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Volume 358, issue 4 (2020), p. 489-495.

<https://doi.org/10.5802/crmath.66>

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A note on flatness of non separable tangent cone at a barycenter

Une note sur la platitude du cône tangeant à un barycentre

Thibaut Le Gouic

Abstract. Given a probability measure \( P \) on an Alexandrov space \( S \) with curvature bounded below, we prove that the support of the pushforward of \( P \) on the tangent cone \( T_{b^*}S \) at its (exponential) barycenter \( b^* \) is a subset of a Hilbert space, without separability of the tangent cone.

Résumé. Étant donné une mesure de probabilité \( P \) sur un espace d’Alexandrov \( S \) avec courbure minorée, nous prouvons que le support de la mesure poussée de \( P \) sur le cône tangent \( T_{b^*}S \) à son barycentre (exponentiel) \( b^* \) est un sous-ensemble d’un espace de Hilbert, sans condition de séparabilité du cône tangent.

Funding. Supported by ONR grant N00014-17-1-2147.

1. Introduction

Barycenter of a probability measure \( P \) (a.k.a. Fréchet means) provides an extension of expectation on Euclidean space to arbitrary metric spaces. We present here a useful tool for the study of barycenters on Alexandrov spaces with curvature bounded below: the support of \( \log_{b^*} \#P \) in the tangent cone at the barycenter is included in a Hilbert space. This rigidity result has been stated in [9] as Theorem 45, however the proof is not written. Moreover, there is an extra assumption of support of \( \log_{b^*} \#P \) being separable, which does not even seem to be a consequence of the support of \( P \) being separable. As pointed out by [7], it is not clear if even \( S \) being proper ensures that the tangent cone is separable. This paper presents a proof of this rigidity result, without this extra separable assumption on the tangent cone. For measurability purposes (see Lemma 6), we suppose however that \( S \) is separable. The proof is essentially the one of Theorem 45 of [9], with needed approximations dealt with a bit differently.
2. Setting and main result

We use a classical notion of curvature bounded below for geodesic spaces, referred to as Alexandrov curvature. We recall several notions whose formal definitions can be found for instance in [3] or in the work in progress [2].

For a metric space \((S,d)\), we denote by \(\mathcal{P}_1(S)\) the set of probability measures on \(S\) with finite moment of order 1 (i.e. such that there exists \(x \in S\) such that \(\int d(x,y) d\mathbf{P}(y) < \infty\)). The support of a measure \(\mathbf{P}\) will be denoted by \(\text{supp}\mathbf{P}\). We use both notation \(\int f d\mathbf{P}\) and \(\mathbf{P}f\) for the integral of \(f\) w.r.t. \(\mathbf{P}\).

A geodesic space is a metric space \((S,d)\) such that every two points \(x,y \in S\) at distance is connected by a curve of length \(d(x,y)\). Such shortest curves are called geodesics. For \(\kappa \in \mathbb{R}\), the model space \((\mathbb{M}_\kappa, d_\kappa)\) denotes the 2-dimensional simply connected complete surface of constant Gauss curvature \(\kappa\). A geodesic space \((S,d)\) is an Alexandrov space with curvature bounded below by \(\kappa\) if for every triangle (3-uple) \((x_0,x_1,y) \in S\), and a constant speed geodesic \((x_t)_{t \in [0;1]}\) there exists an isometric triangle \((\tilde{x}_0,\tilde{x}_1,\tilde{y}) \in \mathbb{M}_\kappa\), such that the geodesic \((\tilde{x}_t)_{t \in [0;1]}\) satisfies for all \(t \in [0;1]\),

\[
d(y,x_t) \geq d_\kappa(\tilde{y},\tilde{x}_t).
\]

For such spaces, angles between two unit-speed geodesics \(\gamma_1,\gamma_2\) starting at the same point \(p \in S\) can be defined as follows:

\[
\cos \angle_p(\gamma_1,\gamma_2) = \lim_{t \to 0} \frac{d^2(\gamma_1(t),p) + d^2(\gamma_2(t),p) - d^2(\gamma_1(t),\gamma_2(t))}{2d(p,\gamma_1(t))d(p,\gamma_2(t))},
\]

where angle \(\angle_p(\gamma_1,\gamma_2) \in [0;\pi]\). Denote by \(\Gamma_p\) the set of all unit-speed geodesics emanating from \(p\). Using angles, we can define the tangent cone \(T_pS\) at \(p \in S\) as follows. First define \(T'_pS\) as the (quotient) set \(\Gamma_p \times \mathbb{R}^+\), equipped with the (pseudo-)metric defined by

\[
\|((y_1,t)-(y_2,s))\|^2_p := s^2 + t^2 - 2st \cos \angle_p(\gamma_1,\gamma_2).
\]

Then, the tangent cone \(T_pS\) is defined as the completion of \(T'_pS\) equipped with the metric \(\| \cdot \|_p\). We will use the notation for \(u,v \in T_pS\),

\[
\langle u,v \rangle_p := \frac{1}{2} (\|u\|_p^2 + \|v\|_p^2 - \|u-v\|_p^2),
\]

We will often identify a point \(\gamma(t) \in S\) with \((\gamma,t) \in T_pS\). Although such \(\gamma\) might not be unique, we will assume a choice of a map \(\log_p : S \to T_pS\), called logarithmic map, such that for all \(x \in S\), there exists a unit-speed geodesic \(\gamma\) emanating from \(p\) such that, for some \(t > 0\), \(\gamma(t) = x\) and

\[
\log_p(x) = (\gamma,t).
\]

This map can be chosen to be \(\mathcal{G}_B\)-measurable, where \(\mathcal{G}_B\) denotes the \(\sigma\)-algebra generated by open balls on the tangent cone \(T_pS\) (see Lemma 6) and this weak measurability is enough for our results to hold and will be assumed for the rest of the paper. Then the pushforward of \(\mathbf{P}\) by \(\log_p\) will be denoted by \(\# \log_p \mathbf{P}\).

The tangent cone is not necessarily a geodesic space (see [4]), however, it is included in a geodesic space, namely the ultratangent space (see for instance Theorem 14.4.2 and 14.4.1 in [2]) that is an Alexandrov space with curvature bounded below by 0.

The tangent cone \(T_pS\) contains the subspace \(\text{Lin}_p\) of all points with an opposite, formally defined as follows. A point \(u\) belongs to \(\text{Lin}_p \subset T_pS\) if and only if there exists \(v \in T_pS\) such that \(\|u\|_p = \|v\|_p\) and

\[
\langle u,v \rangle_p = -\|u\|_p^2.
\]

Our main result is based on the following Theorem.

Theorem (Theorem 14.5.4 in [2]). The set \(\text{Lin}_p\) equipped with the induced metric of \(T_pS\) is a Hilbert space.
A point \( b^* \in S \) is a barycenter of the probability measure \( P \in \mathcal{P}_1(S) \) if for all \( b \in S \)
\[
0 \leq \int d^2(x, b) - d^2(x, b^*) \, dP(x).
\]
Such barycenter might not be unique, neither exists. However, when they exist, they satisfy
\[
\int \langle x, y \rangle_{b^*} \, dP \otimes P(x, y) = 0. \tag{1}
\]
A point \( b^* \in S \) satisfying (1) is called an exponential barycenter of \( P \).
We can now state our main result.

**Theorem 1.** Let \((S, d)\) be an Alexandrov space with curvature bounded below by some \( \kappa \in \mathbb{R} \) and \( P \in \mathcal{P}_1(S) \). If \( b^* \in S \) is an exponential barycenter of \( P \), then \( \text{supplog}_{b^*} \#P \subset \text{Lin}_{b^*} S \). In particular, \( \text{supplog}_{b^*} \#P \) is included in a Hilbert space.

This result allows to prove the following Corollary, that has been implicitly used in [1].

**Corollary 2 (Linearity).** Let \( b \in T_{b^*} S \). Then, the map \( \langle \cdot, b \rangle_{b^*} : \text{Lin}_{b^*} \to \mathbb{R} \) is continuous and linear. In particular, if \( b^* \) is an exponential barycenter of \( P \), then
\[
\int \langle x, b \rangle_{b^*} \, dP(x) = 0.
\]

### 3. Proofs

Recall that we always identify a point in \( S \) and its image in the tangent cone \( T_{b^*} S \) by the \( \log_{b^*} \) map.

**Proof of Corollary 2.** Linearity is obvious from the definition of \( \langle \cdot, b \rangle_{b^*} \). We check that \( x \mapsto \langle x, b \rangle_{b^*} \) is a convex and concave function in \( \text{Lin}_{b^*} S \). Let \( t \in (0, 1), \, x_0, x_1 \in \text{Lin}_{b^*} S \), and set \( x_t = (1 - t)x_0 + tx_1 \). Since the tangent cone is included in an Alexandrov space with curvature bounded below by 0 on one hand, and \( \text{Lin}_{b^*} \) is a Hilbert space on the other hand,
\[
\langle x_t, b \rangle_{b^*} \geq \frac{1}{2} (\|x_t\|_{b^*}^2 + \|b\|_{b^*}^2 - \|x_t - b\|^2)
\]
\[
\quad \leq \frac{1}{2} \left( (1 - t)(\|x_0\|_{b^*}^2 - \|x_0 - b\|^2) + t(\|x_1\|_{b^*}^2 - \|x_1 - b\|^2) + \|b\|^2_{b^*} \right)
\]
\[
\quad = (1 - t)\langle x_0, b \rangle_{b^*} + t\langle x_1, b \rangle_{b^*}.
\]
The same lines applied to \( -x_0 \) and \( -x_1 \) gives the converse inequality
\[
\langle -x_t, b \rangle_{b^*} \leq (1 - t)\langle -x_0, b \rangle_{b^*} + t\langle -x_1, b \rangle_{b^*}.
\]
The second statement follows from the fact that \( b^* \) is a Pettis integral of the pushforward of \( P \)
on \( \text{Lin}_{b^*} \subset T_{b^*} S \), as a direct consequence of Theorem 1.

**Proof of Theorem 1.** Let \( L = \{ x \in S | \int \langle x, \cdot \rangle_{b^*} \, dP = 0 \} \) be a measurable set such that \( P(L) = 1 \) given by Lemma 3. Let \( x \in L \). For \( U = \{ x \} \), use Lemma 5 with \( Q = P \) and \( B_\delta \) a ball of radius \( \delta \) around \( x \) in \( T_{b^*} S \), to get a sequence \( \langle y_{n}^{B_\delta} \rangle_{n} \subset T_{b^*} S \) such that,
\[
\limsup_{n} \cos(1 \frac{x_{n}^{B_\delta}}{\delta_{n}^{B_\delta}}) = \limsup_{n} \frac{\langle x, y_{n}^{B_\delta} \rangle_{b^*}}{d(b^*, x)d(b^*, y_{n}^{B_\delta})} = \frac{1}{d(b^*, x)} \limsup_{n} \frac{\langle x, y_{n}^{B_\delta} \rangle_{b^*}}{d(b^*, y_{n}^{B_\delta})} \leq \frac{1}{d(b^*, x)} \int_{B_\delta} \langle x, y \rangle_{b^*} \, dP(x) \frac{P(B_\delta)}{\left( \int_{B_\delta} \int_{B_\delta} \langle x, y \rangle_{b^*} \, dP \otimes P(x, y) \right)^{1/2}}.
\]
Then, since \( \int \langle x, y \rangle_{b^*} \, dP(y) = 0 \), letting \( \delta \to 0 \), one gets
\[
\frac{1}{P(B_\delta)} \int_{B_\delta} \langle x, y \rangle_{b^*} \, dP(y) = - \frac{1}{P(B_\delta)} \int_{B_\delta} \langle x, y \rangle_{b^*} \, dP(y) \to -d^2(b^*, x),
\]
and
\[
\left( \int_{B_\delta} \int_{B_\delta} \langle x, y \rangle_{b^*} \, dP \otimes P(x, y) \right)^{1/2} \to d(b^*, x).
\]
Thus,
\[
\lim_{\delta \to 0^+} \sup_{n} \cos \angle \left( (1^x_{b^*}, 1^y_{b^*}) \right) = -1
\]
One can thus choose \((\bar{y}^n)_n\) a sequence in \((y^n_\delta)_{n, \delta}\) such that \(\cos \angle (1^x_{b^*}, 1^y_{b^*}) \to -1\). Since \(T_{b^*} S\) is a subspace of an Alexandrov space of curvature bounded below by \(0\), we also have
\[
\angle (1^x_{b^*}, 1^y_{b^*}) \leq 2\pi - \angle (1^{\bar{y}^n}_{b^*}, 1^{\bar{y}^k}_{b^*}) - \angle (1^x_{b^*}, 1^{\bar{y}^k}_{b^*}) \to 0,
\]
as \(n, k \to \infty\). Thus \((\bar{y}^n)_n\) corresponds to a Cauchy sequence in the space of direction, and thus admits a limit in \(T_{b^*} S\), since its “norm” also admits a limit \(d(b^*, x)\). Its limit \(\bar{y}\) satisfies \(\cos \angle (1^x_{b^*}, 1^{\bar{y}}_{b^*}) = -1\), and therefore, it is the opposite \(\bar{y} = -x\).

Finally, by definition of the support, for \(x \in \text{supp}(\log_{b^*} P)\), every ball centered at \(x\) have a positive probability, and thus there exists a sequence \((x_n)_{n \geq 1} \subset L\) such that \(x_n \to x\). We conclude with the completeness of \(\text{Lin}_{b^*}\).

**Lemma 3 (Proposition 1.7 of [8] for non separable metric space).** Suppose \((S, d)\) is an Alexandrov space with curvature bounded below. Then, for any probability measure \(Q \in \mathcal{P}_1(S)\), and any \(b^* \in S\),
\[
\int \langle x, y \rangle_{b^*} \, dQ \otimes Q(x, y) \geq 0.
\]
Moreover, if \(b^*\) is an exponential barycenter of \(Q\), then for \(Q\)-almost all \(x \in S\),
\[
\int \langle x, y \rangle_{b^*} \, dQ(y) = 0.
\]

**Proof.** For brevity, we will adopt the notation \(Q g\) for \(\int g \, dQ\).

The result for \(Q\) finitely supported is the Lang–Schroeder inequality (Proposition 3.2 in [5]). Thus, we just need to approximate \(Q \otimes Q(\cdot, \cdot)_{b^*}\) by some \(Q^1_n \otimes Q^2_n(\cdot, \cdot)_{b^*}\) for some finitely supported \(Q^1_n\).

To approximate \(Q \otimes Q(\cdot, \cdot)_{b^*}\), draw two independent sequences of i.i.d. random variables \((X^1_i)_{i}\) and \((X^2_i)_{i}\) of common law \(Q\), and denote \(Q^1_n\) and \(Q^2_n\) the corresponding empirical measures. In particular, \(Q^1_n \otimes Q^2_n\) and \(Q^2_n \otimes Q^2_n\) are both empirical measures of \(Q \otimes Q\). Since \(S\) is not separable, we can not apply the fundamental theorem of statistics that ensures almost sure weak convergence of \(Q^1_n \otimes Q^2_n\) to \(Q \otimes Q\). However, for a measurable function \(f : S \times S \to \mathbb{R}\), such that \(Q \otimes Q f < \infty\), the law of large number ensures that almost surely
\[
Q^1_n \otimes Q^2_n f \to Q \otimes Q f
\]
and
\[
Q^2_n \otimes Q^2_n f \to Q \otimes Q f.
\]
Since the sequence \((X^1_1, X^1_2, X^2_2, X^2_3, X^1_3, X^2_4, \ldots)\) is also an i.i.d. sequence of random variables of common law \(Q\), the subsequence of the associated empirical measures \((Q^3_n)_n\) defined by
\[
Q^3_n := \frac{1}{2}(Q^1_n + Q^2_n)
\]
also satisfies the almost sure convergence
\[
Q^3_n \otimes Q^3_n f \to Q \otimes Q f.
\]
Now, since
\[ Q_n^1 \otimes Q_n^2 = \frac{1}{4} (Q_n^1 \otimes Q_n^2 + Q_n^1 \otimes Q_n^1 + Q_n^2 \otimes Q_n^1 + Q_n^2 \otimes Q_n^2), \]
we proved that almost surely
\[ Q_n^1 \otimes Q_n^1 f + Q_n^2 \otimes Q_n^2 f \to 2Q \otimes Q f. \]
And since \((Q_n^1)_n\) and \((Q_n^2)_n\) are independent and with same law, it implies that both \(Q_n^1 \otimes Q_n^1 f\) and \(Q_n^2 \otimes Q_n^2 f\) converge to \(Q \otimes Q f\) almost surely. In particular, since \(Q_n^1\) is supported on \(n\) points, there exists a sequence of finitely supported measures (that we rename \((Q_n)_n\)) such that \(Q_n \otimes Q_n f \to Q \otimes Q f\). We thus proved the first result applying \(f = \langle \cdot , \cdot \rangle_{b^*} \).

Now, for any \(x \in S\), applying this first result to the measure \(Q_x := \frac{1}{1+\ve} Q + \frac{\ve}{1+\ve} \delta_x\), we get
\[ 0 \leq (1+\ve)Q_x \otimes Q_x \langle \cdot , \cdot \rangle_{b^*} + 2\ve Q(x, \cdot)_{b^*} + \ve^2 \|x\|_{b^*}^2. \]
Letting \(\ve \to 0^+\), we get
\[ Q(x, \cdot)_{b^*} \geq 0. \]
Then equality follows from the hypothesis \(Q \otimes Q \langle \cdot , \cdot \rangle_{b^*} = 0\) meaning that \(b^*\) is an exponential barycenter.

**Lemma 4 (Subadditivity, Lemma A.6 of [5]).** Let \((S, \delta)\) be an Alexandrov space with curvature bounded below. Take \(b^* \in S\). Let \(x_1, \ldots, x_n \in T_{b^*} S \) and \(U \subset T_{b^*} S \) finite. Then, for all \(\ve > 0\), there exists \(y \in T_{b^*} S\) such that for all \(u \in U\),
\[ \langle y, u \rangle_{b^*} \leq \sum_{i=1}^n \langle x_i, u \rangle_{b^*} + \ve, \]
and
\[ \sum_{i,j=1}^n \langle x_i, x_j \rangle_{b^*} + \ve. \]

**Lemma 5 (Approximation).** Let \(U \subset T_{b^*} S \) finite. Take \(B \subset S\) measurable and a probability measure \(P \in \mathcal{P}_1(S)\) such that \(P \otimes P \langle \cdot , \cdot \rangle_{b^*} = 0\) and \(P(B) > 0\). Then, there exists a sequence \((y^n)_n\) such that for all \(u \in U\)
\[ \frac{1}{P(B)} \int_{B^c} \langle u, x \rangle_{b^*} dP(x) \geq \limsup_n \langle u, y^n \rangle_{b^*} \] (2)
and
\[ \frac{1}{P(B)^2} \int_B \int_B \langle x, y \rangle_{b^*} dP \otimes P(x, y) = \lim_n d^2(b^*, y^n). \] (3)

**Proof.** Using the same arguments as in Lemma 3, we see that the empirical measures \((P_n)_n\) satisfy
\[ P_n \otimes P_n f \to P \otimes P f, \]
almost surely, for any \(f : S \times S \to \mathbb{R} \in L^1(S \times S, P \otimes P)\). In particular, taking \(f(x, y) = \langle x, y \rangle_{b^*} 1_{B \times B}(x, y)\), the following convergence holds in \(L^2(P^{\infty})\),
\[ \int_B \int_B \langle x, y \rangle_{b^*} dP_n \otimes P_n \to \int_B \int_B \langle x, y \rangle_{b^*} dP \otimes P, \] (4)
and similarly for \(B^c\). Also, the law of large number ensures that almost surely, for all \(u \in U\),
\[ \int_B \langle u, x \rangle_{b^*} dP_n \to \int_B \langle u, x \rangle_{b^*} dP, \] (5)
and again, the same for \(B^c\). Thus, there exists a subsequence (of a deterministic realization of) \(P_n\) (that we rename \(P_n\)) such that (4) and (5) both hold for all \(u \in U\).
Then, applying Lemma 4 to finite sum
\[
\frac{1}{P(B)} \int_{B^c} \langle \cdot , u \rangle_{b^*} dP_n,
\]
shows that there exists a sequence \((y^n)_n \in T^*_n S\) such that (2) holds and for a sequence \((\varepsilon_n)_n\) s.t. \(\varepsilon_n \to 0\),
\[
\|y^n\|_{b^*}^2 \leq \frac{1}{P(B)^2} \int_{B^c} \int_{B^c} \int_{B^c} \int_{B^c} \langle x, y \rangle_{b^*} dP_n \otimes P_n + \varepsilon_n.
\]
Then, applying the same Lemma 4 again shows that there exists a sequence \((z^n)_n \in T^*_n S\), such that
\[
0 \leq \frac{1}{P(B)^2} \int_{B^c} \int_{B^c} \langle x, y \rangle_{b^*} dP_n \otimes P_n(x, y)
= \frac{1}{P(B)^2} \left( \int_B \int_{B^c} \int_{B^c} + 2 \int_B \int_{B^c} \right) \langle x, y \rangle_{b^*} dP_n \otimes P_n(x, y)
\geq \|z^n\|_{b^*}^2 + \|y^n\|_{b^*}^2 + 2 \langle y^n, z^n \rangle_{b^*} - \varepsilon_n.
\]
Letting \(n \to \infty\), one obtains
\[
0 \geq \lim_n \|z^n\|_{b^*}^2 + 2 \langle y^n, z^n \rangle_{b^*} + \|y^n\|_{b^*}^2
\geq \lim_n \|z^n\|_{b^*}^2 - 2 \|y^n\|_{b^*} \|z^n\|_{b^*} + \|y^n\|_{b^*}^2
= \lim_n (\|z^n\|_{b^*}^2 - \|y^n\|_{b^*}^2) \geq 0.
\]
and which shows \(\lim_n \|y^n\| = \lim_n \|z^n\|\) and also that (6) becomes an equality at the limit and therefore
\[
\lim_n \|z^n\|_{b^*}^2 = \frac{1}{P(B)^2} \int_B \int_{B^c} \langle x, y \rangle_{b^*} dP \otimes P(x, y)
\]
This Lemma appears in a remark of [6].

Lemma 6 (Measurability of the log map). Let \((S,d)\) be a separable Alexandrov space. Let \(p \in S\). Then \(\log_p : S \to T_p S\) can be chosen to be \(\mathcal{G}_B\)-measurable.

Proof. Denote \(G_p\) the space of all constant speed geodesics emanating from \(p\) equipped with the sup distance \(\| \cdot \|_{\infty}\). Then \((G_p, \| \cdot \|_{\infty})\) is separable and complete too. Using Kuratowski and Ryll-Nardzewski measurable selection theorem, one can choose a Borel map \(g : S \to G_p\) such that \(g\) maps \(x\) to a geodesic from \(p\) to \(x\). Then, using the (proof of) Lemma 4.2 of [7], the map \(l : G_p \to T_p S\) is measurable \(T_p S\) is equipped with the \(\sigma\)-algebra \(\mathcal{G}\) generated by open balls. Therefore, \(\log_p := l \circ g\) is \(\mathcal{G}_B\)-measurable.

Acknowledgements

This note originated from technical issues emanating from the problem considered in [1] and [6]. I would like to thank Quentin Paris and Philippe Rigollet for fruitful discussions that lead to this article. I am also indebted to the anonymous referee for their careful review and comments.

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