Hypersensitive transport in a phase model with multiplicative stimulus

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Abstract

In a simple system with periodic symmetric potential, the phase model under effect of strong multiplicative noise or periodic square wave, we found a giant response, in the form of directed flux, to an ultrasmall dc signal. The resulting flux demonstrates a bell-shaped dependence on multiplicative noise correlation time and occurs even in the case of large (compared to the signal) additive noise.

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The phenomenon of noise-induced transport, when a directed transport of matter emerges in a stochastic system with periodic potential in the absence of directed external force, attracts now a vivid and stable interest. An active theoretical and numerical investigations of such model systems with asymmetric potential (“thermal ratchets”) was stimulated by their potential ability to explain functioning of motor proteins, which are responsible for cell motion. However, the applicability of the concept is more wide. Several experimental groups found rectification of Brownian movement in chemical systems (directed diffusion ( [2], [3]) or rotation ( [2], [3]) of molecules), therefore pointing a possible application of “ratchet effect” in nanoscale mechanical devices and separation technique.

Noise-induced transport was also found in systems with symmetric periodic potential (the term “ratchet” is confusing in this case) ( [2], [3]). The mostly wide known of models for such systems is the phase model \( d\phi/dt = a - b\sin\phi \), which has been applied for the study of various physical systems, e.g., superionic conductors, Josephson junction arrays,
chemical reactions, ring-laser gyroscopes – up to neural networks. The good review of physical applications of phase model can be found in [12]. The model demonstrates, under stochastic stimulus, numerous interesting phenomena including stochastic resonance and noise-induced transport ( [9]–[11]).

The attention of investigators of noise-induced transport was, until now, focused mainly on directed flux of particles induced by noise with zero mean, in the absence of constant external driving. In the present work we consider the response to an ultrasmall external signal in phase model with strong symmetric multiplicative noise. We show that a macroscopic flux of matter appears in such a system under effect of ultrasmall dc driving. This effect resembles the phenomenon of noise-induced hypersensitivity to small signals recently found by us [13], and therefore we call the new phenomenon as hypersensitive transport (HST).

We study the phenomenon of HST for telegraph (dichotomous) and Gaussian colored noise, and observe HST in a wide range of noise correlation times, with bell-shaped dependence of transport velocity (voltage) on the latter. Also we consider the case of entirely deterministic system when a multiplicative periodic stimulus replaces a multiplicative noise; HST in this case occurs as well.

The system under study is described by the following dimensionless equation:

\[
\frac{d\varphi}{dt} = [z(t) - z_c] \sin \varphi + F + \sigma \psi(t).
\] (1)

Here \(z_c, \sigma\) are the positive constants, and \(\psi(t)\) is \(\delta\)-correlated white noise source. The additive dc term \(F\) is called further the signal. The control stimulus \(z(t)\) can be stochastic or deterministic function. We consider here several types of it, namely, exponentially correlated dichotomous noise with amplitude \(\Delta\) and flipping rate \(\gamma/2\), so that \(\langle z(t) z(t') \rangle = \Delta^2 \exp[-\gamma (t - t')]\), colored Gaussian noise with the same autocorrelator, and deterministic square wave with amplitude \(\Delta\) and period \(T\). We restrict ourselves by the case of small signal, i.e., \(z_c, \Delta, \gamma, T^{-1} \gg \sigma, F \sim 10^{-n}, \quad n \gg 1\).

The problem (1) was considered by several authors ([7], [9]–[11]). However, nobody of
them paid a special attention to the region of ultrasmall $\sigma$ and $F$, where the phenomenon of HST occurs.

First, we note that we use sufficiently large stimulus ($\Delta > z_c$), so when it is switched, the stable and unstable fixed points in Eq. (1) are switched also. In fact, Eq. (1) at different time moments becomes:

$$\frac{d\varphi}{dt} = f_{\pm}(\varphi) + F + \sigma \psi(t),$$

$$f_{\pm}(\varphi) = (\pm \Delta - z_c) \sin \varphi = -\frac{\partial U_{\pm}(\varphi)}{\partial \varphi}. \quad \text{(2)}$$

The stable fixed points (FPs) $\varphi_{s+}^n, \varphi_{s-}^n$ and the unstable ones $\varphi_{u+}^n, \varphi_{u-}^n$ of Eq.(2) are:

$$\varphi_{s+}^n = \pi (2n + 1) + \frac{F}{\Delta - z_c}, \quad \varphi_{s-}^n = 2\pi n + \frac{F}{\Delta - z_c},$$

$$\varphi_{u+}^n = 2\pi n - \frac{F}{\Delta - z_c}, \quad \varphi_{u-}^n = \pi (2n + 1) - \frac{F}{\Delta - z_c}. \quad \text{(3)}$$

These FPs together with the potential $U_{\pm}(\varphi) + F$ are depicted in Fig.1. One can see that for small $F$ the unstable FPs of one branch are located left, and very close to, the stable FPs of other branch as well as to the points $\pi n$.

We point out now an interesting feature of the system that results in appearance of HST. We restrict ourselves in further sketch by the case of dichotomous noise.

Let us suppose that $z(t) = \Delta$ and the system is in the stable FP $\varphi_{0+}^0$. When $z(t)$ changes its sign, the system turns to the branch $f_- + F$, therefore moving right (Fig.1). For sufficiently rarely switching stimulus ($\Delta \gg \gamma, T^{-1}$) the system passes through subsequent unstable point $\varphi_{1+}^0$ before stimulus is switched again. As a result, macroscopic transport appears. Such a picture was described in [11], however they did not considered the possibility of macroscopic response to an ultrasmall driving. For frequently switching stimulus, when the condition $\Delta > \gamma$ is not valid, the system may not have enough time to reach $\varphi_{1+}^0$, and after subsequent switching of $z(t)$ it will turn to the branch $f_+ + F$ and will move left. The system obviously can not move left of $\varphi_{0+}^0$, thus the latter point is a reflecting boundary for all $\varphi$ in the interval $(\varphi_{0+}^0, \varphi_{1+}^0)$. For square wave stimulus the system enters a closed loop, with any transport disappearing. But for stochastic $z(t)$, even when it flips rapidly, there is
nonzero probability of passage of the system through the point $\varphi_{1+}$ along the branch $f_{-} + F$. After that the system can not go back left, but easily will go right through $\varphi_{1-}$ along the branch $f_{+} + F$ after next switching of stimulus. Thus we see that the intervals $[\varphi_{n-}, \varphi_{n+}]$ and $[\varphi_{n+}, \varphi_{n-}]$ act as semitransparent mirrors, passing the system through only to the right. The width of these intervals is $2F \Delta (\Delta^2 - z_e^2)^{-1} \sim 10^{-n} \ll 1$. The described picture is valid only in the absence of additive noise. Otherwise, the system is able to pass through mirrors in both directions, however moving right predominantly, and HST still occurs.

Now let us obtain an expression for transport velocity for adiabatic stimulus. In this case, the system should come to equilibrium during the characteristic time of the process ($T$ or $1/\gamma$). This means that the system variable $\varphi$ should decrease to the value of the order of $F$ or $\sigma$, having the relaxation speed of the order of $\Delta - z_c$. Mathematically, this statement can be expressed as: $\exp \left(- (\Delta - z_c) T \right) \ll F$, $\exp \left(- (\Delta - z_c) \gamma^{-1} \right) \ll F$. Thus we get the following criteria:

$$T^{-1}, \gamma \ll \frac{\Delta - z_c}{\ln 1/F} \sim \frac{\Delta - z_c}{n}. \quad (4)$$

From Eq.(4) we see that the macroscopic transport can occur in adiabatic regime at not-so-small stimulus frequency even for ultrasmall $F$.

Let us suppose the system to be at “positive” branch ($z(t) = \Delta$) near stable FP, say $\varphi^*_{0+}$, in the local equilibrium state. The probability density of $\varphi$ in this case is $P_+ (\varphi) = C \exp \left(- \frac{2U_+(\varphi)}{\sigma^2} \right)$ ($C$ is the normalization constant). When $z(t)$ is switched and the system turns to “negative” branch, we see from Fig.1 that, if the system was located right of the point $\varphi_{0-}^u$, it begins to move right and after some time reaches $\varphi_{1-}^u$, increasing $\varphi$ by $\pi$. The probability of this event is $w_{1+}^+ = \int_{\pi - \frac{\Delta - z_c}{F}}^{\infty} P_+ (\varphi) d\varphi$. If the system in the moment of switching appeared at the left of $\varphi_{0-}^u$ it will move left up to the point $\varphi_{0-}^u$ (with $\varphi$ decrease by $\pi$) with the probability $w_{1-}^- = \int_{-\infty}^{\pi - \frac{\Delta - z_c}{F}} P_+ (\varphi) d\varphi$. If the system starts from “negative” branch, we obtain the transport probabilities $w_{2+}^\pm$ in the same way. After calculation of integrals we have:

$$w_{1,2}^{\pm} = \frac{1}{2} (1 \pm \Phi (a_{1,2}),$$

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\[ a_{1,2} \equiv \frac{2F\Delta}{\sigma(\Delta \pm z_c)\sqrt{\Delta + z_c}}, \quad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-x^2} dx. \] (5)

Now we see that for \( \Delta \pm z_c \sim 1, \) \( F, \sigma \sim 10^{-n} \) the transport probabilities \( w_{1,2}^\pm \) depend solely on the ratio \( F/\sigma. \) In zero-temperature case, when the additive noise is absent, the probabilities to move right become equal to unity.

Let us note now that in adiabatic case the system spends most of the time near stable FPs, hopping between them with an average rate \( \gamma/2 \) (or \( 2/T \) for square-wave stimulus), and we can discretize the dynamic variable: \( \varphi(t) = \pi n(t). \) Solving the corresponding master equation, we obtain the following expression for transport velocity \( V = \langle d\varphi/dt \rangle \) for stochastic stimulus:

\[
V = \frac{\pi \gamma}{4} \left( w_1^+ + w_2^+ - w_1^- - w_2^- \right) = \frac{\pi \gamma}{4} \left( \Phi(a_1) + \Phi(a_2) \right),
\]
\[
V = \frac{\pi \gamma}{2}, \quad \sigma = 0. \] (6)

For square-wave stimulus, one should replace \( \gamma \) by \( 4/T \) in Eq.(6).

Note that the HST occurs even for large additive noise \( (\sigma \gg F). \) Indeed, we get from Eqs. (5) and (6), using an asymptotic expression \( \Phi(a) = 2a/\sqrt{\pi}, \) \( a \ll 1, \) and taking into account that \( V(F) \) is an odd function:

\[
V = \frac{F \pi \gamma}{\sigma 2} A,
\]
\[
A = \frac{2\Delta}{\sqrt{\pi(\Delta^2 - z_c^2)}} \left( \sqrt{\Delta + z_c} + \sqrt{\Delta - z_c} \right). \] (7)

Therefore, since \( \sigma \sim 10^{-n}, \) the macroscopic transport still exists.

Numerical integration of Eqs.(1) was performed using an Euler algorithm with the time step \( \Delta t = 0.01. \) Values of \( V \) were obtained by averaging over 100 runs of the model with random initial conditions. The duration of one run was about \( 10^7 \) except the cases when the value of \( \varphi \) grew during that time in such extent that the finite computer precision (20 significant digits) did not allow system to respond to an ultrasmall signal \( F = 10^{-11}. \) In these cases the same statistics was obtained using more runs with shorter duration. In all simulations we take \( z_c = 0 \) and \( \sigma = 0. \)
Fig. 2 demonstrates the dependence of $V$ on noise correlation time $\gamma$ for fixed noise amplitude $\Delta = 10$ and various control stimuli. The results for deterministic one are in excellent agreement with theory. It is seen that HST disappears for stimulus frequency greater than that determined by (4). The results for adiabatic stochastic stimulus demonstrate a good agreement with second Eq.(5) that is shown in Fig.2 with dashed line. One can see also that the transport occurs for high $\gamma$ values, though in a less extent, therefore displaying a resonance on noise correlation time. It is interesting to note that near the adiabatic boundary (just left of the maximum) the deterministic stimulus induces HST more effectively than the stochastic one.

Finally, Fig.3 shows the flux dependence on the amplitude of additive noise $\sigma$. It is evident that the HST is robust to the latter, since even for signal-to-noise ratio 0.01 the transport remains macroscopic, decreasing as $1/\sigma$, in accordance with Eq.(6).

To conclude, we note that in previous publications [13] we reported a related phenomenon, the hypersensitivity to small time-dependent signals in a double-well system that manifests itself as large oscillations synchronized with signal. In the present work we demonstrate that a system with periodic potential, under effect of strong parametric, stochastic or deterministic, stimulus exhibits hypersensitivity to an ultrasmall dc signal, responding by macroscopic transport.

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Figure captions

Fig.1. The potential $U_{\pm}(\varphi) + F$ for “two-state” stimulus (Eq.(2)).

Fig.2. The transport velocity $V$ vs. correlation time $\gamma$ for 1) dichotomous stochastic stimulus (squares), 2) Gaussian one, obtained by Ornstein-Uhlenbeck process simulation (triangles), and 3) square wave (circles). The fixed amplitude $\Delta = 10$.

Fig.3. The transport velocity $V$ vs. intensity of additive white noise at $F = 10^{-11}$. The control noise is dichotomous with $\gamma = 0.02$ and $\Delta = 10$. The line is drawn in accordance with Eqs.(5) and (6).
$V \propto \sigma/F$