Pseudo-forces in quantum mechanics

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Dynamical evolution is described as a parallel section on an infinite dimensional Hilbert bundle over the base manifold of all frames of reference. The parallel section is defined by an operator-valued connection whose components are the generators of the relativity group acting on the base manifold. In the case of Galilean transformations we show that the property that the curvature for the fundamental connection must be zero is just the Heisenberg equations of motion and the canonical commutation relation in geometric language. We then consider linear and circular accelerating frames and show that pseudo-forces must appear naturally in the Hamiltonian.

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I. INTRODUCTION

Evolution of a state vector in quantum mechanics can be viewed as a kind of parallel transport [1]. There have been suggestions to use the geometric language of vector bundles and parallel transport in various situations in quantum mechanics [2, 3, 4, 5, 6]. These ideas are natural in the discussion of the geometric or the Berry phase [7]. Despite these attempts to “geometrize” quantum mechanics there seems to be no common agreement in these approaches about the base space, or the structure group, let alone the connection or the curvature. Moreover it is not clear whether the extra mathematical machinery is justified by a new or clearer physical insight.

In this paper we give the physical reason why the bundle viewpoint is natural in quantum mechanics and illustrate it with application to accelerated frames.

If a physical system is observed in various frames of reference, the states described by them as vectors in their individual Hilbert spaces will form a section in a vector bundle with the Hilbert space as the standard fiber and the set of all frames of reference as the base manifold. There is no canonical identification of the fibers and we need a “connection”, a notion of covariant derivative or that of parallel transport.

We make use of the principle of relativity (all frames of reference are equally suitable for description) to provide the notion of parallelism and make the following assumption: States described by different frames of reference form a parallel section.

As each observer can apply an overall unitary operator on his Hilbert space and still obtain an equivalent description, we see that the structure group should be the group of all unitary operators on the Hilbert space [8]. Thus there is an underlying “gauge freedom” which can be used to transform the natural parallel sections into constant sections and do away with the need to use all Hilbert spaces at once. This is the case in standard quantum mechanics where a single Hilbert space is used by all observers.

In this paper we develop our geometric picture and explicitly consider the case where Galilean group is the underlying relativity group. We find that Heisenberg equations of motion and the canonical commutation relation are contained in a single condition: that the fundamental connection is flat or that its curvature is zero.

Next we apply the geometric construction to accelerated frames and show that pseudo-force terms appear as expected. In the case of linearly accelerated frames we get a linear “gravitational” potential implying that equivalence principle must hold in quantum mechanics. In contrast, in the conventional formalism equivalence principle is obtained by an artificial time dependent phase transformation of the wavefunction. In the case of rotating frames we show that both centrifugal and coriolis forces show up in the Hamiltonian. It is satisfactory to see that the coriolis force does not correspond to a potential because it does no work, being perpendicular to velocity, but naturally appears as a connection term added to the canonical momentum, much like the magnetic force. We are thus able to show that fibre bundles are the natural language in which to discuss quantum mechanical effects of gravity.

II. GEOMETRIC SETTING

A. The bundle

Consider a quantum mechanical system described by observers in different frames of reference. We assume that the set of all frames of reference forms a differentiable manifold. This is physically reasonable because frames of reference are related by symmetry transformations which form a group. This means that the frames can be labelled by coordinates $x$ on the group manifold. A state of the system is described by a vector $\phi(x)$ in a Hilbert space $\mathcal{H}_x$ associated with the observer $x$. We, thus, have the ingredients of a vector bundle [9]. The base is a manifold $M$ with coordinates $x$ and a Hilbert space at each point. To every possible state of the system is associated a section or a mapping $x \rightarrow \phi(x)$ where $\phi(x)$ is the vector describing the state of the system by observer $x$. We assume there exist unitary operators $U(y, x)$ which...
connect the states $\phi(y) = U(y,x)\phi(x)$. These operators must satisfy consistency conditions

$$U(z,y)U(y,x) = U(z,x); \quad U(x,x) = 1$$

We must note that there is no canonical way of choosing states $\phi(x)$ to describe the system in the Hilbert space $\mathcal{H}_x$. If we were to apply a unitary operator to all vectors $\phi(x)$, $\psi(x)$ etc. in $\mathcal{H}_x$, the resulting states are equally well suited to describe the system provided the observables acting in $\mathcal{H}_x$ are similarly changed. In other words, we assume the structure or gauge group acting on the fiber to be the group of all unitary operators.

B. The Connection

Let us choose a complete orthonormal set $\phi_\alpha$ in the Hilbert space of some fixed observer, say at $x = 0$, $\phi_\alpha \equiv \phi_\alpha(0)$. The sets $\phi_\alpha(x) = U(x,0)\phi_\alpha(0)$ then are complete orthonormal sets in all the other spaces $\mathcal{H}_x$. Any arbitrary section $\psi(x)$ can then be written as

$$\psi(x) = \sum_\alpha c_\alpha(x)\phi_\alpha(x)$$

where $c_\alpha(x)$ are the complex coefficients of expansion. Let $\Gamma$ be the set of all sections. They can be added pointwise.

$$(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x)$$

and multiplied with smooth functions

$$(c\psi)(x) = c(x)\psi(x)$$

Let $\Lambda \otimes \Gamma$ be the tensor product of the space $\Lambda$ of all 1-forms on the base $M$ and $\Gamma$. A connection on this bundle is a mapping $D : \Gamma \to \Lambda \otimes \Gamma$ such that

$$D(\psi_1 + \psi_2) = D\psi_1 + D\psi_2 \quad \text{and} \quad D(c\psi) = cD\psi + dc\psi$$

If $\phi_n(x)$ is a basis in $\Gamma$ we can express $D(\phi_n)$ in terms of the basis $dx^\mu \otimes \phi_m$ in $\Lambda \otimes \Gamma$ as

$$(D\phi_n)(x) = \phi_m(x)\Gamma_{\mu mn}dx^\mu$$

where coefficients $\Gamma_{\mu mn}(x)$ are the Christoffel symbols with respect to the basis $dx^\mu \otimes \phi_m$. We write this equation as

$$(D\phi_n)(x) = \phi_m(x)\omega^{\phi}_{mn}(x)$$

where the complex matrix $\omega_{mn}$ can be obtained by taking inner product with $\phi_m$ in Eq.(5).

$$\omega^{\phi}_{mn} = (\phi_m, D\phi_n)$$

This matrix of one-forms is called the connection matrix. We require $D$ to satisfy Leibniz rule

$$D(\phi, \psi) = (D\phi, \psi) + (\phi, D\psi) = d(\phi, \psi)$$

which when applied to $\delta_{mn} = (\phi_m, \phi_n)$ shows that $\omega^{\phi}$ is an anti-hermitian matrix.

Under a change of basis

$$\chi_n(x) = U(x)\phi_n(x)$$

we have

$$\chi_n(x) = \phi_m(x)(\phi_m(x), U(x)\phi_n(x)) = \phi_m(x)U_{mn}(x)$$

Thus, omitting the base point $x$ for simplicity of notation

$$(D\chi_n)(x) = D(\phi_m U_{sn}) = \phi_r \omega^\phi_{rs}U_{sn} + \phi_s dU_{sn} = \chi_m \omega^\chi_{mn} = \phi_r U_{rm}\omega^\chi_{mn}$$

Or,

$$\omega^\chi_{mn} = U^{-1}\omega^\phi_{rs}U_{sn} + U^{-1}dU_{rn}$$

Omitting matrix indices, we have

$$\omega^\chi = U^{-1}\omega^\phi U + U^{-1}dU$$

The curvature 2-form for the connection is given by

$$\Omega^\phi = d\omega^\phi + \omega^\phi \wedge \omega^\phi$$

which transforms as

$$\Omega^\chi = U^{-1}\Omega^\phi U$$

One may also note the Bianchi identity

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$$

III. PARALLEL SECTION AND THE FUNDAMENTAL CONNECTION

We now make the fundamental assumption that a system observed by different observers is represented by parallel sections. Let $\phi_m(x)$ be a family of parallel sections, that is

$$(D\phi)(x) = 0, \quad \text{for all } m$$

This implies

$$\omega^\phi_{mn}(x) = 0$$
sections are constructed by applying transformation
\( U(x,0) \) on \( \phi(0) \) for all \( x \).

We can choose \( \phi_m(0) \) as the new basis
\[
\chi_m(x) \equiv \phi_m(0) = U_x^{-1}\phi_m(x) = \phi_r(x)\left(\phi_r(x), U^{-1}(x)\phi_m(x)\right) = \phi_r(x)U_{rm}^{-1}
\]
(20)

Then
\[
\omega^X = UdU^{-1}
\]
(21)
which, as expected, is pure gauge.

**IV. GALILEAN FRAMES**

Let us consider a particle of mass \( m \) in one space dimension. We use units where \( c = \hbar = 1 \). We consider the basis of sharp momentum states \( |k\rangle \) such that
\[
P|k\rangle = k|k\rangle
\]
(22)
and
\[
\langle k'|k\rangle = \delta(k - k')
\]
(23)
The time and space translations are given by the operators \( U_\tau \) and \( U_\zeta \) respectively,
\[
U_\tau |k\rangle = e^{-iH\tau}|k\rangle = e^{-i\frac{\tau^2}{2m}}|k\rangle
\]
(24)
\[
U_\zeta |k\rangle = e^{iP\zeta}|k\rangle = e^{ik\zeta}|k\rangle
\]
(25)
The boosts act as
\[
U_\eta |k\rangle = |k - m\eta\rangle
\]
(26)
given by
\[
U_\eta = e^{-iK\eta}
\]
(27)
where \( K \) is the boost generator. The lie algebra of the Galilean group is
\[
[P, H] = 0, \quad [K, H] = iP
\]
(28)
V. ACCELERATED FRAMES AND PSEUDO FORCES

Acceleration implies changing from one Galilean frame to another after every infinitesimal amount of time. This can be seen as a curve on the base manifold parametrized by time. We assume that an observer in the accelerating frame uses the same Hilbert space to describe a physical system as the observer at the base manifold point with which it coincides at each instant. Moreover they assign the same state to the system. There is no need for any other condition such as the redefinition of the wavefunction by a time-dependent expression. This means that the Schrödinger equation in the frame $S'$.

A. Linearly accelerated frame and equivalence principle

The question of whether the principle of equivalence in classical mechanics also holds in quantum mechanics was discussed by C.J. Eliezer and P.G. Leach. They studied the transformation of the Schrödinger equation under a change from an inertial frame of reference to another after every infinitesimal amount of time. This is the redefinition of the wavefunction by a time-dependent phase factor.

The phase factor has been chosen precisely to obtain the equivalence principle. There is no explanation put forward for this factor. Thus, the system "sees" an extra potential $X(x)mg$ which is the expected linear "gravitational" potential term. This is manifestation of the equivalence principle in quantum mechanics.

The validity of the equivalence principle in the quantum regime has been experimentally tested in some beautiful experiments done with neutron interference.

B. Rotating frame, Coriolis and centrifugal forces

Consider a frame of reference $S'$ which is rotating with constant angular velocity $\omega$ and radius $r$ about the origin of coordinates on the $xy$ plane of a frame $S$. The two frames are related as follows: we wait for time $t$, translate by $r$ direction, rotate by angle $\theta = \omega t$ and finally give a boost in the $y'$ direction by velocity $v$.

The parallel section is given by

$$ |t, x, v; k\rangle = \frac{1}{\sqrt{2m}} \left( e^{-i\frac{P}{2m}} e^{iX(x)mg} |t, x, v; k\rangle \right) $$

(46)

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where $P(v) = k - mv$ and $X(x) = X + x$. Thus, the system "sees" an extra potential $X(x)mg$ which is the expected linear "gravitational" potential term. This is manifestation of the equivalence principle in quantum mechanics.

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The Hamiltonian $H$ as seen by an observer in the rotating frame is given by the rate of change of the vectors specified along the curve on the base manifold.

$$H = i\frac{dU}{dt}U^{-1}$$

$$= \frac{1}{2m}\left(\left(P_1^2 + (P_2 + m\omega r)^2\right) - \omega (P_2 + m\omega r) \right) - \omega (J + m\omega r)$$

or

$$H = \frac{1}{2m}\left((P_1 + m\omega X_2)^2 + (P_2 - m\omega X_1)^2\right)$$

$$- \frac{1}{2}m\omega^2((X_1 + r)^2 + X_2^2)$$

(49)

Thus the expected centrifugal and coriolis forces appear in the Hamiltonian. Since coriols force does no work it cannot appear as an explicit potential term. Rather it appears as a connection in the canonical momentum.

VI. DISCUSSION

The bundle viewpoint is hinted in Dirac’s work as early as 1932. In a most influential paper *The Lagrangian in quantum mechanics*, Dirac puts forth the following argument: Let $q(t)$ be a complete set of commuting observables in the Heisenberg picture. The set of eigenvalues $q'$ at each $t$ forms a manifold $M$ giving rise to “spacetime” $B \equiv M \times T$ where $T$ represents the time axis.

Evolution is determined by the moving basis $|q', t\rangle$ at each $(q', t)$. This can be interpreted as a section from the base $B$ into a Hilbert space. Let $c : \tau \to (q'(\tau), t(\tau))$ be a curve in $B$. Then the change of basis vectors $|q', t\rangle$ is given by

$$-i\frac{\partial}{\partial q'}|q', t\rangle = |q', t\rangle P(t)$$

$$i\frac{\partial}{\partial t}|q', t\rangle = |q', t\rangle H$$

where $P(t) = e^{iHt}P(0)e^{-iHt}$

Thus the change of a basis vector along the curve is

$$d|q', t\rangle = i|q', t\rangle dS$$

$$dS = P(t) dq' - H dt$$

Thus in the bundle formalism Dirac’s Lagrangian can be seen as an operator -valued 1-form on the Hilbert vector bundle whose base manifold is spanned by the eigenvalues $q'$ of a complete set of commuting operators $q(t)$ specified at all times and the standard fibre is Hilbert space. The components of this 1-form are just the Hamiltonian and momentum operators. If the section $(q', t) \to |q', t\rangle$ is assumed to be parallel then evolution can be seen as parallel transport. This is the theme on which our present work is based.

Asorey et al. consider a Hilbert bundle with positive time axis $R^+$ as the base manifold. Another viewpoint is that of Prugovecki and Drechsler and Tuckey whose bundle is the associated vector bundle for the principle bundle with Poincare group as structure group over spacetime base manifold. The Hilbert space considered by them is the space of square integrable functions over phase space of space coordinates and the mass hyperboloid ($p^2 = m^2$, $p_0 > 0$). This approach allows them to consider parallel transport over curved spaces with possible applications to quantum gravity.

Dirk Graudenz also has a Hilbert bundle with spacetime base. Our approach agrees with that of Graudenz in that description of a physical system is always description by one observer. Yet another construction is given by G. Sardanashvily who consider a $C^*$-algebra at each point of the time axis $R$.

Our geometric construction is different from others in the literature. For us the base manifold consists of all frames of reference. This means actually having a group of symmetry transformations as the base manifold with a frame of reference associated with each point on it. We have considered the case of Galilean group which makes the application specific to quantum mechanics.

Our objective here is to present a new geometric viewpoint from which implies the validity of the equivalence principle in quantum mechanics. We have demonstrated this both for linearly accelerating and rotating frames.
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