Poincaré recurrences as a tool to investigate the statistical properties of dynamical systems with integrable and mixing components

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Abstract. Poincaré recurrences seem able to capture some of the fundamental properties of dynamical systems. In fact, the asymptotic distribution of Poincaré recurrences is exponential for a wide class of mixing systems, even if they are not uniformly hyperbolic. On the other hand, we found strong numerical evidences that for integrable systems such distribution follows an algebraic decay law, showing this behavior for a skew integrable map on a cylinder. For a mixed system, that is a system composed by two or more invariant regions, we proved that the statistics of Poincaré recurrences of points at the boundaries is a linear combination of the spectra characteristic of the various components. We think that these results could allow to understand the behavior of area-preserving maps in the mixed regions where integrable structures and chaotic components coexist. In this respect, the intense numerical studies performed by several authors suggest that in the thin stochastic layer surrounding a chain of islands the decay of Poincaré recurrences could follow a power law due to the sticking phenomenon, which is believed to be responsible for the anomalous diffusion modeled by Levy like processes. Furthermore, such a mixture of exponential and power law decays has been observed in a model of stationary flow with hexagonal symmetry, when the transport is anomalous. Some preliminary investigations show that, at least for the skew and for the mixing maps, the results obtained about the first return times spectra also hold for the successive Poincaré recurrences.

1. Introduction
During the last years the study of Poincaré recurrences (also known as return times) has received a growing attention. So far, rigorous results have been mainly obtained for systems characterized by a strong mixing dynamics and for zero-entropy systems like irrational rotations, where the global properties of Poincaré recurrences are fairly well understood, but besides these two different regimes not very much is known. However we have to consider that even apparently simple systems present unexpected difficulties to analytical treatments, and often the only means to gain some insight is represented by numerical investigations. This is the case, for instance,
of low-dimensional Hamiltonian systems, whose physical importance motivates the interest of obtaining precise results.

Our intention was to make a step in this direction. We decided at first to analyze a rather simple system, given by an integrable skew map defined over a two-dimensional cylinder, which allowed us to prove, for particular subsets of the phase space, that the recurrence spectrum decays with a power law. Moreover, we obtained high accuracy numerical results strongly suggesting that the same holds for a generic domain. Note how this behavior is very different from the one shown by systems having a strong mixing dynamics, where the statistics of return times generally follows an exponential decay. Despite the simplicity of our model, the results we obtained seem to be valid even for the standard map in the quasi-integrable regime.

Furthermore, we proved that for systems composed by two or more invariant regions the statistics of return times computed for points belonging to the boundaries is a linear superposition of the spectra of the adjacent regions, with weights given by the relative sizes of the various components. Again, this seems to describe in a very good way what happens for the standard map in the regions of the phase space where regular and chaotic motions coexist. Poincaré recurrences appear therefore to be able to capture some peculiar feature related to the dynamics of the underlying system.

2. Poincaré recurrences

We wish to briefly recall some of the basic definitions about the statistics of return times. Let us consider a transformation $T$ on a measurable space $\Omega$ which is equipped with a probability measure $\mu$ invariant with respect to the dynamics $T$. Taking a measurable set $A \subseteq \Omega$ and a point $x \in A$, the first return time of $x$ into $A$ is defined as

$$\tau_A(x) = \min \left\{ \left\{ k \in \mathbb{N} : T^k(x) \in A \right\} \cup \{+\infty\} \right\}$$

(1)

and the mean return time is given by

$$\langle \tau_A \rangle = \int_A \tau_A(x) \, d\mu_A,$$

(2)

where $\mu_A$ denotes the conditional measure with respect to $A$: $\mu_A(B) = \mu(B \cap A)/\mu(A)$, for any measurable $B \subseteq \Omega$. We can then introduce the statistics of the first return times

$$F_A(t) = \mu_A \left( \left\{ x \in A : \tau_A(x)/\langle \tau_A \rangle > t \right\} \right).$$

(3)

One of the main questions is whether a limit statistics $F(t) = \lim_{\mu(A) \to 0} F_A(t)$ exists when the set $A$ shrinks around a point $x \in \Omega$. For a wide class of mixing systems, even not uniformly hyperbolic (see [1]–[11]), it has been proved that the limit spectrum decays exponentially, that is $F(t) = e^{-t}$, if $A$ is taken either as a ball or as a cylinder (which are the most natural choices) shrinking around $\mu$-almost every point. On the other hand, in the case of one-dimensional irrational rotations there are at most three possible first returns for each subset of the phase space according to Slater’s theorem [12, 13] and this prevents the existence of a limit statistics, unless the shrinking subsets are chosen in a very particular way [14]. Furthermore, a recent paper [15] shows that for all aperiodic ergodic systems any kind of recurrence spectrum can be obtained, provided that the measure of $A$ decrease suitably around each point, but in general $A$ will not be a ball or a cylinder.
3. Skew map

In our study of the recurrence properties of regular systems we considered the following integrable skew map describing a shear flow and defined over the cylinder $C = T \times [0, 1]$

$$R : \begin{cases} x' = x + y \mod 1 \\ y' = y \end{cases}$$

which is area-preserving, with respect to the usual Lebesgue measure $\mu_L$, and has zero-entropy. The dynamics described by $R$ is rather simple: each point $(x, y) \in C$ is transformed according to a one-dimensional rotation whose rotation number is $y$. Thus the cylinder $C$ appears to be foliated in invariant curves, where the rotation velocities change along the $y$-axis.

Despite its simple behavior, this map presents some interesting features (see [16] for more details). First, it enjoys a sort of “local mixing” property: we proved for a domain $A = [x, x+\epsilon] \times [0, \epsilon]$, $0 < \epsilon < 1$, that the autocorrelation decay goes like $O\left(\frac{1}{n}\right)$, being $n$ the number of iterations

$$|\mu_{\epsilon}(A \cap R^n(A)) - \mu_L^2(A)| = O\left(\frac{1}{n}\right),$$

where $\mu_{\epsilon}$ is the conditional measure defined as $\mu_{\epsilon}(A) = \mu_L(A)/\epsilon$. However, it seems sensible to expect that the same holds for a generic domain. Obviously these results differ form the usual mixing condition, since they have a local character. This local mixing property, caused by filamentation, seems to be responsible for the existence of a continuous limit recurrence statistics, despite the fact that for each irrational $y$-coordinate the corresponding one-dimensional rotation do not admit, in general, a limit spectrum.

Furthermore, even if it is not possible to apply Kac’s theorem (stating that the mean return time into a given domain is equal to the ratio between the measure of the phase space and that of the considered domain) to obtain $\langle \tau_A \rangle$ since $R$ is not ergodic, we showed with a direct computation that, for $\mu_L(A) > 0$, it holds

$$\langle \tau_A \rangle = \mu_L \left( \bigcup_{n=0}^{\infty} R^n(A) \right) \frac{1}{\mu_L(A)}$$

which is similar to Kac’s formula. If, for example, we choose $A = [x, x+\epsilon] \times [y, y+\epsilon]$, the mean return time into $A$ is $\langle \tau_A \rangle = 1/\epsilon$.

By considering a square domain $A = [x, x+\epsilon] \times [y, y+\epsilon]$ we were able to rigorously prove [16], through a simple geometric construction, that the statistics of Poincaré recurrences decays with a power law. In the limit for $\epsilon \to 0$ it becomes

$$F(t) = \begin{cases} 1 & \text{if } t = 0 \\ 1/2 & \text{if } 0 < t < 1 \\ 1/2 \ t^{-2} & \text{if } t \geq 1 \end{cases}$$

In the more general case, when $A = [x, x+\epsilon] \times [y, y+\epsilon]$ with $y \neq 0$, the geometric method is very involved, because the lower side of $A$ is no longer invariant. Therefore in order to compute the statistics of the first return times for such domains we developed a numerical algorithm based on the geometric construction. A least square interpolation of the recurrence spectra numerically obtained showed that, for $t$ sufficiently large, $F(t) \sim t^{-\beta}$ with $|\beta - 2| < 5 \cdot 10^{-4}$.

Lower and upper bounds for the limit statistics in terms of a power law were produced for a model of the hyperbolic part of the phase space of an Hamiltonian system near a hierarchical islands structure [17]. This example worked out a self-similar structure of the phase space, in the same spirit as the model proposed in [18]–[20] for the dynamics of sticky sets in Hamiltonian systems.
4. Mixed dynamical systems

A question could arise at this point: what is the behavior of Poincaré recurrences for domains that intersect two (or more) regions which are invariant with respect to the dynamics? In this regard, let $T$ be a transformation acting on the measurable space $\Omega$ and $\mu$ be a $T$-invariant measure. Moreover let us suppose that the dynamical system $(\Omega, T, \mu)$ splits into two invariant components $(\Omega_1, T_1, \mu)$ and $(\Omega_2, T_2, \mu)$, where

$$\Omega = \Omega_1 \cup \Omega_2, \quad \mu(\Omega_1 \cap \Omega_2) = 0$$

and $T_1$, $T_2$, defined on $\Omega_1$, $\Omega_2$ respectively, are such that

$$T_1 = T|_{\Omega_1\setminus(\Omega_1 \cap \Omega_2)}, \quad T_2 = T|_{\Omega_2\setminus(\Omega_1 \cap \Omega_2)}.$$  

We wish to consider the statistics of return times for a neighborhood $A$ of a point $\bar{x}$ that belongs to the common boundary of the two invariant regions. To this end, we denote with $A_1$ and $A_2$ the two different components of $A$, namely $A = A_1 \cup A_2$ with $A_1 = A \cap \Omega_1$ and $A_2 = A \cap \Omega_2$. It is easy to show that the mean return time in $A$ is related to the ones of $A_1$ and $A_2$ in the following way

$$\langle \tau_A \rangle = w_1(A) \langle \tau_{A_1} \rangle + w_2(A) \langle \tau_{A_2} \rangle,$$

with the relative weights defined as

$$w_1(A) = \frac{\mu(A_1)}{\mu(A)}, \quad w_2(A) = \frac{\mu(A_2)}{\mu(A)}.$$  

Let now $F_1$ and $F_2$ be the return times statistics in $A_1$ and $A_2$ respectively; it is straightforward to prove that

$$F_A(t) = w_1(A) F_1(w'_1(A) t) + w_2(A) F_2(w'_2(A) t)$$

where

$$w'_1(A) = \frac{\langle \tau_A \rangle}{\langle \tau_{A_1} \rangle}, \quad w'_2(A) = \frac{\langle \tau_A \rangle}{\langle \tau_{A_2} \rangle}.$$  

To compute the limit statistics, we choose the sequence of the shrinking neighborhoods of $\bar{x}$ such that the limit of the relative weights exists and is different from zero when $\mu(A) \to 0$, indicating this limit value as $w_1$ and $w_2$ respectively. Furthermore, let us assume the existence of these quantities

$$w'_1 = \lim_{\mu(A) \to 0} w'_1(A), \quad w'_2 = \lim_{\mu(A) \to 0} w'_2(A)$$

and suppose that the limit statistics in $A_1$ and $A_2$

$$F_1(t) = \lim_{\mu(A) \to 0} F_{A_1}(t), \quad F_2(t) = \lim_{\mu(A) \to 0} F_{A_2}(t)$$

are well defined. We proved in [16] that under the above assumptions the limit spectrum of Poincaré recurrences exists and is given by

$$F(t) = w_1 F_1(w'_1 t) + w_2 F_2(w'_2 t)$$

in the points of continuity of both $F_1$ and $F_2$. Note that $F(t)$ appears to be a linear superposition of the spectra characteristic of the two invariant regions, weighted by the relative measure of the various components of $A$ while it shrinks around the point $\bar{x}$ of the boundary.

We performed a first check of Eq. (16) by computing the return times statistics for a system composed of two invariant regions characterized by a strong mixing dynamics: the agreement between the theoretical recurrence spectra and the ones obtained numerically is really good.
5. Coupling of regular and mixing regions

It is interesting to analyze the statistics of return times for a system in which one of the invariant regions has a dynamics represented by the skew integrable map $R$ of Sec. 3. In this regard, let $\Omega_1 = T \times [0,1]$ and $\Omega_2 = T \times [-1,0]$ be the two invariant subspaces; $T_1$ be given by $R$, while we choose $T_2$ as a strong mixing transformation such that the limit recurrence spectrum $F_2(t)$ decays like $e^{-t}$ (for $\mu$-almost every point in $\Omega_2$).

Taking a set $A$ that intersects the boundary $y = 0$ we could use Eq. (12) to obtain the statistics $F_A(t)$; but we have an analytical formula describing the statistics $F_{A_1}(t)$ of the skew map only in a particular case, and we do not know its expression for a generic domain. However, assuming that $F_{A_1}(t) \sim t^{-2}$, at least for large $t$ (as strongly suggested by the numerical computations of the spectra) we expect that, when $\mu(A) \ll 1$, $F_A(t)$ will be a linear superposition of a power and an exponential law, weighted by suitable coefficients. This behavior is confirmed by several numerical analyses, as shown in Figure 1. In particular, note how the polynomial tail is clearly visible for sufficiently large values of the normalized return time $t$.

Up to now we have dealt with domains of finite size; we would like to investigate the limit spectrum $F(t)$ when $\mu(A) \to 0$. However it is not possible to directly use formula (16), because in this case one of the assumptions that assure the existence of $F(t)$ fails. In fact for the skew map $w'_1(A)$ has not a finite limit when $\mu(A) \to 0$, as can be easily verified considering Eq. (6) and Eq. (10). So we have to explicitly compute the limit spectrum, obtaining

$$F(t) = w_1 \xi(t) + (1 - w_1) e^{-(1-w_1)t}$$

where $\xi(0) = 1$ and $\xi(t) = 0$ for $t > 0$. Note that despite the limit statistics follows an exponential decay, as in the pure mixing case, nevertheless the presence of the integrable component of $A$ is revealed by the presence of the weight $w_1$. The statistics of Poincaré recurrences is therefore able to capture both the different qualitative properties of the regular and chaotic regions, and to provide some information about the relative measures of these components.

6. Statistics of return times for the standard map

Although the two-dimensional model just considered is rather simple, the statistics of return times exhibits some features that are found in dynamical systems having an higher interest from a physical point of view. In this respect it is noteworthy to investigate the behavior of Poincaré
Figure 2. Statistics of Poincaré recurrences computed for the standard map with coupling parameter $\epsilon = 3$, using a square domain $A$ wholly contained in the regular region. The dashed line represents the function $1/2 t^{-2}$.

Figure 3. Spectra of return times computed for the standard map with coupling parameter $\epsilon = 3$. The domain $A$ partially lies in the chaotic region, and its mixing component increases from line 1 to 4. The dashed line represents the exponential function $e^{-t}$.

recurrences for the so-called “standard map”

$$
\begin{align*}
    y' &= y - \frac{\epsilon}{2\pi} \sin(2\pi x) \\
    x' &= x + y' \mod 1.
\end{align*}
$$

We considered values of the coupling parameter $\epsilon$ such that the stochastic layer, separating the regular orbits from the surrounding chaotic sea, were sufficiently thin compared to the size of the sets used in the numerical computations of the spectra. At first, we chose domains wholly contained in the regular region: as Figure 2 shows, in this case the recurrence statistics appears asymptotically to follow the power law $1/2 t^{-2}$. Moving the domain in such a way that part of it lies on the chaotic region, the spectrum presents an exponential decay whose slope decreases regularly (Figure 3). Thus the statistics of return times seems to agree with the behavior predicted by our simple model (see Eq. 17), despite the boundary between the regular and the chaotic region is not completely sharp.

This is consistent with several numerical studies suggesting that in the thin stochastic layer surrounding a chain of islands the decay of Poincaré recurrences could follow a power law due to the sticking phenomenon [21, 22], which is believed to be responsible for the anomalous diffusion modeled by Levy like processes. Such a mixture of exponential and power law decays has been
also observed in a model of stationary flow with hexagonal symmetry, when the transport is anomalous [23]. Furthermore, there is some evidence [24] that even for domains of finite size belonging to the chaotic sea and far away from the integrable islands the statistics of first return times exhibits a polynomial tail.

7. Conclusions
From the study of the Poincaré recurrences for an integrable skew map on a cylinder, which is area-preserving and models a shear flow, we obtained strong numerical evidences suggesting that the spectra of return times asymptotically follow a power law like $t^{-2}$, rigorously proving this result in a particular case.

The analysis of the statistics of return times for mixed dynamical systems composed of invariant regions showed that for points belonging to the boundaries the limit spectrum is a linear superposition of the spectra characteristic of the various components, with weights given by the relative size of the intersection of the shrinking neighborhoods with each invariant region. Poincaré recurrences are therefore able to capture some of the fundamental properties of the different components.

The statistics of return times was also investigated for a mixed system constructed by coupling the integrable skew map with a strong mixing transformation. Although this represents a rather simple model, nevertheless the results obtained appear to describe well what happens for systems of higher physical interest, such as the standard map, when we consider domains where regular and chaotic motions coexist. Moreover, this model seems to provide a possible explanation for the existence of the power law tails observed in the spectra of domains lying on the chaotic sea far away from the regular regions.

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