Fermions in an External $SU(2)$ Magnetic Field

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ABSTRACT

We consider Fermions in a constant and uniform external $SU(2)$ magnetic field. We find that the results for the energy levels depend on the choice of gauge potential. Choosing a Landau type gauge potential yields his results. On the other hand in another gauge potential, one obtains a different continuous eigenvalue spectrum.
1 Introduction.

A constant and uniform magnetic field for a non-abelian gauge theory may be produced by at least two inequivalent gauge potentials. This choice is not available for the abelian case. As we shall see below, these two different situations lead to different results for the energy spectrum of a Fermion placed in such a field.

There has been much research done on closely related topics. In reference we cite only that work which impacts strongly on the material presented in this paper.

2 The Two Gauge Potentials.

We confine our attention to $SU(2)$. Let the external magnetic field be in the $z$ direction and in the isotopic 3 direction, i.e., $B_3^3$. It is constant and uniform. Now,

$$F_{12}^3 = \partial_1 A_2^3 - \partial_2 A_1^3 + ge^{abc} A_b^1 A_c^2$$

(1)

2.1 Gauge Potential I:

(let $B_3^3 = B$). One choice of the gauge potential can be $A_2^3 = Bx$ with all other $A_\mu^a = 0$. Then,

$$F_{12}^3 = \partial_1 A_2^3 = B.$$  

(2)

This is the well known Landau gauge potential but for a non-Abelian theory.

2.2 Gauge Potential II:

The second choice is $A_1^1 = \sqrt{\frac{B}{g}}$ and $A_2^3 = \sqrt{\frac{B}{g}}$ with all other $A_\mu^a = 0$. Then,

$$F_{12}^3 = ge^{312} \left( \sqrt{\frac{B}{g}} \right) \left( \sqrt{\frac{B}{g}} \right) = B$$  

(3)

We note that the magnetic field is the same for both gauge potentials.
3 The Two Sets of Solutions to the Dirac Equation.

Let the Fermion have isotopic and ordinary spin $\frac{1}{2}$. The Dirac equation is then given by,

$$\left(\alpha \cdot (p - g\frac{\tau_a}{2}A^a) + m\beta\right)\psi = E\psi$$

for both gauge potentials.

3.1 Choice I:

In this case Eq.(4) becomes,

$$\left(\alpha \cdot (p - g\frac{\tau_3}{2}B\varepsilon_y) + m\beta\right)\psi = E\psi$$

where $\varepsilon_y$ is a unit vector in the $y$ direction. Let $l = gB/2$ and

$$\psi = \Omega(x)\exp\{i(p_yy + p_zz)\}.$$  

Then $\Omega(x)$ satisfies,

$$\left(\alpha_x p_x - \tau_3\alpha_y l x + \alpha_y p_y + \alpha_z p_z + m\beta\right)\Omega = E\Omega$$

where $p_x$ is a q-number, while $p_y$ and $p_z$ are c-numbers.

Let $\tau_3\Omega = t\Omega$, where $t = \pm 1$. Then Eq.(7) becomes

$$\left(\alpha_x p_x - \alpha_y tl x + \alpha_y p_y + \alpha_z p_z + m\beta\right)\Omega = E\Omega.$$  

If we temporarily suppress the dependence on the iso-spinor, we may express $\Omega$ in terms of two component spinors. Thus,

$$\Omega = \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix}$$

Inserting Eq.(9) into (8) and eliminating $\chi$ yields the following equation for $\phi$,

$$\left(\sigma_x p_x - \sigma_y tl x + \sigma_y p_y + \sigma_z p_z\right)^2\phi = \left(E^2 - m^2\right)\phi.$$  

Here, the $\sigma_i$ are the usual Pauli matrices.
We square out the left hand side of Eq.(10) and let $\sigma_z \phi = s \phi$, where $s = \pm 1$, while using $t^2 = 1$, to obtain

$$\left(p_x^2 + t^2 x^2 + p_y^2 + p_z^2 - ts l - 2tl p_y x\right) \phi = \left(E^2 - m^2\right) \phi. \quad (11)$$

We now set

$$\phi = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}. \quad (12)$$

For $s = 1$,

$$u = \lambda h\omega_n \left(x - \frac{tp_y}{l}\right)$$

while

$$v = 0, \quad (13)$$

whereas, for $s = -1$, $u$ and $v$ change roles. $h\omega_n$ is the usual $n^{th}$ order solution to the harmonic oscillator problem with frequency $\omega = l/m = gB/2m$. $\lambda$ is the two component isotopic spinor satisfying $\tau_3 \lambda = t\lambda$, which we had previously suppressed. Hence, the energy spectrum is given by

$$E = \pm \sqrt{m^2 + p_x^2 + (2n + 1 - ts)\frac{gB^2}{2}} \quad (14)$$

where $n = 0, 1, 2 \cdots \infty$ as usual. These are the well known Landau levels, but for a non-abelian theory.

### 3.2 Choice II:

In this case Eq.(4) becomes

$$(\alpha \cdot p - h\alpha_x \tau_1 - h\alpha_y \tau_2 + m\beta) \psi = E\psi \quad (15)$$

where $h = \sqrt{gB}/2$. We may now set

$$\psi = u \exp(i \mathbf{p} \cdot \mathbf{x}) \quad (16)$$

so that all the components of $\mathbf{p}$ in Eq.(15) are c-numbers. Let
Here, $\phi$ and $\chi$ are two component isotopic spin vectors crossed into two component spinors to yield four component objects. Substituting Eq. (17) into (16) into (15) gives two coupled equations for $\phi$ and $\chi$.

\begin{align}
(\sigma \cdot p - h\sigma_x \tau_1 - h\sigma_y \tau_2) \phi &= (E + m) \chi \\
(\sigma \cdot p - h\sigma_x \tau_1 - h\sigma_y \tau_2) \chi &= (E - m) \phi
\end{align}

Eliminating $\chi$ between them and simplifying the result yields,

\begin{align}
(p^2 + 2h^2 - 2h p_x \tau_1 - 2h p_y \tau_2 - 2h^2 \sigma_z \tau_3) \phi &= (E^2 - m^2) \phi \quad (20) \\
\text{Let } \sigma_z \phi &= s\phi \text{ where } s = \pm 1. \text{ We now set } \\
\phi &= \begin{pmatrix} v \\ w \end{pmatrix} \quad (21)
\end{align}

where $v$ and $w$ are the isospin up and down components of $\phi$. We note that $\sigma_z v = sv$ and $\sigma_z w = sw$, with $s$ being the same for these two cases. Inserting Eq. (21) into (20) and eliminating $v$ we obtain

\begin{align}
\left[ \mathcal{E}^2 - p_x^2 - 2h^2(1 + s) \right] \left[ \mathcal{E}^2 - p_y^2 - 2h^2(1 - s) \right] w &= 4h^2(p_x^2 + p_y^2)w \quad (22)
\end{align}

where $\mathcal{E}^2 = E^2 - m^2$. We thus find that $\mathcal{E}$ satisfies

\begin{align}
(\mathcal{E}^2 - p_x^2)^2 - 4h^2\mathcal{E}^2 + 4h^2p_z^2 &= 0. \quad (23)
\end{align}

Solving for $E$ gives,

\begin{align}
E = \pm \sqrt{p_x^2 + m^2 + \frac{gB}{2} + \lambda \sqrt{gB \left( p_x^2 + p_y^2 + \frac{gB}{4} \right)}} \quad (24)
\end{align}

where $\lambda = \pm 1$. It is probably easiest to write the solution for $u$ using the Pauli notation. In this case we need three sets of $2 \times 2$ Pauli matrices. Thus, if $a, b, c$ and $d$ are two-component spinors, we use the notation
\[ \sigma_x \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \sigma_x a \\ \sigma_x b \\ \sigma_x c \\ \sigma_x d \end{pmatrix} \quad \rho_1 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \\ a \\ b \end{pmatrix} \quad \tau_1 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} \]

with corresponding results for the other matrices. Thus,

\[ u = N \left[ 1 + \rho_1 \left( \sigma \cdot \mathbf{p} - \frac{\sqrt{gB}}{2} [\sigma_x \tau_1 + \sigma_y \tau_2] \right) \right] \left[ 1 + \frac{\tau_1 (p_x + ip_y) \sqrt{gB}}{(gB^2 s - \lambda \sqrt{gB \left(p_x^2 + p_y^2 + \frac{gB}{4}\right)}} \right] \begin{pmatrix} u \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

(25)

where \( \sigma_z v = sv \) with \( v \) being a two component spinor.

### 4 Conclusions

For gauge potential I the energy spectrum is given by Eq.(14), while for gauge potential II it is given by Eq.(24). Thus, gauge potential I yields a discrete spectrum while gauge potential II yields a continuous spectrum for the motion in the \( x-y \) plane, even though the \( SU(2) \) magnetic field is the same in both cases. A partial check on these results is obtained by looking at the lowest energy levels in each case. Thus, in Eq.(14) we set \( n = 0 \) and \( ts = 1 \), while in Eq.(24) we set \( \lambda = -1 \) and \( p_x = p_y = 0 \). We then have

\[ E_I = E_{II} = \sqrt{p_z^2 + m^2} \quad (26) \]

If we go up to the next level we set \( n = 0 \) and \( ts = -1 \) in Eq.(14), while in Eq.(24) we set \( \lambda = +1 \) and \( p_x = p_y = 0 \). We then have

\[ E_I = E_{II} = \sqrt{p_z^2 + m^2 + gB} \quad (27) \]

However, the energy spectrum associated with the higher levels involving motion in the \( x-y \) plane are very different.
We do note that the $SU(2)$ currents giving rise to the external magnetic field in the two gauge potentials is also very different. We thus conclude that the motion of a Fermion in an $SU(2)$ external magnetic field depends upon the sources giving rise to the field rather than the field itself.

References

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