DEPENDENCE OF THE HEAVILY COVERED POINT ON PARAMETERS

ALEXEY BALITSKIY♠ AND ROMAN KARASEV♣

Abstract. We examine Gromov’s method of selecting a point “heavily covered” by simplices formed by a given finite point sets, in order to understand the dependence of the heavily covered point on parameters. We have no continuous dependence, but manage to utilize the “homological continuous dependence” of the heavily covered point. This allows us to infer some corollaries in a usual way. We also give an elementary argument to prove the simplest of these corollaries.

1. Introduction

Boros, Füredi, and Bárány in [5, 3] established a theorem on existence of a point “heavily covered” by simplices: For a prescribed N-point set X in \( \mathbb{R}^d \), it is possible to find a point \( c \in \mathbb{R}^d \) that is covered by an essential fraction \( c_d > 0 \) of all the \( \binom{N}{d+1} \) simplices that can be chosen with vertices in X. In another paper [17] Pach established a similar result in a “colorful” version, for selecting the vertices \( \{ x_0, \ldots, x_d \} \) of the simplices in all possible ways from their respective sets \( X_0, \ldots, X_d \) and finding a point that is covered by a certain fraction of all those simplices.

The next important step was made recently, in [10] Gromov has developed a certain topological approach based on the ideas of “bounded cohomology”, and proved the following generalization of the “heavily covered point” results:

**Theorem 1.1** (Gromov, 2010). For any continuous map \( f \) from the \((N−1)\)-dimensional simplex \( \Delta \) to the Euclidean space \( \mathbb{R}^d \), some point \( c \in \mathbb{R}^d \) is covered by a fraction of at least \( \frac{2d}{(d+1)(d+1)} \) of all images of \( d \)-dimensional faces of \( \Delta \).

**Remark 1.2.** In fact, the fraction was \( \frac{2d}{(d+1)(d+1)} - O(1/N) \), but we ignore the correcting term throughout.

The classical case follows from this theorem when \( f \) is chosen to be the linear map taking the vertices of \( \Delta \) to the corresponding points of \( X \). Gromov’s approach allowed not only to generalize the result, but also to improve the known bounds for the fraction \( c_d \) to \( \frac{2d}{(d+1)(d+1)} \). Soon after further improvements were made using Gromov’s method, see [16, 15], for example.

One thing that could facilitate further applications and generalizations of Theorem 1.1 would be the continuous dependence of the point \( c \) (somehow selected from the heavily covered points) on the map \( f \). This is the question that we consider in this note.

In fact, it is not likely to expect any continuous behavior, but instead we want to show the “homologically continuous” behavior. This behavior is very typical in those cases, when the solution of the problem is guaranteed (co)homologically. Namely, we are going
to present a construction of an odd 0-cycle modulo 2 in $\mathbb{R}^d$, that is an odd number of points in $\mathbb{R}^d$, any of which is heavily covered and such that this 0-cycle modulo 2 depends on $f$ continuously.

The continuous dependence of a 0-cycle modulo 2 is informally understood in the following way (see also [1]): Either the points of the cycle move continuously with the change of parameters, or a pair (or several pairs) of them annihilate at the same position, or some pair gets born from nothing at some position.

A more geometric way is to treat an odd 0-cycle modulo 2 in $\mathbb{R}^d$ continuously parameterized by a manifold $P$ to be a subset $C \subset P \times \mathbb{R}^d$ that is an embedded modulo 2 pseudomanifold of codimension $d$ such that its projection onto $P$ is a proper map of degree 1 modulo 2. This kind of continuous dependence is sufficient to apply common methods of generalizing the result by taking a Cartesian product with some other geometric construction. Here a modulo 2 pseudomanifold of dimension $d$ is a $d$-dimensional simplicial complex such that every its $(d-1)$-dimensional face is contained in precisely two $d$-dimensional faces.

Certain consequences of this “homological continuity” are given in Section 3 and Section 4 contains more elementary observations in the planar case using some simple topology.

Acknowledgments. We thank Benjamin Matschke for useful remarks and stimulating questions, and Sergey Melikhov for pointing out the useful review [9].

2. Proof of homological continuity

Before proving the homologically continuous dependence of $c$ on $f$ we recall Gromov’s proof. Though there are simpler proofs for Theorem 1.1, like in [14], we need the original Gromov’s argument here.

2.1. Space of cocycles. The key idea of Gromov is considering the space of $d$-dimensional cocycles modulo 2 in $\Delta$, denoted by $\text{cl}^d(\Delta; \mathbb{F}_2)$. This space is constructed in [10] combinatorially in the following manner: Let $X$ be any topological space, take its cochain complex (let us work with modulo 2 coefficients and omit $\mathbb{F}_2$ from the notation)

\[
\cdots \longrightarrow C^d(X) \longrightarrow C^{d-1}(X) \longrightarrow \cdots
\]

Then we build a certain simplicial abelian group out of this. Its $k$-simplices are the chain maps (reversing the degree): $\sigma : C_k(\Delta^k) \rightarrow C^{d-k}(X)$. These objects constitute a functor from the simplex category to abelian groups (maps) and therefore define a simplicial abelian group, which is natural to call the space of $d$-cocycles of $X$, $\text{cl}^d(X; \mathbb{F}_2)$. Speaking informally, the points of this space are $d$-cocycles themselves, the edges between the points correspond to the “homologous” relation, and the higher-dimensional faces are attached to them according to the chain complex.

Since $\text{cl}^d(X; \mathbb{F}_2)$ is naturally a simplicial abelian group, it is a Kan/fibrant complex (see the nice introductory text [9] or the textbook [18]) and its homotopy groups can be calculated formally as $\pi_k(\text{cl}^d(X; \mathbb{F}_2)) = H^{d-k}(X; \mathbb{F}_2)$ (meaning that the simplicial maps of a simplex $\Delta^k$ to $\text{cl}^d(X; \mathbb{F}_2)$ up to homotopy coincide with the cohomology group of $X$), this is what Gromov [10] calls the Dold–Thom–Almgren theorem. It might be not very clear why it makes sense to call this object a “space of $d$-cocycles”, but intuitively it is very useful to consider it as such, in particular the Dold–Thom theorem for the other version of the cycle space built from currents holds in the same manner, as was shown by Almgren [2]. Actually, we are also going to consider such “spaces of cycles” built from the chain complex of a simplicial complex $X$.

In fact, the above mentioned properties of $\text{cl}^d(X; \mathbb{F}_2)$ are consequences of the following:
Property 2.1. For a simplicial complex $W$ a simplicial map $W \to cl^d(X;\mathbb{F}_2)$ (in the category of simplicial sets) is the same as a chain map $C_*(W;\mathbb{F}_2) \to C^{d-*}(X;\mathbb{F}_2)$.

For any connected $X$, in the $d$-dimensional cohomology of $cl^d(X;\mathbb{F}_2)$, $H^d(cl^d(X;\mathbb{F}_2);\mathbb{F}_2)$, there exists a certain canonical class $\xi$. Its existence can be guessed from the “Dold–Thom–Almgren theorem”, or it can be built explicitly by counting how many times a vertex (as a 0-cocycle) of $X$ participates in a $d$-cycle of cocycles from $cl^d(X;\mathbb{F}_2)$.

More precisely, using Property 2.1 we view a $d$-cycle of cocycles as a simplicial map $c : P \to cl^d(X;\mathbb{F}_2)$, where $P$ is a modulo 2 pseudomanifold of dimension $d$, or equivalently, a chain map $C_*(P;\mathbb{F}_2) \to C^{d-*}(X;\mathbb{F}_2)$.

Definition 2.2. The value of $\xi$ on the cycle of $P$ is the image of the $d$-dimensional fundamental class $[P]$ in $H^0(X;\mathbb{F}_2) = \mathbb{F}_2$.

2.2. Start of Gromov’s proof: The “inverse map”. Return to our particular space $X = \Delta^N$, or just $\Delta$, and build the space of cocycles $C^d = cl^d(\Delta;\mathbb{F}_2)$ from the chain complex of $\Delta$, viewed as a simplicial complex.

Now, in our problem, we compactify the target space $\mathbb{R}^d$ with the point at infinity (denoted by $\infty$) to obtain the sphere $S^d$. So $f : \Delta \to \mathbb{R}^d$ becomes a map to $S^d$ not touching $\infty$. Any point $y \in S^d$ defines a $d$-cycle on $\Delta$ by counting the multiplicity (modulo 2) on the $f$-image of every $d$-face of $\Delta$ at $y$. In a certain sense this cocycle has continuous dependence on $y$ and therefore arises a continuous map $f^c : S^d \to C^d$, mapping the point $\infty$ to the zero $d$-cocycle.

More formally, using Property 2.1, to describe $f^c$ we have to consider a triangulation $T$ of $S^d$ and build the chain map $C_*(T;\mathbb{F}_2) \to C^{d-*}(\Delta;\mathbb{F}_2)$, defined by counting (modulo 2) intersections of faces of $T$ and faces of $f(\Delta)$ of complementary dimension. The crucial fact, noted in [10] is that the canonical class $\xi$ evaluates on the image $f^c(S^d)$ to 1, thus showing that the map $f^c$ is homologically nontrivial. Indeed, the chain map $C_*(T;\mathbb{F}_2) \to C^{d-*}(\Delta;\mathbb{F}_2)$ sends the fundamental class $[S^d]$ to the cochain in $C^0(\Delta;\mathbb{F}_2)$ having coefficient 1 at every vertex of $\Delta$ (since in the general position every vertex of $\Delta$ is mapped by $f$ into precisely one $d$-face of $T$), and we apply Definition 2.2.

2.3. More details: subspace of thin cocycles. The other step of the proof is to note a different thing, which we are going to explain. We consider the subspace $C^d_0 \subset C^d$, corresponding to those cocycles in $C^d(\Delta;\mathbb{F}_2)$ whose supports invoke less than the fraction $\frac{2d}{(d+1)(d+1)}$ of $d$-faces of $\Delta$. We start with a very informal observation that $C^d_0$ can be contracted to the zero cycle $0 \in C^d$ by a certain cocycle filling process. This observation (in [10]) concludes the proof of Theorem 1.1, because $f^c$ must touch $C^d \setminus C^d_0$ in order to be homologically nontrivial (as the cohomology class $\xi$ shows).

Now we add more details to the above sketch. In the process of contraction, in fact, there was another assumption: that the cycles of lower dimension $d-k$ in $\Delta$ constituting the cycle $f^c(T)$ (the same as simplices of positive dimension $k$ of $f^*(T)$) have negligible complexity, that is consist of a tiny $\varepsilon$-fraction of all $(d-k)$-simplices of $\Delta$ for positive $k$. In [10] this assumption was satisfied by taking sufficiently fine triangulation $T$ of $S^d$, then $f^c(T)$ has this property. So we have to include this assumption directly into the definition of the subspace $C^d_\varepsilon$, to define the subspace $C^d_\varepsilon \subset C^d$. More precisely, for $C^d_\varepsilon$ we require that its faces, considered as chain maps $\sigma : C_*(\Delta^k) \to C^{d-*}(\Delta)$, are such that their $(d-k)$-dimensional parts invoke less that the $\varepsilon$-fraction of all $(d-k)$-faces of $\Delta$ in their support for $k > 0$, and the fraction of used $d$-faces of $\Delta$ is at most the magic constant $\frac{2d}{(d+1)(d+1)}$, possibly up to some correction term of order $1/N$ when $N \to \infty$. In fact, our first naive definition of $C^d_\varepsilon$ expressed this set as a collection of vertices of $C^d$, so
we really have to use $C^d_\varepsilon$ for sufficiently small $\varepsilon$ to make this subspace contractible and the whole argument valid.

So what we need from the subspace $C^d_\varepsilon$ is the following:

**Property 2.3.** Every $d$-dimensional modulo 2 cycle in $C^d_\varepsilon$, expressed as a map of a modulo 2 pseudomanifold $P$ to $C^d_\varepsilon$, can be contracted to a single point inside $C^d$.

This property is proved in [10] by extending the map $P \to C^d_\varepsilon$ to the cone over $P$ using the “linear filling profile” in the complex $C_*(\Delta)$ (remember Property 2.1). This linear filling profile (actually having unit norm) allows one to invert the boundary operator $\partial : C_*(\Delta) \to C_*(\Delta)$ economically. This inversion process is iterated over the skeleton of $P$, and finally assures that a chain arising in $C_0(\Delta)$ is actually a cycle because of having a small norm (the size of its support), thus extending the map over a $(d + 1)$-face of the cone over $P$.

The next step is to note that, when we take a sufficiently fine triangulation $T$ of $S^d$ in our argument, the image $f^c(T)$ gets into $C^d_\varepsilon$, provided its “vertex part” is in $C^d_0$. Property 2.3 means that the canonical class $\xi$ vanishes when restricted to the subspace $C^d_\varepsilon \subset C^d$ and therefore (by the exact sequence of the pair) it can be represented by a cocycle $X \in C^d(C^d; \mathbb{F}_2)$ with support outside $C^d_\varepsilon$.

So we assume that the canonical class $\xi$ is represented by a suitable cocycle $X \notin C^d_\varepsilon$. Then the pullback of this canonical cocycle under the map $f^c : S^d \to C^d$ makes a cocycle $X_f \in C^d(S^d; \mathbb{F}_2)$. In order for the pullback to be defined on the level of cocycles, we assume a sufficiently fine triangulation $T$ of $S^d$ (to fit into $C^d_\varepsilon$ from the previous paragraph) and then $X_f$ becomes a $d$-cocycle assigning 0 or 1 to every $d$-face of $T$, and having the following property: The support of $X_f$ in $S^d$ consists of some $d$-faces of $T$, each of which having a “heavily covered” point as its vertex. Here a “heavily covered” point is a point covered by at least $\frac{2d}{(d+1)(d+2)}$ of all images of $d$-faces of $\Delta$.

### 2.4. Homologically continuous dependence

Now we observe that $f^c(S^d)$ depends continuously on $f$ in some intuitive sense, and therefore the cocycle $X_f$ “cut by $X$” on $f^c(S^d)$ depends continuously on $f$, assuming certain topology on the space of cocycles, like the one implicit in the above construction.

The continuous dependence can be understood more precisely as follows: Let everything depend on a parameter space $P$, which we assume to be a PL-manifold; so $\tilde{f} : \Delta \times P \to S^d \times P$ is a family of maps over $P$. Consider $S^d \times P$ as having sufficiently fine triangulation. Every $k$-face $F \subseteq \Delta$ makes a $(k \dim P)$-dimensional subset $F_P = \tilde{f}(F \times P)$ in $S^d \times P$. Then for every vertex $v$ of $S^d \times P$ we count $d$-faces of $\Delta$ whose $F_P$ cover $v$ and obtain a cocycle in $C^d(\Delta; \mathbb{F}_2)$ for every vertex. For every 1-face $\sigma$ of $S^d \times P$ we count $(d - 1)$-faces $F$ of $\Delta$ whose $F_P$ intersect $\sigma$ and obtain the respective cochains in $C^{d-1}(\Delta; \mathbb{F}_2)$, and so on with $k$-faces of $S^d \times P$ and cochains in $C^{d-k}(\Delta; \mathbb{F}_2)$. Those cochains are arranged in a proper way and give a chain map from the complex $C_*(S^d \times P; \mathbb{F}_2)$ to $C^{d-*}(\Delta; \mathbb{F}_2)$, which by Property 2.1 can be viewed as a simplicial map from $S^d \times P$ to $C^d$, this is just the definition of $C^d$ as a simplicial abelian group (see [10]). Therefore our $f^c$ gets extended to a simplicial map from $S^d \times P$, which is a simplicial version of the notion of “a family of maps depending on the parameter $p \in P$ continuously”.

In order to understand the pullback $X_f$, we consider $X$ as a $d$-cocycle on $S^d \times P$ and make a fiberwise (along $S^d$) Poincaré duality, making a cycle $X' \in C_{\dim P}(S^d \times P; \mathbb{F}_2)$ such that its projection $X' \to P$ onto the parameter space has degree 1 modulo 2. Recall that we assume $P$ to be a PL-manifold for simplicity. This $X'$ is therefore interpreted as a family of 0-cycles in $cl_0(S^d; \mathbb{F}_2)$ parameterized by $P$, whose all points represent heavily covered points of $S^d$ under their respective maps $f_p$ for $p \in P$. 
3. Applications

Having established the “homologically continuous” dependence we readily infer the following “transversal” and “dual” analogues of the results in [7, 19] or [13]:

**Theorem 3.1.** Let $P_0, \ldots, P_m$ be finite point sets in $\mathbb{R}^d$, then there exists an $m$-dimensional affine plane $L$ such that for any $i = 0, \ldots, m$ the following holds: The fraction of those $(d - m + 1)$-tuples in $P_i$ whose convex hulls touch $L$ is at least

$$\frac{2(d - m)}{(d - m + 1)!(d - m + 1)}.$$

**Theorem 3.2.** Let $\mathcal{H}$ be a family of hyperplanes in $\mathbb{R}^d$ in general position. Then there exists a point $c \in \mathbb{R}^d$ such that the fraction of those $(d + 1)$-tuples of hyperplanes in $\mathcal{H}$ that surround $c$ is at least $\frac{2d}{(d+1)!}$.

In the last theorem a family of $d + 1$ hyperplanes surround $c$ if $c$ cannot be moved to $\infty$ without touching any of these hyperplanes.

**Remark 3.3.** The proof with a worse constant $\frac{1}{(d+1)!}$ in [3] uses the Tverberg theorem to find the point $c$. Since there are topological analogues of the Tverberg theorem that have a similar homological continuity modulo a prime $p$, Theorems 3.1 and 3.2 with a weaker constant follow immediately. In fact, the constant will be further spoiled because of the requirement that the number of parts in a Tverberg partition is a prime power $p^a$, and by the limitation to even $d - m$ in Theorem 3.1 for odd primes $p$, like in [12]. In other words, the versions of Theorems 3.1 and 3.2 with worse constant directly follow from the corresponding versions of the Tverberg theorem in [12, 13] using the argument from [3].

**Proof of Theorem 3.1.** Choose arbitrary $m$-dimensional linear subspace $V$ of $\mathbb{R}^d$. Then we apply Theorem 1.1 (for linear maps $f_i$, mapping the corresponding simplices linearly following mappings of the vertices to the projected $P_i$’s) to the projections of $P_i$’s onto the orthogonal complement $V^\perp$. We observe the corresponding cycles $c_i(V)$ of heavily covered points. What we need is to find a nonempty intersection of the supports of $c_i(V)$ for some $V$.

Now observe that all possible choices of $V$ and $V^\perp$ constitute the Grassmannian $G_{d,d-m}$, and the family of all possible $V^\perp$ gives the canonical vector bundle $\gamma = \gamma_{d,d-m}$ over $G_{d,d-m}$. It is known (see [7, 19]) that the $m$-th power of the Euler class of $\gamma$ is nonzero modulo 2. Equivalently, the $(m + 1)$th power of the Thom class $\tau(\gamma)$ is nonzero modulo 2 in the cohomology of the Thom space $M(\gamma)$ (the compactified total space of $\gamma$). This makes any $m + 1$ sections of $\gamma$ coincide over some $V \in G_{d,d-m}$.

But this also makes any $m + 1$ “homological sections” coincide somewhere in $\gamma$. Indeed, the homological sections $c_i(V)$ are modulo 2 pseudomanifolds that project with degree 1 mod 2 onto the base. Hence they are all Poincaré dual to the Thom class in the Thom space, and hence their intersection must be nontrivial. This gives the desired $V$ and intersections $c_i$’s, thus completing the proof.

In fact, the last part can be done in less geometric terms: Note that $c_i(V)$’s were defined in Section 2 as $(d - m)$-cocycles on the compactification of $V^\perp$. And the parameterized version of Gromov’s argument means that with varying $V$ those cocycles constitute a single $(d - m)$-coycle of the Thom space $M(\gamma)$, representing the Thom class in the cohomology of $M(\gamma)$. Since the $(m + 1)$th power of the Thom class does not vanish, their supports must have a common point.

**Proof of Theorem 3.2.** We act like in [13] (see its corrected arxiv.org version): Take a convex body $B$ so that every orthogonal projection $\pi_H$ onto a hyperplane $H \in \mathcal{H}$ takes $B$...
into itself. Then for every point \( x \in B \) we consider the point set \( P_x = \{ \pi_H(x) \}_H \). We can find a point \( c \) such that at least \( \frac{2d}{(d+1)(d+1)} \) of all \( d \)-simplices with vertices in \( P_x \) contain \( c \), moreover, by the “homological continuous” dependence we can choose a 0-dimensional cycle \( c(x) \) of such points depending continuously on \( x \). In other words, this cycle is a multivalued map whose graph \( \Gamma_c \subset B \times B \) is a modulo 2 pseudomanifold with boundary projecting onto the first summand \( B \) with degree 1 modulo 2 and taking boundary to boundary.

Then we follow the usual proof of the Brouwer fixed point theorem by showing that \( \Gamma_c \) intersects the diagonal of \( B \times B \), for example, by writing down the classes of \( \Gamma_c \) and the diagonal \( \Delta_B \subset B \times B \) in the relative \( d \)-dimensional homology of \( (B, \partial B) \times (B, \partial B) \). We deduce that for some \( x \in B \) some point in the support of \( c(x) \) coincides with \( x \), then, using the general position assumption, we conclude that those \( d \)-simplices of points \( \pi_H(x) \) that contain \( x \) correspond to those \((d+1)\)-tuples of hyperplanes in \( H \) that surround \( x \).

**Remark 3.4.** Using an appropriate modification of Gromov’s technique, like in [14] with the improvement of the constant in [11], one can prove the “colorful” version of the above theorems. That is, in Theorem 3.1 \( P_c \) can be given colored into \( d - m + 1 \) colors each so that every color covers exactly the fraction \( \frac{1}{d-m+1} \) of its corresponding set \( P_i \), and in the conclusion \( L \) will touch at least the fraction \( \frac{2d}{(d+1)(d+1)} \) of all “rainbow simplices” (having no repetition of colors) of \( P_i \).

Similarly, in Theorem 3.2 \( H \) may be given colored in \( d + 1 \) colors uniformly and in the conclusion the fraction of \((d+1)\)-tuples of hyperplanes surrounding \( c \) in all possible “rainbow \((d+1)\)-tuples” is shown to be at least \( \frac{2d}{(d+1)(d+1)} \). The last claim may be considered as one step of another approach to the result of [4].

**Remark 3.5.** The improvements of the constant \( \frac{2d}{(d+1)(d+1)} \) from [16, 15] are applicable in the above theorems as well, since they actually improve the “filling process” used to contract \( d \)-cycles in what we call \( C^d \).

### 4. Elementary observations about the planar case

It is known that for \( d = 2 \) Theorem 1.1 has very elementary proofs, see [6] or [8], for example. Here we are going to mimick the one in [8] to prove the dual version:

**Elementary proof of the planar case of Theorem 3.2.** Let us have a set of lines in \( \mathbb{R}^2 \). When we are going to speak about a random line, or a random pair of lines, or a random triple of lines, we will choose the lines from the given set uniformly.

Now we consider pairs of a halfplane \( H \) and a point \( q \in \partial H \). We will also parameterize the halfplane \( H \) by its inner unit normal \( p \), considered as a vector at \( q \). Such a pair \((q, H)\) is called exposed if the probability that a pair of lines \( \ell_1 \) and \( \ell_2 \) cuts from \( H \) a triangle having \( q \) on its base is less than \( 2/9 \).

We will usually translate the question about lines to questions about points, having fixed the point \( q \), by replacing a line \( \ell \) with the projection of \( q \) onto \( \ell \), let \( q(\ell) = \pi_\ell(q) \). In terms of points \( q_1 = q(\ell_1) \) and \( q_2 = q(\ell_2) \) the condition of exposed is expressed as follows: The probability that the random segment \( q_1q_2 \) intersects the ray \( \{q + tp\}_{t \geq 0} \) is less than \( 2/9 \); this is almost the same of the definition of “exposed” in [8].

Now we want to find a point \( q \) such that no pair \((q, p)\) is exposed, for any \( p \). This point will be the required point in the theorem, since for any fixed choice of the first line \( \ell_1 \) in the triple, the event “the triple \( \{\ell_1, \ell_2, \ell_3\} \) surrounds \( q \)” is equivalent to the event “the pair \( \{\ell_2, \ell_3\} \) cuts from \( H_{\ell_1} \) a triangle having \( q \) on its base”. Here \( H_{\ell_1} \) is the halfplane with \( \partial H_{\ell_1} \) parallel to \( \ell_1 \), containing \( q \), and disjoint from \( \ell_1 \). If none of \((q, H_{\ell_1})\) is exposed then
we sum the probabilities to have the probability of “the triple \( \{ \ell_1, \ell_2, \ell_3 \} \) surrounds \( q \)” to be at least 2/9.

So we assume that no such \( q \) exists. Now we also follow the proof in [8]. When two pairs \((q, p_1)\) and \((q, p_2)\) with the same \( q \) are exposed, then we consider the rays \( r_1 = \{ q + tp_1 \}_{t \geq 0} \) and \( r_2 = \{ q + tp_2 \}_{t \geq 0} \). The rays divide the plane into two parts \( P_1 \) and \( P_2 \) and one of the parts must have \( < 1/3 \) of the points \( q(\ell) \). Otherwise the probability that the two points \( \{ q(\ell_1), q(\ell_2) \} \) are in different \( P_i \)’s will be at least 4/9, and for one of the rays \( r_i \), the probability that the segment \([q(\ell_1), q(\ell_2)]\) intersects \( r_i \) is at least 2/9, which contradicts the definition of “exposed”. So we conclude that one of \( P_i \)’s contains less than 1/3 of the points \( q(\ell) \). For any \( p \), whose ray \( r = \{ q + tp \}_{t \geq 0} \) is in this \( P_i \), we call the pair \((q, p)\) almost exposed.

As in [8], it is easy to conclude that for any \( q \in \mathbb{R}^2 \) the set \( F_q \) of \( p \) such that the pair \((q, p)\) is almost exposed is a nonempty proper connected subset of the circle \( S^1 \). It is nonempty because of our assumption; and it is proper because for an exposed pair \((q, p_1)\) we can always find another non-exposed pair \((q, p_2)\). It is achieved, when the rays \( r_1 = \{ q + tp_1 \}_{t \geq 0} \) and \( r_2 = \{ q + tp_2 \}_{t \geq 0} \) partition the set \( \{ q(\ell) \} \) into equal parts, this argument is also from [8].

Now the proof is finished by applying an appropriate version of the Brouwer fixed point theorem for convex-valued maps with closed graphs to the map \( q \mapsto S^1 \setminus F_q \), as in [8]. □

Now we give an example showing that the constant 2/9 is the best possible in the planar case of Theorem 3.2. Consider a circle \( C \) and choose a sequence of points \( x_1, \ldots, x_n \) in the given order on an arc of \( C \) of angular measure at most \( \pi/2 \). Let \( \ell_1, \ldots, \ell_n \) be the lines tangent to \( C \) at their respective points \( x_i \)’s.

How a point \( q \) can be surrounded by some three of \( \ell_i \)’s? It is easy to observe that \( q \) must lie outside \( C \) and the three surrounding lines must have three types: One line \( \ell_2 \) must separate \( q \) from \( C \), another line \( \ell_1 \) must have \( q \) and \( C \) on the same side of it and \( q \) must be to the “right” of \( C \) (if we place the arc \([x_1x_n]\) approximately horizontally), and the third line \( \ell_3 \) must also have \( q \) and \( C \) on the same side of it and \( q \) must be to the “left” of \( C \).

In fact, for any given \( q \) outside \( C \) all the lines are separated into such three classes of cardinalities \( n_1, n_2, n_3 \) so that \( n_1 + n_2 + n_3 = n \). So the number of triangles surrounding \( q \) is always bounded by

\[
n_1n_2n_3 \leq \frac{(n_1 + n_2 + n_3)^3}{27} = \frac{n^3}{27},
\]

which approaches \( \frac{2}{9}(\binom{n}{3}) \) for large \( n \).

References

[1] A. Akopyan, R. Karasev, and A. Volovikov. Borsuk–Ulam type theorems for metric spaces. 2012. arXiv:1209.1249.

[2] F. J. Almgren Jr. Homotopy groups of the integral cycle groups. Topology, 1:257–299, 1962.

[3] I. Bárány. A generalization of Carathéodory’s theorem. Discrete Math., 40(2–3):141–152, 1982.

[4] I. Bárány and J. Pach. Homogeneous selections from hyperplanes. Journal of Combinatorial Theory, Series B, 104:81–87, 2014.

[5] Z. Boros, Endre Füredi. The number of triangles covering the center of an \( n \)-set. Geom. Dedicata, 17(1):69–77, 1984.

[6] B. Bukh. A point in many triangles. The Electronic Journal of Combinatorics, 13(10), 2006.

[7] V. Dol’nikov. A generalization of the ham sandwich theorem. Math. Notes, 52:771–779, 1993.

[8] J. Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach. Overlap properties of geometric expanders. Journal für die reine und angewandte Mathematik, 671:49–83, 2012.

[9] G. Friedman. An elementary illustrated introduction to simplicial sets. 2008. arXiv:0809.4221.
[10] M. Gromov. Singularities, expanders and topology of maps. part 2: from combinatorics to topology via algebraic isoperimetry. Geometric and Functional Analysis, 20(2):416–526, 2010.

[11] Z. Jiang. A slight improvement to the colored first selection lemma. 2014. arXiv:1405.2503.

[12] R. Karasev. Tverberg’s transversal conjecture and analogues of nonembeddability theorems for transversals. Discrete & Computational Geometry, 38:513–525, 2007.

[13] R. Karasev. Dual theorems on central points and their generalizations. Sbornik: Mathematics, 199(10):1459–1479, 2008. arXiv:0909.4915.

[14] R. Karasev. A simpler proof of the Boros–Füredi–Bárány–Pach–Gromov theorem. Discrete & Computational Geometry, 47(3):492–495, 2012.

[15] D. Král’, L. Mach, and J.-S. Sereni. A new lower bound based on Gromov’s method of selecting heavily covered points. Discrete & Computational Geometry, 48(2):487–498, 2012.

[16] J. Matoušek and U. Wagner. On Gromov’s method of selecting heavily covered points. Discrete & Computational Geometry, 52(1):1–33, 2014. arXiv:1102.3515.

[17] J. Pach. A Tverberg-type result on multicolored simplices. Comput. Geom., 10:71–76, 1998.

[18] C. A. Weibel. An introduction to homological algebra. Cambridge University Press, 1994.

[19] R. Živaljević and S. Vrećica. An extension of the ham sandwich theorem. Bull. London Math. Soc., 22:183–186, 1990.

E-mail address: alexey.m39@mail.ru

URL: http://www.rkarasev.ru/en/

Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700

Institute for Information Transmission Problems RAS, Bolshoy Karetny per. 19, Moscow, Russia 127994