Jeu de taquin, uniqueness of rectification and ultradiscrete KP

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In this article, we study tropical-theoretic aspects of the ‘rectification algorithm’ on skew Young tableaux. It is shown that the algorithm is interpreted as a time evolution of some tropical integrable system. By using this fact, we construct a new combinatorial map that is essentially equivalent to the rectification algorithm. Some of properties of the rectification can be seen more clearly via this map. For example, the uniqueness of a rectification boils down to an easy combinatorial problem. Our method is mainly based on the two previous researches: the theory of geometric tableaux by Noumi–Yamada, and the study on the relationship between jeu de taquin slides and the ultradiscrete KP equation by Mikami and Katayama–Kakei.

Keywords: tropical integrable systems; Young tableaux; combinatorics; mathematical logic; tropicalization.

1. Introduction

The tropicalization is a procedure to translate mathematical statements (such as propositions, equations, formulas, etc.) written in the ‘language of rings’ into the ‘language of semi-fields’, where the addition, the multiplication, and the multiplicative inverse are transformed as

\[ a + b \mapsto \min[A, B], \quad ab \mapsto A + B, \quad a^{-1} \mapsto -A. \]

For example, rational maps correspond to piecewise linear maps.

In 2001, Berenstein and Kirillov [1] showed that the Robinson–Schensted–Knuth correspondence (RSK correspondence), a crucial bijection in the theory of Young tableaux, can be expressed as a piecewise linear map that is related with the crystal basis. After that, Kirillov [2] introduced the geometric RSK correspondence (originally, tropical RSK correspondence\(^1\)), which is a rational map obtained by ‘lifting’ Berenstein–Krillov’s piecewise linear map. This correspondence was studied further by Noumi–Yamada [3] in terms of tropical integrable systems. In these literature, the method of tropicalization plays a key role in deriving explicit formulas. Such techniques were originally referred to as the ‘tropical approach [2, 3]’.

Recently, Mikami [4] and Katayama–Kakei [5] found an interesting relationship between jeu de taquin slides and the tropical KP equation. What is interesting is the fact that their correspondence is

\(^1\) The word ‘tropical’ nowadays has a different meaning. Many researchers seem to prefer to use the ‘geometric RSK correspondence’ instead.
(probably) independent of Nouni–Yamada’s correspondence. For this reason, Young tableaux are viewed as a significant example that admits (at least) two independent realizations by tropical integrable systems.

By using these facts, we present a new approach for solving problems on combinatorics of Young tableaux. In this article, we introduce a new combinatorial map that is essentially equivalent to the rectification algorithm [6] in terms of tropical (=ultradiscrete) relativistic Toda equation [7]. This tropical-theoretic realization provides a different interpretation, where some properties of the algorithm can be seen clearly. For instance, through this map, a proof of the uniqueness of a rectification, which is probably first non-trivial theorem in the theory of Young tableaux, boils down to an easy combinatorial problem.

For another application, we would mention the author’s related work [8] on the shape equivalence and the Littlewood–Richardson correspondence. We expect further researches will reveal deeper relationships between combinatorics and tropical integrable systems.

1.1 Contents of the article

This work is inspired by the recent studies of Mikami [4] and Katayama–Kakei [5] (see Section 2, Theorem 2.1) on relationships between jeu de taquin slides (cf. Section B) and the ultradiscrete KP equation (2).

Under a change of variables, the ultradiscrete KP equation (2) is transformed to the recursive form (8), which is more suitable for our study. We give a proof of this fact by the method of tropicalization (Section 3). What should be noted here is that the formal tropicalization of a true proposition is not always true. For example, ‘a + b = a + c implies b = c’ does not mean ‘\min(A,B) = \min(A,C) implies B = C’. In many cases, such problems are easily avoided by direct methods—for example, by simplifying expressions—but it sometimes cause errors that are difficult to find. In Section 3.2, we present a systematic approach to such problems by means of mathematical logic. As an application, a formal proof of (8) is given in Section 3.3.

In Section 4, we introduce a diagrammatic algorithm for calculating the time evolution of (8). This algorithm also provides a good reinterpretation of jeu de taquin slides. (See the example given in Section 4.3.)

In Sections 5 and 6, we give an alternative proof of the ‘uniqueness of a rectification [6, Section 1, Claim 2]’, which is probably the first non-trivial theorem in the course of combinatorics of Young tableaux. The key of the proof is the commutative diagram

\[
\begin{array}{ccc}
\{Q, W\} & \xrightarrow{\text{Jeu de taquin}(\S 4.2)} & \{Q', W'\} \\
\Downarrow \text{Row insertion} \quad \text{(\S 5.2, \S A.2)} & & \Downarrow \text{Row insertion} \\
\{P, W\} & \xrightarrow{\text{A new map constructed via tropical integrable systems} \quad (\S 5.4)} & \{P', W'\}
\end{array}
\]

where \(W\) is a skew tableau (Section 4.1), \(W'\) is a rectification of \(W\) (Section 5.1), \(Q\) is a sequence of row numbers from where jeu de taquin slides start (Section 5.1), and \(P\) is the \(P\)-tableau (Section 5.3) associated with \(Q\). See Section 5.5. From this diagram, we find that the rectification depends only on

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2 The Takahashi–Satsuma Box-Ball system [9] is a good example that admits two tropical realizations: one is the ultradiscrete Toda equation, and the other is the ultradiscrete KdV equation.

3 In the paper [5], they considered standard skew tableaux only. However, their proof is valid for general skew tableaux without any changes.
the choice of a $P$-tableau. Then the uniqueness of a rectification boils down to a relatively easy lemma (Corollary 6.3) that states a $P$-tableau of given shape is unique.

In the appendix, we give a short list of basic definitions in mathematical logic in Section A. A brief introduction to combinatorics of Young tableaux is given in Section B.

1.2 Notations

In this article, we follow the convention of Fulton’s textbook [6]. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ be a Young diagram. A semi-standard tableau of shape $\lambda$ is obtained by filling the boxes in $\lambda$ with a number according to the following rules: (i) in each row, the numbers are weakly increasing from left to right, (ii) in each column, the numbers are strongly increasing from top to bottom. A semi-standard tableau is often referred to as a tableau shortly. A tableau with $n$ boxes is called standard if it contains distinct $n$ numbers $1, 2, \ldots, n$. Let $\lambda/\mu$ be a skew diagram, where $\lambda$ and $\mu$ are Young diagrams with $\mu \subset \lambda$. A skew (semi-standard) tableau of shape $\lambda/\mu$ is obtained by filling the boxes with a number according to the same rules as for tableaux. If a skew tableau with $n$ boxes contains distinct $n$ numbers $1, \ldots, n$, it is said to be standard. See Section B for other definitions.

1.3 About this article

Most part of this article (except for Sections 5.4, 5.5 and 6) is an English translation of the author's unpublished manuscript 'S. Iwao, Jeu de taquin, uniqueness of a rectification, and ultradiscrete KP' written in Japanese.

2. Ultradiscrete (tropical) KP equation and jeu de taquin

In this section, we introduce the result of Katayama and Kakei [5] in 2015. The definition of the terms jeu de taquin slide, inside corner, outside corner, etc. can be found in Section B.

Let us consider the discrete KP equation

$$f_{i+1,j+1}^t - f_{i,j}^t f_{i+1,j}^t + f_{i,j+1}^t f_{i+1,j+1}^t = 0. \quad (1)$$

According to the definition of tropicalization introduced in Section 1, the ‘tropicalization of (1)’ should be the following piecewise linear equation:

$$F_{ij}^{t+1} = \max \left[ F_{i+1,j+1}^t + F_{i-1,j}^t, F_{i,j+1}^t + F_{i,j+1}^{t+1} \right]. \quad (2)$$

The following is the main theorem of [5]:

**Theorem 2.1** ([5]. (See also [4])) Let $\{S^t\}_{t=0,1,2,\ldots}$ be a collection of skew tableaux such that $S^{t+1}$ is obtained from $S^t$ by a jeu de taquin slide, carried out from any inside corner. Define

$$F_{ij}^t = \sharp \left( \text{0, 1, 2, \ldots, } j \text{'s contained in the 1st, 2nd, \ldots, } i \text{th rows of } S^t \right),$$

where an empty box is regarded as a box with 0. Then $\{F_{ij}^t\}_{t \geq 1, j \geq 0, i \geq 0}$ satisfies the ultradiscrete KP equation (2).

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4 The original manuscript (in Japanese) had been submitted to *RIMS Kôkyûroku Bessatsu* but was withdrawn (03/07/2019) because the first version of the manuscript had contained a major mathematical error.
Example 2.2 Consider the sequence of jeu de taquin slides displayed below. The grey boxes denote the inside corners from which a jeu de taquin slide is carried out.

Let $S^0, S^1, S^2, S^3$ denote these skew tableaux. The first $3 \times 4$ part of the matrix $F_t = (F_{ij})_{i,j \geq 0}$ is expressed as

$$F^0 = \begin{pmatrix} 2 & 3 & 3 & 3 \\ 3 & 4 & 6 & 6 \\ 3 & 4 & 6 & 8 \end{pmatrix}, \quad F^1 = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 6 \\ 2 & 3 & 5 & 8 \end{pmatrix}.$$ 

$$F^2 = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 4 & 5 \\ 1 & 2 & 4 & 6 \end{pmatrix}, \quad F^3 = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 5 \end{pmatrix}.$$ 

Putting

$$I'_{ij} = \frac{f_{i-1,j} f_{i+1,j}}{f_{i,j} f_{i+1,j}}, \quad V'_{ij} = \frac{f_{i-1,j} f_{i+1,j+1}}{f_{i,j} f_{i+1,j+1}},$$

we rewrite the discrete KP equation (1) as

$$\begin{cases} I'_{ij} V'_{i+1,j} = I'_{i+1,j+1} V'_{ij}, \\ I'_{ij} - V'_{i-1,j} = I'_{i,j+1} - V'_{ij}. \end{cases}$$

Moreover, by putting

$$R'_j := \begin{pmatrix} I'_{1,j} & 1 \\ I'_{2,j} & 1 \\ \vdots & \ddots \end{pmatrix}, \quad L'_j := \begin{pmatrix} 1 & 1 \\ -V'_{1,j} & -V'_{2,j} \\ \vdots & \ddots \end{pmatrix}^{-1},$$

(4) is transformed into the matrix form:

$$R'_j L'_j = L'_{j+1} R'_{j+1}.$$ 

Equation (5) is often referred to as the discrete $(2+1)$-dimensional Toda equation. It is easily checked that (4) is equivalent to the subtraction-free form:

$$I'_{i+1,j+1} = \frac{I'_{i+1,j} + V'_{i+1,j} I'_{i,j}}{I'_{i,j} + V'_{i,j}}, \quad V'_{i,j+1} = \frac{I'_{i+1,j} + V'_{i+1,j+1} I'_{i,j}}{I'_{i,j} + V'_{i,j}}.$$
We now ‘tropicalize’ (3) and (6). Let $Q'_{ij}$ and $W'_{ij}$ be the tropicalization of $I'_{ij}$ and $V'_{ij}$, respectively. Then the tropicalization of (3) is

$Q'_{ij} = F'_{ij} + F'_{i-1,j} - F'_{i-1,j} - F'_{i,j+1},$

$W'_{ij} = F'_{ij} + F'_{i+1,j} - F'_{i-1,j} - F'_{i,j+1},$  \hfill (7)

and that of (6) is

$Q'_{i+1,j+1} = \min(Q'_{i+1,j}, W'_{i,j+1}) - \min(Q'_{ij}, W'_{ij}) + Q'_{ij},$

$W'_{i,j+1} = \min(Q'_{i+1,j}, W'_{i,j+1}) - \min(Q'_{ij}, W'_{ij}) + W'_{ij}.$  \hfill (8)

On the analogy of ‘{(1) and (3)} ⇒ (6)’, it would be natural to expect that the implication ‘{(2) and (7)} ⇒ (8)’ is true. Note, however, that it is not obvious at this stage. (See Section 1.)

Example 2.3 For the skew tableaux in Example 2.2, we have

$Q^0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
Q^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
Q^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
W^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},
W^1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},
W^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

It is directly checked that (8) is satisfied.

3. Tropical approach

As we have seen in the previous section, it would be natural to expect that propositions and theorems written in ‘the language of rings’ imply the similar facts written in ‘the language of semi-fields’, while it is not generally true. In this section, we propose a formal method to deal with such ideas systematically.

We review the ‘naive’ principle of tropicalization in Section 3.1, and introduce its formal counterpart in Section 3.2. As an application, we give a formal proof of ‘{(2) and (7)} ⇒ (8)’ in Section 3.3. This section can be skipped if the reader is interested only in combinatorics.

3.1 ‘Naive’ tropical approach

A real polynomial $f(x) \in \mathbb{R}[x_1, \ldots, x_N]$ is called subtraction-free if it is expressed as $f(x) = \sum c_I x^I$, where $x^I = x_1^{i_1} \cdots x_N^{i_N}$ and $c_I \geq 0$ for all $I = (i_1, \ldots, i_N)$.

Definition 3.1 Let $f(x) = \sum c_I x^I$ be a subtraction-free polynomial. The tropicalization of $f(x)$ is the piecewise linear function $\tilde{f}(X_1, \ldots, X_N)$ defined as

$\tilde{f}(X_1, \ldots, X_N) = \min_{(I : c_I \neq 0)} [I_1 \cdot X_1 + \cdots + I_N \cdot X_N],$

where $\min \emptyset = +\infty.$
Proposition 3.2 (‘Naive’ principle of tropicalization) Let \( f(x_1, \ldots, x_n) \) be a subtraction-free polynomial. By assuming \( x_i = O(e^{-X_i \epsilon}) (\epsilon \downarrow 0) \), where \( \epsilon \) is a positive parameter, we have \(-\lim_{\epsilon \downarrow 0} \epsilon \log f(x_1, \ldots, x_n) = \bar{f}(X_1, \ldots, X_n)\).

Example 3.3 Assume \( a, b, c, d, e, f, g \) satisfy \( a = b + c, d = e + g, \) and \( f = bg \), which imply, for example, \( ad + ce = ae + cd + f \). The tropicalization of this statement is

\[
(A = \min[B, C]) \land (D = \min[E, G]) \land (F = B + G) \\
\implies \min[A + D, C + E] = \min[A + E, C + D, F].
\]

We will give a proof of this proposition by the method of tropicalization. Assume \( A, B, \ldots, F \in \mathbb{R} \) satisfy the assumption of the implication. Set \( b(\epsilon) = e^{-B \epsilon}, c(\epsilon) = e^{-C \epsilon}, e(\epsilon) = e^{-E \epsilon} \) and \( g(\epsilon) = e^{-G \epsilon} \), where \( \epsilon \) is a positive parameter. Let us define \( a(\epsilon) := b(\epsilon) + c(\epsilon), d(\epsilon) := e(\epsilon) + g(\epsilon) \) and \( f(\epsilon) := d(\epsilon)g(\epsilon) \). From the principle of tropicalization (Proposition 3.2), the tropicalization of \( a(\epsilon), b(\epsilon), \ldots, f(\epsilon) \) coincides with \( A, B, \ldots, F \), respectively. Then the desired proposition follows from \( a(\epsilon)d(\epsilon) + c(\epsilon)e(\epsilon) = a(\epsilon)e(\epsilon) + c(\epsilon)d(\epsilon) + f(\epsilon) \).

Example 3.4 The subtraction-free equation \( a + b = a + c \) implies \( b = c \), but \( \min[A, B] = \min[A, C] \) does not imply \( B = C \). Indeed, \( (A, B, C) = (0, 1, 2) \) is a counterexample.

Example 3.4 is a simplest and typical example where the tropicalization causes an error.

3.2 Tropical approach in terms of first-order logic

In this section, we rephrase the method of tropicalization in terms of first-order logic. Basic definitions and notions of first-order logic are introduced in Section A. For readers who are interested in mathematical logic, we recommend the standard textbooks [10, 11].

Let

\[ \mathcal{L} = \{ f_1, f_2, \ldots, R_1, R_2, \ldots, c_1, c_2, \ldots \} \]

be a language, where \( f_i \) is a function symbol, \( R_i \) is a relation symbol, and \( c_i \) is a constant symbol. Consider the two \( \mathcal{L} \)-structures:

\[ \mathcal{M} = (M, f_1, f_2, \ldots, R_1, R_2, \ldots, c_1, c_2, \ldots), \]

\[ \overline{\mathcal{M}} = (\overline{M}, \overline{f_1}, \overline{f_2}, \ldots, \overline{R_1}, \overline{R_2}, \ldots, \overline{c_1}, \overline{c_2}, \ldots) \]

(\( M \) is the domain of \( \mathcal{M} \), and \( \overline{M} \) is the domain of \( \overline{\mathcal{M}} \)) and a homomorphism

\[ M \to \overline{M}, \quad x \mapsto \overline{x} \]

of \( \mathcal{L} \)-structures.

We use the following lemma, which we will prove in the appendix:
Lemma 3.5 For any negation-free \( \mathcal{L} \)-formula \( \psi(x_1, \ldots, x_n) \) and any element \( (a_1, \ldots, a_n) \) of \( M^n \),
\[
\mathcal{M} \models \psi(a_1, \ldots, a_n) \quad \text{implies} \quad \overline{\mathcal{M}} \models \overline{\psi}(a_1, \ldots, a_n).
\]

**Proof.** See Section A.2. \( \square \)

Proposition 3.6 Let \( \mathcal{L} \)-formulas \( \varphi(x_1, \ldots, x_n) \) and \( \psi(x_1, \ldots, x_n) \) satisfy:

1. \( \psi(x_1, \ldots, x_n) \) is negation-free.
2. For any \( (A_1, \ldots, A_n) \in \overline{\mathcal{M}} \), there exists some \( (a_1, \ldots, a_n) \in M^n \) that satisfies \( \overline{a_i} = A_i \) and \( \mathcal{M} \models \varphi(a_1, \ldots, a_n) \).
3. \( \mathcal{M} \models \forall x_1 \ldots \forall x_n (\varphi(x_1, \ldots, x_n) \rightarrow \psi(x_1, \ldots, x_n)) \).

Then, it follows that
\[
\overline{\mathcal{M}} \models \forall X_1 \ldots \forall X_n (\overline{\varphi}(X_1, \ldots, X_n) \rightarrow \overline{\psi}(X_1, \ldots, X_n)).
\]

**Proof.** Assume \( \overline{\mathcal{M}} \models \overline{\varphi}(A_1, \ldots, A_n) \) for some \( (A_1, \ldots, A_n) \in \overline{M^n} \). From (1), there exists \( (a_1, \ldots, a_n) \in M^n \) such that \( \overline{a_i} = A_i \) and \( \mathcal{M} \models \varphi(a_1, \ldots, a_n) \). From (2), we have \( \mathcal{M} \models \psi(a_1, \ldots, a_n) \). As a result, from (0) and Lemma 3.5, we have \( \overline{\mathcal{M}} \models \overline{\psi}(A_1, \ldots, A_n) \). \( \square \)

Example 3.7 The ‘naive’ principle of tropicalization (Proposition 3.2) is a special case of Proposition 3.6. Let \( \mathcal{L} = \{+, \cdot, -^1, 1\} \). Here \( +, \cdot \) are binary function symbols, \( -^1 \) is a unary function symbol, and \( 1 \) is a constant symbol. Define the two \( \mathcal{L} \)-structures \( \mathcal{M} = (M, +, \cdot, -^1, 1) \), \( \overline{\mathcal{M}} = (\overline{M}, \overline{+}, \overline{\cdot}, \overline{-^1}, \overline{1}) \) as follows:

- \( M \) is the set of germs at \( \epsilon = 0 \) of continuous positive functions \( f(\epsilon) \) (\( \epsilon > 0 \)) that satisfy \( -\lim_{\epsilon \downarrow 0} \epsilon \log f(\epsilon) \in \mathbb{R} \).
- \( + \) denotes the standard addition, \( \cdot \) denotes that standard multiplication, and \( -^1 \) denotes the multiplicative inverse. \( 1 = 1(\epsilon) \) is the constant function.
- \( \overline{M} = \mathbb{R} \).
- \( \overline{+} = \min, \overline{\cdot} = +, \overline{-^1} = -, \overline{1} = 0 \).
- The map \( M \rightarrow \overline{M} \) is defined by \( f(\epsilon) \mapsto -\lim_{\epsilon \downarrow 0} \epsilon \log f(\epsilon) \). (This map is usually called ‘ultradiscretization’.)

‘Subtraction-free polynomials’ is now simply rephrased as ‘\( \mathcal{L} \)-terms’. Note that there are other possible choices of \( \mathcal{M} \). For example, one can take \( M \) as the set of real formal power series the lowest coefficient of which is positive, and \( M \rightarrow \overline{M} \) as the valuation map.

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5 See Section A, Definitions A5 and A6.
6 One may add the constant symbol ‘0’ to \( \mathcal{L} \), but it is not mandatory. Since all \( \mathcal{L} \)-formulas that we will use in this manuscript do not contain 0, we can simply omit it.
There exists a useful sufficient condition for (1) in Proposition 3.6. Assume (i) \( M \to \overline{M} \) is onto and (ii) \( \varphi(x_1, \ldots, x_n) \) is ‘a collection of definitions of next variables’, namely, there exist some \( 1 \leq \alpha \leq n \) and \( L \)-terms \( f_i(x_1, \ldots, x_{i-1}) (i = \alpha + 1, \alpha + 2, \ldots, n) \) such that

\[
\varphi(x_1, \ldots, x_n) = \left( \bigwedge_{i=\alpha+1}^n \{ x_i = f_i(x_1, \ldots, x_{i-1}) \} \right),
\]

(9)

where \( x_1, \ldots, x_{i-1} \) are free variables of \( f_i(x_1, \ldots, x_{i-1}) \). Under the assumptions (i–ii), one can find an element \((a_1, \ldots, a_n) \in M^n\) in (1) of Proposition 3.6. In fact, from (i), there exists an element \((a_1, \ldots, a_n) \in M^n\) that satisfies \( \overline{a_1} = A_1, \ldots, \overline{a_n} = A_n \). Putting \( a_i := f_i(a_1, \ldots, a_{i-1}) \) for \( i = \alpha + 1, \alpha + 2, \ldots, n \), we obtain the \( n \)-tuple \((a_1, \ldots, a_n)\), which satisfies \( M \models \varphi(a_1, \ldots, a_n) \). Since \( \overline{M} \models \overline{\varphi}(A_1, \ldots, A_n) \), the equation \( \overline{a_i} = f_i(a_1, \ldots, a_{i-1}) = \overline{A_i} \) holds for \( i > \alpha \).

Definition 3.8 We call an \( L \)-formula \( \varphi(x_1, \ldots, x_n) \) of the form (9) recursive.

Remark 3.9 The condition (ii) can be significantly generalized as

\[
\varphi(x_{i_1}, \ldots, x_{i_k}) = \exists x_{i_1} \exists x_{i_2} \ldots \exists x_{i_k} \varphi(x_1, \ldots, x_n),
\]

where \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \) and \( \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset \). In fact, for any \((A_{i_1}, \ldots, A_{i_k}) \in \overline{M}^k\) with \( \overline{M} \models \overline{\varphi}(A_{i_1}, \ldots, A_{i_k}) \), there exists an \( n \)-tuple \((a_1, \ldots, a_n) \in M^n\) that satisfies \( \overline{M} \models \overline{\varphi}(A_1, \ldots, A_n) \). In a similar way to the quantifier-free case, one can find the desired \((a_1, \ldots, a_n) \in M^n\).

Example 3.10 Example 3.3 follows from Proposition 3.6 if we put

\[
\varphi(a, \ldots, f) = ((a = b + c) \land (d = e + g) \land (f = bg)),
\]

\[
\psi(a, \ldots, f) = (ad + ce = ae + cd + f).
\]

Example 3.11 Consider the proposition ‘for any \( a, b > 0 \), the inequality \( a^2 - 4b > 0 \) implies the existence of \( x > 0 \) that satisfies \( x^2 - ax + b = 0 \)’. This proposition can be rewritten in terms of the language \( L \) as

\[
M \models \forall a \forall b (\varphi(a, b) \to \psi(a, b)),
\]

where \( \varphi(a, b) = \exists y(a^2 = 4b + y) \) and \( \psi(a, b) = \exists x(x^2 + b = ax) \). By Proposition 3.6, we have

\[
\overline{M} \models \forall A \forall B (\exists Y (2A = \min [B, Y]) \to \exists X (\min [2X, B] = A + X)).
\]

3.3 Proof of (8) by formal arguments

We now give a proof of (8) by formal arguments that we have seen in the previous section. Hereafter, we fix the language \( L \), and the \( L \)-structures \( M \) and \( \overline{M} \) as Example 3.7. Let

\[
\varphi(f_{ij}^t, f_{ij+1}^t, f_{i+1,j}^t, f_{i+1,j+1}^t, f_{i-1,j}^t, f_{i,j+1}^t, f_{i,j}^t, f_{i+1}^t, f_{i+1,j+1}^t)
\]
By definition of skew tableaux, we find calculations, is now rephrased as its tropicalization is also true. Then, the statement \{(2) and (7)\} implies (8)’ is true over We give a combinatorial interpretation of (8), which will help us to understand the relationship between 4. Combinatorial interpretation of (8)

4.1 Matrix W

The proposition ‘\{(1) and (3)\} implies (6)’, which can be checked by straightforward algebraic calculations, is now rephrased as

\[ \mathcal{M} = \forall I_{ij}^t, V_{ij}^t, V_{i+1,j}^t, \forall V_{ij}^{t+1}, \forall V_{i+1,j}^{t+1}, \forall I_{ij}^{t+1}, \forall V_{ij}^{t+1}, \forall I_{i+1,j}^{t+1}, \forall V_{i+1,j}^{t+1}, \forall I_{ij}, \forall V_{ij}, \forall I_{i+1,j}, \forall V_{i+1,j} \]

\[ (\varphi_{ij}^t \land \theta_{ij}^t \land \theta_{i+1,j}^t \land \theta_{i+1,j+1}^t \land \chi_{ij}^t \land \chi_{i,j+1}^{t+1} \land \chi_{i+1,j}^t) \rightarrow \Phi_{ij}^t \]

By ordering the variables properly, we find the assumption of the implication is recursive. In fact, it is enough to order them as (any \( I_{s,s}^t, V_{s,s}^t \)) \( f_{i+1,j}^t > f_{ij}^t \) (any other \( f_{s,s}^t \)). Therefore, from Proposition 3.6, its tropicalization is also true. Then, the statement ‘\{(2) and (7)\} implies (8)’ is true over \( \overline{\mathcal{M}} \).

4. Combinatorial interpretation of (8)

We give a combinatorial interpretation of (8), which will help us to understand the relationship between (8) and the jeu de taquin slides.

4.1 Matrix W

Let \( S \) be a skew tableau and \( F_{ij} \) denote the number of 0, 1, \ldots, \( j \)'s in the top \( i \) rows of \( S \). Put

\[ W_{ij} := F_{ij} + F_{ij+1} - F_{i-1,j} - F_{i+1,j+1} \]

\[ = \#\{0, 1, \ldots, j \text{'s in the } i \text{th row} \} - \#\{0, 1, \ldots, (j + 1) \text{'s in the } (i + 1) \text{th row} \}. \tag{10} \]

By definition of skew tableaux, we find \( W_{ij} \) must be non-negative and the sum \( \sum_{p \geq 0} W_{i+p,j+p} \) satisfies

\[ \sum_{p \geq 0} W_{i+p,j+p} = \#\{0, 1, \ldots, j \text{'s in the } i \text{th row} \}. \]
A skew tableau $S$ of shape $\lambda/\mu$ can be identified with the increasing sequence

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(N)} = \lambda,$$

where $\lambda^{(i)}$ is the sub-diagram of $S$ in which one of $0, 1, 2, \ldots, j$ is filled\(^7\). Each skew diagram $\lambda^{(i+1)}/\lambda^{(i)}$ does not contain no two boxes in each column. Obviously, we have

$$\lambda^{(i)}_i = \sum_{p \geq 0} W_{i+p,j+p}, \quad W_{ij} = \lambda^{(i)}_i - \lambda^{(i+1)}_i.$$

The $W_{ij}$ satisfies the following conditions:

1. There exists some $J$ such that $j > J \Rightarrow W_{ij} = W_{ij+1}$ for all $i$. \hfill (11)
2. There exists some $I$ such that $i > I \Rightarrow W_{ij} = 0$ for all $j$. \hfill (12)
3. $\sum_{p \geq 0} W_{i+p,j+p} \geq \sum_{p \geq 0} W_{i+1+p,j+p} \iff \lambda^{(i)}_i \geq \lambda^{(i+1)}_{i+1}$. \hfill (13)

Set

$$\Omega := \text{(the set of skew tableaux)},$$

$$\tilde{\Omega} := \{(\Delta_{ij})_{i \geq 1} \mid W_{ij} \in \mathbb{Z}_{\geq 0} \text{ that satisfies (11), (12), (13)}\}.$$

Then there exists the map $W : \Omega \rightarrow \tilde{\Omega}$ that assigns a skew tableau $S$ with the matrix $(W_{ij})$ defined by (10).

Proposition 4.1 $W$ is bijective.

Proof. By the correspondence $\lambda^{(i)}_i \mapsto \Delta_{ij}$, $\Omega$ can be seen as a subset of

$$\tilde{\Omega} := \{(\Delta_{ij})_{i \geq 1} \mid \Delta_{ij} \in \mathbb{Z}_{\geq 0}, \Delta_{ij} \geq \Delta_{i+1,j+1} \geq \Delta_{i+2,j+2} \geq \cdots \rightarrow 0, (\forall i,j)\}.$$

We also regard $\tilde{\Omega}$ as a subset of

$$\tilde{\Omega} := \{(W_{ij})_{i \geq 1} \mid W_{ij} \in \mathbb{Z}_{\geq 0}, \text{ the sum } \sum_{p \geq 0} W_{i+p,j+p} \text{ converges for each } i,j\}.$$

Then the map

$$\tilde{\Omega} \rightarrow \tilde{\Omega}; \quad (\Delta_{ij})_{ij} \mapsto (\Delta_{ij} - \Delta_{i+1,j+1})_{ij}$$

is obviously bijective (the inverse is $(W_{ij})_{ij} \mapsto (\sum_{p \geq 0} W_{i+p,j+p})_{ij}$). Its restriction to $\Omega$ coincides with $W$. Because $\Omega$ contains the inverse image of $\tilde{\Omega}$, $W$ must be bijective. \qed

\(^7\) An empty box is regarded as a box with 0.
4.2 *Jeu de taquin* $\varphi_k$

From the statement in the previous paragraph, we always identify $\Omega \leftrightarrow \mathcal{X}$. We will construct a map $\varphi_k : \mathcal{X} \rightarrow \mathcal{X}$ for any positive integer $k$ that is a tropical counterpart of the jeu de taquin starting from $k$th row.

For any $W = (W_{ij}) \in \mathcal{X}$, we define $W^+ := \varphi_k(W)$, the image of $W$ by $\varphi_k$, by the following manner:

1. Set $Q_0 = (Q_{1,0}, Q_{2,0}, \ldots) := (0, \ldots, 0, 1, 0, \ldots)$.
2. When the vector $Q_j = (Q_{1,j}, Q_{2,j}, \ldots)$ is already defined for $j \in \mathbb{Z}_{\geq 0}$, define $Q_{j+1} = (Q_{1,j+1}, Q_{2,j+1}, \ldots)$ and $W_j^+ = (W_{1,j}^+, W_{2,j}^+, \ldots)$ by the formula
   \[
   \begin{cases}
   Q_{i+1,j+1} := (\min(Q_{i+1,j}, W_{i+1,j}^+) - \min(Q_{ij}, W_{ij})) + Q_{ij}, \\
   W_{ij}^+ := (\min(Q_{i+1,j}, W_{i+1,j}^+) - \min(Q_{ij}, W_{ij})) + W_{ij},
   \end{cases}
   \] (14)
   where $Q_{0,j} = 0$, $W_{0,j} = +\infty$. (Compare with (8)).
3. Repeat (2) to obtain $Q_{ij}$ and $W_{ij}^+$ for all $i, j$.

Equation (14) can be seen as a kind of recurrence formula, the inputs of which are $Q_j$ and $W_j$, and the outputs are $Q_{j+1}$ and $W_j^+$. To understand this situation, it is convenient to draw the diagram

\[
\begin{array}{c}
W_j \\
Q_j \\
W_{j+} \\
\end{array}
\]

where the inputs are written on the northwest side and the outputs are written on the southeast side. Then the procedure that is presented above can be symbolically displayed as

\[
\begin{array}{c}
W_0 \\
Q_0 \\
W_0^+ \\
\end{array} \quad \begin{array}{c}
W_1 \\
Q_1 \\
W_1^+ \\
\end{array} \quad \begin{array}{c}
W_2 \\
Q_2 \\
W_2^+ \\
\end{array} \quad \begin{array}{c}
W_3 \\
Q_3 \\
W_3^+ \\
\end{array} \quad \cdots
\]

(15)

The map $\varphi_k$ also admits a diagrammatic interpretation as follows:

- Write a matrix $W = (W_{ij})$ down as Fig. 1.
- Draw a path on the matrix by the following rule:
  - The path starts from the $(k, 0)$th position.
  - When the path reaches at the $(i, j)$th position, it extends to the lower right neighbour if $W_{ij} = 0$, or to the right neighbour if $W_{ij} \neq 0$.
  - Decrease all non-zero numbers on the path by one, and increase all the numbers at the upper neighbour of the decreased numbers by one. The matrix given by this procedure coincides with $\varphi_k(W)$.
- The matrix $Q = (Q_{ij})_{ij}$ is given by putting $Q_{ij} = 1$ if the path goes through the $(i, j)$th position, and $Q_{ij} = 0$ otherwise.
The $\varphi_k$ coincides with the jeu de taquin slide that starts from the $k$th row. In fact, the data $W = (W_{ij})$, $W^+ = (W^+_{ij})$, $Q = (Q_{ij})$ in (15) satisfy the relation (8) under the substitution $W_{ij} = W^t_{ij}$, $W^t_{ij} = W^t_{ij}$, $Q_{ij} = Q^t_{ij}$. (Compare (8) with (14).) Since

$$Q^t_{ij} = F^t_{ij} + F^t_{i-1,j} - F^t_{i,j} - F^t_{i,j} = (F^t_{ij} - F^t_{i-1,j}) - (F^t_{i,j} - F^t_{i-1,j})$$

(see (7)), the number of $j$'s in the $i$th row decreases by $Q^t_{ij}$ under the time evolution $t \mapsto t + 1$. This means that substituting $Q_0 = (0, \ldots, 0, 1, 0, \ldots)$ is equivalent to removing an empty box from the $k$th row, and is also equivalent to starting the jeu de taquin slide from the $k$th row.

The $Q_{ij}$ also has a diagrammatic interpretation. Consider the jeu de taquin slide $\varphi_k$. Let $B_j$ be the position of the hole (see Section B) at when all the numbers equal to or less than $j$ have been moved. Then, $Q_{ij} = 1$ if $B_j$ is in the $i$th row, and $Q_{ij} = 0$ otherwise.

4.3 Example

The jeu de taquin slide

$$\begin{array}{ccc}
1 & 2 & 1 \\
1 & 3 & 5 \\
3 & 4 & 4 \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 1 \\
1 & 3 & 5 \\
3 & 4 & 4 \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 1 \\
1 & 3 & 4 \\
3 & 4 & 4 \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 1 \\
1 & 3 & 4 \\
3 & 4 & 4 \\
\end{array}
$$

corresponds with the matrices

$$W = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 \\
\end{pmatrix}, \quad Q = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.$$

The matrix $W^+$ is given by

$$W^+ = \begin{pmatrix}
0 & 2 & 2 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 \\
\end{pmatrix}.$$
5. Tropical interpretation of rectification

In the following two sections, we give an alternative proof of the uniqueness of a rectification [6, Sections 1–3]. First, we introduce the definition of the rectification and its tropical interpretation (Section 5.1). Then, we construct a new useful combinatorial map that is equivalent to the rectification. The construction is based on the theory of Noumi–Yamada’s geometric tableaux (Section 5.2). Finally, we show a diagrammatic realization (Section 5.4) and a commutation relation of this map (Section 5.5). We will see that the commutative diagram in Section 5.5 plays an important role in proving the uniqueness of a rectification.

5.1 Rectification

Any skew tableau of shape $\lambda/\mu$ is led to a (non-skew) tableau by applying a finite sequence of jeu de taquin slides. Repeating jeu de taquin slides is nothing but choosing inside corners repeatedly. By putting numbers in such inside corners in decreasing order, one obtains a standard tableau of shape $\mu$. For example, if we apply the sequence of jeu de taquin slides to the tableau

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & \\
\end{array}
\]

defined by

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 4 \\
\end{array}
\],

we obtain the sequence

\[
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 3 \\
2 & 2 & 3 \\
\end{array}
\]

While, another standard tableau

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 \\
\end{array}
\]

gives the sequence

\[
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 2 & \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 1 & 2 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 3 \\
2 & 2 & 3 \\
\end{array}
\]

It is not a coincidence that the two tableaux at the rightmost are same. In fact, it is known that any choice of standard tableau leads a unique tableau [6, Section 1, Claim 2].

Definition 5.1 Let $S$ be a skew tableau. A rectification of $S$ is a tableau that is obtained by applying a finite sequence of jeu de taquin slides to $S$. 

With diagrammatic expressions as in Section 4.2, the rectification can be displayed as

\[
\begin{array}{ccc}
Q_0^{(1)} & W_0^{(1)} & Q_1^{(1)} \\
W_1^{(1)} & W_0^{(1)} & W_1^{(1)} \\
Q_2^{(1)} & W_1^{(1)} & W_2^{(1)} \\
\vdots & \vdots & \vdots \\
Q_0^{(\ell)} & W_0^{(\ell)} & Q_1^{(\ell)} \\
W_1^{(\ell)} & W_0^{(\ell)} & W_1^{(\ell)} \\
Q_2^{(\ell)} & W_1^{(\ell)} & W_2^{(\ell)} \\
\vdots & \vdots & \vdots \\
\end{array}
\]  

(16)

Each vector \(Q_0^{(\ell-1)}\) corresponds with the jeu de taquin slide starting at \(i\). The vectors \(W_0^{(\ell)}, W_1^{(\ell)}, W_2^{(\ell)}\) at the bottom row correspond with the rectification.

5.2 Noumi-Yamada’s geometric tableau

In [3], Noumi and Yamada introduced an interesting characterization of the row insertion [6] in terms of tropical integrable systems.

For real vectors \(I = (I_1, I_2, \ldots)\) and \(V = (V_1, V_2, \ldots)\), we define the matrices \(E(I), F(V)\) of infinite size as

\[
E(I) = \begin{pmatrix}
I_1 & 1 & & \\
I_2 & 1 & & \\
& & \ddots & \\
& & & I_k \\
& & & \\
& & & \\
\end{pmatrix}, \quad F(V) = \begin{pmatrix}
1 & & & \\
& -V_1 & 1 & \\
& & -V_2 & 1 \\
& & & \ddots \\
& & & \\
\end{pmatrix}.
\]

Moreover, for a vector \(I' = (1, \ldots, 1, I_k, I_{k+1}, \ldots)\) whose first \((k-1)\) entries are 1, we put

\[
E_k(I') = \begin{pmatrix}
\operatorname{Id}_{k-1} & \\
& E(I'') \\
\end{pmatrix}, \quad I'' = (I_k, I_{k+1}, \ldots).
\]

Let us consider the equation

\[
E(I_1) \cdots E(I_2)E(I_1) = E_1(J_1)E_2(J_2) \cdots E_\ell(J_\ell),
\]

(17)

where \(I_k = (I_{1,k}, I_{2,k}, \ldots)\) and \(J_k = (1, \ldots, 1, J_{k,k}, J_{k+1,k}, \ldots)\) are real vectors for \(k = 1, 2, \ldots, \ell\). It is proved by using Gaussian elimination method that (17) determines some rational map that corresponds \(\{I_k\}_k\) to \(\{J_k\}_k\).

The following theorem is given by Noumi–Yamada [3, Section 2]:

Theorem 5.2 (Geometric tableau [3]) Equation (17) possesses the following properties:
The correspondence \( \{I_{ij}\} \mapsto \{J_{ij}\} \) is a subtraction-free rational map, that is, every \( J_{ij} \) is expressed as a subtraction-free rational function of \( \{I_{ij}\} \). This implies the existence of the tropicalization of the map.

Let \( Q_{ij} = I_{ij} \) and \( P_{ij} = J_{ij} \) be tropical variables. Then the ‘tropicalized’ map \( \{Q_{ij}\} \mapsto \{P_{ij}\} \) has the following combinatorial interpretation: let \( Q_j = (Q_1, Q_2, \ldots) \) be the vector whose \( \alpha_j \)th entry is 1 and the others are 0. Then \( P_{ij} \) equals to the number of \( j \)'s in the \( i \)th row of the tableau:

\[
\alpha_1 \leftarrow \alpha_2 \leftarrow \cdots \leftarrow \alpha_j
\]

5.3 P-tableau associated with standard tableau

We now proceed to the discrete Toda equation (5), which is equivalent to

\[
F(V^{i+1})E(I_j^i) = E(I_{j+1}^i)F(V_j^i),
\]

where \( I_j^i = (I_{1,j}^i, I_{2,j}^i, \ldots) \) and \( V_j^i = (V_{1,j}^i, V_{2,j}^i, \ldots) \). With regarding \( I_j^i, V_j^i \) as inputs and \( I_{j+1}^i, V_{j+1}^i \) as outputs, we display (18) diagrammatically as \( I_j^i \xrightarrow{V_j^i} I_{j+1}^i \). The diagram \( Q_j^i \xrightarrow{W_j^i} Q_{j+1}^i \) in Section 4.2 is nothing but its tropicalization.

With this idea, we associate the equation

\[
F(V')E(I^{(k)}) \cdots E(I^{(1)}) = E(I'^{(k)}) \cdots E(I'^{(1)})F(V)
\]

with the vertical diagram

\[
I^{(1)} \xrightarrow{V^{(1)}} I'^{(1)} \\
I^{(2)} \xrightarrow{V^{(2)}} I'^{(2)} \\
\vdots \\
I^{(k)} \xrightarrow{V^{(k)}} I'^{(k)}
\]

Let us transform the upper triangle matrices on the both sides of (19) into the geometric tableaux (17) form:

\[
E(I^{(k)}) \cdots E(I^{(1)}) = E_1(J_1) \cdots E_k(J_k), \quad E(I'^{(k)}) \cdots E(I'^{(1)}) = E_1(J'_1) \cdots E_k(J'_k).
\]

This leads the new equation

\[
F(V')E_1(J_1) \cdots E_k(J_k) = E_1(J'_1) \cdots E_k(J'_k)F(V),
\]
or equivalently,
\[ E_1(J_1) \cdots E_k(J_k) F(V)^{-1} = F(V')^{-1} E_1(J'_1) \cdots E_k(J'_k) \quad (21') \]

which we will display diagrammatically as

\[
\begin{array}{c}
\text{V} \\
(J_1, \ldots, J_k) \end{array} \quad \begin{array}{c}
\text{V'} \\
(J'_1, \ldots, J'_k) \end{array}
\]

**Proposition 5.3** Equation (21) determines a birational map \([J, V] \leftrightarrow [J', V']\), which has the following properties:

1. Each entry of \(J'_i\) and \(V'\) is expressed as a subtraction-free rational function of entries of \(J_i\) and \(V\).
2. Each entry of \(J_i\) and \(V\) is expressed as a subtraction-free rational function of entries of \(J'_i\) and \(V'\).

**Proof.** By straightforward calculations, one verifies that the equation
\[ E_i(A) F(B)^{-1} = F(B')^{-1} E_i(A') \]
determines a birational map \([A, B] \leftrightarrow [A', B']\) for any \(i\). Here each entry of \(A', B'\) is expressed as a subtraction-free rational function of entries of \(A\) and \(B\). With this fact, we have
\[
E_1(J_1) \cdots E_k(J_k) F(V)^{-1} = E_1(J'_1) \cdots E_{k-1}(J'_{k-1}) F(V^{(k)})^{-1} E_k(J'_k)
\]
\[= E_1(J_1) \cdots E_{k-2}(J_{k-2}) F(V^{(2)})^{-1} E_{k-1}(J'_{k-1}) E_k(J'_k)\]
\[= \cdots = F(V^{(k)})^{-1} E_1(J'_1) \cdots E_{k-1}(J'_{k-1}) E_k(J'_k),\]

where each entry of \(J'_i\) and \(V^{(i)}\) are subtraction-free function of positive vectors of \(J_i\) and \(V\). These statements are also true if \([J_i, V]\) and \([J'_i, V']\) are exchanged with each other. \(\square\)

Since all the rational maps in Proposition 5.3 are subtraction-free, the correspondence \([J, V] \leftrightarrow [J', V']\) is one-to-one if we restrict ourselves to real and positive \(J_i, V\). Let \(P_i = \overline{J_i}\) and \(W_i = \overline{V}\) be tropical variables. By the principle of tropicalization (Proposition 3.6), we obtain the one-to-one tropical map \([P_i, W] \leftrightarrow [P'_i, W']\), which will be diagrammatically written as

\[
\begin{array}{c}
\text{W} \\
(P_1, \ldots, P_k) \end{array} \quad \begin{array}{c}
\text{W'} \\
(P'_1, \ldots, P'_k) \end{array}
\]

(22)

Applying Theorem 5.2 (ii) to the data \((P_1, \ldots, P_k)\), we can identify it with some Young tableau, which we will call the \(P\)-tableau. For example, let us consider the tableau at the beginning of Section 5.1:

\[
\begin{array}{ccc}
1 & 2 \\
1 & 2 & 3 \\
1 & 2 \\
\end{array}
\quad W = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 \\
0 & 1 & 2 & 2 \\
\end{pmatrix}
\]
and the sequence of jeu de taquin slides defined by the standard tableau

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
4 & & \\
\end{array}
\]

The order of row numbers from which jeu de taquin slides start is (2nd, 1st, 1st, 1st). The \(P\)-tableau associated with this sequence is

\[
\begin{array}{ccc}
2 & 1 & 1 \\
\hline
1 & 2 & 1 \\
\end{array}
\]

When the jeu de taquin slides are applied, the outside corners in the 3rd, 1st, 2nd, 3rd rows are removed. Finally, we obtain the rectified tableau

\[
\begin{array}{ccc}
1 & 1 & 1 & 2 \\
2 & 2 & 3 & 3 \\
\end{array}
\]

\[
W = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

With use of \(P\)-tableaux, the diagram (16) is now rewritten as

\[
\begin{array}{cccc}
(1,0,0) & (1,0,1) & (1,1,2) & (1,2,2) \\
\hline
(0,0,0) & (1,0,0) & (1,2,0) & (1,3,0) \\
\end{array}
\]

(23)

The tableau on the rightmost position corresponds to the sequence of row insertions

\[
\begin{array}{ccc}
3 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\end{array}
\]

Note that the \(P\)-tableau does not change if one replaces the standard tableau with

\[
\begin{array}{ccc}
1 & 2 & 4 \\
\hline
3 & & \\
\end{array}
\]

(= 1st, 2nd, 1st, 1st) or \[
\begin{array}{ccc}
1 & 3 & 4 \\
\hline
2 & & \\
\end{array}
\]

(= 1st, 1st, 2nd, 1st).

This is the essential reason why the rectification is unique.

### 5.4 Diagrammatic expression of the map (22)

As we have seen above, the matrix equation (21') boils down to the simple equation

\[
E_p(J)F(V)^{-1} = F(V')^{-1}E_p(J'), \quad V = (V_1, V_2, \ldots), \quad J = (1, \ldots, 1, J_p, J_p+1, \ldots)
\]

(25)

for \(1 \leq p \leq k\) (see the proof of Proposition 5.3). Let \((J, V) \mapsto \{J', V'\}\) be the subtraction-free rational map defined by (25). Then, its tropicalization \(\{P, W\} \mapsto \{P', W'\}\) can be obtained by almost same calculations
that we have done in Section 2. In fact, we have

\[
P'_{i+1} = \begin{cases} 
\min[P_{i+1}, W_{i+1}], & i = p - 1 \\
\left(\min[P_{i+1}, W_{i+1}] - \min[P_i, W_i]\right) + P_i, & i \geq p
\end{cases},
\]

\[
W'_i = \begin{cases} 
W_i, & i < p - 1 \\
\min[P_{i+1}, W_{i+1}] + W_i, & i = p - 1 \\
\left(\min[P_{i+1}, W_{i+1}] - \min[P_i, W_i]\right) + W_i, & i \geq p
\end{cases}.
\]

Note that, by putting \(P_i \equiv 0\) for all \(1 \leq i < p\), they are simplified as

\[
P'_{i+1} = \left(\min[P_{i+1}, W_{i+1}] - \min[P_i, W_i]\right) + P_i,
\]

\[
W'_i = \left(\min[P_{i+1}, W_{i+1}] - \min[P_i, W_i]\right) + W_i.
\] (26)

(Compare with (8).)

The system (26) can be realized by the kicker-and-ball model [7], which was first introduced as a tropicalization (=ultradiscretization) of the discrete relativistic Toda equation. Let \(P = (0, \ldots, 0, P_p, P_{p+1}, \ldots)\) and \(W = (W_1, W_2, \ldots)\) be sequences of non-negative integers. Consider infinitely many boxes aligned in a half-line towards the right and put the following objects on them:

- \(P_i\) kickers at the \(i\)th site from the left.
- \(W_i\) balls at the \(i\)th site from the left.

For example, if \(P = (0, 2, 0, 2, 1, 0, 0, \ldots)\) and \(W = (0, 3, 1, 1, 0, 0, 0, \ldots)\), we draw

\[
\begin{array}{cccccccc}
& & & & & & & \\
& k & k & o & o & o & k & k & k & \ldots
\end{array}
\]

To obtain \(\{P', W'\}\), we move the kickers \((k)\) and the balls \((o)\) by the following rules:

- Kickers who stand nearby a ball kick one out into the box on their left. (A ball that is kicked out from the leftmost box will disappear.)
- Kickers who have no balls to kick out proceed to the box on their right.

For the example above, we obtain

\[
\begin{array}{cccccccc}
& & & & & & & \\
& o & o & k & k & o & o & k & k & k & \ldots
\end{array}
\]

and then we find \(P' = (0, 2, 0, 1, 1, 0, \ldots)\) and \(W' = (2, 1, 2, 0, 0, 0, 0, \ldots)\). This situation is expressed as

\[
\begin{array}{cccccccc}
(0, 3, 1, 1, 0, 0, 0) \\
(0, 2, 0, 2, 1, 0, 0) & + & (0, 2, 0, 1, 1, 0) \\
(2, 1, 2, 0, 0, 0, 0)
\end{array}
\]
or equivalently,

\[
(0, 3, 1, 1, 0, 0, 0)
\begin{array}{cccc}
2 & 2 & 4 & 4 \\
2 & 2 & 4 & 5 \\
2, 1, 2, 0, 0, 0, 0
\end{array}
\begin{array}{cccc}
2 & 2 & 4 & 4 \\
2 & 2 & 4 & 5 \\
2, 1, 2, 0, 0, 0, 0
\end{array}
\]

According to (21'), the whole procedure to obtain \( \{P', W'\} \) from \( \{P, W\} \) is written diagrammatically as

\[
P_k \xrightarrow{W} P'_k \\
P_{k-1} \xrightarrow{W'} P'_{k-1} \\
\vdots \\
P_1 \xrightarrow{W'} P'_1
\]

(27)

Here, each step is a map described in the previous paragraph. For example, the diagram

\[
\begin{array}{cccc}
1 & 1 & 1 \\
2 & 2 & 2 \\
0, 0, 0
\end{array}
\]

is decomposed as

\[
\begin{array}{cccc}
1 & 1 & 1 \\
2 & 3 \\
(1, 0, 0)
\end{array}
\begin{array}{cccc}
1 & 2 & 2 \\
2 & 2 & 2 \\
0, 0, 0
\end{array}
\]

5.5 Summary: a commutative diagram

Our results are summed up by the commutative diagram:

\[
\begin{array}{cccc}
\{Q, W\} \xrightarrow{(16)} \{Q', W'\} \\
\text{Row insertion} \downarrow \quad \downarrow \text{Row insertion} \\
\{P, W\} \xrightarrow{(22)} \{P', W'\} \\
(\text{equivalently, (27)})
\end{array}
\]

(28)

Here the matrix \( W \) is equivalent to a skew tableau (Proposition 4.1), and \( Q \) represents a sequence of row numbers at where jeu de taquin slides start (Section 5.1).
For example, we have

\[
\begin{align*}
\{ (2\text{nd}, 1\text{st}, 1\text{st}, 1\text{st}), & \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & \\
& & \\
\end{array} \} & \rightarrow \{ (3\text{rd}, 1\text{st}, 2\text{nd}, 3\text{rd}), & \begin{array}{ccc}
1 & 1 & 2 \\
& 2 & 3 \\
& & \\
\end{array} \} .
\end{align*}
\]

Note that if one replaces \((2\text{nd}, 1\text{st}, 1\text{st}, 1\text{st})\) at the top left of this diagram with \((1\text{st}, 2\text{nd}, 1\text{st}, 1\text{st})\) or \((1\text{st}, 1\text{st}, 2\text{nd}, 1\text{st})\), the bottom row does not change at all (see (24)).

6. Application: proof of the uniqueness of a rectification

By using the results in the previous section, the uniqueness of a rectification now boils down to an relatively easy and purely combinatorial lemma, which we will show below.

A reverse lattice word (or a Yamanouchi word) is a sequence of positive integers \(t_1, t_2, \ldots, t_N\) that satisfies the following inequality for each \(p\) and \(i\):

\[
\#(i's \text{ contained in } t_p, t_p+1, \ldots, t_N) \geq \#((i + 1)'s \text{ contained in } t_p, t_p+1, \ldots, t_N).
\]

Definition 6.1 Let \(U(\mu)\) denote the tableau of shape \(\mu\) whose \(i\)th row contains only \(i\) for all \(i\).

Lemma 6.2 (See Fulton [6, Section 5.2, Lemma 1]) If \(t_1, t_2, \ldots, t_N\) is a reverse lattice word, we have

\[
\begin{align*}
&f_1 \leftarrow f_2 \leftarrow \cdots \leftarrow f_N = U(\lambda)
\end{align*}
\]

for some Young diagram \(\lambda\).

Proof. We first focus on the 1st row. Row-inserting \(t_1, \ldots, t_N\), we obtain a 1st row \([a_1, a_2, \ldots, a_r]\) and a series \(s_1, s_2, \ldots, s_q\) that are bumped from the 1st row \(([a_1, \ldots, a_r] \cup \{s_1, \ldots, s_q\} = \{t_1, \ldots, t_N\})\). We will prove \(a_1 = a_2 = \cdots = a_r = 1\) and

\[
\#(i's \text{ contained in } s_p, s_{p+1}, \ldots, s_q) \geq \#((i + 1)'s \text{ contained in } s_p, s_{p+1}, \ldots, s_q)
\]

for all \(i > 1\) and \(p\). (In other words, \(s_1 - 1, s_2 - 1, \ldots, s_q - 1\) is a reverse lattice word.) Let \(T_k := t_1 \leftarrow t_2 \leftarrow \cdots \leftarrow t_k\) and

\[
\begin{align*}
&L_i^k := \#(i's \text{ contained in } t_{k+1}, t_{k+2}, \ldots, t_N), \\
&P_i^k := \#(i's \text{ contained in the 1st row of } T_k).
\end{align*}
\]

An excellent proof of Lemma 6.2 can be found in Fulton's book [6], but we give another elementary proof in this article to avoid the possibility of circular reasoning. See Remark 6.4.
Since $t_1, \ldots, t_N$ is a reverse lattice word, the sequence $L^1_1, L^2_2, \ldots$ is weakly decreasing for each $k$. Let $X^k_i := L^k_i + P^k_i$. By the definition, the row insertion algorithm is explicitly characterized by $L^k_{i+1} = L^k_i - \delta_{i,k}$ and $X^k_{i+1} = X^k_i - \delta_{i,a}$, where $\alpha$ is the minimum number with $(\alpha > t_k$ and $P^k_\alpha > 0)$. ($\delta_{i,a} \equiv 0$ if there exists no such $\alpha$.) Obviously, $X^k_i \geq L^k_i$ and $X^k_i \geq X^k_{i+1}$ hold. We will show $X^k_{i+1} \leq L^k_i$, which implies that $X^k_1, X^k_2, \ldots$ is weakly decreasing. When $k = 0$, the claim is trivial because $P^0_i = 0$ for all $i$. Assume that the claim is true for some $k \geq 0$. Therefore, we have (i) $i \neq t_k \Rightarrow L^k_{i+1} - X^k_{i+1} = L^k_i - X^k_{i+1} \geq L^k_i - X^k_{i+1}$, (ii) $(i = t_k$ and $P^k_{i+1} > 0) \Rightarrow \alpha = i + 1 = t_k + 1 \Rightarrow L^k_{i+1} - X^k_{i+1} = L^k_i - X^k_{i+1}$, and (iii) $(i = t_k$ and $P^k_{i+1} = 0) \Rightarrow P^k_{i+1} = 0 \Rightarrow L^k_{i+1} - X^k_{i+1} = L^k_i - X^k_{i+1}$. In each case, we obtain $L^k_{i+1} - X^k_{i+1} \leq 0$ by the induction hypothesis. Then $X^k_{i+1} \leq L^k_i$ is proved for all $k \geq 0$.

As $L^0_i = 0$ for all $i$, we have $X^0_i = X^0_i = \cdots = 0$. On the other hand, we have $X^1_i = L^1_i$ because any 1 cannot be bumped from the 1st row. This means $s_1, \ldots, s_q$ contains no 1’s, and the row $[a_1 \ a_2 \ \cdots \ a_p]$ satisfies $a_1 = \cdots = a_p = 1$. Moreover, because $X^1_i$ is equal to the number of $i$’s contained in $s_{p+1}, \ldots, s_q$ ($p$ is the number of numbers bumped from the 1st row in the first $k$ steps) for $i \geq 2$, and $X^1_1, X^1_2, \ldots$ is weakly decreasing, we find $s_1 - 1, s_2 - 1, \ldots, s_q - 1$ is a reverse lattice word.

Repeating this procedure, we show that the $j$th row of $T_N$ contains only $j$ for all $j$. We conclude $T_N = U(\lambda)$ with $\lambda_i = L^0_i$. \hfill \Box

Corollary 6.3 The $P$-tableau associated with any standard tableau of shape $\mu$ must be $U(\mu)$.

Proof. Let $S$ be a standard tableau of shape $\mu$ and size $N$. Assume $(N - i + 1)$ is contained in the $r$-th row of $S$. The sequence $t_1, t_2, \ldots, t_N$ should be a reverse lattice word because the subdiagram of $S$ that consists of $1, \ldots, N - i$ is still a Young diagram for any $i$. Therefore, the $P$-tableau associated with $S$ is $U(\lambda)$ for some $\lambda$. Obviously, $\lambda = \mu$. \hfill \Box

The uniqueness of a rectification is now almost trivial from the diagram (28) and Corollary 6.3.

Remark 6.4 The notion of the ‘$P$-tableau associated with a standard tableau’ is equivalent to the ‘$P$-tableau of a reverse lattice word’, which is well-known in the context of combinatorial. For example, in Fulton’s textbook [6, Section 5.3], the standard tableau whose $i$th row contains $i$ is denoted by $U(w)$ for a reverse lattice word $w = t_1 \ t_2 \ \ldots \ t_N$. Therein, the $P$-tableau associated with $U(w)$ is denoted by $P(w)$.

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Appendix A. Notes on mathematical logic

In this appendix, we shortly review a few of basic notations of mathematical logic and we give a proof of Lemma 3.5.

A.1 Basic definitions

In this section, we follow the notations in the textbooks of mathematical logic [10, 11].
Definition A1 A language $\mathcal{L}$ is a set of function symbols, relation symbols and constant symbols. Each function symbol $f$ is associated with a natural number $n_f$, and each relation symbol $R$ is associated with a natural number $n_R$.

We say that ‘$f$ is an $n_f$-ary function’ and ‘$R$ is an $n_R$-ary relation’.

Definition A2 An $\mathcal{L}$-structure $\mathcal{M}$ is a collection of the following objects:

- An non-empty set $M$, which is called the domain.
- A map $f^\mathcal{M} : M^{n_f} \rightarrow M$ for each function symbol $f \in \mathcal{L}$.
- A set $R^\mathcal{M} \subset M^{n_R}$ for each relation symbol $R \in \mathcal{L}$.
- An element $c^\mathcal{M} \in M$ for each constant symbol $c \in \mathcal{L}$.

These $f^\mathcal{M}, R^\mathcal{M}, c^\mathcal{M}$ are called an interpretation of $f, R, c$, respectively.

We often write ‘$R^\mathcal{M}(m_1, \ldots, m_n)$’ instead of ‘$(m_1, \ldots, m_n) \in R^\mathcal{M}$’.

Definition A3 Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures and $M, N$ be their domains. A map $h : M \rightarrow N$ is called a morphism of $\mathcal{L}$-structures if:

- $h(f^\mathcal{M}(m_1, \ldots, m_n)) = f^\mathcal{N}(h(m_1), \ldots, h(m_n))$ for any $m_1, \ldots, m_n \in M$,
- $h(R^\mathcal{M}(m_1, \ldots, m_n)) \Rightarrow R^\mathcal{N}(h(m_1), \ldots, h(m_n))$ for any $m_1, \ldots, m_n \in M$,
- $h(c^\mathcal{M}) = c^\mathcal{N}$ for any constant symbol $c \in \mathcal{L}$.

Definition A4 An $\mathcal{L}$-term is a sequence of constant symbols, function symbols and variables $x_1, x_2, \ldots$ that is defined recursively as follows:

- All constant symbols and variables are $\mathcal{L}$-terms.
- If $t_1, \ldots, t_n$ are $\mathcal{L}$-terms and $f$ is a $n_f$-ary function symbol, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.

Definition A5 An $\mathcal{L}$-formula is a sequence of $\mathcal{L}$-terms, $=, \neg, \land$ and $\exists$ that is defined recursively as follows:

(i) If $t_1$ and $t_2$ are $\mathcal{L}$-terms, then $t_1 = t_2$ is an $\mathcal{L}$-formula.
(ii) If $t_1, \ldots, t_n$ are $\mathcal{L}$-terms and $R$ is an $n$-ary relation symbol, then $R(t_1, \ldots, t_n)$ is an $\mathcal{L}$-formula.
(iii) If $\Psi_1$ and $\Psi_2$ are $\mathcal{L}$-formulas, then $\Psi_1 \land \Psi_2$ is an $\mathcal{L}$-formula.
(iv) If $\Psi$ is an $\mathcal{L}$-formula and $x$ is a variable, $\exists x \Psi$ is an $\mathcal{L}$-formula.
(v) If $\Psi$ is an $\mathcal{L}$-formula, then $\neg \Psi$ is an $\mathcal{L}$-formula.

The following abbreviations are often used:

- $\Psi_1 \lor \Psi_2$ denotes $\neg (\neg \Psi_1 \land \neg \Psi_2)$.
• $\Psi_1 \rightarrow \Psi_2$ denotes $\neg(\Psi_1 \land \neg \Psi_2)$.

• $\Psi_1 \leftrightarrow \Psi_2$ denotes $(\Psi_1 \rightarrow \Psi_2) \land (\Psi_2 \rightarrow \Psi_1)$.

• $\forall x \Psi$ denotes $\neg(\exists x \neg \Psi)$.

Definition A6 (Negation-free formula) An $\mathcal{L}$-formula is called negation-free\(^9\) if it is consisted of $\mathcal{L}$-terms, $\neg$, $\land$, $\lor$ and $\exists$.

A variable $x$ is said to be free if it does not occur within the scope of a quantifier $\exists x$. If an $\mathcal{L}$-formula $\varphi$ contains free variables $x_1, x_2, \ldots, x_n$, we often denote it by $\varphi(x_1, \ldots, x_n)$.

Definition A7 For an $\mathcal{L}$-structure $\mathcal{M}$, an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, and an element $(m_1, \ldots, m_n) \in M^n$, we define

$$\mathcal{M} \models \varphi(m_1, \ldots, m_n)$$

recursively as follows:

• If $t_1^\mathcal{M}(m_1, \ldots, m_n) = t_2^\mathcal{M}(m_1, \ldots, m_n)$, then $\mathcal{M} \models (t_1 = t_2)(m_1, \ldots, m_n)$.

• If $R^\mathcal{M}(t_1^\mathcal{M}(m_1, \ldots, m_n), \ldots, t_2^\mathcal{M}(m_1, \ldots, m_n))$, then $\mathcal{M} \models (R(t_1, \ldots, t_i))(m_1, \ldots, m_n)$.

• If both $\mathcal{M} \models \Psi_1(m_1, \ldots, m_n)$ and $\mathcal{M} \models \Psi_2(m_1, \ldots, m_n)$ are satisfied, then $\mathcal{M} \models (\Psi_1 \land \Psi_2)(m_1, \ldots, m_n)$.

• If there exists $a \in M$ with $\mathcal{M} \models \Psi(m_1, \ldots, a, \ldots, m_n)$, then $\mathcal{M} \models \exists x \Psi(m_1, \ldots, x, \ldots, m_n)$.

• If $\mathcal{M} \not\models \Psi(m_1, \ldots, m_n)$, then $\mathcal{M} \models \neg \Psi(m_1, \ldots, m_n)$.

If $\mathcal{M} \models \Psi(m_1, \ldots, m_n)$, we say ‘$\Psi(m_1, \ldots, m_n)$ is true over $\mathcal{M}$’.

A.2 Proof of Lemma 3.5.

Let $\mathcal{L}$ be a language, and $\mathcal{M}, \overline{\mathcal{M}}$ be $\mathcal{L}$-structures. We let $M$ and $\overline{M}$ denote the domain of $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively. Consider a morphism $\mathcal{M} \rightarrow \overline{\mathcal{M}}$ of $\mathcal{L}$-structures.

Assume $\mathcal{M} \models \psi(a_1, \ldots, a_n)$ for a negation-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n) \in M^n$. We prove $\overline{\mathcal{M}} \models \overline{\psi}(\overline{a_1}, \ldots, \overline{a}_n)$ by induction (see Definitions A5 and A6). First, for $t_1 = t_2$, we have

$$\mathcal{M} \models (t_1 = t_2)(a_1, \ldots, a_n) \Rightarrow t_1^\mathcal{M}(a_1, \ldots, a_m) = t_2^\mathcal{M}(a_1, \ldots, a_m) \Rightarrow t_1^\overline{\mathcal{M}}(\overline{a_1}, \ldots, \overline{a}_m) = t_2^\overline{\mathcal{M}}(\overline{a_1}, \ldots, \overline{a}_m) \quad (\therefore x \mapsto \overline{x} \text{ is a } \mathcal{L}\text{-morphism})$$

$$\overline{\mathcal{M}} \models (\overline{t_1} = \overline{t_2})(\overline{a_1}, \ldots, \overline{a}_n).$$

---

\(^9\) Note that our ‘negation-free formulas’ do not admit the quantifier $\forall$. 
The same argument works in the case of $R(t_1, \ldots, t_n)$. Next assume that the assertion holds for $\Psi_1(x_1, \ldots, x_n)$ and $\Psi_2(x_1, \ldots, x_n)$. Then, we have

$$
\mathcal{M} \models (\Psi_1 \land \Psi_2)(a_1, \ldots, a_n) \Rightarrow \mathcal{M} \models \Psi_1(a_1, \ldots, a_n) \quad \text{and} \quad \mathcal{M} \models \Psi_2(a_1, \ldots, a_n)
$$

$$
\Rightarrow \mathcal{M} \models \overline{\Psi}_1(a_1, \ldots, a_n) \quad \text{and} \quad \mathcal{M} \models \overline{\Psi}_2(a_1, \ldots, a_n)
$$

$$
\Rightarrow \mathcal{M} \models (\overline{\Psi}_1 \land \overline{\Psi}_2)(a_1, \ldots, a_n),
$$

and

$$
\mathcal{M} \models (\Psi_1 \lor \Psi_2)(a_1, \ldots, a_n) \Rightarrow \mathcal{M} \models (\neg \Psi_1 \land \neg \Psi_2)(a_1, \ldots, a_n)
$$

$$
\Rightarrow \mathcal{M} \models \Psi_1(a_1, \ldots, a_n) \quad \text{or} \quad \mathcal{M} \models \Psi_2(a_1, \ldots, a_n)
$$

$$
\Rightarrow \mathcal{M} \models \overline{\Psi}_1(a_1, \ldots, a_n) \quad \text{or} \quad \mathcal{M} \models \overline{\Psi}_2(a_1, \ldots, a_n)
$$

$$
\Rightarrow \mathcal{M} \models (\neg \Psi_1 \land \neg \Psi_2)(a_1, \ldots, a_n)
$$

$$
\Rightarrow \mathcal{M} \models (\overline{\Psi}_1 \lor \overline{\Psi}_2)(a_1, \ldots, a_n).
$$

Moreover,

$$
\mathcal{M} \models \exists x \Psi(x, a_2, \ldots, a_n) \Rightarrow \text{there exists some } a_1 \in M \text{ with } \mathcal{M} \models \Psi(a_1, a_2, \ldots, a_n)
$$

$$
\Rightarrow \text{there exists some } a_1 \in M \text{ with } \mathcal{M} \models \overline{\Psi}(a_1, a_2, \ldots, a_n)
$$

$$
\Rightarrow \text{there exists some } A \in \overline{M} \text{ with } \overline{\mathcal{M}} \models \overline{\Psi}(A, a_2, \ldots, a_n)
$$

$$
\Rightarrow \overline{\mathcal{M}} \models \exists X \overline{\Psi}(X, a_2, \ldots, a_n).
$$

Therefore, the assertion holds for any negation-free formula.

**Appendix B. Jeu de taquin and row insertion**

A box $B$ in a Young diagram is said to be a *corner* if there exist no boxes below nor on the right to $B$. For a skew diagram $\lambda/\mu$, a corner of $\lambda$ is called an *outside corner of $\lambda/\mu$* and a corner of $\mu$ is called an *inside corner of $\lambda/\mu$*.

A location at where no box exists is called a *hole*. For a skew tableau $T$ and an inside corner $b$, the *jeu de taquin slide starting from $b$* is defined as follows: (i) compare the two entries in the boxes below and on the right to the hole $b$, and slide a box with smaller number to $b$. If these two entries are equal, slide the box below $b$. (ii) Repeat (i) until the hole reaches to an outside corner.

The following is an example of a jeu de taquin slide.

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
2 & 3 & \hline \\
1 & 2 & 3 & 4 \\
2 & 4 & 5 \\
\end{array}
\quad
\begin{array}{c|c|c}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 3 & \hline \\
2 & 4 & 5 \\
2 & 4 & 5 \\
\end{array}
\quad
\begin{array}{c|c|c}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 3 & \hline \\
1 & 3 & \hline \\
2 & 4 & 5 \\
2 & 4 & 5 \\
\end{array}
\quad
\begin{array}{c|c|c}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 3 & \hline \\
1 & 3 & \hline \\
2 & 4 & 5 \\
2 & 4 & 5 \\
\end{array}
\]

Let $T$ be a tableau and $t$ be a number. The *row insertion* (or *row bumping*) of $t$ to $T$ is a procedure that is defined as follows: (i) If $t$ is equal to or greater than all the entries in the 1st row of $T$, put a new
box filled with $t$ at the end of this row. If not, $t$ ‘bumps’ the leftmost entry greater than $t$. The bumped number proceeds to the next row. (ii) Apply the same procedure as (i) to the next row and the bumped number. (iii) Repeat (ii) until the bumped number is put at the end of some row.

The following is an example of a row insertion, where 3 is row-inserted to a Young tableau

$\begin{array}{ccc}
1 & 3 & 4 & 5 \\
2 & 4 & 6 & 6 \\
4 & 5 \\
6 
\end{array} \leftarrow 3
\quad
\begin{array}{ccc}
1 & 3 & 3 & 5 \\
2 & 4 & 6 & 6 \\
4 & 5 \\
6 
\end{array} \leftarrow 4
\quad
\begin{array}{ccc}
1 & 3 & 3 & 5 \\
2 & 4 & 4 & 6 \\
4 & 5 \\
6 
\end{array} \leftarrow 6

The tableau obtained by the row insertion of $t$ to $T$ is denoted by $T \leftarrow t$ or $T \leftarrow \{t\}$. 

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