On the stable configuration of ultra-relativistic material spheres.  
The solution for the extremely hot gas

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During the last stage of collapse of a compact object into the horizon of events, the potential energy of its surface layer decreases to a negative value below all limits. The energy-conservation law requires an appearance of a positive-valued energy to balance the decrease. We derive the internal-state properties of the ideal gas situated in an extremely strong, ultra-relativistic gravitational field and suggest to apply our result to a compact object with the radius which is slightly larger than or equal to the Schwarzschild’s gravitational radius. On the surface of the object, we find that the extreme attractivity of the gravity is accompanied with an extremely high internal, heat energy. This internal energy implies a correspondingly high pressure, the gradient of which has such a behavior that it can compete with the gravity. In a more detail, we find the equation of state in the case when the magnitude of the potential-type energy of constituting gas particles is much larger than their rest energy. This equation appears to be identical with the general-relativity condition of the equilibrium between the gravity and pressure gradient. The consequences of the identity are discussed.

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Motto: "Chandrasekhar shows that a star of mass greater than a certain limit remains a perfect gas and can never cool down. The star has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few kilometers’ radius when gravity becomes strong enough to hold the radiation and the star can at least find peace. I felt driven to the conclusion that this was almost a reductio ad absurdum of the relativistic degeneracy formula. Various accidents may intervene to save the star, but I want more protection than that. I think that there should be a law of Nature to prevent the star from behaving in this absurd way.”

Sir Arthur Eddington

I. INTRODUCTION

After a star with a mass lower than the Oppenheimer-Volkoff limit has spent its nuclear fuel, it becomes either the white dwarf or neutron star. The rest of its thermal energy is irradiated into the neighbouring space and, thus, its temperature permanently decreases, down to a value approaching the absolute zero. Its internal structure can be described with the help of theory of cold degenerated electron or neutron gas.

In this work, we polemize with the current theory of the supplementary case: the end-state of the star with the final mass exceeding the Oppenheimer-Volkoff limit. Namely, there are several serious arguments that such a star cannot end as a cold object. Even if it was cooled enough during the first stage of the collapse, its internal energy and temperature can be expected to increase over all limits, when its radius approaches the horizon of events, which was found at the Schwarzschild gravitational radius, $R_g$. More specifically, the increase of internal energy can be expected when the proper radius of the object is reduced below, say, $1.01R_g$ or $1.001R_g$.

Our considerations are based on the classical quantum statistics and classical general relativity (with no-hair theorems valid), which both are well-known a long time. Only new element concerning the assumptions in this work is a term corresponding with a potential-type energy in the formulas of quantum-statistics equations of state for the extremely hot gas. In fact, our idea is not new, in principle. The potential-energy term is well known in the classical Maxwell-Boltzmann impulse-distribution law of ideal gases having a low internal energy. In the latter, it is identical to the classical potential energy and situated in the Boltzmann part of the distribution law.

In our description of gas properties, we use the formula for the relativistic energy which is suitable in a situation when the rest energy of gas particles can be neglected. In the classical works by Landau, Chandrasekhar, or Oppenheimer & Volkoff, the influence of conservative gravitational field on the state properties of the gas was not taken into account. A step toward an inclusion of this influence occurred in more recent studies of stellar collapse and neutron stars. Prakash et al. considered the potential-energy term in course to provide the equation of state of a hot gas, which constitutes a neutron star a short time after its formation. The authors, however, considered...
that approximative form of the relativistic formula for energy, which is applicable at the values much lower than the rest energy. Their work was focused especially on the nuclear reactions and alternative forms of matter constituting the star. They did not deal with any collapse going on down to the gravitational radius. Very recently, Moustakidis & Panos, who continued in the work of Prakash et al., provided an equation of state for a hot nuclear matter with the $\beta$-decay equilibrium. This kind of matter was again assumed to constitute a neutron star in less extreme conditions, allowing neutrons to be unstable, than we are going to consider.

The following arguments can be presented to support the concept of the occurrence of extremely hot gas in an object with radius approaching $R_g$.

(1) Let us consider a radiation coming from an ultra-relativistic compact object to an observer at a relatively large distance. In his own reference frame, the observer detects the radiation which is red-shifted. In other words, its energy is lower in the reference frame of the observer than was at the moment of its emission, in the reference frame of the source. In a reverse situation, when a radiation comes onto the surface of the compact objects from a distant source, its wave-length is blue-shifted. This time, its energy is increasing with time. It is higher in the object-surface reference frame than in the frame of the distant source.

The numerical calculation reveals that the frequency of the radiation coming to the horizon at $R_g$ from an outer space increases over all limits implying the corresponding energy to increase over all limits. Such the increase must, however, occur not only in the case of radiation, but in the case of non-zero rest-mass particles, when these are falling down, as well. The requirement of the local conservation of energy in a volume of the compact-object-surface layer means that the increase must be, on the other hand-side, balanced by a decrease of that type energy of the surface layer, which has a character of potential energy. We refer to this energy as the "potential-type energy".

Therefore, when the collapsing-object radius approaches the gravitational radius, its internal thermal energy must increase over all limits, because we do not know any efficient mechanism of the extreme-rate cooling. A fraction of internal energy is irradiated in form of photons and carried out by neutrinos only from the object’s surface. Even in this case, the efficiency of both cooling processes decreases down to zero when the gravitational radius is approached (see our more specific considerations in Sect. V). Thus, an extreme accumulation of the internal energy seems to be inevitable in the last stage of the collapse to the horizon and can be the cause of a strong pressure with a steep gradient resisting to the gravity. After the object’s radius reaches the horizon, the accumulated energy is trapped inside the object forever. (Some exceptions resulting from our work are discussed in Sect. VII.) In the context of our work, we do not consider fine effects as, e.g., Hawking’s evaporation of black holes.)

(2) Tolman and Tolman & Ehrenfest found that inside the sphere filled in with an ideal material fluid, the product of both proper temperature of the fluid, $T$, and square root of component $g_{44}$ of metric tensor is a finite constant, i.e.

$$ T \sqrt{g_{44}} = T \exp(\nu/2) = T_c = \text{const.} $$

$(\exp(\nu)$ is an alternative expression of $g_{44}$; see Eq. (2) and the text below this equation). It means that if a volume of the fluid is situated near the surface of the sphere, the radius of which approaches $R_g$, then $g_{44} \to 0$ and the temperature of the fluid must approach infinity the product of both could be the finite constant.

(3) At the same time, Tolman & Ehrenfest demonstrated that the quantity $P_{\text{rad}} g_{44}$ is equal to a finite constant inside the sphere filled in with the black-body radiation. $P_{\text{rad}}$ is the radiation pressure. When a volume of gaseous material fluid sinks to a more and more curved space-time at the gravitational radius, radiation pressure $P_{\text{rad}}$ must be magnified over all limits to keep the constancy of $P_{\text{rad}} g_{44}$. The requirement of this finite constancy also supports the hot concept of final stage of very massive stars.

Considering all these facts, we search for the "hot" solution of the static configuration of a spherically symmetric object with no generated energy. Specifically, we give a brief summary of the theory and results that have been achieved up to date in Sect. III. The introduction of a new modification of concerning fundamental theory with some obvious steps toward the solution of established fundamental integrals is presented in Sect. III. In Sect. IV, we derive the solution, which is applicable on an extremely hot gas. In more detail, we derive the state quantities of the ideal gas situated in a region of extremely curved space-time of the stable spherically symmetric object, whereby the curvature is so large that the magnitude of the potential-type energy of a given particle is much greater than its rest energy. In such a gas, a radiation must necessarily occur. So, an influence of the radiation pressure on the object's stability is discussed in Sect. V. In Sect. VI, we attempt roughly sketch a behavior of the collapse of object before it disappears below horizon. Finally, we summarize our most important results and discuss some implications toward a potential observational evidence of this kind of objects, in Sect. VII.

II. THE OLD THEORY

The theory of the object contracting to approach a zero radius was completed by Oppenheimer & Volkoff. They went on in the work started by Landau, who showed that for a model of the star consisting of a cold degenerate Fermi-Dirac gas, after all the elements inside available for thermonuclear reactions are
used up, there exist no stable equilibrium configurations for masses greater than a certain critical mass. In contrast to Landau, who considered the Newtonian gravitational theory, Oppenheimer & Volkoff accounted for the general relativistic effects, which were expected for the very high masses and densities.

Oppenheimer & Volkoff \[4\] considered the static line element known for the spherical symmetry in the form

\[ds^2 = -e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^\nu dt^2, \]

where \(r\), \(\theta\), and \(\phi\) are common spherical coordinates, \(t\) is time, and \(\lambda\), \(\nu\) are functions of \(r\). They further considered the Einstein equations of field written for the case of spherical symmetry \[9\], in which they put the expressions of the non-zero components of energy tensor \(T^1_1 = T^2_2 = T^3_3 = -P_{\text{gas}}\) and \(T^4_4 = E_{\text{gas}}\) expressed in this form by Tolman \[10\] (see Eqs. (17), (18), and (19), in Sect. 3). \(P_{\text{gas}}\) and \(E_{\text{gas}}\) are the gas pressure and its energy density (internal energy of gas per unit volume), respectively. The contra-variant components of the energy tensor for an ideal material fluid, \(T^{11} = T^{22} = T^{33} = -P_{\text{gas}}\) and \(T^{44} = E_{\text{gas}}\), were specified earlier by Eddington \[9\].

To solve the field equations, Oppenheimer & Volkoff established the auxiliary function \(u = u(r)\) equal to

\[u = \frac{1}{2} r(1 - e^{-\lambda}), \]

with the help of which the equations of field could be written in form (in SI units used throughout our paper)

\[\frac{du}{dr} = \frac{4\pi G}{c^4} E_{\text{gas}} r^2, \]

\[\frac{dP_{\text{gas}}}{dr} = -\frac{P_{\text{gas}} + E_{\text{gas}}}{r(r - 2u)} \left[\frac{4\pi G}{c^4} P_{\text{gas}} r^3 + u\right], \]

where \(G\) is the gravitational constant and \(c\) is the velocity of light. For \(P_{\text{gas}}\) and \(E_{\text{gas}}\), they supplied the parametric form of the equation of state, for the cold degenerated Fermi-Dirac gas, found by Chandrasekhar \[[3], \text{appendix}]\). After the analysis of the possible solutions of final equations, Oppenheimer & Volkoff concluded that there was not any stable solution for the object having the mass exceeding the finite limit they specified.

If their conclusion was indisputable, our re-opening the problem on the stable configuration would be meaningless. However, we find three of their assumptions disputable. At first, they assumed that the collapsing object is cold enough for the theory of the cold degenerated Fermi-Dirac-statistics gas to be applicable. As we argued in Sect. 4 the decreasing potential-type energy should be balanced by an increase of the thermal energy and this process leads, instead, to a hot gas. The extreme increase occurs especially at the last stage of collapse in proper time, but above the event horizon, when the physical radius of collapsing object is close to \(R_g\) and metriquiv-tensor components \(g_{11}\) and \(g_{44}\) start significantly deviate from unity. So, even if the object was cold enough at a proper radius several times larger than \(R_g\), it must be highly heated during this last stage.

Interestingly, Oppenheimer & Volkoff noted, in their paper \[4\], that for those singular solutions in which \(g_{44}\) vanishes, it is conceivable that the temperature may be high with the respect to the Tolman & Ehrenfest \[5\] relation \(T \sqrt{g_{44}} = \text{const}\). And, the equation of state reduces to \(E_{\text{gas}} = 3P_{\text{gas}}\) in the limit of \(T \rightarrow \infty\) (which we analyse in Sect. (IV)). However, they supposed the appropriate conditions for the very high temperature only in the center of the object. They did not noticed that the appropriate conditions may also exist in the region around the border of the sphere of radius equal to \(R_g\).

At second, they required the pressure approached zero at the surface of the object, i.e. for \(r = R_b\) (\(R_b\) is the proper radius of the object, i.e. the distance of the object’s surface from its center measured in the object’s surface reference frame). In everyday experience, we know, however, a lot of examples of pressure or density discontinuities: the border between a rocky earth surface and air, the sea level and air, etc. Another example can be found in the outer space, where neutron stars are expected to have a very large density at the surface, but, of course, zero density above the surface. We cannot a priori exclude the possibility that the pressure (or energy density or particle concentration) is finite in the surface layer between radii \(R_b - dr\) and \(R_b\) and zero in the adjacent layer above, between \(R_b\) and \(R_b + dr\).

At third, they started the numerical integration of the final equations with the initial value \(u_o = u(r = 0)\) being in the interval \(0 \geq u_o \geq -\infty\). They excluded the possibility \(u_o > 0\) arguing that the component of metriquiv tensor \(g_{11} = -e^\lambda\) must never be positive, therefore \(u_o = 0\) for all finite values of \(e^{-\lambda}\) and \(u_o \leq 0\) for infinite values of \(e^{-\lambda}\) at the origin. The argument of not positive \(-e^{-\lambda}\) is, however, questionable. Let us consider the object which already collapsed below the gravitational radius. In the interval of distances between \(R_b\) and \(R_g\), the well-known outer Schwarzschild solution is applicable. According to this solution \(-e^{-\lambda} = -1/(1 - R_g/r) > 0\) for \(R_b < r < R_g\). Therefore, if we rejected the possibility \(-e^{-\lambda} > 0\), there would not exist not only any solution for the stable configuration of an extinct star, but we would not have any acceptable description of the field for \(r < R_g\), either. Consequently, the reason for the infinite collapse due to the gravitational attraction would disappear. But we believe that it is reasonable to retain the possibility \(-e^{-\lambda} > 0\). And, if we accept this possibility for \(r < R_g\), then why we should not doing so for \(r \geq R_g\)?

### III. THE BASIC CONSIDERATIONS

Let us consider a material object, which is spherically symmetric and ultra-compact, i.e. its proper physical radius, \(R_b\), is only slightly larger than its Schwarzschild gravitational radius, \(R_g\). To better explain the conse-
quences resulting from the new formulas we find below, we consider the idealized, non-rotating object that is in or near its supposed stable configuration. Hereinafter, we refer to this object as the "ultra-relativistic material sphere" (URMS).

In this section, we summarize the well-known fundamental equations of the problem of URMS stable configuration and describe several first steps to obtain the solution for the stable URMS. Within the summary, we introduce our own modification of the fundamental equations necessary for taking the strong, ultra-relativistic gravitational field into account. The steps toward the solution can be made generally for the gas consisting of a single kind of particles having spin 1/2 (e.g. neutrons). No other assumptions are needed.

In the following, we consider the reference frame related to a volume of URMS' surface layer. If the URMS is stable, then the number density, $n_{gas}$, pressure, $P_{gas}$, and internal energy per unit volume, $E_{gas}$, of the gas in the surface layer can generally be calculated, as the functions of the impulse $p$, with the help of well-known integrals (e.g. \[ n_{gas} = \frac{1}{4\pi} \int_{p} n_{gas}(p) 4\pi p^2 dp, \] \[ P_{gas} = \frac{1}{3} \int_{p} n_{gas}(p) p^2 4\pi p^2 dp, \] \[ E_{gas} = \int_{p} n_{gas}(p) E(p) 4\pi p^2 dp, \] where the kinetic energy $E$ and velocity $V$ of a particle can be calculated by the well-know relativistic formula \[ \mathcal{E}(p) = \sqrt{c^2 p^2 + W^2} - W_0, \] \[ V = \frac{\partial \mathcal{E}}{\partial p} = \frac{c^2 p}{\sqrt{c^2 p^2 + W^2}}. \] Symbol $W_0$ stands for the particle’s rest energy.

The number density $n_{gas}(p)$ for the ideal gas is \[ n_{gas}(p) = K_N \sum_j f_j \frac{1}{\exp\left(\frac{W_j(p) - \mu}{(kT)}\right) + 1}, \] where $K_N$ is a normalization constant, $k$ is the Boltzmann constant, $T$ is the temperature of the gas, $f_j$ is the degeneracy of state $j$ (i.e., it gives the number of states having the same energy $W_j$), and $\mu$ is the chemical potential.

The energy state of a particle $W_j$ in the argument of exponential in Eq. (1) must be of the type varying with the impulse $p$ and with the force field in which the gas is situated. In the limit of a weak field, the part $W_j/(kT)$ of the argument in Eq. (1) must converge to expression $p^2/(2mkT) + W_0/(kT)$ figuring in the Maxwell-Boltzmann distribution law. So, the energy does not contain the rest-energy term, $W_0$. It is only the sum of kinetic and potential-type energy, i.e.

\[ W_{k+p} = \sqrt{c^2 p^2 + W^2} - W_0 - \chi, \] where $-\chi$ is the potential-type energy of the particle. By the convention used in our work, the fact that the gravity is attractive force is expressed with sign minus in front of $\chi$ in Eq. (12). Within this convention, it is always valid that $\chi \geq 0$. Completing the description of the energy state with term $-\chi$ is an essential point of our work. In Sects. VII, VIII we derive some important consequences of this additional term.

The relativistic formula (12), but with the rest-energy term included, is well-known in quantum physics. Its validity was proved, for example, in the construction of the Klein-Gordon equation, which is the basis to derive the well-known Dirac equations providing the energy terms of hydrogen atom, which are in a very good agreement with the experimental values. In this usage, $-\chi$ is identified to the classical Coulomb energy for a static system. In the strong, ultra-relativistic gravitational field, the function $\chi = \chi(r)$ cannot be, of course, identified with the well-known formula for the potential energy in the Newtonian gravitational field. The explicit form of this function for a strong field is unknown. We discuss its dependence on the quantity $v$ (in metric tensor; see Eq. 2) in the case of the extremely strong field in the next section. For now, we profit from working with the implicit $\chi$. Fortunately, a more specific expression of $\chi$ appears not necessary.

In the context of function $\chi$, the "Fermi impulse", $p_F$, corresponding to the Fermi energy, is

\[ p_F = \frac{1}{c} \sqrt{(\chi + W_0 + \mu)^2 - W_0^2}. \]

This relation can be derived requiring the argument of exponential in Eq. (11) to be zero and identifying the energy $W_{k+p}$ given by Eq. (12) to the Fermi energy of the particle. We note that $p_F = \chi/c$ for $\chi \gg W_0$ and, of course, $p_F \to \infty$ for $\chi \to \infty$.

The energy states, $W_j$, in Eq. (11) differ each other by their rest energies, $W_{j\alpha}$, for various kinds of the particles. Since we consider, in this section, only the gas consisting of one-half spin particles of one kind (neutrons), the sum in Eq. (11) consists of only a single term ($j = 1$) and $f_1 = 2$.

The size of impulse $p$ can vary from zero to infinity. So, the number density, pressure, and internal energy per unit volume are the integrals

\[ n_{gas} = K \int_{0}^{\infty} \frac{p^2}{\exp\left(\frac{\sqrt{c^2 p^2 + W^2} - W_0 - \chi - \mu}{kT}\right) + 1} dp. \]
\[ P_{\text{gas}} = \frac{Kc^2}{3} \int_0^\infty \frac{p^4}{\sqrt{c^2p^2 + W_o^2}} \cdot \exp \left( \frac{\sqrt{c^2p^2 + W_o^2 - W_o - \chi - \mu}}{kT} \right) + 1 \]^{-1} dp, \quad (15) \]

\[ E_{\text{gas}} = K \int_0^\infty - \frac{p^2}{\exp \left( \frac{\sqrt{c^2p^2 + W_o^2 - W_o - \chi - \mu}}{kT} \right) + 1} \cdot \left( \frac{\sqrt{c^2p^2 + W_o^2 - W_o - \chi - \mu}}{kT} \right) + 1 \]^{-1} dp, \quad (16) \]

respectively. We denoted \( K = 8\pi K_N \).

For the URMS, the Einstein’s equations of field can be written in their form for the spherical symmetry \( [9], [10] \) (see also \( [12], \) §95):

\[ \frac{8\pi G}{c^4} P = \exp(-\lambda) \left( \frac{1}{r^2} \right) - \frac{1}{r^2}, \quad (17) \]

\[ \frac{8\pi G}{c^4} P = \exp(-\lambda) \left( \frac{1}{r^2} \right) - \frac{1}{r^2}, \quad (18) \]

\[ \frac{8\pi G}{c^4} E = \exp(-\lambda) \left( \frac{1}{r^2} \right) - \frac{1}{r^2}, \quad (19) \]

in the SI units. \( P \) and \( E \) are the total pressure and total energy density in a given volume of space in the proper reference frame. The cosmological constant, \( \Lambda \), is put zero, since we assume that the URMS can occupy a volume many orders of magnitude smaller than the cosmological distance-scale.

It is well-known \( [10], [12] \) that we can derive the equation

\[ \frac{dP}{dr} = -\frac{E + P}{2} \frac{d\nu}{dr}, \quad (20) \]

from Eqs.\( (17) - (19) \). This is the relativistic analogue of the Newtonian relation

\[ \frac{dP}{dr} = \rho \frac{d\Psi}{dr} = -\rho \frac{GM_r}{r^2}, \quad (21) \]

giving the dependence of total-pressure gradient, \( dP/dr \), on the gravitational potential, \( \Psi \) \( [12], \) §95). This equation is also known as the equation of the hydrostatic equilibrium in the theory of internal structure of stars (e.g. 11, chap. 1). Quantity \( \rho \) is the density of stellar plasma at the distance \( r \) from the stellar center and \( M_r \) is the mass within radius \( r \). There is an essential difference between the relativistic Eq.\( (20) \) and classical-physics Eq.\( (21) \). The latter does not contain any term corresponding to term \(- (P/2) d\nu/dr\) in Eq.\( (20) \). Recently, Rappaport et al. \( [13] \) and Schwab et al. \( [14] \) represented this term as the self-gravity of pressure.

Our task to find a stable configuration of the URMS will be solved, when we find a reasonable combination of functions \( P, E, \) and \( \nu \), for which Eq.\( (20) \) is satisfied. For the gas, we calculate the corresponding, gas-related part of the left-hand side of Eq.\( (20) \) derivating Eq.\( (15) \), in which the temperature, \( T \), and potential-type energy, \( -\chi \), are expected to be the functions of radial distance, \( r \), from the center of the URMS. (Since we assume the mono-particle gas, the chemical potential, \( \mu \), is obviously not any function of \( r \).) We obtain

\[ \frac{dP_{\text{gas}}}{dr} = \frac{Kc^2}{3kT^2} I_{P1} + \frac{Kc^2}{3kT} \left( \frac{d\chi}{dr} \right) - W_o + \chi + \mu \frac{dT}{T} \frac{dT}{dr} I_{P2}, \quad (22) \]

where

\[ I_{P1} = \int_0^\infty p^4 \exp \left( \frac{\sqrt{c^2p^2 + W_o^2 - W_o - \chi - \mu}}{kT} \right) \right) \right) + 1 \]^{-2} dp, \quad (23) \]

and

\[ I_{P2} = \int_0^\infty p^4 \exp \left( \frac{\sqrt{c^2p^2 + W_o^2 - W_o - \chi - \mu}}{kT} \right) \right) \right) + 1 \]^{-1} dp, \quad (24) \]

Denoting further

\[ S_n = \int_0^\infty \frac{p^2}{\exp \left( \frac{\sqrt{c^2p^2 + W_o^2 - W_o - \chi - \mu}}{kT} \right) + 1} dp, \quad (25) \]

\[ S_P = \int_0^\infty \frac{p^4}{\sqrt{c^2p^2 + W_o^2}} \right) \right) + 1 \]^{-1} dp, \quad (26) \]

and

\[ S_E = \int_0^\infty \frac{p^2}{\sqrt{c^2p^2 + W_o^2}} \right) \right) + 1 \]^{-1} dp, \quad (27) \]
integrals (23) and (24) can be expressed (after one step of the per partes integration) as

\[ I_{P1} = 4kTS_P + \frac{3kTW^2}{c^2}S_E, \quad (28) \]
\[ I_{P2} = \frac{3kT}{c^2}S_n. \quad (29) \]

Since the integrals \( S_n, S_P, \) and \( S_E \) can also be given as \( S_n = n_{gas}/K, S_P = 3P_{gas}/(Kc^2), \) and \( S_E = (E_{gas} - 3P_{gas} + W_o n_{gas})/(KW_o^2), \) the gradient of pressure (22) can be expressed as

\[ \frac{dP_{gas}}{dr} = n_{gas} \frac{d\chi}{dr} + B \frac{dT}{dr}. \quad (30) \]

In the last equation, we denoted

\[ B = P_{gas} + E_{gas} - \chi n_{gas} - \mu n_{gas}. \quad (31) \]

IV. THE SOLUTION FOR AN EXTREMELY HOT GAS

Let us consider the reference frame, to which the line element (2) is referred. It means, it is the proper reference frame of an infinitesimally small gas volume in the URMS surface layer at the distance \( r \) from the center of URMS. We further consider the static case, i.e. the components of the metrique tensor \( g_{11} = \exp(\lambda) \) and \( g_{44} = \exp(\nu) \) are the functions of only the space coordinate \( r. \) According the outer Schwarzschild solution of the Einstein’s equations of field, it is well-known that

\[ g_{11} = -\frac{1}{g_{44}} = -\frac{1}{1 - R_g/r}. \quad (32) \]

For \( r \to R_g, \) there appears the singularity \( |1/g_{44}| \to \infty, \) which is essential in our deductions. The behavior of the state quantities of gas are derived in the coordinate system, in which this singularity appears. Its existence implies that the conditions for the extremely high pressure and temperature must not only at the center of the URMS (at \( r \to 0, \) therefore \( 1/r \to \infty, \)) but around the sphere of radius \( R_g \) as well. This indicates that it is reasonable to investigate, in the search for the URMS’ stability, the properties of the extremely hot, ultra-relativistic gas, the existence of which appears to be very probable in the surface layer of the URMS with the proper physical radius not much different from its gravitational radius.

Close to \( R_g, \) we can obviously expect that \( \chi \to \infty. \) So, let us search for the solution of integrals (14)-(16) assuming that the gas is situated in the gravitational field characterized with the strong inequality \( \chi \gg W_o. \) In other words, we neglect the rest energy of the gas particles with respect to the magnitude of their relativistic potential-type energy, \( \chi. \) Consequently, the relation (12) reduces to

\[ W = cp - \chi. \quad (33) \]

As well, we neglect the chemical potential, \( \mu, \) because any chemistry cannot be important in the considered extreme conditions.

If both \( W_o \) and \( \mu \) are neglected, integrals (14)-(16) are simplified to

\[ n_{gas} = K \int_0^\infty \frac{p^2}{\exp(\frac{cp}{kT}) + 1} dp, \quad (34) \]
\[ P_{gas} = \frac{Kc}{3} \int_0^\infty \frac{p^3}{\exp(\frac{cp}{kT}) + 1} dp, \quad (35) \]
\[ E_{gas} = 3P_{gas}, \quad (36) \]

respectively. One can notice that we also neglect \( W_o \) in the argument of square root \( \sqrt{c^2p^2 + W_o}\) in the denominator of Eq. (15), where the rest energy \( W_o \) is comparable to \( cp \) at the down limit of this integration. Fortunately, this neglect is possible, because the contribution of the integration in this region of \( p \) to the total integral is well-negligible.

Eqs. (34) and (35) do not contain any particle characteristics, except of constant of proportionality \( K. \) Since all following relations are valid for every constant \( K, \) our considerations concerning the hot gas can be generalized for an arbitrary Fermi-Dirac-statistics gas consisting of several kinds of particles characterized with different degeneracy states \( f_j \) and having the partial concentrations \( n_{g,j}. \) In the generalization, the constant \( K = 4\pi f_1, \) for the 1/2-spin mono-particle gas, must only be replaced with constant \( K = (4\pi/n_{gas}) \sum_j n_{g,j} f_j \) for the various-half-spin multi-particle gas.

The integrals of type

\[ I_\gamma = \int_0^\infty \frac{a^\gamma}{\exp(x - b) + 1} dx \quad (37) \]

are well-known in quantum statistics. The approximative solution can be found in form (e.g. 13),

\[ I_\gamma = b^{\gamma+1} \left[ 2 \sum_{j=1}^{\infty} \frac{\alpha_{2j} \Pi(\gamma)}{\Pi(\gamma - 2j + 1)} \Pi(\gamma) \Pi(\gamma + 1) \right] + \Pi(\gamma) R(b, \gamma), \quad (38) \]

where

\[ \alpha_{2j} = \sum_{l=0}^{\infty} \frac{(-1)^l}{(l + 1)^{2j}}, \quad |R(b, \gamma)| \leq \exp(-b), \]
and \( \Pi \) is the factorial function. Below in this section we however find the possibility that exponent \( b \) in the condition delimiting \( \left| R \right| \) may not be large, therefore \( \left| R \right| \) cannot be neglected. For the values of \( \gamma \) figuring in the considered integrals \([31] \) and \([33] \), i.e. \( \gamma = 2 \) and \( 3 \), we utilize the full results of integrations yielding

\[
E = \frac{K}{3e^2} \left[ 1 + \frac{12\alpha_2(kT)^2}{\chi^2} + \frac{6(kT)^2}{\chi^4} \right],
\]

where \( \alpha_2 = \pi^2/12 = 0.822467033 \) and \( \alpha_1 = 0.947032829 \).

Using Eq. \([36] \) and neglecting \( \mu \), the auxiliary quantity \( B \) (see Eq. \([43] \)) is

\[
B = 4P_{gas} - \chi n_{gas}.
\]

Using further Eq. \([11] \), we can find

\[
\frac{1}{T} \frac{dT}{dr} = -\frac{1}{2} \frac{d\mu}{dr}.
\]

With the help of Eqs. \([41] \) and \([42] \), we can write the relation \([40] \) in form

\[
\frac{dP_{gas}}{dr} = -2P_{gas} \frac{d\mu}{dr} + \left( \frac{d\chi}{dr} + \frac{1}{2} \frac{d\nu}{dr} \right) n_{gas}.
\]

For the strong field characterized with condition \( \chi \gg W_o \), function \( \chi = \chi(\nu) \) can be found assuming that the URMS was formed as a product of collapse of an object with so large radius, that the magnitude of potential-type energy of gas particles was much lower than their rest energy, i.e. \( \chi \ll W_o \) for a single particle. And, it was not extremely hot, therefore the kinetic energy of the particles was also much lower than their rest energy, i.e. \( \sqrt{c^2p^2 - W_o^2} \ll W_o \) for a single particle. It means, whole energy of a particle consisted only of the rest energy and we can, in the given context, well approximate the sum of kinetic and potential-type energies by zero (\( W_{k+p} = 0 \)) before the collapse. The kinetic energy per one particle can also be given as \( E_{gas}/n_{gas} \). Since the total energy is expected to be conserved, energy \( W_{k+p} \) must be zero also after the collapse in the situation, when \( \chi \gg W_o \). This requires the validity of \( E_{gas}/n_{gas} - \chi = 0 \).

The energy density, \( E_{gas} \), is proportional to the temperature. Specifically, we express it as

\[
E_{gas} = s n_{gas} kT,
\]

where \( s \) is a constant parameter. Using relation \([11] \) and fact that \( E_{gas}/n_{gas} - \chi = 0 \), function \( \chi \) is

\[
\chi = skT.
\]

If we now use the Tolman & Ehrenfest’s relation \([11] \), we obtain

\[
\chi = skT_e \exp(-\nu/2).
\]

The constant \( skT_e \) can be identified to \( W_o \) because of the following requirement. Let us consider the quantum-physics concept of the wave associated with the particle. If the angular frequency of this associated wave in the free space, far from the URMS, is \( \omega_o \), the wave-related energy of the particle is identical to its rest energy and given by the well-known formula \( W_o = \hbar \omega_o \), where \( \hbar \) is the Planck constant divided by \( 2\pi \). In the place of a considered gas volume, where the potential-type energy is \( \chi \), the angular frequency is \( \omega \) and the corresponding wave-related energy of the particle is \( W_w = \hbar \omega \). Energy \( W_w \) is positive and increases with the increasing magnitude of the gravity. It is obviously a consequence of the fact that just this increase of energy balances the corresponding decrease of the particle’s potential-type energy. Therefore, we can put \( \chi = W_w - W_o \). Using further the well-known transformation between \( \omega \) and \( \omega_o \), i.e. \( \omega = \omega_o/\sqrt{4\pi} \), we finally find

\[
W_w - W_o = \hbar \omega_o \exp(-\nu/2) - \hbar \omega_o,
\]

or, after neglecting \( W_o \),

\[
\chi = W_o \exp(-\nu/2),
\]

where we put \( \hbar \omega_o = W_o \). Comparing the right-hand side of this equation with the right-hand side of Eq. \([40] \), it is clear that constant \( skT_e \) must equal \( W_o \).

In the last step in the search for the URMS stability, let us look for such a configuration of quantities that the quantum-statistics solution for the gradient of pressure satisfies the equation of the equilibrium \([20] \). When Eq. \([35] \) is also the valid relation between the total internal energy density and total pressure, i.e. it is valid that \( E = 3P \) (this validity can be deduced from Eqs. \([30] \) and \([33] \), Eq. \([20] \) changes to

\[
\frac{dP}{dr} = -2P \frac{d\nu}{dr}.
\]

We can see that this functional form for the gradient of pressure, derived from the Einstein’s field equations, is identical to that derived within the quantum Fermi-Dirac statistics (see Eq. \([13] \)), if

\[
\frac{d\chi}{dr} + \frac{1}{2} \frac{d\nu}{dr} = 0.
\]

The validity of this condition can easily be shown differentiating Eq. \([13] \), therefore Eq. \([13] \) becomes simplified to

\[
\frac{dP_{gas}}{dr} = -2P_{gas} \frac{d\nu}{dr}.
\]
If the radiation was completely absent, this would prove that the URMS would be close to its stable configuration at whatever value of the component $\exp(\nu)$ of metrique tensor or whatever radius $R_o$ for which the condition $\chi \gg W_o$ (or $W_o/\chi \ll 1$) is satisfied. (The exact stability would appear for $W_o/\chi \rightarrow 0$. A further discussion of the stability is given in Sect. VI.)

The continuity of the gravity behavior implies that the stability would appear for $\chi \gg W_o$ (or $W_o/\chi \ll 1$) as well as the derivative $d\nu/dr$ at the surface of the URMS should approach their values calculated by the outer Schwarzschild solution, according to

$$\exp(\nu) = 1 - \frac{R_o}{r}
$$

and, subsequently,

$$\frac{d\nu}{dr} = \left(1 - \frac{R_o}{r}\right)^{-1} \frac{R_o}{r^2}.
$$

We can see that $d\nu/dr \rightarrow \infty$ for $r \rightarrow R_o$, therefore also $dP_{gas}/dr \rightarrow \infty$ (see Eq. (51) for $r \rightarrow R_o$).

Integrating Eq. (51), we can find

$$P_{gas} = P_o \exp(-2\nu),
$$

where $P_o$ is an integration constant. This solution transparently implies that $P_{gas} \rightarrow \infty$ when $\exp(-\nu) \rightarrow \infty$. Constant $P_o$ can be found realizing the fact that the solutions (50) and (51) must be identical. Analysing these functional forms, we can find

$$P_o = \frac{K W_o}{12c^3} \left[1 + \frac{24\alpha_2}{s^2} + \frac{48\alpha_4}{s^4} - \frac{24}{s^2} \sum_{j=1}^{\text{max}} \frac{(-1)^{j+1}}{j^2} \exp(-js)\right],
$$

where

$$s = \frac{W_o}{kT_c} = \text{const.}
$$

with respect to the relations (1) and (48).

For the cold, degenerated relativistic Fermi-Dirac gas, the relation between the pressure, $P_{gas}$, and concentration of particles, $n_{gas}$, was found in the form $P_{gas} \propto n_{gas}^{4/3}$ (see, e.g., [16]). It is interesting that this proportionality is also valid in the case of the extremely hot gas. From Eqs. (48) and (54), we can obtain

$$P_{gas} = \frac{P_o}{W_o^4} \chi^4.
$$

Denoting further

$$n_o = \frac{K}{3c^3} \left[1 + \frac{12\alpha_2}{s^2} + \frac{6}{s^3} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} \exp(-js)\right],
$$

we can re-write Eq. (53) as

$$n_{gas} = n_o \chi^3,
$$

from which $\chi = (n_{gas}/n_o)^{1/3}$. Supplying this into Eq. (57), we obtain

$$P_{gas} = \frac{P_o}{W_o^4 n_o^{4/3} n_{gas}^{4/3}} = \text{const.} \times n_{gas}^{4/3}.
$$

V. THE INFLUENCE OF RADIATION

Since the strong gravitational field of the URMS implies a high internal energy of gas and, therefore, a high temperature, we can expect an occurrence of an intensive radiation inside such a gas. The pressure of this radiation has obviously to contribute to the total pressure. Let us estimate an influence of this contribution on the stability of the URMS, in this section.

The statistical behavior of photons of electromagnetic radiation can be described by the Bose-Einstein statistics. Because of a larger context, we give a review of the well-known derivation of basic relations for an arbitrary kind of bosons.

In an analogy to Eqs. (31)–(36) of the Fermi-Dirac statistics, the concentration, pressure, and internal energy per unit of volume can be written as

$$n_{BE} = K \int_0^\infty \frac{p^2}{\exp \left(\frac{p^2 - \chi}{kT}\right) - 1} dp,
$$

$$P_{BE} = \frac{Kc}{3} \int_0^\infty \frac{p^3}{\exp \left(\frac{p^2 - \chi}{kT}\right) - 1} dp, \tag{62}
$$

$$E_{BE} = \frac{Kc}{3} \int_0^\infty \frac{p^3}{\exp \left(\frac{p^2 - \chi}{kT}\right) - 1} dp = 3P_{BE}, \tag{63}
$$

for the photons, or other Bose-Einstein-statistics particles, when their rest energy, $W_o$, is zero or can be neglected. Again, we inserted the potential-type-energy term, $-\chi$, into the arguments of the exponentials to completely characterize the energy state.

Calculating integrals (61) and (62), we can find

$$n_{BE} = \frac{2K}{c^3} (kT)^3 \sum_{j=1}^{\infty} \frac{1}{j^3} \exp \left(\frac{j \chi}{kT}\right), \tag{64}
$$

$$P_{BE} = \frac{2K}{c^3} (kT)^4 \sum_{j=1}^{\infty} \frac{1}{j^3} \exp \left(\frac{j \chi}{kT}\right). \tag{65}
$$

The solutions (64) and (65) for $n_{BE}$ and $P_{BE}$ contain the sums, which are divergent for $\chi/(kT) > 0$. In an expectation of a stable gas volume, this result implies that either the same initial assumptions cannot be valid for
both kinds of particle statistics, Fermi-Dirac and Bose-Einstein, or \( \chi/(kT) = 0 \). It is actually known that the bosons are not permanent constituents of matter. In experiments, they are found to be unstable, except for photons, therefore no law of the conservation of their number is satisfied, in a common reality.

For photons with zero rest mass and, thus, zero rest energy, Eqs. (61)–(63) are exact, not only approximative equations. Consequently, the result of integrations expressed by relations (64) and (65) is exact. Eq. (65) implies that \( \chi = 0 \) for the photons, since \( W_o = 0 \) for them. As mentioned above, the sums in relations (64) and (65) are convergent for \( \chi = 0 \), therefore we have the regular solutions for \( n_{BE}, P_{BE} \), as well as \( E_{BE} \) (\( E_{BE} = 3P_{BE} \)).

For the radiation, the normalization constant \( K_N \) is equal to \( K_N = 1/(2\pi)^3 \). The latter implies that \( K = 1/(\pi^2\hbar^3) \). If \( \chi = 0 \), the pressure given by Eq. (65) becomes \( P_{BE} = P_{rad} = 2\beta_k(kT)^4/(\pi^2\hbar^3c^3) = \pi^2(kT)^4/(45\hbar^3c^3) \) (where \( \beta_k = \sum_{j=1}^{\infty} j^{-4} = \pi^4/90 \)).

Or, with the help of the radiation density constant, \( a = \pi^2k^4/(15\hbar^3c^3) \), we can rewrite this relation to the form of another well-known formula for the radiation pressure

\[
P_{rad} = \frac{1}{3}aT^4. \tag{66}
\]

The gradient of this pressure is

\[
\frac{dP_{rad}}{dr} = B_{rad} \frac{dT}{T} \frac{dT}{dr}, \tag{67}
\]

where

\[
B_{rad} = \frac{4}{3}aT^4. \tag{68}
\]

Using Eqs. (1) and (66), the quantum statistics Eq. (67) can be re-written to

\[
\frac{dP_{rad}}{dr} = -2P_{rad} \frac{d\nu}{dr}, \tag{69}
\]

i.e. to the functional form identical to Eq. (19) derived from the Einstein’s equations of field. As well, it is identical to the functional form in Eq. (51) valid for the gas consisting of fermions. We see that the gradient of the radiation pressure behaves in the same way as the gradient of the fermion-gas pressure.

VI. THE LAST STAGE OF THE COLLAPSE

ABOVE THE EVENT HORIZON

Since the individual gradients of both gas pressure and radiation pressure acquire the values approaching those which are needed to balance the gravity, as we demonstrated in Sects. IV and V, the sum of these gradients also approaches that at which these gradients would balance the gravitational attraction in the extremely strong potential close to the Schwarzschild gravitational radius.

In terms of mathematics, the equation of the "asymptotic" equilibrium between the total pressure gradient and gravity is the sum of Eqs. (51) and (69), i.e.

\[
\frac{d(P_{gas} + P_{rad})}{dr} = -2(P_{gas} + P_{rad}) \frac{d\nu}{dr}. \tag{70}
\]

We emphasize that this-type requirement of the stable configuration of URMS is set by Eq. (19) yielding from the Einstein’s equations of field. Our proof of the balance (in the limit \( W_o/\chi \to 0 \)) is the fact that the same equation as Eq. (49) also occurs as the necessary description of state quantities in the quantum statistics, which is represented by Eq. (70). (We note \( P_{gas} + P_{rad} = P \), in this equation.) Both gravity and internal quantum state of the considered gas/radiation converge to the same behavior close to the horizon, at the gravitational radius.

When applying the above conclusion about the balance in reality, we must, however, keep in the mind that the condition of the balance is \( \chi \gg W_o \), or \( W_o/\chi \ll 1 \). The exact balance is set, when \( W_o/\chi \to 0 \). This can be expected for \( r \to R_g \). For a larger \( r \), we can expect a minute break of the exact balance: in contrast to the before-URMS-stage fast collapse (according to the theory of the last stages of stellar evolution), the URMS either very slowly expands or collapses. If an expansion appeared, it could be fuelled only by the internal energy of gas. Since there is still a certain energy loss (see our considerations below), such the expansion would earlier or later turn into a collapse.

In this section, we discuss some aspects of the last stage of the assumed final, slow collapse of the URMS. We regard as the last that stage of the collapse, when quantity \( \exp(-\nu) \gg 1 \), but before the moment when all the URMS' matter disappears under the horizon of events. Unfortunately, we do not know the exact solution of general equations (14), (15), and (16) giving the quantities of state as the functions of distance. Moreover, we would need a solution of Einstein’s equations of field for non-static case, which is also unknown. A certain information can be deduced from a rough comparison of relative ratios of the internal-energy loss of two URMSs with different masses due to the escape of the energy to the outer space.

Considering the relation \( E_{rad} = 3P_{rad} \) as well as Eq. (66), in which \( T = T_c \exp(-\nu) \) with respect to Eq. (1), the energy of radiation in a unit surface volume, \( V_1 \), is

\[
W_{rad} = V_1 aT_c^4 \exp(-2\nu). \tag{71}
\]

The energy irradiated through the unit surface area, \( S_1 \), of this volume per a time unit can be given with help of the well-known radiation law as

\[
\Phi = S_1 \sigma T^4 = S_1 \sigma T_c^4 \exp(-2\nu), \tag{72}
\]

where \( \sigma \) is the Stefan-Boltzmann constant.

When we assume that the URMS’ gas consists of neutrons with the rest mass \( m_n \), we can calculate \( T_c \) using
Eq. \([56]\), i.e. \(T_c = n_o c^2/(sk)\). In the thermodynamical equilibrium, the temperature of gas and radiation is the same. In our ultra-relativistic case, \(s = 3\), therefore \(T_c \approx 3.63 \times 10^{12}\) K.

Eq. \((73)\) gives the radiation flux in the coordinate frame of the emission source, i.e. in the frame of a volume in the URMS surface layer. A distant observer detects the original radiation, having the angular frequency \(\omega\), with the red-shifted angular frequency \(\omega_o\), whereby

\[
\omega_o = \omega \exp(\nu/2).
\]

So, the flux measured by the distant observer is

\[
\Phi_o = S_1 \sigma T_c^4 \exp\left(-\frac{3}{2} \nu\right).
\]

The fraction \(f_{\text{rad}}\) of radiation energy detected by the distant observer per time \(t\), which can be regarded as the loss of internal energy of the surface volume \(V_1\) per time \(t\), is

\[
f_{\text{rad}} = \frac{S_1 \sigma}{V_1 \sigma} t \exp(\nu/2),
\]

in the simple case when \(t\) is short enough that \(f_{\text{rad}} \ll 1\). To provide a specific numerical estimate of \(f_{\text{rad}}\), we would need to specify volume \(V_1\). Unfortunately, it is impossible without an explicit knowledge of functions \(\lambda = \lambda(r)\) and \(\nu = \nu(r)\) for the URMS’ interior. Nevertheless, we can expect the continuous behavior of quantity \(\nu\). At the URMS surface, it must approach to that known from the outer Schwarzschild solution, i.e. \(\exp(\nu) = 1 - R_0/\bar{R}_0\). Therefore, \(\exp(\nu/2) \rightarrow 0\) for \(R_0 \rightarrow R_g\). For a finite time \(t\), the fraction of the radiation-energy loss must also \(f_{\text{rad}} \rightarrow 0\), when \(R_0 \rightarrow R_g\). The URMS collapse can be expected to be stopped at \(R_0 = R_g\).

In a weak gravitational field with no significant redshift, both internal energy per a unit volume and irradiated energy from the volume per a unit time are proportional to \(T^4\), therefore the appropriate \(f_{\text{rad}}\) does not depend on temperature. There is no such the decrease in the efficiency of cooling as in the above mentioned, \(\exp(\nu/2)\)-proportional, ultra-relativistic case.

We can expect a certain dependence of \(f_{\text{rad}}\) and, thus, the efficiency of cooling on the mass of the URMS. To compare, at least very roughly, this dependence for the URMSs of two different masses, let us consider a naive concept based on the assumption that the whole internal energy of URMS is uniformly distributed in the volume \(4\pi R_0^3/3\). Considering further the corresponding surface \(4\pi R_0^2\) and identifying \(R_0 = R_g\) (the difference between \(R_0\) and \(R_g\) is a small fraction of \(R_g\) during the whole last stage of the URMS collapse), we can put \(S_1/V_1 \approx 3/R_g\) implying \(f_{\text{rad}} \approx 3t \exp(\nu/2)/(aR_g)\). Or, the time-scale, on which fraction \(f_{\text{rad}}\) of radiation energy is lost, can be estimated as

\[
t \approx \frac{a R_g f_{\text{rad}}}{3 \sigma} \exp(-\nu/2) \approx \frac{2G a f_{\text{rad}}}{3c^2 \sigma} M \exp(-\nu/2).
\]

The whole context of our study intuitively implies a replacement of the old concept of the black hole by the concept of the URMS. Thus, not only stellar-sized objects, but also the massive compact objects in the galactic centers can be identified with the URMSs. The linear dependence of the time-scale \(t\) on mass \(M\) given by Eq.\((76)\) indicates that the time-scale is about \(10^5\) to \(10^{10}\) times longer for a super-massive URMS in a galactic center, with correspondingly larger mass, than that for a stellar-sized URMS. This large factor indicates that the collapse of the potential large URMSs in the universe cannot obviously be the matter of a moment or a short period.

Another potential mechanism of the internal energy loss is the escape of neutrinos carrying energy. This mechanism is well-known to occur at the first stage of the collapse of a massive stellar object. It is verified observationally, so far, by the detection of neutrinos preceding the optical outburst of supernova 1987A in the Large Magellanic Cloud \([17]\).

Although the extremely high pressure inside the URMS is keeping the particle constituents of the gas, neutrons, stable, the neutrinos can appear as a final result of collisions of photons, which are abundant and very energetic in the URMS environment. Specifically, the collision of two photons can produce an electron-positron pair, which can again annihilate to produce the neutrino-antineutrino pair. Or, neutrino (antineutrino) can occur in a collision of photon with a lepton (occurred in the first reaction), which behaves as a catalysts. Another reactions, including other elementary particles, can also occur in a collision of photons. The important fact in our context are the photons entering the reaction and neutrinos/antineutrinos as the result of the reaction.

The mean free path of neutrino in the gas with concentration of screening particles \(n_{\text{gas}}\) can be given as

\[
l_\nu \approx \frac{1}{\sigma_o n_{\text{gas}}},
\]

If the energy of neutrino is \(E_\nu\), the cross-section in the collision of a neutrino with a particle of the gas can be estimated as \([11]\)

\[
\sigma_\nu \approx \sigma_{ov} E_\nu^2,
\]

where the constant of proportionality, in SI units, was found to equal \(\sigma_{ov} \sim 10^{-11}\) m\(^2\) J\(^{-2}\).

The neutrino cooling can only be sufficient if the energy carried out by a single neutrino is comparable or higher than the internal energy of gas per one gas particle (neutron), i.e. \(E_\nu \gtrsim E/n_{\text{gas}} = skT\). Supplying Eqs.\([59]\) and \((78)\) into Eq. \((77)\), whereby \(\chi\) is expressed using Eq. \((18)\), we easily find for the neutron gas

\[
l_\nu \approx \frac{1}{\sigma_{ov} n_o W_0} \exp(5\nu/2) \sim 10^{-16} \exp(5\nu/2)
\]

(in meters). Identifying constant \(K\) to \(K = 1/(\pi^2\hbar^3)\) in relation \([58]\), the value of constant \(n_o\) is \(n_o = 2.253 \times 10^{27}\) (per cubic
The efficiency of the neutrino cooling of the thin surface layer correlates with that of radiation. The neutrinos also lose their energy during their escape from the URMS surface due to the red-shift as photons. And, either these particles cannot escape when the URMS radius reaches the gravitational radius.

VII. SUMMARY AND CONCLUSION

In Sects. IV and V, we found the dependence of state quantities on the auxiliary function $\nu$, for an extremely energetic environment. This dependence on $\nu$ appears to be the same for both particles satisfying the Fermi-Dirac statistics and photons satisfying the Bose-Einstein statistics. In the case of the high energies, the rest mass of fermions is negligible, therefore they seem to be as massless particles as photons, and the identical dependence on $\nu$ is not very surprising.

In the beginning of Sect. VI, we demonstrated that the behavior of the gradient of total pressure, given by Eq. (49), is identical to the relativistic equation of state (19), being a simplification of Eq. (20) in the limit of infinite energy. In this point, we can see a unification of quantum-physics description of state of ideal gas/radiation with the general-relativistic description of space-time filled in with a gas and radiation.

Although we demonstrated that the efficiency of the URMS cooling is low when the condition $\chi \gg W_\nu$ is valid, a certain energy still escapes away and this energy loss obviously leads to shrinking of the URMS until its proper physical radius becomes $R_\nu = R_g$. For $R_\nu = R_g$, no regular energy loss is possible and the equilibrium set by the identity of Eqs. (19) and (70) is exact. This equilibrium seems to be the law of Nature, suposed by Eddington, which prevents very massive stars from the reductio ad absurdum. According to our result, the final stage of an object with a mass exceeding the Oppenheimer-Volkoff limit is the stable object consisting of a matter and radiation and having radius exactly equal to $R_g$. In other words, the outer border of the object is situated just at the horizon of events.

The concept of the space-time surrounding the URMS does not differ from the classical concept of the space-time surrounding the non-rotating, Schwarzschild black hole. Also the existence of the horizon of events is predicted within both concepts. In our concept, this horizon cannot be, however, regarded as an absolute, insuperable barrier for any matter or radiation to escape. Just at the horizon, there are permanently situated the gas and radiation having the internal energy approaching infinity. In this context, it is necessary to note that the derived equations are related to a perfectly quiet fluid with no local phenomena. But in the reality, we do not know any perfectly quiet object. Also on the URMS surface, some fluctuations of quantities of state can be expected. Due to these fluctuations, some matter cannot be excluded to occur above the horizon. If this happens, an emission of a radiation can be expected. It must, of course, be extremely red-shifted, but its eventual detection cannot be excluded. In this aspect, our concept of the URMS is different to the concept of classical black holes. (We started to use the term ”URMS”, instead of the traditional ”black hole”, especially because of this reason.)

To explain the basic principles of our new concept of the extremely hot compact object more transparently, we constrained our considerations to the simplest case of spherically symmetric, non-rotating object. Despite this simplicity, we believe that the concept of URMS will be helpful to explain some observational phenomena related to the concerning objects.

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