Texture of Single Vanishing Subtrace in Neutrino Mass Matrix

A. Ismael$^1$, M. AlKhateeb$^2$†, N. Chamoun$^3$‡ and E. I. Lashin$^{1,4,5}$§

1 Department of Physics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt.
2 Département de Physique, Université de Cergy-Pontoise, F-95302 Cergy-Pontoise, France.
3 Physics Department, HIAST, P.O.Box 31983, Damascus, Syria.
4 Centre for Fundamental Physics, Zewail City of Science and Technology, 6 October City, Giza 12588, Egypt.
5 The Abdus Salam, ICTP, P. O. Box 586, 34100 Trieste, Italy.

June 22, 2020

Abstract

We consider a texture for the neutrino mass matrix characterized by one vanishing $2 \times 2$ subtrace. We analyze phenomenologically and analytically all the six possible patterns, and show that all non-singular ones are able to accommodate the experimental bounds, whereas singular patterns allow only for four inverted-hierarchy type textures. We then present some possible realizations of this texture, within seesaw scenarios, either directly or indirectly by relating it to zero-textures.

Keywords: Neutrino Physics; Flavor Symmetry;
PACS numbers: 14.60.Pq; 11.30.Hv;

1 Introduction

The fact that neutrinos are massive was the first firm sign of physics beyond standard model [1]. Many flavor models for neutrino mass matrix were conceived, motivated by phenomenological data on neutrino oscillations [2]. Zero textures were studied extensively [3,4,5], but other forms of textures were equally studied, such as zero minors [6], and partial $\mu-\tau$ symmetry textures [7]. In [8], a particular texture of vanishing two subtraces was studied. There, analytical expressions for the measurable neutrino observables were derived, and numerical analysis was done to show that 8 patterns, out of the 15 independent ones, can accommodate data. The objective of this work is to study the texture characterized by one vanishing subtrace. Actually, the one vanishing $2 \times 2$ subtrace texture can be seen also as a generalization of the zero texture when the latter is regarded as a vanishing $1 \times 1$ subtrace texture. We implement the new experimental bounds of [9], with the new updates on the non-vanishing value of the smallest mixing matrix [10], and carry out a complete numerical analysis, where all the free parameters are scanned within their experimentally accepted ranges. We discuss non-singular patterns having all the neutrino masses non vanishing and singular patterns where one of the masses is zero. We find all six

$^*$ahmedEhusien@sci.asu.edu.eg
$^\dagger$mohkha88@gmail.com
$^\ddagger$nidal.chamoun@hiast.edu.sy, nchamoun@th.physik.uni-bonn.de
$^\S$slashin@zewailcity.edu.eg, elashin@ictp.it
non-singular textures able to accommodate the experimental data. As to singular textures, only
four textures can accommodate data of inverted hierarchy type. We then address the question
of how to realize such a vanishing subtrace texture. First, we relate the symmetry imposing
the vanishing subtrace pattern to another symmetry which forces zeros at specific locations.
The former pattern arises upon rotating the zero-texture form. Although the texture form is
imposed at high scale, however many arguments [11] were presented in favor of keeping the
form when running into low scale. The method we suggest for realizing the vanishing subtrace
texture applies only to four out of six possible patterns. However, it is generic enough to be
applicable to any specific texture related by unitary transformation to zero textures, and we
apply it successfully within type I and type II seesaw scenarios. Second, we present direct
realizations of the textures within type I and type II seesaw scenarios, without relating them to
to zero textures, based on discrete symmetries.

The plan of the paper is as follows. In section 2, we present the notation and adopted con-
ventions. In section 3, we explain the method to follow for the phenomenology. In section 4, we
present the analysis of all the non-singular six patterns of one vanishing subtrace supplemented
with one single table summarizing all the predictions of the various patterns. Subsections therein
correspond to these different patterns where for each one we report the relevant defining quan-
tities, correlation plots and one representative point for each type of hierarchy. We repeat the
analysis in section 5 for the singular patterns. Section 6 outlines the problem of how to build
models implementing the vanishing subtrace texture whether it be directly or indirectly. In
section 7, we develop the generic method relating the symmetry of vanishing subtraces to that
of zero textures and use it as an indirect method for getting the vanishing subtrace texture. In
section 8, we clarify the notion of flavor basis which is of paramount relevance in our discussion.
The aforementioned indirect method is applied within type I seesaw scenarios in order to real-
ize invertible (singular) vanishing subtrace textures in section 9 (10). We reapply this indirect
method in section 11 but within type II seesaw scenario. In sections 12 and 13, we present
respectively a direct way to impose the textures within type I and type II seesaw scenarios.
Summary and conclusion are presented in section 14.

2 Notation

In the ‘flavor’ basis, where the charged lepton mass matrix is diagonal and thus the observed
neutrino mixing matrix comes solely from the neutrino sector, we have

\[ M_\nu = V_{PMNS} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (V_{PMNS})^T, \]  

\[ P = \text{diag} (e^{i\rho}, e^{i\sigma}, 1), \quad U = R_{23} (\theta_{23}) R_{13} (\theta_{13}) \text{ diag} (1, e^{-i\delta}, 1) R_{12} (\theta_{12}), \]  

\[ V_{PMNS} = U P = \begin{pmatrix} c_{12} c_{13} e^{i\rho} & s_{12} c_{13} e^{i\sigma} & s_{13} \\ (-c_{12} s_{23} s_{13} - s_{12} c_{23} e^{-i\delta}) e^{i\rho} & (-s_{12} s_{23} s_{13} + c_{12} c_{23} e^{-i\delta}) e^{i\sigma} & s_{23} c_{13} \\ (-c_{12} c_{23} s_{13} + s_{12} s_{23} e^{-i\delta}) e^{i\rho} & (-s_{12} c_{23} s_{13} - c_{12} s_{23} e^{-i\delta}) e^{i\sigma} & c_{23} c_{13} \end{pmatrix}, \]  

(2)

where \( R_{ij} (\theta_{ij}) \) is the rotation matrix in the \((i, j)\)-plane by angle \( \theta_{ij} \), and \( s_{12} \equiv \sin \theta_{12} \ldots \) Note that in this adopted parametrization, the third column of \( V_{PMNS} \) is real.

The mass spectrum is classified into two categories:

- Normal hierarchy: characterized by \( m_1 < m_2 < m_3 \) and is denoted by \( N \).
• Inverted hierarchy: characterized by $m_3 < m_1 < m_2$ and is denoted by $I$.

The neutrino mass-squared differences, characterizing respectively solar and atmospheric neutrino mass-squared differences together with their ratio $R_\nu$, are defined as,

$$\delta m^2 \equiv m_2^2 - m_1^2, \quad \Delta m^2 \equiv \left|m_3^2 - \frac{1}{2}(m_1^2 + m_2^2)\right|, \quad R_\nu = \delta m^2/\Delta m^2. \quad (3)$$

Two parameters which put bounds on the neutrino mass scales, through studying beta-decay kinematics and neutrinoless double-beta decay, are the effective electron-neutrino mass:

$$m_e = \sqrt{\sum_{i=1}^{3} (|V_{ei}|^2 m_i^2)}, \quad (4)$$

and the effective Majorana mass term $m_{ee}$:

$$m_{ee} = |m_1 V_{e1}^2 + m_2 V_{e2}^2 + m_3 V_{e3}^2| = |M_{\nu11}|. \quad (5)$$

Cosmological observations put bounds on the ‘sum’ parameter $\Sigma$:

$$\Sigma = \sum_{i=1}^{3} m_i. \quad (6)$$

The last measurable quantity we shall consider is the Jarlskog rephasing invariant defined by:

$$J = s_{12} c_{12} s_{23} c_{23} s_{13} c_{13} \sin \delta. \quad (7)$$

The experimental bounds for the oscillation parameters are summarized in Table 1.

| Parameter | Hierarchy | Best fit | 1σ      | 2σ      | 3σ      |
|----------|-----------|----------|---------|---------|---------|
| $\delta m^2$ (10^{-3} eV^2) | NH, IH | 7.37 | [7.21,7.54] | [7.07,7.73] | [6.93,7.96] |
| $\Delta m^2$ (10^{-3} eV^2) | NH | 2.53 | [2.50,2.57] | [2.45,2.61] | [2.41,2.65] |
| | IH | 2.51 | [2.47,2.54] | [2.43,2.58] | [2.39,2.62] |
| $R_\nu$ | NH | 0.029 | [0.028,0.030] | [0.027,0.031] | [0.026,0.033] |
| | IH | 0.029 | [0.028,0.030] | [0.027,0.032] | [0.026,0.033] |
| $\theta_{12}$ (°) | NH, IH | 33.02 | [32.02,34.09] | [30.98,35.30] | [30.00,36.51] |
| $\theta_{13}$ (°) | NH | 8.43 | [8.30,8.55] | [8.11,8.74] | [7.92,8.90] |
| | IH | 8.45 | [8.27,8.59] | [8.08,8.78] | [7.92,8.94] |
| $\theta_{23}$ (°) | NH | 40.69 | [39.82,41.89] | [38.93,43.29] | [38.10,51.66] |
| | IH | 42.42 | [40.23,42.02] | [39.18,44.02] | [38.29,52.90] |
| $\delta$ (°) | NH | 248.40 | [212.40,289.80] | [180.00,342.00] | [0.00,30.60] |
| | IH | 235.80 | [201.60,291.60] | [165.60,338.40] | [0.27,0.00] |

Table 1: Allowed 1-2-3σ-ranges for the neutrino oscillation parameters: mixing angles, Dirac phase $\delta$, mass-square differences together with the $R_\nu$ parameter, taken from the global fit to neutrino oscillation data. The quantities $\delta m^2$, $\Delta m^2$ and $R_\nu$ are respectively defined as $m_2^2 - m_1^2$, $|m_3^2 - (m_1^2 + m_2^2)/2|$ and $\delta m^2/\Delta m^2$. Normal and Inverted Hierarchies are respectively denoted by NH and IH.
For the non oscillation parameters $\Sigma$, $m_{ee}$ and $m_{e}$, we adopt the ranges reported in the recent reference [12] for the first two, while for $m_{e}$ we use more stringent values found in the earlier reference [13]:

\[
\begin{aligned}
\Sigma &< 0.7 \text{ eV}, \\
m_{ee} &< 0.3 \text{ eV}, \\
m_{e} &< 1.8 \text{ eV}.
\end{aligned}
\]  

(8)

### 3 Texture of one-vanishing subtrace

We denote by $C_{ij}$ the texture where the subtrace corresponding to the $ij$th element (i.e. the trace of the sub-matrix obtained by deleting the $i$th row and the $j$th column of $M_\nu$) is equal to zero. We have six possibilities of having one subtrace vanishing. Let the diagonal elements of the trace-free submatrix corresponding to $C_{ij}$ be the elements at the $(a, b)$ and $(c, d)$ entries of $M_\nu$, then the vanishing subtrace condition is written as:

\[
M_{\nu \, ab} + M_{\nu \, cd} = 0,
\]  

(9)

then we have

\[
\sum_{\ell=1}^{3} (U_{a\ell}U_{b\ell} + U_{c\ell}U_{d\ell}) \lambda_{\ell} = 0.
\]  

(10)

with $\lambda_{1} = m_{1}e^{2i\rho}$, $\lambda_{2} = m_{2}e^{2i\sigma}$ and $\lambda_{3} = m_{3}$. This leads to:

\[
\begin{aligned}
\frac{m_{1}}{m_{3}} &= \frac{\text{Re}(A_{3})\text{Im}(A_{2}e^{2i\sigma}) - \text{Re}(A_{2}e^{2i\sigma})\text{Im}(A_{3})}{\text{Im}(A_{1}e^{2i\rho})\text{Re}(A_{2}e^{2i\sigma}) - \text{Re}(A_{1}e^{2i\rho})\text{Im}(A_{2}e^{2i\sigma})}, \\
\frac{m_{2}}{m_{3}} &= \frac{\text{Im}(A_{3})\text{Re}(A_{1}e^{2i\rho}) - \text{Re}(A_{3})\text{Im}(A_{1}e^{2i\rho})}{\text{Im}(A_{1}e^{2i\rho})\text{Re}(A_{2}e^{2i\sigma}) - \text{Re}(A_{1}e^{2i\rho})\text{Im}(A_{2}e^{2i\sigma})},
\end{aligned}
\]  

(11)

where $A_{\alpha}$ is defined as,

\[
A_{\alpha} = (U_{a\alpha}U_{b\alpha} + U_{c\alpha}U_{d\alpha}), \quad \alpha = 1, 2, 3.
\]  

(12)

We see that knowing the mixing and phase angles we can get mass ratios. Considering now

\[
m_{3} = \sqrt{\frac{\delta m^{2}}{(m_{4}/m_{3})^{2}}}, \quad m_{1} = m_{3} \times \frac{m_{1}}{m_{3}}, m_{2} = m_{3} \times \frac{m_{2}}{m_{3}}
\]  

(13)

we see that knowing $\delta m^{2}$ will allow us now to compute the mass spectrum and all the neutrino observables. Thus our input parameters will be the seven parameters (three mixing angles + three phase angles + solar mass squared difference) which for the texture imposing one complex condition (two real conditions) allow us to determine the 9 degrees of freedom of the neutrino mass matrix. We then can compute all the observable quantities and test the experimental bounds in Table (1) of $\Delta m^{2}$ and in Eqs. (8) of the remaining mass bounds, and draw correlation plots of the accepted points.

Also, one should investigate the possibility, for each pattern, to have singular (non-invertible) mass matrix. The viable singular mass matrix is characterized by one of the two masses ($m_{1}$ for N hierarchy, and $m_{3}$ for I hierarchy) being equal to zero, as compatibility with the data prevents the simultaneous vanishing of two masses:
The vanishing of \(m_1\) together with Eqs. (3,10,12) imply that the mass spectrum of \(m_2\) and \(m_3\) takes the values \(\sqrt{\delta m^2}\) and \(\sqrt{\Delta m^2 + \delta m^2/2}\) respectively, and we get

\[
\Delta m^2 = \delta m^2 \left( \left| \frac{A_2}{A_3} \right|^2 - \frac{1}{2} \right),
\]

\[
e^{2i\sigma} = -\frac{A_3 m_3}{A_2 m_2}.
\]

The vanishing of \(m_3\) together with Eqs. (3,10,12) imply that the mass spectrum of \(m_2\) and \(m_1\) takes the values \(\sqrt{\Delta m^2 + \delta m^2/2}\) and \(\sqrt{\Delta m^2 - \delta m^2/2}\) respectively, and we get

\[
\Delta m^2 = \frac{1}{2} \delta m^2 \left( \left| \frac{A_1}{A_2} \right|^2 + 1 \right),
\]

\[
e^{2i(\rho-\sigma)} = -\frac{A_2 m_2}{A_1 m_1}.
\]

4 Phenomenological analysis for non-singular textures

The parameter space is seven-dimensional representing the parameters \((\theta_{12}, \theta_{13}, \theta_{23}, \delta, \rho, \sigma, \delta m^2)\) within their allowed experimental ranges, where we throw \(N\) points uniformly in the corresponding parameter space and test using the Eqs. (11,13) first to check the hierarchy type, then to see whether or not the bounds of \(\Delta m^2\) with those of Eq. (8) are satisfied. Since the experimental bounds stated in Table (1) are not identical for the two types of hierarchy, then the parameter spaces in both cases are different, and one is obliged to repeat the sampling in the two cases, imposing the desired type of hierarchy with the other experimental bounds on the accepted points. The number of points \(N\) needed for a statistically significant sampling is found be at least of the order \(10^7 - 10^{10}\).

In each of the following subsections, labeled by the textures \(C_{ij}\), and for each corresponding pattern we provide the analytic expressions of the quantities \(A_{ij}\), defined in Eq.(12), which characterize the pattern. We find that all the textures accommodate data for all types of hierarchy and at all statistical levels. All various predictions concerning the ranges spanned by mixing angles, phase angles, neutrino masses, \(m_e\), \(m_{\mu}\) and \(J\) are summarized in Table (2). No signature is apparent in the case of normal ordering for the spanned ranges of neutrino masses presented in Table (2). However, in the case of inverted ordering of the neutrino masses, we see that \(m_3\) can reach a vanishing value for the textures \(C_{12}, C_{13}\) at all \(\sigma\)-levels, and only at 2-3-\(\sigma\)-levels for the textures \(C_{22}\) and \(C_{33}\). In contrast, \(m_3\) is never vanishing for the textures \(C_{11}\) and \(C_{23}\). Thus, the textures \(C_{12}, C_{13}, C_{22}\) and \(C_{33}\) are predicted to allow for singular mass matrix, as will be shown later to be the case. The ranges spanned by the parameter \(J\), in Table (2), show that \(J\) at the 1-2-\(\sigma\)-levels for normal ordering and 1-\(\sigma\)-level for inverted ordering is negative in all textures, which puts the Dirac phase \(\delta\) in the third and fourth quarters. Also from Table (2), the ranges spanned by the phase angles \((\rho, \sigma)\) indicate that for the texture \(C_{12}\) in case of normal hierarchy and at the 2\(\sigma\)-level there are gaps \((\sigma \notin [90^0, 150^0]\) and \(\rho \notin [34^0, 101^0]\)), and a similar gap \((\rho \notin [0^0, 18^0])\) for the texture \(C_{13}\) which becomes \((\rho \notin [0^0, 5^0])\) at the 3\(\sigma\)-level. However, in the case of inverted ordering, we see at all levels that the phase \(\rho\) for the texture \(C_{23}\) is bound to be in the interval \(([60^0, 120^0])\).

We present for each texture with either hierarchy type the neutrino mass matrix obtained at one representative point chosen from the points accepted out of those generated randomly.
in the corresponding parameter space at the 3-σ-level. The choice of the representative point is made in such a way to be close as possible to the best fit values for mixing and Dirac phase angles.

Finally, we plot all the possible correlations at the 2-σ-level. We show for each texture with either ordering twenty correlations. All correlations for each texture are organized in a single figure divided into left and right panels. The left panel of the figure consists of two columns where the first (second) column is devoted for normal (inverted) hierarchy and shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of \(\delta\) with \((J, m_{ee}, \text{LNM})\) (LNM=least neutrino mass) and finally the correlation \((m_3, m_{23} \equiv \frac{m_2}{m_3})\).

On the other hand, we follow for the right panel of the figure the same division strategy as in the left one, but each column includes all the nine inter-correlations between the phase angles and the mixing angles, and the correlation \((m_3, m_{21} \equiv \frac{m_2}{m_1})\). For the sake of convenience and easy referencing, each subfigure is labeled by three letters which indicate the vertical positioning \((a, b, c, \cdots)\), the type of ordering \((N=\text{Normal}, I=\text{Inverted})\) and the paneling \((L=\text{Left}, R=\text{Right})\). The last row in the figure thus gives information on the severity of the mass hierarchy.

Irrespective of the ordering (normal or inverted), we find in all the textures a sinusoidal correlation between \((\delta, J)\) which is a direct consequence of Eq.(7) where \(J\) depends on mixing angles and Dirac phase \(\delta\). The variations due to the mixing angles in this relation are tiny because of the tight range allowed for the mixing angles, and thus \(J \propto \sin \delta\). The appearing sinusoidal curve is not a full sine curve which would have covered a complete cycle, rather it is a portion depending on the admissible range for \(\delta\). Another generic feature that we find is the quasi degeneracy of the first two neutrino masses characterized by \(m_1 \approx m_2\).

In the case of normal ordering, we see, for the textures \(C_{12}\) and \(C_{13}\), sizable forbidden bands for both \(\sigma\) and \(\rho\) that tend to diminish as the statistical level increases, and a quasi degenerate spectrum for all neutrino masses with \((0.7 \leq m_{23} < 1)\). As to the textures \(C_{11}\) an \(C_{22}\), we see that there remain persistent forbidden bands for \((\sigma, \rho)\) at all statistical levels, and that we can have a mild or moderate mass hierarchy characterized by \((0.4 \leq m_{23} < 1)\) in texture \(C_{22}\), whereas we have a quasi degenerate spectrum for all masses with \(m_{23} \approx 1\) in texture \(C_{11}\). Moreover, for the latter texture \(C_{11}\), we find two ribbons for the correlation \((\delta, \sigma)\). Regarding the texture \(C_{23}\), we see that there are forbidden bands for \(\rho\), and that the mass hierarchy can be mild or moderate \((0.4 \leq m_{23} \leq 0.9)\). This situation repeats itself for the texture \(C_{33}\) where we have \((0.35 \leq m_{23} \leq 0.9)\).

In the case of inverted ordering, we find for the texture \(C_{11}\) two ribbons for the correlation \((\delta, \sigma)\) and a mild or moderate mass hierarchy characterized by \((1 < m_{23} \leq 3)\). For the textures \(C_{12}, C_{13}\) and \(C_{22}\), we may get an acute hierarchy reaching a strength \(m_{23} \approx 10^4\) for \(C_{12}\) and \(m_{23} \approx 10^3\) for both \(C_{13}\) and \(C_{22}\). We get for the mass spectrum a mild hierarchy characterized by \((1.2 \leq m_{23} \leq 3)\) in the texture \(C_{23}\), where, in addition, we find forbidden bands for \((\sigma, \rho)\). Finally for the texture \(C_{33}\), we have again forbidden bands for \((\sigma, \rho)\), but the hierarchy can be severe reaching a strength \(m_{23} \approx 10^4\).
| Parameter | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ | $\nu_7$ | $\nu_8$ | $\nu_9$ | $\nu_{10}$ | $\nu_{11}$ | $\nu_{12}$ | $\nu_{13}$ | $\nu_{14}$ | $\nu_{15}$ | $\nu_{16}$ | $\nu_{17}$ | $\nu_{18}$ | $\nu_{19}$ | $\nu_{20}$ | $\nu_{21}$ | $\nu_{22}$ | $\nu_{23}$ | $\nu_{24}$ | $\nu_{25}$ | $\nu_{26}$ | $\nu_{27}$ | $\nu_{28}$ | $\nu_{29}$ |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1        | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   | 0.00   |
| 2        | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   | 0.02   |
| 3        | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   | 0.04   |
| 4        | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   | 0.06   |
| 5        | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   | 0.08   |
| 6        | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   | 0.10   |
| 7        | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   | 0.12   |
| 8        | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   | 0.14   |
| 9        | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   | 0.16   |
| 10       | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   | 0.18   |
| 11       | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   | 0.20   |
| 12       | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   | 0.22   |

Table 2: The various prediction for the patterns of one vanishing subtrace textures designated by $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}$ and $C_{33}$. 
4.1 Pattern C\textsubscript{11}: Vanishing of $M_{\nu \nu 22} + M_{\nu \nu 33}$

The relevant expressions for $A_1$, $A_2$ and $A_3$, as defined in Eq.\[12\] for this pattern, are

\[
\begin{align*}
A_1 &= \left( c_{12} s_{23} s_{13} + c_{12} c_{23} e^{-i\delta} \right)^2 + \left( c_{12} c_{23} s_{13} - s_{12} s_{23} e^{-i\delta} \right)^2, \\
A_2 &= \left( s_{12} s_{23} s_{13} - c_{12} c_{23} e^{-i\delta} \right)^2 + \left( s_{12} c_{23} s_{13} + c_{12} s_{23} e^{-i\delta} \right)^2, \\
A_3 &= c_{13}^2. 
\end{align*}
\]

(16)

For a representative point with normal ordering, we take $\theta_{12} = 33.2327^\circ$, $\theta_{23} = 41.7746^\circ$, $\theta_{13} = 8.5625^\circ$, $\delta = 189.5139^\circ$, $\rho = 91.0971^\circ$, $\sigma = 102.9376^\circ$, $m_1 = 0.2162 \text{ eV}$, $m_2 = 0.21642 \text{ eV}$, $m_3 = 0.22212 \text{ eV}$, $m_\ell = 0.21642 \text{ eV}$, $m_{ee} = 0.20292 \text{ eV}$ with the corresponding mass matrix (in eV):

\[
M_\nu = \begin{pmatrix}
-0.2001 - 0.0334i & 0.0400 + 0.0333i & 0.0495 - 0.0230i \\
0.0400 + 0.0333i & -0.0234 - 0.0059i & 0.2112 - 0.0014i \\
0.0495 - 0.0230i & 0.2112 - 0.0014i & 0.0234 + 0.0059i
\end{pmatrix}.
\]

(17)

For an inverted hierarchy representative point we take $\theta_{12} = 33.8335^\circ$, $\theta_{23} = 42.3044^\circ$, $\theta_{13} = 8.7834^\circ$, $\delta = 246.8983^\circ$, $\rho = 51.0968^\circ$, $\sigma = 150.5728^\circ$, $m_1 = 0.0553 \text{ eV}$, $m_2 = 0.0560 \text{ eV}$, $m_3 = 0.0236 \text{ eV}$, $m_\ell = 0.0550 \text{ eV}$, $m_{ee} = 0.0220 \text{ eV}$ with the corresponding mass matrix (in eV):

\[
M_\nu = \begin{pmatrix}
0.0014 + 0.0219i & 0.0286 + 0.0239i & -0.0214 - 0.0263i \\
0.0286 + 0.0239i & -0.0076 - 0.0055i & 0.0225 + 0.0000i \\
-0.0214 - 0.0263i & 0.0225 + 0.0000i & 0.0076 + 0.0055i
\end{pmatrix}.
\]

(18)

We see, from Table(2), that $m_3$ does not approach a vanishing value in inverted type which indicates that no corresponding singular matrix exists. We see also that $J$ at 1-2$\sigma$-levels for normal ordering and 1$\sigma$ for inverted ordering is negative so the corresponding $\delta$ is at third or fourth quarters. For normal ordering, the allowed ranges for $\rho$ and $\sigma$ tend to increases as the statistical level increases reaching $[46.41^\circ, 131.49^\circ]$ ([47.41$^\circ$, 135.33$^\circ$]) for $\rho$ ($\sigma$) at 3$\sigma$-level.

For the plots of Fig.\[1\], two ribbons for the correlation ($\delta, \sigma$) exist for both types of hierarchy. The mass spectrum has a moderate mass hierarchy characterized by ($1 < m_{23} \leq 3$) in the inverted ordering. In contrast, the mass spectrum is quasi degenerate in the case of normal ordering where $m_1 \approx m_2 \approx m_3$. 

8
Figure 1: Pattern $C_{11} \equiv M_{e22} + M_{e33} = 0$ for non-singular mass matrices: The left panel (the left two columns) shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of $\delta$ with $J$, $m_{ee}$, LNM (LNM=least neutrino mass) and finally the correlation $(m_{32}, m_{23} \equiv \frac{m_{23}}{m_{ee}})$ for normal (N) and Inverted (I) hierarchy. The right panel (the right two columns) shows all the nine inter-correlations between phase angles and mixing angles, and the correlation $(m_{32}, m_{23} \equiv \frac{m_{23}}{m_{ee}})$ for normal (N) and Inverted (I) hierarchy.
4.2 Pattern C_{12}: Vanishing of $M_{\nu_{21}} + M_{\nu_{33}}$

The relevant expressions for $A_1$, $A_2$ and $A_3$, as defined in Eq. (12) for this pattern, are

$$A_1 = - \left( c_{12} s_{23} s_{13} + s_{12} c_{23} e^{-i\delta} \right) c_{12} c_{13} + \left( c_{12} c_{23} s_{13} - s_{12} s_{23} e^{-i\delta} \right)^2,$$

$$A_2 = - \left( s_{12} s_{23} s_{13} - c_{12} c_{23} e^{-i\delta} \right) s_{12} c_{13} + \left( s_{12} c_{23} s_{13} + c_{12} s_{23} e^{-i\delta} \right)^2,$$

$$A_3 = s_{13} s_{23} c_{13} + c_{23}^2 c_{13}. \quad (19)$$

For a representative point with normal ordering, we take $\theta_{12} = 33.7367^\circ$, $\theta_{23} = 41.7468^\circ$, $\theta_{13} = 8.4134^\circ$, $\delta = 312.5765^\circ$, $\rho = 151.9557^\circ$, $\sigma = 63.7910^\circ$, $m_1 = 0.0458$ eV, $m_2 = 0.0466$ eV, $m_3 = 0.0684$ eV, $m_e = 0.0466$ eV, $m_{ee} = 0.0178$ eV, with the corresponding mass matrix (in eV):

$$M_{\nu} = \begin{pmatrix}
0.0102 - 0.0146i & -0.0255 + 0.0052i & 0.0343 - 0.0017i \\
-0.0255 + 0.0052i & 0.0288 - 0.0081i & 0.0404 + 0.0062i \\
0.0343 - 0.0017i & 0.0404 + 0.0062i & 0.0255 - 0.0052i 
\end{pmatrix}. \quad (20)$$

For an inverted hierarchy representative point we take $\theta_{12} = 33.2436^\circ$, $\theta_{23} = 41.4914^\circ$, $\theta_{13} = 8.6242^\circ$, $\delta = 236.7486^\circ$, $\rho = 149.9638^\circ$, $\sigma = 123.9278^\circ$, $m_1 = 0.0507$ eV, $m_2 = 0.0515$ eV, $m_3 = 0.0115$ eV, $m_e = 0.0504$ eV, $m_{ee} = 0.0456$ eV with the corresponding mass matrix (in eV):

$$M_{\nu} = \begin{pmatrix}
0.0118 - 0.0440i & 0.0093 - 0.0076i & -0.0083 + 0.0156i \\
0.0093 - 0.0076i & -0.0194 + 0.0154i & 0.0255 - 0.0121i \\
-0.0083 + 0.0156i & 0.0255 - 0.0121i & -0.0093 + 0.0076i 
\end{pmatrix}. \quad (21)$$

We see, from Table (2), that $m_3$ can reach zero in inverted type, so we expect a possible singular texture existing. Again $J$ at 1-2$\sigma$-levels for normal ordering and 1$\sigma$ for inverted ordering is negative so the corresponding $\delta$ is at third or fourth quarters. For normal ordering, at 1$\sigma$-level there is a gap $[6^\circ, 103^\circ]$ $([56^\circ, 157^\circ])$ for $\rho$ ($\sigma$) which becomes at 2$\sigma$-level $[34^\circ, 102^\circ]$ $([91^\circ, 152^\circ])$.

For the plots of Fig. (2) in normal ordering, we find large forbidden gaps for $\rho$ and $\sigma$ and a quasi degenerate mass spectrum where $0.6 \leq m_{23} \leq 0.95$. As to the plots of Fig. (2) in inverted type, we see that we may get an acute hierarchy with $m_{23}$ reaching up to $10^4$ which reveals the possibility of vanishing $m_3$. 

10
Figure 2: Pattern $C_{12} \equiv M_{21} + M_{33} = 0$ for non-singular mass matrices: The left panel (the left two columns) shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of $\delta$ with $J, m_{ee}$, LNM (LNM=least neutrino mass) and finally the correlation $(m_3, m_{23} \equiv \frac{m_3}{m_2})$ for normal (N) and Inverted (I) hierarchy. The right panel (the right two columns) shows all the nine inter-correlations between phase angles and mixing angles, and the correlation $(m_3, m_{21} \equiv \frac{m_3}{m_1})$ for normal (N) and Inverted (I) hierarchy.
4.3 Pattern C_{13}: Vanishing of $M_{\nu 21} + M_{\nu 23}$

The relevant expressions for $A_1$, $A_2$ and $A_3$, as defined in Eq. (12) for this pattern, are

$$
\begin{align*}
A_1 &= (c_{12}s_{23}s_{13} + s_{12}c_{23}e^{-i\delta})
(s_{12}c_{23}s_{13} - c_{12}c_{13} - s_{12}s_{23}e^{-i\delta}), \\
A_2 &= (s_{12}s_{23}s_{13} - c_{12}c_{23}e^{-i\delta})
(s_{12}c_{23}s_{13} - s_{12}s_{13} - c_{12}s_{23}e^{-i\delta}), \\
A_3 &= c_{13}s_{23}(c_{23}c_{13} + s_{13}).
\end{align*}
$$

For a representative point with normal ordering, we take $\theta_{12} = 33.8222^\circ$, $\theta_{23} = 40.4289^\circ$, $\theta_{13} = 8.7721^\circ$, $\delta = 243.7429^\circ$, $\rho = 148.0834^\circ$, $\sigma = 34.0333^\circ$, $m_1 = 0.1821$ eV, $m_2 = 0.1823$ eV, $m_3 = 0.1888$ eV, $m_e = 0.1823$ eV, $m_{ee} = 0.0987$ eV with the corresponding mass matrix (in eV):

$$
M_{\nu} = \begin{pmatrix}
0.0791 - 0.0590i & -0.0909 - 0.0491i & 0.0997 + 0.0538i \\
-0.0909 - 0.0491i & 0.1037 - 0.0459i & 0.0909 + 0.0491i \\
0.0997 + 0.0538i & 0.0909 + 0.0491i & 0.0911 - 0.0527i
\end{pmatrix}.
$$

Equally, for an inverted hierarchy we can take a representative point as follows: $\theta_{12} = 33.8850^\circ$, $\theta_{23} = 42.2823^\circ$, $\theta_{13} = 8.5649^\circ$, $\delta = 244.3791^\circ$, $\rho = 48.7884^\circ$, $\sigma = 57.5072^\circ$, $m_1 = 0.0644$ eV, $m_2 = 0.0650$ eV, $m_3 = 0.0421$ eV, $m_e = 0.0642$ eV, $m_{ee} = 0.0623$ eV with the corresponding mass matrix (in eV):

$$
M_{\nu} = \begin{pmatrix}
-0.0131 + 0.0609i & 0.0099 - 0.0112i & 0.0023 - 0.0022i \\
0.0099 - 0.0112i & 0.0508 - 0.0098i & -0.0099 + 0.0112i \\
0.0023 - 0.0022i & -0.0099 + 0.0112i & 0.0507 - 0.0098i
\end{pmatrix}.
$$

We see, from Table (2), that $m_3$ can reach zero in inverted type, so we expect a possible singular texture existing. Table (2) also reveals that $J$, at 1-2$\sigma$-levels for both normal and inverted ordering, is negative so the corresponding $\delta$ is in third or fourth quarters. For normal ordering, the ranges for $\rho$ ($\sigma$) are restricted to be $[42^\circ, 155^\circ]$ ($[17^\circ, 102^\circ]$) at 1$\sigma$-level, whereas they tend to be wider at 3$\sigma$-level covering $[6^\circ, 175^\circ]$ ($[0.01^\circ, 180^\circ]$).

For the plots of Fig. (3) in normal ordering, we find a quasi degenerate mass spectrum where ($0.65 \leq m_{23} \leq 0.95$). As to the plots of Fig. (3) in inverted type, we may get an acute hierarchy with $m_{23}$ reaching up to $10^3$, so a vanishing $m_3$ is possible.
Figure 3: Pattern $C_{13} \equiv M_{e21} + M_{e23} = 0$ for non-singular mass matrices: The left panel (the left two columns) shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of $\delta$ with $J, m_{ee}$, LNM (LNM=least neutrino mass) and finally the correlation ($m_3, m_{23} \equiv \sigma$) for normal (N) and Inverted (I) hierarchy. The right panel (the right two columns) shows all the nine inter-correlations between phase angles and mixing angles, and the correlation ($m_3, m_{21} \equiv \frac{\rho}{m_1}$) for normal (N) and Inverted (I) hierarchy.
4.4 Pattern C\textsubscript{22}: Vanishing of $M_{\nu_{11}} + M_{\nu_{33}}$

The relevant expressions for $A_1$, $A_2$ and $A_3$, as defined in Eq.(12) for this pattern, are

$$A_1 = c_{12}^2 c_{13}^2 + \left(c_{12} c_{23} s_{13} - s_{12} s_{23} e^{-i\delta}\right)^2,$$

$$A_2 = s_{12}^2 c_{13}^2 + \left(s_{12} c_{23} s_{13} + c_{12} s_{23} e^{-i\delta}\right)^2,$$

$$A_3 = s_{13}^2 + c_{23}^2 c_{13}^2. \quad (25)$$

For a representative point with normal ordering, we take $\theta_{12} = 33.8006^\circ$, $\theta_{23} = 40.7648^\circ$, $\theta_{13} = 8.4791^\circ$, $\delta = 300.9481^\circ$, $\rho = 81.5950^\circ$, $\sigma = 69.8454^\circ$, $m_1 = 0.0398$ eV, $m_2 = 0.0407$ eV, $m_3 = 0.0647$ eV, $m_e = 0.0408$ eV, $m_{ee} = 0.0372$ eV with the corresponding mass matrix (in eV):

$$M_{\nu} = \begin{pmatrix}
-0.0337 + 0.0157i & 0.0064 + 0.0032i & 0.0138 - 0.0059i \\
0.0064 + 0.0032i & 0.0252 - 0.0235i & 0.0327 + 0.0196i \\
0.0138 - 0.0059i & 0.0327 + 0.0196i & 0.0337 - 0.0157i
\end{pmatrix}. \quad (26)$$

For an inverted hierarchy representative point, we take $\theta_{12} = 33.5774^\circ$, $\theta_{23} = 42.7607^\circ$, $\theta_{13} = 8.7549^\circ$, $\delta = 281.0485^\circ$, $\rho = 99.6048^\circ$, $\sigma = 167.4130^\circ$, $m_1 = 0.0749$ eV, $m_2 = 0.0754$ eV, $m_3 = 0.0571$ eV, $m_e = 0.0747$ eV, $m_{ee} = 0.0372$ eV with the corresponding mass matrix (in eV):

$$M_{\nu} = \begin{pmatrix}
-0.0263 - 0.0263i & 0.0201 + 0.0479i & -0.0011 - 0.0388i \\
0.0201 + 0.0479i & 0.0162 + 0.0104i & 0.0336 - 0.0196i \\
-0.0011 + 0.0388i & 0.0336 - 0.0196i & 0.0263 + 0.0263i
\end{pmatrix}. \quad (27)$$

We see, from Table[2], that $m_3$ can reach zero in inverted type, so we expect a possible singular texture existing. Again, from Table[2], $J$ at 1-2-$\sigma$-levels for normal ordering and 1-$\sigma$ for inverted ordering is negative so the corresponding $\delta$ is in third or fourth quarters. For normal ordering, values of $\rho$ are restricted to fall in the range $[52^\circ, 128^\circ]$ at the 3-$\sigma$-level.

For the plots, Fig.(4), in normal ordering, we find forbidden bands for $(\sigma, \rho)$ and a moderate mass hierarchy where $(0.45 \leq m_{23} < 1)$. As to the plots of Fig.(4) in inverted type, there are forbidden bands for $(\sigma, \rho)$ and the mass hierarchy can become acute with $m_{23}$ reaching up to $10^3$ making a vanishing $m_3$ possible.
Figure 4: Pattern $C_{22} \equiv M_{21,1} + M_{23,3} = 0$ for non-singular mass matrices. The left panel (the left two columns) shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of $\delta$ with $J, m_{ee}$, LNM (LNM=least neutrino mass) and finally the correlation $(m_3, m_{23} \equiv \frac{m_3}{m_2})$ for normal (N) and Inverted (I) hierarchy. The right panel (the right two columns) shows all the nine inter-correlations between phase angles and mixing angles, and the correlation $(m_3, m_{23} \equiv \frac{m_3}{m_2})$ for normal (N) and Inverted (I) hierarchy.
4.5 Pattern C$_{23}$: Vanishing of $M_{\nu 11} + M_{\nu 23}$

The relevant expressions for $A_1$, $A_2$ and $A_3$, as defined in Eq.(12) for this pattern, are

$$A_1 = c_{12}^2 c_{13}^2 + \left( c_{12} s_{23} s_{13} + s_{12} c_{23} e^{-i\delta} \right) \left( c_{12} c_{23} s_{13} - s_{12} s_{23} e^{-i\delta} \right),$$

$$A_2 = s_{12}^2 c_{13}^2 + \left( s_{12} s_{23} s_{13} - c_{12} c_{23} e^{-i\delta} \right) \left( s_{12} c_{23} s_{13} + c_{12} s_{23} e^{-i\delta} \right),$$

$$A_3 = s_{13}^2 + s_{23} c_{23} c_{13}^2. \tag{28}$$

For a representative point with normal ordering, we take $\theta_{12} = 33.4546^\circ$, $\theta_{23} = 42.2981^\circ$, $\theta_{13} = 8.4653^\circ$, $\delta = 248.6157^\circ$, $\rho = 94.3533^\circ$, $\sigma = 68.6630^\circ$, $m_1 = 0.0203$ eV, $m_2 = 0.0221$ eV, $m_3 = 0.0554$ eV, $m_e = 0.0222$ eV, $m_{ee} = 0.0175$ eV with the corresponding mass matrix (in eV):

$$M_\nu = \begin{pmatrix}
-0.0173 + 0.0024i & 0.0012 - 0.0013i & 0.0136 + 0.0007i \\
0.0012 - 0.0013i & 0.0361 + 0.0029i & 0.0173 - 0.0024i \\
0.0136 + 0.0007i & 0.0173 - 0.0024i & 0.0369 + 0.0020i
\end{pmatrix}. \tag{29}$$

For an inverted hierarchy representative point we take $\theta_{12} = 33.2679^\circ$, $\theta_{23} = 42.8064^\circ$, $\theta_{13} = 8.6838^\circ$, $\delta = 236.7459^\circ$, $\rho = 98.4416^\circ$, $\sigma = 2.2660^\circ$, $m_1 = 0.0573$ eV, $m_2 = 0.0579$ eV, $m_3 = 0.0288$ eV, $m_e = 0.0570$ eV, $m_{ee} = 0.0222$ eV with the corresponding mass matrix (in eV):

$$M_\nu = \begin{pmatrix}
-0.0198 - 0.0100i & -0.0214 + 0.0285i & 0.0299 - 0.0243i \\
-0.0214 + 0.0285i & 0.0122 - 0.0172i & 0.0198 + 0.0100i \\
0.0299 - 0.0243i & 0.0198 + 0.0100i & 0.0042 - 0.0042i
\end{pmatrix}. \tag{30}$$

We see, from Table(2), that $m_3$ can not reach zero, so we expect no viable corresponding singular pattern. Again, from Table(2), $J$ at 1-2$\sigma$-levels for normal ordering and $1\sigma$ for inverted ordering is negative so the corresponding $\delta$ is in third or fourth quarters. For both normal and inverted ordering, the phase $\rho$ is bound at all $\sigma$-levels to be nearly in the interval $([60^\circ,120^\circ])$.

For the plots, Fig.(5), in both normal and inverted ordering, we get an approximately degenerate spectrum characterized respectively by $0.4 \leq m_{23} \leq 0.9$ and $1.2 \leq m_{23} \leq 3$. The plots in Fig.(4), also reveal that the phase $\rho$ is bound to fall approximately in the interval $([60^\circ,120^\circ])$, while there are forbidden bands for the phase $\sigma$ for both type of hierarchies.
Figure 5: Pattern $C_{23} \equiv M_{21} + M_{23} = 0$ for non-singular mass matrices: The left panel (the left two columns) shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of $\delta$ with $J, m_{ee}$, LNM (LNM=least neutrino mass) and finally the correlation ($m_{32}, m_{23} \equiv \frac{m_{32}}{m_{23}}$) for normal (N) and Inverted (I) hierarchy. The right panel (the right two columns) shows all the nine inter-correlations between phase angles and mixing angles, and the correlation ($m_{32}, m_{23} \equiv \frac{m_{32}}{m_{23}}$) for normal (N) and Inverted (I) hierarchy.
4.6 Pattern C$_{33}$: Vanishing of $M_{\nu_{11}} + M_{\nu_{22}}$

The relevant expressions for $A_1$, $A_2$ and $A_3$, as defined in Eq.(12) for this pattern, are

\[
A_1 = c_{12}^2 c_{13}^2 + (c_{12} s_{23} s_{13} + s_{12} c_{23} e^{-i\delta})^2, \\
A_2 = s_{12}^2 c_{13}^2 + (s_{12} s_{23} s_{13} - c_{12} c_{23} e^{-i\delta})^2, \\
A_3 = s_{13}^2 + s_{23}^2 c_{13}^2. \quad (31)
\]

As for a normal type representative point, we take $\theta_{12} = 33.5935^\circ$, $\theta_{23} = 40.7528^\circ$, $\theta_{13} = 8.7162^\circ$, $\delta = 252.1164^\circ$, $\rho = 77.1818^\circ$, $\sigma = 164.3730^\circ$, $m_1 = 0.0530$ eV, $m_2 = 0.0537$ eV, $m_3 = 0.0736$ eV, $m_e = 0.0537$ eV, $m_{ee} = 0.0184$ eV with the corresponding mass matrix (in eV):

\[
M_\nu = \begin{pmatrix}
-0.0170 + 0.0072i & 0.0158 + 0.0354i & 0.0047 - 0.0320i \\
0.0158 + 0.0354i & 0.0170 - 0.0072i & 0.0456 - 0.0010i \\
0.0047 - 0.0320i & 0.0456 - 0.0010i & 0.0333 + 0.0073i
\end{pmatrix}. \quad (32)
\]

For an inverted type representative point, we can take $\theta_{12} = 33.0850^\circ$, $\theta_{23} = 42.6054^\circ$, $\theta_{13} = 8.7610^\circ$, $\delta = 221.0642^\circ$, $\rho = 123.4419^\circ$, $\sigma = 60.2387^\circ$, $m_1 = 0.0540$ eV, $m_2 = 0.0547$ eV, $m_3 = 0.0231$ eV, $m_e = 0.0537$ eV, $m_{ee} = 0.0300$ eV with the corresponding mass matrix (in eV):

\[
M_\nu = \begin{pmatrix}
-0.0221 - 0.0204i & -0.0148 - 0.0236i & 0.0231 + 0.0260i \\
-0.0148 + 0.0236i & 0.0221 + 0.0204i & 0.0041 - 0.0138i \\
0.0231 + 0.0260i & 0.0041 - 0.0138i & 0.0146 + 0.0072i
\end{pmatrix}. \quad (33)
\]

We see, from Table[2], that $m_3$ can reach zero in inverted type, so we expect a viable corresponding singular pattern. Again, from Table[2], $J$ at 1-$\sigma$-levels for normal ordering and 1$\sigma$ for inverted ordering is negative so the corresponding $\delta$ is in third or fourth quarters. For normal ordering, the values of the phase $\rho$ are restricted to fall in the range $[64^\circ, 126^\circ]$ at the 1-$\sigma$-level, and in $[56^\circ, 129^\circ]$ at the 2-$\sigma$-level, and in $[52^\circ, 130^\circ]$ at the 3-$\sigma$-level, but, in contrast, there is almost no restriction for $\sigma$. For inverted ordering, there is a restriction for the phase $\rho$ range: $[57^\circ, 132^\circ]$ at 1-$\sigma$-level, $[15^\circ, 170^\circ]$ at 2-$\sigma$-level and $[1.5^\circ, 173^\circ]$ at 3-$\sigma$-level. In contrast, there is a forbidden gap for $\sigma$ which is $[94^\circ, 142^\circ]$ at 1-$\sigma$-level, $[86^\circ, 106^\circ]$ at 2-$\sigma$-level and $[83^\circ, 100^\circ]$ at 3-$\sigma$-level.

For the plots, Fig.(6), in normal ordering, we find narrow forbidden bands for ($\rho$) and a mild mass hierarchy characterized by ($0.35 \leq m_{23} \leq 0.9$). As to the plots, Fig.(6), in inverted type, we also find forbidden bands for both $\rho$ and $\sigma$, but the hierarchy can be severe with $m_{23}$ reaching up to $10^4$ indicating the possibility of vanishing $m_3$. 

Figure 6: Pattern \( C_{33} \equiv M_{311} + M_{222} = 0 \) for non-singular mass matrices: The left panel (the left two columns) shows three correlations amidst the mixing angles, three correlations amidst the phase angles and three correlations of \( \delta \) with \( J, m_{ee}, \) LNM (LNM=least neutrino mass) and finally the correlation (\( m_{23}, m_{23} \equiv \frac{m_2}{m_4} \)) for normal (N) and Inverted (I) hierarchy. The right panel (the right two columns) shows all the nine inter-correlations phase angles and mixing angles, and the correlation (\( m_{31}, m_{23} \equiv \frac{m_3}{m_4} \)) for normal (N) and Inverted (I) hierarchy.
5 Phenomenological analysis for singular textures

Experimental data allow for one neutrino mass to vanish. The Eqs. (11) are not valid when the neutrino mass matrix is singular, where instead we should use Eq. (14, 15) to calculate the mass spectrum given the mixing and phase (Dirac and one Majorana) angles and the solar squared mass splitting. The analytic formulae we get are simpler than when the mass matrix is invertible, but still they are too cumbersome to write them down, even if one restricts to first order in powers of $s_z$.

The mass spectrum in the normal ordering is given by

$$m_1 = 0, \quad m_2 = \sqrt{\delta m^2}, \quad m_3 = \sqrt{\Delta m^2 + \delta m^2 / 2}, \quad \Delta m^2 = \delta m^2 \left(\frac{A_2}{A_3} - \frac{1}{2}\right).$$

(34)

Numerically, no singular texture of normal type could accommodate data.

In the inverted ordering the mass spectrum is given by

$$m_3 = 0, \quad m_1 = \sqrt{\Delta m^2 - \delta m^2 / 2}, \quad m_2 = \sqrt{\Delta m^2 + \delta m^2 / 2}, \quad \Delta m^2 = \frac{1}{2} \delta m^2 \left(\frac{A_1}{A_2}^2 + 1\right) - \left(\frac{A_1}{A_2}^2 - 1\right).$$

(35)

Four “acceptable” textures ($C_{12}, C_{13}, C_{22}, C_{33}$) are found able to accommodate data.

We follow the same methodology in generating numerical results (random sampling) and the same nomenclature in presenting results as in the case of non-singular mass matrices. All various predictions concerning the ranges spanned by mixing angles, phase angles, neutrino masses, $m_e, m_{ee}$ and $J$ are summarized in Table (3). We note that the textures $C_{22}, C_{33}$ do not pass the experimental constraints at 1σ-level. We present for each viable singular texture the neutrino mass matrix obtained at one representative point chosen from the accepted points out of those generated randomly in the corresponding parameter space at the 3-σ-level. The choice of the representative point is made in such a way to be as close as possible to the best fit values for mixing and Dirac phase angles.

Briefly, we see that $J < 0$ at all σ-levels for the texture $C_{13}$, putting $\delta$ in the third and fourth quarters. The same applies for the texture $C_{12}$ at 1-2σ-levels, and for the texture $C_{22}$ at 2σ-level, specifying equally the $\delta$-quarters for these acceptable textures. Positive values for $J$ can be achieved at 3σ-level for the textures $C_{12}$ and $C_{22}$ and also at 1-2σ-levels for the texture $C_{33}$.

Finally, we plot for each texture the possible correlations at the 2σ-level showing eighteen correlations grouped into two panels. The left panel shows three correlations amongst the mixing angles, three correlations amongst the phase angles and two correlations of $\delta$ with $(J, m_{ee})$ and finally the correlation $(m_{12} = m_1/m_2, m_2)$. The right panel includes all the nine inter-correlations between phase-angles and mixing angles.

In all four acceptable textures, the mass spectrum is almost degenerate ($m_1 \approx m_2$), and there is a strong linear correlation between $(\rho, \sigma)$ depicting two linear ribbons of positive slope. Also, there is a linear correlation between $(J, \delta)$ in the four textures and this is due to the small allowed range for $\delta$ which renders the sine curve ($J \propto \sin \delta$) looking like a linear one. In this respect, especially clear is the positive (negative) slope in the texture $C_{22}$ ($C_{33}$).

5.1 Singular Pattern of $C_{12}$: Vanishing of $M_{\nu_{21}} + M_{\nu_{33}}$ and $m_3$

We see, from Table [3], that $J$ is negative at 1 − 2σ-levels and the corresponding $\delta$ is in the third quarter.
For a representative point we take with $m_3 = 0$: $\theta_{12} = 33.8683^\circ$, $\theta_{23} = 40.9412^\circ$, $\theta_{13} = 8.7098^\circ$, $\delta = 255.0672^\circ$, $\rho = 26.7494^\circ$, $\sigma = 174.1569^\circ$, $m_1 = 0.0490\text{ eV}$, $m_3 = 0.0498\text{ eV}$, $m_e = 0.0487\text{ eV}$, $m_{ee} = 0.0417\text{ eV}$ with the corresponding mass matrix (in eV):

$$M_\nu = \begin{pmatrix}
0.0344 + 0.0235i & 0.0113 + 0.0086i & -0.0168 - 0.0122i \\
0.0113 + 0.0086i & -0.0222 - 0.0167i & 0.0170 + 0.0127i \\
-0.0168 - 0.0122i & 0.0170 + 0.0127i & -0.0113 - 0.0086i
\end{pmatrix}. \quad (36)$$

For the plots, Fig.(7), when $J$ increases $\delta$ tends to decrease in a linear manner. A strong positive linear correlation between $(\rho, \sigma)$ exists with two ribbons. There is a forbidden gap for $m_{ee}$: $[0.0395, 0.0400]$ eV. The mass spectrum is almost degenerate ($m_1 \approx m_2$).
Figure 7: Pattern $C_{12}$ for singular mass matrices with inverted ordering: The left panel shows three correlations amidst the mixing angles, three correlations amidst the phase angles and two correlations of $\delta$ with $J, m_{ee}$ and finally the correlation ($m_{12} \equiv \frac{m_1}{m_2}$). The right panel shows all the nine correlations inter phase-angles and mixing angles. Angles (masses) are evaluated in degrees (eV).
5.2 Singular Pattern of $C_{13}$: Vanishing of $M_{\nu_{12}} + M_{\nu_{23}}$ and $m_3$

We see, from Table(3), that $J$ is negative at all levels and the corresponding $\delta$ is in the fourth quarter.

For a representative point we take with $m_3 = 0$: $\theta_{12} = 33.8148^\circ$, $\theta_{23} = 40.7781^\circ$, $\theta_{13} = 8.4919^\circ$, $\delta = 284.0999^\circ$, $\rho = 53.2226^\circ$, $\sigma = 85.4376^\circ$, $m_1 = 0.0496 \text{eV}$, $m_2 = 0.0504 \text{eV}$, $m_e = 0.0493 \text{eV}$, $m_{ee} = 0.0424 \text{eV}$ with the corresponding mass matrix (in eV):

$$M_{\nu} = \begin{pmatrix}
-0.0246 + 0.0346i & 0.0127 - 0.0187i & -0.0061 + 0.0093i \\
0.0127 - 0.0187i & 0.0118 - 0.0174i & -0.0127 + 0.0187i \\
-0.0061 + 0.0093i & -0.0127 + 0.0187i & 0.0122 - 0.0180i
\end{pmatrix}.$$  \hspace{1cm} (37)

For the plots in Fig.(8), a linear correlation between $J$ and $\delta$ exists where $J$ tends to increase as $\delta$ increases. The plots in Fig.(8) also reveal a strong linear correlation between $(\rho, \sigma)$ with two narrow ribbons exists. Also, there is a negative-slope linear dependence between $(\delta, \theta_{12})$. The neutrino masses are almost degenerate ($m_1 \approx m_2$).
Figure 8: Pattern C_{13} for singular mass matrices with inverted ordering: The left panel shows three correlations amidst the mixing angles, three correlations amidst the phase angles and two correlations of $\delta$ with $J$, $m_{ee}$ and finally the correlation ($m_{12} \equiv \frac{m_{1}}{m_{2}}$, $m_{2}$). The right panel shows all the nine correlations inter phase-angles and mixing angles. Angles (masses) are evaluated in degrees (eV).
5.3 Singular Pattern of $C_{22}$: Vanishing of $M_{\nu 11} + M_{\nu 33}$ and $m_3$

We see, from Table(3), that at 1\textsigma-level, the singular pattern is not viable. We also note that $J$ is negative at 2\textsigma-level and the corresponding $\delta$ is in the fourth quarter.

For a representative point we take with $m_3 = 0$: $\theta_{12} = 34.5161^\circ$, $\theta_{23} = 50.8655^\circ$, $\theta_{13} = 8.5346^\circ$, $\delta = 339.6445^\circ$, $\rho = 124.3227^\circ$, $\sigma = 26.7978^\circ$, $m_1 = 0.0492\,\text{eV}$, $m_2 = 0.0499\,\text{eV}$, $m_e = 0.0489\,\text{eV}$, $m_{ee} = 0.0180\,\text{eV}$ with the corresponding mass matrix (in eV):

$$M_{\nu} = \begin{pmatrix} -0.0026 - 0.0178i & 0.0046 + 0.0304i & -0.0050 - 0.0331i \\ 0.0046 + 0.0304i & 0.0000 + 0.0007i & -0.0011 - 0.0081i \\ -0.0050 - 0.0331i & -0.0011 - 0.0081i & 0.0026 + 0.0178i \end{pmatrix}. \quad (38)$$

For the plots in Fig.(9), $J$ and $\delta$ are correlated quasi linearly and positively. A strong linear correlation with two ribbons between $(\rho, \sigma)$ exists. The mass spectrum is almost degenerate ($m_1 \approx m_2$).
Figure 9: Pattern $C_{22}$ for singular mass matrices with inverted ordering: The left panel shows three correlations amidst the mixing angles, three correlations amidst the phase angles and two correlations of $\delta$ with $J, m_{ee}$ and finally the correlation $(m_{12} \equiv m_1, m_2)$. The right panel shows all the nine correlations inter phase-angles and mixing angles. Angles (masses) are evaluated in degrees (eV).
5.4 Singular Pattern of $C_{33}$: Vanishing of $M_{\nu_{11}} + M_{\nu_{22}}$ and $m_3$

As in the previous case $C_{22}$, the singular pattern $C_{33}$ is not viable at 1\,$\sigma$-level as evident from Table(3). In contrast to the previous case $C_{22}$, $J$ can assume positive as well as negative values at 2\,$\sigma$-level and the corresponding $\delta$ lies in the second and third quarters.

For a representative point we take with $m_3 = 0$: $\theta_{12} = 35.9702^\circ$, $\theta_{23} = 42.1759^\circ$, $\theta_{13} = 8.4675^\circ$, $\delta = 204.6858^\circ$, $\rho = 127.4906^\circ$, $\sigma = 45.5467^\circ$, $m_1 = 0.0487$\,eV, $m_2 = 0.0495$\,eV, $m_e = 0.0485$\,eV, $m_{ee} = 0.0159$\,eV with the corresponding mass matrix (in eV):

$$M_\nu = \begin{pmatrix} -0.0084 - 0.0135i & -0.0169 - 0.0275i & 0.0170 + 0.0276i \\ -0.0169 - 0.0275i & 0.0084 + 0.0135i & -0.0042 - 0.0067i \\ 0.0170 + 0.0276i & -0.0042 - 0.0067i & 0.0004 + 0.0005i \end{pmatrix}. \quad (39)$$

For the plots in Fig.10, we see that $(J, \delta)$ are strongly correlated linearly and negatively. A strong linear correlation between $(\rho, \sigma)$ exists with two ribbons. The masses $(m_1, m_2)$ are almost degenerate.
Figure 10: Pattern $C_{33}$ for singular mass matrices with inverted ordering: The left panel shows three correlations amongst the mixing angles, three correlations amongst the phase angles and two correlations of $\delta$ with $J, m_{ee}$ and finally the correlation ($m_{12} \equiv m_{13}, m_{23}$). The right panel shows all the nine correlations inter phase-angles and mixing angles. Angles (masses) are evaluated in degrees (eV).
Table 3. The various predictions for the patterns of one vanishing subtrace textures and vanishing $m_\nu$ designated by $C_{zx}, C_{za}, C_{zz}$ and $C_{zs}$. 

The table presents data related to the models designated by $C_{zx}, C_{za}, C_{zz}$ and $C_{zs}$, showing various predictions for the patterns of one vanishing subtrace textures and vanishing $m_\nu$. The table includes columns for different models and parameters, with values for each corresponding to the various predictions.
6 Theoretical Realization of the textures

We present in this section, theoretical realizations of some of the one vanishing subtrace textures, where symmetry assignments at high scale impose this texture in the “gauge” basis. However, one way to find these assignments is to start from another symmetry imposing zero textures and relate these two symmetries by a rotation. As to the symmetry responsible for imposing the zero elements at high scale, we can just follow the analysis of [5]. We shall find that four vanishing subtrace textures, out of six, are able to be amended by “rotating” zero-textures. In section 7, we explain the general strategy of relating the two symmetries, which would be of great help in this method of indirect realization. In section 8, we discuss the notion of flavor basis due to its paramount relevance into our study. In section 9, making use of “rotating” zero-textures, we adopt a type I seesaw scenario with discrete symmetry ($Z_8 \times Z_2$) in order to generate nonsingular vanishing subtrace textures. We repeat the work for singular vanishing subtrace textures in section 10, but with discrete symmetry ($Z_{12} \times Z_2$). In section 11, we present an implementation of one vanishing subtrace texture using type II seesaw scenario supplemented with ($Z_{12} \times Z_2$) discrete symmetry, and following the same strategy of ‘rotation’ from zero textures to vanishing subtrace. In section 12, we present a direct way of realization for type I seesaw scenario implementation with ($Z_6 \times Z_2$) discrete symmetry not related to zero textures. In section 13, we pursue the direct method of realization but now for type II seesaw scenario implementation with ($Z_{12} \times Z_2$) discrete symmetry. One can consider these outlined sections as an exercise in model building aiming to show that the studied texture of vanishing subtrace can be generated at the Lagrangian level by symmetry considerations in which the symmetry is exact but broken spontaneously. As a final remark, the presented method of ‘rotation’ is applicable to any specific pattern which can be generated from the zero-texture pattern via a unitary transformation.

7 ‘Rotating’ Strategy: from zero-texture to vanishing subtrace texture

We need to find a unitary matrix $S$ which when acted on the symmetric neutrino matrix:

$$M_\nu = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix}.$$  \hspace{1cm} (40)

gives the combination which define the subtrace patterns $[(C_{11}) : (D + F)]$, $[(C_{12}) : (B + F)]$, $[(C_{13}) : (B + E)]$, $[(C_{22}) : (A + F)]$, $[(C_{23}) : (A + E)]$, and $[(C_{33}) : (A + D)]$ in one of the elements of the transformed matrix ($\tilde{M} = S^T \ M_\nu \ S$), where $M_\nu$ is the effective Majorana neutrino mass matrix. More specifically, for the texture $C_{33}$, if we take the unitary matrix

$$S_{33} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$ \hspace{1cm} (41)

then we find that

$$S_{33} \ M_\nu \ S_{33}^T = -\frac{1}{2} \begin{pmatrix} A + 2iB - D & A + D & -\sqrt{2} \ (iC - E) \\ A + D & A - 2iB - D & -\sqrt{2} \ (iC + E) \\ -\sqrt{2} \ (iC - E) & -\sqrt{2} \ (iC + E) & -2F \end{pmatrix}.$$ \hspace{1cm} (42)

30
and so the combination \((A + D)\) appears in the element \((12)\) of the transformed matrix

\[
M_{\nu 0} = S_{33} \, M_{\nu} \, S_{33}^T
\]  

(43)

Thus, if by some symmetry \(S_{Y0}\) applied on the transformed matrix \(M_{\nu 0}\) one can impose a zero element:

\[
S_{Y0}^T \, M_{\nu 0} \, S_{Y0} = M_{\nu 0} \Rightarrow M_{\nu 012} = 0,
\]  

(44)

then we see that we have

\[
S_{Y}^T \, M_{\nu} \, S_{Y} = M_{\nu} \Rightarrow M_{\nu 11} + M_{\nu 22} = 0
\]  

(45)

where the new symmetry implementing the vanishing subtrace of the texture \(C_{33}\) is

\[
S_{Y} = S_{33}^T \, S_{Y0} \, S_{33}^T.
\]  

(46)

Let’s define \(u^{ij}\) as the matrix resulting by swapping the \(i^\text{th}\) and the \(j^\text{th}\) columns of the identity matrix \((I)\). Then we have the properties:

\[
u^{ij} = \nu^{ijT} = u^{ij}, \quad u^{ij} \, u^{ij} = I
\]  

(47)

Then, for any matrix \(M\), we see that \((M \, u^{ij})\) swaps the \(i^\text{th}\) and \(j^\text{th}\) columns of \((M)\), whereas \((u^{ij} \, M)\) swaps the \(i^\text{th}\) and \(j^\text{th}\) rows of \((M)\). Note that \(u^{ij} \, M \, u^{ij}\) has the effect of swapping first the \((i^\text{th}\) and \(j^\text{th}\)\) columns, followed by the \((i^\text{th}\) and \(j^\text{th}\)\) rows, or the other way round. We note now that the six vanishing one subtrace textures can be divided into three classes

- **Class of textures** \(\{C_{11}, C_{22}, C_{33}\}\): In the sense that if I find a unitary transformation \(\tilde{S}\) giving me one of them, then directly I get the unitary transformation giving me the other two textures. This comes because

\[
(u^{13} \, M \, u^{13})_{22} = M_{22}, \quad (u^{13} \, M \, u^{13})_{11} = M_{33} \Rightarrow u^{13} \, \tilde{S}_{33} \, u^{13} = \tilde{S}_{11},
\]

where \(\tilde{S}_{ij}\) is a unitary matrix which, provided its action on \(M_{\nu}\) keeps the latter invariant, imposes the texture defined by the subtrace \((C_{i,j})\) (see Eq. \([45]\) where \(S_{Y}\) plays the role of \(\tilde{S}_{33}\)).

- **Class of textures** \(\{C_{13}\}\): Actually, we can take:

\[
S_{13} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}
\]  

(49)

because the \((B + E)\) combination appears in the \((1,2)\) element of

\[
S_{13} \, M_{\nu} \, S_{13}^T = \frac{1}{2} \begin{pmatrix} A + 2 \, C + F & \sqrt{3} \, (B + E) & F - A \\ \sqrt{2} \, (B + E) & 2D & \sqrt{2} \, (E - B) \\ F - A & \sqrt{2} \, (E - B) & A - 2 \, C + F \end{pmatrix}
\]  

(50)

- **Class of textures** \(\{C_{12}, C_{32}\}\):

\[
(u^{13} \, M \, u^{13})_{33} = M_{11}, \quad (u^{13} \, M \, u^{13})_{21} = M_{23} \Rightarrow u^{13} \, \tilde{S}_{12} \, u^{13} = \tilde{S}_{32}
\]  

(51)

However, one can algebraically show that the transformation \(S \, M_{\nu} \, S^T\) can not bring in the sole combination \(A + E\), corresponding to the texture \(C_{32}\), at any entry.
Our strategy for the realization of the vanishing subtrace texture is that one imposes a starting symmetry, with corresponding transformations on the Higgs and the lepton fields at the Lagrangian level (“gauge” basis), known to impose some zero elements for the neutrino mass matrix. We then transform this symmetry by applying some “rotation” so that to get a new second symmetry, with new transformations on the fields (also at the gauge basis), which would imply the vanishing subtrace texture. By following the previous discussion, we may find the rotation which, when applied on neutrino mass matrix, allows to go from zero texture to vanishing subtrace, so now this rotation would help to define, by (Eq. 52), how to move from the first symmetry field transformations to the second symmetry ones by the following “adjoint action” rule:

\[ T_f = S^T T_f^0 S^{T\dagger}, \]

where \( T_f(T_f^0) \) defines the transformation on the field \( f \) satisfying the new (old) symmetry, and \( S \) is the unitary transformation relating the two symmetries.

However, one should be sure that the new symmetry transformations assure that we are at the flavor basis, or approximately so. The point is that we should get a generic charged lepton mass matrix by the first symmetry, so that to get also a generic one by the second symmetry. Then, by adopting some ‘natural’ assumptions on the fields vacuum expectation values (vevs), without the need of unnatural constraints on the Yukawa couplings, one can diagonalize the ‘generic’ charged lepton mass matrix by an infinitesimal rotation, and so one can, with a good approximation, assume that the new symmetry puts us in the flavor basis. We shall give some examples for this strategy within both type I and II seesaw scenarios, where the symbol 0 will be consigned for quantities corresponding to the first ‘unrotated’ symmetry. The method based on ‘rotated’ symmetry can be considered as an indirect method for realizing the texture of vanishing subtrace.

8 Flavor basis

The notion of basis is intricate and needs to be clarified since a variety of bases could arise in our discussion such as gauge, flavor and mass basis. In order to delve into the notion of different types of basis, we take for simplicity the following Lagrangian piece responsible for the mass terms in the leptonic sector expressed in the gauge basis as:

\[ \mathcal{L}_M \supset Y_{1ij}^g \overline{D_{Li}} H e_{Rj} + Y_{2ij}^g \overline{D_{Li}} \tilde{H} \nu_{Rj} + Y_{3ij}^g \nu_{Ri} C^{-1} \chi \nu_{Rj}, \]

where \( D_{Li} \) is the left-handed lepton doublet \( (\nu_{Li}, e_{Li})^T \), \( e_{Ri} \) is the right-handed charged lepton, \( \nu_{Ri} \) is the right-handed neutrino, \( \chi \) is a scalar singlet, \( H \) is the Higgs doublet and \( \tilde{H} \equiv i \sigma_2 H^* \). The relevant Yukawa coupling matrices are denoted by \( (Y_{1i}^g, Y_{2i}^g, Y_{3i}^g) \) which are defined in this “gauge” basis, whence the superscript \( (g) \). The indices \( i, j \) are the family ones while \( C \) is the charge conjugation.

When the Higgs doublet \( H \) and the singlet \( \chi \) take a vev, then we get the mass term which can be cast into the form,

\[ \overline{e_{Li}} M_{\ell \ell} e_{Rj} + \overline{e_{Ri}} M_{\nu \nu} \nu_{Rj} + \left( \nu_{Li}^T \nu_{Rj}^T \right) C^{-1} \begin{pmatrix} 0 & M_{\ell \nu} \\ M_{\nu \ell}^T & M_{\nu \nu} \end{pmatrix} \begin{pmatrix} \nu_{Li} \\ \nu_{Rj} \end{pmatrix}, \]

\[ (54) \]

*We stress here that \( M \) and \( M_0 \) are not mass matrices for the same system at different bases related by rotation. Rather, they are mass matrices of two systems, satisfying two different symmetries, where the matrices are defined in the same Lagrangian gauge basis. The two symmetries are related by ‘rotation’.
which, via the seesaw mechanism, gives approximately, after decoupling the right-handed neutrinos,

\[ \mathcal{L}_M \supset \bar{e}_L M_{\ell ij} e_{Rj} + \nu_{\ell i}^T C^{-1} M_{\nu ij} \nu_{ij}, \]

with \( M_\nu = M_D M_{R}^{-1} M_{D}^T \), and \( \nu_{\ell i} \) are approximately left-handed \( (\approx \nu_{Li}) \).

By diagonalizing, we get the “mass” basis which is denoted by the superscript \( m \):

\[
\left\{
\begin{align*}
\mathcal{L}_M & \supset \bar{e}_L^m U_L^\dagger M_\ell U_R \, e_R^m + \nu_{\ell i}^m V^T M_\nu \, \nu_i^m, \\
& \supset \bar{e}_L^m M_\ell \text{diag} \, e_R^m + \nu_{\ell i}^m M_\nu \text{diag} \, \nu_i^m, \\
& e_L^m = U_L^\dagger e_L, \ e_R^m = U_R^\dagger e_R, \ \nu_i^m = V^\dagger \nu_i,
\end{align*}
\right.
\]

where \( U_L^\dagger M_\ell U_R \) and \( V^T M_\nu V \) are diagonal.

In the gauge basis, the interaction (say, \( \bar{e}_L W^- \nu_i \)) between the charged lepton sector and the neutrino sector, when expressed in terms of the mass bases \( (\bar{e}_L^m U_L^\dagger V W^- \nu_i^m) \) would involve the experimentally measurable \( V_{\text{PMNS}} = U_L^\dagger V \) expressing the mismatch between the rotations of the left-handed charged leptons and of the left-handed neutrinos. The “flavor basis”, by definition, occurs when by convention we assume, without loss of generality, the left-handed charged leptons to be pure states, i.e. \( U_L = 1 \) and \( e_L^m = e_L \). This can always be taken, since one can use the freedom in defining the fields in a way to attribute the whole rotation, appearing when expressing the interaction term in terms of mass states, entirely to the left-handed neutrinos. The situation is exactly the same for the quark sectors when one can take by convention the up sector as pure states and the flavor mixing is described in terms of the rotation CKM matrix operating on the down sector only \[14\].

9 Indirect realization of type I seesaw with \( Z_8 \times Z_2 \) symmetry for nonsingular textures

We implement here a discrete symmetry within type I seesaw scenario in order to generate one vanishing subtrace texture following the ‘Rotating’ strategy.

9.1 Indirect realization of \( C_{33} \) (Type I nonsingular): Vanishing of \( M_{\nu 11} + M_{\nu 22} \)

We saw that the matrix \( S_{33} \) conjures, when acted on \( M_\nu \), the combination \( M_{\nu 22} + M_{\nu 33} \) in the element \((1, 2)\) of the transformed \( \bar{M}_\nu \). Thus we follow \[23\] and impose \( Z_8 \times Z_2 \) symmetry to have a zero in the \((1, 2)\) entry of the ‘unrotated’ \( M_{\nu 0} \) and check that the ‘rotated’ mass matrix \( M_\nu = S^T M_\nu S \) has a texture \( C_{33} \) with \( S = S_{33} \)

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \Rightarrow S^T = \begin{pmatrix} A & 0 & C \\ 0 & D & E \\ C & E & F \end{pmatrix} S = \begin{pmatrix} -\frac{1}{\sqrt{2}}(A+D) & \frac{1}{\sqrt{2}}(D-A) & \frac{i}{\sqrt{2}}(E+C) \\ \frac{i}{\sqrt{2}}(D-A) & \frac{1}{\sqrt{2}}(A+D) & \frac{1}{\sqrt{2}}(E-C) \\ \frac{1}{\sqrt{2}}(E+C) & \frac{1}{\sqrt{2}}(E-C) & F \end{pmatrix}.
\]

First, we show how one can impose the zero texture. We introduce five SM Higgs doublets \( \Phi_a (a = 1, \ldots, 5) \), three real scalar singlets \( \chi_i (i = 1, 2, 3) \), and denote the left-handed lepton Doublet of the first (second, third) family by \( D_{L1} \) \( (D_{L2}, D_{L3}) \). The right-handed charged lepton and neutrino singlets are denoted by \( (\ell_R, \nu_R) \). We assume the following transformations in Table \[1\] under \( Z_8 \times Z_2 \) for the fields:
By forming bilinear terms of $\mathcal{D}L_i \ell_R$ and $\mathcal{D}L_i \nu_R$, relevant for Dirac mass matrices of neutrino and charged leptons, and of $\nu_R \nu_R$, relevant for the Majorana neutrino mass matrix $M_R$ in the Lagrangian ($Y_{ij}^a$ are the Yukawa coupling constants, the indices $(i, j)$ are flavor ones, the indices $(a, b)$ run respectively over the Higgs doublet and Scalar singlet fields, $C$ is the charge conjugation matrix and $\tilde{\Phi} = i \sigma_2 \Phi^*$):

$$L_M \supset \sum_{i,j=1}^{3} \sum_{a=1}^{3} \sum_{b=1}^{3} Y_{0i}^{b} \chi_{b} \nu_{Ri}^{T} C^{-1} \nu_{Rj} + Y_{0Di}^{a} \mathcal{D}L_{i} \tilde{\Phi}_{a} \nu_{Rj} + Y_{0Dij}^{a} \mathcal{D}L_{i} \Phi_{a} \ell_{Rj} \quad (58)$$

and examining how they transform under $Z_8 \times Z_2$, we see that the invariance under the symmetry implies the following forms

$$M_{D0} = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \quad M_{R0} = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} \quad \Rightarrow M_{\nu 0} = M_{D0} M_{R0}^{-1} M_{D0}^{T} = \begin{pmatrix} \times & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix}. \quad (59)$$

Note that, in contrast to $[5]$ where we introduced only three Higgs doublets, we introduce here five Higgs doublets otherwise we would have got as in $[5]$ a diagonal charged lepton mass matrix before proceeding to the ‘rotation’ defined by $S$ of Eq. [57]. Had we done this then we shall get field transformations corresponding to the ‘rotated’ symmetry by adjoint acting on the ‘unrotated’ transformations by the rotation $S$, which will produce a non-diagonal matrix for the charged leptons, which means that upon ‘rotating’ and getting the vanishing subtrace texture we would have left the flavor basis. Actually, we added the extra Higgs fields exactly in order to get a ‘generic’ charged lepton mass matrix in the ‘unrotated’ basis while keeping the form of the Dirac neutrino mass matrix. The fields $\Phi_{4,5}$ are responsible for the desired form of $M_{D0}$, whereas the fields $(\Phi_{1,2,3})$ produce generic $M_{0}$. 

In order to find the new ‘rotated’ symmetry, we need to find then all the fields would transform. Thus, we should explicitly write down the form of the mass matrices in terms of the Yukawa couplings when the Higgs and singlet scalar fields get vevs. Actually the invariance of the Majorana term under $Z_8 \times Z_2$ implies the following constraint

$$Y_{0}^{b \chi} = T_{\chi \alpha}^{0Z} (T_{\nu R}^{0Z})^{T} Y_{0i}^{a} (T_{\nu R}^{0Z}) \quad (60)$$

where $(a, b = 1, 2, 3), (Y_{0i}^{b \chi})$ is a matrix in flavor space with element $Y_{0i}^{b \chi}$ at its $(i, j)^{th}$ entry, and $T_{f}^{0Z}(f = \chi, \nu_R)$ is a matrix (diagonal by construction) defining the transformation of the
field $f$ under the considered symmetry factor $Z$ ($Z = Z_8$ or $Z_2$). This constraint (Eq. 60) can be solved for both symmetry factors and leads to the following form, when $\chi_a$ gets a vev $v_{\chi a}$:

$$M_{R0} = \begin{pmatrix}
Y^1_{\chi_{11}} v_{\chi 1} & 0 & 0 \\
0 & Y^1_{\chi_{22}} v_{\chi 1} & Y^2_{\chi_{23}} v_{\chi 2} \\
0 & Y^2_{\chi_{23}} v_{\chi 2} & Y^3_{\chi_{33}} v_{\chi 3}
\end{pmatrix}.$$  

(61)

The invariance of the Dirac neutrino mass term under $Z_8 \times Z_2$ implies the following constraint,

$$(Y^b_{\ell \ell'}) = \begin{pmatrix} (T^0_{\Phi})^\dagger ab (T^0_{D_L})^\dagger (Y^a_{\ell \ell'}) (T^0_{\nu_R})^\dagger \end{pmatrix}$$

(62)

where $(a, b = 1, \ldots, 5), (Y^a_{\ell \ell'})$ is a matrix in flavor space with element $Y^a_{\ell \ell ij}$ at its $(i, j)^{th}$ entry, and $(T^0_{\ell \ell})^\dagger (f = \Phi, D_L, \nu_R)$ is a diagonal -by construction- matrix defining the transformation of the field $f$ under the considered symmetry factor $Z$ ($Z = Z_8$ or $Z_2$) which leads when solved for both $Z_8$ and $Z_2$ to the following form, when $\Phi_a$ gets a vev $v_{\Phi a}$:

$$M_{D0} = \begin{pmatrix}
Y^4_{\Phi_{11}} v_{\Phi 1} & 0 & 0 \\
0 & Y^4_{\Phi_{22}} v_{\Phi 4} & 0 \\
Y^5_{\Phi_{31}} v_{\Phi 5} & 0 & Y^4_{\Phi_{33}} v_{\Phi 4}
\end{pmatrix}.$$  

(63)

And we get $M_{\ell \ell} = M_{D0} M_{R0}^{-1} M_{D0}^T$ of the desired form with vanishing element at the $(2, 1)^{th}$ entry. As to the charged lepton mass matrix, the invariance of the corresponding mass term give

$$(Y^b_{\ell \ell'}) = \begin{pmatrix} (T^0_{\ell \ell})^\dagger \end{pmatrix}$$

(64)

where $(a, b = 1, \ldots, 5), (Y^a_{\ell \ell ij})$ is a matrix in flavor space with element $Y^a_{\ell \ell ij}$ at its $(i, j)^{th}$ entry, and $T^0_{\ell \ell}(f = \Phi, D_L, \ell_R)$ is a matrix defining the transformation of the field $f$ under the considered symmetry factor $Z$ ($Z = Z_8$ or $Z_2$) which leads to a generic form for the charged lepton matrix:

$$M_{\ell \ell} = \begin{pmatrix}
Y^1_{\ell_{11}} v_{\ell 1} & Y^1_{\ell_{12}} v_{\ell 1} & Y^1_{\ell_{13}} v_{\ell 1} \\
Y^2_{\ell_{21}} v_{\ell 2} & Y^2_{\ell_{22}} v_{\ell 2} & Y^2_{\ell_{23}} v_{\ell 2} \\
Y^3_{\ell_{31}} v_{\ell 3} & Y^3_{\ell_{32}} v_{\ell 3} & Y^3_{\ell_{33}} v_{\ell 3}
\end{pmatrix}.$$  

(65)

In order to find the field transformations corresponding to the new ‘rotated’ symmetry defined by $S$ (Eq. 57), we apply the same rule as in Eq. (46) or Eq.(52), with caution, for all the fields $f$: 

$$T^Z_f = S^\dagger T^Z_f S$$

(66)

and extending in the case of the 5-dimensional $\Phi$ the matrix $S$ to be $S_{ex} = \text{diag} (S, 1_{2 \times 2})$. Note that we do not get generally diagonal matrices $T^Z_f$ because of the rotation $S$. Thus one can write down similar constraints to those of Eqs. (60) (62) (64) corresponding to the rotated symmetry, albeit with Yukawa couplings and vevs without the subscript 0, and by solving them we get:

$$M_R = \begin{pmatrix}
-Y^1_{\chi_{22}} (v_{\chi 1} + i v_{\chi 2}) & -Y^1_{\chi_{12}} (v_{\chi 1} + i v_{\chi 2}) & -Y^2_{\chi_{23}} (v_{\chi 1} + i v_{\chi 2}) \\
-Y^2_{\chi_{22}} (v_{\chi 1} + i v_{\chi 2}) & -Y^2_{\chi_{23}} (v_{\chi 1} + i v_{\chi 2}) & Y^3_{\chi_{33}} v_{\chi 3}
\end{pmatrix}.$$  

(67)

where $\boxed{\text{denotes an element deduced by symmetry property of the matrix}}$ ($M = M_T$) and this convention will be used from now on.

$$M_{D} = \begin{pmatrix}
Y^4_{D_{22}} v_{\chi 4} & -Y^4_{D_{21}} v_{\chi 4} & 0 \\
Y^4_{D_{21}} v_{\chi 4} & Y^4_{D_{21}} v_{\chi 4} & 0 \\
-i Y^5_{D_{32}} v_{\chi 5} & Y^5_{D_{32}} v_{\chi 5} & Y^4_{D_{33}} v_{\chi 4}
\end{pmatrix}.$$  

(68)
One can check that the resulting $M_\nu$ satisfies the texture $C_{33}$. Note also that all the Yukawa couplings and the vevs in Eqs (67-68) are different from those in Eqs (61-63) since each set of Yukawa couplings and vevs correspond to the Lagrangian under a specific symmetry. However, they are related through the transformation:

$$M = S^T M_0 S$$

(69)

which should be valid for $(M_\nu, M_R, M_D)$, and one can check that the form of $M_{(\nu, D, R)}$ is the same as that of $S^T M_{(\nu_0, D_0, R_0)} S$.

As to $M_\ell$, we get a generic mass matrix:

$$M_\ell = \left( \begin{array}{ccc} Y_{\ell 21} v_{\phi_1} - Y_{\ell 21} v_{\phi_2} & Y_{\ell 22} v_{\phi_1} - Y_{\ell 22} v_{\phi_2} & Y_{\ell 23} v_{\phi_1} - Y_{\ell 23} v_{\phi_2} \\ Y_{\ell 21} v_{\phi_1} + Y_{\ell 21} v_{\phi_2} & Y_{\ell 22} v_{\phi_1} + Y_{\ell 22} v_{\phi_2} & Y_{\ell 23} v_{\phi_1} + Y_{\ell 23} v_{\phi_2} \\ Y_{\ell 31} v_{\phi_3} & Y_{\ell 32} v_{\phi_3} & Y_{\ell 33} v_{\phi_3} \end{array} \right),$$

(70)

so if assume the related vevs are comparable $v_{\phi_1} \approx v_{\phi_2} \approx v_{\phi_3} \approx v$ then we get

$$M_\ell \approx v \left( \begin{array}{ccc} Y_{\ell 21} - Y_{\ell 21} & Y_{\ell 22} - Y_{\ell 22} & Y_{\ell 23} - Y_{\ell 23} \\ Y_{\ell 21} + Y_{\ell 21} & Y_{\ell 22} + Y_{\ell 22} & Y_{\ell 23} + Y_{\ell 23} \\ Y_{\ell 31} & Y_{\ell 32} & Y_{\ell 33} \end{array} \right) = v \left( \begin{array}{cc} a^T & b^T \\ b \cdot c & c^T \end{array} \right),$$

(71)

where $a$, $b$ and $c$ stand for column vectors extracted from the corresponding rows, formed of Yukawa couplings, in the matrix $M_\ell$, and this abbreviation will be used from now on. The dot product refers to the usual Hermitian inner product defined as $a \cdot b = \sum_{i=1}^{3} a_i b_i$. Thus

$$M_\ell M_\ell^\dagger \approx v^2 \left( \begin{array}{ccc} a \cdot a & a \cdot b & a \cdot c \\ b \cdot a & b \cdot b & b \cdot c \\ c \cdot a & c \cdot b & c \cdot c \end{array} \right),$$

(72)

so taking only the following natural assumption on the norms of the vectors

$$\|a\|/\|c\| = m_e/m_\tau \sim 3 \times 10^{-4}, \quad \|b\|/\|c\| = m_\mu/m_\tau \sim 6 \times 10^{-2}$$

(73)

one can diagonalize $M_\ell M_\ell^\dagger$ by an infinitesimal rotation as was done in [5], which proves that we are to a good approximation in the flavor basis.

### 9.2 Indirect realization of $C_{11}$ (Type I nonsingular): Vanishing of $M_{\nu 22} + M_{\nu 33}$

Following the same procedure as for the case $C_{33}$, we just state briefly the results. The ‘rotation’ matrix which moves a zero texture at $(2, 3)$ to the texture $C_{11}$ is given by:

$$S = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} \sqrt{2} & 0 & 0 \\ 0 & i & -1 \\ 0 & i & 1 \end{array} \right) \Rightarrow S^T \left( \begin{array}{ccc} A & B & C \\ B & D & 0 \\ C & 0 & F \end{array} \right) S = \left( \begin{array}{ccc} A - \frac{1}{\sqrt{2}}(B + C) & -\frac{1}{\sqrt{2}}(B - C) \\ -\frac{1}{\sqrt{2}}(D + F) & -\frac{1}{\sqrt{2}}(D - F) \\ \frac{1}{\sqrt{2}}(D + F) & \frac{1}{\sqrt{2}}(D - F) \end{array} \right),$$

(74)

and we check that the sum of elements at $(2, 2)$ and $(3, 3)$ vanishes. At the Lagrangian level, the symmetry transformations for the fields which imposes a zero texture neutrino mass matrix with generic charged lepton mass matrix are given in Table 5.

By forming bilinear terms of the fields we see that the above transformations force a neutrino mass matrix with zero texture at $(2, 3)$ entry. Again we define the new transformations for the fields corresponding to the new symmetry imposing the vanishing subtrace by the rule in Eq.
Table 5: The $Z^0_0 \times Z^0_0$ symmetry realization of the one zero texture at $(2, 3)$-entry corresponding upon rotation to vanishing subtrace $C_{11}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\omega$ denotes $e^{i\pi/4}$.

\[ M_R = \begin{pmatrix}
Y^{\chi_{11}}_{11} v_{\chi_{11}} & Y^{\chi_{12}}_{12} (v_{\chi_2} + i v_{\chi_3}) \\
& Y^{\chi_{12}}_{12} (v_{\chi_2} + i v_{\chi_3}) & i Y^{\chi_{12}}_{12} (v_{\chi_2} + i v_{\chi_3}) \\
i (-Y^{\chi_{23}}_{23} v_{\chi_1} + i Y^{\chi_{23}}_{23} v_{\chi_2} + Y^{\chi_{23}}_{23} v_{\chi_3}) & Y^{\chi_{23}}_{23} v_{\chi_1} + i Y^{\chi_{23}}_{23} v_{\chi_2} + Y^{\chi_{23}}_{23} v_{\chi_3} \\
& -i (-Y^{\chi_{23}}_{23} v_{\chi_1} + i Y^{\chi_{23}}_{23} v_{\chi_2} + Y^{\chi_{23}}_{23} v_{\chi_3}) & 
\end{pmatrix}, \]

and,

\[ M_D = \begin{pmatrix}
Y^{D_{11}}_{11} v_{\Phi_4} & i Y^{D_{13}}_{13} v_{\Phi_5} & Y^{D_{13}}_{13} v_{\Phi_5} \\
0 & Y^{D_{22}}_{22} v_{\Phi_4} & Y^{D_{23}}_{23} v_{\Phi_4} \\
0 & -Y^{D_{23}}_{23} v_{\Phi_4} & Y^{D_{22}}_{22} v_{\Phi_4} 
\end{pmatrix}. \]

One can check that the resulting $M_\nu$ satisfies the texture $C_{11}$.

As to $M_\ell$ we get,

\[ M_\ell = \begin{pmatrix}
Y^{\ell_{11}}_{11} v_{\phi_1} & Y^{\ell_{12}}_{12} v_{\phi_1} \\
Y^{\ell_{13}}_{13} v_{\phi_2} + Y^{\ell_{31}}_{31} v_{\phi_3} & Y^{\ell_{12}}_{12} v_{\phi_2} - Y^{\ell_{32}}_{32} v_{\phi_3} \\
-2 Y^{\ell_{21}}_{21} v_{\phi_2} + Y^{\ell_{31}}_{31} v_{\phi_2} + Y^{\ell_{22}}_{22} v_{\phi_2} - Y^{\ell_{23}}_{23} v_{\phi_2} - Y^{\ell_{33}}_{33} v_{\phi_2} + Y^{\ell_{23}}_{23} v_{\phi_3} \n\end{pmatrix}, \]

then we see that if assume all the related vevs are comparable $v_{\phi_1} \approx v_{\phi_2} \approx v_{\phi_3} \approx v$ then we get

\[ M_\ell \approx v \begin{pmatrix}
Y^{\ell_{11}}_{11} & Y^{\ell_{12}}_{12} & Y^{\ell_{13}}_{13} \\
Y^{\ell_{13}}_{13} + Y^{\ell_{31}}_{31} & Y^{\ell_{22}}_{22} - Y^{\ell_{32}}_{32} & Y^{\ell_{33}}_{33} + Y^{\ell_{23}}_{23} \\
-2 Y^{\ell_{21}}_{21} + Y^{\ell_{31}}_{31} & Y^{\ell_{22}}_{22} + Y^{\ell_{23}}_{23} & -Y^{\ell_{33}}_{33} + Y^{\ell_{23}}_{23} \n\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T \n\end{pmatrix}, \]

which can be diagonalized by an infinitesimal rotation under some natural assumptions on the amplitudes of the Yukawa vectors as done for the case of $C_{33}$.

9.3 Indirect realization of $C_{22}$ (Type I nonsingular): Vanishing of $M_{\nu_{11}} + M_{\nu_{33}}$

The ‘rotation’ matrix which moves a zero texture at $(1, 3)$ to the texture $C_{22}$ is given by:

\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix}
i & 0 & -1 \\
0 & \sqrt{2} & 0 \\
i & 0 & 1 \n\end{pmatrix} \Rightarrow S^T \begin{pmatrix}
A & B & 0 \\
B & D & E \\
0 & E & F \n\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2}(A + F) & \frac{i}{\sqrt{2}}(B + E) & \frac{i}{\sqrt{2}}(F - A) \\
\frac{1}{\sqrt{2}}(B + E) & D & \frac{1}{\sqrt{2}}(E - B) \\
\frac{1}{2}(A + F) & \frac{1}{2}(E - B) & \frac{1}{2}(A + F) \n\end{pmatrix}, \]

and we check that the sum of elements at $(1, 1)$ and $(3, 3)$ vanishes.
Table 6: The $Z_3^0 \times Z_2^0$ symmetry realization of the one zero texture at $(1,3)$-entry corresponding upon rotation to vanishing subtrace $C_{22}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\omega$ denotes $e^{i\pi/4}$.

At the Lagrangian level, the symmetry transformations for the fields which imposes a zero texture $M_{\ell 0}$ with generic $M_{\ell 0}$ are given in Table (6)

By forming bilinear terms of the fields we see that the above transformations force the $(1,3)$ entry in $M_{\ell 0}$ to vanish. Again we define the new transformations for the fields corresponding to the new symmetry imposing the vanishing subtrace by the rule in Eq. (60), but with $S$ given by Eq. (79). The ‘rotated’ symmetry imposes some constraints on the Yukawa couplings and the vevs, which when solved give the following results,

$$M_R = \begin{pmatrix}
  i [Y_{\chi_{33}}^3 (v_{\chi_1} + i v_{\chi_3}) - Y_{\chi_{13}}^2 (v_{\chi_1} + i v_{\chi_3}) ] & Y_{\chi_{33}}^3 (v_{\chi_1} + i v_{\chi_3}) + Y_{\chi_{13}}^2 (v_{\chi_1} + i v_{\chi_3}) \\
  -i Y_{\chi_{23}}^3 (v_{\chi_1} + i v_{\chi_3}) & -i Y_{\chi_{33}}^3 (v_{\chi_1} + i v_{\chi_3}) - Y_{\chi_{13}}^2 (v_{\chi_1} + i v_{\chi_3})
\end{pmatrix},$$

and

$$M_D = \begin{pmatrix}
  Y_{D_{11}}^4 \nu_{\Phi_4} & 0 & -Y_{D_{31}}^4 \nu_{\Phi_4} \\
  -i Y_{D_{23}}^4 \nu_{\Phi_5} & Y_{D_{22}}^4 \nu_{\Phi_4} & Y_{D_{23}}^4 \nu_{\Phi_5} \\
  Y_{D_{31}}^4 \nu_{\Phi_4} & 0 & Y_{D_{11}}^4 \nu_{\Phi_4}
\end{pmatrix}. \quad (81)$$

One can check that the resulting $M_\ell$ satisfies the texture $C_{22}$. As to $M_\ell$ one gets,

$$M_\ell = \begin{pmatrix}
  Y_{\ell_{11}}^1 \nu_{\Phi_1} + Y_{\ell_{11}}^3 \nu_{\Phi_3} & Y_{\ell_{12}}^1 \nu_{\Phi_2} + Y_{\ell_{12}}^3 \nu_{\Phi_3} & Y_{\ell_{13}}^1 \nu_{\Phi_3} + Y_{\ell_{13}}^3 \nu_{\Phi_1} \\
  Y_{\ell_{21}}^3 \nu_{\Phi_2} & Y_{\ell_{22}}^3 \nu_{\Phi_2} & Y_{\ell_{23}}^3 \nu_{\Phi_2} \\
  Y_{\ell_{11}}^3 \nu_{\Phi_1} - Y_{\ell_{11}}^1 \nu_{\Phi_3} & Y_{\ell_{12}}^3 \nu_{\Phi_1} - Y_{\ell_{12}}^1 \nu_{\Phi_3} & Y_{\ell_{13}}^3 \nu_{\Phi_1} - Y_{\ell_{13}}^1 \nu_{\Phi_3}
\end{pmatrix}, \quad (82)$$

then we see that if assume all the related vevs are comparable $\nu_{\Phi_1} \approx \nu_{\Phi_2} \approx \nu_{\Phi_3} \approx v$ then we get

$$M_\ell \approx v \begin{pmatrix}
  Y_{\ell_{11}}^1 + Y_{\ell_{11}}^3 & Y_{\ell_{12}}^1 + Y_{\ell_{12}}^3 & Y_{\ell_{13}}^1 + Y_{\ell_{13}}^3 \\
  Y_{\ell_{21}}^3 & Y_{\ell_{22}}^3 & Y_{\ell_{23}}^3 \\
  Y_{\ell_{11}}^3 - Y_{\ell_{11}}^1 & Y_{\ell_{12}}^3 - Y_{\ell_{12}}^1 & Y_{\ell_{13}}^3 - Y_{\ell_{13}}^1
\end{pmatrix} = v \begin{pmatrix}
  a^T \\
  b^T \\
  c^T
\end{pmatrix}, \quad (83)$$

which can be diagonalized by an infinitesimal rotation under some natural assumptions on the amplitudes of the Yukawa vectors as done for the previous two cases.
9.4 Indirect realization of $C_{31}$ (Type I nonsingular): Vanishing of $M_{12} + M_{23}$

The ‘rotation’ matrix which moves a zero texture at $(2, 3)$ to the texture $C_{31}$ is given by:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow S^T \begin{pmatrix} A & B & C \\ B & D & 0 \\ C & 0 & F \end{pmatrix} S = \begin{pmatrix} \frac{i}{2}(A + F) + C & \frac{1}{\sqrt{2}}B & \frac{1}{\sqrt{2}}(F - A) \\ \frac{1}{\sqrt{2}}B & \frac{i}{2}(A + F) - C & 0 \\ -\frac{1}{\sqrt{2}}B & 0 & \frac{i}{2}(A + F) + C \end{pmatrix},$$

and we check that the sum of elements at $(1, 2)$ and $(2, 3)$ vanishes.

At the Lagrangian level, the symmetry transformations for the fields which imposes a zero texture $M_{10}$ at $(2, 3)$ entry with generic $M_{10}$ are given in Table 7.

| Symmetry under $Z_3^0$ factor | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Phi_4$ | $\Phi_5$ | $D_{L1}$ | $D_{L2}$ | $D_{L3}$ | $\nu_{R1}$ | $\nu_{R2}$ | $\nu_{R3}$ | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\ell_R$ |
|--------------------------------|---------|---------|---------|---------|---------|--------|--------|--------|----------|----------|----------|--------|--------|--------|--------|
| 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 |

| Symmetry under $Z_2^0$ factor | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 | $\omega^4$ | $\omega$ | 1 |

Table 7: The $Z_3^0 \times Z_2^0$ symmetry realization of the one zero texture at $(2, 3)$-entry corresponding upon rotation to vanishing subtrace $C_{31}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\omega$ denotes $e^{i\pi/4}$.

Again we define the new transformations for the fields corresponding to the new symmetry imposing the vanishing subtrace by the adjoint action rule (Eq. 86), but with $S$ given by Eq. 84. The ‘rotated’ symmetry imposes some constraints on the Yukawa couplings and the vevs, which when solved give the following results.

$$M_R = \begin{pmatrix} Y_{\chi_{33}}^1 v_{\chi_1} + Y_{\chi_{13}}^1 v_{\chi_3} & -Y_{\chi_{23}}^2 v_{\chi_2} & Y_{\chi_{13}}^1 v_{\chi_1} + Y_{\chi_{33}}^1 v_{\chi_3} \\ -Y_{\chi_{23}}^2 v_{\chi_2} & Y_{\chi_{33}}^1 v_{\chi_3} + Y_{\chi_{13}}^1 v_{\chi_1} & \\ Y_{\chi_{13}}^1 v_{\chi_1} + Y_{\chi_{33}}^1 v_{\chi_3} & \end{pmatrix},$$

and

$$M_D = \begin{pmatrix} Y_{D_{11}}^4 v_{\Phi_4} + Y_{D_{11}}^5 v_{\Phi_5} & 0 & Y_{D_{31}}^4 v_{\Phi_4} + Y_{D_{11}}^5 v_{\Phi_5} \\ 0 & Y_{D_{22}}^4 v_{\Phi_4} & 0 \\ -Y_{D_{11}}^5 v_{\Phi_5} + Y_{D_{31}}^4 v_{\Phi_4} & 0 & Y_{D_{11}}^4 v_{\Phi_4} - Y_{D_{11}}^5 v_{\Phi_5} \end{pmatrix}.$$ 

One can check that the resulting $M_D$ satisfies the texture $C_{31}$. As to $M_\ell$, we get

$$M_\ell = \begin{pmatrix} Y_{\ell_{11}}^1 v_{\ell_{11}} + Y_{\ell_{31}}^1 v_{\ell_{31}} & Y_{\ell_{12}}^1 v_{\ell_{12}} + Y_{\ell_{32}}^1 v_{\ell_{32}} & Y_{\ell_{13}}^1 v_{\ell_{13}} + Y_{\ell_{33}}^1 v_{\ell_{33}} \\ Y_{\ell_{21}}^1 v_{\ell_{21}} & Y_{\ell_{22}}^1 v_{\ell_{22}} & Y_{\ell_{23}}^1 v_{\ell_{23}} \\ Y_{\ell_{31}}^1 v_{\ell_{31}} + Y_{\ell_{12}}^1 v_{\ell_{12}} + Y_{\ell_{13}}^1 v_{\ell_{13}} + Y_{\ell_{33}}^1 v_{\ell_{33}} & \end{pmatrix},$$

then we see that if assume $v \approx v_{\Phi_1} \approx v_{\Phi_2} \gg v_{\Phi_3}$ then we get

$$M_\ell \approx v \begin{pmatrix} Y_{\ell_{11}}^1 & Y_{\ell_{12}}^1 & Y_{\ell_{13}}^1 \\ Y_{\ell_{21}}^1 & Y_{\ell_{22}}^1 & Y_{\ell_{23}}^1 \\ Y_{\ell_{31}}^1 & Y_{\ell_{32}}^1 & Y_{\ell_{33}}^1 \end{pmatrix} = v \begin{pmatrix} a^T \\ b^T \\ c^T \end{pmatrix},$$

which can be diagonalized by an infinitesimal rotation under some natural assumptions on the amplitudes of the vectors as done in the previous cases.
10 Indirect realization of type I seesaw with $Z_{12} \times Z_2$ symmetry for singular textures

We shall adopt the same strategy of moving from the symmetry imposing a zero texture where $M_D$ is singular to the symmetry imposing a vanishing subtrace with again $M_D$ singular, which gives via seesaw type I a singular neutrino mass matrix. Again, we follow [5] to find the symmetry transformations leading to zero elements at singular $M_\nu$, but will add in new fields so that to get a generic charged lepton mass matrix and not a diagonal one as was the case in [5], in such a way that the new ‘rotated’ symmetry, as defined in Eq.(66), leads to vanishing subtraces at singular $M_\nu$ and to another generic $M_\ell$. The latter under some reasonable assumptions can be diagonalized via infinitesimal rotations, which put us to a good approximation in the flavor basis.

10.1 Indirect realization of $C_{33}$ (Type I singular): Vanishing of $M_{\nu 11} + M_{\nu 22}$

As in the nonsingular cases, we move from zero texture at $(1, 2)$ to the texture $C_{33}$ by $S$ of Eq. (57).

At the Lagrangian level, the symmetry transformations for the fields which imposes a zero texture neutrino mass matrix with generic charged lepton mass matrix and singular Dirac neutrino mass matrix are given in Table 8.

| Symmetry under $Z_{12}^0$ factor |
|----------------------------------|
| $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7, D_{L1}, D_{L2}, D_{L3}, \nu_{R1}, \nu_{R2}, \nu_{R3}, \chi_1, \chi_2, \chi_3, \ell_R$ |
| $\theta^{11}, \theta^9, \theta^4, \theta^2, \theta^8, \theta^9, \theta, \theta^{11}, \theta^9, \theta^4, \theta^2, \theta^5, \theta^{10}, \theta^8, \theta^2, 1$ |

| Symmetry under $Z_2^0$ factor |
|--------------------------------|
| $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$ |

Table 8: The $Z_{12}^0 \times Z_2^0$ symmetry realization of the one zero singular texture at $(1, 2)$-entry corresponding upon rotation to singular vanishing subtrace $C_{33}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\theta$ denotes $e^{i \pi/6}$.

By forming bilinear terms of the fields we see that the above transformations force a neutrino mass matrix with zero texture at $(1, 2)$ entry. Actually, we get:

$$M_{R0} = \begin{pmatrix}
Y_{1\chi_1}^0 & v_0 & 0 & 0 \\
0 & Y_{2\chi_2}^0 & v_0 & 0 \\
0 & 0 & Y_{3\chi_3}^0 & v_0 \\
0 & 0 & 0 & v_0 \\
\end{pmatrix}, \quad M_{D0} = \begin{pmatrix}
Y_{4\phi_1}^0 & v_0 & 0 & 0 \\
0 & 0 & Y_{5\phi_2}^0 & v_0 \\
Y_{6\phi_3}^0 & v_0 & 0 & Y_{7\phi_4}^0 \\
0 & 0 & 0 & v_0 \\
\end{pmatrix}. \quad (89)$$

We see that $M_{D0}$ is singular, and $M_{\nu 0} = M_{D0} M_{R0}^{-1} M_{D0}^T$ is singular with the desired form of vanishing element at the $(1, 2)^{th}$ entry. We can check that $M_{R0}$ is of generic form as the one presented in Eq.(55).

Again, in order to find the field transformations corresponding to the new ‘rotated’ symmetry defined by $S$ (Eq. 57), we apply the rule in Eq. (66) for all the fields $f$ and extending in the case of the 7-dimensional $\Phi$ the matrix $S$ to be $S_{ex} = \text{diag}(S, 1_{4\times 4})$, in such a way that we do not get generally diagonal matrices $T_f^Z$ because of the rotation $S$. As in nonsingular cases, one
can write down constraints involving the Yukawa couplings and vevs (now without the subscript 0), and by solving them we get:

\[
MR = \begin{pmatrix}
-Y_{X2}^1 v_{x1} - Y_{X2}^2 v_{x2} - Y_{X2}^2 v_{x1} + Y_{X2}^1 v_{x2} & 0 \\
-Y_{X2}^2 v_{x1} + Y_{X2}^1 v_{x2} & Y_{X2}^1 v_{x1} + Y_{X2}^2 v_{x2}
\end{pmatrix},
\]

\[
MD = \begin{pmatrix}
-i Y_{D12}^3 v_{\phi_4} & Y_{D12}^3 v_{\phi_4} & -i Y_{D23}^3 v_{\phi_5} \\
-i Y_{D12}^4 v_{\phi_4} & -i Y_{D12}^4 v_{\phi_4} & Y_{D23}^3 v_{\phi_5} \\
-i Y_{D32}^3 v_{\phi_6} & Y_{D32}^3 v_{\phi_6} & Y_{D33}^3 v_{\phi_7}
\end{pmatrix}.
\]  

One can check that the resulting \( M_R \) is singular and satisfies the texture \( C_{33} \).

As to \( M_\ell \), we get a generic mass matrix:

\[
M_\ell = \begin{pmatrix}
Y_{\ell21}^2 v_{\phi_1} + Y_{\ell11}^2 v_{\phi_2} & Y_{\ell22}^2 v_{\phi_1} + Y_{\ell12}^2 v_{\phi_2} & Y_{\ell23}^2 v_{\phi_1} + Y_{\ell13}^2 v_{\phi_2} \\
-Y_{\ell11}^2 v_{\phi_1} + Y_{\ell21}^2 v_{\phi_2} & -Y_{\ell12}^2 v_{\phi_1} + Y_{\ell22}^2 v_{\phi_2} & -Y_{\ell13}^2 v_{\phi_1} + Y_{\ell23}^2 v_{\phi_2} \\
Y_{\ell31}^3 v_{\phi_3} & Y_{\ell32}^3 v_{\phi_3} & Y_{\ell33}^3 v_{\phi_3}
\end{pmatrix}.
\]  

If we assume the related vevs are comparable \( v_{\phi_1} \approx v_{\phi_2} \approx v_{\phi_3} \approx v \) then we get

\[
M_\ell \approx v \begin{pmatrix}
Y_{\ell21}^2 + Y_{\ell11}^1 & Y_{\ell22}^2 + Y_{\ell12}^1 & Y_{\ell23}^2 + Y_{\ell13}^1 \\
-Y_{\ell11}^2 + Y_{\ell21}^1 & -Y_{\ell12}^2 + Y_{\ell22}^1 & -Y_{\ell13}^2 + Y_{\ell23}^1 \\
Y_{\ell31}^3 & Y_{\ell32}^3 & Y_{\ell33}^3
\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T
\end{pmatrix},
\]  

whereas if we assume \( v \approx v_{\phi_1} \approx v_{\phi_3} \gg v_{\phi_2} \) we get

\[
M_\ell \approx v \begin{pmatrix}
Y_{\ell21}^2 & Y_{\ell22}^2 & Y_{\ell23}^2 \\
-Y_{\ell11}^2 & -Y_{\ell12}^2 & -Y_{\ell13}^2 \\
Y_{\ell31}^3 & Y_{\ell32}^3 & Y_{\ell33}^3
\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T
\end{pmatrix}
\]  

In both cases, one can naturally diagonalize \( M_\ell \) by an infinitesimal rotation, which means that we are to a good approximation in the flavor basis.

### 10.2 Indirect realization of \( C_{11} \) (Type I singular): Vanishing of \( M_{\nu 22} + M_{\nu 33} \)

We move from zero texture at (2, 3) to the texture \( C_{11} \) by \( S \) of Eq. (74).

The symmetry transformations for the fields which imposes a zero texture \( M_{\ell 0} \) with generic \( M_{\ell 0} \) and singular \( M_{\ell 0} \) are given in Table 9. By forming bilinear terms of the fields we see that the above transformations force a neutrino mass matrix with zero texture at (2, 3) entry. Again we define the new transformations for the fields corresponding to the new symmetry imposing the vanishing subtracte by the rule in Eq. (66) with \( S \) given by Eq. (74) or its extension \( S_{ex} \) to the 7-dim space of \( \Phi \)'s. The ‘rotated’ symmetry imposes some constraints on the Yukawa couplings and the vevs, which when solved give the following results,

\[
MR = \begin{pmatrix}
Y_{X11}^1 v_{x1} & 0 & 0 \\
0 & Y_{X23}^3 v_{x1} + Y_{X23}^2 v_{x3} & Y_{X23}^2 v_{x2} + Y_{X23}^3 v_{x3} \\
0 & Y_{X23}^2 v_{x1} + Y_{X23}^3 v_{x3} & Y_{X23}^1 v_{x2} - Y_{X23}^2 v_{x3}
\end{pmatrix},
\]

\[
MD = \begin{pmatrix}
Y_{D11}^4 v_{\phi_4} & Y_{D13}^5 v_{\phi_5} & Y_{D13}^5 v_{\phi_5} \\
i Y_{D13}^5 v_{\phi_5} & Y_{D33}^7 v_{\phi_7} & -i Y_{D33}^7 v_{\phi_7} \\
i Y_{D31}^6 v_{\phi_6} & Y_{D33}^7 v_{\phi_7} & Y_{D33}^7 v_{\phi_7}
\end{pmatrix}.
\]
Table 9: The $Z_{12}^0 \times Z_2^0$ symmetry realization of the one zero singular texture at $(2,3)$-entry corresponding upon rotation to singular vanishing subtrace $C_{11}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\theta$ denotes $e^i\pi/6$.

One can check that $\det(M_D) = 0$, and that the resulting $M_\nu$ is singular and satisfies the texture $C_{11}$.

As to $M_\ell$, we get a generic mass matrix:

$$M_\ell = \begin{pmatrix}
Y_{e11} v_{\phi_1} & Y_{e12} v_{\phi_3} & Y_{e13} v_{\phi_1} \\
Y_{e21} v_{\phi_2} + Y_{e22} v_{\phi_3} & Y_{e22} v_{\phi_2} + Y_{e23} v_{\phi_3} & Y_{e23} v_{\phi_2} + Y_{e23} v_{\phi_3} \\
-Y_{e21} v_{\phi_2} + Y_{e21} v_{\phi_3} & -Y_{e22} v_{\phi_2} + Y_{e22} v_{\phi_3} & -Y_{e23} v_{\phi_2} + Y_{e23} v_{\phi_3}
\end{pmatrix}.$$  \hspace{1cm} (95)

Assuming the related vevs are comparable $v_{\phi_1} \approx v_{\phi_2} \approx v_{\phi_3} \approx v$ then we get

$$M_\ell \approx v \begin{pmatrix}
Y_{e11} & Y_{e12} & Y_{e13} \\
Y_{e21} + Y_{e22} & Y_{e22} + Y_{e23} & Y_{e23} + Y_{e23} \\
-Y_{e21} + Y_{e21} & -Y_{e22} + Y_{e22} & -Y_{e23} + Y_{e23}
\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T
\end{pmatrix},$$  \hspace{1cm} (96)

whereas if we assume $v \approx v_{\phi_1} \approx v_{\phi_3} \gg v_{\phi_2}$ we get

$$M_\ell \approx v \begin{pmatrix}
Y_{e11} & Y_{e12} & Y_{e13} \\
Y_{e21} & Y_{e22} & Y_{e23} \\
Y_{e31} & Y_{e32} & Y_{e33}
\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T
\end{pmatrix}.$$  \hspace{1cm} (97)

Again, in both cases, one can naturally diagonalize $M_\ell$ by an infinitesimal rotation, which means that we are to a good approximation in the flavor basis.

### 10.3 Indirect realization of $C_{22}$ (Type I singular): Vanishing of $M_{\nu11} + M_{\nu33}$

We move from zero texture at $(1,3)$ to the texture $C_{22}$ by $S$ of Eq.(79). The symmetry transformations for the fields which imposes a zero texture $M_{\nu0}$ at entry $(1,3)$, with generic $M_{\nu0}$ and singular $M_{D0}$ are given in Table 10.

Once more, we define the new transformations for the fields corresponding to the new symmetry imposing the vanishing subtrace by applying the rule in Eq.(66) with $S$ given by Eq.(79) or its extension $S_{ex}$ to the 7-dim space of $\Phi$’s. The ‘rotated’ symmetry imposes some constraints...
Table 10: The $Z^0_{12} \times Z^0_2$ symmetry realization of the one zero singular texture at (1, 3)-entry corresponding upon rotation to singular vanishing subtrace $C_{22}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\theta$ denotes $e^{i\pi/6}$.

On the Yukawa couplings and the vevs, which when solved give the following results,

$$M_R = \begin{pmatrix}
-Y^3_{\chi13}v_{x1} + Y^3_{\chi11}v_{x3} & 0 & Y^3_{\chi11}v_{x1} + Y^3_{\chi13}v_{x3} \\
0 & Y^2_{\chi22}v_{x2} & 0 \\
Y^3_{\chi11}v_{x1} + Y^3_{\chi13}v_{x3} & 0 & Y^3_{\chi13}v_{x1} - Y^3_{\chi11}v_{x3}
\end{pmatrix},$$

$$M_D = \begin{pmatrix}
Y^7_{D33}v_{\phi7} + Y^4_{D33}v_{\phi4} & 0 & -iY^7_{D33}v_{\phi7} + iY^4_{D33}v_{\phi4} \\
-iY^5_{D23}v_{\phi5} + iY^6_{D23}v_{\phi6} & 0 & Y^5_{D23}v_{\phi5} + Y^6_{D23}v_{\phi5} \\
iY^7_{D33}v_{\phi7} - iY^4_{D33}v_{\phi4} & 0 & Y^7_{D33}v_{\phi7} + Y^4_{D33}v_{\phi4}
\end{pmatrix}. \quad (98)$$

One can check that $\det(M_D) = 0$, and that the resulting $M_\nu$ is singular and satisfies the texture $C_{22}$.

As to $M_\ell$, we get a generic mass matrix:

$$M_\ell = \begin{pmatrix}
Y^3_{\ell31}v_{\phi1} + Y^3_{\ell11}v_{\phi3} & Y^3_{\ell32}v_{\phi1} + Y^3_{\ell12}v_{\phi3} & Y^3_{\ell33}v_{\phi1} + Y^3_{\ell13}v_{\phi3} \\
Y^2_{\ell21}v_{\phi2} & Y^2_{\ell22}v_{\phi2} & Y^2_{\ell23}v_{\phi2} \\
-Y^3_{\ell11}v_{\phi1} + Y^3_{\ell31}v_{\phi3} & -Y^3_{\ell12}v_{\phi1} + Y^3_{\ell32}v_{\phi3} & -Y^3_{\ell13}v_{\phi1} + Y^3_{\ell33}v_{\phi3}
\end{pmatrix}. \quad (99)$$

When $v_{\phi1} \approx v_{\phi2} \approx v_{\phi3} \approx v$ then we get

$$M_\ell \approx v \begin{pmatrix}
Y^3_{\ell11} + Y^3_{\ell11} & Y^3_{\ell12} + Y^3_{\ell12} & Y^3_{\ell13} + Y^3_{\ell13} \\
Y^2_{\ell21} & Y^2_{\ell22} & Y^2_{\ell23} \\
-Y^3_{\ell11} + Y^3_{\ell31} & -Y^3_{\ell12} + Y^3_{\ell32} & -Y^3_{\ell13} + Y^3_{\ell33}
\end{pmatrix} = v \begin{pmatrix} a^T \\ b^T \\ c^T \end{pmatrix}, \quad (100)$$

whereas when $v \approx v_{\phi3} \approx v_{\phi2} \gg v_{\phi1}$ we get

$$M_\ell \approx v \begin{pmatrix}
Y^3_{\ell11} & Y^3_{\ell12} & Y^3_{\ell13} \\
Y^2_{\ell21} & Y^2_{\ell22} & Y^2_{\ell23} \\
Y^3_{\ell31} & Y^3_{\ell32} & Y^3_{\ell33}
\end{pmatrix} = v \begin{pmatrix} a^T \\ b^T \\ c^T \end{pmatrix}. \quad (101)$$

In both cases, one can naturally diagonalize $M_\ell$ by an infinitesimal rotation, which means that we are approximately in the flavor basis.

10.4 Indirect realization of $C_{31}$ (Type I singular): Vanishing of $M_{\nu12} + M_{\nu23}$

We move from zero texture at (2, 3) to the texture $C_{31}$ by $S$ of Eq. [84]. The symmetry transformations for the fields which imposes a zero texture at entry (2, 3) of $M_{\nu0}$ with generic $M_\ell$ and singular $M_{D0}$ are given in Table 11.
Table 11: The $Z_{12}^0 \times Z_{12}^0$ symmetry realization of the one zero singular texture at $(2,3)$-entry corresponding upon rotation to singular vanishing subtrace $C_{31}$. The index $D_{L1}$ indicates the left-handed lepton doublet first family and so on. The $\chi_k$ denotes a scalar singlet which produces an entry in the right-handed Majorana mass matrix when acquiring a VEV at the see-saw scale. $\theta$ denotes $e^{i \pi/6}$.

In order to define the new transformations for the fields corresponding to the new symmetry imposing the vanishing subtrace, we apply the rule of Eq. (103) with $S$ given by Eq. (84) or its extension $S_{\text{zzv}}$ to the 7-dim space of $\Phi$’s. Solving the constraints on the Yukawa couplings and the vevs resulting from the ‘rotated’ symmetry, we get:

$$M_R = \begin{pmatrix} Y^1_{\chi 11} v_{\chi 1} + Y^1_{\chi 13} v_{\chi 3} & 0 & Y^1_{\chi 13} v_{\chi 1} + Y^1_{\chi 11} v_{\chi 3} \\ 0 & Y^2_{\chi 22} v_{\chi 2} & 0 \\ Y^1_{\chi 13} v_{\chi 1} + Y^1_{\chi 11} v_{\chi 3} & 0 & Y^1_{\chi 11} v_{\chi 1} + Y^1_{\chi 13} v_{\chi 3} \end{pmatrix},$$

$$M_D = \begin{pmatrix} Y^4_{D33} v_{\phi 4} - Y^5_{D31} v_{\phi 5} + Y^7_{D33} v_{\phi 7} & 0 & -Y^4_{D33} v_{\phi 4} - Y^5_{D31} v_{\phi 5} + Y^7_{D33} v_{\phi 7} \\ -Y^6_{D23} v_{\phi 6} & 0 & Y^6_{D23} v_{\phi 6} \\ -Y^4_{D33} v_{\phi 4} + Y^5_{D31} v_{\phi 5} + Y^7_{D33} v_{\phi 7} & 0 & Y^4_{D33} v_{\phi 4} + Y^5_{D31} v_{\phi 5} + Y^7_{D33} v_{\phi 7} \end{pmatrix}. \quad (102)$$

One can check that $\det(M_D) = 0$, and that the resulting $M_\ell$ is singular and satisfies the texture $C_{31}$.

As to $M_\ell$, we get a generic mass matrix:

$$M_\ell = \begin{pmatrix} Y^3_{\ell 31} v_{\phi 1} + Y^3_{\ell 11} v_{\phi 3} & Y^3_{\ell 32} v_{\phi 1} + Y^3_{\ell 12} v_{\phi 3} & Y^3_{\ell 33} v_{\phi 1} + Y^3_{\ell 13} v_{\phi 3} \\ Y^2_{\ell 21} v_{\phi 2} & Y^2_{\ell 22} v_{\phi 2} & Y^2_{\ell 23} v_{\phi 2} \\ Y^3_{\ell 11} v_{\phi 1} + Y^3_{\ell 13} v_{\phi 3} & Y^3_{\ell 12} v_{\phi 1} + Y^3_{\ell 13} v_{\phi 3} & Y^3_{\ell 33} v_{\phi 1} + Y^3_{\ell 33} v_{\phi 3} \end{pmatrix}. \quad (103)$$

When $v \approx v_{\phi 1} \approx v_{\phi 2} \gg v_{\phi 3}$ then we get

$$M_\ell \approx v \begin{pmatrix} Y^3_{\ell 31} & Y^3_{\ell 32} & Y^3_{\ell 33} \\ Y^3_{\ell 21} & Y^3_{\ell 22} & Y^3_{\ell 23} \\ Y^3_{\ell 11} & Y^3_{\ell 12} & Y^3_{\ell 13} \end{pmatrix} = v \begin{pmatrix} a^T \\ b^T \\ c^T \end{pmatrix}, \quad (104)$$

whereas if we assume $v \approx v_{\phi 3} \approx v_{\phi 2} \gg v_{\phi 1}$ we get

$$M_\ell \approx v \begin{pmatrix} Y^3_{\ell 11} & Y^3_{\ell 12} & Y^3_{\ell 13} \\ Y^3_{\ell 21} & Y^3_{\ell 22} & Y^3_{\ell 23} \\ Y^3_{\ell 31} & Y^3_{\ell 32} & Y^3_{\ell 33} \end{pmatrix} = v \begin{pmatrix} a^T \\ b^T \\ c^T \end{pmatrix}. \quad (105)$$

In both cases, one can naturally diagonalize $M_\ell$ by an infinitesimal rotation, which means that we are in the flavor basis approximately.
11 Indirect realization of type II seesaw with $Z_5$ symmetry

To fix the ideas, we treat here in some details the case of $C_{33}$ vanishing subtrace which can be related to zero texture, noting that the procedure can be generalized to all other textures ($C_{11}, C_{22}$ and $C_{31}$) that also can be related to zero textures. We follow the same ‘Rotating’ strategy outlined in section (7).

As we saw in Eq. (57), the matrix $S$ allows to move from one zero texture at $\nu_1 \ell_2$ to vanishing subtrace texture $C_{33}$. Again, we use a subscript (or superscript) 0 to denote the gauge basis satisfying the ‘unrotated’ symmetry $Z_5^0$, whereas we drop this subscript (superscript) for the ‘rotated’ $Z_5$.

11.1 Matter Content

Following the conventions of [5], we extend the SM extended by introducing several $SU(2)_L$ scalar triplets $H_a$, $(a = 1, 2, \cdots N)$,

$$H_a \equiv \left[ H_a^{++}, H_a^+, H_a^0 \right].$$

The gauge invariant Yukawa interaction relevant for neutrino mass takes the form,

$$\mathcal{L}_{H,L} = \sum_{i,j=1}^{N} \sum_{a=1}^{N} Y^{\nu a}_{ij} \left[ H_a^0 \nu_L^i C^{-1} \nu_L^j + H_a^+ \left( \nu_L^i C^{-1} \ell_L^j + \ell_L^i C^{-1} \nu_L^j \right) + H_a^+ \ell_L^i C^{-1} \nu_L^j \right],$$

where $Y^{\nu a}_{ij}$ are the corresponding Yukawa coupling constants, the indices $i,j$ are flavor ones, and $C$ is the charge conjugation matrix.

The field $H_a^0$ could acquire a small vev, $\langle H_a^0 \rangle = v_a^H$ that gives rise to a Majorana neutrino mass matrix of the following form,

$$M_{\nu ij} = \sum_{a=1}^{N} Y^{\nu a}_{ij} v_a^H.$$

The smallness of the vev $v_a^H$ is attributed to the largeness of the triplet scalar mass scale[15].

As to the charged lepton mass, we introduce, in contrast to [5], various Higgs doublets $\Phi_a$, $a = 1, \ldots, K$

$$\mathcal{L}_\ell = \sum_{i,j=1}^{K} \sum_{a=1}^{K} Y^\ell_{ij}^{a} \mathcal{D}_{Li} \Phi_a \ell_R^j,$$

Note that we did not consider only one SM Higgs, otherwise we would have got, as in [5], a diagonal charged lepton mass matrix $M_{\ell 0}$ when the neutrino mass matrix $M_{\nu 0}$ had a zero texture. We would like to get a generic $M_{\ell 0}$ corresponding to zero texture $M_{\ell 0}$, so that when we ‘rotate’ and get a vanishing subtrace texture for the neutrino mass matrix $M_{\nu}$ we get also another generic charged lepton mass matrix $M_{\ell}$. This latter can under suitable assumptions be diagonalized by infinitesimal rotations. Had we restricted our SM Higgs to only one Higgs doublet, then the diagonal $M_{\ell 0}$ corresponding to zero texture $M_{\ell 0}$ will give, upon ‘rotation’ by $S$, a nondiagonal charged mass matrix $M_{\ell}$ that is diagonalizable by a finite rotation $S$, which means that the vanishing subtrace texture does not correspond to the flavor basis.
11.2 \( Z_5^0 \) symmetry for zero texture \( M_{\nu 0} \) characterized by \( M_{\nu 12} = 0 \)

In order to impose a zero texture \( M_{\nu 0} \) by \( Z_5^0 \) symmetry with a generic \( M_{\nu 0} \), we introduce four scalar triplets \( H_a \) and three Higgs doublets \( \Phi_b \), with the following assignments under \( Z_5^0 \) defined in Table 12.

| Symmetry under \( Z_5^0 \) |
|-----------------------------|
| \( H_1 \) | \( H_2 \) | \( H_3 \) | \( H_4 \) | \( 1_F \) | \( 2_F \) | \( 3_F \) | \( \ell_R \) | \( \Phi_1 \) | \( \Phi_2 \) | \( \Phi_3 \) |
| 1 | \( \Omega^3 \) | \( \Omega^2 \) | \( \Omega \) | 1 | \( \Omega \) | \( \Omega^2 \) | 1 | 1 | \( \Omega \) | \( \Omega^2 \) |

Table 12: The \( Z_5^0 \) symmetry seesaw type II realization of the one zero texture at (1,2)-entry corresponding upon rotation to vanishing subtrace \( C_{33} \). \( H_a \) are triplet scalars, whereas \( 1_F \) refers to the fermions, apart from the right-handed charged leptons \( \ell_R \), in the first generation and so on. The \( \Phi_b \) denote SM Higgs doublets. \( \Omega \) denotes \( e^{i2\pi/5} \).

By forming bilinear terms of \( H_a \nu^T L \nu_L \) we can find out the invariant Lagrangian terms under \( Z_5^0 \), which gives

\[
M_{\nu 0} = \begin{pmatrix}
\times & 0 & \times \\
0 & \times & \times \\
\times & \times & \times
\end{pmatrix}
\]  

(110)

11.3 \( Z_5 \) symmetry for \( C_{33} \) texture \( (M_{\nu 11} + M_{\nu 22} = 0) \) and Yukawa couplings constraints

In order to find the new ‘rotated’ symmetry \( Z_5 \), we need first to find how all the fields would transform. Here, we carefully use the rule of (Eq. 46 or 52) for all the fields \( f \), in that if \( f \) transforms under \( Z_5^0 \) according to the diagonal, by construction, matrix \( T_f^{0Z} \), then it transforms under \( Z_5 \) according to \( T_f^Z = S_{ex}^{f} T_f^{0Z} S_{ex}^{f} \) (cf. Eq. 66) with \( S_{ex} = \text{diag} (S, 1_{r \times r}) \) possibly an extension of \( S \) to match the finite-dimensional space of the field \( f \) of dimensions \( (3 + r) \).

The invariance of the Lagrangian terms under the symmetry will impose constraints on the Yukawa couplings that one can in principle solve to give the form of the mass matrices when the Higgs/scalar fields get a vev.

Actually, one can check that under both \( Z_5^0 \) and \( Z_5 \), defined by the transformations \( T_f^{0Z} \) and \( T_f^Z \) respectively, we have the following constraints:

\[
(Y_{0}^{ab}) = T_{Hab}^{0Z} (T_{\nu_L}^{0Z})^T (Y_{0}^{nu}) (T_{\nu_L}^{0Z}),
\]  

(111)

and

\[
(Y^{ab}) = T_{Hab}^Z (T_{\nu_L}^Z)^T (Y^{nu}) (T_{\nu_L}^Z),
\]  

(112)

where \( a, b = 1, \ldots, 4 \), \( (Y_{0}^{ab}) \) is a matrix in flavor space with element \( Y_{0}^{ab} \) at its \( (i,j) \)th entry. The two constraints of Eqs. (111, 112) are related in that if we know the solution to one constraint we know it for the other. More specifically, one can check that if \( (Y_{0}^{ab}) \) was a solution of Eq. (111) then

\[
(Y^{ce}) = (S^T) (Y_{0}^{ab}) (S) (S_{ex}^{H})_{bc}.
\]  

(113)

is a solution of Eq.(112).
11.4 $M_{\nu 0}$ and $M_{\nu}$ resulting respectively from $Z_5^0$ and $Z_5$ invariance

By solving Eqs. (111,112) we get when $H^{\nu}_{a}$'s get the vevs $v^H_a$ under $Z_5^0$:

$$M_{\nu 0} = \begin{pmatrix} Y^{\nu 1}_{011} v^H_{01} & 0 & Y^{\nu 2}_{013} v^H_{02} \\ Y^{\nu 2}_{022} v^H_{02} & Y^{\nu 3}_{023} v^H_{03} & Y^{\nu 4}_{033} v^H_{04} \\ \end{pmatrix},$$

$$M_{\nu} = \begin{pmatrix} -Y^{\nu 1}_{23} v^H_1 - Y^{\nu 2}_{22} v^H_2 & Y^{\nu 2}_{12} v^H_1 + Y^{\nu 3}_{12} v^H_2 & Y^{\nu 3}_{23} v^H_1 + Y^{\nu 4}_{23} v^H_2 \\ -Y^{\nu 1}_{12} v^H_1 + Y^{\nu 2}_{22} v^H_2 & Y^{\nu 2}_{12} v^H_1 + Y^{\nu 3}_{12} v^H_2 & Y^{\nu 3}_{23} v^H_1 + Y^{\nu 4}_{23} v^H_2 \\ \end{pmatrix} \ \text{(114)}$$

We see that the texture $C_{33}$ is met in $M_{\nu}$ while ($M_{\nu 0 12} = 0$) for $M_{\nu 0}$. One can deduce the relations between the Higgs vevs in the “unrotated” system ($v^0_a$) and the Higgs vevs in the “rotated” system ($v^H_a$) by writing Eq.(109) and considering Eq.(113).

11.5 $M_{\theta 0}$ and $M_{\ell}$ resulting respectively from $Z_5^0$ and $Z_5$ invariance

The introduction of three SM Higgs $\Phi$’s was needed essentially to produce a generic charged lepton matrix. Actually the bilinear of the relevant term $\mathcal{T}_{Li} \ell_j$ transforms under $Z_5^0$ as,

$$\mathcal{D}_{Li} \ell_j \ \cong \ \begin{pmatrix} 1 & 1 & 1 \\ \Omega^4 & \Omega^4 & \Omega^4 \\ \Omega^3 & \Omega^3 & \Omega^3 \end{pmatrix} \ \text{(115)}$$

We see now that the transformations of $\Phi$’s in Table [12] were chosen exactly to make all the entries in $M_{\theta 0}$ eligible. Again expressing the invariance under $Z_5^0$ give constraints on the Yukawa couplings:

$$(Y^{\ell \theta}) = T^{D^0} \Phi_{ab} (T^{D^0}_L)^{i} (Y^{\nu \alpha}_0) \ (T^{D^0}_R)^{i}, \ \text{(116)}$$

and

$$(Y^{\ell \varphi}) = T^{D^0} \Phi_{ab} (T^{D^0}_L)^{j} (Y^{\nu \alpha}_0) \ (T^{D^0}_R)^{j}, \ \text{(117)}$$

where $(a, b = 1, \ldots, 3)$, $(Y^{\ell \theta}_0)$ is a matrix in flavor space with element $Y^{\ell \theta}_0_{ij}$ at its $(i,j)^{th}$ entry. The two constraints of Eqs. (116,117) are related in that if $(Y^{\ell \theta})$ was a solution of Eq.(113), then

$$(Y^{\ell \varphi}) = (S^T) \ (Y^{\ell \theta}) \ (S \ S^\varphi_{\varphi \nu}) \ \text{(118)}$$

is a solution of Eq.(117) where $S^\varphi_{\varphi \nu} = S$ since we have three $\Phi$’s.

Solving the Yukawa constraints in Eqs.(116,117) we see that when the $\Phi_a$’s get vevs $v^\varphi_{0a}$ we get the following $M_{\theta 0}$ and $M_{\ell}$,

$$M_{\theta 0} = \begin{pmatrix} Y^{\ell 1}_{011} v^\varphi_{01} & Y^{\ell 1}_{012} v^\varphi_{02} & Y^{\ell 1}_{013} v^\varphi_{03} \\ Y^{\ell 2}_{021} v^\varphi_{02} & Y^{\ell 2}_{022} v^\varphi_{02} & Y^{\ell 2}_{023} v^\varphi_{03} \\ Y^{\ell 3}_{031} v^\varphi_{03} & Y^{\ell 3}_{032} v^\varphi_{03} & Y^{\ell 3}_{033} v^\varphi_{03} \end{pmatrix} \ \text{(119)}$$

and

$$M_{\ell} = \begin{pmatrix} Y^{\ell 2}_{21} v^\varphi_{1} + Y^{\ell 2}_{11} v^\varphi_{2} & Y^{\ell 2}_{23} v^\varphi_{1} + Y^{\ell 2}_{13} v^\varphi_{2} & Y^{\ell 2}_{21} v^\varphi_{1} + Y^{\ell 2}_{13} v^\varphi_{2} \\ -Y^{\ell 2}_{11} v^\varphi_{1} + Y^{\ell 2}_{21} v^\varphi_{2} & -Y^{\ell 2}_{12} v^\varphi_{1} + Y^{\ell 2}_{22} v^\varphi_{2} & -Y^{\ell 2}_{12} v^\varphi_{1} + Y^{\ell 2}_{22} v^\varphi_{2} \\ Y^{\ell 3}_{31} v^\varphi_{3} & Y^{\ell 3}_{31} v^\varphi_{3} & Y^{\ell 3}_{33} v^\varphi_{3} \end{pmatrix} \ \text{(120)}$$

47
Again, one can deduce the relations between the “unrotated” vevs \( (v_{\Phi_a}^\text{\text{unrot}}) \) and the “rotated” vevs \( (v_{\Phi_a}^\text{\text{rot}}) \) by writing (cf. Eq. [63]) \( M_v = S^T M_{\ell o} S \) and considering Eq. (118).

Thus if we assume the related vevs are comparable \( v_{\Phi_1} \approx v_{\Phi_2} \approx v_{\Phi_3} = v \) then we get

\[
M_v \approx v \begin{pmatrix}
Y_{11}^{\nu_\ell} + Y_{12}^{\nu_\ell} & Y_{11}^{\nu_\ell} + Y_{12}^{\nu_\ell} & Y_{11}^{\nu_\ell} + Y_{12}^{\nu_\ell} \\
Y_{21}^{\nu_\ell} + Y_{22}^{\nu_\ell} & Y_{22}^{\nu_\ell} + Y_{23}^{\nu_\ell} & Y_{23}^{\nu_\ell} + Y_{23}^{\nu_\ell} \\
Y_{31}^{\nu_\ell} + Y_{32}^{\nu_\ell} & Y_{31}^{\nu_\ell} + Y_{32}^{\nu_\ell} & Y_{33}^{\nu_\ell} + Y_{33}^{\nu_\ell}
\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T
\end{pmatrix}
\]

Consequently,

\[
M_v M_v^\dagger \approx v^2 \begin{pmatrix}
a.a & a.b & a.c \\
b.a & b.b & b.c \\
c.a & c.b & c.c
\end{pmatrix}
\]

so taking only the following natural assumption on the norms of the vectors

\[
\| a \| / \| c \|= m_e/m_\tau \sim 3 \times 10^{-4}, \quad \| b \| / \| c \|= m_\mu/m_\tau \sim 6 \times 10^{-2}
\]

one can diagonalize \( M_v M_v^\dagger \) by an infinitesimal rotation as was done in [5], which proves that we are to a good approximation in the flavor basis.

### 12 Direct realization of type I seesaw with \( Z_6 \times Z_2 \)-symmetry

We present now another method which leads directly to the vanishing subtrace texture without relating it to zero textures by ‘rotation’. It is applicable again only for the four textures \( (C_{33}, C_{11}, C_{22} \text{ and } C_{31}) \).

#### 12.1 Type I seesaw Direct realization of \( C_{33} \): Vanishing of \( M_{\nu_{11}} + M_{\nu_{22}} \)

Within type I seesaw scenario, the Lagrangian responsible for mass is similar to the one given in Eq. [63] which, after conveniently simplifying the notations by dropping the Yukawa 0-subscript and the summation signs, is rewritten here

\[
\mathcal{L}_M \supset Y_{\nu_{ij}}^b \chi_b \nu_{Ri}^T C^{-1} \nu_{Rj} + Y_{\ell_{ij}}^a D_{Li} \Phi_a \nu_{Rj} + Y_{\ell_{ij}}^o D_{Li} \Phi_o \ell_{Rj}.
\]

We have for the pattern \( C_{33} \) the relation \( M_{\nu_{11}} + M_{\nu_{22}} = 0 \), which can give a hint motivating the search for solutions involving a permutation symmetry \( (1 \leftrightarrow 2) \). Actually, we can think of the vanishing subtrace constraint as arising from symmetry considerations leading to textures implementing these “permutation” restrictions at the level of \( M_R \) and \( M_D \), which by seesaw scenario resurface at the level of \( M_\nu \) which inherits the “permutation” structure. One can try simple forms for both \( M_R \) and \( M_D \) with enough number of parameters in order to produce generic \( M_\nu \) having the sole constraint \( M_{\nu_{11}} + M_{\nu_{22}} = 0 \). To be concrete, one can assume the following forms for \( M_R \) and \( M_D \) as shown below together with the derived \( M_\nu \) (through seesaw mechanism),

\[
M_R = \begin{pmatrix}
x & y & 0 \\
y & -x & 0 \\
0 & 0 & z
\end{pmatrix}, \quad M_D = \begin{pmatrix}
A & -B & iC \\
B & A & -C \\
-iD & D & E
\end{pmatrix}, \quad M_\nu = M_D M_R^{-1} M_D^T = \begin{pmatrix}
\Delta & \times & \times \\
-\Delta & \times & \times \\
\times & \times & \times
\end{pmatrix}
\]

where \( A, B, C, D, E, x, y \) and \( z \) are generic independent parameters, the \( \times \) and \( \Delta \) signs denote generic independent nonvanishing entries. We stress here that these forms proposed for \( M_R \)
and $M_D$ are not necessarily the simplest choice, but they are just mere possibilities that can be derived from symmetry considerations.

The fields and their assigned symmetry transformations under $Z_6 \times Z_2$ are presented in Table 13.

| Matter Content and symmetry transformation (Pattern $C_{33}$) |
|-------------------------------------------------------------|
| **Symmetry under $Z_6$**                                    |
| $\nu_{R1} \rightarrow \nu_{R1}$  | $\nu_{R2} \rightarrow \nu_{R2}$  | $\nu_{R3} \rightarrow \omega \nu_{R3}$  | $\chi_1 \rightarrow \omega^4 \chi_1$  | $\chi_2 \rightarrow \chi_2$  | $\chi_3 \rightarrow \chi_3$  |
| $D_{L1} \rightarrow D_{L1}$  | $D_{L2} \rightarrow D_{L2}$  | $D_{L3} \rightarrow \omega D_{L3}$  | $\Phi_1 \rightarrow \Phi_1$  | $\Phi_2 \rightarrow \omega \Phi_2$  | $\Phi_3 \rightarrow \omega^5 \Phi_3$  |
| $\Phi_4 \rightarrow \omega^3 \Phi_4$  | $\Phi_5 \rightarrow \omega^3 \Phi_5$  | $\ell_{R1} \rightarrow \omega^3 \ell_{R1}$  | $\ell_{R2} \rightarrow \omega^3 \ell_{R2}$  | $\ell_{R3} \rightarrow \omega^4 \ell_{R3}$  |

| **Symmetry under $Z_2$**                                    |
| $\nu_{R1} \rightarrow i \nu_{R2}$  | $\nu_{R2} \rightarrow -i \nu_{R1}$  | $\nu_{R3} \rightarrow \nu_{R3}$  | $\chi_1 \rightarrow \chi_1$  | $\chi_2 \rightarrow \chi_2$  | $\chi_3 \rightarrow \chi_3$  |
| $D_{L1} \rightarrow i D_{L2}$  | $D_{L2} \rightarrow -i D_{L1}$  | $D_{L3} \rightarrow D_{L3}$  | $\Phi_1 \rightarrow \Phi_1$  | $\Phi_2 \rightarrow -\Phi_2$  | $\Phi_3 \rightarrow \Phi_3$  |
| $\Phi_4 \rightarrow i \Phi_5$  | $\Phi_5 \rightarrow -i \Phi_4$  | $\ell_{R1} \rightarrow \ell_{R1}$  | $\ell_{R2} \rightarrow \ell_{R2}$  | $\ell_{R3} \rightarrow \ell_{R3}$  |

Table 13: The $Z_6 \times Z_2$ symmetry seesaw type I realization of the vanishing subtrace $C_{33}$. $\Phi$ are five SM Higgs doublets, $D_L$ refers to the flavor three left handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. $\omega$ denotes $e^{i \pi/3}$.

Forming the required bilinears dictated by $Z_6$ symmetry, we obtain

$$\nu_{R1}^T \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^2 \\ \omega & \omega & \omega^3 \end{pmatrix}, \quad \overline{D}_{Li} \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^5 \\ \omega^5 & \omega^4 & \omega \end{pmatrix}, \quad \ell_{Rj} \equiv_{Z_6} \begin{pmatrix} -1 & -1 & \omega^4 \\ 1 & 1 & \omega^4 \\ \omega^2 & \omega^2 & -1 \end{pmatrix}. \quad (126)$$

When the resulting bilinears combine with the appropriate scalar fields, we get under $Z_6$, keeping only the combinations that produce singlets, the following:

$$\chi_1 \nu_{R1}^T \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} \omega^4 & \omega^4 & \omega^5 \\ \omega^4 & \omega^4 & \omega^5 \\ \omega^5 & \omega^5 & 1 \end{pmatrix}, \quad \chi_2 \nu_{R1}^T \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^5 \\ \omega & \omega & \omega^2 \end{pmatrix}, \quad \chi_3 \nu_{R1}^T \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega & \omega & \omega^2 \end{pmatrix}, \quad (127)$$

$$\Phi_1 \overline{D}_{Li} \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^5 \\ \omega & \omega & \omega \end{pmatrix}, \quad \Phi_2 \overline{D}_{Li} \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} \omega & \omega & \omega \end{pmatrix}, \quad \Phi_3 \overline{D}_{Li} \nu_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega & \omega & \omega \end{pmatrix}, \quad (127)$$

$$\Phi_4 \overline{D}_{Li} \ell_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \end{pmatrix}, \quad \Phi_5 \overline{D}_{Li} \ell_{Rj} \equiv_{Z_6} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \end{pmatrix}. \quad (127)$$
Thus the resulting Lagrangian dictated by $Z_6$ symmetry takes the form,

\[
\mathcal{L}_{M}^{Z_6} \propto Y_{\chi_33}^{1} \chi_{1} T_{R_3} C^{-1} \nu_{R_3} + Y_{\chi_33}^{1} \chi_{1} T_{R_1} C^{-1} \nu_{R_1} + Y_{\chi_33}^{1} \chi_{1} T_{R_2} C^{-1} \nu_{R_2} + Y_{\chi_33}^{1} \chi_{1} T_{R_3} C^{-1} \nu_{R_3} + Y_{\chi_33}^{1} \chi_{1} T_{R_1} C^{-1} \nu_{R_1} + Y_{\chi_33}^{1} \chi_{1} T_{R_2} C^{-1} \nu_{R_2} + Y_{\chi_33}^{1} \chi_{1} T_{R_3} C^{-1} \nu_{R_3}
\]

which transforms under $Z_2$ as

\[
\mathcal{L}_{M}^{Z_2} \propto Y_{\chi_33}^{1} \chi_{1} T_{R_3} C^{-1} \nu_{R_3}
\]

Thus, invariance under $Z_6 \times Z_2$ implies the following constraints on the Yukawa couplings:

\[
Y_{\chi_33}^{1} = Y_{\chi_33}^{1}, \quad Y_{\chi_12}^{2} = Y_{\chi_12}^{2}, \quad Y_{\chi_12}^{3} = Y_{\chi_12}^{3}, \quad Y_{\chi_11}^{2} = -Y_{\chi_22}^{2}, \quad Y_{\chi_11}^{3} = -Y_{\chi_22}^{3},
\]

\[
Y_{\chi_33}^{1} = Y_{\chi_33}^{1}, \quad Y_{\chi_12}^{1} = Y_{\chi_12}^{1}, \quad Y_{\chi_12}^{2} = Y_{\chi_12}^{2}, \quad Y_{\chi_12}^{3} = Y_{\chi_12}^{3}, \quad Y_{\chi_11}^{1} = -Y_{\chi_22}^{1}, \quad Y_{\chi_11}^{2} = -Y_{\chi_22}^{2}, \quad Y_{\chi_11}^{3} = -Y_{\chi_22}^{3},
\]

\[
Y_{\chi_33}^{1} = Y_{\chi_33}^{1}, \quad Y_{\chi_12}^{1} = Y_{\chi_12}^{1}, \quad Y_{\chi_12}^{2} = Y_{\chi_12}^{2}, \quad Y_{\chi_12}^{3} = Y_{\chi_12}^{3}, \quad Y_{\chi_11}^{1} = -Y_{\chi_22}^{1}, \quad Y_{\chi_11}^{2} = -Y_{\chi_22}^{2}, \quad Y_{\chi_11}^{3} = -Y_{\chi_22}^{3},
\]

where all vanishing Yukawa couplings are omitted. In fact and as was done for the ‘rotated’ symmetry (indirect realization), by brute force, one also could have used all the machinery encoded in the invariance equations, as given in Eqs. (128, 129), in order to obtain a system of linear equations involving Yukawa coupling constants. Solving this resulting system of linear equations would have provided us then with the symmetry constraints (Eq. 130).

Thus, the $Z_6 \times Z_2$ symmetry imposes some constraints on the Yukawa couplings which have to be taken into consideration when constructing mass terms after the relevant scalar fields acquire vevs. The emergent $M_R$ and $M_D$ turn out to be,

\[
M_R = \begin{pmatrix}
-Y_{\chi_22}^{2} v_{\chi_2} - Y_{\chi_23}^{2} v_{\chi_3} & Y_{\chi_12}^{2} v_{\chi_2} + Y_{\chi_12}^{3} v_{\chi_3} & 0 \\
Y_{\chi_22}^{2} v_{\chi_2} + Y_{\chi_23}^{3} v_{\chi_3} & 0 & 0 \\
Y_{\chi_33}^{1} v_{\chi_3} & 0 & 0
\end{pmatrix}
\]

(131)
Following the same method outlined in case M₁, one can check that the resulting $M_\nu$, through seesaw mechanism, satisfies the texture $C_{33}$.

As to $M_\ell$ we get,

$$M_\ell = \begin{pmatrix}
Y_{\ell 21}^5 v_{\Phi 4} + Y_{\ell 11}^5 v_{\Phi 5} & Y_{\ell 22}^5 v_{\Phi 4} + Y_{\ell 12}^5 v_{\Phi 5} & 0 \\
-Y_{\ell 11}^5 v_{\Phi 4} + Y_{\ell 21}^5 v_{\Phi 5} & Y_{\ell 12}^5 v_{\Phi 4} + Y_{\ell 22}^5 v_{\Phi 5} & 0 \\
0 & 0 & -i Y_{\ell 33}^5 v_{\Phi 4} + Y_{\ell 32}^5 v_{\Phi 5}
\end{pmatrix}. \quad (133)$$

Thus, and as an example, one can assume $v \approx v_{\Phi 5} \gg v_{\Phi 4}$ so that to get

$$M_\ell \approx v \begin{pmatrix}
Y_{\ell 11}^5 & Y_{\ell 12}^5 & 0 \\
Y_{\ell 21}^5 & Y_{\ell 22}^5 & 0 \\
0 & 0 & Y_{\ell 33}^5
\end{pmatrix} = v \begin{pmatrix}
a^T \\
b^T \\
c^T
\end{pmatrix}. \quad (134)$$

Another time, one can by just imposing some reasonable assumptions on the ratios of the ‘free’ vectors diagonalize $M_\ell M_\ell^T$ by an infinitesimal rotation, which puts us thus to a good approximation in the flavor basis, as desired.

12.2 Type I seesaw Direct realization of $C_{11}$, $C_{22}$ and $C_{13}$

Following the same method outlined in case $C_{33}$, we state briefly the results of the cases $C_{11}$, $C_{22}$ and $C_{13}$, in Tables (14,15) and (16) respectively.

| Matter Content and symmetry transformation (Pattern $C_{11}$) |
|---------------------------------------------------------------|
| Symmetry under $Z_6$ |
| $\nu_{R1} \rightarrow \omega \nu_{R1}$ & $\nu_{R2} \rightarrow \nu_{R2}$ & $\nu_{R3} \rightarrow \nu_{R3}$ & $\chi_1 \rightarrow \omega^4 \chi_1$ & $\chi_2 \rightarrow \chi_2$ & $\chi_3 \rightarrow \chi_3$ |
| $D_{L1} \rightarrow \omega D_{L1}$ & $D_{L2} \rightarrow \omega D_{L2}$ & $D_{L3} \rightarrow \omega D_{L3}$ & $\Phi_1 \rightarrow \Phi_1$ & $\Phi_2 \rightarrow \omega^5 \Phi_2$ | $\Phi_3 \rightarrow \omega \Phi_3$ |
| $\Phi_4 \rightarrow \omega^3 \Phi_4$ & $\Phi_5 \rightarrow \omega^3 \Phi_5$ & $\ell_{R1} \rightarrow \omega^4 \ell_{R1}$ & $\ell_{R2} \rightarrow \omega^3 \ell_{R2}$ & $\ell_{R3} \rightarrow \omega^3 \ell_{R3}$ |
| Symmetry under $Z_2$ |
| $\nu_{R1} \rightarrow \nu_{R1}$ & $\nu_{R2} \rightarrow i \nu_{R3}$ & $\nu_{R3} \rightarrow -i \nu_{R2}$ & $\chi_1 \rightarrow \chi_1$ & $\chi_2 \rightarrow \chi_2$ & $\chi_3 \rightarrow \chi_3$ |
| $D_{L1} \rightarrow D_{L1}$ & $D_{L2} \rightarrow i D_{L3}$ & $D_{L3} \rightarrow -i D_{L2}$ & $\Phi_1 \rightarrow \Phi_1$ & $\Phi_2 \rightarrow \Phi_2$ & $\Phi_3 \rightarrow -\Phi_3$ |
| $\Phi_4 \rightarrow i \Phi_5$ & $\Phi_5 \rightarrow -i \Phi_4$ & $\ell_{R1} \rightarrow \ell_{R1}$ & $\ell_{R2} \rightarrow \ell_{R2}$ & $\ell_{R3} \rightarrow \ell_{R3}$ |

| Mass matrices $M_R$, $M_D$, $M_\nu$, and $M_\ell$ |
|---------------------------------------------------------------|
| $M_R = \begin{pmatrix}
x & 0 & 0 \\
0 & y & z \\
0 & z & -g
\end{pmatrix}$, $M_D = \begin{pmatrix}
A & -iD & D \\
-E & B & -C \\
E & C & B
\end{pmatrix}$, $M_\nu = \begin{pmatrix}
\times & \times & \times \\
-\Delta & \times & \times \\
\times & -\Delta & \times
\end{pmatrix}$, $M_\ell v_{\Phi 5} \gg v_{\Phi 4}$ $v_{\Phi 5}$ $y_{\ell 41}$ $0$ $0$ $y_{\ell 22}$ $y_{\ell 23}$ $y_{\ell 32}$ $y_{\ell 33}$ |

Table 14: The $Z_6 \times Z_2$ symmetry seesaw type I realization of the vanishing subtrace $C_{11}$. $\Phi_a$ are five SM Higgs doublets ($a = 1 \cdots 5$), $D_L$ refers to the flavor three left handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. $\omega$ denotes $e^{i\pi/3}$
behind this realization is having a permutation performed through the group factor \( \ell \). Right-handed charged lepton singlets are denoted by \( \nu \). Table 15: The \( Z_6 \times Z_2 \) symmetry seesaw type I realization of the vanishing subtrace \( C_{22} \). \( \Phi_a \) are five SM Higgs doublets \((a = 1 \cdots 5)\), \( D_1 \) refers to the flavor three left handed lepton doublets, while the three right-handed charged lepton singlets are denoted by \( \ell _R \). \( \omega \) denotes \( e^{i \pi/3} \).

Table 16: The \( Z_6 \times Z_2 \) symmetry seesaw type I realization of the vanishing subtrace \( C_{13} \). \( \Phi_a \) are four SM Higgs doublets \((a = 1 \cdots 4)\), \( D_1 \) refers to the flavor three left handed lepton doublets, while the three right-handed charged lepton singlets are denoted by \( \ell _R \). \( \omega \) denotes \( e^{i \pi/3} \) and \( u \) is a generic independent parameter present in \( M_R \).

13 Direct realization of type II seesaw with \( Z'_2 \times Z_2 \)-symmetry

By the same token, we present now, within type II seesaw scenario, a “direct” method which leads straightforwardly to the vanishing subtrace texture without relating it to zero textures by ‘rotation’. It is applicable again only for the four textures \((C_{33}, C_{11}, C_{22} \text{ and } C_{31})\). Besides, the key idea behind this realization is having a permutation performed through the group factor \( Z'_2 \).
13.1 Type II seesaw Direct realization of $C_{33}$

Within type II seesaw scenario, the term $Y^{\nu a}_{ij} H^0_d \nu_{Li}^T C^{-1} \nu_{Lj}$ in the Lagrangian of Eq. (107) is the term responsible for $M_\ell$, where we introduced three Higgs triplets. We introduce two Higgs doublets $\Phi_b$ responsible for $M_\ell$ through the term $Y^{\nu a}_{ij} \overline{\Phi}_b \Phi_a \ell_{Rj}$ (see Eq.(109)). We assume the field transformations defined in Table (17).

| Matter Content (Pattern $C_{33}$) |
|-----------------------------------|
| $H$ | $D_L$ | $\ell_R$ | $\Phi$ |
| Symmetry under $Z'_2$ |
| $G'_H H$ | $G'_D D_L$ | $G'_L \ell_R$ | $G'_\Phi \Phi$ |
| $G'_H = \text{diag} (1, 1, -1)$ | $G'_D = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $G'_L = \text{diag} (1, 1, 1)$ | $G'_\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| Symmetry under $Z_2$ |
| $G_H H$ | $G_D D_L$ | $G_L \ell_R$ | $G_\Phi \Phi$ |
| $G_H = \text{diag} (1, -1, -1)$ | $G_D = \text{diag} (-1, -1, 1)$ | $G_L = \text{diag} (1, 1, -1)$ | $G_\Phi = \text{diag} (-1, -1)$ |

Table 17: The $Z'_2 \times Z_2$ symmetry seesaw type II realization of the vanishing subtrace $C_{33}$. $H$ are three triplet scalars, $D_L$ refers to the flavor three left handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. The $\Phi$ denotes two SM Higgs doublets.

By forming bilinear terms in order to see the transformations under $Z_2$, we get for $\nu_{Li}^T \nu_{Lj}$,

$$\nu_{Li}^T \nu_{Lj} \xrightarrow{Z_2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

(135)

and so, when the bilinear $\nu_{Li}^T \nu_{Lj}$ is combined with the transformation of $H_a$ under $Z_2$, we get

$$H^0_d \nu_{Li}^T \nu_{Lj} \xrightarrow{Z_2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad H^0_d \nu_{Li}^T \nu_{Lj} \xrightarrow{Z_2} \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad H^0_d \nu_{Li}^T \nu_{Lj} \xrightarrow{Z_2} \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(136)

Thus, the Lagrangian terms in Eq.(107), responsible for $M_\ell$, which are due to the interaction with $H^0_d$ and are consistent with $Z_2$ symmetry are

$$L_{H_1 \nu}^Z \times H^0_d \left( Y_{11}^{\nu a} \nu_{Li}^T \nu_{L1} + Y_{22}^{\nu a} \nu_{Li}^T \nu_{L2} C^{-1} \nu_{L2} + Y_{33}^{\nu a} \nu_{Li}^T \nu_{L3} C^{-1} \nu_{L3} + Y_{12}^{\nu a} \nu_{Li}^T \nu_{L1} C^{-1} \nu_{L2} + Y_{12}^{\nu a} \nu_{Li}^T \nu_{L2} C^{-1} \nu_{L1} \right),$$

(137)

which transforms under $Z'_2$ as

$$L_{H_1 \nu}^Z \times H^0_d \left( -Y_{11}^{\nu a} \nu_{Li}^T \nu_{L1} C^{-1} \nu_{L2} - Y_{22}^{\nu a} \nu_{Li}^T \nu_{L2} C^{-1} \nu_{L1} + Y_{33}^{\nu a} \nu_{Li}^T \nu_{L3} C^{-1} \nu_{L3} + Y_{12}^{\nu a} \nu_{Li}^T \nu_{L1} C^{-1} \nu_{L2} + Y_{12}^{\nu a} \nu_{Li}^T \nu_{L2} C^{-1} \nu_{L1} \right).$$

(138)

Thus, invariance under $Z_2 \times Z'_2$ implies the constraint:

$$Y_{11}^{\nu a} = -Y_{22}^{\nu a}, \quad Y_{13}^{\nu a} = Y_{23}^{\nu a} = 0.$$
By the same way, one can see the constraints on the Yukawa couplings due to interaction with $H_2^0$ and $H_3^0$, and we get

$$
\begin{align*}
Y_{13}^{\nu 2} &= i Y_{23}^{\nu 2}, & Y_{11}^{\nu 2} &= Y_{22}^{\nu 2} = Y_{12}^{\nu 2} = 0,
Y_{13}^{\nu 3} &= -i Y_{23}^{\nu 3}, & Y_{11}^{\nu 3} &= Y_{22}^{\nu 3} = Y_{12}^{\nu 3} = 0.
\end{align*}
$$

(140)

So when $H_a^0$ gets a vev $v_a^H$ we get $M_\nu$ in the form

$$
M_\nu = \begin{pmatrix}
- Y_{22}^{1} v_1^{H} & Y_{12}^{1} v_1^{H} & i \left( Y_{23}^{\nu 2} v_2^{H} - Y_{23}^{\nu 3} v_3^{H} \right) \\
- Y_{22}^{1} v_1^{H} & Y_{22}^{1} v_1^{H} & \left( Y_{23}^{\nu 2} v_2^{H} + Y_{23}^{\nu 3} v_3^{H} \right) \\
- Y_{33}^{\nu 1} v_1^{H} & - Y_{33}^{\nu 1} v_1^{H} & Y_{33}^{\nu 1} v_1^{H} \\
\end{pmatrix}.
$$

(141)

We see that the texture $C_{33}$ is realized.

For the charged lepton mass matrix $M_\ell$, we follow the same procedure by forming bilinear terms in order to see the transformations under $Z_2$:

$$
\begin{align*}
D_{Li} \ell_{Rj} \delta Z_2 \Rightarrow \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}
\end{align*}

\Rightarrow
\begin{align*}
D_{Li} \ell_{Rj} \phi_1 \delta Z_2 \phi_2 \delta Z_2
\end{align*}

(142)

Thus, the Lagrangian terms in Eq. (109), responsible for $M_\ell$, which are due to the interaction with $\phi_1, \phi_2$ and are consistent with $Z_2$ symmetry are

$$
L_{\phi \ell} \propto \phi_1 ( Y_{11}^{\nu \ell 1} D_{L1} \ell_{R1} + Y_{12}^{\nu \ell 1} D_{L2} \ell_{R2} + Y_{21}^{\nu \ell 1} D_{L1} \ell_{R1} + Y_{22}^{\nu \ell 1} D_{L2} \ell_{R2} + Y_{33}^{\nu \ell 1} D_{L3} \ell_{R3} ) +
$$

$$
\phi_2 ( Y_{11}^{\nu \ell 2} D_{L1} \ell_{R1} + Y_{12}^{\nu \ell 2} D_{L2} \ell_{R2} + Y_{21}^{\nu \ell 2} D_{L1} \ell_{R1} + Y_{22}^{\nu \ell 2} D_{L2} \ell_{R2} + Y_{33}^{\nu \ell 2} D_{L3} \ell_{R3} ),
$$

(143)

which transforms under $Z_2$ as

$$
L_{\phi \ell} \delta Z_2 \phi_2 \delta Z_2
\Rightarrow \begin{pmatrix} -i Y_{11}^{\nu \ell 1} D_{L2} \ell_{R1} - i Y_{12}^{\nu \ell 1} D_{L2} \ell_{R2} + i Y_{21}^{\nu \ell 1} D_{L1} \ell_{R1} + i Y_{22}^{\nu \ell 1} D_{L1} \ell_{R2} + Y_{33}^{\nu \ell 1} D_{L3} \ell_{R3} \\ -i Y_{11}^{\nu \ell 2} D_{L2} \ell_{R1} - i Y_{12}^{\nu \ell 2} D_{L2} \ell_{R2} + i Y_{21}^{\nu \ell 2} D_{L1} \ell_{R1} + i Y_{22}^{\nu \ell 2} D_{L1} \ell_{R2} + Y_{33}^{\nu \ell 2} D_{L3} \ell_{R3} \\ -i Y_{11}^{\nu \ell 2} D_{L2} \ell_{R1} - i Y_{12}^{\nu \ell 2} D_{L2} \ell_{R2} + i Y_{21}^{\nu \ell 2} D_{L1} \ell_{R1} + i Y_{22}^{\nu \ell 2} D_{L1} \ell_{R2} + Y_{33}^{\nu \ell 2} D_{L3} \ell_{R3} \\ \end{pmatrix}.
$$

(144)

Thus, invariance under $Z_2 \times Z_2'$ implies the constraint:

$$
\begin{align*}
Y_{11}^{\nu \ell 1} &= Y_{11}^{\nu \ell 2}, & Y_{12}^{\nu \ell 1} &= Y_{12}^{\nu \ell 2}, & Y_{21}^{\nu \ell 1} &= Y_{22}^{\nu \ell 1}, & Y_{21}^{\nu \ell 2} &= Y_{22}^{\nu \ell 2}, & Y_{33}^{\nu \ell 1} &= Y_{33}^{\nu \ell 2}, \\
Y_{11}^{\nu (2)} &= Y_{12}^{\nu (2)} = Y_{33}^{\nu (2)} = Y_{32}^{\nu (2)} = 0.
\end{align*}
$$

(145)

So when $\Phi_0^0$ gets a vev $v_a^\phi$ we get $M_\ell$ in the form

$$
M_\ell = \begin{pmatrix}
Y_{11}^{\nu \ell 1} v_1^\phi & i Y_{21}^{\nu \ell 1} v_2^\phi & 0 \\
Y_{12}^{\nu \ell 1} v_1^\phi & Y_{12}^{\nu \ell 2} v_2^\phi & 0 \\
0 & 0 & Y_{33}^{\nu \ell 1} (v_1^\phi + v_2^\phi)
\end{pmatrix}.
$$

(146)

Thus, when $v \approx v_2^\phi \gg v_1^\phi$ we get

$$
M_\ell = v \begin{pmatrix}
i Y_{21}^{\nu \ell 1} & i Y_{22}^{\nu \ell 1} & 0 \\
- i Y_{11}^{\nu \ell 2} & - i Y_{12}^{\nu \ell 2} & 0 \\
0 & 0 & Y_{33}^{\nu \ell 1}
\end{pmatrix} = v \begin{pmatrix} a^T \\
b^T \\
c^T
\end{pmatrix}.
$$

(147)

One can by just imposing some reasonable assumptions on the ratios of the ‘free’ vectors diagonalize $M_\ell M_\ell^T$ by an infinitesimal rotation, which puts us thus to a good approximation in the flavor basis, as desired.
13.2 Type II seesaw Direct realization of $C_{11}$, $C_{22}$ and $C_{13}$

Following the same method outlined in case $C_{33}$, we state briefly the results of the cases $C_{11}$, $C_{22}$ and $C_{13}$, in Tables (18,19) and (20) respectively.

| Matter content (Pattern $C_{11}$) |
|-----------------------------------|
| $H$ | $D_L$ | $\ell_R$ | $\Phi$ |
|---|---|---|---|

Symmetry under $Z_2'$

| $G'_H H$ | $G'_D D_L$ | $G'_\ell \ell_R$ | $G'_\Phi \Phi$ |
|---|---|---|---|
| $G'_H = \text{diag} (1,1,-1)$ | $G'_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ | $G'_\ell = \text{diag} (1,1,1)$ | $G'_\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |

Symmetry under $Z_2$

| $G_H H$ | $G_D D_L$ | $G_\ell \ell_R$ | $G_\Phi \Phi$ |
|---|---|---|---|
| $G_H = \text{diag} (1,-1,-1)$ | $G_D = \text{diag} (1,-1,-1)$ | $G_\ell = \text{diag} (-1,1,1)$ | $G_\Phi = \text{diag} (-1,-1,1)$ |

Mass matrices

$$M_\nu = \begin{pmatrix} Y^{\nu 1}_{11} v_1^H & i \left( Y^{\nu 2}_{13} v_2^H - Y^{\nu 3}_{13} v_3^H \right) & Y^{\nu 2}_{13} v_2^H + Y^{\nu 3}_{13} v_3^H \\ -Y^{\nu 2}_{33} v_2^H & Y^{\nu 3}_{33} v_3^H & \end{pmatrix} \text{diag} (v_1^\Phi, v_2^\Phi) \approx \begin{pmatrix} Y^{\nu 1}_{11} & 0 & 0 \\ 0 & Y^{\nu 2}_{22} & Y^{\nu 2}_{23} \\ 0 & Y^{\nu 2}_{32} & Y^{\nu 2}_{33} \end{pmatrix}$$

Table 18: The $Z'_2 \times Z_2$ symmetry seesaw type II realization of the vanishing subtrace $C_{11}$. $H$ are three triplet scalars, $D_L$ refers to the flavor three left-handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. The $\Phi$ denotes two SM Higgs doublets.
charged lepton singlets are denoted by $\ell$.

Table 20: The $Z'_2 \times Z_2$ symmetry seesaw type II realization of the vanishing subtrace $C_{22}$. $H$ are three triplet scalars, $D_L$ refers to the flavor three left-handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. The $\Phi$ denotes two SM Higgs doublets.

| $H$ | $D_L$ | $\ell_R$ | $\Phi$ |
|-----|-------|----------|-------|
| $G'_H H$ | $G'_D D_L$ | $G'_{\ell R}$ | $G'_\Phi$ |
| $G'_H = \text{diag} (1,1,-1)$ | $G'_D = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix}$ | $G'_{\ell} = \text{diag} (1,1,1)$ | $G'_\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |

Symmetry under $Z'_2$

| $G_H H$ | $G_D D_L$ | $G_{\ell} \ell_R$ | $G_{\Phi} \Phi$ |
|----------|----------|------------------|------------------|
| $G_H = \text{diag} (1,-1,-1)$ | $G_D = \text{diag} (-1,1,-1)$ | $G_{\ell} = \text{diag} (1,-1,1)$ | $G_{\Phi} = \text{diag} (-1,-1)$ |

Mass matrices

$$M_\nu = \begin{pmatrix} -Y_{33}^{\nu_1} v_1^H & i \left( Y_{23}^{\nu_2} v_2^H - Y_{33}^{\nu_3} v_3^H \right) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \approx v_1^H, \quad v_2^H, \quad v_3^H, \quad v_4^H, \quad v_5^H, \quad v_6^H, \quad v_7^H, \quad v_8^H, \quad v_9^H, \quad v_10^H, \quad v_11^H, \quad v_12^H, \quad v_13^H$$

Table 19: The $Z'_2 \times Z_2$ symmetry seesaw type II realization of the vanishing subtrace $C_{13}$. $H$ are three triplet scalars, $D_L$ refers to the flavor three left-handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. The $\Phi$ denotes two SM Higgs doublets.

| $H$ | $D_L$ | $\ell_R$ | $\Phi$ |
|-----|-------|----------|-------|
| $G'_H H$ | $G'_D D_L$ | $G'_{\ell R}$ | $G'_\Phi$ |
| $G'_H = \text{diag} (1,-1,-1)$ | $G'_D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | $G'_{\ell} = \text{diag} (1,1,1)$ | $G'_\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |

Symmetry under $Z'_2$

| $G_H H$ | $G_D D_L$ | $G_{\ell} \ell_R$ | $G_{\Phi} \Phi$ |
|----------|----------|------------------|------------------|
| $G_H = \text{diag} (1,1,-1)$ | $G_D = \text{diag} (-1,1,-1)$ | $G_{\ell} = \text{diag} (1,-1,1)$ | $G_{\Phi} = \text{diag} (-1,-1)$ |

Mass matrices

$$M_\nu = \begin{pmatrix} Y_{33}^{\nu_1} v_1^H & Y_{33}^{\nu_2} v_2^H & Y_{33}^{\nu_3} v_3^H \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \approx v_1^H, \quad v_2^H, \quad v_3^H, \quad v_4^H, \quad v_5^H, \quad v_6^H, \quad v_7^H, \quad v_8^H, \quad v_9^H, \quad v_10^H, \quad v_11^H, \quad v_12^H, \quad v_13^H$$

Table 20: The $Z'_2 \times Z_2$ symmetry seesaw type II realization of the vanishing subtrace $C_{13}$. $H$ are three triplet scalars, $D_L$ refers to the flavor three left-handed lepton doublets, while the three right-handed charged lepton singlets are denoted by $\ell_R$. The $\Phi$ denotes two SM Higgs doublets.

56
14 Summary and Conclusion

We have studied a specific texture characterized by one vanishing subtrace of the neutrino mass matrix. We found that all textures, whether they be of inverted or normal type, can accommodate the recent experimental bounds. Moreover, four textures of inverted type can accommodate data in case one neutrino mass is zero. We have carried out a complete phenomenological and analytic analysis, but did not state the analytic expressions, as they are too cumbersome, even the first terms in a Taylor expansion in powers of $s_{13}$. Finally, for the model building of the texture, we first proposed a ‘generic’ strategy to justify such a specific texture form based on finding a corresponding symmetry implying certain zeros at $M_{\nu 0}$, which when ‘rotated’ to a new ‘rotated symmetry’ leads to the desired form of vanishing subtrace in $M_{\nu}$. We applied this strategy for both types of seesaw scenarios and in both invertible and singular neutrino mass matrices. We also presented a direct method to realize the textures without ‘rotation’ for both types of seesaw scenarios based on discrete symmetry.

Acknowledgements

M.A. thanks the staff members of the CPPM-Marseille and LPTM-Cergy-Pontoise University Laboratories. N.C. acknowledges support from ICTP-Associate program, ITP-PIFI program and Humboldt Foundation.

References

[1] see e.g. K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010).

[2] see e.g. Y. Fukuda et al., Phys. Lett. B 436, 33 (1998); Phys. Rev. Lett. 81, 1562 (1998); C. K. Jung, C. McGrew, T. Kajita, and T. Mann, Annu. Rev. Nucl. Part. Sci. 51, 451 (2001), and references therein.

[3] P. H. Frampton, S. L. Glashow, and D. Marfatia, Phys. Lett. B 536, 79 (2002),
Z. Z. Xing, Phys. Lett. B 530, 159 (2002).
H. Fritzsch, Z. Z. Xing, and S. Zhou, J. High Energy Phys. 09 (2011) 083.

[4] A. Merle and W. Rodejohann, Phys. Rev. D 73, 073012 (2006).

[5] E. I. Lashin and N. Chamoun, Phys. Rev. D 85, 113011 (2012)

[6] L. Lavoura, Phys. Lett. B 609, 317 (2005).
E. I. Lashin and N. Chamoun, Phys. Rev. D 78, 073002 (2008) and Phys. Rev. D 80, 093004 (2009).

[7] E. I. Lashin and N. Chamoun, Phys. Rev. D 89, 093004 (2014), Phys. Rev. D 91, 113014 (2015) and Phys. Rev. D 96, 015003 (2017)

[8] H. A. Alhendi, E. I. Lashin, and A. A. Mudlej, Phys. Rev. D 77, 013009 (2008); [arXiv:0708.2007].

[9] F. Capozzi, E. D. Valentino, E. Lisi, A. Marrone, A. Melchiorri and A. Palazzo, Phys. Rev. D 95, 096014 (2017)
[10] F. P. An et al. (DAYA-BAY Collaboration), Phys. Rev. Lett. 108, 171803 (2012).

[11] W. Grimus and L. Lavoura, Eur. Phys. J. C 39, 219 (2005); A. Dighe, S. Goswami, and P. Roy, Phys. Rev. D 76, 096005 (2007); S. Luo and Z.-z. Xing, Phys. Rev. D 86, 073003 (2012).

[12] Heinrich Päs and Werner Rodejohann, (2015) New J. Phys. 17 115010

[13] E. Andreotti et al., Astropart. Phys. 34, 822 (2011).

[14] M. Bargiotti, et al., Riv.Nuovo Cim.23N3:1,2000, arXiv:hep-ph/0001293

[15] T. P. Cheng and L. F. Li, Phys. Rev. D 22, 2860 (1980); R. N. Mohapatra and G. Senjanovic, Phys. Rev. D 23, 165 (1981).