Abstract

Starting from the Kac–Moody structure of the WZNW model for $SL(2, R)$ and using the general canonical formalism, we formulate a gauge theory invariant under local $SL(2, R) \times SL(2, R)$ and diffeomorphisms. This theory represents a gauge extension of the WZNW system, defined by a difference of two simple WZNW actions. By performing a partial gauge fixing and integrating out some dynamical variables, we prove that the resulting effective theory coincides with the induced gravity in 2D. The geometric properties of the induced gravity are obtained out of the gauge properties of the WZNW system with the help of the Dirac bracket formalism.

1 Introduction

The subject of two–dimensional (2D) gravity has two–fold interest: first, it describes important dynamical aspects of string theory, as an effective theory induced by quantum string fluctuations, and second, it represents a useful theoretical model for the realistic theory of gravity in four dimensions. Being closely related to the Weyl anomaly in string theory [1], the induced gravity features a deep analogy with the usual Wess–Zumino action in gauge theories, and represents its gravitational analogue [2]. The effective action for 2D gravity was originally calculated in the conformal gauge, where it has the form of the Liouville theory [1, 3]. Analyzing the dynamical structure of this theory in the light–cone gauge Polyakov found an unexpected connection with $SL(2, R)$ current algebra [2]. The importance of this result has been confirmed by the existence of a canonical formulation of the theory in terms of gauge independent variables, the $SL(2, R)$ currents [4, 5].

Inspired by the above results, Polyakov studied the connection between the Wess–Zumino–Novikov–Witten (WZNW) model for $SL(2, R)$ and the induced gravity in the light–cone gauge, trying to understand how the geometric structure of spacetime can be obtained out of the chiral $SL(2, R)$ symmetry of the WZNW model [6] (see also [7]). Similar approach based on
the conformal gauge showed that the related form of 2D induced gravity, the Liouville theory, may be obtained from the $SL(2, R)$ WZNW model by imposing certain conformally invariant constraints \[8\]. A consistent approach to this reduction procedure has been formulated using a gauge extension of the original WZNW model, based on two gauge fields \[9\].

In the present paper we shall use the general canonical formalism to formulate a gauge theory invariant under local $SL(2, R) \times SL(2, R)$ transformations and diffeomorphisms, which represents a gauge extension of the WZNW system,

$$I(g_1, g_2) = I(g_1) - I(g_2) \quad g_1, g_2 \in SL(2, R) ,$$

defined by a difference of two simple WZNW actions for $SL(2, R)$ group; then, we shall show, by performing a suitable gauge fixing and integrating out some dynamical variables, that the resulting effective theory coincides with the induced gravity in 2D:

$$I_G(\phi, g_{\mu\nu}) = \int d^2\xi \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \alpha \phi R + M \left( e^{2\phi/\alpha} - 1 \right) \right] ,$$

We are able to demonstrate this connection in a covariant way, fully respecting the diffeomorphism invariance of the induced gravity, generalizing thereby the results of Polyakov and others \[6, 7, 8, 9\].

We are going to use the general canonical method of constructing gauge invariant actions \[10\]. It is based on the fact that the Lagrangian equations of motions are equivalent to the Hamiltonian equations derived from the action

$$I(q, \pi, u) = \int d\xi (\pi \dot{q} - H_0 - u^m G_m) ,$$

where $G_m$ are primary constraints, and $H_0$ is the canonical Hamiltonian. If $G_m$ are first class constraints, satisfying the Poisson bracket algebra

$$\{G_m, G_n\} = U_{mn}^r G_r , \quad \{G_m, H_0\} = V_m^r G_r ,$$

than the canonical action $I(q, \pi, u)$ is invariant under the following gauge transformations:

$$\delta F = \varepsilon^m \{F, G_m\} , \quad F = F(q, \pi)$$

$$\delta u^m = \varepsilon^m + u^r \varepsilon^s U_{sr}^m + \varepsilon^r V_r^m .$$

This paper represents not only an extension of the results obtained in the previous letter \[11\], but also a significant simplification of the basic dynamical structure; it also gives a natural explanation of the gauge origin of the geometry of spacetime. The Hamiltonian approach presented here is in complete agreement with the results of the Lagrangian analysis \[12\].

We begin our exposition in Section 2 by recalling some basic facts about the WZNW model for $SL(2, R)$. Then, we use the Hamiltonian formalism to analyze chiral symmetries of the model by choosing $\tau = \xi^-$ and $\tau = \xi^+$ as the time variables, and derive the related $SL(2, R)$ currents. In Section 3 we use these currents to define the energy–momentum components $T_{\pm}$ as the first class constraints satisfying two independent Virasoro algebras, whereupon the application of the
general canonical formalism leads to the covariant extension (with respect to diffeomorphisms) of
the WZNW model. In Section 4 we study the problem of gauging the internal $SL(2, R) \times SL(2, R)$
symmetry of the WZNW theory by doubling the number of phase space variables. After defining
a new set of currents $I_{\pm a}$, satisfying two independent $SL(2, R)$ algebras without central charges,
we apply the canonical gauge procedure to the set of first class constraints $G_m = (T_\pm, I_{\pm a})$, and
obtain our basic model — canonically gauged action of the WZNW system. In Section 5 we
define a restriction of the theory based on a subset of first class constraints $G_m$, then we choose
a set of gauge fixing conditions that does not affect the diffeomorphism invariance, formulate the
quantum action using the BRST formalism, and finally integrate out some variables to obtain
an effective theory that coincides with the induced gravity. In Section 6 we use the Dirac
brackets to show how geometric properties of the induced gravity follow from gauge properties
of the WZNW system, and Section 6 is devoted to concluding remarks. Geometric properties
of the group $SL(2, R)$ and spacetime manifold $\Sigma$, as well as some other technical details, are
presented in the Appendix.

2 Chiral symmetries of the WZNW model for $SL(2, R)$

Chiral symmetries of the $SL(2, R)$ WZNW model can be naturally analyzed in the Hamiltonian
formalism based on $\tau = \xi^-$ or $\xi^+$ as the evolution parameters. As a result, one finds that
these symmetries are closely related to the Kac–Moody (KM) structure of the theory, which
plays an essential role in the canonical formalism for constructing gauge invariant theories.

2.1 Construction of the action

Two–dimensional WZNW model is a field theory in which the basic field $g$ is a mapping from $\Sigma$
to $G$, $\Sigma$ being a two–dimensional Riemannian spacetime, and $G$ being a semisimple Lie group.
The model is defined by the action

$$I(g) = I_0 + n\Gamma = \frac{1}{2}\kappa \int_\Sigma (^*v, v) + \frac{1}{3}\kappa \int_M (v, v^2), \quad v = g^{-1}dg,$$

(2.1)

where the first term is the action of the non–linear $\sigma$–model, while the second one is the topo-
logical Wess–Zumino term, defined over a three–manifold $M$ whose boundary is the spacetime:
$\partial M = \Sigma$. Here, $n$ is an integer, $\kappa = n\kappa_0$, $\kappa_0$ being a normalization constant, $v$ is the Maurer–
Cartan (Lie algebra valued) one–form, $^*v$ is the dual of $v$, and $(X, Y) = \frac{1}{2}\text{Tr}(XY)$ is the
Cartan–Killing bilinear form (the trace operation is taken in the adjoint representation of $G$).
With a suitable choice of $\kappa_0$ the Wess–Zumino term is well defined modulo a multiple of $2\pi$,
which is irrelevant in the functional integral $I = \int Dg \exp[iI(g)]$.

Using the variation $\delta g = gu$ and the first structural equation $dv + v^2 = 0$, one obtains the
equations of motion in the form $d(^*v - v) = 0$. In local coordinates $\xi^\pm$ on $\Sigma$ these equations can
be written as $\partial_-(g^{-1}\partial_+ g) = 0$, or, equivalently, $\partial_+(g\partial_- g^{-1}) = 0$.

It should be noted that the WZNW model is invariant under chiral transformations

$$g \rightarrow g' = \Omega(\xi^-)g\Omega^{-1}(\xi^+),$$

(2.2)
where \((\Omega, \bar{\Omega})\) belongs to \(G \times G\).

If the group elements are parametrized by some local coordinates \(q^\alpha\), \(g = g(q^\alpha)\), one can use the expansion \(v = E^a t_a = dq^\alpha E^a_{\alpha} t_a\), where \(t_a\) are the generators of \(G\), and derive the relations
\[
(*)v, v = dq^\alpha dq^\beta \gamma_{\alpha\beta}, \quad \gamma_{\alpha\beta}(q) \equiv E^a_{\alpha} E^b_{\beta} \gamma_{ab},
\]
\[
(v, v^2) = \frac{1}{2} E^a E^b E^c f_{abc} = -6 d\tau.
\]

Here, \(\gamma_{ab}\) is the Cartan metric on \(G\), \(f_{abc} = f_{abe} \gamma_{ec}\) are totally antisymmetric structure constants, and the form of the last equation follows from \(d(v, v^2) = 0\), using the theorem that any closed form is locally exact. Then, the WZNW action (2.1) takes the form
\[
I(q) = \kappa \int_\Sigma \left( \frac{1}{2} dq^\alpha dq^\beta \gamma_{\alpha\beta} - dq^\alpha dq^\beta \tau_{\alpha\beta} \right),
\]
where we used the Stokes theorem to transform the second term into an integral over \(\Sigma\), and \(\tau = dq^\alpha dq^\beta \tau_{\alpha\beta}/2\). Choosing the Minkowskian structure for spacetime one can introduce the inertial coordinates \(\xi^\mu (\mu = 0, 1)\) on \(\Sigma\), and write the action as
\[
I(q) = \kappa \int_\Sigma d^2\xi \left( \frac{1}{2} \eta^{\mu\nu} \partial_\mu q^\alpha \partial_\nu q^\beta \gamma_{\alpha\beta} - \varepsilon^{\mu\nu} \partial_\mu q^\alpha \partial_\nu q^\beta \tau_{\alpha\beta} \right).
\]

Now, we turn our attention to \(G = SL(2, R)\). Starting from the fact that any element \(g\) of \(SL(2, R)\) admits the Gauss decomposition, defined by equation (A.3), one can introduce the related group coordinates \(q^\alpha = (x, \varphi, y)\), use the expressions (A.5) and (A.6) for \(\gamma_{\alpha\beta}\) and \(\tau\), respectively, and derive the following local form of the WZNW action:
\[
I = \kappa \int_\Sigma d^2\xi \left[ \frac{1}{2} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 2(\gamma^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_\mu x \partial_\nu ye^{-\varphi} \right],
\]
\[
= \kappa \int_\Sigma d^2\xi \left( \partial_+ \varphi \partial_- \varphi + 4 \partial_+ x \partial_- ye^{-\varphi} \right), \quad (2.3)
\]
where \(\xi^\pm = (\xi^0 \pm \xi^1)/\sqrt{2}\).

### 2.2 Chiral symmetries and KM currents

Chiral symmetries (2.2) are not the standard gauge symmetries: parameters of the transformations are not arbitrary functions of both coordinates, but depend only on \(\xi^-\) or \(\xi^+\). Usually, gauge symmetries in the Hamiltonian framework are related to the presence of first class constraints. However, if we take \(\tau = \xi^0\) as the time variable in the action (2.3), it is easily seen that there are no first class constraints in the theory. The solution to this puzzle lies in the observation that the Hamiltonian definition of gauge symmetries is based upon a definite choice of time. The absence of gauge symmetries for the choice \(\tau = \xi^0\) does not mean that these symmetries are absent for any other choice. Investigations of 2D induced gravity [14, 3] and the WZNW model [13] showed that the correct approach to understanding chiral symmetries in the Hamiltonian approach is to use the light–cone coordinate, \(\xi^-\) or \(\xi^+\), as the time variable. Following the approach of reference [13], we shall be able to detect the chiral symmetry (2.2).
of the $SL(2,R)$ WZNW action (2.3) and find out its close relationship to a set of currents, satisfying an $SL(2,R)$ KM algebra. These currents represent basic objects in our approach: they will enable us to make a covariant and gauge extension of the WZNW system (1.1), whereupon one can dynamically reduce the whole structure by fixing the gauge and integrating out some dynamical variables, and obtain the induced gravity action (1.2).

1. Let us first consider the choice $\tau = \xi^-, \sigma = \xi^+$. The basic Lagrangian dynamical variables in the action (2.3) are $q^a = (x, \varphi, y)$. The definition of the corresponding conjugate momenta $(\pi_x, \pi_\varphi, \pi_y)$ leads to the following primary constraints:

$$
\begin{align*}
-J_{-x} & \equiv \pi_x \approx 0, \\
-J_{-\varphi} & \equiv \pi_\varphi - \kappa \varphi' \approx 0, \\
-J_{-y} & \equiv \pi_y - 4\kappa x' e^{-\varphi} \approx 0,
\end{align*}
$$

where prime denotes the space ($\sigma$) derivative. It is convenient to transform the constraints $J_{-a} = (J_{-x}, J_{-\varphi}, J_{-y})$ into the tangent space basis by writing $J_{-a} = \bar{E}_a^\alpha J_{-\alpha}$, where $\bar{E}_a^\alpha$ are the vielbein components on the $SL(2,R)$ manifold (Appendix A):

$$
\begin{align*}
J_{-(+)} & = \pi_x , \\
J_{-(0)} & = x\pi_x + (\pi_\varphi - \kappa \varphi') , \\
J_{-(-)} & = -x^2 \pi_x - 2x(\pi_\varphi - \kappa \varphi') - 4\kappa x' + \pi_y e^\varphi. \\
\end{align*}
$$

(2.4)

Poisson brackets of the primary constraints define an $SL(2,R)$ KM algebra with central charge $c_- = -2\kappa$:

$$
\{J_{-a}, J_{-b}\} = f_{abc} J_{-c} \delta.
$$

Since the Lagrangian is linear in velocities, the canonical Hamiltonian vanishes, and the total Hamiltonian takes the form $\bar{H}_T = \int d\sigma u^a J_{-a}$. The consistency conditions of the primary constraints $J_{-a}$ are

$$
\frac{d}{d\tau} J_{-a} = \{J_{-a}, \bar{H}_T\} \approx 2\kappa u'_a \approx 0,
$$

which implies $u^a(\tau, \sigma) = u^a(\tau)$, i.e. $u^a$ is an arbitrary multiplier depending on $\tau = \xi^-$ only. Therefore,

$$
\bar{H}_T = u^a(\tau) j_{-a}, \quad j_{-a} \equiv \int d\sigma J_{-a}(\tau, \sigma).
$$

It follows from (2.5) that the constraints $j_{-a}$, the zero modes of $J_{-a}$, are of the first class:

$$
\{j_{-a}, j_{-b}\} = f_{abc} j_{-c} \delta. 
$$

The presence of arbitrary multipliers $u^a(\tau)$ in $\bar{H}_T$ means that the theory possesses a specific gauge symmetry, the chiral symmetry, characterized by parameters $\omega^a = \omega^a(\tau)$. Since there are no secondary constraints, the symmetry generator takes the simple form: $G = \omega^a(\tau) j_{-a}$ [15]. The symmetry transformations of $q^a = (x, \varphi, y)$ are given as

$$
\delta q^a = \{q^a, G\} = -E_a^\alpha \omega^a.
$$

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The related symmetry transformations of \( g(q^\alpha) \) are:
\[
g \delta g^{-1} = t_a \bar{E}^a_\alpha \delta q^\alpha = -\omega \quad \Rightarrow \quad \delta g = \omega g,
\]
where \( \omega \equiv \omega^a t_a \). Since \( \omega \) is an infinitesimal parameter, we can write
\[
g \rightarrow g + \delta g \approx \Omega(\xi^-)g, \quad \Omega(\xi^-) = e^{\omega(\xi^-)},
\]
which is equivalent to the \( \Omega(\xi^-) \) piece of (2.2). Thus, \( \xi^- \) chiral symmetry is produced by the zero modes of the KM currents \( J_{-a} \).

The Hamiltonian equations of motion are
\[
\dot{q}^\alpha = \{q^\alpha, \bar{H}_T\} = -\bar{E}^a_\alpha u^a(\tau),
\]
\[
g \frac{d}{d\tau} g^{-1} = t_a \bar{E}^a_\alpha \dot{q}^\alpha = -u \quad \Rightarrow \quad \dot{g} = u g,
\]
where \( u = u^a t_a \). This implies \( \partial_+(g\partial_- g^{-1}) = 0 \), in conformity with the Lagrangian result.

2. Now, we consider the second choice, \( \tau = \xi^+, \; \sigma = -\xi^- \) (the minus sign is adopted in order to preserve the orientation of the manifold), and find the following primary constraints:
\[
-J_{+x} \equiv \pi_x + 4\kappa \gamma e^{-\varphi} \approx 0, \\
-J_{+\varphi} \equiv \pi_{\varphi} + \kappa \varphi' \approx 0, \\
-J_{+y} \equiv \pi_y \approx 0.
\]
They can be transformed into the tangent space basis with the help of \( J_{+a} = E^\alpha_a J_{+\alpha} \):
\[
J_{+(+)} = y^2 \pi_y + 2y(\pi_{\varphi} + \kappa \varphi') - 4\kappa \gamma e^{-\varphi}, \\
J_{+(0)} = -y \pi_y - (\pi_{\varphi} + \kappa \varphi'), \\
J_{+(\pm)} = -\pi_y.
\]
(2.6)
The related Poisson bracket algebra has the form of the KM algebra with central charge \( c_+ = 2\kappa \):
\[
\{J_{+a}, J_{+b}\} = f_{ab} c J_{+c} + 2\kappa \gamma_{ab} \delta^c.
\]
(2.7)
The canonical Hamiltonian vanishes, while the total Hamiltonian is linear in \( J_{+a} \). The rest of the Hamiltonian analysis can be done in a similar manner, leading to the \( \bar{\Omega}(\xi^+) \) piece of the chiral symmetry (2.2).

3 Covariant extension of the WZNW model

In the previous analysis we obtained chiral symmetries of the WZNW model by using \( \tau = \xi^\pm \) as the evolution parameter in the Hamiltonian approach. These symmetries are generated by the zero modes of the KM currents \( J_{\mp a} \). Now, we return to the usual formulation with \( \tau = \xi^0 \), and
discuss how the KM structure of the WZNW model can be used to build the covariant extension (with respect to diffeomorphisms) of the WZNW model (2.3).

Using the explicit, canonical expressions for the KM currents, given by equations (2.4) and (2.7), we can construct the related $SL(2, R)$ invariant expressions,

$$
T_-(q, \pi) = \frac{1}{4\kappa} \gamma^{ab} J_(-)a J_(-)b = \frac{1}{4\kappa} \left[ \pi_x \pi_y e^\varphi + (\pi_\varphi - \kappa \varphi')^2 \right] - x' \pi_x , \\
T_+(q, \pi) = -\frac{1}{4\kappa} \gamma^{ab} J_+a J_+b = -\frac{1}{4\kappa} \left[ \pi_x \pi_y e^\varphi + (\pi_\varphi + \kappa \varphi')^2 \right] - y' \pi_y , \tag{3.1}
$$

representing the components of the energy–momentum tensor (the Hamiltonian of the action (2.3) for $\tau = \xi^0$ is given by $T_- - T_+$). These components satisfy two independent Virasoro algebras:

$$
\{T_+ (\sigma_1), T_+ (\sigma_2)\} = -[T_+ (\sigma_1) + T_+ (\sigma_2)] \partial_1 \delta . \tag{3.2}
$$

The above result shows that the KM currents of the WZNW model can be used to construct the Virasoro algebra, which is equivalent to the algebra of diffeomorphisms (see, e.g., Ref. [5]). In the next step we shall use the general canonical formalism, expressed by equations (1.3), to construct a covariant theory, in which

$$
H_0 = 0 , \quad G_m = (T_-, T_+) . \tag{3.3a}
$$

This is done by introducing the canonical Lagrangian

$$
\mathcal{L}(q, \pi, h) = \pi_\alpha \dot{q}^\alpha - h^- T_- - h^+ T_+ . \tag{3.3b}
$$

To see the usual content of this Lagrangian, one can eliminate the momentum variables with the help of the equations of motion:

$$
\pi_x = \frac{4\kappa}{h^- - h^+} e^{-\varphi} (\partial_0 + h^+ \partial_1) y , \\
\pi_\varphi \pm \kappa \varphi' = \frac{2\kappa}{h^- - h^+} (\partial_0 + h^\pm \partial_1) \varphi , \\
\pi_y = \frac{4\kappa}{h^- - h^+} e^{-\varphi} (\partial_0 + h^- \partial_1) x .
$$

Then, after introducing new variables $(h^-, h^+) \to \tilde{g}^{\mu\nu},$

$$
\tilde{g}^{00} = \frac{2}{h^- - h^+} , \quad \tilde{g}^{01} = \frac{h^- + h^+}{h^- - h^+} , \quad \tilde{g}^{11} = \frac{2h^- h^+}{h^- - h^+} ,
$$

with $\det(\tilde{g}^{\mu\nu}) = -1,$ one obtains

$$
\mathcal{L}(q, h) = \kappa \left[ \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 2(\tilde{g}^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_\mu x \partial_\nu ye^{-\varphi} \right] . \tag{3.4}
$$

It is now natural to identify $\tilde{g}^{\mu\nu}$ with the metric density, whereupon the above expression is seen to represent the covariant generalization of the WZNW theory.
The transformation properties of $\tilde{g}^{\mu\nu}$ are consistent with this interpretation. Indeed, using the general form of the gauge transformations (1.4), where $V_r^m = 0$ and $U_{sr}^m$ is calculated from the algebra (3.2) (Appendix C), one obtains

$$\delta h^\mp = \partial_0 \varepsilon^\mp + h^\mp \partial_1 \varepsilon^\mp - \varepsilon^\mp \partial_1 h^\mp,$$

(3.5a)

Then, after introducing new parameters $\varepsilon^\mp = \varepsilon^1 - \varepsilon^0 h^\mp$ one finds

$$\delta \tilde{g}^{\mu\nu} = \tilde{g}^{\mu\rho} \partial_\rho \varepsilon^\nu + \tilde{g}^{\nu\rho} \partial_\rho \varepsilon^\mu - \partial_\rho (\varepsilon^\rho \tilde{g}^{\mu\nu}),$$

(3.5b)

which is the diffeomorphism transformation of the metric density $\tilde{g}^{\mu\nu} = \sqrt{-gg^{\mu\nu}}$, as follows from (3.5).

Let us note that the Lagrangian (3.4) is also invariant under conformal rescalings of the metric. Of particular importance for latter considerations is the light–cone basis, defined in Appendix B, in which the Lagrangian (3.4) can be written as

$$L(q, h) = \kappa \sqrt{-\hat{g}} \left[ \hat{\partial}_+ \varphi \hat{\partial}_- \varphi + 4 \hat{\partial}_+ x \hat{\partial}_- ye^{-\varphi} \right],$$

(3.6)

where

$$\hat{\partial}_\pm \equiv \hat{e}_\pm^\mu \partial_\mu = \frac{\sqrt{2}}{h^--h^+}(\partial_0 + h^\mp \partial_1), \quad \sqrt{-\hat{g}} = \frac{1}{2} (h^- - h^+).$$

4 Gauging $SL(2, R) \times SL(2, R)$ and the WZNW system

In the previous section we obtained the covariant extension of the WZNW model, using the energy–momentum components $T_\pm$ as the generators of diffeomorphisms. Since the components $T_\pm$ satisfy the Virasoro algebras (3.2) without central charges, they are first class constraints, and one is able to apply directly the general canonical method for constructing gauge invariant actions, presented in the introduction.

Our next task is to consider the possibility of gauging the internal $SL(2, R) \times SL(2, R)$ symmetry. One should observe that the currents $J_\pm a$ are not of the first class, since the related KM algebras have central charges $c_\pm = \pm 2\kappa$; therefore, they can not be used as the gauge generators. We wish to find a set of generators satisfying two independent $SL(2, R)$ algebras without central charges. To this end we double the number of dynamical variables,

$$q \rightarrow (q_1, q_2), \quad \pi \rightarrow (\pi_1, \pi_2),$$

and introduce two sets of currents,

$$J^{(1)}_{\pm a} = J_{\pm a}(q_1, \pi_1), \quad J^{(2)}_{\pm a} = J_{\pm a}(q_2, \pi_2)|_{\kappa \rightarrow -\kappa},$$

(4.1)

satisfying two $SL(2, R)$ KM algebras with opposite central charges:

$$c^{(1)}_\pm = \pm 2\kappa, \quad c^{(2)}_\pm = \mp 2\kappa.$$

Now, we introduce new currents,

$$I_{\pm a} = J^{(1)}_{\pm a} + J^{(2)}_{\pm a},$$

(4.2)
which are easily seen to satisfy two independent $SL(2, R)$ algebras with vanishing central charges. The new currents are of the first class, and can be used to gauge the internal $SL(2, R) \times SL(2, R)$ symmetry.

In order to include the diffeomorphisms into this procedure, we introduce the energy–momentum components of two sectors, defined in terms of $J^{(1)}$ and $J^{(2)}$ as in (3.1),

$$ T^{(1)}_{\pm} = T_{\pm}(q_1, \pi_1), \quad T^{(2)}_{\pm} = T_{\pm}(q_2, \pi_2)|_{\kappa \to -\kappa}, $$

which obey the Poisson bracket algebra (3.2). The complete energy–momentum is defined by

$$ T_{\pm} = T^{(1)}_{\pm} + T^{(2)}_{\pm}, $$

and it also satisfies the Virasoro algebra (3.2).

The Poisson bracket algebra between $I_{\pm a}$ and $T_{\pm}$ has the form

$$ \{I_{\pm a}(\sigma_1), I_{\pm b}(\sigma_2)\} = f_{abc} I_{\pm c}(\sigma_2) \delta^a, $$

$$ \{T_{\pm}(\sigma_1), I_{\pm a}(\sigma_2)\} = -I_{\pm a}(\sigma_1) \delta^\prime, $$

$$ \{T_{\pm}(\sigma_1), T_{\pm}(\sigma_2)\} = -[T_{\pm}(\sigma_1) + T_{\pm}(\sigma_2)] \delta^\prime, $$

(4.5)

and represents two copies of the semi–direct product of the $SL(2, R)$ and Virasoro algebras. Together with diffeomorphisms, described in the previous section, we have here an additional $SL(2, R) \times SL(2, R)$ structure. Therefore, the collection $(T_{\pm}, I_{\pm a})$ can be taken as a set of first class constraints in the general canonical construction based on (1.3). The related dynamical system will be called the WZNW system.

We display here the complete set of constraints, multipliers and gauge parameters:

$$ G_m = T_-, \quad T_+, \quad I_{-a}, \quad I_{+a}, $$

$$ u^m = h^-, \quad h^+, \quad a^+_a, \quad a^-_a, $$

$$ \varepsilon^m = \varepsilon^-, \quad \varepsilon^+, \quad \eta^+_a, \quad \eta^-_a. $$

Now, using

$$ H_0 = 0, \quad G_m = (T_-, T_+, I_{-a}, I_{+a}), $$

(4.6a)

one can construct the related canonical Lagrangian:

$$ \mathcal{L}(q_i, \pi_i, h) = \pi_1 a_1^a + \pi_2 a_2^a - h^T_- h^+ T_+ - a^+_a T_{-a} - a^-_a I_{+a}. $$

(4.6b)

It represents a gauge theory invariant under both local $SL(2, R) \times SL(2, R)$ transformations and diffeomorphisms.

Using the general rule (1.4), with $V_{r m} = 0$ and $U_{sr m}$ calculated from the algebra (1.5) (Appendix C), one finds the following gauge transformations of the multipliers:

$$ \delta h^\pm = (\partial_0 + h^\mp \partial_1) \varepsilon^\pm - \varepsilon^\pm \partial_1 h^\pm, $$

$$ \delta a^\pm_a = (\partial_0 + h^\mp \partial_1) \eta^\pm_a - f_{ab} c_{a b}^\pm \eta^\pm b - \varepsilon^\pm \partial_1 a^c_a. $$

(4.7)
Gauge transformations of the dynamical variables are
\[
\delta q^\alpha = -\tilde{E}^\alpha_+ \eta^\alpha_+ - E^{\alpha}_- \eta^\alpha_- + \frac{1}{2\kappa} \left( \varepsilon^+ J^\alpha_+ - \varepsilon^- J^\alpha_- \right), \quad q = q_1 , \tag{4.8}
\]
while \( \delta q_2 \) is obtained by changing \( \kappa \) to \(-\kappa\).

As before, we can eliminate the momenta \( \pi_{1\alpha} \) and \( \pi_{2\alpha} \) in order to clarify the usual Lagrangian content of the theory. Explicit calculation shows the complete agreement with the Lagrangian treatment of reference [12]: the resulting Lagrangian describes the gauge extension of the WZNW system (1.1), invariant under local \( SL(2, R) \times SL(2, R) \) and diffeomorphisms (Appendix D).

In the canonical approach, the WZNW system is introduced in the process of constructing the first class constraints \( I_{\pm\alpha} \), used to gauge the whole \( SL(2, R) \times SL(2, R) \) group. This approach closely parallels the related gauge procedure in the Lagrangian formalism [12]. Namely, it is well known that one can not gauge the simple WZNW model (2.1) consistently for an arbitrary gauge group \( H \subseteq SL(2, R) \times SL(2, R) \), since the Wess–Zumino term \( \Gamma \) does not have a gauge invariant extension that can be expressed as an integral over spacetime \( \Sigma \) [16]. However, the problem can be solved by going over to the WZNW system (1.1), where the problematic, nonlocal term appearing in the first sector during the gauge procedure, cancels the corresponding term in the second sector, producing thus the consistent gauge theory for every \( H \). There is a clear analogy between the cancellation of nonlocal terms in the Lagrangian procedure, and the elimination of central charges in the Hamiltonian approach.

5 Gauge extension of the WZNW system and induced gravity

In this section we shall show that 2D induced gravity can be obtained from the canonical gauge extension of the WZNW system, by

(a) performing a suitable gauge fixing, and

(b) integrating out some dynamical variables in the functional integral.

5.1 Canonical \( H_+ \times H_- \) gauge theory

Let us consider a restriction of the canonical theory (4.6), defined by the following subset of first class constraints:
\[
G'_m = \left( T_-, T_+, I_n \right) , \quad I_n \equiv \left[ I_{-(+)}, I_{-(0)}, I_{-(+)} \right] , \tag{5.1}
\]
representing a subalgebra of (4.5). This restriction can be obtained from the full canonical theory (4.6) by imposing the following gauge conditions:
\[
a^{(-)}_+ = 0 , \quad a^{(+)}_- = 0 . \tag{5.2}
\]
The restricted algebra based on (5.1) describes diffeomorphisms combined with the internal symmetry

\[ H = H_+ \times H_-, \quad (5.3) \]

where \( H_+ \) and \( H_- \) are subgroups of \( SL(2, R) \) defined by the generators \((t_+, t_0)\) and \((t_0, t_-)\), respectively.

The canonical action of the restricted theory takes the form

\[ L(q_i, \pi_i, h) = \pi_{1\alpha} \dot{q}^1_\alpha + \pi_{2\alpha} \dot{q}^2_\alpha - h^- T^- - h^+ T^+ - a^n I_n, \quad (5.4) \]

where \( a^n \equiv [a_+^{(+)}, a_+^{(0)}, a_-^{(-)}, a_-^{(0)}] \). Here, explicit expressions for the energy–momentum components are given by equation (4.4), in conjunction with (4.3) and (3.1), while the currents \( I_n \) are of the form

\[
\begin{align*}
I_{- (+)} &= \pi_{x_1} + \pi_{x_2}, \\
I_{- (0)} &= \left[ x_1 \pi_{x_1} + (\pi \varphi_1 - \kappa \varphi'_1) \right] + \left[ x_2 \pi_{x_2} + (\pi \varphi_2 + \kappa \varphi'_2) \right], \\
I_{+ (-)} &= -\pi_{y_1} - \pi_{y_2}, \\
I_{+ (0)} &= \left[ -y_1 \pi_{y_1} - (\pi \varphi_1 + \kappa \varphi'_1) \right] + \left[ -y_2 \pi_{y_2} - (\pi \varphi_2 - \kappa \varphi'_2) \right].
\end{align*}
\]

It is clear that the canonical action (5.4) represents a gauge extension of the WZNW system (1.1). Indeed, by choosing the gauge fixing \( a^n = 0 \), and eliminating the momenta \( \pi_{1\alpha} \) and \( \pi_{2\alpha} \), the action (5.4) reduces to the form

\[ L(q_1, q_2, h) = L(q_1, h) - L(q_2, h), \]

where \( L(q, h) \) is given by equation (3.6), representing the covariant extension of (1.1) (see Appendix D). In what follows we shall demonstrate that the action (5.4) can be effectively reduced to the induced gravity (1.2).

### 5.2 Effective theory in the canonical form

**Quantum action.** In order to demonstrate the connection of the canonical theory (5.4) to the induced gravity, we begin by choosing the gauge conditions corresponding to the first class constraints \( I_n \):

\[
\Omega_n = \begin{bmatrix} \Omega_{- (+)}, \Omega_{- (0)}, \Omega_{+ (-)}, \Omega_{+ (0)} \end{bmatrix},
\]

\[
\Omega_{\pm (+)} = J^{(1)}_{\pm (+)} - \mu_{\pm} = 0, \quad \Omega_{\pm (0)} = J^{(2)}_{\pm (0)} - \lambda_{\pm} = 0. \quad (5.5)
\]

To impose these conditions in the functional integral, we use the BRST formalism and introduce the following set of ghosts, antighosts and new multipliers:

- **Ghost fields:** \( e^-, e^+, e^n \)
- **Antighosts:** \( \bar{e}^n \)
- **Multipliers:** \( b^n \)
where \( c^n \equiv [c^{-(+)} , c^{(-0)} , c^{+(-)} , c^{+(0)}] \), and similarly for \( \tilde{c}^n \) and \( b^n \). While ghost fields correspond to gauge parameters, antighosts and multipliers are associated to the gauge conditions. Since the diffeomorphisms are not gauge fixed, the related antighosts and multipliers are not present in the formalism. The BRST transformation \( sX \) of a dynamical variable \( X = (q_1 , q_2 , h^\pm) \), is obtained from the gauge transformation \( \delta X \) by replacing gauge parameters with ghosts; for the new fields we have \( s\tilde{c}^n = b^n \), \( s\bar{b}^n = 0 \), while \( sc^n \) is not needed here \( (sc^n \text{ follows from the nilpotency condition: } s^2 X = 0) \).

Then, we introduce the gauge fermion \( \Psi = \bar{c}^n \Omega_n \), and define the quantum action in the usual way:

\[
L_Q = L(q_i , \pi_i , h) + s\Psi = L(q_i , \pi_i , h) + L_{GF} + L_{FP} .
\]

The gauge fixing and the Faddeev–Popov parts are given by

\[
L_{GF} = b^\Omega_n \Omega_n , \quad L_{FP} = -\bar{c}^n [s\Omega_n] ,
\]

where

\[
\begin{align*}
    s\Omega_{\mp}(\pm) &= - \left[ e^\mp J_{\mp}(\pm) \right]' \mp c^{\mp(0)} J_{\mp}(\pm) , \\
    s\Omega_{\mp}(0) &= - \left[ e^\mp J_{\mp}(0) \right]' \pm c^{\mp(0)} J_{\mp}(\pm) \mp 2\kappa \left[c^{\mp(0)}\right]' .
\end{align*}
\]

**Effective theory.** Having derived the quantum action, we are now going to show that it can be effectively reduced to the induced gravity, by integrating out all the variables except \( \varphi_1 , \varphi_2 \), and the related momenta. To simplify the exposition technically, we shall divide it into several smaller steps.

(a) The integration over the multipliers \( b^\pm , a_+ \) and \( a_- \) transforms \( L_Q \) into the effective Lagrangian

\[
L_E(\varphi , \pi , h) = \left[ \pi_{1\alpha} \dot{q}_1^\alpha + \pi_{2\alpha} \dot{q}_2^\alpha - h^- T_- - h^+ T_+ + L_{FP} \right]_{I=\Omega=0} .
\]

It is now convenient to rewrite the first class constraints \( I_n = 0 \) and the related gauge conditions \( \Omega_n = 0 \) in the form

\[
J_{\mp}(\pm) = \mu_\mp = -J_{\mp}(\pm) , \quad -J_{(\mp)(0)} = \lambda_\mp = J_{(\mp)(0)} ,
\]

or, more explicitly,

\[
\begin{align*}
    \pi_{x_1} &= \mu_- = -\pi_{x_2} , \\
    -\pi_{y_1} &= \mu_+ = \pi_{y_2} , \\
    x_1 \pi_{x_1} + 2K_{1-} &= \lambda_- = -(x_2 \pi_{x_2} + 2K_{2+}) , \\
    -y_1 \pi_{y_1} + 2K_{1+} &= \lambda_+ = y_2 \pi_{y_2} + 2K_{2-} ,
\end{align*}
\]

where \( K_{\pm} = (\pi_{\varphi} \pm \kappa \pi')/2 \).

(b) The momentum variables \( \pi_{x_1} , \pi_{y_1} \) and \( \pi_{x_2} , \pi_{y_2} \) are constant, so that the related \( \pi \dot{q} \) terms in the action can be ignored as total time derivatives.
where we used equation (B.8).

In order to find out the usual dynamical content of the previous result, we shall eliminate the remaining momentum variables from (5.7) by using their equations of motion:

\[ \mathcal{L}_E(\varphi_1, \pi_1, h) = \pi_1 \dot{\varphi}_1 + \pi_2 \dot{\varphi}_2 - h^- \dot{T}_- - h^+ \dot{T}_+ . \]  

(5.7)

### 5.3 Transition to the induced gravity

In order to find out the usual dynamical content of the previous result, we shall eliminate the remaining momentum variables from (5.7) by using their equations of motion:

\[ \pi_1 = \frac{\kappa}{\sqrt{2}} \left[ \hat{\varphi}_1 + \dot{\varphi}_1 + 2(\hat{\varphi}_1 - \varphi_1) \right] , \]

\[ \pi_2 = \kappa \varphi_1 = \sqrt{2}\kappa \left( \hat{\varphi}_1 + \varphi_1 - \hat{\varphi}_1 - \varphi_1 \right) , \]  

(5.8)

while \( \pi_2 \) is obtained by the replacement \( \varphi_1 \rightarrow \varphi_2, \kappa \rightarrow -\kappa \) (\( \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \) are defined in Appendix B). The effective theory is described by the Lagrangian

\[ \mathcal{L}_E(\varphi_1, \pi_2, h) = \Lambda(\varphi_1, h) - \Lambda(\varphi_2, h) , \]

\[ \Lambda(\varphi, h) = \sqrt{-g} \left[ \kappa \hat{\varphi}_1 \varphi_2 + 2\kappa \left( \hat{\varphi}_1 - \hat{\varphi}_2 \right) \right] + M e^\varphi \]

(5.9)

where \( M = \mu/2\kappa \). If we now change the variables according to

\[ \phi = \sqrt{\kappa} (\varphi_1 - \varphi_2) , \quad 2F = \varphi_2 , \]

the effective Lagrangian takes the final form:

\[ \mathcal{L}_E(\phi, F, h) = \sqrt{-g} \left[ \hat{\varphi}_1 \varphi_2 + 2\sqrt{\kappa} \left( \hat{\varphi}_1 + \hat{\varphi}_2 \right) \right] + M e^{2F} \left( e^{\phi/\sqrt{\kappa}} - 1 \right) \]

(5.10)

The geometric meaning of this Lagrangian becomes more transparent if we use conformally rescaled metric (Appendix B),

\[ g_{\mu\nu} = e^{2F} g_{\mu\nu} , \quad \omega_\pm = e^{-F} \left( \hat{\varphi}_\pm \mp \hat{\varphi}_\pm \right) , \quad \partial_\pm = e^{-F} \partial_\pm , \]

whereupon the effective Lagrangian is easily seen to transform into the expression that coincides with the induced gravity action (1.2):

\[ \mathcal{L}_E(\phi, g_{\mu\nu}) = \sqrt{-g} \left[ \hat{\varphi}_1 \varphi_2 + 2\sqrt{\kappa} \left( \hat{\varphi}_1 \varphi_2 - \varphi_1 \varphi_2 \right) + M \left( e^{\phi/\sqrt{\kappa}} - 1 \right) \right] \]

\[ = \sqrt{-g} \left[ \hat{\varphi}_1 \varphi_2 + \sqrt{\kappa} \phi R + M \left( e^{\phi/\sqrt{\kappa}} - 1 \right) \right] \]

\[ = \mathcal{L}_G(\phi, g_{\mu\nu}) , \]  

(5.10)

where we used equation (B.8)
6 Geometric properties from gauge transformations

In the process of constructing the induced gravity action from the gauged WZNW system, one expects the original gauge transformations of dynamical variables to go over into geometric transformations of the final, gravitational theory. We have already seen that gauge transformations of canonical multipliers \( h^\pm \) produce correct geometric transformations of the metric density \( \tilde{g}^{\mu \nu} \). Complete interpretation of the induced gravity demands to clarify the nature of two additional fields, \( \sqrt{-g} \) and \( \phi \), given by

\[
\sqrt{-g} = \frac{1}{2}(h^- - h^+)e^{\varphi^2}, \quad \phi = \sqrt{\kappa}(\varphi_1 - \varphi_2) .
\] (6.1)

We begin by noting that the transformation rule (4.8) of the WZNW variables \( q_i = (x_i, \varphi_i, y_i), \quad i = 1, 2 \), describes the \( SL(2, R) \) gauge transformations, defined by parameters \( \eta_\pm \) [12], and the \( \varepsilon_\pm \) transformations, which we expect to be related to diffeomorphisms. In particular, the \( \varepsilon_\pm \) transformation of \( \varphi_1 \) has the form

\[
\delta_{\varepsilon_+} \varphi = -\frac{1}{2\kappa} \left[ \varepsilon^+ (\pi_\varphi + \kappa \varphi') - \varepsilon^- (\pi_\varphi - \kappa \varphi') \right] , \quad \varphi = \varphi_1 ,
\] (6.2)

while \( \delta_{\varepsilon_-} \varphi_2 \) is obtained by replacing \( \kappa \to -\kappa \).

Now, let us go to the gauge fixed, effective theory, expressed by equation (5.7). While the gauge transformations in the WZNW theory are defined using the Poisson brackets in (1.4), the related transformation rules in the gauge fixed theory (induced gravity) should be calculated with the help of the Dirac brackets, determined by \( (I_n, \Omega_n) \).

In order to check whether \( \delta(\sqrt{-g}) \) has the correct geometric form (B.5), we use the results of Appendix E, in particular equation (E.3), and find that the above transformation law for \( \varphi \) should be replaced with the Dirac bracket expression:

\[
\delta^*_\varepsilon \varphi = \delta_{\varepsilon_+} \varphi - \partial_1 (\varepsilon^- + \varepsilon^+) .
\] (6.3)

where, after eliminating \( \pi_\varphi \) with the help of (5.8), \( \delta_{\varepsilon_+} \varphi \) takes the form

\[
\delta_{\varepsilon_+} \varphi = \frac{1}{\sqrt{2}} \left[ - (\varepsilon^+ \hat{\partial}_+ \varphi - \varepsilon^- \hat{\partial}_- \varphi) + (\varepsilon^- - \varepsilon^+) (\hat{\omega}_- - \hat{\omega}_+) \right] = -\varepsilon \cdot \partial \varphi - \varepsilon^0 \partial_1 (h^- + h^+) .
\]

Comparing the expression (6.3) with equation (B.6), one concludes that \( \delta^*_\varepsilon \varphi \) yields the correct transformation law for \( \sqrt{-g} \).

It is now easy to see that the variable \( \phi \) behaves as a scalar field,

\[
\delta^*_\varepsilon \phi = -\varepsilon \cdot \partial \phi ,
\] (6.4)
in agreement with its geometric role.

The following relations make the geometric structure of the effective theory particularly transparent:

\[
\{ T_{\pm}^{(1)}(\sigma_1), T_{\pm}^{(1)}(\sigma_2) \} = -[ T_{\pm}^{(1)}(\sigma_1) + T_{\pm}^{(1)}(\sigma_2) ] \delta \mp 2\kappa \delta'' ,
\]

\[
\{ T_{\pm}^{(2)}(\sigma_1), T_{\pm}^{(2)}(\sigma_2) \} = -[ T_{\pm}^{(2)}(\sigma_1) + T_{\pm}^{(2)}(\sigma_2) ] \delta \mp 2\kappa \delta'' ,
\]

\[
\{ T_{\pm}(\sigma_1), T_{\pm}(\sigma_2) \} = -[ T_{\pm}(\sigma_1) + T_{\pm}(\sigma_2) ] \delta .
\] (6.5)
We see that the energy–momentum components of the WZNW sectors 1 and 2, $T^{(1)}$ and $T^{(2)}$, are not first class constraints in the gauge fixed, effective theory, since their algebras contain central charges, while the complete energy–momentum tensor, $T = T^{(1)} + T^{(2)}$, is of the first class.

7 Concluding remarks

In the present paper we used the canonical approach to elucidate how the induced gravity action, together with its geometric properties, can be obtained from the dynamical structure of the $SL(2, R)$ WZNW system.

We first analyzed chiral symmetries of the $SL(2, R)$ WZNW model (2.3), using the Hamiltonian formalism based on the choice of time $\tau = \xi^{\pm}$, which led us naturally to the currents $J_{\pm a}$, satisfying two independent $SL(2, R)$ KM algebras. These currents are basic objects in our canonical approach. They are used to construct quadratic $SL(2, R)$ invariants, the energy–momentum components $T_{\pm}$, that satisfy two independent Virasoro algebras and represent first class constraints corresponding to diffeomorphisms, in the canonical gauge formalism defined by (1.3). Then, the gauge procedure is generalized by introducing two sets of KM currents, $J_{\pm a}^{(1)}$ and $J_{\pm a}^{(2)}$, corresponding to two sectors of the WZNW system (1.1), which are used to define the new first class constraints $I_{\pm a} = J_{\pm a}^{(1)} + J_{\pm a}^{(2)}$, satisfying an $SL(2, R) \times SL(2, R)$ algebra without central charge, and the energy–momentum components $T_{\pm}$ corresponding to the whole WZNW system. The resulting theory is clearly gauge equivalent to the WZNW system (1.1), being its canonical gauge extension. The Hamiltonian process of elimination of central charges in the algebra of new currents $I_{\pm a}$ closely parallels the cancellation of the nonlocal terms in the Lagrangian approach [12]. As the main result of our analysis, we showed, (a) by choosing a suitable gauge fixing, and (b) integrating out some dynamical variables, that this gauge theory reduces effectively to the induced gravity (1.2). Geometric properties of the gravitational theory are derived from gauge properties of the gauge extended WZNW system, with the help of the Dirac brackets.

The results obtained here supplement those of the recent Lagrangian analysis [12], and improve our understanding of geometric properties of 2D spacetime in terms of the related gauge structure. They can be used to better understand singular solutions of the induced gravity in terms of globally regular solutions of the WZNW system, and clarify the nature of black holes [9, 17].

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A Geometric properties of $SL(2, R)$

In this Appendix we present some useful results concerning the Riemannian structure of the group manifold $SL(2, R)$. 

15
Choosing the generators of $SL(2, R)$ as $t_{(\pm)} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, $t_{(0)} = \frac{1}{2}\sigma_3$, where $\sigma_k$ are the Pauli matrices, one finds that the Lie algebra $[t_a, t_b] = f_{ab}^c t_c$ takes the form

$$[t_{(+)}, t_{(-)}] = 2t_{(0)}, \quad [t_{(\pm)}, t_{(0)}] = \mp t_{(\pm)}. \quad (A.1)$$

Explicit evaluation of the Cartan metric $\gamma_{ab} = (t_a, t_b) = \frac{1}{2}f_{ac}^d f_{bd}^e$ yields

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad a, b = (+), (0), (-). \quad (A.2)$$

The Cartan metric $\gamma_{ab}$ and its inverse $\gamma^{ab}$ are used to lower and raise the tangent space indices $(a, b, ...)$. Any element $g$ of $SL(2, R)$ in a neighborhood of identity can be parametrized by using the Gauss decomposition:

$$g = e^{xt(+)e^{y(0)}}e^{yt(-)} = e^{-\varphi/2} \begin{pmatrix} e^\varphi + xy & x \\ y & 1 \end{pmatrix}, \quad (A.3)$$

where $q^\alpha = (x, \varphi, y)$ are group coordinates.

Now, the Lie algebra valued 1–form $v = g^{-1}dg = t_a E^a = t_a E^a_\alpha dq^\alpha$ defines the quantity $E^a_\alpha$, the vielbein on the group manifold. The above expression for $g$ leads to

$$E^{(+)a} = e^{-\varphi} dx, \quad E^{(0)a} = 2y e^{-\varphi} dx + d\varphi, \quad E^{(-)a} = -y^2 e^{-\varphi} dx - y d\varphi + dy,$$

so that the vielbein $E^a_\alpha$ and its inverse $E^\alpha_a$ are given as

$$E^a_\alpha = \begin{pmatrix} e^{-\varphi} & 0 & 0 \\ 2y e^{-\varphi} & 1 & 0 \\ -y^2 e^{-\varphi} & -y & 1 \end{pmatrix}, \quad E^\alpha_a = \begin{pmatrix} e^\varphi & 0 & 0 \\ -2y & 1 & 0 \\ -y^2 & y & 1 \end{pmatrix}. \quad (A.4)$$

The Cartan metric in the coordinate basis, $\gamma_{\alpha\beta} = E^a_\alpha E^b_\beta \gamma_{ab}$, has the form

$$\gamma_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 2e^{-\varphi} \\ 0 & 1 & 0 \\ 2e^{-\varphi} & 0 & 0 \end{pmatrix}, \quad \alpha, \beta = x, \varphi, y. \quad (A.5)$$

Using the property $d(v, v^2) = 0$, one can write locally $(v, v^2) = -6d\tau$, where

$$d\tau = E^{(+)} E^{(0)} E^{(-)} = d(e^{-\varphi} dx dy). \quad (A.6)$$

Similarly, the calculation of $\bar{v} = gdg^{-1} = t_a \bar{E}^a = t_a E^a_\alpha dq^\alpha$ leads to

$$\bar{E}^a_\alpha = \begin{pmatrix} -1 & x & x^2 e^{-\varphi} \\ 0 & -1 & -2xe^{-\varphi} \\ 0 & 0 & -e^{-\varphi} \end{pmatrix}, \quad \bar{E}^\alpha_a = \begin{pmatrix} -1 & -x & x^2 \\ 0 & -1 & 2x \\ 0 & 0 & -e^\varphi \end{pmatrix}. \quad (A.7)$$

The metric $\bar{\gamma}_{\alpha\beta}$ is the same as $\gamma_{\alpha\beta}$. 

16
B Riemannian structure on $\Sigma$

Here, we present some basic features of the Riemannian geometry on two–dimensional spacetime $\Sigma$.

**Light–cone basis.** Starting from the interval on $\Sigma$,

$$ds^2 = g_{\mu\nu}d\xi^\mu d\xi^\nu = (d\xi^0)^2 \left[ g_{00} + 2g_{01}u + g_{11}u^2 \right], \quad u \equiv d\xi^1/d\xi^0,$$

we can solve the equation $ds^2 = 0$ for $u$,

$$u_{1,2} = \frac{-g_{01} \pm \sqrt{-g}}{g_{11}} \equiv h^{\pm},$$

and obtain

$$ds^2 = (d\xi^0)^2 g_{11}(u - h^+)(u - h^-) = 2d\xi^+ d\xi^-.$$

Here,

$$d\xi^+ = \sqrt{-g_{11}/2} (-h^+ d\xi^0 + d\xi^1) = e^+_{\mu} d\xi^\mu,$$

$$d\xi^- = \sqrt{-g_{11}/2} (h^- d\xi^0 - d\xi^1) = e^-_{\mu} d\xi^\mu.$$

If we introduce $-g_{11} = e^{2F}$, three independent components of the metric $g_{\mu\nu}$ can be expressed in terms of the new, light–cone variables $(h^-, h^+, F)$. In particular,

$$\sqrt{-g} = e^{2F} \sqrt{-\hat{g}}, \quad \sqrt{-\hat{g}} \equiv \frac{1}{2}(h^- - h^+).$$

At each point of $\Sigma$ the quantities

$$e^i_\mu = e^F \hat{e}^i_\mu, \quad \hat{e}^i_\mu \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -h^+ & 1 \\ h^- & -1 \end{pmatrix} \quad (i = +, -) \quad (B.1)$$

define an orthonormal, light–cone basis of 1–forms, $\theta^i = d\xi^i = e^i_\mu d\xi^\mu$. Note that $\det(e^i_\mu) = -\sqrt{-g}$. We also introduce the related basis of tangent vectors, $e_i \equiv \partial_i = e_i^\mu \partial_\mu$,

$$e_i^\mu = e^{-F} \hat{e}_i^\mu, \quad \hat{e}_i^\mu \equiv \frac{\sqrt{2}}{h^- - h^+} \begin{pmatrix} 1 \\ h^- \\ h^+ \end{pmatrix}. \quad (B.2)$$

The metric $\eta_{ij}$ in the tangent space has the light–cone form: $\eta_{++} = \eta_{--} = 1$. Tangent space components of an arbitrary vector $V_\mu$ are

$$V_\pm = e_\pm^\mu V_\mu = e^{-F} \frac{\sqrt{2}}{h^- - h^+} (V_0 + h^+ V_1) = e^{-F} \hat{V}_\pm. \quad (B.3)$$

The components of the metric $\hat{g}_{\mu\nu}$ and its inverse $\hat{g}^{\mu\nu}$ are

$$\hat{g}_{\mu\nu} = \frac{1}{2} \begin{pmatrix} -2h^-h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix},$$

$$\hat{g}^{\mu\nu} = \frac{2}{(h^- - h^+)^2} \begin{pmatrix} 2 & h^- + h^+ \\ h^- + h^+ & 2h^-h^+ \end{pmatrix}.$$
**Diffeomorphisms.** The standard transformation rule of the zweibein $e^i_\mu$ under the diffeomorphisms, $\xi^\mu \rightarrow \xi^\mu + \epsilon^\mu(\xi)$, implies
\[
\delta h^\pm = \partial_0 \epsilon^\pm + h^\pm \partial_1 \epsilon^\pm - \epsilon^\pm \partial_1 h^\pm .
\] (B.4)
where $\epsilon^\pm = \epsilon^1 - \epsilon^0 h^\pm$. The transformation rule of the metric is:
\[
\delta g^{\mu\nu} = g^\rho^\mu \partial_\rho \epsilon^\nu + g^\rho^\nu \partial_\rho \epsilon^\mu - \epsilon^\rho \partial_\rho g^{\mu\nu},
\]
\[
\delta \sqrt{-g} = -\partial_\rho (\epsilon^\rho \sqrt{-g}).
\] (B.5)
The conformal rescaling $g_{\mu\nu} = e^{2F} \hat{g}_{\mu\nu}$ leads to
\[
\delta F = -\partial_1 \epsilon^1 + \partial_1 \epsilon^0 (h^- + h^+) - \epsilon^\rho \partial_\rho F .
\]
Transition to $\epsilon^\pm$ yields
\[
\delta (2F) = -\partial_1 (\epsilon^+ + \epsilon^-) + (\epsilon^- - \epsilon^+) \frac{\partial_1 (h^- + h^+)}{h^- - h^+} - \frac{1}{\sqrt{2}} (\epsilon^+ \hat{\partial}_+ - \epsilon^- \hat{\partial}_- )2F .
\] (B.6)
The algebra of diffeomorphisms has the form
\[
[\delta(\epsilon_1), \delta(\epsilon_2)] h^\pm = \delta(\epsilon_3) h^\pm , \quad \epsilon^\pm_3 = \epsilon^\pm_1 \partial_1 \epsilon^\pm_2 - \epsilon^\pm_2 \partial_1 \epsilon^\pm_1 .
\]
It is similar to the Virasoro algebra, but not the same, since $\epsilon^\pm = \epsilon^\pm(\xi^+, \xi^-)$. In the limit of conformally flat space, $h^\pm \rightarrow \mp 1$, coordinate transformations are restricted to two sets of conformal transformations, $\epsilon^\pm = \epsilon^\pm(\xi^\pm)$, with two independent Virasoro algebras. The light-cone gauge is defined by $h^+ = -1$ (or $h^- = 1$) and $\sqrt{-g} = 1$.

**Connection and curvature.** Riemannian connection on $\Sigma$ is defined by the first structural equation:
\[
d\theta^i + \omega^i_j \wedge \theta^j = 0 , \quad \omega^i_j = \epsilon^i_j \omega .
\]
For the connection 1–form $\omega = \omega_i \theta^i$ we find
\[
\omega_+ = e^{-F} (\hat{\omega}_+ - \hat{\partial}_+ F) , \quad \hat{\omega}_+ = -\frac{\sqrt{2}}{h^- - h^+} (h^-)' ,
\]
\[
\omega_- = e^{-F} (\hat{\omega}_- + \hat{\partial}_+ F) , \quad \hat{\omega}_- = \frac{\sqrt{2}}{h^- - h^+} (h^+)' .
\] (B.7)
The curvature is defined by the second structural equation:
\[
d\omega^i_j = \frac{1}{2} R^i_{jkl} \theta^k \wedge \theta^l ,
\]
where we used $\omega^i_k \wedge \omega^k_j = 0$. Since $d\omega = (\nabla_- \omega_+ - \nabla_+ \omega_-) \theta^- \wedge \theta^+$, one finds
\[
R = 2R_{+-} = 2(\nabla_- \omega_+ - \nabla_+ \omega_-) .
\] (B.8)
C Structure functions

The Poisson bracket algebra (4.5) implies the relations

\[
\{ (\eta^\mp a I_{\mp a})_\sigma, (a_\pm I_{\mp b})_{\sigma_2} \} = [f_{ab}^c \eta^\mp a a^b_\pm]_\sigma (I_{\mp c})_\sigma \delta,
\]
\[
\{ (\eta^\mp a I_{\mp a})_\sigma, (h^\mp T_\mp)_{\sigma_2} \} = [(\eta^\mp a)^h h^\mp]_\sigma (I_{\mp a})_\sigma \delta,
\]
\[
\{ (\varepsilon^\mp T_\mp)_{\sigma_1}, (a_\mp I_{\mp a})_{\sigma_2} \} = -[\varepsilon^\mp (a^a_\mp)^\prime]_\sigma (I_{\mp a})_\sigma \delta,
\]
\[
\{ (\varepsilon^\mp T_\mp)_{\sigma_1}, (h^\mp T_\mp)_{\sigma_2} \} = -[(h^\mp)^\prime \varepsilon^\mp + (\varepsilon^\mp)^h h^\mp]_\sigma (T_\mp)_\sigma \delta,
\]

needed to calculate gauge transformations of the multipiers \( u^m = (h^-, h^+, a^a_+, a^a_-) \), according to the general rule (1.4). Here, \( \delta = \delta(\sigma - \sigma_2) \), and one understands that an integration over \( \sigma \) and \( \sigma_2 \) is to be performed.

D Lagrangian form of the gauged WZNW system

In this Appendix we find the usual Lagrangian description of the canonical gauge action for the WZNW system.

First, we focus our attention on the restricted action (5.4), which can be written as a sum of two terms, \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \), describing two sectors of the theory. The variation over \( \pi_{x_1}, \pi_{\varphi_1} \) and \( \pi_{y_1} \) yields:

\[
\pi_{x_1} = 2\sqrt{2} \kappa e^{-\varphi_1} \dot{D}_- y_1,
\]
\[
\pi_{\varphi_1} = \kappa \varphi_1' = \sqrt{2} \kappa \left[ \dot{\varphi}_1 + \dot{A}_+^0 - \dot{B}_-^0 \right],
\]
\[
\pi_{y_1} = 2\sqrt{2} \kappa e^{-\varphi_1} \dot{D}_+ x_1,
\]

where

\[
\dot{A}_c^0 = \frac{-\sqrt{2}}{h^- - h^+} a^c_+,
\]
\[
\dot{B}_c^- = \frac{-\sqrt{2}}{h^- - h^+} a^c_-,
\]
\[
\dot{D}_+ x_1 = \left[ \dot{\varphi}_1 + \dot{A}_+^0 \right] x_1 + \dot{A}_+^0,
\]
\[
\dot{D}_- y_1 = \left[ \dot{\varphi}_1 - \dot{B}_-^0 \right] y_1 - \dot{B}_-^0.
\]

Replacing this into \( \mathcal{L}_1 \) leads to

\[
\mathcal{L}_1 = \mathcal{L}^r(q_1, \dot{A}, \dot{B}) + \kappa \sqrt{-g} \left[ \dot{B}_-^0 - \dot{A}_+^0 \right]^2,
\]
\[
\mathcal{L}^r(q_1, \dot{A}, \dot{B}) = \kappa \sqrt{-g} \left[ \dot{\varphi}_1 \dot{\varphi}_1 + 2\dot{A}_+^0 \dot{\varphi}_1 - 2\dot{B}_-^0 \dot{\varphi}_1 + 4\dot{D}_+ x_1 \dot{D}_- y_1 e^{-\varphi_1} \right].
\]

In a similar manner one finds

\[
\mathcal{L}_2 = -\mathcal{L}^r(q_2, \dot{A}, \dot{B}) - \kappa \sqrt{-g} \left[ \dot{B}_-^0 - \dot{A}_+^0 \right]^2,
\]

so that the complete Lagrangian of the gauged WZNW system (5.3) is given by

\[
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}^r(q_1, \dot{A}, \dot{B}) - \mathcal{L}^r(q_2, \dot{A}, \dot{B}).
\]
Comparing this result with equation (4.5) in Ref. [12], one finds that (D.2) gives the standard Lagrangian description of the gauged WZNW system, provided the canonical multipliers $\hat{A}_+$ and $\hat{B}_-$ are identified with the related gauge potentials. The correctness of this identification is checked by comparing their transformation laws. Thus, e.g., $\eta_\pm$ gauge transformations of the canonical multipliers, obtained with the help of (4.7),

$$\delta_\eta \hat{A}_+ = -\partial_+ \eta_+ - f_{ab} \hat{A}_+ \eta_+^b, \quad \delta_\eta \hat{B}_- = -\partial_- \eta_- - f_{ab} \hat{B}_- \eta_+^b,$$

are identical to the $SL(2, R) \times SL(2, R)$ transformations of Lagrangian gauge potentials.

Similar analysis can be done for the complete gauge action (4.6b), with the same conclusion.

### E Dirac brackets

In order to have a clear geometric interpretation of the transition from the gauged WZNW system to the induced gravity, we shall calculate here the Dirac brackets corresponding the set of first class constraints and gauge conditions:

$$\theta_{+a} = \left( I_{+(+)}, \Omega_{+(+)} \right), \quad \theta_{-a} = \left( I_{-(+)}, \Omega_{-(+)} \right). \quad (E.1)$$

The calculation of $\Delta_{+ab} = \{\theta_{+a}, \theta_{+b}\}$ yields:

$$\Delta_{+ab} = \begin{pmatrix}
0 & -\mu_+ \delta & 0 & 0 \\
-\mu_+ \delta & 2\kappa \delta' & 2\kappa \delta' & 0 \\
0 & 2\kappa \delta' & 0 & -\mu_+ \delta \\
0 & 0 & \mu_+ \delta & 0
\end{pmatrix}.$$

The inverse is:

$$\Delta_{+}^{-1ab} = \frac{1}{\mu_+^2} \begin{pmatrix}
2\kappa \delta' & \mu_+ \delta & 0 & -2\kappa \delta' \\
-\mu_+ \delta & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_+ \delta \\
-2\kappa \delta' & 0 & -\mu_+ \delta & 0
\end{pmatrix}.$$

Similarly,

$$\Delta_{-ab} = -(\Delta_{+})_{ab} |_{\mu_+ \to -\mu_-}, \quad (\Delta_{-})^{-1ab} = -(\Delta_{+})^{-1ab} |_{\mu_+ \to -\mu_-}.$$

Since $\{\theta_{+a}, \theta_{-a}\} = 0$, the Dirac bracket of $X$ and $Y$ is defined by

$$\{X, Y\}^* = \{X, Y\} - \{X, \theta_{+a}\}(\Delta_{+})^{-1ab}\{\theta_{+b}, Y\} - \{X, \theta_{-a}\}(\Delta_{-})^{-1ab}\{\theta_{-b}, Y\}. \quad (E.2)$$

We display here several useful results:

$$\{\varphi_1, \pi_{\varphi_1}\}^* = \{\varphi_1, \pi_{\varphi_1}\}, \quad \{\varphi_1, T^{(1)}_{\pm}\}^* = \{\varphi_1, T^{(1)}_{\pm}\} - \delta',$$

$$\{\varphi_2, \pi_{\varphi_2}\}^* = \{\varphi_2, \pi_{\varphi_2}\}, \quad \{\varphi_2, T^{(2)}_{\pm}\}^* = \{\varphi_2, T^{(2)}_{\pm}\} - \delta'. \quad (E.3)$$
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