Chapter 1

Diameter of some monomial digraphs.

A. Kodess
University of Rhode Island, Kingston, RI, USA kodess@uri.edu

F. Lazebnik
University of Delaware, Newark, DE, USA fellaz@udel.edu

S. Smith
University of Delaware, Newark, DE, USA smithsj@udel.edu

J. Sporre
University of Delaware, Newark, DE, USA jsporre@udel.edu

1.1 Introduction

For all terms related to digraphs which are not defined below, see Bang-Jensen and Gutin [1]. In this paper, by a directed graph (or simply digraph) \( D \) we mean a pair \((V, A)\), where \( V = V(D) \) is the set of vertices and \( A = A(D) \subseteq V \times V \) is the set of arcs. For an arc \((u, v)\), the first vertex \( u \) is called its tail and the second vertex \( v \) is called its head; we also denote such an arc by \( u \rightarrow v \). If \((u, v)\) is an arc, we call \( v \) an out-neighbor of \( u \), and \( u \) an in-neighbor of \( v \). The number of out-neighbors of \( u \) is called the out-degree of \( u \), and the number of in-neighbors of \( u \) — the in-degree of \( u \). For an integer \( k \geq 2 \), a walk \( W \) from \( x_1 \) to \( x_k \) in \( D \) is an alternating sequence \( W = x_1a_1x_2a_2x_3 \ldots x_{k-1}a_{k-1}x_k \) of vertices \( x_i \in V \) and arcs \( a_i \in A \) such that the tail of \( a_i \) is \( x_i \) and the head of \( a_i \) is \( x_{i+1} \) for every \( i, 1 \leq i \leq k-1 \). Whenever the labels of the arcs of a walk are not important, we use the
notation $x_1 \to x_2 \to \cdots \to x_k$ for the walk, and say that we have an $x_1x_k$-walk. In a digraph $D$, a vertex $y$ is reachable from a vertex $x$ if $D$ has a walk from $x$ to $y$. In particular, a vertex is reachable from itself. A digraph $D$ is strongly connected (or, just strong) if, for every pair $x, y$ of distinct vertices in $D$, $y$ is reachable from $x$ and $x$ is reachable from $y$. A strong component of a digraph $D$ is a maximal induced subdigraph of $D$ that is strong. If $x$ and $y$ are vertices of a digraph $D$, then the distance from $x$ to $y$ in $D$, denoted $\text{dist}(x, y)$, is the minimum length of an $xy$-walk, if $y$ is reachable from $x$, and otherwise $\text{dist}(x, y) = \infty$. The distance from a set $X$ to a set $Y$ of vertices in $D$ is $\text{dist}(X, Y) = \max\{\text{dist}(x, y) : x \in X, y \in Y\}$.

The diameter of $D$ is $\text{diam}(D) = \text{dist}(V, V)$. Let $p$ be a prime, $e$ a positive integer, and $q = p^e$. Let $\mathbb{F}_q$ denote the finite field of $q$ elements, and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$.

Let $\mathbb{F}_q^2$ denote the Cartesian product $\mathbb{F}_q \times \mathbb{F}_q$, and let $f : \mathbb{F}_q^2 \to \mathbb{F}_q$ be an arbitrary function. We define a digraph $D = D(q; f)$ as follows: $V(D) = \mathbb{F}_q^2$, and there is an arc from a vertex $x = (x_1, x_2)$ to a vertex $y = (y_1, y_2)$ if and only if

$$x_2 + y_2 = f(x_1, y_1).$$

If $(x, y)$ is an arc in $D$, then $y$ is uniquely determined by $x$ and $y_1$, and $x$ is uniquely determined by $y$ and $x_1$. Hence, each vertex of $D$ has both its in-degree and out-degree equal to $q$.

By Lagrange’s interpolation, $f$ can be uniquely represented by a bivariate polynomial of degree at most $q - 1$ in each of the variables. If $f(x, y) = x^m y^n$, $1 \leq m, n \leq q - 1$, we call $D$ a monomial digraph, and denote it also by $D(q; m, n)$. Digraph $D(3; 1, 2)$ is depicted in Fig. 1.1. It is clear, that $x \to y$ in $D(q; m, n)$ if and only if $y \to x$ in $D(q; n, m)$. Hence, one digraph is obtained from the other by reversing the direction of every arc. In general, these digraphs are not isomorphic, but if one of them is strong then so is the other and their diameters are equal. As this paper is concerned only with the diameter of $D(q; m, n)$, it is sufficient to assume that $1 \leq m \leq n \leq q - 1$.

The digraphs $D(q; f)$ and $D(q; m, n)$ are directed analogues of some algebraically defined graphs, which have been studied extensively and have many applications. See Lazebnik and Woldar [18] and references therein; for some subsequent work see Viglione [24], Lazebnik and Mubayi [13], Lazebnik and Viglione [17], Lazebnik and Verstraëte [16], Lazebnik and Thomason [15], Dmytrenko, Lazebnik and Viglione [7], Dmytrenko, Lazebnik and
Diameter of some monomial digraphs

Williford [8], Ustimenko [23], Viglione [25], Terlep and Williford [22], Kronenthal [13], Cioab˘a, Lazebnik and Li [3], Kodess [11], and Kodess and Lazebnik [12].

The questions of strong connectivity of digraphs $D(q; f)$ and $D(q; m, n)$ and descriptions of their components were completely answered in [12]. Determining the diameter of a component of $D(q; f)$ for an arbitrary prime power $q$ and an arbitrary $f$ seems to be out of reach, and most of our results below are concerned with some instances of this problem for strong monomial digraphs. The following theorems are the main results of this paper.

**Theorem 1.1.1.** Let $p$ be a prime, $e, m, n$ be positive integers, $q = p^e$, $1 \leq m \leq n \leq q - 1$, and $D_q = D(q; m, n)$. Then the following statements hold.

1. If $D_q$ is strong, then $\text{diam}(D_q) \geq 3$.
2. If $D_q$ is strong, then
   - for $e = 2$, $\text{diam}(D_q) \leq 96\sqrt{n} + 1 + 1$;
   - for $e \geq 3$, $\text{diam}(D_q) \leq 60\sqrt{n} + 1 + 1$.
3. If $\gcd(m, q - 1) = 1$ or $\gcd(n, q - 1) = 1$, then $\text{diam}(D_q) \leq 4$. If $\gcd(m, q - 1) = \gcd(n, q - 1) = 1$, then $\text{diam}(D_q) = 3$.
4. If $p$ does not divide $n$, and $q > (n^2 - n + 1)^2$, then $\text{diam}(D(q; 1, n)) = 3$.
5. If $D_q$ is strong, then:
   - (a) If $q > n^2$, then $\text{diam}(D_q) \leq 49$.
   - (b) If $q > (m - 1)^4$, then $\text{diam}(D_q) \leq 13$.
(c) If \( q > (n - 1)^4 \), then \( \text{diam}(D(q; n, n)) \leq 9 \).

**Remark 1.** The converse to either of the statements in part (b) of Theorem 1.1.1 is not true. Consider, for instance, \( D(9; 2, 2) \) of diameter 4, or \( D(29; 7, 12) \) of diameter 3.

**Remark 2.** The result of part 5a can hold for some \( q \leq m^2 \).

For prime \( q \), some of the results of Theorem 1.1.1 can be strengthened.

**Theorem 1.1.2.** Let \( p \) be a prime, \( 1 \leq m \leq n \leq p - 1 \), and \( D_p = D(p; m, n) \). Then \( D_p \) is strongly connected and the following statements hold.

1. \( \text{diam}(D_p) \leq 2p - 1 \) with equality if and only if \( m = n = p - 1 \).
2. If \((m, n) \not\in \{(p-1)/2, (p-1)/2, (p-1)/2, p-1\}, \) then \( \text{diam}(D_p) \leq 120\sqrt{m} + 1 \).
3. If \( p > (m-1)^3 \), then \( \text{diam}(D_p) \leq 19 \).

The paper is organized as follows. In section 1.2 we present all results which are needed for our proofs of Theorems 1.1.1 and 1.1.2 in sections 1.3 and 1.4, respectively. Section 1.5 contains concluding remarks and open problems.

### 1.2 Preliminary results.

We begin with a general result that gives necessary and sufficient conditions for a digraph \( D(q; m, n) \) to be strongly connected.

**Theorem 1.2.1.** ([12], Theorem 2) \( D(q; m, n) \) is strongly connected if and only if \( \gcd(q-1, m, n) \) is not divisible by any \( q_d = (q-1)/(p^d-1) \) for any positive divisor \( d \) of \( e \), \( d < e \). In particular, \( D(p; m, n) \) is strongly connected for any \( m, n \).

Every walk of length \( k \) in \( D = D(q; m, n) \) originating at \((a, b)\) is of the form

\[
(a, b) \rightarrow (x_1, -b + a^m x_1^n)
\rightarrow (x_2, -a^m x_1^n + x_2^m x_2^n)
\rightarrow \ldots
\rightarrow (x_k, x_k^m x_{k-1}^m - x_k^m x_{k-1}^n + \cdots + (-1)^{k-1} a^m x_1^n + (-1)^k b).
\]

Therefore, in order to prove that \( \text{diam}(D) \leq k \), one can show that for any choice of \( a, b, u, v \in \mathbb{F}_q \), there exists \((x_1, \ldots, x_k) \in \mathbb{F}_q^k\) so that

\[
(u, v) = (x_k, x_k^m x_{k-1}^n + \cdots + (-1)^{k-1} a^m x_1^n + (-1)^k b).
\]
In order to show that diam($D$) ≥ $l$, one can show that there exist $a, b, u, v \in F_q$ such that (1.1) has no solution in $F_q^k$ for any $k < l$.

1.2.1 Waring’s Problem

In order to obtain an upper bound on diam($D(q; m, n)$) we will use some results concerning Waring’s problem over finite fields.

Waring’s number $\gamma(r, q)$ over $F_q$ is defined as the smallest positive integer $s$ (should it exist) such that the equation

$$x_1^r + x_2^r + \cdots + x_s^r = a$$

has a solution $(x_1, \ldots, x_s) \in F_q^s$ for any $a \in F_q$. Similarly, $\delta(r, q)$ is defined as the smallest positive integer $s$ (should it exist) such that for any $a \in F_q$, there exists $(\epsilon_1, \ldots, \epsilon_s)$, each $\epsilon_i \in \{-1, 1\} \subseteq F_q$, for which the equation

$$\epsilon_1 x_1^r + \epsilon_2 x_2^r + \cdots + \epsilon_s x_s^r = a$$

has a solution $(x_1, \ldots, x_s) \in F_q^s$. It is easy to argue that $\delta(r, q)$ exists if and only if $\gamma(r, q)$ exists, and in this case $\delta(r, q) \leq \gamma(r, q)$.

A criterion on the existence of $\gamma(r, q)$ is the following theorem by Bhashkaran [2].

Theorem 1.2.2. [2, Theorem G] Waring’s number $\gamma(r, q)$ exists if and only if $r$ is not divisible by any $q_d = (q-1)/(p^d-1)$ for any positive divisor $d$ of $e$, $d < e$.

The study of various bounds on $\gamma(r, q)$ has drawn considerable attention. We will use the following two upper bounds on Waring’s number due to J. Cipra [5].

Theorem 1.2.3. [5, Theorem 4] If $e = 2$ and $\gamma(r, q)$ exists, then $\gamma(r, q) \leq 16\sqrt{r + 1}$. Also, if $e \geq 3$ and $\gamma(r, q)$ exists, then $\gamma(r, q) \leq 10\sqrt{r + 1}$.

Corollary 1.2.1. [5, Corollary 7] If $\gamma(r, q)$ exists and $r < \sqrt{q}$, then $\gamma(r, q) \leq 8$.

For the case $q = p$, the following bound will be of interest.

Theorem 1.2.4. [Cochrane, Pinner [6], Corollary 10.3] If $|\{x^k : x \in F_p^*\}| > 2$, then $\delta(k, p) \leq 20\sqrt{p}$.

The next two statements concerning very strong bounds on Waring’s number in large fields follow from the work of Weil [26], and Hua and Vandiver [10].
Theorem 1.2.5. [Small 20] If $q > (k-1)^4$, then $\gamma(k,q) \leq 2$.

Theorem 1.2.6. [Cipra 4, p. 4] If $p > (k-1)^3$, then $\gamma(k,p) \leq 3$.

For a survey on Waring’s number over finite fields, see Castro and Rubio (Section 7.3.4, p. 211), and Ostafe and Winterhof (Section 6.3.2.3, p. 175) in Mullen and Panario [19]. See also Cipra [4].

We will need the following technical lemma.

Lemma 1.2.1. Let $\delta = \delta(r,q)$ exist, and $k \geq 2\delta$. Then for every $a \in \mathbb{F}_q$ the equation
\[ x_1^r - x_2^r + x_3^r - \cdots + (-1)^{k+1}x_k^r = a \] (1.2)
has a solution $(x_1, \ldots, x_k) \in \mathbb{F}_q^k$.

Proof. Let $a \in \mathbb{F}_q$ be arbitrary. There exist $\varepsilon_1, \ldots, \varepsilon_\delta$, each $\varepsilon_i \in \{-1,1\} \subseteq \mathbb{F}_q$, such that the equation $\sum_{i=1}^{\delta} \varepsilon_i y_i^r = a$ has a solution $(y_1, \ldots, y_\delta) \in \mathbb{F}_q^\delta$. As $k \geq 2\delta$, the alternating sequence $1, -1, 1, \ldots, (-1)^k$ with $k$ terms contains the sequence $\varepsilon_1, \ldots, \varepsilon_\delta$ as a subsequence. Let the indices of this subsequence be $j_1, j_2, \ldots, j_\delta$. For each $l$, $1 \leq l \leq k$, let $x_l = 0$ if $l \neq j_i$ for any $i$, and $x_l = y_i$ for $l = j_i$. Then $(x_1, \ldots, x_k)$ is a solution of (1.2). \qed

1.2.2 The Hasse-Weil bound

In the next section we will use the Hasse-Weil bound, which provides a bound on the number of $\mathbb{F}_q$-points on a plane non-singular absolutely irreducible projective curve over a finite field $\mathbb{F}_q$. If the number of points on the curve $C$ of genus $g$ over the finite field $\mathbb{F}_q$ is $|C(\mathbb{F}_q)|$, then
\[ ||C(\mathbb{F}_q)| - q - 1| \leq 2g\sqrt{q}. \] (1.3)
It is also known that for a non-singular curve defined by a homogeneous polynomial of degree $k$, $g = (k-1)(k-2)/2$. Discussion of all related notions and a proof of this result can be found in Hirschfeld, Korchmáros, Torres [9] (Theorem 9.18, p. 343) or in Szőnyi [21] (p. 197).

1.3 Proof of Theorem 1.1.1

As there is a loop at $(0,0)$, and there are arcs between $(0,0)$ and $(x,0)$ in either direction, for every $x \in \mathbb{F}_q^*$, the number of vertices in $D_q$ which are at distance at most 2 from $(0,0)$ is at most $1 + (q-1) + (q-1)^2 < q^2$. Thus, there are vertices in $D_q$ which are at distance at least 3 from $(0,0)$, and so $\text{diam}(D_q) \geq 3$. 


As $D_q$ is strong, by Theorem 1.2.1 for any positive divisor $d$ of $e$, $d < e$, $q_d \neq \gcd(p^e - 1, m, n)$. As, clearly, $q_d \mid (p^e - 1)$, either $q_d \mid m$ or $q_d \mid n$. This implies by Theorem 1.2.2 that either $\gamma(m, q)$ or $\gamma(n, q)$ exists.

Let $(a, b)$ and $(u, v)$ be arbitrary vertices of $D_q$. By (1.1), there exists a walk of length at most $k$ from $(a, b)$ to $(u, v)$ if the equation

$$v = x_k^{m_1} - x_k^{m_2} + \cdots + (-1)^{k-1}a^m + (-1)^kb$$

has a solution $(x_1, \ldots, x_k) \in \mathbb{F}^k_q$.

Assume first that $\gamma_m = \gamma(m, q)$ exists. Taking $k = 6\gamma_m + 1$, and $x_i = 0$ for $i \equiv 1 \mod 3$, and $x_i = 1$ for $i \equiv 0 \mod 3$, we have that (1.3) is equivalent to

$$-x_5^m + x_3^m - \cdots + (-1)^{k-1}x_2^m = v - (-1)^kb - u^n.$$ 

As the number of terms on the left is $(k - 1)/3 = 2\gamma_m$, this equation has a solution in $\mathbb{F}_q^{2\gamma_m}$ by Lemma 1.2.1. Hence, (1.3) has a solution in $\mathbb{F}_q^{2\gamma_m}$.

If $\gamma_n = \gamma(n, q)$ exists, then the argument is similar: take $k = 6\gamma_n + 1$, $x_i = 0$ for $i \equiv 0 \mod 3$, and $x_i = 1$ for $i \equiv 1 \mod 3$.

The result now follows from the bounds on $\gamma(r, q)$ in Theorem 1.2.3.

**Remark 3.** As $m \leq n$, if $\gamma(m, q)$ exists, the upper bounds in Theorem 1.1.1 part (2), can be improved by replacing $n$ by $m$. Also, if a better upper bound on $\delta(m, q)$ than $\gamma(m, q)$ (respectively, on $\delta(n, q)$ than $\gamma(n, q)$) is known, the upper bounds in Theorem 1.1.1 (2), can be further improved: use $k = 6\delta(m, q) + 1$ (respectively, $k = 6\delta(n, q) + 1$) in the proof. Similar comments apply to other parts of Theorem 1.1.1 as well as Theorem 1.1.2.

**Remark 4.** Recall the basic fact $\gcd(r, q - 1) = 1 \iff \{x^r : x \in \mathbb{F}_q\} = \mathbb{F}_q$.

Let $k = 4$. If $\gcd(m, q - 1) = 1$, a solution to (1.1) of the form $(0, x_2, 1, u)$ is seen to exist for any choice of $a, b, u, v \in \mathbb{F}_q$. If $\gcd(n, q - 1) = 1$, there exists a solution of the form $(1, x_2, 0, u)$. Hence, $\text{diam}(D_q) \leq 4$.

Let $k = 3$, and $\gcd(m, q - 1) = \gcd(n, q - 1) = 1$. If $a = 0$, then a solution to (1.1) of the form $(x_1, 0, u)$ exists. If $a \neq 0$, a solution of the form $(x_1, 0, u)$ exists. Hence, $D_q$ is strong and $\text{diam}(D_q) \leq 3$. Using the lower bound from part (11), we conclude that $\text{diam}(D_q) = 3$.

**Remark 4.** As was shown in part (3) for any $n$, $\text{diam}(D(q, 1, n)) \leq 4$. If, additionally, $\gcd(n, q - 1) = 1$, then $\text{diam}(D(q, 1, n)) = 3$. It turns out that if $p$ does not divide $n$, then only for finitely many $q$ is the diameter of $D(q, 1, n)$ actually 4.
For $k = 3$, (1.1) is equivalent to
\[(u, v) = (x_3, x_2 x_3^n - x_1 x_2^n + a x_1^n - b),\]
which has solution $(x_1, x_2, x_3) = (0, u^{-n}(b + v), u)$, provided $u \neq 0$.

Suppose now that $u = 0$. Aside from the trivial case $a = 0$, the question of the existence of a solution to (1.5) shall be resolved if we prove that the equation
\[a x^n - x y^n + c = 0\]
has a solution for any $a, c \in \mathbb{F}_q$ (for $c = 0$, (1.6) has solutions). The projective curve corresponding to this equation is the zero locus of the homogeneous polynomial
\[F(X, Y, Z) = a X^n Z - X Y^n + c Z^{n+1}.\]
It is easy to see that, provided $p$ does not divide $n$, \[F = F_X = F_Y = F_Z = 0 \iff X = Y = Z = 0,\]
and thus the curve has no singularities and is absolutely irreducible.

Counting the two points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ on the line at infinity $Z = 0$, we obtain from (1.3), the inequality
\[N \geq q - 1 - 2g \sqrt{q},\]
where $N = N(c)$ is the number of solutions of (1.6). As $g = n(n - 1)/2$, solving the inequality $q - 1 - n(n - 1) \sqrt{q} > 0$ for $q$, we obtain a lower bound on $q$ for which $N \geq 1$.

(5a). The result follows from Corollary 1.2.1 by an argument similar to that of the proof of part (2).

(5b). For $k = 13$, (1.1) is equivalent to
\[(u, v) = (x_{13}, -b + a^m x_1^n - x_1^m x_2^n + x_2^m x_3^n - \cdots - x_{12}^m x_{12}^n + x_{13}^m x_{13}^n).\]
If $q > (m-1)^4$, set $x_1 = x_4 = x_7 = x_{10} = 0, x_3 = x_6 = x_9 = x_{12} = 1$. Then $v - u^n + b = -x_{11}^m + x_8^m - x_5^m + x_2^m$, which has a solution $(x_2, x_5, x_8, x_{11}) \in \mathbb{F}_q^4$ by Theorem 1.2.5 and Lemma 1.2.4.

(5c). For $k = 9$, (1.1) is equivalent to
\[(u, v) = (x_9, -b + a^m x_1^n - x_1^m x_2^n + x_2^m x_3^n - \cdots - x_7^m x_8^n + x_9^m x_9^n).\]
If $q > (n-1)^3$, set $x_1 = x_4 = x_5 = x_8 = 0, x_3 = x_7 = 1$. Then $v + b = x_2^n + x_6^n$, which has a solution $(x_2, x_6) \in \mathbb{F}_q^2$ by Theorem 1.2.4.
1.4 Proofs of Theorem 1.1.2

Lemma 1.4.1. Let $D = D(q; m, n)$. Then, for any $\lambda \in F_q^*$, the function $\phi : V(D) \to V(D)$ given by $\phi((a, b)) = (\lambda a, \lambda^{m+n} b)$ is a digraph automorphism of $D$.

The proof of the lemma is straightforward. It amounts to showing that $\phi$ is a bijection and that it preserves adjacency: $x \to y$ if and only if $\phi(x) \to \phi(y)$. We omit the details. Due to Lemma 1.4.1, any walk in $D$ initiated at a vertex $(a, b)$ corresponds to a walk initiated at a vertex $(0, b)$ if $a = 0$, or at a vertex $(1, b')$, where $b' = a^{m+n} b$, if $a \neq 0$. This implies that if we wish to show that diam($D_p$) $\leq 2p - 1$, it is sufficient to show that the distance from any vertex $(0, b)$ to any other vertex is at most $2p - 1$, and that the distance from any vertex $(1, b)$ to any other vertex is at most $2p - 1$.

First we note that by Theorem 1.2.1, $D_p = D(p; m, n)$ is strong for any choice of $m, n$.

For $a \in F_p$, let integer $\overline{a}$, $0 \leq \overline{a} \leq p - 1$, be the representative of the residue class $a$.

It is easy to check that diam($D(2; 1, 1)$) $= 3$. Therefore, for the remainder of the proof, we may assume that $p$ is odd.

1. In order to show that diam($D_p$) $\leq 2p - 1$, we use (1) with $k = 2p - 1$, and prove that for any two vertices $(a, b)$ and $(u, v)$ of $D_p$ there is always a solution $(x_1, \ldots, x_{2p-1}) \in F_q^{2p-1}$ of

$$(u, v) = (x_{2p-1}, -b + a^m x_1^n - x_1 x_2^n + x_2 x_3^n - \cdots - x_{2p-3} x_{2p-2}^n + x_{2p-2} x_{2p-1}^n),$$

or, equivalently, a solution $x = (x_1, \ldots, x_{2p-2}) \in F_q^{2p-2}$ of

$$a^m x_1^n - x_1 x_2^n + x_2 x_3^n - \cdots - x_{2p-3} x_{2p-2}^n + x_{2p-2} x_{2p-1}^n = b + v. \quad (1.7)$$

As the upper bound $2p - 1$ on the diameter is exact and holds for all $p$, we need a more subtle argument compared to the ones we used before. The only way we can make it is (unfortunately) by performing a case analysis on $b + v$ with a nested case structure. In most of the cases we just exhibit a solution $x$ of (1.7) by describing its components $x_i$. It is always a straightforward verification that $x$ satisfies (1.7), and we will suppress our comments as cases proceed.

Our first observation is that if $b + v = 0$, then $x = (0, \ldots, 0)$ is a solution to (1.7). We may assume now that $b + v \neq 0$.\]
Case 1.1: $b + v \geq \frac{p - 1}{2} + 2$

We define the components of $x$ as follows:

- if $1 \leq i \leq 4(p - (b + v))$, then $x_i = 0$ for $i \equiv 1, 2 \mod 4$, and $x_i = 1$ for $i \equiv 0, 3 \mod 4$;
- if $4(p - (b + v)) < i \leq 2p - 2$, then $x_i = 0$.

Note that $x_i^m x_{i+1}^n = 0$ unless $i \equiv 3 \mod 4$, in which case $x_i^m x_{i+1}^n = 1$.

If we group the terms in groups of four so that each group is of the form $-x_i^m x_{i+1}^n + x_i^{m+1} x_{i+2}^n - x_i^{2m} x_{i+3}^n + x_i^{m+3} x_{i+4}^n$, where $i \equiv 1 \mod 4$, then assuming $i, i + 1, i + 2, i + 3$, and $i + 4$ are within the range of $1 \leq i \leq 4(p - (b + v))$, it is easily seen that one group contributes $-1$ to $a x_1^m x_2^m + x_2^m x_3^m - \cdots - x_{2p-3}^m x_{2p-2}^m + x_{2p-2}^m x_{2p-1}^m$. There are $\frac{4(p - (b + v))}{4} = p - (b + v)$ such groups, and so the solution provided adds $-1$ exactly $p - (b + v)$ times. Hence, $x$ is a solution to (1.7).

For the remainder of the proof, solutions to (1.7) will be given without justification as the justification is similar to what’s been done above.

Case 1.2: $b + v \leq \frac{p - 1}{2}$

We define the components of $x$ as follows:

- if $1 \leq i \leq 4(b + v) - 1$, then $x_i = 0$ for $i \equiv 0, 1 \mod 4$, and $x_i = 1$ for $i \equiv 2, 3 \mod 4$;
- if $4(b + v) - 1 < i \leq 2p - 2$, then $x_i = 0$.

Case 1.3: $b + v = \frac{p - 1}{2} + 1$

This case requires several nested subcases.

Case 1.3.1: $u = x_{2p-1} = 0$

Here, there is no need to restrict $x_{2p-2}$ to be 0. The components of a solution $x$ of (1.7) are defined as:

- if $1 \leq i \leq 2p - 2$, then $x_i = 0$ for $i \equiv 1, 2 \mod 4$, and $x_i = 1$ for $i \equiv 0, 3 \mod 4$.

Case 1.3.2: $a = 0$

Here, there is no need to restrict $x_1$ to be 0. Therefore, the components of a solution $x$ of (1.7) are defined as:

- if $1 \leq i \leq 2p - 2$, then $x_i = 0$ for $i \equiv 0, 3 \mod 4$, and $x_i = 1$ for $i \equiv 1, 2 \mod 4$. 
Case 1.3.3: $u \neq 0$ and $a \neq 0$

Because of Lemma 1.4.1, we may assume without loss of generality that $a = 1$. Let $x_{2p-2} = 1$, so that $x_{2p-2}^n = u^n \neq 0$ and let $t = b + v - u^n$. Note that $t \neq \frac{p-1}{2} + 1$.

Case 1.3.3.1: $t = 0$

The components of a solution $x$ of (1.7) are defined as: $x_{2p-2} = 1$, and if $1 \leq i < 2p - 2$, then $x_i = 0$.

Case 1.3.3.2: $0 < t \leq \frac{p-1}{2}$

The components of a solution $x$ of (1.7) are defined as: $x_{2p-2} = 1$, and if $1 \leq i \leq 4(t - 1) + 1$, then $x_i = 0$ for $i \equiv 2, 3 \mod 4$, and $x_i = 1$ for $i \equiv 0, 1 \mod 4$;

if $4(t - 1) + 1 < i < 2p - 2$, then $x_i = 0$.

Case 1.3.3.3: $t \geq \frac{p-1}{2} + 2$

The components of a solution $x$ of (1.7) are defined as: $x_{2p-2} = 1$, and if $1 \leq i \leq 4(p - t)$, then $x_i = 0$ for $i \equiv 1, 2 \mod 4$, and $x_i = 1$ for $i \equiv 0, 3 \mod 4$;

if $4(p - t) < i < 2p - 2$, then $x_i = 0$.

The whole range of possible values $b + v$ has been checked. Hence, $\text{diam}(D) \leq 2p - 1$.

We now show that if $\text{diam}(D) = 2p - 1$, then $m = n = p - 1$. To do so, we assume that $m \neq p - 1$ or $n \neq p - 1$ and prove the contrapositive. Specifically, we show that $\text{diam}(D) \leq 2p - 2 < 2p - 1$ by again using (1.1) but with $k = 2p - 2$. We prove that for any two vertices $(a, b)$ and $(u, v)$ of $D_p$ there is always a solution $(x_1, \ldots, x_{2p-2}) \in \mathbb{F}_q^{2p-2}$ of

$$(u, v) = (x_{2p-2}, b - a^m x_1^n + x_1^n x_2^n - \cdots - x_{2p-4}^m x_{2p-3}^n + x_{2p-3}^m x_{2p-2}^n).$$

or, equivalently, a solution $x = (x_1, \ldots, x_{2p-3}) \in \mathbb{F}_q^{2p-3}$ of

$-a^m x_1^n + x_1^n x_2^n - x_2^n x_3^n + \cdots - x_{2p-4}^m x_{2p-3}^n + x_{2p-3}^m u^n = -b + v.$ (1.8)

We perform a case analysis on $-b + v$.

Our first observation is that if $-b + v = 0$, then $x = (0, \ldots, 0)$ is a solution to (1.8). We may assume for the remainder of the proof that $-b + v \neq 0$.

Case 2.1: $-b + v \leq \frac{p-1}{2} - 1$
We define the components of \( x \) as follows:
if \( 1 \leq i \leq 4(\frac{1}{2}b - v) \), then \( x_i = 0 \) for \( i \equiv 1, 2 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 0, 3 \mod 4 \);
if \( 4(\frac{1}{2}b - v) < i \leq 2p - 3 \), then \( x_i = 0 \).

**Case 2.2:** \( -b + v \geq \frac{p-1}{2} + 2 \)
We define the components of \( x \) as follows:
if \( 1 \leq i \leq 4(p - (\frac{1}{2}b + v)) - 1 \), then \( x_i = 0 \) for \( i \equiv 0, 1 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 2, 3 \mod 4 \);
if \( 4(p - (\frac{1}{2}b + v)) - 1 < i \leq 2p - 3 \), then \( x_i = 0 \).

**Case 2.3:** \( -b + v = \frac{p-1}{2} \)

**Case 2.3.1:** \( a = 0 \)
We define the components of \( x \) as:
if \( 1 \leq i \leq 2p - 3 \), then \( x_i = 0 \) for \( i \equiv 0, 3 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 1, 2 \mod 4 \).

**Case 2.3.2:** \( a \neq 0 \)
Here, we may assume without loss of generality that \( a = 1 \) by Lemma 1.3.1.

**Case 2.3.2.1:** \( n \neq p - 1 \)
If \( n \neq p - 1 \), then there exists \( \beta \in \mathbb{F}_p^* \) such that \( \beta^n \notin \{0, 1\} \). For such a \( \beta \), let \( x_1 = \beta \) and consider \( t = -\frac{1}{2}b + v + a^\frac{1}{2}x_1^2 = -\frac{1}{2}b + v + \beta^\frac{1}{2} \notin \{\frac{1}{2}b, \frac{1}{2}\} + 1 \} \).

**Case 2.3.2.1.1:** \( t = 0 \)
We define the components of \( x \) as: \( x_1 = \beta \) and
if \( 2 \leq i \leq 2p - 3 \), then \( x_i = 0 \).

**Case 2.3.2.1.2:** \( t \leq \frac{p-1}{4} - 1 \)
We define the components of \( x \) as: \( x_1 = \beta \) and
if \( 2 \leq i \leq 4t \), then \( x_i = 0 \) for \( i \equiv 1, 2 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 0, 3 \mod 4 \);
if \( 4t < i \leq 2p - 3 \), then \( x_i = 0 \).

**Case 2.3.2.1.3:** \( t \geq \frac{p-1}{4} + 2 \)
We define the components of \( x \) as: \( x_1 = \beta \) and
if \( 2 \leq i \leq 4(p - t) + 1 \), then \( x_i = 0 \) for \( i \equiv 2, 3 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 0, 1 \mod 4 \);
if \( 4(p - t) + 1 < i \leq 2p - 3 \), then \( x_i = 0 \).


**Case 2.3.2.2:** $n = p - 1$

**Case 2.3.2.2.1:** $u \in \mathbb{F}_p^*$

Here, we have that $u^n = 1$, so that the components of a solution $x$ of (1.5) are defined as:

- if $1 \leq i \leq 2p - 3$, then $x_i = 0$ for $i \equiv 1, 2 \mod 4$, and $x_i = 1$ for $i \equiv 0, 3 \mod 4$.

**Case 2.3.2.2.2:** $u = 0$

Since $n = p - 1$, it must be the case that $m \neq p - 1$ so that there exists $\alpha \in \mathbb{F}_p^*$ such that $\alpha^m \not\in \{0, 1\}$. For such an $\alpha$, let $x_2 = \alpha, x_3 = 1$ and consider $t = -b + v + x_2^2x_3 = -b + v + \alpha^m \not\in \{\frac{p-1}{2}, \frac{p+1}{2} + 1\}$.

**Case 2.3.2.2.2.1:** $t = 0$

We define the components of $x$ as: $x_1 = 0, x_2 = \alpha, x_3 = 1$ and

- if $4 \leq i \leq 2p - 3$, then $x_i = 0$.

**Case 2.3.2.2.2.2:** $t \leq \frac{p-1}{2} - 1$

We define the components of $x$ as: $x_1 = 0, x_2 = \alpha, x_3 = 1$ and

- if $4 \leq i \leq 4t$, then $x_i = 0$ for $i \equiv 1, 2 \mod 4$, and $x_i = 1$ for $i \equiv 0, 3 \mod 4$;

- if $4t < i \leq 2p - 3$, then $x_i = 0$.

**Case 2.3.2.2.2.3:** $t \geq \frac{p-1}{2} + 2$

We define the components of $x$ as: $x_1 = 0, x_2 = \alpha, x_3 = 1$ and

- if $4 \leq i \leq 4(p - t) + 3$, then $x_i = 0$ for $i \equiv 0, 1 \mod 4$, and $x_i = 1$ for $i \equiv 2, 3 \mod 4$;

- if $4(p - t) + 3 < i \leq 2p - 3$, then $x_i = 0$.

**Case 2.4:** $-b + v = \frac{p-1}{2} + 1$

**Case 2.4.1:** $u = 0$

We define the components of $x$ as:

- if $1 \leq i \leq 2p - 3$, then $x_i = 0$ for $i \equiv 0, 1 \mod 4$, and $x_i = 1$ for $i \equiv 2, 3 \mod 4$.

**Case 2.4.2:** $u \neq 0$

Here, we may assume without loss of generality that $u = 1$ by Lemma (1.3.1).

**Case 2.4.2.1:** $m \neq p - 1$

If $m \neq p - 1$, then there exists $\alpha \in \mathbb{F}_p^*$ such that $\alpha^m \not\in \{0, 1\}$. For such
an \( \alpha \), let \( x_{2p-3} = \alpha \) and consider \( t = -b + v - x_{2p-3}^m = -b + v - \alpha^m \notin \{ \frac{p-1}{2}, \frac{p+1}{2} + 1 \} \).

Case 2.4.2.1.1: \( t = 0 \)
We define the components of \( x \) as: \( x_{2p-3} = \alpha \) and
if \( 1 \leq i \leq 2p - 4 \), then \( x_i = 0 \).

Case 2.4.2.1.2: \( t \leq \frac{p-1}{2} - 1 \)
We define the components of \( x \) as: \( x_{2p-3} = \alpha \) and
if \( 1 \leq i \leq 4t \), then \( x_i = 0 \) for \( i \equiv 1, 2 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 0, 3 \mod 4 \);
if \( 4t < i \leq 2p - 4 \), then \( x_i = 0 \).

Case 2.4.2.1.3: \( t \geq \frac{p-1}{2} + 2 \)
We define the components of \( x \) as: \( x_{2p-3} = \alpha \) and
if \( 1 \leq i \leq 4(p - t) - 1 \), then \( x_i = 0 \) for \( i \equiv 0, 1 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 2, 3 \mod 4 \);
if \( 4(p - t) - 1 < i \leq 2p - 4 \), then \( x_i = 0 \).

Case 2.4.2.2: \( m = p - 1 \)

Case 2.4.2.2.1: \( a \in \mathbb{F}_p^* \)
Here, we have that \( a^m = 1 \), so that the components of a solution \( x \) of (LS) are defined as:
if \( 1 \leq i \leq 2p - 5 \), then \( x_i = 0 \) for \( i \equiv 2, 3 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 0, 1 \mod 4 \).

Case 2.4.2.2.2: \( a = 0 \)
Since \( m = p - 1 \), it must be the case that \( n \neq p - 1 \) so that there exists \( \beta \in \mathbb{F}_p^* \) such that \( \beta^n \notin \{ 0, 1 \} \). For such a \( \beta \), let \( x_{2p-5} = 1, x_{2p-4} = \beta \) and consider \( t = -b + v - x_{2p-5}^n x_{2p-4}^m = -b + v - \beta^m \notin \{ \frac{p-1}{2}, \frac{p+1}{2} + 1 \} \).

Case 2.4.2.2.2.1: \( t = 0 \)
We define the components of \( x \) as: \( x_{2p-5} = 1, x_{2p-4} = \beta, x_{2p-3} = 0 \) and
if \( 1 \leq i \leq 2p - 6 \), then \( x_i = 0 \).

Case 2.4.2.2.2.2: \( t \leq \frac{p-1}{2} - 1 \)
We define the components of \( x \) as: \( x_{2p-5} = 1, x_{2p-4} = \beta, x_{2p-3} = 0 \) and
if \( 1 \leq i \leq 4t - 2 \), then \( x_i = 0 \) for \( i \equiv 0, 3 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 1, 2 \mod 4 \);
if \( 4t - 2 < i \leq 2p - 6 \), then \( x_i = 0 \).
Diameter of some monomial digraphs

Case 2.4.2.2.2.3: \( t \geq \frac{p+1}{2} + 2 \)

We define the components of \( x \) as: \( x_{2p-5} = 1, x_{2p-4} = \beta, x_{2p-3} = 0 \) and if \( 1 \leq i \leq 4(p - t) - 1 \), then \( x_i = 0 \) for \( i \equiv 0, 1 \mod 4 \), and \( x_i = 1 \) for \( i \equiv 2, 3 \mod 4 \);

if \( 4(p - t) - 1 < i \leq 2p - 6 \), then \( x_i = 0 \).

All cases have been checked, so if \( m \neq p - 1 \) or \( n \neq p - 1 \), then \( \text{diam}(D) < 2p - 1 \).

We now prove that if \( m = n = p - 1 \), then \( d := \text{diam}(D(p; m, n)) = 2p - 1 \). In order to do this, we explicitly describe the structure of the digraph \( D(p; p - 1, p - 1) \), from which the diameter becomes clear. In this description, we look at sets of vertices of a given distance from the vertex \((0, 0)\), and show that some of them are at distance \( 2p - 1 \). We recall the following important general properties of our digraphs that will be used in the proof.

- Every out-neighbor \((u, v)\) of a vertex \((a, b)\) of \( D(q; m, n) \) is completely determined by its first component \( u \).
- Every vertex of \( D(q; m, n) \) has its out-degree and in-degree equal \( q \).
- In \( D(q; m, m) \), \( x \rightarrow y \) if and only if \( y \rightarrow x \).

In \( D(p; p - 1, p - 1) \), we have that \((x_1, y_1) \rightarrow (x_2, y_2)\) if and only if

\[
y_1 + y_2 = x_1^{p-1}x_2^{p-1} = \begin{cases} 0 & \text{if } x_1 = 0 \text{ or } x_2 = 0, \\ 1 & \text{if } x_1 \text{ and } x_2 \text{ are non-zero}. \end{cases}
\]

For notational convenience, we set

\[(*, a) = \{(x, a) : x \in \mathbb{F}_p^*\}\]

and, for \( 1 \leq k \leq d \), let

\[N_k = \{v \in V(D(p; m, n)) : \text{dist}((0, 0), v) = k\}\]

We assume that \( N_0 = \{(0, 0)\} \). It is clear from this definition that these \( d+1 \) sets \( N_k \) partition the vertex set of \( D(p; p-1, p-1) \); for every \( k, 1 \leq k \leq d-1 \), every out-neighbor of a vertex from \( N_k \) belongs to \( N_{k-1} \cup N_k \cup N_{k+1} \), and \( N_{k+1} \) is the set of all out-neighbors of all vertices from \( N_k \) which are not in \( N_{k-1} \cup N_k \).

Thus we have \( N_0 = \{(0, 0)\}, N_1 = (*, 0), N_2 = (*, 1), N_3 = \{(0, -1)\} \). If \( p > 2 \), \( N_4 = \{(0, 1)\}, N_5 = (*, -1) \). As there exist two (opposite) arcs between each vertex of \((*, x)\) and each vertex \((*, -x + 1)\), these subsets of
vertices induce the complete bipartite subdigraph $\overrightarrow{K}_{p-1,p-1}$ if $x \neq -x + 1$, and the complete subdigraph $\overrightarrow{K}_{p-1}$ if $x = -x + 1$. Note that our $\overrightarrow{K}_{p-1,p-1}$ has no loops, but $\overrightarrow{K}_{p-1}$ has a loop on every vertex. Digraph $D(5; 4, 4)$ is depicted in Fig. 1.2.

The structure of $D(p; p-1, p-1)$ for any other prime $p$ is similar. We can describe it as follows: for each $t \in \{0, 1, \ldots, (p-1)/2\}$, let

$$N_{\mathcal{F}} = \{(0, t)\}, \quad N_{\mathcal{F}+1} = (\ast , -t),$$

and for each $t \in \{0, 1, \ldots, (p-3)/2\}$, let

$$N_{\mathcal{F}+2} = (\ast , t+1), \quad N_{\mathcal{F}+3} = \{(0, -t-1)\}.$$

Note that for $0 \leq \overline{t} < (p-1)/2$, $N_{\mathcal{F}+1} \neq N_{\mathcal{F}+2}$, and for $\overline{t} = (p-1)/2$, $N_{2p-1} = (\ast , (p+1)/2)$. Therefore, for $p \geq 3$, $D(p; p-1, p-1)$ contains $(p-1)/2$ induced copies of $\overrightarrow{K}_{p-1,p-1}$ with partitions $N_{\mathcal{F}+1}$ and $N_{\mathcal{F}+2}$, and a copy of $\overrightarrow{K}_{p-1}$ induced by $N_{2p-1}$. The proof is a trivial induction on $\overline{t}$. Hence, $\text{diam}(D(p; p-1, p-1)) = 2p - 1$. This ends the proof of Theorem 1.1.2 (1).

(2). We follow the argument of the proof of Theorem 1.1.1 part (2) and use Lemma 1.2.1 with $k = 6\delta(m, p) + 1$. We note, additionally, that if $m \notin \{p, (p-1)/2\}$, then $\gcd(m, p-1) < (p-1)/2$, which implies $|\{x^m : x \in \mathbb{F}_p^*\}| > 2$. The result then follows from Theorem 1.2.4.

(3). We follow the argument of the proof of Theorem 1.1.1 part (5b) and use Lemma 1.2.1 and Theorem 1.2.6.

This ends the proof of Theorem 1.1.2.
1.5 Concluding remarks.

Many results in this paper follow the same pattern: if Waring’s number \( \delta(r, q) \) exists and is bounded above by \( \delta \), then one can show that \( \text{diam}(D(q; m, n)) \leq 6\delta + 1 \). Determining the exact value of \( \delta(r, q) \) is an open problem, and it is likely to be very hard. Also, the upper bound \( 6\delta + 1 \) is not exact in general. Out of all partial results concerning \( \delta(r, q) \), we used only those ones which helped us deal with the cases of the diameter of \( D(q; m, n) \) that we considered, especially where the diameter was small. We left out applications of all asymptotic bounds on \( \delta(r, q) \). Our computer work demonstrates that some upper bounds on the diameter mentioned in this paper are still far from being tight. Here we wish to mention only a few strong patterns that we observed but have not been able to prove so far. We state them as problems.

**Problem 1.** Let \( p \) be prime, \( q = p^e \), \( e \geq 2 \), and suppose \( D(q; m, n) \) is strong. Let \( r \) be the largest divisor of \( q - 1 \) not divisible by any \( q_d = (p^e - 1)/(q^d - 1) \) where \( d \) is a positive divisor of \( e \) smaller than \( e \). Is it true that

\[
\max_{1 \leq m \leq n \leq q-1} \{\text{diam}(D(q; m, n))\} = \text{diam}(D(q; r, r))?
\]

Find an upper bound on \( \text{diam}(D(q; r, r)) \) better than the one of Theorem 1.1.1 part (5c).

**Problem 2.** Is it true that for every prime \( p \) and \( 1 \leq m \leq n \), \( (m, n) \neq (p - 1, p - 1) \), \( \text{diam}(D(p; m, n)) \leq (p + 3)/2 \) with the equality if and only if \( (m, n) = ((p - 1)/2, (p - 1)/2) \) or \( (m, n) = ((p - 1)/2, p - 1) \)?

**Problem 3.** Is it true that for every prime \( p \), \( \text{diam}(D(p; m, n)) \) takes only one of two consecutive values which are completely determined by \( \gcd((p - 1, m, n)) \)?

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