On the second-order Fréchet derivatives of eigenvalues of Sturm–Liouville problems in potentials

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Abstract. The works of V.A. Vinokurov have shown that eigenvalues and normalized eigenfunctions of Sturm–Liouville problems are analytic in potentials, considered as mappings from the Lebesgue space to the space of real numbers and the Banach space of continuous functions respectively. Moreover, the first-order Fréchet derivatives are known and play an important role in many problems. In this paper, we will find the second-order Fréchet derivatives of eigenvalues in potentials, which are also proved to be negative definite quadratic forms for some cases.

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1. Introduction. It is an important issue and has a long history to study the dependence of eigenvalues of differential operators on coefficients, boundary data, and domains involved in the problems [3,7,8,10]. In this paper, we are concerned with the dependence of eigenvalues on potentials. More precisely, let $q = q(x) \in L^1 := L^1(I, \mathbb{R})$ be an integrable potential, where $I = [0, \ell]$, $\ell > 0$, is a closed interval. We consider the Sturm–Liouville eigenvalue problem

$$-z'' + q(x)z = \lambda z, \quad x \in I,$$

with the boundary conditions

$$z(0) \cos \alpha - z'(0) \sin \alpha = 0,$$

$$z(\ell) \cos \beta - z'(\ell) \sin \beta = 0,$$

where $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$ are given parameters. It is well-known that problem (1.1)–(1.3) has a discrete spectrum consisting of an increasing infinite
sequence of (real, simple) eigenvalues $\lambda_n$ such that $\lambda_n \to +\infty$ as $n \to \infty$. See, for example, [17]. The eigenvalues $\lambda_n$ depend on the potential $q$ and on the boundary data $(\alpha, \beta)$ as well.

Let us fix $(\alpha, \beta)$ and consider $\lambda_n = \lambda_n(q)$ as (nonlinear) functionals of potentials $q \in L^1$. Associated with $\lambda_n$ is the corresponding eigenfunction $E_n(x) = E_n(x; q)$, normalized as

$$\|E_n\|_2 = \left( \int_I E_n^2(x) \, dx \right)^{1/2} = 1. \quad (1.4)$$

Moreover, in order that $E_n(x)$ is uniquely determined, it is required that $E_n(x) > 0$ on some neighborhood of the type $(0, \delta)$. \quad (1.5)

Among many studies on Sturm–Liouville problems, around 2005, Vinokurov and his collaborator [12–14] have systematically studied the dependence of solutions and eigenvalues on potentials. One of their results in [13] is that when $L^1$ and $C(I, \mathbb{R})$ are respectively endowed with the $L^1$ norm $\| \cdot \|_1$ and the supremum norm $\| \cdot \|_\infty$, the nonlinear mappings

$$\begin{cases} (L^1, \| \cdot \|_1) \to \mathbb{R}, & q \mapsto \lambda_n(q), \\ (L^1, \| \cdot \|_1) \to (C(I, \mathbb{R}), \| \cdot \|_\infty), & q \mapsto E_n(\cdot; q), \end{cases} \quad (1.6)$$

are proved to be analytic in the sense of [1,2]. Moreover, the Fréchet derivative of the eigenvalue $\lambda_n(q)$ is given by

$$\partial_q \lambda_n(q)(h) := \partial_s \lambda_n(q + sh)_{|s=0} = \int_I (E_n(x; q))^2 h(x) \, dx \quad (1.7)$$

for $h \in L^1$. The Fréchet derivative of the eigenfunction $E_n(\cdot; q)$ in $q \in L^1$ can also be found in [13]. In fact, the result (1.7) for the Dirichlet eigenvalues and potentials in $L^2(I, \mathbb{R})$ was already obtained in [11]. As for the first Dirichlet eigenvalue of the Laplacian with potentials, a similar formula of the Fréchet derivative has been derived very recently in [6]. Formula (1.7) can also be written as

$$\partial_q \lambda_n(q) = (E_n(\cdot; q))^2, \quad (1.8)$$

understood as a kernel function in the dual space $(L^1, \| \cdot \|_1)^* = L^\infty(I, \mathbb{R})$.

One simple implication of (1.8) is that the eigenvalues $\lambda_n(q)$ are strictly increasing in $q \in L^1$, because the derivatives are positive. As for the continuous dependence, about ten years ago, one of the authors of this paper and his collaborator have obtained a further result. That is, when the norm topology $\| \cdot \|_1$ in $L^1$ is replaced by the topology $w_1$ of weak convergence in $L^1$, the mappings in (1.6) are still continuous [9,18]. Based on such a complete continuity of eigenvalues in potentials, the derivatives (1.8) of eigenvalues are applied in [15,19] to study some typical optimization problems on eigenvalues of Sturm–Liouville operators. This leads to some connection between the linear and nonlinear stationary Schrödinger equations. It is interesting to note that in very recent papers [5,6], such a connection is also established for some inverse spectral problems.
Because of these applications, it is convincing that the higher-order Fréchet derivatives of eigenvalues and eigenfunctions will be useful. However, though $\lambda_n(q)$ and $E_n(x; q)$ are analytic in $q \in \mathcal{L}^1$, as far as we know, even their second-order Fréchet derivatives in potentials are not available in the literature. The aim of this paper is to fill this gap.

For $q, h \in \mathcal{L}^1$, let us define the second-order Fréchet derivative \cite{16} of $\lambda_n(q)$ by

$$\partial_{qq} \lambda_n(q)(h) := \partial_{ss} \lambda_n(q + sh)|_{s=0} \in \mathbb{R}. \quad (1.9)$$

We will obtain the following results.

**Theorem 1.1.** For $q, h \in \mathcal{L}^1$, let $L$ be as in (1.9) and $U_n(x) = U_n(x; q, h)$ be the unique solution of the following inhomogeneous linear ODE

$$-z'' + (q(x) - \lambda_n(q)) z = -E_n(x) (h(x) - L), \quad x \in I, \quad (E)$$

satisfying the initial conditions

$$z(0) = z'(0) = 0. \quad (I)$$

Then the second-order Fréchet derivative of the eigenvalues is given by

$$\partial_{qq} \lambda_n(q)(h) = 2 \int_I E_n(x) (h(x) - L) U_n(x) \, dx. \quad (1.10)$$

Moreover, it can also be expressed as a quadratic form of $h$

$$\partial_{qq} \lambda_n(q)(h) = \int_{I^2} J_n(x, y) h(x) h(y) \, dx \, dy. \quad (1.11)$$

Here $U_n(x; q, h)$ and $J_n(x, y) = J_n(x, y; q)$ are explicitly given in formula (2.24) and in (2.29)–(2.30) respectively.

Next, we are still using the solutions $U_n(x)$, but with some restriction on the boundary condition (1.3) at the right end-point $x = \ell$.

**Theorem 1.2.** For any $q, h \in \mathcal{L}^1$, let $U_n(x)$ be as in Theorem 1.1. Then

- (i) $U_n(x)$ also satisfies the boundary conditions (1.2)–(1.3).
- (ii) Assume that (1.3) takes the following boundary condition

$$z(\ell) = 0 \quad \text{or} \quad z'(\ell) = 0. \quad (1.12)$$

Then, in this case, the second-order Fréchet derivative can also be expressed as

$$\partial_{qq} \lambda_n(q)(h) = -2 \int_I (U'_n)^2 + (q(x) - \lambda_n(q)) U^2_n \, dx. \quad (1.13)$$
The paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.2 will be given in Sect. 2 and in Sect. 3 respectively. As for the first Dirichlet eigenvalues $\lambda_1^D(q)$ of (1.1), we will use (1.13) to prove in Theorem 3.2 that $\partial_q\lambda_1^D(q)(h)$ is a negative definite quadratic form of $h \in \mathcal{L}^1$. This result is consistent with the concavity of $\lambda_1^D(q)$ in $q \in \mathcal{L}^1$. See the discussion in Remark 3.3. At the end of the paper, we will propose some further problems on the second-order Fréchet derivatives.

2. The second-order Fréchet derivatives of eigenvalues. Let $q \in \mathcal{L}^1$. For $h \in \mathcal{L}^1$ and $s \in \mathbb{R}$, let us denote

$$Q(x,s) := q(x) + sh(x) - \lambda_n(q + sh) \in \mathcal{L}^1.$$  

(2.1)

We consider the linear ODE

$$-z'' + Q(x,s)z = 0, \quad x \in I.$$  

(2.2)

Here $' = \frac{d}{dx}$ and $'' = \frac{d^2}{dx^2}$ are also written as $\partial_x$ and $\partial_{xx}$ respectively. For each $s$, let $z = \varphi(x,s)$ be the solution of Eq. (2.2) satisfying the initial condition

$$(z(0),z'(0)) = (\sin \alpha, \cos \alpha).$$  

(2.3)

That is, for each $s$, one has

$$-\partial_{xx}\varphi(x,s) + Q(x,s)\varphi(x,s) = 0,$$  

(2.4)

$$(\varphi(0,s), \partial_x\varphi(0,s)) = (\sin \alpha, \cos \alpha).$$  

(2.5)

Moreover, from the definition of the eigenvalues, for each $s$, $\varphi(\cdot,s)$ also satisfies the boundary condition (1.3)

$$\varphi(\ell,s) \cos \beta - \partial_x\varphi(\ell,s) \sin \beta = 0.$$  

(2.6)

Since $\lambda_n(q + sh)$ is analytic in $s$, as a mapping from $\mathbb{R}$ to $\mathcal{L}^1$, $Q(\cdot,s)$ is then analytic in $s$. Moreover, since the solution of a initial value problem of a linear ODE is analytic in coefficient potential [12], $\varphi(x,s)$ and $\partial_x\varphi(x,s)$ are also analytic in $s$. Thus, by differentiating Eq. (2.4) with respect to $s$ twice, we obtain the following inhomogeneous linear ODEs

$$-\partial_{xx}\partial_s\varphi(x,s) + Q(x,s)\partial_s\varphi(x,s) = -\partial_sQ(x,s) \cdot \varphi(x,s),$$  

(2.7)

$$-\partial_{xx}\partial_{ss}\varphi(x,s) + Q(x,s)\partial_{ss}\varphi(x,s) = -\partial_{ss}Q(x,s) \cdot \varphi(x,s) - 2\partial_sQ(x,s) \cdot \partial_s\varphi(x,s).$$  

(2.8)

Let us consider these equations at $s = 0$. For simplicity, denote

\[
\begin{align*}
L &:= \partial_s\lambda_n(q + sh)|_{s=0} = \partial_q\lambda_n(q)(h) \in \mathbb{R}, \\
M &:= \partial_{ss}\lambda_n(q + sh)|_{s=0} = \partial_{qq}\lambda_n(q)(h) \in \mathbb{R}, \\
Q(x) &:= Q(x,0) = q(x) - \lambda_n(q), \\
\varphi(x) &:= \varphi(x,0), \\
\varphi_k(x) &:= \partial_s^k\varphi(x,s)|_{s=0}, \quad k = 1,2.
\end{align*}
\]

(2.9)

By (2.1), one has $\partial_sQ(x,s)|_{s=0} = h(x) - L$ and $\partial_{ss}Q(x,s)|_{s=0} = -M$. Thus Eqs. (2.7) and (2.8) mean that $z = \varphi_k(x)$ is a solution of the inhomogeneous linear ODE

$$-z'' + Q(x)z = f_k(x), \quad x \in I,$$  

(2.10)
where
\[ f_1(x) = (L - h(x))\varphi(x) \quad \text{and} \quad f_2(x) = M\varphi(x) - 2(h(x) - L)\varphi_1(x). \] (2.11)
Moreover, by differentiating initial condition (2.5) and boundary condition (2.6) with respect to \( s \) twice, we know that \( z = \varphi_k(x) \), \( k = 1, 2 \), satisfy
\[
\begin{align*}
(z(0), z'(0)) &= (0, 0), \\
z(\ell) \cos \beta - z'(\ell) \sin \beta &= 0.
\end{align*}
\] (2.12) (2.13)

**Lemma 2.1.** Let \( q \in L^1 \) be given. For any \( h \in L^1 \), one has
(i) \( \varphi_k(x) = \varphi_k(x; q, h) \), \( k = 1, 2 \), are uniquely determined. Actually, they are the solutions of the initial value problems (2.10)–(2.12).
(ii) \( \varphi_k(x) = \partial_{s^k}\varphi(x, s)|_{s=0} \), \( k = 1, 2 \), are given by
\[
\varphi_k(x) = \int_0^x W(x, y)f_k(y)\,dy, \quad x \in I, \tag{2.14}
\]
where \( f_1 \) and \( f_2 \) are as in (2.11), and \( W(x, y) \) is as in (2.17) below. In particular,
\[
\varphi_1(x) = \int_0^x W(x, y)\varphi(y)(L - h(y))\,dy, \quad x \in I. \tag{2.15}
\]
(iii) \( z = \varphi(x) \), \( \varphi_1(x) \), and \( \varphi_2(x) \) satisfy the boundary conditions (1.2)–(1.3).

**Proof.** (i) This is clear from the above deductions.
(ii) Let \( \psi_i(x) = \psi_i(x; q) \), \( i = 1, 2 \), be the fundamental solutions of the homogeneous linear ODE
\[
-\ z'' + Q(x)z = -z'' + (q(x) - \lambda_n(q))z = 0, \tag{2.16}
\]
i.e. the solutions of Eq. (2.16) satisfying the initial conditions \( (z(0), z'(0)) = e_1 := (1, 0) \) and \( (z(0), z'(0)) = e_2 := (0, 1) \) respectively. Define
\[
W(x, y) := \begin{vmatrix} \psi_1(x) & \psi_2(x) \\ \psi_1(y) & \psi_2(y) \end{vmatrix} = \psi_1(x)\psi_2(y) - \psi_2(x)\psi_1(y). \tag{2.17}
\]
By applying the formula of constant variant to (2.10)–(2.12), we know that \( \varphi_k(x) \) is given by (2.14). In particular, (2.15) follows from (2.11) and (2.14). Here one can notice that \( \psi_i(x) \) and \( W(x, y) \) depend only on \( q \), not on \( h \).
(iii) Due to the choice (2.3) of the initial conditions for \( \varphi(x) \), \( \varphi(x) \) satisfies boundary condition (1.2). Moreover, by the definition of the eigenvalues \( \lambda_n(q) \), one knows that \( \varphi(x) \) satisfies (1.3) as well.

For \( k = 1, 2 \), it is clear from (2.12) and (2.13) that \( z = \varphi_k(x) \) must satisfy the boundary conditions (1.2)–(1.3). \( \square \)

Lemma 2.1 (iii) shows that \( \varphi(x) \) is an eigenfunction associated with \( \lambda_n(q) \). One then sees that the normalized eigenfunction satisfying (1.4) and (1.5) is
\[
E_n(x) = E_n(x; q) \equiv \varphi(x)/\|\varphi\|_2. \tag{2.18}
\]
To derive the formulas for the Fréchet derivatives $L$ and $M$, we can exploit the Fredholm principle [4].

**Lemma 2.2.** Consider the following inhomogeneous linear ODE

$$- z'' + Q(x)z = f(x), \quad x \in I, \quad (2.19)$$

where $f(x) \in L^1(I)$. If Eq. (2.19) admits a solution $z(x)$ satisfying the boundary conditions (1.2)–(1.3), it is necessary that

$$\int_I \varphi(x) f(x) \, dx = 0. \quad (2.20)$$

**Proof.** Let $\varphi(x)$ be as in (2.9). From Lemma 2.1 (iii), $z = \varphi(x)$ is a solution of boundary value problem (2.16)–(1.2)–(1.3), where Eq. (2.16) is the corresponding homogeneous linear ODE of Eq. (2.19). Hence the solvability of problem (2.19)–(1.2)–(1.3) is actually equivalent to condition (2.20).

**Proof of Theorem 1.1.** Applying Lemma 2.2 to Eq. (2.10), we obtain

$$\int_I \varphi(x) f_k(x) \, dx = 0, \quad k = 1, 2. \quad (2.21)$$

For $k = 1$, it follows from (2.11) that Eq. (2.21) is $\int_I (L - h(x)) \varphi^2(x) \, dx = 0$, i.e.

$$L = \frac{\int_I \varphi^2(x) h(x) \, dx}{\int_I \varphi^2(x) \, dx} = \int_I \left( \frac{\varphi(x)}{\|\varphi\|_2} \right)^2 h(x) \, dx = \int_I E_n^2(x) h(x) \, dx \quad (2.22)$$

(see (2.18)). This gives another deduction of formula (1.7), which is different from that in [13].

Let now $L$ be as in (2.22). For $k = 2$, it follows from (2.11) that Eq. (2.21) is

$$\int_I (M \varphi^2(x) - 2 \varphi(x)(h(x) - L)\varphi_1(x)) \, dx = 0.$$ 

By using (2.18), this yields

$$M = 2 \int_I E_n(x) (h(x) - L) \frac{\varphi_1(x)}{\|\varphi\|_2} \, dx.$$ 

Thus we have obtained (1.10), where

$$U_n(x) = U_n(x; q, h) := \frac{\varphi_1(x)}{\|\varphi\|_2} \equiv \frac{\partial_s \varphi(x, s)|_{s=0}}{\|\varphi\|_2}. \quad (2.23)$$

Dividing Eq. (2.10) by the factor $\|\varphi\|_2$ and making use of conditions (2.11) and (2.12), one sees that $U_n(x)$ is just the solution of the initial value problem (E)-(I). From (2.15), (2.18), and (2.23), it is easy to see that
\[ U_n(x) = U_n(x; q, h) \equiv \int_0^x W(x, y) E_n(y)(L - h(y)) \, dy, \quad x \in I, \quad (2.24) \]

where \( W(x, y) \) and \( L \) are in (2.17) and (2.22) respectively.

Now we are using formulas (1.10) and (2.24) to deduce formula (1.11).

Define
\[
\hat{h}(x) := E_n(x)(h(x) - L) \quad (2.25)
\]
\[
= \int_I E_n(x) E_n^2(u)(h(x) - h(u)) \, du, \quad (2.26)
\]
because \( \int_I E_n^2(u) \, du = 1 \). By (2.24), one has
\[
U_n(x) = -\int_0^x W(x, y) \hat{h}(y) \, dy.
\]
Hence (1.10) gives
\[
M = 2 \int_0^\ell \hat{h}(x) U_n(x) \, dx = 2 \int_0^\ell \hat{h}(x) \left( -\int_0^x W(x, y) \hat{h}(y) \, dy \right) \, dx
\]
\[
= \int_{I^2} G(x, y) \hat{h}(x) \hat{h}(y) \, dx \, dy. \quad (2.27)
\]

Here \( G(x, y) : I^2 \to \mathbb{R} \) is the following symmetrization of \( W(x, y) \)
\[
G(x, y) := \begin{cases} 
W(x, y), & 0 \leq x \leq y \leq \ell, \\
-W(x, y), & 0 \leq y \leq x \leq \ell.
\end{cases} \quad (2.28)
\]

Obviously, we have \( G(x, y) = G(y, x) \), i.e. \( G(x, y) \) is symmetric.

By (2.25), we have
\[
\hat{h}(x) \hat{h}(y) = E_n(x) E_n(y) h(x) h(y) - E_n(x) E_n(y) \cdot L h(x)
\]
\[
- E_n(x) E_n(y) \cdot L h(y) + E_n(x) E_n(y) \cdot L \cdot L.
\]
Denote
\[
\mathcal{E}(x, y, u, v) := E_n(x) E_n(y) E_n(u) E_n(v).
\]
Then (2.27) is
\[
M = M_1 - M_2 - M_3 + M_4,
\]
where, by using (2.26),
\[
M_1 = \int_{I^2} \left( \int_{I^2} G(x, y) E_n^2(u) E_n^2(v) \, du \, dv \right) E_n(x) E_n(y) h(x) h(y) \, dx \, dy
\]
\[
= \int_{I^2} \left( \int_{I^2} G(x, y) E_n(u) E_n(v) \mathcal{E}(x, y, u, v) \, du \, dv \right) h(x) h(y) \, dx \, dy,
\]
\[ M_2 = \int_{I^2} G(x,v)E_n(x)E_n(v) \cdot Lh(x) \, dx \, dv \]
\[ = \int_{I^2} G(x,v)E_n(x)E_n(v) \left( \int_{I} E_n^2(y)h(y) \, dy \right) h(x) \, dx \, dv \]
\[ = \int_{I^2} \left( \int_{I^2} G(x,v)E_n(u)E_n(y)E(x,y,u,v) \, du \right) h(x)h(y) \, dx \, dy. \]

Similarly,
\[ M_3 = \int_{I^2} \left( \int_{I^2} G(u,y)E_n(v)E_n(x)E(x,y,u,v) \, du \right) h(x)h(y) \, dx \, dy, \]
\[ M_4 = \int_{I^2} \left( \int_{I^2} G(u,v)E_n(x)E_n(y)E(x,y,u,v) \, du \right) h(x)h(y) \, dx \, dy. \]

Thus, by defining \( J_n(x,y,u,v) : I^4 \to \mathbb{R} \) as
\[ J_n(x,y,u,v) := G(x,y)E_n(u)E_n(v) - G(x,v)E_n(u)E_n(y) - G(u,y)E_n(x)E_n(v) + G(u,v)E_n(x)E_n(y), \] (2.29)
and by defining \( J_n(x,y) : I^2 \to \mathbb{R} \) as
\[ J_n(x,y) := \left( \int_{I^2} J_n(x,y,u,v)E_n(u)E_n(v) \, du \right) E_n(x)E_n(y), \] (2.30)
we know that \( M \) is expressed as the integral form (1.11).

The proof of Theorem 1.1 is completed. \( \square \)

**Remark 2.3.** The kernel \( J_n(x,y,u,v) \) of (2.30), a continuous function defined on \( I^4 \), is determined from (2.28) and (2.29). These kernels have the following symmetries
\[ J_n(x,y,u,v) \equiv J_n(u,v,x,y) \quad \text{and} \quad J_n(x,y) \equiv J_n(y,x). \]

**3. The concavity of eigenvalues in potentials.** We will derive formula (1.13) for the second-order Fréchet derivatives of eigenvalues in potentials.

**Proof of Theorem 1.2.** (i) For the general boundary conditions (1.2)–(1.3), it is clear from Lemma 2.1 (iii) that the solution \( U_n(x) \) of (E) also satisfies the boundary conditions (1.2)–(1.3).

(ii) Recall that \( U_n(x) \) satisfies the ODE
\[ -U_n'' + Q(x)U_n = -E_n(x)(h(x) - L). \] (3.1)

Since we are now considering the boundary conditions (1.2)–(1.12), from the proof of Lemma 2.1, we know that \( U_n(x) = \varphi_1(x)/\|\varphi\|_2 \) satisfies
\[ U_n(0) = U_n'(0) = 0 \quad \text{and} \quad U_n(\ell)U_n'(\ell) = 0. \] (3.2)
Multiplying Eq. (3.1) by $U_n(x)$ and then integrating on $I$, we obtain
\[
\int_I E_n(x)(h(x) - L)U_n(x) \, dx = -\int_I (-U_n'' + Q(x)U_n) \, dx \\
= U_n(x)U_n'(x)|_x^\ell - \int_I (U_n'' + Q(x)U_n^2) \, dx \\
= -\int_I (U_n'' + Q(x)U_n^2) \, dx,
\]
due to (3.2). Hence formula (1.13) can be deduced from (1.10) and the above equality.

**Remark 3.1.** In boundary conditions (1.2)–(1.3), if (1.2) is restricted to be either $z(0) = 0$ or $z'(0) = 0$, dually we have
\[
\partial_{qq} \lambda_n(q)(h) = -2 \int_I (V_n'' + (q(x) - \lambda_n(q)) V_n^2) \, dx.
\]
Here $z = V_n(x)$ is the solution of $(E)$ satisfying the initial conditions $z(\ell) = z'(\ell) = 0$.

As an example, let us consider the Dirichlet boundary conditions
\[z(0) = z(1) = 0.\]

For $q \in \mathcal{L}^1$, we use $\lambda_n^D(q)$, $n \in \mathbb{N}$, to denote the eigenvalues of problem (1.1)–(D). It is known from [3] that $\lambda_1^D(q)$ has the following minimization characterization
\[
\lambda_1^D(q) = \min_{z \in H_0^1(I), \ z \neq 0} \frac{\int_I (z''^2 + q(x)z^2) \, dx}{\int_I z^2 \, dx}.
\] (3.3)

For any $n \geq 2$, $\lambda_n^D(q)$ has the following maximin characterization
\[
\lambda_n^D(q) = \max_{W_{n-1}} \min_{z \in W_{n-1}^\perp, \ z \neq 0} \frac{\int_I (z''^2 + q(x)z^2) \, dx}{\int_I z^2 \, dx},
\] (3.4)

where the maximum is taken over all subspaces $W_{n-1}$ of $H_0^1(I)$ of dimension $n - 1$, and
\[
W_{n-1}^\perp := \left\{ z \in H_0^1(I) : \int_I w(x)z(x) \, dx = 0 \text{ for all } w \in W_{n-1} \right\}.
\]

**Theorem 3.2.** For the first Dirichlet eigenvalues $\lambda_1^D(q)$, the second-order Fréchet derivatives satisfy $\partial_{qq} \lambda_1^D(q)(h) \leq 0$ for all $h \in \mathcal{L}^1$. 

Proof. Since the Dirichlet boundary conditions satisfy (1.2)–(1.12), we can use formula (1.13) for $\partial_{qq}\lambda^D_1(q)(h)$. By Theorem 1.2, $U_1(x) = U^D_1(x)$ satisfies (D), i.e. $U_1(x) \in H^1_0(I)$. From the minimization characterization (3.3), one has

$$
\int_I (U^2_1 + q(x)U^2_1) \, dx \geq \lambda^D_1(q) \int_I U^2_1 \, dx.
$$

Thus (1.13) shows that $\partial_{qq}\lambda^D_1(q)(h) \leq 0$. \hfill \Box

Remark 3.3. (i) It is standard from nonlinear analysis [16] that the negative definiteness of the second-order Fréchet derivatives is the same as the concavity. As a result, Theorem 3.2 implies that

$$
\lambda^D_1(\tau q_1 + (1-\tau)q_2) \geq \tau \lambda^D_1(q_1) + (1-\tau)\lambda^D_1(q_2)
$$

for all $q_i \in \mathcal{L}^1$ and $\tau \in [0,1]$.

(ii) The concavity (3.5) of the first eigenvalue $\lambda^D_1(q)$ in $q$ can also be directly deduced from the minimization characterization (3.3). In fact, one has

$$
\lambda^D_1(\tau q_1 + (1-\tau)q_2) = \min_{\|z\|_2=1 \atop z \in H^1_0(I)} \int_I \left( \tau \int_I (z'^2 + q_1(x)z^2) \, dx + (1-\tau) \int_I (z'^2 + q_2(x)z^2) \, dx \right)
$$

$$
\geq \tau \cdot \min_{\|z\|_2=1 \atop z \in H^1_0(I)} \int_I (z'^2 + q_1(x)z^2) \, dx + (1-\tau) \cdot \min_{\|z\|_2=1 \atop z \in H^1_0(I)} \int_I (z'^2 + q_2(x)z^2) \, dx
$$

$$
= \tau \lambda^D_1(q_1) + (1-\tau)\lambda^D_1(q_2).
$$

(iii) For the zeroth Neumann eigenvalue $\lambda^N_0(q)$ of problem (1.1), it can be proved that $\partial_{qq}\lambda^N_0(q)(h) \leq 0$ for all $h \in \mathcal{L}^1$.

We end the paper with two problems.

1. Arguing as in the deduction of (3.6), one can use the maximin characterization (3.4) to obtain the concavity of $\lambda^D_n(q)$ in $q \in \mathcal{L}^1$. Therefore $\partial_{qq}\lambda^D_n(q)(h)$ is also negative definite. It is an interesting problem to give a direct proof for the negative definiteness of $\partial_{qq}\lambda^D_n(q)(h)$ for the case $n \geq 2$.

2. We have known from [13] that the eigenfunctions $E_n(x;q)$ are also analytic in $q \in \mathcal{L}^1$. It is then an important problem to find the second-order Fréchet derivatives of $E_n(x;q)$ in $q \in \mathcal{L}^1$.

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