SCALING LIMITS OF THE UNIFORM SPANNING TREE AND LOOP-ERASED RANDOM WALK ON FINITE GRAPHS

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Abstract. Let $x$ and $y$ be chosen uniformly in a graph $G$. We find the limiting distribution of the length of a loop-erased random walk from $x$ to $y$ on a large class of graphs that include the torus $\mathbb{Z}^d_n$ for $d \geq 5$. Moreover, on this family of graphs we show that a suitably normalized finite-dimensional scaling limit of the uniform spanning tree is a Brownian continuum random tree.

1. Introduction

The uniform spanning tree (UST) $T$ on a graph $G$ is a random tree, uniformly distributed among all spanning trees of $G$. For two points $x$ and $y$ in a graph, let $d_T(x,y)$ be the distance from $x$ to $y$ in $T$. For the complete graph $K_m$, the distance $d_T(x,y)$ is on the order of $\sqrt{m}$ and the distribution satisfies

$$P[d_T(x,y) > \lambda \sqrt{m}] = \exp[-\lambda^2/2] + o(1).$$

Moreover, rescaling by dividing edge lengths by $\sqrt{m}$ results in a scaling limit for the UST on $K_m$ that is the Brownian continuum random tree (CRT) [1]. (We will recall the construction of the Brownian CRT in $\S$ 1.1.) Pitman [16] conjectured that the Brownian CRT should also be the scaling limit of the UST in certain other graphs, and in particular should be the scaling limit of the UST on the $d$-dimensional discrete torus $\mathbb{Z}^d_n$ for large $d$. This conjecture is supported by a recent result of Benjamini and Kozma [5], who showed that the expected distance of loop-erased random walk (LERW) between two uniformly chosen points on $\mathbb{Z}^d_n$ is on the order of $n^{d/2}$ for $d \geq 5$. Pemantle [15] proved that for any $x$ and $y$, $d_T(x,y)$ is equal in distribution to the length of a loop-erased random walk from $x$ to $y$, so Benjamini and Kozma’s result also gives some information about $T$. In Theorem 1.1 we confirm that Pitman’s conjecture holds for $d \geq 5$.

Theorem 1.1. Let $x$ and $y$ be independently and uniformly chosen in $\mathbb{Z}^d_n$, and let $d_T(x,y)$ denote the distance between $x$ and $y$ in the UST. For $d \geq 5,$
there exists a constant $\beta(d) > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}[d_T(x, y) > \beta(d) \lambda n^{d/2}] = \exp\left[-\frac{\lambda^2}{2}\right].$$

Moreover, if \{x_1, \ldots, x_k\} are uniformly chosen from $Z_n^d$, then as $n$ tends to infinity, the joint distribution of

$$\frac{d_T(x_i, x_j)}{\beta(d)n^{d/2}}$$

converges in distribution to $F_k$, the joint distribution of distances between the first $k$ points of the Poisson line breaking construction of the Brownian CRT.

In Section 8, we express $\beta(d)$ in the form $\gamma(d)/\sqrt{\alpha(d)}$, where $\gamma(d)$ and $\alpha(d)$ are probabilities of events involving random walk on $Z^d$ with $\alpha(d) \to 1$ and $\gamma(d) \to 1$ as $d \to \infty$.

Using Pemantle’s [15] result that $d_T(x, y)$ is equal in distribution to the length of a loop-erased random walk from $x$ to $y$, (1) gives the limiting distribution of the length of LERW from $x$ to $y$ in $Z_n^d$. In Subsection 1.3 we will recall Wilson’s algorithm for constructing the UST, which gives even stronger connections between the UST and LERW.

1.1. Construction of the CRT and $F_k$. We define the distribution $F_k$ referred to in Theorem 1.1 via the Poisson line breaking construction of the Brownian CRT. Let $s_1, s_2, \ldots$ be the arrival times for an inhomogeneous Poisson process whose arrival rate at time $t$ is $t$. Draw an initial segment of length $s_1$, and label its ends $y_1$ and $y_2$. Pick a point uniformly on this segment, attach a new segment of length $s_2 - s_1$, and label the end of this segment $y_3$. To continue inductively, given a tree with $k$ ends $y_1, y_2, \ldots, y_k$ and total edge length $s_{k-1}$, pick a point uniformly on the tree and attach to this point a segment of length $s_k - s_{k-1}$, and label the new end $y_{k+1}$. We let $F_k$ denote the joint distribution of the distances between the $k$ points $y_1, \ldots, y_k$. Note that $F_2$ is simply the distribution of $s_1$, which is given by

$$\mathbb{P}[d(y_1, y_2) > \lambda] = \exp\left[-\frac{\lambda^2}{2}\right].$$

As we are only interested in $k$-point distributions of the UST on graphs, we can stop our construction at time $s_{k-1}$. More generally, as $t$ tends to infinity, the resulting sequence of trees, viewed as a sequence of metric spaces, converges to a random, compact metric space. The limit is known as the Brownian CRT. As shown in [1], for $y_1, \ldots, y_k \in K_m$, the joint distribution of $d_T(y_i, y_j)/\sqrt{m}$ converges to $F_k$ as $m \to \infty$. Our main arguments involve coupling LERW on $G_n$ with LERW on $K_m$, and we will obtain the results about the scaling limit of UST on $G_n$ via this coupling.

For a further discussion of the Brownian CRT, see the original papers of Aldous [1, 2, 3] or the lecture notes of Pitman [17].
1.2. **Other Graphs.** The limiting behavior in (1) is not universal. On a cycle, for instance, the typical distance is on the order of $|G_n|^{1/2}$ rather than $|G_n|^{1/2}$. The case of $\mathbb{Z}^d_n$ is quite hard, but recently Kenyon [9, 10] showed that the typical distance of LERW is on the order of $|G_n|^{5/8}$, and Schramm [19] and Lawler, Schramm, and Werner [12] have studied the scaling limit of the UST.

There are also examples of graphs that are not vertex transitive, such as a star, in which the typical distance can be much less than $|G_n|^{1/2}$. Nevertheless, our methods also apply to a broader class of graphs, including the hypercubes $\mathbb{Z}^d_n$ and expander graphs. As a generalization of Theorem 1.1, we will give a set of three conditions such that whenever all three hold, a suitable scaling limit is again the Brownian CRT. The three assumptions that we will make are vertex transitivity, a bounded number of local intersections of two independent random walks, and relatively fast mixing of the random walk.

More formally, let $p^t(x)$ denote the distribution of simple random walk at time $t$ started at a basepoint $o \in G_n$. Our first assumption is that there exists a constant $\theta$ such that

$$(2) \quad \sup_n \sup_{x \in G_n} \sum_{t=0}^{\frac{|G_n|^{1/2}}{2}} (t + 1)p^t(x) \leq \theta < \infty.$$ 

On $\mathbb{Z}^d$, $p^t(0) \text{ decays like } t^{-d/2}$, so for $d \geq 5$, a condition similar to (2) holds in that $\sum_{t=0}^{\infty} tp^t(0)$ is bounded. The implication on $\mathbb{Z}^d$ is that two random walks with the same starting point intersect each other finitely often (see e.g., [11], Theorem 3.5.1). Lemma 4.1 will show that (2) is an analog for finite graphs that says that two independent random walks starting at the same point only intersect a bounded number of times in the first $|G_n|^{1/2}$ steps.

Let $\pi = \pi_n(\cdot)$ denote the stationary distribution of the walk on $G_n$. Denote the (uniform) mixing time by

$$\tau = \tau_n = \inf \left\{ t : \sup_{x \in G_n} \left| \frac{p^t(x)}{\pi(x)} - 1 \right| \leq \frac{1}{2} \right\}.$$ 

Our second assumption is that for some $\delta > 0$,

$$(3) \quad \tau_n = o(|G_n|^{1/2-\delta}).$$

Note that this mixing time assumption implicitly requires that the walk be aperiodic, as otherwise no such $\tau$ exists. As adding a holding probability of $1/2$ to a random walk does not affect LERW, this aperiodicity assumption does not affect the final LERW or the UST. We will show in Sections 8 and 9 that examples of graphs satisfying (2) and (3) include the tori $\mathbb{Z}^d_n$ for $d \geq 5$, the hypercube $\mathbb{Z}^2_n$, expanders, and the complete graph $K_n$. For vertex transitive graphs, conditions (2) and (3) are sufficient to generalize [11].
Theorem 1.2. Suppose that \( \{G_n\} \) is a sequence of vertex transitive graphs satisfying (2) and (3) and such that \( |G_n| \to \infty \). Then there exists a sequence of constants \( \{\beta_n\} \) with

\[
0 < \inf \beta_n \leq \sup \beta_n < \infty
\]
such that for \( x \) and \( y \) uniformly chosen from \( G_n \),

\[
\lim_{n \to \infty} \mathbb{P}[d_T(x, y) > \beta_n \lambda |G_n|^{1/2}] = \exp \left[ -\frac{\lambda^2}{2} \right].
\]

Moreover, if \( k \) points \( \{x_1, \ldots, x_k\} \) are chosen independently and uniformly from \( G_n \), then the joint distribution of

\[
\frac{d_T(x_i, x_j)}{\beta_n |G_n|^{1/2}}
\]
converges to \( F_k \) as \( n \to \infty \).

Note that the difference between Theorem 1.1 and Theorem 1.2 is that we potentially need a bounded sequence \( \beta_n \), rather than having a single scaling constant \( \beta \). If we strengthen assumption (2), then the factor of \( \beta_n \) in the rescaling turns out to be identically 1. More precisely, (2) says that the number of intersections of two walks run for \( |G_n|^{1/2} \) steps is bounded. To strengthen that, let \( \{X_t\} \) and \( \{Y_u\} \) be independent random walks on \( G_n \), take \( q = (\tau|G_n|^{1/2})^{1/2} \), and suppose that

\[
\sup_{x \neq y} \mathbb{E}[\|\{X_t\}_{t=0}^q \cap \{Y_u\}_{u=0}^q| X_0 = x, Y_0 = y | X_0 = x, Y_0 = y] \to 0
\]
as \( n \to \infty \). Note that condition (6) does not hold if the graphs \( \{G_n\} \) have uniformly bounded degree as then the probability that two walks started at the same point will intersect after 1 step is bounded away from 0. It does hold, however, for the hypercubes or the complete graphs. The local intersections are what caused us to require that the points \( \{x_i\} \) be chosen uniformly, and that \( \beta_n \neq 1 \). Thus assumption (6) yields the following, stronger result.

Theorem 1.3. Let \( \{G_n\} \) be a sequence of vertex transitive graphs such that (3) and (6) hold. Then for any choices of \( k \) distinct vertices \( \{x_1^{(n)}, \ldots, x_k^{(n)}\} \) in \( G_n \), the joint distributions of

\[
\frac{d_T(x_i^{(n)}, x_j^{(n)})}{|G_n|^{1/2}}
\]
converge to \( F_k \) when \( n \to \infty \).

In Section 1.3 we define LERW and recall Wilson’s algorithm. In Section 2 we give an outline of the proof of Theorem 1.1 and explain the significance of the size \( |G_n|^{1/2} \), and discuss the structure of the rest of the paper.
1.3. **Loop-erased random walks and Wilson’s algorithm.** Here and throughout this paper, we will use \( \langle \cdot \rangle \) to denote sequences when order is important, and \( \{ \cdot \} \) to denote sets.

Given a finite path \( \gamma = \langle u_0, u_1, \ldots, u_\ell \rangle \) in a graph \( G \), let \( \text{LE}(\gamma) \) denote the loop-erasure of \( \gamma \) with loops erased in chronological order. Formally, \( \text{LE}(\gamma) \) is the sequence \( \langle v_0, v_1, \ldots \rangle \) constructed recursively as follows: first, let \( v_0 = u_0 \); then, given \( v_r \), let \( k \) be the last time that \( u_k = v_r \), and let \( v_{r+1} = u_{k+1} \), with the convention that if \( k = \ell \), then \( v_r \) is the last term of \( \text{LE}(\gamma) \). In the case when \( \gamma \) is the path of a random walk that starts at \( x \) and is stopped when it reaches \( y \), we call \( \text{LE}(\gamma) \) a loop-erased random walk (LERW) from \( x \) to \( y \). More generally, if \( \gamma \) is a random walk from \( x \) stopped when it hits a set \( S \), then we say \( \text{LE}(\gamma) \) is LERW from \( x \) to \( S \).

For an unweighted graph \( G \), the UST on \( G \) is connected to LERW on \( G \) by Wilson’s algorithm for constructing a UST: pick a root vertex \( \rho \), and form an initial subtree \( T_1 \) by picking another vertex \( x_1 \) and running LERW from \( x_1 \) to \( \rho \). We then proceed recursively as follows: given a subtree \( T_i \), pick a vertex \( x_{i+1} \) and run LERW from \( x_{i+1} \) to \( T_i \). Let \( T_{i+1} \) be the union of \( T_i \) with this new path. Proceeding until \( T_i \) is a spanning tree yields a UST \[20\].

Taking \( \rho = y \) and \( x_1 = x \) yields Pemantle’s result \[15\] that the distribution of the path from \( x \) to \( y \) within the UST is the same as the distribution of LERW from \( x \) to \( y \). We will use the fact that Wilson’s algorithm is very robust—indeed, the sequence \( \{x_i\} \) may be chosen arbitrarily, and it also applies to weighted graphs. To apply Wilson’s algorithm to weighted graphs, we let LERW denote the loop-erasure of a weighted random walk that moves according to the weights on the graphs. The probability of any given spanning tree resulting from Wilson’s algorithm then is proportional to the product of the weights of the edges of the tree, so Wilson’s algorithm yields a weighted UST \[20\].

As mentioned earlier, the limiting distribution of \( d_T(x, y) \) on the complete graph \( K_m \) is given by

\[
\mathbb{P}[d_T(x, y) > \lambda \sqrt{m}] = \exp\left[-\frac{\lambda^2}{2}\right] + o(1).
\]

Because there is no geometry on the complete graph, there are a number of derivations of this limit (c.f., \[6\]). One such argument that uses the connection between the UST and the LERW is as follows: Let \( \gamma = \langle x_1, \ldots, x_\ell \rangle \) denote the loop-erased path from \( x \) to \( y \), so \( x_1 = x \) and \( x_\ell = y \). Conditioned on \( \gamma_i = \langle x_1, \ldots, x_i \rangle \), we want to compute \( \mathbb{P}[x_{i+1} = z] \). Considering the walk after the last visit to \( \gamma_i \), we want to condition on never returning to \( \gamma_i \). Let \( f(z) \) be the probability that the next step is to \( z \) and the walk then reaches \( y \) before hitting \( \gamma_i \). We have \( f(y) = 1/(m - 1) \), and by symmetry, \( f(z) = 1/[(m - 1) (i + 1)] \) for \( z \in K_m \setminus \{\gamma_i \cup y\} \). As \( f \equiv 0 \) on \( \gamma_i \), conditioning on the union of these events occurring show that the probability of stepping
to \(y\) at step \((i + 1)\) is thus \((i + 1)/m\). In particular,
\[
\mathbb{P}[d_T(x, y) = k] = \frac{k + 1}{m} \left[ \prod_{i=1}^{k-1} \left(1 - \frac{i + 1}{m}\right) \right].
\]
Rescaling by \(m^{-1/2}\) and letting \(m \to \infty\), this distribution converges to the Rayleigh distribution. Generalizing this argument for graphs with more structure is difficult, so we will use other techniques.

2. Outline of Proof

To prove Theorem 1.2, we would like to use Wilson’s algorithm and make a comparison with the Poisson line breaking construction. Unfortunately, if \(x\) and \(y\) are uniformly chosen from a graph \(G_n\) that satisfies the hypotheses of Theorem 1.2, it turns out that the typical random walk from \(x\) to \(y\) takes on the order of \(|G_n|\) steps. In the special case of \(G_n = \mathbb{Z}^d_n\), Benjamini and Kozma [5] showed that the typical length of the loop-erased path is only of the order of \(|G_n|^{1/2}\), meaning that almost all of the original path is erased. Theorem 1.2 implies that this problem occurs more generally, so to avoid it we would prefer to only run a random walk for the order of \(|G_n|^{1/2}\) steps.

To do this, for a given \(L\), consider the extension \(G_{n,L}^*\) of \(G_n\) formed by adding a vertex \(\rho\), with an edge from every vertex of \(G_n\) to \(\rho\) of weight such that a weighted random walk on \(G_{n,L}^*\) is simple random walk on \(G_n\) modified to move to \(\rho\) after a geometric number of steps with mean \(L|G_n|^{1/2}\). We will run Wilson’s algorithm on \(G_{n,L}^*\) with root vertex \(\rho\). This results in a spanning tree \(T^*\) on \(G_{n,L}^*\), rather than the uniform spanning tree \(T\) on \(G_n\).

The tree \(T^*\) induces a spanning forest \(\tilde{T}\) on \(G_n\) formed by restricting \(T^*\) to \(G_n\). Let \(d_T(x, y)\) and \(d_{\tilde{T}}(x, y)\) denote the distance between \(x\) and \(y\) in \(T\) and \(\tilde{T}\) respectively, with the convention that \(d_{\tilde{T}}(x, y) = \infty\) if \(x\) and \(y\) are in different components of \(\tilde{T}\). This spanning forest \(\tilde{T}\) is comparable in the following sense to the UST \(T^*\):

**Lemma 2.1.** Let \(\{G_n\}\) be a sequence of vertex transitive graphs such that (2) and (3) hold with constants \(\theta\) and \(\delta\) respectively, and such that \(|G_n| \to \infty\). Pick \(k\) points \(\{x_1, \ldots, x_k\}\) uniformly and independently from \(G_n\). For any \(\varepsilon > 0\), there exist constants \(L_0\) and \(N\) such that if \(L > L_0\) and \(n > N\), then the total variation distance between the joint distribution of the distances \(d_T(x_i, x_j)\) and the joint distribution of the distances \(d_{\tilde{T}}(x_i, x_j)\) is less than \(\varepsilon\).

We will prove Lemma 2.1 in Section 4. The point of Lemma 2.1 is that it now suffices to understand \(\tilde{T}\) rather than \(T\). To do so, we will use a comparison with an extension of the complete graph \(K_m\). Let \(K_m\) denote the complete graph with a self-loop at every vertex, and construct an extension \(K_{m,L}^*\) of \(K_m\) in the same way that \(G_{n,L}^*\) was constructed, meaning that \(K_{m,L}^*\) is formed by adding a vertex \(\rho\) that is connected to every vertex of \(K_m\) with
an edge of weight \(m/(\bar{L}\sqrt{m} - 1)\) and so weighted random walk on \(K^*_{m,\bar{L}}\) is simple random walk on \(K_m\) that jumps to \(\rho\) after a geometric number of steps with mean \(\bar{L}\sqrt{m}\).

On \(G_n\), pick \(\{x_1, \ldots, x_k\}\) uniformly and independently. Running Wilson’s algorithm on \(G^*_{n,\bar{L}}\) with root vertex \(\rho\), let \(\mathcal{T}_0 = \{\rho\}\). Given \(\mathcal{T}_\ell\), let \(\mathcal{T}_{\ell+1}\) be formed by running LERW from \(x_{\ell+1}\) to \(\mathcal{T}_\ell\). Note that \(\mathcal{T}_1\) is LERW on \(G^*_{n,\bar{L}}\) from \(x_1\) to \(\rho\). We call \(\mathcal{T}_\ell\) the \(\ell\)-th partial spanning tree. Likewise, let \(\tilde{\mathcal{T}}_\ell\) denote the \(\ell\)-th partial spanning tree on \(K^*_{m,\bar{L}}\). We want to select \(m\) and \(\bar{L}\) in such a way to couple \(\mathcal{T}_\ell\) and \(\tilde{\mathcal{T}}_\ell\). To couple both hitting probabilities and length, we need to consider the following definition of capacity.

**Definition 1.** Let \(\{X_t\}\) be simple random walk on \(G_n\). For \(S \subset G_n\), let \(T_S = \min\{t \geq 0 : X_t \in S\}\) denote the hitting time of \(S\). The \(r\)-capacity of \(S\) is given by

\[
\text{Cap}_r(S) = \mathbb{P}_\pi[T_S < r].
\]

By considering the expected number of visits to \(S\) in the interval \([0, r)\),

\[
\text{Cap}_r(S) \leq r \pi(S) = \frac{r|S|}{|G_n|}
\]

for any set \(S\). When \(S\) is a segment of a random walk, we will show in Lemma 5.3 that the bound in (8) is, with high probability, sharp up to constants.

**Definition 2.** Let \(\{x_1, \ldots, x_k\}\) and \(\{y_1, \ldots, y_k\}\) be uniformly and independently chosen from \(G_n\) and \(K_m\) respectively. For any \(C, L, \bar{L}, \alpha, \beta, \delta, \) and \(r\), the partial spanning trees \(\mathcal{T}_k\) and \(\tilde{\mathcal{T}}_k\) on \(G^*_{n,\bar{L}}\) and \(K^*_{m,\bar{L}}\) are said to be successfully coupled if the following three conditions hold:

\[
\left| \frac{\text{Cap}_r(\mathcal{T}_k)|G_n|^{1/2}}{\alpha^{1/2}r} - \frac{|\tilde{\mathcal{T}}_k|}{\sqrt{m}} \right| \leq C|G_n|^{-\delta/32}
\]

for all \(i, j \leq k\),

\[
\left| \frac{d_{\mathcal{T}_k}(x_i, x_j)}{\beta|G_n|^{1/2}} - \frac{d_{\tilde{\mathcal{T}}_k}(y_i, y_j)}{\sqrt{m}} \right| \leq C|G_n|^{-\delta/32}
\]

Let \(\mu\) be the uniform measure on \(\mathcal{T}_k\) and \(\nu\) the hitting measure, meaning that if \(\{X_t\}\) is simple random walk on \(G_n\) and \(T = \inf\{t \geq 0 : X_t \in \mathcal{T}_k\}\), then \(\nu(x) = \mathbb{P}[X_T = x]\). There is a coupling of \(\mu\) and \(\nu\) such that if \(\xi\) is chosen according to \(\mu\) and \(\eta\) according to \(\nu\), then

\[
\mathbb{P}[d_{\mathcal{T}_k}(\xi, \eta) < C|G_n|^{1/2-\delta/32}] \geq 1 - C(|G_n|^{-\delta/32}).
\]

**Lemma 2.2.** Let \(\{G_n\}\) be a sequence of vertex transitive graphs with \(|G_n| \to \infty\) such that (2) and (3) hold with constants \(\theta\) and \(\delta\) respectively. Then there are sequences of constants \(\alpha = \alpha(n), \beta = \beta(n)\) and \(r = r(n)\), and
for any $L$ and $\varepsilon > 0$, there are sequences $\bar{L} = \bar{L}(L, n)$ and $m = m(n)$, and a constant $C = C(k)$ such that if $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are chosen independently without replacement from $G_n$ and $K_m$ respectively, then for any $n > N$, there is a coupling of $\bar{T}_k$ and $\bar{T}_k$ that is successful with probability $1 - C(k)\left(|G_n|^{-\delta/32}\right)$. Moreover, $m$ and $\bar{L}$ can be made arbitrarily large by first taking $L$ and then $n$ sufficiently large.

Theorem 1.2 is then a consequence of (10), along with Lemma 2.1 and Aldous’ theorem that the Brownian CRT is the scaling limit of UST on $K_m$. Although only (10) is needed to imply Theorem 1.2, we add conditions (9) and (11) to Lemma 2.2 because the proof will be by induction, and the inductive step requires those two pieces.

The key to the proof is a rescaling argument that makes use of the facts that loops formed by a typical random walk are either very short or very long. More formally, let $\{X_t\}$ be a simple random walk on $G_n$ run for $L|G_n|^{1/2}$ steps. Call a loop short if it is of length at most $\tau$, and long if it is of length at least $|G_n|^{1/2-\delta}$. If $t - u \geq \tau$, then $P[X_t = X_u] \leq 2/|G_n|$, so the expected number of loops of length between $\tau$ and $|G_n|^{1/2-\delta}$ formed by time $L|G_n|^{1/2}$ is bounded by $2L|G_n|^{-\delta}$. In particular, the probability of having loops of this type of intermediate length tends to 0.

In Section 3 we will describe a rescaling argument that takes advantage of the absence of intermediate length loops. In order to make this rescaling argument work, we will show in Section 4 that short pieces of a loop-erased random walk retain a positive proportion of their length. Section 5 then combines these pieces of the walk to show that longer lengths of loop-erased walk have a behavior that is tightly concentrated around the mean behavior. Section 6 is then devoted to using these pieces to prove Lemma 2.2. In Section 7 we prove Lemma 2.1, thus concluding the proof of Theorem 1.2. In Section 8 we then prove Theorem 1.1, and in Section 9 we prove Theorem 1.3.

3. Key definitions and introduction of rescaling

As the inductive step in the proof of Lemma 2.2 requires running LERW from $x_{k+1}$ to $\bar{T}_k$ and showing that this new length is similar to what occurs on $K_{m,\bar{L}}$, we begin by considering what happens on $K_{m,\bar{L}}^*$. Suppose that $S \subset K_{m,\bar{L}}^*$ is a subset such that $\rho \in S$. Pick a point $y \in K_{m,\bar{L}}^*$ with $y \neq \rho$, and let $\{Y_u\}$ denote weighted random walk on $K_{m,\bar{L}}^*$ started at $y$ and stopped at time $T = \min\{u \geq 0 : Y_u \in S\}$. We wish to understand $\mathbf{LE}(Y_u)_{u=0}^T$, and in particular its length. To do this, let

$$\tilde{I}_{ij} = \begin{cases} 1 & Y_i = Y_j \\ 0 & Y_i \neq Y_j \end{cases}$$

(12)
be a collection of indicator random variables that keep track of intersections. One special property of the complete graph is that, for fixed \( j \), the joint distribution of \( \tilde{I}_{ij} \) as \( i \) varies conditioned on the values of \( \tilde{I}_{k\ell} \) for \( k, \ell < j \) is the same as the joint distribution conditioned on \( \{Y_u\}_u<j \). Since we are only interested in the length of LERW from \( y \) to \( S \), the lack of geometry on \( K_m \) means that we do not lose information by only keeping track of the time indices instead of locations of \( \{Y_u\} \). Moreover, keeping track of times instead of locations will generalize more appropriately for our rescaling argument when we consider LERW on \( G^*_{n,L} \).

We now inductively construct a family of sequences \( \{\tilde{S}_j\} \) that record what time indices have survived loop-erasure up to time \( j \). Let our initial sequence of length one be given by \( \tilde{S}_0 = \langle 0 \rangle \). Letting \( \ast \) denote concatenation of sequences, for \( 0 < j \leq T \) let

\[
\tilde{S}_j = \langle k \in \tilde{S}_{j-1} : \tilde{I}_{ij} = 0 \forall i \leq k, i \in \tilde{S}_{j-1} \rangle \ast \{j\}.
\]

The result of this definition is that \( \tilde{S}_j \) consists of the original time indices of the walk that have survived loop-erasure at time \( j \), with the convention that when a loop is formed, the original time index is removed and the new time is retained. One consequence of this is that for \( j \leq T \),

\[
\text{LE}(Y_u|u=0) = \langle Y_u \rangle_{u \in \tilde{S}_j}.
\]

For the random walk on \( G^*_{n,L} \), we will again define a collection of indicator random variables \( \{I_{ij}\} \) and index sets \( \{S_j\} \), but the difference now is that the indices will represent a moderately long segment of the loop-erased random walk instead of individual points.

In the rest of this paper, we will be using a variety of different time scales; for ease of reference we give here a summary of the meanings of the different scales on which we will be working. First, weighted random walk on the extension \( G^*_{n,L} \) started in \( G_n \) moves to \( \rho \) after a geometric number of steps that is on the order of \( |G_n|^{1/2} \). Recall that \( \tau = \tau_n \) denotes the mixing time for random walk on \( G_n \). We will break up \( |G_n|^{1/2} \) into shorter segments \( A_i \) of length that is asymptotically \( r = [\tau^{1/4}|G_n|^{3/8}] \). By assumption (3), \( \tau \), and thus \( r \), is of a lower order than \( |G_n|^{1/2} \). To show that the behavior of the walk on each run of length \( r \) is close to its mean behavior, we will further break these runs into smaller pieces of length \( q = [\tau^{1/2}|G_n|^{1/4}] \) and then sum the pieces to get large deviation estimates. These estimates require independence between the segments, so instead of considering the loop-erased path, we will consider a local loop-erasure, with a window size \( s = [\tau^{3/4}|G_n|^{1/8}] \) that is much smaller than \( q \). To justify that the restriction to local loop-erasure does not throw away too much information, we will then finally show that there are a number of local cutpoints, for which we will only look at path segments whose length is on the order of \( \tau \). As a summary, see Table 1.
To remember these relative sizes, note that \( \{\tau, s, q, r, |G_n|^{1/2}\} \) is approximately a geometric sequence with common ratio \( \tau^{-1/4}|G_n|^{1/8} \).

As mentioned above, in order to get the independence between various pieces that we need for our large deviation estimates, we will need to use a local loop-erasure instead of the original loop-erasure.

**Definition 3.** Let \( \{X_t\} \) denote a weighted random walk on \( G_{n,L}^* \) with \( X_0 \neq \rho \) that is stopped at time \( T \). A time \( u \) is **locally retained** if

\[
\text{LE}\langle X_t \rangle^u \cap \{X_t\}_{t=u+1}^{\min\{T,u+s\}} = \emptyset.
\]

**Definition 4.** For a stopping time \( T \), let \( U \) be the set of all times \( t \leq T \) that are locally retained. The **local loop-erasure** \( \text{LE}_s\langle X_t \rangle^T_{t=0} \) is the subsequence of \( \langle X_t \rangle^T_{t=0} \) such that

\[
\text{LE}_s\langle X_t \rangle^T_{t=0} := \langle X_u \rangle_{u \in U}.
\]

When the stopping time \( T \) is not specified, we will take \( T = \infty \) in the definition of the local loop-erasure.

It is not *a priori* true that either the local loop-erasure or the loop-erasure contains the other. For example, if the original path has a loop of length \( s - 1 \), the local loop-erasure could have a jump, while if the original path has a loop of length just longer than \( s \), then the local loop-erasure can have short loops. These differences raise a problem that we will have to deal with later. We will later formalize the notion that, with high probability, the local loop-erasure has no jumps, and that the main difference between the local loop-erasure and the loop-erasure comes from having long loops (meaning of length greater than \( r \)). Our coupling with LERW on the complete graph will keep track of the long loops.

Given a set \( S \subset G_{n,L}^* \) with \( \rho \in S \), pick a point \( x \in G_n \), and let \( \{X_t\} \) be weighted random walk on \( G_{n,L}^* \) with \( X_0 = x \) and run until time \( T = \min\{t \geq 0 : X_t \in S\} \).

For \( i < T/r \), let

\[
A_i = A_i(r,s) = \mathbb{Z} \cap [(i-1)r + 2s + 1, ir - s].
\]

As \( s = o(r) \), the number of elements of \( A_i \) is asymptotically \( r \). Adding a buffer of length \( s \) at the beginning and end of \( A_i \) means that the times that are locally retained within the different \( A_i \) are independent. The second delay of \( s \) at the start of \( A_i \) will also mean that the locations of the path on

| \(|G_n|^{1/2}\) | typical length of LERW |
| --- | --- |
| \( r = |G_n|^{1/4}|G_n|^{5/8} \) | length of \( A_i \) |
| \( q = |G_n|^{1/4} \) | subdivisions of \( A_i \) |
| \( s = |G_n|^{1/8} \) | window for local loop-erasure |
| \( \tau = \tau_n \) | mixing time, and window for local cutpoints |

**Table 1.** Definitions of scales
different $A_i$ are close to independent. We will let $\mathbf{LE}_s(A_i)$ denote the part of the local loop-erasure of $\{X_t\}$ whose original times were in $A_i$, that is to say

$$\mathbf{LE}_s(A_i) := \langle X_t \rangle_{t \in A_i \cap U}.$$  

Since $\rho \in S$, the hitting time $T$ is on the order of at most $|G_n|^{1/2}$, so the number of intervals $A_i$ is at most on the order of $|G_n|^{1/2}/r$. We now wish to keep track of non-local loops, meaning loops that somehow involve two of the $A_i$. For $i < j$, let $I_{ij}$ denote indicator random variables

$$I_{ij} = \begin{cases} 1 & \{\mathbf{LE}_s(A_i) \cap \{X_t\}_{t \in A_j} \neq \emptyset\}, \\ 0 & \{\mathbf{LE}_s(A_i) \cap \{X_t\}_{t \in A_j} = \emptyset\}. \end{cases}$$

Unlike the case of the complete graph, here the joint distribution of $I_{ij}$ conditioned on $\{X_t\}_{t < r_j}$ is different from the joint distribution of $I_{ij}$ conditioned on $\{k_{kl}, k, \ell < j\}$. Despite this, we can still recursively construct a family of sequences $S_j$ that in some sense records which runs $\mathbf{LE}_s(A_i)$ survive loop-erasure. The construction will implicitly take the entire path $\{X_t\}$ into consideration. To begin, let $S_0 = \langle 0 \rangle$. For $0 < j \leq \lceil T/r \rceil$, let $*$ denote concatenation of sequences, and let

$$S_j = \langle k \in S_{j-1} : I_{ij} = 0 \forall i \leq k, i \in S_{j-1} * \{j\}. \rangle$$

These sequences are intended to play much the same role as $\{\tilde{S}_j\}$, and can be thought of as keeping track of indices $i$ such that $\mathbf{LE}_s(A_i)$ is completely contained inside $\mathbf{LE}(X_t)^{T}_{t=0}$. There are some problems: if $\mathbf{LE}_s(A_i)$ is involved in a long loop, then part of it is erased. In particular, it is not true that $\mathbf{LE}_s(A_i) \subset \mathbf{LE}(X_t)^{T}_{t=0}$, and also if the long loop involves $\mathbf{LE}_s(A_j)$, then only one of $i$ or $j$ is retained in $S_j$. There is also a problem when one end of a long loop is in the gap of length $3s$ between the $A_i$. We will later prove that these differences are sufficiently rare that their contribution is of a lower order of magnitude than the length of LERW.

In Section 5 we will show that the length of $\mathbf{LE}_s(A_i)$ is tightly concentrated about its mean $\gamma r$ for some $\gamma = \gamma(n)$ that is bounded away from 0. This implies that $|\mathbf{LE}(X_t)^{T}_{t=0}|$ is roughly $\gamma r |S_{\lfloor T/r \rfloor}|$. We will prove the induction step of Lemma 2.2 in Section 6 but to understand the idea, take $S = \mathcal{T}_k$. We want to pick an $m$ that lets us couple LERW on $G^\ast_{n,L}$ and $K^\ast_{m,\mathcal{L}}$ in such a way that $|S_{\lfloor T/r \rfloor}| = |\tilde{S}_{\mathcal{T}}|$, where $\tilde{T}$ is the hitting time for $\mathcal{T}_k \subset K^\ast_{m,\mathcal{L}}$. To do this, we need the distribution of $I_{ij}$ to be close to the distribution of $\tilde{I}_{ij}$. Let $\text{Cap}_r(S)$ denote the capacity of $S$ as defined in Definition 11

Define $\gamma = \gamma(n)$ and $\alpha = \alpha(n)$ by

$$\gamma = \mathbb{E}\frac{|\mathbf{LE}_s(A_i)|}{r}, \quad \alpha = \mathbb{E}\text{Cap}_r[\mathbf{LE}_s(A_i)]\frac{|G_n|}{r^2}. \quad (17)$$

In Lemma 5.3 we will show that both $|\mathbf{LE}_s(A_i)|$ and $\text{Cap}_r[\mathbf{LE}_s(A_i)]$ are tightly concentrated about their means. The capacity bound implies that
$\mathbb{E} I_{ij}$ is approximately $\alpha r^2/|G_n|$. On $K_m$, $\mathbb{E} I_{ij} = (1 + o(1))/m$, so in our rescaling we will take the size of the complete graph to be

$$m = \left\lceil \frac{|G_n|}{\alpha r^2} \right\rceil.$$

and let

$$\beta = \beta_n = \frac{\gamma}{\alpha^{1/2}}.$$

For the special case of $S = T_0 = \{\rho\}$, taking $\tilde{T}$ to be the hitting time of $\rho$ on $K^*_m$, the two point distribution reduces to showing that

$$\mathbb{P}\left[\mathbf{L}E_{s}(X_t)_{t=0}^{T} > \frac{\gamma}{\alpha^{1/2}} |G_n|^{1/2} \right] = \mathbb{P}\left[|S_{T/r}| > \frac{\lambda |G_n|^{1/2}}{\alpha^{1/2}r} \right] + o(1)$$

$$= \mathbb{P}\left[|\tilde{S}_{\tilde{T}}| > \lambda \sqrt{m} \right] + o(1),$$

which tends to $\exp\left[-\lambda^2/2\right]$ as $m, \tilde{L} \to \infty$.

To obtain $\mathbb{P}$, in Section 5 we will show that, on the torus, the limits of $\gamma(n)$ and $\alpha(n)$ exist. Computing these limits and replacing $\beta_n$ by the limit of $\beta_n$ will then prove Theorem 1.1.

In the next few sections, we will develop the tools needed to prove Lemma 2.2.

### 4. Positive length of small pieces

The aim of this section is to study local-loop erasure of runs of length roughly $q \approx \tau^{1/2} |G_n|^{1/4}$ and show that $\mathbf{L}E_{s}(X_t)_{t=2s+1}^{q-s}$ retains a positive proportion of the original walk. Note that the buffers of length $s$ and $2s$ at the start and end of these pieces of length $q$ are the same size as in $A_i$ and serve the same role. The fact that erasing loops shortens the path, along with (8), gives the upper bounds

$$\mathbb{E} \left| \mathbf{L}E_{s}(X_t)_{t=2s}^{q-s} \right| \leq 1,$$

$$\mathbb{E} \operatorname{Cap}_r(\mathbf{L}E_{s}(X_t)_{t=2s}^{q-s}) \frac{|G_n|}{q^r} \leq 1,$$

where $\operatorname{Cap}_r(S)$ is as in Definition 1. The focus of this section will be giving lower bounds for these quantities, and in particular showing that they are bounded away from $0$.

For random walks on $\mathbb{Z}^d$, the fact that $\sum k \mathbb{P}[X_k = 0] < \infty$ for $d \geq 5$ is equivalent to the fact that two simple random walks on $\mathbb{Z}^d$ will intersect each other finitely often in dimensions $5$ and higher. The next lemma makes more precise the fact that condition (2) gives a local analog.

**Lemma 4.1.** Suppose that $\{G_n\}$ is a sequence of vertex transitive graphs satisfying (2) with constant $\theta$. Let $\{X_t\}$ and $\{Y_t\}$ be independent random
walks started at the same point \( o \). Let \( \tilde{X}_t \) and \( \tilde{Y}_t \) be the walks \( \{ X_t \} \) and \( \{ Y_t \} \) killed at random times \( T_X \) and \( T_Y \), which are geometrically distributed random variables with mean \((1 - \lambda)^{-1} \). Letting \( \theta \) be the constant from (2),

\[
(18) \quad \mathbb{P}(\{ \tilde{X}_t \}_{t\geq 0} \cap \{ \tilde{Y}_t \}_{t\geq 1} = \emptyset) \geq \left( \theta + \frac{2}{|G_n|(1 - \lambda)^2} \right)^{-1}. 
\]

Proof. We use the central idea of the proof of Proposition 3.2.2 in [11]. Call a pair of times \((i, j)\) a *-last intersection if

\[
\{(t, u) : \tilde{X}_t = \tilde{Y}_u, t \geq i, u \geq j\} = \{(i, j)\}.
\]

Abbreviate

\[
f(\lambda) = \mathbb{P}(\{ \tilde{X}_t \}_{t\geq 0} \cap \{ \tilde{Y}_t \}_{t\geq 1} = \emptyset).
\]

Note that \( \mathbb{P}[\tilde{X}_i = \tilde{Y}_j] = \lambda^{i+j}\mathbb{P}[X_i = Y_j] \), so by vertex transitivity and the memoryless property of exponential random variables, the probability that \((i, j)\) is a *-last intersection is bounded by

\[
\lambda^{i+j}\mathbb{P}[X_i = Y_j]f(\lambda).
\]

Because the killed paths are finite in length, there is at least one *-last intersection. By symmetry of the walk, \( \mathbb{P}[X_i = Y_j] = \mathbb{P}[X_{i+j} = o] \), so considering the expected number of *-last intersections gives

\[
1 \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lambda^{i+j}\mathbb{P}[X_i = Y_j]f(\lambda)
= \sum_{k=0}^{\infty} \lambda^k(k + 1)\mathbb{P}_o[X_k = o]f(\lambda)
\leq \left( \sum_{k=0}^{\infty} (k + 1)\mathbb{P}_o[X_k = o] + \sum_{k=\lceil|G_n|^{1/2}\rceil}^{\infty} \frac{2(k + 1)}{|G_n|}\lambda^k \right) f(\lambda)
\leq \left( \theta + \frac{2}{|G_n|(1 - \lambda)^2} \right) f(\lambda).
\]

\[ \square \]

For a path \( \langle X_t \rangle_{t=0}^{T} \), a point \( X_u \) is a local cutpoint if \( \{ X_{u-\tau}, \ldots, X_{u-1} \} \cap \{ X_{u+1}, \ldots, X_{u+\tau} \} = \emptyset \). Lemma [11] and condition [3] imply that the expected number of local cutpoints of random walk is a positive proportion of the length of the path: taking \( 1 - \lambda = (\tau|G_n|^{1/2})^{-1/2} \) means that \([(1 - \lambda)^2|G_n|]^{-1} \) tends to 0, and also that \( \mathbb{P}[\min\{ T_X, T_Y \} > \tau] = 1 - o(1) \). Lemma [11] then implies that the probability that a given point is a local cutpoint is at least \( 1/\theta + o(1) \). The significance of local cutpoints is that conditioned on not having long loops (meaning loops of length greater than \( \tau \)), local cutpoints are also retained in the loop-erasure.
Lemma 4.2. Suppose that \( \{G_n\} \) is a sequence of graphs satisfying the assumptions of Theorem 1.2 and let \( \{X_t\} \) be simple random walk on \( G_n \) with geometric killing rate \( (L|G_n|^{1/2})^{-1} \). Let \( T \) be the killing time, and let \( \Gamma_n \) be the event that for all \( t \in [0,T-s] \), the interval \([t,t+s]\) contains a local cutpoint. Then there is a constant \( C \) such that for any \( s \), \( \mathbb{P}[\Gamma_n] \geq 1-L|G_n|^{1/2} \text{exp}(-Cs/\tau) \).

Note that taking \( s \geq (\log |G_n|)^2 \tau \), as the choice in Table 1 does, gives \( \mathbb{P}[\Gamma_n] \geq 1 - o(|G_n|^{-1}) \).

Proof. By Lemma 4.1, any point is a local cutpoint with probability bounded away from 0. Whether or not \( X_t \) and \( X_{t+2\tau} \) are local cutpoints are independent events, so the probability of having no local cutpoint in an interval of length \( 2k\tau \) decays exponentially in \( k \). The result then follows from summing over all start times \( t \in [0,T-s] \).

Repeating the argument that the the probability of a point being a local cutpoint is bounded below, but considering times that are locally retained rather than local cutpoints, shows that the probability that any given time is locally retained is at least \( (1+o(1))/\theta \). In particular, this gives

Corollary 4.2. Suppose that \( \{G_n\} \) is a sequence of vertex transitive graphs satisfying \( \mathbb{2} \) with constant \( \theta \), and suppose that \( s = s(n) \) and \( q = q(n) \) are such that \( q \leq |G_n|^{1/2-\delta/4} \) and \( s = o(q) \). If \( \{X_t\} \) is simple random walk on \( G_n \), then

\[
\mathbb{E} \left[ \text{LE}_s\langle X_t \rangle_{t=2s}^{q-s} \right] \geq \frac{q + o(q)}{\theta}.
\]

In addition to knowing that a positive fraction of the walk is retained by local loop-erasure, we also want to know that the probability of two random walks intersecting is not too greatly reduced by local loop-erasure of one of the paths. For two i.i.d., transient random walks \( \{X_t\} \) and \( \{Y_u\} \), this holds quite generally. For example,

\[
\mathbb{P}[\text{LE}(X_t) \cap \{Y_u\} \neq \emptyset] \geq 2^{-8}\mathbb{P}[(X_t) \cap \{Y_u\} \neq \emptyset]
\]

(see \[\mathbb{4}\]). Markov chains with geometric killing on a finite state space are transient Markov chains, but \( \mathbb{20} \) does not apply in our case, partly because we need deterministic (rather than geometric) killing, but mostly because we are interested in cases when the killing times are on different orders of magnitude and so the killed walks are not i.i.d.

Lemma 4.2. Suppose that \( \{G_n\} \) is a sequence of vertex transitive graphs satisfying \( \mathbb{2} \) with constant \( \theta \). Let \( \{X_t\} \) and \( \{Y_u\} \) be two independent random walks run with deterministic killing times \( T_X \) and \( T_Y \) respectively, run from uniformly chosen starting locations \( x,y \in G_n \). If \( T_X,T_Y \leq |G_n|^{1/2}/2 \) and \( s \leq \varepsilon|G_n|^{1/2} \), then

\[
\mathbb{P}[\text{LE}_s\langle X_t \rangle_{t=0}^{T_X} \cap \{Y_u\}_{u=0}^{T_Y} \neq \emptyset] \geq \frac{T_X T_Y}{|G_n| \theta^2(4\theta-3)} \left[ \frac{1}{1+2\varepsilon} - 2(1 - \sqrt{\varepsilon}) \right]
\]
where \( \theta \) is the constant in (2).

Proof. The proof relies on the second moment bound

\[
\mathbb{P}[Z > 0] \geq \frac{(\mathbb{E}Z)^2}{\mathbb{E}(Z^2)}
\]

for non-negative random variables \( Z \). For \( i \leq T_X \) and \( j \leq T_Y \), let \( J_{ij} \) be an indicator random variable for the event

\[
\{ X_i = Y_j \in \text{LE}_s(X_t)_{t=0}^{T_X}, \}
\]

and take \( Z = \sum_{i,j} J_{ij} \) to be the number of ordered pairs \((i, j)\) corresponding to intersections of \( \text{LE}_s(X_t) \) with \( \{ Y_u \} \).

To bound \( \mathbb{E}Z \), let \( \hat{T}_1 \) and \( \hat{T}_2 \) be independent geometric random variables with mean \((1 - \lambda)^{-1} = (\varepsilon|G_n|)^{1/2}\), and let \( U \) denote the set of times \( 0 \leq t \leq T_X \) that are locally retained. As \( \langle X_t \rangle \) is symmetric, we can extend it to a doubly infinite chain \( \langle X_t \rangle_{t=-\infty}^{\infty} \). Since \( Y_0 \) is uniformly distributed,

\[
\mathbb{E}J_{ij} = \frac{1}{|G_n|} \mathbb{P}[i \in U] \geq \frac{1}{|G_n|} \left( \mathbb{P}\{\langle X_t \rangle_{t=\hat{T}_1}^{0} \cap \langle X_t \rangle_{t=\hat{T}_2}^{T_X} = \emptyset \} - 2\mathbb{P}[\hat{T}_1 \leq s] \right).
\]

Using Lemma 4.1 and observing that \( \theta \geq 1 \) yields

\[
\mathbb{E}Z \geq \frac{T_X T_Y}{\theta|G_n|} \left[ \frac{1}{1 + 2\varepsilon} - 2(\sqrt{\varepsilon}) \right].
\]

To bound the second moment, note that the number of intersections of the local loop-erasure starting at \( x \) with the path from \( y \) is bounded above by the number of intersections of the original walk, so it suffices to bound the second moment of the number of such intersections. To do so, let \( I_{ij} \) be an indicator random variable for the event \( \{ X_i = Y_j \} \) and fix a basepoint \( o \in G_n \). Because the starting points of our walks are uniform,

\[
\mathbb{E}Z^2 \leq \sum_{i=0}^{T_X} \sum_{j=0}^{T_Y} \sum_{k=i}^{T_X} \sum_{\ell=j}^{T_Y} 4\mathbb{E}I_{ij}I_{k\ell} - 3 \sum_{i=0}^{T_X} \sum_{j=0}^{T_Y} \mathbb{E}I_{ij}
\]

\[
= \sum_{i=0}^{T_X} \sum_{j=0}^{T_Y} \sum_{k=i}^{T_X} \sum_{\ell=j}^{T_Y} 4\mathbb{P}_o[X_{k-i+\ell-j} = o]\frac{|G_n|}{|G_n|} - 3\frac{T_X T_Y}{|G_n|}
\]

\[
= \frac{4T_X T_Y}{|G_n|} \sum_{k=0}^{T_X} \sum_{\ell=0}^{T_Y} \mathbb{P}_o[X_{k+\ell} = o]\frac{|G_n|}{|G_n|} - 3\frac{T_X T_Y}{|G_n|}
\]

\[
\leq \frac{4T_X T_Y}{|G_n|} \sum_{k=0}^{T_X} \sum_{\ell=0}^{T_Y} \mathbb{P}_o[X_{k+\ell} = o] - 3\frac{T_X T_Y}{|G_n|}
\]

\[
\leq [4\theta - 3]\frac{T_X T_Y}{|G_n|}
\]

where \( \theta \) is as in (2). Using these quantities to lower bound \( (\mathbb{E}Z)^2/\mathbb{E}(Z^2) \) gives the desired result. \( \square \)

Let \( \text{Cap}_r(S) \) be as in Definition 1. Taking \( T_X = q - 3s \) and \( T_Y = r \) gives:
Corollary 4.3. Suppose that \( \{G_n\} \) is a sequence of graphs satisfying the assumptions of Theorem 1.2. If \( q, r \leq |G_n|^{1/2 - \delta/4} \), and \( s = o(q) \), then for simple random walk \( \{X_t\} \) on \( G_n \),

\[
\mathbb{E} \left[ \text{Cap}_r(\text{LE}_s(X_t)_{t=2s}) \right] \geq \frac{qr(1 + o(1))}{|G_n|^{\theta^2(4\theta - 3)}}.
\]

5. Tight concentration of longer segments

We now combine the results of the previous section with large deviation estimates to show that \( |\text{LE}_s(A_i)| \) and \( \text{Cap}_r(\text{LE}_s(A_i)) \) are tightly concentrated about their means. We will do this by viewing \( A_i \) as a union of smaller pieces and using the following large deviation bound:

**Lemma 5.1** (Hoeffding). Suppose \( \{Z_i\} \) are a family of independent random variables such that \( 0 \leq Z_i \leq b \). Then

\[
P \left[ \sum_{i=1}^n Z_i - \mathbb{E}Z_i > nt \right] \leq \exp \left[ -2n \left( \frac{t}{b} \right)^2 \right].
\]

This lemma is from [8], Theorem 1, equation (2.3), and is proved using standard arguments.

Our applications of this large deviation bound include showing that, with high probability, the capacity and length of \( \text{LE}_s(A_i) \) are close to their means for all \( A_i \), as well as the fact that for different \( i \), the sequences \( \text{LE}_s(A_i) \) are relatively far apart in the graph. Much of the difficulty in doing this for \( \text{Cap}_r(\text{LE}_s(A_i)) \) involves the possibility that a run of length \( r \) of a walk might hit more than one subsegment of \( \text{LE}_s(A_i) \). These multiple intersections are important because they mean that capacity is subadditive.

**Definition 5.** Let \( \{Y_u\} \) be simple random walk on a graph \( G \), and let \( T_S = \min \{u \geq 0 : Y_u \in S\} \) denote the hitting time for \( S \). For sets \( U \) and \( V \), the closeness of \( U \) and \( V \) is given by

\[
\text{Close}(U, V) = \mathbb{P}_\pi\left[T_U < r, T_V < r\right].
\]

Note that \( \text{Close}(U, V) \) will be small if \( U \) and \( V \) intersect in a single point yet are otherwise very far away. On the other hand, if \( U \) and \( V \) are disjoint, but \( V \) is a translation by a small fixed distance, then \( \text{Close}(U, V) \) will be large. Closeness is primarily a measure of whether or not typical points of \( U \) are near \( V \), and vice versa, and is maximized when the two sets coincide.

In the case when \( V \) is a segment of a random walk, \( \text{Close}(U, V) \) is a random variable whose mean is bounded by the following lemma:

**Lemma 5.2.** Let \( \{G_n\} \) be a sequence of graphs satisfying the assumptions of Theorem 1.2 and let \( r = r(n) \) be a sequence of positive integers. Let \( \{X_t\} \) be a random walk on \( G_n \) started with \( X_0 = x \), let \( T \geq 0 \) be a random time that is independent of \( \{X_t\} \), and let \( V = \{X_t\}_{t=T}^T \). Then for any set \( U \) and
starting position $x$.

(24) \[ \mathbb{E} \text{Close}(U, V) \leq 4 \frac{r \text{Cap}_r[U] ET}{|G_n|}. \]

Proof. Let $\{Y_u\}$ be a simple random walk on $G_n$ started in the stationary distribution $\pi$. By the strong Markov property, for any fixed set $W$, \[ \text{Close}(U, W) \leq \mathbb{P}_\pi[T_U < r] \mathbb{P}_{Y_{U}}[T_W < r \mid T_U < r] \]

To apply this to our random $V$, let $\mathbb{E}_x$ denote expectation given $X_0 = x$. \[ \mathbb{E}_x \left[ \mathbb{P}_\pi[T_V < r] \mathbb{P}_{Y_{V}}[T_U < r \mid T_V < r] \right] \]

Likewise, $\mathbb{E}_x \left[ \mathbb{P}_{Y_{U}}[T_V < r \mid T_U < r] \right]$ is bounded by \[ \sum_y \mathbb{P}_\pi[Y_{U} = y \mid T_U < r] \sum_{u=0}^{r-1} \sum_{t=\tau}^{t=\tau} \mathbb{P}[Y_u = X_t, t \leq T \mid Y_0 = y, X_0 = x] \]

where the second inequality uses the fact that as $t \geq \tau$, \[ \mathbb{P}[Y_U = X_t \mid Y_0 = y, X_0 = x] \leq 2/|G_n|. \]

As $\mathbb{P}_\pi[T_U < r] = \text{Cap}_r[U]$, this completes the proof. \hfill \Box

Let $\alpha = \alpha(n)$ and $\gamma = \gamma(n)$ be as in (17), implying that

\[ \mathbb{E}[\text{LE}_s\{X_t\}_{t=2s}] = \gamma r, \quad \mathbb{E} \text{Cap}_r[\text{LE}_s\{X_t\}_{t=2s}] = \frac{\alpha r^2}{|G_n|}. \]

Lemma 5.3. Let $\{G_n\}$ be a sequence of graphs satisfying the assumptions of Theorem 1.2 and let $\{X_t\}$ be simple random walk on $G_n$. Suppose that $r = r(n)$ and $s = s(n)$ are sequences such that

\[ 4s^{1/2}|G_n|^{1/4} \log |G_n| \leq r \leq |G_n|^{1/2-\delta/4} \]
Taking $A_i$ as in (14), then whenever $|G_n| \geq 2^8$,

\begin{equation}
\Pr \left[ |\text{LE}_s(A_i)| - \gamma r \right] > 2r \left( \frac{s}{r} \right)^{1/6} \right) \leq \exp \left[ -2 \left( \frac{r}{s} \right)^{1/6} \right],
\end{equation}

\begin{equation}
\Pr \left[ \left| \text{Cap}_r(\text{LE}_s(A_i)) - \frac{\alpha r^2}{|G_n|} \right| > \left( \frac{r}{|G_n|^{1/2}} \right)^{9/4} \right] \leq 9 \left( \frac{r^{7/4}}{|G_n|^{7/8}} \right),
\end{equation}

Moreover, $\alpha$ and $\gamma$ satisfy the bounds

\begin{equation}
\gamma \geq \frac{1 + o(1)}{\theta}, \quad \alpha = \frac{1 + o(1)}{\theta^2(4\theta - 3)}.
\end{equation}

The situation that we are most interested in is when $s$ and $r$ are as in Table 1 in which case assumption (3) shows that Lemma 5.3 applies. More generally, note that the assumptions on $s$ and $r$ imply that as $|G_n| \to \infty$, $s = o(r)$ and also $s|G_n|^{-1/2} = o(r^2|G_n|^{-1})$.

**Proof.** Let $q = \lfloor (rs)^{1/2} \rfloor$. To prove all three parts of this lemma, we will break the interval $A_i$ down into $|r/q|$ smaller pieces $B_{i,k}$ whose length is asymptotically $q$, and use Lemma 5.1 and the results of Section 4. Let $B_{i,k} = Z \cap [ir + kq + 2s, ir + (k + 1)q - s]$, and denote

$$\text{LE}_s(B_{i,k}) = \langle X_t \rangle_{t \in B_{i,k} \cap U},$$

where $U$ is the set of times that are locally retained.

We begin with (25). Breaking $\text{LE}_s(A_i)$ into pieces and summing gives

\begin{align*}
\sum_{k=0}^{\lfloor r/q \rfloor - 1} |\text{LE}_s(B_{i,k})| &\leq |\text{LE}_s(A_i)| \\
&\leq \sum_{k=0}^{\lfloor r/q \rfloor} |\text{LE}_s(B_{i,k})| + 3s \left( \frac{r}{q} + 1 \right).
\end{align*}

By our choice of $q$, $rs/q \sim s(r/s)^{1/2}$, which is of a lower order than $r(s/r)^{1/6}$. The restrictions on $r$ and $|G_n|$ insure that $3s(r+q)/q \leq r(s/r)^{1/6}$. The spacing between the $B_{i,k}$ is such that $|\text{LE}_s(B_{i,k})|$ are i.i.d, so applying Lemma 5.1 with $Z_k = |\text{LE}_s(B_{i,k})|$, $b = q$, and $t = q^{1/3}r^{-1/3}$ shows that

\begin{align*}
\Pr \left[ \sum_{k=0}^{\lfloor r/q \rfloor - 1} (Z_k - \mathbb{E}Z_k) > r(s/r)^{1/6} \right] &\leq \exp \left[ -2 \frac{r}{q} \left( \frac{q^{1/3}}{r^{1/3}} \right)^2 \right] \\
&\leq \exp \left[ -2 \left( \frac{r}{s} \right)^{1/6} \right].
\end{align*}

The argument for (26) is similar, but the naive upper and lower bounds are farther apart. The capacity is bounded above by the sum of the capacities of the pieces, plus a little extra since the $B_{i,k}$ are spaced 3s steps apart in time. Summing the capacity of the pieces overcounts by the probability of
hitting at least two pieces. For a lower bound, we subtract the probability of having double hits.

\[
\sum_{k=0}^{\lfloor r/q \rfloor - 1} \text{Cap}_r[\text{LE}_s(B_{i,k})] - \sum_{k,j,k \neq j} \text{Close}(B_{i,k}, B_{i,j}) \leq \text{Cap}_r[\text{LE}_s(A_i)]
\]

\[
\leq \sum_{k=0}^{\lfloor r/q \rfloor} \text{Cap}_r[\text{LE}_s(B_{i,k})] + 3 \left( \frac{r}{q} + 1 \right) \frac{sr}{|G_n|}.
\]

Applying Lemma 5.1 with \( Z_k = \text{Cap}_r[\text{LE}_s(B_{i,k})], b = qr/|G_n|, \) and \( t = (qr^{5/4}/|G_n|^{9/8})/2 \) gives

\[
P \left[ \sum_k Z_k - \mathbb{E}Z_k > \frac{1}{2} \left( \frac{r}{|G_n|^{1/2}} \right)^{9/4} \right] \leq \exp \left[ - \frac{1}{2} \frac{r^{3/2}}{s |G_n|^{1/4}} \right],
\]

which is at most \((1/2)|G_n|^{-2} \leq r^{7/4} |G_n|^{-7/8}\) by the assumptions on \( r \). By Lemma 5.2

\[
\sum_{k,j,k \neq j} \mathbb{E} \text{Close}(B_{i,k}, B_{i,j}) \leq 4 \left( \frac{r}{q} + 1 \right) \frac{q^2 r^2}{|G_n|^2}.
\]

By Markov’s inequality,

\[
P \left[ \sum_{k,j,k \neq j} \text{Close}(B_{i,k}, B_{i,j}) \geq \frac{r^{9/4}}{2 |G_n|^{9/8}} \right] \leq \frac{17}{2} \left( \frac{r^{7/4} |G_n|^{-7/8}}{2|G_n|^{9/8}} \right).
\]

This bound proves (26).

Finally, the lower bounds of (27) are those of (16) and (22). \( \square \)

6. Completion of the coupling argument

The aim of this section is to prove Lemma 2.2. First, we will show that \( \text{LE}(X_t)_{t=0}^{T} \) can be decomposed into runs of length \( r \) without losing too much information, then we will prove two lemmas that allow us to couple random walk on \( G^*_{n,L} \) with \( K^{*}_{m,L} \) for suitable \( m \) and \( L \), and then we will end by proving Lemma 2.2.

Given \( T, r \) and \( s \), a time index \( i \leq \ell = \lfloor T/r \rfloor \) is called \textit{good} by time \( T \) if

\[
\{X_t\}_{t \in A_i} \cap \{X_t\}_{t \leq ir} \cup \{X_t\}_{t \in ([i+1)r,T]} = \emptyset,
\]

where \( A_i \) is as in (14). Intuitively, this means that there are no loops longer than length \( s \) with one endpoint inside \( A_i \) and one outside, although there are some slight differences at times near the endpoints of \( A_i \). Similarly, a time index \( i \leq \ell \) is called a \textit{single intersection} at time \( T \) if there exists \( j \leq \ell \) such that \( \{X_t\}_{t \in A_i} \cap \{X_t\}_{t \in A_j} \neq \emptyset \), but \( \{X_t\}_{t \in A_i} \cap \{X_t\}_{t \in I} = \emptyset \), where \( I = \{0,T \} \setminus \{(i-1)r,ir\} \cup \{(j-1)r,jr\} \} \cap \mathbb{Z} \).

Finally, a time index is called \textit{bad} if it is neither a single intersection nor good. Let \( B_T \) denote the collection of time indices \( i \leq \ell \) that are bad at
time \( T \), let \( C_T \) be those that are single intersections, and \( G_T \) be those that are good.

**Definition 6.** Given \( T, r \) and \( s \), let \( \ell = \lceil T/r \rceil \). A run \( \langle X_t \rangle_{t=0}^T \) is called locally decomposable if:

1. \( B_T = \emptyset \)
2. for all \( i \leq \ell \), \(|\text{LE}_a(A_i)| - \gamma r| \leq 2r \left( \frac{4}{r} \right)^{1/6} \)
3. for all \( i \leq \ell \), \( |\text{Cap}_a(\text{LE}_a(A_i))| - \alpha r^2|G_n|^{-1}| \leq 4r^{9/4}|G_n|^{-9/8} \)
4. There is no pair \( (t, u) \in \mathbb{Z}^2 \) with \( t, u \in [0, T] \) and \(|t - u| \in [\tau, r]\) such that \( X_t = X_u \).
5. There is no pair \( (t, u) \in \mathbb{Z}^2 \) with \( t, u \in [0, T] \) and \(|t - u| \geq s\) such that \( t \notin \bigcup_i A_i \) and \( X_t = X_u \).
6. For all \( t \in [0, T - s] \), the interval \([t, t + s]\) contains a local cutpoint.

**Lemma 6.1.** Let \( \{X_t\} \) be simple random walk on \( G_n \) and \( T \) a geometric random variable with mean \( L|G_n|^{1/2} \) that is independent of \( \{X_t\} \). Suppose that \( r \) and \( s \) satisfy the hypotheses of Lemma 5.3 and also that \( s > \tau \log |G_n|^{2} \). Then the probability that \( \langle X_t \rangle_{t=0}^T \) is locally decomposable is \( 1 - O(L^2r^{3/4}|G_n|^{-3/8}) \). Moreover,

\[
\mathbb{E}[B_T \cup C_T] = O(L^2).
\]

When \( s, q, \) and \( r \) are as in Table 1, condition (3) implies that Lemma 6.1 applies and also that \( r^{3/4}|G_n|^{-3/8} = o(|G_n|^{-38/16}) \).

**Proof.** Counting the expected number of long loops involving \( \text{LE}_a(A_i) \) gives

\[
P[i \in B_T \cup C_T \mid T] \leq \frac{2Tr}{|G_n|}.
\]

As \( \ell \leq (T/r) + 1 \), this implies that

\[
\mathbb{E}[[B_T \cup C_T] \mid T] \leq \frac{2(T^2 + rT)}{|G_n|}.
\]

Using the restrictions on \( r \) and taking the expected value of \( [29] \) yields \( [28] \).

Likewise, the probability that an index \( i \) is bad is bounded by the expected number of pairs \( j \) and \( k \) such that \( \{X_t\}_{t \in A_i} \) intersects both \( \{X_t\}_{(j-1)r \leq t \leq jr} \) and \( \{X_t\}_{(k-1) \leq t \leq k} \), plus the expected number of times such that \( \{X_t\}_{t \in A_i} \) intersects the gaps between the \( A_j \), yielding

\[
P[i \in B_T \mid T] \leq \left( \frac{T/r}{2} \right) \left( \frac{2r^2}{|G_n|} \right)^2 + \left( \frac{T}{r} \right) \frac{6sr}{|G_n|}
\leq \frac{2(T + r)^2r^2}{|G_n|^2} + \frac{6(T + r)s}{|G_n|},
\]

and as \( \mathbb{E}T = L|G_n|^{1/2} \) and \( s|G_n|^{-1/2} = o(r^2|G_n|^{-1}) \), we see that \( \mathbb{E}[B_T] = O \left( L^3r|G_n|^{-1/2} \right) \). In particular, \( \mathbb{P}[B_T = \emptyset] = 1 - O(L^3r|G_n|^{-1/2}) \).
Consider the sequence of events \( \mathcal{A}_i \) given by:
\[
\mathcal{A}_i = \left\{ \begin{array}{l}
|\text{LE}_s(A_j)| - \gamma r \leq 2r \left( \frac{2}{\ell} \right)^{1/6}, \\
\text{Cap}_r(\text{LE}_s(A_j)) - \alpha r^2 |G_n|^{-1} \leq r^{9/4}|G_n|^{-9/8},
\end{array} \right. \quad j \leq i
\]
Using Lemma 5.3 and the facts that \( \ell \) and \( \text{Cap}_r(\text{LE}_s(A_j)) \) can be bounded.

Using Lemma 6.2, suppose Condition 5 holds with probability at least 1.

Let \( \Gamma \) be the event that for all \( t, \ell \) contains a local cutpoint. By Corollary 4.1, there exists a coupling of \( \xi \) respectively. There exist constants \( C \) so that \( \text{Cap}_r(\text{LE}_s(A_j)) \) can be bounded.

To bound the probability of Condition 4 of Definition 6 holding, conditioned on \( T \), the expected number of loops of length in the interval is
\[
\sum_{i=0}^{T-r} \sum_{j=i+r} \mathbb{P}[X_i = X_j] \leq 2Tr |G_n|^{-1/2}.
\]
As \( \mathbb{E}T = L|G_n|^{-1/2} \), Condition 4 holds with probability \( 1 - O(Lr/|G_n|^{1/2}) \).

For Condition 5, conditioning on \( T \) and counting the expected number of pairs \( t \) and \( u \) that we wish to avoid bounds the probability of failing by
\[
3s \left( \frac{T}{r} + 1 \right) \frac{2T}{|G_n|},
\]
so Condition 5 holds with probability at least \( 1 - O(sL^2/r) \). Combining all six of these bounds proves the claim.

For a finite tree \( T \), call a vertex \( v \in T \) a leaf if the degree of \( v \) is 1.

**Lemma 6.2.** Suppose \( T \) and \( \hat{T} \) are trees with \( k \) leaves \( \{x_1, \ldots, x_k\} \) and \( \{y_1, \ldots, y_k\} \) respectively. Let \( \xi \) and \( \eta \) be uniformly chosen from \( T \) and \( \hat{T} \) respectively. There exist constants \( C_1, C_2 \) depending on \( k \) such that for some \( \varepsilon > 0 \) and \( R \geq 1 \), \( |d(x_i, x_j) - Rd(y_i, y_j)| \leq \varepsilon d(x_1, x_2) \) for all \( i, j \leq k \), then there exists a coupling of \( \xi \) and \( \eta \) such that
\[
\mathbb{P} \left[ \forall i \leq k, |d(\xi, x_i) - Rd(\eta, y_i)| \leq C_1[\varepsilon d(x_1, x_2) + R] \right] \geq 1 - C_2\varepsilon.
\]

**Proof.** In the case when \( k = 2 \), take \( \mu \) to be a uniform random variable on \([0, 1]\), take \( \xi \) so that \( d(\xi, x_1) = [\mu d(x_1, x_2) + 1] \), and take \( \eta \) so that \( d(\eta, y_1) = [\mu d(y_1, y_2) + 1] \). We then have \( |Rd(\eta, y_1) - d(\xi, x_1)| \leq \varepsilon d(x_1, x_2) + R + 1 \). As \( R \geq 1 \), this shows that we can take \( C_1 = 2 \) and \( C_2 = 0 \).
Proceeding by induction, assume that we know the result for $k = n$. Given a tree with $n + 1$ ends, the vertex $x_{n+1}$ is connected to the subtree spanned by $\{x_1, \ldots, x_n\}$ by a segment of length

$$A = \min_{i,j \leq n} |d(x_i, x_{n+1}) + d(x_j, x_{n+1}) - d(x_i, x_j)|.$$ 

Likewise, let $\hat{A}$ denote the length of the segment connecting $y_{n+1}$ to the tree spanned by $\{y_1, \ldots, y_n\}$. By repeated application of the triangle inequality, our assumption on the distances shows that $|A - R\hat{A}| \leq 6\varepsilon d(x_1, x_2)$ and also that

$$\left| \mathbb{P}[d(\xi, x_{n+1}) < A] - \mathbb{P}[d(\eta, y_{n+1}) < \hat{A}] \right| \leq C\varepsilon$$

for some constant $C = C(n)$. If both $\xi$ and $\eta$ are chosen on the segment connecting $x_{n+1}$ and $y_{n+1}$ to the rest of the tree, then we can couple them so that $|d(\xi, x_{n+1}) - Rd(\eta, y_{n+1})| \leq 2R + 6\varepsilon d(x_1, x_2)$, and so by the triangle inequality, for all $i \leq n$, $|d(\xi, x_i) - Rd(\eta, y_i)| \leq 2R + 8\varepsilon d(x_1, x_2)$. If both are in the rest of the tree, then the induction hypothesis shows that with probability $1 - C_2(n)\varepsilon$, we can couple them so that $|d(\xi, x_i) - Rd(\eta, y_i)| \leq C_1(n)\varepsilon d(x_1, x_2) + R$ for all $i \leq n$, and thus by the triangle inequality, $|d(\xi, x_{n+1}) - Rd(\eta, y_{n+1})| \leq (C_1(n) + 6)\varepsilon d(x_1, x_2) + C_1(n)R$. We have thus proved the lemma with $C_1(n+1) = C_1(n) + 6$, $C_1(1) = 2$, and $C_2(n+1) = C_2(n) + C(n)$, $C_2(1) = 0$. \hfill $\square$

We now begin constructing the coupling. Let $G_{n,L}^*$ and $K_{m,L}^*$ be as in Lemma 2.2 with $m$ and $\tilde{L}$ given by

$$(31) \quad m = \left[ \frac{|G_n|}{\alpha r^2} \right] \quad \text{and}$$

$$(32) \quad \tilde{L} = \frac{1}{\sqrt{m}} \left[ 1 - \left( 1 - \frac{1}{L|G_n|^{1/2}} \right)^2 \right]^{-1}.$$ 

For fixed $L$, note that $\tilde{L}\alpha^{-1/2} \to L$ as $n \to \infty$ since $r = o(|G_n|^{1/2})$.

**Lemma 6.3.** Suppose that $\{G_n\}$ is a sequence of graphs satisfying the assumptions of Theorem 1.2, and $r = r(n)$ and $s = s(n)$ are sequences of constants satisfying the assumptions of Lemma 6.7. For fixed $L$, let $\{X_i\}$ be weighted random walk on $G_{n,L}^*$ with $X_0$ chosen uniformly from $G_n$. Taking $m$ and $\tilde{L}$ as in (31) and (32), let $\{Y_i\}$ be weighted random walk on $K_{m,L}^*$ with $Y_0 \in K_m$, and let $\tilde{S}_i$ and $S_i$ be as in (13) and (10). Suppose that $\mathcal{T}_k \subset G_{n,L}^*$ and $\tilde{\mathcal{T}}_k \subset K_{m,L}^*$ are such that $\rho \in \mathcal{T}_k$ and $\tilde{\rho} \in \tilde{\mathcal{T}}_k$. Then there exists a coupling of $\{X_i\}$ and $\{Y_i\}$ such that

$$\mathbb{P} \left[ \tilde{T} = \left[ \frac{T}{r} \right], S_j = \tilde{S}_j \forall j \leq \tilde{T} \right] = 1 - O \left( \frac{L^2r^{1/4}}{|G_n|^{1/2}} \right) - O \left( \frac{Ls|\tilde{T}_k|}{r|G_n|^{1/2}} \right)$$
where $T$ is the hitting time of $T_k$ and $\tilde{T}$ is the hitting time of $\tilde{T}_k$.

Note that for $s, q$ and $r$ as in Table 1 when (3) and (9) hold, the hypotheses are met and also that the bound obtained is $1 - O(|G_n|^{-\delta/32})$.

Proof. To establish the coupling, we will use the fact that while the distribution of $S_j$ depends on more than just $S_j$, the fact that $Y_u$ is uniformly chosen from $K_m$ means that the distribution of $\tilde{S}_j$ depends only on the value of $\tilde{S}_j$. To this end, we will run the process $\{X_t\}$ on $G^*_n$, use that to evaluate the indicator random variables $\{I_{ij}\}_{i \in S_j}$ (and hence compute $S_j$), couple the indicators $\{I_{ij}\}_{i \in S_j - 1}$ with $\{\tilde{I}_{ij}\}_{i \in \tilde{S}_j - 1}$, and use the fact that we can construct $Y_j$ given the values of the indicators $\{\tilde{I}_{ij}\}_{i \in \tilde{S}_j - 1}$.

We need to show two things: first, that we can couple $[T/r]$ and $\tilde{T}$, and second, that we can couple the walks until these hitting times.

To couple the hitting times, let $T_\rho$ denote the first time that $X_t = \rho$, and note that $T_\rho \geq T$. Until time $T_\rho$, the path of $\{X_t\}$ is a simple random walk on $G_n$, that is independent of $T_\rho$, so a first moment estimate gives

$$\Pr[T \notin \cup A_i | T_\rho] \leq \frac{|T_k|}{|G_n|} \left( \frac{T_\rho}{r} \right).$$

As $ET_\rho = L |G_n|^{1/2}$, this yields $\Pr[T \notin \cup A_i] = O \left( Ls |T_k|r^{-1} |G_n|^{-1/2} \right)$.

Likewise, let $\tilde{T}_\rho$ denote the hitting time of $\tilde{\rho}$. By the definition of $L$,

$$\Pr[\tilde{T} = j | \tilde{T} > j - 1] = \frac{|\tilde{T}_k \cap K_m|}{m} \left( 1 - \frac{1}{L\sqrt{m}} \right) + \frac{1}{L\sqrt{m}} = \frac{|\tilde{T}_k \cap K_m|}{m} \left( 1 - \frac{1}{L |G_n|^{1/2}} \right)^r + 1 - \left( 1 - \frac{1}{L |G_n|^{1/2}} \right)^r.$$

To estimate what happens on $G^*_n$, we will use the fact that $s$ is much larger than $\tau$. In particular, as the $L^\infty$ distance between the distribution of $\{X_t\}$ and the stationary distribution $\pi$, given by $\sup_x |p_t(x) - \pi|$, is submultiplicative (see e.g., [18] Proposition 2.2), we have $\Pr[x = y] = [1 + O(2^{-s/r})]/|G_n|$, which is $[1 + o(|G_n|^{-1})]/|G_n|$ by the lower bound on $s$. Using this to correct the fact that our walk is not quite uniform in distribution gives

$$\text{Cap}_{r-3s}(T_k \cap G_n) [1 + O(2^{-s/r})] + 1 - \left( 1 - \frac{1}{L |G_n|^{1/2}} \right)^r - 3s$$

$$\geq \Pr[\{X_t\} \cap \{T_k \cap G_n \neq \emptyset \} | T > (j - 1)r - s]$$

$$\geq \left( 1 - \frac{1}{L |G_n|^{1/2}} \right)^{3s} \left[ \frac{\text{Cap}_{r-3s}(T_k \cap G_n)}{1 + O(2^{-s/r})} + 1 - \left( 1 - \frac{1}{L |G_n|^{1/2}} \right)^r - 3s \right].$$
Combining the estimates on $G_{n,L}^*$ and $K_{m,L}^*$, we obtain

$$\left| \mathbb{P}(\{X_t\}_{t \in A_j} \cap T_k \neq \emptyset \mid T > (j-1)r - s) - \mathbb{P}(\tilde{T} = j \mid \tilde{T} > j-1) \right| \leq O(2^{-s/\tau}) + O\left(\frac{s}{L|G_n|^{1/2}}\right) + \left| \text{Cap}_{r-3s}(T_k \cap G_n) - \frac{[T_k \cap K_m]}{m} \right|.$$

Note that $\text{Cap}_{r}[T_k] - \text{Cap}_{r-3s}[T_k] = O(s|T_k||G_n|^{-1})$ by considering the expected number of intersections in the interval $[r-3s, r]$. Using the lower bound on $s$ to bound $2^{-s/\tau}$ and the fact that $T$ is bounded by a geometric random variable with mean $L|G_n|^{1/2}$, we have thus shown that the hitting times can be coupled with the claimed probability.

We now show that, with high probability, that the two walks are coupled up until time $T$. Consider the event that the first step at which the coupling breaks is step $j$, which in turn is bounded by

$$\mathbb{P}[S_{j-1} = \tilde{S}_{j-1}, S_j \neq \tilde{S}_j].$$

To bound (33), we will bound the total variation distance between the joint distributions of $\{I_{ij}\}_{i \in S_{j-1}}$ and $\{\tilde{I}_{ij}\}_{i \in \tilde{S}_{j-1}}$ conditioned on the event $\tilde{S}_{j-1} = S_{j-1}$ as well as the past of the walk $\{X_t\}_{t < r(j-1)}$. To do so, consider three types of cases for the values of $\{I_{ij}\}_{i \in S_{j-1}}$.

Let Case 1 be the case when no long loop is formed at step $j$, meaning that we need to bound

$$\left| \mathbb{P}[I_{ij} = 0 \forall i \in S_{j-1}] - \mathbb{P}[\tilde{I}_{ij} = 0 \forall i \in \tilde{S}_{j-1}] \right|.$$

Conditioned on $S_{j-1} = \tilde{S}_{j-1}$,

$$\mathbb{P}[I_{ij} = 0 \forall i \in \tilde{S}_{j-1}] = 1 - \frac{|S_{j-1}|}{m}.$$

Again using a factor of $1 + O(2^{-s/\tau}) = 1 + o(|G_n|^{-1})$ to correct for the fact that our walk is not quite uniform in distribution,

$$1 - \left( \sum_{i \in S_{j-1}} \text{Cap}_{r-3s}[\text{LE}_s(A_i)] \right) \leq \mathbb{P}[I_{ij} = 0 \forall i \in S_{j-1}] [1 + o(|G_n|^{-1})] \leq 1 - \left( \sum_{i \in S_{j-1}} \text{Cap}_{r-3s}[\text{LE}_s(A_i)] \right) + \sum_{i, k \in S_{j-1}, i \neq k} \text{Close}(\text{LE}_s(A_i), \text{LE}_s(A_j)) .$$

Combining these and using the triangle inequality, we see that (34) is, up to a factor of $1 + o(|G_n|^{-1})$, bounded by

$$\left| \sum_{i, k \in S_{j-1}, i \neq k} \text{Close}(\text{LE}_s(A_i), \text{LE}_s(A_j)) + \sum_{i \in S_{j-1}} \text{Cap}_{r-3s}[\text{LE}_s(A_i)] - \frac{1}{m} \right|.$$
Let Case 2 be the case when there is a long loop formed involving exactly one \( A_i \), meaning that we need to bound
\[
\mathbb{P} [ \tilde{I}_{ij} = 1, \tilde{I}_{kj} = 0 \mid k \in \mathcal{S}_{j-1} \setminus \{i\} ] - \mathbb{P} [ I_{ij} = 1, I_{kj} = 0 \mid k \in \mathcal{S}_{j-1} \setminus \{i\} ].
\]
But
\[
\text{Cap}_{r-3s} [ \text{LE}_s(A_i)] \geq \mathbb{P} [ I_{ij} = 1, I_{kj} = 0 \mid k \in \mathcal{S}_{j-1} \setminus \{i\} ] (1 + o(|G_n|^{-1}))
\]
\[
\geq \text{Cap}_{r-3s} [ \text{LE}_s(A_i)] - \sum_{k \in \mathcal{S}_{j-1} \setminus \{i\} } \text{Close}(\text{LE}_s(A_i), \text{LE}_s(A_k))
\]
and \( \mathbb{P} [ \tilde{I}_{ij} = 1, \tilde{I}_{kj} = 0 \mid k \in \mathcal{S}_{j-1} \setminus \{i\} ] = \frac{1}{m} \), so (36) is, up to a factor of \( 1 + o(|G_n|^{-1}) \), bounded by
\[
\mathbb{P} [ \tilde{I}_{ij} = 1, \tilde{I}_{kj} = 0 \mid k \in \mathcal{S}_{j-1} \setminus \{i\} ] = \frac{1}{m},
\]
so (36) is, up to a factor of \( 1 + o(1) \),
\[
3 \sum_{i,k \in \mathcal{S}_{j-1}, i \neq k} \text{Close}(\text{LE}_s(A_i), \text{LE}_s(A_k)) + 2 \sum_{i \in \mathcal{S}_{j-1}} \left| \text{Cap}_{r-3s} [ \text{LE}_s(A_i)] - \frac{1}{m} \right|.
\]
Taking expectation to remove the conditioning, \( \text{Cap}_{r-3s} [ \text{LE}_s(A_i)] \leq r^2 / |G_n| \) by equation (8). Using Lemma 5.2, \( \mathbb{E} \text{Close}(\text{LE}_s(A_i), \text{LE}_s(A_k)) \leq 4r^4 / |G_n|^2 \).

Since \( T_{\rho} \) is independent of the path of \( \{X_t\} \) for \( t \leq T_{\rho} \),
\[
\mathbb{E} \sum_{j=1}^{[T_{\rho}/r]} \sum_{i,k \in \mathcal{S}_{j-1}, i \neq k} \text{Close}(\text{LE}_s(A_i), \text{LE}_s(A_k)) \leq \frac{4r^4}{|G_n|^2} \frac{\mathbb{E}(T_{\rho} + r)^3}{r^3}
\]
\[
= O \left( \frac{rL^3}{|G_n|^{1/2}} \right).
\]
As \( \alpha(r - 3s)^2 |G_n|^{-1} - m^{-1} | = O(r^4 |G_n|^{-2}) \), Lemma 5.3 and the bound from (8) show that
\[
\mathbb{E} \left| \text{Cap}_{r-3s} [ \text{LE}_s(A_i)] - \frac{1}{m} \right| \leq \left( \frac{r}{|G_n|^{1/2}} \right)^{9/4} + 9 \frac{r^2}{|G_n|} \frac{r^{7/4}}{|G_n|^{7/8}}
\]
\[
= O \left( \frac{r}{|G_n|^{1/2}} \right)^{9/4}.
\]
This leads to the bound
\[
\mathbb{E} \sum_{j=1}^{[T_\ell/r]} \sum_{i \in S_{j-1}} \left| \text{Cap}_{r-3s}[LE_s(A_i)] - \frac{1}{m} \right| \leq \frac{\mathbb{E}T_\ell^2}{r^2} O \left( \frac{r}{|G_n|^{1/2}} \right)^{9/4} = O \left( \frac{L^2r^{1/4}}{|G_n|^{1/8}} \right),
\]
which completes the proof. \(\square\)

We now use Lemmas 6.1 – 6.3 to prove the induction step of Lemma 2.2.

**Proof of Lemma 2.2.** We proceed by induction. Let \{X_i\} be a weighted random walk on \(G_{n,L}\) with \(X_0 = x_{k+1}\), and let \(T\) denote the hitting time of \(T_k\). Take \(q, r\) and \(s\) as in Table 1, and assume that Lemma 2.2 holds for \(k\). As before, take \(\ell = [T/r]\). If \(i \in G_T \cap S_t\), and the path \(X_{t}^T\) is locally decomposable, then \(LE_s(A_i) \subset LE(X_{t}^T)\) and, in particular,
\[
\sum_{i \in S_T \cap G_T} |LE_s(A_i)| \leq |LE(X_{t}^T)|.
\]

Moreover, when \(X_{t}^T\) is locally decomposable, the only difference between \(\bigcup_{i \in S_t} LE_s(A_i)\) and \(LE(X_{t}^T)\) are from the gaps of length \(3s\) between the \(A_i\), the runs that were erased during the single intersections \(C_T\), and the final run \(LE_s(X_{t}^T)\). Thus, when \(X_{t}^T\) is locally decomposable and \(S_\ell = S_{\ell-1} \cap \{\ell\}\),
\[
|LE(X_{t}^T)| \leq \sum_{i \in S_\ell} |LE_s(A_i)| + \left( \frac{3sT}{r} \right) + r(|C_T| + 1).
\]

Combining (39) and (40),
\[
|LE(X_{t}^T)| - \gamma r|S_\ell| \leq \left( \frac{3sT}{r} \right) + r(|C_T| + 1) + \sum_{i \in S_\ell} |LE_s(A_i)| - \gamma r.
\]

By the definitions of Table 1, equation (28), and condition 3,
\[
\mathbb{E} \left[ \frac{3sT}{r} \right] = O \left( L_T^{1/2}|G_n|^{1/4} \right) = o \left( |G_n|^{1/2-\delta/2} \right) \quad \text{and} \quad \mathbb{E}[r|C_T|] = O(L^2r) = o \left( |G_n|^{1/2-\delta/4} \right).
\]

When \(X_{t}^T\) is locally decomposable,
\[
\sum_{i \in S_\ell} \left| LE_s(A_i) \right| - \gamma r \leq 2 \left[ \frac{T + r}{r} \right] r \left( \frac{s}{r} \right)^{1/6} = o \left( T|G_n|^{-\delta/12} \right).
\]

As \(T\) is geometric with mean \(L|G_n|^{1/2}\), for large enough \(n\), this is with probability \(1 - o(|G_n|^{-1})\) less than \(|G_n|^{1/2-\delta/24}\). Note that
\[
P(S_\ell = S_{\ell-1} \cap \{\ell\} | T) \geq T r |G_n|^{-1},
\]
and applying this along with Lemma 6.3 and Markov’s inequality gives

$$\mathbb{P} \left[ \frac{\text{LE}(X_t)_{t=0}^T}{\beta |G_n|^{1/2}} - \gamma r |S_t| \right] > |G_n|^{-\delta/24} = O( |G_n|^{-3\delta/16} ).$$

To compare with the complete graph,

$$\left| \frac{\text{LE}(X_t)_{t=0}^T}{\beta |G_n|^{1/2}} - |S_t| \right| \leq \left| \frac{\text{LE}(X_t)_{t=0}^T}{\beta |G_n|^{1/2}} - \gamma r |S_t| \right| + \left| |S_t| \right| \frac{1}{\sqrt{m}} - \frac{\alpha r}{|G_n|^{1/2}}$$

But \(|m^{-1/2} - \alpha r |G_n|^{-1/2}| = O(r |G_n|^{-1})\), and by Lemma 6.3 with probability \(1 - O(|G_n|^{-\delta/32})\), \(|S_t| = |\tilde{S}_t|\) is the length of the segment added to \(\tilde{T}_k\) to form \(\tilde{T}_{k+1}\). Let \(\xi\) be chosen uniformly from \(\tilde{T}_k\) and according to the coupling of (11). Then

$$d(x_i, x_{k+1}) = d(x_i, \xi) + |\text{LE}(X_t)_{t=0}^T| + d(x_i, X_T) - d(x_i, \xi).$$

Using the decomposition

$$|d(x_i, x_{k+1}) - d(y_i, y_{k+1})| \leq \left| \frac{\text{LE}(X_t)_{t=0}^T}{\beta |G_n|^{1/2}} - \gamma r |S_t| \right| + \frac{d(x_i, \xi)}{\beta |G_n|^{1/2}} + \frac{d(x_i, X_T) - d(x_i, \xi)}{\beta |G_n|^{1/2}},$$

we have already controlled the first term, equation (11) implies that

$$\mathbb{P}[d(x_i, X_T) - d(x_i, \xi)] > |G_n|^{1/2-\delta/32} \leq O( |G_n|^{-\delta/32} ),$$

and applying Lemma 6.2 to couple \(\xi\) and \(Y_T\), we see that there is a coupling such that (11) holds with probability \(1 - O(|G_n|^{-\delta/32})\).

To prove (9), we first bound \(\text{Close}(T_k, \text{LE}(X_t)_{t=0}^T)\). By Lemma 5.2

$$\mathbb{E} \left[ \text{Close}(T_k, (X_t)_{t=0}^T) \mid T_k \right] \leq 4 \frac{L r \text{Cap}(T_k)}{|G_n|^{1/2}}.$$

Markov’s inequality gives

$$\mathbb{P} \left[ \text{Close}(T_k, (X_t)_{t=0}^T) > 4 \frac{\text{Cap}(T_k) r^{1/2}}{|G_n|^{1/4}} \mid T_k \right] \leq \frac{L r^{1/2}}{|G_n|^{1/4}}.$$

The fact that \(\text{LE}(X_t)_{t=0}^T \subset \{ X_t \}_{t=0}^\infty \) and monotonicity of closeness, along with the fact that \(r^{1/2} |G_n|^{-1/4} = o(|G_n|^{-\delta/8})\), implies that

$$\mathbb{P}[\text{Close}(T_k, \text{LE}(X_t)_{t=0}^T) \leq |G_n|^{-\delta/8} \text{Cap}(T_k) \mid T_k] = 1 - o(|G_n|^{-\delta/8}).$$

Now estimating \(\text{Cap}(\text{LE}(X_t)_{t=0}^T)\), \(\langle X_t \rangle_{t=0}^\infty\) is locally decomposable,

$$\sum_{i \in S_r} \text{Cap}(\text{LE}(A_i)) - \sum_{i, j \in S_r} \text{Close}(\text{LE}(A_i), \text{LE}(A_j)) \leq \text{Cap}(\text{LE}(X_t)_{t=0}^T)$$

$$\leq \sum_{i \in S_r} [\text{Cap}(\text{LE}(A_i))] + \left(3sT \frac{\text{Cap}}{|G_n|} \right) + \frac{r^2(|C_T| + 1)}{|G_n|}.$$
In particular, we have

\[
\left| \text{Cap}_r [\mathbf{LE}(X_t)_{t=0}^T] - \frac{|S_t|}{m} \right| \leq \sum_{i \in S_t} \left| \text{Cap}_r [\mathbf{LE}_s(A_i)] - \frac{1}{m} \right|
\]

\[+ \sum_{i,j \in S_t} \text{Close}(\mathbf{LE}_s(A_i), \mathbf{LE}_s(A_j)) + \left( 3sT \frac{|G_n|}{|G_n|^1/2} \right) + \frac{r^2(|G_n| + 1)}{|G_n|}.
\]

The expected value of the sum of the last three of these terms is bounded by

\[
\frac{4r^2 \mathbb{E}(\tilde{T})^2}{|G_n|^1/2} + \frac{3sL}{|G_n|^1/2} + \frac{2r^2[\mathbb{E}(\tilde{T})^2 + 1]}{|G_n|^2} = O \left( \frac{r^2}{|G_n|} \right).
\]

By Markov’s inequality, the sum of these three terms is therefore less than \( |G_n|^{-\delta/16} \) with probability \( 1 - o(|G_n|^{-\delta/32}) \). On the event \( \mathcal{A}_t \),

\[
\sum_{i \in S_t} \left| \text{Cap}_r [\mathbf{LE}_s(A_i)] - \frac{1}{m} \right| \leq \frac{\tilde{T} + r}{r} \cdot \left[ \frac{r^{3/4}}{|G_n|^{3/8}} + O \left( \frac{r^4}{|G_n|^2} \right) \right],
\]

which is less than \( |G_n|^{-\delta/20}(r/|G_n|^{1/2}) \) with probability \( 1 - o(|G_n|^{-1}) \) because \( \tilde{T} \) is geometric with mean \( L|G_n|^{1/2} \). By Lemma \ref{lem:decomposable} the values of Table \ref{table:decomposable} and Condition \ref{cond:decomposable}, \( \mathbf{LE}(X_t)_{t=0}^T \) is locally decomposable with probability \( 1 - o(|G_n|^{-35/16}) \). By Lemma \ref{lem:decomposable} and our inductive assumption that \ref{cond:decomposable} holds with probability \( 1 - O(|G_n|^{-\delta/32}) \) on \( \mathcal{T}_k \), with probability \( 1 - O(|G_n|^{-\delta/32}) \),

\[
\left| \text{Cap}_r(\mathcal{T}_{k+1}) - \text{Cap}_r(\mathcal{T}_k) \right| \frac{|G_n|^{1/2}}{\alpha^{1/2}r} \leq \left( \text{Cap}_r [\mathbf{LE}(X_t)_{t=0}^T] + \text{Close}(\mathcal{T}_k, \mathbf{LE}(X_t)_{t=0}^T) \right) \frac{|G_n|^{1/2}}{\alpha^{1/2}r}
\]

\[= o(|G_n|^{-\delta/24}) + \frac{|S_t||G_n|^{1/2}}{m\alpha^{1/2}r}
\]

\[= o(|G_n|^{-\delta/24}) + \frac{\tilde{T}_{k+1} - |\tilde{T}_k|}{\sqrt{m}}
\]

and hence by induction \ref{cond:decomposable} holds with probability \( 1 - O(|G_n|^{-\delta/32}) \).

Finally, we need to show that the uniform measure \( \mu \) and the hitting measure \( \nu \) on \( \mathcal{T}_{k+1} \) can be coupled in such a way that \ref{cond:coupling} holds. Let \( \xi \) be chosen according to \( \mu \) and \( \eta \) according to \( \nu \). Note that

\[
(43) \quad d_{\mathcal{T}_{k+1}}(\xi, \eta) \leq \max_{i,j \leq k+1} d_{\mathcal{T}_{k+1}}(x_i, x_j).
\]

We will use \ref{cond:coupling} as an upper bound whenever our procedure for constructing the coupling fails. To make this coupling, consider the event \( F \) that either \( \xi \) and \( \eta \) are both in \( \mathcal{T}_k \) or both are in \( \mathbf{LE}_s(A_i) \) for some \( i \in S_t \). If both are in \( \mathcal{T}_k \), we wish to use the induction hypothesis to establish the coupling. For a random walk \( \langle Y_u \rangle \) started in the uniform distribution on \( G_n \), let \( T_{\text{old}} \) denote
the hitting time of $T_k$ and $T_{\text{new}}$ the hitting time of $\text{LE}(X_t)_{t=0}^T$. We know that with probability $1 - O(|G_n|^{-\delta/32})$, that $X_{T_{\text{old}}}$ can be coupled with a uniformly chosen point. But

$$P[T_{\text{new}} < T_{\text{old}} < T_{\text{new}} + r] \leq \frac{\text{Cap}_r(T_{k+1})}{\text{Cap}_r(T_{k+1})} = O(r|G_n|^{-1/2})$$

and so

$$\sum_{x \in T_k} [P[X_{T_{\text{old}}} = x] - P[X_{T_{\text{old}}} = x \mid T_{\text{new}} < T_{\text{old}}]] \leq O(2^{-r/r}) + O(r|G_n|^{-1/2}).$$

This means that conditioning on hitting $T_k$ first has a small enough effect that we can still couple $\xi$ and $\eta$ with the claimed probability. If both $\xi$ and $\eta$ are in $\text{LE}(A_i)$ for $i \in G_T$, then $d_{T_{k+1}}(\xi, \eta) \leq r = o(|G_n|^{1/2})$. We thus need to show that $P[F^c] = O(|G_n|^{-\delta/32}).$

To do so, by Lemma 6.1, $\langle X_t \rangle_{t=0}^T$ is locally decomposable with probability $1 - o(|G_n|^{-36/16}).$ On the event $A_t,$

$$\text{Cap}_r[\text{LE}_s(A_i)] = \frac{\alpha r^2}{|G_n|} [1 + O(|G_n|^{-\delta/16})]$$

and so

$$|\text{LE}_s(A_i)| = \gamma r [1 + O(|G_n|^{-\delta/12})].$$

and so

$$\text{Cap}_r[\text{LE}_s(A_i)] = \frac{\gamma |G_n|}{r} \left[ 1 + O \left( |G_n|^{-\delta/16} \right) \right].$$

Likewise, we have already seen that, with probability $1 - O(|G_n|^{-\delta/32}),$ equations (9) and (10) hold on $T_{k+1},$ so

$$\text{Cap}_r(T_{k+1}) = |T_{k+1}|m [1 + O(|G_n|^{-\delta/32})]$$

and

$$|T_{k+1}| = |T_{k+1}|(|G_n|/m)^{1/2} [1 + O(|G_n|^{-\delta/32})],$$

which gives

$$\text{Cap}_r(T_{k+1})/|T_{k+1}| = \frac{\gamma |G_n|}{r} \left[ 1 + O \left( |G_n|^{-\delta/32} \right) \right].$$

By considering the total variation distance between $\mu$ and $\nu$, there is a coupling such that

$$P[F^c \mid T_k, \langle X_t \rangle_{t=0}^T] \leq \sum_{i \in S \cap \partial T} \left| \mu[\text{LE}_s(A_i)] - \nu[\text{LE}_s(A_i)] \right| + |\mu(T_k) - \nu(T_k)| + O \left( \frac{r |C_T|}{|T_{k+1}|} \right) + O \left( \frac{s T}{r |T_{k+1}|} \right).$$

To use $P[F^c] = \mathbb{E}P[F^c \mid T_k, \langle X_t \rangle_{t=0}^T],$ note that the expected value of these last two terms is $O(|G_n|^{-\delta/4}).$ For the other two terms, for any $S,$

$$|\mu(S) - \nu(S)| \leq |\mu(S) - \frac{\text{Cap}_r(S)}{\text{Cap}_r(T_{k+1})}| + |\nu(S) - \frac{\text{Cap}_r(S)}{\text{Cap}_r(T_{k+1})}|.$$
On the event \( \mathcal{A}_k \), \( \mathbf{12} \) and \( \mathbf{13} \) imply that for \( S = \mathbf{LE}_s(A_i) \) or \( \mathcal{T}_k \\
\mu(S) - \frac{\operatorname{Cap}_r(S)}{\operatorname{Cap}_r(\mathcal{T}_{k+1})} \leq \mu(S) o \left( |G_n|^{-\delta/32} \right) . \\
Moreover, \\
\left| \nu(S) - \frac{\operatorname{Cap}_r(S)}{\operatorname{Cap}_r(\mathcal{T}_{k+1})} \right| \leq \frac{\operatorname{Close}(S, \mathcal{T}_{k+1} \setminus S)}{\operatorname{Cap}_r(\mathcal{T}_{k+1})} . \\
By Lemma 5.2 \( \mathbf{5.2} \)
(46) \[
\frac{\mathbb{E}[\operatorname{Close}(\mathbf{LE}_s(A_i), \mathcal{T}_{k+1} \setminus \mathbf{LE}_s(A_i)) | \mathcal{T}_{k+1} \setminus \mathbf{LE}_s(A_i)]}{\operatorname{Cap}_r(\mathcal{T}_{k+1})} \leq 4 \frac{r^2}{|G_n|} .
\]
Combining \( \mathbf{46} \) with \( \mathbf{42} \), we thus obtain \\
\[ \mathbb{P}[F] \leq \mu(\mathcal{T}_{k+1}) o \left( |G_n|^{-\delta/32} \right) + O \left( \frac{Lr}{|G_n|^{1/2}} \right) + O \left( |G_n|^{-\delta/4} \right) \]
which is \( o \left( |G_n|^{-\delta/32} \right) \) as required. \( \square \)

7. Stochastic domination of spanning forests by trees

We now prove Lemma 2.1. The heart of the proof relies on the stochastic domination of \( \tilde{\mathcal{T}} \) by \( \mathcal{T}^* \). Suppose that \( G \) is a graph with vertex set \( V \) and edges \( E \). Let \( G_\lambda \) denote the graph that is the extension of \( G \) formed by adding an additional vertex \( \rho \), and from every vertex \( v \in V \), an edge \((v, \rho)\) of weight \( 1 - \lambda \). Let \( \mathcal{T}_\lambda \) be a weighted spanning tree on \( G_\lambda \) as generated by Wilson’s algorithm with root vertex \( \rho \). The graph \( \mathcal{T}_\lambda \) induces a forest \( \tilde{\mathcal{T}} \subset G \) simply by restricting to edges in \( E \cap \mathcal{T}_\lambda \). We will show that \( \tilde{\mathcal{T}} \) is stochastically dominated by the uniform spanning tree \( \mathcal{T} \).

An event \( A \) is said to be an increasing event on a graph \( G \) if for any subgraph \( a \subseteq A \), if \( a' \) is another subgraph of \( G \) formed by adding edges to \( a \), then \( a' \subseteq A \) as well. We say that an event \( A \) is supported on a set of edges \( E_1 \) if determining whether or not \( a \) is in \( A \) only requires looking at the edges \( E_1 \). (Equivalently, if \( a \subseteq A \), then \( a' \subseteq A \), where \( a' \) is the subgraph whose edges are in both \( a \) and \( E_1 \).)

**Lemma 7.1** (Feder and Mihail). *For increasing events \( A \) and \( B \) supported on disjoint edge sets of \( G \), \( \mathbb{P}[\mathcal{T} \in A \mid \mathcal{T} \in B] \leq \mathbb{P}[\mathcal{T} \in A] \).*

Let \( E_1 \) and \( E_2 \) be disjoint edge sets such that \( A \) is supported on \( E_1 \), \( B \) is supported on \( E_2 \), and \( E_1 \cup E_2 = E \). The case when \(|E_2| = 1\) was originally proved by Feder and Mihail (\( \mathbf{7} \), Lemma 3.2), and they remark that iterating their proof implies that the general case is also true. A proof of the general case appears in the solution to Exercise 8.10 in \( \mathbf{13} \).

Fix integers \( M_{i,j} \geq 1 \) for \( i, j \leq k \). Let \( B \) denote the event that the degree of \( \rho \) is at least 2, and let \( A \) denote the event that there is a path of length exactly \( M_{i,j} \) in \( G \) from \( x_i \) to \( x_j \) for all pairs \( i, j \leq k \). Clearly the event \( A \) is increasing and requiring that the path be in \( G \) means that the event is supported on \( E \), the original edge set of \( G \). On the other hand, the event \( B \)
is supported on edges from $G$ to $\rho$. As adding edges from $\rho$ to $G$ increases the degree of $\rho$, the event $B$ is increasing. There are no loops in a forest, so the event $A$ implies that $d_\hat{T}(x_i, x_j) = M_{i,j}$ for all $i,j$.

By Lemma 7.1 the events $A$ and $B$ are negatively correlated. Moreover, $B^c$ is equivalent to $\hat{T}$ being a spanning tree. In particular, conditioned on $B^c$, $\hat{T}$ is equal to $T$ in distribution. This shows that the probability of having the right lengths in $\hat{T}$ is a lower bound for the probability of having the right lengths in $T$, i.e., for any collection of values $M_{i,j} < \infty$,

$$P[d_\hat{T}(x_i, x_j) = M_{i,j} \forall i, j \leq k] \leq P[d_T(x_i, x_j) = M_{i,j} \forall i, j \leq k].$$

Using the convention that $d_\hat{T}(x_i, x_j) = \infty$ whenever $x_i$ and $x_j$ are in different components of $\hat{T}$, the total variation distance between the joint distribution of $d_\hat{T}(x_i, x_j)$ and the joint distribution of $d_T(x_i, x_j)$ is bounded by

$$P[\exists (i,j) : d_\hat{T}(x_i, x_j) = \infty].$$

The key step is now controlling the probability that $\{x_1, \ldots, x_k\}$ are in the same component of $\hat{T}$.

**Lemma 7.2.** Suppose that $\{G_n\}$ is a sequence of graphs satisfying the assumptions of Theorem 1.2. For any $\varepsilon \in (0,1)$, let $\{X_t\}$ and $\{Y_t\}$ be two independent random walks on $G_n$, with (possibly different) starting points $x$ and $y$, and let $T_X$ and $T_Y$ be independent geometric random variables with mean $\varepsilon^{-2}|G_n|^{1/2}$. Then

$$P[|E(X_t)_{t=0}^{T_X} \cap \{Y_u\}_{u=0}^{T_Y} = \emptyset| \geq 1 - 2\varepsilon - a(n, \varepsilon),$$

where $a(n, \varepsilon) \to b(\varepsilon)$ as $n \to \infty$ and $b(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$.

**Proof.** Let $s$ and $r$ be as in Table 1. For any set $S \subseteq G_n$, let $T_S$ denote the time $\{Y_u\}_{u=0}^{\infty}$ first hits $S$. If $\text{Cap}_r(S) \geq r\varepsilon/|G_n|^{1/2}$, then

$$P[T_Y < T_S] \leq \sum_{i=0}^{\infty} P[ir < T_Y \leq (i+1)r \mid T_S, T_Y > ir]P[T_S, T_Y > ir]$$

$$\leq \sum_{i=0}^{\infty} \frac{\varepsilon^2 r^i}{|G_n|^{1/2}} P[T_S > ir]P[T_Y > ir]$$

$$\leq \frac{\varepsilon^2 r}{|G_n|^{1/2}} \sum_{i=0}^{\infty} \left(1 - \frac{r\varepsilon}{2|G_n|^{1/2}}\right)^i$$

$$= \frac{\varepsilon^2 r}{|G_n|^{1/2}} \frac{2|G_n|^{1/2}}{r\varepsilon}$$

$$= 2\varepsilon.$$

The proof thus reduces to finding $a(n, \varepsilon)$ such that

$$P \left[ \text{Cap}_r(\text{LE}(X_t)_{t=0}^{\infty}) > \frac{r\varepsilon}{|G_n|^{1/2}} \right] \geq 1 - a(n, \varepsilon).$$
Let $T = (2\varepsilon/\alpha)|G_n|^{1/2}$, where $\alpha$ is as in $\text{(17)}$. Since $T_X$ is geometric with mean $\varepsilon^{-2}|G_n|^{1/2}$, we have $\mathbb{P}[T_X < T] \leq 2\varepsilon^3/\alpha$. Restricting to the event \{$T_X \geq T$\}, we will now view the path $\langle X_t \rangle_{t=0}^{T_X}$ as being in two parts: an initial run of length $T_X - T$ and a final tail of length $T$. The idea of the proof is that if the initial segment has a large capacity, then because the tail is short, it probably misses enough of the initial segment that the capacity remains high. Conversely, if the initial segment has low capacity, then the tail will probably survive and is long enough to be of high capacity itself.

More formally, consider two possibilities based on whether or not the event
\begin{equation}
\{\text{Cap}_r[\text{LE}(X_t)_{t=0}^{T_X-T}] > 2\varepsilon r/|G_n|^{1/2}\}
\end{equation}
occurs. Let $\langle \eta(k) \rangle$ be an increasing sequence such that $\text{LE}(X_t)_{t=0}^{T_X-T} = \langle X_{\eta(k)} \rangle$. When \text{(19)} holds, let $M$ be the smallest number such that
\[
\text{Cap}_r[\langle X_{\eta(k)} \rangle_{k \leq M}] > \frac{\varepsilon r}{|G_n|^{1/2}}.
\]
Denote this initial segment by $U := \langle X_{\eta(k)} \rangle_{k \leq M}$. Because capacity is subadditive, $M \geq \varepsilon|G_n|^{1/2}$ and
\[
\frac{\varepsilon r}{|G_n|^{1/2}} < \text{Cap}_r U < \frac{\varepsilon r}{|G_n|^{1/2}} + \frac{r}{|G_n|} < \frac{2\varepsilon r}{|G_n|^{1/2}}.
\]
Subadditivity of capacity also implies that $T_X - T - M > \varepsilon |G_n|^{1/2} - 1$, which for large enough $n$ is greater than $s$. But
\[
\mathbb{P}[(X_t)_{t=T_X-T}^{T_X-T+s} \cap U \neq \emptyset] \leq \mathbb{E}[\{X_t\}_{t=T_X-T}^{T_X-T+s} \cap \{X_t\}_{t=0}^{T_X-T-\varepsilon |G_n|^{1/2}}]
\leq \frac{2s\mathbb{E}T_X}{|G_n|} = \frac{2s}{\varepsilon^2 |G_n|^{1/2}}.
\]
Subdividing the final $T - s$ steps of the walk into pieces of length $r$, using the fact that $X_{T_X-T+s}$ is close to uniform, and considering the expected number of those pieces that intersect $U$ gives
\[
\mathbb{P}[(X_t)_{t=T_X-T}^{T_X-T+s} \cap U \neq \emptyset] \leq \frac{2\varepsilon r}{|G_n|^{1/2}} \frac{2T}{r} = \frac{8\varepsilon^2}{\alpha}.
\]
If $\{X_t\}_{t=T_X-T} \cap U = \emptyset$, then $U$ survives loop-erasure, so this shows that, conditioned on \text{(19)}, with probability $1 - a(n, \varepsilon)$ for $a(n, \varepsilon)$ of the suitable form, $U$ survives loop-erasure and yields the desired capacity for $\text{LE}(X_t)$.

When \text{(19)} fails, we have $\text{Cap}_r[\text{LE}(X_t)_{t<T_X-T}] \leq 2\varepsilon r/|G_n|^{1/2}$. The probability of a segment of length $T$ started at uniform intersecting this initial piece is then $O(\varepsilon^2)$ by the same argument as before. Adding a buffer of $s$ steps to get close to a uniform position, we thus have
\[
\mathbb{P}[(X_t)_{t>T_X-T+s} \cap \text{LE}(X_t)_{t=0}^{T_X-T} = \emptyset] = 1 - O(\varepsilon^2) - (1/2)^{s/r}.
\]
Moreover, $T$ is small enough such that the expected number of loops longer than $\tau$ within the final $T - \tau$ steps is bounded by $4\varepsilon^2/\alpha^2$. Thus with probability $1 - f(n, \varepsilon)$, where $f(n, \varepsilon) = O(\varepsilon^2) + (1/2)^{s/\tau}$, $\text{LE}(X_t)_{t=0}^{T_X}$ contains $W = \text{LE}(X_t)_{t=T_X - T + s}$, and so

$$
P \left[ \text{Cap}_r(\text{LE}(X_t)_{t=0}^{T_X}) > \frac{r\varepsilon}{|G_n|^{1/2}} \right] \geq P \left[ \text{Cap}_r(W) > \frac{r\varepsilon}{|G_n|^{1/2}} \right] - f(n, \varepsilon)$$

$$= P \left[ \text{Cap}_r(W) > \frac{\alpha T r}{2 |G_n|} \right] - f(n, \varepsilon).$$

By again breaking the final $T - s$ steps up into runs of length $r$, using the concentration of capacity about its mean on such a run, and bounding the closeness between these runs as in the proof of Lemma 2.2, this final probability is of the form $a(n, \varepsilon)$ as required, with $b(\varepsilon) = O(\varepsilon^2)$. \hfill \Box

**Proof of Lemma 2.1.** As mentioned above, we need to bound

$$(50) \quad P \left[ \exists (i, j) : d_\hat{\tau}(x_i, x_j) = \infty \right],$$

and in particular show that it is less than $\varepsilon$ for sufficiently large $n$ and $L$. But using Lemma 7.2 gives

$$P \left[ \exists (i, j) : d_\hat{\tau}(x_i, x_j) = \infty \right] \leq \frac{k^2}{2} E[d_\hat{\tau}(x_1, x_2) = \infty]$$

$$\leq \frac{k^2}{2} L^{1/2} + a(n, L^{-1/2}),$$

where $a(n, x)$ is as in Lemma 7.2. Taking $L = k^4\varepsilon^{-2}$ bounds (50) by $\varepsilon + a(n, \varepsilon k^{-2})$, which for small enough $\varepsilon$ and large enough $n$ yields the required bound. \hfill \Box

8. **Constants on the torus**

This section is devoted to proving Theorem 1.1. To see that $\mathbb{Z}_n^d$ satisfies the hypotheses of Theorem 1.2, note that for simple random walk on $\mathbb{Z}_n^d$ with holding probability $1/2$, $|P_0[X_t = o] - n^{-d}| \leq C \tau^{-d/2}$ for a suitable constant $C$ (see e.g., [4], Chapter 5). This means that the only way in which Theorem 1.1 is not a special case of Theorem 1.2 is that there is a single rescaling constant $\beta$ rather than a sequence of constants $\beta_n$ that (possibly) depend on $n$. Thus, what we need to show in this section is that $\lim \beta_n$ exists. We will do so by giving an expression for the limit.

**Lemma 8.1.** Let $\{\tilde{Y}_t\}$, $\{\tilde{Z}_u\}$, and $\{\tilde{W}_v\}$ be independent simple random walks on $\mathbb{Z}_n^d$, all starting at the origin. Let $G_n = \mathbb{Z}_n^d$, and take $\alpha = \alpha(n)$ and $\gamma = \gamma(n)$ to be as in (17). Then for $d \geq 5$,

$$(51) \quad \lim_{n \to \infty} \gamma(n) = \mathbb{P} \left[ \text{LE}(\tilde{Y}_t)_{t=0}^{\infty} \cap \{\tilde{Z}_u\}_{u=1}^{\infty} = 0 \right]$$

$$(52) \quad \lim_{n \to \infty} \alpha(n) = L^{1/2} + a(n, L^{-1/2}).$$
\[ \mathbb{P}\left[ \text{LE}(\hat{Y}_t)_{\infty} \cap \{ \hat{Z}_u \}_{\infty} = \emptyset, \left( \text{LE}(\hat{Y}_t)_{\infty} \cup \text{LE}(\hat{Z}_u)_{\infty} \right) \cap \{ \hat{W}_v \}_{\infty} = \emptyset \right] = 0. \]

**Proof.** Take \( s, q, \) and \( r \) as in Table 1. To understand scales, recall that on \( \mathbb{Z}^d_n, \tau \) is on the order of \( n^2 \), meaning that \( s, q \) and \( r \) are on the order of \( n^{(d+12)/8}, n^{(d+4)/4}, \) and \( n^{(3d+4)/8} \) respectively.

Let \( U \subset [2s + 1, r - s] \) be the set of times \( t \) in the interval \([2s + 1, r - s]\) that are locally retained (see Definition 3). As the probability of each time being locally retained is constant on \([2s + 1, r - s]\),

\[
\gamma r = \mathbb{E}\left[ \text{LE}(X_t)_{t=2s+1}^{r-s} \right]
\]

\[
= \sum_{t=2s+1}^{r-s} \mathbb{P}[t \in U]
\]

\[
= (r - 3s)\mathbb{P}[q \in U].
\]

As \( s = o(r) \), proving \( \gamma \) reduces to computing \( \lim_{n \to \infty} \mathbb{P}[q \in U] \). On the torus in dimension \( d \geq 5 \), the expected number of loops of length greater than \( n^{7/4} \) in a run of length \( r \) is bounded by

\[
\sum_{i=0}^{r} \sum_{j=i+n^{7/4}}^{r} \mathbb{P}[X_i = X_j] \leq \sum_{k=n^{7/4}}^{r} r\mathbb{P}_o[X_k = o]
\]

\[
= O\left( r^2 n^{-d} + r \left( n^{7/4} \right)^{1-d/2} \right)
\]

\[
= O\left( n^{(4-d)/4 + (18-4d)/8} \right),
\]

For \( d \geq 5 \), this expression is \( o(1) \). Moreover, Lemma 4.1 and the proof of Corollary 4.1 imply that

\[
\mathbb{P}[\exists T \in [n^{7/4}, n^{9/5}]: \langle X_t \rangle_{t=T-n^{7/4}}^T \cap \langle X_t \rangle_{t=T+1}^T = 0] = 1 - o(1).
\]

Combining these two facts, the probability that \( \langle X_t \rangle_{t=0}^{s} \) has a cutpoint in the time interval \( n^{7/4} \leq t \leq n^{9/5} \) is \( 1 - o(1) \). The importance of having cutpoints is that whether or not the point \( X_j \) survives loop-erasure can be determined from only considering what happens between two cutpoints, one at a time before \( j \), and one at a time after \( j \). As \( 9/5 < 2 \), with probability \( 1 - o(1) \), any run of length \( n^{9/5} \) remains inside a cube of edge length \( n \), and in particular does not see the difference between the torus and the full lattice \( \mathbb{Z}^d \). Combining these facts,

\[
\mathbb{P}[q \in U] = \mathbb{P}[\text{LE}(X_t)_{t=0}^{-s} \cap \{ X_t \}_{t=1}^{s} = 0]
\]

\[
= \mathbb{P}[\text{LE}(X_t)_{t=0}^{-n^{9/5}} \cap \{ X_t \}_{t=1}^{n^{9/5}} = 0] + o(1)
\]

\[
= \mathbb{P}[\text{LE}(\hat{Y}_t)_{t=0}^{n^{9/5}} \cap \{ \hat{Z}_u \}_{u=1}^{n^{9/5}} = 0] + o(1).
\]

Note that if \( \langle \text{LE}(\hat{Y}_t)_{t=0}^{n^{9/5}} \cap \{ \hat{Z}_u \}_{u=1}^{n^{9/5}} \rangle \neq \langle \text{LE}(\hat{Y}_t)_{t=0}^{n^{9/5}} \cap \{ \hat{Z}_u \}_{u=1}^{n^{9/5}} \rangle \), then either \( \langle \hat{Y}_t \rangle_{t>n^{9/5}} \cap \{ \hat{Z}_u \}_{u>0} \neq 0 \) or \( \langle \hat{Y}_t \rangle_{t>0} \cap \{ \hat{Z}_u \}_{u>n^{9/5}} \neq 0 \). But a first
moment argument shows that
\[ \Pr \left[ \left( \{ \tilde{Z}_u \}_{u=\binom{n}{6}} \cap \{ \tilde{Y}_t \}_{t=1}^\infty \right) \right] = \left( \{ \tilde{Z}_u \}_{u=\binom{n}{6}} \cap \{ \tilde{Y}_t \}_{t=1}^\infty \right) = 0 \right] = 1 - o(1), \]
which completes the proof of (51).

To prove (52), let \( S = \mathbf{LE}_s <X_t\rangle_{t=1}^{r-s} \). As \( \alpha = r^{-2} \mathbb{E} \mathcal{C}_r S | G_n \), we need to compute \( \mathbb{E} \mathcal{C}_r S \). For a simple random walk \( \{ Y_k \}_{k=0}^\infty \) on \( \mathbb{Z}^d \), let \( \tau_S = \inf \{ k > 0 : Y_k \in S \} \), and \( T_S = \inf \{ k > 0 : Y_k \in S \} \). Considering the time reversal and again letting \( U \subset [2s+1, r-s] \) denote the locally retained times,

\[
\mathcal{C}_r S = \sum_{j \in U} \sum_{k=0}^r \Pr \left[ \tau_S = k, X_j = Y_k \right]
\]

\[
= \sum_{j \in U} \sum_{k=0}^r \sum_{z \in \mathbb{Z}^d} \Pr [ \tau_S = k, X_j = Y_k ] n^{-d}
\]

\[
= \sum_{k=0}^r \sum_{j \in U} \sum_{z \in \mathbb{Z}^d} \Pr [ T_S > k, Y_k = z ] n^{-d}
\]

\[
= \sum_{k=0}^r \sum_{j \in U} n^{-d} \Pr [ T_S > k ].
\]

Let \( \mathbf{1}_{U}(\cdot) \) be an indicator function for \( U \) and let \( W \) denote the event \( \{ \mathbf{LE}(X_t)_{t=1}^{r-s} \cap \{ X_t \}_{t=1}^\infty = \emptyset \} \). Then

\[
\mathbb{E} \mathcal{C}_r S = \sum_{k=0}^r \sum_{j=2s+1}^{r-s} \mathbb{E} \left[ \mathbf{1}_{U}(j) \Pr [ X_j > k ] \langle X_t \rangle_{t=0}^r \right] n^{-d}
\]

\[
= \sum_{k=0}^r \sum_{j=2s+1}^{r-s} n^{-d} \left[ \Pr [ W, \mathbf{LE}(X_t)_{2s+1}^{r-s-j} \cap \{ Y_u \}_{1}^k = \emptyset \mid Y_0 = X_0 ] \right].
\]

There are fewer than \( 3rs \) terms in which \( k < s \) or \( j \notin [3s, r-2s] \), each of which is bounded by \( n^{-d} \). The sum of these terms thus contributes at most \( 3rsn^{-d} \), which is of a lower order than \( \mathbb{E} \mathcal{C}_r S \) (which is on the order of \( r^2n^{-d} \)). As \( \mathbb{E} \mathcal{C}_r S = \alpha r^{-2}n^{-d} \), it thus suffices to show that for the \( r^2(1 + o(1)) \) terms with \( k \geq s \) and \( j \in [3s, r-2s] \), we uniformly obtain

\[
\lim_{n \to \infty} \mathbb{E} \left[ \Pr [ W, \mathbf{LE}(X_t)_{2s+1}^{r-s-j} \cap \{ Y_u \}_{1}^k = \emptyset \mid Y_0 = X_0 ] \right]
\]

\[
= \mathbb{P} \left[ \mathbf{LE}(Y_1)_{1}^\infty \cap \{ Z_u \}_{1}^\infty = \emptyset, \left( \mathbf{LE}(Y_1)_{1}^\infty \cup \mathbf{LE}(Z_u)_{1}^\infty \right) \cap \{ \tilde{W}_e \}_{1}^\infty = \emptyset \right].
\]

As before, since we are only running the walk for times on the order of \( r \), the probability that there are no loops of length longer than \( n^{7/4} \) is \( 1 - o(1) \). We then convert from a statement on the torus to one on the full lattice exactly as before. The convergence is uniform because the analogous conversions to (53) rely only on the existence of these cutpoints. \( \square \)


9. Expanders, Hypercubes, and Proof of Theorem 1.3

We stated in Section 1 that sequences of expander graphs satisfy the assumptions of Theorem 1.2. This is immediate from the fact that, for a sequence of expanders, there exist $C > 0$ and $\lambda < 1$ such that the bound $|\mathbb{P}_o[X_t = o] - |G_n|^{-1}| \leq C \lambda^t$ holds for the entire sequence.

We likewise claimed that (53) applies on the hypercubes $\mathbb{Z}_n^2$. To see this, consider two random walks $\{X_t\}$ and $\{Y_u\}$ with different starting points, both run in continuous time with rate 1. The expected amount of time of intersection is the expected number of intersections for two discrete time walks. In continuous time,

$$
\mathbb{P}[X_t = Y_u] \leq \left(\frac{1}{2}\right)^n \left(1 + e^{-2(t+u)/n}\right)^{n-1} \left(1 - e^{-2(t+u)/n}\right).
$$

For $t + u \leq n^{1/4}$, bound (54) by $(1 - \exp[-2(t+u)/n])/2 < 2(t+u)/n$. As $e^{-x} \leq 1 - x/2$ for $0 \leq x \leq 1$, we obtain the bound

$$
\frac{1}{2} \left(1 + \exp\left[-\frac{2(t+u)}{n}\right]\right) < 1 - \frac{t+u}{n} < \exp\left[-\frac{t+u}{n}\right]
$$

for $n^{1/4} < t + u \leq n/2$, and then finally

$$
\left(\frac{1}{2}\right)^n \left(1 + e^{-2(t+u)/n}\right)^{n-1} \leq \left(\frac{1}{2}\right)^n (1 + e^{-1})^n
$$

for $(t + u) > n/2$ yields

$$
\int_0^r \int_0^r \mathbb{P}[X_t = Y_u] \, dt \, du = o(1),
$$

which in turn says that the expected time of overlap of the two paths is $o(1)$. As we ran the continuous time walks for time $r \gg q$, the expected number of intersections in the first $q$ steps of the discrete walks is also $o(1)$.

Turning now to the proof of Theorem 1.3, note that Theorem 1.3 differs from Theorem 1.2 in two ways: first, we need to show that assumption (3) allows us to omit the hypothesis that $\{x_1, \ldots, x_k\} = \{x_1^{(n)}, \ldots, x_k^{(n)}\}$ are chosen uniformly, and second, we need to show that $\lim \beta_n = 1$.

To show that we can choose $\{x_1, \ldots, x_k\}$ freely under assumption (3), let $\mathcal{T}_0 = \{\rho\}$, and for $1 \leq i \leq k$, let $\{X_i^t\}_{t=0}^\infty$ be i.i.d., weighted random walks on $G_{n,L}$ with $X_i^0 = x_i$. Let $T_i = \min\{t \geq 0 : X_i^t \in \mathcal{T}_{i-1}\}$, and take $\mathcal{T}_i = \mathcal{T}_{i-1} \cup \text{LE}(X_i^t)_{t=T_i}$.

Let $s$ be as in Table 1. As the $L^\infty$ distance between the distribution of $X_i^t$ and the uniform is $o(|G_n|^{-1})$ for $t \geq s$, we only need to show that $\mathbb{P}[T_k < s] = o(1)$. The main concern is intersections that might occur from $x_k$ being close to $\{x_1, \ldots, x_{k-1}\}$. After running for $s$ steps, the $X_i^t$ are close to uniform. Using the expected number of intersections between $\{X_i^{k}\}_{t<s}$ and $\{X_i^t\}_{t\geq s}$ to bound the probability of such an intersection occurring, we
obtain
\[ P[T_k < s] \leq k \max_{i < k} P \left[ \{X^i_t\}_{t < s} \cap \{X^k_t\}_{t < s} \neq \emptyset \right] + \frac{2sE \sum_{i=1}^{k-1} T_i}{|G_n|}. \]

But assumption (6) is the fact that \( P \left[ \{X^i_t\}_{t < s} \cap \{X^k_t\}_{t < s} \neq \emptyset \right] = o(1) \), so as \( ET_i \leq L|G_n|^{1/2} \), equation (55) is exactly what we need.

For the second part, assumption (6) implies that any point is a local cutpoint with probability \( 1 - o(1) \), so \( \gamma = 1 - o(1) \). Likewise, (6) implies that the probability of a run of length \( r \) intersecting \( LE_s(A_i) \) more than once, even conditioned on there being an intersection, is \( o(1) \). This means that the probability of an intersection is, up to a factor of \( 1 + o(1) \), the same as the expected number of intersections, and so we also have \( \alpha = 1 - o(1) \).

In particular, \( \beta_n = 1 + o(1) \), which completes the proof of Theorem 1.3.

10. FURTHER QUESTIONS

Although the main results of this paper give a good picture of the scaling limit of UST on many graphs, there are still a number of questions that remain.

(1) Is the UST on the complete graph in some sense smaller than on any other vertex transitive graph? More precisely, if \( \{G_n\} \) are vertex transitive, and \( x \) and \( y \) are uniformly chosen from \( G_n \), is there a constant \( C \) such that
\[ P[d_T(x, y) > \lambda |G_n|^{1/2}] \geq \exp \left[ -C \frac{\lambda^2}{2} \right] (1 + o(1))? \]

Benjamini and Kozma [5] asked an averaged form of this question, asking if \( E d_T(x, y) \geq C |G_n|^{1/2} \) holds.

(2) Theorems 1.1-1.3 only prove that the scaling limit of the UST is the Brownian CRT in the sense that the finite dimensional distributions converge. Does this convergence also hold in a stronger topology?

(3) Our theorems do not apply to the torus \( \mathbb{Z}_4^n \) because \( \tau \) and \( |G_n|^{1/2} \) are on the same order of magnitude in dimension 4. After taking into account a logarithmic correction factor, the scaling limit of LERW on \( \mathbb{Z}^4 \), however, is still Brownian motion [11]. As discussed in [5], heuristics suggest that \( E d_T(x, y) \) is on the order of \( n^2 \log^{1/6} n \). If so, what is the limiting distribution of \( d_T(x, y) \)? What is the scaling limit of the UST on \( \mathbb{Z}_4^n \)?

(4) In this paper, we have focused on the intrinsic geometry of the UST, discussing distances in the UST. We can also ask about the existence of a scaling limit in the extrinsic geometry induced by embedding our graph in the torus of side length 1. The path of the LERW is asymptotically dense in this embedding, but lifting to the universal cover \( \mathbb{R}^d \) and dividing lengths by \( n^{d/4} \) (the square root of the typical length of a path), Theorem 1.1 suggests the following scaling limit for the lifted UST: first, the LERW from \( x \) to \( y \) lifts to a Brownian
motion on $\mathbb{R}^d$ run for a random amount of time $T$ that is Rayleigh distributed. Lifting the partial spanning tree defined by $k$-points in this way, we obtain an embedding in $\mathbb{R}^d$ of the first $k$ steps of the Poisson line breaking constructing for the Brownian CRT, where an edge length of length $\ell$ in the CRT corresponds to a Brownian path run for time $\ell$. (This is a version of Le-Gall’s Brownian snake.) However, establishing this picture requires further work.

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