Certification of a non-projective qudit measurement using multiport beamsplitters

The most common form of measurement in quantum mechanics projects a wavefunction onto orthogonal states that correspond to definite outcomes. However, generalized quantum measurements that do not fully project quantum states are possible and have an important role in quantum information tasks. Unfortunately, it is difficult to certify that an experiment harvests the advantages made possible by generalized measurements, especially beyond the simplest two-level qubit system. Here we show that multiport beamsplitters allow for the robust realization of high-quality generalized measurements in higher-dimensional systems with more than two levels. Using multicore optical fibre technology, we implement a seven-outcome generalized measurement in a four-dimensional Hilbert space with a fidelity of 99.7%. We present a practical quantum communication task and demonstrate a success rate that cannot be simulated in any conceivable quantum protocol based on standard projective measurements on quantum messages of the same dimension. Our approach, which is compatible with modern photonic platforms, showcases an avenue for faithful and high-quality implementation of genuinely non-projective quantum measurements beyond qubit systems.
higher-dimensional quantum information (see, for example, refs. 32,33), it is natural to go beyond the simplest quantum system and investigate
generalized measurements of higher dimension.

Owing to the possibility of Neumark dilations2, the minimal setting for investigating generalized measurements fixes the Hilbert space
dimension, that is, the experimental degrees of freedom are known. Therefore, a forceful, and most elementary, framework for asserting
the advantages of generalized measurements is to view all quantum devices as uncharacterized up to their dimension34. However, certifying
a generalized measurement when only the dimension is known and
global classical randomness is free requires highly precise experimental
apparatuses. For instance, the smallest known total visibility required
to certify a qubit generalized measurements is 97.0% (ref. 35). For \( d > 2 \)
it becomes even higher, and the number of setup configurations also
rises rapidly36. Thus, the stringent requirements concerning the complexity and quality of the experimental device make it challenging
to implement and certify generalized measurements.

Here we demonstrate a new path to realizing high-quality general-
ized quantum measurements on \( d \)-dimensional systems. Our approach
uses multiport beamsplitters (MBSs), which are devices for which \( D \)
input optical modes interfere at a single interface. As an experimental
platform, we adopt beamsplitters built with multicore optical fibres
(MCFs)38–43 that are compatible with modern space-division multi-
plexing optical fibre technology39 and implement a seven-outcome
generalized measurement of dimension \( d = 4 \). These devices, for
which a fabrication technique was recently presented in ref. 37, pro-
duce ultra-high-quality interference. To showcase the method and the
quality of the measurements achievable, we present a new concrete
quantum communication task in which the non-projective nature of
the measurement is a resource. This protocol is made practically viable
both in terms of its noise requirements and its required number of
measurement configurations. In it, an uncharacterized sender emits
four-dimensional states and an uncharacterized receiver performs
measurements with the aim of accessing the encoded information.
We prove that an optimal quantum implementation requires the use
of generalized measurements. Achieving an average visibility greater
than 99.7% in our high-quality device, we experimentally certify
(by more than three standard deviations) that our measurement is
non-projective by achieving a success rate that surpasses those possi-
ble in any quantum protocol based on projective measurements.
Our approach expands the reach of state-of-the-art photonic devices
to the faithful implementation of higher-dimensional generalized
measurements at nearly perfect fidelity. Moreover, it is fully compatible
with integrated photonics43–45 and can, therefore, find applications in
different quantum information scenarios.

A MBS is a device described by unitary matrix \( U_D \), which couples
\( D \) input optical modes to \( D \) output modes. We now show that, when
used as a measurement device, they can be used to implement general-
ized measurements on a \( d \)-dimensional quantum system. This is
achieved using a subset of \( d \) input modes as the principal system and
the remaining \( D - d \) modes as an ancilla system46, as illustrated in
Fig. 1. The case \( D = d \) corresponds to projective measurement. For \( D < d \),
different types of generalized measurements (here, rank-one positive
operator-valued measures (POVMs)) can be realized. The total number of
different possible POVMs is given by the number of ways to choose
\( d \) out of \( D \) elements without repetition, which is \( \frac{D!}{d!(D-d)!} \). Taking
into account the fact that the phase on each input mode of the system
can be modulated, the rank-one POVM elements are given by
\( \Pi_j = |q_j\rangle \langle q_j| \), with

\[
|q_j\rangle = \phi_{k_1 \ldots k_d} M_{k_1 \ldots k_d} |j\rangle,
\]

where \( k_1 \ldots k_d \) are indices corresponding to the input modes defining
the quantum system, \( \phi_{k_1 \ldots k_d} \) is a diagonal matrix containing the relative
phases and \( M_{k_1 \ldots k_d} \) is the portion of the matrix \( U_D^j \) restricted to the
system space. In this way, the device is reconfigurable, in that a number
of different generalized measurements can be implemented on a
\( d \)-dimensional photonic quantum system by connecting the optical
modes in different ways and modulating the input phases. Examples
for 35 different types of POVMs when \( D = 7 \) and \( d = 4 \) can be found in
the Supplementary Information Section III. As is shown in our experi-
ment, the use of new fibre-based MBS devices allows for the implemen-
tation of generalized measurements of nearly perfect quality.

**Certification protocol**

We introduce a quantum communication task and show that general-
ized measurements are necessary for an optimal implementation. Then,
a sufficiently near-optimal implementation serves as a certification of
the genuine need for generalized measurements in the experiment.
Notably, such a certification can already be performed without signif-
ificant distance between the preparation and measurement devices.
Once the certification of the measurement is achieved, the device can be
utilized for generic long-distance quantum communication scenarios
in which generalized measurements are desirable.

Consider the scenario illustrated in Fig. 2. A sender, Alice, selects
one of seven possible classical inputs \( x \in \{1, \ldots, 7\} \) and encodes it into a
four-dimensional quantum state, \( \rho_x \), that is sent to Bob. Bob chooses
two between actions. First, he can select some \( y \in \{1, \ldots, 7\} \) and perform
a dichotomic quantum measurement \( \{E_{xy}\} \) with the goal of learning
whether \( x = y \) (\( b = 1 \)) or \( x \neq y \) (\( b = 0 \)). A correct answer in the former or
latter is rewarded with two or one points, respectively. Second, Bob
can select an eighth input, labelled \texttt{povm}, corresponding to a
seven-outcome measurement \( \{M_{bx}\} \) for \( b' \in \{1, \ldots, 7\} \), with the goal of
learning Alice’s input (\( b' = x \)). A correct answer is rewarded with three
points. From the Born rule,\(^4\) the total number of points in the commu-
nication task is

\[
W = \sum_{x,y \in \{1, \ldots, 7\}} (1 + \delta_{xy}) \text{tr}(\rho_x E_{xy}) + 3 \sum_{x \in \{1, \ldots, 7\}} \text{tr}(\rho_x M_x),
\]

(2)

where \( \delta_{xy} \) is the Kronecker delta function. Moreover, Alice and Bob
are allowed to stochastically synchronize their quantum operations
via a shared source of classical randomness. However, because of the
linearity of the scoring function, the optimal strategy is deterministic.

We have determined the largest value of \( W \) (labelled \( W_{\text{proj}} \)) achieve-
able in a quantum protocol where Bob’s measurements are projective
and four-dimensional. To this end, we first note that a deterministic
projective measurement in this scenario can have at most four out-
comes. An upper bound on \( W_{\text{proj}} \) is then obtained via the method
of semidefinite programming relaxations of fixed-dimensional quantum
correlations\(^46\). See Methods for details. To obtain a tight bound, we have
additionally exploited the symmetries of the \( W \) (ref. 19). Tightness is proven
by saturating (up to solver precision) the upper bound with a lower

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**Fig. 1 | Generalized measurement scheme using a MBS.** The horizontal lines
represent the optical modes, the black circles represent phase shifts \( \phi_j \) and
the box represents the multicore beamsplitter. We encode the input quantum state
only in the first \( d \) modes. Each output mode is associated with a POVM element.

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**Footnotes**

1. [Link]

2. [Link]

3. [Link]

4. [Link]
bound obtained from an explicit quantum model found via a numerical search. We thus find that projective quantum models are limited by

$$W_{\text{proj}} = 62.5152.$$  \hfill (3)

To certify that the setting povm corresponds to a generalized measurement, we must show that there exists a quantum protocol that outperforms the bound (3). Again using the above methods, one can show that the largest quantum value is $W_{\text{quant}} = 62.75$. However, we focus on a protocol based on a generalized measurement that is particularly suitable for implementation in our photonic platform. Choosing povm as this measurement, we have numerically found states $|\rho_i\rangle$ and measurements $|E_j\rangle$ that achieve the desired certification, specifically reaching $W_{\text{opt}} = 62.6982 > W_{\text{proj}}$. The states and measurements are given in Supplementary Information Section I.

**Experimental implementation**

Our experiment is based on modern MCFs created for space division multiplexing in classical telecommunication, which is a technique that uses multiple spatial optical modes for increasing data communication capacity\(^{16}\). These fibres are composed of several 8 μm diameter cores that reside within the same cladding material. The core separation ensures that the light coupling between them is greatly reduced ($<40$ dB km$^{-1}$), such that each core works as an independent and isolated transmission channel. Multicore fibre technology was recently introduced as a toolbox for high-dimensional quantum information processing\(^{17}\). The basic idea is to build multi-arm interferometers within a multicore fibre, taking advantage of the low loss, intrinsic stability and high-quality optical mode provided by the fibre, to implement high-fidelity unitary operations in higher-dimensional quantum systems. These interferometers have been proven to be useful not only for new fundamental investigations\(^{18,19}\) but also for quantum communication systems\(^{20,21}\), quantum randomness generation\(^{22}\) and quantum computing\(^{23}\). Yet all of these works have been limited to a fixed core geometry and, therefore, are capable of only implementing standard projective measurements. Motivated by the potential of generalized measurements for quantum information processing, we now show how multicore fibres of different core geometries can be combined together for implementing high-fidelity general quantum measurements in higher dimensions. Generalized measurements were implemented in a dimension $d > 2$ with a fidelity that allows for their semidefinite independent certification.

The experimental setup is illustrated in Fig. 3. It is essentially a four-path interferometer consisting of two main blocks: the preparation stage (Alice) and the measurement stage (Bob), who can implement projective or generalized measurements. To prepare the states considered in the protocol, we use as the laser source a continuous-wave telecommunications laser operating at 1.546 nm (Fig. 3a). It is connected to a fibre-pigtailed amplitude modulator (FMZ), which generates 5 ns long pulses at a repetition rate of 2 MHz. Attenuators (ATTs) are then used to set the average number of photons per pulse to $\mu = 0.2$, such that our source can be seen as a good approximation of a non-deterministic source of single photons\(^{25}\). In this configuration, the probability of having pulses containing at least one photon is $P(\text{one pulse} = 0.2) = 18\%$ and 90.3% of the non-null pulses contain only one photon. The photons are sent through a single-mode fibre (SMF) connected to a built-in fibre demultiplexer (DMUX), connecting four independent SMFs to one core of a four-core MCF (4C-MCF)\(^{26}\). With this, light can be sent from a standard single-mode fibre into one core of a MCF or vice versa. Here, only one core of the 4C-MCF is illuminated. The DMUX is then connected to a 4C-MCF-based MBS (4CF-MBS)\(^{27}\) that coherently distributes the signal over the four fibre cores that define a four-dimensional quantum state of a single photon. Owing to the symmetry of the 4C-MCF structure, there is a close to ideal 25% coupling between all four cores of the fibre. The 4CF-MBS corresponds to a unitary Hadamard matrix described by$$U_4 = \frac{1}{2} \left[ \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right].$$

Thus, the 4CF-MBS takes a photon in the logical core state $|j\rangle$ ($j = 1,...,4$) to an equally weighted four-dimensional ‘ququart’ of the form $|\psi_j\rangle = \frac{1}{\sqrt{4}} \sum_{k=1}^{4} \beta_k |k\rangle$, where $\beta_k = \pm 1$ are the entries of the matrix (4). Likewise, it can be used to map superposition states into logical basis states: $U_4 |\psi_j\rangle = |j\rangle$. In the experiment, the 4CF-MBS has a mean fidelity with equation (4) of $F = 0.995 \pm 0.003$ (ref. 37). The output defines four paths of a Mach–Zehnder interferometer that will be used to prepare the states and implement the measurements required to violate the bound of equation (3) in the protocol described above. This is done in the same spirit of phase-coding quantum cryptography where the states are prepared by Alice controlling the initial part of the interferometer, while Bob performs the measurements by applying a second set of phases at the final part of the interferometer\(^{1}\). To control the initial quantum state entering the interferometer, a second DMUX is used to couple the 4CF-MBS to four SMFs, so that each path can be sent through a phase and amplitude fibre-pigtailed modulator, as depicted in Fig. 3a. The general ququart state that Alice prepares is then given by $|\chi\rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^{4} a_j e^{i\phi_j} |j\rangle$, where $a_j$ and $\phi_j$ are the transmissivity and relative phase of core $j$, respectively, and $|\chi\rangle$ is a normalization constant. The seven required states in the protocol are prepared by applying specific voltage values to these modulators.

To implement the required projective and generalized measurements, Bob can resort to two different measurement procedures schematically shown in Fig. 3b,c, respectively. The projective measurement bases are implemented using a system that is similar to Alice’s preparation stage. A second set of intensity and phase modulators is used to adjust the amplitude and phase and is then connected to another 4CF-MBS. This thus closes a four-arm, four-output interferometer (Fig. 3b). Bob can thus project onto a basis defined by states $|\zeta\rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^{4} \beta_j |\psi_j\rangle$, where $\beta_j$ and $\phi_j$ are the transmissivities and relative phases defined by Bob, and $|\zeta\rangle$ is a normalization constant.

To detect the photons, the outputs of the 4CF-MBS are connected through a DMUX and SMFs to four avalanche photo Detectors (APDs). They are commercial InGaAs single-photon detection modules that are triggered by the laser modulation signal, and they are configured with a 5 ns detection gate and 10% detection efficiency. In reference to the certification scheme, the outcome $b = 1$ corresponds to the APD in output mode 1, while the outcome $b = 0$ statistics are given in terms of the sum of APD counts in outputs 2, 3 and 4.
The extra loss introduced by the $7 \times 7$ MBS (compared to the $4 \times 4$ MBS) is compensated by the intensity modulators used for the projective measurements.

The seven-outcome POVM measurement adopted in the protocol can be realized using a $7 \times 7$ MBS built in a seven-core MCF (7C-MCF)\(^{37}\). The core geometry of the 7C-MCF is shown schematically in the inset of Fig. 3c. Even though the different core geometries of the 4C- and 7C-MCFs prohibit them from being connected directly, the $7 \times 7$ MBS can still be used to close the four-arm interferometer by using DMUX devices that are compatible with these seven core fibres as shown in Fig. 3c as well as in Fig. 5. In this case, the measurement of the four-dimensional quantum state will have seven different possible outcomes (registered by seven APDs) and, thus, the measurement can be only described as non-projective. The exact POVM that is implemented depends on the unitary matrix representing the $7 \times 7$ MBS, as well as those of the four input cores are connected. There are a total of 35 inequivalent POVMs that can be implemented in the configuration given in Fig. 3c. The $7 \times 7$ MBS used in our experiment was characterized in ref. \(^{16}\), and the corresponding unitary matrix $U$ is given explicitly in the Methods section and the 35 possible POVMs in Supplementary Information Section III. To implement the protocol described above and violate the projective bound of equation (3), we optimized the value of $W$ (equation (2)) over all possible states and projections for each one of the 35 POVMs we can implement in our scheme. We found that the POVM implemented with input cores 4, 5, 6 and 7, namely, $M_{4567}$, combined with the states and measurement projections also given explicitly in Supplementary Information Section I, corresponds to the POVM measurement that allows a bound of $W_{\text{proj}} = 62.6982$ to be reached, which is larger than the projective bound in equation (3).

The proximity of $W_{\text{proj}}$ to $W_{\text{opt}}$ requires high-quality phase stabilization and synchronization of preparation and measurement. To achieve this, the entire system is automatically controlled by two field-programmable gate array electronic units (FPGA1 and FPGA2). Alice’s FPGA1 controls the FMZ used to generate the optical pulses and control her intensity and phase modulators, capable of preparing different predefined states for each optical pulse sent through the interferometer. Bob’s FPGA2 is used to control his intensity and phase modulators and to record the counts of all detectors involved in a given measurement configuration. To achieve high-quality measurements, the system is switched between a stabilization mode and a data acquisition mode. In the stabilization mode, FPGA1 searches for a near-zero phase relation as reference. Once found, the system switches to the acquisition mode and records data for 0.1 s and then returns to the stabilization mode. This procedure allows us to achieve greaterthan 99% interference visibility required for the certification scheme and is described in more detail in the Methods.

Figure 4a,b shows examples of the recorded statistics and compares them with the theoretical predictions for the projective measurement configuration of Fig. 3b and the generalized measurement of Fig. 3c, respectively. One can clearly see that a high-quality implementation of the protocol is obtained, which is a consequence of the fact that the MCF devices have the same single optical mode as the SMFs, allowing for near perfect mode overlap in all MBSs. All the recorded statistics, related to the many different states and measurements settings required for measuring the value $W$, are given explicitly in Supplementary Information Section II. In Fig. 4c we show the experimental value obtained for $W_{\exp} = 62.6208 \pm 0.0036$, which represents a clear violation of the projective bound with a corresponding $P$ value of $2.793 \times 10^{-4}$, corresponding to 3.45 standard deviations. We can thus certify that generalized quantum measurement in a dimension $d = 4$ has been implemented.

**Conclusion**

We have demonstrated that generalized quantum measurements in photonic quantum systems can be implemented in a simple and robust way using MBSs. In particular, we have certified a generalized measurement using beamsplitters constructed within MCFs, which have the additional advantage that they can be easily integrated into optical fibre systems. In addition to configurability provided by the different input mode configurations possible, the high-quality mode overlap provided by multicore beamsplitters leads to very-high-quality measurement fidelity. The latter is key for near-optimal implementations of many quantum information protocols and we demonstrated it in the particularly demanding task of certifying a higher-dimensional generalized quantum measurement in a scenario where only the Hilbert space dimension is known. Our experiment used a seven-core beamsplitter to implement a seven-outcome measurement on the four-dimensional...
quantum system, achieving measurement fidelity greater than 99.7%, certifying that the output results cannot be achieved with projective measurements in a four-dimensional Hilbert space.

Our results pave the way towards the realization and use of generalized measurements in higher-dimensional quantum information protocols, a task that has been difficult to achieve so far. The measurement scheme is quite general and can be realized in any MBS architecture, including those involving other photonic degrees of freedom. In terms of scalability, multicore fibres should allow for the realization of POVMs with many more outcomes, as fibres with a few tens of cores have already been produced, which could lead to multiports involving more modes. Nonetheless, it remains to be determined what type of multiports, and thus generalized measurements, could be implemented. We believe that this is a very promising path for future work.

**Online content**

Any methods, additional references, Nature Portfolio reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41567-022-01845-z.

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Methods

Determining $W_{\text{proj}}$

Owing to the symmetries of $W$, we can without loss of generality restrict the measurement $\{M_{b}\}$ to the form $M_{b'} = |\psi_{b'}\rangle\langle \psi_{b'}|$, for $b' \in \{1, 2, 3, 4\}$, where $\{|\psi_{b}\rangle\}_{b}$ is an orthonormal basis of $\mathbb{C}^d$ and $M_{b'} = 0$ for $b' \in \{5, 6, 7\}$. The fact that we can restrict to a rank-one projectors for $b' \in \{1, 2, 3, 4\}$ follows from the optimality of Alice preparing pure states $\rho_{b} = |\psi_{b}\rangle\langle \psi_{b}|$ and that the only relevant contribution to the total score is $\langle \psi_{b'}|M_{b'}|\psi_{b}\rangle$. Hence, we can write

$$W = \sum_{x=1}^{7} \left(1 + \delta_{x,y}\right) \langle \psi_{x}|E_{b',y}|\psi_{x}\rangle + \frac{3}{7} \sum_{x=1}^{4} \langle \psi_{x}|\psi_{y}\rangle^2.$$  \hspace{1cm} (5)

Our optimization problem becomes

$$W_{\text{proj}} = \max_{\{\psi_{b}\}_{b}} W,$$

s.t. $\langle \psi_{b}|\psi_{y}\rangle = 1$, $\sum_{b'=1}^{4} \langle \psi_{b'}|\psi_{y}\rangle = I$, $E_{b',y} \geq 0$, $E_{b',y} + E_{b,y} = I$, $E_{b,y} = E_{y,b}$. \hspace{1cm} (6)

Solving this is difficult. However, increasingly precise upper bounds can be obtained using a hierarchy of semidefinite programming relaxations developed in ref. 45. This method is based on first sampling a basis of moment matrices and then evaluating a semidefinite program over this basis. We refer to ref. 45 for details.

In our case, we have used a moment matrix of size 176, which corresponds to the hierarchy level given by

$$\{\{\langle \psi_{x}|\psi_{y}\rangle|E_{b',y}|\psi_{x}\rangle, \{\langle \psi_{x}|\psi_{y}\rangle|E_{b',y}|\psi_{x}\rangle, \{\langle \psi_{x}|\psi_{y}\rangle|E_{b',y}|\psi_{x}\rangle\} \}.$$ \hspace{1cm} (7)

However, this implementation requires over 2,000 sampled moment matrices (at which point the computation was terminated) followed by a correspondingly large semidefinite program. To remedy this, we have employed the method of symmetrization 36. Specifically, we perform a variable reduction by averaging each sampled moment matrix over the symmetry group $S_4$ of the scoring function, which is simply a permutation of $\{1, 2, 3, 4\}$ applied to each of $x, y$ and $b$ (when $x, y, b \in \{5, 6, 7\}$ we leave them unchanged). We denote by $T_{ab}$ the action of a permutation $\pi \in S_4$ on the moment matrix $\Gamma$. The new moment matrix is then

$$\Gamma' = \frac{1}{|S_4|} \sum_{\pi \in S_4} T_{\pi b} \Gamma'. $$ \hspace{1cm} (8)

This procedure shrinks the size of the moment matrix basis to just 93 elements and the computation of the semidefinite program can then be efficiently achieved. The upper bound we obtain is $W_{\text{proj}} \leq 62.5152$.

To prove that the bound is tight, we use a seesaw approach. That is, we first sample a random set of states $\{|\psi_{x}\rangle\}$ and evaluate the largest value of $W$ as a semidefinite program over the measurements $\{E_{b,x}\}$ and $\{M_{b}\}$. Then, for the optimal measurements, we determine the new set of optimal states through an eigenvalue computation. The process is repeated until it appears to converge on a stable value of $W$. The corresponding quantum strategy can then be extracted. Note that since $\{E_{b,x}\}$ is dichotomic and all extremal dichotomic measurements are projective, we need not enforce an additional constraint of projectivity. However, if we impose that $M_{b'} = M_{b}$, then the optimization over the measurements is no longer a semidefinite program. To circumvent this, we relax the set of projective measurements (recall that we need only to consider the first four outcomes) to the set of all measurements of four outcomes. Running the seesaw procedure we find $W = 62.5152$. We can verify that, in the corresponding quantum strategy, $\{M_{b}\}$ indeed is projective. Thus, we conclude that the bound is tight.

Seven-outcome POVM implemented by the 7CF-MBS

To implement the seven-outcome measurement required to estimate the value of $W$, we resort to using a 7CF-MBS built-in 7C-MCF. We also use a spatial multiplexing/multiplexer unit (DMUX) for accessing each of the seven cores of the 7CF-MBS independently. These devices allow us to implement the unitary transformation demanded to realize a seven-element POVM. The optical characterization of the 7CF-MBS involves intensity and phase measurements, for which the results are numerically optimized to obtain a genuine unitary operation 37. In the logical basis, the unitary matrix $U_{7}$ for the MBS is given by

$$U_{7} = \begin{pmatrix}
0.5639 & 0.2010 & 0.3019 & 0.3749 \\
0.2222 & -0.0065 & 0.1874 & -0.5700 & -0.3060i & 0.3558 & -0.0865i \\
0.3487 & -0.6271 & -0.3102i & 0.1178 & -0.0994i & -0.2245 & -0.2686i \\
0.3929 & 0.3320 & 0.0156i & -0.1620 & -0.2950i & -0.1267 & -0.3353i \\
0.3709 & -0.1842 & 0.2868i & -0.1199 & 0.1069i & -0.0224 & -0.2699i \\
0.1468 & 0.3709 & 0.2029i & 0.3572 & -0.0915i & -0.0936 & -0.4318i \\
0.4444 & -0.0220 & -0.1704i & -0.0651 & 0.4328i & -0.3159 & 0.3254i
\end{pmatrix}$$ \hspace{1cm} (9)

Here $U_{7}$ is written considering the logical basis $|\psi_{j}\rangle$ as the path-encoding strategy shown in Fig. 3, where $j = 1, \ldots, 7$ denotes the $j$th core. Taking into account the unitary operation $U_{7}$, our MCF-based devices can implement finite sets of inequivalent rank-one POVM elements, as we explain next. We consider the scenario depicted in Fig. 1 in the main text, where a POVM is realized acting upon an input quantum state $|\psi_{j}\rangle$ of dimension $d$ lower than $D$. In our experiment, $D = 7$ corresponds to all the spatial modes available in the $7 \times 7$ MBS. Then, the remaining $D - d$ cores define the ancilla system used to extend the Hilbert space dimension of the input state. Likewise, according to Neumark’s dilation theorem, the action of the POVM is obtained by performing a projective measurement over the total system (that is, detecting all cores). The total number of feasible POVMs we can implement is determined by the number of subsets of $d$ elements chosen from a set of $D$ elements, that is,

$$\binom{D}{d} = \frac{D!}{d!(D-d)!}.$$  

Then, the rank-one POVM elements can be written as $H_{j} = |\psi_{j}\rangle\langle \psi_{j}|$ where

$$|\psi_{j}\rangle = M_{k_{1}, \ldots, k_{d}} |\psi_{i}$$

are unnormalized states. The matrix $M_{k_{1}, \ldots, k_{d}}$ is the $U_{7}$ restriction on the subspace spanned by the $d$ input modes labelled by $k_{1}, \ldots, k_{d}$. Furthermore, we are able to modify local phases connecting phase modulators at the excited input cores. This operation can be seen as a diagonal matrix $\Phi_{k_{1}, \ldots, k_{d}}^{T}$ whose entries represent the phase applied on the input modes, as is also depicted in Fig. 1 in the main text. Phase modulation expands the possibilities to implement even more general POVMs. Nonetheless, no phase modulation at the POVM implementation is required for our task, where we use the stabilization system to ensure this zero-phase condition at the detection stage. We recall that this stabilization scheme actively controls any long-term phase drifts.
that may appear during an experimental run (see the main text for more details).

Then, there are 35 different POVMs that we can implement while four-dimensional states \((d = 4)\) are considered. For the specific task to estimate the value of \(W\), it is necessary to find the best POVM to surpass the projective limit \(W_{\text{proj}}\); we have run optimizations considering these 35 POVMs, finding that the optimal one is

\[
M_{\text{opt}} = 
\begin{bmatrix}
0.3749 & 0.3662 & 0.3501 & 0.3584 \\
0.4918 & -0.2263 - 0.3188i & -0.05239 + 0.1049i & -0.3021 - 0.1664i \\
0.09054 & 0.3586 + 0.2097i & 0.1473 - 0.2053i & -0.01789 - 0.3615i \\
0.3998 & -0.06171 + 0.1833i & -0.3041 + 0.2693i & 0.4521 - 0.1977i \\
0.2709 & 0.4418 & 0.4535 & -0.05134 - 0.02569i - 0.1856 + 0.5787i - 0.2055 - 0.2405i \\
-0.7187 + 0.07419i & 0.1132 + 0.261i & -0.145 - 0.02382i & 0.09788 + 0.335i - 0.03915 - 0.1297i - 0.4235 - 0.2733i \\
\end{bmatrix}
\]  

which is implemented encoding the input four-dimensional quantum state exciting the cores 4, 5, 6, and 7 at the input of the MBS (Fig. 5). The full list of the 35 different POVMs is available in Supplementary Information Section III.

Phase stabilization of Interferometer

Independently of the measurement configuration adopted, the four-arm interferometer is distributed in a small area of 30 cm \(\times\) 30 cm with all components thermally insulated to minimize random phase drifts. Nonetheless, phase drifts are always present and a control system is required to actively compensate for them. The stabilization system relies on a perturb and observe power point tracking method applied to Alice’s phase modulators and is described in more detail in ref. \(^{38}\). The control FPGA devices toggle between the stabilization mode and the experiment mode. In the stabilization mode, Alice’s FPGA monitors and compensates for phase drift of the interferometer every 0.2 s until a near-zero phase relation is achieved. When this occurs, the system is switched to the experiment mode in which the desired states are prepared and measured over 0.1 s. Then, the system returns to the stabilization mode and the process is repeated. Typically, a total of approximately 10,000 single counts are observed over 0.1 s for each measurement configuration. This procedure allows us to achieve the greater than 99% interference visibility required for the certification scheme. This phase stabilization process was sufficient for our tabletop device. In a quantum communication scenario, faster compensation and/or monitoring via auxiliary lasers might be required to achieve adequate stability.

Data availability

Source data are provided with this paper. Any additional data related to the findings of this paper are available upon reasonable request.

Code availability

Computer codes related to the findings of this paper are available upon request.

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Author contributions

D.M., E.S.G. and J.C. performed the experiment and analyzed the data under the supervision of S.P.W. and G.L. A.D. and L.P. theoretically modelled the quantum device. A.T developed the theory of the protocol. All authors contributed to the writing of the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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Correspondence and requests for materials should be addressed to Armin Tavakoli.

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