Logical Quantum Field Theory

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Abstract

We consider the statistical mechanical ensemble of bit string histories that are computed by a universal Turing machine. The role of the energy is played by the program size. We show that this ensemble has a first-order phase transition at a critical temperature, at which the partition function equals Chaitin’s halting probability Ω. This phase transition is almost zeroth-order in the sense that the free energy is continuous near the critical temperature, but almost jumps: it converges more slowly to its finite critical value than any computable function. We define a non-universal Turing machine that approximates this behavior of the partition function in a computable way by a super-logarithmic singularity, and study its statistical mechanical properties. For universal Turing machines, we conjecture that the ensemble of bit string histories at the critical temperature can be formulated as a string theory.

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1 Introduction

In 1975, G. Chaitin [1] introduced a constant associated with a given universal Turing machine $U$ that is often called the "halting probability" $\Omega$. It is computed as a weighted sum over all prefix-free input programs $p$ for $U$ that halt:

$$\Omega_U = \sum_{\text{halting } p(U)} 2^{-l(p)} = \sum_{l=1}^{\infty} N(l) \cdot 2^{-l}$$

(1)

where $p$ is a "program" (a bit string made up of 0’s and 1’s), $l(p)$ is its length (the number of bits), and $N(l)$ is the number of prefix-free programs of length $l$ for which $U$ halts. Turing machines and prefix-free bit strings are briefly reviewed in section 2 and in the appendix. For a general introduction to information theory, see [3].

Until now, much of the discussion around $\Omega$ has focused on its first few digits, which are determined by the function $N(l)$ for small program length $l$. As every mathematical hypothesis can be translated into a halting problem (the question whether a given program halts for a given Turing machine), many long-standing mathematical problems could be solved if only one could compute $\Omega$ digit by digit. Unfortunately, $\Omega$ is not computable by any halting program, precisely because knowing $\Omega$ would imply that one could decide mathematical problems that are known to be undecidable in the sense of Gödel's incompleteness theorem [4]. Moreover, even when the first few digits of $\Omega$ are computable for a given universal Turing machine $U$, they are not universal: they depend on the choice of $U$.

In this note, we will therefore not be concerned with the contribution of short programs to $\Omega$, nor will we dwell much on the issues of incompleteness and undecidability. Instead, we will focus on the contribution of very long programs to $\Omega$, i.e., on the behaviour of $N(l) \cdot 2^{-l}$ as $l \to \infty$. More precisely, in a generalization of (1), we consider the statistical mechanical ensemble of bit string histories with partition function

$$Z_U(\beta) = \sum_{\text{halting } p(U)} \exp\{-\beta \cdot l(p)\} \quad \text{with} \quad \beta = \frac{1}{kT}$$

(2)

where $k$ is the Boltzmann constant and $T$ the temperature as usual in statistical mechanics (see, e.g., [5] for a review of statistical mechanics and field theory).
We will study $Z$ as a function of $\beta$ and show that the ”Chaitin point” $\beta = \beta_c \equiv \ln 2$ corresponds to a critical temperature, at which a first-order phase transition occurs. Near this point, the free energy almost jumps: it converges more slowly to its finite critical value than any computable function. We also provide a non-universal Turing machine as a toy model that approximates this behavior of the free energy by a super-logarithmic singularity.

We will refer to such theories of bit string ensembles, in which the action is defined as the length of programs by which a Turing machine computes bit strings, as ”Logical Quantum Field Theories”, reflecting the relationship between mathematical logic and Turing machines on the one hand, and between statistical mechanics and quantum field theory on the other hand. One could also study a simpler variant of such theories, in which the sum runs over all input bit strings, and not only over those that are prefix-free. However, the partition function would then diverge at $\beta = \ln 2$, instead of converging to Chaitin’s $\Omega$. We will therefore focus on the variant with prefix-free programs in this note.

In a follow-up paper, we will discuss Logical Quantum Field Theory as a string theory, in which the string world-sheet is spanned by the bit string and the computation time.

2 Turing Machines and Prefix-free Programs

We follow Chaitin’s definition of a Turing machine, which is reviewed in appendix A. A bit string $y$ is called a prefix of a bit string $x$, if $x$ can be written as a concatenation $x = yz$, with a third bit string $z$. A set of bit strings is called prefix-free, if no bit string is a prefix of another. For the current argument, it is sufficient to think of a Turing machine $T$ as a map (“computation”) from a set $P$ of prefix-free input bit strings $p \in P$ (”programs”), for which the computation halts, to the set $O$ of arbitrary output bit strings of any length:

$$T : p \in P \rightarrow T(p) \in O$$

The output bit strings are written on a ”work tape” that extends infinitely in both directions. The computation manipulates them until it halts. The prefix-free input bit strings are written on a finite read-only ”program tape” (see appendix A for details). The prefix-free input programs $p$ are what we sum over in (2), and whose lengths $l$ play the role of the energy in the Boltzmann factor.
Figure 1: Tree representation of prefix-free bit strings

One may represent a set $X$ of prefix-free bit strings by a tree (fig. 1). The vertices in the $(l+1)$-th line (or $l$-th generation) of the graph represent the $2^l$ binary numbers $b_l$ with $l$ digits. Branching to the left appends a 0, branching to the right appends a 1 at the end of $b_l$ to yield the next generation of $b_{l+1}$. At each vertex, the corresponding number $b_l$ is either added to the set $X_l \subset X$ of prefix-free programs $p_l$ of size $l$ (red dots) or not (black dots). Black dots are prefixes (parents, grand-parents, ...) of red dots and give birth to two children; we assume that each black dot is the prefix of at least one red dot. Red dots have no children. In the figure, white dots represent bit strings that are never born.

Let $n_l$ be the number of red dots (prefix-free programs) of length $l$. Let $m_l$ be the number of black dots (prefixes) of length $l$. Let $w_l = 2^l - n_l - m_l$ be the number of white dots of length $l$. We define the percentages $Q_l = w_l \cdot 2^{-l}$ of white dots and $P_l = n_l \cdot 2^{-l}$ of red dots in the $l$-th generation and get

$$P_l = Q_{l+1} - Q_l \quad \text{with} \quad \lim_{l \to \infty} Q_l = \sum_{l=1}^{\infty} P_l = \sum_{p_l \in X} 2^{-l} = 1,$$

where the last equation states that "Kraft’s inequality is satisfied with equality" (see [3]). As an example of a set of prefix-free programs, consider "Fibonacci coding": a child is a member of $X_l \subset X$ of prefix-free programs $p_l$ of size $l$ (red dots) or not (black dots). In this case, one easily verifies that $P_l$ falls off exponentially as $l \to \infty$.

For a given Turing machine $T$, there are two kinds of red dots: $\tilde{n}_l$ halting programs and $n_l - \tilde{n}_l$ non-halting programs. We denote by $h_l = \tilde{n}_l/n_l$ the fraction of programs in the $l$-th generation that halt. Then the partition function [2] can be written as

$$Z_U(\beta) = \sum_{l=1}^{\infty} P_l \cdot h_l \cdot e^{-\epsilon l} \quad \text{with} \quad \epsilon = \beta - \beta_c, \quad \beta_c = \ln 2. \quad (4)$$
3 Universality of the Singularity

At the critical point $\beta = \beta_c = \ln 2$, our partition function (4) is Chaitin’s $\Omega$ (1):

$$Z_U(\beta_c) = \sum_{l=1}^{\infty} P_l \cdot h_l = \Omega < 1$$

For $\beta < \beta_c$, the partition function diverges, as long as $P_l \cdot h_l$ falls off more slowly than exponentially as $l \to \infty$, which is the case for any universal Turing machine $U$ (see below). How exactly does $Z_U$ approach $\Omega$ as $\beta$ approaches $\beta_c$ from above? Let us first discuss in how far this singularity near $\beta = \beta_c$ is universal, i.e., independent of $U$.

A universal Turing Machine (“UTM”) $U$ is one that can simulate any other Turing machine $T_i$ in the following sense: there is a finite bit string (“translator program”) $c_i$ such that for each program $p$, $U(c_i p) = T_i(p)$. I.e., if $p$ makes $T_i$ compute an output bit string, the concatenation $c_i p$ makes $U$ compute the same output bit string. Let $C_i$ be the finite length of the program $c_i$. Then the partition function $Z_U(\beta)$ of the UTM contains the partition function $Z_i(\beta)$ of $T_i$ as a subset:

$$Z_U(\beta) \geq e^{-\beta C_i} \cdot Z_i(\beta)$$

This applies to all Turing machines $T_i$. Thus, as $\epsilon = \beta - \beta_c \to 0$, the Turing machine $T_i$ with the strongest singularity (i.e., with the largest derivative $Z_i'(\epsilon)$ at $\epsilon \sim 0$) dominates the singularity of the partition function $Z_U(\beta)$ at $\beta = \ln 2$. As this applies to all $U$, we conclude that this singularity is universal, i.e., independent of the choice of the UTM, up to an overall pre-factor $2^{-C_i}$.

Our ensemble (2) includes only programs that halt. This makes it intractable, as it is generally an undecidable question whether a given Turing machine halts for a given program. Thus, the factor $h_l$ in (4), Chaitin’s $\Omega$, and the partition function $Z_U(\beta)$ are actually not computable by any halting program. These issues around un-decidability and non-computability, fascinating as they may be, will not play a major role here. It is clear from (4) that the strongest singularity in $\epsilon$ corresponds to the product $P_l \cdot h_l$ that decays most slowly as $l \to \infty$. Thus, the non-computable factor $h_l$ can only make this singularity weaker. We will therefore first discuss non-universal Turing machines $T_i$ for which all programs halt (i.e. $h_l = 1$), and then return to universal Turing machines in the last section.
As an example of a function $P_l$ that converges more slowly than that from Fibonacci coding, let the $N$ of Fibonacci coding grow with the program length: $N(l) = \text{int}(1 + \lg l)$, where $\lg \equiv \log_2$. In this case, it is not difficult to see that $P_l$ decays like a power of $l$:

$$P_l \propto l^{-\alpha} \quad \text{as} \quad l \to \infty \quad \text{with} \quad \alpha > 1 \quad \Rightarrow \quad 1 - Z(\epsilon) \propto \epsilon^{\alpha-1} \quad \text{as} \quad \epsilon = \beta - \beta_c \to 0$$

Next, we present a machine that yields a much stronger, super-logarithmic singularity.

### 4 The Counting Machine

We now define a Turing machine $T_0$ that we call the "counting machine", corresponding to a particular set of prefix-free programs, that always halts. We will then show that its partition function $[\square]$ has a computable, super-logarithmic singularity that, for our purposes, serves as a good model of the singularity of UTM’s.

Let us first describe the output of the machine $T_0$. Given any infinite input bit string $p$ on the program tape, $T_0$ writes a number $N$ of 1’s in a row on its otherwise blank work tape and then halts. We call $p_N$ the prefix of $p$ consisting only of those bits of $p$ that have been read by the time the machine halts, i.e., the machine halts on the last bit of $p_N$. This defines a set $P$ of prefix-free input programs $p_N \in P$. We will construct $T_0$ such that any number $N \in \mathbb{N}_0$ of 1’s appears as the output bit string of exactly one such $p_N$, namely:

$$\begin{align*}
\text{for } N < 3 : & \quad p_0 = 00, \quad p_1 = 01, \quad p_2 = 10 \quad \text{with length } \quad l_N = 2 \\
\text{for } N = 3 : & \quad p_3 = 110 \quad \text{with length } \quad l_N = 3 \\
\text{for } N > 3 : & \quad p_N = 11n_2...n_kN0 \quad \text{with length } \quad l_N = 6 + n_2 + ... + n_k
\end{align*}$$

(5)

where $n_k$ is the binary length of $N$, $n_{k-1}$ is the binary length of $n_k$, and so on, until a length $n_1 = 3 = 11_2$ is reached. For $N > 2$, $p_N$ begins with "11" and ends with "0". The number of iterations $k$ can be recursively expressed as follows:

$$k(N) = \begin{cases} 
0 \quad \text{if } N < 4 \\
1 + k(1 + \lg N) \quad \text{if } N \geq 4 
\end{cases}$$

(6)

This yields, e.g., $k(4) = k(7) = 1, k(8) = k(127) = 2, k(128) = 3$, and so on.
Next, we describe how $T_0$ reconstructs $N$ from $p_N$. Given an infinite string $p$ on the program tape, $T_0$ proceeds as follows:

1. $T_0$ reads the first two digits $n_1 = p_1p_2$ of $p$. If $n_1 = 00$, $T_0$ leaves the work tape blank and halts; if $n_1 = 01$, $T_0$ writes 1 on the work tape and halts; if $n_1 = 10$, $T_0$ writes 11 on the work tape and halts. If $n_1 = 11$, $T_0$ reads the next digit $p_3$ and defines the new integer $m_1 = 3$.

2. If $p_{m_1} = p_3 = 0$, $T_0$ writes $1^{n_1} = 111$ on the work tape, then halts. If $p_{m_1} = p_3 = 1$, $T_0$ reads the next $n_1 = 3$ digits $p_{m_1+1}, \ldots, p_{m_1+n_1}$ of $p$, i.e., $p_4, p_5, p_6$. $T_0$ defines $m_2 = m_1 + n_1 = 6$ and $n_2 = p_3p_4p_5$ (the concatenation with $p_3$ but without $p_6$).

   i. In the $i$-th step, if $p_{m_{i-1}} = 0$, $T_0$ writes $1^{n_{i-1}}$ on the work tape and halts. If $p_{m_{i-1}} = 1$, $T_0$ reads in the next $n_{i-1}$ digits. It defines $m_i = m_{i-1} + n_{i-1}$ and the concatenation $n_i = p_{m_{i-1}} \ldots p_{m_i-1}$ and moves on to step $(i + 1)$, until $T_0$ halts. If $T_0$ halts in the $i$-th step, then $i$ is related to $k$ of [6] by $k = i - 2$, and $N = n_{i-1}$.

As an example, consider the input bit string 11100110100. Then $n_1 = 112$, so in step 2, $T_0$ defines $m_2 = 6, n_2 = 100_2 = 4$. In step 3, since $p_6 = 1$, $T_0$ sets $m_3 = 10$ and reads in the 4-digit number $n_3 = 1101_2 = 13$. In step 4, since the next digit $p_{10}$ is a 0, $T_0$ writes $N = 13$ digits 1 in a row and halts. Only the first 10 digits 1110011010 of the input bit string constitute an element of $P$. More generally, if the counting machine halts in step $k$, the first $m_{k-1}$ digits of the input string constitute an element of $P$. The first elements are:

$$P = \{00, 01, 10, 110, 111000, 111010, 111100, 111110, 1110010000, \ldots\}$$

One may verify that any number $N$ of 1’s in a row appears as the output bit string of exactly one program $p_N \in P$, as claimed above. It is also clear that $P$ is complete in the sense that it cannot be enlarged by any additional bit string without spoiling its property of being prefix-free. As a result, [3] implies that $Z(\beta_c) = 1$.

The counting machine $T_0$ provides a highly compact specification of large numbers $N$ by prefix-free programs. Of course, other sets of prefix-free programs may give a shorter description of individual large numbers, such as $2^{2^{1024}}$, at the expense of the average large number. Appendix A4 presents a concrete implementation of the counting machine $T_0$.  

7
5 Super-logarithmic Singularity

In this section, we compute how the partition function \((4)\) approaches its critical value as \(\beta\) approaches \(\beta_c\) from above in the case of the counting machine. The counting machine halts for every input program \((h_l = 1)\) and therefore has a computable partition function

\[
\hat{Z}(\beta) = \sum_{\text{all } p} \exp\{-\beta \cdot l(p)\} = \sum_{k=0}^{\infty} \hat{Z}_k(\beta) \quad \text{with} \quad \hat{Z}(\beta_c) = 1, \quad (7)
\]

where \(\hat{Z}_k(\beta)\) is the contribution from programs \(p\) that halt after \(k\) iterations, \(k\) being defined in \((6)\). Using \((5)\), we expand:

\[
\hat{Z}_0(\beta) = 3e^{-2\beta} + e^{-3\beta}, \quad \hat{Z}_1(\beta) = 4e^{-6\beta}
\]

\[
\hat{Z}_2(\beta) = 8e^{-10\beta} + 16e^{-11\beta} + 32e^{-12\beta} + 64e^{-13\beta}
\]

\[
\hat{Z}_k(\beta) = \sum_{n_2, \ldots, n_k, N} e^{-\beta(6+n_2+\ldots+n_k)} 
\sim \sum_{n_2, \ldots, n_k} 2^{-(6+n_2+\ldots+n_k-1)} \cdot \frac{1}{2} e^{-\epsilon n_k} \quad \text{with} \quad \epsilon = \beta - \beta_c. \quad (8)
\]

where \(n_2\) runs from 4 to 7, \(n_{i+1}\) runs from \(2^{n_i-1}\) to \(2^{n_i} - 1\), and \(N\) runs from \(2^{n_k-1}\) to \(2^{n_k} - 1\). In the last line, we have expanded near \(\beta = \beta_c = \ln 2\), and kept only the leading term in \(\epsilon\), noting that \(n_k \gg n_{k-1}\). For a given \(k\), let \(\Lambda_k\) be the largest possible value of \(n_k\):

\[
\Lambda_1 = 3, \quad \Lambda_2 = 7, \quad \Lambda_3 = 127, \quad \Lambda_{k+1} = 2\Lambda_k - 1. \quad (9)
\]

If \(\Lambda_{k-1} \gg 1/\epsilon\), we can approximate \(\hat{Z}_k\) in \((8)\) by 0, since the minimum value of \(n_k\) is \(\Lambda_{k-1} + 1\). On the other hand, if \(\Lambda_k \ll 1/\epsilon\), we can approximate \(\epsilon\) by 0 in \(\hat{Z}_k\). This yields \(\hat{Z}_0 = 7/8, \hat{Z}_1 = 1/16\). Noting that there are always \(2^{n_i}/2\) possible values for \(n_{i+1}\), in the case \(\Lambda_k \ll 1/\epsilon\) we can iteratively perform the sum over \(n_2, \ldots, n_k\) for \(k > 1\) to obtain

\[
\hat{Z}_k = \frac{1}{2} \sum_{n_2, \ldots, n_k} 2^{-6-n_2-\ldots-n_k-1} = \frac{1}{4} \sum_{n_2, \ldots, n_{k-1}} 2^{-6-n_2-\ldots-n_{k-2}} = \ldots = \frac{1}{2^{k+3}}
\]

We now perform the sum \((7)\) over \(k\) and first consider the (rare) case where \(1/\epsilon = \Lambda_K\) for some \(K\). In appendix A5, it is shown that, in this case, \(\hat{Z}_K = 2^{-K-3}, Z_{K+1} = 0\) to high accuracy already for \(K \geq 4\). Thus,

\[
\hat{Z}(\epsilon) = \frac{7}{8} + \frac{1}{16} + \sum_{k=2}^{K} \hat{Z}_k = 1 - 2^{-K-3} \quad \text{with} \quad \frac{1}{\epsilon} = \Lambda_K \quad (10)
\]
The singularity in $\epsilon$ comes from the dependence of $K$ on $\epsilon$. To continue \((10)\) to general $\epsilon$, we use the "super-logarithm" $\text{slog}_2(x)$ with basis 2 in the so-called "linear approximation":

\[
\text{slog}_2(x) = \begin{cases} 
  x - 1 & \text{if } 0 < x \leq 1 \\
  \text{slog}_2(\lg(x)) + 1 & \text{if } x > 1
\end{cases}
\]  

(11)

Its integer values are $\text{slog}_2(1) = 0$, $\text{slog}_2(2) = 1$, $\text{slog}_2(4) = 2$, $\text{slog}_2(2^x) = \text{slog}_2(x) + 1$. Real values of $\text{slog}_2$ are interpolated from its integer part $\lg^-(x) = \text{int}(\text{slog}_2(x))$ by

\[
\text{slog}_2(x) = \lg^-(x) + \lg \ldots \lg x \quad \text{with } 1 + \lg^-(x) \text{ iterations}
\]

We can now express $K(\epsilon)$ in terms of the super-logarithm by noting from (9) that $\text{slog}_2(\Lambda_{k+1}) \to \text{slog}_2(\Lambda_k)+1$ to very high accuracy already for $k > 2$: $\text{slog}_2(\Lambda_1) = 1+\lg \lg 3 \sim 1.66$, $\text{slog}_2(\Lambda_2) = 2 + \lg \lg \lg 7 \sim 2.57$, $\text{slog}_2(\Lambda_k) = k + 0.57$

\[
\Rightarrow \quad K(\epsilon) \sim \text{slog}_2(1/\epsilon) - \phi \quad \text{with } \phi = 0.57\ldots
\]

\[
\hat{Z}(\epsilon) \sim 1 - \lambda \cdot 2^{-\text{slog}_2(1/\epsilon)} = 1 - \lambda \cdot 2^{-\lg^-(1/\epsilon)} \cdot \{\lg \ldots \lg \{1/\epsilon\}\}^{-1},
\]

(12)

where $\lambda = 2^\phi - 3 \sim 0.186$, and there are $\lg^-(1/\epsilon)$ iterations of the logarithm in the last line.

This continues \((10)\) to any $\epsilon$. Although the continuation \((11)\) of the super-logarithm to real values, and thus the continuation \((12)\) of $\hat{Z}(\epsilon)$, is not unique, different continuations differ only by sub-leading orders in $\epsilon$. Thus, \((12)\) is the leading singularity of the partition function $\hat{Z}(\epsilon)$ at the critical point. This partition function is plotted in fig. 5. It converges extremely slowly to 1 as $\epsilon \to 0$, and is continuous but "almost" discontinuous.

![Figure 2: $\hat{Z}(\epsilon)$ as a function of $1/\epsilon$ (left) and $\beta$ (right)](image-url)
6 Critical Behavior

Armed with the results of section 5, we would now like to examine the phase transition near the critical point $\beta \sim \beta_c = \ln 2$ for the counting machine. The free energy $F$ and average program length $<l>$ are:

$$\hat{Z} = e^{-\beta F} = \sum_p e^{-\beta l(p)} \Rightarrow F = -\frac{1}{\beta} \ln \hat{Z}, \quad <l> = -\partial_\beta \ln \hat{Z}$$

The program length is the energy in our case. The heat capacity is (using $T \partial_T = -\beta \partial_\beta$):

$$C(T) = -T \frac{\partial^2 F}{\partial T^2} \sim -\partial_\beta \ln <l> + \text{higher orders in } \epsilon$$

Generally, in a 0-th order phase transition the free energy $F(T)$ is discontinuous at a critical point $T = T_c$. In a first order transition, $F(T)$ is continuous but $\partial_T F(T)$ is discontinuous, the gap being the latent heat. In a second order transition, $\partial_T F(T)$ is also continuous, but some higher-order derivative of $F(T)$ is discontinuous [5]. In our case,

$$\hat{Z}(\epsilon) = 1 - \lambda \cdot 2^{-\log_2(1/\epsilon)} = 1 - \lambda \cdot 2^{-\log_2(1/\epsilon)} \cdot \{\log \log (1/\epsilon)\}^{-1}$$

where $\lambda \sim 0.186$, $\log_2$ is the integer part of the super-logarithm and we have $\log_2(1/\epsilon)$ iterations of the logarithm. Thus, in the limit $\epsilon \to 0$, we have

$$F(\epsilon) \propto -\lambda \cdot 2^{-\log_2(1/\epsilon)} \cdot \{\log \log (1/\epsilon)\}^{-1} \quad (13)$$

$$<l> \propto [\epsilon \cdot \frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon} \cdot \log \log \frac{1}{\epsilon} \cdot \cdots \cdot (\log \log \frac{1}{\epsilon})^{-2}]^{-1} \quad (14)$$

$F(\epsilon)$ is finite at the critical point. It is continuous, but almost discontinuous. Thus, the phase transition is first-order, but almost zeroth order. We also see that the latent heat is infinite, and that the average program size $<l>$ diverges at the critical point. The average size $N$ of the output strings also diverges, as $l$ is of the order $\log N$.

To put things into perspective, the size of the observable universe in Planck lengths is about $2^{200}$. The super-logarithm of this number is about 4.6, so for $\epsilon$ of order $2^{-200}$, $Z$ is still about 0.76% away from 1. To get a super-logarithm of 5, we need a universe of size $2^{65.536}$ Planck lengths. Even then, $Z$ is still 0.58% away from 1. In this sense, the super-logarithmic singularity is indistinguishable from a discontinuity of the partition function at least for all bit string ensembles that can be hosted by our universe.
7 Singularity for Universal Turing Machines

In the previous section, we have discussed the singularity of the partition function (4) near $\epsilon = 0$ for the non-universal counting machine. How does it compare with the singularity for a universal Turing machine?

Since it is generally an undecidable question whether a given Turing machine halts for a given program, for a UTM the function $h_l$ in (4) and Chaitin’s $\Omega$ are not computable by any halting program. Neither is the singularity of $Z(\epsilon)$ at the critical point computable. In fact, $Z(\epsilon)$ converges towards $\Omega$ more slowly than any computable function.

To see this, let us slightly modify the last step of the counting machine of section 4: if, in the $i$-th step, $p_{m_{i-1}} = 0$, the modified $T_0$ switches into a new mode: instead of writing $1^{n_{i-1}}$ on the work tape, it reads the next $\Sigma(n_{i-1})$ digits of the program $p$ from the program tape, where $\Sigma(n)$ is the busy-beaver function. The modified machine $\tilde{T}_0$ writes those digits on the work tape and then halts. Formula (8) thus gets replaced by

$$\tilde{Z}_k(\beta) = \sum_{n_2, \ldots, n_k} \Sigma(n_k) \sum_{N=0}^{2\Sigma(n_k)} e^{-\beta(6+n_2+\ldots+n_k+\Sigma(n_k))} \sim \sum_{n_2, \ldots, n_k} 2^{-(6+n_2+\ldots+n_k)} \frac{1}{2} e^{-\epsilon \Sigma(n_k)}$$

$\Sigma(n)$ is known to diverge faster than any computable function as $n \to \infty$. This implies that $\tilde{Z}(\epsilon)$ converges more slowly than any computable function to its critical value $1$ for the modified machine $\tilde{T}_0$. Now, any UTM $U$ simulates the modified machine $\tilde{T}_0$, if it is fed with all possible input programs. This implies that, for any UTM, $Z_U(\epsilon)$ converges more slowly than any computable function to its critical value $\Omega$.

The conclusion for UTM’s is thus similar as for the counting machine: at the critical point, the phase transition is first-order but almost zeroth-order, and the average program size diverges. The average size of the output bit strings also diverges, as $U$ simulates $\tilde{T}_0$, among other machines. Strictly speaking, $Z_U(\epsilon)$ for a UTM is continuous at $\epsilon = 0$, but in practise, the behavior is indistinguishable from a discontinuity. As we have seen, at least for all bit string ensembles within our universe, the super-logarithmic singularity of the counting machine already has such an effective discontinuity. In this respect, the counting machine provides a simplified toy model for our ensemble.
8 Outlook

We have shown that our ensemble of bit string histories has a first-order phase transition at the Chaitin point. This phase transition is almost zeroth-order in the sense that the free energy is continuous but "almost" discontinuous: it converges to its critical value, namely Chaitin’s Ω, more slowly than any computable function. At this critical point, the average size of the input programs and the average size of the output bit strings both diverge.

There are a number of immediate questions. In phase transitions in physical systems there is usually an order parameter, which transforms as a representation of some symmetry group that is spontaneously broken below the critical temperature. What - if any - are the order parameter and the symmetry group in this case?

One is also tempted to regard the two-dimensional space spanned by the bitstring and the computation time as a string world sheet. It is then natural to ask what kind of string theory describes our bit string ensemble at the critical point. We conjecture that it has a continuum formulation in terms of two-dimensional quantum field theory, and that it is thereby related to superstring theory [6].

Usually, quantum field theories describe statistical mechanical ensembles at second-order, not at first-order phase transitions. However, there are precedents of systems that exhibit zeroth- or first-order phase transitions as a function of one parameter, and second-order phase transitions as a function of other parameters, and that do have continuum formulations in terms of two-dimensional field theory. Examples include the Ising model on a random surface [7] or the Kosterlitz-Thouless transition on a random surface [8]. In these cases, the first parameter is the two-dimensional cosmological constant, which we expect to be the analog of β of this note. We will address these issues in future work.

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Appendix A: Review of Turing Machines

The appendix is organized as follows. A1 presents a Turing machine that contains only a work tape and no program tape. An example is given in A2. In A3, Chaitin’s definition of a Turing machine is recalled. Using the example of A2 as a building block, we realize the counting machine of section 4 in A4. A5 contains a supplementary argument to section 5.

A1. A simple Turing Machine

Our first example of a Turing machine contains a "work tape" that extends infinitely in both directions. It consists of cells that are blank, except for a finite, contingent bit string of 0’s and 1’s (the "input string"). A blank cannot be written between 0’s or 1’s, so it is not equivalent to a third letter in addition to 0 and 1. Rather, blank areas mark the beginning and end of the string on the work tape. On the first cell of the input string sits a head, which can read, write, and move in both directions. The head can be in one of several states, labelled by 1, 2, 3, ... , H. At each step, the machine operates as follows:

1. it reads the bit on the work tape on which the head sits (0, 1 or a blank)
2. depending on that bit and on its internal state, it writes a 0, 1 or a blank in that cell on the work tape. It may only write a blank if the cell has a blank neighbour, to ensure that the binary string remains contingent
3. it moves the head either one cell to the left or one cell to the right
4. it may or may not change its internal state
5. If and when it reaches the state "H", it halts

A2. An example

As an example, consider a Turing machine with 5 states 1, 2, 3, 4, H. The table in fig. (left) defines how this particular machine writes 0, 1 or 2 (2 denoting a blank), then moves left (−1) or right (+1), and then switches to a new state, depending on the input bit it reads (left column) and the state it is in (top row).

First, let the input string be "01". Fig (center) graph shows a two-dimensional graph of the evolution of the bit string, with a new row added for each time step. The machine
Figure 3: A simple Turing machine

- starts in state 1 on the first bit of the work tape, which is 0.

- writes a 0, remains in state 1 and moves right to the next bit, whose value is 1.

- writes a 1, remains in state 1, and moves right to the next bit, which is blank.

- switches to state 2, and moves back left to the previous bit, whose value is 1.

- overwrites it with 0, switches to state 3, and moves left, and so on.

At some point, the machine lands on a blank bit in state 2, moves one bit to the right and halts. The output string are two 1’s in a row. By modifying the input string, other output strings can be produced. The reader may verify that if the input string is the number $b$ in binary code, then the output string of this particular Turing machine always consists of $b$ 1’s in a row, with the head halting on its first bit. The right graph shows this computation in condensed form for $b = 1001_2 = 8_{10}$.

As an example of a universal Turing Machine (UTM) that can simulate all other Turing machines, consider our brain: given the above table for any Turing machine, we can read it and use it to simulate the machine as you have just done if you have followed the exercise. Essentially, the table becomes part of the input, rather than being hard-coded into the Turing machine. A UTM is arbitrarily flexible and can quickly compute strings with one Turing machine that take a long time or are impossible to compute with another machine.

There are many alternative, but equivalent definitions of Turing machines. E.g., one can introduce other symbols in addition to 0 and 1, or more states, or one can work with several parallel work tapes instead of just one.
A3. Chaitin’s Machine

In Chaitin’s definition, there is a read-only "program tape" of finite length, in addition to the work tape. The program tape begins with a blank cell followed by a finite bit string of 0’s and 1’s, the "program". On the program tape sits another head, the "program head". Initially, it sits on the blank cell. At each step, the machine performs the following operations in addition to steps 1-5 of subsection A1:

- initial step: it reads the bit on the program tape on which the head sits
- last step: it moves the program head either one cell to the left or leaves it where it is

The machine either halts or runs forever without reading any more program bits. As a result, the set of input programs, from the first to the last bit that has been read by the machine, is prefix-free.

A4. The Counting Machine

As an example within Chaitin’s framework, we present an implementation of the counting machine of section 4. We begin with the Turing machine of appendix (A1), and add a finite read-only program tape, on which the programs of section 4 are written. We start with a work tape that is initially blank.

We first add three additional states 6, 7, 8, whose role is to read the first two bits on the program tape and get the machine started (steps 1 and 2 of section 4). The operations in states 6, 7, 8 depend only on the bit "pin" on which the program head sits, and not on the bit "in" on which the work head sits. They are defined in the table in fig. 4. The machine is initially in state 8 (in states 6 and 7, "pin" is then never 2).

Next, we slightly modify state 2 of appendix A1 as follows: if the machine is in state 2, and the head on the work tape sits on a blank, then it switches to state \( H \) only if the head of the program tape sits on a 0. Otherwise, it moves to a new state 5.

The operations of the new state 5 are also defined in the table below. Its role is to write a new portion from the program tape onto the work tape, thereby over-writing the contingent sequence of 1’s. Its operations depend both on the current work tape bit "in"
and on the current program bit "pin" (the machine is never in state 5 when the program
head is on a blank or when the work head is on a 0).

| State | 5 in=0 | 5 in=1 | 5 in=B | 6 | 7 | 8 | Operation |
|-------|--------|--------|--------|---|---|---|-----------|
| pin   | 0      | 0      | 2      | 2 | 1 | 2 | write     |
| pin   | 0      | 5      | 2      | 8 | H | H | new state |
| pin   | 0      | 1      | -1     | 1 | -1| -1| move (work)|
| pin   | 0      | 1      | 0      | 1 | 0 | 0 | move (program) |
| pin   | 1      | 1      | 2      | 1 | 1 | 1 | write     |
| pin   | 1      | 5      | 2      | 7 | 1 | H | new state |
| pin   | 1      | 1      | -1     | 1 | -1| -1| move (work) |
| pin   | 1      | 1      | 0      | 1 | 1 | 0 | move (program) |
| pin   | 2      | 2      | 2      | 2 |    |    | write     |
| pin   | 2      | 2      | 6      | 6 |    |    | new state |
| pin   | 2      | 2      | 1      | 1 |    |    | move (work) |
| pin   | 2      | 2      | 1      | 1 |    |    | move (program) |

Figure 4: New states of the base machine

It is straightforward to verify that this machine indeed represents the counting machine
of section 4.

A5. A Supplementary Argument

In section 5, we want to evaluate the $K$-th part of the partition function

$$
\hat{Z}_K(\epsilon) = \sum_{n_2,\ldots,n_K} 2^{-(6+n_2+\ldots+n_{K-1})} \cdot \frac{1}{2} e^{-\epsilon n_K} \quad \text{in the case } \frac{1}{\epsilon} = \Lambda_K
$$

where $K \geq 4$, $n_2$ runs from 4 to 7, $n_{i+1}$ runs from $2^{n_i-1}$ to $2^{n_i} - 1$, and $\Lambda_K$ is the largest possible value of $n_K$. Specifically, $\Lambda_3 = 127$, $\Lambda_4 = 2^{127} - 1$, and therefore $\Lambda_{K-1} \sim \lg(\Lambda_K) = \lg(1/\epsilon)$ to high accuracy for $K \geq 4$.

Defining $M = 2^{n_{K-1}}$ and performing the sum over $n_K$ yields

$$
\hat{Z}_K(\epsilon) = \sum_{n_1,n_2,\ldots,n_{K-1}} 2^{-(6+n_2+\ldots+n_{K-2})} \cdot A(n_{K-1}, \epsilon)
$$

$$
A(n_{K-1}, \epsilon) = \frac{1}{2M} \sum_{n_K=M/2}^{M-1} e^{-\epsilon n_K} = \frac{1}{2M\epsilon} (e^{-M\epsilon/2} - e^{-M\epsilon})
$$
$A(x, \epsilon)$ is plotted in fig. 5. It is a monotonously decaying function with

$$A(x, \epsilon) \rightarrow \begin{cases} 
\frac{1}{4} & \text{for } x \ll \log \frac{1}{\epsilon} \sim \Lambda_{K-1} \\
0 & \text{for } x \gg \log \frac{1}{\epsilon} \sim \Lambda_{K-1}
\end{cases}$$  \hspace{1cm} (18)$$

$n_{K-1}$ runs from $2^{n_{K-2} - 1}$ to $2^{n_{K-2}} - 1$. Only for the maximal value of $n_{K-2}$ are there a few values of $n_{K-1}$ near $\Lambda_{K-1}$, for which $A$ differs significantly from $1/4$. Even in this case, the contribution of these differences is

- small (of order 1%) for $K = 4$: for the highest value $n_2 = 7$, $n_3$ runs from 64 to 127. Only the last few of these $n_3$ contribute significantly to the difference
- practically zero for $K \geq 5$: e.g., for $K = 5$ and the highest value $n_3 = 127$, $n_4$ runs from $2^{126}$ to $2^{127} - 1$. Only a tiny portion of these $n_4$ contribute to the difference

As long as $K \geq 4$, we can thus approximate $A$ by $1/4$ for $x \leq \Lambda_{K-1}$ to obtain $\hat{Z}_K = 2^{-K-3}$ as claimed in section 5. An analogous argument, not repeated here, shows that we can approximate $A$ by 0 for $x > \Lambda_{K-1}$ to obtain $\hat{Z}_{K+1} = 0$, as long as $K \geq 4$.

Figure 5: The function $A(x)$