Communication

An Efficient 3-D Stochastic HIE-FDTD Algorithm for Investigation of Statistical Variation in Electromagnetic Field

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Abstract—We propose a stochastic hybrid implicit–explicit finite-difference time-domain method (S-HIE-FDTD) to compute the mean and variance of the electromagnetic (EM) fields using a single simulation, given those of the conductivity and permittivity in the computation domain. The mean and variance field update equations underlying the proposed method are derived from the field update equations of the “traditional” deterministic HIE-FDTD. The Courant–Friedrichs–Lewy condition of the S-HIE-FDTD depends on the spatial discretization sizes only in two dimensions; therefore, for computation domains with fine geometric features only in the remaining dimension, it uses a time step size that is larger than that of fully explicit schemes. Indeed, numerical results demonstrate that the proposed method is faster than the previously developed stochastic FDTD in computing the mean and variance of the EM fields in two different problems: wave propagation through a multilayer human tissue and transmission through a frequency selective surface.

Index Terms—Hybrid implicit–explicit finite-difference time-domain (HIE-FDTD) method, statistical analysis, stochastic parameter, weakly conditional stability.

I. INTRODUCTION

The statistical electromagnetic (EM) analysis [1] is indispensable in various fields of engineering and physics, including, but not limited to, bioelectromagnetics [2]–[4], atmospheric wave propagation [5], remote sensing [6], [7], and integrated circuit design [8], where materials’ electrical properties are not deterministically known (often due to measurement limitations) and/or scatterer and antenna dimensions are only known within a range of values (often due to fabrication tolerances). The purpose of such an analysis is to predict the mean and variance [and, sometimes, the probability distribution function (pdf)] of a quantity of interest (QoI) (e.g., induced voltage and transmitted power) when the mean and variance (and the pdf) of input/simulation parameters (e.g., conductivity, permittivity, scatterer, and antenna dimensions) are provided.

A common way to statistically characterize a QoI is the Monte Carlo (MC) method [9]. The MC method samples the domain of input parameters and computes the QoI at every sampling point using a deterministic EM solver. These samples of the QoI are then used to compute its mean, variance, and pdf. The MC method is easy to implement since it does not call for any modifications to existing EM solvers (i.e., it is nonintrusive), but it requires a large number of samples (EM solver executions) to produce accurate results.

To avoid sampling in the domain of input parameters, recently, a novel stochastic finite-difference time-domain method (S-FDTD) has been developed [10]. This method is built upon the deterministic FDTD [11]–[13], but it is reformulated and implemented to produce the mean and variance of the EM fields given those of the conductivity and permittivity in the simulation domain. However, just like its deterministic counterpart, the time step size of the S-FDTD is determined by the finest spatial discretization size through the Courant–Friedrichs–Lewy (CFL) condition [11], [14] but not the desired level of accuracy, especially for structures that have fine and coarse geometric features at the same time.

To this end, various unconditionally stable (deterministic) FDTD schemes [15]–[17] have been developed to remove the CFL restriction. These schemes can use larger time step sizes, but their computational cost per time step is increased due to the implicit field updates. Therefore, they might unnecessarily increase the overall computational cost for problems involving structures with fine geometric features only in one or two dimensions (e.g., radiation from patch antennas, transmission through thin layers, and scattering in a layered medium). To address this shortcoming, several FDTD schemes, such as weakly conditionally stable (WCS) method [18], [19] and hybrid implicit-explicit method [20]–[23], have been developed. These schemes use implicit updates only in one or two dimensions (ideally where fine geometric features exist). This has two consequences: 1) the spatial discretization size along these dimensions does not “contribute” to the CFL condition, which results in a larger time step size compared with fully explicit schemes and 2) their computational cost per time step is lower compared with fully implicit schemes. Therefore, the overall computational cost of these schemes is expected to be lower than that of the fully explicit or fully implicit FDTD methods when analyzing structures with fine geometric features only in one or two dimensions.

In this communication, we develop a stochastic hybrid implicit–explicit FDTD (S-HIE-FDTD) to efficiently compute the mean and variance of the EM fields given those of the conductivity and permittivity in a simulation with fine geometric features in one or
Indeed, the numerical results demonstrate that the S-HIE-FDTD is as its deterministic counterpart as described in the previous paragraph. A short stability analysis shows that the S-HIE-FDTD has a CFL condition similar to that of the HIE-FDTD, and as a result, the S-HIE-FDTD has the same computational benefits as its deterministic counterpart as described in the previous paragraph. Indeed, the numerical results demonstrate that the S-HIE-FDTD is more efficient than the S-FDTD in the statistical analysis of wave propagation through a multilayer biological tissue and transmission through a frequency selective surface (FSS).

II. FORMULATION

The full set of equations, which are solved/updated by the HIE-FDTD for all electric and magnetic field components, can be found in [23]. In this communication, only the equations, which update the x-component of the electric field and the z-component of the magnetic field, namely, $E_x$ and $H_z$, are used to demonstrate how the HIE-FDTD is adopted to compute the mean and variance of the fields given those of the relative permittivity $\varepsilon_r$ and conductivity $\sigma$.

The HIE-FDTD updates $E_x$ and $H_z$ using

$$E_x^{n+1}_{i,j,k} = E_x^{n}_{i,j,k} - \frac{\nu_1}{\Delta y} \left( H_y^{n+1/2}_{i+1/2,j+1,k} - H_y^{n+1/2}_{i-1/2,j+1,k} \right)$$

$$H_z^{n+1}_{i,j+1/2,k} = H_z^{n}_{i,j+1/2,k} - \frac{\nu_3}{2\Delta y} \left( H_x^{n+1}_{i,j+1/2,k} - H_x^{n+1}_{i,j-1/2,k} \right)$$

where $\nu_1$ and $\nu_2$ are the permittivity and permeability in free space, $\nu_1 = (2\varepsilon_0 \varepsilon_r - \sigma \Delta t)/(2\varepsilon_0 \varepsilon_r + \sigma \Delta t)$, $\nu_2 = 2\Delta t/(2\varepsilon_0 \varepsilon_r + \sigma \Delta t)$, $\nu_3 = \Delta z / \Delta t$, $\Delta x$, $\Delta y$, and $\Delta z$ are the space steps. In (1) and (2) and in the rest of the text, the superscript $n$ and the subscript $i, j, k$ mean that the pertinent field variables are sampled at time $n\Delta t$ and location $(i \Delta x, j \Delta y, k \Delta z)$.

A. Stochastic Model

Let $g(X_1, \ldots, X_N)$ represent a function of random variables $X_1, \ldots, X_N$ with mean values $\mu_{X_1}, \ldots, \mu_{X_N}$. Using the linearity of expected value operator $E[.]$, i.e., $E[\sum_{i=1}^{N} a_i X_i] = \sum_{i=1}^{N} a_i E[X_i]$, where $a_i$ are constants, the Delta method [24], and following the derivation in [10], we obtain:

$$E[g(X_1, \ldots, X_N)] = g(\mu_{X_1}, \ldots, \mu_{X_N}).$$

The variance of $g(X_1, \ldots, X_N)$ is expressed as

$$\text{Var}(g(X_1, \ldots, X_N)) = E[\{g(X_1, \ldots, X_N) - E[g(X_1, \ldots, X_N)]\}^2]$$

where $\text{Var}[]$ represents the standard deviation and variance operators. Similarly, using the Delta method [24] and following the derivation in [10], we obtain:

$$\text{Var}(g(X_1, \ldots, X_N)) = \sum_{i=1}^{N} \sum_{j=1}^{N} \text{cov}(X_i, X_j)\mu_{X_i}\mu_{X_j} \times \text{Var}(g(X_i, X_j)).$$

B. Mean Field Equations

Let the function $g(.)$ be defined by (1) or (2). Using (3) in (1) and (2), we obtain

$$E\left[ g(X_1^{n+1/2}, X_2^{n+1/2}) \right] = \mu_1 E\left[ E_x^{n+1/2} \right]$$

$$= -\frac{\mu_{\varepsilon_r}}{\Delta z} \left( H_y^{n+1/2}_{i+1/2,j+1,k+1/2} - E[H_y^{n+1/2}_{i+1/2,j+1,k-1/2}] \right)$$

$$+ \frac{\mu_2}{\Delta y} \left( H_x^{n+1/2}_{i,j+1/2,k+1/2} - E[H_x^{n+1/2}_{i,j+1/2,k-1/2}] \right)$$

As a result, the mean and variance field update equations of the proposed S-HIE-FDTD involve only the correlation coefficients between field samples, $\varepsilon_r$ and $\sigma$, in $\varepsilon_r$, $\sigma$, and $\rho$, as the mean values of $\varepsilon_r$ and $\sigma$. Comparing (8) and (9) with (1) and (2), we can see that the mean field update equation is the same as the “classical” field update equation with the variables replaced by their mean values. In (8) and (9), the mean values of the field samples are the unknowns to be updated during marching in time.
C. Variance Field Equations

Let the function \( g() \) be defined by (1). The first step in the derivation of the variance field equation for \( E_x \) is to take the variance of both sides of (1)

\[
S^2 \left[ E_{x_{i+1/2,j,k}}^{n+1} - v_1 E_{x_{i+1/2,j,k}}^n \right] = S^2 \left[ -\frac{v_2}{\Delta z} (H_y_{i+1/2,j,k}^{n+1/2} - H_y_{i+1/2,j,k}^n) + \frac{v_2}{2\Delta y} (H_x_{i+1/2,j+1/2,k}^{n+1} - H_x_{i+1/2,j+1/2,k}^n) + H_y_{i+1/2,j+1/2,k}^{n+1} - H_y_{i+1/2,j+1/2,k}^n) \right].
\]

(10)

Applying (6) to the left-hand side of (10) and using (7) while noticing \( v_1 \) is a function of random variables \( \epsilon_r \) and \( \sigma \), we obtain

\[
S^2 \left[ E_{x_{i+1/2,j,k}}^n - v_1 E_{x_{i+1/2,j,k}}^n \right] = \left( S\left[ E_{x_{i+1/2,j,k}}^n \right] - S[v_1 E_{x_{i+1/2,j,k}}^n] \right)^2.
\]

(11)

Applying (5) into \( S^2[v_1 E_{x_{i+1/2,j,k}}^n] \) and using (7), we obtain

\[
S[v_1 E_{x_{i+1/2,j,k}}^n] = \mu v_1 S[E_{x_{i+1/2,j,k}}^n] + \kappa_1 E_s E_{x_{i+1/2,j,k}}^n.
\]

(12)

Applying (5) and (6) multiple times to the right-hand side of (10) and using (7), we obtain

\[
S^2[v_2 h] = (\mu v_2 h_E)^2 + 2\mu v_2^{\delta_2} E_s h_s + \mu v_2^{\delta_3} E[h_s].
\]

(13)

In (12) and (13)

\[
\kappa_1 = 4\Delta t \epsilon_r \epsilon_r E_s \sigma - \mu v_1 \sigma \Delta t \right)^2
\]

\[
\kappa_2 = 4\Delta t (\mu_1 \mu_2 \epsilon_r E_s \sigma - \mu_2 \mu_2 \sigma \Delta t \right)^2
\]

\[
\kappa_3 = 4\Delta t^{\frac{1}{2}} + 4\Delta t^{\frac{3}{2}} + 4\Delta t^{\frac{5}{2}} - \mu v_1 \sigma \Delta t
\]

\[
h_{i+1/2,j+1/2,k} = \left( H_y_{i+1/2,j+1/2,k}^{n+1} - H_y_{i+1/2,j+1/2,k}^n \right)
\]

\[
S[H_y_{i+1/2,j+1/2,k}^{n+1/2}] - S[H_y_{i+1/2,j+1/2,k}^n] \right)^2.
\]

(17)

Applying (6) to the left- and right-hand sides of (17) and using (7) in the resulting equation, we obtain

\[
S[H_y_{i+1/2,j+1/2,k}^{n+1} - S[H_y_{i+1/2,j+1/2,k}^n] \right)^2.
\]

(18)

Note that \( v_3 \) is a constant (not a function of a random variable). Taking the square root of (18) and rearranging the terms yield

\[
S[H_y_{i+1/2,j+1/2,k}^{n+1/2}] - S[H_y_{i+1/2,j+1/2,k}^n] \right)^2.
\]

(19)

In (19), the standard deviation values of the field samples are the unknowns to be updated during marching in time.

A closer look at (16) and (19) reveals that the unknowns that are most advanced in time, i.e., \( S[H_y_{i+1/2,j+1/2,k}^n] \) and \( S[E_{x_{i+1/2,j,k}}^n] \), are “coupled.” To “decouple” them, we insert (19) into (16) and an explicit equation in \( S[E_{x_{i+1/2,j,k}}^n] \)

\[
(1 + \kappa_4) S[E_{x_{i+1/2,j,k}}^n] + 0.5\kappa_4 \left( S[E_{x_{i+1/2,j,k}}^n] + S[E_{x_{i+1/2,j,k}}^n] \right)
\]

\[
+ \mu v_1 \left( S[H_y_{i+1/2,j+1/2,k}^n] - S[H_y_{i+1/2,j+1/2,k}^n] \right)^2.
\]

(14)

Assuming \( \mu v_2 h_E + \mu v_2 \kappa_2 h_s^2 \leq h_s^2 \), we obtain

\[
S[v_2 h] = (\mu v_2 h_E + \mu v_2 \kappa_2 E_s h_s)^2 + \mu v_2 \kappa_2 E_s h_s^2 + \mu v_2 \kappa_3 E_s h_s^2.
\]

(15)

In (16), the standard deviation values of the field samples are the unknowns to be updated during marching in time.

D. Numerical Stability Analysis

For the sake of simplicity of the numerical stability analysis, the conductivity is assumed deterministic and set to zero. Following the Fourier method [10], the CFL condition of the S-HIE-FDTD is
TABLE I

| Dielectric | \( \mu_{\text{eff}} \) | \( \mu_{\text{r}} \) | \( \mu_{\text{r}} \) (S/m) | \( \sigma_{\text{r}} \) |
|------------|----------------|----------------|-----------------|----------------|
| Skin       | 35.0           | 4.6            | 0.87            | 0.10          |
| Fat        | 16.2           | 2.7            | 0.214           | 0.06          |
| Muscle     | 39.0           | 3.4            | 0.43            | 0.10          |

In this section, the accuracy and efficiency of the S-HIE-FDTD are compared with those of a traditional MC method [9] and/or the S-FDTD [10] for two numerical examples. In both the examples, absorbing boundaries enforce the first-order standard Mur absorbing boundary condition (ABC), as described in [10] and [11], and the plane wave excitation is introduced using the total-field/scattered-field (TF/SF) technique described in [11]. All the computations are performed on a desktop computer with a 32-GB RAM and a 3.90-GHz Intel Core i7-3770 processor.

A. Wave-Propagation Through Multilayer Biological Tissue

For the first example, we use the proposed method to analyze EM wave propagation through human tissue consisting of three layers: skin, fat, and muscle (see Fig. 1). The mean and standard deviation of the conductivity and permittivity of these layers are given in Table I. The simulation domain used in this problem is shown in Fig. 1. The dimensions of the simulation domain are given in Table I. The simulation domain used in this problem is shown in Fig. 1. The dimensions of the simulation domain are given in Table I. The simulation domain used in this problem is shown in Fig. 1. The dimensions of the simulation domain are given in Table I.

The simulation domain used for the analysis of wave propagation through a multilayer human tissue.

![Image](image.png)

where \( c \) denotes the speed of light in free space. This CFL condition uses only the mean value of the permittivity but not its variance. In other words, the CFL of the S-HIE-FDTD is the same as that of the deterministic HIE-FDTD with permittivity replaced with its mean.

\[
\Delta t \leq \frac{\sqrt{\mu_{\text{eff}}}}{c (\sqrt{(\Delta x)^{-2} + (\Delta z)^{-2}})} \tag{21}
\]

III. NUMERICAL EXAMPLES

The results obtained by the MC method, S-FDTD, and S-HIE-FDTD agree very well with each other for both values of \( \rho \), and are compared with those of a traditional MC method [9] and/or the S-FDTD [10] for two numerical examples. In both the examples, absorbing boundaries enforce the first-order standard Mur absorbing boundary condition (ABC), as described in [10] and [11], and the plane wave excitation is introduced using the total-field/scattered-field (TF/SF) technique described in [11]. All the computations are performed on a desktop computer with a 32-GB RAM and a 3.90-GHz Intel Core i7-3770 processor.

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The simulation domain used for the analysis of wave propagation through a multilayer human tissue.

![Image](image.png)

For each execution, the permittivity and conductivity of the tissue layers are selected independently of the Gaussian distributions with the mean and standard deviation, as provided in Table I. The simulation domain used in this problem is shown in Fig. 1. The dimensions of the simulation domain are given in Table I. The simulation domain used in this problem is shown in Fig. 1. The dimensions of the simulation domain are given in Table I.

In the second set of simulations, the thickness of the skin, fat, and muscle layers is 5.4, 54, and 42 mm, respectively. The spatial mesh uses \( \Delta t = \Delta x = \Delta z = 1 \text{ mm} \) and \( \Delta t = 6.67 \times 10^{-3} \text{ ns} \) for the S-HIE-FDTD and S-FDTD. The MC method executes the (deterministic) FDTD for 10,000 times. For each execution, the permittivity and conductivity of the tissue layers are selected independently of the Gaussian distributions with the mean and standard deviation, as provided in Table I. The S-HIE-FDTD, S-FDTD, and MC method are used to compute the mean and variance of the electric field’s \( x \)-component at \( x = 5 \text{ mm}, y \in [0, 500] \text{ mm}, \) and \( z = 5 \text{ mm} \). Fig. 2(a) and (b) shows the plots of the amplitude and phase of this mean (after transformed into frequency domain at 2 GHz) versus \( y \) and demonstrates that all three methods produce practically the same mean values everywhere in the simulation domain. The fact that the result from the MC method matches those obtained by the S-FDTD and S-HIE-FDTD shows that the assumption of the high correlation between the samples of the electric field and between the samples of the magnetic field [see (7)] is accurate. Fig. 3 shows the plots of the time-domain amplitude of the variance versus \( y \). The results obtained by the S-HIE-FDTD and S-FDTD agree very well with each other for both values of \( \rho \). However, results with \( \rho = 0.5 \) agree better with the result obtained by the MC method. We believe this means that setting \( \rho = 1 \) overestimates the correlation.

In the second set of simulations, the thickness of the skin, fat, and muscle layers is 5.4, 54, and 42 mm, respectively. The spatial mesh uses \( \Delta x / 3 = \Delta y = \Delta z / 3 = 0.5 \text{ mm} \) and \( \Delta t = 3 \times 10^{-3} \text{ ns} \) for the S-HIE-FDTD and S-FDTD. Note that \( \Delta y \) is smaller than \( \Delta x \) and \( \Delta z \) because the layers are located along the \( y \)-direction and skin layer is very thin. Also, note that the CFL condition of the S-HIE-FDTD has no constraint in the \( y \)-direction, that is why \( \Delta t \) of the S-HIE-FDTD does not have to be reduced and is larger than \( \Delta t \) of the S-FDTD. The rest of the simulation parameters is the same as those used in the first set of simulations.
B. Transmission Through an FSS

In the second example, we use the proposed method to analyze transmission through an FSS [21] supported by a dielectric substrate (see Fig. 6). The top view of FSS unit cell is shown in Fig. 6(a).

Table II

| Method          | Total iterations | Simulation time(s) | Memory storage(GB) |
|-----------------|------------------|--------------------|--------------------|
| MC(FDTD)        | 15000            | 630503.23          | 139.43             |
| S-FDTD          | 15000            | 163.88             | 10.38              |
| S-HIE-FDTD      | 5000             | 96.76              | 12.80              |

The S-HIE-FDTD, S-FDTD, and MC method are used to compute the mean and variance of the electric field’s x-component at \( x = 5 \) mm, \( y \in [0, 500] \) mm, and \( z = 5 \) mm. Fig. 4(a) and (b) shows the plots of the amplitude and phase of this mean (after transformed to the frequency domain at 2 GHz) versus \( y \). The same methods are used to compute the variance of the electric field’s x-component at \( x = 5 \) mm, \( y \in [0, 500] \) mm, and \( z = 5 \) mm. Fig. 5 shows the plots of the time-domain amplitude of this variance versus \( y \). The conclusions drawn for the results of the first set of simulations are also valid here.

Table II presents the computational requirements of the MC method, S-HIE-FDTD, and S-FDTD. The S-FDTD and S-HIE-FDTD are significantly faster than the MC method. The number of (total) iterations is equal to the number of time steps multiplied by the number of executions. The number of executions is 1 for the S-FDTD and S-HIE-FDTD and 10000 for the MC method. The S-HIE-FDTD is roughly 1.7 times faster than the S-FDTD with only a 20% increase in the memory requirement. The memory requirement of the MC method is significantly larger than those of the S-HIE-FDTD and S-FDTD since it stores field samples needed to compute the mean and variance.

We note here that the Fourier transform of the variance of a time-domain field is not equal to the variance of its frequency-domain counterpart (variance is nonlinear in field values). That is why we plot the variance in the time domain in Figs. 3 and 5. This is also the approach adopted in several other communications [10], [25].
of the substrates results in a shift in the resonance frequency of the transmittance. Fig. 8 shows the plots of M obtained using the S-HIE-FDTD with \( \mu_\varepsilon \in \{1.9, 2.2, 2.4, 2.6\} \). A red/blue shift is observed with the increase/decrease of the permittivity that coincides with the results of M+4/S−M−S. It should be noted that the shift of the peak in Fig. 8 is more obvious than that in Fig. 7. The reason is that \( \mu_\varepsilon \) is fixed, and the plus and minus S (see Fig. 7) only leads to a weak influence compared with the change of mean (see Fig. 8).

Table III presents the computational requirements of the S-HIE-FDTD with different means of substrate’s permittivity.

| Method          | Total iterations | Simulation time(s) | Memory storage(MB) |
|-----------------|------------------|--------------------|--------------------|
| S-FDTD          | 10000            | 1213.36            | 45.83              |
| S-HIE-FDTD      | 2500             | 528.32             | 55.57              |

IV. CONCLUSION

An S-HIE-FDTD to compute the mean and variance of the EM fields by a single simulation given those of the conductivity and permittivity in the computation domain is presented. The mean and variance field update equations are derived. The CFL condition of the S-HIE-FDTD method resembles that of the HIE-FDTD method, resulting in a time step size larger than that would be used by an explicit scheme.

Two numerical examples are provided to show the accuracy, efficiency, and applicability of the proposed S-HIE-FDTD. In the first example, EM wave propagation through a human tissue consisting of three layers is analyzed. In each layer, the permittivity and conductivity are defined by their mean and standard deviation. In the second example, transmission through an FSS supported by a dielectric substrate is analyzed. The permittivity of this substrate is defined by its mean and variance. The results show that the proposed method can dramatically reduce the computation time without any reduction in the accuracy level compared with the S-FDTD method. However, the variance of the EM fields still demonstrates a difference with respect to results obtained using the MC method; we believe this is due to the approximation of the correlation coefficient between the field samples and the permittivity/conductivity.

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