LOCAL PROPERTIES ON THE REMAINDERS OF THE TOPOLOGICAL GROUPS

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Abstract

When does a topological group $G$ have a Hausdorff compactification $bG$ with a remainder belonging to a given class of spaces? In this paper, we mainly improve some results of A. V. Arhangel’skii and C. Liu’s. Let $G$ be a non-locally compact topological group and $bG$ be a compactification of $G$. The following facts are established: (1) If $bG\setminus G$ has locally a $k$-space with a point-countable $k$-network and $\pi$-character of $bG\setminus G$ is countable, then $G$ and $bG$ are separable and metrizable; (2) If $bG\setminus G$ has locally a $\theta$-base, then $G$ and $bG$ are separable and metrizable; (3) If $bG\setminus G$ has locally a quasi-$G_\delta$-diagonal, then $G$ and $bG$ are separable and metrizable. Finally, we give a partial answer for a question, which was posed by C. Liu in [16].

1. Introduction

By a remainder of a space $X$ we understand the subspace $bX\setminus X$ of a Hausdorff compactification $bX$ of $X$. In [3, 4, 5, 13, 16], many topologists studied the following question of a Hausdorff compactification: When does a Tychonoff space $X$ have a Hausdorff compactification $bX$ with a remainder belonging to a given class of spaces? A famous classical result in this study is the following theorem of M. Henriksen and J. Isbell [13]:

(M. Henriksen and J. Isbell) A space $X$ is of countable type if and only if the remainder in any (in some) compactification of $X$ is Lindelöf

Recall that a space $X$ is of countable type [10] if every compact subspace $F$ of $X$ is contained in a compact subspace $K \subseteq X$ with a countable base of open neighborhoods in $X$. Suppose that $X$ is a non-locally compact topological
group, and that $bX$ is a compactification of $X$. In [4], A. V. Arhangel’skii showed that if the remainder $Y = bX\setminus X$ has a $G_δ$-diagonal or a point-countable base, then both $X$ and $Y$ are separable and metrizable. In [16], C. Liu improved the results of A. V. Arhangel’skii, and proved that if $Y$ satisfies one of the following conditions (i) and (ii), then $X$ and $bX$ are separable and metrizable.

(i) $Y = bX\setminus X$ is a quotient $s$-image of a metrizable space, and $π$-character of $Y$ is countable;
(ii) $Y = bX\setminus X$ has locally a $G_δ$-diagonal.

In this paper, we mainly concerned with the following statement, and under what condition it is true.

**Statement** Suppose that $G$ is a non-locally compact topological group, and that $Y = bG\setminus G$ has locally a property-$\Phi$. Then $G$ and $bG$ are separable and metrizable.

Recall that a space $X$ has *locally a property-$\Phi$* if for each point $x \in X$ there exists an open set $U$ with $x \in U$ such that $U$ has a property-$\Phi$.

In Section 2 we mainly study some local properties on the remainders of the topological group $G$ such that $G$ and $bG$ are separable and metrizable if the $π$-character of $bG\setminus G$ is countable. Therefore, we extend some results of A. V. Arhangel’skii and C. Liu.

In Section 3 we prove that if the remainders of a topological group $G$ has locally a quasi-$G_δ$-diagonal, then $G$ and $bG$ are separable and metrizable. Therefore, we improve a result of C. Liu in [16]. Also, we study the remainders that are the unions of $G_δ$-diagonals.

In Section 4 we mainly give a partial answer for a question, which was posed by C. Liu in [16]. Finally, we also study the remainders that are locally hereditarily $D$-spaces.

Recall that a family $\mathcal{U}$ of non-empty open sets of a space $X$ is called a $π$-base if for each non-empty open set $V$ of $X$, there exists an $U \in \mathcal{U}$ such that $V \subset U$. The $π$-character of $x$ in $X$ is defined by $\pi_x(\mathcal{U}, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a local } π\text{-base at } x \text{ in } X\}$. The $π$-character of $X$ is defined by $\pi_X(\mathcal{U}) = \sup\{\pi_x(\mathcal{U}, X) : x \in X\}$.

The $p$-spaces are a class of generalized metric spaces [1]. It is well-known that every metrizable space is a $p$-space, and every $p$-space is of countable type.

Throughout this paper, all spaces are assumed to be Hausdorff. The positively natural numbers is denoted by $N$. We refer the readers to [10, 11] for notations and terminology not explicitly given here.

### 2. Remainders with the countable $π$-characters

Let $\mathcal{A}$ be a collection of subsets of $X$. $\mathcal{A}$ is a $p$-network [7] for $X$ if for distinct points $x, y \in X$, there exists an $A \in \mathcal{A}$ such that $x \in A \subset X - \{y\}$. The collection $\mathcal{A}$ is called a $p$-base (i.e., $T_1$-point-separating open cover) [7] for $X$ if $\mathcal{A}$
is a $p$-network and each element of $\mathcal{A}$ is an open subset of $X$. The collection $\mathcal{A}$ is a $p$-metabase [15] (in [7]), $p$-metabase is denoted by the condition (1.5)) for $X$ if for distinct points $x, y \in X$, there exists an $\mathcal{F} \in \mathcal{A}^{<\omega}$ such that $x \in (\bigcup \mathcal{F})^o \subseteq \bigcup \mathcal{F} \subseteq X \setminus \{y\}$. The collection $\mathcal{A}$ is a $p$-network [15] (in [12]), $p$-network is denoted by the condition (1.4)) for $X$ if, whenever $K \subseteq X \setminus \{y\}$ with $K$ compact in $X$, then $K \subseteq \bigcup \mathcal{F} \subseteq X \setminus \{y\}$ for some $\mathcal{F} \in \mathcal{A}^{<\omega}$.

First, we give some technique lemmas.

**Lemma 2.1** [3]. If $X$ is a Lindelöf $p$-space, then any remainder of $X$ is a Lindelöf $p$-space.

**Lemma 2.2** [16]. Let $G$ be a non-locally compact topological group. Then $G$ is locally separable and metrizable if for each point $y \in Y = bG \setminus G$, there is an open neighborhood $U(y)$ of $y$ such that every countably compact subset of $U(y)$ is metrizable and $\pi$-character of $Y$ is countable.

**Lemma 2.3.** Suppose that $X$ has a point-countable $p$-metabase. Then each countably compact subset of $X$ is a compact, metrizable, $G_\delta$-subset\(^1\) of $X$.

**Proof.** Suppose that $\mathcal{U}$ is a point-countable $p$-metabase of $X$, and that $K$ is a countably compact subset of $X$. Then $K$ is compact by [7]. According to a generalized Miščenko’s Lemma in [22, Lemma 6], there are only countably many minimal neighborhood-covers\(^2\) of $K$ by finite elements of $\mathcal{U}$, say $\{r(n) : n \in \mathbb{N}\}$. Let $V(n) = \bigcup r(n)$. Then $K \subseteq \bigcap\{V(n) : n \in \mathbb{N}\}$. Suppose that $x \in X \setminus K$. For each point $y \in K$, there is an $\mathcal{F}_y \in \mathcal{U}^{<\omega}$ with $y \in (\bigcup \mathcal{F}_y)^o \subseteq \bigcup \mathcal{F}_y \subseteq X \setminus \{x\}$. Then there is some sub-collection of $\bigcup\{\mathcal{F}_y : y \in K\}$ is a minimal finite neighborhood-covers of $K$ since $K$ is compact. Therefore, we obtain one of the collections $r(n)$ with $K \subseteq V(n) = \bigcup r(n) \subseteq X \setminus \{x\}$.]

**Lemma 2.4.** Suppose that $X$ has a point-countable $p$-metabase. Then $X$ has a point-countable $p$-metabase.

**Proof.** For each point $x \in X$, there is an open neighborhood $U(x)$ with $x \in U(x)$ such that $U(x)$ has a point-countable $p$-metabase $\mathcal{F}_x$. Let $\mathcal{U} = \{U(x) : x \in X\}$. Since $X$ is Lindelöf, it follows that there exists a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $X = \bigcup \mathcal{U}'$. Denoted $\mathcal{U}'$ by $\{U_x : i \in \mathbb{N}\}$. Obviously, $\mathcal{F} = \bigcup_{i} \mathcal{F}_x$ is a point-countable $p$-metabase for $X$. □

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\(^1\)A subset $K$ of $X$ is called a $G_\delta$-subset of $X$ if $K$ is the intersection of countably open subsets of $X$.

\(^2\)Let $\mathcal{P}$ be a collection of subsets of $X$ and $A \subseteq X$. The collection $\mathcal{P}$ is a neighborhood-cover of $A$ if $A \subseteq (\bigcup \mathcal{P})^o$. A neighborhood-cover $\mathcal{P}$ of $A$ is a minimal neighborhood-cover if for each $P \in \mathcal{P}$, $\mathcal{P}\setminus \{P\}$ is not a neighborhood-cover of $A$. 
**Theorem 2.5.** Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ has locally a point-countable $p$-metabase. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

**Proof.** It is easy to see that $G$ is locally separable and metrizable by Lemmas 2.2 and 2.3. Then $G$ is a $p$-space. Hence $Y$ is Lindelöf by Henriksen and Isbell's theorem. From Lemma 2.4 it follows that $Y = bG \setminus G$ has a point-countable $p$-metabase.

**Claim:** The space $Y$ has a $G\delta$-diagonal.

Put $G = \bigoplus_{s \in A} G_s$, where $G_s$ is a separable and metrizable subset for each $s \in A$. Let $\zeta = \{G_s : s \in A\}$, and let $F$ be the set of all points of $bG$ at which $\zeta$ is not locally finite. Since $\zeta$ is discrete in $G$, it follows that $F \subset bG \setminus G$. It is easy to see that $F$ is compact. Therefore, it follows from Lemma 2.3 that $F$ is separable and metrizable. Hence $F$ has a countable network.

Let $M = Y \setminus F$. For each point $y \in M$, there is an open neighborhood $O_y$ in $bG$ such that $O_y \cap F = \emptyset$. Since $\zeta$ is discrete, $O_y$ meets at most finitely many $G_s$. Let $L = \bigcup\{G_s : G_s \cap O_y \neq \emptyset\}$. Then $L$ is separable and metrizable. By Lemma 2.1, $L \setminus L$ is a Lindelöf $p$-space. Obviously, $L \setminus L \subset Y$. Therefore, $L \setminus L$ has a point-countable $p$-metabase. Hence $L \setminus L$ is separable and metrizable by [12], which implies that $L$ has a countable network. It follows that $L$ is separable and metrizable. Clearly, $O_y \subset L$ and $O_y \cap M$ is separable and metrizable. Therefore, $M$ is locally separable and metrizable. From Lemma 2.3 it follows that each compact subset of $Y$ is a $G\delta$-subset of $Y$. Since $F$ is compact and $Y$ is Lindelöf, it follows that $M$ is Lindelöf. Therefore, $M$ is separable. Then $M$ has a countable network. So $Y$ has a countable network, which implies that $Y$ has a $G\delta$-diagonal. Thus, Claim is verified.

Therefore, $G$ and $bG$ are separable and metrizable by [4, Theorem 5].

**Corollary 2.6.** Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ has locally a point-countable $p$-base. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

**Corollary 2.7.** Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ is locally a $k$-space with a point-countable $p$-$k$-network. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

**Proof.** Note that if $\mathcal{P}$ is a point-countable $p$-$k$-network for a $k$-space $X$, then $\mathcal{P}$ is a point-countable $p$-metabase for $X$ by [12].

A collection $\mathcal{P}$ of subsets of a space $X$ is a $k$-network [11] for $X$ if, whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \bigcup \mathcal{F} \subset U$ for some $\mathcal{F} \in \mathcal{P}^{<\omega}$.

Obviously, if a space $X$ has a point-countable $k$-network, then $X$ has a point-countable $p$-$k$-network. So we have the following Theorem 2.8, which improves the result [16, Theorem 4] of C. Liu.
THEOREM 2.8. Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ is locally a k-space with a point-countable k-network. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

COROLLARY 2.9 [4]. Suppose that $G$ is a non-locally compact topological group. If $Y = bG \setminus G$ has a point-countable base, then $G$ and $bG$ are separable and metrizable.

Next, we consider the remainders with locally a $\delta \theta$-base$^3$ of the topological groups.

LEMMA 2.10. Let $X$ be a Lindelöf space with locally a $\delta \theta$-base. Then $X$ has a $\delta \theta$-base.

Proof. For each point $x \in X$, there is an open neighborhood $U(x)$ with $x \in U(x)$ such that $U(x)$ has a $\delta \theta$-base $\mathcal{B}_x = \bigcup_n \mathcal{B}_{n,x}$. Let $\mathcal{U} = \{U(x) : x \in X\}$. Since $X$ is Lindelöf, it follows that there exists a countable subfamily $\mathcal{U}' \subset \mathcal{U}$ such that $X = \bigcup \mathcal{U}'$. Denoted $\mathcal{U}'$ by $\{U_x : i \in \mathbb{N}\}$. Obviously, $\mathcal{B} = \bigcup_{i,n} \mathcal{B}_{n,x_i}$ is a $\delta \theta$-base for $X$.

THEOREM 2.11. Let $G$ be a non-locally compact topological group. If $Y = bG \setminus G$ has locally a $\delta \theta$-base. Then $G$ and $bG$ are separable and metrizable.

Proof. Obviously, $Y$ is first countable. By [8, Proposition 2.1], each countably compact subset of $Y$ is a compact, metrizable, $G_\delta$-subset of $Y$. From Lemma 2.2 it follows that $G$ is locally separable and metrizable. Then $G$ is a $p$-space. Hence $Y$ is Lindelöf by Henriksen and Isbell’s theorem. From Lemma 2.10 it follows that $Y = bG \setminus G$ has a $\delta \theta$-base.

By the same notations in Theorem 2.5, it is easy to see from [8, Propostion 2.1] that $F \subset bG \setminus G$ is compact and metrizable in view of the proof of Theorem 2.5. By [11, Corollary 8.3] and Lemma 2.1, $\mathbb{L} \setminus \mathbb{L}$ is separable and metrizable. In view of the proof of Theorem 2.5, $G$ and $bG$ are separable and metrizable by [8, Proposition 2.1].

COROLLARY 2.12 [16]. Let $G$ be a non-locally compact topological group. If $Y = bG \setminus G$ is locally a quasi-developable$^4$. Then $G$ and $bG$ are separable and metrizable.

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$^3$Recall that a collection $\mathcal{B} = \bigcup_n \mathcal{B}_n$ of open subsets of a space $X$ is a $\delta \theta$-base [11] if whenever $x \in U$ with $U$ open, there exist an $n \in \mathbb{N}$ and a $B \in \mathcal{B}$ such that

(i) $1 \leq \text{ord}(x, \mathcal{B}_n) \leq \omega$;

(ii) $x \in B \subset U$.

$^4$A space $X$ is quasi-developable if there exists a sequence $\{\mathcal{G}_n\}_n$ of families of open subsets of $X$ such that for each point $x \in X$, $\{\text{st}(x, \mathcal{G}_n) : n \in \mathbb{N}, \text{st}(x, \mathcal{G}_n) \neq \emptyset\}$ is a base at $x$. 

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Finally, we consider the remainders with locally a c-semistratifiable space of the topological group.

Let $X$ be a topological space. $X$ is called a \textit{c-semistratifiable space} (CSS) \cite{17} if for each compact subset $K$ of $X$ and each $n \in \mathbb{N}$ there is an open set $G(n, K)$ in $X$ such that:

(i) $\bigcap \{G(n, K) : n \in \mathbb{N}\} = K$;

(ii) $G(n + 1, K) \subseteq G(n, K)$ for each $n \in \mathbb{N}$; and

(iii) if for any compact subsets $K, L$ of $X$ with $K \subseteq L$, then $G(n, K) \subseteq G(n, L)$ for each $n \in \mathbb{N}$.

\textbf{Theorem 2.13.} Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ is locally a CSS-space. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

\textit{Proof.} By \cite[Proposition 3.8(c)]{8} and the definition of CSS-spaces, it is easy to see that each countably compact subset of $Y$ is a compact, metrizable, $G_\delta$-subset of $Y$. From Lemma 2.2 it follows that $G$ is locally separable and metrizable. Then $G$ is a $\rho$-space. Hence $Y$ is Lindel"of by Henriksen and Isbell's theorem. From Lemma 2.10 it follows that $Y = bG \setminus G$ is a CSS-space by \cite[Proposition 3.5]{8}.

By the same notations in Theorem 2.5, it is easy to see from \cite[Proposition 3.8]{8} that $F \subseteq bG \setminus G$ is compact and metrizable in view of the proof of Theorem 2.5. By \cite[Proposition 3.8]{8}, $L \setminus L$ is separable and metrizable. In view of the proof of Theorem 2.5, it is easy to see that $G$ and $bG$ are separable and metrizable.

\textbf{Corollary 2.14.} Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ is locally a $\sigma^\#$-space\footnote{A space $X$ is called a $\sigma^\#$-space \cite{17} if $X$ has a $\sigma$-closure-preserving closed $\rho$-network.}. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

\textit{Proof.} By \cite[Lemma 3.1]{8}, it follows that every $\sigma^\#$-space is a CSS-space. Hence $G$ and $bG$ are separable and metrizable by Theorem 2.13.

\textbf{Question 2.15.} Let $G$ be a non-locally compact topological group. If $Y = bG \setminus G$ satisfies the following conditions (1) and (2), are $G$ and $bG$ separable and metrizable?

(1) For each point $y \in Y$, there exists an open neighborhood $U(y)$ of $y$ such that every countably compact subset of $U(y)$ is metrizable and $G_\delta$-subset of $U(y)$;

(2) $\pi$-character of $Y$ is countable.
3. Remainders that are locally quasi-\(G_\delta\)-diagonals, and that are unions

First, we study the remainders with locally a quasi-\(G_\delta\)-diagonal\(^6\) and improve a result of C. Liu.

We call a space \(X\) is Ohio complete \([3,\ Corollary \ 3.7]\) if in each compactification \(bX\) of \(X\) there is a \(G_\delta\)-subset \(Z\) such that \(X \subset Z\) and each point \(y \in Z \setminus X\) is separated from \(X\) by a \(G_\delta\)-subset of \(Z\).

**Lemma 3.1.** Let \(X\) be a \(p\)-space and every compact subset of \(bX\) is metrizable. Then there exists a \(G_\delta\)-subset \(Y\) of \(bX\) such that \(X \subset Y\) and satisfies the following conditions:

1. \(bX\) is first countable at every point \(y \in Y \setminus X\);
2. If \(X\) is a topological group and \(y \in Y \setminus X\) then \(X\) is metrizable.

**Proof.** Since \(X\) is a \(p\)-space, \(X\) is Ohio complete \([3,\ Corollary \ 3.7]\). It follows that there is a \(G_\delta\)-subset \(Y\) of \(bX\) such that \(X \subset Y\). We now prove that \(Y\) satisfies the conditions (1) and (2).

1. From the choice of \(Y\), it is easy to see that for every point \(y \in Y \setminus X\) there exists a compact \(G_\delta\)-subset \(C\) of \(bX\) such that \(y \in C \subset Y \setminus X \subset bX \setminus X\). Since \(C\) is compact, the compact subset \(C\) is metrizable. Therefore, \(y\) is a \(G_\delta\)-point in \(bX\) and hence, \(bX\) is first countable at \(y\).

2. We choose a point \(a \in Y \setminus X\). Since \(X\) is a \(p\)-space, there exists a compact subset \(F\) of \(X\) such that \(a \in F\) and \(F\) has a countable base of neighborhoods in \(X\). Since \(X\) is dense in \(bX\), the set \(F\) has a countable base of open neighborhoods \(\phi = \{ U_n : n \in \omega \}\) in \(bX\). Since \(a \in Y \setminus X\), we can fix a \(b_n \in U_n \cap (Y \setminus X)\) for each \(n \in \omega\). Obviously, there is a point \(c \in F\) which is a limit point for the sequence \(\{ b_n \}\). By (1), we know that \(bX\) is first countable at \(b_n\) for every \(n \in \omega\). We can fix a countable base \(\eta_n\) of \(bX\) at \(b_n\). Then \(\bigcup \{ \eta_n : n \in \omega \}\) is a countable \(\pi\)-base of \(bX\) at \(c\). Then the space \(X\) also has a countable \(\pi\)-base at \(c\), since \(c \in X\) and \(X\) is dense in \(bX\). Since \(X\) is a topological group, the space \(X\) is metrizable. \(\square\)

**Theorem 3.2.** Let \(G\) be a non-locally compact topological group. If \(Y = bG \setminus G\) has a quasi-\(G_\delta\)-diagonal. Then \(G\) and \(bG\) are separable and metrizable.

**Proof.** Obviously, \(Y\) has a countable pseudocharacter. By \([5,\ Theorem \ 5.1]\), \(G\) is a paracompact \(p\)-space or \(Y\) is first countable.

Case 1: The space \(Y\) is first countable.

From \([8,\ Proposition \ 2.3]\) it follows that each countably compact subset of \(Y\) is a compact, metrizable, \(G_\delta\)-subset of \(Y\). Note that a Lindelöf \(p\)-space with a

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\(^6\)A space \(X\) has a quasi-\(G_\delta\)-diagonal \([14]\) if there exists a sequence \(\{ \%_n \}_n\) of families of open subsets of \(X\) such that for each point \(x \in X\), \(\{ \text{st}(x, \%_n) : n \in \mathbb{N}, \text{st}(x, \%_n) \neq \emptyset \}\) is a \(p\)-network at point \(x\).
quasi-$G_\beta$-diagonal is metrizable by [14, Corollary 3.6]. In view of the proof of Theorem 2.5, it is easy to see that $G$ and $bG$ are separable and metrizable.

Case 2: The space $G$ is a paracompact $p$-space.

By [3, Corollary 3.7], $G$ is Ohio complete. Therefore, there exists a $G_\beta$-subset $X$ of $bG$ such that $G \subset X$ and every point $x \in X \setminus G$ can be separated from $G$ by a $G_\delta$-set of $X$. Let $M = X \setminus G$. Then $bG$ is first countable at every point $y \in M$ by Lemma 3.1.

Subcase 1: $M \cap G = \emptyset$. Then $X \setminus M = G$. Hence $G$ is a $G_\delta$-subset of $bG$. It follows that $Y$ is $\sigma$-compact. Since $Y$ has a quasi-$G_\beta$-diagonal, every compact subspace of $Y$ is separable and metrizable by [8, Proposition 2.3]. Hence $Y$ is separable. Since both $Y$ and $G$ are dense in $bG$, it follows that the Souslin number of $G$ is countable. The space $G$ is Lindelöf, since $G$ is paracompact. Therefore, $G$ is a Lindelöf $p$-space. Then $Y$ is a Lindelöf $p$-space by Lemma 2.1. Since $Y$ is separable, which implies that the Souslin number of $G$ is countable. Since $G$ is metrizable, the space $G$ is separable. Then $Y$ is a Lindelöf $p$-space by Lemma 2.1. Hence $Y$ is metrizable by [14, Corollary 3.6]. It follows that $Y$ has a $G_\delta$-diagonal. Therefore, $G$ and $bG$ are separable and metrizable by [4, Theorem 5].

Subcase 2: $M \cap G \neq \emptyset$. Then $G$ is metrizable by Lemma 3.1.

Subcase 2(a): $G$ is locally separable. By [8, Proposition 2.3], it is easy to see that $G$ and $bG$ are separable and metrizable by the proof of Theorem 2.5.

Subcase 2(b): $G$ is nowhere locally separable. Fix a base $\mathcal{B} = \bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \}$ of $G$ such that each $\mathcal{U}_n$ is discrete in $G$. Let $F_n$ be the set of all accumulation points for $\mathcal{U}_n$ in $bG$ for each $n \in \mathbb{N}$. Put $Z = \bigcup \{ F_n : n \in \mathbb{N} \}$. Then $Z$ is dense in $Y$ and $\sigma$-compact by [4, Proposition 4]. Since every compact space with a quasi-$G_\beta$-diagonal is separable and metrizable by [8, Proposition 2.3], the space $Z$ has a countable network. Because $G$ is nowhere locally compact, the space $Y$ is dense in $bG$. It follows that $Z$ is dense in $bG$. Hence $bG$ is separable, which implies that the Souslin number of $G$ is countable. Since $G$ is metrizable, the space $G$ is separable. Then $Y$ is a Lindelöf $p$-space by Lemma 2.1. Hence $Y$ is metrizable by [14, Corollary 3.6]. It follows that $Y$ is separable and metrizable, which implies that $G$ and $bG$ are separable and metrizable.

**Lemma 3.3.** Let $X$ be a Lindelöf space with locally a quasi-$G_\beta$-diagonal. Then $X$ has a quasi-$G_\beta$-diagonal.

**Proof.** For each point $x \in X$, there exists an open neighborhood $U(x)$ such that $x \in U(x)$ and $U(x)$ has a quasi-$G_\beta$-diagonal. Then $\mathcal{U} = \{ U(x) : x \in X \}$ is an open cover of $X$. Since $X$ is a Lindelöf space, there exists a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $X = \bigcup \mathcal{V}$. Denoted $\mathcal{V}^*$ by $\{ U_n : n \in \mathbb{N} \}$. For each $n \in \mathbb{N}$, let $\{ \mathcal{U}_{nk} \}_{k \in \mathbb{N}}$ be a quasi-$G_\beta$-diagonal sequence of $U_n$. Let $\mathcal{F} = \{ \mathcal{U}_{nk} \}_{n,k \in \mathbb{N}}$. Then $\mathcal{F}$ is a quasi-$G_\beta$-diagonal sequence of $X$.

Indeed, for distinct points $x,y \in X$, there exists an $n \in \mathbb{N}$ such that $x \in U_n$ and $y \notin U_n$, then $x \in U_n \subset X \setminus \{ y \}$. Since $\{ \mathcal{U}_{nk} \}_{k \in \mathbb{N}}$ is a quasi-$G_\beta$-diagonal sequence of $U_n$, there exists a $k \in \mathbb{N}$ such that $x \in \bigcup \mathcal{U}_{nk}$. Hence $x \in \text{st}(x, \mathcal{U}_{nk}) \subset \bigcup \mathcal{U}_{nk} \subset U_n \subset X \setminus \{ y \}$. 


If \( y \in U_n \), then \( x \in U_n - \{ y \} \subset X - \{ y \} \). Since \( \{ U_{nk} \}_{k \in \mathbb{N}} \) is a quasi-
\( G_\delta \)-diagonal sequence of \( U_n \), there exists a \( k \in \mathbb{N} \) such that \( x \in \text{st}(x, U_{nk}) \subset U_n - \{ y \} \subset X - \{ y \} \).

Therefore, \( \mathcal{F} \) is a quasi-\( G_\delta \)-diagonal sequence of \( X \).

**Theorem 3.4.** Let \( G \) be a non-locally compact topological group. If \( Y = bG \setminus G \) has locally a quasi-\( G_\delta \)-diagonal, then \( G \) and \( bG \) are separable and metrizable.

**Proof.** By [8, Proposition 2.1 and 2.5] and Lemma 2.2, it is easy to see that \( G \) is locally a separable and metrizable space. Then \( Y \) is a Lindelöf space by Henriksen and Isbell’s theorem. From Lemma 3.3 it follows that \( Y \) has a quasi-
\( G_\delta \)-diagonal. Then \( G \) and \( bG \) are separable and metrizable by Theorem 3.2.

**Question 3.5.** Is there a topological group \( G \) such that the \( Y = bG \setminus G \) has a \( W_\delta \)-diagonal\(^7\), \( G \) is not reparable and metrizable?

**Corollary 3.6** [16]. Let \( G \) be a non-locally compact topological group. If \( Y = bG \setminus G \) has locally a \( G_\delta \)-diagonal, then \( G \) and \( bG \) are separable and metrizable.

Next, we study the remainder that are the unions of the \( G_\delta \)-diagonals.

**Lemma 3.7.** Let \( G \) be a non-locally compact topological group. If there exists a point \( a \in Y = bG \setminus G \) such that \( \{ a \} \) is a \( G_\delta \)-set in \( Y \), then at least one of the following conditions holds:

1. \( G \) is a paracompact \( p \)-space;
2. \( Y \) is first-countable at some point.

**Proof.** Suppose that \( Y \) is not first-countable at point \( a \). Since \( a \) is a \( G_\delta \)-point in \( Y \), there exists a compact subset \( F \subset bG \) with a countable base at \( F \) in \( bG \) such that \( \{ a \} = F \cap (bG \setminus G) \). We have \( F \setminus \{ a \} \neq \emptyset \), since \( Y \) is not first-countable at point \( a \). Therefore, there exists a non-empty compact subset \( B \subset F \) with a countable base at \( B \) in \( bG \). Obviously, \( B \subset G \). It follows that \( G \) is a topological group of countable type [18]. Therefore, \( G \) is a paracompact \( p \)-space [18].

**Lemma 3.8.** Let \( G \) be a non-locally compact topological group, and \( Y = bG \setminus G = Y_1 \cup Y_2 \), where both \( Y_1 \) and \( Y_2 \) have a countable pseudocharacter. If at most one of the \( Y_1 \) and \( Y_2 \) is dense in \( bG \), then at least one of the following conditions holds:

1. \( G \) is a paracompact \( p \)-space;
2. \( Y \) is first-countable at some point.

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\(^7\)A space \( X \) is said to have a \( W_\delta \)-diagonal if there is a sequence \( (\mathcal{B}_n) \) of bases for \( X \) such that whenever \( x \in B_n \in \mathcal{B}_n \), and \( (B_n) \) is decreasing (by set inclusion), then \( \{ x \} = \cap \{ B_n : n \in \omega \} \).
Proof. Without loss of generality, we can assume that $Y_1 \neq bG$. Let $U = bG \setminus Y_1$. Then $V = U \cap Y = U \cap Y_2 \neq \emptyset$. It follows that $V$ is an open subset of $Y$ and each point of $V$ is a $G_{\delta}$-point. By Lemma 3.7, we complete the proof.

Theorem 3.9. Let $G$ be a non-locally compact topological group, and $Y = bG \setminus G = Y_1 \cup Y_2$, where both $Y_1$ and $Y_2$ have a countable pseudocharacter. If both $Y_1$ and $Y_2$ are Ohio complete, then at least one of the following conditions holds:

1. $G$ is a paracompact p-space;
2. $Y$ is first-countable at some point.

Proof. Case 1: $Y_1 \neq bG$ or $Y_2 \neq bG$.

It is easy to see by Lemma 3.8.

Case 2: $Y_1 = bG$ and $Y_2 = bG$.

Then $bG$ is the Hausdorff compactification of $Y_1$ and $Y_2$. Since $Y_1$ and $Y_2$ are Ohio complete, there exist $G_{\delta}$-subsets $X_1$ and $X_2$ satisfy the definition of Ohio complete, respectively.

Case 2(a): $Y_1 = X_1$ and $Y_2 = X_2$.

Then $Y$ has countable pseudocharacter. By [5, Theorem 5.1], we complete the proof.

Case 2(b): $Y_1 \neq X_1$ or $Y_2 \neq X_2$.

Without loss of generality, we can assume that $Y_1 \neq X_1$. If $(X_1 \setminus Y_1) \cap Y_2 \neq \emptyset$, then for each $y \in (X_1 \setminus Y_1) \cap Y_2$ there exists a compact subset $C$ such that $y \in C$ and $C \cap Y_1 = \emptyset$. Obviously, $y$ is a $G_{\delta}$-point of $Y$. By Lemma 3.7, we also complete the proof. If $(X_1 \setminus Y_1) \cap Y_2 = \emptyset$, then there exists a compact subset $C \subset G$ with a countable base at $C$ in $bG$. It follows that $G$ is a topological group of countable type [18]. Therefore, $G$ is a paracompact p-space [18].

A space with a $G_{\delta}$-diagonal is Ohio complete [2]. Therefore, by Theorem 3.9, we have the following result.

Theorem 3.10. Let $G$ be a non-locally compact topological group, and $Y = bG \setminus G = Y_1 \cup Y_2$, where both $Y_1$ and $Y_2$ have a $G_{\delta}$-diagonal. Then at least one of the following conditions holds:

1. $G$ is a paracompact p-space;
2. $Y$ is first-countable at some point.

Question 3.11. Let $G$ be a non-locally compact topological group, and $Y = bG \setminus G = \bigcup_{i=1}^{n} Y_i$, where $Y_i$ has a $G_{\delta}$-diagonal for every $1 \leq i \leq n$. Is $G$ a paracompact p-space or is $Y$ first-countable at some point?

Question 3.12. Let $G$ be a non-locally compact topological group, and $Y = bG \setminus G = Y_1 \cup Y_2$, where both $Y_1$ and $Y_2$ have quasi-$G_{\delta}$-diagonal. Is $G$ a paracompact p-space or is $Y$ first-countable at some point?
4. Remainders of locally BCO and locally hereditarily D-spaces

First, we study the following question, which was posed by C. Liu in [16].

**Question 4.1.** Let \( G \) be a non-locally compact topological group, and \( Y = bG \setminus G \) have a BCO\(^8\). Are \( G \) and \( bG \) separable and metrizable?

Now we give a partial answer for Question 4.1.

**Theorem 4.2.** Let \( G \) be a non-locally compact topological group, and \( Y = bG \setminus G \) has a BCO. If \( Y \) is Ohio complete, then \( G \) and \( bG \) are separable and metrizable.

**Proof.** Since \( Y \) is Ohio complete, \( G \) is a paracompact \( p \)-space or \( s \)-compact space by [3, Theorem 4.3].

Case 1: The space \( G \) is a paracompact \( p \)-space.

Since \( G \) is a \( p \)-space, the space \( Y \) is Lindelöf by Henriksen and Isbell’s theorem. Hence \( Y \) is developable by [11, Theorem 6.6]. Then \( G \) and \( bG \) are separable and metrizable by Theorem 3.4.

Case 2: The space \( G \) is a \( s \)-compact space.

We claim that \( G \) is metrizable. Suppose that \( G \) is not metrizable. Then \( Y \) is \( \omega \)-bounded\(^9\) by [5, Theorem 3.12]. Since \( G \) is a \( \sigma \)-compact topological group, the Souslin number \( c(G) \) of \( G \) is countable by a theorem of Tkachenko [21, Corollary 2]. Therefore, \( c(bG) \leq \omega \). \( Y \) is dense in \( bG \), since \( G \) is non-locally compact. It follows that \( c(Y) \leq \omega \) as well. Since \( Y \) is Čech-complete, there exists a dense subspace \( Z \subset Y \) such that \( Z \) is a paracompact and Čech-complete subspace of \( Y \) by [19]. Then \( Z \) is a paracompact space with a BCO. Therefore, \( Z \) is metrizable by [11, Theorem 1.2 and 6.6]. Since \( c(Y) \leq \omega \) and \( Z \) is dense for \( Y \), \( c(Z) \leq \omega \) as well. It follows that \( Z \) is separable. Since \( Y \) is \( \omega \)-bounded, it is compact. Therefore, \( G \) is locally compact, which is a contradiction. It follows that \( G \) is metrizable. Therefore, \( G \) and \( bG \) are separable and metrizable by Case 1.

**Theorem 4.3.** Let \( G \) be a non-locally compact topological group, and \( Y = bG \setminus G \) have a BCO. If \( G \) is an \( \Sigma \)-space, then \( G \) and \( bG \) are separable and metrizable.

**Proof.** From [6, Theorem 2.8] it follows that every compact subspace of \( Y \) has countable character in \( Y \). Since \( G \) is non-locally compact, \( Y \) is also a dense subset of \( bG \). Hence \( G \) is Lindelöf space by Henriksen and Isbell’s theorem. If

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\(^8\)A space \( X \) is said to have a base of countable order (BCO) [11] if there is a sequence \( \{ B_n \} \) of base for \( X \) such that whenever \( x \in b_n \in B_n \) and \( (b_n) \) is decreasing (by set inclusion), then \( \{ b_n : n \in \mathbb{N} \} \) is a base at \( x \).

\(^9\)A space \( X \) is said to be \( \omega \)-bounded if the closure of every countable subset of \( X \) is compact.
$G$ is a $\sigma$-compact space, then $G$ and $bG$ are separable and metrizable by Case 2 in Theorem 4.2. Hence we assume that $G$ is non-$\sigma$-compact. Since $G$ is a Lindelöf $\Sigma$-space, it is easy to see that $G$ is a Lindelöf $p$-space by the proof of [5, Theorem 4.2]. It follows that $G$ and $bG$ are separable and metrizable by Case 1 in Theorem 4.2.

Finally, we study the remainders of topological groups with locally a hereditarily D-space.

**Theorem 4.4.** Let $G$ be a topological group. If for each $y \in Y = bG \setminus G$ there exists an open neighborhood $U(y)$ of $y$ such that every $\omega$-bounded subset of $U(y)$ is compact, then at least one of the following conditions holds:

1. $G$ is metrizable;
2. $bG$ can be continuously mapped onto the Tychonoff cube $I^{\omega_1}$.

**Proof.** Case 1: The space $G$ is locally compact.

If $G$ is not metrizable, then $G$ contains a topological copy of $D^{\omega_1}$. Since the space $G$ is normal, the space $G$ can be continuously mapped onto the Tychonoff cube $I^{\omega_1}$.

Case 2: The space $G$ is not locally compact.

Obviously, both $G$ and $Y$ are dense in $bG$. Suppose that the condition (2) doesn’t hold. Then, by a theorem of Šapirovskii in [20], the set $A$ of all points $x \in bG$ such that the $\pi$-character of $bG$ at $x$ is countable is dense in $bG$. Since $G$ is dense in $bG$, it can follow that the $\pi$-character of $G$ is countable at each point of $A \cap G$.

Subcase 2(a): $A \cap G \neq \emptyset$.

Since $G$ is a topological group, it follows that $G$ is first countable, which implies that $G$ is metrizable.

Subcase 2(b): $A \cap G = \emptyset$.

Obviously, $A \subset Y$. For each $y \in Y$, there exists an open neighborhood $U(y)$ in $Y$ such that $y \in U(y)$ and every $\omega$-bounded subset of $U(y)$ is compact. Obviously, $A \cap U(y)$ is dense of $U(y)$. Also, it is easy to see that $A \cap U(y)$ is $\omega$-bounded subset for $U(y)$. Therefore, $A \cap U(y)$ is compact. Then $A \cap U(y) = U(y)$, since $A \cap U(y)$ is dense of $U(y)$. Hence $Y$ is locally compact, a contradiction. 

A neighborhood assignment for a space $X$ is a function $\varphi$ from $X$ to the topology of $X$ such that $x \in \varphi(x)$ for each point $x \in X$. A space $X$ is a D-space [9], if for any neighborhood assignment $\varphi$ for $X$ there is a closed discrete subset $D$ of $X$ such that $X = \bigcup_{d \in D} \varphi(d)$.

It is easy to see that every countably compact D-space is compact. Hence we have the following result by Theorem 4.4.

**Theorem 4.5.** Let $G$ be a topological group. If $Y = bG \setminus G$ is locally a hereditarily D-space, then at least one of the following conditions holds:
1. $G$ is metrizable;
2. $bG$ can be continuously mapped onto the Tychonoff cube $I^{\omega_1}$.

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