Adiabatic formation of high-Q modes by suppression of chaotic diffusion in deformed microdiscs

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Abstract. Resonant modes with high-Q factors in a two-dimensional deformed microdisc cavity are analyzed by using a dynamical and semiclassical approach. The analysis focuses particularly on the ultra-small cavity regime, where the scale of a resonant free-space wavelength is comparable with that of the microdisc size. Although the deformed microcavity has strongly chaotic internal ray dynamics, modes with high-Q factors in this regime show unexpectedly regular distributions in configuration space and adiabatic features in phase space. By tracing the evolution process of such high-Q modes through the deformation from a circular cavity, it is uncovered that the high-Q modes are formed adiabatically on cantori. Due to the openness of microcavities, such adiabatic formation of high-Q modes around cantori is enabled, in spite of strong chaos in ray dynamics. Since the cantori are in close contact with short periodic orbits, their influence on the modes, such as localization patterns in phase space, can be also clarified. In order to quantitatively analyze the spectral range where high-Q modes appear, the phase space section of the deformed microcavity is partitioned by partial barriers of short periodic orbits, and the semiclassical quantization scheme is applied to the partitioned areas and their action fluxes. The derived spectral ranges for the high-Q modes show a good agreement with a numerically observed spectrum. In the course of semiclassical quantization, it is shown that the chaotic diffusion in the system that we investigate can be resolved by the

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scale of a quarter effective Planck’s constant, and the topological structure of the manifolds in phase space allows for this resolution higher than a Planck constant scale. By analyzing flux Farey trees, the role of short periodic orbits in chaotic diffusion and their connection to cantori are verified.

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1. Introduction

If a dielectric structure is spherically or cylindrically shaped and a ray is launched inside (assuming the total internal reflection condition holds), this ray can eternally run around the boundary of the structure due to the rotational symmetry. In its wave mechanical counterpart, such a ray dynamical feature manifests itself in an optical mode that is called a ‘whispering gallery mode (WGM)’ (figure 1(a)) [1]. As the reflectivity of the total internal reflection is higher than that of conventional optical mirrors, light can be trapped in WGMs for a remarkably long time. For optical applications, this is a very valuable property, because a micrometer-sized cavity with an extremely high quality factor ($Q$ factor) can be experimentally realized. Such tiny cavities with high-$Q$ factors can support superfine spectral resolution and low threshold lasing [2]. This sort of dielectric cavity is called a ‘microcavity.’ Microcavities have naturally attracted research interest in recent years, and have been applied to various fields, from quantum optics to optical engineering [3, 4].

However, the isotropic radiational emission from WGMs due to the rotational symmetry of the cavities is an obstacle for some applications. To overcome this problem, the idea of boundary deformation was proposed [5–7]. Figure 1(b) shows an example of a two-dimensional deformed microcavity, the boundary of which is given by a Limaçon shape [8, 9]. The inset in figure 1(b) shows the far-field emission of the mode presented in figure 1(b). As the emission has a roughly three times stronger intensity in the right-hand side direction than others, the emission is regarded as having unidirectional emission. Also as this unidirectionality is commonly observed in other modes, this is taken as a property of the microcavity with a Limaçon shape.

Recently, many studies demonstrated that the emission directionality of deformed microcavities can be predicted by a ray dynamical calculation [10]. However, the ray dynamics cannot explain every optical property of a deformed microcavity in the small-cavity regime, because an excited resonant mode in a deformed microcavity cannot completely follow the fine dynamical structures generated by the chaos [11]. Thus, we often encounter situations in which wave effects should be considered. For that reason, quantum chaos theory has been extensively
Figure 1. (a) WGM in a circular cavity and (b) optical resonant mode in a Limaçon-shaped microcavity with deformation parameter $\varepsilon = 0.43$. (inset) Far-field distribution of the mode: the emission directionality is unidirectional to the right-hand side.

applied to the research on deformed microcavities so far, and many pronounced phenomena of quantum chaos have been confirmed in deformed microcavity experiments [5, 12, 13].

Currently, micro-optical experiments are pursuing the realization of two-dimensional semiconductor microdisc lasers in the ultra-small sized regime, where the characteristic size of the cavity is comparable with the wavelength in vacuum [9, 14, 15]. Since the ultra-small microdisc lasers are fabricated to have a very thin active layer, supporting a single wave-guide mode, they demonstrate good agreement with two-dimensional numerical analysis. As the size of optical cavities gets smaller, the need for wave mechanical consideration, including quantum chaos theory, is imperative, because the discrepancy between ray and wave dynamics is larger and wave effects such as tunneling become more dominant [11, 16–18].

In this paper, we focus on high-$Q$ modes in the ultra-small cavity regime with respect to the free-space wavelength. Taking the Limaçon cavity as a deformed microcavity, its optical properties such as its spectrum and resonant modes are numerically calculated. In the ultra-small regime, the high-$Q$ modes always appear with a curved series of modes on the complex wave number plane, and all the modes belonging to a common mode series exhibit interesting and unexpected features in both configuration space and phase space. In configuration space, all high-$Q$ modes share a regular distribution with well defined mode numbers, although the deformed microcavity generates very strong chaos in its internal ray dynamics. In phase space, every mode from one mode series has a common localized pattern. However, these modes are different from the conventional ‘Scar’, because there is no classical periodic orbit which exactly coincides with the localization in phase space [19].

By investigating the evolution of the high-$Q$ modes and the corresponding ray dynamics through the deformation, we first make clear the adiabatic formation of high-$Q$ modes around partial barriers. A partial barrier is the remnant of a broken invariant structure in phase space. In this study, we observe that they can effectively play a role of an adiabatic invariant by trapping a mode, even if they are considerably broken [20]. Based on the open property of microcavity, it can be reasoned that such a quantization of a broken tori is possible through the suppression of chaotic diffusion. Since the high-$Q$ modes appearing with curved mode series exhibits the strong influence of short periodic orbits, we partition the phase space with the partial barriers of the short periodic orbits. By introducing the concept of action flux through a partial barrier and
semiclassically quantizing it, the spectral conditions for the high-$Q$ modes are derived. These conditions are verified by a good agreement with a numerically calculated spectrum. In the process of the theoretical development, it is observed that the topological structure of manifolds enables a mode to resolve a quarter wavelength scale in chaotic diffusion. In addition, the strong influence of short periodic orbits on the modes residing on broken tori is proved by using the action farey trees.

This paper is structured as follows: in the next section, the model system is introduced, and its internal ray dynamics is analyzed within the framework of Hamiltonian chaos theory. In section 3, the optical properties of the model system, such as its spectrum and modes, are numerically investigated and the characteristics of the high-$Q$ modes are presented. Then, the adiabatic features in the formation of high-$Q$ modes are discussed in section 4. In section 5, by evaluating the action fluxes of the partial barriers, quantitative conditions for the high-$Q$ modes are derived and compared with numerical results.

2. Model system and internal ray dynamics

As a deformed microcavity, we choose the ‘Limaçon cavity’, the boundary of which is described by the lowest order term of the multipolar expansion [8]

$$r(\phi) = R(1 + \varepsilon \cos \phi).$$  (1)

As the deformation $\varepsilon$ is increased, the internal ray dynamics of the microcavity becomes transiently chaotic. This chaotic transition can be visualized on a phase space section, using Birkhoff coordinates [21]. The Birkhoff coordinates are a tool to analyze dynamics in a billiard, and comprise an arc length along the billiard boundary as the position coordinate and the sine value of a reflection angle of a bouncing ray on the boundary as the corresponding canonical momentum. By recording these two values at every bounce of a ray, a two-dimensional section of phase space is constructed. On the phase space section represented by the Birkhoff coordinates, the condition for the critical angle of total internal reflection is set as a horizontal line on the phase space, the vertical position of which is at $\sin \chi = 1/n$. Then, the phase space section is divided into two parts, the region above this line where a ray is confined by the total internal reflection and the region under this line where a ray can escape to the outside of the cavity through refraction.

At $\varepsilon = 0$, the boundary shape is a circle. Accordingly, the angular momentum $L$ is a conserved quantity, and each ray trajectory with total momentum $p_{\text{tot}}$ has a circular caustic line with radius $r_0 = R \sin \chi$ ($= L/p_{\text{tot}}$) which is concentric to the boundary (figure 2(b)). Due to the rotational symmetry, the equations of motion can be separated into each degree of freedom. The separated radial motion can be described by the effective potential

$$V_{\text{eff}}(r) = \left(\frac{r_0}{r}\right)^2 + \frac{1 - \bar{n}^2(r)}{n^2}, \quad \text{where} \quad \bar{n}(r) = \begin{cases} n & \text{for} \quad r \leq R, \\ 1 & \text{for} \quad r > R. \end{cases}$$  (2)

In figure 2(a), the total effective energy of the ray is taken to be $E_{\text{tot}} = 1$. Then, the motion projected onto the radial direction corresponds to an oscillation in the confining potential.

If the phase space section of the system is represented by Birkhoff coordinates, the angular momentum conservation manifests itself as horizontal manifolds (figure 3(a)), which are sections of tori in the higher dimensional phase space. The WGMs such as in figure 1(a) can be understood as semiclassically quantized modes of these manifolds. Each manifold on the
Figure 2. (a) Effective potential in the radial degree of freedom. (b) Two consecutive bounces of a trajectory on a single manifold. From this configuration, the relation between the change of the central angle ($\theta$) and the characteristic incident angle ($\chi$) is found: $\theta = \pi - 2\chi$. The continued trajectory forms an internal circle (Caustic) with radius $r_0 = R \sin \chi$ (red dotted line).

Figure 3. Chaotic transition of the ray dynamics in the Limaçon billiard. The phase space sections are represented by the Birkhoff coordinates in the unit of the total perimeter length, $S_0$ (inset of (a)). The critical condition for total internal reflection is marked by a red horizontal line on each phase space section ($\sin \chi = 1/n$, $n = 3.0$). The deformation ($\epsilon$) for each figure is (a) 0.0 (circle), (b) 0.2, (c) 0.25 and (d) 0.43. At the top of the phase space section of (d), one can notice a small blank area which contains regular trajectories.

Phase space is characterized by a winding number ($q/p$) which is given by the ratio between the rotational period ($p$) and the radial period ($q$). As two consecutive bounces corresponds to one period of the radial oscillation, the winding number can be derived by using the change
of the central angle between two consecutive bounces ($\theta$), and then reformulated with the characteristic incident angle of each manifold ($\chi$):

$$q/p = \frac{\theta}{2\pi} = \frac{1}{2} - \frac{\chi}{\pi}.$$  

(3)

On the phase space section in figure 3(a), the winding number of a manifold is continuously decreased from $1/2$ to 0 with increasing the value of $\sin \chi$.

As the rotational symmetry is broken by deformation $\epsilon > 0$, the phase space exhibits a transient evolution to chaos. Because the deformation term in equation (1) is given by a smooth function $\cos \phi$, the chaotic transition follows the Kolmogorov–Arnold–Moser (KAM) scenario [22, 23]. With a slight deformation, namely $\epsilon = 0.2$ in figure 3(b), the manifolds (torus) with rational winding numbers are first broken and turn into island chains. As the Poincaré–Birkhoff theorem proves [22, 23], there remain only two periodic orbits on an island chain. One of them is called a ‘minimizing orbit’ and is located at the separatrices in the stable island chain. The other one is located at the centers of the stable islands, and is called a ‘minimax orbit’ [23]. Around the separatrices at the boundary of low-periods island chains, chaotic dynamics starts to be generated, as the manifolds of the separatrices form tangles of stable and unstable manifolds. On the other hand, the manifolds with irrational winding numbers are meandering, but still remain unbroken.

With increasing deformation, the chaotic regions around the separatrices get wider. At $\epsilon = 0.25$, the chaotic regions around different island chains are unified (figure 3(c)). Thereby, the manifolds with irrational winding numbers which exist between island chains are broken, and turn into a fractal structure like a Cantor set. This broken manifold with an irrational winding number is called a ‘cantorus’ [24, 25].

Through the broken structures of the manifolds such as island chains and cantori, new passing trajectories are opened. The amount of them can be measured by an action flux and can be assigned to each broken structure. In the sense that the broken structures have a definite action flux through themselves, they are called ‘partial barriers’ [23].

If the deformation parameter $\epsilon$ is increased further up to 0.43, most of phase space section’s area is covered by chaotic trajectories. At this point, one single trajectory can reach most of the phase space section by chaotic diffusion. Nevertheless, there are still regular trajectories, i.e. trajectories on unbroken tori in a tiny region on the top of the phase space section. The existence of this regular region is supported by Lazutkin’s theorem [26], which proves that a convex billiard with a smooth perturbation on the boundary always has regular trajectories near the top of the phase space section.

From the dynamical features of the Limaçon billiard system, some advantages for applications can be found. The first one is that the unbroken manifolds in a regular phase space region can form WGMs regardless of the deformation, as long as the boundary is convex. The second one is that the chaotic diffusion in the phase space guarantees directional emission. The emission of Limaçon shaped microcavities is especially unidirectional due to the one-fold reflectional symmetry. These advantageous features have already been experimentally verified, and have proven to be very robust against small fabrication discrepancies [9, 15].

In order that such dynamical features emerge, the modes should be able to resolve the structure in phase space. However, if a mode corresponding to a long wavelength is excited, in other words, if the microcavity is too small in comparison with the wavelength scale, it is not clear how the chaotic dynamics affects the optical properties of the microcavity. In the following sections, this point will be clarified based on numerical observations.

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3. Spectral and modal properties

In this section, the optical properties of a Limaçon-shaped microcavity with chaotic ray dynamics are numerically investigated. In order to induce chaotic dynamics, the deformation parameter $\varepsilon$ is set equal to 0.43 (cf figure 3(d)). The effective refractive index of the microcavity is set to be $n = 3.0$ assuming a semiconductor cavity. Because the effective refractive index is defined by a waveguide mode in the vertical direction of microdiscs, it has normally mode-dependence. However, since the expected change of effective refractive index in the spectral range that we investigate is not large enough to induce a new mode-coupling, it does not affect the physics that we will present. Hence, without loss of generality and for the sake of computational convenience, we take it as a constant. With those given conditions, the stationary electromagnetic wave equation is solved numerically, using the boundary element method [27], and the spectrum and the modes are obtained. For this study, the transverse magnetic polarized modes are considered.

Figure 4(a) presents the numerically obtained spectrum with the distribution of complex resonant wavenumbers in the ultra-small cavity regime, $kR = 2–15$, where $kR$ is the normalized frequency, $kR = R\omega/c$. The corresponding cavity $Q$ factors can be calculated by using $Q = -\text{Re}(kR)/2\text{Im}(kR)$. Consequently, the closer a wavenumber is to the real axis, the higher its $Q$ factor is. Thus, one can notice that the high-$Q$ modes have an interesting feature in their distribution: they always appear in a group with a curved series on the complex wavenumber plane. In figure 4(a), the curved mode series are denoted by different symbols and connected by dotted lines. The real parts of the modes’ wave numbers in a curved series are very regularly given, as though they have a well defined free spectral range.

More interestingly, all the high-$Q$ modes belonging to curved series have an unexpected common feature in configuration space. Figures 5(a)–(d) show the configurational distributions of resonant modes which are chosen from each mode series. All of the modes exhibit a very regular distribution with well defined mode numbers, the radial mode number $l$ and the rotational mode number $m$. The radial mode number of the modes is $l = 1$ in common, and the rotational mode numbers are (a) $m = 8$, (b) $m = 16$, (c) $m = 35$ and (d) $m = 72$. The rotational mode number $m$ increases by 1 over a free spectral range. This observation can be viewed as beyond the conventional knowledge of dynamics, because there is no dynamical invariant corresponding to the well-defined mode numbers or the regular mode distribution. More concretely speaking, since the angular momentum $L$ of ray is not a constant of motion in the deformed cavity, the rotational mode number $m$ cannot be a good mode number in conventional systems. Also, the wavenumbers of the modes are too small to resolve the regular tori which are supported by Lazutkin’s theorem.

In the phase space distribution functions, we can find another noticeable feature which is able to characterize each mode series. To look into the mode features on the phase space section, the Husimi distribution function on the Birkhoff coordinates is employed. The Husimi distribution can be derived by calculating the overlap between a Gaussian wave packet with a varying wave number and a resonant mode on the boundary [29]:

$$H(s, \chi) = \sum_{j=-\infty}^{\infty} \int_{S_0} E(s')e^{i\chi s'} \sin(\chi - s - 2j\pi)^2/4\sigma^2 ds',$$

where $E(s)$ is the field value of a chosen resonant mode on the cavity boundary, $\sigma$ is the width of a basis Gaussian wave packet and $S_0$ is the perimeter of the boundary. In order to keep the
Figure 4. (a) Distribution of resonant wavenumbers in the complex plane in the ultrasmall regime \(2 < kR < 15\). It is noticeable that the local maximum of high-\(Q\) factors are reached by curved modes which are denoted by colored symbols. Each mode curve has common features in both configuration and phase space. (b) The same distribution in a broader spectral regime \(1.5 < kR < 30\). The curved mode series appear in a higher spectral regime, as well. The appearing and disappearing points for each curve are in good agreement with the semiclassical predictions of equations (9) and (13), which are denoted by arrows with \(N_n\) and \(E_n\). One mode from each curve is picked out (denoted by surrounding circles), and its configuration and phase space distribution are shown in figure 5. (c) Distribution of \(U\) which measures the degree of unidirectional emission. It shows the opposite tendency to the \(Q\) factors.

ratio of the basis Gaussian wave packet on the phase space section, \(\sigma\) is given as a function depending on \(nkR\), \(\sigma = \sqrt{2}/nkR\).

From the Husimi distributions of the high-\(Q\) modes, common localization patterns can be found in each mode series. The Husimi distributions shown in figures 5(e)–(h) for the resonant modes presented in figures 5(a)–(d) show (a) three, (b) four (c) five, and (d) six localizations, respectively. The localization positions on the phase space sections are characteristic in each curved mode series, i.e. all the modes in one series have a common localization pattern on the phase space section, and this pattern is distinct from those of other mode series.

The most probable interpretation for this observation might be the ‘scarred mode’, which has a localized amplitude profile on a certain unstable periodic orbit in the chaotic classical counterpart [19]. The periodic orbits which supposedly correspond to the modes present in

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Figure 5. (a)–(d) Configurational and (e)–(h) Husimi distributions of the modes denoted by circles in figure 4(b). Each distribution shows the common features of each curved mode series in configuration and phase space. On the configurational distributions, (a) period 3, (b) period 4, (c) period 5 and (d) period 6 minimizing (dotted line) and minimax (solid line) SPOs are superimposed. The internal segments of these periodic orbits seemingly limit the internal side of the amplitude distributions of the corresponding modes like a caustic in a circular cavity in figure 2(b). Also, they show very well defined mode numbers. All the modes have $l = 1$ as radial mode number, and (a) $m = 8$, (b) $m = 16$, (c) $m = 35$ and (d) $m = 72$ as rotational mode numbers. On the phase space section, a Husimi distribution of each mode has the same number of localizations with the period of the corresponding SPOs, e.g. there are (a) three, (b) four, (c) five and (d) six localizations. The fixed points of the SPOs are marked by ⊙ (minimax orbits) and × (minimizing orbits).

Figure 5 are short simply closed periodic orbits with $q = 1$ (SPO, hereafter), which has a closed polygonal geometry in configuration space. To test the scarring property of the modes precisely, the pairs of the SPOs are computed and the fixed points of them on the phase space section are superimposed on the Husimi distributions. As figures 5(e)–(h) show, the localizations do not coincide with the fixed points of any periodic orbit, but they are shifted up from the minimax orbits.

In previous studies [15, 30], attempts were undertaken to explain an upward shift of the localization similar to that in figure 5 by an effective single ray description using the Goos–Hänchen effect in the reflection of a beam at a dielectric interface. However, in this approach, the correspondence of the wave mechanical modes and the classical manifolds that we obtain from deformation of the cavity boundary remains unclear.

Consistently with the phase space distribution, the configurational distribution of each mode does not follow the SPOs, either. Instead, they are located between the physical boundary of the cavity and the internal segments of the SPOs. Hence, the SPOs seemingly confine the internal ends of the mode distributions like caustics in a circular billiard. The high-$Q$ modes
with such features emerge also in the spectral regime higher than the ultra-small regime, even though the curvature of the mode series is smeared in the crowd of other modes (figure 4(b)).

The radiational emissions from the high-\(Q\) modes show another interesting correlation with the \(Q\) factors. To quantify the emission directionality, we introduce the measure

\[
U = \int_0^{2\pi} \cos \theta f(\theta) \, d\theta,
\]

where \(f(\theta)\) is the normalized farfield distribution of a given mode \((\int_0^{2\pi} f(\theta) \, d\theta = 1)\). Because the conventional measure for a directional emission [31] is not appropriate for the unidirectionality of our system, we customize a new measure for this property by equation (5). To measure how much the emission directionality deviates from unidirectionality, \(\cos \theta\) is inserted in the integration as a window function.

From the numerical result of this measure for each mode (figure 4(c)), it can be noticed that the high-\(Q\) modes have a lower measure than the other modes. The emission of the high-\(Q\) modes is not unidirectional, but bidirectional at angles 0 and \(\pi\). Due to the window function, \(\cos \theta\), the measure introduced in equation (5) is useful for a bidirectional emission as well [15].

As mentioned in section 2, the unidirectional emission is a feature resulted from chaotic diffusion. Hence, the bidirectional emission of high-\(Q\) modes implies that the chaotic diffusion does not work as a channel of emission. From this observation along with the regular distribution of the modes, we can deduce that there is some ray dynamical structure, which plays a role to form a mode. To verify this deduction, we will investigate the correspondence between the high-\(Q\) modes and ray dynamical manifolds in the following sections.

4. Evolution of modes with deformation

In order to identify ray dynamical structures corresponding to high-\(Q\) modes, the evolution process of a resonant mode through the deformation of the cavity’s boundary is investigated. First of all, the high-\(Q\) modes in a deformed microcavity are brought back to a circular cavity. Then, the modes reach the WGMs with the same mode numbers in the circular cavity. Since the circular cavity is rotationally symmetric and integrable, a single classical manifold, i.e. a single horizontal line in figure 3(a), is specifically determined as a corresponding ray motion for the resonant mode. The manifold and the mode are connected by the following equation [6]:

\[
\sin \chi = \frac{m}{nkR},
\]

where \(\chi\) is the incident angle and \(m\) is the rotational mode number. As the WGM is the semiclassical quantization of the manifold, the resonant wavenumber \(nkR\) can be also represented by the following quantization condition on the phase space volume enclosed by the corresponding manifold [32]

\[
\text{Re}(nkR) = \frac{\pi/4 + \alpha_p}{\sqrt{1 - X_p^2 - X_p \cos^{-1} X_p}},
\]

where \(X_p = 1 - R_p\), \(R_p\) is the ratio between the area above the specified manifold on the phase space section and the total area of the phase space section, and \(\alpha_p\) is the phase shift from Fresnel’s factor.

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Although equation (7) is based on the semiclassical approximation, this is even valid for the ultrasmall spectral regime (as in figure 4(a)). Because the effective Planck’s constant, $\hbar$ is equivalent to $1/\text{nk} R$ in a microcavity, and the error boundary of the semiclassical quantization is normally given by $\mathcal{O}(h^2)$, the maximal error in the spectral regime under investigation is less than 2%.

The connection between the mode and the classical manifold also appears in the Husimi distribution. By putting $E(s') \propto e^{-i m s' R}$ in equation (4), the Husimi distribution for a mode in a circular cavity is derived as follows:

$$H(s, \sin \chi) = \sqrt{\frac{\sigma}{2\pi}} e^{i(nk \sin \chi - \frac{m}{\pi})} e^{-\sigma (nk \sin \chi - \frac{m}{\pi})^2}. \quad (8)$$

As the given wave function has one directional rotation along the boundary which can be viewed as a translational motion, the computed Husimi function in Birkhoff coordinates has the localization on the corresponding ray-dynamical manifold, as figure 6(e) shows.

Using the Husimi distribution and the corresponding classical manifold, the evolution of a mode and its ray-dynamical counterpart can be compared in phase space throughout the deformation of the cavity boundary. With the mode and the corresponding classical manifold in figures 6(a) and (e) as the initial condition, their evolutions are simultaneously traced.

At $\varepsilon = 0.2$, the manifold remains unbroken, therefore the manifold is able to sustain the evolved regular mode adiabatically (figure 6(b)) [33]. Accordingly, the classical trajectory belonging to the manifold still shows a smooth caustic line in configuration space (upper part of figure 6(b)), and the corresponding mode fits this caustic line with the regular distribution.
As the winding number of the manifold \( q/p \sim 0.169 \) lies between that of the period 5 and the period 6 orbit, the corresponding manifold has five wiggles and the mode shows five localizations (figure 6(f)).

The adiabatic feature of the mode can be also understood with the effective radial potential in figure 2(a). As the rotational symmetry is broken by the deformation, the effective radial potential is not constant during the rotation anymore, but supposed to be modulated [34]. This modulation of course is caused by the nonlinear coupling between the radial motion and the rotational motion, and the period of the modulation is the same as that of the rotational motion. The localizations in the Husimi distributions can be also understood with this modulation. Since the ray dynamics for all high-\( Q \) modes have a radial motion faster than its rotational motion, the system can be adiabatically analyzed. Especially when the modulation is small enough, the system can be addressed by the first-order adiabatic approximation, i.e. the radial mode number does not change during the rotational motion. The regular distribution of the mode exhibits this adiabatic property, explicitly.

As the deformation is increased further up to 0.25, the caustic line of the classical trajectory gets broken and the density of the trajectory in configuration space gets more uneven (figure 6(c)). The broken feature of the caustic appears as a fractal structure of the manifold in phase space (figure 6(g)). Although the manifold is broken, the mode still keeps an adiabatic feature such as the regular configurational distribution (figure 6(c)). Considering that the manifold is just slightly broken and the action flux through it is accordingly very tiny, such an adiabatic feature is understandable. Such modes quantized by partial barriers have been already observed in other closed chaotic systems [28]. As the manifold’s wiggling gets larger, the localizations in the corresponding Husimi distribution become more evident (figure 6(g)).

When the cavity boundary is deformed to \( \varepsilon = 0.43 \), both the manifold in phase space and the caustic line in configuration space are completely broken (figures 6(d) and (h)). Such dynamical features are signatures that the system is far beyond the adiabatic regime. Nevertheless, the corresponding mode still maintains a regularity with well defined mode numbers, and the Husimi distribution is still localized on the broken manifold, as though the broken manifold serve as an adiabatic invariant. If the radial momentum of the mode is represented in the Husimi distribution, the confining feature of the cantori can be shown, due to the radial oscillation given by the effective potential in figure 2(a) [35].

Another interesting factor can be noticed in the relationship between cantori and SPOs. As the dynamical status on the phase space gets transiently chaotic, the cantorus in figure 6(h) converges to the upper homoclinic orbit of the period 5 SPO, which is located at the crossings of chaotic manifolds of the periodic orbit [36]. Not only this particular cantorus, but also other cantori belonging to the same curved mode series in figure 4 show the same convergence. As figures 7(a) and (b) show, three manifolds which form \( m = 18, 24 \) and 32 in a circular cavity converge to the upper homoclinic orbit of the period 5 SPO with chaotic transition. Therefore, the location of their cantori and Husimi distributions have a close contact with the period 5 SPO. Thereby, the period 5 orbit can have a strong influence on these mode (figures 6, 7(a) and (b)).

For this adiabatic formation of a mode around cantori, the openness of the microcavity plays an important role by suppressing dynamical coupling. When a closed dynamical system takes a chaotic transition, the quantum mechanical counterpart exhibits a complicated spectral dynamics including avoided crossings. Behind the avoided crossings, there are mode couplings carried by chaotic dynamics [37]. However, if the same system is opened and experiences the same chaotic transition, the situation is substantially changed, because the openness brings
Figure 7. Breakup of manifolds in phase space: (a) three manifolds which form high-$Q$ WGMs with mode numbers $(18,1)$ (black), $(24,1)$ (green) and $(32,1)$ (yellow) by action quantization. With deformation, all the three modes evolve to modes in the green curve in figure 4. (b) With deformation, the three manifolds get broken and form cantori. When the deformation $\varepsilon$ reaches 0.43 and strong chaos is generated, the cantori converge to the upper homoclinic orbit of the period 5 orbit, which is located at the crossing of the stable (blue) and the unstable (red) manifolds of the period 5 orbits. (c) Evolution of modes with small imaginary wave numbers. The overall evolution shows only a few avoided crossings. In particular, the mode in figure 6 (red thick curve) undergoes no avoided crossing, which implies that this mode has no strong interaction with other modes.

about a time scale competition between the coupling and the relative attenuation through the open channel. When the attenuation is faster than the coupling, the two modes can behave as two independent modes, exhibiting less or no avoided crossings in the transition process. This relation is equivalent to the criterion for the strong coupling, derived from a $2 \times 2$ non-Hermitian matrix model, $|\Delta| > |\text{Im}(E_1) - \text{Im}(E_2)|$, where $\Delta$ is the coupling between two modes and $\text{Im}(E_i)$ is the attenuation rate of each mode $[38, 39]$. Figure 6 presents a good example of chaotic transition in an open system. In the evolution process of figures 6(a)–(d), the mode does not experience coupling with other modes. Correspondingly, the spectrum does not show avoided crossings, either (figure 7(c)).

As noticed, a microcavity is an open and leaky system. Even if its shape is circular, attenuation through evanescent tunneling exists. Therefore, the dynamical coupling between two different modes can be suppressed by the attenuation of the modes. Within the limited mode lifetimes, the modes can effectively exist as two non-interacting modes, and correspondingly remain adiabatically around the broken classical manifold with regular distribution [40].

5. Semiclassical analysis of the spectrum

In the previous section, we have shown that high-$Q$ modes lie on cantori, as though the cantori plays a role of adiabatic invariants. As the cantori converge to homoclinic orbits of SPOs with chaotic transition, the cantori are connected very closely to a SPO. Since the SPO has much
Figure 8. (a) Description of how to determine the action flux through the period 4 SPO. A line segment connecting two consecutive fixed points (red dot-dash line) is iterated forward and backward to the predetermined principal gap (marked by two black triangles). Then, the Turnstiles (marked by $W_4$) are formed by the forward (black solid line) and the backward image (green dotted line). By connecting all iterated images, the whole phase can be divided into an upper part (gray) and a lower part (white). (b) Turnstile and partial barrier structures of period ($p$) 3, 4, 5 and 6 SPOs. The upper parts of each periodic orbit are denoted by $A_p$. (c) Upper part area of each SPO ($A_p$). (d) Turnstile area of each SPO ($W_p$).

Figure 8(a) shows how to quantify the action flux of a period 4 SPO, schematically. First, an initial condition is set with a line segment which connects the two minimizing fixed points (1 and 4 in figure 8(a)) through one minimax fixed point (1$'$ in figure 8(a)). This initial condition is iterated forward and backward until it reaches the principal gap which is marked by two triangles in figure 8(a). Then, a line connecting all the images of the initial condition (black lines in figures 8(a) and (b)) divides the phase space into two parts, the upper and the lower part. As the connected line is very closely situated to an adiabatic curve including the period 4 SPO [41], the partial barrier of the period 4 SPO can be defined with this line. Meanwhile, at the principal gap, there are two areas enclosed by the backward and forward iterated images. These two areas are called Turnstiles [23, 42] and their area quantifies the action fluxes of the periodic larger action transport than the cantori as will be shown at the end of this section, one can assume that the chaotic diffusion of a high-$Q$ mode can be limited by action transport of SPOs. In this section, we will thus partition the phase space section with respect to SPOs, and attempt to analyze the spectral range for high-$Q$ mode series by applying semiclassical quantization to the partial barriers of SPOs. The obtained result will be verified by comparison with numerical results. Then, using action farey tree, we will tackle the overall distribution of action flux in chaotic phase space, and verify the locally dominant chaotic diffusion of SPOs and the convergence of cantori to homoclinic orbits of SPOs.
orbits for one period. The area of a turnstile, i.e. the action flux is assigned to each SPO, as a characteristic quantity independent of the choice of the initial condition. For other SPOs, the partial barriers and the turnstiles are defined in the same way, and the upper part of the partial barriers and the turnstile areas are referred to as $A_p s$ and $W_p s$, respectively, where $p$ means the radial period of each SPO.

By numerical calculation, we already observed that the high-$Q$ modes are formed on a cantorus. As a cantorus touches a SPO below it, the $A_p s$ can be regarded as the maximum area which confines the high-$Q$ modes in a mode series. With the assumption that the semiclassical quantization for the cylindrical cavity is still valid for the partial barriers (equation (7)), the following condition for the minimum wavelength to form a mode in $A_p$ can be derived:

$$\text{Re}(k R) > \frac{1}{n} \frac{\pi/4 + \alpha_p}{\sqrt{1 - X_p^2 - X_p \cos^{-1} X_p}}.$$  \hfill (9)

From this $k R$ value in the spectrum, one curved mode series under the influence of one SPO is supposed to arise.

In a similar way, the spectral point where one mode series disappears can be determined by the minimum wavenumber to resolve a turnstile area. For this purpose, we use the relationship between the turnstile area and the action difference between the minimizing and minimax orbits. In Hamiltonian chaos theory, it is well known that the area of a turnstile ($W_p$) is equal to the action difference of the paired minimizing and minimax orbits on an island chain \cite{23}. Since the action of a periodic orbit in a billiard system is the same as the geometrical length of the trajectory, the turnstile area is equal to the length difference of the paired trajectories. To take the pair of period 4 SPOs as an example again, the area of the turnstile ($W_4$ in figure 8(a)) is the same as the difference in lengths between the square shaped and the diamond shaped orbits in configuration space (figure 9). Using the fact that the action in the ray dynamics corresponds to the optical length in wave mechanics, the minimum wavenumber to resolve a turnstile area can be formulated as $k \Delta L_p = kW_p = 1$. From this formula, one can also notice that the propagation of a wave function along the two periodic orbits would constructively interfere after one round trip, when the turnstile is resolvable.

However, the phase shift in association with the Maslov index must be taken into account in addition to the above analysis. In semiclassical physics, it is known that a $\pi/2$ phase shift is associated with the Maslov index. The Maslov index is defined by number of conjugate points, and the conjugate points on a given trajectory can be visualized by ray tracing. For the ray tracing, the initial conditions are set as a small bundle of rays distributed around the periodic orbits and iterated by one period. The calculated ray trajectories are superimposed on the two period 4 orbits in figure 9. As the figure shows, some focal points are formed inside the cavity, and each focal point corresponds to one conjugate point. Then, one interesting difference from the two period 4 orbits can be noticed, which is that the diamond shaped period 4 orbit has three conjugate points inside the cavity, while the square shaped orbit has four. This is due to the small curvature on the left-hand side of the Limaçon billiard boundary in figure 9. After the rays’ bouncing on this side, they cannot be focused until the next bounce. To avoid unnecessary ambiguities in the ray-dynamical calculation, the bundle of distributed rays are chosen on the unstable manifold of the period 4 orbits, and the result of the conjugate point is confirmed by a stability matrix calculation.

In the case that a mode with resonant wavenumber $k R$ is excited, the optical lengths of the square shaped ($S_{41}$) and the diamond shaped period 4 orbit ($S_{42}$) can be written with the
Figure 9. Ray tracing around (a) the minimizing and (b) the minimax period 4 SPOs. Conjugate points (focal points of rays, denoted by red circles on the figures) are observed (a) four times on the minimizing orbit (square shaped) and (b) three times on the minimax orbit (diamond shaped).

geometrical lengths of the orbits ($L_{41}$ and $L_{42}$), considering the phase shifts on the conjugate points, as follows:

\[ S_{41} = L_{41} + (4\pi + \alpha_{41})/2n \Re(kR), \]  
\[ S_{42} = L_{42} + (3\pi + \alpha_{42})/2n \Re(kR). \]  

In equations (10) and (11), the phase shift by reflections on the boundary are taken into account by including $\alpha_{41}$ and $\alpha_{42}$, which can be assigned by Fresnel’s factor. Since the geometrical length of a minimax periodic orbit ($L_{41}$) is shorter than that of a minimizing periodic orbit ($L_{42}$) in a convex billiard, the optical length difference of the two periodic orbits ($\Delta S_4 = S_{42} - S_{41}$) can be expressed by the turnstile area $W_4$ ($= \Delta L_4 = L_{42} - L_{41}$):

\[ \Delta S_4 = \Delta L_4 - \pi/2n \Re(kR) = W_4 - \pi/2n \Re(kR), \]  

where $\Delta \alpha_4$ ($= \alpha_{42} - \alpha_{41}$) is neglected, because the fixed points of the two period 4 orbits have almost same average values of reflection angles ($\chi$) (figures 8(a) and (b)). From equation (12), we can see a striking effect caused by conjugate points. When a mode with a shorter wavelength than the length difference, $\Delta L_4$, is excited, the mode sees those two orbits as though the diamond shaped one is optically longer than the square shaped. If the wavenumber is further increased in the spectrum, a mode can resolve the turnstile area for the first time, when the two optical lengths of the two periodic orbits become equal. At this point, the geometrical length difference of the two period 4 trajectories is the quarter wavelength of the mode. To bring this situation to quantum mechanics, the turnstile area at this point corresponds to one-quarter Planck’s constant, $\hbar/4$. Considering that the conjugate points are the configurational projections of the twists on manifolds from a phase space [43], we can see that the manifolds of the diamond shaped period 4 orbit have a Möbius-like topology, whereas the other period 4 orbit has a normal connectivity. This topological distinction leads to the resolution of a quarter wavelength or a quarter Planck’s constant scale, as shown above.
Figure 10. Flux Farey trees of ray dynamics in (a) $\varepsilon = 0.2$ and (b) $\varepsilon = 0.43$ Limaçon-shaped microcavities. The $X$-axis corresponds to a winding number of a periodic orbit $(q/p)$ and the $Y$-axis corresponds to the action flux of a periodic orbit. Two branches which asymptotically approach to $1/3$ (denoted by arrows) are converging to the action fluxes of (left) the upper and (right) lower homoclinic orbits of the period 3 SPO.

As the Maslov indices differ by one between minimizing and minimax periodic orbits for all other SPOs, the condition to resolve the turnstile area of a $p$-period orbit can be generalized:

$$\text{Re}(k R) \leq \frac{\pi}{2n W_p}.$$  \hspace{1cm} (13)

As mentioned, the minimum wavenumber indicated by equation (13) corresponds to the longest wavelength which can induce the constructive interference of two $p$-period orbits by one roundtrip.

The conditions of equations (9) and (13) for a $p$-period periodic orbit are denoted by black arrows with $N_p$ and red arrows with $E_p$, respectively, in figure 4(b). As the figure shows, the two conditions of each periodic orbit point out the emerging and disappearing points of each curved mode series on the spectrum in perfect agreement with the numerical data.

The reason why SPOs play such a decisive role in determining the spectral and modal properties can be found in the flux Farey tree [23, 42]. The Farey tree is originally a number theoretical binary tree expansion based on a continued fraction, which can asymptotically expand all rational and irrational numbers in a given range. If all periodic orbits with winding numbers of co-prime numbers on the Farey tree are found and the associated action fluxes are calculated in a dynamical system, a ‘flux Farey tree’ can be constructed, and this provides comprehensive information about the system.

Figure 10 shows two flux Farey trees, one for the $\varepsilon = 0.2$ Limaçon billiard dynamics, and the other one for $\varepsilon = 0.43$. The connectivity of the tree is inherited from the number theoretical trees. The $X$ and $Y$ coordinate of each node are, respectively, a co-prime fractional number expanded by the Farey tree and the action flux of an orbit with the $X$ coordinate as its winding number. In a convex billiard, the winding numbers of periodic orbits $(q/p)$ are monotonically decreasing with average $\sin \chi$ values of their fixed points on the phase space section. Therefore, two nodes on the Farey tree with a small difference in their winding numbers are very closely located on the phase space section. The action fluxes of SPOs are encoded on the leftmost branch as denoted in figure 10. Action fluxes of periodic orbits with irrational winding...
numbers can be also extracted from the action Farey tree. As shown in figure 10, many branches are extended from a node, and every irrational number has a corresponding branch. By following the limiting value of such a branch, the action flux of a periodic orbit with irrational winding number can be found. Among branches of a flux farey tree, there are also two approaching branches for every rational number, for instance the branches approaching to $1/3$ denoted by arrows in figure 10(b). The limiting values of such branches corresponds to upper (approaching from left) and lower (from right) homoclinic orbits of a SPO.

As presented in figure 3, the ray dynamics in a $\epsilon = 0.2$ Limaçon billiard has a very regular feature on the phase space section, which means that many tori in phase space are still unbroken. Even if they are broken, the action fluxes through them are tiny. Correspondingly, the flux Farey tree exhibits many branches converging rapidly to 0 (figure 10(a)).

In contrast, all branches stemming from a SPO in the flux Farey tree for $\epsilon = 0.43$ are flattened and their limiting values converge to the action fluxes of homoclinic orbits of SPOs. This feature is consistent with the convergence of cantori to homoclinic orbits in phase space, which we noted in figures 7(a) and (b). Because the chaotic dynamics is generated through a resonance overlap in phase space [44], it can be reasoned that the resonance areas of two consecutive SPOs are almost adjacent in a strongly chaotic regime. Therefore, the lower homoclinic orbit of the higher periodic SPO and the upper homoclinic orbit of the lower periodic SPO are very close to each other. For instance, the lower homoclinic orbit of the period 3 orbit and the upper homoclinic orbit of the period 4 orbit are very close at $\epsilon = 0.43$. Consequently, cantori which have irrational winding numbers between $1/3$ and $1/4$ are very densely distributed between those two homoclinic orbits.

From the features of the flux Farey tree given above, we can draw conclusions about the formation of the high-$Q$ modes in a Limaçon microcavity: all the SPOs lie on the leftmost branch, and their action fluxes are larger than those of other branches expanded from them. If one mode is located on a cantorus and the resolution of it does not reach the action flux of the nearest SPO on its right-hand side on the flux Farey tree, this SPO can constrain the spreading or chaotic diffusion of the mode and lead to the adiabatic formation of the mode. As all cantori are densely located around the homoclinic orbits of SPOs, high-$Q$ modes confined by homoclinic orbits of a common SPO are supposed to be under the strong influence of the SPO, such as the localization pattern on the phase space section and the spectral condition for the mode formation.

6. Conclusion

In this paper, the characteristics of high-$Q$ modes in a deformed microcavity were studied in accordance with a chaotic internal ray dynamics. As a model system, we chose the Limaçon-shaped microcavity, which is known to have many valuable features in applications and to follow the generic KAM scenario in its chaotic transition.

In the ultra-small regime with respect to the free-space wavelength, high-$Q$ modes in the Limaçon microcavity appear with curved mode series on the complex wavenumber space, and all modes belonging to one mode series have common features in both configuration and phase space. The common feature in configuration space is the regular formation with well defined mode numbers despite strong chaos, and that in the phase space is the common localization structure in their Husimi distributions. At the same time, the high-$Q$ modes exhibit worse emission directionality, while other modes have unidirectional radiational emissions through...
chaotic diffusion. By comparing the wave mechanical result with the corresponding classical
dynamics, it was observed that the high-$Q$ modes have adiabatic properties, which feature the
well-defined mode numbers in the configurational mode distribution and the localization on a
cantorus in the phase space distribution. Therefore, the modes seem to have the corresponding
cantori as adiabatic invariants. This observation is striking and beyond the conventional
knowledge of quantum chaos, because the classical dynamics corresponding to the modes is
strongly chaotic and thus beyond the adiabatic regime.

The origin of such adiabatic properties can be found in the openness of the system. Since
a microcavity is an open system, modes in the cavity have intrinsic attenuations. If two modes
have the larger difference in their attenuation rates than their coupling, the strong coupling is
suppressed, and consequently these two modes can behave like two independent modes. As
the corresponding manifolds in ray dynamics have normally very small action flux, the modes
cannot resolve the flux within their mode lifetime, and remain around it adiabatically.

When the internal ray dynamics of a microcavity becomes chaotic, the cantori which
confine modes converge to homoclinic orbits of the SPOs. This convergence of cantori to
homoclinic orbits causes the common pattern in the Husimi distributions of modes which are
located between two consecutive SPOs. Also, the spectral conditions for high-$Q$ modes can be
derived on the basis of this fact. Because homoclinic orbits have very close contacts to SPOs and
the SPOs have relatively larger amount of action fluxes than other partial barriers, it was first
assumed that they play a limiting role to suppress the chaotic diffusion of modes. More precisely,
when an SPO has a smaller action flux than the resolvability of a mode slightly above the SPO,
this mode can be confined by a partial barrier around the SPO. Based on this assumption, the
phase space section was partitioned by the partial barriers of the short periodic orbits and the
action fluxes through them were quantified. From the action flux, the spectral range for high-$Q$
modes was theoretically derived, and good agreement with the numerically observed spectrum
was found.

In the formulation of the condition for the high-$Q$ modes, we additionally found that
conjugate points influence the resolution of the action flux through the associated phase shifts.
First, it was observed that the minimizing and minimax orbits of an SPO have a difference of one
in the number of their conjugate points, i.e. the Maslov indices, which implies that these two
periodic orbits with the same period have different topological structures on their manifolds,
one with a normal connectivity and the other with a Möbius-strip connectivity. Noting that
the resolution of the action flux of a periodic orbit on the phase space section implies the
constructive interference of two paired periodic orbits (minimizing and minimax orbits) by
one period, this topological distinction results in a discrepancy between the optical and the
geometrical length of periodic orbits and affects the resolution of the action flux. Consequently,
an optical mode can resolve a quarter wavelength scaled action flux on a chaotic phase space
section. Quantum mechanically speaking, this phenomenon corresponds to the quarter Planck
constant scale resolution in chaotic phase space.

The investigation of the overall distribution of the action fluxes was also performed through
the flux Farey tree. This investigation obtained some convincing clues for the reason why the
SPOs play a limiting role in mode confinement. The first one is that the SPOs have a local
maximum of the action flux. Thus, if one mode has less resolution than the action flux of an SPO,
all other periodic orbits or cantori branched out of the SPO can certainly block the diffusion of
the mode. The second one is that the flux Farey tree shows explicitly the relationship between
the SPO and the cantori, that is, all cantori with co-prime winding numbers converge closely to
the homoclinic orbits of the SPOs. Hence, they have close contacts in a phase space section and a strong correlation between them is expected.

Using the findings of this paper, the controllability and the predictability of the optical properties of a deformed microcavity can be improved. For instance, by choosing a mode at the boundary of a high-$Q$ mode range, the $Q$ factor and the emission directionality can be optimized. By providing a connection between a mode and a ray trajectory, this work can serve a ray dynamical basis, on which a subtle optical property of a deformed microcavity such as the sensitive dependence of emission directionality on boundary deformation [45], can be analyzed. As the theoretical analysis presented in this work deals with KAM systems, we expect that the methodology developed in this paper can be generalized to other open quantum or wave mechanical systems with underlying classical chaotic dynamics.

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