Optimising Rolling Stock Planning including Maintenance with Constraint Programming and Quantum Annealing

Patricia Bickert, Cristian Grozea, Ronny Hans, Matthias Koch, Christina Riehn, and Armin Wolf

1 DB Systel GmbH, Jürgen-Ponto-Platz 1, 60329 Frankfurt am Main, Germany
firstname dot lastname @deutschebahn.com

2 Fraunhofer FOKUS, Kaiserin-Augusta-Allee 31, 10589 Berlin, Germany
firstname dot lastname @fokus.fraunhofer.de

Abstract. We propose and compare Constraint Programming (CP) and Quantum Annealing (QA) approaches for rolling stock assignment optimisation considering necessary maintenance tasks. In the CP approach, we model the problem with an alldifferent constraint, extensions of the element constraint, and logical implications, among others. For the QA approach, we develop a quadratic unconstrained binary optimisation (QUBO) model. For evaluation, we use data sets based on real data from Deutsche Bahn and run the QA approach on real quantum computers from D-Wave. Classical computers are used to evaluate the CP approach as well as tabu search for the QUBO model. At the current development stage of the physical quantum annealers, we find that both approaches tend to produce comparable results.

1 Introduction and Motivation

Every day, 40,000 trains travel on the Deutsche Bahn rail network, heading to 5,700 stations. DB Fernverkehr AG, a subsidiary company of Deutsche Bahn, provides 315 Intercity-Express (ICE) trains and carries about 220,000 passengers between 140 ICE train stations every day in 2020. Although mathematical optimisation is a key opportunity to improve the economic viability of railroad companies, the problem sizes are tremendous. The requirements to guarantee a safe operation, e.g. periodic and non-periodic maintenance constraints, increase the complexity even further. Furthermore, railroad companies must be able to respond quickly to disruptions, e.g., technical fault on a train, to ensure an operation in accordance with the timetable. Many real timetable-related assignment problems are NP-hard, which makes short-term planning very challenging.

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** The authors are listed in alphabetical order.

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Footnote 1: https://www.deutschebahn.com/de/konzern/konzernprofil/zahlen_fakten
The acceleration promised by quantum computation could be a game changer for logistics companies. Quantum-based optimisation of industrial problems became an active field of research, even more after Google claimed quantum supremacy in 2019 [1]. One of the most promising devices for this now is the D-Wave annealer, a special purpose optimisation machine. On the other side, Constraint Programming (CP) is a well established programming paradigm for solving combinatorial search and optimisation problems [2].

The paper is structured as follows: In the following section, we discuss related approaches based on a literature review. Subsequently, we present and explain the rolling stock planning problem in detail. Then, in Section 4 we present our CP-based solution. Afterwards, in Section 5 we describe our quantum computing approach. In Section 6 we explain the data we use for the evaluation and present the results of our two approaches. In Section 7 we discuss the results. The paper closes with conclusions and future research directions.

2 Related Work

A comparative survey on research activities concerning optimised rolling stock assignment and maintenance can be found in [3]. Similar to our case, there, the focus lies on passenger transportation, where the schedule of the trips is fixed in advance. We consider rolling stock rotation maximising the number of operated trips, minimising empty runs, and performing maintenance tasks, as in [3]. Here, as in our approach, a pre-processing is performed to determine feasible sequences of train services possibly including empty runs and maintenance tasks. In [5], rolling stock rescheduling is considered together with depot re-planning in order to handle short-term disruptions in railway traffic. There, a specialised branch-and-price-and-cut approach is used to handle such problems extending previous work [6]. In [3, 4, 7, 12], Mixed Integer Programming (MIP) is applied to model and solve rolling stock and related problems — like locomotive scheduling — which seems to be a common approach for such problems. In [13], constraint propagation together with depth-first search based on backtracking was used to perform reactive scheduling for rolling stock operations. Similarly, in [14] CP modelling and solving was applied for capacity maximisation of an Australian railway system transporting coal from mines to harbours. There, CP was able to solve this problem finding coal train schedules which are close to optimistic upper capacity bounds computed analytically. Both CP approaches encourage us to consider CP for our rolling stock optimisation problem.

A routing problem concerning railway carriages in a railway network, which is similar to rolling stock problems, is addressed in [15]. There, a local search approach, namely Simulated Annealing (SA), is applied to solve the problem, because an integer programming approach fails. Quantum optimisation based on a QUBO model has been employed by [16], where delay and conflict management on a single-track railway line is considered. A similar QUBO approach has also been used, e.g., for flight assignment tasks [17, 18] or traffic flow optimisation [19].
3 Rolling Stock Planning with Maintenance

Here, all parameters of rolling stock planning problems are presented, i.e. the stations and the trains performing trips between them. Here a trip is an entire end-to-end journey between two stations with potentially many stops along the way performed by one train. The trains are potentially maintained between their trips – either periodic or non-periodic. The time resolution is in minutes \([\text{min}]\) and distances are in kilometres \([\text{km}]\). The considered scheduling horizon is defined by an ordered set \(H \subset \mathbb{N}\) of time points in minutes. Typically \(H = \{0, \ldots, e-1\}\) where 0 is the canonical begin of the scheduling horizon and \(e-1\) its end, e.g. \(H = \{0, \ldots, 1439\}\) represents one day.

3.1 Data Objects, Attributes, Constraints and Objectives

**Stations:** let \(B = \{b_0, \ldots, b_{l-1}\}\) be the set of stations. For each pair of stations \(b, c \in B, b \neq c\) let
- \(\text{travelDistance}(b, c) \in \mathbb{N}^+\) and \(\text{edriveDistance}(b, c) \in \mathbb{N}^+\) be the distances for travelling resp. for driving empty from station \(b\) to station \(c\) in \([\text{km}]\).
- \(\text{travelDuration}(b, c) \in \mathbb{N}^+\) and \(\text{edriveDuration}(b, c) \in \mathbb{N}^+\) be the durations for travelling resp. for driving empty from station \(b\) to station \(c\) in \([\text{min}]\).

Obviously, we assume that \(\text{travelDistance}(b, b) = \text{edriveDistance}(b, b) = 0\) and \(\text{travelDuration}(b, b) = \text{edriveDuration}(b, b) = 0\) holds for each station \(b \in B\).

Sometimes we abbreviate \(\text{edriveDistance}(b, c)\) by \(\text{edD}(b, c)\).

**Trips:** let \(F = \{f_0, \ldots, f_{n-1}\}\) be the set of trips, where \(n\) is the number of trips. For each trip \(f \in F\) let
- \(f.\text{departureStation} \in B\) (abbr. \(\text{dStation}\)) and \(f.\text{arrivalStation} \in B\) (abbr. \(\text{aStation}\)) be the departure resp. the arrival station where the trip \(f\) starts resp. ends.
- \(f.\text{departureTime} \in H\) and \(f.\text{arrivalTime} \in H\) be the departure resp. the arrival time when the trip \(f\) starts resp. ends.
- \(f.\text{travelDistance} \in \mathbb{N}^+\) be the travelling distance of the trip \(f\).
- \(f.\text{travelDuration} \in H\) be the travelling duration of the trip \(f\).
- \(f.\text{postProcessing} \in H\) be some post-processing time used for preparing the train after the trip \(f\).

It is assumed that for each trip \(f \in F\) it holds that \(f.\text{departureTime} < f.\text{arrivalTime}\) and that \(\max_{f \in F} f.\text{arrivalTime} = e-1\), i.e. the end of the scheduling horizon is defined by the latest arriving trip.

**Maintenance types:** they are either periodic or non-periodic. Let \(P\) be the set of periodic maintenance types, let \(S\) be the set of non-periodic maintenance types. Obviously, the both sets are disjoint: \(P \cap S = \emptyset\). Further let \(W = P \cup S\) be all maintenance types. For each maintenance type \(w \in W\) let

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4 For the sake of simplicity we assume a homogeneous train fleet, i.e. such that the durations for travelling between stations do not depend on the trains.

5 In the considered scenarios, we used 120 min. for each trip, e.g. for cleaning etc.
– \( w.\text{stations} \subseteq B \) be the set of \( \text{stations} \) where maintenance tasks of type \( w \) can be performed.
– \( w.\text{duration} \in H \) be the \( \text{duration} \) of the maintenance tasks of type \( w \).
– \( w.\text{limit} \in \mathbb{N}^+ \) be either the length of the maintenance interval of a periodic maintenance or the threshold of a non-periodic maintenance – both in [\( \text{km} \)].

Within this limit the according maintenance task \( w \) has to be performed.

**Trains:** let \( Z = \{ z_0, \ldots, z_{m-1} \} \) be the trains potentially performing the trips in \( F \), where \( m \) is the number of trains. For each train \( z \in Z \) let

– \( z.\text{initialStation} \in B \) be the \textit{initial station} from where the train \( z \) starts its first trip, if the train performs any trips.
– \( z.\text{earliestTime} \in H \) be the \textit{earliest available time} of the train \( z \).
– \( z.\text{initialKm}(p) \in \mathbb{N}^+ \) be the initial \textit{kilometre reading} of the train \( z \) since the last periodic maintenance of type \( p \in P \) performed on this train (cf. Sec. 3.1).
– \( z.\text{initialKm} \in \mathbb{N}^+ \) be the initial \textit{kilometre reading} of the train \( z \) used for non-periodic maintenance types performed on this train (cf. Sec. 3.1).

**Constraints and Objectives:** the rolling stock problem including maintenance is characterised by the following constraints:

– each train performing a trip must be available in time at the departure station of the trip.
– all maintenance intervals – either periodic or non-periodic – of the trains must be respected, for regular trips as well as for empty trips.

The objective of the rolling stock problem including maintenance and idle times is to allocate as much as possible trips to trains, to reduce the number of empty driven kilometres such that the specified constraints are satisfied.

4 **Constraint-Programming-Based Solution**

In this section a formal model of the considered rolling stock problem is presented which is appropriate for Constraint Programming.

4.1 **Abstractions and Variables**

For each (real) trip \( f \in F \) we define the parameters

\[
\begin{align*}
\text{start}(f) &= f.\text{departureTime} \quad \text{and} \quad \text{end}(f) = f.\text{arrivalTime} \\
\text{travelDuration}(f) &= f.\text{travelDuration} \quad \text{and} \quad \text{postProcessing}(f) = f.\text{postProcessing} \\
\text{origination}(f) &= f.\text{departureStation} \quad \text{and} \quad \text{destination}(f) = f.\text{arrivalStation} .
\end{align*}
\]

For each train \( z \in Z \) we define a ‘virtual’ zero-duration \textit{initial trip} \( f_z \), where \( f_z = f_{n+i} \) by convention such the all trips have unique numbers (indices), with

\[
\begin{align*}
\text{start}(f_z) &= z.\text{earliestTime} \quad \text{and} \quad \text{end}(f_z) = z.\text{earliestTime} \\
\text{travelDuration}(f_z) &= 0 \quad \text{and} \quad \text{postProcessing}(f) = 0 \\
\text{origination}(f_z) &= z.\text{initialStation} \quad \text{and} \quad \text{destination}(f_z) = z.\text{initialStation} .
\end{align*}
\]
For each train \( z \in Z \) and for \( k = 0, \ldots, n - 1 \) we define the sets of trips:

\[
    F(z)_0 = \{ f_z \} \quad \text{and} \quad F(z)_{k+1} = F(z)_k \cup \{ f \in F | \text{ready}(f) = \min_{g \in F \setminus \{ f_z \}} \text{ready}(g) \}, \quad (1)
\]

where \( \text{ready}(f) = \text{end}(f) + \text{postProcessing}(f) \) for each \( f \in F \setminus \{ f_z \} \) with \( z \in Z \). Now for each \( z_i \in Z \) let \( q(i) \) be the smallest number such that \( F(z_i)_{q(i)+1} = F(z_i)_{q(i)} \) holds. Then \( q(i) \) is a non-trivial upper bound of the number of trips performed by the train \( z_i \) or of the number of slots for trips for each train that maximally fit into the scheduling horizon \( H \).

We would like to point out that \( q(i) \) is the smallest number of steps in order to extend the set \( F(z)_0 \) based on Equation (1) until a fixed point is reached. By complete induction we can prove that \( F(z)_k \) contains \( k \) different 'real' trips that can be scheduled after the 'virtual' initial trip \( f_z \) of train \( z \) without temporal overlapping, without considering time for any maintenance nor any empty drives between different locations if \( F(z)_k \) is a proper subset of \( F(Z)_k + 1 \). If \( F(z)_k = F(z)_k + 1 \) holds then there are at most \( k \) 'real' trips that can be scheduled in that way after the 'virtual' initial trip \( f_z \) of train \( z \). Considering additional maintenance tasks or empty drives between the trips performed by any train \( z_i \) the number of trips that can be performed by this train is not greater than \( q(i) \) such that \( q(i) \) is a non-trivial upper bound of the number of 'real' trips the can be performed by train \( z_i \). Due to the fact that the sets \( F(z)_k \) are only used for the calculation of these bounds, no solutions are lost.

Based on \( q(i) \) we define for each train \( z_i \in Z \) a sequence of finite domain variables \( \text{Trip}_{i,0} = n + i \) (i.e. the initial trip first) and

\[
    \text{Trip}_{i,1} \in \{-(iq + 1), 0, \ldots, n - 1\}, \ldots, \text{Trip}_{i,q(i)} \in \{-(iq + q), 0, \ldots, n - 1\},
\]

where \( q = \max_{i \in \{0, m-1\}} q(i) \) is the greatest number of trips for any train. These variables are presenting the indices of potential trips that will be performed by the train \( z_i \) according to the order of the sequence, i.e. \( \text{Trip}_{i,j} \) indicates the potential trip performed by train \( z_i \) in time slot \( j \). By definition \( \text{Trip}_{i,j} < 0 \) indicates that train \( z_i \) will not perform any 'regular' (neither 'real' nor 'virtual') trip in slot \( j \). We have chosen unique negative values within the domains of these variables such they can be pairwise different (see Equation (2) below). In addition to these \( \text{Trip} \) variables we use further indexed variables:

- \( \text{Start}_{i,j} \in H \): the start time of the 'trip' performed in slot \( j \) of train \( z_i \).
- \( \text{End}_{i,j} \in H \): the end time of the 'trip' performed in slot \( j \) of train \( z_i \).
- \( \text{PostProcessing}_{i,j} \in H \): the post processing time of the 'trip' in \( j \) of \( z_i \).
- \( \text{Origination}_{i,j} \in \{-1, 0, \ldots, l - 1\} \): the origination of the 'trip' in \( j \) of \( z_i \).
- \( \text{Destination}_{i,j} \in \{-1, 0, \ldots, l - 1\} \): the destination of the 'trip' in \( j \) of \( z_i \).
- \( \text{OptDuration}_{i,j} \in H \): the optional duration of a maintenance task performed after the 'trip' in slot \( j \) of train \( z_i \).
- \( \text{OptLocation}_{i,j} \in \{-1, 0, \ldots, l - 1\} \): either the optional location of a maintenance task performed after the 'trip' performed in slot \( j \) of train \( z_i \) or the destination of this trip.
– \(\text{DurationTo}_{i,j} \in H\) and \(\text{DurationFrom}_{i,j} \in H\): the durations after the ‘trip’ in slot \(j\) to resp. from an optional maintenance of train \(z_i\).

– \(\text{DistanceTo}_{i,j} \in H\) and \(\text{DistanceFrom}_{i,j} \in H\): the distance after the ‘trip’ in slot \(j\) to resp. from an optional maintenance task of train \(z_i\).

– \(\text{Distance}_{i,j} \in \mathbb{N}^+\): the distance of the ‘trip’ in slot \(j\) of train \(z_i\).

– \(\text{IsRegularTrip}_{i,j} \in \{0, 1\}\): signals whether the ‘trip’ in slot \(j\) of train \(z_i\) is ‘regular’.

– \(\text{IsMaintained}(w)_{i,j} \in \{0, 1\}\): signals whether there is a maintenance task of type \(w\) after the ‘trip’ in slot \(j\) of train \(z_i\).

– \(\text{KmReading}(p)_{i,j} \in \mathbb{N}^+\): the kilometre reading for a periodic maintenance task of type \(p\) of train \(z_i\) directly after the ‘trip’ in slot \(j\).

– \(\text{KmReading}_{i,j}\): the kilometre reading for all non-periodic maintenance tasks of train \(z_i\) directly after the ‘trip’ in slot \(j\).

4.2 Constraints

Due to the fact that each trip will be performed by at most one train all these trip variables must have pairwise different values:

\[
\text{allDifferent}\left(\{\text{Trip}_{i,j} | i \in \{0, \ldots, n - 1\} \land j \in \{1, \ldots, q(i)\}\}\right). \tag{2}
\]

Due to the fact that each sequence of trips performed by a train \(z_i\) is without gaps, it must hold that

\[
\text{Trip}_{i,j-1} < 0 \Rightarrow \text{Trip}_{i,j} < 0 \quad \text{resp.} \quad \text{Trip}_{i,j} \geq 0 \Rightarrow \text{Trip}_{i,j-1} \geq 0 \tag{3}
\]

for \(i = 0, \ldots, n - 1\) and \(j = 1, \ldots, q(i)\) and further

\[
\text{IsRegularTrip}_{i,j} \iff \text{Trip}_{i,j} \geq 0 \quad \text{for} \quad i = 0, \ldots, n - 1 \quad \text{and} \quad j = 0, \ldots, q(i). \tag{4}
\]

For the set of indices \(I = \{0, \ldots, n + m - 1\}\) of all trips additional (extended) element constraints\(^7\) must hold on indexed variables, when \(k = \text{Trip}_{i,j}\):

\[
\text{Start}_{i,j} = \text{Start}[\text{Trip}_{i,j}] = \begin{cases} \text{start}(f_k) & \text{for} \ k \in I \\ e - 1 & \text{for} \ k < 0 \end{cases} \tag{5}
\]

and analogously \(\text{End}_{i,j} = \text{End}[\text{Trip}_{i,j}]\) where \(\text{start}(f_k)\) is replaced by \(\text{end}(f_k)\) and \(\text{PostProcessing}_{i,j} = \text{PostProcessing}[\text{Trip}_{i,j}]\) where

\[
\text{PostProcessing}[\text{Trip}_{i,j}] = \begin{cases} \text{postProcessing}(f_k) & \text{for} \ k \in I \\ 0 & \text{for} \ k < 0 \end{cases}
\]

\[
\text{Destination}_{i,j} = \text{Destination}[\text{Trip}_{i,j}] = \begin{cases} \text{destination}(f_k) & \text{for} \ k \in I \\ -1 & \text{for} \ k < 0 \end{cases}
\]

as well as \(\text{Origination}_{i,j}\) where \(\text{destination}(f_k)\) is replaced by \(\text{origination}(f_k)\).

\(^6\) cf. https://sofdem.github.io/gccat/gccat/Calldifferent.html

\(^7\) cf. https://sofdem.github.io/gccat/gccat/Celement.html
There is at most one maintenance task even after each ‘real’ trip performed by a train, i.e. for each train $z_i$ and each slot $j$ it must hold that

$$\sum_{w \in W} \text{IsMaintained}(w)_{i,j} \leq 1.$$  \hfill (6)

If there is not any trip performed by train $i$ in slot $j$ or in slot $j+1$ then there will not be any maintenance task afterwards:

$$\text{Trip}_{i,j} < 0 \lor \text{Trip}_{i,j+1} < 0 \Rightarrow \text{IsMaintained}(w)_{i,j} = 0 \text{ for each } w \in W$$

resp. $\text{Trip}_{i,j} < 0 \lor \text{Trip}_{i,j+1} < 0 \Rightarrow \sum_{w \in W} \text{IsMaintained}(w)_{i,j} = 0$. \hfill (7)

For each type of maintenance $w \in W$, each train $z_i \in Z$ and each time slot $j$ it holds that the optional duration and the optional locations for the maintenance of this type are defined if the maintenance occurs or not:

$$\text{IsMaintained}(w)_{i,j} = 1 \Rightarrow \text{OptDuration}_{i,j} = w.\text{duration}$$

$$\land \text{OptLocation}_{i,j} \in w.\text{stations}$$

$$\sum_{w \in W} \text{IsMaintained}(w)_{i,j} = 0 \Rightarrow \text{OptDuration}_{i,j} = 0$$

$$\land \text{OptLocation}_{i,j} = \text{Destination}_{i,j}.$$  \hfill (8)

The trips of each train are performed in linear order and there must be enough time for empty transition drives and optional maintenance tasks, i.e. for each train $z_i \in Z$ and each time slot $j = 0, \ldots, q(i) - 1$ it must hold that

$$\text{Start}_{i,j+1} \geq \text{End}_{i,j} + \text{PostProcessing}_{i,j}$$

$$+ \text{OptDuration}_{i,j} + \text{DurationTo}_{i,j} + \text{DurationFrom}_{i,j}.$$  \hfill (10)

Here we use two-dimensional extended element constraints:

$$\text{DurationTo}_{i,j} = EDriveDuration[\text{Destination}_{i,j}][\text{OptLocation}_{i,j}].$$

$$\text{DurationFrom}_{i,j} = EDriveDuration[\text{OptLocation}_{i,j}][\text{Origination}_{i,j+1}].$$

where $EDriveDuration[r][s]$ represents the duration for empty drivings, i.e.

$$EDriveDuration[r][s] = \begin{cases} 
edriveDuration(b_r, b_s) & \text{for } r, s \in 0, \ldots, l - 1 \\
0 & \text{for } r < 0 \text{ or } s < 0 \end{cases}$$  \hfill (13)

and analogously $\text{TravelDuration}[r][s]$ where $\text{driveDuration}(b_r, b_s)$ is replaced by $\text{travelDuration}(b_r, b_s)$ representing the travelling durations between stations.

The kilometre reading for all non-periodic maintenance tasks of train $z_i$ are defined by $KmReading_{i,0} = z_i.\text{initialKm}$ and directly after the trip in slot $j$ by

$$KmReading_{i,j+1} = KmReading_{i,j} + \text{DistanceTo}_{i,j}$$

$$+ \text{DistanceFrom}_{i,j} + \text{Distance}_{i,j+1}.$$  \hfill (14)
Here we use two-dimensional extended element constraints, too:

\[
\text{DistanceTo}_{i,j} = EDriveDistance[\text{Destination}_{i,j}][\text{OptLocation}_{i,j}] \quad (15)
\]

\[
\text{DistanceFrom}_{i,j} = EDriveDistance[\text{OptLocation}_{i,j}][\text{Origination}_{i,j+1}] \quad (16)
\]

\[
\text{Distance}_{i,j+1} = \text{TravelDistance}[\text{Origination}_{i,j+1}][\text{Destination}_{i,j+1}] \quad (17)
\]

where \( EDriveDistance[] \) represents the empty driving distances, i.e.

\[
EDriveDistance[r][s] = \begin{cases} 
\text{eDriveDistance}(b_r, b_s) & \text{for } r, s \in 0, \ldots, l-1 \\
0 & \text{for } r < 0 \text{ or } s < 0 
\end{cases} \quad (18)
\]

and analogously \( \text{TravelDistance}[r][s] \) where \( \text{eDriveDistance}(b_r, b_s) \) is replaced by \( \text{travelDistance}(b_r, b_s) \) representing the travelling distances.

After the ‘initial’ trip where the kilometre reading is defined by the initial kilometre reading for all successive slots \( j+1 \) with \( j \in \{0, \ldots, q-1\} \) it holds that the kilometre reading is the kilometre reading of the previous slot \( j \) plus the distances for all trips performed in between. Furthermore, the limits of all non-periodic maintenance tasks must be respected, i.e. if for any train \( z_i \) the kilometre reading exceeds the threshold of a non-periodic maintenance of type \( s \) after a trip in slot \( j \geq 0 \) then this maintenance must be performed directly before this trip:

\[
\text{kmReading}_{i,j} \leq s.\text{limit} \land \text{kmReading}_{i,j+1} > s.\text{limit} \Rightarrow \text{IsMaintained}(s)_{i,j} = 1.
\]

For the kilometre reading of a periodic maintenance \( p \) of train \( z_i \) let

\[
\text{KmReading}(p)_{i,0} = z_i.\text{initialKm}(p) \quad \text{where } z_i.\text{initialKm}(p) \leq p.\text{limit}
\]

and directly after the trip in slot \( j \geq 0 \) it must hold that

\[
\text{KmReading}(p)_{i,j+1} = \text{DistanceFrom}_{i,j} + \text{Distance}_{i,j+1} + (1 - \text{IsMaintained}(p)_{i,j}) \cdot (\text{KmReading}(p)_{i,j} + \text{DistanceTo}_{i,j})
\]

and \( \text{KmReading}(p)_{i,j+1} \leq p.\text{limit} \) to satisfy the maintenance limits.

### 4.3 Simplifications

If there are similar trains then there is a maximum distance \( \maxD(H) \) that each train can drive during the scheduling horizon \( H \). Then for each train \( z_i \in Z \) we can decide in advance whether there are any maintenance tasks to be performed: If \( \text{KmReading}_{i,0} + \maxD(H) \leq s.\text{limit} \) holds for each non-periodic maintenance of type \( s \) and \( \text{KmReading}(p)_{i,0} + \maxD(H) \leq p.\text{limit} \) holds for each periodic maintenance of type \( p \) then the variables \( \text{IsMaintained}(w)_{i,j} \) as well as Conditions (6–8) can be omitted and Condition (9) can be simplified to:

\[
\text{OptDuration}_{i,j} = 0 \quad \text{and } \text{OptLocation}_{i,j} = \text{Destination}_{i,j}.
\]
4.4 Objective

The objective of the rolling stock planning with maintenance tasks is to maximise the number of performed trips while minimising the empty driven kilometres. Therefore we maximise the weighted sum

\[
2 \cdot D \sum_{i=0}^{m-1} q(i) \sum_{j=1}^{q(i)} \text{IsRegularTrip}_{i,j} - \sum_{i=0}^{m-1} \sum_{j=0}^{q(i)-1} (\text{DistanceTo}_{i,j} + \text{DistanceFrom}_{i,j}) \tag{19}
\]

where the factor \(2 \cdot D = 2 \cdot \max_{(b,c) \in B \times B} \text{driveDistance}(b,c)\) is twice the maximal distance of all empty drives assuming that \(\text{DistanceTo}_{i,j} + \text{DistanceFrom}_{i,j} < 2 \cdot D\) holds for any trip \(i\) and any slot \(j\)\[^8\]. This ensures that in the objective (19) the first addend dominates the second subtrahend such that the number of performed trips is the major objective and empty driven kilometres is the subordinate objective, as business requirements specify. The choice of the factor \(2 \cdot D\) is sufficient. It ensures that if a trip takes place, then the possibly negative contribution due to empty trips is less than \(2 \cdot D\), i.e. for a trip the amount to the objective is always positive. It is greater, the shorter any empty runs. If no trip takes place in the slot \(j\) of a train \(i\), then, by definition, \(\text{DistanceTo}_{i,j} = \text{DistanceFrom}_{i,j} = 0\) holds.

The number of maintenance tasks are not explicitly minimised because a maintenance task is only performed if necessary and minimising the empty driven kilometres implicitly covers the minimisation of empty driven kilometres to and from maintenance tasks, too.

5 Quantum-Computing-Based Solution

For a quantum optimisation approach, we developed a model based on quadratic unconstrained binary optimisation (QUBO) \[^{20,21}\]. The QUBO models have the advantage of being hardware independent, approachable both on the gate-based universal quantum computers with QAOA \[^{22}\] and on adiabatic quantum computers such as D-Wave machines \[^{23}\] with quantum annealing (QA).

5.1 QUBO Model

The heart of our QUBO model are the assignment variables, which link the available trains to the trips. In our model, each train is able to operate at most \(q\) trips, with \(q\) being defined beforehand. \[^9\] This leads to the binary decision variable \(X[i,f,z]\), where \(f \in F\) defines the trip, \(z \in Z\) the train and \(i \in \{0, \ldots, q - 1\}\).

\[^8\] otherwise use \(2 \cdot D + 1\) or \(2 \cdot (D + 1)\) instead.

\[^9\] This is a simplification w.r.t. the “train-specific” value \(q(i)\) introduced in Section 4.1.
corresponds to the time slot. If a trip \( f \) is operated by train \( z \) in time slot \( i \), \( X[i, f, z] \) then it is equal to 1, otherwise 0. We used \( q = 3 \) for all trains. In order to obtain a valid timetable, we have to enforce the following three constraints:

1. Each train operates at most one trip in each time slot.
2. Each trip is operated at most once and by a single train.
3. Successive trips operated by the same train do not overlap (there is sufficient time for a possible necessary empty trip and the required post-processing).

The next step is the inclusion of maintenance. To this end, we integrate maintenance actions \( w \in W \) in a new set of (extended) trips \( F'_{f,w} \). This new set of trips consists of duplicates of the original trip \( f \in F \) from the timetable, but for each trip \( f' \in F'_{f,w} \), a maintenance action \( w \) is conducted before the regular trip \( f \) takes place. In total, for each service station able to conduct a maintenance \( w \) an optional maintenance trip is created and added to \( F'_{f,w} \). Consequently, each trip \( f' \in F'_{f,w} \) starts from one of the possible maintenance stations, such that \( f'.maintenanceStation \in w.stations \) (abbr. \( mStation \)), \( f'.departureStation = f'.maintenanceStation \), and its duration exceeds that of \( f \) by the maintenance duration and the travel duration from the maintenance station to the start station of \( f \), \( f'.duration = f.travelDuration + w.duration + eDriveDuration(w.station, f.departureStation) \). Also the departure time is adapted accordingly, \( f'.departureTime = f.departureTime - w.duration - eDriveDuration(w.station, f.departureStation) \). The integration of the maintenance actions increases the number of decision variables to \( n \cdot m \cdot q \cdot \sum_{w \in W} |w.stations| \). This approach requires Constraint 2 to be modified, as at most one trip of \( F'_{f} \cup \{ f \} \) needs to be operated, where \( F'_{f} = \cup_{w \in W} F'_{f,w} \) contains all optional maintenance trips that are obtained from the regular trip \( f \in F \) for all maintenance \( w \in W \). The set \( F_{all} = \cup_{f \in F} (F'_{f} \cup \{ f \}) \) contains all trips, with and without maintenance, where \( F \) corresponds to the regular trips from the original timetable (cf. Sec. 3.1).

In the proposed QUBO model, the Constraints 1-3 are implemented as penalty terms in the objective function (Eq. (27)). If the constraint is fulfilled, the associated term \((c1-c3)\) evaluates to zero:

\[
c1 = \sum_{i=0}^{q-1} \sum_{z \in Z} \sum_{f_1, f_2 \in F_{all}} X[i, f_1, z] \cdot X[i, f_2, z],
\]

\[
c2 = \sum_{f \in F} \sum_{f_1, f_2 \in F_{f} \cup \{ f \}} \sum_{i_1, i_2 = 0}^{q-1} \sum_{z_1, z_2 \in Z} X[i_1, f_1, z_1] \cdot X[i_2, f_2, z_2],
\]

\[
c3 = \sum_{i_1, i_2 = 0}^{q-1} \sum_{z \in Z} \sum_{f_1, f_2 \in F_{all}} \sum_{\text{overlap}(f_1, f_2)} X[i_1, f_1, z] \cdot X[i_2, f_2, z],
\]

where the condition \( \text{overlap}(f_1, f_2) \) determines if two trips \( f_1, f_2 \) overlap:

\[
\text{overlap}(f_1, f_2) := (f_2.departureTime < f_1.arrivalTime)
\]
+ f1.postProcessing + edriveDuration(f1.arrivalStation, f2.departureStation)) .

The maintenance constraints are included heuristically in our model. We distinguish two cases: First, when a train \( z \) is close to the maintenance limit for a maintenance type \( w \), an immediate maintenance action is required, \( \text{immediateAction}(w, z) := w.\text{limit} < z.\text{initialKm} + 500 \), which should be preferably carried out before the first trip; 500 km is a data-based threshold chosen such that trains with less available range, with high probability, will not be able to perform regular trips. Second, if no immediate maintenance is necessary, maintenance actions \( w \) should be performed within the scheduling horizon depending on how close a train \( z \) is to the maintenance limit at the beginning of the optimisation. To be cost effective, we want to delay the maintenance as long as possible. However, conducting the maintenance just before reaching the maintenance limit strongly restricts the possible solutions of the problem. Therefore, we integrate the need of maintenance heuristically and slowly increase the maintenance penalty with increasing kilometre reading of the train. To this end, we calculate a heuristic weight \( \alpha(w, z) \) for the maintenance penalties discussed below and fix it to

\[
\alpha(w, z) := \frac{1}{e^{0.002 \cdot (w.\text{limit} - 1300 - z.\text{initialKm} + 500)} + 1},
\]

which increases continuously from 0 to 1 with increasing \( z.\text{initialKm} \). The constant 0.002 is a damping factor reducing the weight for small \( z.\text{initialKm} \), and 1300 has been chosen, because it corresponds to the average driven kilometres of a train being in operation for three days. This way, we increase the chances that the trains are not too close to the limit and have enough remaining range left for a potential next planning period. The cost function is then extended with the following terms:

\[
\begin{align*}
\text{cm1} &= \sum_{z \in Z} \sum_{w \in W} \text{immediateAction}(w, z) \alpha(w, z) \left( \sum_{f \in F} \sum_{f_1 \in F'_{f,w}} X[0, f_1, z] - 1 \right)^2, \\
\text{cm2} &= \sum_{z \in Z} \sum_{w \in W} \neg\text{immediateAction}(w, z) \alpha(w, z) \left( \sum_{i=0}^{q} \sum_{f \in F} \sum_{f_1 \in F'_{f,w}} X[i, f_1, z] - 1 \right)^2, \\
\text{cm3} &= \sum_{z \in Z} \sum_{w \in W} (1 - \alpha(w, z)) \sum_{i=0}^{q-1} \sum_{f \in F} \sum_{f_1 \in F'_{f,w}} X[i, f_1, z].
\end{align*}
\]

The first two terms promote that a maintenance action is performed. The first term forces maintenance to take place in the first time slot, while the second term does not distinguish between the time slots, but maintenance should be within the optimisation horizon. The third term, on the other hand, tries to prevent unnecessary maintenance. The significance of the two contradicting optimisation goals is controlled by \( \alpha(w, z) \).
It is important to note that while we steer indeed heuristically with the soft constraints $cm1$-$3$ the pressure towards maintenance (and simultaneously against too much maintenance), the choice whether to perform maintenance or not, its time, and its place, is still left to the optimisation for all trains, except for those very close to exceeding a maintenance interval - here we apply more pressure that the maintenance happens before the first trip.

As discussed in Section 4.4, our aim is not only to maximise the number of performed trips but also to minimise the empty driven kilometres. To this end, we sum up the empty driven kilometres by

$$\text{totalEmptyKM} = \sum_{z \in Z} \left[ \sum_{f \in F} ^{} \left( X[0,f,z] \cdot \text{edD}(z.\text{initialStation}, f.\text{departureStation}) \right) \right. $$

$$ + \sum_{f', z \in F} ^{} X[0,f',z] \cdot \left( \text{edD}(z.\text{initialStation}, f'.\text{maintenanceStation}) \right.$$

$$ + \text{edD}(f'.\text{maintenanceStation}, f.\text{departureStation})) \right) $$

$$ + \sum_{i=0} ^{q-2} \sum_{f_1 \neq f_2} ^{} \left( X[i,f_1,z] \cdot X[i+1,f_2,z] \cdot \text{edD}(f_1.\text{aStation}, f_2.\text{dStation}) \right.$$  

$$ + \sum_{f'_2 \in F_2} ^{} \left( X[i,f_1,z] \cdot X[i+1,f'_2,z] \cdot \left( \text{edD}(f_1.\text{aStation}, f'_2.\text{mStation}) \right.$$  

$$ + \text{edD}(f'_2.\text{mStation}, f_2.\text{dStation})) \right) \right]$$

Finally, our actual optimisation goal is to maximise the number of trips operated by each train, which is equivalent to minimising its negative:

$$\text{optimizationGoal} = -\sum_{z \in Z} \sum_{f \in F} ^{} X[0,f,z]$$

$$-\sum_{i=0} ^{q-2} \sum_{f_1 \neq f_2} ^{} \sum_{f_1 \neq f_2} ^{} X[i,f_1,z] \cdot X[i+1,f_2,z].$$

To force the optimiser to select a trip for each time slot, we reward successive trips in particular. Otherwise, the optimiser might select only a trip for every second time slot, to avoid the $\text{totalEmptyKM}$ penalty (e.g. $x[0,f_1,z] = 1$, $x[1,f,z] = 0 \forall f \in F_{\text{all}}$, and $x[2,f_2,z] = 1$).

The complete cost function is then given by

$$w_g^{\text{reward}} \cdot \text{optimizationGoal} + w_g^{\text{penalty}} \cdot (c1 + c2 + c3)$$

$$+ w_g^{\text{maintenance}} \cdot (cm1 + cm2 + cm3) + w_g^{\text{km}} \cdot \text{totalEmptyKM}. $$
We use the weights

\[ w_{\text{reward}} = 100, w_{\text{km}} = 100/(2D), w_{\text{penalty}} = 1000, w_{\text{maintenance}} = 100. \]

The weight \( w_{\text{km}} \) was chosen such that the ratio with \( w_{\text{reward}} \) is consistent with Equation (19), in order to synchronise the objective functions of the CP and QUBO approaches.

To reduce the problem complexity and the number of necessary qubits, we identify those variables which can be neglected already before the optimisation. To this end, we consider two cases. First, we check if a trip \( f \in F_{\text{all}} \) is feasible for a train \( z \) by evaluating the following inequality

\[
\text{departureTime} < \text{edriveDuration}(z.\text{initialStation}, f.\text{departureStation}) (28) + z.\text{earliestTime}
\]

If this is true, the train cannot operate the trip in time and we do not consider the variable for this combination, since \( X[0, f, z] = 0 \). Second, we neglect maintenance trips \( f' \in F'_{j,w} \) for a train \( z \), if the train is far away from the maintenance limit, i.e. \( z.\text{initialKm} + 3000 < w.\text{limit} \), where 3000 km is a threshold chosen based the distance a train can drive within the scheduling horizon, which is one day in our case. If this expression is true, the train will not reach the maintenance limit within the scheduling horizon and \( X[i, f', z] = 0 \) \( \forall \) \( i \in \{0, \ldots, q - 1\} \).

6 Empirical Examination of Both Approaches

To be able to properly compare our CP and QUBO solutions, we started from the same mathematical model of a real-world problem described in Section 3. We tried to keep both approaches as similar as possible, but not identically due to technical reasons. The differences often result from the technical limitations of current quantum hardware and they are necessary to keep the number of qubits low.

- We restrict the solution to three trips per train. This makes almost no difference on our real data and our 24h planning horizon, because considering the trip length and the starting times, it was barely possible for a train to operate more than three trips during the optimisation horizon. The CP solution produce for very few trains 4 trips assignments before limiting the train specific upper bound to 3. This is enforced for the CP as well as the QUBO model.
- The maintenance constraints are implemented as a soft constraints for QUBO vs. hard constraint for CP. Thus for QUBO some of those might be violated, which is addressed in postprocessing, possibly reducing the number of covered trips. Both the number before and after postprocessing are shown.
For empirical examination of both approaches we have generated different-sized data sets considering a subset of the German rail network with trips between the major cities Berlin, Frankfurt, Hamburg, Munich, and Cologne, shown in Figure 1. The data sets are based on a real train schedule for one day from Deutsche Bahn. We then simplified the timetable under the following aspects: first, we only consider direct trips between these cities and ignore intermediate stations. This simplification yields a timetable with 284 trips for one day. Furthermore, we used standardised distances and travel times between the cities, ignoring the variations that depend on the actual paths. We consider two maintenance types, one periodic and one non-periodic maintenance, using realistic intervals. Finally, we obtain different-sized subsets of our data set by varying the total number of trips and trains (cf. Table 1).

We then employ our two approaches to search for good or even best trip allocations w.r.t. to the defined objective functions, cf. (19) and (27), respectively. We use a Constraint Programming library \texttt{firstCS} \cite{24} to implement and solve our CP approach. For this purpose we implemented straightforward filtering methods for the extended element constraints and the logical implications. Further, we applied monotonous branch and bound (B&B) using a depth-first tree search with a first-fail heuristic: Trains with smaller numbers of potential trips or the same number of potential trips but with a smaller numbers of potential slots are considered first. Then for each train the slots are labelled with trips: first slot first, last slot last. The trips are chosen according to their index: greatest index first, smallest index last such that 'regular' trips are considered first. For the first slot the 'virtual' initial trip is fixed and for the other slots the 'real' trips are chosen first. Subsequently the maintenance tasks are labelled between trips starting with the “no maintenance” index \((-1)\) first to avoid maintenance tasks if not required. Tree search and constraint processing, i.e. mainly filtering, was performed on an Intel\(^{(R)}\) Xeon\(^{(R)}\) CPU E5-2695 v4 \(\@\) 2.10 GHz (in single core mode) running Ubuntu 20.04.2 and using OpenJDK 1.8.0, which is the basis of the implemented CP approach. In particular, the first-fail heuristic only influences the order of variable selection for assignment during search. There are no branches cut in the search tree.

The QUBO models are evaluated on the D-Wave hybrid (classical+quantum) cloud system named LEAP, which integrates the 5760 qubit machine “Advantage”, as well as on a classical computer employing tabu search. The classical computer used has 512 GB of RAM and 36 cores (72 with hyper-threading).

Table 1 shows the results of the empirical examination of both approaches on the generated data sets. For better alignment of the CP and QUBO approaches...
Table 1. Results of the empirical examination.

| data set     | # qubits | # trips alloc. | # trips used | empty rides [km] | method         | run-time       |
|--------------|----------|----------------|--------------|------------------|----------------|----------------|
| real-small   | 4662     | 52(51)         | 34           | 3769             | CP first       | < 1 sec        |
| 70 trips     |          |                |              |                  | CP improved    | 1 sec          |
| 38 trains    |          |                |              |                  | LEAP           | 1+2 min        |
| real-50%     | 20256    | 121(120)       | 70           | 8694             | CP first       | 2 sec          |
| 141 trips    |          |                |              |                  | LEAP           | 8+60 min       |
| 75 trains    |          |                |              |                  | tabu search    | 8+28 min       |
| real-75%     | 47358    | 189(177)       | 106          | 8310             | CP first       | 7 sec          |
| 212 trips    |          |                |              |                  | CP improved    | 10 sec         |
| 112 trains   |          |                |              |                  | LEAP           | 0.5+1 hours    |
| real-100%    | 87069    | 252(240)       | 139          | 14841            | CP first       | 11 sec         |
| 284 trips    |          |                |              |                  | CP improved    | 17 sec         |
| 150 trains   |          |                |              |                  | LEAP           | 1.25+2 hours   |
|              |          |                |              |                  | tabu search    | 1.25+5.5 hours |

For the QUBO approaches, i.e. LEAP and tabu search, two numbers for the allocated trips are given. The first one is the number provided by the solution, the second one, in parenthesis, is lower and represents the number of trips after removing the trips which violate maintenance constraints. This can also be seen in Figure 2, where an extract of a QUBO solution is shown. The time for computing the solution is split into pre-processing and the actual search time. Here, the main pre-processing task is the calculation of the QUBO matrix, which can be very time-consuming for a large number of qubits, but still polynomial. In contrast to other quantum computing cloud providers such as IBM, in the case of D-Wave the time spent queuing was negligible, the time we reported almost equals the processing time.
Fig. 2. Extract from a QUBO-based solution, showing regular trips (black), maintenance (blue), empty travel (green), unavailability (yellow) and conflict with the maintenance requirements (red).

7 Discussion

For CP, in all cases the number of allocated trips in the first solution (found within seconds) are rather good, i.e. it is not significantly increased while searching for better solutions. Spending hours searching for better solutions only results in either one or two additional allocated trips or in moderate reductions (2% resp. 25%) of empty driven kilometres. For the “real-small” subset only one improved solution was found rather quickly after 1 second, however ongoing search for several days failed to find a better solution. For each examined data set, it was impossible to prove with the CP approach that one of the found solutions is optimal. We used a CP solver for the examinations that implements state-of-the-art filtering algorithms and techniques of modern CP solvers (cf. [25]) because there we were able to implement special filtering algorithms for the logical implications and the extended element constraints used in the CP model. This CP solver was sufficient to show that a CP approach is currently able to outperform a Quantum / QUBO approach.

The numerical results presented in Table 1 are of similar quality, as the QUBO solutions cover only up to 3 % fewer trips than the CP solutions (up to 7 % fewer trips when only the LEAP solutions are considered), while the empty driven kilometres are of the same order of magnitude. This way the QUBO approach is able to produce qualitatively similar results as the CP approach. Of course, the run times for both approaches differ widely. We noticed that the empty driven kilometres have the lowest value for classical tabu search.

The run times for the QUBO approaches are much longer than for the CP cases. However, the LEAP cloud hybrid solver from D-Wave shows in the online dashboard that the amount of time spent on the QPU never surpassed three seconds, even when the total time spent by LEAP was up to 2 hours.

8 Conclusion and Future Work

We found once again that problem-specific constraints and heuristics are required to being able to handle realistic problem sizes. We were thus able to optimise the high-speed Intercity-Express (ICE) railway traffic in Germany that goes through
5 major cities. Surprisingly, the QUBO-based method was able to handle up to almost 100,000 qubits. In general, the CP solution outperformed the QUBO solution except for the real small subset. Considering the current stage of the quantum annealer and of the hybrid solver from D-Wave, the results obtained on classical computers and using the quantum annealer are fairly comparable. We plan to extend the amount of cities covered and handle more details of the real problem (e.g. the intermediary stations of the trips) towards a practice-ready prototype. For a stable operation, it is also advantageous to optimise the time slots when trains are not in operation, which, ideally, are as long as possible, instead of scheduling many short break time slots. This increases the chance that one train can replace another cancelled train. Hopefully, in the near future, the performance of the quantum annealers will increase beyond the capacity of classical computing.

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