Minimum-weight double-tree shortcutting for Metric TSP: Bounding the approximation ratio

Vladimir Deineko\textsuperscript{a,b}, Alexander Tiskin\textsuperscript{a,c}

\textsuperscript{a}Centre for Discrete Mathematics and Its Applications (DIMAP), University of Warwick, Coventry, CV4 7AL, UK.
\textsuperscript{b}Warwick Business School, University of Warwick, Coventry CV4 7AL, UK.
\textsuperscript{c}Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK.

Abstract

The Metric Traveling Salesman Problem (TSP) is a classical NP-hard optimization problem. The double-tree shortcutting method for Metric TSP yields an exponentially-sized space of TSP tours, each of which approximates the optimal solution within at most a factor of 2. We consider the problem of finding among these tours the one that gives the closest approximation, i.e. the minimum-weight double-tree shortcutting. Previously, we gave an efficient algorithm for this problem, and carried out its experimental analysis. In this paper, we address the related question of the worst-case approximation ratio for the minimum-weight double-tree shortcutting method. In particular, we give lower bounds on the approximation ratio in some specific metric spaces: the ratio of 2 in the discrete shortest path metric, 1.622 in the planar Euclidean metric, and 1.666 in the planar Minkowski metric. The first of these lower bounds is tight; we conjecture that the other two bounds are also tight, and in particular that the minimum-weight double-tree method provides a 1.622-approximation for planar Euclidean TSP.

1. Introduction

The Metric Travelling Salesman Problem (TSP) is a classical combinatorial optimization problem. We represent a set of \( n \) points in a metric space by a complete weighted graph on \( n \) nodes, where the weight of an edge is defined by the distance between the corresponding points. The objective of Metric TSP is to find in this graph a minimum-weight Hamiltonian cycle...
(equivalently, a minimum-weight tour visiting every node at least once). The most common example of Metric TSP is the planar Euclidean TSP, where the points lie in the two-dimensional Euclidean plane, and the distances are measured according to the Euclidean metric.

Metric TSP, even restricted to planar Euclidean TSP, is well-known to be NP-hard [10]. Metric TSP is also known to be NP-hard to approximate to within a ratio 1.00456, but polynomial-time approximable to within a ratio 1.5. Fixed-dimension Euclidean TSP is known to have a PTAS (i.e. a family of algorithms with approximation ratio arbitrarily close to 1) [1]; this generalises to any metric defined by a fixed-dimension Minkowski vector norm.

Two simple methods, double-tree shortcutting [12] and Christofides’ [4, 13], allow one to approximate the solution of Metric TSP within a factor of 2 and 1.5, respectively. Both these methods belong to the class of tour-constructing heuristics, i.e. “heuristics that incrementally construct a tour and stop as soon as a valid tour is created” [7]. In both methods, we build an Eulerian graph on the given point set, select an Euler tour of the graph, and then perform shortcutting on this tour by removing repeated nodes, until all node repetitions are removed. In general, it is not prescribed which one of several occurrences of a particular node to remove. Therefore, the methods yield an exponentially-sized space of TSP tours (shortcuttings of a specific Euler tour in a specific Eulerian graph), each of which approximates the optimal solution within at most a factor of 2 (respectively, 1.5).

The two methods differ in the way the initial weighted Eulerian graph is constructed. Both start by finding the graph’s minimum-weight spanning tree (MST). The double-tree method then doubles every edge in the MST, while the Christofides method adds to the MST a minimum-weight matching built on the set of odd-degree nodes. The weight of the resulting Euler tour is higher than the optimal TSP tour at most by a factor of 2 (respectively, 1.5), and the subsequent shortcutting can only decrease the tour weight.

While any tour obtained by shortcutting of the original Euler tour approximates the optimal solution within at most a factor of 2 (respectively, 1.5), clearly, it is still desirable to find the shortcutting that gives the closest approximation. Given an Eulerian graph on a set of points, we will consider its minimum-weight shortcutting across all shortcuttings of all possible Euler tours of the graph. We shall correspondingly speak about the minimum-weight double-tree and the minimum-weight Christofides methods.

Unfortunately, for the general Metric TSP (i.e. an arbitrary complete weighted graph with the triangle inequality), the corresponding double-tree and Christofides minimum-weight shortcutting problems are both NP-hard.
The minimum-weight double-tree shortcutting problem was also believed for a long time to be NP-hard for planar Euclidean TSP, until a polynomial-time algorithm was given by Burkard et al. [3]. In [6], we gave an improved algorithm running in time $O(4^d n^2)$, where $d$ is the maximum node degree in the rooted minimum spanning tree (e.g. in the non-degenerate planar Euclidean case, $d \leq 4$). In contrast, the Christofides version of the problem remains NP-hard even for planar Euclidean TSP [11].

A natural question about the properties of the two approximation methods and their variants is whether the approximation ratios 2 and 1.5 are tight, i.e. whether there is a problem instance where the approximate solution has approximation ratio 2 (respectively, 1.5), or a family of problem instances where the approximate solutions approach these ratios arbitrarily closely.

For the minimum-weight double-tree method, the answer to this question is unknown, as observed e.g. in [9]. The only existing lower bounds for the double-tree method apply to a shortcutting that is performed in some suboptimal, easily computable order. An example of such an order is depth-first tree traversal; we shall call the resulting method depth-first double-tree shortcutting. A tight lower bound for this method is given by the standard Euclidean lower-bound construction shown in Figure 1, which adapted from [8]. Figure 1a shows an instance point set and the (unique) minimum spanning tree. We assume that $\epsilon = o(1)$; for example, we can take $\epsilon = 1/n$. The vertical size of the instance set is 1, and the horizontal size is $(1 + o(1))n$. The weight of the unique MST is $(2 + o(1))n$; the double-tree weight is $(4 + o(1))n$. The double tree undergoes no significant shortcutting, and the resulting tour (Figure 1b) still has weight $(4 + o(1))n$. The absolute minimum-weight tour (Figure 1c) has weight $(2 + o(1))n$, therefore the approximation ratio on the given instance set is 2.

For the minimum-weight Christofides algorithm, a tight lower bound is given by the standard Euclidean lower-bound construction shown in Figure 2 which is adapted from [5] and uses the same conventions as Figure 1.
The minimum spanning tree has exactly two odd-degree nodes, therefore the additional matching consists of a single edge. The resulting Eulerian graph (Figure 2b) is already a Hamiltonian cycle, hence no shortcutting is required. The weight of the cycle is \((3 + o(1))n\). As before, the absolute minimum-weight tour (Figure 2c) has weight \((2 + o(1))n\), therefore the approximation ratio on the given instance sets is 1.5.

In the rest of this paper, we address the question of the worst-case approximation ratio for the minimum-weight double-tree shortcutting method in some specific metric spaces. In particular, we give a lower bound on the approximation ratio in the discrete shortest path metric\(^1\); this bound is tight, and can be regarded as a lower bound for a generic metric space. We also give the first non-trivial lower bound for the planar Euclidean and planar Minkowski metrics.

2. The lower bounds

2.1. The discrete shortest path metric

The worst-case approximation ratio of the double-tree and Christofides methods can clearly be dependent on the metric in which the TSP problem is defined. In the Introduction, we described a tight lower bound of 2 on the worst-case approximation ratio in the planar Euclidean metric, both for the depth-first version of the double-tree method and for the minimum-weight Christofides method. In contrast, no non-trivial lower bounds have been known, to our knowledge, for the minimum-weight double-tree method in any metric. A tight bound in the Euclidean metric seems difficult to obtain; however, it can be established that the upper bound of 2 is tight in some non-Euclidean metrics, and therefore is tight for the generic Metric TSP.

\(^1\)The same result has been obtained independently by Bilò at al. \(^2\).
Given a weighted undirected graph, consider the *discrete shortest path metric* on its node set. The distance between two nodes in this metric is defined as the weight of the shortest path connecting them in the graph. Let $n$ be a power of 2. Let $T_n$ be a rooted tree on $n$ nodes, where the root has a single child, which branches off into a complete binary tree with $n/2$ leaves. We construct an instance graph on $2^n$ nodes as follows. First, we create two copies of the tree $T_n$, keeping track of corresponding pairs of nodes (i.e. pairs of nodes which are copies of the same node in $T_n$). We then give all the edges in each tree weight 1, and connect the roots of the two trees by a *root edge* of weight 1. Finally, we connect every pair of corresponding non-root nodes in both trees by a *cross-edge* of weight $1 + \epsilon$. We assume that $\epsilon = o(1)$; for example, we can take $\epsilon = 1/n$. The instance graph corresponding to $n = 8$ is shown in Figure 3a, where edges of weight 1 and $1 + \epsilon$ are represented, respectively, by solid and dotted lines.

The unique MST consists of both copies of $T_n$ plus the root edge, and has weight $(2 - o(1))n$; the double-tree weight is $(4 - o(1))n$. Note that for any two nodes $a, b$ within the same copy of $T_n$, the distance between $a$ and $b$ is equal to the weight of the path connecting these nodes in the tree. Hence, a shortcutting from $a, b, c$ to $a, c$ can reduce the tour weight, only if $a$ and $c$ belong to different copies of $T_n$. Also note that any double-tree Euler tour of $T_n$ has weight $2n - 2$. Any Hamiltonian cycle of the complete weighted graph obtained by shortcutting the double-tree Euler tour will contain a Hamiltonian path in a complete weighted subgraph induced by each copy of $T_n$. The weight of a such a Hamiltonian path can differ from the weight of the double-tree tour of $T_n$ by at most the weight of a single edge, which cannot exceed $2\log n = o(n)$. Therefore, the resulting Hamiltonian cycle still has weight $(4 - o(1))n$. 

![Figure 3: The minimum-weight double-tree method: a lower-bound instance in the discrete shortest path metric](image)
Figure 4: The minimum-weight double-tree method: a lower-bound instance in the Euclidean and Minkowski metrics

The minimum-weight double-tree tour for our example is shown in Figure 3b, where straight edges have weights 1 and $1 + \epsilon$, and curved edges have integer weights greater than 1. An edge’s curvature indicates the layout of the shortest path along which the edge weight is measured. The absolute minimum-weight tour has weight $(2 + o(1))n$, and consists of the root edge and all the cross-edges, linked together by edge-disjoint paths in the two trees. The absolute minimum-weight tour for our example is shown in Figure 3c, using the same graphic conventions as in Figure 3b. The approximation ratio of the minimum-weight double-tree method on the given instance set is $4/2 = 2$, which matches the generic upper bound$^2$.

2.2. Euclidean and Minkowski metrics

Compared with the above construction for the discrete shortest path metric, it appears to be much more difficult to obtain a tight bound in planar Euclidean-type metrics. We describe a construction that provides the first non-trivial lower bound on the approximation ratio of the minimum-weight double-tree method in the planar Euclidean and Minkowski metrics.

The proposed construction consists of $6n + 1$ points, and is shown in Figure 4a for $n = 4$. The instance point set consists of seven points forming a symmetric three-way central star of arbitrary constant size, and six rows of points extending from the star’s ends in three symmetric directions in steps of length 1. Figure 4a shows the (unique) minimum spanning tree, which has weight $(6 + o(1))n$. Figure 4b shows the minimum-weight double-tree

\[2^\text{Many variations on the described construction are possible. We have chosen a variant that is easy to visualise.}\]
tour, which has weight \((8 + \sqrt{3} + o(1))n\). Figure 4c shows the absolute minimum-weight tour, which has weight \((6 + o(1))n\). The approximation ratio of the minimum-weight double-tree method on the given instance set is \((8 + \sqrt{3})/6 \approx 1.622\). There remains a substantial gap between this lower bound and the generic upper bound of 2, which is also the best known upper bound in the planar Euclidean metric.

The same construction provides a somewhat stronger lower bound in a metric defined by the hexagonal norm — a Minkowski vector norm with the unit disc in the shape of a regular hexagon (see Figure 4a). In this metric, the distance between two points is measured along a polygonal path composed from segments parallel to the edges of the unit disc. The weights of the minimum spanning tree (Figure 4a) and of the absolute minimum-weight tour (Figure 4c) on the above instance set remain asymptotically unchanged in the new metric. However, the weight of the minimum-weight double-tree tour (Figure 4b) increases to \((10 + o(1))n\). Therefore, the lower bound in the hexagonal metric is \(10/6 \approx 1.666\).

3. Conclusions

In the previous section, we presented lower bounds on the minimum-weight double-tree method. We have shown that the trivial upper bound of 2 is tight in at least some metrics (in particular, the discrete shortest path metric). However, in the important cases of the Euclidean and Minkowski metrics, a substantial gap remains between our lower bounds of 1.622 (respectively, 1.666) and the trivial upper bound of 2. Considering the apparent difficulty of improving on these lower bounds, and the good approximation behaviour of the minimum-weight double-tree algorithm on typical Euclidean TSP instances [4], we conjecture that these lower bounds are tight, and that the minimum-weight double-tree method provides a 1.622-approximation for planar Euclidean TSP.

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