Generation for Lagrangian cobordisms in Weinstein manifolds

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Abstract

We prove that Lagrangian cocores and Lagrangian linking disks of a stopped Weinstein manifold generate the Lagrangian cobordism ∞-category. This is another part of a program attempting to show that Lagrangian cobordisms are a stable-homotopy-theory lift of partially wrapped Fukaya categories in the exact setting. Moreover, we prove that the homotopy category of the Lagrangian cobordism ∞-category of a Liouville domain is a module over the homotopy category of the corresponding category for a point—together with the main result, this gives evidence of an enrichment conjecture from [NT11]: The spectrum associated to a point is a coefficient ring spectrum over which the theory of Lagrangian cobordisms is linear.

1 Introduction

Let $M$ be Liouville manifold—that is, an exact symplectic manifold whose Liouville flow induces a conical end outside some compact set. Fix also some subset $\Lambda \subset M$.

To this data, one can associate an ∞-category $\text{Lag}_\Lambda (M)$ whose objects are certain exact Lagrangian branes in $M \times T^*\mathbb{R}^n$ for $n \geq 0$, and whose morphisms are Lagrangian cobordisms satisfying a non-characteristic condition with respect to $\Lambda$. (See Definition 2.11.)

Notation 1.1. One can choose different brane structures to decorate Lagrangians, and each choice of brane convention results in a different ∞-category of Lagrangian cobordisms. When relevant, we denote a choice of
brane convention by $\mathcal{B}$ and we indicate the dependence by $\text{Lag}_\Lambda^\mathcal{B}(M)$. When the choice of $\mathcal{B}$ is immaterial, we will write $\text{Lag}_\Lambda(M)$. A typical choice of $\mathcal{B}$ is to demand that every Lagrangian in sight is equipped with a primitive vanishing at infinity, a relative Pin structure, and a grading. At the bare minimum, in this work, we demand that every Lagrangian in sight is equipped with a primitive which vanishes at infinity. (In particular, every brane is conical at infinity.)

Also recall that if $Z$ is the Liouville vector field (pointing outward along the conical end of $M$) and if its backward flow is complete (which we assume to be the case), one can define the core, or skeleton of $M$ to be the set

$$\text{core}(M) := \bigcap_{t > 0} \text{Flow}^{-t}_Z(M^o)$$

where $M^o$ is the (compact) complement of a conical end. Equivalently, $\text{core}(M)$ is the set of points that do not escape $M^o$ under the flow of $Z$.

In this work, we will be dealing with the special case that $Z$ is gradient-like for a Morse function; we will call such an $M$ Weinstein. (We caution that the definition of Weinstein tends to vary from work to work in the literature, even in works by the same author.)

**Remark 1.2.** This main difference between the Weinstein and the general Liouville case is that Weinstein manifolds, like ordinary manifolds, can be built out of (symplectic) handle attachments. As a result, Weinstein manifolds also have natural Lagrangians submanifolds given as cocores of the attaching handles of index $n$ (where $\dim\mathbb{R} M = 2n$). It is more difficult to find Lagrangians inside arbitrary Liouville manifolds.

We first prove the following:

**Theorem 1.** Assume $Z$ is gradient-like for a Morse function. Let $\{D\}$ denote the collection of Lagrangian cocores, and let $\{D^\alpha\}$ denote the collection of Lagrangian cocores equipped with brane structures. Then for $\Lambda = \text{core}(M)$, the collection $\{D^\alpha\}$ generates $\text{Lag}_\Lambda(M)$ as a stable $\infty$-category.

More generally, choose a stop $f \subset \partial_\infty M$. Let $\{D\}$ denote the collection of Lagrangian cocores and Lagrangian linking disks of the critical components of $f$. Then for $\Lambda = \text{core}(M) \cup \bigcup_t \text{Flow}^{-t}_Z(f)$, the collection $\{D^\alpha\}$ generates $\text{Lag}_\Lambda(M)$ as a stable $\infty$-category.
Remark 1.3. Because $\text{Lag}_\Lambda(M)$ in general is not known to be idempotent-closed, the generation statement is stronger than split-generation.

Recall that every stable $\infty$-category has a shift operation $L \mapsto L[1]$, where $L[1]$ is defined as the cofiber of the zero map $L \to 0$. As a corollary of Theorem 1, we find:

**Theorem 2.** Let $M = \Lambda = pt$ be a point, and let $L = pt$ be the unique non-empty Lagrangian submanifold. Fix a choice of brane convention $\mathcal{B}$ for which the set of brane structures on $L$ is acted upon transitively by the shift operation. Then $\text{Lag}_{\text{pt}}^\mathcal{B}(pt)$ is generated by a single object: the point.

Let $\mathcal{L}^\mathcal{B} = \text{End}(pt)$ denote its ring of endomorphisms. Then $\text{Lag}_{\text{pt}}^\mathcal{B}(pt)$ is equivalent to a full subcategory of modules over $\mathcal{L}^\mathcal{B}$, consisting of those modules that can be presented as finite extensions of (shifts of) $\mathcal{L}^\mathcal{B}$.

(Note that by definition, $\text{Lag}_{\text{pt}}^\mathcal{B}(pt)$ has objects branes in $T^*\mathbb{R}^N$ for any $N$, so the theorem is a non-trivial statement about non-compact, exact branes in Euclidean space: Any brane can be constructed via iterated mapping cones beginning with Lagrangian cobordisms between cotangent fibers.)

**Proof.** The point is the unique cocore of $M = pt$. By hypothesis on $\mathcal{B}$, any brane structure on the point can be obtained from any other by shifts; so any brane structure on any cocore can be obtained from a single object. Moreover, these cocores generate $\text{Lag}_{\text{pt}}^\mathcal{B}(pt)$ by Theorem 1, so the first claim is proven.

The second statement is immediate from the definition of generation and the fact that $\text{Lag}$ is a stable $\infty$-category. 

**Remark 1.4.** Recall that $\text{Lag}_{\Lambda}^\mathcal{B}(M)$ is a stable $\infty$-category by [NT11]. In particular, $\mathcal{L}^\mathcal{B}$ is an $A_\infty$ ring spectrum. By modules over $\mathcal{L}^\mathcal{B}$, we of course mean modules of spectra.

Finally, we prove a $\pi_0$ version of a statement we hope to enrich in later work:

**Theorem 3.** For any $M$ and $\Lambda$, let $\text{ho}\text{Lag}_{\Lambda}^\mathcal{B}(M)$ denote the homotopy category of $\text{Lag}_{\Lambda}^\mathcal{B}(M)$. Then $\text{ho}\text{Lag}_{\text{pt}}^\mathcal{B}(pt)$ is a symmetric monoidal category, $\text{ho}\text{Lag}_{\Lambda}^\mathcal{B}(M)$ is a module over $\text{ho}\text{Lag}_{\text{pt}}^\mathcal{B}(pt)$. Moreover, the action preserves exact triangles in each variable.
A higher version of Theorem 3 would imply that for any \( M \), \( \text{Lag}^B_\Lambda(M) \) is linear over \( \mathcal{L}^B \).

**Remark 1.5.** Because \( \text{Lag}^B_\Lambda(M) \) is stable, its homotopy category is naturally triangulated—exact triangles in \( \text{hoLag}^B_\Lambda(M) \) are those sequences of morphisms that arise from a co/fiber sequence in \( \text{Lag}^B_\Lambda(M) \).

**Remark 1.6** (Motivation). The first theorems are of course inspired by the Fukaya-categorical analogues. For example, the generation result of Theorem 1 is known for the wrapped Fukaya category of a Weinstein manifold or a Weinstein sector (which is equivalent to the partially wrapped case) [CDRGG17], [GPS18].

Lagrangian cobordisms represent one of three present strategies to enrich the Fukaya category over stable homotopy theory. (The other two use microlocal methods or deformation-theoretic methods.) Indeed, in joint work with David Nadler [NT11], we conjectured that \( \text{Lag}^B_\Lambda(M) \otimes \mathcal{L}^B \mathbb{Z} \) is equivalent to the partially wrapped Fukaya category of \( M \) (with objects having brane structure in class \( \mathcal{B} \)) with respect to a subset \( \Lambda \), which in this paper we think of as a (singular) Lagrangian filling of a stop \( \mathfrak{f} \) [Syl16]. Theorem 1 is further evidence of this conjecture.

Let us explain how Theorem 3 also works toward the conjecture. One ought to ask what the base ring spectrum of Lagrangian cobordism theory is, and Theorem 3 is further evidence that the whole theory is linear over \( \mathcal{L}^B = \text{End}_{\text{Lag}^B}(pt) \). In particular, a higher version of Theorem 3 would show that the expression \( \text{Lag}^B_\Lambda(M) \otimes \mathcal{L}^B \mathbb{Z} \) makes sense.

**1.1 Remarks on the proofs**

**1.1.1. Proving generation.** The proof method for Theorem 1 is a strategy we learned (for the wrapped Fukaya setting) from conversations with Shende at the KIAS Higher Categories and Mirror Symmetry conference, where it was suggested by Shende that the same proof should work for Lagrangian cobordisms; see [GPS18] for the argument in the wrapped Fukaya setting. In the present work, we need not perform any computations in the Fukaya category. As has been the trend in recent years, if the necessary lemmata can be reduced to statements with no mention of holomorphic disk computations, one expects to be able to carry out the proof in the Lagrangian cobordism setting. This is what we have done.
The proof is as follows: By design, a brane $L$ and its stabilization $L \times T^*_0\mathbb{R}$ are equivalent objects in the Lagrangian cobordism $\infty$-category. Via genericity, one can flow $L \times T^*_0\mathbb{R}$ past Lagrangian cocores in $M \times T^*\mathbb{R}$ in such a way that (i) we only introduce one intersection point at a time, and (ii) the result is contained entirely in $M \times T^*\mathbb{R}_{\tau<0}$ (i.e., where the cotangent component is negative). The first condition guarantees that we can resolve each intersection point via Polterovich surgery to obtain a mapping cone sequence [Tan18], while the latter states that the end result is a zero object in the Lagrangian cobordism $\infty$-category [NT11]. There are some details to be filled in, of course—see Section 3. The main new ingredient is that attaching exact handles at infinity does not change the equivalence class of an object in $\text{Lag}_{\Lambda}(M)$. (The corresponding statement for the partially wrapped Fukaya category was shown in [GPS18].)

**Remark 1.7** (Contrast with [CDRGG17].) The reader can find a proof of the same wrapped Fukaya category result in [CDRGG17] for Weinstein sectors (which can always be obtained from a Weinstein manifold with a chosen stop by removing a standard neighborhood of the stop); we highlight a difference in proof method for the reader.

In [CDRGG17], one tries to construct a Lagrangian $L'$ out of an initial Lagrangian $L$ such that (i) $L'$ is obtained by surgering $L$ along cocores, and (ii) $L'$ does not intersect the skeleton (hence is a zero object). In loc. cit., this strategy produces an $L'$ immersed in $M$, and the authors apply Floer theory for immersed objects by lifting to a Legendrian and passing to a Lagrangian cobordism for Legendrians. Computations utilizing an augmentation algebra show that one produces a twisted complex in the Fukaya category equivalent to $L$, constructed out of cocores.

When proving results about the Lagrangian cobordism category, Floer-theoretic techniques usually fall short of proving structural statements of this sort. In particular, the augmentation algebra is not available in the present paper, as it is yet unclear how the differentials in the Floer chain complex (and other counts of holomorphic disks) relate to the topology of the space of Lagrangian cobordisms.

The only step in which Floer-theoretic tools are crucially used in [CDRGG17], and hence the main obstruction to extending their strategy, is the immersed nature of the Lagrangian surgeries: If a cocore disk intersects $L$ in more points than along the skeleton, we do not know how to construct a Lagrangian cobordism via surgeries while staying within the embedded setting.
The main difference in our work (and in the original strategy of [GPS18]) is that we need not consider immersed objects. This is one utility of stabilization.

1.1.2. Proving the point acts. The proof of Theorem 3 is different, and utilizes some simple observations about the theory of cobordisms. To give the reader an idea of what the proof feels like: A proper proof in similar spirit would show that the sphere spectrum (modeled as an infinite loop space of framed cobordisms) admits a commutative ring structure in the ∞-category of spectra. In this paper, we do not give a proper proof, in that we only prove results at the level of homotopy categories—as a result, our arguments are quite elementary and only involve simpleton observations about isotopies in \( \mathbb{R}^N \).

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2 Preliminaries
With the exception of Lemma 2.25, every fact below is old news (i.e., has proofs in previous papers).

2.1 Geometric set-up and notation
Notation 2.1. We will often use the notation \( E = F = \mathbb{R} \). This is to disambiguate the many roles that \( \mathbb{R} \) plays for the Lagrangian cobordism category—\( E \) is a stabilization direction, while \( F \) is a time/propagation/morphism direction.

Notation 2.2. We will often write \( E^N \subset T^* E^N \) to mean the zero section.
2.1.1. Liouville manifolds and stops  We engage in some rapid-fire recollections; details can be found, for example, in [GPS17, GPS18].

A Liouville manifold \((M, \theta)\) is an exact symplectic manifold equivalent to a symplectization outside some compact set. We let \(\partial_\infty M\) denote the associated contact manifold, so that \(M \cong X^o \cup_{\partial_\infty M} \partial_\infty M \times [1, \infty)\) with \(X^o \subset X\) compact. A stop is any closed subset \(f \subset \partial_\infty M\). (Note that this definition of stop is more general than the definition originally given by Sylvan [Syl16].) The subset \(X^o\) is called a Liouville domain.

We let \(Z\) denote the Liouville vector field of \(M\), defined by \((d\theta)(Z, -) = \theta\).

If \(M\) and \(Y\) are Liouville manifolds, so is \(M \times Y\). (Note that the product of Liouville domains \(M^o \times Y^o\) requires a corner-smoothing to be realized as a Liouville domain inside the Liouville manifold \(M \times Y\).)

Finally, we say something is true at infinity if it is true inside a conical collar—equivalently, outside a sufficiently large compact set of \(M\).

Construction 2.3 (Products of stops). If \(f \subset \partial_\infty M\) and \(g \subset \partial_\infty Y\) are choices of stops, we endow \(M \times Y\) with the stop \(f \times \text{core}(Y) \cup \text{core}(M) \times g \cup f \times g \times \mathbb{R}\).

Remark 2.4. Consider Liouville domains (which are compact, with boundary) \(M^o\) and \(Y^o\), so \(M^o \times Y^o\) is a manifold with corners. In particular, \(\partial M^o \times \partial Y^o \subset \partial (M^o \times Y^o)\) admits a collar neighborhood topologically, but one must choose a smoothing to render \(\partial (M^o \times Y^o)\) smooth; this smoothing then contains a smooth copy of a collar neighborhood \((\partial M^o \times \partial Y^o) \times \mathbb{R}\). This explains the \(\mathbb{R}\) factor in the term \(f \times g \times \mathbb{R} \subset \partial_\infty M \times \partial_\infty Y \times \mathbb{R}\) in Construction 2.3.

Example 2.5. Here are the examples of interest to us.

- Let \(M\) be a Weinstein manifold, so that \(Z\) is gradient-like for some Morse function \(f\). Then \(\text{core}(M)\) is the union of the stable manifolds that ascend to \(\text{Crit}(f)\).

- Let \(E = \mathbb{R}\) and let \(T^*E\) be equipped with \(Z = 1/2(q \partial_q + p \partial_p)\). We choose a stop \(g = \{\pm \infty\} \in [-\infty, \infty]\). Or, if one thinks of the unit disk as the corresponding Liouville domain, the stop \(g\) is given by the points \(\{\pm 1\}\) in the boundary of the unit disk.

- The product \(T^*E \mathbb{N}\) can be thought of as associated to a Liouville domain diffeomorphic to \(D^{2\mathbb{N}} \subset \mathbb{C}^\mathbb{N}\), whose stop is the real equator \(S^{N-1} \subset \mathbb{R}^N \cap D^{2n}\).
• Then $M \times T^*E$ has a stop given by $\text{core}(M) \times g$.

If $M = T^*Q$ is a cotangent bundle with the usual Liouville form $\theta$, the core is the zero section.

**Remark 2.6.** Let $M$ be Weinstein, possibly with a stop $f$, and consider $T^*E$ with stop $g$ as above. As we will see, in this paper, the invariants we care about for $M$ are equivalent to the invariants we care about for $M \times T^*E$. The reader may benefit from a reminder that, when we prove something for a Weinstein manifold $M$, we will have also proven it for all of its stabilizations $M \times T^*E^N, N \geq 0$.

### 2.1.2. Lagrangians

A Lagrangian $L \subset M$ is *eventually conical* if, outside some compact set, $L$ is closed under the Liouville flow.

A *primitive* for $L$ is a smooth function $f : L \to \mathbb{R}$ such that $df = \theta|_L$. We will always demand $L$ to be equipped with a primitive $f$ such that $f$ vanishes outside a compact subset; in particular, this guarantees that $L$ is eventually conical. Given any eventually conical Lagrangian, such an $f$ can be obtained after a deformation of the Lagrangian.

A *brane* is a vague term in this paper. A brane is an eventually conical Lagrangian $L$, equipped with a primitive vanishing outside a compact set, with possibly more tangential data, such as a grading or a relative Pin structure. Our theorems are true as stated without specifying what kind of brane convention $\mathcal{B}$ we work with, so long as the Polterovich surgery construction works as in the framework of [Tan18].

### 2.2 Stable infinity-categories

**Warning 2.7** (Fiber sequences are more than just sequences). In a stable $\infty$-category, a fiber sequence is not just the data of morphisms $A \to B \to C$; a fiber sequence must come equipped with a homotopy from a composite $A \to C$ to the zero map $A \to 0 \to C$. Regardless, to save space and to evoke intuition from the theory of triangulated categories, we will often write “let $A \to B \to C$ be a fiber sequence” with the choice of null-homotopy non-explicit in the notation. A more proper convention would be: “Let

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C
\end{array}
$$

8
be a fiber sequence,” but paper is precious.

**Notation 2.8.** Let $\mathcal{C}$ be a stable $\infty$-category and fix a collection of objects $\{D\} \subset \text{Ob} \mathcal{C}$. We let $\langle \{D\} \rangle \subset \mathcal{C}$ denote the smallest full subcategory of $\mathcal{C}$ which is stable, contains 0, and contains $\{D\}$.

**Remark 2.9.** Of course, “the” smallest is a notion which only makes sense up to essential image.

For the record, we state the following straightforward result:

**Proposition 2.10.** Let $\mathcal{C}$ be a stable $\infty$-category and fix a collection of objects $\{D\} \subset \text{Ob} \mathcal{C}$. Then $X$ is in the essential image of the inclusion $\langle \{D\} \rangle \to \mathcal{C}$ if and only if there exists objects $X = X_0, \ldots, X_n \simeq 0$ such that for each $i$, there exist integers $a_i$ and objects $D_i \in \{D\}$ fitting into a fiber sequence

$$D_i[a_i] \to X_{i+1} \to X_i.$$ 

**Proof.** If $n = 1$, the statement is obvious, as the fiber sequence $D_1[a_1] \to 0 \to X_0$ implies that $X_0$ is a shift of $D_1[a_1]$, hence $X_0 \in \mathcal{D}$. The proof is finished by induction on $n$. \qed

### 2.3 The infinity-category of Lagrangian cobordisms

Fix $N \geq 0$. Let us first sketch the definition of an $\infty$-category $\text{Lag}_N^\Lambda(M)$.

An object $L \in \text{Ob} \text{Lag}_N^\Lambda(M)$ is the data of an eventually conical Lagrangian $L \subset M \times T^*E^N$, together with a brane structure which we shall not make explicit in the notation. Moreover, $L$ must satisfy the following condition: The complement of the conical ends of $L$ must be contained in a compact region of $M \times T^*E^N$ (this means that, on the core of $M \times T^*E^N$, $L$ has compact support), and $L$ must avoid the stop of $M \times T^*E^N$ at infinity.

**Definition 2.11 (Morphisms).** Given two branes $L_0, L_1 \subset M \times T^*E^N$, a morphism is a choice of eventually conical Lagrangian brane $Q \subset M \times T^*E^N \times T^*F$, satisfying the following:

1. (Collaring.) There exists $t_0 \leq t_1 \in F$ so that $Q|_{t \leq t_0} = L_0 \times (-\infty, t_0] \subset (M \times T^*E^N) \times T^*F$, where $(-\infty, t_0] \subset T^*F$ is a subset of the zero section. Likewise, we require $Q|_{t \geq t_1} = L_1 \times [t_1, \infty)$. 

2. (Λ-avoiding.) Let $\tau$ denote the cotangent coordinate of $T^*F$. When $\tau << 0$, we demand that $Q$ does not intersect $\Lambda$. Because $Q$ is eventually conical, this means that if $Q$ has any conical part where $\tau \to -\infty$, the region $E^N \times F \times \{\tau << 0\} \times \Lambda$ does not intersect $Q$. We say that such a $Q$ is $\Lambda$-avoiding, or $\Lambda$-non-characteristic.

Such a $Q$ is a morphism from $L_0$ to $L_1$, and is called a Lagrangian cobordism from $L_0$ to $L_1$.

**Example 2.12.** The identity morphism of $L$ to itself is given by $L \times F \subset (M \times T^*E^N) \times T^*F$.

Higher morphisms are defined as collared, eventually conical Lagrangians $Q \subset M \times T^*E^N \times T^*F^M$ that are also $\Lambda$-avoiding. The collection of all (higher) morphisms and objects fit together to form a simplicial set; we model $\text{Lag}_\Lambda(M)$ as a quasi-category in this way. For details, see [NT11] and [Tan16].

Importantly, one has a simplicial set $\text{Lag}_\Lambda^N(M)$ for every $N \geq 0$ consisting of objects contained in $M \times T^*E^N$ and morphisms between them; there are moreover stabilization maps

$$\diamond : \text{Lag}_\Lambda^N(M) \to \text{Lag}_\Lambda^{N+1}(M), \quad L \mapsto L \times T_0^*E$$

given by taking the direct product of a brane (or a cobordism) with the cotangent fiber at the origin of $E$. We define $\text{Lag}_\Lambda(M)$ to be the sequential colimit under these stabilization functors; in particular, in $\text{Lag}_\Lambda(M)$, an object $L$ is equivalent to its stabilization $L \times T_0^*E^N$ for any $N$.

**Notation 2.13.** Given $L \in \text{Ob} \text{Lag}_\Lambda(M)$, modeled as $L \subset M \times T^*E^N$ for some $N$, we let $L^0 = L \times T_0^*E$ be the stabilization, and we let $L^{on} = L \times T_0^*E^n$ be the $n$-fold stabilization.

**Remark 2.14.** Note that the $\Lambda$-avoiding condition is only imposed on the part of a cobordism $Q$ with negative $\tau$ coordinate, with no restrictions on the positive bit. This asymmetry is what makes $\text{Lag}_\Lambda(M)$ not an $\infty$-groupoid.

The main theorem of [NT11] is that $\text{Lag}_\Lambda(M)$ is a stable $\infty$-category, meaning there is a zero object (given by the empty Lagrangian), there co/fiber sequences, and taking co/fibers of zero maps defines an autoequivalence of the category.
Construction 2.15 (The mapping cone Cone(Q)). Let \( Q : L_0 \to L_1 \) be a morphism; we model it by \( Q \subset M \times T^*E^N \times F \). Then we can construct two connected, eventually conical curves \( \gamma_0, \gamma_1 \subset T^*F \) such that

- \( \gamma_0 \subset T^*(-\infty, t_0], \gamma_0 = (t_0 - \epsilon, t_0] \) near \( t_0 \), and \( \gamma \) is equal to a vertical curve \( \{ t = \text{const} \} \) for \( \tau << 0 \), for some \( t < t_0 \). Further, \( \gamma_0 \) is equipped with a primitive vanishing where \( \tau << 0 \) and near \( t_0 \in F \).

- Likewise, \( \gamma_1 \subset T^*[t_1, \infty) \) with \( \gamma_1 \) equal to the zero section near \( t_1 \), and equal to a vertical curve \( \{ t = \text{const} \} \) for \( \tau << 0 \) and for some \( t > t_1 \). We demand \( \gamma_1 \) is also equipped with a primitive vanishing where \( \tau << 0 \) and near \( t_1 \).

Then the Lagrangian

\[
\text{Cone}(Q) := (\gamma_0 \times L_0) \cup Q_{[t_0, t_1]} \cup (\gamma_1 \times L_1) \subset M \times T^*E^{N+1}
\]

is a model for the mapping cone of \( Q \). We have a fiber sequence Cone(Q) \( \to L_0 \xrightarrow{Q} L_1 \), see [NT11].

Example 2.16. Let \( \gamma \subset T^*F \) be a connected curve such that \( \gamma \cap T^*(-\infty, t_0] = (-\infty, t_0] \) for some \( t_0 \), and such that \( \gamma \) is equal to a vertical curve \( \{ t = \text{const} \} \) for \( \tau >> 0 \) and for some \( t > t_0 \); we also assume \( \gamma \) admits a compactly supported primitive. Then \( L \times \gamma \) is a morphism from \( L \) to the empty Lagrangian, and is a zero morphism. The cone of this zero morphism is, by construction, linearly Hamiltonian isotopic to a grading shift of \( L^\circ \); in particular, the shift functor \( L \mapsto L[1] \) is the same on objects as in the partially wrapped Fukaya category with graded branes.

Below are some useful results about this \( \infty \)-category: Polterovich surgery (also knows as Lagrangian surgery) induces mapping cone sequences, and linear Hamiltonian isotopies induce equivalences.

Theorem 4 ([Tan18]). Let \( L_0, L_1 \subset M \times T^*E^\alpha \) be Lagrangian branes. Suppose that \( L_0 \) and \( L_1 \) intersect at a unique point, and the intersection is transverse. Then for every brane structure on \( L_0 \), there is a brane structure \( \alpha \) on \( L_1 \), along with a fiber sequence

\[
L_1^\alpha \to L_1^{\alpha \cup} L_0 \to L_0
\]

where the middle term is a Polterovich surgery.
Proposition 2.17 ([NT11]). Let $L \subset M \times T^*E^n$ be a brane such that, for some neighborhood $U$ of $sk(M) \times E^n$, we have that $L \cap U = \emptyset$. Then $L$ is a zero object in $\text{Lag}(M)$.

Finally, wrappings are not a necessary component of defining the $\infty$-category of Lagrangian cobordisms; but wrapping is a useful way of producing equivalences.

Definition 2.18. A Hamiltonian $H : M \to \mathbb{R}$ is called \textit{eventually linear} if $Z(H) = H$ near infinity.

Proposition 2.19. Let $H(-, -) : \mathbb{R} \times M \to \mathbb{R}$ be a time-dependent function on $M$ such that for every $t \in \mathbb{R}$, $H(t, -)$ is eventually linear, and such that $H_t = 0$ for $|t| \gg 0$. Then for any exact, eventually conical Lagrangian $L$, one has a Lagrangian cobordism from $L$ to the flow of $L$ at time $t \gg 0$ as follows:

$L \times \mathbb{R} \to M \times T^*F; \quad (x, t) \mapsto (\text{Flow}_t(x), t, -H_t(\text{Flow}_t(x)))$

Remark 2.20. If the value of $H_t(x)$ is very positive or very negative, the “eventually linear” assumption implies that $x$ is far from the core of $M$. Thus so long as $H$ vanishes in a neighborhood of the stops in $\partial \infty M$, this cobordism is $\Lambda$-avoiding, and in fact, an equivalence, because it is bounded away from $\Lambda$ at $dt \to \infty$ too. (See Section 2.6 of [Tan].) For example, Figure 1 exhibits the higher cobordism showing that a composition of a morphism $Q$, and its “$dt \mapsto -dt$” flip $\overline{Q}$, compose to be higher-cobordant to the identity cobordism. Note that when $Q$ is vertically bounded (i.e., has bounded $dt$ coordinate), or avoids $\Lambda$ in both the $+\infty dt$ and $-\infty dt$ directions, then $\overline{Q}$ is indeed a morphism in $\text{Lag}_\Lambda(M)$, and the depicted higher cobordism is indeed a homotopy in $\text{Lag}_\Lambda(M)$.

Thus we have

Proposition 2.21. If two branes $L_0$ and $L_1$ in $\text{Lag}_\Lambda(M)$ are related by eventually linear Hamiltonian isotopies avoiding $f$, then $L_0$ and $L_1$ are equivalent objects.

Example 2.22. Let $V$ be a contact vector field on $\partial \infty M$, meaning $(\text{Flow}^V_t)^*\theta|_{\partial \infty M} = f_t\theta|_{\partial \infty M}$ for some smooth, nowhere vanishing function $f_t : \partial \infty M \to \mathbb{R}$. Choosing a Liouville flow coordinate $s$ near infinity, so that we have the identification $\lambda = e^s\lambda|_{\partial \infty M}$ let us define $H = e^s\lambda|_{\partial \infty M}(V)$ near infinity, and extend
Figure 1: A homotopy showing $\overline{Q} \circ Q \sim \text{id}_{L_0}$. A rotation in the other direction shows $Q \circ \overline{Q} \sim \text{id}_{L_1}$. Thus, if $Q : L_0 \to L_1$ is vertically bounded, $Q$ is an equivalence.

$H$ arbitrarily into the interior of $M$. Then $H$ is eventually linear. Our main source of contact vector fields will be those extended from Legendrian isotopies.

2.4 Attaching handles exactly

We refer the reader to [GPS18] for the notion of attaching an exact embedded Lagrangian k-handle; we recall some basics here. As before we will let $M^o \subset M$ denote a compact set given by the complement of some cylindrical end.

If $L \subset M$ is a brane which is eventually conical with Legendrian boundary $A = \partial_\infty L \subset \partial_\infty M$, let $U \subset \partial_\infty M$ be a Darboux-Weinstein neighborhood of $A$—then $U$ is contactomorphic to a neighborhood of the zero section of the jet bundle $J^1(A) \cong \mathbb{R}_s \times T^*A$. The symplectization $\mathbb{R}_{s>0} \times U \subset M$ is isomorphic to a neighborhood of the zero section of the cotangent bundle $T^*(\mathbb{R}_{s>0} \times A)$.

The result of attaching an index $k$ exact embedded Lagrangian handle is a Lagrangian submanifold $L_{1} \subset \mathbb{R}_{s>0} \times U$, collared by $A$ at small $s$, and collared by a Legendrian $A'$ at larger $s$, where $A'$ is topologically the result of an index $k$ surgery of $A$. The union $L := (L \cap M^o) \cup_A L_{1}$ is an eventually conical Lagrangian brane inside $M$, and we call $L$ the result of attaching an exact embedded Lagrangian k-handle to $L$.  

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We emphasize that $L_1$ is contained entirely in $\mathbb{R}_{s>0} \times U \subset M$. Moreover, if $L$ avoids $\Lambda$ and $\mathfrak{f}$ at infinity, then $L_1$ also avoids $\Lambda$ and $\mathfrak{f} \times \mathbb{R}_s$ entirely. Finally, $L_1$ admits a primitive which vanishes at $A$ and at $A'$. (This is the reason for the adjective *exact* for this handle.)

**Remark 2.23.** $L_1$ is topologically an elementary cobordism, and has been naturally called a Lagrangian cobordism from $A$ to $A'$ in the literature (see for example [DR16]). Note that $L_1$ “propagates” in the conical direction of $\mathbb{R}_s$, within the Liouville manifold $M$ itself; in particular it is not a Lagrangian cobordism in our sense, and does not represent a morphism in our category. It is simply a subset of a new object $\mathbb{L}$.

In the partially wrapped setting, the Viterbo restriction functor, and the corestriction functor of completing Liouville subdomains, yield the following:

**Proposition 2.24.** $L$ and $\mathbb{L}$ are equivalent objects in the partially wrapped Fukaya category.

*Proof.* See Section 1.8 of [GPS18].

The following is the analogue of Proposition 2.24 in the setting of Lagrangian cobordisms. We do not have, as far as we know, a Viterbo restriction functor for the $\infty$-category of Lagrangian cobordisms, so we present an alternate proof by constructing an explicit equivalence.

**Lemma 2.25.** There is an equivalence $\mathbb{L} \to L$ in $\text{Lag}(M)$.

*Proof.* We utilize the isomorphism $\mathbb{R}_s \times J^1(A) \cong T^*(\mathbb{R}_s \times A)$—this can be realized, for example, by $T^*(\mathbb{R}_s \times A) \to \mathbb{R}_s \times J^1(A), \quad (s, a, \sigma, \alpha) \mapsto (\log s, a, \alpha/s, \sigma)$

where $s \in \mathbb{R}_s, a \in A, \sigma \in T^*\mathbb{R}_s, \alpha \in T^*A$, and the last coordinate of $(\log s, a, \alpha/s, \sigma)$ is the 1-jet direction. (See Section 4 of [DR16], or Section 3 of [GPS18] for a different map.)

Note that, crucially, one can choose a symplectomorphism as above so that a brane in $\mathbb{R}_s \times J^1(A)$ is exact whenever it is exact in $T^*(\mathbb{R}_s \times A)$ under the standard Liouville structures on both.

Let $F = \mathbb{R}_t$ (i.e., we will write an element of $F$ by $t$), and consider the Lagrangian $L_1 \times \mathbb{R}_t \subset T^*(\mathbb{R}_s \times A \times \mathbb{R}_t)$, and for appropriate choices of real numbers $S, T > 0$, let $\phi$ be a smooth embedding $(0, S) \times (0, T), t \mapsto \cdots$
(0, S)_s \times (0, S)_t \subset \mathbb{R}_s \times \mathbb{R}_t as depicted in Figure 2; informally, \( \phi \) is a collared version of an orientation-reversing rotation of \( \mathbb{R}_s \) into \( \mathbb{R}_t \). We let \( \Phi \) denote the induced symplectic embedding \( T^*(\mathbb{R}_s \times A) \to T^*(\mathbb{R}_s \times A \times \mathbb{R}_t) \).

Because \( L_1 \) is collared by \( A \) for \( s \) small, consider \( Q_1 = \Phi(L_1 \times \mathbb{R}_t) \cup A \times V \) where \( V \subset (0, \infty)_s \times (0, \infty)_t \) is the complement of the image of \( \Phi \). Via the inclusion \( T^*(\mathbb{R}_s \times A) \cong \mathbb{R}_s \times J^1(A) \supset \mathbb{R}_s \times U \subset M \), we consider \( Q_1 \) as a subset of \( M \times T^*\mathbb{R}_t \). Then the union \( Q := Q_1 \cup (L \times \mathbb{R}_t) \cap (M \times T^*\mathbb{R}_t)^\circ \) is a Lagrangian cobordism from \( L \) to \( L \). It is vertically bounded (i.e., the cotangent coordinate in the \( \mathbb{R}_t \) direction is bounded), so it is an invertible morphism in \( \text{Lag}_\Lambda(M) \). (This is the same reasoning as in Remark 2.20.)

Finally, note that one can as usual deform this Lagrangian in \( M \times T^*F \) so its primitive vanishes near infinity, which (because the cobordism is vertically bounded) is where \( |s| >> 0 \). On the other hand, we already know that \( L_1 \) was equipped with a vanishing primitive in this region, as did \( \partial_{\infty} L \), so we see that the resulting deformed Lagrangian is indeed still collared by \( L \) and by \( L \), as desired.

\[ \phi((0, S) \times (0, T)) \]

Figure 2: On the left, an open rectangle \((0, S)_s \times (0, T)_t \). On the right, the image of the open embedding \( \phi \) taking the open rectangle to a quarter-disk-shaped region in \((0, S)_s \times (0, S)_t \).

3 The proof of Theorem 1

We follow the notation of [GPS18].
Let \((M, f)\) be a Weinstein manifold with a stop \(f = f^{\text{crit}} \cup f^{\text{subcrit}} \subset \partial_\infty M\), where \(f^{\text{crit}} \subset \partial_\infty M\) is a Legendrian.

Let \(Y = (\mathbb{R}^2, \{\pm \infty\})\) denote \(\mathbb{R}^2\) with the radial Liouville structure, and two non-origin points chosen as stops, which we will call \(\pm \infty\). This stopped Liouville manifold is equivalent to \(T^*\mathbb{R}\) with its usual cotangent Liouville structure. Up to isotopy, there is a unique Lagrangian linking disk in \(Y\), which under this correspondence one can model as the cotangent fiber of \(T^*\mathbb{R}\).

Then \(M \times Y\), with the product stop, has the following property:

**Lemma 3.1** ([GPS18].) The product of a linking disk in \(M\) and a linking disk in \(Y\) is a linking disk in \(M \times Y\), and the product of a cocore in \(M\) with a linking disk in \(Y\) is a linking disk in \(M \times Y\).

**Proof.** See Section 7.2 of [GPS18]. \(\Box\)

Now we let \(f\) be a stop and let \(\Lambda = \text{core}(M) \cup \bigcup_{t \in \mathbb{R}} \text{Flow}^Z_t(f)\).

Note that because a linking disk in \(Y\) is equivalent to a cotangent fiber of \(T^*\mathbb{R}\), the product of a Lagrangian \(L\) in \(M\) with a linking disk in \(Y\) is the stabilization of \(L\)—in particular, the equivalence class of \(L\) in \(\text{Lag}_\Lambda(M)\) is unchanged.

Given an object \(L \in \text{Lag}_\Lambda(M)\), let \(L^\circ = L \times \mathbb{R}_y\) be the stabilization. There is an eventually linear, positive Hamiltonian isotopy that isotopes \(\mathbb{R}_y\) to a (non-compact) arc with \(y\)-coordinates strictly less than 0. An induced, eventually linear isotopy on \(M \times Y\) moves \(L^\circ\) to a brane whose \(y\)-coordinates are strictly less than zero. This mean that this brane is a zero object in \(\text{Lag}(M)\). (Proposition 2.17.)

By genericity, one can arrange that this isotopy is modeled by passing through Lagrangian linking disks of \(M \times Y\), one point at a time, and transversally. (Lemmas 2.2 and 2.3 of [GPS18].) These positive isotopies at infinity can be realized by eventually linear Hamiltonians by Example 2.22.

To set notation, let us time-order the intersection points \(y_1, \ldots, y_m\), so \(y_i \in \text{Flow}_{t_i}(L^\circ) \cap D_i\) where \(D_i\) is some Lagrangian linking disk, and \(0 < t_1 < \ldots < t_m\). By Section 4 of [Tan18], there exists a Polterovich surgery \(D_i \#_{y_i} \text{Flow}_{t_i}(L^\circ)\) equipped with a brane structure restricting to that of \(L^\circ\) away from \(y_i \cup D_i\).

**Lemma 3.2.** The surgery \(D_i \#_{y_i} \text{Flow}_{t_i}(L^\circ)\) is equivalent to \(\text{Flow}_{t_i+\epsilon}(L^\circ)\) in \(\text{Lag}_\Lambda(M)\).
Proof. For this proof only, let $\gamma_i$ denote the Reeb chord realizing the intersection $y_i$, and let $\text{Flow}_{t_i-\epsilon}(L^o)\sharp_\gamma D_i$ denote the result of attaching a non-exact embedded Lagrangian 1-handle as in [GPS18]. (Note that the object with subscript $\sharp_\gamma$ is a different submanifold than the object with subscript $\sharp y_i$.) Also, this non-exact handle attachment does result in an exact Lagrangian brane, but after potentially changing the brane structures of its constituents.) Let us attach an exact embedded $(n-1)$-handle to $\text{Flow}_{t_i-\epsilon}(L^o)\sharp_\gamma D_i$, and call the result $\mathbb{L}_i$. By Remark 3.5 of loc. cit., $\mathbb{L}_i$ is linearly Hamiltonian isotopic to the surgery $D_i \sharp y_i \text{Flow}_{t_i}(L^o)$. Further, by Lemma 2.25, $\mathbb{L}_i$ and $\text{Flow}_{t_i-\epsilon}(L^o)\sharp_\gamma D_i$ are equivalent.

Moreover, by Proposition 1.27 of [GPS18], we know that there is an eventually linear Hamiltonian isotopy between $\text{Flow}_{t_i+\epsilon}(L^o)$ and $D_i \sharp_\gamma \text{Flow}_{t_i}(L^o)$. Thus we have the equivalences

$$D_i \sharp_\gamma \text{Flow}_{t_i}(L^o) \sim \mathbb{L}_i \sim \text{Flow}_{t_i-\epsilon}(L^o) \sim \text{Flow}_{t_i+\epsilon}(L^o).$$

By Theorem 1.3 of [Tan18], each $D_i \sharp_\gamma \mathbb{L}_i$ admits a fiber sequence $D_i^\alpha \to D_i \sharp_\gamma \mathbb{L}_i \to \text{Flow}_{t_i-\epsilon}(L^o)$, where $D_i^\alpha$ is our notation for the disk $D_i$, equipped with a well-chosen brane structure $\alpha$. So by Lemma 3.2, we obtain fiber sequences

$$D_i^\alpha \to \text{Flow}_{t_i+\epsilon}(L^o) \to \text{Flow}_{t_i-\epsilon}(L^o), \quad i = 1, \ldots, m.$$

Moreover, if the flow of $L^o$ does not pass through any linking disk in the time interval $[s,t]$, then $\text{Flow}_{s}(L^o)$ and $\text{Flow}_{t}(L^o)$ are equivalent objects in $\text{Lag}(M)$ by Remark 2.20.

Because we know $\text{Flow}_{0}(L^o) = L^o$ and $\text{Flow}_{t_m+\epsilon}(L^o)$ is a zero object, the theorem is proven by Proposition 2.10.

4 The symmetric monoidal structure

4.1 The definition

We now define a functor $\otimes : \text{ho Lag}_pt \times \text{ho Lag}_pt \to \text{ho Lag}_pt$.

Recall that any $\infty$-category $\mathcal{C}$ defines a category (in the usual sense, for instance, of MacLane [ML98]) $\text{ho} \mathcal{C}$ called its homotopy category. Its objects
Figure 3: The Lagrangian \( P \times P' \subset T^*E^n \times T^*E'^r \times T^*F^2 \). We have projected everything to \( F^2 \).

are the same as that of \( C \), and the set of morphisms \( \text{hom}_{\text{ho}C}(L, L') \) is given by \( \pi_0 \text{hom}_C(L, L') \).

Fix two objects \( L \subset T^*E^n \) and \( L' \subset T^*E'^r \). As mentioned in [Tan16, Tan], the product of these need not be eventually conical. But assuming that their primitives vanish away from a compact set, one can always choose a deformation \( L \otimes L' \) of \( L \times L' \) which is eventually conical. (See for example [Tan] or Section 6.2 of [GPS18].) We will assume we have made such a choice \( L \otimes L' \) for every ordered pair \( (L, L') \), and define \( \otimes \) on objects as \( (L, L') \mapsto L \otimes L' \). Note that any two such choices are Hamiltonian isotopic by an eventually linear Hamiltonian.

Fix two morphisms \( P : L_0 \rightarrow L_1 \) and \( P' : L'_0 \rightarrow L'_1 \) in \( \text{Lag}_\text{pt}(pt) \). We let \([P], [P']\) denote their morphisms in \( \text{ho}\text{Lag}_\text{pt}(pt) \), and we define

\[
[P] \otimes [P'] := [(P \times \text{id}_{L'_1}) \circ (\text{id}_{L_0} \times P')] \in \pi_0 \text{hom}_{\text{ho}\text{Lag}_\text{pt}(pt)}(L_0 \otimes L'_0, L_1 \otimes L'_1)
\]

to be the equivalence class of a chosen, eventually conical deformation of the composed cobordism \((P \times \text{id}_{L'_1}) \circ (\text{id}_{L_0} \times P')\). We choose this deformation so that the domain and codomain of this cobordism are indeed the objects \( L_0 \otimes L'_0 \) and \( L_1 \otimes L'_1 \), respectively.

**Remark 4.1.** Given \( P \) and \( P' \), note that \( P \times P' \) most naturally lives over a rectangle, not over a line. (See Figure 3.) The natural guess to extract a 1-morphism (i.e., something living over a line) would be to take a cobordism living along the diagonal of this rectangle, but the resulting submanifold may be singular.
What instead $P \times P'$ encodes is a higher cobordism (hence, in $\text{Lag}_{pt}(pt)$, a homotopy) between two natural morphisms:

$$(P \times \text{id}_{L'_1}) \circ (\text{id}_{L_0} \times P') \quad \text{and} \quad (\text{id}_{L'_1} \times P') \circ (P \times \text{id}_{L'_0})$$

which we can read off of the rectangle by considering the cobordisms collaring the boundary edges of the rectangle. Indeed, resolving the corners of the rectangle, we obtain a 2-morphism (a homotopy) showing that the two natural morphisms are homotopic in $\text{Lag}_{pt}(pt)$. Thus, either of these compositions is equally deserving of the title “$P \otimes P'$.” Indeed, it follows that $[(P \times \text{id}_{L'_1}) \circ (\text{id}_{L_0} \times P')] = [(\text{id}_{L'_1} \times P') \circ (P \times \text{id}_{L'_0})]$ in $\text{hoLag}$.

A similar argument shows that if $[P] = [Q]$ and $[P'] = [Q']$ in $\text{hom}_{\text{hoLag}}$, then $[P \otimes P'] = [Q] \otimes [Q']$.

**Remark 4.2.** The reader can anticipate a host of coherence questions that would arise by taking products of higher morphisms; and hence we hope we have conveyed a glimpse of the intricacies of defining the symmetric monoidal structure at the $\infty$-categorical level of $\text{Lag}$ (as opposed to the categorical level of $\text{hoLag}$).

The following is now straightforward:

**Proposition 4.3.** $\otimes : \text{hoLag}_{pt}(pt) \times \text{hoLag}_{pt}(pt) \to \text{hoLag}_{pt}(pt)$ is a functor.

### 4.2 The swap map

Any symmetric monoidal category comes equipped with a natural isomorphism

$$s : L \otimes L' \cong L' \otimes L.$$  

We define such a swap map for $\text{hoLag}_{pt}(pt)$.

**Remark 4.4.** Recall that given any time-dependent vector field $V_t$ on a smooth manifold $Q$, one has an induced Hamiltonian flow via the function

$$H_t : T^*Q \to \mathbb{R}, \quad \alpha \mapsto \alpha(V_t).$$

The flow of $L_{H_t}$ and the flow of $V_t$ make the following diagram commute:

$$
\begin{array}{ccc}
T^*Q \times \mathbb{R} & \xrightarrow{\text{Flow}_{L_{H_t}}} & T^*Q \\
\downarrow{\pi \times \text{id}_{\mathbb{R}}} & & \downarrow{\pi} \\
Q \times \mathbb{R} & \xrightarrow{\text{Flow}_{V_t}} & Q.
\end{array}
$$

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Here, $\pi$ is the projection map from the cotangent bundle to the zero section. Note also that spin structures and gradings are preserved by Hamiltonian isotopies of a cotangent bundle induced by flows on the zero section.

Let $L$ and $L'$ be branes in $T^*\mathbb{R}^n$ and $T^*\mathbb{R}^{n'}$, respectively. Stabilizing if necessary, we may assume $n = n'$. We define a morphism

$$s : L \times L' \simeq L' \times L.$$ 

1. By assumption, the reverse Liouville flow of $L \times L'$ is contained in some compact subset $K \subset \mathbb{R}^n \times \mathbb{R}^{n'}$. Choose some vector $(v, v') \in \mathbb{R}^n \times \mathbb{R}^{n'}$ so that the translate $K + (v, v')$ does not intersect the origin of $\mathbb{R}^n \times \mathbb{R}^{n'}$. Consider the object $(L \times L')^{\diamond n} \subset (T^*\mathbb{R}^n)^3$. Translation by $(v, v', 0)$ defines a Hamiltonian isotopy of $(T^*\mathbb{R}^n)^3$, which leaves the $\diamond n$ coordinate unaffected. Perform this translation on $(L \times L')^{\diamond n}$.

2. Now consider the vector field

$$\partial_{\theta} := \psi(r) \sum_{1 \leq i \leq n} \frac{x_i \partial_i - x_{n+i} \partial_{n+i}}{x_i^2 + x_{n+i}^2}$$

where $r : \mathbb{R}^{3n} \to \mathbb{R}$ is the distance from the origin, and $\psi(r)$ is a bump function which equals 1 on $r(K)$, and zero outside a small neighborhood of $r(K)$. We rotate by 90 degrees via this vector field. This has the effect of fixing the $\diamond n$ coordinate, but rotating the coordinates $x_i$ into $x_{n+i}$. Applying the resulting Hamiltonian vector field on $T^*\mathbb{R}^{3n}$, one obtains a Lagrangian which is equal to $(L' \times L)^{\diamond n}$, translated. Note that $x_{n+i}$ is mapped to $-x_i$, hence the overline on $L'$.

3. Now we likewise consider a vector field which rotates the $n'$ coordinate into the $\diamond n$ coordinate, and apply a rotation of 180 degrees. The resulting Hamiltonian sends the Lagrangian from the previous step to one equaling a translate of $(L' \times L)^{\diamond n}$.

4. Translating back, we obtain the equivalence $(L \times L')^{\diamond n} \rightarrow (L' \times L)^{\diamond n}$.

The only choices involved here were $(v, v')$, and the bump function $\psi$. The space of choices of $\psi$ is obviously contractible. Moreover, as $n, n' \rightarrow \infty$, the
choice of vectors \((v, v') \in \mathbb{R}^{n+n'} \setminus K\) is also contractible. Since the rotations and translations are determined completely by these two choices, we see that the map \(s\) is well-defined up to contractible choice.

**Remark 4.5.** It is an easy exercise to perform \(s \circ s\) and see that the resulting Lagrangian is equivalent to \(L \times L'\), and that moreover \(s \circ s\), being a result of Hamiltonian isotopies induced by vector fields on \(\mathbb{R}^N\), is homotopic to the identity cobordism.

**Remark 4.6.** Naturality of \(s\) also follows easily from the fact that \(s\) is constructed out of eventually linear Hamiltonian isotopies.

### 4.3 The proof of Theorem 3

We now show that \(\otimes\) and the swap map induce a symmetric monoidal structure

\[
\text{hoLag} \times \text{hoLag} \to \text{hoLag}
\]

on the homotopy category of \(\text{Lag}\); what remains is to construct associators and verify compatibilities.

The natural associators

\[
\alpha : (L \otimes L') \otimes L'' \cong L \otimes (L' \otimes L'')
\]

and natural isomorphism

\[
\rho : L \otimes 1 \cong L
\]

are given by the obvious morphisms, using that the Cartesian product \(L \times L' \times L''\) is associative (up to natural bijections of sets). The other natural isomorphism

\[
\lambda : 1 \otimes L \cong L,
\]

is given by the swap map from Section 4.2 showing \(1 \otimes L \cong L \otimes 1\).

And in general, we take the braiding \(s\) to be the swap map defined in Section 4.2. We must show that the following diagrams commute:

\[
\begin{array}{ccc}
L \otimes 1 & \xrightarrow{s} & 1 \otimes L \\
\downarrow{\rho} & & \downarrow{\lambda} \\
L & \quad & L \otimes 1
\end{array}
\]
The bottom diagram commutes by Remark 4.5, and by definition, the top diagram requires commentary. Without loss of generality, assume \(L, L', L''\) are all submanifolds of \(T^*\mathbb{R}^n\) for the same \(n\). Then the left vertical column is the swap map applied to \(L \otimes L\) and \(L \otimes L' \times L''\). This is induced by the rotating vector field sending \(x_i\) to \(x_{2n+i}\) in \(\mathbb{R}^{4n}\).

The righthand column (together with the top horizontal arrow) is in contrast induced by two rotation maps on \(\mathbb{R}^{3n}\). First, the swap rotation sending \(x_i\) to \(x_{n+i}\), then the rotation map sending \(x_{n+i}\) to \(x_{2n+i}\). The end effect is a rotation map sending \(x_i\) to \(x_{2n+i}\), just as above. Moreover, \((L' \times L'' \times L)^{on}\) is equivalent exactly to \((L' \times L'') \times L^{on}\). Hence the middle diagram commutes as well.

So \(\otimes\) lifts to a symmetric monoidal functor.

The bimodule action \(\text{hoLag}_{\text{pt}}(pt) \times \text{hoLag}_A(M) \rightarrow \text{hoLag}_A(M) \times \text{hoLag}_{\text{pt}}(pt) \rightarrow \text{hoLag}_A(M)\) is defined similarly, sending

\[
(L, L') \mapsto L' \otimes L, \quad (L', L) \mapsto L' \otimes L
\]

for \(L \subset T^*E, L' \subset M \times T^*E'\). The compatibilities to be verified are analogous to the earlier bits of this section, so we do not repeat them.

So to finish the proof of the Theorem, we need only verify that mapping cones in each variable are sent to mapping cones. This is obvious, as given \(P : L_0 \rightarrow L_1\), and an object \(L'\), then \(P \otimes \text{id}_{L'} \simeq P \times L'\) as a manifold (up to Hamiltonian deformation), hence \([P \otimes \text{id}_{L'}] = [P \times L']\), and the mapping cone of \(P \times L'\) is geometrically constructed as \(\text{Cone}(P) \times L'\). (See Construction 2.15.) This completes the proof.
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