Abstract

The purpose of this paper is to give some solutions for the classification problem in fibration theory by using the homotopy sequences of fibrations (sequences of $n$-th homotopy groups $\pi_n(S, s_0)$ of total spaces of fibrations). In particular, to show the role of homotopy sequence of $n$-th homotopy to get the required fiber map in Fadell-Dold theorem such that the restriction of this fiber map on some fiber spaces is a homotopy equivalence.

Keywords: Fibration; homotopy group; homotopy equivalence.

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1 Introduction

The homotopy theory of topological spaces attempts to classify weak homotopy types of spaces and homotopy classes of maps. The classification of maps within a homotopy is a central problem in topology and several authors contributed in this area, see for example the related works in [19].

The concepts of Hurewicz fibrations have played very important roles for investigating the mutual relations of among the objects. For this purpose Coram and Duvall [5] introduced an approximate fibration as a map having the approximate homotopy lifting property for every space, which is a generalization of a Hurewicz fibration having valuable properties similar to the Hurewicz fibration and is widely applicable to the maps whose fibers are nontrivial shapes. Thus it is very essential to examine whether a given decomposition map is an approximate fibration, for exact homotopy sequence that will provide us structural informations about any one object by means of their interrelations with the others, Coram and Duvall [4] gave several characterizations for an approximate fibration.

In [7], Dwyer and Kan followed the simplicial model category and introduced weak equivalences between the objects. Further, Dwyer and Kan in [9] define a notion of equivalence of simplicial localizations by using simplicial sets for the diagrams, which provide an answer to the question that posed by Quillen on the equivalence of homotopy theories in [20]. In fact, the category of simplicial localizations together with this notion of equivalence gives rise to a "homotopy theory of homotopy theory", see [11].

There has also been some further developments leading to classification in the homotopy type of newer manifolds (Wall’s manifolds, Milnor manifolds, etc.) which form generators of several different groups of manifolds, see more details [19], [24] and [25] by using the $\Pi$-algebras, see
fibration theory. Thus in this paper, we show the role of homotopy sequence of Fadell-Dold theorem in Hurewicz homotopy groups
\[ \pi_n \]
where \( r \) is homotopy equivalence into \( f \). Let \( f : S \rightarrow O \) be a fibration with a base \( O \), total space \( S \) and fiber space \( F_{r_0} = f^{-1}(r_0) \), where \( r_0 \in O \). A map \( L_f : \triangle f \rightarrow S^I \) is called a lifting function for \( f \) if \( L_f(s,\alpha)(0) = s \) and \( f[L_f(s,\alpha)] = \alpha \) for all \( (s,\alpha) \in \triangle f \), where \( \triangle f = \{ (s,\alpha) \in S \times O^I : f(s) = \alpha(0) \} \). If \( L_f(s,f \circ \bar{s}) = \bar{s} \) for all \( s \in S \), then the lifting function is called a regular lifting function. A fibration \( f \) is called regular fibration if it has regular lifting function.

Let \( f : S \rightarrow O \) be a fibration with a base \( O \), total space \( S \) and fiber space \( F_{r_0} = f^{-1}(r_0) \), where \( r_0 \in O \). A map \( L_f : \triangle f \rightarrow S^I \) is called a lifting function for \( f \) if \( L_f(s,\alpha)(0) = s \) and \( f[L_f(s,\alpha)] = \alpha \) for all \( (s,\alpha) \in \triangle f \), where \( \triangle f = \{ (s,\alpha) \in S \times O^I : f(s) = \alpha(0) \} \). If \( L_f(s,f \circ \bar{s}) = \bar{s} \) for all \( s \in S \), then the lifting function is called a regular lifting function. A fibration \( f \) is called regular fibration if it has regular lifting function.

Recall the Curtis-Hurewicz theorem, [15], which is one of the famous theorems in fibration theory which shows that any map is regular fibration if and only if it has regular lifting function. One of the main problems in fibration theory is a classification problem which is given by:

Under what conditions two fibrations, over a common base, will be fiber homotopy equivalent? Fadell-Dold theorem, [12], is one of the solutions of this problem which clarifies that if the common base \( O \) of two fibration \( f_1 : S_1 \rightarrow O \) and \( f_2 : S_2 \rightarrow O \) is a pathwise connected and an absolute neighborhood retract (ANR), then \( f_1 \) and \( f_2 \) are fiber homotopy equivalent if and only if there is a fiber map \( h : S_1 \rightarrow S_2 \) such that the restriction map of \( h \) on \( f_1^{-1}(r_0) \) is homotopy equivalence into \( f_2^{-1}(r_0) \), for some \( r_0 \in O \).

In general it is difficult to find the required fiber map of Fadell-Dold theorem in Hurewicz fibration theory. Thus in this paper, we show the role of homotopy sequence of \( n \)-th homotopy groups \( \pi_n(S,s_0) \) of two fibrations to get this required fiber map in Fadell-Dold theorem such that the restriction of this fiber map on some fiber spaces is a homotopy equivalence. That is, we give some solutions for the classification problem by using the \( n \)-th homotopy groups \( \pi_n(S,s_0) \) of total spaces of fibrations.

2 Preliminaries

For the \( n \)-th homotopy groups \( \pi_n(S,s_0) \) and \( n \)-th relative homotopy groups \( \pi_n(S,A,s_0) \), recall [22] that:

1. If \( h : S \rightarrow O \) is a map then for a positive integer \( n > 0 \), there is a homomorphism \( \hat{h} : \pi_n(S,s_0) \rightarrow \pi_n(O,h(s_0)) \) defined by \( \hat{h}([\alpha]) = [h \circ \alpha] \). At \( n = 0 \), \( \hat{h} \) sends the path-components of \( S \) into those of \( O \). \( \hat{h} \) is called a homomorphism induced by \( h \).

2. For a positive integer \( n > 1 \), there is a homomorphism (called boundary operator) \( \partial : \pi_n(S,A,s_0) \rightarrow \pi_{n-1}(A,s_0) \) defined by \( \partial([\alpha]) = [\alpha|_{r_0-1x\{0\}}] \). At \( n = 1 \), \( \alpha(I^n) \) is a point in \( A \) which determines a path-component \( C \in \pi_0(S,s_0) \) and \( \partial([\alpha]) = C \).
3. $\pi_n(S, s_o)$ is isomorphic to $\pi_{n-1}(\Omega(S, s_o), \tilde{s}_o)$, for a positive integer $n > 0$.

4. $\pi_n(S, s_o)$ is abelian group for a positive integer $n > 1$.

**Theorem 2.1.** \[22\] Let $h : (S, s_o) \to (O, h(s_o))$ be a homotopy equivalence. Then the induced homomorphisms $\tilde{h} : \pi_n(S, s_o) \to \pi_n(O, h(s_o))$ are isomorphisms for a positive integer $n > 0$.

Recall \[22\] that if $h$ is a fibration, then Theorem 2.1 remains valid. The following theorem is the consequences of Whitehead’s \[23\] and Hurewicz’s theorems, see \[2\].

**Theorem 2.2.** Let $S$ and $O$ be simply connected spaces which are dominated by ANR’s. If there is a map $f : S \to O$ induces isomorphism between the $n$–th homotopy groups of $S$ and $O$, then $f$ is homotopy equivalence.

Basically, Whitehead’s Theorem says that for CW-complexes, if a map $f : X \to Y$ induces an isomorphism on all homotopy groups then it is a homotopy equivalence. But, as the example above shows, you need the map. Such a map is called a weak homotopy equivalence. We note that Whitehead’s Theorem is not true for spaces wilder than CW-complexes for example, the Warsaw circle has all of its homotopy groups trivial but the unique map to a point is not a homotopy equivalence.

**Theorem 2.3.** \[3\] Let $f : S \to O$ be a fibration and a fiber space $F_{r_o}$ be a pathwise connected ANR for some $r_o \in O$. If $\Omega(O, r_o) \simeq ANR$, then $\Omega(S, s_o)$ is dominated by ANR for any $s_o \in F_{r_o}$. If $S$ is a simply connected then $\Omega(S, s_o)$ is of the same homotopy type with ANR space.

The definition of homotopy sequence of a fibration $f : S \to O$ is given as follow:

Let $s_o \in F_{r_o}$. We can consider $f$ as a map of a triple $(S, F_{r_o}, s_o)$ into a pair $(O, r_o)$. Then the following sequence is called a *homotopy sequence* of a fibration $f$:

$$
\begin{align*}
\cdots \pi_n(S, s_o) & \xrightarrow{\hat{f}} \pi_n(O, r_o) \xrightarrow{\partial_*} \pi_{n-1}(F_{r_o}, s_o) \xrightarrow{\hat{i}} \pi_{n-1}(S, s_o) \\
\cdots \pi_2(O, r_o) & \xrightarrow{\partial_*} \pi_1(F_{r_o}, s_o) \xrightarrow{\hat{i}} \pi_1(S, s_o) \xrightarrow{\hat{f}} \pi_1(O, r_o) \\
\pi_0(F_{r_o}, s_o) & \xrightarrow{\partial_*} \pi_0(S, s_o),
\end{align*}
$$

where $i, j$ are inclusion maps and $\partial_* = \partial \circ (\hat{f})^{-1}$. Recall \[16\] that this sequence is exact, that is, the kernel of each homomorphism is equal to the image of the previous one.

**Theorem 2.4.** \[13\] Let $f : S \to O$ be a fibration with pathwise connected space $O$. Then $f^{-1}(r_1)$ and $f^{-1}(r_2)$ are of the same homotopy type for any $r_1, r_2 \in O$.

**Lemma 2.5.** \[22\] Let $R_i, M_j : S^I \to S^I$ be maps, where $i, j = 1, 2, 3, 4$, which are defined by $R_1(\alpha) = \alpha(0)$, $M_1(\alpha) = \alpha \star \alpha$, $R_2(\alpha) = \alpha(1)$, $M_2(\alpha) = \alpha \star \alpha$, $M_3(\alpha) = \alpha$, $R_3(\alpha) = \alpha(0) \star \alpha$, $M_4(\alpha) = \alpha$ and $R_4(\alpha) = \alpha \star \alpha(1)$ for all $\alpha \in S^I$. Then $R_1, M_1$ are homotopic by homotopy $H$ which has the following property:

$$
[H(\alpha, r)](1) = \alpha(0) \quad \text{for} \quad r \in I, \alpha \in S^I,
$$

(1)
and $R_i$, $M_i$, $(i = 2, 3, 4)$, are homotopic by homotopy $G_i$ which has the following property:

$$[G_i(\alpha, r)](1) = \alpha(1) \quad \text{for} \quad r \in I, \alpha \in S^I. \quad (2)$$

Lemma 2.6. Consider Figure 1 which involves abelian groups and homomorphisms

![Diagram](image)

such that $\psi_{i+1} \circ h_i = h'_i \circ \psi_i$ for all $i = 1, 2, 3, 4$. If $\psi_1, \psi_2, \psi_4$ and $\psi_5$ are isomorphisms, then $\psi_3$ is an isomorphism.

3 Fibrations $\Gamma(f, s_o)$ and $\Sigma(f)$

In this section, we shall introduce the notions of fibrations $\Sigma(f)$ and $\Gamma(f, s_o)$ which are induced by fibration $f : S \longrightarrow O$ with some results about their properties.

The functors $\Gamma$ and $\Sigma$ are defined as follows:

$$\Gamma(S, F, s_o) = \{\alpha \in S^I : \alpha(0) = s_o, \alpha(1) \in F\}$$

and

$$\Sigma(S, F) = \{\alpha \in S^I : \alpha(0) \in F, \alpha(1) \in F\}$$

for any subspace $F$ of any topological space $S$, where $s_o \in F$.

Let $f : S \longrightarrow O$ be a fibration with a fiber space $F_{r_o}$. We will define two fibrations $\Gamma(f, s_o)$ and $\Sigma(f)$ on the functors $\Gamma$ and $\Sigma$, respectively, induced by $f$ as follow:

$\Gamma(f, s_o)$ will denote the fibration $\Psi_{s_o} : \Gamma(S, F_{r_o}, s_o) \longrightarrow F_{r_o}$ given by

$$\Psi_{s_o}(\alpha) = \alpha(1) \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o}, s_o)$$

and we say $\Gamma(f, s_o)$ is a fibration induced by $f$, which has fiber space $\Psi_{s_o}^{-1}(s_o) = \Omega(S, s_o)$ over a point $s_o \in F_{r_o}$.

$\Sigma(f)$ will be denote the fibration $\Phi : \Sigma(S, F_{r_o}) \longrightarrow F_{r_o} \times F_{r_o}$ given by

$$\Phi(\alpha) = [\alpha(0), \alpha(1)] \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o})$$

and we say $\Sigma(f)$ is a fibration induced by $f$, which has fiber space $\Phi^{-1}[(s_o, s_o)] = \Omega(S, s_o)$ over a point $(s_o, s_o) \in F_{r_o} \times F_{r_o}$.

Lemma 3.1. Let $f : S \longrightarrow O$ be a fibration. Then the maps $D, D_o : \triangle f \longrightarrow S$ defined by

$$D(s, \alpha) = L_f[L_f(s, \alpha)(1), \overline{\alpha}(1)] \quad \text{and} \quad D_o(s, \alpha) = s,$$

for all $(s, \alpha) \in \triangle f$, are homotopic.
Proof. For $\alpha \in O^I$ and $r \in I$, define paths $\alpha_r$, $\alpha'_r$ and $\alpha''_r$ in $O$ by
\[
\alpha_r(t) = \alpha(rt), \quad \alpha'_r(t) = \alpha[r + (1 - r)t] \quad \text{and} \quad \alpha''_r(t) = \alpha[2r(1 - t)],
\]
for all $t \in I$. Define two homotopies $H : \triangle f \times I \rightarrow S$ by
\[
H[(s, \alpha), t] = L_f[L_f(s, \alpha_t)(1), \alpha'_t](1) \quad \text{for} \ t \in I, (s, \alpha) \in \triangle f,
\]
and a homotopy $G : O^I \times I \rightarrow O^I$ by
\[
[G(\alpha, r)](t) = \begin{cases} 
\alpha_r(t) & \text{for} \ 0 \leq t \leq 1/2, \\
\alpha''_r(t) & \text{for} \ 1/2 \leq t \leq 1,
\end{cases}
\]
for all $\alpha \in O^I, r \in I$. Hence define a homotopy $F : \triangle f \times I \rightarrow S$ by
\[
F[(s, \alpha), t] = H[(s, G(\alpha, t)), 1/2] \quad \text{for} \ t \in I, (s, \alpha) \in \triangle f.
\]
By the regularity for $L_f$ we observe that for $(s, \alpha) \in \triangle f$,
\[
F[(s, \alpha), 1] = H[(s, G(\alpha, 1)), 1/2] = H[(s, \alpha \star \overline{\alpha}), 1/2] = L_f[K[s, (\alpha \star \overline{\alpha})_{1/2}, (\alpha \star \overline{\alpha})']_{1/2}]_{1/2}(1) = L_f[K(s, \alpha), \overline{\alpha}]_{1/2} = L_f[L_f(s, \alpha)(1), \overline{\alpha}]_{1/2} = D(s, \alpha)
\]
for all $(s, \alpha) \in \triangle f$, and
\[
F[(s, \alpha), 0] = H[(s, G(\alpha), 0)[1/2] = H[(s, \overline{\alpha}(0)), 1/2] = L_f[K[s, \overline{\alpha}(0)_{1/2}, \overline{\alpha}(0)]_{1/2}]_{1/2} = L_f[K[s, \overline{\alpha}(0)], \overline{\alpha}(0)]_{1/2} = L_f[L_f(s, f \circ \overline{s})(1), f \circ \overline{s}]_{1/2} = L_f(s, f \circ \overline{s})_{1/2} = s = D_o(s, \alpha)
\]
for all $(s, \alpha) \in \triangle f$. Hence $D$ and $D_o$ are homotopic.

In the proof of Lemma above we get that the homotopy $F$ has the following property:
\[
f\{F[(s, \alpha), t]\} = \alpha(0) \quad \text{for} \ (s, \alpha) \in \triangle f. \quad (3)
\]

Proposition 3.2. For any fibration $f : S \rightarrow O$ with fiber space $F_{r_o}$, the following statements are true:
1. $\Sigma(S, F_{r_o}) \simeq \Omega(O, r_o) \times F_{r_o}$;
2. $\Gamma(S, F_{r_o}, s_o) \simeq \Omega(O, r_o)$ \quad for all $s_o \in F_{r_o}$.
Proof. 1. Define a map $N : \Sigma(S, F_{r_o}) \rightarrow \Omega(O, r_o) \times F_{r_o}$ by

$$N(\alpha) = [f \circ \alpha, \alpha(0)] \quad \text{for} \quad \alpha \in \Sigma(S, F_{r_o}),$$

and a map $M : \Omega(O, r_o) \times F_{r_o} \rightarrow \Sigma(S, F_{r_o})$ by

$$M(\alpha, s) = L_f(s, \alpha) \quad \text{for} \quad (\alpha, s) \in \Omega(O, r_o) \times F_{r_o}.$$

Then we have that

$$(N \circ M)(\alpha, s) = N[L_f(s, \alpha)]$$

$$= \{f[L_f(s, \alpha)], L_f(s, \alpha)(0)\}$$

$$= (\alpha, s) = id_{\Omega(O, r_o) \times F_{r_o}}(\alpha, s)$$

for all $(\alpha, s) \in \Omega(O, r_o) \times F_{r_o}$. That is, $N \circ M = id_{\Omega(O, r_o) \times F_{r_o}}$. By Lemma 3.1 we have that the composition map $M \circ N : \Sigma(S, F_{r_o}) \rightarrow \Sigma(S, F_{r_o})$ given by

$$(M \circ N)(\alpha) = L_f[\alpha(0), f \circ \alpha] \quad \text{for} \quad \alpha \in \Sigma(S, F_{r_o})$$

is homotopic to the identity map $id_{\Sigma(S, F_{r_o})}$. Therefore

$$\Sigma(S, F_{r_o}) \simeq \Omega(O, r_o) \times F_{r_o}.$$  

2. Let $s_o \in F_{r_o}$. Define a map $R : \Gamma(S, F_{r_o}, s_o) \rightarrow \Omega(O, r_o)$ by

$$R(\alpha) = f \circ \alpha \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o}, s_o),$$

and a map $D : \Omega(O, r_o) \rightarrow \Gamma(S, F_{r_o}, s_o)$ by

$$D(\alpha) = L_f(s_o, \alpha) \quad \text{for} \quad \alpha \in \Omega(O, r_o).$$

Then we have

$$(R \circ D)(\alpha) = R(L_f(s_o, \alpha))$$

$$= f[L_f(s_o, \alpha)]$$

$$= \alpha = id_{\Omega(O, r_o)}(\alpha)$$

for all $\alpha \in \Omega(O, r_o)$. That is, $R \circ D = id_{\Omega(O, r_o)}$. By Lemma 3.1 we get that the composition map $D \circ R : \Gamma(S, F_{r_o}, s_o) \rightarrow \Gamma(S, F_{r_o}, s_o)$ given by

$$(D \circ R)(\alpha) = L_f(s_o, f \circ \alpha) = L_f(\alpha(0), f \circ \alpha)$$

for all $\alpha \in \Gamma(S, F_{r_o}, s_o)$ is homotopic to the identity map $id_{\Gamma(S, F_{r_o}, s_o)}$. Therefore

$$\Gamma(S, F_{r_o}, s_o) \simeq \Omega(O, r_o),$$

for all $s_o \in F_{r_o}$. \qed

There are several fibrations $\Gamma(f, s_o)$ according to the number of points in $F_{r_o}$. But when we let $F_{r_o}$ pathwise connected, then the set of fiber homotopy equivalence classes of the collection set of all these fibrations will be a single. As it is clear in the following theorem.
**Theorem 3.3.** Let \( f : S \longrightarrow O \) be a fibration with a pathwise connected fiber space \( F_{r_o} \). Then the fibration \( \Gamma(f, s_o) \) is determined up to a fiber homotopy equivalence class. That is, \( \Gamma(f, s_o) \) and \( \Gamma(f, s'_o) \) are fiber homotopy equivalent for all \( s_o, s'_o \in F_{r_o} \).

**Proof.** Let \( s_o, s'_o \in F_{r_o} \). Since \( F_{r_o} \) is a pathwise connected then there is path \( \beta : I \longrightarrow F_{r_o} \) between \( s_o \) and \( s'_o \). Now let us to define two fiber maps, we can define the map \( h : \Gamma(S, F_{r_o}, s_o) \longrightarrow \Gamma(S, F_{r_o}, s'_o) \) by

\[
h(\alpha) = \overline{\beta} \ast \alpha \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o}, s_o),
\]

and a map \( g : \Gamma(S, F_{r_o}, s'_o) \longrightarrow \Gamma(S, F_{r_o}, s_o) \) by

\[
g(\alpha) = \beta \ast \alpha \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o}, s'_o).
\]

Then we have

\[
\Psi_{s'_o}[h(\alpha)] = (\overline{\beta} \ast \alpha)(1) = \alpha(1) = \Psi_{s_o}(\alpha)
\]

and

\[
\Psi_{s_o}[g(\alpha)] = (\beta \ast \alpha)(1) = \alpha(1) = \Psi_{s'_o}(\alpha)
\]

for all \( \alpha \in \Gamma(S, F_{r_o}, s_o) \). That is, \( h \) and \( g \) are fiber maps.

Now from Lemma 2.5 and Equations 11 12 we observe that the composition map \( g \circ h : \Gamma(S, F_{r_o}, s_o) \longrightarrow \Gamma(S, F_{r_o}, s'_o) \) given by

\[
(g \circ h)(\alpha) = (\beta \ast \overline{\beta}) \ast \alpha \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o}, s_o)
\]

is fiber homotopic to the identity map \( \text{id}_{\Gamma(S, F_{r_o}, s_o)} \) and the composition map \( h \circ g : \Gamma(S, F_{r_o}, s'_o) \longrightarrow \Gamma(S, F_{r_o}, s'_o) \) given by

\[
(h \circ g)(\alpha) = (\overline{\beta} \ast \beta) \ast \alpha \quad \text{for} \quad \alpha \in \Gamma(S, F_{r_o}, s'_o)
\]

is fiber homotopic to the identity map \( \text{id}_{\Gamma(S, F_{r_o}, s'_o)} \). Therefore \( \Gamma(f, s_o) \) and \( \Gamma(f, s'_o) \) are fiber homotopy equivalent. \( \square \)

Here we give some concepts which will be used in the next sections.

**Definition 3.4.** Let \( f : S \longrightarrow O \) be a fibration with fiber space \( F_{r_o} = f^{-1}(r_o) \), where \( r_o \in O \). By the \( Lf \)-function for fibration \( f \) induced by a lifting function \( L_f \) we mean a map \( \Theta_{L_f} : \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o} \) which is defined by

\[
\Theta_{L_f}(\alpha, s) = L_f(s, \alpha)(1) \quad \text{for} \quad s \in F_{r_o}, \alpha \in \Omega(O, r_o).
\]

Henceforth, we will denote by \([S, f, O, F_{r_o}, \Theta_{L_f}]\) the regular fibration \( f : S \longrightarrow O \) with an \( Lf \)-function \( \Theta_{L_f} : \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o} \), induced by the lifting function \( L_f \) and with a fiber space \( F_{r_o} = f^{-1}(r_o) \), where \( r_o \in O \).

**Definition 3.5.** Let \([S, f, O, F_{r_o}, \Theta_{L_f}]\) be a fibration. For \( s_o \in F_{r_o} \), the map \( R : \Omega(O, r_o) \longrightarrow F_{r_o} \) defined by

\[
R(\alpha) = \Theta_{L_f}(\alpha, s_o) \quad \text{for} \quad \alpha \in \Omega(O, r_o)
\]

is called an \( Lf \)-restriction for the fibration \( f \) and we denote it by \( f^{s_o} \).
Example 3.6. The first fibration $P_1 : O \times S \rightarrow O$ has a regular lifting function $L_{P_1} : \Delta P_1 \rightarrow (O \times S)^I$ defined by

$$L_{P_1}[(b, s), \alpha](t) = (\alpha(t), s) \text{ for } t \in I, [(b, s), \alpha] \in \Delta L_{P_1}.$$  

Then the $Lf$–function $\Theta_{L_{P_1}}$ for fibration $P_1$ induced by $L_{P_1}$ will be given by

$$\Theta_{L_{P_1}}(\alpha, x) = x \text{ for } x \in F_{r_o}, \alpha \in \Omega(O, r_o).$$

The $Lf$–restriction $P_1^s_o$ for the fibration $P_1$ will be given by

$$P_1^{s_o}(\alpha) = s_o \text{ for all } \alpha \in \Omega(O, r_o).$$

Definition 3.7. Let $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ be two fibrations. The $Lf$–functions $\Theta_{L_{f_1}}$ and $\Theta_{L_{f_2}}$ are said to be conjugate if there is $g \in H(F_{r_o}^1, F_{r_o}^2)$ such that

$$\Theta_{L_{f_1}} \simeq S \bar{\gamma} \circ \Theta_{L_{f_2}} \circ (id_{\Omega(O, r_o)} \times g).$$

We say that $f_1$ and $f_2$ have conjugate $Lf$–restrictions if there is $g \in H(F_{r_o}^1, F_{r_o}^2)$ such that

$$f_1^{s_o} \simeq S \bar{\gamma} \circ f_2^{g(s_o)}$$

where $s_o \in F_{r_o}$, $H(F_{r_o}^1, F_{r_o}^2)$ is the set of all homotopy equivalences from $F_{r_o}^1$ into $F_{r_o}^2$ and $\bar{\gamma}$ is the inverse homotopy of $g$.

If two fibrations have conjugate $Lf$–functions, they also have conjugate $Lf$–restrictions.

4 Fibration $\Gamma(f, s_o)$ and $Lf$–restriction

In this section, we are going to introduce the role of homotopy sequences of fibrations (using $Lf$–restriction ) in satisfying FHE between two fibrations $\Gamma(f_1, s_o)$ and $\Gamma(f_2, s_o)$ which are induced by two fibrations $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ over a common base $O$.

In the following theorem, we show that for two fibrations $f_1$ and $f_2$ with conjugate $Lf$–restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F_{r_o}^1, F_{r_o}^2)$, there are two fiber maps between two fibrations $\Gamma(f_1, s_o)$ and $\Gamma(f_2, g(s_o))$.

Theorem 4.1. Let $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ be two fibrations with conjugate $Lf$–restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F_{r_o}^1, F_{r_o}^2)$, where $s_o \in F_{r_o}^1$. Then there are two fiber maps

$$h : \Gamma(S_1, F_{r_o}^1, s_o) \rightarrow \Gamma(S_2, F_{r_o}^2, g(s_o))$$

and

$$k : \Gamma(S_2, F_{r_o}^2, g(s_o)) \rightarrow \Gamma(S_1, F_{r_o}^1, \bar{\gamma} \circ g)(s_o))$$

over $g$ and $\bar{\gamma}$, respectively. That is, Figure 2 is commutative. Further the groups $\pi_n(S_1, s_o)$ and $\pi_n(S_2, g(s_o))$ are isomorphic for a positive integer $n > 1$. 

Proof. By hypothesis, \( f^s_1 \simeq \overline{g} \circ f^g_2 \). This implies \( g \circ f^g_1 \simeq f^g_2 \). Then there is a homotopy \( G : \Omega(O, r_o) \times I \to F^2_{r_o} \) such that

\[
G(\alpha, 0) = f^g_2(\alpha) = \Theta f_{f_2}(\alpha, g(s_o))
\]

and

\[
G(\alpha, 1) = (g \circ f^s_1)(\alpha) = g[\Theta f_{f_1}(\alpha, s_o)]
\]

for all \( \alpha \in \Omega(O, r_o) \). For \( \alpha \in S^I_1 \) and \( r \in I \), we can define a path \( \alpha_r \in S^I_1 \) by

\[
\alpha_r(t) = \begin{cases}
\alpha(t) & \text{for } 0 \leq t \leq r, \\
\alpha(r) & \text{for } r \leq t \leq 1.
\end{cases}
\]

For \( \beta = f_1 \circ \alpha \) and \( r \in I \), we can define the path \( \beta^{1-r} \in O^I \) by

\[
\beta^{1-r}(t) = \begin{cases}
\beta(t + r) & \text{for } 0 \leq t \leq 1 - r, \\
\beta(1) & \text{for } 1 - r \leq t \leq 1.
\end{cases}
\]

Define a homotopy \( H : S^I_1 \times I \to S^I_1 \) by

\[
[H(\alpha, r)](t) = \begin{cases}
\alpha_r(t), & \text{for } 0 \leq t \leq r, \\
L_f_1(\alpha(r), \beta^{1-r})(t - r), & \text{for } r \leq t \leq 1,
\end{cases}
\]

for all \( r \in I, \alpha \in S^I_1 \). We get that

\[
H(\alpha, 0) = L_f_1(\alpha(0), f_1 \circ \alpha) \quad \text{and} \quad H(\alpha, 1) = \alpha \quad \text{for } \alpha \in S^I_1.
\]

For \( \alpha \in \Gamma(S_1, F^1_{r_o}, s_o) \), let \( M(\alpha) \) be a path in \( \Gamma(S_1, F^1_{r_o}, s_o) \) defined by

\[
M(\alpha)(r) = [H(\alpha, r)](1) \quad \text{for } r \in I.
\]

Hence we can define a map \( M' : \Gamma(S_1, F^1_{r_o}, s_o) \to (F^2_{r_o})^I \) by

\[
M'(\alpha) = g[M(\alpha)] \quad \text{for } \alpha \in \Gamma(S_1, F^1_{r_o}, s_o),
\]

and a map \( L'_{f_2} : \Gamma(S_1, F^1_{r_o}, s_o) \to \Gamma(S_2, F^2_{r_o}, g(s_o)) \) by

\[
L'_{f_2}(\alpha) = L_{f_2}(g(s_o), f_1 \circ \alpha) \quad \text{for } \alpha \in \Gamma(S_1, F^1_{r_o}, s_o).
\]

Now define the map \( h : \Gamma(S_1, F^1_{r_o}, s_o) \to \Gamma(S_2, F^2_{r_o}, g(s_o)) \) by

\[
h(\alpha) = [L'_{f_2}(\alpha) \ast G(f_1 \circ \alpha)] \ast M'(\alpha) \quad \text{for } \alpha \in \Gamma(S_1, F^1_{r_o}, s_o).
\]
Then $h$ is a well-defined as a continuous map. Since

$$(\Psi g_{s_o} \circ h)(\alpha) = \{[L'_f(\alpha) \ast G(f_1 \circ \alpha)] \ast M'(\alpha)\}(1)$$

$$= M'(\alpha)(1) = g[M(\alpha)(1)] = g[\alpha(1)]$$

$$(g \circ \Psi_{s_o})(\alpha)$$

for all $\alpha \in \Gamma(S_1, F_{r_o}, s_o)$. That is, $\Psi g_{s_o} \circ h = g \circ \Psi_{s_o}$. Hence $h$ is fiber map over $g$.

Now we prove that $\pi_n(S_1, s_o)$ and $\pi_n(S_2, g(s_o))$ are isomorphic for a positive integer $n > 1$. In Figure 3, define the map $N_1 : \Gamma(S_1, F_{r_o}, s_o) \rightarrow \Omega(O, r_o)$ by

$$N_1(\alpha) = f_1 \circ \alpha \quad \text{for } \alpha \in \Gamma(S_1, F_{r_o}, s_o),$$

and a map $N_2 : \Gamma(S_2, F_{r_o}, g(s_o)) \rightarrow \Omega(O, r_o)$ by

$$N_2(\alpha) = f_2 \circ \alpha \quad \text{for } \alpha \in \Gamma(S_2, F_{r_o}, g(s_o)).$$

From the definition of fiber map $h$, we observe that $[(N_2 \circ h)(\alpha)](t) = r_o$ at $t = 1/4$ and for

$$[(N_2 \circ h)(\alpha)](t) = [N_2(h(\alpha))](t) = f_2[L'_f(h(\alpha))](4t)$$

$$= f_2[L'_f(g(s_o), f_1 \circ \alpha)](4t)$$

$$= (f_1 \circ \alpha)(4t) = [N_1(\alpha)](4t).$$

That is, $(N_2 \circ h)(\alpha) \neq N_1(\alpha)$ concentrated on the interval $[0, 1]$ but $(N_2 \circ h)(\alpha) = N_1(\alpha)$ concentrated on the interval $[0, 1/4]$. This implies that $N_2 \circ h \simeq N_1$. Hence Figure 3 is not commutative in the usual sense but it is a homotopy commutative. That is, Figure 4 is a commutative i.e.,

$$\pi_n(\Gamma(S_1, F_{r_o}, s_o), \tilde{s_o})$$

$$\tilde{h}$$

$$\pi_n(\Gamma(S_2, F_{r_o}, g(s_o)), \tilde{g}(s_o))$$

$$\tilde{N}_2$$

$$\pi_n(\Omega(O, r_o), \tilde{r}_o)$$

Figure 4

$$\tilde{N}_2 \circ \tilde{h} = \tilde{N}_1.$$

(4)
By the part 2 in Proposition 3.2, the maps \( N_1 \) and \( N_2 \) are homotopy equivalences of \( \Gamma(S_1, F^1_{r_o}, s_o) \) into \( \Omega(O, r_o) \) and of \( \Gamma(S_2, F^2_{r_o}, g(s_o)) \) into \( \Omega(O, r_o) \), respectively. Hence

\[
\hat{N}_1 : \pi_n(\Gamma(S_1, F^1_{r_o}, s_o), \tilde{s}_o) \rightarrow \pi_n(\Omega(O, r_o), \tilde{r}_o)
\]

and

\[
\hat{N}_2 : \pi_n(\Gamma(S_2, F^2_{r_o}, g(s_o)), \tilde{g}(s_o)) \rightarrow \pi_n(\Omega(O, r_o), \tilde{r}_o)
\]

are isomorphisms for a positive integer \( n > 0 \). By Equation 4 we get that

\[
\hat{h} : \pi_n(\Gamma(S_1, F^1_{r_o}, s_o)) \rightarrow \pi_n(\Gamma(S_2, F^2_{r_o}, g(s_o)))
\]

is an isomorphism for a positive integer \( n > 0 \). Consider

\[
h_o : \Omega(S_1, s_o) \rightarrow \Omega(S_2, g(s_o))
\]

is a restriction of \( h \) on \( \Psi^{-1}_{s_o}(s) = \Omega(S_1, s_o) \). Now we can integrate the homotopy sequences of fibrations \( \Psi_{s_o} \) and \( \Psi_{g(s_o)} \) in Figure 5,

![Homotopy Diagram](image)

where \( j_1, j_2 \) are inclusion maps, \( \partial_1 \) and \( \partial_2 \) are boundary operators and

\[
(\partial_1)_* \quad = \quad \partial_1 \circ (\Psi_{s_o})^{-1},
\]

\[
(\partial_2)_* \quad = \quad \partial_2 \circ (\Psi_{g(s_o)})^{-1},
\]

\[
\pi_n(\Gamma(S_1)) \quad = \quad \pi_n(\Gamma(S_1, F^1_{r_o}, s_o), \tilde{s}_o),
\]

\[
\pi_n(\Gamma(S_2)) \quad = \quad \pi_n(\Gamma(S_2, F^2_{r_o}, g(s_o)), \tilde{g}(s_o)),
\]

for a positive integer \( n > 0 \).

In Figure 5, we observe that \( \hat{j}_1, \hat{j}_2, (\partial_1)_*, (\partial_2)_*, \Psi_{s_o} \) and \( \Psi_{g(s_o)} \) are homomorphisms. Since \( \hat{h} \) and \( \hat{g} \) are isomorphisms, then Lemma 2.6 shows that for a positive integer \( n > 0 \),

\[
\hat{h}_o : \pi_n(\Omega(S_1, s_o), \tilde{s}_o) \rightarrow \pi_n(\Omega(S_2, g(s_o)), \tilde{g}(s_o))
\]
is an isomorphism. Since $\pi_{n+1}(S_1, s_0)$ is isomorphic to $\pi_n(\Omega(S_1, s_o), s_o)$ and $\pi_{n+1}(S_2, g(s_o))$ is isomorphic to $\pi_n(\Omega(S_2, g(s_o)), g(s_o))$ for a positive integer $n > 0$, then $\pi_n(S_1, s_o)$ is isomorphic to $\pi_n(S_2, g(s_o))$ for a positive integer $n > 1$.

Finally, since $g \circ f_1^{s_o} \simeq f_2^{g(s_o)} \implies g \circ f_1^{(s_0)} \simeq f_2^{g(s_o)}$, then similarly, there is a fiber map $k$ satisfying above requirement properties for $h$.

**Corollary 4.2.** If two fibrations $[S_1, f_1, O, F^1_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F^2_{r_o}, \Theta_{L_{f_2}}]$ have conjugate $L_f$–functions by $g \in H(F^1_{r_o}, F^2_{r_o})$, then Theorem 4.1 holds for any $s_o \in F^1_{r_o}$.

**Proof.** It is clear that if two fibrations $f_1$ and $f_2$ have conjugate $L_f$–functions by $g \in H(F^1_{r_o}, F^2_{r_o})$, then they have conjugate $L_f$–restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F^1_{r_o}, F^2_{r_o})$, for any $s_o \in F^1_{r_o}$. Hence Theorem 4.1 holds for any $s_o \in F^1_{r_o}$.

We explain in the following corollary that if $S_1$, $S_2$ are simply connected in Theorem 4.1 then two loop spaces $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are of the same homotopy type.

**Corollary 4.3.** Let $[S_1, f_1, O, F^1_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F^2_{r_o}, \Theta_{L_{f_2}}]$ be fibrations with conjugate $L_f$–restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F^1_{r_o}, F^2_{r_o})$ and $S_1$, $S_2$ be simply connected spaces. Let $\Omega(O, r_o) \simeq$ ANR. If $F^1_{r_o}$ and $F^2_{r_o}$ are pathwise connected and ANR’s, then

$$\Omega(S_1, s_o) \simeq \Omega(S_2, g(s_o)).$$

**Proof.** Since $S_1$ and $S_2$ are simply connected, then it is clear that $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are pathwise connected. Since $\Omega(O, r_o) \simeq$ ANR then Theorem 2.3 shows that $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are dominated by ANR’s. By Theorem 4.1 there is a map

$$h_o : \Omega(S_1, s_o) \rightarrow \Omega(S_2, g(s_o))$$

induces isomorphisms between the homotopy groups. Hence by Theorem 2.2, $h_o$ is homotopy equivalence.

In the next step, we employ Theorem 4.1 to satisfy FHE relation for fibrations $\Gamma(f, s_o)$. Figure 2 in Theorem 4.1 suggests that perhaps in some sense there is FHE relation between $\Gamma(f_1, s_o)$ and $\Gamma(f_2, g(s_o))$. But the notion of the FHE relation applied to fibrations having a common base. One might try fiberizing $\Gamma(S_1, F^1_{r_o}, s_o)$ over $F^1_{r_o}$ using the map $g \circ \Psi_{s_o}$ but in general this will not give rise to fibering since it might happen that $g(F^1_{r_o}) = g(s_o)$ and in this case $g \circ \Psi_{s_o}$ would not be onto if $F^2_{r_o}$ consisted of more than one point. Hence we give the restrictions such as $F^1_{r_o} = F^2_{r_o} = F_{r_o}$ and $g : F_{r_o} \rightarrow F_{r_o}$ is a homeomorphism map.

Let $[S, f, O, F_{r_o}, \Theta_{L_f}]$ be a fibration and $g$ be a homeomorphism map of $O$ onto a topological semigroup $O'$. Then the composition $g \circ f : S \rightarrow O'$ is also a fibration denoted by $[f]_g$.

**Theorem 4.4.** Let $[S_1, f_1, O, F_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}, \Theta_{L_{f_2}}]$ be fibrations with conjugate $L_f$–restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by a homeomorphism $g \in H(F_{r_o}, F_{r_o})$, where $s_o \in F_{r_o}$, and $F_{r_o}$ be a pathwise connected ANR. If $S_1$ and $S_2$ are simply connected and $\Omega(O, r_o) \simeq$ ANR, then $[\Gamma(f_1, s_o)]_g$ and $\Gamma(f_2, g(s_o))$ are fiber homotopy equivalent.
Proof. By Theorem [4.1] in Figure 6, there is a fiber map
\[ h : \Gamma(S_1, F_{r_o}, s_o) \rightarrow \Gamma(S_2, F_{r_o}, g(s_o)). \]
Let
\[ A_1 = (g \circ \Psi_{s_o})^{-1}(s_o) = \{ \alpha \in S_1^I : \alpha(0) = s_o, \ \alpha(1) = g^{-1}(s_o) \}, \]
and
\[ A_2 = \Psi_{g(s_o)}^{-1}(s_o) = \{ \alpha \in S_2^I : \alpha(0) = g(s_o), \ \alpha(1) = s_o \}. \]

![Diagram](figure6.png)

Figure 6

We observe that \( A_1 \) and \( A_2 \) are fiber spaces for fibrations \([\Gamma(f_1, s_o)]_g\) and \([\Gamma(f_2, g(s_o))]_g\) over \( s_o \), respectively. Hence homotopy sequences for the fibrations \([\Gamma(f_1, s_o)]_g\) and \([\Gamma(f_2, g(s_o))]_g\) in Theorem 4.1 show that \( h|_{A_1} : A_1 \rightarrow A_2 \) induces isomorphisms between \( \pi_n(A_1) \) and \( \pi_n(A_2) \) for a positive integer \( n > 0 \).

Now since \( S_1 \) and \( S_2 \) are simply connected spaces, then \( \Omega(S_1, s_o) \) and \( \Omega(S_2, g(s_o)) \) are pathwise connected. Since \( \Omega(O, r_o) \simeq ANR \), then by Theorem 2.3 \( \Omega(S_1, s_o) \) and \( \Omega(S_2, g(s_o)) \) are dominated by ANR's. Also these loop spaces are fiber spaces for the fibrations \( \Gamma(f_1, s_o) \) and \( \Gamma(f_2, g(s_o)) \) over \( s_o \), respectively. Since \( F_{r_o} \) is a pathwise connected and by Theorem 2.4 all fiber spaces are of the same homotopy type, then \( A_1 \) and \( A_2 \) are pathwise connected and dominated by ANR's. Since \( h|_{A_1} : A_1 \rightarrow A_2 \) induces isomorphisms between \( \pi_n(A_1) \) and \( \pi_n(A_2) \) for a positive integer \( n > 0 \), then by Theorem 2.2 \( h|_{A_1} : A_1 \rightarrow A_2 \) is a homotopy equivalence. Therefore since \( F_{r_o} \) is pathwise connected ANR, then by Fadell-Dold theorem, we get that \([\Gamma(f_1, s_o)]_g\) and \([\Gamma(f_2, g(s_o))]_g\) are fiber homotopy equivalent.

Corollary 4.5. Let \([S, f, O, F_{r_o}, \Theta_{f,r_o}]\) be a fibration with simply connected ANR fiber space \( F_{r_o} \) and with simply connected base \( O \) such that \( \Omega(O, r_o) \simeq ANR \). If there is a map \( k : O \rightarrow S \) such that \( f \circ k = id_O \), then \( \Gamma(f, k(r_o)) \) and \( \Gamma(P_1, (r_o, k(r_o))) \) are fiber homotopy equivalent, where \( P_1 : O \times F_{r_o} \rightarrow O \) is the first fibration.

Proof. Since \( k(r_o) \in S \) and \( f \circ k = id_O \), then
\[ f^{k(r_o)}(\alpha) = \Theta_{L_f}(\alpha, k(r_o)) = L_f(k(r_o), \alpha)(1) = L_f[k(r_o), f \circ (k \circ \alpha)](1) \]
for all \( \alpha \in \Omega(O, r_o) \). We observe easily that the \( L_f \)-restriction \( f^{k(r_o)} \) is homotopic to the map \( L : \Omega(O, r_o) \rightarrow F_{r_o} \) which is defined by
\[ \Omega(\alpha) = (k \circ \alpha)(1) = k(r_o) \quad \text{for} \quad \alpha \in \Omega(O, r_o) \]
by using the form of homotopy $H$ in the proof of Theorem 5.1. Consider $F_{r_0}$ as fiber space for $P_1$ because there is a homeomorphism map between it and $P_1^{-1}(r_0) = \{r_0\} \times F_{r_0}$. Now from Example 3.6, we get that fibration $P_1$ has $Lf$–restriction $P_1^{(r_0,k(r_0))} : \Omega(O, r_0) \rightarrow F_{r_0}$ given by

$$P_1^{(r_0,k(r_0))}(\alpha) = k(r_0) \quad \text{for} \quad \alpha \in \Omega(O, r_0).$$

Hence $f^{k(r_0)} \simeq L = \Phi_1^{k(r_0)}$, that is, fibrations $f$ and $P_1$ have conjugate $Lf$–restrictions $f^{k(r_0)}$ and $P_1^{(r_0,k(r_0))}$ by a homeomorphism $g = id_{F_{r_0}} \in H(F_{r_0}, F_{r_0})$. Hence by theorem above, $\Gamma(f, k(r_0))$ and $\Gamma(P_1, (r_0, k(r_0)))$ are fiber homotopy equivalent.

5 Fibration $\Sigma(f)$ and $Lf$–function

Here, we will introduce the role of homotopy sequences of fibrations (using $Lf$–function ) in satisfying FHE between two fibrations $\Sigma(f_1)$ and $\Sigma(f_2)$ which are induced by two fibrations $[S_1, f_1, O, F^1_{r_0}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F^2_{r_0}, \Theta_{L_{f_2}}]$ over a common base $O$.

In the following theorem, we show that for two fibrations $f_1$ and $f_2$ with conjugate $Lf$–functions, there are two fiber maps between two fibrations $\Sigma(f_1)$ and $\Gamma(f_2)$.

**Theorem 5.1.** Let $[S_1, f_1, O, F^1_{r_0}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F^2_{r_0}, \Theta_{L_{f_2}}]$ be fibrations with conjugate $Lf$–functions by $g \in H(F^1_{r_0}, F^2_{r_0})$. Then there are two fiber maps

$$D : \Sigma(S_1, F^1_{r_0}) \rightarrow \Sigma(S_2, F^2_{r_0}) \quad \text{and} \quad R : \Sigma(S_2, F^2_{r_0}) \rightarrow \Sigma(S_1, F^1_{r_0})$$

over $g \times g$ and $\overline{g} \times \overline{g}$, respectively. That is, Figure 7 is a commutative

\[
\begin{array}{ccc}
\Sigma(S_1, F^1_{r_0}) & \xrightarrow{D} & \Sigma(S_2, F^2_{r_0}) & \xrightarrow{R} & \Sigma(S_1, F^1_{r_0}) \\
F^1_{r_0} \times F^1_{r_0} & \xrightarrow{\Phi_1} & F^2_{r_0} \times F^2_{r_0} & \xrightarrow{\Phi_2} & F^1_{r_0} \times F^1_{r_0} \\
g \times g & & \overline{g} \times \overline{g} & & \end{array}
\]

Figure 7

**Proof.** Firstly, we will define fiber map $D$. By the hypothesis we get that

$$\Theta_{L_{f_1}} \simeq \overline{g} \circ \Theta_{L_{f_2}} \circ (id_{\Omega(O, r_0)} \times g).$$

This implies

$$g \circ \Theta_{L_{f_1}} \simeq \Theta_{L_{f_2}} \circ (id_{\Omega(O, r_0)} \times g).$$

Hence there is a homotopy $T : \Omega(O, r_0) \times F^1_{r_0} \rightarrow (F^2_{r_0})^l$ such that

$$T(\alpha, s)(0) = (\Theta_{L_{f_2}} \circ (id_{\Omega(O, r_0)} \times g))(\alpha, s)$$

$$= \Theta_{L_{f_2}}(\alpha, g(s)).$$
and

\[ T(\alpha, s)(1) = [g \circ \Theta_{L_f}] (\alpha, s) = g(\Theta_{L_f}(\alpha, s)) \]

for all \( \alpha \in \Omega(O, s), s \in F^1_{r_o} \). Define a map \( L''_{f_2} : \Sigma(S_1, F^1_{r_o}) \rightarrow \Sigma(S_2, F^2_{r_o}) \) by

\[ L''_{f_2}(\alpha) = L_{f_2}(g(\alpha(0), f_1 \circ \alpha)) \quad \text{for} \quad \alpha \in \Sigma(S_1, F^1_{r_o}), \]

and for \( \alpha \in \Sigma(S_1, F^1_{r_o}) \), we can use the homotopy \( H \) (which is defined in the proof of Theorem 4.1) to define the path \( W(\alpha) \in (F^2_{r_o})^I \) by

\[ W(\alpha)(t) = g\{[H(\alpha, t)](1)\} \quad \text{for} \quad t \in I. \]

Now we can define a map \( D : \Sigma(S_1, F^1_{r_o}) \rightarrow \Sigma(S_2, F^2_{r_o}) \) by

\[ D(\alpha) = [L''_{f_2}(\alpha) \ast T(f_1 \circ \alpha)] \ast W(\alpha) \quad \text{for} \quad \alpha \in \Sigma(S_1, F^1_{r_o}). \] (5)

Hence it is clear that \( D \) is well defined as a continuous map. We get that

\[ [\Phi_2 \circ D](\alpha) = [D(\alpha)(0), D(\alpha)(1)] = [L''_{f_2}(\alpha)(0), W(\alpha)(1)] = [g(\alpha(0)), g(\alpha(1))] = (g \times g)(\alpha(0), \alpha(1)) = [(g \times g) \times \Phi_1](\alpha) \]

for all \( \alpha \in \Sigma(S_1, F^1_{r_o}) \). That is, \( D \) is a fiber map over \( g \times g \). Secondly, we can find a fiber map \( R \) by above similar manner. \( \square \)

In the proof of Theorem 5.1 the two fiber maps have properties:

\[ D|_{\Omega(S_1, s_o)} = h_o \quad \text{and} \quad R|_{\Omega(S_2, g(s_o))} = k_o, \]

where \( h_o \) and \( k_o \) are defined in Theorem 4.1 and \( s_o \in F^1_{r_o} \). In proof of Theorem 4.1 it is clear that the map \( G \) is a restriction of a map \( T \) on \( \Omega(O, r_o) \), the map \( L'_{f_2} \) is a restriction of a map \( L''_{f_2} \) on \( \Gamma(S_1, F^1_{r_o}, s_o) \), and the map \( M' \) is a restriction of a map \( W \) on \( \Gamma(S_1, F^1_{r_o}, s_o) \). Hence from Equations 4 and 5 we get that the map \( h \) is a restriction of a map \( D \) on \( \Gamma(S_1, F^1_{r_o}, s_o) \), that is, \( D|_{\Omega(S_1, s_o)} = h_o \). Similarly, for \( R|_{\Omega(S_2, g(s_o))} = k_o \).

Also we introduce theorem about the functor \( \Sigma \) which is similar of Theorem 4.4.

**Theorem 5.2.** Let \( [S_1, f_1, O, F_{r_o}, \Theta_{L_{f_1}}] \) and \( [S_2, f_2, O, F_{r_o}, \Theta_{L_{f_2}}] \) be fibrations with conjugate \( Lf - \) functions by a homeomorphism \( g \in H(F_{r_o}, F_{r_o}) \), where \( s_o \in F_{r_o} \), and \( F_{r_o} \) be a common pathwise connected ANR. If \( S_1, S_2 \) are simply connected and \( \Omega(O, r_o) \simeq \text{ANR} \), then \( [\Sigma(f_1)]_{g \times g} \) and \( \Sigma(f_2) \) are fiber homotopy equivalent.
\[ \Sigma(S_1, F_{r_0}) \xrightarrow{D} \Sigma(S_2, F_{r_0}) \xrightarrow{\Phi_2} F_{r_0} \times F_{r_0} \]

Figure 8

Proof. By Theorem above, there is a fiber map \( D : \Sigma(S_1, F_{r_0}) \to \Sigma(S_2, F_{r_0}) \) in Figure 8. Let \( B_1 = [(g \times g) \circ \Phi_1]^{-1}(s_o, s_o) \) and \( B_2 = \Phi_2^{-1}(s_o, s_o) \), then

\[
B_1 = [(g \times g) \circ \Phi_1]^{-1}(s_o, s_o) \\
= \Phi_1^{-1}[(g^{-1} \times g^{-1})(s_o, s_o)] \\
= \{ \alpha \in S_1^I : \alpha(0) = g^{-1}(s_o), \alpha(1) = g^{-1}(s_o) \} \\
= \Omega(S_1, g^{-1}(s_o)),
\]

and

\[
B_2 = \Phi_2^{-1}(s_o, s_o) = \{ \alpha \in S_2^I : \alpha(0) = s_o, \alpha(1) = s_o \} = \Omega(S_2, s_o).
\]

We observe that \( B_1 \) and \( B_2 \) are fiber spaces for two fibrations \( \Sigma(f_1)|_{g \times g} \) and \( \Sigma(f_2) \) over \((s_o, s_o)\), respectively. Hence homotopy sequences for two fibrations \( \Sigma(f_1)|_{g \times g} \) and \( \Sigma(f_2) \) in Theorem 4.1 show that \( D|_{B_1} : B_1 \to B_2 \) induces isomorphisms between \( \pi_n(B_1) \) and \( \pi_n(B_2) \) for a positive integer \( n > 0 \). Since \( S_1 \) and \( S_2 \) are simply connected spaces, then \( B_1 \) and \( B_2 \) are pathwise \( S^N \)-connected. Since \( \Omega(O, r_o) \simeq ANR \), then by Theorem 2.3 we get that \( B_1 \) and \( B_2 \) are dominated by ANR’s. And since \( D|_{B_1} : B_1 \to B_2 \) induces isomorphisms between \( \pi_n(B_1) \) and \( \pi_n(B_2) \) for a positive integer \( n > 0 \), then by Theorem 2.2, \( D|_{B_1} : B_1 \to B_2 \) is a homotopy equivalence. Hence since \( F_{r_0} \times F_{r_0} \) is pathwise connected ANR, then by Fadell-Dold theorem, we get that \( \Sigma(f_1)|_{g \times g} \) and \( \Sigma(f_2) \) are fiber homotopy equivalent. \( \square \)

Conclusion: Further we also prove some theorems related to fiber homotopy equivalent classes by using the fiber homotopy sequences of homotopy groups. Thus we show the role of these fiber homotopy sequences in order to get the required fiber map in Fadell-Dold theorem. Further, the possible practical use of our theorems as applications will provide some solutions for the classification problem in Hurewicz fibration theory by using Fadell-Dold theorem.

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