Abstract
Quantum theory (QT) has been confirmed by numerous experiments, yet we still cannot fully grasp the meaning of the theory. As a consequence, the quantum world appears paradoxical. Here we shed new light on QT by having it follow from two main postulates: (i) the theory should be logically consistent; (ii) inferences in the theory should be computable in polynomial time. The first postulate is what we require to each well-founded mathematical theory. The computation postulate defines the physical component of the theory. We show that the computation postulate is the only true divide between QT, seen as a generalised theory of probability, and classical probability. All quantum paradoxes, and entanglement in particular, arise from the clash of trying to reconcile a computationally intractable, somewhat idealised, theory (classical physics) with a computationally tractable theory (QT) or, in other words, from regarding physics as fundamental rather than computation.

1 Introduction
Quantum theory (QT) is one of the most fundamental, and accurate, mathematical descriptions of our physical world. It dates back to the 1920s, and in spite of nearly one century passed by since its inception, we do not have a clear understanding of such a theory yet. In particular, we cannot fully grasp the meaning of the theory: why it is the way it is. As a consequence, we cannot come to terms with the many paradoxes it appears to lead to — its so-called “quantum weirdness”.

This paper aims at finally explaining QT while giving a unified reason for its many paradoxes. We pursue this goal by having QT follow from two main postulates:

(Coherence) The theory should be logically consistent.

(Computation) Inferences in the theory should be computable in polynomial time.

The first postulate is what we essentially require to each well-founded mathematical theory, be it physical or not: it has to be based on a few axioms and rules from which we can unambiguously derive its mathematical truths. The second postulate will turn out to be central. It requires that there should be an efficient way to execute the theory in a computer.

QT is an abstract theory that can be studied detached from its physical applications. For this reason, people often wonder which part of QT actually pertains to physics. In our representation, the answer to this question shows itself naturally: the computation postulate defines the physical component of the theory. But it is actually stronger than that: it states that computation is more primitive than physics.

Let us recall that QT is widely regarded as a “generalised” theory of probability. In this paper we make the adjective “generalised” precise. In fact, our coherence postulate leads to a theory of probability, in the sense that it disallows “Dutch books”: this means, in gambling terms, that a bettor on a quantum experiment cannot be made a sure loser by exploiting inconsistencies in their probabilistic assessments.

But probabilistic inference is in general NP-hard. By imposing the additional postulate of computation, the
theory becomes one of “computational rationality”: one that is consistent (or coherent), up to the degree that polynomial computation allows. This weaker, and hence more general, theory of probability is QT.

As a result, for a subject living inside QT, all is coherent. For us, living in the classical, and somewhat idealised, probabilistic world (not restricted by the computation postulate), QT displays some inconsistencies: precisely those that cannot be fixed in polynomial time. All quantum paradoaxes, and entanglement in particular, arise from the clash of these two world views: i.e., from trying to reconcile an unrestricted theory (i.e., classical physics) with a theory of computational rationality (quantum theory). Or, in other words, from regarding physics as fundamental rather than computation.

But there is more to it. We show that the theory is “generalised” also in another direction, as QT turns out to be a theory of “imprecise” probability: in fact, requiring the computation postulate is similar to defining a probabilistic model using only a finite number of moments; and therefore, implicitly, to defining the model as the set of all probabilities compatible with the given moments. In QT, some of these compatible probabilities can actually be signed, that is, they allow for “negative probabilities”. In our setting, these have no meaning per se, they are just a mathematical consequence of polynomially bounded coherence (or rationality).

2 Results

Coherence postulate

De Finetti’s subjective foundation of probability [1] is based on the notion of rationality (consistency or coherence). This approach has then been further developed in [2, 3], giving rise to the so-called theory of desirable gambles (TDG). This is an equivalent reformulation of the well-known Bayesian decision theory (à la Anscombe-Aumann [4]) once it is extended to deal with incomplete preferences [5, 6]. In this setting probability is a derived notion in the sense that it can be inferred via mathematical duality from a set of logical axioms that one can interpret as rationality requirements in the way a subject, let us call her Alice, accepts gambles on the results of an uncertain experiment. It goes as follow.

Let \( \Omega \) denote the possibility space of an experiment (e.g., \{Head, Tail\} or \( C^n \) in QT). A gamble \( g \) on \( \Omega \) is a bounded real-valued function of \( \Omega \), interpreted as an uncertain reward. It plays the traditional role of variables or, using a physical parlance, of observables. In the context we are considering, accepting a gamble \( g \) by an agent is regarded as a commitment to receive, or pay (depending on the sign), \( g(\omega) \) utiles (abstract units of utility, we can approximately identify it with money provided we deal with small amounts of it [7, Sec. 3.2.5]) whenever \( \omega \) occurs. If by \( \mathcal{L} \) we denote the set of all the gambles on \( \Omega \), the subset of all non-negative gambles, that is, of gambles for which Alice is never expected to lose utiles, is given by \( \mathcal{L}^\geq := \{ g \in \mathcal{L} : \inf g \geq 0 \} \). Analogously, negative gambles, those gambles for which Alice will certainly lose some utiles, even an epsilon, is defined as \( \mathcal{L}^< := \{ g \in \mathcal{L} : \sup g < 0 \} \). In what follows, with \( \mathcal{G} := \{ g_1, g_2, \ldots, g_{|\mathcal{G}|} \} \subset \mathcal{L} \) we denote a finite set of gambles that Alice finds desirable (we will comment on the case when \( \mathcal{G} \) may not be finite): these are the gambles that she is willing to accept and thus commits herself to the corresponding transactions.

The crucial question is now to provide a criterion for a set \( \mathcal{G} \) of gambles representing assessments of desirability to be called rational. Intuitively Alice is rational if she avoids sure losses: that is, if, by considering the implications of what she finds desirable, she is not forced to find desirable a negative gamble. This postulate of rationality is called “no arbitrage” in economics and “no Dutch book” in the subjective foundation of probability. In TDG we formulate it thorough the notion of logical consistency which, despite the informal interpretation given above, is a purely syntactical (structural) notion. To show this, we need to define an appropriate logical calculus (characterising the set of gambles that Alice must find desirable as a consequence of having desired \( \mathcal{G} \) in the first place) and based on it to characterise the family of consistent sets of assessments.

For the former, since non-negative gambles may increase Alice’s utility without ever decreasing it, we first have that:

A0. \( \mathcal{L}^\geq \) should always be desirable.
This defines the tautologies of the calculus. Moreover, whenever \( f,g \) are desirable for Alice, then any positive linear combination of them should also be desirable (this amounts to assuming that Alice has a linear utility scale, which is a standard assumption in probability). Hence the corresponding deductive closure of a set \( \mathcal{F} \) is given by:

\[
A1. \quad \mathcal{K} := \text{posi}(\mathcal{L}^\geq \cup \mathcal{G}).
\]

Here “posi” denotes the conic hull operators. When \( \mathcal{G} \) is not finite, \( A1 \) requires in addition that \( \mathcal{K} \) is closed.

In the betting interpretation given above, a sure loss for an agent is represented by a negative gamble. We therefore say that:

**Definition 1** (Coherence postulate). A set \( \mathcal{K} \) of desirable gambles is coherent if and only if

\[
A2. \quad \mathcal{L}^\leq \cap \mathcal{K} = \emptyset.
\]

Note that \( \mathcal{K} \) is coherent if and only if \(-1 \in \mathcal{K}\); therefore \(-1 \) can be regarded as playing the role of the Falsum and \( A2 \) can be reformulated as \(-1 \notin \mathcal{K} \).

Postulate \( A2 \), which presupposes postulates \( A0 \) and \( A1 \), provides the normative definition of TDG, referred to by \( \mathcal{T} \). Based on it we derive the axioms of classical probability theory. Assume \( \mathcal{K} \) is coherent.

We give \( \mathcal{K} \) a probabilistic interpretation by observing that the mathematical dual of \( \mathcal{K} \) is a closed convex set of probabilities:

\[
\mathcal{P} = \left\{ \mu \in \mathcal{S} \mid \int_{\Omega} g(\Omega)d\mu(\Omega) \geq 0, \forall g \in \mathcal{G} \right\},
\]

where \( \mathcal{S} = \{ \mu \in \mathcal{K} \mid \inf \mu \geq 0, \int_{\Omega} d\mu(\Omega) = 1 \} \) is the set of all probabilities in \( \Omega \), and \( \mathcal{M} \) the set of all charges (a charge is a finitely additive signed-measure \([8, \text{Ch.11}]\) on \( \Omega \). We have derived the axioms of probability—a non-negative function that integrates to one—from the the coherence postulate \( A2 \). Hence, whenever an agent is coherent, Equation (1) states that desirability corresponds to non-negative expectation (for all probabilities in \( \mathcal{P} \)). When \( \mathcal{K} \) is incoherent, \( \mathcal{P} \) turns out to be empty—there is no probability compatible with the assessments in \( \mathcal{K} \). As simple as it looks, expression \( A2 \) alone therefore captures the coherence postulate as formulated in the Introduction in case of classical probability theory.

**Computation postulate**

The problem of checking whether or not \( \mathcal{K} \) is coherent can be formulated as the following decision problem:

\[
\exists \lambda_i \geq 0 : -1 - \sum_{i=1}^{\vert \mathcal{G} \vert} \lambda_i g_i \in \mathcal{L}^\geq.
\]

If the answer is “yes”, then the gamble \(-1 \) belongs to \( \mathcal{K} \), proving \( \mathcal{K} \)’s incoherence. Actually any inference task can ultimately be reduced to a problem of the form (2), as discussed in the Supplementary 2.2. Hence, the above decision problem unveils a crucial fact: the hardness of inference in classical probability corresponds to the hardness of evaluating the non-negativity of a function in the considered space (let us call this the “non-negativity decision problem”).

When \( \Omega \) is infinite (in this paper we consider the case \( \Omega \subset \mathbb{C}^n \)) and for generic functions, the non-negativity decision problem is undecidable. To avoid such an issue, we may impose restrictions on the class of allowed gambles and thus define \( \mathcal{T} \) on a appropriate subspace \( \mathcal{L}_K \) of \( \mathcal{L} \) (see Appendix B in Supplementary). For instance, instead of \( \mathcal{L} \), we may consider \( \mathcal{L}_K \); the class of multivariate polynomials of degree at most \( d \) (we denote by \( \mathcal{L}_K^+ \subset \mathcal{L}_K \) the subset of non-negative polynomials and by \( \mathcal{L}_K^- \subset \mathcal{L}_K \) the negative ones). In doing so, by Tarski-Seidenberg quantifier elimination theory \([9, 10] \), the decision problem becomes decidable, but still intractable, being in general NP-hard. If we accept the so-called “Exponential Time Hypothesis” (that \( P \neq \text{NP} \)) and we require that inference should be tractable (in P), we are stuck. What to do? A solution is to change the meaning of “being non-negative” for a function by considering a subset \( \Sigma^\geq \subset \mathcal{L}_K^- \) for which the membership problem in (2) is in P.

In other words, a computationally efficient TDG, which we denote by \( \mathcal{T}^* \), should be based on a logical redefinition of the tautologies, i.e., by stating that
B0. $\Sigma^\geq$ should always be desirable,
in the place of $A_0$. The rest of the theory can develop following the footprints of $\mathcal{T}$. In particular, the
deductive closure for $\mathcal{T}^\ast$ is defined by:

B1. $\mathcal{C} := \text{posi}(\Sigma^\geq \cup \mathcal{T})$.

And the coherence postulate, which now naturally encompasses the computation postulate, states that:

Definition 2 (P-coherence). A set $\mathcal{C}$ of desirable gambles is P-coherent if and only if

B2. $\Sigma^\geq \bigcap K = \emptyset$.

P-coherence owes its name to the fact that, whenever $\Sigma^\geq$ contains all positive constant gambles, B2 can
be checked in polynomial time by solving:

$$\exists \lambda_i \geq 0 \text{ such that } -1 - \sum_{i=1}^{[\mathcal{G}]} \lambda_i g_i \in \Sigma^\geq,$$

where $-1$ denotes the constant function $f \in \mathcal{L}_R$ such that $f(\omega) = -1$ for all $\omega \in \Omega$.

Hence, $\mathcal{T}^\ast$ and $\mathcal{T}$ have the same deductive apparatus; they just differ in the considered set of tautologies,
and thus in their (in)consistencies.

Interestingly, we can associate a “probabilistic” interpretation as before to the calculus defined by B0–B2 by computing the dual of a P-coherent set. Since $\mathcal{L}_R$ is a topological vector space, we can consider
its dual space $\mathcal{L}_R^\ast$ of all bounded linear functionals $L : \mathcal{L}_R \rightarrow \mathbb{R}$. Hence, with the additional condition that
linear functionals preserve the unitary gamble, the dual cone of a P-coherent $\mathcal{C} \subset \mathcal{L}_R$ is given by

$$\mathcal{C}^\circ = \{ L \in \mathcal{L}_R^\ast | L(g) \geq 0, \ L(1) = 1, \ \forall g \in \mathcal{C} \}.$$  \hspace{1cm} (4)

To $\mathcal{C}^\circ$ we can then associate its extension $\mathcal{C}^\ast$ in $\mathcal{M}$, that is, the set of all charges on $\Omega$ extending an
element in $\mathcal{C}^\circ$. In general however this set does not yield a classical probabilistic interpretation to $\mathcal{T}^\ast$. This is because, whenever $\Sigma^\geq \subset \mathcal{L}_R^\geq$, there are negative gambles that cannot be proved to be negative in polynomial time:

Theorem 1. Assume that $\Sigma^\geq$ includes all positive constant gambles and that it is closed (in $\mathcal{L}_R$). Let
$\mathcal{C} \subset \mathcal{L}_R$ be a P-coherent set of desirable gambles. The following statements are equivalent:

1. $\mathcal{C}$ includes a negative gamble that is not in $\Sigma^\leq$.
2. $\text{posi}(\Sigma^\geq \cup \mathcal{T})$ is incoherent, and thus $\mathcal{P}$ is empty.
3. $\mathcal{C}^\circ$ is not (the restriction to $\mathcal{L}_R$ of) a closed convex set of mixtures of classical evaluation functionals.
4. The extension $\mathcal{C}^\ast$ of $\mathcal{C}^\circ$ in the space $\mathcal{M}$ of all charges in $\Omega$ includes only charges that are not
   probabilities (they have some negative value).

Theorem 1 is the central result of this paper. It states that whenever $\mathcal{C}$ includes a negative gamble
(item 1), there is no classical probabilistic interpretation for it (item 2). The other points suggest alternatives
solutions to overcome this deadlock: either to change the notion of evaluation functional (item 3) or to
use quasi-probabilities (probability distributions that admit negative values) a means for interpreting $\mathcal{T}^\ast$
(item 4).

In what follows, we are going to show that QT can be deduced from a particular instance of the theory
$\mathcal{T}^\ast$. As a consequence, we get that the computation postulate, and in particular B0, is not only the unique
non-classical postulate of QT, regarded as a theory of probability, but also the unique reason for all its
paradoxes, which all boil down to a rephrasing of the various statements of Theorem 1 in the considered
quantum context.
QT as computational rationality

Consider first a single particle system with $n$-degrees of freedom and

$$\mathbb{C}^n := \{ x \in \mathbb{C}^n : x^\dagger x = 1 \}.$$  

We can interpret an element $\tilde{x} \in \mathbb{C}^n$ as “input data” for some classical preparation procedure. For instance, in the case of the spin-1/2 particle ($n = 2$), if $\theta = [\theta_1, \theta_2, \theta_3]$ is the direction of a filter in the Stern-Gerlach experiment, then $\tilde{x}$ is its one-to-one mapping into $\mathbb{C}^2$ (apart from a phase term). For spin greater than 1/2, the variable $\tilde{x} \in \mathbb{C}^n$ associated to the preparation procedure cannot directly be interpreted in terms only of “filter direction”. Nevertheless, at least on the formal level, $\tilde{x}$ plays the role of a “hidden variable” in our model and $\mathbb{C}^n$ of the possibility space $\Omega$. This hidden-variable model for QT is also discussed in [11, Sec. 1.7], where the author explains why this model does not contradict the existing “no-go” theorems for hidden-variables, see also Supplementary 7.3 and Supplementary 7.4.

In QT any real-valued observable is described by a Hermitian operator. This naturally imposes restrictions on the type of functions $g$ in (2):

$$g(x) = x^\dagger G x,$$

where $x \in \Omega$ and $G \in \mathcal{H}^{n \times n}$, with $\mathcal{H}^{n \times n}$ being the set of Hermitian matrices of dimension $n \times n$. Since $G$ is Hermitian and $x$ is bounded ($x^\dagger x = 1$), $g$ is a real-valued bounded function ($g(x) = \langle x | G | x \rangle$ in bra-ket notation).

More generally speaking, we can consider composite systems of $m$ particles, each one with $n_j$ degrees of freedom. The possibility space is the Cartesian product $\Omega = \times_{j=1}^m \mathbb{C}^{n_j}$ and the functions are $m$-quadratic forms:

$$g(x_1, \ldots, x_m) = (\otimes_{j=1}^m x_j) \dagger G (\otimes_{j=1}^m x_j),$$

with $G \in \mathcal{H}^{n_1 \times n_1} \times \cdots \times \mathcal{H}^{n_m \times n_m}$ and where $\otimes$ denotes the tensor product between vectors regarded as column matrices. Notice that in our setting the tensor product is ultimately a derived notion, not a primitive one (see Supplementary 7.4), as it follows by the properties of $m$-quadratic forms.

For $m = 1$ (a single particle), evaluating the non-negativity of the quadratic form $x^\dagger G x$ boils down to checking whether the matrix $G$ is Positive Semi-Definite (PSD) and therefore can be performed in polynomial time. This is no longer true for $m \geq 2$: indeed, in this case there exist polynomials of type (5) that are non-negative, but whose matrix $G$ is indefinite (it has at least one negative eigenvalue). Moreover, it turns out that problem (2) is not tractable:

**Proposition 1** ([12]). *The problem of checking the non-negativity of functions of type (5) is NP-hard for $m \geq 2$.*

What to do? As discussed previously, a solution is to change the meaning of “being non-negative” by considering a subset $\Sigma^g \subseteq \mathcal{L}^g$ for which the membership problem, and thus (2), is in $P$. For functions of type (5), we can extend the notion of non-negativity that holds for a single particle to $m > 1$ particles:

$$\Sigma^g := \{ g(x_1, \ldots, x_m) = (\otimes_{j=1}^m x_j) \dagger G (\otimes_{j=1}^m x_j) : G \geq 0 \}.$$  

That is, the function is “non-negative” whenever $G$ is PSD (note that $\Sigma^g$ is the so-called cone of *Hermitian sum-of-squares* polynomials, see Supplementary 7.2, and that, in $\Sigma^g$ the non-negative constant functions have the form $g(x_1, \ldots, x_m) = c (\otimes_{j=1}^m x_j) \dagger I (\otimes_{j=1}^m x_j)$ with $c \geq 0$). Now, consider any set of desirable gambles $\mathcal{G}$ satisfying B0–B2 with the given definition of $\Sigma^g$: Eureka! We have just derived the first postulate of QT (see Postulate 1 in [13, p. 110]). Indeed, let $\mathcal{F}$ be a finite set of assessments, and $\mathcal{K}$ the deductive closure as defined by B1; it is not difficult to prove that the dual of $\mathcal{K}$ is

$$\mathcal{L} = \{ \rho \in \mathcal{S} | Tr(G\rho) \geq 0, \ \forall g \in \mathcal{F} \},$$

where $\mathcal{S} = \{ \rho \in \mathcal{H}^{n \times n} | Tr(\rho) \geq 0, \ Tr(\rho) = 1 \}$ is the set of all density matrices. As before, whenever the set $\mathcal{G}$ representing Alice’s beliefs about the experiment is coherent, Equation (6) means that desirability implies non-negative “expected value” for all models in $\mathcal{L}$. Note that in QT the expectation of $g$ is $Tr(G\rho)$. This
follows by Born’s rule, a law giving the probability that a measurement on a quantum system will yield a given result. The agreement with Born’s rule is an important constraint in any alternative axiomatisation of QT. Our theory agrees with it although this is a derived notion in our setting. In fact, in the view of a density matrix as a dual operator, \( \rho \) is formally equal to

\[
\rho = L\left( (\otimes_{j=1}^m x_j) (\otimes_{j=1}^m x_j)^\dagger \right),
\]

with \( L \) defined in (3). Hence, when a projection-valued measurement characterised by the projectors \( \Pi_1, \ldots, \Pi_n \) is considered, then

\[
L((\otimes_{j=1}^m x_j)^\dagger \Pi_i (\otimes_{j=1}^m x_j)) = \text{Tr}(\Pi_i L((\otimes_{j=1}^m x_j) (\otimes_{j=1}^m x_j)^\dagger)) = \text{Tr}(\Pi_i \rho).
\]

Since \( \Pi_i \geq 0 \) and the polynomials \((\otimes_{j=1}^m x_j)^\dagger \Pi_i (\otimes_{j=1}^m x_j)\) for \( i = 1, \ldots, n \) form a partition of unity, i.e.:

\[
\sum_{i=1}^n (\otimes_{j=1}^m x_j)^\dagger \Pi_i (\otimes_{j=1}^m x_j) = (\otimes_{j=1}^m x_j)^\dagger I (\otimes_{j=1}^m x_j) = 1,
\]

we have that

\[
\text{Tr}(\Pi_i \rho) \in [0, 1] \text{ and } \sum_{i=1}^n \text{Tr}(\Pi_i \rho) = 1.
\]

For this reason, \( \text{Tr}(\Pi_i \rho) \) is usually interpreted as a probability. But the projectors \( \Pi_i \)'s are not indicator functions, whence, strictly speaking, the traditional interpretation is incorrect. This can be seen clearly in the special case where postulates A0 and B0 coincide, as in the case of a single particle, that is, where the theory can be given a classical probabilistic interpretation, see Supplementary 7.3. In such a case, the corresponding \( \rho \) is just a (truncated) moment matrix, i.e., one for which there is at least one probability such that \( E[(\otimes_{j=1}^m x_j)(\otimes_{j=1}^m x_j)^\dagger] = \rho \). In summary, our standpoint here is that \( \text{Tr}(\Pi_i \rho) \) should rather be interpreted as the expectation of the \( m \)-quadratic form \((\otimes_{j=1}^m x_j)^\dagger \Pi_i (\otimes_{j=1}^m x_j)\). This makes quite a difference with the traditional interpretation since in our case there can be (and usually there will be) more than one charge compatible with such an expectation, as we will point out more precisely later on.

Hence again, since both B1,B2 and A1,A2 are the same logical postulates parametrised by the appropriate meaning of “being negative/non-negative”, the only axiom truly separating classical probability theory from the quantum one is B0 (with the specific form of \( \Sigma^E \)), thus implementing the requirement of computational efficiency.

What about the other postulates and rules of QT: Lüders rule (measurement updating) and Schrödinger rule (time evolution)? In Supplementary 7.5, by providing a connection between the present work and [14], we show that also these postulates follow consistently from B0–B2.

**Entanglement**

Entanglement is usually presented as a characteristic of QT. In this section we are going to show that it is actually an immediate consequence of computational tractability, meaning that entanglement phenomena are not confined to QT but can be observed in other contexts too. An example of a non-QT entanglement is provided in Supplementary 6.

To illustrate the emergence of entanglement from P-coherence, we verify that the set of desirable gambles whose dual is an entangled density matrix \( \rho_e \) includes a negative gamble that is not in \( \Sigma^e \), and thus, although being logically coherent, it cannot be given a classical probabilistic interpretation.

In what follows we focus only on bipartite systems \( \Omega_A \times \Omega_B \), with \( n = m = 2 \). The results are nevertheless general.

Let \((x, y) \in \Omega_A \times \Omega_B \), where \( x = [x_1, x_2]^T \) and \( y = [y_1, y_2]^T \). We aim at showing that there exists a gamble \( h(x, y) = (x \otimes y)^\dagger H(x \otimes y) \) satisfying:

\[
\text{Tr}(H \rho_e) \geq 0 \quad \text{and} \quad h(x, y) = (x \otimes y)^\dagger H(x \otimes y) < 0 \text{ for all } (x, y) \in \Omega_A \times \Omega_B.
\]
By B0–B2, the inequalities in (8) imply that \( \rho \). The first inequality says that \( h \). The second inequality says that \( h \) is negative and, therefore, leads to a sure loss in \( \mathcal{F} \). By B0–B2, the inequalities in (8) imply that \( H \) must be an indefinite Hermitian matrix.

Assume that \( n = m = 2 \) and consider the entangled density matrix:

\[
\rho_e = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \end{bmatrix},
\]

and the Hermitian matrix:

\[
H = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & -2.0 & 1.0 & 0.0 \\
0.0 & 1.0 & -2.0 & 0.0 \\
1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.
\]

This matrix is indefinite (its eigenvalues are \{1, -1, -1, -3\}) and is such that \( Tr(H \rho_e) = 1 \). Since \( Tr(H \rho_e) \geq 0 \), the gamble

\[
(x \otimes y)^\dagger H (x \otimes y) = -2x_1y_1^2 + x_1y_1^2 + x_1y_1^2 + x_1y_1^2 - 2x_2y_1y_1^2,
\]

is desirable for Alice in \( \mathcal{F}^* \).

Let \( x_i = x_{ia} + i_x ib \) and \( y_j = y_{ja} + i y_{jb} \) with \( x_{ia}, x_{ib}, y_{ja}, y_{jb} \in \mathbb{R} \), for \( i = 1, 2 \), denote the real and imaginary components of \( x, y \). Then

\[
(x \otimes y)^\dagger H (x \otimes y) = -2x_1^2y_1^2 - 2x_2^2y_2^2 + 4x_1y_1y_2y_2 + 4x_1y_1y_1y_2 - 2x_2y_1y_2y_2 + 4x_1y_2y_1y_2
\]

\[
= - (\sqrt{2}x_{ia}y_{ia} - \sqrt{2}x_{ia}y_{ia} - \sqrt{2}y_{ia}y_{ia})^2 - (\sqrt{2}x_{ia}y_{ia} - \sqrt{2}y_{ia}y_{ia} - \sqrt{2}y_{ia}y_{ia})^2 < 0.
\]

This is the essence of the quantum puzzle: \( \mathcal{E} \) is P-coherent but (Theorem 1) there is no \( \mathcal{F} \) associated to it and therefore, from the point of view of a classical probabilistic interpretation, it is not coherent (in any classical description of the composite quantum system, the variables \( x, y \) appear to be entangled in a way unusual for classical subsystems).

As previously mentioned, there are two possible ways out from this impasse: to claim the existence of either non-classical evaluation functionals or of negative probabilities. Let us examine them in turn.

1. **Existence of non-classical evaluation functionals:** From an informal betting perspective, the effect of a quantum experiment on \( h(x, y) \) is to evaluate this polynomial to return the payoff for Alice. By Theorem 1, there is no compatible classical evaluation functional, and thus in particular no value of the variables \( x, y \in \Omega_A \times \Omega_B \), such that \( h(x, y) = 1 \). Hence, if we adopt this point of view, we have to find another, non-classical, explanation for \( h(x, y) = 1 \). The following evaluation functional, denoted as \( ev(\cdot) \), may do the job:

\[
ev \begin{bmatrix} x_1y_1 \\
x_2y_1 \\
x_1y_2 \\
x_2y_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\
0 \\
0 \\
\frac{-\sqrt{2}}{2} \end{bmatrix},
\]

which implies \( ev((x \otimes y)^\dagger H (x \otimes y)) = 1 \).

Note that, \( x_1y_1 = \frac{\sqrt{2}}{2} \) and \( x_2y_1 = 0 \) together imply that \( x_2 = 0 \), which contradicts \( x_2y_2 = \frac{\sqrt{2}}{2} \). Similarly, \( x_2y_2 = \frac{\sqrt{2}}{2} \) and \( x_1y_2 = 0 \) together imply that \( x_1 = 0 \), which contradicts \( x_1y_1 = \frac{\sqrt{2}}{2} \). Hence, as expected, the above evaluation functional is non-classical. It amounts to assigning a value to the products \( x_iy_j \) but not to the single components of \( x \) and \( y \) separately. Quoting [11, Supplement 3.4], “entangled states are holistic entities in which the single components only exist virtually”.
(2) Existence of negative probabilities: Negative probabilities are not an intrinsic characteristic of QT. They appear whenever one attempts to explain QT “classically” by looking at the space of charges on Ω. To see this, consider \( \rho_c \) and assume that, based on (11), one calculates:

\[
\int \prod_{i=1}^{9} \left(\int \delta_{\left\{ (x^{(i)}, y^{(i)}) \right\}}(x, y) \right) d\mu(x, y) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Because of Theorem 1, there is no probability charge \( \mu \) satisfying these moment constraints, the only compatible being signed ones. Box 1 reports the nine components and corresponding weights of one of them:

\[
\mu(x, y) = \sum_{i=1}^{9} w_i \delta_{\left\{ (x^{(i)}, y^{(i)}) \right\}}(x, y) \quad \text{with} \quad \sum_{i=1}^{9} w_i = 1
\]

Note that some of the weights are negative but \( \sum_{i=1}^{9} w_i = 1 \), meaning that we have an affine combination of atomic charges (Dirac’s deltas).

**Box 1: charge compatible with (11)**

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| \( x \) | 0.0082 | 0.0565 | 0.0564 | 0.1084 | 0.1254 |
| \( y \) | 0.1285 | 0.1576 | 0.0816 | 0.0827 | 0.0672 |
| \( \lambda \) | 0.0205 | 0.0131 | 0.0080 | 0.0087 | 0.0061 |

The table reports the components of a charge \( \sum_{i=1}^{9} w_i \delta_{\left\{ (x^{(i)}, y^{(i)}) \right\}}(x, y) \) that satisfies (11). The \( i \)-th column of the row denoted as \( x \) (resp. \( y \)) corresponds to the element \( x^{(i)} \) (resp. \( y^{(i)} \)). The \( i \)-th column of the vector \( \lambda \) corresponds to \( w_i \).

Consider for instance the first monomial \( x_1 x_1 y_1 y_1 \) in (11), its expectation w.r.t. the above charge is

\[
\int x_1 x_1 y_1 y_1 \left( \sum_{i=1}^{9} w_i \delta_{\left\{ (x^{(i)}, y^{(i)}) \right\}}(x, y) \right) d\mu = \sum_{i=1}^{9} w_i x_1^{(i)} y_1^{(i)} y_1^{(i)} y_1^{(i)}
\]

The charge described in Box 1 is one among the many that satisfy (11) and has been derived numerically. Explicit procedure for constructing such negative-probability representations have been developed in [15, 16].

Again, we want to stress that the two above paradoxical interpretations are a consequence of Theorem 1, and therefore can emerge when considering any instance of a theory of P-coherence in which the hypotheses of this result hold.

**Local realism**

The issue of *local realism* in QT arises when one performs measurements on a pair of separated but entangled particles. This again shows the impossibility of a peaceful agreement between the internal logical
in Box 2. In this interpretation, the CHSH experiment is an CHSH-like experiment, which aims at experimentally reproducing a situation where (8) holds, as explained

We can now ask Nature, by means of a real experiment, to decide between our common-sense notions of how a meaning, i.e., they can be explained through probabilistic mixtures of classical evaluation functionals. We

by summing up some of the components of the matrix (13), we can recover the marginal linear operator

where the last equality holds when \( L((x \otimes y)(x \otimes y)) = \rho_c \), i.e., \( M_i = \frac{1}{2} I_2 \) is the reduced density matrix of \( \rho_c \) on system \( B \). The operation we have just described, when applied to a density matrix, is known in QT as partial trace. Given the interpretation of \( \rho \) as a dual operator, the operation of partial trace simply follows by Equation (14).

Similarly, we can obtain

Matrix \( M_i \) (analogously to \( M_e \)) is compatible with probability: there are marginal probabilities whose \( M_e \) is the moment matrix, an example being

In other words, we are brought to believe that marginally the physical properties \( x, y \) of the two particles have a meaning, i.e., they can be explained through probabilistic mixtures of classical evaluation functionals. We can now ask Nature, by means of a real experiment, to decide between our common-sense notions of how the world works, and Alice’s one. Experimental verification of this phenomenon can be obtained by a CHSH-like experiment, which aims at experimentally reproducing a situation where (8) holds, as explained in Box 2. In this interpretation, the CHSH experiment is an entanglement witness, we discuss the connection between (8) and the entanglement witness theorem in Section 2.1.

Box 2: CHSH experiment
The source produces pairs of entangled photons, sent in opposite directions. Each photon encounters a two-channel polariser whose orientations can be set by the experimenter. Emerging signals from each channel are captured by detectors. Four possible orientations $\alpha_i, \beta_j$ for $i, j = 1, 2$ of the polarisers are tested. Consider the Hermitian matrices $G_{\alpha_i} = \sin(\alpha_i) \sigma_i + \cos(\alpha_i) \sigma_2$, $G_{\beta_j} = \sin(\beta_j) \sigma_i - \cos(\beta_j) \sigma_2$, where $\sigma_i, \sigma_2$ are the 2D Pauli’s matrices, and define the gamble

$$(x^1 G_{\alpha_i}(x) y^1 G_{\beta_j}(y) = (x \otimes y)^1 G_{\alpha_i \beta_j}(x \otimes y)$$

on the result of experiment with $G_{\alpha_i \beta_j} = G_{\alpha_i} \otimes G_{\beta_j}$. Consider then the sum gamble

$$h(x, y) = (x \otimes y)^1 (G_{\alpha_i} \otimes (G_{\beta_j} - G_{\beta_0})) (x \otimes y) + (x \otimes y)^1 (G_{\alpha_i} \otimes (G_{\beta_i} + G_{\beta_0})) (x \otimes y)$$

and observe that

$$h(x, y) = (x \otimes y)^1 (G_{\alpha_i} \otimes (G_{\beta_j} - G_{\beta_0})) (x \otimes y) + (x \otimes y)^1 (G_{\alpha_i} \otimes (G_{\beta_i} + G_{\beta_0})) (x \otimes y)$$

$$\leq y^1 (G_{\beta_i} + G_{\beta_0}) y$$

$$\leq 2,$$

this is the CHSH inequality. For a small $\epsilon > 0$, we have that

$$h(x, y) - 2 - \epsilon = (x \otimes y)^1 ((-2 + \epsilon) I_4 + G_{\alpha_i \beta_i} - G_{\alpha_i \beta_0} + G_{\alpha_i \beta_i} + G_{\alpha_i \beta_0}) (x \otimes y) < 0$$

but

$$Tr(((-2 + \epsilon) I_4 + G_{\alpha_i \beta_i} - G_{\alpha_i \beta_0} + G_{\alpha_i \beta_i} + G_{\alpha_i \beta_0}) \rho_x) = -2 + \epsilon + 2\sqrt{2} \geq 0$$

for $\alpha_i = \pi/2, \alpha_0 = 0, \beta_0 = \pi/4, \beta_i = -\pi/4$. We are again in a situation like (8). The experiment in the figure certifies the entanglement by measuring the QT expectation of the four components of $h(x, y)$.

The situation we have just described is the playground of Bell’s theorem, stating the impossibility of Einstein’s postulate of local realism.

The argument goes as follows. If we assume that the physical properties $x, y$ of the two particles (the polarisations of the photons) have definite values $\tilde{x}, \tilde{y}$ that exist independently of observation (realism), then the measurement on the first qubit must influence the result of the measurement on the second qubit. Vice versa if we assume locality, then $x, y$ cannot exist independently of the observations. To sum up, a local hidden variable that is compatible with QT results cannot exist [17].

But is there really anything contradictory here? The message we want to convey is that this is not the case. Indeed, since Theorem 1 applies, there is no probability compatible with the moment matrix $\rho_x$. Ergo, although they may seem to be compatible with probabilities, the marginal matrices $M_x, M_y$ are not moment matrices of any probability.

The conceptual mistake in the situation we are considering is to forget that $M_x, M_y$ come from $\rho_x$. A joint linear operator uniquely defines its marginals but not the other way round. There are infinitely many joint probability charges whose $M_x, M_y$ are the marginals, e.g.,

$$d\mu(x, y) = \left(\frac{1}{2} \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)(x) \frac{1}{2} \delta \begin{pmatrix} 0 \\ 1 \end{pmatrix} (x) + \frac{1}{2} \delta \begin{pmatrix} 1 \\ 1 \end{pmatrix} (y) + \frac{1}{2} \delta \begin{pmatrix} 0 \\ 1 \end{pmatrix} (y),$$

but none of them satisfy Equation (11). In fact, such an intrinsic non-uniqueness of the compatible joints is another amazing characteristic of QT: from this perspective, it is not only a theory of probability, but a theory of imprecise probability, see Supplementary 4.3 and Supplementary 7.1.

Instead, the reader can verify that the charge in Equation (12) satisfies both Equation (11) and:

$$\int \frac{x_1 x_2 y_1 y_2}{x_1 y_1 x_2 y_2} d\mu(x, y) = \int \frac{y_1 x_2 y_2}{x_1 y_1} d\mu(x, y) = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The take-away message of this subsection is that we should only interpret $M_x, M_y$ as marginal operators and keep in mind that QT is a logical theory of P-coherence. We see paradoxes when we try to force a physical interpretation upon QT, whose nature is instead computational. In other words, if we accept that computation is more primitive than our classical interpretation of physics, all paradoxes disappear.
2.1 Entanglement witness theorem

In the previous Subsections, we have seen that all paradoxes of QT emerge because of disagreement between its internal coherence and the attempt to force on it a classical coherent interpretation.

Do quantum and classical probability sometimes agree? Yes they do, but when at play there are density matrices $\rho$ such that Equation (8) does not hold, and thus in particular for separable density matrices. We make this claim precise by providing a link between Equation (8) and the entanglement witness theorem [18, 19].

We first report the definition of entanglement witness [20, Sec. 6.3.1]:

**Definition 3** (Entanglement witness). A Hermitian operator $W \in \mathcal{H}^{n_1 \times n_2}$ is an entanglement witness if and only if $W$ is not a positive operator but $(x_1 \otimes x_2)^\dagger W (x_1 \otimes x_2) \geq 0$ for all vectors $(x_1, x_2) \in \Omega_1 \times \Omega_2$.

The next well-known result (see, e.g., [20, Theorem 6.39, Corollary 6.40]) provides a characterisation of entanglement and separable states in terms of entanglement witness.

**Proposition 2.** A state $\rho_e$ is entangled if and only if there exists an entanglement witness $W$ such that $Tr(\rho_e W) < 0$. A state is separable if and only if $Tr(\rho, W) \geq 0$ for all entanglement witnesses $W$.

Assume that $W$ is an entanglement witness for $\rho_e$ and consider $W' = -W$. By Definition 3 and Proposition 2, it follows that

$$Tr(\rho_e W') > 0 \text{ and } (x_1 \otimes x_2)^\dagger W' (x_1 \otimes x_2) \leq 0. \tag{15}$$

The first inequality states that the gamble $(x_1 \otimes x_2)^\dagger W' (x_1 \otimes x_2)$ is strictly desirable for Alice (in theory $\mathcal{F}^*$) given her belief $\rho_e$. Since the set of desirable gambles (B1) associated to $\rho_e$ is closed, there exists $\varepsilon > 0$ such that $W'' = W' - \varepsilon I$ is still desirable, i.e., $Tr(\rho_e W'') \geq 0$ and

$$(x_1 \otimes x_2)^\dagger W'' (x_1 \otimes x_2) = (x_1 \otimes x_2)^\dagger W' (x_1 \otimes x_2) - \varepsilon < 0,$$

where we have exploited that $(x_1 \otimes x_2)^\dagger \varepsilon I (x_1 \otimes x_2) = \varepsilon$. Therefore, (15) is equivalent to

$$Tr(\rho_e W'') \geq 0 \text{ and } (x_1 \otimes x_2)^\dagger W'' (x_1 \otimes x_2) < 0, \tag{16}$$

which is the same as (8).

Hence, by Theorem 1, we can equivalently formulate the entanglement witness theorem as an arbitrage/Dutch book:

**Theorem 2.** Let $\mathcal{G} = \{g_1, \ldots, g_m\} = (\otimes_{j=1}^m x_j)^\dagger G (\otimes_{j=1}^m x_j) \mid Tr(G \hat{\rho}) \geq 0\}$ be the set of desirable gambles corresponding to some density matrix $\hat{\rho}$. The following claims are equivalent:

1. $\hat{\rho}$ is entangled;
2. $\text{posi}(\mathcal{G} \cup \mathcal{L}^\succeq)$ is not coherent in $\mathcal{F}$.

This result provides another view of the entanglement witness theorem in light of P-coherence. In particular, it tells us that the existence of a witness satisfying Equation (15) boils down to the disagreement between the classical probabilistic interpretation and the theory $\mathcal{F}^*$ on the rationality (coherence) of Alice, and therefore that whenever they agree on her rationality it means that $\rho_e$ is separable.

This connection explains why the problem of characterising entanglement is hard in QT: it amounts to proving the negativity of a function, which is NP-hard.

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1In [20, Sec. 6.3.1], the last part of this definition says “for all factorized vectors $x_1 \otimes x_2$”. This is equivalent to considering the pair $(x_1, x_2)$. 

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11
3 Discussions

Since its foundation, there have been two main ways to explain the differences between QT and classical probability. The first one, that goes back to Birkhoff and von Neumann [21], explains this differences with the premise that, in QT, the Boolean algebra of events is taken over by the “quantum logic” of projection operators on a Hilbert space. The second one is based on the view that the quantum-classical clash is due to the appearance of negative probabilities [22, 23].

Recently, there has been a research effort, the so-called “quantum reconstruction”, which amounts to trying to rebuild the theory from more primitive postulates. The search for alternative axiomatisations of QT has been approached following different avenues: extending Boolean logic [21, 24, 25], using operational primitives [26, 27, 28, 29], using information-theoretic postulates [28, 30, 31, 32, 33, 34, 35, 36], building upon the subjective foundation of probability [37, 38, 39, 40, 41, 42, 43, 44, 45, 14, 46] and starting from the phenomenon of quantum nonlocality [28, 31, 32, 47, 48].

A common trait of all these approaches is that of regarding QT as a generalised theory of probability. But why is probability generalised in such a way, and what does it mean? We have shown that the answer to this question rests in the computational intractability of classical probability theory contrasted to the polynomial-time complexity of QT.

Note that there have been previous investigations into the computational nature of QT but they have mostly focused on topics of undecidability (these results are usually obtained via a limiting argument, as the number of particles goes to infinity, see, e.g., [49]; this does not apply to our setting as we rather take the stance that the Universe is a finite physical system) and of potential computational advantages of non-standard theories involving modifications of quantum theory [50, 51, 52, 53].

The key postulate that separates classical probability and QT is B0: the computation postulate. Because of B0, Theorem 1 applies and thus the “weirdness” of QT follows: negative probabilities, existence of non-classical evaluation functionals and, therefore, irreconcilability with the classical probabilistic view. The formulation of Theorem 1 points to the fact that there are three possible ways out to provide a theoretical foundation of QT: (1) redefining the notion of evaluation functionals (algebra of the events), which is the approach adopted within the Quantum Logic [24, Axiom VII]; (2) the algebra of the events is classical but probabilities are replaced by quasi-probabilities (allowing negative values), see for instance [15, 54]; (3) the quantum-classical contrast has a purely computational character. The last approach starts by accepting P ≠ NP to justify the separation between the microscopic quantum system and the macroscopic world. We quote Aaronson [55]:

... while experiment will always be the last appeal, the presumed intractability of NP-complete problems might be taken as a useful constraint in the search for new physical theories.

The postulate of computational efficiency embodied by B0 (through Σ2) may indeed be the fundamental law in QT, similar to the second law of thermodynamics, or the impossibility of superluminal signalling.

A Supplementary

The Supplementary Material of this manuscript can be found at https://arxiv.org/abs/1902.03513

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