Inquiry of $P$-reduction in Cook’s 1971 Paper  
- from Oracle machine to Turing machine

JianMing ZHOU, Yu LI

(1) MIS, Université de Picardie Jules Verne, 33 rue Saint-Leu, 80090 Amiens, France
(2) Institut of Computational Theory and Application, Huazhong University of Science and Technology, Wuhan, China

Abstract

In this paper, we inquire the key concept $P$-reduction in Cook’s theorem and reveal that there exists the fallacy of definition in $P$-reduction caused by the disguised displacement of NDTM from Oracle machine to Turing machine. The definition or derivation of $P$-reduction is essentially equivalent to Turing’s computability. Whether NP problems might been reduced to logical forms (tautology or SAT) or NP problems might been reduced each other, they have not been really proven in Cook’s 1971 paper.

1 Introduction

NP problems and computational complexity theory are based on Cook’s theorem proposed by Cook in the paper entitled The Complexity of Theorem-Proving Procedures in 1971 [1].

Cook’s theorem claims that any problem decidable by NDTM (Non-Deterministic Turing Machine) can be reduced to the SAT (SATisfiability) problem in polynomial time by TM (Turing Machine).

From Cook’s theorem, on the one hand, it produces out two popular definitions of NP (Non-deterministic Polynomial time) that are considered to be equivalent:

**Definition 1** NP is the class of problems decidable (or solvable) by NDTM in polynomial time.

**Definition 2** NP is the class of problems verifiable by TM in polynomial time.

On the other hand, it deduces out the NP-completeness: if there is a polynomial time algorithm to solve the SAT problem, then each problem in NP can be solved by a polynomial time algorithm. The question of whether such a polynomial time algorithm exists for solving a NP problem is expressed as the $P$ vs $NP$ problem that is widely considered as the most important open problem in computer science, also selected as one of the seven millennial challenges by the Clay Mathematics Institute in 2001 [2].

However, people who come across the theory of NP-completeness would have more or less such impressions, although the definition of NP seems simple and clear, but if one slightly investigates

Email address: yu.li@u-picardie.fr (JianMing ZHOU, Yu LI).
it, one would feel some confusion that is difficult to express out [3]. Scott Aaroson who perennially works on $P \text{ vs } NP$ expressed his perplexity: There seems to be an “invisible electric fence” that separates the $P$ problem from the $NP$ complete problem [4][5]. Moreover, if one wants to go deeper into $NP$ completeness theory, one would have to go back to study or think this theory again and again, finally one is either disappointed or obligated himself to obey the authoritative explications. On the other hand, $NP$ completeness theory is almost completely out of touch with the practical resolution of $NP$ problems, some people even say that this theory can be abandoned. In fact, the content of $NP$ completeness theory is getting lighter and lighter in basic computer theory courses in universities. These situations prevent computer theory from further developing. In fact, the $P \text{ vs } NP$ problem has implicitly become the gap between the computer theory and the booming artificial intelligence theory.

We have made some preliminary interpretations of several issues related to $Cook’s theorem$ from different perspectives [6][7][8][9]. In this paper, we inquire the key concept $P$-reduction of $Cook’s theorem$ and reveal the cognitive and theoretical errors in $Cook’s theorem$. We hope to attract the attention of the academic community.

2 Overview of $Cook’s theorem$

The original statement of $Cook’s theorem$ was presented in Cook’s 1971 paper [1]:

**Theorem 1** If a set $S$ of strings is accepted by some nondeterministic Turing machine within polynomial time, then $S$ is $P$-reducible to $\{DNF \text{ tautologies}\}$.

The main idea of the proof of Theorem 1 was described in [1]:

*Proof of the theorem. Suppose a nondeterministic Turing machine $M$ accepts a set $S$ of strings within time $Q(n)$, where $Q(n)$ is a polynomial. Given an input $w$ for $M$, we will construct a propositional formula $A(w)$ in conjunctive normal form (CNF) such that $A(w)$ is satisfiable iff $M$ accepts $w$. Thus $\neg A(w)$ is easily put in disjunctive normal form (using De Morgan’s laws), and $\neg A(w)$ is a tautology if and only if $w \notin S$. Since the whole construction can be carried out in time bounded by a polynomial in $|w|$ (the length of $w$), the theorem will be proved.*

As an intuitive interpretation, Cook attempted to prove that any problem solvable by a nondeterministic Turing machine at level of cognition is $P$-reducible to the problem of determining whether a given propositional formula is satisfiable at level of computation.

3 $P$-reduction and Turing reduction: disguised displacement of $NDTM$

$P$-reduction is defined as follows [1]:

**Definition.** A set $S$ of strings is $P$-reducible ($P$ for polynomial) to a set $T$ of strings iff there is some query machine $M$ and a polynomial $Q(n)$ such that for each input string $w$, the $T$-computation of $M$ with input $w$ halts within $Q(|w|)$ steps ($|w|$ is the length of $w$) and ends in an accepting state iff $w \in S$.

About the query machine, Cook said [1]:

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In order to make this notion precise, we introduce query machines, which are like Turing machines with oracles in [1].

A query machine is a multitape Turing machine with a distinguished tape called the query tape, and three distinguished states called the query state, yes state, and no state, respectively. If \( M \) is a query machine and \( T \) is a set of strings, then a \( T \)-computation of \( M \) is a computation of \( M \) in which initially \( M \) is in the initial state and has an input string \( w \) on its input tape, and each time \( M \) assures the query state there is a string \( u \) on the query tape, and the next state \( M \) assumes is the yes state if \( u \in T \) and the no state if \( u \notin T \). We think of an 'oracle', which knows \( T \), placing \( M \) in the yes state or no state.

It can be seen that the NDTM in Cook's theorem refers to the query machine, and the query machine refers to the Oracle Machine. Thus, \( NP \) is decidable by NDTM, actually that is to say, \( NP \) is decidable by the Oracle Machine (Definition 1), and \( P \)-reduction refers to Turing reduction based on Oracle Machine (Figure 1).

Concerning Turing reduction, Martin Davis said in his paper entitled what is Turing Reducibility? [11]:

The concept of Turing reducibility has to do with the question: can one non-computable set be more non-computable than another? In a rather incidental aside to the main topic of Alan Turing’s doctoral dissertation (the subject of Soloman Fefermans article in this issue of the Notices), he introduced the idea of a computation with respect to an oracle. An oracle for a particular set of natural numbers may be visualized as a black box that will correctly answer questions about whether specific numbers belong to that set. We can then imagine an oracle algorithm whose operations can be interrupted to query such an oracle with its further progress dependent on the reply obtained. Then for sets \( A, B \) of natural numbers, \( A \) is said to be Turing Reducible to \( B \) if there is an oracle algorithm for testing membership in \( A \) having full recourse to an oracle for \( B \). The notation used is: \( A \leq_t B \). Of course, if \( B \) is itself a computable set, then nothing new happens; in such a case \( A \leq_t B \) just means that \( A \) is computable. But if \( B \) is non-computable, then interesting things happen.

According to Martin Davis’s understanding, Turing reduction can be defined as follows: a set \( A \) (in number theory) is reduced to a set \( B \) (in number theory), if \( B \) can be decided by some oracle (black box), then \( A \) can be decided. However, some sets in number theory themself are undecidable, and only an oracle can decide them, thus Martin Davis said, if \( B \) is itself a
computable set, then nothing new happens, that is, Turing reduction is unconsciously considered to deal with uncomputable problems.

Therefore, Oracle should be understood as an omniscient logic or an omnipotent algorithm. In this sense, Oracle Machine is not a constructive model of machine, but an imaginary theoretical model, as Turing said [12], *We shall not go any further into the nature of this oracle apart from saying that it cannot be a machine.* In other words, Oracle Machine is NTM (Non Turing Machine) as opposed to TM (Turing Machine).

From the perspective of the relativity to computability, the introduction of NP by Oracle Machine to study its undecidability is meaningful. Unfortunately, Cook completely misunderstood Turing's original idea about Oracle Machine, and tried to use Oracle Machine as an actual machine in his constructive proof. However, the Oracle Machine is just a theoretical model, so it is impossible to actually represent the solution to a problem given by Oracle, that is, the string $u$ (Figure 1) in the *query machine*, so Cook had to secretly throw away Oracle before he began his proof:

We may as well assume the Turing machine $M$ has only one tape, which is infinite to the right but has a left-most square. Let us number the squares from left to right 1, 2, ..., L et us fix an input $w$ to $M$ of length $n$, and suppose $w \in S$. Then there is a computation of $M$ with input $w$ that ends in an accepting state within $T = Q(n)$ steps.

In other words, Cook used an invisible hand to directly incorporate $u$ obtained by Oracle from the query tape onto the tape of Turing Machine without any justification, so that the *query machine* is directly transformed into *Turing Machine* from Oracle Machine (Figure 2):

![Figure 2. Query Machine transformed to TM](image)

Later scholars maybe thought that such disguised displacement of NDTM was not appropriate. In order to save Cook’s theorem, a guessing module was used to replace the Oracle in the query machine (Figure 3) [13][14]:

This NDTM with guessing module is just the current NDTM [15]:

At any point in a computation the machine may proceed according to several possibilities. The computation of a nondeterministic Turing machine is a tree whose branches correspond to different possibilities for the machine. If some branch of the computation leads to the accept state, the machine accepts its input.

And this NDTM is in essence TM [15]:

Every nondeterministic Turing machine has an equivalent deterministic Turing machine. (Theorem 3.16 in [9])

In this way, the solution $u$ to a problem $w$ that should been given by the Oracle of the query
machine becomes a guess solution $u$ (certificate) given by the guessing module $TM'$ and then $u$ is verified by an other Turing machine $TM$, which leads to the second popular definition of $NP$: $NP$ is polynomial time verifiable (Definition 2).

So far, Cook’s theorem completed the disguised displacement of $NDTM$ from Oracle machine to Turing Machine, and caused the confusion between the two mutually exclusive concepts $NTM$ and $TM$! Henceforth, the undecidable problems ($NP$) expressed by Oracle Machine became decidable problems ($P$) expressed by Turing Machine, which leads to the loss of nondeterminism from $NP$.

$P$-reduction can no longer refer to Turing reduction, then Cook’s theorem has lost any support in mathematical and logical sense.

4 Conclusion

Our analysis shows that there exists the fallacy of definition in $P$-reduction of Cook’s theorem, caused by the disguised displacement of $NDTM$ from Oracle machine to Turing machine. Consequently Cook’s theorem declares that $NP$ problems can be reduced to logical forms (tautology or $SAT$) through $TM$, which is equivalent to affirming the computability of $NP$ problems, thus the loss of nondeterminism from $NP$. A brief analysis about the logical form of $A(w)$, a key of the proof of Cook’s theorem, can be found in [10].

The impact of Cook’s theorem is enormous, it concerns $NP$, $NP$-completeness, and the relationship between computability theory and computational complexity theory, so that $P$ vs $NP$ has become one of millennial challenges. This issue is not only an academic theoretical one, but also a historical event with great influence in modern academic history.

In introducing the second poll about $P$ vs $NP$ conducted by Gasarch in 2012, Hemaspaandra said [16]:

*I hope that people in the distant future will look at these four articles to help get a sense of peoples thoughts back in the dark ages when $P$ versus $NP$ had not yet been resolved.*

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