Characteristic decay of the autocorrelation functions prescribed by the Aharonov-Bohm time operator

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The wave functions, the autocorrelation functions of which decay faster than $t^{-2}$, for both the one-dimensional free particle system and the repulsive-potential system are examined. It is then shown that such wave functions constitute a dense subset of $L^2(\mathbb{R}^1)$, under several conditions that are particularly satisfied by the square barrier potential system. It implies that the faster than $t^{-2}$-decay character of the autocorrelation function persists against the perturbation of potential. It is also seen that the denseness of the above subset is guaranteed by that of the domain of the Aharonov-Bohm time operator.

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I. INTRODUCTION

The time operator $T$ is usually defined as an operator that satisfies the canonical commutation relation (CCR) with the Hamiltonian $H$ for the system considered

$$[T, H] = i.$$  \hfill (1)

This operator has been widely studied in the attempt of deriving the time-energy uncertainty relation within the framework of quantum mechanics (see Ref. 1 and the references therein), and of prescribing the time-of-arrival in the quantum theory. For the latter, see a detailed review written by Muga and Leavens. In the efforts to solve these problems, a criticism by Pauli against the existence of the time operator has been reexamined. Furthermore, there is a possibility that the investigation of the time operator may reveal its connection to the quantum dynamics. This is expected firstly from the definition of the time operator itself, i.e. the CCR (1), which is algebraically so strong that their operator characters are mutually restricted. We have a possibility to obtain the information of the quantum dynamics through a study of the time operator. In this context, the investigation of the time operator is focused on the commutator (not necessarily canonical). In particular, it should be remembered that the connection between the commutator of the form $[H, iA] = C$ ($C \geq 0$) and the spectral property of self-adjoint operators, $H$, $A$ and $C$, has been widely studied by Putnam and Kato, and applied to the Schrödinger operators by Lavine and others. Another implication is seen in a statement about the ergodic theory in classical mechanics, which can be formulated in terms of the Liouville operator $L$. It states that, for a classical system to be $K$-flow, there necessarily exists a self-adjoint operator $Q$ satisfying the CCR with the absolutely continuous part of $L$: $[Q, L] = i$. Since $L$ and $H$ are mathematically equivalent in the role of the generator of time evolution, we may have such a statement in quantum mechanics although this expectation has to remain in a naive sense.

In order to examine this possibility, a symmetric operator $T$ is introduced in Ref. 6 on a Hilbert space $\mathcal{H}$. It satisfies the following operator relation with the Hamiltonian operator $H$ on $\mathcal{H}$:

$$Te^{-itH} = e^{-itH}(T + t)$$ \hfill (2)

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for all \( t \in \mathbb{R}^1 \). Then the CCR (\[1\]) holds on the domain \( \text{Dom}(TH) \cap \text{Dom}(HT) \), and so this operator is called the time operator. Mathematically there is no reason why we must restrict the relation (\[2\]) to the case of the time operator and the Hamiltonian. Actually this relation was investigated in detail by Schmüdgen, who showed that the above \( T \) and \( H \) are unitary equivalent to the momentum operator satisfying some boundary condition and the position operator, respectively. In the case of the time operator, the Aharonov-Bohm time operator \( T_0 \) (more precisely, its symmetric extension defined later) and the free Hamiltonian \( H_0 \) for the one-dimensional free-particle system (1DFPS) satisfy the above relation (\[2\]). They are defined as,

\[
T_0 := \frac{1}{4}(QP^{-1} + P^{-1}Q), \quad H_0 := P^2,
\]

where the domain of \( T_0 \) is defined as \( \text{Dom}(T_0) := \text{Dom}(QP^{-1}) \cap \text{Dom}(P^{-1}Q) \subset L^2(\mathbb{R}^1) \). The operators \( P \) and \( P^{-1} \) are the momentum operator on \( L^2(\mathbb{R}^1) \) and its inverse, where \( P \) is defined as \( P := -iD_x, \) \( D_x \) being a differential operator on \( L^2(\mathbb{R}^1) \). Note that \( P^{-1} \) is self-adjoint for \( P \) is an injection. The position operator \( Q \) on \( L^2(\mathbb{R}^1) \) is defined as an operator of multiplication by \( x \). Then it is known that \( T_0 \) is a symmetric operator on \( L^2(\mathbb{R}^1) \). For the later convenience, we here introduce a symmetric extension of \( T_0 \), denoted by \( \widetilde{T_0} \), which is defined in the momentum representation:

\[
\text{Dom}(\widetilde{T_0}) := \left\{ \psi \in L^2(\mathbb{R}^1) \mid \hat{\psi} \in AC(\mathbb{R}_k \setminus \{0\}), \lim_{k \to 0} \frac{\hat{\psi}(k)}{|k|^{1/2}} = 0, \quad \int_{\mathbb{R}_k \setminus \{0\}} \left( \frac{d\hat{\psi}(k)}{dk} + \frac{1}{k} \frac{d\hat{\psi}(k)}{dk} \right)^2 \, dk < \infty \right\},
\]

\[
(\widetilde{T_0} \psi)(k) = \frac{i}{4} \left( \frac{d\hat{\psi}(k)}{dk} + \frac{1}{k} \frac{d\hat{\psi}(k)}{dk} \right), \text{ a.e. } k \in \mathbb{R}_k \setminus \{0\}, \quad \psi \in \text{Dom}(\widetilde{T_0})
\]

where for any \( \psi \in L^2(\mathbb{R}^1) \) the symbol \( \hat{\psi} \) denotes \( F\psi, \) \( F \) being the Fourier transform from \( L^2(\mathbb{R}^1) \) to \( L^2(\mathbb{R}_k^1) \). One can see that the operators \( \widetilde{T_0} \) and \( H_0 \) satisfy the relation (\[2\]), i.e.

\[
\widetilde{T_0} e^{-itH_0} = e^{-itH_0} (\widetilde{T_0} + t)
\]

for all \( t \in \mathbb{R}^1 \). Here our interest is in the connection between the time operator and the quantum dynamics. Actually, for every \( \psi \in \text{Dom}(T) \) and for every \( t \in \mathbb{R}^1 \setminus \{0\} \), an inequality

\[
|\langle \psi, e^{-itH} \psi \rangle|^2 \leq \frac{4}{t^2} \langle \Delta T \rangle^2 \psi \|\psi\|^2
\]

was recently proved in Ref. 6 from the relation (\[2\]). Here \( \langle \Delta T \rangle \psi := \langle |T - \langle \psi, T\psi \rangle| \psi \rangle \) is the standard deviation of \( T \) in the state \( \psi \), and \( |\langle \psi, e^{-itH} \psi \rangle|^2 \) is the survival probability at time \( t \) in the initial state \( \psi \). We call the latter the autocorrelation function of the initial state \( \psi \) for the system considered. This quantity \( |\langle \psi, e^{-itH} \psi \rangle|^2 \) is considered as an indicator of the overlap of the state \( e^{-itH} \psi \) at time \( t \) with the initial state \( \psi \), and thus, the name, the autocorrelation function, is here appropriate. The inequality (\[2\]) is very important because the inequality directly connects the familiar quantity, the autocorrelation function and the not-familiar one, \( \langle \Delta T \rangle \psi \). It should be noticed that there are many attempts to relate the time-energy uncertainty relation to the autocorrelation function (see e.g. Refs. 15, 16, and 17 ). We can easily see from the inequality that \( 2\sqrt{2} \langle \Delta T \rangle \psi \|\psi\| \) gives an upper bound of the half-time \( \tau_h(\psi) := \sup \left\{ t \geq 0 \right\} \left| \langle \psi, e^{-itH} \psi \rangle \right|^2 = 1/2 \). Since the above inequality
originates only from the algebraic relation (8), we can also obtain, for the operators \( \bar{T}_0 \) and \( H_0 \),

\[
|\langle \psi, e^{-itH_0} \psi \rangle|^2 \leq \frac{4 (\Delta \bar{T}_0)^2 \|\psi\|^2}{t^2}
\]

for all \( \psi \in \text{Dom}(\bar{T}_0) \) and for all \( t \in \mathbb{R}^1 \setminus \{0\} \). It is interesting to ask how the inequality (8) characterizes the decay-dynamics of the autocorrelation function for the 1DFPS. With respect to this question, we first see that

\[
\psi \in \text{Dom}(\bar{T}_0) \implies \exists C > 0, \forall t \in \mathbb{R}^1 \setminus \{0\}, \|\langle \psi, e^{-itH_0} \psi \rangle\|^2 \leq \frac{C}{t^2},
\]

however the converse does not hold. We can easily find a counterexample for the latter. Consider the wave function \( \psi_1(k) = (4/\pi)^{1/4}ke^{-k^2/2} \in L^2(\mathbb{R}_k^1) \). Then one can see that \( |\langle \psi_1, e^{-itH_0} \psi_1 \rangle|^2 \) decays like \( |t|^{-3} \), however \( \psi_1 \notin \text{Dom}(\bar{T}_0) \) from the definition (8). Note that in spite of this example, \( \text{Dom}(\bar{T}_0) \) is dense in \( L^2(\mathbb{R}^1) \).

Our aim in the present paper is to answer the question about what the faster than \( t^{-2} \)-decay property of the autocorrelation function as in (8) implies to the quantum dynamics, and to clarify the role played by the time operator or its domain in this respect. To this end, we attempt to consider two one-dimensional systems, i.e. a system of a free particle and another system of a particle with a repulsive potential, described by the Hamiltonian \( H_1 \).

In particular, we focus on the square-integrable wave functions, autocorrelation functions of which decay faster than \( t^{-2} \) for both of the two systems. Then it shall be found, under the several conditions on the eigenfunctions of \( H_1 \), that the subset of such square-integrable wave functions, denoted by \( C(H_0, 2) \cap C(H_1, 2) \), is dense in \( L^2(\mathbb{R}^1) \). The denseness of such a subset implies that the faster than \( t^{-2} \)-decay character of the autocorrelation function is persistent against the perturbation of potentials, among which the square barrier potential is shown to be included. These statements are our main results where given in Sec. II. We will also see that the subset \( C(H_0, 2) \cap C(H_1, 2) \) includes the dense subspace of \( \text{Dom}(\bar{T}_0) \), and thus the denseness of \( C(H_0, 2) \cap C(H_1, 2) \) is guaranteed by that of \( \text{Dom}(\bar{T}_0) \).

In the next section, we mention the motivation to consider the two systems, as an approach to the examination of the inequality (8). The definitions of the sets \( C(H_i, 2) \) \((i = 0, 1)\) are also given there. Our results mentioned above are referred to as Theorem 3.3 and Theorem 3.4 in Sec. II, where the latter theorem is applied to the square barrier potential system.

In order to prove these theorems, we also put Proposition 3.1 that is proved in Sec. IV, after its preparation in Sec. III. In Sec. IV, the wave operator, the eigenfunction expansions, and the one-dimensional Lippmann-Schwinger equation are introduced. The concluding remarks are given in Sec. V. In the Appendix, the eigenfunctions for the system of square barrier potential are shown to satisfy the conditions needed in applying Theorem 3.4.

II. INTERSECTION \( C(H_0, 2) \cap C(H_1, 2) \)

In this section, we mention the reason for considering the two systems, i.e. a system of a free particle and another system of a particle with a repulsive potential. We first introduce the time operators for systems with the repulsive potentials \( V(x) \) :

\[
0 \leq V(x) \leq \text{const.} < \infty, \quad V \in L^1(\mathbb{R}^1).
\]

Then, it follows from Putnam’s results in the theory of Schrödinger operators, that for any potential \( V(x) \) satisfying (10), the Hamiltonian for the potential system defined by \( H_1 := H_0 + V \) is a self-adjoint operator on \( L^2(\mathbb{R}^1) \) and the wave operators \( W \) defined by
\[ W_{\pm} := s- \lim_{t \to \pm \infty} e^{itH_1} e^{-itH_0} \]  
(11)

exist and are unitary operators on \( L^2(\mathbb{R}^1) \). Here "s-" stands for strong limit. Furthermore they satisfy

\[ H_1 = W_{\pm} H_0 W_{\pm}^*. \]  
(12)

The time operators \( T_{\pm} \) for the potential system with the potential \( V(x) \) satisfying \( |t| \) are constructed along the similar relation to the above.

\[ T_{1,\pm} := W_{\pm} \tilde{\mathcal{T}}_0 W_{\pm}^*, \]  
(13)

where \( \text{Dom}(T_{1,\pm}) = W_{\pm} \text{Dom}(\tilde{T}_0) \). It should be noted that the construction of the time operator for the potential system in the above sense has been already proposed by León et al. \[20\] Several properties of such time operators were revealed by them \[20\] and Baute et al. \[21\].

Remembering that \( \tilde{T}_0 \) and \( H_0 \) satisfy the relation \( (1) \), and unitarily transforming \( (1) \) by \( W_{\pm} \) through \( (12) \) and \( (13) \), we have

\[ T_{1,\pm} e^{-i t H_1} = e^{-i t H_1} (T_{1,\pm} + t), \quad \forall t \in \mathbb{R}^1. \]  
(14)

Thus \( T_{1,\pm} \) is just regarded as the time operators for the repulsive potential system specified by \( (11) \). We also remember that the inequality \( (14) \) are derived only from the algebraic relation \( (2) \), so that

\[ |\langle \psi, e^{-i t H_1} \psi \rangle|^2 \leq \frac{4 (\Delta T_{1,\pm})_0^2 \| \psi \|^2}{t^2}, \quad \forall \psi \in \text{Dom}(T_{1,\pm}), \forall t \in \mathbb{R}^1 \backslash \{0\} \]  
(15)

must hold. It is important to notice that the inequalities \( (15) \) mean the existence of a dense subset of \( L^2(\mathbb{R}^1) \), for each of the potential systems, that is composed by the wave functions where autocorrelation functions decay faster than \( t^{-2} \).

In order to get rid of the restriction of the faster than \( t^{-2} \)-decay dynamics to the domain of the time operator as in \( (11) \), let us introduce a subset of \( L^2(\mathbb{R}^1) \), denoted by \( C(H, n) \), that is defined, for an arbitrary self-adjoint operator \( H \) on \( L^2(\mathbb{R}^1) \) and a non-negative real number \( n \), as

\[ C(H, n) := \left\{ \psi \in L^2(\mathbb{R}^1) \mid \exists C > 0, \forall t \in \mathbb{R}^1 \backslash \{0\}, \quad |\langle \psi, e^{-i t H} \psi \rangle|^2 \leq \frac{C^2}{|t|^n} \right\}. \]  
(16)

Clearly we have

\[ \text{Dom}(\tilde{T}_0) \subset C(H_0, 2) \subset L^2(\mathbb{R}^1), \]  
(17)

and

\[ \text{Dom}(T_{1,\pm}) \subset C(H_1, 2) \subset L^2(\mathbb{R}^1). \]  
(18)

Here it should be noticed that \( C(H_0, 2) \neq L^2(\mathbb{R}^1) \), since, e. g., a Gaussian wave packet \( \tilde{\psi}_0(k) = (\pi)^{-1/4} e^{-k^2/2} \in L^2(\mathbb{R}^1) \), has its autocorrelation function decaying like \( |t|^{-1} \). We also see that \( \text{Dom}(\tilde{T}_0) \neq C(H_0, 2) \) for the relation \( (11) \).

Now we attempt to combine the relations \( (17) \) and \( (18) \), i.e.

\[ \text{Dom}(\tilde{T}_0) \cap \text{Dom}(T_{1,\pm}) \subset C(H_0, 2) \cap C(H_1, 2) \subset L^2(\mathbb{R}^1). \]  
(19)

One can find that \( C(H_0, 2) \cap C(H_1, 2) \) has an interesting character, that is, if an initial state belongs to \( C(H_0, 2) \cap C(H_1, 2) \), then its autocorrelation function must decay faster than \( t^{-2} \), irrespectively of the presence of potential. One may have a question whether such a state can exist. In fact, the intersection is not necessarily dense in \( L^2(\mathbb{R}^1) \), though \( C(H_0, 2) \) and \( C(H_1, 2) \) are respectively dense in \( L^2(\mathbb{R}^1) \). The primary motivation for the present paper is to answer this question and the latter can bring us a further understanding of the faster than \( t^{-2} \)-decay dynamics. In the following sections, it will be seen that \( C(H_0, 2) \cap C(H_1, 2) \) is possible to be dense in \( L^2(\mathbb{R}^1) \), under the several conditions.
III. DENSENESS OF $C(H_0, 2) \cap C(H_1, 2)$

Consider one-dimensional systems with the potential of the class:

$$\exists \delta > 2, \exists c > 0, \forall x \in \mathbb{R}^1, |V(x)| \leq \frac{c}{(1 + |x|)^\delta}.$$  \hfill (20)

The Hamiltonian for the potential system, denoted by $H_1 := H_0 + V$ on $L^2(\mathbb{R}^1)$, where $H_0$ is the free Hamiltonian on $L^2(\mathbb{R}^1)$. For such potentials, $H_1$ has no positive eigenvalue. \footnote{We also see that the above potential is an Agmon potential and thus that, for each $\Phi$ such that $\Phi \in C(\mathbb{R}^1 \setminus \{0\})$ in Sec. IV belong to $C^1(\mathbb{R}^1 \setminus \{0\})$, and satisfy the following three conditions:

$$\gamma_{\pm} := \sup_{x \in \mathbb{R}^1, k \in \mathbb{R}^1 \setminus \{0\}} |\varphi_{\pm}(x, k)| < \infty,$$

$$\delta_{\pm} := \sup_{x \in \mathbb{R}^1, k \in \mathbb{R}^1 \setminus \{0\}} \frac{|\varphi_{\pm}(x, k)|}{k} < \infty,$$

$$\gamma_{\partial, \pm} := \sup_{x \in \mathbb{R}^1, k \in \mathbb{R}^1 \setminus \{0\}} \left| \frac{\partial \varphi_{\pm}(x, k)}{\partial k} \right| < \infty,$$  \hfill (23)

where $I := \text{supp}V(x)$. Then for any $\psi \in \text{Dom}(\widetilde{T}_0) \cap S(\mathbb{R}^1)$ there is some constant $C > 0$ such that

$$|\langle P_{ac}(H_1)\psi, e^{-itH_1}P_{ac}(H_1)\psi \rangle| \leq \frac{C}{|t|}$$  \hfill (24)

for all $t \in \mathbb{R}^1 \setminus \{0\}$, that is

$$P_{ac}(H_1)(\text{Dom}(\widetilde{T}_0) \cap S(\mathbb{R}^1)) \subset C(H_1, 2).$$  \hfill (25)

In the above statement, $S(\mathbb{R}^1)$ denotes the subspace of rapidly decreasing functions, and $P_{ac}(H_1)$ the projection operator associated with the absolutely continuous subspace of $H_1$ (e.g. see Refs. 23 and 24). Use of the projection $P_{ac}(H_1)$ in the statement comes from the possibility of the existence of bound states of the Hamiltonian $H_1$, depending on the potential in the class (20). Proposition 3.1 is applicable to the system with such a Hamiltonian. Here we remark that the above conditions (21) and (22) ensure the existence of the following functions $\Phi_{\pm}(k)$:

$$\Phi_{\pm}(k) := \begin{cases} \delta_{\pm} |k| & (|k| \leq 1) \\ \gamma_{\pm} & (|k| > 1) \end{cases},$$  \hfill (26)

satisfying

$$\Phi_{\pm}/|k| \in L^2(\mathbb{R}^1_k), \quad \forall k \in \mathbb{R}^1_k \setminus \{0\}, \sup_{x \in \mathbb{R}^1} |\varphi_{\pm}(x, k)| \leq \Phi_{\pm}(k).$$  \hfill (27)
It should be noticed that under the condition (20),
\[ C(H_1, 2) \subset P_{ac}(H_1)L^2(\mathbb{R}^1) \]  
holds. This follows from the next lemma. Before proving it, let us remember the results from the theory of Schrödinger operators (e.g. see Refs. 23 and 24): Suppose that \( H \) is a self-adjoint operator on a certain Hilbert space, and define the three subspaces as \( \mathcal{H}_{pp}(H) := \{ \psi \in \mathcal{H} \mid \|E(\cdot)\psi\|^2 \text{ is pure point } \} \), \( \mathcal{H}_{ac}(H) := \{ \psi \in \mathcal{H} \mid \|E(\cdot)\psi\|^2 \text{ is absolutely continuous } \} \), and \( \mathcal{H}_{sing}(H) := \{ \psi \in \mathcal{H} \mid \|E(\cdot)\psi\|^2 \text{ is continuous singular} \} \), where \( \{E(B) \mid B \in \mathcal{B}^1 \} \) is the spectral measure of \( H \) and \( \mathcal{B}^1 \) is the Borel sets of \( \mathbb{R}^1 \). Then these subspaces are closed and orthogonally decompose \( \mathcal{H} \), i.e. \( \mathcal{H} = \mathcal{H}_{pp}(H) \oplus \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{sing}(H) \). Furthermore, defining the projection operators \( P_{pp}(H), P_{ac}(H), \) and \( P_{sing}(H) \), associated with the closed subspaces \( \mathcal{H}_{pp}(H), \mathcal{H}_{ac}(H), \) and \( \mathcal{H}_{sing}(H) \), respectively, then these projection operators and the spectral measure \( E(B) \) are mutually commutative.

**Lemma 3.2** Suppose that \( \mathcal{H} \) is a Hilbert space and \( H \) is a self-adjoint operator on \( \mathcal{H} \), then \( C(H) \subset P_c(H)\mathcal{H} \), where
\[
C(H) := \left\{ \psi \in \mathcal{H} \mid \exists C > 0, \forall n > 0, \forall t \in \mathbb{R}^1 \setminus \{ 0 \}, \left| \langle \psi, e^{-iHt} \psi \rangle \right|^2 \leq \frac{C^2}{|t|^n} \right\},
\]
and \( P_c(H) := P_{ac}(H) + P_{sing}(H) \).

**Proof:** For notational simplicity, we abbreviate \( P_c \psi \) to \( \psi_c \), and \( P_{pp} \psi \) to \( \psi_{pp} \), for all \( \psi \in \mathcal{H} \). Then, by the virtue of \( E(B)P_c = P_cE(B) \) and \( E(B)P_{pp} = P_{pp}E(B) \), we have \( \langle \psi, e^{-iHt} \psi \rangle = \langle \psi_c, e^{-iHt} \psi_c \rangle + \langle \psi_{pp}, e^{-iHt} \psi_{pp} \rangle \), and the following inequality
\[
\left| \langle \psi_{pp}, e^{-iHt} \psi_{pp} \rangle \right|^2 = \left| \langle \psi, e^{-iHt} \psi \rangle - \langle \psi_c, e^{-iHt} \psi_c \rangle \right|^2 \\
\leq 2 \left( \left| \langle \psi, e^{-iHt} \psi \rangle \right|^2 + \left| \langle \psi_c, e^{-iHt} \psi_c \rangle \right|^2 \right).
\]
Let us now consider a particular \( \psi \in C(H) \) \((\psi \neq 0)\). From the definition of \( C(H) \), we can define for this \( \psi \) a continuous function \( G(t) \) of \( t \in \mathbb{R}^1 \):
\[
G(t) := \left\{ \begin{array}{c}
\|\psi\|^2 \\
C|t|^{-n/2} \end{array} \right. \begin{array}{c}
(|t| \leq (\|\psi\|^2/C)^{-2/n} \\
(|t| > (\|\psi\|^2/C)^{-2/n}) \end{array},
\]
satisfying \( \left| \langle \psi, e^{-iHt} \psi \rangle \right| \leq G(t), \forall t \in \mathbb{R}^1 \setminus \{0\} \). Then we have
\[
\left| \langle \psi_{pp}, e^{-iHt} \psi_{pp} \rangle \right|^2 \leq 2 \left( G(t)^2 + \left| \langle \psi_c, e^{-iHt} \psi_c \rangle \right|^2 \right).
\]
Integrate both sides of this inequality on \([0, T]\) \((T > 0)\) and divide them by \( T \). First, we easily see
\[
\lim_{T \to \infty} \frac{1}{T} \int_{[0, T]} G(t)^2 dt = 0.
\]
Secondly notice that \( \|E(\cdot)\psi_c\|^2 \) is finite and continuous, that is, \( \|E(\mathbb{R}^1)\psi_c\| = \|\psi_c\| < \infty \) and \( \|E(A)\psi_c\|^2 \neq 0 \) for any countable set \( A \in \mathcal{B}^1 \), since \( \psi_c \in P_c(H)\mathcal{H} \). Then by using Wiener’s theorem, we see
\[
\lim_{T \to \infty} \frac{1}{T} \int_{[0, T]} \left| \langle \psi_c, e^{-iHt} \psi_c \rangle \right|^2 dt = 0.
\]
Thus, we obtain
\[
\limsup_{T \to \infty} \frac{1}{T} \int_{[0,T]} |\langle \psi_{pp}, e^{-iHt} \psi_{pp} \rangle|^2 \, dt = 0.
\]

Furthermore we have
\[
0 = \lim_{T \to \infty} \frac{1}{T} \int_{[0,T]} |\langle \psi_{pp}, e^{-iHt} \psi_{pp} \rangle|^2 \, dt
= \int \int \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda = \mu \} \, d\langle \psi_{pp}, E(\lambda)\psi_{pp} \rangle \, d \langle \psi_{pp}, E(\mu)\psi_{pp} \rangle
= \sum_{\lambda \in \mathbb{R}^1} \| E(\{\lambda\})\psi_{pp} \|^4 \geq \sum_{\lambda \in A} \| E(\{\lambda\})\psi_{pp} \|^4,
\]
where \( A \) is any countable set in \( \mathbb{R}^1 \). Thus \( \psi_{pp} \) must belong to \( P_c(H)\mathcal{H} \). This implies that \( P_{pp}\psi = \psi_{pp} = 0 \), i.e. \( \psi = P_c\psi \), \( \forall \psi \in C(H) \), and the proof is completed. \( \square \)

Now the relation (28) is understood from the following argument. From the definition of \( C(H_1, 2) \) in (16) and the above lemma, we have \( C(H_1, 2) \subset P_c(H_1)L^2(\mathbb{R}^1) \). On the other hand, as stated just after (20), the Hamiltonian \( H_1 \) defined in Proposition 3.1 has no continuous singular spectrum, i.e. \( P_{cs}(H_1) = 0 \), and thus we obtain the relation (28).

From the relations (24) and (28), we have
\[
P_{ac}(H_1)(\text{Dom}(\tilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1)) \subset C(H_1, 2) \subset P_{ac}(H_1)L^2(\mathbb{R}^1).
\]

Then we see that above subsets are densely connected to each other, because \( \text{Dom}(\tilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \) is dense in \( L^2(\mathbb{R}^1) \). The latter follows from the existence of the dense subspace \( F^{-1}\mathcal{C}_1 \) satisfying \( F^{-1}\mathcal{C}_1 \subset \text{Dom}(\tilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \), where \( \mathcal{C}_1 \) is defined in the momentum space as
\[
\mathcal{C}_1 := \left\{ \hat{\psi} \in C_c^\infty(\mathbb{R}^1) \mid \exists \delta > 0, \forall k \in (-\delta, \delta), \hat{\psi}(k) = 0 \right\}.
\]

As a summary, we have from the relations (17) and (29)

**Theorem 3.3:** If the conditions in Proposition 3.1 are satisfied, then \( P_{ac}(H_1)C(H_0, 2) \cap C(H_1, 2) \) is dense in \( P_{ac}(H_1)L^2(\mathbb{R}^1) \).

Disregarding the singular continuous part of \( H_1 \), \( P_{ac}(H_1)L^2(\mathbb{R}^1) \) is generally considered as the subspace spanned by all scattering states of \( H_1 \). If we assume that \( V(x) \geq 0 \), i.e. the repulsive type, in addition to the potential condition (20), then \( V(x) \) clearly satisfies the condition (16). In such a case, \( H_0 \) and \( H_1 \) is unitary equivalent and thus \( H_1 \) is (spectrally) absolutely continuous, i.e. \( P_{ac}(H_1) \) is an identity operator on \( L^2(\mathbb{R}^1) \). This consideration leads us to the next theorem, that was announced in Secs. I and II.

**Theorem 3.4:** If the conditions in Proposition 3.1 and \( V(x) \geq 0 \) are satisfied, then \( C(H_0, 2) \cap C(H_1, 2) \) is dense in \( L^2(\mathbb{R}^1) \).

We here mention an application of Proposition 3.1 and Theorem 3.4 to the square barrier potential system.

**Example 1:** Consider the square barrier potential system with the potential:
\[
V(x) := \left\{ \begin{array}{ll}
V_0 & (|x| \leq a/2) \\
0 & (|x| > a/2)
\end{array} \right.
\]
where \( a > 0 \) and \( V_0 > 0 \). The Hamiltonian \( H_1 \) for the system is defined by \( H_1 := H_0 + V \) on \( L^2(\mathbb{R}^1) \) and is self-adjoint. Clearly \( V(x) \) satisfies the condition (20) and \( V(x) \geq 0 \), and thus \( P_{ac}(H_1) = 1 \), i.e. \( H_1 \) has no bound states or point spectrum. As shown explicitly in the
Appendix, the eigenfunctions of $H_1$ satisfy the conditions in Proposition 3.1, (11), and (23). Thus Proposition 3.1 is applicable to this system and for any $\psi \in \text{Dom}(T_0) \cap \mathcal{S}(\mathbb{R}^1)$, for any $V_0 > 0$, and for any $a > 0$ there is some constant $C > 0$ such that

$$\left| \langle \psi, e^{-itH_1} \psi \rangle \right| \leq \frac{C}{|t|}$$

for all $t \in \mathbb{R}^1 \setminus \{0\}$. In particular $C(H_0, 2) \cap C(H_1, 2)$ is dense in $L^2(\mathbb{R}^1)$.

IV. AUTOCORRELATION FUNCTION FOR POTENTIAL SYSTEMS

For the preparation of the proof of Proposition 3.1, we shall introduce the wave operator, the eigenfunction expansions, and the one-dimensional Lippmann-Schwinger equation. The wave operators $W_\pm$ are defined by

$$W_\pm := \lim_{t \to \pm \infty} e^{itH_1} e^{-itH_0},$$

where $H_1 := H_0 + V$ with the potential $V(x)$ of the class $[29]$, different from $[10]$. Since $V(x) \in L^1(\mathbb{R}^1) \cap L^2(\mathbb{R}^1)$, these wave operators exist and are complete. However, in this case, $W_\pm$ are generally partially-isometric rather than unitary, i.e.

$$W_\pm^* W_\pm = 1, \quad W_\pm W_\pm^* = P_{ac}(H_1),$$

and in particular

$$e^{-itH_1} W_\pm = W_\pm e^{-itH_0}, \forall t \in \mathbb{R}^1.$$  

Then it follows from the use of (32) and (33) that

$$\langle P_{ac}(H_1) \psi, e^{-itH_1} P_{ac}(H_1) \psi \rangle = \langle W_\pm^* \psi, e^{-itH_0} W_\pm^* \psi \rangle$$

for all $\psi \in L^2(\mathbb{R}^1)$. By the use of the eigenfunction expansions, we can explicitly express $W_\pm^* \psi$ in the momentum representation. In particular, for some fixed numbers $s$ and $s'$ satisfying

$$s + s' = \delta, \quad s > 3/2, \quad s' > 1/2, \quad \text{and} \quad s' \leq s,$$

we have, without resort to the $L^2$-convergence,

$$(W_\pm^* \psi)(k) = \int_{\mathbb{R}^1} \varphi_\pm(x, k) \psi(x) \, dx, \quad \forall \psi \in L^{2,s}(\mathbb{R}^1),$$

where bar ($\overline{\phantom{a}}$) stands for complex conjugate. $L^{2,s}(\mathbb{R}^1)$ is a weighted $L^2$-space that consists of all $L^2$-functions $\psi$ satisfying $\int_{\mathbb{R}^1} (1 + |x|^2)^s |\psi(x)|^2 \, dx < \infty$, and then we have under the conditions $[23]$ the inclusion relation $L^{2,s}(\mathbb{R}^1) \subset L^1(\mathbb{R}^1)$. The functions $\varphi_\pm(x, k)$, defined on $\mathbb{R}^1 \times \mathbb{R}_k \setminus \{0\}$, are the eigenfunctions of $H_1$ (more precisely the absolutely continuous part of $H_1$). Under the conditions $[23]$ and $[35]$, $\varphi_\pm(x, k)$ are guaranteed to be in $L^{2,-s}(\mathbb{R}^1)$, so that the integral in (36) is finite, and satisfy the one-dimensional Lippmann-Schwinger equation: They are explicitly given by

$$\varphi_\pm(x, k) = (2\pi)^{-1/2} e^{ikx} + g_\pm(x, k),$$

where

$$g_\pm(x, k) := \mp \frac{1}{2\pi |k|} \int_{\mathbb{R}^1} e^{\mp ik|y|} V(y) \varphi_\pm(y, k) \, dy$$

and $\varphi_\pm(x, k)$ is the absolute continuous part of $\varphi_\pm(x, k)$, different from $\varphi_\pm(x, k)$.
for all \( x \in \mathbb{R}^3 \) and all \( k \in \mathbb{R}^3 \setminus \{0\} \). The same conditions ensure that \( \varphi_\pm(x,k) \) are \( C^1 \)-functions of \( x \) for each \( k \in \mathbb{R}_0^3 \setminus \{0\} \) fixed and satisfy the time-independent Schrödinger equation,

\[
\left(-\frac{d^2}{dx^2} + V(x)\right) \varphi_\pm(x,k) = k^2 \varphi_\pm(x,k),
\]

(39)

in the sense of distribution. Here we don’t prove the eigenfunction expansions stated above, however our proof essentially follows the references 27 and 28. By the use of (36) and (37), the probability amplitude (34) is divided into four terms,

\[
\langle P_{ac}(H_1) \psi, e^{-itH_1} P_{ac}(H_1) \psi \rangle = \langle \tilde{\psi}, e^{-itk^2} \tilde{\psi} \rangle \\
+ \langle \int_{\mathbb{R}^1} g_{\pm}(y,k) \psi(y) \, dy, e^{-itk^2} \tilde{\psi} \rangle \\
+ \langle e^{itk^2} \tilde{\psi}, \int_{\mathbb{R}^1} g_{\pm}(y,k) \psi(y) \, dy \rangle \\
+ \langle \int_{\mathbb{R}^1} g_{\pm}(y,k) \psi(y) \, dy, e^{-itk^2} \int_{\mathbb{R}^1} g_{\pm}(y,k) \psi(y) \, dy \rangle,
\]

(40)

where \( \psi \in L^{2,\infty}(\mathbb{R}^1) \). This expression will be used in the proof of Proposition 3.1 in the next section.

V. PROOF OF PROPOSITION 3.1

Supposing \( \psi \in \text{Dom}(\hat{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \), one can find the following statement, which is used in the proof of Proposition 3.1.

**Lemma 5.1**: If \( \psi \in \text{Dom}(\hat{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \), then \( \hat{\psi}(0) = \hat{\psi}'(0) = 0 \) in which \( \hat{\psi}'(k) := d\hat{\psi}(k)/dk \). In particular, \( \text{Dom}(\hat{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \subset \text{Dom}(QP^{-1}) \cap \text{Dom}(P^{-1}Q) \subset \text{Dom}(P^{-2}) \).

**Proof**: Let us prove this in the momentum representation. We first note that, for every \( \hat{\psi} \in \mathcal{S}(\mathbb{R}^3), \hat{\psi} \in C^\infty(\mathbb{R}^3) \) and \( \sup_{k \in \mathbb{R}^3} |\hat{\psi}''(k)| < \infty \). It also follows that \( \hat{\psi}(0) = 0 \) because of \( \psi \in \text{Dom}(\hat{T}_0) \) and the definition (4). Then one can show that for any \( k \in (-1,1) \), there are some numbers \( \theta_0(k) \) and \( \theta_1(k) \) such that \( 0 < \theta_0(k) < 1, 0 < \theta_1(k) < 1 \) and

\[
\hat{\psi}(k) = \hat{\psi}'(0)k + \frac{i}{2} \hat{\psi}''(\theta_0(k)k)k^2, \\
\hat{\psi}'(k) = \hat{\psi}'(0) + \hat{\psi}''(\theta_1(k)k)k.
\]

It follows from the above equations that

\[
(\hat{T}_0 \psi)(k) = -\frac{i}{2k^2} \hat{\psi}(k) + \frac{i}{k} \hat{\psi}'(k) \\
= \frac{i}{2} \hat{\psi}'(0) - \frac{i}{2} \hat{\psi}''(\theta_0(k)k)k^2 + i \hat{\psi}''(\theta_1(k)k),
\]

and thus

\[
\frac{\hat{\psi}'(0)}{k} \leq 2 \left| (\hat{T}_0 \psi)(k) \right| + \frac{5}{2} \sup_{k \in \mathbb{R}_0^3} \left| \hat{\psi}''(k) \right|.
\]

This result implies that \( \hat{\psi}'(0) = 0 \), because \( (\hat{T}_0 \psi)(k) \) is integrable on \((-1,1)\) for \( \hat{T}_0 \psi \in L^{2,\infty}(\mathbb{R}^3) \). Thus the first part of the lemma has been proved. We now easily see that if
\[ \psi \in \text{Dom}(\widetilde{D}_0) \cap \mathcal{S}(\mathbb{R}^1), \text{ then } \tilde{\psi} \in \text{Dom}(iD_kk^{-1}) \cap \text{Dom}(k^{-1}iD_k) \subset \text{Dom}(k^{-2}). \] Since \( iD_k = FQF^{-1} \) and \( k^{-1} = FP^{-1}F^{-1} \), this means the latter in the lemma is valid. \( \Box \)

One can see from (3) and this lemma that \( \text{Dom}(\widetilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1) = \text{Dom}(T_0) \cap \mathcal{S}(\mathbb{R}^1) \). Examples of wave functions in \( \text{Dom}(\widetilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \) include \( \delta_n(k) := k^n e^{-ak^2} (a > 0 \text{ and integers } n \geq 2) \) which were used in Ref. 1 by Kobe.

Proof of Proposition 3.1: In this proof, we suppose \( \psi \in \text{Dom}(\widetilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \) as prescribed in the conditions in Proposition 3.1. Then clearly \( \mathcal{S}(\mathbb{R}^1) \subset L^{2,\infty}(\mathbb{R}^1) \) and the expression (40) is applicable to the proof. Furthermore, since \( \psi \in \text{Dom}(\widetilde{T}_0) \), the first term in the right-hand side of (40) has to decay faster than \( |t|^{-1} \) by virtue of the inequality \( 8 \). Therefore, in order to prove the proposition, it is sufficient to show that the remaining three terms in the right-hand side of (40) decay, at least like \( |t|^{-1} \), or more rapidly.

We first examine the integral \( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \) in (40) and its derivative \( d \left( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right) /dk \). We note from Lemma 5.1 that \( \psi \in \text{Dom}(Q\psi^{-1}) \). It then follows that \( \psi(x) = (P^{-1}\psi)(x) = -i\text{d}(P^{-1}\psi)(x)/dx \text{ a.e. } x, \text{ and } \lim_{x \to \pm \infty}(P^{-1}\psi)(x) = 0 \) for \( P^{-1}\psi \in \text{Dom}(P) \). Moreover \( P^{-1}\psi \in L^1(\mathbb{R}^1) \) since \( \text{Dom}(Q) \subset L^1(\mathbb{R}^1) \), which follows from \( \int_{\mathbb{R}^1} |f(x)| dx \leq \left( \int_{\mathbb{R}^1} |1 + |x||^{-2} \right) \int_{\mathbb{R}^1} (1 + |x|^2|f(x)|^2 dx)^{1/2} < \infty \), \( \forall f \in \text{Dom}(Q) \). These properties enable us to derive

\[
\int_{\mathbb{R}^1} e^{\pm ik|y-x|}\psi(y) dy = \int_{[x, \infty)} e^{\pm ik|y-x|}\psi(y) dy + \int_{(-\infty, x]} e^{\mp ik|y-x|}\psi(y) dy
\]

\[
= \lim_{y \to \infty} -ie^{\pm ik|y-x|}(P^{-1}\psi)(y) + i(P^{-1}\psi)(x) + k \int_{[x, \infty)} e^{\pm ik|y-x|}(P^{-1}\psi)(y) dy
\]

\[
- i(P^{-1}\psi)(x) + \lim_{y \to x^-} i\epsilon e^{\mp ik|y-x|}(P^{-1}\psi)(y) \pm k \int_{(-\infty,x]} e^{\mp ik|y-x|}(P^{-1}\psi)(y) dy
\]

\[
= \mp k \int_{\mathbb{R}^1} \frac{y-x}{|y-x|} e^{\mp ik|y-x|}(P^{-1}\psi)(y) dy. \tag{41}
\]

Recalling the assumption (21) which implies \( \int_{\mathbb{R}^1} |V(x)| dx < \infty \), and (21), we can obtain from Fubini’s theorem and the above result that

\[
\int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy = \frac{\pm 1}{2ik} \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^1} e^{\pm ik|y-x|}\psi(y) dx \right) V(x) \varphi_{\pm}(x, k) \ dx
\]

\[
= \frac{i}{2} \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^1} \frac{y-x}{|y-x|} e^{\mp ik|y-x|}(P^{-1}\psi)(y) dy \right) V(x) \varphi_{\pm}(x, k) \ dx, \tag{42}
\]

and thus

\[
\left| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right| \leq \frac{1}{2} \int_{\mathbb{R}^1} |V(x)| dx \int_{\mathbb{R}^1} |(P^{-1}\psi)(y)| dy \varphi_{\pm}(x, k), \tag{43}
\]

where \( \varphi_{\pm}(k) \) are defined by (20). Then, from (27) and (13), we see that \( \left( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right) /k \) is a square integrable function of \( k \), and is uniformly bounded on \( \mathbb{R}^*_+ \setminus \{0\} \). We next examine the following derivative term which is well defined for all \( k \in \mathbb{R}^*_+ \setminus \{0\} \) under the assumptions in the proposition: For \( k > 0, \)

\[
\frac{d}{dk} \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy
\]
from the assumption (20). One can have the similar inequality to the above for different integration and in the second equality, (41) have been used at partial differentiation and integration, and in the second equality, (41) have been used at partial derivatives.

Let us now consider the second and third terms in (40). Since the third term is essentially \( \int \psi \left( \int \frac{\partial \varphi (x,k)}{\partial k} dx \right) \psi (y)dy \),
\[
\int \frac{d}{dk} \int \varphi (y,k) \psi (y)dy \leq \frac{1}{2} \int |V(x)|dx \int |(P^{-1}\psi)(y)|dy \left| \frac{\Phi (k)}{k} \right|
\]

and the potential. The second integral has to be a square integrable function of \( k \), \( \Phi (k) \) is also seen to be uniformly bounded on \( \mathbb{R} \) for all \( k \), and is used in exchanging the order of differentiation and integration, and in the second equality, \( \Phi (k) \) have been used at partial integrations. It then follows that

\[ | \int \varphi (y,k) \psi (y)dy | \leq C_1 \frac{|t|}{|t|^2}, \quad \text{for all } k \text{ where } \mathbb{R} \setminus \{0\}, \text{ and } \int |V(x)||x|dx < \infty \text{ have been used. The last relation follows from the assumption (20). One can have the similar inequality to the above for } k < 0. \]

Therefore \( \int \varphi (y,k) \psi (y)dy \) has to be a square integrable function of \( k \) from (27), and is also seen to be uniformly bounded on \( \mathbb{R} \) \( \setminus \{0\} \).

Let us now consider the second and third terms in (40). Since the third term is essentially equivalent to the second, it suffices to examine the second term in (40). Then we will show that

\[ \left| \int \frac{\partial \varphi (x,k)}{\partial k} \psi (y)dy \right| \leq C_1 \frac{|t|}{|t|^2}, \quad \text{for all } k \text{ where } \mathbb{R} \setminus \{0\}, \text{ and } \int |V(x)||x|dx < \infty \text{ have been used. The last relation follows from the assumption (20). One can have the similar inequality to the above for } k < 0. \]

Therefore \( \int \varphi (y,k) \psi (y)dy \) has to be a square integrable function of \( k \) from (27), and is also seen to be uniformly bounded on \( \mathbb{R} \) \( \setminus \{0\} \).

Let us now consider the second and third terms in (40). Since the third term is essentially equivalent to the second, it suffices to examine the second term in (40). Then we will show that

\[ \left| \int \varphi (y,k) \psi (y)dy \right| \leq C_1 \frac{|t|}{|t|^2}, \quad \text{for all } k \text{ where } \mathbb{R} \setminus \{0\}, \text{ and } \int |V(x)||x|dx < \infty \text{ have been used. The last relation follows from the assumption (20). One can have the similar inequality to the above for } k < 0. \]

Therefore \( \int \varphi (y,k) \psi (y)dy \) has to be a square integrable function of \( k \) from (27), and is also seen to be uniformly bounded on \( \mathbb{R} \) \( \setminus \{0\} \).

Let us now consider the second and third terms in (40). Since the third term is essentially equivalent to the second, it suffices to examine the second term in (40). Then we will show that

\[ \left| \int \varphi (y,k) \psi (y)dy \right| \leq C_1 \frac{|t|}{|t|^2}, \quad \text{for all } k \text{ where } \mathbb{R} \setminus \{0\}, \text{ and } \int |V(x)||x|dx < \infty \text{ have been used. The last relation follows from the assumption (20). One can have the similar inequality to the above for } k < 0. \]

Therefore \( \int \varphi (y,k) \psi (y)dy \) has to be a square integrable function of \( k \) from (27), and is also seen to be uniformly bounded on \( \mathbb{R} \) \( \setminus \{0\} \).

Let us now consider the second and third terms in (40). Since the third term is essentially equivalent to the second, it suffices to examine the second term in (40). Then we will show that

\[ \left| \int \frac{\partial \varphi (x,k)}{\partial k} \psi (y)dy \right| \leq C_1 \frac{|t|}{|t|^2}, \quad \text{for all } k \text{ where } \mathbb{R} \setminus \{0\}, \text{ and } \int |V(x)||x|dx < \infty \text{ have been used. The last relation follows from the assumption (20). One can have the similar inequality to the above for } k < 0. \]
where

\[ \lim_{k \to \infty} e^{-itk^2} \frac{\hat{\psi}(k)}{k} \int_{\mathbb{R}^1} g_{\pm}(y, k) \overline{\psi(y)} dy = 0 \text{ and } \lim_{k \to 0} e^{-itk^2} \frac{\hat{\psi}(k)}{k} \int_{\mathbb{R}^1} g_{\pm}(y, k) \overline{\psi(y)} dy = 0 \]

have been used in the last equality, following from (43), (26), and the assumption \( \psi \in \text{Dom}(\tilde{T}_0) \cap \mathcal{S}(\mathbb{R}^1) \). Moreover it should be noticed that the partial integration used in the second equality is justified because both of the two \( k \)-integrals in (47) have finite values. To see the latter, we note that \( \hat{\psi}/k \) and \( d\hat{\psi}/dk \), are square integrable for Lemma 5.1, and \( \left( \int_{\mathbb{R}^1} g - (y, k) \psi(y) dy \right)/k \) and \( d \left( \int_{\mathbb{R}^1} g - (y, k) \psi(y) dy \right)/dk \) are also square integrable functions of \( k \), from the results obtained before. Therefore we see that the two integrals are bounded so that partial integration is well performed. These arguments are also applied to the \( k \)-integration over \((-\infty, 0]\), and we can obtain (44).

We will next prove that the last term in the right-hand side of (10) satisfies

\[ \left| \left\langle \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy, e^{-itk^2} \int_{\mathbb{R}^1} g_{\pm}(y, k) \overline{\psi(y)} dy \right\rangle \right| \leq C_2 \frac{1}{|t|}, \quad (46) \]

where \( C_2 \) is some positive constant which depends only on \( \psi \) and the potential. In order to see this, we first estimate the following one,

\[ \int_0^\infty \left\| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right\|^2 e^{-itk^2} dk \]

\[ = \frac{i}{2t} \int_0^\infty \left\| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right\|^2 \frac{1}{k} |e^{-itk^2}| dk \]

\[ = \frac{i}{2t} \left[ \lim_{k \to \infty} \left\| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right\|^2 \frac{1}{k} |e^{-itk^2}| - \lim_{k \to 0} \left\| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right\|^2 \frac{1}{k} |e^{-itk^2}| \right] \]

\[ - \frac{i}{2t} \int_0^\infty \left[ \left( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right) \frac{d}{dk} \left( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right) + \text{c.c.} \right] \]

\[ - \frac{1}{k^2} \left\| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right\|^2 e^{-itk^2} dk \]

\[ = \frac{i}{2t} \int_0^\infty \left[ \left( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right) \frac{d}{dk} \left( \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right) + \text{c.c.} \right] \]

\[ - \frac{1}{k^2} \left\| \int_{\mathbb{R}^1} g_{\pm}(y, k) \psi(y) dy \right\|^2 e^{-itk^2} dk. \quad (47) \]

In the last equality, we have used that
Decay dynamics of the autocorrelation functions

\[
\lim_{k \to \infty} \left| \int_{\mathbb{R}^1} g_\pm(y, k) \psi(y) \, dy \right| = \frac{1}{k} e^{-ik^2} = 0, \quad \lim_{k \downarrow 0} \left| \int_{\mathbb{R}^1} g_\pm(y, k) \psi(y) \, dy \right| = \frac{1}{k} e^{-ik^2} = 0.
\]

These follow from (43) and (26). We already know that \( \left( \int_{\mathbb{R}^1} g_\pm(y, k) \psi(y) \, dy \right) / k \) and \( \frac{d}{dk} \left( \int_{\mathbb{R}^1} g_\pm(y, k) \psi(y) \, dy \right) \) are square integrable functions of \( k \), and thus the integrand in (47) is integrable on \( \mathbb{R}^1 \). The same argument is also applicable to \( k \)-integration over \(( -\infty, 0 ] \) as in (47), and hence we obtain (46). By putting (40), (44), and (46) together, we can finally derive (24) and the proof has been completed. \( \square \)

VI. CONCLUDING REMARKS

We would first like to remember Theorem 3.4, rather than its slightly generalized form, Theorem 3.3, because of its clarity for explanation. We have considered two one-dimensional systems, i.e. the free particle system with the free-Hamiltonian \( H_0 \) and the repulsive-potential system with the Hamiltonian \( H_1 \), and examined the wave functions \( \psi \in L^2(\mathbb{R}^1) \) where autocorrelation functions decay faster than \( t^{-2} \) for both systems. Theorem 3.4 states that, under the conditions in the statement, such wave functions compose a dense subset of \( L^2(\mathbb{R}^1) \), denoted by \( C(H_0, 2) \cap C(H_1, 2) \). The denseness of this subset seems to imply that the faster than \( t^{-2} \)-decay property of the autocorrelation function is persistent against the perturbation of potential. For example, the square-barrier-potential system is found to satisfy the conditions in the theorem. However these conditions are given in terms of the eigenfunctions of the Hamiltonian \( H_1 \), and thus our statement is not straightforward for the practical use. Reducing the conditions to explicitly that of the potential is a left problem, and needed to know how extensive the class of potential such that the faster than \( t^{-2} \)-decay character remains persistent is. Furthermore our conditions is perhaps too strong and expected to be relaxed.

Theorem 3.4 may be thought of as a statement independent of the time operator. However, from its derivation, we understand that the existence of the Aharonov-Bohm time operator \( \tilde{T}_0 \) or \( \text{Dom}(\tilde{T}_0) \) guarantees the denseness of \( C(H_0, 2) \cap C(H_1, 2) \) in \( L^2(\mathbb{R}^1) \). Theorem 3.4 can be regarded as a remarkable sign of the existence of the connection between the time operator and the quantum dynamics.

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APPENDIX: EIGENFUNCTIONS FOR THE SQUARE BARRIER POTENTIAL SYSTEM

In this section, we show that eigenfunctions for the square barrier potential system satisfy the conditions in Proposition 3.1, i.e. (21), (22), and (23). We here focus on \( \phi_-(x, k) \) in (37), however, the same argument is applicable to \( \phi_+(x, k) \). One can see that \( \phi_-(x, k) \) for \( k > 0 \) is the stationary solution for the case where a particle approaches from the left of the barrier and is reflected or transmitted (see, e.g. Ref. 29). We define \( k_0 := \sqrt{V_0} \), and the support of \( V(x) \) as \( I := [-a/2, a/2] \) for convenience. It follows from (37) and (39) that for \( k > k_0 \).
\[
\varphi_-(x, k) = \begin{cases} 
A e^{ikx} + B(k)e^{-ikx} & (x < -a/2) \\
C(k)e^{ikx} + D(k)e^{-ikx} & (-a/2 \leq x \leq a/2) \\
F(k)e^{ikx} & (a/2 < x),
\end{cases}
\] (A1)

where \( A := (2\pi)^{-1/2} \) and \( \kappa := \sqrt{k^2 - V_0} \). Each coefficient is determined by the property stated just before (39) that \( \varphi_-(x, k) \) belongs to \( C^1(\mathbb{R}^1) \) for each \( k \in \mathbb{R}_+^1 \setminus \{0\} \) fixed. For \( x \in I \) and \( k > k_0 \) we have

\[
C(k) = \frac{\kappa + k}{2\kappa} e^{i(k-\kappa)a/2} F(k),
\]

\[
D(k) = \frac{-\kappa - k}{2\kappa} e^{i(k+\kappa)a/2} F(k),
\]

\[
F(k) = \left( \cos \kappa a - i \frac{k^2 + \kappa^2}{2\kappa} \sin \kappa a \right) \left( e^{-ika} \right)^{-1} e^{-ika} A.
\] (A2)

In the case \( k_0 > k > 0 \), one obtain the corresponding equations, through replacing \( i\kappa \) with \( \rho := \sqrt{V_0 - k^2} \) in (A1) and (A2). At \( k = k_0 \), we also have

\[
\varphi_-(x, k_0) = (ik_0 x + 1 - ik_0a/2)e^{ik_0a/2} F(k_0)
\]

for all \( x \in I \) where \( F(k_0) = (1 - ik_0a/2)^{-1} e^{-ik_0a} A \). Then one can check that \( \varphi_-(x, k) \) is continuous and differentiable in \( k \in (0, \infty) \), and \( \varphi_-(x, \cdot) \in C^1(\mathbb{R}_+^1 \setminus \{0\}) \), for each fixed \( x \in I \). For \( k < 0 \), one can substitute -x for x, and -k for k, in (A1) and (A2) respectively. As long as the conditions (21), (22), and (23) are concerned, it suffices to examine \( \varphi_-(x, k) \) for all \( x \in I \). It is noted that for any positive \( k \) ( \( \neq k_0 \) ), \( |F(k)| \leq A \). Then we have

\[
|\varphi_-(x, k)|^2 = \left| \cos[\kappa(x-a/2)] + i \frac{k}{\kappa} \sin[\kappa(x-a/2)] \right|^2 \left| e^{ika/2} F(k) \right|^2
\]

\[
= \left| \cos[\kappa(x-a/2)] + i \left( \frac{\sqrt{k^2 - k_0^2}}{\sqrt{k^2 + k_0^2}} + \frac{k_0}{\kappa} \right) \sin[\kappa(x-a/2)] \right|^2 \left| F(k) \right|^2
\]

\[
\leq \left| 2 + k_0 |x-a/2| \right|^2 \left| F(k) \right|^2
\]

\[
\leq \left| 2 + k_0 a \right|^2 A^2,
\] (A3)

for every \( x \in I \) and \( k > k_0 \). When \( k_0 > k > 0 \), we have

\[
|\varphi_-(x, k)|^2 = \left| \cosh[\rho(x-a/2)] + i \frac{k}{\rho} \sinh[\rho(x-a/2)] \right|^2 \left| e^{ika/2} F(k) \right|^2
\]

\[
= \left| \cosh[\rho(x-a/2)] - i \left( \frac{k_0^2 - k}{k_0^2 + k} - \frac{k_0}{\rho} \right) \sinh[\rho(x-a/2)] \right|^2 \left| F(k) \right|^2
\]

\[
\leq \left| \cosh(k_0 a) + \sinh(k_0 a) + k_0 |x-a/2| \frac{\sinh(k_0 a)}{k_0 a} \right|^2 \left| F(k) \right|^2
\]

\[
\leq \left| \cosh(k_0 a) + 2 \sinh(k_0 a) \right|^2 A^2,
\] (A4)

for every \( x \in I \). As in the same way in the above, one can have a similar result for \( k < 0 \). Thus \( \varphi_-(x, k) \) is bounded on \( I \times \mathbb{R}_+^1 \setminus \{0\} \) and satisfies (21), i.e.

\[
\sup_{x \in I, k \in \mathbb{R}_+^1 \setminus \{0\}} |\varphi_-(x, k)| < \infty.
\]

In order to verify the condition (22), we should note that

\[
\lim_{k \to 0} \frac{F(k)}{k} = A \left( \frac{k_0}{2} \sinh k_0 a \right)^{-1} \quad \text{and} \quad \lim_{k \to \infty} \frac{F(k)}{k} = 0,
\] (A5)

and thus \( \sup_{k \in \mathbb{R}_+^1 \setminus \{0\}} |F(k)|/|k| < \infty \), because \( F(k) \) is continuous in \( \mathbb{R}_+^1 \setminus \{0\} \). It follows, from this result and (A3) and (A4), that \( \varphi_-(x, k) \) satisfies (22).
Let us next consider the derivative of \( \varphi_-(x, k) \) with respect to \( k \). When \( k > k_0 \), it is differentiable at \( k \) for each fixed \( x \in I \), and we have

\[
\frac{\partial \varphi_-(x, k)}{\partial k} = e^{ika/2} F(k) \frac{\partial}{\partial k} \left( \cos[k(x-a/2)] + \frac{k}{\kappa} \sin[k(x-a/2)] \right) \\
+ \frac{\partial e^{i k a / 2} F(k)}{\partial k} \left( \cos[k(x-a/2)] + \frac{k}{\kappa} \sin[k(x-a/2)] \right),
\]

where

\[
\frac{\partial e^{i k a / 2} F(k)}{\partial k} = - \left( -\frac{k}{\kappa} \sin \kappa a - i \frac{4 k^2 \kappa^2 - (k^2 + \kappa^2)^2}{2 k^2 \kappa^3} \sin \kappa a - i a \frac{k^2 + \kappa^2}{2 \kappa^2} \cos \kappa a \right) \\
\times e^{i k a / 2} F^2(k) A^{-1} - i (a/2) e^{i k a / 2} F(k).
\]

As for the first term in (A6), it follows that

\[
\left| e^{i k a / 2} F(k) \frac{\partial}{\partial k} \left( \cos[k(x-a/2)] + \frac{k}{\kappa} \sin[k(x-a/2)] \right) \right| \leq |F(k)| \left| \left| x - a/2 \right| (1 + k_0 |x - a/2|) + |x - a/2| + 2|x - a/2| + C_h |x - a/2|^3 \right| \\
\leq (4 + k_0 a + C_h a^2) a A
\]

for all \( x \in I \) and \( k > k_0 \), and also for the second term in (A6),

\[
\left| \left( \cos[k(x-a/2)] + \frac{k}{\kappa} \sin[k(x-a/2)] \right) \frac{\partial e^{i k a / 2} F(k)}{\partial k} \right| \leq (2 + k_0) |x - a/2| \right| \left[ (a(1 + k_0 a) + 5a^3 V_0 C_h/2) |F^2(k)| A^{-1} + (a/2) |F(k)| \right] \\
\leq (2 + k_0) \left[ 13/2 + k_0 a + a^2 V_0 C_h/2 \right] a A
\]

for all \( x \in I \) and \( k > k_0 \). Here we have used the facts

\[
\left| \frac{k}{\kappa} \sin[k(x-a/2)] \right| \leq 1 + k_0 |x - a/2|
\]

for every \( k > k_0 \), and \( C_h := \sup_{x \geq 0} |h(x)| \), where \( h(x) := (\sin x - x \cos x)/x^3 \). It should be noted that \( C_h < \infty \), because \( \lim_{x \to 0} h(x) = 0 \), \( \lim_{x \to \infty} h(x) = 0 \) and \( h(x) \) is continuous at each \( x > 0 \). Hence it can be seen, from the above results, that \( \partial \varphi_-(x, k) / \partial k \) is bounded on \( I \times (k_0, \infty) \). If \( k_0 > k > 0 \), \( \varphi_-(x, k) \) is differentiable with respect to \( k \) for each fixed \( x \in I \) and we have

\[
\frac{\partial \varphi_-(x, k)}{\partial k} = e^{i k a / 2} F(k) \frac{\partial}{\partial k} \left( \cosh[p(x-a/2)] + \frac{k}{\rho} \sinh[p(x-a/2)] \right) \\
+ \frac{\partial e^{i k a / 2} F(k)}{\partial k} \left( \cosh[p(x-a/2)] + \frac{k}{\rho} \sinh[p(x-a/2)] \right),
\]

where
shown (23), i.e.

Concerning the first term in the right-hand side of (A7), we obtain, for and

and

and

and

and

and

Concerning the first term in the right-hand side of (A7), we obtain, for \( k_0 > k > 0 \),

for all \( x \in I \), and also for the second term in (A7)

Here we have used the fact that \((\sinh x)/x\) is a monotonically increasing function of \( x \) for \( x > 0 \) and an inequality

Furthermore, we have defined \( C_l := \sup_{0 < x < k_0 a} l(x) \), where \( l(x) := (\sinh x - x \cosh x)/x^3 \). Then \( C_l < \infty \) because of the relation \( \lim_{x \to 0} l(x) = -\frac{1}{2} \) and the continuity of \( l(x) \) on \( (0, \infty) \). Thus, \( \partial \varphi_- / \partial k \) has to be bounded on \( I \times (0, k_0) \). See (X3). Hence we have obtained that \( \varphi_-(x, k) \) is differentiable in \( R^*_k \backslash \{0\} \) of \( k \) and its derivative with respect to \( k \) is bounded on \( I \times (0, \infty) \). A similar result can be also obtained for \( I \times (-\infty, 0) \), and thus we have finally shown (23), i.e.

\[
\sup_{x \in I, k \in R^*_k \backslash \{0\}} \left| \frac{\partial \varphi_- (x, k)}{\partial k} \right| < \infty .
\]
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