THE ALGEBRAIC FUNDAMENTAL GROUP
OF A REDUCTIVE GROUP SCHEME
OVER AN ARBITRARY BASE SCHEME

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ABSTRACT. We define the algebraic fundamental group $\pi_1(G)$ of a reductive group scheme $G$ over an arbitrary non-empty base scheme and show that the resulting functor $G \mapsto \pi_1(G)$ is exact.

1. Introduction

If $G$ is a (connected) reductive algebraic group over a field $k$ of characteristic 0 and $T$ is a maximal $k$-torus of $G$, the algebraic fundamental group $\pi_1(G, T)$ of the pair $(G, T)$ was defined by the first-named author [1] and shown there to be independent (up to a canonical isomorphism) of the choice of $T$ and useful in the study of the first Galois cohomology set of $G$. See Definition 3.11 below for a generalization of the original definition of $\pi_1(G, T)$. Independently, and at about the same time, Merkurjev [12, §10.1] defined the algebraic fundamental group of $G$ over an arbitrary field. Later, Colliot-Thélène [4, Proposition-Definition 6.1] defined the algebraic fundamental group $\pi_1(G)$ of $G$ in terms of a flasque resolution of $G$, showed that his definition was independent (up to a canonical isomorphism) of the choice of the resolution, and established the existence of a canonical isomorphism $\pi_1(G) \cong \pi_1(G, T)$, see [4, Proposition A.2]. Recall that a flasque resolution of $G$ is a central extension

$$1 \to F \to H \to G \to 1$$

where the derived group $H^{\text{der}}$ of $H$ is simply connected, $H^{\text{tor}} := H/H^{\text{der}}$ is a quasi-trivial $k$-torus, and $F$ is a flasque $k$-torus, i.e., the group of cocharacters of $F$ is an $H^1$-trivial Galois module. It turns out that flasque resolutions of reductive group schemes exist over bases that are more general than spectra of fields, and the second-named author has used such resolutions to generalize Colliot-Thélène’s definition of $\pi_1(G)$ to reductive group schemes $G$ over any non-empty, reduced, connected, locally Noetherian and geometrically unibranch scheme. See [9, Definition 3.7].

In the present paper we extend the definition of [9] to reductive group schemes $G$ over an arbitrary non-empty scheme. Since flasque resolutions are not available in this general setting (see [9, Remark 2.3]), we shall use

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instead \( t \)-resolutions, which exist over any non-empty base scheme \( S \). A \( t \)-resolution of \( G \) is a central extension

\[
1 \to T \to H \to G \to 1,
\]

where \( T \) is an \( S \)-torus and \( H \) is a reductive \( S \)-group scheme such that the derived group \( H^{\text{der}} \) is simply connected. Since a flasque resolution is a particular type of \( t \)-resolution, the definition of \( \pi_1(G) \) given here (Definition 2.11) does indeed extend the definition of the second-named author [9]. Further, since the choice of a maximal \( S \)-torus of \( G \) (when one exists) canonically determines a \( t \)-resolution of \( G \) (see Lemma 3.9), our Definition 2.11 turns out to be a common generalization of the definitions of [1] and of [4] and [9].

Once the general definition of \( \pi_1(G) \) is in place, we proceed to study some of the basic properties of the resulting functor \( G \mapsto \pi_1(G) \), culminating in a proof of its exactness (Theorem 3.8). We give, in fact, two proofs of Theorem 3.8, the second of which makes use of the étale-local existence of maximal tori in reductive \( S \)-group schemes and generalizes [3, proof of Lemma 3.7].

In the final section of the paper we use \( t \)-resolutions to relate the (flat) abelian cohomology of \( G \) over \( S \) introduced in [8] to the cohomology of \( S \)-tori, thereby generalizing [9, §4].

Remark 1.1. Let \( G \) be a (connected) reductive group over the field of complex numbers \( \mathbb{C} \). Here we comment on the interrelation between the algebraic fundamental group \( \pi_1(G) \), the topological fundamental group \( \pi_1^{\text{top}}(G(\mathbb{C})) \), and the étale fundamental group \( \pi_1^{\text{ét}}(G) \). By [1, Prop. 1.11] the algebraic fundamental group \( \pi_1(G) \) is canonically isomorphic to the group

\[
\pi_1^{\text{top}}(G(\mathbb{C}))(-1) := \text{Hom}(\pi_1^{\text{top}}(\mathbb{C}^\times), \pi_1^{\text{top}}(G(\mathbb{C}))).
\]

It follows that if \( G \) is a reductive \( k \)-group \( G \) over an algebraically closed field \( k \) of characteristic zero, then the profinite completion of \( \pi_1(G) \) is canonically isomorphic to the group

\[
\pi_1^{\text{ét}}(G)(-1) := \text{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(G_{m,k}), \pi_1^{\text{ét}}(G)).
\]

where \( \text{Hom}_{\text{cont}} \) denotes the group of continuous homomorphisms and \( G_{m,k} \) denotes the multiplicative group over \( k \). See [2] for details and for a generalization of the algebraic fundamental group \( \pi_1(G) \) to arbitrary homogeneous spaces of connected linear algebraic groups.

**Notation and terminology.** Throughout this paper, \( S \) denotes a non-empty scheme. An \( S \)-torus is an \( S \)-group scheme which is fpqc-locally isomorphic to a group of the form \( G^n_{m,S} \) for some integer \( n \geq 0 \) [6, Exp. IX, Definition 1.3]. An \( S \)-torus is affine, smooth and of finite presentation over \( S \) [6, Exp. IX, Proposition 2.1(a), (b) and (c)]. An \( S \)-group scheme \( G \) is called reductive (respectively, semisimple, simply connected) if it is affine and smooth over \( S \) and its geometric fibers are connected reductive (respectively, semisimple, simply connected) algebraic groups [6, Exp. XIX, Definition 2.7].
An S-torus is reductive, and any reductive S-group scheme is of finite presentation over S [6, Exp. XIX, 2.1]. Now, if G is a reductive group scheme over S, \( \text{rad}(G) \) will denote the radical of G, i.e., the identity component of the center \( Z(G) \) of G. Further, \( G^{\text{der}} \) will denote the derived group of G. Thus \( G^{\text{der}} \) is a normal semisimple subgroup scheme of \( G \) and \( G^{\text{tor}} := G/G^{\text{der}} \) is the largest quotient of \( G \) which is an S-torus. We shall write \( \tilde{G} \) for the simply connected central cover of \( G^{\text{der}} \) and \( \mu := \text{Ker}[\tilde{G} \to G^{\text{der}}] \) for the fundamental group of \( G^{\text{der}} \). See [9, §2] for the existence and basic properties of \( \tilde{G} \). There exists a canonical homomorphism \( \partial: \tilde{G} \to G \) which factors as \( \tilde{G} \twoheadrightarrow G^{\text{der}} \hookrightarrow G \). In particular, \( \text{Ker}\ \partial = \mu \) and \( \text{Coker}\ \partial = G^{\text{tor}} \).

If \( X \) is a (commutative) finitely generated twisted constant S-group scheme [6, Exp. X, Definition 5.1], then \( X \) is quasi-isotrivial, i.e., there exists a surjective étale morphism \( S' \to S \) such that \( X \times_S S' \) is constant. Further, the functors

\[
X \mapsto X^* := \text{Hom}_{S}\text{-gr}(X, \mathbb{G}_{m,S}) \quad \text{and} \quad M \mapsto M^* := \text{Hom}_{S}\text{-gr}(M, \mathbb{G}_{m,S})
\]

are mutually quasi-inverse anti-equivalences between the categories of finitely generated twisted constant S-group schemes and S-group schemes of finite type and of multiplicative type [6, Exp. X, Corollary 5.9]. Further, \( M \to M^* \) and \( X \to X^* \) are exact functors (see [6, Exp. VIII, Theorem 3.1] and use faithfully flat descent). If \( G \) is a reductive S-group scheme, its group of characters \( G^* \) equals \( (G^{\text{tor}})^* \) (see [6, Exp. XXII, proof of Theorem 6.2.1(i)]). Now, if \( T \) is an S-torus, the functor \( \text{Hom}_{S}\text{-gr}(\mathbb{G}_{m,S}, T) \) is represented by a (free and finitely generated) twisted constant S-group scheme which is denoted by \( T_* \) and called the group of cocharacters of \( T \) (see [6, Exp. X, Corollary 4.5 and Theorem 5.6]). There exists a canonical isomorphism of free and finitely generated twisted constant S-group schemes

\[
T^* \simeq (T_*)^\vee := \text{Hom}_{S}\text{-gr}(T_*, \mathbb{Z}_S).
\]

A sequence

\[
0 \to T \to H \to G \to 0
\]

of reductive S-group schemes and S-homomorphisms is called exact if it is exact as a sequence of sheaves for the fppf topology on S. In this case the sequence (2) will be called an extension of \( G \) by \( T \).

If \( G \) is a reductive S-group scheme, the identity homomorphism \( G \to G \) will be denoted \( \text{id}_G \). Further, if \( T \) is an S-torus, the inversion automorphism \( T \to T \) will be denoted \( \text{inv}_T \).

\[\text{1}\text{Although [6, Exp. IX, Definition 1.4] allows for groups of multiplicative type which may not be of finite type over S, such groups will play no role in this paper.}\]
2. Definition of $\pi_1$

**Definition 2.1.** Let $G$ be a reductive $S$-group scheme. A *t-resolution of $G$* is a central extension

$$1 \to T \to H \to G \to 1,$$

where $T$ is an $S$-torus and $H$ is a reductive $S$-group scheme such that $H^{\text{der}}$ is simply connected.

**Proposition 2.2.** Every reductive $S$-group scheme admits a $t$-resolution.

*Proof.* By [6, Exp. XXII, 6.2.3], the product in $G$ defines a faithfully flat homomorphism $\text{rad}(G) \times_S G^{\text{der}} \to G$ which induces a faithfully flat homomorphism $\text{rad}(G) \times_S \tilde{G} \to G$. Let $\mu_1 = \ker[\text{rad}(G) \times_S \tilde{G} \to G]$, which is a finite $S$-group scheme of multiplicative type contained in the center of $\text{rad}(G) \times_S \tilde{G}$ (see [9] proof of Proposition 3.2, p. 9). By [5, Proposition B.3.8], there exist an $S$-torus $T$ and a closed immersion $\psi: \mu_1 \hookrightarrow T$. Let $H$ be the pushout of $\varphi: \mu_1 \hookrightarrow \text{rad}(G) \times_S \tilde{G}$ and $\psi: \mu_1 \hookrightarrow T$, i.e., the cokernel of the central embedding

$$\begin{align*}
(\varphi, \text{inv}_T \circ \psi)_S: \mu_1 &\hookrightarrow (\text{rad}(G) \times_S \tilde{G}) \times_S T.
\end{align*}$$

Then $H$ is a reductive $S$-group scheme, cf. [6, Exp. XXII, Corollary 4.3.2], which fits into an exact sequence

$$1 \to T \to H \to G \to 1,$$

where $T$ is central in $H$. Now, as in [4] proof of Proposition-Definition 3.1 and [9] proof of Proposition 3.2, p. 10], there exists an embedding of $\tilde{G}$ into $H$ which identifies $\tilde{G}$ with $H^{\text{der}}$. Thus $H^{\text{der}}$ is simply connected, which completes the proof. \(\square\)

As in [4] p. 93] and [9] (3.3)], a $t$-resolution

$$1 \to T \to H \to G \to 1$$

induces a “fundamental diagram”

$$\begin{array}{c}
1 \\
\downarrow \\
1 \hookrightarrow \mu \hookrightarrow \tilde{G} \hookrightarrow G^{\text{der}} \hookrightarrow 1 \\
\downarrow \\
1 \hookrightarrow T \hookrightarrow H \hookrightarrow G \hookrightarrow 1 \\
\downarrow \\
1 \hookrightarrow M \hookrightarrow R \hookrightarrow G^{\text{for}} \hookrightarrow 1, \\
\downarrow \\
1 \\
\end{array}$$
where $M = T/\mu$ and $R = H^{\text{tor}}$. This diagram induces, in turn, a canonical isomorphism in the derived category

$$(Z(\tilde{G}) \xrightarrow{\partial_2} Z(G)) \approx (T \to R)$$

(cf. [9, Proposition 3.4]) and a canonical exact sequence

$$(1) 1 \to \mu \to T \to R \to G^{\text{tor}} \to 1,$$

where $\mu$ is the fundamental group of $G^{\text{der}}$. Since $\mu$ is finite, (6) shows that the induced homomorphism $T_\ast \to R_\ast$ is injective. Set

$$\pi_1(\mathcal{R}) = \text{Coker}[T_\ast \to R_\ast].$$

Thus there exists an exact sequence of (étale, finitely generated) twisted constant $S$-group schemes

$$(7) 1 \to T_\ast \to R_\ast \to \pi_1(\mathcal{R}) \to 1.$$

Set

$$\mu(-1) := \text{Hom}_{S\text{-gr}}(\mu^\ast, (\mathbb{Q}/\mathbb{Z})S).$$

**Proposition 2.3.** A $t$-resolution $\mathcal{R}$ of a reductive $S$-group scheme $G$ induces an exact sequence of finitely generated twisted constant $S$-group schemes

$$(8) 1 \to \mu(-1) \to \pi_1(\mathcal{R}) \to (G^{\text{tor}})_\ast \to 1.$$

**Proof.** The proof is similar to that of [4, Proposition 6.4], using (6). \qed

**Definition 2.4.** Let $G$ be a reductive $S$-group scheme and let

$$\begin{align*}
(\mathcal{R}') & \quad 1 \to T' \to H' \to G \to 1 \\
(\mathcal{R}) & \quad 1 \to T \to H \to G \to 1
\end{align*}$$

be two $t$-resolutions of $G$. A **morphism from** $\mathcal{R}'$ **to** $\mathcal{R}$, written $\phi: \mathcal{R}' \to \mathcal{R}$, **is a commutative diagram**

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T' & \longrightarrow & H' & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow{\phi_T} & & \downarrow{\phi_H} & & \downarrow{\text{id}_G} & & & & \\
1 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1,
\end{array}$$

where $\phi_T$ and $\phi_H$ are $S$-homomorphisms. Note that, if $R' = (H')^{\text{tor}}$ and $R = H^{\text{tor}}$, then $\phi_H$ induces an $S$-homomorphism $\phi_R: R' \to R$.

We shall say that a $t$-resolution $\mathcal{R}'$ of $G$ **dominates** another $t$-resolution $\mathcal{R}$ of $G$ if there exists a morphism $\mathcal{R}' \to \mathcal{R}$.

The following lemma is well-known.

**Lemma 2.5.** A **morphism of complexes** $f: P \to Q$ in an abelian category is a quasi-isomorphism if and only its cone $\text{C}(f)$ is acyclic (i.e., has trivial cohomology).
Proof. By [7, Lemma III.3.3] there exists a short exact sequence of complexes
\[ 0 \to P \to \text{Cyl}(f) \to C(f) \to 0, \]
where \( \text{Cyl}(f) \) is the cylinder of \( f \). Further, the complex \( \text{Cyl}(f) \) is canonically isomorphic to \( Q \) in the derived category. Now the short exact sequence (10) induces a cohomology exact sequence
\[ \cdots \to H^i(P) \to H^i(Q) \to H^i(C(f)) \to H^{i+1}(P) \to \cdots \]
from which the lemma is immediate. \( \square \)

Lemma 2.6. Let \( g: C \to D \) be a quasi-isomorphism of bounded complexes of split \( S \)-tori. Then the induced morphism of complexes of cocharacter \( S \)-group schemes \( g_*: C_* \to D_* \) is a quasi-isomorphism.

Proof. Since the assertion is local in the étale topology, we may and do assume that \( S \) is connected. The given quasi-isomorphism induces a quasi-isomorphism \( g^*: D^* \to C^* \) of bounded complexes of free and finitely generated constant \( S \)-group schemes. Thus, by (11), it suffices to check that the functor \( X \mapsto X^\vee \) on the category of bounded complexes of free and finitely generated constant \( S \)-group schemes preserves quasi-isomorphisms. We thank Joseph Bernstein for the following argument. By Lemma 2.5 a morphism \( f: P \to Q \) of bounded complexes in the (abelian) category of finitely generated constant \( S \)-group schemes is a quasi-isomorphism if and only if its cone \( C(f) \) is acyclic. Now, if \( f: P \to Q \) is a quasi-isomorphism and \( P \) and \( Q \) are bounded complexes of free and finitely generated constant \( S \)-group schemes, then \( C(f) \) is an acyclic complex of free and finitely generated constant \( S \)-group schemes. We see immediately that the dual complex
\[ C(f)^\vee = C(f^\vee)[−1] \]
is acyclic, whence \( f^\vee \) is a quasi-isomorphism by Lemma 2.5. \( \square \)

Lemma 2.7. Let \( G \) be a reductive \( S \)-group scheme and let \( \mathcal{R}' \) be a \( t \)-resolution of \( G \) which dominates another \( t \)-resolution \( \mathcal{R} \) of \( G \). Then a morphism of \( t \)-resolutions \( \phi: \mathcal{R}' \to \mathcal{R} \) induces an isomorphism of finitely generated twisted constant \( S \)-group schemes \( \pi_1(\phi): \pi_1(\mathcal{R}') \sim \pi_1(\mathcal{R}) \) which is independent of the choice of \( \phi \).

Proof. Let \( \mathcal{R}': 1 \to T' \to H' \to G \to 1 \) and \( \mathcal{R}: 1 \to T \to H \to G \to 1 \) be the given \( t \)-resolutions of \( G \), as in Definition 2.4 and set \( R = H^\text{tor} \) and \( R' = (H')^\text{tor} \). Since the assertion is local in the étale topology, we may and do assume that the tori \( T, T', R \) and \( R' \) are split and that \( S \) is connected. From (9) we see that the morphism of complexes of split tori (in degrees 0 and 1)
\[ (\phi_T, \phi_R): (T' \to R') \to (T \to R) \]
is a quasi-isomorphism. Now by Lemma 2.6
\[ \pi_1(\phi) := H^1((\phi_T, \phi_R)_*): \pi_1(\mathcal{R}') \sim \pi_1(\mathcal{R}) \]
Proof. We follow an idea of Kottwitz [11, Proof of Lemma 2.4.4]. Let \( \phi \) be a morphism of \( t \)-resolutions. It is clear from diagram (11) that \( \psi_H \) differs from \( \phi_H \) by some homomorphism \( H' \to T \) which factors through \( R' = (H')^{\text{tor}} \). It follows that the induced homomorphisms \( (\psi_H)_*, (\phi_H)_*: R'_* \to R_* \) differ by a homomorphism which factors through \( T_* \). Consequently, the induced homomorphisms
\[
\pi_1(\phi), \pi_1(\psi): \text{Coker} [T'_* \to R'_*] \to \text{Coker} [T_* \to R_*]
\]
coincide.

\[\square\]

**Proposition 2.8.** Let \( \varkappa: G_1 \to G_2 \) be a homomorphism of reductive \( S \)-group schemes and let
\[
(\mathcal{R}_1) \quad 1 \to T_1 \to H_1 \to G_1 \to 1
\]
\[
(\mathcal{R}_2) \quad 1 \to T_2 \to H_2 \to G_2 \to 1
\]
be \( t \)-resolutions of \( G_1 \) and \( G_2 \), respectively. Then there exists an exact commutative diagram
\[
\begin{array}{ccc}
1 & \longrightarrow & T_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & T'_1 & \longrightarrow & H'_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & T_2 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & 1
\end{array}
\]
where the middle row is a \( t \)-resolution of \( G_1 \).

**Proof.** We follow an idea of Kottwitz [11, Proof of Lemma 2.4.4]. Let \( H'_1 = H_1 \times_{G_2} H_2 \), where the morphism \( H_1 \to G_2 \) is the composition \( H_1 \to G_1 \xrightarrow{\varkappa} G_2 \). Clearly, there are canonical morphisms \( H'_1 \to H_1 \) and \( H'_1 \to H_2 \). Now, since \( H_2 \to G_2 \) is faithfully flat, so also is \( H'_1 \to H_1 \). Consequently the composition \( H'_1 \to H_1 \to G_1 \) is faithfully flat as well. Let \( T'_1 \) denote its kernel, i.e., \( T'_1 = S \times_{G_1} H'_1 \). Then
\[
T'_1 = (S \times_{G_1} H_1) \times_{G_2} H_2 = T_1 \times_S (S \times_{G_2} H_2) = T_1 \times_S T_2,
\]
which is an \( S \)-torus. The existence of diagram (11) is now clear. Further, since \( T_i \) is central in \( H_i \) (i = 1, 2), \( T'_1 = T_1 \times_S T_2 \) is central in \( H'_1 = H_1 \times_{G_2} H_2 \). The \( S \)-group scheme \( H'_1 \) is affine and smooth over \( S \) and has connected reductive fibers, i.e., is a reductive \( S \)-group scheme. Further, the faithfully flat morphism \( H'_1 \to G_1 \) induces a surjection \( (H'_1)^\text{der} \to G_1^{\text{der}} \) with (central) kernel \( T'_1 \cap (H'_1)^\text{der} \). Since \( (H'_1)^\text{der} \) is semisimple, the last map is in fact a central isogeny. Consequently, \( (H'_1)^\text{der} \to H_1^{\text{der}} = \tilde{G}_1 \) is a central isogeny as well, whence \( (H'_1)^\text{der} = \tilde{G}_1 \) is simply connected. Thus the middle row of (11) is indeed a \( t \)-resolution of \( G_1 \). \[\square\]
Corollary 2.9. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two $t$-resolutions of a reductive $S$-group scheme $G$. Then there exists a $t$-resolution $\mathcal{R}_3$ of $G$ which dominates both $\mathcal{R}_1$ and $\mathcal{R}_2$.

Proof. This is immediate from Proposition 2.8 (with $G_1 = G_2 = G$ and $\kappa = \text{id}_G$ there). □

Lemma 2.10. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two $t$-resolutions of a reductive $S$-group scheme $G$. Then there exists a canonical isomorphism of finitely generated twisted constant $S$-group schemes $\pi_1(\mathcal{R}_1) \cong \pi_1(\mathcal{R}_2)$.

Proof. By Corollary 2.9, there exists a $t$-resolution $\mathcal{R}_3$ of $G$ and morphisms of resolutions $\mathcal{R}_3 \to \mathcal{R}_1$ and $\mathcal{R}_3 \to \mathcal{R}_2$. Thus, Lemma 2.7 gives a composite isomorphism $\psi_{\mathcal{R}_3}: \pi_1(\mathcal{R}_1) \sim \pi_1(\mathcal{R}_3) \sim \pi_1(\mathcal{R}_2)$. Let $\mathcal{R}_4$ be another $t$-resolution of $G$ which dominates both $\mathcal{R}_1$ and $\mathcal{R}_2$ and let $\psi_{\mathcal{R}_4}: \pi_1(\mathcal{R}_1) \sim \pi_1(\mathcal{R}_4) \sim \pi_1(\mathcal{R}_2)$ be the corresponding composite isomorphism. There exists a $t$-resolution $\mathcal{R}_5$ which dominates both $\mathcal{R}_3$ and $\mathcal{R}_4$. Then $\mathcal{R}_5$ dominates $\mathcal{R}_1$ and $\mathcal{R}_2$ and we obtain a composite isomorphism $\psi_{\mathcal{R}_5}: \pi_1(\mathcal{R}_1) \sim \pi_1(\mathcal{R}_5) \sim \pi_1(\mathcal{R}_2)$. We have a diagram of $t$-resolutions

\[
\begin{array}{ccc}
\mathcal{R}_5 & \to & \mathcal{R}_1 \\
\downarrow & & \downarrow \\
\mathcal{R}_3 & \to & \mathcal{R}_2 \\
\end{array}
\]

which may not commute. However, by Lemma 2.7, this diagram induces a commutative diagram of twisted constant $S$-group schemes and their isomorphisms

\[
\begin{array}{ccc}
\pi_1(\mathcal{R}_5) & \to & \pi_1(\mathcal{R}_1) \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{R}_3) & \to & \pi_1(\mathcal{R}_2) \\
\end{array}
\]

We conclude that $\psi_{\mathcal{R}_3} = \psi_{\mathcal{R}_5} = \psi_{\mathcal{R}_4}: \pi_1(\mathcal{R}_1) \sim \pi_1(\mathcal{R}_2)$, from which we deduce the existence of a canonical isomorphism $\psi: \pi_1(\mathcal{R}_1) \sim \pi_1(\mathcal{R}_2)$. □

Definition 2.11. Let $G$ be a reductive $S$-group scheme. Using the preceding lemma, we shall henceforth identify the $S$-group schemes $\pi_1(\mathcal{R})$ as $\mathcal{R}$ ranges
over the family of all \( t \)-resolutions of \( G \). Their common value will be denoted by \( \pi_1(G) \) and called the \textit{algebraic fundamental group} of \( G \). Thus
\[
\pi_1(G) = \pi_1(\mathcal{R})
\]
for any \( t \)-resolution \( \mathcal{R} \) of \( G \).

Note that, by (8), a \( t \)-resolution
\[
1 \rightarrow T_s \rightarrow (H_{tor})_s \rightarrow \pi_1(G) \rightarrow 1.
\]
Further, by Proposition 2.3, there exists a canonical exact sequence
\[
1 \rightarrow \mu(-1) \rightarrow \pi_1(G) \rightarrow (G_{tor})_s \rightarrow 1.
\]

Remark 2.12. One can also define \( \pi_1(G) \) using \( m \)-resolutions. By an \( m \)-resolution of \( G \) we mean a short exact sequence
\[
1 \rightarrow M \rightarrow H \rightarrow G \rightarrow 1,
\]
where \( H \) is a reductive \( S \)-group scheme such that \( H_{\text{der}} \) is simply connected, and \( M \) is an \( S \)-group scheme of multiplicative type. Clearly, a \( t \)-resolution of \( G \) is in particular an \( m \)-resolution of \( G \). It is very easy to see that any reductive \( S \)-group scheme \( G \) admits an \( m \)-resolution: we can take \( H := \text{rad}(G) \times_S \hat{G}, \) with the homomorphism \( H \rightarrow G \) from the beginning of the proof of Proposition 2.2, and set \( M := \mu_1 = \text{Ker}[H \rightarrow G], \) which is a finite \( S \)-group scheme of multiplicative type.

Now let \( \mathcal{R} \) be an \( m \)-resolution of \( G \) and consider the induced homomorphism \( M \rightarrow H_{tor} \). We claim that there exists a complex of \( S \)-tori \( T \rightarrow R \) which is isomorphic to \( M \rightarrow H_{tor} \) in the derived category. Indeed, by [5, Proposition B.3.8] there exists an embedding \( M \hookrightarrow T \) of \( M \) into an \( S \)-torus \( T \). Denote by \( R \) the pushout of the homomorphisms \( M \rightarrow H_{tor} \) and \( M \rightarrow T \). Then the complex of \( S \)-tori \( T \rightarrow R \) is quasi-isomorphic to the complex \( M \rightarrow H_{tor} \), as claimed.

Now we choose an \( m \)-resolution \( \mathcal{R} \) of \( G \), a complex of \( S \)-tori \( T \rightarrow R \) which is isomorphic to \( M \rightarrow H_{tor} \) in the derived category, and set \( \pi_1(G) = \pi_1(\mathcal{R}) := \text{Coker}[T_s \rightarrow R_s] \).

3. \textbf{Functoriality and exactness of} \( \pi_1 \)

In this section we show that \( \pi_1 \) is an exact covariant functor from the category of reductive \( S \)-group schemes to the category of finitely generated twisted constant \( S \)-group schemes.

\textbf{Definition 3.1.} Let \( \kappa: G_1 \rightarrow G_2 \) be a homomorphism of reductive \( S \)-group schemes. A \textit{t-resolution of} \( \kappa \), written \( \kappa_{\mathcal{R}}: \mathcal{R}_1 \rightarrow \mathcal{R}_2 \), is an exact
commutative diagram

\[
\begin{array}{ccccccccc}
(\mathcal{R}_1) & 1 & \longrightarrow & T_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \longrightarrow & & \downarrow & & \\
(\mathcal{R}_2) & 1 & \longrightarrow & T_2 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & 1,
\end{array}
\]

where \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are t-resolutions of \(G_1\) and \(G_2\), respectively.

Thus, if \(G\) is a reductive \(S\)-group scheme and \(\mathcal{R}'\) and \(\mathcal{R}\) are two t-resolutions of \(G\), then a morphism from \(\mathcal{R}'\) to \(\mathcal{R}\) (as in Definition 2.4) is a t-resolution of \(\text{id}_G: G \to G\).

Remark 3.2. A t-resolution \(\kappa_{\mathcal{R}}: R_1 \to R_2\) of \(\kappa: G_1 \to G_2\) induces a homomorphism of finitely generated twisted constant \(S\)-group schemes

\[
\pi_1(\kappa_{\mathcal{R}}): \pi_1(R_1) \to \pi_1(R_2).
\]

If \(G_3\) is a third reductive \(S\)-group scheme, \(\lambda: G_2 \to G_3\) is an \(S\)-homomorphism and \(\lambda_{\mathcal{R}}: \mathcal{R}_2 \to \mathcal{R}_3\) is a t-resolution of \(\lambda\), then \(\lambda_{\mathcal{R}} \circ \kappa_{\mathcal{R}}: \mathcal{R}_1 \to \mathcal{R}_3\) is a t-resolution of \(\lambda \circ \kappa\) and

\[
\pi_1(\lambda_{\mathcal{R}} \circ \kappa_{\mathcal{R}}) = \pi_1(\lambda_{\mathcal{R}}) \circ \pi_1(\kappa_{\mathcal{R}}).
\]

Lemma 3.3. Let \(\kappa: G_1 \to G_2\) be a homomorphism of reductive \(S\)-group schemes and let \(\mathcal{R}_2\) be a t-resolution of \(G_2\). Then there exists a t-resolution \(\kappa_{\mathcal{R}}: \mathcal{R}_1 \to \mathcal{R}_2\) of \(\kappa\) for a suitable choice of t-resolution \(\mathcal{R}_1\) of \(G_1\). In particular, every homomorphism of reductive \(S\)-group schemes admits a t-resolution.

Proof. Choose any t-resolution \(\mathcal{R}_1'\) of \(G_1\) and apply Proposition 2.8 to \(\kappa\), \(\mathcal{R}_1'\) and \(\mathcal{R}_2\). \(\square\)

Definition 3.4. Let \(\kappa: G_1 \to G_2\) be a homomorphism of reductive \(S\)-group schemes and let \(\mathcal{R}_2\) be a t-resolution of \(G_2\). Then there exists a t-resolution \(\kappa_{\mathcal{R}}: \mathcal{R}_1 \to \mathcal{R}_2\) of \(\kappa\) for a suitable choice of t-resolution \(\mathcal{R}_1\) of \(G_1\). In particular, every homomorphism of reductive \(S\)-group schemes admits a t-resolution.

A morphism from \(\kappa'_{\mathcal{R}}\) to \(\kappa_{\mathcal{R}}\), written \(\kappa'_R \to \kappa_R\), is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{R}_1 & \overset{\kappa'_{\mathcal{R}}}{\longrightarrow} & \mathcal{R}_2' \\
\downarrow & & \downarrow \\
\mathcal{R}_1 & \overset{\kappa_{\mathcal{R}}}{\longrightarrow} & \mathcal{R}_2,
\end{array}
\]

where the left-hand vertical arrow is a t-resolution of \(\text{id}_{G_1}\) and the right-hand vertical arrow is a t-resolution of \(\text{id}_{G_2}\). By a t-resolution dominating a t-resolution \(\kappa_{\mathcal{R}}\) of \(\kappa\) we mean a t-resolution \(\kappa'_{\mathcal{R}}\) of \(\kappa\) admitting a morphism \(\kappa'_R \to \kappa_R\).

Lemma 3.5. If \(\kappa_{\mathcal{R}}: \mathcal{R}_1 \to \mathcal{R}_2\) and \(\kappa'_{\mathcal{R}}: \mathcal{R}_1' \to \mathcal{R}_2'\) are two t-resolutions of a morphism \(\kappa: G_1 \to G_2\), then there exists a third t-resolution \(\kappa''_{\mathcal{R}}\) of \(\kappa\) which dominates both \(\kappa_{\mathcal{R}}\) and \(\kappa'_{\mathcal{R}}\).
Proof. By Corollary 2.9, there exists a $t$-resolution $R''_2$ of $G_2$ which dominates both $R_2$ and $R'_2$. On the other hand, by Lemma 3.3, there exists a $t$-resolution $\tilde{x}_G$: $R'''_1 \rightarrow R''_2$ of $x$ for a suitable choice of $t$-resolution $R''_1$ of $G_1$. Now a second application of Corollary 2.9 yields a $t$-resolution $R''_1$ of $G_1$ which dominates $R'_1$, $R'_2$ and $R'''_1$. Let $\phi: R''_1 \rightarrow R''_2$ be the corresponding morphism, which is a $t$-resolution of id$_{G_1}$. Then $\tilde{x}_{G}'' = \tilde{x}_G \circ \phi: R''_1 \rightarrow R''_2$ is a $t$-resolution of $x$ which dominates both $\tilde{x}_G$ and $\tilde{x}'_G$.

Construction 3.6. Let $x: G_1 \rightarrow G_2$ be a homomorphism of reductive $S$-group schemes. By Lemma 3.3 there exists a $t$-resolution $x_G: R_1 \rightarrow R_2$ of $x$, which induces a homomorphism $\pi_1(x_G): \pi_1(R_1) \rightarrow \pi_1(R_2)$ of finitely generated twisted constant $S$-group schemes. Thus, if we identify $\pi_1(G_i)$ with $\pi_1(R_i)$ for $i = 1, 2$, we obtain an $S$-homomorphism $\pi_1(x_G): \pi_1(G_1) \rightarrow \pi_1(G_2)$ which, by Lemma 3.5, can be shown to be independent of the chosen $t$-resolution $x_G$ of $x$. We denote it by

$$\pi_1(x): \pi_1(G_1) \rightarrow \pi_1(G_2).$$

Lemma 3.7. Let $G_1 \xrightarrow{x} G_2 \xrightarrow{\lambda} G_3$ be homomorphisms of reductive $S$-group schemes. Then

$$\pi_1(\lambda \circ x) = \pi_1(\lambda) \circ \pi_1(x).$$

Proof. Choose a $t$-resolution $R_3$ of $G_3$. Applying Lemma 3.3 first to $\lambda$ and then to $x$, we obtain $t$-resolutions $R_1 \xrightarrow{x_G} R_2 \xrightarrow{\lambda_G} R_3$ of $x$ and $\lambda$, and the composition $\lambda_G \circ x_G$ is a $t$-resolution of $\lambda \circ x$. Thus, by Remark 3.2,

$$\pi_1(\lambda \circ x) = \pi_1(\lambda_G \circ x_G) = \pi_1(\lambda_G) \circ \pi_1(x_G) = \pi_1(\lambda) \circ \pi_1(x),$$

as claimed.

Summarizing, for any non-empty scheme $S$, we have constructed a co-variant functor $\pi_1$ from the category of reductive $S$-group schemes to the category of finitely generated twisted constant $S$-group schemes. Now assume that $S$ is admissible in the sense of [9, Definition 2.1] (i.e., reduced, connected, locally Noetherian and geometrically unibranch), so that every reductive $S$-group scheme admits a flasque resolution [9, Proposition 3.2]. In this case the functor $\pi_1$ defined here in terms of $t$-resolutions coincides with the functor $\pi_1$ defined in [9, Definition 3.7] in terms of flasque resolutions, because a flasque resolution is a particular case of a $t$-resolution. A basic example of a non-admissible scheme $S$ to which the constructions of the present paper apply, but not those of [9], is an algebraic curve over a field having an ordinary double point. See [9, Remark 2.3].

The following result generalizes [3, Lemma 3.7], [4, Proposition 6.8] and [9, Theorem 3.14].

Theorem 3.8. Let $1 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 1$ be an exact sequence of reductive $S$-group schemes. Then the induced sequence of finitely generated twisted constant $S$-group schemes

$$0 \rightarrow \pi_1(G_1) \rightarrow \pi_1(G_2) \rightarrow \pi_1(G_3) \rightarrow 0$$

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is exact.

Proof. The proof is similar to that of [9, Theorem 3.14] using the exact sequence (13). Namely, one first proves the theorem when $G_1$ is semisimple using the same arguments as in the proof of [9, Lemma 3.12] (those arguments rely on [9, Proposition 2.8], which is valid over any non-empty base scheme $S$). Secondly, one proves the theorem when $G_1$ is an $S$-torus using the same arguments as in the proof of [9, Lemma 3.13] (which rely on [9, Proposition 2.9], which again holds over any non-empty base scheme $S$). Finally, the theorem is obtained by combining these two particular cases as in the proof of [9, Theorem 3.14].

We shall now present a second proof of Theorem 3.8 which relies on the étale-local existence of maximal tori in reductive $S$-group schemes. To this end, we shall first show that if $G$ is a reductive $S$-group scheme which contains a maximal torus $T$, then $T$ canonically determines a $t$-resolution of $G$.

Lemma 3.9. Let $G$ be a reductive $S$-group scheme having a maximal $S$-torus $T$, and set $\tilde{T} := G \times_G T$, it is a maximal $S$-torus of $\tilde{G}$. Then there exists a $t$-resolution of $G$

$$1 \to \tilde{T} \to H \to G \to 1$$

such that $H^{tor}$ is canonically isomorphic to $T$.

Proof. By [9, proof of Proposition 3.2], the product in $G$ and the canonical epimorphism $\tilde{G} \to G^{der}$ induce a faithfully flat homomorphism $rad(G) \times_S \tilde{G} \to G$ whose (central) kernel $\mu_1$ embeds into $Z(\tilde{G})$ via the canonical projection $rad(G) \times_S \tilde{G} \to \tilde{G}$. In particular, we have a central extension

$$1 \to \mu_1 \to rad(G) \times_S \tilde{G} \to G \to 1. \tag{14}$$

Since $Z(\tilde{G}) \subset \tilde{T}$ by [6, Exp. XXII, Corollary 4.1.7], we obtain an embedding $\psi: \mu_1 \to \tilde{T}$. Let $H$ be the pushout of $\varphi: \mu_1 \to rad(G) \times_S \tilde{G}$ and $\psi: \mu_1 \to \tilde{T}$, i.e., the cokernel of the central embedding

$$\varphi, \text{inv}_T \circ \psi)_S: \mu_1 \to (rad(G) \times_S \tilde{G}) \times_S \tilde{T}. \tag{15}$$

Now let $\varepsilon: S \to rad(G) \times_S \tilde{G}$ be the unit section of $rad(G) \times_S \tilde{G}$ and set $j = (\varepsilon, \text{id}_T)_S: S \times S \tilde{T} \to (rad(G) \times_S \tilde{G}) \times_S \tilde{T}$.

Composing $j$ with the canonical isomorphism $\tilde{T} \simeq S \times_S \tilde{T}$, we obtain an $S$-morphism $\tilde{T} \to rad(G) \times_S \tilde{G} \times_S \tilde{T}$ which induces an embedding $\iota_T: \tilde{T} \to H$. Further, let $\pi_T: H \to G$ be the homomorphism which is induced by the projection

$$rad(G) \times_S \tilde{G} \times_S \tilde{T} \to rad(G) \times_S \tilde{G}.$$ 

Then we obtain a $t$-resolution of $G$

$$1 \to \tilde{T} \to H \to G \to 1$$
which is canonically determined by $T$ (cf. the proof of Proposition 2.2). It remains to show that $H^\text{tor}$ is canonically isomorphic to $T$. Let $\varepsilon_{\text{rad}} : S \to \text{rad}(G)$ and $\varepsilon_{\tilde{T}} : S \to \tilde{T}$ be the unit sections of $\text{rad}(G)$ and $\tilde{T}$, respectively, and consider the homomorphism

$$(\varepsilon_{\text{rad}}, \text{id}_{\tilde{G}}, \varepsilon_{\tilde{T}}) : S \times_S \tilde{G} \times_S S \to \text{rad}(G) \times_S \tilde{G} \times_S \tilde{T}.$$ 

Composing this homomorphism with the canonical isomorphism $\tilde{G} \simeq S \times_S \tilde{G} \times_S S$, we obtain a canonical embedding $\tilde{G} \hookrightarrow \text{rad}(G) \times_S \tilde{G} \times_S \tilde{T}$. The latter map induces a homomorphism $\tilde{G} \to H$ which identifies $\tilde{G}$ with $H^\text{der}$. Now consider the composite homomorphism

$$\varphi_{\text{rad}} : \mu_1 \varphi \to \text{rad}(G) \times_S \tilde{G} \overset{\text{pr}_1}{\to} \text{rad}(G).$$

Then $H^\text{tor} := H/H^\text{der} = H/\tilde{G}$ is isomorphic to the cokernel of the central embedding

$(\varphi_{\text{rad}}, \text{inv}_{\tilde{T}} \circ \psi) : \mu_1 \hookrightarrow \text{rad}(G) \times_S \tilde{T}.$

Compare (15). Finally, the canonical embedding $\tilde{T} \hookrightarrow \tilde{G}$ induces an embedding $H^\text{tor} \hookrightarrow G$ (see (14) and (16)) whose image is $\text{rad}(G) \cdot (T \cap G^\text{der}) = T$ [6, Exp. XXII, proof of Proposition 6.2.8(i)]. This completes the proof. 

**Remark 3.10.** It is clear from the above proof that the homomorphism $\tilde{T} \to H^\text{tor} = T$ induced by the $t$-resolution $\mathcal{R}_T$ of Lemma 3.9 is the canonical homomorphism $\partial : \tilde{T} \to T$.

**Definition 3.11.** Let $G$ be a reductive $S$-group scheme containing a maximal $S$-torus $T$. The *algebraic fundamental group of the pair $(G, T)$* is the $S$-group scheme $\pi_1(G, T) := \text{Coker} [\partial_\ast : \tilde{T}_* \to T_*]$.

By Lemma 3.9 and Definition 2.11 we have a canonical isomorphism

$$\vartheta_T : \pi_1(G, T) \xrightarrow{\cong} \pi_1(\mathcal{R}_T) = \pi_1(G).$$

Further, any morphism of pairs $\kappa : (G_1, T_1) \to (G_2, T_2)$ (in the obvious sense) induces an $S$-homomorphism $\kappa_* : \pi_1(G_1, T_1) \to \pi_1(G_2, T_2)$. It can be shown that the following diagram commutes:

$$\begin{array}{ccc}
\pi_1(G_1, T_1) & \xrightarrow{\kappa_*} & \pi_1(G_2, T_2) \\
\vartheta_{T_1} \downarrow & & \vartheta_{T_2} \downarrow \\
\pi_1(G_1) & \xrightarrow{\vartheta_{T_1}} & \pi_1(G_2). \\
\end{array}$$

This is immediate in the case where $\kappa$ is a *normal* homomorphism, i.e. $\kappa(G_1)$ is normal in $G_2$ (this is the only case needed in this paper). Indeed, in this case we have $\kappa(\text{rad}(G_1)) \subset \text{rad}(G_2)$ and therefore $\kappa$ induces a morphism of $t$-resolutions $\kappa_{\mathcal{R}_T} : \mathcal{R}_{T_1} \to \mathcal{R}_{T_2}$. See the proof of Lemma 3.9.
Remark 3.12. The preceding considerations and Lemma 2.10 show that, if $S$ is an admissible scheme in the sense of [9, Definition 2.1], so that every reductive $S$-group scheme $G$ admits a flasque resolution $\mathcal{F}$, and $G$ contains a maximal $S$-torus $T$, then there exists a canonical isomorphism $\pi_1(\mathcal{F}) \cong \text{Coker } [\partial_* : \tilde{T}_s \to T_s]$. This fact generalizes [4, Proposition A.2], which is the case $S = \text{Spec } k$, where $k$ is a field, of the present remark.

Lemma 3.13. Let
$$1 \to (G_1, T_1) \xrightarrow{\kappa} (G_2, T_2) \xrightarrow{\lambda} (G_3, T_3) \to 1$$
be an exact sequence of reductive $S$-group schemes with maximal tori. Then the sequence of étale, finitely generated twisted constant $S$-group schemes
$$0 \to \pi_1(G_1, T_1) \xrightarrow{\kappa_*} \pi_1(G_2, T_2) \xrightarrow{\lambda_*} \pi_1(G_3, T_3) \to 0$$
is exact.

Proof. The assertion of the lemma is local for the étale topology, so we may and do assume that $T_1$, $T_2$, and $T_3$ are split. By [9, Proposition 2.10], there exists an exact commutative diagram of reductive $S$-group schemes
$$
\begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & \tilde{G}_1 \\
\downarrow \partial_1 & & \downarrow \partial_1 \\
1 & \rightarrow & G_1
\end{array}
\begin{array}{ccc}
\rightarrow & \tilde{G}_2 \\
\downarrow \partial_2 & & \downarrow \partial_2 \\
\rightarrow & G_2
\end{array}
\begin{array}{ccc}
\rightarrow & \tilde{G}_3 \\
\downarrow \partial_3 & & \downarrow \partial_3 \\
\rightarrow & G_3
\end{array}
\rightarrow 1
\end{array}
$$
which induces an exact commutative diagram of split $S$-tori
$$\begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & \tilde{T}_1 \\
\downarrow \partial_1 & & \downarrow \partial_1 \\
1 & \rightarrow & T_1
\end{array}
\begin{array}{ccc}
\rightarrow & \tilde{T}_2 \\
\downarrow \partial_2 & & \downarrow \partial_2 \\
\rightarrow & T_2
\end{array}
\begin{array}{ccc}
\rightarrow & \tilde{T}_3 \\
\downarrow \partial_3 & & \downarrow \partial_3 \\
\rightarrow & T_3
\end{array}
\rightarrow 1
\end{array}
$$
(19)

where $\tilde{T}_i := \tilde{G}_i \times_G T_i$ $(i = 1, 2, 3)$. Now, as in [9, Proof of Lemma 3.7], diagram (19) induces an exact commutative diagram of constant $S$-group schemes
$$
\begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & \tilde{T}_{1s} \\
\downarrow \partial_{1s} & & \downarrow \partial_{1s} \\
1 & \rightarrow & T_{1s}
\end{array}
\begin{array}{ccc}
\rightarrow & \tilde{T}_{2s} \\
\downarrow \partial_{2s} & & \downarrow \partial_{2s} \\
\rightarrow & T_{2s}
\end{array}
\begin{array}{ccc}
\rightarrow & \tilde{T}_{3s} \\
\downarrow \partial_{3s} & & \downarrow \partial_{3s} \\
\rightarrow & T_{3s}
\end{array}
\rightarrow 1
\end{array}
$$
with injective vertical arrows. An application of the snake lemma to the last diagram now yields the exact sequence
$$0 \to \text{Coker } \partial_{1s} \to \text{Coker } \partial_{2s} \to \text{Coker } \partial_{3s} \to 0,$$
which is the assertion of the lemma. □
Second proof of Theorem 3.8. Let \(1 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 1\) be an exact sequence of reductive \(S\)-group schemes. By [3] Exp. XIX, Proposition 6.1, for any reductive \(S\)-group scheme \(G\) there exists an étale covering \(\{S_\alpha \rightarrow S\}_{\alpha \in A}\) such that each \(G_{S_\alpha} := G \times_S S_\alpha\) contains a split maximal \(S_\alpha\)-torus \(T_\alpha\). Thus, since the assertion of the theorem is local for the étale topology, we may and do assume that \(G_2\) contains a split maximal \(S\)-torus \(T_2\). Let \(T_1 = G_1 \times G_2, T_2\) and let \(T_3\) be the cokernel of \(T_1 \rightarrow T_2\). Then \(T_i\) is a split maximal \(S\)-torus of \(G_i\) for \(i = 1, 2, 3\) and we have an exact sequence of pairs

\[
1 \rightarrow (G_1, T_1) \rightarrow (G_2, T_2) \rightarrow (G_3, T_3) \rightarrow 1.
\]

Now the theorem follows from Lemma 3.13 (17) and (18). \(\square\)

4. Abelian cohomology and \(t\)-resolutions

Let \(S_{\text{fl}}\) (respectively, \(S_{\text{ét}}\)) be the small fppf (respectively, étale) site over \(S\). If \(F_1\) and \(F_2\) are abelian sheaves on \(S_{\text{fl}}\) (regarded as complexes concentrated in degree 0), \(F_1 \otimes^L F_2\) (respectively, \(\text{RHom}(F_1, F_2)\)) will denote the total tensor product (respectively, right derived Hom functor) of \(F_1\) and \(F_2\) in the derived category of the category of abelian sheaves on \(S_{\text{fl}}\).

Let \(G\) be a reductive group scheme over \(S\). For any integer \(i \geq -1\), the \(i\)-th abelian (flat) cohomology group of \(G\) is by definition the hypercohomology group

\[
H^i_{\text{ab}}(S_{\text{fl}}, G) = \mathbb{H}^i(S_{\text{fl}}, Z(\widetilde{G})) \xrightarrow{\partial_2} Z(G).
\]

On the other hand, the \(i\)-th dual abelian cohomology group of \(G\) is the group

\[
H^i_{\text{ab}}(S_{\text{ét}}, G^*) = \mathbb{H}^i(S_{\text{ét}}, Z(G)^* \xrightarrow{\partial_2} Z(\widetilde{G})^*).
\]

Here all the complexes of length 2 are in degrees \((-1, 0)\). See [3] beginning of §4 for basic properties of these cohomology groups and [11, 3, 10] for some of their arithmetical applications.

The following result is an immediate consequence of (5).

Proposition 4.1. Let \(G\) be a reductive \(S\)-group scheme and let \(1 \rightarrow T \rightarrow H \rightarrow G \rightarrow 1\) be a \(t\)-resolution of \(G\). Then the given \(t\)-resolution defines isomorphisms \(H^i_{\text{ab}}(S_{\text{fl}}, G) \simeq \mathbb{H}^i(S_{\text{fl}}, T \rightarrow R)\) and \(H^i_{\text{ab}}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i(S_{\text{ét}}, R^* \rightarrow T^*)\), where \(R = H^{\text{tor}}\). Further, there exist exact sequences

\[
\ldots \rightarrow H^i(S_{\text{ét}}, T) \rightarrow H^i(S_{\text{ét}}, R) \rightarrow H^i_{\text{ab}}(S_{\text{fl}}, G) \rightarrow H^{i+1}(S_{\text{ét}}, T) \rightarrow \ldots
\]

and

\[
\ldots \rightarrow H^i(S_{\text{ét}}, R^*) \rightarrow H^i(S_{\text{ét}}, T^*) \rightarrow H^i_{\text{ab}}(S_{\text{ét}}, G^*) \rightarrow H^{i+1}(S_{\text{ét}}, R^*) \rightarrow \ldots \ \square
\]

Corollary 4.2. Let \(G\) be a reductive \(S\)-group scheme. Then, for every integer \(i \geq -1\), there exist isomorphisms

\[
H^i_{\text{ab}}(S_{\text{fl}}, G) \simeq \mathbb{H}^i(S_{\text{fl}}, \pi_1(G) \otimes^L G_{m, S})
\]

and

\[
H^i_{\text{ab}}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i(S_{\text{ét}}, \text{RHom}(\pi_1(G), \mathbb{Z}_S)).
\]
Proof. This follows from Proposition 4.1 in the same way as [9, Corollary 4.3] follows from [9, Proposition 4.2]. □

Proposition 4.3. Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive $S$-group schemes. Then there exist exact sequences of abelian groups

$$\ldots \to H^i_{ab}(S_{\text{fl}}, G_1) \to H^i_{ab}(S_{\text{fl}}, G_2) \to H^i_{ab}(S_{\text{fl}}, G_3) \to H^{i+1}_{ab}(S_{\text{fl}}, G_1) \to \ldots$$

and

$$\ldots \to H^i_{ab}(S_{\text{ét}}, G_3^*) \to H^i_{ab}(S_{\text{ét}}, G_2^*) \to H^i_{ab}(S_{\text{ét}}, G_1^*) \to H^{i+1}_{ab}(S_{\text{ét}}, G_3^*) \to \ldots$$

Proof. This follows from Corollary 4.2 and Theorem 3.8. □

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References

[1] Borovoi, M., Abelian Galois cohomology of reductive groups, Mem. Amer. Math. Soc., 1998, 132(626), 1–50.
[2] Borovoi, M., Demarche, C., Le groupe fondamental d’un espace homogène d’un groupe algébrique linéaire, preprint available at http://arxiv.org/abs/1301.1046.
[3] Borovoi, M., Kunyavskiĭ B., Gille, P., Arithmetical birational invariants of linear algebraic groups over two-dimensional geometric fields, J. Algebra, 2004, 276(1), 292–339.
[4] Colliot-Thélène, J.-L., Résolutions flasques des groupes linéaires connexes, J. reine angew. Math., 2008, 618, 77–133.
[5] Conrad, B., Reductive group schemes (SGA3 Summer School, 2011), preprint available at http://math.stanford.edu/~conrad/papers/luminysga3.pdf.
[6] Demazure, M., Grothendieck, A. (Eds.), Schémas en groupes. Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Augmented and corrected 2008-2011 re-edition of the original by P. Gille and P. Polo. Available at http://www.math.jussieu.fr/~polo/SGA3. Volumes 1 and 3 have been published: Documents mathématiques 7 and 8, Société Mathématique de France, Paris, 2011.
[7] Gelfand, S.I., Manin, Yu.I., Methods of Homological Algebra, 2nd ed., Springer-Verlag, Berlin, 2003.
[8] González-Avilés, C.D., Quasi-abelian crossed modules and nonabelian cohomology, J. Algebra, 2012, 369, 235–255.
[9] González-Avilés, C.D., Flasque resolutions of reductive group schemes, Cent. Eur. J. Math., 2013, 11(7), 1159–1176.
[10] González-Avilés, C.D., Abelian class groups of reductive group schemes, Israel J. Math. (in press), preprint available at http://arxiv.org/abs/1108.3264.
[11] Kottwitz, R.E., Stable trace formula: cuspidal tempered terms, Duke Math. J., 1984, 51(3), 611–650.
[12] Merkurjev, A.S., K-theory and algebraic groups, European Congress of Mathematics (Budapest, 1996), Vol. II, Progr. Math., 169, Birkhäuser, Basel, 1998, 43–72.

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