Research Article

Numerical Scheme for Finding Roots of Interval-Valued Fuzzy Nonlinear Equation with Application in Optimization

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In this research article, we propose efficient numerical iterative methods for estimating roots of interval-valued trapezoidal fuzzy nonlinear equations. Convergence analysis proves that the order of convergence of numerical schemes is 3. Some real-life applications are considered from optimization as numerical test problems which contain interval-valued trapezoidal fuzzy quantities in parametric form. Numerical illustrations are given to show the dominance efficiency of the newly constructed iterative schemes as compared to existing methods in literature.

1. Introduction

One of the ancient problems of science and engineering in general and in mathematics is to approximate roots of a nonlinear equation. The nonlinear equations play a major role in the field of engineering, mathematics, physics, chemistry, economics, medicine, and finance, and in optimization. Many times the particular realization of such type of nonlinear problems involves imprecise and nonprobabilistic uncertainties in the parameter, where the approximations are known due to expert knowledge or due to some experimental data. Due to these reasons, several real-world applications contain vagueness and uncertainties. Therefore, in most of real-world problems, the parameter involved in the system or variables of the nonlinear functions are presented by a fuzzy number or interval-valued trapezoidal fuzzy number. The concept of fuzzy numbers and arithmetic operation with fuzzy numbers were first introduced and investigated in [1–8]. Hence, it is necessary to approximate the roots of fuzzy nonlinear equation.

\[ F(r) = c. \]  

(1)

The standard analytical technique like the Buckley and Qu method [9–12] is not suitable for solving the equations like

\[ ar^6 + br^4 - cr^3 + dr - e = f, \quad r + \cos (r) = g, \quad r \ln (r) + e' \]

\[ \frac{1}{1 + r^2} + \tan (r) = h, \]  

(2)

where \( a, b, c, d, e, f, g, \) and \( h \) are fuzzy numbers and \( r \) is a fuzzy variable.
We therefore look towards numerical iterative schemes which approximate the roots of fuzzy nonlinear equations. To approximate roots of fuzzy nonlinear equations, Abbasbandy and Asady [13] used Newton’s method, Allahviranloo and Asari [14] used the Newton-Raphson method, Mosleh [15] used the Adomian decomposition method, and Ibrahim et al. give the Levenberg-Marquest method (see also [16–23]).

This research article is aimed at proposing efficient higher order iterative method as compared to well-known classical method, such as the Newton-Raphson method. Numerical test results, CPU time, and log of residual show the dominance efficiency of our newly constructed method over the classical Newton’s method.

This paper is organized in five sections. In Section 2, we recall some fundamental results of interval-valued trapezoidal fuzzy numbers. In Section 3, we propose numerical iterative scheme for approximating roots of interval-valued trapezoidal fuzzy nonlinear equations and its convergence analysis. In Section 4, we illustrate some real-world applications from optimization as numerical test examples to show the performance and efficiency of the constructed method and conclusions in the last section. Section 5 is a conclusion section.

2. Preliminaries

Definition 1. A fuzzy number is a fuzzy set like $r : \mathbb{R} \rightarrow I = [0, 1]$ which satisfies [24–27].

(1) $r$ is upper semicontinuous

(2) $r(a) = 0$ outside some interval $[a_1, a_2]$

(3) There are real numbers $b_1, b_2$ such that $a_1 \leq b_1 \leq a_2 \leq b_2$ and

(i) $r(a)$ is monotonic increasing on $[a_1, b_1]$

(ii) $r(a)$ is monotonic decreasing on $[b_2, a_2]$

(iii) $r(a) = 1$, for $b_1 \leq a \leq b_2$

We denote by $E$, the set of all fuzzy numbers. An equivalent parametric form is also given in [19] as follows.

Definition 2 [28]. A fuzzy number $r$ in parametric form is a pair $(r_L, r_U)$ of function $r_L(\tau), r_U(\tau), 0 \leq \tau \leq 1$, which satisfies the following requirements:

(1) $r_L(\tau)$ is a bounded monotonic increasing left continuous function

(2) $r_U(\tau)$ is a bounded monotonic decreasing left continuous function

(3) $r_L(\tau) \leq r_U(\tau), 0 \leq \tau \leq 1$

A popular fuzzy number is the generalized interval-valued trapezoidal fuzzy number $A$, denoted by $A = (a_1, a_2, a_3, a_4; \tilde{\omega})$, $0 < \tilde{\omega} < 1$, a fuzzy number with membership function as follows:

$$A(\tau) = \begin{cases} \tilde{\omega} \frac{r - a_1}{a_2 - a_1} & \text{if } a_1 < r < a_2, \\ \tilde{\omega} & \text{if } a_2 \leq r \leq a_3, \\ \tilde{\omega} \frac{a_4 - r}{a_4 - a_3} & \text{if } a_3 \leq r < a_4, \\ 0 & \text{otherwise.} \end{cases}$$

Assume $F_{TN}(\tilde{\omega})$ be the family of all $\tilde{\omega}$-trapezoidal fuzzy number, i.e.,

$$F_{TN}(\tilde{\omega}) = \left\{ A = (a_1, a_2, a_3, a_4; \tilde{\omega}), a_1 \leq a_2 \leq a_3 \leq a_4; 0 < \tilde{\omega} < 1 \right\}.$$
Analytical solution of example 1

Analytical solution of example 2

Analytical solution of example 3

Figure 2: Analytical solution of Examples 1–3.

Figure 3: Computational order of convergence.
Figure 4: Computational time in seconds.

Figure 5: Initial guessed values, analytical, and numerical approximate solution.
Figure 6: Initial guessed values, analytical, and numerical approximate solution.

Figure 7: Initial guessed values, analytical, and numerical approximate solution.
Figure 8: Error graph of iterative methods MM and NN.
The family of all interval-valued trapezoidal fuzzy number $\cal{A}$ level $(w, \lambda, w, \lambda)$ — interval-valued trapezoidal fuzzy number $A$, denoted by

$$A = [A^L, A^U] = (a^L_1, a^L_2, a^L_3, a^L_4; w, \lambda, w, \lambda),$$

is an interval-valued fuzzy number on set $R$ with

$$A^L(r) = \begin{cases} \frac{w - \lambda}{\lambda^2 - \lambda} a^L_1 - \frac{\lambda}{\lambda - 1} a^L_2 & \text{if } a^L_1 < r < a^L_2, \\ \frac{w - \lambda}{\lambda^2 - \lambda} a^L_3 - \frac{\lambda}{\lambda - 1} a^L_4 & \text{if } a^L_3 < r < a^L_4, \\ 0 & \text{otherwise}, \end{cases}$$

$$A^U(r) = \begin{cases} \frac{w - \lambda}{\lambda^2 - \lambda} a^U_2 - \frac{\lambda}{\lambda - 1} a^U_1 & \text{if } a^U_1 < r < a^U_2, \\ \frac{w - \lambda}{\lambda^2 - \lambda} a^U_4 - \frac{\lambda}{\lambda - 1} a^U_3 & \text{if } a^U_3 < r < a^U_4, \\ 0 & \text{otherwise}, \end{cases}$$

where $a^L_1 \leq a^L_2 \leq a^L_3 \leq a^L_4$, $a^L_2 \leq a^L_3 \leq a^L_4$, $a^U_1 \leq a^U_2 \leq a^U_3 \leq a^U_4$, $0 \leq w, \lambda \leq w, \lambda \leq 1$, $a^L_1 \leq a^L_2$, and $a^L_2 \leq a^L_4$. This interval-valued trapezoidal fuzzy number is shown in Figure 1. Moreover, $A^L(r) \leq A^U(r)$, which means the grade of membership $r \in A = [A^L(r), A^U(r)]$, and the latest and greatest grade of membership at $r$ are $A^L(r)$ and $A^U(r)$, respectively. We therefore denote the family of all interval-valued trapezoidal fuzzy number

**Algorithm 1: (MM method).**

**Step 4.** Use MM to compute next iteration

$$j_{k+1} = j_k - (\nabla F_j)^{-1} J,$$

where $Z = (4J_s - 2J_s^-1) * (J_s - J)$.

**Step 5.** For given $\varepsilon > 0$, if (i) $e_\mu = ||F(r) - \mu|| < \varepsilon$ and (ii) $e_\nu = ||r_{n+1} - r_n|| < \varepsilon$, then stop.

**Step 6.** Set $k = k + 1$ and go to step 1.

**Definition 3 [29].** Let $A^L \in F_{TN}(w, \lambda^L)$ and $A^U \in F_{TN}(w, \lambda^U)$ be level $(w, \lambda^L, w, \lambda^U)$ — interval-valued trapezoidal fuzzy number $A$, denoted by

$$A = [A^L, A^U] = (a^L_1, a^L_2, a^L_3, a^L_4; w, \lambda^L, w, \lambda^U),$$

is an interval-valued fuzzy number on set $R$ with

$$A^L(r) = \begin{cases} \frac{w - \lambda}{\lambda^2 - \lambda} a^L_1 - \frac{\lambda}{\lambda - 1} a^L_2 & \text{if } a^L_1 < r < a^L_2, \\ \frac{w - \lambda}{\lambda^2 - \lambda} a^L_3 - \frac{\lambda}{\lambda - 1} a^L_4 & \text{if } a^L_3 < r < a^L_4, \\ 0 & \text{otherwise}, \end{cases}$$

$$A^U(r) = \begin{cases} \frac{w - \lambda}{\lambda^2 - \lambda} a^U_2 - \frac{\lambda}{\lambda - 1} a^U_1 & \text{if } a^U_1 < r < a^U_2, \\ \frac{w - \lambda}{\lambda^2 - \lambda} a^U_4 - \frac{\lambda}{\lambda - 1} a^U_3 & \text{if } a^U_3 < r < a^U_4, \\ 0 & \text{otherwise}, \end{cases}$$

where $a^L_1 \leq a^L_2 \leq a^L_3 \leq a^L_4$, $a^L_2 \leq a^L_3 \leq a^L_4$, $a^L_1 \leq a^L_2$, and $a^L_2 \leq a^L_4$. This interval-valued trapezoidal fuzzy number is shown in Figure 1. Moreover, $A^L(r) \leq A^U(r)$, which means the grade of membership $r \in A = [A^L(r), A^U(r)]$, and the latest and greatest grade of membership at $r$ are $A^L(r)$ and $A^U(r)$, respectively. We therefore denote the family of all interval-valued trapezoidal fuzzy number

**Algorithm 1: (MM method).**

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**Step 5.** For given $\varepsilon > 0$, if (i) $e_\mu = ||F(r) - \mu|| < \varepsilon$ and (ii) $e_\nu = ||r_{n+1} - r_n|| < \varepsilon$, then stop.

**Step 6.** Set $k = k + 1$ and go to step 1.
be equal iff \( A = B \), i.e., \( a_i^L = b_i^L \) and \( a_i^U = b_i^U \) for all \( i = 1, 2, 3, 4 \).

**Definition 6.** [30]. Extend addition, scalar multiplication, and extend multiplication in \((\omega^L, \omega^U)\) interval-valued trapezoidal fuzzy number are defined as if \( A = (a_1^L, a_1^U; a_2^L, a_2^U; a_3^L, a_3^U; a_4^L, a_4^U; w^L, w^U)\) and \( B = (b_1^L, b_1^U; b_2^L, b_2^U; b_3^L, b_3^U; b_4^L, b_4^U; w^L, w^U)\) then, \( A \oplus B = \left( a_1^L + b_1^L, a_1^U + b_1^U; a_2^L + b_2^L, a_2^U + b_2^U; a_3^L + b_3^L, a_3^U + b_3^U; a_4^L + b_4^L, a_4^U + b_4^U; w^L \right) \),

\[
A \odot B = \left\{ \begin{array}{ll}
(k_{a_1^L}, k_{a_1^U} \cdot [k_{a_2^L}, k_{a_2^U} \cdot [k_{a_3^L}, k_{a_3^U} \cdot [k_{a_4^L}, k_{a_4^U} \cdot \omega^L, w^U]) \), & k > 0, \\
(k_{a_1^L}, k_{a_1^U} \cdot [k_{a_2^L}, k_{a_2^U} \cdot [k_{a_3^L}, k_{a_3^U} \cdot [k_{a_4^L}, k_{a_4^U} \cdot \omega^L, w^U]) \), & k < 0, \\
((0, 0, 0, 0; \omega^L, 0, 0, 0, 0; w^U)) \), & k = 0,
\end{array} \right.
\]

**Table 2: Analytical solution for Example 1.**

| \( r \) | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( r_1^U \) | 0.9172 | 0.9268 | 0.9368 | 0.9473 | 0.9581 | 0.9694 | 0.9812 | 0.9936 | 1.0065 | 1.0200 | 1.0342 |
| \( r_1^L \) | 1.0000 | 1.0096 | 1.0197 | 1.0304 | 1.0418 | 1.0538 | 1.0666 | 1.0802 | 1.0947 | 1.1102 | 1.1269 |
| \( r_2^U \) | 1.2749 | 1.2394 | 1.2080 | 1.1800 | 1.1547 | 1.1318 | 1.1109 | 1.0918 | 1.0743 | 1.0581 | 1.0430 |
| \( r_2^L \) | 1.4371 | 1.3974 | 1.3620 | 1.3301 | 1.3012 | 1.2749 | 1.2507 | 1.2285 | 1.2080 | 1.1890 | 1.1712 |

**Table 3**

| \( r \) | \( \| r_{n+1} - r_n \| \) | \( \| F(r_n) \| \) | \( \| r_{n+1} - r_n \| \) | \( \| F(r_n) \| \) |
|-----|-----|-----|-----|-----|
| 0.0 | 4.0e-49 | 4.0e-33 | 1.7e-13 | 1.7e-11 |
| 0.1 | 5.6e-47 | 5.6e-32 | 3.4e-13 | 1.2e-11 |
| 0.2 | 3.1e-43 | 8.8e-33 | 6.1e-14 | 3.6e-11 |
| 0.3 | 5.4e-48 | 6.1e-36 | 5.1e-13 | 7.1e-11 |
| 0.4 | 6.1e-49 | 7.9e-35 | 4.4e-14 | 5.6e-12 |
| 0.5 | 3.7e-48 | 4.0e-33 | 4.3e-13 | 3.4e-11 |
| 0.6 | 1.6e-48 | 8.2e-33 | 1.7e-13 | 4.6e-11 |
| 0.7 | 3.6e-49 | 4.3e-33 | 3.7e-13 | 6.7e-11 |
| 0.8 | 1.2e-49 | 3.6e-34 | 7.1e-14 | 5.6e-11 |
| 0.9 | 7.5e-46 | 8.9e-34 | 8.7e-14 | 1.2e-11 |
| 1.0 | 8.1e-49 | 6.1e-35 | 9.1e-14 | 5.1e-11 |

**3. Construction of Iterative Scheme (MM)**

In order to approximate the roots of interval-valued trapezoidal fuzzy nonlinear equation \( F(r) = c \), we propose the following two-step iterative scheme as follows:

\[
\begin{align*}
F_1^L (r_1^L, r_1^U; r_2^L, r_2^U; \tau) &= c_1^L (\tau), \\
F_1^U (r_1^L, r_1^U; r_2^L, r_2^U; \tau) &= c_1^U (\tau), \\
F_1^U (r_2^L, r_2^U; r_1^L, r_1^U; \tau) &= c_2^U (\tau), \\
F_1^L (r_2^L, r_2^U; r_1^L, r_1^U; \tau) &= c_2^L (\tau).
\end{align*}
\]

Suppose that \( r = (a_1^L, a_1^U; a_2^L, a_2^U; a_3^L, a_3^U) \) is the solution of above system and \( \hat{r}_0 = (r_{0L}, r_{0U}, r_{0L}, r_{0U}) \) is approximate solutions of the system, \( t \) denote the alpha-cut parameter; then,

\[
\begin{align*}
\alpha_1^L (\tau) &= r_{0L}^L (\tau) + h_1 (\tau), \\
\alpha_1^U (\tau) &= r_{0L}^U (\tau) + k_1 (\tau), \\
\alpha_2^L (\tau) &= r_{0L}^L (\tau) + h_2 (\tau), \\
\alpha_2^U (\tau) &= r_{0L}^U (\tau) + k_2 (\tau).
\end{align*}
\]

By using Taylor’s series of \( F_1^L, F_1^U, F_2^L, F_2^U \) about \( (r_{0L}, r_{0U}, r_{0L}, r_{0U}) \), then we have the following:
If \((r_{10}^{L}(\tau), r_{10}^{U}(\tau), r_{10}^{L}(\tau), r_{10}^{U}(\tau))\) are close to \((a^{L}_{1}(\tau), a^{L}_{1}(\tau), a^{L}_{1}(\tau), a^{L}_{1}(\tau))\), then \(h_{1}(\tau), k_{1}(\tau), h_{2}(\tau), k_{2}(\tau)\) are small enough. Assume all partial derivatives of \(h_{1}(\tau), k_{1}(\tau), h_{2}(\tau), k_{2}(\tau)\) exist and bounded; then, we have the following:

\[
\begin{align*}
F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + h_{1}F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + h_{2}F_{L}^{U}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + k_{1}F_{U}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + k_{2}F_{U}^{U}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) &= c_{1}(\tau), \\
F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + h_{1}F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + h_{2}F_{L}^{U}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + k_{1}F_{U}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + k_{2}F_{U}^{U}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) &= c_{2}(\tau), \\
F_{L}^{U}(r_{10}^{L}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + h_{1}F_{U}^{L}(r_{10}^{L}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + h_{2}F_{U}^{L}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + k_{1}F_{U}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + k_{2}F_{U}^{U}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) &= c_{3}(\tau), \\
F_{L}^{U}(r_{10}^{L}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + h_{1}F_{U}^{L}(r_{10}^{L}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + h_{2}F_{U}^{L}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) + k_{1}F_{U}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) + k_{2}F_{U}^{U}(r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, r_{10}^{U}, \tau) &= c_{4}(\tau).
\end{align*}
\]

Since \(h_{1}(\tau), k_{1}(\tau), h_{2}(\tau), k_{2}(\tau)\) are unknown quantities, they are obtained by solving the following equations:

\[
\begin{bmatrix}
J_{*}(r_{10}^{L}, r_{10}^{L}, r_{10}^{U}, r_{10}^{U}, \tau) \cdot \begin{bmatrix}
h_{1}(\tau) \\
k_{1}(\tau) \\
h_{2}(\tau) \\
k_{2}(\tau)
\end{bmatrix} = \begin{bmatrix}
c_{1}(\tau) - F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) \\
c_{2}(\tau) - F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) \\
c_{3}(\tau) - F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) \\
c_{4}(\tau) - F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau)
\end{bmatrix},
\end{align*}
\]

where

\[
J_{*} = \begin{bmatrix}
F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) \\
F_{L}^{L}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) \\
F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) \\
F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau) & F_{L}^{U}(r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, r_{10}^{L}, \tau)
\end{bmatrix}.
\]

\[
\begin{bmatrix}
y_{0}(\tau) \\
y_{0}(\tau) \\
y_{0}(\tau) \\
y_{0}(\tau)
\end{bmatrix} = \begin{bmatrix}
r_{10}^{L}(\tau) \\
r_{10}^{U}(\tau) \\
r_{10}^{L}(\tau) \\
r_{10}^{U}(\tau)
\end{bmatrix} + \begin{bmatrix}
h_{1}(\tau) \\
k_{1}(\tau) \\
h_{2}(\tau) \\
k_{2}(\tau)
\end{bmatrix}.
\]
found by using recursive scheme as follows:

\[
J_{\ast} = J_{\ast} (r^L_{n,m}, r^U_{n,m}, r^L_{n,m}, \tau), \quad J_{\ast} = J_{\ast} (y^L_{n,m}, y^U_{n,m}, y^U_{n,m}, \tau),
\]

and the next approximation for \( r^L_{n}(\tau), r^U_{n}(\tau), r^L_{n}(\tau), r^U_{n}(\tau) \) is found by using recursive scheme as follows:

\[
Z = (4J_{\ast} (y^L_{0,n}, y^U_{0,n}, y^U_{0,n}, \tau) - 2J_{\ast} (r^L_{0,n}, r^U_{0,n}, r^U_{0,n}, \tau))^{-1} \ast (J_{\ast} (y^L_{0,n}, y^U_{0,n}, y^U_{0,n}, \tau) - J_{\ast} (r^L_{0,n}, r^U_{0,n}, r^U_{0,n}, \tau)),
\]

\[
J_{\ast} = J_{\ast} (r^L_{n,m}, r^U_{n,m}, r^L_{n,m}, \tau)
\]

\[
\begin{bmatrix}
  r^L_{n+1}(\tau) \\
  r^U_{n+1}(\tau) \\
  r^L_{n}(\tau) \\
  r^U_{n}(\tau)
\end{bmatrix} =
\begin{bmatrix}
  y^L_{n}(\tau) \\
  y^U_{n}(\tau) \\
  y^U_{n}(\tau) \\
  y^U_{n}(\tau)
\end{bmatrix} + Z \ast
\begin{bmatrix}
  h_{1n}(\tau) \\
  k_{1n}(\tau) \\
  h_{2n}(\tau) \\
  k_{2n}(\tau)
\end{bmatrix},
\]

\[
Z = 4J_{\ast} (y^L_{1,n}, y^U_{1,n}, y^U_{1,n}, \tau) - 2J_{\ast} (r^L_{1,n}, r^U_{1,n}, r^U_{1,n}, \tau))^{-1} \ast (J_{\ast} (y^L_{1,n}, y^U_{1,n}, y^U_{1,n}, \tau) - J_{\ast} (r^L_{1,n}, r^U_{1,n}, r^U_{1,n}, \tau)),
\]

\[
J_{\ast} (r^L_{n,m}, r^U_{n,m}, r^L_{n,m}, \tau) =
\begin{bmatrix}
  h_{1n}(\tau) \\
  k_{1n}(\tau) \\
  h_{2n}(\tau) \\
  k_{2n}(\tau)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  r^L_{n}(\tau) \\
  r^U_{n}(\tau) \\
  r^L_{n}(\tau) \\
  r^U_{n}(\tau)
\end{bmatrix} =
\begin{bmatrix}
  y^L_{n}(\tau) \\
  y^U_{n}(\tau) \\
  y^U_{n}(\tau) \\
  y^U_{n}(\tau)
\end{bmatrix} + Z \ast
\begin{bmatrix}
  h_{1n}(\tau) \\
  k_{1n}(\tau) \\
  h_{2n}(\tau) \\
  k_{2n}(\tau)
\end{bmatrix},
\]

\[
J_{\ast} (r^L_{n,m}, r^U_{n,m}, r^L_{n,m}, \tau) =
\begin{bmatrix}
  h_{1n}(\tau) \\
  k_{1n}(\tau) \\
  h_{2n}(\tau) \\
  k_{2n}(\tau)
\end{bmatrix}
\]
Lemma 10. Let $F(\alpha^L, \alpha^U) = (c^L, c^U, \xi^L, \xi^U)$, and if the sequence of \{(r^L, r^U, \xi^L, \xi^U)\}_{n=0}^\infty converges to $(\alpha^L, \alpha^U, \xi^L, \xi^U)$ according to Newton's method, then

$$\lim_{n \to \infty} P_n = 0,$$

where

$$P_n = \sup_{0 \leq r \leq 1} \max \{ h_{1n}(r), k_{1n}(r), h_{2n}(r), k_{2n}(r) \}.$$  

Proof. It is obvious, because for all $\forall r \in [0, 1]$ in convergent case,

$$\lim_{n \to \infty} h_{1n}(r) = \lim_{n \to \infty} k_{1n}(r) = \lim_{n \to \infty} h_{2n}(r) = \lim_{n \to \infty} k_{2n}(r) = 0.$$  

(21)

Hence, it is proved. □

Finally, it is shown that under certain condition, the MM method for fuzzy equation $F(r) = 0$ is cubic convergent. In compact form for $F(r) = 0$, the MM method can be written as follows:

$$\begin{align*}
y_{n+1}(r) &= r_n(r) - \left(F'(r_n(r))\right)^{-1}F(r_n(r)), \\
\forall r \in [0, 1],
\end{align*}$$

(22)

where

$$Z = (4J_{🔒} - 2I_{🔒})^{-1} (J_{🔒} - J_{🔒}).$$

(23)

Theorem 11. Let $F : H \subseteq R^n \to R^n$ be $u$-times Fréchet differential function on a convex set $H$ containing the root $\alpha$ of $F(r) = 0$; then, the MM method has cubic convergence and satisfies the following error equation:

$$e_{n+1} = 2 \left( (A_2)^2 - 4(A_2^2) \right) (e_n)^3 + ||O(e_n)||,$$

(24)

where $A_n = 1/2l \cdot F(r_n(r)) / F'(r_n(r), n = 2, 3, \ldots$. 

Proof. Let $e_n = r_n - \alpha$ and $e_{n+1} = r_{n+1} - \alpha$, then by Taylor series of $F(r_n(r))$ in the neighborhood of $\alpha$ if $J_{ Locke}$ and $J_{ Locke}$ exist. Then,

$$F(r, x) = F(r_n, r) + F'(r_n, r)(x - r_n) + \frac{1}{2} F''(r_n, r)(x - r_n)^2 + \cdots$$

(25)

and $F(r, \alpha) = 0$.

$$F(r_n, \tau) = F'(r, \tau) (e_n + A_2(e_n)^2 + A_3(e_n)^3) + ||O(e_n)||.$$  

(26)
This gives

\[
\left( F'(r_n, \tau) \right)^{-1} F(r_n, \tau) = e_n + A_2(e_n)^2 + (2A_2 + 2A_3)(e_n)^3 + \ldots ,
\]

\[ y_n - a = A_2(e_n)^2 + (-2A_2 + 2A_3)(e_n)^3 + \ldots \]  \hspace{1cm} (27)

Expanding \( F'(y_n, \tau) \) about \( a \), we have the following:

\[
F'(y_n, \tau) = 1 + 2(A_2)^2(e_n)^2 + 2(-2(A_2)^2 + 2A_3)(e_n)^3 + \ldots ,
\]

\[ Z \cdot \left( F'(r_n, \tau) \right)^{-1} F(r_n, \tau) = -A_2(e_n)^2 + \left( 4(A_2)^2 - \frac{3}{2}A_3 - 4(A_2)^2 \right)(e_n)^3 + \ldots ,
\]

\[ r_{n+1} - a = y_n - a - A_2(e_n)^2 + \left( 4(A_2)^2 - \frac{3}{2}A_3 - 4(A_2)^2 \right)(e_n)^3 + \ldots , \]

\[ e_{n+1} = 2 \cdot \left( (A_2)^2 - \frac{1}{2}A_3 - 4(A_2)^2 \right)(e_n)^3 + \| O(e_n)^3 \| . \]  \hspace{1cm} (28)

Hence, the theorem is proved.

#### 4. Numerical Applications

Here, we present examples to illustrate the performance and efficiency of MM and NN methods for approximating roots of interval-valued trapezoidal fuzzy nonlinear equations. Examples 1–3 are considered from Buckley and Qu [9]. All the computations are performed using CAS Maple 18 with 64 digits floating point arithmetic with stopping criteria as follows. Analytical, numerical approximate solutions, computational order of convergence [32], computational time in second, and residual error graph of interval-valued trapezoidal fuzzy nonlinear equation used in Examples 1–3 are shown in Figures 2–8(a) and 8(c), respectively. Algorithm 1 shows the implementation of MM iterative method on CAS Maple 18.

\[ (i) \, e_n = \| F(r_n, \tau) \| < \epsilon \quad (ii) \, e_n = \| r_{n+1}(\tau) - r_n(\tau) \| < \epsilon \]  \hspace{1cm} (31)

where \( e_n \) represents the absolute error. We take \( \epsilon = 10^{-15} \).

In Figure 2, left shows analytical solution of interval-valued trapezoidal fuzzy nonlinear equation used in Example
where \( B \) or \( r \) lies between an interval-valued trapezoidal fuzzy number whose support lies between \([0, 1]\). After a year, the amount in the account will be

\[
(A_1 - S_1) + A_1 r(t). \tag{32}
\]

At the end of second year, total amount left is

\[
(A_1 - S_1) + A_1 * r(t) + ((A_1 - S_1) - A_1 * r(t))r(t), \tag{33}
\]

or

\[
A_1(r(t)^2) + B * r(t) + D, \tag{34}
\]

where \( B = 2A_1 - S_1 \) and \( D = A_1 - S_1 \). Therefore, we have to solve

\[
A_1 * (r(t))^2 + B * r(t) + D = S_2, \tag{35}
\]

or

\[
A_1 * (r(t))^2 + B * r(t) = C, \tag{36}
\]

where \( C = S_2 - D \). For fuzzy interest rate substituting values of \( A_1, B, \) and \( C \) in above equation, we have the following:

\[
\begin{align*}
&\left(10, 20, 30, 40; \frac{2}{3}\right), (5, 15, 35, 45; 1) \right) (r(t))^2 \\
&+ \left(50, 60, 70, 80; \frac{2}{3}\right), (45, 55, 75, 95; 1) r(t) \\
&= \left(80, 90, 110, 120; \frac{2}{3}\right), (75, 85, 115, 125; 1).
\end{align*}
\]

Figure 5 shows initial guessed values, analytical, and numerical approximate solution graph of iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Example 1.
To obtain initial guess, we use above system for $\tau = 0$ and $\tau = 1$; therefore,

\[
\begin{align*}
10 \left( r^L_1 \right)^2(0) + 50 r^L_1(0) &= 80, \\
40 \left( r^L_1 \right)^2(0) + 80 r^L_1(0) &= 120, \\
5 \left( r^U_1 \right)^2(0) + 45 r^U_1(0) &= 75, \\
45 \left( r^L_1 \right)^2(0) + 95 r^L_1(0) &= 125, \\
25 \left( r^L_1 \right)^2(1) + 65 r^L_1(1) &= 95, \\
25 \left( r^L_1 \right)^2(1) + 65 r^L_1(1) &= 105, \\
15 \left( r^U_1 \right)^2(1) + 55 r^U_1(1) &= 85, \\
35 \left( r^U_1 \right)^2(1) + 75 r^U_1(1) &= 115.
\end{align*}
\tag{39}
\]

Consequently, $r^L_1(0) = 0.5$, $r^L_1(0) = 0.5$, $r^L_1(0) = 0.5$, $r^U_1(0)$ = 0.5, and $r^U_1(0) = r^L_1(0) = r^U_1(0) = 1/2$. After 4 iterations, we obtain the solution with the maximum error less than 10^{-30}. Now suppose $r$ is negative, we have the following:

\[
\begin{align*}
(10 + 15r) \left( r^L_1(\tau) \right)^2 + (50 + 15r) r^L_1(\tau) &= (80 + 15r), \\
(40 - 15r) \left( r^L_1(\tau) \right)^2 + (80 - 15r) r^L_1(\tau) &= (120 - 15r), \\
(5 + 10r)(r^U_1(\tau))^2 + (45 + 10r)r^U_1(\tau) &= (75 + 10r), \\
(45 - 10r)(r^L_1(\tau))^2 + (95 - 20r)r^U_1(\tau) &= (125 - 10r).
\end{align*}
\tag{40}
\]

For $\tau = 0$, we have $r^L_1(0) > r^L_1(0)$; therefore negative root does not exist.

**Example 2.** Consider the interval-valued trapezoidal fuzzy nonlinear equation

\[
\begin{align*}
\left\langle \left( \begin{array}{c} 0.65, 0.73, 0.87, 0.95; \frac{2}{3} \end{array} \right), \left( \begin{array}{c} 0.6, 0.7, 0.9, 1; 1 \end{array} \right) \right\rangle (r(\tau))^2 &+ \left\langle \left( \begin{array}{c} 0.25, 0.33, 0.47, 0.55; \frac{2}{3} \end{array} \right), \left( \begin{array}{c} 0.2, 0.3, 0.5, 0.6; 1 \end{array} \right) \right\rangle r(\tau) \\
= \left\langle \left( \begin{array}{c} 0.45, 0.53, 0.67, 0.75; \frac{2}{3} \end{array} \right), \left( \begin{array}{c} 0.4, 0.5, 0.7, 0.8; 1 \end{array} \right) \right\rangle.
\end{align*}
\tag{41}
\]

Without any loss of generality, assume that $r$ is positive; then, the parametric form of this equation is as follows:

\[
\begin{align*}
\left\langle \left( \begin{array}{c} 0.65 + 0.12r, 0.95 - 0.12r; 0.6 + 0.1r, 1 - 0.1r \end{array} \right) \right\rangle (r(\tau))^2 &+ \left\langle \left( \begin{array}{c} 0.25 + 0.12r, 0.55 - 0.12r; 0.2 + 0.1r, 0.6 - 0.1r \end{array} \right) \right\rangle r(\tau) \\
= \left\langle \left( \begin{array}{c} 0.45 + 0.12r, 0.75 - 0.12r; 0.4 + 0.1r, 0.8 - 0.1r \end{array} \right) \right\rangle.
\end{align*}
\tag{42}
\]

Consequently, $r^L_1(0) = 0.6$, $r^L_1(0) = 0.6$, $r^L_1(0) = 0.6$, $r^U_1(0)$ = 0.6, and $r^U_1(0) = r^L_1(0) = r^U_1(0) = 1/2$. After 4 iterations, we obtain the solution with the maximum error less than 10^{-30}. Now suppose $r$ is negative, we have

\[
\begin{align*}
(0.65 + 0.12r)(r^L_1(\tau))^2 + (0.25 + 0.12r)r^L_1(\tau) &= (0.45 + 0.12r), \\
(0.95 - 0.12r)(r^L_1(\tau))^2 + (0.55 - 0.12r)r^L_1(\tau) &= (0.75 - 0.12r), \\
(0.6 + 0.1r)(r^U_1(\tau))^2 + (0.2 + 0.1r)r^U_1(\tau) &= (0.4 + 0.1r), \\
(1 - 0.1r)(r^U_1(\tau))^2 + (0.6 - 0.1r)r^U_1(\tau) &= (0.8 - 0.1r).
\end{align*}
\tag{43}
\]

Table 3 clearly shows the dominance behavior of MM over NN in terms of absolute error on the same number of iterations $n = 4$ for Example 2.

Table 4 shows analytical solutions for Example 2.

Figure 6 shows initial guessed values, analytical, and numerical approximate solution graph of iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Example 2.

To obtain initial guess, we use above system for $\tau = 0$ and $\tau = 1$; therefore,

\[
\begin{align*}
0.65(r^L_1)^2(0) + 0.2r^L_1(0) &= 0.45, \\
0.95(r^L_1)^2(0) + 0.55r^L_1(0) &= 0.75, \\
0.6(r^U_1)^2(0) + 0.2r^U_1(0) &= 0.4, \\
1.0(r^L_1)^2(0) + 0.6r^L_1(0) &= 0.8, \\
0.77(r^U_1)^2(1) + 0.37r^U_1(1) &= 0.57, \\
0.83(r^U_1)^2(1) + 0.43r^U_1(1) &= 0.63, \\
0.7(r^U_1)^2(1) + 0.3r^U_1(1) &= 0.5, \\
0.9(r^U_1)^2(1) + 0.5r^U_1(1) &= 0.7.
\end{align*}
\tag{44}
\]

Consequently, $r^L_1(0) = 0.6$, $r^L_1(0) = 0.6$, $r^L_1(0) = 0.6$, $r^U_1(0)$ = 0.6, and $r^U_1(0) = r^L_1(0) = r^U_1(0) = 1/2$. After 4 iterations, we obtain the solution with the maximum error less than 10^{-30}. Now suppose $r$ is negative, we have

\[
\begin{align*}
(0.65 + 0.12r)(r^L_1(\tau))^2 + (0.25 + 0.12r)r^L_1(\tau) &= (0.45 + 0.12r), \\
(0.95 - 0.12r)(r^L_1(\tau))^2 + (0.55 - 0.12r)r^L_1(\tau) &= (0.75 - 0.12r), \\
(0.6 + 0.1r)(r^U_1(\tau))^2 + (0.2 + 0.1r)r^U_1(\tau) &= (0.4 + 0.1r), \\
(1 - 0.1r)(r^U_1(\tau))^2 + (0.6 - 0.1r)r^U_1(\tau) &= (0.8 - 0.1r).
\end{align*}
\tag{45}
\]

For $\tau = 0$, we have $r^L_1(0) > r^L_1(0)$, therefore, negative root does not exist.
Example 3. Consider the interval-valued trapezoidal fuzzy nonlinear equation

\[
\left\langle \left( 0.45, 0.53, 0.67, 0.75 ; \frac{2}{3} \right), \left( 0.4, 0.5, 0.7, 0.8 ; 1 \right) \right\rangle (r(\tau)) \sin (r(\tau))
\]

\[+ \left\langle \left( 0.65, 0.73, 0.87, 0.95 ; \frac{2}{3} \right), \left( 0.6, 0.7, 0.9, 1 ; 1 \right) \right\rangle \sin (r(\tau))
\]

\[= \left\langle \left( 0.25, 0.33, 0.47, 0.55 ; \frac{2}{3} \right), \left( 0.2, 0.3, 0.5, 0.6 ; 1 \right) \right\rangle \sin (r(\tau))
\]

\[
(46)
\]

Without any loss of generality, assume that \( r \) is positive; then, the parametric form of this equation is as follows:

\[
\left\langle \left( 0.45 + 0.12r, 0.75 - 0.12r, 0.4 + 0.1r, 0.8 - 0.1r \right) \right\rangle \cdot (r(\tau))^3 + \left\langle \left( 0.65 + 0.12r, 0.95 - 0.12r, 0.6 + 0.1r, 0.6 - 0.1r \right) \right\rangle r(\tau)
\]

\[= \left\langle \left( 0.25 + 0.12r, 0.55 - 0.12r, 0.2 + 0.1r, 0.6 - 0.1r \right) \right\rangle \sin (r(\tau))
\]

\[
(47)
\]

or

\[
\left\langle (0.45 + 0.12r) (r(\tau))^3 + (0.65 + 0.12r) \sin (r(\tau)) \right\rangle = (0.25 + 0.12r)
\]

\[
(0.75 - 0.12r) (r(\tau))^3 + (0.95 - 0.12r) \sin (r(\tau)) = (0.55 - 0.12r)
\]

\[
(0.4 + 0.1r) (r(\tau))^3 + (0.6 + 0.1r) \sin (r(\tau)) = (0.2 + 0.1r)
\]

\[
(0.8 - 0.1r) (r(\tau))^3 + (1.0 - 0.1r) \sin (r(\tau)) = (0.6 - 0.1r)
\]

\[
(48)
\]

Consequently, \( r_{1L}^U (0) = 0.5, r_{1L}^L (0) = 0.3, r_{1L}^L (0) = 0.5, r_{1L}^U (0) = 0.3, r_{1L}^U (0) = 0.5, r_{1L}^L (0) = 1/2 \). After 4 iterations, we obtain the solution with the maximum error less than \( 10^{-30} \). Now suppose \( r \) is negative, we have

\[
\left\langle (0.45 + 0.12r) (r(\tau))^3 + (0.65 + 0.12r) \sin (r(\tau)) \right\rangle = (0.25 + 0.12r),
\]

\[
(0.75 - 0.12r) (r(\tau))^3 + (0.95 - 0.12r) \sin (r(\tau)) = (0.55 - 0.12r),
\]

\[
(0.4 + 0.1r) (r(\tau))^3 + (0.6 + 0.1r) \sin (r(\tau)) = (0.2 + 0.1r),
\]

\[
(0.8 - 0.1r) (r(\tau))^3 + (1.0 - 0.1r) \sin (r(\tau)) = (0.6 - 0.1r)
\]

\[
(50)
\]

For \( r = 0 \), we have hence \( r_{1L}^U (0) > r_{1L}^L (0) \); therefore, negative root does not exist.

Figures 8(a)–8(c) show residual falls for iterative methods MM and NN for interval-valued trapezoidal fuzzy nonlinear equation used in Examples 1–3, respectively.

5. Conclusion

In this research paper, we constructed highly efficient two-step numerical iterative method to approximate roots of interval-valued trapezoidal fuzzy nonlinear equations. A set of real-life applications from optimization are considered as a numerical test examples showing the practical performance and dominance efficiency of MM over NN method on the same number of iterations. From Tables 1–6 and Figures 1–8, we observe that numerical results of MM methods are better in terms of absolute error and CPU time as compared to NN method. Considering the same ways as in this article, we can establish higher order and efficient numerical iterative methods for solving system of fuzzy nonlinear equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors’ Contributions

All authors contributed equally in the preparation of this manuscript.

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