On $G^1$ stitched bi-cubic Bézier patches with arbitrary topology

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Abstract

Lower bounds on the generation of smooth bi-cubic surfaces imply that geometrically smooth ($G^1$) constructions need to satisfy conditions on the connectivity and layout. In particular, quadrilateral meshes of arbitrary topology can not in general be covered with $G^1$-connected Bézier patches of bi-degree 3 using the layout proposed in [ASC17]. This paper analyzes whether the pre-refinement of the input mesh by repeated Doo-Sabin subdivision proposed in that paper yields an exception.

1 Introduction

For many applications, for example artistic rendering and sculpting, a few subdivision steps provide a pleasing rounding of the original polyhedral shape. The simplicity of subdivision with small, local stencils (refinement rules) is appealing and, in particular Catmull-Clark subdivision [CC78] is a staple of geometric modeling environments for creating computer graphics assets. However Catmull-Clark subdivision has also been demonstrated to lead to shape deficiencies, such as pinching of highlight lines, that can be traced back to the simple stencil-based rules [KPR04, KP17].

The algorithm of [ASC17] proposes an approach to obtaining $C^2$ continuous Bi-Cubic Bézier patches that are guaranteed to be stitched with $G^1$ continuity regardless of the underlying mesh topology. This approach consists of applying not Catmull-Clark but Doo-Sabin subdivision to an initial polyhedral input mesh. The approach then derives quadrilateral facets and Bézier control points from the refined mesh and constructs $n$ bi-cubic patches for each $n$-sided facet.

Beyond demonstrating pleasant rounding, [ASC17] emphasizes that the result is a ‘smooth surface with $G^1$ continuity’. If true, this result would be remarkable. It would contradict the restrictions on bi-cubic $G^1$ spline complexes that [PF09] Section 3] derived and that prompted the special constructions in [PS15, SP16]. If [ASC17] were correct then these special constructions published earlier in the same conference series would be superfluous!

$^1G^1$ is typeset as $G_1$ in several places in [ASC17].
Below we show that, while the surfaces generated by the approach of [ASC17] often appear to be smooth, in general they are not.

**Overview.** Section 2 summarizes the algorithm in [ASC17] and the lower bound result of [PF09] as it pertains to bi-cubic $G^1$ constructions. Section 3 provides an explicit, minimal counterexample to the claim that the approach in [ASC17] generates $G^1$ surfaces. Section 4 discusses options for constructing both formally smooth and near-smooth bi-3 constructions.

2 $G^1$ continuity, the construction of [ASC17] and a theorem

The construction of [ASC17] applies two steps of Doo-Sabin subdivision to an initial polyhedral input mesh $M$ and then places the corners of bicubic patches at the Doo-Sabin limit points of the facets obtained in the initial subdivision (Fig 5 of [ASC17]). That is every vertex and every face of $M$ has a corner of a bi-3 patch associated with it. This layout looks more general, and therefore more challenging than the one in [HBC08] which assumed that the input mesh has quadrilateral faces and used $2 \times 2$ bi-cubics to cover them.

Denote by $v$ and $w$ limit points associated with adjacent facets of $M$ (see Fig. 3). Since $M$ is unrestricted and $v$ and $w$ and their tangent planes can be freely adjusted. This independence is typically desirable for flexibility of modeling. The construction in [ASC17] therefore $G^1$ vertex-localized in the sense that the Taylor expansion at $v$ is not tightly linked to that at $w$. Since it does not matter in the construction whether $v$ or $w$ is listed first, the construction along the common boundary is also unbiased, and this is also typically desirable. The unbiased $G^1$ constraints between two patches $p, q : (u, v) \rightarrow \mathbb{R}^3$ along $p(u, 0) = q(u, 0)$ are

\[
\partial_2 p(u, 0) + \partial_2 q(u, 0) = \alpha(u) \partial_1 p(u, 0). \tag{1}
\]

When, at the split point $m$ of the edge $v$, $w$, the four bicubic patches join $C^1$ (see e.g. Fig 8 of [ASC17]) the following theorem applies.

**Theorem 1 ([PF09]: two double edge knots needed)** In general, using splines of degree bi-3 for a vertex-localized unbiased $G^1$ construction without forced linear boundary segments, the splines must have at least two internal double knots.

In other words, Theorem 1 states that to satisfy $G^1$ constraints along $v$ and $w$ (and not have straight line segments embedded in the surface), three rather than the constructed two polynomial boundary segments are needed to connect $v$ and $w$. One might hope that the initialization via Doo-Sabin or adding or leaving out some adjustment does side-step the assumptions of Theorem 1. The next section therefore looks more closely at the construction of [ASC17].

Below the bicubic tensor-product polynomial surface patches $p, q$ of bi-degree 3 are in the following expressed in Bernstein-Bézier (BB) form, e.g.

\[
p(u, v) := \sum_{i=0}^{3} \sum_{j=0}^{3} p_{ij} B_i^3(u) B_j^3(v), \quad (u, v) \in \square := [0..1]^2,
\]
where $B_3^k(t) := \binom{3}{k}(1-t)^{3-k}t^k$ is the Bernstein-Bézier (BB) polynomials of degree 3 and $p_{ij} \in \mathbb{R}^3$ are the BB coefficients [Far02, PBP02].

3 A Counterexample: an input mesh where [ASC17] does not yield a $G^1$ output

Figure 1: Counterexample: the input is regular tetrahedron, only one of whose faces is shown with a wood texture. The grey quad-mesh is the result of applying three steps of Doo-Sabin subdivision. The subnet of 12 Bernstein-Bézier control points of interest are sketched on the refined mesh: from the 4-valent point $m$ to the 3-valent point $v$, these are the BB-coefficients of (3) that influence the $G^1$ continuity between the two bi-3 patches $p$ and $q$.

Since the algorithm of [ASC17] applies initially multiple steps of Doo-Sabin subdivision, the challenge of finding a simple explicit counterexample seems formidable. Yet, the simplest example, $\mathcal{M}$ a regular tetrahedron with vertices

$$A := \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad B := \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad D := \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

(2)

suffices to show that the construction of [ASC17] as stated can not in general generate $G^1$ surfaces. Let $m$ be the point where the curves connecting the limit points associated with $C$ and $D$ meet the curves connecting $v$, the center of the face $B, C, D$, to the center of $A, C, D$ (see Fig. 3). We consider $G^1$ continuity along the edge from $v$ to $m$.

To compute with integers throughout we scale $\mathcal{M}$ by $2^23^2 \cdot 5 \cdot 7$. Following the algorithm of [ASC17] up to the claim ‘Our calculation of the control points guarantees $G_1$ continuity’, the mesh points and BB-coefficients can then be computed as integers. Three rows of BB-coefficients determine the $G^1$ continuity constraints [1] between the
resulting two adjacent bi-3 patches $p$ and $q$. We focus on the BB-coefficients of $p_{ij}$ for $i = 0, 1, 2, 3$ and $j = 0, 1$ using $\sim$ to indicate proportionality after scaling the coefficients to the right of $\sim$ to the smallest integer values: (after multiplication by 210)

$$
\begin{align*}
    p_{i1} &= \begin{bmatrix} 7 & 10 \\ 7 & 8 \\ 10 & 10 \\ 4 & 10 \end{bmatrix}, \\
    q_{i3} &= \begin{bmatrix} 16 & 3 \\ 3 & 8 \\ 4 & 10 \\ 2 & 10 \end{bmatrix}, \\
    p_{i0} &= q_{i,3} \sim \begin{bmatrix} 7 & 10 \\ 10 & 10 \\ 4 & 10 \\ 3 & 3 \end{bmatrix}, \\
    q_{i2} &= \begin{bmatrix} 16 & 3 \\ 3 & 8 \\ 4 & 10 \\ 2 & 10 \end{bmatrix},
\end{align*}
$$

(3)

Then the coefficients of the derivatives across and along the common edge are (after multiplication by 630)

$$
\begin{align*}
    \partial_2 p &= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \\
    \partial_1 p &= \partial_1 q = \begin{bmatrix} 0 \\ -1 \\ 6 \\ -6 \end{bmatrix}, \\
    \partial_2 q &= \begin{bmatrix} 1 \\ -2 \\ 3 \\ -3 \end{bmatrix}, \sim \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\end{align*}
$$

(4)

We compute the determinant (scaled by 10716300)

$$
|\partial_2 p, \partial_1 p, \partial_2 q| \sim [0, 105, 185, 105, 36, 5, 0, 0, 0]. \quad \text{(5)}
$$

Taking taking the dot-product of \( \Pi \) with $\partial_1 p(u, 0) \times \partial_2 q(u, 0)$ implies that $|\partial_2 p(u, 0), \partial_1 p(u, 0), \partial_2 q(u, 0)| = 0$. However in the counterexample the determinant is non-zero and hence the two patches do not join with $G^1$ continuity.

4 Alternative bi-3 constructions

Attempts to generalize bi-cubic splines to irregular layouts have a long history and include [Bez77], [Bee86], [CC78], [Sab68], [Sar87], [GZ94], [Pet91], [Pet94], [vW86] to list just a few. The $3 \times 3$ bi-3 patches per quad construction of [FP08] achieves the lower bound determined by Theorem 1 and has been used to implement the multi-sided caps included in bi-cubic T-spline constructions. [SP16] focuses on restricted input mesh to ensure that $G^1$ bi-3 surfaces can be built with fewer pieces.

Use of a guide shape (of higher polynomial degree) appears to be necessary to construct bi-3 surfaces with a good distribution of highlight lines, as required for car styling and many other outer surfaces. For example, the guided approach improves the shape of bi-3 singularly parameterized surfaces [KP16]. The paper “Can bi-cubic surfaces be class $A$?” [KP15] emphasizes the distinction between exact $G^1$ continuity and acceptable shape in terms of curvature distribution and highlight lines. This distinction, accompanied by mathematical estimates of the jump in normals, could also be useful in the context of [ASC17]. Since proving surface ‘fairness’ is typically not possible, it is recommended to test new surface construction algorithms on the obstacle course [KP] of local input meshes.
5 Conclusion

The approach of [ASC17] rounds shapes but cannot guarantee $G^1$ continuity. A number of alternative finite bi-3 surface constructions exist in the literature. Depending on the valence they require more or fewer pieces than [ASC17]. There are many constructions using few patches but of higher degree than bi-3.

References

[ASC17] Ergun Akleman, Vinod Srinivasan, and Jianer Chen. Interactive modeling of smooth manifold meshes with arbitrary topology: $G^1$ stitched bi-cubic bzier patches. *Computer & Graphics, Special Issue on SMI 2017*, 66:64–73, 2017.

[Bee86] E. Beeker. Smoothing of shapes designed with free-form surfaces. *Computer-Aided Design*, 18(4):224–232, May 1986.

[Bez77] Pierre E. Bezier. *Essai de definition numerique des courbes et des surfaces experimentales*. Ph.d. thesis, Universite Pierre et Marie Curie, February 1977.

[CC78] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, 10:350–355, 1978.

[Far02] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*. Academic Press, San Diego, 2002.

[FP08] Jianhua Fan and Jörg Peters. On smooth bicubic surfaces from quad meshes. In G. Bebis et al., editor, *ISVC (1)*, volume 5358 of *Lecture Notes in Computer Science*, pages 87–96. Springer, 2008.

[GZ94] John A. Gregory and Jianwei Zhou. Filling polygonal holes with bicubic patches. *Computer Aided Geometric Design*, 11(4):391–410, 1994.

[HBC08] Stefanie Hahmann, Georges-Pierre Bonneau, and Baptiste Caramiaux. Bicubic $G^1$ interpolation of irregular quad meshes using a 4-split. In Falai Chen and Bert Jüttler, editors, *Advances in Geometric Modeling and Processing, 5th International Conference, GMP 2008, Hangzhou, China, April 23-25, 2008. Proceedings*, volume 4975 of *Lecture Notes in Computer Science*, pages 17–32. Springer, 2008.

[KP] Kęstutis Karčiauskas and Jörg Peters. Quad-net obstacle course. [http://www.cise.ufl.edu/research/SurfLab/shape_gallery.shtml](http://www.cise.ufl.edu/research/SurfLab/shape_gallery.shtml) Accessed: 2017-09-05.

[KP15] Kęstutis Karčiauskas and Jörg Peters. Can bi-cubic surfaces be class A? In *SGP 2015, Symposium Geometry Processing 2015, Graz*, July 6-8 2015.
[KP16] Kęstutis Karčiauskas and Jörg Peters. Curvature continuous bi-4 constructions for scaffold- and sphere-like surfaces. Computer Aided Design (SPM 2016), 78:48–59, June 17 2016.

[KP17] Kęstutis Karčiauskas and Jörg Peters. Improved shape for refinable surfaces with singularly parameterized irregularities. Computer Aided Design, pages xx–xx, jun 2017.

[KPR04] K. Karčiauskas, J. Peters, and U. Reif. Shape characterization of subdivision surfaces – case studies. Computer-Aided Geometric Design, 21(6):601–614, july 2004.

[PBP02] Hartmut Prautzsch, Wolfgang Boehm, and Marco Paluszny. Bézier and B-spline techniques. Springer Verlag, 2002.

[Pet91] J. Peters. Smooth interpolation of a mesh of curves. Constructive Approximation, 7:221–247, 1991.

[Pet94] J. Peters. Surfaces of arbitrary topology constructed from biquadratics and bicubics. In Nickolas S. Sapidis, editor, Designing fair curves and surfaces: shape quality in geometric modeling and computer-aided design, pages 277–293. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1994.

[PF09] J. Peters and Jianhua Fan. On the complexity of smooth spline surfaces from quad meshes. Computer-Aided Geometric Design, 27:96–105, 2009.

[PS15] Jörg Peters and Martin Sarov. Polynomial spline surfaces with rational linear transitions. Computers & Graphics, 51:43–51, Oct 2015.

[Sab68] M. Sabin. Conditions for continuity of surface normals between adjacent parametric surfaces. Technical Report VTO/MS/151, British Aircraft Corporation, 1968.

[Sar87] R. Sarraga. $G^1$ interpolation of generally unrestricted cubic Bézier curves. Computer Aided Geometric Design, 4(1-2):23–40, 1987.

[SP16] Martin Sarov and Jörg Peters. Refinable polycube g-splines. Computers & Graphics (SMI 2016), 58:92–101, August 2016.

[vW86] J. van Wijk. Bicubic patches for approximating non-rectangular control-point meshes. Computer Aided Geometric Design, 3(1):1–13, 1986.