BICROSSPRODUCT STRUCTURE OF AFFINE QUANTUM GROUPS

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July – revised November, 1995

Abstract

We show that the affine quantum group $U_q(\hat{sl}_2)$ is isomorphic to a bicrossproduct central extension $\mathbb{C}Z \vartriangleright U_q(Lsl_2)$ of the quantum loop group $U_q(Lsl_2)$ by a quantum cocycle $\chi$, which we construct. We prove the same result for $U_q(\hat{g})$ in R-matrix form.

Keywords: affine quantum group – central extension – quantum cocycle – loop group – R-matrix.

1 Introduction

Affine quantum groups figure prominently as the quantum ‘non-Abelian symmetries’ of q-deformed conformal field theories and certain statistical models[1]. Their representation theory has been extensively studied via the techniques of vertex algebras[2]. A result which has been missing, however, is the sense in which these quantum groups are central extensions of quantum loop groups. We recall that this is important for the correct geometrical picture in the classical theory[3], hence should be important for a geometrical picture in the $q$-deformed case as well.

We provide this result in the present paper, constructing the appropriate ‘quantum cocycle’ $\chi : U_q(Lg) \otimes^2 \rightarrow \mathbb{C}Z$, from which $U_q(\hat{g})$ is then obtained as the corresponding extension. Here $U_q(Lg)$ is the level 0 version of the affine quantum group, and $\mathbb{C}Z$ denotes the group algebra of $\mathbb{Z}$, i.e. polynomials in a generator $c$ and its inverse.

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The required theory of (non-Abelian) Hopf algebra extensions was already introduced (by the author) in [4], where it is shown that extensions generally have the form of a \textit{cocycle bicrossproduct}. The quantum Weyl group is already known to be of this form [3]. In general however, it can be hard to come up with the cocycle data required in the construction. Our first result, in Section 2, provides a general solution to the construction problem in the case of a central extension.

We give the case of $U_q(\hat{sl}_2)$ in detail, in Section 3. In Section 4 we give the result more generally using the $R$-matrix formalism with generators $l^\pm(z)$ in [3][4].

Acknowledgements

These results were obtained during a visit in June 1995 to R.I.M.S. in Kyoto under a joint programme with the Isaac Newton Institute in Cambridge and the J.S.P.S. I would like to thank M. Jimbo for some useful discussions.

2 General Construction of Quantum Cocycles

In this section we introduce a general construction for quantum cocycles appropriate to the central extensions which concern us. We begin by recalling the more abstract (but not constructive) theory of cocyle bicrossproducts and Hopf algebras from [4][3] in the form now required.

We assume standard notations for Hopf algebras $H, A$, coproducts $\Delta h = h^{(1)} \otimes h^{(2)}$, antipodes $S$, left and right actions $\triangleright, \triangleleft$, coaction $\beta(h) = h^{(1)} \otimes h^{(2)}$, etc.; see the texts [7][8]. A module algebra means an algebra which is acted upon covariantly. Likewise, a comodule coalgebra means a coalgebra which is coacted upon covariantly.

Let $H$ be a Hopf algebra and $A$ an algebra. A quantum 2-cocycle on $H$ with values in $A$ means a map $\chi : H \otimes H \to A$ obeying

$$\chi(g_{(1)} \otimes f_{(1)})\chi(h \otimes g_{(2)}f_{(2)}) = \chi(h_{(1)} \otimes g_{(1)})\chi(h_{(2)}g_{(2)} \otimes f), \quad \chi(1 \otimes (\ )) = \chi((\ ) \otimes 1) = \epsilon. \quad (1)$$

It is also possible to allow here an action of $H$ on $A$, but we will not need this. Such cocycles are known to allow [3] the formation of an algebra structure $A_\chi \bowtie H$ with product

$$(a \otimes h)(b \otimes g) = ab\chi(h_{(1)} \otimes g_{(1)}) \otimes h_{(2)}g_{(2)}. \quad (2)$$
To this older theory, we add\(^4\) the supposition that \(\beta : H \to H \otimes A\) makes \(H\) a right comodule coalgebra compatible with \(\chi\) in the sense

\[
\beta(hg) = (1 \otimes \chi^{-1}(h_{(1)} \otimes g_{(1)}))\beta(h_{(2)})\beta(g_{(2)})(1 \otimes \chi(h_{(3)} \otimes g_{(3)})), \quad \beta(1) = 1 \otimes 1
\] (3)

\[
[\beta(h), 1 \otimes a] = 0
\] (4)

\[
\Delta \chi(h \otimes g) = \chi(h_{(1)}^{(1)} \otimes g_{(1)}^{(1)}) \otimes h_{(1)}^{(2)} g_{(1)}^{(2)} \chi(h_{(2)} \otimes g_{(2)}), \quad \epsilon \chi = \epsilon \otimes \epsilon.
\] (5)

In this case the standard cross coproduct coalgebra \(A \triangleleft H\), with coproduct

\[
\Delta(a \otimes h) = a_{(1)} \otimes h_{(1)}^{(1)} \otimes a_{(2)} h_{(1)}^{(2)} \otimes h_{(2)}
\] (6)

forms with the algebra \((2)\) a Hopf algebra, denoted \(A_\chi \triangleleft \triangleright H\), the cocycle bicrossproduct. Projection to \(H\) by applying the counit \(\epsilon\) of \(A\), and the inclusion of \(A\) as \(A \otimes 1\) provide Hopf algebra maps

\[
A \hookrightarrow A_\chi \triangleleft \triangleright H \to H
\] (7)

obeying certain properties. It is possible to characterise cocycle bicrossproducts more abstractly as Hopf algebras \(A \hookrightarrow E \to H\) with the maps obeying certain properties making \(E\) an abstract (cleft and cocleft) extension of Hopf algebras. Then \(E \cong A_\chi \triangleright \triangleright \psi H\) of the general type in \((3)\) with, possibly, an action of \(H\) on \(A\) and a dual cocycle \(\psi\) in the coalgebra. The special case \((1)-(6)\) has the additional features that \(A\) commutes with the elements of \(H\) and application of the counit of \(H\) is a coalgebra map \(E \to A\).

There remains the problem of how to actually construct \(\chi, \beta\) obeying \((1), (3)-(6)\). Part of this, the coaction \(\beta\), is usually easy to identify. The following lemma shows that once this is done, the remaining cocycle can be found, and provides a formula for it.

**Proposition 2.1** Let \(A, E, H\) be Hopf algebras with \(E = A \triangleleft H\) as a coalgebra for a coaction \(H \to H \otimes A\). Suppose that \(E \to H\) by the counit of \(H\) and \(A \subset E\) by the canonical inclusion are algebra homomorphisms and that the linear map \(j : H \to E\) defined by \(j(h) = 1 \otimes h\) obeys \(aj(h) = a \otimes h = j(h)a\) for all \(h \in H\) and \(a \in A\). Then

\[
\chi(h \otimes g) = j(h_{(1)}) j(g_{(1)})(S j((h_{(2)} g_{(2)})^{(1)}))(h_{(2)} g_{(2)})^{(2)}
\]

is a quantum 2-cocycle on \(H\) with values in \(A\), and \(E = A_\chi \triangleleft \triangleright H\).
**Proof**  We assume that $H$ is a right $A$-comodule coalgebra so that we can form the coproduct $\Delta$ on $A \otimes H$. Then both $A \subset E$ and $\pi : E \to H$ defined by $a \mapsto a \otimes 1$ and $a \otimes h \mapsto \epsilon(a)h$ become Hopf algebra maps. There is also a coaction $\Delta_R : E \to E \otimes H$ given by $(\text{id} \otimes \pi) \circ \Delta$, such that $A = E^H = \{ e \in E | \Delta_R(e) = e \otimes 1 \}$, the fixed point subalgebra. We show now that the linear map $j$ is convolution-invertible, i.e. we find a linear map $j^{-1} : H \to E$ such that $j(h_{(1)})j^{-1}(h_{(2)}) = \epsilon(h) = j^{-1}(h_{(1)})j(h_{(2)})$ for all $h \in H$. Indeed, we set

$$j^{-1}(h) = (Sj(h^{(i)}))h^{(2)}$$

where $S$ denotes the antipode of $E$ and the products are in $E$. Then

$$j^{-1}(h_{(1)})j(h_{(2)}) = (Sj(h_{(1)}^{(i)}))h_{(1)}^{(2)}j(h_{(2)}) = (S(1 \otimes h_{(1)}^{(i)}))(h_{(1)}^{(2)} \otimes h_{(2)}) = \epsilon(1 \otimes h) = \epsilon(h)$$

using the form of the coproduct $\Delta$ of $E$. On the other side, we have

$$j(h_{(1)})j^{-1}(h_{(2)}) = (1 \otimes h_{(1)})(S(1 \otimes h_{(2)}^{(i)}))h_{(2)}^{(2)}$$

$$= (1 \otimes h_{(1)}^{(i)})(S(1 \otimes h_{(2)}^{(i)}))(S(h_{(1)}^{(2)}h_{(1)}^{(2)}h_{(2)}^{(2)})$$

$$= (1 \otimes h_{(1)}^{(i)})(S(1 \otimes h_{(2)}^{(i)}))(S(h_{(1)}^{(2)})h_{(2)}^{(2)}$$

$$= (1 \otimes h_{(1)}^{(i)})(S(1 \otimes h_{(2)}^{(i)}))(S(h_{(1)}^{(2)}h_{(2)}^{(2)})$$

$$= (1 \otimes h_{(1)}^{(i)})(S(h_{(1)}^{(2)} \otimes h_{(2)}^{(2)}))h^{(2)} = \epsilon(h^{(1)})h^{(2)} = \epsilon(h)$$

where the second equality uses the antipode axiom in $A$ to insert $(S(h_{(1)}^{(2)}h_{(1)}^{(2)}h_{(2)}^{(2)})$, the third uses the coaction axiom, the fourth uses covariance of the coproduct of $H$ under the coaction and the fifth that $A$ is a sub-Hopf algebra. We can then use that $S$ is the antipode in $E$ to collapse the expression. Next, from the form of the coproduct it is clear that the map $j$ intertwines the above coaction $\Delta_R$ and the right regular coaction of $H$ on itself. Hence all the conditions for a cleft extension are satisfied and we know from general theory of algebra extensions that $\chi(h \otimes g) = j(h_{(1)})j(g_{(3)})j^{-1}(h_{(2)}g_{(2)})$ has values in $A$ and obeys if $aj(h) = j(h)a$. This is also easy enough to verify directly along the lines above. We can now make the cocycle product $\Delta$ and identify this as the algebra of $E$. We do not need to verify directly since we already know that $E$ is a Hopf algebra; these conditions are equivalent to the bialgebra homomorphism property $\Delta$. □
Extensions of this type can be viewed as quantum principal bundles in the sense of \cite{11}, see \cite{12} for a discussion. We do not require that $A$ is actually commutative, but when this is so it appears in the center of $E$ in the setting above.

### 3 $U_q(\mathfrak{sl}_2)$ as a quantum central extension

In this section we show that the affine quantum group $U_q(\mathfrak{sl}_2)$ in \cite{13} is a quantum group central extension. This quantum group plays a central role in an approach to certain quantum statistical systems\cite{3}. In the conventions of the latter, we have generators $E_i, K_i, F_i$, where $i = 0, 1 \mod 2$, and relations\cite{13}

\begin{align*}
  K_i E_i &= q E_i K_i, & K_i F_i &= q^{-1} F_i K_i, & K_i E_{i+1} &= q^{-1} E_{i+1} K_i, & K_i F_{i+1} &= q F_{i+1} K_i \\
  K_i K_j &= K_j K_i, & [E_i, F_j] &= \delta_{ij} \frac{K_i - K_j^{-1}}{q - q^{-1}},
\end{align*}

(8)

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, and the coproduct

\begin{align*}
  \Delta K_i &= K_i \otimes K_i, & \Delta E_i &= E_i \otimes K_i + 1 \otimes E_i, & \Delta F_i &= F_i \otimes 1 + K_i^{-1} \otimes F_i.
\end{align*}

(9)

Our first step is to work with new generators $K_i F_i \mapsto F_i$, $K_1 \mapsto K$ and $K_0 \mapsto c K^{-1}$ so that the relations become $c$ central and

\begin{align*}
  K E_0 &= q^{-1} E_0 K, & K E_1 &= q E_1 K, & K F_0 &= q F_0 K, & K F_1 &= q^{-1} F_1 K, \\
  q E_0 F_0 - F_0 E_0 &= \frac{k^{2} - 1}{q - q^{-1}}, & q E_1 F_1 - F_1 E_1 &= \frac{k^{2} - 1}{q - q^{-1}}
\end{align*}

(10)

\begin{align*}
  \Delta K &= K \otimes K, & \Delta E_0 &= E_0 \otimes c K^{-1} + 1 \otimes E_0, & \Delta E_1 \otimes K + 1 \otimes E_1, \\
  \Delta c &= c \otimes c, & \Delta F_0 &= F_0 \otimes c K^{-1} + 1 \otimes F_0, & \Delta F_1 \otimes K + 1 \otimes F_1,
\end{align*}

along with the $q$-Serre relations of the same form as in\cite{3} in terms of our new generators $E_i, F_i$; the $c, K$ cancel from these.

Let $U_q(\mathfrak{l} \mathfrak{s} \mathfrak{l}_2)$ denote the quotient of $U_q(\mathfrak{sl}_2)$ obtained by setting $c = 1$. We let $e_i, f_i, k$ denote the images of the generators $E_i, F_i, K$. Thus, $q e_0 f_0 - f_0 e_0 = \frac{k^{2} - 1}{q - q^{-1}}$ and $\Delta e_0 = e_0 \otimes k^{-1} + 1 \otimes e_0$, etc. Since $c$ is central and group-like, this quotient remains a Hopf algebra. It is clear that we have Hopf algebra maps

\begin{align*}
  CZ \mapsto U_q(\mathfrak{sl}_2) \mapsto U_q(\mathfrak{l} \mathfrak{s} \mathfrak{l}_2)
\end{align*}

(11)
where \( \mathbb{C}Z = \mathbb{C}[c, c^{-1}] \) is the sub-Hopf algebra generated by \( c, c^{-1} \). By moving the \( c \)'s to the left, we can clearly identify \( U_q(sl_2) = \mathbb{C}Z \otimes U_q(Lsl_2) \) as linear spaces. This identification restricted to \( U_q(Lsl_2) \) is the linear map

\[
\Delta(k^n g(e_0, e_1) h(f_0, f_1)) = K^n g(E_0, E_1) h(F_0, F_1), \quad j : U_q(Lsl_2) \to U_q(sl_2).
\]

(12)

Here a general element of \( U_q(Lsl_2) \) is clearly a linear combination of terms of this type (i.e. the quantum group has a triangular decomposition), where \( g \) is a polynomial in the non-commuting generators \( e_i \), and \( h \) in the \( f_i \).

Next, because the \( q \)-Serre relations in (8) are homogeneous in the \( E_0, F_0 \), the same is true for the \( e_0, f_0 \). Hence we have a well-defined \( \mathbb{Z} \)-grading defined for polynomials \( g, h \) which are homogeneous in \( e_0, f_0 \) respectively. We write \( |g|, |h| \) for the total degree of \( e_0, f_0 \) respectively (i.e. each term in \( g \) has \( |g| \) occurrences of \( e_0 \), etc.). From this grading, we can define a coaction

\[
\beta(k^n g(e_0, e_1) h(f_0, f_1)) = k^n g(e_0, e_1) h(f_0, f_1) \otimes c^{|g|+|h|}, \quad \beta : U_q(Lsl_2) \to U_q(Lsl_2) \otimes \mathbb{C}Z \quad (13)
\]

on homogeneous polynomials. This is a coaction because applying it again gives \( c^{|g|+|h|} \otimes c^{|g|+|h|} = \Delta c^{|g|+|h|} \). Note that it does not respect the algebra structure, so (as for the map \( j \)) it is not enough to give it on generators. It does, however, respect the coproduct because the coproduct preserves our above ordering (in which we write any \( e_i \) to the left of any \( f_i \)), and manifestly preserves the \( e_0 \) and \( f_0 \) degrees when acting on generators. Hence the product of the coactions on \( \Delta(k^n g(e_0, e_1) h(f_0, f_1)) \) gives the same result as applying the coaction first as in (13) and then \( \Delta \).

We are therefore in a position to make the cross coproduct coalgebra as in (8). For example,

\[
\Delta(1 \otimes e_0) = (1 \otimes e_0) \otimes (c \otimes k^{-1}) + (1 \otimes 1) \otimes (1 \otimes e_0)
\]

from (8). This is the coproduct of \( U_q(sl_2) \) on its identification with \( \mathbb{C}Z \otimes U_q(Lsl_2) \). Similarly for \( \Delta(1 \otimes f_0) \). Hence we are in the setting of Proposition 2.1. We conclude:

**Proposition 3.1** \( U_q(sl_2) \) has the structure of a cocycle bicrossproduct \( \mathbb{C}Z \triangleright \bowtie U_q(Lsl_2) \) with cross coproduct using coaction \( \beta \) from (13) and the cocycle product (8) with respect to \( \chi : U_q(Lsl_2)^{\otimes 2} \to \mathbb{C}Z \) defined by Proposition 2.1 and \( j \) from (12).
For example,

\[
\chi(e_0^a \otimes f_0^b) = \sum_{r=a,s=b}^{r=a,s=b} \left[ \begin{array}{c} a \\ r \\ q^{-1} \end{array} \right] \left[ \begin{array}{c} b \\ s \\ q \end{array} \right] j(e_0^a)j(f_0^b)(Sj(k^{-r}e_0^a r^{-s} f_0^{b-s}))c^{a+b-r-s}
\]

\[
= \sum_{r=a,s=b}^{r=a,s=b} \left[ \begin{array}{c} a \\ r \\ q^{-1} \end{array} \right] \left[ \begin{array}{c} b \\ s \\ q \end{array} \right] E_0^a F_0^b (S(K^{-r}E_0^a r^{-s} F_0^{b-s}))c^{a+b-r-s} = \delta_{a_0} \delta_{b_0}
\]

since we arrive at \( \cdot \circ (\text{id} \otimes S) \circ \Delta(E_0^a F_0^b) \) in this case. Here \( \left[ \begin{array}{c} a \\ r \\ q^{-1} \end{array} \right] \) denotes the appropriate \( q \)-binomial coefficient in \( \Delta e_0^a \), etc. However,

\[
\chi(f_0 \otimes e_0) = F_0 E_0 K^2 + F_0 (Sj(k^{-1}e_0))c + E_0 (Sj(f_0 k^{-1}))c + (Sj((f_0 e_0) J)^{[1]}))(f_0 J_0)^{[2]}
\]

\[
= F_0 E_0 K^2 + F_0 c K S E_0 + E_0 c K S F_0 + (Sj(qe_0 f_0))c^2 + Sj \left( \frac{1 - K^{-2}}{q - q^{-1}} \right)
\]

\[
= F_0 E_0 K^2 + F_0 c K S E_0 + E_0 c K S F_0 + c^2 S(qE_0 F_0) + \frac{1 - K^2}{q - q^{-1}}
\]

\[
= F_0 E_0 K^2 + F_0 c K S E_0 + E_0 c K S F_0 + c^2 S(F_0 E_0) + \frac{K^2 - c^2}{q - q^{-1}} + \frac{1 - K^2}{q - q^{-1}}
\]

\[
= \frac{1 - c^2}{q - q^{-1}}
\]

Similarly for more general \( \chi(f_0^a \otimes e_0^b) \). This is the nontrivial part of the cocycle \( \chi \).

One can check that the cocycle product from \( [2] \) indeed recovers the correct one for \( U_q(\hat{sl}_2) \).

For example,

\[
F_0 E_0 = \chi(f_0 \otimes e_0) K^2 + j(f_0 e_0) = \frac{1 - c^2}{q - q^{-1}} K^2 + j(qe_0 f_0) + \frac{1 - K^{-2}}{q - q^{-1}} = qE_0 F_0 + \frac{1 - c^2 K^{-2}}{q - q^{-1}}
\]

as required.

### 4 \( R \)-matrix form of the quantum cocycle

Here we show that \( U_q(\hat{g}) \) is likewise a quantum group central extension, at least for those cases which can be treated using the \( R \)-matrix formalism in \( [1] [2] \). These authors identified suitable generators \( \mathbf{1}^{\pm}(z), c \), and the relations

\[
\mathbf{1}^{\pm}_1(z) \mathbf{1}^{\pm}_2(w) R(\frac{z}{w}) = R(\frac{z}{w}) \mathbf{1}^{\pm}_2(w) \mathbf{1}^{\pm}_1(z), \quad \mathbf{1}^{\pm}_1(z) \mathbf{1}^{\pm}_2(w) R(\frac{z}{w} q^c) = R(\frac{z}{w} q^{-c}) \mathbf{1}^{\pm}_2(w) \mathbf{1}^{\pm}_1(z)
\]

\[
[c, \mathbf{1}^{\pm}(z)] = 0, \quad \Delta \mathbf{1}^c = \mathbf{1}^c \otimes \mathbf{1}^c, \quad \Delta \mathbf{1}^c = \mathbf{1}^{\pm}(zq^{c_2/2}) \otimes \mathbf{1}^{\pm}(zq^{-c_1/2})
\]

where \( c_1 = c \otimes 1 \) and \( c_2 = 1 \otimes c \) and \( R(z) \) is a suitable solution of the parametrized quantum Yang-Baxter equation \( R_{12}(\frac{z}{w}) R_{13}(z) R_{23}(w) = R_{23}(w) R_{13}(z) R_{12}(\frac{z}{w}) \). There are still further
relations obeyed by the $1^\pm$ which, together with the above ‘quadratic’ ones, provide a definition of $U_q(\hat{g})$ in these terms. Note that the conventions we use here are not quite those in [8] but are more in line with the established conventions for $U_q(g)$ in [14].

First, we move to new matrix generators

$$M^\pm(z) = 1^\pm(zq^{\pm \frac{c}{2}})$$

so that the mixed relations and coproduct become

$$M^-_1(z)M^+_2(w)R(\frac{z}{w}) = R(\frac{z}{w}q^{-2c})M^+_2(w)M^-_1(z), \quad \Delta M^\pm(z) = M^\pm(zq^{\pm c_2}) \otimes M^\pm(z).$$

(16)

The $M^+_1 M^-_2$ relations are unaffected. The antipode is

$$SM^\pm(z) = (M^\pm)^{-1}(zq^{\mp c})$$

(17)

where $(M^\pm)^{-1}(z)$ is the ‘pointwise’ inverse matrix-valued powerseries to $M^\pm(z)$.

Let $U_q(Lg)$ be the quotient of $U_q(\hat{g})$ obtained by setting $q^c = 1$. This is a Hopf algebra since $q^c$ is grouplike and central. We denote the matrix generators in this quotient by $m^\pm$. We have Hopf algebra maps

$$\mathbb{C} Z \to U_q(\hat{g}) \to U_q(Lg).$$

(18)

Here $\mathbb{C} Z$ is the group algebra of $\mathbb{Z}$ with group-like generator $q^c$ (or the Lie algebra $U(1)$ with primitive $c$). It is clear that we can identify $U_q(\hat{g}) = \mathbb{C} Z \otimes U_q(Lg)$ as linear spaces by putting $q^c$ to the left in all normal-ordered expressions. Normal ordered means for us that all $M^-$ modes are put to the right of all $M^+$ modes using the cross relations [14]. Once ordered correctly, we identify products of $M^\pm$ are corresponding to products of $m^\pm$ in $U_q(Lg)$, i.e. restriction of this identification gives the linear map $j : U_q(Lg) \to U_q(\hat{g})$,

$$j(m^+_i(z_1) \cdots m^+_iz_i m^-_{i+1}(z_{i+1}) \cdots m^-_j(z_j)) = M^+_i(z_1) \cdots M^+_iz_i M^-_{i+1}(z_{i+1}) \cdots M^-_j(z_j).$$

(19)

Similarly, we define a coaction $U_q(Lg) \to U_q(Lg) \otimes \mathbb{C} Z$

$$\beta(m^+_i(z_1) \cdots m^+_iz_i m^-_{i+1}(z_{i+1}) \cdots m^-_j(z_j)) = m^+_i(z_1q^{c_2}) \cdots m^+_iz_i q^{c_2} m^-_{i+1}(z_{i+1}q^{-c_2}) \cdots m^-_j(z_j q^{-c_2}),$$

(20)

where $c_2 = 1 \otimes c$. It is easy to see that this is a coaction and respects the matrix form $\Delta m^\pm(z) = m^\pm(z) \otimes m^\pm(z)$ of the coproduct of $U_q(Lg)$ (making it a comodule coalgebra).
The cross coproduct coalgebra structure by this coaction recovers the coproduct of $U_q(\hat{g})$. Thus,

$$\Delta M^\pm(z) = \Delta(1 \otimes m^\pm(z)) = (1 \otimes m^\pm(q^c z)) \otimes (1 \otimes m^\pm(z)) = M^\pm(q^c z) \otimes M^\pm(z),$$

where $c_3 = 1 \otimes 1 \otimes c \otimes 1$ in the middle expression. Hence we are in the setting of Proposition 2.1 and conclude:

**Proposition 4.1** $U_q(\hat{g})$ has the structure of a cocycle bicrossproduct $\mathbb{C}Z_{\chi} \triangleright U_q(Lg)$ with cross coproduct from (9) using coaction (20) and cocycle product (2) with respect to $\chi : U_q(Lg)^\otimes 2 \to \mathbb{C}Z$ defined from Proposition 2.1 and $j$ from (14).

For example, we have

$$\chi(m^+_1(z) \otimes m^-_2(w)) = M^+_1(z)M^-_2(w)S(M^+_1(zq^{c_2})M^-_2(wq^{-c_2})) = 1,$$

where $c_2$ denotes $c$ placed to the far right (outside the range of $S$). These factors cancel $q^{\pm c}$ in (17) when we compute $S$, giving the identity matrix. In the other order, we have

$$\chi(m^-_1(z) \otimes m^+_2(w)) = M^-_1(z)M^+_2(w)Sj((m^-_1(z)m^+_2(w))^{(i)})(m^-_1(z)m^+_2(w))^{(\bar{i})}$$

$$= M^-_1(z)M^+_2(w)Sj(R(-\bar{w})m^+_2(wq^{c_2})m^-_1(zq^{-c_2})R^{-1}(-\bar{w}))$$

$$= M^-_1(z)M^+_2(w)R(-\bar{w})(M^-)^{-1}_1(z)(M^+_1)^{-1}(w)R^{-1}(-\bar{w}) = R(-\bar{w}q^{-2c})R^{-1}(-\bar{w})$$

where we used the relations (16) to normal order before applying the coaction $\beta$ and the map $j$. The effect of the coaction cancels the $q^{\pm c}$ factor from the action of $S$. The final step uses the relations (16) in reverse.

It should be clear that the same computation works in general and gives us $\chi$ on a general product of $m^\pm$ in terms of products of $R$ and $R^{-1}$ with $q^{\pm c}$ in the arguments. For example,

$$\chi(m^-_1(z_1)m^-_2(z_2) \otimes m^+_3(z_3)m^+_4(z_4))$$

$$= R_{23}(\bar{z}_2q^{-2c})R_{13}(\bar{z}_3^{-1})R_{24}(\bar{z}_4q^{-2c})R_{14}(\bar{z}_4^{-1})R_{23}^{-1}(\bar{z}_3q^{-2c})R_{13}^{-1}(\bar{z}_3q^{-2c})R_{23}^{-1}(\bar{z}_3q^{-2c})R_{13}^{-1}(\bar{z}_3q^{-2c})R_{23}^{-1}(\bar{z}_3q^{-2c}).$$

The general case consists of the $R$-matrices arising on one side in the normal ordering of the product of the arguments of $\chi$, with factors $q^{-2c}$, followed by the inverse pattern of $R$-matrices without the $q^{-2c}$ factor.
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