A stochastic method of solution of the Parker transport equation

A. Wawrzynczak\textsuperscript{1}, R. Modzelewsk\textsuperscript{a}, A. Gil\textsuperscript{2}

\textsuperscript{1}Institute of Computer Sciences, Siedlce University, Poland, \textsuperscript{2}Institute of Mathematics and Physics, Siedlce University, Poland.

E-mail: awawrzynczak@uph.edu.pl, renatam@uph.edu.pl, gila@uph.edu.pl

Abstract. We present the stochastic model of the transport of galactic cosmic ray (GCR) particles in the heliosphere. Based on a solution of the Parker transport equation we developed models of the short-term variation of the GCR intensity, i.e. the Forbush decrease (Fd) and the 27-day variations of the GCR intensity. Parker’s transport equation, being the Fokker-Planck type equation, delineates non-stationary transport of charged particles in a turbulent medium. The presented approach of the numerical solution is grounded on solving the set equivalent stochastic differential equations (SDEs). We demonstrate the method of deriving from Parker’s transport equation the corresponding SDEs in the heliocentric spherical coordinate system for the backward approach. Features indicative of the preeminence of the backward approach over the forward are stressed. We compare the outcomes of the stochastic models of the Fd and 27-day variation of the GCR intensity with our former models established by the finite difference method. Both models are in agreement with the experimental data.

Introduction

The first stochastic equation (Langevin’s) was linked to Newton’s principle [1]. From the beginning of the XX century, the stochastic approach became more useful for describing random physical processes. Stochastic differential equations (SDEs), in conjunction with various Monte Carlo technics, are broadly used in many fields, such as physics, finance, biology, chemistry, engineering, or management science. The main statistical characteristic is the representation of the solution of the Fokker-Planck type equation as a probability density distribution. Employment of probabilistic description with Monte Carlo simulations allows to reduce the solution of the partial differential equation (PDE) describing the analyzed phenomena to the integration of SDEs.

We apply stochastic methodology to model the galactic cosmic rays (GCR) transport in the heliosphere. Algorithms used in the particle transport simulations are mainly based on finite difference methods (e.g. [2]). Galerkin methods are used for solving time-dependent high order PDEs (e.g. [3]). The homotopy perturbation method [4] was recently developed for the numerical solution of various linear and nonlinear PDEs. Also, there exists an approach to the PDE solution grounded on particle methods, characterized by low numerical dispersion [5].

During the propagation through the heliosphere, GCR particles are modulated by the solar wind and heliospheric magnetic field (HMF). Modulation of the GCR is a result of an action of four primary processes: convection by the solar wind, diffusion on irregularities of HMF, particles drifts in the non-uniform magnetic field and adiabatic cooling. Transport of the GCR
particles in the heliosphere can be described by the Parker transport equation [6]:

$$\frac{\partial f}{\partial t} = \vec{\nabla} \cdot (K_{ij}^S \cdot \vec{\nabla} f) - (\vec{v}_d + \vec{U}) \cdot \vec{\nabla} f + \frac{R}{3}(\vec{\nabla} \cdot \vec{U}) \frac{\partial f}{\partial R},$$  

(1)

where $f = f(\vec{r}, R, t)$ is an omnidirectional distribution function of three spatial coordinates $\vec{r} = (r, \theta, \varphi)$, particles rigidity $R$ and time $t$; $\vec{U}$ is the solar wind velocity, $\vec{v}_d$ the drift velocity, and $K_{ij}^S$ is the symmetric part of the diffusion tensor of the GCR particles.

Employing the stochastic approach to solving the Parker transport equation is not the latest idea [7, 8]. However, the majority of models presented in the literature are used to determine the simulated spectra and compare it with experimental observations carried out by space probes such as Voyagers, AMS, BESS and PAMELA (e.g. [9]-[13]). In this paper, we present models in which we do not only reproduce the proton spectra but also simulate the short- time GCR variations i.e. the Forbush decrease (Fd) and 27-day variation. Additionally, we compare the results of the stochastic modeling with our previous well grounded models, developed by using the finite difference method (FDM) to solve the Parker transport equation (e.g. [14]-[16]). To perform the reliable comparison between the two models, we consider the same coefficients and parameters as in the baseline Parker transport equation. Furthermore, the parameters included are obtained by approximation of the experimental data.

The aim of our paper is twofold: at first, to compose a consistent mathematical model of the GCR transport in the heliosphere by means of the SDEs. And then, to employ the created model to simulate the short-term variations being in an agreement with the experimental data.

2. The stochastic approach

In order to model the GCR transport in the heliosphere using stochastic methods equivalent SDEs must be obtained. In the first step, the Parker transport equation (Eq. 1) must be converted to a general form of the Fokker-Planck equation (FPE). Depending on the direction of integration, FPE must be expressed in two forms [17].

Figure 1. The sample pseudoparticles trajectories within the heliosphere.
Figure 2. Trajectories of the pseudoparticles with rigidity 10 GV for the $A>0$ and $A<0$ solar magnetic cycle. The specific colors highlight the trajectories of the sample pseudoparticles traced backward in time from the heliosphere boundary until they reach the position $r = 1$ AU, $\theta = 90^\circ$, $\varphi = 180^\circ$.

Figure 3. The histograms of the particles rigidity and exit time for the pseudoparticles initialized with rigidity 10 GV from position $r = 1$ AU, $\theta = 90^\circ$, $\varphi = 180^\circ$ for the $A>0$ and $A<0$ solar magnetic cycle.

the time-backward:

$$\frac{\partial F}{\partial t} = \sum_i A_i \frac{\partial F}{\partial x_i} + \frac{1}{2} \sum_{i,j} B_{ij} B_{ij}^T \frac{\partial^2 F}{\partial x_i \partial x_j},$$

(2)
and the time-forward:

$$\frac{\partial F}{\partial t} = \sum_i \frac{\partial}{\partial x_i} (A_i \cdot F) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (B_{ij} B_{ij}^T \cdot F).$$  \hspace{1cm} (3)$$

SDEs equivalent to Eq. 2 and 3 has the form (e.g. [17]):

$$d\vec{r} = \vec{A}_i \cdot dt + B_{ij} \cdot d\vec{W},$$  \hspace{1cm} (4)$$

where $\vec{r}$ is the individual pseudoparticle trajectory in the phase space and $d\vec{W}_i$ is the Wiener process, commonly written as $d\vec{W}_i = \sqrt{dt} \cdot dw_i$, with $dw_i$ being the randomly fluctuating term with Gaussian distribution.

To solve Eq. 4 in both cases (backward and forward), at the onset, we initiate pseudoparticles at some starting point in space and time and integrate alongside the pseudoparticles trajectories until they reach the boundary. Choosing between the forward or backward in time approach we need to bear in mind the problem that we want to solve. In the time-forward integration, pseudoparticles start from diverse boundary points, being for the GCR particles the entrance to the heliosphere. After that, their trajectories are traced up to the target position, e.g. 1 AU. Thus, a high number of pseudoparticles has to be initialized in order to obtain a robust statistic because plenty of them do not reach the target position. The time-backward integration is much more effective in the case of the GCR propagation in the heliosphere. In the time-backward approach, the number of ‘useless’ particles is reduced. Pseudoparticles start from point of interest (e.g. 1 AU) and are traced backward in time until crossing the heliospheric boundary (in this paper this boundary is assumed at 100 AU, Fig. 1). During their travel throughout the heliosphere the pseudoparticles gain/lose their energy/rigidity proportionally to their travel time. In result the value of the particle distribution function, $f(\vec{r}, R)$, for the starting point can be found as an average of $f_{LIS}(R)$ values for pseudoparticles characteristics at the entry positions, $f(\vec{r}, R) = \frac{1}{N} \sum_{n=1}^{N} f_{LIS}(R)$, where $f_{LIS}(R)$ is the cosmic ray local interstellar spectrum taken as in [18] for rigidity $R$ of the $n^{th}$ particle at the exit/entrance point.

The Parker equation (Eq. 1), in the 3-D spherical coordinate system ($r, \theta, \varphi$), as the time-backward FPE diffusion equation, has the form:

$$\frac{\partial f}{\partial t} = a_1 \frac{\partial^2 f}{\partial r^2} + a_2 \frac{\partial^2 f}{\partial \theta^2} + a_3 \frac{\partial^2 f}{\partial \varphi^2} + a_4 \frac{\partial^2 f}{\partial r \partial \theta} + a_5 \frac{\partial^2 f}{\partial r \partial \varphi} + a_6 \frac{\partial^2 f}{\partial \theta \partial \varphi} + a_7 \frac{\partial f}{\partial r} + a_8 \frac{\partial f}{\partial \theta} + a_9 \frac{\partial f}{\partial \varphi} + a_{10} \frac{\partial f}{\partial R},$$  \hspace{1cm} (5)$$

where coefficients have the form:

$$a_1 = K^S_{rr}, a_2 = \frac{K^S_{\theta \theta}}{r^2}, a_3 = \frac{K^S_{\varphi \varphi}}{r^2 \sin^2 \theta}, a_4 = \frac{2K^S_{r \theta}}{r}, a_5 = \frac{2K^S_{r \varphi}}{r \sin \theta}, a_6 = \frac{2K^S_{\theta \varphi}}{r^2 \sin^2 \theta}.$$
rigidity equals 10 GV, is rather low. The exit time is a bit shorter for the $A > 0$ pseudoparticles varies around 25 days and the energy loss of pseudoparticles, with an initial.

Fig. 3. Moreover, Fig. 3 presents that the most frequently occurring exit time for simulated.

In order to validate our newly developed numerical model, we demonstrate the binned pseudoparticles versus backward time, while the right panels show the position in latitude versus.

r position (bottom panels), traced backward in time from the heliospheric boundary until they reach the

The equivalent to Eq. 5 set of SDEs, with matrix $B_{ij}$, $(i, j = r, \theta, \varphi)$, has the following form (the same can be found in [22]):

$$\begin{align*}
ad_t &= a_7 \cdot dt + [B \cdot dW]_r \\
\begin{bmatrix}
\frac{\partial B}{\partial \theta} \\
\frac{\partial B}{\partial \varphi}
\end{bmatrix}
\end{align*}$$

$B_{ij} =$

$$\begin{bmatrix}
0 & 0 \\
\frac{a_2}{\sqrt{2\alpha_{1}}} \sqrt{2a_2 - \frac{a_1^2}{2\alpha_{1}}} & 0 \\
\frac{a_6 - \frac{a_1 a_5}{2\alpha_{1}}}{B_{00}} & \sqrt{2a_3 - B_{rr}^2 - B_{\varphi\varphi}^2}
\end{bmatrix}.$$  

The Euler-Maruyama scheme is used in order to integrate Eq. 6 backward in time. To solve Eq. 6 in spherical coordinates system we are using boundary conditions of the form:

$\varphi_i < 0 \to \varphi_i = \varphi_i + 2\pi$; $\varphi_i > 2\pi \to \varphi_i = \varphi_i - 2\pi$; $\theta_i < 0 \to \theta_i = -\theta_i$ & $\varphi_i = \varphi_i \pm \pi$ and

$\theta_i > \pi \to \theta_i = 2\pi - \theta_i$ & $\varphi_i = \varphi_i \pm \pi$ [22]. The reflecting boundary is considered to be the inner radial boundary, $\frac{\partial f}{\partial r} = 0$ at $r = 0.001$ AU. An empty heliosphere constitutes an initial condition, $f_i(0.01 AU < r < 100 AU, \theta, \varphi, R, 0) = 0$, as was shown in [11]. The performed tests proved that the 3000 pseudoparticles were enough to simulate the short-time variations of GCR particles with rigidity 10 GV.

Fig. 2 shows the trajectories of simulated pseudoparticles with rigidity 10 GV (galactic protons) for the positive ($A > 0$) (upper panels) and for the negative ($A < 0$) polarity epoch (bottom panels), traced backward in time from the heliospheric boundary until they reach the position $r = 1$ AU, $\theta = 90^\circ$, $\varphi = 180^\circ$. The left panels in Fig. 2 display the radial position of pseudoparticles versus backward time, while the right panels show the position in latitude versus time. In order to validate our newly developed numerical model, we demonstrate the binned exit rigidity and the corresponding binned propagation times for simulated pseudoparticles in Fig. 3. Moreover, Fig. 3 presents that the most frequently occurring exit time for simulated pseudoparticles varies around 25 days and the energy loss of pseudoparticles, with an initial rigidity equals 10 GV, is rather low. The exit time is a bit shorter for the $A > 0$ than for the

Figure 5. Simulated galactic protons rigidity spectra at Earth with respect to the LIS (blue dotted line) at 100 AU for two cases: the $A > 0$ (red solid line) and the $A < 0$ (green dashed line) polarity epochs.
A < 0 because of the large drift speed in the polar regions for A > 0. This is in agreement with [7]. Fig. 4 displays latitude vs. longitude distribution of simulated pseudoparticles initiated at Earth’s orbit with rigidity 10 GV for A > 0 and A < 0 cycles. Fig. 4 shows the different character of heliospheric transport depending on various drift patterns in the A > 0 and A < 0 cycles. In the A > 0 cycle, pseudoparticles are transported mainly toward higher latitudes, while in the A < 0 epoch pseudoparticles are drifting outward mainly along the neutral sheet at low latitudes (Fig. 2, right panel). Fig. 5 demonstrates simulated rigidity spectra at the Earth’s orbit for the A > 0 (red solid line), and for the A < 0 (green dashed line) polarity epoch. The blue dotted line represents the unmodulated spectrum (LIS) [18] at 100 AU. However, for the rigidity greater than 1 GV the difference is subtle, but still visible, being in agreement with Zhang results (compare with [7], Fig. 3).

In all presented models the heliospheric magnetic field vector $\boldsymbol{B}$ is taken as $\boldsymbol{B} = (1 - 2H(\theta - \theta')) (B_r \hat{e}_r + B_\varphi \hat{e}_\varphi)$ [19, 2] where $H$ is the Heaviside step function changing the sign of the global magnetic field in each hemisphere, $\theta'$ corresponds to the heliolatitudinal position of the heliospheric neutral sheet, $\hat{e}_r$ and $\hat{e}_\varphi$ are the unite vectors directed along the components $B_r$ and $B_\varphi$ of the HMF for the two-dimensional Parker field [26]. Furthermore, we assume that perpendicular diffusion coefficient $K_\perp$ is proportional to parallel one $K_\parallel$: $\frac{K_\perp}{K_\parallel} = \frac{1}{1 + (\omega \tau)^2}$, where $\omega$ is a particle’s angular velocity and $\tau$ is the time between two sequences of GCR particle collisions. For the GCR particles to which the neutron monitor responds the $\omega \tau$ can change in the range $3 \leq \omega \tau \leq 5$.

3. The model of the Forbush decrease of the GCR intensity

We present the model of the recurrent Fd taking place due to established corotating heliolongitudinal disturbances in the interplanetary space. Corotating interaction regions (CIR) passing the Earth gradually diminish the diffusion at the Earth’s orbit, causing larger scattering of the GCR particles, and in effect fewer GCR particles reach the Earth. We simulate this process by the gradual decrease and then the increase of the diffusion coefficient at the Earth’s orbit with respect to the heliolongitudes. The diffusion coefficient $K_\parallel$ of cosmic ray particles has the form: $K_\parallel = K_0 \cdot K(r) \cdot K(R, \nu)$, where $K_0 = 10^{21}$ cm$^2$/s, $K(r) = 1 + 0.5 \cdot (r/1AU)$. The dependence of the diffusion coefficient on the particles rigidity $R$, based on the quasilinear theory, e.g. [23, 24], takes the form: $K(R, \nu) = (R/R_0)^{2-\nu}$, where $R_0 = 1$ GV and $\nu$ is the exponent of the power spectral density (PSD) of the HMF turbulence. The exponent $\nu$ pronounces the increase of the HMF turbulence in the vicinity of space where the Fd is created (for details see e.g. [14, 25]), and is taken as: $\nu = 1 + 0.3 \sin(\varphi - 90^\circ)$ for $r < 30$AU and $90^\circ \leq \varphi \leq 270^\circ$.

We assume the existence of two dimensional Parker’s spiral heliospheric magnetic field $B$ [26] implemented through the angle $\psi = \arctan(-B_\varphi/B_r)$ in the 3D anisotropic diffusion tensor $K_{ij}$.

Figure 6. Changes of the expected amplitudes of the Fd of the GCR intensity at the Earth orbit, for the rigidity of 10 and 20 GV based on the solutions of the Parker transport equation by SDEs and FDM in comparison with the GCR intensity registered by Potchefstroom and Moscow neutron monitors during the Fd from 18 March to 4 April 2002.
of GCR particles [21]. We compare the results obtained by the solution of the Parker transport equation by SDEs with our previous method of solution by FDM [25] assuming the same changes of all included parameters.

The expected changes of the GCR intensity for the rigidity of 10 and 20 GV during the simulated Fd in comparison with the profiles of the daily GCR intensities, recorded by the two neutron monitors with different cut off rigidities from 18 March to 4 April 2002 presents Fig. 6. One can see that the proposed models are in an acceptable coincidence with the experimental data and, as is expected, the amplitude of the Fd decreases for higher rigidities. Moreover, the model of the Fd obtained based on the solution of the SDEs allows reflecting the stochastic character of the GCR particles distribution in the heliosphere and analyze the pseudoparticle trajectory through the 3D heliosphere (Fig. 1), which is not possible based on the solution of the Parker transport equation by the FDM (e.g. [14]).

4. The model of the 27-day variation of the GCR intensity
The recurrence of the GCR intensity connected with the solar rotation is commonly called the 27-day variation, although the durations can slightly differ. The 27-day variation of the GCR intensity is connected with the heliolongitudinal asymmetry of the electromagnetic conditions in the heliosphere. The recent minimum of solar activity between solar cycles No. 23 and 24 was quite exceptional. Recurrent variations connected with corotating structures (∼ 27 days), at the end of 2007 and almost for the whole year 2008 were clearly established in all solar wind and interplanetary parameters. Consequently, the 27-day variation of cosmic ray intensity was clearly visible in a variety of cosmic ray counts of neutron monitors (e.g., [28, 15]) and space probes (e.g., [27]). We present the model of the 27-day variation of the GCR intensity considering an individual period of solar rotation starting on 2007.09.07. As it was stated by [28, 29], the 27-day variation of the GCR intensity during all epochs is preferentially related to the heliolongitudinal asymmetry of the solar wind velocity.

In the model we apply approximation of the in situ measurements of the solar wind speed as source of the 27-day variation of the GCR intensity described by the formula: 

$$ U = U_0 (1 - 0.31 \sin(\varphi + 6.10) + 0.06 \sin(2\varphi + 0.82) - 0.10 \sin(3\varphi - 1.04)), \quad U_0 = 400 \text{km/s}. $$

The diffusion coefficient $K_\parallel$ of cosmic ray particles has the form: 

$$ K_\parallel = K_0 \cdot K(r) \cdot K(R), $$

where $K_0 = 10^{22} \text{cm}^2/\text{s}$, $K(r) = 1 + 0.5 \cdot (r/1AU)$ and $K(R) = R^{0.5}$. Fig. 7 compares the results of the solutions of the Parker transport equation by SDEs and by FDM, considering the same changes of all parameters taken into account, for the GCR particles with rigidity R=10 GV. The model of the 27-day wave of the GCR intensity obtained by the SDEs and FDM at the Earth’s orbit (1 AU, $\theta = 90^\circ$) is in reliable agreement with the data of Moscow NM (Fig. 7).

The reason for some discrepancies between the SDE and FDM models lies in the various
numerical approaches to solve the Parker transport equation. In the case of the stochastic models we must remember about the role of the randomly fluctuating component. From the other side our previously published models based on the FDM contain the initial condition with respect to the rigidity (e.g., [28]), which cannot be passed one to one into the stochastic solution.

5. Conclusion

- We presented the model of the Fd and the 27-day variation of the GCR intensity obtained based on the stochastic approach to the solution of the Parker transport equation. The modeling results are in a good agreement with the data of neutron monitors. We showed a quite reliable agreement between the stochastic approach results and grounded in our previous papers the finite difference method results.

- The SDEs were integrated backward in time in the spherical coordinates applying the full 3D anisotropic diffusion tensor. The models obtained based on the solution of the SDEs reflect the stochastic character of the pseudoparticles distribution in the heliosphere. Additionally, this approach allows to observe the statistically possible changes of the pseudoparticles trajectories, which is not possible based on the solution of the Parker transport equation by finite difference methods.

Acknowledgments

This work is supported by The Polish National Science Centre grant awarded by decision number DEC-2012/07/D/ST6/02488. We thank the principal investigators of Potchefstroom and Moscow neutron monitors and OMNIweb for the ability to use their data. We thank the referees for the thorough review.

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