GENERAL STABILITY OF ABSTRACT THERMOELASTIC SYSTEM WITH INFINITE MEMORY AND DELAY

JIANGHAO HAO AND JUNNA ZHANG

School of Mathematical Sciences
Shanxi University, Taiyuan, Shanxi, 030006, China

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Abstract. In this paper we study an abstract thermoelastic system in Hilbert space with infinite memory and time delay. Under some suitable conditions, we prove the well-posedness by invoking semigroup theory. Since the damping may stabilize the system while the delay may destabilize it, we discuss the interaction between the damping and the delay term, and obtain that the system is uniformly stable when the effect of damping is greater than that of time delay. By establishing suitable Lyapunov functionals which are equivalent to the energy of system we also establish the general energy decay results for abstract thermoelastic system.

1. Introduction. In this paper, we investigate the following abstract thermoelastic system with infinite memory and time delay,

\[
\begin{align*}
  u_{tt} + Au + Bu_t - \int_0^\infty g(s) Au(t-s)ds - A^\alpha \theta + \mu u_t(t-\tau) &= 0, & t > 0, \\
  \theta_t + kA^\beta \theta + A^\alpha u_t &= 0, & t > 0, \\
  u_t(t-\tau) &= f_0(t-\tau), & t \in (0,\tau), \\
  u(-t) &= u_0(t), & t \geq 0, \\
  u_t(0) &= u_1, \quad \theta(0) = \theta_0,
\end{align*}
\]

(1.1)

in which \( u \) is the displacement vector, \( \theta \) is the temperature difference and \( \alpha \in [0,1), \beta \in (0,1), k \) and \( \mu \) are positive constants. \( H \) is a real Hilbert space equipped with the inner product \( \langle \cdot, \cdot \rangle \) and the related norm \( \| \cdot \| \). The operators \( A : D(A) \to H \) and \( B : D(B) \to H \) are self-adjoint linear positive definite operators with domain \( D(A) \subset D(B) \subset H \) such that the embedding is dense and compact. Moreover, \( \tau > 0 \) represents the time delay and \( u_0, u_1, \theta_0 \) and \( f_0 \) are given initial data belonging to a suitable space and the convolution kernel \( g \) is a given function which represents the term of dissipation.

The thermoelastic systems are known as the equations of thermoelastic mechanics, they are mathematical models established by the temperature distribution and

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Corresponding author: Jianghao Hao.
the deformation of thermoelastic bodies, such models had been investigated in many articles, we can see [1, 5, 11, 17, 22] and their references.

It is well known that the abstract evolution equation has been widely and actively analyzed in the last few decades, so a large number of relevant results of these problems had been obtained, such as the well-posedness and stability results, see for instance [7, 12, 13, 23] and their references. Let us recall some related problems.

Alabau-Boussouira et al. [2] studied the dissipative equation with finite memory

$$u_{tt} + Au - \int_0^t g(t-s)Au(s)ds = 0,$$  \hfill (1.2)

in which $g'(s) \leq -\chi(g(s))$, $s \geq 0$, $\chi$ is a nonnegative measurable function on $[0,k_0]$ for some constant $k_0 > 0$, and strictly increasing of class $C^1$ function on $[0,k_1]$ for some constant $k_1 \in (0,k_0]$, such that $\chi(0) = \chi'(0) = 0$, there exists $\chi_0 > 0$, such that $\chi \geq \chi_0$ on $[k_1,k_0]$ and $\int_0^{k_0} \frac{dx}{\chi(x)} = \infty$, $\int_0^{k_0} \frac{x}{\chi(x)} dx < 1$, in light of these conditions the authors showed the energy of solution decay at least as fast as the kernel $g$ at infinity. Later, Lasiecka et al. [15] studied (1.2) with the memory kernel satisfying the inequality $g'(s) + G(g(s)) \leq 0$, where $G(s)$ is a strictly increasing convex function such that $G(0) = 0$, and the authors developed a method different from [2] to proved that the decay rates of energy given in terms of function $G(s)$. When the finite memory term in (1.2) was replaced by infinite memory $\int_0^\infty g(s)Au(t-s)ds$, Guesmia et al. [10] established a general decay result depending only on the behaviour of the relaxation function $g$. When $\int_0^\infty g(s)Au(t-s)ds$ in [10] was replaced by $\int_0^\infty g(s)Bu(t-s)ds$, Guesmia [8] obtained that the decay rates of energy corresponding to function $G(s)$ in two cases respectively, in which $G(s)$ satisfies suitable conditions. Recently, Youkno [29] improved the results of [8] with the condition $g'(s) \leq -\xi(s)g^p(s)$, for $1 \leq p < \frac{3}{2}$ and obtained a better rate of decay in the polynomial case.

Muñoz Rivera et al. [21] considered the following weakly dissipative second-order system with infinite memory

$$u_{tt} + Au - \int_0^\infty g(s)A^\alpha u(t-s)ds = 0,$$  \hfill (1.3)

the main focus was on the case when $0 \leq \alpha < 1$, the system is polynomially stable and the optimal rate of decay is $1/t^{\frac{\alpha}{1-\alpha}}$. When adding the term $\beta u(t)$ $(\beta \geq 0)$ in (1.3), Muñoz Rivera et al. [20] showed that if $0 < \alpha < 1$ and $\beta \geq 0$, the energy of system decays polynomially even if the kernel $g$ decays exponentially.

When the damping mechanical is given not only by the memory term, Muñoz Rivera et al. [19] discussed the linear weak dissipative equation with infinite memory

$$Du_{tt} + Au - \int_0^\infty g(s)Bu(t-s)ds + Cu_t = 0,$$  \hfill (1.4)

in which $D$ is self-adjoint operator with inverse $D^{-1}$, and the operators $\tilde{A} = D^{-1} A$, $\tilde{B} = D^{-1} B$, $\tilde{C} = D^{-1} C$ satisfy $\tilde{B} = f(\tilde{A})$, $\tilde{C} = h(\tilde{A})$ and $f(s) = o(s^\beta)$, $h(s) = o(s^\gamma)$, and showed that if $\beta \leq 1$ and $0 \leq \gamma \leq 1$, the system decays exponentially, but if $\beta < 1$ and $\gamma < 0$, the system is polynomially stable.

As we all know, the effects of time delay had been analyzed in numerous articles, such as [3, 16, 24, 27] and their references, from those we may figure out that time delay is the source of instability whenever it in the interior of the system or on the boundary feedback and an arbitrarily small delay may destroy the stability of
system that is asymptotically stable in the absence of delay term. Therefore, how to add new control conditions to reduce the influences of time delay and re-stabilize a system with time delay has aroused the interest of many authors.

Guesmia [9] investigated the abstract second-order intergo-differential equation with delay

\[ u_{tt} + Au - \int_0^\infty g(s)Au(t-s)ds + \mu u_t(t-\tau) = 0, \quad (1.5) \]

and showed the well-posedness of system by semigroup theory when \( g \) is integrable and \( g'(s) + \delta g(s) \leq 0 \) for some constant \( \delta > 0 \), it was then proved the solution of system tends to zero exponentially provided that \( |\mu| \) is small enough. Later, Pignotti [26] considered the second-order evolution equation with memory and switching time-delay

\[ u_{tt} + Au - \int_0^\infty g(s)Au(t-s)ds + b(t)u_t(t-\tau) = 0, \quad (1.6) \]

and showed that the asymptotic or exponential stability are preserved if feedback coefficient and memory kernel have suitable conditions, in particular, asymptotic stability is ensured if \( b(\cdot) \in L^1(0, +\infty) \) and the time intervals where the delay feedback is off are sufficient large.

Meanwhile, when the abstract operator is given specifically, the problem with delay had been extensively studied. Kirance et al. [14] investigated the viscoelastic wave equation with delay term

\[ u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1u_t(x,t) + \mu_2u_t(x,t-\tau) = 0, \quad (1.7) \]

and used Fadeo-Galerkin approximations together with energy estimates to obtain the global existence of solution and the exponential stable of system when \( \mu_2 \leq \mu_1 \). Later, when the operator \( -\Delta \) was replaced by \( \Delta^2 \), the problem (1.7) had been investigated by Yang [28], the exponential decay result of energy for the concerned system had been obtained when \( 0 < |\mu_2| < \mu_1 \).

However, as far as we know, there are few works about the stability for the time delay effects on an abstract thermoelastic system with infinite memory. Based on this reason, we investigate the stability of system (1.1) by using multiplier method and constructing suitable Lyapunov functionals. The paper is organized as follows.

In section 2, we give some preliminaries needed in this paper. Next, we also establish the well-posedness of system (1.1) by using semigroup theory in section 3. Section 4 is devoted to the general stability of system (1.1) by constructing suitable Lyapunov functions. Finally, we give a application to the abstract system (1.1) in section 5.

2. Preliminaries.

The aim of this section is to introduce some materials which are needed in our paper and set our problem in an appropriate function space. To proceed, we assume the operators \( A \) and \( B \), kernel function \( g \), time delay \( \tau \) and positive constant \( \mu \) satisfy the following hypotheses.

2.1. Assumptions

(H1) There exist positive constants \( c_i \) (i = 1, 2, 3) such that

\[ \|\omega\|^2 \leq c_1\|B^2\omega\|^2 \leq c_2\|A_1^2\omega\|^2, \quad c_3\|\omega\|^2 \leq \|A_2^2\omega\|^2, \quad \omega \in D(A_3^\frac{1}{2}). \]
By simple calculations, we get

$$\|A^\frac{1}{2}\omega\|^2 \leq c_4\|A^\frac{1}{2}\omega\|^2 \leq c_5\|A^\frac{1}{2}\omega\|^2, \quad \omega \in D(A^\frac{1}{2}).$$

(H2) The constants $\alpha$ and $\beta$ satisfy $\alpha \leq \beta$, and there exist positive constants $c_4$, $c_5$ satisfying

$$\|A^\frac{1}{2}\omega\|^2 \leq c_4\|A^\frac{1}{2}\omega\|^2 \leq c_5\|A^\frac{1}{2}\omega\|^2, \quad \omega \in D(A^\frac{1}{2}).$$

(H3) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing $C^1$ function, and there exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ and $1 \leq p < \frac{3}{2}$ such that

$$g'(s) \leq -\xi(s)g^p(s), \quad s \in \mathbb{R}^+,$$

and

$$g_0 = \int_0^\infty g(s)ds \in (0, 1).$$

(H4) The positive constant $\mu$ satisfies

$$\mu < \frac{1}{c_1}.$$

Remark 2.1 Assume $A = B = -\Delta$ with Dirichlet boundary conditions, $H = L^2(\Omega)$ endowed with its natural inner product, then $D(A^\frac{1}{2}) = H^1_0(\Omega)$, there exist a embedding constant $c_1 > 0$ satisfy $\|\omega\|^2 \leq c_1\|B^\frac{1}{2}\omega\|^2$, we may choose suitable positive constant $\mu$ satisfy (H4).

In order to solve our problem effectively, firstly, we introduce a new variable $\eta$, which is the relative history of $u$ and it was first introduced in [6],

$$\eta^t(s) := u(t) - u(t - s), \quad t, s \in \mathbb{R}^+,$$

$$\eta_0(s) := \eta^0(s) = u_0(0) - u_0(s), \quad s \in \mathbb{R}^+.$$  \hfill (2.1)

By simple calculations, we can get

$$\frac{\partial \eta^t}{\partial t} + \frac{\partial \eta^t}{\partial s} = u_t,$$ \hfill (2.2)

and

$$\eta^t(0) = 0.$$

To deal with the new variable $\eta^t$, we introduce a weighted $L^2$-space

$$L^2_\omega(\mathbb{R}^+; D(A^\frac{1}{2})) = \left\{ \omega : \mathbb{R}^+ \to D(A^\frac{1}{2}) | \int_0^\infty g(s)\|A^\frac{1}{2}\omega(s)\|_2^2 ds < +\infty \right\},$$

which is a Hilbert space endowed with the inner product

$$\langle \omega_1, \omega_2 \rangle_{L^2_\omega} = \int_0^\infty g(s)\langle A^\frac{1}{2}\omega_1(s), A^\frac{1}{2}\omega_2(s) \rangle ds, \quad \omega_1, \omega_2 \in L^2_\omega.$$

Secondly, in order to deal with the time delay term, we introduce as [2], another variable

$$z(\cdot, \rho, t) = u_t(\cdot, t - \tau \rho), \quad \rho \in (0, 1), \quad t > 0,$$

then we have

$$\tau z_t(\cdot, \rho, t) + z_\rho(\cdot, \rho, t) = 0, \quad \rho \in (0, 1), \quad t > 0,$$ \hfill (2.4)

and

$$u_t(\cdot, t - \tau) = z(\cdot, 1, t), \quad u_t(\cdot, t) = z(\cdot, 0, t).$$

At the same time, we introduce a weighted space $L^2((0, 1); H)$ as in [9], defined by

$$L^2((0, 1); H) = \left\{ \omega : (0, 1) \to H | \int_0^1 \|\omega(\rho)\|^2 d\rho < +\infty \right\},$$
endowed with the inner product
\[ \langle \omega_1, \omega_2 \rangle_{L^2((0,1);H)} = \int_0^1 \langle \omega_1(\rho), \omega_2(\rho) \rangle d\rho. \]

Now, we introduce the state space
\[ \mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L^2_\rho \times L^2 ((0,1); H), \]

it is well know that \( \mathcal{H} \) equipped with the inner product
\[ \langle (u,v,\theta, w, z), (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{z}) \rangle_{\mathcal{H}} = (1 - g_0) \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}} \bar{u} \rangle + \langle v, \bar{v} \rangle + \langle \theta, \bar{\theta} \rangle + \langle w, \bar{w} \rangle_{L^2} + \tau \mu \int_0^1 \langle z, \bar{z} \rangle d\rho, \]

is a Hilbert space.

2.2. Equivalent system

According to the new variables (2.1) and (2.3), system (1.1) is equivalent to the following system,

\[
\begin{aligned}
&u_{tt} + (1 - g_0)Au + Bu_t + \int_0^\infty g(s)A\eta^t(s)ds - A^\alpha \theta + \mu z(\cdot, 1, t) = 0, \\ &\theta_t + kA^\beta \theta + A^\alpha u_t = 0, \\ &\tau z_t(\cdot, \rho, t) + z_\rho(\cdot, \rho, t) = 0, \\ &z(\cdot, \rho, 0) = f_0(\cdot, -\tau \rho), \\ &u(-t) = u_0(t), \\ &u_t(0) = u_1 \theta(0) = \theta_0,
\end{aligned}
\]

therefore, system (2.6) is formally equivalent to the following evolution system in the Hilbert state space \( \mathcal{H} \),

\[
\begin{aligned}
U'(t) &= AU(t), \\ U(0) &= U_0,
\end{aligned}
\]

where \( ' \) denotes the derivative with respect to time \( t \), \( U = (u, v, \theta, w, z)^T, U_0 = (u_0(0), u_1, \theta_0, u_0(0) - u_0(s), f_0(\cdot, -\tau \rho))^T \), the operator \( A \) is defined by

\[
AU = \begin{pmatrix}
 v, \\
 -(1 - g_0)Au - Bv - \int_0^\infty g(s)Aw(s)ds + A^\alpha \theta - \mu z(\cdot, 1), \\
-kA^\beta \theta - A^\alpha v, \\
 v - \frac{\partial w}{\partial s} - \frac{1}{\tau \rho} z_\rho
\end{pmatrix}^T,
\]

in which

\[
U = (u, v, \theta, w, z)^T \in D(A) = \left\{ (u, v, \theta, w, z)^T \in \mathcal{H} \mid (1 - g_0)u + \int_0^\infty g(s)w(s)ds \in D(A), \\
v \in D(A^\frac{1}{2}) \cap D(B) \cap D(A^\alpha), \\
\theta \in D(A^\alpha) \cap D(A^\beta), \\
\frac{\partial w}{\partial s} \in L^2_\rho, w(0) = 0, \frac{\partial z}{\partial \rho} \in L^2 ((0,1); H), z(0) = v \right\}.
\]

Next, we use the semigroup theory to give the well-posedness of system (2.7) which is equivalent to system (1.1).
3. Well-posedness of system. In this section we give the existence and uniqueness of solution \((u, \theta)\) of system (1.1), which is equivalent to the existence and uniqueness of solution \(U\) of system (2.7). For this purpose, we shall evoke the semigroup theory.

**Theorem 3.1.** Suppose that the assumptions \((H1)-(H4)\) hold. Then, the linear operator \(A\) generates a \(C_0\) semigroup of contraction \(T(t)\) on \(\mathcal{H}\). Additionally, for any initial data \(U_0 \in D(A)\), system (2.7) has a unique strong solution \(U(t) = T(t)U_0 \in D(A)\) for all \(t \geq 0\) such that

\[
U \in C^1(\mathbb{R}^+; \mathcal{H}) \cap C(\mathbb{R}^+; D(A)).
\]

In turn, if \(U_0 \in \mathcal{H}\), then system (2.7) has a unique weak solution \(U(t) = T(t)U_0 \in \mathcal{H}\) for all \(t \geq 0\) such that

\[
U \in C(\mathbb{R}^+; \mathcal{H}).
\]

**Proof.** It suffices to prove the dissipativity and maximality of the operator \(A\). First of all, we prove that \(A\) is dissipative. Let \(U = (u, v, \theta, w, z)^T \in D(A)\). Using (2.5) and (2.8), we obtain

\[
\langle AU, U \rangle_{\mathcal{H}} = (1 - g_0)\langle A^\frac{3}{2} v, A^\frac{3}{2} u \rangle - (1 - g_0)\langle Au, v \rangle - \langle Bv, v \rangle
\]

\[
- \langle \int_0^\infty g(s)Aw(s)ds, v \rangle
\]

\[
+ \langle A^\alpha \theta, v \rangle - \mu(\langle z(\cdot, 1), v \rangle - kA^\alpha \theta - \langle A^\alpha v, \theta \rangle)
\]

\[
+ \int_0^\infty g(s)\langle A^\frac{3}{2} (v - \partial w / \partial s), A^\frac{3}{2} w \rangle ds + \tau \mu \int_0^1 \langle -\frac{1}{\tau} z_\rho, z \rangle d\rho.
\]

Since the operators \(A\) and \(B\) are positive defined self-adjoint operators and integrating by parts, we have

\[
\langle AU, U \rangle_{\mathcal{H}} = -\|B^\frac{3}{2} v\|^2 - k\|A^\frac{3}{2} \theta\|^2 + \frac{1}{2} \int_0^\infty g'(s)\|A^\frac{3}{2} w\|^2 ds
\]

\[
- \mu \int_0^1 \langle z_\rho, z \rangle d\rho - \mu(\langle z(\cdot, 1), z(\cdot, 0) \rangle).
\]

(3.1)

Since

\[
\mu \int_0^1 \langle z_\rho, z \rangle d\rho = \frac{\mu}{2} \int_0^1 \frac{d}{d\rho} \|z(\cdot, \rho)\|^2 d\rho = \frac{\mu}{2} \|z(\cdot, 1)\|^2 - \frac{\mu}{2} \|z(\cdot, 0)\|^2,
\]

again applying Young’s inequality, we deduce from (3.2)

\[
\langle AU, U \rangle_{\mathcal{H}} \leq -\|B^\frac{3}{2} v\|^2 - k\|A^\frac{3}{2} \theta\|^2 + \frac{1}{2} \int_0^\infty g'(s)\|A^\frac{3}{2} w\|^2 ds + \mu \|z(\cdot, 0)\|^2.
\]

(3.3)

According to \((H1)\), we have

\[
\mu \|z(\cdot, 0)\|^2 \leq \mu c_1 \|B^\frac{3}{2} v\|^2,
\]

(3.4)

thus from (3.3) we arrive at

\[
\langle AU, U \rangle_{\mathcal{H}} \leq -k\|A^\frac{3}{2} \theta\|^2 + \frac{1}{2} \int_0^\infty g'(s)\|A^\frac{3}{2} w\|^2 + (\mu c_1 - 1)\|B^\frac{3}{2} v\|^2.
\]

(3.5)

Exploring the assumptions \((H3)\) and \((H4)\), we can claim that the operator \(A\) is dissipative.
Secondly, we will show that the operator \((I - A)\) is onto, which is equivalent to prove that for given \((f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}\), we can seek \((u, v, \theta, w, z)^T \in D(\mathcal{A})\) such that \((I - A)(u, v, \theta, w, z)^T = (f_1, f_2, f_3, f_4, f_5)^T\), namely,

\[
\begin{align*}
\begin{cases}
  u - v &= f_1 \in D(A^{\frac 12}), \\
  v + (1 - g_0)Au + Bv + \int_0^\infty g(s)Aw(s)ds - A\alpha \theta + \mu z(\cdot, 1) &= f_2 \in H, \\
  \theta + kA^2 \theta + A^\alpha v &= f_3 \in \mathcal{H}, \\
  w - v + \frac{\partial w}{\partial s} &= f_4 \in L_g^2, \\
  z + \frac{1}{\tau}z_\rho &= f_5 \in L^2((0, 1); H) \tag{3.6}
\end{cases}
\end{align*}
\]

Suppose that \(u\) and \(\theta\) are given with the appropriate regularity. Then, from the first equation in (3.6), we have

\[v = u - f_1.\] (3.7)

Moreover, solving the fourth equation of (3.6) and recalling that \(w(0) = 0\), we get

\[w = u(1 - e^{-s}) - f_1(1 - e^{-s}) + \int_0^s e^{\gamma - \eta}f_4(\cdot; \eta)d\eta.\] (3.8)

According to the last equation in (3.6) and together with the fact \(z(\cdot, 0, t) = v\), we obtain

\[z = (u - f_1)e^{-\rho \tau} + \tau \int_0^\rho e^{\tau(\eta - \rho)}f_5(\cdot; \eta)d\eta.\] (3.9)

The variational formulation corresponding to system (3.6) takes the form

\[\mathcal{B}((u, \theta), (\bar{u}, \bar{\theta})) = P(\bar{u}, \bar{\theta}),\] (3.10)

in which \(\mathcal{B} : D(A^{\frac 12}) \times H \rightarrow \mathbb{R}\) is the bilinear form defined by

\[\begin{align*}
\mathcal{B}((u, \theta), (\bar{u}, \bar{\theta})) &= (1 + \mu e^{-\tau})\langle u, \bar{u} \rangle + \left(1 - \int_0^\infty g(s)e^{-s}ds\right) \langle A^{\frac 12}u, A^{\frac 12}\bar{u} \rangle \\
&\quad + \langle B^\alpha u, B^\alpha \bar{u} \rangle - \langle A^\frac 12 \theta, A^\frac 12 \bar{\theta} \rangle + k\langle A^2 \theta, A^\frac 12 \bar{\theta} \rangle + \langle A^\frac 12 u, A^\frac 12 \bar{\theta} \rangle,
\end{align*}\]

and \(P : D(A^{\frac 12}) \times H \rightarrow \mathbb{R}\) is the linear functional given by

\[\begin{align*}
P(\bar{u}, \bar{\theta}) &= \langle f_1 + f_2, \bar{u} \rangle + \left(g_0 - \int_0^\infty g(s)e^{-s}ds\right) \langle A^\frac 12 f_1, A^\frac 12 \bar{u} \rangle \\
&\quad - \int_0^\infty g(s) \int_0^s e^{\gamma - s}\langle A^\frac 12 f_4(\cdot, \eta), A^\frac 12 \bar{u} \rangle d\eta ds + \langle B^\alpha f_1, B^\alpha \bar{u} \rangle \\
&\quad + \langle f_3, \bar{\theta} \rangle + \langle A^\frac 12 f_1, A^\frac 12 \bar{\theta} \rangle + \mu e^{-\tau}\langle f_1, \bar{\theta} \rangle - \mu \tau \int_0^1 e^{\tau(\eta - 1)}f_5(\cdot; \eta)d\eta, \bar{\theta} \rangle.
\end{align*}\] (3.11)

Thus, we obtain the bilinear form \(\mathcal{B}\) is coercive on \(D(A^{\frac 12}) \times H\). Exploring Lax-Milgram theorem [4], we deduce that (3.11) has a unique solution \((u, \theta) \in D(A^{\frac 12}) \times H\).

Besides, if we choose \(\bar{\theta} = 0\), then equation (3.10) can reduces that \(u\) is the solution of

\[\begin{align*}
&(1 + \mu e^{-\tau})u + \left(1 - \int_0^\infty g(s)e^{-s}ds\right)Au + Bu - A^\alpha \theta \\
&\quad = f_1 + f_2 + Bf_1 + \left(g_0 - \int_0^\infty g(s)e^{-s}ds\right)Af_1 - \int_0^\infty g(s) \int_0^s e^{\gamma - s}d\eta ds \\
&\quad - \mu e^{-\tau}f_1 - \mu \tau \int_0^1 e^{\tau(\eta - 1)}f_5(\cdot; \eta)d\eta,
\end{align*}\]
lemmas which are needed in our proof. In the sequel, \( C \) of system (1.4) denotes a generic positive constant which may vary from line to line.

4. Stability results. The aim of this section is to prove the general decay results of system (1.1) by constructing suitable Lyapunov functionals. Here we give some lemmas which are needed in our proof. In the sequel, \( C \) denotes a generic positive constant which may vary from line to line.

Firstly, we define the energy functional of system (1.1) as follow,

\[
E(t) = \frac{1}{2}(1-g_0)\|A^\frac{1}{2}u\|^2 + \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\int_0^\infty g(s)\|A^\frac{1}{2}\eta^t(s)\|^2ds + \frac{T\mu}{2}\int_0^1 \|z(\cdot,p,t)\|^2dp.
\]

(4.1)

Next we state our main stability results of system (1.1).

Theorem 4.1. Under the conditions of Theorem 3.1 and \( \|A^\frac{1}{2}u_0\| \) is bounded uniformly on \( \mathbb{R}^+ \), the energy of system (1.1) satisfies the following general decay rates,

\[
E(t) \leq \zeta \left( 1 + \int_0^t g_1^{1-K_3}(s)ds \right) e^{-K_3\int_0^t \xi(s)ds} + \zeta \int_t^\infty g(s)ds, \quad p = 1,
\]

(4.2)
in which \( \zeta \) and \( K_3 \) are positive constants, and \( K_3 \in (0,1) \), and

\[
E(t) \leq C(1+t)^{-\frac{1}{2p-1}}\xi^{-\frac{2p-1}{2p-2}}(t) \left[ 1 + \int_0^t (1+s)^{\frac{1}{p-1}}\xi^{\frac{2p-1}{p-1}}(s)h^{2p-1}(s)ds \right], \quad 1 < p < \frac{3}{2},
\]

(4.3)

where \( h(t) = \xi(t)\int_t^\infty g(s)\|A^\frac{1}{2}\eta^t(s)\|^2ds \). Moreover, for \( 1 < p < \frac{3}{2} \), if

\[
\int_0^\infty \left[ (1+t)^{-\frac{1}{2p-1}}\xi^{-\frac{2p-1}{2p-2}}(t) \left( 1 + \int_0^t (1+s)^{\frac{1}{p-1}}\xi^{\frac{2p-1}{p-1}}(s)h^{2p-1}(s)ds \right) \right] dt < +\infty,
\]

(4.4)

then we have

\[
E(t) \leq C(1+t)^{-\frac{1}{2p}}\xi^{-\frac{2p}{2p-1}}(t) \left[ 1 + \int_0^t (1+s)^{\frac{1}{p-1}}\xi^{\frac{2p}{p-1}}(s)h^p(s)ds \right], \quad 1 < p < \frac{3}{2},
\]

(4.5)
Remark 4.2. (1) The estimate (4.5) is a special case of estimate (4.3).
(2) When \( p = 1 \), assumption (H3) can be satisfied for many kernel functions, such as the exponential type
\[ g(s) = e^{-ms}, \]
with \( \xi(s) = m \), \( m \) is a positive constant, then (4.2) implies that the energy functional satisfies
\[ E(t) \leq re^{-K_3mt}, \]
in which \( r \) is a positive constant depending on \( m \) and \( K_3 \).
If kernel function \( g \) satisfies polynomial type, such as
\[ g(s) = \frac{m-1}{(1+s)^m}, \]
with \( \xi(s) = \frac{m}{1+s} \), \( m > 1 \) is a positive constant, thus from (4.2) we conclude that there exist some positive constants \( r_3 \) and \( r_4 \) such that
\[ E(t) \leq r_3(1 + t)^{-r_4}. \]
When \( 1 < p < \frac{3}{2} \), let
\[ g(s) = \frac{a}{(1+s)^q}, \]
where \( a \) is chosen so that (H3) remains valid and \( q = \frac{1}{p-1} > 2 \). By simple calculations we have
\[ g'(s) = -bg^p(s), \]
in which \( \xi(s) = b = qa^{-\frac{1}{q}} \). Let us compute
\[ h(t) = \xi(t) \int_t^\infty g(s)ds = \frac{ab}{q-1} (1 + t)^{1-q}, \]
\[ \int_0^t (1 + s)^{1-p} \xi^p(\frac{t}{s})h^p(s)ds = C (1 + t)^{1+p(1-q) + \frac{1}{p-1} - 1}, \]
then, (4.5) yields
\[ E(t) \leq r_5(1 + t)^{-\frac{q^2+q+1}{q}}, \]
in which \( r_5 \) is a positive constant.

Lemma 4.3. Assume that the conditions of Theorem 4.1 hold. Then the energy functional \( E(t) \) is non-increasing and satisfies
\[ E'(t) \leq \frac{1}{2} \int_0^\infty g'(s)\|A^\frac{1}{2}\eta^t(s)\|^2 ds - k\|A^\frac{\alpha}{2}\theta\|^2 + (\mu c_1 - 1)\|B^\frac{1}{2}\theta\|^2 \leq 0. \]

Proof. Multiplying the first equation in (2.6) by \( u_t \) and integrating over \( H \), we obtain
\[ \langle u_{tt}, u_t \rangle + (1 - g_0)\langle A^\frac{1}{2}u, A^\frac{1}{2}u_t \rangle + \langle B^\frac{1}{2}u_t, B^\frac{1}{2}u_t \rangle + \langle \int_0^\infty g(s)A^\frac{1}{2}\eta^t(s)ds, A^\frac{1}{2}u_t \rangle \\
+ \langle \int_0^\infty g(s)A^\frac{1}{2}\eta^t(s)ds, A^\frac{1}{2}u_t \rangle - \langle A^{\frac{\alpha}{2}}\theta, A^{\frac{\alpha}{2}}u_t \rangle + \mu \langle z(\cdot, 1, t), u_t \rangle = 0. \]
Lemma 4.4. Suppose that the conditions of Theorem 4.1 hold, then the functional

$$\phi(t) := -\left( u_t, \int_0^\infty g(s)\eta^t(s)ds \right)$$

satisfies for $t \in \mathbb{R}^+$

$$\phi'(t) \leq g_0 \left( \frac{(1-g_0)^2}{2} + \frac{c_2}{2c_1} + 1 + \frac{c_5}{2} + \frac{\mu c_2}{2} \right) \int_0^\infty g(s)\|A^{\frac{3}{2}}\eta^t(s)\|^2ds - (g_0 - \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|A^{\frac{3}{2}}u_t\|^2 + \frac{1}{2}\|B^{\frac{3}{2}}\eta^t(s)\|^2 - \frac{\mu}{2}\|z(\cdot, 1, t)\|^2.$$
Proof. Using Hölder inequality, we obtain
\[
\| \int_0^\infty g'(s)A^\frac{1}{2} \eta'(s) ds \|^2 \le \int_0^\infty (g'(s)) ds \int_0^\infty -g'(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds,
\]
and
\[
\| \int_0^\infty g(s)A^\frac{1}{2} \eta'(s) ds \|^2 \le \left( \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \| ds \right)^2 \le g_0 \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds.
\]
Then using (2.2), (2.6), (4.14), (4.15) and Young’s inequality, we can get
\[
\phi'(t) = -\langle u_t, \int_0^\infty g(s) \eta'(s) ds \rangle - \langle u_t, \int_0^\infty g(s)(u_t - \frac{\partial \eta'}{\partial s}) ds \rangle \\
= (1 - g_0) \langle A^\frac{1}{2} u, \int_0^\infty g(s)A^\frac{1}{2} \eta'(s) ds \rangle + \langle B^\frac{1}{2} u_t, \int_0^\infty g(s)B^\frac{1}{2} \eta'(s) ds \rangle \\
+ \| \int_0^\infty g(s)A^\frac{1}{2} \eta'(s) ds \|^2 - \langle A^\frac{1}{2} \theta, \int_0^\infty g(s)A^\frac{1}{2} \eta'(s) ds \rangle \\
+ \mu \langle z(\cdot, 1, t), \int_0^\infty g(s) \eta'(s) ds \rangle \\
\le -(g_0 - \frac{1}{2}) \| u_t \|^2 + \frac{1}{2} \| A^\frac{1}{2} u \|^2 + \frac{1}{2} \| B^\frac{1}{2} u_t \|^2 + \frac{1}{2} \| A^\frac{1}{2} \theta \|^2 \\
+ g_0 \left( 1 - g_0 \right)^2 + 1 \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds + \frac{g_0 c_2}{2} \int_0^\infty g(s) \| B^\frac{1}{2} \eta'(s) \|^2 ds \\
+ \frac{g_0}{2} \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds - \frac{g(0)c_2}{2} \int_0^\infty g'(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds \\
+ \mu \frac{\| z(\cdot, 1, t) \|^2}{2} + \frac{\mu g_0 c_2}{2} \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds.
\]
Thanks to assumptions (H1) and (H2), from (4.16) we have
\[
\phi'(t) \le -(g_0 - \frac{1}{2}) \| u_t \|^2 + \frac{1}{2} \| A^\frac{1}{2} u \|^2 + \frac{1}{2} \| B^\frac{1}{2} u_t \|^2 + \frac{1}{2} \| A^\frac{1}{2} \theta \|^2 \\
+ g_0 \left( 1 - g_0 \right)^2 + 1 \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds + \frac{g_0 c_2}{2c_1} \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds \\
+ \frac{g_0 c_5}{2} \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds - \frac{g(0)c_2}{2} \int_0^\infty g'(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds \\
+ \mu \frac{\| z(\cdot, 1, t) \|^2}{2} + \frac{\mu g_0 c_2}{2} \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds \\
\le g_0 \left( 1 - g_0 \right)^2 + \frac{c_2}{2c_1} + 1 + \frac{c_5}{2} + \frac{\mu c_2}{2} \int_0^\infty g(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds \\
-(g_0 - \frac{1}{2}) \| u_t \|^2 + \frac{1}{2} \| A^\frac{1}{2} u \|^2 + \frac{1}{2} \| B^\frac{1}{2} u_t \|^2 + \frac{1}{2} \| A^\frac{1}{2} \theta \|^2 \\
- \frac{c_2 g(0)}{2} \int_0^\infty g'(s) \| A^\frac{1}{2} \eta'(s) \|^2 ds + \frac{\mu}{2} \| z(\cdot, 1, t) \|^2.
\]
This completes the proof of Lemma 4.4. \(\square\)

**Lemma 4.5.** Suppose that the conditions of Theorem 4.1 hold, then the functional
\[
\psi(t) := \langle u_t, u \rangle
\]
satisfies for \( t \in \mathbb{R}^+ \)
\[
\psi'(t) \leq -\left(1 - g_0 - \frac{1}{2} - \frac{c_2}{2} - \frac{c_2}{2c_1} - \frac{\mu c_2}{2}\right) \|A^s u\|^2 + \frac{1}{2} \|B^s u_t\|^2 + \frac{1}{2} \|A^s \theta\|^2 + \frac{g_0}{2} \int_0^\infty g(s) \|A^s \eta'(s)\|^2 ds + \|u_t\|^2 + \frac{\mu}{2} \|z(\cdot, 1, t)\|^2.
\]

**Proof.** Using (2.6), (4.15) and Young’s inequality, we get
\[
\psi'(t) = -(1 - g_0) \langle Au, u \rangle - \langle Bu_t, u \rangle - \left(\int_0^\infty g(s) A\eta'(s) ds, u \right) + (A^s \theta, u)
\]
\[
\leq -(1 - g_0) \|A^s u\|^2 + \frac{1}{2} \|B^s u_t\|^2 + \frac{1}{2} \|B^s u_t\|^2 + \mu \|u_t\|^2
\]
\[
+ \frac{1}{2} \|A^s \theta\|^2 + \frac{1}{2} \|A^s \eta'(s)\|^2 ds + \frac{g_0}{2} \int_0^\infty g(s) \|A^s \eta'(s)\|^2 ds
\]
\[
\leq -(1 - g_0 - \frac{1}{2} - \frac{c_2}{2} - \frac{c_2}{2c_1} - \frac{\mu c_2}{2}) \|A^s u\|^2 + \frac{1}{2} \|B^s u_t\|^2 + \frac{1}{2} \|A^s \theta\|^2
\]
\[
+ \frac{g_0}{2} \int_0^\infty g(s) \|A^s \eta'(s)\|^2 ds + \frac{\mu}{2} \|z(\cdot, 1, t)\|^2 + \|u_t\|^2.
\]
This completes the proof of Lemma 4.5. \( \square \)

**Lemma 4.6.** Suppose that the conditions of Theorem 4.1 hold, then the functional
\[
I(t) := \tau \int_0^1 e^{-\tau \rho} \|z(\cdot, \rho, t)\|^2 d\rho,
\]
satisfy for \( t \in \mathbb{R}^+ \)
\[
I'(t) \leq -\tau \int_0^1 e^{-\tau \rho} \|z(\cdot, \rho, t)\|^2 d\rho - C \|z(\cdot, 1, t)\|^2 + \|u_t\|^2.
\]

**Proof.** Differentiating (4.19) with respect to \( t \) and using (2.4), we have
\[
\frac{d}{dt} I(t) = 2\tau \int_0^1 e^{-\tau \rho} \langle z, z_t \rangle d\rho = -2 \int_0^1 e^{-\tau \rho} \langle z, z_\rho \rangle d\rho
\]
\[
= -\tau \int_0^1 e^{-\tau \rho} \|z(\cdot, \rho, t)\|^2 d\rho - \int_0^1 \frac{\partial}{\partial \rho} (e^{-\tau \rho} \|z(\cdot, \rho, t)\|^2) d\rho
\]
\[
= -\tau \int_0^1 e^{-\tau \rho} \|z(\cdot, \rho, t)\|^2 d\rho - e^{-\tau} \|z(\cdot, 1, t)\|^2 + \|z(\cdot, 0, t)\|^2
\]
\[
\leq -\tau \int_0^1 e^{-\tau \rho} \|z(\cdot, \rho, t)\|^2 d\rho - C \|z(\cdot, 1, t)\|^2 + \|u_t\|^2.
\]
This completes the proof of Lemma 4.6. \( \square \)

**Lemma 4.7.** Assume that the function \( g \) satisfies \((H3)\), then for \( \sigma < 2 - p \), we have
\[
\int_0^\infty \xi(t) g^{1-\alpha}(t) dt < +\infty.
\]

**Lemma 4.8.** Assume that \( g \) satisfies \((H3)\), then for \( 0 < \sigma < 1 \), we have
\[
\int_0^t g(s) \|A^s \eta'(s)\|^2 ds \leq C \left[ E(0) \int_0^t g^{1-\sigma}(s) ds \right]^{\frac{p-\frac{1}{2}}{p-1+\sigma}} \left[ \int_0^t g^{p}(s) \|A^s \eta'(s)\|^2 ds \right]^{\frac{\sigma}{p-1+\sigma}}.
\]
in particular, if \( \sigma = \frac{1}{2} \), we have

\[
\int_0^t g(s)\|A^1\eta(s)\|^2 ds \leq C \left[ E(0) \int_0^t g^\frac{2}{p+2} ds \right]^{\frac{2}{p+2}} \left[ \int_0^t g^p(s)\|A^\frac{1}{2}\eta^p(s)\|^2 ds \right]^{\frac{2}{p+2}}. \tag{4.21}
\]

**Lemma 4.9.** (Jensen’s inequality [18]). Let \( G: [a, b] \to R \) be a concave function, the functions \( f: \Omega \to [a, b] \) and \( h: \Omega \to R \) are integrable such \( h(x) \geq 0 \), and for any \( x \in \Omega \) and \( \int_\Omega h(x)dx = k > 0 \), then

\[
\frac{1}{k} \int_\Omega G[f(x)]h(x)dx \leq G \left[ \frac{1}{k} \int_\Omega f(x)h(x)dx \right].
\]

**Lemma 4.10.** ([29]) Let \( S \) and \( h \) be two positive functions, and \( \delta, c_6 \) and \( c_7 \) are three positive constants, such that

\[
S'(t) \leq -c_6 \xi^{\delta+1}(t)S^{\delta+1}(t) + c_7 h^{\delta+1}(t), \quad t \geq 0,
\]

then, for any \( t \geq 0 \) and some constant \( C > 0 \), we have

\[
S(t) \leq C(1 + t)^{-\frac{1}{2}} \xi^{-\frac{\delta+1}{\delta+1}} (t) \left[ 1 + \int_0^t (1 + s)^{\frac{1}{2}} \xi^{\frac{\delta+1}{\delta+1}} (s) h^{\delta+1}(s)ds \right].
\]

In order to prove the decay results of system (1.1), we construct the following Lyapunov functional

\[
L(t) := NE(t) + \phi(t) + \varepsilon_1 \psi(t) + \varepsilon_2 I(t), \tag{4.22}
\]

where \( N \), \( \varepsilon_1 \), \( \varepsilon_2 \) are positive constants which will be chosen later. Now, we will show that \( L(t) \) is equivalent to \( E(t) \).

In fact, since (4.12), (4.15), (4.17), (4.19) and (H1), we can get some estimates easily as follows,

\[
|\phi(t)| \leq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \int_0^\infty g(s)\|u(s)\|^2 ds \leq \frac{1}{2} \|u_t\|^2 + \frac{c_2 g_0}{2} \int_0^\infty g(s)\|A^{\frac{1}{2}}\eta^p(s)\|^2 ds, \tag{4.23}
\]

\[
|\psi(t)| \leq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u_t\|^2 \leq \frac{1}{2} \|u_t\|^2 + \frac{c_2}{2} \|A^{\frac{1}{2}}u\|^2, \tag{4.24}
\]

and

\[
|I(t)| \leq C \int_0^1 \|z(\cdot, \rho, t)\|^2 d\rho. \tag{4.25}
\]

Together with (4.23) – (4.25), we get

\[
|L(t)| \leq \frac{1}{2} + \varepsilon_1 \|u_t\|^2 + \frac{c_2 g_0}{2} \int_0^\infty g(s)\|A^{\frac{1}{2}}\eta^p(s)\|^2 ds + \frac{c_2 \varepsilon_1}{2} \|A^{\frac{1}{2}}u\|^2
\]

\[
+ C \varepsilon_2 \int_0^1 \|z(\cdot, \rho, t)\|^2 d\rho + NE(t),
\]

by letting \( m_0 := \max \{ \frac{1+\varepsilon_1}{2}, \frac{c_2 g_0}{2}, \frac{c_2 \varepsilon_1}{2}, C \varepsilon_2 \} \), we deduce that

\[
(N - m_0)E(t) \leq L(t) \leq (N + m_0)E(t). \tag{4.26}
\]
Lemma 4.11. Suppose that the conditions of Theorem 4.1 hold, then $L(t)$ satisfies for $t \in \mathbb{R}^+$

$$L'(t) \leq -2dE(t) + K_1 \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds,$$  

(4.27)
in which $d$, $K_1$ are positive constants.

Proof. Using (4.6), (4.13), (4.18), (4.20) and (4.22), we have

$$L'(t) \leq \frac{N}{2} \int_0^\infty g'(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds - kN\|A^\frac{1}{2}\theta\|^2 + N(\mu c_1 - 1)\|B^\frac{1}{2}u_t\|^2$$

$$+ g_0 \left( \frac{(1-g_0)^2}{2} + \frac{c_2}{2c_1} + 1 + \frac{c_5}{2} + \frac{\mu c_2}{2} \right) \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$- \left( g_0 - \frac{1}{2} \right)\|u_t\|^2 + \frac{1}{2}\|A^\frac{1}{2}u_t\|^2 + \frac{1}{2}\|B^\frac{1}{2}u_t\|^2 + \frac{1}{2}\|A^\frac{1}{2}\theta\|^2$$

$$- \frac{c_2 g(0)}{2} \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds + \mu g_0 \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$+ \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds + \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$- \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds + \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$\leq - \left( N(1 - \mu c_1) - \frac{1}{2} - \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \right) \|B^\frac{1}{2} u_t\|^2 - \left( g_0 - \frac{1}{2} - \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \right)\|u_t\|^2$$

$$- c_3 \left( kN - c_4 \left( \frac{1}{2} + \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \right) \right)\|\theta\|^2 + \left( \frac{N}{2} - \frac{c_2 g(0)}{2} \right) \int_0^\infty g'(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$+ g_0 \left( \frac{(1-g_0)^2}{2} + \frac{c_2}{2c_1} + 1 + \frac{c_5}{2} \right) \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$- \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds - \left( \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \right) \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

$$- \left( \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \right) \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds$$

(4.28)

At this point, we choose suitable $c_1$, $c_2$, $c_5$, $\varepsilon_1$ satisfying

$$c_5 + \frac{2c_2}{c_1} < 1 - g_0, \quad 0 < \varepsilon_1 < g_0 - \frac{1}{2}$$

such that

$$\varepsilon_1(1 - g_0 - \frac{1}{2} - c_5 - \frac{c_2}{2c_1} - \frac{\mu c_2}{2} - \frac{1}{2}) > 0.$$  

For fixed $\varepsilon_1 > 0$, we take $\varepsilon_2 > 0$ so that (4.26) remains valid and further

$$\frac{\mu(1 + \varepsilon_1)}{2C} < \varepsilon_2 < g_0 - \frac{1}{2} - \varepsilon_1.$$  

Next, for fixed $\varepsilon_1$, $\varepsilon_2$, we pick $N > 0$ large enough, such that

$$N(1 - \mu c_1 - \frac{1}{2} - \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} > 0, \quad c_3 \left( kN - c_4 \left( \frac{1}{2} + \frac{\|\|z(\cdot, 1, t)\|^2}{c_1} \right) \right) > 0, \quad \frac{N}{2} - \frac{c_2 g(0)}{2} > 0.$$  

Since $I(t) \geq 0$ and $g(s)$ is a non-increasing function, then from (4.1) and (4.28) we get that there exists positive constant $d$ such that

$$L'(t) \leq -2dE(t) + K_1 \int_0^\infty g(s)\|A^\frac{1}{2}\eta'(s)\|^2 ds.$$
in which
\[ K_1 := g_0 \left( \frac{1 - g_0}{2} + \frac{c_2}{2c_1} + 1 + \frac{c_5}{2} + \frac{\mu c_2}{2} + \frac{\varepsilon_1}{2} \right) + d. \] (4.29)
This completes the proof of Lemma 4.11.

\[ \square \]

Proof of Theorem 4.1. Multiplying (4.27) by \( \xi(t) \), we arrive at
\[ \xi(t)L'(t) \leq -2d\xi(t)E(t) + K_1\xi(t) \int_0^\infty g(s)\|A^1/2 \eta'(s)\|^2 ds \]
\[ \leq -2d\xi(t)E(t) + K_1 \int_0^t \xi(s)g(s)\|A^1/2 \eta'(s)\|^2 ds + K_1\xi(t) \int_t^\infty g(s)\|A^1/2 \eta'(s)\|^2 ds. \] (4.30)

Due to (2.1) and (4.6), we can obtain
\[ \int_t^\infty g(s)\|A^1/2 \eta'(s)\|^2 ds = \int_t^\infty g(s)\|A^1/2 u(t) - A^1/2 u(t - s)\|^2 ds \]
\[ \leq 2 \int_t^\infty g(s)\|A^1/2 u(t)\|^2 ds + 2 \int_t^\infty g(s)\|A^1/2 u(t - s)\|^2 ds \]
\[ \leq 2\|A^1/2 u(t)\|^2 \int_t^\infty g(s) ds + 2 \sup_{\sigma \geq 0} \|A^1/2 u(\sigma)\|^2 \int_t^\infty g(s) ds \]
\[ \leq \left( \frac{2}{1 - g_0} E(0) + 2 \sup_{\sigma \geq 0} \|A^1/2 u(\sigma)\|^2 \right) \int_t^\infty g(s) ds \]
\[ := K_2 \int_t^\infty g(s) ds. \] (4.31)

Next, we distinguish the following two cases to prove Theorem 4.1.

Case of \( p = 1 \). Recalling (H3), and from (4.30) we can get
\[ \int_0^t \xi(s)g(s)\|A^1/2 \eta'(s)\|^2 ds \leq -\int_0^t g'(s)\|A^1/2 \eta'(s)\|^2 ds \]
\[ \leq -\int_0^t g'(s)\|A^1/2 \eta'(s)\|^2 ds \leq -2E'(t). \] (4.32)

Since \( \xi'(t) \leq 0 \), then from (4.30) - (4.32) we can see that
\[ \frac{d}{dt} (\xi(t)L(t)) \leq -2d\xi(t)E(t) - 2K_1 E'(t) + K_1 K_2 \xi(t) \int_t^\infty g(s) ds. \] (4.33)

Now, we define
\[ F(t) := \xi(t)L(t) + 2K_1 E(t), \] (4.34)
and together with (4.26), we can deduce that \( F(t) \) and \( E(t) \) are equivalent, that is
\[ 2K_1 E(t) \leq F(t) \leq \left[ (N + m_0)\xi(0) + 2K_1 \right] E(t). \] (4.35)

Thus, from (4.33)-(4.35) we have
\[ F'(t) \leq -2d\xi(t)E(t) + K_1 K_2 \xi(t) \int_t^\infty g(s) ds \]
\[ \leq -\frac{2d}{(N + m_0)\xi(0) + 2K_1} \xi(t)F(t) + K_1 K_2 \xi(t) \int_t^\infty g(s) ds. \]

Moreover, we have
\[ F'(t) + K_2 \xi(t)F(t) \leq K_4 \xi(t) \int_t^\infty g(s) ds, \] (4.36)
in which \( K_3 := 2d/[(N + m_0)\xi(0) + 2K_1] \in (0, 1) \). In fact from (4.35), we know
\( 2K_1 \leq (N + m_0)\xi(0) + 2K_1 \), and (4.29) implies that \( K_1 \geq d \), due to these inequalities,
we have \( K_3 \in (0, 1) \) and \( K_4 := K_1K_2 \).

Integrating (4.36) over \([0, t]\), we conclude
\[
F(t) \leq e^{-K_3 \int_0^t \xi(s)ds} \left[ F(0) + K_4 \int_0^t e^{K_3 \int_0^s \xi(r)dr} \int_\sigma^\infty g(s)dsd\sigma \right].
\] (4.37)

On the other hand, by integration by part, we get
\[
\int_0^t e^{K_3 \int_0^s \xi(r)dr} \int_\sigma^\infty g(s)dsd\sigma = \int_0^t \frac{1}{K_3} \frac{d}{d\sigma} \left( e^{K_3 \int_0^s \xi(r)dr} \right) \int_\sigma^\infty g(s)dsd\sigma

= \frac{1}{K_3} \left( e^{K_3 \int_0^t \xi(s)ds} \int_\sigma^\infty g(s)ds - \int_0^t g(s)ds + \int_0^t g(\sigma)e^{K_3 \int_0^\sigma \xi(s)ds}d\sigma \right),
\] (4.38)

and (H3) implies that \((g(\sigma)e^{\int_0^{\sigma} \xi(s)ds})' \leq 0\), thus
\[
\int_0^t g(\sigma)e^{K_3 \int_0^\sigma \xi(s)ds}d\sigma = \int_0^t g^{1-K_3}(\sigma) \left( g(\sigma)e^{\int_0^\sigma \xi(s)ds} \right)^K_3 d\sigma

\leq g^{K_3}(0) \int_0^t g^{1-K_3}(\sigma)d\sigma.
\] (4.39)

Inserting (4.38) and (4.39) into (4.37), we can get
\[
F(t) \leq e^{-K_3 \int_0^t \xi(s)ds} F(0) + \frac{K_4}{K_3} \left( \int_t^\infty g(s)ds + g^{K_3}(0) \int_0^t g^{1-K_3}(\sigma)d\sigma e^{-K_3 \int_0^\sigma \xi(s)ds} \right)

\leq \left( F(0) + \frac{K_4}{K_3} g^{K_3}(0) \int_0^t g^{1-K_3}(\sigma)d\sigma \right) e^{-K_3 \int_0^t \xi(s)ds} + \frac{K_4}{K_3} \int_t^\infty g(s)ds.
\] (4.40)

Finally, we have (4.2) with
\[
\xi := \frac{1}{2K_1} \max \left\{ F(0), \frac{K_4}{K_3} g^{K_3}(0), \frac{K_4}{K_3} \right\}.
\] (4.41)

**Case of** \( 1 < p \leq \frac{3}{2} \). Use (4.30) and recall Lemma 4.7, Lemma 4.8, (H3) and (4.6) to get
\[
\xi(t) \int_0^t g(s)\|A^{\frac{2}{3}} \eta(t)(s)\|^2 ds

\leq C \xi^{\frac{2p-2}{2p-1}}(t) \left[ \int_0^t g^{\frac{1}{2}}(s)ds \right]^{\frac{2p-2}{2p-1}} \xi^{\frac{1}{2p-1}}(t) \left[ \int_0^t g^{p}(s)\|A^{\frac{2}{3}} \eta(t)(s)\|^2 ds \right]^{\frac{1}{2p-1}}

\leq C \left[ \int_0^t \xi(s)g^{\frac{1}{2}}(s)ds \right]^{\frac{2p-2}{2p-1}} \left[ \int_0^t \xi(s)g^p(s)\|A^{\frac{2}{3}} \eta(s)\|^2 ds \right]^{\frac{1}{2p-1}}

\leq C \left[ -g^{\prime}(s)\|A^{\frac{2}{3}} \eta(s)\|^2 ds \right]^{\frac{1}{2p-1}} \leq C [-E'(t)]^{\frac{1}{2p-1}},
\] (4.42)

and from (4.31), letting
\[
h(t) := \xi(t) \int_t^\infty g(s)\|A^{\frac{2}{3}} \eta(t)(s)\|^2 ds.
\] (4.43)

We again consider (4.30) and use (4.42) and (4.43) to obtain
\[
\xi(t)\xi'(t) \leq -2d\xi(t)E(t) + C[-E'(t)]^{\frac{1}{2p-1}} + Ch(t).
\] (4.44)
Multiplication of (4.44) by $\xi^\delta(t)E^\delta(t)$, where $\delta = 2p - 2$, gives

$$\xi^{\delta+1}(t)E^\delta(t)L'(t) \leq -2d\xi^{\delta+1}(t)E^{\delta+1}(t) + C\xi^{\delta}(t)E^\delta(t)[E'(t)]^{\frac{1}{2p-2}} + C\xi^\delta(t)E^\delta(t)h(t).$$

Using of Young’s inequality yields for any $\varepsilon > 0$

$$\xi^{\delta+1}(t)E^\delta(t)L'(t) \leq -2d\xi^{\delta+1}(t)E^{\delta+1}(t) + 2\varepsilon\xi^{\delta+1}(t)E^{\delta+1}(t) - C_\varepsilon E'(t) + C_\varepsilon h^{\delta+1}(t),$$

in which $C_\varepsilon$ is a positive constant depending on $\varepsilon$. Next, let

$$M(t) := \xi^{\delta+1}(t)E^\delta(t)L(t) + C_\varepsilon E'(t).$$

It is easy to show that $M(t)$ is equivalent to $E(t)$. Then for $\varepsilon$ small enough, there exists two positive constants $\gamma_1$ and $\gamma_2$, such that

$$M'(t) \leq -\gamma_1 \xi^{\delta+1}(t)M^{\delta+1}(t) + \gamma_2 h^{\delta+1}(t).$$

(4.45)

In view of Lemma 4.10 and taking into account the equivalence of $M(t)$ and $E(t)$, and choosing $\delta = 2p - 2$, we obtain

$$E(t) \leq C(1 + t)^{-\frac{1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(t) \left[ 1 + \int_0^t (1 + s)^{-\frac{1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(s)h^{2p-1}(s)ds \right],$$

which gives (4.3).

Next we establish (4.5). From (4.3) and (4.4), it yields

$$\int_0^\infty E(t)dt < +\infty.$$  

(4.46)

Again considering (4.30), together with (4.43) and (4.46), we have

$$\xi(t)L'(t) \leq -2d\xi(t)E(t) + K_1 \int_0^t \xi(s)g(s)\|A^\frac{1}{2} \eta(s)\|^2 ds + Ch(t)
\leq -2d\xi(t)E(t) + K_1 \int_0^t [\xi^p(s)g^p(s)]^\frac{1}{p} \|A^\frac{1}{2} \eta(s)\|^2 ds + Ch(t),$$

(4.47)

where

$$J(t) = \int_0^t \|A^\frac{1}{2}u(t) - A^\frac{1}{2}u(t-s)\|^2 ds \leq C \int_0^t \left( \|A^\frac{1}{2}u(t)\|^2 + \|A^\frac{1}{2}u(t-s)\|^2 \right) ds
\leq C \int_0^t (E(t) + E(t-s)) ds \leq 2C \int_0^t E(s)ds \leq C \int_0^\infty E(s)ds < \infty.$$  

(4.48)

Applying Lemma 4.9 for the second term on the right hand side of (4.47), with $G(y) = y^\frac{1}{p}$, $f(s) = \xi^p(s)g^p(s)$, $h(s) = \|A^\frac{1}{2} \eta(s)\|^2$, and since $\xi$ is non-increasing, then from (4.6) and (4.48) we deduce

$$\xi(t)L'(t) \leq -2d\xi(t)E(t) + CJ(t) \left( \frac{1}{J(t)} \int_0^t \xi^p(s)g^p(s)\|A^\frac{1}{2} \eta(s)\|^2 ds \right)^\frac{1}{p} + Ch(t)
\leq -2d\xi(t)E(t) + CJ(t)^{1-\frac{1}{p}} \left( \int_0^t \xi^p(s)g^p(s)\|A^\frac{1}{2} \eta(s)\|^2 ds \right)^\frac{1}{p} + Ch(t)
\leq -2d\xi(t)E(t) + CJ(t)^{1-\frac{1}{p}} \xi^{1-\frac{1}{p}}(0) \left( \int_0^t \xi(s)g^p(s)\|A^\frac{1}{2} \eta(s)\|^2 ds \right)^\frac{1}{p} + Ch(t)
\leq -2d\xi(t)E(t) + C \left( \int_0^t -g'(s)\|A^\frac{1}{2} \eta(s)\|^2 ds \right)^\frac{1}{p} + Ch(t)
\leq -2d\xi(t)E(t) + C[-E'(t)]^\frac{1}{p} + Ch(t).$$

(4.49)
Multiplying (4.49) by \( E^\delta(t)^\xi(t) \) for \( \delta = p - 1 \), and repeating the same computations as in above, we arrive at
\[
E(t) \leq C(1 + t)^{-\frac{2p}{p-1}} \xi^{-\frac{2p}{p-1}}(t) \left[ 1 + \int_0^t (1 + s)^{\frac{2p}{p-1}} \xi^{\frac{2p}{p-1}}(s) h^p(s)ds \right].
\]
Thus we obtain the general decay result (4.5) for abstract thermoelastic system (1.1). This completes the proof of the Theorem 4.1. \( \square \)

5. Application. In this section, we present a application for the stability results of our abstract system (1.1). For this, we let \( \Omega \subset \mathbb{R}^n (n \geq 1) \) be a open bounded domain with smooth boundary \( \Gamma \).

Let parameter \( \alpha = \beta = 1 \), and the operators \( A = B = -\Delta \), where \( H = L^2(\Omega) \) with its natural inner product \( (u_1, u_2)_{L^2(\Omega)} = \int_\Omega u_1u_2dx \), \( D(A) = D(B) = H^2(\Omega) \cap H_0^1(\Omega) \).

Then we consider the following system
\[
\begin{align*}
&u_{tt} - \Delta u - \Delta u_t + \int_0^\infty g(s)\Delta u(t-s)ds + \Delta \theta + \mu u_t(t-\tau) = 0, \quad x \in \Omega, t > 0, \\
&\theta_t - k\Delta \theta - \Delta u_t = 0, \quad x \in \Omega, t > 0, \\
&u = \theta = 0, \quad x \in \Gamma, t > 0, \\
&u(t-\tau) = f_0(t-\tau), \quad t \in (0, \tau), \\
&w(-t) = u_0(t), \quad t \geq 0, \\
&u_t(0) = u_1, \quad \theta(0) = \theta_0.
\end{align*}
\]

Let \( g(s) = \frac{q}{(1+s)^p}, q = \frac{1}{p-1} > 2 \), the assumptions (H1)–(H4) hold, then Theorem 4.1 implies that when \( p = 1 \) and \( q \neq \frac{1}{1-K_3} \), there exists a positive constant \( \tilde{c} = \min\{2a, qK_3\} \), such that
\[
E(t) \leq C(1 + t)^{-\tilde{c}}
\]
when \( 1 < p < \frac{3}{2} \),
\[
E(t) \leq C(1 + t)^{-\frac{q^2+1}{4}}.
\]

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E-mail address: hjhao@sxu.edu.cn
E-mail address: 183406153@qq.com