ENERGETIC VARIATIONAL APPROACHES FOR MULTIPHASE FLOW SYSTEMS WITH PHASE TRANSITION

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Abstract. We study the governing equations for the motion of the fluid particles near air-water interface from an energetic point of view. Since evaporation and condensation phenomena occur at the interface, we have to consider phase transition. This paper applies an energetic variational approach to derive multiphase flow systems with phase transition, where a multiphase flow means compressible and incompressible two-phase flow. We also research the conservation and energy laws of our system. The key ideas of deriving our systems are to acknowledge the existence of the interface and to apply an energetic variational approach. More precisely, we assume that both the coefficient of surface tension and the density of the interface are constants, and we apply an energetic variational approach to look for the dominant equations for the densities of our multiphase flow systems with phase transition. As applications, we can derive the usual Euler and Navier-Stokes systems, or a two-phase flow system with surface tension by our methods.

1. Introduction

We are interested in the motion of the fluid particles near the boundary between the atmosphere and the ocean. We call the boundary the air-water interface. Since evaporation and condensation phenomena occur at the interface, we have to study multiphase flow with phase transition in order to understand air-sea interaction. This paper considers the governing equations for the motion of the fluid particles in two moving domains and the interface from an energetic point of view. We employ an energetic variational approach to derive our multiphase flow systems with phase transition. Of course, this paper proposes our system as one of the models for phase transition.

Let us first introduce fundamental notations. Let \( t \geq 0 \) be the time variable, and \( x(= t(x_1, x_2, x_3)) \in \mathbb{R}^3 \) the spatial variable. Fix \( T > 0 \). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with a smooth boundary \( \partial \Omega \). The symbol \( n_\Omega = n_\Omega(x) = \mathbf{i}(n^1_\Omega, n^2_\Omega, n^3_\Omega) \) denotes the unit outer normal vector at \( x \in \partial \Omega \). Let \( \Omega_A(t)(= \{ \Omega_A(t) \}_{0 \leq t < T}) \) be a bounded domain in \( \mathbb{R}^3 \) with a moving boundary \( \Gamma(t) \). Assume that \( \Gamma(t)(= \{ \Gamma(t) \}_{0 \leq t < T}) \) is a smoothly evolving surface and is a closed Riemannian 2-dimensional manifold. The symbol \( n_\Gamma = n_\Gamma(x, t) = \mathbf{i}(n^1_\Gamma, n^2_\Gamma, n^3_\Gamma) \) denotes the unit outer normal vector at \( x \in \Gamma(t) \). For each \( t \in [0, T) \), assume that \( \Omega_A(t) \subset \Omega \). Set

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\( \Omega_B(t) = \Omega \setminus \overline{\Omega_A(t)} \). It is clear that \( \Omega = \Omega_A(t) \cup \Gamma(t) \cup \Omega_B(t) \) (see Figure 1). Set
\[
(1.1) \quad \Omega_{A,T} = \bigcup_{0 < t < T} \{ \Omega_A(t) \times \{ t \} \}, \quad \Omega_{B,T} = \bigcup_{0 < t < T} \{ \Omega_B(t) \times \{ t \} \},
\]
\[
\Gamma_T = \bigcup_{0 < t < T} \{ \Gamma(t) \times \{ t \} \}, \quad \Omega_T = \Omega \times (0, T), \quad \partial \Omega_T = \partial \Omega \times (0, T).
\]

In this paper we assume that the fluid in \( \Omega_{A,T} \) is an incompressible one, and that the fluid in \( \Omega_{B,T} \) is a compressible one. Let us state physical notations. Let \( \rho_A = \rho_A(x, t), v_A = v_A(x, t) = (v_1^A, v_2^A, v_3^A), \pi_A = \pi_A(x, t), \) and \( \mu_A = \mu_A(x, t) \) be the density, the velocity, the pressure, and the viscosity of the fluid in \( \Omega_A(t) \), respectively. Let \( \rho_B = \rho_B(x, t), v_B = v_B(x, t) = (v_1^B, v_2^B, v_3^B), \pi_B = \pi_B(x, t), \) and \( \mu_B = \mu_B(x, t), \lambda_B = \lambda_B(x, t) \) be the density, the velocity, the pressure, and two viscosities of the fluid in \( \Omega_B(t) \), respectively. Let \( v_S = v_S(x, t) = (v_1^S, v_2^S, v_3^S) \) be the motion velocity of the evolving surface \( \Gamma(t) \). The symbol \( \rho_0 > 0 \) denotes the density of the interface \( \Gamma(t) \), and \( \pi_0 \in \mathbb{R} \setminus \{ 0 \} \) denotes the surface tension (coefficient) at \( x \in \Gamma(t) \). We assume that \( \rho_A, v_A, \pi_A, \mu_A, \rho_B, v_B, \pi_B, \mu_B, \lambda_B, v_S \) are smooth functions in \( \mathbb{R}^4 \), and that \( \rho_0, \pi_0 \) are constants.

Let us explain the background and the ansatz of this study. Let us now consider the phase transition phenomenon on ice melting. There exists a layer between the ice and the air. The layer is called a quasi-liquid layer (see Kuroda-Lacmann\[13\], Furukawa-Yamamoto-Kuroda\[6\]). It is well-known that a quasi-liquid layer has both liquid and solid properties. Experiments in Sazaki-Zepeda-Nakatsubo-Yokoyama-Furukawa\[14\] showed that ice particles change into particles in the quasi-liquid layer and then the particles in the layer change into water vapor. A similar process occurs when ice forms. Therefore, we can consider a two-phase problem with a phase transition as a three-phase problem. In this paper, we admit the existence of a surface mass at the interface \( \Gamma(t) \), and assume that the particles at
the interface can change into both particles in $\Omega_A(t)$ and particles in $\Omega_B(t)$ (see Figure 1).

Let us explain the key restriction of mathematical modeling of multiphase flow systems with phase transition. We assume that

\begin{equation}
\begin{aligned}
\text{div} v_A &= 0 \quad \text{in } \Omega_{A,T}, \\
v_B \cdot n_\Omega &= 0 \quad \text{on } \partial \Omega_T,
\end{aligned}
\end{equation}

The condition $\text{div} v_A = 0$ means the incompressibility condition of the fluid in $\Omega_{A,T}$, and $v_B \cdot n_\Omega = 0$ means that fluid particles do not go out of the domain $\Omega$. In general, we assume several jump conditions when we make models for multiphase flow with phase transition. This paper does not use assumptions on jump conditions to derive our systems. See Slattery-Sagis-Oh [19] for several jump conditions on interfacial phenomena.

This paper has three purposes. The first one is to make the following inviscid model for multiphase flow with phase transition:

\begin{equation}
\begin{aligned}
D_t D_A^A \rho_A &= \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_A v_A \right\} \quad \text{in } \Omega_{A,T}, \\
D_t D_B^B \rho_B + (\text{div} v_B)\rho_B &= \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_B v_B \right\} \quad \text{in } \Omega_{B,T}, \\
\rho_A D_t D_A^A v_A + \text{grad} \pi_A &= -\text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_A v_A \right\} v_A \quad \text{in } \Omega_{A,T}, \\
\rho_B D_t D_B^B v_B + \text{grad} \pi_B &= -\text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_B v_B \right\} v_B \quad \text{in } \Omega_{B,T}, \\
\pi_0 H_T n_T - \pi_A n_T + \pi_B n_T &= \delta(0,0,0) \quad \text{on } \Gamma_T
\end{aligned}
\end{equation}

with (1.2). Here $D_t f = \partial_t f + (v_A \cdot \nabla) f$, $D_B f = \partial_t f + (v_B \cdot \nabla) f$, $(v_A \cdot \nabla) f = v_1^A \partial_1 f + v_2^A \partial_2 f + v_3^A \partial_3 f$, $(v_B \cdot \nabla) f = v_1^B \partial_1 f + v_2^B \partial_2 f + v_3^B \partial_3 f$, $\text{div} v_A = \nabla \cdot v_A$, $\text{div} v_B = \nabla \cdot v_B$, $\text{grad} f = \nabla f$, $\nabla^f = \delta(\partial_1, \partial_1, \partial_3)$, $\partial_i = \partial_i / \partial x_i$, and $\partial_i = \partial_i / \partial t$. The symbol $H_T = H_T(x,t)$ denotes the mean curvature in the direction $n_T$ defined by $H_T = -\text{div} T T n_T = -(\partial_1^f n_1^T + \partial_2^f n_2^T + \partial_3^f n_3^T)$, where $\partial_i^f f := \sum_{j=1}^3 (\delta_{ij} - n_i^T n_j^T) \partial_j f = \partial_j f - n_j^T (n_T \cdot \nabla) f$. More precisely, under the restriction (1.2) we apply an energetic variational approach to derive system (1.3). See subsection 4.2 for details. Remark that the motion velocity $v_S$ is given by

$v_S = \frac{1}{\pi_0 H_T} \left\{ \pi_A (v_A \cdot n_T) - \pi_B (v_B \cdot n_T) \right\} n_T$ on $\Gamma_T$

if $H_T \neq 0$ and $P_T v_S = \delta(0,0,0)$. See the proof of Theorem 2.2 in Section 3 for details.

The second one is to make the following viscous model for multiphase flow with phase transition:

\begin{equation}
\begin{aligned}
D_t D_A^A \rho_A &= -\frac{\rho_0}{\pi_0} \text{div} \left( T_A v_A \right) \quad \text{in } \Omega_{A,T}, \\
D_t D_B^B \rho_B + (\text{div} v_B)\rho_B &= -\frac{\rho_0}{\pi_0} \text{div} \left( T_B v_B \right) \quad \text{in } \Omega_{B,T}, \\
\rho_A D_t D_A^A v_A &= \text{div} T_A + \frac{\rho_0}{\pi_0} \text{div} \left( T_A v_A \right) v_A \quad \text{in } \Omega_{A,T}, \\
\rho_B D_t D_B^B v_B &= \text{div} T_B + \frac{\rho_0}{\pi_0} \text{div} \left( T_B v_B \right) v_B \quad \text{in } \Omega_{B,T}, \\
\pi_0 H_T n_T + \bar{T}_A n_T - \bar{T}_B n_T &= \delta(0,0,0) \quad \text{on } \Gamma_T
\end{aligned}
\end{equation}

with (1.4). Here $\bar{T}_A = T_A - \nabla \pi_A$ and $\bar{T}_B = T_B - \nabla \pi_B$. See subsection 4.2 for details.
with
\[
\begin{aligned}
\text{div} v_A &= 0 & \text{in } \Omega_{A,T}, \\
v_A \cdot n_T &= v_S \cdot n_T & \text{on } \Gamma_T, \\
v_B \cdot n_T &= v_S \cdot n_T & \text{on } \Gamma_T,
\end{aligned}
\]
for \(1.5\)
\[
\begin{aligned}
\{T_A &= T_A(\pi_A, v_A) := \mu_A D(v_A) - \pi_A I_{3 \times 3}, \\
T_B &= T_B(\pi_B, v_B) := \mu_B D(v_B) + \lambda_B \text{div} v_B) I_{3 \times 3} - \pi_B I_{3 \times 3}, \\
\tilde{T}_A &= \tilde{T}_A(\pi_A, v_A) := \mu_A (n_T \cdot (n_T \cdot \nabla) v_A) - \pi_A, \\
\tilde{T}_B &= \tilde{T}_B(\pi_B, v_B) := \mu_B (n_T \cdot (n_T \cdot \nabla) v_B) + \lambda_B (\text{div} v_B) - \pi_B.
\end{aligned}
\]
for \(1.6\)

Here \(D(v_A) = \{[\text{div} v_A] + ([\nabla v_A] / 2)\} / 2\) and \(D(v_B) = \{[\text{div} v_B] + ([\nabla v_B] / 2)\} / 2\). The symbol \(I_{3 \times 3}\) denotes the \(3 \times 3\) identity matrix, and \(P_T = P_T(x, t)\) the orthogonal projection to a tangent space defined by \(P_T = I_{3 \times 3} - n_T \otimes n_T\), where \(\otimes\) is the tensor product. More precisely, under the restriction \(1.5\) we apply an energetic variational approach to derive system \(1.4\). See subsection \(4.3\) for details. Remark that the motion velocity \(v_S\) is given by
\[
v_S = \frac{1}{\sigma_0 H_T} (-\tilde{T}_A(v_A \cdot n_T) + \tilde{T}_B(v_B \cdot n_T)) n_T \text{ on } \Gamma_T
\]
if \(H_T \neq 0\) and \(P_T v_S = 4(0, 0, 0)\). See the proof of Theorem \(2.2\) in Section \(3\) for details. See the proof of Theorem \(2.2\) in Section \(3\) for details.

The third one is to investigate the conservation and energy laws, and the conservative form of our systems \(1.3\) and \(1.4\). In fact, any solution to system \(1.4\) with \(1.5\) satisfies that for \(t_1 < t_2\),
\[
\int_{\Omega_A(t_2)} \rho_A(x, t_2) \, dx + \int_{\Omega_B(t_2)} \rho_B(x, t_2) \, dx + \int_{\Gamma(t_2)} \rho_0 \, dH^2_x
\]
\[
= \int_{\Omega_A(t_1)} \rho_A(x, t_1) \, dx + \int_{\Omega_B(t_1)} \rho_B(x, t_1) \, dx + \int_{\Gamma(t_1)} \rho_0 \, dH^2_x,
\]
and
\[
\int_{\Omega_A(t_2)} \frac{1}{2} \rho_A |v_A|^2 \, dx + \int_{\Omega_B(t_2)} \frac{1}{2} \rho_B |v_B|^2 \, dx + \int_{t_1}^{t_2} \int_{\Omega_A(t)} \mu_A |D(v_A)|^2 \, dxdt
\]
\[
+ \int_{t_1}^{t_2} \int_{\Omega_B(t)} (\mu_B |D(v_B)|^2 + \lambda_B |\text{div} v_B|^2) \, dxdt
\]
\[
= \int_{\Omega_A(t_1)} \frac{1}{2} \rho_A |v_A|^2 \, dx + \int_{\Omega_B(t_1)} \frac{1}{2} \rho_B |v_B|^2 \, dx
\]
\[
+ \int_{t_1}^{t_2} \int_{\Gamma(t)} (\text{div} v_S) \rho_0 \, dH^2_x \, dt + \int_{t_1}^{t_2} \int_{\Omega_A(t)} \left( \frac{\rho_0}{2\pi_0} \text{div}(T_A v_A)|v_A|^2 \right) \, dxdt
\]
\[
+ \int_{t_1}^{t_2} \int_{\Omega_B(t)} \left( \text{div} v_B) \pi_B + \frac{\rho_0}{2\pi_0} \text{div}(T_B v_B)|v_B|^2 \right) \, dxdt.
\]
Moreover, any solution to system \(1.3\) with \(1.2\) satisfies \(1.7\) and \(1.8\) with \(\mu_A \equiv \mu_B \equiv \lambda_B \equiv 0\). Here \(|D(v_A)|^2 = D(v_A) : D(v_A)\), \(|D(v_B)|^2 = D(v_B) : D(v_B)\), and \(dH^2_x\) denotes the 2-dimensional Hausdorff measure. We often call \(1.7\) and \(1.8\), the law of conservation of mass and the energy equality, respectively. We
easily check that system (1.4) with $\text{div} v_A = 0$ satisfies the following conservative form:

\[
\begin{aligned}
\partial_t \rho_A + \text{div} \left( \rho_A v_A + \frac{\rho_0}{\pi_0} T_A v_A \right) &= 0 & \quad & \text{in } \Omega_{A,T}, \\
\partial_t \rho_B + \text{div} \left( \rho_B v_B + \frac{\rho_0}{\pi_0} T_B v_B \right) &= 0 & \quad & \text{in } \Omega_{B,T}, \\
\partial_t (\rho_A v_A) + \text{div} (\rho_A v_A \otimes v_A - T_A) &= \tau(t,0,0,0) & \quad & \text{in } \Omega_{A,T}, \\
\partial_t (\rho_B v_B) + \text{div} (\rho_B v_B \otimes v_B - T_B) &= \tau(t,0,0,0) & \quad & \text{in } \Omega_{B,T}.
\end{aligned}
\]

(1.9)

**Remark 1.1.** (i) If we change (1.2) or (1.5) to another restriction, then we can derive another system by applying our approaches.

(ii) If we choose $\rho_0 = 0$, then we derive the usual Euler and Navier-Stokes systems, or a two-phase flow system with surface tension by our approaches.

Let us explain three key ideas of deriving our multiphase flow systems with phase transition. The first point is to acknowledge the existence of the interface, that is, we assume that the density of the interface is a positive constant $\rho_0$. The second point is to divide the condition $(v_A - v_B) \cdot n_T = 0$ into $v_A \cdot n_T = v_S \cdot n_T$ and $v_B \cdot n_T = v_S \cdot n_T$. The third point is to make use of an energetic variational approach. More precisely, we apply an energetic variational approach in order to look for functions $\Phi_A$ and $\Phi_B$ satisfying

\[
\begin{aligned}
D_t^A \rho_A &= \Phi_A & \quad & \text{in } \Omega_{A,T}, \\
D_t^B \rho_B + (\text{div} v_B)\rho_B &= \Phi_B & \quad & \text{in } \Omega_{B,T},
\end{aligned}
\]

and

\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_0 \, dH^2_x \right) = 0.
\]

An energetic variational approach is a method for deriving PDEs by using the forces derived from a variation of energies. Gyarmati [7] applied an energetic variational approach, which had been studied by Strutt [21] and Onsager [14, 15], to make several models for fluid dynamics in domains. Hyon-Kwak-Liu [8] made use of their energetic variational approach to study complex fluid in domains. Koba-Sato [12] applied their energetic variational approach to make their non-Newtonian fluid systems in domains. Koba-Liu-Giga [11] and Koba [9] employed their energetic variational approaches to derive their fluid systems on an evolving closed surface. However, these papers [8, 12, 11, 9] did not consider multiphase flow. This paper improves and modifies their methods in [8, 12, 11, 9] to derive our multiphase flow systems. See Section 4 for details.

Finally, we introduce the results related to this paper. Bothe-Prüss [3, 2] considered multiphase flow with interface effects. In [5], they made their models for multiphase flow with surface tension and viscosities by applying the Boussinesq-Sriven law. In [2], they made use of their jump conditions to make models for multi-component two-phase flow system with phase transition, and to study the individual mass densities of an isothermal mixture of $N$-species in a domain. Although this paper does not consider surface viscosity (surface flow), this paper considers interface effects such as surface tension and phase transition. Note that our models are different from the ones in [3, 2]. See also [11, 9] for models for surface flow.
The outline of this paper is as follows: In Section 2 we state the main results of this paper. In Section 3 we study the law of conservation of mass for multiphase flow with phase transition. In Section 4 we apply an energetic variational approach to make mathematical models for multiphase flow with phase transition. In Section 5 we investigate the conservation and energy laws of our systems. In Appendix, we provide two useful lemmas to derive our systems.

2. Main Results

We first introduce the transport theorems. Then we state the main results.

**Definition 2.1** ($\Omega_T$ is flowed by the velocity fields $(v_A, v_B, v_S)$). We say that $\Omega_T$ is **flowed by the velocity fields** $(v_A, v_B, v_S)$ if for each $0 < t < T$, $f \in C^1(\mathbb{R}^4)$, and $\Lambda \subset \Omega$

\begin{align}
\frac{d}{dt} \int_{\Omega_A(t) \cap \Lambda} f(x, t) \, dx &= \int_{\Omega_A(t) \cap \Lambda} \{D_t^A f + (\text{div} v_A) f\} \, dx, \\
\frac{d}{dt} \int_{\Omega_B(t) \cap \Lambda} f(x, t) \, dx &= \int_{\Omega_B(t) \cap \Lambda} \{D_t^B f + (\text{div} v_B) f\} \, dx, \\
\frac{d}{dt} \int_{\Gamma(t) \cap \Lambda} f(x, t) \, d\mathcal{H}^2 = &\int_{\Gamma(t) \cap \Lambda} \{D_t^S f + (\text{div} v_S) f\} \, d\mathcal{H}^2.
\end{align}

Here $D_t^f = \partial_t f + (v + \nabla)f$, $\text{div} v = \partial_1^f v_1^f + \partial_2^f v_2^f + \partial_3^f v_3^f$, $\partial_3^f f = \partial_3 f - n_j^\Gamma (n_\Gamma \cdot \nabla) f$, where $\ell = A, B, S$, and $j = 1, 2, 3$. Note that $\text{div} v_A = 0$ in this paper.

We often call (2.1), (2.2) the **transport theorems**, and (2.3) the **surface transport theorem**. The derivation of the surface transport theorem can be found in [1, 5, 6]. Throughout this paper we assume that $\Omega_T$ is flowed by the velocity fields $(v_A, v_B, v_S)$.

Now we state the main results of this paper.

**Theorem 2.2** (Laws of conservation of mass).

(i) Assume that $(\rho_A, \rho_B, \rho_0, v_A, v_B, v_S, \pi_A, \pi_B, \pi_0)$ satisfy

\begin{align}
D_t^A \rho_A &= \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_A v_A \right\} \quad \text{in } \Omega_{A,T}, \\
D_t^B \rho_B + (\text{div} v_B) \rho_B &= \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_B v_B \right\} \quad \text{in } \Omega_{B,T}, \\
\pi_0 n_\Gamma \cdot \nu_\Gamma - \pi_A n_\Gamma + \pi_B n_\Gamma &= l(0, 0, 0) \quad \text{on } \Gamma_T,
\end{align}

and (1.6). Then (1.7) holds for all $0 < t_1 < t_2 < T$.

(ii) Assume that $(\rho_A, \rho_B, \rho_0, v_A, v_B, v_S, \pi_A, \pi_B, \pi_0, \mu_A, \mu_B, \lambda_B)$ satisfy

\begin{align}
D_t^A \rho_A &= -\frac{\rho_0}{\pi_0} \text{div} (T_A v_A) \quad \text{in } \Omega_{A,T}, \\
D_t^B \rho_B + (\text{div} v_B) \rho_B &= -\frac{\rho_0}{\pi_0} \text{div} (T_B v_B) \quad \text{in } \Omega_{B,T}, \\
\pi_0 n_\Gamma \cdot \nu_\Gamma - T_A n_\Gamma - T_B n_\Gamma &= l(0, 0, 0) \quad \text{on } \Gamma_T,
\end{align}

and (1.2), where $(T_A, T_B, \tilde{T}_A, \tilde{T}_B)$ are defined by (1.10). Then (1.7) holds for all $0 < t_1 < t_2 < T$.

**Theorem 2.3** (Conservative form, conservation and energy Laws).

(i) Any solution to system (1.3) with (1.2) satisfies (1.7) and (1.8) with $\mu_A \equiv
\[ \mu_B \equiv \lambda_B \equiv 0. \]

(ii) Any solution to system (1.4) with (1.5) satisfies (1.7) and (1.8).

(iii) If \( \text{div} v_A = 0 \) in \( \Omega_{A,T} \), then system (1.4) satisfies the conservative form (1.4). We prove Theorem 2.2 in Section 3 and Theorem 2.3 in Section 5. In Section 4, we derive our systems (1.3) and (1.4).

3. Laws of Conservation of Mass

Let us immediately derive the one of the main results of this paper.

Proof of Theorem 2.2. We first show (i). From (2.4), we have

\[ \pi_0 H_T(v_S \cdot n_T) - \pi_A(v_S \cdot n_T) + \pi_B(v_S \cdot n_T) = 0 \] on \( \Gamma_T \).

Since \( \pi_0 \neq 0 \) by assumption, we use (1.2) to derive

\[ H_T(v_S \cdot n_T) = \frac{1}{\pi_0} \pi_A(v_A \cdot n_T) - \frac{1}{\pi_0} \pi_B(v_B \cdot n_T) \] on \( \Gamma_T \).

Using the transport theorems (2.2) and (2.2), we check that

\[ \frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}^2_x \right) = \int_{\Omega_A(t)} D_t^A \rho_A \, dx + \int_{\Omega_B(t)} \{ D_t^B \rho_B + (\text{div} v_B) \rho_B \} \, dx + \frac{d}{dt} \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}^2_x. \]

Applying the surface transport and divergence theorems (2.3) with (3.1), we find that

\[ \frac{d}{dt} \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}^2_x = \int_{\Gamma(t)} (\text{div} v_S) \rho_0 \, d\mathcal{H}^2_x = - \int_{\Gamma(t)} \rho_0 H_T(v_S \cdot n_T) \, d\mathcal{H}^2_x \]

\[ = - \int_{\Gamma(t)} \frac{\rho_0}{\pi_0} \pi_A(v_A \cdot n_T) \, d\mathcal{H}^2_x - \int_{\partial \Omega_T} \frac{\rho_0}{\pi_0} \pi_B(v_B \cdot n_T) \, d\mathcal{H}^2_x + \int_{\Gamma(t)} \frac{\rho_0}{\pi_0} \pi_B(v_B \cdot n_T) \, d\mathcal{H}^2_x. \]

Note that \( v_B \cdot n_\Omega = 0 \). Using the divergence theorem, we have

\[ \frac{d}{dt} \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}^2_x = - \int_{\Omega_A(t)} \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_A v_A \right\} \, dx - \int_{\Omega_B(t)} \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_B v_B \right\} \, dx. \]

By (3.2), (3.3), and (2.4), we see that

\[ \frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}^2_x \right) = \int_{\Omega_A(t)} D_t^A \rho_A - \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_A v_A \right\} \, dx \]

\[ + \int_{\Omega_B(t)} D_t^B \rho_B + (\text{div} v_B) \rho_B - \text{div} \left\{ \frac{\rho_0}{\pi_0} \pi_B v_B \right\} \, dx = 0. \]

Integrating with respect to \( t \), we have (1.7). Therefore, we see (i).

Before proving (ii) we prepare the following lemma.

Lemma 3.1. If \( P_T v_A = t'(0,0,0) \) and \( P_T v_B = t'(0,0,0) \) on \( \Gamma(t) \), then

\[ D(v_A) v_A \cdot n_T = (n_T \cdot (n_T \cdot \nabla) v_A) (v_A \cdot n_T) \] on \( \Gamma(t) \),

\[ D(v_B) v_B \cdot n_T = (n_T \cdot (n_T \cdot \nabla) v_B) (v_B \cdot n_T) \] on \( \Gamma(t) \).
Proof of Lemma 3.1. We now drive (3.4). Since $P_tv_A = t(0,0,0)$, we find that $v_A = P_tv_A + (v_A \cdot n_T)n_T = (v_A \cdot n_T)n_T$ on $\Gamma(t)$.

From $D(v_A) = \{t(\nabla v_A), (\nabla v_A)/2\}$, we easily check that

\[
D(v_A)v_A \cdot n_T = \{((v_A \cdot \nabla)v_A) \cdot n_T + ((n_T \cdot \nabla)v_A) \cdot v_A\}/2
\]

\[
= \{(v_A \cdot n_T)\nabla(v_A) \cdot n_T + ((n_T \cdot \nabla)v_A) \cdot [(v_A \cdot n_T)n_T]\}/2
\]

\[
= (n_T \cdot (n_T \cdot \nabla)v_A)(v_A \cdot n_T)
\]

on $\Gamma(t)$,

which is (3.4). Similarly, we see (3.5). Therefore, the lemma follows. □

Now we attack (ii). By Lemma 3.1 and (1.5), we see that

\[
\begin{aligned}
\mathcal{T}_Av_A \cdot n_T &= \tilde{T}_A(v_A \cdot n_T) \text{ on } \Gamma(t), \\
\mathcal{T}_Bv_B \cdot n_T &= \tilde{T}_B(v_B \cdot n_T) \text{ on } \Gamma(t).
\end{aligned}
\]

(3.6)

From (2.5) and (1.5), we check that

\[
\pi_0H_G(v_S \cdot n_T) = -\tilde{T}_A(v_S \cdot n_T) + \tilde{T}_B(v_S \cdot n_T)
\]

\[
= -\tilde{T}_A(v_A \cdot n_T) + \tilde{T}_B(v_B \cdot n_T) \text{ on } \Gamma(t).
\]

(3.7)

By (3.6), we have

\[

H_G(v_S \cdot n_T) = -\frac{1}{\pi_0}(\mathcal{T}_Av_A \cdot n_T) + \frac{1}{\pi_0}(\mathcal{T}_Bv_B \cdot n_T) \text{ on } \Gamma(t).
\]

Applying the surface transport theorem (2.3), the surface divergence theorem (6.1), (3.7), and $v_B|_{\partial\Omega} = t(0,0,0)$, we see that

\[
\frac{d}{dt} \int_{\Gamma(t)} \rho_0 d\mathcal{H}_x^2 = \int_{\Gamma(t)} \rho_0(\text{div}_Gv_S) d\mathcal{H}_x^2 = \int_{\Gamma(t)} \rho_0H_G(v_S \cdot n_T) d\mathcal{H}_x^2
\]

\[
= \frac{\rho_0}{\pi_0} \int_{\Gamma(t)} \mathcal{T}_Av_A \cdot n_T d\mathcal{H}_x^2 + \frac{\rho_0}{\pi_0} \int_{\partial\Omega} \mathcal{T}_Bv_B \cdot n_T d\mathcal{H}_x^2 - \frac{\rho_0}{\pi_0} \int_{\partial\Omega} \mathcal{T}_Bv_B \cdot n_T d\mathcal{H}_x^2.
\]

Using the divergence theorem, we check that

\[
\frac{d}{dt} \int_{\Gamma(t)} \rho_0 d\mathcal{H}_x^2 = \int_{\Omega_A(t)} \frac{\rho_0}{\pi_0} \text{div}(\mathcal{T}_Av_A) dx + \int_{\Omega_B(t)} \frac{\rho_0}{\pi_0} \text{div}(\mathcal{T}_Bv_B) dx.
\]

By the same argument as in (i), we see (ii). Therefore, Theorem 2.2 is proved. □

4. Mathematical Modeling

In this section we make mathematical models for multiphase flow with phase transition. We apply our energetic variational approaches to derive system (1.4) in subsection 4.1 and system (1.3) in subsection 4.2.

4.1. Viscous Model. Under the restriction (1.5), we apply an energetic variational approach to derive system (1.4). We assume that $(v_A, v_B, v_S)$ satisfies (1.5).

Let $\Phi_A, \Phi_B \in C(\mathbb{R}^4)$. We assume that the dominant equations for the densities of our system are written by

\[
\begin{aligned}
D_t^\Omega \rho_A &= \Phi_A \quad \text{ in } \Omega_A, \\
D_t^\Omega \rho_B + (\text{div}_G\rho_B) &= \Phi_B \quad \text{ in } \Omega_B.
\end{aligned}
\]

(4.1)
From now we look for \( \Phi_A, \Phi_B \) satisfying
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_o \, d\mathcal{H}^2_x \right) = 0
\]
for all \( t \in (0,T) \) by applying an energetic variational approach.

In order to derive the momentum equations of our system, we now discuss the variation of the velocities \((v_A, v_B, v_S)\) to the work and dissipation energies for our viscous model. Fix \( 0 < t < T \). We set the work \( E_W \) done by pressures \( \pi_B \) and \( \pi_o \), and the dissipation energies \( E_D \) due to viscosities \((\mu_A, \mu_B, \lambda_B)\) as follows:
\[
E_W[v_A, v_B, v_S] = \int_{\Omega_A(t)} (\nabla v_B) \pi_B \, dx + \int_{\Gamma(t)} (\nabla v_S) \pi_o \, d\mathcal{H}^2_x,
\]
\[
E_D[v_A, v_B, v_S] = \int_{\Omega_A(t)} \left( -\frac{\mu_A}{2} |D(v_A)|^2 \right) \, dx + \int_{\Omega_B(t)} \left( -\frac{\mu_B}{2} |D(v_B)|^2 - \frac{\lambda_B}{2} |\nabla v_B|^2 \right) \, dx.
\]

Set \( E_{D+W}[] = E_D[] + E_W[] \).

**Remark 4.1.**
(i) From \( \nabla v_A = 0 \) we see that \( (\nabla v_A) \pi_A = 0 \).
(ii) Collectively, we call \((\nabla v_B) \pi_B, (\nabla v_S) \pi_o, \mu_A |D(v_A)|^2, \mu_B |D(v_B)|^2, \lambda_B |\nabla v_B|^2\) the energy densities. See [12] and [9] for mathematical validity of the energy densities.

We consider the variation of \( E_{D+W} \) with respect to the velocities \((v_A, v_B, v_S)\).

Let \( \varphi_A, \varphi_B, \varphi_S \in [C^\infty(\mathbb{R}^3)]^3 \). For \( -1 < \varepsilon < 1 \), \( v_A^\varepsilon := v_A + \varepsilon \varphi_A, \ v_B^\varepsilon := v_B + \varepsilon \varphi_B, \ v_S^\varepsilon := v_S + \varepsilon \varphi_S \). We call \((v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon)\) a variation of \((v_A, v_B, v_S)\).

Let \( E_{D+W}[v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon] := \int_{\Omega_B(t)} \left\{ (\nabla v_B^\varepsilon) \pi_B - \frac{\mu_B}{2} |D(v_B^\varepsilon)|^2 - \frac{\lambda_B}{2} |\nabla v_B^\varepsilon|^2 \right\} \, dx \]
\[
+ \int_{\Omega_A(t)} \left\{ -\frac{\mu_A}{2} |D(v_A^\varepsilon)|^2 \right\} \, dx + \int_{\Gamma(t)} (\nabla v_S^\varepsilon) \pi_o \, d\mathcal{H}^2_x.
\]

A direct calculation gives
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} E_{D+W}[v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon] = \int_{\Omega_B(t)} \{(\nabla \varphi_B) \pi_B - \mu_B D(v_B) : D(\varphi_B) - \lambda_B (\nabla v_B)(\nabla \varphi_B)\} \, dx
\]
\[
+ \int_{\Omega_A(t)} \{-\mu_A D(v_A) : D(\varphi_A)\} \, dx + \int_{\Gamma(t)} (\nabla \varphi_S) \pi_o \, d\mathcal{H}^2_x.
\]

From [15], we assume that for \( -1 < \varepsilon < 1 \),
\[
\begin{align*}
\text{in } \Omega_A(t), & \quad v_A^\varepsilon = 0, \\
\text{on } \Gamma(t), & \quad v_A^\varepsilon \cdot n = v_S^\varepsilon \cdot n, \\
\text{on } \Gamma(t), & \quad P_tv_A^\varepsilon = t(0,0,0) 
\end{align*}
\]

\[
\begin{align*}
\text{and } \Omega_B(t), & \quad v_B^\varepsilon = 0, \\
\text{on } \Gamma(t), & \quad v_B^\varepsilon \cdot n = v_S^\varepsilon \cdot n, \\
\text{on } \Gamma(t), & \quad P_tv_B^\varepsilon = t(0,0,0)
\end{align*}
\]

\[
\begin{align*}
\text{in } \Omega_A(t), & \quad v_B^\varepsilon = 0, \\
\text{on } \Gamma(t), & \quad v_B^\varepsilon \cdot n = v_S^\varepsilon \cdot n, \\
\text{on } \Gamma(t), & \quad P_tv_B^\varepsilon = t(0,0,0)
\end{align*}
\]

\[
\begin{align*}
\text{in } \Omega_A(t), & \quad v_S^\varepsilon = 0, \\
\text{on } \Gamma(t), & \quad v_S^\varepsilon \cdot n = v_S^\varepsilon \cdot n, \\
\text{on } \Gamma(t), & \quad P_tv_S^\varepsilon = t(0,0,0)
\end{align*}
\]

\[
\begin{align*}
\text{in } \Omega_A(t), & \quad v_S^\varepsilon = 0, \\
\text{on } \Gamma(t), & \quad v_S^\varepsilon \cdot n = v_S^\varepsilon \cdot n, \\
\text{on } \Gamma(t), & \quad P_tv_S^\varepsilon = t(0,0,0)
\end{align*}
\]
Then we have
\[
\begin{aligned}
\begin{cases}
\text{div} \varphi_A = 0 & \text{in } \Omega_A(t), \\
\varphi_A \cdot n_T = \varphi_S \cdot n_T & \text{on } \Gamma(t), \\
\varphi_B \cdot n_T = \varphi_S \cdot n_T & \text{on } \Gamma(t),
\end{cases}
\end{aligned}
\]
(4.2)
\[
\begin{cases}
\varphi_B = \mathbf{t}(0,0,0) & \text{on } \partial \Omega,
\end{cases}
\]
Proof of Lemma 4.2. By assumption, we find that for all \(\varphi_A, \varphi_B, \varphi_S \in [C^\infty(\mathbb{R}^3)]^3\) satisfying (4.2),
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} E_{D+W}[v_A^{\varepsilon}, v_B^{\varepsilon}, \varphi_S^{\varepsilon}] = \int_{\Omega_A(t)} F_A \cdot \varphi_A \, dx + \int_{\Omega_B(t)} F_B \cdot \varphi_B \, dx + \int_{\Gamma(t)} F_S \cdot \varphi_S \, d\mathcal{H}_2.
\]
Then there is \(\pi_A \in C^1(\overline{\Omega_A(t)})\) such that
\[
\begin{aligned}
\begin{cases}
F_A = \text{div} \mathcal{T}_A(v_A, \pi_A) & \text{in } \Omega_A(t), \\
F_B = \text{div} \mathcal{T}_B(v_B, \pi_B) & \text{in } \Omega_B(t), \\
F_S = -\pi_0 H_T n_T - \mathcal{T}_A(v_A, \pi_A) n_T + \mathcal{T}_B(v_B, \pi_B) n_T & \text{on } \Gamma(t).
\end{cases}
\end{aligned}
\]
(4.3)
Here \((\mathcal{T}_A, \mathcal{T}_B, \tilde{\mathcal{T}}_A, \tilde{\mathcal{T}}_B)\) is defined by (1.6).
Proof of Lemma 4.2. By assumption, we find that for all \(\varphi_A, \varphi_B, \varphi_S \in [C^\infty(\mathbb{R}^3)]^3\) satisfying (4.2),
\[
\int_{\Omega_A(t)} \{(\text{div} \varphi_B) \pi_B - \mu_B D(v_B) : D(\varphi_B) - \lambda_B (\text{div} v_B) (\text{div} \varphi_B)\} \, dx
\]
\[
+ \int_{\Omega_A(t)} -\mu_A D(v_A) : D(\varphi_A) \, dx + \int_{\Gamma(t)} (\text{div} \varphi_S) \pi_0 \, d\mathcal{H}_2
\]
\[
= \int_{\Omega_A(t)} F_A \cdot \varphi_A \, dx + \int_{\Omega_B(t)} F_B \cdot \varphi_B \, dx + \int_{\Gamma(t)} F_S \cdot \varphi_S \, d\mathcal{H}_2.
\]
Using the surface divergence theorem (6.1), integration by parts, and (4.2), we have
\[
\int_{\Omega_B(t)} (-F_B - \nabla \pi_B + \text{div} \{\mu_B D(v_B) + \lambda_B (\text{div} v_B) I_{3 \times 3}\}) \cdot \varphi_B \, dx
\]
\[
+ \int_{\Omega_A(t)} (-F_A + \text{div} \{\mu_A D(v_A)\}) \cdot \varphi_A \, dx
\]
\[
+ \int_{\Gamma(t)} \{-F_S - \pi_0 H_T n_T + \tilde{\mathcal{T}}_B n_T - \mu_A (n_T \cdot (n_T \cdot \nabla) v_A) n_T\} \cdot \varphi_S \, d\mathcal{H}_2 = 0.
\]
Here we used the facts that
\[
D(v_A) n_T \cdot \varphi_A = (n_T \cdot (n_T \cdot \nabla) v_A) (n_T \cdot \varphi_A) \text{ on } \Gamma(t),
\]
\[
D(v_B) n_T \cdot \varphi_B = (n_T \cdot (n_T \cdot \nabla) v_B) (n_T \cdot \varphi_B) \text{ on } \Gamma(t).
\]
We now consider the case when \(\varphi_A = \mathbf{t}(0,0,0)\) and \(\varphi_S = \mathbf{t}(0,0,0)\), that is, for every \(\varphi_B \in [C^\infty(\mathbb{R}^3)]^3\) satisfying \(\varphi_B = \mathbf{t}(0,0,0)\) on \(\partial \Omega\) and \(\varphi_B = \mathbf{t}(0,0,0)\) on \(\Gamma(t),
\]
\[
\int_{\Omega_B(t)} (-F_B - \nabla \pi_B + \text{div} \{\mu_B D(v_B) + \lambda_B (\text{div} v_B) I_{3 \times 3}\}) \cdot \varphi_B \, dx = 0.
\]
This shows that
\[(4.5) \quad F_B = \text{div} T_B(v_B, \pi_B) \text{ in } \Omega_B(t). \]

Next we consider the case when \( \varphi_B = t'(0,0,0) \) and \( \varphi_S = t'(0,0,0) \), that is, for every \( \varphi_A \in \mathcal{C}^\infty(\mathbb{R}^3) \) satisfying \( \varphi_A = t'(0,0,0) \) on \( \Gamma(t) \) and \( \text{div} \varphi_A = 0 \) in \( \Omega_A(t) \),
\[\int_{\Omega_A(t)} (-F_A + \text{div} \{\mu_A D(v_A)\}) \cdot \varphi_A \, dx = 0.\]

Since \( \text{div} \varphi_A = 0 \) in \( \Omega_A(t) \), we apply the Helmholtz-Weyl decomposition (Lemma \[0,2\]) to find that there exists \( \pi_A \in C^1(\overline{\Omega_A(t)}) \) such that
\[(4.6) \quad -F_A + \text{div} \{\mu_A D(v_A)\} = \nabla \pi_A \text{ in } \Omega_A(t).\]

Finally, we consider the case when \( \varphi_S \neq t'(0,0,0) \). By \[4.4\], \[4.5\], \[4.6\], we see that
\[\int_{\Omega_A(t)} (\nabla \pi_A) \cdot \varphi_A \, dx + \int_{\Gamma(t)} \{-F_S - \pi_0 H_T n_T + \overline{T}_B n_B - \mu_A (n_A \cdot (n_A \cdot \nabla) v_A) n_A\} \cdot \varphi_S \, d\mathcal{H}_x^2 = 0.\]

Using integration by parts with \( \text{div} \varphi_A = 0 \) and \( \varphi_A \cdot n_A = \varphi_S \cdot n_A \), we have
\[\int_{\Gamma(t)} (-F_S - \pi_0 H_T n_T - \overline{T}_A n_A + \overline{T}_B n_B) \cdot \varphi_S \, d\mathcal{H}_x^2 = 0.\]

Since the above equality holds for all \( \varphi_S \in \mathcal{C}^\infty(\mathbb{R}^3) \), we see that
\[(4.7) \quad F_S = -\pi_0 H_T n_T - \overline{T}_A n_A + \overline{T}_B n_B \text{ on } \Gamma(t).\]

Therefore, Lemma \[4.2\] is proved. \( \square \)

Now we return to derive our momentum equations. We admit the fundamental principle of the dynamics of fluid motion (see Chapter B in \[17\]). We assume that the time rate of change of the momentum equals to the forces derived from the variation of the work done by pressures and the energies dissipation due to viscosities, that is, suppose that for every \( 0 < t < T \) and \( \Lambda \subset \Omega \),
\[
\frac{d}{dt} \int_{\Omega_A(t) \cap \Lambda} \rho_A v_A \, dx = \int_{\Omega_A(t) \cap \Lambda} F_A \, dx,
\frac{d}{dt} \int_{\Omega_B(t) \cap \Lambda} \rho_B v_B \, dx = \int_{\Omega_B(t) \cap \Lambda} F_B \, dx,
0 = \int_{\Gamma(t) \cap \Lambda} F_S \, d\mathcal{H}_x^2.
\]

Here \( F_A, F_B, F_S \) is defined by \[4.3\]. Remark that we do not consider the momentum on the surface \( \Gamma(t) \) since we do not consider surface flow. Applying the transport theorems with \[4.1\], we have
\[
\begin{align*}
\rho_A D^A_t v_A + \Phi_A v_A = F_A & \quad \text{ in } \Omega_{A,T}, \\
\rho_B D^B_t v_B + \Phi_B v_B = F_B & \quad \text{ in } \Omega_{B,T}, \\
\pi_0 H_T n_T + \overline{T}_A n_A - \overline{T}_B n_B = t'(0,0,0) & \quad \text{ in } \Gamma_T.
\end{align*}
\]
Using the transport theorems \(2.1, 2.3\) with \(4.1\), we check that
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x, t) \, dx + \int_{\Omega_B(t)} \rho_B(x, t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}_x^2 \right)
= \int_{\Omega_A(t)} \Phi_A \, dx + \int_{\Omega_B(t)} \Phi_B \, dx + \int_{\Gamma(t)} \rho_0 \text{div}_\Gamma v_S \, d\mathcal{H}_x^2.
\]
By the same argument in the proof of Theorem \(2.2\), we see that
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x, t) \, dx + \int_{\Omega_B(t)} \rho_B(x, t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}_x^2 \right)
= \int_{\Omega_A(t)} \left( \Phi_A + \frac{\rho_0}{\pi_0} \text{div}(T_A v_A) \right) \, dx + \int_{\Omega_B(t)} \left( \Phi_B + \frac{\rho_0}{\pi_0} \text{div}(T_B v_B) \right) \, dx.
\]
Thus, we set \(\Phi_A = -\frac{\rho_0}{\pi_0} \text{div}(T_A v_A)\) and \(\Phi_B = -\frac{\rho_0}{\pi_0} \text{div}(T_B v_B)\) to derive
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x, t) \, dx + \int_{\Omega_B(t)} \rho_B(x, t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}_x^2 \right) = 0.
\]
Therefore, combining \(4.1, 4.3,\) and \(4.8\), we have system \(1.4\).

4.2. Inviscid Model. Under the restriction \(1.2\) we apply an energetic variational approach to derive system \(1.3\). We assume that \((v_A, v_B, v_S)\) satisfy \(1.2\).

Let \(\Psi_A, \Psi_B \in C(\mathbb{R}^4)\). We assume that the dominant equations for the densities of our system are written by
\[
(4.9) \quad \begin{cases} D_t^A \rho_A = \Psi_A & \text{in } \Omega_{A,T}, \\ D_t^B \rho_B + \text{div}_B \rho_B = \Psi_B & \text{in } \Omega_{B,T}. \end{cases}
\]

From now we look for \(\Psi_A, \Psi_B\) satisfying
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x, t) \, dx + \int_{\Omega_B(t)} \rho_B(x, t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}_x^2 \right) = 0
\]
by applying an energetic variational approach.

In order to derive the momentum equations of our system, we now discuss the variation of the velocities \((v_A, v_B, v_S)\) to the work for our inviscid model. Fix \(0 < t < T\). We set the work \(E_W\) done by pressures \(\pi_B\) and \(\pi_0\) as follows:
\[
E_W[v_A, v_B, v_S] = \int_{\Omega_B(t)} (\text{div} v_B) \pi_B \, dx + \int_{\Gamma(t)} (\text{div}_\Gamma v_S) \pi_0 \, d\mathcal{H}_x^2.
\]
We consider the variation of the work \(E_W\) with respect to the velocities \((v_A, v_B, v_S)\). Let \(0 < t < T\). Let \(\varphi_A, \varphi_B, \varphi_S \in C^\infty(\mathbb{R}^3)^3\). For \(-1 < \varepsilon < 1\), \(v_A^\varepsilon := v_A + \varepsilon \varphi_A\), \(v_B^\varepsilon := v_B + \varepsilon \varphi_B\), \(v_S^\varepsilon := v_S + \varepsilon \varphi_S\). We call \((v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon)\) a variation of \((v_A, v_B, v_S)\).

For each variation \((v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon)\),
\[
E_W[v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon] := \int_{\Omega_B(t)} (\text{div} v_B^\varepsilon) \pi_B \, dx + \int_{\Gamma(t)} (\text{div}_\Gamma v_S^\varepsilon) \pi_0 \, d\mathcal{H}_x^2.
\]
From \(1.2\), we assume that for \(-1 < \varepsilon < 1\),
\[
\begin{cases} \text{div} v_A^\varepsilon = 0 & \text{in } \Omega_A(t), \\ v_B^\varepsilon \cdot n_\Omega = 0 & \text{on } \partial \Omega, \\ v_B^\varepsilon \cdot n_\Gamma = v_S^\varepsilon \cdot n_\Gamma & \text{on } \Gamma(t), \end{cases}
\]
Then we have
\[
\begin{align*}
\text{(4.10)} \quad & \begin{cases}
\text{div} \varphi_A = 0 & \text{in } \Omega_A(t), \\
\varphi_A \cdot n_G = \varphi_S \cdot n_G & \text{on } \Gamma(t), \\
\varphi_B \cdot n_G = 0 & \text{on } \partial \Omega , \\
\varphi_B \cdot n_G = \varphi_S \cdot n_G & \text{on } \Gamma(t).
\end{cases}
\end{align*}
\]

Now we study the forces derived from a variation of the work.

**Lemma 4.3.** Let \( 0 < t < T \), and let \( G_A, G_B, G_S \in [C(\mathbb{R}^3)]^3 \). Assume that for every \( \varphi_A, \varphi_B, \varphi_S \in [C^\infty(\mathbb{R}^3)]^3 \) satisfying (4.10),
\[
\left. \frac{d}{d\varepsilon} \right|_{\varepsilon = 0} E_W[v_A^\varepsilon, v_B^\varepsilon, v_S^\varepsilon] = \int_{\Omega_A(t)} G_A \cdot \varphi_A \, dx + \int_{\Omega_B(t)} G_B \cdot \varphi_B \, dx + \int_{\Gamma(t)} G_S \cdot \varphi_S \, d\mathcal{H}_2^2.
\]

Then there is \( \pi_A \in C^1(\overline{\Omega_A(t)}) \) such that
\[
\left\{ \begin{array}{ll}
G_A = -\text{grad} \pi_A & \text{in } \Omega_A(t), \\
G_B = -\text{grad} \pi_B & \text{in } \Omega_B(t), \\
G_S = -\pi_0 H_G n_G + \pi_A n_G - \pi_B n_G & \text{on } \Gamma(t).
\end{array} \right.
\]

By the same arguments in the proof of Lemma 4.2, we can prove Lemma 4.3.

Now we derive our momentum equations. We admit the principle of conservation of linear momentum (see Chapter B in [17]). We assume that the time rate of change of the momentum equals to the forces derived from the work done by pressures, that is, suppose that for every \( 0 < t < T \) and \( \Lambda \subset \Omega \),
\[
\begin{align*}
\frac{d}{dt} \int_{\Omega_A(t) \cap \Lambda} \rho_A v_A \, dx &= \int_{\Omega_A(t) \cap \Lambda} G_A \, dx, \\
\frac{d}{dt} \int_{\Omega_B(t) \cap \Lambda} \rho_B v_B \, dx &= \int_{\Omega_B(t) \cap \Lambda} G_B \, dx, \\
0 &= \int_{\Gamma(t) \cap \Lambda} G_S \, d\mathcal{H}_2^2.
\end{align*}
\]

Here \( (G_A, G_B, G_S) \) is defined by (4.11). Remark that we do not consider the momentum on the surface \( \Gamma(t) \) since we do not consider surface flow. Applying the transport theorems with (4.9), we have
\[
\left\{ \begin{array}{ll}
\rho_A \text{div}^I v_A + \Psi_A v_A = G_A & \text{in } \Omega_{A,T}, \\
\rho_B \text{div}^I v_B + \Psi_B v_B = G_B & \text{in } \Omega_{B,T}, \\
\pi_0 H_G n_G - \pi_A n_G + \pi_B n_G = \mathcal{I}(0,0,0) & \text{in } \Gamma_T.
\end{array} \right.
\]

Using the transport theorems (2.1)-(2.3) with (4.9), we check that
\[
\begin{align*}
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}_2^2 \right) &= \int_{\Omega_A(t)} \Psi_A \, dx + \int_{\Omega_B(t)} \Psi_B \, dx + \int_{\Gamma(t)} \rho_0 (\text{div}_G v_S) \, d\mathcal{H}_2^2.
\end{align*}
\]

By the same argument in the proof of Theorem 2.22 we see that
\[
\begin{align*}
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}_2^2 \right) &= \int_{\Omega_A(t)} \Psi_A - \text{div} \left( \frac{\rho_0}{\pi_A} \pi_A v_A \right) \, dx + \int_{\Omega_B(t)} \Psi_B - \text{div} \left( \frac{\rho_0}{\pi_B} \pi_B v_B \right) \, dx.
\end{align*}
\]
Thus, we set $\Psi_A = \text{div}\{\frac{\partial}{\partial_0} \pi_A v_A\}$ and $\Psi_B = \text{div}\{\frac{\partial}{\partial_0} \pi_B v_B\}$ to derive
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x,t) \, dx + \int_{\Omega_B(t)} \rho_B(x,t) \, dx + \int_{\Gamma(t)} \rho_0 \, d\mathcal{H}^2_x \right) = 0.
\]
Combining (4.9), (4.11), and (4.12), therefore, we have system (1.3).

5. CONSERVATION AND ENERGY LAWS

Applying the transport theorems and integration by parts, we can prove Theorem 2.3. Therefore, we only show that any solution to system (1.4) with (1.5) satisfies (1.8).

Applying the transport theorems (2.1), (2.2), and system (1.4), we see that

\begin{equation}
\frac{d}{dt} \left( \int_{\Omega_A(t)} \frac{1}{2} \rho_A |v_A|^2 \, dx + \int_{\Omega_B(t)} \frac{1}{2} \rho_B |v_B|^2 \, dx \right) \\
= \int_{\Omega_A(t)} \left( \frac{1}{2} |v_A|^2 (D_t^A \rho_A) + \rho_A D_t^A v_A \cdot v_A \right) \, dx \\
+ \int_{\Omega_B(t)} \left( \frac{1}{2} |v_B|^2 (D_t^B \rho_B) + (\text{div} v_B) \rho_B + \rho_B D_t^B v_B \cdot v_B \right) \, dx \\
= \int_{\Omega_A(t)} \left( (\text{div} \tilde{T}_A) \cdot v_A + \frac{\rho_0}{2\pi_0} \text{div}(\tilde{T}_A v_A) |v_A|^2 \right) \, dx \\
+ \int_{\Omega_B(t)} \left( (\text{div} \tilde{T}_B) \cdot v_B + \frac{\rho_0}{2\pi_0} \text{div}(\tilde{T}_B v_B) |v_B|^2 \right) \, dx.
\end{equation}

Using integration by parts with $v_B|_{\partial\Omega} = \tilde{v}(0,0,0)$, we observe that (R.H.S.) of (5.1)
\[
= \int_{\Omega_B(t)} \left( (\text{div} v_B) \pi_B + \frac{\rho_0}{2\pi_0} \text{div}(\tilde{T}_B v_B) |v_B|^2 - \mu_B |D(v_B)|^2 - \lambda_B |\text{div} v_B|^2 \right) \, dx \\
+ \int_{\Omega_A(t)} \left( \frac{\rho_0}{2\pi_0} \text{div}(\tilde{T}_A v_A) |v_A|^2 - \mu_A |D(v_A)|^2 \right) \, dx + \int_{\Gamma(t)} (\text{div} v_S) \pi_0 \, d\mathcal{H}^2_x.
\]

Here we used the fact that
\[
\int_{\Gamma(t)} (\tilde{T}_A n_{\Gamma} \cdot v_A - \tilde{T}_B n_{\Gamma} \cdot v_B) \, d\mathcal{H}^2_x = \int_{\Gamma(t)} (\tilde{T}_A - \tilde{T}_B) v_S \cdot n_{\Gamma} \, d\mathcal{H}^2_x \\
= - \int_{\Gamma(t)} \pi_0 H_{\Gamma} (v_S \cdot n_{\Gamma}) \, d\mathcal{H}^2_x = \int_{\Gamma(t)} (\text{div} v_S) \pi_0 \, d\mathcal{H}^2_x.
\]

Integrating with respect to $t$, we have (1.8). Therefore, Theorem 2.3 is proved.

6. APPENDIX: TOOLS

We introduce useful tools to make a mathematical model for multiphase flow.

Lemma 6.1 (Surface divergence theorem). Let $\Gamma_\ast \subset \mathbb{R}^3$ be a smooth closed 2-dimensional surface. Then for each $V_S \in \mathcal{C}^1(\Gamma_\ast)^3$,
\begin{equation}
\int_{\Gamma_\ast} \text{div}_\gamma V_S \, d\mathcal{H}^2_x = - \int_{\Gamma_\ast} H_{\Gamma_\ast} (V_S \cdot n_{\Gamma_\ast}) \, d\mathcal{H}^2_x.
\end{equation}
Here $H_{\Gamma_*}$ is the mean curvature in the direction $n_{\Gamma_*}$ defined by $H_{\Gamma_*} = -\text{div}_{\Gamma_*} n_{\Gamma_*}$, where $n_{\Gamma_*}(x) = \frac{1}{3}(n_1^\ast, n_2^\ast, n_3^\ast)$ denotes the unit outer normal vector at $x \in \Gamma_*$. The proof of Lemma 6.1 can be founded in Simon [18] and Koba [10].

**Lemma 6.2** (Helmholtz-Weyl decomposition). Let $\Omega_*$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial \Omega_*$. Set

$$C_0^{\infty, \text{div}}(\Omega_*) = \{ \varphi \in [C_0^{\infty}(\Omega_*)]^3; \text{div}\varphi = 0 \}.$$

Let $F_* \in [C(\Omega_*)]^3$. Assume that for each $\varphi \in C_0^{\infty, \text{div}}(\Omega_*)$

$$\int_{\Omega_*} F_* \cdot \varphi \, dx = 0.$$

Then there is $\Pi_* \in C^1(\Omega_*)$ such that $F_* = \nabla \Pi_*$ in $\Omega_*$. The proof of Lemma 6.2 can be founded in Temam [22] and Sohr [20].

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**REFERENCES**

[1] David E. Betounes, *Kinematics of submanifolds and the mean curvature normal*. Arch. Rational Mech. Anal. 96 (1986), no. 1, 1–27. MR0853973
[2] Dieter Bothe and Jan Prüss, *Modeling and analysis of reactive multi-component two-phase flows with mass transfer and phase transition—the isothermal incompressible case*. Discrete Contin. Dyn. Syst. Ser. S 10 (2017), no. 4, 673–696.
[3] Dieter Bothe and Jan Prüss, *On the two-phase Navier-Stokes equations with Boussinesq-Smoluchowski surface fluid*. J. Math. Fluid Mech. 12 (2010), no. 1, 133–150. MR2602917.
[4] Gerhard Dziuk and Charles M. Elliott, *Finite elements on evolving surfaces*. IMA J. Numer. Anal. 27 (2007), no. 2, 262–292. MR2317005.
[5] Morton E. Gurtin, Allan Struthers, and William O. Williams, *A transport theorem for moving interfaces*. Quart. Appl. Math. 47 (1989), no. 4, 737–777. MR1031691
[6] Yoshinori Furukawa, Masaki Yamamoto, and Toshio Kuroda, *Ellipsometric study of the transition layer on the surface of an ice crystal*. Journal of Crystal Growth, 82 (1987), 665-677.
[7] István Gyarmati. *Non-equilibrium Thermodynamics*. Springer, 1970. ISBN: 978-3-642-51067-0
[8] Yunkyong Hyon, Do Y. Kwak, and Chun Liu, *Energetic variational approach in complex fluids: maximum dissipation principle*. Discrete Contin. Dyn. Syst. 26 (2010), no. 4, 1291–1304. MR2600746
[9] Hajime Koba, *On Derivation of Compressible Fluid Systems on an Evolving Surface*. Quart. Appl. Math. 76 (2018), no. 2, 303–359.
[10] Hajime Koba, *On Generalized Diffusion and Heat Systems on an Evolving Surface with a Boundary*. Quart. Appl. Math. 78 (2020), 617-640
[11] Hajime Koba, Chun Liu, and Yoshikazu Giga *Energetic variational approaches for incompressible fluid systems on an evolving surface*. Quart. Appl. Math. 75 (2017), no 2, 359–389. MR3614501. Errata to Energetic variational approaches for incompressible fluid systems on an evolving surface. Quart. Appl. Math. 76 (2018), no 1, 147–152.
[12] Hajime Koba and Kazuki Sato, *Energetic variational approaches for non-Newtonian fluid systems*. Z. Angew. Math. Phys. (2018) 69: 643. https://doi.org/10.1007/s00033-018-1039-1.
[13] T. Kuroda and R. Lacmann, *Growth kinetics of ice from the vapour phase and its growth forms*. Journal of Crystal Growth 56 (1982), 189–205.
[14] Lars Onsager *Reciprocal Relations in Irreversible Processes*. I. Physical Review. (1931):37:405-109 DOI:https://doi.org/10.1103/PhysRev.37.405
Lars Onsager, *Reciprocal Relations in Irreversible Processes*. II. Physical Review. (1931);38:2265-79 DOI:https://doi.org/10.1103/PhysRev.38.2265

Gen Sazaki, Salvador Zepeda, Shunichi Nakatsuho, Etsuro Yokoyama, and Yoshinori Furukawa, *Elementary steps at the surface of ice crystals visualized by advanced optical microscopy*, PNAS 107 (2010), 19702-19707.

J. Serrin, *Mathematical principles of classical fluid mechanics*. 1959 Handbuch der Physik (herausgegeben von S. Flugge), Bd. 8/1, Stromungsmechanik I (Mitherausgeber C. Truesdell) pp. 125–263 Springer-Verlag, Berlin-Göttingen-Heidelberg MR0108116. [Fluid Dynamics I / Stroemungsmechanik I (Handbuch der Physik Encyclopedia of Physics) 2013]. MR0108116

Leon Simon, *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, 3. Australian National University, Centre for Mathematical Analysis, Canberra, 1983. vii+272 pp. ISBN: 0-86784-429-9 MR0756417.

John C. Slattery, Leonard. Sagis, and Eun-Suok Oh, *Interfacial transport phenomena. Second edition*. Springer, New York, 2007. xviii+827 pp. ISBN: 978-0-387-38438-2; 0-387-38438-3 MR2284654.

H. Sohr, *The Navier-Stokes equations. An elementary functional analytic approach*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2001. x+367 pp. MR1928881

Hon. J.W. Strutt M.A., *Some General Theorems Relating to Vibrations*. Proc. London. Math. Soc. (1873);IV:357-68. MR1575554

R. Temam, *Navier-Stokes equations, Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. x+500 pp. ISBN: 0-7204-2840-8. MR0669732

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