Hidden regular variation, copula models, and the limit behavior of conditional excess risk measures

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Abstract

Risk measures like Marginal Expected Shortfall and Marginal Mean Excess quantify conditional risk and in particular, aid in the understanding of systemic risk. In many such scenarios, models exhibiting heavy tails in the margins and asymptotic tail independence in the joint behavior play a fundamental role. The notion of hidden regular variation has the advantage that it models both properties: asymptotic tail independence as well as heavy tails. An alternative approach to addressing these features is via copulas. First, we elicit connections between hidden regular variation and the behavior of tail copula parameters extending previous works in this area. Then we study the asymptotic behavior of the aforementioned conditional excess risk measures; first under hidden regular variation and then under restrictions on the tail copula parameters, not necessarily assuming hidden regular variation. We provide a broad variety of examples of models admitting heavy tails and asymptotic tail independence along with hidden regular variation and with the appropriate limit behavior for the risk measures of interest.

Keywords: asymptotic tail independence, copula models, expected shortfall, heavy-tail, hidden regular variation, mean excess, multivariate regular variation, systemic risk

1. Introduction

In practice, one often encounters risk factors which are heavy tailed in nature; which means values further away from the mean have a relatively high probability of occurring than for example for exponentially tailed distributions like normal or exponential; see \cite{2, 14, 18, 39, 48} for details. The joint behavior of such multi-dimensional heavy-tailed random variables are often studied using the notion of multivariate regular variation (MRV); see \cite{3, 44}. For our paper we resort to this approach.

In certain scenarios, such multivariate models exhibit a phenomenon called asymptotic tail independence; see Poon et al. \cite{41} for empirical evidence of asymptotic tail independence and heavy tails in five major stock market indices. In this paper we restrict to study non-negative variables and hence, our interest is in the upper tail dependence between different variables. To that end we interchangeably use the names asymptotic tail independence and asymptotic upper tail independence for the same notion. For any random variable $Z$, let $F_Z$ denote its probability distribution function. For a bivariate random vector $Z = (Z_1, Z_2)$ we can define asymptotic tail independence (in the upper tails) as

$$\lim_{p \to 0} \Pr(Z_1 > F_{Z_1}^{-}(1-p)|Z_2 > F_{Z_2}^{-}(1-p)) = 0 \quad (1)$$

where $F_{Z_i}^{-}(1-p) = \inf\{x \in \mathbb{R} : F_{Z_i}(x) \geq 1-p\}$ is the generalized inverse of $F_Z$. Hence, the presence of asymptotic tail independence among $Z_1$ and $Z_2$ implies that it is highly unlikely for the two random variables to take extreme values together. This phenomenon aptly noted by Sibuya \cite{47} more than half a century back, especially in the context of the very popular and useful bivariate normal distribution has been a source of intrigue and further research by many.

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A key notion in defining multivariate regularly varying distributions under the assumption of asymptotic tail independence is hidden regular variation; see \cite{17, 34, 43}. The family of hidden regular varying distributions is a semi-parametric subfamily of the full-family of distributions possessing multivariate regular variation. In the past few years, researchers have explored the connection between hidden regular variation and copula models via tail dependence functions \cite{23, 29, 34, 53}, and weak tail dependence functions \cite{49}. In this paper we explore this further and provide insight into the connections between hidden regular variation and copula models. Furthermore, we construct multivariate models exhibiting hidden regular variation as used in a systemic risk context using both additive models as well as copula models.

In the univariate context a popular risk measure is the Expected Shortfall (ES), also known as Conditional Tail Expectation (CTE) and Tail-Value-at-Risk (TVaR). It is widely used in practice and also incorporated in the regulatory frameworks of Basel III for banks and Solvency II for insurances. In systems with more than one variable, it is often of interest to judge the risk behavior of one component given a high risk or stress in the others. Hence, we resort to computing conditional excess risk measures which include the Marginal Expected Shortfall (MES) and the Marginal Mean Excess (MME). The MES is well-known in many contexts, and has been especially proposed for measuring systemic risk \cite{1, 6, 52}.

In order to define the above risk measures, recall that for a random variable $Z$ and $p \in (0,1)$ the Value-at-Risk (VaR) at level $p$ is the quantile function

$$\text{VaR}_p(Z) = \inf\{x \in \mathbb{R} : \Pr(Z > x) \leq 1 - p\} = \inf\{x \in \mathbb{R} : \Pr(Z \leq x) \geq p\}.$$ 

Note that smaller values of $p$ lead to higher values of VaR. Now suppose that $Z = (Z_1, Z_2) \in [0, \infty)^2$ denotes the risk exposure of a financial institution, and $Z_1$ and $Z_2$ are the marginal risks of two risk factors. We intend to study the expected behavior of one risk, given that the other risk is high; and the following two measures achieve this. For $\mathbb{E}[|Z_1| < \infty$ the MES at level $p \in (0,1)$ is defined as

$$\text{MES}(p) = \mathbb{E}[|Z_1| Z_2 > \text{VaR}_{1-p}(Z_2)],$$  \hspace{1cm} (2)

and the MME at level $p \in (0,1)$ is defined as

$$\text{MME}(p) = \mathbb{E}[(Z_1 - \text{VaR}_{1-p}(Z_2))_+ Z_2 > \text{VaR}_{1-p}(Z_2)].$$  \hspace{1cm} (3)

The measure MES represents the expected shortfall of one risk given that the other risk is higher than its Value-at-Risk at level $1 - p$, whereas the measure MME represents the expected excess of risk $Z_1$ over the Value-at-Risk of $Z_2$ at level $1 - p$ given that the value of $Z_2$ is already greater than the same Value-at-Risk; see Das and Fasen-Hartmann \cite{15} for details where the measure MME is also defined. Note that the measure MES$(p)$ is equal to the expected shortfall (ES) if $Z_1 \equiv Z_2$. In this context we also consider a few other extensions of ES:

$$\text{MES}^*(p) = \mathbb{E}[|Z_1| Z_1 + Z_2 > \text{VaR}_{1-p}(Z_1 + Z_2)],$$

$$\text{MES}^\max(p) = \mathbb{E}[|Z_1| \max(Z_1, Z_2) > \text{VaR}_{1-p}(\max(Z_1, Z_2))], \hspace{1cm} (4)$$

see Cai and Li \cite{7}, Cousin and Di Bernardino \cite{13}. The idea is that $Z_1 + Z_2$ is the aggregate risk of the institution, and $\min(Z_1, Z_2)$ and $\max(Z_1, Z_2)$ are the extremal risks. The risk measure MES$^*$ is associated with the Euler allocation rule; see McNeil et al. \cite{39}, Section 6.3. Further interpretations of these risk measures in finance and insurance are elaborated in Cai and Li \cite{7}.

The asymptotic tail behavior of MES was discussed under the assumption of regular variation and asymptotic tail dependence in Cai et al. \cite{3}, Hua and Joe \cite{22}, and under asymptotic tail independence in Cai and Musta \cite{8}, Das and Fasen-Hartmann \cite{15}. Such risk metrics have also been studied in a time-series context in Kulik and Soulier \cite{32}, and under a copula-framework in Hua and Joe \cite{24}. In the asymptotic tail-dependent case it has been observed that

$$\text{MES}(p) \sim \text{MME}(p) \sim \text{VaR}_{1-p}(Z_1) \quad {\text{as}} \quad p \downarrow 0,$$

independent of the structure of the dependence. Interestingly, in the asymptotically tail-independent case these risk measures may have different rates of convergence or even converge to a constant, e.g., for independent random variables $Z_1, Z_2$ we have $\text{MES}(p) = \mathbb{E}[Z_1]$. The asymptotic behavior of MME and MES for a hidden regularly varying
random vector $Z = (Z_1, Z_2)$ has been investigated in Das and Fasen-Hartmann [15], furthermore, consistent estimators for these risk measures based on methods from extreme value theory have been proposed; for asymptotic normality of MES see Cai and Musta [8]. On the other hand, the asymptotic behavior of the risk measures as defined in [4] are not particularly well-studied, especially in the context of heavy-tailed asymptotically tail independent risks. Explicit formulas for MES are given in Chiragiev and Landsman [12] for multivariate Pareto distributions, in Cai and Li [7] for multivariate phase-type distributions, in Bargès et al. [4] for light tailed risks with Farlie-Gumbel-Morgenstern copula and in Landsman and Valdez [33] for elliptical distributions.

In this paper we extend the work of Das and Fasen-Hartmann [15] for the asymptotic tail behavior of MME and MES and set it in a more general framework. On the one hand, we derive the limit behavior of MES and MME for a variety of hidden regularly varying models using the results from [15]. Asymptotic limits are also obtained for the risk measures given in [4] pursuing a similar approach. On the other hand, we formulate the asymptotic behavior of these conditional risk measures under very general assumptions on the copula tail parameters showing that the limit behavior for $p \downarrow 0$ of MES($p$) and MME($p$) are similar to those in the case of hidden regular variation. In particular, we show that the asymptotic behavior of MES($p$) depends only on the tail of $Z_1$ and the tail copula; neither the presence of hidden regular variation nor the tail of $Z_2$ have any influence on the limit. We compare the conclusions for MES($p$) with those of Cai and Musta [8]. Finally, we provide several examples for models satisfying our assumptions.

The paper is organized as follows. In Section 2 we give a brief introduction into copulas and survival copulas along with notions of multivariate regular variation and hidden regular variation. Then, in Section 3 we construct models exhibiting hidden regular variation on the one hand, by additive models as used in the systemic risk context and on the other hand, by copula models. The asymptotic behavior of MME, MES and the measures in [4] under the models of Section 3 are content of Section 4.1. Finally, in Section 4.2 we develop sufficient conditions on the copula tail parameters to obtain the asymptotic behavior for MME($p$) and MES($p$) as in Section 4.1 without assuming hidden regular variation. Conclusions are drawn in Section 5 along with ideas for future directions of research in this domain.

2. Preliminaries

In this section we discuss necessary tools and definitions for regular variation and copula theory which are used in the subsequent sections. Details on regular variation defined using $M$-convergence are available in [17, 26, 37] and details on copulas and survival copulas can be found in [40]. Unless otherwise stated all random variables take non-negative values and we discuss copulas and regular variation in two-dimensions. Moreover, for vectors $x = (x_1, x_2) \in \mathbb{R}^2$, we denote by $\|x\|$ any suitable norm in $\mathbb{R}^2$.

2.1. Copulas and survival copulas

Copula theory is popularly used to separate out the marginal behavior of random variables from their dependence structure. In two dimensions, a copula is a distribution function on $[0, 1]^2$ with uniformly distributed margins. Using Sklar’s theorem [40], we know that for every bivariate distribution function $F$ with marginal distribution functions $F_i$ ($i = 1, 2$), there exists a copula $C$ such that

$$F(x, y) = C(F_1(x), F_2(y)) \quad \text{for } (x, y) \in \mathbb{R}^2. \quad (5)$$

If $F_1, F_2$ are continuous then $C$ is uniquely defined by

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)) \quad \text{for } (u, v) \in [0, 1]^2.$$

Denoting the survival or tail distribution of $F_i$ by $\bar{F}_i(x) := 1 - F_i(x)$, a version of (5) applies also to the joint survival function $\bar{F}(x, y) := \text{Pr}(Z_1 > x, Z_2 > y) = \bar{F}_1(x) + \bar{F}_2(y) - (1 - F(x, y))$ of the bivariate random vector $Z = (Z_1, Z_2)$ with distribution function $F$ and margins $F_1, F_2$. In this case, there exists again a copula $\bar{C}$, the survival copula, such that

$$\bar{F}(x, y) = \bar{C}(\bar{F}_1(x), \bar{F}_2(y)) \quad \text{for } (x, y) \in \mathbb{R}^2.$$
Moreover, in the bivariate case \( C \) and \( \hat{C} \) are related by
\[
\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v) \quad \text{for} \quad (u, v) \in [0, 1]^2.
\]
Since we are interested in the dependence (as well as independence) in the upper tails, the behavior of the survival copula \( \hat{C}(u, v) \) for \( u, v \) close to 1 will be of significance in this paper. The relationship between the survival copula and hidden regular variation is discussed further in Section 3.2.

2.2. Regular variation and hidden regular variation

Recall that a measurable function \( f : (0, \infty) \to (0, \infty) \) is regularly varying at \( \infty \) with index \( \rho \in \mathbb{R} \) if 
\[
\lim_{x \to \infty} f(tx)/f(t) = x^\rho \quad \text{for any} \quad x > 0 \quad \text{and we write} \quad f \in \mathcal{RV}_\rho; \quad \text{in contrast, we say} \quad f \quad \text{is regularly varying at} \quad 0 \quad \text{with index} \quad \rho \quad \text{if} \quad \lim_{x \to 0} f(tx)/f(t) = x^\rho \quad \text{for any} \quad x > 0.
\]
In this paper, unless otherwise specified, regular variation means regular variation at infinity. A random variable \( Z \) with distribution function \( F_Z \) has a regularly varying tail if 
\[
\rho_Z = 1 - F_Z \in \mathcal{RV}_\rho \quad \text{for some} \quad \rho \geq 0.
\]
We often write \( Z \in \mathcal{RV}_\rho \) by abuse of notation. We define multivariate regular variation using \( \hat{M} \)-convergence; see Lindskog et al. [37]. All notions are restricted to \((0, \infty)^2 \) and its' subspaces. Suppose \( C_0 \subset C \subset [0, \infty)^2 \) where \( C_0 \) and \( C \) are closed cones containing \((0, 0)\) \( \in \mathbb{R}^2 \). Denote by \( \mathcal{M}(C \setminus C_0) \) the class of Borel measures on \( C \setminus C_0 \) which are finite on subsets bounded away from \( C_0 \). For functions \( f : \{0, \infty\}^2 \to \mathbb{R} \), denote by \( \mu(f) := \int f \, d\mu \). Then \( \mu_n \overset{M}{\to} \mu \) in \( \mathcal{M}(C \setminus C_0) \) if \( \mu_n(f) \to \mu(f) \) for all continuous and bounded functions \( f : C \setminus C_0 \to \mathbb{R} \) whose supports are bounded away from \( C_0 \).

**Definition 2.1 (Multivariate regular variation)** A random vector \( Z = (Z_1, Z_2) \in C \) is (multivariate) regularly varying on \( C \setminus C_0 \), if there exist a function \( b(t) \uparrow \infty \) and a non-zero measure \( \nu(\cdot) \in \mathcal{M}(C \setminus C_0) \) such that as \( t \to \infty \),
\[
\nu(\cdot) = t \Pr(Z/b(t) \in \cdot) \overset{M}{\to} \nu(\cdot) \quad \text{in} \quad \mathcal{M}(C \setminus C_0).
\]

The limit measure has the homogeneity property: \( \nu(cA) = c^{-\alpha} \nu(A) \) for some \( \alpha > 0 \). We write \( Z \in \mathcal{MRV}(\alpha, b, \nu, C\setminus C_0) \) and sometimes write MRV for multivariate regular variation.

Classically, MRV is defined on the space \( \mathbb{E} = [0, \infty)^2 \setminus \{(0, 0)\} = C \setminus C_0 \) where \( C = [0, \infty)^2 \) and \( C_0 = \{(0, 0)\} \). Sometimes it is possible and perhaps necessary to define further regular variation on subspaces of \( \mathbb{E} \), since the limit measure \( \nu \) as obtained in (5) concentrates on a proper subspace of \( \mathbb{E} \). The most likely way this happens is through asymptotic tail independence of a random vector as defined in (1). The property can be nicely described by using the survival copula function (see McNeil et al. [39]). Note that, we say that \( Z_1 \) and \( Z_2 \) are tail-equivalent if \( \lim_{t \to \infty} \Pr(Z_1 > t)/\Pr(Z_2 > t) \) exists in \((0, \infty)\).

**Lemma 2.2.** Suppose \( Z = (Z_1, Z_2) \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \) having continuous marginals and survival copula \( \hat{C} \). Consider the following statements.

(a) \( Z_1, Z_2 \) are asymptotically tail independent.

(b) \( \lim_{p \to 0} \hat{C}(p, p)/p = 0 \).

(c) \( \nu((0, \infty) \times (0, \infty)) = 0 \).

Then (a) \( \iff \) (b), \( (b) \implies (c) \), and if we assume that \( Z_1 \) and \( Z_2 \) are tail-equivalent then (c) \( \implies \) (a).

The lemma is easy to verify; more details can be found in Reiss [42, Chapter 7] and Resnick [45, Proposition 5.27]. Independent random vectors are trivially asymptotically tail independent. Note that fact (c) does not necessarily imply (a); this can be verified using the following counterexample: let \( Z \in \mathcal{RV}_\rho \) for \( \rho > 0 \), then the random vector \( (Z, Z^2) \) is multivariate regularly varying with limit measure \( \nu \) with \( \nu((0, \infty) \times (0, \infty)) = 0 \); but of course \( (Z, Z^2) \) is asymptotically tail dependent.

Consequently, if \( (Z_1, Z_2) \) are MRV, asymptotically tail independent and the margins are tail-equivalent we would approximate \( \Pr(Z_2 > x | Z_1 > x) \approx 0 \) for large thresholds \( x \) and conclude that risk contagion between \( Z_1 \) and \( Z_2 \) is absent. This conclusion may be naive and hence, the concept of hidden regular variation on \( E_0 = [0, \infty)^2 \setminus \{(0) \times [0, \infty) \cup [0, \infty) \times (0)\} \) was introduced in Resnick [43]. Note that we do not assume that the marginal tails of \( Z \) are necessarily tail-equivalent in order to define hidden regular variation, which is usually done in [43].
Definition 2.3 (Hidden regular variation) A regularly varying random vector $Z$ on $\mathbb{E}$ possesses hidden regular variation on $E_0 = (0, \infty)^2$ with index $\alpha_0(\geq \alpha > 0)$ if there exist scaling functions $b(t) \in RV_{1/\alpha}$ and $b_0(t) \in RV_{1/\alpha_0}$ with $b(t)/b_0(t) \to \infty$ and limit measures $\nu, \nu_0$ such that

$$Z \in MRV(\alpha, b, \nu, E) \cap MRV(\alpha_0, b_0, \nu_0, E_0).$$

We write $Z \in HRV(\alpha_0, b_0, \nu_0)$ and sometimes write HRV for hidden regular variation.

For example, say $Z_1, Z_2$ are iid random variables with distribution function $F(x) = 1 - x^{-1}, x > 1$. Here $Z = (Z_1, Z_2)$ possesses MRV on $E$, asymptotic tail independence and HRV on $E_0$. Specifically,

$$Z \in MRV(\alpha = 1, b(t) = t, \nu, E) \cap MRV(\alpha_0 = 2, b_0(t) = t, \nu_0, E_0)$$

where for $x > 0, y > 0$,

$$\nu([[0, 0]], (x, y]) = \frac{1}{x} + \frac{1}{y} \quad \text{and} \quad \nu_0([x, \infty) \times [y, \infty)) = \frac{1}{xy}.$$

A combination of Maulik and Resnick [38], Resnick [43] and Das and Fasen-Hartmann [15, Lemma 1] is the following.

Lemma 2.4. Let $Z \in MRV(\alpha, b, \nu, E) \cap MRV(\alpha_0, b_0, \nu_0, E_0)$. Then

$$Z \in MRV(\alpha, b, \nu, E) \cap HRV(\alpha_0, b_0, \nu_0, E_0)$$

iff $Z$ is asymptotically tail independent.

3. Hidden regular variation in additive models and copula models

In this section we investigate hidden regular variation properties of models that are generated using different methods; on one hand, we investigate additive models and on the other hand, we investigate copula models. The models discussed in this section are then used in Section 4 to compute the asymptotic limits of the conditional excess measures MME, MES and the measures in (4). Note that we concentrate on multivariate regularly varying models in a non-standard sense. Hence, $Z = (Z_1, Z_2) \in MRV(\alpha)$ does not necessarily imply that both marginal variables have equivalent (or equal) tails; in fact, both margins need not be regularly varying either.

3.1. Hidden regular variation of additive models

Hidden regular variation properties of additive models (sometimes called mixture models) have been discussed in Weller and Cooley [51] and Das and Resnick [16] where the authors concentrate on adding two standard regularly varying models to get an additive structure with hidden regular variation. The class of models we consider are more general in the sense that the marginal tails of the additive components are not necessarily tail-equivalent. We establish the presence of hidden regular variation in these models under certain regularity conditions. First, we state a result on hidden regular variation of independent regularly varying random variables.

Lemma 3.1. Suppose $Y = (Y_1, Y_2) \in [0, \infty)^2$ where $Y_1$ and $Y_2$ are independent random variables with $F_{Y_1} \in RV_{-\alpha}$ and $F_{Y_2} \in RV_{-\alpha'}$. Then $Y \in MRV(\min(\alpha, \alpha')) \cap RV(\alpha + \alpha')$.

Proof. Either $F_{Y_1}$ and $F_{Y_2}$ are tail-equivalent or one tail is lighter tailed than the other. Without loss of generality we assume that $Y_2$ has a tail lighter than or is tail-equivalent to that of $Y_1$, in particular $\alpha \leq \alpha'$. and

$$\lim_{t \to \infty} \frac{\Pr(Y_2 > t)}{\Pr(Y_1 > t)} = C,$$

where $C \in [0, \infty)$. Then for $x, y > 0$ we have with $A_{x,y} = ([0, x] \times [0, y])$, $x > 0$.

$$\Pr(Y \in A_{x,y}) = \Pr(Y_1 > x) + \Pr(Y_2 > y) - \Pr(Y_1 > x) \Pr(Y_2 > y).$$
Hence,

$$\lim_{t \to \infty} \frac{\Pr(Y \in tA_{x,y})}{\Pr(Y \in tA_{1,1})} = \frac{x^{-\alpha} + Cy^{-\alpha}}{1 + C}$$

where either $\alpha = \alpha^*$ or $C = 0$. This implies $Y \in \mathcal{M}RV(\alpha)$. We also have

$$\Pr(Y_1 > t, Y_2 > t) = \Pr(Y_1 > t) \Pr(Y_2 > t) \in \mathcal{R}V - (\alpha + \alpha^*),$$

and for $x, y > 0$,

$$\frac{\Pr(Y_1 > xt, Y_2 > yt)}{\Pr(Y_1 > t, Y_2 > t)} \to x^{-\alpha}y^{-\alpha^*}.$$  

Hence, $Y$ is hidden regularly varying with index $\alpha + \alpha^*$. \hfill \Box

**Remark 3.2** For the additive models we consider next, one of the summands behaves like $Y$ of Lemma [3.1] and hence, the component to be added must have regular variation index smaller than $\alpha + \alpha^*$ to provide a non-trivial generative set of limit models.

**Model A** Suppose $Y = (Y_1, Y_2)$, $V = (V_1, V_2)$, and $Z = (Z_1, Z_2)$ are random vectors in $[0, \infty)^2$ such that $Z = Y + V$. Assume the following holds:

(A1) $Y \in \mathcal{M}RV(\alpha, b, \nu, E)$ and $Y_1, Y_2$ are independent.

(A2) $V \in \mathcal{M}RV(\alpha_0, b_0, \nu_0, E)$ with $\alpha_0 \geq \alpha$ and $V$ does not possess asymptotic tail independence. Moreover,

$$\lim_{t \to \infty} \frac{\Pr(\|V\| > t)}{\Pr(\|Y\| > t)} = 0.$$

(A3) $Y$ and $V$ are independent.

(A4) Suppose $\overline{F}_{Y_1} \in \mathcal{R}V_{-\alpha}$ or $\overline{F}_{Y_2}(t) = o(t^{-\alpha})$ where $\alpha^* > \alpha_0 - \alpha$.

Obviously, (A1) implies that either $\overline{F}_{Y_1} \in \mathcal{R}V_{-\alpha}$, or $\overline{F}_{Y_1}(t) \in \mathcal{R}V_{-\alpha}$ where $\lim_{t \to \infty} \overline{F}_{Y_1}(t)/\overline{F}_{Y_1}(t) = 0$. However, to use Lemma [3.1] we impose the additional assumption (A4) with $\alpha^* > \alpha_0 - \alpha$. Note that $Y$ and $V$ are independent of one another and both are multivariate regularly varying. The tail of $\|Y\|$ is heavier than that of $\|V\|$; $V$ does not have asymptotic tail independence and hence, in turn no hidden regular variation on $E_0 = (0, \infty)^2$ can be defined for $V$. The following theorem shows the existence of hidden regular variation in Model $[\text{A}]$. A version of the theorem is stated in Das and Fasen-Hartmann [13, Theorem 3] referring to the proof in the present paper.

**Theorem 3.3.** Let $Z = Y + V$ be as in Model [\text{A}]. Then the following statements hold:

(a) $Z \in \mathcal{M}RV(\alpha, b, \nu, E) \cap \mathcal{H}RV_\nu(\alpha_0, b_0, \nu_0, E_0)$. 

(b) If $Y_2 \equiv 0$, then $Z^* = (Z_1 + Z_2, Z_2) \in \mathcal{M}RV(\alpha, b, \nu, E) \cap \mathcal{H}RV(\alpha_0, b_0, \nu_0, E_0)$ with

$$\nu^0_0(A) = \nu_0((v_1, v_2) \in E_0 : (v_1 + v_2) \in A)) \quad \text{for } A \in \mathcal{B}(E_0).$$

(c) If $\lim_{t \to \infty} \Pr(Y_1 > t) / \Pr(Y_2 > t) > 0$, then $Z^{\min} = (Z_1, \min(Z_1, Z_2)) \in \mathcal{M}RV(\alpha, b, \nu^{\min}, E) \cap \mathcal{H}RV(\alpha_0, b_0, \nu_0^{\min}, E_0)$ with

$$\nu^{\min}(A) = \nu((y_1, 0) \in E_0 : (y_1, 0) \in A)) \quad \text{for } A \in \mathcal{B}(E_0),$$

$$\nu_0^{\min}(A) = \nu_0((v_1, v_2) \in E_0 : (v_1, \min(v_1, v_2)) \in A)) \quad \text{for } A \in \mathcal{B}(E_0).$$

(d) If $Y_2 \equiv 0$, then $Z^{\max} = (\max(Z_1, Z_2), Z_2) \in \mathcal{M}RV(\alpha, b, \nu, E) \cap \mathcal{H}RV(\alpha_0, b_0, \nu_0^{\max}, E_0)$ with

$$\nu^{\max}(A) = \nu_0((v_1, v_2) \in E_0 : (\max(v_1, v_2)) \in A)) \quad \text{for } A \in \mathcal{B}(E_0).$$
Since our aim is often to find and compare systemic risk in the presence of risk factors pertaining to two institutions, Theorem 3.3 addresses different kinds of measures for systemic risk in this context. If risk is additive and we compare risk of one with that of the portfolio of the system we refer to (b), if risk is measured in terms of both institutions being at risk we refer to (c), and in case systemic risk pertains to any of the institutions being in risk, we refer to part (d). Thus, a gamut of systemic risk measurement can be addressed under Model A.

Proof of Theorem 3.3

(a) **Step 1.** We get $Z \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ using Jessen and Mikosch [28, Lemma 3.12].

**Step 2.** The proof of $Z \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ is analogous to the proof of Proposition 3.2 in Das and Resnick [16] and skipped here.

(b) Define $Y^* = (Y_1, 0)$, $V^* = (V_1 + V_2, V_2)$ and write $Z^* = Y^* + V^*$. Now, we can check that $Y^* \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ and $V^* \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$. Moreover, conditions (A2), (A3) and (A4) for Model A with $Y^*$ and $V^*$ are also satisfied. Hence, $Z^*$ can be considered to be an example of case (a) again.

(c) **Step 1.** First, we show that $Z_{\min} = (Z_1, \min(Z_1, Z_2)) \in \mathcal{MRV}(\alpha, b, \nu_{\min}, \mathbb{E})$. Define

$$R := Z_{\min} - (Y_1, 0) = (V_1, \min(Y_1 + V_1, V_2 + V_2)).$$

Then

$$\Pr(|R| > t) \leq \Pr(\min(Y_1, Y_2) > t/2) + \Pr(\max(V_1, V_2) > t/2) = o(\Pr(Y_1 > t)) \quad \text{as } t \to \infty,$$

since $F_{\min Y_1, Y_2}(t) \in \mathcal{RV}_{\mu(\alpha + \alpha')} \ (t) \ or \ F_{\min Y_1, Y_2}(t) = o(F_{Y_1}(t)\alpha) \ (t) \ and \ F_{\max Y_1, Y_2}(t) \in \mathcal{RV}_{\alpha_0}$. (Y_1, 0) \in \mathcal{MRV}(\alpha, b, \nu_{\min}, \mathbb{E})$, using [28, Lemma 3.12] we have $Z_{\min} \in \mathcal{MRV}(\alpha, b, \nu_{\min}, \mathbb{E})$.

**Step 2.** Next, we prove $Z_{\min} \in \mathcal{MRV}(\alpha_0, b_0, \nu_{\min}, \mathbb{E}_0)$. We apply criterion (ii) of the Portmanteau Theorem 2.1 in Lindskog et al. [37] to show that

$$\nu_t(\cdot) = \frac{t}{\nu_{\min}(\cdot)} \rightarrow M(\mathbb{E}_0).$$

Let $f$ be in $C((0, \infty)^2)$ and without loss of generality suppose that $f$ is bounded by a constant $||f||$, is uniformly continuous and

$$f(x) = 0 \quad \text{if } x_1 \wedge x_2 < \eta,$$

for some $\eta > 0$. Uniform continuity of $f$ means that the modulus of continuity

$$\omega_f(\delta) := \sup\{|f(x) - f(y)| : \|x - y\| < \delta\} \rightarrow 0.$$

Since $V_{\min} := (V_1, \min(V_1, V_2)) \in \mathcal{MRV}(\alpha_0, b_0, \nu_{\min}, \mathbb{E})$ we have

$$\lim_{t \to \infty} t \mathbb{E}[f(V_{\min}/b_0(t))] = \nu_{\min}(f),$$

and so it suffices to show that as $t \to \infty$,

$$\lim_{t \to \infty} t \mathbb{E}[f(Z_{\min}/b_0(t))] - t \mathbb{E}[f(V_{\min}/b_0(t))] = 0.$$

Let $0 < \delta < \eta$. Then

$$t \mathbb{E}[f(Z_{\min}/b_0(t))] - t \mathbb{E}[f(V_{\min}/b_0(t))]$$

$$= t \mathbb{E}\left[\left[f(Z_{\min}/b_0(t)) - f(V_{\min}/b_0(t))\right] \mathbb{I}_{\{Y_1, Y_2 > b_0(t)\}}\right]$$

$$\quad + t \mathbb{E}\left[\left[f(Z_{\min}/b_0(t)) - f(V_{\min}/b_0(t))\right] \mathbb{I}_{\{Y_1, Y_2 \leq b_0(t)\}}\right]$$

$$= I_1(t, \delta) + I_2(t, \delta).$$

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For $I_t(t, \delta)$ we take the upper bound

$$|I_t(t, \delta)| \leq t \left[ \mathbb{E}\left[ \left| f(\mathbf{Z}^{\min}/b_0(t)) - f(\mathbf{V}^{\min}/b_0(t)) \right| I_{Y_1 \wedge Y_2 > b_0(t)} \right] \right] + t \mathbb{E}\left[ \left| f(\mathbf{Z}^{\min}/b_0(t)) - f(\mathbf{V}^{\min}/b_0(t)) \right| I_{Y_1 > b_0(t), Y_2 \leq b_0(t)} \right] + t \mathbb{E}\left[ \left| f(\mathbf{Z}^{\min}/b_0(t)) - f(\mathbf{V}^{\min}/b_0(t)) \right| I_{Y_1 \leq b_0(t), Y_2 > b_0(t)} \right] \]

$$

$$= J_1(t, \delta) + J_2(t, \delta) + J_3(t, \delta).$$

First, by Potter’s Theorem (see Bingham et al. [5, Theorem 1.5.6]) for any $\epsilon \in (0, \alpha^* + \alpha - \alpha_0)$ there exists a constant $C_1 > 0$ such that for large $t$

$$J_1(t, \delta) \leq 2\|/\| t \Pr(Y_1 \wedge Y_2 > b_0(t)) \delta \leq 2\|/\| t \Pr(Y_1 > b_0(t)) \delta \Pr(Y_2 > b_0(t)) \delta \leq C_1 t^{\omega_0-\omega_1} \rightarrow 0.$$

Similarly, by Potter’s Theorem for $\epsilon \in (0, \alpha)$ there exists a constant $C_2 > 0$ such that

$$J_2(t, \delta) \leq 2\|/\| t \Pr(Y_1 \wedge Y_2 > b_0(t) \min((\eta - \delta), \delta)) \leq C_2 t^{\omega_0-\omega_1} \rightarrow 0.$$

Finally, again, by Potter’s Theorem for $\epsilon \in (0, \alpha^*)$ there exists a constant $C_3 > 0$ such that

$$J_3(t, \delta) \leq 2\|/\| t \Pr(Y_1 \wedge Y_2 > b_0(t) \min((\eta - \delta), \delta)) \leq C_3 t^{\omega_0-\omega_1} \rightarrow 0.$$

Hence, we have

$$\lim_{t \rightarrow \infty} |I_t(t, \delta)| = 0.$$

Since by definition $f(x) = 0$ if $x_1 \wedge x_2 < \eta$, we have

$$|I_2(t, \delta)| = t \mathbb{E}\left[ \left| f(\mathbf{Z}^{\min}/b_0(t)) - f(\mathbf{V}^{\min}/b_0(t)) \right| I_{Y_1 \wedge Y_2 \leq b_0(t), Y_1 \wedge Y_2 > (\eta - \delta)b_0(t)} \right] \leq \omega_f(\delta) t \Pr(Y_1 \wedge Y_2 > (\eta - \delta)b_0(t)).$$

Hence,

$$\lim_{\delta \rightarrow 0} |I_2(t, \delta)| \leq \limsup_{\delta \rightarrow 0} \omega_f(\delta)(\eta - \delta)^{-\omega_0} = 0.$$

Therefore, we have $\mathbf{Z}^{\min} = (Z_1, \min(Z_1, Z_2)) \in \mathcal{M}R\nu(a_0, b_0, 0, E_0, \nu, \mathbb{E}).$

(d) **Step 1.** Note that $Y_2 \equiv 0$. First, we show that $\mathbf{Z}^{\max} = (\max(Z_1, Z_2), Z_2) \in \mathcal{M}R\nu(a, b, \nu, \mathbb{E})$. Define

$$R := \mathbf{Z}^{\max} - Y.$$ Then

$$\Pr(||R|| > t) \leq \Pr(\max(V_1, V_2) > t/2) = o(\Pr(V_1 > t)) \quad \text{as } t \rightarrow \infty.$$

Since $Y = (Y_1, 0) \in \mathcal{M}R\nu(a, b, \nu^*, \mathbb{E})$, using [28, Lemma 3.12] we have $\mathbf{Z}^{\max} \in \mathcal{M}R\nu(a, b, \nu^*, \mathbb{E}).$

**Step 2.** Next, we show that $\mathbf{Z}^{\max} = (\max(Z_1, Z_2), Z_2) \in \mathcal{M}R\nu(a_0, b_0, \nu^*_0, \mathbb{E}).$

Since $V \in \mathcal{M}R\nu(a_0, b_0, \nu_0, \mathbb{E})$ we have $\nu^*_{\max} := (\max(V_1, V_2), V_2) \in \mathcal{M}R\nu(a_0, b_0, \nu^*_0, \mathbb{E})$ and with the notations and definitions of (c),

$$\lim_{t \rightarrow \infty} t \mathbb{E}[f(\mathbf{Z}^{\max}/b_0(t))] = \nu^*_{\max}(f).$$

So it suffices to show that

$$\lim_{t \rightarrow \infty} t \mathbb{E}[f(\mathbf{Z}^{\max}/b_0(t))] - t \mathbb{E}[f(\mathbf{V}^{\max}/b_0(t))] = 0.$$
Let $0 < \delta < \eta$. Then
\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E}[f(Z_{\max} / b_0(t)) - f(V_{\max} / b_0(t))] = \lim_{t \to \infty} \mathbb{E}\left[\left|f(Z_{\max} / b_0(t)) - f(V_{\max} / b_0(t))\right| I_{[Y_1 > b_0(t)\delta]}\right] + \lim_{t \to \infty} \mathbb{E}\left[\left|f(Z_{\max} / b_0(t)) - f(V_{\max} / b_0(t))\right| I_{[Y_1 \leq b_0(t)\delta]}\right] =: I_1(t, \delta) + I_2(t, \delta).
\]

Again by Potter’s Theorem for any fixed $\epsilon \in (0, \alpha)$ there exists a constant $C > 0$ such that for large $t$,
\[
|I_1(t, \delta)| \leq 2\|f\| t \Pr(Y_1 > b_0(t)\delta) \Pr(V_2 > b_0(t)\eta) \leq C t \frac{\max}{\min} \rightarrow 0,
\]
and moreover,
\[
|I_2(t, \delta)| = \mathbb{E}\left[\left|f(Z_{\max} / b_0(t)) - f(V_{\max} / b_0(t))\right| I_{[Y_1 \leq b_0(t)\delta, V_2 > \eta b_0(t)]}\right] \leq \omega_f(\delta) t \Pr(V_2 > \eta b_0(t))
\]

implying
\[
\lim_{\delta \to 0} \lim_{t \to \infty} |I_2(t, \delta)| \leq \limsup_{\delta \to 0} \omega_f(\delta) \eta b_0(t) = 0.
\]

Hence, we can conclude as in (c) that $Z_{\max} = (\max(Z_1, Z_2), Z_2) \in \mathcal{MRV}(\alpha_0, b_0, \nu_{\max}, \mathbb{E}_0)$.

Since in Model[A] we know either $F_{Y_1} \in \mathcal{RV}_\alpha$, or $F_{Y_1}(t) \in \mathcal{RV}_{-\alpha}$ where $\lim_{t \to \infty} F_{Y_1}(t)/F_{Y_1}(t) = 0$, we obtain the following special case as well.

**Corollary 3.4.** Let $Z = Y + V$ be as in Model[A] and assume $2\alpha > \alpha_0$. Then the assumptions and hence, the conclusions of Theorem 3.3 are satisfied.

### 3.2. Hidden regular variation and the survival copula

We present a characterization of hidden regular variation via the behavior of the survival copula. First, we introduce a generalized version of the upper tail order function (see Hua and Joe [21, 23]) along with an upper tail order pair. The notion of upper tail order pair is related also to operator tail dependence in Li [35], and to the generalized upper tail index $\kappa$ in Wadsworth and Tawn [51].

**Definition 3.5 (Upper Tail Order)** Let $F$ be a bivariate distribution function with survival copula $\hat{C}$. For a given constant $\tau > 0$, if there exist a real constant $\kappa > 0$, and a slowly varying function $\ell$ at 0 with
\[
\hat{C}(s, s') \sim s^{\ell(s)} \text{ as } s \downarrow 0,
\]
the pair $(\kappa, \tau)$ is called an upper tail order pair of $F$. The upper tail order function $T : \mathbb{E}_0 \to \mathbb{R}_+$ with respect to $(\kappa, \tau)$ is defined as
\[
T(x, y) = \lim_{s \downarrow 0} \frac{\hat{C}(sx, sy)}{s^{\ell(s)}} \text{ for } x, y > 0
\]

provided that the limit function exists.

**Remark 3.6** Note that the pair $(\tau, \kappa)$ need not be a unique for the definition to hold. Albeit this fact, introducing the quantity $\tau$ helps in rescaling marginal tails when they are not equivalent (see Theorem 3.10 below). Since $0 \leq \hat{C}(s, s') \leq \hat{C}(1, 1) = s^{\ell(1)}$ for $s \in (0, 1)$ we have $\kappa \geq \tau$ and similarly we obtain $\kappa \geq 1$ as well. Note that the existence of the upper tail order pair is not a sufficient assumption for the existence of the upper tail order function. We often provide examples fixing $\tau = 1$, which also fixes the value of $\kappa$.

**Lemma 3.7.** Suppose the bivariate distribution function $F$ with survival copula $\hat{C}$ exhibits asymptotic upper tail independence and the upper tail order pair $(\kappa, \tau)$ exists with $\tau \geq 1$ and $\hat{C}(s, s') \sim s^{\ell(s)}$ as $s \downarrow 0$. Then
\[
\lim_{s \downarrow 0} s^{\ell(s)} = 0.
\]
Proof. Note that, for $\tau \geq 1$,

$$1 = \lim_{s \downarrow 0} \frac{\widehat{C}(s, s^\tau)}{s^\tau \ell(s)} \leq \lim_{s \downarrow 0} \frac{\widehat{C}(s, s)}{s} \lim_{s \downarrow 0} \frac{1}{s^{1-\tau} \ell(s)}. \tag{9}$$

Since $F$ exhibits asymptotic upper tail independence, using Lemma 2.2, we have $\lim_{s \downarrow 0} \widehat{C}(s, s)/s = 0$. Hence, the inequality in (9) is only possible if $\lim \inf_{s \downarrow 0} 1/(s^{1-\tau} \ell(s)) = \infty$. \hfill \qed

Remark 3.8 The classical definition of upper tail order is for $\tau = 1$ and it is equivalent to the definition of coefficient of tail dependence in Ledford and Tawn [34].

1. If $\kappa = \tau = 1$ and $\lim_{s \downarrow 0} \ell(s) = c$ for some finite constant $c$, then we get asymptotic dependence in the upper tail. In this case $T$ is the upper tail dependence function introduced in Jaworski [27]. However, if $\kappa = \tau = 1$ and $\lim_{s \downarrow 0} \ell(s) = 0$ then again we observe asymptotic tail independence.

2. The case $1 < \kappa < 2$, $\tau = 1$ is between tail dependence ($\kappa = 1$ and $c \neq 0$) and tail independence ($\kappa = 2$) and indicates some positive tail dependence although the tails are asymptotically tail independent. It is called intermediate tail dependence by Hua and Joe [21, 23].

3. Note that it is possible to have $\kappa > 2$ which often signifies negative tail dependence; see Example 3.9(a) below.

Example 3.9 We compute upper tail order functions and upper tail order pairs for a few well-known (survival) copula models here.

(a) The Gaussian copula turns out to be one of the most famous, if not infamous copula models, especially in financial risk management; see Salmon [46]. It is given by

$$C_{\Phi, \rho}(u, v) = \Phi_2(\Phi^-(u), \Phi^-(v)) \quad \text{for } (u, v) \in [0, 1]^2,$$

where $\Phi$ is the standard-normal distribution function and $\Phi_2$ is a bivariate normal distribution function with standard normally distributed margins and correlation $\rho$. Then the survival copula satisfies:

$$\widehat{C}_{\Phi, \rho}(s, s) = C_{\Phi, \rho}(s, s) \sim s^{\rho} \ell(s) \quad \text{as } s \downarrow 0.$$ (see Ledford and Tawn [34], Reiss [43]). For $\rho \in (-1, 1)$ we have $\kappa = 2/(\rho + 1) > 1$ and $\tau = 1$ so that a distribution with Gaussian copula has still some kind of dependence although it exhibits asymptotic upper tail independence. The upper tail order function is given by (see Reiss [43], Example 7.2.7)

$$T(x, y) = x^{\gamma_1} y^{\gamma_2} \quad \text{for } x, y > 0.$$

Note that $0 < \rho < 1$ implies $1 < \kappa < 2$ relating to positive intermediate tail dependence, $\rho = 0$ implies $\kappa = 2$ which is the independent case and $\rho < 0$ implies $\kappa > 2$ which is the case of negative tail dependence.

(b) If the survival copula is a Marshall-Olkin copula then:

$$\widehat{C}_{\gamma_1, \gamma_2}(u, v) = uv \min(u^{-\gamma_1}, v^{-\gamma_2}) \quad \text{for } (u, v) \in [0, 1]^2,$$

some $\gamma_1, \gamma_2 \in (0, 1)$. For a fixed $\tau \geq 1$, the upper tail order is $\kappa = \max(\tau + 1 - \gamma_1, \tau + 1 - \gamma \gamma_2)$, and the upper tail order function is

$$T(x, y) = \begin{cases} xy^{1-\gamma_1}, & \text{if } \gamma_1 > \gamma_2, \\ xy \max(x, y^{1/\gamma_1})^{-\gamma_2}, & \text{if } \gamma_1 = \gamma_2, \\ x^{1-\gamma_2} y, & \text{if } \gamma_1 < \gamma_2, \end{cases} \quad \text{for } x, y > 0.$$

The Marshall-Olkin copula belongs to the class of extreme value copulas. This structure of $T(x, y)$ holds in general for bivariate extreme value copulas with discrete Pickands dependence function; see Hua and Joe [21, Example 2].
(c) If the survival copula is a *Morgenstern copula* with parameter \(-1 \leq \theta \leq 1\) then:

\[
\bar{C}_\theta(u, v) = uv(1 + \theta(1 - u)(1 - v)) \quad \text{for } (u, v) \in [0, 1]^2.
\]

Hence, for \(-1 < \theta < \leq 1\) and fixed \(\tau \geq 1\) we get \(\kappa = \tau + 1\) with upper tail order function \(T(x, y) = xy^\tau\) for \(x, y > 0\).

For \(\theta = -1\) we have the upper tail order pair \((\kappa = 3, \tau = 1)\) with upper tail order function \(T(x, y) = xy(x + y)\) for \(x, y > 0\). If we fix \(\tau > 1\) then \(\kappa = 1 + 2\tau\) and the upper tail order function is \(T(x, y) = xy^2\) for \(x, y > 0\).

(d) A copula is called an *Archimedean copula* if it is of the form

\[
C(u, v) = \phi^{-1}(\phi(u) + \phi(v)) \quad \text{for } (u, v) \in [0, 1]^2,
\]

where the Archimedean generator \(\phi : [0, 1] \to [0, 1]\) is convex, decreasing and satisfies \(\phi(1) = 0\). In the bivariate case this is a necessary and sufficient condition for \(C\) to be a copula. If the generator \(\phi\) is regularly varying near 0 the lower tail is asymptotically independent and if \(\phi\) is regularly varying at 1 the upper tail is asymptotically independent (see Ballerini [3], Capéraà et al. [10]). Charpentier and Segers [11, Section 4] provide tail order coefficients and functions for Archimedean copulas specifically under asymptotic tail independence. We use them to generalize results for the tail order pair here; see also Hua and Joe [21].

1. Let \(\phi^{-\gamma}\) be twice continuously differentiable with \(\phi^{-\gamma}(0) < \infty\). Then an upper tail order pair is \((\kappa = 2, \tau = 1)\) with upper tail order function

\[
T(x, y) = xy.
\]

2. Let \(\phi'(1) = 0\) and let the function \(-\phi'(1 - s) - s^{-1}\phi(1 - s)\) be positive and slowly varying around 0. Then the upper tail order pair is \((\kappa = 1, \tau = 1)\) with upper tail order function

\[
T(x, y) = (x + y) \log(x + y) - x \log x - y \log y)/(2 \log 2).
\]

3. Assume that \(\bar{C}\) is an Archimedean copula with generator \(\phi\) satisfying \(\phi(0) = \infty\), \(\lim_{s \to 0} s\phi'(s)/\phi(s) = 0\) and \(-1/(\log \phi^{-\gamma}'\) is regularly varying with some index \(\rho \leq 1\). Then the upper tail order pair is \((\kappa = 2^{1-\rho}, \tau = 1)\) with upper tail order function

\[
T(x, y) = x^{2^{1-\rho}} y^{2^\rho}.
\]

We are now able to present a connection between hidden regular variation and the upper tail order function; these results are extensions of Hua et al. [25] and Li and Hua [32] which include several examples as well. The first result is a generalization of Hua et al. [25, Proposition 3.2] for \(\tau = 1\).

**Theorem 3.10.** Let \(Z \in \mathcal{MRV}(\alpha, b, \nu, E) \cap \mathcal{HRV}(\alpha_0, b_0, v_0)\) with continuous margins \(F_1, F_2\) satisfying \(\lim_{t \to \infty} F_2(t)/F_1(t) = \eta \in (0, \infty)\) where \(\alpha_0/\alpha \geq \tau \geq 1\). Then an upper tail order pair exists with \((\kappa = \alpha_0/\alpha, \tau)\), and the corresponding upper tail order function is given by

\[
T(x, y) = C_{\nu_0}\left(\left(\frac{x^{1/\alpha}}{\alpha}, \infty\right) \times \left(\frac{y^{1/(\tau\alpha)}}{(\tau\alpha), \infty}\right)\right)^{-1} < \infty.
\]

**Proof.** The proof follows with similar arguments as in Hua et al. [25, Proposition 3.2] if we can show the following:

(a) \(\tau_1 \in \mathcal{RV}_{-\alpha}\).

(b) If for any \(y > 0\) and \(\varepsilon > 0\) small there exists an \(s_0 > 0\) such that for any \(0 < s < s_0\) the inequality

\[
(1 - \varepsilon)\eta^{1/(\tau\alpha)} y^{-1/(\tau\alpha)} < \frac{F_2^{\tau_1}(yx)}{F_1(s)} < (1 + \varepsilon)\eta^{1/(\tau\alpha)} y^{-1/(\tau\alpha)}
\]

holds.
We prove (a) and (b) in the following.

(a) Case 1: \( \tau > 1 \). Then \( \lim_{t \to \infty} \frac{F_2(t)}{F_1(t)} = 0 \). Since \( Z = (Z_1, Z_2) \in \mathcal{MRV}(\alpha, b, v, \mathbb{E}) \) in particular, \( \max(Z_1, Z_2) \in \mathcal{RV}_{\tau} \). Moreover,

\[
1 \leq \frac{\Pr(\max(Z_1, Z_2) > t)}{\frac{t}{F_1(t)}} \leq 1 + \frac{F_2(t)}{F_1(t)} \xrightarrow{t \to \infty} 1.
\]

Hence, \( F_1 \in \mathcal{RV}_{\tau} \).

Case 2: \( \tau = 1 \). Since \( Z = (Z_1, Z_2) \in \mathcal{MRV}(\alpha, b, v, \mathbb{E}) \) either \( F_1 \in \mathcal{RV}_{\tau} \) or \( F_2 \in \mathcal{RV}_{\tau} \). Since \( \lim_{t \to \infty} \frac{F_2(t)}{F_1(t)} = \eta \in (0, \infty) \) we have that \( F_1 \) and \( F_2 \) are tail-equivalent and in particular, both are \( \mathcal{RV}_{\tau} \).

(b) Let \( y > 0 \). Define \( G_1(x) = 1/F_2(x) \) and \( G_2(x) = 1/\eta F_1(x) \). Then \( G_1(x) \sim G_2(x) \) as \( x \to \infty \), \( G_1, G_2 \in \mathcal{RV}_{\tau} \) and both are non-decreasing. Using Resnick [44], Proposition 2.6] we have \( G_1^{-1}, G_2^{-1} \in \mathcal{RV}_{1/(\alpha \tau)} \) and \( G_1^{-1}(z) \sim G_2^{-1}(z) \) as \( z \to \infty \). Here \( G_1^{-1}(z) = F_2^{-1}(1/z) \) and \( G_2^{-1}(z) = F_1^{-1}(1/\eta z) \). Hence,

\[
\frac{F_2^{-1}(y^{s\tau})}{F_1(s)} = \frac{F_2^{-1}(y^{s\tau})}{F_1(\eta^{-1/\tau}y^{s\tau})} \frac{F_1(\eta^{-1/\tau}y^{s\tau})}{F_1(s)} \xrightarrow{s \downarrow 0} \eta^{1/(\alpha \tau)}y^{-1/(\alpha \tau)},
\]

where the first term converges to 1 due to (10) and the second term converges to \( \eta^{1/(\alpha \tau)}y^{-1/(\alpha \tau)} \) since \( F_1 \) is regularly varying with index \(-1/\alpha\) near 0. Hence, (b) holds.

\[\Box\]

Remark 3.11 Note that in the presence of hidden regular variation as above the pair \((b_0, v_0)\) is not exactly uniquely defined since if \( Z \in \mathcal{HRV}(a_0, b_0, v_0) \) then \( Z \in \mathcal{HRV}(a, Cb_0, C^{-a_0}v_0) \) for \( 0 < C < \infty \) as well. But we can consider this to be uniqueness up to a scale.

The converse of Theorem 3.10 also holds; this is an extension of Hua et al. [25], Proposition 3.3] for \( \tau = 1 \).

Theorem 3.12. Let \( Z \in [0, \infty)^2 \) with continuous margins \( F_1, F_2 \), survival copula \( \hat{C} \), \( \mathbb{E}[Z_1] < \infty \) and \( F_1 \in \mathcal{RV}_{\tau} \). Suppose \( \hat{C} \) has upper tail order pair \((\kappa, \tau)\) with \( \kappa \geq 1 \) and some slowly varying function \( \ell \) at 0 with \( \lim_{x \downarrow 0} x^{-\kappa+1}\ell(s) = 0 \) satisfying (7) and (8). Moreover, assume that \( \lim_{t \to \infty} \frac{F_2(t)}{F_1(t)} = \eta \in (0, \infty) \). Then \( Z \in \mathcal{MRV}(\alpha, b, v, \mathbb{E}) \cap \mathcal{HRV}(a, b_0, v_0) \) with \( a_0 = \ell \kappa \),

\[
v_0((x, \infty] \times (y, \infty)) = T(x^{-\alpha}, y^{\tau})y^{-\alpha} \quad \text{for } x, y > 0,
\]

and some properly chosen \( b_0 \in \mathcal{RV}_{1/\alpha} \).

Proof. The proof follows similar arguments as that of Hua et al. [25], Proposition 3.2] using (11) and is omitted here.

\[\Box\]

Remark 3.13 It is possible to have \( \kappa = \tau = 1 \) as well but then \( \lim_{t \to \infty} \ell(s) = 0 \) which excludes the case of asymptotic upper tail dependence.

A conclusion from these results is that hidden regular variation is not only the effect of the copula of the joint distribution but also of the ratio of the individual marginal tails. This may seem surprising since copulas, in theory, are supposed to decouple the marginal distributions from the dependence structure of random vectors. Clearly, this is not the case for tail dependence especially for regularly varying tails as observed here.
4. Asymptotic behavior of risk measures

The asymptotic behavior of the MME and the MES under hidden regular variation were obtained in Das and Fasen-Hartmann [15]. Under these constraints consistent estimators for MME and MES were derived; moreover Cai and Musta [8] have also shown asymptotic normality for MES. In this section we present a variety of examples of model classes satisfying the assumptions of Das and Fasen-Hartmann [15] and relate these assumptions in particular, to copula models. We recall in brief the results from [15] and then present additive models in Section 4.1 and copula model classes satisfying the assumptions of Das and Fasen-Hartmann [15] and relate these assumptions in particular.

(1) From Das and Fasen-Hartmann [15, Remark 6] we know that under condition (C) the limit \( \lim_{M \to \infty} \int_M^\infty \Pr(Z_1 > xt, Z_2 > t) \, dx = 0. \) (B)

Then

\[
\lim_{p \to 0} \frac{p b_0^0 [\text{VaR}_{1-p}(Z_2)]}{\text{VaR}_{1-p}(Z_2)} \text{MME}(p) = \int_0^\infty v_0((x, \infty) \times (1, \infty)) \, dx.
\]

Moreover, \( 0 < \int_0^\infty v_0((x, \infty) \times (1, \infty)) \, dx < \infty. \)

(b) Suppose the following condition holds:

\[
\lim_{M \to \infty} \int_M^\infty + \int_0^{1/M} \Pr(Z_1 > xt, Z_2 > t) \, dx = 0.
\]

Then

\[
\lim_{p \to 0} \frac{p b_0^0 [\text{VaR}_{1-p}(Z_2)]}{\text{VaR}_{1-p}(Z_2)} \text{MES}(p) = \int_0^\infty v_0((x, \infty) \times (1, \infty)) \, dx.
\]

Moreover, \( 0 < \int_0^\infty v_0((x, \infty) \times (1, \infty)) \, dx < \infty. \)

Clearly, condition (C) implies condition (B). In the face of it, it appears that the rate of increase of MES which is governed by the function \( p b_0^0 [\text{VaR}_{1-p}(Z_2)]/\text{VaR}_{1-p}(Z_2) \) is determined by the tail behavior of the marginal distribution \( F_2 \). However, we notice in Section 4.2 that this is not true for MES; the rate is in fact governed by the joint tail behavior of the copula of \((Z_1, Z_2)\) and that of the marginal tail of \( F_1 \).

Remark 4.2 Define the function governing the limit behavior of MES(\( p \)) and MME(\( p \)) in Theorem 4.1 as

\[
a(t) := \frac{b_0^0 [\text{VaR}_{1-1/(Z_2)}]}{\text{VaR}_{1-1/(Z_2)}} \tag{12}
\]

(1) From Das and Fasen-Hartmann [15, Remark 6] we know that under condition (C) the limit \( \lim_{p \to 0} a(1/p) = 0 \) is valid and hence, under the conditions of Theorem 4.1 \( \lim_{p \to 0} \text{MME}(p) = \infty. \) Thus, even under the presence of asymptotic upper tail independence, the tail dependence is still strong enough for \( \lim_{p \to 0} \text{MES}(p) = \infty. \) In contrast, if \( Z_1 \) and \( Z_2 \) are independent we have \( \text{MSE}(p) = \mathbb{E}(Z_1) \) (then condition (C) is not satisfied).

(2) Consider the case \( F_{Z_2} \in \mathcal{R}_{\mathcal{V},-\alpha}. \) Then \( (Z_1, Z_2) \in \mathcal{M\mathcal{R\mathcal{V}}}(\alpha, b, \nu, E) \) implies \( \alpha^* \geq \alpha. \) Furthermore, if additionally \( (Z_1, Z_2) \in \mathcal{M\mathcal{R\mathcal{V}}}(\alpha_0, b_0, \nu_0, E_0) \) then \( \alpha^* \leq \alpha_0 \) as well (see [15, Lemma 2.7]). In this case, \( a(t) \in \mathcal{R}_{\mathcal{V}(\alpha_0-\alpha^*-1)/\alpha^*}. \) Therefore, a necessary condition for \( \lim_{p \to 0} a(1/p) = 0 \) is \( \alpha_0 \leq \alpha^* + 1 \) and a sufficient condition is \( \alpha_0 < \alpha^* + 1. \) Finally, \( \alpha_0 \leq \alpha^* + 1 \) is as well a necessary assumption for (C).

(3) If \( Z_1 \) and \( Z_2 \) are independent then \( \alpha_0 = \alpha + \alpha^* \) and hence, \( \alpha \geq 1 \) and \( \alpha_0 < \alpha^* + 1 \) is not possible. Thus, the independent case does not satisfy (C) and a scaled limit for MES(\( p \)) cannot be calculated using Theorem 4.1.

(4) Under condition (B) both \( \lim_{p \to 0} a(1/p) = \infty \) and \( \lim_{p \to 0} \text{MME}(p) = 0 \) are possible. For the independent margin case the asymptotic behavior of MME(\( p \)) can still be calculated using Theorem 4.1 since with \( \alpha > 1 \) and \( Z_1, Z_2 \) independent, condition (B) is satisfied.
Proposition 4.5. Let \( Z = Z_1 + Z_2 \). Then \( Z \) is independent, and the following statements hold.

\[
\begin{align*}
(a) & \quad Z = (Z_1, Z_2). \\
(b) & \quad Z^* = (Z_1^* + Z_2^*) \text{ and } \bar{Z}^* = (Z_1^* + Z_2^*) \text{ if } Y_2 \equiv 0. \\
(c) & \quad Z_{\min} = (Z_1, \min(Z_1, Z_2)) \text{ if } \lim_{t \to \infty} \text{Pr}(Y_1 > t) / \text{Pr}(Y_2 > t) > 0. \\
(d) & \quad Z_{\max} = (\max(Z_1, Z_2), Z_2) \text{ and } \bar{Z}_{\max} = (Z_2, \max(Z_1, Z_2)) \text{ if } Y_2 \equiv 0.
\end{align*}
\]

As a consequence, the asymptotic limits of the scaled MME(\( p \)) and MES(\( p \)) can be obtained in each case using Theorem 4.3.

4.1. Asymptotic behavior of MME and MES for additive models

In Section 4.1, we introduced with Model A a general additive model for multivariate regular variation and discussed in Theorem 3.3 the existence of hidden regular variation. Do such models satisfy conditions (C) and hence (B) as well? The following result provides an answer.

Theorem 4.3. Let \( Z = Y + V \) be as in Model A and \( a(t) \) be defined as in (12). Suppose that \( \text{E}[Z_1] < \infty \) and \( \alpha \leq a_0 < 1 + \alpha^* \). Then the following models satisfy condition (C):

\[
\begin{align*}
(a) & \quad \text{MME}(p) = K^* \text{ if } Y_2 \equiv 0. \\
(b) & \quad \text{MES}(p) = K^* \text{ if } \lim_{t \to \infty} \text{Pr}(Y_1 > t) / \text{Pr}(Y_2 > t) > 0. \\
(c) & \quad \text{MES}(p) = K^* \text{ if } Y_2 \equiv 0.
\end{align*}
\]

A direct consequence of Theorem 4.3 is the ability to compute asymptotic limits of MES, MES\(^{\min}\), and MES\(^{\max}\) as defined in (1) and is summarized in the following corollary.

Corollary 4.4. Let \( Z = Y + V \) be as in Model A and \( a(t) \) be defined as in (12). Suppose that \( \text{E}[Z_1] < \infty \) and \( \alpha \leq a_0 < 1 + \alpha^* \). Then there exist finite constants \( K^*, K^{\min}, K^{\max} > 0 \) so that following statements hold.

\[
\begin{align*}
(a) & \quad \lim_{p \to 0} a(1/p)\text{MES}(p) = K^* \text{ if } Y_2 \equiv 0. \\
(b) & \quad \lim_{p \to 0} a(1/p)\text{MES}(p) = K^{\min} \text{ if } \lim_{t \to \infty} \text{Pr}(Y_1 > t) / \text{Pr}(Y_2 > t) > 0. \\
(c) & \quad \lim_{p \to 0} a(1/p)\text{MES}(p) = K^{\max} \text{ if } Y_2 \equiv 0.
\end{align*}
\]

We prove some auxiliary results first which are used to prove Theorem 4.3. The following proposition provides sufficient conditions under Model A for condition (C) to hold.

Proposition 4.5. Let \( Z = Y + V \) be as in Model A. Suppose that \( \text{E}[Z_1] < \infty \) implying \( \alpha \geq 1 \) and that the following conditions are satisfied:

\[
\begin{align*}
(i) & \quad Z \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_0, b_0, v_0, \mathcal{E}_0). \\
(ii) & \quad \lim_{t \to \infty} \text{Pr}(Y_2 > t) = 0. \\
(iii) & \quad \lim_{M \to \infty} \lim_{t \to \infty} \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\text{Pr}(Y_1 > tx, Y_2 > t)}{\text{Pr}(V_1 > t, V_2 > t)} dx = 0.
\end{align*}
\]

Then \( Z \) satisfies condition (C).

Proof. By the assumptions of Model A and Theorem 4.3 \( V \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_0, b_0, v_0, \mathcal{E}) \), \( V \) does not possess asymptotic tail independence, and \( Z \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_0, b_0, v_0, \mathcal{E}_0) \). Hence,

\[
\text{Pr}(V_1 > t) \sim C_1 \text{Pr}(V_2 > t) \sim C_2 \text{Pr}(V_1 > t, V_2 > t) \sim C_3 \text{Pr}(Z_1 > t, Z_2 > t) \quad \text{as } t \to \infty,
\]
for some finite constants $C_1, C_2, C_3 > 0$. Let $0 < \delta < 1$. For some $M > 0$ we have
\[
\left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Z_1 > x \delta t, Z_2 > t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[
= \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Z_1 > x \delta t, Z_2 > t, Y_1 > x \delta t, Y_2 > \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[
+ \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Z_1 > x \delta t, Z_2 > t, Y_1 > x \delta t, Y_2 \leq \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[
+ \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Z_1 > x \delta t, Z_2 > t, Y_1 \leq x \delta t, Y_2 > \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[
+ \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Z_1 > x \delta t, Z_2 > t, Y_1 \leq x \delta t, Y_2 \leq \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[=: I_1(M, t) + I_2(M, t) + I_3(M, t) + I_4(M, t).\]

Now, we investigate all four terms separately. Using (13) for large enough $t$,
\[I_1(M, t) \leq \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Y_1 > x \delta t, Y_2 > \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[\leq \frac{2C_1 \Pr(Y_1 > \delta t, Y_2 > \delta t)}{C_2 \Pr(V_1 > t, V_2 > t)} \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(Y_1 > x \delta t, Y_2 > \delta t)}{\Pr(V_1 > t, V_2 > t)} \, dx.
\]

A consequence of assumption (ii) and $V \in \mathcal{MRV}(a_0, b_0, \nu_0, \mathbb{E})$ is that $\lim_{M \to \infty} \lim_{t \to \infty} I_1(M, t) = 0$. For the second term $I_2(M, t)$ we have by the independence of $Y_1$ and $V_2$, and by Potter’s bound for some $0 < \epsilon < \alpha - 1$, the upper bound
\[I_2(M, t) \leq \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(V_2 > (1 - \delta) t, Y_1 > x \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[= \frac{\Pr(V_2 > (1 - \delta) t)}{\Pr(Z_1 > t, Z_2 > t)} \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(V_1 > x \delta t)}{\Pr(V_1 > t, V_2 > t)} \, dx
\]
\[\leq \frac{1}{M} + C M^{-\alpha + \epsilon}.
\]

Now using (13) results in $\lim_{M \to \infty} \lim_{t \to \infty} I_2(M, t) = 0$. The third term $I_3(M, t)$ satisfies
\[I_3(M, t) \leq \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(V_1 > x(1 - \delta) t, Y_2 > \delta t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[= \frac{\Pr(Y_2 > \delta t)}{(1 - \delta) \Pr(Z_1 > t, Z_2 > t)} \left[ \int_0^{1/M} + \int_M^{\infty} \right] (1 - \delta) \Pr(V_1 > x(1 - \delta) t) \, dx
\]
\[\leq C \frac{\Pr(Y_2 > \delta t)}{\Pr(V_1 > t, V_2 > t)} \mathbb{E}[V_1] \lim_{t \to \infty} = 0
\]
by assumption (ii) and $\mathbb{E}[V_1] \leq \mathbb{E}[Z_1] < \infty$. Finally, using (13) and by Potter’s theorem for some $0 < \epsilon < \alpha_0 - 1$, the last term $I_4(M, t)$ has the upper bound
\[I_4(M, t) \leq \left[ \int_0^{1/M} + \int_M^{\infty} \right] \frac{\Pr(V_1 > x(1 - \delta) t, V_2 > (1 - \delta) t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx
\]
\[\leq C \frac{\Pr(V_2 > (1 - \delta) t)}{\Pr(V_2 > t)} + C \int_M^{\infty} \frac{\Pr(V_1 > x(1 - \delta) t)}{\Pr(V_1 > t)} \, dx
\]
\[\leq C M^{-\alpha} + C M^{-\alpha + \epsilon},
\]
which implies $\lim_{M \to \infty} \lim_{t \to \infty} I_4(M, t) = 0$ as well. 

\[\Box\]
The following lemma provides necessary and sufficient conditions for assumption $\text{(iii)}$ of Proposition 4.5 to hold.

**Lemma 4.6.** Let $Z = Y + V$ be as in Model $A$. Suppose that $\overline{F}_{Y_{Z}} \in \mathcal{RV}_{+\infty}$ for some $\alpha^* \geq \alpha \geq 1$. Then $\alpha_0 \leq 1 + \alpha^*$ is a necessary condition and $\alpha_0 < 1 + \alpha^*$ is a sufficient condition for assumption $\text{(iii)}$ of Proposition 4.5 to hold.

**Proof.** Since $\Pr(Y_2 > t) / [t \Pr(V_1 > t, V_2 > t)] \in \mathcal{RV}_{+\infty}$, assumption $\text{(iii)}$ of Proposition 4.5 can only hold if $-\alpha^* - 1 + \alpha_0 \leq 0$. On the other hand, $-\alpha^* - 1 + \alpha_0 < 0$ implies assumption $\text{(iii)}$.

Now we are able to prove Theorem 4.3.

**Proof of Theorem 4.3**

(a) First of all, due to Theorem 3.3, Assumption 4.5(i) is valid. Assumption 4.5 (ii) is already satisfied due to Lemma 4.6. Finally, Assumption 4.5 (iii) follows from

\[
\mathbb{E}(Y_1) \xrightarrow{t \to \infty} 0.
\]

Hence, we obtain that $Z$ satisfies $\text{(C)}$.

(b) This can be seen as special case of $\text{(a)}$ where

\[
Z^+ = (Z_1 + Z_2, Z_1) = (Y_1, 0) + (V_1 + V_2, V_1) =: Y^+ + V^+,
\]

\[
\tilde{Z}^+ = (Z_1, Z_1 + Z_2) = (0, Y_1) + (V_1, V_1 + V_2) =: \tilde{Y}^+ + \tilde{V}^+.
\]

(c) For $x > 1$,

\[
\frac{\Pr(Z_1 > x, \min(Z_1, Z_2) > t)}{\Pr(Z_1 > t, \min(Z_1, Z_2) > t)} \leq \frac{\Pr(Z_1 > x, Z_2 > t)}{\Pr(Z_1 > t, Z_2 > t)}
\]

so that $Z^\text{min}$ satisfies $\text{(iii)}$ due to $\text{(a)}$. For $x \leq 1$,

\[
\frac{\Pr(Z_1 > x, \min(Z_1, Z_2) > t)}{\Pr(Z_1 > t, \min(Z_1, Z_2) > t)} = 1
\]

so that $\text{(C)}$ holds for $Z^\text{min}$ as well.

(d) For $x > 0$,

\[
\frac{\Pr(\max(Z_1, Z_2) > x, Z_2 > t)}{\Pr(\max(Z_1, Z_2) > t, Z_2 > t)} \leq \frac{\Pr(Z_1 + Z_2 > x, Z_2 > t)}{\Pr(Z_2 > t)}
\]

\[
= \frac{\Pr(Z_1 + Z_2 > x, Z_2 > t)}{\Pr(Z_1 + Z_2 > t, Z_2 > t)}.
\]

Then $Z^\text{max}$ satisfies $\text{(C)}$ due to (b). Analogous arguments show that condition $\text{(C)}$ is satisfied for $\tilde{Z}^\text{max} = (Z_2, \max(Z_1, Z_2))$ as well.

\[\square\]

The following corollary is now easy to verify and provides a ready check for a model conforming to the premise of Theorem 4.3.

**Corollary 4.7.** Let $Z = Y + V$ be as in Model $A$. Suppose that $\mathbb{E}|Z_1| < \infty$ and $\alpha \leq \alpha_0 < 1 + \alpha$. Then the assumptions of Theorem 4.3 are satisfied.
4.2. Asymptotic behavior of MME and MES for copula models

Theorem 4.1 provides conditions under which we can compute the asymptotic behavior of MME and MES in an additive model which possesses hidden regular variation. A question to ask here is whether a similar result would hold for heavy-tailed multivariate distributions with dependence governed by certain copulas or survival copulas exhibiting asymptotic tail independence. It turns out that the answer is positive and we can provide a suitable generalization of Theorem 4.1 without necessarily assuming either HRV or a tail behavior for the distribution function $F_2$ of $Z_2$. The outcomes for MME and MES require mildly different conditions and hence, are stated separately.

4.2.1. Asymptotic behavior of MME for copula models

**Theorem 4.8.** Let $Z \in [0, \infty)^2$ with continuous margins $F_1, F_2$, survival copula $\hat{C}$, $\mathbb{E}|Z_1| < \infty$ and $\bar{F}_t \in \mathcal{RV}_{-\alpha}$ for some $\alpha \geq 1$. Suppose $C$ has upper tail order pair $(s, \tau)$ with $s \geq \tau \geq 1$ and some slowly varying function $\ell$ at 0 with $\lim_{s\to0} s^{\kappa-1} \ell(s) = 0$ satisfying \((7)\) and \((8)\). Also assume that

$$\lim_{M \to \infty} \lim_{t \to \infty} \int_M^{\infty} \frac{\hat{C}(x^{-\alpha}s, s')}{C(s, s')} \, dx = 0$$

holds. Moreover, suppose that $\lim_{t \to \infty} \frac{F_2(t)}{F_1(t)} = \eta \in (0, \infty)$. Then there exists a function $a(t) \in \mathcal{RV}_{-\alpha}$ and a constant $K \in (0, \infty)$ such that

$$\lim_{t \to \infty} a(1/p)\text{MME}(p) = K.$$

**Proof.** Due to Theorem 3.12 we have that $\mathcal{Z} \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha, b, \eta, \mathbb{E}) \cap \mathcal{H}\mathcal{R}\mathcal{V}(\alpha_0, b_0, \eta_0)$ with $\alpha_0 = \alpha b$. The only part we need to show here is that condition \((D)\) implies condition \((B)\) of Theorem 4.1(a). Then the stated result is a consequence of Theorem 4.1(a).

**Proof of \((D)\) implies \((B)\):** We need to show that

$$\lim_{M \to \infty} \lim_{t \to \infty} \int_M^{\infty} \frac{\Pr(Z_2 > xt, Z_2 > t)}{\Pr(Z_1 > t, Z_2 > t)} \, dx = \lim_{M \to \infty} \lim_{t \to \infty} \int_M^{\infty} \frac{\hat{C}(F_1(xt), F_2(t))}{C(F_1(t), F_2(t))} = 0. \tag{14}$$

For notational ease, without loss of generality we assume $\eta = 1$. Let $0 < \epsilon < 1$. Using Potter’s bound (see Resnick 1987 Proposition 2.6(ii)) there exists a $t_0 > 0$ such that for $t \geq t_0, x \geq 1$, we have

$$(1 - \epsilon)x^{-\alpha} \leq \frac{F_1(xt)}{F_1(t)} \leq (1 + \epsilon)x^{-\alpha} \quad \text{and} \quad (1 - \epsilon)^{-1} \geq \frac{F_2(t)}{F_1(t)} \leq (1 + \epsilon)^{-1}. \tag{15}$$

Hence,

$$\begin{align*}
\limsup_{t \to \infty} \int_M^{\infty} &\frac{\hat{C}(F_1(xt), F_2(t))}{C(F_1(t), F_2(t))} \, dx \\
&\leq \limsup_{t \to \infty} \int_M^{\infty} \frac{\hat{C}((1 + \epsilon)x^{-\alpha}F_1(t), (1 + \epsilon)^{-1}F_1(t))}{C(F_1(t), (1 - \epsilon)F_1(t))} \, dx \\
&= \limsup_{t \to \infty} \int_M^{\infty} \frac{\hat{C}((1 + \epsilon)x^{-\alpha}s, (1 + \epsilon)^{-1}s')} {C(s, (1 - \epsilon)s')} \, dx \\
&\leq \limsup_{t \to \infty} \int_M^{\infty} \frac{\hat{C}(x^{-\alpha}s, s')}{C(s, s')} \, dx \\
&\leq \frac{T(1, 1)}{T((1 + \epsilon)^{-1}, (1 + \epsilon)^{-1})} \limsup_{t \to \infty} \int_M^{\infty} \frac{\hat{C}(x^{-\alpha}s, s')}{C(s, s')} \, dx. \tag{16}
\end{align*}$$

Similarly, we have

$$\begin{align*}
\liminf_{t \to \infty} \int_M^{\infty} &\frac{\hat{C}(F_1(xt), F_2(t))}{C(F_1(t), F_2(t))} \, dx \\
&\geq \frac{T(1, 1)}{T((1 - \epsilon)^{-1}, (1 - \epsilon)^{-1})} \liminf_{t \to \infty} \int_M^{\infty} \frac{\hat{C}(x^{-\alpha}s, s')}{C(s, s')} \, dx. \tag{17}
\end{align*}$$
Since $T$ is strictly positive, using (13) along with (16) and (17), we can conclude that (14) holds. In fact, we can show in a similar fashion that (15) (or (14)) implies (10), too.

**Remark 4.9** Let $Z \in [0, \infty)^2$ with continuous margins $F_1, F_2$, $\mathbb{E}|Z_1| < \infty$ and $\bar{F}_1 \in \mathcal{RV}_{\alpha}$ for some $\alpha \geq 1$. Suppose that the survival copula $\hat{C}$ of $Z$ is either a Gaussian copula, a Marshall-Olkin copula or a Morgenstern copula as given in Example [3.9]. Then the assumptions of Theorem 4.8 hold.

### 4.2.2. Asymptotic behavior of MES for copula models

The next result complements as well as generalizes the results of Hua and Joe [24] where the asymptotic behavior of the MES was investigated for special copula families.

**Theorem 4.10.** Let $Z \in [0, \infty)^2$ with continuous margins $F_1, F_2$, survival copula $\hat{C}$, $\mathbb{E}|Z_1| < \infty$ and $\bar{F}_1 \in \mathcal{RV}_{\alpha}$ for some $\alpha \geq 1$. Suppose $\hat{C}$ has upper tail order pair $(\kappa, \tau)$ with $\kappa \geq \tau \geq 1$ and some slowly varying function $\ell$ at 0 with $\lim_{s \to 0} s^{\kappa-1} \ell(s) = 0$ satisfying (7) and (8). Moreover, assume that for some continuous distribution function $F_2$ with $\lim_{s \to 0} F_2(t)/F_1(t) = \eta \in (0, \infty)$ the asymptotic behavior

$$
\lim_{M \to \infty} \lim_{r \to 0^+} \int_0^{r^M} \int_0^{\infty} \hat{C}\left(\frac{F_1(t), F_2(t)}{F(t)}\right) \text{d}x = 0
$$

holds. Then $(\kappa - \tau) \alpha < 1$ and there exists a function $a(t) \in \mathcal{RV}_{\alpha \kappa \tau}$ and a constant $K \in (0, \infty)$ such that

$$
\lim_{p \to 0^+} a(1/p) \text{MES}(p) = K.
$$

**Proof.** We can assume w.l.o.g. that the tail of $Z_2$ is $\bar{F}_2$ (otherwise apply the monotone transformation $F_2^{\star \alpha} \circ F_2$ on $Z_2$ which does not change the MES and the copula). If the tail of $Z_2$ is $\bar{F}_2$ then the equivalence of (E) and (C) is easy to check. Thus, the conclusion for the asymptotic behavior of MES follows from Theorem 3.12 and Theorem 4.1(b). Finally, Remark 4.12(2) and $\bar{F}_2(t) \sim \eta \bar{F}_2(t) \in \mathcal{RV}_{\alpha \kappa \tau}$ implies $(\kappa - \tau) \alpha < 1$.

**Corollary 4.11.** Let $Z = (Z_1, Z_2) \in [0, \infty)^2$ with survival copula $\hat{C}$, continuous margins $F_1, F_2$, $\mathbb{E}|Z_1| < \infty$ and $\bar{F}_1(t) \sim K_1 t^{-\alpha}$ for some $\alpha > 1$ and constant $K_1 \in (0, \infty)$. Suppose $\hat{C}$ has upper tail order pair $(\kappa, \tau)$ with $\kappa \geq \tau \geq 1$ and some slowly varying function $\ell$ at 0 with $\lim_{s \to 0} s^{\kappa-1} \ell(s) = 0$ satisfying (7) and (8). Moreover,

$$
\lim_{M \to \infty} \lim_{s \to 0^+} \int_0^{r^M} \int_0^{\infty} \hat{C}\left(\frac{\kappa \alpha s, s^\tau}{C(s, s^\tau)}\right) \text{d}x = 0
$$

holds. Then $(\kappa - \tau) \alpha < 1$ and there exists a function $a(t) \in \mathcal{RV}_{\alpha \kappa \tau}$ and a constant $K \in (0, \infty)$ such that

$$
\lim_{p \to 0^+} a(1/p) \text{MES}(p) = K.
$$

**Proof.** Similar to the proof of Theorem 4.8 the proof here follows easily if we show that conditions (13) and (14) are equivalent. However, since Potter’s bounds hold only for $x \geq 1$ we require the additional assumption that the slowly varying part in the tail of $F_1$ behaves like a constant to obtain a similar bound as (15) for $0 < x \leq 1$. Then the result is a direct consequence of Theorem 4.10.

**Remark 4.12** A few observations from the above results are noted below.

(1) The result shows that the asymptotic behavior of the MES is determined only by the dependence structure and the tail behavior of $Z_1$; the tail behavior of $Z_2$ has no influence. Particularly, we see that HRV is not a necessary assumption.

(2) An analogous result for the MME does not hold, since a monotone transformation of $Z_2$ will in fact change the MME in contrast to the MES; the tail of $Z_2$ has an influence on the limit behavior of MME. Further, note that (13) is only an assumption on the upper tail dependence in contrast to (E) where the whole dependence structure plays a role as well.
A result similar to Corollary 4.11 under stronger assumptions has been discussed in Cai and Musta [8, Proposition 2.1]. Inter alia they assume the slowly varying function $\ell$ to be a constant, $x \mapsto T(x, 1)$ to be continuous and $\tau = 1$.

The copula examples in Example 3.9 only satisfy (D) but not (E) and hence, Corollary 4.11 cannot be applied. However, such examples are covered in Hua and Joe [24, Section 3.4] for either Pareto or Weibull-margins. In these examples the rate of increase of the MES is slower than in the asymptotic tail dependent case but faster than under condition (E).

The rest of this section is dedicated to construct examples of survival copulas that satisfy the assumptions of Theorem 4.10. The examples are created using the additive structure in Model A and Bernoulli mixture models as discussed in Hua et al. [23, Section 5] and Das and Fasen-Hartmann [13, Example 2]. First, we propose a result which we apply on the suggested models. Note that the models in the examples are not created using copulas apriori but we use the inherent copula structure governing the generation method in order to obtain the examples.

**Example 4.14**

A result similar to Corollary 4.11 under stronger assumptions has been discussed in Cai and Musta [8, Proposition 2.1]. Inter alia they assume the slowly varying function $\ell$ to be a constant, $x \mapsto T(x, 1)$ to be continuous and $\tau = 1$.

**Example 4.15**

The copula examples in Example 3.9 only satisfy (D) but not (E) and hence, Corollary 4.11 cannot be applied. However, such examples are covered in Hua and Joe [24, Section 3.4] for either Pareto or Weibull-margins. In these examples the rate of increase of the MES is slower than in the asymptotic tail dependent case but faster than under condition (E).

The rest of this section is dedicated to construct examples of survival copulas that satisfy the assumptions of Theorem 4.10. The examples are created using the additive structure in Model A and Bernoulli mixture models as discussed in Hua et al. [23, Section 5] and Das and Fasen-Hartmann [13, Example 2]. First, we propose a result which we apply on the suggested models. Note that the models in the examples are not created using copulas apriori but we use the inherent copula structure governing the generation method in order to obtain the examples.

**Proposition 4.13.** Let $Z \in \mathcal{MRV}(\alpha, b, v, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, v_0, \mathbb{E}_0)$ with continuous margins $F_1, F_2$, $\mathbb{E}|Z| < \infty$, $\lim_{t \to -\infty} F_1(t)/F_2(t) = 1$ for some $\alpha_0/\alpha \geq \tau \geq 1$ and suppose (E) holds. Denote by

\[
\hat{C}(u, v) = u + v - 1 + R(x, \mathbb{E} x)(1 - u), F_2^{-1}(1 - v))
\]

the survival copula of $Z$. Furthermore, let $Z^* = (Z_1^*, Z_2^*) \in [0, \infty)^2$ be a random vector with survival copula $\hat{C}$ and marginal distribution function $F_1$ of $Z_1^*$ and some continuous distribution function $F_2^*$ of $Z_2^*$. Then with $a(t) = b_0^\nu [\text{VaR}_{t-1,\nu}(Z_2)]/[\text{VaR}_{t-1,\nu}(Z_2)] \in \mathcal{RV}(\alpha_0, \alpha, \tau - 1)/\alpha$, we have

\[
\lim_{p \to 0} a(1/p)\mathbb{E}(Z_1|Z_2^*) = \text{VaR}_{1-p}(Z_2^*) = \int_0^\infty \mathbb{E}((x, \infty) \times (1, \infty))) dx
\]

where $0 < \int_0^\infty \mathbb{E}((x, \infty) \times (1, \infty))) dx < \infty$.

**Proof.** Using Theorem 3.10 the upper tail order function of $\hat{C}$ exists with upper tail order pair $(\kappa, \tau) = (\alpha_0/\alpha, \tau)$, i.e., $\hat{C}(s, s') \sim s' \ell(s)$ as $s \downarrow 0$. Further, $\lim_{q \to 0} s^{s-1}\ell(s) = 0$ due to Lemma 2.4 and Lemma 3.7. Moreover, (E) proved in Theorem 4.10 implies (E). Hence, the result is a consequence of Theorem 4.10.

**Example 4.14**

Let $Z = Y + V$ be as in Model A with continuous margins for $Y_1, Y_2, V_1, V_2$ and suppose $F_2 \in \mathcal{RV}_{\alpha-1}$ with $\alpha + \alpha' > \alpha_0$. Further, let $Z^* = (Z_1^*, Z_2^*)$ be defined as in Proposition 4.13. Then there exists a function $a(t) \in \mathcal{RV}(\alpha_0, \alpha, \tau - 1)/\alpha$ and a constant $K \in (0, \infty)$ such that

\[
\lim_{p \to 0} a(1/p)\mathbb{E}(Z_1|Z_2^*) = \text{VaR}_{1-p}(Z_2^*) = K.
\]

**Proof.** The conclusion is easy to see since by Theorem 3.3 and Theorem 4.3 we already know that $Z \in \mathcal{MRV}(\alpha, b, v, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, v_0, \mathbb{E}_0)$ and (E) holds. The rest is a consequence of Proposition 4.13.

Clearly, analogous results hold if $\hat{C}$ is the copula of the other examples of vectors defined in Theorem 4.13.

**Example 4.15** This model is motivated by the Bernoulli mixture type models discussed in Hua et al. [23, Section 5] and Das and Fasen-Hartmann [13, Example 2]. Suppose that $X_1, X_2, X_3$ are independent Pareto random variables with parameters $\alpha, \alpha_0$ and $\gamma$, respectively, where $1 < \alpha < \alpha_0 < \gamma, \alpha + \gamma > \alpha_0$. Let $B$ be a Bernoulli($q$) random variable with $q \in (0, 1)$. $R = (R_1, R_2)$ be a random vector with each margin defined on $[1, \infty)$ and $\mathbb{E}|R| = 0 < \infty$. We also assume $X_1, X_2, X_3, B, R$ are independent of each other. Now define

\[
Z = (Z_1, Z_2) = B(X_1, X_3) + (1 - B)(R_1, R_2),
\]

and let $Z^* = (Z_1^*, Z_2^*)$ be defined as in Proposition 4.13. Then

\[
\lim_{p \to 0} p\mathbb{E}(Z_1|Z_2^*) = \text{VaR}_{1-p}(Z_2^*) = (1 - q) \int_0^\infty \mathbb{E}((x, \infty) \times (1, \infty))) dx.
\]
Proof. We only need to verify that $Z \in \mathcal{MRV}(a, b, \nu, \mathbb{E}) \cap \mathcal{HRV}(a_0, b_0, \nu_0, \mathbb{E}_0)$ and that (C) is satisfied since the rest is a consequence of Proposition 4.13. However, it is easy to check (cf. the similar models in [25, Section 5] and [13, Example 2]) that $Z \in \mathcal{MRV}(a, b, \nu, \mathbb{E}) \cap \mathcal{HRV}(a_0, b_0, \nu_0, \mathbb{E}_0)$ with $b(t) = t^{1/\alpha}$, $\nu(dx, dy) = g(x^{-\alpha-1} dx \times e_0(dy)$, $b_0(t) = t^{1/\alpha_0}$ and $\nu_0((\times, \infty) \times (y, \infty)) = (1 - q) \mathbb{E}(\min(x^{-1}R_1, y^{-1}R_2)x^{-\alpha})$ for $x, y > 0$. Moreover, for $t, x \geq 1$, we have the inequality
\[
(1 - q)^{-\gamma_0} x^{-\alpha_0} \mathbb{E}(\min(R_1, R_2)^{\gamma_0}) \leq \Pr(Z_1 > xt, Z_2 > t) \leq (q + (1 - q) \mathbb{E}(R_1^{\gamma_0})) t^{-\alpha_0} x^{-\alpha}
\]
and for $0 < x \leq 1$,
\[
\Pr(Z_1 > xt, Z_2 > t) \leq \Pr(Z_2 > t) \leq (q + (1 - q) \mathbb{E}(R_2^{\gamma_0})) t^{-\alpha_0}.
\]
Thus, condition (C) is also satisfied. \qed

5. Conclusion

Our goal in this paper was to investigate certain conditional excess measures for bivariate models with asymptotic tail independence and heavy tails in the margins. We have been able to find asymptotic rates of convergence for the measures MES, MME as well as $\text{MES}^+, \text{MES}_{\text{min}}, \text{MES}_{\text{max}}$ for a variety of copula models, additive models and Bernoulli mixture models. We particularly note that the limit behavior of MES only depends on the tail of the survival copula and the tail behavior of the variable which is not-conditioned (denoted by $Z_1$ in most of our examples). The asymptotic behavior of MME involves further information on the copula as well as the tail of the conditioning variable ($Z_2$ in our examples). In addition we constructed a large class of hidden regularly varying models useful in the context of systemic risks which were not known or used hitherto up to our knowledge. Interesting extensions of our results to multivariate structures beyond $d = 2$ (see Hoffmann [19], Hoffmann et al. [20]) as well as graphical and network structures (see Kley et al. [30, 31]) are possible and are topics of future research.

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