A Spacetime Area Law Bound on Quantum Correlations

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Area laws are a far-reaching consequence of the locality of physical interactions, and they are relevant in a range of systems, from black holes to quantum many-body systems. Typically, these laws concern the entanglement entropy or the quantum mutual information of a subsystem at a single time. However, when considering information propagating in spacetime, while carried by a physical system with local interactions, it is intuitive to expect area laws to hold for spacetime regions; in this work, we prove such a law in the case of quantum lattice systems. We consider a sequence of local quantum operations performed at discrete times on a spin-lattice, such that each operation is associated to a point in spacetime. In the time between operations, the time evolution of the spins is governed by a finite range Hamiltonian. By considering a purification of the quantum instruments and analyzing the quantum mutual information between the ancillas used in the purification, we obtain a spacetime area law bound for the correlations between the measurement outcomes inside a spacetime region, and those outside of it.

I. INTRODUCTION

The investigation of the correlations between results of spatially separated measurements performed on a quantum system has paved the way, through Bell’s inequality and its violation, to our current understanding of fundamental aspects of quantum theory.

Characterizing the correlations between temporally separated events is the main task in the study of dynamical systems. In classical systems, temporal correlations are fully described in terms of a stochastic process. The generalization of this notion to quantum mechanics is not straightforward due to the nontrivial effect observations have on the state of the system. Solutions to this problem date back to Lindblad [1] and Accardi et al. [2]; there, a quantum stochastic process is described by a multi-time correlation matrix which encodes, for a given choice of measurements, the probabilities for all the possible outcomes at different times.

In recent elaborations on these ideas, quantum processes are described in terms of objects referred to as quantum combs [3, 4], operator tensors [5], or process tensors [6] (also see Ref. [7]). In order to fully probe the dynamics of the system at hand, the measurement procedures considered in the analysis include general quantum instruments. The process matrix formalism [8] is a generalization of the above approaches which allows dealing with both spatially and temporally separated measurement scenarios in a unified framework [7, 9, 10]; also notable is Hardy’s pioneering operational framework for indefinite causal structure [11–13].

Space and time assume starkly different roles in non-relativistic quantum mechanics. However, for a many-body system evolving under a local Hamiltonian, relativistic notions emerge. The Lieb-Robinson bound [14] provides a limit on the speed of propagation of information and gives rise to an effective light cone structure. In this setting it has been shown that neither information can be sent outside the light cone, nor can correlations between spatial regions be created in a time shorter than that given by the distance between the regions divided by the Lieb-Robinson velocity [15].

Another typical consequence of the locality of interactions are area laws for the entanglement entropy. First studied in relation to black hole thermodynamics [16–19], area laws were observed to hold in ground states of non-critical quantum lattice systems [20]. They are of considerable importance because they imply an efficient classical representation of such states [21]. An intuitive explanation of the area law in non-critical systems comes from the decay of two point correlation functions [22], it is, however, not trivial to make this argument rigorous [23]. Another quantity of interest in the context of local systems is the quantum mutual information, for which an area law was shown to hold in thermal states [24]. Furthermore, quantum mutual information measures the total amount of information of one system about another [25].

In this paper, we prove a spacetime area law bound on correlations in the presence of local dy-
dynamics (i.e. systems governed by a local Hamiltonian). We consider multiple agents acting locally on a many-body system at different spacetime points with general quantum instruments. We show that the correlations between the outcomes of operations performed within a spacetime region and those performed in its complement are bounded by the area of the boundary of that region (a codimension 1 surface in spacetime). We prove the bound for finite-dimensional quantum systems, with time evolution governed by finite range Hamiltonians, and for one-dimensional quantum cellular automata.

As a computational aid, we shall consider the purification of the instruments used to probe the system. This will reduce the problem of characterizing spacetime correlation between the measurement outcomes to studying the final many-body state of the ancillas used for the purification. Our bound on correlations will be obtained as the consequence of an area law bound on the quantum mutual information between the ancillary systems corresponding to the operations performed inside the spacetime region and the ones corresponding to the operations outside of it. This bound is independent of the instruments used by the different agents, the details of the purification and of the dimensions of the ancillary systems, and can be, therefore, seen as an intrinsic property of the dynamics.

Note that the mentioned results regarding entropy area laws for spacetime regions [16–19] refer to the area of a codimension 2 surface in spacetime, whereas in this paper the boundary of a region is a codimension 1 surface. Several approaches have been suggested with the aim of reconstructing the geometry of spacetime from entanglement structure or from correlations of certain quantum states [26–29]. Based on the quantity we bound in this paper, we suggest a definition for a measure which quantifies the maximum bipartite correlation generated by a general process matrix. This may allow the application of similar methods of reconstruction to general quantum processes.

The paper is organized as follows: In Section II we specify the purification scheme which maps measurements in spacetime to a many-body state. In Section III we describe the setting of the problem. In Section IV we state the main result of the paper. Section V elaborates on the relation between correlations and quantum mutual information. In Section VI and Section VII the proofs are provided for the one-dimensional and D-dimensional cases respectively. In Section VIII we suggest a generalization of the quantity studied in this paper to general quantum processes. We conclude in Section IX with a summary and a discussion.

II. PURIFICATION OF QUANTUM INSTRUMENTS

Any quantum instrument can be implemented by introducing an ancillary quantum system in a pure state (which we denote by $|0\rangle$), applying a unitary on both system and ancilla and performing a projective measurement on the ancilla to obtain the recorded measurement outcome and the corresponding post-measurement state of the system [30].

When performing a sequence of measurements, each one involving a fresh ancillary system, the projective measurements can be deferred to the end of the overall process. Up to that point, the physical system and the ancillas undergo a unitary transformation. The resulting state of the ancillas (before the projective measurements) can be represented by a matrix product state [31], as illustrated in Figs. 1 and 2. In [32] this representation was used to show that consecutive measurements on a single quantum system can exhibit temporal correlations as complex as the ones between outcomes of projective measurements performed on a spin chain in an entangled state (e.g. W, GHZ or $1D$ cluster states). In what follows we shall represent the state of the ancillas after the measurement process as a tensor network state [21]. The graphical notation we shall use is explained in Figs. 1 and 2.

Throughout the following, we shall use 'quantum instruments', 'quantum operations' and simply 'measurements' interchangeably.

![Diagram](https://example.com/diagram.png)

**FIG. 1.** Purification of a quantum instrument. After a unitary interaction between the system, in the state $|\Psi\rangle$, and the ancilla, in the state $|0\rangle$, the ancilla is measured to produce an outcome and a post measurement state, which continues down the ‘wire’ on the right (solid line). Such projective measurements can be deferred to later times, as indicated by the dashed line.
III. THE SETTING

For the sake of clarity we shall first formulate the problem for a system in one spatial dimension. In this case, the graphical representation of the problem is instructive and easy to follow (see Fig. 3). The generalization of the problem to any spatial dimension is straightforward and we shall state the result in full generality in the next section.

Consider a ‘spin’ chain, i.e. identical d-dimensional quantum systems positioned on a one-dimensional lattice, with each system labeled by its position \((x_1, x_2, \ldots, x_N)\) (we shall refer to the physical systems as spins throughout the following). Let \(\mathcal{H}\) be the Hilbert space representing one such spin, we denote by \(\mathcal{L}(\mathcal{H})\) the space of operators on \(\mathcal{H}\). Let the chain initially be in an arbitrary state \(\rho_0 \in \mathcal{L}(\mathcal{H}^\otimes N)\) and evolve in time according to a unitary time evolution generated by a finite range interaction Hamiltonian \(H = \sum_i h_i\), where each term \(h_i\) acts on at most \(r\) adjacent spins \((r\) is a function of the range of the Hamiltonian and of the spatial dimension). Let the interaction terms be uniformly bounded by \(\|h_i\| := \sup_i \|h_i\|\).

At times \((t_1, t_2, \ldots)\) a quantum instrument acts on each spin. We assume that the measurements are performed instantaneously, i.e. either they happen on a negligible time scale compared to the typical time scale of the Hamiltonian or that one has the ability to pause the Hamiltonian time evolution and perform the operations (e.g. in an experimental setting). The different measurements are performed at spacetime points \((x, t)\), where \(x\) is the position of the spin in the chain and \(t\) is the time of the measurement. With each measurement, we associate an ancillary system. At the end of the process, we can perform a general quantum measurement (POVM) on the physical spins. The state of the ancillas at the end of the measurement process is given by the tensor network state shown in Fig. 3. Let \(A\) be a spacetime region comprised of \(X\) spins (we measure the spatial extent of a system in units of the lattice spacing, so that in one dimension length is equal to the number of spins) and spanning \(T = \tau \Delta t\), where \(\tau\) is the number of time steps and \(\Delta t\) is the length of time evolution between measurements (for ease of notation the time intervals between measurements are taken to be equal; from the proof method it shall become clear that this plays no role, the result holds for arbitrary time intervals). The region \(A\) is the area encircled by the red dashed line in Fig. 3.

IV. MAIN RESULT

We state our results in terms of a bound on the quantum mutual information between the ancillas of the measurements performed inside the spacetime region \(A\) and the rest of the system, which includes the ancillas of the measurements outside \(A\) as well as the physical spins at the end of the measurement process. The quantum mutual information of a bipartite quantum system in a state \(\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)\) is given by [30]:

\[
I(A : B)_\rho = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) ,
\]

where \(S(\rho) = -\text{Tr} \rho \log \rho\) is the von Neumann entropy and \(\rho_A = \text{Tr}_B \rho_{AB}\) is the reduced state of the system \(A\).

We shall now state the main result of the paper for a spin lattice of any spatial dimension. We fix a measure of distance on the lattice in terms of which we define finite range Hamiltonians with range \(R\) as such in which each interaction term acts inside a ball of radius \(R\).

**Theorem 1.** Let a \(D\)-dimensional spin lattice, with each spin described by a \(d\)-dimensional Hilbert space, initially be in a state \(\rho_0\), and let the spins evolve in time according to a finite range Hamiltonian \(H = \sum h_i\) with range \(R\), where all the terms are bounded by \(\|h_i\| = \sup_i \|h_i\|\). Let arbitrary quantum instruments be applied individually on each spin at times \((t_1, t_2, \ldots)\). Let \(\Sigma\) be a connected subset of the \(D\)-dimensional spin lattice and let \((t_m, t_n)\) be a time interval so that together they define a spacetime region \(A = \Sigma \times (t_m, t_n)\). Let \(\rho\) be the state of the combined system of spins and ancillas at the end of the measurement process, then there exists a constant \(C > 0\) which depends only on \(D\) and \(R\), such
that the following bound holds for the quantum mutual information between the ancillas corresponding to measurements performed inside the region $A$ and the rest of the system $\bar{A}$:

$$I(A : \bar{A})_\rho \leq C \|h\| \log d |\partial A|,$$

where $|\partial A| = 2|\Sigma| + T|\partial \Sigma|$, with $T = t_n - t_m$ and where $| \cdot |$ counts the number of elements in a subsystem.

V. MUTUAL INFORMATION AND CORRELATIONS

Before proceeding with the proof of Theorem 1 we elaborate on the relation between quantum mutual information and the correlations between the recorded measurement outcomes.

The quantum mutual information has a well defined operational meaning as the total amount of correlations between two systems - measured by the amount of local noise necessary to erase those correlations [25]. Furthermore, the quantum mutual information in a bipartite system bounds all connected correlation functions between observables on the two parts [24]. More precisely:

**Proposition 1.** Let the system $AB$ be in the state $\rho_{AB}$, and let $O_A$ and $O_B$ be Hermitian operators on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, then

$$\frac{|\langle O_A \otimes I \rangle \otimes \rho_{AB} - \langle O_A \otimes O_B \rangle|}{2\|O_A\|^2\|O_B\|^2} \leq I(A : B)_{\rho_{AB}},$$

where $\langle X \rangle := \text{Tr}(X \rho_{AB})$ and $I(A : B)$ is the mutual information defined in Eq. (1).

In order to bound correlation functions between the measurement outcomes recorded by the instruments in two different regions in spacetime, it is, therefore, sufficient to bound the quantum mutual information between the ancillas associated with those regions. Note, however, that such a bound on the quantum mutual information implies a bound on correlation functions for a larger class of measurements than those obtained by performing projective measurements of individual ancillas separately. As quantum mutual information bounds correlations between any two operators supported on the Hilbert spaces of the ancillas in regions $A$ and $\bar{A}$, the bound applies to correlations between outcomes of “spacetime entangled measurements” (e.g. measurements in the Bell basis of two ancillary qubits corresponding to operations performed on different spins at different times).

Quantum mutual information is non-increasing with respect to applying completely positive and trace preserving (CPTP) maps separately on each subsystem [30]. Let $\mathcal{C}$ be a CPTP map on $\mathcal{L(H}_A)$,
then
\[ I(A : B)_{(C \otimes i)\rho_{AB}} \leq I(A : B)_{\rho_{AB}}. \tag{3} \]

In particular, tracing out parts of subsystems does not increase mutual information. Let \( \rho \in \mathcal{L}(H_A \otimes H_{A'} \otimes H_B) \) be a state of a tripartite system, then
\[ I(A : B)_{\text{Tr}_{A'} \rho} \leq I(AA' : B)_\rho. \tag{4} \]

Eq. (4) is equivalent to the strong subadditivity of the von Neumann entropy \([33]\). This property entails the following. First, we can purify the initial state of the spin lattice \( \rho_0 \), and consider a pure initial joint state of the spins together with the ancillas and the purifying system. As both the instruments and the time evolution are implemented unitarily, the state remains pure throughout the process. The mutual information between the ancillas corresponding to a spacetime region and the rest of the system is then simply given by twice the von Neumann entropy of the reduced state. It is therefore sufficient to bound the entropy of the ancillas in the region \( A \) in order to bound the mutual information. We denote it by \( S_A := S(\rho_A) \).

Secondly, we can bound the classical mutual information between the probability distributions for the recorded outcomes of measurements performed in the region \( A \) and those performed outside of it, in terms of the quantum mutual information. Let \( \rho \) be a state of the ancilla system used to purify a quantum measurement, and let the measurement outcomes be obtained by a POVM \( \{E_i\} \). The following CPTP map transforms \( \rho \) into a diagonal density matrix with entries equal to the probabilities \( p_i = \text{Tr}(\rho E_i) \):
\[ \rho \mapsto \sum_i \text{Tr}(\rho E_i) |i\rangle \langle i|. \]

After applying this channel on both sides of a bipartite state, the quantum mutual information is equal to the classical mutual information of the probabilities for the outcomes of both parties. The classical mutual information is, therefore, bounded, due to Eq. (3), by the quantum mutual information of the bipartite state before applying the channel.

This bound on the classical mutual information emphasizes an operational consequence of our result: when restricted to access the dynamical system only in a given spacetime region, the information an observer can gain about possible measurement outcomes outside of the region is bounded by the area of the region’s boundary.

VI. PROOF IN ONE SPATIAL DIMENSION

The representation of the state of the ancillas as a tensor network state in Fig. 3 allows for a direct proof of an area law bound on the mutual information, in the case when the evolution between time steps is given by a matrix product operator \([34]\). This is due to the fact that tensor network states have, by construction, an area law bound on the entanglement entropy of subsystems. Precisely

**Lemma 1.** The entanglement entropy of a pure tensor network state with respect to a bipartition \( A : B \) is bounded by the logarithm of the product of the bond dimensions severed by the cut which defines the bipartition
\[ S_A \leq \log \left( \prod_{i \in \partial A} D^{(i)} \right), \]

where \( D^{(i)} \) is the dimension of the bond \( i \).

See \([21]\) for a proof.

Quantum cellular automata are a form of translationally invariant discrete dynamics of a spin chain whose defining characteristic is that they map local operators to local operators \([35]\). They have been shown to exactly coincide with the class of translationally invariant matrix product unitary operators \([36]\), with the bond dimension determined by the quantum cellular automaton’s propagation speed. This characterization, combined with Fig. 3 and Lemma 1 proves the following (replacing the time evolution operator in Fig. 3 by a matrix product operator allows to apply Lemma 1 when “cutting through” it).

**Proposition 2.** Let the time evolution operator \( U \) in Fig. 3 be a quantum cellular automaton, then an area law for the quantum mutual information holds:
\[ I(A : \bar{A})_\rho \leq C \log d |\partial A|, \]

where \( \rho, d \) and \( A \) are defined as in Thm. 1, and where \( C > 0 \) is a constant that depends on the cellular automaton’s propagation speed.

Even though it is true that in 1D time evolution operators generated by finite range Hamiltonians are well approximated by matrix product operators \([37]\), pursuing this method of proof resulted in the desired bound up to a correction which scales like \( \tau \log(|A|) \). In what follows, we prove an exact area law bound by using a bound on the entanglement generation rate of local Hamiltonians \([38]\).
Theorem 1 follows from Proposition 3 given below, where we consider a fixed spacetime region and allow for general instruments to be applied inside and outside of the region. This is a stronger result in the sense that it corresponds to agents given full control over a subset of the spins at each time step, whereas in Theorem 1 all operations are strictly local. The new setting is depicted in Fig. 4 where the instruments are allowed to act non-locally in space as long as they do not cross the region’s boundaries.

**Proposition 3.** Let $A$ be a spacetime rectangle with side lengths $X$ and $T = \tau \Delta t$ (the region encircled by the red dashed line in Fig. 4), where $\tau$ is the number of time steps and $\Delta t$ is the length of time evolution between measurements. Let instruments be applied at each time step such that they do not cross the boundary of $A$ (compare Fig. 3), then there exists a constant $C > 0$, depending only on $r$ (the maximal number of spins in the support of each term in the interaction Hamiltonian) such that the quantum mutual information of the ancilla systems in $A$ and the rest of the system $\bar{A}$ satisfies

$$I(A : \bar{A}) \leq C \| h \| \log d (2X + 2T) = C \| h \| \log d |\partial A| ,$$

where $d$ is the dimension of the Hilbert space of a single spin.

To set the stage for the proof we divide the spins and ancillas into the following three sets:

- **A:** The ancillas inside the region $A$ (encircled by the dashed red line in Fig. 4).
- **B:** The spins in the chain on which the measurements in $A$ are performed (there are $X$ of them).
- **C:** The rest of the spins and ancillas.

With the above definitions of subsystems $A$, $B$, and $C$, the sequence of measurements can be represented by the circuit diagram in Fig. 5, which is key to understanding how the system $A$ gets entangled with the rest of the system.

Denote by $t_0$ the time of the last measurement which precedes the measurements inside $A$, let $t_n := t_0 + n\Delta t$ and let $t_f$ be the time of the final measurement. From $t_f$ onward, the system $A$ does not interact with the system $BC$ (see Fig. 5), therefore the entropy of $A$ at the end of the entire process, $S_A(t_f)$ (the quantity we wish to bound), is equal to $S_A(t_r)

Next, we require the following result [38, 39] which bounds the entropy generation rate for local interactions between quantum systems $A$ and $B$, equipped with ancillary systems $a$ and $b$ respectively.

**Theorem 2** (Small Incremental Entangling). Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_b$, and let $|\psi(t)\rangle = \exp(-itH)|\psi\rangle$, where $H = \mathds{1}_a \otimes H_{AB} \otimes \mathds{1}_b$ is an interaction Hamiltonian between $A$ and $B$. Denote $d = \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$. There exists a constant $c > 0$, independent of the sizes of the Hilbert spaces, the initial state and of the details of the interaction Hamiltonian, such that the rate of change of the entropy of the system $aA$ is bounded by

$$\left| \frac{dS(\rho_{aA}(t))}{dt} \right| \leq c \| H_{AB} \| \log d ,$$

where $\rho_{aA}(t) = \text{Tr}_{BB} |\psi(t)\rangle\langle \psi(t)|$.

As in the time interval $(t_0, t_f)$ systems $AB$ and $C$ interact only via the time evolution operators acting on the physical spins, we shall now use Theorem 2 in order to bound the increase of $S_C = S_{AB}$ with each time step (the overall state $\rho_{ABC}$ is pure at all times). The following shows that the only terms in the Hamiltonian able to generate entanglement (increase $S_C$) are the ones that intersect the boundary of the system $B$ (the set of spins on which the measurements in $A$ are performed; in the one-dimensional case, $\partial B$ consists of two points).

**Lemma 2.** Let $|\psi\rangle \in \mathcal{H}_C \otimes \mathcal{H}_B$, where $\mathcal{H}_B = \mathcal{H}_C^{\otimes |B|}$, $\mathcal{H}_C = \mathcal{H}_C^{\otimes |C|}$ and $|B|$ and $|C|$ are the numbers of spins constituting the subsystems. Let $H = H_C + H_B + \sum_{i=1}^M H_{CB}^i$ where each one of the $M$ interaction terms $\{H_{CB}^i\}$ is supported on at most $r$ spins. The increase in the entropy $S_C$ after evolving for a period of time $t$ is bounded by

$$|S_C(t) - S_C(0)| \leq ctM(r - 1)\| h \| \log d,$$

where $c$ is the constant from Theorem 2, $\| h \| := \max_i (\| H_{CB}^i \|)$ and $d = \dim \mathcal{H}$ is the Hilbert space dimension of a single spin.

**Proof.** Approximate $U = \exp(-itH)$ by an order $n$ Trotter product [40]

$$\tilde{U}_n = e^{\left( \frac{-t}{n} H_C \right)} e^{\left( \frac{-t}{n} H_B \right)} \prod_{i=1}^M e^{\left( \frac{-t}{n} H_{CB}^i \right)} \right] .$$

Consider the entropy increase due to the approximate time evolution. Only the terms $\{ \exp(\frac{-t}{n} H_{CB}^i) \}$ can generate entanglement between $B$ and $C$. Assume w.l.o.g. $|C| > r$ (we can always add spins with trivial evolution to $C$), and consider the worst case for $d$ in Theorem 2, when each term $H_{CB}^i$ acts on one spin in $C$ and $r - 1$ spins in $B$. Integrating the bound in Theorem 2
we find that each term’s contribution to the entropy is bounded by $cn^{-1} \log(d'^{-1}) \|h\|$ (apply the theorem with “a” taken to be the ancillas in $C$, “A” the spins in $C$, “B” - the support of the interaction term intersected with $B$ and “b” - the rest of $B$). Summing up the $M \times n$ different contributions, we find that, independently of $n$, the entropy increase due to the approximate time evolution is bounded by the RHS of Eq. (5). The approximate reduced state $\tilde{\rho}_C(t) = \Tr_B \tilde{\rho}(t) = \Tr_B \tilde{U}_n \rho \tilde{U}_n^\dagger$ approximates the real reduced state arbitrarily well for increasingly larger $n$. Precisely, from the Fuchs-Van de Graaf inequality [41], and the fact that partial trace is contractive with respect to the trace distance, it follows that

$$\| \Tr_B \tilde{\rho}(t) - \Tr_B \tilde{\rho}(t) \|_1 \leq 2 \| U - \tilde{U}_n \| \longrightarrow 0 \ ,$$

where the right hand side vanishes as $n \rightarrow \infty$ because the Trotter product approximates the time evolution operator up to $O(n^{-1})$ in operator norm [40]. Finally, by the Fannes inequality [30], the entropy is continuous with respect to the trace norm. It, therefore, follows that the entropy increase due to the real time evolution also satisfies the stated bound.

We now have the required ingredients for the proof
of Proposition 3.

Proof. (Proposition 3) According to Lemma 2, the increase in $S_C$ with each time step is bounded by 
$$\Delta S = 2c\Delta t \|h\|(r-1)^2 \log(d)$$
($M$ - the number of interaction terms between systems $B$ and $C$ is equal to $2(r-1)$). Bound the total increase in $S_C$ in the time interval of interest:

$$S_{AB}(t_f) = S_C(t_\tau) \leq S_C(t_{\tau-1}) + \Delta S$$
$$\leq S_C(t_{\tau-2}) + 2\Delta S \leq \ldots$$
$$\leq S_C(t_1) + \tau\Delta S$$

(6)
$$= S_{AB}(t_1) + \tau\Delta S$$
$$= S_B(t_1) + \tau\Delta S$$

where in the last step we used the fact that at time $t_1$ the system $AB$ is in a product state and the state of $A$ is pure. Using the triangle inequality for $S_{AB}$ and Eq. (6) we obtain

$$|S_A(t_f) - S_B(t_f)| \leq S_{AB}(t_f) \leq S_B(t_1) + \tau\Delta S$$.

Therefore,

$$S_A(t_f) \leq S_B(t_1) + S_B(t_f) + \tau\Delta S \leq 2\log(d_B) + \tau\Delta S$$,

where $d_B$ is the dimension of $H_B$, i.e. $d_B = d^N$, and we bounded $S_B(t)$ by its maximum possible value. Plugging in the bound $\Delta S$ we obtain the desired area law

$$S_A(t_f) \leq 2\log(d_B) + 2c\tau\Delta t \|h\|(r-1)^2 \log(d)$$
$$= 2\log(d^N) + 2cT(r-1)^2\|h\| \log(d)$$
$$\leq C(r)\|h\| \log(d)(2X + 2T)$$
$$= C(r)\|h\| \log(d)\|O\!A\|$$,

where $C(r) := 2c(r-1)^2$ and $c$ from Theorem 2
(w.l.o.g. $\|h\|e > 1$).

We comment that examples that saturate the area law bound can be readily constructed by choosing the time evolution such that in each interval $\Delta t$ a product state of two spins across the boundary $|0\rangle \otimes |0\rangle$ is transformed into a maximally entangled state, and using swap gates between the spins and ancillas at each measurement. This way each ancilla pair on the temporal boundary will become maximally entangled. Maximal entanglement across the spatial boundary can be achieved by inputting a specific entangled state of the spin chain, and by swapping half of a maximally entangled ancilla state with each spin at the last measurement inside $A$.

VII. PROOF IN HIGHER SPATIAL DIMENSIONS

The proof is essentially the same when space is $D$-dimensional. The circuit diagram representation in Fig. 5 still holds for the appropriately defined subsystems. It remains only to compute the number of interaction terms $M$ in Lemma 2. Let $n(D, R)$ be the number of spins inside a ball of radius $R$. For a finite range Hamiltonian with range $R$, the number of interaction terms acting on a single spin in the lattice is then at most $n(D, R)$. Ignoring multiple counting of the same terms, we can bound $M$ by $|\partial S| \cdot n(D, R)$ which gives the desired result, thus proving Theorem 1.

VIII. A GENERAL MEASURE OF CORRELATIONS FOR QUANTUM PROCESSES

In this section we generalise some of the previous reasoning, which was for the specific case of local Hamiltonian evolution of lattice systems, to the case of general process matrices [8]. In particular, we use the quantity that is bounded in Theorem 1 to provide an intrinsic measure of the amount of correlations achievable by a multipartite process matrix.

The process matrix formalism allows to compute the joint probabilities for the outcomes of quantum experiments performed in local laboratories, without needing to pre-assume a causal order between the different laboratories. The process formalism is general enough to describe any causally ordered scenario [4, 42], as well as experimentally relevant non-causal processes [43–49]; it also predicts counterintuitive – and so far unobserved – phenomena, such as the violation of causal inequalities [8, 49–52]. The formalism is cursory reviewed in Appendix A.

Let $W$ be a $N$-partite process matrix, with parties labeled from 1 to $N$. To each party $j$, associate an input Hilbert space $H_{I_j}$, an output Hilbert space $H_{O_j}$ and an ancillary output Hilbert space $H_{O_j'}$ of arbitrary dimensions. Let each of the parties make an ancilla interact with the process, i.e. perform a local CPTP map $M_j : \mathcal{L}(H_{I_j}) \rightarrow \mathcal{L}(H_{O_j} \otimes H_{O_j'})$. The final state of all the ancillas is obtained from Eq. (8), whose graphical representation is Figure 6, and we denote this state by $\rho_M$ with a subscript to emphasize its dependence on the choice of local CPTP maps $M : \{M_j\}_{j=1}^N$.

Given any bipartition of the $N$ parties into two sets $A$ and $\bar{A}$, one can calculate $I_{\rho_M}(A : A)$, the
mutual information, across the bipartition, of the final state of the ancillas. By maximising the mutual information, over all probing schemes \( \mathcal{M} \), we get a bound on the maximal amount of correlations that the process allows across this bipartition. This leads us to define the following intrinsic measure of the strength of correlations allowed by a process:

\[
\mathcal{C}_W(A : \bar{A}) := \sup_{\mathcal{M}} I_{\mathcal{P}, \mathcal{M}}(A : \bar{A}),
\]

where the maximisation is taken over all CPTP maps \( \{\mathcal{M}_j\}_{j=1}^N \) and all dimensions for the ancillary output Hilbert spaces. Since the process matrix formalism is closely related to many other operational approaches for dealing with multipartite signalling quantum correlations [3–7]; our definition of \( \mathcal{C}_W \) thus directly applies as an intrinsic measure of correlations in those formalisms as well.

The quantity \( \mathcal{C}_W(A : \bar{A}) \) is exactly what is being bounded by the spacetime area law of Theorem 1, in the case when \( W \) describes a local Hamiltonian evolution. For arbitrary processes \( W \) however, the mutual information cannot be expected to scale with the boundary area, and will rather only be bounded by the spacetime volume of the region.

We finish this section by discussing some examples that illustrate the meaning of \( \mathcal{C}_W \), in case of two parties, which we label by \( A \) and \( B \). Let \( W = \omega^{A_1B_1} \otimes I_{A_0B_0} \) be a state shared between the two parties. In this case it is straightforward to see that \( \mathcal{C}_W(A : B) \) is just the usual quantum mutual information of the state \( \omega \).

Another important example is when the process \( W^{A_0B_1} \) (here \( A_1 \) and \( B_0 \) are assumed to be the trivial Hilbert space) is the Choi state of a channel \( \mathcal{W} : \mathcal{L}(\mathcal{H}_{A_0}) \rightarrow \mathcal{L}(\mathcal{H}_{B_1}) \). Let \( \omega^{A_0A_0'} \) be any state in \( \mathcal{H}_{A_0} \otimes \mathcal{H}_{A_0'} \), and consider the state \( \Omega^{A_0B_1} \) that is obtained by using \( W \) to send the \( A_0 \) subsystem to \( B_1 \). Then \( \mathcal{C}_W(A : B) \) is the maximum mutual information between \( A_0' \) and \( B_1 \) that one can get when \( \Omega^{A_0B_1} \) is obtained in that way. This quantity is known to be equal to the entanglement-assisted classical capacity of the channel \( \mathcal{W} \) [53], which is evidently an upper bound to the ordinary classical capacity of \( \mathcal{W} \).

One interesting probing scheme consists of all parties performing \( \mathcal{M}_A(\rho) = \rho^{A_0A_0'} \otimes |\Phi^+\rangle\langle\Phi^+|^{A_0A_0'} \), \( A_0' := A_1^B \otimes A_2^B \cong A_1 \otimes A_0 \), and appears in Refs. \([7, 54]\). In this case, the final state of the ancillas is equal (up to normalisation), to the process matrix \( W \). This shows that \( \mathcal{C}_W \) is at least as large as the mutual information of the state \( \rho = W/\text{Tr}(W) \). However, one can find examples showing that \( \mathcal{C}_W \) is in general strictly larger than that. Indeed, let \( d_{A_1} = d_{A_0} = d_{B_1} = 2, d_{B_0} = 1 \), and take the (causally-ordered) process

\[
W = |\Phi^+\rangle\langle\Phi^+|^{A_1B_1} \otimes |0\rangle\langle 0|^{A_0} + \frac{1}{4}|1\rangle\langle 1|^{A_0}. 
\]  

The mutual information of \( W/\text{Tr}(W) \), across the \( (A : B) \) bipartition is 1 bit, while for the probing scheme \( \mathcal{M}_A(\rho) = |0\rangle\langle 0|^{A_0} \otimes \rho^{A_0'}, \mathcal{M}_B(\rho) = \rho^{B_0'}, \) the final state of the ancillas has a mutual information of 2 bits.

**IX. Conclusion**

We have considered agents acting locally on a quantum system which undergoes time evolution governed by local dynamics. We showed that the amount of information that an agent localised within a spacetime region can acquire about the outside is at most proportional to the area of the region’s boundary (Proposition 3). We further showed that when all operations are local in spacetime, the correlations between their outcomes are bounded by the same spacetime area law (Theorem 1).

These results make precise the intuition that when both dynamics and operations are local, the information content of a spacetime region should scale like its boundary. We derived this result relying on the principles of operationalism and locality. The same principles are at the basis of recent suggestions for a generalization of quantum theory to arbitrary background spacetimes [55]. There, quantum states are indeed associated with boundaries of spacetime regions.

It has been suggested that spatial geometry can be reconstructed from the entanglement structure of
certain quantum states [26]. In this paper we proposed a measure of bipartite correlation which is intrinsic to general quantum processes. Further study shall investigate the possibility of applying similar methods to a set of such correlations in order to reconstruct the spacetime geometry of the underlying process.

We have restricted our analysis to finite-dimensional systems. This allowed us to obtain bounds on the entropy of subsystems simply by considering their dimensions and to ignore information about initial states, the particular type of interactions etc. This simplification, however, makes our result inapplicable to infinite-dimensional systems, for example quantum fields, which are necessary for Lorentz invariant local interactions. Further work shall consider the same questions in a setting with infinite-dimensional systems.

X. ACKNOWLEDGMENTS

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[1] G. Lindblad, “Non-Markovian quantum stochastic processes and their entropy,” Comm. Math. Phys. 65, 281–294 (1979).
[2] L. Accardi, A. Frigerio, and J. T. Lewis, “Quantum Stochastic Processes,” Publications of the Research Institute for Mathematical Sciences 18, 97–133 (1982).
[3] G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Quantum Circuit Architecture,” Phys. Rev. Lett. 101, 060401 (2008). arXiv:0712.1325.
[4] G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Theoretical framework for quantum networks,” Phys. Rev. A 80, 022339 (2009), arXiv:0904.4483 [quant-ph].
[5] L. Hardy, “The operator tensor formulation of quantum theory,” Philos. Trans. R. Soc. A 370, 3385–3417 (2012), arXiv:1201.4390.
[6] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, “Non-Markovian quantum processes: Complete framework and efficient characterization,” Phys. Rev. A 97, 012127 (2018).
[7] J. Cotler, C.-M. Jian, X.-L. Qi, and F. Wilczek, “Superdensity Operators for Spacetime Quantum Mechanics,” arXiv:1711.03119 [quant-ph].
[8] O. Oreshkov, F. Costa, and Č. Brukner, “Quantum correlations with no causal order,” Nat. Commun. 3, 1092 (2012), arXiv:1105.4464 [quant-ph].
[9] J. Fitzsimons, J. Jones, and V. Vedral, “Quantum correlations which imply causation,” arXiv:1302.2731 [quant-ph].
[10] D. Horsman, C. Heunen, M. F. Pusey, J. Barrett, and R. W. Spekkens, “Can a quantum state over time resemble a quantum state at a single time?,” Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 473, (2017).
[11] L. Hardy, “Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity,” arXiv:gr-qc/0509120 [gr-qc].
[12] L. Hardy, “Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure,” J. Phys. A: Math. Theor. 40, 3081 (2007), arXiv:gr-qc/0608043 [gr-qc].
[13] L. Hardy, “Quantum Gravity Computers: On the Theory of Computation with Indefinite Causal Structure,” in Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle: Essays in Honour of Abner Shimony, pp. 379–401. Springer Netherlands, Dordrecht, 2009. arXiv:quant-ph/0701019 [quant-ph].
[14] E. H. Lieb and D. W. Robinson, “The finite group velocity of quantum spin systems,” Communications in Mathematical Physics 28, 251–257 (1972).
[15] S. Bravyi, M. B. Hastings, and F. Verstraete, “Lieb-Robinson bounds and the generation of correlations and topological quantum order,” arXiv:0603121 [quant-ph].
[16] M. Srednicki, “Entropy and area,” Phys. Rev. Lett. 71, 666–669 (1993).
[17] C. G. Callan, Jr. and F. Wilczek, “On geometric entropy,” Phys. Lett. B333, 55–61 (1994), arXiv:hep-th/9401072 [hep-th].
[18] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, “Quantum source of entropy for black holes,”
Appendix A  THE PROCESS MATRIX
FORMALISM

We briefly review the basics of the process matrix formalism. We refer the reader to the original reference [8], as well as Refs. [44, 54] for more complete introductions. Our choice of convention for the Choi-Jamiolkowski isomorphism [56, 57] follows that of Ref. [54].

If \( \mathcal{H} \) is a (finite-dimensional) Hilbert space, we denote by \( \mathcal{L}(\mathcal{H}) \) the space of linear operators acting on \( \mathcal{H} \). If \( \mathcal{H}_A \) and \( \mathcal{H}_B \) are two Hilbert spaces, we use \( \mathcal{H}_{AB} \) to denote their tensor product. For every linear map \( M : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A) \) we define its Choi matrix \( M \in \mathcal{L}(\mathcal{H}_A;\mathcal{H}_A) \) as

\[
M^{A_1A_0} = \sum_{ij=1}^{\dim \mathcal{H}_A} |i\rangle\langle j|^A_1 \otimes \mathcal{M}(|i\rangle\langle j|)^A_0.
\]

One may check that the isomorphism can be inverted by using the formula

\[
\mathcal{M}(\rho) = \text{Tr}_{A_1} \left( M^{A_1A_0} \cdot (\rho^A)^T \otimes I^{A_0} \right),
\]

where \( T \) denotes the transposition in the computational basis. The above equation can be used to show that \( \mathcal{M} \) is trace-preserving iff \( \text{Tr}_{A_0} M^{A_1A_0} = I^{A_1} \). Choi’s theorem [57], states that \( M^{A_1A_0} \) corresponds to a completely-positive (CP) map if and only if \( M \geq 0 \).

We define process matrices in the case of two parties, whose local laboratories respectively have input Hilbert spaces \( \mathcal{H}_{A_1}, \mathcal{H}_{B_1} \), and output Hilbert spaces \( \mathcal{H}_{A_0}, \mathcal{H}_{B_0} \). The generalisation of the definition to more parties is straightforward.

**Definition 1.** (Process matrix) An operator \( W^{A_1A_0B_1B_0} \in \mathcal{L}(\mathcal{H}_{A_1A_0B_1B_0}) \) is a process matrix if for all CPTP maps \( M : \mathcal{L}(\mathcal{H}_{A_1A_0}) \rightarrow \mathcal{L}(\mathcal{H}_{A_1A_0}^{A'_0}) \), \( N : \mathcal{L}(\mathcal{H}_{B_1B_0}) \rightarrow \mathcal{L}(\mathcal{H}_{B_1B_0}^{B'_0}) \), where \( \mathcal{H}_{A_1}, \mathcal{H}_{A_0}, \mathcal{H}_{B_1}, \mathcal{H}_{B_0} \) are ancillary Hilbert spaces of arbitrary dimension, the operator

\[
G = \text{Tr}_{A_1A_0B_1B_0} \left( W^T (M^{A_1A_0A'_0} \otimes N^{B_1B_0B'_0}) \right)
\]

is the Choi state of a CPTP map from \( \mathcal{H}_{A_1B_1} \) to \( \mathcal{H}_{A'_0B'_0} \), i.e. \( \text{Tr}_{A_0B_0} G = I^{A'_1B'_1} \). In the above, \( W^T \) is transpose of \( W \), while \( M \) and \( N \) are the Choi operators corresponding to the CPTP maps \( M^{A_1A_0A'_0} \) and \( N^{B_1B_0B'_0} \).

A process matrix allows one to calculate probabilities for local quantum instruments. A quantum instrument is defined as a collection \( \{ M_{a|x} \} \) of CP maps which sum to a CPTP map; here \( x \) is a set of settings, while \( a \) is the outcome recorded by the instrument. When the ancillary Hilbert spaces are trivial, Eq. (8) reduces to

\[
p(ab|xy) = \text{Tr} \left[ \left( M_{a|x}^{A_1A_0} \otimes M_{b|y}^{B_1B_0} \right) W^T \right].
\]

In order for \( W \) to produce valid probabilities (positive and normalised), for all choices of local quantum instruments, one may show that the following constraints must be satisfied [44, 54]

\[
W \geq 0,
\]

\[
\text{Tr} W = d_{A_0} d_{B_0},
\]

\[
B_1B_0 W = A_0B_1B_0 W,
\]

\[
A_1A_0 W = A_1A_0B_0 W,
\]

\[
W = B_0 W + A_0 W - A_0B_0 W,
\]

where \( xW := \frac{\partial x}{\partial x} \otimes \text{Tr}_X W \).