Three dimensional vortices in Abelian Gauge Theories

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Abstract
In this paper we consider an Abelian Gauge Theory in $\mathbb{R}^4$ equipped with the Minkowski metric. This theory leads to a system of equations, the Klein-Gordon-Maxwell equations, which provide models for the interaction between the electromagnetic field and matter. A three dimensional vortex is a finite energy solution of these equations in which the magnetic field looks like the field created by a finite solenoid. Under suitable assumptions, we prove the existence of vortex-solutions.

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1 Introduction

Abelian gauge theories (in $\mathbb{R}^4$ equipped with the Minkowski metric), provide models for the interaction between the electromagnetic field and matter. Actually an Abelian gauge theory leads to a system of equations, the Klein-Gordon-Maxwell equations (KGM), which occurs in various physical problems (elementary particles, superconductivity, ...); see e.g. [14], [16].

The properties of the solutions of the KGM depend on the lower order term $W$ (see (2)). Actually the choice of this term determines the peculiarities of the various models.

In this paper we show that a suitable choice of the term $W$ guarantees the existence of finite energy vortices in three space dimensions. Roughly speaking, a vortex is a finite energy solution in which the magnetic field looks like the field created by a solenoid.

As far as we know, the existence of vortex-solutions for Abelian gauge theories has been studied only in the case of two space dimensions (see the pioneering papers [1], [12] and the books [11], [13], [14], [16] with their references). Clearly the two dimensional vortices in the $x_1, x_2$ plane can be extended to $\mathbb{R}^3$ as constant maps in the $x_3$-direction. Of course these solutions have infinite energy. We point out that, in the 2-dimensional models, the functions $W$ that have been considered are of the type

$$W(s) = (1 - s^2)^2$$

name double well shaped and positive functions.

Here the lower order term which we have considered is the following one:

$$W(s) = \frac{1}{2} s^2 - \frac{s^p}{p}, \quad 2 < p < 6, \quad s \geq 0$$

The KGM with the lower order term (1) has been studied in [3], [4], [6], [7], [8], [9]. In these papers the existence of stationary solutions having spherical symmetry has been proved. It is easy to see that the spherical symmetry implies that both the magnetic field and the angular momentum vanish. On the contrary, a vortex-solution breaks the spherical symmetry (cf. Remark 5), moreover the magnetic field and the angular momentum do not vanish.

Since the KGM are invariant for the Lorentz group, a Lorentz transformation of a vortex creates a travelling solitary wave. By solitary wave, we mean a solution of a field equation whose energy travels as a localized packet. In this respect solitary waves have a particle-like behavior (see e.g. [11], [12]). A detailed discussion of solitary waves for KGM can be found in [4]. Solitary waves obtained by vortices behave as particles having a magnetic field and a kind of spin. When considering evolution problems relative to KGM, the request $W(s) \geq 0$ seems necessary to have "good" solutions (cf. [5]). In [4], the existence of solitary waves relative to stationary solutions having spherical symmetry has been
proved also for a class of functions $W \geq 0$. The existence of three dimensional vortices for positive $W$ is still an open problem.

The paper is organized as follows: In the first section we introduce the KGM-equations, we give a precise definition of three dimensional vortex solution and we state the main existence theorem. In the second section we introduce the functional framework. In the third and forth sections we prove the existence theorem.

2 Statement of the problem

2.1 The Klein-Gordon-Maxwell system

Let $\psi : \mathbb{R}^4 \to \mathbb{C}$ be a complex scalar field on the space-time $\mathbb{R}^4$ and $\Gamma$ be a 1-form on $\mathbb{R}^4$ whose coefficients $\Gamma_j$ are in the Lie algebra $u(1)$ of the group $U(1) = S^1$, i.e. $\Gamma_j = -iA_j$, where $i$ is the imaginary unit and $A_j$ ($j = 0, \ldots, 3$) are real maps defined in $\mathbb{R}^3$.

Consider the Abelian gauge theory related to $\psi$ and to $\Gamma$ and described by the Lagrangian density (see e.g. [10], [14])

$$L = L_0 + L_1 - W(|\psi|)$$

(2)

where

$$L_0 = -\frac{1}{2} \langle d_A \psi, d_A \psi \rangle, \quad L_1 = -\frac{1}{2} \langle d_A A, d_A A \rangle, \quad A = \sum_{j=0}^3 A_j dx^j$$

and $W$ is a real $C^1$-function. Here

$$d_A = d - iA = \sum_{j=0}^3 \left( \frac{\partial}{\partial x^j} - iA_j \right) dx^j$$

denotes the Weyl covariant differential and $\langle \cdot, \cdot \rangle$ denotes the scalar product between forms with respect the Minkowski metric in $\mathbb{R}^4$.

Since $A_j$ are real maps,

$$d_A A = dA - iA \wedge A = dA$$

Now we set

$$A = (A_1, A_2, A_3) \text{ and } \phi = -A_0$$

If we set $t = -x_0$ and $x = (x_1, x_2, x_3)$ and use vector notation, the Lagrangian densities $L_0, L_1$ can be written as follows

$$L_0 = \frac{1}{2} \left[ |(\partial_t + i\phi) \psi|^2 - |(\nabla - iA) \psi|^2 \right].$$

(3)

$$L_1 = \frac{1}{2} |\partial_t A + \nabla \phi|^2 - \frac{1}{2} |\nabla \times A|^2$$
Here $\nabla \times$ and $\nabla$ denote respectively the curl and the gradient operators with respect to the $x$ variable and $\partial_t$ denotes the derivative with respect to $t$ variable.

Observe that $L_1$ is the Maxwell Lagrangian density of the electromagnetic field

$$E = -\partial_t A - \nabla \phi, \quad H = \nabla \times A$$

Now consider the total action of the Abelian gauge theory

$$S = \int (L_0 + L_1 - W(|\psi|)) \, dxdt$$

Making the variation of $S$ with respect to $\psi$, $\phi$ and $A$ we get the system of equations

$$\begin{align*}
(\partial_t + i\phi)^2 \psi - (\nabla - iA)^2 \psi + W'(|\psi|)\frac{\psi}{|\psi|} &= 0 \\
\nabla \cdot (\partial_t A + \nabla \phi) &= \left( \text{Im} \frac{\partial_t \psi}{\psi} + \phi \right) |\psi|^2 \\
\nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \phi) &= \left( \text{Im} \frac{\nabla \psi}{\psi} - A \right) |\psi|^2
\end{align*}$$

Here $\nabla \cdot$ denotes the divergence operator. We recall that these equations are gauge invariant i.e. if $\psi$, $A$, $\phi$ solve $(5)$, $(6)$, $(7)$, then also $e^{i\chi} \psi$, $A + \nabla \chi$, $\phi - \partial_t \chi$ (with $\chi \in C^\infty(\mathbb{R}^4)$) solve the same equations.

### 2.2 Stationary solutions and vortices

We look for stationary solutions of $(5)$, $(6)$, $(7)$, namely solutions of the form

$$\psi(t, x) = u(x) e^{i(S(x) - \omega t)}, \quad u \in \mathbb{R}^+, \quad \omega \in \mathbb{R}, \quad S \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$$

$$\partial_t A = 0, \quad \partial_t \phi = 0$$

A stationary solution solves the following set of equations:

$$\begin{align*}
- \Delta u + \left[ |\nabla S - A|^2 - (\phi - \omega)^2 \right] u + W'(u) &= 0 \\
- \nabla \cdot \left[ |\nabla S - A|^2 \right] u &= 0 \\
- \Delta \phi &= (\omega - \phi) u^2 \\
\nabla \times (\nabla \times A) &= (\nabla S - A) u^2
\end{align*}$$

Observe that equation $(9)$ easily follows from equation $(11)$. Then we are reduced to study the system $(8)$, $(10)$, $(11)$. Clearly when $u = 0$, the only finite energy gauge potentials which solve $(10)$, $(11)$ are the trivial ones $A = 0$, $\phi = 0$.

It is possible to have three types of stationary non trivial solutions:

- electrostatic solutions: $A = 0$, $\phi \neq 0$;
• magneto-static solutions: \( A \neq 0, \phi = 0 \);

• electro-magneto-static solutions: \( A \neq 0, \phi \neq 0 \).

Under suitable assumptions, all these types of solutions exist. The existence and the non existence of electrostatic solutions for a system like (8),..., (10) has been proved under different assumptions on \( W \) (see [3], [4], [6], [7], [8], [9]). In particular the existence of radially symmetric, finite energy electrostatic solutions has been analyzed.

Here we are interested in magneto-static and electro-magneto-static solutions, in particular we shall study the existence of vortices in the sense of the definition stated below. We set

\[
\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}
\]

and we define the map

\[
\theta : \mathbb{R}^3 \setminus \Sigma \rightarrow \mathbb{R} / 2\pi \mathbb{Z}
\]

\[
\theta(x_1, x_2, x_3) = \text{Im} \log(x_1 + ix_2).
\]

Since there are three equations (8), (10), (11) in the unknowns \( u, S, A, \phi \), we shall take \( S = k\vartheta \) (\( k \) integer) and we shall solve with respect \( u, A, \phi \). So we give the following definition. A solution of Eq. (8), (10), (11) is called vortex if \( \psi \) has the following form

\[
\psi(t, x) = u(x) e^{i(k\theta(x) - \omega t)}, \quad k \in \mathbb{Z} - \{0\}.
\]

We shall see in Theorem 11 that if \( \psi \) has the above form, then \( A \) and \( H \) look like the fields created by a finite solenoid.

Observe that \( \theta \in C^\infty (\mathbb{R}^3 \setminus \Sigma, \mathbb{R} / 2\pi \mathbb{Z}) \) and \( \nabla \theta \in C^\infty (\mathbb{R}^3 \setminus \Sigma, \mathbb{R}^3) \), namely

\[
\nabla \theta(x) = \left( \frac{x_2}{x_1^2 + x_2^2}, \frac{-x_1}{x_1^2 + x_2^2}, 0 \right).
\]

Using this ansatz equations (8), (10), (11) become

\[
-\Delta u + \left[ k \nabla \theta - A \right]^2 - (\phi - \omega)^2 u + W'(u) = 0 \quad \text{(13)}
\]

\[
-\Delta \phi = (\omega - \phi) u^2 \quad \text{(14)}
\]

\[
\nabla \times (\nabla \times A) = (k \nabla \theta - A) u^2 \quad \text{(15)}
\]

It can be shown (see [4]) that the energy of a solution \( (u, \phi, A) \) of equations (13), (14), (15) has the following expression

\[
E = \frac{1}{2} \int \left( |\nabla u|^2 + |\nabla \phi|^2 + |\nabla A|^2 + (|A - k \nabla \theta|^2 + (\phi - \omega)^2) u^2 \right) + \int W(u)
\]
2.3 The main results

Let

\[ W(s) = \frac{1}{2}s^2 - F(s) \]  

where \( F \) is a \( C^2 \) real function satisfying the following assumptions:

\[ F(0) = F'(0) = F''(0) = 0 \]  

(17)

There are constants \( c > 0 \) and \( p \) with \( 2 < p < 6 \) such that

\[ |F'(s)| \leq cs^{p-1}, \quad s \geq 0 \]  

(18)

\[ sF'(s) \geq pF(s) > 0 \]  

for \( s > 0 \)  

(19)

Thus a typical function \( W \) satisfying our assumptions is

\[ W(s) = \frac{s^2}{2} - \frac{sp}{p} s \geq 0, \quad 2 < p < 6. \]  

(20)

Moreover, for technical reasons it is useful to assume that \( W \) is defined for all \( s \in \mathbb{R} \) just setting

\[ W(s) = \frac{s^2}{2} \]  

for \( s < 0 \)

We shall prove the following existence result for vortex solutions

**Theorem 1** Assume that the function \( W \) satisfy assumptions (16)…(19) with \( 2 < p < 6 \) and set

\[ \omega_p = \min \left( 1, \sqrt{\frac{p-2}{2}} \right) \]

Then for any \( \omega \in (-\omega_p, \omega_p) \) and any \( k \in \mathbb{Z} \) the equations (13), (14), (15) admit a solution (in the sense of distributions on \( \mathbb{R}^3 \)) \( (u, \phi, A) \) with \( u \neq 0 \) which satisfy the following properties:

- (a) \( \int \left[ |\nabla u|^2 + (1 + \frac{1}{r}) u^2 \right] dx < +\infty \), \( r^2 = x_1^2 + x_2^2 \)
- (b) \( \int |\nabla \phi|^2 + |\nabla A|^2 < +\infty \)
- (c) \( u \geq 0 \)
- (d) there exists a real function \( b \) such that \( A = b\nabla \theta \).
- (e) \( u, \phi \) and \( |A| \) have cylindrical symmetry, i.e. they depend only on \( r \) and \( x_3 \)

Moreover

i) if \( \omega \neq 0 \) and \( k = 0 \), then \( \phi \neq 0 \) and \( A = 0 \) (electrostatic solutions).

ii) if \( \omega = 0 \) and \( k \neq 0 \), then \( \phi = 0 \) and \( A \neq 0 \) (magnetostatic vortices).

iii) if \( \omega \neq 0 \) and \( k \neq 0 \), then \( \phi \neq 0 \) and \( A \neq 0 \) (electromagnetostatic vortices)
Remark 2 The properties (a) and (b) guarantee that the energy is finite.

Remark 3 If \((u, \phi, A)\) with \(u \neq 0\) solves (13), (14), (15), then assertions i), ii), iii) in Theorem 1 follow immediately from (14), (15).

Remark 4 The magnetostatic vortices are the critical points of the functional
\[
\int \frac{1}{2} |(\nabla - iA) \psi|^2 + \frac{1}{2} |\nabla \times A|^2 + W(|\psi|)
\]
Thus, they can be interpreted as solutions of a Euclidean gauge theory in dimension 3.

Remark 5 By the presence of the term \(\nabla \theta\) equations (13), (15) are not invariant under the \(O(3)\) group action as it happens for the equations (3), (5), (7) we started from. Indeed there is a breaking of radial symmetry and the solutions \(u, \phi\) and \(A\) have only a \(O(1) = S^1\) symmetry (see (d) and (e) in Theorem 1).

Finally we point out that the solutions in Theorem 1 corresponding to different \(k\) or different \(\omega\) cannot be obtained one from the other by means of a smooth gauge transformation. In fact the following proposition holds

**Proposition 6** Let \((u_1(x) e^{i(k_1 \theta(x) - \omega_1 t)}, \phi_1, A_1)\) and \((u_2(x) e^{i(k_2 \theta(x) - \omega_2 t)}, \phi_2, A_2)\) be two vortex solutions with \((\phi_1, A_1), (\phi_2, A_2) \in D^{1,2} \times (D^{1,2})^3\) (see (27)). Assume that \(\omega_2 \neq \omega_1\) or \(k_2 \neq k_1\). Then these solutions cannot be obtained one from the other by means of a gauge transformation.

**Proof.** By assumption there exists a map \(\chi\) such that
\[
u_1(x) e^{i(k_1 \theta(x) - \omega_1 t + \chi)} = u_2(x) e^{i(k_2 \theta(x) - \omega_2 t)} \tag{21}\]

\[
\phi_1 - \partial_t \chi = \phi_2, \ A_1 + \nabla \chi = A_2 \tag{22}\]
then by (21) we have

\[
u_1 = u_2, \ \chi(x, t) = (k_2 - k_1) \theta(x) - (\omega_2 - \omega_1) t \tag{23}\]
from which

\[
\partial_t \chi = \omega_1 - \omega_2, \ \nabla \chi = (k_2 - k_1) \nabla \theta(x) \tag{24}\]
By (21), (22) we get

\[
\phi_2 - \phi_1 = \omega_2 - \omega_1 \tag{25}\]

\[
A_2 - A_1 = (k_2 - k_1) \nabla \theta(x) \tag{26}\]
Since \((\phi_1, A_1), (\phi_2, A_2) \in D^{1,2} \times (D^{1,2})^3\), by (25) and (26) we deduce \(\omega_2 = \omega_1\) and \(k_2 = k_1\) and this contradicts our assumptions. \(\blacksquare\)
3 The functional setting

3.1 Weak solutions

Let $H^1$ denote the usual Sobolev space with norm

$$\|u\|_{H^1}^2 = \int (|\nabla u|^2 + u^2) \, dx;$$

moreover we need to use also the weighted Sobolev space $\hat{H}^1$ whose norm is given by

$$\|u\|_{\hat{H}^1}^2 = \int \left[ |\nabla u|^2 + \left(1 + \frac{1}{r^2}\right) u^2 \right] \, dx$$

where $r = \sqrt{x_1^2 + x_2^2}$.

We set $\mathcal{D} = C_0^\infty(\mathbb{R}^3)$ and we denote by $\mathcal{D}^{1,2}$ the completion of $\mathcal{D}$ with respect to the inner product

$$(v \mid w)_{\mathcal{D}^{1,2}} = \int \nabla v \cdot \nabla w \, dx \quad (27)$$

Here and in the following the dot $\cdot$ will denote the Euclidean inner product in $\mathbb{R}^3$.

We set

$$H = \hat{H}^1 \times \mathcal{D}^{1,2} \times (\mathcal{D}^{1,2})^3$$

$$\|(u, \phi, A)\|_H^2 = \int |\nabla u|^2 + \left(1 + \frac{1}{r^2}\right) u^2 + |\nabla \phi|^2 + |\nabla A|^2. \quad (28)$$

Now we consider the functional

$$J(u, \phi, A) = \frac{1}{2} \int |\nabla u|^2 - |\nabla \phi|^2 + |\nabla A|^2$$

$$+ \frac{1}{2} \int \left[ |A - k\nabla \theta|^2 - (\phi - \omega)^2 \right] u^2 + \int W(u) \quad (29)$$

where $(u, \phi, A) \in H$. The equations (13), (14) and (15) are the Euler-Lagrange equations of the functional $J$. In fact the following lemma holds:

**Lemma 7** The functional $J$ is $C^1$ on $H$.

**Proof.** The variations of $J$ with respect to $u, \phi, A$ are respectively

$$J'_u = -\Delta u + \left[k\nabla \theta - A\right]^2 - (\phi - \omega)^2 + W'(u) \quad (30)$$

$$J'_\phi = \Delta \phi - (\phi - \omega) u^2 \quad (31)$$

$$J'_A = \nabla \times \nabla \times A + A - k\nabla \theta \quad (32)$$
We need to prove that the maps

\[ (u, \phi, A) \in H \mapsto J'_u (u, \phi, A) \in \left( \mathring{H}^1 \right)' \]  
\[ (u, \phi, A) \in H \mapsto J'_\phi (u, \phi, A) \in (D^{1,2})' \]  
\[ (u, \phi, A) \in H \mapsto J'_A (u, \phi, A) \in ((D^{1,2})^3)' \]

are continuous. First we show that the map

\[ (u, A) \rightarrow |A|^2 u \]  

is continuous from \( \mathring{H}^1 \times (D^{1,2})^3 \) to the Lebesgue space \( L^\infty \) (and therefore to \( \left( \mathring{H}^1 \right)' \)).

In fact, if \((u_n, A_n) \rightarrow (u, A)\) in \( \mathring{H}^1 \times (D^{1,2})^3 \), then, by Sobolev embedding and Hölder inequalities, we easily obtain:

\[
\left\| |A_n|^2 u_n - |A|^2 u \right\|_{L^\infty} \leq \\
\left\| |A_n|^2 (u_n - u) \right\|_{L^\infty} + \left\| (|A_n|^2 - |A|^2) u \right\|_{L^\infty} \leq \\
\leq \|A_n\|^2_{L^\infty} \|u_n - u\|_{L^2} + \left\| |A_n|^2 - |A|^2 \right\|_{L^3} \|u\|_{L^2}
\]

from which we get the conclusion.

Analogously it can be shown that the maps

\[ (u, \phi) \in \mathring{H}^1 \times D^{1,2} \rightarrow \phi^2 u \in L^\infty \subset \left( \mathring{H}^1 \right)' \]  
\[ (u, A) \in \mathring{H}^1 \times (D^{1,2})^3 \rightarrow A u^2 \in (L^\infty)^3 \subset ((D^{1,2})^3)' \]  
\[ (u, \phi) \in \mathring{H}^1 \times D^{1,2} \rightarrow \phi u^2 \in L^\infty \subset (D^{1,2})' \]

are continuous. Moreover it is immediate to see that the linear map

\[ u \in \mathring{H}^1 \rightarrow |\nabla \theta|^2 u = \frac{u}{r^2} \in \left( \mathring{H}^1 \right)' \]

is continuous.

Also the map

\[ (u, A) \in \mathring{H}^1 \times (D^{1,2})^3 \rightarrow u A \cdot \nabla \theta \in L^2 \subset \left( \mathring{H}^1 \right)' \]

is continuous. In fact, let \((u_n, A_n) \rightarrow (u, A)\) in \( \mathring{H}^1 \times (D^{1,2})^3 \). For all \( \varphi \in H^1 \), we have that

\[
\left| \int (u_n A_n \cdot \nabla \theta - u A \cdot \nabla \theta) \varphi \right| \leq 
\]
\[
\leq \left| \int u_n (A_n - A) \cdot \nabla \theta \varphi \right| + \left| \int A \cdot \nabla \varphi (u_n - u) \right| \leq \\
\leq \int \frac{|u_n \varphi|}{r} |A_n - A| + \int \frac{|(u_n - u) \varphi|}{r} |A| \leq \\
\leq \|u_n\|_{\dot{H}^1} \|\varphi\|_{L^3} \|A_n - A\|_{L^6} + \|u_n - u\|_{\dot{H}^1} \|\varphi\|_{L^3} \|A\|_{L^6}
\]
from which the conclusion easily follows.

Arguing as before it is possible to show that also the map
\[
u \in \dot{H}^1 \rightarrow u^2 \nabla \theta \in \left( \mathcal{D}^{1,2} \right)'
\]
is continuous. So by the continuity of \(36, 40, 42\) we deduce that \(33, 34\) and \(35\) are continuous.

By the above lemma it follows that the critical points \((u, \varphi, A) \in H\) of \(J\) are weak solutions of eq. \(13\), \(14\) and \(15\), namely
\[
\int \nabla u \cdot \nabla v + \left[ |A - k \nabla \theta|^2 - (\phi - \omega)^2 \right] uv + W'(u) v = 0, \forall v \in \dot{H}^1
\]
\[
\int \nabla \varphi \cdot \nabla w + (\phi - \omega) u^2 w = 0, \forall w \in \mathcal{D}^{1,2}
\]
\[
\int \nabla A \cdot \nabla V + u^2 (A - k \nabla \theta) \cdot V = 0, \forall V \in \left( \mathcal{D}^{1,2} \right)^3.
\]

### 3.2 Solutions in the sense of distributions

Since \(\mathcal{D}\) is not contained in \(\dot{H}^1\), a solution \((u, \varphi, A) \in H\) of \(33, 34, 35\) need not to be a solution of \(13, 14, 15\) in the sense of distributions on \(\mathbb{R}^3\). In fact, since \(\nabla \theta(x)\) is singular on \(\Sigma\), it might be that for some test function \(v \in \mathcal{D}\), the integral \(\int |A - k \nabla \theta|^2 uv\) diverges, unless \(u\) is sufficiently small as \(x \rightarrow \Sigma\).

In this section we will show that this fact does not occur, namely the singularity is removable in the sense of the following theorem:

**Theorem 8** Let \((u_0, \varphi_0, A_0) \in H\) be a solution of \(33, 34, 35\) (i.e. a critical point of \(J\)). Then \((u_0, \varphi_0, A_0)\) is a solution of eq. \(33, 34, 35\) in the sense of distribution, namely
\[
\int \nabla u_0 \cdot \nabla v + \left[ |A_0 - k \nabla \theta|^2 - (\phi_0 - \omega)^2 \right] u_0 v + W'(u_0) v = 0, \forall v \in \mathcal{D}
\]
\[
\int \nabla \varphi_0 \cdot \nabla w + (\phi_0 - \omega) u_0^2 w = 0, \forall w \in \mathcal{D}
\]
\[
\int \nabla A_0 \cdot \nabla V + (A_0 - k \nabla \theta) u_0^2 \cdot V = 0, \forall V \in \mathcal{D}^3.
\]
Before proving Theorem 8 we need some lemmas.

**Lemma 9** Let \((u_0, \phi_0, A)\) be a solution of (43), (44), (45). Then \(u_0(x) \geq 0\) a.e.

**Proof.** Set \(u_0^- = \min\{u_0(x), 0\}\). Then, if in eq. (43) we set \(v = u_0^-\), we get

\[
\int |\nabla u_0^-|^2 + \left[|A - k\nabla \theta|^2 - (\phi_0 - \omega)^2\right] (u_0^-)^2 + W'(u_0^-)u_0^- = 0
\]

and since \(W'(s) = s\) for \(s \leq 0\), we get

\[
\int |\nabla u_0^-|^2 + \left[|A - k\nabla \theta|^2 + 1 - (\phi_0 - \omega)^2\right] (u_0^-)^2 = 0
\]

We shall show (see Lemma 15) that

\[
0 \leq \frac{\phi_0}{\omega} \leq 1
\]

So, since \(\omega^2 < 1\), we have \((\phi_0 - \omega)^2 \leq 1\). Then

\[
\int \left[1 - (\phi_0 - \omega)^2\right] (u_0^-)^2 \geq 0
\]

From (50) and (51) we deduce that \(u_0^- = 0\). \(\blacksquare\)

Let \(\chi_n (n\ \text{positive integer})\) be a family of smooth functions depending only on \(r = \sqrt{x_1^2 + x_2^2}\) and \(x_3\) and which satisfy the following assumptions:

- \(\chi_n (r, x_3) = 1\) for \(r \geq \frac{2}{n}\)
- \(\chi_n (r, x_3) = 0\) for \(r \leq \frac{1}{n}\)
- \(|\chi_n (r, x_3)| \leq 1\)
- \(|\nabla \chi_n (r, x_3)| \leq 2n\)
- \(\chi_{n+1} (r, x_3) \geq \chi_n (r, x_3)\)

**Lemma 10** Let \(\varphi\) be a function in \(H^1 \cap L^\infty\) with bounded support and set \(\varphi_n = \varphi \cdot \chi_n\). Then, up to a subsequence, we have that

\[
\varphi_n \rightharpoonup \varphi \text{ weakly in } H^1
\]

**Proof.** Clearly \(\varphi_n \rightarrow \varphi\) a.e. Then, by standard arguments, the conclusion holds if we show that \(\{\varphi_n\}\) is bounded in \(H^1\). Clearly \(\{\varphi_n\}\) is bounded in \(L^2\). Let us now prove that

\[
\left\{ \int |\nabla \varphi_n|^2 \right\} \text{ is bounded}
\]
We have
\[
\int |\nabla \varphi_n|^2 \leq 2 \int |\nabla \varphi \cdot \chi_n|^2 + |\varphi \cdot \nabla \chi_n|^2
\]
\[
\leq 2 \int |\nabla \varphi|^2 + 2 \int_{\Gamma_\varepsilon} |\varphi \cdot \nabla \chi_n|^2
\]
where
\[
\Gamma_\varepsilon = \{ x \in \mathbb{R}^3 : \varphi \neq 0 \text{ and } |\nabla \chi_n(r,z)| \neq 0 \}
\]
By our construction, |\Gamma_\varepsilon| \leq c/n^2 where c depends only on \varphi. Thus
\[
\int |\nabla \varphi_n|^2 \leq 2 \int |\nabla \varphi|^2 + 2 \| \varphi \|_{L^\infty}^2 \int_{\Gamma_\varepsilon} |\nabla \chi_n|^2
\]
\[
\leq 2 \int |\nabla \varphi|^2 + 8c \| \varphi \|_{L^\infty}^2
\]
Thus, \varphi_n is bounded in \( H^1 \) and \( \varphi_n \to \varphi \) weakly in \( H^1 \).

Now we are ready to prove Theorem 8

**Proof.** Clearly (47) and (48) immediately follows by (44) and (45). Let us prove (46). We take any \( v \in D \) and set \( \varphi_n = v^+ \chi_n \). Then taking \( \varphi_n \) as test function in Eq. (43), we have
\[
\int \nabla u_0 \cdot \nabla \varphi_n + \left[ |A_0 - k \nabla \theta|^2 - (\phi_0 - \omega)^2 \right] u_0 \varphi_n + W'(u_0) \varphi_n = 0 \quad (52)
\]
Equation (52) can be written as follows
\[
A_n + B_n + C_n + D_n = 0 \quad (53)
\]
where
\[
A_n = \int \nabla u_0 \cdot \nabla \varphi_n, \quad B_n = \int \left( A_0^2 u_0 - (\phi_0 - \omega)^2 u_0 + W'(u_0) \right) \varphi_n \quad (54)
\]
\[
C_n = -2 \int A_0 \cdot k \nabla \theta u_0 \varphi_n, \quad D_n = \int |k \nabla \theta|^2 u_0 \varphi_n \quad (55)
\]
By Lemma 10
\[
\varphi_n \to v^+ \text{ weakly in } H^1 \quad (56)
\]
Then we have
\[
A_n \to \int \nabla u_0 \cdot \nabla v^+ \quad (57)
\]
Now
\[
\left( A_0^2u_0 - (\phi_0 - \omega)^2 u_0 + W'(u_0) \right) \in L^{6/5} = (L^6)' 
\]
Then, using again (56) and by the embedding \( H^1 \subset L^6 \), we have
\[
B_n \to \int \left( A_0^2u_0 - (\phi_0 - \omega)^2 u_0 + W'(u_0) \right) v^+ < \infty \quad (58)
\]
Now we shall prove that
\[
C_n \to -2 \int A_0 \cdot k\nabla \theta u_0 v^+ < \infty \quad (59)
\]
Set
\[
C = B_R \times [-l, l], \quad B_R = \{(x_1, x_2) \in \mathbb{R}^2 : r^2 = x_1^2 + x_2^2 < R\}
\]
where \( l, R > 0 \) are so large that the cylinder \( C \) contains the support of \( v^+ \).

Then
\[
\int \left( \frac{\varphi_n}{r} \right)^\frac{2}{3} dx = \int_C \left( \frac{v^+e_n}{r} \right)^\frac{2}{3} dx \leq c_1 \int_{-l}^l \int_0^R \left( \frac{1}{r} \right)^\frac{2}{3} r dr dx_3 = M < \infty \quad (60)
\]
where \( c_1 = 2\pi \sup (v^+) \frac{2}{3} \). By (61) we have
\[
\int |A_0 \cdot \nabla \theta u_0 \varphi_n| \, dx \leq \|u_0 A_0\|_{L^3} \left\| \frac{\varphi_n}{r} \right\|_{L^{\frac{6}{5}}} \leq \|u_0 A_0\|_{L^3} M \frac{2}{3} \quad (62)
\]
Now
\[
|A_0 \cdot \nabla \theta u_0 \varphi_n| \to |A_0 \cdot \nabla \theta u_0 v^+| \quad \text{a.e. in } \mathbb{R}^3
\]
and the sequence \( \{ |A_0 \cdot \nabla \theta u_0 \varphi_n| \} \) is monotone. Then, by the monotone convergence theorem, we get
\[
\int |A_0 \cdot k\nabla \theta u_0 \varphi_n| \, dx \to \int |A_0 \cdot k\nabla \theta u_0 v^+| \, dx \quad (63)
\]
By (62) and (63) we deduce that
\[
\int |A_0 \cdot k\nabla \theta u_0 v^+| \, dx < \infty \quad (64)
\]
Then, since
\[
|A_0 \cdot \nabla \theta u_0 \varphi_n| \leq |A_0 \cdot \nabla \theta u_0 v^+| \in L^1
\]
by the dominated convergence Theorem, we get (59). Finally we prove that
\[
D_n \to \int |k\nabla \theta|^2 u_0 v^+ < \infty \quad (65)
\]
By (53), (57), (58) and (59) we have that
\[ D_n = \int |k\nabla\theta|^2 u_0 \varphi_n \text{ is bounded} \] (66)

Moreover, by Lemma 9, \( u_0 \geq 0 \). Then the sequence \( |\nabla \theta|^2 u_0 \varphi_n \) is monotone and it converges a.e. to \( |\nabla \theta|^2 u_0 v^+ \). Then, by the monotone convergence theorem, we get
\[ \int |k\nabla\theta|^2 u_0 \varphi_n dx \rightarrow \int |k\nabla\theta|^2 u_0 v^+ dx \] (67)

By (66) and (67) we get (65).

Taking the limit in (53) and by using (57), (58), (59), (65) we have
\[ \int \nabla u_0 \cdot \nabla v^+ + \left[ |A_0 - k\nabla\theta|^2 - (\phi_0 - \omega)^2 \right] u_0 v^+ + W'(u_0) v^+ = 0 \]

Taking \( \varphi_n = v^- \chi_n \) and arguing in the same way as before, we get
\[ \int \nabla u_0 \cdot \nabla v^- + \left[ |A_0 - k\nabla\theta|^2 - (\phi_0 - \omega)^2 \right] u_0 v^- + W'(u_0) v^- = 0. \]

Then
\[ \int \nabla u_0 \cdot \nabla v + \left[ |A_0 - k\nabla\theta|^2 - (\phi_0 - \omega)^2 \right] u_0 v + W'(u_0) v = 0. \]

Since \( v \in \mathcal{D} \) is arbitrary, we get that equation (46) is solved. 

4 The natural constraints

The functional \( J \) presents two main difficulties:

1) The term \( \int |\nabla \times A|^2 \) is not a Sobolev norm and it does not yield a control on \( \int |\nabla A|^2 = \| A \|_{(D^1, 2)^3}^2 \).

2) The presence of the term \( -\int |\nabla \phi|^2 \) gives to the functional \( J \) a strong indefiniteness, namely any critical point of \( J \) has infinite Morse index: this fact is a great obstacle to a direct study of its critical points.

In order to avoid these difficulties we introduce a suitable manifold \( \mathcal{M} \subset H \) such that:

- the critical points of \( J \) restricted to \( \mathcal{M} \) satisfy Eq. (13), (14), (15); namely \( \mathcal{M} \) is a "natural constraint" for \( J \).
- The components \( A \) of the elements in \( \mathcal{M} \) are divergence free, then the term \( \int |\nabla \times A|^2 \) can be replaced by \( \| A \|_{(D^1, 2)^3}^2 = \int |\nabla A|^2 \).
- The functional \( I := J \mid_\mathcal{M} \) does not exhibit the severe indefiniteness before mentioned.
4.1 The manifold of divergence free vector fields

We shall denote by $u = u(r, x_3)$ the maps having cylindrical symmetry, i.e. those maps in $\mathbb{R}^3$ which depends only from $r = \sqrt{x_1^2 + x_2^2}$ and $x_3$. We set

$$A_0 := \{ \mathbf{X} \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}^3) : \mathbf{X} = b(r, x_3) \nabla \theta; \ b \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}) \}$$

(68)

Let $A$ denote the closure of $A_0$ with respect to the norm of $(D^{1,2})^3$. We set

$$D_r = \{ u \in D : u = u(r, x_3) \}$$

(69)

and we shall consider the following space

$$V := \hat{H}_r^1 \times A$$

(70)

where $\hat{H}_r^1$, is the closure of $D_r$ with respect to the $\hat{H}_r^1$ norm.

We need some technical preliminaries

**Lemma 11** If $A \in A$, then $\nabla \times \nabla \times A \in A'$

**Proof.** Let $a \nabla \theta \in A_0$, where $a \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R})$, $a = a(r, x_3)$. Then a straight computation shows that

$$\nabla \times \nabla \times (a \nabla \theta) = b \nabla \theta$$

(71)

where

$$b = -\frac{\partial^2 a}{\partial r^2} + \frac{1}{r} \frac{\partial a}{\partial r} - \frac{\partial^2 a}{\partial x_3^2}$$

Now let $a_n \nabla \theta$ be a sequence in $A_0$ converging in $(D^{1,2})^3$ to $A \in A$. By continuity, $\nabla \times \nabla \times (a_n \nabla \theta)$ converges in $((D^{1,2})^3)'$ to $\nabla \times \nabla \times A$. On the other hand by (71) we have

$$\nabla \times \nabla \times (a_n \nabla \theta) = b_n \nabla \theta \in A_0$$

Then, by definition, $\nabla \times \nabla \times A \in A'$.

**Lemma 12** If $A \in A$ and $u \in \hat{H}_r^1$,

$$(A - \nabla \theta) u^2 \in A'$$

(72)

**Proof.** Set

$$D_r (\mathbb{R}^3 \setminus \Sigma) = \{ u \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma) : u = u(r, x_3) \}$$

Since $u \in \hat{H}_r^1$, there exists a sequence $\{ u_n \} \subset D_r (\mathbb{R}^3 \setminus \Sigma)$ such that

$$u_n \rightharpoonup u \text{ in } \hat{H}_r^1$$

(73)

Since $A \in A$, there exists a sequence $\{ b_n \} \subset D_r (\mathbb{R}^3 \setminus \Sigma)$ such that

$$b_n \nabla \theta \rightharpoonup A \text{ in } (D^{1,2}_r)^3$$

(74)
By (74) and (73) we deduce, following analogous arguments as those used in proving lemma 7, that

\[(b_n - 1)u_n^2 \nabla \theta = (b_n \nabla \theta - \nabla \theta) u_n^2 \rightarrow (A - \nabla \theta) u^2 \text{ in } ((D_r^{1,2})') \quad (75)\]

Clearly

\[(b_n - 1)u_n^2 \in D_r (\mathbb{R}^3 \setminus \Sigma) \quad (76)\]

Then, by (75), (76), we get (72). ■

**Lemma 13** If \(A \in \mathcal{A}\), then

\[
\int |\nabla \times A|^2 = \int |\nabla A|^2
\]

and hence

\[-\Delta A = \nabla \times \nabla \times A\]

**Proof.** Let \(A = b \nabla \theta \in \mathcal{A}_0\). Since \(b\) depends only on \(r\) and \(x_3\), it is easy to check that

\[\nabla b \cdot \nabla \theta = 0\]

Since \(\theta\) is harmonic in \(\mathbb{R}^3 \setminus \Sigma\) and \(b\) has support in \(\mathbb{R}^3 \setminus \Sigma\)

\[b \Delta \theta = 0\]

Then

\[\nabla \cdot A = \nabla \cdot (b \nabla \theta) = \nabla b \cdot \nabla \theta + b \Delta \theta = 0\]

Thus, by continuity, we get

\[\int (\nabla \cdot A)^2 = 0 \text{ for any } A \in \mathcal{A}\]

Then

\[\int |\nabla \times A|^2 = \int (\nabla \cdot A)^2 + \int |\nabla \times A|^2 = \int |\nabla A|^2\]

and clearly

\[-\Delta A = \nabla \times \nabla \times A\]

■
4.2 The Gauss equation

Equation (14), which we call Gauss equation, can be written as follows

\[- \Delta \phi + u^2 \phi = \omega u^2 \]  

(77)

**Lemma 14** Let \( u \in H^1(\mathbb{R}^3) \), then there exists a unique solution \( \phi \in D^{1,2} \) of (77).

**Proof.** \( H^1(\mathbb{R}^3) \) is continuously embedded into \( L^6(\mathbb{R}^3) \), then clearly

\( u^2 \in L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \) (78)

and, by interpolation, we have

\( u^2 \in L^{\frac{2}{3}}(\mathbb{R}^3) \) (79)

Now consider the bilinear form

\[ a(\phi, v) = \int \{(\nabla \phi \cdot \nabla v) + u^2 \phi v\} \, dx, \ u, \phi \in D^{1,2} \]

By (79) we easily derive that \( a(\phi, v) \) is equivalent to the standard inner product in \( D^{1,2} \), i.e.

\[ c_1 \| \phi \|^2_{D^{1,2}} \leq a(\phi, \phi) \leq c_2 \| \phi \|^2_{D^{1,2}}, \ c_1, c_2 > 0 \]

On the other hand, using again (78), we have

\( u^2 \in L^{\frac{2}{3}}(\mathbb{R}^3) \subset (D^{1,2})' \)

So there exists a unique \( \phi \in D^{1,2} \) such that

\[ a(\phi, v) = -\omega \int u^2 v \text{ for all } v \in D^{1,2} . \]

\( \phi \) clearly solves (77). \( \blacksquare \)

Clearly, if \( u \in \dot{H}^1(\mathbb{R}^3) \), the solution \( \phi = \phi_u \) of (77) belongs to \( D^{1,2}_r \). Then we can define the map

\[ u \in \dot{H}^1(\mathbb{R}^3) \rightarrow Z(u) = \phi_u \in D^{1,2}_r \text{ solution of (77)} \]

(80)

Standard arguments show that the map \( Z \) is \( C^1 \). Since \( \phi_u \) solves (77), clearly we have

\[ d_\phi J(u, \phi_u, A) = 0 \]

(81)

For \( u \in H^1(\mathbb{R}^3) \), let \( \Phi = \Phi_u \) be the solution of the equation (77) with \( \omega = 1 \), then \( \Phi_u \) solves the equation

\[- \Delta \Phi_u + u^2 \Phi_u = u^2 \]

(82)

Clearly

\[ \phi_u = \omega \Phi_u \]

(83)
**Lemma 15** For any $u \in H^1(\mathbb{R}^3)$,

$$0 \leq \Phi_u \leq 1$$

**Proof.** By the maximum principle

$$\Phi_u \geq 0$$

Moreover, arguing by contradiction assume that there is a region $\Omega$ where $\Phi_u > 1$ and $\Phi_u = 1$ on $\partial \Omega$. Clearly

$$-\Delta(\Phi_u - 1) + u^2(\Phi_u - 1) = -\Delta\Phi_u + u^2\Phi_u - u^2 = 0$$

Then $v = \Phi_u - 1$ solves the Dirichlet problem

$$-\Delta v + u^2 v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega$$

Multiplying by $v$ and integrating in $\Omega$ we get

$$\int_{\Omega} \left( |\nabla v|^2 + u^2 v^2 \right) dx = 0$$

Then $v = \Phi_u - 1 = 0$ in $\Omega$ contradicting $\Phi_u > 1$ in $\Omega$. ■

### 4.3 The reduced functional

Now, if $(u, A) \in \hat{H}^1_x \times A$, we set

$$I(u, A) = J(u, Z(u), A)$$

where $J$ is defined in (29). Thus $I$ can be regarded as $J$ restricted to the manifold

$$\mathcal{M} = \{(u, Z(u), A) : (u, A) \in \hat{H}^1_x \times A\}$$

We will refer to $I(u, A)$ as the reduced functional.

From (82) we have

$$\int u^2 \Phi_u dx = \int |\nabla \Phi_u|^2 dx + \int u^2 \Phi_u^2 dx$$

(85)

Now, using lemma 13 and Eq. (83), (85), we have:

$$I(u, A) = \frac{1}{2} \int |\nabla u|^2 - |\nabla \Phi_u|^2 + |\nabla \times A|^2$$

$$+ \frac{1}{2} \int \left( |A - k\nabla \theta|^2 - (\phi_u - \omega)^2 \right) u^2 + \int W(u)$$

$$= \frac{1}{2} \int \left( |\nabla u|^2 + |\nabla A|^2 + |A - k\nabla \theta|^2 \right) u^2$$

$$- \frac{1}{2} \omega^2 \int \left( |\nabla \Phi_u|^2 + u^2 \Phi_u^2 + u^2 - 2u^2 \Phi_u \right) + \int W(u)$$

$$= \frac{1}{2} \int |\nabla u|^2 + |\nabla A|^2 + |A - k\nabla \theta|^2 u^2$$

$$+ \frac{1}{2} \int (1 - \omega^2 [1 - \Phi_u]) u^2 - \int F(u)$$

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Since $\Phi_u \leq 1$ and $\omega^2 < 1$, the functional $I$ contains only one negative term ($-\int F(u)$). Moreover $\int |\nabla A|^2$ replaces the term $\int |\nabla \times A|^2$. As a consequence the functional $I$ does not exhibit the difficulties mentioned at the beginning of this section.

It can be shown also that $\mathcal{M}$ defined in [84] is a natural constraint for the criticizing sequences of $J$, namely the following theorem holds

**Theorem 16** Let $(u_n, A_n) \in \hat{H}^1 \times A$ be a sequence such that $\forall (v, V) \in \hat{H}^1 \times A$

$$dI(u_n, A_n)[v, V] \to 0$$

(86)

Then, we have that

$$d_u J(u_n, Z(u_n), A_n)[v] \to 0 \text{ for all } v \in \hat{H}^1$$

(87)

$$d_\phi J(u_n, Z(u_n), A_n)[w] = 0 \text{ for all } n \in \mathbb{N} \text{ and } w \in D^{1,2}$$

(88)

$$d_A J(u_n, Z(u_n), A_n)[V] \to 0 \text{ for all } V \in (D^{1,2})^3$$

(89)

where $d_u, d_\phi, d_A$ denote the partial differentials with respect to $u, \phi, A$.

**Proof.** Using the chain rule, we have for all $(v, V) \in \hat{H}^1 \times A$

$$dI(u_n, A_n)[v, V] = d_u I(u_n, A_n)[v] + d_A I(u_n, A_n)[V]$$

$$= d_u J(u_n, Z(u_n), A_n)[v]$$

$$+ d_\phi J(u_n, Z(u_n), A_n)[d_u Z(u_n)[w]]$$

$$+ d_A J(u_n, Z(u_n), A_n)[V]$$

By equation (81),

$$d_\phi J(u_n, Z(u_n), A_n) = 0$$

(90)

then for all $(v, V) \in \hat{H}^1 \times A$

$$dI(u_n, A_n)[v, V] = d_u J(u_n, Z(u_n), A_n)[v] + d_A J(u_n, Z(u_n), A_n)[V]$$

(91)

So by (80) we have for all $(v, V) \in \hat{H}^1 \times A$

$$d_u J(u_n, Z(u_n), A_n)[v] \to 0$$

(91)

$$d_A J(u_n, Z(u_n), A_n)[V] \to 0$$

(92)

Equations (90), (91), (92), written explicitly take the following form

$$\int \nabla u_n \cdot \nabla v + \left[ |A_n - k\nabla \theta|^2 - (\phi_n - \omega)^2 \right] u_n v + W'(u_n) v = \langle \eta_n, v \rangle_{\hat{H}^1}$$

$$\int \nabla \phi_n \cdot \nabla w + (\phi_n - \omega) u_n^2 w = 0$$

$$\int \nabla A_n \cdot \nabla V + (A_n - k\nabla \theta) u_n^2 \cdot V = \langle \zeta_n, V \rangle_{(D^{1,2})^3}$$

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where \((v, V) \in \dot{H}^1_r \times \mathcal{A}, w \in D^{1,2}, \phi_n = Z(u_n), \eta_n \to 0\) in \((\dot{H}^1)^\prime\) and \(\zeta_n \to 0\) in \([(D^{1,2})^3]^\prime\).

\(\eta_n\) and \(\zeta_n\) have the following expression:

\[
\eta_n = -\Delta u_n + \left[|A_n - k \nabla \theta|^2 - (\phi_n - \omega)^2\right] u_n + W'(u_n) \quad (93)
\]

\[
\zeta_n = -\Delta A_n + (A_n - k \nabla \theta) u_n^2 \quad (94)
\]

Since the Laplace operator preserves the cylindrical symmetry and \((u_n, A_n) \in \dot{H}^1_r \times \mathcal{A},\) we have that \(\eta_n \in (\dot{H}^1)^\prime\). Moreover, by lemma (12), we have that \(\zeta_n \in \mathcal{A}\).

Now take any \((v, V) \in \dot{H}^1 \times (D^{1,2})^3\) and we set

\[
v = v_1 + v_2 \text{ with } v_1 \in \dot{H}^1_r, v_2 \in (\dot{H}^1)^\perp
\]

and

\[
V = V_1 + V_2 \text{ with } V_1 \in \mathcal{A}, V_2 \in \mathcal{A}^\perp
\]

Then, since \(\eta_n \in (\dot{H}^1)^\prime\) and \(v_2 \in (\dot{H}^1)^\perp\), we have

\[
d_u J(u_n, Z(u_n), A_n)[v] = \langle \eta_n, v_1 \rangle_{\dot{H}^1} + \langle \eta_n, v_2 \rangle_{\dot{H}^1} = \langle \eta_n, v_1 \rangle_{\dot{H}^1}
\]

So, since \(\langle \eta_n, v_1 \rangle_{\dot{H}^1} \to 0\), we get

\[
d_u J(u_n, Z(u_n), A_n)[v] \to 0
\]

Analogously it can be shown that

\[
d_A J(u_n, Z(u_n), A_n)[V] \to 0
\]

\[\blacksquare\]

## 5 The existence proof

For the existence proof we need to construct suitable Palais-Smale sequences for the functional

\[
I(u, A) = \frac{1}{2} \int |\nabla u|^2 + |\nabla A|^2 + |A - k \nabla \theta|^2 u^2 - \frac{\omega^2}{2} \int [1 - \Phi u] u^2 + \int W(u) \quad (95)
\]

in the space \(V := \dot{H}^1_r \times \mathcal{A} \).

To simplify the notations, we set

\[
U = (u, A)
\]

\[
\|U\|_V = \|u\|_{\dot{H}^1} + \|A\|_{(D^{1,2})^3}
\]

First we prove the existence of a Palais-Smale sequence for \(I\), namely the following lemma holds.
Lemma 17 Assume that the function \( W \) satisfies assumptions (16)... (19) and that \( \omega^2 < 1 \). Then there exits a PS sequence at some level \( c > 0 \), namely a sequence \( \{U_n\} \subset V \) such that

\[
I(U_n) \to c > 0 \text{ and } dI(U_n) \to 0 \text{ in } V'
\]

Proof. By a variant of the well known mountain pass theorem [2], the existence of a PS sequence will be guaranteed if we show that the functional (95) has the mountain pass geometry, namely if there exist \( \alpha, \rho > 0 \) and \( U_0 \in V \) with \( \|U_0\|_V > \rho \), such that:

\[
I(U) \geq \alpha \text{ for } \|U\|_V = \rho \tag{96}
\]

and

\[
I(U_0) \leq 0 \tag{97}
\]

Let us first prove (96). In the following \( C_1, ..., C_4 \) will denote positive constants.

We have that

\[
\int |A - k\nabla \theta|^2 u^2 \geq \int \left(|A|^2 - 2\left|\frac{A}{r} + \frac{1}{r^2}\right| + \frac{1}{r^2}\right) u^2
\]

\[
= \int \left(|A|^2 - 2\left(\frac{|A|}{\sqrt{2}}\frac{1}{\sqrt{2}r} + \frac{1}{r^2}\right) + \frac{1}{r^2}\right) u^2
\]

\[
\geq \int \left(|A|^2 - \left(\frac{|A|}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}r}\right)^2 + \frac{1}{r^2}\right) u^2
\]

\[
= \int \left(|A|^2 - 2|A|^2 - \frac{1}{2r^2} + \frac{1}{r^2}\right) u^2
\]

\[
= \int -|A|^2 u^2 + \frac{u^2}{2r^2}
\]

\[
\geq \int \frac{u^2}{2r^2} - \|A\|_{L^6}^2 \|u\|_{L^3}^2
\]

\[
\geq \int \frac{u^2}{2r^2} - \frac{1}{2} \|A\|_{L^6}^4 - \frac{1}{2} \|u\|_{L^3}^4
\]

\[
\geq \frac{1}{2} \int \frac{u^2}{r^2} - \frac{C_1}{2} \|A\|_{D^{1,2}}^4 - \frac{C_2}{2} \|u\|_{H^1}^4. \tag{98}
\]

Moreover by (17), (18), we have that

\[
\int F(u) \leq \frac{c}{p - 1} \|u\|_{L^p}^p \leq C_3 \|u\|_{H^1}^p. \tag{99}
\]
For any $U = (u, A) \in V$ we have

$$I(U) = \frac{1}{2} \int |\nabla u|^2 + |\nabla A|^2 + |A - k \nabla \theta|^2 u^2 - \frac{\omega^2}{2} \int [1 - \Phi_u] u^2 + \int W(u)$$

(by (98) and since $\omega \geq \phi_u \geq 0$) $\geq \frac{1}{2} \int |\nabla u|^2 + |\nabla A|^2 + \frac{1}{2} \int \left( |\nabla u|^2 + \frac{u^2}{r^2} \right) - \frac{C_2}{2} \|u\|^4_{H^1} + \frac{1}{2} \int u^2 - C_3 \|u\|^p_{H^1}$

(by (99)) $\geq \frac{1}{2} \left( \|A\|^2_{D^{1,2}} - C_1 \|A\|^4_{D^{1,2}} \right) + \frac{1}{2} - \frac{\omega^2}{2} \|u\|^2_{H^1} - \frac{C_2}{2} \|u\|^4_{H^1} - C_3 \|u\|^p_{H^1}$

( since $\omega < 1$) $\geq \frac{1}{2} \left( \|A\|^2_{D^{1,2}} - C_1 \|A\|^4_{D^{1,2}} \right) + \frac{1 - \omega^2}{2} \|u\|^2_{H^1} - \frac{C_2}{2} \|u\|^4_{H^1} - C_3 \|u\|^p_{H^1}

= \frac{1}{2} \left( 1 - C_1 \|A\|^2_{D^{1,2}} \right) \|A\|^2_{D^{1,2}} + \left( \frac{1 - \omega^2}{2} - \frac{C_2}{2} \|u\|^2_{H^1} - C_3 \|u\|^p_{H^1} \right) \|u\|^2_{H^1}$

Then, since $\omega < 1$, there exist $\alpha, \rho > 0$ such that

$$I(U) \geq \alpha \text{ for } \|U\|_V = \rho \quad (100)$$

Let us now prove (97). Take $u_0 \in C_0^\infty (\mathbb{R}^3 \setminus \Sigma)$ $u_0 \geq 0$, $u_0 \neq 0$. By (10)

$$F(s) \geq F(1)s^p, \ s > 0;$$

then

$$\int F(tu_0) \geq \int F(1)|tu_0|^p \geq C_4 t^p \quad (101)$$

Now we have

$$I(tu_0, 0) = \frac{t^2}{2} \int \left( |\nabla u_0|^2 + \frac{u_0^2}{r^2} \right) dx - \frac{1}{2} \omega^2 t^2 \int [1 - \Phi_{tu_0}] u_0^2 + \int W(tu_0) \leq$$

(since $1 \geq \Phi_{tu_0}$) $\leq \frac{t^2}{2} \int \left( |\nabla u_0|^2 + \frac{u_0^2}{r^2} \right) dx + \frac{t^2}{2} \int u_0^2 - \int F(tu_0) \leq$

(by (101)) $\leq C_5 t^2 - C_4 t^p$.

Then (97) is satisfied if we take $U_0 = (tu_0, 0)$ with $t$ sufficiently large. $\blacksquare$

Our functional is invariant for translations in the $x_3$-direction, namely for $U \in V$ and $L \in \mathbb{R}$ we have

$$I(T_L U) = I(U)$$

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where
\[ T_L(U)(x_1, x_2, x_3) = U(x_1, x_2, x_3 + L) \]

As consequence of this invariance we have that a PS sequence for the functional \( I \) does not contain in general a (strongly) convergent subsequence. So, in order to prove the existence of non trivial critical points of \( I \), we shall use the following strategy:

(i) we prove that any PS sequence \((u_n, A_n)\) of the functional \( I \) is bounded in \( \dot{H}^1_r \times (D^{1,2})^3 \).

(ii) we prove that there exists a PS sequence \((u_n, A_n)\) whose weak limit \((u_0, A_0)\) gives rise to a non trivial critical point \((u_0, \phi_{u_0}, A_0)\) of \( J \).

**Lemma 18** Assume that the function \( W \) satisfies assumptions (16)... (19) and assume that \( \omega^2 < \min(1, \frac{p-2}{2}) \). Let \( \{U_n\} = \{(u_n, A_n)\} \subset \dot{H}^1_r \times A \) be a PS sequence for the functional \( I \). Then \( \{U_n\} \) is bounded in \( \dot{H}^1_r \times (D^{1,2})^3 \).

**Proof.** Since \( \{(u_n, A_n)\} \subset V \) is a PS sequence for \( I \), we have
\[ dI(U_n) = \eta_n \to 0 \text{ in } V' \] (102)

and
\[ I(U_n) = c_n \to c \] (103)

From (102) we get
\[ \int |\nabla u_n|^2 + \left[ |A_n - k\nabla \theta|^2 - (\phi_n - \omega)^2 \right] u_n^2 + W'(u_n) u_n = \langle \eta_n, u_n \rangle \] (104)

where
\[ \phi_n = \omega \Phi_{u_n} \]

The expression (103) can be written
\[ \frac{1}{2} \int |\nabla u_n|^2 + |\nabla A_n|^2 + |A_n - k\nabla \theta|^2 u_n^2 - \frac{1}{2} \int \omega [\omega - \phi_n] u_n^2 + \int W(u_n) = c_n \] (105)

Multiplying (102) by \( \frac{1}{p} \) and subtracting from (105) we have
\[
\left( \frac{1}{2} - \frac{1}{p} \right) \int \left( |\nabla u_n|^2 + |A_n - k\nabla \theta|^2 u_n^2 \right) dx + \left( \frac{1}{2} - \frac{1}{p} \right) \int |\nabla A_n|^2 + \int \left( W(u_n) - \frac{1}{p} W'(u_n) u_n + \frac{1}{p} (\phi_n - \omega)^2 u_n^2 - \frac{1}{2} \omega [\omega - \phi_n] u_n^2 \right) dx
\]
\[ = c_n - \frac{1}{p} \langle \eta_n, u_n \rangle \] (106)
Now
\[ W(u_n) - \frac{1}{p} W'(u_n) u_n + \frac{1}{p} (\phi_n - \omega)^2 u_n^2 - \frac{1}{2} \omega [\omega - \phi_n] u_n^2 = (\text{by } 106) \]
\[ = \left( \frac{1}{2} - \frac{1}{p} \right) u_n^2 + \frac{1}{p} F'(u_n) u_n - F(u_n) + \omega^2 \left( \frac{1}{p} - \frac{1}{2} \right) u_n^2 + \left( \frac{1}{2} - \frac{2}{p} \right) \phi_n \omega u_n^2 \]

Then, since
\[ \omega \geq \frac{1}{2} - \frac{1}{p} \] and assume \( \omega^2 < 1 \), from (108) we have that
\[ W(u_n) - \frac{1}{p} W'(u_n) u_n + \frac{1}{p} (\phi_n - \omega)^2 u_n^2 - \frac{1}{2} \omega [\omega - \phi_n] u_n^2 \geq \left( \frac{1}{2} - \frac{1}{p} \right) (1 - \omega^2) u_n^2 \]

Since \( \omega^2 < 1 \), from (109) we deduce that
\[ \int \left( W(u_n) - \frac{1}{p} W'(u_n) u_n + \frac{1}{p} (\phi_n - \omega)^2 u_n^2 - \frac{1}{2} \omega [\omega - \phi_n] u_n^2 \right) dx \geq C_1 \| u_n \|_{L^2}^2 \]

where
\[ C_1 = \left( \frac{1}{2} - \frac{1}{p} \right) (1 - \omega^2) > 0 \]

By (109) and (110) we get
\[ \left( \frac{1}{2} - \frac{1}{p} \right) \int \left( \| \nabla u_n \|^2 + \| A_n - \kappa \nabla \theta \|^2 \right) + \frac{1}{2} \int \| \nabla A_n \|^2 + C_1 \| u_n \|_{L^2}^2 \leq \]
\[ \leq C_1 \] \( \| \eta_n \|_{L^2} \)

Then
\[ C_2 \| u_n \|_{H^1}^2 + \frac{1}{2} \| A_n \|_{(D^1, 2)^3}^2 \leq C_n + \frac{1}{p} \| \eta_n \|_{H^{-1}} \| u_n \|_{H^1} \]

So we conclude that the sequences \( \{ \| u_n \|_{H^1} \} \) and \( \{ \| A_n \|_{(D^1, 2)^3} \} \) are bounded.

Now consider the case \( p < 4 \) and assume \( \omega^2 < \frac{p-2}{2} \).

By (108), we easily have
\[ W(u_n) - \frac{1}{p} W'(u_n) u_n + \frac{1}{p} (\phi_n - \omega)^2 u_n^2 - \frac{1}{2} \omega [\omega - \phi_n] u_n^2 \geq (\text{since } \Phi_{u_n} \leq 1) \]
\[ \geq \left[ \left( \frac{1}{2} - \frac{1}{p} \right) (1 - \omega^2) + \left( \frac{1}{2} - \frac{2}{p} \right) \omega^2 \right] u_n^2 \geq C_3 u_n^2 \]

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Since $\omega^2 < \frac{p^2}{2}$, we have

$$C_3 = \left(\frac{1}{2} - \frac{1}{p}\right) (1 - \omega^2) + \left(\frac{1}{2} - \frac{2}{p}\right) \omega^2 > 0$$

Then

$$\int \left( W(u_n) - \frac{1}{p} W'(u_n) u_n + \frac{1}{p} (\phi_n - \omega)^2 u_n^2 - \frac{1}{2} \omega [\omega - \phi_n] u_n^2 \right) \geq C_3 \|u_n\|_{L^2}^2$$

Arguing as in the first case, by (106) and (111), we again conclude that the sequences $\{\|u_n\|^2_{H^1}\}$ and $\{\|A_n\|^2_{(D_1,2,3)}\}$ are bounded.

It remains to prove that $u_n$ is bounded also in $\hat{H}^1$. We have that

$$c + 1 \geq \frac{1}{2} \int |\nabla u_n|^2 + \frac{|\nabla A_n|^2}{2} + |A_n - k\nabla \theta|^2 u_n^2 - \frac{\omega^2}{2} \int [1 - \Phi_{u_n}] u_n^2 + \int W(u_n)$$

$$\geq \frac{1}{2} \int |\nabla u_n|^2 + \frac{|\nabla A_n|^2}{2} + |A_n - k\nabla \theta|^2 u_n^2 + u_n^2 - C_4$$

$$\geq \frac{1}{2} \int |\nabla u_n|^2 + \frac{|\nabla A_n|^2}{2} + |A_n|^2 u_n^2 + |k\nabla \theta|^2 u_n^2 - 2 |A_n| |k\nabla \theta| u_n^2 + u_n^2 - C_4$$

$$\geq \frac{1}{2} \|u_n\|^2_{\hat{H}^1} - \int \frac{k}{r} |A_n| u_n^2 - C_4$$

Also, we have that

$$\int \frac{k}{r} |A_n| u_n^2 \leq \int \frac{1}{2} \left( 4 k^2 |A_n|^2 + \frac{1}{4r^2} \right) u_n^2 = 2k^2 \int |A_n|^2 u_n^2 + \frac{1}{8} \int \frac{1}{r^2} u_n^2$$

$$\leq \frac{1}{8} \|u_n\|^2_{\hat{H}^1} + C_5$$

Then

$$c + 1 \geq \frac{1}{2} \|u_n\|^2_{\hat{H}^1} - \frac{1}{8} \|u_n\|^2_{\hat{H}^1} - C_5 - C_4$$

$$= \frac{3}{8} \|u_n\|^2_{\hat{H}^1} - C_5 - C_4$$

$$\blacksquare$$

**Lemma 19** Let the assumptions of Lemma 17 be satisfied. Then there is a PS sequence $U_n = (u_n, A_n) \subset V$ for the functional $I$ such that for $n$ large enough

$$\|u_n\|_{L^p}^p \geq C_5 > 0$$

(112)
Proof. By Lemma 17 there is a sequence \( U_n = (u_n, A_n) \subset V \) satisfying assumptions (102) and (103) with \( c > 0 \). Clearly we have

\[
\int \left( |\nabla u_n|^2 + \left[ A_n - k \nabla \theta \right] - (\phi_n - \omega)^2 \right) u_n^2 + W'(u_n) u_n \right) \, dx = \langle \eta_n, u_n \rangle
\]

(113)

where \( \| \eta_n \|_{\dot{H}^{-1}} \to 0 \).

By Lemma 18 the sequence \( \{ \| u_n \|_{H^1} \} \) is bounded. Then

\[
\int \left( |\nabla u_n|^2 + \left[ A_n - k \nabla \theta \right] - (\phi_n - \omega)^2 \right) u_n^2 + W'(u_n) u_n \right) \, dx \to 0 \quad (114)
\]

Moreover, by (16), (18) and (19), we get from (113)

\[
\| u_n \|_{H^1}^2 + \int | A_n - k \nabla \theta |^2 u_n^2 - \omega^2 \int (\Phi_{u_n} - 1)^2 u_n^2 \leq \varepsilon_n + \| u_n \|_{L^p}^p, \quad \varepsilon_n \to 0
\]

Then, since \( 0 \leq \Phi_{u_n} \leq 1 \), we have

\[
(1 - \omega^2) \| u_n \|_{H^1}^2 \leq \varepsilon_n + \| u_n \|_{L^p}^p \quad (115)
\]

Clearly, since \( \omega^2 < 1 \), (112) will be a consequence of (115) if we show that

\[
0 < C_3 \leq \| u_n \|_{H^1}^2, \quad C_3 > 0 \quad (116)
\]

We argue by contradiction and assume that (up to a subsequence)

\[
\| u_n \|_{H^1}^2 \to 0 \quad (117)
\]

Since \( \{ (u_n, A_n) \} \) is a PS sequence for the functional \( I \), we have

\[
-\Delta \phi_n - (\omega - \phi_n) \, u_n^2 = 0, \quad \phi_n = Z(u_n)
\]

(118)

\[
-\Delta A_n - (k \nabla \theta - A_n) \, u_n^2 = \eta_n \to 0 \text{ in } \left( (D^{1/2})^3 \right)'
\]

(119)

From these two equations we get

\[
\| \phi_n \|_{D^{1/2},2}^2 = \int (\omega - \phi_n) \, u_n^2 \phi_n \quad (120)
\]

\[
\| A_n \|_{(D^{1/2})^3}^2 = \int \left( A_n - k \nabla \theta \right) \cdot A_n u_n^2 + \delta_n \quad (121)
\]

where \( \delta_n = \langle \eta_n, A_n \rangle \). Since, by Lemma 18 \( \{ \| A_n \|_{(D^{1/2})^3}^2 \} \) is bounded, we have

\[
\delta_n \to 0 \quad (122)
\]

From (120) we have

\[
\| \phi_n \|_{D^{1/2},2}^2 \leq \omega \int u_n^2 \phi_n \leq \omega \| u_n \|_{L^6}^2 \| \phi_n \|_{L^6} \leq C \| u_n \|_{L^6}^2 \| \phi_n \|_{D^{1/2}} \quad (123)
\]
By (117) and (123) we deduce
\[ \| \phi_n \|_{D^{1,2}} \to 0 \quad (124) \]

By (124) and (117) we easily have
\[ \int \left( |\nabla u_n|^2 - (\phi_n - \omega)^2 u_n^2 + W'(u_n) u_n \right) \to 0 \quad (125) \]
then by (114) and (125) we get
\[ \int |A_n - k\nabla \theta|^2 u_n^2 \to 0 \quad (126) \]

Let us show that we have also
\[ \| A_n \|_{(D^{1,2})^3} \to 0 \quad (127) \]

Now
\[ \left| \int (A_n - k\nabla \theta) \cdot A_n u_n^2 \right| \leq \int |A_n - k\nabla \theta| |A_n| u_n^2 \quad (128) \]
\[ \leq \left( \int |A_n - k\nabla \theta|^2 u_n^2 \right)^{\frac{1}{2}} \left( \int |A_n|^2 u_n^2 \right)^{\frac{1}{2}} \]

Since the sequence \( \{ \| A_n \|_{(D^{1,2})^3} \} \) is bounded, we easily get from (117) that
\[ \int |A_n|^2 u_n^2 \to 0 \quad (129) \]

By (129) and (129), we deduce that
\[ \int (A_n - k\nabla \theta) \cdot A_n u_n^2 \to 0 \quad (130) \]

By (130) and (121) we get (127).

By (117) and (124) we have
\[ \int (1 - \omega^2 [1 - \Phi_{u_n}]) u_n^2 - F(u_n) \to 0, \quad (131) \]

Then, by (131), (117), (127) and (126), we deduce that
\[ I(u_n, A_n) = \frac{1}{2} \int |\nabla u_n|^2 + |\nabla A_n|^2 + |A_n - k\nabla \theta|^2 u_n^2 \]
\[ + \frac{1}{2} \int (1 - \omega^2 [1 - \Phi_{u_n}]) u_n^2 - \int F(u_n) \to 0 \]
This contradicts the assumption \( I(U_n) \to c > 0 \). Then (116) holds. \( \blacksquare \)
Lemma 20 Let the assumptions of Lemma 17 be satisfied. Then there is a PS sequence \( U_n = (u_n, A_n) \) for the functional \( I \) such that \( U_n \to U_0 = (u_0, A_0) \) weakly in \( \hat{H}^1 \times (D_0)^3 \) and \( u_0 \neq 0 \).

Proof. By Lemma 19 we can take a PS sequence \( \text{Proof. By Lemma 19 we can take a PS sequence } U_n = (u_n, A_n) \subset V \) for the functional \( I \) such that

\[
\|u_n\|_{L^p} \geq C_5 > 0 \text{ for } n \text{ large} \tag{132}
\]

By Lemma 18 the sequence \( \{U_n\} \) is bounded in \( \hat{H}^1 \times (D_0)^3 \). So we can assume that \( U_n \to U_0 \) weakly in \( \hat{H}^1 \times (D_0)^3 \). If \( U_0 \neq 0 \) we have finished. If not we will show that there is a sequence of integers \( j_n \) such that (up to a subsequence) \( V_n := T_{j_n} U_n \to U_0 \) weakly in \( H^1 \times (D_0)^3 \).

We set

\[
\Omega_j = \{(x_1, x_2, x_3) : j \leq x_3 < j + 1\}, \quad j \text{ integer}
\]

We have for \( n \) large

\[
0 < C_5 \leq \|u_n\|_{L^p}^p = \sum_j \int_{\Omega_j} |u_n|^p = \sum_j \left( \int_{\Omega_j} |u_n|^p \right)^{\frac{(p-2)}{p}} \cdot \left( \int_{\Omega_j} |u_n|^p \right)^{\frac{2}{p}}
\]

\[
\leq \sup_j \|u_n\|_{L^p(\Omega_j)}^{p-2} \sum_j \left( \int_{\Omega_j} |u_n|^p \right)^{\frac{2}{p}} \leq C_6 \cdot \sup_j \|u_n\|_{L^p(\Omega_j)}^{p-2} \cdot \sum_j \|u_n\|^2_{H^1(\Omega_j)}
\]

\[
\leq C_6 \cdot \sup_j \|u_n\|_{L^p(\Omega_j)}^{p-2} \cdot \|u_n\|^2_{H^1(\Omega_j)} \leq (\text{since } \|u_n\|^2_{H^1(\Omega_j)} \leq C_7)
\]

\[
\leq C_6 C_7 \sup_j \|u_n\|_{L^p(\Omega_j)}^{p-2}
\]

Then, for \( n \) large, there exists an integer \( j_n \) such that

\[
\|u_n\|_{L^p(\Omega_{j_n})}^{p-2} \geq \frac{C_4}{2C_5 C_6} := C_7 > 0 \tag{133}
\]

Now set

\[
\left(u'_n, A'_n\right) = U'_n(x_1, x_2, x_3) = U_n(x_1, x_2, x_3 + j_n) = T_{j_n}(U_n)
\]

By lemma 18 the sequence \( u'_n \) is bounded \( \hat{H}^1(\mathbb{R}^3) \), then (up to a subsequence) it converges weakly to \( u_0 \in \hat{H}^1(\mathbb{R}^3) \). We want to show that \( u_0 \neq 0 \). Now, let \( \varphi = \varphi(x_3) \) be a nonnegative, \( C^\infty \)-function whose value is 1 for \( 0 < x_3 < 1 \) and 0 for \( |x_3| > 2 \). Then, the sequence \( \varphi u'_n \) is bounded in \( H^1_0(\mathbb{R}^2 \times (-2, 2)) \), moreover \( \varphi u'_n \) has cylindrical symmetry. Then, using the compactness result proved in [10], we have

\[
\varphi u'_n \to \chi \text{ strongly in } L^p(\mathbb{R}^2 \times (-2, 2))
\]

On the other hand

\[
\varphi u'_n \to \varphi u_0 \text{ a.e.} \tag{134}
\]
Then
\[
\phi'u'_{n} \to \phi u_{0} \text{ strongly in } L^{p}(\mathbb{R}^{2} \times (-2,2)) \quad (135)
\]
Moreover by (133)
\[
\|\phi'u'_{n}\|_{L^{p}(\mathbb{R}^{2} \times (-2,2))} \geq \|u'_{n}\|_{L^{p}(\Omega_{n})} \geq C^{1/(p-2)} (136)
\]
Then by (135) and (136)
\[
\|\phi u_{0}\|_{L^{p}(\mathbb{R}^{2} \times (-2,2))} \geq C^{1/(p-2)} > 0.
\]
Thus we have that \(u_{0} \neq 0\).

Now we complete the proof Theorem 1.

By Lemma 20 there is a PS sequence \(U_{n} = (u_{n}, A_{n})\) for the functional \(I\) such that \(U_{n} \to U_{0} = (u_{0}, A_{0})\) weakly in \(\dot{H}^{1} \times (D^{1,2})^{3}\) and \(u_{0} \neq 0\). We shall show that \((u_{0}, Z(u_{0}), A_{0})\) is a critical point of \(J\).

Since \((u_{n}, A_{n})\) is a PS sequence for \(I\) we have for any \((v, V) \in \dot{H}^{1} \times A\)
\[
d_{u}I(u_{n}, A_{n})[v] \to 0 \quad (137)
\]
\[
d_{A}I(u_{n}, A_{n})[V] \to 0 \quad (138)
\]
Then, since the manifold \(M\) defined in (84) is a natural constraint for the Palais-Smale sequences (see Theorem 16), for any \(v, w \in C_{0}^{\infty}(\mathbb{R}^{3} \setminus \Sigma)\) and \(V \in (C_{0}^{\infty}(\mathbb{R}^{3} \setminus \Sigma))^{3}\), we get
\[
d_{u}J(u_{n}, Z(u_{n}), A_{n})[v] \to 0 \quad (139)
\]
\[
d_{u}J(u_{n}, Z(u_{n}), A_{n})[w] = 0 \quad (140)
\]
\[
d_{A}J(u_{n}, Z(u_{n}), A_{n})[V] \to 0 \quad (141)
\]
Since \(v, w\) and \(V\) have compact support in \(\mathbb{R}^{3} \setminus \Sigma\) and \((u_{n}, A_{n}) \to (u_{0}, A_{0})\) weakly in \(\dot{H}^{1} \times (D^{1,2})^{3}\), by standard argument we can take the limit in (139), (140), (141) and we get that \(u_{0}, Z(u_{0}), A_{0}\) solve the equations
\[
d_{u}J(u_{0}, Z(u_{0}), A_{0})[v] = 0 \quad (142)
\]
\[
d_{u}J(u_{0}, Z(u_{0}), A_{0})[w] = 0 \quad (143)
\]
\[
d_{A}J(u_{0}, Z(u_{0}), A_{0})[V] = 0 \quad (144)
\]
Now by a density and continuity argument the test functions \(v, w\) and \(V\) in (142), (143) and (144) can be taken in \(\dot{H}^{1}, D^{1,2}\) and \((D^{1,2})^{3}\) respectively. Then \((u_{0}, Z(u_{0}), A_{0})\) is a critical point of \(J\). Therefore, by using Theorem 1 we get that \(u_{0}, Z(u_{0}), A_{0}\) solve (13), (14), (15) in the sense of distributions in \(\mathbb{R}^{3}\). Finally, since \(u_{0} \neq 0\), the conclusion follows (see Remark 3).
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