Multi-photon coherent states via a Lie algebra method

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Abstract. Using Lie algebra methods, we find a new class of Bogoliubov transformations which generalize the notion of squeezed states. The Hamiltonians for the simple harmonic and anharmonic oscillators, turn out to be the generators of a Lie group, whose other generators may be found exactly, or up to any desired order of the perturbation parameter. An element of this Lie group, which is realized as the multi-photon operator, transforms the anharmonic Hamiltonian to the harmonic one. The transformation of the ordinary annihilation and creation operators under this unitary transformation leads to the introduction of multi-photon coherent states. We specifically consider four-photon coherent states in detail and study the time dependent position and momentum uncertainties in these states.

1. Introduction
When fields interact with matter, that is characterized by nonlinear properties and can be described by anharmonic oscillators, nonlinear optical processes occur [1, 2]. Such processes correspond to the production of two or k-photon coherent states [3, 4]. One expects to generalize squeezed states or two-photon coherent states to k-photon ones, by means of straightforward generalization of the two-photon unitary operator; however, this task encounters divergence difficulties [5] and several approaches [6, 7, 8, 9, 10, 11, 12] have been taken toward the alleviation of this problem.

In this work, we present a simple method to generalize squeezed states. It also bridges between the notion of the latter and the anharmonic oscillators. We use a Lie algebra method to relate the Hamiltonian of an anharmonic oscillator to that of a harmonic one, in a perturbation sense. This is achieved via a canonical transformation, which is unique to the structure of the specific Hamiltonian under study. We introduce our method in section 2, where we apply it to three examples: a harmonic oscillator with a linear perturbation term, one with a quadratic perturbation term that just brings about a frequency shift and also the quartic anharmonic oscillator. Section 3 will be devoted to the study of four-photon coherent states that emerge from our method. Time dependent position and momentum uncertainties are obtained in these states and their oscillating nature are studied. Finally, we deal with discussion and conclusions in section 4.
2. Our method

The anharmonic oscillator Hamiltonian with a perturbation term of order \( n \) may be expressed by

\[
H_n = a^\dagger a + \frac{1}{2} + \lambda \left( \frac{a + a^\dagger}{\sqrt{2}} \right)^n = H_0 + \lambda \left( \frac{a + a^\dagger}{\sqrt{2}} \right)^n, \tag{1}
\]

where, \( H_0 \) is the Hamiltonian for a simple harmonic oscillator. If \( N \) hermitian generators \( L_i \), including \( L_1 = H_0 \) and \( L_2 = H_n \), form a closed commutator algebra, the operators \( L_i \) specify a Lie group, whose generators they are. The elements of this group are given by

\[
U = \exp(-i \sum_{i=1}^{N} \alpha_i L_i), \tag{2}
\]

where, \( \alpha_i \)'s are real parameters.

The case \( n = 1 \) in equation (1) may be considered as a harmonic oscillator in an external electric field. Considering the commutator \([H_0, H_1] = \lambda (a^\dagger - a)/\sqrt{2} \), we realize that the operators \( H_0, H_1, i(a^\dagger - a) \) and the identity operator \( I \), form a closed commutator algebra and are the generators of a Lie group. The Glauber operator \( U_1 = \exp(\alpha a^\dagger - \alpha^* a) \), with real \( \alpha \), is an element of this group which transforms \( H_0 \) to \( H_1 \) in the following manner

\[
H_1 = U_1^\dagger H_0 U_1 - \frac{\lambda^2}{2}, \tag{3}
\]

where \( \lambda \) has been replaced by \( \lambda/\sqrt{2} \) in \( U_1 \). Using Eq. (3), the eigenfunctions of \( H_1 \) are given by \( U_1^\dagger |n\rangle_a \) where \( |n\rangle_a \) are \( a \)-mode or the displaced number states [13].

Inserting \( n = 2 \) in Eq.(1)gives the Hamiltonian of a shifted frequency harmonic oscillator. In this case the set of hermitian operators \( \{H_0, H_2, i(a^{12} - a^2), I\} \) are the generators of a Lie group. The squeezing operator, \( U_2 = \exp(\beta a^{12} - \beta^* a^2) \) with \( \beta \) assumed real, is an element of this group. The transformation of the fundamental mode operators \( a \) and \( a^\dagger \) into the new ones, \( b \) and \( b^\dagger \) under \( U_2 \), is carried out by the following Bogoliubov transformations

\[
\begin{align*}
b &= U_2^\dagger a U_2 = \mu a + \nu a^\dagger, \\
b^\dagger &= U_2^\dagger a^\dagger U_2 = \mu a^\dagger + \nu a,
\end{align*}
\]

where, \( \mu = \cosh(2\beta) \) and \( \nu = \sinh(2\beta) \). As \( \mu^2 - \nu^2 = 1 \), we have also \( [b, b^\dagger] = 1 \). The squeezing operator transforms \( H_0 \) to \( H_2 \) in the following manner

\[
H_2 = \sqrt{1 + 2\lambda} U_2^\dagger H_0 U_2 = \sqrt{1 + 2\lambda(b^\dagger a + a b^\dagger)}, \tag{4}
\]

where, \( \lambda = 2\mu \nu/\mu - \nu^2 \) has been assumed. Using equation (4), the eigenfunctions of \( H_2 \), called \( b \)-mode or squeezed number states [13, 14] are given by \( |n\rangle_b = U_2^\dagger |n\rangle_a \).

We now deal with the quartic anharmonic oscillator by inserting \( n = 4 \) in Eq. (1). For small values of \( \lambda \), the set of operators \( \{H_0, H_4, i\lambda (a^{14} - a^4), i\lambda (a^{12} - a^2), i\lambda (a^{13} a - a^3 a^3), \lambda (a^{14} + a^4), \lambda (a^{12} + a^2), \lambda (a^{13} a + a^3 a^3), I\} \) form a closed commutator algebra, up to first order, and therefore they are the generators of a Lie group up to that order. Let’s consider the following element of this Lie group, known as the four photon operator [5]

\[
U_4 = \exp[A\lambda (a^{14} - a^4) + B\lambda (a^{12} - a^2) + C\lambda (a^{13} a - a^3 a^3)], \tag{5}
\]
where \( A, B \) and \( C \) are real parameters. This operator transforms the fundamental mode annihilation and creation operators, \( a \) and \( a^\dagger \) to the new ones \( c \) and \( c^\dagger \), which may be expressed up to the first order in the parameter \( \lambda \), by the following expressions

\[
c = U_4^\dagger a U_4 = a + \lambda (4AA^3 + 2Ba^\dagger + 3Ca^2a - Ca^3),
\]

\[
c^\dagger = U_4^\dagger a^\dagger U_4 = a^\dagger + \lambda (4AA^3 + 2Ba + 3Ca^\dagger a^2 - Ca^3). \tag{7}
\]

It should be emphasized that \( c \) and \( c^\dagger \) also obey the canonical commutation relation \( \{c, c^\dagger\} = 1 \); therefore, we have introduced a new class of Bogoliubov transformations by (6) and (7).

Assuming \( A = 1/16, B = 3/4 \) and \( C = 1/2 \), the four photon operator transforms \( H_0 \) to \( H_4 \), up to the first order in the parameter \( \lambda \), as follows

\[
U_4^\dagger H_0 U_4 = c^\dagger c + \frac{1}{2} = H_4 - \frac{3\lambda}{2}(N_a^2 + N_a) - \frac{3\lambda}{4}, \tag{8}
\]

where, \( N_a = a^\dagger a \) is the normal number operator. Using equation (8), the eigenstates of the quartic anharmonic oscillator, our \( c \)-mode number states, are given by

\[
|n\rangle_c = U_4^\dagger |n\rangle_a = e^{-\lambda(n+\frac{1}{2})} e^{\frac{3\lambda n}{4}} |n\rangle_a, \tag{9}
\]

and the first order perturbed energies are expressed by

\[
E_n = n + \frac{1}{2} + \frac{3\lambda}{4} + \frac{3\lambda}{2}(n + 1).
\]

The above result is in complete agreement with that obtained, using first order Rayleigh-Schrödinger perturbation theory [15].

3. Four photon coherent states

Ordinary coherent states, or so called one photon coherent states, are eigenstates of the ordinary annihilation operator \( a \)

\[
a|\alpha\rangle = \alpha|\alpha\rangle, \quad |\alpha\rangle = U_1(\alpha)|0\rangle_a = e^{\alpha a^\dagger - \alpha^* a}|0\rangle_a,
\]

entailing a Poisson distribution, in terms of the ordinary number states. Similarly, defining \( b \)-mode vacuum state \( |0\rangle_b \), by the relation \( b|0\rangle_b = 0 \), we may construct the so called \( b \)-mode number states by repeated application of \( b \)-mode creation operator \( b^\dagger \) on the \( b \)-mode vacuum state as follows

\[
|n\rangle_b = b^n \sqrt{n!} |0\rangle_b, \quad |n\rangle_b \equiv U_2^\dagger |n\rangle_a.
\]

Eigenstates of the operator \( b \), \( |\beta\rangle \) are called the \( b \)-mode coherent states, two photon coherent states or squeezed states

\[
b|\beta\rangle = \beta|\beta\rangle, \quad |\beta\rangle = U_2^\dagger U_1(\beta)|0\rangle_a = e^{\beta b^\dagger - \beta^* b}|0\rangle_b.
\]

In a similar way, we may introduce the \( c \)-mode number states, by the relation

\[
|n\rangle_c = c^n \sqrt{n!} |0\rangle_c, \quad |n\rangle_c \equiv U_3^\dagger |n\rangle_a,
\]
where, $|0\rangle_c$ is the $c$-mode vacuum state and it is defined by $c|0\rangle_c = 0$. The four photon or $c$-mode coherent states, or the generalized squeezed states $|\gamma\rangle$, are defined as the eigenstates of the $c$-mode annihilation operator as follows
\begin{equation}
|\gamma\rangle = \frac{1}{\sqrt{2}}((\cos t - i\sin t)a(0) - \lambda[A_1a(0) + A_2a^3(0)] + A_1^*a^2(0)a(0)) + h.c. ,
\end{equation}
where $h.c.$ stands for the Hermitian conjugate of the other terms, and the parameters $A_1$ and $A_2$ are defined as follows
\begin{align}
A_1 &= \frac{3}{4}[t\sin t + i(t\cos t - \sin t)], \\
A_2 &= \frac{1}{16}[(\cos t - \cos 3t) + i(\sin 3t - 3\sin t)].
\end{align}
We also write down the momentum operator
\begin{equation}
p(t) = -\frac{1}{\sqrt{2}}((\sin t + i\cos t)a(0) + \lambda[A_1a(0) + A_2a^3(0)] + \hat{A}_1^*a^2(0)a(0)) + h.c. ,
\end{equation}
where $\hat{A}_1$ and $\hat{A}_2$ are the time derivatives of $A_1$ and $A_2$ respectively.

Using (13) and (14), we obtain the position and momentum uncertainties in the four photon coherent state $|\gamma\rangle$ as follows
\begin{align}
\Delta^2 x(t) &= \frac{1}{2} + \frac{\lambda}{2}\left(\frac{3}{2} + 3|\gamma|^2\right)\sin^2 t + 2Re[\frac{3}{4}\gamma^2 + 3|\gamma|^2 + \frac{3}{2}]e^{2it} \\
&+ A_1\gamma^2e^{-it} + 3A_2\gamma^2e^{it}],
\end{align}
\begin{align}
\Delta^2 p(t) &= \frac{1}{2} + \frac{\lambda}{2}\left(\frac{3}{2} + 3|\gamma|^2\right)\sin^2 t + 2Re[\frac{3}{4}\gamma^2 + 3|\gamma|^2 + \frac{3}{2}]e^{2it} \\
&+ iA_1\gamma^2e^{-it} - 3iA_2\gamma^2e^{it}],
\end{align}
where, all the expressions of order two and higher in terms of $\lambda$ are neglected. The product of uncertainties (15) and (16) reduces to $\frac{1}{2}$, up to the first order in $\lambda$ at $t \to 0$ limit. Thus we conclude that the four photon coherent state (12) is also a minimum uncertainty state at $t = 0$. As time passes, one of the uncertainties dips well more and more, below the value $\frac{1}{2}$ in successive periods; thus revealing the squeezing properties of such states. We have illustrated the uncertainties of $\dot{x}$ and $\dot{p}$ versus time in figure 1. These quantities depend on $\gamma$ and $\lambda$, oscillate in opposite directions and their overall range of variation increase with time.
Figure 1. Plot of the $\hat{x}$-uncertainty (solid line) and $\hat{p}$-uncertainty (dashed line) for four-photon coherent state versus time. $\lambda = 0.01, \gamma = 1$.

4. Discussion

We have presented multi-photon coherent states via anharmonic oscillators. We have constructed a Lie group, whose generators include $H_0$ and $H_n$. The generator of the group $H_0$ may be transformed to the generator $H_n$ by a unitary transformation, endowed by an element of the group. In the cases of $H_1$ and $H_2$, the familiar Glauber and squeezed operators, being canonical transformations, emerge respectively; while in the case of $H_4$, the four photon operators are generated. The transformation of the annihilation operator $a$, under the squeezed operator $U_2$, leads to the Bogoliubov transformation, and the associated squeezed states. Keeping terms up to the first order, the transformation of the ordinary annihilation operator under four photon operator, leads to a new class of Bogoliubov transformations, in which the associated eigenstates are four photon coherent states. The uncertainties for the four photon coherent state $|\gamma\rangle$, oscillate around $\frac{1}{2}$ in time, similar to the squeezed states, but the overall range of their variation grows with time. Finally, one may generalize this procedure to six photon coherent states, via sextic anharmonic oscillators, eight photon coherent states, via octic anharmonic oscillator and so on.

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