GENERALIZED IDEAL TRANSFORMS

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Abstract. We study basic properties of the generalized ideal transforms $D_I(M, N)$ and the set of associated primes of the modules $R^iD_I(M, N)$.

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1. Introduction

Throughout this paper, $R$ is a Noetherian commutative ring with non-zero identity and $I$ is an ideal of $R$. In [6], Brodmann defined ideal transform $D_I(M)$ of an $R$–module $M$ with respect to $I$ by

$$D_I(M) = \lim_{n \to} \hom_R(I^n, M).$$

Ideal transforms turn out to be a powerful tool in various fields of commutative algebra and they are closed to local cohomology modules of Grothendieck.

In [11], Herzog introduced the definition of generalized local cohomology modules which is an extension of local cohomology modules of Grothendieck. The $i$-th generalized local cohomology module of modules $M$ and $N$ with respect to $I$ was given as

$$H_I^i(M, N) = \lim_{n \to} \text{Ext}_R^i(M/I^nM, N).$$

A natural way, we have a generalization of the ideal transform. In [10], the generalized ideal transform functor with respect to an ideal $I$ is defined by

$$D_I(M, -) = \lim_{n \to} \hom_R(I^nM, -).$$

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Also in [10] they used the generalized ideal transforms to study the cofiniteness of generalized local cohomology modules. Let $R^iD_I(M, -)$ denote the $i$-th right derived functor of $D_I(M, -)$. It is clear that

$$R^iD_I(M, -) \cong \lim_{n \to \infty} \text{Ext}^i_R(I^n, M, -)$$

for all $i \geq 0$.

The organization of our paper is as follows. In the next section we study basic properties of the generalized ideal transform functor $D_I(M, -)$ and its right derived functors $R^iD_I(M, -)$. The first result is Theorem 2.1 which says that if $M$ is a finitely generated $R$-module and $N$ is an $I$-torsion $R$-module, then $R^iD_I(M, N) = 0$ for all $i \geq 0$. Next, Theorem 2.7 gives us isomorphisms $D_I(M, N) \cong D_I(M, N)$ and $D_{aR}(M, N) \cong D_{aR}(M, N)$. In Theorem 2.10 we see that the module $\text{Hom}_R(R/I, R^tD_I(M, N))$ is a finitely generated $R$-module provided the modules $R^iD_I(M, N)$ are finitely generated for all $i < t$. The section is closed by Theorem 2.12 which shows the Artinianness of the modules $R^iD_I(M, N)$.

The last section is devoted to study the set of associated primes $\text{Ass}(R^iD_I(M, N))$. Theorem 3.3 shows that if $M$ is a finitely generated $R$-module and $N$ is a weakly Laskerian $R$-module, then $\text{Supp}(H^i_M(M, N))$ and $\text{Ass}(R^iD_I(M, N))$ are finite sets for all $i \geq 0$. Finally, Theorem 3.5 gives us two interesting consequences about the finiteness of the sets $\text{Ass}(R^iD_I(M, N))$ (Corollary 3.6) and $\text{Supp}_R(R^iD_I(M, N))$ (Corollary 3.7).

2. SOME BASIC PROPERTIES OF GENERALIZED IDEAL TRANSFORMS

An $R$-module $N$ is called $I$-torsion if $\Gamma_I(N) \cong N$. We have the following result.

**Theorem 2.1.** Let $M$ be a finitely generated $R$-module and $N$ an $I$-torsion $R$-module. Then $R^iD_I(M, N) = 0$ for all $i \geq 0$.

**Proof.** We first prove $D_I(M, N) = 0$.

Consider the $n$-th injection $\lambda_n : \text{Hom}_R(I^nM, N) \to \bigoplus \text{Hom}_R(I^nM, N)$ and the homomorphisms $\varphi^i_j : \text{Hom}_R(I^iM, N) \to \text{Hom}_R(I^jM, N)$ such that $\varphi^i_j(f_i) = f_i|_{I^jM}$ for all $i \leq j$.

Let $S$ be an $R$-submodule of $\bigoplus \text{Hom}_R(I^nM, N)$ which is generated by elements $\lambda_j\varphi^i_j(f_i) - \lambda_i f_i, f_i \in \text{Hom}_R(I^iM, N)$ and $i \leq j$. Then

$$\lim_{n} \text{Hom}_R(I^nM, N) = (\bigoplus \text{Hom}_R(I^nM, N))/S.$$
For any $u \in D_I(M, N) = \lim_{\to} \text{Hom}_R(I^n M, N)$, we have $u = \lambda t f_t + S$, where $f_t \in \text{Hom}_R(I^t M, N)$.

Since $I^t M$ is a finitely generated $R$-module and $N$ is an $I$-torsion $R$-module, there exists a positive integer $p$ such that $\varphi^t_{p+1}(f_t) = 0$.

It follows from [17, 2.17 (ii)] that $u = 0$ and then $D_I(M, N) = 0$.

The proof will be complete if we show $R^i D_I(M, N) = 0$ for all $i > 0$.

As $N$ is $I$-torsion, there is an injective resolution $E^\bullet$ of $N$ such that each term of the resolution is an $I$-torsion $R$-module. By the above proof, we have $\lim_{\to} \text{Hom}_R(I^n M, E^i) = 0$ for all $i \geq 0$. Therefore $R^i D_I(M, N) = 0$ for all $i \geq 0$. \hfill $\square$

**Corollary 2.2.** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module such that $D_I(N) = 0$. Then $R^i D_I(M, N) = 0$ for all $i \geq 0$.

**Proof.** We consider the exact sequence

$$0 \to \Gamma_I(N) \to N \to D_I(N) \to H^1_I(N) \to 0$$

From the hypothesis, we have $\Gamma_I(N) \cong N$ that means $N$ is $I$-torsion. From 2.1 we have the conclusion. \hfill $\square$

The following lemmas will be used to prove the next propositions.

**Lemma 2.3.** ([10, 2.2]) Let $M, N$ be $R$-modules. Then, there is an exact sequence

$$0 \to H^0_I(M, N) \to \text{Hom}_R(M, N) \to D_I(M, N) \to H^1_I(M, N) \to \cdots$$

$$\cdots H^i_I(M, N) \to \text{Ext}_R^i(M, N) \to R^i D_I(M, N) \to H^{i+1}_I(M, N) \to \cdots$$

Moreover, if $\text{pd}(M) < \infty$, then $R^i D_I(M, N) \cong H^{i+1}_I(M, N)$ for all $i \geq \text{pd}(M) + 1$.

**Lemma 2.4.** ([5, Theorem 1]) The following conditions on an $R$-module $M$ are equivalent:

(i) $M$ admits a resolution by finitely generated projectives;
(ii) The functors $\text{Ext}^n_R(M, -)$ preserve direct limits for all $n$;
(iii) The functors $\text{Tor}^R_n(-, M)$ preserve products for all $n$.

The following lemma shows some basic properties of generalized ideal transforms that we shall use.

**Lemma 2.5.** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Then

(i) $D_I(M, N)$ is an $I$-torsion-free $R$-module;
(ii) $R^i D_I(M, N) \cong R^i D_I(M, N/\Gamma_I(N))$ for all $i \geq 0$;
(iii) $R^i D_I(M, N) \cong R^i D_I(M, D_I(N))$ for all $i \geq 0$;
(iv) $D_I(D_I(M, N)) \cong D_I(M, N)$;
(v) $D_I(\text{Hom}_R(M, N)) \cong \text{Hom}_R(M, D_I(N))$.

Proof. (i) We have by 2.4
\[
\Gamma_I(D_I(M, N)) = \lim_{n \to \infty} \text{Hom}_R(R/I^n, D_I(M, N)) \\
\cong \lim_{n \to \infty} \lim_{t \to \infty} \text{Hom}_R(R/I^n, \text{Hom}_R(I^t M, N)) \\
\cong \lim_{t \to \infty} \lim_{n \to \infty} \text{Hom}_R(I^t M, \Gamma_I(N)) \\
\cong \Gamma_I(M, \Gamma_I(N)).
\]
Since $\Gamma_I(N)$ is an $I$-torsion $R$-module and from 2.1, we get $D_I(M, \Gamma_I(N)) = 0$. Thus $D_I(M, N)$ is an $I$-torsion-free $R$-module.

(ii) The short exact sequence
\[
0 \to \Gamma_I(N) \to N \to N/\Gamma_I(N) \to 0
\]
deduces the long exact sequence
\[
0 \to D_I(M, \Gamma_I(N)) \to D_I(M, N) \to D_I(M, N/\Gamma_I(N)) \to \cdots
\]
\[
\cdots \to R^i D_I(M, N) \to R^i D_I(M, N/\Gamma_I(N)) \to R^{i+1} D_I(M, \Gamma_I(N)) \cdots
\]
Then $R^i D_I(M, N) \cong R^i D_I(M, N/\Gamma_I(N))$ for all $i \geq 0$, as $R^i D_I(M, \Gamma_I(N)) = 0$.

(iii) The short exact sequence
\[
0 \to N/\Gamma_I(N) \to D_I(N) \to H^1_I(N) \to 0
\]
deduces a long exact sequence
\[
0 \to D_I(M, N/\Gamma_I(N)) \to D_I(M, D_I(N)) \to D_I(M, H^1_I(N)) \to \cdots
\]
\[
\to R^i D_I(M, N/\Gamma_I(N)) \to R^i D_I(M, D_I(N)) \to R^i D_I(M, H^1_I(N)) \to
\]
As $R^i D_I(M, H^1_I(N)) = 0$, $R^i D_I(M, N/\Gamma_I(N)) \cong R^i D_I(M, D_I(N))$ for all $i \geq 0$.

(iv) We have
\begin{align*}
\text{DI}(\text{DI}(M, N)) &= \lim_{n} \text{Hom}_R(I^n, \text{DI}(M, N)) \\
&\cong \lim_{n} \lim_{t} \text{Hom}_R(I^n, \text{Hom}_R(I^tM, N)) \\
&\cong \lim_{t} \lim_{n} \text{Hom}_R(I^n \otimes I^tM, N) \\
&\cong \lim_{t} \lim_{n} \text{Hom}_R(I^tM, \text{Hom}_R(I^n, N)) \\
&\cong \lim_{t} \text{Hom}_R(I^tM, \text{DI}(N)) \\
&\cong \text{DI}(M, \text{DI}(N)) \\
&\cong \text{DI}(M, N).
\end{align*}

(v) From 2.4 we have
\begin{align*}
\text{DI}(\text{Hom}_R(M, N)) &= \lim_{n} \text{Hom}_R(I^n, \text{Hom}_R(M, N)) \\
&\cong \lim_{n} \text{Hom}_R(M, \text{Hom}_R(I^n, N)) \\
&\cong \text{Hom}_R(M, \text{DI}(N))
\end{align*}
as required. \hfill \Box

If \( f : N \to N' \) is an \( R \)-module homomorphism such that \( \text{Ker} f \) and \( \text{Coker} f \) are both \( I \)-torsion \( R \)-modules, then \( R^i \text{DI}(N) \cong R^i \text{DI}(N') \) for all \( i \geq 0 \) (see [6]). We have a similar property in the case of generalized ideal transforms.

**Proposition 2.6.** Let \( f : N \to N' \) be an \( R \)-module homomorphism such that \( \text{Ker} f \) and \( \text{Coker} f \) are both \( I \)-torsion \( R \)-modules. Then
\[
R^i \text{DI}(M, N) \cong R^i \text{DI}(M, N')
\]
for all non-negative integer \( i \).

**Proof.** Two short exact sequences
\[
0 \to \text{Ker} f \to N \to \text{Im} f \to 0
\]
\[
0 \to \text{Im} f \to N' \to \text{Coker} f \to 0
\]
deduce two long exact sequences
\[
0 \to \text{DI}(M, \text{Ker} f) \to \text{DI}(M, N) \to \text{DI}(M, \text{Im} f) \to R^1 \text{DI}(M, \text{Ker} f) \ldots
\]
\[
0 \to \text{DI}(M, \text{Im} f) \to \text{DI}(M, N') \to \text{DI}(M, \text{Coker} f) \to R^1 \text{DI}(M, \text{Im} f) \ldots
\]
Since \( \text{Ker} f \) and \( \text{Coker} f \) are both \( I \)-torsion \( R \)-modules, \( R^i \text{DI}(M, \text{Ker} f) = 0 \) and \( R^i \text{DI}(M, \text{Coker} f) = 0 \) for all \( i \geq 0 \). Hence \( R^i \text{DI}(M, N) \cong R^i \text{DI}(M, \text{Im} f) \) and \( R^i \text{DI}(M, \text{Im} f) \cong R^i \text{DI}(M, N') \). Finally, we get \( R^i \text{DI}(M, N) \cong R^i \text{DI}(M, N') \) for all \( i \geq 0 \). \hfill \Box

Let \( N_a \) denote the localization of \( N \) respect to the multiplicatively closed subset \( S = \{a^i \mid i \in \mathbb{N} \} \). We have the following theorem.
Theorem 2.7. Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Then

(i) $D_I(\text{Hom}_R(M, N)) \cong D_I(M, N)$;
(ii) If $I = aR$ is a principal ideal of $R$, then

$$D_{aR}(M, N) \cong D_{aR}(M, N)_a.$$ 

Proof. (i). The long exact sequence

$$0 \to \Gamma_I(M, N) \to \text{Hom}_R(M, N) \to D_I(M, N) \xrightarrow{f} H^1_I(M, N) \to \cdots$$

deduces an exact sequence

$$0 \to \Gamma_I(M, N) \to \text{Hom}_R(M, N) \to D_I(M, N) \to \text{Im} f \to 0.$$ 

Note that $\text{Im} f$ is an $R$-submodule of $H^1_I(M, N)$, then $\text{Im} f$ is an $I$-torsion $R$-module.

Since $\Gamma_I(M, N)$ and $\text{Im} f$ are both $I$-torsion $R$-modules, there are isomorphisms

$$D_I(\text{Hom}_R(M, N)) \cong D_I(D_I(M, N)) \cong D_I(M, N).$$

(ii). From [2.5] we have $D_I(M, N) \cong D_I(M, D_I(N))$. We now consider the module $D_I(M, D_I(N))$. Since $I = aR$ is a principal ideal, it follows $D_I(N) \cong N_a$. As $I^n M$ is finitely generated, we have by [17, 3.83]

$$\text{Hom}_R(I^n M, N \otimes S^{-1}R) \cong S^{-1}R \otimes \text{Hom}_R(I^n M, N).$$

It follows

$$\lim_{n \to \infty} \text{Hom}_R(I^n M, N \otimes S^{-1}R) \cong \lim_{n \to \infty} S^{-1}R \otimes \text{Hom}_R(I^n M, N).$$

Hence

$$D_I(M, D_I(N)) \cong S^{-1}R \otimes D_I(M, N).$$

Finally, we get $D_{aR}(M, N) \cong D_{aR}(M, N)_a$. □

If $E$ is an injective $R$-module, then $\Gamma_I(E)$ is also injective and $H^1_I(E) = 0$. Hence, the short exact sequence

$$0 \to \Gamma_I(E) \to E \to D_I(E) \to 0$$

is split. It implies that $D_I(E)$ is an injective $R$-module.

Proposition 2.8. Let $M$ be a finitely generated $R$-module, $N$ an $R$-module and $J^\bullet$ an injective resolution of $N$. Then

$$R^i D_I(M, N) \cong H^i(\text{Hom}_R(M, D_I(J^\bullet))).$$
Proof. Combining 2.7 with 2.5 yields
\[ R^iD_I(M, N) = H^i(D_I(M, J^*)) \]
\[ \cong H^i(D_I(Hom(M, J^*))) \]
\[ \cong H^i(Hom(M, D_I(J^*))) \]
as required. \(\square\)

Next, we study the finiteness of generalized ideal transforms which relates to generalized local cohomology modules.

Proposition 2.9. Let \( M, N \) be two finitely generated \( R \)-modules and \( i \) a positive integer. Then \( H^i_I(M, N) \) is finitely generated if and only if \( R^{i-1}D_I(M, N) \) is finitely generated.

Proof. Since \( M, N \) are finitely generated \( R \)-modules, \( \text{Ext}_R^i(M, N) \) is also a finitely generated \( R \)-module for all \( i \geq 0 \). By 2.3 we have the conclusion. \(\square\)

Theorem 2.10. Let \( M \) be a finitely generated \( R \)-module and \( N \) an \( R \)-module. If \( t \) is a non-negative integer such that \( R^iD_I(M, N) \) is finitely generated for all \( i < t \), then \( \text{Hom}_R(R/I, R^tD_I(M, N)) \) is a finitely generated \( R \)-module.

Proof. We use induction on \( t \).

Let \( t = 0 \). We have \( \text{Hom}_R(R/I, D_I(M, N)) = 0 \), since \( D_I(M, N) \) is \( I \)-torsion-free.

When \( t > 0 \), from 2.5 it follows
\[ R^iD_I(M, N) \cong R^iD_I(M, D_I(N)) \]
for all \( i \geq 0 \).

It is sufficient to prove that \( \text{Hom}_R(R/I, R^iD_I(M, D_I(N))) \) is finitely generated.

Since \( D_I(N) \) is an \( I \)-torsion-free \( R \)-module, there is a \( D_I(N) \)-regular element \( x \in I \). Now the short exact sequence
\[ 0 \rightarrow D_I(N) \overset{x}{\rightarrow} D_I(N) \rightarrow D_I(N) \rightarrow 0, \]
where \( D_I(N) = D_I(N)/xD_I(N) \) gives rise a long exact sequence
\[ 0 \rightarrow D_I(M, D_I(N)) \overset{x}{\rightarrow} D_I(M, D_I(N)) \rightarrow D_I(M, D_I(N)) \rightarrow \cdots \]
\[ \cdots R^iD_I(M, D_I(N)) \overset{h}{\rightarrow} R^iD_I(M, D_I(N)) \rightarrow \cdots. \]
It induces a short exact sequence
\[ 0 \rightarrow \text{Im}h \rightarrow R^{i-1}D_I(M, D_I(N)) \rightarrow \text{Im}k \rightarrow 0. \]
As \( R^iD_I(M, D_I(N)) \) is finitely generated for all \( i < t \), \( \text{Im}h \) and
$R^iD_I(M, D_I(N))$ are both finitely generated $R$-modules for all $i < t-1$.

By the inductive hypothesis, $\text{Hom}_R(R/I, R^{t-1}D_I(M, D_I(N)))$ is finitely generated. Hence $\text{Hom}_R(R/I, \text{Im} k)$ is finitely generated.

Next, the exact sequence

$$0 \rightarrow \text{Im} k \rightarrow R^tD_I(M, D_I(N)) \rightarrow R^tD_I(M, D_I(N)) \rightarrow$$

deduces a long exact sequence

$$0 \rightarrow \text{Hom}_R(R/I, \text{Im} k) \rightarrow \text{Hom}_R(R/I, R^tD_I(M, D_I(N))) \rightarrow \cdots$$

As $x \in I$, $\text{Hom}_R(R/I, R^tD_I(M, D_I(N)))$ is finitely generated. □

In [7, 2.5] Tang and Chu proved that if $H^r_I(M, R/\mathfrak{p})$ is Artinian for any $\mathfrak{p} \in \text{Supp}(N)$ and $r \geq \text{pd}(M)$, then $H^i_I(M, N)$ is Artinian for all $i \geq r$. We show a similar following proposition.

**Proposition 2.11.** Let $M, N$ be two finitely generated $R$-modules with $\text{pd}(M) < \infty$. Assume that $t$ is a positive integer such that $t > \text{pd}(M)$.

(i) If $R^iD_I(M, R/\mathfrak{p})$ is Artinian for all $\mathfrak{p} \in \text{Supp}(N)$, then $R^iD_I(M, N)$ is also an Artinian $R$-module for all $i \geq t$.

(ii) If $R^tD_I(M, R/\mathfrak{p})$ and $H^t_I(M, R/\mathfrak{p})$ are Artinian for all $\mathfrak{p} \in \text{Supp}(N)$, then $\text{Ext}^i_R(M, N)$ is also an Artinian $R$-module for all $i \geq t$.

**Proof.** (i). The proof of (i) is similar to that in the proof of [7, 2.5].

(ii). By 2.3 there is an exact sequence

$$0 \rightarrow H^0_I(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow D_I(M, N) \rightarrow H^1_I(M, N) \rightarrow \cdots$$

$$\cdots H^i_I(M, N) \rightarrow \text{Ext}^i_R(M, N) \rightarrow R^iD_I(M, N) \rightarrow H^{i+1}_I(M, N) \rightarrow \cdots$$

Thus the conclusion follows from (i) and [7, 2.5]. □

In the following theorem we study the Artinianness of modules $R^iD_I(M, N)$ when $N$ is Artinian or finitely generated.

**Theorem 2.12.** Let $M$ be a finitely generated $R$-module.

(i) If $N$ is an Artinian $R$-module, then $R^iD_I(M, N)$ is Artinian for all $i \geq 0$.

(ii) If $N$ is a finitely generated $R$-module such that $p = \text{pd}(M)$ and $d = \text{dim}(N)$ are finite, then $R^{p+d}D_I(M, N)$ is an Artinian $R$-module.
Proof. (i). It follows from [15, 2.6] that $H_I^i(M, N)$ is Artinian for all $i \geq 0$. On the other hand, Ext$_R^i(M, N)$ is Artinian for all $i \geq 0$. Thus the claim follows from the exact sequence of [2,3]

(ii). When $d = \text{dim}(N) = 0$. It is clear that $N$ is an Artinian $R$-module. Hence Ext$_R^i(M, N)$ is Artinian for all $i \geq 0$.

By [2, 5.1], $H_I^i(M, N) = 0$ for all $i > \text{pd}(M)$ and $H_I^p(M, N)$ is Artinian. Now we have the exact sequence by [2,3]

$\cdots \rightarrow R^{p-1}D_I(M, N) \rightarrow H_I^p(M, N) \rightarrow \text{Ext}_R^p(M, N) \rightarrow R^pD_I(M, N) \rightarrow 0$.

It follows that $R^pD_I(M, N)$ is an Artinian $R$-module.

Let $d = \text{dim}(N) > 0$. Since Ext$_R^i(M, N) = 0$ for all $i > \text{pd}(M)$,

$R^{p+d}D_I(M, N) \cong H_I^{p+d+1}(M, N) = 0$.

This finishes the proof. □

3. Associated primes of the modules $R^iD_I(M, N)$

To study some properties of associated primes of $R^iD_I(M, N)$ we recall the concepts of weakly Laskerian modules [9] and FSF modules [16]. An $R$-module $M$ is called weakly Laskerian if the set of associated of prime ideals of any quotient module of $M$ is finite. An $R$-module $M$ is called a FSF module if there is a finitely generated submodule $N$ of $M$ such that the support of $M/N$ is a finite set. Note that a module $M$ is a weakly Laskerian module if and only if $M$ is a FSF module (see [1, 2.5]).

Lemma 3.1 ([9]).

(i) Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence. Then $M$ is weakly Laskerian if and only if $M'$ and $M''$ are both weakly Laskerian.

(ii) If $M$ is a finitely generated $R$-module and $N$ is a weakly Laskerian, then Ext$_R^i(M, N)$ and Tor$_R^i(M, N)$ are weakly Laskerian for all $i \geq 0$.

(iii) Artinian modules and finitely generated modules are weakly Laskerian modules.

Proposition 3.2. Let $M, N$ be finitely generated $R$-modules. Then

$\text{Ass}(D_I(M, N)) = (\text{Supp}(M) \cap \text{Ass}(N)) \setminus V(I)$.

Proof. It follows from [18, 3.1] that $\text{Ass}(D_I(N)) = \text{Ass}(N) \setminus V(I)$. Then we have by [2,7]
\[ \text{Ass}(D_I(M, N)) = \text{Ass}(D_I(\text{Hom}_R(M, N))) = \text{Ass}(\text{Hom}_R(M, N)) \setminus V(I) = (\text{Supp}(M) \cap \text{Ass}(N)) \setminus V(I) \]
as required. \[\square\]

It is well-known that, if \( M, N \) are finitely generated \( R \)-modules, then \( H^i_m(M, N) \) is Artinian for all \( i \geq 0 \) (see [8, 2.2]). It implies that \( \text{Supp}(H^i_m(M, N)) \) is a finite set. When \( N \) is a weakly Laskerian \( R \)-module we have the following theorem.

**Theorem 3.3.** Let \( M \) be a finitely generated \( R \)-module and \( N \) a weakly Laskerian \( R \)-module. If \( \mathfrak{m} \) is a maximal ideal of \( R \), then \( \text{Supp}(H^i_m(M, N)) \) and \( \text{Ass}(R^iD_m(M, N)) \) are finite sets for all \( i \geq 0 \).

**Proof.** As \( N \) is weakly Laskerian, there exists a finitely generated submodule \( L \) of \( N \) such that \( \text{Supp}(N/L) \) is a finite set. Then the short exact sequence
\[
0 \to L \to N \to N/L \to 0
\]
deduces a long exact sequence
\[
\cdots \to H^i_m(M, L) \xrightarrow{f} H^i_m(M, N) \xrightarrow{g} H^i_m(M, N/L) \to \cdots
\]

Since \( H^i_m(M, L) \) is an Artinian \( R \)-module, \( \text{Supp}(H^i_m(M, L)) \) is a finite set and \( \text{Im}f \) is an Artinian \( R \)-module.

Note that \( \text{Supp}(\text{Im}g) \) is a finite set because
\[
\text{Supp}(\text{Im}g) \subset \text{Supp}(H^i_m(M, N/L)) \subset \text{Supp}(N/L).
\]

From the long exact sequence, we obtain a short exact sequence
\[
0 \to \text{Im}f \to H^i_m(M, N) \to \text{Im}g \to 0
\]
which implies \( \text{Supp}(H^i_m(M, N)) = \text{Supp}(\text{Im}f) \cup \text{Supp}(\text{Im}g) \). Thus \( \text{Supp}(H^i_m(M, N)) \) is a finite set and then \( \text{Ass}(H^i_m(M, N)) \) is a finite set. We now consider the exact sequence
\[
0 \to \Gamma_m(M, N) \to \text{Hom}_R(M, N) \to D_m(M, N) \to \cdots
\]
\[
\cdots \to \text{Ext}^i_R(M, N) \to R^iD_m(M, N) \to H^{i+1}_m(M, N) \to \cdots
\]
Since \( \text{Ass}(\text{Ext}^i_R(M, N)) \) is a finite set, we have the conclusion. \[\square\]

**Proposition 3.4.** Let \( M \) be a finitely generated module and \( N \) a weakly Laskerian module over a local ring \((R, \mathfrak{m})\). If \( \dim(N) \leq 2 \), then \( R^iD_1(M, N) \) is weakly Laskerian for all \( i \geq 0 \).
Proof. From [2.3] there is an exact sequence
\[ 0 \to H^0_I(M, N) \to \text{Hom}_R(M, N) \to D_I(M, N) \to H^1_I(M, N) \to \cdots \]
\[ \cdots \to D^i_I(M, N) \to \text{Ext}^i_R(M, N) \to H^{i+1}_I(M, N) \to \cdots \]
If \( \dim(N) \leq 2 \), then \( H^i_I(M, N) \) is weakly Laskerian for all \( i \geq 0 \) by \([12, 3.1]\). Note that \( \text{Ext}^i_R(M, N) \) is weakly Laskerian for all \( i \geq 0 \). Therefore \( R^iD_I(M, N) \) is weakly Laskerian for all \( i \geq 0 \). □

Theorem 3.5. Let \( M \) be a finitely generated \( R \)-module and \( N \) an \( R \)-module. Then

(i) There is a Grothendieck spectral sequence
\[ E_2^{pq} = \text{Ext}^p_R(M, R^qD_I(N)) \Rightarrow R^{p+q}D_I(M, N); \]

(ii) \( \text{Ass}_R(R^tD_I(M, N)) \subseteq (\bigcup_{i=1}^t \text{Ass}_R(E_{i+2}^{i,t-i})) \cup \text{Ass}_R(\text{Hom}(M, R^tD_I(N))); \)

(iii) \( \text{Supp}_R(R^tD_I(M, N)) \subseteq \bigcup_{i=0}^t \text{Supp}_R(\text{Ext}_R^t(M, R^{t-i}D_I(N))). \)

Proof. (i). Let us consider functors \( F(-) = \text{Hom}_R(M, -) \) and \( G(-) = D_I(-) \). The functor \( F(-) \) is left exact. For any injective module \( E \), \( G(E) \) is also an injective module and then is right \( F \)-acyclic. On the other hand, there is a natural equivalence by [2.7] \( D_I(M, -) \cong \text{Hom}_R(M, D_I(-)) \). Thus from [17, 11.38] we have the Grothendieck spectral sequence
\[ E_2^{pq} = \text{Ext}^p_R(M, R^qD_I(N)) \Rightarrow R^{p+q}D_I(M, N). \]

(ii). From the spectral of (i) there is a finite filtration \( \Phi \) of \( R^{p+q}D_I(M, N) \) with
\[ 0 = \Phi^{i+1}H^t \subset \Phi^iH^t \subset \cdots \subset \Phi^1H^t \subset \Phi^0H^t = R^tD_I(M, N) \]
and
\[ E_{\infty}^{i,t-i} = \Phi^iH^t/\Phi^{i+1}H^t, \] where \( t = p + q \), \( 0 \leq i \leq t \).

Exact sequences for all \( 0 \leq i \leq t \)
\[ 0 \to \Phi^{i+1}H^t \to \Phi^iH^t \to E_{\infty}^{i,t-i} \to 0 \]
gives
\[ \text{Ass}(\Phi^iH^t) \subseteq \text{Ass}(\Phi^{i+1}H^t) \cup \text{Ass}(E_{\infty}^{i,t-i}). \]

We may integrate this for \( i = 0, 1, \ldots, t \) to conclude that
\[ \text{Ass}(R^tD_I(M, N)) \subseteq \bigcup_{i=0}^t \text{Ass}_R(E_{\infty}^{i,t-i}). \]
We now consider homomorphisms of the spectral
\[ E_{t+2}^{i-t-2,2t-i+1} \rightarrow E_{t+2}^{i-t} \rightarrow E_{t+2}^{i+t+2,-i-1}. \]
Note that \( E_{t+2}^{i-t-2,2t-i+1} = E_{t+2}^{i+t+2,-i-1} = 0 \) for \( i = 0, 1, \ldots, t \). It follows
\[ E_{t+2}^{i,t-i} = E_{t+3}^{i,t-i} = \ldots = E_{t+2}^{i,t-i}. \]
In particular,
\[ E_0^{0,t} = \ldots = E_{t+3}^{0,t} \subseteq E_{t+2}^{0,t} \subseteq E_{t+1}^{0,t} \subseteq \ldots \subseteq E_2^{0,t}. \]
Therefore
\[ \text{Ass}_R(R^tD_I(M, N)) \subseteq (\bigcup_{i=1}^{t} \text{Ass}_R(E_{t+2}^{i,t-i})) \bigcup \text{Ass}_R(\text{Hom}(M, R^tD_I(N))). \]

(iii). Analysis similar to that in the proof of (ii) shows that
\[ \text{Supp}(R^tD_I(M, N)) \subseteq \bigcup_{i=0}^{t} \text{Supp}_R(E_{t+1}^{i,t-i}) \]
and
\[ E_{t+2}^{i,t-i} = E_{t+3}^{i,t-i} = \ldots = E_{t+2}^{i,t-i}. \]
Thus \( E_0^{i,t-i} \) is a subquotient of \( E_2^{i,t-i} \) and then
\[ \text{Supp}_R(E_{t+1}^{i,t-i}) \subseteq \text{Supp}_R(E_{2}^{i,t-i}) = \text{Supp}_R(\text{Ext}_R^i(M, R^{t-i}D_I(N))). \]
This finishes the proof. \( \square \)

**Corollary 3.6.** Let \( M \) be a finitely generated \( R \)-module, \( N \) a weakly Laskerian \( R \)-module and \( t \) a non-negative integer. If \( R^tD_I(N) \) is weakly Laskerian for all \( i < t \), then \( \text{Ass}(R^tD_I(M, N)) \) is a finite set.

**Proof.** We have by 3.3
\[ \text{Ass}_R(R^tD_I(M, N)) \subseteq (\bigcup_{i=1}^{t} \text{Ass}_R(E_{t+2}^{i,t-i})) \bigcup \text{Ass}_R(\text{Hom}(M, R^tD_I(N))). \]
As \( R^tD_I(N) \) is weakly Laskerian for all \( i < t \), \( \text{Ext}_R^i(M, R^{t-i}D_I(N)) \) is also weakly Laskerian for all \( 1 \leq i < t \). Since \( E_{t+2}^{i,t-i} \) is a subquotient of \( E_2^{i,t-i} = \text{Ext}_R^i(M, R^{t-i}D_I(N)) \), \( E_{t+2}^{i,t-i} \) is weakly Laskerian for all \( 1 \leq i < t \). It follows that \( \bigcup_{i=1}^{t} \text{Ass}_R(E_{t+2}^{i,t-i}) \) is finite. Note that \( R^tD_I(N) \cong H_{t+1}^{i+1}(N) \) for \( i > 0 \). Thus \( H_{t}^{i}(N) \) is weakly Laskerian for all \( i < t+1 \). By [3, 2.2], \( \text{Ass}_R(H_{t}^{i+1}(N)) \) is finite and then \( \text{Ass}_R(\text{Hom}(M, R^tD_I(N))) \) is finite. The proof is complete. \( \square \)
Corollary 3.7. Let $M$ be a finitely generated $R$-module, $N$ an $R$-module and $t$ a non-negative integer. If $\text{Supp}_R(R^iD_I(N))$ is a finite set for all $i \leq t$, then $\text{Supp}_R(R^tD_I(M, N))$ is also a finite set.

Proof. It follows from 3.5 that

$$\text{Supp}_R(R^tD_I(M, N)) \subseteq \bigcup_{i=0}^{t} \text{Supp}_R(\text{Ext}_R^i(M, R^{t-i}D_I(N)))$$

$$\subseteq \bigcup_{i=0}^{t} \text{Supp}_R(R^{t-i}D_I(N)).$$

By the hypothesis we have the conclusion. \square

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