Lie algebra of Ashtekar-Barbero connections

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Holonomies of the Ashtekar-Barbero connection can be considered as abstract elements of a Lie group exponentially mapped from their connections representation. This idea provides a possibility to compare the geometric and algebraic properties of these objects. The result allows to identify the next-to-the-leading-order terms in the geometric expansion of the holonomy with a simple polynomial structure of connections. This relation takes the form of the Maurer-Cartan equation. In consequence, it leads to an algebraic constraint on the geometrically-motivated operations of fixing and perturbing local gauge symmetries.

I. MOTIVATION

The Ashtekar variables \cite{1} are the representation of the gravitational degrees of freedom and allow to express a particular action \cite{2}, leading to the Einstein field equations \cite{3} in a form of a gauge theory, analogous to the Yang-Mills model \cite{4}. In the case of the real Ashtekar variables, they lead to the Hamiltonian \cite{5}, which may be considered as a candidate for the classical limit of the background-independent formulation of quantum gravity called loop quantum gravity (LQG) \cite{6}. It is worth bringing up that the complex Ashtekar variables would lead to rather nonphysical candidates for observables — see \cite{8}. In this case, auxiliary effective methods \cite{7,8} of reality conditions implementation on the classical phase space would be needed.

It is known that the symmetry transformations of operators, which preserve probabilities at the quantum level, are related to a Lie algebra — cf. \cite{9}. The same algebra describes the symmetries of the candidates for field theory operators. We are going to analyze the particular candidates forming the canonical pair for the background-independent tetrad-based models of quantum gravity with a time gauge \cite{10} (mainly most variants of LQG). These variables are the real Ashtekar-Barbero connection and the densitized dreibein \cite{6,11}. However, they are not directly quantized but require rather special procedures of regularization \cite{2,11}, in which the Ashtekar-Barbero connection is replaced by a holonomy — see \cite{11,13}.

Apart from the reality of the Lie algebra-valued variables, it is required that the related generators are either unitary and linear or antilinear and antilinear \cite{12}. As in the case of the standard operators in quantum mechanics and quantum field theory, they are unitary and linear in general \cite{9}. All of the aforementioned facts encouraged this investigation of the holonomy representation algebra.

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II. THE HOLONOMY OF THE ASHTEKAR-BARBERO CONNECTION

The parallel transport of a a vector bundle element over the manifold $M$ to another bundle along a smooth path $\ell(s) : [0, 1] \to M$ is determined by the expression

$$h_{[0,1]}^{-1}[A] = \mathcal{P} \exp\left( -\int_0^1 ds \dot{\ell}(s) A_\alpha(\ell(s)) \right). \quad (1)$$

This object is called the holonomy of the vector potential $A := A^I t_I$ along the path $\ell$, where $t_I$ are the generators of a Lie algebra. The ‘inverse notation’, $h^{-1}$, follows the convention present in the LQG-related literature \cite{11}. Consequently, $h$ denotes the inverse holonomy. Taking a piecewise smooth path $\mathcal{L} := \ell^1 \circ \ell^2 \circ ...$, the holonomy is defined as a composite functional of smooth pieces $\ell^1, \ell^2, ...$. The Ashtekar-Barbero connection coefficients $A^I \in \mathbb{R}$ are specified by the following normalization of the Lie bracket,

$$[t_J, t_K] = \frac{1}{2} C^I_{JK} t_I, \quad (2)$$

where $C^I_{JK} = - C^I_{KJ} \in \mathbb{R}$ are the structure constants. By definition, this normalization resolves the issue of reality conditions implementation.

The formalism of LQG (cf. \cite{11}) introduces a graph structure, allowing to define the associated scalar product as a relation between the positions of variables on this graph, hence without using a metric. This technique is based on the framework constructed in \cite{13,14}. The holonomy $h_{\ell}^{-1} = \mathcal{P} \exp( - \int_0^\ell A )$, adjusted to a particular graph's edge (of length $l^p := L_0 \varepsilon_p$) can be expanded around the infinitesimal value of the dimensionless regularization parameter $\varepsilon_p := \varepsilon p$,

$$h_{\ell}^{-1}[A] = 1 \mp \mathbb{L}_0 \varepsilon_p A_{(p)} + \frac{1}{2} (\mathbb{L}_0 \varepsilon_p)^2 A_{(p)} A_{(p)}$$

$$\mp \frac{1}{2} (\mathbb{L}_0 \varepsilon_p)^2 \partial_{(p)} A_{(p)} + \mathcal{O}(s^3). \quad (3)$$

The indices written in the brackets are not summed and $L_0$ is a fiducial length scale. Analogously, by expanding the holonomy $h_{qr}$ around the smallest loop (closing two
edges, $l^q$ and $l^r$ that emanate from a single point) one finds

$$h^{-1}_{(q)(r)}[A] = 1 + \alpha \mathbb{L}_0 \varepsilon \mathcal{F}_{(q)(r)} + O(\varepsilon^3). \quad (4)$$

The parameter $\alpha$ takes value $1/2$ in the case of a triangular loop and $1$ in the case of a quadrilateral one (the graph’s structure depends on a selected tessellation of $M$), while $\mathcal{F}_{(q)(r)} := F_{(q)(r)} t_j$ is the curvature of $A$.

In the LQG’s regularization procedure $[6,11]$, one is required to invert relations (5, 6) to replace the Ashtekar-Barbero connection and its curvature with appropriate functionals of holonomies. Both these relations, are expanded up to the same order, however, the former consists of the derivative of $A$. Therefore, in this standard procedure, the terms of order $\varepsilon^2$ in expression (4) are omitted. Although the application of these expansions is unbalanced, by expanding holonomies and taking the limit $\varepsilon \to 0$ one recovers the original formula. However, finding an equivalent of (4) in the form of a polynomial of $A$, inverting it, and applying in the same order of $\varepsilon$, would provide a preferred formulation.

### III. LIE ALGEBRA

Let $U(\theta)$ denotes a representation of a connected Lie group described by a finite set of real continuous parameters $\theta^I$ and Hermitian generators $s_I$. The Wigner theorem $[12]$ tells that any representation of a symmetry transformation of a ray space is either a unitary and linear or else antiunitary and antilinear transformation of a Hilbert space. By expanding $U(\theta)$ around a trivial transformation, i.e. the identity, one can focus only on the unitary generators $[12]$. Consequently, in a finite neighborhood of the identity, one obtains the expansion

$$U(\theta) = 1 + i \theta^I s_I - \frac{1}{2} \theta^I \theta^K s_J s_K - \frac{i}{2} \theta^I \theta^K C^I_{J,K} s_I + O(\theta^3). \quad (5)$$

Assuming the representation of the same Lie group as in section $[11]$ $C^I_{J,K}$ are the same real structure constants, resulting from the following Lie bracket,

$$[s_J, s_K] = i C^I_{J,K} s_I. \quad (6)$$

By comparing expressions (2) and (6), one finds the explicit form of the internal representation generators of $A$,

$$t_I = -\frac{1}{2} s_I, \quad (7)$$

where $s_I$ is Hermitian and unitary.

The next natural step is to compare formulas (4) and (5). The second element of each of these sums adjusts the following parameters, $2\theta^I = \mathbb{L}_0 \varepsilon \mathcal{A}^I$. Let us remind that by expanding an abstract holonomy in (11), the physical quantities simplify as follows, $\mathbb{L}_0 \varepsilon = 1$. The last element in both considered power series connects the Ashtekar-Barbero connection derivative with the pair of connections contracted with the structure constants in the relation

$$\partial_{(p)} A_I^{(q)} = -\frac{1}{2} C^I_{J,K} A_J^{(q)} A_K^{(p)}. \quad (8)$$

This result allows reexpressing the expansion in (4) as a polynomial of $A_{(p)}$ up to the terms of order $\varepsilon^2$, therefore reaching the accuracy of the expansion in (4). Let it be reminded that the established geometrical procedure in $[6,11]$, although unbalanced, is precise enough to reproduce the pre-regularized Hamiltonian in the limit $\varepsilon \to 0$.

### IV. CONCLUSIONS

The standard procedure of regularization of the connections-related degrees of freedom in LQG can be formulated more precisely. Concerning this issue, it is worth recalling yet another formula that is constructed in a way, that geometrically expands the derivative in (4) — see $[13,10]$. However, this expansion is also based on the short path approximation and it can improve the standard approximation of a holonomy, $h^{-1}_{(p)}[A] \approx 1 + \mathbb{L}_0 \varepsilon A_{(p)}$, maximally by one order. The exact inverse of the relation in (5) (in the Ashtekar-Barbero formalism) is motivated by the power series expansion of a Lie group, hence it shares the precision of the gauge symmetry. Therefore, any theory founded on this latter approximation is as accurate as the Standard Model of particle physics.

So far, the application of the latter method has been implemented only for the model, in which (densitized) dreibeins are gauge-fixed to a diagonal form — cf. $[17]$. In this case, the identification of the representations entails the identification of the group elements

$$U(\theta) = h^{-1}[A], \quad (9)$$

where $\mathcal{A}_{(p)}$ is constant along $l^r$. The question of whether it is possible to apply the result in (8) into a physical model without gauge-fixing, remains open.

Furthermore, by rewriting the relation in (8) in terms of Lie brackets, one can obtain a new formula

$$\partial_{(p)} A_{(q)} = -[A_{(p)},A_{(q)}]. \quad (10)$$

This is the Maurer-Cartan equation $[18,19]$ for the internal space of the Ashtekar-Barbero connection. The vanishing curvature determines the geometry of the internal space as the flat case. This outcome has an important consequence concerning some gauge-fixing methods in the theories based on the holonomy representation defined in (11). In particular, this strengthens the argument in $[20]$ raised against the idea of a partial gauge fixing.
(see for instance \[21\], \[22\]). The procedures involving the correlation
\[
t_I \propto e^{\mu I}
\] (11)
of the basis vectors \(t_I\) and \(e^\mu\) of the internal and external spaces, respectively, become constrained. These conclusions are worth to be expressed as a universal theorem.

Considering a gauge theory with a set of observables given in terms of the holonomies of connections \(A := A^I_\mu t_I e^\mu\), where \(t_I\) are generators of a Lie algebra, while \(e^\mu\) are any linear combinations of these generators, both the spaces spanned by \(t_I\) and \(e^\mu\) have locally vanishing curvatures. This property is transferred from observables to the corresponding, canonically-constructed, gauge-valued operators.

This theorem puts constraints on non-standard procedures fixing the metric tensor symmetry applied to a wide class of curved space(-time) theories relying on the formalism of holonomies. In particular, it forbids to locally perturb some gauge-fixing constraints. This concerns, for instance, any gauge fixing of LQG analogous to loop quantum cosmology \[23\], \[24\] or any similar symmetry-fixing methods applied to curved space(-time) generalizations of Wilson loops-based models \[25\].

Finally, considering the \(su(2)\) representation, the set of the Hermitian and unitary generators, \(s_I = \sigma_I\), is composed of the Pauli matrices, and the structure constants \(C_{JK} = 2e^{I}_{JK}\) are the totally antisymmetric Levi-Civita tensor coefficients. This is the standard choice in LQG. In this case, the \(su(2)\)-valued Ashtekar-Barbero connection becomes a canonically conjugate partner to the densitized dreibein. Together they form the real Ashtekar variables, thus allowing the variables to represent the Holst Hamiltonian \[2\] in a quantizable form after the lattice regularization.

**ACKNOWLEDGMENTS**

This work was supported by the National Natural Science Foundation of China, through the grants No. 11675145 and 11975203. The author thanks Piotr Latasiewicz for language editing.

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