Rényi generalizations of the conditional quantum mutual information

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Abstract

The conditional quantum mutual information $I(A;B|C)$ of a tripartite state $\rho_{ABC}$ is an information quantity which lies at the center of many problems in quantum information theory. Three of its main properties are that it is non-negative for any tripartite state, that it decreases under local operations applied to systems $A$ and $B$, and that it obeys the duality relation $I(A;B|C) = I(A;B|D)$ for a four-party pure state on systems $ABCD$. The conditional mutual information also underlies the squashed entanglement, an entanglement measure that satisfies all of the axioms desired for an entanglement measure. As such, it has been an open question to find Rényi generalizations of the conditional mutual information, that would allow for a deeper understanding of the original quantity and find applications beyond the traditional memoryless setting of quantum information theory. The present paper addresses this question, by defining different $\alpha$-Rényi generalizations $I_\alpha(A;B|C)$ of the conditional mutual information that all converge to the conditional mutual information in the limit $\alpha \to 1$. Furthermore, we prove that many of these generalizations satisfy the aforementioned properties. As such, the quantities defined here should find applications in quantum information theory and perhaps even in other areas of physics, but we leave this for future work. Finally, we state a conjecture regarding the monotonicity of the Rényi conditional mutual informations defined here with respect to the Rényi parameter $\alpha$. We prove that this conjecture is true in some special cases and provide evidence that it is true when $\alpha$ is in a neighborhood of one.

1 Introduction

How much correlation do two parties have from the perspective of a third? This kind of correlation is what the conditional quantum mutual information quantifies \cite{16,62}. Indeed, let $\rho_{ABC}$ be a density operator corresponding to a quantum state shared between three parties, say, Alice, Bob, and Charlie. Then the conditional quantum mutual information is defined as

$$I(A;B|C)_{\rho} \equiv H(AC)_{\rho} + H(BC)_{\rho} - H(C)_{\rho} - H(ABC)_{\rho},$$

where $H(F)_{\sigma} \equiv -\text{Tr}\{\sigma_F \log \sigma_F\}$ is the von Neumann entropy of a state $\sigma_F$ on system $F$ and we unambiguously let $\rho_C \equiv \text{Tr}_{AB} \{\rho_{ABC}\}$ denote the reduced density operator on system $C$, for

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example. Devetak and Yard provided a compelling operational interpretation of the conditional quantum mutual information in terms of the quantum state redistribution protocol [16, 62]: given a four-party pure state $\psi_{ADBC}$, with a sender possessing systems $D$ and $B$ and a receiver possessing system $C$, the optimal rate of quantum communication necessary to transfer the system $B$ to the receiver is given by $\frac{1}{2} I(A;B|C)_{\psi}$.

It is a nontrivial fact, known as strong subadditivity of quantum entropy [37, 38], that the conditional quantum mutual information of any tripartite quantum state is non-negative. This can be viewed as a general constraint imposed on the marginal entropy values of tripartite quantum states, of which there are many more for general many-body states [49, 40, 7]. Strong subadditivity also implies that the conditional mutual information can only decrease under local quantum operations performed on the systems $A$ and $B$ [10], so that $I(A;B|C)_{\rho}$ is a sensible measure of the correlations present between systems $A$ and $B$, from the perspective of $C$. That is, the following inequality holds

$$I(A;B|C)_{\rho} \geq I(A;B|C)_{\omega}, \quad (1.2)$$

where $\omega_{ABC} \equiv (N_{A} \otimes M_{B}) (\rho_{ABC})$ with $N_{A}$ and $M_{B}$ arbitrary local quantum operations performed on the systems $A$ and $B$, respectively. Inequalities like these are extremely useful in applications, with nearly all coding theorems in quantum information theory invoking the strong subadditivity inequality in their proofs.

One of the most fruitful avenues of research in quantum information theory has been the program of generalizing entropies beyond those that are linear combinations of the von Neumann entropy [47, 50, 13, 56, 43, 60, 17]. Not only is this interesting from a theoretical perspective, but more importantly, these generalizations have found application in operational settings in which there is no assumption of an independent and identically distributed (i.i.d.) resource, so that the law of large numbers does not come into play. In particular, the family of Rényi entropies has proved to possess a wide variety of applications in these non-i.i.d. settings. More recently, researchers have shown that nearly all of the known information quantities being employed in the non-i.i.d. setting are special cases of a Rényi family of quantum entropies [33, 3].

However, in spite of this aforementioned progress, it has been a vexing open question to determine a Rényi generalization of the conditional quantum mutual information that can be useful in applications. One naive Rényi generalization of the conditional mutual information of a tripartite state $\rho_{ABC}$ is simply to take a linear combination of Rényi entropies. E.g., in analogy with the definition in (1.1),

$$I'_{\alpha}(A;B|C)_{\rho} \equiv H_{\alpha}(AC)_{\rho} + H_{\alpha}(BC)_{\rho} - H_{\alpha}(C)_{\rho} - H_{\alpha}(ABC)_{\rho}, \quad (1.3)$$

where $H_{\alpha}(F)_{\sigma} \equiv [1 - \alpha]^{-1} \log \text{Tr}\{\sigma_{F}^{\alpha}\}$ is the Rényi entropy of a state $\sigma_{F}$ on system $F$, with parameter $\alpha \in [0, \infty]$ (with the Rényi entropy being defined for $\alpha \in \{0, 1, \infty\}$ in the limit as $\alpha$ approaches 0, 1, and $\infty$, respectively). Although this quantity is non-negative in some very special cases [1], in general, $I'_{\alpha}(A;B|C)_{\rho}$ can be negative, and in fact there are some simple examples of states for which this occurs. Furthermore, the results of [39] imply that there are generally no constraints on the marginal Rényi entropies of a multiparty quantum state other than non-negativity when $\alpha \in (0, 1)$ and that there are no linear inequality constraints when $\alpha \in (1, \infty)$. This implies that monotonicity under local quantum operations generally does not hold for $I'_{\alpha}(A;B|C)_{\rho}$, and [39] provides many examples of four-party states $\rho_{ABCD}$ such that $I'_{\alpha}(A;BD|C)_{\rho} < I'_{\alpha}(A;B|C)_{\rho}$. For these reasons, we feel that formulas like in (1.3) should not be considered as Rényi generalizations of the conditional quantum mutual information, given that non-negativity and monotonicity under
local operations are two of the basic properties of the conditional quantum mutual information which are consistently employed in applications\footnote{However, one could certainly argue that the case $\alpha = 2$ is useful for the class of Gaussian quantum states, as done in \cite{1}.}

On the other hand, the standard approach for generalizing information quantities such as entropy, conditional entropy, and mutual information beyond the von Neumann setting begins with the realization that these quantities can be written in terms of the relative entropy $D(\rho||\sigma)$,

$$H(A)_{\rho} = - D(\rho_A||I_A),$$

$$H(A|B)_{\rho} \equiv H(AB)_{\rho} - H(B)_{\rho} = - \min_{\sigma_B} D(\rho_{AB}||I_A \otimes \sigma_B),$$

$$I(A;B)_{\rho} \equiv H(A)_{\rho} + H(B)_{\rho} - H(AB)_{\rho} = \min_{\sigma_B} D(\rho_{AB}||\rho_A \otimes \sigma_B),$$

where

$$D(\rho||\sigma) \equiv \left\{ \begin{array}{ll}
[\text{Tr}\{\rho\}^{-1} \text{Tr}\{\rho \log \rho\} - \text{Tr}\{\rho \log \sigma\}] & \text{if supp}(\rho) \subseteq \text{supp}(\sigma) \\
+\infty & \text{otherwise}
\end{array} \right..$$

With the Rényi relative entropy of order $\alpha$ defined as

$$D_\alpha(\rho||\sigma) \equiv \left\{ \begin{array}{ll}
\frac{1}{\alpha-1} \log \text{Tr}\{[\text{Tr}\{\rho\}]^{-1}\rho^\alpha \sigma^{1-\alpha}\} & \text{if supp}(\rho) \subseteq \text{supp}(\sigma) \text{ or } \alpha \in (0,1) \\
+\infty & \text{otherwise}
\end{array} \right.,$$

one can easily define Rényi generalizations of entropy, conditional entropy, and mutual information in analogy with the above formulations,

$$H_\alpha(A)_{\rho} = - D_\alpha(\rho_A||I_A),$$

$$H_\alpha(A|B)_{\rho} \equiv - \min_{\sigma_B} D_\alpha(\rho_{AB}||I_A \otimes \sigma_B),$$

$$I_\alpha(A;B)_{\rho} \equiv \min_{\sigma_B} D_\alpha(\rho_{AB}||\rho_A \otimes \sigma_B).$$

Since the Rényi relative entropy obeys monotonicity under quantum operations for $\alpha \in [0,2]$, in the sense that $D_\alpha(\rho||\sigma) \geq D_\alpha(N(\rho)||N(\sigma))$ for a quantum operation $N$, the above generalizations have proven useful in several applications (see \cite{34, 41, 56} and references therein).

### 2 Overview of results

The main purpose of the present paper is to develop Rényi generalizations of the conditional quantum mutual information that satisfy the aforementioned properties of non-negativity and monotonicity under local quantum operations. As a result, the quantities we define here should be useful in applications in quantum information theory, and other areas of physics \cite{26, 28, 25, 30, 33, 23, 32}.

After establishing some notation and recalling definitions in the next section, our starting point is in Section 4 with the realization that the conditional quantum mutual information of a tripartite state $\rho_{ABC}$ can be written in terms of the relative entropy as follows (see, e.g., page 55 of \cite{48}),

$$I(A;B|C)_{\rho} = D(\rho_{ABC}||\exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_{C}\}).$$

(2.1)
By recalling the Lie-Trotter product formula
\[ \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_{C} \} = \lim_{\alpha \to 1} \left[ \rho_{AC}^{(1-\alpha)/2} \rho_{BC}^{\alpha-1} \rho_{C}^{(1-\alpha)/2} \rho_{AC}^{1-\alpha} \right]^{1/(1-\alpha)}, \quad (2.2) \]
Section 5 then establishes a number of Rényi generalizations of the relative entropy formulation in (2.1), one of which is
\[ D_{\alpha} \left( \rho_{ABC} \| \left[ \rho_{AC}^{(1-\alpha)/2} \rho_{BC}^{\alpha-1} \rho_{C}^{(1-\alpha)/2} \rho_{AC}^{1-\alpha} \right]^{1/(1-\alpha)} \right). \quad (2.3) \]
We prove that several of these Rényi conditional mutual informations are non-negative for \( \alpha \in [0, 2] \) and obey monotonicity under local quantum operations on systems A and B in the same range of \( \alpha \) (with the proof following from the Lieb concavity theorem [36] and the Ando convexity theorem [2]). We also consider several limiting cases, the most important of which is that the \( \alpha \)-Rényi conditional mutual informations converge to \( I(A; B|C)_\rho \) in the limit as \( \alpha \to 1 \). Note that classical and even quantum quantities related to these have been explored in prior work [4, 19].

The sandwiched Rényi relative entropy [43, 60] is another variant of the Rényi relative entropy which has found a number of applications recently in the context of strong converse theorems [60, 12, 22, 12]. It is defined as follows,
\[ \tilde{D}_{\alpha}(\rho|\sigma) \equiv \begin{cases} \frac{1}{\alpha-1} \log \left[ \frac{1}{\text{Tr} \{ \rho \}} \text{Tr} \left\{ (\sigma^{(1-\alpha)/2} \rho \sigma^{(1-\alpha)/2})^\alpha \right\} \right] & \text{if supp} (\rho) \subseteq \text{supp} (\sigma) \\ +\infty & \text{or/and } \alpha \in (0, 1) \\ \text{otherwise} \end{cases}. \quad (2.4) \]
In Section 6, we use this sandwiched Rényi relative entropy to establish a number of sandwiched Rényi generalizations of the conditional mutual information, one of which is
\[ \tilde{D}_{\alpha} \left( \rho_{ABC} \| \left[ \rho_{AC}^{(1-\alpha)/2} \rho_{BC}^{\alpha-1} \rho_{C}^{(1-\alpha)/2} \rho_{AC}^{1-\alpha} \right]^{\alpha/(1-\alpha)} \right). \quad (2.5) \]
We prove that several of these sandwiched Rényi conditional mutual informations are non-negative for all \( \alpha \in [1/2, \infty] \) and that they are monotone under local quantum operations on systems A and B for the same range of \( \alpha \) (with the proof following from recent work of Frank and Lieb [20]). They also converge to \( I(A; B|C)_\rho \) in the limit as \( \alpha \to 1 \), and there are other interesting limits such as when \( \alpha \to 1/2 \) or \( \alpha \to \infty \).

One of the most curious properties of the conditional mutual information is that it obeys a duality relation [16, 62]. That is, for a four-party pure state \( \psi_{ABCD} \), the following equality holds
\[ I(A; B|C)_\psi = I(A; B|D)_\psi. \quad (2.6) \]
In Section 7 we prove that some variants of the Rényi conditional mutual information obey duality relations analogous to the above one (and furthermore, that the relations reduce to the above one in the limit as \( \alpha \to 1 \)).

A well known property of both the traditional and the sandwiched Rényi relative entropy is that they are monotone increasing in \( \alpha \). That is, for \( 0 \leq \alpha \leq \beta \), we have the following inequalities
\[ D_{\alpha}(\rho|\sigma) \leq D_{\beta}(\rho|\sigma), \quad (2.7) \]
\[ \tilde{D}_{\alpha}(\rho|\sigma) \leq \tilde{D}_{\beta}(\rho|\sigma). \quad (2.8) \]
Section \textsection{8} states an open conjecture, that the Rényi generalizations of the conditional mutual information obey a similar monotonicity. We prove that this conjecture is true in some special cases, provide evidence that it is true when \( \alpha \) is in a neighborhood of one, and numerical evidence indicates that it is true in general.

In Section \textsection{9} we detail some dimension bounds on the Rényi conditional mutual informations and their relations to other Rényi information quantities. Section \textsection{10} discusses some quantities derived from the Rényi conditional mutual informations and generalizations of it to parties beyond three. We finally conclude in Section \textsection{11} with a summary of our results and a discussion of directions for future research.

### 3 Notation and definitions

#### Norms, states, channels, and measurements.

Let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded linear operators acting on a Hilbert space \( \mathcal{H} \). We restrict ourselves to finite-dimensional Hilbert spaces throughout this paper. The \( \alpha \)-norm of an operator \( X \) is defined as

\[
\|X\|_\alpha \equiv \text{Tr}\{ (\sqrt{X^\dagger X})^\alpha \}^{1/\alpha}.
\] (3.1)

Let \( \mathcal{B}(\mathcal{H})_+ \) denote the subset of positive semi-definite operators (we often simply say that an operator is “positive” if it is positive semi-definite). We also write \( X \geq 0 \) if \( X \in \mathcal{B}(\mathcal{H})_+ \). An operator \( \rho \) is in the set \( \mathcal{S}(\mathcal{H}) \) of density operators if \( \rho \in \mathcal{B}(\mathcal{H})_+ \) and \( \text{Tr}\{\rho\} = 1 \). The tensor product of two Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) is denoted by \( \mathcal{H}_A \otimes \mathcal{H}_B \). Given a multipartite density operator \( \rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B \), we unambiguously write \( \rho_A = \text{Tr}_B \{\rho_{AB}\} \) for the reduced density operator on system \( A \). A linear map \( \mathcal{N}_{A\rightarrow B} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B) \) is positive if \( \mathcal{N}_{A\rightarrow B}(\sigma_A) \in \mathcal{B}(\mathcal{H}_B)_+ \) whenever \( \sigma_A \in \mathcal{B}(\mathcal{H}_A)_+ \). Let \( \text{id}_A \) denote the identity map acting on a system \( A \). A linear map \( \mathcal{N}_{A\rightarrow B} \) is completely positive if the map \( \text{id}_R \otimes \mathcal{N}_{A\rightarrow B} \) is positive for a reference system \( R \) of arbitrary size. A linear map \( \mathcal{N}_{A\rightarrow B} \) is trace-preserving if \( \text{Tr}\{\mathcal{N}_{A\rightarrow B}(\tau_A)\} = \text{Tr}\{\tau_A\} \) for all input operators \( \tau_A \in \mathcal{B}(\mathcal{H}_A) \). If a linear map is completely positive and trace-preserving (CPTP), we say that it is a quantum channel or quantum operation. A positive operator-valued measure (POVM) is a set \( \{\Lambda^n\} \) of positive operators such that \( \sum_n \Lambda^n = I \). The POVM formalism gives the most general way of expressing a quantum measurement mathematically if one is only concerned with the classical outcomes of the measurement and not with the post-measurement quantum state.

#### Relative entropies.

We defined the relative entropy between two positive operators \( P \) and \( Q \) in \( \text{(1.8)} \). This quantity is non-negative if \( \text{Tr}\{P\} \geq \text{Tr}\{Q\} \), a result known as Klein’s inequality. Thus, for density operators \( \rho \) and \( \sigma \), the relative entropy is non-negative, and furthermore, it is equal to zero if and only if \( \rho = \sigma \).

We defined the Rényi relative entropy in \( \text{(1.8)} \). This quantity obeys the following monotonicity inequality for all \( \alpha \in [0,2] \),

\[
D_\alpha(P\parallel Q) \geq D_\alpha(\mathcal{N}(P)\parallel \mathcal{N}(Q)) ,
\] (3.2)

where \( P \) and \( Q \) are positive operators and \( \mathcal{N} \) is a CPTP map. Thus, by applying this, we find that \( D_\alpha(P\parallel Q) \) is non-negative for all \( \alpha \in [0,2] \) whenever \( \text{Tr}\{P\} \geq \text{Tr}\{Q\} \), so that it is always non-negative for density operators \( \rho \) and \( \sigma \). Furthermore, it is equal to zero if and only if \( \rho = \sigma \).

We also defined the sandwiched Rényi relative entropy in \( \text{(2.3)} \). Whenever \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \)
or \( \alpha \in (0, 1) \), it admits the following alternate forms,

\[
\tilde{D}_\alpha (\rho || \sigma) = \frac{1}{\alpha - 1} \log \left[ \text{Tr} \{ \rho \}^{-1} \text{Tr} \left\{ \left( \sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha} \right)^\alpha \right\} \right]
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \left( \sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha} \right)^\alpha \right\} - \frac{1}{\alpha - 1} \log \text{Tr} \{ \rho \}.
\]

\[
= \frac{\alpha}{\alpha - 1} \log \left\| \sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha} \right\|_\alpha \left( 1 - \frac{1}{\alpha - 1} \log \text{Tr} \{ \rho \} \right).
\]

It obeys the following monotonicity inequality for all \( \alpha \in [1/2, \infty] \),

\[
\tilde{D}_\alpha (P || Q) \geq \tilde{D}_\alpha (\mathcal{N} (P) || \mathcal{N} (Q)),
\]

where \( P \) and \( Q \) are positive operators and \( \mathcal{N} \) is a CPTP map \([20]\) (see also \([5, 60, 43]\) for other proofs of this for more limited ranges of \( \alpha \)). Thus, by applying this, we find that \( D_\alpha (P || Q) \) is non-negative for all \( \alpha \in [1/2, \infty) \) whenever \( \text{Tr} \{ P \} \geq \text{Tr} \{ Q \} \), so that it is always non-negative for density operators \( \rho \) and \( \sigma \). Furthermore, it is equal to zero if and only if \( \rho = \sigma \).

**Quantum Markov states.** An important class of tripartite quantum states are the quantum Markov states \([24]\). A state \( \rho_{ABC} \) is defined to be a quantum Markov state if \( I (A; B | C)_\rho = 0 \). Let \( \mathcal{M}_{A-C-B} \) denote this class of states. The main result of \([24]\) is that such a state has the following explicit form,

\[
\rho_{ABC} = \bigoplus_j q (j) \sigma_{AC_j^L} \otimes \sigma_{C_j^R B},
\]

for some probability distribution \( q (j) \), density operators \( \{ \sigma_{AC_j^L}, \sigma_{C_j^R B} \} \), and a decomposition of the Hilbert space for \( C \) as

\[
\mathcal{H}_C = \bigoplus_j \mathcal{H}_{C_j^L} \otimes \mathcal{H}_{C_j^R}.
\]

We also know that a state \( \rho_{ABC} \) is a quantum Markov state if any of the following conditions hold \([51, 35, 63]\),

\[
\rho_{ABC} = \rho_{AC}^{1/2} \rho_C^{-1/2} \rho_{BC} \rho_C^{-1/2} \rho_{AC}^{1/2},
\]

\[
\rho_{ABC} = \rho_{BC}^{1/2} \rho_C^{-1/2} \rho_{AC} \rho_C^{-1/2} \rho_{BC}^{1/2},
\]

\[
\rho_{ABC} = \exp \left\{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \right\}.
\]

Now, if we take an arbitrary state \( \sigma_{ABC} \) and compute the operator

\[
\exp \left\{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \right\},
\]

we are not guaranteed that \( \exp \left\{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \right\} \) is a quantum Markov state (or even a normalized state for that matter), but we do know that

\[
\text{Tr} \left\{ \exp \left\{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \right\} \right\} \leq 1,
\]

following from a triple matrix generalization of the Golden-Thompson inequality \([36]\). Similarly, if we compute

\[
\sigma_{AC}^{1/2} \sigma_C^{-1/2} \sigma_{BC} \sigma_C^{-1/2} \sigma_{AC}^{1/2},
\]
we are not guaranteed that this is a quantum Markov state [64]. However, in this case, we can conclude that

$$\text{Tr}\left\{ \sigma_{AC}^{1/2} \sigma_{C}^{-1/2} \sigma_{BC}^{1/2} \sigma_{C}^{-1/2} \sigma_{AC}^{1/2} \right\} = 1,$$

(3.16)

and we have similar conclusions for $\sigma_{BC}^{1/2} \sigma_{C}^{-1/2} \sigma_{AC}^{1/2} \sigma_{C}^{-1/2} \sigma_{BC}^{1/2}$.

4 Conditional quantum mutual information based on von Neumann entropy

In this section, we prove that the conditional quantum mutual information has no less than ten different representations in terms of the relative entropy (all of them being equal). This paves the way for designing different Rényi generalizations of the conditional quantum mutual information. Furthermore, we give an alternate proof of the fact that the conditional quantum mutual information $I(A;B|C)$ is monotone under local quantum operations on systems $A$ and $B$. This alternate proof will be the basis for similar proofs when we consider Rényi generalizations in Sections 5 and 6. Finally, we discuss how representing $I(A;B|C)$ as we do in Theorem 3 allows for a straightforward comparison of it with the $\Delta$ quantity defined in [29].

4.1 Various formulations of the conditional quantum mutual information

**Proposition 1** The following equality holds for a tripartite density operator $\rho_{ABC}$

$$I(A;B|C)_\rho = D(\rho_{ABC} || \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_{C} \}).$$

(4.1)

**Proof.** This follows simply by substituting into the relative entropy,

$$D(\rho_{ABC} || \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_{C} \})$$

$$= \text{Tr} \left\{ \rho_{ABC} \log \rho_{ABC} \right\} - \text{Tr} \left\{ \rho_{ABC} \log \left\{ \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_{C} \} \right\} \right\}$$

$$= -H(A;B|C)_\rho - \text{Tr} \left\{ \rho_{ABC} \log \rho_{AC} + \log \rho_{BC} - \log \rho_{C} \right\}$$

$$= -H(A;B|C)_\rho - \text{Tr} \left\{ \rho_{AC} \log \rho_{AC} \right\} - \text{Tr} \left\{ \rho_{BC} \log \rho_{BC} \right\} + \text{Tr} \left\{ \rho_{C} \log \rho_{C} \right\}$$

$$= -H(A;B|C)_\rho + H(AC)_\rho + H(BC)_\rho - H(C)_\rho$$

$$= I(A;B|C)_\rho.$$  

(4.6)

**Lemma 2** The following equality holds for a tripartite density operator $\rho_{ABC}$ and states $\sigma_{AC}$, $\theta_{BC}$, and $\omega_{C}$

$$D(\rho_{ABC} || \exp \{ \log \sigma_{AC} + \log \theta_{BC} - \log \omega_{C} \})$$

$$= I(A;B|C)_\rho + D(\rho_{AC} || \sigma_{AC}) + D(\rho_{BC} || \theta_{BC}) - D(\rho_{C} || \omega_{C}).$$

(4.7)
Hence, we can write

\[
D (\rho_{ABC} | \exp \{ \log \sigma_{AC} + \log \theta_{BC} - \log \omega_C \}) \\
= D (\rho_{ABC} | \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}) \\
+ \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \}] \}
- \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \}] \}
= \text{Tr} \{ \rho_{ABC} \log \rho_{ABC} \} - \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}] \}
- \text{Tr} \{ \rho_{ABC} \log \rho_{ABC} \}
+ \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \}] \}
= I (A; B|C)_\rho + \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \}] \}
- \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}] \}.
\]

(4.8)

Hence, we can write

\[
\text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \}] \}
- \text{Tr} \{ \rho_{ABC} \log [\exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}] \}
= D (\rho_{AC} | \tau_{AC}) + D (\rho_{BC} | \theta_{BC}) - D (\rho_C | \omega_C) \tag{4.11}
\]

For the mutual information, there are four seemingly different ways of writing it as a relative entropy \[11\]. However, for the conditional mutual information, there are no less than ten ways of doing so, as stated in the following theorem.

**Theorem 3** The following equalities holds for a tripartite density operator \( \rho_{ABC} \), and where the optimizations are over states on the indicated Hilbert spaces,

\[
I (A; B|C)_\rho = \min_{\sigma_{ABC}} D (\rho_{ABC} | \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \}),
\]

(4.12)

\[
= \min_{\tau_{AC}, \theta_{BC}, \omega_C} \max \{ D (\rho_{ABC} | \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}) \},
\]

(4.13)

\[
= \min_{\tau_{AC}, \theta_{BC}} D (\rho_{ABC} | \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}),
\]

(4.14)

\[
= \min_{\tau_{AC}, \theta_{BC}} D (\rho_{ABC} | \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}),
\]

(4.15)

\[
= \min_{\sigma_{ABC}} \max \{ D (\rho_{ABC} | \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \omega_C \}) \},
\]

(4.16)

\[
= \min_{\tau_{AC}} D (\rho_{ABC} | \exp \{ \log \tau_{AC} + \log \rho_{BC} - \log \omega_C \}),
\]

(4.17)

\[
= \min_{\theta_{BC}} D (\rho_{ABC} | \exp \{ \log \rho_{AC} + \log \theta_{BC} - \log \omega_C \}),
\]

(4.18)

\[
= \max_{\omega_C} D (\rho_{ABC} | \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \omega_C \}),
\]

(4.19)

\[
= \min_{\tau_{AC}, \theta_{BC}} D (\rho_{ABC} | \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}),
\]

(4.20)

\[
= \min_{\sigma_{ABC}} D (\rho_{ABC} | \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \rho_C \}).
\]

(4.21)
Proof. We only prove two of these relations, noting that the rest follow from similar ideas. We first prove (4.13). Invoking Lemma 2, we have that

\[
\min_{\tau_{AC}, \theta_{BC}} \max_{\omega_C} D(\rho_{ABC} \| \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \})
= I(A;B|C)_\rho + \min_{\tau_{AC}} D(\rho_{AC} \| \tau_{AC}) + \min_{\theta_{BC}} D(\rho_{BC} \| \theta_{BC}) - \min_{\omega_C} D(\rho_C \| \omega_C). \tag{4.22}
\]

Invoking the properties of the relative entropy, we see that the right hand side is equal to \( I(A;B|C)_\rho \).

We now prove (4.12). Consider choosing a state \( \sigma_{ABC} \) and setting \( \tau_{AC} = \sigma_{AC} \), \( \theta_{BC} = \sigma_{BC} \), and \( \omega_C = \sigma_C \). Then by Lemma 2, we have that

\[
D(\rho_{ABC} \| \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \})
= I(A;B|C)_\rho + D(\rho_{AC} \| \sigma_{AC}) + D(\rho_{BC} \| \sigma_{BC}) - D(\rho_C \| \sigma_C). \tag{4.23}
\]

But it is known that

\[
D(\rho_{AC} \| \sigma_{AC}) \geq D(\rho_C \| \sigma_C) \tag{4.24}
\]

from monotonicity of the relative entropy under partial trace. Thus, we have that

\[
D(\rho_{AC} \| \sigma_{AC}) + D(\rho_{BC} \| \sigma_{BC}) - D(\rho_C \| \sigma_C) \geq 0. \tag{4.25}
\]

This implies that

\[
\min_{\sigma_{ABC} \in S(\mathcal{H}_{ABC})} D(\rho_{ABC} \| \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \})
= I(A;B|C)_\rho + \min_{\sigma_{ABC} \in S(\mathcal{H}_{ABC})} [D(\rho_{AC} \| \sigma_{AC}) + D(\rho_{BC} \| \sigma_{BC}) - D(\rho_C \| \sigma_C)] \tag{4.26}
\]

The three rightmost terms are non-negative (as shown above), so that we can minimize it (to its absolute minimum of zero) by picking a state \( \sigma_{ABC} \) such that

\[
\sigma_{AC} = \rho_{AC}, \tag{4.27}
\]

\[
\log \sigma_{BC} - \log \sigma_C = \log \rho_{BC} - \log \rho_C, \tag{4.28}
\]

or by symmetry, one such that

\[
\sigma_{BC} = \rho_{BC}, \tag{4.29}
\]

\[
\log \sigma_{AC} - \log \sigma_C = \log \rho_{AC} - \log \rho_C. \tag{4.30}
\]

One clear choice satisfying this is \( \sigma_{ABC} = \rho_{ABC} \), but there could be others. \( \blacksquare \)

Corollary 4 As an immediate corollary, we find many Pinsker-like lower bounds on the conditional mutual information, the tightest of which is

\[
I(A;B|C)_\rho \geq \frac{1}{2} \max_{\omega_C} \| \rho_{ABC} - \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \omega_C \} \|_1^2.
\]

Proof. The idea is to apply the usual Pinsker inequality for relative entropy along with the identities in Theorem 3 \( \blacksquare \)
4.2 Monotonicity of the conditional quantum mutual information under local quantum operations

One of the crucial properties of the conditional quantum mutual information is that it is monotone under local CPTP maps acting on the systems $A$ and $B$, respectively. That is,

$$I(A; B|C)_\rho \geq I(A; B|C)_\xi,$$

where $\xi_{ABC} \equiv (\mathcal{N}_A \otimes \mathcal{M}_B)(\rho_{ABC})$. We can now see that this is a consequence of the following theorem.

**Theorem 5** Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$. Let $\mathcal{N}_A$ and $\mathcal{M}_B$ be local CPTP maps acting on the systems $A$ and $B$, respectively. Then the following monotonicity inequality holds

$$D(\rho_{ABC}||\exp\{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}) \geq D((\mathcal{N}_A \otimes \mathcal{M}_B)(\rho_{ABC})||\exp\{\log \mathcal{N}_A(\tau_{AC}) + \log \mathcal{M}_B(\theta_{BC}) - \log \omega_C\}), \quad (4.31)$$

**Proof.** We first prove the inequality

$$D(\rho_{ABC}||\exp\{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}) \geq D(\mathcal{N}_A(\rho_{ABC})||\exp\{\log \mathcal{N}_A(\tau_{AC}) + \log \theta_{BC} - \log \omega_C\}), \quad (4.32)$$

To prove this, we simply expand out the terms,

$$D(\rho_{ABC}||\exp\{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}) = \text{Tr}\{\rho_{ABC}\log \rho_{ABC}\} - \text{Tr}\{\rho_{ABC}[\log \tau_{AC} + \log \theta_{BC} - \log \omega_C]\} \quad (4.33)$$

$$= D(\rho_{ABC}||\tau_{AC} \otimes I_B) - \text{Tr}\{\rho_{BC}\log \theta_{BC}\} + \text{Tr}\{\rho_C\log \omega_C\} \quad (4.34)$$

Similarly,

$$D(\mathcal{N}_A(\rho_{ABC})||\exp\{\log \mathcal{N}_A(\tau_{AC}) + \log \theta_{BC} - \log \omega_C\}) = \text{Tr}\{\mathcal{N}_A(\rho_{ABC})\log \mathcal{N}_A(\rho_{ABC})\} - \text{Tr}\{\mathcal{N}_A(\rho_{ABC})[\log \mathcal{N}_A(\tau_{AC}) + \log \theta_{BC} - \log \omega_C]\} \quad (4.35)$$

$$= D(\mathcal{N}_A(\rho_{ABC})||\mathcal{N}_A(\tau_{AC}) \otimes I_B) - \text{Tr}\{\rho_{BC}\log \theta_{BC}\} + \text{Tr}\{\rho_C\log \omega_C\}. \quad (4.36)$$

Then the inequality in (4.32) follows from the ordinary monotonicity of relative entropy,

$$D(\rho_{ABC}||\tau_{AC} \otimes I_B) \geq D(\mathcal{N}_A(\rho_{ABC})||\mathcal{N}_A(\tau_{AC}) \otimes I_B). \quad (4.37)$$

An essentially identical approach gives us the following inequality,

$$D(\mathcal{N}_A(\rho_{ABC})||\exp\{\log \mathcal{N}_A(\tau_{AC}) + \log \theta_{BC} - \log \omega_C\}) \geq D(\mathcal{M}_B(\rho_{ABC})||\exp\{\log \mathcal{M}_B(\theta_{BC}) - \log \omega_C\}). \quad (4.38)$$

Combining this one with (4.32) gives us the inequality in the statement of the theorem. ■

From this, we can conclude with a conceptually different proof (other than directly making use of strong subadditivity) that the conditional mutual information is monotone under tensor product maps acting on systems $A$ and $B$. 

10
**Theorem 6** The following inequality holds

\[ I(A; B|C)_\rho \geq I(A; B|C)_{(N_A \otimes M_B)(\rho_{ABC})}. \]  

**Proof.** Using the formula in (4.12) for \( I(A; B|C) \), consider that

\[
I(A; B|C)_{(N_A \otimes M_B)(\rho_{ABC})} = \min_{\omega_{ABC}} D((N_A \otimes M_B)(\rho_{ABC}) \| \exp \{ \log \omega_{AC} + \log \omega_{BC} - \log \omega_C \})
\]  

\[
\leq D((N_A \otimes M_B)(\rho_{ABC}) \| \exp \{ \log N_A(\sigma_{AC}) + \log M_B(\sigma_{BC}) - \log \sigma_C \})
\]  

\[
\leq D(\rho_{ABC} \| \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \}).
\]  

But since the inequality holds for an arbitrary choice of \( \sigma_{ABC} \), we can conclude that

\[
I(A; B|C)_{(N_A \otimes M_B)(\rho_{ABC})} \leq \min_{\sigma_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})} D(\rho_{ABC} \| \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \}),
\]  

which is equivalent to the statement of the theorem. Proofs of this theorem using the other representation formulas in Theorem 3 follow in a similar way. □

### 4.3 Comparison with the minimum relative entropy to quantum Markov states

We can now easily compare \( I(A; B|C) \) with the \( \Delta \) quantity defined in [29]. First, recall that \( \Delta(\rho_{ABC}) \) is defined as

\[
\Delta(\rho_{ABC}) \equiv \min_{\sigma_{ABC} \in \mathcal{M}_{A-C-B}} D(\rho_{ABC} \| \sigma_{ABC}).
\]

Since every Markov state satisfies the condition \( \sigma_{ABC} = \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \} \), we see that this formula is equivalent to

\[
\Delta(\rho_{ABC}) = \min_{\sigma_{ABC} \in \mathcal{M}_{A-C-B}} D(\rho_{ABC} \| \exp \{ \log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C \}),
\]

from which we obtain the following inequality,

\[
\Delta(\rho_{ABC}) \geq I(A; B|C)_\rho,
\]

because the minimization of the formula in (4.12) is over all states \( \sigma_{ABC} \), rather than just the Markov states. The above inequality was already proved in [29], but one of the main contributions of [29] was to show that there are tripartite states \( \omega_{ABC} \) for which there is a strict inequality

\[
\Delta(\omega_{ABC}) > I(A; B|C)_\omega,
\]

and in fact these authors showed that the gap can be arbitrarily large.
5 Rényi conditional mutual information

In this section, we define several Rényi conditional mutual informations that bear some properties similar to the conditional mutual information and converge to it in a limit. We are motivated to define the Rényi conditional mutual information by making use of the following fact

\[
\exp\{\log \tau_{AC} + \log \theta_{BC} - \log \omega_{C}\} = \lim_{\alpha \to 1}\left[\frac{\tau_{AC}^{(1-\alpha)/2} \omega_{C}^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_{C}^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}}{\tau_{AC}^{(1-\alpha)/2} \omega_{C}^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_{C}^{(\alpha-1)/2}}\right]^{1/(1-\alpha)}. \tag{5.1}
\]

We supply a proof for this fact in the appendix—however, note that the proof follows from well known proofs for the Lie-Trotter product formula.

We can then use the above observation and the definition of the Rényi relative entropy to define Rényi generalizations of the conditional mutual information. There are many variations that we could take for a Rényi conditional mutual information by using the different expansions given in Theorem 3. However, we choose the following one because it obeys several properties which we would expect to hold for a Rényi generalization of the conditional mutual information.

**Definition 7** The Rényi conditional mutual information of a tripartite state \(\rho_{ABC}\) (or more generally a tripartite positive operator) is defined as

\[
I_\alpha(A;B|C)_{\rho} = \min_{\sigma_{BC} \in S(H_{BC})} D_\alpha\left(\rho_{ABC} \left\| \left[\frac{\rho_{AC}^{(1-\alpha)/2} \rho_{AC}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_{BC}^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2}}{\rho_{AC}^{(1-\alpha)/2} \rho_{AC}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_{BC}^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2}}\right]^{1/(1-\alpha)}\right)\). \tag{5.2}
\]

5.1 Limiting cases of the Rényi conditional mutual information

In this section, we prove that several variations of the Rényi conditional mutual information converge to the conditional mutual information in the limit as \(\alpha \to 1\). We also consider limiting cases of it when \(\alpha \to 0\) and \(\alpha \to 2\).

**Theorem 8** All variations of the Rényi conditional mutual information converge to the conditional mutual information in the limit as \(\alpha \to 1\) for any tripartite state \(\rho_{ABC}\).

**Proof.** First we consider the limit of

\[
D_\alpha\left(\rho_{ABC} \left\| \left[\frac{\tau_{AC}^{(1-\alpha)/2} \omega_{C}^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_{C}^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}}{\tau_{AC}^{(1-\alpha)/2} \omega_{C}^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_{C}^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}}\right]^{1/(1-\alpha)}\right)\right) \tag{5.3}
\]

as \(\alpha \to 1\) (without taking any minimum over states \(\sigma_{ABC}\)). Consider that

\[
\lim_{\alpha \to 1} \alpha - 1 = 0, \tag{5.4}
\]

and

\[
\lim_{\alpha \to 1} \log \text{Tr}\left\{\rho_{ABC}^{\alpha} \tau_{AC}^{(1-\alpha)/2} \omega_{C}^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_{C}^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}\right\} = 0, \tag{5.5}
\]

whenever \(\text{supp}(\rho_{ABC}) \subseteq \text{supp}(\tau_{AC}^{0} \omega_{C}^{0} \theta_{BC}^{0} \omega_{C}^{0} \tau_{AC}^{0})\), which holds since we are taking a minimum over settings of the states \(\tau_{AC}, \omega_{C}, \theta_{BC}\) and one such choice is always \(\rho_{AC}, \rho_{C}, \text{and } \rho_{BC}\). To figure out the limit, we need to apply L’Hopital’s rule. In order to do so, we use Taylor expansions to aid in
evaluating the limit. Let $\alpha = 1 + t$, so that we think of $t$ as a small positive number and we will be taking $t \to 0$. Consider that

\[
X^{1+t} = X + tX \log X + O(t^2), \quad (5.6)
\]
\[
X^{-t} = I - t \log X + O(t^2), \quad (5.7)
\]
so that

\[
X^{-t/2} = I - (t/2) \log X + O(t^2), \quad (5.8)
\]
\[
X^{t/2} = I + (t/2) \log X + O(t^2). \quad (5.9)
\]

Plugging in, we find that

\[
\rho_{ABC}^\alpha = \rho_{ABC} + t \rho_{ABC} \log \rho_{ABC} + O(t^2), \quad (5.10)
\]
\[
\tau_{AC}^{(1-\alpha)/2} = I - (t/2) \log \tau_{AC} + O(t^2), \quad (5.11)
\]
\[
\omega_C^{(\alpha-1)/2} = I + (t/2) \log \omega_C + O(t^2), \quad (5.12)
\]
\[
\theta_{BC}^{1-\alpha} = I - t \log \theta_{BC} + O(t^2). \quad (5.13)
\]

So we can write the quantity of interest as

\[
\text{Tr} \left\{ \rho_{ABC}^{\alpha} \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2} \right\}
= \text{Tr} \left\{ (\rho + t \rho \log \rho) (I - (t/2) \log \tau_{AC}) (I + (t/2) \log \omega_C)
(I - t \log \theta_{BC}) (I + (t/2) \log \omega_C) (I - (t/2) \log \tau_{AC}) \right\}, \quad (5.14)
\]

where we have omitted the terms $O(t^2)$ for simplicity. We then find after some algebra that this is equal to

\[
\text{Tr} \{ \rho_{ABC} \} +
\]
\[
t \left[ \text{Tr} \{ \rho_{ABC} \log \rho_{ABC} \} - \text{Tr} \{ \rho_{ABC} \log \tau_{AC} \} - \text{Tr} \{ \rho_{ABC} \log \theta_{BC} \} + \text{Tr} \{ \rho_{ABC} \log \omega_C \} \right] + O(t^2)
= 1 + t \left( D(\rho_{ABC}||\exp \{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}) + O(t^2) \right). \quad (5.15)
\]

So taking the derivative with respect to $t$ and setting $t = 0$ gives

\[
D(\rho_{ABC}||\exp \{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}). \quad (5.16)
\]

So this implies that

\[
\lim_{\alpha \to 1} D_{\alpha} \left( \rho_{ABC}|| \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right)
= D(\rho_{ABC}||\exp \{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}) \quad (5.17)
\]

Since $I_{\alpha} (A; B|C)_{\rho}$ can be phrased in terms of various optimizations along the lines of Theorem 4.12, for a given $\alpha$, in the limit as $\alpha \to 1$, we will be considering the optimization of

\[
D(\rho_{ABC}||\exp \{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\})
\]
over various settings for $\tau_{AC}$, $\theta_{BC}$, and $\omega_C$, which by Theorem 4.12 is equal to $I (A; B|C)_{\rho}$. ■
5.1.1 Other limiting cases

We also have the following limit,

\[
\lim_{\alpha \to 0} \min_{\sigma_{ABC}} D_\alpha \left( \rho_{ABC} \| \left[ \sigma_{AC}^{(1-\alpha)/2} \sigma_C^{(\alpha-1)/2}, \sigma_{BC}^{(\alpha-1)/2} \right] \right)^{1/(1-\alpha)}
\]

\[
= \min_{\sigma_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})} -\log \text{Tr} \left\{ \Pi_{\rho_{ABC}} \sigma_{BC}^{1/2} \sigma_{AC}^{1/2} \sigma_{BC}^{1/2} \right\},
\]

(5.18)

where \( \Pi_{\rho_{ABC}} \) is the projection onto the support of the state \( \rho_{ABC} \). By inspection, this quantity is related to the zero-Rényi relative entropy considered in [13].

We can also consider the limit as \( \alpha \to 2 \),

\[
\lim_{\alpha \to 2} \min_{\sigma_{ABC}} D_\alpha \left( \rho_{ABC} \| \left[ \sigma_{AC}^{(1-\alpha)/2} \sigma_C^{(\alpha-1)/2}, \sigma_{BC}^{(\alpha-1)/2} \right] \right)^{1/(1-\alpha)}
\]

\[
= \min_{\sigma_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})} \log \text{Tr} \left\{ 2 \rho_{ABC} \left[ \sigma_{BC}^{1/2} \sigma_{AC}^{1/2} \sigma_{BC}^{1/2} \right] \right\}
\]

\[
= \min_{\sigma_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})} \log \text{Tr} \left\{ 2 \rho_{ABC} \left[ \sigma_{BC}^{1/2} \sigma_{AC}^{1/2} \sigma_{BC}^{1/2} \right] \right\}.
\]

(5.19)

5.2 Monotonicity with respect to local quantum operations for \( \alpha \in [0, 2] \)

The following lemma is the critical one which will allow us to conclude that all variations of the Rényi conditional mutual information are monotone with respect to local CPTP maps acting on systems \( A \) and \( B \).

**Lemma 9** The following monotonicity inequality holds with respect to local quantum operations (CPTP maps) \( \mathcal{N}_A \) and \( \mathcal{M}_B \) acting on systems \( A \) and \( B \), respectively, and for any positive operators \( \rho_{ABC}, \tau_{AC}, \omega_C, \) and \( \theta_{BC} \),

\[
D_\alpha \left( \rho_{ABC} \| \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \right] \right)^{1/(1-\alpha)}
\]

\[
\geq D_\alpha \left( \rho'_{ABC} \| \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \right] \right)^{1/(1-\alpha)},
\]

(5.21)

where \( \rho'_{ABC} \equiv (\mathcal{N}_A \otimes \mathcal{M}_B)(\rho_{ABC}), \tau_{AC}' = \mathcal{N}_A(\tau_{AC}), \theta_{BC}' = \mathcal{M}_B(\theta_{BC}) \).

**Proof.** We first prove monotonicity under local maps on system \( A \). Consider that the following quantity is jointly concave in \( \rho_{ABC} \) and \( \theta_{BC} \) when \( \alpha \in [0, 1] \)

\[
\text{Tr} \left\{ \rho_{ABC}^{1-\alpha/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2} \right\}.
\]

(5.22)

This is a result of Lieb’s concavity theorem [30], a special case of which is the statement that the function

\[
\text{Tr} \left\{ S^{\lambda} X^{1-\lambda} X^\dagger \right\}
\]

(5.23)

is jointly concave in \( S \) and \( R \), with \( R, S \geq 0 \) and \( \lambda \in [0, 1] \). (We apply the theorem by choosing \( S = \rho_{ABC}, R = \theta_{BC} \), and \( X = \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2}. \) Furthermore, by an application of Ando’s
convexity theorem \[2\], we know that the quantity in \(\text{(5.22)}\) is jointly convex in \(\rho_{ABC}\) and \(\theta_{BC}\) when \(\alpha \in [1, 2]\).

By a standard (well known) argument due to Uhlmann \[59\], we then have the following monotonicity,

\[
D_\alpha \left( \rho_{ABC} \parallel \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right)
\geq D_\alpha \left( \mathcal{N}_A \left( \rho_{ABC} \right) \parallel \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right). \tag{5.24}
\]

(The standard argument is to add an ancilla to system \(A\), apply a unitary to \(A\) and the ancilla, and then use joint convexity or joint concavity when mixing over the ancilla with Heisenberg-Weyl unitaries or Haar-random unitaries. See the “Proof of Theorem 1 given Proposition 3” in \[20\], for example.)

To prove the result for local maps on system \(A\), let \(F\) be the following operator

\[
F = \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}, \tag{5.25}
\]

so that

\[
F^\dagger F = \tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}, \tag{5.26}
\]

\[
FF^\dagger = \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2}. \tag{5.27}
\]

The operators \(F^\dagger F\) and \(FF^\dagger\) are isospectral, so that there is a unitary \(V\) relating them

\[
F^\dagger F = V F F^\dagger V^\dagger, \tag{5.28}
\]

which implies that

\[
\text{Tr} \left\{ \rho_{ABC}^{\alpha} \tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2} \right\}
\geq \text{Tr} \left\{ \rho_{ABC}^{\alpha} V \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} V^\dagger \right\}. \tag{5.29}
\]

But now we can see that Lieb’s concavity theorem applies again with

\[
S = \rho_{ABC}, \tag{5.30}
\]

\[
R = \tau_{AC}, \tag{5.31}
\]

\[
X = V \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2}, \tag{5.32}
\]

so that this function is jointly concave in \(\rho_{ABC}\) and \(\tau_{AC}\) for \(\alpha \in [0, 1]\). By a similar argument, Ando’s convexity theorem allows to conclude that this function is jointly convex in \(\rho_{ABC}\) and \(\tau_{AC}\) for \(\alpha \in [1, 2]\). By the same argument stated above, we obtain the following monotonicity,

\[
D_\alpha \left( \rho_{ABC} \parallel \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right)
\geq D_\alpha \left( \mathcal{N}_A \left( \rho_{ABC} \right) \parallel \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{1-\alpha} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right). \tag{5.33}
\]

Combining the inequality in \(\text{(5.24)}\) with the inequality above gives the monotonicity inequality given in the statement of the theorem. ■
Theorem 10 All variations of the Rényi conditional mutual information are monotone with respect to local quantum operations on systems A and B, for any tripartite positive operator $\rho_{ABC}$, in the sense that

$$I_\alpha(A;B|C)_{\rho} \geq I_\alpha(A;B|C)_{\omega},$$

(5.34)

where $\omega_{ABC} = (N_A \otimes M_B)(\rho_{ABC})$, with $N_A$ and $M_B$ CPTP maps that act on systems A and B, respectively.

Proof. We prove that a variation derived from (4.13) obeys the monotonicity (with all the others following from similar ideas). Beginning with the inequality in Lemma 9, we find that

$$\max_{\omega_C} D_\alpha\left(\rho_{ABC}\left|\| (\tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right)$$

(5.35)

$$\geq \max_{\omega_C} D_\alpha\left(\rho'_{ABC}\left|\| (\tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right),$$

(5.36)

$$\geq \min_{\tau_{AC}', \theta_{BC}''} \max_{\omega_C} D_\alpha\left(\rho''_{ABC}\left|\| (\tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right)$$

(5.37)

Since this inequality holds for all $\tau_{AC}$ and $\theta_{BC}$, it holds in particular for the minimum of the first line over all such states, establishing the monotonicity for the variation of the Rényi conditional mutual information derived from (4.13).

Corollary 11 The variations of the Rényi conditional mutual information derived from (4.12)-(4.19) are non-negative for all $\alpha \in [0,2]$ for any density operator $\rho_{ABC}$.

Proof. A similar proof technique applies to the quantities derived from (4.12)-(4.19). We apply Lemma 9 choosing the local maps on systems A and B to be partial traces, to conclude that

$$D_\alpha\left(\rho_{ABC}\left|\| (\tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right)$$

$$\geq D_\alpha\left(\rho_C\left|\| (\tau_C^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_C^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_C^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right).$$

(5.38)

Then, we can conclude that

$$\max_{\omega_C} D_\alpha\left(\rho_{ABC}\left|\| (\tau_{AC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_{BC}^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_{AC}^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right)$$

(5.39)

$$\geq \max_{\omega_C} D_\alpha\left(\rho_C\left|\| (\tau_C^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \theta_C^{(1-\alpha)/2} \omega_C^{(1-\alpha)/2} \tau_C^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right),$$

(5.40)

$$\geq D_\alpha\left(\rho_C\left|\| (\tau_C^{(1-\alpha)/2} \theta_C^{(1-\alpha)/2} \theta_C^{(1-\alpha)/2} \tau_C^{(1-\alpha)/2}) \right|^{1/(1-\alpha)} \right)$$

(5.41)

$$= D_\alpha(\rho_C|\tau_C)$$

(5.42)

$$\geq 0,$$

(5.43)

with the last inequality following from non-negativity of the Rényi relative entropy as discussed in Section 3. Using this, we can conclude that all variations mentioned in the statement of the corollary are non-negative.
Remark 12 It is an open question whether the following variations of the Rényi conditional mutual information are non-negative,

\[
\min_{\tau_{AC}, \theta_{BC}} D_\alpha \left( \rho_{ABC} \left\| \left( \tau_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \theta_{BC} \rho_C \rho_{AC} \right)^{1/(1-\alpha)} \right\| \right),
\]

(5.44)

\[
\min_{\sigma_{ABC}} D_\alpha \left( \rho_{ABC} \left\| \left( \sigma_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC} \rho_C \rho_{AC} \right)^{1/(1-\alpha)} \right\| \right),
\]

(5.45)

Remark 13 Corollary [11] combined with Theorem [8] constitutes another proof of the strong subadditivity of quantum entropy. Many other such proofs have appeared in the literature [44, 15, 52, 28, 62, 18] since the original proof in [38].

5.3 Sibson identity for the Rényi conditional mutual information

The Rényi conditional mutual information in Definition [7] has an explicit form, much like other Rényi information quantities [51, 53, 22]. We prove this in two steps, first by proving the following Sibson identity [54].

Lemma 14 The following quantum Sibson identity holds

\[
D_\alpha \left( \rho_{ABC} \left\| \left( \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC} \rho_C \rho_{AC} \right)^{1/(1-\alpha)} \right\| \right) = D_\alpha \left( \rho_{ABC} \left\| \left( \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC} \rho_C \rho_{AC} \right)^{1/(1-\alpha)} \right\| \right) + D_\alpha (\sigma_{BC}^* || \sigma_{BC}),
\]

(5.46)

with the state \( \sigma_{BC}^* \) having the form

\[
\sigma_{BC}^* = \frac{\left( \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \rho_{ABC} \rho_C \rho_{AC} \right\} \right)^{1/\alpha}}{\text{Tr} \left( \left\{ \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \rho_{ABC} \rho_C \rho_{AC} \right\} \right\} \right)^{1/\alpha}}.
\]

(5.47)

Proof. The relation for \( \sigma_{BC}^* \) implies that

\[
\left[ \sigma_{BC}^* \text{Tr} \left\{ \left( \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \rho_{ABC} \rho_C \rho_{AC} \right\} \right)^{1/\alpha} \right\} \right]^{\alpha} = \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \rho_{ABC} \rho_C \rho_{AC} \right\}. \]

(5.48)

Then consider that

\[
D_\alpha \left( \rho_{ABC} \left\| \left( \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC} \rho_C \rho_{AC} \right)^{1/(1-\alpha)} \right\| \right) = \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC} \rho_C \rho_{AC} \sigma_{BC} \rho_C \rho_{AC} \right\} (5.49)
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC} \rho_C \rho_{AC} \right\}, \]

(5.50)

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \rho_{ABC} \rho_C \rho_{AC} \right\} \sigma_{BC} \rho_C \rho_{AC} \right\} \sigma_{BC} \rho_C \rho_{AC} \right\} \right\}, (5.51)
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \left[ \sigma_{BC}^* \right]^{\alpha} \sigma_{BC}^{1-\alpha} \right\} + \frac{\alpha}{\alpha - 1} \log \text{Tr} \left\{ \left( \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \rho_{ABC} \rho_C \rho_{AC} \right\} \right)^{1/\alpha} \right\}. (5.52)
\]
Now consider expanding the following,

\[
D_\alpha \left( \rho_{ABC} \| \left[ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right)
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \left[ \sigma_{BC}^{\alpha} \right]^{1-\alpha} \right\}  \hspace{1cm} (5.53)
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \text{Tr} A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\} \left[ \sigma_{BC}^{\alpha} \right]^{1-\alpha} \right\} \hspace{1cm} (5.54)
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \text{Tr} A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\} \right\}^{1/\alpha} \hspace{1cm} (5.55)
\]

\[
+ \log \text{Tr} \left\{ \text{Tr} A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\} \right\}^{1/\alpha} \hspace{1cm} (5.56)
\]

\[
= \frac{\alpha}{\alpha - 1} \log \text{Tr} \left\{ \text{Tr} A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\} \right\}^{1/\alpha} \hspace{1cm} (5.57)
\]

Putting everything together, we can conclude the statement of the lemma. ■

**Corollary 15** The Rényi conditional mutual information has the following explicit form,

\[
I_\alpha (A; B|C) = \frac{\alpha}{\alpha - 1} \log \text{Tr} \left\{ \text{Tr} A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\} \right\}^{1/\alpha} \hspace{1cm} (5.58)
\]

**Proof.** This follows by the previous lemma,

\[
I_\alpha (A; B|C)
\]

\[
= \min_{\sigma_{BC}} D_\alpha \left( \rho_{ABC} \| \left[ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right) \hspace{1cm} (5.59)
\]

\[
= \min_{\sigma_{BC}} \left\{ D_\alpha \left( \rho_{ABC} \| \left[ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right) \right\} + D_\alpha (\sigma_{BC} || \sigma_{BC}) \hspace{1cm} (5.60)
\]

\[
= D_\alpha \left( \rho_{ABC} \| \left[ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right) \hspace{1cm} (5.61)
\]

\[
= \frac{\alpha}{\alpha - 1} \log \text{Tr} \left\{ \text{Tr} A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\} \right\}^{1/\alpha} \hspace{1cm} (5.62)
\]

■

**6 Sandwhiched Rényi conditional mutual information**

As in the previous section, there are many ways in which we can define a sandwiched Rényi conditional mutual information in terms of the following general quantity

\[
\tilde{D}_\alpha \left( \rho_{ABC} \| \left( \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \omega_{BC}^{(\alpha-1)/2} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2} \right)^{\alpha/(1-\alpha)} \right). \hspace{1cm} (6.1)
\]

We choose the definition given below because it obeys many of the properties that the conditional mutual information does.
Definition 16 The sandwiched Rényi conditional mutual information is defined as
\[
\tilde{I}_\alpha (A; B|C)\equiv \min_{\sigma_{BC}} \max_{\omega_C} \tilde{D}_\alpha \left( \rho_{ABC} \| \left( \rho_{AC}^{(1-\alpha)/2\alpha} \omega_C^{(\alpha-1)/2\alpha} \sigma_{BC}^{(1-\alpha)/\alpha} \omega_C^{(\alpha-1)/2\alpha} \rho_{AC}^{(1-\alpha)/2\alpha} \right)^{\alpha/(1-\alpha)} \right). \tag{6.2}
\]

Proposition 17 In the above definition, we can exchange the minimum and the maximum whenever \(\alpha \in (1, 2]\).

Proof. The proof idea is identical to the idea of the proof of Lemma 4 in [22], so we omit it. \(\blacksquare\)

6.1 Limiting cases of the sandwiched Rényi conditional mutual information

Proposition 18 In the limit as \(\alpha\) approaches one, all variations of the sandwiched Rényi conditional quantum mutual information converge to the conditional quantum mutual information \(I(A; B|C)\).

Proof. Let
\[
\sigma'_{ABC} \equiv \tau_{AC}^{(1-\alpha)/2\alpha} \omega_C^{(\alpha-1)/2\alpha} \theta_{BC}^{(1-\alpha)/\alpha} \omega_C^{(\alpha-1)/2\alpha} \tau_{AC}^{(1-\alpha)/2\alpha}.
\tag{6.3}
\]
We will show that
\[
\lim_{\alpha \to 1} \tilde{D}_\alpha \left( \rho_{ABC} \| (\sigma'_{ABC})^{\alpha/(1-\alpha)} \right) = D(\rho_{ABC} \| \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}). \tag{6.4}
\]
Let \(F\) be the operator \(F = (\sigma'_{ABC})^{1/2} (\rho_{ABC})^{1/2}\), so that
\[
FF^\dagger = (\sigma'_{ABC})^{1/2} \rho_{ABC} (\sigma'_{ABC})^{1/2}, \tag{6.5}
\]
\[
F^\dagger F = (\rho_{ABC})^{1/2} \sigma'_{ABC} (\rho_{ABC})^{1/2}. \tag{6.6}
\]
Since \(FF^\dagger\) and \(F^\dagger F\) are isospectral, we know that \(\text{Tr}\{[FF^\dagger]^\alpha\} = \text{Tr}\{[F^\dagger F]^\alpha\}\). Therefore,
\[
\tilde{D}_\alpha \left( \rho_{ABC} \| (\sigma'_{ABC})^{\alpha/(1-\alpha)} \right)
\]
can also be written as
\[
\tilde{D}_\alpha \left( \rho_{ABC} \| (\sigma'_{ABC})^{\alpha/(1-\alpha)} \right) = \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \left( (\rho_{ABC})^{1/2} \sigma'_{ABC} (\rho_{ABC})^{1/2} \right)^\alpha \right\}. \tag{6.8}
\]
Suppose that the support of \(\rho_{ABC}\) is contained in the support of the operator \((\sigma'_{ABC})^{\alpha/(1-\alpha)}\). It suffices to show that
\[
\frac{\partial}{\partial \alpha} \text{Tr} \left\{ \left[ (\rho_{ABC})^{1/2} \sigma'_{ABC} (\rho_{ABC})^{1/2} \right]^\alpha \right\}\bigg|_{\alpha = 1} = D(\rho_{ABC} \| \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}). \tag{6.9}
\]
The rest of the proof employs some ideas of [45], [9]. Let us rewrite the expression inside the trace by expanding \(\sigma'_{ABC}\), and using \(\alpha = 1 + \varepsilon\), as
\[
\text{Tr} \left\{ \left[ (\rho_{ABC})^{1/2} \tau_{AC}^{\varepsilon/(1+\varepsilon)} \omega_C^{\varepsilon/(1+\varepsilon)} \theta_{BC}^{\varepsilon/(1+\varepsilon)} \omega_C^{\varepsilon/(1+\varepsilon)} \tau_{AC}^{\varepsilon/(1+\varepsilon)} \rho_{ABC}^{1/(1+\varepsilon)} \right]^{1+\varepsilon} \right\}. \tag{6.10}
\]
Furthermore, we can use two parameters $\varepsilon_1$ and $\varepsilon_2$ so that the above expression is a special case of

$$f(\varepsilon_1, \varepsilon_2) \equiv \text{Tr} \left\{ \left( \rho_{ABC} \right)^{1/2} \tau_{AC}^{1/2} \omega_C^{1/2} \theta_{BC}^{1/2} \omega_C^{1/2} \tau_{AC}^{1/2} \left( \rho_{ABC} \right)^{1/2} \right\}. \quad (6.11)$$

We then have that

$$\frac{\partial}{\partial \alpha} \text{Tr} \left\{ \left( \rho_{ABC} \right)^{1/2} \tau_{AC}^{1/2} \omega_C^{1/2} \theta_{BC}^{1/2} \omega_C^{1/2} \tau_{AC}^{1/2} \left( \rho_{ABC} \right)^{1/2} \right\} \bigg|_{\alpha=1} = \frac{\partial}{\partial \varepsilon} f(\varepsilon, \varepsilon) \bigg|_{\varepsilon=0}, \quad (6.12)$$

which can be evaluated as

$$\frac{\partial}{\partial \varepsilon} f(\varepsilon, \varepsilon) \bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon_1} f(\varepsilon_1, 0) \bigg|_{\varepsilon_1=0} + \frac{\partial}{\partial \varepsilon_2} f(0, \varepsilon_2) \bigg|_{\varepsilon_2=0}. \quad (6.13)$$

Consider the following Taylor expansions around $\varepsilon = 0$,

$$X^{1+\varepsilon} = X + \varepsilon X \log X + O(\varepsilon^2), \quad (6.14)$$

$$X^{1+\varepsilon} = I - \varepsilon \log X + O(\varepsilon^2). \quad (6.15)$$

From these, we calculate $f(\varepsilon_1, 0)$ as

$$f(\varepsilon_1, 0) = \text{Tr} \left\{ (\rho_{ABC})^{1/2} \sigma_{ABC} (\rho_{ABC})^{1/2} \right\} \quad (6.16)$$

$$= \text{Tr} \left\{ \rho_{ABC} \sigma_{ABC} \right\} \quad (6.17)$$

$$= \text{Tr} \left\{ \rho_{ABC} (I - \varepsilon_1 \log \tau_{AC} - \varepsilon_1 \log \theta_{BC} + \varepsilon_1 \log \omega_C) \right\} + O(\varepsilon_1^2). \quad (6.18)$$

$$= \text{Tr} \left\{ \rho_{ABC} - \varepsilon_1 \text{Tr} \left\{ \rho_{ABC} (\log \tau_{AC} + \log \theta_{BC} - \log \omega_C) \right\} + O(\varepsilon_1^2) \right\}. \quad (6.19)$$

It then follows that

$$\frac{\partial}{\partial \varepsilon_1} f(\varepsilon_1, 0) \bigg|_{\varepsilon_1=0} = -\text{Tr} \left\{ \rho_{ABC} (\log \tau_{AC} + \log \theta_{BC} - \log \omega_C) \right\}. \quad (6.20)$$

We then calculate $f(0, \varepsilon_2)$ as

$$f(0, \varepsilon_2) = \text{Tr} \left\{ \rho_{ABC}^{1+\varepsilon_2} \right\} \quad (6.21)$$

$$= \text{Tr} \left\{ \rho_{ABC} \right\} + \varepsilon_2 \text{Tr} \left\{ \rho_{ABC} \log \rho_{ABC} \right\} + O(\varepsilon_2^2). \quad (6.22)$$

It then follows that

$$\frac{\partial}{\partial \varepsilon_2} f(0, \varepsilon_2) \bigg|_{\varepsilon_2=0} = \text{Tr} \left\{ \rho_{ABC} \log \rho_{ABC} \right\}. \quad (6.23)$$

Putting these together, we find that

$$\frac{\partial}{\partial \varepsilon} f(\varepsilon, \varepsilon) \bigg|_{\varepsilon=0} = \text{Tr} \left\{ \rho_{ABC} \log \rho_{ABC} \right\} - \text{Tr} \left\{ \rho_{ABC} (\log \tau_{AC} + \log \theta_{BC} - \log \omega_C) \right\} \quad (6.24)$$

$$= D(\rho_{ABC}, \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}). \quad (6.25)$$

Since $\bar{I}_\alpha (A; B|C)_\rho$ can be phrased in terms of various optimizations along the lines of Theorem 4.12 for a given $\alpha$, in the limit as $\alpha \to 1$, we will be considering the optimization of

$$D(\rho_{ABC}, \exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \}) \quad (6.26)$$

over various settings for $\tau_{AC}$, $\theta_{BC}$, and $\omega_C$, which by Theorem 4.12 is equal to $I(A; B|C)_\rho$. ■
6.1.1 Other limiting cases

We define the max-conditional mutual information as follows,

\[
I_{\text{max}}(A;B|C)_{\rho} \equiv \lim_{\alpha \to \infty} \min_{\sigma_{ABC}} \tilde{D}_{\alpha} \left( \rho_{ABC} \middle\| \left( \sigma_{AC}^{(1-\alpha)/2} \sigma_{C}^{(1-\alpha)/2} \right)_{\alpha/(1-\alpha)} \right)
\]

\[
= \min_{\sigma_{ABC}} \log \left\| \left( \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \right)_{-1/2} \rho_{ABC} \left( \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \right)_{-1/2} \right\|_{\infty}
\]

\[
= \inf \left\{ \lambda : \rho_{ABC} \leq 2^{\lambda} \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \right\}
\]

\[
= \min_{\sigma_{ABC}} D_{\text{max}} \left( \rho_{ABC} \| \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \right),
\]

where every line is an equivalent formulation. This generalizes the max-mutual information defined in [6].

Also, we can define the min-conditional mutual information as follows,

\[
I_{\text{min}}(A;B|C)_{\rho} \equiv \min_{\sigma_{ABC}} \tilde{D}_{1/2} \left( \rho_{ABC} \middle\| \left( \sigma_{AC}^{(1-\alpha)/2} \sigma_{C}^{(1-\alpha)/2} \right)_{\alpha/(1-\alpha)} \right)
\]

\[
= \min_{\sigma_{ABC}} -2 \log \left\| \sqrt{\rho_{ABC}} \sqrt{\sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \sigma_{BC}^{-1/2}} \right\|_1
\]

\[
= -\log \max_{\sigma_{ABC}} F \left( \rho_{ABC}, \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \right),
\]

with the fidelity between two operators \( P \) and \( Q \) defined as \( F(P,Q) \equiv \left\| \sqrt{P} \sqrt{Q} \right\|_1^2 \).

The forms given above seem quite natural, as the operators \( \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \sigma_{BC}^{-1/2} \) appear in our review of quantum Markov states in Section 3 (however, note again that this operator is not a Markov state unless \( \sigma_{ABC} = \sigma_{BC}^{1/2} \sigma_{AC}^{-1/2} \sigma_{BC}^{-1/2} \)).

6.2 Monotonicity under local quantum operations for \( \alpha \in [1/2, \infty] \)

**Lemma 19** The following monotonicity inequality holds with respect to local quantum operations (CPTP maps) \( \mathcal{N}_A \) and \( \mathcal{M}_B \) acting on systems \( A \) and \( B \), respectively, and for any positive operators \( \rho_{ABC}, \tau_{AC}, \omega_C, \) and \( \theta_{BC} \) for all \( \alpha \in [1/2, \infty] \),

\[
\tilde{D}_{\alpha} \left( \rho_{ABC} \mid \frac{1}{\omega_C} (\tau_{AC})^{(1-\alpha)/2} \omega_{C}^{(1-\alpha)/2} \right) \geq \tilde{D}_{\alpha} \left( \rho'_{ABC} \mid \frac{1}{\omega_C} (\tau'_{AC})^{(1-\alpha)/2} \omega_{C}^{(1-\alpha)/2} \right),
\]

where \( \rho'_{ABC} \equiv (\mathcal{N}_A \otimes \mathcal{M}_B) (\rho_{ABC}) \), \( \tau'_{AC} \equiv \mathcal{N}_A (\tau_{AC}) \), and \( \theta'_{BC} \equiv \mathcal{M}_B (\theta_{BC}) \).

**Proof.** Hiai has proven in part 1) of Theorem 1.1 of [27] that the following function is jointly concave in \( S \) and \( T \),

\[
(S,T) \mapsto \text{Tr} \left\{ \left[ \Phi (S^p)^{1/2} \Psi (T^q) \Phi (S^p)^{1/2} \right]^{\alpha/(1-\alpha)} \right\},
\]

(6.35)
for $S,T \geq 0$, positive maps $\Phi(\cdot)$ and $\Psi(\cdot)$, $0 < p, q \leq 1$, and $1/2 \leq s \leq 1/(p + q)$. We can then see that our trace function of interest

$$\text{Tr} \left\{ \left[ \frac{1}{\tau_{AC}} \omega_{C}^{\alpha_{1}} \theta_{BC}^{\alpha_{2}} \omega_{C}^{\alpha_{3}} \tau_{AC} \right]^{1/2} \rho_{ABC} \left[ \frac{1}{\tau_{AC}} \omega_{C}^{\alpha_{1}} \theta_{BC}^{\alpha_{2}} \omega_{C}^{\alpha_{3}} \tau_{AC} \right]^{1/2} \right\}^{\alpha} \right\} (6.36)$$

is of this form, with

$$\Psi = \text{id}, \quad q = 1, \quad (6.37)$$

$$\Phi(\cdot) = \tau_{AC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{\alpha-1} / 2^\alpha \omega_{C}^{\alpha-1} / 2^\alpha \tau_{AC}^{(1-\alpha)/2\alpha}, \quad (6.39)$$

$$p = \frac{1-\alpha}{\alpha}, \quad (6.40)$$

$$s = \alpha. \quad (6.41)$$

For the range $\alpha \in [1/2, 1)$, we have that $p \in (0,1]$ and $1/(p + q) = \alpha$, so that the conditions of Hiai’s theorem are satisfied. We conclude that the function in (6.36) is jointly concave in $\theta_{BC}$ and $\rho_{ABC}$. From this, we can conclude monotonicity with respect to local CPTP maps acting on system $B$ for $\alpha \in [1/2, 1)$.

To prove that

$$\tilde{D}_\alpha \left( \rho_{ABC} \vert \vert \left[ \tau_{AC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{\alpha-1} / 2^\alpha \omega_{C}^{\alpha-1} / 2^\alpha \tau_{AC}^{(1-\alpha)/2\alpha} \right]^{\alpha/(1-\alpha)} \right) \quad (6.42)$$

is monotone with respect to maps on system $A$, consider that

$$\tau_{AC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{\alpha-1} / 2^\alpha \omega_{C}^{\alpha-1} / 2^\alpha \tau_{AC}^{(1-\alpha)/2\alpha} = V \theta_{BC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{(1-\alpha)/2\alpha} V^\dagger \quad (6.43)$$

for some unitary $V$. Then the same proof as above applies, expect that we take

$$\Phi(\cdot) = V \theta_{BC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1} / 2^\alpha \theta_{BC}^{(1-\alpha)/2\alpha} V^\dagger. \quad (6.44)$$

We then conclude that (6.36) is jointly concave in $\tau_{AC}$ and $\rho_{ABC}$. From this, we can conclude monotonicity with respect to quantum operations acting on system $A$.

The proof for $\alpha \in (1, \infty]$ is a straightforward generalization of the technique used to prove Proposition 3 in Frank and Lieb’s paper [20]. To prove monotonicity with respect to local maps on system $B$, it suffices to prove that the following function

$$(\rho_{ABC}, \theta_{BC}) \mapsto \text{Tr} \left\{ \left[ \left( K^{1/2} \theta_{BC} \right) \right]^{1/2} \rho_{ABC} \left[ \left( K^{1/2} \theta_{BC} \right) \right]^{1/2} \right\} \quad (6.45)$$

is jointly convex for $\alpha > 1$, where

$$K = \tau_{AC}^{(1-\alpha)/2\alpha} \omega_{C}^{\alpha-1}/2^\alpha. \quad (6.46)$$

To this end, consider that we can write the trace function in (6.45) as

$$\text{Tr} \left\{ \left[ \left( K^{1/2} \theta_{BC} \right) \right]^{1/2} \rho_{ABC} \left[ \left( K^{1/2} \theta_{BC} \right) \right]^{1/2} \right\} \quad (6.47)$$

$$= \sup_{H \geq 0} \alpha \text{Tr} \left\{ H \rho_{ABC} \right\} - (\alpha - 1) \text{Tr} \left\{ \left[ H^{1/2} L^{1/2} \theta_{BC} L^{1/2} H^{1/2} \right]^{\alpha/(1-\alpha)} \right\}$$
where
\[ L \equiv \tau_{AC}^{(\alpha-1)/2\alpha} \omega_{C}^{(1-\alpha)/2\alpha}. \quad (6.48) \]

Observe that
\[ K^{1-\alpha}_{BC} K^\dagger = \left( L^{\alpha-1}_{BC} L^\dagger \right)^{-1}. \quad (6.49) \]

From the fact that the following map
\[ S \mapsto \text{Tr} \left\{ [T^\dagger S^p T]^{1/p} \right\} \]

is concave in \( S \) for a fixed \( T \) and for \(-1 \leq p \leq 1 \) (Lemma 5 of [20]) and the representation formula given in (6.47), we can then conclude that the function in (6.45) is jointly convex in \( \rho_{ABC} \) and \( \theta_{BC} \).

So it remains to prove the representation formula in (6.47). Since \( H^{1/2} L^{\alpha-1}_{BC} L^\dagger H^{1/2} \) and
\[ \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} H \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} \]
have the same nonzero eigenvalues, the expression on the second line of (6.47) is equal to
\[ \sup_{H \geq 0} \alpha \text{Tr} \{ H \rho_{ABC} \} - (\alpha - 1) \text{Tr} \left\{ \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} H \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} \right\}^{\alpha/(\alpha-1)}. \quad (6.52) \]

Let us show that this supremum is given by the expression on the first line of (6.47). To prove this, we observe that the supremum is attained and that the Euler-Lagrange equation for the optimal \( H \) reads
\[ \alpha \rho_{ABC} - \alpha \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} H \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} \left[ \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} \right]^{1/(\alpha-1)} \left( L^{\alpha-1}_{BC} L^\dagger \right)^{1/2} = 0, \quad (6.53) \]

so that the optimal \( H \) is equal to
\[ \left( K^{1-\alpha}_{BC} K^\dagger \right)^{1/2} \left( K^{1-\alpha}_{BC} K^\dagger \right)^{1/2} \rho_{ABC} \left( K^{1-\alpha}_{BC} K^\dagger \right)^{1/2} \rho_{ABC} \left( K^{1-\alpha}_{BC} K^\dagger \right)^{1/2} \], \quad (6.54) \]

where we recall (6.49). Plugging this optimal \( H \) into (6.52) gives the desired equality in (6.47).

To obtain monotonicity with respect to local maps on system \( A \), it suffices to prove joint convexity of the following function for \( \alpha > 1 \),
\[ (\rho_{ABC}, \tau_{AC}) \mapsto \text{Tr} \left\{ \left[ \tau_{AC}^{\alpha} \omega_{C}^{\alpha} \theta_{BC}^{\alpha} \omega_{C}^{\alpha} \tau_{AC}^{\alpha} \right]^{1/2} \rho_{ABC} \left[ \tau_{AC}^{\alpha} \omega_{C}^{\alpha} \theta_{BC}^{\alpha} \omega_{C}^{\alpha} \tau_{AC}^{\alpha} \right]^{1/2} \right\}^\alpha. \quad (6.55) \]

However, as in previous proofs, we observe that the RHS of the above equation is equal to
\[ \text{Tr} \left\{ \left( K'_{AC}^{1-\alpha} (K')^\dagger \right)^{1/2} \rho_{ABC} \left( K'_{AC}^{1-\alpha} (K')^\dagger \right)^{1/2} \right\}^\alpha, \quad (6.56) \]
where
\[ K' = V^{\frac{1-\alpha}{2\alpha}} \omega^{\frac{\alpha-1}{2\alpha}}_C, \]  
(6.57)
with \( V \) a unitary chosen so that (6.43) holds. The proof then proceeds similarly as in the first case given above.

**Theorem 20** All variations of the sandwiched Rényi conditional mutual information are monotone with respect to quantum operations on systems \( A \) and \( B \) for all \( \alpha \in [1/2, \infty) \),

\[ \tilde{I}_\alpha (A; B|C)_\rho \geq \tilde{I}_\alpha (A; B|C)_\omega, \]  
(6.58)
where \( \rho_{ABC} \) is a tripartite quantum state, and \( \omega_{ABC} \equiv (N_A \otimes M_B) (\rho_{ABC}) \), with \( N_A \) and \( M_B \) CPTP maps that act on systems \( A \) and \( B \), respectively.

**Proof.** The argument is exactly the same as that in the proof of Theorem 10.

**Corollary 21** The variations of the sandwiched Rényi conditional mutual information derived from (4.12)-(4.19) are non-negative for all \( \alpha \in [1/2, \infty) \) for any density operator \( \rho_{ABC} \).

**Proof.** The argument is exactly the same as that in the proof of Corollary 11.

**Remark 22** It is an open question whether the following variations of the sandwiched Rényi conditional mutual information are non-negative,

\[ \min_{\tau_{AC}, \theta_{BC}} \tilde{D}_\alpha \left( \rho_{ABC} \left\| \left( \tau_{AC}^{(1-\alpha)/2} \rho_C^{(1-\alpha)/\alpha} \theta_{BC}^{(1-\alpha)/\alpha} \tau_{AC}^{(1-\alpha)/2} \right)^{\alpha/(1-\alpha)} \right) \right), \]  
(6.59)
\[ \min_{\sigma_{ABC}} \tilde{D}_\alpha \left( \rho_{ABC} \left\| \left( \sigma_{AC}^{(1-\alpha)/2} \rho_C^{(1-\alpha)/\alpha} \sigma_{BC}^{(1-\alpha)/\alpha} \sigma_{AC}^{(1-\alpha)/2} \right)^{\alpha/(1-\alpha)} \right) \right). \]  
(6.60)

### 7 Duality

A fundamental property of the conditional mutual information is a duality relation: For a four-party pure state \( \psi_{ABCD} \), the following equality holds

\[ I (A; B|C)_\psi = I (A; B|D)_\psi. \]

#### 7.1 Duality for the Rényi conditional mutual information

We show that duality holds for the following variation of the definition of the Rényi conditional mutual information,

\[ I_\alpha (A; B|C)_\rho = \min_{\sigma_{BC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha/(\alpha-1)} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{(\alpha-1)/2} \sigma_{BC}^{(\alpha-1)/2} \rho_{AC}^{(\alpha-1)/2} \right\}. \]  
(7.1)

**Theorem 23** The following duality relation holds for all \( \alpha \in [0, \infty) \) for a pure four-party state \( \psi_{ABCD} \):

\[ I_\alpha (A; B|C)_\psi = I_\alpha (B; A|D)_\psi. \]
Proof. Our proof exploits ideas used in the proof of Lemma 6 in [58]. It suffices to prove the following equivalent statement,

\[
\min_{\sigma_{BC}} \text{Tr} \left\{ \psi_{ABC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \right\} = \min_{\sigma_{AD}} \text{Tr} \left\{ \psi_{ABD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{AD}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \right\}. \quad (7.2)
\]

In fact, we will prove that

\[
\text{Tr} \left\{ \psi_{ABC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \right\} = \text{Tr} \left\{ \psi_{ABD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{AD}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \right\}, \quad (7.3)
\]

for any operator \(\sigma_{BC}^{1-\alpha}\) and its transposition \((\sigma_{AD}^{T})^{1-\alpha}\) with respect to a basis on the joint system AD. So this means that

\[
\text{Tr} \left\{ \psi_{ABC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \right\} = \text{Tr} \left\{ \psi_{ABD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{AD}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \right\}. \quad (7.4)
\]

But since this inequality holds for all \(\sigma_{BC}\), we can conclude

\[
\min_{\sigma_{BC}} \text{Tr} \left\{ \psi_{ABC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \right\} \geq \min_{\sigma_{AD}} \text{Tr} \left\{ \psi_{ABD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{AD}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \right\}. \quad (7.7)
\]

A similar line of reasoning gets the reverse inequality, so that we can conclude (7.2).

We now prove (7.3). For any four-party pure state, we have the following identities,

\[
\psi_{ABC} \psi_{AC} = \psi_{ABD} \psi_{BD}, \quad (7.8)
\]
\[
\psi_{AC} \psi_{AC} = \psi_{BD} \psi_{BD}, \quad (7.9)
\]
\[
\psi_{BC} \psi_{ABC} = \psi_{AD} \psi_{ABC}, \quad (7.10)
\]
\[
\psi_{C} \psi_{ABC} = \psi_{ABD} \psi_{ABC}. \quad (7.11)
\]

Then consider the following chain of equalities,

\[
\text{Tr} \left\{ \psi_{ABC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \right\} = \text{Tr} \left\{ \psi_{ABD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{AD}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \right\}, \quad (7.12)
\]
\[
= \langle \psi \psi_{ABC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \psi_{ABC} \rangle \langle \psi \rangle \quad (7.13)
\]
\[
= \langle \psi \psi_{AC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \psi_{AC} \rangle \langle \psi \rangle \quad (7.14)
\]
\[
= \langle \psi \psi_{AC}^{\alpha} \psi_{AC}^{(1-\alpha)/2} \psi_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{C}^{(\alpha-1)/2} \psi_{AC} \rangle \langle \psi \rangle \quad (7.15)
\]
\[
= \langle \psi \psi_{BD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \psi_{BD} \rangle \langle \psi \rangle \quad (7.16)
\]
\[
= \langle \psi \psi_{BD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \psi_{BD} \rangle \langle \psi \rangle \quad (7.17)
\]
\[
= \langle \psi \psi_{ABD}^{\alpha} \psi_{BD}^{(1-\alpha)/2} \psi_{D}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \psi_{D}^{(\alpha-1)/2} \psi_{ABD} \rangle \langle \psi \rangle \quad (7.18)
\]
The above expression can be written in terms of the maximally entangled vector $|\Gamma\rangle_{BC|AD}$ across the $BC|AD$ bipartite cut, as

$$\langle \Gamma |_{BC|AD}^{1/2} \psi^{(a-1)/2}_A \psi^{(a-1)/2}_{BD} |\psi_D^{(a-1)/2} \psi_{ABD}^{-1/2} \psi_{BD}^{-1/2} |\Gamma\rangle_{BC|AD} = \langle \Gamma |_{BC|AD} X_{ABD} \sigma_{BC}^{1/2} X_{ABD}^\dagger |\Gamma\rangle_{BC|AD},$$

(7.19)

where we define $X_{ABD} \equiv \psi^{(a-1)/2}_A \psi^{(a-1)/2}_{BD} |\psi_D^{(a-1)/2}$. Now, consider the vector $|\varphi\rangle_{ABCD} = X_{ABD} |\Gamma\rangle_{BC|AD}$. This vector can also be written as

$$|\varphi\rangle = \tau_{AD}^{1/2} |\Gamma\rangle_{BC|AD},$$

(7.20)

$$\tau_{AD} \equiv \text{Tr}_{BC} \left\{ X_{ABD}^\dagger |\Gamma\rangle |\Gamma\rangle_{BC|AD} X_{ABD} \right\}.$$  

(7.21)

Therefore, we have

$$\langle \Gamma |_{BC|AD} X_{ABD} \sigma_{BC}^{1/2} X_{ABD}^\dagger |\Gamma\rangle_{BC|AD}$$

(7.22)

$$= \langle \Gamma |_{BC|AD} \left( \tau_{AD}^{1/2} \otimes I_{BC} \right) \left( \sigma_{AD} \otimes \sigma_{BC}^{1/2} \otimes I_{BC} \right) |\Gamma\rangle_{BC|AD},$$

(7.23)

$$= \langle \Gamma |_{BC|AD} \tau_{AD} (\sigma_{AD})^{1/2} \otimes I_{BC} |\Gamma\rangle_{BC|AD},$$

(7.24)

$$= \text{Tr} \left\{ \tau_{AD} \left( \sigma_{AD}^{T} \right)^{1/2} \right\}.$$

(7.25)

Substituting back for $|\Gamma\rangle_{BC|AD}$ based on (7.21), we have

$$\text{Tr} \left\{ \sigma_{AD}^{T} \right\}$$

(7.26)

$$= \text{Tr} \left\{ \left( \sigma_{AD}^{T} \right)^{1/2} \text{Tr}_{BC} \left\{ X_{ABD}^\dagger |\Gamma\rangle |\Gamma\rangle_{BC|AD} X_{ABD} \right\} \right\}$$

(7.27)

$$= \text{Tr} \left\{ \left( \sigma_{AD}^{T} \right)^{1/2} X_{ABD}^\dagger |\Gamma\rangle |\Gamma\rangle_{BC|AD} X_{ABD} \right\}$$

(7.28)

$$= \langle \Gamma |_{BC|AD} X_{ABD} \left( \sigma_{AD}^{T} \right)^{1/2} X_{ABD}^\dagger |\Gamma\rangle_{BC|AD}$$

(7.29)

$$= \langle \Gamma |_{BC|AD} \psi^{(a-1)/2}_A \psi^{(a-1)/2}_{BD} |\psi_D^{(a-1)/2} \psi_{ABD}^{-1/2} \psi_{BD}^{-1/2} \psi_{ABD}^{-1/2} \psi_{BD}^{-1/2} |\Gamma\rangle_{BC|AD}$$

(7.30)

$$= \langle \psi | \psi^{(a-1)/2}_A \psi^{(a-1)/2}_{BD} |\psi_D^{(a-1)/2} \psi_{ABD}^{-1/2} \psi_{BD}^{-1/2} \psi_{ABD}^{-1/2} \psi_{BD}^{-1/2} |\psi\rangle$$

(7.31)

$$= \text{Tr} \left\{ \psi^{(a-1)/2}_A \psi^{(a-1)/2}_{BD} \psi^{(a-1)/2}_D \left( \sigma_{AD}^{T} \right)^{1/2} \psi^{(a-1)/2}_{ABD} \psi^{(a-1)/2}_{BD} \psi^{(a-1)/2}_{ABD} \psi^{(a-1)/2}_{BD} \right\}$$

(7.32)

$$= \text{Tr} \left\{ \psi^{(a-1)/2}_A \psi^{(a-1)/2}_{BD} \psi^{(a-1)/2}_D \left( \sigma_{AD}^{T} \right)^{1/2} \psi^{(a-1)/2}_{ABD} \psi^{(a-1)/2}_{BD} \psi^{(a-1)/2}_{ABD} \psi^{(a-1)/2}_{BD} \right\}.$$

(7.33)

\subsection*{7.2 Duality for the sandwiched Rényi conditional mutual information}

We show that duality holds for the following variation of the definition of the sandwiched Rényi conditional mutual information,

$$\overline{I}_{\alpha} (A; B|C) = \min_{\sigma_{BC}} \max_{\omega_C} \frac{\alpha}{\alpha - 1} \log \left\| \rho_{ABC}^{\alpha} \rho_{AC}^{\alpha} \omega_C^{\alpha} \rho_{ABD}^{\alpha} \rho_{AC}^{\alpha} \omega_C^{\alpha} \rho_{ABD}^{\alpha} \right\|_{\alpha},$$

(7.34)
Theorem 24 The following duality relation holds for all $\alpha \in [0, \infty]$ for a pure four-party state $\psi_{ABCD}$, 

$$\tilde{I}_\alpha(A;B|C)_{\psi} = \tilde{I}_\alpha(B;A|D)_{\psi}.$$ 

Proof. Our proof uses ideas similar to those in the proof of Theorem 23 and those in the proof of Theorem 10 in [43]. We begin by considering the case when $\alpha > 1$. We begin by recalling that it is possible to express the $\alpha$-norm with its dual norm (see, e.g., Lemma 12 of [43]),

$$\min_{\sigma_{BC}} \max_{\omega_C} \frac{1}{2} \left( \frac{1}{2} \langle (1-\alpha)/2\sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} - \frac{1}{2} \langle (1-\alpha)/2\sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}} \right) =$$

$$\min_{\sigma_{BC}} \max_{\omega_C} \max_{\tau_{ABCD}} \text{Tr} \left\{ \frac{1}{2} (1-\alpha)/2 \sigma_{BC} \omega_C^2 \sigma_{BC} \omega_C^2 \rho_{AC} \rho_{ABC} \rho_{ABC}^\dagger \right\} =$$

$$\min_{\sigma_{ABD}} \max_{\omega_{ABD}} \max_{\tau_D} \text{Tr} \left\{ \frac{1}{2} (1-\alpha)/2 \sigma_{ABD} \omega_{ABD}^2 \sigma_{ABD} \omega_{ABD}^2 \rho_{AD} \rho_{BD} \rho_{ABC} \rho_{ABC}^\dagger \right\}. \quad (7.35)$$

So it suffices to prove the following relation,

$$\min_{\sigma_{BC}} \max_{\omega_C} \max_{\tau_{ABCD}} \text{Tr} \left\{ \frac{1}{2} (1-\alpha)/2 \sigma_{BC} \omega_C^2 \sigma_{BC} \omega_C^2 \rho_{AC} \rho_{ABC} \rho_{ABC}^\dagger \right\} =$$

$$\min_{\sigma_{ABD}} \max_{\omega_{ABD}} \max_{\tau_D} \text{Tr} \left\{ \frac{1}{2} (1-\alpha)/2 \sigma_{ABD} \omega_{ABD}^2 \sigma_{ABD} \omega_{ABD}^2 \rho_{AD} \rho_{BD} \rho_{ABC} \rho_{ABC}^\dagger \right\}. \quad (7.36)$$

Indeed, we will prove that

$$\text{Tr} \left\{ \frac{1}{2} (1-\alpha)/2 \sigma_{BC} \omega_C^2 \sigma_{BC} \omega_C^2 \rho_{AC} \rho_{ABC} \rho_{ABC}^\dagger \right\} =$$

$$\text{Tr} \left\{ \frac{1}{2} (1-\alpha)/2 \sigma_{ABD} \omega_{ABD}^2 \sigma_{ABD} \omega_{ABD}^2 \rho_{AD} \rho_{BD} \rho_{ABC} \rho_{ABC}^\dagger \right\}, \quad (7.37)$$

from which one can conclude the equality above with optimizations (the reasoning here being similar to that near the beginning of the proof of the previous duality theorem).

Proceeding, we observe that

$$\text{Tr} \left\{ \frac{1/2}{(1-\alpha)/2} \sigma_{BC} \omega_C^2 \sigma_{BC} \omega_C^2 \rho_{AC} \rho_{ABC} \rho_{ABC}^\dagger \right\} =$$

$$= \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.38)$$

$$= \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.39)$$

$$= \langle \psi | \rho_{AC} \frac{1}{2} \sigma_{AC} \omega_C^2 \left( \tau_{D} \right) \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \psi | \rho_{BD} \frac{1}{2} \sigma_{BD} \omega_{ABD}^2 \left( \tau_{D} \right) \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.40)$$

$$= \langle \psi | \rho_{AC} \frac{1}{2} \sigma_{AC} \omega_C^2 \left( \tau_{D} \right) \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \psi | \rho_{BD} \frac{1}{2} \sigma_{BD} \omega_{ABD}^2 \left( \tau_{D} \right) \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.41)$$

$$= \langle \psi | \rho_{AC} \frac{1}{2} \sigma_{AC} \omega_C^2 \left( \tau_{D} \right) \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \psi | \rho_{BD} \frac{1}{2} \sigma_{BD} \omega_{ABD}^2 \left( \tau_{D} \right) \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.42)$$

$$= \langle \psi | \rho_{AC} \frac{1}{2} \sigma_{AC} \omega_C^2 \left( \tau_{D} \right) \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \psi | \rho_{BD} \frac{1}{2} \sigma_{BD} \omega_{ABD}^2 \left( \tau_{D} \right) \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.43)$$

$$= \langle \psi | \rho_{AC} \frac{1}{2} \sigma_{AC} \omega_C^2 \left( \tau_{D} \right) \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \psi | \rho_{BD} \frac{1}{2} \sigma_{BD} \omega_{ABD}^2 \left( \tau_{D} \right) \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.44)$$

$$= \langle \psi | \rho_{AC} \frac{1}{2} \sigma_{AC} \omega_C^2 \left( \tau_{D} \right) \langle \sigma_{AC} \sigma_{BC} \omega_C^2 \rangle_{\tilde{I}_\alpha(A;B|C)_{\psi}} = \langle \psi | \rho_{BD} \frac{1}{2} \sigma_{BD} \omega_{ABD}^2 \left( \tau_{D} \right) \langle \sigma_{AD} \sigma_{BD} \omega_{ABD}^2 \rangle_{\tilde{I}_\alpha(B;A|D)_{\psi}}. \quad (7.45)$$

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Let $|\varphi\rangle_{ABCD}$ be the following vector,

$$|\varphi\rangle_{ABCD} \equiv X_{ABD} |\Gamma\rangle_{ABD|C} ; \tag{7.46}$$

$$X_{ABD} \equiv \left( \tau_T^{(a-1)/2a} \rho_{BD}^{(a-1)/2a} \rho_{ABD}^T \omega_{ABD}^{(a-1)/2a} \right). \tag{7.47}$$

Then taking $|\Gamma\rangle_{AD|BC}$, we can write

$$|\varphi\rangle_{ABCD} \equiv \varsigma_{AD} \otimes I_{BC} |\Gamma\rangle_{AD|BC} ; \tag{7.48}$$

$$\varsigma_{AD} \equiv \text{Tr}_{BC} \left\{ X_{ABD} |\Gamma\rangle (|\Gamma\rangle_{ABD|C} X_{ABD}^\dagger \right\}. \tag{7.49}$$

Thus, we have that

$$\langle \Gamma |_{ABD|C} \left( \omega_{ABD}^T \right)^{a-1/2a} \rho_{ABD}^{1/2} \rho_{BD}^{1-\alpha} \left( \tau_T^{(a-1)/2a} \sigma_{BC}^{1-\alpha/2} \left( \tau_T^{(a-1)/2a} \rho_{BD}^{1/2} \rho_{ABD} \left( \omega_{ABD}^{(a-1)/2a} \right) \right)^{a-1/2a} |\Gamma\rangle_{ABD|C} \right)$$

$$= \langle \Gamma |_{ABD|C} X_{ABD}^\dagger \sigma_{BC}^{(1-\alpha)/\alpha} X_{ABD} |\Gamma\rangle_{ABD|C} \tag{7.50}$$

$$= \langle \Gamma |_{AD|BC} \left( \varsigma_{AD} \otimes I_{BC} \right) \sigma_{BC}^{(1-\alpha)/\alpha} \left( \varsigma_{AD} \otimes I_{BC} \right) |\Gamma\rangle_{AD|BC} \tag{7.51}$$

$$= \langle \Gamma |_{AD|BC} \varsigma_{AD} \left( \sigma_{AD}^{(1-\alpha)/\alpha} \right) |\Gamma\rangle_{AD|BC} \tag{7.52}$$

$$= \text{Tr} \left\{ \varsigma_{AD} \left( \sigma_{AD}^{(1-\alpha)/\alpha} \right) \right\} \tag{7.53}$$

$$= \text{Tr} \left\{ X_{ABD} |\Gamma\rangle \langle \Gamma |_{ABD|C} X_{ABD}^\dagger \left( \sigma_{AD}^{(1-\alpha)/\alpha} \right) \right\} \tag{7.54}$$

$$= \text{Tr} \left\{ X_{ABD} |\Gamma\rangle \langle \Gamma |_{ABD|C} X_{ABD}^\dagger \left( \sigma_{AD}^{(1-\alpha)/\alpha} \right) \right\} \tag{7.55}$$

$$= \langle \Gamma |_{ABD|C} X_{ABD}^\dagger \left( \sigma_{AD}^{(1-\alpha)/\alpha} \right) X_{ABD} |\Gamma\rangle_{ABD|C} \tag{7.56}$$

$$= \text{Tr} \left\{ X_{ABD}^\dagger \left( \sigma_{AD}^{(1-\alpha)/\alpha} \right) X_{ABD} \right\} \tag{7.57}$$

$$= \text{Tr} \left\{ \left( \omega_{ABD}^T \right)^{a-1/2a} \rho_{BD}^{1/2} \rho_{ABD} \left( \tau_T^{(a-1)/2a} \sigma_{AD}^{(1-\alpha)/\alpha} \left( \tau_T^{(a-1)/2a} \rho_{BD}^{1/2} \rho_{ABD} \left( \omega_{ABD}^{(a-1)/2a} \right) \right)^{a-1/2a} \right) \right\} \tag{7.58}$$

$$= \text{Tr} \left\{ \rho_{BD}^{1/2} \rho_{ABD} \left( \tau_T^{(a-1)/2a} \sigma_{AD}^{(1-\alpha)/\alpha} \left( \tau_T^{(a-1)/2a} \rho_{BD}^{1/2} \rho_{ABD} \left( \omega_{ABD}^{(a-1)/2a} \right) \right)^{a-1/2a} \right) \right\} \tag{7.59}$$

8 Monotonicity in $\alpha$

From numerical evidence and analytical evidence given below, we think it is natural to put forward the following conjecture.

**Conjecture 25** Both kinds of Rényi conditional mutual information are monotone increasing in $\alpha$, in the sense that, for $0 \leq \alpha \leq \beta$

$$I_\alpha (A; B|C)_\rho \leq I_\beta (A; B|C)_\rho , \tag{8.1}$$

$$\overline{I}_\alpha (A; B|C)_\rho \leq \overline{I}_\beta (A; B|C)_\rho . \tag{8.2}$$
Note that this conjecture does not follow straightforwardly from the following monotonicity
\[ D_\alpha (\rho \| \sigma) \leq D_\beta (\rho \| \sigma), \] (8.3)
\[ \tilde{D}_\alpha (\rho \| \sigma) \leq \tilde{D}_\beta (\rho \| \sigma), \] (8.4)
which holds for \( 0 \leq \alpha \leq \beta \). This is because the second operator in both \( I_\alpha (A; B|C) \) and \( \tilde{I}_\alpha (A; B|C) \) changes with \( \alpha \).

We can provide evidence that the monotonicity in \( \alpha \) holds within a neighborhood of \( \alpha = 1 \), by following a proof technique of Tomamichel et al. (Lemma 3 in [58]; see also the proof of Theorem 7 in [33]). For simplicity, we consider the following variant of the Rényi conditional mutual information,
\[
\frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho^{\alpha}_{\text{ABC}} \rho^{\frac{\alpha-1}{2}}_{\text{BC}} \rho^{\frac{\alpha-1}{2}}_{\text{AC}} \rho^{\frac{\alpha-1}{2}}_{\text{C}} \rho^{\frac{\alpha-1}{2}}_{\text{BC}} \right\}.
\] (8.5)
If the first derivative of this function with respect to \( \alpha \) is non-negative, then we can conclude that this function is monotone increasing. So, we introduce a variable \( \beta = \alpha - 1 \), and with
\[
Y(\beta) \equiv \rho^{1+\beta}_{\text{ABC}} \rho^{\frac{\beta}{2}}_{\text{BC}} \rho^{\frac{\beta}{2}}_{\text{AC}} \rho^{\frac{\beta}{2}}_{\text{C}} \rho^{\frac{\beta}{2}}_{\text{BC}}
\] (8.6)
it follows that
\[
\frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho^{\alpha}_{\text{ABC}} \rho^{\frac{\alpha-1}{2}}_{\text{BC}} \rho^{\frac{\alpha-1}{2}}_{\text{AC}} \rho^{\frac{\alpha-1}{2}}_{\text{C}} \rho^{\frac{\alpha-1}{2}}_{\text{BC}} \right\} = \frac{1}{\beta} \log \text{Tr} \{ Y(\beta) \}.
\] (8.7)
Then since \( d\beta/d\alpha = 1 \),
\[
\frac{d}{d\alpha} \left[ \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho^{\alpha}_{\text{ABC}} \rho^{\frac{\alpha-1}{2}}_{\text{BC}} \rho^{\frac{\alpha-1}{2}}_{\text{AC}} \rho^{\frac{\alpha-1}{2}}_{\text{C}} \rho^{\frac{\alpha-1}{2}}_{\text{BC}} \right\} \right] = \frac{d}{d\beta} \left[ \frac{1}{\beta} \log \text{Tr} \{ Y(\beta) \} \right].
\] (8.8)
We first explicitly compute the derivative before giving our evidence that the monotonicity holds in a neighborhood of \( \alpha = 1 \),
\[
\frac{d}{d\beta} \left[ \frac{1}{\beta} \log \text{Tr} \{ Y(\beta) \} \right] = -\frac{1}{\beta^2} \log \text{Tr} \{ Y(\beta) \} + \frac{\text{Tr} \{ \frac{d}{d\beta} Y(\beta) \}}{\beta \text{Tr} \{ Y(\beta) \}}
\] (8.9)
Then
\[
\beta \frac{d}{d\beta} Y(\beta) = \beta \frac{d}{d\beta} \left[ \rho^{1+\beta}_{\text{ABC}} \rho^{\frac{\beta}{2}}_{\text{BC}} \rho^{\frac{\beta}{2}}_{\text{AC}} \rho^{\frac{\beta}{2}}_{\text{C}} \rho^{\frac{\beta}{2}}_{\text{BC}} \right]
\] (8.11)
\[
= \beta \left[ \log \rho_{\text{ABC}} \right] Y(\beta) + \beta \rho^{1+\beta}_{\text{ABC}} \left[ \log \rho^{\frac{1}{2}}_{\text{BC}} \right] \rho^{\frac{\beta}{2}}_{\text{BC}} \rho^{\frac{\beta}{2}}_{\text{AC}} \rho^{\frac{\beta}{2}}_{\text{C}} \rho^{\frac{\beta}{2}}_{\text{BC}} + \beta \rho^{1+\beta}_{\text{ABC}} \rho^{\frac{\beta}{2}}_{\text{BC}} \rho^{\frac{\beta}{2}}_{\text{AC}} \rho^{\frac{\beta}{2}}_{\text{C}} \rho^{\frac{\beta}{2}}_{\text{BC}}
\] (8.12)
\[
= \log \rho_{\text{ABC}} Y(\beta) + \rho^{1+\beta}_{\text{ABC}} \left[ \log \rho^{\frac{1}{2}}_{\text{BC}} \right] \rho^{\frac{\beta}{2}}_{\text{BC}} \rho^{\frac{\beta}{2}}_{\text{AC}} \rho^{\frac{\beta}{2}}_{\text{C}} \rho^{\frac{\beta}{2}}_{\text{BC}}
\] (8.13)
Our evidence in support of the conjecture is to consider a Taylor expansion of $Y(\beta)$ in a neighborhood of $\beta$ near zero. Indeed, consider that

$$X^{1+\beta} = X + \beta X \log X + \frac{\beta^2}{2} X \log^2 X + O(\beta^3), \quad (8.14)$$

$$X^\beta = I + \beta \log X + \frac{\beta^2}{2} \log^2 X + O(\beta^3). \quad (8.15)$$

For our case, we make the following substitutions into $\text{Tr}\{Y(\beta)\}$,

$$\rho_{ABC}^{1+\beta} = \rho_{ABC} + \beta \rho_{ABC} \log \rho_{ABC} + \frac{\beta^2}{2} \rho_{ABC} \log^2 \rho_{ABC} + O(\beta^3), \quad (8.16)$$

$$\sigma_{BC}^\beta = I - \frac{\beta}{2} \log \sigma_{BC} + \frac{\beta^2}{8} \log^2 \sigma_{BC} + O(\beta^3), \quad (8.17)$$

$$\sigma_C^{\beta} = I + \frac{\beta}{2} \log X + \frac{\beta^2}{8} \log^2 X + O(\beta^3), \quad (8.18)$$

$$\sigma_{AC}^{-\beta} = I - \beta \log X + \frac{\beta^2}{2} \log^2 X + O(\beta^3). \quad (8.19)$$

After a rather tedious calculation, we find that

$$\text{Tr}\{Y(\beta)\} = \text{Tr}\{\rho_{ABC}\} + \beta I(A;B|C)_{\rho} + \frac{\beta^2}{2} \left[V(A;B|C)_{\rho} + \left[I(A;B|C)_{\rho}\right]^2\right] + O(\beta^3), \quad (8.20)$$

where $V(A;B|C)_{\rho}$ is quantity for which it seems natural to call the conditional mutual information variance of a tripartite state $\rho_{ABC}$,

$$V(A;B|C)_{\rho} \equiv \text{Tr}\left\{\rho_{ABC} \left[\log \rho_{ABC} - \log \rho_{AC} - \log \rho_{BC} + \log \rho_C - I(A;B|C)_{\rho}\right]^2\right\}. \quad (8.21)$$

For any Hermitian operator $H$, we have that

$$\langle H^2 \rangle_{\rho} - \langle H \rangle_{\rho}^2 \geq 0. \quad (8.22)$$

So taking $H \equiv \log \rho_{ABC} - \log \rho_{AC} - \log \rho_{BC} + \log \rho_C$, we conclude that $V(A;B|C)_{\rho} \geq 0$, an observation central to our development here. We will make the abbreviations $I \equiv I(A;B|C)_{\rho}$ and $V \equiv V(A;B|C)_{\rho}$ from here forward, so that

$$\text{Tr}\{Y(\beta)\} = 1 + \beta I + \frac{\beta^2}{2} [V + I^2] + O(\beta^3) \quad (8.23)$$

So this implies that

$$\beta \text{Tr}\left\{\frac{d}{d\beta} Y(\beta)\right\} = \beta I + \beta^2 [V + I^2] + O(\beta^3), \quad (8.24)$$

$$\text{Tr}\{Y(\beta)\} \log \text{Tr}\{Y(\beta)\} = \left[1 + \beta I + \frac{\beta^2}{2} [V + I^2] + O(\beta^3)\right] \log \left[1 + \beta I + \frac{\beta^2}{2} [V + I^2] + O(\beta^3)\right]. \quad (8.25)$$
Then for small $\beta$, we have the following Taylor expansion for the logarithm,
\[
\log \left[ 1 + \beta I + \frac{\beta^2}{2} [V + I^2] + O(\beta^3) \right] = \beta I + \frac{\beta^2}{2} [V + I^2] - \frac{\beta^2 I^2}{2} + O(\beta^3) \tag{8.26}
\]
which gives
\[
\text{Tr} \{ Y(\beta) \} \log \text{Tr} \{ Y(\beta) \} = \left[ 1 + \beta I + \frac{\beta^2}{2} [V + I^2] + O(\beta^3) \right] \left[ \beta I + \frac{\beta^2}{2} V + O(\beta^3) \right] = \beta I + \frac{\beta^2}{2} V + \beta^2 I^2 + O(\beta^3).
\tag{8.28}
\]
Finally, we can say that
\[
\beta \text{Tr} \left\{ \frac{d}{d\beta} Y(\beta) \right\} - \text{Tr} \{ Y(\beta) \} \log \text{Tr} \{ Y(\beta) \} = \beta I + \beta^2 [V + I^2] - \left[ \beta I + \frac{\beta^2}{2} V + \beta^2 I^2 \right] + O(\beta^3) = \frac{\beta^2}{2} V + O(\beta^3).
\tag{8.29}
\]
Since $V \geq 0$, we can conclude that as long as $\beta$ is very near to zero, all terms $O(\beta^3)$ are negligible in comparison to $\frac{\beta^2}{2} V$, and the monotonicity holds in such a regime.

### 8.1 Special cases of the conjecture

We can prove that the conjecture is true in a number of special cases, due to the special form that the Rényi conditional mutual information takes in these cases. For simplicity, we will redefine the quantities for the rest of this section as follows,
\[
I_\alpha (A; B|C)_\rho = \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha - 1} \rho_{BC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{AC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{CB}^{\frac{1}{2} - \frac{1}{\alpha}} \right\}, \tag{8.30}
\]
and we redefine
\[
I_0 (A; B|C)_\rho = - \log \text{Tr} \left\{ \Pi^{\rho_{ABC}} \rho_{BC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{AC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{CB}^{\frac{1}{2} - \frac{1}{\alpha}} \right\}, \tag{8.32}
\]
\[
I_{\min} (A; B|C)_\rho = I_{1/2} (A; B|C)_\rho = - \log F \left( \rho_{ABC}, \rho_{BC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{AC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{CB}^{\frac{1}{2} - \frac{1}{\alpha}} \right), \tag{8.33}
\]
\[
I_2 (A; B|C)_\rho = \log \text{Tr} \left\{ \rho_{ABC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{BC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{AC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{CB}^{\frac{1}{2} - \frac{1}{\alpha}} \right\}, \tag{8.34}
\]
\[
I_{\max} (A; B|C)_\rho = \bar{I}_{\infty} (A; B|C)_\rho = \inf \left\{ \lambda : \rho_{ABC} \leq 2^\lambda \rho_{BC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{AC}^{\frac{1}{2} - \frac{1}{\alpha}} \rho_{CB}^{\frac{1}{2} - \frac{1}{\alpha}} \right\}. \tag{8.35}
\]
Recall that the following inequality holds for all $\alpha > 0$ \[14\]:
\[
\bar{D}_\alpha (\rho||\sigma) \leq D_\alpha (\rho||\sigma). \tag{8.36}
\]
Using the monotonicity given in (8.4), we can conclude that

$$I_0 (A; B|C)_\rho \leq I_2 (A; B|C)_\rho$$  \hfill (8.37) \\
$$I_{\min} (A; B|C)_\rho \leq I_{\max} (A; B|C)_\rho.$$  \hfill (8.38)

However, we cannot relate to the (von Neumann entropy based) conditional mutual information because its representation in terms of the relative entropy does not feature the operator $\rho_{BC} \rho_{C}^{-\frac{1}{2}} \rho_{AC} \rho_{C}^{-\frac{1}{2}} \rho_{BC}$ as its second argument but instead has $\exp \{ \log \rho_{BC} + \log \rho_{AC} - \log \rho_C \}$.

### 8.2 Implications of the conjecture

The implications of Conjecture 25 are highly nontrivial. For example, if it were true, then we could conclude the following relation,

$$I (A; B|C)_\rho \geq I_{\min} (A; B|C)_\rho$$  \hfill (8.39) \\
$$= - \log F \left( \rho_{ABC}, \frac{1}{2} \rho_{BC} \rho_{C}^{-\frac{1}{2}} \rho_{AC} \rho_{C}^{-\frac{1}{2}} \rho_{BC} \right)$$  \hfill (8.40) \\
$$\geq - \log \left[ 1 - \frac{1}{2} \left\| \rho_{ABC} - \frac{1}{2} \rho_{BC} \rho_{C}^{-\frac{1}{2}} \rho_{AC} \rho_{C}^{-\frac{1}{2}} \rho_{BC} \right\|_1 \right]$$  \hfill (8.41) \\
$$\geq \frac{1}{4} \left\| \rho_{ABC} - \frac{1}{2} \rho_{BC} \rho_{C}^{-\frac{1}{2}} \rho_{AC} \rho_{C}^{-\frac{1}{2}} \rho_{BC} \right\|_1^2,$$  \hfill (8.42)

where the first inequality would follow from Conjecture 25, the second is a result of well known relations between trace distance and fidelity 21, and the last is a consequence of the inequality $- \log (1 - x) \geq x$, valid for $x \leq 1$. Thus, the truth of Conjecture 25 would establish the truth of an open conjecture from 31 (up to a constant). As pointed out in 31, this would then imply that for tripartite states $\rho_{ABC}$ with conditional mutual information $I (A; B|C)_\rho$ small (i.e. states that fulfill strong subadditivity with near equality), Petz’s transpose map 24

$$R_{AC \rightarrow ABC}(\cdot) = \frac{1}{2} \rho_{BC} \rho_{C}^{-\frac{1}{2}} \cdot \rho_{C}^{-\frac{1}{2}} \rho_{BC}$$  \hfill (8.43)

for the partial trace over $B$ is good for recovering $\rho_{ABC}$ from $\rho_{AC}$. Hence, even though $\rho_{ABC}$ does not have to be close to a quantum Markov state in relative entropy distance if $I (A; B|C)_\rho$ is small (as shown in 29), $B$ would still be nearly independent of $A$ from the perspective of $C$ in the sense that $\rho_{ABC}$ can be approximately recovered from $\rho_{AC}$ alone. This would give a characterization of states that fulfill strong subadditivity with near equality.

### 9 Dimension bounds and other inequalities

For the bounds in this section, we take the following definitions:

$$I_\alpha (A; B|C)_\rho = \min_{\sigma_{ABC}} D_{\alpha} \left( \rho_{ABC} \left\| \left[ \sigma_{AC}^{(1-\alpha)/2} \sigma_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_{AC}^{(\alpha-1)/2} \right]^{1/\alpha} \right\|_{\infty} \right),$$

$$\bar{I}_\alpha (A; B|C)_\rho = \min_{\sigma_{ABC}} \bar{D}_{\alpha} \left( \rho_{ABC} \left\| \left[ \sigma_{AC}^{(1-\alpha)/2} \sigma_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_{AC}^{(\alpha-1)/2} \right]^{1/\alpha} \right\|_{\infty} \right).$$
Proposition 26 The following dimension bounds hold

\[ I_\alpha (A; B|C)_\rho \leq \log d_A - H_\alpha (A|BC)_\rho , \]  
\[ I_\alpha (A; B|C)_\rho \leq \log d_B - H_\alpha (B|AC)_\rho , \]  
\[ \overline{I}_\alpha (A; B|C)_\rho \leq \log d_A - \overline{H}_\alpha (A|BC)_\rho , \]  
\[ \overline{I}_\alpha (A; B|C)_\rho \leq \log d_B - \overline{H}_\alpha (B|AC)_\rho . \]

Proof. The inequality in (9.2) follows from

\[ I_\alpha (A; B|C)_\rho = \min_{\sigma_{ABC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{1-\alpha} \sigma_{AC}^{(1-\alpha)/2} \sigma_{BC}^{(\alpha-1)/2} \right\} \]  
\[ \leq \min_{\sigma_{AC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha} \sigma_{AC}^{(1-\alpha)/2} \sigma_{BC}^{(\alpha-1)/2} \left( \pi_B \otimes \sigma_C \right)^{1-\alpha} \sigma_C^{(\alpha-1)/2} \sigma_{AC}^{(1-\alpha)/2} \right\} \]  
\[ = \min_{\sigma_{AC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha} \sigma_{AC}^{1-\alpha} \right\} \]  
\[ = \log d_B - \left( \min_{\sigma_{AC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha} \sigma_{AC}^{1-\alpha} \right\} \right) \]  
\[ = \log d_B - H_\alpha (B|AC)_\rho , \]

where \( \pi_B = I_B/d_B \) and \( H_\alpha (B|AC)_\rho \equiv -\min_{\sigma_{AC}} \frac{1}{\alpha - 1} \log \text{Tr} \{ \rho_{ABC}^{\alpha} \sigma_{AC}^{1-\alpha} \} \). The bound in (9.1) follows similarly by choosing \( \sigma_{ABC} = \pi_A \otimes \sigma_{BC} \). The proofs for the sandwiched Rényi CMIs follow similarly, except we end up with the sandwiched Rényi conditional entropy in the upper bound.

Corollary 27 The following dimension bound holds for \( \alpha \in [0, 2] \)

\[ I_\alpha (A; B|C)_\rho \leq 2 \min \{ \log d_A, \log d_B \} , \]  

and the following one holds for \( \alpha \in [1/2, \infty] \),

\[ \overline{I}_\alpha (A; B|C)_\rho \leq 2 \min \{ \log d_A, \log d_B \} . \]

Proof. To prove the first one, we use the duality relation proved in [58]. From the above theorem, we know that

\[ I_\alpha (A; B|C)_\rho \leq \log d_A + \min_{\sigma_{BC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha} \sigma_{BC}^{1-\alpha} \right\} \]  
\[ \leq \log d_A + \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^{\alpha} \sigma_{BC}^{1-\alpha} \right\} \]  
\[ \equiv \log d_A - H_\alpha^A (A|BC)_\rho \]  
\[ = \log d_A + H_\beta^A (A|D)_\rho \]  
\[ \leq \log d_A + H_\beta (A)_\rho \]  
\[ \leq 2 \log d_A , \]

where we are using the notation from [57]. The third equality follows from the duality relation mentioned above, i.e.,

\[ H_\alpha^A (A|BC)_\rho = -H_\beta^A (A|D)_\rho , \]  

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where $\rho_{ABCD}$ is a purification of $\rho_{ABC}$ and $\beta$ is chosen so that $\alpha + \beta = 2$. The third inequality follows from data processing and the last from a dimension bound on the Rényi entropy.

The second inequality in the statement of the corollary follows from the duality of the sandwiched conditional Rényi entropy,

$$\tilde{H}_\alpha (A|BC)_\rho = -\tilde{H}_\beta (A|D)_\rho,$$

where $\rho_{ABCD}$ is a purification of $\rho_{ABC}$ and $\beta$ is chosen so that $\frac{1}{\alpha} + \frac{1}{\beta} = 2$. So this means that

$$\tilde{I}_\alpha (A; B|C)_\rho \leq \log d_A - \tilde{H}_\alpha (A|BC)_\rho,$$

$$= \log d_A + \tilde{H}_\beta (A|D)_\rho,$$

$$\leq \log d_A + \tilde{H}_\beta (A)_\rho,$$

$$\leq 2 \log d_A,$$

where the second inequality follows from data processing and the last is a universal bound on the Rényi entropy.

**Proposition 28** The following bounds hold

$$I_\alpha (A; B|C)_\rho \leq I_\alpha (A; BC)_\rho,$$

$$I_\alpha (A; B|C)_\rho \leq I_\alpha (B; AC)_\rho,$$

$$\tilde{I}_\alpha (A; B|C)_\rho \leq \tilde{I}_\alpha (A; BC)_\rho,$$

$$\tilde{I}_\alpha (A; B|C)_\rho \leq \tilde{I}_\alpha (B; AC)_\rho.$$

**Proof.** A proof for the first inequality follows from

$$I_\alpha (A; B|C)_\rho = \min_{\sigma_{ABC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^\alpha \sigma_{AC}^{(1-\alpha)/2} \sigma_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_{C}^{(\alpha-1)/2} \sigma_{AC}^{(1-\alpha)/2} \right\},$$

$$\leq \min_{\sigma_{BC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^\alpha (\rho_A \otimes \sigma_C)^{(1-\alpha)/2} \sigma_{C}^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_{C}^{(\alpha-1)/2} (\rho_A \otimes \sigma_C)^{(1-\alpha)/2} \right\},$$

$$= \min_{\sigma_{BC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^\alpha (\rho_A^{1-\alpha} \otimes \sigma_{BC}^{1-\alpha}) \right\},$$

$$\equiv I_\alpha (A; BC)_\rho.$$  

A proof for the second inequality follows similarly by choosing $\sigma_{ABC} = \rho_B \otimes \sigma_{AC}$. Proofs for the last two inequalities are similar, except the sandwiched Rényi mutual information is defined for a bipartite state $\rho_{AB}$ as

$$\tilde{I}_\alpha (A; B)_\rho \equiv \min_{\sigma_B} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \left[ (\rho_A \otimes \sigma_B)^{(1-\alpha)/2} \rho_{AB} (\rho_A \otimes \sigma_B)^{(1-\alpha)/2} \right]^{\alpha} \right\}.$$  

\[34\]
10 Quantities derived from the Rényi conditional quantum mutual information

This section delineates several Rényi quantities that can be derived from the Rényi conditional mutual informations.

10.1 Rényi quantum discord

We obtain a Rényi generalization of the quantum discord [46], since we can write the discord of a bipartite state \( \rho_{AB} \) as

\[
I(A;B) - \max_{\{A\}} I(A;X) = \min_{\{A\}} I(A;B) - I(A;X) \quad (10.1)
\]

\[
= \min_{\{A\}} I(A;EX) - I(A;X) \quad (10.2)
\]

\[
= \min_{\{A\}} I(A;E|X) \quad (10.3)
\]

where the optimization is over all POVMs acting on the system \( B \), with classical output \( X \). We can then find an isometric extension of any such measurement, where we label the environment system as \( E \). So we just define the Rényi quantum discord in the following way,

\[
\min_{\{A\}} I_\alpha(A;E|X) \quad (10.4)
\]

10.2 Rényi squashed entanglement

The squashed entanglement of a bipartite state \( \rho_{AB} \) is defined as

\[
E_{sq}^\alpha(A;B)_\rho \equiv \frac{1}{2} \inf_{\rho_{ABE}} \{ I(A;B|E) : \rho_{AB} = \text{Tr}_E \{ \rho_{ABE} \} \}.
\]

Thus, a straightforward Rényi generalization is as follows,

\[
E_{sq}^\alpha(A;B)_\rho \equiv \frac{1}{2} \inf_{\rho_{ABE}} \{ I_\alpha(A;B|E) : \rho_{AB} = \text{Tr}_E \{ \rho_{ABE} \} \}.
\]

It remains the topic of future research to investigate this quantity in full detail.

10.3 Multipartite Information

We also obtain generalizations of the multipartite information of a state \( \rho_{A_1\cdots A_l} \)

\[
I(A_1; A_2; \cdots ; A_l)_\rho \equiv \sum_{i=1}^l H(A_i)_\rho - H(A_1 A_2 \cdots A_l)_\rho,
\]

which has appeared in various contexts (see [61] and references therein). This quantity can be written as a relative entropy as follows,

\[
I(A_1; A_2; \cdots ; A_l)_\rho = D(\rho_{A_1\cdots A_l} \| \exp \left\{ \sum_{i=1}^l \log \rho_{A_i} \right\}).
\]
Thus, we obtain two natural Rényi generalizations of this measure, defined as

\[
I_{\alpha} (A_1; A_2; \ldots ; A_l)_{\rho} \equiv D_\alpha \left( \rho_{A_1 \cdots A_l} \| \left[ \rho_{A_1}^{(1-\alpha)/2} \cdots \rho_{A_2}^{(1-\alpha)/2} \rho_{A_1} \cdots \rho_{A_l}^{(1-\alpha)/2} \right]^{1/(1-\alpha)} \right),
\]

\[
\tilde{I}_{\alpha} (A_1; A_2; \ldots ; A_l)_{\rho} \equiv \tilde{D}_\alpha \left( \rho_{A_1 \cdots A_l} \| \left[ \rho_{A_1}^{(1-\alpha)/2} \cdots \rho_{A_2}^{(1-\alpha)/2} \rho_{A_1} \cdots \rho_{A_l}^{(1-\alpha)/2} \right]^{\alpha/(1-\alpha)} \right).
\]

These quantities satisfy monotonicity under local quantum operations for \( \alpha \in [0, 2] \) and \( \alpha \in [1/2, \infty] \), respectively, and they reduce to the von Neumann entropy in the limit as \( \alpha \to 1 \). The proofs for these facts follow the approaches outlined in previous sections.

## 11 Discussion

This paper has defined several Rényi generalizations of the conditional quantum mutual information (CQMI) quantities that satisfy properties that should find use in applications. Namely, we showed that these generalizations are non-negative and are monotone under local quantum operations. Some of them satisfy a generalization of the duality relation \( I(A; B|C) = I(A; B|D) \), which holds for a four-party pure state \( \psi_{ABCD} \). We conjecture that these Rényi generalizations of the CQMI are monotone in the Rényi parameter \( \alpha \) and provided evidence that this conjecture is true when \( \alpha \) is in a neighborhood of one. The truth of this conjecture would have implications in condensed matter physics, as detailed in \([31]\).

There are many directions to consider going forward from here. One could explore a Rényi squashed entanglement and determine if several properties hold which are analogous to the squashed entanglement \([10]\). Such a quantity might be helpful in strengthening the main result of \([55]\), so that the upper bound established on the two-way assisted quantum capacity could be interpreted as a strong converse rate. The quantities defined here might be useful in the context of one-shot information theory, for example, to establish a one-shot state redistribution protocol as an extension of the main result of \([16]\). One could also explore applications of the Rényi conditional mutual informations in the context of condensed matter physics, as the Rényi entropy has been employed extensively in this context \([9]\).

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## A Lie-Trotter product formula

**Proposition 29** The following equivalence holds

\[
\exp \{ \log \tau_{AC} + \log \theta_{BC} - \log \omega_C \} = \lim_{\alpha \to 1} \left[ \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2} \right]^{1/(1-\alpha)}.
\]  

(A.1)
Proof. The proof is a straightforward generalization of the well known proof of the Lie-Trotter product formula. Let $N$ be identified with $1/(1-\alpha)$, so that the limit $\alpha \to 1^-$ corresponds to the limit $N \to \infty$. Also, let $\log \tau_{AC} = A$, $\log \theta_{BC} = B$ and $\log \omega_C = C$. We then have

$$
\exp \{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\} = \exp \{A + B - C\} \tag{A.2}
$$

$$
\left[\tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{(1-\alpha)/2} \tau_{AC}^{-(1-\alpha)/2}\right]^{1/(1-\alpha)} = \left[\exp \frac{A}{2N} \exp \frac{-C}{2N} \exp \frac{B}{N} \exp \frac{-C}{2N} \exp \frac{A}{2N}\right]^N, \tag{A.3}
$$

with the second equality holding whenever $1/(1-\alpha)$ is an integer.

We shall first compare the following two operators,

$$
X_N = \exp \{(A + B - C)/N\}, \tag{A.4}
$$

$$
Y_N = \exp \frac{A}{2N} \exp \frac{-C}{2N} \exp \frac{B}{N} \exp \frac{-C}{2N} \exp \frac{A}{2N}. \tag{A.5}
$$

Consider the Taylor expansions of $X_N$ and $Y_N$ up to first order in $N^{-1}$.

$$
X_N = I + \frac{A + B - C}{N} + O(N^{-2}), \tag{A.6}
$$

$$
Y_N = (I + \frac{A}{N} + O(N^{-2}))(I - \frac{C}{2N} + O(N^{-2}))(I + \frac{B}{2N} + O(N^{-2}))
\times (I - \frac{C}{2N} + O(N^{-2}))(I + \frac{A}{N} + O(N^{-2})) \tag{A.7}
$$

$$
= I + \frac{A + B - C}{N} + O(N^{-2}). \tag{A.8}
$$

Therefore, clearly, $X_N - Y_N = O(N^{-2})$.

Now, consider $X_N^N - Y_N^N$. For arbitrary matrices $X$ and $Y$, it can be shown that

$$
\|X^N - Y^N\| \leq NM^{N-1} \|X - Y\|, \tag{A.9}
$$

where $M = \max(\|X\|, \|Y\|)$. For the case $X = X_N$ and $Y = Y_N$, we have

$$
\|X_N^N - Y_N^N\| \leq NM^{N-1} \|X_N - Y_N\|. \tag{A.10}
$$

Given that

$$
\|X_N\| \leq \|\exp \{(A + B - C)/N\}\| \leq \exp \{\|A\| + \|B\| + \|C\|)/N\}, \tag{A.11}
$$

$$
\|Y_N\| \leq \left\|\exp \frac{A}{2N} \exp \frac{-C}{2N} \exp \frac{B}{N} \exp \frac{-C}{2N} \exp \frac{A}{2N}\right\| \leq \exp \{\|A\| + \|B\| + \|C\|)/N\}, \tag{A.12}
$$

we have $M^N \leq \exp \{\|A\| + \|B\| + \|C\|\}$. Therefore,

$$
\|X_N^N - Y_N^N\| \leq N \exp \{\|A\| + \|B\| + \|C\|\} O(N^{-2}) = O(N^{-1}). \tag{A.13}
$$

That is, in the limit $N \to \infty$, $\|X_N^N - Y_N^N\| = 0$. The proof for the limit when $\alpha \to +1$ is similar, so that we can conclude the statement of the proposition. ■
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