Weighted Fréchet–Kolmogorov Theorem and Compactness of Vector-Valued Multilinear Operators

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Abstract
In this paper, we establish a weighted version of the well-known Fréchet–Kolmogorov theorem, which holds for weights beyond $A_\infty$. This weighted theory extends the previous known unsatisfactory results in the terms of relaxing the index to the natural range. As applications, we obtain the weighted compactness theory for the commutators of multilinear vector-valued Calderón–Zygmund type operators, including the commutators of multilinear Littlewood–Paley type operators. It is worthy to point out that the commutators we considered contain almost all the commutators formerly studied in this literature.

Keywords Fréchet–Kolmogorov theorem · Vector-valued multilinear operators · Compactness · Commutators

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# 1 Introduction

Let $1 \leq p < \infty$ and $F$ be a subset in $L^p(\mathbb{R}^n)$. The well-known Fréchet–Kolmogorov theorem ([22], [34, p. 275]) was first proved by Riesz [22] in 1933. It states that:

**Theorem A** [22], [34, p. 275]. $F$ is sequentially compact in $L^p(\mathbb{R}^n)$ if and only if the following three conditions are satisfied:

(i) $\sup_{f \in F} \| f \|_{L^p} < \infty$;

(ii) $\lim_{N \to \infty} \sup_{f \in F} \int_{|x| \geq N} |f(x)|^p \, dx = 0$;

(iii) $\lim_{|t| \to 0} \sup_{f \in F} \int_{\mathbb{R}^n} |f(\cdot + t) - f(\cdot)|^p \, dx = 0$.

The weighted Fréchet-Kolmogorov theorem was given by Clop and Cruz in [8].

**Theorem B** [8]. Let $1 \leq p < \infty$, $w \in A_p$, $F \subset L^p(w)$. Then $F$ is a compact set in $L^p(w)$ if and only if the following three conditions hold:

(i) $\sup_{f \in F} \| f \|_{L^p(w)} < \infty$;

(ii) $\lim_{N \to \infty} \sup_{f \in F} \int_{|x| \geq N} |f(x)|^p w(x) \, dx = 0$;

(iii) $\lim_{|t| \to 0} \sup_{f \in F} \int_{\mathbb{R}^n} |f(\cdot + t) - f(\cdot)|^p w(x) \, dx = 0$.

It was actually only pointed out in [8] that conditions (i)–(iii) in Theorem B were sufficient for the set $A$ to be a compact set in $L^p(\omega)$ when $p > 1$. But it is easy to see that their argument also holds for $p = 1$. The necessary part in Theorem B follows from the proof of the unweighted case in [29].

It is natural to ask whether Fréchet–Kolmogorov theorem is true or not for $0 < p < 1$. In 1951, Tsuji [28] showed that the unweighted Fréchet–Kolmogorov theorem can be extended to $0 < p < 1$. This article hasn’t received much attention for a long time. Based on this somehow unnoticed paper [28], the first aim of this paper is to show that the weighted Fréchet–Kolmogorov theorem can also be extended to the case $0 < p < 1$. Moreover, we actually showed that the following weighted Fréchet–Kolmogorov theorem holds even for more general weights than $A_\infty$, which could be of interest in its own.

**Theorem 1.1** Let $w$ be a weight on $\mathbb{R}^n$. Assume that $w^{-1/(p_0-1)}$ is also a weight on $\mathbb{R}^n$ for some $p_0 > 1$. Let $0 < p < \infty$ and $F$ be a subset in $L^p(w)$. Then $F$ is sequentially compact in $L^p(w)$ if and only if the following three conditions are satisfied:

(i) $\sup_{f \in F} \| f \|_{L^p(w)} < \infty$;

(ii) $\lim_{N \to \infty} \sup_{f \in F} \int_{|x| \geq N} |f(x)|^p w(x) \, dx = 0$;

(iii) $\lim_{|t| \to 0} \sup_{f \in F} \int_{\mathbb{R}^n} |f(\cdot + t) - f(\cdot)|^p w(x) \, dx = 0$.

With Theorem 1.1 in hand, the second main aim of this paper is to apply Theorem 1.1 in the study of compactness of the commutators of Calderón–Zygmund type operators. We begin by recalling some known results.
It was well known that the boundedness of the linear commutators of Calderón–Zygmund operators was given by Coifman, Rochberg and Weiss [9] when the symbol is in $BMO(\mathbb{R}^n)$. Later on, commutators of some classical operators, such as the Calderón–Zygmund operators [29], the Fourier multipliers [10] and certain Littlewood–Paley square functions [7], have been shown that they are not only bounded on $L^p$ spaces but also compact when they were multiplied with functions in $CMO(\mathbb{R}^n)$.

The bilinear commutators of the Calderón–Zygmund operator $T$ were first studied by Pérez and Torres [20], which are defined by

\[ [T, b]_1(f, g)(x) = (T(bf, g) - bT(f, g))(x) \quad (1.1) \]

\[ [T, b]_2(f, g)(x) = (T(f, bg) - bT(f, g))(x). \quad (1.2) \]

Related topics were later further studied by Tang [23], Lerner et al. [17] and Xue [30]. The concept of compactness for a bilinear operator was first introduced by Bényi and Torres [3]. They proved that the commutators of bilinear Calderón–Zygmund operator were also compact from $L^{p_1} \times L^{p_2}$ to $L^p$ if $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 < p_1, p_2 < \infty$ and $p \geq 1$. Later on, this compactness property has been extended to the following operators: the maximal bilinear Calderón–Zygmund operators [12], the bilinear Fourier multiplier operators [16] and the bilinear pseudodifferential operators [2]. The working spaces have also been extended to weighted Lebesgue spaces [4] and Morry spaces [11].

All the previously mentioned results of compactness need to assume that $1 < p_1, p_2 < \infty$ and $p \geq 1$. However, the boundedness of the commutators of the above operators shows that $p \geq 1$ is unnecessary for the boundedness to be hold. Moreover, the condition $p \geq 1$ comes from the application of Fréchet–Kolmogorov theorem [34], as well as its weighted analogue [8]. However, Tsuji [28] had already showed that the unweighted Fréchet–Kolmogorov theorem can be extended to $p > 0$. Recently, the region of $p$ was extended to the case $1/2 < p \leq 1$ by Torres et al. [26]. It was shown that if $b \in CMO$, $1 < p_1, p_2 < \infty$ and $1/2 < p < \infty$ with $1/p_1 + 1/p_2 = 1/p$, then the commutator defined in (1.1) or (1.2), $[T, b]_j : L^{p_1} \times L^{p_2} \rightarrow L^p$ is a compact bilinear operator. Still more recently, Chaffee et al. [5] further proved that the commutators of certain kinds of homogeneous bilinear Calderón–Zygmund operators enjoy the compactness property if and only if $b \in CMO$ and $p > 1$.

We will consider the compactness of the following generalized commutator of multilinear Calderón–Zygmund operator.

**Definition 1.2** [32] Let $T$ be an $m$-linear Calderón–Zygmund operator and $S$ be a finite subset of $\mathbb{Z}^+ \times \{1, \ldots, m\}$. For any $f_j \in \mathcal{S}$, $j = 1, \ldots, m$, the generalized commutator $T_{b,S}^{\tilde{f}}$ of $T$ is defined by

\[ T_{b,S}^{\tilde{f}}(x) = \int_{(\mathbb{R}^n)^m} \prod_{(i,j) \in S} (b_i(x) - b_i(y_j)) K(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy_1, \ldots, dy_m. \]
$T_{b,S}$ is called to be the generalized commutator of $T$ due to its capacity and flexibility with respect to $S$. For example by choosing $S = \{(i,i) : i \in \{1, \ldots, m\}\}$, or $S = \{(j,j) : j \in \{1, \ldots, m\}\}$, then $T_{b,S}$ coincides with $T_b(f)$ [17] and $T_{[1]}b$ [18], respectively. Other choice of selection may lead to new type of commutators of $T$.

Instead of considering the compactness for the commutators of operator $T$, we will try to establish directly a compactness theory for the commutators of vector-valued multilinear operators. This is mainly because multilinear Littlewood-Paley operators, such as multilinear $g$-function, Marcinkiewicz integral and $g_x^*-$function (see Sect. 3 for the definitions) can be regarded as multilinear vector-valued Calderón–Zygmund operators [33]. Therefore, the extension of the compactness result to vector-valued operators does make sense and it will enable us to study the generalized commutators of these multilinear square operators similarly defined as in Definition 1.2 and will enable us to show that the commutators of them are all compact. For more recent works related to the compactness of other operators, such as the bilinear Fourier multipliers and the bilinear pseudodifferential operators, we refer the readers to [25] and [27].

We begin with some definitions. Let $0 < r < \infty$. For any quasi-Banach space, denote $B_{r,X} = \{x \in X : \|x\|_X \leq r\}$. Then there is a natural extension of the corresponding definition in [3] ($m = 2$).

**Definition 1.3** Let $X_j (j = 1, \ldots, m), Y$ be quasi-Banach spaces. A multilinear operator $T : X_1 \times \cdots \times X_m \to Y$ is called a compact operator, if $T(B_{1,X_1} \times \cdots \times B_{1,X_m})$ is relatively compact in $Y$.

**Remark 1** In the above definition, it is equivalent to require $T(B_{1,X_1} \times \cdots \times B_{1,X_m})$ to be sequentially compact in $Y$, as $Y$ is a quasi-Banach space.

Let $B(X_1 \times \cdots \times X_m, Y)$ be the set of all bounded multilinear operators from $X_1 \times \cdots \times X_m$ to $Y$ and let $K(X_1 \times \cdots \times X_m, Y)$ be the set of all compact operators from $X_1 \times \cdots \times X_m$ to $Y$. Then, we have $K(X_1 \times \cdots \times X_m, Y)$ is closed in $B(X_1 \times \cdots \times X_m, Y)$. To see this, when $Y$ is a Banach space and $m = 2$, this property is shown in [3]. As a matter of fact, it is sufficient to assume that $Y$ is a quasi-Banach space. The extension from $m = 2$ to general $m$ needs only simple modification in the proof given in [3].

**Definition 1.4** Let $1 < p_j < \infty, j = 1, \ldots, m$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$, $\overrightarrow{\omega} = (\omega_1, \ldots, \omega_m)$, $A$ is a nonempty subset of $\{1, \ldots, m\}$. Denote $\frac{1}{p_{A}} = \sum_{j \in A} \frac{1}{p_j}, \nu_{\overrightarrow{\omega},A} = \prod_{j \in A} \omega_j^{p_{j}} / p_{j}$. We say that $\overrightarrow{\omega}$ satisfies the $A_{\overrightarrow{p},A}$ condition, or $\overrightarrow{\omega}_A \in A_{\overrightarrow{p},A}$, if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q v_{\overrightarrow{\omega},A} \right)^{\frac{1}{p_{A}}} \prod_{j \in A} \left( \frac{1}{|Q|} \int_Q \omega_j^{1-p_j} \right)^{\frac{1}{p_j}} < \infty.$$  

**Remark 2** If $\omega_i \in A_{p_i}$, $i \in \{1, \ldots, m\}$, then for any nonempty subset $A$, it follows that $v_{\overrightarrow{\omega},A} \in A_{p_{|A|}}$ and $\overrightarrow{\omega}_A \in A_{\overrightarrow{p},A}$.

We denote by $BMO(\mathbb{R}^n)$ the John–Nirenberg space of function of bounded mean oscillation endowed with its usual norm. The space of $C^\infty$ functions with compact
support is denoted by $C^\infty_c$, we define

$$CMO = \overline{C^\infty_c BMO},$$

the closure of $C^\infty_c$ in the $BMO$ norm.

Our second main result is as follows:

**Theorem 1.5** Let $1 < p_j < \infty$, $1 \leq j \leq m$ and $S_j = \{i : (i, j) \in S\}$ for $1 \leq j \leq m$. Let $B$ be a Banach space, $T$ be a $B$-valued $m$-linear Calderón–Zygmund operator, and for any $i \in \bigcup_{j=1}^m S_j$, $b_i \in CMO$. If $\vec{\omega} \in A_{\vec{p}}$, $\vec{\omega}_{A^c} \in A_{\vec{p}_{A^c}}$, and $\nu_{\vec{\omega}, A} \in A_{\mu_{A^c}|A|}$, then $T_{\vec{b}, S}$ is a compact operator from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p_B(\nu_{\vec{\omega}})$.

**Remark 3** When $A = \{1, \ldots, m\}$, $v_{\vec{\omega}, A} = v_{\vec{\omega}}$. Then, in Theorem 1.5, it is enough to assume that $\vec{\omega} \in A_{\vec{p}}$.

Theorem 1.5 implies not only the compactness of generalized commutators of scalar-valued multilinear Calderón–Zygmund operator, but also the compactness of generalized commutators of multilinear square operators, we summarize these results in two corollaries.

**Corollary 1.6** Let $S_j = \{i : (i, j) \in S\}$ for $1 \leq j \leq m$. Let $1 < p_j < \infty$, $j = 1, \ldots, m$, $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, $\vec{\omega} \in A_{\vec{p}}$, and for any $i \in \bigcup_{j=1}^m S_j$, $b_i \in CMO$. Then $T_{\vec{b}, S}$ is compact from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$.

**Corollary 1.7** Let $T$ be any one of the following three operators: multilinear Littlewood–Paley $g$-function, multilinear Marcinkiewicz integral, and multilinear $g^*_\lambda$-function (see Sect. 3 for the definitions). Assume $1 < p_i < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and for any $i \in \bigcup_{j=1}^m S_j$, $b_i \in CMO$. If $\vec{\omega} \in A_{\vec{p}}$, then $T_{\vec{b}, S}$ is a compact operator from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$.

This paper will be organized as follows. We will recall some definitions and known results about multilinear Calderón–Zygmund operators, multiple weights, generalized commutators and multilinear square operators in Sect. 2. Section 3 is devoted to present the theory of the vector-valued multilinear Calderón–Zygmund operators, including their generalized commutators. The proofs of Theorem 1.1 and Theorem 1.5 will be given in Sects. 4 and 5, respectively.

### 2 Preliminaries

#### 2.1 Multilinear C–Z Operator and Multiple Weights

For $f_j \in S(\mathbb{R}^n)$, $1 \leq j \leq m$, and $x \notin \bigcap_{j=1}^m \text{supp } f_j$, the multilinear operator $T$ is defined by

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \ldots dy_m,$$
where the kernel $K(x, y_1, \ldots, y_m)$ is a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying, for some $\varepsilon$, $A_\varepsilon > 0$,

(i) $|K(x, y_1, \ldots, y_m)| \leq \frac{C}{(\sum_{j=1}^{m} |x-y_j|)^{mn}}$;

(ii) $|K(x, y_1, \ldots, y_i, \ldots, y_m) - K(x, y_1, \ldots, y'_i, \ldots, y_m)| \leq \frac{C|y_i-y'_i|^\varepsilon}{(\sum_{j=1}^{m} |x-y_j|)^{mn+\varepsilon}}$

whenever $|y_i - y'_i| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j|$;

(iii) $|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq \frac{C|x-x'|^\varepsilon}{(\sum_{j=1}^{m} |x-y_j|)^{mn+\varepsilon}}$

whenever $|x - x'| \leq \frac{1}{2} \sum_{j=1}^{m} |x - y_j|$.

An $m$-linear (or quasi-linear) operator is called bounded at some point if it is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^p$ for some $1 \leq p_i \leq \infty$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$ and $p < \infty$.

Following the work of Grafakos and Torres [13], we call the above $T$ is a mutilinear Calderón–Zygmund operator if it is bounded at some point. Grafakos and Torres [13] showed that a mutilinear Calderón–Zygmund operator is always bounded from $L^1 \times \cdots \times L^1$ to $L^\frac{m}{m-\varepsilon}$ after introducing the multiple weights associated with the so-called new maximal functions, the authors in [17] established a nice weighted theory for $T$. We first recall the definition of $A_{\vec{\rho}}$ weight class.

**Definition 2.1** [17]. Let $1 \leq p_1, \ldots, p_m < \infty$, $p$ satisfies $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. For any $i = 1, \ldots, m$, let $\omega_i$ be a weight, which is nonnegative and locally integrable function. Given $\vec{\omega} = (\omega_1, \ldots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{1/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{\rho}}$ condition if

$$
\sup_B \left( \frac{1}{|B|} \int_B \prod_{i=1}^{m} \omega_i^{p_i/p_i} \right)^{1/p} \prod_{i=1}^{m} \left( \frac{1}{|B|} \int_B \omega_i^{1-p_i} \right)^{1/p_i} < \infty,
$$

when $p_i = 1$, $\left( \frac{1}{|B|} \int_B \omega_i^{1-p_i} \right)^{1/p_i}$ is understood as $(\inf_B \omega_i)^{-1}$.

Given a nonempty set $A \subset \{1, \ldots, m\}$, define

$$
\mathcal{M}_A(\vec{f})(x) = \sup_{Q \ni x} \prod_{j \in A} \frac{1}{|Q|} \int_Q |f_j(y_j)|dy_j
$$

and denote $\mathcal{M}_{\vec{\omega}}(\vec{f})(x) = 1$. The related results in [17] can be summarized as follows.

**Theorem 2.2** [17] Let $A = \{1, 2, \ldots, m\}$, $1 < p_i < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\vec{\omega}$ satisfy the $A_{\vec{\rho}}$ condition. Then

$$
\|\mathcal{M}_A(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}.
$$
Theorem 2.3 [17] Let $T$ be an $m$-linear Calderón–Zygmund operator and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition, $1 < p_i < \infty$. Then

$$
\left\| T(f) \right\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \| f_i \|_{L^{p_i}(\omega_i)}.
$$

2.2 Generalized Commutators

The following three kinds of commutators were firstly introduced and studied in [19], [17] and [18], respectively.

$$
\begin{align*}
T_{\vec{b}}(f)(x) &= \int_{\mathbb{R}^n} \left[ \prod_{j=1}^m (b_j(x) - b_j(y)) \right] K(x, y) f(y) dy, \quad (2.1) \\
T_{\vec{b}}(\vec{f})(x) &= \sum_{i=1}^m (b_i T(\vec{f})(x) - T(f_1, \ldots, b_i f_i, \ldots, f_m)(x)), \quad (2.2) \\
T_{\prod b}(\vec{f})(x) &= \int_{\mathbb{R}^{nm}} \prod_{j=1}^m (b_j(x) - b_j(y_j)) K(x, y_1, \ldots, y_m) dy_1 \ldots dy_m. \quad (2.3)
\end{align*}
$$

The other main results obtained by these papers are that all these commutators enjoy a natural weighted strong and weighted endpoint boundedness. Although these commutators are of course different with each other and subsequently the proofs and the results are independent there. However, one can always see that their proofs are all in similar patterns. In [32], we introduced a kind of generalized commutators which contains the three type of commutators mentioned above.

Definition 2.4 [32] Let $T$ be an $m$-linear Calderón–Zygmund operator with kernel $K$. Let $S$ be a finite subset of $Z^+ \times \{1, \ldots, m\}$. The commutators of $T$ are defined by

$$
T_{\vec{b}, S}(\vec{f})(x) = \int_{\mathbb{R}^{nm}} \prod_{(i, j) \in S} (b_i(x) - b_i(y_j)) K(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy, \quad (2.4)
$$

for all $f_j \in S$, $j = 1, \ldots, m$, and all $x \notin \bigcap_{j=1}^m \text{supp } f_j$.

Not very surprisingly, this kind of commutators also enjoys the natural weighted strong and weighted endpoint boundedness. For instance, it holds that

Theorem 2.5 [32] Let $\vec{\omega} \in A_{\vec{p}}$ with $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, $1 < p_j < \infty$, $j = 1, \ldots, m$. Then there exists a constant $C$ such that for any $f_j \in L^{p_j}(\omega_j)$, it holds that

$$
\left\| T_{\vec{b}, S}(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{(i, j) \in S} \| b_i \|_{BMO} \prod_{j=1}^m \| f_j \|_{L^{p_j}(\omega_j)}.
$$
2.3 Multilinear Square Operators

Recall that, when it comes to the linear theory, we have basically three typical square operators, the Littlewood–Paley $g$-function, the Marcinkiewicz integral and the $g^*_κ$-function. Correspondingly, we may define three kinds of multilinear square operators. To begin with, we first introduce two kinds of kernels.

**Definition 2.6** Let $K$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^{mn}$ with $\text{supp} \, K \subseteq B := \{(x, y_1, \ldots, y_m) : \sum_{j=1}^{m} |x - y_j|^2 \leq 1\}$. $K$ is called a multilinear Marcinkiewicz kernel if for some $0 < \delta < mn$ and some positive constants $A$, $γ_0$, and $B_1$,

(a) $|K(x, y)| \leq \frac{A}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\delta}}$;

(b) $|K(x, y) - K(x, y_1, \ldots, y_i', \ldots, y_m)| \leq \frac{A|y_j - y'_j|^{\gamma_0}}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\delta+\gamma_0}}$;

(c) $|K(x, y) - K(x', y_1, \ldots, y_m)| \leq \frac{A|x - x'|^{\gamma_0}}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\delta+\gamma_0}}$,

where (b) holds whenever $(x, y_1, \ldots, y_m) \in B$ and $|y_j - y'_j| \leq \frac{1}{B_1} |x - y_j|$ for all $0 \leq i \leq m$, and (c) holds whenever $(x, y_1, \ldots, y_m) \in B$ and $|x - x'| \leq \frac{1}{B_1} \max_{1 \leq j \leq m} |x - y_j|$.

**Definition 2.7** Let $K(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. $K$ is called a multilinear Littlewood–Paley kernel if for some positive constants $A$, $γ_0$, $δ$, and $B_1$, it holds that

(d) $|K(x, y)| \leq \frac{A}{(1 + \sum_{j=1}^{m} |x - y_j|)^{mn+\delta}}$;

(e) $|K(x, y) - K(x, y_1, \ldots, y_i', \ldots, y_m)| \leq \frac{A|y_j - y'_j|^{\gamma_0}}{(1 + \sum_{j=1}^{m} |x - y_j|)^{mn+\delta+\gamma_0}}$;

(f) $|K(x, y) - K(x', y_1, \ldots, y_m)| \leq \frac{A|x - x'|^{\gamma_0}}{(1 + \sum_{j=1}^{m} |x - y_j|)^{mn+\delta+\gamma_0}}$,

where (e) holds whenever $|y_j - y'_j| \leq \frac{1}{B_1} |x - y_j|$ and for all $0 \leq i \leq m$, and (f) holds whenever $|x - x'| \leq \frac{1}{B_1} \max_{1 \leq j \leq m} |x - y_j|$.

Given a kernel $K$, denote $K_t(x, y_1, \ldots, y_m) = t^{-mn} K\left(\frac{x}{t}, \frac{y_1}{t}, \ldots, \frac{y_m}{t}\right)$. Define the multilinear square function by

$$G(\vec{f})(x) = \left( \int_0^\infty \left| \int_{(\mathbb{R}^n)^m} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j) dy_1 \ldots dy_m \left| \frac{dt}{t} \right|^2 \right)^{1/2},$$

for any $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^{m} \text{supp} \, f_j$.

Suppose that $G$ is bounded at some point. $G$ is called a multilinear Marcinkiewicz operator when $K$ is a multilinear Marcinkiewicz kernel. $G$ is called a multilinear Littlewood–Paley $g$-function when $K$ is a multilinear Littlewood–Paley kernel.

Meanwhile, the multilinear square $g^*_κ$-function associated with the above kernel $K$ is defined by

$$G^*_κ(\vec{f})(x) = \left( \int \int_{\mathbb{R}^{n+1}} \left( \frac{t}{|x - z| + t} \right)^{nκ} \left| \int_{\mathbb{R}^{nm}} K_t(z, \vec{y}) \prod_{j=1}^{m} f_j(y_j) d\vec{y} \right|^2 \frac{dz dt}{t^{n+1}} \right)^{1/2}.$$
whenever \( \tilde{f} = (f_1, \ldots, f_m) \in S(\mathbb{R}^n) \times S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \) and \( x \notin \bigcap_{j=1}^{m} \text{supp } f_j \), with itself bounded at some point.

See [6,24,31] respectively for the convolution type of the above three kinds of multilinear square operators, where endpoint estimates as well as the weighted boundedness, like Theorem 2.3, were obtained. Although each of the proofs is complete and independent and somehow seems quite different, once again, we can actually tackle these square operators in a unified manner [33], by viewing all of them as vector-valued multilinear square operators. The following two lemmas are the crucial estimates.

**Lemma 2.1** [33] When \( K \) is either a multilinear Littlewood–Paley kernel or multilinear Marcinkiewicz kernel, there exists some positive constants \( \gamma, A, \) and \( B \), such that

\[
\left( \int_0^\infty |K_t(x, \tilde{y})|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq \frac{A}{(\sum_{j=1}^{m} |x - y_j|)^{mn}},
\]

\[
\left( \int_0^\infty |K_t(z, \tilde{y}) - K_t(x, \tilde{y})|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq \frac{A|z - x|^{\gamma}}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\gamma}},
\]

whenever \( |z - x| \leq \frac{1}{B} \max_{j=1}^{m} \{|x - y_j|\} \); and

\[
\left( \int_0^\infty |K_t(x, \tilde{y}) - K_t(x, y_1, \ldots, y_i', \ldots, y_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq \frac{A|y_i - y_i'|^{\gamma}}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\gamma}}
\]

for any \( i \in \{1, \ldots, m\} \), whenever \( |y_i - y_i'| \leq \frac{|x - y_j|}{B} \).

**Lemma 2.2** [33] When \( K \) is a multilinear Littlewood–Paley kernel, there exists some positive constants \( \gamma, A, \) and \( B \), such that

\[
\left( \int_{\mathbb{R}_{+}^{n+1}} \left( \frac{t}{|x - z| + t} \right)^{n\lambda} |K_t(z, \tilde{y})|^2 \frac{dz \, dr}{t^{n+1}} \right)^{\frac{1}{2}} \leq \frac{A}{(\sum_{j=1}^{m} |x - y_j|)^{mn}};
\]

\[
\left( \int_{\mathbb{R}_{+}^{n+1}} \left( \frac{t}{|z| + t} \right)^{n\lambda} |K_t(x, \tilde{y}) - K_t(x', \tilde{y})|^2 \frac{dz \, dr}{t^{n+1}} \right)^{\frac{1}{2}} \leq \frac{A|x - x'|^{\gamma}}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\gamma}},
\]

whenever \( |x - x'| \leq \frac{1}{B} \max_{j=1}^{m} \{|x - y_j|\} \); and if \( |y_i - y_i'| \leq \frac{|x - y_j|}{B} \), it holds that

\[
\left( \int_{\mathbb{R}_{+}^{n+1}} \left( \frac{t}{|x - z| + t} \right)^{n\lambda} |K_t(z, \tilde{y}) - K_t(z, y_1, \ldots, y_i', \ldots, y_m)|^2 \frac{dz \, dr}{t^{n+1}} \right)^{\frac{1}{2}} \leq \frac{A|y_i - y_i'|^{\gamma}}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\gamma}}.
\]
Now, we define the generalized commutators of multilinear square operators.

\[ G_{\vec{b},S}(\vec{f})(x) = \left( \int_0^\infty \left| \int_{\mathbb{R}^n} \prod_{(i,j) \in S} (b_i(x) - b_i(y_j)) K_t(x, y) \right| \frac{dt}{t} \right)^{1/2} \times \prod_{j=1}^m f_j(y_j) dy_1 \cdots dy_m 1^{1/2}, \]

and

\[ G^*_{\lambda,\vec{b},S}(\vec{f})(x) = \left( \int_{\mathbb{R}^{n+1}} \left( \frac{t}{|x - z| + t} \right)^{n\lambda} \left| \prod_{(i,j) \in S} (b_i(x) - b_i(y_j)) K_t(z, y) \right| \right)^{1/2} \times \prod_{j=1}^m f_j(y_j) d\vec{y} \left| \frac{dz dt}{t^{n+1}} \right|^{1/2}. \]

### 3 Vector-Valued Theory

Let \( B \) be a quasi-Banach space, \( 0 < p < \infty \). For a \( B \)-valued strongly measurable function defined on \( \mathbb{R}^n \), define

\[ L^p_B = \left\{ f : \left( \int_{\mathbb{R}^n} \| f(x) \|^p_B dx \right)^{1/p} < \infty \right\} = \left\{ f : \| f \|_{L^p_B} < \infty \right\}. \]

We can extend the multilinear Calderón–Zygmund theory to the vector-valued case without much extra efforts. Let \( K(x, y_1, \ldots, y_m) \) be a \( B \)-valued locally integrable function defined away from the diagonal \( x = y_1 = \cdots = y_m \) in \( (\mathbb{R}^n)^{m+1} \). We define the \( B \)-valued multilinear Calderón-Zygmund operator \( T \) in the way that

\[ T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m \]

for \( f_j \in \mathcal{S}(\mathbb{R}^n), 1 \leq j \leq m, \) and \( x \not\in \bigcup_{j=1}^m \text{supp } f_j \), with the kernel satisfying, for some \( \varepsilon, A_\varepsilon > 0, \)

(i) \( \| K(x, y_1, \ldots, y_m) \|_B \leq \frac{C}{\sum_{j=1}^m |x - y_j|^\varepsilon} \);

(ii) \( \| K(x, y_1, \ldots, y_i, \ldots, y_m) - K(x, y_1, \ldots, y_i', \ldots, y_m) \|_B \leq \frac{C|y_i - y_i'|^\varepsilon}{(\sum_{j=1}^m |x - y_j|)^{m+\varepsilon}} \)

whenever \( |x - x'| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j| \);

(iii) \( \| K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m) \|_B \leq \frac{C|x - x'|^\varepsilon}{(\sum_{j=1}^m |x - y_j|)^{m+\varepsilon}} \)

whenever \( |x - x'| \leq \frac{1}{2} \sum_{j=1}^m |x - y_j| \),
and if $T$ is bounded at some point.

Similarly, such a $B$-valued operator $T$ is said to be bounded at some point if it is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^p_B$ for some $1 \leq p_i \leq \infty$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$ and $p < \infty$.

We may get the vector-valued version of the Theorem 2.3. As this is almost a step by step copies of the original proof, except for just adding the norm $\| \cdot \|_B$ step by step, we omit the proof. One can see [1,14,15,21] for part of the ideas.

**Theorem 3.1** Let $T$ be a $B$-valued $m$-linear Calderón–Zygmund operator, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition, $1 < p_i < \infty$. Then there exists a constant $C$ such that for any $f_j \in L^{p_i}(\omega_j)$, it holds that

$$
\| T(\vec{f}) \|_{L^p(\vec{\nu}),B} \leq C \prod_{i=1}^{m} \| f_i \|_{L^{p_i}(\omega_i)}.
$$

Now we may define the generalized commutators of the vector-valued multilinear Calderón–Zygmund operators. Once again, this generalized commutator also satisfies the natural weighted strong and weighted endpoint boundedness, and we omit the proof.

**Theorem 3.2** Let $\vec{\omega} \in A_{\vec{p}}$ with $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$ with $1 < p_j < \infty$, $j = 1, \ldots, m$. Then there exists a constant $C$ such that for any $f_j \in L^p(\omega_j)$, it holds that

$$
\| T_{B,S}(\vec{f}) \|_{L^p(\vec{\nu}),B} \leq C \prod_{(i,j) \in S} \| b_i \|_{BMO} \prod_{j=1}^{m} \| f_j \|_{L^{p_j}(\omega_j)}.
$$

**Remark 4** The boundedness in the above two theorems can both be extended to the endpoint cases, just as they hold for scalar-valued multilinear operators, and the proofs will be mostly copies of scalar-valued ones. We omit them.

For functions $f_1(t)$ defined on $\mathbb{R}^+$ and functions $f_2(t, z)$ defined on $\mathbb{R}^{n+1}_+$, define their norm respectively by

$$
\| f_1 \|_{H_1} = \left( \int_{0}^{\infty} \frac{|f_1(t)|^2}{t} \, dt \right)^{\frac{1}{2}},
$$

$$
\| f_2 \|_{H_2} = \left( \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{|x-z|+t} \right)^{n\lambda} |f_2(t,z)|^2 \, dxdz \right)^{\frac{1}{2}}.
$$

Lemmas 2.1 and 2.2 are thus leading to the following facts: A multilinear Marcinkiewicz operator $\mu_{\Omega}$ or a multilinear Littlewood–Paley operator $g$ is a $H_1$-valued multilinear Calderón–Zygmund operator. And a multilinear square function $g^*_\lambda$ is a $H_2$-valued multilinear Calderón–Zygmund operator.

Combining Theorem 2.5 with the above facts, we have
Corollary 3.3 Assume $T$ be any one of the following three multilinear operators $\mu_{\Omega}$, $g$ or $g^*_\lambda$. Let $T_{\bar{b}, S}$ be its generalized commutator defined similarly as in (2.4), $1 \leq p_j < \infty, j = 1, \ldots, m, \frac{1}{p_j} = \sum_{j=1}^{m} \frac{1}{p_j}, \omega_i \in A_{p_i}$. Then there exists a constant $C > 0$, such that for any $b_i \in BMO, f_j \in C_c^\infty$, it holds that

$$\|T_{\bar{b}, S}(\vec{f})\|_{L^p(v_\vec{\omega})} \leq C \prod_{(i, j) \in S} \|b_i\|_{BMO} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)}.$$  

Remark 5 The above result is new not only because it considered the generalized commutators, or the weighted case, but also for the facts that, there is no literature about compactness of commutators of multilinear square operators. The linear case is indeed known [7].

Remark 6 We can obtain the weighted endpoint boundedness for the generalized commutators of multilinear square operators, just as Remark 1 implied.

4 Proof of Theorem 1.1

We need the following Lemma.

Lemma 4.1 Let $1 \leq p < \infty$. Let $w$ be a weight on $\mathbb{R}^n$ such that $w^{-p'/p} = w^{-1/(p-1)}$ is also a weight on $\mathbb{R}^n$ (ess inf |x| \leq A > 0 for every $A > 0$ when $p = 1$). Let $G$ be a subset of $L^p(w)$. Then $G$ is relatively compact in $L^p(w)$ if it satisfies the following three conditions:

(i) There exists $K > 0$ such that $\|f\|_{L^p(w)} \leq K$ for all $f \in G$;

(ii) For any $\varepsilon > 0$ there exists $A > 0$ such that

$$\left( \int_{|x| > A} |f(x)|^p w(x) dx \right)^{1/p} < \varepsilon \text{ for any } f \in G;$$

(iii) For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left( \int_{\mathbb{R}^n} |f(x + u) - f(x)|^p w(x) dx \right)^{1/p} < \varepsilon$$
for any $f \in G$ and $|u| < \delta$.

Remark 7 Since $G' = \{f w^{1/p}; f \in G\}$ is a subset of $L^p(\mathbb{R}^n)$, from the Fréchet–Kolmogorov theorem (cf. [34]), it follows that $G$ is relatively compact in $L^p(w)$ if and only if it satisfies the following three conditions:

(i) There exists $K > 0$ such that $\|f\|_{L^p(w)} \leq K$ for all $f \in G$;

(ii) For any $\varepsilon > 0$, there exists $A > 0$ such that

$$\left( \int_{|x| > A} |f(x)|^p w(x) dx \right)^{1/p} < \varepsilon, \text{ for any } f \in G;$$
For any $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
\left( \int_{\mathbb{R}^n} |f(x + u)w(x + u)^{1/p} - f(x)w(x)^{1/p}|^p \, dx \right)^{1/p} < \varepsilon, \text{ for any } f \in G \text{ and } |u| < \delta.
\]

**Remark 8** If $w \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $w$ satisfies the conditions in Lemma 4.1.

**Proof** We shall prove only for the case $p > 1$. The case $p = 1$ can be proved with a minor change.

For any measurable set $E \subset \mathbb{R}^n$ and a locally integrable function $f$ on $\mathbb{R}^n$, we set $f_E = |E|^{-1} \int_E f(x) \, dx$. Then, for $B = B(x, t)$ we have
\[
\|f - f_B\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} \frac{1}{|B(x, t)|} \int_{B(x,t)} (f(x) - f(y))^p w(x) \, dy \right)^{1/p} \leq \left( \int_{\mathbb{R}^n} \frac{1}{|B(0, t)|} \int_{B(0,t)} (f(x) - f(y + x))^p w(x) \, dx \right)^{1/p},
\]
and so by Minkowski’s inequality, we have
\[
\|f - f_B\|_{L^p(w)} \leq \frac{1}{|B(0, t)|} \int_{B(0,t)} \left( \int_{\mathbb{R}^n} |f(x) - f(y + x)|^p w(x) \, dx \right)^{1/p} dy.
\]
Thus for any fixed $\varepsilon > 0$, choosing $\delta > 0$ in the assumption (iii) in Lemma 4.1, we see that for $0 < t < \delta$, it holds that
\[
\|f - f_{B(\cdot , t)}\|_{L^p(w)} < \varepsilon \text{ for any } f \in G. \tag{4.1}
\]
We fix such a $t > 0$. We next estimate $f_B$ and $f_B - f_{B'}$.
\[
|f_B(x, t)| \leq \left( \frac{1}{|B(x, t)|} \int_{B(x,t)} |f(y)|^p w(y) \, dy \right)^{1/p} \left( \frac{1}{|B(x, t)|} \int_{B(x,t)} w(y)^{\frac{p'}{p}} \, dy \right)^{\frac{1}{p'}}. \tag{4.2}
\]
For $f_B - f_{B'}$, we get
\[
|f_B(x, t) - f_{B'(y, t)}| = \left| \frac{1}{|B(y, t)|} \int_{B(y,t)} f(u - y + x) \, du - \frac{1}{|B(y, t)|} \int_{B(y,t)} f(u) \, du \right|
\leq \left( \frac{1}{|B(y, t)|} \int_{B(y,t)} |f(u - y + x) - f(u)| \, du \right)^{1/p}
\leq \left( \frac{1}{|B(y, t)|} \int_{B(y,t)} |f(u - y + x) - f(u)|^p w(u) \, du \right)^{1/p}
\times \left( \frac{1}{|B(y, t)|} \int_{B(y,t)} w(u)^{\frac{p'}{p}} \, du \right)^{\frac{1}{p'}}. \tag{4.3}
\]

\[ \text{Springer} \]
Now choose $A > 0$ in (ii). Since we have assumed that $w^{-p'/p}$ is a weight on $\mathbb{R}^n$, we see that there exists $c_0 > 0$ such that $\int_{B(x,t)} w(u)^{-\frac{p'}{p}} \, du \leq c_0$ for all $|x| \leq A$. Hence, $\{f_{B(x,t)}\}_{f \in G}$ is equi-bounded and equi-continuous on the closed ball $B(0, A)$. So, by Ascoli–Arzelá theorem, it is relatively compact and so totally bounded in $C(B(0, A))$. Thus, there exist a finite number of $f_1, f_2, \ldots, f_k \in G$ such that

$$
\inf \sup_{|x| \leq A} |f_{B(x,t)} - f_{j,B(x,t)}| < \varepsilon / w(B(0, A))^{1/p} \quad \text{for all } f \in G.
$$

It follows that for $f \in G$ there exists $1 \leq j \leq k$ such that

$$
\sup_{|x| \leq A} |f_{B(x,t)} - f_{j,B(x,t)}| < \varepsilon / w(B(0, A))^{1/p}.
$$

For these $f, f_j$ we have

$$
\|f - f_j\|_{L^p(w)} \leq \|(f - f_j)\chi_{|x| \leq A}\|_{L^p(w)} + \|(f - f_j)\chi_{|x| > A}\|_{L^p(w)}
$$

$$
\leq \|(f - f_{B(x,t)})\chi_{|x| \leq A}\|_{L^p(w)} + \|(f_{B(x,t)} - f_j,B(x,t))\chi_{|x| \leq A}\|_{L^p(w)}
$$

$$
+ \|(f_j,B(x,t) - f_j)\chi_{|x| \leq A}\|_{L^p(w)} + \|f\chi_{|x| > A}\|_{L^p(w)}
$$

$$
+ \|f_j\chi_{|x| > A}\|_{L^p(w)}.
$$

Hence by (iii), (4.1) and (4.4) we obtain

$$
\|f - f_j\|_{L^p(w)} \leq 5\varepsilon,
$$

which means that $G$ is totally bounded and hence relatively compact in $L^p(w)$. This completes the proof.

We now restate Theorem 1.1 in the following form:

**Lemma 4.2** Let $w$ be a weight on $\mathbb{R}^n$. Assume that $w^{-1/(p_0-1)}$ is also a weight on $\mathbb{R}^n$ for some $p_0 > 1$. Let $0 < p < \infty$ and $F$ be a subset in $L^p(w)$. Then $F$ is relatively compact in $L^p(w)$ if and only if the following three conditions are satisfied:

(i) There exists $K > 0$ such that $\|f\|_{L^p(w)} \leq K$ for all $f \in F$;

(ii) For any $\varepsilon > 0$ there exists $A > 0$ such that

$$
\left(\int_{|x| > A} |f(x)|^p w(x) \, dx\right)^{1/p} < \varepsilon \quad \text{for any } f \in F;
$$

(iii) For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$
\left(\int_{\mathbb{R}^n} |f(x + u) - f(x)|^p w(x) \, dx\right)^{1/p} < \varepsilon \quad \text{for any } f \in F \text{ and } |u| < \delta.
$$

We note that if $0 < p < 1$, the metric is defined by $\|\cdot\|_{L^p(w)}^p$. Springer
Using the idea in Tsuji [28], we may demonstrate Lemma 4.2 now.

**Proof** If \( p \geq p_0 \), we get the conclusion by Lemma 4.1. So, we assume \( p < p_0 \) and set \( 0 < a = p/p_0 < 1 \). Without loss of generality, we assume every \( f \in F \) is nonnegative function. By an elementary calculation (see [28]) it holds that

\[
|s^a - t^a| \leq |s - t|^a \quad \text{for} \ s, t > 0,
\]

and

\[
|s - t|^a \leq \frac{1}{a} \left( \frac{s + t}{|s - t|} \right)^{1-a}|s^a - t^a| \quad \text{for} \ s, t > 0.
\]

Using (4.5), we have

\[
\int_{\mathbb{R}^n} |f(x + u)^a - f(x)^a|^{p_0} w(x) \, dx \\
\leq \int_{\mathbb{R}^n} |f(x + u) - f(x)|^{ap_0} w(x) \, dx = \int_{\mathbb{R}^n} |f(x + u) - f(x)|^{p} w(x) \, dx,
\]

which implies that \( F^a := \{ f^a; f \in F \} \) satisfies the condition (iii) in Lemma 4.1 for \( p_0 \). Also, \( F^a \) satisfies the conditions (i) and (ii) in Lemma 4.1 for \( p_0 \). Hence, by Lemma 4.1, we see that \( F^a \) is relatively compact in \( L^{p_0}(w) \). Now let \( \{ f_j \} \) be a sequence of functions in \( F \). Since \( F^a \) is relatively compact in \( L^{p_0}(w) \), there exists a Cauchy subsequence of \( \{ f^a_j \} \), which we denote again by \( \{ f^a_j \} \) for simplicity. Then for any \( \varepsilon > 0 \), there exists an integer \( N \) such that for \( i, j \geq N \), it follows that

\[
\int_{\mathbb{R}^n} |f^a_i(x) - f^a_j(x)|^{p_0} w(x) \, dx < \varepsilon^{p_0}.
\]

Let \( E_\varepsilon \) be the set in \( \mathbb{R}^n \) such that

\[
\frac{f_i(x) + f_j(x)}{|f_i(x) - f_j(x)|} < \frac{1}{\varepsilon}.
\]

Then, noting \( ap_0 = p \) and using (4.6), (4.7), we have

\[
\int_{E_\varepsilon} |f_i(x) - f_j(x)|^{p} w(x) \, dx \leq a^{-p_0 \varepsilon^{(a-1)p_0}} \int_{E_\varepsilon} |f^a_i(x) - f^a_j(x)|^{p_0} w(x) \, dx \\
\leq a^{-p_0 \varepsilon^{(a-1)p_0}} \varepsilon^{p_0} = a^{-p_0 \varepsilon^p}.
\]

On \( E_\varepsilon \), by (4.6) and (i) we have

\[
\int_{E_\varepsilon} |f_i(x) - f_j(x)|^{p} w(x) \, dx \leq \int_{E_\varepsilon} |\varepsilon(f_i(x) + f_j(x))|^{p} w(x) \, dx \\
\leq \varepsilon^p \left( \int_{E_\varepsilon} |f_i(x)|^{p} w(x) \, dx + \int_{E_\varepsilon} |f_j(x)|^{p} w(x) \, dx \right) \\
\leq 2K p \varepsilon^p.
\]
By the above two estimates, it follows that \( \{ f_j \} \) is a Cauchy sequence in \( F \subset L^p(w) \). Thus \( F \) is relatively compact in \( L^p(w) \), which completes the proof. \( \square \)

**Remark 9** If \( w \in A_\infty(\mathbb{R}^n) \), \( w \) satisfies the assumption in Lemma 4.2 for some \( 1 < p_0 < \infty \).

Finally in this section we present a counter-example for the necessity of the condition (iii) in the above results. Let \( 1 < p_0 < p < \infty \) and \( \frac{1}{p} < \alpha < \frac{p}{p_0} \). Set

\[
    w(x) = |x|^{p_0 - 1} \quad \text{and} \quad f(x) = |x|^{-\alpha} \chi_{\{|x| \leq 1\}}.
\]

Then we get \( p_0 - 1 - p\alpha > -1 \) and \( p\alpha > 1 \), and hence

\[
    w \in A_p(\mathbb{R}), \quad f \in L^p(w), \quad \text{but} \quad f(\cdot + h) \notin L^p(w), \quad \forall h \neq 0.
\]

So, letting \( \mathcal{F} = \{ f \} \), we see that \( \mathcal{F} \) is a compact set in \( L^p(w) \). But \( \mathcal{F} \) does not satisfy (iii).

### 5 Proof of Theorem 1.5

Several lemmas will be needed to prove results for square operators.

**Lemma 5.1** There exists constant \( C > 0 \), such that for any \( \delta > 0 \), \( f_j \in C_\infty, \ j = 1, \ldots, m \), and any \( a \in \{1, \ldots, m\} \), it holds that

\[
    \int_{\sum_{j=1}^m |x-y_j| \leq \delta} \prod_{j=1}^m |f_j(y)| \left( \sum_{j=1}^m |x-y_j| \right)^{nm-1} d\vec{y} \leq C \delta \mathcal{M}(\vec{f})(x);
\]

\[
    \int_{\sum_{j=1}^m |x-y_j| \geq \delta} \prod_{j=1}^m |f_j(y)| \left( \sum_{j=1}^m |x-y_j| \right)^{nm+1} d\vec{y} \leq \frac{C}{\delta} \mathcal{M}(\vec{f})(x).
\]

The above estimate when \( m = 2 \) is shown in [4]. Its idea however can be applied to general \( m \in \mathbb{N} \). Once again we omit the proof.

**Lemma 5.2** For any nonempty set \( A \subset \{1, \ldots, m\} \), any constant \( N > 0 \), there exists a constant \( C > 0 \), such that for any \( f_j \in C_\infty, \ j = 1, \ldots, m \), supp \( f_k \subset B(0, N), \ k \in A \),

\[
    \int_{\mathbb{R}^nm} \prod_{j=1}^m |f_j(y_j)| \left( \sum_{j=1}^m |x-y_j| \right)^{nm} d\vec{y} \leq \frac{C}{|x||A|} \prod_{j \in A} \| f_j \|_{L^1} \mathcal{M}_{A^c}(\vec{f})(x).
\]
Proof of Lemma 5.2. Since \( \text{supp } f_k \subset B(0, N), k \in A, |x| \geq 2N \), then

\[
\int_{\mathbb{R}^{nm}} \prod_{j=1}^{m} |f_j(y_j)| \left( \sum_{j=1}^{m} |x - y_j| \right)^{-mn} d\vec{y}
\leq C \prod_{j \in A} \|f_j\|_{L^1} \int_{\mathbb{R}^{n|A^c|}} \prod_{j \in A^c} |f_j(y_j)| \left( |x| + \sum_{j \in A^c} |x - y_j| \right)^{-mn} \prod_{j \in A^c} dy_j.
\]

So it suffices to show that

\[
I := \int_{\mathbb{R}^{n|A^c|}} \prod_{j \in A^c} |f_j(y_j)| \left( |x| + \sum_{j \in A^c} |x - y_j| \right)^{-mn} \prod_{j \in A^c} dy_j \leq C \frac{C}{|x| |A|} \mathcal{M}_{A^c}(\tilde{f})(x).
\]

Decompose \( R^n |A^c| \), we get

\[
I \leq C \frac{C}{|x|^{mn}} \int_{B(x, |x|)} \prod_{j \in A^c} |f_j(y_j)| dy_j + \sum_{k=0}^{\infty} \frac{C}{(2^k |x|)^{mn}} \int_{B(x, 2^{k+1} |x|)} \prod_{j \in A^c} |f_j(y_j)| dy_j \leq C \frac{1}{|x|^{n|A^c|}} \int_{B(x, |x|)} |f_j(y_j)| dy_j + \sum_{k=0}^{\infty} \frac{C}{(2^k |x|)^n} \int_{B(x, 2^{k+1} |x|)} |f_j(y_j)| dy_j.
\]

By the definition of \( \mathcal{M}_{A^c}(\tilde{f}) \), one can get Lemma 5.2. \( \square \)

**Lemma 5.3.** Let \( B \) be a Banach space, \( T \) be a \( B \)-valued \( m \)-linear Calderón-Zygmund operator, \( 1 < p_i < \infty \), and \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \). Let \( S \) be a finite set in \( \mathbb{Z}^+ \times \{1, \ldots, m\} \), \( A = \{ j \in \{1, \ldots, m\} : (i, j) \in S \text{ for some } i \in \mathbb{Z}^+ \} \) and \( S_j = \{ i : (i, j) \in S \} \) for \( 1 \leq j \leq m \). Then, if \( \tilde{\omega} \in A_p \), \( \tilde{\omega}_{A^c} \in A_{p, A^c} \), \( v_{\tilde{\omega}, A} \in A_{pA|A|} \), and \( b_i \in C^\infty_{\mathbb{R}^n} \) for \( i \in \bigcup_{j=1}^{m} S_j \), then \( T_{\tilde{\omega}, S, \delta} \) converges to \( T_{\tilde{\omega}, S} \) in \( B(L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m), L_B^{p}(v_{\tilde{\omega}})) \) when \( \delta \to 0 \).
Proof. It follows that
\[
\|T_{b,S}(\tilde{f})(x) - T_{b,S,\delta}(\tilde{f})(x)\|_B \\
\leq \int \sum_{j=1}^m |x - y_j| \prod_{(i,j) \in S} |b_i(x) - b_i(y_j)| \|K(x, y_1, \ldots, y_m)\|_B \prod_{j=1}^m f_j(y_j) dy_1 
\]
\[\ldots \prod_{j=1}^m f_j(y_j) dy_m.\]

Let \((i_0, j_0) \in S\), then
\[
\|T_{b,S}(\tilde{f})(x) - T_{b,S,\delta}(\tilde{f})(x)\|_B \\
\leq C \|\nabla b_{i_0}\|_{L^\infty} \int \sum_{j=1}^m |x - y_j| \prod_{j=1}^m f_j(y_j) \left( \sum_{j=1}^m |x - y_j| \right)^{n-1} dy.
\]

By Lemma 5.1, one may obtain that
\[
\|T_{b,S}(\tilde{f})(x) - T_{b,S,\delta}(\tilde{f})(x)\|_B \leq C \|\nabla b_{i_0}\|_{L^\infty} \delta \mathcal{M}(\tilde{f})(x). \tag{5.1}
\]

By Theorem 2.2, Lemma 5.3 is proved. \(\square\)

Note that if \(T_n \in K(X_1 \times \cdots \times X_m, Y), n = 1, \ldots,\) and \(T_n\) converge to some \(T \in B(X_1 \times \cdots \times X_m, Y)\) in \(B(X_1 \times \cdots \times X_m, Y)\), then \(T \in K(X_1 \times \cdots \times X_m, Y)\). In order to prove Theorem 1.5, we may assume \(b_i \in C_c^\infty\) for \(i \in \bigcup_{j=1}^m S_j\) by Theorem 3.2 and the density of \(C_c^\infty\) in \(CMO\). Therefore, we only need to prove under the conditions of Theorem 1.5, for any \(\delta > 0\), \(T_{b,S,\delta}\) is a compact operator from \(L^p_1(\omega_1) \times \cdots \times L^p_m(\omega_m)\) to \(L^p_B(\nu\bar{\omega})\).

By Lemma 5.3 and Theorem 1.1, it suffices to show the following theorem:

**Theorem 5.1** Let \(T_{b,S,\delta}\) be defined as the same in Lemma 5.3. Then

(i) There exists a constant \(C > 0\), such that for any \(f_i \in C_c^\infty, i = 1, \ldots, m,\)
\[
\|T_{b,S,\delta}(\tilde{f})(x)\|_{L^p_B(\nu\bar{\omega})} \leq C \prod_{j=1}^m \|f_j\|_{L^p(\omega_j)}; \tag{5.2}
\]

(ii) For any \(\epsilon > 0\), there exists a constant \(N > 0\), such that for any \(f_i \in C_c^\infty, i = 1, \ldots, m,\)
\[
\int_{|x| \geq N} \|T_{b,S,\delta}(\tilde{f})(x)\|_{L^p_B(\nu\bar{\omega})} dx \leq \epsilon \prod_{j=1}^m \|f_j\|_{L^p(\omega_j)};
\]
(iii) For any \( \epsilon > 0 \), there exists a constant \( N > 0 \), such that for any \( f_i \in C_c^\infty, i = 1, \ldots, m, |t| < \delta \),

\[
\int_{\mathbb{R}^n} \|T_{\tilde{b},S,\delta}(\tilde{f})(x + t) - T_{\tilde{b},S,\delta,1}(\tilde{f})(x)\|_B v_\bar{\omega} dx \leq \epsilon \prod_{j=1}^m \|f_j\|_{L^p(\omega_j)}.
\]

**Proof** (i) follows directly from (5.1), Theorems 3.2 and 2.2. Now we prove (ii). Since \( b_i \in C_c^\infty(\mathbb{R}^n) \) for any \( i \in \bigcup_{j=1}^m S_j \), there exists \( N_0 \in \mathbb{N} \) such that \( \text{supp} b_j \subset B(0,N_0) \) for any \( i \in \bigcup_{j=1}^m S_j \). So for \( |x| \geq N \geq 2N_0 \), it holds that

\[
\|T_{\tilde{b},S,\delta}(\tilde{f})(x)\|_B \leq C \prod_{(i,j) \in S} \|b_i\|_{L^\infty} \int_{\mathbb{R}^n} \frac{1}{|x|^{|A|}} \prod_{j \in A} \|f_j\|_{L^1} \mathcal{M}_{A^c}(\tilde{f})(x).
\]

By Lemma 5.2, one obtains that

\[
\|T_{\tilde{b},S,\delta}(\tilde{f})(x)\|_B \leq C \prod_{(i,j) \in S} \|b_i\|_{L^\infty} \prod_{j \in A} \|f_j\|_{L^1} \mathcal{M}_{A^c}(\tilde{f})(x).
\]

By the fact that \( \frac{1}{p} = \frac{1}{p_A} + \frac{1}{p_{A^c}}, v_\bar{\omega} = v_{\bar{\omega},A}^p v_{\bar{\omega},A^c}^p \), we have

\[
\left( \int_{|x| \geq N} \|T_{\tilde{b},S,\delta}(\tilde{f})(x)\|_B^{p} v_\bar{\omega} dx \right)^{\frac{1}{p}} \leq C \prod_{(i,j) \in S} \|b_i\|_{L^\infty} \prod_{j \in A} \|f_j\|_{L^1} \left( \int_{|x| \geq N} \frac{1}{|x|^{|A|p_A}} \mathcal{M}_{A^c}(\tilde{f})(x)^p v_\bar{\omega} dx \right)^{\frac{1}{p}}
\]

\[
\leq C \prod_{j \in A} (\omega_j)^{1/p_j'} \left( B(0,N_0)) \right)^{1/p_j'} \prod_{(i,j) \in S} \|b_i\|_{L^\infty} \prod_{j \in A} \|f_j\|_{L^{p_j}(\omega_j)} \times \left( \int_{|x| \geq N} \frac{1}{|x|^{|A|p_A}} v_{\bar{\omega},A} dx \right)^{\frac{1}{p_A}} \mathcal{M}_{A^c}(\tilde{f})_{L^{p_{A^c}}(v_{\bar{\omega},A^c})}.
\]

Notice that \( \bar{\omega}_{A^c} \in A_{\bar{p}_{A^c}}, v_{\bar{\omega},A} \in A_{p_A|A|} \), it follows that

\[
\left( \int_{|x| \geq N} \|T_{\tilde{b},S,\delta}(\tilde{f})(x)\|_B^{p} v_\bar{\omega} dx \right)^{\frac{1}{p}} \leq C \prod_{j \in A} (\omega_j)^{1/p_j'} \left( B(0,N_0)) \right)^{1/p_j'} \prod_{(i,j) \in S} \|b_i\|_{L^\infty} \prod_{j \in A} \|f_j\|_{L^{p_j}(\omega_j)} \left( \int_{|x| \geq N} \frac{1}{|x|^{|A|p_A}} v_{\bar{\omega},A} dx \right)^{\frac{1}{p_A}}.
\]

\( \square \) Springer
Note for any $1 < q < \infty$, any $\omega \in A_q$, it holds that
\[
\int_{\mathbb{R}^n} \frac{\omega(x)}{(1 + |x|)^n q} dx < \infty.
\]
and hence $\int_{|x| \geq N} \frac{1}{|x|^n p_A} v_{\omega, A} dx \to 0$ as $N \to \infty$. Indeed, since $v_{\omega, A} \in A_{p_A} [A]$, there exists $1 < r_A < p_A [A]$ such that $v_{\omega, A} \in A_{r_A}$. So we get
\[
\int_{|x| \geq N} \frac{v_{\omega, A}(x)}{|x|^n p_A [A]} dx = \sum_{k=0}^{\infty} \int_{2^k N \leq |x| < 2^{k+1} N} \frac{v_{\omega, A}(x)}{|x|^n p_A [A]} dx
\leq \sum_{k=0}^{\infty} \frac{1}{(2^k N)^n p_A [A]} \int_{|x| < 2^{k+1} N} v_{\omega, A}(x) dx.
\]
It is known that if $1 \leq p < \infty$ and $\omega \in A_p (\mathbb{R}^n)$, then $\omega(\lambda B) \leq [\omega]_{A_p} \lambda^n p \omega(B)$ for any $\lambda > 0$ and any ball $B$. (see, for example, Proposition 7.1.5 (9), p. 504, Grafakos book, Classical Fourier Analysis). Then, it gives that Using this, we get
\[
\sum_{k=0}^{\infty} \frac{1}{(2^k N)^n p_A [A]} \int_{|x| < 2^{k+1} N} v_{\omega, A}(x) dx \leq \sum_{k=0}^{\infty} \frac{[v_{\omega, A}]_{A_{r_A}}}{(2^k N)^n p_A [A]} (2^{k+1} N)^n r_A v_{\omega, A}(B(0, 1))
\leq C \frac{1}{N^n p_A [A] - nr_A}.
\]
Thus, (ii) is obtained.
So, it suffices to show (iii). Note that
\[
\|T_{B, \delta, \delta}(f_1, \ldots, f_m)(x + t) - T_{B, \delta, \delta}(f_1, \ldots, f_m)(x)\|_B
= \| \int_{\mathbb{R}^n} \left( \prod_{(i, j) \in S} (b_i(x + t) - b_i(y_j)) K_\delta(x + t, y_1, \ldots, y_m) - \prod_{(i, j) \in S} (b_i(x) - b_i(y_j)) K_\delta(x, y_1, \ldots, y_m) \right) f_1(y_1) \cdots f_m(y_m) dy \|_B
\leq I + II,
\]
where
\[
I = \| \int_{\mathbb{R}^n} \left( \prod_{(i, j) \in S} (b_i(x + t) - b_i(y_j)) - \prod_{(i, j) \in S} (b_i(x) - b_i(y_j)) \right) K_\delta(x, y_1, \ldots, y_m) \times f_1(y_1) \cdots f_m(y_m) dy \|_B;
\]
\[
II = \| \int_{\mathbb{R}^n} \prod_{(i, j) \in S} (b_i(x + t) - b_i(y_j)) \left( K_\delta(x + t, y_1, \ldots, y_m) - K_\delta(x, y_1, \ldots, y_m) \right) \times f_1(y_1) \cdots f_m(y_m) dy \|_B.
\]
Weighted Fréchet–Kolmogorov theorem and compactness

Let \( a_{i,j} = b_i(x + t) - b_i(x), \) \( b_{i,j} = b_i(x) - b_i(y_j), \) one has

\[
\prod_{(i,j) \in D} (b_i(x + t) - b_i(y_j)) - \prod_{(i,j) \in S} (b_i(x) - b_i(y_j)) = \sum_{D \subseteq S} \prod_{(i,j) \in D} (b_i(x) - b_i(y_j)) \prod_{(i,j) \in S \setminus D} (b_i(x + t) - b_i(x)).
\]

Therefore, we may continue to estimate \( I. \)

\[
I \leq \sum_{D \subseteq S} \prod_{(i,j) \in S \setminus D} |b_i(x + t) - b_i(x)| \int_{\mathbb{R}^n} \prod_{(i,j) \in D} (b_i(x + t) - b_i(y_j)) K_\delta(x, y_1, \ldots, y_m) \times f_1(y_1) \ldots f_m(y_m) d\vec{y} \|_B.
\]

Furthermore, since

\[
\prod_{(i,j) \in D} (b_i(x + t) - b_i(y_j)) = \sum_{E \subseteq D} (-1)^{|E|'} \prod_{(i,j) \in E} b_i(x + h) \prod_{(i,j) \in D \setminus E} b_i(y_j),
\]

we have

\[
I \leq \sum_{D \subseteq S} \sum_{E \subseteq D} \prod_{(i,j) \in S \setminus D} |b_i(x + t) - b_i(x)| \prod_{(i,j) \in E} |b_i(x + h)|
\times \left\| \int_{\mathbb{R}^n} K_\delta(x, y_1, \ldots, y_m) \prod_{(i,j) \in D \setminus E} b_i(y_j) \prod_{j=1}^m f_j(y_j) d\vec{y} \right\|_B
\]

\[
= \sum_{D \subseteq S} \sum_{E \subseteq D} \prod_{(i,j) \in S \setminus D} |b_i(x + t) - b_i(x)| \prod_{(i,j) \in E} |b_i(x + h)|
\times \left\| T_\delta(f_1 \prod_{i : (i,1) \in D \setminus E} b_i, \ldots, f_m \prod_{i : (i,m) \in D \setminus E} b_i) \right\|_B.
\]
By (5.2), we have
\[
\left\| I \right\|_{L^p(\nu_{\vec{\omega}})} \leq \sum_{D \subseteq S} \sum_{E \subseteq D} \prod_{(i,j) \in S \setminus D} |t| \left\| \nabla b_i \right\|_{L^\infty} \prod_{(i,j) \in E} \left\| b_i \right\|_{L^\infty} \\
\times \| T_\delta (f_1 \prod_{i : (i,1) \in D \setminus E} b_i, \ldots, f_m \prod_{i : (i,m) \in D \setminus E} b_i) \|_{L^p_{\vec{\omega}}(\nu_{\vec{\omega}})} \\
\leq C \sum_{D \subseteq S} \sum_{E \subseteq D} \prod_{(i,j) \in S \setminus D} |t| \left\| \nabla b_i \right\|_{L^\infty} \prod_{(i,j) \in E} \left\| b_i \right\|_{L^\infty} \\
\times \prod_{j=1}^m \left\| f_j \prod_{i : (i,j) \in D \setminus E} b_i \right\|_{L^{p_j}(\omega_j)}.
\] (5.3)

If \(|t| \leq \frac{1}{2} \delta\), the smoothness of \(K_\delta\) yields that
\[
II \leq C \prod_{(i,j) \in S} \left\| b_i \right\|_{L^\infty} |t| \int \frac{\prod_{j=1}^m |f_j(y_j)|}{\sum_{j=1}^m |x - y_j| \geq \delta} \left( \sum_{j=1}^m |x - y_j| \right)^{nm+1} d\vec{y}.
\]

By Theorem 5.1, we have
\[
II \leq C \prod_{(i,j) \in S} \left\| b_i \right\|_{L^\infty} |t| \frac{1}{\delta} \mathcal{M}(f)(x).
\]

Thus
\[
\| II \|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{(i,j) \in S} \left\| b_i \right\|_{L^\infty} |t| \frac{1}{\delta} \prod_{j=1}^m \left\| f_j \right\|_{L^{p_j}(\omega_j)}.
\] (5.4)

(iii) follows by combining (5.3) and (5.4). Hence, we completed the proof of Theorem 5.1, and finished the proof of Theorem 1.5.

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