ELASTIC CONSISTENCY: A GENERAL CONSISTENCY MODEL FOR DISTRIBUTED STOCHASTIC GRADIENT DESCENT

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ABSTRACT

Machine learning has made tremendous progress in recent years, with models matching or even surpassing humans on a series of specialized tasks. One key element behind the progress of machine learning in recent years has been the ability to train machine learning models in large-scale distributed shared-memory and message-passing environments. Many of these models are trained employing variants of stochastic gradient descent (SGD) based optimization.

In this paper, we introduce a general consistency condition covering communication-reduced and asynchronous distributed SGD implementations. Our framework, called elastic consistency enables us to derive convergence bounds for a variety of distributed SGD methods used in practice to train large-scale machine learning models. The proposed framework de-clutters the implementation-specific convergence analysis and provides an abstraction to derive convergence bounds. We utilize the framework to analyze a sparsification scheme for distributed SGD methods in an asynchronous setting for convex and non-convex objectives. We implement the distributed SGD variant to train deep CNN models in an asynchronous shared-memory setting. Empirical results show that error-feedback may not necessarily help in improving the convergence of sparsified asynchronous distributed SGD, which corroborates an insight suggested by our convergence analysis.

1 Introduction

Machine learning models are now able to match or even surpass humans on specialized tasks such as image classification [24, 20], speech recognition [37], or complex games [38]. At the core of this progress lies a classic method in optimization called stochastic gradient descent (SGD), which is by and large the method of choice for training large-scale machine learning models, from standard regression to complex deep models.

SGD is a stochastic method for minimizing a function \( f : \mathbb{R}^d \to \mathbb{R} \), where \( d \) is the dimension parameter. It is common to assume that this function is differentiable, and that we are given access to (possibly noisy) gradients of this function, denoted by \( \bar{G} \). Sequential SGD will start at a randomly chosen point \( \bar{x}_0 \), commonly assumed by convention to be \( 0^d \), and converge towards a (possibly local) minimum of the function by iterating the following procedure:

\[
\bar{x}_{t+1} = \bar{x}_t - \alpha_t \bar{G}(\bar{x}_t)
\]  

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We address this question in this paper.

which can have millions of parameters [24, 5]. At the same time, consistency also induces a synchronization cost, see e.g. [11]. Since samples are now processed in parallel, parallelization should result in a reduction of the wall-clock running time, since the number of samples processed per second is in theory multiplied by p.

In practice, maintaining perfect consistency of $\bar{x}_t$ can negate the benefits of parallelization. Keeping the parameters perfectly consistent has a clear communication cost: since the size of gradient updates is linear in the size of the parameter, the resulting communication may easily become a system bottleneck for large-scale neural networks which can have millions of parameters [24, 5]. At the same time, consistency also induces a synchronization cost, since processors need to synchronize in a barrier-like fashion upon each iteration update, which can occur every few milliseconds. For this reason, there have been several proposals for relaxing the consistency requirements of SGD-like iterations, under various system constraints. These proposals can be roughly categorized as:

- **Asynchronous Methods**: These methods allow processors to forgo the barrier-like synchronization step performed at each iteration, and move forward with computation without waiting for straggler processors. Moreover, implementations such as Hogwild! [34] can completely forgo synchronization on the updates of the individual parameter components, removing most of the overhead of coordination.

- **Communication Compression**: These methods aim to reduce the bandwidth cost of exchanging the gradients. This usually entails performing (possibly lossy) compression of the gradients before transmission, followed by efficient encoding and reduction/summation, and decoding on the receiver end. This can be either via bit-width reduction (quantization), e.g. [37, 5], or via structured sparsification of the updates [28, 2].

There has been a significant amount of work aiming to provide convergence bounds for asynchronous and communication-compressed variants of SGD [43], starting with the seminal work of Bertsekas and Tsitsiklis [10]. However, many of these proofs are ad-hoc, in the sense that they are specialized to specific variants of the algorithm, and to the specific model definition. Thus, they are hard to generalize beyond the specific scheme considered.

In this context, it is natural to ask: are there generic conditions covering natural consistency relaxations for SGD, under which one can prove convergence? More broadly, is there a general consistency condition which can cover both shared-memory and message-passing systems as well as both asynchronous and communication-compression methods? We address this question in this paper.

We introduce a general consistency condition for distributed SGD called elastic consistency, which provides a convergence condition independent of the distributed system model — shared-memory or message-passing — and can be specialized to cover both asynchronous and communication-reduced methods. To our knowledge, elastic consistency is
We prove that this condition is sufficient to ensure SGD convergence for a wide range of loss functions, system models, although proving this property formally is not always immediate.

In a nutshell, elastic consistency bounds the expected norm distance between the view of the parameter perceived by a processor and the “true” view corresponding to all the updates to the parameter generated up to that point, by all processors. To our knowledge, elastic consistency is satisfied by most models under which asynchronous or communication-reduced methods have been analyzed so far, in particular by delay-bounded models, although proving this property formally is not always immediate. Elastic consistency allows us to provide a unified analysis framework which enables to analyze for the first time or to re-prove convergence of inconsistent SGD iterations for both message-passing and shared-memory models, for both convex and non-convex objective functions.

Elastic consistency at a high level is specified as follows.

Consider a system with \( p \) processors executing possibly inconsistent SGD iterations. Each processor has access to a **parameter oracle**, used to obtain estimates of the set of (shared) parameters, and to a **gradient oracle**, used to obtain stochastic gradients with respect to a parameter. To execute a step \( t \), the processor first invokes the parameter oracle to obtain a view of the parameter at step \( t \), denoted by \( \vec{v}_t \). Then, the processor queries its gradient oracle, computing a stochastic gradient \( \vec{G}(\vec{v}_t) \) corresponding to \( \vec{v}_t \) with respect to a randomly chosen data sample. Third, the processor registers this update at the parameter oracle, which may enable other processors to perceive its update, fully or partially. We will assume that these three steps are individually atomic, and ordered at each processor, although their global interleaving is controlled by an adversarial scheduler.

The parameter oracle plays the key role in this construction, as it “stores” all updates submitted by processors, and decides a parameter vector returned to a query. To formalize this, we introduce an auxiliary **true parameter** variable \( \bar{x}_t \), defined as follows\(^3\). By convention, the true parameter starts at \( \bar{x}_0 = \vec{0} \), and is updated by summing all the gradients submitted by the processors to the oracle, modulated by the learning rate:

\[
\bar{x}_{t+1} = \bar{x}_t - \alpha_t \bar{G}(\vec{v}_t) = -\alpha_t \sum_{k=0}^{t-1} \bar{G}(v_k), \forall t \geq 0. \tag{3}
\]

With these definitions in place, intuitively, elastic consistency specifies that there must exist an absolute bound \( B \) such that, at any time \( t \), the expected norm distance, where the expectation is taken over the randomness of sampling, between the true parameter \( \bar{x}_t \) and any view \( \vec{v}_t \) returned by the parameter oracle is bounded by \( \alpha_t B \), where \( \alpha_t \) is the learning rate in effect at the current iteration. Crucially, the bound \( B \) must be uniform, and independent of the time \( t \). Formally, we require

\[
\mathbb{E}[\|\bar{x}_t - \vec{v}_t\|] \leq \alpha_t B, \forall t \geq 0,
\]

where \( B > 0 \) is constant with respect to time, and the expectation is taken over the randomness of the samples used to generate the stochastic gradients \( \bar{G} \).

We prove that this condition is sufficient to ensure SGD convergence for a wide range of loss functions, system models, and practical consistency relaxations. More precisely:

1. Under standard smoothness assumptions on the loss function, elastic consistency is sufficient to guarantee convergence rates for inconsistent SGD iterations for both convex problems (such as solving regression), and non-convex problems (such as optimizing neural networks).
2. We show that elastic consistency is satisfied by both asynchronous message-passing and shared-memory models with faults, and by communication-reduced methods.
3. In particular, we provide convergence bounds for SGD in the classic asynchronous and semi-synchronous message-passing models [8]. We also provide convergence bounds for SGD in the asynchronous shared-memory model. Our framework thus simplifies the analysis frameworks of [36, 4] to message-passing systems and non-convex objectives.
4. We analyze a sparsified asynchronous SGD targeting the reduction of write-write conflicts in shared memory, and in particular, study the impact of error-correction on the convergence. We verify the analysis in the context of practical performance of this method when training CNNs.

We summarize our results in Table 1.

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\(^3\)Note that we are slightly abusing notation here, in that we are denoting by \( \bar{x}_t \) both the parameter in the sequential iteration, and the parameter maintained by the parameter oracle. These are not the same in a general execution, as gradients in our case are computed with respect to the views \( \vec{v}_t \) returned by the oracle. However, they would be equivalent in a sequential execution.
From the technical perspective, we provide an analysis technique for inconsistent iterations which is general enough to cover an extremely wide range of system models and objectives. While some of these results could be re-derived individually for each model (message-passing/shared-memory) and objective (convex/non-convex) the key advantage of our approach is that it allows all of these bounds to be derived uniformly, in a modular fashion. Importantly, elastic consistency allows a clean separation between the system / consistency model assumptions, and the techniques used to derive convergence, which may differ significantly depending on the properties of the objective function.

One fundamental question is whether our notion of consistency is necessary: that is, if increasing the elastic consistency constant $B$ does indeed increase the convergence time of SGD. We provide a simple argument showing this is the case both analytically—by providing a worst-case function and schedule instance where SGD convergence becomes linear in $B^2$—and empirically—by exhibiting similar behaviour of our proposed optimization regime. In particular we argue that the asynchronous SGD with error-feedback has a slower convergence. Thus, except in certain pathological cases [22] where error-feedback is necessary for convergence, our empirical observations as well as the analytical arguments suggest that error-feedback may not always help in improving convergence of a sparsiﬁed distributed SGD in an asynchronous setting.

2 Related Work

The literature studying the convergence of optimization algorithms under inconsistent iterations is extremely vast, tracing back to Bertsekas and Tsitsiklis [10]. It has gained new momentum recently following the surge of interest in distributed optimization. A landmark recent result is the asynchronous Hogwild! analysis of [34], which showed, under convexity and gradient sparsity assumptions, that SGD in asynchronous shared-memory can converge even in the absence of synchronization. Follow-up research worked on relaxing these technical assumptions in the convex case, e.g. [12, 36, 25, 4] or on obtaining convergence bounds in the more general non-convex case, e.g. [26, 36, 47, 27].

Ample empirical evidence for the efficiency of distributed SGD with inconsistent asynchronous updates can be found in e.g. [16, 21, 13, 26]. [21] proposed a stale synchronous parallel model that synchronizes a fast worker with a straggler with respect to the state of parameter to compute stochastic gradients. [46] proposed tuning the learning rate w.r.t. the staleness in asynchronous SGD. Recently, Dai et al. ([15]) analyzed how the staleness in such methods influences the convergence in a wide array of learning problems. Separately, a line of work towards the same goal with regards the communication–efﬁcient methods can be traced to [14, 19, 29, 44, 45, 42, 7, 39].

De Sa et al. [36] were among the first to approach a uniﬁed analysis framework for asynchronous and communication compressed iterations. Relative to it, our framework has three key advantages: (i) it does not require the restrictive gradient sparsity assumptions which they consider; (ii) it is also able to analyze the case where the updates are not unbiased estimators of the gradient, which allows us to analyze communication-reduction methods such as one-bit SGD [37] and TopK SGD [7], and (3) our technique also allows us to tackle the convergence of general non-convex objectives, as it does not require the existence of super-martingale behavior for the sequential variant of the studied processes.

Furthermore, De Sa et al. [36] also proposed and analyzed Buckwild!, a shared-memory based distributed SGD for smooth convex problems, that compresses the computed stochastic gradient at a ﬁxed bit-level, before adding it to the shared parameter vector. In comparison, we propose and analyze stochastic sparsiﬁcation, with and without error-feedback, for both convex and general non-convex smooth objectives. Wangni et al. [43] presented a distributed sparsiﬁed SGD for convex and non-convex smooth objectives, however, unlike our work, they do not consider asynchrony or error-feedback which require a more involved analysis. Reference [26] presented the ﬁrst general analysis of asynchronous SGD for non-convex objectives, without considering communication-reduction.
Recent work by Qiao et al. ([33]) models asynchronous staleness or communication reduction as a perturbations of the SGD iteration. They bound a “rework cost” for an inconsistent series of iterations with respect to an unperturbed series corresponding to a supposedly consistent iteration trajectory. Their framework generalizes Ho et al. [21], and a similar work by Zhang et al. [46], which suggested a bound on the inconsistency error in order to make a distributed optimization converge. Our work, technique and overall aim are different: we propose a unified analysis of an inconsistent distributed SGD, which gets specified on a given distributed system.

Karimireddy et al. [22] provided a general analysis of communication-reduced methods in synchronous message-passing system, which they call δ-compressor. A δ-compressor represents a quantization or sparsification scheme in which the norm of the residue which is kept back and fed to a future communication is δ percentage of the norm of the computed stochastic gradient. The proposed δ-compressor does not apply to asynchronous settings, but can be analyzed in our framework, since the δ-compressor condition would imply the elastic consistency condition we require. (The reverse implication does not hold.) Recent work [9] specializes the δ-compressor framework to design and analyze a distributed SGD containing combination of quantization, sparsification and controlled asynchrony. While the goals are similar, the result is not directly comparable to our framework: the analysis in [9] focuses on communication-compression in message-passing systems, whereas our analysis framework is more general, as it can cover distributed system settings such as shared-memory, and various other algorithms. We believe our framework can be extended to provide a simplified version of this analysis.

Elastic consistency can be used to obtain convergence rates which match the state-of-the-art for delay-bounded asynchronous shared-memory [4] and message-passing [27]. At the same time, it allows us to approach new system and fault models: for instance, we are the first to provide bounds for crash and message-omission faults in message-passing. In our view, the resulting arguments are simpler and more modular.

3 Elastic Consistency

3.1 Preliminaries

Distributed System Model and Adversarial Scheduling. We consider distributed systems consisting of P processors, some of which may be faulty, where communication happens either by message-passing, or via shared-memory. We analyze a range of system, communication and synchrony assumptions, which we will specify in the respective sections.

We assume that the scheduling of communication steps (e.g. reads/writes in shared-memory, or message delivery in message-passing) is controlled by an entity called the adversary. We assume that the adversary is oblivious, in the sense that its scheduling decisions are independent of the randomness in the algorithm, in particular the data sampling. Practically, the adversary has knowledge of the algorithm structure, but not of the values of random flips. The independence between stochastic gradients and scheduling implies that the conditioning on any event involving previous iteration s < t does not change its stochastic properties.

Distributed Optimization. We assume that each of the processors is given access to the random samples coming from an unknown d-dimensional data distribution D. The processors collaborate to jointly minimize a function f : X^d → R over the distribution D, where X is a compact subset of R^d. In practice, nodes optimize over a finite set of samples S = {S_1, S_2, ..., S_M}. Then, the function f is defined as

\[ f(\bar{x}) = \frac{1}{m} \sum_i \ell(S_i, \bar{x}) \]  

(4)

The goal is to find a parameter value \( \bar{x}^* \in \mathbb{R}^d \), which minimizes the expected loss over samples, defined as:

\[ \bar{x}^* = \arg\min_{\bar{x}} f(\bar{x}) = \arg\min_{\bar{x}} \mathbb{E}_{s \sim D}[\ell(s, \bar{x})], \]

where \( \ell \) is the loss function at a sample \( s \).

Distributed Stochastic Gradient Descent (SGD). As discussed, SGD is an iterative procedure which starts with an initial parameter value \( \bar{x}_0 \in \mathbb{R}^d \), which it proceeds to update by applying stochastic gradients to it.

In a distributed setting, the standard way to implement SGD is by partitioning the data across processors (commonly called data-parallelism). Each processor generates stochastic gradients with respect to its data, and then these local gradients are summed globally. Processors update their local parameters by the resulting sum, leading to iteration (2). Note that, there is no inconsistency across processors in this distributed implementation.
In our subsequent discussion we consider a distributed implementation of SGD in which the processors have access to the entire data, execute iterations independently, and communicate to synchronize with other processors in the distributed system in a range of specified manner.

Properties of Stochastic Gradients. Let \( \mathcal{A} \) be the \( \sigma \)-algebra over the space of random events – arising from the randomness of stochastic gradients due to random sampling – for all iterations \( t \) and parameter vector \( \bar{x} \). Let \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \geq 0} \) be a filtration of \( \mathcal{A} \) with respect to the iterations \( t \), wherein \( \mathcal{F}_t \) pertains to all events up until and including the iteration \( t \). We work under the following standard assumptions about stochastic gradients [36]:

1. **Unbiasedness.** The i.i.d. stochastic gradients are unbiased estimators of the true gradient of the function \( f \), that is:
   \[
   \forall \bar{x} \in \mathbb{R}^d, \quad E_{\bar{s} \sim \mathcal{D}} \left[ \hat{G}(\bar{x}) | \mathcal{F}_t \right] = \nabla f(\bar{x}).
   \]  

2. **Bounded Second Moment.** Furthermore, it is standard to assume that the second moment of the stochastic gradients over the sample space is bounded:
   \[
   \forall \bar{x} \in \mathbb{R}^d, \quad E_{\bar{s} \sim \mathcal{D}} \left[ \| \hat{G}(\bar{x}) \|^2 | \mathcal{F}_t \right] \leq M^2.
   \]  

Hereafter, unless otherwise specified, we shall denote \( E_{\bar{s} \sim \mathcal{D}}[.] \) by \( E[.]. \)

Properties of the Objective Function. We will make use of the following standard definitions regarding the objective function \( f: \forall \bar{x}, \bar{y} \in \mathbb{R}^d, \)

1. **Smoothness.** The function \( f: \mathbb{R}^d \to \mathbb{R} \) is smooth iff:
   \[
   \| \nabla f(\bar{x}) - \nabla f(\bar{y}) \| \leq L \| \bar{x} - \bar{y} \| \text{ for } L > 0.
   \]  

2. **Strongly convex problems.** The problems such as linear regression have a strongly convex objective:
   \[
   (\bar{x} - \bar{y})^T (\nabla f(\bar{x}) - \nabla f(\bar{y})) \geq c \| \bar{x} - \bar{y} \|^2 \text{ for } c > 0.
   \]  

   Note that, for strongly convex functions, the bound over second moment of stochastic gradients does not hold \( \forall \bar{x} \in \mathbb{R}^d \) [30]. Therefore, for a strongly convex function \( f \), we assume that \( f: \mathcal{X} \to \mathbb{R} \) for a convex \( \mathcal{X} \subset \mathbb{R} \), such that \( \forall \bar{x} \in \mathcal{X}, \) (6) is satisfied.

3. **Lower bound of non-strongly-convex functions.** In many settings, such as training of neural networks, the objective function is not necessarily strongly-convex. In such “non-strongly-convex” settings, it is necessary to assume that \( f \) is bounded from below:
   \[
   \exists f^* \text{ finite s.t. } \forall \bar{x} \in \mathbb{R}^d, \quad f(\bar{x}) \geq f^*.
   \]  

3.2 Elastic Consistency Definition

In this section, we will abstract the synchronization and communication details of the model, and present a general consistency model for SGD iterations from the view of a single processor.

We assume a partial ordering of the gradients generated by processors at the iterations 1, 2, ..., \( t \). For example, in the sequential setting described by Equation 1, the gradient at iteration \( t + 1 \) is always generated with respect to the consistent version of the parameter vector \( \bar{x}_t \) at iteration \( t \). Whereas, in a parallel setting multiple processors may generate gradients at the iteration \( t + 1 \) with respect to the same consistent version of \( \bar{x}_t \). In the inconsistent settings that we consider, the parameters with respect to which gradients are generated may itself be inconsistent.

We abstract this process as follows. We assume that each processor possesses a parameter oracle \( \mathcal{O} \). In each iteration, each processor invokes this oracle, and receives a local view \( \bar{v}_t \) of the parameter. The processor then uses this view of the parameter to generate a new update (stochastic gradient) \( \hat{G}(\bar{v}_t) \) using its local gradient oracle \( \mathcal{G} \), with respect to this view.

The key question is how to express the consistency of the view \( \bar{v}_t \) across processors. For this, we introduce an auxiliary variable \( x_t \), which we call the global parameter, which simply sums the updates performed by the algorithm up to iteration \( t \), which leads to the iteration described by (3).

With this in place, our consistency condition will track the difference (in terms of the expected norm distance) between the gradient view \( \bar{v}_t \) returned by the parameter oracle at an iteration \( t \), and all the gradient updates to the initial parameter generated so far:
We prove it in Appendix A.1.

We begin with the more general case where the objective function is not necessarily convex. In this case, since $A.1$, after this, we follow the standard arguments to arrive at the result. The complete proof is available in Appendix

At this step we take expectation given $F$.

Lemma 4.1. Consider a smooth non-convex objective function with lower bound as described by (9). For an SGD iteration defined in (3), satisfying the elastic consistency (10) with constant $B$,

$$\min_{t \in [T-1]} \mathbb{E} \left[ \| \nabla f(x_t) \|^2 \right] \leq \frac{2(f(x_0) - f^*)}{\sum_{t=0}^{T-1} \alpha_t} + \frac{LM^2 \sum_{t=0}^{T-1} \alpha_t^2}{\sum_{t=0}^{T-1} \alpha_t} + \frac{L^2 B^2 \sum_{t=0}^{T-1} \alpha_t^3}{\sum_{t=0}^{T-1} \alpha_t}.$$  \hspace{1cm} (11)

Proof-Sketch. Using the Descent Lemma [32][Lemma 1.30], we get:

$$f(x_{t+1}) \leq f(x_t) - \alpha_t \nabla f(x_t)^T \nabla f(x_t) + \alpha_t \left( \nabla f(x_t) - \bar{G}(\bar{v}_i) \right)^T \nabla f(x_t) + \frac{\alpha_t^2 L}{2} \| \bar{G}(\bar{v}_i) \| ^2.$$

Taking expectation on randomness of samples given $F_t$ and $v_t$, using assumptions (5), (6), (7), applying Cauchy-Schwarz inequality,

$$\mathbb{E} \left[ f(x_{t+1}) | F_t, v_t \right] \leq f(x_t) - \alpha_t \mathbb{E} \| \nabla f(x_t) \|^2 + \alpha_t L \| x_t - \bar{v}_i \| \| \nabla f(x_t) \| + \frac{\alpha_t^2 L M^2}{2}.$$

At this step we take expectation given $F_t$ to bound $\mathbb{E} \left[ \| x_t - \bar{v}_i \| | F_t \right]$ following the definition of elastic consistency and apply Young’s inequality; thereby we obtain the following

$$\mathbb{E} \left[ f(x_{t+1}) | F_t \right] \leq f(x_t) - \frac{\alpha_t}{2} \mathbb{E} \| \nabla f(x_t) \|^2 + \frac{\alpha_t^2 L B^2}{2} + \frac{\alpha_t^2 L M^2}{2}.$$

After this, we follow the standard arguments to arrive at the result. The complete proof is available in Appendix A.1.

Theorem 4.2. Consider SGD iterations defined in (3) for a smooth non-convex objective function. Fix a success parameter $\epsilon > 0$. Then for a constant learning rate $\alpha = \frac{\nu}{LM^3 + L^2 B^2}$, for some $\nu \in (0, 1)$, where $B$ is the elastic consistency constant, $\min_{t \in [T-1]} \mathbb{E} \left[ \| \nabla f(x_t) \|^2 \right] \leq \epsilon$, for every iteration $T \geq \frac{2(LM^2 + L^2 B^2)(f(x_0) - f^*)}{\epsilon(B + 1 - \nu)}.$

We prove it in Appendix A.1.
4.2 Convergence in the Convex Case

If the objective function is strongly convex, we can provide the following stronger notion of convergence to the unique global minimum:

**Lemma 4.3.** Consider a $c$-strongly convex smooth objective function (8). For an inconsistent distributed SGD iteration defined in (3), that satisfies the elastic consistency (10) with constant $B$, we have

$$
\mathbb{E} \left[ \left\| \bar{x}_T - \bar{x}^* \right\|^2 \right] \leq \left\| \bar{x}_0 - \bar{x}^* \right\|^2 \prod_{t=0}^{T-1} (1 - \alpha_t) c \\
+ \sum_{t=0}^{T-1} \left( \alpha_t^2 M^2 + \alpha_t^2 L^2 c^{-2} B^2 \right) \prod_{j=0}^{t-1} (1 - \alpha_j) c. 
$$

(12)

**Proof-Sketch.** Using the properties (7) and (8) and applying Cauchy-Schwarz inequality:

$$
\mathbb{E} \left[ \left\| x_{t+1} - x^* \right\|^2 \right] \leq \left( 1 - 2\alpha_t c \right) \left\| x_t - x^* \right\|^2 + 2\alpha_t L \left\| x_t - x^* \right\| \mathbb{E} \left[ \left\| x_t - v_t \right\| \right] + \alpha_t^2 M^2.
$$

At this step take expectation given $\mathcal{F}_t$ to bound $\mathbb{E} \left[ \left\| x_{t+1} - \bar{v}_t \right\| \right]$ using the definition of elastic consistency and apply Young’s inequality to obtain

$$
\mathbb{E} \left[ \left\| x_{t+1} - x^* \right\|^2 \mid \mathcal{F}_t, v_t \right] \leq \left( 1 - \alpha_t c \right) \left\| x_t - x^* \right\|^2 + \alpha_t^2 M^2 + L^2 c^{-2} \alpha_t B^2.
$$

After that standard algebraic manipulations lead to the result. The complete proof is available in [3].

**Theorem 4.4.** Consider SGD iterations defined in (3) for a $c$-strongly-convex smooth objective function. Fix a success parameter $\epsilon > 0$. Then for a constant learning rate $\alpha = \frac{\epsilon^2 L^2 c^{-2} B^2}{M^2 + c^2 + L^2 B^2}$, for some $\nu \in (0, 1)$, where $B$ is the elastic consistency constant, $\mathbb{E} \left[ \left\| x_T - x^* \right\|^2 \right] \leq \epsilon$ for every iteration $T \geq \frac{M^2 + c^2 + L^2 B^2}{\epsilon^2 c^2 \nu} \log \left( \frac{\left\| x_0 - x^* \right\|^2}{\epsilon} \right)$.

For proof see Appendix A.2.

**Remark 1.** [Convergence rate.] $B$ abstracts away a number of distributed-system specific parameters to provide a clean derivation of the convergence theory. $B$ linearly depends on $M$, which we specify in the individual applications later. Theorems 4.2 and 4.4 show that the convergence of distributed SGD depends linearly on $B^2$. More specifically, an ergodic convergence of the non-convex objectives at a rate of $\mathcal{O}(1/t)$, and a linear convergence of a strongly-convex objective, depend linearly on $B^2$. Notice that, if $M = 0$, i.e. there was no variance in the dataset, our formulation imply that we could achieve these convergence rates even if there is an unbounded delay.

**Remark 2.** [Speedup by parallel updates.] Note that in the above analysis an iteration pertains to an update submitted by an agent (process/node) in a distributed setting. Thus, if there are $p$ agents in the system, it amounts to $p$ concurrent – possibly overlapping – updates in Theorems 4.2 and 4.4, thereby the convergence becomes roughly $p$ times faster.

**Remark 3.** [Mini-batch updates.] In practice, an agent processes a mini-batch containing $b$ samples from the dataset to compute a stochastic gradient. With that, $M^2$ and $B^2$ are discounted to $M^2/b$ and $B^2/b$, respectively, in the above discussion.

**Remark 4.** [Local Convergence.] We began by observing that in a distributed system the perfect consistency across processors is costly and therefore inconsistent iterations are preferred. Notwithstanding, in practice, in an asynchronous system we track the convergence trajectory only for the inconsistent views $\tilde{v}_t$ of the true parameter vector $\bar{x}_t$. Therefore, it is pertinent to discuss the convergence of $f(\tilde{v}_t)$. Having described the convergence of a distributed SGD in Theorems 4.2 and 4.4, following our definition of elastic consistency, the trajectory of $\tilde{v}_t$ can be traced by the expected norm distance of $\tilde{v}_t$ from $\bar{x}_t$, which is bounded by $B\alpha_t$. Thus, for standard diminishing learning rate regimens, we infer that the definition of elastic consistency provides a sequence of balls of decreasing radii containing the local optimization trajectory in the parameter space.

4.3 Lower Bound on Convergence

It is interesting to ask whether the elastic consistency bound condition is necessary, in the sense that the bound $B$ could actually directly influence the convergence in worst-case instances. We address this question here, and show that, in the worst case, convergence slow-down can be linear in $B$.
Consider a one-dimensional minimization of the function \( f(x) = \frac{1}{2}x^2 \). For simplicity, we will assume that gradients are non-stochastic, which renders our updates deterministic. In this case, the gradient descent update is:

\[
x_{t+1} = (1 - \alpha)x_t, \quad \text{and therefore} \quad x_{t+1} = (1 - \alpha)^t x_0.
\]

This implies linear convergence at a rate of \( 1 - \alpha \).

Now imagine an adversary which, for any iteration index \( t \), requires the procedure to evaluate the gradient not at \( x_t \), but at \( x_t + B \). Then, unwinding the recurrence, we obtain

\[
x_{t+1} = (1 - \alpha)^t x_0 + \alpha B t.
\]

The extra \( \alpha B t \) term will decrease the rate of convergence linearly in \( B \), since it forces the algorithm to scale down the learning rate by a \( B \) factor (in order for the second term in the recurrence above to be below \( \epsilon \)). At the same time, this reduces the exponential decrease rate for the first term linearly in \( B \). We sum up this discussion as Lemma 4.5:

**Lemma 4.5 (Convergence Lower Bound).** There exists a convex function \( f \) and an adversarial parameter oracle \( O \) such that the number of iterations required to achieve \( \mathbb{E} \left[ \| \hat{x}_T - \bar{x}^* \|^2 \right] \leq \epsilon \) is \( \Omega \left( \frac{B^2}{\epsilon^2} \log \left( \frac{1}{\epsilon} \right) \right) \).

Later in the paper, our experiments also verify this finding empirically, where we observe the decrease in convergence as \( B \) increases. Please see Figure 1.

## 5 Elastic Consistency Bounds for Distributed Systems

### 5.1 Fault-Tolerant Message-Passing Systems

We consider a fully connected message-passing (MP) system of a set of \( p \) nodes \( P = \{P^1, P^2, \ldots, P^p\} \) executing SGD iterations. For an illustration. Our analysis technique can encompass both parameter-server systems (in which a node or a subset of the nodes maintain consistent parameter copies) as well as decentralized systems (in which each node maintains its own parameter copy). For this reason, we assume a system where each node functions both as a worker (generating stochastic gradients) and as a parameter-server (PS) (maintaining a parameter copy). More precisely, we assume that, initially, each of the nodes \( P^i \in P \) have identical copies \( \bar{x}_0^i \) of the parameter \( \bar{x} \), commonly \( \bar{x} \). A node \( P^i \) works as a parameter server to update the parameter vector \( \bar{x}_0^i \) at iteration \( t \), and as a worker to compute the stochastic gradients \( \hat{G}(\bar{x}_t^i) \). The computed stochastic gradients are broadcasted to the PSs over the links (or channels [8]). The computation and broadcast of stochastic gradients are ordered by the iterations.

**Fault model.** In the following discussion, we consider an MP system under the following benign fault models:

(a) **Crash.** A node \( P^i \in P \) can crash during computation of stochastic gradients or during broadcast after computation. In the case it crashes during computation, no message from it is received by any peer in any subsequent broadcast. However, if it crashes during a broadcast, it can possibly send the computed stochastic gradient to some of the peers, but not to all of them, resulting in asymmetric updates of parameter \( \bar{x} \) across the nodes. Hereafter, by a crash fault we will indicate one which results in asymmetric updates. We assume that no more than \( f < p \) number of nodes can crash in the MP system.

(b) **Message-drop.** (also known as omission failures [8]) After a (wall-clock) time \( T_{\max} \) if message from a peer \( P^j \) is not received, a node \( P^i \) discards the stochastic gradient from \( P^j \) while updating the parameter \( \bar{x} \) in the current iteration; however, it can include the dropped stochastic gradient in a later iteration if received by then. We assume that eventually all messages that are sent will reach their destination, and that the maximum delay in terms of the number of iterations in receiving a stale gradient from a worker at iteration \( t \) is upper bounded by \( \tau_t \leq \tau_{\max} \), where \( \tau_t \) is the maximum delay at iteration \( t \). We note that this matches common assumptions from the literature, e.g. [26]. We assume that a message-drop fault can not happen on more than \( f \) out of \( p - 1 \) peers.

**Update model.** Let \( P^i \) denote the PS at node \( P^i \). Let \( L_t^i \subseteq \{1, \ldots, p\} \) denote the set of indices of workers of \( P^i \) at the iteration \( t \). Note that the local worker of \( P^i \) always belongs to its set of workers. We consider the following update models:

(a) **Synchronous.** While updating \( \bar{x}_t \), \( P^i \) excludes a stale stochastic gradient: one computed by a worker at an iteration previous to \( t \) but received delayed due to a message-drop fault. Such an update scheme is called synchronous.
(b) **Asynchronous.** To avoid the exclusion of computed stochastic gradients, the value of $T_{\text{max}}$ may be increased in order to wait for the stragglers, thereby potentially slowing down the parameter update per unit wall-clock time. To mitigate this, an asynchronous approach includes the stale gradients to update $\tilde{x}_t$, while ensuring faster parameter update per iteration.

**Decentralized SGD.** Computations of stochastic gradients at the nodes $P_i \in \mathcal{P}$ are executed independently without a relative time synchronization. A typical iteration $t$ at a node $P_i$ is shown in Algorithm 1. At the start of an iteration $t$, $P_i$ gets a local view $\vec{v}_i$ of the parameter vector $\vec{x}$ and computes the stochastic gradient $\vec{G}( \vec{v}_i )$. It broadcasts $\vec{G}(\vec{v}_i)$ to all its peers as a worker to them. After that, it receives the stochastic gradients $\vec{G}(\vec{v}_j)$ from the workers until all of them are reachable or there is a timeout, say at $T_{\text{max}}$, and updates the parameter vector $\vec{x}_t$. For simplicity, we use a constant learning rate $\alpha_t = \alpha$ for all $t \geq 0$. The update at an iteration $t$ at a PS $P_i$ is given by the following recursion:

$$\vec{x}_{t+1}^i = \vec{x}_t^i - \alpha_t \left( \sum_{j \in \mathcal{L}_i^t} \vec{G}(\vec{v}_j^i) \right),$$

where $\vec{x}_0^i$ is the parameter vector at the PS $P_i$ at the $t^{th}$ iteration. We now prove that the distributed SGD satisfies elastic consistency in various message-passing system settings.

**Algorithm 1:** Iteration $t$ at a node $P_i \in \mathcal{P}$

1. $\vec{v}_i^t \leftarrow \vec{x}_i^t$; // Get the local view.
2. Compute $\vec{G}(\vec{v}_i^t)$; Broadcast $\vec{G}(\vec{v}_i^t)$; // Compute and Broadcast SG to peers.
3. $\text{time} \leftarrow 0$; $\tilde{G} \leftarrow 0$; // Prepare to collect stochastic gradients from workers.
4. **while** $\text{time} < T_{\text{max}}$ **do** // Collect stochastic gradients till it is timeout.
5. **if** Not done for $W_{t,j} \in \mathcal{L}_i^t$ **then** // Collect SGs from each of the workers.
6. **if** $\vec{G}(\vec{v}_j^t)$ is available **then** $\tilde{G} \leftarrow \tilde{G} + \vec{G}(\vec{v}_j^t)$;
7. $\vec{x}_{t+1}^i \leftarrow \vec{x}_t^i - \alpha \tilde{G}$; // Update parameter vector.

To simplify the analysis we decompose each iteration $t$ to $|\mathcal{L}_i^t|$ sub-iterations. Let $(t,j), j \in \mathcal{L}_i^t$, denote the sub-iteration at which the stochastic gradient from worker $W_j$ is used. The stochastic gradients obtained from workers are partially ordered by the iterations.

In an asynchronous setting, at a slow PS, even if a stochastic gradient from a latter iteration is available from a faster worker, it is used only in a latter iteration. To implement this mechanism, we assume the availability of buffers at the PSs. In a synchronous setting, the latest stochastic gradient available from a worker is used to update the parameter.

**Synchronous Updates.** In this case, $P_i$’s use the stochastic gradients computed at the current iteration to update the parameter. Essentially, $\vec{v}_i^t = \vec{x}_i^t$, for $1 \leq i, j \leq p, t \geq 0$. Lemma 5.1 describes elastic consistency for SGD with synchronous updates.

**Lemma 5.1.** In a synchronous message-passing system consisting of $p$ nodes with failure bound $f$, we have $\mathbb{E} \left[ \| \vec{x}_t^i - \vec{v}_t^i \| | \mathcal{F}_t \right] \leq \alpha f M$, at a PS $P_i$ at any iteration $t \geq 0$.

**Proof.** First consider the crash fault setting. Once a worker crashes it can not join the list of workers ever after, thereby $|\mathcal{L}_i^t| \leq |\mathcal{L}_i^s|$ for $u \geq t$. Suppose that at iteration $s$, crash faults occurred for the first time at some of the nodes. Therefore, $\vec{x}_t^i = \vec{x}_s^i$ for all $1 \leq i, j \leq p$ and $0 \leq r \leq s$. Let the set of indices of the crashed nodes that caused asymmetric stochastic gradient updates, up until sub-iteration $(t,j)$, between $P_i$ and a peer $P_j$, be given by $\mathcal{C}_{t,j}^{i,j} \subset \{1, 2, \ldots, p\}$.

By bounded failure assumption, $\bigcup_{t \geq 0} |\mathcal{C}_{t,j}^{i,j}| \leq f$. Considering the sub-iteration $(t,j)$,

$$\mathbb{E} \left[ \| \vec{x}_t^i - \vec{v}_t^i \| | \mathcal{F}_t \right] = \mathbb{E} \left[ \| \vec{x}_s^i - \vec{v}_s^i \| | \mathcal{F}_t \right] \leq \alpha \sum_{k \in \mathcal{C}_{t,j}^{i,j}} \sqrt{\mathbb{E} \left[ \| \vec{G}(\vec{v}_k^i) \|^2 | \mathcal{F}_s \right]} \leq \alpha f M.$$

Now consider message-drop fault. In this setting as well, the stochastic gradients from previous iterations are not used. However, it could be possible that a previously faulty peer joins the list of workers at the current iteration and the latest
stochastic gradient from it is available. For example, a PS itself can become slower or a worker can catch up with the PS. However, as we partially order the stochastic gradients, the extra or dropped messages in the current iteration do not change the upper bound.

Therefore, we set $B = fM$ as the elastic consistency constant in (10).

**Asynchronous Updates.** In an asynchronous setting, for $1 \leq i, j \leq p$, $\vec{v}^t_i = \vec{x}^t_j$, for some $s$ such that $s \leq t$. Note that, $t - s < \tau_{\max}$ by bounded delay assumption. Suppose that the first fault – crash or message-drop – occurred during an iteration $s \leq t$, so that, $\vec{x}^r_t = \vec{x}^t_j$ for all $1 \leq i, j \leq p$ and $0 \leq r \leq s$. Now, consider $P^i$ updating its parameter at sub-iteration $(t, j)$, using a stale stochastic gradient computed at iteration $s$. Irrespective of the fault model, the worst case of the difference in the view of parameter at sub-iteration $t$ is $\tau_{\max}$.

**Lemma 5.2.** In an asynchronous message-passing system with delay bound $\tau_{\max}$, $E \left[ \| \vec{x}^t_i - \vec{v}^t_i \| | F_t \right] \leq \alpha \tau_{\max} (p - 1) M$.

**Proof.** Having described the worst case scenario in terms of iterations $t$ and $s \leq t$, we have,

$$E \left[ \| \vec{x}^t_i - \vec{v}^t_i \| | F_t \right] \leq \alpha E \left[ \| \vec{x}^t_i - \vec{x}^s_j \| | F_t \right] \leq \alpha \frac{E \left[ \sum_{k \in \mathcal{L}^p - (j)} |G_k(v_w)| | F_t \right]}{Jensen \& (6)} \leq \alpha \tau_{\max} (p - 1) M.$$

Therefore, we set $B = \tau_{\max} (p - 1) M$ as the elastic consistency constant in (10).

### 5.2 Shared-Memory Systems

We consider a shared-memory system with $p$ processes $\mathcal{P} = \{P^1, P^2, \ldots, P^p\}$ that supports atomic read and fetch&add (faa). The parameter vector $\vec{x} \in \mathbb{R}^d$ is shared by the processes for concurrent lock-free read and write or update. The read/update at each of the indices $\vec{x}_i$, $1 \leq i \leq d$, of parameter, are atomic. Consider a process $P \in \mathcal{P}$. By design, $P$ reads as well as writes over an inconsistent snapshot of $\vec{x}$, see [4]. We order the iterations of SGD by the atomic faa over the first index of $\vec{x}$. The inconsistent snapshot of $\vec{x}$ which is updated at iteration $t$ is denoted by $\vec{x}_t$, whereas, the one which is read to compute the stochastic gradient is denoted by $\vec{v}_t$. Note that, iterations by the processes $P \in \mathcal{P}$ are collectively ordered. Iteration $t$ is shown in Algorithm 2. At iteration $t$, let $\tau_t^i$ be the delay in the stochastic gradient update at an arbitrary index $\vec{x}_i$, $1 \leq i \leq d$, which is essentially the gap in terms of the number of iterations between the $t^{th}$ atomic read of $\vec{x}_i$ and the $t^{th}$ faa over $\vec{x}_i$.

**Algorithm 2: Iteration $t$.**

1. for $1 \leq i \leq d$ do $\vec{v}_i \leftarrow \text{read}(\vec{x}_i)$; // Lock-free read.
2. Compute $\vec{G}(\vec{v}_t)$;
3. for $1 \leq i \leq d$ do faa$(\vec{x}_i, \alpha \vec{G}(\vec{v}_t)_i)$; // Lock-free update.

**Lemma 5.3.** Given an asynchronous shared-memory system with maximum delay bound $\tau_{\max}$, we have $E \left[ \| \vec{x} - \vec{v} \| | F_t \right] \leq \sqrt{d \alpha \tau_{\max} M}$.

---

4In this setting, there is no parameter server or worker, therefore the analysis does not require indexing the processes.
Elastic Consistency: A General Consistency Model for Distributed Stochastic Gradient Descent

Proof. The 1-norm of the difference between the inconsistent snapshots $\bar{x}_t$ and $\bar{v}_t$ is bounded as the following:

$$\|\bar{x}_t - \bar{v}_t\|_1 \leq \sum_{j=1}^{\tau_{\text{max}}} \|\bar{x}_{t-j+1} - \bar{x}_{t-j}\|_1 \leq \sqrt{d} \sum_{j=1}^{\tau_{\text{max}}} \|\bar{x}_{t-j+1} - \bar{x}_{t-j}\| \quad (\text{as } \|\bar{x}\|_1 \leq \sqrt{d} \|\bar{x}\|, \text{ for } \bar{x} \in \mathbb{R}^d).$$

Then,

$$E \|\bar{x}_t - \bar{v}_t\|_{\mathcal{F}_t} \leq E \|\bar{x}_t - \bar{v}_t\|_{\mathcal{F}_1} \quad \text{using } \|\bar{x}\| \leq \|\bar{x}\|_1, \text{ for } \bar{x} \in \mathbb{R}^d$$

$$\leq E \left[ \sqrt{d} \sum_{j=1}^{\tau_{\text{max}}} \|\bar{x}_{t-j+1} - \bar{x}_{t-j}\| \mathbb{P}_{\mathcal{F}_t} \right]$$

$$\leq \sqrt{d} \alpha \sum_{j=1}^{\tau_{\text{max}}} E \left[ \|G(\bar{v}_{t-j})\|_{\mathbb{P}_{\mathcal{F}_t}} \right]$$

$$= \sqrt{d} \alpha \sum_{j=1}^{\tau_{\text{max}}} E \left[ \|G(\bar{v}_{t-j})\|_{\mathbb{P}_{\mathcal{F}_t}} \right]$$

$$\leq \sqrt{d} \alpha \sum_{j=1}^{\tau_{\text{max}}} \sqrt{E \left[ \|G(\bar{v}_{t-j})\|^2_{\mathbb{P}_{\mathcal{F}_t}} \right]} \quad (6)$$

$$\leq \sqrt{d} \alpha \tau_{\text{max}} M.$$

We set the elastic consistency constant $B = \sqrt{d} \tau_{\text{max}} M$. 

5.3 Communication-Efficient Methods

In this section, we consider a synchronous message-passing system of $p$ nodes $\mathcal{P} = \{P^1, P^2, \ldots, P^p\}$ executing SGD iterations. For simplicity, we assume that the system is synchronous and fault-free, although our analysis would work in the absence of these assumptions as well.

As described in section 5.1, each of the nodes work both as a parameter server (maintaining a parameter) and as a worker (computing gradients). In each iteration, the workers communicate with parameter servers. However, instead of communicating an exact computed stochastic gradient, workers broadcast a compressed version of the gradient, in order to reduce communication costs. Communication-efficiency can be achieved in two ways: for a computed stochastic gradient, (a) quantization: broadcast a quantized vector that requires fewer bits [37, 6], or, (b) sparsification: broadcast a sparse vector [7, 39].

Note that, our message-passing system encompasses both centralized and decentralized settings. In this sense, it provides a more generic setting than previous related works [37, 6, 7, 39, 22], who consider a centralized distributed system.

In a shared-memory setting, such a distributed system can be simulated by a set of processes (or threads) as the following. Each of the processes maintain a copy of the parameter, over the shared memory, and a buffer to which each of them can push their computed stochastic gradients. For a synchronous update, we can use barrier synchronization after each iteration. Lower write cost analogizes communication efficiency: low-precision computation over shared-memory [36, 35] can counterpart quantization, whereas, a partial update along fewer number of co-ordinates exemplifies sparsification; see the next subsection for a variant of such a scheme.

Clearly, this strategy leads to losses in the parameter updates at the parameter server at each iteration. A popular approach to control this error is using error-feedback: the accumulated residual error from the previous broadcasts is added to the stochastic gradient at the current iteration before applying quantization or sparsification. We will show that the residual error can be modelled in the context of elastic consistency, and that it stays bounded in the case of popular communication-efficient techniques, which implies their convergence.

In the description below, we will be using the term lossy compression collectively for both quantization and sparsification. A vector which undergoes a lossy compression will be called a compressed vector.

A typical iteration at a node $P^i$ is shown in Algorithm 3. The algorithm works as the following. Each node $P^i$ maintains a local model $\tilde{v}_t^i$, and a local error accumulation $\tilde{\epsilon}_t^i$, starting from $\tilde{v}_0^i = \tilde{\epsilon}_0^i = 0$.

At each iteration, as a worker, $P^i$ computes a stochastic gradient with respect to its local view $\tilde{x}_t^i$, adds to it the accumulated error from the previous iterations, applies a lossy compression function $Q$ to the result, and broadcasts...
Algorithm 3: Iteration $t$ at a node $P^i \in \mathcal{P}$

1. Get $\vec{v}_t^i$; // Get local inconsistent view.
2. Compute $\vec{G}(\vec{v}_t^i)$;
3. Compute $\vec{w}_t^i \leftarrow \vec{v}_t^i + \vec{G}(\vec{v}_t^i)$; // Add the accumulated error to the computed SG.
4. Compute and Broadcast $Q(\vec{w}_t^i)$; // Apply lossy compression and broadcast.
5. Compute $\vec{e}_{t+1} \leftarrow \vec{w}_t^i - Q(\vec{w}_t^i)$; // Update the accumulated error.
6. $\vec{G} \leftarrow 0$;
7. for each $\vec{w}_t^j \in W_t^j$ do
   8. Receive $Q(\vec{w}_t^j)$; $\vec{G} \leftarrow \vec{G} + Q(\vec{w}_t^j)$;
   9. $\vec{v}_{t+1}^i \leftarrow \vec{v}_t^i - \alpha \frac{\vec{G}}{p}$; // Update parameter vector.

10. Get $\vec{v}_t^i$; // Get local inconsistent view.
11. Compute $\vec{w}_t^i \leftarrow \vec{v}_t^i + \vec{G}(\vec{v}_t^i)$; // Add the accumulated error to the computed SG.
12. Compute and Broadcast $Q(\vec{w}_t^i)$; // Apply lossy compression and broadcast.
13. Compute $\vec{e}_{t+1} \leftarrow \vec{w}_t^i - Q(\vec{w}_t^i)$; // Update the accumulated error.
14. $\vec{G} \leftarrow 0$;
15. for each $\vec{w}_t^j \in W_t^j$ do
   16. Receive $Q(\vec{w}_t^j)$; $\vec{G} \leftarrow \vec{G} + Q(\vec{w}_t^j)$;
   17. $\vec{v}_{t+1}^i \leftarrow \vec{v}_t^i - \alpha \frac{\vec{G}}{p}$; // Update parameter vector.

The one-bit SGD quantization was first described in [37]. We use the notation $\vec{x}\dagger$ to denote the $i$'th component of $\vec{x}$. Consider the vector $\vec{w}$ and let $S^+(\vec{w})$ be its index of positive components and $S^-(\vec{w})$ that of its negative components, i.e., $S^+(\vec{w}) = \{i : |\vec{w}_i| \geq 0\}$ and $S^-(\vec{w}) = \{i : |\vec{w}_i| < 0\}$. Then let $\vec{w}^+ = \frac{1}{|S^+(\vec{w})|} \sum_{i \in S^+(\vec{w})} |\vec{w}_i|$, and $\vec{w}^- = \frac{1}{|S^-(\vec{w})|} \sum_{i \in S^-(\vec{w})} |\vec{w}_i|$. Now define the one-bit quantization operation $Q(\cdot)$ as,

$$[Q(\vec{w})]_i = \begin{cases} 
\vec{w}^+ & \text{for } i \in S^+(\vec{w}) \\
\vec{w}^- & \text{for } i \in S^-(\vec{w}) 
\end{cases}$$

(16)
It is easy to verify that one-bit quantization satisfies inequality (15) for $c = 1$:

$$
\|Q(\vec{w})\| = \sqrt{\sum_{i=1}^{\|S^+(\vec{w})\|} (\vec{w}^+)_i^2 + \sum_{i=1}^{\|S^-(\vec{w})\|} (\vec{w}^-)_i^2}
$$

$$
= \sqrt{\sum_{i=1}^{\|S^+(\vec{w})\|} \left(\frac{1}{\|S^+(\vec{w})\|} \sum_{i \in S^+(\vec{w})} [\vec{w}]_i\right)^2 + \sum_{i=1}^{\|S^-(\vec{w})\|} \left(\frac{1}{\|S^-(\vec{w})\|} \sum_{i \in S^-(\vec{w})} [\vec{w}]_i\right)^2}
$$

$$
\leq \sqrt{\sum_{i \in S^+(\vec{w})} ([\vec{w}]_i^+)^2 + \sum_{i \in S^-(\vec{w})} ([\vec{w}]_i^-)^2} = \|\vec{w}\|
$$

To obtain the elastic consistency constant $B$ via Lemma 5.4, we show the following in this case:

**Lemma 5.5.** $E[\|\vec{e}_t\| | F_t] \leq 2M$.

**Proof.** The quantization formula (16) implies that

$$
\|\vec{w}_t - Q(\vec{w}_t)\| \leq \frac{1}{2} \|\vec{w}_t\|.
$$

(17)

Therefore by (14),

$$
\|\vec{e}_t\| = \|\vec{w}_{t-1} - Q(\vec{w}_{t-1})\| \leq \frac{1}{2} \|\vec{w}_{t-1}\| = \frac{1}{2} \|\vec{e}_{t-1} + \vec{G}(\vec{\nu}_{t-1})\| = \frac{1}{2} \|\vec{e}_{t-1}\| + \frac{1}{2} \|\vec{G}(\vec{\nu}_{t-1})\|.
$$

Unwinding the recursion and taking expectation given $F_t$,

$$
E[\|\vec{e}_t\| | F_t] \leq \sum_{s=0}^{t-1} \frac{1}{2^{s-2}} E[\|\vec{G}(\vec{\nu}_s)\| | F_t] \leq \sum_{s=0}^{t-1} \frac{1}{2^{s-2}} \sqrt{E[\|\vec{G}(\vec{\nu}_s)\|^2 | F_t]} \leq 2M.
$$

(6)

Substituting in Lemma 5.4, $E[\|\vec{e}_t^2 - \vec{v}_{t(j)}\| | F_t] \leq 2\alpha M$. Selecting $B = 2M$, One-bit quantization scheme satisfies elastic consistency (10).

**TopK quantization**

The TopK algorithm [40] presents a sparsification scheme for the stochastic gradients. Essentially, we select the top $K$ of the indices of $\vec{w}$ sorted by their absolute value. Because $d - k$ indices of a vector, which are not the top ones by their absolute value, are discarded in this method, it clearly satisfies inequality (15) for $c = 1$. To obtain the elastic consistency constant $B$ via Lemma 5.4, we show the following:

**Lemma 5.6.** $E[\|\vec{e}_t\| | F_t] \leq M \frac{d-K}{K}$.

**Proof.** We note that for any $\vec{w} \in \mathbb{R}^d$,

$$
\|\vec{w} - Q(\vec{w})\| \leq \sqrt{\frac{d-K}{d}} \|\vec{w}\|.
$$

Therefore by (14),

$$
\|\vec{e}_t\| = \|\vec{w}_{t-1} - Q(\vec{w}_{t-1})\| \leq \left(\frac{d-K}{d}\right)^{1/2} \|\vec{w}_{t-1}\| = \left(\frac{d-K}{d}\right)^{1/2} \|\vec{e}_{t-1} + \vec{G}(\vec{\nu}_{t-1})\| \leq \left(\frac{d-K}{d}\right)^{1/2} \left(\|\vec{e}_{t-1}\| + \|\vec{G}(\vec{\nu}_{t-1})\|\right).
$$

Unwinding the recursion we get

$$
E[\|\vec{e}_t\| | F_t] \leq M \sum_{s=0}^{t-1} \left(\frac{d-K}{d}\right)^{\left[\frac{s}{2}\right]} \leq M \frac{1 - (K/d)}{K/d}.
$$

(7)
Substituting in Lemma 5.4, \( \mathbb{E} \left[ \| \tilde{x}_{(t,j)} - \tilde{v}_{(t,j)} \|_{F} \right] \leq M \alpha \frac{d-K}{K} \). Selecting \( B = M \frac{d-K}{K} \), TopK quantization scheme satisfies elastic consistency (10).

We could now use these bounds in Theorems 4.2 and 4.4 to obtain convergence guarantees for these two methods.

Below we describe an experiment that verifies the effect of varying \( B \) over the convergence of TopK algorithm, see Figure 1.

![Figure 1: We trained a linear regression model with a single worker varying a bound, denoted by \( B \), on the accumulated error in the TopK algorithm. This is a synchronous setting with a single GPU being used as a worker. We plotted the training error vs. number of epochs. As expected, convergence is slower if we allow higher error. This experiment demonstrates that we can trade-off the convergence of SGD with the quantization error.](image)

### 5.4 Asynchronous Communication-Efficient Methods

We consider applying a lossy compression \( Q(\tilde{x}) \) over the stochastic gradients computed by a node in an asynchronous message passing or by a process in an asynchronous shared-memory setting, before it is either sent over a link or applied to the parameter vector over the shared memory.

The main purpose of error-feedback is to minimize the loss in convergence due to lossily compressed stochastic gradients. However, as soon as asynchrony comes in the scenario, the error-feedback particularly becomes stale. In effect, a delayed stochastic gradient with its error-feedback component entails a higher elastic consistency constant \( B \) to ensure convergence. Here we describe a method and show analytically that asynchronous communication-compressed distributed SGD has a worse convergence with error-feedback in a shared-memory setting. We verify experimentally that error-feedback in an asynchronous setting worsens convergence.

Consider an asynchronous distributed system with \( p \) workers. For example, a workstation with a multi-processor CPU, or, a multi-GPU workstation, where one of the GPUs allows concurrent read/write to multiple CPU processes, simulates such an asynchronous system.

Consider a training problem with a blocked parameter vector \( \tilde{x} = \{ \tilde{x}_i \}_{i \in [k]} \) such that \( \tilde{x}_i \)s are blocks of parameter components. Consider a simple schemes of distributed asynchronous SGD to train such a parameter stored over the shared memory:

**EF-Rand** \( \beta \): Maintain a local tensor-list \( \tilde{E} := \{ \tilde{E}_i \}_{i \in [k]} \) for error-accumulation. Initially \( \{ \tilde{E}_i \} = 0 \forall i \in [k] \), where each \( \tilde{E}_i \) is of the same size as \( \tilde{x}_i \). Sample (a minibatch) from the dataset. Scan through the blocks \( \tilde{x}_i \)s to do the following:

1. Compute the stochastic gradient \( d(\tilde{x}_i) \) for the sample (mini-batch) and update \( \tilde{E}_i \leftarrow \tilde{E}_i + d(\tilde{x}_i) \).
2. With a fixed probability \( \beta \) do \( \tilde{x}_i \leftarrow \tilde{x}_i - \alpha \tilde{E}_i \).

The aim of this methods is to save on write-conflicts in a shared-memory system, though in a multi-GPU setting the cross-GPU communication would way outweigh such a saving.

Lemma 5.7 presents the elastic consistency of EF-Rand \( \beta \).

**Lemma 5.7.** Given an asynchronous shared-memory system with maximum delay bound \( \tau_{\text{max}} \), we have \( \mathbb{E} \left[ \| \tilde{x}_t - \tilde{v}_t \|_{F} \right] \leq \left( \sqrt{d} \tau_{\text{max}} \sqrt{\beta} + p(\beta^{-1} - 1) \right) \alpha M \) for the method EF-Rand \( \beta \).

**Proof.** First, we note that this compression scheme is similar to TopK, and satisfies \( \| \tilde{w} - Q(\tilde{w}) \| \leq \sqrt{1-\beta} \| \tilde{w} \| \). Furthermore, the second moment bound (6) changes to \( \forall \tilde{v} \in \mathbb{R}^d, \mathbb{E} \left[ ||Q(\tilde{G}(\tilde{v}))||^2_{\mathcal{F}} \right] = \mathbb{E} \left[ \beta \| \tilde{G}(\tilde{v}) \|_{\mathcal{F}}^2 \right] \leq \sqrt{\beta} M^2 \).

Now, if there were no stochastic gradient delays up until an iteration \( s \), using Lemmas 5.4 and 5.6, and the fact that here we do not average the stochastic gradients received from the workers, \( \mathbb{E} \left[ || \tilde{x}_s - \tilde{v}_s ||_{\mathcal{F}} \right] \leq p\alpha M \beta^{-1} - 1 \).
Without loss of generality, suppose that at an iteration $t \geq s$, we update the an index of the parameter vector using the delayed stochastic gradient computed at iteration $s$. Then, applying the derivation steps of Lemma 5.3,

\[
\mathbb{E} \left[ \| \bar{x}_t - \bar{v}_t \| \| F_t \right] \leq \mathbb{E} \left[ \| \bar{x}_t - \bar{x}_s \| \| F_t \right] + \mathbb{E} \left[ \| \bar{x}_s - \bar{v}_s \| \| F_s \right] \\
\leq \mathbb{E} \left[ \| \bar{x}_t - \bar{x}_s \| \| F_t \right] + \sqrt{d} \sum_{j=1}^{\tau_{\text{max}}} \| \bar{x}_{t-j+1} - \bar{x}_{t-j} \| \| F_j \right] \\
\leq \sqrt{d} \alpha \sum_{j=1}^{\tau_{\text{max}}} \mathbb{E} \left[ \| Q \left( \bar{G} \left( \bar{x}_{t-j} \right) \right) \| \| F_j \right] + \alpha M (\beta^{-1} - 1) \\
= \sqrt{d} \alpha \sum_{j=1}^{\tau_{\text{max}}} \mathbb{E} \left[ \| Q \left( \bar{G} \left( \bar{x}_{t-j} \right) \right) \| \| F_j \right] + \alpha M (\beta^{-1} - 1) \\
\leq \sqrt{d} \alpha \sum_{j=1}^{\tau_{\text{max}}} \sqrt{\mathbb{E} \left[ \| \bar{G} \left( \bar{x}_{t-j} \right) \| F_j \right]} + \alpha M (\beta^{-1} - 1) \\
\leq \sqrt{d} \alpha \tau_{\text{max}} \sqrt{\beta} M + \alpha M (\beta^{-1} - 1).
\]

We select $B = \left( \sqrt{d} \tau_{\text{max}} \sqrt{\beta} + p(\beta^{-1} - 1) \right) M$ as the elastic consistency constant. Thus, if $\sqrt{d} \tau_{\text{max}} \sqrt{\beta} + p(\beta^{-1} - 1) \geq \sqrt{d} \tau_{\text{max}}$, $B$ increases due to the error-feedback terms potentially slowing down the convergence in comparison to full updates. In practice, $\tau_{\text{max}}$ may not be too high and therefore EF-Rand/β performs worse than HW.

Now, consider a multi-layer convolutional neural network (CNN) in which each layer consists of tensors corresponding to the weights and biases thereof. In the computation graph of the CNN output, these tensors correspond to the leaves. Effectively, they form a “block” of the parameter vector. Thus, the scheme proposed above can be effectively used for a CNN. In the experiments in next section, we verify the above theoretical arguments for a CNN.

6 Experiments

To experimentally verify the convergence behavior of EF-Rand/β we use training of three well-known CNN/Dataset combinations: (a) Resnet20 [20] over CIFAR-10 [23], (b) EfficientNet-B0 [41] over CIFAR-100 [23], and, (c) EfficientNet-B3 [41] over CIFAR-100 [23]. These combinations provide a variation of optimization tasks. Our experiments are set up over a multicore workstation with 12 hardware threads packing 8 GeForce 2080Ti Nvidia GPUs. GeForce 2080Ti is based on Nvidia’s Turing architecture and allows concurrent access to a GPU by multiple processes using Nvidia’s MPS runtime service. To investigate the impact of varying asynchrony we trained the CNNs with 4, 6 and 8 GPUs, where each GPU is used by a single CPU process to compute stochastic gradient via backpropagation. Each of the processes perform 16 epochs, where each epoch pertains to update due to a full pass over the entire dataset. We used the popular open-source Pytorch framework [31] and Python multiprocessing to implement the entire computation.

In the results, HW pertains to the full gradient update as described in Sub-section 5.2. To implement EF-Rand/β (ER/β), we compute the stochastic gradient for all the layers and thereafter apply the random sparsification and error accumulation. We plot the train-loss minimization trajectory and record the test-accuracy and end-to-end running time, which includes time incurred in loading data to GPUs.

See the results and discussions in Figures 2, 3 and 4.

7 Conclusion

In this paper we presented a general elastic consistency condition which ensures competitive convergence rates for inconsistent stochastic gradient methods. We showed that this condition holds for some practically useful and effective methods that make use of distributed computing to parallelize computation but are subject to crashes and discrepancies in computational power, as well as some methods that reduce communication time by sparsifying or quantizing the computed stochastic gradient before updating the optimal parameter estimates. One key line of extension which we plan
to pursue in future work is to study whether this consistency condition can be applied to other first-order distributed optimization methods, such as stochastic coordinate descent, or zeroth- or second-order optimization methods.

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Figure 4: This set of plots show the convergence of EfficientNetB3 over CIFAR100 Dataset. This is a relatively bigger CNN architecture. Hyperparameter selections across the methods are identical to the previous set of experiments. Interestingly, EF-Rand$\beta$ tends to be performing competitively for higher values of $\beta$. However, we observe that as asynchrony increases—from 4 to 6 to 8 processes—the relative performance of EF-Rand$\beta$ deteriorates.

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A Detailed Proofs

A.1 Complete Proof of Convergence in the Non-Convex Case

Lemma A.1 (Restatement of Lemma 4.1). Consider an objective function with lower bound as described by (9). For an inconsistent distributed SGD iteration defined in (3), that satisfies the elastic consistency (10) with constant $B$, \[
\min_{t \in [T-1]} \mathbb{E} \left[ \| \nabla f(\bar{x}_t) \|^2 \right] \leq \frac{2(f(\bar{x}_0) - f^*)}{\sum_{t=0}^{T-1} \alpha_t} + \frac{L M^2 \sum_{t=0}^{T-1} \alpha_t^2}{\sum_{t=0}^{T-1} \alpha_t} + \frac{L^2 B^2 \sum_{t=0}^{T-1} \alpha_t^3}{\sum_{t=0}^{T-1} \alpha_t}.
\]

Proof. Using Descent Lemma:

\[
f(\bar{x}_{t+1}) = f(\bar{x}_t) - \alpha_t \nabla f(\bar{x}_t)^T \nabla f(\bar{x}_t) + \frac{\alpha_t L}{2} \| \nabla f(\bar{x}_t) \|^2
\]

Taking expectation on randomness of samples given $F_t$ and $v_t$, using assumptions (5) and (6), and applying Cauchy-Schwarz inequality,

\[
\mathbb{E} \left[ f(\bar{x}_{t+1}) \big| F_t, v_t \right] \leq f(\bar{x}_t) - \alpha_t \| \nabla f(\bar{x}_t) \|^2 + \alpha_t \| \nabla f(\bar{x}_t) \| \| \nabla f(\bar{x}_t) \| + \frac{\alpha_t^2 L M^2}{2}
\]

Now taking the expectation given $F_t$,

\[
\mathbb{E} \left[ f(\bar{x}_{t+1}) \big| F_t \right] \leq f(\bar{x}_t) - \alpha_t \| \nabla f(\bar{x}_t) \|^2 + \alpha_t L \mathbb{E} \left[ \| \bar{x}_t - v_t \| \| F_t \| \| \nabla f(\bar{x}_t) \| \right] + \frac{\alpha_t^2 L M^2}{2}
\]

Taking full expectation (w.r.t. the randomness in optimization trajectory),

\[
\mathbb{E} \left[ f(\bar{x}_{t+1}) - f(\bar{x}_t) \right] \leq -\alpha_t \mathbb{E} \left[ \| \nabla f(\bar{x}_t) \|^2 \right] + \frac{\alpha_t^2 L^2 B^2}{2} + \frac{\alpha_t^2 L M^2}{2}.
\]

Now for any $T \geq 1$,

\[
f(\bar{x}_0) - f^* \geq f(\bar{x}_0) - f(\bar{x}_T) = \sum_{t=0}^{T-1} \left[ f(\bar{x}_t) - f(\bar{x}_{t+1}) \right] \text{winding the recursion}.
\]

Taking full expectations (w.r.t. the randomness in optimization trajectory),

\[
f(\bar{x}_0) - f^* \geq \sum_{t=0}^{T-1} \mathbb{E} \left[ f(\bar{x}_t) - f(\bar{x}_{t+1}) \right]
\]

\[
\geq \frac{\sum_{t=0}^{T-1} \alpha_t \mathbb{E} \left[ \| \nabla f(\bar{x}_t) \|^2 \right]}{2} - \frac{L^2 B^2 \sum_{t=0}^{T-1} \alpha_t^3}{2} - \frac{L M^2 \sum_{t=0}^{T-1} \alpha_t^2}{2}
\]

\[\Rightarrow \min_{t \in [T-1]} \mathbb{E} \left[ \| \nabla f(\bar{x}_t) \|^2 \right] \leq \frac{2(f(\bar{x}_0) - f^*)}{\sum_{t=0}^{T-1} \alpha_t} + \frac{L M^2 \sum_{t=0}^{T-1} \alpha_t^2}{\sum_{t=0}^{T-1} \alpha_t} + \frac{L^2 B^2 \sum_{t=0}^{T-1} \alpha_t^3}{\sum_{t=0}^{T-1} \alpha_t}. \]
Proof of Theorem 4.2

Theorem A.2 (Restatement of Theorem 4.2). Consider SGD iterations defined in (3) for a smooth non-convex objective function. Fix a success parameter $\epsilon > 0$. Then for a constant learning rate $\alpha = \frac{\nu}{LM^2 + L^2B^2}$, for some $\nu \in (0, 1)$, where $B$ is the elastic consistency constant, $\min_{t \in [T-1]} \mathbb{E} \| \nabla f(x_t) \|^2 \leq \epsilon$, for every iteration $T \geq \frac{2(LM^2 + L^2B^2)(f(x_0) - f^*)}{\epsilon \nu (1 - \nu)}$.

Proof. From (19) we have,

$$
\min_{t \in [T-1]} \mathbb{E} \| \nabla f(x_t) \|^2 \leq \frac{2(f(x_0) - f^*)}{\alpha T} + \frac{LM^2 \sum_{t=0}^{T-1} \alpha_t^2}{K_1} + \frac{L^2B^2 \sum_{t=0}^{T-1} \alpha_t^2}{K_2}.
$$

Substitute $\alpha_t = \alpha \forall t$ in (20). We get

$$
\min_{t \in [T-1]} \mathbb{E} \| \nabla f(x_t) \|^2 \leq \frac{2(f(x_0) - f^*)}{\nu \alpha T} + \frac{LM^2 + L^2B^2}{K_2},
$$

For

$$
\alpha = \frac{\nu}{LM^2 + L^2B^2},
$$

we have $K_2 = \epsilon \nu$.

Considering $K_1$, if $K_1 \leq (1 - \nu)$, or,

$$
\frac{2(f(x_0) - f^*)}{\alpha T} \leq \epsilon (1 - \nu)
$$

or, $T \geq \frac{2(LM^2 + L^2B^2)(f(x_0) - f^*)}{\epsilon \nu (1 - \nu)},$

we have the convergence as

$$
\min_{t \in [T-1]} \mathbb{E} \| \nabla f(x_t) \|^2 \leq \epsilon.
$$

A.2 Complete Proof of Convergence in the Convex Case

Lemma A.3 (Restatement of Lemma 4.3). Consider a $c$-strongly convex objective function (8). For an inconsistent distributed SGD iteration defined in (3), that satisfies the elastic consistency (10) with constant $B$, we have

$$
\mathbb{E} \| x_T - x^* \|^2 \leq \| x_0 - x^* \|^2 \prod_{t=0}^{T-1} (1 - \alpha_t c) + \sum_{t=0}^{T-1} \left( \alpha_t^2 M^2 + \alpha_t^2 L^2 c^{-2} B^2 \right) \prod_{j=0}^{t-1} (1 - \alpha_j c).
$$

Proof. We begin by simply expanding the iteration at step $t$. A simple sequence of derivations leads to:

$$
\| x_{t+1} - x^* \|^2 = \| x_t - \alpha_t \bar{G}(\bar{v}_t) - x^* \|^2 = \| x_t - x^* \|^2 - 2\alpha_t (x_t - x^*)^T \bar{G}(\bar{v}_t) + \alpha_t^2 \| \bar{G}(\bar{v}_t) \|^2
$$

Taking the expectation given $F_t$ and $v_t$, up until and including the iteration $t$,

$$
\mathbb{E} \| x_{t+1} - x^* \|^2 | F_t, v_t = \| x_t - x^* \|^2 - 2\alpha_t (x_t - x^*)^T (\nabla f(x_t) - \nabla f(x^*)) + 2\alpha_t (x_t - x^*)^T \left( \nabla f(x_t) - \mathbb{E} \bar{G}(\bar{v}_t) | F_t \right) + \alpha_t^2 \mathbb{E} [\| \bar{G}(\bar{v}_t) \|^2 | F_t],
$$

which completes the proof.

\[\Box\]
Applying Cauchy-Schwarz inequality to bound the inner product by product of norms,
\[ \mathbb{E} \left[ \|x_{t+1} - x^*\|^2 | F_t, v_t \right] \leq \|\tilde{x}_t - \tilde{x}^*\|^2 - 2\alpha_t (\tilde{x}_t - \tilde{x}^*)^T (\nabla f(\tilde{x}_t) - \nabla f(\tilde{x}^*)) + 2\alpha_t \|\tilde{x}_t - \tilde{x}^*\| \|\nabla f(\tilde{x}_t) - \nabla f(\tilde{v}_t)\| + \alpha_t^2 M^2. \]

Given \( \alpha_t > 0 \), using the strong convexity property (8) on the second term and the smoothness property (7) on the third term,
\[ \mathbb{E} \left[ \|x_{t+1} - x^*\|^2 | F_t, v_t \right] \leq \|x_t - x^*\|^2 - 2\alpha_t c \|x_t - x^*\|^2 + 2\alpha_t L \|x_t - x^*\| + \alpha_t^2 M^2. \]

Now taking the expectation given \( F_t \),
\[ \mathbb{E} \left[ \|x_{t+1} - x^*\|^2 | F_t \right] \leq \|x_t - x^*\|^2 - 2\alpha_t \|x_t - x^*\|^2 + 2\alpha_t L \|x_t - x^*\| + \alpha_t^2 M^2 \]
\[ \begin{align*}
& \leq \|x_t - x^*\|^2 - 2\alpha_t \|x_t - x^*\|^2 + 2\alpha_t^2 LB \|x_t - x^*\| + \alpha_t^2 M^2 \\
& \leq \|x_t - x^*\|^2 - 2\alpha_t \|x_t - x^*\|^2 + \alpha_t^2 \|x_t - x^*\|^2 + 2L^2 e^{-2\alpha_t^3 B^2} + \alpha_t^2 M^2 \\
& = (1 - \alpha_t c) \|x_t - x^*\|^2 + 2L^2 e^{-2\alpha_t^3 B^2} + \alpha_t^2 M^2. 
\end{align*} \]

Taking full expectation (w.r.t. randomness of optimization trajectory),
\[ \mathbb{E} \left[ \|x_{t+1} - x^*\|^2 \right] \leq (1 - \alpha_t c) \mathbb{E} \left[ \|x_t - x^*\|^2 \right] + \alpha_t^2 M^2 + \alpha_t^2 L^2 B^2 e^{-2}. \quad \tag{21} \]

Unwinding the recursion and using \( \mathbb{E} \left[ \|x_0 - x^*\|^2 \right] = \|x_0 - x^*\|^2 \),
\[ \mathbb{E} \left[ \|\tilde{x}_T - \tilde{x}^*\|^2 \right] \leq \|\tilde{x}_0 - \tilde{x}^*\|^2 \prod_{t=0}^{T-1} (1 - \alpha_t c) + \sum_{t=0}^{T-1} (\alpha_t^2 M^2 + \alpha_t^2 L^2 c^{-2} B^2) \prod_{j=0}^{t-1} (1 - \alpha_j c). \quad \tag{22} \]

\[ \square \]

**Proof of Theorem 4.4**

**Theorem A.4 (Restatement of Theorem 4.4).** Consider SGD iterations defined in (3) for a c-strongly-convex smooth objective function. Fix a success parameter \( \epsilon > 0 \). Then for a constant learning rate \( \alpha = \frac{cc^3 \epsilon \nu}{M^2 c^2 + L^2 B^2} \), for some \( \nu \in (0, 1) \), where \( B \) is the elastic consistency constant, \( \mathbb{E} \left[ \|\tilde{x}_T - \tilde{x}^*\|^2 \right] \leq \epsilon \) for every iteration \( T \geq \frac{M^2 c^2 + L^2 B^2}{c \epsilon^2} \log \left( \frac{\|x_0 - x^*\|^2}{\epsilon^2 (1 - \nu^2)} \right). \)

**Proof.** From (22),
\[ \mathbb{E} \left[ \|\tilde{x}_T - \tilde{x}^*\|^2 \right] \leq \|\tilde{x}_0 - \tilde{x}^*\|^2 \prod_{t=0}^{T-1} (1 - \alpha_t c) + \sum_{t=0}^{T-1} (\alpha_t^2 M^2 + \alpha_t^2 L^2 c^{-2} B^2) \prod_{j=0}^{t-1} (1 - \alpha_j c) \]
\[ \leq \|\tilde{x}_0 - \tilde{x}^*\|^2 \prod_{t=0}^{T-1} (1 - \alpha_t c) + \sum_{t=0}^{T-1} \alpha_t^2 (M^2 + L^2 c^{-2} B^2) \prod_{j=0}^{t-1} (1 - \alpha_j c) \quad \tag{23} \]

Fix \( \alpha_t = \alpha \forall t \). Then,
\[ \mathbb{E} \left[ \|\tilde{x}_T - \tilde{x}^*\|^2 \right] \leq \|\tilde{x}_0 - \tilde{x}^*\|^2 (1 - \alpha)^T + \alpha^2 (M^2 + L^2 c^{-2} B^2) \sum_{t=0}^{T-1} (1 - \alpha)^t \]
\[ \leq \|\tilde{x}_0 - \tilde{x}^*\|^2 \exp(-\alpha T) + \alpha^{-1} (M^2 + L^2 c^{-2} B^2). \quad \tag{24} \]

The second inequality above comes from \( 1 + x \leq \exp(x) \forall x \in \mathbb{R} \) and using sum of a geometric series. Now, for
\[ \alpha = \frac{cc^3 \nu}{M^2 c^2 + L^2 B^2}, \text{ for some } \nu \in (0, 1), \]
we have \( K_2 = \nu \epsilon \).

Considering \( K_1 \), if \( K_1 \leq \epsilon (1 - \nu) \), or,
$$\|\vec{x}_0 - \vec{x}^*\|^2 \exp(-\alpha c T) \leq \epsilon(1 - \nu),$$
or,
$$-\alpha c T \leq \log \left( \frac{\epsilon(1 - \nu)}{\|\vec{x}_0 - \vec{x}^*\|^2} \right),$$
or,
$$T \geq \frac{1}{\alpha c} \log \left( \frac{\|\vec{x}_0 - \vec{x}^*\|^2}{\epsilon(1 - \nu)} \right)$$
$$= \frac{M^2 \epsilon^2 + L^2 B^2}{\epsilon c^4 \nu} \log \left( \frac{\|\vec{x}_0 - \vec{x}^*\|^2}{\epsilon(1 - \nu)} \right),$$
we have the convergence using $K_1 \leq \epsilon(1 - \nu)$ and $K_2 = \nu \epsilon$, we get $E \left[ ||\vec{x}_T - \vec{x}^*||^2 \right] \leq \epsilon.$