Quantum radiation from a classical point source

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Abstract

We study the radiation of photons from a classical charged particle. We particularly consider a situation where the particle has a constant velocity in the distant past, then is accelerated, and then has a constant velocity in the distant future. Starting with no photons in the distant past we seek to characterize the quantum state of the photon field in the distant future. Working in the Coulomb gauge and in a C* algebra formulation, we give sharp conditions on whether this state is or is not in Fock space.

1 Introduction

We study the radiation of the quantum electromagnetic field in the presence of a classical source. With a specified four-current consisting of functions \((j^0, j)\) satisfying \(\partial_\mu j^\mu = 0\), one seeks associated quantum field operators \(A = (A^0, A)\). In the Lorentz gauge \(\partial_\mu A^\mu = 0\) these are connected by the wave equation with source \(\Box A^\mu = j^\mu\). A textbook treatment of this problem can be found in Itzykson-Zuber [4] whose sign conventions we follow. A mathematical treatment is due to Naudts-DeRoeck [6]. A characteristic phenomena is that if one starts in a state with no photons in the distant past, the vacuum in Fock space, it may evolve into a state in the distant future with infinitely many photons radiated and not in Fock space. This is sometimes known as the infrared catastrophe.

In this paper we are interested in characterizing exactly when this happens. We particularly study the case of a point source where the four-current has the form

\[
\begin{align*}
  j^0(x, t) &= \delta(x - x(t)) \\
  j(x, t) &= x'(t) \delta(x - x(t))
\end{align*}
\]

(1)

and \(x(t)\) is the trajectory of the charged particle in \(\mathbb{R}^3\). We suppose that the particle has a velocity \(v_{in}\) in the distant past, then accelerates for a finite amount of time, and
then continues with a velocity \( v_{\text{out}} \) in the distant future. We start with the Fock vacuum in the distant past considered as a state \( \omega_{\text{in}} \) on the \( C^\ast \)-algebra generated by the fields. It evolves to state \( \omega_{\text{out}} \) in the distant future. We establish the following results

- If \( v_{\text{out}} = v_{\text{in}} = 0 \) and the trajectory \( x(t) \) is smooth then \( \omega_{\text{out}} \) is a Fock state.
- If \( v_{\text{out}} = v_{\text{in}} = 0 \) and the velocity \( x'(t) \) has a discontinuity then \( \omega_{\text{out}} \) is not a Fock state.
- In general for a smooth trajectory \( \omega_{\text{out}} \) is a Fock state if and only if \( v_{\text{in}} = v_{\text{out}} \).

We work in the Coulomb gauge rather than the Lorentz gauge. The advantage is that we avoid working with indefinite metric Hilbert spaces. The disadvantage is that relativistic properties like Lorentz invariance and locality are not manifest, although they are still there. (See Skagerstam, Erikson, Rekdal [7] for a discussion of locality in the Coulomb gauge.)

The Coulomb gauge is defined by the condition \( \nabla \cdot A = 0 \) on the magnetic potential. In this gauge Maxwells equations become

\[
-\Delta A^0 = j^0
\]
\[
\frac{\partial^2 A}{\partial t^2} - \Delta A + \nabla \frac{\partial}{\partial t} A^0 = j
\]

The first equation for the electrostatic potential \( A^0 \) has the solution

\[
A^0(x,t) = \int V(x - y)j^0(t,y)dy
\]

where \( V(x) = (4\pi)^{-1}|x|^{-1} \) is the Coulomb potential. In the second equation we call the term \( \nabla(\partial A^0/\partial t) \) the longitudinal current. Since \( \partial j^0/\partial t + \nabla \cdot j = 0 \) this can be expressed as

\[
j_L(x,t) = \nabla(\partial A^0/\partial t)(x,t) = -\nabla \int V(x - y)(\nabla \cdot j)(y,t)dy
\]

It satisfies \( \nabla \cdot j_L = \nabla \cdot j \). The transverse current is defined by \( j_T = j - j_L \) and it satisfies \( \nabla \cdot j_T = 0 \). The second equation in (2) and the constraint can now be written

\[
\frac{\partial^2 A}{\partial t^2} - \Delta A = j_T \quad \nabla \cdot A = 0
\]

or just \( \Box A = j_T \) with \( \Box = \partial_t^2 - \Delta \).

The electrostatic potential \( A^0 \) is given by (3) and is not quantized. We seek quantum field operators \( A \) satisfying (5).
2 Preliminaries

2.1 fundamental solutions

We review some facts about solutions of the wave equation with source. At first this is just classical so we seek functions $A$ satisfying $\Box A = j$. We are interested in solutions of the form $A = G_{\pm} * j$ where $G_{\pm}$ are retarded and advanced fundamental solutions which we define shortly. We want to allow $j$ to be certain distributions so $G_{\pm} * j$ is defined as $< G_{\pm} * j, \phi > = < j, G_{\pm} * \phi >$ for $\phi$ in (vector-valued) $\mathcal{S}(\mathbb{R}^4)$, the Schwartz space of smooth rapidly decreasing functions. However we also want to work at sharp time so the test functions $\phi$ will be replaced by $f \otimes \delta_t$ with $f$ in (vector-valued) $\mathcal{S}(\mathbb{R}^3)$.

The upshot is that we want to define functions $G_{\pm} *(f \otimes \delta_t)$, or without the convolution $G_{\pm} (f \otimes \delta_t) \equiv G_{\pm} (f, t)$.

The $G_{\pm}$ are given by

$$G_{\pm}(x, t) = -\frac{1}{(2\pi)^2} \int_{p_0 \in \mathbb{R} \pm i \epsilon} e^{i(p_0 t + px)} \frac{1}{p_0^2 - |p|^2} dp_0 dp \quad (6)$$

For $f \in \mathcal{S}(\mathbb{R}^3)$ this is to be understood as for $G_{\pm}(f, t) = \int f(x) G_{\pm}(x, t) dx$ given by

$$G_{\pm}(f, t) = -2\pi^{-\frac{3}{2}} \int_{p_0 \in \mathbb{R} \pm i \epsilon} \frac{1}{p_0^2 - |p|^2} e^{-i p_0 t} \tilde{f}(-p) \ dp_0 dp \quad (7)$$

where $\tilde{f}(p) = (2\pi)^{-\frac{3}{2}} \int e^{-i px} f(x) dx$ is the Fourier transform of $f$.

Consider $G_+(f, t)$. We have $|e^{-ip_0 t}| = e^{-lm(p_0) t}$ so if $t < 0$ we can close the $p_0$-contour in the upper half plane and get zero. On the other hand If $t > 0$ we close the contour in the lower half plane, picking up poles at $p_0 = \pm \omega$ where $\omega(p) = |p|$. Similarly for $G_-(f, t)$. We obtain

$$G_{\pm}(f, t) = \pm (2\pi)^{-\frac{3}{2}} \theta(\pm t) i \int \left( e^{-i\omega t} - e^{i\omega t} \right) \tilde{f}(-p) \frac{dp}{2\omega} \quad (8)$$

where $\theta$ is the Heaviside function.

The convolution is defined by $(G_{\pm} * (f \otimes \delta_{t_0}))(x, t) = G_{\pm} \left( f(x - \cdot), \delta_{t_0}(t - \cdot) \right)$. But the Fourier transform of $f(x - \cdot)$ at $-p$ is $e^{ipx} \tilde{f}(p)$ and $\delta_{t_0}(t - \cdot) = \delta_{t-t_0}$. Thus we have

$$(G_{\pm} * (f \otimes \delta_{t_0}))(x, t) = \pm (2\pi)^{-\frac{3}{2}} \theta(\pm (t-t_0)) i \int \left( e^{-i\omega(t-t_0)} - e^{i\omega(t-t_0)} \right) e^{ipx} \tilde{f}(p) \frac{dp}{2\omega} \quad (9)$$

We also will want to consider the propagator $G = G_+ - G_-$. In this expression the Heaviside functions $\theta$ disappear and we have

$$(G * (f \otimes \delta_{t_0}))(x, t) = u_f(x, t - t_0) \quad (10)$$

where

$$u_f(x, t) = (2\pi)^{-\frac{3}{2}} i \int \left( e^{-i\omega t} - e^{i\omega t} \right) e^{ipx} \tilde{f}(p) \frac{dp}{2\omega} \quad (11)$$
is a smooth solution of the wave equation with data at \( t = 0 \)

\[
uf(x,0) = 0 \quad \frac{\partial uf}{\partial t}(x,0) = f(x)
\]  

(12)

### 2.2 free quantum field

We review the mathematics of the quantized free field in the Coulomb gauge. In this case we can take \( A^0 = 0 \), and study the field equation

\[
\frac{\partial^2 A}{\partial t^2} - \Delta A = 0
\]

(13)

We want to define a quantum field operator satisfying the equation, the constraint \( \nabla \cdot A = 0 \), and the canonical commutation relations.

It is convenient to use the relativistic Hilbert space which is (vector valued)

\[
\mathcal{H} = L^2(\mathbb{R}^3, (2\omega)^{-1} dp)
\]

(14)

again with \( \omega(p) = |p| \). On this space define a projection operator.

\[
(Ph)_j(p) = \sum_{k=1}^{3} \left( \delta_{jk} - \frac{p_j p_k}{|p|^2} \right) h_k(p)
\]

(15)

which is the orthogonal projection onto the space of transverse functions satisfying \( p \cdot h(p) = 0 \). The one photon Hilbert space is \( \mathcal{H}_T = P\mathcal{H} \), the n-photon Hilbert space \( \mathcal{H}_T^n \) is the n-fold symmetric tensor product of \( \mathcal{H}_T \) with itself, and the Fock space over \( \mathcal{H}_T \) is

\[
\mathcal{F}(\mathcal{H}_T) = \bigoplus_{n=0}^{\infty} \mathcal{H}_T^n
\]

(16)

For \( h \in \mathcal{H}_T \) let \( a^*(h) \) and \( a(h) \) be the creation and annihilation operators on the Fock space, with \( a(h) \) anti-linear in \( h \) and \( a^*(h) \) linear in \( h \). These operators (and all operators in this section) are defined on the dense subspace \( \mathcal{D} \) consisting elements of Fock space with a finite number of particles. They satisfy \([a(h_1), a^*(h_2)] = (h_1, h_2)\). We also define momentum space fields \( A_\circ(h) \) and \( \Pi_\circ(h) = A_\circ(ih) \) by

\[
A_\circ(h) = \frac{1}{\sqrt{2}} \left( a^*(h) + a(h) \right)
\]

\[
\Pi_\circ(h) = \frac{i}{\sqrt{2}} \left( a^*(h) - a(h) \right)
\]

(17)

These are real linear in \( h \), symmetric, and satisfy \([A_\circ(h_1), A_\circ(h_2)] = i\text{Im}(h_1, h_2)\).

The field operator is defined as a distribution. For a real test function \( f \in \mathcal{S}(\mathbb{R}^3) \) define \( A(f,t) \) (formally \( \int f(x)A(x,t)dx \)) by

\[
A(f,t) = \sqrt{2} A_\circ(e^{i\omega t} \tilde{f}_{T}) = a^*(e^{i\omega t} \tilde{f}_{T}) + a(e^{i\omega t} \tilde{f}_{T})
\]

(18)
defines a state on the algebra. In particular the Fock vacuum $\Omega_0$ the condition that

We want to allow states for this system which are not Fock states, and for this we focus on the standard material. We consider operators $A$ on the algebra of the fields. See for example Bratteli-Robinson [1] for more details on this standard material. We consider operators $\Pi(h, t) = \partial / \partial \tau A(h, t)$ given by

$$\Pi(f, t) = \sqrt{2} \Pi_c \left( e^{i\omega t} \Omega_\tau \right) = i \left( a^*(e^{i\omega t} \Omega_\tau) - a(e^{i\omega t} \Omega_\tau) \right)$$

The $\Pi(f, t)$ commute with each other and satisfy the canonical commutation relations (CCR)

$$[A(f_1, t), \Pi(f_2, t)] = i(\tilde{f}_1, T, 2\omega \tilde{f}_2, T) = i(\tilde{f}_1, T, \tilde{f}_2, T)_{L^2(\mathbb{R}^3, dp)}$$

2.3 the $C^*$ algebra and coherent states

We want to allow states for this system which are not Fock states, and for this we focus on the algebra of the fields. See for example Bratteli-Robinson [1] for more details on this standard material. We consider operators $A_0(h)$ for $h$ in a dense subspace $\mathcal{S} \subset \mathcal{H}_T$. The operator is essentially self-adjoint on the domain $\mathcal{D} \subset \mathcal{F}(\mathcal{H}_T)$, and so we can define

$$W_0(h) = e^{iA_0(h)} \quad h \in \mathcal{S}$$

This satisfies

$$W_0(h_1)W_0(h_2) = e^{-\frac{i}{2}\text{Im}(h_1, h_2)}W_0(h_1 + h_2)$$

This is the Weyl form of the CCR. The operators $W_0(h)$ generate a $C^*$ algebra, denoted $\mathfrak{A}$, which is the closure of the set of operators which are finite sums $\sum_n c_n W_0(h_n)$ in the Banach space of bounded operators on $\mathcal{F}(\mathcal{H}_T)$. Different choices of the dense subspace $\mathcal{S}$ give different algebras, and we will make a specific choice later.

A state $\omega$ on the algebra is a positive linear functional of norm one. Positivity is the condition that $\omega(B^*B) \geq 0$ for any $B \in \mathfrak{A}$. Any normalized state in Fock space defines a state on the algebra. In particular the Fock vacuum $\Omega_0$ defines a state by $\omega_0(\cdot) = (\Omega_0, [\cdot] \Omega_0)$. It satisfies $a(h)\Omega_0 = 0$ and is characterized by

$$\omega_0(W_0(h)) = (\Omega_0, e^{iA_0(h)}\Omega_0) = (\Omega_0, e^{ia^*(h)/\sqrt{2}}e^{ia(h)/\sqrt{2}}\Omega_0) e^{-\frac{1}{4}\|h\|^2} = e^{-\frac{1}{4}\|h\|^2}$$

Another class of states are the coherent states. Given $J \in \mathcal{H}_T$ and the Fock vacuum $\Omega_0$ we define the coherent state

$$\Omega_J = e^{-\frac{1}{2}\|J\|^2}e^{a^*(J)}\Omega_0 = e^{-\frac{1}{2}\|J\|^2}\sum_{n=0}^{\infty} \frac{1}{n!} (a^*(J))^n \Omega_0$$
The series converges in Fock space since \( \| (a^*(J))^n \Omega_0 \| = \sqrt{n!} \| J \|^n \). It is normalized to \( \| \Omega_J \| = 1 \) and is an eigenstate of the annihilation operator:

\[
a(h) \Omega_J = (h, J) \Omega_J
\]  

(27)

As a state on the algebra \( \omega_h = (\Omega_h, [\cdot ] \Omega_h) \) it satisfies

\[
\omega_J(W_0(h)) = (\Omega_J, e^{iA_0(h)} \Omega_J) = (\Omega_J, e^{i\alpha(h)} / \sqrt{2} e^{i\alpha(h)} / \sqrt{2} \Omega_J) e^{-\frac{1}{4} \| h \|^2} = e^{i\alpha(h) / \sqrt{2} \Omega_J} e^{-\frac{1}{4} \| h \|^2} = e^{-\frac{1}{4} \| h \|^2 + i \sqrt{2} \text{Re}(J, h)}
\]  

(28)

More generally we define a coherent state on \( \mathfrak{A} \) to be a state satisfying

\[
\omega(W_0(h)) = e^{-\frac{1}{4} \| h \|^2 + i \sqrt{2} \text{Re}(L(h))}
\]  

(29)

for some linear function \( L \) on the dense domain \( S \). It can also be written

\[
\omega(W_0(h)) = \omega_0(W_0(h)) e^{i \sqrt{2} \text{Re}(L(h))}
\]  

(30)

The positivity condition is satisfied since for any finite sequence of complex numbers \( c_n \) and \( h_n \in S \) and \( B = \sum_n c_n W_0(h_n) \) we have

\[
\omega(B^* B) = \sum_{n,m} \bar{c}_m c_n \omega(W_0(h_n - h_m)) e^{\frac{i}{2} \text{Im}(h_n, h_m)}
\]

\[
= \sum_{n,m} \left( \bar{c}_m e^{-i \sqrt{2} \text{Re}(L(f_n))} \right) \left( c_n e^{i \sqrt{2} \text{Re}(L(f_n))} \right) \omega_0(W_0(h_n - h_m)) e^{i \sqrt{2} \text{Re}(L(f_n))} e^{\frac{i}{2} \text{Im}(h_n, h_m)}
\]  

(31)

\[
= \omega_0(B_0^* B_0) \geq 0
\]

where

\[
B_0 = \sum_n c_n e^{i \sqrt{2} \text{Re}(L(f_n))} W_0(h_n)
\]  

(32)

The coherent state \( \omega \) defined by \( L \) is known to be a pure state (Honegger-Rapp [3]). Furthermore \( \omega \) is a Fock state if and only if \( L \) is continuous on the pre-Hilbert space \( S \) (Honegger-Rieckers [2]). This is the criterion we will use in the following.
3 Regular source

To warm up we first consider our problem with a regular source. We take \( j \in C_0^\infty(\mathbb{R}^4) \) and seek quantum field operators which solve \( \Box A = j_T \) and \( \nabla \cdot A = 0 \). Note that partial Fourier transform \( \tilde{j}(p,t) \) is bounded and rapidly decreasing in \( p \) as is \( \tilde{j}_T(p,t) = P(p)\tilde{j}(p,t) \) The inverse Fourier transform \( j_T(x,t) \) then has compact support in \( t \), and for each \( t \) is bounded and continuous in \( x \) (since \( \tilde{j}(p,t) \) is integrable in \( p \)) and square integrable in \( x \) (since \( \tilde{j}(p,t) \) is square integrable in \( p \)). The localization of \( j_T(x,t) \) in \( x \) is considerably weaker than that of \( j(x,t) \).

We want a solution which is a free field in the distant past and the distant future. Therefore we take \( A_{\text{in}}(f,t) \) to be a free field defined from creation and annihilation operators \( a_{\text{in}}^*(h), a_{\text{in}}(h) \) as in (13). Then for real \( f \in S(\mathbb{R}^3) \) we define

\[
A(f,t) = A_{\text{in}}(f,t) + (G_+ * j_T)(f,t)
\] (33)

The expression \( G_+ * j_T \) can be analyzed pointwise. Indeed we have explicitly

\[
(G_\pm * j_T)(x,t) = \pm \left( 2\pi \right)^{-\frac{3}{2}} i \int \theta(\pm(t-s)) i \left( e^{-i\omega(t-s)} - e^{i\omega(t-s)} \right) e^{ipx} \tilde{j}_T(p,s) \frac{dp}{2\omega} ds
\] (34)

Then \( (G_+ * j_T)(x,t) \) vanishes in the distant past since \( j_T(p,s) \) does, and hence \( A(f,t) = A_{\text{in}}(f,t) \) in the distant past. We have \( \Box (G * j_T) = j_T \), also in the sense of distributions, and hence \( \Box A = j_T \). Furthermore \( \nabla \cdot (G * j_T) \), also in the sense of distributions, and hence \( \nabla \cdot A = 0 \).

Next we define \( A_{\text{out}}(f,t) \) by

\[
A(f,t) = A_{\text{out}}(f,t) + (G_- * j_T)(f,t)
\] (35)

Then \( A(f,t) = A_{\text{out}}(f,t) \) in the distant future since \( (G_- * j_T)(f,t) \) vanishes in the distant future. Combining (33) and (35) and identifying \( G = G_+ - G_- \) gives

\[
A_{\text{out}}(f,t) = A_{\text{in}}(f,t) + (G * j_T)(f,t)
\] (36)

where now

\[
(G * j_T)(x,t) = \left( 2\pi \right)^{-\frac{3}{2}} i \int \left( e^{-i\omega(t-s)} - e^{i\omega(t-s)} \right) e^{ipx} \tilde{j}_T(p,s) \frac{dp}{2\omega} ds
\]

\[
= (2\pi)^{-1} i \int \left( e^{-i\omega t} \hat{j}_T(\omega,p) - e^{i\omega t} \hat{j}_T(-\omega,p) \right) e^{ipx} \frac{dp}{2\omega}
\] (37)

where \( \hat{j} \) is the four dimensional Fourier transform with Lorentz inner product:

\[
\hat{j}(p_0,p) = (2\pi)^{-2} \int e^{ip_0(t-p_0)} j(x,t) \, dx dt = (2\pi)^{-\frac{3}{2}} \int e^{ip_0 t} \hat{j}(p,t) \, dt
\] (38)

Then \( G * j_T \) satisfies \( \Box (G * j_T) = 0 \) and \( \nabla \cdot (G * j_T) = 0 \), and hence so does \( A_{\text{out}}(f,t) \) Furthermore together with \( \Pi_{\text{out}}(f,t) = \partial A_{\text{out}}(f,t)/\partial t \) it satisfies the CCR. Thus \( A_{\text{out}}(f,t) \) is a free field.
In the second term in \((37)\) make the change of variable \(p \rightarrow -p\). Since \(j\) is real we have \(j_T(\omega, p) = j_T(-\omega, -p)\) and the second term is identified as the complex conjugate of the first term. Thus if we define

\[
J(p) = -\sqrt{2\pi} i \tilde{j}(\omega, p)
\]

and \(J_T(p) = P(p)J(p)\) then we have

\[
(G \ast j_T)(x, t) = -(2\pi)^{-\frac{3}{2}} 2 \text{ Re} \int e^{-i(\omega t - px)} J_T(p) \frac{dp}{2\omega}
\]

This gives

\[
(G \ast j_T)(f, t) = -2 \text{ Re} \int \tilde{f}(-p)e^{-i\omega t} J_T(p) \frac{dp}{2\omega} = -2 \text{ Re} (e^{i\omega t} \tilde{f}_T, J_T)
\]

Now we have

\[
A_{\text{out}}(f, t) = A_{\text{in}}(f, t) - 2 \text{ Re} (e^{i\omega t} \tilde{f}_T, J_T)
\]

We define creation and annihilation operators by taking what amounts to the positive and negative frequency parts of the last expression. We define for \(h \in \mathcal{H}_T\)

\[
a_{\text{out}}^*(h) = a_{\text{in}}^*(h) - (J_T, h)
\]

\[
a_{\text{out}}(h) = a_{\text{in}}(h) - (h, J_T)
\]

These satisfy the standard \([a_{\text{out}}(h_2), a_{\text{out}}^*(h_1)] = (h_1, h_2)\). Then

\[
a_{\text{out}}^*(e^{i\omega t} \tilde{f}_T) = a_{\text{in}}^*(e^{i\omega t} \tilde{f}_T) - (J_T, e^{i\omega t} \tilde{f}_T)
\]

\[
a_{\text{out}}(e^{i\omega t} \tilde{f}_T) = a_{\text{in}}(e^{i\omega t} \tilde{f}_T) - (e^{i\omega t} \tilde{f}_T, J_T)
\]

If we add these we get \(A_{\text{in}}(f, t) - 2 \text{ Re} (e^{i\omega t} \tilde{f}_T, J_T) = A_{\text{out}}(f, t)\) on the right. Thus we must get the same on the left which says

\[
A_{\text{out}}(f, t) = a_{\text{out}}^*(e^{i\omega t} \tilde{f}_T) + a_{\text{out}}(e^{i\omega t} \tilde{f}_T)
\]

just as for \(A_{\text{in}}(f, t)\).

The out vacuum in Fock space should satisfy \(a_{\text{out}}(h)\Omega_{\text{out}} = 0\). Note that \(j \in C_0^\infty(\mathbb{R}^4)\) implies that \(\tilde{j}\) is bounded and rapidly decreasing, hence \(J_T \in \mathcal{H}_T\). Therefore we can construct the coherent state

\[
\Omega_{\text{out}} = \Omega_{J_T} = e^{-\frac{1}{2}||J_T||^2} \exp(a_{\text{in}}^*(J_T))\Omega_{\text{in}}
\]

This gives the the desired

\[
a_{\text{out}}(h)\Omega_{\text{out}} = (a_{\text{in}}(h) - (h, J_T))\Omega_{\text{out}} = 0
\]

We summarize:
Proposition 1. Let \( j \in C_0^\infty(\mathbb{R}^4) \). Then \( J_T(p) = -\sqrt{2\pi} i \hat{j}_T(\omega, p) \) is in \( \mathcal{H}_T \) and the out vacuum \( \Omega_{out} \) exists in Fock space.

Remarks.

1. To obtain this result can certainly relax the condition that \( j \in C_0^\infty(\mathbb{R}^3) \). The key condition is \( J_T \in \mathcal{H}_T \) which follows from \( J \in \mathcal{H} \) which says

\[
\int |\hat{j}(\omega, p)|^2 \frac{dp}{2\omega} < \infty
\]  

(48)

This is similar to a condition that Naudts - De Roeck \([6]\) found in the Lorentz gauge.

2. We sketch some further developments of the scattering theory associated with a regular source. As in (17) define \( \Pi_{\text{in}}(h) = i\sqrt{2}(a_{\text{in}}^*(h) - a_{\text{in}}(h)) \). This is essentially self-adjoint on \( \mathcal{D} \) and we define a unitary operator by

\[
S = \exp \left( i\sqrt{2}\Pi_{\text{in}}(J_T) \right) = \exp \left( -a_{\text{in}}^*(J_T) + a_{\text{in}}(J_T) \right)
\]  

(49)

We have

\[
S^{-1}\Omega_{\text{in}} = e^{-\frac{1}{2}\|J_T\|^2} \exp \left( a_{\text{in}}^*(J_T) \right) \exp \left( -a_{\text{in}}(J_T) \right) \Omega_{\text{in}}
\]  

(50)

and

\[
S^{-1}a_{\text{in}}(h)S = a_{\text{in}}(h) + [a_{\text{in}}^*(J_T), a_{\text{in}}(h)] = a_{\text{in}}(h) - (h, J_T) = a_{\text{out}}(h)
\]  

(51)

So the in and out fields are unitarily equivalent.

One can form other asymptotic states by applying creation operators \( a_{\text{out}}^*(h) \) to \( \Omega_{\text{out}} \) and then form scattering amplitudes like

\[
\left( a_{\text{out}}^*(h_1') \cdots a_{\text{out}}^*(h_m') \Omega_{\text{out}}, a_{\text{in}}^*(h_1) \cdots a_{\text{in}}^*(h_n) \Omega_{\text{in}} \right)
\]  

(52)

This can also be written

\[
\left( a_{\text{in}}^*(h_1') \cdots a_{\text{in}}^*(h_m') \Omega_{\text{in}}, S a_{\text{in}}^*(h_1) \cdots a_{\text{in}}^*(h_n) \Omega_{\text{in}} \right)
\]  

(53)

and \( S \) is revealed as a scattering operator.
4 Point source - I

Again we study the equations $\Box A = j_T$ and $\nabla \cdot A = 0$ and look for solutions which are free in the distant past and future. We now specialize to a point source with current $j(t, x) = x'(t)\delta(x - x(t))$. We assume in this section that $x'(t)$ has compact support. So the particle is at rest in the distant past, accelerates for finite amount of time, and then stops in the distant future.

Again we take $A_{\text{in}}(f, t)$ to be a free field as defined in (18) for real $f \in \mathcal{S}(\mathbb{R}^3)$. Then we define

$$A(f, t) = A_{\text{in}}(f, t) + (G_+ * j_T)(f, t) \quad (54)$$

Now we do not define $G_+ * j_T$ pointwise, but treat it as a distribution and proceed by throwing everything onto the test function $f$. Since $G_+(x, t) = G_-(-x, -t)$ we interpret $(G_+ * j_T)(f, t)$ as

$$\langle G_+ * j_T, f \otimes \delta_t \rangle = \langle j_T, G_- * (f \otimes \delta_t) \rangle \quad (55)$$

Then formally we can transfer the transverse projection from $j$ and to $f$ and get

$$\langle j_T, G_- * (f \otimes \delta_t) \rangle = \langle j, G_- * (f_T \otimes \delta_t) \rangle \quad (56)$$

The latter is well-defined for our current since $G_- * (f_T \otimes \delta_t)$ as given by (9) is a bounded function continuous in the spatial variable. With this interpretation (54) becomes

$$A(f, t) = A_{\text{in}}(f, t) + \langle j, G_- * (f_T \otimes \delta_t) \rangle \quad (57)$$

Since $j$ has compact support and $G_- * (f_T \otimes \delta_t)$ vanishes to the future of $t$, we have that $\langle j, G_- * (f_T \otimes \delta_t) \rangle$ vanishes for $t$ sufficiently negative and so $A(f, t) = A_{\text{in}}(f, t)$ in the distant past. Since $A_{\text{in}}(f, t)$ is free and since $G_-$ is a fundamental solution we have

$$\Box A_{\text{out}}(f, t) = \frac{\partial^2 A_{\text{out}}}{\partial t^2}(f, t) - A_{\text{out}}(\Delta f, t) = \langle j, \frac{\partial^2}{\partial t^2}G_- * (f_T \otimes \delta_t) - G_- * ((\Delta f)_T \otimes \delta_t) \rangle$$

$$= \langle j, \Box(G_- * (f_T \otimes \delta_t)) \rangle = \langle j, f_T \otimes \delta_t \rangle \equiv \langle j_T, f \otimes \delta_t \rangle \quad (58)$$

So $A$ satisfies $\Box A = j_T$ in the sense of distributions. Similarly $\nabla \cdot A = 0$.

Next we define $A_{\text{out}}(f, t)$ by

$$A(f, t) = A_{\text{out}}(f, t) + G_- j_T(f, t) \quad (59)$$

which is interpreted as

$$A(f, t) = A_{\text{out}}(f, t) + \langle j, G_+ * (f_T \otimes \delta_t) \rangle \quad (60)$$
Then $A(f, t) = A_{out}(f, t)$ in the distant future.

Combining (57) and (60) gives

\[ A_{out}(f, t) = A_{in}(f, t) - \langle j, G \ast (f_T \otimes \delta_t) \rangle \]

Then $A_{out}$ satisfies $\Box A_{out} = 0$ and $\nabla \cdot A_{out} = 0$ in the sense of distributions and is a free field.

Now take the expression $(G \ast (f_T \otimes \delta_t))(x, s) = u_{f_T}(x, s - t)$ from (10) and find

\[ \langle j, G \ast (f_T \otimes \delta_t) \rangle = \int x'(s) \delta(x - x(s))u_{f_T}(x, s - t)dxds \]

\[ = \int x'(s)u_{f_T}(x(s), s - t)ds \]

\[ = (2\pi)^{-\frac{3}{2}}i \int x'(s) \left( e^{-i\omega(s-t)} - e^{i\omega(s-t)} \right) e^{ipx(s)} \tilde{f}_T(p) \frac{dp}{2\omega} ds \]

\[ = (2\pi)^{-\frac{3}{2}} \int x'(s) \left( e^{i\omega(s-t)+ipx(s)} \tilde{f}_T(p) - e^{i\omega(s-t)-ipx(s)} \tilde{f}_T(-p) \right) \frac{dp}{2\omega} ds \]

\[ = (2\pi)^{-\frac{3}{2}} 2 \text{Re} \int x'(s) \left( -ie^{i\omega(s-t)-ipx(s)} \tilde{f}_T(-p) \right) \frac{dp}{2\omega} ds \]

(62)

Now change the order of integration (easily justified), and define (new definition)

\[ J(p) = -i(2\pi)^{-\frac{3}{2}} \int x'(s)e^{i\omega s-ipx(s)} ds \]

(63)

Then

\[ \langle j, G \ast (f_T \otimes \delta_t) \rangle = 2 \text{Re} \int e^{-i\omega t} \tilde{f}_T(-p)J(p) \frac{dp}{2\omega} = 2 \text{Re} (e^{i\omega t} \tilde{f}_T, J_T) \]

(64)

Then integral converges since $J_T(p)$ is bounded and $\tilde{f}_T(p)$ is bounded and rapidly decreasing. We have written it as an inner product in $\mathcal{H}$ but are not asserting that $J_T \in \mathcal{H}_T$, at least not yet. Now we have as before

\[ A_{out}(f, t) = A_{in}(f, t) - 2 \text{Re} (J_T, e^{i\omega t} \tilde{f}_T) \]

(65)

We consider the linear function $h \rightarrow (J_T, h)$ for $h$ in a dense domain $\mathcal{S} \subset \mathcal{H}_T$ containing the functions $e^{i\omega t} f_T$ for $f \in \mathcal{S}(\mathbb{R}^3)$. It is the domain of of rapidly decreasing functions:

\[ \mathcal{S} = \{ h \in \mathcal{H}_T : \text{ for } k \geq 0 \text{ there is } C \text{ so } |h((p))| \leq C(1 + |p|^2)^{-k} \} \]

(66)

For $h \in \mathcal{S}$ we define

\[ a^*_out(h) = a^*_in(h) - (J_T, h) \]

\[ a_{out}(h) = a_{in}(h) - (h, J_T) \]

(67)
Instead of changing operators we have changed states. The form of the equation shows that these are elements of $C$ fields are $A$ and then as in (63) - (65)

$$A_{\text{out}}(f, t) = a^*_\text{out}(e^{i\omega t} \tilde{f}_T) + a_{\text{out}}(e^{i\omega t} \tilde{f}_T)$$

just as for $A_{\text{in}}(f, t)$.

Now we can write $A_{\text{in/out}}(f, t) = \sqrt{2}A_{\text{in/out}}(e^{i\omega t} \tilde{f}_T)$ where the momentum space fields are $A_{\text{in/out}}(h) = \frac{1}{\sqrt{2}}(a^*_{\text{in/out}}(h) + a_{\text{in/out}}(h))$. In terms of these fields the defining relation (67) can be written

$$A_{\text{in/out}}(h) = A_{\text{in}}(h) - \sqrt{2} \Re(J_T, h)$$

With this identity we pass to the $C^*$ algebra. Define

$$W_{\text{out}}(h) = e^{iA_\omega \text{out}(h)} \quad W_{\text{in}}(h) = e^{iA_\omega \text{in}(h)}$$

and then

$$W_{\text{out}}(h) = W_{\text{in}}(h)e^{-i\sqrt{2} \Re(J_T, h)}$$

These are elements of $C^*$ algebra $A$ generated by $W_{\text{in}}(h), h \in S$. In the Fock vacuum $\omega_{\text{in}} = (\Omega_{0, \text{in}}, \cdots )\Omega_{0, \text{in}}$ we have

$$\omega_{\text{in}}(W_{\text{out}}(h)) = \omega_{\text{in}}(W_{\text{in}}(h))e^{-i\sqrt{2} \Re(J_T, h)}$$

But $W_{\text{out}}(h)$ is again a representation of the CCR and generates the same $C^*$ algebra $A$ as $W_{\text{in}}(h)$. Then there is a unique $*$-automorphism $\alpha$ on $A$ such that $\alpha(W_{\text{out}}(h)) = W_{\text{in}}(h)$ (see for example [1]). This satisfies $\alpha(W_{\text{in}}(h)) = W_{\text{in}}(h)e^{i\sqrt{2} \Re(J_T, h)}$. If we define a new state by $\omega_{\text{out}} \equiv \omega_{\text{in}} \circ \alpha$ then

$$\omega_{\text{out}}(W_{\text{in}}(h)) = \omega_{\text{in}}(W_{\text{in}}(h))e^{i\sqrt{2} \Re(J_T, h)}$$

Instead of changing operators we have changed states. The form of the equation shows that $\omega_{\text{out}}$ is a coherent state by our $C^*$ algebra definition.

The question is now whether $\omega_{\text{out}}$ is a Fock state. Before answering the question we sharpen our criterion for when this is the case.

**Proposition 2.** $\omega_{\text{out}}$ is a Fock state if and only if $J_T \in \mathcal{H}_T$.

**Proof.** We have already noted that $\omega_{\text{out}}$ is a Fock state if and only if the linear function $h \to (J_T, h)$ is continuous on the pre-Hilbert space $S$. We now claim this is equivalent to $J_T \in \mathcal{H}_T$. If $J_T \in \mathcal{H}_T$ then continuity is immediate. If the continuity holds then there is $J' \in \mathcal{H}_T$ such that $(J_T, h) = (J', h)$ for $h \in S$. If $h$ rapidly decreasing then $h_T \in S$ and $(J_T - J', h) = (J_T - J', h_T) = 0$. Then $(J_T - J', h) = 0$ for $h \in S(\mathbb{R}^3)$. Hence $(J_T - J')/\omega$ as distributions and hence almost everywhere as functions. Thus $J_T = J'$ almost everywhere and so $J_T \in \mathcal{H}_T$. This completes the proof.

In the following we assume that the velocity of the charged particle is less than the speed of light which is here one.

12
Proposition 3. Let \( x(t) \) be \( C^3 \) with \( |x'(t)| \leq v_0 < 1 \), and suppose \( x'(t) \) has compact support. Then \( J_T \in \mathcal{H}_T \) and \( \omega_{out} \) is the Fock state defined by \( \Omega_{out} = \Omega_{J_T} \).

Proof. It suffices to show \( J \in \mathcal{H} \). Ignoring the multiplicative constants in (63) we might as well assume that

\[
J(p) = \int x'(t) e^{i\psi(p,t)} dt \quad \psi(p,t) = \omega t - p \cdot x(t) \tag{74}
\]

Since this is a bounded continuous function of \( p \), to decide whether it is in \( \mathcal{H} \) we need to study asymptotics as \( p \to \infty \). Our first thought might be the stationary phase method. So we look for points where

\[
\psi_t(p,t) = \omega - p \cdot x'(t) = 0 \tag{75}
\]

However \( |p \cdot x'(t)| < |p|v_0 = \omega v_0 \) and

\[
\omega(1 - v_0) \leq |\psi_t(p,t)| \leq \omega(1 + v_0) \tag{76}
\]

There is no point of stationary phase for \( p \neq 0 \). Instead we can integrate by parts and write with \( \psi_{tt} = -p \cdot x''(t) \)

\[
J(p) = \int \frac{\psi_t}{\psi_t^2} x'(t) - \frac{1}{\psi_t} x''(t) e^{i\psi(p,t)} dt = \frac{1}{i} \int \left[ \psi_{tt} x'(t) - \frac{1}{\psi_t^2} x''(t) \right] e^{i\psi(p,t)} dt \tag{77}
\]

Now \( x', x'' \) are bounded and \( \psi_t^{-1} \) and \( \psi_{tt}/\psi_t^2 \) are \( O(\omega^{-1}) \) for \( |p| \geq 1 \). Hence \( J(p) = O(\omega^{-1}) \). This is not quite enough for convergence, but we can repeat the last step and get

\[
J(p) = \frac{1}{i} \int \left[ \frac{\psi_{tt}}{\psi_t^2} x'(t) - \frac{3 \psi_{tt}^2}{\psi_t^4} x'(t) + \frac{3 \psi_{tt}}{\psi_t^3} x''(t) - \frac{1}{\psi_t^2} x'''(t) \right] e^{i\psi(p,t)} dt \tag{78}
\]

Now \( x', x'', x''' \) are all bounded and the coefficients are all \( O(\omega^{-2}) \) for \( |p| \geq 1 \). Thus \( J(p) = O(\omega^{-2}) \) and we have

\[
\int_{|p| \geq 1} |J(p)|^2 \frac{dp}{2\omega} \leq \text{const} \int_{|p| \geq 1} \omega^{-5} dp < \infty \tag{79}
\]

Hence \( J \in \mathcal{H} \) and the proof is complete.

Remark. We can allow jump discontinuities in \( x''(t) \) without changing the result. Suppose we allow a jump discontinuity in \( x''(t) \), say at \( t = 0 \). Then in the integration by parts in (78) we get a boundary term which is

\[
\left[ \frac{p \cdot (x''(0^-) - x''(0^+)) x'(0)}{\psi_t^3(p,0)} + \frac{x''(0^-) - x''(0^+)}{\psi_t^2(p,0)} \right] e^{i\psi(p,0)} \tag{80}
\]
This is still $\mathcal{O}(\omega^{-2})$ and our results still hold. However a discontinuity in the velocity $x'(t)$ is more serious as we now show.

**Proposition 4.** Suppose $x(t)$ is continuous on $\mathbb{R}$ and $C^3$ on $(-\infty, 0]$ and $[0, \infty)$ with
\[
x'(0^-) - x'(0^+) \neq 0 \tag{81}
\]
Suppose also $x'(t)$ has compact support and $|x'(t)| \leq v_0 < 1$. Then $J_T \notin \mathcal{H}_T$ and $\omega_{out}$ is not a Fock state.

**Proof.** Since $J(p)$ is still bounded, it again suffices to consider asymptotics as $p \to \infty$. In the integration by parts in (77) there is now a boundary term and we have
\[
J(p) = \int x'(t) \frac{1}{\psi_t} \frac{\partial}{\partial t} e^{i\psi(p,t)} dt
= \frac{1}{i} \int \left[ \frac{\psi_t}{\psi_t^2} x'(t) - \frac{1}{\psi_t} x''(t) \right] e^{i\psi(p,t)} dt + \frac{1}{i} \Delta(p) e^{i\psi(p,0)} \tag{82}
\]
where
\[
\Delta(p) = \frac{x'(0^-)}{\omega - p \cdot x'(0^-)} - \frac{x'(0^+)}{\omega - p \cdot x'(0^+)} \tag{83}
\]
The first term in (82) is analyzed as in the previous proposition, unaffected by possible discontinuities in $x''$, $x'''$, and the transverse part is in $\mathcal{H}_T$. Thus to prove our result it suffices to show that $\Delta_T$ is not in $\mathcal{H}_T$.

Let $v^- = x'(0^-)$ and $v^+ = x'(0^+)$. Then this can be written
\[
\Delta(p) = \frac{\omega(v^- - v^+) + (v^+ p \cdot v^- - v^- p \cdot v^+)}{(\omega - p \cdot v^-)(\omega - p \cdot v^+)} \tag{84}
\]
Now $v^+ p \cdot v^- - v^- p \cdot v^+ = p \times (v^+ \times v^-)$. If we define $\delta v = v^- - v^+ \neq 0$ and $n_p = p/\omega = p/|p|$ then $\Delta(p) = \Delta_1(p) + \Delta_2(p)$ where
\[
\Delta_1(p) = \frac{1}{\omega} \left[ \frac{\delta v}{(1 - n_p \cdot v^-)(1 - n_p \cdot v^+)} \right] \tag{85}
\]
\[
\Delta_2(p) = \frac{1}{\omega} \left[ \frac{n_p \times (v^+ \times v^-)}{(1 - n_p \cdot v^-)(1 - n_p \cdot v^+)} \right]
\]
Note that the denominators here are bounded below since $|v^\pm| < 1$. Now $\Delta_2(p)$ is already transverse, and the transverse part of $\Delta_1(p)$ is with $n_{\delta v} = \delta v/|\delta v|$
\[
\Delta_{1,T}(p) = \frac{1}{\omega} \left[ \frac{\delta v - n_p(n_p \cdot \delta v)}{(1 - n_p \cdot v^-)(1 - n_p \cdot v^+)} \right] = \frac{\delta v}{\omega} \left[ \frac{n_{\delta v} - n_p(n_p \cdot n_{\delta v})}{(1 - n_p \cdot v^-)(1 - n_p \cdot v^+)} \right] \tag{86}
\]
We have now $\Delta_T(p) = \Delta_{1,T}(p) + \Delta_2(p)$. 

14
Now we divide into two cases. Either \( v^\pm \) are colinear or not. If they are colinear then \( \Delta_2(p) = 0 \), and we compute

\[
\frac{1}{2\omega} \int_{|p| \geq 1} |\Delta_T(p)|^2 dp \geq \frac{1}{2\omega} \int_{|p| \geq 1, |n_p \cdot n_{\delta v}| \leq \frac{1}{2}} |\Delta_{1,T}(p)|^2 dp \\
\geq \text{const} |\delta v|^2 \int_{|p| \geq 1, |n_p \cdot n_{\delta v}| \leq \frac{1}{2}} \frac{1}{\omega^3} dp = \infty
\]  

(87)

If \( v^\pm \) are not colinear then \( v_\times \equiv v^+ \times v^- \neq 0 \). We look at the component of \( \Delta_T(p) \) along \( v_\times \). Again \( \Delta_2(p) \) does not contribute, nor does the \( n_{\delta v} \) term in \( \Delta_{1,T}(p) \). We have

\[
n_{v_\times} \cdot \Delta_T(p) = \frac{|\delta v|}{\omega} \left[ \frac{-(n_p \cdot n_{v_\times})(n_p \cdot n_{\delta v})}{(1 - n_p \cdot v^-)(1 - n_p \cdot v^+)} \right]
\]

(88)

Then

\[
\frac{1}{2\omega} \int_{|p| \geq 1} |\Delta_T(p)|^2 dp \geq \frac{1}{2\omega} \int_{|p| \geq 1, |n_p \cdot n_{v_\times}| \geq \frac{1}{2}, |n_p \cdot n_{\delta v}| \geq \frac{1}{2}} |n_{v_\times} \cdot \Delta_T(p)|^2 dp \\
\geq \text{const} |\delta v|^2 \int_{|p| \geq 1, |n_p \cdot n_{v_\times}| \geq \frac{1}{2}, |n_p \cdot n_{\delta v}| \geq \frac{1}{2}} \frac{1}{\omega^3} dp = \infty
\]

(89)

The last step follows since \( \delta v \) orthogonal to \( v_\times \) implies the cones \( |n_{v_\times} \cdot n_p| \geq \frac{1}{2} \) and \( |n_p \cdot n_{\delta v}| \geq \frac{1}{2} \) have a non-empty intersection. This completes the proof.

5 Point source - II

Now we consider a trajectory \( x(t) \) which has constant velocity \( v_{\text{in}} \) in the distant past, and constant velocity \( v_{\text{out}} \) in the distant future, and is otherwise smooth. So we no longer have compact support for \( x'(t) \), but we do have compact support for \( x''(t) \).

There is now a problem with the convergence of integrals like (63). To avoid this we approximate the current \( j \) by

\[
j_{R}(x,t) = \begin{cases} 
  x'(t)\delta(x - x(t)) & |t| \leq R \\
  0 & \text{otherwise}
\end{cases}
\]

(90)

and seek to take the limit \( R \to \infty \). The analysis proceeds as in the the previous section and we have \( A_{\text{out}}(f,t) = A_{\text{in}}(f,t) - \left< j_R, G * (f_T \otimes \delta_t) \right> \) where

\[
\left< j_R, G * (f_T \otimes \delta_t) \right> = (2\pi)^{-\frac{3}{2}} 2 \text{ Re} \int_{-R}^{R} x'(s) \int \left[ \int \frac{-i \epsilon^{i\omega(s-t)-ipx(s)} f_T(-p)}{2\omega} dp \right] ds
\]

(91)
Proposition 5. Let $x(t)$ be $C^3$ with $|x'(t)| \leq v_b < 1$, and suppose $x''(t)$ has compact support. Then the limit $R \to \infty$ exists on the right side of (77) and defines the left side at $R = \infty$. We have as in (63)

$$\langle j, G \ast (f_T \otimes \delta_t) \rangle = 2 \Re \int e^{-i \omega t} \tilde{f}_T(-p) J(p) \frac{dp}{2\omega} = 2 \Re (e^{i \omega t} \tilde{f}_T, J_T)$$

(92)

where up to a multiplicative constant

$$J(p) = \frac{1}{i} \int \left[ \frac{\psi_{tt}}{\psi_t^2} x'(t) - \frac{1}{\psi_t} x''(t) \right] e^{i \psi(p,t)} dt$$

(93)

as in (77), and the representation (78) for $J(p)$ holds as well.

Proof. Note that since $\psi_{tt} = -p \cdot x''(t)$ the integrand on the right side of (93) has compact support in $t$ and that integral exists. Furthermore $J(p)$ is $O(\omega^{-1})$ everywhere and $\tilde{f}_T(p)$ is bounded and rapidly decreasing so the integral (92) converges. Once (93) is established (78) holds by a further integration by parts.

To derive (92) it suffices to consider the case $t = 0$. Then in (91) with $\psi(p,s) = \omega s - p \cdot x(s)$ we need to find an $R \to \infty$ limit for

$$\int_{-R}^R x'(s) \left[ \int e^{i \psi(p,s)} \tilde{f}_T(-p) \frac{dp}{2\omega} \right] ds$$

$$= \int_{-R}^R x'(s) \frac{d}{ds} \left[ \int e^{i \psi(p,s)} \tilde{f}_T(-p) \frac{dp}{2\omega} \right] ds + \int_{-R}^R x'(s) \left[ \int e^{i \psi(p,s)} \tilde{f}_T(-p) \frac{dp}{2\omega} \right] ds$$

$$= \frac{1}{i} \int_{-R}^R \left[ \frac{e^{i \psi(p,s)} x'(s)}{\psi_s^2} x'(s) - \frac{1}{\psi_s} x''(s) \right] e^{i \psi(p,s)} \tilde{f}_T(-p) \frac{dp}{2\omega} ds$$

$$+ \left[ x'(s) \int \frac{e^{i \psi(p,s)} \tilde{f}_T(-p) \frac{dp}{2\omega}}{\psi_s^2} \right]_{s=-R}^{s=R}$$

(94)

In the double integral the $R \to \infty$ limit exists by the support of $x''(t)$. The integrand is $O(\omega^{-2})$ as $|p| \to 0$ and rapidly decreasing as $|p| \to \infty$, so the integral is absolutely convergent. Changing the order of integration gives the result (92), provided the endpoint term goes to zero.

For the endpoint term we need to show $\int \psi_{t}^{-1} e^{i \psi(t,p)} \tilde{f}_T(-p) dp/2\omega$ goes to zero as $|t| \to \infty$. As $t \to \infty$ we have $x'(t) \to v_{out}$ so it suffices to show

$$\int \frac{1}{\omega - p \cdot v_{out}} e^{i(\omega - p \cdot v_{out})t} \tilde{f}_T(-p) \frac{dp}{2\omega}$$

(95)

goes to zero as $t \to \infty$. Let $p \to -p$ and then write this in polar coordinates $(|p|, \theta, \phi)$ with $v_{out}$ as the $z$-axis. Then $\omega + p \cdot v_{out} = \omega(1 + \cos \theta \ v_{out})$ and $(1 + \cos \theta \ v_{out})$ is bounded above and below. The integral is now

$$\int \frac{1}{2(1 + \cos \theta \ v_{out})} \left[ \int e^{i|p|(1 + \cos \theta \ v_{out})t} \tilde{f}_T(|p|, \theta, \phi) \ |d|p| \right] \sin \theta d\theta d\phi$$

(96)
Now the interior integral goes to zero by the Riemann-Lebesgue lemma, and the overall integral goes to zero by dominated convergence. The limit $t \to -\infty$ is entirely similar. This completes the proof.

Having established (92), (93) we proceed as in (65) - (73) and again find

$$\omega_{\text{out}}(W_{\text{in}}(h)) = \omega_{\text{in}}(W_{\text{in}}(h))e^{i\sqrt{2}\Re(J_T,h)}$$

(97)

**Proposition 6.** Let $x(t)$ be $C^3$ with $|x'(t)| \leq v_0 < 1$, and suppose $x'(t) = v_{\text{in}}$ in the distant past and $x'(t) = v_{\text{out}}$ in the distant future so $x''(t)$ has compact support. Then $\omega_{\text{out}}$ is a Fock state if and only if $v_{\text{in}} = v_{\text{out}}$.

**Proof.** We must show that $J_T \in \mathcal{H}_T$ if and only if $v_{\text{in}} = v_{\text{out}}$. We can use the representation (78) as before to get convergence for $|p| \geq 1$. So the issue is whether the integral for $|p| \leq 1$ converges. (In proposition 3 convergence for $|p| \leq 1$ was easy.) Our general bound is that $J(p)$ is $O(\omega^{-1})$, as in $J_T(p)$. We show that in case $v_{\text{in}} \neq v_{\text{out}}$ this is actually sharp and gives a logarithmic divergence.

In the representation (84) for $|p| \leq 1$ we write $e^{ir(p,t)} = 1 + O(\omega)$ and correspondingly $J(p) = F(p) + O(1)$. The $O(1)$ term is in $\mathcal{H}$ and the transverse part in in $\mathcal{H}_T$, so it suffices to consider $F(p)$ and show that $F_T(p)$ either is or is not square integrable for $|p| \leq 1$ with respect to $(2\omega)^{-1}dp$.

We have up to a multiplicative constant

$$F(p) = \int \left[ \frac{x'(p, x'')}{(\omega - p \cdot x'(t))^2} + \frac{x''}{(\omega - p \cdot x'(t))} \right] dt$$

$$= \int \frac{d}{dt} \left[ \frac{x'(t)}{(\omega - p \cdot x'(t))} \right] dt$$

$$= \frac{v_{\text{out}}}{(\omega - p \cdot v_{\text{out}})} - \frac{v_{\text{in}}}{(\omega - p \cdot v_{\text{in}})}$$

$$= \frac{\omega(v_{\text{out}} - v_{\text{in}}) + (v_{\text{in}} p \cdot v_{\text{out}} - v_{\text{out}} p \cdot v_{\text{in}})}{(\omega - p \cdot v_{\text{out}})(\omega - p \cdot v_{\text{in}})}$$

(98)

where $\delta v = v_{\text{out}} - v_{\text{in}}$. We see that if $v_{\text{in}} = v_{\text{out}}$ then $F = 0$ and $F_T = 0$ and hence $J \in \mathcal{H}$ and $J_T \in \mathcal{H}_T$ and $\omega_{\text{out}}$ is Fock.

We proceed with the case $v_{\text{out}} \neq v_{\text{in}}$. We are now almost exactly in the situation that occurred in the proof of Proposition 4. The main difference is that we consider $|p| \leq 1$ rather than $|p| \geq 1$. We write $F_T(p) = F_{1,T}(p) + F_2(p)$ where with $\delta v \equiv v_{\text{out}} - v_{\text{in}}$ and $v_x = v_{\text{in}} \times v_{\text{out}}$

$$F_{1,T}(p) = \frac{|\delta v|}{\omega} \left[ \frac{n_{\delta v} - n_p(n_p \cdot v_{\delta v})}{(1 - n_p \cdot v_{\text{out}})(1 - n_p \cdot v_{\text{in}})} \right]$$

$$F_2(p) = \frac{1}{\omega} \left[ \frac{n_p \times v_x}{(1 - n_p \cdot v_{\text{out}})(1 - n_p \cdot v_{\text{in}})} \right]$$

(99)
If \( v_{\text{in}} \) and \( v_{\text{out}} \) are colinear (which includes the case where one of them is zero), then only \( F_{1,T} \) contributes and we have as in (87)

\[
\int_{|p|\leq 1} |F_T(p)|^2 \frac{dp}{2\omega} \geq \text{const} |\delta v|^2 \int_{|p|\leq 1, |n_p n_{\delta v}| \leq \frac{1}{2}} \frac{1}{\omega^3} dp = \infty \quad (100)
\]

If \( v_{\text{in}} \) and \( v_{\text{out}} \) are not colinear we look at the component along \( v_{\times} \) and find as in (88)

\[
n_{v_{\times}} \cdot F_T(p) = \frac{|\delta v|}{\omega} \left[ -\frac{(n_p \cdot n_{v_{\times}})(n_p \cdot n_{\delta v})}{(1 - n_p \cdot v_{\text{out}})(1 - n_p \cdot v_{\text{in}})} \right] \quad (101)
\]

Then as in (89)

\[
\int_{|p|\leq 1} |F_T(p)|^2 \frac{dp}{2\omega} \geq \text{const} |\delta v|^2 \int_{|p|\leq 1, |n_p n_{v_{\times}}| \geq \frac{1}{2}, |n_p n_{\delta v}| \geq \frac{1}{2}} \frac{1}{\omega^3} dp = \infty \quad (102)
\]

Thus the \( F_T \) condition fails, and \( \omega_{\text{out}} \) is not Fock.

**Remarks.**

1. Note that if the particle has constant velocity then \( x''(t) = 0 \), hence \( J(p) = 0 \), and \( \omega_{\text{out}} = \omega_{\text{in}} \). No photons are radiated.

2. The result does not depend on the complete history of the trajectory, only on the initial and final states. Except in the special circumstance \( v_{\text{in}} = v_{\text{out}} \) we always leave Fock space.

3. It is noteworthy that the departure from Fock space comes from a UV divergence in the case of discontinuities in the velocity (proposition 4), and from an IR divergence in the case \( v_{\text{in}} \neq v_{\text{out}} \) (proposition 6).

4. It might be of interest to obtain results of this kind for the radiation of gravitons by a classical source in linearized quantum gravity. See Skagerstam, Eriksson, Rekdal [8], for a formulation of this model in an analog of the Coulomb gauge. In this connection we also mention the work of Kegeles, Oriti, Tomlin [5] who suggest non-Fock coherent states of the type used here as appropriate for a group field theory model of quantum gravity.
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