A note on the Assmus–Mattson theorem for some binary codes II

Eiichi Bannai · Tsuyoshi Miezaki · Hiroyuki Nakasora

Received: 5 October 2022 / Revised: 10 March 2023 / Accepted: 13 March 2023 / Published online: 30 March 2023 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract

Let $C$ be a four-weight binary code, which has all one vector. Furthermore, we assume that $C$ supports $t$-designs for all weights obtained from the Assmus–Mattson theorem. We previously showed that $t \leq 5$. In the present paper, we show an analogue of this result in the cases of five and six-weight codes.

Keywords Assmus–Mattson theorem · $t$-Designs · Harmonic weight enumerator

Mathematics Subject Classification Primary 05B05 · Secondary 94B05 · 20B25

1 Introduction

Let $D_w$ be the support design of a binary code $C$ for weight $w$ and

$$\delta(C) := \max \{ t \in \mathbb{N} \mid \forall w, D_w \text{ is a } t\text{-design} \},$$

$$s(C) := \max \{ t \in \mathbb{N} \mid \exists w \text{ s.t. } D_w \text{ is a } t\text{-design} \}.$$

We note that $\delta(C) \leq s(C)$. In the previous papers [11, 18–21], we considered the possible occurrence of $\delta(C) < s(C)$. This was motivated by Lehmer’s conjecture, which is an analogue of $\delta(C) < s(C)$ in the theory of lattices and vertex operator algebras. For the details, see [4, 5, 7, 8, 12, 14, 16, 17, 21–23].
Let us explain our results. Throughout this paper, $C$ denotes a binary $[n, k, d]$ code and $1_n \in C$. Let $C^\perp$ be a binary $[n, n-k, d^\perp]$ dual code of $C$. We set $C_u := \{c \in C \mid \text{wt}(c) = u\}$. Note that $d^\perp$ is even because $1_n \in C$. We always assume that there exists $t \in \mathbb{N}$ that satisfies the following condition:

$$d^\perp - t = \sharp\{u \mid C_u \neq \emptyset, 0 \leq u \leq n-t\}. \quad (1.1)$$

This is a condition of the Assmus–Mattson theorem (see Theorem 2.1), and say the AM-condition. Let $C$ satisfy the AM-condition with $d^\perp - t$.

For cases in which $d^\perp - t = 4$, Theorem 1.1 (1), and Theorem 1.2 (1). In Sect. 4, we provide proofs of the

**Theorem 1.1** (1) If $C$ satisfies the AM-condition with $d^\perp - t = 4$, then $(d^\perp, t) = (6, 2)$ or $(8, 4)$.

(2) If $C$ satisfies the AM-condition with $d^\perp - t = 5$, then $(d^\perp, t) = (6, 1), (8, 3)$, or $(10, 5)$.

For cases in which $d^\perp - t = 4$ or 5, the following theorem provides a criterion for $n$ and $d$ such that $\delta(C^\perp) < s(C^\perp)$ occurs. Let $d = d_1$ and $d_2$ be the second smallest weight of $C$.

**Theorem 1.2** (1) Let $C$ satisfy the AM-condition with $(d^\perp, t) = (6, 2), (8, 4)$. Let $w \in \mathbb{N}$ such that

$$\sum_{i=0}^{w} (-1)^{w-i} \binom{d_1 - (t + 1)}{w - i} \left(\frac{n - 2d_1}{2i + 1}\right) - \frac{n - 2d_1}{n - 2d_2} \sum_{j=0}^{w} (-1)^{w-j} \binom{d_2 - (t + 1)}{w - j} \left(\frac{n - 2d_2}{2j + 1}\right) = 0.$$

Then $D_{2w+t+2}^\perp$ is a $(t+1)$-design. Hence, we have $\delta(C^\perp) < s(C^\perp)$.

(2) Let $C$ satisfy the AM-condition with $(d^\perp, t) = (6, 1), (8, 3)$, or $(10, 5)$. Let $w \in \mathbb{N}$ such that

$$\sum_{i=0}^{w} (-1)^{w-i} \binom{d_1 - (t + 1)}{w - i} \left(\frac{n - 2d_1}{2i}\right) - \frac{(n - 2d_1)(n - 2d_1 - 2)}{(n - 2d_2)(n - 2d_2 - 2)} \sum_{j=0}^{w} (-1)^{w-j} \binom{d_2 - (t + 1)}{w - j} \left(\frac{n - 2d_2}{2j}\right) + \frac{8(d_2 - d_1)(n - d_1 - d_2 - 1)}{(n - 2d_2)(n - 2d_2 - 2)} (-1)^{w+1} \binom{n/2 - (t + 1)}{w} = 0.$$

Then $D_{2w+t+1}^\perp$ is a $(t+1)$-design. Hence, we have $\delta(C^\perp) < s(C^\perp)$.

This paper is organized as follows: In Sect. 2, we provide background material and terminology. We review the concept of harmonic weight enumerators and some theorems of designs, which are used in the proof of the main results. In Sect. 3, we provide proofs of the case $d^\perp - t = 4$, Theorem 1.1 (1), and Theorem 1.2 (1). In Sect. 4, we provide proofs of the case $d^\perp - t = 5$, Theorem 1.1 (2), and Theorem 1.2 (2). Finally, in Sect. 5, we conclude the paper with some remarks.

We performed all the computer calculations in this paper with the help of MAGMA [9] and MATHEMATICA [24].
2 Preliminaries

2.1 Background material and terminology

A binary linear code $C$ of length $n$ is a subspace of $\mathbb{F}_2^n$. An inner product $(x, y)$ on $\mathbb{F}_2^n$ is given by

$$(x, y) = \sum_{i=1}^{n} x_i y_i,$$

where $x, y \in \mathbb{F}_2^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. The dual of a linear code $C$ is defined as follows:

$$C^\perp = \{ y \in \mathbb{F}_2^n \mid (x, y) = 0 \text{ for all } x \in C \}.$$

A linear code $C$ is self-dual if $C = C^\perp$. For $x \in \mathbb{F}_2^n$, the weight $\operatorname{wt}(x)$ is the number of its nonzero components. The minimum distance of the code $C$ is $\min\{\operatorname{wt}(x) \mid x \in C, x \neq 0\}$.

A linear code of length $n$, dimension $k$, and minimum distance $d$ is called an $[n, k, d]$ code (or $[n, k]$ code) and the dual code is called an $[n, n-k, d^\perp]$ code.

A $t$-$(v, k, \lambda)$ design (or $t$-design, for short) is a pair $\mathcal{D} = (X, \mathcal{B})$, where $X$ is a set of points of cardinality $v$, and $\mathcal{B}$ is a collection of $k$-element subsets of $X$ called blocks, with the property that any $t$ points are contained in precisely $\lambda$ blocks.

The support of a vector $x := (x_1, \ldots, x_n), x_i \in \mathbb{F}_2$ is the set of indices of its nonzero coordinates: $\operatorname{supp}(x) = \{i \mid x_i \neq 0\}$. Let $\Omega := \{1, \ldots, n\}$ and $\mathcal{B}(C_t) := \{\operatorname{supp}(x) \mid x \in C_t\}$. Then for a code $C$ of length $n$, we say that $C_t$ is a $t$-design if $\mathcal{D}_t = (\Omega, \mathcal{B}(C_t))$ is a $t$-design. We call $\mathcal{D}_t$ a support design of $C$.

The following theorem is from Assmus and Mattson [2]. It is one of the most important theorems in coding theory and design theory:

**Theorem 2.1** [2] Let $C$ be a binary $[n, k, d]$ linear code and $C^\perp$ be the $[n, n-k, d^\perp]$ dual code. Let $t$ be an integer less than $d$. Let $C$ have at most $d^\perp - t$ non-zero weights less than or equal to $n - t$. Then, for each weight $w$ with $d \leq w \leq n - t$, the support design in $C$ is a $t$-design, and for each weight $w$ with $d^\perp \leq w \leq n$, the support design in $C^\perp$ is a $t$-design.

2.2 Harmonic weight enumerators

In this subsection, we review the concept of harmonic weight enumerators.

Let $C$ be a code of length $n$. The weight distribution of the code $C$ is the sequence $\{A_i \mid i = 0, 1, \ldots, n\}$, where $A_i$ is the number of codewords of weight $i$. The polynomial

$$W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$$

is called the weight enumerator of $C$. The weight enumerator of the code $C$ and its dual $C^\perp$ are related. The following theorem, proposed by MacWilliams, is called the MacWilliams identity:

**Theorem 2.2** [13] Let $W_C(x, y)$ be the weight enumerator of an $[n, k]$ code $C$ over $\mathbb{F}_q$ and let $W_{C^\perp}(x, y)$ be the weight enumerator of the dual code $C^\perp$. Then

$$W_{C^\perp}(x, y) = q^{-k} W_C(x + (q - 1)y, x - y).$$
A striking generalization of the MacWilliams identity was provided by Bachoc [3], who proposed the concept of harmonic weight enumerators and a generalization of the MacWilliams identity. Harmonic weight enumerators have many applications; in particular, the relations between coding theory and design theory are reinterpreted and progressed by harmonic weight enumerators [3, 6]. For the reader’s convenience, we quote the definitions and properties of discrete harmonic functions from [3, 10].

Let \( \Omega = \{1, 2, \ldots, n\} \) be a finite set (which is the set of coordinates of the code) and let \( X \) be the set of its subsets, where for all \( h = 0, 1, \ldots, n \), \( X_h \) is the set of its \( h \)-subsets. Let \( \mathbb{R}X \) and \( \mathbb{R}X_h \) denote the free real vector spaces spanned by the elements of \( X \) and \( X_h \), respectively. An element of \( \mathbb{R}X_h \) is denoted by

\[
  f = \sum_{z \in X_h} f(z)z
\]

and is identified with the real-valued function on \( X_h \) given by \( z \mapsto f(z) \).

Such an element \( f \in \mathbb{R}X_h \) can be extended to an element \( \tilde{f} \in \mathbb{R}X \) by setting, for all \( u \in X \),

\[
  \tilde{f}(u) = \sum_{z \in X_h, z \subset u} f(z).
\]

If an element \( g \in \mathbb{R}X \) is equal to some \( \tilde{f} \), for \( f \in \mathbb{R}X_h \), we say that \( g \) has degree \( h \). The differentiation \( \gamma \) is the operator defined by linearity from

\[
  \gamma(z) = \sum_{y \in X_{h-1}, y \subset z} y
\]

for all \( z \in X_h \) and for all \( h = 0, 1, \ldots, n \), and \( \text{Harm}_h \) is the kernel of \( \gamma \):

\[
  \text{Harm}_h = \ker(\gamma|_{\mathbb{R}X_h}).
\]

**Theorem 2.3** [10, Theorem 7] A set \( B \subset X_m \), where \( m \leq n \) of blocks is a \( t \)-design if and only if \( \sum_{b \in B} \tilde{f}(b) = 0 \) for all \( f \in \text{Harm}_h \), \( 1 \leq h \leq t \).

In [3], the harmonic weight enumerator associated with a binary linear code \( C \) was defined as follows:

**Definition 2.4** Let \( C \) be a binary code of length \( n \) and let \( f \in \text{Harm}_h \). The harmonic weight enumerator associated with \( C \) and \( f \) is

\[
  W_{C,f}(x, y) = \sum_{c \in C} \tilde{f}(c)x^{n-\text{wt}(c)}y^{\text{wt}(c)}.
\]

Bachoc proved the following MacWilliams-type equality:

**Theorem 2.5** [3] Let \( W_{C,f}(x, y) \) be the harmonic weight enumerator associated with the code \( C \) and the harmonic function \( f \) of degree \( h \). Then

\[
  W_{C,f}(x, y) = (xy)^h Z_{C,f}(x, y),
\]

where \( Z_{C,f} \) is a homogeneous polynomial of degree \( n - 2h \), and satisfies

\[
  Z_{C_{\perp},f}(x, y) = (-1)^h \frac{2^{n/2}}{|C|} Z_{C,f} \left( \frac{x + y}{\sqrt{2}}, \frac{x - y}{\sqrt{2}} \right).
\]
3 Case $d^t = 4$

In this section, we always assume that $d^t = 4$. Then the weights of $C$ are $0, d_1, d_2, n - d_2, n - d_1, n$. Before providing the proof, we show the following lemma, which will be used in the proof of Theorem 3.2:

**Lemma 3.1** Let $n, k \in \mathbb{Z}_{\geq 0}$. The solutions of the following equation

$$\sum_{i=0}^{4} \binom{n-1}{i} = \frac{1}{24} (24 - 18n + 23n^2 - 6n^3 + n^4) = 2^k$$

are as follows:

$$(n, k) = (0, 0), (1, 0), (2, 1), (3, 2), (4, 3), (5, 4), (10, 8).$$

**Proof** We assume that $k \equiv 0 \pmod{2}$ and $y = 2^{k/2}$. Then

$$\frac{1}{24} (24 - 18n + 23n^2 - 6n^3 + n^4) = y^2$$

$$\Leftrightarrow 6 \times (24 - 18n + 23n^2 - 6n^3 + n^4) = 6 \times 24 \times y^2.$$

Let $Y = 12y$. Then

$$6 \times (24 - 18n + 23n^2 - 6n^3 + n^4) = Y^2.$$

By “IntegerQuarticPoints” (a command in MAGMA), we obtain the solutions $(n, Y)$:

$$(-12, 456), (10, -192), (5, 48), (3, -24), (36, 2928), (1, 12), (-2, -36), (0, -12), (-237, 139344).$$

If

$$\frac{1}{24} (24 - 18n + 23n^2 - 6n^3 + n^4) = 2^k$$

is a power of 2, then $n = 10, 5, 3, 1, 0, k = 8, 4, 2, 0, 0$, respectively.

We assume that $k \equiv 1 \pmod{2}$, $y = 2^{(k-1)/2}$. Then

$$\frac{1}{24} (24 - 18n + 23n^2 - 6n^3 + n^4) = 2y^2$$

$$\Leftrightarrow 3 \times (24 - 18n + 23n^2 - 6n^3 + n^4) = 3 \times 24 \times 2y^2.$$

Let $Y = 12y$. Then

$$3 \times (24 - 18n + 23n^2 - 6n^3 + n^4) = Y^2.$$

Let $n = X + 2$. Then

$$(144 + 102X + 33X^2 + 6X^3 + 3X^4) = Y^2.$$

By “IntegerQuarticPoints” (a command in MAGMA), we obtain the solutions $(X, Y)$:

$$(2, -24), (0, -12).$$

We have $n = 4, 2$. If

$$\frac{1}{24} (24 - 18n + 23n^2 - 6n^3 + n^4) = 2^k$$

is a power of 2, then $n = 4, 2, k = 3, 1$, respectively. \square
3.1 Proof of Theorem 1.1 (1)

In this subsection, we provide the proof of Theorem 1.1 (1). Let

\[ W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2} + \beta x^{d_2} y^{n-d_2} + \alpha x^{d_1} y^{n-d_1} + y^n \]

be the weight enumerator of \( C \). We remark that

\[ 1 + \alpha + \beta = 2^{k-1}. \]

We show that if \( d_\perp \geq 10 \), then we have the constraints Eqs. (3.1). By Theorem 2.2,

\[ W_{C_\perp}(x, y) = 2^{-k} W_C(x + y, x - y) = 2^{-k} \sum_{i \geq 0} A_i x^{n-2i} y^{2i}. \]

If \( d_\perp \geq 10 \), then we have the following constraints:

\[ A_2 = A_4 = A_6 = A_8 = 0. \] (3.1)

Using Eqs. (3.1), we show the following theorem:

**Theorem 3.2** There is no code \( C \) with \( d_\perp \geq 10 \).

**Proof** We assume that \( C \) has \( d_\perp \geq 10 \). Using Eqs. (3.1), we delete their constant terms as follows:

\[
\begin{align*}
\binom{n}{4} A_2 - \binom{n}{2} A_4 &= 0 \iff X_1 \alpha + Y_1 \beta = 0, \\
\binom{n}{6} A_2 - \binom{n}{2} A_6 &= 0 \iff X_2 \alpha + Y_2 \beta = 0, \\
\binom{n}{8} A_2 - \binom{n}{2} A_8 &= 0 \iff X_3 \alpha + Y_3 \beta = 0,
\end{align*}
\]

where \( X_i \) (1 \( \leq i \leq 3 \)) and \( Y_i \) (1 \( \leq i \leq 3 \)) are the coefficients of \( \alpha \) and \( \beta \), respectively. Let

\[ M_1 = \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} X_1 & Y_1 \\ X_3 & Y_3 \end{pmatrix}. \]

By \( \alpha \neq 0 \) and \( \beta \neq 0 \),

\[ \det(M_1) = 0, \det(M_2) = 0. \]

Then using **MATHEMATICA** [24], we obtain the solutions, which are listed on the homepage of one of the authors [15].

Because 0 \( < d_1 < d_2 < n/2 \) and 5 \( < n \), the solutions (1a)–(20a), (23a), (24a), (27a), and (28a) in [15] are impossible. We show that (25a) in [15] is impossible. The other cases (21a),(22a), and (26a) can be proved similarly.

Using Eqs. (3.1), and (25a) in [15], using **MATHEMATICA**, we obtain

\[ 1 + \alpha + \beta = \frac{1}{24} (24 - 18n + 23n^2 - 6n^3 + n^4). \]

By Lemma 3.1,

\[ (n, k) = (0, 0), (1, 0), (2, 1), (3, 2), (4, 3), (5, 4), (10, 8), \]

and it is clear that these cases are impossible. \( \square \)
Proof of Theorem 1.1 (1) By Theorem 3.2, $d^\perp \leq 8$. Hence, $(d^\perp, t) = (6, 2)$ or $(8, 4)$. \hfill \square

3.2 Proof of Theorem 1.2 (1)

In this subsection, we provide the proof of Theorem 1.2 (1).

Proof The harmonic weight enumerator of $f \in \text{Harm}_{t+1}^t$ is

$$W_{C, f} = a_1 x^{n-d_1} y^{d_1} + a_2 x^{n-d_2} y^{d_2} + b_1 x^{d_1} y^{n-d_1}$$

$$= (xy)^{t+1}(a_1 x^{n-d_1-(t+1)} y^{d_1-(t+1)} + a_2 x^{n-d_2-(t+1)} y^{d_2-(t+1)}$$

$$+ b_2 x^{d_2-(t+1)} y^{n-d_2-(t+1)} + b_1 x^{d_1-(t+1)} y^{n-d_1-(t+1)}),$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. We set

$$Z_{C, f} = a_1 x^{n-d_1-(t+1)} y^{d_1-(t+1)}$$

$$+ a_2 x^{n-d_2-(t+1)} y^{d_2-(t+1)}$$

$$+ b_1 x^{d_1-(t+1)} y^{n-d_1-(t+1)}$$

$$+ b_2 x^{d_2-(t+1)} y^{n-d_2-(t+1)}.$$ 

Then, by Theorem 2.5,

$$Z_{C^\perp, f} = a'_1 (x + y)^{n-d_1-(t+1)} (x - y)^{d_1-(t+1)}$$

$$+ a'_2 (x + y)^{n-d_2-(t+1)} (x - y)^{d_2-(t+1)}$$

$$+ b_1 (x + y)^{d_1-(t+1)} (x - y)^{n-d_1-(t+1)}$$

$$+ b_2 (x + y)^{d_2-(t+1)} (x - y)^{n-d_2-(t+1)}.$$ 

Since the coefficient of $x^{n-2t-2}$ in $Z_{C^\perp, f}$ is zero, $a'_1 + b'_1 = 0$ and $a'_2 + b'_2 = 0$.

By $d^\perp \neq t + 2$, the coefficient of $x^{n-2t-3} y$ in $Z_{C^\perp, f}$ is zero. Then,

$$a'_1 (n - d_1 - (t + 1) - (d_1 - (t + 1)))$$

$$+ a'_2 (n - d_2 - (t + 1) - (d_2 - (t + 1))) = 0.$$ 

Hence,

$$a'_2 = \frac{n - 2d_1}{n - 2d_2} a'_1.$$

Then,

$$Z_{C^\perp, f} = a'_1 ((x + y)^{n-d_1-(t+1)} (x - y)^{d_1-(t+1)}$$

$$- \frac{n - 2d_1}{n - 2d_2} (x + y)^{n-d_2-(t+1)} (x - y)^{d_2-(t+1)}$$

$$+ \frac{n - 2d_1}{n - 2d_2} (x + y)^{d_1-(t+1)} (x - y)^{n-d_1-(t+1)}$$

$$- (x + y)^{d_1-(t+1)} (x - y)^{n-d_1-(t+1)})$$

$$= a'_1 ((x^2 - y^2)^{d_1-(t+1)} (x + y)^{n-2d_1}$$

$$- \frac{n - 2d_1}{n - 2d_2} (x^2 - y^2)^{d_2-(t+1)} (x + y)^{n-2d_2}$$

$$+ \frac{n - 2d_1}{n - 2d_2} (x^2 - y^2)^{d_1-(t+1)} (x - y)^{n-2d_2}$$

$$- (x^2 - y^2)^{d_1-(t+1)} (x - y)^{n-2d_1}).$$ (3.2)
Let 

\[ W_{C^\perp, f} = (xy)^{t+1} Z_{C^\perp, f} = \sum p_i x^{n-i} y^i. \]

Recall that if \( p_{2w+t+2} = 0 \) then \( D_{2w+t+2}^\perp \) is a \((t+1)\)-design. By (3.2),

\[
p_{2w+t+2} = \text{(constant)} \times \left( \sum_{i=0}^{w} (-1)^{w-i} \binom{d_1 - (t+1)}{w-i} (n-2d_1) \right) 2i + 1
- \frac{n - 2d_1}{n - 2d_2} \sum_{j=0}^{w} (-1)^{w-j} \binom{d_2 - (t+1)}{w-j} (n-2d_2) \right) 2j + 1)\right).
\]

By Theorem 2.3, if the equation

\[
\sum_{i=0}^{w} (-1)^{w-i} \binom{d_1 - (t+1)}{w-i} (n-2d_1) \] 2i + 1
- \frac{n - 2d_1}{n - 2d_2} \sum_{j=0}^{w} (-1)^{w-j} \binom{d_2 - (t+1)}{w-j} (n-2d_2) \right) 2j + 1) = 0,
\]

\( D_{2w+t+2}^\perp \) is a \((t+1)\)-design. \( \Box \)

4 Case \( d^\perp - t = 5 \)

In this section, we always assume that \( d^\perp - t = 5 \). Then the weights of \( C \) are \( 0, d_1, d_2, n/2, n - d_2, n - d_1, n \). Before providing the proof, we show the following lemma, which will be used in the proof of Theorem 4.2:

**Lemma 4.1** Let \( n, m \in \mathbb{Z}_{\geq 0} \). The solutions of the following equation

\[
\sum_{i=0}^{5} \binom{n-1}{i} = \frac{1}{120} (184n - 110n^2 + 55n^3 - 10n^4 + n^5) = 2^m
\]

are as follows:

\( (n, m) = (1, 0), (2, 1), (3, 2), (4, 3), (5, 4), (6, 5), (12, 10). \)

**Proof** The following argument was made by Professor Max Alekseyev in [1]. We seek that the integer solutions of

\[
S(n, 5) = \sum_{i=0}^{5} \binom{n}{i} = 2^m.
\]

Then, \( 5! S(n, 5) = (n + 1) g_5(n) \), where

\[
g_5(n) = 120 - 26n + 31n^2 - 6n^3 + n^4.
\]

Hence,

\[
n + 1 = 2^k, 3 \times 2^k, 5 \times 2^k, 15 \times 2^k.
\]
First, we assume that \( n + 1 = 2^k \). Then
\[
g_5(2^k - 1) = 184 + 31 \times 2^{2k} + 2^{4k} - 9 \times 2^{1+k} - 2^{2+k} \\
- 11 \times 2^{3+k} + 3 \times 2^{3+2k} - 3 \times 2^{1+3k} - 2^{2+3k} = 15 \times 2^\ell.
\]
We note that 184 = 2^3 \times 23.

If \( k \leq 5 \), then we obtain the solution \((n, \ell) = (0, 3), (1, 3), (3, 4)\). If \( k > 5 \), then considering modulo 2^7, the following equation has no solutions:
\[
g_5(2^k - 1) = 184 + 31 \times 2^{2k} + 2^{4k} - 9 \times 2^{1+k} - 2^{2+k} \\
- 11 \times 2^{3+k} + 3 \times 2^{3+2k} - 3 \times 2^{1+3k} - 2^{2+3k} = 15 \times 2^3.
\]

For the other cases \( n + 1 = 3 \times 2^k, 5 \times 2^k, 15 \times 2^k \), we obtain the solutions similarly as follows:
\[
(n, \ell) = (0, 3), (1, 3), (2, 5), (3, 4), (4, 7), (5, 7), (11, 11).
\]

\[\square\]

4.1 Proof of Theorem 1.1 (2)

In this subsection, we provide the proof of Theorem 1.1 (2). Let
\[
W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2} + \gamma x^2 y^2 + \alpha x d_1 y^{n-d_1} + y^n
\]
be the weight enumerator of \( C \). We remark that
\[
1 + \alpha + \beta + \frac{\gamma}{2} = 2^{k-1}.
\]

First, we show that if \( d^\perp \geq 12 \), then we have the constraints Eqs. (4.1). By Theorem 2.2,
\[
W_{C^\perp}(x, y) = 2^{-k} W_C(x + y, x - y) = \sum_{i \geq 0} A_i x^{n-2i} y^{2i},
\]
If the coefficient of \( x^{n-2i} y^{2i} \) (1 \( \leq i \leq 5 \)) in \( W_{C^\perp}(x, y) \) is zero, then
\[
A_2 = A_4 = A_6 = A_8 = A_{10} = 0.
\] (4.1)

Therefore, if \( d^\perp \geq 12 \), then we have the constraints Eqs. (4.1). Using Eqs. (4.1), we show that the following theorem:

Theorem 4.2 There is no code \( C \) with \( d^\perp \geq 12 \).

Proof We assume that \( C \) has \( d^\perp \geq 12 \). Using Eqs. (4.1), we write \( \alpha, \beta, \) and \( \gamma \) in terms of \( n, d_1, \) and \( d_2 \), that is,
\[
\alpha = \alpha_1 = Y_{11}(n, d_1, d_2), \\
\beta = \beta_1 = Y_{12}(n, d_1, d_2), \\
\gamma = \gamma_1 = Y_{13}(n, d_1, d_2).
\]
Similarly, using Eqs. (4.1), we write $\alpha$, $\beta$, and $\gamma$ in terms of $n$, $d_1$, and $d_2$, that is,

\[
\begin{align*}
\alpha &= \alpha_2 = Y_{21}(n, d_1, d_2), \\
\beta &= \beta_2 = Y_{22}(n, d_1, d_2), \\
\gamma &= \gamma_2 = Y_{23}(n, d_1, d_2),
\end{align*}
\]

and using Eqs. (4.1), we write $\alpha$, $\beta$, and $\gamma$ in terms of $n$, $d_1$, and $d_2$, that is,

\[
\begin{align*}
\alpha &= \alpha_3 = Y_{31}(n, d_1, d_2), \\
\beta &= \beta_3 = Y_{32}(n, d_1, d_2), \\
\gamma &= \gamma_3 = Y_{33}(n, d_1, d_2).
\end{align*}
\]

Using MATHEMATICA, we obtain the solutions of

\[
\begin{align*}
\alpha_1 &= \alpha_2, \quad \alpha_1 = \alpha_3, \quad \beta_1 = \beta_2, \quad \beta_1 = \beta_3, \quad \gamma_1 = \gamma_2, \quad \gamma_1 = \gamma_3.
\end{align*}
\]

We note that these solutions are listed on the homepage of one of the authors [15]. Because $0 < d_1 < d_2 < n/2$ and $5 < n$, the solutions (1b)–(19b), (22b), (23b), (26b), and (27b) in [15] are impossible. We show that (25b) in [15] is impossible. The other cases (20b), (21b), and (24b) can be proved similarly.

Then using the solution (25b) in [15] and Eqs. (4.1), using MATHEMATICA, we obtain

\[
1 + \alpha + \beta + \frac{\gamma}{2} = \frac{1}{120}(184n - 110n^2 + 55n^3 - 10n^4 + n^5).
\]

By Lemma 4.1,

\[
(n, k) = (1, 0), \ (2, 1), \ (3, 2), \ (4, 3), \ (5, 4), \ (6, 5), \ (12, 10),
\]

and it is clear that these cases are impossible. \hfill \square

**Proof of Theorem 1.1 (1)** By Theorem 4.2, $d^\perp \leq 10$. Hence, $(d^\perp, t) = (6, 1), \ (8, 3), \ or \ \ (10, 5). \hfill \square

### 4.2 Proof of Theorem 1.2 (2)

In this subsection, we provide the proof of Theorem 1.2 (2).

**Proof** The harmonic weight enumerator of $f \in \text{Harm}_{r+1}$ is

\[
W_{C,f} = a_1 x^{n-d_1} y^{d_1} + a_2 x^{n-d_2} y^{d_2} + bx^{\frac{n}{2}} y^{\frac{n}{2}} + c_2 x^{d_2} y^{n-d_2} + c_1 x^{d_1} y^{n-d_1}
\]

\[
= (xy)^{t+1} (a_1 x^{n-d_1-(t+1)} y^{d_1-(t+1)} + a_2 x^{n-d_2-(t+1)} y^{d_2-(t+1)}
\]

\[
+ bx^{\frac{n}{2}-(t+1)} y^{\frac{n}{2}-(t+1)} + c_2 x^{d_2-(t+1)} y^{n-d_2-(t+1)}
\]

\[
+ c_1 x^{d_1-(t+1)} y^{n-d_1-(t+1)}),
\]

where $a_1, \ a_2, \ b, \ c_1, \ c_2 \in \mathbb{R}$. We set

\[
Z_{C,f} = a_1 x^{n-d_1-(t+1)} y^{d_1-(t+1)} + a_2 x^{n-d_2-(t+1)} y^{d_2-(t+1)}
\]

\[
+ bx^{\frac{n}{2}-(t+1)} y^{\frac{n}{2}-(t+1)}
\]

\[
+ c_2 x^{d_2-(t+1)} y^{n-d_2-(t+1)}
\]

\[
+ c_1 x^{d_1-(t+1)} y^{n-d_1-(t+1)}.
\]
Then by Theorem 2.5,
\[
Z_{C^\perp, f} = a'_1(x + y)^{n-d_1-(t+1)}(x - y)^{d_1-(t+1)}
\]
\[
+ a'_2(x + y)^{n-d_2-(t+1)}(x - y)^{d_2-(t+1)}
\]
\[
+ b'(x + y)^{\frac{n}{2}-(t+1)}(x - y)^{\frac{n}{2}-(t+1)}
\]
\[
+ c'_2(x + y)^{d_2-(t+1)}(x - y)^{n-d_2-2-(t+1)}
\]
\[
+ c'_1(x + y)^{d_1-(t+1)}(x - y)^{n-d_1-1-(t+1)}
\]
\[
= a'_1(x^2 - y^2)^{d_1-(t+1)}(x + y)^{n-2d_1}
\]
\[
+ a'_2(x^2 - y^2)^{d_2-(t+1)}(x + y)^{n-2d_2}
\]
\[
+ b'(2 - y^2)^{\frac{n}{2}-(t+1)}
\]
\[
+ c'_2(x^2 - y^2)^{d_2-(t+1)}(x - y)^{n-2d_2}
\]
\[
+ c'_1(x^2 - y^2)^{d_1-(t+1)}(x - y)^{n-2d_1}.
\]

Since \(C^\perp\) does not have an odd weight, \(a'_1 - c'_1 = 0\) and \(a'_2 - c'_2 = 0\). By \(d^\perp \neq t + 1\) and \(t + 3\), the coefficients of \(x^{n-2(t+1)}\) and \(x^{n-2(t+2)}y^2\) are zero. Then,
\[
2a'_1 + 2a'_2 + b' = 0,
\]
\[
2a'_1 \left(d_1 - (t + 1) + \left(\frac{n-2d_1}{2}\right)\right) + 2a'_2 \left(d_2 - (t + 1) + \left(\frac{n-2d_2}{2}\right)\right)
\]
\[
+ b' \left(\frac{n}{2} - (t + 1)\right) = 0.
\] (4.2)

By (4.2) and (4.3),
\[
a'_2 = -\frac{(n-2d_1)(n-2d_1-2)}{(n-2d_2)(n-2d_2-2)} \cdot a'_1,
\]
\[
b' = \frac{8(d_2-d_1)(n-d_1-d_2-1)}{(n-2d_2)(n-2d_2-2)} \cdot a'_1.
\]

Then,
\[
Z_{C^\perp, f} = a'_1\left(x^2 - y^2\right)^{d_1-(t+1)}(x + y)^{n-2d_1}
\]
\[
- \frac{(n-2d_1)(n-2d_1-2)}{(n-2d_2)(n-2d_2-2)} \cdot (x^2 - y^2)^{d_2-(t+1)}(x + y)^{n-2d_2}
\]
\[
+ \frac{8(d_2-d_1)(n-d_1-d_2-1)}{(n-2d_2)(n-2d_2-2)} \cdot (x^2 - y^2)^{\frac{n}{2}-(t+1)}
\]
\[
- \frac{(n-2d_1)(n-2d_1-2)}{(n-2d_2)(n-2d_2-2)} \cdot (x^2 - y^2)^{d_2-(t+1)}(x - y)^{n-2d_2}
\]
\[
+ (x^2 - y^2)^{d_1-(t+1)}(x - y)^{n-2d_1}.
\] (4.4)

Let
\[
W_{C^\perp, f} = (xy)^{t+1}Z_{C^\perp, f} = \sum p_i x^{n-i} y^i.
\]

\[\text{Springer} \]
Recall that if \( p_{2w+t+1} = 0 \) then \( D_{2w+t+1}^\perp \) is a \((t+1)\)-design. By (4.4),

\[
p_{2w+t+1} = (\text{constant}) \times \left( \sum_{i=0}^{w} (-1)^{w-i} \binom{d_1 - (t+1)}{w-i} \binom{n-2d_1}{2i} \right)
\]

\[
- \frac{(n-2d_1)(n-2d_1-2)}{(n-2d_2)(n-2d_2-2)} \sum_{j=0}^{w} (-1)^{w-j} \binom{d_2 - (t+1)}{w-j} \binom{n-2d_2}{2j}
\]

\[
+ \frac{8(d_2-d_1)(n-d_1-d_2-1)}{(n-2d_2)(n-2d_2-2)} (-1)^{w+1} \binom{n/2 - (t+1)}{w}
\]

By Theorem 2.3, if the equation

\[
\sum_{i=0}^{w} (-1)^{w-i} \binom{d_1 - (t+1)}{w-i} \binom{n-2d_1}{2i} \left( \begin{array}{c} n-2d_1 \\ w-i \end{array} \right) 
\]

\[
- \frac{(n-2d_1)(n-2d_1-2)}{(n-2d_2)(n-2d_2-2)} \sum_{j=0}^{w} (-1)^{w-j} \binom{d_2 - (t+1)}{w-j} \binom{n-2d_2}{2j}
\]

\[
+ \frac{8(d_2-d_1)(n-d_1-d_2-1)}{(n-2d_2)(n-2d_2-2)} (-1)^{w+1} \binom{n/2 - (t+1)}{w} = 0,
\]

then \( D_{2w+t+1}^\perp \) is a \((t+1)\)-design. \( \Box \)

5 Concluding remarks

Remark 5.1

(1) Are there examples that satisfy the condition of Theorem 1.2?

(2) For the case \( d^\perp - t = 4 \), if we assume that \( d^\perp \geq 10 \) and

\[
W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2} + \beta x^{d_2} y^{n-d_2} + \alpha x^{d_1} y^{n-d_1} + y^n,
\]

then we have

\[
1 + \alpha + \beta = \sum_{i=0}^{4} \binom{n-1}{i}.
\]

Similarly, for the case \( d^\perp - t = 5 \), if we assume that \( d^\perp \geq 12 \) and

\[
W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2} + \gamma x^2 y^2 + \beta x^{d_2} y^{n-d_2} + \alpha x^{d_1} y^{n-d_1} + y^n,
\]

then we have

\[
1 + \alpha + \beta + \frac{\gamma}{2} = \sum_{i=0}^{5} \binom{n-1}{i}.
\]

This suggests the following conjecture:
Conjecture 5.2 Let $C$ be a binary antipodal $[n, k]$ code and 

$$W_C(x, y) = x^n + \sum_{i \geq 1} \alpha_i x^{n-d_i} y^{d_i} + y^n,$$

be the weight enumerator of $C$. We assume that $C$ satisfies the AM-condition with $d^\perp - t = \ell$ ($\ell \geq 6$). If we assume that $d^\perp \geq 2\ell + 2$, then we have

$$\begin{cases} 1 + \alpha_1 + \cdots + \alpha_{\ell/2} = \sum_{i=0}^{\ell} \binom{n-1}{i} = 2^{k-1} \ (if \ \ell \equiv 0 \ (mod \ 2)), \\ 1 + \alpha_1 + \cdots + \frac{\alpha_{\ell+1}/2}{2} = \sum_{i=0}^{\ell} \binom{n-1}{i} = 2^{k-1} \ (if \ \ell \equiv 1 \ (mod \ 2)). \end{cases}$$

Moreover, we have $t \leq \ell$.

To date, we do not have a proof of this conjecture.

Acknowledgements The authors would like to thank the anonymous reviewers for their beneficial comments on an earlier version of the manuscript. The second named author was supported by JSPS KAKENHI (22K03277). We thank Maxine Garcia, PhD, from Edanz (https://jp.edanz.com/ac) for editing a draft of this manuscript.

References

1. Alekseyev M.: When do binomial coefficients sum to a power of 2? Mathoverflow. https://mathoverflow.net/questions/412940/
2. Assmus E.F. Jr., Mattson H.F. Jr.: New 5-designs. J. Comb. Theory 6, 122–151 (1969).
3. Bachoc C.: On harmonic weight enumerators of binary codes. Des. Codes Cryptogr. 18(1–3), 11–28 (1999).
4. Bannai E., Miezaki T.: Toy models for D. H. Lehmer’s conjecture. J. Math. Soc. Jpn. 62(3), 687–705 (2010).
5. Bannai E., Miezaki T.: Toy models for D. H. Lehmer’s conjecture II: quadratic and higher degree forms. Dev. Math. 31, 1–27 (2013).
6. Bannai E., Koike M., Shinohara M., Tagami M.: Spherical designs attached to extremal lattices and the modulo p property of Fourier coefficients of extremal modular forms. Mosc. Math. J. 6, 225–264 (2006).
7. Bannai E., Miezaki T., Yudin V.A.: An elementary approach to toy models for Lehmer’s conjecture. Russ. Izv. Ross. Akad. Nauk Ser. Mat. 75(6), 3–16 (2011).
8. Bannai E., Miezaki T., Yudin V.A.: An elementary approach to toy models for Lehmer’s conjecture. Russ. Izv. Math. 75(6), 1093–1106 (2011).
9. Bosma W., Cannon J., Playoust C.: The Magma algebra system. I. The user language. J. Symb. Comput. 24, 235–265 (1997).
10. Delsarte P.: Hahn polynomials, discrete harmonics, and $t$-designs. SIAM J. Appl. Math. 34(1), 157–166 (1978).
11. Horiguchi N., Miezaki T., Nakasora H.: On the support designs of extremal binary doubly even self-dual codes. Des. Codes Cryptogr. 72, 529–537 (2014).
12. Lehmer D.H.: The vanishing of Ramanujan’s $\tau(n)$. Duke Math. J. 14, 429–433 (1947).
13. Macwilliams J.: A theorem on the distribution of weights in a systematic code. Bell Syst. Tech. J. 42, 79–84 (1963).
14. Miezaki T., Nakasora H.: On the Assmus–Mattson type theorem for Type I and even formally self-dual codes, submitted.
15. Miezaki T.: Tsuyoshi Miezaki’s website. http://www.f.waseda.jp/miezaki/data.html.
16. Miezaki T.: Conformal designs and D. H. Lehmer’s conjecture. J. Algebra 374, 59–65 (2013).
17. Miezaki T.: Design-theoretic analogies between codes, lattices, and vertex operator algebras. Des. Codes Cryptogr. 89(5), 763–780 (2021).
18. Miezaki T., Nakasora H.: An upper bound of the value of $t$ of the support $t$-designs of extremal binary doubly even self-dual codes. Des. Codes Cryptogr. 79, 37–46 (2016).
19. Miezaki T., Nakasora H.: The support designs of the triply even binary codes of length 48. J. Comb. Des. 27, 673–681 (2019).
20. Miezaki T., Nakasora H.: A note on the Assmus–Mattson theorem for some binary codes. Des. Codes Cryptogr. 90(6), 1485–1502 (2022).
21. Miezaki T., Munemasa A., Nakasora H.: A note on Assmus–Mattson type theorems. Des. Codes Cryptogr. 89, 843–858 (2021).
22. Venkov B.B.: Even unimodular extremal lattices (Russian), Algebraic geometry and its applications (Translation in Proc. Steklov Inst. Math. 165 (1985) 47–52). Trudy Mat. Inst. Steklov. 165, 43–48 (1984).
23. Venkov B.B.: Réseaux et designs sphériques, (French) [Lattices and spherical designs] Réseaux euclidiens, designs sphériques et formes modulaires. Monogr. Enseign. Math. 37, 10–86 (2001).
24. Wolfram Research, Inc.: Mathematica, Version 11.2, Champaign, IL (2017).

Publisher's Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.