}^n) \tran (\tilde{\beta}' - \hat{\beta}_{k,raw}^n).$$

(16)

Under Assumption 1, using eq. (10) from Lemma 5 and the lower bound on $\lambda$ from Lemma 7, the difference between the components of interest is bounded by

$$\Delta \geq f(\tilde{\beta}') - f(\hat{\beta}_{k,raw}^n) \geq \frac{c}{5} \lambda J \|\hat{\beta}_{k,raw}^n - \tilde{\beta}'_{(-k)}\|^2$$

with probability $1 - \gamma$. Using the expression for the expectation of $\Delta$ from Lemma 5,

$$\mathbb{E}_\epsilon (\|\hat{\beta}_{k,raw}^n - \tilde{\beta}'_{k,raw}^n\|^2) \leq \frac{5}{c\lambda J} \left(\frac{1}{(1 - \rho)^2} - 1\right) R(X\hat{\beta}_{k,raw}^n)$$

with probability $1 - (\delta + \frac{p - \tau}{c\tau} + \gamma)$. Lastly, we use this expression to find a bound for the difference between the full ridge solution $\hat{\beta}_n^r$ and the estimate $\hat{\beta}$ returned by LOCO

$$\mathbb{E}_\epsilon (\|\hat{\beta}_n^r - \hat{\beta}\|^2) \leq \frac{5K}{c\lambda J} \left(\frac{1}{(1 - \rho)^2} - 1\right) R(X\hat{\beta}_n^r)$$

with probability $1 - \xi = 1 - K(\delta + \frac{p - \tau}{c\tau} + \gamma)$.

Remark 1 (Discussion of bound). Combining eq. (10) and the lower bound on $\lambda$ from Lemma 7 shows that the gradient is bounded away from zero under Assumption 1. In figure 3, $\hat{\beta}_0^0$ denotes the least squares solution with minimal $\ell_2$ norm. Then, under (A1) through (A3) we can ensure that the distance $d$ shown in figure 3 does not go to zero which would translate into a weaker bound.

Figure 3: Illustration of bound.
Bounding the error when \( \| \beta^0 \|^2 \leq t \) and further considerations.

Above, we considered the case where the constraint \( t \) was active, i.e., where regularization was needed to achieve best predictive accuracy. In this setting, the least squares solution with minimal \( \ell_2 \) norm \( \beta^0 \) has a larger squared \( \ell_2 \) norm than \( t \). For completeness, consider the case where \( \| \beta^0 \|^2 < t \). Then the constraint \( t \) is large enough such that the ridge estimate coincides with \( \hat{\beta}^0 \) and \( \nabla f(\beta^r) = 0 \). Then
\[
 f(\beta') - f(\beta^r) = (\beta' - \beta^r) \top \frac{X \top X}{n} (\beta' - \beta^r)
\]
where we made use of the fact that the second-order Taylor expansion holds exactly for the squared error loss. If \( \beta' = \beta^r \), worker \( k \) estimates the coefficients of the raw features optimally such that we do not have to consider this case further. If \( \beta' \neq \beta^r \), the following result relates \( \| X (\beta' - \beta^r) \|^2 \) to \( \| \beta' - \beta^r \|^2 \).

We first make an assumption on the smallest eigenvalue of the covariance matrix \( \Sigma = \frac{1}{n} X \top X \).

**Assumption 2** (Restricted minimum eigenvalue condition),
\[
 \min \psi \left\{ \left( \psi - \gamma \right) \top \Sigma \left( \psi - \gamma \right) / \| \psi - \gamma \|_2^2 \right\}; \text{ such that } \| \psi \| = 1, \gamma = \frac{\beta^r}{\| \beta^r \|} \geq \varphi > 0.
\]

**Theorem 8.** If for some constant \( \varphi > 0 \),
\[
 \| \beta' - \beta^r \|^2 \leq (\beta' - \beta^r) \top \Sigma (\beta' - \beta^r) / \varphi,
\]
we can upper bound \( \| \beta' - \beta^r \|^2 \) by \( \Delta / \varphi \) so that
\[
 \frac{1}{\varphi} \left( \frac{1}{(1 - \rho)^2} - 1 \right) R(X \beta^r)
\]
holds with probability at least \( 1 - \left( \delta + \frac{p - n}{e^2} \right) \) and \( \rho = C \sqrt{\frac{r \log(2r/\delta)}{(r-1)} \rho_{u,v}} \).

**Proof.** If Assumption 2 holds, then
\[
 \frac{\Delta}{\varphi} \geq (\beta' - \beta^r) \top \Sigma (\beta' - \beta^r) / \varphi \geq \| \beta' - \beta^r \|^2 \geq \| \beta^r_{\text{raw}} - \beta^r \|^2
\]
where \( \varphi \) is the smallest eigenvalue of \( \Sigma \), and
\[
 \| \beta^r_{\text{raw}} - \beta^r \|^2 \leq \frac{\Delta}{\varphi}.
\]

Using the expression for the expectation of \( \Delta \) from Lemma \( \Box \) yields the result.

**Remark 2.** Assumption 2 implies that the smallest eigenvalue of the \( \Sigma \) is bounded away from zero. Of course, this would not be satisfied if \( p > n \). On the other hand, if \( p > n \) cross validation would always yield a value for \( t \) such that \( \| \beta^0 \|^2 > t \) because the least squares solution has bad statistical properties in this setting and regularization is needed to achieve best predictive accuracy. Furthermore, if \( n \geq p \) and the covariance matrix is close to being singular, we would again choose a constraint such that \( \| \beta^0 \|^2 > t \). Therefore, we can assume that the smallest eigenvalue of the covariance matrix is bounded away from zero in all relevant cases, namely if cross validation yields a value of \( t \) such that \( \| \beta^0 \|^2 \leq t \).

**SI.2.2 Ordinary Least Squares**

For OLS regression, \( f(\beta) = \| y - X \beta \|^2 \) is the squared error loss while there is no regularizer. Here, \( \beta^{rh} \) denotes the solution which results from solving the OLS problem using the original data while each worker solves least squares using \( X_k \) instead of \( X \) and returns \( \beta^{rh}_k \).
Definition 3. Let \( \tilde{z}_j \) denote the residual which results from regressing the feature vector \( j, X^j \), onto the remaining features \( X^{-j} \)

\[
\tilde{z}_j = X^j - X^{-j} \tilde{\gamma}
\]

where \( \tilde{\gamma} = \arg \min_{\gamma} \| X^j - X^{-j} \gamma \|^2 \)

and let \( \hat{z}_j \) denote the residual which results from regressing feature \( j, X^j \), onto the randomized approximation of the remaining features \( \hat{X}_k^{-j} \)

\[
\hat{z}_j = X^j - \hat{X}_k^{-j} \hat{\gamma}_k
\]

where \( \hat{\gamma}_k = \arg \min_{\gamma_k} \| X^j - \hat{X}_k^{-j} \gamma_k \|^2 \).

The following result relies on Lemma 18 by [11] which can be found in §3.1.3

Theorem 9. Let \( \tilde{z}_j, \hat{z}_j \) and \( \tilde{\gamma} \) be as given in Definition 3 and let \( \Sigma = X^{-j\top} X^{-j} / n \). Furthermore, let all other quantities be as defined in Lemma 18. Then the difference between the residuals is bounded by

\[
\frac{1}{n} \sum_{i=1}^{n} (\| \tilde{z}_j - \hat{z}_j \|_2^2) \leq \frac{1}{\tau_{\text{subs}}} \| \tilde{\gamma} \|_2^2 \Sigma + \text{tr} (\Sigma) I_{p-1} + \kappa \sum_{i=1}^{n} e_i B_i
\]

where \( \| u \|_M = \sqrt{u^\top M u} \) is the Mahalanobis norm.

Corollary 10. As the \( j \)-th OLS regression coefficient can be expressed as

\[
\hat{\beta}^j = \frac{\tilde{z}_j \cdot \mathbf{y}}{\tilde{z}_j \cdot X^j} \quad \text{resp.} \quad \hat{\beta}^j_{k,j} = \frac{\hat{z}_j \cdot \mathbf{y}}{\hat{z}_j \cdot X^j},
\]

the bound on the difference between \( \tilde{z}_j \) and \( \hat{z}_j \) implies that the difference between \( \hat{\beta}^j \) and \( \hat{\beta}^j_{k,j} \) is bounded as well. Therefore, the estimate for the raw feature \( j \) returned by worker \( k \), \( \hat{\beta}^j_{k,j} \), is sufficiently close to the global estimate \( \hat{\beta}^j \).

In order to present the proof of Theorem 9 we need the following lemmata.

Lemma 11. Consider the OLS regression problem in the original space

\[
\hat{\beta}^j = \arg \min_{\beta} f(\beta) = \arg \min_{\beta} \| \mathbf{y} - X \beta \|^2,
\]

yielding the fitted values \( \hat{\mathbf{y}} = X \hat{\beta} \). Furthermore, let \( \hat{X}_k \) be the design matrix of the least squares problem worker \( k \) solves. Then the squared \( \ell_2 \) norm of the residual can be decomposed as follows

\[
\| \mathbf{y} - \hat{\mathbf{y}} \|^2 = \| \mathbf{P}_{\perp X} (\mathbf{y} - \hat{\mathbf{y}}) \|^2 + \| \mathbf{P}_{\perp X \perp \hat{X}_k} (\mathbf{y} - \hat{\mathbf{y}}) \|^2 + \| \mathbf{P}_{\perp \hat{X}_k} (\mathbf{y} - \hat{\mathbf{y}}) \|^2,
\]

where \( \mathbf{P}_{\perp X} = X (X^\top X)^{-1} X^\top \) denotes the orthogonal projection onto the column space of the matrix \( X \) and \( \mathbf{P}_{\perp X} = \mathbf{I} - \mathbf{P}_{\perp X} \) is the projection onto the orthogonal complement of the column space of \( X \).

Proof. The decomposition follows from the orthogonality of the considered spaces. Note that \( \hat{X}_k \) lies in the column space of \( X \) since the random features are linear combination of the raw features not in \( \mathcal{P}_k \) and the features in \( \mathcal{P}_k \) are contained in both matrices. The first summand in eq. (18) is the part of \( \mathbf{y} \) that cannot be accounted for in the space spanned by the original design matrix \( X \). Therefore, it is the error that the OLS fit cannot avoid to incur. The remaining two terms are zero due to the definition of the OLS estimator

\[
\| \mathbf{P}_{\perp X \perp \hat{X}_k} (\mathbf{y} - \hat{\mathbf{y}}) \|^2 + \| \mathbf{P}_{\perp \hat{X}_k} (\mathbf{y} - \hat{\mathbf{y}}) \|^2 = \| \mathbf{P}_{\perp X} (\mathbf{y} - \hat{\mathbf{y}}) \|^2 = \| X (X^\top X)^{-1} X^\top \mathbf{y} - \hat{\mathbf{y}} \|^2 = \| X \hat{\beta} - \hat{\mathbf{y}} \|^2 = 0.
\]
Lemma 12. Consider the OLS regression problem in the original space as given in eq. (17) and in the compressed space

\[ \hat{\beta}_k^l = \arg\min_{\beta_k} f_k(\beta_k) = \arg\min_{\beta_k} \|y - \bar{X}_k \beta_k\|^2, \]

yielding the fitted values \( \bar{y} = \bar{X}_k \hat{\beta}_k^l \). The squared \( \ell_2 \) norm of the difference between \( \bar{y} \) and \( \tilde{y} \) can be expressed as

\[ \| \bar{y} - \tilde{y} \|^2 = \| X \hat{\beta}^l - \bar{X}_k \hat{\beta}_k^l \|^2 = \| P_{\|X\perp X_k}(y - \hat{y}) \|^2 = \| P_{\|X\perp X_k}(y - \bar{y}) \|^2, \]

which corresponds to the approximation error which results from fitting the model in the compressed space instead of the original space.

Proof. According to Lemma 11, the residual of the fit in the compressed domain can be decomposed as follows

\[ \| y - \bar{y} \|^2 = \| \bar{P}_{\perp X}(y - \bar{y}) \|^2 + \| \bar{P}_{\|X\perp X_k}(y - \tilde{y}) \|^2 + \| \bar{P}_{\|X\perp X_k}(y - \hat{y}) \|^2. \] \hspace{1cm} (19)

Now, the first and the second term form the part of \( y \) that cannot be accounted for in the space spanned by the projected design matrix while the third term vanishes due to the definition of the least squares estimator. Thus, we see that the difference between eq. (18) and eq. (19) is given by the second term in eq. (19) and due to the orthogonality structure, we have

\[ \| \bar{y} - \tilde{y} \|^2 = \| X \hat{\beta}^l - \bar{X}_k \hat{\beta}_k^l \|^2 = \| \bar{P}_{\|X\perp X_k}(y - \bar{y}) \|^2 = \| \bar{P}_{\|X\perp X_k}(y - \hat{y}) \|^2. \]

This expression corresponds to the approximation error which results from fitting the model in the compressed space instead of the original space.

Lemma 13. Let \( \hat{y} \) denote the fitted values resulting from the OLS problem in the original space and let \( \tilde{y} \) be the fitted values resulting from the OLS problem in the compressed space of worker \( k \). Furthermore, let all other quantities be as defined in Lemma 11. Then

\[ \frac{1}{n} \mathbb{E}_{\Pi}(\| \bar{y} - \hat{y} \|^2) \leq \frac{1}{\tau_{subs}} \| \beta \|^2 \| \Sigma + T_{\mathcal{R}(\Sigma)}I_p + \kappa \sum_{i=\ell} c_i B_i \]

where \( \| u \|_M = \sqrt{u^T M u} \) is the Mahalanobis norm.

Proof. From Lemma 12, we see that the difference between the two residuals is equal to the approximation error which results from fitting the model in the compressed domain. This approximation error measures the distance between the compressed space and the optimal regression function in the original space. It can be bounded as follows

\[ \| \bar{y} - \tilde{y} \|^2 = \| X \hat{\beta}^l - \bar{X}_k \hat{\beta}_k^l \|^2 \leq \| X \hat{\beta}^l - X \Pi \bar{X}_k \hat{\beta}_k^l \|^2. \]

This bound follows from the fact that using \( \Pi \mathbb{X} \) implies a larger degree of approximation than using \( \bar{X}_k \) which also contains raw features. In other words, the column space of \( \Pi \mathbb{X} \) is contained in the column space of \( \bar{X}_k \) and \( \mathbb{X} \) and the distance between \( \Pi \mathbb{X} \) and \( \mathbb{X} \) is larger than the distance between \( \bar{X}_k \) and \( \mathbb{X} \). Additionally, \( \hat{\beta}_k^l \) minimizes the objective in the compressed space such that the coefficients given by \( \Pi \bar{X}_k \hat{\beta}_k^l \) cannot be associated with a smaller approximation error.

Applying Lemma 18 yields the desired result.

Corollary 14. If \( \Pi \) is a Gaussian (or some other random projection with excess kurtosis, \( \kappa = 0 \)), then the bound in Lemma 13 reduces to

\[ \frac{1}{n} \mathbb{E}_{\Pi}(\| \bar{y} - \hat{y} \|^2) \leq \frac{1}{\tau_{subs}} \| \hat{\beta}_k^l \|^2 \| \Sigma + T_{\mathcal{R}(\Sigma)}I_p \].
Lemma 15. The $j$-th OLS regression coefficient is proportional to the inner product of a residual and the response, where the residual results from regressing the $j$-th feature vector onto the remaining part of the design matrix. More formally, let $X^j$ be the feature vector of interest and let $X^{-j}$ denote the matrix containing the remaining features. Then the residual $\tilde{z}_j$ results from regressing $X^j$ onto $X^{-j}$

$$\tilde{z}_j = X^j - X^{-j}\hat{\gamma} \quad \text{where} \quad \hat{\gamma} = \arg \min_{\gamma} \|X^j - X^{-j}\gamma\|^2,$$

and the $j$-th regression coefficient can be written as

$$\beta^l_{j} = \frac{\tilde{z}_j \cdot y}{z_j \cdot x_j^j}.$$

Proof. This result reflects the fact that the $\beta^l_j$ contains the additional contribution of feature $j$ on $y$ after having accounted for the remaining features. The formula follows from “regression by successive orthogonalization”, for more details see [10].

Proof of Theorem 9. From Lemma [15] we have a closed-form expression for the $j$-th OLS regression coefficient in the original space. Now consider the regression problem worker $k$ has to solve and let $j'$ be a raw feature in $X_k$ which corresponds to feature $j$ in $X$. For ease of notation, set $j' = j$ such that we have $X^j = X^j_k$. In the problem worker $k$ solves the expressions in eq. (20) become

$$\tilde{z}_j = X^j - X_k^{-j}\hat{\gamma} \quad \text{where} \quad \hat{\gamma} = \arg \min_{\gamma_k} \|X^j - X_k^{-j}\gamma_k\|^2 \quad \text{and} \quad \beta^l_{j,k} = \frac{\tilde{z}_j \cdot y}{z_j \cdot x_{j,k}^j}.$$

The only change between the expressions for the $j$-th coefficient results from replacing $\tilde{z}_j$ by $\tilde{z}_j$. From Lemma [13] we have that the difference between the residuals $\tilde{z}_j$ by $\tilde{z}_j$ is bounded such that the difference between the estimated coefficients $\beta^l_{j}$ and $\beta^l_{j,k}$ is bounded as well.

SI.3 Supporting results

SI.3.1 Ridge Regression

Lemma 16 (Lemma 1 of [13]). Let $W$ be an $p \times r$ ($p > r$) matrix where $W^TW = I_r$. Let $\Pi$ be a $p \times p_{subs}$ SRHT matrix where $p_{subs}$ is the subsampling size and $p > p_{subs} > r$. Then with failure probability at most $\delta + p/e^r$

$$\|W^T \Pi \Pi^T W - W^T W\| \leq \sqrt{\frac{c \log(2r/\delta)r}{p_{subs}}}.$$

We rely on the following theorem from [13] which bounds the risk of the subsampled approximation to the ridge regression estimator. We provide an alternative proof of this result based on the bias-variance decomposition of the regularized kernel ridge regression estimator from [4].

Lemma 17 (Risk Inflation [13]). Consider solving ridge regression in the dual space with $\beta = X^\top \alpha$, $\alpha \in \mathbb{R}^n$ and letting $K = XX^\top$ be the kernel matrix, the dual optimization problem is

$$\hat{\alpha} = \arg \min_{\alpha \in \mathbb{R}^n} n^{-1}\|y - K\alpha\|_2^2 + \lambda \alpha^\top K \alpha$$

with solution $\hat{\alpha} = (K + n\lambda I)^{-1}y$ and $\hat{\beta}^\top = X^\top \hat{\alpha}$. Using $K_H = \Pi K \Pi^\top$ instead of $K = XX^\top$, where $\Pi \in \mathbb{R}^{p \times p_{subs}}$ is a SRHT matrix, allows for a faster computation. Let $\hat{\beta}^\top_H$ denote the resulting estimate from this randomized approximation and let $r$ be the rank of the $X$ matrix. With probability at least $1 - (\delta + p/e^r)$ we have the following relation between the risk of ridge regression and the randomized approximation

$$R(X\Pi \hat{\beta}^\top_H) \leq (1 - \rho)^{-2} R(X\hat{\beta}^\top)$$

where $\rho = C \sqrt{\frac{r \log(2r/\delta)}{p_{subs}}}$.  

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Proof. We follow the proof of Theorem 1 in [4] and consider the regularized approximation to $K$

$$L_\gamma = XN_\gamma X^\top$$

where $\gamma > 0$ and

$$N_\gamma = \frac{1}{1 + \gamma} \Pi \Pi^\top$$

where $\Pi = \sqrt{\frac{p}{p_{subs}}} \text{DHS}$ is the SRHT. With $\gamma = 0$, we have $N_0 = \Pi \Pi^\top$ so $L_0 = K_H$ which is the quantity of interest.

Lemma 16 implies that with probability at least $1 - (\delta + p/e^r)$

$$(1 - \rho)K \preceq L_0 \preceq (1 + \rho)K$$

where $\rho = C' \sqrt{\frac{r \log (2r/\delta)}{p_{subs}}}$. Since $X_\hat{\beta}^\top = K_\hat{\alpha}$, we have that $R(X_\hat{\beta}^\top) = R(K_\hat{\alpha})$. Setting $z = E_\varepsilon [y] = X_\beta^*$ the risk can be decomposed in the following way [4]

$$R(X_\hat{\beta}^\top) = R(K_\hat{\alpha}) = n^{-1}E_\varepsilon \|z - K_\hat{\alpha}\|^2 = \frac{\sigma^2}{n} \text{tr} \left( \left[ K(K + n\lambda I)^{-1} \right]^2 \right) + \frac{n\lambda^2 z^\top (K + n\lambda I)^{-2} z}{\mathbb{V}(K)}$$

where $\mathbb{V}(\cdot)$ is the variance and $\mathbb{B}(\cdot)$ is the bias. When learning with $L_\gamma$ instead of $K$, the variance term of the risk is given by

$$\mathbb{V}(L_\gamma) = \frac{\sigma^2}{n} \text{tr} \left( \left[ L_\gamma (L_\gamma + n\lambda I)^{-1} \right]^2 \right). \quad (21)$$

The function $\gamma \mapsto N_\gamma$ is matrix-non-increasing (i.e., if $\gamma \geq \gamma'$ then $N_\gamma \preceq N_{\gamma'}$). Therefore, we have $0 \preceq N_\gamma \preceq N_0$. Since the variance $\mathbb{V}(L_\gamma)$ is non-decreasing in $N_\gamma$, this implies $\mathbb{V}(L_\gamma) \leq \mathbb{V}(L_0)$. Furthermore, as $L_0 \preceq (1 + \rho)K$

$$X\Pi\Pi^\top X^\top \preceq (1 + \rho)K$$

$$N_0 = \Pi\Pi^\top \preceq (1 + \rho)I$$

Thus, we have

$$\mathbb{V}(K_H) \leq \mathbb{V}((1 + \rho)K).$$

For the bias term, we have

$$\mathbb{B}(L_\gamma) = n\lambda^2 z^\top (L_\gamma + n\lambda I)^{-2} z \quad (22)$$

which is a non-decreasing function of $\gamma$. For $\gamma = 0$, $N_0$ is lower-bounded as follows

$$X\Pi\Pi^\top X^\top \succeq (1 - \rho)K$$

$$N_0 = \Pi\Pi^\top \succeq (1 - \rho)I$$

and as the bias is non-increasing in $N_\gamma$

$$\mathbb{B}(K_H) \leq \mathbb{B}((1 - \rho)K).$$

Finally, the risk can be bounded as

$$R(X\hat{\beta}^\top_H) = \mathbb{V}(K_H) + \mathbb{B}(K_H) \leq \mathbb{V}((1 + \rho)K) + \mathbb{B}((1 - \rho)K) \leq (1 - \rho)^{-2} (\mathbb{V}(K) + \mathbb{B}(K)) = (1 - \rho)^{-2} R(X\hat{\beta}^\top_H).$$

$\square$
SI.3.2 Ordinary Least Squares

Here we state a result by [11] which we rely on to state Theorem [9]. It bounds the approximation error that is incurred by fitting the model in the compressed space as opposed to the initial space. We defer the proof to the original paper.

**Lemma 18 (Compressive least squares [11])**. Let \( \Pi \) be a \( p \times p_{subs} \) random matrix, \( p_{subs} < p \), with entries drawn i.i.d. from a zero-mean symmetric distribution with variance \( 1/p_{subs} \) and excess kurtosis \( \kappa = \mathbb{E}(\Pi_{ij}^4)/\mathbb{E}(\Pi_{ij}^2)^2 - 3 \). Let \( \Sigma = X^\top X/n \) be fixed with eigenvalues \( e_1, \ldots, e_p \) and let \( B_i \) be a \( p \times p \) diagonal matrix with the \( j \)-th diagonal element being \( \sum_{a=1}^p U_{ai}^2 U_{aj}^2 \) where \( U_{ai} \) is the \( \alpha \)-th entry of the \( i \)-th eigenvector of \( \Sigma \). Finally, let \( \hat{\beta}^l \) contain the optimal regression coefficients in \( \mathbb{R}^p \). Then

\[
\frac{1}{n} \mathbb{E}_\Pi(\|X\hat{\beta}^l - X\Pi^{\top}\hat{\beta}^l\|^2) = \frac{1}{p_{subs}} \|\hat{\beta}^l\|^2 \sum_{a=1}^p U_{ai}^2 + \kappa \sum_{i=1}^p e_i B_i
\]

where \( \|u\|_M = \sqrt{u^\top M u} \) is the Mahalanobis norm.

Note that for Gaussian random projections and sparse random projections \( \kappa = 0 \) while for random sign random projections \( \kappa = -2 \) such that the bound tightens [11].

SI.3.3 Consequences of concatenating random projections.

The following lemma may seem obvious to the reader but we provide it as confirmation of an intuitive result. The lemma is minor reformulation of the row sampling lemma (Lemma 3.4) from [20].

What this lemma confirms is that concatenating \((K-1)\) lots of \( \tau_{subs} \) random projections as in Step 4. of LOCO is equivalent to computing a single, SRHT \( \Pi \in \mathbb{R}^{(p - \tau) \times (K-1)\tau_{subs}} \).

The consequence is that the computation of the SRHT can be divided among \( K \) workers and computed in parallel provided the \( p \) coordinates are uniformly distributed among the workers.

The proof is provided below and is very similar to the proof of the original lemma.

**Lemma 19 (Concatenated row sampling)**. Let \( W \) be an \( n \times p \) matrix with orthonormal columns, and define the quantity \( M := n \cdot \max_{j=1,\ldots,n} \|e_j^l W\|^2 \). Let \( W_1, \ldots, W_K \) be a balanced, random partitioning of the rows of \( W \) where each matrix \( W_k \) has exactly \( \tau = n/K \) rows. For a positive parameter \( \alpha \), select the subsample size

\[
l \geq \alpha M \log(p).
\]

Let \( S_{Tk} \in \mathbb{R}^{l \times \tau} \) denote the operation of uniformly at random sampling a subset, \( T_k \) of the rows of \( W_k \) by sampling \( l \) coordinates from \( \{1, 2, \ldots, \tau\} \) without replacement. Now denote \( SW \) as the concatenation of the subsampled rows

\[
\left[(S_{T_1} W_1)^\top, \ldots, (S_{T_K} W_K)^\top\right]^\top,
\]

Then

\[
\sqrt{(1 - \delta) l \cdot K \over n} \leq \sigma_p(SW) \quad \text{and} \quad \sigma_p(SW) \leq \sqrt{(1 + \eta) l \cdot K \over n}
\]

with failure probability at most

\[
p \cdot \left[ e^{-\delta} \over (1 - \delta)^{1 - \delta} \right]^{\alpha \log p} + p \cdot \left[ e^\eta \over (1 + \eta)^{1 + \eta} \right]^{\alpha \log p}
\]

**Proof.** Define \( w_j^l \) as the \( j^{th} \) row of \( W \) and \( M := n \cdot \max_j \|w_j\|^2 \).

\[
G := (SW)^\top (SW) = \sum_{k=1}^K \sum_{j \in T_k} w_j w_j^\top.
\]

We can consider \( G \) as a sum of \( l \cdot K \) random matrices

\[
X_1^{(1)}, \ldots, X_1^{(K)}, \ldots, X_l^{(1)}, \ldots, X_l^{(K)}
\]
sampled uniformly at random from the family $\mathcal{X} := \{w_i w_i^T : i = 1, \ldots, \tau\}$.

To use the matrix Chernoff bound from Lemma 20, we require the quantities $\mu_{\text{min}}, \mu_{\text{max}}$ and $B$. Noticing that $\lambda_{\text{max}}(w_j w_j^T) = \|w_j\|^2 \leq \frac{M}{n}$, we can set $B \leq M/n$.

Taking expectations with respect to the random partitioning $(E_P)$ and subsampling within each partition $(E_S)$, using linearity of expectation and the fact that columns of $W$ are orthonormal we obtain
\[
\mathbb{E} \left[ X^{(k)} \right] = E_S E_P X^{(k)} = \frac{1}{\tau} \sum_{i=1}^{\tau} \frac{1}{K} w_j w_j^T = \frac{K}{Kn} I = \frac{1}{n} I.
\]

Recall that we take $l$ samples in $K$ blocks so we can define
\[
\mu_{\text{min}} = \frac{l \cdot K}{n} \quad \text{and} \quad \mu_{\text{max}} = \frac{l \cdot K}{n}.\]

Plugging these values into Lemma 20, the lower and upper Chernoff bounds respectively yield
\[
P \left\{ \lambda_{\text{min}}(G) \leq (1 - \delta) \frac{l \cdot K}{n} \right\} \leq p \cdot \left[ \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^{l \cdot K/M} \quad \text{for } \delta \in [0, 1), \text{ and}
\]
\[
P \left\{ \lambda_{\text{max}}(G) \geq (1 + \delta) \frac{l \cdot K}{n} \right\} \leq p \cdot \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^{l \cdot K/M} \quad \text{for } \delta \geq 0.
\]

Noting that $\lambda_{\text{min}}(G) = \sigma_p(G)^2$, similarly for $\lambda_{\text{max}}$ and using the identity for $G$ above obtains the desired result.

For ease of reference, we also restate the Matrix Chernoff bound from [20, 21] but defer its proof to the original papers.

**Lemma 20** (Matrix Chernoff from [20]). Let $\mathcal{X}$ be a finite set of positive-semidefinite matrices with dimension $p$, and suppose that
\[
\max_{X \in \mathcal{X}} \lambda_{\text{max}}(X) \leq B
\]

Sample $\{X_1, \ldots, X_1\}$ uniformly at random from $\mathcal{X}$ without replacement. Compute
\[
\mu_{\text{min}} = l \cdot \lambda_{\text{min}}(E X_i) \quad \text{and} \quad \mu_{\text{max}} = l \cdot \lambda_{\text{max}}(E X_i)
\]

Then
\[
P \left\{ \lambda_{\text{min}} \left( \sum_i X_i \right) \leq (1 - \delta) \mu_{\text{min}} \right\} \leq p \cdot \left[ \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^{\mu_{\text{min}}/B} \quad \text{for } \delta \in [0, 1), \text{ and}
\]
\[
P \left\{ \lambda_{\text{max}} \left( \sum_i X_i \right) \geq (1 + \delta) \mu_{\text{max}} \right\} \leq p \cdot \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^{\mu_{\text{max}}/B} \quad \text{for } \delta \geq 0.
\]